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Applications de la théorie des jeux à champ moyen

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À ma mère et à toute ma famille.

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CHAPTER 1

Introduction générale

La théorie des jeux de champ moyen est une branche nouvelle de la théorie des jeux introduite par Lasry and Lions 2006a, 2006b et par Huang et al. 2006, 2007. Cette théorie s'attache à l'analyse du comportement limite des jeux différentiels stochastiques impliquant un grand nombre de petits joueurs rationnels, chaque joueur ayant très peu d'influence sur les stratégies utilisées par les autres. L'équilibre de champ moyen est défini comme la limite de l'équilibre de Nash d'un jeu différentiel à N joueurs. Afin de formaliser le comportement d'un continuum d'agents rationnels, la théorie des jeux à champ moyen s'inspire des méthodes de la physique statistique dans la modélisation de l'interaction d'un grand nombre de particules. Les physiciens considèrent chaque particule comme étant influencée par un "champ moyen" exercé par toutes les autres particules, tout en prenant en compte l'influence de chaque particule sur le champ moyen.

Appliquée en économie, cette théorie suppose que chaque agent est influencé par le champ moyen fait de la distribution du comportement des autres joueurs et considère les conséquences de chaque décision individuelle sur ce champ moyen.

Dans un jeu champ moyen standard, la dynamique du système est régi par deux équations : une équation de Hamilton-Jacobi-Bellman rétrograde décrivant la stratégie optimale des agents la répartition des autres joueurs et une équation de transport de type Kolmogorov qui décrit l'évolution de la distribution de la population en prenant en compte l'influence de chaque joueur sur le champ moyen. L'équilibre de Nash du jeu apparaît comme la solution de ces deux équations.

Le but de cette thèse est de présenter des applications économiques de cette nouvelle théorie.

Dans le deuxième chapitre de cette thèse, nous présentons formellement la théorie des jeux à champ moyen. Nous commençons par les jeux statiques et traitons le cas dérivant d'un potentiel où la résolution est particulièrement simple. Nous décrivons ensuite le cadre dynamique et mettons en évidence les propriétés de cette classe de jeux.

Dans le troisième chapitre, nous nous intéressons à la diffusion des logiciels dans un environnement marqué à la fois par des pratiques de piratage et la montée des logiciels libres. Le choix des utilisateurs évolue ainsi entre trois types de logiciels : propriétaires, libres et piratés. Nous étudions les stratégies optimales de l'éditeur du logiciel propriétaire et leurs influences sur les décisions des utilisateurs des logiciels. Ce modèle n'est pas à proprement parler de type champ moyen du fait de la rationalité limitée des agents. Seul l'éditeur du logiciel propriétaire est considéré rationnel vu ses connaissances du marché qui lui permettent d'anticiper son évolution.

Dans le quatrième chapitre, nous proposons un modèle de stationnement urbain avec une population de consommateurs hétérogènes. Dans ce modèle statique à deux états, chaque consommateur, suivant sa propension à payer, choisit entre le stationnement sur rue ou le stationnement dans un garage. Nous évaluons différentes politiques publiques concernant les tarifs et la durée de stationnement.

Le cinquième chapitre développe un modèle dynamique à champ moyen de compétition entre deux paradigmes scientifiques en se basant sur la théorie Kuhnienne des révolutions scientifiques. La dynamique du modèle est guidée par le choix scientifique des jeunes chercheurs au début de leur carrière. Selon les valeurs initiales des paramètres, le modèle présente un ou deux équilibres stables. Dans chaque équilibre, les deux paradigmes coexistent toujours, un paradigme est dominant et attire la plupart des chercheurs. Le changement de paradigme apparaît comme la conséquence de deux types de chocs imprévisibles. Un choc sur les fonctions de production scientifiques peut modifier les interactions entre les paradigmes et favoriser un paradigme au détriment de l'autre. De même, les politiques publiques peuvent favoriser le changement de paradigme en offrant des incitations temporaires aux jeunes chercheurs.

CHAPTER 2

Présentation des jeux à champ moyen

2.1 Introduction

Nous présentons dans ce chapitre de manière formelle la théorie des jeux à champ moyen et rappelons quelques résultats d'existence et d'unicité des équilibres de Nash. Nous commençons par les jeux statiques en détaillant la résolution du cas dérivant d'un potentiel. Nous exposons, dans un deuxième temps, les jeux dynamiques et le système d'équations qui caractérisent l'équilibre de Nash. Nous nous inspirons dans cet exposé de Cardaliaguet 2010 et du cours de Pierre Louis Lions au collège de France.

2.2 Jeux à champ moyen statiques

Nous considérons dans cette section un jeu avec un grand nombre N de joueurs. Les joueurs vivent une seule période pendant laquelle ils doivent choisir une stratégie $x \in Q$ où Q est un ensemble compact de $\mathbb{R}$. Notons $F_i^N = F_i^N(x_1, ..., x_N)$ le coût de chaque joueur $i \in 1, ...N$. Nous supposons que les joueurs sont symétriques ou indistinguables, cela se traduit mathématiquement par la relation suivante :

$$F^N_{\sigma(i)}(x_{\sigma(1)}, ..., x_{\sigma(N)}) = F_i(x_1, ..., x_N)$$

pour toute permutation σ sur l'ensemble $\{1, ..., N\}$.

Cette propriété de symétrie nous permet d'étudier et de caractériser les équilibres de Nash pour un très grand nombre de joueurs.

2.2.1 Résultats généraux

Avant d'étudier le comportement des équilibres de Nash pour N tendant vers l'infini, nous allons examiner le comportement asymptotique de ces fonctions de coût symétriques.

Definition. *Une fonction U est dite symétique si pour toute permutation σ sur l'ensemble $\{1, ..., N\}$, on a :*

$$U(x_{\sigma(1)}, ..., x_{\sigma(N)}) = U(x_1, ..., x_N)$$

Theorem 1. *Soit $(U_N)_{N\in\mathbb{N}}$ une suite de fonctions symétriques. Supposons que les fonctions U_N sont bornées et uniformément continues alors il existe une fonction continue et définie sur l'ensemble des mesures sur l'espace Q, $U : \mathcal{P}(Q) \to \mathbb{R}$ telle que*

$$\lim_{N\to\infty} \sup_{X\in Q^N} |U_N(X) - U(m^N_X)| = 0$$

où $m^N_X = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ est la mesure empirique associée à la stratégie des joueurs.

Ce théorème permet de simplifier considérablement l'écriture des fonctions F^N_i à la limite sous la forme $F(x, m)$ où m est la densité des joueurs.

Le théorème caractérise les équilibres de Nash quand N tend vers l'infini.

Theorem 2. *Supposons que, pour tout N, $X^N = (\bar{x}^N_1, ..., \bar{x}^N_N)$ soit un équilibre de Nash pour le jeu $F^N_1, ..., F^N_N$. Alors la suite de mesures empiriques $\bar{m}^N$ converge vers une limite $\bar{m} \in P(Q)$ qui vérifie*

$$\int_Q F(y, \bar{m}) d\bar{m}(y) = \inf_{m \in \mathcal{P}(Q)} \int_Q F(y, \bar{m}) dm(y) \tag{2.1}$$

Cette équation de champ moyen montre que la distribution d'équilibre $\bar{m}$ ne charge que les points de minimum de la fonction de coût $F(y, \bar{m})$.

L'unicité de la mesure $\bar{m}$ est assurée si la fonction de coût F vérifie la condition de monotonicité suivante

Theorem 3. *Supposons que la fonction F vérifie la condition de monotonie suivante :*

$$\int_Q (F(y, m_1) - F(y, m_2) d(m_1 - m_2)(y) > 0 \tag{2.2}$$

pour toutes mesures m_1 et m_2 définies sur l'espace Q, alors il existe au plus une mesure qui vérifie l'équation 2.1.

2.2.2 Un exemple : Le cas dérivant d'un potentiel

La résolution du jeu à champ moyen et spécifiquement de l'équation 2.1 devient particulièrement simple dans le cas dérivant d'un potentiel. Cela correspond au cas où la fonction F s'écrit $F(x, m) = \Phi'(m)(x)$. Prenons l'exemple d'un coût sous la forme $F(x, m) = V(x) + G(m(x))$. Chaque joueur cherche à minimiser son coût $F(x, m)$. La fonction composante V est liée à la position du joueur et exprime ses préférences géographiques alors que la deuxième composante, G, exprime les préférences vis-à-vis de la répartition totale des joueurs (attirance ou aversion à la foule). La résolution du problème revient à trouver la répartition optimale $\bar{m}$.

Dans ce cas, l'équation 2.1 s'écrit

$$\int_Q F(x, \bar{m}) dm \geq \int_Q F(x, \bar{m}) d\bar{m}, \text{ pour tout distribution m}$$

ce qui équivaut à

$$\int_Q \Phi'(x,\bar{m})(m-\bar{m}) \geq 0$$

L'équation de champ moyen 2.1 montre que la distribution $\bar{m}$ ne charge que les points de minimum de $V(x)+G(m(x))$. Par conséquent, nous avons :

$$V(x)+G(\bar{m}(x)) \geq \lambda, \text{ pour tout } x \in Q$$
$$V(x)+G(\bar{m}(x)) = \lambda, \text{ pour tout } x \in supp(\bar{m})$$

où $\lambda = \min_y V(y)+G(\bar{m}(y))$ et vérifie $\int_Q d\bar{m} = 1$

Nous pouvons en déduire la forme de la distribution $\bar{m}$ à l'équilibre:

$$\bar{m}(x) = G^{-1}((\lambda - V(x))_+)$$

Dans le cas particulier où

– $V(x)=x^2$ et $G(m)=cm$ avec $c>0$, la solution est

$$m^*(x) = \frac{(\lambda - x^2)_+}{c}$$

– $V(x)=x^2$ et $G(m)=log(m)$, la solution est

$$m^*(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

2.3 Jeux à champ moyen dynamiques

Nous présentons dans cette partie les jeux à champ moyen dynamiques et le système d'équilibre qui en découle.

Soit le problème d'optimisation individuel suivant :

$$\begin{cases} \inf_\alpha \mathbb{E}\left[\int_0^T f(X_t^x,\alpha_t) + g(X_t^x,m_t)dt + \Phi(X_T^x,m_T)\right] \\ dX_t^x = \alpha(t,X_t^x)dt + \sigma dW_t \end{cases} \tag{2.3}$$

Chaque joueur minimise, sur la période $[0,T]$, un coût convexe f associé à la caractéristique du joueur x et au contrôle α, un coût g lié à la distribution globale des joueurs et un coût final Φ.

Soit un joueur ayant la caractéristique x à l'instant t, sa fonction valeur s'écrit de la manière suivante :

$$v(t,x) = \inf_{(\alpha_s)_{s>t}, X_t^x = x} \mathbb{E}\left[\int_t^T f(X_s^x, \alpha_s) + g(X_s^x, m_s) ds + \Phi(X_T^x, m_T)\right] \tag{2.4}$$

La résolution du probléme de minimisation 2.3 donne le systéme d'équations suivant :

$$\begin{cases} \partial_t m - \frac{\sigma^2}{2}\Delta m + div(m\partial_p H(x, \nabla v)) = 0, m(0,x) = m_0(x) \\ \partial_t v + \frac{\sigma^2}{2}\Delta v + H(x, \nabla v) = g(m), v(T,x) = \Phi(x, m_T) \end{cases} \tag{2.5}$$

où le Hamiltonien $H(x,p) = f^*(x,p) = \sup_\alpha (p\alpha - f(x,\alpha))$.

La première équation de type kolmogorov est une équation de transport et décrit l'évolution de la distribution des joueurs au cours du temps. La deuxième est de type Hamilton-Jacobi-Bellman et donne la stratégie individuelle à l'équilibre.

CHAPTER 3

Could Tolerance of Software Piracy Reduce Free Software Diffusion?

This chapter is based on the paper "Could Tolerance of Software Piracy Reduce Free Software Diffusion?" which is a joint work with Heger Attaya.

Abstract

Past research classifies software users in two groups: legal users of a firm's proprietary software and illicit users or pirates of that software. However, this categorization de facto miscounts users of free/open source software. In this chapter, we introduce the free software users and study the dynamics of the software market when anti-piracy action is applied. Notably, we study the influence of anti-piracy action on the shifting of users from one software to another. Based on the market share of free software and the growth of the market, the proprietary producer uses optimally a lax policy on piracy (piracy tolerance) and then hardens its anti-piracy measures after a certain period.

3.1 Introduction

In the presence of free software alternative, catching back pirate users and integrating them into the legal market of proprietary software represents a big challenge for the proprietary editor. In this context, a strategy of piracy tolerance could sustain a larger user-base for the proprietary software. As *Bill Gates* said in 2008 about software piracy on China: "*It is easier for our software to compete with Linux when there's piracy rather than when there is not*". [1] According to Business Software Alliance (BSA) annual report in 2008, only 44 % of the installed software on personal computers are proprietary and properly licensed while 41 % of them are pirated and the remaining 15 % are free software. [2]

In this paper, we aim to display the difference that free software creates in the market and model how a producer of proprietary software can optimally *tolerate* a certain degree of piracy of his software in response to the confrontation with free software. We study the case with three types of software users: legal users who regularly pay for the license, illegal users who do not pay for the proprietary software and finally, users of the open-source software which is generally available for free. We show that the piracy control strategy is applied immediately only when the global network size of proprietary software, including the illicit versions of that software, is larger than the free software one. As long as tolerance of piracy is applied on the market and free software diffusion is low, a fraction of users will prefer using pirated software than free alternative.

The remainder of this paper is organized as follows. In the first section, we review the literature on piracy effect, the different anti-piracy measures and the beneficial effects of piracy tolerance strategy. The second section presents the model of maximization of the proprietary of software producer when there is a risk that some pirate users shift to the free software. The optimal solution is resolved in the third section. The fourth section provides a numerical simulation of the

1. Piracy: the Silver Lining. The Economist, July 19th-25th, 2008 ed. p.23.
2. Sixth annual BSA-IDC Global Software, 2008, Piracy study. The study is about 110 individual countries. Available at : http://global.bsa.org/idcglobalstudy2008/studies/globalpiracy2008.

model for various values of the network size and discusses the consequences of this anti-piracy strategy on the software market. The optimal anti-piracy strategy will be discussed in three different cases under the consideration for a certain expansion for the market growth. The last section concludes the paper.

3.2 Literature review of piracy effect

It is frequently claimed by proprietary software producers that software piracy is a theft causing important losses. Actually, the global piracy rate is accounted by BSA for 42% which causes estimated losses to software industry about 63.4 billion dollars. [3]

The classic way to reduce piracy is by lowering the software price as high prices induce piracy. But that reduces profits as well as investment incentives. Consequently, a producer of proprietary software has to weigh cost piracy control and potential benefits which suggests that sometimes he decides not to control piracy (opportunity cost). Research about piracy of digital products suggests reinforcement of copyright protection as a way to control piracy (Novos and Waldman 1984; Johnson 1985). The main anti-piracy measures include formal laws enforcement. The French Intellectual Property Code in its article L.335-3 for instance, defines piracy of software as any violation of intellectual property rights: "*All violation of author rights of software is (...) a counterfeiting crime.*" [4] In the United States, the "*Copyright Act*" of 1976 is the federal statute governing copyright law and granting "Congress" the power to promote copyright law such as the "Digital Millennium Copyright Act" introduced in 1998. However, enforcing the law is not an easy task. Recently, two proposed laws against piracy were rejected by the Congress during 2012: the "*SOPA*" (Stop Online Piracy Act) and "*PIPA*" (Protect IP Act). In France, the main law against piracy, the Hadopi law, introduced in 2009, is still also difficult to apply. [5]

3. Global Study 2011, Ninth Annual BSA and IDC Global Software Piracy Study `http://globalstudy.bsa.org/2011/`

4. Free translation. *La propriété littéraire et artistique*, Livre Ier, Le Droit D'Auteur, Titre II, Droits des Auteurs, Chapitre II , Droits patrimoniaux.

5. Haute autorité pour la diffusion des oeuvres et la protection des droits sur internet. See the

Different methods and various anti-piracy measures have been therefore devised to prevent illegal reproduction of software. Firstly, some researches propose that producers should price discriminate and charge some customers more than others King and Lampe 2003. Takeyama 1994 proposes to distinguish between "high-valuation" consumers and "low-valuation" and notes that considering price discrimination simultaneously with the anti-piracy strategy is an optimal strategy to lock all users.[6] According to Shy and Thisse 1999 producers should price discriminate between consumers who need the software supported services "support-oriented" and those who do not "support-independent". They show that the first ones are attracted by the original product and the second ones by the unauthorized copying.[7] Moreover, Slive and Dan 1998 note that strategies of price discrimination between "*business and home*" users seem to combat software piracy by charging business pirates higher prices.

Secondly, detecting the unauthorized copies can be managed by elaborating technical measures. In the case of illegal downloading music for example, Peitz and Waelbroeck 2006 propose to eliminate piracy by insetting additional resources in a technology that increases the detection probability of piracy. They note that for digital products (e.g. music, video and computer games), a targeted enforcement policy and additional investment on technology can reduce piracy without affecting the firm's profit. More precisely, Peitz and Waelbroeck 2006 suggest that the effect of piracy depends on the characteristics of the industry. They propose two possibilities either by technical measures or by lowering the price and conclude that for the specific case of software industry the effect of piracy is less than in the music and movie industries. Haruvy et al. 2004 suggest furthermore that protection of software can be managed by developing a hardware design that makes use of illegal software difficult. Actually, the known technical protection for software piracy is mainly enforced by Digital Rights Management

inventory of Hadopi application in the French Congress website, published the 25th december 2012. `http://questions.assemblee-nationale.fr/q14/14-3096QE.htm`

6. Takeyama 1994 provides a similar analysis of a price-setting monopoly than the one of Conner and Rumelt 1991.

7. Shy and Thisse 1999 model how piracy can be used strategically to confront competitors when network effects are not large.

(DRM) technologies used by hardware producers and proprietary software editors to limit the copy of the software. [8] There exists more sophisticated means of material protection including the "Online license key" that check instantly with the web site if the software has a registration code, such as "Windows Genuine Advantage".

Finally, legal actions against software piracy are backed up by the governments international alliances, as Business Software Alliance (BSA), *Software and Information Industry Association* (SIIA) or *International Federation of Phonographic Industry* (IFPI).

However, several researchers find that software producers can benefit from piracy, because a firm whose software product is pirated benefits from a larger *network size* (Conner and Rumelt 1991; Takeyama 1994; Shy and Thisse 1999). The *network size* of software implies that each additional user raises the value for the existing users as well as the future adopters of the software (Katz and Shapiro 1985). Under these conditions, this line of research shows that *piracy tolerance* strategy creates a higher user-base due to the *lock-in* effects and maximizes the value of the network that balances the *opportunity cost* of no controling piracy.

Shy and Thisse 1999 claim that when network effects are strong, tolerance of piracy is an equilibrium for a competitive software industry. [9] According to Haruvy et al. 2004, piracy tolerance is specifically useful for newly launched software when it is strategically managed through price and protection measures. They observe for instance that some firms often allow the use of shareware versions after their expiration date (e.g. WinZip). Givon et al. 1997 mentioned the example of the spreadsheet program MS Excel and argue that the large user-base is probably due to Microsoft's tolerance for piracy. Furthermore, Prasda and Mahajan 2003 model the optimal number of pirates to tolerate and show that a monopoly should start with a minimum protection and then impose a maximum protection once the software has diffused half way. Here, we argue that tolerating some piracy can also

8. *Apple* and *Iphone* applications is the most knowing example of rendering programs hard to be reproduced illegally.

9. This result obtained in duopoly framework is an extension of the one found by Conner and Rumelt 1991 in a price-setting monopoly.

be justified by the presence of free software on the market (Gaudeul 2008; Economides and Katsamakas 2006). Partly, due to the voluntary contribution of the free communities: more opportunities to fix bugs and to improve the functionality and user interface of the software (Bonaccorsi and Ross 2003). In this context, what should a producer of proprietary software do against piracy? For the producer, it is important to understand the effect of an anti-piracy action as it is still difficult to deter pirates when there is a risk that they switch to free software alternative (*stealing market* effect of free software). We examine therefore, the optimal decisions of a proprietary software producer regarding piracy in the context of free software competition.

3.3 A model of software market

3.3.1 The model

We consider one firm producing a particular proprietary software. We model the market continuously throughout the finite life of the software $[0, T]$. The producer of proprietary software is in a monopoly situation. But there is a threat of piracy and one free alternative for that software. [10] The free software is available at no cost. The production of proprietary software is assumed to be constant per period and the software price equal to 1. The producer of proprietary software seeks to maximize his market share by choosing an appropriate anti-piracy policy to confront both piray and free software. For simplicity, the cost of the anti-piracy strategy is assumed to be zero [11].

We consider a continuum of size N of users that grows at a fixed rate r, namely $\dot{N}(t) = rN(t)$. We include three groups of software users: legal users of proprietary software who are willing to purchase it, illegal users who prefer to risk and obtain the pirated software, and users who are not willing to pay the price of software and

10. Others cases are possible: many free alternatives for the proprietary software and many other proprietary software competitors.

11. This assumption is not very restrictive because if there is cost, the incentive to be tolerant is enhanced.

prefer the free software alternative.[12] The switching of users between the three types of software impacts the market share of proprietary software.

The solution used here is an optimal control over a finite horizon. This methodology is suitable because the producer of proprietary software is able to vary its anti-piracy strategy over the life cycle of the software according to the evolution of his market share.[13]

Description of the proprietary software producer program

The objective of a producer is to maximize his market share over the lifetime of the software or, equivalently here, the number of legal users purchasing the proprietary software.

The efforts devoted to piracy, as producer of proprietary software applies control or tolerance of piracy, are represented by the variable "s" with $s(t) \in [0,1]$. The situation $s = 1$ corresponds to a rigid anti-piracy strategy and conversely $s = 0$ corresponds to a total tolerance of piracy. In case of increases in anti-piracy actions such as heavier prosecutions of pirates or more sophisticated means of material protection, the share of pirates user will decrease and the risk that some of pirates would switch to free software alternative will increase. The proprietary software firm is willing to attract pirates by relaxing the anti-piracy measures so that in the future more users adopt their product. The problem of the producer is to decide to what degree tolerate piracy over time in order to increase the number of his legal users and thereby confront both piracy and free software substitution.

The maximization program of the proprietary producer depends thereby both on the number of legal users and the strategy devoted to piracy. The maximization problem is detailed in section 4.

12. The most important example of such a model is the market of PC operating system and the competition between Windows and Linux.

13. Dynamic models of markets with network externalities have been frequently used in the literature. For instance, Haruvy et al. 2004, 2008 extend the static analysis on software market to a dynamic framework.

Description of the evolution of the population of users

We analyze herein the market shares of the software market according to the movement of the population of users. The population of users is described in continuous time during the life time of the software: [14]

$$N(t) = N_P(t) + N_I(t) + N_F(t), \forall t \in [0, T]$$

Where:

- N_P indicates the users of proprietary software
- N_I indicates the users of pirated software
- N_F indicates the users of the free software.

The evolution of the population of users is analysed under the assumptions that:

- the learning effects explain the relative attraction of the different populations.
- the learning effects are proportionnal to the size of the population of users of a given software, as established expertise among users raises the value for the new adopters.

We have simultaneously users going from N_I to N_P and reversely from N_P to N_I (for any given level of control). The attractions depend on the schemes of the encounter between users which may present diverse configurations. [15]

Specifically, the evolution of the population of users of proprietary software N_P is expressed by the following equation:

$$\dot{N}_P(t) = a_1(N_P(t) + N_I(t)) - a_2 N_F(t) + a_3 s(t) N_I(t) - a_4(1 - s(t)) N_P(t) \quad (3.1)$$

The learning effects in the population of users of free software will lead to attract $a_2 N_F$, who will leave the population of users of proprietary software. Similarly, the high attraction of the proprietary software explain its encounter between some legal and illegal users ($a_1(N_P + N_I)$). In consequence of a control piracy strategy implementation, some pirates will increase the network size of proprietary software

14. In this paper, we chose to focus on the global population evolution regarding network size and anti-piarcy policy. The individual preferences are not explicity modeled here.
15. These encounters may also concern new users and population of old users.

(a_3N_I) and conversely because of piracy tolerance, a proportion of legal users are unwilling to pay for the software (a_4N_P).

Looking at the population of users of pirated software, the evolution of this population is as follows:

$$\dot{N}_I(t) = b_1(N_P(t)+N_I(t))-b_2N_F(t)-b_3(t)s(t)N_I(t)+a_4(1-s(t))N_P(t)+b_4(1-s(t))N_F(t) \tag{3.2}$$

Similar learning effects and parameters interpretations about b_1 and b_2 are assumed. The learning effects in the population of users of free software will lead to attract b_2N_F, who will leave the population of illegal users of proprietary software and join the community of free software users. The benefit from the proprietary network is given by $(b_1(N_P+N_I))$. When an anti-piracy strategy is applied (s=1), some pirates will increase the network size of both the proprietary software and the free software (b_3N_I). Conversely, because of piracy tolerance (s=0), some legal and free software users are unwilling to pay for the software and become illegal users in proportion a_4N_P and b_4N_F.

We interpret the equations above as the global evolution for each population group .

Description of the shifting market share

We can try to express the system of equations above in terms of market shares, so as to be able to simulate the distribution of market shares between the three populations of users, given a certain law of expansion for the overall market.

Let $S_P(t)$, $S_I(t)$ and $S_F(t)$ be the relative market shares, then $S_P(t)+ S_I(t) + S_F(t)=1$ and let $N(t)$ be the overall size of the market.
For a given type of software $j \in P, I, F$, we have :

$$\dot{S}_j(t) = \frac{d}{dt}(\frac{N_j(t)}{N(t)}) = \frac{N(t)\dot{N}_j(t) - N_j(t)\dot{N}(t)}{N^2(t)} = \frac{\dot{N}_j(t)}{N(t)} - \frac{N_j(t)}{N(t)}(\frac{\dot{N}(t)}{N(t)})$$

hence

$$\frac{\dot{N}_j(t)}{N(t)} = \dot{S}_j(t) + (S_j(t))(\frac{\dot{N}(t)}{N(t)})$$

The rate of growth of the market $\frac{\dot{N}}{N}$ is assumed to be equal to r. Then the expression of the evolution of the market share S_j is given by :

$$\dot{S}_j(t) = \frac{\dot{N}_j(t)}{N(t)} - rS_j(t) \tag{3.3}$$

The equations (3.1) and (3.2) can therefore be expressed in terms of market shares S_j as follows :

$$\begin{cases} \dot{S}_P(t) = a_1(S_P(t) + S_I(t)) - a_2 S_F(t) + a_3 s(t) S_I(t) - a_4(1 - s(t))S_P - rS_P(t) \\ \dot{S}_I(t) = b_1(S_P(t) + S_I(t)) - b_2 S_F(t) - b_3 s(t) S_I(t) + a_4(1 - s(t))S_P(t) \\ +b_4(1 - s(t))S_F(t) - rS_I(t) \end{cases} \tag{3.4}$$

3.3.2 Profit-maximizing strategy against piracy

We can now write the proprietary producer's maximization program as follows:

$$max_{s(t)\in[0,1]} \int_0^T S_P\left(t, s\left(t\right)\right) \mathrm{d}t. \tag{3.5}$$

under the constraints :

$$\begin{cases} \dot{S}_P(t) = a_1(S_P(t) + S_I(t)) - a_2 S_F(t) + a_3 s(t) S_I(t) - a_4(1 - s(t))S_P(t) - rS_P(t) \\ \dot{S}_I(t) = b_1(S_P(t) + S_I(t)) - b_2 S_F(t) - b_3 s(t) S_I(t) + a_4(1 - s(t))S_P(t) \\ +b_4(1 - s(t))S_F(t) - rS_I(t) \end{cases} \tag{3.6}$$

The maximization problem (3.5) with constraints (3.6) is solvable by the *Pontryagin's Maximum Principle* which is commonly used in optimal control theory

(Seierstad and Sydsaeter 1987).

The following proposition provides the main result of the paper and describes the optimal strategy of the proprietary software producer. It states that the anti-piracy action is applied only after a certain time denoted t^*.

Proposition. *There is $t^* \in [0, T]$ such that the optimal strategy s^* is given by:*

$$s^*(t) = \begin{cases} 0 \ \text{if } t \leq t^* \\ 1 \ \text{if } t > t^* \end{cases} \tag{3.7}$$

Proof. The proof of this proposition is based on the Pontryagin theorem. We will first recall the statement of the theorem and then we will turn to the demonstration of the proposition.

The Pontryagin theorem and the necessary conditions

Let T $\in]0, \infty[$, Ω is a non empty open set of $\mathbb{R}^n, U \subset \mathbb{R}^m$ and $\eta \in \Omega$.
$f_0 : [0, T] \times \Omega \times U \to \mathbb{R}$, and $f : [0, T] \times \Omega \times U \to \mathbb{R}^n$.
Let f_0 and f be regular functions. The maximization problem is given by

$$max \int_0^T f_0(t, x(t), u(t))\mathrm{d}t$$

$$\begin{cases} \forall t \in [0, T], \dot{x}(t) = f(t, x(t), u(t)) \\ x\,(0) = \eta \end{cases}$$

In our case, the function f_0 is equal to the function $N_P\,(t)$, u to the strategy s and x to the number of users N.

The Hamiltonian H: $[0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}$ is defined by :

$$H\,(t, x, u, p, p_0) = p_0 f_0\,(t, x, u) + p \cdot f\,(t, x, u)$$

Let $(x\,(t)\,, u\,(t))$ be an optimal process, then there exists a function $p : [0, T] \to \mathbb{R}^n$ such that:

1. $p(T) = 0$

2. $\forall t \in [0,T]$, $\dot{p}(t) = -H_x(t, x(t), u(t), p(t), 1)$
   We denote the derivative of H with respect to x by H_x.

3. $\forall t \in [0,T]$, $\forall u \in U$, $H(t, x(t), u(t), p(t), 1) \geq H(t, x(t), u, p(t), 1)$

Proof of the proposition We first calculate the adjoint function P(t) and present then the optimal control solution s^* that maximizes the market share proprietary software N_P^* during the lifetime of the software.
Let $a = a_1 + a_2$ and $b = b_1 + b_2$.
From the necessary conditions theorem, the adjoint variables verify the following system of differential equations:

$$\begin{cases} \dot{p}_1(t) & = -H_{S_P}(t, S(t), s^*(t), P(t)) \\ & = (-a + r + a_4(1 - s^*(t)))p_1(t) - (b - b_4(1 - s^*(t)) \\ & +a_4(1 - s^*(t))p_2(t) - 1 \\ \\ \dot{p}_2(t) & = -H_{S_I}(t, S(t), s^*(t), P(t)) \\ & = -\left(a + a_3 s^*(t)\right) p_1(t) - \left(b - r - b_3 s^*(t) - b_4\left(1 - s^*(t)\right)\right) p_2(t) \\ p_1(T) = p_2(T) = 0 \end{cases}$$

Hence, the adjoint function P(t) verifies :

$$\dot{P}(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} = A(t) \times P(t) + B$$
$$P(T) = 0$$

where
$\mathbf{B} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $\mathbf{A}(t) = \begin{pmatrix} -a + r + a_4\left(1 - s^*(t)\right) & -b + (b_4 - a_4)(1 - s^*(t)) \\ -a - a_3 s^*(t) & -b + r + b_3 s^*(t) + b_4\left(1 - s^*(t)\right) \end{pmatrix}$

The solution of this equation is :

$$P(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} = -\int_t^T (e^{-\int_t^\tau \mathbf{A}(r)dr}\mathbf{B})d\tau$$

We turn now to the evolution of the market shares $N(t)$. The equation (3.6) can be written as follow:

$$\dot{S}(t) = C(t) * S(t) + D(t)$$

Where $C(t) = \begin{pmatrix} a - r - a_4(1 - s^*(t)) & a + a_3 s^*(t) \\ b - (b_4 - a_4)(1 - s^*(t)) & b - r - b_3 s^*(t) - b_4(1 - s^*(t)) \end{pmatrix}$
and
$D(t) = \begin{pmatrix} -a_2 \\ -b_2 + b_4(1 - s^*(t)) \end{pmatrix}$

The solution of this equation is given by:

$$S^*(t) = e^{\int_0^t C(r)dr} S_0 + \int_0^t e^{\int_\tau^t C(r)dr} D(\tau)\, d\tau$$

Finally, the optimal control s^* maximizes the Hamiltonian H, then:

$$\begin{aligned} H(t, S(t), s, P(t)) &= f_0(t, S(t), s) + p(t) \cdot f(t, S(t), s) \\ &= p_1 \left[a(S_P + S_I) - a_2 + a_3 s S_I - a_4(1 - s) S_P - r S_P \right] \\ &+ p_2 [(b - b_4(1 - s))(S_P + S_I) - b_2 - b_3 s S_I + a_4(1 - s) S_P + b_4(1 - s) \\ &- r S_I] + S_P \\ &= \underbrace{[(a_3 S_I + a_4 S_P) p_1 - ((b_3 - b_4) S_I + (a_4 - b_4) S_P + b_4) p_2]}_{=g(t)} s \\ &+ \underbrace{k(S(t), p(t))}_{\text{independent of s}} \end{aligned}$$

hence

if $g(t) > 0$ then $s^* = 1$.
if $g(t) \leq 0$ then $s^* = 0$
The threshold t^* is the solution of the equation $g(t) = 0$.

It will be interesting to see how a market in expansion (r>0) or in recession (r<0) modifies the conclusions.

3.4 Simulation of the effect of anti-piracy action on the shifting market share

We simulate in this section the effect of the anti-piracy action on the population of users and the market shares of the three types of software. We show that piracy tolerance strategy reduces efficiently software piracy when the network size of free software is weak. More precisely, we distinguish three cases depending on the initial population of users as follows: [16]

	Case 1	Case 2	Case 3
N_P	0.3	0.3	0.05
N_I	0.35	0.1	0.35
N_F	0.35	0.6	0.6

Table 3.1: Diverse configurations of initial market shares

We extend the study to a situation of a certain expansion for the market with a growth rate of $\pm 5\%$.

3.4.1 Case 1: Equal market shares

We suppose in this first case that the eco-system of the software market is approximately equally shared between proprietary software, free software alternative and pirated versions. As an example the case of *Adobe Photoshop* which is estimated in 2010 by BSA to be the most pirated software on the Internet. The free alternative competing with *Adobe Photoshop* is the *GNU Image Manipulation Program* (GIMP) which works with numerous operating systems including Mac OS X and Microsoft Windows.

The simulation points that for both configurations of a market in expansion or in recession, the optimal solution are an immediate anti-piracy action applied at $t^* = 0$. The proprietary software producer succeed to create a high *lock-in* effect of pirate users and a high *stealing market* effect of free users.

This first result is confirmed by classic research suggesting severe actions on digital products piracy. Under the condition of equal initial market shares between the three groups of users, an immediate piracy control can be then profitable for the producer as it prevents both the size of piracy and the free software network from growing. The global network size of proprietary software plays herein an

16. We fix the values of a_i and b_i as follows: $a_1 = 0.45, a_2 = 0.4, a_3 = 0.4, b_1 = 0.25, b_2 = 0.3, b_3 = 0.8, b_4 = 0.5$.

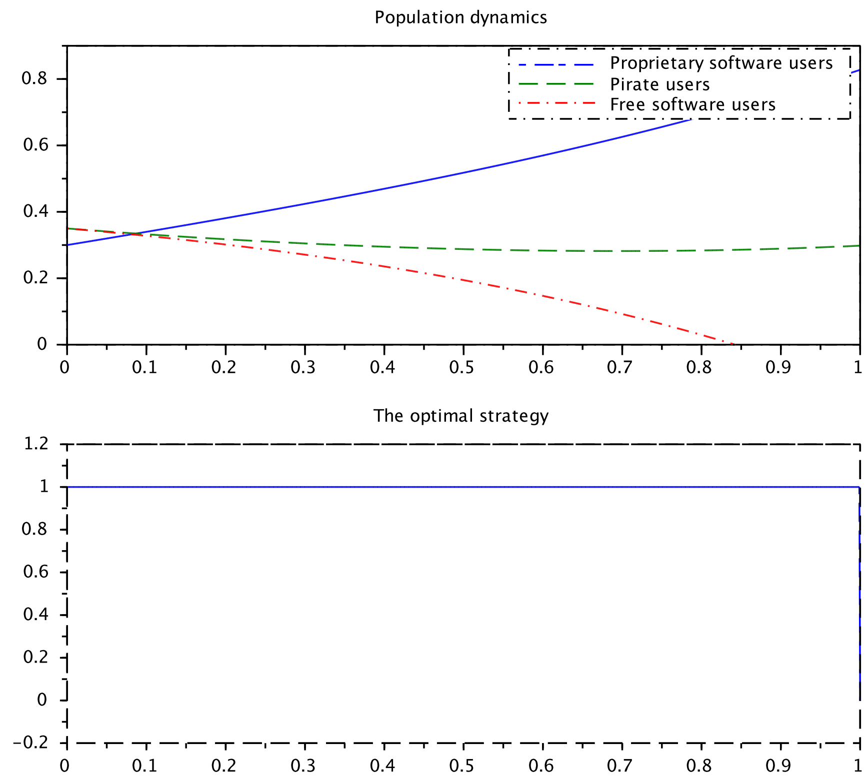

Figure 3.1: Case 1 when market decreases

important role in lock-in the market for new launched free/open source software even if the total number of software users decreases .

3.4.2 Case 2: When free software is preferred to piracy and proprietary software

In this second case, we depict the dynamic of software market when the free software is in a quasi monopoly situation, such as the use of the free web browser *Mozilla-firefox* which was the first competitor to Microsoft's *Internet Explorer*.

Simulations show that at the first period, consequent to the piracy tolerance, the number of pirate users increases independently of the trend of the market growth. In the second period when anti-piracy is applied, the market share of

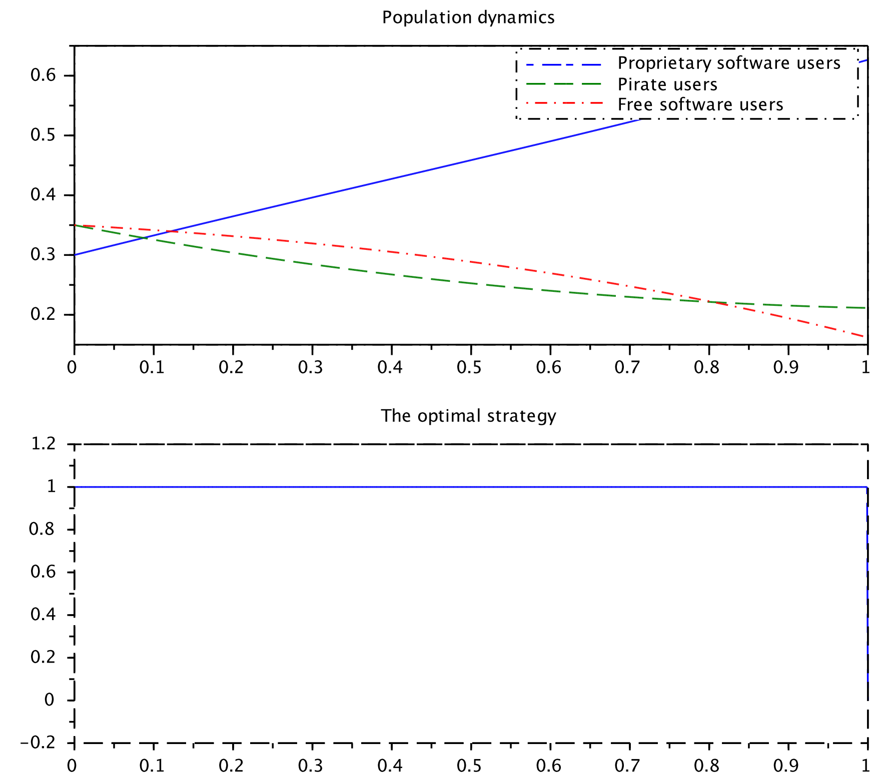

Figure 3.2: Case 1 when market increases

pirated software decreases relatively only when the market is in expansion. When the market is in recession, an early control strategy cannot involve reducing piracy enforcement over time because in the early period of the life cycle of software the producer aims at gaining from network effects and at the later stages of the life cycle this incentive diminishes. Pirate users are then locked momentarily only because it is tolerated to use pirated versions. The application of a piracy tolerance strategy, allows therefore producer of proprietary software to increase his market share in a recessed market. The *stealing market* effect, by comparison to the case 1, is low in this case. By opposition, when the market is in expansion (r>0), we observed that pirate users will prefer to use free software than proprietary software. The producer of proprietary software does not succeed in catching pirate users. In parallel, the market share of free software increases. Thus the application of

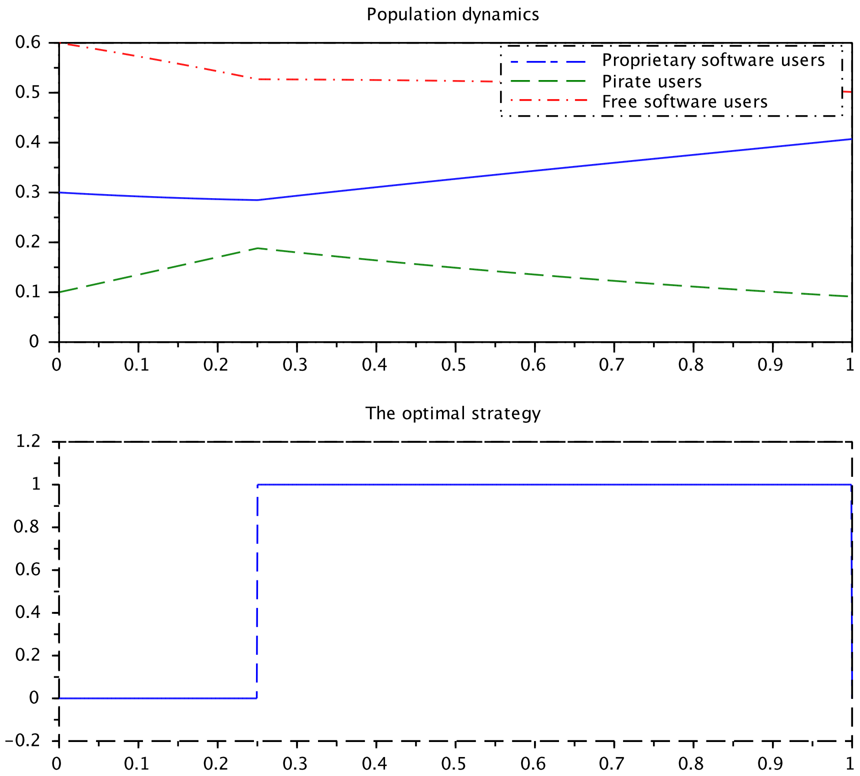

Figure 3.3: Case 2 when market decreases

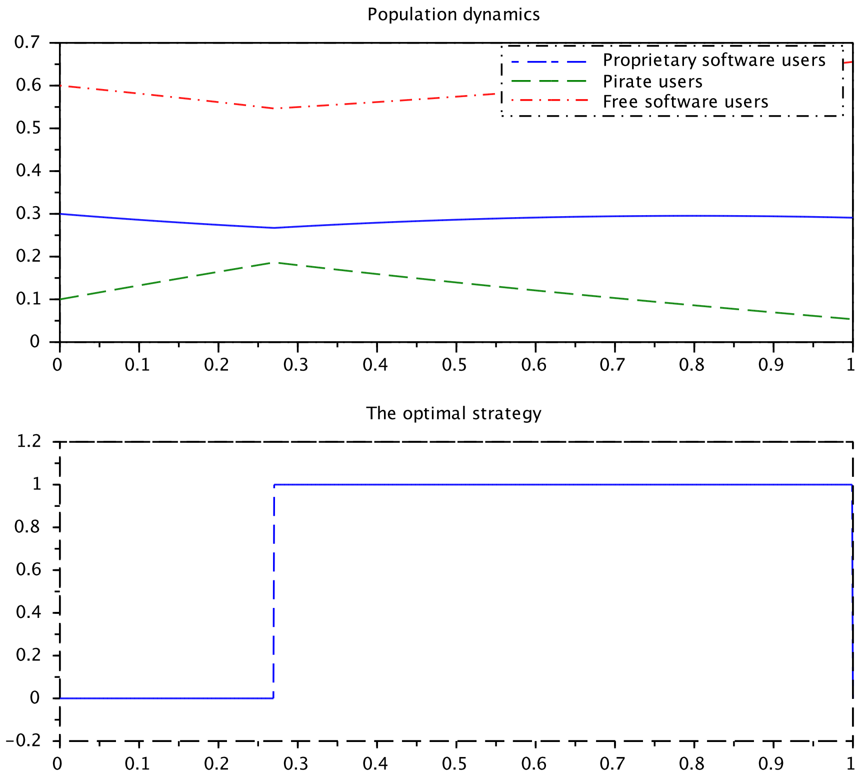

Figure 3.4: Case 2 when market increases

piracy tolerance strategy allows producer only to maintain relatively his position in the market.

3.4.3 Case 3: When free software and piracy are preferred to proprietary software

We suppose in this last case, that free software and pirated versions are preferred to proprietary software. Hence, we assume that widespread unauthorized copies of proprietary software is similar to the case of *Windows* in China as announced by *Bill Gates* in 2008.

The simulation show that initially, the piracy tolerance is applied for a longer time until t^* (close to 0.3). The control action later leads to reduce the proportion of pirate users when the market is in recession. But the market share of free software remains nearly identical after controlling is applied. Under these considerations, proprietary software seems to have a high *lock-in* effect and a low *stealing market* effect when the market is in decrease. Instead, piracy tolerance strategy has no effect on reducing the group of free software users when the market is in expansion as free software market share increases after controlling is applied. This last result shows that the dynamics of the software market impacts the free software progression probably supported by the free community. It seems then that in a context of diffusion of piracy, the tolerance of piracy strategy is efficient in maintaining the market share of proprietary software.

3.5 Conclusion

In this chapter we study the impact of anti-piracy action on the market share of a given software when free software is an alternative to proprietary software. The model makes two principal contributions. First, we have considered the software market as a combination of three competitive products: proprietary, free software and pirated software. The analysis extends the previously piracy literature on network externalities to a free software framework. Comparing these results with

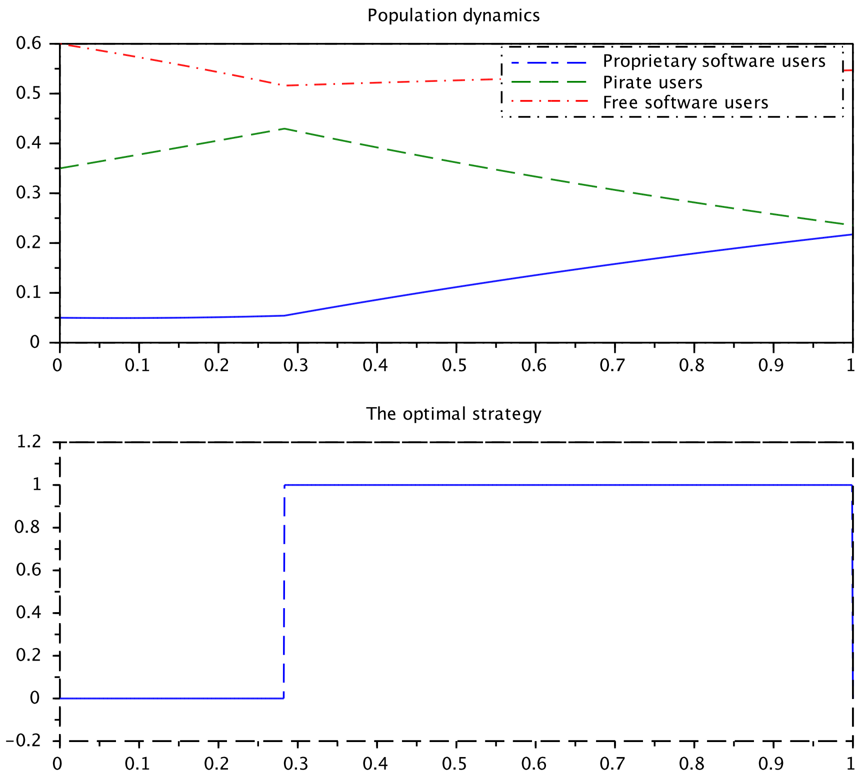

Figure 3.5: Case 3 when market decreases

standard assumptions in the literature about piracy and about free software, the model is more realistic in its analysis of the three users group. The optimal anti-piracy strategy is discussed in three different cases under the consideration for a certain expansion for the market growth. It shows under certain conditions, that early piracy tolerance can be profitable for the proprietary software producer because it prevents the size of the free software network from growing. In cases where the free software market share is very large and the piracy diffusion is low within a dynamic software market, the application of anti-piracy action has no effect on reducing free software competition. It also shows that larger the pirates group is, the more difficult it becomes to control piracy and the more tolerant a strategy will be applied.

Finally, it should be reemphasized that the software market consists of three

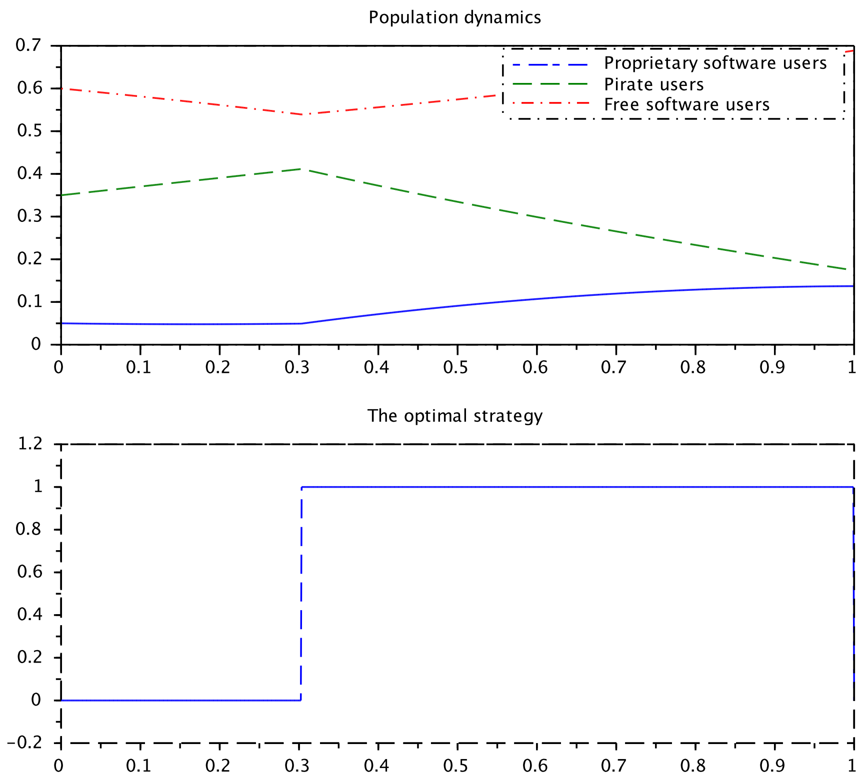

Figure 3.6: Case 3 when market increases

competitive products and that the producer has to decide between controlling or allowing piracy by taking account of the free software substitution risk and the dynamic growth of the market. This analysis, of course, does not take into account the price strategy of the proprietary software's producer. Extension of our model to endogenize price variable is an important complement to the current analysis.

CHAPTER 4

Downtown Parking with Heterogeneous Consumers

Abstract

This chapter studies a parking model with a population of heterogeneous consumers. Each consumer, according to her willingness to pay, chooses between parking on-street or parking in a garage. Formulating the model as a two-state mean field game, we prove the existence and uniqueness of a Nash equilibrium. We also prove the existence of a social optimum and study its properties, in particular we show that this optimum does not depend on the on-street price. A numerical example with a uniform distribution of willingness to pay investigates the optimal parking policy.

4.1 Introduction

It is a well known fact that all big cities in the world confront parking problems. Several measures can be initiated to remedy to this problem. For example, the investigation conducted by the CERTU [1] in 2005 in the French cities with over 20 000 inhabitants shows that local governments intervene in the market not only by imposing parking fees and maximum length of stay but also develop formulas for specific users : residents, mobile professionals ... (CERTU 2008).

Since the seminal work of Vickrey 1969, the theoretical literature of parking policy extensively treated the problems related to parking search, choice behavior and optimal pricing policies (Anderson and De Palma 2004; Calthrop and Proost 2006). It is suggested that when the off-street market is supplied under constant returns to scale, street prices must equal garage prices in order to achieve optimality (Calthrop and Proost 2006).

However, these models do not take into account the heterogeneity of agents. This paper makes a first step in this direction by assuming heterogeneity among agents in their willingness to pay and investigates the effect of this kind of heterogeneity on the pricing policy.

We construct a simple model in the line of Calthrop and Proost 2006 in which a consumer goes downtown for leisure or shopping activities and decides where to park. Each agent has two strategies and decides according to his willingness to pay to either look for a vacant spot on-street or to go directly to a parking garage. This equilibrium parking model with a continuum of heterogeneous agents enables us to analyze the optimal parking policy and its effects on the allocation of consumers between on-street parking and parking garage.

Our model is formulated as a game with a continuum of players (i.e. consumers). Each player has to make a binary choice between looking for a vacant spot on-street or going directly to a parking garage. This model is a simple case of the " Mean Field Games ". In our specific model, we prove the existence and the uniqueness of a Nash equilibrium. At equilibrium, a consumer whose willingness

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to pay is less than a threshold value searches on-street, and a consumer whose willingness to pay is greater than this threshold goes to the parking garage. We also establish the existence of a social optimal threshold. An interesting result of the social optimal threshold is that it does not depend on the on-street price.

We also evaluate different policies including pricing policies and maximum length of stay restrictions. We show that an increase of on-street price may alleviate congestion and reduce the inefficiency of the equilibrium but has a negative impact on the social welfare. The reduction of the garage parking price is an effective policy to mitigate congestion and to improve the social welfare. Finally, reducing the maximum length of stay does not help reduce congestion in cities but has a positive impact on social welfare.

There is a growing interest in economics for games with discrete choice involving a large number of heterogeneous players. An and Zhang 2012 developed a similar model to study traffic congestion. Commuters make a binary choice between riding a bus and driving private vehicles. They addressed the issue of a gasoline tax and its efficiency. Daniels et al. 2013 used Global Games techniques to study the allocation of order flow between a crossing network and a dealer market. Such models are very useful to understand the effect of heterogeneity and allow the removal of the multiplicity of equilibria that emerged in models assuming homogeneity.

The paper is organized as follows. Section 2 presents our main assumptions. In Section 3, we show the existence and uniqueness of the equilibrium and discuss some of its properties. Section 4 is devoted to the welfare analysis and the properties of the social optimum. In section 5, we study the case of the uniform distribution of willingness to pay. A last section concludes.

4.2 The model

We consider a continuum of measure one of risk-neutral consumers. Each consumer, who might be a shopper or a tourist driver, goes to downtown and tries to park for a period of time t, and then exits. Once he arrives downtown, two strategies are available : looking for a vacant spot on-street where the meter rate is p_1 per unit of time (though this outcome is uncertain since the total supply of on-street parking is finite) or parking in a parking garage at rate p_2 per unit of time. We assume that, $p_1 < p_2$.

Each consumer has her/his own willingness to pay θ in the interval $[\underline{\theta}, \overline{\theta}]$, and is distributed according to a differential cumulative distribution function F. We denote f the probability density function.

4.2.1 Rationing rule

The stock of on-street parking is fixed such that the total demand equals on-street supply when each individual parks for Q units of time and the number of garage parking spaces is assumed to be sufficiently high. We assume in our model that the time of parking t is larger than Q, so that the total on-street supply is not sufficient to satisfy the total demand of the consumers. If demand for on-street parking is larger than supply, the excess demand is rationed stochastically. The probability of finding a vacant spot is given by

$$\pi(\alpha, Q) = \min\{1, \frac{Q}{\alpha t}\}$$

where, α is the proportion of consumers who choose to search for a spot on-street.

As stressed by Calthrop and Proost 2006 "This rationing rule is justified if drivers arrive in downtown area more or less at random. We consider this well suited to modeling parking in city centers for shopping, tourism and leisure activities. It is clearly far less suited to modeling workplace parking, where spots are often reserved."

4.2.2 Decision rule

For a consumer with a willingness to pay θ_i, the certain payoff from parking in a garage parking is given by

$$U_1(\theta_i, p_2) = \theta_i - p_2$$

and the expected payoff from searching for a vacant spot on-street is given by

$$U_0(\theta_i, \alpha, p_1) = \pi(\alpha, Q)(\theta_i - p_1)$$

Where α is the proportion of consumers who choose to look for a vacant spot on-street. U_0 is equal to the probability of finding a vacant spot times the surplus of the consumer. We assume here that a consumer who cannot find a spot gets a utility equal to zero and quits downtown.

Moreover, we consider two assumptions in our model.

Assumption A1 The meter rate per unit of time $p_1 < \underline{\theta}$

This assumption ensures that the consumer with the minimum willingness to pay $\underline{\theta}$ can at least park his car on-street. Otherwise, he gets a negative utility and she/he cannot choose any of the two possibilities.

Assumption A2 The meter rate of garage parking $p_2 < \overline{\theta}$

This assumption is also quite natural since it guarantees that a fraction of the population receives a positive utility by parking in the garage.

4.3 Equilibrium

In this section, we show the existence and the uniqueness of a non-trivial equilibrium and discuss some of its properties.

4.3.1 Existence and uniqueness of the equilibrium

Definition 1 A strategy is a function $g : [\underline{\theta}, \overline{\theta}] \to \{0, 1\}$ such that

$$g(\theta_i) = \begin{cases} 1 \text{ if the consumer chooses to search for a spot on-street,} \\ 0 \text{ if the consumer decides to park in a garage parking.} \end{cases}$$

Definition 2 Given (p_1, p_2), a Nash equilibrium is a pair $(g, \alpha^\star)$ such that

– For all $\theta_i \in [\underline{\theta}, \overline{\theta}]$ the optimal strategy $g(\theta_i)$ is given by

$$g(\theta_i) = \begin{cases} 1 \text{ if } U_0(\theta_i, \alpha^\star, p_1) \geq U_1(\theta_i, p_2), \\ 0 \text{ if } U_1(\theta_i, p_2) > U_0(\theta_i, \alpha^\star, p_1). \end{cases}$$

– The proportion $\alpha^\star$ verifies $\alpha^\star = \mathbb{E}[g(\theta)]$.

An equivalent definition of an equilibrium is a pair $(\theta^\star, \alpha^\star)$ where

– The threshold $\theta^\star$ verifies $U_0(\theta^\star, \alpha^\star, p_1) = U_1(\theta^\star, p_2)$ or $\theta^\star = \overline{\theta}$

– The proportion $\alpha^\star$ satisfies $\alpha^\star = F(\theta^\star)$

This means that consumers whose willingness to pay is less than $\theta^\star$ decide to search for a vacant spot on-street and those whose willingness to pay is greater than $\theta^\star$ decide to park in the garage.

The following theorem states the existence and the uniqueness of a nontrivial Nash equilibrium.

Theorem 4. *Given p_1,p_2,Q,t and the distribution F, if we assume that $\overline{\theta} > \frac{p_2 - \frac{Q}{t}p_1}{1-\frac{Q}{t}}$ then there exists a unique non-trivial Nash equilibrium $(\theta^\star, \alpha^\star)$ which satisfies*

$$\theta^\star = \frac{p_2 - \frac{Q}{\alpha^\star t}p_1}{1 - \frac{Q}{\alpha^\star t}} \tag{4.1}$$

and the probability of finding a spot on-street is given by

$$\pi(\alpha^\star, Q) = \frac{Q}{\alpha^\star t} < 1 \tag{4.2}$$

Proof. See Appendix □

Remark : If $\overline{\theta} \leq \frac{p_2 - \frac{Q}{t}p_1}{1-\frac{Q}{t}}$ then the only Nash Equilibrium is the trivial equilibrium $(\theta^\star, \alpha^\star) = (\overline{\theta}, 1)$ where all consumers reject the parking garage and decide to look for a vacant spot on-street.

4.3.2 Properties of the equilibrium

In the previous section, we established the existence and uniqueness of a non-trivial equilibrium. It is interesting to study its properties and conduct comparative statics.

Proposition 1 The equilibrium threshold $\theta^\star$ is strictly greater than p_2.

Proof. Suppose that the equilibrium threshold $\theta^\star \leq p_2$. For a consumer with a willingness to pay $\theta_i = \theta^\star$:

$$U_0(\theta_i, \alpha) = \pi(\alpha, Q)(\theta_i - p_1) > 0$$

since $\pi(\alpha, Q) = \min\{1, \dfrac{Q}{\alpha t}\} > 0$ and $\theta_i = \theta^\star > p_1$ by assumption A1, and

$$U_1(\theta_i, \alpha) = (\theta_i - p_2) \leq 0$$

Then, the condition $U_0(\theta_i, \alpha) = U_1(\theta_i, \alpha)$ is not satisfied and $\theta^\star$ is not an equilibrium. □

The two following propositions study the effect of the on-street rate p_1 and the parking garage rate p_2 on the equilibrium proportion of consumers $\alpha^\star$.

Proposition 2 The equilibrium proportion of consumers who choose to search for a spot in the street $\alpha^\star$ is decreasing in p_1.

Proof. To show this result we differentiate Eq(4.1) with respect to p_1 which yields to

$$\frac{d\theta^\star}{dp_1} - 1 = \frac{t}{Q}F'(\theta^\star)\frac{d\theta^\star}{dp_1}(\theta^\star - p_2) + \frac{t}{Q}F(\theta^\star)\frac{d\theta^\star}{dp_1}$$

Hence

$$(1 - \frac{t}{Q}F'(\theta^\star)(\theta^\star - p_2) - \frac{t}{Q}F(\theta^\star))\frac{d\theta^\star}{dp_1} = 1$$

Then

$$\frac{d\theta^\star}{dp_1} = \frac{1}{1 - \frac{t}{Q}F(\theta^\star) - \frac{t}{Q}F'(\theta^\star)(\theta^\star - p_2)}$$

We have shown that $\pi(\alpha, Q) = \frac{Q}{\alpha t} < 1$ and $\theta^\star > p_2$, then the denominator of the right hand side is negative. As a result, $\frac{d\theta^\star}{dp_1} < 0$. Recall that $\alpha^\star = F(\theta^\star)$, we have

$$\frac{d\alpha^\star}{dp_1} = \frac{d\alpha^\star}{d\theta^\star}\frac{\theta^\star}{dp_1} = F'(\theta^\star)\frac{\theta^\star}{dp_1} < 0$$

$\square$

Proposition 3 The equilibrium proportion of consumers who choose to search for a spot in the street $\alpha^\star$ is increasing in p_2.

Proof. To show this result, we differentiate equation(4.1) with respect to p_2 which yields

$$\frac{d\theta^\star}{dp_2} = \frac{t}{Q}F'(\theta^\star)\frac{d\theta^\star}{dp_2}(\theta^\star - p_2) + \frac{t}{Q}F(\theta^\star)(\frac{d\theta^\star}{dp_2} - 1)$$

Hence

$$(1 - \frac{t}{Q}F(\theta^\star) - \frac{t}{Q}F'(\theta^\star)(\theta^\star - p_2))\frac{d\theta^\star}{dp_2} = -\frac{t}{Q}F(\theta^\star)$$

Then

$$\frac{d\theta^\star}{dp_2} = \frac{-\frac{t}{Q}F(\theta^\star)}{1 - \frac{t}{Q}F(\theta^\star) - \frac{t}{Q}F'(\theta^\star)(\theta^\star - p_2)} > 0$$

$\square$

This result states that the number of consumers whose willingness to pay is greater than p_2 decreases, implying that the proportion of consumers who decide to look for a vacant spot on-street will increase.

4.4 Welfare analysis

Let us consider the optimal proportion of consumers $\tilde{\alpha}$, or equivalently, the optimal threshold $\tilde{\theta}$, that maximizes the expected total payoff. If $\tilde{\theta}$ is a threshold value, then consumers whose willingness to pay is less than $\tilde{\theta}$ will choose to park in the street and those whose willingness to pay is greater than $\tilde{\theta}$ will choose to go to the garage parking. Then the total expected payoff is given by

$$U(\tilde{\theta}) = \int_{\underline{\theta}}^{\tilde{\theta}} U_0(\theta, \tilde{\alpha}) m(\theta) d\theta + \int_{\tilde{\theta}}^{\overline{\theta}} U_1(\theta, \tilde{\alpha}) m(\theta) d\theta.$$

The first term of the right hand side corresponds to the total expected payoff of consumers who choose to search for a vacant spot on-street and the second term is equal to the total expected payoff of consumers who choose to go to garage parking.

4.4.1 Existence of a social optimum

The theorem below states the existence of an optimal proportion $\tilde{\alpha}$ or equivalently an optimal threshold $\tilde{\theta}$.

Theorem 5. *Suppose the assumption A1 holds, then*

- *An optimal threshold value $\tilde{\theta}$ exists.*
- *The equilibrium is inefficient : the equilibrium threshold $\theta^{\star}$ is larger than the optimal threshold $\tilde{\theta}$ or equivalently the equilibrium proportion $\alpha^{\star}$ is larger than $\tilde{\alpha}$.*

Proof. See Appendix □

4.4.2 Properties of the social optimum

Similar to the equilibrium, we conduct some comparative statics on the optimal threshold $\tilde{\theta}$. We show that an increase in the garage rate p_2 leads to the same effect on the equilibrium threshold $\theta^{\star}$ and the optimal threshold $\tilde{\theta}$ but the latter does not depend on the on-street rate p_1.

Proposition 4 The optimal threshold $\tilde{\theta}$ does not depend on p_1.

Proof. It is straightforward from the definition of the total expected payoff U. Indeed,

$$\begin{aligned} U(\theta) &= \tfrac{Q}{F(\theta)t}[\int_{\underline{\theta}}^{\theta} sf(s)ds - p_1 F(\theta)] + \int_{\theta}^{\overline{\theta}} sf(s)ds - p_2(1-F(\theta)) \\ &= \tfrac{Q}{F(\theta)t}(\int_{\underline{\theta}}^{\theta} sf(s)ds) - \tfrac{Q}{t}p_1 + \int_{\theta}^{\overline{\theta}} sf(s)ds - p_2(1-F(\theta)) \end{aligned}$$

Therefore, $argmax_\theta\{U(\theta)\} = argmax_\theta\{U(\theta) + \frac{Q}{t}p_1\}$ and $U(\theta) + \frac{Q}{t}p_1$ does not depend on p_1 then $\tilde{\theta}$ which maximizes the total expected utility U does not depend on p_1.

□

Proposition 5 The optimal threshold $\tilde{\theta}$ is increasing in p_2.

Proof. To show that $\tilde{\theta}$ is increasing in p_2, we distinguish three cases :

- If $\tilde{\theta}$ is such that $\frac{Q}{\tilde{\alpha}t} = 1$ then $\tilde{\theta}$ does not depend in p_2 since $\tilde{\alpha} = F(\tilde{\theta})$.
- If $\tilde{\theta}$ is equal to the upper bound $\overline{\theta}$ then $\tilde{\theta}$ does not depend in p_2.
- If $\tilde{\theta}$ is an interior solution then it is sufficient to show that $\frac{\partial\tilde{\theta}}{\partial p_2} \leq 0$ by applying the implicit function theorem. Since $\tilde{\theta}$ is an interior solution, it satisfies $U'(\tilde{\theta}) = 0$ then

$$U'(\tilde{\theta}) = U'(\tilde{\theta}, p_2) = 0$$

Hence

$$\frac{\partial\tilde{\theta}}{\partial p_2} = -\frac{\frac{\partial U'}{\partial p_2}}{\frac{\partial U'}{\partial\tilde{\theta}}}$$

Theorem 5 states that that $\tilde{\theta}$ is a maximum then $\frac{\partial U'}{\partial\tilde{\theta}} = U''(\tilde{\theta}) \leq 0$. From the definition of $U'(\tilde{\theta}, p_2)$ we have

$$U'(\tilde{\theta}, p_2) = [-\frac{Q}{F^2(\theta)t}(\int_{\underline{\theta}}^{\theta} sf(s)ds - p_1F(\theta)) + (-1 + \frac{Q}{F(\theta)t})(\theta - p_1) + (p_2 - p_1)]f(\theta).$$

then

$$\frac{\partial U'}{\partial p_2} = f(\tilde{\theta}) > 0.$$

This concludes the proof.

□

4.5 The uniform distribution

In this section, we treat the particular case of a uniform distribution $\mathcal{U}[\underline{\theta}, \overline{\theta}]$. It allows us to have explicit solutions for the equilibrium threshold $\theta^\star$ and the optimal threshold $\tilde{\theta}$.

In the two following propositions, we calculate the values of $\theta^\star$ and $\tilde{\theta}$. This will allow us to study the implications of a change in the prices of parking and of the maximum length of stay.

Let $T = \overline{\theta} - \underline{\theta}$.

Proposition 6 The equilibrium $(\theta^\star, \alpha^\star)$ is given by

$$\begin{cases} \theta^\star = \frac{TQ+t(p_2+\underline{\theta})+T\sqrt{(Q+\frac{t}{T}(\underline{\theta}+p_2))^2-4\frac{t}{T}(\frac{t}{T}\underline{\theta}p_2+Qp_1)}}{2t}, \\ \alpha^\star = F(\theta^\star) = \frac{\theta^\star-\underline{\theta}}{\overline{\theta}-\underline{\theta}}. \end{cases} \tag{4.3}$$

Proof. We consider here a uniform distribution supported on the intervall $[\underline{\theta}, \overline{\theta}]$. The cumulative distribution function $F(\theta) = \frac{1}{T}(\theta - \underline{\theta})$. From Theorem 4, we have

$$\theta^\star = \frac{p_2 - \frac{Q}{\alpha^\star t}p_1}{1 - \frac{Q}{\alpha^\star t}}$$

Then

$$(1 - \frac{Q}{\alpha^\star t})\theta^\star = p_2 - \frac{Q}{\alpha^\star t}p_1$$

$$(\alpha^\star t - Q)\theta^\star = \alpha^\star t p_2 - Qp_1$$

Since $\alpha^\star = F(\theta^\star) = \frac{\theta^\star-\underline{\theta}}{\overline{\theta}-\underline{\theta}}$, we have

$$(\frac{\theta^\star - \underline{\theta}}{T}tQ)\theta^\star = \frac{\theta^\star - \underline{\theta}}{T}tp_2 - Qp_1$$

Hence

$$\frac{t}{T}(\theta^\star)^2 - (\frac{t}{T}(p_2 + \underline{\theta}) + Q)\theta^\star + (\frac{\underline{\theta}}{T}tp_2 + Qp_1) = 0.$$

Since $\theta^\star > 0$ then the equilibrium threshold is given by

$$\theta^\star = \frac{TQ + t(p_2 + \underline{\theta}) + T\sqrt{(Q + \frac{t}{T}(\underline{\theta} + p_2))^2 - 4\frac{t}{T}(\frac{t}{T}\underline{\theta}p_2 + Qp_1)}}{2t}$$

□

Proposition 7 The social optimum $\tilde{\theta}$ is given by

$$\tilde{\theta} = \frac{QT}{2t} + p_2.$$

Proof. In the case of a uniform distribution, the total expected payoff is given by

$$\begin{aligned} U(\theta) &= \tfrac{Q}{F(\theta)t}(\tfrac{1}{T}\int_{\underline{\theta}}^{\theta} sds) - \tfrac{Q}{t}p_1 + \tfrac{1}{T}\int_{\theta}^{\overline{\theta}} sds - p_2(1 - F(\theta)) \\ &= \tfrac{Q}{2t}(\theta + \underline{\theta}) - \tfrac{Q}{t}p_1 + \tfrac{1}{2T}(\overline{\theta}^2 - \theta^2) - p_2(\tfrac{\overline{\theta}-\theta}{T}). \end{aligned}$$

The FOC gives

$$U'(\theta) = \frac{Q}{2t} - \frac{1}{T}\theta + \frac{p_2}{T}$$

hence

$$\tilde{\theta} = \frac{QT}{2t} + p_2$$

□

4.5.1 Comparative statics and policy implications

The results of the comparative statics are helpful to determine policy effects. The goal of the analysis is to understand the impacts of the p_1, p_2 and t on the equilibrium distribution of the population $\alpha^\star$, the socially optimal distribution $\tilde{\alpha}$ and the total expected payoff of the population. The result is given in the following proposition.

Proposition 8 The effects of changes in p_1, p_2 and t are summarized in Table 4.1.

Where the sign '+' represents a positive impact, '-' represents a negative impact

	$\theta^\star$	$\tilde{\theta}$	Total expected utility U
p_1	-	0	-
p_2	+	+	-
t	-	-	-

Table 4.1: Comparative statics regarding p_1, p_2 and t

Figure 4.1: Equilibrium and optimal proportion of searchers with respect to p_1

and '0' represents no impact.

Effect of changes in p_1

One interesting result of this model is that p_1 has no impact on the optimal threshold $\tilde{\theta}$. However, the table 1 shows that the equilibrium distribution $\alpha^\star$ is decreasing in p_1. Therefore, increasing the on-street price achieves two goals: alleviating congestion in cities and reducing the inefficiency of the equilibrium as described in Figure 4.1. However, this policy leads to a decrease of the total expected payoff.

Figure 4.2: Equilibrium and optimal proportion of searchers with respect to p_2

Effect of changes in p_2

Proposition 3 and proposition 5 show that $\alpha^\star$ and $\tilde{\alpha}$ are both increasing in p_2. Hence, reduction of the congestion can be achieved by lowering the price off-street. This result is consistent with that of Calthrop and Proost 2006. They showed that, when off-street market is supplied under constant returns to scale, on-street price and off-street price should be equal. However, this policy does not affect the inefficiency of the equilibrium as shown in Figure 4.2.

Effect of changes in t

Table 1 shows that parking time t has a negative impact on the three variables studied. The reduction of parking time is not an effective policy to alleviate congestion in cities and to enhance consumers to use off-street parkings as shown in Figure 4.3. However, it improves social welfare.

Figure 4.3: Equilibrium and optimal proportion of searchers with respect to the maximum length of stay t

4.6 Conclusion

The allocation of heterogeneous consumers between on-street parking and garage parking is investigated in this paper. It aims at studying the effect of heterogeneity in the choice of consumers. The model shows two important results. First, under fairly weak assumptions, we prove the existence and uniqueness of a non-trivial equilibrium. Second, we show the existence of a social optimum. The study of its properties leads to an important result since it is insensitive to the on-street price.

We also studied the effect of different policies in our model, including pricing policies and maximum length of stay restrictions. Increasing on-street price allows to reduce congestion but does not improve social welfare. Lowering the garage parking price seems to be more effective both reducing congestion and improving social welfare. However, imposing restrictions on parking time does not help mitigate congestion.

The model presents some simplifications. It could be improved by considering instead of a public garage parking with a fixed fee, an oligopolistic market structure. It will also be interesting to integrate traffic congestion and introduce cruising for parking access in the model.

The type of model used in this paper is very useful for modeling situations with heterogeneous agents such as road traffic or the allocation of order flow between a crossing network and a dealer market. It can also be used in other fields such as economics of science. Suppose, for example, a continuum of researchers with different abilities must choose between publishing their work in a journal with a selection committee and a collective book without selection but much less valued in academic field.

Appendix

Theorem 4

An agent with a willingness to pay θ_i will choose to look for a spot in the street if

$$U_0(\theta_i, \alpha^\star, p_1) > U_1(\theta_i, p_2) \tag{4.4}$$

That is

$$(\theta_i - p_1)\pi(\alpha, Q) > \theta_i - p_2$$

then

$$p_2 - p_1\pi(\alpha, Q) > \theta_i(1 - \pi(\alpha, Q)).$$

We will first show that the equilibrium can not be trivial $\alpha^\star \neq 0, 1$ and $\pi(\alpha^\star, Q) < 1$.

- First case : Suppose that $\alpha^\star = 1$.
  if $\alpha^\star = 1$ then all consumers have decided to search for a spot in the street which means that, at equilibrium, $\forall\theta_i$, $U_0(\theta_i, \alpha^\star, p_1) > U_1(\theta_i, p_2)$, that is

  $$\frac{Q}{t}(\theta_i - p_1) > \theta_i - p_2$$

  Hence

  $$\forall\theta_i, \theta_i < \frac{p_2 - \frac{Q}{t}p_1}{1 - \frac{Q}{t}}.$$

  Since $\overline{\theta}$ is strictly greater than $\frac{p_2 - \frac{Q}{t}p_1}{1 - \frac{Q}{t}}$, this condition does not hold and $\alpha^\star = 1$ is not an equilibrium.
- Second case : Suppose that $\alpha^\star$ is close to zero.
  If $\alpha^\star$ is close to 0, then $\pi(\alpha^\star, Q) = min\{1, \frac{Q}{\alpha^\star t}\} = 1$.
  If $\pi(\alpha^\star, Q) = 1$ then we can write $U_0(\theta_i, \alpha^\star, p_1) = \theta_i - p_1$ and $U_1(\theta_i, p_2)) =$

$\theta_i - p_2$. Hence, for all θ_i, $U_0 > U_1$ which leads to $\alpha^\star = 1$. This contradicts the fact that $\alpha^\star$ is close to zero.

Hence, if an equilibrium exists, it is not trivial and $\pi(\alpha^\star, Q) = \frac{Q}{\alpha^\star t} < 1$.
Eq(4.4) can be rewritten

$$(\theta_i - p_1)\frac{Q}{\alpha t} > \theta_i - p_2$$

$$\theta_i < \frac{p_2 - p_1\frac{Q}{\alpha t}}{1 - \frac{Q}{\alpha t}} = \theta^\star(\alpha). \tag{4.5}$$

Thus, for all $\theta_i \in [\underline{\theta}, \theta^\star(\alpha)[$, $U_0(\theta_i, \alpha, p_1) > U_1(\theta_i, p_2)$ and $\forall \theta_i \in]\theta^\star(\alpha), \overline{\theta}]$, $U_0(\theta_i, \alpha, p_1) < U_1(\theta_i, p_2)$. Then, for a given proportion of consumers choosing to park in the street, the optimal strategy for a consumer with a willingness to pay θ_i is given by

$$g(\theta_i, \alpha) = \{ \begin{array}{l} 1 \text{ if } U_0(\theta_i, \alpha, p_1) \geq U_1(\theta_i, p_2), \\ 0 \text{ if } U_1(\theta_i, p_2) > U_0(\theta_i, \alpha, p_1). \end{array}$$

For $\theta_i \leq \theta^\star(\alpha)$, the agent chooses to search a spot in the street and for $\theta_i > \theta^\star(\alpha)$, she directly goes to the garage parking. At the equilibrium, the proportion of consumers choosing to search, α, is equal to

$$\alpha = \mathbb{E}[g(\theta, \alpha)] = \int_{\underline{\theta}}^{\theta^\star(\alpha)} g(\theta, \alpha) m(\theta) d\theta + \int_{\theta^\star(\alpha)}^{\overline{\theta}} g(\theta, \alpha) m(\theta) d\theta = F(\theta^\star(\alpha)) \tag{4.6}$$

The two equations (4.5) and (4.6) guarantee the existence and the uniqueness of the equilibrium. □

Theorem 5

Let us first prove the existence of a socially optimal threshold $\tilde{\theta}$.

The function U is defined on a compact set $[\underline{\theta}, \overline{\theta}]$ and is continuous on $\tilde{\theta}$. The existence of a solution of $\min_{\tilde{\theta} \in [\underline{\theta}, \overline{\theta}]} U(\tilde{\theta})$ is guaranteed by the Weierstrass Theorem. Moreover, the social optimal $\tilde{\theta}$ is such that $\frac{Q}{\tilde{\alpha} t} \leq 1$ where $\tilde{\alpha} = F(\tilde{\theta})$.

Suppose that $\frac{Q}{\tilde{\alpha}t} > 1$, then $\exists \theta_1 > \tilde{\theta}$ such that $\frac{Q}{\tilde{\alpha}t} > \frac{Q}{\alpha_1 t} > 1$

$$\begin{aligned} U(\theta_1) &= \int_{\underline{\theta}}^{\theta_1} (\theta - p_1) m(\theta) d\theta + \int_{\theta_1}^{\overline{\theta}} (\theta - p_2) m(\theta) d\theta \\ &= \mathbb{E}(\theta) - p_1 F(\theta_1) - p_2(1 - F(\theta_1) \\ &= \mathbb{E}(\theta) + (p_2 - p_1) F(\theta_1) - p_2. \end{aligned}$$

For $\theta_1 > \tilde{\theta}$, $F(\theta_1) > F(\tilde{\theta})$, then $U(\theta_1) > U(\tilde{\theta})$ which contradicts the fact that $\tilde{\theta}$ is optimal.

The inefficiency of the equilibrium :
To show that the equilibrium $\theta^\star$ is not efficient, we can distinguish two cases

- $\tilde{\theta}$ is such that $\frac{Q}{F(\tilde{\theta})t} = 1$. In this case $\tilde{\theta} < \theta^\star$ since from Proposition 1, $\frac{Q}{F(\theta^\star)t} < 1$ therefore $F(\theta^\star) > F(\tilde{\theta})$ which proves the result.
- $\tilde{\theta}$ is such that $\frac{Q}{F(\tilde{\theta})t} < 1$. In this case, it is sufficient to prove that for all $\theta \geq \theta^\star, U'(\theta) < 0$. That is, for all $\theta \geq \theta^\star$, $U(\theta) < U(\theta^\star)$ and it exists $\theta < \theta^\star$, $U(\theta) > U(\theta^\star)$ which means that $\theta^\star$ is inefficient and the social optimum $\tilde{\theta} = argmax\{U(\theta)\} < \theta^\star$. For $\theta \in [\underline{\theta}, \overline{\theta}]$, we have

$$\begin{aligned} U(\theta) &= \int_{\underline{\theta}}^{\theta} U_0(s) f(s) ds + \int_{\theta}^{\overline{\theta}} U_1(s) f(s) ds \\ &= \pi(\theta, Q) \int_{\underline{\theta}}^{\theta} (s - p_1) f(s) ds + \int_{\theta}^{\overline{\theta}} (s - p_2) f(s) ds \\ &= \frac{Q}{F(\theta)t} [\int_{\underline{\theta}}^{\theta} s f(s) ds - p_1 F(\theta)] + \int_{\theta}^{\overline{\theta}} s f(s) ds - p_2(1 - F(\theta)) \\ &= (-1 + \frac{Q}{F(\theta)t}) [\int_{\underline{\theta}}^{\theta} s f(s) ds - p_1 F(\theta)] + \mathbb{E}(\theta) + (p_2 - p_1) F(\theta) - p_2. \end{aligned}$$

The derivative of this expression with respect to θ is

$$\begin{aligned} U'(\theta) &= -\frac{Q f(\theta)}{F^2(\theta)t} (\int_{\underline{\theta}}^{\theta} s f(s) ds - p_1 F(\theta)) + (-1 + \frac{Q}{F(\theta)t})(\theta f(\theta) - p_1 f(\theta)) + (p_2 - p_1) f(\theta) \\ &= [-\frac{Q}{F^2(\theta)t} (\int_{\underline{\theta}}^{\theta} s f(s) ds - p_1 F(\theta)) + (-1 + \frac{Q}{F(\theta)t})(\theta - p_1) + (p_2 - p_1)] f(\theta). \end{aligned}$$

The equilibrium threshold verifies the following equation

$$\theta^\star = \frac{p_2 - \frac{Q}{F(\theta^\star)t} p_1}{1 - \frac{Q}{F(\theta^\star)t}}$$

this condition can be expressed as

$$(1-\frac{Q}{F(\theta^\star)t})(\theta^\star-p_1)=p_2-p_1$$

and for all $\theta \geq \theta^\star$,

$$(1-\frac{Q}{F(\theta)t})(\theta-p_1)\geq p_2-p_1$$

.

By assumption A1, $\underline{\theta} > p_1$ then

$$-\frac{Q}{F^2(\theta)t}(\int_{\underline{\theta}}^{\theta} sf(s)ds - p_1F(\theta)) < 0$$

Hence, for all $\theta \in [\theta^\star, \overline{\theta}]$

$$U'(\theta) = [\underbrace{-\frac{Q}{F^2(\theta)t}(\int_{\underline{\theta}}^{\theta} sf(s)ds - p_1F(\theta))}_{<0} + \underbrace{(-1+\frac{Q}{F(\theta)t})(\theta-p_1)+(p_2-p_1)}_{<0}]f(\theta) < 0$$

which concludes the proof.

□

Proposition 9

First, we recall the expression of the total expected payoff in the case of a uniform distribution over the segment $[\underline{\theta}, \overline{\theta}]$.

$$U(\theta^\star) = \tfrac{Q}{2t}(\theta^\star+\underline{\theta}) - \tfrac{1}{2T}(\overline{\theta}^2 - \theta^{\star 2}) + p_2(1-F(\theta^\star)) - \tfrac{Q}{t}p_1$$

Differentiation with respect to p_1

Let us calculate the partial derivative of $U(\theta^\star)$ with respect to p_1.

$$\begin{aligned}\frac{\partial U(\theta^\star)}{\partial p_1} &= \frac{Q}{2t}\frac{\partial\theta^\star}{\partial p_1} + \frac{1}{T}\theta^\star\frac{\partial\theta^\star}{\partial p_1} - \frac{1}{T}p_2\frac{\partial\theta^\star}{\partial p_1} - \frac{Q}{t}\\ &= \frac{Q}{2t}\frac{\partial\theta^\star}{\partial p_1} + \frac{1}{T}\frac{\partial\theta^\star}{\partial p_1}(\theta^\star - p_2) - \frac{Q}{t}.\end{aligned}$$

We already shown that $\frac{\partial\theta^\star}{\partial p_1} < 0$ and $\theta^\star > p_2$ then $\frac{\partial U(\theta^\star)}{\partial p_1} < 0$

Differentiation with respect to p_2

Let us calculate the partial derivative of $U(\theta^\star)$ with respect to p_1

$$\begin{aligned}\frac{\partial U(\theta^\star)}{\partial p_2} &= \frac{Q}{2t}\frac{\partial\theta^\star}{\partial p_2} + \frac{1}{T}\theta^\star\frac{\partial\theta^\star}{\partial p_2} + (1 - F(\theta^\star)) - \frac{1}{T}p_2\frac{\partial\theta^\star}{\partial p_2}\\ &= \frac{Q}{2t}\frac{\partial\theta^\star}{\partial p_2} + \frac{1}{T}\frac{\partial\theta^\star}{\partial p_2}(\theta^\star - p_2) + (1 - F(\theta^\star)).\end{aligned}$$

Since $\frac{\partial\theta^\star}{\partial p_2} > 0$ and $\theta^\star - p_2 > 0$ then $\frac{\partial U(\theta^\star)}{\partial p_2} > 0$.

Differentiation with respect to t

The derivative of the total expected payoff with respect to t is given by :

$$\begin{aligned}\frac{\partial U(\theta^\star)}{\partial t} &= \frac{Q}{2t}\frac{\partial\theta^\star}{\partial t} - \frac{Q}{2t^2}(\theta^\star + \underline{\theta}) + \frac{1}{T}\theta^\star\frac{\partial\theta^\star}{\partial t} - \frac{1}{T}p_2\frac{\partial\theta^\star}{\partial t} + \frac{Q}{2t^2}p_1\\ &= \frac{1}{T}\frac{\partial\theta^\star}{\partial t}(\theta^\star - p_2) + \frac{Q}{2t}\frac{\partial\theta^\star}{\partial t} + \frac{Q}{2t^2}(p_1 - \theta^\star - \underline{\theta})\end{aligned}$$

Since $\theta^\star > p_2$ and $\theta^\star > p_1$ and $\frac{\partial\theta^\star}{\partial t} < 0$ then $\frac{\partial U(\theta^\star)}{\partial t} < 0$.

CHAPTER 5

Paradigm Shift

This chapter is based on the paper "Paradigm shift : A mean field game approach" which is a joint work with Professor Damien Besancenot (Besancenot and Dogguy 2014).

Abstract

This chapter analyses the consequences of young researchers' scientific choice on the dynamics of sciences. We develop a simple two-state mean field game model to analyze the competition between two paradigms based on Kuhn's theory of scientific revolution. The dynamics of the model is driven by the scientific choice of young researchers at the beginning of their career. Despite the possibility of multiple equilibria, the model exhibits at least one stable solution in which both paradigms coexist. The occurrence of shocks on the parameters may induce the shift from one paradigm to the other. During this shift, researchers' choice is proved to be having a great impact on the evolution of sciences.

5.1 Introduction

Social sciences, among other disciplines, are consistently subject to conceptual or methodological swings. In economics for instance, Transaction Costs analysis gave way to Agency Theory, Endogenous Growth appeared at the expense of Standard Growth Theory and, more recently, Behavioral Finance deeply challenged the standard Efficient Market Hypothesis. Such an evolution suggests the existence of life cycles affecting research agendas or paradigm shifts, a core concept developed in Kuhn 1970.

In a broad sense, a paradigm may be defined as a set of theories and empirical methodologies which allow a scientific community to identify, frame and solve problems and serve as a foundation for future scientific discoveries. During periods of normal science, the dominant paradigm helps report interesting or surprising findings and remains dominant as long as it stays attractive for the large majority of researchers. [1] During a paradigm shift, two simultaneous changes are supposed to occur: the decline of the old paradigm, when the paradigm begins to fail solving problems and explaining anomalies and the emergence of a new one when a new theoretical corpus leads to the publication of promising results. During these changes, the hope of new discoveries modifies the scientific choices of researchers who progressively abandon the traditional fields of research in favor of a new set of assumptions.

Driven first by scientific considerations, the paradigm shift also appears as a social fact involving the entire community of scientists. During crises, the increase in the number of researchers involved in the new scientific approach induces a social phenomenon which cumulatively fosters its attractivity. The presence of more researchers in an academic field simultaneously increases the potential audience for a given research, makes it easier to find efficient co-writers, guarantees an easier access to publication mediums and contributes to simplifying the funding of research. The expansion of the scientific community interested in a scientific

1. Hereafter, we will refer to as "dominant paradigm", the paradigm which attracts the majority of researchers and thus defines what can be considered as orthodoxy. Other approaches, sustained by a minority of researchers, will be qualified as "dominated paradigms".

field thus influences –per se– the researchers' scientific choice. When the new set of assumptions attracts most of the new generation of scientists, researchers who stay working in the old school see their influence diminished and their contribution rapidly marginalized.

Demographic elements also contribute to the dynamics of science. History of sciences provides various examples to illustrate the fact that the retirement of one generation of elite scientists and their replacement by a new generation allows the latter to develop new theories or approaches more easily (Barber 1961).

Besides, one cannot neglect the stimulus brought to researchers through paradigm competition. According to Kuhn 1970, "*Competition between segments of the scientific community is the only historical process that ever actually results in the rejection of one previously accepted theory or in the adoption of another*". During periods of normal science, opponents to the dominant approach highlight the existence of anomalies which seem inconsistent with the leading paradigm. In answer, supporters of the paradigm spend a large part of their career in the process of puzzle solving, an activity which allows to comfort the established framework. Paradigm competition appears as one additional driving forces of scientific productivity.

We aim at considering the various determinants of paradigm shifts. In particular, we mainly focuse on the researchers' choice of their scientific agenda and the consequences of such choices in the general evolution of science.

If this approach clearly deals with various aspects of Kuhn's work, we do not claim to formalize his theory. Our purpose is to focus on the various conditions that contribute to the decline of a paradigm and the shift to a new one. For this purpose, we build a highly stylized mean field game closely related to Guéant's (2009) description of the workers' choices in a two-sector economy.

We consider, here, an economy with a continuum of researchers and two competing paradigms. Researchers produce homogeneous papers according to a production function which reflects the competition between the two paradigms. At each point in time, a fraction of researchers quit academia. They are replaced by an equivalent number of young researchers. Each of them has to choose in which paradigm he or she wants to carry out his or her work. Two factors motivate the choice of these young researchers at the beginning of their career: the intertemporal remuneration scheme (social or monetary) and personal preferences.

A priori, the young researchers' scientific choice is first influenced by their affinity with various topics. They will choose according to their taste, their greater or lesser reluctance to treat the issues at stake or their desire to engage in riskier issues (Alon 2009). However, in their choice, young scientists cannot ignore the influence of the remuneration scheme offered by each of the two paradigms. As any scientist, a young researcher seeks social recognition, a recognition which comes with the publication of new results and is dramatically linked to the possibility of creating and disseminating new knowledge (Stephan 1996). Besides, monetary compensation is highly related to the academic resume and the individual scientific production of the researcher (see for instance A. M. Diamond 1986 and Swidler and Goldreyer 1998). As this scientific production is influenced by the proportion of researchers working in the same paradigm, the dynamics of the population distribution between the two paradigms has a crucial influence on the young researchers' choice.

According to the initial values of the parameters, our model may exhibit one or two stable equilibria. In each equilibrium, the two paradigms always coexist; one paradigm attracts the majority of the researchers (it is therefore dominant) while the other remains in the minority (and is dominated). In these equilibria, coexistence is due to the voluntary choice by some young researchers of a research agenda in the dominated paradigm, even if this agenda is not intended to lead to major innovations.

When the model allows for two stable equilibria, the equations give no indication as to which of the two competing paradigms should become predominant. Both paradigms could possibly become dominant and the hierarchy is inherited from the history of the scientific field which led to the initial distribution of researchers between the two paradigms (for instance, one of the paradigms may have existed for some time and is partially depleted while the other is just emerging). In this case, a paradigm shift may occur if random shocks on the parameters contribute to eliminating the dominant paradigm as a stable equilibrium. After such shocks, the vast majority of young researchers will be attracted by the new paradigm which allows for a rising remuneration as long as the number of researchers involved in the paradigm increases.

While analyses of the dynamics of sciences belong now to a well established field of research in economics, there are only few theoretical analyses that offer a formal model of paradigms evolution. As a related work, we can refer to Sterman and Wittenberg 1999 who provide a Kuhnian dynamic model in which paradigm changes are conditioned by positive feedback loops. Bramoullé and Saint Paul 2010 developed an overlapping generation model in which researchers allocate their working time between old or new fields of research in order to maximize the authors' reward. At each period, one paper published in a given paradigm yields both a citation premium increasing with the future number of contributions to the paradigm and a direct remuneration linked to the intrinsic value of the paper. The model exhibits solutions with various properties according to the values of the parameters. The model allows for periods of emergence of new paradigms and periods of exploitation of old ones. In some cases, sunspots may occur where expectations of a high payoff in investment in a scientific field attract lots of researchers and allow for self-fulfilling expectations. More recently, Faria et al. 2011 worked out a hierarchical differential game between editors and authors. The production of scientific knowledge is analyzed as the extraction of potential knowledge seen as an exhaustible resource. Editors can accelerate or slow down knowledge production and paradigm depletion may occur when editors allow for a fast rate of knowledge extraction. This model considers paradigm depletion as the result of an optimal process but does not directly deal with the problem of paradigm shift. Within the literature available, our model presents a greater affinity with the work of Brock and Durlauf 1999 who developed a model in which researchers' scientific choice is made by reference to conformity. Their model puts a special emphasis on the tendency for individual scientists to place a greater weight on theories accepted by the majority of the academic community. Under this assumption, the authors put forward a multiplicity of equilibria and the possibility of jumping from one equilibrium to the other in case of shock on the parameters. Our approach differs from this work in three ways. First we develop a model in which the arguments of the scientific choice are directly linked to the scientific reward scheme. In their choice, researchers perfectly take into account the future possibilities of paper publications and the social and monetary rewards that come with the academic resume. Second, our model allows for the taking into account of the demographic dimension of the problem and its influence on the paradigm shift. Third, the model is built on the mean field game approach introduced in the first chapter.

In a standard Mean Field Game, the dynamics of the system is governed by two equations: a backward Hamilton-Jacobi-Bellman equation describing the optimal behavior of agents given the distribution of the other players and a forward Kolmogorov equation which takes into account the influence of each player on the mean field. The Nash equilibrium of the game appears as the solution of these two equations. In this model, we consider a simplified model based upon a system of ordinary differential equations while keeping the general characteristics of a mean field game. Here the mean field is formalized by the distribution of researchers between two competing paradigms. It reflects the historical choices made by the successive generations of researchers and influences the young researcher's choice through the reward scheme in a dynamic framework.

The chapter is organized as follows. Section 2 presents some stylized facts about paradigm shifts in growth theory. Section 3 introduces our main assumptions about the researcher's payoffs, their productivity and the dynamics of the model, given the young researchers' choice. Section 4 provides a numerical simulation of the model for various values of the parameters and discusses the results in terms of paradigm shifts. A last section concludes the chapter.

5.2 An example of paradigm shifts: Growth Theory

Since the second half of the last century, growth theory has given rise to four different approaches, each new theory challenging the previous one. In order to

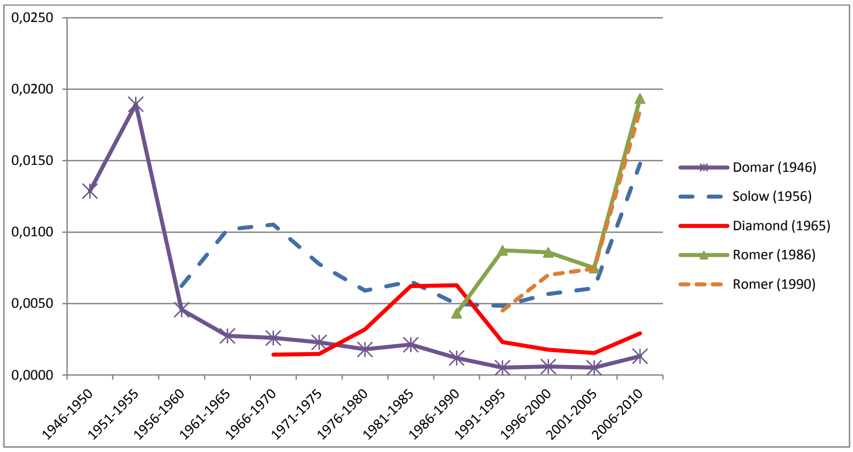

Figure 5.1: Citations of seminal papers in Growth Theory

emphasize their interaction, Figure 5.1 considers five seminal papers and measures the influence of the paradigm that they helped to found through the relative number of their citations. For each period, Figure 5.1 gives the ratio between the number of citations received by a paper and the number of papers linked to "Economic Growth" according to Google Scholar. In the WWII aftermath, growth theory was mostly influenced by the main contributions of Harrod 1939 and Domar 1946. Growth was considered in a Keynesian perspective and, if the authors showed that the economic system could follow an equilibrium growth path (on a knife-edge), they also stated that there was no natural reason for an economy to achieve balanced growth since the system has no equilibrating force. The paper by Solow 1956 brought an opposite conception of growth. According to the "neoclassical" model, long-run growth must be stable. However, as growth is linked to the accumulation of capital, the diminishing returns of capital imply that economies must eventually reach a steady state. At this stage, any increase in capital will no longer induce growth and economies can only continue growing by inventing new technologies. In this approach, the process by which countries continue growing is "exogenous". Figure 1 illustrates both the growing influence of this new approach from the mid-60s to the late 70s and, during the same period, the relative loss of interest of the economists for the Harrod-Domar model. It is worth noting that even if from the mid-60s the relative number of papers explicitly referring to Domar's contribution follow a stable decreasing path, the paper has still been receiving an important number of citations. Even though it is a minority, there are still a significant number of researchers working on the topic of the old dominant paradigm.

To a lesser extent, the overlapping generation model initiated by P. A. Diamond 1965 underlaid a new strand of research. Relaxing the assumption of infinitely lived agents, this approach highlights the possibility for the decentralized competitive equilibrium to be different from that of the social planner's choice and indeed even not to be Pareto efficient. Once again, Figure 5.1 shows that the progressive success of this approach came simultaneously with a relative decline in the influence of Solow's neoclassical model.

A more interesting phenomenon occured in the late 80s and the early 90s with the first papers unveiling the possibility of endogenous growth. During this period, economists unsatisfied with Solow's explanation worked to "endogenize" technology. They developed a new growth theory that includes an explanation of technological advancement. Research in this area focused on education, innovation and technological change. In this new paradigm, economic growth became an endogenous outcome of the economic system. Figure 5.1 considers the influence of this new paradigm through two contributions by Romer 1986, 1990. After the publication of the first paper, more economists started working on this new theory at the expense of the OLG model which lost attractivity: a new paradigm shift occurred. However, as Romer 1986 explicitly traced his work back to Solow's model and recognizing it as a continuation of the previous work, the interest of Solow's seminal paper was clearly renewed and its relative number of citations exploded. Here, the old paradigm clearly benefited from the development of the new one and the new paradigm developed by reference to the old one.

The model in the next section will attempt to capture the stylized facts highlighted by the evolution of growth theory.

5.3 The model

We consider an academic world made up of a continuum of researchers of size 1. Each researcher practices his/her skills in one of the two available paradigms. Hereafter, a researcher working in paradigm i will be referred to as an $i - researcher$. Except for their preferences, researchers are assumed to be homogeneous.

At each point in time, a fraction of researchers quit the academic world (through voluntary departure, or involuntarily through retirement or death) and are replaced by an equivalent number of new researchers. Young researchers have then to decide in which paradigm they want to carry out their research. This decision is final. The choice will depend on the researchers' reward structure which includes two different items: an intrinsic remuneration linked to the researcher's affinity with his/her research agenda, and an extrinsic one which results from his/her research activity. The assumption that a young researcher makes a definitive choice of his/her research topic at the beginning of his/her career is purely technical. However, it perfectly matches with Kuhn's quotation of Max Plank: "a new scientific truth does not triumph by convincing its opponents [...], but rather because its opponents eventually die, and a new generation grows up that is familiar with it" (Kuhn 1970, p.150). It reflects the resistance by senior researchers to scientific changes. [2]

5.3.1 Researchers extrinsic remuneration

The extrinsic reward of an academic work is composed of two different elements: a social reward linked to the interest paid by the scientific community to the researcher's work and a financial reward, typically the salary of the researcher. These elements will be formalized through three main variables:

– Let us denote by $Q_i(t)$ the number of papers published at date t by a representative $i-$researcher (papers quality is assumed to be homogeneous and

2. After a paradigm shift, an academic resume incorporating mostly papers written in the former dominant paradigm would be highly devaluated. The more the time spent in a dominant paradigm, the higher the cost in case of paradigm shift.

$Q_i(t)$ also gives a qualitative measure of the scientific production of the $i-researchers$). According to Merton 1957, the scientific community awards recognition for being the first to communicate a new knowledge. Publication, which is a necessary step in establishing priorities, thus appears as a proof of efficiency and the larger the number of publications in an academic resume the higher the peer social recognition (Stephan 1996). Moreover, the financial part of the researchers' reward is largely influenced by his/her academic resume. The role played by the number of publications or citations in an academic career has largely been documented in the academic literature (A. M. Diamond 1986 or Swidler and Goldreyer 1998). Hereafter, social and monetary rewards will thus be assumed increasing with $Q_i(t)$.

- Let $N_i(t)$ denote the number of $i - researchers$ at date t. The greater the population of researchers potentially interested in a scientist's work, the more his/her work will be cited and the larger will be his/her scientific reputation. Thus, the researcher's social reward in paradigm i increases with $N_i(t)$. [3]
- As funding agencies may want to promote some specific research, they may offer special subsidies to researchers involved in this field of research. In the paper at hand, $m_i \in [1, \infty]$ measures the level of these monetary incentives. When $m_i = 1$, funding agencies provide no incentive for researchers to work in the scientific area i. For $m_i > 1$, the higher is m_i and the higher are the incentives to become a i-researcher.

Finally, we assume that the instantaneous value, $\omega_i(t)$, of the researchers' extrinsic remuneration (social and monetary) presents a multiplicative shape and is given by:

$$\omega_i(t) = \omega_i(m_i, N_i(t), Q_i(t)) = m_i N_i(t) Q_i(t)$$

3. Remunerations usually depend on the relative position of researchers among peers and should not necessarily be increasing with $N_i(t)$: A more realistic model should take into account this dimension (for instance, see Besancenot et al. 2012 for a model in which rankings influence the researchers' remuneration). However, this cannot be taken into account in this model where all $i - researchers$ present the same instantaneous academic production.

and, at date t the intertemporal expected remuneration for an $i - researcher$ is given by:

$$u_i(t) = \mathbb{E}\left[\int_t^{t+T} \omega_i(m_i, N_i(s), Q_i(s))\, e^{-\alpha(s-t)} ds\right] \tag{5.1}$$

Here α is the discount rate and T is a random variable that indicates the time spent in the research field i by an $i - researcher$. Assuming that the variable T follows an exponential law of rate λ, Eq.(5.1) takes the simplified shape [4]:

$$u_i(t) = \int_t^{\infty} \omega_i(m_i, N_i(s), Q_i(s))\, e^{-(\alpha+\lambda)(s-t)} ds. \tag{5.2}$$

5.3.2 Specific assumptions

In order to obtain tractable solutions, the $i - researcher$'s production function will be formalized through a classical CES function [5]:

$$Q_i(t) = \left(a_i N_i^r(t) + (1 - a_i)\left(N_{-i}(t)\, Q_{-i}(t)\right)^r\right)^{1/r}. \tag{5.3}$$

where:

1. $N_i(t)$ is the number of i-researchers.
2. $N_{-i}(t)$ is the number of researchers in the competing paradigm. As the continuum of researchers is of size one, we have $N_{-i}(t) + N_i(t) = 1$.

4. The exponential law describes the life of an individual when the death occurs at a constant average rate. The rate parameter λ measures the probability of death of the researcher at each point in time and represents the reciprocal of the average life of individuals (Balakrishnan and Basu 1996).

5. Under this assumption, the case $N_i = 0$ could rise a formal problem as the production function would allow for some scientific production in the field of research i while no researcher would be involved in this specific field. In our model, however, this difficulty is avoided as the case $N_i = 0$ is inconsistent with the equilibrium solution.

3. $N_{-i}(t)\,Q_{-i}(t)$ measures the number of papers published within the competing paradigm.

4. a_i is a specific constant measuring the dependence of paradigm i with respect to its rival. A high level of a_i reveals an autonomous field of research in which researchers are poorly influenced by the scientific activity of the other field.

The rationale behind such a function is straightforward. Other things remaining the same, an i-researcher's productivity is fostered by the number $N_i(t)$ of researchers involved in the same paradigm. More researchers means more conferences in which one can receive critics about his work and discuss with other academic fellows the new scientific developments of the paradigm. More researchers involved in a scientific field also means more opportunity of collaborations which increase productivity (see for instance Mcdowell and Melvin 1983, Landry et al. 1996 or Abrahamson 2009) and induces a greater number of reviews in which one can publish his/her work (Stigler et al. 1995).

Besides, competition between paradigms plays a crucial role on scientific productivity. During periods of normal science, while opponents to the dominant approach highlight the existence of anomalies which seem inconsistent with the leading paradigm, supporters of the paradigm spend a large part of their career to comfort the established framework. In economics, a good illustration of such a phenomenon can be found in the evolution of the efficient market hypothesis in reaction to the systematic research of anomalies in the financial market by supporters of behavioral finance (Schwert 2003).

This opposition is formalized by the specific constant a_i which captures the intrinsic dynamism of the paradigm i and its stage in the paradigm shift. From its rise until its decline, a paradigm's life is subject to random shocks that affects its relation *vis-à-vis* its competitor. In the early years of the new paradigm i, some researchers are disappointed by the results of the dominant concepts and start pursuing alternative topics or methodology in the hope that a new set of tools or assumptions would bring better results. At this stage, the new approach defines itself by opposition to the dominant paradigm and a_i is rather low. However, a shock on a_i can occur if the new set of assumptions starts allowing to report interesting or surprising findings. In such a case, a_i increases as authors become more interested in the development of the new results than by the criticism of the old ones. Finally, a_i may decrease when the most important problems of the field are solved or proven to be unsolvable. In this case, new papers in the field bring fewer innovations and researchers will spend most of their time trying to answer the critics raised by the competing paradigm.

5.3.3 Intrinsic remuneration and the young researchers' choice

At the beginning of his academic life, each researcher has to choose the sector in which he/she will work for the rest of his/her life. In this choice, the remuneration offered by each field of research plays a determining role; however, the young researchers will also take into account their personal preferences among the various academic fields (Alon 2009, Stephan 1996). Here, researcher's preferences are modeled by a random variable μ which measures the value for a young researcher of building his/her career in the first paradigm. By assumption each researcher is characterized by his/her own μ, and this value is distributed over the researchers' population according to a standard normal law.

When the two research agendas bring the same intertemporal remuneration $u_1(t) = u_2(t)$, Cf. Eq. (5.2), a researcher will choose the first paradigm for any μ positive and the second one for a negative μ. When the intertemporal remunerations exhibit significant differences, a young researcher may nevertheless choose the less remunerative if he/she exhibits strong preferences for this field of research. Formally, the decision rule for a young researcher will choose the first area if and only if[6]:

$$u_1(t) + \mu \geq u_2(t). \tag{5.4}$$

Let us consider an infinitesimal interval $[t, t+dt]$. According to the previous assumptions, during this time period a proportion λdt of researchers retires both for

6. We made the assumption that the young researchers have perfect foresight.

sector 1 and sector 2 and a population of size λdt enters the academic world. The proportion of new researchers that choose sector 1 is given by:

$$\mathbb{P}\left(u_1\left(t\right)+\mu \geq u_2\left(t\right)\right)=F\left(u_1\left(t\right)-u_2\left(t\right)\right), \tag{5.5}$$

where F is the cumulative distribution function of a standard normal variable.

Thus, the system is governed by the following two equations:

$$\left\{\begin{array}{l}\dot{N}_1\left(t\right)=-\lambda N_1\left(t\right)+\lambda F\left(u_1\left(t\right)-u_2\left(t\right)\right)\\ \dot{N}_2\left(t\right)=-\lambda N_2\left(t\right)+\lambda F\left(u_2\left(t\right)-u_1\left(t\right)\right)\end{array}\right. \tag{5.6}$$

Hereafter, we will use the variable $\Delta u = u_1 - u_2$.

From Eq.(5.2) and (5.6), we can now describe the dynamics of the model:

Proposition 1 An equilibrium of the mean field game is defined by any couple $(N_1(t), \Delta u(t))$ which satisfies the two dynamic equations [7]:

$$\left\{\begin{array}{c}\dfrac{dN_1\left(t\right)}{dt}=-\lambda N_1\left(t\right)+\lambda F\left(\Delta u\left(t\right)\right)\\ \dfrac{d\Delta u\left(t\right)}{dt}=\left(\alpha+\lambda\right)\Delta u\left(t\right)-\left[\omega_1\left(N_1\left(t\right),N_2\left(t\right)\right)-\omega_2\left(N_1\left(t\right),N_2\left(t\right)\right)\right]\end{array}\right. \tag{5.7}$$

with an initial condition on N_1, $N_1\left(0\right)$, and a terminal condition on Δu, $\lim_{t\to\infty} e^{-(\alpha+\lambda)t}\Delta u\left(t\right)=0$.

Proof. Remark that the first equation of the system (5.7) and (5.6) are the same. In a same way, (5.7) and the terminal condition verified by Δu are equivalent to the integral form (5.2) above. Indeed, after subtraction of the term $\left(\alpha+\lambda\right)\Delta u\left(t\right)$ from both sides of (5.7) and multiplication by $-e^{-(\alpha+\lambda)t}$ we get:

7. The system of differential equations presented above is typical of mean field game. The first equation which is forward can be identified to the Kolmogorov equation whereas the second one, backward, replaces the Hamilton-Jacobi-Bellman equation (Guéant 2009).

$$\frac{d\left[e^{-(\alpha+\lambda)t}\Delta u\left(t\right)\right]}{dt}=-e^{-(\alpha+\lambda)t}\left[\omega_{1}\left(N_{1}\left(t\right),N_{2}\left(t\right)\right)-\omega_{2}\left(N_{1}\left(t\right),N_{2}\left(t\right)\right)\right] \quad (5.8)$$

After integration with respect to t, and under the terminal condition, this is equivalent to:

$$-e^{-(\alpha+\lambda)t}\Delta u\left(t\right)=-\int_{t}^{\infty}e^{-(\alpha+\lambda)s}\left[\omega_{1}\left(N_{1}(s),N_{2}(s)\right)-\omega_{2}\left(N_{s}(t),N_{2}(s)\right)\right]ds, \quad (5.9)$$

which finally leads to Eq.(5.2):

$$\Delta u\left(t\right)=\int_{t}^{\infty}e^{-(\alpha+\lambda)(s-t)}\left[\omega_{1}\left(N_{1}\left(s\right),N_{2}\left(s\right)\right)-\omega_{2}\left(N_{1}\left(s\right),N_{2}\left(s\right)\right)\right]ds. \quad (5.10)$$

□

5.3.4 Properties of the steady states

The next proposition states the main result of the paper :

Proposition 2

The dynamical system of Eq.(5.7) admits at least one steady state equilibrium given by:

$$\begin{cases} N_{1}^{*}=F\left(\dfrac{\omega_{1}\left(N_{1}^{*},1-N_{1}^{*}\right)-\omega_{2}\left(N_{1}^{*},1-N_{1}^{*}\right)}{\alpha+\lambda}\right) \\ \Delta u^{*}=\dfrac{\omega_{1}\left(N_{1}^{*},1-N_{1}^{*}\right)-\omega_{2}\left(N_{1}^{*},1-N_{1}^{*}\right)}{\alpha+\lambda} \end{cases} \quad (5.11)$$

Proof. From Eq.(5.7) a steady state satisfies the following condition :

$$\begin{cases} 0=-\lambda N_{1}+\lambda F\left(\Delta u\right) \\ 0=\left(\alpha+\lambda\right)\Delta u-\left[\omega_{1}\left(N_{1},1-N_{1}\right)-\omega_{2}\left(N_{1},1-N_{1}\right)\right] \end{cases} \quad (5.12)$$

The second equation gives Δu^{*}; plugging Δu^{*} in the first equation leads to N_{1}^{*}. The existence of this solution is a simple application of the intermediate value theorem. Indeed, as $\omega_i = m_i N_i Q_i$, the difference $\omega_1 - \omega_2$ is bounded, hence if we consider the function :

$f(N_1) = N_1 - F\left(\frac{1}{\alpha+\lambda}\left(\omega_1(N_1, 1-N_1) - \omega_2(N_1, 1-N_1)\right)\right)$

we get $f(0) < 0$ and $f(1) > 0$. This concludes the proof. □

It now remains to study the dynamical properties of the system and the nature of each steady state. Let us linearize the system (5.7) in the neighborhood of each steady state $(N_1^*, \Delta u^*)$:

$$\begin{cases} \frac{dN_1}{dt}(t) = -\lambda N_1(t) + \lambda \Delta u(t) F'(\Delta u^*) \\ \frac{d\Delta u}{dt}(t) = (\alpha+\lambda)\Delta u(t) - [\partial_1\omega_1 - \partial_2\omega_2 - \partial_1\omega_2 + \partial_2\omega_2](N_1^*, 1-N_1^*) N_1(t) \end{cases}$$

The nature of the steady state is given by the sign of the eigenvalues of the following matrix :

$$M = \begin{pmatrix} -\lambda & \lambda F'(\Delta u^*) \\ -[\partial_1\omega_1 - \partial_2\omega_2 - \partial_1\omega_2 + \partial_2\omega_2](N_1^*, 1-N_1^*) & \alpha+\lambda \end{pmatrix}$$

Proposition 3

Under the transversality condition : $\lim_{t\to\infty} e^{-(\alpha+\lambda)t}\Delta u(t) = 0$, any initial condition on $N_1(0)$ leads to a convergent path towards a stable steady state equilibrium of the model.

Proof. The mean field equation Eq.(5.7) presents the evolution of the system at the equilibrium. It is a coupled Forward/Backward system of equations which is difficult to study since we do not know the value of Δu at $t = 0$. Transversality condition links Δu at $t = 0$ with the value of $N_1(t)$ at equilibrium and guarantees that for any initial value $N_1(0)$, the equilibrium $(N_1(t), \Delta u(t))$ converges to a stable steady state. Imagine that the terminal condition is verified on a trajectory that diverges. Since ω_i is bounded there exists $C > 0$ such that $\forall N \in [0,1]$, $|\omega_1(N, 1-N) - \omega_2(N, 1-N)| \leq C$. From proposition 1 and under the transversality condition, $\Delta u(t)$ is given by :

$$\begin{aligned}
\Delta u\left(t\right) &= \int_t^{\infty} \omega_1\left(N_1\left(s\right), 1-N_1\left(s\right)\right) - \omega_2\left(N_1\left(s\right), 1-N_1\left(s\right)\right) e^{-(\alpha+\lambda)(s-t)} ds \\
&\Rightarrow \quad \left|\Delta u\left(t\right)\right| \leq \int_t^{\infty} \left|\omega_1\left(N_1\left(s\right), 1-N_1\left(s\right)\right) - \omega_2\left(N_1\left(s\right), 1-N_1\left(s\right)\right)\right| e^{-(\alpha+\lambda)(s-t)} ds \\
&\Rightarrow \quad \left|\Delta u\left(t\right)\right| \leq C \int_t^{\infty} e^{-(\alpha+\lambda)(s-t)} ds \\
&\Rightarrow \quad \left|\Delta u\left(t\right)\right| \leq \frac{C}{\alpha+\lambda}
\end{aligned}$$

But, by assumption, $\lim_{t \to \infty} \left|\Delta u\left(t\right)\right| = +\infty$. This is not possible then this trajectory is not compatible with the terminal condition on Δu. □

5.4 Numerical simulations

We have seen above that the differential system admits at least one steady state Eq.(5.11) but the number of solutions depends upon the value of the variables a_1, a_2, m_1 and m_2. In this section, we consider three important cases presented in table 5.1. Hereafter, we will take $r = 0.25$.

case	a_1	a_2	m_1	m_2	Number of stationary solutions
Case 1	0.5	0.5	1	1	3
Case 2	0.2	0.6	1	1	1
Case 3	0.6	0.2	1	1	1
			1	1.5	3

Table 5.1: Table of parameter values

5.4.1 Case 1 : The two paradigms are symmetric

In Figure 5.2 we plot the graph of the identity function on $[0, 1]$ and the function $N_1 \mapsto F\left(\frac{\omega_1 - \omega_2}{\alpha+\lambda}\right)$. It shows the existence of three fixed points which are $N_1^{*,1} = 0.0493$, $N_1^{*,2} = 0.5$ and $N_1^{*,3} = 0.9506$.

Fixed point	value	Determinant	Nature
$N_1^{*,1}$	0.0493	-0.0023	Saddle point
$N_1^{*,2}$	0.5	0.0020	Repulsive point
$N_1^{*,3}$	0.9506	-0.0023	Saddle point

Table 5.2: Dynamical properties of stationary solutions

To study the nature of each steady states, we have to compute the determinant of the matrix M in each steady state. The results in table 5.2 indicate that equilibria $N_1^{*,1}$ and $N_1^{*,3}$ are stable.

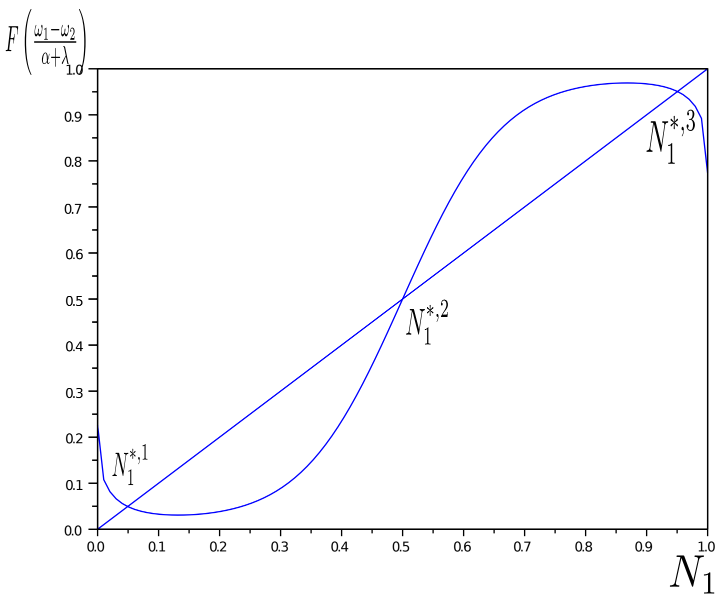

Figure 5.2: Case 1 : The two paradigms are symmetric

5.4.2 Case 2 : Paradigm 2 is more active

In this case $a_1 < a_2$. The unique fixed point is equal to $N_1^* = 0.0083$ as shown in figure. The stationary solution of the system is a saddle point since the determinant of the matrix M is equal to -0.0047.

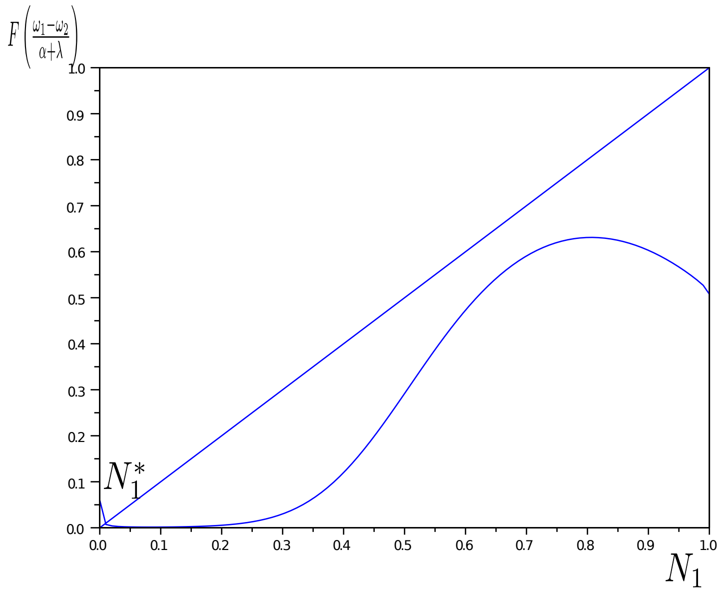

Figure 5.3: Case 2 : paradigm 2 is more active

Fixed point	value	Determinant	Nature
$N_1^{*,1}$	0.2461	-0.003	Saddle point
$N_1^{*,2}$	0.5024	0.0034	Repulsive point
$N_1^{*,3}$	0.9916	-0.0046	Saddle point

Table 5.3: Dynamical properties of stationary solutions

5.4.3 Case 3 : Paradigm 1 is more active

In this case, $a_1 > a_2$. When $m_1 = 1$ and $m_2 = 1.5$, there exist three steady states as represented by the dashed curve in figure 5.4. However, the institutional factor may considerably change the dynamic properties of the system. Indeed, when $m_1 = m_2 = 1$, the system admits a unique fixed point with $N_1^{*,3} = 0.9916$ and $\Delta u^* = 2.3953$. This solution is a saddle point since the determinant of the matrix M is equal to -0.0047. The results are summarized in table 5.3.

If paradigm 2 was initially dominant, the disparition of equilibrium $N_1^{*,1}$ implies a paradigm shift with a dynamic convergence toward equilibrium $N_1^{*,3}$.

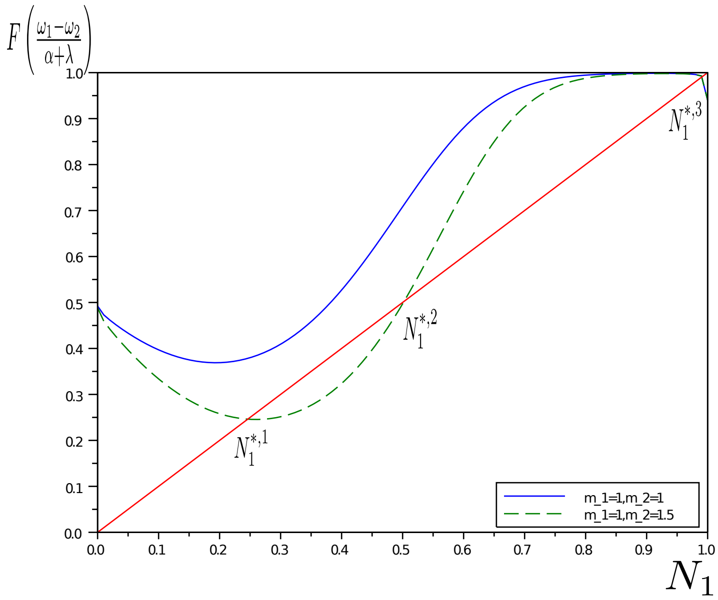

Figure 5.4: Case 3 : paradigm 1 is more active

5.4.4 The paradigm shift

Whatever be the case considered in the previous section, the two competing paradigms always coexist in equilibrium. However, in the stable equilibria the academic landscape is asymmetric by nature. One scientific approach appears as dominant, attracting a large majority of researchers, while the other, clearly dominated, is in minority. Coexistence is guaranteed in this equilibrium because each paradigm is the complement of the other. The dominant field of research is stimulated by the researchers' critics from the competing research field while these researchers find easily matters for criticism in the massive scientific production of the dominant paradigm. Note that, in case 1, the two paradigms could be potentially dominant. The hierarchy between the two paradigms is only due to historical choices made by past researchers who mostly chose one of the two paradigms.

In this model, paradigm shifts may appear as the consequence of successive and unanticipated shocks on the values of the parameters. Shocks may affect the relative values of a_i or be the consequence of political choices that affect the m_i

values.

Note first that the number of stable equilibria is highly dependent on the relative a_i values. Figure 5.5 maps the total number of steady states of the model according to the values of the variables a_1 and a_2 when $m_1 = m_2 = 1$. With a small a_i paradigm i has only little chance to become dominant. During periods

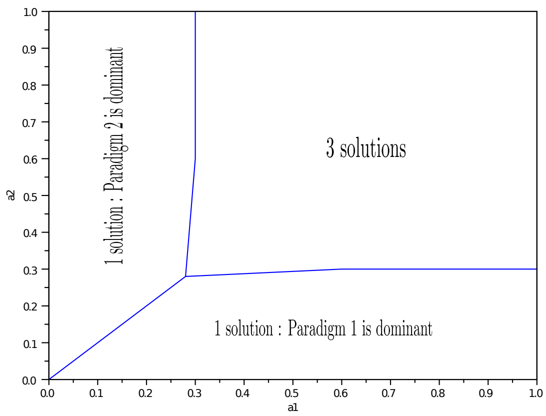

Figure 5.5: Number of steady states of the model

of normal science, researchers only focus on the development of the dominant paradigm (hereafter paradigm 2, with $N_1 = N_1^{*,1}$ in Figure 5.2). Researchers are mainly interested in improving the assumptions or methodology inside paradigm 2 and a_2 is close to one. In such a period, results from the dominated field of research are neglected in the scientific debate and researchers involved in these topics have to define themselves in opposition to the dominant paradigm, a_1 is low.

Apparition of anomalies brings an important shock to the model and changes the nature of the equilibrium. As more puzzles appear inconsistent with dominant concepts, new possibilities of analysis are considered by young researchers who start studying these problems with a greater autonomy, a_1 rises. At the same time, researchers from the dominant paradigm have to spend more time to address the criticisms of their challengers: a_2 drops. Under our specific assumptions this implies a lower number of publications in the dominant field and a slide in the social and monetary remuneration for researchers involved in this paradigm. As the opposite effects are at work in the other field, it becomes more attractive for young researchers. When a radical change affects a_1 and a_2, the model can reach the situation described by the continuous curve of Case 2. In this situation, the steady state equilibrium with $N_1^{*,1}$ disappears and the model presents a unique stable equilibrium in which the old paradigm 2 is dominated. There is a paradigm shift during which the number of researchers attracted by new topics raises continuously. Greater social recognition and higher wages are the two incentives that attract the young scientists in the new paradigm and the impact of this massive attraction. At the end of the adjustment process, a new steady state is reached in which proponents of the paradigm 2 remain active - but with a minority status.

Note that the paradigm shift may also be caused or hindered by public policy. Political decisions may change the relative values of m_1 and m_2 and consolidate or reduce the dominance of one paradigm. For instance, by granting salary rises or bonuses in case of publication in top tier journals, the public authorities mainly increase the reward of researchers working within the dominant paradigm. The m_i value associated with this paradigm increases which reinforces its dominant status.

5.5 Conclusion

The two-state mean field game developed in this chapter formalizes the competition between two paradigms in an academic field, giving a central role to the young researchers' choice in the dynamics of science.

Three major insights emerge from the model. First, for any set of parameters, there always exists a stable steady state equilibrium. In this equilibrium, both paradigms coexist in a hierarchical order. Second, changes in the reward schemes are able to challenge this hierarchical order. An increase in the productivity in one paradigm or the implementation of incentives in favor of one of the two paradigms clearly contributes to the reinforcement of this particular set of assumptions and tools. Third, important shocks on the parameters may cause the equilibrium with the dominating paradigm to disappear. In this case, one can observe a paradigm shift with the progressive replacement of former major scientists involved in the old paradigm by new generations of researchers, an increasing number of whom will be choosing the new paradigm.

In order to keep the analysis tractable, this model is built on some restrictive assumptions. For instance, the model considers that young researchers make a definitive choice at the beginning of their academic life; future work should consider the possibility of a radical revision of the researchers' agenda. Moreover, in order to obtain an analytical characterization of the equilibrium solution, some specific assumptions have been made about the reward structure and the functional forms of academic production. These assumptions may be questioned in order to assess the accuracy of the model.

Bibliography

Abrahamson, Eric. 2009. “Necessary conditions for the study of fads and fashion in science”. *Scandinavian Journal of Management* 25 (2): 235–239.

Alon, Uri. 2009. “How to choose a good scientific problem”. *Molecular Cell* 35 (6): 726–728.

An, Yonghong, and Zhixiang Zhang. 2012. “Congestion with heterogeneous commuters”. *Economic Modelling* 29:557–565.

Anderson, Simon P., and André De Palma. 2004. “The economics of pricing parking”. *Journal of Urban Economics* 55 (1): 1–20.

Balakrishnan, Narayanswamy, and Asit P. Basu. 1996. *Exponential Distribution : Theory, Methods and Applications.* Gordon / Breach Publishers.

Barber, Bernard. 1961. “Resistence by scientists to scientific discovery”. *Science* 134:596–602.

Besancenot, Damien, Jean Michel Courtault, and Khaled El Dika. 2012. “Piecework versus merit pay : a mean field game approach to academic behavior”. *Revue d'Economie Politique* 122 (4): 547–563.

Besancenot, Damien, and Habib Dogguy. 2014. “Paradigm shift : a mean field game approach”. *Bulletin of Economic research.*

Bonaccorsi, Andrea, and Cristina Ross. 2003. “Why Open source software can succeed”. *Research Policy* 32 (7): 1243–1258.

Bramoullé, Yann, and Gilles Saint Paul. 2010. “Research cycles”. *Journal of Economic Theory* 145:1890–1920.

Brock, William A., and Steven N. Durlauf. 1999. “A formal model of theory choice in science”. *Economic Theory* 14:113–130.

Calthrop, Edward, and Stef Proost. 2006. “Regulating on-street parking”. *Regional Science and Urban Economics* 36:29–48.

Cardaliaguet, Pierre. 2010. *Notes On Mean Field Games.* Technical report. Université Paris-Dauphine.

CERTU. 2008. *Le stationnement public en France en 2005.* Technical report.

Conner, Kathleen R., and Richard P. Rumelt. 1991. “Software piracy : An analysis of protection startegies”. *Management Science* 37:125–139.

Daniels, Tijmen R., Donges Jutta, and Frank Heinemann. 2013. “Crossing network versus dealer market : Unique equilibrium in the allocation of order flow”. *European Economic Review* 62:41–57.

Diamond, Arthur M. 1986. “What is a citation worth”. *Journal of Human Ressources* 21 (2): 200–215.

Diamond, Peter A. 1965. “National Debt in a Neoclassical Growth Model”. *American Economic Reveiw* 55 (5): 1126–1150.

Domar, Evsey. 1946. “Capital Expansion, Rate of Growth, and Employment”. *Econometrica* 14 (2): 137–47.

Economides, Nicholas, and Evangelos Katsamakas. 2006. *Linux vs. Windows: A Comparison of Application and Platform Innovation Incentives for Open Source and Proprietary Software Platforms.*

Faria, Joao Ricardo, Damien Besancenot, and Andreas J. Novak. 2011. “Paradigm depletion, Knowledge production and research effort”. *Metroeconomica* 62 (4): 587–604.

Gaudeul, Alexia. 2008. “Consumer welfare and market structure in a model of competition between open source and proprietary software”. *International Journal of Open Source and Processes* 1 (2): 43–65.

Givon, Moshe, Vijay Mahajan, and Eitan Muller. 1997. “Assessing the relationship between the user-based market share and unit-based market share for pirated software brands in competitive markets”. *Technological Forecasting and Social Change* 55 (2).

Guéant, Olivier. 2009. “Théorie des jeux à champs moyens et applications économiques”. PhD thesis, University Paris Dauphine.

Harrod, Roy F. 1939. “An Essay in Dynamic Theory”. *The Economic Journal* 49 (139): 14–33.

Haruvy, Ernan, Vijay Mahajan, and Ashutosh Prasad. 2004. “The effect op piracy on the market penetration of subscription sotware”. *The Journal of Business* 77 (2): 81–107.

Haruvy, Ernan, Suresh Sethi, and Jing Zhou. 2008. “Open source development with a commercial complementary product or service”. *Production and Operations Management* 17 (1): 29–43.

Huang, Minyi, Peter E. Caines, and Roland P. Malhamé. 2007. “Large-Population Cost-Coupled LQG Problems With Nonuniform Agents: Individual-Mass Behavior and Decentralized $\epsilon - NashEquilibria$”. *IEEE Transactions on Automatic Control* 52, number 9 (): 1560–1571.

Huang, Minyi, Roland P. Malhamé, and Peter E. Caines. 2006. “Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle”. *Communications in Information Systems* 6 (3): 221–252.

Johnson, William R. 1985. “The economics of copying”. *Journal of Political Economics* 93:158–174.

Katz, Michael L., and Carl Shapiro. 1985. “Network externalities, competion and compatibility”. *American Economic Review* 75 (3): 424–440.

King, Stephen P., and Ryan Lampe. 2003. “Network externalities, price discrimination and profitable piracy”. *Information Economics and Piracy* 15 (3): 271–290.

Kuhn, Thomas. 1970. *The strucutre of scientific revolution.* University of chicago press.

Landry, Réjean, Namatie Traore, and Benoît Godin. 1996. “An econometric analysis of the effect of collaboration on academic research productivity”. *Higher Education* 32 (3): 283–301.

Lasry, Jean Michel, and Pierre Louis Lions. 2006a. “Jeux à champ moyen. I. Le cas stationnaire”. *Comptes Rendus Mathématiques* 343 (9): 619–625.

———. 2006b. “Jeux à champ moyen. II. Horizon fini et contrôle optimal.” *Comptes Rendus Mathématiques* 343 (10): 679–684.

Mcdowell, John M., and Michael Melvin. 1983. “The determinant of coauthorship : an analysis of the economic litterature”. *The Review of Economics and Statistics* 65 (1): 155–160.

Merton, Robert K. 1957. “Priorities in scientific discovery : a chapter in the sociology of science”. *American Sociological Review* 22 (6): 635–659.

Novos, Ian E., and Michael Waldman. 1984. “The effects of increased copyright protection : an analytic approach”. *Journal of Political Economy* 92 (236-246).

Peitz, Martin, and Patrick Waelbroeck. 2006. “Piracy of digital products : A critical review of theoretical literature”. *In frmation Economics and Policy* 18 (4): 449–476.

Prasda, Ashutosh, and Vijay Mahajan. 2003. "How many pirates should a software firm tolerate?: An analysis of piracy protection on the diffusion of software". *International Journal of research in Marketing* 20 (4): 337–353.

Romer, Paul M. 1986. "Increasing Returns and Long Run Growth". *The Journal of Political Economy* 94 (5): 1002–1037.

———. 1990. "Endogenous Technological Change". *The Journal of Political Economy* 98 (5): S71–S102.

Schwert, William G. 2003. "Handbook of the Economics of Finance". Chapter Anomalies and market efficiency, 1:939–974. 2. Elsevier.

Seierstad, Atle, and Knut Sydsaeter. 1987. *Optimal control theory with economic applications.* Edited by North-Holland.

Shy, Oz, and Jacques-François Thisse. 1999. "A strategic approach to software protection". *Journal of Economics and Management Strategy* 8 (2): 163–190.

Slive, Joshua, and Bernhardt Dan. 1998. "Pirated for profit". *Canadian Journal of Economics* 31 (4): 886–899.

Solow, Robert M. 1956. "A Contribution to the Theory of Economic Growth". *The Quartely Journal of Economics* 25 (2): 235–239.

Stephan, Paula. 1996. "The economics of science". *Journal of Economic Litterature* 34 (3): 1199–1235.

Sterman, John D., and Jason Wittenberg. 1999. "Path dependence, competition and succession in the dynamics of scientific revolution". *Organisation Science* 10 (3): 322–341.

Stigler, George J., Stephen M. Stigler, and Claire Friedland. 1995. "The Journals of Economics". *The Journal of Political Economy* 103 (2): 331–359.

Swidler, Steve, and Elizabeth Goldreyer. 1998. "The value of a finance journal publication". *The Journal of Finance* 53 (1): 351–363.

Takeyama, Lisa N. 1994. "The welfare implications of unauthorized reproduction of intellectual property in the presence of demand network externalities". *Journal of Industrial Economics* 42 (2): 155–166.

Vickrey, William S. 1969. "Congestion theory and transport investment". *American Economic Review* 59:251–261.

Résumé

Dans cette thèse nous nous intéressons à l'application de la théorie des jeux à champ moyen en économie. Cette nouvelle branche de la théorie des jeux permet d'étudier les systémes impliquant un grand nombre d'agents en utilisant la notion de champ moyen empruntée à la physique statistique. Cette méthode réduit considérablement la compléxité des interactions. Le premier modèle est consacré à l'étude des logiciles et montre que la tolérance du piratage peut être un moyen efficace contre la propagation des logiciels libres. Le deuxième modèle est un modèle de champ moyen statique et traite du problème de stationnement dans les villes en introduisant de l'hétérogénéité dans la population des consommateurs. Cela nous permet de mieux évaluer les politiques publiques mises en oeuvre. Le troisième modèle analyse, dans un cadre dynamique, les conséquences du choix des jeunes chercheurs sur la dynamique des sciences.

Abstract

In this thesis we study the application of Mean Field Game Theory in economics. This new branch of game theory is devoted to the study of systems involving a large number of interacting agents using the notion of mean field from Statistical Physics. This method reduces greatly the complexity of interactions. The first model is devoted to the study of Software market and shows that tolerance of piracy can be an effective strategy in order to limit the diffusion of free softwares. The second model is a static mean field game and addresses the problem of parking in cities by introducing heterogeneity among agents. This allows us to evaluate public policies. The third model analysis, in a dynamic setting, the consequences of the choice of young researchers on the dynamics of science.