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Marches aléatoires sur les arbres aléatoires

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soutenue publiquement devant les membres du jury :

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Thèse de doctorat

Marches aléatoires sur les arbres aléatoires

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Je commence par Maman. Tu me manques. Un peu avant que tu partes en 2013, j'avais acheté un livre de maths pour m'occuper l'esprit. C'était un livre sur les modules de Grégory Berhuy. Beau refuge que les mathématiques. Je sentais que la dose nécessaire serait forte et que j'aurais besoin d'accompagnement (mathématique). Au lycée Jacques Feyder d'Épinay-sur-Seine où j'enseignais, je donnais aussi des colles en CPES, une classe préparatoire où les enseignements étaient donnés pour moitié par l'université Paris 13. J'ai choisi (un peu au hasard) une adresse électronique `math.univ-paris13.fr` sur un mail collectif concernant ces colles. C'était celle de Farell Brumley. Bouteille à la mer reçue. Il s'est montré très compréhensif et m'a redirigé vers Pascal Boyer qui m'a immédiatement proposé un rendez-vous et rassuré sur le fait que je pourrais m'inscrire en M1 malgré mon travail à plein temps. Je les remercie, eux et tous ceux, au LAGA ou à l'institut Galilée (je pense notamment à Séverine Girod) qui ont rendu possible ma reprise d'étude.

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Je finis par Maman. Donc, tu vois, tu me manques mais je ne suis pas seul pour autant. Et en partant, sans le savoir, tu m'as fait ce dernier cadeau en me remettant en selle

sur le chemin des mathématiques. Je ne sais pas combien de temps encore je pourrai le suivre, mais je suis déjà content du périple accompli. Je sais que Papa et toi auriez été fiers et heureux pour moi.

Introduction (en français)

Une *courbe de Jordan* est une injection continue du cercle unité dans le plan complexe $\mathbb{C}$. Un théorème de Camille Jordan affirme que le plan complexe privé d'une telle courbe comporte deux composantes connexes, dont l'une est bornée et l'autre non. Arthur Moritz Schoenflies affine ce résultat en montrant qu'une courbe de Jordan peut être prolongée en un homéomorphisme du disque unité fermé dans $\mathbb{C}$ dont l'image du disque ouvert est la composante bornée. En particulier, celle-ci est simplement connexe. Appelons domaine de Jordan une telle partie et notons-la Ω.

Le *problème de Dirichlet* sur Ω est la recherche, pour une fonction continue donnée u sur la courbe de Jordan $\partial\Omega$ d'une fonction $\hat{u}$ sur la fermeture $\overline{\Omega}$ de Ω qui prolonge u et est harmonique sur Ω, c'est-à-dire vérifie, pour tout x dans Ω et tout $r > 0$ tel que la boule fermée $\mathscr{B}(x,r)$ est incluse dans Ω, $\hat{u}(x) = \mathbb{E}[\hat{u}(X)]$, où la variable aléatoire X est de loi uniforme sur le cercle $\partial\mathscr{B}(x,r)$. Le domaine Ω étant simplement connexe, il vérifie la condition du cône de Poincaré (voir par exemple [9, Chapitre II, Proposition 1.14]) et l'unique solution du problème de Dirichlet est donnée par

$$\hat{u}(x) = \mathbb{E}_x[u(B_\tau)],$$

où $(B_t)_{t\geq 0}$ est, sous $\mathbb{P}_x$, un mouvement brownien dans le plan issu de x et τ est le temps d'atteinte par (B_t) de $\partial\Omega$.

Le lien entre mouvement brownien et équation de la chaleur remonte au moins à 1905, *annus mirabilis* d'Einstein et à son article *Sur le mouvement de petites particules en suspension dans un liquide immobile, comme requis par la théorie cinétique moléculaire de la chaleur* ([16]) ou à Bachelier et sa théorie de la spéculation ([8]). Ce lien est progressivement exploré par les mathématiciens pendant la première moitié du XX$^{\text{ème}}$ siècle, voir les notes de [48, Chapitre 3] pour un historique plus détaillé.

D'après le théorème de représentation de Riesz-Markov, pour x_0 fixé dans Ω, la fonction $u \mapsto \hat{u}(x_0)$ peut s'écrire comme l'intégrale de u par rapport à une mesure borélienne μ_{x_0} et cette mesure s'appelle la *mesure harmonique* sur $\partial\Omega$. Le point de vue probabiliste est d'identifier μ_{x_0} à la loi de B_τ sous $\mathbb{P}_{x_0}$. Lorsque la courbe de Jordan $\partial\Omega$ est rectifiable, un théorème des frères Riesz affirme que pour tout x_0 dans Ω, μ_{x_0} est équivalente à la longueur d'arc (c'est-à-dire la mesure de Hausdorff 1-dimensionnelle) sur $\partial\Omega$. Cependant, on peut vouloir considérer des courbes non rectifiables.

Un exemple de telle courbe est le *flocon de Koch* inventé par le mathématicien suédois Helge von Koch en 1904. La *courbe de Koch* est construite itérativement de la façon suivante. Initialement, on considère un segment $[AB]$ du plan que l'on découpe en trois parties égales $[AC]$, $[CE]$ et $[EB]$ puis on construit le triangle équilatéral CDE, de sorte que la ligne brisée $ACDEB$ est composée de 4 segments de même longueur, voir figure 1. On itère la construction en recommançant à chaque étape les mêmes opérations

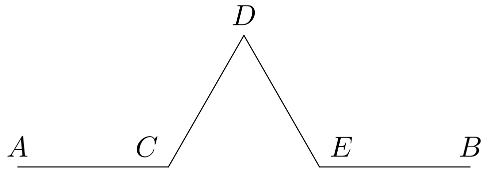

FIGURE 1 – Première étape de la construction de la courbe de Koch

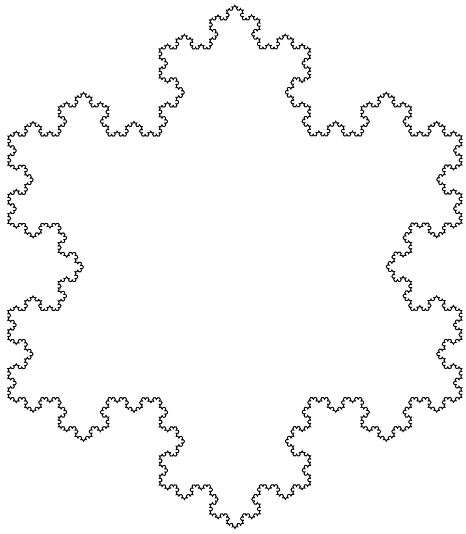

FIGURE 2 – Flocon de Koch

sur chaque segment de la ligne brisée. Pour obtenir le flocon, une courbe fermée, il suffit de joindre 3 copies de la courbe de von Koch, voir figure 2. Dans l'article [59], von Koch prouve que son flocon est bien une courbe de Jordan mais, faisant ainsi écho aux fonctions de Weierstrass, que celle-ci n'admet nulle part de tangente. À cette époque, un objet comme celui-ci était vu comme une curiosité (voire une monstruosité) mais les travaux de Benoît Mandelbrot ont contribué à faire de ces objets, qu'il a qualifiés de *fractals*, des sujets d'étude de premier plan. En particulier, Mandelbrot propose de quantifier le degré d'irrégularité d'un tel objet par sa dimension de Hausdorff. Pour le flocon, la valeur qu'il calcule en utilisant des arguments géométriques est $\log(4)/\log(3)$. Plus tard, le mathématicien australien John Hutchinson dans l'article fondateur [29] donnera un autre point de vue sur le flocon en le faisant entrer dans la catégorie des attracteurs de systèmes de similitudes itérées pour lesquelles, sous la condition dite de l'ensemble ouvert, il établit que la dimension de similarité, facile à calculer, est égale à la dimension de Hausdorff.

Revenons au problème de Dirichlet. On pourrait s'attendre à ce que la mesure harmonique μ_{x_0} sur une courbe de Jordan non rectifiable soit absolument continue par rapport à une mesure de Hausdorff s-dimensionnelles pour $s > 1$ (ou un raffinement de celle-ci). Nikolai Georgievich Makarov, dans [47] montre qu'il n'en est rien. Son théorème, d'une précision remarquable énonce que *quelle que soit la courbe de Jordan*, la mesure harmonique est absolument continue par rapport à la mesure de Hausdorff de fonction de jauge φ (voir 1.10 pour une définition) définie par

$$\varphi(t) = t \exp\left(C \sqrt{\log(1/t) \log\log\log(1/t)} \right),$$

où C est une constante universelle et que, par ailleurs, pour toute fonction de jauge ψ négligeable devant t quand t tend vers 0, la mesure harmonique et la mesure de Hausdorff relativement à ψ sont mutuellement singulières. De la première partie de cet énoncé, il découle que la dimension de Hausdorff inférieure de la mesure μ_{x_0} (voir section 1.11 pour une définition) est supérieure ou égale à 1 et de la deuxième partie, que sa dimension de Hausdorff supérieure est inférieure ou égale à 1. Ainsi, on a $\dim_{\mathrm{H}} \mu_{x_0} = 1$ tandis que la mesure μ_{x_0} a un support plein dans $\partial\Omega$ qui peut être de dimension strictement plus grande que 1, comme c'est le cas, par exemple, avec le flocon de Koch.

Ce phénomène plutôt surprenant a été appelé *chute de dimension* (*dimension drop*) par Russell Lyons, Robin Pemantle et Yuval Peres dans [43] en 1995. L'objet de cet article, qui par bien des aspects est à la base du présent travail, est l'étude de la marche aléatoire simple sur un arbre de Galton-Watson surcritique conditionné à survivre, cadre que nous allons introduire dans les prochains paragraphes.

Pour définir un arbre de Galton-Watson T, on se donne une loi de reproduction, c'est-à-dire une suite $(p_k)_{k\geq 0}$ de réels positifs ou nuls de somme 1 et de façon informelle (pour une définition plus précise, voir le chapitre 2) la racine $\varnothing$ de l'arbre a un nombre aléatoire d'enfants suivant (p_k), puis ses enfants eux-même se reproduisent de façon indépendante suivant la même loi et ainsi de suite, voir la figure 3. Si l'on note q la probabilité que T soit fini et $m = \sum_{k\geq 0} kp_k$ la moyenne de la loi de reproduction , il est bien connu que $q < 1$ équivaut à $m > 1$. La loi de reproduction (p_k) est dite surcritique (respectivement critique, sous-critique) lorsque $m > 1$ (respectivement $m = 1$, $m < 1$).

Pour un réel $\lambda > 0$, la marche aléatoire λ-biaisée sur un arbre enraciné est une marche aléatoire aux plus proches voisins avec poids λ vers le parent et poids 1 vers les enfants, voir figure 4. On parle de marche aléatoire simple lorsque $\lambda = 1$. Sur l'événement de non-extinction, c'est-à-dire lorsque T est infini, une marche aléatoire aux plus proches voisins, toujours irréductible, est soit récurrente soit transiente. Dans [39], Russell Lyons montre que la marche λ-biaisée sur T conditionné à survivre est récurrente si et seulement si $\lambda \geq m$. Le *bord* ∂T de l'arbre T est l'ensemble des rayons dans T, c'est-à-dire les chemins infinis $(\xi_0 = \varnothing, \xi_1, \xi_2, \dots)$ partant de la racine et tels que pour tout $i \geq 1$, ξ_{i+1} est un enfant de ξ_i. On peut le munir d'une distance naturelle d en posant, pour deux rayons ξ et η, $d(\xi, \eta) = \exp(-n)$ si n est le maximum des indices i tels que $\xi_i = \eta_i$. Cela fait de ∂T un espace ultramétrique compact. La dimension de Hausdorff de cet espace vaut presque sûrement $\log m$, d'après un théorème de John Hawkes ([25]) amélioré par Lyons (toujours dans [39]). Lorsqu'une marche aléatoire (X_n) sur T est transiente, la *mesure harmonique* associée à cette marche est la loi de l'unique rayon Ξ partageant une infinité de sommets avec la trajectoire (X_n). C'est une mesure de probabilité borélienne sur ∂T.

On peut maintenant énoncer les principaux résultats de [43]. Dans le cas d'un arbre

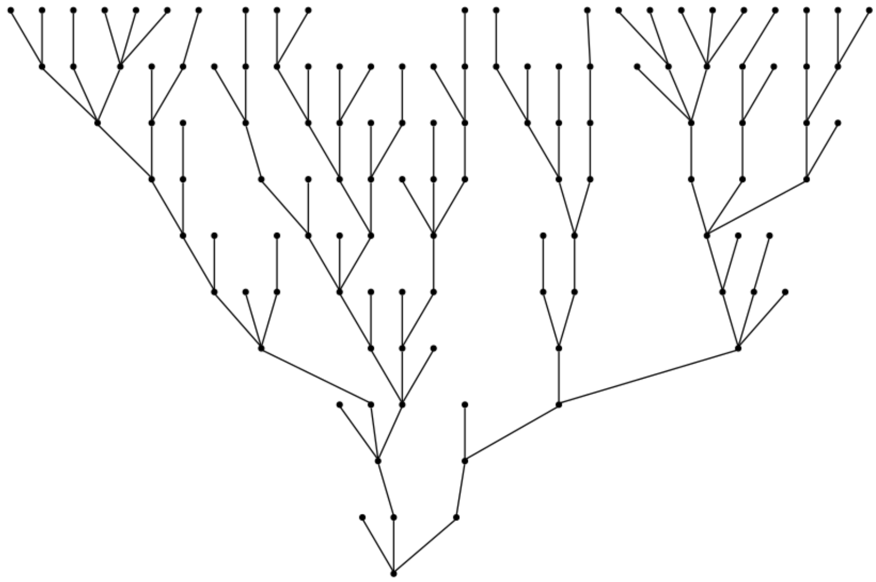

Figure 3 – 10 générations d'une simulation d'un arbre de Galton-Watson de loi de reproduction uniforme sur $\{0, 1, 2, 3\}$

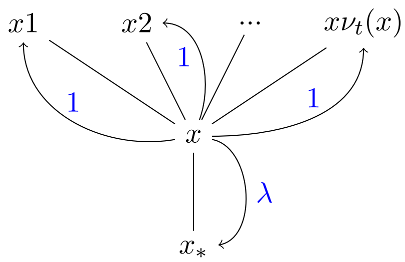

Figure 4 – Poids associés à la marche λ-biaisée : ici x est un sommet d'un arbre enraciné t, x_* est son parent et $x1$, ..., $x\nu_t(x)$ sont ses enfants.

de Galton-Watson tel que $m < \infty$, $p_1 < 1$ et $p_0 = 0$ [1], la marche aléatoire simple (qui est nécessairement transiente) va vers l'infini à vitesse linéaire : presque sûrement, on a

$$\lim_{n\to\infty} \frac{|X_n|}{n} = \mathbb{E}\left[\frac{\nu-1}{\nu+1}\right],$$

où $|X_n|$ désigne la hauteur de la n-ième position de la marche aléatoire et ν est une variable aléatoire de loi (p_k). De plus, si la loi de reproduction n'est pas dégénérée, la mesure harmonique a une dimension strictement inférieure à $\log m$, la dimension de ∂T. Autrement dit, le phénomène de chute de dimension se produit également dans ce cadre. Sur le plan des outils développés pour démontrer ces résultats, les apports de cet article sont également considérables. Notons en particulier l'invention des arbres de Galton-Watson *augmentés* (*augmented Galton-Watson trees*) dont la racine a un enfant supplémentaire pour obtenir un environnement stationnaire par rapport à la marche aléatoire et le concept de *règle cohérente de flot* permettant d'utiliser la théorie ergodique sur des arbres de Galton-Watson munis d'un rayon. Cette théorie sera détaillée dans le chapitre 2 de cet ouvrage.

L'année suivante, ces mêmes auteurs traitent le cas de la marche λ-biaisée transiente. Ils démontrent dans [44] que le phénomène de chute de dimension a encore lieu et que lorsque λ est plus grand qu'une valeur critique λ_c dépendant de la probabilité d'extinction q (et nulle si $p_0 = 0$), la marche aléatoire va encore vers l'infini à vitesse linéaire. Cependant, le manque d'un environnement stationnaire par rapport à la marche aléatoire fait que d'une part, les résultats obtenus sont moins précis (pas de formule explicite pour la vitesse et la dimension) et que d'autre part les techniques utilisées sont très différentes du cas $\lambda = 1$. C'est l'étude de temps particuliers de la marche appelés temps de régénérations qui permet finalement d'obtenir la chute de dimension. Nous détaillerons ces arguments dans le chapitre 5, dans un cadre plus général. L'article se termine sur des questions qui sont toujours largement ouvertes :

1. Si $p_0 = 0$, est-ce que la vitesse de la marche est une fonction monotone de λ ?
2. Est-ce que la dimension de la mesure harmonique est une fonction monotone de λ ?
3. Quelle est la régularité de ces fonctions ?

L'année 2013 voit avec l'article [3] un grand progrès dans la compréhension des marches λ-biaisées transientes sur les arbres de Galton-Watson infinis. Dans cet article, Élie Aïdékon utilise certaines particularités de ces marches (de façon cruciale, le fait qu'à un temps frais, c'est-à-dire un temps où la marche découvre un nouveau sommet, retourner la trajectoire et le temps s'annulent en un certain sens), pour obtenir un environnement asymptotique vu de la particule. Il en déduit une formule exprimant la vitesse en fonction de la loi de la conductance de l'arbre

$$\beta(T) = \mathrm{P}^T_{\varnothing}(\tau_{\varnothing_*} = \infty),$$

1. Cette hypothèse n'est présente que pour simplifier l'exposition, les résultats de [43] sont plus généraux.

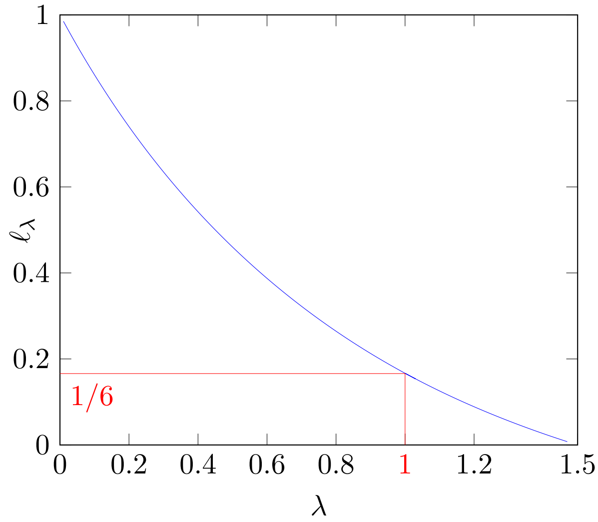

FIGURE 5 – Vitesse ℓ_λ de la marche λ-biaisée en fonction de λ dans le cas $p_1 = p_2 = 1/2$.

où $\mathrm{P}^T_\varnothing$ est la loi de la marche λ-biaisée sur T partant de $\varnothing$, $\varnothing_*$ est un parent artificiel de $\varnothing$ et $\tau_{\varnothing_*}$ est son temps d'atteinte par la marche aléatoire. Si ℓ_λ désigne la vitesse de la marche sur T conditionné à survivre, pour $\lambda_c < \lambda < m$, alors

$$\ell_\lambda = \mathbb{E}\left[\frac{(\nu-\lambda)\beta_0}{\lambda - 1 + \beta_0 + \sum_{j=1}^{\nu}\beta_j}\right] \Bigg/ \mathbb{E}\left[\frac{(\nu+\lambda)\beta_0}{\lambda - 1 + \beta_0 + \sum_{j=1}^{\nu}\beta_j}\right].$$

Dans cette expression, les variables aléatoires β_0, β_1, ... sont des copies i.i.d. de $\beta(T)$ et ν est distribuée suivant (p_k) et est indépendante des β_i. Malheureusement, la loi de $\beta(T)$ est encore très mal comprise et cette formule n'a pour le moment permis de prouver la monotonie de la vitesse que sur l'intervalle $[0, 1/2]$. On peut cependant vérifier numériquement la validité de la conjecture de Lyons, Pemantle et Peres sur la vitesse. En effet, la conductance est la plus grande solution de l'équation distributionnelle récursive

$$\beta(T) \stackrel{\mathrm{d}}{=} \frac{\sum_{j=1}^{\nu}\beta_j}{\lambda + \sum_{j=1}^{\nu}\beta_j}.$$

On peut donc, à λ fixé se servir de cette égalité en loi pour calculer numériquement la loi de $\beta(T)$ et ainsi obtenir une valeur numérique de ℓ_λ. Pour $p_1 = p_2 = 1/2$, la courbe obtenue est la figure 5 qui semble confirmer la conjecture (la valeur $1/6$, pour $\lambda = 1$, est donnée par la formule de Lyons, Pemantle et Peres).

Notre premier résultat, obtenu de façon indépendante par Shen Lin ([35]) est une formule similaire à celle d'Aïdékon pour calculer la dimension de la mesure harmonique. En utilisant une idée de Nicolas Curien et Jean-François Le Gall dans [10] [2] nous avons

2. Ces auteurs précisent que cette idée provient d'un « passage à la limite non rigoureux » de l'environnement d'Aïdékon

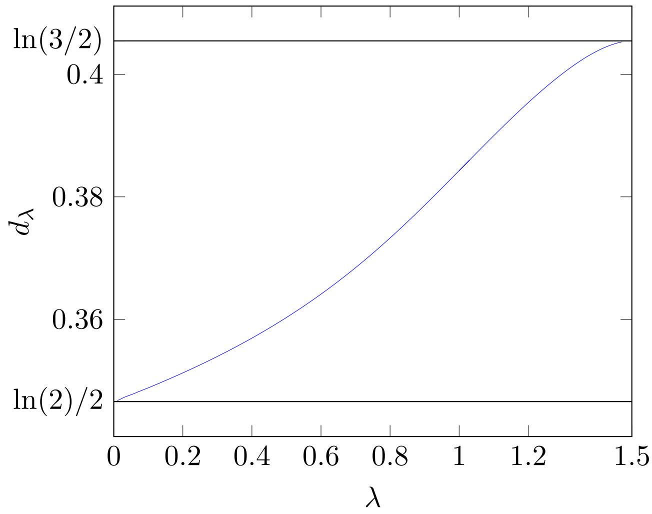

FIGURE 6 – Dimension d_λ de la mesure harmonique en fonction de λ, pour $p_1 = p_2 = 1/2$.

donné des conditions algébriques suffisantes pour qu'une règle de flot admette une probabilité invariante de densité explicite par rapport à la loi de T. Ce résultat abstrait est donné à la fin du chapitre 2. Il s'applique au cas de la marche λ-biaisé (voir le chapitre 4 ou [55]) et, avec la théorie ergodique sur les arbres de Galton-Watson, permet d'obtenir le résultat suivant sur la dimension d_λ de la mesure harmonique en fonction du biais :

$$d_\lambda = \log(\lambda) - C^{-1}\mathbb{E}\left[\log(1-\beta_0)\frac{\beta_0\sum_{j=1}^{\nu}\beta_j}{\lambda - 1 + \beta_0 + \sum_{j=1}^{\nu}\beta_j}\right],$$

où la constante de renormalisation C est égale à l'espérance de la fraction ci-dessus. De façon similaire, on peut mener des expérimentations numériques pour tester la validité de certaines conjectures apparaissant dans [44, 45]. Dans la figure 6, les deux limites en 0 et m sont données par un théorème de Shen Lin et valent respectivement $\mathbb{E}[\log\nu]$ et $\log\mathbb{E}[\nu] = \log m$. S'il est vrai que $\log m$ est une borne supérieure stricte (d'après le théorème de chute de dimension de Lyons, Pemantle et Peres), il n'est pas prouvé en toute généralité que $\mathbb{E}[\log\nu]$ est une borne inférieure. Bien sûr une preuve de la continuité et de la monotonie de la fonction $\lambda \mapsto d_\lambda$ impliquerait cette propriété.

Nous continuons cette histoire en revenant sur l'article [10] de Curien et Le Gall. Il est ici question de marche aléatoire simple (X_k) sur un arbre de Galton-Watson $T^{(n)}$ critique conditionné à survivre au moins jusqu'à la génération n. On suppose ici que la variance de la loi de reproduction est finie. Si l'on note $\tau^{(n)}$ le premier temps où la marche atteint la hauteur n, et, pour x à hauteur n dans $T^{(n)}$, $\mu_n(\{x\}) = \mathrm{P}_\varnothing^{T^{(n)}}(X_{\tau^{(n)}} = x)$, alors on peut voir en μ_n l'analogue de la mesure harmonique pour ce modèle. Les auteurs démontrent que, bien que l'ensemble des sommets à hauteur n ait de l'ordre de n éléments, la mesure harmonique est presque entièrement portée par une partie ayant de l'ordre de n^δ éléments, où $\delta \approx 0,78$ est une constante universelle, ce qui est encore un résultat de chute de dimension. Pour ce faire, ils réduisent (de façon hautement non triviale) le problème à un problème de chute de dimension sur un arbre infini muni de longueurs d'arêtes (nous avons dans [55] appelé ces arbres « arbres à longueurs récursives »). Notons $\widetilde{T}^{(n)}$ l'arbre obtenu à partir de $T^{(n)}$ après élagage (on ne garde que les arêtes appartenant à un plus court chemin entre la racine et un sommet de hauteur n), et réduction (on ne conserve que les sommets à hauteur n, la racine et les points de branchement dans l'arbre) mais en gardant la distance de graphe héritée de $T^{(n)}$. Alors, au sens de Gromov-Hausdorff, on a convergence de $\frac{1}{n}\widetilde{T}^{(n)}$ vers ce que les auteurs appellent l'arbre réduit continu défini de la façon suivante : la loi de reproduction est donnée par $p_2 = 1$ et pour chaque mot x sur l'alphabet $\{1, 2\}$ on tire une variable aléatoire U_x uniforme sur $(0, 1)$, puis on donne à l'arête reliant x à son parent x_* la longueur

$$U_x(1 - U_{x_*}) \cdots (1 - U_\varnothing),$$

où le produit ci-dessus est indexé par les ancêtres stricts de x. La marche aléatoire simple sur $T^{(n)}$ correspond à la limite à une marche aléatoire aux plus proches voisins sur cet arbre réduit continu, où les poids de transitions sont les inverses des longueurs des arêtes (pour plus de précision voir le chapitre 3). Cette marche est transiente. Curien et Le Gall parviennent ensuite, par deux méthodes très différentes (l'une d'elle utilisant le calcul stochastique) à obtenir une mesure invariante par rapport au flot harmonique puis montrent que la dimension de la mesure harmonique est $\delta < 1$ tandis que la dimension du bord (pour la métrique associée aux longueurs d'arêtes) vaut 1. Dans [34], Shen Lin, alors en thèse sous la direction de Le Gall, étend ce résultat au cas où la loi de reproduction de l'arbre de départ est dans le bassin d'attraction d'une loi α-stable avec $\alpha \in (0, 1]$, le principal changement étant le fait que les sommets de l'arbre réduit continu peuvent avoir plus de 2 enfants.

Dans le chapitre 3, nous généralisons une partie de ces résultats et nous intéressons au cas où la loi de reproduction dans l'arbre à longueur récursive vérifie uniquement l'hypothèse $p_0 = 0$ (et $p_1 < 1$) et en remplaçant la distribution uniforme sur $(0, 1)$ par une distribution quelconque sur $(0, 1)$, voir la figure 7 (où pour une raison pratique on considère les variables i.i.d. à valeurs dans $(0, 1)$ comme les inverses de variables i.i.d. $(\Gamma_x)_{x\in T}$ à valeurs dans $(1, \infty)$). Nous montrons, en réécrivant une partie de la théorie ergodique sur les arbres de Galton-Watson dans ce cadre, que sous des hypothèses peu restrictives, le phénomène de chute de dimension a encore lieu et la dimension du bord de l'arbre pour la distance associée aux longueurs est le paramètre de Malthus α défini par

$$\mathbf{E}[(1 - \Gamma_\varnothing^{-1})^\alpha] = 1/m.$$

Les deux derniers chapitres de cette thèse sont consacrés à l'étude d'un modèle très étudié (et très riche par la diversité des comportements possibles) appelé *marche aléatoire en milieu aléatoire sur un arbre de Galton-Watson*[3] et introduit dans [41]. Il s'agit de considérer non pas seulement une loi de reproduction mais une variable aléatoire $\mathbf{A}$ à valeurs dans l'ensemble $\mathsf{Tuples} = \bigcup_{k\geq 0}(0, \infty)^k$ des suites finies de réels strictement

3. Cette appélation est assez maladroite car une marche aléatoire sur un arbre de Galton-Watson est, de fait, une marche aléatoire en mileu aléatoire.

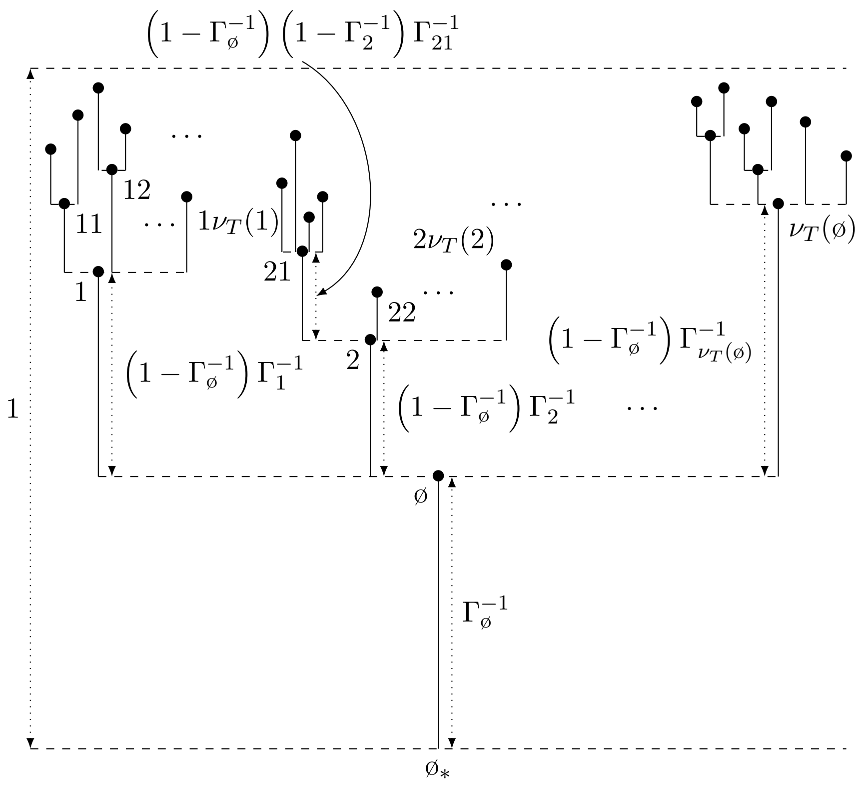

FIGURE 7 – Une représentations schématique d'un arbre de Galton-Watson à longueurs récursives

positifs, où l'on convient que $(0,\infty)^0$ contient la suite vide (). On construit un arbre pondéré aléatoire de la façon suivante : on commence par tirer $\mathbf{A}^{\varnothing}$ de même loi que $\mathbf{A}$, on donne à la racine $\varnothing$ un nombre d'enfants égal à la longueur de la suite $\mathbf{A}^{\varnothing}$ et pour chaque enfant i de la racine, on donne à l'arête $\{\varnothing, i\}$ un poids $\mathsf{A}_T(i)$ égal à la i-ième composante de $\mathbf{A}^{\varnothing}$, puis on réitère ce procédé de façon indépendante pour chaque enfant i de $\varnothing$ et ainsi de suite. On obtient ainsi un arbre de Galton-Watson T ainsi qu'une fonction $\mathsf{A}_T : T \setminus \{\varnothing\} \to (0,\infty)$ donnant, pour chaque x de $T \setminus \{\varnothing\}$, le poids de l'arête le reliant à son parent. On peut alors définir une marche aléatoire aux plus proches voisins en posant

$$\mathrm{P}^T(x,y) = \begin{cases} \dfrac{\mathsf{A}_T(xi)}{1+\sum_{j=1}^{\nu_t(\varnothing)} \mathsf{A}_T(xj)} & \text{si } y = xi, \text{ pour un } 1 \le i \le \nu_T(x)\,; \\ \dfrac{1}{1+\sum_{j=1}^{\nu_T(\varnothing)} \mathsf{A}_t(xj)} & \text{if } y = x_*, \end{cases}$$

où $\nu_T(x)$ désigne le nombre d'enfants de x dans T et x_* son parent, voir la figure 8. La

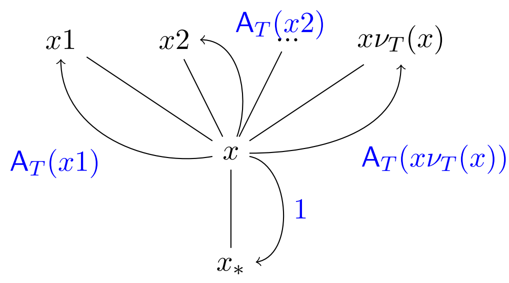

FIGURE 8 – Poids de la marche aléatoire sur l'arbre pondéré aléatoire T

fonction $\psi : [0, \infty) \to (-\infty, \infty]$ définie par

$$\psi(s) = \log \mathbb{E}\Big[\sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i)^s\Big]$$

joue un rôle important dans l'étude de ce modèle. En particulier, si $\min_{[0,1]} \psi > 0$, alors la marche aléatoire sur l'arbre pondéré T est presque sûrement transiente. Se pose alors la question de la chute de dimension. Dans le chapitre 5, nous appliquons la méthode esquissée dans [44] pour démontrer que sous l'hypothèse $m < \infty$, la chute de dimension a toujours lieu, sauf si le modèle se réduit à une marche λ-biaisée sur un arbre régulier. Pour présenter brièvement cette méthode, on doit introduire les temps de sortie et les temps de régénération. Pour une trajectoire (X_k) de la marche aléatoire, un entier $s \geq 1$ est un temps de sortie si X_{s-1} est le parent de X_s et pour tout $k > s$, $X_k \neq (X_s)_*$. Après un tel temps de sortie, la marche aléatoire est condamnée à rester dans le sous-arbre issu de X_s. Si de plus, s est un temps frais, c'est-à-dire que s est le premier temps de visite du sommet X_s, alors s est appelé un temps de régénération de (X_k). On montre qu'il y a presque sûrement une infinité de temps de régénérations et qu'à ces temps l'environnement qui est devant la marche est stationnaire. Pour revenir aux temps de sortie, qui sont ceux qui nous intéressent, suivant l'idée de [44], on construit une tour de Rokhlin. L'examen minutieux de cette tour nous conduit à introduire une nouvelle fonction κ sur l'espace des arbres pondérés qui sera la densité par rapport à la loi de T d'une mesure invariante par rapport au flot harmonique, nous permettant de conclure.

Dans le dernier chapitre de cette thèse, nous quittons les marches transientes pour nous consacrer à un cas récurrent dans le modèle précédent appelé le cas *sous-diffusif*. Nous travaillons sous les hypothèses suivantes : on suppose qu'on est dans le cas dit *normalisé*

$$\psi(1) = \log \mathbf{E} \sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i) = 0.$$

et pour pouvoir appliquer le théorème de Biggins (voir aussi [32] de Kahane et Peyrière,

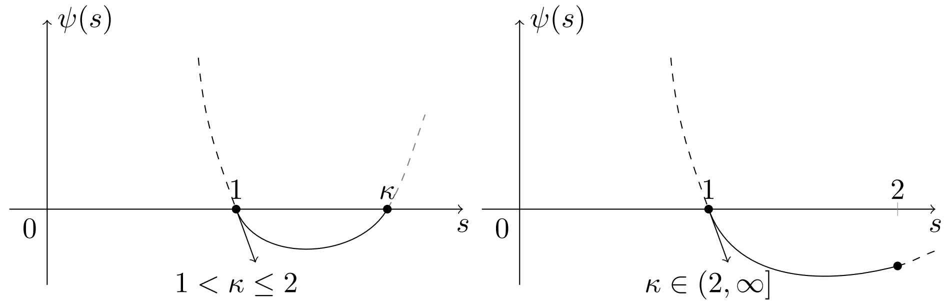

FIGURE 9 – Comportement schématique de ψ sous les hypothèses du chapitre 6

ou la preuve de Lyons [40]), on fait d'abord l'hypothèse

$$\psi'(1) := \mathbf{E}\left[\sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i) \log \mathsf{A}_T(i)\right] \in [-\infty, 0);$$

et, si

$$\kappa = \inf\{s > 1 : \psi(s) = 0\} \in (1, \infty],$$

on suppose que

$$\mathbf{E}\Big[\Big(\sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i)\Big)^{\kappa}\Big] + \mathbf{E}\Big[\sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i)^{\kappa} \log^{+} \mathsf{A}_T(i)\Big] < \infty, \quad \text{si } 1 < \kappa \le 2,$$

$$\mathbf{E}\Big[\Big(\sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i)\Big)^{2}\Big] < \infty, \quad \text{si } \kappa \in (2, \infty].$$

Ces hypothèses sont rappelées dans la figure 9. La martingale additive $(M_n(T))_{n\ge 0}$ (parfois aussi appelée martingale de Mandelbrot ou martingale de Biggins) est définie par

$$M_n(T) = \sum_{|x|=n} \prod_{\varnothing \prec y \preceq x} \mathsf{A}_T(y).$$

D'après le théorème de Biggins, elle converge presque sûrement et dans L^1 vers une variable aléatoire $M_\infty(T)$ non dégénérée. Si l'on désigne par $\mathscr{C}_n(T)$ la conductance entre la racine de T et ses sommets à hauteur n, la récurrence de la marche implique que $\mathscr{C}_n(T)$ tend presque sûrement vers 0 et nous nous demandons à quelle vitesse. Nous montrons que

$$0 < \liminf_{n\to\infty} n^{1/(\kappa-1)} \mathbb{E}[\mathscr{C}_n(T)] \le \limsup_{n\to\infty} n^{1/(\kappa-1)} \mathbb{E}[\mathscr{C}_n(T)] < \infty \quad \text{si } 1 < \kappa < 2\,;$$

$$0 < \liminf_{n\to\infty} n \log n \mathbb{E}[\mathscr{C}_n(T)] \le \limsup_{n\to\infty} n \log n \mathbb{E}[\mathscr{C}_n(T)] < \infty \quad \text{si } \kappa = 2 \text{ et}$$

$$\lim_{n\to\infty} n \mathbb{E}[\mathscr{C}_n(T)] = \|M_\infty(T)\|_2 \quad \text{si } \kappa > 2.$$

et dans tous les cas que, presque sûrement et dans L^p pour $p \in [1, \kappa)$, si $1 < \kappa \leq 2$ et dans L^2 si $\kappa > 2$,

$$\lim_{n\to\infty} \mathscr{C}_n(T)/\mathbb{E}[\mathscr{C}_n(T)] = M_\infty(T).$$

Le reste de cette thèse ne dépend pas de cette introduction et toutes les définitions seront rappelées dans un cadre plus formel.

Le chapitre 1 présente les arbres planaires tels qu'ils ont été formalisés par Jacques Neveu ([50]) et leurs bords. On y caractérise une famille de « bonnes » distances sur le bord d'un arbre et on donne une définition un peu originale des mesures de Hausdorff et de packing sur ces espaces en les motivant par la recherche de théorèmes de densité. Ensuite, on rappelle les définitions classiques de ces mesures sur les espaces métriques et on fait le lien avec celles que nous avons introduites en utilisant notre caractérisation des distances. Enfin on définit et étudie différentes notions de dimensions pour les mesure boréliennes sur le bord d'un arbre.

Le chapitre 2 est quant à lui consacré à la théorie ergodique sur les arbres de Galton-Watson marqués. On y établit avec beaucoup de précision les résultats principaux parus dans [43] ou [46]. Certains détails sont un peu originaux, au moins dans leurs formes. La dernière partie de ce chapitre est consacrée au critère algébrique d'existence de mesure invariante dont nous avons déjà parlé précédemment.

Les chapitres 3 à 6, déjà présentés, comportent l'essentiel des nouveautés de cette thèse, mais nous n'avons pas hésité à présenter des démonstrations de théorèmes déjà connus, pour que ce texte soit aussi autonome que possible.

Introduction (in English)

A *Jordan curve* is a continuous one-to-one mapping from the unit circle to the complex plane $\mathbb{C}$. A theorem of Camille Jordan claims that the complement of such a curve in $\mathbb{C}$ is made of two connected components, one being bounded, the other one unbounded. Arthur Moritz Schoenflies sharpens this result by showing that a Jordan curve can be extended to a homeomorphism from the closed unit circle to $\mathbb{C}$ whose image of the open unit disk is the bounded connected component. As a consequence, this component is simply connected. We call it a Jordan domain and denote it by Ω.

The *Dirichlet problem* on Ω is to find, for a given function u on the Jordan curve $\partial\Omega$, a function $\hat{u}$ on the closure $\overline{\Omega}$ of Ω, which extends u and is harmonic on Ω, that is, satisfies, for all x in Ω and all $r > 0$ such that the ball $\mathscr{B}(x, r)$ is included in Ω, $\hat{u}(x) = \mathbb{E}[\hat{u}(X)]$, where the random variable X is uniform on the circle $\partial\mathscr{B}(x, r)$. Since Ω is simply connected, it satisfies the Poincaré cone condition (see [9, Chapitre II, Proposition 1.14]) and the unique solution of the Dirichlet problem is given by

$$\hat{u}(x) = \mathbb{E}_x[u(B_\tau)],$$

where $(B_t)_{t\geq 0}$ is, under $\mathbb{P}_x$, a Brownian motion in the plane starting from x and τ is the first hitting time by (B_t) of $\partial\Omega$.

The link between Brownian motion and the heat equation goes back at least to 1905, *annus mirabilis* of Einstein and his article *On the Motion of Small Particles Suspended in a Stationary Liquid, as Required by the Molecular Kinetic Theory of Heat* ([16]) or to Bachelier and his theory of speculation ([8]). This link is progressively expored by mathematicians during the first half of the twentieth century. See the notes of [48, Chapitre 3] for a more detailed historical account.

By Riesz-Markov's representation theorem, for a fixed x_0 in Ω, the function $u \mapsto \hat{u}(x_0)$ may be written as the integral of u with respect to a Borel measure μ_{x_0} and this measure is called the *harmonic measure*. With a more probabilistic point of view, μ_{x_0} is the distribution of B_τ under $\mathbb{P}_{x_0}$. When the Jordan curve $\partial\Omega$ is rectifiable, a theorem by the Riesz brothers asserts that for all x_0 in Ω, μ_{x_0} is equivalent to arc length (that is, the 1-dimensional Hausdorff measure). We may however consider non rectifiable curves.

An example of such a curve is the *Koch snowflake* invented by the swedish mathematician Helge von Koch in 1904. The *Koch curve* is iteratively built in the following way. First consider a segment $[AB]$ of the plane and cut it in three equal parts $[AC]$, $[CE]$ and $[EB]$ and build the equilateral triangle CDE so that the broken line $ACDEB$ is made of 4 segments of the same length, see 1. Iterate the previous construction by doing the same operations on each segment of the broken line, and so on and so forth. To obtain a closed curve, it then suffices to glue together 3 Koch curves, see figure 2. In the article [59] von Koch shows that his snowflake is indeed a Jordan curve, but, in

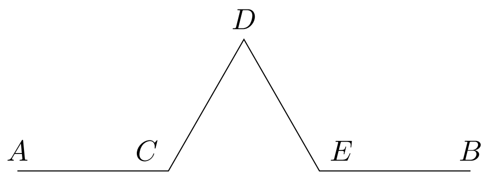

Figure 1 – First step of the construction of the Koch curve

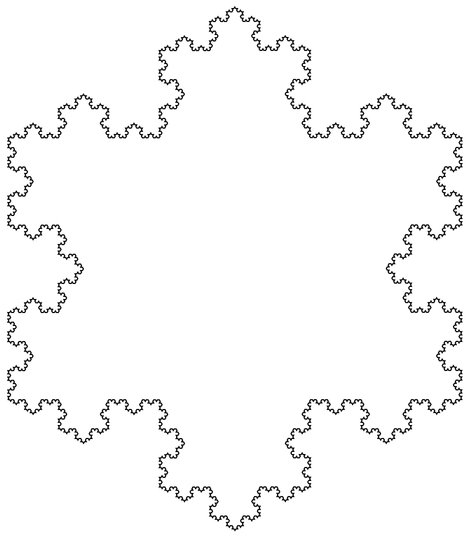

Figure 2 – Koch snowflake

a way reminiscent to the Weierstrass functions, there is no point of this curve at which there exists a tangent. At that time, such an object was seen as a curiosity (if not a monstruosity) but the work of Benoît Mandelbrot made these objects, which he called *fractals*, subjects of study in their own right. In particular, Mandelbrot suggests to quantify the degree of irregularity of such an object by its Hausdorff dimension. For the snowflakes, he computes, using geometric arguments, the value $\log(4)/\log(3)$. Later, the australian mathematician John Hutchinson in the seminal paper [29] will give another point of view on the snowflake, by classifying it in the category of attractors of iterated function systems for which, under a condition called the open set condition, he shows that the similarity dimension, easy to compute, is equal to the Hausdorff dimension.

Let us go back to the Dirichlet problem. We could expect the harmonic measure μ_{x_0} on a non-rectifiable Jordan curve to be absolutely continuous with respect to a s-dimensional Hausdorff measure (or a slight modification of it), with $s > 1$. Nikolai Georgievich Makarov, in [47] proves this to be wrong. His remarkably precise theorem states that *for any Jordan curve* the harmonic measure is absolutely continuous with respect to the Hausdorff measure associated to the gauge function

$$\varphi(t) = t \exp\left(C\sqrt{\log(1/t)\log\log\log(1/t)}\right),$$

where C is a universal constant and that, on the other hand, for any gauge function ψ such that $\psi(t) = o_{t\to 0}(t)$, the harmonic measure and the ψ-Hausdorff measure are mutually singular. The first part of this statement entails that the lower Hausdorff dimension of the measure μ_{x_0} (see section 1.11 for a definition) is greater or equal to 1 and the second part of the statement implies that its upper Hausdorff dimension is less or equal to 1. So we have $\dim_{\mathrm{H}} \mu_{x_0} = 1$ while the measure μ_{x_0} has full support in $\partial\Omega$ which may have dimension greater than 1, as is the case with the Koch snowflake.

This rather surprising phenomenon has been called *dimension drop* by Russell Lyons, Robin Pemantle and Yuval Peres in [43] in 1995. The subject of this article, which in many ways is the foundation of the present work, is the study of the simple random walk on a supercritical Galton-Watson tree conditioned to survive. We describe this setting in the next two paragraphs.

In order to define a Galton-Watson random tree T, we consider a reproduction law, that is a sequence $(p_k)_{k\geq 1}$ of non-negative real numbers adding up to 1 and in an informal way (a more precise definition is given in Chapter 2), the root $\varnothing$ of T has a random number of children with distribution (p_k), then its children reproduce independently with the same distribution and so on and so forth, see Figure 3. Denoting by q the probability that T is finite and by $m = \sum_{k\geq 0} kp_k$ the mean of the reproduction law, it is well known that $q < 1$ if and only if $m > 1$. The reproduction law (p_k) is called supercritical (respectively critical, subcritical) when $m > 1$ (respectively $m = 1$, $m < 1$).

Now for a real number $\lambda > 0$, the λ-biased random walk on a rooted tree is a nearest-neighbor random walk with weight λ to the parent and 1 to the children, see figure 4. When $\lambda = 1$, we call this walk a simple random walk. On the event of non-extinction, that is, when T is infinite, a nearest-neighbor random walk, which is always irreducible, is either recurrent or transient. In [39], Russell Lyons shows that the λ-biased random

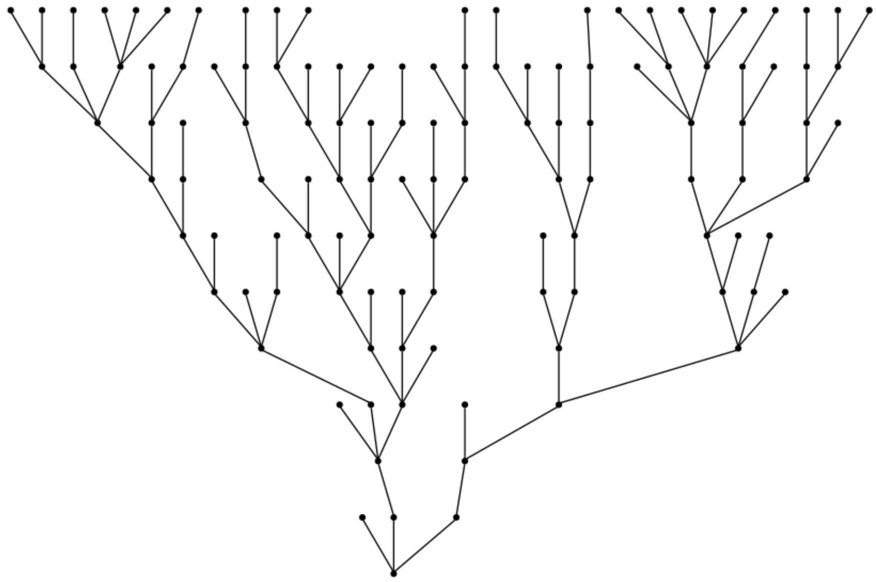

Figure 3 – 10 generation of a simulated Galton-Watson tree of uniform reproduction law on $\{0, 1, 2, 3\}$

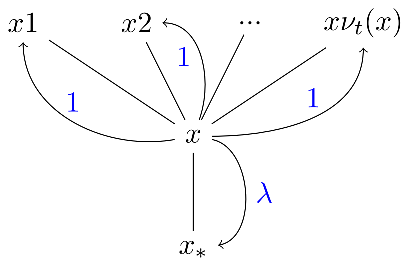

Figure 4 – Weights associated λ-biased : here x is a vertex of a rooted tree t, x_* is its parent and $x1$, ..., $x\nu_t(x)$ are its children.

walk on T conditioned to survive is recurrent if and only if $\lambda \geq m$. The *boundary* ∂T of the tree T is the set of all the rays in T, that is the infinite paths $(\xi_0 = \emptyset, \xi_1, \xi_2, \dots)$ starting from the root and such that for all $i \geq 1$, ξ_{i+1} is a child of ξ_i. We endow it with a natural distance by setting for two distinct rays ξ and η, $d(\xi, \eta) = \exp(-n)$ if n is the greatest index such that $\xi_n = \eta_n$. It makes ∂T into a compact ultrametric space. The Hausdorff dimension of this space is almost surely $\log m$, by a theorem of John Hawkes ([25]) sharpened by Lyons (again in [39]). When a random walk (X_n) on T is transient, the *harmonic measure* associated to this walk is the distribution of the unique ray Ξ sharing infinitely many vertices with the trajectory (X_n). It is a Borel probability measure on ∂T.

We may now state the main results of [43]. In the case of a Gaton-Watson tree T such that $m < \infty$, $p_0 = 0$ [4], and $p_1 < 1$, simple random walk (which is necessarily transient) goes to infinity with linear speed: almost surely,

$$\lim_{n\to\infty} \frac{|X_n|}{n} = \mathbb{E}\left[\frac{\nu - 1}{\nu + 1}\right],$$

where $|X_n|$ denotes the height of the n-th position of the random walk and ν is a random variable distributed as (p_k). Furthermore, if the reproduction law is not degenerated, the dimension of the harmonic measure is strictly less than $\log m$, the dimension of ∂T. In other words, the dimension drop phenomenon also occurs in this setting. The tools developped to prove these results are also remarkable. In particular, the authors invent the *augmented Galton-Watson tree* whose root has one more child to obtain an invariant environment with respect to the random walk. They also create the concept of *consistent flow rules* in order to build an ergodic theory on Galton-Watson trees with a distinguished ray. This theory will be detailed in the second chapter of the thesis.

The following year, the same authors treat the case of transient λ-biased walks. They show in [44] that the dimension drop phenomenon still holds and that when λ is greater than a critical value λ_c (which equals 0 if $p_0 = 0$) the random walk again goes to infinity at linear speed. However, the lack of an invariant environment with respect to the random walk entails that, on the one hand, the results they obtain are less precise (no explicit formula for the speed or the dimension) and on the other hand that the techniques they use are very different from the case $\lambda = 1$. The dimension drop is proved by considering some particular random times associated to the trajectory called regeneration times. These arguments will be detailes in Chapter 5, in a more general setting. The paper [44] ends with several questions which are still vastly open:

1. If $p_0 = 0$, is the speed a monotonic function of λ?
2. Is the dimension of the harmonic measure a monotonic function of λ?
3. What is the regularity of these functions?

In 2013 is published [3] which makes an important progress in the understanding of transient λ-biased random walks on infinite Galton-Watson trees. In this paper, Élie Aïdékon uses specific features of theses walks (most crucially that at a fresh time, that

4. This hypothesis is present only for simplicity, the results in [43] are more general.

is, at a time where the walk discovers a new vertex, time reversal and trajectory reversal cancel out each other in some way) to obtain an asymptotic environment seen from the particle. From this, he deduces a formula expressing the speed as a function of the distribution of the conductance of the tree

$$\beta(T) = \mathrm{P}^T_{\varnothing}(\tau_{\varnothing_*} = \infty),$$

where under $\mathrm{P}^T_{\varnothing}$, (X_k) is a λ-biased random walk on T starting from $\varnothing$, $\varnothing_*$ is an artificial parent of $\varnothing$ and $\tau_{\varnothing_*}$ is its first hitting time by the walk (X_k). If we denote by ℓ_λ the speed of the walk on T conditioned to survive, then for $\lambda_c < \lambda < m$,

$$\ell_\lambda = \mathbb{E}\left[\frac{(\nu-\lambda)\beta_0}{\lambda - 1 + \beta_0 + \sum_{j=1}^{\nu}\beta_j}\right] \Bigg/ \mathbb{E}\left[\frac{(\nu+\lambda)\beta_0}{\lambda - 1 + \beta_0 + \sum_{j=1}^{\nu}\beta_j}\right],$$

In the above expression, the random variables β_0, β_1, ..., are i.i.d. copies of $\beta(T)$ and ν has distribution (p_k) and is independent of the β_i's. Unfortunately, the distribution of $\beta(T)$ is still very mysterious and this formula allowed only to prove the monotonicity of the speed in the interval $[0, 1/2]$. One may however check numerically the validity of the conjecture about the speed of Lyons, Pemantle and Peres. Indeed, the conductance satisfies the following recursive distributional equation:

$$\beta(T) \overset{\mathrm{d}}{=} \frac{\sum_{j=1}^{\nu}\beta_j}{\lambda + \sum_{j=1}^{\nu}\beta_j}.$$

and is the greatest (for the stochastic partial order) solution of this equation. We may then, for any fixed λ use this equality in distribution to compute numerically the distribution of $\beta(T)$ and thus obtain with Aidékon's formula a numerical value for the speed. When $p_1 = p_2 = 1/2$, we obtain the graph in figure 5 which seems to confirm the conjecture. The value $1/6$ is given by Lyons, Pemantle and Peres' formula for simple random walk.

Our first result, independently found by Shen Lin ([35]) is a formula to compute the dimension of the harmonic measure, similar to that of Aïdékon. Using an idea from Nicolas Curien and Jean-François Le Gall in [10] [5], we have given a set of algebraic conditions that, if satisfied by a flow rule, allow us to build an invariant measure with respect to this flow rule, with an explicit density with respect to the distribution of T. This abstract result is stated at the end of Chapter 2. It can be applied to the case of transient λ-biased walk (see Chapter 4 or [55]) and, together with the ergodic theory on Galton-Watson trees, allows us to obtain the following formula for the dimension d_λ of the harmonic measure:

$$d_\lambda = \log(\lambda) - C^{-1}\mathbb{E}\left[\log(1-\beta_0)\frac{\beta_0\sum_{j=1}^{\nu}\beta_j}{\lambda - 1 + \beta_0 + \sum_{j=1}^{\nu}\beta_j}\right],$$

where the renormalization constant C equalis the expectation of the quotient above. In a similar way, we may conduct some numerical experiments to check the validity of some

5. The authors indicate that this idea follows from a "non-rigorous passing to the limit" in the environment of Aïdékon.

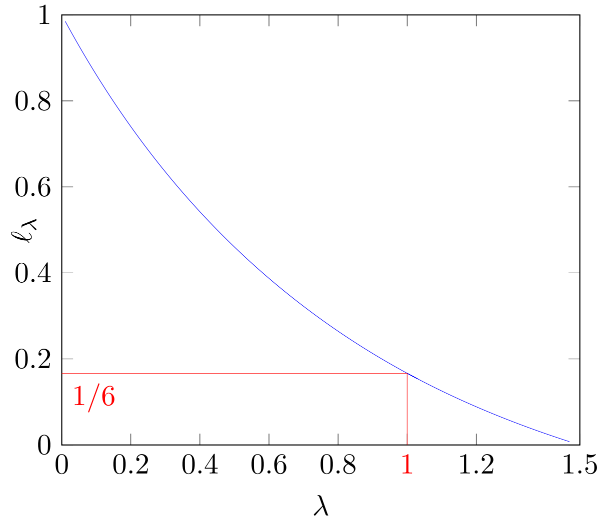

Figure 5 – Speed ℓ_λ of the λ-biased random walk as a function of λ in the case $p_1 = p_2 = 1/2$.

conjectures in [44, 45]. In figure 6, the limits at 0 and m are given by a theorem of Shen Lin and equal, respectively, $\mathbb{E}[\log \nu]$ and $\log \mathbb{E}[\nu] = \log m$. We know that $\log m$ is a strict upper bound (by the dimension drop theorem of Lyons, Pemantle et Peres), but it is not proved in general that $\mathbb{E}[\log \nu]$ is a lower bound. Of course this would be a consequence of continuity and monotonicity of the function $\lambda \mapsto d_\lambda$ if we could prove it.

We continue this story by going back to the paper [10] of Curien and Le Gall. This paper is about simple random walk (X_k) on a critical Galton-Watson tree $T^{(n)}$ conditioned to survive at least until generation n, with the assumption that the variance of the reproduction law is finite. Denoting by $\tau^{(n)}$ the first hitting time of height n and letting for x at height n in $T^{(n)}$, $\mu_n(\{x\}) = \mathrm{P}_{\varnothing}^{T^{(n)}}(X_{\tau^{(n)}} = x)$, we have in μ_n an analogue of the harmonic measure in this setting. The authors show that, while it is well-known that $T^{(n)}$ has a number of vertices at height n of order n, the harmonic measure is almost completely carried by a part of cardinality of order n^δ, where $\delta \approx 0,78$ is a universal constant. This is to be understood as another dimension drop result. To prove it, they reduce (not easily) the problem to a problem of dimension drop on an infinite tree with random edge lengths. (we have in [55] called these trees "Galton-Watson trees with recursive lengths"). Denote by $\widetilde{T}^{(n)}$ the tree obtained from $T^{(n)}$ after pruning (only keeping the vertices that belong to a smallest path between the root and a vertex at height n) and reduction (only keeping the vertices at height n, the root and the branching points of the tree) but still keeping the graph distance inherited from $T^{(n)}$. Then, in the Gromov-Hausdorff sense, the metric space $\widetilde{T}^{(n)}$ converges to what the authors call the continuous reduced tree defined in the following way: the reproduction law is given by $p_2 = 1$ and for each word x on the alphabet $\{1, 2\}$ pick a random variable U_x uniform on $(0, 1)$ in an independent way, then give to the edge connecting x to its parent x_* the

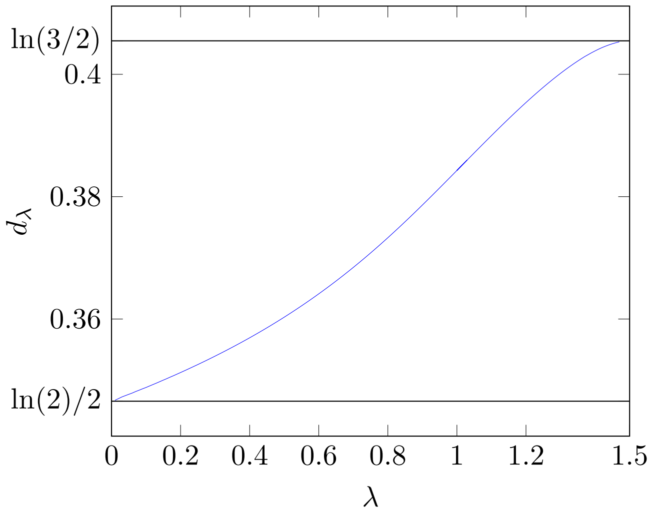

Figure 6 – Dimension d_λ of the harmonic measure as a function of λ, for $p_1 = p_2 = 1/2$.

length

$$U_x(1 - U_{x_*}) \cdots (1 - U_\varnothing),$$

where the product above is indexed by the strict ancestors of x. The simple random walk on $T^{(n)}$ corresponds in the limit to a nearest-neighbor random walk in the continuous reduced tree, where the transition weights are the inverses of the edge lengths (see Chapter 3 for a more precise definition). This random walk is transient. Then, Curien and Legall, with two very different methods (one of which using stochastic calculus) obtain an invariant measure with respect to the harmonic flow and show that the dimension of the harmonic measure is $\delta < 1$ whereas the dimension of the boundary (for the metric associated to the edge lengths) is equal 1. In [34], Shen Lin, who was then a PhD student of Le Gall, extended this result to the case where the reproduction law of the original tree is in the basin of attraction of an α-stable law, with $\alpha \in (0, 1]$, the main difference being the fact that in the continuous reduced tree, the vertices may have more than 2 children.

In Chapter 3, we generalize a part of these results to the case where the reproduction law in the Galton-Watson tree with recursive lengths only satisfies $p_0 = 0$ and $p_1 < 1$ and we replace the uniform distribution on $(0, 1)$ by an arbitrary distribution on $(0, 1)$. see figure 7 (where, for a practical reason the random variables used for the recursive lengths are the inverse of the the random variables $(\Gamma_x)_{x \in T}$ with values in $(1, \infty)$). We show, by rewriting a part of the ergodic theory on Galton-Watson trees in this setting that, under some fairly non-restricting hypotheses, the dimension drop phenomenon still holds and the dimension of the boundary of the tree for the metric associated to the lengths is the Malthusian parameter α defined by

$$\mathbf{E}[(1 - \Gamma_\varnothing^{-1})^\alpha] = 1/m.$$

The last two chapters of this thesis are devoted to the study of a model, which has been

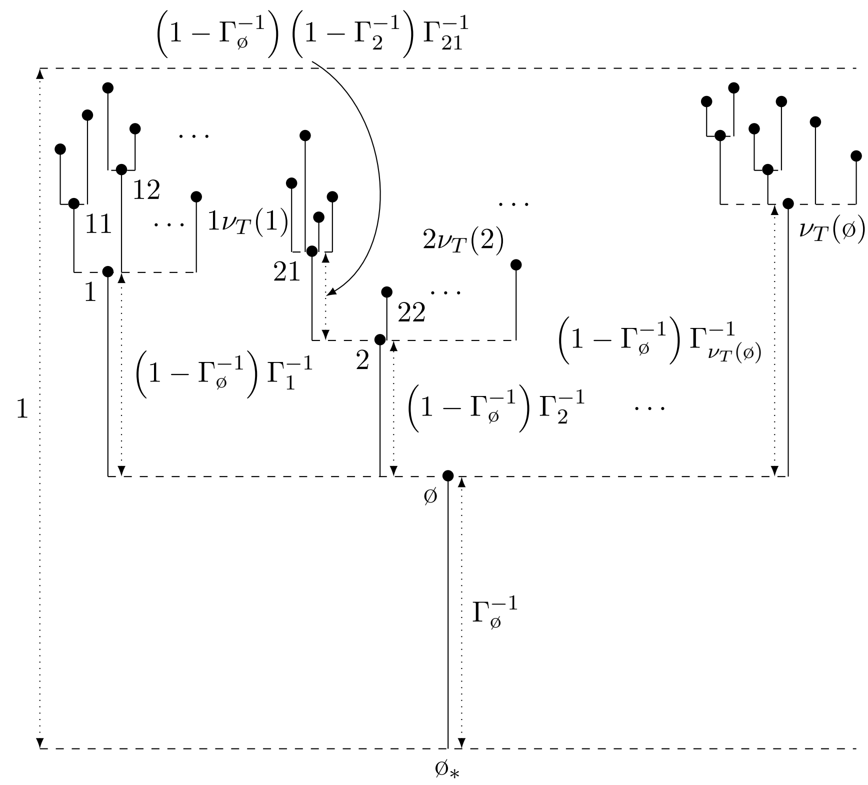

Figure 7 – A schematic representation of a Galton-Watson tree with recursive lengths

the subject of many papers in the past two decades and, called *Random walk in a random environment on a Galton-Watson tree* [6] introduced in [41]. To define this model, consider not only a reproduction law, but a random element $\mathbf{A}$ of the space $\mathsf{Tuples} = \bigcup_{k\geq 0}(0,\infty)^k$ of finite sequences of positive numbers, where we agree that $(0,\infty)^0$ contains only the empty sequence $()$. We then build a random weighted tree in the following way: we first pick $\mathbf{A}^{\varnothing}$ distributed as $\mathbf{A}$, give to the root $\varnothing$ a number of children equal to the length of the sequence $\mathbf{A}^{\varnothing}$ and for each child i of the root, we give to the edge $\{\varnothing, i\}$ a weight $\mathsf{A}_T(i)$ equal to the i-th component of $\mathbf{A}^{\varnothing}$, then we iterate this process in an independent way for each child i of the root, and so on and so forth. We thus obtain a Galton-Watson tree T, as well as a function $\mathsf{A}_T : T \setminus \{\varnothing\} \to (0,\infty)$ giving for each x of $T \setminus \{\varnothing\}$, the weight of the edge connecting it to its parent. We may then define a nearest-neighbor

6. This name is perhaps a bit ill-chosen since a random walk on a Galton-Watson tree is already a random walk in a random environment.

random walk by setting:

$$\mathrm{P}^T(x,y) = \begin{cases} \dfrac{\mathsf{A}_T(xi)}{1+\sum_{j=1}^{\nu_t(\varnothing)} \mathsf{A}_T(xj)} & \text{si } y = xi, \text{ pour un } 1 \le i \le \nu_T(x); \\ \dfrac{1}{1+\sum_{j=1}^{\nu_T(\varnothing)} \mathsf{A}_t(xj)} & \text{if } y = x_*, \end{cases}$$

where $\nu_T(x)$ is the number of children of x in T and x_* is its parent, see figure 8. The

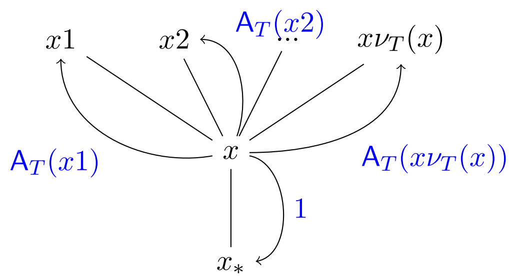

Figure 8 – Transition weights of the random walk on the weighted Galton-Watson tree T

function $\psi : [0,\infty) \to (-\infty,\infty]$ defined by

$$\psi(s) = \log \mathbb{E}\Big[\sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i)^s\Big]$$

plays an important role in this model. In particular, if $\min_{[0,1]} \psi > 0$, then the random walk on the weighted tree T is transient and we may ask whether the dimension drop phenomenon holds in this setting. In Chapter 5 we apply the method sketched in [44] to show that, under the hypothesis $m < \infty$, dimension drop occurs, unless the model reduces to a λ-biased walk on a regular tree. To describe briefly this method, we introduce the *exit times* and the *regeneration times*. For a trajectory (X_k) of the random walk, an integer $s \ge 1$ is an exit time if X_{s-1} is the parent of X_s and for all $k > s$, $X_k \ne (X_s)_*$. After such a time, the random walk is condemned to stay in the subtree starting from X_s. If, furthermore, s is a fresh time, meaning that s is the first passage of the walk at the vertex X_s, then we say that s is a regeneration time of (X_k). We show that there are almost surely infinitely many regeneration times and that at these times the forward environment from the particle is stationary. To get back to the exit times, which are the ones that interest us, following the idea in [44], we build a Rokhlin tower. A careful examination of this tower leads us to introduce a new function κ on the space of weighted trees and this function is the density with respect to the distribution of T of an invariant measure for the harmonic flow, entailing the result.

In the last chapter of this thesis, we leave transient random walks and study, still in this setting of weighted Galton-Watson trees a recurrent case known as *the subdiffusive*

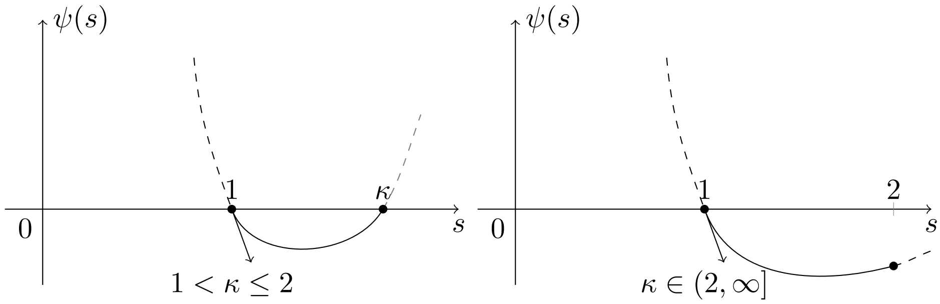

Figure 9 – Schematic behavior of ψ under the hypotheses in Chapter 6

case. We work under the following hypotheses: first assume that we are in the *normalized* setting

$$\psi(1) = \log \mathbf{E} \sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i) = 0.$$

and in order to apply Biggins' theorem, that

$$\psi'(1) := \mathbf{E}\left[\sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i) \log \mathsf{A}_T(i)\right] \in [-\infty, 0).$$

In this case, the following parameter plays a crucial role:

$$\kappa = \inf\{s > 1 : \psi(s) = 0\} \in (1, \infty],$$

and we assume that

$$\mathbf{E}\Big[\Big(\sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i)\Big)^{\kappa}\Big] + \mathbf{E}\Big[\sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i)^{\kappa} \log^+ \mathsf{A}_T(i)\Big] < \infty, \quad \text{si } 1 < \kappa \leq 2,$$

$$\mathbf{E}\Big[\Big(\sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i)\Big)^{2}\Big] < \infty, \quad \text{si } \kappa \in (2, \infty].$$

These hypotheses are summed up in figure 9. The additive martingale $(M_n(T))_{n \geq 0}$ (also called sometimes Mandelbrot's martingale or Biggins' martingale) is defined by:

$$M_n(T) = \sum_{|x|=n} \prod_{\varnothing \prec y \preceq x} \mathsf{A}_T(y).$$

By Biggins' theorem (see also [32] by Kahane and Peyrière, or [40]), it converges almost surely and in L^1 to a non-degenerate random variable $M_\infty(T)$. Denoting by $\mathscr{C}_n(T)$ the conductance between the root of T and its vertices at height n, by recurrence of the walk, $\mathscr{C}_n(T)$ converges almost surely to 0 and we are interested in its rate of decay. We show that

$$0 < \liminf_{n\to\infty} n^{1/(\kappa-1)}\mathbb{E}[\mathscr{C}_n(T)] \leq \limsup_{n\to\infty} n^{1/(\kappa-1)}\mathbb{E}[\mathscr{C}_n(T)] < \infty \quad \text{if } 1<\kappa<2 \text{ ;}$$
$$0 < \liminf_{n\to\infty} n\log n\mathbb{E}[\mathscr{C}_n(T)] \leq \limsup_{n\to\infty} n\log n\mathbb{E}[\mathscr{C}_n(T)] < \infty \quad \text{if } \kappa = 2 \text{ et}$$
$$\lim_{n\to\infty} n\mathbb{E}[\mathscr{C}_n(T)] = \|M_\infty(T)\|_2 \quad \text{if } \kappa > 2.$$

And, in any case, almost surely,

$$\lim_{n\to\infty} \mathscr{C}_n(T)/\mathbb{E}[\mathscr{C}_n(T)] = M_\infty(T).$$

Moreover, this convergence also holds in L^p for $p \in [1, \kappa/2)$, if $1 < \kappa \leq 2$ and in L^2 if $\kappa > 2$,

The rest of this thesis is independent of this introduction and all the definitions will be recalled in a more formal setting.

Chapter 1 introduces the planar trees as they were formalized by Jacques Neveu ([50]) as well as their boundaries. We characterize a family of “good” metrics on the boundary of a tree and we give a slightly original definition of Hausdorff and packing measures with the density theorems as a starting point. We then recall the classical definitions of these measures on metric spaces and connect them with our definition by using the aforementioned characterisation of good metrics. Finally, we define and study different notions of dimensions for the Borel measures on the boundary of a tree.

Chapter 2 is devoted to the ergodic theory on marked Galton-Watson trees. We establish with many details the main results of [43] or [46, Chapter 17]. Some of these details are slighlty original, at least in their forms. In the last section of this chapter, we state and prove the already mentioned algebraic criterion of existence of an invariant measure.

Chapters 3 to 6, already introduced, include most of the new results of this thesis, but we did not hesitate to present proofs of known theorems, to make this text as self-contained as possible.

1 Trees and their boundaries

In this chapter, we define our (rooted, planar) trees, using Neveu's formalism which identifies trees with some particular sets of finite words on the alphabet $\mathbb{N}^* = \{1, 2, \dots\}$. The boundary ∂t of an infinite tree t is the set of all non-backtracking, infinite paths starting from its root. This boundary may be seen as a subset of the set of all infinite words on $\mathbb{N}^*$. We equip this set with a topology (two infinite words are "close" to each other if they have a "long" common prefix) and the associated Borel σ-algebra. At this point, we characterize a class of metrics on the boundary of an infinite tree which satisfy a natural property and introduce the notion of diameter function.

We then proceed to describe the Borel probability measures on the boundary of a tree by associating to each of them a unique unit flow. Random walks on trees are briefly introduced, with an emphasis on the transient ones since they lead to the harmonic measures on the boundary of a tree.

We then go back to the study of general unit flows on an infinite tree and see how Hausdorff and packing measures arise from the study of the upper and lower densities of a flow. We then proceed to connect our somewhat personnal definition of these measures with the classical definitions of the literature, using the diameter functions. The study of the local dimensions of a flow and their links with the Hausdorff and packing dimension concludes this chapter.

1.1 Finite and infinite words

Let $\mathbb{N}^* = \{1, 2, \dots\}$ be our alphabet. For an integer $n \geq 1$, a word of length n, on the alphabet $\mathbb{N}^*$ is a finite sequence

$$x = (i_1, i_2, \dots, i_n) \overset{\text{(not.)}}{=} i_1 i_2 \cdots i_n,$$

of elements of $\mathbb{N}^*$. The empty word is denoted by $\varnothing$ and has length 0. We write $|x|$ for the length of a finite word x. The set of all finite words on $\mathbb{N}^*$ is denoted by

$$\mathcal{U} = \bigsqcup_{k=0}^{\infty} (\mathbb{N}^*)^k,$$

with the convention $(\mathbb{N}^*)^0 = \{\varnothing\}$. The *concatenation* of the non-empty words $x = i_1 i_2 \cdots i_m$ and $y = j_1 \cdots j_n$ is the finite word of length $m + n$, $xy = i_1 i_2 \cdots i_m j_1 \cdots j_m$. The concatenation with the empty word is defined by $x\varnothing = \varnothing x = x$, for all x in $\mathcal{U}$. A word x is called a *prefix*, or an *ancestor*, of a word y if and only if there exists a word z such that $y = xz$. In this case, we write $x \preceq y$ ($x \prec y$ if $x \neq y$) and $z = x^{-1}y$. The relation $\preceq$ is a partial order and we denote by $x \wedge y$ the greatest common prefix of x and y. Two words are said to be *incomparable* if neither of them is a prefix of the other. The *parent* of a non-empty word $x = i_1 i_2 \cdots i_{|x|}$ is its greater strict prefix $x_* = i_1 i_2 \cdots i_{|x|-1}$ if $|x| \geq 2$; otherwise it is the empty word $\varnothing$. We also say that x is *a child* of x_*.

We will often need to add an artificial parent of the empty word, denoted by $\varnothing_*$, and set $\mathcal{U}_* = \mathcal{U} \sqcup \{\varnothing_*\}$. We let $|\varnothing_*| = -1$ and, for $x \in \mathcal{U}$, the concatenation of $\varnothing_*$ with x is defined by $\varnothing_* x = x \varnothing_* = x_*$.

A *ray* in $\mathcal{U}$ is an infinite sequence $\rho = (\rho_0, \rho_1, \dots)$ of words such that $\rho_0 = \varnothing$ and for each $k \geq 0$, ρ_{k+1} is a child of ρ_k. In particular, for each $k \geq 0$, $|\rho_k| = k$. Let $\mathcal{U}_\infty = (\mathbb{N}^*)^{\mathbb{N}^*}$ be the set of all infinite words on the alphabet $\mathbb{N}^*$. The concatenation of a finite word x with an infinite word ξ is denoted by $x\xi$. For a non-negative integer k, The *k-th truncation* of an infinite word ξ is the finite word composed of its k first letters and is denoted by ξ_k, with $\xi_0 = \varnothing$. The mapping $\xi \mapsto (\xi_0, \xi_1, \xi_2, \dots)$ is a bijection between infinite words and rays, therefore we will abuse notation and write ξ for both the infinite word and the ray associated to it. When a finite word x is a truncation of an infinite word ξ, we still say that x is a prefix of ξ and write $x \prec \xi$, in this case, define $x^{-1}\xi$ as the unique infinite word η such that $x\eta = \xi$. For two distinct infinite words ξ and η, we may again consider their *greatest common prefix* $\xi \wedge \eta \in \mathcal{U}$. For x in $\mathcal{U}$, the *cylinder* generated by x is the set

$$[x] = \{\xi \in \mathcal{U}_\infty : x \prec \xi\} = \{x\eta : \eta \in \mathcal{U}_\infty\}.$$

The set of all infinite words is endowed with the topology generated by all the cylinders. We first remark that for a sequence $(\xi^{(n)})$ of rays to converge to a ray ξ, it is necessary and sufficient that

$$\forall N \geq 0,\ \exists n_0 \geq 0,\ \forall n \geq n_0, \quad \xi^{(n)} \in [\xi_N]. \tag{1.1}$$

Another important topological property is the fact that , for two finite words x and y,

— if $x \preceq y$, then $[y] \subset [x]$;

— if $y \preceq x$, then $[x] \subset [y]$;

— otherwise, $[x] \cap [y] = \emptyset$.

In particular, the set

$$\{[x] : x \in \mathcal{U}\} \cup \{\emptyset\}$$

covers $\mathcal{U}_\infty$ and is stable under (finite) intersections thus forms a base of our topology and any open set may be written as a union of cylinders. We actually have more: any (non-empty) open set may be written as a union of *pairwise disjoint* cylinders. To prove this, we use the following lemma, which plays the role of a covering lemma in this theory.

Lemma 1.1. *Let Q be a non-empty subset of $\mathcal{U}$. There exists a subset $\widetilde{Q}$ of Q such that:*
1. any two distinct elements of $\widetilde{Q}$ are incomparable (for the prefix order $\preceq$) and
2. any element of Q has a unique ancestor in $\widetilde{Q}$.

Proof. On Q we define the equivalence relation:

$$w \sim_Q w' \iff \exists u \in Q,\ u \preceq w \wedge w'$$

It is clearly reflexive and symmetric. To see that it is transitive, let w, w' and w'' be such that $w \sim_Q w'$ and $w' \sim_Q w''$. By definition, this means that there exist u and u' in Q such that $u \preceq w \wedge w'$ and $u' \preceq w' \wedge w''$. Now, since u and u' are both ancestors of w', they can be compared. Therefore, if u'' is the minimum (for the order $\preceq$) of u and u', we have $u'' \preceq w$ and $u'' \preceq w''$, hence $w \sim_Q w''$.

Let, for an equivalence class C in the quotient set $Q \big/ {\sim_Q}$, $\iota_Q(C)$ be the element of the class C which has the smallest length. To see that this element is indeed unique, assume that w and w' are two such elements. Since they are both in the same class, there exists an element u in C such that $u \preceq w$ and $u \preceq w'$, thus, $u = w = w'$. Let $\widetilde{Q} = \iota_Q\left(Q \big/ {\sim_Q}\right)$ to conclude the proof. □

Lemma 1.2. *Any non-empty open set of $\mathcal{U}_\infty$ may be written as a countable union of pairwise disjoint cylinders.*

Proof. Let U be a non-empty open set of $\mathcal{U}_\infty$. The cylinders form a base of the topology, so we may write $U = \bigcup_{x \in Q}[x]$, for a subset Q of $\mathcal{U}$. Let $\widetilde{Q}$ be a subset of Q as in the preceding lemma. The fact that any element of Q has an ancestor in $\widetilde{Q}$ implies that $U = \bigcup_{x \in \widetilde{Q}}[x]$ and, since the element of $\widetilde{Q}$ are pairwise incomparable, those cylinders are pairwise disjoint. □

The topology generated by the cylinders is metrizable by a large family of distances. We will mostly (but not always) use the *natural distance* $\mathrm{d}_{\mathcal{U}_\infty}$ defined by

$$\mathrm{d}_{\mathcal{U}_\infty}(\xi, \eta) = e^{-|\xi \wedge \eta|}, \quad \text{for any } \xi \neq \eta \text{ in } \mathcal{U}_\infty. \tag{1.2}$$

This makes $\mathcal{U}_\infty$ into a complete, separable, ultrametric space.

1.2 Trees and Neveu's formalism

Following Neveu ([50]), we represent our trees as subsets of $\mathcal{U}$.

Definition 1.1. A (rooted, planar, locally finite) tree t, is a subset of $\mathcal{U}$ such that:
1. $\varnothing$ is in t;
2. for any $x \neq \varnothing$ in t, x_* is in t;
3. for any x in t, there exists a unique non-negative integer, denoted by $\nu_t(x)$ and called the number of children of x in t, such that for any $i \in \mathbb{N}^*$, xi is in t if and only if $1 \leq i \leq \nu_t(x)$.

A tree is endowed with the undirected graph structure obtained by drawing an edge between each word and its children, see Figure 2.1. In this context, we call $\varnothing$ the *root* of t, and if x is an element of t, its length $|x|$ is rather called its *height* in the tree. A vertex x of t such that $\nu_t(x) = 0$ is called a *leaf* of t. For convenience, we add an empty tree, denoted by †, to the family of all the trees. When, for a tree t, we need to add an artificial parent to the root, we write $t_* = t \cup \{\emptyset_*\} \subset \mathcal{U}_*$.

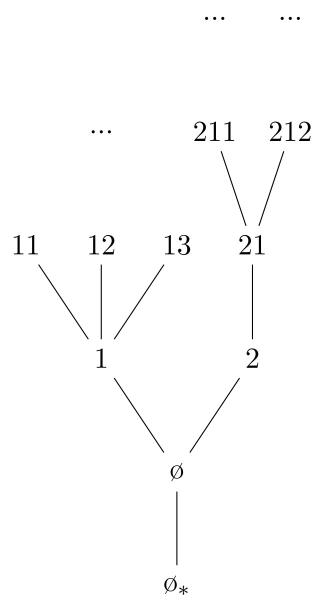

Figure 2.1 – A representation of a tree t_* with an artificial parent of the root

As a simple example of an infinite tree, for an integer $m \geq 1$, define the m-regular tree as the tree in which each vertex has exactly m children. As a set in Neveu's formalism, it consists of ø, and all finite sequences of elements of $\{1, 2, \ldots, m\}$.

An important operation on trees is to extract a subtree originated from a given vertex x of a tree t. However, we need to reindex it if we want it to be a tree in our formalism. The reindexed subtree $t[x]$ is the empty tree † if x is not in t, and is

$$t[x] = \{y \in \mathcal{U}_* : xy \in t\};$$

otherwise. For example, if t is an m-regular tree, the reindexed subtrees $t[x]$, for x in t, are all equal to t.

1.3 The boundary of a tree

Recall that we identify rays in $\mathcal{U}$ with infinite words. The *boundary* of t is the set of all rays ξ in $\mathcal{U}_\infty$ such that for any $k \geq 0$, ξ_k is in t. It is denoted by ∂t. For instance, in the case of an m-regular tree, the boundary of the tree is the set of all infinite sequences of elements of $\{1, 2, \ldots, m\}$.

The boundary ∂t of a tree t is endowed with the trace topology from $\mathcal{U}_\infty$. Since our trees are locally finite by definition, ∂t is a compact subspace of $\mathcal{U}_\infty$. For any vertex x

in t, the *cylinder* $[x]_t$ is the set

$$[x]_t = \{\xi \in \partial t : x \prec \xi\} = [x] \cap \partial t.$$

Notice that $[x]_t$ is empty if and only if the subtree $t[x]$ is finite. We define the pruned tree (from infinity) as

$$t^* = \{x \in t : t[x] \text{ is infinite}\} = \{x \in t : \exists \xi \in \partial t,\ x \prec \xi\}.$$

If need be, it can be made into a tree in Neveu's formalism by reindexing its vertices in such a way that the lexicographical order is preserved (and such a reindexing is unique).

A ray ξ in ∂t is isolated if and only if $[\xi_n]_t = \{\xi\}$ for some $n \geq 1$. We denote by $\mathrm{Isolated}(\partial t)$ the set of isolated rays in ∂t. It is countable, since there are countably many distinct cylinders. We extend the definition of the cylinders to the isolated rays by setting

$$\forall \xi \in \mathrm{Isolated}(t),\ [\xi]_t = \{\xi\}.$$

For x in t, define the *number of infinite lineages* from x in t by

$$\nu_t^*(x) = \#\{1 \leq i \leq \nu_t(x) : t[xi] \text{ is infinite}\}.$$

If $t[x]$ is infinite, this is the number of children of x in the pruned tree. One should beware that, if $\nu_t^*(x) \leq 1$, we have $[x]_t = [x1]_t$. Thus we introduce, if $\partial t[x]$ has at least 2 elements, the first branching point above x in t,

$$\mathsf{bp}_t^{\uparrow}(x) = \min\{y \succeq x : \nu_t^*(y) \geq 2\},$$

and extend it to the case where $t[x]$ has only one ray ξ by setting $\mathsf{bp}_t^{\uparrow}(x) = \xi$. This gives $[x]_t = [\mathsf{bp}_t^{\uparrow}(x)]_t$, for any x in t^*. It is convenient here to introduce the *reduced* tree (from infinity)

$$\mathsf{reduced}(t) = \{y \in t : \nu_t^*(y) \geq 2\} = \{\xi \wedge \eta : \xi \neq \eta \in \partial t\},$$

and the *skeleton* of the tree t:

$$\mathsf{skel}(t) = \mathsf{reduced}(t) \cup \mathrm{Isolated}(\partial t) \subset t \cup \partial t.$$

From the skeleton of a tree we could rebuild the pruned tree. Furthermore,

$$\mathsf{bp}_t^{\uparrow}(x) = \min\{y \in \mathsf{skel}(t) : y \succeq x\},$$

thus the mapping $x \mapsto [x]_t$ is a one-to-one correspondance between $\mathsf{skel}(t)$ and the set of all non-empty cylinders of ∂t.

1.4 A class of metrics on the boundary of a tree

We may again use the distance $d_{\mathcal{U}_\infty}$ on ∂t to metrize its topology. Denoting by $\mathscr{B}(\xi, r)$ the closed ball centered at $\xi \in \partial t$ of radius $r \in (0,1]$, we have

$$\mathscr{B}(\xi, r) = \{\eta \in \partial t : |\xi \wedge \eta| \geq \lceil -\log r \rceil\} = [\xi_{\lceil -\log r \rceil}]_t,$$

and we see that the balls for this metric are the cylinders of this topology:

$$\{\mathscr{B}(\xi, r) : \xi \in \partial t,\ r \geq 0\} \cup \{\emptyset\} = \{[x]_t : x \in t\} \cup \{\emptyset\}. \tag{1.3}$$

As remarked in [13], this property leads to nice results when we deal with Hausdorff measures, and we want to characterize the metrics on ∂t that have this property (the random metrics in Chapter 3 will satisfy it). Although they are not all the metrics that metrize the topology of ∂t, we denote by $\mathsf{Metrics}(\partial t)$ the set of all such metrics.

Let d be in $\mathsf{Metrics}(\partial t)$. For ξ in ∂t and $r \geq 0$, write $\mathscr{B}^d(\xi, r)$ for the closed ball of radius r, centered at ξ, and diam^d for the diameter with respect to d. First notice that for any rays $\xi \neq \eta$ in ∂t,

$$\mathscr{B}^d(\xi, d(\xi,\eta)) = [\xi \wedge \eta]_t = \mathscr{B}^d(\eta, d(\xi,\eta)),$$

since it is the smallest ball that contains both ξ and η. This entails that d is ultrametric. Indeed, let $\zeta \in \partial t$ and assume that $d(\xi,\zeta) \geq d(\eta,\zeta)$. Since $\mathscr{B}^d(\zeta, d(\xi,\zeta))$ contains $\mathscr{B}^d(\zeta, d(\eta,\zeta))$, we have $[\xi \wedge \zeta]_t \supset [\zeta \wedge \eta]_t$, thus $\xi \wedge \zeta \preceq \eta \wedge \zeta \prec \eta$. Hence, $\xi \wedge \zeta \preceq \xi \wedge \eta$, which implies that $\mathscr{B}^d(\xi, d(\xi,\eta))$ is contained in $\mathscr{B}^d(\xi, d(\xi,\zeta))$. Now, since $\eta \in \mathscr{B}^d(\xi, d(\xi,\zeta))$, we have $d(\xi,\eta) \leq d(\xi,\zeta)$, as claimed. Next we see that for any $\xi \neq \eta$ in ∂t,

$$d(\xi,\eta) = \mathrm{diam}^d [\xi \wedge \eta]_t.$$

Indeed, if ζ and ζ' are in $[\xi \wedge \eta]_t$, then, by the ultrametric inequality,

$$d(\zeta,\zeta') \leq \max(d(\xi,\zeta), d(\xi,\zeta')) \leq d(\xi,\eta),$$

since ζ and ζ' both belong to $\mathscr{B}^d(\xi, d(\xi,\eta))$. Hence, $\mathrm{diam}^d([\xi \wedge \eta]_t) \leq d(\xi,\eta)$. The other inequality is obvious. Let, for x in $t \cup \mathrm{Isolated}(\partial t)$,

$$\varphi_d(x) = \mathrm{diam}^d [x]_t, \tag{1.4}$$

so that, for any $\xi \neq \eta$ in ∂t,

$$d(\xi,\eta) = \varphi_d(\xi \wedge \eta).$$

The function $\varphi_d : t \cup \mathrm{Isolated}(\partial t) \to \mathbb{R}_+$ satisfies the following properties:

1. The function φ_d vanishes on $\mathrm{Isolated}(t)$ and is (strictly) decreasing on $\mathsf{skel}(t)$.
2. If ξ is in $\partial t \setminus \mathrm{Isolated}(t)$, and $\xi_{n_0} \prec \xi_{n_1} \prec \ldots$ is the infinite sequence of prefixes of ξ that are in $\mathsf{reduced}(t)$, then as k goes to infinity, $\varphi_d(\xi_{n_k})$ goes to 0.
3. For x in $t \setminus t^*$, $\varphi_d(x) = 0$, and for $x \in t^*$, $\varphi_d(x) = \varphi_d(\mathsf{bp}_t^{\uparrow}(x))$.

The second property holds because the family of compact sets $([\xi_{n_k}])$ is decreasing and has intersection $\{\xi\}$. Of course, we also have, for any ξ in ∂t, $\lim_{n\to\infty}\varphi(\xi_n)=0$. When a function $\varphi : t \cup \text{Isolated}(\partial t) \to \mathbb{R}_+$ satisfies these three conditions we say that φ is a *diameter function on* t. A function φ defined only on $\mathsf{skel}(t)$ and satisfying the first two properties may be extended in a unique way to $t \cup \text{Isolated}(\partial t)$ by using the third one.

It is clear that, conversely, if φ is a diameter function on t, then, setting for $\xi \neq \eta$ in ∂t,

$$d_\varphi(\xi,\eta)=\varphi(\xi\wedge\eta), \tag{1.5}$$

one obtains a metric that satisfies (1.3). As a conclusion we have obtained:

Lemma 1.3. *The mapping $d \mapsto \varphi_d$ defined by* (1.4) *is a bijection between* $\mathsf{Metrics}(\partial t)$ *and the set of all diameter functions on t, with inverse bijection $\varphi \mapsto d_\varphi$ defined by* (1.5)*.*

For future reference we prove here the following lemma.

Lemma 1.4. *Let t be an infinite tree and let φ be a diameter function on t. For $n \geq 0$, let*

$$\delta_n=\max\{\varphi(x) : |x|=n\}.$$

Then, $\lim_{n\to\infty}\delta_n=0$*. For $\delta>0$, let*

$$n_\delta=\min\{n\geq 0 : \exists|x|\leq n,\ 0<\varphi(x)\leq\delta\},$$

and assume that ∂t is infinite. Then, $\lim_{\delta\to 0} n_\delta=\infty$*.*

Proof. The sequence (δ_n) is non-increasing. For any $n \geq 0$, let $x_n \in t$ be such that $|x_n|=n$ and $\delta_n=\varphi(x_n)$. We may assume that for all $n\geq 0$, $\varphi(x_n)>0$, otherwise there is nothing to prove. Since φ vanishes outside of t^*, we may pick an element $\xi^{(n)}$ in $[x_n]_t$, for each $n \geq 0$. By compactness of ∂t, we may extract from $\xi^{(n)}$ a subsequence $\xi^{(n_k)}$ such that $\lim_{k\to\infty}\xi^{(n_k)}=\xi$, and extracting again if necessary, we may assume that for each k, $\xi_k^{(n_k)}=\xi_k$. Since $\xi^{(n_k)}\in[x_{n_k}]_t$, we have $\varphi(x_{n_k})=\varphi(\xi_k)$, thus the sequence $(\varphi(x_{n_k}))$ converges to 0 and so does $(\varphi(x_n))$ by monotony.

Now assume that ∂t is infinite and let the sequence (v_n) be defined by

$$v_n=\min\{\varphi(x) : |x|\leq n \text{ and } \varphi(x)>0\}.$$

This sequence converges to 0, is non-increasing and positive. $\square$

1.5 Flows on a tree

A flow θ on the tree t is a function $\theta : t \to \mathbb{R}_+$ such that, for any x in t,

$$\theta(x)=\sum_{i=1}^{\nu_t(x)}\theta(xi).$$

If additionally $\theta(\varnothing)=1$, one says that θ is a *unit flow*. Notice that θ vanishes outside of t^*. If M is a finite Borel measure on ∂t, we may define a flow θ_M on t by setting, for all x in t, $\theta_M(x) = M([x]_t)$. Indeed, for any vertex x of t, the cylinder $[x]_t$ may be partitioned as

$$[x]_t = \bigsqcup_{i=1}^{\nu_t(x)} [xi]_t,$$

so we have

$$\theta_M(x) = M([x]_t) = \sum_{i=1}^{\nu_t(x)} M([xi]_t) = \sum_{i=1}^{\nu_t(x)} \theta_M(xi).$$

By a monotone class argument (the set of all cylinders is stable by intersection and generates the Borel σ-algebra of ∂t) the mapping $M \mapsto \theta_M$ is a one-to-one correspondance and we will write θ for both the flow on t and the associated finite Borel measure on ∂t. Unit flows correspond to Borel probability measures.

Notice that if $t[x]$ is infinite,

$$\theta(x) = \theta(\mathsf{bp}_t^{\uparrow}(x)),$$

so that a flow is completely determined by its values on $\mathsf{skel}(t)$.

Example 1.1 (visibility measure)**.** As a simple (yet important) first example of a unit flow, let us consider the *visibility measure* VIS_t on an infinite tree. We name it after [43], it was also studied at the same time in [38]. Informally, say a unit amount of water enters in the tree from the root with a unit flow, and that every time the water reaches a vertex it evenly spreads through the edges connecting this vertex to its children who have an infinite lineage. For any vertex x of t^*, define $\mathsf{VIS}_t(x)$ as the water flow through x, see Figure 5.2. More precisely,

$$\mathsf{VIS}_t(x) = \prod_{\varnothing \preceq y \prec x} \frac{1}{\nu_t^*(y)},$$

with the usual convention that a product over an empty set is equal to 1.

The measure point of view of VIS_t is as follows: VIS_t is the law of a random ray Ξ, such that, for any $n \geq 0$, for any vertex x of height n in t, conditionally on the event $\{\Xi_n = x\}$, the vertex Ξ_{n+1} is chosen uniformly among the children of x who have an infinite lineage. In other words, the probablity measure VIS_t can be seen as the law of the trajectory of a non-backtracking simple random walk on t^*, starting from $\varnothing$.

Notice that, for any vertex xz in t, we have by definition $\nu_t^*(xz) = \nu_{t[x]}^*(z)$, so we may write, for any vertex xy in t,

$$\mathsf{VIS}_t(xy) = \prod_{\varnothing \preceq z \preceq x} \frac{1}{\nu_t^*(z)} \prod_{\varnothing \preceq z \prec y} \frac{1}{\nu_{t[x]}^*(z)} = \mathsf{VIS}_t(x)\mathsf{VIS}_{t[x]}(y).$$

In other words, conditionally on the event that the random ray Ξ (of law VIS_t) goes through x, Ξ has the law of $x\Xi'$, where Ξ' has distribution $\mathsf{VIS}_{t[x]}$. One says that VIS satisfies the *flow rule property*. This will be made more precise and studied at length in the next chapter.

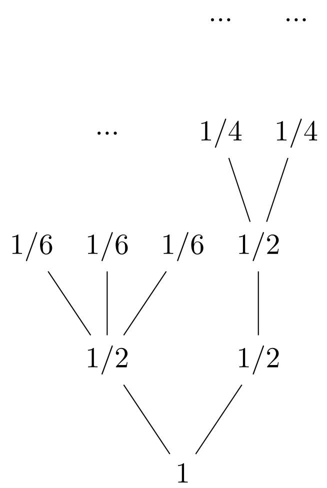

Figure 5.2 – The visibility measure VIS_t as a unit flow on an infinite tree without leaves

1.6 Random walks on trees

We now want to study random walks which are allowed to backtrack. We fix an infinite tree t and consider a transition matrix P^t on $t_* = t \cup \{\varnothing_*\}$. We restrict ourselves to nearest-neighbor random walks, that is we impose that, for x and y in t_*, $\mathrm{P}^t(x, y)$ is positive if and only if either x is the parent of y or y is the parent of x. In particular, $\mathrm{P}^t(\varnothing_*, \varnothing) = 1$. For x in t_*, we denote by P^t_x the probability measure under which the random path $\mathbf{X} = (X_0, X_1, \dots)$ in t_* is a Markov chain starting from x with transition matrix P^t. The associated expectation is denoted by E^t_x. Since we will later consider random trees, P^t_x and E^t_x will often be referred to as the "quenched" probabilities and expectations. Notice that, by our assumption that the transition matrix is always positive between neighbors, the Markov chain (t_*, P^t) is irreducible.

Example 1.2 (simple random walk)**.** Our first example is the *simple random walk* on an infinite tree t_*. For x in t, define

$$\mathrm{P}^t(x, y) = \frac{1}{\nu_t(x) + 1}$$

whenever y is either the parent of x, or one of its children. In words, the walker chooses its next position uniformly among the neighbors of its current position.

Example 1.3 (λ-biased random walk)**.** Our second example, introduced in [39], is called the λ-biased random walk, where the parameter λ is a positive real number. For $x \neq \varnothing_*$ in t let

$$\mathrm{P}^t(x, y) = \begin{cases} \dfrac{1}{\lambda + \nu_t(x)} & \text{if } y \text{ is a child of } x \\ \dfrac{\lambda}{\lambda + \nu_t(x)} & \text{if } y = x_*. \end{cases}$$

Now the walker chooses a child of its current position with weight 1 and the parent of its current position with weight λ, see Figure 6.3. If $\lambda = 1$, we recover the simple random walk, while if we allowed $\lambda = 0$ (which we do not) and the t had no leaf, the model would reduce to the study of the visibility measure VIS_t.

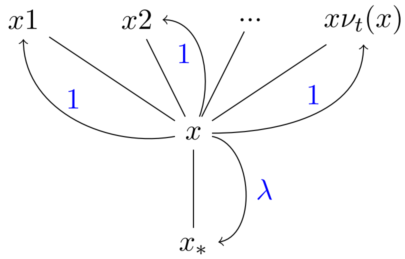

Figure 6.3 – Weights of the λ-biased random walk at a vertex x of a tree t

Our last example is the most general model of nearest-neighbor random walk on a tree.

Example 1.4 (random walk on a weighted tree)**.** Let t be a tree and consider a *weight function* $\mathsf{A}_t : t \setminus \{\varnothing\} \to (0, \infty)$. For any vertex $x \neq \varnothing_*$, define

$$\mathrm{P}^t(x,y) = \begin{cases} \dfrac{\mathsf{A}_t(xi)}{1 + \sum_{j=1}^{\nu_t(\varnothing)} \mathsf{A}_t(xj)} & \text{if } y = xi, \text{ for } 1 \leq i \leq \nu_t(x); \\ \dfrac{1}{1 + \sum_{j=1}^{\nu_t(\varnothing)} \mathsf{A}_t(xj)} & \text{if } y = x_*. \end{cases} \tag{1.6}$$

For instance, we recover the λ-biased random walk if the weight function is constant equal to λ^{-1}. This model was introduced in [41], in the context of random weights.

In the other way around, assume that P^t is a transition kernel on t satisfying our nearest-neighbor condition. Let, for all xi in $t \setminus \{\varnothing\}$,

$$\mathsf{A}_t(xi) = \frac{\mathrm{P}^t(x, xi)}{\mathrm{P}^t(x, x_*)}.$$

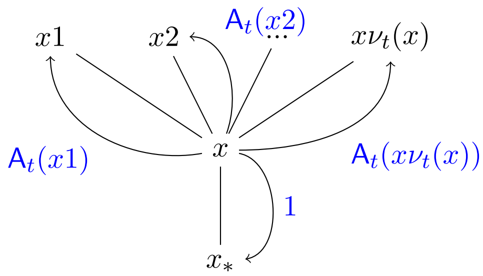

Figure 6.4 – Random walk on a weighted tree t with weight function A_t

Summing over the children of x, we obtain

$$\sum_{j=1}^{\nu_t(x)} \mathsf{A}_t(xj) = \frac{1}{\mathrm{P}^t(x, x_*)} - 1,$$

which shows that the transition kernel P^t satisfies (1.6).

For x in t, define the weight function $\mathsf{A}_{t[x]}$ on $t[x]$ by

$$\mathsf{A}_{t[x]}(y) = \mathsf{A}_t(xy), \qquad \text{for all } xy \text{ in } t,$$

and define the transition kernel $\mathrm{P}^{t[x]}$ accordingly. We see that, for any $n \geq 1$ and any xy_0, xy_1, ..., xy_n in $t \setminus \{\varnothing_*\}$

$$\mathrm{P}^t_{xy_0}(X_1 = xy_1, \ldots X_n = xy_n) = \mathrm{P}^{t[x]}(X_1 = y_1, \ldots, X_n = y_n).$$

In general, when one studies *reversible* nearest-neighbor random walks on countable graphs, the theory of electric networks comes in handy. Since a tree has no non-trivial cycles, any nearest-neighbor random walk is reversible and we may express the probability transitions in terms of conductance.

For a weighted tree t, and a vertex x in $t \setminus \{\varnothing\}$, define the conductance of the (undirected) edge $\{x_*, x\}$ by

$$\mathsf{c}_t(x) = \prod_{\varnothing \prec y \preceq x} \mathsf{A}_t(y).$$

The edge $\{\varnothing_*, \varnothing\}$ has conductance $\mathsf{c}_t(\varnothing) = 1$. With this notation, for any vertex x in t, and any $1 \leq i \leq \nu_t(x)$,

$$\mathrm{P}^t(x, xi) = \mathsf{c}(xi)/\pi_t(x) \quad \text{ and } \quad \mathrm{P}^t(x, x_*) = \mathsf{c}_t(x)/\pi_t(x),$$

where π_t is the usual reversible measure

$$\pi_t(x) = \mathsf{c}_t(x) + \sum_{i=1}^{\nu_t(x)} \mathsf{c}_t(xi).$$

Example 1.5. For the λ-biased random walk on a tree t, any undirected edge $\{x, x_*\}$, for $x \in t$, has conductance $\lambda^{-|x|}$.

For a vertex x of t, define the first hitting time and the first return time of x by

$$\tau_x = \inf\{n \geq 0 : X_n = x\} \quad \text{ and } \quad \tau_x^+ = \inf\{n \geq 1 : X_n = x\},$$

with the convention that $\inf \emptyset = \infty$.

Define the *conductance* $\beta(t)$ of the tree t by

$$\beta(t) = \mathrm{P}^t_{\varnothing}(\tau_{\varnothing_*} = \infty).$$

In electric networks terms this is the conductance between $\varnothing_*$ and infinity, see [12], or [46, chapter 2] for more information.

By the Markov property at times 1 and $\tau_{\varnothing}$ (or electric networks considerations), we have

$$\beta(t) = \frac{\sum_{i=1}^{\nu_t(\varnothing)} \mathrm{P}^t(\varnothing, i)\beta(t[i])}{\mathrm{P}^t(\varnothing, \varnothing_*) + \sum_{i=1}^{\nu_t(\varnothing)} \mathrm{P}^t(\varnothing, i)\beta(t[i])} = \frac{\sum_{i=1}^{\nu_t(\varnothing)} A_t(i)\beta(t[i])}{1 + \sum_{i=1}^{\nu_t(\varnothing)} A_t(i)\beta(t[i])}. \tag{1.7}$$

The same argument yields

$$\mathrm{P}^t_{\varnothing}(\tau^+_{\varnothing} = \infty) = \sum_{i=1}^{\nu_t(\varnothing)} \mathrm{P}^t(\varnothing, i)\beta(t[i]), \tag{1.8}$$

hence

$$\beta(t) = \frac{\mathrm{P}^t_{\varnothing}(\tau^+_{\varnothing} = \infty)}{\mathrm{P}(\varnothing, \varnothing_*) + \mathrm{P}^t_{\varnothing}(\tau^+_{\varnothing} = \infty)},$$

and we see that the Markov chain $\left(t, \mathrm{P}^t\right)$ is transient if and only if $\beta(t) > 0$.

In [39], Lyons invented the *branching number* of a tree to give a recurrence/transience criterion of the λ-biased random walk.

Definition 1.2. Let t be an infinite tree. A *cutset* in t (between the root and infinity) is a finite set Π of vertices of $t \setminus \{\varnothing\}$ such that any ray in t has an element in Π. The *branching number* of t is

$$\mathsf{br}(t) = \inf\{\lambda > 1 : \inf_{\Pi} \sum_{x \in \Pi} \lambda^{-|x|} = 0\},$$

where the infimum is over all cutsets of t.

Informally, the branching number is, in some way, the average number of children in a tree. For instance, if t is m-regular, its branching number is m. We will later relate this number to the Hausdorff dimension, with respect to $\mathrm{d}_{\mathcal{U}_\infty}$, of the boundary of the tree.

Theorem 1.5 (Lyons, 1990)**.** *Let t be an infinite tree. If $\lambda > \mathsf{br}(t)$, then the λ-biased random walk is recurrent on t. If $\lambda < \mathsf{br}(t)$, then the λ-biased random walk is transient on t.*

When λ is equal to the branching number, both cases may happen (but for the random, Galton-Watson trees we will be interested in, the case $\lambda = \mathsf{br}(t)$ is known to be recurrent).

The proof of this theorem (see [39, Theorem 4.3] or [46, Chapters 2 and 3]) uses electric networks theory in connection with Ford and Fulkerson's max-flow/min-cut theorem.

1.7 Transient random walks and the harmonic measure

From now on, we assume that the Markov chain (t_*, P^t) is transient. Then, the random *exit times* defined by

$$\mathsf{et}_n = \inf\{s \geq 0 : \forall k \geq s,\ |X_k| \geq n\}$$

are $\mathrm{P}^t_{\varnothing}$-almost surely finite. We call $\Xi = (X_{\mathsf{et}_n})_{n \geq 0}$ the harmonic ray et denote by HARM_t its distribution. Another point of view is that Ξ is the only ray that shares infinitely many vertices with the random trajectory $X_0, X_1, \dots$. By definition, for x in $t \setminus \{\varnothing_*\}$, the harmonic measure of the cylinder $[x]_t$ is

$$\mathsf{HARM}_t(x) = \mathrm{P}^t_\varnothing(x \prec \Xi) = \mathrm{P}^t_\varnothing(\exists s \geq 0,\, X_s = x, \forall k \geq s, X_s \neq x_*).$$

Define the *Green function* associated to the transient weighted tree t by

$$\mathrm{G}^t(x,y) = \sum_{k \geq 0} \mathrm{P}^t_x(X_k = y), \qquad \forall x, y \in t.$$

Applying successively the Markov property at times τ_y and τ_y^+, we obtain

$$\mathrm{G}^t(x,y) = \frac{\mathrm{P}^t_x(\tau_y < \infty)}{\mathrm{P}^t_y(\tau_y^+ = \infty)}. \tag{1.9}$$

Now, for x in t, decomposing with respect to the last passage of the walk in x,

$$\begin{aligned}
\mathsf{HARM}_t(x) &= \sum_{s \geq 0} \mathrm{P}^t_\varnothing(X_s = x,\, X_{s+1} \neq x_*,\, \forall k > s,\, X_k \neq x) \\
&= \sum_{s \geq 0} \mathrm{P}^t_\varnothing(X_s = x) \sum_{i=1}^{\nu_t(x)} \mathrm{P}^t(x, xi) \mathrm{P}^t_{xi}(\forall k \geq 0,\, X_k \neq x) \\
&= \mathrm{G}^t(\varnothing, x) \mathrm{P}^{t[x]}_\varnothing(\tau_\varnothing^+ = \infty).
\end{aligned} \tag{1.10}$$

While, for $1 \leq i \leq \nu_t(x)$,

$$\begin{aligned}
\mathsf{HARM}_t(xi) &= \sum_{s \geq 0} \mathrm{P}^t_\varnothing\left(X_s = x, X_{s+1} = xi,\, \forall k > s,\, X_k \neq x\right) \\
&= \sum_{s \geq 0} \mathrm{P}^t_\varnothing(X_s = x) \mathrm{P}^t(x, xi) \beta(t[xi]) \\
&= \mathrm{G}^t(\varnothing, x) \mathrm{P}^t(x, xi) \beta(t[xi]).
\end{aligned} \tag{1.11}$$

In particular, when $x = \varnothing$, using (1.9) and (1.8), for all $1 \leq i \leq \nu_t(\varnothing)$,

$$\mathsf{HARM}_t(i) = \frac{\mathrm{P}^t(\varnothing, i)\beta(t[i])}{\sum_{j=1}^{\nu_t(\varnothing)} \mathrm{P}^t(\varnothing, j)\beta(t[j])} = \frac{\mathsf{A}_t(i)\beta(t[i])}{\sum_{j=1}^{\nu_t(\varnothing)} \mathsf{A}_t(j)\beta(t[j])}. \tag{1.12}$$

Now, equations (1.12), (1.11), (1.10) imply, for all x in t, and all $1 \leq i \leq \nu_t(xi)$,

$$\mathsf{HARM}_t(xi) = \mathsf{HARM}_t(x)\mathsf{HARM}_{t[x]}(i),$$

and by induction, for all xy in t,

$$\mathsf{HARM}_t(xy) = \mathsf{HARM}_t(x)\mathsf{HARM}_{t[x]}(y) \tag{1.13}$$

and we see that HARM satisfies the flow rule property. This property will play a crucial role in this work.

1.8 Upper density theorem and Hausdorff measures

The aim of most of this thesis is to establish qualitative and quantitative results about the behavior of $\theta(\xi_n)$ as n goes to infinity, when θ is a particular flow on an infinite, "typical" tree t and ξ is a "typical" ray in t.

Such questions include the computation of

$$\lim_{n\to\infty} \left\{ \begin{matrix} \sup \\ \inf \end{matrix} \right\} \frac{\theta(\xi_n)}{\phi(\xi_n)} \text{ or, more realistically, } \lim_{n\to\infty} \left\{ \begin{matrix} \sup \\ \inf \end{matrix} \right\} \frac{\log \theta(\xi_n)}{\log \phi(\xi_n)},$$

where ϕ is a, more or less, "natural" rate of decay function. These include, for instance, $\phi : x \mapsto e^{-\alpha|x|}$ for $\alpha > 0$, or $\phi : x \mapsto (\mathrm{diam}[x]_t)^\alpha$ where the diameter is computed with respect to some metric d, or other quantities of interest in the model (we have in mind the random multiplicative structure involved in Section 3.2).

As it is the case in the more classical setting of the euclidean space, the theory of Hausdorff and packing measures and dimensions is the right tool to relate these kinds of *local* properties to more *global* ones. Unfortunately, the boundaries of trees are often treated as second-class citizens when it comes to this theory. The general opinion seems to be that since they are easier to deal with, everything should behave as in the euclidean case. This is, more or less, true. But we think they deserve a proper treatment. With this in mind, we try in the rest of this chapter to give a setting which is quite specific to the case of the boundary of an infinite tree, but general enough to contain most of the many different definitions of the litterature.

We reverse here the traditionnal point of view. Instead of defining first the Hausdorff and packing measures and then establishing density theorem, we try to establish density theorems and, by doing so, produce some measures which should, hopefully, look natural to the reader. They will be related in Section 10 to the more classical definitions of Hausdorff and packing measures on metric spaces.

This part of this chapter was mostly inspired by [11, 24, 56] and by [18, 19].

Definition 1.3. Let t be an infinite tree and let θ be a flow on t. Let $\phi : t \to \mathbb{R}_+$. Let $\xi \in \partial t$ and assume that ξ is in the support of θ or that $\phi(\xi_n) > 0$ for all $n \geq 0$. The *lower* and *upper* ϕ*-densities* of the flow θ at ξ are, respectively,

$$\underline{\mathrm{d}}_\theta^\phi(\xi) = \liminf_{n\to\infty} \frac{\theta(\xi_n)}{\phi(\xi_n)} \quad \text{and} \quad \overline{\mathrm{d}}_\theta^\phi(\xi) = \limsup_{n\to\infty} \frac{\theta(\xi_n)}{\phi(\xi_n)},$$

If E is a Borel set of ∂t, we want to recover some information about $\theta(E)$ from the upper and lower densities of θ on E. Since those densities involve the measure of cylinders, the most natural idea is to cover E by small cylinders.

Definition 1.4. Let t be an infinite tree, and E any subset of ∂t. Let $n \geq 0$, and $\mathcal{C} \subset t$ such that

$$\forall x \in \mathcal{C},\ |x| \geq n \text{ and } E \cap [x]_t \neq \emptyset.$$

We say that $\mathcal{C}$ is a *(centered) n-cover of E (by cylinders)* if $E \subset \bigcup_{x\in\mathcal{C}}[x]_t$. We call $\mathcal{C}$ an *n-packing of E* if for any distinct elements x and y in $\mathcal{C}$, $[x]_t \cap [y]_t = \emptyset$. The set of all n-covers of E is denoted by $\mathsf{Cov}_n(E)$ and the set of all n-packings of E by $\mathsf{Pack}_n(E)$.

By Lemma 1.1, for any element $\mathcal{C}$ of $\mathsf{Cov}_n(E)$, there exists $\mathcal{C}' \subset \mathcal{C}$ such that $\mathcal{C}' \in \mathsf{Pack}_n(E) \cap \mathsf{Cov}_n(E)$.

From now on we assume that E is included in the support of θ or that $\phi(\xi_n)$ is positive for all $\xi \in E$ and all $n \geq 0$, so that the upper and lower ϕ-densities of θ are well-defined on E. Now, let $b = \sup_{\xi\in E} \overline{\mathrm{d}}_{\mu}^{\phi}(\xi)$ and assume that $b < \infty$. We want to integrate on E, in some sense, the inequality $\overline{\mathrm{d}}_{\mu}^{\phi}\xi \leq b$. Fix $\varepsilon > 0$ and let, for $n \geq 1$,

$$E_{n,\varepsilon} = \{\xi \in E : \forall i \geq n,\ \frac{\theta(\xi_i)}{\phi(\xi_i)} \leq b + \varepsilon\}.$$

By assumption, as n goes to infinity, $E_{n,\varepsilon} \uparrow E$. For any family $\mathcal{C}$ in $\mathsf{Cov}_n(E_{n,\varepsilon})$,

$$\theta(E_{n,\varepsilon}) \leq \theta(\cup_{x\in\mathcal{C}}[x]_t) \leq \sum_{x\in\mathcal{C}} \theta(x) \leq (b+\varepsilon) \sum_{x\in\mathcal{C}} \theta(x).$$

In the right hand side of this inequality, we want to take the infimum over all n-covers of $E_{n,\varepsilon}$. This motivates the following definition: for any $F \subset \partial t$,

$$\mathscr{H}_n^{\phi}(F) = \inf\Big\{\sum_{x\in\mathcal{C}} \phi(x) : \mathcal{C} \in \mathsf{Cov}_n(F)\Big\}.$$

The set function $\mathscr{H}_n^{\phi}$ is non-decreasing. To prove this, let $F \subset G \subset \partial t$, and let $\mathcal{C}$ be an n-cover of G. By Lemma 1.1, we may find $\mathcal{C}' \subset \mathcal{C}$ such that $\mathcal{C}'$ is both an n-packing and an n-cover of G. Let $\mathcal{C}'_F$ be the set of all x in $\mathcal{C}'$ such that $F \cap [x]_t \neq \emptyset$. Let $\xi \in F$. There is a unique x in $\mathcal{C}'$ such that ξ is in $[x]_t$, and by definition of $\mathcal{C}'_F$, x is in $\mathcal{C}'_F$. This proves that $\mathcal{C}'_F$ is an n-cover of F. Hence,

$$\mathscr{H}_n^{\phi}(F) \leq \sum_{x\in\mathcal{C}'_F} \phi(x) \leq \sum_{x\in\mathcal{C}'} \phi(x) \leq \sum_{x\in\mathcal{C}} \phi(x),$$

and we can conclude by taking the infimum over all n-covers of G.

By monotonicity, we have, for all $n \geq 1$,

$$\theta(E_{n,\varepsilon}) \leq (b+\varepsilon)\, \mathscr{H}_n^{\phi}(E).$$

Now, it is clear that for any fixed set $F \subset \partial t$, the sequence $(\mathscr{H}_n^{\phi}(F))$ is non-decreasing. Since we want to let n go to infinity, we define for any $F \subset \partial t$,

$$\mathscr{H}^{\phi}(F) = \lim_{n\to\infty}\!\!{}^{\uparrow}\, \mathscr{H}_n^{\phi}(F) \in [0,\infty].$$

Letting first n go to infinity and then ε go to 0, we obtain

$$\theta(E) \leq \sup_{\xi\in E} \overline{\mathrm{d}}_{\theta}^{\phi}(\xi)\, \mathscr{H}^{\phi}(E),$$

which is the first half of the upper density theorem on the boundary of a tree.

In the other direction, assume that $a = \inf_{\xi\in E} \overline{\mathrm{d}}_\theta^\phi(\xi)$ is positive and let $\varepsilon > 0$ be so small that $a - \varepsilon > 0$. Let $n \ge 0$. By definition of a, for any $\xi \in E$, we may choose an integer $k(\xi) \ge n$ such that $\theta(\xi_{k(\xi)}) \ge (a-\varepsilon)\phi(\xi_{k(\xi)})$. The set of all $\xi_{k(\xi)}$ for ξ in E is an n-cover of E, and by Lemma 1.1, we may extract from it a sub-cover $\mathcal{C}$ of E by pairwise disjoint cylinders. Define the n-enlargement of E by:

$$E^{(n)} = \bigcup_{|x|=n,\ [x]_t\cap E\neq\emptyset} [x]_t = \bigcup_{\xi\in E} [\xi_n]_t.$$

and notice that, as n goes to infinity, $E^{(n)} \downarrow \overline{E}$, the closure of E.

Since $\sqcup_{x\in\mathcal{C}}[x]_t \subset E^{(n)}$,

$$\theta(E^{(n)}) \ge \sum_{x\in\mathcal{C}} \theta(x) \ge (a-\varepsilon)\,\mathscr{H}_n^\phi(E).$$

Letting n go to infinity, we obtain

$$\theta(\overline{E}) \ge \inf_{\xi\in E} \overline{\mathrm{d}}_\theta^\phi(\xi)\,\mathscr{H}^\phi(E).$$

The other half of the upper density theorem is thus proved for closed sets. We will prove later that $\mathscr{H}^\phi$ is a Borel measure on ∂t and that

$$\mathscr{H}^\phi(E) = \sup\{\mathscr{H}^\phi(F) : F \subset E,\ F \text{ closed}\}, \tag{1.14}$$

because the previous inequality implies that $\mathscr{H}^\phi(E) < \infty$. Now θ is a finite Borel measure and ∂t is metrizable, thus θ also satisfies this property (see for instance [53, Chapter 2, Theorem 1.2]). For any closed subset F of E,

$$\theta(F) \ge \inf_{\xi\in F} \overline{\mathrm{d}}_\theta^\phi(\xi)\,\mathscr{H}^\phi(F) \ge \inf_{\xi\in E} \overline{\mathrm{d}}_\theta^\phi(\xi)\,\mathscr{H}^\phi(F).$$

Thus, taking the supremum over all closed subsets of E,

$$\theta(E) \ge \inf_{\xi\in E} \overline{\mathrm{d}}_\theta^\phi(\xi)\,\mathscr{H}^\phi(E),$$

which is the second half of the upper density theorem.

Let us sum up the previous discussion.

Definition 1.5 (Hausdorff measure)**.** Let t be an infinite tree and E be any subset of ∂t. Let $\phi : t \to \mathbb{R}_+$. For any $n \ge 0$, let

$$\mathscr{H}_n^\phi(E) = \inf\Big\{\sum_{x\in\mathcal{C}} \phi(x) : \mathcal{C} \in \mathsf{Cov}_n(E)\Big\}.$$

The Hausdorff ϕ-measure of E is

$$\mathscr{H}^\phi(E) = \lim_{n\to\infty}{}^{\uparrow}\, \mathscr{H}_n(E).$$

We will later see how this definition of the Hausdorff measures is related with the usual definition of the literature.

Theorem 1.6 (upper density theorem). *Let t be an infinite tree, $\phi : t \to \mathbb{R}_+$, θ a flow on t and E a Borel subset of ∂t. Assume that E is included in the support of θ or that $\phi(\xi_n)$ is positive for all $\xi \in E$ and all $n \geq 0$. Then,*

$$\inf_{\xi\in E} \overline{\mathrm{d}}^{\phi}_{\theta}(\xi)\, \mathscr{H}^{\phi}(E) \leq \theta(E) \leq \sup_{\xi\in E} \overline{\mathrm{d}}^{\phi}_{\theta}(\xi)\, \mathscr{H}^{\phi}(E),$$

where we agree that the lower bound is 0 if one of the two terms of the product is 0 and that the upper bound is infinite if one of the two terms of the product is infinite.

Before we move on to the lower density we need to prove that eq. (1.14) holds.

Proposition 1.7. *Let t be an infinite tree, $\phi : t \to \mathbb{R}_+$ and $n \geq 0$. The following assertions hold.*
1. The set functions $\mathscr{H}^{\phi}_n$ and $\mathscr{H}^{\phi}$ are outer measures on ∂t.
2. The Borel sets are $\mathscr{H}^{\phi}$-measurable.
3. For any set $E \subset \partial t$, we may find a sequence $(U_i)_{i\geq 1}$ of open sets containing E such that $\mathscr{H}^{\phi}(E) = \mathscr{H}^{\phi}(\bigcap_{i\geq 1} U_i)$.
4. For any $\mathscr{H}^{\phi}$-measurable set E such that $\mathscr{H}^{\phi}(E) < \infty$,

$$\mathscr{H}^{\phi}(E) = \sup\{\mathscr{H}^{\phi}(F) : F \subset E,\ F \text{ closed}\}.$$

Proof. 1. It is clear that $\mathscr{H}^{\phi}_n(\emptyset) = 0$ and we have already proved the monotonicity. To see that $\mathscr{H}^{\phi}_n$ is countably subadditive, let $(E_k)_{k\geq 1}$ be a sequence of subsets of ∂t and $E = \bigcup_{k\geq 1} E_k$. We may assume that $\sum_{k\geq 1} \mathscr{H}^{\phi}(E_k) < \infty$ otherwise there is nothing to prove. Let $\varepsilon > 0$. For each k, let $\mathcal{C}_k$ be an n-cover of E_k such that

$$\sum_{x\in\mathcal{C}_k} \phi(x) \leq \mathscr{H}^{\phi}_n(E_k) + 2^{-k}\varepsilon.$$

The set $\mathcal{C} = \bigcup_{k\geq 1} \mathcal{C}_k$ is an n-cover of E, thus

$$\mathscr{H}^{\phi}_n(E) \leq \sum_{x\in\mathcal{C}} \phi(x) \leq \sum_{k\geq 1}\sum_{x\in\mathcal{C}_k} \phi(x) \leq \sum_{k\geq 1} \mathscr{H}^{\phi}_n(E_k) + \varepsilon.$$

Letting ε go to 0, we obtain the countable subadditivity of $\mathscr{H}^{\phi}_n$. By monotone convergence, it also holds for $\mathscr{H}^{\phi}$.
2. It suffices to show that the cylinders are measurable by Caratheodory's restriction theorem. Let $E \subset \partial t$ and $x \in t$. Let $n \geq |x|$. If $\mathcal{C}$ is an n-cover of E, then we may write

$$\mathcal{C} = \underbrace{\{y \in \mathcal{C} : y \succeq x\}}_{\mathcal{C}_1} \sqcup \underbrace{\{y \in \mathcal{C} : y \not\succeq x\}}_{\mathcal{C}_2}.$$

The sets $\mathcal{C}_1$ and $\mathcal{C}_2$ are n-covers of $E \cap [x]_t$ and $E \cap {[x]_t}^{\mathrm{c}}$ respectively and

$$\sum_{x\in\mathcal{C}} \phi(x) = \sum_{x\in\mathcal{C}_1} \phi(x) + \sum_{x\in\mathcal{C}_2} \phi(x) \geq \mathscr{H}^{\phi}_n(E \cap [x]_t) + \mathscr{H}^{\phi}_n(E \cap {[x]_t}^{\mathrm{c}}).$$

To conclude, take the infimum over all n-covers on the left hand side and let n go to infinity.
3. Let $E \subset \partial t$. If $\mathscr{H}^\phi(E) = \infty$, by monotonicity, $\mathscr{H}^\phi(E) = \mathscr{H}^\phi(\partial t) = \infty$ and the assertion is true with $U_i = \partial t$ for all $i \geq 1$. Now, assume that $\mathscr{H}^\phi(E) < \infty$. For any $n \geq 1$, we may find an n-cover $\mathcal{C}$ of E such that

$$\sum_{x\in\mathcal{C}} \phi(x) \leq \mathscr{H}^\phi_n(E) + \frac{1}{n}.$$

Let $U_n = \bigcup_{x\in\mathcal{C}} [x]_t$. Then U_n is open, contains E and since $\mathcal{C}$ is also an n-cover of U_n,

$$\mathscr{H}^\phi_n(U_n) \leq \sum_{x\in\mathcal{C}} \phi(x) \leq \mathscr{H}^\phi_n(E) + \frac{1}{n}.$$

Then, by monotony, for any $n \geq 1$,

$$\mathscr{H}^\phi_n(E) \leq \mathscr{H}^\phi_n(\textstyle\bigcap_{i\geq 1} U_i) \leq \mathscr{H}^\phi_n(U_n) \leq \mathscr{H}^\phi_n(E) + 1/n,$$

and we conclude by letting n go to infinity.
4. We proceed as in [57, Lemma 5.1]. For a more direct proof see [17, Theorem 1.6]. The previous assertion implies in particular that $\mathscr{H}^\phi$ is Borel-regular, that is, for any subset E of ∂t, there exists a Borel set $B \supset E$ such that $\mathscr{H}^\phi(E) = \mathscr{H}^\phi(B)$. Now, let E be as in the assertion and let B be as above. Since E is $\mathscr{H}^\phi$-measurable,

$$\mathscr{H}^\phi(B) = \mathscr{H}^\phi(E) + \mathscr{H}^\phi(B \setminus E),$$

thus $\mathscr{H}^\phi(B \setminus E) = 0$ because $\mathscr{H}^\phi(E) < \infty$. Now let $N \supset B \setminus E$ be a Borel set such that $\mathscr{H}^\phi(N) = 0$ and let $\widetilde{B} = B \setminus N$. Then $\widetilde{B}$ is a Borel set contained in E and $\mathscr{H}^\phi(\widetilde{B}) = \mathscr{H}^\phi(E)$. Let μ be the Borel finite measure defined by $\mu(A) = \mathscr{H}(A \cap \widetilde{B})$ for any Borel set A. By metrizability of ∂t, for any ε, we may find a closed set F included in $\widetilde{B}$ such that $\mu(F) \geq \mu(\widetilde{B}) - \varepsilon$ which implies that

$$\mathscr{H}^\phi(F) = \mathscr{H}^\phi(F \cap \widetilde{B}) = \mu(F) \geq \mu(\widetilde{B}) - \varepsilon = \mathscr{H}^\phi(\widetilde{B}) - \varepsilon = \mathscr{H}^\phi(E) - \varepsilon. \qquad \square$$

1.9 Lower density theorem and packing measure

We are now interested in the lower density. We work under the same assumptions as in Theorem 1.6. Let $a = \inf_{\xi\in E} \underline{\mathrm{d}}^\phi_\theta(\xi)$ and assume that $a > 0$. Let ε be so small that $a - \varepsilon > 0$. For $n \geq 0$, let

$$E_{n,\varepsilon} = \{\xi \in E : \forall i \geq n,\ \theta(\xi_i) \geq (a-\varepsilon)\phi(\xi_i)\},$$

Let $\mathcal{C}$ be an n-packing of $E_{n,\varepsilon}$. Notice that $\bigsqcup_{x\in\mathcal{C}} [x]_t \subset E^{(n)}$, the n-enlargement of E. Thus

$$\theta(E^{(n)}) \geq \theta\Big(\bigsqcup_{x\in\mathcal{C}} [x]_t\Big) = \sum_{x\in\mathcal{C}} \theta(x) \geq (c-\varepsilon) \sum_{x\in\mathcal{C}} \phi(x).$$

We want to take the supremum over all n-packings in the lower bound, so we introduce, for any subset F of ∂t,

$$\mathscr{P}^{\phi}_{n}(F) = \sup\Big\{\sum_{x\in\mathcal{C}} \phi(x) : \mathcal{C} \in \mathsf{Pack}_n(F)\Big\}.$$

The sequence $(\mathscr{P}^{\phi}_{n}(F))_{n\geq 0}$ is non-increasing. We define for any $F \subset \partial t$,

$$\mathscr{P}^{\phi}_{\infty}(F) = \lim_{n\to\infty}{}^{\downarrow}\, \mathscr{P}^{\phi}_{n}(F),$$

and we obtain, after we let n go to ∞ and ε go to 0,

$$\theta(\overline{E}) \geq \inf_{\xi\in E} \overline{\mathrm{d}}^{\phi}_{\theta}(\xi)\, \mathscr{P}^{\phi}_{\infty}(E). \tag{1.15}$$

We would like to proceed similarly as the proof of the upper density theorem. Unfortunately, $\mathscr{P}^{\phi}_{\infty}$ is not an outer measure. However, if for any subset F of ∂t, we set

$$\mathscr{P}^{\phi}(F) = \inf\Big\{\sum_{k\geq 1} \mathscr{P}^{\phi}_{\infty}(F_k) : F \subset \bigcup_{k\geq 1} F_k\Big\},$$

we obtain an outer measure called the *packing ϕ-measure*. We will later prove this fact and show that the packing ϕ-measure also satisfies (1.14). To conclude this half of the lower-density theorem, it suffices to see that, for any non-empty closed set $F \subset E$,

$$\theta(F) \geq \inf_{\xi\in F} \underline{\mathrm{d}}^{\phi}_{\theta}(\xi)\, \mathscr{P}^{\phi}_{\infty}(F) \geq \inf_{\xi\in F} \underline{\mathrm{d}}^{\phi}_{\theta}(\xi)\, \mathscr{P}^{\phi}(F) \geq \inf_{\xi\in E} \underline{\mathrm{d}}^{\phi}_{\theta}(\xi)\, \mathscr{P}^{\phi}(F)$$

and to take the supremum over closed subsets of E on both sides (by (1.15) and the assumption $a > 0$, $\mathscr{P}^{\phi}(E) < \infty$).

We need to know more about the packing measures before we turn to the upper bound.

Definition 1.6 (packing measure)**.** Let t be an infinite tree, $\phi : t \to \mathbb{R}_+$ and E any subset of ∂t. For any $n \geq 1$, let

$$\mathscr{P}^{\phi}_{n}(E) = \sup\Big\{\sum_{x\in\mathcal{C}} \phi(x) : \mathcal{C} \in \mathsf{Pack}_n(E)\Big\} \qquad \text{and}$$
$$\mathscr{P}^{\phi}_{\infty}(E) = \lim_{n\to\infty}{}^{\downarrow}\, \mathscr{P}^{\phi}_{n}(E).$$

The packing ϕ-measure of E is

$$\mathscr{P}^{\phi}(E) = \inf\Big\{\sum_{k\geq 1} \mathscr{P}^{\phi}_{\infty}(E_k) : E \subset \bigcup_{k\geq 1} E_k\Big\}.$$

Proposition 1.8. *The following assertions hold.*
1. *The set functions $\mathscr{P}^{\phi}_{n}$ and $\mathscr{P}^{\phi}_{\infty}$ are non-decreasing and finitely subadditive.*
2. *For any $E \subset \partial t$, $\mathscr{P}^{\phi}_{\infty}(E) = \mathscr{P}^{\phi}_{\infty}(\overline{E})$.*
3. *The set function $\mathscr{P}^{\phi}$ is an outer measure on ∂t.*

4. The Borel sets are $\mathscr{P}^\phi$-measurable.
5. For any $E \subset \partial t$, there is an $\mathcal{F}_{\sigma\delta}$ set $B \supset E$ such that $\mathscr{P}^\phi(B) = \mathscr{P}^\phi(E)$. In particular, $\mathscr{P}^\phi$ is Borel-regular.
6. For any $\mathscr{P}^\phi$-measurable set E such that $\mathscr{P}^\phi(E) < \infty$,

$$\mathscr{P}^\phi(E) = \sup\{\mathscr{P}^\phi(F) : F \subset E,\ F \text{ closed}\}.$$

7. For any $E \subset \partial t$, $\mathscr{P}^\phi(E) = \inf\{\lim\nolimits^{\uparrow}_{k\to\infty} \mathscr{P}^\phi_\infty(E_k) : E_k \uparrow E\}$.
8. For any $E \subset \partial t$, $\mathscr{P}^\phi(E) \geq \mathscr{H}^\phi(E)$.

Proof. 1. If $E \subset F \subset \partial t$, then, for any $n \geq 0$, any n-packing of E is also an n-packing of F, thus $\mathscr{P}^\phi_n$ is monotonic. Furthermore, if $\mathcal{C}$ is an n-packing of E, letting $\mathcal{C}_E = \{x \in \mathcal{C} : [x]_t \cap E \neq \emptyset\}$ and $\mathcal{C}_F = \mathcal{C} \setminus \mathcal{C}_E$, we obtain n-packings of E and F. This shows that

$$\sum_{x\in\mathcal{C}} \phi(x) \leq \mathscr{P}^\phi_n(E) + \mathscr{P}^\phi_n(F),$$

hence the subadditivity of $\mathscr{P}^\phi_n$. Letting n go to infinity, we obtain the same properties for $\mathscr{P}^\phi_\infty$.
2. A cylinder intersects $\overline{E}$ if and only if it intersects E, therefore, for any $n \geq 0$, any n-packing of $\overline{E}$ is also an n-packing of E.
3. The construction of $\mathscr{P}^\phi$ from the set function $\mathscr{P}^\phi_\infty$ is classical (it is called “Method 1.” in [49, p. 47]) and leads to outer measures.
4. Let $x \in t$, $E \subset \partial t$ and $n \geq |x|$. Let $\mathcal{C}_1 \in \mathsf{Pack}_n(E \cap [x]_t)$ and $\mathcal{C}_2 \in \mathsf{Pack}_n(E \cap {[x]_t}^{\mathrm{c}})$. Then $\mathcal{C} = \mathcal{C}_1 \sqcup \mathcal{C}_2$ is an n-packing of E. This gives

$$\mathsf{Pack}_n(E) \geq \sum_{x\in\mathcal{C}_1} \phi(x) + \sum_{x\in\mathcal{C}_2} \phi(x),$$

and after first taking the supremum over $\mathcal{C}_1$ and $\mathcal{C}_2$, and then letting n go to infinity,

$$\mathsf{Pack}_\infty(E) \geq \mathsf{Pack}_\infty(E \cap [x]_t) + \mathsf{Pack}_\infty(E \cap {[x]_t}^{\mathrm{c}}).$$

Now let (E_k) be a sequence of sets such that $E \subset \bigcup E_k$. By the previous inequality,

$$\sum_k \mathscr{P}^\phi_\infty(E_k) \geq \sum_k \mathscr{P}^\phi_\infty(E_k \cap [x]_t) + \sum_k \mathscr{P}^\phi_\infty(E_k \cap {[x]_t}^{\mathrm{c}}) \geq \mathscr{P}^\phi(E \cap [x]_t) + \mathscr{P}^\phi(E \cap [x]_t).$$

Taking the infimum over all such sequences in the left hand side finishes the proof.
5. Let $E \subset \partial t$. We may assume $\mathscr{P}^\phi(E) < \infty$. By the second property,

$$\mathscr{P}^\phi(E) = \inf\{\sum_k \mathscr{P}^\phi_\infty(E_k) : \bigcup E_k \supset E,\ E_k \text{ closed}\}.$$

For $l \geq 1$, let $(E^l_k)_{k\geq 1}$ be a sequence of closed sets such that

$$E \subset \bigcup_{k\geq 1} E^l_k \quad \text{and} \quad \sum_{k\geq 1} \mathscr{P}^\infty_\phi(E^l_k) \leq \mathscr{P}^\phi(E) + 1/l.$$

Let $F^l = \bigcup_k E_k^l$ and $F = \bigcap_l F^l$. Then, for all $l \geq 1$,

$$\mathscr{P}^\phi(E) \leq \mathscr{P}^\phi(F) \leq \mathscr{P}^\phi(F^l) \leq \mathscr{P}^\phi(E) + 1/l.$$

Thus $\mathscr{P}^\phi(F) = \mathscr{P}^\phi(E)$ and by construction, F is an $\mathcal{F}_{\sigma\delta}$, hence a Borel set.
6. This property follows from the Borel regularity and the metrizability of ∂t in the same way as in Proposition 1.8.
7. Let $E_k \uparrow E$. Since $\mathscr{P}^\phi$ is Borel-regular,

$$\mathscr{P}^\phi(E) = \lim_{k\to\infty}^{\uparrow} \mathscr{P}^\phi(E_k) \leq \lim_{k\to\infty}^{\uparrow} \mathscr{P}^\phi_\infty(E_k).$$

For the other inequality, let $\bigcup_{k\geq 1} E_k \supset E$. Set for $k \geq 1$, $F_k = \bigcup_{i=1}^k E_k \cap E$. Then, as k goes to infinity, $F_k \uparrow E$. By finite subadditivity, for each $k \geq 1$,

$$\mathscr{P}^\phi_\infty(F_k) \leq \sum_{i=1}^k \mathscr{P}^\phi_\infty(E_i \cap E) \leq \sum_{i=1}^k \mathscr{P}^\phi_\infty(E_i) \leq \sum_{i=1}^\infty \mathscr{P}^\phi_\infty(E_i).$$

This shows that

$$\inf\{\lim_{k\to\infty}^{\uparrow} \mathscr{P}^\phi_\infty(F_k) : F_k \uparrow E\} \leq \sum_{i=1}^\infty \mathscr{P}^\phi_\infty(E_i).$$

8. Let $n \geq 0$ and $\mathcal{C}$ be an n-cover of E which is also an n-packing. Then we have

$$\mathscr{H}^\phi_n(E) \leq \sum_{x\in\mathcal{C}} \phi(x) \leq \mathscr{P}^\phi_n(E),$$

and $\mathscr{H}^\phi(E) \leq \mathscr{P}^\phi_\infty(E)$. Let $\bigcup_{k\geq 1} E_k \supset E$. By subadditivity of $\mathscr{H}^\phi$,

$$\mathscr{H}^\phi(E) \leq \sum_{k\geq 1} \mathscr{H}^\phi(E_k) \leq \sum_{k\geq 1} \mathscr{P}^\phi_\infty(E_k). \qquad \square$$

We now state the lower density theorem and prove its remaining inequality.

Theorem 1.9 (lower density theorem)**.** *Let t be an infinite tree, $\phi : t \to \mathbb{R}_+$, θ a flow on t and E a Borel subset of ∂t. Assume that E is included in the support of θ or that $\phi(\xi_n)$ is positive for all $\xi \in E$ and all $n \geq 0$. Then,*

$$\inf_{\xi\in E} \underline{\mathrm{d}}^\phi_\theta(\xi)\, \mathscr{P}^\phi(E) \leq \theta(E) \leq \sup_{\xi\in E} \underline{\mathrm{d}}^\phi_\theta(\xi)\, \mathscr{P}^\phi(E),$$

where we agree that the lower bound is 0 if one of the two terms of the product is 0 and that the upper bound is infinite if one of the two terms of the product is infinite.

Proof. We write θ^* for the outer measure extension of θ, that is the outer measure defined for any $A \subset \partial t$ by

$$\theta^*(A) = \inf\{\sum_{k\geq 1} \theta(A_k) : A \subset \bigcup_{k\geq 1} A_k,\ A_k \text{ Borel}\} = \inf\{\theta(B) : B \supset A,\ B \text{ Borel}\}.$$

Let $d = \sup_{\xi \in E} \underline{\mathrm{d}}_\theta^\phi(\xi)$ and assume that $d < \infty$. Let $\varepsilon > 0$. By assumption,

$$\forall \xi \in E,\ \exists n \geq 0,\ \forall i \geq n, \quad \theta(\xi_i) \leq (b+\varepsilon)\phi(\xi_i).$$

Let F be any subset of E and $n \geq 0$. Consider the family $\mathcal{C} \subset t$ defined by

$$x \in \mathcal{C} \iff |x| \geq n \text{ and } \exists \xi \in F\ x \prec \xi \text{ and } \theta(x) \geq (b+\varepsilon)\phi(x).$$

Since $F \subset E$, $\mathcal{C}$ is an n-cover of F and we may extract from it $\mathcal{C}' \in \mathsf{Cov}_n(F) \cap \mathsf{Pack}_n(F)$. By subadditivity of θ^*,

$$\theta^*(F) \leq \theta^*\Big(\bigcup_{x \in \mathcal{C}'} [x]_t\Big) \leq \sum_{x \in \mathcal{C}'} \theta^*([x]_t) = \sum_{x \in \mathcal{C}'} \theta(x) \leq (b+\varepsilon) \sum_{x \in \mathcal{C}'} \phi(x) \leq (b+\varepsilon)\, \mathscr{P}_\infty^\phi(F).$$

Now, let $F_k \uparrow E$. Since θ^* is also Borel-regular and E is a Borel set,

$$\theta(E) = \theta^*(E) = \lim_{k \to \infty}^{\uparrow} \theta^*(F_k) \leq (b+\varepsilon) \lim_{k \to \infty}^{\uparrow} \mathscr{P}_\infty^\phi(F_k).$$

By property 7. of the previous proposition, this is enough to conclude. $\square$

Remark 1.1. We have used to a great extent the fact that any n-cover of a subset E of ∂t contains minimal n-covers which are also n-packings. It feels natural to ask, in the other direction, if any n-packing $\mathcal{C}$ can be completed into an n-cover. By Zorn's lemma, there exists a maximal n-packing $\mathcal{C}_{\max}$ containing $\mathcal{C}$. Now let $\xi \in E$ and assume that $\xi \notin \bigcup_{x \in \mathcal{C}_{\max}}$. Then, by maximality of $\mathcal{C}_{\max}$, for all $i \geq n$, there exists $x \in \mathcal{C}_{\max}$ such that $[x]_t \cap [\xi_i]_t \neq \emptyset$. This proves that $E \subset \overline{\bigcup_{x \in \mathcal{C}_{\max}} [x]_t}$.

This is all we can get. Indeed, consider on the 2-regular tree the 1-packing of ∂t $\{1; 21; 221; \dots\}$. This packing is maximal but does not cover the element $222\cdots$.

1.10 Hausdorff and packing measures on a metric space

In this section, $(\mathscr{X}, d)$ denotes a metric space, endowed with its Borel σ-algebra. The closed ball of center x in $\mathscr{X}$ and radius $r \geq 0$ is denoted by $\mathscr{B}(x,r)$. For $\delta > 0$, a δ-cover of a subset E of $\mathscr{X}$ is a family E_1, E_2, ... of subsets of $\mathscr{X}$ such that $E \subset \bigcup_{i=1}^\infty E_i$ and for all $i \geq 1$, $\operatorname{diam} E_i \leq \delta$. The set of all those δ-covers is denoted by $\operatorname{cov}_\delta(E)$. A countable family (x_1, r_1), (x_2, r_2), ... of elements of $E \times [0, \delta]$ is

— a δ-packing of E if for any $i \neq j$, $\mathscr{B}(x_i, r_i) \cap \mathscr{B}(x_j, r_j) = \emptyset$;

— a centered δ-cover of E if $E \subset \bigcup_{i \geq 1} \mathscr{B}(x_i, r_i)$.

We denote the set of all δ-packings of E by $\operatorname{pack}_\delta(E)$ and the set of all centered δ-covers by $\text{c-cov}_\delta(E)$.

A *gauge function* is a non-decreasing function $g : \mathbb{R}_+ \to [0,\infty]$ such that $g(0) = 0$, $g(q) > 0$, for all $q > 0$ and $\lim_{q\downarrow 0} g(q) = 0$. Let $E \subset \mathscr{X}$ and define

$$\mathsf{C}_0^g(E) = \lim_{\delta\downarrow 0}{}^{\uparrow} \inf\Big\{ \sum_{(x,r)\in\mathcal{C}} g(\operatorname{diam}\mathscr{B}(x,r)) : \mathcal{C} \in \text{c-cov}_\delta(E)\Big\}$$
$$\mathsf{P}_0^g(E) = \lim_{\delta\downarrow 0}{}^{\downarrow} \sup\Big\{ \sum_{(x,r)\in\mathcal{C}} g(\operatorname{diam}\mathscr{B}(x,r)) : \mathcal{C} \in \operatorname{pack}_\delta(E)\Big\}$$
$$\mathsf{H}^g(E) = \lim_{\delta\downarrow 0}{}^{\uparrow} \inf\Big\{ \sum_{F\in\mathcal{C}} g(\operatorname{diam}(F)) : \mathcal{C} \in \operatorname{cov}_\delta(E)\Big\}.$$

It is well known that H^h, which is called the *Hausdorff h-measure*, is an outer measure on $\mathscr{X}$ and that it is even a *metric* outer measure (that is, $\mathsf{H}^h(A\cup B) = \mathsf{H}^h(A) + \mathsf{H}^h(B)$ whenever $d(A,B) > 0$) and as such is a measure on the Borel σ-algebra of $\mathscr{X}$. It is not hard to see that C_0^h is countably subadditive and that P_0^h is monotonic, but they are not outer measures in general. However, if we define

$$\mathsf{C}^g(E) = \sup\Big\{\mathsf{C}_0^g(F) : F \subset E\Big\} \qquad \text{and}$$
$$\mathsf{P}^g(E) = \inf\Big\{\sum_{k\geq 1} \mathsf{P}_0^g(E_k) : E \subset \bigcup_{k\geq 1} E_k\Big\},$$

then we obtain metric outer measures as well. These are called, respectively, the *centered covering g-measure* and the *packing g-measure*.

The packing measures were first introduced in [58] and [57], in the context of the euclidean space. Some authors point our that replacing $\operatorname{diam}\mathscr{B}(x,r)$ by $2r$ in the definition can lead to better properties in general metric space. See [24], who studied packing measures on ultrametric spaces and [11] for a detailed study on general metric spaces. The centered covering measures appeared in [56] in order to obtain, in the euclidean space, an upper density theorem symetric to the lower density theorem. Once again, a radius-based definition is possible (see [52]) and may lead to different measures. A thourough study of these radius-based measures and their many variations may be found in [14].

When the function g is defined by $g(q) = q^s$, one simply writes H^s, C^s and P^s for the associated so-called *s-dimensional* measures. Then, it is well known that for any $E \subset \mathscr{X}$, there exist numbers α and β in $[0,\infty]$ such that

$$\forall s < \alpha,\ \mathsf{H}^s(E) = \infty \quad \text{and} \quad \forall s > \alpha,\ \mathsf{H}^s(E) = 0;$$
$$\forall s < \beta,\ \mathsf{P}^s(E) = \infty \quad \text{and} \quad \forall s > \beta,\ \mathsf{P}^s(E) = 0.$$

These are called, respectively, the *Hausdorff dimension* of E and *the packing dimension* of E, denoted by $\dim_{\mathrm{H}}(E)$ and $\dim_{\mathrm{p}}(E)$ [1].

Let us get back to our trees. To define Haudorff, packing and covering measures on the boundary of an infinite tree t, we need to choose a metric on ∂t. The distance $\mathrm{d}_{\mathcal{U}_\infty}$ is the standard metric in the literature, it conveys the idea that "most infinite trees have

1. The covering dimension is absent of this theory because it always equals the Hausdorff dimension.

an exponential growth" and only depends on the heights in the tree. An other choice would be for instance

$$d(\xi,\eta) = \frac{1}{1+\max\{k \geq 0 : \xi_k = \eta_k\}},$$

which hardly seems less natural but would give infinite Hausdorff dimension to all trees with exponential growth (it would however seem to be a reasonable choice to work on trees with polynomial growth). Our conclusion is that the choice of a metric should reflect the properties of the class of trees we work on.

Now we proceed to show how our previous metric-agnostic definition of Haudorff and packing measures agrees with those usual metric definitions.

Proposition 1.10. *Let t be an infinite tree and $d \in \mathsf{Metrics}(\partial t)$, with associated diameter function φ. Let g be a gauge function. Then, for all $E \subset \partial t$,*

$$\mathsf{P}_0^g(E) = \mathscr{P}_\infty^{g\circ\varphi}(E) \quad \textit{and} \quad \mathsf{C}_0^g(E) = \mathsf{H}^g(E) = \mathscr{H}^{g\circ\varphi}(E).$$

Proof. For short, we write $\phi = g \circ \varphi$. First let $\xi \in \partial t$. Since $\lim_{n\to\infty}\varphi(\xi_n) = 0$ and $\lim_{q\to 0} g(q) = 0$, then

$$\mathsf{P}_0^g(\{\xi\}) = \mathscr{P}_\infty^\phi(\{\xi\}) = \mathsf{C}_0^g(\{\xi\}) = \mathsf{H}^g(\{\xi\}) = \mathscr{H}^\phi(\{\xi\}) = 0.$$

By finite subadditivity of all these set functions, this proves the proposition for finite subsets of ∂t. Now we assume that ∂t is infinite. We use the notations δ_n and n_δ from Lemma 1.4. For ξ in ∂t and $r > 0$, we write $f(\xi, r)$ for the unique element of $\mathsf{skel}(t)$ such that $\mathscr{B}(\xi, r) = [f(\xi, r)]_t$.

First, we prove that $\mathsf{P}_0^g(E) \leq \mathscr{P}_\infty^\phi(E)$. Let $C \in \mathrm{pack}_\delta(E)$. For (ξ, r) in C,

$$\mathrm{diam}(\mathscr{B}(\xi, r)) = \mathrm{diam}[f(\xi, r)]_t = \varphi(f(\xi, r)).$$

Since $\mathrm{diam}\,\mathscr{B}(\xi, r) \leq r$, then $\varphi(f(\xi, r)) \leq \delta$. Now the set

$$\mathcal{C} = \{f(\xi, r) : (\xi, r) \in C,\ \varphi(f(\xi, r)) > 0\}$$

is an n_δ-packing of E, thus

$$\sum_{(\xi,r)\in C} g(\mathrm{diam}\,\mathscr{B}(\xi, r)) = \sum_{x\in\mathcal{C}} \phi(x) \leq \mathscr{P}_{n_\delta}^\phi(E).$$

In turn this implies that $\mathsf{P}_\delta^g(E) \leq \mathscr{P}_{n_\delta}^\phi(E)$ and by Lemma 1.4, the result follows.

Now we prove that $\mathscr{P}_\infty^\phi(E) \leq \mathsf{P}_0^g(E)$. Let $\mathcal{C} \in \mathsf{Pack}_n(E)$. For each x in $\mathcal{C}$ pick an arbitrary ray $\xi^{(x)} \in [x]_t$ and let

$$C = \{(\xi^{(x)}, \varphi(x)) : x \in \mathcal{C}\} \in \mathrm{pack}_{\delta_n}(E).$$

Since $\mathscr{B}(\xi^{(x)}, \varphi(x)) = [x]_t$ we have

$$\sum_{x\in\mathcal{C}} \phi(x) = \sum_{(\xi,r)\in C} g(\mathrm{diam}\,\mathscr{B}(\xi^{(x)}, \phi(x))) \leq \mathsf{P}_{\delta_n}^g(E),$$

which gives $\mathscr{P}^{\phi}_{n}(E) \leq \mathsf{P}^{g}_{\delta_n}(E)$ and by Lemma 1.4 proves this point.

To prove the inequality $\mathscr{H}^{\phi}_{\infty}(E) \leq \mathsf{H}^{g}(E)$, we need to introduce, for any non-empty $F \subset \partial t$,

$$\bigwedge F = \max\{x \in t \cup \partial t : \forall \xi \in F,\ x \preceq \xi\}.$$

To see that this set is indeed totally ordered it suffices to see that it is a subset of the totally ordered set $\{\xi_0, \xi_1, \dots\} \cup \{\xi\}$, for any $\xi \in F$. This remark also shows that as long as F has at least two elements, there exist $\xi \neq \eta$ in F such that $\bigwedge F = \xi \wedge \eta$. Now, let $\delta > 0$ and $C \in \mathrm{cov}_\delta(E)$. Consider

$$\mathcal{C} = \{\bigwedge F : F \in C,\ \mathrm{diam}(F) > 0\} \quad \text{and} \quad G = \{\xi : \exists F \in C,\ F = \{\xi\}\}.$$

We claim that $\mathcal{C} \in \mathsf{Cov}_{n_\delta}(E \setminus G)$. Indeed, assume that $F \in C$ is not a singleton. Let $\xi \neq \eta \in F$ be such that $\bigwedge F = \xi \wedge \eta$. Then we have

$$0 < \varphi(\xi \wedge \eta) = d(\xi, \eta) \leq \mathrm{diam}\, F \leq \delta.$$

thus $|\xi \wedge \eta| \geq n_\delta$. Finally,

$$\sum_{F \in C} g(\mathrm{diam}\, F) = \sum_{F \in C} g(\varphi(\bigwedge F)) = \sum_{x \in \mathcal{C}} \phi(x) \geq \mathscr{H}^{\phi}_{n_\delta}(E \setminus G) = \mathscr{H}^{\phi}_{n_\delta}(E),$$

since G is at most countable. Taking the infimum over all δ-covers gives $\mathsf{H}^{g}_{\delta}(E) \geq \mathscr{H}^{\phi}_{n_\delta}(E)$.

The inequality $\mathsf{H}^{g}(E) \leq \mathsf{C}^{g}_{0}(E)$ is obvious so we are left with $\mathsf{C}^{g}_{0}(E) \leq \mathscr{H}^{\phi}(E)$. Let $n \geq 0$ and $\mathcal{C} \in \mathsf{Cov}_n(E)$. For each $x \in C$, let $\xi^{(x)} \in [x]_t$ and recall that $\mathscr{B}(\xi^{(x)}, \varphi(x)) = [x]_t$. Let

$$C = \{(\xi^{(x)}, \phi(x)) : x \in \mathcal{C}\} \in \text{c-cov}_{\delta_n}(E).$$

Then we have

$$\sum_{x \in \mathcal{C}} \phi(x) = \sum_{(\xi, r) \in C} g(\mathrm{diam}\, \mathscr{B}(\xi^{(x)}, \phi(x))) \geq \mathsf{C}^{g}_{\delta_n}(E).$$

Taking the infimum over all n-covers and letting n go to infinity finishes the proof. $\square$

Remark 1.2. Our point of view is to study the space ∂t in an intrinsic way. In [60], the author defines the packing measure on the whole metric space $(\mathcal{U}_\infty, \mathrm{d}_{\mathcal{U}_\infty})$ and then views subsets of ∂t as subsets of $\mathcal{U}_\infty$. This yields *a priori* different packing measures (because $\mathrm{diam}[x]_t$ is not always equal to $\mathrm{diam}[x]$), but may still be expressed as $\mathscr{P}^{\phi}$, with $\phi(x) = g(e^{-|x|})$ for $x \in t$ and g the gauge function which is used.

We may still simplify the expression of Hausdorff measures when the metric is simple enough. Indeed, consider a function $f : \mathbb{N} \to (0, \infty)$ which is decreasing and vanishing at infinity (for instance $f(n) = e^{-n}$). Consider the metric d_f defined by

$$\forall \xi \neq \eta \in \partial t, \quad d_f(\xi, \eta) = f(|\xi \wedge \eta|).$$

Then we can get rid of the diameter in the definition of the Hausdorff measures.

Proposition 1.11. *Let t be an infinite tree, f and d_f as in the previous discussion, g a gauge function. Endow ∂t with the metric d_f. Let, for x in t, $\phi_f(x) = f(|x|)$ and $\varphi_f(x) = \operatorname{diam}^{d_f}[x]_t$. Then, for all $n \geq 0$, $\mathscr{H}_n^{g\circ\phi_f} = \mathscr{H}_n^{g\circ\varphi_f}$.*

Proof. Let E be a subset of ∂t. The inequality $\mathscr{H}_n^{g\circ\varphi_f}(E) \leq \mathscr{H}_n^{g\circ\phi_f}(E)$ comes from the fact that $\operatorname{diam}^{d_f}[x]_t \leq f(|x|)$ for all x in t. In the other direction, we may assume that E has no isolated rays since $\operatorname{Isolated}(\partial t)$ is at most countable, hence has 0 measure. Let $\mathcal{C}$ be an n-cover of E and consider

$$\mathcal{C}' = \{\mathsf{bp}_t^{\uparrow}(x) : x \in \mathcal{C}\}.$$

Then $\mathcal{C}'$ is again an n-cover of E and

$$\sum_{x\in\mathcal{C}} g(\operatorname{diam}[x]_t) = \sum_{x\in\mathcal{C}'} g(\operatorname{diam}[x]_t) = \sum_{x\in\mathcal{C}'} g(f(|x|)) \geq \mathscr{H}_n^{g\circ\phi}(E).$$

Taking the infimum over all n-covers yields $\mathscr{H}_n^{g\circ\varphi}(E) \geq \mathscr{H}_n^{g\circ\varphi}(E)$. $\square$

As already announced, there is a relation between the branching number of a tree and the Hausdorff dimension of its boundary.

Proposition 1.12. *Let t be an infinite tree. If its boundary ∂t is equipped with the distance $\mathrm{d}_{\mathcal{U}_\infty}$, then $\dim_{\mathrm{H}}(\partial t) = \log \mathsf{br}(t)$.*

Proof. For $\lambda > 1$, let

$$L_\lambda(\partial t) = \inf\Big\{\sum_{x\in\pi} \lambda^{-|x|} : \pi \text{ cutset of } t\Big\}.$$

For x in t, let $\phi_\lambda(x) = \lambda^{-|x|}$. By compactness of ∂t, we may extract of any 1-cover of ∂t a finite 1-cover of ∂t, that is, a cutset. Hence we see that $L_\lambda(\partial t) = \mathscr{H}_1^{\phi_\lambda}(\partial t)$. Next we claim that

$$\mathscr{H}_1^{\phi_\lambda}(\partial t) > 0 \iff \mathscr{H}^{\phi_\lambda}(\partial t) > 0.$$

One of the implications is obvious. Now assume that $\mathscr{H}^{\phi_\lambda}(\partial t) > 0$. Hence, there exists $n_0 \geq 1$ such that $\mathscr{H}_{n_0}^{\phi_\lambda}(\partial t) > 0$. Now if $\mathcal{C}$ is a 1-cover,

$$\sum_{x\in\mathcal{C}} \phi_\lambda(x) \geq \begin{cases} \mathscr{H}_{n_0}^{\phi_\lambda}(\partial t) & \text{if } \mathcal{C} \in \mathsf{Cov}_{n_0}(\partial t) \\ \min\limits_{1\leq|x|\leq n_0} \phi_\lambda(x) & \text{otherwise,} \end{cases}$$

which proves the claim.

Finally, when ∂t is endowed with the metric $\mathrm{d}_{\mathcal{U}_\infty}$, $\mathsf{H}^{\log\lambda} = \mathscr{H}^{\phi_\lambda}$, by the two previous propositions. $\square$

1.11 Dimension(s) of a flow

In this section, unless specified otherwise, t is an infinite tree, d is a distance in $\mathsf{Metrics}(\partial t)$ with associated diameter function φ and θ is a (not identically 0) flow on t. So far, we have related upper and lower densities of θ on an infinite tree with Hausdorff and packing measures. Unfortunately, knowing with precision the rate of decay of $\theta(\xi_n)$ as n goes to infinity is often a difficult problem. A less ambitious approach may be to study the functions

$$\underline{\dim}_{\mathrm{loc}}\theta(\xi) = \liminf_{n\to\infty} \frac{\log\theta(\xi_n)}{\log \mathrm{diam}[\xi_n]_t}, \quad \text{and} \quad \overline{\dim}_{\mathrm{loc}}\theta(\xi) = \limsup_{n\to\infty} \frac{\log\theta(\xi_n)}{\log \mathrm{diam}[\xi_n]_t}$$

when they make sense (for instance when ξ is not isolated or is in the support of θ). These are called, respectively, the *lower and upper dimensions* of θ at ξ. We will see that they are related to the Hausdorff and packing dimensions.

A more global notion of dimension for the flow θ is to consider

$$\begin{aligned}\underline{\dim}_{\mathrm{H}}\theta &= \inf\{\dim_{\mathrm{H}}(E) : E \text{ Borel}, \theta(E) > 0\} \quad \text{and}\\ \overline{\dim}_{\mathrm{H}}\theta &= \inf\{\dim_{\mathrm{H}}(E) : E \text{ Borel}, \theta(E) = \theta(\partial t)\},\end{aligned}$$

which are the *lower and upper* Hausdorff dimensions of the flow θ. The lower and upper packing dimensions of θ are defined in the same way.

As we did in Sections 8 and 9, we want to relate the local properties with the global ones.

Proposition 1.13. *The following equalities hold:*

$$\begin{aligned}\theta\text{-}\operatorname{ess\,inf} \underline{\dim}_{\mathrm{loc}}\theta = \underline{\dim}_{\mathrm{H}}\theta \quad &\textit{and} \quad \theta\text{-}\operatorname{ess\,sup} \underline{\dim}_{\mathrm{loc}}\theta = \overline{\dim}_{\mathrm{H}}\theta;\\ \theta\text{-}\operatorname{ess\,inf} \overline{\dim}_{\mathrm{loc}}\theta = \underline{\dim}_{\mathrm{p}}\theta \quad &\textit{and} \quad \theta\text{-}\operatorname{ess\,sup} \overline{\dim}_{\mathrm{loc}}\theta = \overline{\dim}_{\mathrm{p}}\theta.\end{aligned}$$

Proof. Let $\alpha = \theta\text{-}\operatorname{ess\,inf} \underline{\dim}_{\mathrm{loc}}\theta$. We first show that $\alpha \le \underline{\dim}_{\mathrm{H}}\theta$. We may assume $\alpha > 0$ Let $s \in (0, \alpha)$ and $F_s = \{\xi \in \operatorname{supp}\theta : \underline{\dim}_{\mathrm{loc}}\theta(\xi) > s\}$. Then, $\theta(F_s) = \theta(\partial t)$. Let E be a Borel set such that $\theta(E) > 0$. Then, for any $\xi \in E \cap F_s$,

$$\liminf_{n\to\infty} \frac{\log(\theta(\xi_n))}{\log(\varphi(\xi_n))} > s, \quad \text{hence} \quad \limsup_{n\to\infty} \frac{\theta(\xi_n)}{\varphi(\xi_n)^s} < 1.$$

This proves that $\sup_{\xi\in E\cap F_s} \overline{\mathrm{d}}_\theta^{\varphi^s}(\xi) < 1$, thus by the upper density theorem that

$$\mathsf{H}^s(E \cap F_s) > \theta(E \cap F_s) > 0.$$

As a consequence, $\dim_{\mathrm{H}}(E) \ge \dim_{\mathrm{H}}(E \cap F_s) \ge s$, for all E such that $\theta(E) > 0$, hence $\underline{\dim}_{\mathrm{H}}\theta \ge s$ for all $s < \alpha$.

Now we show that $\underline{\dim}_{\mathrm{H}}\theta \le \alpha$. We may assume $\alpha < \infty$. Let $s > \alpha$. Let $E_s = \{\xi \in \operatorname{supp}\theta : \underline{\dim}_{\mathrm{loc}}\theta(\xi) < s\}$. Then, $\theta(E_s) > 0$ and for any $\xi \in E_s$,

$$\overline{\mathrm{d}}_\theta^{\varphi^s}(\xi) > 1.$$

Again by the upper density theorem, this proves that

$$\mathsf{H}^s(E_s) < \theta(E_s) < \infty.$$

Hence $\underline{\dim}_{\mathrm{H}}\theta \leq \dim_{\mathrm{H}}(E_s) \leq s$. Letting s go to α finishes the proof of this inequality.

The proof of the second equality is so similar that we feel free to omit it, and it suffices to replace the upper density theorem by the lower density theorem to obtain the results about packing measures. ☐

Definition 1.7. If there exists $\alpha \in [0, \infty]$ such that,

$$\text{for } \theta\text{-almost every } \xi \in \partial t,\ \lim_{n\to\infty} \frac{\log(\theta(\xi_n))}{\log(\varphi(\xi_n))} = \alpha,$$

we say that θ is *exact-dimensional* on ∂t for the metric d and we simply write $\dim\theta = \alpha$, since all our definitions of dimension of the flow θ coincide.

Remark 1.3. Some authors (see [20]), in a more general setting, reserve this term to the case where for θ-almost every ξ,

$$\lim_{r\to 0} \frac{\log(\theta\,\mathscr{B}(\xi, r))}{\log(r)} = \alpha.$$

It is a stronger condition than ours. Indeed, if ξ is not isolated and such that the previous limit converges to α, consider the sequence $\xi_{n_1} \prec \xi_{n_2} \prec \dots$ of ancestors of ξ in $\mathsf{reduced}(t)$. Then, for all $i \geq 1$, $\mathscr{B}(\xi, \varphi(\xi_{n_i})) = [\xi_{n_i}]_t$, hence

$$\lim_{i\to\infty} \frac{\log(\theta(\xi_{n_i}))}{\log(\varphi(\xi_{n_i}))} = \lim_{r\to 0} \frac{\log(\theta\,\mathscr{B}(\xi, r))}{\log(r)} = \alpha.$$

Since for $n_i \leq k < n_{i+1}$, we have $\theta(\xi_k) = \theta(\xi_{n_i})$ and $\varphi(\xi_k) = \varphi(\xi_{n_i})$ the whole sequence $(\log(\theta(\xi_n))/\log(\varphi(\xi_n)))$ also converges to α.

Remark 1.4. In the case $d = \mathrm{d}_{\mathcal{U}_\infty}$, we have, for all $\xi \in \partial t$, $\varphi(\xi_n) \leq e^{-n}$, thus

$$\frac{\log(\theta(\xi_n))}{\log(\varphi(\xi_n))} \leq \frac{-1}{n} \log(\theta(\xi_n)).$$

If ξ is not isolated, with the same notations as in the previous remark, we have, for all $i \geq 1$,

$$\frac{\log(\theta(\xi_{n_i}))}{\log(\varphi(\xi_{n_i}))} = \frac{-1}{n_i} \log(\theta(\xi_{n_i})),$$

hence

$$\liminf_{n\to\infty} \frac{-1}{n} \log(\theta(\xi_n)) \leq \liminf_{n\to\infty} \frac{\log(\theta(\xi_n))}{\log(\varphi(\xi_n))},$$

which shows that these inferior limits are equal. The left hand side is called the Hölder exponent of θ at ξ (see [46, Section 15.4]).

Definition 1.8. When, for a metric $d \in \mathsf{Metrics}(\partial t)$, we have $\overline{\dim}_{\mathrm{H}}\theta < \dim_{\mathrm{H}} \partial t$, then θ-almost every ξ lies in a subset of ∂t which has a smaller dimension. In this case, we shall say that the dimension drop phenomenon occurs for θ (with the metric d).

Such a phenomenon was first observed in the context of the harmonic measure in the euclidean space by Makarov (see [47]).

2 Ergodic theory on marked Galton-Watson trees

The first two sections of this chapter introduce the Galton-Watson trees and, when they are infinite, the boundaries of these random trees, with an emphasis on their Hausdorff and Packing dimension. The next sections of this chapter are devoted to the ergodic theory on Galton-Watson trees, developed in [43]. There are some small differences between our treatment of this theory and the original. We work on the space of marked trees (we need it for our applications) and we do not assume that $p_0 = 0$. Fortunately, this does not change the main ideas. Our more formal treatment leads to the definition of the inherited part of a set. Some minor additional results are also presented. The last section presents a theorem of the author which, though not as general as the rest of the theory, seemed to belong in this chapter.

2.1 Galton-Watson trees

Since we want to define random trees, we wish to endow the set $\mathscr{T}$ of all trees with a metric. Informally, we want to say that two trees are close to each other if they agree up to a large height. Thus we set for all distinct trees t and t',

$$\mathrm{d}(t,t') = \sum_{r\geq 0} 2^{-r-1}\delta^{(r)}\left(t,t'\right), \quad \text{where}$$

$$\delta^{(r)}(t,t') = \begin{cases} 0 & \text{if } \forall |x| \leq r,\, x \in t \iff x \in t'; \\ 1 & \text{otherwise.} \end{cases}$$

The space $(\mathscr{T}, \mathrm{d})$ is then a complete, ultrametric, separable space. We denote by $\mathscr{T}^*$ the set of all infinite trees.

Let $\mathbf{p} = (p_k)_{k\in\mathbb{N}}$ be a non-negative sequence of real numbers such that $\sum_{k\geq 0} p_k = 1$ and $p_1 < 1$. We denote by g the generated function of $\mathbf{p}$ defined on $[0,1]$ by

$$\mathsf{g}(s) = \sum_{k\geq 0} p_k s^k.$$

Let $\Omega = \mathbb{N}^{\mathcal{U}}$ be the set of all functions from $\mathcal{U}$ to $\mathbb{N}$, endowed with its product σ-algebra $\mathcal{F}$. Let $\mathbf{P}$ be the probability measure defined by

$$\mathbf{P}\{\omega \in \Omega : \omega(x_1) = n_1, \omega(x_2) = n_2, \dots, \omega(x_k) = n_k\} = p_{n_1} p_{n_2} \cdots p_{n_k},$$

for all $k \geq 1$, all pairwise distinct x_1, x_2, ..., x_k in $\mathcal{U}$ and all n_1, n_2, ..., n_k in $\mathbb{N}$. We denote the associated expectation by $\mathbf{E}$.

Now for $\omega \in \Omega$, let $T(\omega)$ be constructed as follows:

— $T_0(\omega) = \{\varnothing\}$;
— for all $n \geq 0$, $T_{n+1}(\omega) = \{xi : x \in T_n(\omega),\ 1 \leq i \leq \omega(i)\}$;
— $T = \bigcup_{n=0}^{\infty} T_n(\omega)$.

Then we see that $T(\omega)$ is a tree and that for all x in $T(\omega)$, the number of children of x in $T(\omega)$ is $\nu_{T(\omega)}(x) = \omega(x)$. It is not hard to check that $T : (\Omega, \mathcal{F}, \mathbf{P}) \to (\mathscr{T}, d_{\mathscr{T}})$ is measurable. The random tree T is called *a Galton-Watson tree of reproduction law* $\mathbf{p}$ and its distribution is denoted by $\mathbf{GW_p}$ (or simply $\mathbf{GW}$ when there is no risk of confusion).

For $x \in \mathcal{U}$ and $\omega \in \mathbb{N}^{\mathcal{U}}$, define the translated function ω^x by $\omega^x(u) = \omega(xu)$ for all $u \in \mathcal{U}$, and the Galton-Watson tree from x by $T^x(\omega) = T(\omega^x)$. On the event that $x \in T$, we have $T^x = T[x]$. It is clear that all the random trees T^x have the same distribution as T and moreover, whenever a subset Q of $\mathcal{U}$ is made of pairwise incomparable words, the random trees $(T^x)_{x \in Q}$ are independent and independent of the family of random variables $\{\omega \mapsto \omega(y) : y \in \mathcal{U},\ \forall x \in Q,\ y \not\geq x\}$. We call this property the *branching property* whose most common avatar is the following proposition.

Proposition 2.1. *Let T be a Galton-Watson tree of reproduction law* $\mathbf{p}$. *Then for all $k \geq 0$, for all Borel sets $B_1, \ldots, B_k$ of $\mathscr{T}$,*

$$\mathbf{P}(\nu_T(\varnothing) = k, T[1] \in B_1, \ldots, T[k] \in B_k) = p_k \prod_{i=1}^{k} \mathbf{GW}(B_i).$$

For $n \geq 0$ and any tree t, we set

$$Z_n(t) = \#\{x \in t : |x| = n\}.$$

The stochastic process $(Z_0(T), Z_1(T), \ldots, Z_n(T), \ldots)$ is called *a Galton-Watson process.*

These processes were first introduced to model the survival of the family names of the nobility, independently by Bienaymé and Galton and Watson during the 19$^{\text{th}}$ century. The idea of considering not only the numerical process but the whole tree has progressively emerged and has been formalised by Neveu ([50]). A sharp criterion to know whether or not T can be infinite with positive probability may go back to Bienaymé (see [33]). For a proof, we refer to [7, Section 1.3].

Theorem 2.2. *Let T be a Galton-Watson tree of reproduction law* $\mathbf{p}$. *Let $m = \sum_{k \geq 0} p_k k$ and $q = \mathbf{P}(T \text{ is finite})$. Then we have $q < 1 \iff m > 1$.*

Now assume that $1 < m < \infty$. What can we say of the rate of growth of $Z_n(T)$ on the event of non-extinction? It is easy to see that the process $(Z_n(T)/m^n)_{n \geq 0}$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma\{\omega(x) : |x| \leq n,\ \omega \in \Omega\}$. Let $W(T)$ be its almost-sure limit. In 1966, Kesten and Stigum published a sharp criterion on the uniform integrability of this martingale.

Theorem 2.3 (Kesten-Stigum)**.** *Assume that $1 < m < \infty$. Then,*

$$\mathbf{E}[W(T)] > 0 \iff \mathbf{E}[W(T)] = 1 \iff \mathbf{P}(W = 0) = q \iff \sum_{k \geq 1} p_k k \log k < \infty$$

The first proofs of this theorem were analytical, see for instance [7, Section 1.10]. A strikingly simpler proof was found by Lyons, Pemantle and Peres in [42], where the authors defined the size-biased Galton-Watson trees. When $\sum_{k\geq 1} p_k k \log k$ is infinite, the rate of growth is slightly less than m^n but a normalizing sequence still exists.

Theorem 2.4 (Seneta-Heyde)**.** *Assume that $1 < m < \infty$. Then there exists a deterministic sequence $(c_n)_{n\geq 0}$ of positive real numbers such that $\lim_{n\to\infty} c_n/c_{n+1} = m$ and $(Z_n(T)/c_n)_{n\geq 0}$ converges almost surely to a random variable $\widetilde{W}(T)$ which is positive and finite on the event of non-extinction.*

A theorem of Athreya ([6]) asserts that

$$\mathbf{E}[\widetilde{W}(T)] < \infty \iff \sum_{k\geq 1} p_k k \log k < \infty.$$

In fact, when $\sum_{k\geq 1} p_k k \log k < \infty$, by asymptotic uniqueness of normalization sequences (see [23, Section 10]), for any sequence (c_n) as in the previous theorem, there is a constant $C \in (0,\infty)$ such that $\lim_{n\to\infty} c_n/m^n = C$ and $W(T) = C\widetilde{W}(T)$ almost surely. Since there is no risk of confusion (up to a constant factor) we will always write $W(T)$.

2.2 The boundary of an infinite Galton-Watson tree

From now on we assume that $m > 1$. Let $\mathbf{GW}^*$ be the distribution of a Galton-Watson tree conditioned on non-extinction. We also write $\mathbf{P}^*$ for the probability measure $\mathbf{P}$ conditioned on the event on non-extinction and $\mathbf{E}^*$ for the associated expectation.

Lemma 2.5. *For $\mathbf{GW}^*$-almost every t, ∂t has no isolated rays.*

Proof. First we claim that

$$\alpha \coloneqq \mathbf{P}(\partial T \text{ has exactly one ray}) = 0.$$

By the branching property, this probability equals

$$\begin{aligned}\alpha &= \sum_{k\geq 1} p_k \sum_{i=1}^{k} \mathbf{P}(T^i \text{ has exactly one ray and the trees } T^j \text{ are finite for } j \neq i \in [\![1,k]\!])\\ &= \sum_{k\geq 1} p_k k q^{k-1} \alpha = \mathbf{P}(\nu_T^*(\varnothing) \leq 1)\alpha,\end{aligned}$$

where we recall that $\nu_T^*(\varnothing)$ denotes the number of infinite lineages from the root. Since $\mathbf{P}(\nu_T^*(\varnothing) \leq 1) < 1$, this proves the claim. Now the probability that ∂T has isolated rays is

$$\mathbf{P}(\exists x \in T,\ \#[x]_t = 1) \leq \sum_{x\in\mathcal{U}} \mathbf{P}(x \in T,\ \#[x]_t = 1) = \sum_{x\in\mathcal{U}} \mathbf{P}(x\in T)\alpha = 0. \qquad \square$$

We endow ∂T with the metric $\mathrm{d}_{\mathcal{U}_\infty}$ and want to compute its Haudorff and packing dimensions.

Theorem 2.6 (Hawkes, Liu, Lyons, Watanabe)**.** *Almost surely on the event of non-extinction,*

$$\dim_{\mathrm{H}} \partial T = \dim_{\mathrm{p}} \partial T = \log m.$$

We begin with the upper bound of the packing dimension. This is due to Watanabe ([60]). Liu proved it first in [36] under an additional integrability assumption. We shall see in the proof that it does not matter whether we define the packing measure as the restriction of the packing measure on $\mathcal{U}_\infty$ or intrinsically on ∂T.

Lemma 2.7. *Almost surely on the event of non-extinction,* $\dim_{\mathrm{p}} \partial T \leq \log m$.

Proof. If $m = \infty$, there is nothing to prove. Assume $m < \infty$ and let $s > \log m$. Consider for $n \geq 0$, the random variable

$$S_n(T) = \sum_{k \geq n} e^{-ks} Z_k(T).$$

The sequence $(S_n(T))$ is non-increasing. Let $S_\infty(T)$ be its limit. Since for any $k \geq 0$, $\mathbf{E}[Z_k] = m^k$, we have

$$\mathbf{E}[S_n] = \sum_{k \geq n} e^{-k(s - \log m)} \xrightarrow[n \to \infty]{} 0,$$

hence by Fatou's lemma, $S_\infty = 0$ almost surely. Now if $\mathcal{C}$ is an n-packing and ϕ_s is defined on T by $\phi_s(x) = e^{-s|x|}$ or by $\phi_s(x) = (\operatorname{diam}[x]_t)^s$,

$$\sum_{x \in \mathcal{C}} \phi_s(x) \leq \sum_{x \in \mathcal{C}} e^{-s|x|} \leq \sum_{|x| \geq n} e^{-s|x|} = S_n.$$

Hence we see that $\mathscr{P}_n^{\phi_s}(\partial T) \leq S_n$ and $\mathscr{P}_\infty^{\phi_s}(\partial T) = 0$ almost surely. Since $\mathscr{P}^{\phi_s}(\partial T) \leq \mathscr{P}_\infty^{\phi_s}(\partial T)$, this concludes the proof. $\square$

We now turn to the Hausdorff dimension. We shall give two proofs of the fact that almost surely on the event of non-extinction, $\dim_{\mathrm{H}} \partial T \geq \log m$.

The first proof, due to Lyons [39], "from the book", uses percolation on trees. Let $p \in (0,1)$. Consider, under a probability $\mathbb{P}$, i.i.d. random variables $(B_x)_{x \in \mathcal{U}}$ with Bernoulli distribution of parameter p and let t be an infinite tree. The random tree $\Gamma(t)$ is the connected component of the root in the forest $\{x \in t : B_x = 1\}$. The critical parameter $p_{\mathrm{c}}(t) \in [0,1]$ is defined by

$$p_{\mathrm{c}}(t) = \sup\{p > 0 : \mathbb{P}(\Gamma(t) \text{ is infinite}) = 0\}.$$

Lyons showed ([39, Theorem 6.2]) that for any infinite tree t, $p_c(t) = 1/\mathsf{br}(t)$.

Now if T is a Galton-Watson tree whose reproduction law has mean $m > 1$, and if $p > \frac{1}{m}$, then the tree $\Gamma(T)$ (suitably reindexed) has the distribution of a Galton-Watson tree whose reproduction law has mean $mp > 1$ hence can be infinite with positive probability. This shows that $p_{\mathrm{c}}(T) \leq 1/m$, hence that $\mathsf{br}(T) \geq m$ and by Proposition 1.12 that $\dim_{\mathrm{H}}(T) \geq \log m$, almost surely on the event of non-extinction.

In order to give a second proof of this inequality, we need to introduce the limit uniform measure. Assume $1 < m < \infty$. Recall the definition of $W(T)$ using the Seneta-Heyde norming sequence $(c_n)_{n\geq 0}$. Let $k \geq 0$. We may decompose $Z_{n+k}(T)$ as the sum of the terms $Z_n(T[x])$ for $|x| = k$ in T, therefore

$$W(T) = \lim_{n\to\infty} \frac{Z_{n+k}(T)}{c_{n+k}} = \lim_{n\to\infty} \frac{c_{n+k}}{c_n} \sum_{|x|=k} \frac{Z_n(T[x])}{c_n} = \frac{1}{m^k} \sum_{|x|=k} W(T[x]). \tag{2.1}$$

Therefore the function UNIF_T defined by

$$\mathsf{UNIF}_T(x) = \frac{W(T[x])}{m^{|x|} W(T)}$$

is a unit flow, which we call the *limit uniform measure* after [43]. Some authors consider the flow $W(T)\mathsf{UNIF}_T$ instead and call it the *branching measure*.

The first computation of $\dim_{\mathrm{H}} \partial T$, due to Hawkes ([25]) used the following asymptotic property:

Theorem 2.8 (Hawkes, Lyon-Pemantle-Peres)**.** *Assume that $\sum_{k\geq 1} p_k k \log k < \infty$, then for $\mathbf{GW}^*$-almost every t, for UNIF_t-almost every ray ξ,*

$$\lim_{n\to\infty} \frac{-1}{n} \log(\mathsf{UNIF}_t(\xi_n)) = \log m.$$

Actually, Hawkes obtained this result with probabilistic methods under the assumption $\sum_{k\geq 1} p_k k (\log k)^2 < \infty$ and this stronger version was proved in [43] using ergodic theory. We shall prove it later, once we have the right tools.

Let us just notice that it entails our lower bound. Indeed, by Proposition 1.13, this implies that for $\mathbf{GW}^*$-almost every t, $\overline{\dim}_{\mathrm{H}} \mathsf{UNIF}_t = \underline{\dim}_{\mathrm{H}} \mathsf{UNIF}_t = \log m$. By definition of these dimensions, this shows that for $\mathbf{GW}^*$-almost every t, $\dim_{\mathrm{H}} \partial t \geq \log m$ as long as $\sum_{k\geq 1} p_k k \log k < \infty$. To remove this hypothesis, we may proceed as follows. Let $\ell \geq 1$ and, for a tree t, define $t^{(\ell)}$ as the tree t in which all the children of ranks greater than ℓ have been removed. Formally, $t_0^{(\ell)} = \varnothing$; for all $n \geq 0$,

$$t_{n+1}^{(\ell)} = \{xi : x \in t_n^{(\ell)},\ 1 \leq i \leq \nu_t(x) \wedge \ell\}$$

and $t^{(\ell)} = \bigcup_{n\geq 0} t_n^{(\ell)}$. Then $T^{(\ell)}$ is a Galton-Watson tree of reproduction law $(p_k^{(\ell)})$ given by

$$p_k^{(\ell)} = \begin{cases} p_k & \text{if } k < \ell; \\ \sum_{k\geq \ell} p_k & \text{if } k = \ell; \\ 0 & \text{if } k > \ell. \end{cases}$$

Its mean is

$$m^{(\ell)} = \sum_{k=1}^{\ell} p_k k + \ell \sum_{k\geq \ell+1} p_k,$$

which goes to m as ℓ goes to infinity and in particular is greater than 1 for ℓ large enough. Now we want to prove that

$$\mathbf{P}^*(\forall \ell \geq 1,\ T^{(\ell)} \text{ is finite}) = 0.$$

At this point we need the concept of inherited set. An *inherited set* is a Borel subset $\mathcal{I}$ of $\mathscr{T}^*$ such that

$$\forall t \in \mathcal{I},\ \forall x \in t^*,\ t[x] \in \mathcal{I}.$$

Lemma 2.9. *Any inherited set has* $\mathbf{GW}^*$ *measure* 0 *or* 1*.*

Proof. Let $\mathcal{I}$ be an inherited set and $\mathcal{J}$ be the union of $\mathcal{I}$ and the set of all finite trees. Then, by the branching property:

$$\begin{aligned}\mathbf{GW}(\mathcal{J}) = \mathbf{P}(T \in \mathcal{J}) &\leq \sum_{k=0}^{\infty} p_k \mathbf{P}(T^1 \in \mathcal{J}, ..., T^k \in \mathcal{J}) \\ &\leq \sum_{k=0}^{\infty} p_k \mathbf{GW}(\mathcal{J})^k = \mathsf{g}(\mathbf{GW}(\mathcal{J})).\end{aligned}$$

By strict convexity of g ($p_1 < 1$), for all $s \in (q,1)$, $\mathsf{g}(s) < s$. Since $\mathbf{GW}(\mathcal{J}) \geq q$, we must have $\mathbf{GW}(\mathcal{J}) \in \{q, 1\}$. Now if $\mathbf{GW}(\mathcal{J}) = q$, then $\mathbf{GW}^*(\mathcal{J}) = 0$, otherwise $\mathbf{GW}^*(\mathcal{J}) = 1$. ☐

Going back to our second proof, we see that the event $\{\forall \ell \geq 1,\ T^{(\ell)} \text{ is finite}\}$ is inherited. Its probability is less than 1 since whenever $m^{(\ell)} > 1$, $\mathbf{P}(T^{(\ell)} \text{ is infinite}) > 0$.

Hence, almost surely on the event of non-extinction, there exists a minimum random $N \geq 1$ such that for all $\ell \geq N$, $T^{(\ell)}$ is infinite. Now for any $k \geq 1$, on the event $N = k$ we know that almost surely for all $\ell \geq k$, $\dim_{\mathrm{H}} \partial T^{(\ell)} \geq \log m^{(\ell)}$ (the reproduction law of $T^{(\ell)}$ has finite support) therefore $\dim_{\mathrm{H}} \partial T \geq \log m^{(\ell)}$. Letting ℓ go to infinity finishes this second proof (modulo the proof of Theorem 2.8).

2.3 Marked trees and inheritance

Let Marks be a Polish space with distance d_{Marks} in which our marks will live. We follow Neveu ([50]) and let it be completely abstract.

A marked tree t is a tree together with a function $\mathsf{mk}_t : t \to \mathsf{Marks}$. To lighten notations, we will write t when we should write (t, mk_t). We still, however, write $x \in t$ when we mean that a word x is a vertex of the marked tree t. Let $\mathscr{T}_{\mathrm{m}}$ be the set of all marked trees, with the distance

$$d_{\mathrm{m}}\left(t, t'\right) = \sum_{r \geq 0} 2^{-r-1} \delta_{\mathrm{m}}^{(r)}\left(t, t'\right),$$

where $\delta_m^{(r)}$ is defined by

$$\delta_{\mathrm{m}}^{(r)}\left(t, t'\right) = \begin{cases} 1 \text{ if } t \text{ and } t' \text{ (without their marks) do not agree up to height } r; \\ \min\left(1, \max\{d_{\mathsf{Marks}}\left(\mathsf{mk}_t(x), \mathsf{mk}_{t'}(x)\right) : x \in t,\ 0 \leq |x| \leq r\}\right) \text{ otherwise.}\end{cases}$$

The set $\mathscr{T}_{\mathrm{m}}$ is then a Polish space. Recall that † denotes the empty tree. We extend the distance d_m on $\mathscr{T}_{\mathrm{m}} \sqcup \{\dagger\}$ by setting $d_m(\dagger, t) = 1$ for any t in $\mathscr{T}_{\mathrm{m}}$.

For any marked tree t and any vertex x in t, recall that

$$t[x] = \{y \in \mathcal{U} : xy \in t\},$$

is the reindexed subtree of t starting from x. Its marks are inherited from the original tree:

$$\mathsf{mk}_{t[x]}(y) = \mathsf{mk}_t(xy), \quad \forall y \in t[x].$$

Recall that when a word x in $\mathcal{U}$ does not belong to t, we set $t[x] = \dagger$. For any x in $\mathcal{U}$, the map

$$\begin{aligned} \mathsf{Sub}_x : \mathscr{T}_{\mathrm{m}} &\to \mathscr{T}_{\mathrm{m}} \cup \{\dagger\} \\ t &\mapsto t[x] \end{aligned}$$

is continuous, hence measurable. We denote by $\mathscr{T}_{\mathrm{m}}^*$ the set of all *infinite* marked trees.

Definition 2.1. Let A be a Borel subset of $\mathscr{T}_{\mathrm{m}}$. The *inherited part* of A is

$$A^{\mathrm{o}} = \{t \in \mathscr{T}_{\mathrm{m}} : \forall x \in t,\ t[x] \text{ is finite or } t[x] \in A\}.$$

We say that A is *inherited* if $A = A^{\mathrm{o}}$. For a Borel probability measure μ on $\mathscr{T}_{\mathrm{m}}^*$, we say that A is *μ-inherited* when $\mu(A) = \mu(A^{\mathrm{o}})$.

In particular, A^{o} contains all finite marked trees (thus it is possible that A^{o} is not contained in A; however, every infinite tree in A^{o} belongs to the set A). Notice that, by continuity of the maps Sub_x, for x in $\mathcal{U}$, the set

$$A^{\mathrm{o}} = \bigcap_{x \in \mathcal{U}} \mathsf{Sub}_x^{-1}\left(\{\dagger\} \cup A \cup (\mathscr{T}_{\mathrm{m}} \setminus \mathscr{T}_{\mathrm{m}}^*)\right), \tag{2.2}$$

is again a Borel subset of $\mathscr{T}_{\mathrm{m}}$. The inherited part satisfies nice set-theoretical properties.

Lemma 2.10. *Let I be any set, A, B and $(A_i)_{i \in I}$ be Borel subsets of $\mathscr{T}_m$. Then,*
1. $(A^{\mathrm{o}})^{\mathrm{o}} = A^{\mathrm{o}}$ *;*
2. if $A \subset B$*, then* $A^{\mathrm{o}} \subset B^{\mathrm{o}}$*;*
3. $(\bigcap_{i \in I} A_i)^{\mathrm{o}} = \bigcap_{i \in I} {A_i}^{\mathrm{o}}$*;*
4. $(\bigcup_{i \in I} {A_i}^{\mathrm{o}})^{\mathrm{o}} = \bigcup_{i \in I} {A_i}^{\mathrm{o}}$*.*

Proof. 1. Since A^{o} contains all finite trees, it is clear that $(A^{\mathrm{o}})^{\mathrm{o}}$ is contained in A^{o}. On the other hand, if t is in A^{o} and x is in t, then, for all xy in t, $t[xy]$ is in A or is finite. Thus, for all y in $t[x]$, $t[x][y]$ is in A or is finite, that is, $t[x]$ is in A^{o}. Since this holds for all x in t, t is in $\left(A^{\mathrm{o}}\right)^{\mathrm{o}}$.

2. Straightforward.

3. It is a consequence of (2.2).

4. The second property implies that for all $i \in I$, $({A_i}^{\mathrm{o}})^{\mathrm{o}} \subset \left(\bigcup_{j\in I} {A_j}^{\mathrm{o}}\right)^{\mathrm{o}}$, hence the first inclusion, by the first property.

Now, we have

$$\mathscr{T}_{\mathrm{m}}^* \cap \Big(\bigcup_{i\in I} {A_i}^{\mathrm{o}}\Big)^{\mathrm{o}} \subset \mathscr{T}_{\mathrm{m}}^* \cap \bigcup_{i\in I} {A_i}^{\mathrm{o}},$$

hence the result, since both $\left(\bigcup_{i\in I} {A_i}^{\mathrm{o}}\right)^{\mathrm{o}}$ and $\bigcup_{i\in I} {A_i}^{\mathrm{o}}$ contain the set of finite trees. □

Notice that in general we only have $\bigcup_{i\in I} {A_i}^{\mathrm{o}} \subset (\bigcup_{i\in I} A_i)^{\mathrm{o}}$. The last two statements imply that the set of all inherited subsets of $\mathscr{T}_{\mathrm{m}}$ is stable under any union and any intersection.

2.4 Flow rules and harmonicity

We endow the set $\mathbb{R}^{\mathcal{U}}$ of all functions from $\mathcal{U}$ to $\mathbb{R}$ with its product σ-algebra, that is the smallest σ-algebra that makes all the functions $\theta \mapsto \theta(x)$, for $x \in \mathcal{U}$ measurable.

Definition 2.2. Let B be a (Borel) inherited subset of $\mathscr{T}_{\mathrm{m}}$. We say that a measurable function Θ from B to $\mathbb{R}^{\mathcal{U}}$ is a (consistent, unit) *flow rule* if for all $t \in B$,
1. Θ_t is a unit flow on t;
2. for any xy in t,

$$\Theta_t(xy) = \Theta_t(x)\Theta_{t[x]}(y). \tag{2.3}$$

We call B the *domain* of Θ and write $\operatorname{dom}\Theta = B$. If additionally, for all $t \in B$, for all $x \in t^*$, $\Theta_t(x) > 0$, we say that Θ is a *positive flow rule* (it always has full support).

If μ is a Borel probability measure on $\mathscr{T}_{\mathrm{m}}^*$, we say that Θ is a μ*-flow* rule, or that μ is a Θ*-probability measure* whenever $\mu \operatorname{dom}\Theta = 1$.

Remark 2.1. To construct a flow rule, it suffices to consider a measurable function $\phi : \mathscr{T}_{\mathrm{m}} \to [0,\infty]$ such that $\phi(t) = 0$ if t is finite. Then set $A = \{t \in \mathscr{T}_{\mathrm{m}} : \phi(t) \in (0,\infty)\}$. Define a flow rule Θ of domain A^{o} by setting for all $t \in A^{\mathrm{o}}$, for all $1 \le i \le \nu_t(\varnothing)$,

$$\Theta_t(i) = \frac{\phi(t[i])}{\sum_{j=1}^{\nu_t(\varnothing)} \phi(t[j])},$$

and continuing recursively, giving for all $x \in t^*$,

$$\Theta_t(x) = \prod_{\varnothing \prec y \preceq x} \frac{\phi(t[y])}{\sum_{z_* = y_*} \phi(t[z])}.$$

We have already seen some examples of flow rules: the three first examples in the list below (since they do not involve marks, let $\mathsf{Marks} = \{1\}$ for them). The last example shows a flow rule which depends on the marks.

Example 2.1.
1. As we have seen before in example 1.1, VIS is positive flow rule on $\mathscr{T}_{\mathrm{m}}^*$.

2. Let $\lambda > 0$ and for an infinite tree t, let $\beta(t)$ be the conductance of t for the λ-biased random walk as in Section 1.6. Let $A = \{t \in \mathscr{T}^* : \beta(t) > 0\}$. Then $A^{\mathrm{o}} = \{t \in \mathscr{T}^* : \forall x \in t^*,\ \beta(t[x]) > 0\}$. A tree in A^{o} is said to be *everywhere transient.* On A^{o}, as we have seen in Section 1.7, HARM is a positive flow rule. If $\mathbf{p}$ is a reproduction law of mean $m > \max(1, \lambda)$, then $\mathbf{GW}^*_{\mathbf{p}}$ is a HARM-probability measure.

3. Let $\mathbf{p}$ be a reproduction law of finite mean $m > 1$. Let (c_n) be a Seneta-Heyde normalizing sequence. let B be the set of all infinite trees t such that, for all $x \in t^*$, $Z_n(t[x])/c_n$ converges to an element of $(0, \infty)$. Then UNIF is a positive flow rule on B and $\mathbf{GW}^*_{\mathbf{p}}$ is a UNIF-probability measure.

4. Let $\mathsf{Marks} = (0, \infty)$ and B be the set of all infinite marked trees without leaves. Define $\overline{\mathsf{VIS}}$ on B by letting, for all $t \in B$, for all $1 \le i \le \nu_t(\varnothing)$,

$$\overline{\mathsf{VIS}}_t(i) = \frac{\mathsf{mk}_t(i)}{\sum_{j=1}^{\nu_t(\varnothing)} \mathsf{mk}_t(j)},$$

and continuing recursively, as in the previous discussion. This flow rule has been studied in [38]. It is denoted there by $\overline{\nu}$.

By standard arguments, we may rewrite the flow rule property as the following change of variable formula: for all $t \in \operatorname{dom}\Theta$, $x \in t^*$, f measurable functions from ∂t to $\mathbb{R}_+$:

$$\int \mathbf{1}_{\{x \prec \xi\}} f(\xi) \Theta_t(\mathrm{d}\xi) = \Theta_t(x) \int f(x\widetilde{\xi}) \Theta_{t[x]}(\mathrm{d}\widetilde{\xi}). \tag{2.4}$$

We introduce the notation, for a tree t and $i \ge 0$,

$$t^*_i = \{x \in t : |x| = i \text{ and } t[x] \text{ is infinite}\}.$$

A flow rule Θ defines an operator on functions. Let f be a real measurable function on $\operatorname{dom}\Theta$. The function Θf is defined by

$$\Theta f(t) = \sum_{i=1}^{\nu_t(\varnothing)} \Theta_t(i) f(t[i]) = \sum_{i \in t^*_1} \Theta_t(i) f(t[i]), \quad \forall t \in \operatorname{dom}\Theta.$$

If μ is a Θ-probability measure, the probability measure $\mu\Theta$ is defined by

$$\int f \,\mathrm{d}(\mu\Theta) = \int \Theta f \,\mathrm{d}\mu,$$

for all non-negative measurable functions $f : \mathscr{T}^*_{\mathrm{m}} \to \mathbb{R}$ (since $\mu \operatorname{dom}\Theta = 1$, the integrand in the right-hand side almost surely makes sense). We say that μ is Θ-*invariant* if $\mu\Theta = \mu$. A measurable function $f : \mathscr{T}^*_{\mathrm{m}} \to \mathbb{R}$ is called (μ, Θ)-*harmonic* when $\Theta f = f$, μ-almost surely. A Borel subset A of $\mathscr{T}^*_{\mathrm{m}}$ is called (μ, Θ)-invariant when the indicator function $\mathbf{1}_A$ is (μ, Θ)-harmonic.

The set of all (μ, Θ)-invariant sets is a σ-algebra which clearly contains all Borel subsets of $\mathscr{T}^*_{\mathrm{m}}$ which have 0 or full μ-measure. We denote it by $\mathrm{Inv}(\mu, \Theta)$. It can be completely described in the case where Θ is positive and μ is Θ-invariant.

Proposition 2.11. *Let Θ be a positive flow rule and μ a Θ-invariant probability measure. Then, a non-negative or bounded measurable function f on $\mathscr{T}_m^*$ is (μ,Θ)-harmonic if and only if for μ-almost every $t \in \mathscr{T}_m^*$, for all x in t^*, $f(t[x]) = f(t)$.*

In particular, a Borel subset A of $\operatorname{dom}\Theta$ is in $\operatorname{Inv}(\mu,\Theta)$ if and only if,

$$\mu(A^{\mathrm{o}} \sqcup (A^{\mathrm{c}})^{\mathrm{o}}) = 1.$$

Proof. The "if" part is immediate, so let us assume that f is a bounded (μ,Θ)-harmonic function. Let α be a rational number. By positivity of the operator Θ, we have

$$\Theta(f \wedge \alpha) \leq f \wedge \alpha, \quad \mu\text{-a.s.}$$

Integrating with respect to μ and using the fact that μ is Θ-invariant by assumption, we obtain

$$\Theta(f \wedge \alpha) = f \wedge \alpha, \quad \mu\text{-a.s.}$$

In particular, for μ-almost every $t \in \operatorname{dom}\Theta$,

$$(f \wedge \alpha)(t) \leq \sum_{i \in t_1^*} \Theta_t(i) f(t[i]) \wedge \alpha.$$

Since for $i \in t_1^*$, $\Theta_t(i) > 0$ and $\sum_{i \in t_1^*} \Theta_t(i) = 1$, this implies that μ-almost surely, for every rational number α,

$$f(t) \geq \alpha \implies \forall i \in t_1^*,\ f(t[i]) \geq \alpha,$$

which entails that, for μ-almost every tree t, for all i in t_1^*, $f(t) \leq f(t[i])$. Considering $-f$ instead of f shows that these inequalities are equalities.

To complete the proof by induction, consider, for $n \geq 1$,

$$A_n = \{t \in \operatorname{dom}\Theta : \forall x \in \bigcup_{k \leq n} t_k^*,\ f(t[x]) = f(t)\} \quad \text{and}$$
$$B_n = \{t \in \operatorname{dom}\Theta : \forall i \in t_1^*,\ t[i] \in A_n\}.$$

We have just proved that $\mu(A_1) = 1$. Notice that $A_n \cap B_n = A_{n+1}$. Now if $\mu(A_n) = 1$, then

$$1 = \mu(A_n) = \int \Theta_t \mathbf{1}_{A_n} \mu(\mathrm{d}t) = \int \sum_{i \in t_1^*} \Theta_t(i) \mathbf{1}_{A_n}(t[i]) \mu(\mathrm{d}t),$$

which shows that $\mu(B_n) = 1$, hence that $\mu(A_{n+1}) = \mu(A_n \cap B_n) = 1$. $\square$

2.5 Marked trees with rays : exact-dimensionality for a class of flow rules

We now turn to the space of infinite marked trees with a distinguished ray. Let

$$\mathscr{T}_{\mathrm{m,r}} = \{(t,\xi) : t \in \mathscr{T}_{\mathrm{m}}^*,\ \xi \in \partial t\}.$$

We endow this space with the distance $\mathrm{d_{m,r}}$ defined by

$$\mathrm{d_{m,r}}\big((t,\xi),(t',\xi')\big) = \sum_{r\geq 0} 2^{-r-1}\delta^{(r)}_{\mathrm{m,r}}\big((t,\xi),(t',\xi')\big),$$

for all (t,ξ) and (t',ξ') in $\mathscr{T}_{\mathrm{m,r}}$, where the sequence of functions $\delta^{(r)}_{\mathrm{m,r}}$, for r in $\mathbb{N}$ is defined by

$$\delta^{(r)}_{\mathrm{m,r}}\big((t,\xi),(t',\xi')\big) = \begin{cases} \delta^{(r)}_{\mathrm{m}}(t,t') & \text{if } \xi_k = \xi'_k,\ \forall 0\leq k\leq r; \\ 1 & \text{otherwise.}\end{cases}$$

The metric space $(\mathscr{T}_{\mathrm{m,r}}, \mathrm{d_{m,r}})$ is again a Polish space. The *shift* operator S on $\mathscr{T}_{\mathrm{m,r}}$ is defined by

$$\mathrm{S}(t,\xi) = (t[\xi_1], \xi_1^{-1}\xi).$$

In words, we look at the subtree selected by the ray ξ and the rest of the ray on it. The shift is continuous with respect to $d_{m,r}$.

Let Θ be a flow rule and μ a Θ-probability on $\mathscr{T}^*_{\mathrm{m}}$. Recall that for any marked tree t in $\operatorname{dom}\Theta$, we may see Θ_t as a Borel probability measure on ∂t. Hence, we may build a Borel probability measure $\mu\ltimes\Theta$ on $\mathscr{T}_{\mathrm{m,r}}$ in the following way: for any non-negative measurable function f on $\mathscr{T}_{\mathrm{m,r}}$,

$$\int f\,\mathrm{d}(\mu\ltimes\Theta) = \int\Big(\int f(t,\xi)\,\mathrm{d}\Theta_t(\xi)\Big)\,\mathrm{d}\mu(t). \tag{2.5}$$

Lemma 2.12. *The system $(\mathscr{T}_{m,r}, \mathrm{S}, \mu\ltimes\Theta)$ is measure-preserving if and only if μ is Θ-invariant.*

Proof. To prove the direct implication, let $f : \mathscr{T}^*_{\mathrm{m}} \to \mathbb{R}_+$ be a measurable function and define $g(t,\xi) = f(t)$ for all $(t,\xi)\in\mathscr{T}_{\mathrm{m,r}}$. Then we have

$$\int f\,\mathrm{d}\mu = \int\Big(\int g(t,\xi)\Theta_t(\mathrm{d}\xi)\Big)\mu(\mathrm{d}t) = \int\Big(\int g\circ S(t,\xi)\Theta_t(\mathrm{d}\xi)\Big)\mu(\mathrm{d}t).$$

Since $g\circ\mathrm{S}(t,\xi) = f(t[\xi_1])$, decomposing with respect to the value of ξ_1 gives, for μ-almost every t,

$$\int g\circ\mathrm{S}(t,\xi)\Theta_t(\mathrm{d}\xi) = \sum_{i\in t^*_1} f(t[i])\Theta_t(i) = \Theta f(t),$$

showing that $\mu\Theta = \mu$.

For the converse implication, let f be a non-negative measurable function on $\mathscr{T}_{\mathrm{m,r}}$. For $t\in\operatorname{dom}\Theta$,

$$\begin{aligned}\int f\circ\mathrm{S}(t,\xi)\,\mathrm{d}\Theta_t(\xi)\,\mathrm{d}\mu(t) &= \sum_{i\in t^*_1}\int \mathbf{1}_{\{\xi_1=i\}} f(t[i], i^{-1}\xi)\,\mathrm{d}\Theta_t(\xi) \\ &= \sum_{i\in t^*_1}\Theta_t(i)\int f(t[i],\widetilde{\xi})\,\mathrm{d}\Theta_{t[i]}(\widetilde{\xi}),\end{aligned}$$

by the flow rule property (2.4). Hence, if we set for t in $\operatorname{dom}\Theta$,

$$g(t) = \int f(t,\xi)\,\mathrm{d}\Theta_t(\xi),$$

the fact that μ is Θ-invariant yields

$$\int\left(\int f\circ \mathrm{S}(t,\xi)\,\mathrm{d}\Theta_t(\xi)\right)\mathrm{d}\mu(t) = \int \Theta g(t)\,\mathrm{d}\mu(t) = \int g(t)\,\mathrm{d}\mu(t). \qquad \square$$

Definition 2.3. Let h be a Borel non-negative or integrable function on $\mathscr{T}_{\mathrm{m,r}}$. One says that h is $(S,\mu\ltimes\Theta)$-invariant, when $h\circ\mathrm{S} = h$, $\mu\ltimes\Theta$-almost surely. A Borel set A of $\mathscr{T}_{\mathrm{m,r}}$ is called $(S,\mu\ltimes\Theta)$-invariant when its indicator function $\mathbf{1}_A$ is.

The family of $(S,\mu\ltimes\Theta)$-invariant sets is a σ-algebra. We denote it by $\mathrm{Inv}(S,\mu\ltimes\Theta)$. In the case where μ is Θ-invariant and Θ is positive, it does not contain much more information than the σ-algebra $\mathrm{Inv}(\mu,\Theta)$ that we have seen before:

Proposition 2.13. *Let Θ be a positive flow rule and μ a Θ-invariant probability. A Borel, non-negative or integrable function h on $\mathscr{T}_{m,r}$ is $(S,\mu\ltimes\Theta)$-invariant if and only if there exists a (μ,Θ)-*harmonic *function f on $\mathscr{T}_m$ such that*

$$h(t,\xi) = f(t), \quad \textit{for } \mu\ltimes\Theta\textit{-almost every } (t,\xi).$$

In particular, a Borel set A is $\mu\ltimes\Theta$-invariant if and only if there exists a set $\widetilde{A}$ in $\mathrm{Inv}(\mu,\Theta)$ *such that,*

$$(t,\xi)\in A \iff t\in\widetilde{A}, \quad \mu\ltimes\Theta\text{-}a.s. \tag{2.6}$$

Proof. Denoting, for $k\geq 0$, by S^k the k-th iterate of S, we first notice that

$$\mathbf{1}_{\{h\circ\mathrm{S}=h\}}\circ\mathrm{S} = \mathbf{1}_{\{h\circ\mathrm{S}^2=h\circ\mathrm{S}\}}.$$

Hence, by the fact that $\mu\ltimes\Theta$ is S-invariant, we also have

$$h = h\circ\mathrm{S}^k, \quad \forall k\geq 0, \quad \mu\ltimes\Theta\text{-a.s.} \tag{2.7}$$

For a fixed infinite marked tree $t\in\operatorname{dom}\Theta$ and $n\geq 0$, let $\mathcal{G}_n$ be the σ-algebra on ∂t generated by all the cylinders $[x]_t$, for $x\in t_n^*$. Then we have $\mathcal{G}_0\subset\mathcal{G}_1\subset\mathcal{G}_2\subset\cdots$, and the σ-algebra generated by the union $\bigcup_{n\geq 0}\mathcal{G}_n$ is the Borel σ-algebra of ∂t. Under some probability $\mathbb{P}$ with associated expectation $\mathbb{E}$, let Ξ be a random ray on t of distribution Θ_t. Then we have for all $n\geq 0$,

$$\mathbb{E}[h(t,\Xi)\,|\,\mathcal{G}_n] = \mathbb{E}[h(t,\Xi)\,|\,\Xi_n] = \sum_{x\in t_n^*}\mathbb{E}[h(t,\Xi)\,|\,\Xi_n = x]\mathbf{1}_{\{\Xi_n=x\}}.$$

By the flow rule property (eq. (2.4)), for $x\in t_n^*$,

$$\mathbb{E}[h(t,\Xi)\,|\,\Xi_n = x] = \frac{\int\mathbf{1}_{\{x\prec\xi\}}h(t,\xi)\Theta_t(\mathrm{d}\xi)}{\Theta_t(x)} = \int h(t,x\widetilde{\xi})\Theta_{t[x]}(\mathrm{d}\widetilde{\xi}),$$

showing that

$$\mathbb{E}[h(t,\Xi)\,|\,\mathcal{G}_n] = \int h(t,\Xi_n\widetilde{\xi})\Theta_{t[\Xi_n]}(\mathrm{d}\widetilde{\xi}).$$

Hence, by the regular martingale convergence theorem, for Θ_t-almost every ray ξ,

$$h(t,\xi) = \lim_{n\to\infty}\int h(t,\xi_n\widetilde{\xi})\Theta_{t[\xi_n]}(\mathrm{d}\widetilde{\xi}). \tag{2.8}$$

Observe that, by the flow rule property, if a random ray Ξ in t is distributed according to Θ_t and, conditionally on Ξ_n, $\widetilde{\Xi}$ is a random ray in $t[\Xi_n]$ distributed according to $\Theta_{t[\Xi_n]}$, then the ray $\Xi_n\widetilde{\Xi}$ is distributed according to Θ_t. Thus, if t is such that (2.7) holds, we have, for any $n \geq 0$, for Θ_t-almost every ray ξ, for $\Theta_{t[\xi_n]}$-almost every ray $\widetilde{\xi}$,

$$h(t,\xi_n\widetilde{\xi}) = h\circ \mathrm{S}^n(t,\xi_n\widetilde{\xi}) = h(t[\xi_n],\widetilde{\xi}). \tag{2.9}$$

Plugging this into the previous limit, we obtain, for $\mu \ltimes \Theta$-almost every (t,ξ),

$$h(t,\xi) = \lim_{n\to\infty}\int h(t[\xi_n],\widetilde{\xi})\Theta_{t[\xi_n]}(\mathrm{d}\widetilde{\xi}).$$

Now define when it makes sense (it does on a set of full μ-measure), for $t \in \mathscr{T}^*_{\mathrm{m}}$,

$$f(t) = \int h(t,\xi)\,\mathrm{d}\Theta_t(\xi).$$

Then, we claim that f is (μ,Θ)-harmonic. Indeed, for μ-almost every $t \in \mathscr{T}^*_{\mathrm{m}}$,

$$\begin{aligned} f(t) &= \int h(t,\xi)\,\mathrm{d}\Theta_t(\xi) = \int h\circ \mathrm{S}(t,\xi)\Theta_t(\mathrm{d}\xi) \\ &= \sum_{i\in t^*_1}\int h(t[i],i^{-1}\xi)\mathbf{1}_{\{\xi_1=i\}}\Theta_t(\mathrm{d}\xi) \\ &= \sum_{i\in t^*_1}\Theta_t(i)\int h(t[i],\widetilde{\xi})\Theta_{t[i]}(\mathrm{d}\widetilde{\xi}) = \Theta f(t). \end{aligned}$$

Getting back to (2.8), and using Proposition 2.11, we finally obtain, for μ-almost every tree t, for Θ_t-almost every ray ξ,

$$h(t,\xi) = \lim_{n\to\infty} f(t[\xi_n]) = \lim_{n\to\infty} f(t) = f(t). \qquad \square$$

Remark 2.2. It is clearly possible to write versions of Proposition 2.11 and Proposition 2.13 without the assumption that the flow rule is positive. This would however complicate the statements of the propositions and all the applications we can think of involve positive flow rules.

Corollary 2.14. *Let Θ be a positive flow rule and μ be a Θ-invariant probability measure on $\mathscr{T}^*_m$. Then the measure-preserving system $(\mathscr{T}^*_{m,r},\mathrm{S},\mu\ltimes\Theta)$ is ergodic if and only if, for all Borel subsets A of $\mathscr{T}^*_m$,*

$$\mu(A^{\mathrm{o}}\cup(A^{\mathrm{c}})^{\mathrm{o}}) = 1 \implies \mu(A)\in\{0;1\}. \tag{2.10}$$

Proof. Let $B \in \mathrm{Inv}(\mathrm{S}, \mu \ltimes \Theta)$. By the previous proposition, there exists a set $\widetilde{B}$ in $\mathrm{Inv}(\mu, \Theta)$ such that $\mu \ltimes \Theta(B) = \mu(\widetilde{B})$. By Proposition 2.11, $\mu(\widetilde{B}^{\mathrm{o}} \cup (\widetilde{B}^{\mathrm{c}})^{\mathrm{o}}) = 1$. Hence, if μ satisfies the above condition, the system is ergodic.

In the other direction, if $A \subset \mathscr{T}_{\mathrm{m}}^*$ is such that $\mu(A^{\mathrm{o}} \cup (A^{\mathrm{c}})^{\mathrm{o}}) = 1$ and $0 < \mu(A) < 1$, then we set

$$B = \{(t, \xi) : t \in A,\ \xi \in \partial t\},$$

and by the previous proposition, B is in $\mathrm{Inv}(\mathrm{S}, \mu \ltimes \Theta)$ but has measure $\mu \ltimes \Theta(B) = \mu(A) \in (0, 1)$. □

Thus we shall say that a Borel probability measure μ on $\mathscr{T}_{\mathrm{m}}^*$ is Θ-*ergodic* when it is Θ-invariant and satisfies (2.10) (this condition says that any Borel set A such that A and its complement A^{c} are both μ-inherited must have measure 0 or 1).

Corollary 2.15. *Let Θ be a positive flow rule and μ a Θ-ergodic probability measure. Then, for μ-almost every $t \in \mathscr{T}_m^*$, for Θ_t-almost every $\xi \in \partial t$,*

$$\lim_{n\to\infty} \frac{-1}{n} \log(\Theta_t(\xi_n)) = \int -\log\Theta_t(\xi_1)\, \mathrm{d}(\mu \ltimes \Theta)(t, \xi).$$

Proof. Let $t \in \operatorname{dom}\Theta$ and $\xi \in \partial t$. For $n \geq 1$, we have, by the flow rule property,

$$\Theta_t(\xi_n) = \prod_{i=0}^{n-1} \frac{\Theta_t(\xi_{i+1})}{\Theta_t(\xi_i)} = \prod_{i=0}^{n-1} \Theta_{t[\xi_i]}(\xi_i^{-1}\xi_{i+1}) = \prod_{i=0}^{n-1} \Theta_{t[\xi_i]}(\xi^{(i+1)}),$$

where $\xi^{(1)}$, $\xi^{(2)}$, ... is the sequence of letters of the infinite word ξ. Set

$$f(t, \xi) = -\log(\Theta_t(\xi^{(1)})).$$

We have for any $i \geq 0$,

$$f \circ \mathrm{S}^i(t, \xi) = -\log \Theta_{t[\xi_i]}((\xi_i^{-1}\xi)^{(1)}) = -\log \Theta_{t[\xi_i]}(\xi^{(i+1)}),$$

so that, for any $n \geq 1$,

$$\frac{-1}{n} \log \Theta_t(\xi_n) = \frac{1}{n} \sum_{i=0}^{n-1} f \circ \mathrm{S}^i(t, \xi),$$

and the pointwise ergodic theorem finishes the proof. □

2.6 Marked Galton-Watson trees and flow rules

Under some probability $\mathbb{P}$, let (N, M) be a random variable with values in $\mathbb{N}^* \times \mathsf{Marks}$ such that $\mathbb{P}(N = 1) < 1$. We build a marked Galton-Watson tree such that the joint reproduction and mark law is the law of (N, M) in much the same way as in the first section of this chapter. First we consider the set $\Omega = (\mathbb{N} \times \mathsf{Marks})^{\mathcal{U}}$ endowed with its product σ-algebra $\mathcal{F}$. For an element $\omega \in \Omega$, we write $\omega = (\nu, \mathsf{mk})$, with $\nu : \mathcal{U} \to \mathbb{N}$ and

$\mathsf{mk} : \mathcal{U} \to \mathsf{Marks}$ We then define the probability measure $\mathbf{P}$ on the measurable space $(\Omega, \mathcal{F})$ by:

$$\mathbf{P}\{\omega \in \Omega : \nu(x_1) = n_1,\ \mathsf{mk}(x_1) \in A_1,\ \ldots,\ \nu(x_k) = n_k,\ \mathsf{mk}(x_k) \in A_k\}$$
$$= \prod_{1 \le i \le k} \mathbb{P}(N = n_i, M \in A_i),$$

for all $k \ge 0$, all distinct $x_1, \ldots, x_k$ in $\mathcal{U}$, all $n_1, \ldots, n_k$ in $\mathbb{N}$ and all borel sets A_1, ..., A_k of Marks. Now $T(\omega)$ is defined as the Galton-Watson tree associated to the function ν, together with the restriction to the vertices of T of the mark function mk. We call the random marked tree T a marked Galton-Watson tree, and denote its law again by $\mathbf{GW}$ in order not to add yet another notation. The random marked trees T^x, for $x \in \mathcal{U}$ are defined in the same way as in Section 2.1. Of course, if we forget the marks, we obtain a Galton-Watson tree as in the first section whose reproduction law is the law of N and will again be denoted by $\mathbf{p} = (p_0, p_1, \dots)$. The branching property is still valid in this setting of marked trees. For instance we have:

Proposition 2.16. *Let k be a non-negative integer, A a Borel set of* Marks*, and B_1, B_2, …, B_k Borel sets of $\mathscr{T}_m$. Then,*

$$\mathbf{P}(\nu_T(\varnothing) = k,\, \mathsf{mk}_T(\varnothing) \in A,\, T[1] \in B_1, \ldots, T[k] \in B_k)$$
$$= \mathbb{P}(N = k, M \in A) \prod_{i=1}^{k} \mathbf{GW}(B_i).$$

As before, we denote by m the mean of $\mathbf{p}$ and assume from now on that $m > 1$, so that T is infinite with positive probability. We again denote by $\mathbf{GW}^*$ the distribution of T on the event of non-extinction. We still have the following crucial 0-1 law.

Proposition 2.17. *Any $\mathbf{GW}^*$-inherited subset of $\mathscr{T}_m$ has $\mathbf{GW}^*$-measure 0 or 1. Moreover, for any Borel subset A of $\mathscr{T}_m^*$, $\mathbf{GW}^*(A) = 1 \iff \mathbf{GW}^*(A^{\mathrm{o}}) = 1$.*

Proof. By definition, if A is $\mathbf{GW}^*$-inherited, then $\mathbf{GW}^*(A) = \mathbf{GW}^*(A^{\mathrm{o}})$. Proceed as in the proof of Lemma 2.9 to show that $\mathbf{GW}^*(A^{\mathrm{o}}) \in \{0, 1\}$.

Now assume that $\mathbf{GW}^*(A) = 1$. Then, $\mathbf{P}(T \in \mathscr{T}_{\mathrm{m}}^* \cap A^{\mathrm{c}}) = 0$ and by the union bound and the branching property,

$$\mathbf{P}(\exists x \in T,\ T[x] \in \mathscr{T}_{\mathrm{m}}^* \cap A^{\mathrm{c}}) \le \sum_{x \in \mathcal{U}} \mathbf{P}(x \in T)\mathbf{P}(T \in \mathscr{T}_{\mathrm{m}}^* \cap A^{\mathrm{c}}) = 0. \qquad \square$$

We are now ready to prove the central theorem of the ergodic theory on marked Galton-Watson trees.

Theorem 2.18. *Let Θ be a flow rule on marked trees. Let μ be a Θ-invariant probability. If μ is absolutely continuous with respect to $\mathbf{GW}^*$ (notation: $\mu \ll \mathbf{GW}^*$), then also $\mathbf{GW}^* \ll \mu$ and the measure-preserving system $(\mathscr{T}_{m,r}, \mathrm{S}, \mu \ltimes \Theta)$ is ergodic. Moreover, such a measure μ, if it exists, is unique.*

Proof. First we claim that if $A \in \mathrm{Inv}(\mu, \Theta)$, then $\mathbf{GW}^*(A)$ is 0 or 1. Indeed, by Proposition 2.11, this entails that $\mu(A^{\mathrm{o}} \cup (A^{\mathrm{c}})^{\mathrm{o}}) = 1$. Since $\mu \ll \mathbf{GW}^*$, necessarily $\mathbf{GW}^*(A^{\mathrm{o}} \cup (A^{\mathrm{c}})^{\mathrm{o}}) > 0$. The rather subtle point is that the set $A^{\mathrm{o}} \cup (A^{\mathrm{c}})^{\mathrm{o}}$ is inherited by the last point of Lemma 2.10, therefore by Proposition 2.17 we have $\mathbf{GW}^*(A^{\mathrm{o}} \cup (A^{\mathrm{c}})^{\mathrm{o}}) = 1$. Hence,

$$\mathbf{GW}^*(A) = \mathbf{GW}^*(A \cap (A^{\mathrm{o}} \cup (A^{\mathrm{c}})^{\mathrm{o}})) = \mathbf{GW}^*(A^{\mathrm{o}}),$$

meaning that A is $\mathbf{GW}^*$-inherited, hence by a second use of Proposition 2.17, $\mathbf{GW}^*(A) \in \{0, 1\}$.

Now, if N is a Borel subset of $\mathscr{T}_{\mathrm{m}}^*$ such that $\mu(N) = 0$, then $\mu \ll \mathbf{GW}^*$ implies that $\mathbf{GW}^*(N) < 1$. But since N is in $\mathrm{Inv}(\mu, \Theta)$, this means that $\mathbf{GW}^*(N) = 0$, which proves that $\mathbf{GW}^* \ll \mu$.

To prove the ergodicity, by Corollary 2.14, we only need to check that any set A in $\mathrm{Inv}(\mu, \Theta)$ has μ-measure 0 or 1. But since it has $\mathbf{GW}^*$-measure 0 or 1 and μ is absolutely continuous with respect to $\mathbf{GW}^*$, it is now established.

Finally[1], assume that a probability measure $\mu' \ll \mathbf{GW}^*$ is Θ-invariant. Let A be a Borel set of $\mathscr{T}_{\mathrm{m}}$. By the pointwise ergodic theorem and equivalence to $\mathbf{GW}^*$, we have for $\mathbf{GW}^*$-almost every t, for Θ_t-almost every ξ,

$$\frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_A \left(t[\xi_k] \right) \xrightarrow[n \to \infty]{} \mu(A).$$

Since the same holds for μ', we must have $\mu(A) = \mu'(A)$. □

Corollary 2.19. *Under the same assumptions as the previous theorem, for* $\mathbf{GW}^*$*-every tree* t*, if we endow* ∂t *with the metric* $\mathrm{d}_{\mathcal{U}_\infty}$*, the flow* Θ_t *is exact-dimensional of dimension*

$$\dim \Theta_t = \int -\log \Theta_t(\xi_1) \, \mathrm{d}(\mu \rtimes \Theta)(t, \xi).$$

Proof. By the previous theorem, μ is Θ-ergodic and equivalent to $\mathbf{GW}^*$. Hence, by Corollary 2.15, for $\mathbf{GW}^*$-almost every t, for Θ_t-almost every $\xi \in \partial t$,

$$\lim_{n \to \infty} \frac{-1}{n} \log \Theta_t(\xi_n) = \int -\log \Theta_t(\xi_1) \, \mathrm{d}(\mu \rtimes \Theta)(t, \xi) =: \alpha.$$

By Lemma 2.5, we may assume that t has no isolated rays. Therefore, by remark 1.4, we have for Θ_t-almost every ξ,

$$\liminf_{n \to \infty} \frac{\log \Theta_t(\xi_n)}{\log \operatorname{diam}[\xi_n]_t} = \liminf_{n \to \infty} \frac{-1}{n} \log \Theta_t(\xi_n) = \alpha.$$

On the other hand, the inequality $\operatorname{diam}[\xi_n]_t \leq e^{-n}$ yields for Θ_t-almost every ξ,

$$\limsup_{n \to \infty} \frac{\log \Theta_t(\xi_n)}{\log \operatorname{diam}[\xi_n]_t} \leq \limsup_{n \to \infty} \frac{-1}{n} \log \Theta_t(\xi_n) = \alpha.$$ □

1. The last point is reminiscent of the classical fact that two distinct ergodic measures are mutually singular, but does not quite fit in this box since it is the measure $\mu \ltimes \Theta$ which is ergodic. However, we prove it with the same classical arguments.

We now examine the situation where two flow rules cohabit on the same Galton-Watson tree.

Proposition 2.20. *Let Θ and Θ' be two $\mathbf{GW}^*$-flow rules. Then, $\mathbf{GW}^*\{t \in \mathscr{T}_m : \Theta_t = \Theta'_t\} \in \{0;1\}$. Furthermore, this probability is 1 if and only if*

$$\mathbf{P}^*(\forall i \in T_1^*,\ \Theta_T(i) = \Theta'_T(i)) = 1.$$

Finally, if there exist a Θ-invariant probability $\mu \ll \mathbf{GW}^*$ and a Θ'-invariant probability $\mu' \ll \mathbf{GW}^*$, then, either for $\mathbf{GW}^*$-almost every t, $\Theta_t \perp \Theta'_t$, or $\Theta_t = \Theta'_t$ for $\mathbf{GW}^*$-almost every t.

Proof. First consider the Borel subset of $\mathscr{T}_\mathrm{m}^*$:

$$A = \{t \in \operatorname{dom}\Theta \cap \operatorname{dom}\Theta' : \forall i \in t_1^*,\ \Theta_t(i) = \Theta'_t(i)\}.$$

By the flow rule property,

$$A^\mathrm{o} = \{t \in \operatorname{dom}\Theta \cap \operatorname{dom}\Theta' : \Theta_t = \Theta'_t\},$$

hence the first part of the proposition, by Proposition 2.17.

For the last part, we use the pointwise ergodic theorem. Let g be a non-negative measurable function. By Theorem 2.18, for $\mathbf{GW}^*$-almost every marked tree t, for Θ_t-almost every ray ξ,

$$\frac{1}{n}\sum_{k=0}^{n-1} g(\mathrm{S}^k(t,\xi)) \xrightarrow[n\to\infty]{} \int \left(\int g(t,\xi)\Theta_t(\mathrm{d}\xi)\right)\mu(\mathrm{d}t),$$

and the same is true when we replace Θ by Θ' and μ by μ'. Thus we are done if we can prove that, if Θ_t and Θ'_t are not almost surely equal, then for some non-negative measurable function g,

$$\int \left(\int g\,(t,\xi)\,\Theta_t(\mathrm{d}\xi)\right)\mu(\mathrm{d}t) \neq \int \left(\int g\,(t,\xi)\,\Theta'_t(\mathrm{d}\xi)\right)\mu'(\mathrm{d}t).$$

If this is not the case, then the probability measures $\mu \ltimes \Theta$ and $\mu' \ltimes \Theta'$ on $\mathscr{T}_{\mathrm{m,r}}$ are equal. Projecting on the space $\mathscr{T}_\mathrm{m}^*$, we obtain that necessarily $\mu = \mu'$ and by equivalence with $\mathbf{GW}^*$, that $\mathbf{GW}^* \ltimes \Theta = \mathbf{GW}^* \ltimes \Theta'$.

This entails that for any x in $\mathcal{U}$ and any non-negative measurable function h on $\mathscr{T}_\mathrm{m}^*$,

$$\int h(t)\mathbf{1}_{\{x\in t\}}\Big(\int \mathbf{1}_{\{x\prec\xi\}}\,\mathrm{d}\Theta_t(\xi)\Big)\,\mathrm{d}\mathbf{GW}^*(t) = \int h(t)\mathbf{1}_{\{x\in t\}}\Big(\int \mathbf{1}_{\{x\prec\xi\}}\,\mathrm{d}\Theta'_t(\xi)\Big)\,\mathrm{d}\mathbf{GW}^*(t),$$

thus for $\mathbf{GW}^*$-almost every marked tree t,

$$\mathbf{1}_{\{x\in t\}}\Theta_t(x) = \mathbf{1}_{\{x\in t'\}}\Theta'_t(x).$$

Since this holds for any x in $\mathcal{U}$ and $\mathcal{U}$ is countable, this implies that for $\mathbf{GW}^*$-almost every tree t, $\Theta_t = \Theta'_t$. $\square$

At this point, we go back to the construction in remark 2.1. That is, we consider a function $\phi : \mathscr{T}_{\mathrm{m}} \to [0,\infty]$ such that $\phi(t) = 0$ whenever t is finite. We let $A = \{t \in \mathscr{T}_{\mathrm{m}}^* : \phi(t) \in (0,\infty)\}$, and assume that $\mathbf{GW}^*(A) = 1$, so that, by Proposition 2.17, $\mathbf{GW}^*(A^{\mathrm{o}}) = 1$. Then we define the flow rule Θ of domain $\operatorname{dom}\Theta = A^{\mathrm{o}}$ by: for all $t \in \operatorname{dom}\Theta$, for all $x \in t^*$,

$$\Theta_t(x) = \prod_{\varnothing \prec y \preceq x} \frac{\phi(t[y])}{\sum_{z_* = y_*} \phi(t[z])}.$$

The first examples that come to mind are UNIF, with $\phi(t) = \limsup_{n\to\infty} Z_n(t)/c_n$, VIS, with $\phi(t) = \mathbf{1}_{\{t \text{ is infinite}\}}$ and HARM for the λ-biased random walk of bias $\lambda < m$ with $\phi(t) = \beta(t)$, the conductance of the tree as defined in Section 1.7. Notice that in general, ϕ may also depend on the marks of the marked tree.

First we prove a criterion for two flow rules built this way to be equal, with arguments similar to those in the proof of [43, Proposition 8.3].

Lemma 2.21. *Assume that Θ and Θ' are two $\mathbf{GW}^*$-flow rules constructed respectively from ϕ and ϕ' as above. Then, $\mathbf{P}^*(\Theta_T = \Theta'_T) = 1$ if and only if, there is a constant $C \in (0,\infty)$ such that $\mathbf{P}^*$-almost surely, $\phi(T) = C\phi'(T)$.*

Proof. The converse implication is obvious. So assume that

$$\mathbf{P}^*(\forall i \in T_1^*,\ \Theta_T(i) = \Theta'_T(i)) = 1.$$

We reason conditionally on the event A that $\nu_T(\varnothing) \geq 2$ and $T[1]$ and $T[2]$ are infinite. Let $\mathbf{P}_A$ be the conditional probability with respect to A. Under $\mathbf{P}_A$, $T[1]$ and $T[2]$ are independent and have distribution $\mathbf{GW}^*$. However, $\mathbf{P}_A$-almost surely,

$$\frac{\phi(T[1])}{\phi'(T[1])} = \frac{\sum_{j=1}^{\nu_T(\varnothing)} \phi(T[j])}{\sum_{j=1}^{\nu_T(\varnothing)} \phi'(T[j])} = \frac{\phi(T[2])}{\phi'(T[2])}.$$

Hence, by independence, the random variable $\phi(T[1])/\phi'(T[1])$ is degenerated, that is there exists $C > 0$ such that $\mathbf{P}_A$-almost surely $\phi(T[1]) = C\phi'(T[1])$. Since the law of $T[1]$ under $\mathbf{P}_A$ is the law of T under $\mathbf{P}^*$, the proof is complete. □

The next proposition gives the local dimension of a flow Θ'_T on a set of full Θ_T-measure.

Proposition 2.22. *Let Θ and Θ' be as in the previous lemma. Let Ξ be distributed according to Θ_T. Assume that there exists a probability measure $\mu \ll \mathbf{GW}^*$ which is Θ-invariant and that the random variable*

$$\log \phi'(T[\Xi_1]) - \log \phi'(T)$$

is bounded from below by a $(\mu \ltimes \Theta)$-integrable random variable then, for $\mathbf{GW}^*$-almost every tree t, for Θ_t-almost every ray ξ, we have

$$\frac{-1}{n} \log \Theta'_t(\xi_n) \xrightarrow[n\to\infty]{} \mathbf{E}_\mu^*\Big[\log \frac{\sum_{i=1}^{\nu_T(\varnothing)} \phi'(T[i])}{\phi'(T)}\Big],$$

and if $\mathbf{P}^*(\Theta_T = \Theta'_T) < 1$*, then this limit is almost surely greater than* $\dim \Theta_T$*, the exact dimension of* Θ_T *with respect to* $\mathrm{d}_{\mathcal{U}_\infty}$.

Proof. By the same arguments as in the proof of Corollary 2.15, we have for $\mathbf{GW}^*$-almost every tree t, for Θ_t-almost every ray ξ,

$$\frac{-1}{n}\log \Theta'_t(\xi_n) \xrightarrow[n\to\infty]{} \mathbf{E}^*_\mu\Big[\log \frac{1}{\Theta'_T(\Xi_1)}\Big],$$

Now it suffices to write

$$\Theta'_T(\Xi_1) = \frac{\phi'(T[\Xi_1])}{\phi'(T)} \frac{\phi'(T)}{\sum_{i=1}^{\nu_T(\varnothing)} \phi'(T[i])}$$

and to apply Lemma 2.24 to obtain the first part of the proposition. The final part of the proposition is simply an application of Gibbs' inequality as in Proposition 2.27. □

2.7 The limit uniform measure

In this section, we assume that $m > 1$ and that $\sum_{k\geq 1} p_k k \log k < \infty$. Recall that we want to prove that for $\mathbf{GW}^*$-almost every t, for UNIF_t-almost every $\xi \in \partial t$,

$$\lim_{n\to\infty} \frac{-1}{n} \log \mathsf{UNIF}_t(\xi_n) = \log m.$$

In light of what we have seen, we want to find a UNIF-invariant measure $\mu_{\mathsf{UNIF}} \ll \mathbf{GW}^*$. Not so surprisingly, this task is handled by the size-biased Galton-Watson tree.

Lemma 2.23. *Let* T *be a Galton-Watson tree such that* $m > 1$ *and* $\sum_{k\geq 1} p_k k \log k < \infty$*. Then, the Borel probability measure* μ_{UNIF} *defined by*

$$\int f(t)\mu_{\mathsf{UNIF}}(\mathrm{d}t) = (1-q)\mathbf{E}^*[W(T)f(T)],$$

for all measurable functions $f : \mathscr{T}^*_m \to \mathbb{R}_+$ *is* UNIF*-invariant.*

Proof. We need to show that for all measurable $f : \mathscr{T}^*_{\mathrm{m}} \to \mathbb{R}_+$,

$$\mathbf{E}[W(T)f(T(\Xi_1))] = \mathbf{E}[W(T)f(T)],$$

where Ξ is a random ray in ∂T of law UNIF_T. Recall that $W(T) = W(T)\mathbf{1}_{\{T \text{ is infinite}\}}$. Decomposing with respect to the values of $\nu_T(\varnothing)$ and Ξ_1, we have

$$\mathbf{E}[W(T)f(T[\Xi_1])] = \sum_{k\geq 1} \mathbf{E}\Big[\mathbf{1}_{\{\nu_T(\varnothing)=k,\ T\in\mathscr{T}^*_{\mathrm{m}}\}} \sum_{i=1}^{k} \mathbf{1}_{\{\Xi_1=i\}} W(T)f(T[i])\Big].$$

We may replace $\mathbf{1}_{\{\Xi_1=i\}}$ by its conditional expectation given T, which is

$$\mathbf{1}_{\{T[i] \text{ is infinite}\}} \frac{W(T[i])}{\sum_{j=1}^{\nu_T(\varnothing)} W(T[j])} = \frac{W(T[i])}{\sum_{j=1}^{\nu_T(\varnothing)} W(T[j])}.$$

Since $W(T) = \frac{1}{m}\sum_{i=1}^{\nu_T(\varnothing)} W(T[j])$, this gives

$$\begin{aligned}\mathbf{E}[W(T)f(T[\Xi_1])] &= \frac{1}{m}\sum_{k\geq 1}\sum_{i=1}^{k}\mathbf{E}\Big[\mathbf{1}_{\{\nu_T(\varnothing)=k,\ T\in\mathscr{T}_{\mathrm{m}}^*\}}W(T[i])f(T[i])\Big]\\ &= \frac{1}{m}\sum_{k\geq 1}k\mathbf{E}\Big[\mathbf{1}_{\{\nu_T(\varnothing)=k,\ T\in\mathscr{T}_{\mathrm{m}}^*\}}W(T[1])f(T[1])\Big],\end{aligned}$$

by symmetry. Now observe that for any $k\geq 1$,

$$\begin{aligned}\mathbf{1}_{\{\nu_T(\varnothing)=k,\ T\in\mathscr{T}_{\mathrm{m}}^*\}}W(T[1]) &= \mathbf{1}_{\{\nu_T(\varnothing)=k,\ T\in\mathscr{T}_{\mathrm{m}}^*,\ T[1]\in\mathscr{T}_{\mathrm{m}}^*\}}W(T[1]).\\ &= \mathbf{1}_{\{\nu_T(\varnothing)=k,\ T^1\in\mathscr{T}_{\mathrm{m}}^*\}}W(T^1).\end{aligned}$$

So we may use the branching property and obtain

$$\begin{aligned}\mathbf{E}[W(T)f(T[\Xi_1])] &= \frac{1}{m}\sum_{k\geq 1}kp_k\mathbf{E}\Big[\mathbf{1}_{\{T^1\in\mathscr{T}_{\mathrm{m}}^*\}}W(T^1)f(T^1)\Big],\\ &= \mathbf{E}[W(T)f(T)].\end{aligned}$$ □

By Theorem 2.18, μ_{UNIF} is also UNIF-ergodic. By Corollary 2.19, we already know that, for $\mathbf{GW}^*$-almost every t, for UNIF_t-almost every ξ,

$$\begin{aligned}\lim_{n\to\infty}\frac{-1}{n}\log\mathsf{UNIF}_t(\xi_n) &= -\int\log\mathsf{UNIF}_t(\xi_1)\,\mathrm{d}(\mu\ltimes\Theta)(t,\xi)\\ &= \int(\log m+\log W(t)-\log W(t[\xi_1])\,\mathrm{d}(\mu\ltimes\Theta)(t,\xi).\end{aligned}$$

We are done with the proof of Theorem 2.8 if we can show that, by stationarity,

$$\int(\log W(t)-\log W(t[\xi_1]))\,\mathrm{d}(\mu_{\mathsf{UNIF}}\ltimes\mathsf{UNIF})(t,\xi)=0.$$

This is the purpose of [46, Lemma 17.20], whose statement and proof are reproduced here for the reader's convenience.

Lemma 2.24. *Let $(\mathscr{X},\mathcal{A},\mathrm{S},\mu)$ be a measure-preserving system and g a measurable function from $\mathscr{X}$ to $\mathbb{R}$. Then, $\int(g-g\circ\mathrm{S})^+\,\mathrm{d}\mu=\int(g-g\circ\mathrm{S})^-\,\mathrm{d}\mu$. In particular, if $g-g\circ\mathrm{S}$ is bounded from below by an integrable function, then both these integrals are finite and $\int(g-g\circ\mathrm{S})\,\mathrm{d}\mu=0$.*

Proof. When g is integrable, the result is a direct consequence of the assumption that the system is measure-preserving.

To prove the general case, we remark that if $f:\mathbb{R}\to\mathbb{R}$ is a contraction, then, for all x and y in $\mathbb{R}$,

$$(f(x)-f(y))^+ + (f(x)-f(y))^- \leq (x-y)^+ + (x-y)^-,$$

and if additionally f is non-decreasing, the positive (resp. negative) parts vanish at the same time in the two sides of the inequality, thus

$$(f(x)-f(y))^+ \leq (x-y)^+ \quad \text{and} \quad (f(x)-f(y))^- \leq (x-y)^-.$$

Now, for $n \in \mathbb{N}^*$ the function $F_n : \mathbb{R} \to \mathbb{R}$ defined by

$$F_n(x) = \begin{cases} n & \text{if } x > n; \\ -n & \text{if } x < -n; \\ x & \text{otherwise,} \end{cases}$$

is a non-decreasing contraction. Since the function $g_n = F_n \circ g$ is bounded, we have for all $n \geq 1$,

$$\int (g_n - g_n \circ \mathrm{S})^+ \,\mathrm{d}\mu = \int (g_n - g_n \circ \mathrm{S})^- \,\mathrm{d}\mu.$$

Since $F_n = F_n \circ F_{n+1}$ the previous inequalities yield that the sequences $(g_n - g_n \circ \mathrm{S})^+$ and $(g_n - g_n \circ \mathrm{S})^-$ are non-decreasing and we can conclude by the monotone convergence theorem. □

Going back to the proof of Theorem 2.8, we set for $t \in \operatorname{dom} \mathsf{UNIF}$ and $\xi \in \partial t$, $g(t,\xi) = \log(W(t))$. Then,

$$\begin{aligned} g(t,\xi) - g \circ \mathrm{S}(t,\xi) &= \log W(t) - \log W(t[\xi_1]) \\ &\geq \log \frac{W(t[\xi_1])}{m} - \log W(t[\xi_1]) \geq -\log m, \end{aligned} \tag{2.11}$$

and the proof of Theorem 2.8 is now complete.

A lot more could be said about UNIF. Under more assumptions, it can be shown (see [13] and [61]) that there exists a constant $c \in (0,\infty)$ such that for $\mathbf{GW}^*$-almost every T, the Hausdorff measure $\mathscr{H}^\phi$ on ∂T equals $cW(T)\mathsf{UNIF}_T$, where $\phi(x) = m^{-|x|}\alpha_{|x|}$ and (α_n) are the quantiles of W defined by $\mathbf{P}(W > \alpha_n) = 1/n$. The article [60] contains a result in the other direction with the proportionality to a non-trivial packing measure under the assumption that $p_0 = p_1 = 0$.

2.8 Non-uniform flow rules

We first turn to the flow rule VIS.

Proposition 2.25. *The probability measure $\mathbf{GW}^*$ is VIS-invariant (hence also VIS-ergodic) and for $\mathbf{GW}^*$-almost every tree t, for VIS_t-almost every ξ,*

$$\lim_{n\to\infty} \frac{-1}{n} \log \mathsf{VIS}_t(\xi_n) = \mathbf{E}^*[\log(\nu_T^*(\varnothing))].$$

Proof. Let Ξ be a random ray on T, distributed according to VIS_T and $f : \mathscr{T}^*_{\mathrm{m}} \to \mathbb{R}_+$ be a measurable function. Then by definition of the flow rule VIS,

$$\begin{aligned}\mathbf{E}[f(T[\Xi_1])\mathbf{1}_{\{T\in\mathscr{T}^*_{\mathrm{m}}\}}] &= \sum_{k\geq 1}\mathbf{E}\Big[\mathbf{1}_{\{\nu_T(\varnothing)=k,\ T\in\mathscr{T}^*_{\mathrm{m}}\}}\sum_{i=1}^{k}\frac{\mathbf{1}_{\{T[i]\in\mathscr{T}^*_{\mathrm{m}}\}}}{\sum_{j=1}^{k}\mathbf{1}_{\{T[j]\in\mathscr{T}^*_{\mathrm{m}}\}}}f(T[i])\Big]\\ &= \sum_{k\geq 1}k\mathbf{E}\Big[\mathbf{1}_{\{\nu_T(\varnothing)=k,\ T\in\mathscr{T}^*_{\mathrm{m}},\ T[1]\in\mathscr{T}^*_{\mathrm{m}}\}}\frac{1}{1+\sum_{j=2}^{k}\mathbf{1}_{\{T[j]\in\mathscr{T}^*_{\mathrm{m}}\}}}f(T[1])\Big],\end{aligned}$$

by symmetry. Now we first observe that for any $k \geq 1$,

$$\mathbf{1}_{\{\nu_T(\varnothing)=k,\ T\in\mathscr{T}^*_{\mathrm{m}},\ T[1]\in\mathscr{T}^*_{\mathrm{m}}\}} = \mathbf{1}_{\{\nu_T(\varnothing)=k,\ T^1\in\mathscr{T}^*_{\mathrm{m}}\}},$$

so by the branching property,

$$\mathbf{E}[f(T[\Xi_1])\mathbf{1}_{\{T\in\mathscr{T}^*_{\mathrm{m}}\}}] = \sum_{k\geq 1}p_k k\mathbf{E}\Big[\mathbf{1}_{\{T^1\in\mathscr{T}^*_{\mathrm{m}}\}}\frac{1}{1+\sum_{j=2}^{k}\mathbf{1}_{\{T^j\in\mathscr{T}^*_{\mathrm{m}}\}}}f(T^1)\Big]$$

Then we see that for any $k \geq 1$, the random variable

$$\frac{1}{1+\sum_{j=2}^{k}\mathbf{1}_{\{T^j\in\mathscr{T}^*_{\mathrm{m}}\}}}$$

is independent of T^1, thus may be replaced by its expectation which equals

$$\frac{1}{k}\frac{1-q^k}{1-q},$$

because the random variable $\sum_{j=2}^{k}\mathbf{1}_{\{T[j]\in\mathscr{T}^*_{\mathrm{m}}\}}$ has a binomial distribution of parameters $k-1$ and $(1-q)$. So finally, we obtain

$$\begin{aligned}\mathbf{E}[f(T[\Xi_1])\mathbf{1}_{\{T\in\mathscr{T}^*_{\mathrm{m}}\}}] &= \frac{1}{1-q}\mathbf{E}[f(T)\mathbf{1}_{\{T\in\mathscr{T}^*_{\mathrm{m}}\}}](\mathsf{g}(1)-\mathsf{g}(0)-(\mathsf{g}(q)-\mathsf{g}(0))\\ &= \mathbf{E}[f(T)\mathbf{1}_{\{T\in\mathscr{T}^*_{\mathrm{m}}\}}].\end{aligned}$$

Now, by Corollary 2.15, we have for $\mathbf{GW}^*$-almost every t, for VIS_t-almost every ξ,

$$\lim_{n\to\infty}\frac{-1}{n}\log\mathsf{VIS}_t(\xi_n) = \mathbf{E}^*[-\log(\mathsf{VIS}_T(\Xi_1))] = \mathbf{E}^*[\log\nu^*_T(\varnothing)].$$ □

Now we remark that, by Jensen's inequality, as long as the reproduction law is not degenerated and $\sum_{k\geq 1}p_k\log k < \infty$, $\mathbf{E}^*[\log\nu^*_T(\varnothing)] < \log\mathbf{E}^*[\nu^*_T(\varnothing)] = \log m$ (the fact that the law of ν^*_T under $\mathbf{P}^*$ also has mean m is classical and very well explained in [1]). Hence we have $\dim\mathsf{VIS}_T < \dim_{\mathrm{H}}\partial T$ almost surely and the dimension drop phenomenon occurs for VIS_T, although it has full support.

Now we assume that $m < \infty$. We will establish that it is in fact a general fact for any flow rule Θ such that there exists a Θ-invariant probability $\mu \ll \mathbf{GW}^*$. First we need Gibbs' inequality.

Lemma 2.26 (Gibbs' inequality)**.** *Let $n \in \mathbb{N}^*$. Assume that the finite sequences p_1, p_2, $\dots$, p_n and q_1, q_2, $\dots$, q_n of positive real numbers both add up to* 1. *Then,*

$$\sum_{i=1}^{n} p_i(-\log)(p_i) \leq \sum_{i=1}^{n} p_i(-\log)(q_i),$$

with equality if and only if $p_i = q_i$ for all $1 \leq i \leq n$.

Proof. Under some probability $\mathbb{P}$, let X be a random variable such that, for any $1 \leq i \leq n$, $\mathbb{P}(X = i) = p_i$. By Jensen's inequality and strict concavity of log,

$$\sum_{1 \leq i \leq n} p_i(-\log \frac{q_i}{p_i}) = \mathbb{E}[-\log(q_X/p_X)] \geq -\log \mathbb{E}[q_X/p_X] = -\log(\sum_{i=1}^{n} q_i) = 0,$$

giving our inequality. The equality may only happen when the random variable q_X/p_X is degenerated. $\square$

Proposition 2.27. *Assume that $1 < m < \infty$. Let Θ be a flow rule such that there exists a Θ-invariant probability $\mu \ll \mathbf{GW}^*$. Then, if $\mathbf{P}^*(\Theta_T = \mathsf{UNIF}_T) < 1$, we have for $\mathbf{GW}^*$-almost every t,*

$$\dim \Theta_t < \log m.$$

Proof. By Proposition 2.20, we have

$$\mathbf{P}^*(\forall i \in T_1^*,\ \Theta_T(i) = \mathsf{UNIF}_T(i)) < 1.$$

Let Ξ be a random ray of distribution Θ_T on ∂T. We denote by κ the density of μ with respect to $\mathbf{GW}^*$. Then, almost surely

$$\begin{aligned}
\dim \Theta_T &= \mathbf{E}^*[\kappa(T)(-\log \Theta_T(\Xi_1))] \\
&= \mathbf{E}^*\Big[\kappa(T) \sum_{i \in T_1^*} \Theta_T(i)(-\log \Theta_T(i))\Big] \\
&< \mathbf{E}^*\Big[\kappa(T) \sum_{i \in T_1^*} \Theta_T(i)(-\log \mathsf{UNIF}_T(i))\Big],
\end{aligned}$$

by Gibbs' inequality. The upper bound is

$$\mathbf{E}^*[\kappa(T)(-\log \mathsf{UNIF}_T(\Xi_1))] = \mathbf{E}^*[\kappa(T)(\log m + \log W(T) - \log W(T[\Xi_1]))],$$

which equals $\log m$ by the same arguments as those used in the end of Section 2.7. $\square$

2.9 Invariant measures for a class of flow rules

We have seen that much can be said about a $\mathbf{GW}^*$-flow rule Θ, as long as we can prove that there exists a Θ-invariant probability $\mu \ll \mathbf{GW}^*$. Unfortunately, we know no necessary and sufficient condition for such an invariant measure to exist. We exhibit here a class of $\mathbf{GW}^*$-flow rules for which this problem is solved. This will have practical applications in the following two chapters. The content of this section is largely taken from [55, Section 3], with the small improvement that we do not assume that $p_0 = 0$.

We work in the context of remark 2.1 with the flow rule Θ associated to a function $\phi : \mathscr{T}_{\mathrm{m}} \to [0, \infty]$ and $\operatorname{dom}\Theta = \{t \in \mathscr{T}^*_{\mathrm{m}} : \phi(t) \in (0,\infty)\}^{\mathrm{o}}$. We assume that $\mathbf{GW}^* \operatorname{dom}\Theta$ equals 1.

We now make some assumptions on the model. First we assume that *the marks and the number of children are independent*, that is, with the notations of the beginning of Section 2.6, we assume that N and M are independent. Second we assume that the marks and the function ϕ have their values in the same sub-interval J of $(0, \infty)$ and that there exists a measurable function h from $J \times J$ to J such that, $\mathbf{P}^*$-almost surely,

$$\phi(T) = h\Big(\mathsf{mk}_T(\varnothing), \sum_{i=1}^{\nu_T(\varnothing)} \phi(T[i])\Big).$$

In words, ϕ is an observation on the tree T which can be recovered from the mark of the root and the sum of such observations on the subtrees $T[1], \ldots, T[\nu_T(\varnothing)]$. It will be convenient to extend the definition of h by saying that it is 0 whenever one of its argument is.

We now make algebraic assumptions on the function h:

symmetry $\forall u, v \in J,\ h(u,v) = h(v,u)$;

associativity $\forall u, v, w \in J,\ h(h(u,v), w) = h(u, h(v,w))$;

position of summand $\forall u, v \in J,\ \forall a > 0,\ \dfrac{h(u+a, v)}{(u+a)v} = \dfrac{h(u, a+v)}{u(a+v)}$.

These assumptions, as well as the next theorem, are inspired by the proofs of [10, Proposition 25] and [34, Proposition 8]. Here are examples of functions satisfying these properties:

1. $J = (0,\infty)$ and $h(u,v) = \alpha uv$, for some $\alpha > 0$;
2. for $c > 0$, $J = (c, \infty)$ and $h(u,v) = \frac{uv}{u+v-c}$;
3. for $d \geq 0$, $J = (0,\infty)$ and $h(u,v) = \frac{uv}{u+v+d}$.

We treat the second case. By writing

$$h(u,v) = \left(u^{-1} + v^{-1} - cu^{-1}v^{-1}\right)^{-1} = \left(u^{-1}\left(1 - cv^{-1}\right) + v^{-1}\right)^{-1},$$

and noticing that $(1 - cv^{-1}) > 0$, we see that

$$h(u,v) > \left(c^{-1}\left(1 - cv^{-1}\right) + v^{-1}\right)^{-1} > c.$$

Symmetry is clear, so is the last property because $h(u,v)/(uv)$ only depends on the sum of u and v. Associativity follows from the following identity :

$$h(h(u,v),w) = \left(\left(u^{-1}+v^{-1}-cu^{-1}v^{-1}\right)+w^{-1}-c\left(u^{-1}+v^{-1}-cu^{-1}v^{-1}\right)w^{-1}\right)^{-1}$$
$$= \left(u^{-1}+v^{-1}+w^{-1}-c\left(u^{-1}v^{-1}+u^{-1}w^{-1}+v^{-1}w^{-1}\right)+c^2u^{-1}v^{-1}w^{-1}\right)^{-1}.$$

The fist example is satisfied by UNIF, with $\alpha = 1/m$ and the marks all equal to 1. Another application of this first example with random marks will be given in the next chapter. The second and third examples have applications to harmonic measures in the following two chapters.

We enlarge the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ so that we may consider random variables $(\widetilde{\nu}, \widetilde{T}^1, \widetilde{T}^2, ...)$ which are independent and independent of T, $\widetilde{\nu}$ having the same law as N and $\widetilde{T}^1$, $\widetilde{T}^2$, ... having law $\mathbf{GW}$. For u in J, define

$$\kappa(u) = \mathbf{E}\Big[h\Big(u, \sum\nolimits_{i=1}^{\widetilde{\nu}} \phi(\widetilde{T}^i)\Big)\Big].$$

Theorem 2.28. *Assume that $C \coloneqq \mathbf{E}^*[\kappa(\phi(T))] < \infty$. Then the probability measure μ with density $C^{-1}\kappa(\phi(T))$ with respect to $\mathbf{GW}^*$ is Θ-invariant.*

Before we start the proof, we adopt some notations to make it more digest. For short, we let $\nu = \nu_T(\varnothing)$ and write ϕ_1, ϕ_2, $\widetilde{\phi}_1$, ... instead of $\phi(T[1])$, $\phi(T[2])$, $\phi(\widetilde{T}^1)$, ... And we denote, for u and v in J, $u \odot v = h(u,v)$, with the extension $0 \odot u = u \odot 0 = 0$. Recall that, by assumption, $\mathbf{1}_{\{T \text{ is infinite}\}} = \mathbf{1}_{\{\phi(T)>0\}}$, almost surely.

Now let f be a measurable function from $\mathscr{T}_{\mathrm{m}}^*$ to $\mathbb{R}_+$ and Ξ a random ray distributed according to Θ_T. We need to show that

$$\mathbf{E}[\mathbf{1}_{\{T\in\mathscr{T}_{\mathrm{m}}^*\}} f(T[\Xi_1])\kappa(\phi(T))] = \mathbf{E}[\mathbf{1}_{\{T\in\mathscr{T}_{\mathrm{m}}^*\}} f(T)\kappa(\phi(T))].$$

By definition of κ, Tonelli's theorem yields

$$\mathbf{E}[\mathbf{1}_{\{T\in\mathscr{T}_{\mathrm{m}}^*\}} f(T[\Xi_1])\kappa(\phi(T))] = \sum_{k\geq 1}\mathbf{E}\Big[\mathbf{1}_{\{T\in\mathscr{T}_{\mathrm{m}}^*,\nu=k\}}\sum_{i=1}^k f(T[i])\frac{\phi_i}{\sum_{j=1}^k\phi_j}\kappa(\phi(T))\Big]$$
$$= \sum_{k\geq 1}\mathbf{E}\Big[\mathbf{1}_{\{T\in\mathscr{T}_{\mathrm{m}}^*,\ \nu=k\}}\sum_{i=1}^k f(T[i])\frac{\phi_i}{\sum_{j=1}^k\phi_j}\mathsf{mk}_T(\varnothing)\odot\Big(\sum_{i=1}^k\phi_i\Big)\odot\Big(\sum_{j=1}^{\widetilde{\nu}}\widetilde{\phi}_j\Big)\Big].$$

By symmetry, for all $k \geq 1$,

$$\mathbf{E}\Big[\mathbf{1}_{\{T\in\mathscr{T}_{\mathrm{m}}^*,\ \nu=k\}}\sum_{i=1}^k f(T^i)\frac{\phi_i}{\sum_{j=1}^k\phi_j}\mathsf{mk}_T(\varnothing)\odot\Big(\sum_{i=1}^k\phi_i\Big)\odot\Big(\sum_{j=1}^{\widetilde{\nu}}\widetilde{\phi}_j\Big)\Big]$$
$$= k\mathbf{E}\Big[\mathbf{1}_{\{T\in\mathscr{T}_{\mathrm{m}}^*,\ \nu=k,\ T^1\in\mathscr{T}_{\mathrm{m}}^*\}} f(T^1)\frac{\phi_1}{\sum_{j=1}^k\phi_j}\mathsf{mk}_T(\varnothing)\odot\Big(\sum_{i=1}^k\phi_i\Big)\odot\Big(\sum_{j=1}^{\widetilde{\nu}}\widetilde{\phi}_j\Big)\Big]$$
$$= kp_k\mathbf{E}\Big[\mathbf{1}_{\{T^1\in\mathscr{T}_{\mathrm{m}}^*\}} f(T^1)\frac{\phi_1}{\sum_{j=1}^k\phi_j}\mathsf{mk}_T(\varnothing)\odot\Big(\sum_{i=1}^k\phi_i\Big)\odot\Big(\sum_{j=1}^{\widetilde{\nu}}\widetilde{\phi}_j\Big)\Big],$$

by independence of ν and T^1. Now consider the marked tree $\widetilde{T}$ defined by: $\mathsf{mk}_{\widetilde{T}}(\varnothing) = \mathsf{mk}_T(\varnothing)$, $\nu_{\widetilde{T}}(\varnothing) = \widetilde{\nu}$ and $\widetilde{T}[i] = \widetilde{T}^i$ for all $1 \le i \le \widetilde{\nu}$. Then we have

$$\widetilde{\phi} := \phi(\widetilde{T}) = \mathsf{mk}_T(\varnothing) \odot \Big(\sum_{j=1}^{\widetilde{\nu}} \phi_j\Big),$$

and the random marked trees $(\widetilde{T}, T[1], T[2], \dots, T[k])$ are i.i.d.. Using the commutativity and the associativity of $\odot$, we first have

$$\mathsf{mk}_T(\varnothing) \odot \Big(\phi_1 + \sum_{i=2}^{k} \phi_i\Big) \odot \Big(\sum_{j=1}^{\widetilde{\nu}} \widetilde{\phi}_j\Big) = \widetilde{\phi} \odot \Big(\phi_1 + \sum_{i=2}^{k} \phi_i\Big).$$

Then by the third property ("position of summand") of $\odot$,

$$\mathbf{1}_{\{\widetilde{\phi}>0,\ \phi_1>0\}} \widetilde{\phi} \odot \Big(\phi_1 + \sum_{i=2}^{k} \phi_i\Big) = \mathbf{1}_{\{\widetilde{\phi}>0,\ \phi_1>0\}} \phi_1 \odot \Big(\widetilde{\phi} + \sum_{j=2}^{k} \phi_j\Big) \frac{\widetilde{\phi}\Big(\phi_1 + \sum_{i=2}^{k} \phi_i\Big)}{\phi_1\Big(\widetilde{\phi} + \sum_{j=2}^{k} \phi_j\Big)},$$

and the last expectation now becomes

$$\mathbf{E}\Big[f(T[1])\mathbf{1}_{\{T[1]\in\mathscr{T}^*_{\mathrm{m}},\ \widetilde{T}\in\mathscr{T}^*_{\mathrm{m}}\}} \frac{\widetilde{\phi}}{\widetilde{\phi} + \sum_{j=2}^{k} \phi_j} \phi_1 \odot \Big(\widetilde{\phi} + \sum_{j=2}^{k} \phi_j\Big)\Big].$$

Now set $T' = T[1]$, $\overline{T}^1 = \widetilde{T}$ and $\overline{T}^j = T[j]$ for $2 \le j \le k$ with similar notations for the function ϕ. By symmetry, for all $1 \le i \le \nu$,

$$\begin{aligned}
&\mathbf{E}\Big[f(T')\mathbf{1}_{\{T'\in\mathscr{T}^*_{\mathrm{m}},\ \overline{T}\in\mathscr{T}^*_{\mathrm{m}}\}} \frac{\overline{\phi}_i}{\sum_{j=1}^{k} \overline{\phi}_j} \phi' \odot \Big(\sum_{j=1}^{k} \overline{\phi}_j\Big)\Big] \\
&= \mathbf{E}\Big[f(T')\mathbf{1}_{\{T'\in\mathscr{T}^*_{\mathrm{m}},\ \overline{T}\in\mathscr{T}^*_{\mathrm{m}}\}} \frac{\overline{\phi}_1}{\sum_{j=1}^{k} \overline{\phi}_j} \phi' \odot \Big(\sum_{j=1}^{k} \overline{\phi}_j\Big)\Big],
\end{aligned}$$

so that finally,

$$\begin{aligned}
&k\mathbf{E}\Big[f(T')\mathbf{1}_{\{T'\in\mathscr{T}^*_{\mathrm{m}},\ \overline{T}\in\mathscr{T}^*_{\mathrm{m}}\}} \frac{\overline{\phi}_1}{\sum_{j=1}^{k} \overline{\phi}_j} \phi' \odot \Big(\sum_{j=1}^{k} \overline{\phi}_j\Big)\Big], \\
&= \mathbf{E}\Big[f(T')\mathbf{1}_{\{T'\in\mathscr{T}^*_{\mathrm{m}},\ \overline{T}\in\mathscr{T}^*_{\mathrm{m}}\}} \sum_{i=1}^{k} \frac{\overline{\phi}_i}{\sum_{j=1}^{k} \overline{\phi}_j} \phi' \odot \Big(\sum_{j=1}^{k} \overline{\phi}_j\Big)\Big] \\
&= \mathbf{E}\Big[f(T')\mathbf{1}_{\{T'\in\mathscr{T}^*_{\mathrm{m}}\}} \phi' \odot \Big(\sum_{j=1}^{k} \overline{\phi}_j\Big)\Big].
\end{aligned}$$

Summing over k, we obtain

$$\begin{aligned}
\mathbf{E}[\mathbf{1}_{\{T\in\mathscr{T}_{\mathrm{m}}^*\}} f(T[\Xi_1])\kappa(\phi(T))] &= \sum_{k\geq 1} p_k \mathbf{E}\Big[f(T')\mathbf{1}_{\{T'\in\mathscr{T}_{\mathrm{m}}^*\}}\phi' \odot \Big(\sum_{j=1}^{k}\overline{\phi}_j\Big)\Big] \\
&= \sum_{k\geq 1} \mathbf{E}\Big[f(T')\mathbf{1}_{\{T'\in\mathscr{T}_{\mathrm{m}}^*,\ \overline{\nu}=k\}}\phi' \odot \Big(\sum_{j=1}^{\overline{\nu}}\overline{\phi}_j\Big)\Big] \\
&= \mathbf{E}\Big[f(T')\mathbf{1}_{\{T'\in\mathscr{T}_{\mathrm{m}}^*\}}\phi' \odot \Big(\sum_{j=1}^{\overline{\nu}}\overline{\phi}_j\Big)\Big].
\end{aligned}$$

To conclude the proof, we replace $\phi' \odot \Big(\sum_{j=1}^{\overline{\nu}}\overline{\phi}_j\Big)$ by its conditional expectation given T', which is $\kappa(\phi')$.

3 Galton-Watson trees with recursive lengths

3.1 Description of the model

We generalize a model of trees with random lengths (or resistances) that can be found in [10, Section 2] and [34, Section 2]. It appeared as the scaling limit of the sequence $(T_n/n)_{n\geq 1}$, where T_n is a reduced critical Galton-Watson tree conditioned to survive at the n^{th} generation.

In this chapter, we work on marked trees and the space of marks is $\mathsf{Marks} = (1, \infty)$. The marks should be interpreted as conductances so, as a reminder, we will write $\gamma_t(x)$ (instead of $\mathsf{mk}_t(x)$) for the mark of a vertex x in a marked tree t. We build a marked Galton-Watson tree T under the following assumptions:

1. The reproduction law $\mathbf{p} = (p_k)_{k\geq 1}$ of the Galton-Watson tree T satisfies $p_0 = 0$ [1] and $p_1 < 1$.

2. For the marks, we consider a random variable Γ with values in $(0, 1)$ and let $(\Gamma_x)_{x\in\mathcal{U}}$ be i.i.d. with the same distribution as Γ. The mark of a vertex x in T is then

$$\gamma_T(x) = \Gamma_x$$

(in the notations of Section 2.6, this corresponds to the case where $M = \Gamma$ is independent of N).

In this chapter, we call T a $(\Gamma, \mathbf{p})$-Galton-Watson tree and denote its distribution by **GW**.

In both [10] and [34], the marks have the law of the inverse of a uniform random variable on $(0, 1)$. The reproduction law is $p_2 = 1$ in [10], whereas in [34] it is given by

$$p_k = \begin{cases} 0 & \text{if } k \leq 1; \\ \dfrac{\alpha\,\mathsf{\Gamma}(k-\alpha)}{k!\,\mathsf{\Gamma}(2-\alpha)} & \text{otherwise,} \end{cases} \tag{3.1}$$

where α is a parameter in $(1, 2)$, and $\mathsf{\Gamma}$ is the standard Gamma function.

Let t be a marked infinite tree with marks in $(1, \infty)$. We associate to each vertex x in t, the resistance, or length, of the edge (x_*, x):

$$r_t(x) = \gamma_t(x)^{-1} \prod_{y \prec x} \left(1 - \gamma_t(y)^{-1}\right).$$

1. Actually, if $p_0 > 0$ and $m > 1$, we can, conditionally on non-extinction, consider the pruned tree T^*. It is then (modulo reindexing) a Galton-Watson tree without leaves and much, if not all, of what we say in this chapter still holds for T conditioned on non-extinction.

Informally, the edge between the root and its parent has length $\gamma_t(\varnothing)^{-1} \in (0,1)$. Then we multiply all the lengths in the subtrees $t[1]$, $t[2]$, ..., $t[\nu_t(\varnothing)]$ by $\left(1-\gamma_t(\varnothing)^{-1}\right)$ and we continue recursively see Figure 1.1.

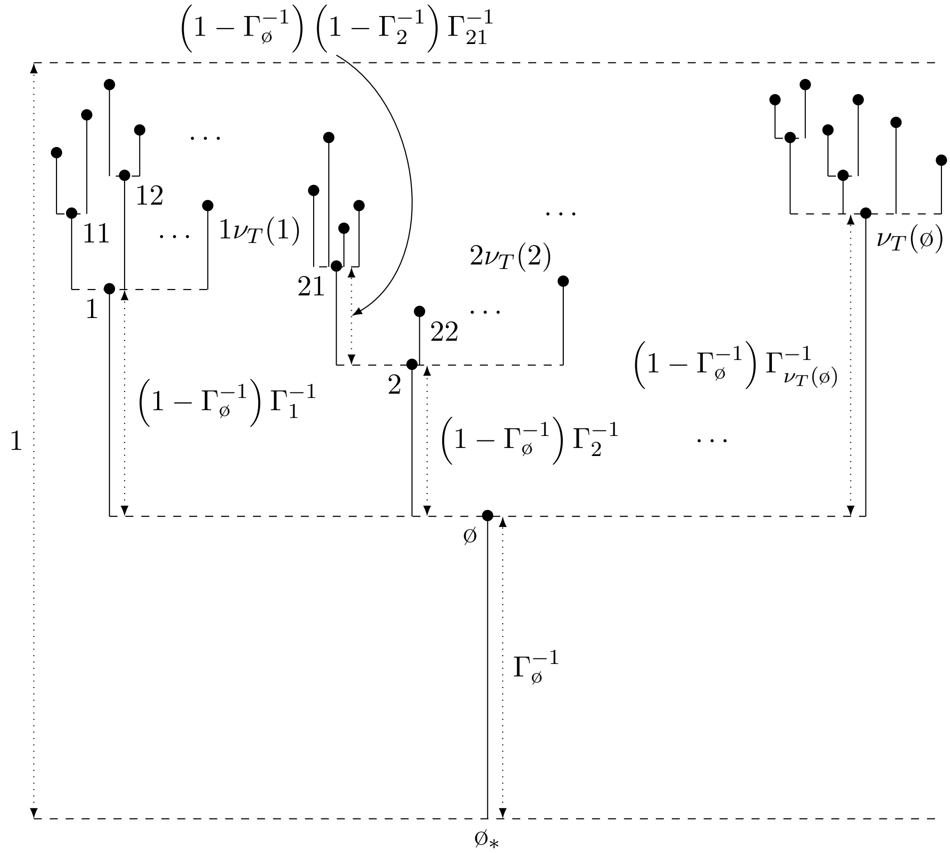

Figure 1.1 – A schematic representation of a Galton-Watson tree with recursive lengths.

We run a nearest-neighbour random walk on the tree with transition probabilities inversely proportional to the lengths of the edges (further neighbours are less likely to be visited). To make this more precise, the random walk in t_*, associated to this set of resistances has the following transition probabilities:

$$\mathrm{P}^t(x,y) = \begin{cases} 1 & \text{if } x=\varnothing_* \text{ and } y=\varnothing; \\ \gamma_t(xi)/\left(\gamma_t(x)-1+\sum_{j=1}^{\nu_t(x)}\gamma_t(xj)\right) & \text{if } y=xi \text{ for some } i\le\nu_t(x) \\ (\gamma_t(x)-1)\,/\left(\gamma_t(x)-1+\sum_{j=1}^{\nu_t(x)}\gamma_t(xj)\right) & \text{if } y=x_*; \\ 0 & \text{otherwise.} \end{cases}$$

When we reindex a subtree, we also change the resistances to gain stationarity. For $x \in t$ and $y \in t[x]$, we define

$$r_{t[x]}(y) = \frac{r_t(xy)}{\prod_{z\prec x}\left(1-\gamma_t(z)\right)^{-1}}.$$

This is consistent with the marks of the reindexed subtree $t[x]$ and does not change the probability transitions of the random walk inside this subtree. For x in t_*, let P_x^t be a probability measure under which the process $(X_n)_{n\geq 0}$ is the random walk on t starting from x with probability transitions given by P^t. To prove that this walk is almost surely transient, we use Rayleigh's principle and compare the resistance of the whole tree between $\varnothing_*$ and infinity to the resistance of a single ray ξ of t. If, for n greater or equal to one, we denote by $r_n(\xi_n)$ the resistance in the ray between $\varnothing_*$ and the vertex ξ_n , we have that

$$1 - r_n(\xi_n) = \left(1 - \gamma_t(\varnothing)^{-1}\right)\left(1 - \gamma_t(\xi_1)^{-1}\right)\cdots\left(1 - \gamma_t(\xi_n)^{-1}\right), \tag{3.2}$$

so the resistance of the whole ray is less or equal to 1. In particular, it is finite and so is the resistance of the whole tree. We denote by HARM_t^Γ the law of the exit ray of this random walk. For x in t_*, let

$$\tau_x = \inf\{k \geq 0 : X_k = x\},$$

with $\inf \emptyset = \infty$ and

$$\beta(t) = Q_\varnothing^t(\tau_{\varnothing_*} = \infty).$$

Applying equations (1.7) and (1.12) to this model, we obtain:

$$\mathsf{HARM}_t^\Gamma(i) = \frac{\gamma_t(i)\beta(t[i])}{\sum_{i=1}^{\nu_t(\varnothing)} \gamma_t(j)\beta(t[j])}, \quad \forall i \leq \nu_t(\varnothing), \tag{3.3}$$

$$\gamma_t(\varnothing)\beta(t) = \frac{\gamma_t(\varnothing)\sum_{j=1}^{\nu_t(\varnothing)} \gamma_t(j)\beta(t[j])}{\gamma_t(\varnothing) - 1 + \sum_{j=1}^{\nu_t(\varnothing)} \gamma_t(j)\beta(t[j])}. \tag{3.4}$$

The effective conductance between $\varnothing_*$ and infinity is

$$\phi(t) = \gamma_t(\varnothing)\beta(t) = \gamma_t(\varnothing)Q_\varnothing^t(\tau_{\varnothing_*} = \infty), \tag{3.5}$$

because the edge between $\varnothing_*$ and $\varnothing$ now has conductance $\gamma_t(\varnothing)$. From the identity (3.2), the Rayleigh principle and the law of parallel conductances, whenever t has at least two rays, $\phi(t) > 1$. Thus we can write

$$\phi(t) = h\Big(\gamma_t(\varnothing), \sum_{i=1}^{\nu_t(\varnothing)} \phi(t[i])\Big), \tag{3.6}$$

with $h(u,v) = uv/(u+v-1)$ for all u and v in $J = (1,\infty)$.

3.2 Invariant measure and dimension drop for the natural distance

We set for all $u > 1$,

$$\kappa(u) = \mathbf{E}\left[h\Big(u, \sum_{j=1}^{\widetilde{\nu}} \phi(\widetilde{T}^j)\Big)\right] = \mathbf{E}\left[\frac{u\sum_{j=1}^{\widetilde{\nu}} \phi(\widetilde{T}^j)}{u - 1 + \sum_{j=1}^{\widetilde{\nu}} \phi(\widetilde{T}^j)}\right]. \tag{3.7}$$

where $(\widetilde{\nu}, \widetilde{T}^1, \widetilde{T}^2, ...)$ are independent random variables independent of T, $\widetilde{\nu}$ being distributed as $\mathbf{p}$ and each $\widetilde{T}^i$ being distributed as T. We will be able to use Theorem 2.28 if we can prove that $\kappa(\phi(T))$ is integrable. To this end, one needs some information about the distribution of Γ and/or about $\mathbf{p}$. The following criterion is certainly not sharp but it might suffice in some practical cases. For its proof, we rely on ideas from [10, Proposition 6].

Proposition 3.1. *Assume that there exist two positive numbers a and C such that for all numbers s in $(1,\infty)$, $\mathbf{P}(\Gamma_\varnothing \geq s) \leq Cs^{-a}$. Then, $\mathbf{E}[\phi(T)]$ and $\mathbf{E}[\kappa(\phi(T))]$ are finite whenever one of the following conditions occurs:*

1. *$a > 1$;*
2. *$a = 1$ and $\sum_{k\geq 1} p_k k \log k < \infty$;*
3. *$0 < a < 1$ and $\sum_{k\geq 1} p_k k^{2-a} < \infty$.*

Proof. From the fact that for all real numbers u and v greater that 1, $h(u,v) < u$, we deduce that $\mathbf{E}[\kappa(\phi(T))]$ is finite as soon as $\mathbf{E}[\phi(T)]$ is, and we also conclude in the first case.

Let $\mathcal{M}$ be the set of all Borel probability measures on $(1,\infty]$. For any distribution μ in $\mathcal{M}$, let $\Psi(\mu)$ be the distribution of $h(\Gamma, \sum_{i=1}^{\nu} X_i)$, where ν, Γ and X_1, X_2, ... are independent, each X_i having distribution μ and ν having distribution $\mathbf{p}$. To handle the case where $\mu(\{\infty\}) > 0$, we define by continuity $h(u,\infty) = u$ for all $u > 1$. Consider for any $s \in (1,\infty)$, $F_\mu(s) = \mu[s,\infty]$, with $F_\mu(s) = 1$ if $s \leq 1$. On $\mathcal{M}$, the stochastic partial order $\preceq$ is defined as follows: $\mu \preceq \mu'$ if and only if there exists a coupling (X, X') in some probability space, with X distributed as μ, X' distributed as μ' such that $X \leq X'$ almost surely. This is equivalent to $F_\mu \leq F_{\mu'}$. From the identity

$$h(u,v) - h(u,v') = (v - v')\frac{u(u-1)}{(u+v-1)(u+v'-1)}, \tag{3.8}$$

we see that Ψ is increasing with respect to the stochastic partial order.

Let us denote by φ the distribution of $\phi(T)$ and by γ the distribution of Γ. Since $\Psi(\delta_\infty) = \gamma$ and $\Psi(\varphi) = \varphi$, we have $\varphi \preceq \Psi^n(\gamma)$ for all $n \geq 1$. We are done if we can show that $\Psi^n(\gamma)$ has a finite first moment for some $n \geq 1$.

For any μ in $\mathcal{M}$ and $s \in (1,\infty)$,

$$F_{\Psi(\mu)}(s) = \mathbf{P}\Big(\Gamma \geq s,\, \sum_{i=1}^{\nu} X_i \geq s\frac{\Gamma - 1}{\Gamma - s}\Big) \leq \mathbf{P}(\Gamma \geq s)\mathbf{P}\Big(\sum_{i=1}^{\nu} X_i \geq s\Big),$$

by independence of ν and Γ.

Decomposing with respect to the value of ν yields

$$\begin{aligned} F_{\Psi(\mu)}(s) &\leq F_\gamma(s) \sum_{k\geq 1} p_k \mathbf{P}\Big(\sum_{i=1}^{k} X_i \geq s\Big) \\ &\leq F_\gamma(s) \sum_{k\geq 1} k p_k F_\mu\Big(\frac{s}{k}\Big). \end{aligned} \tag{3.9}$$

We may apply it to γ, to get

$$\begin{aligned}\int_1^\infty F_{\Psi(\gamma)}(s)\,\mathrm{d}s &\le \sum_{k\ge 1} kp_k \left(\int_1^k F_\gamma(s)\,\mathrm{d}s + \int_k^\infty F_\gamma(s) F_\gamma\left(\frac{s}{k}\right)\mathrm{d}s\right)\\ &= \sum_{k\ge 1} kp_k \left(\int_1^k F_\gamma(s)\,\mathrm{d}s + k\int_1^\infty F_\gamma(s)F_\gamma(ks)\,\mathrm{d}s\right).\end{aligned}$$

In the second case, where $F_\gamma(s) \le Cs^{-1}$ and $\sum_{k\ge1} p_k k\log k < \infty$, this is enough to conclude.

In the third case, we need to play this game a little bit longer. Let $N \ge 1$ be the smallest integer such that $a\,(N+1) > 1$. Notice that this implies that $aN \le 1$. Iterating on the inequality (3.9) and applying it to γ, we get

$$\begin{aligned}&F_{\Psi^N(\gamma)}(s)\\ &\le \sum_{k_1,k_2,\dots,k_N\ge 1} k_1k_2\cdots k_N p_{k_1}p_{k_2}\cdots p_{k_N} F_\gamma\Big(s\Big)F_\gamma\Big(\frac{s}{k_1}\Big)F_\gamma\Big(\frac{s}{k_1k_2}\Big)\cdots F_\gamma\Big(\frac{s}{k_1k_2\cdots k_N}\Big).\end{aligned}$$

Hence, we may write an upper bound of $\int_1^\infty F_{\Psi^N(\gamma)}(s)\,\mathrm{d}s$ as

$$\begin{aligned}\sum_{k_1,\dots,k_N\ge1} k_1\cdots k_N p_{k_1}\cdots p_{k_N}\Big[I_1(k_1) + I_2(k_1,k_2) + \cdots + I_N(k_1,\dots,k_N)&\\ + J(k_1,\dots,k_N)\Big]&,\end{aligned}$$

where I_1, I_2, ..., I_N and J are defined as follows:

$$I_1(k_1) = \int_1^{k_1} F_\gamma(s)\,\mathrm{d}s \le \frac{C}{1-a}k_1^{1-a}.$$

For r between 2 and N,

$$\begin{aligned}I_r(k_1,\dots,k_r) &= \int_{k_1\cdots k_{r-1}}^{k_1\dots k_r} F_\gamma(s)F_\gamma(s/k_1)\cdots F_\gamma\big(s/(k_1\cdots k_{r-1})\big)\,\mathrm{d}s\\ &= k_1\cdots k_{r-1}\int_1^{k_r} F_\gamma(s)F_\gamma(sk_{r-1})\cdots F_\gamma(sk_{r-1}\cdots k_1)\,\mathrm{d}s\\ &\le \begin{cases} k_1\cdots k_{r-1}C^r\log(k_r)k_{r-1}^{-a(r-1)}\cdots k_1^{-a} & \text{if } r=N \text{ and } aN=1;\\ k_1\cdots k_{r-1}\frac{1}{1-ar}C^r k_r^{1-ar}k_{r-1}^{-a(r-1)}\cdots k_1^{-a} & \text{otherwise;}\end{cases}\\ &\le \widetilde{C}k_1^{1-a}\dots k_r^{1-a},\end{aligned}$$

where $\widetilde{C}$ is the positive constant defined by

$$\widetilde{C} = \begin{cases}\max_{2\le r\le N}\left(C^r/(1-ar)\right) & \text{if } aN<1;\\ \max\Big(\max_{2\le r\le N-1}\left(C^r/(1-ar)\right), C^N\sup_{k\ge1}\left(k^{a-1}\log(k)\right)\Big) & \text{if } aN=1.\end{cases}$$

Finally,

$$\begin{aligned} J(k_1,\dots,k_N) &= \int_{k_1\cdots k_N}^{\infty} F_\gamma(s)F_\gamma\left(s/k_1\right)\cdots F_\gamma(s/(k_1\cdots k_N))\,\mathrm{d}s \\ &\le C^{N+1}k_1^{1-a}\cdots k_N^{1-a}\int_1^\infty s^{-a(N+1)}\,\mathrm{d}s \\ &= \frac{C^{N+1}}{a(N+1)-1}k_1^{1-a}\cdots k_N^{1-a}, \quad \text{by our assumption that } a(N+1)>1. \end{aligned}$$

The condition $\sum_{k\geq 1} p_k k^{2-a} < \infty$ ensures that all the above sums are finite. □

Example 3.1. If the law of Γ^{-1} is uniform on $(0,1)$ (as in [10] and [34]), we have, for any s in $(1,\infty)$, $\mathbf{P}(\Gamma \geq s) = s^{-1}$ and the previous proposition shows that $\mathbf{E}[\kappa(\phi(T))]$ is finite if $\sum_{k=1} p_k k \log k < \infty$. If the reproduction law is the same as in [34], that is, is given by (3.1), then by a well-known equivalent on gamma function ratios (see for instance [31]), we have

$$p_k = \frac{\alpha}{\Gamma(2-\alpha)}\frac{\Gamma(k-\alpha)}{\Gamma(k+1)} \sim_{k\to\infty} \frac{\alpha}{\Gamma(2-\alpha)}k^{-1-\alpha},$$

with α in $(1,2)$. Thus $\sum_{k\geq 1} p_k k\log(k)$ is finite and so is $\mathbf{E}[\kappa(\phi(T))]$.

With some more knowledge of $\mathbf{p}$ and/or the law of Γ, it could be possible to describe more precisely the law of $\phi(T)$. See for instance [34, Proposition 5] or [10, Proposition 6]. However, in general, it is often very difficult to establish properties (for instance, absolute continuity) of probability measures defined by distributional recursive equations like (3.6).

We now apply Theorem 2.28 to our problem and prove that the dimension drop phenomenon occurs when the metric is the natural distance $\mathrm{d}_{\mathcal{U}_\infty}$, defined by (1.2).

Theorem 3.2. *Let T be a $(\Gamma,\mathbf{p})$-Galton-Watson tree. Let $\phi(T)$ and κ be defined respectively by* (3.5) *and* (3.7)*. Assume that $C = \mathbf{E}[\kappa\left(\phi(T)\right)]$ is finite. Then, the probability measure of density $C^{-1}\kappa(\phi(T))$ with respect to* **GW** *is invariant and ergodic with respect to the flow rule* HARM^Γ.

The dimension of the measure HARM_T^Γ *on ∂T with respect to* $\mathrm{d}_{\mathcal{U}_\infty}$ *equals almost surely*

$$C^{-1}\mathbf{E}\left[\log\left(\frac{1-\Gamma_\varnothing^{-1}}{1-\Gamma_\varnothing^{-1}\phi(T)}\right)\kappa(\phi(T))\right]. \tag{3.10}$$

It is almost surely strictly less than $\log m$ *unless both* $\mathbf{p}$ *and the distribution of* Γ *are degenerate.*

Proof. The first part of the theorem is a direct consequence of Theorem 2.28.

Write μ for the probability measure with density $C^{-1}\kappa(\phi(T))$ with respect to **GW** $\mathbf{P}_\mu$ for the probability measure with density $C^{-1}\kappa(\phi(T))$ with respect to $\mathbf{P}$ and $\mathbf{E}_\mu$ for the associated expectation. Then by Corollary 2.19, invariance of μ and equality (3.3) the dimension of HARM_T^Γ equals almost surely

$$\dim^{\mathrm{d}_{\mathcal{U}_\infty}} \mathsf{HARM}_T^\Gamma = \mathbf{E}_\mu\left[\log \frac{1}{\mathsf{HARM}_T^\Gamma(\Xi_1)}\right] = \mathbf{E}_\mu\left[\log \frac{\sum_{i=1}^{\nu_T(\varnothing)} \phi(T[i])}{\phi(T[\Xi_1])}\right].$$

From formula (3.4), we deduce that

$$\sum_{i=1}^{\nu_T(\varnothing)} \phi(T[i]) = \frac{\phi(T)(1-\Gamma_\varnothing^{-1})}{1-\Gamma_\varnothing^{-1}\phi(T)},$$

so that almost surely,

$$\dim \mathsf{HARM}_T^\Gamma = \mathbf{E}_\mu\Big[\log(1-\Gamma_\varnothing^{-1}) - \log(1-\Gamma_\varnothing^{-1}\phi(T)) + \log(\phi\,(T)) - \log(\phi(T[\Xi_1]))\Big].$$

By Lemma 2.24, it suffices to prove that $\log(\phi(T)) - \log(\phi(T[\Xi_1]))$ is bounded from below by an integrable random variable to conclude. Using again formula (3.4) yields

$$\begin{aligned}\frac{\phi(T)}{\phi(T[\Xi_1])} &= \frac{1+\phi(T[\Xi_1])^{-1}\sum_{i=1,\, i\neq\Xi_1}^{\nu_T(\varnothing)} \phi(T[i])}{1-\Gamma_\varnothing^{-1}+\Gamma_\varnothing^{-1}\phi(T[\Xi_1])+\Gamma_\varnothing^{-1}\sum_{i=1,\, i\neq\Xi_1}^{\nu_T(\varnothing)}\phi(T[i])}\\ &\geq \frac{1}{1-\Gamma_\varnothing^{-1}+\Gamma_\varnothing^{-1}\phi(T[\Xi_1])}.\end{aligned}$$

Hence, since $1-\Gamma_\varnothing^{-1} \leq 1$ and $\Gamma_\varnothing^{-1}\phi(T[\Xi_1]) = \beta(T[\Xi_1]) \leq 1$, we have

$$\frac{\phi(T)}{\phi(T[\Xi_1])} \geq \frac{1}{2}.$$

To prove the dimension drop, i.e. the fact that almost surely $\dim \mathsf{HARM}^\Gamma < \log m$, we do not use the formula (3.10) since we know so little about the distribution of $\phi(T)$. Instead, we compare the flow rule HARM^Γ to the uniform flow UNIF defined in Section 2.7.

By Proposition 2.27 and Lemma 2.21 we only need to prove that if there exists a positive real number K such that, for $\mathbf{GW}$-almost every tree t, $W(t) = K\phi(t)$, then both the reproduction law and the mark law are degenerate.

Under this assumption, by the recursive equation (2.1) satisfied by W, we have almost surely,

$$\phi\,(T) = \frac{1}{m}\sum_{i=1}^{\nu_T(\varnothing)} \phi(T[i]).$$

Plugging this into the recursive equation (3.4), we first obtain that

$$\phi(T) = \frac{m\Gamma_\varnothing\phi(T)}{\Gamma_\varnothing + m\phi(T) - 1},$$

so that almost surely

$$\phi(T) = \frac{1}{m}\left((m-1)\Gamma_\varnothing + 1\right).$$

In turn, using again (3.4), this implies that

$$\frac{1}{m}\left[(m-1)\,\Gamma_\varnothing+1\right]=\frac{\Gamma_\varnothing\sum_{i=1}^{\nu_T(\varnothing)}\frac{1}{m}\left[(m-1)\Gamma_i+1\right]}{\Gamma_\varnothing-1+\sum_{i=1}^{\nu_T(\varnothing)}\frac{1}{m}\left[(m-1)\Gamma_i+1\right]}.$$

Now, if we denote by S the random variable $\sum_{i=1}^{\nu_T(\varnothing)}\frac{1}{m}\left[(m-1)\Gamma_i+1\right]$, elementary algebra leads to the second degree polynomial equation

$$(m-1)\Gamma_\varnothing^2+\Gamma_\varnothing(2-m-S)+S-1=0,$$

whose discriminant is equal to $(S-m)^2$. Hence, we always have

$$\Gamma_\varnothing=\frac{m+S-2\pm(S-m)}{2(m-1)}.$$

We must choose the solution $\Gamma_\varnothing=(S-1)/(m-1)$, because the other solution is 1, which we forbid. As a consequence,

$$\Gamma_\varnothing=\frac{1}{m}\sum_{i=1}^{\nu_T(\varnothing)}\Gamma_i+\frac{1}{m-1}\left(\frac{\nu_T(\varnothing)}{m}-1\right).$$

which, by independence of $\nu_T(\varnothing)$, $\Gamma_\varnothing$, Γ_1, Γ_2, …, imposes that both $\mathbf{p}$ and the law of Γ are degenerate. □

3.3 Dimension and dimension drop for the length metric

Note that in the previous theorem, the dimension is computed with respect to the natural distance $\mathrm{d}_{\mathcal{U}_\infty}$. This distance does not take into account the marks $(\Gamma_x)_{x\in T}$, so we do not compute the same dimension as in [10] and [34], where the distance between two points in the tree is the sum of all the resistances (or lengths) of the edges between these two points.

To make this definition more precise, let us introduce, for $x\in T$, the Γ-height of x:

$$|x|^\Gamma=\sum_{y\preceq x}\Big(\prod_{z\prec y}\left(1-\Gamma_z^{-1}\right)\Big)\Gamma_y^{-1}.$$

We then have

$$1-|x|^\Gamma=\prod_{y\preceq x}\left(1-\Gamma_y^{-1}\right).$$

For two distinct rays η and ξ, let

$$\mathrm{d}^\Gamma(\xi,\eta)=1-|\xi\wedge\eta|^\Gamma.$$

Notice that, for any rays ξ and η, and all integer $n\geq 1$, we have:

$$\mathrm{d}^\Gamma(\xi,\eta)\leq 1-|\xi_n|^\Gamma \iff \eta\in[\xi_n]_T, \tag{3.11}$$

where we recall that $[\xi_n]_t$ is the set of all rays whose ξ_n is a prefix.

We will compute the dimension of HARM_T^Γ with respect to this distance d^Γ and show that in this case too, we observe a dimension drop phenomenon, but we begin with more general statements. We want to build a theory similar to the one we have presented in Chapter 2, in our setting of trees with recursive lengths with the length metric d^Γ.

We will need the following elementary lemma.

Lemma 3.3. *Let f and g be two positive non-increasing functions defined on $(0,1)$. Let (r_n) be a decreasing sequence of positive numbers converging to 0. Assume that*

$$\frac{f(r_n)}{g(r_n)} \xrightarrow[n\to\infty]{} \ell \in [0,\infty) \text{ and } \frac{f(r_{n+1})}{f(r_n)} \xrightarrow[n\to\infty]{} 1.$$

Then, we have $\lim_{r\downarrow 0} f(r)/g(r) = \ell$.

Proof. Let $\varepsilon > 0$ and n_0 be large enough so that for all $n \geq n_0$,

$$\frac{f(r_n)}{g(r_n)}\frac{f(r_{n+1})}{f(r_n)} \leq \ell + \varepsilon \text{ and } \frac{f(r_{n+1})}{g(r_{n+1})}\frac{f(r_n)}{f(r_{n+1})} \geq \ell - \varepsilon.$$

Then, using the assumption that (r_n) is decreasing to 0, for all $r \leq r_{n_0}$, there exists $n \geq n_0$ such that $r_{n+1} < r \leq r_n$ and we have

$$\ell - \varepsilon \leq \frac{f(r_{n+1})}{g(r_{n+1})}\frac{f(r_n)}{f(r_{n+1})} = \frac{f(r_n)}{g(r_{n+1})} \leq \frac{f(r)}{g(r)} \leq \frac{f(r_{n+1})}{g(r_n)} = \frac{f(r_n)}{g(r_n)}\frac{f(r_{n+1})}{f(r_n)} \leq \ell + \varepsilon. \quad \square$$

Proposition 3.4 (dimension of a flow rule)**.** *Let Θ be a $\mathbf{GW}$-flow rule such that there exists a Θ-invariant probability measure $\mu \ll \mathbf{GW}$. Then, almost surely, the probability measure Θ_T is exact-dimensional on the metric space $(\partial T, \mathrm{d}^\Gamma)$, with deterministic dimension*

$$\dim^{\mathrm{d}^\Gamma} \Theta_T = \frac{\dim^{\mathrm{d}_{\mathcal{U}_\infty}} \Theta_T}{\mathbf{E}_\mu[-\log(1-\Gamma_\varnothing^{-1})]}. \tag{3.12}$$

Proof. We first prove that, for $\mathbf{GW}$-almost every tree t, for Θ_t-almost every ray ξ,

$$\lim_{n\to\infty} \frac{-\log \Theta_t(\xi_n)}{-\log(1-|\xi_n|^\Gamma)} = \frac{\dim^{\mathrm{d}_{\mathcal{U}_\infty}} \Theta_T}{\mathbf{E}_\mu[-\log(1-\Gamma_\varnothing^{-1})]}. \tag{3.13}$$

The numerator equals

$$\sum_{k=0}^{n-1} -\log \frac{\Theta_t\,(\xi_{k+1})}{\Theta_t\,(\xi_k)},$$

so, by the ergodic theorem (for non-negative functions), recalling that $\mu \ltimes \Theta$ is ergodic and μ is equivalent to $\mathbf{GW}$, for $\mathbf{GW}$-almost every t and Θ_t-almost every ξ,

$$\frac{1}{n}\sum_{k=0}^{n-1} -\log \frac{\Theta_t\,(\xi_{k+1})}{\Theta_t\,(\xi_k)} \xrightarrow[n\to\infty]{} \mathbf{E}_\mu[-\log \Theta_T\,(\Xi_1)] = \dim^{\mathrm{d}_{\mathcal{U}_\infty}} \Theta_T \in (0, \log m]. \tag{3.14}$$

For the denominator, we have, for any ξ in ∂t and any $n \geq 1$,

$$\frac{1}{n+1}(-\log)(1-|\xi_n|^\Gamma) = \frac{1}{n+1}\sum_{i=0}^{n} -\log(1-\gamma_t(\xi_i)^{-1}).$$

Again by the pointwise ergodic theorem, we have, for **GW**-almost every t and Θ_t-almost every ξ,

$$\frac{1}{n+1}(-\log)\left(1-|\xi_n|^\Gamma\right) \xrightarrow[n\to\infty]{} \mathbf{E}_\mu[-\log(1-\Gamma_\varnothing^{-1})] \in (0,\infty]. \tag{3.15}$$

Thus, the convergence (3.13) is proved.

Now we will show the following (*a priori* stronger) statement: for **GW**-almost every t and Θ_t-almost every ξ,

$$\lim_{r\downarrow 0}\frac{-\log\Theta_t\mathscr{B}(\xi,r)}{-\log r} = \frac{\dim^{\mathrm{d}_{\mathcal{U}_\infty}}\Theta_T}{\mathbf{E}_\mu[-\log(1-\Gamma_\varnothing^{-1})]}, \tag{3.16}$$

where $\mathscr{B}(\xi,r)$ is the closed ball of center ξ and radius r in the metric space $(\partial t, \mathrm{d}^\Gamma)$, so that almost surely, Θ_T is exact-dimensional in the strong sense of remark 1.3.

Now let t be a marked tree and ξ be a ray in t such that (3.14) and (3.15) hold. Denote, for $n \geq 0$, $r_n = 1-|\xi_n|^\Gamma$. The sequence (r_n) is positive, decreasing, and converges to 0 by (3.15). For r in $(0,1)$, define $f(r) = -\log\Theta_t\mathscr{B}(\xi,r)$ and $g(r) = -\log(r)$. The functions f and g are positive and non-increasing. Furthermore,

$$\frac{f(r_{n+1})}{f(r_n)} = \frac{-\log\Theta_t(\xi_{n+1})}{-\log\Theta_t(\xi_n)} = 1 + \frac{-\log\frac{\Theta_t(\xi_{n+1})}{\Theta_t(\xi_n)}}{-\log\Theta_t(\xi_n)} = 1 + \frac{-\frac{1}{n}\log\frac{\Theta_t(\xi_{n+1})}{\Theta_t(\xi_n)}}{-\frac{1}{n}\log\Theta_t(\xi_n)}.$$

Using (3.14), we obtain $\lim_{n\to\infty} f(r_{n+1})/f(r_n) = 1$, and conclude by the preceding lemma. □

We now associate to the random marked tree T an *age-dependent process* (in the definition of [7, chapter 4]). For any $x \in T$, let $\Lambda_x = -\log\left(1-\Gamma_x^{-1}\right)$ be the lifetime of particle x. Informally, the root lives for time $\Lambda_\varnothing$, then simultaneously dies and gives birth to $\nu_T(\varnothing)$ children who all have i.i.d. lifetimes $\Lambda_1, \Lambda_2, \ldots, \Lambda_{\nu_T(\varnothing)}$, and then independently live and produce their own offspring and die, and so on. We are interested in the number $Z_u(T)$ of living individuals at time $u > 0$, that is

$$Z_u(T) = \#\Big\{x \in T : \sum_{y\preceq x_*}\Lambda_y < u \leq \sum_{y\preceq x}\Lambda_y\Big\}.$$

The *Malthusian parameter* of this process is the unique real number $\alpha > 0$ such that

$$\mathbf{E}\left[e^{-\alpha\Lambda_\varnothing}\right] = \frac{1}{m}. \tag{3.17}$$

We now assume that $\sum_{k=1}^{\infty} p_k k\log k$ is finite. Under this assumption, we know from [30, Theorem 5.3] [2] that there exists a positive random variable $W^\Gamma(T)$ of expectation

2. See also Section 3.4 of the preliminary Saint-Flour 2017 lecture notes by Remco Van Der Hoffstad.

1, such that, almost surely,

$$\lim_{u\to\infty} e^{-u\alpha} Z_u(T) = W^\Gamma(T). \tag{3.18}$$

By definition, we obtain the recursive equation

$$W^\Gamma(T) = e^{-\alpha\Lambda_\varnothing} \sum_{j=1}^{\nu_T(\varnothing)} W^\Gamma(T[i]). \tag{3.19}$$

We now go back to our original tree with recursive lengths. Equations (3.17), (3.18) and (3.19) become

$$\mathbf{E}[(1-\Gamma_\varnothing^{-1})^\alpha] = 1/m; \tag{3.20}$$

$$\lim_{\varepsilon\to 0} \varepsilon^\alpha Z_{-\log(\varepsilon)}(T) = W^\Gamma(T); \tag{3.21}$$

$$W^\Gamma(T) = (1-\Gamma_\varnothing^{-1})^\alpha \sum_{j=1}^{\nu_T(\varnothing)} W^\Gamma(T[i]). \tag{3.22}$$

We define the $\mathbf{GW}$-flow rule UNIF^Γ by

$$\mathsf{UNIF}_T^\Gamma(i) = \frac{W^\Gamma(T[i])}{\sum_{j=1}^{\nu_T(\varnothing)} W^\Gamma(T[j])}, \quad \forall\, 1 \le i \le \nu_T(\varnothing).$$

Proposition 3.5 (dimension of the limit uniform measure)**.** *Assume that* $\sum_{k\geq 1} p_k k \log k$ *is finite. Then, both the dimension of* UNIF_T^Γ *and the Hausdorff dimension of the boundary* ∂T*, with respect to the metric* d^Γ*, are almost surely equal to the Malthusian parameter* α*.*

Proof. We can use Theorem 2.28, with $h(u,v) = uv$ and the marks equal to $((1-\Gamma_x^{-1})^\alpha)_{x\in T}$ (or a direct computation) to show that the probability measure with density W^Γ with respect to $\mathbf{GW}$ is UNIF^Γ-invariant. So we may apply Proposition 3.4 to obtain that the dimension of UNIF^Γ with respect to the metric d^Γ equals

$$\dim^{d^\Gamma} \mathsf{UNIF}_T^\Gamma = \frac{\dim^{\mathrm{d}_{\mathcal{U}_\infty}} \mathsf{UNIF}_T^\Gamma}{\mathbf{E}[-\log(1-\Gamma_\varnothing^{-1})W^\Gamma(T)]}.$$

The numerator equals, by Corollary 2.19 and the recursive equation (3.22),

$$\begin{aligned}
\dim^{\mathrm{d}_{\mathcal{U}_\infty}} \mathsf{UNIF}^\Gamma &= \mathbf{E}\Big[\Big(-\log W^\Gamma(T[\Xi_1]) + \log \sum_{j=1}^{\nu_T(\varnothing)} W^\Gamma\,(T[j])\Big)W^\Gamma(T)\Big] \\
&= \mathbf{E}\Big[\Big(\log((1-\Gamma_\varnothing^{-1})^{-\alpha}) + \log W^\Gamma(T) - \log W^\Gamma(T[\Xi_1])\Big)W^\Gamma(T)\Big] \\
&= \alpha\mathbf{E}\Big[-\log(1-\Gamma_\varnothing^{-1})W^\Gamma(T)\Big],
\end{aligned}$$

provided we can show that the term $(\log W^\Gamma(T) - \log W^\Gamma(T[\Xi_1]))W^\Gamma(T)$ is bounded from below by an integrable random variable.

To prove this, we first use the recursive equation (3.22), to obtain

$$\log W^\Gamma(T) - \log W^\Gamma(T[\Xi_1]) = \alpha \log(1-\Gamma_\varnothing^{-1}) + \log\left(\frac{\sum_{i=1}^{\nu_t \varnothing} W^\Gamma(T[i])}{W^\Gamma(T[\Xi_1])}\right).$$

Since Ξ_1 is one of the children of the root, we have

$$\log\left(\frac{\sum_{i=1}^{\nu_t \varnothing} W^\Gamma(T[i])}{W^\Gamma(T[\Xi_1])}\right) \geq 0.$$

Using again (3.22), we obtain

$$\begin{aligned}
&\Big(\log W^\Gamma(T) - \log W^\Gamma(T[\Xi_1])\Big)\, W^\Gamma(T) \\
&\geq \alpha \log(1-\Gamma_\varnothing^{-1})(1-\Gamma_\varnothing^{-1})^\alpha \sum_{i=1}^{\nu_T(\varnothing)} W^\Gamma(T[i]) \geq -\frac{1}{e} \sum_{i=1}^{\nu_T(\varnothing)} W^\Gamma(T[i]),
\end{aligned}$$

where, for the last inequality, we have used the fact that the minimum of the function $x \mapsto x^\alpha \log(x)$ on the interval $(0,1)$ is $-1/(\alpha e)$. Since $\mathbf{E}[\sum_{i=1}^{\nu_T(\varnothing)} W^\Gamma(T[i])] = m < \infty$, this concludes the proof that $\dim^{\mathrm{d}\mathcal{U}_\infty} \mathsf{UNIF}^\Gamma = \alpha \mathbf{E}[-\log(1-\Gamma_\varnothing^{-1})W^\Gamma(T)]$.

We remark that $\mathbf{E}[\log(1-\Gamma_\varnothing^{-1})W^\Gamma(T)]$ is finite, because $\dim^{\mathrm{d}\mathcal{U}_\infty} \mathsf{UNIF}^\Gamma \leq \log m$. Thus we have

$$\dim^{d^\Gamma} \mathsf{UNIF}_T^\Gamma = \frac{\dim^{\mathrm{d}\mathcal{U}_\infty} \mathsf{UNIF}_T^\Gamma}{\mathbf{E}[-\log(1-\Gamma_\varnothing^{-1})W^\Gamma(T)]} = \frac{\alpha\mathbf{E}[-\log(1-\Gamma_\varnothing^{-1})W^\Gamma(T)]}{\mathbf{E}[-\log(1-\Gamma_\varnothing^{-1})W^\Gamma(T)]} = \alpha.$$

We now know that the Hausdorff dimension of the boundary ∂T (with respect to d^Γ) is almost surely greater or equal to α, so we just need the upper bound. Recall the definition of the Hausdorff measures in metric spaces in Section 1.10. We let

$$A_\varepsilon = \{x \in T : 1 - |x|^\Gamma \leq \varepsilon < 1 - |x_*|^\Gamma\},$$

whose number of elements is $Z_{-\log(\varepsilon)}(T)$. By the limit (3.21), we have

$$\mathsf{H}_\varepsilon^\alpha(\partial T) \leq \sum_{x \in A_\varepsilon} (\operatorname{diam}^\Gamma[x]_T)^\alpha \leq \varepsilon^\alpha Z_{-\log(\varepsilon)}(T) \xrightarrow[\varepsilon\to 0]{\text{a.s.}} W^\Gamma(T),$$

so $\mathsf{H}^\alpha(\partial T) \leq W^\Gamma(T) < \infty$, which concludes the proof. □

Proposition 3.6 (dimension drop for other flow rules)**.** *Assume that $\sum_{k\geq 1} p_k k \log k$ is finite. Let T be a $(\Gamma, \mathbf{p})$-Galton-Watson tree and Θ be a $\mathbf{GW}$-flow rule such that Θ_T and UNIF_T^Γ are not almost surely equal and there exists a Θ-invariant probability measure $\mu \ll \mathbf{GW}$. Then the dimension of Θ with respect to the distance d^Γ is almost surely strictly less than the Malthusian parameter α.*

Proof. First, we remark that if $\mathbf{E}_\mu[-\log(1-\Gamma_\varnothing^{-1})]$ is infinite, then the Hausdorff dimension of Θ_T with respect to the distance d^Γ is almost surely equal to 0, so there is nothing to prove.

So we assume that $\mathbf{E}_\mu[-\log(1-\Gamma_\varnothing^{-1})]$ is finite. Let Ξ be a random ray in ∂T with distribution Θ_T. Using Corollary 2.19 and conditioning on the value of Ξ_1 gives

$$\dim^{\mathrm{d}\mathcal{U}_\infty}(\Theta_T) = \mathbf{E}_\mu\Big[\sum_{i=1}^{\nu_T(\varnothing)} -\Theta_T(i)\log(\Theta_T(i))\Big] < \mathbf{E}_\mu\Big[\sum_{i=1}^{\nu_T(\varnothing)} -\Theta_T(i)\log \mathsf{UNIF}_T^\Gamma(i)\Big],$$

where, for the strict inequality we have used Gibb's inequality. This upper bound is equal to

$$\mathbf{E}_\mu\Big[\alpha(-\log(1-\Gamma_\varnothing^{-1})) + \log W^\Gamma(T) - \log W^\Gamma(T[\Xi_1])\Big].$$

Once again, all that is left to prove is that the last two terms are bounded from below by an integrable random variable, and this is the case, because

$$\log\frac{W^\Gamma(T)}{W^\Gamma(T[\Xi_1])} \geq \alpha\log(1-\Gamma_\varnothing^{-1}),$$

and by our assumption that $\mathbf{E}_\mu[\log(1-\Gamma_\varnothing^{-1})]$ is finite. Cancelling out this term in equation (3.12), we finally obtain $\dim^{d^\Gamma}\Theta_T < \alpha$. $\square$

Before we state and prove the main theorem of this subsection, we want to know when the dimension (with respect to d^Γ) of the harmonic measure equals 0.

Lemma 3.7. *Let T be a $(\Gamma,\mathbf{p})$-Galton-Watson marked tree. Assume that $\mathbf{E}[\kappa(\phi(T))]$ and $\sum_{k\geq 1} p_k k$ are finite. Then, we have*

$$\mathbf{E}[\log(1-\Gamma_\varnothing^{-1})\kappa(\phi(T))] < \infty \iff \mathbf{E}[\log(1-\Gamma_\varnothing^{-1})] < \infty.$$

Proof. By Tonelli's theorem, the definition of κ, and the associativity property of the function h, we have

$$\mathbf{E}[\log(1-\Gamma_\varnothing^{-1})\kappa(\phi(T))] = \mathbf{E}\left[\log(1-\Gamma_\varnothing^{-1})h\Big(\Gamma_\varnothing, h(\sum_{i=1}^{\nu_T(\varnothing)}\phi(T[i]), \sum_{j=1}^{\widetilde{\nu}}\phi(\widetilde{T}^i))\Big)\right],$$

where $\widetilde{\nu}$, $\widetilde{T}^1$, $\widetilde{T}^2$, ..., are as in the beginning of Section 3.2. Since for any u and v greater than 1, $h(u,v) > 1$, the direct implication is proved. For the reciprocal implication, recall that for u and v in $(1,\infty)$, $h(u,v) < v$, hence

$$h\Big(\Gamma_\varnothing, h(\sum_{i=1}^{\nu_T(\varnothing)}\phi(T[i]), \sum_{j=1}^{\widetilde{\nu}}\phi(\widetilde{T}^j))\Big) < h\Big(\sum_{i=1}^{\nu_T(\varnothing)}\phi(T[i]), \sum_{j=1}^{\widetilde{\nu}}\phi(\widetilde{T}^j)\Big).$$

The right-hand side of the previous inequality is integrable. Indeed,

$$\begin{aligned} h\Big(\sum_{i=1}^{\nu_T(\varnothing)} \phi(T[i]), \sum_{j=1}^{\widetilde{\nu}} \phi(\widetilde{T}^j)\Big) &= \frac{\big(\sum_{i=1}^{\nu_T(\varnothing)} \phi(T[i])\big)\big(\sum_{j=1}^{\widetilde{\nu}} \phi(\widetilde{T}^j)\big)}{\sum_{i=1}^{\nu_T(\varnothing)} \phi(T[i]) + \sum_{j=1}^{\widetilde{\nu}} \phi(\widetilde{T}^j) - 1} \\ &\leq \sum_{i=1}^{\nu_T(\varnothing)} \frac{\phi(T[i])\big(\sum_{j=1}^{\widetilde{\nu}} \phi(\widetilde{T}^j)\big)}{\phi(T[i]) + \sum_{j=1}^{\widetilde{\nu}} \phi(\widetilde{T}^j) - 1} \end{aligned}$$

and the expectation of this upper bound equals, by independence,

$$\mathbf{E}[\nu_T(\varnothing)]\mathbf{E}[\kappa(\phi(T))],$$

which is finite by assumption. Thus, using the fact that

$$\Gamma_\varnothing \text{ and } h\Big(\sum_{i=1}^{\nu_T(\varnothing)} \phi(T[i]), \sum_{j=1}^{\widetilde{\nu}} \phi(\widetilde{T}^j)\Big) \text{ are independent,}$$

we have

$$\mathbf{E}[\log(1-\Gamma_\varnothing^{-1})\kappa(\phi(T))] \leq \mathbf{E}[\log(1-\Gamma_\varnothing^{-1})]\mathbf{E}[\nu_T(\varnothing)]\mathbf{E}[\kappa(\phi(T))],$$

which proves the reciprocal implication of the lemma. □

Putting everything together, we finally obtain the dimension drop for the flow rule HARM^Γ, with respect to the distance d^Γ.

Theorem 3.8. *Let T be a $(\Gamma, \mathbf{p})$-Galton-Watson marked tree, with metric d^Γ on its boundary. Assume that both $\mathbf{E}[\kappa(\phi(T))]$ and $\sum_{k\geq 1} p_k k \log k$ are finite. Then, almost surely, the flow HARM_T^Γ is exact-dimensional, of deterministic dimension*

$$\dim^{d^\Gamma} \mathsf{HARM}^\Gamma(T) = \frac{\mathbf{E}[\log(1-\Gamma_\varnothing^{-1}\phi(T))\kappa(\phi(T))]}{\mathbf{E}[\log(1-\Gamma_\varnothing^{-1})\kappa(\phi(T))]} - 1, \tag{3.23}$$

except in the case $\mathbf{E}[-\log(1-\Gamma_\varnothing^{-1})] = \infty$, where it is 0. This deterministic dimension is strictly less than the Malthusian parameter α (which is almost surely the Hausdorff dimension of the boundary ∂T with respect to the distance d^Γ) as soon as the mark law and the reproduction law are not both degenerate.

Proof. The formula for the Hausdorff dimension is just a rewriting using equations (3.12) and (3.10). All that is left to prove is that if there exists a positive real number K, such that, for $\mathbf{GW}$-almost every tree t, $W^\Gamma(t) = K \times \phi(t)$, then both the mark law and the reproduction law are degenerate.

We assume that the latter assertion holds, and we proceed similarly as in the proof of Theorem 3.2. From the recursive equation (3.22) for W^Γ, we deduce that almost surely

$$\sum_{i=1}^{\nu_T(\varnothing)} \phi(T[i]) = (1-\Gamma_\varnothing^{-1})^{-\alpha}\phi(T), \tag{3.24}$$

and plugging it into the recursive equation (3.4) for ϕ, we obtain that, almost surely,

$$\phi(T) = \Gamma_{\varnothing}(1 - (1 - \Gamma_{\varnothing}^{-1})^{\alpha+1}).$$

This implies that each $\phi(T[i])$ depends only on Γ_i and

$$\sum_{i=1}^{\nu_T(\varnothing)} \phi(T[i]) = \Gamma_{\varnothing}(1 - \Gamma_{\varnothing}^{-1})^{-\alpha} + 1 - \Gamma_{\varnothing},$$

so, by independence, $\sum_{i=1}^{\nu_T(\varnothing)} \phi(T[i])$ must be constant, which imposes that $\nu_T(\varnothing)$ must be constant (equal to m) and that the law of $\phi(T)$ is degenerate. From (3.24), we now see that this implies that $(1 - \Gamma_{\varnothing}^{-1})^{\alpha} = 1/m$ and the law of $\Gamma_{\varnothing}$ is degenerate. □

To conclude this work, we want to check that our formula (3.23) is consistent with the formula obtained in [10]. From now on, we work under the following hypotheses:

1. the reproduction law is given by $p_2 = 1$;
2. the common law of the marks is the law of U^{-1}, where the law of U is uniform on $(0, 1)$.

Remark 3.1. The function denoted by $t \mapsto \kappa(\phi(t))$, in [10, Proposition 25] is slightly different (it differs by a factor 1/2) from our function also denoted by $\kappa(\phi(t))$.

Under these hypotheses, Curien and Le Gall proved that the dimension (with respect to the metric d^{Γ}) of the harmonic measure is almost surely (see [10, Proposition 4]):

$$\dim^{\mathrm{d}^{\Gamma}} \mathsf{HARM}^{\Gamma}(T) = 2\mathbf{E}\left[\log\left(\frac{\phi_1 + \phi_2}{\phi_1}\right)\frac{\phi_1\widetilde{\phi}}{\widetilde{\phi} + \phi_1 + \phi_2 - 1}\right] \Big/ \mathbf{E}\left[\frac{\phi_1\phi_2}{\phi_1 + \phi_2 - 1}\right], \tag{3.25}$$

where ϕ_1, ϕ_2 and $\widetilde{\phi}$ are independent copies of $\phi(T)$. For short, we write $U = \Gamma_{\varnothing}^{-1}$, $\phi = \phi(T)$, $\phi_1 = \phi(T[1])$ and $\phi_2 = \phi(T[2])$.

We first show that

$$\mathbf{E}\left[-\log\left(\frac{1 - U\phi}{1 - U}\right)\kappa(\phi)\right] = 2\mathbf{E}\left[\log\left(\frac{\phi_1 + \phi_2}{\phi_1}\right)\frac{\phi_1\widetilde{\phi}}{\widetilde{\phi} + \phi_1 + \phi_2 - 1}\right]. \tag{3.26}$$

Recall from the proof of Theorem 3.2 that, by stationarity,

$$\mathbf{E}[\log(\phi)\kappa(\phi)] = \mathbf{E}[\log(\phi(T[\Xi_1]))\kappa(\phi)].$$

By the recursive formula (3.6),

$$\frac{1 - U\phi}{1 - U} = \frac{U^{-1}}{\phi_1 + \phi_2 + U^{-1} - 1} = \frac{\phi}{\phi_1 + \phi_2},$$

thus we obtain

$$\begin{aligned}
\mathbf{E}\left[\log\left(\frac{1-U\phi}{1-U}\right)\kappa(\phi)\right] &= \mathbf{E}\left[\log\left(\frac{\phi}{\phi_1+\phi_2}\right)\kappa(\phi)\right] \\
&= \mathbf{E}\left[\log\left(\frac{\phi(T[\Xi_1])}{\phi_1+\phi_2}\right)\kappa(\phi)\right] \\
&= \mathbf{E}\left[\frac{\phi_1}{\phi_1+\phi_2}\log\left(\frac{\phi_1}{\phi_1+\phi_2}\right)\kappa(\phi)+\frac{\phi_2}{\phi_1+\phi_2}\log\left(\frac{\phi_2}{\phi_1+\phi_2}\right)\kappa(\phi)\right] \\
&= 2\mathbf{E}\left[\frac{\phi_1}{\phi_1+\phi_2}\log\left(\frac{\phi_1}{\phi_1+\phi_2}\right)\kappa(\phi)\right],
\end{aligned}$$

by symmetry. Let $\widetilde{T}$ be a $(\Gamma,\mathbf{p})$-Galton-Watson tree such that the mark of the root is U^{-1}, and $\widetilde{T}[1]$ and $\widetilde{T}[2]$ are independent of $T[1]$ and $T[2]$. Write $\widetilde{\phi}$ for the conductance of $\widetilde{T}$ and $\widetilde{\phi}_i=\phi(\widetilde{T}[i])$ for $i=1,2$. By Tonelli's theorem and the definition of κ, we have

$$\begin{aligned}
\mathbf{E}\left[\log\left(\frac{1-U\phi}{1-U}\right)\kappa(\phi)\right] &= 2\mathbf{E}\left[\frac{\phi_1}{\phi_1+\phi_2}\log\left(\frac{\phi_1}{\phi_1+\phi_2}\right)h\Big(h\Big(U^{-1},\phi_1+\phi_2\Big),\widetilde{\phi}_1+\widetilde{\phi}_2\Big)\right] \\
&= 2\mathbf{E}\left[\frac{\phi_1}{\phi_1+\phi_2}\log\left(\frac{\phi_1}{\phi_1+\phi_2}\right)h\Big(h\Big(U^{-1},\widetilde{\phi}_1+\widetilde{\phi}_2\Big),\phi_1+\phi_2\Big)\right] \\
&= 2\mathbf{E}\left[\frac{\phi_1}{\phi_1+\phi_2}\log\left(\frac{\phi_1}{\phi_1+\phi_2}\right)h\Big(\widetilde{\phi},\phi_1+\phi_2\Big)\right] \\
&= 2\mathbf{E}\left[\frac{\phi_1}{\phi_1+\phi_2}\log\left(\frac{\phi_1}{\phi_1+\phi_2}\right)\frac{\widetilde{\phi}(\phi_1+\phi_2)}{\widetilde{\phi}+\phi_1+\phi_2-1}\right],
\end{aligned}$$

where, between the first and the second line, we have used the associativity and the symmetry of the function h. The proof of (3.26) is complete.

Now, we want to show that

$$\mathbf{E}[-\log(1-U)\kappa(\phi)] = \mathbf{E}\left[\frac{\phi_1\phi_2}{\phi_1+\phi_2-1}\right]. \tag{3.27}$$

Here, we rely heavily on the fact that U is uniform on $(0,1)$. From [10, equation (13)], we know that, for any function $g:[1,\infty)\to\mathbb{R}_+$ such that $g(x)$ and $g'(x)$ are both $o(x^a)$ for some a in $(0,\infty)$, we have

$$\mathbf{E}[g(\phi_1+\phi_2)] = \mathbf{E}[\phi_1(\phi_1-1)g'(\phi_1)]+\mathbf{E}[g(\phi_1)]. \tag{3.28}$$

As before, let ϕ_1, ϕ_2, $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$ be independent copies of $\phi(T)$, independent of U. Let $\psi_1:(1,\infty)^3\to(1,\infty)$ be defined by

$$\psi_1(x,y,z)=h(x,h(y,z))=\frac{xyz}{xy+yz+xz-x-y-z+1}.$$

By definition of κ, we have

$$\mathbf{E}[-\log(1-U)\kappa(\phi)] = \mathbf{E}\Big[-\log(1-U)\psi_1\Big(U^{-1},\phi_1+\phi_2,\widetilde{\phi}_1+\widetilde{\phi_2}\Big)\Big].$$

For x, y, z in $(1,\infty)$, let

$$\psi_2(x,y,z) = \psi_1(x,y,z) + y(y-1)\partial_y \psi_1(x,y,z) = \frac{x^2y^2z^2}{(xy+xz+yz-x-y-z+1)^2}.$$

Reason conditionally on U, $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$ and apply the identity (3.28) to the function $y \mapsto \psi_1(x,y,z)$, to obtain

$$\mathbf{E}[-\log(1-U)\kappa(\phi)] = \mathbf{E}\Big[-\log(1-U)\psi_2\Big(U^{-1},\phi_1,\widetilde{\phi}_1+\widetilde{\phi}_2\Big)\Big].$$

Playing the same game again, we obtain

$$\mathbf{E}[-\log(1-U)\kappa(\phi)] = \mathbf{E}\Big[-\log(1-U)\psi_3\Big(U^{-1},\phi_1,\widetilde{\phi}_1\Big)\Big],$$

with the function ψ_3 defined by

$$\begin{aligned}\psi_3(x,y,z) &= \psi_2(x,y,z) + z(z-1)\partial_z\psi_2(x,y,z)\\ &= (xyz)^2\left[\frac{2xyz}{(xy+xz+yz-x-y-z+1)^3} - \frac{1}{(xy+xz+yz-x-y-z+1)^2}\right].\end{aligned}$$

Fix y and z in $(1,\infty)$ and let, for u in $(0,1)$,

$$\begin{aligned}\psi_4(u) &= \psi_3(u^{-1},y,z)\\ &= y^2z^2\left[\frac{2yz}{[(yz+1-y-z)u+(y+z-1)]^3} - \frac{1}{[(yz+1-y-z)u+(y+z-1)]^2}\right]\\ &= (a+b)^2\left[\frac{2(a+b)}{(au+b)^3} - \frac{1}{(au+b)^2}\right],\end{aligned}$$

with $a=(yz+1-y-z)$ and $b=(y+z-1)$. Finally, integrating by parts gives

$$\int_0^1 -\log(1-u)\psi_4(u)\,\mathrm{d}u = \frac{a+b}{b} = \frac{yz}{y+z-1},$$

so that, by independence of U, ϕ_1 and $\widetilde{\phi}_1$,

$$\mathbf{E}\left[-\log(1-U)\psi_3\left(U^{-1},\phi_1,\widetilde{\phi_1}\right)\Big|\phi_1,\widetilde{\phi}_1\right] = \frac{\phi_1\widetilde{\phi}_1}{\phi_1+\widetilde{\phi}_1-1},$$

which completes the proof of (3.27), and the verification of the consistency of formula (3.25) with (3.23).

4 Transient λ-biased random walk on a Galton-Watson tree

4.1 The dimension of the harmonic measure

Let $t \in \mathscr{T}^*$, the set of all infinite trees, and fix $\lambda > 0$. In this chapter, we always see trees as marked trees with all marks equal to 1 and endow their boundaries with the metric $\mathrm{d}_{\mathcal{U}_\infty}$. Recall that the λ-biased random walk on t_* is the Markov chain whose transition probabilities are the following :

$$\mathrm{P}^t(x,y) = \begin{cases} 1 & \text{if } x = \varnothing_* \text{ and } y = \varnothing; \\ \dfrac{\lambda}{\lambda + \nu_t(x)} & \text{if } y = x_*; \\ \dfrac{1}{\lambda + \nu_t(x)} & \text{if } y \text{ is a child of } x; \\ 0 & \text{otherwise.} \end{cases}$$

For x in t, we write P^t_x for a probability measure under which the process $(X_n)_{n\geq 0}$ is the Markov chain on t, starting from x, with transition kernel P^t. Recall the definition of the conductance $\beta(t)$ from Section 1.7:

$$\beta(t) = \mathrm{P}^t_{\varnothing}(\tau_{\varnothing_*} = \infty)$$

and the fact that $\beta(t) > 0$ if and only if the random walk $(X_n)_{n\geq 1}$ is transient on t. We have also established in Section 1.7 the recursive equation

$$\beta(t) = \frac{\sum_{i=1}^{\nu_t(\varnothing)} \beta(t[i])}{\lambda + \sum_{i=1}^{\nu_t(\varnothing)} \beta(t[i])}. \tag{4.1}$$

Consider the Borel set

$$A = \{t \in \mathscr{T}^* : \beta(t) > 0\},$$

and the set A^{o} of everywhere transient trees (for the λ-biased random walk). For $t \in A^{\mathrm{o}}$, denote by HARM_t the harmonic measure on ∂t associated to the λ-biased random walk and recall that HARM is a flow rule on A^{o}. Equation (1.12) now becomes

$$\forall 1 \leq i \leq \nu_t(\varnothing),\ \mathsf{HARM}_t(i) = \frac{\beta(t[i])}{\sum_{j=1}^{\nu_t(\varnothing)} \beta(t[j])}. \tag{4.2}$$

Under a probability measure $\mathbf{P}$, let T be a Galton-Watson tree, of supercritical reproduction law $\mathbf{p}$, whose *finite* mean is $m > \lambda$. We use the notations $\mathbf{P}^*$, $\mathbf{GW}^*$, ..., from Chapter 2. Since $m > \lambda$, we know from [39], that $\mathbf{P}^*$-almost surely, the λ-biased random walk on T is transient, so that $\mathbf{GW}^*(A) = \mathbf{GW}^*(A^{\mathrm{o}}) = 1$. We are now ready to use the machinery described in Section 2.9. Set $J = (0, \infty)$ if $\lambda \geq 1$ and $J = (1-\lambda, \infty)$ if $\lambda < 1$. In the latter case, the fact that $\mathbf{P}^*$-almost surely $\beta(T) > 1 - \lambda$ comes from the Rayleigh principle and the fact that $p_1 < 1$, comparing the conductance of the whole tree to the one of a tree with a unique ray. Set, for u and v in J,

$$h(u,v) = \frac{uv}{u + v + \lambda - 1},$$

with, the extension that $h(u,v) = 0$ whenever at least one of its argument is null. Notice that we are in the setting of the second example of Section 2.9 if $\lambda < 1$ and of the third if $\lambda \geq 1$, so that h fulfills the algebraic assumptions stated in Section 2.9. Moreover, since for $u \in J$, $u + \lambda - 1 \geq 0$, we have for all $u, v \in J$,

$$h(u,v) \leq \frac{uv}{v} \leq u. \tag{4.3}$$

By equation (4.1), for all $t \in A^{\mathrm{o}}$,

$$\beta(t) = h\Big(1, \sum\nolimits_{i=1}^{\nu_t(\varnothing)} \beta(t[i])\Big) = h\Big(\mathsf{mk}_t(\varnothing), \sum\nolimits_{i=1}^{\nu_t(\varnothing)} \beta(t[i])\Big).$$

For $u \in J$, let

$$\kappa(u) = \mathbf{E}\Big[h\Big(u, \sum\nolimits_{i=1}^{\widetilde{\nu}} \beta(\widetilde{T}^i)\Big)\Big] = \mathbf{E}\left[\frac{u \sum_{i=1}^{\widetilde{\nu}} \beta(\widetilde{T}^i)}{\lambda - 1 + u + \sum_{i=1}^{\nu_{\widetilde{T}}(\varnothing)} \beta(\widetilde{T}^i)}\right], \tag{4.4}$$

where $\widetilde{\nu}$, $\widetilde{T}^1$, $\widetilde{T}^2$, ..., are independent and independent of T, with ν distributed as $\nu_T(\varnothing)$ and $\widetilde{T}^1$, $\widetilde{T}^2$, ..., distributed as T.

To use Theorem 2.28, we remark that, by (4.3)

$$\kappa(\beta(T)) \leq \beta(T) \leq 1, \tag{4.5}$$

and therefore is integrable. The following theorem was independently discovered by Lin in [35].

Theorem 4.1. *The probability measure μ_{HARM} of density $C^{-1}\kappa(\beta(T))$, with respect to $\mathbf{GW}^*$, where κ is defined by (4.4) and $C = \mathbf{E}^*[\kappa(\beta(T))]$ is HARM-invariant. The dimension of HARM_T equals $\mathbf{P}^*$-almost surely*

$$d_\lambda = \log(\lambda) - C^{-1}\mathbf{E}^*\left[\log(1 - \beta(T)) \frac{\beta(T) \sum_{i=1}^{\widetilde{\nu}} \beta(\widetilde{T}^i)}{\lambda - 1 + \beta(T) + \sum_{i=1}^{\widetilde{\nu}} \beta(\widetilde{T}^i)}\right]. \tag{4.6}$$

Proof. The only statement we still need to prove is the formula for the dimension. We let $\mu = \mu_{\mathsf{HARM}}$ for short and $\mathbf{E}^*_\mu[\,\cdot\,] \coloneqq \mathbf{E}^*[\,\cdot\, C^{-1}\kappa(\beta(T))]$. By Corollary 2.19 and equation (4.2),

$$d_\lambda = \mathbf{E}^*_\mu\left[\log \frac{1}{\mathsf{HARM}_T(\Xi_1)}\right] = \mathbf{E}^*_\mu\left[\log \frac{\sum_{i=1}^{\nu_T(\varnothing)} \beta(T[i])}{\beta(T[\Xi_1])}\right].$$

Using equation (4.1), we see that

$$\sum\nolimits_{i=1}^{\nu_T(\varnothing)} \beta(T[i]) = \frac{\lambda\beta(T)}{1-\beta(T)}.$$

Therefore,

$$d_\lambda = \log\lambda + \mathbf{E}^*_\mu[-\log(1-\beta(T)) + \log(\beta(T)) - \log(\beta(T[\Xi_1]))].$$

We are done if we can prove that $\log(\beta(T)) - \log(\beta(T[\Xi_1]))$ is integrable with integral 0 with respect to $\mathbf{E}^*_\mu$. By invariance and Lemma 2.24, it is enough to show that it is bounded from below by an integrable function. We compute, using again formula (1.7):

$$\frac{\beta(T)}{\beta(T[\Xi_1])} = \frac{1+\beta(T[\Xi_1])^{-1}\sum_{i=1,\,i\neq\Xi_1}^{\nu_T(\varnothing)} \beta(T[i])}{\lambda + \beta(T[\Xi_1]) + \sum_{i=1,\,i\neq\Xi_1}^{\nu_T(\varnothing)} \beta(T[i])} \geq \frac{1}{\lambda+\beta(T[\Xi_1])} \geq \frac{1}{\lambda+1}, \tag{4.7}$$

where, for the first inequality, we used the fact that the function

$$x \longmapsto \frac{\beta(T[\Xi_1])^{-1}x+1}{x+\lambda+\beta(T[\Xi_1])}$$

is increasing on $[0,\infty)$. □

4.2 Comparison of flow rules on a Galton-Watson tree

Since we want to compare the harmonic measures for different values of λ, we now denote by HARM^λ the harmonic measure with respect to the λ-biased random walk, for $0 < \lambda < m$ and by $\beta^\lambda(T)$ for the associated conductance. The density of the HARM^λ-invariant measure in Theorem 4.1 with respect to $\mathbf{GW}^*$ is denoted here by f^λ.

Together with VIS and UNIF, the harmonic measures form a class of flow rules for which we know explicit invariant probability measures absolutely continuous with respect to $\mathbf{GW}^*$.

Proposition 4.2. *Unless the reproduction law is degenerated, for any $\Theta \neq \Theta'$ in the class of flow rules*

$$\{\mathsf{VIS},\ \mathsf{UNIF}\} \cup \{\mathsf{HARM}^\lambda : \lambda \in (0,m)\},$$

we have $\mathbf{P}^*(\Theta_T = \Theta'_T) = 0$ and additionally $\mathbf{P}^*(\Theta_T \perp \Theta'_T) = 1$ whenever Θ and Θ' are not UNIF or $\sum_{k\geq1} p_k k \log k < \infty$.

Proof. We first show that for $0 < \lambda \neq \lambda' < m$, $\mathbf{P}^*(\mathsf{HARM}_T^\lambda = \mathsf{HARM}_T^{\lambda'}) < 1$. This will be enough to conclude by Proposition 2.20 and the previous theorem. So assume that $\mathbf{P}^*(\mathsf{HARM}_T^\lambda = \mathsf{HARM}_T^{\lambda'}) = 1$. By Lemma 2.21, there exists $C \in (0,\infty)$ such that $\mathbf{P}^*$-almost surely, $\beta^\lambda(T) = C\beta^{\lambda'}(T)$. The recursive equation (4.1) gives that, $\mathbf{P}^*$-almost surely,

$$\frac{\lambda}{\sum_{i\in T_1^*}\beta^{\lambda'}(T[i])} + C = \frac{\lambda'}{\sum_{i\in T_1^*}\beta^{\lambda'}(T[i])} + 1,$$

thus $C \neq 1$. This implies that, $\mathbf{P}^*$-almost surely,

$$\sum_{i\in T_1^*}\beta^{\lambda'}(T[i]) = \frac{\lambda-\lambda'}{1-C}.$$

Taking the conditional expectation with respect to $\nu_T^*(\varnothing)$, we obtain

$$\nu_T^*(\varnothing)\mathbf{E}^*[\beta(T)] = \frac{\lambda-\lambda'}{1-C},$$

hence the distribution of $\nu_T^*(\varnothing)$ under $\mathbf{P}^*$ is degenerated and, since $m > 1$, this may only happen when the reproduction law is itself degenerated.

Now we turn to VIS and HARM^λ and assume that $\mathbf{P}^*(\mathsf{VIS}_T = \mathsf{HARM}_T^\lambda) = 1$. Again by Lemma 2.21, we may assume that there exists $C \in (0,\infty)$ such that $\mathbf{P}^*$-almost surely, $\beta^\lambda(T) = C\mathbf{1}_{\{T\in\mathscr{T}^*\}} = C$. Equation (4.1) now yields

$$\frac{1}{C} = \frac{\lambda}{C\nu_T^*(\varnothing)} + 1,$$

which again implies that the reproduction law is degenerated.

Finally, if $\sum_{k\geq 1} p_k k\log k < \infty$ and there exists $C \in (0,\infty)$ such that $\beta^\lambda(T) = CW(T)$, then we find that, by eq. (2.1),

$$\beta^\lambda(T) = CW(T) = \frac{C}{m}\sum_{i\in T_1^*} C\beta^\lambda(T[i]),$$

hence, by eq. (4.1),

$$\frac{1}{\lambda + \sum_{i\in T_1^*}\beta^\lambda(T[i])} = \frac{C^2}{m},$$

and we may conclude by similar arguments.

The comparison of UNIF and VIS is not necessary since we already know that they have distinct dimensions. $\square$

Next, given two flow rules Θ and Θ' in this class, we compute the local dimension of Θ_T for Θ'_T-almost every ray. Some of the assertions in the following proposition are minor improvements of [38, Theorem 3].

Proposition 4.3. *Assume that the reproduction law is not degenerated. Let λ and λ' be distinct elements of $(0,m)$. For $\mathbf{GW}^*$-almost every t:*

1. *If* $\sum_{k\geq 1} p_k k \log k < \infty$, *then for* UNIF_t*-almost every* ξ *in* ∂t,

$$\lim_{n\to\infty} \frac{-1}{n} \log \mathsf{VIS}_t(\xi_n) = \frac{1}{m}\mathbf{E}^*[\nu_T^*(\varnothing)\log \nu_T^*(\varnothing)] \in \Big(\log m, \infty\Big) \quad \textit{and}$$

$$\lim_{n\to\infty} \frac{-1}{n} \log \mathsf{HARM}_t^\lambda(\xi_n) = \log\lambda - \mathbf{E}[\log(1-\beta^\lambda(T))W(T)] \in (\log m, \infty).$$

2. *With the only assumption that* $1 < m < \infty$, *for* VIS_t*-almost every* ξ *in* ∂t,

$$\lim_{n\to\infty} \frac{-1}{n} \log \mathsf{UNIF}_t(\xi_n) = \log m \quad \textit{and}$$

$$\lim_{n\to\infty} \frac{-1}{n} \log \mathsf{HARM}_t^\lambda(\xi_n) = \log\lambda - \mathbf{E}^*[\log(1-\beta^\lambda(T))] \in (\mathbf{E}^*[\log(\nu_T^*(\varnothing))], \infty).$$

3. *With the only assumption that* $1 < m < \infty$, *for* HARM_t^λ*-almost every* ξ *in* ∂t,

$$\lim_{n\to\infty} \frac{-1}{n} \log \mathsf{UNIF}_t(\xi_n) = \log m;$$

$$\lim_{n\to\infty} \frac{-1}{n} \log \mathsf{HARM}_t^{\lambda'}(\xi_n) = \log\lambda' - \mathbf{E}^*[\log(1-\beta^{\lambda'}(T))f^\lambda(T)] \in (d_\lambda, \infty) \quad \textit{and}$$

$$\lim_{n\to\infty} \frac{-1}{n} \log \mathsf{VIS}_t(\xi_n) = \mathbf{E}^*[\log(\nu_T^*(\varnothing))f^\lambda(T)] \in (d_\lambda, \infty).$$

Proof. For the first point, recall that the flow rule VIS is associated to the function $\phi : t \mapsto \mathbf{1}_{\{t\in\mathscr{T}^*\}}$, so we have, by Proposition 2.22, for $\mathbf{GW}^*$-almost every t, for UNIF_t-almost every ξ,

$$\lim_{n\to\infty} \frac{-1}{n} \log \mathsf{VIS}_t(\xi_n) \xrightarrow[n\to\infty]{} \mathbf{E}[W(T)\log \nu_T^*(\varnothing)].$$

Decomposing with respect to the value of $\nu_T(\varnothing)$ and using the recursive equation for W yield

$$\begin{aligned}\mathbf{E}[W(T)\log \nu_T^*(\varnothing)] &= \sum_{k\geq 1} \frac{1}{m} \sum_{i=1}^{k} \mathbf{E}[\mathbf{1}_{\{\nu_T(\varnothing)=k\}} W(T[i]) \log \nu_T^*(\varnothing)] \\ &= \sum_{k\geq 1} \frac{1}{m} k \mathbf{E}[\mathbf{1}_{\{\nu_T(\varnothing)=k\}} W(T[1]) \log \nu_T^*(\varnothing)], \end{aligned} \tag{4.8}$$

by symmetry. Now, for $k_1 \leq k_2$ and l integers, consider the event

$$A_{k_1,k_2,l} = \{\#\{k_1 \leq i \leq k_2 : T^i \in \mathscr{T}^*\} = l\}.$$

We have, for $k \geq 1$ and $l \geq 1$,

$$\{\nu_T(\varnothing) = k,\ \nu_T^*(\varnothing) = l,\ T^1 \in \mathscr{T}^*\} = \{\nu_T(\varnothing) = k,\ T^1 \in \mathscr{T}^*,\ A_{2,k,l-1}\},$$

so we can use the branching property in the last expectation:

$$\begin{aligned}\mathbf{E}[\mathbf{1}_{\{\nu_T(\varnothing)=k\}} W(T[1]) \log \nu_T^*(\varnothing)] &= p_k \sum_{l=1}^{k} \log(l) \mathbf{E}[\mathbf{1}_{\{T^1\in\mathscr{T}^*\}} W(T^1) \mathbf{1}_{A_{2,k,l-1}}] \\ &= p_k \sum_{l=1}^{k} \log(l) \mathbf{E}[\mathbf{1}_{\{T^1\in\mathscr{T}^*\}} W(T^1)] \mathbf{P}(A_{2,k,l-1})\end{aligned}$$

Again by the branching property,

$$\mathbf{P}(A_{2,k,l-1}) = \binom{k-1}{l-1} q^{k-l}(1-q)^{l-1} = \frac{l}{k(1-q)}\binom{k}{l} q^{k-l}(1-q)^{l-1}$$
$$= \frac{l}{k(1-q)}\mathbf{P}(\nu_T^*(\varnothing) = l \mid \nu_T(\varnothing) = k).$$

This implies that

$$\mathbf{E}[\mathbf{1}_{\{\nu_T(\varnothing)=k\}} W(T[1]) \log \nu_T^*(\varnothing)] = \frac{p_k}{(1-q)k}\sum_{l=1}^{k} l \log l \binom{k}{l} q^{k-l}(1-q)^l$$
$$= \frac{p_k}{(1-q)k}\mathbf{E}[\nu_T^*(\varnothing) \log \nu_T^*(\varnothing) \mid \nu_T(\varnothing) = k],$$

with the usual convention $0 \log 0 = 0$. Getting back to (4.8), we obtain

$$\mathbf{E}[W(T) \log \nu_T^*(\varnothing)] = \frac{1}{m(1-q)}\mathbf{E}[\nu_T^*(\varnothing) \log \nu_T^*(\varnothing)] = \frac{1}{m}\mathbf{E}^*[\nu_T^*(\varnothing) \log \nu_T^*(\varnothing)].$$

This number is strictly greater than $\log m$ by Proposition 2.22 (or more directly by Jensen's inequality) and is finite because under $\mathbf{P}$, $\nu_T^*(\varnothing) \leq \nu_T(\varnothing)$ and we made the assumption that $\sum_{k\geq 1} p_k k \log k < \infty$.

The limit in the second assertion of the first point is an immediate consequence of the inequality (4.7) and Proposition 2.22. To see that it is finite, it suffices to remark that

$$\log \lambda - \log(1 - \beta(T)) = \log(\lambda + \sum_{i=1}^{\nu_T(\varnothing)} \beta(T[i])) \leq \log(\lambda + \nu_T(\varnothing)),$$

and using the fact that $W(T)$ is the limit of the regular martingale $(Z_n(T)/m^n)$,

$$\mathbf{E}[\log(\lambda + \nu_T(\varnothing))W(T)] = \mathbf{E}[\log(\lambda + \nu_T(\varnothing))\frac{\nu_T(\varnothing)}{m}] < \infty,$$

by assumption.

The second point is a direct consequence of inequalities (2.11) and (4.7) together with Proposition 2.22.

So is the last point (use inequality (4.5) to prove the last two integrabilities). □

4.3 Numerical results

We will now use our formula to conduct numerical experiments about the dimension d_λ as a function of λ.

It was asked in [45] whether d_λ is a monotonic function of λ, for λ in $(0, m)$. To the best of our knowledge, this question is still open. We were not able to find a theoretical answer. However, using formula (4.6) together with the recursive equation (1.7), we are able to draw a credible enough graph of d_λ versus λ, for a given (computationally reasonable) reproduction law.

The idea is the following. Fix a reproduction law $\mathbf{p}$ of finite mean $m > 1$ and a bias λ in $(0, m)$. For any non-negative integer n, and a Galton-Watson tree T, let $\beta_n(T) = \mathrm{P}_\varnothing^T\left(\tau^{(n)} < \tau_{\varnothing_*}\right)$, where $\tau^{(n)}$ is the first hitting time of level n by the λ-biased random walk $(X_n)_{n\geq 0}$. Since the family of events $\{\tau^{(n)} < \tau_{\varnothing_*}\}$ is decreasing, we have

$$\beta(T) = \mathrm{P}_\varnothing^T\Big(\bigcap_{n\geq 1}\left\{\tau^{(n)} < \tau_{\varnothing_*}\right\}\Big) = \lim_{n\to\infty}\beta_n(T).$$

Using the Markov property as in (1.7) yields the recursive equation

$$\beta_{n+1}(T) = \frac{\sum_{i=1}^{\nu_T(\varnothing)}\beta_n(T[i])}{\lambda + \sum_{i=1}^{\nu_T(\varnothing)}\beta_n(T[i])}.$$

By definition, $\beta_0(T)$ is equal to one. Hence, we may use the following algorithm to compute the law of $\beta_n \coloneqq \beta_n(T)$:

— the law of β_0 is the Dirac measure δ_1;

— for any $n \geq 0$, assuming we know the law of β_n, the law of β_{n+1} is the law of the random variable

$$\frac{\sum_{i=1}^{\nu}\beta_n^{(i)}}{\lambda + \sum_{i=1}^{\nu}\beta_n^{(i)}},$$

where ν, $\beta_n^{(1)}$, $\beta_n^{(2)}$, ... are independent, ν has the law $\mathbf{p}$ and each $\beta_n^{(i)}$ has the law β_n.

Using the preceding algorithm, after n iteration, we obtain the law of $\beta_n(T)$. Since $\beta_n(T) \to \beta(T)$, almost surely, we also have convergence in law.

Remark 4.1. The preceding discussion shows that the law of β is the greatest (for the stochastic partial order) solution of the recursive equation (1.7). In [45, Theorem 4.1], for $\lambda = 1$, the authors show that the only solutions to this recursive equation are the Dirac measure δ_0 and the law of β. However, their proof cannot be adapted to the more general case $\lambda \in (0, m)$. That is why, here, we had to choose for our initial measure, the Dirac measure δ_1.

For the numerical computations, we discretize the interval $[0, 1]$ and apply the preceding algorithm with some fixed final value of n. See Figure 3.1 for an example of what one can obtain with 100 iterations and a discretization step equal to 1/20000.

Before we compute the dimension, we simplify a little bit the formula (4.6). First, notice that we may write

$$\sum_{j=1}^{\nu}\beta(\widetilde{T}^j) = \frac{\lambda\widetilde{\beta}}{1-\widetilde{\beta}}, \tag{4.9}$$

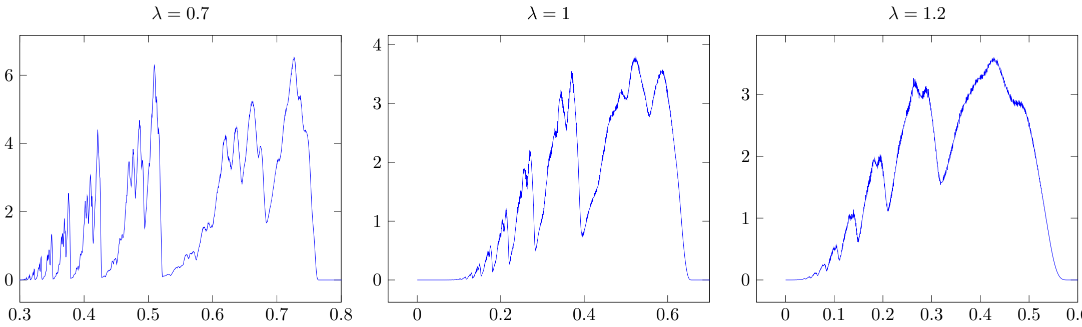

Figure 3.1 – The apparent density of the conductance β, for $p_1 = p_2 = p_3 = \frac{1}{3}$ and λ in $\{0.7, 1, 1.2\}$.

where $\widetilde{\beta}$ is an independent copy of $\beta = \beta(T)$. Recalling that the constant C in (4.6) is the expectation of $\kappa(\beta)$, we obtain

$$d_\lambda = \log(\lambda) - \mathbf{E}^*\left[\frac{\log(1-\beta)\beta\widetilde{\beta}}{\lambda - 1 + \beta + \widetilde{\beta} - \beta\widetilde{\beta}}\right] \Big/ \mathbf{E}^*\left[\frac{\beta\widetilde{\beta}}{\lambda - 1 + \beta + \widetilde{\beta} - \beta\widetilde{\beta}}\right]. \tag{4.10}$$

From there, computing d_λ from a dicretized approximation of the law of β is straightforward.

From [35], we know that d_λ goes to $\mathbf{E}^*\left[\log(\nu_T^*(\emptyset))\right]$ (the almost sure dimension of the visibility measure, see [43, Section 4]) as λ goes to 0, and to $\log m$ as λ goes to m.

We also compute numerically the speed. Recall from [3], that the speed of the λ-biased transient random walk for $\lambda \in (\mathrm{g}'(q), m)$ is given by

$$\ell_\lambda = \mathbf{E}^*\left[\frac{(\nu - \lambda)\beta_0}{\lambda - 1 + \beta_0 + \sum_{j=1}^{\nu}\beta_j}\right] \Big/ \mathbf{E}^*\left[\frac{(\nu + \lambda)\beta_0}{\lambda - 1 + \beta_0 + \sum_{j=1}^{\nu}\beta_j}\right],$$

where ν, β_0, β_1, ... are independent and ν has law $\mathbf{p}$, while for each i, β_i has law $\beta(T)$. Using first symmetry and then (4.9), one obtains

$$\mathbf{E}^*\left[\frac{(\nu \pm \lambda)\beta_0}{\lambda - 1 + \beta_0 + \sum_{j=1}^{\nu}\beta_i}\right] = \mathbf{E}^*\left[\frac{\left(\sum_{j=1}^{\nu}\beta_j\right) \pm \lambda\beta_0}{\lambda - 1 + \beta_0 + \sum_{j=1}^{\nu}\beta_j}\right] = \mathbf{E}^*\left[\frac{\lambda\left(\widetilde{\beta} \pm \beta \mp \beta\widetilde{\beta}\right)}{\lambda - 1 + \beta + \widetilde{\beta} - \widetilde{\beta}\beta}\right],$$

where we have denoted $\beta = \beta_0$ and $\sum_{j=1}^{\nu}\beta_i = \lambda\widetilde{\beta}/(1 - \widetilde{\beta})$. Finally, we may express the speed as

$$\ell_\lambda = \mathbf{E}^*\left[\frac{\beta\widetilde{\beta}}{\lambda - 1 + \beta + \widetilde{\beta} - \beta\widetilde{\beta}}\right] \Big/ \mathbf{E}^*\left[\frac{\beta + \widetilde{\beta} - \beta\widetilde{\beta}}{\lambda - 1 + \beta + \widetilde{\beta} - \beta\widetilde{\beta}}\right]. \tag{4.11}$$

We also recall that, in the case $\lambda = 1$, it was shown in [43] that the speed of the random walk equals

$$\ell_1 = \mathbf{E}^*\left[\frac{\nu - 1}{\nu + 1}\right].$$

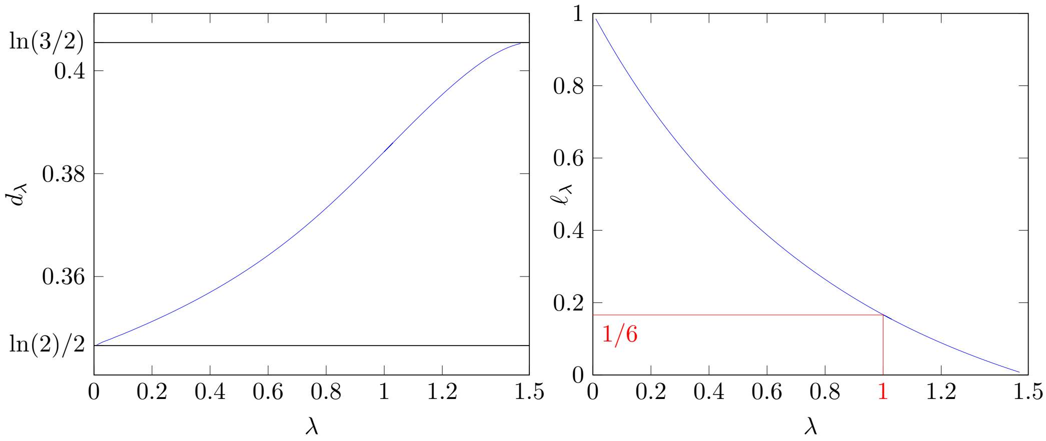

Figure 3.2 – The dimension and the speed of the λ-biased random walk on a Galton-Watson tree as functions of λ, for $p_1 = p_2 = 1/2$.

We have made the numerical computations in two cases, the first one is when the reproduction law is given by $p_1 = p_2 = 1/2$, see Figure 3.2 and the second one is for $\mathbf{p}$ given by $p_1 = p_2 = p_3 = 1/3$, see Figure 3.3.

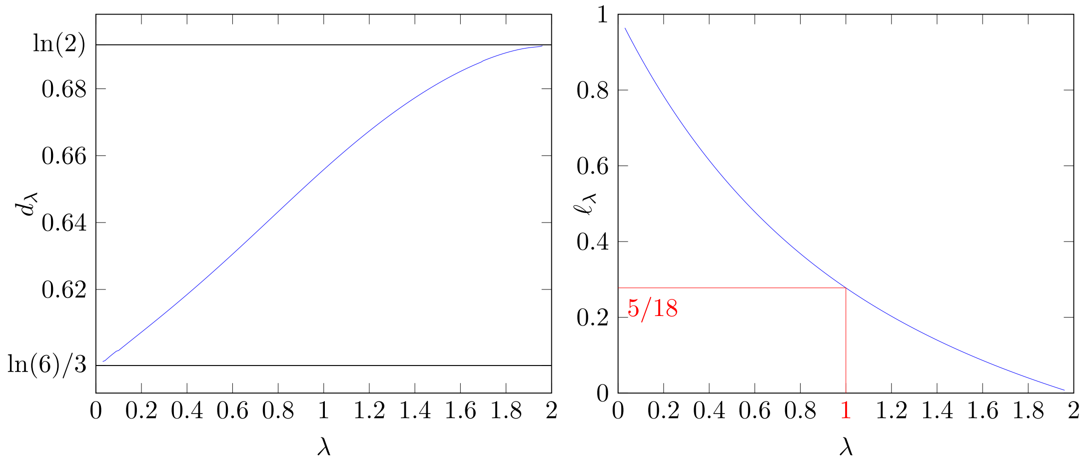

Figure 3.3 – The dimension and the speed of the λ-biased random walk on a Galton-Watson tree as functions of λ, for $p_1 = p_2 = p_3 = 1/3$.

These figures suggest that the speed and the dimension are indeed monotonic with respect to λ. Furthermore, the speed looks convex, while the dimension seems to be neither convex nor concave (there might be an inflection point at 1 but it is perhaps too bold a conjecture to make at this point).

5 Transient random walk on a weighted Galton-Watson tree

5.1 Presentation of the model

A weighted tree is a tree t together with a weight function $\mathsf{A}_t : t \setminus \{\varnothing\} \to (0, \infty)$ and that we associate to it a probability kernel P^t on $t_* = t \cup \{\varnothing_*\}$ defined by

$$\mathrm{P}^t(x, y) = \begin{cases} \dfrac{\mathsf{A}_t(xi)}{1 + \sum_{j=1}^{\nu_t(\varnothing)} \mathsf{A}_t(xj)} & \text{if } y = xi, \text{ for } 1 \leq i \leq \nu_t(x); \\ \dfrac{1}{1 + \sum_{j=1}^{\nu_t(\varnothing)} \mathsf{A}_t(xj)} & \text{if } y = x_*. \end{cases}$$

In this chapter, we will only work with weighted trees, so to lighten notations, we will write t when we should write (t, A_t). We still, however, write $x \in t$ when we mean that a word x is a vertex of the weighted tree t.

We define the (local) distance between two weighted trees t and t' by

$$d_w(t, t') = \sum_{r \geq 0} 2^{-r-1} \delta_m^{(r)}(t, t'),$$

where $\delta_w^{(r)}$ is defined by

$$\delta_w^{(r)}(t, t') = \begin{cases} 1 \text{ if } t \text{ and } t' \text{ (without their weights) do not agree up to height } r; \\ \min(1, \sup\{|\mathsf{A}_t(x) - \mathsf{A}_{t'}(x)| : x \in t,\ 1 \leq |x| \leq r\}) \text{ otherwise.} \end{cases}$$

We denote by $\mathscr{T}_w$ the metric space of all *infinite* weighted trees. It is a Polish space. For a weighted tree t and a vertex $x \in t$, we denote by

$$t[x] = \{u \in \mathcal{U}_* : xu \in t\},$$

the reindexed subtree starting from x with weights

$$\mathsf{A}_{t[x]}(y) = \mathsf{A}_t(xy), \quad \forall y \in t[x] \setminus \{\varnothing, \varnothing_*\}.$$

For a weighted tree t and a vertex x in t, define $t^{\leq x}$ as the weighted tree

$$t^{\leq x} = \{y \in t : x \not< y\}.$$

together with the restriction of A_t to $t^{\leq x} \setminus \{\varnothing\}$. Notice that x is in $t^{\leq x}$.

For two weighted trees t and t', and $x \neq ø_*$ in t, we define the glued weighted tree $t^{\leq x} \triangleleft t'$ as the tree

$$t^{\leq x} \triangleleft t' = t^{\leq x} \cup \{xy : y \in t' \setminus \{ø_*, ø\}\}$$

together with the weights :

$$\mathsf{A}_{t^{\leq x} \triangleleft t'}(z) = \begin{cases} \mathsf{A}_t(z) & \text{if } x \not< z; \\ \mathsf{A}_{t'}(x^{-1}z) & \text{otherwise.} \end{cases}$$

Notice that in particular the weight of x in $t^{\leq x} \triangleleft t'$ is still $\mathsf{A}_t(x)$.

Next we introduce Galton-Watson weighted trees. This model has been introduced in [41] and has been extensively studied (see for instance [2, 28, 5, 4]). We will use the definition from [21], which is a generalization and can be described as follows. Under a probability $\mathbf{P}$, let A and $(\mathsf{A}^x)_{x\in\mathcal{U}}$ be i.i.d. random elements of the set

$$\mathsf{Tuples} = \bigcup_{k\geq 1} (0, \infty)^k$$

of all finite sequences of positive real numbers, with the convention that $(0, \infty)$ contains only the empty sequence $()$. Define the length of a sequence in the obvious way. Define the random Galton-Watson weighted tree T by

1. $ø \in T_0$;

2. for $i \geq 0$ and $j \in \mathbb{N}^*$, $xj \in \mathcal{U}$ is in T_{i+1} if and only if x is in T_i and j is not greater that the length of A^x, in which case, we let $\mathsf{A}_T(xj) = \mathsf{A}^x(j)$, the j-th component of the sequence A^x.

3. $T = \bigcup_{k\geq 1} T_i$.

For $x \in \mathcal{U}$, define T^x as the tree associated to the sequences $(\mathsf{A}^{xy})_{y\in\mathcal{U}}$ and remark that

1. On the event that $x \in T$, $T[x] = T^x$.

2. If Q is a subset of $\mathcal{U}$ made of pairwise incomparable elements, then the weighted trees $(T^x)_{x\in Q}$ are i.i.d. and independent of the random sequences $\{\mathsf{A}^y : y \not\geq x\}$.

The previous property will be referred to as *the branching property.*

Weighted Galton-Watson trees can easily be represented as *marked Galton-Watson trees* as they are defined in Section 2.6, so that all the results of chapter 2 are available to us in this context of weighted Galton-Watson trees. We again denote by $\mathbf{p} = (p_k)_{k\geq 0}$ the reproduction law of T. We assume for simplicity that $p_0 = 0$. We also assume that $p_1 < 1$ and that the mean m of $\mathbf{p}$ is finite. As before we denote by $\mathbf{GW}$ the distribution of T.

In [41, Theorem 3], we may find a transience criterion for the random walk $\mathbf{X}$ on T with transition matrix P^T, when the weights are i.i.d. It is generalized for our setting in [21, Theorem 1.1] (the integrability assumptions are not needed for the proof of the transient case).

Fact 5.1. *If* $\min_{s\in[0,1]} \mathbf{E}\Big[\sum_{i=1}^{\nu_T(ø)} \mathsf{A}_T(i)^s\Big] > 1$*, then for* $\mathbf{GW}$*-almost every weighted tree* t*, the random walk defined by* P^t *is transient.*

We will assume throughout this work that we are in this regime.

An infinite path in $\mathcal{U}_*$ is a sequence $\mathbf{x} = (x_0, x_1, \dots)$ such that for any $k \geq 0$, x_{k+1} is either a child of x_k or its parent. A *transient path* is an infinite path $\mathbf{x}$ such that $\lim_{k\to\infty}|x_k| = \infty$.

For such a path $\mathbf{x}$, we define:

— the set of *fresh times*:

$$\mathsf{ft}(\mathbf{x}) = \{s \geq 0 : \forall k < s,\ x_k \neq x_s\} = \{\mathsf{ft}_0(\mathbf{x}), \mathsf{ft}_1(\mathbf{x}), \dots\},$$

where $\mathsf{ft}_0(\mathbf{x}) < \mathsf{ft}_1(\mathbf{x}) < \cdots$;

— the set of *exit times*:

$$\mathsf{et}(\mathbf{x}) = \{s \geq 0 : \forall k > s,\ x_k \neq (x_s)_*\} = \{\mathsf{et}_0(\mathbf{x}), \mathsf{et}_1(\mathbf{x}), \dots\},$$

where $\mathsf{et}_0(\mathbf{x}) < \mathsf{et}_1(\mathbf{x}) < \cdots$;

— the *exit points*, $\mathsf{ep}_k(\mathbf{x}) \coloneqq x_{\mathsf{et}_k(\mathbf{x})}$, for $k = 0, 1, \dots$;

— the set of *regeneration times*:

$$\mathsf{rt}(\mathbf{x}) = \mathsf{ft}(\mathbf{x}) \cap \mathsf{et}(\mathbf{x}).$$

Likewise, the regeneration times (if there are any) are ordered $\mathsf{rt}_0(\mathbf{x}) < \mathsf{rt}_1(\mathbf{x}) < \cdots$, and if there are at least k regeneration times, $\mathsf{rp}_k(\mathbf{x}) = x_{\mathsf{rt}_k(\mathbf{x})}$ is the k-th *regeneration point* and $\mathsf{rh}_k(\mathbf{x}) = |\mathsf{rp}_k(\mathbf{x})|$ is the k-th *regeneration height.*

— For $u \in \mathcal{U}_*$, the *first hitting time* and the *first return time* of the path $\mathbf{x}$ to u are respectively

$$\tau_u(\mathbf{x}) = \inf\{s \geq 0 : x_s = u\}, \quad \tau_u^+(\mathbf{x}) = \inf\{s \geq 1 : x_s = u\},$$

with the convention $\inf \emptyset = +\infty$.

Any transient path $\mathbf{x}$ starting from $x_0 = \varnothing$ defines a ray

$$\mathrm{ray}(\mathbf{x}) = (\mathsf{ep}_0(\mathbf{x}), \mathsf{ep}_1(\mathbf{x}), \dots) \in \mathcal{U}_\infty.$$

In this chapter, we always endow $\mathcal{U}_\infty$ and its subsets with the natural distance $\mathrm{d}_{\mathcal{U}_\infty}$.

As before, we denote by $(X_k)_{k\geq 0}$ a random walk on $T_* = T \cup \{\varnothing_*\}$ with transition kernel P^T, and by Ξ the ray associated to it and by HARM_T the distribution of Ξ. Recall that HARM is a flow rule.

Let $\mathscr{T}_{w,p}$ be the space of all infinite trees t in $\mathscr{T}_w$ with a distinguished transient path $\mathbf{x}$ starting from the root. On $\mathscr{T}_{w,p}$, we define the distance $d_{w,p}$ by

$$d_{w,p}((t, \mathbf{x}), (t', \mathbf{x}')) = \sum_{r\geq 0} 2^{-r-1} \delta_{w,p}^{(r)}((t, \mathbf{x}), (t', \mathbf{x}')),$$

where $\delta_{w,p}^{(r)}((t, \mathbf{x}), (t', \mathbf{x}')) = 1$ if the vertices of t and of t' do not agree up to height r or if the paths $\mathbf{x}$ and $\mathbf{x}'$ do not coincide before the first time they reach height $r + 1$. Otherwise, $\delta_{w,p}^{(r)}((t, \mathbf{x}), (t', \mathbf{x}')) = \delta_w^{(r)}(t, t')$. The metric space $\mathscr{T}_{w,p}$ is again Polish.

We denote by $\mathbb{P}$ the "annealed" probability, that is, the probability associated to the expectation $\mathbb{E}$ defined by

$$\mathbb{E}[f(T,\mathbf{X})] = \mathbf{E}[\mathrm{E}_{\varnothing}^{T}[f(T,\mathbf{X})]],$$

for all suitable measurable functions f on $\mathscr{T}_{w,p}$.

The main result of this chapter is the following theorem:

Theorem 5.2 (Dimension drop for HARM)**.** *Let T be a random weighted Galton-Watson tree. The harmonic measure HARM_T is almost surely exact-dimensional and its Hausdorff dimension is almost surely a constant that equals*

$$\dim_{\mathrm{H}} \mathsf{HARM}_T = \mathbb{E}[-\log(\mathsf{HARM}_T(\Xi_1))\kappa(T)],$$

with κ defined on the space $\mathscr{T}_w$ by

$$\kappa(t) = \mathbf{E}\Big[\sum_{y\in T} \mathrm{P}_{\varnothing}^{T^{\leq y}\triangleleft t}(\mathsf{rp}_1 \succ y,\, \tau_{\varnothing_*} = \infty)\,.\Big] \tag{5.1}$$

It is almost surely strictly less than the Hausdorff dimension of the whole boundary ∂T (which is almost surely $\log m$), unless the model reduces to a transient λ-biased random walk (with a deterministic and constant $\lambda < m$) on an m-regular tree.

Our results are inspired by the work of Lyons, Peres and Pemantle on transient λ-biased random walks on Galton-Watson trees ([44]). We use in the same way the notions of exit times and regeneration times to build an invariant measure for the forward environment seen by the particle at exit times. The construction of this measure, via a Rokhlin tower, was already suggested in [44].

The chapter is organized as follows. In Section 5.2, we recall some basic results from ergodic theory. In Section 5.3, we show that there are almost surely infinitely many regeneration times and find an invariant measure for the forward environment seen by the particle at such times. Again, this follows the ideas of [44], but we give detailed proofs in our setting of weighted trees for completeness. The heart of this work is Section 5.4, where we give a detailed "tower construction" over the preceding dynamical system to build an invariant measure for the forward environment seen by the particle at exit times. We then show that this measure has a density with respect to the joint law of the tree and the random path on it and give an expression of this density. To conclude in Section 5.5, we project this measure on the space of trees with a random ray on it and we use the general theory of flow rules on Galton-Watson trees developed in [43].

5.2 Basic facts of ergodic theory

We recall here some definitions and basic properties which are used in this paper. The notations of this section are local to this section.

Definition 5.1. Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be two measurable spaces and let $S_X : X \to X$, $S_Y : Y \to Y$ be two measurable transformations. A *semi-conjugacy* between $(X, \mathcal{F}_X, S_X)$ and $(Y, \mathcal{F}_Y, S_Y)$ is a surjective measurable mapping $h : X \twoheadrightarrow Y$ such that $h \circ S_X = S_Y \circ h$.

One says that h is a *conjugacy* between $(X, \mathcal{F}_X, S_X)$ and $(Y, \mathcal{F}_Y, S_Y)$ if, in addition, the semi-conjugacy h is also injective.

The following well-known fact can be checked very directly, so we omit the proof.

Fact 5.3. *Let $(X, \mathcal{F}_X, S_X)$ and $(Y, \mathcal{F}_Y, S_Y)$ be two measurable spaces endowed with a measurable transformation. Let $h : X \twoheadrightarrow Y$ be a semi-conjugacy and μ_X be a probability measure on $\mathcal{F}_X$. Then, if the system $(X, \mathcal{F}_X, S_X, \mu_X)$ is measure-preserving (resp. ergodic, mixing), so is $(Y, \mathcal{F}_Y, S_Y, \mu_X \circ h^{-1})$.*

Definition 5.2. Let $(X, \mathcal{F}, S, \mu)$ be a measure-preserving system (with $\mu(X) = 1$) and A be in $\mathcal{F}$ such that $\mu(A) > 0$. For x in X, let

$$n_A(x) = \inf\{k \geq 1 : S^k(x) \in A\},$$

with the convention $\inf \emptyset := +\infty$. For B in $\mathcal{F}$, let $\mu_A(B) = \mu(A \cap B)/\mu(A)$ and for x in X, let $S_A(x) = S^{n_A(x)}(x)$ if $n_A(x)$ is finite and (say) $S_A(x) = x$ if $n_A(x) = \infty$. The *induced system* on A is defined as $(A, \mathcal{F} \cap A, S_A, \mu_A)$.

Lemma 5.4. *With the notations and assumptions of the previous definition, the system $(A, \mathcal{F} \cap A, S_A, \mu_A)$ is measure-preserving. Moreover, the whole system $(X, \mathcal{F}, S, \mu)$ is ergodic if and only if $\mu(\bigcup_{k \geq 1} S^{-k}(A)) = 1$ and $(A, \mathcal{F} \cap A, S_A, \mu_A)$ is ergodic.*

We provide a short proof of the "if" part, since we did not find it in the litterature (although it is well-known). For the other assertions, see for instance [15, Lemma 2.43].

Proof. For k in $\mathbb{N}^* \cup \{\infty\}$, let $A_k = \{x \in A : n_A(x) = k\}$. Notice that

$$A = A_\infty \sqcup \bigsqcup_{k \geq 1} A_k.$$

Let B be in $\mathcal{F}$ such that $S^{-1}(B) = B$. We prove that $A \cap B$ is S_A-invariant. Indeed,

$$S_A^{-1} = (A_\infty \cap B \cap A) \sqcup \bigsqcup_{k \geq 1} A_k \cap S^{-k}(B \cap A).$$

For $k \geq 1$, using the fact that $A_k \subset S^{-k}(A)$,

$$A_k \cap S^{-k}(B \cap A) = A_k \cap S^{-k}(B) \cap S^{-k}(A) = A_k \cap B,$$

thus we have

$$S_A^{-1}(B \cap A) = (A_\infty \cap B) \sqcup \bigsqcup_{k \geq 1} (A_k \cap B) = B \cap A.$$

Now, assume that $\mu(\bigcup_{k\geq 1} S^{-k}(A)) = 1$ and $(A, \mathcal{F}\cap A, S_A, \mu_A)$ is ergodic. By ergodicity and S_A-invariance, $\mu\,(B\cap A)$ equals 0 or $\mu(A)$. If it is 0, then

$$\mu(B) = \mu\Big(B\cap \bigcup_{k\geq 1} S^{-k}(A)\Big) \leq \sum_{k\geq 1} \mu\Big(B\cap S^{-k}(A)\Big) = 0,$$

since, for any $k\geq 1$,

$$\mu(B\cap S^{-k}(A)) = \mu(S^{-k}(B)\cap S^{-k}(A)) = \mu(B\cap A).$$

If $\mu(B\cap A) = \mu(A)$, we reason on the complement B^{c} of B, which is still invariant by S and satisfies $\mu(B^{\mathrm{c}}\cap A) = 0$. □

5.3 Regeneration Times

Let $(t, \mathbf{x})$ be in $\mathscr{T}_{w,p}$. Following [44, proof of Proposition 3.4], for any s in $\mathsf{ft}\,(\mathbf{x})$, we consider the tree and the path before time s:

$$\Phi_s(t, \mathbf{x}) = (t^{\leq x_s}, (x_i)_{0\leq i\leq s}).$$

Likewise if s is in $\mathsf{et}\,(\mathbf{x})$, the reindexed tree and path after time s is

$$\Psi_s(t, \mathbf{x}) = (t[x_s], \mathbf{x}[s]),$$

where

$$\mathbf{x}[s] = (x_s^{-1} x_{s+k})_{k\geq 0}.$$

By definition of fresh times and exit times, each path belongs to the corresponding tree.

For short, we write ft for $\mathsf{ft}(\mathbf{X})$, fp for $\mathsf{fp}(\mathbf{X})$, etc, and Ψ_s, Φ_s for $\Psi_s(T, \mathbf{X})$ and $\Phi_s(T, \mathbf{X})$. The following key lemma states that, at regeneration times, the branching property implies independence between the past (trajectory and environment) and the future.

Lemma 5.5 (Branching property at regeneration times)**.** *For s in $\mathbb{N}^*$, f and g measurable and non-negative functions,*

$$\mathbb{E}[\mathbf{1}_{\{s\in \mathsf{rt}\}} f\,(\Phi_s)\, g\,(\Psi_s)] = \mathbb{E}[\mathbf{1}_{\{s\in \mathsf{ft}\}} f(\Phi_s)]\mathbb{E}[g(T, \mathbf{X})\mathbf{1}_{\{\tau_{\varnothing_*}=\infty\}}].$$

Proof. First notice that

$$\{s\in \mathsf{rt}\} = \{s\in \mathsf{ft}\} \cap \{\forall k > s,\ X_k \neq (X_s)_*\}.$$

We decompose the expectation according to the value of X_s.

$$\begin{aligned}&\mathbb{E}[\mathbf{1}_{\{s\in \mathsf{rt}\}} f(\Phi_s) g(\Psi_s)]\\ &\quad = \sum_{x\in\mathcal{U}} \mathbf{E}\Big[\mathbf{1}_{\{x\in T\}} \mathrm{E}^T_{\varnothing}\Big[\mathbf{1}_{\{X_s = x,\, s\in \mathsf{ft}\}} f(T^{\leq x}, (X_i)_{0\leq i\leq s}) \mathbf{1}_{\{s\in \mathsf{et}\}} g\big(T[x], (x^{-1}X_{s+k})_{k\geq 0}\big)\Big]\Big].\end{aligned}$$

By the Markov property at time s, for any fixed x in T, the quenched expectation can be rewritten as

$$\begin{aligned}
&\mathrm{E}_{\varnothing}^{T}\Big[\mathbf{1}_{\{X_s=x,\, s\in\mathsf{ft}\}}f(T^{\leq x},(X_i)_{0\leq i\leq s})\mathbf{1}_{\{s\in\mathsf{et}\}}g\big(T[x],(x^{-1}X_{s+k})_{k\geq 0}\big)\Big] \\
&\qquad = \mathrm{E}_{\varnothing}^{T}\big[\mathbf{1}_{\{X_s=x,\, s\in\mathsf{ft}\}}f(T^{\leq x},(X_i)_{0\leq i\leq s})\big]\mathrm{E}_{x}^{T}\big[\mathbf{1}_{\{\tau_{x*}=\infty\}}g\big(T[x],(x^{-1}X_k)_{k\geq 0}\big)\big].
\end{aligned}$$

Now, the first quenched expectation is only a function of the weighted tree $T^{\leq x}$ while the second is only a function of $T[x]$. So we can use the branching property and sum over x in $\mathcal{U}$ to get the result. □

Lemma 5.6. *For* $\mathbf{GW}$*-almost every weighted tree* t*, for* $\mathrm{P}_{\varnothing}^{t}$*-almost every path* $\mathbf{x}$*, the set* $\mathsf{rt}\,(\mathbf{x})$ *is infinite.*

Proof. This proof is very similar to [44, Lemma 3.3]. For $k\geq 1$, let $\mathcal{F}_k$ be the σ-algebra on $\mathscr{T}_{w,p}$ generated by $X_0, X_1, \dots, X_k$ and $\mathcal{F}_\infty$ the σ-algebra generated by the whole path $\mathbf{X}$. For N in $\mathbb{N}$, let $\mathsf{ft}^{(N)}$ be the first fresh time after (or at) time N. Then,

$$\mathbb{P}[\bigcup_{s\geq N}\{s\in\mathsf{rt}\}\,|\,\mathcal{F}_N]\geq\mathbb{P}[\mathsf{ft}^{(N)}\in\mathsf{rt}\,|\,\mathcal{F}_N]=\sum_{s\geq N}\mathbb{E}[\mathbf{1}_{\{\mathsf{ft}^{(N)}=s\}}\mathbf{1}_{\{s\in\mathsf{rt}\}}\,|\,\mathcal{F}_N].$$

Thus we can use Lemma 5.5 to obtain

$$\mathbb{P}[\bigcup_{s\geq N}\{s\in\mathsf{rt}\}\,|\,\mathcal{F}_N]\geq\sum_{s\geq N}\mathbb{E}[\mathbf{1}_{\{\mathsf{ft}^{(N)}=s\}}\,|\,\mathcal{F}_N]\mathbb{E}[\mathbf{1}_{\{\tau_{\varnothing*}=\infty\}}]=\mathbf{E}[\beta\,(T)]>0,$$

By regular martingale convergence theorem and the fact that for any N in $\mathbb{N}$, the event $\bigcup_{s\geq N}\{s\in\mathsf{rt}\}$ is in $\mathcal{F}_\infty$, we have almost surely,

$$\begin{aligned}
\mathbf{1}_{\bigcup_{s\geq N}\{s\in\mathsf{rt}\}} &= \lim_{k\to\infty}\mathbb{P}[\bigcup_{s\geq N}\{s\in\mathsf{rt}\}\,\mid\mathcal{F}_{N+k}] \\
&\geq \mathbb{P}[\bigcup_{s\geq N+k}\{s\in\mathsf{rt}\}\,\mid\mathcal{F}_{N+k}]\geq\mathbf{E}[\beta(T)]>0.
\end{aligned}$$

Hence, $\mathbf{1}_{\bigcup_{s\geq N}\{s\in\mathsf{rt}\}}=1$, almost surely. □

We will now work on the space of weighted trees with transient paths that have infinitely many regeneration times. We still denote it $\mathscr{T}_{w,p}$ in order not to add another notation.

Proposition 5.7. *Let* f *and* g *be measurable non-negative functions. For any* $n\geq 1$*,*

$$\begin{aligned}
\mathbb{E}[f(\Phi_{\mathsf{rt}_n})g(\Psi_{\mathsf{rt}_n})] &= \mathbb{E}[f(\Phi_{\mathsf{rt}_n})]\mathbb{E}[g(T,\mathbf{X})\,|\,\tau_{\varnothing*}=\infty] \\
&= \mathbb{E}[f\,(\Phi_{\mathsf{rt}_n})]\mathbb{E}[g\,(\Psi_{\mathsf{rt}_n})].
\end{aligned}\tag{5.2}$$

Proof. Again, this proof is similar to [44, p. 255]. For $1 \le n \le s$, let C_n^s be the event that exactly n edges have been crossed exactly one time before time s, so that

$$\{\mathsf{rt}_n = s\} = C_{n-1}^{s-1} \cap \{s \in \mathsf{rt}\}.$$

Reasoning on the value of the n-th regeneration time, we first obtain

$$\mathbb{E}[f(\Phi_{\mathsf{rt}_n})g(\Psi_{\mathsf{rt}_n})] = \sum_{s\ge n} \mathbb{E}[\mathbf{1}_{\{\mathsf{rt}_n = s\}} f(\Phi_s)\, g(\Psi_s)] = \sum_{s\ge n} \mathbb{E}[\mathbf{1}_{\{s\in\mathsf{rt}\}}\mathbf{1}_{C_{n-1}^{s-1}} f(\Phi_s) g(\Psi_s)].$$

On the event $\{s \in \mathsf{rt}\}$, the indicator $\mathbf{1}_{C_{n-1}^{s-1}}$ is a function of the past Φ_s, thus, using Lemma 5.5, we obtain

$$\begin{aligned}\mathbb{E}[f(\Phi_{\mathsf{rt}_n})g(\Psi_{\mathsf{rt}_n})] &= \mathbb{E}[\mathbf{1}_{\{\tau_{\emptyset_*}=\infty\}} g(T,\mathbf{X})] \sum_{s\ge n} \mathbb{E}[\mathbf{1}_{\{s\in\mathsf{ft}\}}\mathbf{1}_{C_{n-1}^{s-1}} f(\Phi_s)] \\ &= \mathbb{E}[g(T,\mathbf{X}) \,|\, \tau_{\emptyset_*}=\infty] \sum_{s\ge n} \mathbb{P}[\tau_{\emptyset_*}=\infty]\mathbb{E}[\mathbf{1}_{\{s\in\mathsf{ft}\}}\mathbf{1}_{C_{n-1}^{s-1}} f(\Phi_s)].\end{aligned}$$

Using Lemma 5.5 the other way around,

$$\begin{aligned}\mathbb{E}[f(\Phi_{\mathsf{rt}_n})g(\Psi_{\mathsf{rt}_n})] &= \mathbb{E}[g(T,\mathbf{X}) \,|\, \tau_{\emptyset_*}=\infty] \sum_{s\ge n} \mathbb{E}[\mathbf{1}_{\{s\in\mathsf{et}\}}\mathbf{1}_{\{s\in\mathsf{ft}\}}\mathbf{1}_{C_{n-1}^{s-1}} f(\Phi_s)] \\ &= \mathbb{E}[g(T,\mathbf{X}) \,|\, \tau_{\emptyset_*}=\infty]\mathbb{E}[f(\Phi_{\mathsf{rt}_n})].\end{aligned}$$

Finally, taking f constant equal to one yields the last equality. $\square$

We define on $\mathscr{T}_{w,p}$ the shift at exit times

$$\mathrm{S}_{\mathsf{e}} : (t,\mathbf{x}) \mapsto \Psi_{\mathsf{et}_1(\mathbf{x})}(t,\mathbf{x}) = (t[\mathsf{ep}_1], \mathbf{x}[\mathsf{et}_1]),$$

and the shift at regeneration times

$$\mathrm{S}_{\mathsf{r}} : (t,\mathbf{x}) \mapsto \Psi_{\mathsf{rt}_1(\mathbf{x})}(t,\mathbf{x}) = (t[\mathsf{rp}_1], \mathbf{x}[\mathsf{rt}_1]).$$

For $k \ge 1$, let

$$\mathrm{S}_{\mathsf{e}}^k = \underbrace{\mathrm{S}_{\mathsf{e}} \circ \cdots \circ \mathrm{S}_{\mathsf{e}}}_{k \text{ times}}.$$

The exit time number rh_1 is the first regeneration time, thus the shifts S_{e} and S_{r} are related by

$$\mathrm{S}_{\mathsf{r}}(t,\mathbf{x}) = \mathrm{S}_{\mathsf{e}}^{\mathsf{rh}_1(\mathbf{x})}(t,\mathbf{x}), \qquad \forall (t,\mathbf{x}) \in \mathscr{T}_{w,p}.$$

As a corollary to the previous proposition, we obtain our first measure-preserving system.

Corollary 5.8. *The law μ_{r} of $(T,\mathbf{X})$ on $\mathscr{T}_{w,p}$, under the probability measure*

$$\mathbb{P}^* = \mathbb{P}[\cdot \,|\, \tau_{\emptyset_*} = \infty]$$

is invariant and mixing with respect to the shift S_{r}.

Proof. For the invariance, take f constant equal to one in (5.2).

Now let f and g be non-negative measurable functions on $\mathscr{T}_{w,p}$. By a monotone class argument, we may assume that g only depends on the N first generations of the weighted tree and on the path until it escapes these generations for the first time.

Since the N-th regeneration point is at least of height N, we get, using (5.2), for all $k \geq N$,

$$\begin{aligned}\mathbb{E}^*[f \circ \mathrm{S}_{\mathsf{r}}^k(T,\mathbf{X})g(T,\mathbf{X})] &= \mathbb{E}^*[f(T[\mathsf{rp}_k],\mathbf{X}[\mathsf{rt}_k])g(T,\mathbf{X})] \\ &= \mathbb{E}^*[f(T,\mathbf{X})]\mathbb{E}^*[g(T,\mathbf{X})],\end{aligned}$$

thus the system is mixing. □

5.4 Tower construction of an invariant measure for the shift at exit times

We now build a Rokhlin tower over the system $(\mathscr{T}_{w,p}, \mathrm{S}_{\mathsf{r}}, \mu_r)$ in order to obtain a probability measure that is invariant with respect to the shift S_{e}. This is a classical and general construction but we provide details in our specific case for the reader's convenience.

For any $i \geq 1$, let

$$E_i = \{(t,\mathbf{x}) \in \mathscr{T}_{w,p} : \mathsf{rh}_1(\mathbf{x}) \geq i\}.$$

We then have

$$\mathscr{T}_{w,p} = E_1 \supset E_2 \supset \cdots .$$

For $i \geq 1$, let $\widetilde{E}_i = E_i \times \{i\}$ and $\widetilde{E} = \bigsqcup_{i\geq 1} \widetilde{E}_i$. Let $\phi_i : E_i \to \widetilde{E}_i$ be the natural bijection. We define the measure $\widetilde{\mu_{\mathsf{r}}}^0$ by : for any measurable $\widetilde{A}$ in $\widetilde{E}$,

$$\widetilde{\mu_{\mathsf{r}}}^0(\widetilde{A}) \coloneqq \sum_{i\geq 1} \mu_{\mathsf{r}}(\phi_i^{-1}(\widetilde{A} \cap \widetilde{E}_i)).$$

The total mass of $\widetilde{\mu_{\mathsf{r}}}^0$ is

$$\widetilde{\mu_{\mathsf{r}}}^0(\widetilde{E}) = \sum_{i\geq 1} \mathbb{P}^*(\mathsf{rh}_1 \geq i) = \mathbb{E}^*[\mathsf{rh}_1].$$

Lemma 5.9. *The expectation $\mathbb{E}^*[\mathsf{rh}_1]$ is finite and equals $\mathbf{E}[\beta(T)]^{-1}$.*

We will prove this lemma later. We now write $\widetilde{\mu_{\mathsf{r}}} \coloneqq \widetilde{\mu_{\mathsf{r}}}^0 / \widetilde{\mu_{\mathsf{r}}}^0(\widetilde{E})$. We define the shift $\widetilde{S}$ on $\widetilde{E}$ by :

$$\widetilde{S}(t,\mathbf{x},i) \coloneqq \begin{cases}(t,\mathbf{x},i+1) & \text{if } \mathsf{rh}_1(\mathbf{x}) \geq i+1; \\ (\mathrm{S}_{\mathsf{r}}(t,\mathbf{x}),1) & \text{if } \mathsf{rh}_1(\mathbf{x}) = i.\end{cases} \tag{5.3}$$

Lemma 5.10. *The measure $\widetilde{\mu_{\mathsf{r}}}$ is invariant and ergodic with respect to the shift $\widetilde{S}$.*

Proof. Let $f : \widetilde{E} \to \mathbb{R}_+$ be a measurable function.

$$\int f \circ \widetilde{S}(t,\mathbf{x},i)\,\mathrm{d}\widetilde{\mu_{\mathsf{r}}}(t,\mathbf{x},i) = \sum_{j\geq 1}\int_{\widetilde{E}_j} f \circ \widetilde{S}(t,\mathbf{x},i)\,\mathrm{d}\widetilde{\mu_{\mathsf{r}}}(t,\mathbf{x},i)$$
$$= \mathbb{E}^*[\mathsf{rh}_1]^{-1}\sum_{j\geq 1}\int_{E_j} f\circ \widetilde{S}(t,\mathbf{x},j)\,\mathrm{d}\mu_{\mathsf{r}}(t,\mathbf{x})$$
$$= \mathbb{E}^*[\mathsf{rh}_1]^{-1}\Big(\sum_{j\geq 1}\int_{E_j\setminus E_{j+1}} f\,(\mathrm{S}_{\mathsf{r}}(t,\mathbf{x}),1)\,\mathrm{d}\mu_{\mathsf{r}}\,(t,\mathbf{x}) + \int_{E_{j+1}} f(t,\mathbf{x},j+1)\,\mathrm{d}\mu_{\mathsf{r}}(t,\mathbf{x})\Big)$$
$$= \mathbb{E}^*[\mathsf{rh}_1]^{-1}\Big(\int_{E_1} f(\mathrm{S}_{\mathsf{r}}(t,\mathbf{x}),1)\,\mathrm{d}\mu_{\mathsf{r}}(t,\mathbf{x}) + \int_{\widetilde{E}\setminus\widetilde{E}_1} f(t,\mathbf{x},i)\,\mathrm{d}\widetilde{\mu_{\mathsf{r}}}(t,\mathbf{x},i)\Big).$$

The fact that S_{r} is invariant with respect to μ_{r} concludes the proof of the invariance.

For the ergodicity, we remark that, by construction,

$$\bigcup_{k=1}^{\infty}\widetilde{S}^{-k}(\widetilde{E}_1) = \widetilde{E}.$$

and the induction of the system on $\widetilde{E}_1$ is canonically conjugated to $(\mathscr{T}_{w,p}, \mathrm{S}_{\mathsf{r}}, \mu_{\mathsf{r}})$, thus is ergodic and (see Section 5.2) so is the whole system. □

Proof of Lemma 5.9. This proof can be found in [2, Subsection 3.1]. We reproduce it with our notations for the reader's convenience. From Proposition 5.7, we know that under $\mathbb{P}^*$, we have $\mathsf{rh}_0 = 0$ and the increments rh_1, $\mathsf{rh}_2 - \mathsf{rh}_1$, ..., $\mathsf{rh}_{k+1} - \mathsf{rh}_k$, ... are i.i.d. For $n \geq 1$,

$$\mathbb{P}^*(n \in \mathsf{rh}) = \mathbb{E}^*[\#\mathsf{rh}\cap\{0,1,\dots,n\}] - \mathbb{E}^*[\#\mathsf{rh}\cap\{0,1,\dots,n-1\}].$$

So, by the renewal theorem ([22, p 360]),

$$\mathbb{P}^*(n\in\mathsf{rh}) \xrightarrow[n\to\infty]{} 1/\mathbb{E}^*[\mathsf{rh}_1].$$

On the other hand, for $n \geq 1$, let

$$\tau^{(n)} = \inf\{k \geq 0 : |X_k| = n\}.$$

By transience, for **GW**-almost every t, the stopping time $\tau^{(n)}$ is $\mathrm{P}^t_{\varnothing}$-almost surely finite. Notice that n is a regeneration height if and only if $\tau^{(n)}$ is also an exit time. In particular, if n is a regeneration height, then, exactly one vertex at height n is hit by the random walk. Hence we have the following decomposition :

$$\{n\in\mathsf{rh}, \tau_{\varnothing_*} = \infty\} = \bigsqcup_{|x|=n}\{X_{\tau^{(n)}} = x,\ \tau_{\varnothing_*} > \tau^{(n)},\ \forall k \geq \tau^{(n)},\ X_k \neq x_*\}.$$

By the Markov property at time $\tau^{(n)}$,

$$\begin{aligned}&\mathrm{P}^T_{\varnothing}(X_{\tau^{(n)}} = x,\ \tau_{\varnothing_*} > \tau^{(n)},\ \forall k\geq\tau^{(n)},\ X_k\neq x_*)\\ &\quad= \mathrm{P}^T_{\varnothing}(X_{\tau^{(n)}} = x,\ \tau_{\varnothing_*} > \tau^{(n)})\mathrm{P}^T_x(\forall k\geq 0,\ X_k \neq x_*)\\ &\quad= \mathrm{P}^T_{\varnothing}(X_{\tau^{(n)}} = x,\ \tau_{\varnothing_*} > \tau^{(n)})\beta(T[x]).\end{aligned}$$

Therefore, we may write $\mathbb{P}^*(n \in \mathsf{rh})$ as

$$\begin{aligned}\mathbb{P}^*(n \in \mathsf{rh}) &= \mathbf{E}[\beta\,(T)]^{-1} \sum_{|x|=n} \mathbf{E}\big[\mathbf{1}_{\{x\in T\}} \mathrm{P}^T_{\varnothing}(X_{\tau^{(n)}} = x,\ \tau_{\varnothing_*} > \tau^{(n)},\ \forall k \geq \tau^{(n)},\ X_k \neq x_*)\big] \\ &= \mathbf{E}[\beta\,(T)]^{-1} \sum_{|x|=n} \mathbf{E}\big[\mathbf{1}_{\{x\in T\}} \mathrm{P}^T_{\varnothing}(X_{\tau^{(n)}} = x,\ \tau_{\varnothing_*} > \tau^{(n)})\beta(T[x])\big].\end{aligned}$$

Now notice that the random variable

$$\mathbf{1}_{\{x\in T\}} \mathrm{P}^T_{\varnothing}(X_{\tau^{(n)}} = x,\ \tau_{\varnothing_*} > \tau^{(n)})$$

is a function of $T^{\leq x}$, so by the branching property,

$$\begin{aligned}\mathbb{P}^*(n \in \mathsf{rh}) &= \mathbf{E}[\beta\,(T)]^{-1} \sum_{|x|=n} \mathbf{E}\big[\mathbf{1}_{\{x\in T\}} \mathrm{P}^T_{\varnothing}(X_{\tau^{(n)}} = x,\ \tau_{\varnothing_*} > \tau^{(n)})\big]\mathbf{E}[\beta(T^x)] \\ &= \sum_{|x|=n} \mathbf{E}\big[\mathbf{1}_{\{x\in T\}} \mathrm{P}^T_{\varnothing}(X_{\tau^{(n)}} = x,\ \tau_{\varnothing_*} > \tau^{(n)})\big] = \mathbb{P}(\tau_{\varnothing_*} > \tau^{(n)}).\end{aligned}$$

By dominated convergence,

$$\mathbb{P}[\tau_{\varnothing_*} > \tau^{(n)}] \xrightarrow[n\to\infty]{} \mathbb{P}[\tau_{\varnothing_*} = \infty] = \mathbf{E}[\beta(T)]. \qquad \square$$

In order to construct an S_{e}-invariant measure on $\mathscr{T}_{w,p}$, all we need now is the right semi-conjugacy. Let $h_{\mathsf{e}} : \widetilde{E} \to \mathscr{T}_{w,p}$ be defined by

$$h_{\mathsf{e}}\,(t, \mathbf{x}, i) := \big(t\,[\mathsf{ep}_{i-1}]\,, \mathbf{x}\,[\mathsf{et}_{i-1}]\big)\,.$$

By construction,

$$h_{\mathsf{e}} \circ \widetilde{S} = \mathrm{S}_{\mathsf{e}} \circ h_{\mathsf{e}},$$

that is h_{e} is a semi-conjugacy on its image, so we get the desired ergodic system.

Corollary 5.11. *The probability measure $\mu_{\mathsf{e}} := \widetilde{\mu_{\mathsf{r}}} \circ h_{\mathsf{e}}^{-1}$ on $\mathscr{T}_{w,p}$ is invariant and ergodic with respect to the shift S_{e}.*

We now investigate further the law μ_{e}. Let $f : \mathscr{T}_{w,p} \to \mathbb{R}_+$ be a measurable function. By definition,

$$\begin{aligned}\int f(t,\mathbf{x})\,\mathrm{d}\mu_{\mathsf{e}}(t,\mathbf{x}) &= \int f(t[\mathsf{ep}_{i-1}], \mathbf{x}[\mathsf{et}_{i-1}])\,\mathrm{d}\widetilde{\mu_{\mathsf{r}}}(t,\mathbf{x},i) \\ &= \mathbf{E}[\beta(T)] \sum_{i\geq 1} \int \mathbf{1}_{\{\mathsf{rh}_1 \geq i\}} f(t[\mathsf{ep}_{i-1}], \mathbf{x}[\mathsf{et}_{i-1}])\,\mathrm{d}\mu_{\mathsf{r}}(t,\mathbf{x}) \\ &= \sum_{j\geq 1} \sum_{i=0}^{j-1} \mathbb{E}[\mathbf{1}_{\{\mathsf{rh}_1 = j,\, \tau_{\varnothing_*} = \infty\}} f(T[\mathsf{ep}_i], \mathbf{X}[\mathsf{et}_i])].\end{aligned}$$

Reasoning on the value of rp_1 and its strict ancestors, we get for all $j \geq 1$,

$$\sum_{i=0}^{j-1} \mathbb{E}\Big[\mathbf{1}_{\{\mathsf{rh}_1 = j,\, \tau_{\varnothing_*}=\infty\}} f(T[\mathsf{ep}_i], \mathbf{X}[\mathsf{et}_i])\Big]$$
$$= \sum_{|x|=j} \sum_{\varnothing \preceq y \prec x} \sum_{s\geq 1} \mathbb{E}\Big[\mathbf{1}_{\left\{x\in T,\, \mathsf{rp}_1 = x,\, \tau_{\varnothing_*}=\infty,\, \mathsf{ep}_{|y|}=y,\, \mathsf{et}_{|y|}=s\right\}} f(T[y], \mathbf{X}[s])\Big].$$

Summing over j, we obtain

$$\sum_{j\geq 1}\sum_{i=0}^{j-1} \mathbb{E}\Big[\mathbf{1}_{\{\mathsf{rh}_1 = j,\, \tau_{\varnothing_*}=\infty\}} f(T[\mathsf{ep}_i], \mathbf{X}[\mathsf{et}_i])\Big]$$
$$= \sum_{y\in\mathcal{U}} \sum_{s\geq 1} \mathbb{E}\Big[\mathbf{1}_{\left\{y\in T,\, \tau_{\varnothing_*}=\infty,\, \mathsf{ep}_{|y|}=y,\, \mathsf{et}_{|y|}=s\right\}} f(T[y], \mathbf{X}[s]) \sum_{x\succ y} \mathbf{1}_{\{x\in T,\, \mathsf{rp}_1 = x\}}\Big]$$
$$= \sum_{y\in\mathcal{U}} \sum_{s\geq 1} \mathbb{E}\Big[\mathbf{1}_{\left\{y\in T,\, \tau_{\varnothing_*}=\infty,\, \mathsf{et}_{|y|}=s\right\}} f(T[y], \mathbf{X}[s]) \mathbf{1}_{\{\mathsf{rp}_1 \succ y\}}\Big].$$

We want to use the Markov property at time s. For $s \geq 1$, and y in $\mathcal{U}$, let $D_s(y)$ be the event that:

— the walk has not hit $\varnothing_*$ before time s ;

— the walk hits y at time s and y_* at time $s-1$;

— for all $\varnothing \prec z \preceq y$, there exist $1 \leq i < j \leq s$ such that $X_i = z$ and $X_j = z_*$.

Notice that

$$D_s(y) \cap \{X_k \neq y_*,\, \forall k > s\} = \{\mathsf{et}_{|y|} = s,\, \tau_{\varnothing_*} = \infty,\, \mathsf{rp}_1 \succ y\}.$$

For fixed $y \in \mathcal{U}$ and $s \geq 1$, on the event $\{y \in T\}$, we denote by $\mathbf{X}'$ a random walk in T starting from y independent of $X_0, X_1, \ldots, X_s$ and let $y^{-1}\mathbf{X}' = (y^{-1}X'_0, y^{-1}X'_1, \ldots)$. By the Markov property at time s,

$$\mathbf{1}_{\{y\in T\}} \mathrm{E}^T_{\varnothing}\Big[f(T[y], \mathbf{X}[s]) \mathbf{1}_{\{\forall k\geq s,\, X_k \neq y_*\}} \mathbf{1}_{D_s(y)}\Big]$$
$$= \mathbf{1}_{\{y\in T\}} \mathrm{E}^T_{y}\Big[f(T[y], y^{-1}\mathbf{X}') \mathbf{1}_{\left\{\forall k\geq 0,\, X'_k \neq y_*\right\}}\Big] \mathrm{P}^T_{\varnothing}[D_s(y)].$$

We remark that the law of $y^{-1}\mathbf{X}'$ on the event $\{\forall k \geq 0,\, X'_k \neq y_*\}$ is the same as the law of a random walk $\mathbf{Y}$ in the weighted tree $T[y]$, starting from $\varnothing$, on the event $\{\forall k \geq 0,\, Y_k \neq \varnothing_*\}$, thus

$$\mathbf{1}_{\{y\in T\}} \mathrm{E}^T_{y}\Big[f(T[y], y^{-1}\mathbf{X}') \mathbf{1}_{\left\{\forall k\geq 0,\, X'_k \neq y_*\right\}}\Big] \mathrm{P}^T_{\varnothing}[D_s(y)]$$
$$= \mathbf{1}_{\{y\in T\}} \mathrm{E}^{T[y]}_{\varnothing}\Big[f(T[y], \mathbf{Y}) \mathbf{1}_{\{\forall k\geq 0,\, Y_k \neq y_*\}}\Big] \mathrm{P}^T_{\varnothing}[D_s(y)]$$
$$= \mathbf{1}_{\{y\in T\}} \mathrm{E}^{T[y]}_{\varnothing}\Big[f(T[y], \mathbf{Y}) \,\Big|\, \tau_{\varnothing_*}(\mathbf{Y}) = \infty\Big] \mathrm{P}^T_{\varnothing}\left[D_s(y), \forall k \geq s, X_k \neq y_*\right]$$
$$= \mathbf{1}_{\{y\in T\}} \mathrm{E}^{T[y]}_{\varnothing}\Big[f(T[y], \mathbf{Y}) \,\Big|\, \tau_{\varnothing_*}(\mathbf{Y}) = \infty\Big] \mathrm{P}^T_{\varnothing}[\mathsf{rp}_1 \succ y,\, \mathsf{et}_{|y|} = s,\, \tau_{\varnothing_*} = \infty].$$

Summing over s, we obtain

$$\int f(t,\mathbf{x})\,\mathrm{d}\mu_{\mathsf{e}}(t,\mathbf{x})$$
$$= \sum_{y\in\mathcal{U}} \mathbb{E}\Big[\mathbf{1}_{\{y\in T\}}\mathrm{E}_{\varnothing}^{T[y]}[f(T[y],\mathbf{Y})\,|\,\tau_{\varnothing*}(\mathbf{Y})=\infty]\mathrm{P}_{\varnothing}^{T}[\mathsf{rp}_1\succ y,\,\tau_{\varnothing*}=\infty]\Big].$$

We may write

$$\mathrm{P}_{\varnothing}^{T}[\mathsf{rp}_1\succ y,\tau_{\varnothing*}=\infty]=\mathrm{P}_{\varnothing}^{T^{\leq y}\triangleleft T[y]}[\mathsf{rp}_1\succ y,\tau_{\varnothing*}=\infty]=:h(T^{\leq y},T[y]).$$

By the branching property, for any y in $\mathcal{U}$,

$$\mathbb{E}\left[\mathbf{1}_{\{y\in T\}}\mathrm{E}_{\varnothing}^{T[y]}[f(T[y],\mathbf{Y})\,|\,\tau_{\varnothing*}(\mathbf{Y})=\infty]\mathrm{P}_{\varnothing}^{T}[\mathsf{rp}_1\succ y,\,\tau_{\varnothing*}=\infty]\right]$$
$$=\mathbb{E}[\mathbf{1}_{\{y\in T\}}\mathrm{E}_{\varnothing}^{\widetilde{T}}[f(\widetilde{T},\mathbf{Y})\,|\,\tau_{\varnothing*}(\mathbf{Y})=\infty]h(T^{\leq y},\widetilde{T})],$$

where $\widetilde{T}$ is a weighted tree whose law is $\mathbf{GW}$, independant of $T^{\leq y}$ and $\mathbf{1}_{\{y\in T\}}$. As a consequence, the conditional expectation of $\mathbf{1}_{\{y\in T\}}h(T^{\leq y},\widetilde{T})$ given $\widetilde{T}=t$ equals

$$\mathbf{E}[\mathbf{1}_{\{y\in T\}}h\left(T^{\leq y},t\right)].$$

Summing over $y\in\mathcal{U}$, we finally obtain the following theorem which summarizes the results of this section.

Theorem 5.12. *The system $(\mathscr{T}_{w,p},\mathrm{S}_{\mathsf{e}},\mu_{\mathsf{e}})$ is measure-preserving and ergodic. The probability measure μ_{e} has the following expression : for all non-negative measurable functions f,*

$$\int f(t,\mathbf{x})\,\mathrm{d}\mu_{\mathsf{e}}(t,\mathbf{x})=\mathbb{E}[f\,(T,\mathbf{X})\,\kappa(T)\beta(T)^{-1}\mathbf{1}_{\{\tau_{\varnothing*}=\infty\}},]$$

where, for all weighted trees t,

$$\kappa(t)=\mathbf{E}\Big[\sum_{y\in T}\mathrm{P}_{\varnothing}^{T^{\leq y}\triangleleft t}(\mathsf{rp}_1\succ y,\,\tau_{\varnothing*}=\infty)\Big]\quad\textit{and}\quad\beta(t)=\mathrm{P}_{\varnothing}^{t}(\tau_{\varnothing*}=\infty).\tag{5.4}$$

5.5 Invariant Measure for the Harmonic Flow Rule

We now slightly change our point of view. We will forget everything about the random path $\mathbf{X}$, except the ray it defines. Let μ_{HARM} be the projection on $\mathscr{T}_w$ of the probability measure μ_{e}, that is, the probability defined by:

$$\int f(t)\,\mathrm{d}\mu_{\mathsf{HARM}}(t)=\mathbf{E}[f(T)\kappa(T)],\tag{5.5}$$

for all non-negative measurable functions f on $\mathscr{T}_w$. We denote by $\mathscr{T}_{w,r}$ the space of all weighted trees with a distinguished ray, that is :

$$\mathscr{T}_{w,r}:=\{(t,\xi):t\in\mathscr{T}_w,\,\xi\in\partial t\}.$$

We view it as a metric subspace of $\mathscr{T}_{w,p}$.

We build a Borel probability measure $\mu_{\mathsf{HARM}} \ltimes \mathsf{HARM}$ on $\mathscr{T}_{w,r}$ by :

$$\int_{\mathscr{T}_{w,r}} f(t,\xi)\,\mathrm{d}(\mu_{\mathsf{HARM}} \ltimes \mathsf{HARM})(t,\xi) = \int_{\mathscr{T}_w} \Big(\int_{\partial t} f(t,\xi)\,\mathrm{d}\mathsf{HARM}_t(\xi)\Big)\,\mathrm{d}\mu_{\mathsf{HARM}}(t),$$

for all positive measurable functions $f : \mathscr{T}_{w,r} \to \mathbb{R}_+$. The *shift* S on $\mathscr{T}_{w,r}$ is

$$\mathrm{S}(t,\xi) = (t[\xi_1], \xi_1^{-1}\xi).$$

To check that this new system is (canonically) semi-conjugated to the one of Theorem 5.12 we need the following lemma.

Lemma 5.13. *Let t be in $\mathscr{T}_w$. Under the probability $\mathrm{E}^t_{\emptyset}$, $\mathrm{ray}(\mathbf{X})$ is independent of the event $\{\tau_{\emptyset_*} = \infty\}$.*

Proof. Let x be in t. By the Markov property, first at time $\tau_{\emptyset_*}$ and then at time 1, we have

$$\mathrm{P}^t_{\emptyset}(x \in \mathrm{ray}\,(\mathbf{X})\,,\, \tau_{\emptyset_*} < \infty) = \mathrm{P}^t_{\emptyset}(x \in \mathrm{ray}\,(\mathbf{X}))\mathrm{P}^t_{\emptyset}(\tau_{\emptyset_*} < \infty).$$

Since the cylinders $\{\xi \in \partial t : x \prec \xi\}$, for x in t, generate the Borel σ-algebra of ∂t, we conclude by a monotone class argument. □

Proposition 5.14. *The system $(\mathscr{T}_{w,r}, \mathrm{S}, \mu_{\mathsf{HARM}} \ltimes \mathsf{HARM})$ is measure-preserving and ergodic. Furthermore, the probability measures μ_{HARM} and $\mathbf{GW}$ are mutually absolutely continuous.*

Proof. Let $h_{p\to r} : \mathscr{T}_{w,p} \to \mathscr{T}_{w,r}$ be defined by $h_{p\to r}(t,\mathbf{x}) = (t, \mathrm{ray}(\mathbf{x}))$. The mapping $h_{p\to r}$ is surjective and satisfies $h_{p\to r} \circ \mathrm{S}_{\mathsf{e}} = \mathrm{S} \circ h_{p\to r}$,so is a semi-conjugacy. By the previous lemma, the probability measure $\mu_{\mathsf{HARM}} \ltimes \mathsf{HARM}$ equals $\mu_{\mathsf{e}} \circ h^{-1}_{p\to r}$.

We already know that μ_{HARM} is absolutely continuous with respect to $\mathbf{GW}$. We only need to show that, for $\mathbf{GW}$-almost every tree t, the density $\kappa(t)$ is positive. This is the case, because

$$\begin{aligned}\kappa(t) &= \mathbf{E}\Big[\sum_{y\in T} \mathrm{P}^{T^{\leq y}\triangleleft t}_{\emptyset}(\mathsf{rp}_1 \succ y,\, \tau_{\emptyset_*} = \infty)\Big] \\ &\geq \mathbf{E}\Big[\mathrm{P}^{T^{\leq \emptyset}\triangleleft t}_{\emptyset}(\mathsf{rp}_1 \succ \emptyset,\, \tau_{\emptyset_*} = \infty)\Big] = \mathrm{P}^t_{\emptyset}(\tau_{\emptyset_*} = \infty) = \beta(t),\end{aligned}$$

and $\mathbf{GW}$-almost every tree t is transient, thus is such that $\beta(t) > 0$. □

We could also have used [43, Proposition 5.2] to prove the ergodicity and the absolute continuity of $\mathbf{GW}$ with respect to μ_{HARM} since our measure μ_{HARM} was already known to be absolutely continuous with respect to $\mathbf{GW}$. To conclude that there indeed is a dimension drop phenomenon, we proceed as in [43, Theorem 7.1] and compare our flow rule HARM to UNIF.

Lemma 5.15. *For $\mathbf{GW}$-almost any weighted tree t, $\mathsf{HARM}_t \neq \mathsf{UNIF}_t$, unless $p_m = 1$ for some integer $m \geq 2$ and the weights are all deterministic and equal.*

Proof. We prove it by contradiction. By [43, proposition 5.1], we may assume that almost surely, $\mathsf{HARM}_T = \mathsf{UNIF}_T$. Let $k \geq 2$ and i and j be distinct integers in $[1,k]$. We consider the event $\{\nu_T(\varnothing) = k\}$, assuming it has positive probability (we recall that, by assumption, $\mathbf{P}(\nu_T(\varnothing) = 1) < 1$). Since $\mathsf{HARM}_T(i) = \mathsf{UNIF}_T(i)$ and $\mathsf{HARM}_T(j) = \mathsf{UNIF}_T(j)$, we have

$$\frac{\mathsf{A}_T(i)\beta(T[i])}{W(T[i])} = \frac{\sum_{\ell=1}^k \mathsf{A}_T(\ell)\beta(T[\ell])}{\sum_{\ell=1}^k W(T[\ell])} = \frac{\mathsf{A}_T(j)\beta(T[j])}{W(T[j])}.$$

In particular,

$$\mathsf{A}_T(i)\beta(T[i])W(T[j]) = \mathsf{A}_T(j)\beta(T[j])W(T[i]). \tag{5.6}$$

We first take the conditional expectation with respect to the σ-algebra generated by $\mathsf{A}_T(i)$, $\mathsf{A}_T(j)$ and the tree $T[i]$ to get that $\mathbf{E}[W(T)] < \infty$, so it is 1. Then, conditioning only with respect to $\mathsf{A}_T(i)$ and $\mathsf{A}_T(j)$, we get $\mathsf{A}_T(i) = \mathsf{A}_T(j)$. Let us denote by A_k the common value of $\mathsf{A}_T(1), \ldots, \mathsf{A}_T(k)$. Simplifying in (5.6) and taking the conditional expectation with respect to the subtree $T[i]$ gives $\beta(T[i]) = \alpha W(T[i])$, for $\alpha = \mathbf{E}\left[\beta(T)\right] \in (0,1)$. Since the law of $T[i]$ is itself $\mathbf{GW}$, we have, for $\mathbf{GW}$-amost every tree t,

$$\beta(t) = \alpha W(t).$$

We reason again on the event $\{\nu_T(\varnothing) = k\}$. Using the recursive equations (1.7) and (2.1), we have

$$\alpha W(T) = \beta(T) = \frac{A_k \alpha \sum_{j=1}^k W(T[j])}{1 + A_k \alpha \sum_{j=1}^k W(T[j])} = \frac{A_k \alpha m W(T)}{1 + A_k \alpha m W(T)}.$$

This implies

$$m A_k(\alpha W(T) - 1) = 1,$$

which, by independence, can only happen if W and A_k are almost surely constant. This is possible only if the law $\mathbf{p}$ is degenerated and $k = m$. In this case, we have $A_k = \frac{1}{k(\alpha-1)} > \frac{1}{k}$, that is, our random walk model reduces to transient λ-biased random walk on a regular tree, with deterministic $\lambda < m$. □

We now have all the ingredients to prove Theorem 5.2.

Proof of Theorem 5.2. Since the flow rule HARM admits an invariant measure μ_{HARM} which is absolutely continuous with respect to $\mathbf{GW}$, we may use Corollary 2.19 to obtain the formula for the (exact) dimension of HARM_T. Furthermore, since almost surely, $\mathsf{HARM}_T \neq \mathsf{UNIF}_T$ by the previous lemma, Proposition 2.27 implies that this dimension is almost surely strictly less that $\log m$, the dimension of ∂T. □

6 Subdiffusive random walk on a weighted Galton-Watson tree

6.1 Introduction

In this chapter, we again work in the setting of weighted Galton-Watson trees as described in the previous chapter. However, we study here a recurrent regime. Define the cumulant generating function associated to the intensity measure of the point process $\sum_{i=1}^{N} \delta_{\log \mathsf{A}(i)}$ by

$$\psi(s) = \log \mathbf{E} \sum_{i=1}^{N} \mathsf{A}(i)^s, \qquad \forall s \in \mathbb{R}.$$

We assume throughout this chapter that we are in the normalized case

$$\psi(1) = \log \mathbf{E} \sum_{i=1}^{N} \mathsf{A}(i) = 0. \tag{H_{norm}}$$

We will need Biggins' theorem, thus we also assume

$$\psi'(1) \coloneqq \mathbf{E}\left[\sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i) \log \mathsf{A}_T(i)\right] \in [-\infty, 0). \tag{$H_{\mathrm{derivative}}$}$$

Define

$$\kappa = \inf\{s > 1 : \psi(s) = 0\} \in (1, \infty],$$

and assume that

$$\begin{aligned} \mathbf{E}\Big[\Big(\sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i)\Big)^{\kappa}\Big] + \mathbf{E}\Big[\sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i)^{\kappa} \log^+ \mathsf{A}_T(i)\Big] < \infty, &\quad \text{if } 1 < \kappa \leq 2, \\ \mathbf{E}\Big[\Big(\sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i)\Big)^{2}\Big] < \infty, &\quad \text{if } \kappa \in (2, \infty]. \end{aligned} \tag{H_κ}$$

These assumptions are summed up in Figure 1.1. The additive martingale $(M_n(T))_{n\geq 0}$ is defined by

$$M_n(T) = \sum_{|x|=n} \prod_{\varnothing \prec y \preceq x} \mathsf{A}_T(y).$$

By Biggins' theorem (see also [32]) it converges almost surely and in L^1 to a random variable $M_\infty(T)$ which is positive on the event of non-extinction. Define by $\mathscr{C}_n(T)$ the

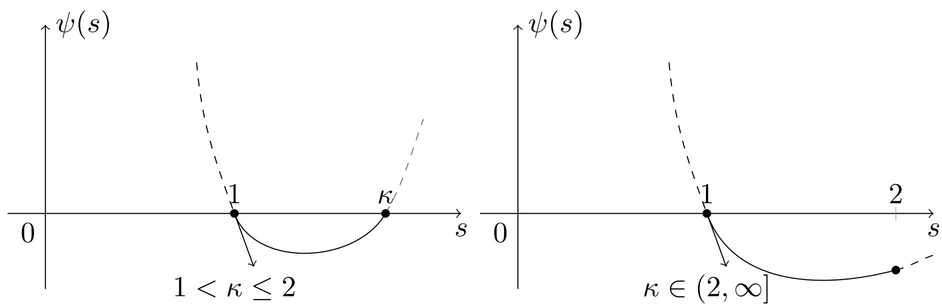

Figure 1.1 – Schematic behavior of ψ under our hypotheses

conductance between the root of T and its vertices at height n. The fact that the random walk is recurrent implies that $\mathscr{C}_n(T)$ converges to 0 and we want to study its speed of convergence. The main result of this chapter is the following:

Theorem 6.1. *Under the hypotheses* (H_{norm}), $(H_{\text{derivative}})$ *and* (H_κ),

$$0 < \liminf_{n\to\infty} n^{1/(\kappa-1)}\mathbf{E}[\mathscr{C}_n(T)] \leq \limsup_{n\to\infty} n^{1/(\kappa-1)}\mathbf{E}[\mathscr{C}_n(T)] < \infty \quad \textit{if } 1 < \kappa < 2\ ;$$
$$0 < \liminf_{n\to\infty} n\log n\mathbf{E}[\mathscr{C}_n(T)] \leq \limsup_{n\to\infty} n\log n\mathbf{E}[\mathscr{C}_n(T)] < \infty \quad \textit{if } \kappa = 2 \textit{ and}$$
$$\lim_{n\to\infty} n\mathbf{E}[\mathscr{C}_n(T)] = \|M_\infty(T)\|_2 \quad \textit{if } \kappa > 2.$$

And in any case, almost surely,

$$\lim_{n\to\infty} \mathscr{C}_n(T)/\mathbf{E}[\mathscr{C}_n(T)] = M_\infty(T).$$

Moreover, the above convergence also holds in L^p *for* $p \in [1,\kappa)$ *if* $1 < \kappa \leq 2$ *and in* L^2 *if* $\kappa > 2$.

6.2 Subdiffusive weighted Galton-Watson trees

6.2.1 Weighted trees and effective conductance

We work on the space $\mathscr{T}_w$ of trees t equipped with a weight function A_t from the set $t \setminus \{\varnothing, \varnothing_*\}$ to $(0,\infty)$. For a weighted tree t, and a vertex x in $t \setminus \{\varnothing, \varnothing_*\}$, define the conductance of the (undirected) edge $\{x_*, x\}$, by

$$\mathsf{c}_t(x) = \prod_{\varnothing \prec y \preceq x} \mathsf{A}_t(y).$$

The edge $\{\varnothing_*, \varnothing\}$ has conductance $\mathsf{c}_t(\varnothing) = 1$. We define a nearest-neighbor random walk on t in the usual way: for any vertex x distinct from $\varnothing_*$ in t, and any $1 \leq i \leq \nu_t(x)$,

$$P^t(x, xi) = \mathsf{c}(xi)/\pi_t(x) \quad \text{and} \quad P^t(x, x_*) = \mathsf{c}_t(x)/\pi_t(x),$$

where π_t is the usual reversible measure

$$\pi_t(x) = \mathsf{c}_t(x) + \sum_{i=1}^{\nu_t(x)} \mathsf{c}_t(xi).$$

In terms of weights,

$$P^t(x, xi) = \frac{\mathsf{A}_t(xi)}{1 + \sum_{j=1}^{\nu_t(x)} \mathsf{A}_t(xj)} \quad \text{and} \quad P^t(x, x_*) = \frac{1}{1 + \sum_{j=1}^{\nu_t(x)} \mathsf{A}_t(xj)}.$$

For a vertex x of t, define the first hitting time and the first return time of x by

$$\tau_x = \inf\{n \geq 0 : X_n = x\} \quad \text{and} \quad \tau_x^+ = \inf\{n \geq 1 : X_n = x\},$$

with the convention that $\inf \emptyset = \infty$. For $n \geq 0$, the first hitting time of the n-th level is

$$\tau^{(n)} = \inf\{n \geq 0 : |X_n| = n\}.$$

The effective conductance β_n of the weighted tree t between $\varnothing_*$ and the n-th level of the tree satisfies

$$\beta_n(t) = P^t_\varnothing(\tau^{(n)} < \tau_{\varnothing_*}),$$

while the effective conductance $\mathscr{C}_n(t)$ of t between $\varnothing$ and the n-th level of the tree has the following probabilistic interpretation

$$\mathscr{C}_n(t) = \frac{P^t_\varnothing(\tau^{(n)} < \tau^+_\varnothing)}{P^t(\varnothing, \varnothing_*)}.$$

Using either the Markov property or the laws of series and parallel circuits leads to the following relations: for $n \geq 1$,

$$\mathscr{C}_n(t) = \frac{\beta_n(t)}{1 - \beta_n(t)}, \tag{6.1}$$

$$\mathscr{C}_n(t) = \sum_{i=1}^{\nu_t(\varnothing)} \mathsf{A}_t(i) \beta_{n-1}(t[i]). \tag{6.2}$$

Thus, for $n \geq 2$,

$$\mathscr{C}_n(t) = \sum_{i=1}^{\nu_t(\varnothing)} \mathsf{A}_t(i) \frac{\mathscr{C}_{n-1}(t[i])}{1 + \mathscr{C}_{n-1}(t[i])} = \sum_{i=1}^{\nu_t(\varnothing)} \mathsf{A}_t(i) \left(\mathscr{C}_{n-1}(t[i]) - \frac{\mathscr{C}_{n-1}(t[i])^2}{1 + \mathscr{C}_{n-1}(t[i])} \right).$$

Finally we define the family of functions $(M_n)_{n \geq 0}$ by

$$M_n(t) = \sum_{|x|=n} \mathsf{c}_t(x) = \sum_{|x|=n} \prod_{\varnothing \prec y \preceq x} \mathsf{A}_t(y). \tag{6.3}$$

By definition, for all $n \geq 1$, one has

$$M_n(t) = \sum_{i=1}^{\nu_t(\varnothing)} \mathsf{A}_t(i) M_{n-1}(t[i]). \tag{6.4}$$

6.2.2 The additive martingale on a weighted Galton-Watson tree

For convenience, let $\mathbf{A} = (\mathsf{A}(1), \ldots, \mathsf{A}(N))$ be a random tuple distributed as $\mathbf{A}^{\varnothing}$. For $n \geq 0$, let $\mathcal{F}_n$ be the σ-algebra generated by the random variables $\mathbf{A}^x$, for $|x| \leq n-1$ (with $\mathcal{F}_0$ defined as the trivial σ-algebra). Under the previous assumption, the process $(M_n(T))$ is a martingale with respect to the filtration $(\mathcal{F}_n)$, and thus converges as n goes to infinity to a non-negative random variable denoted by $M_\infty(T)$.

Consider the following assumption:

$$\mathbf{E}\left[\left(\sum_{i=1}^{N} \mathsf{A}(i)\right) \log^+\left(\sum_{i=1}^{N} \mathsf{A}(i)\right)\right] < \infty, \tag{$H_{X\log X}$}$$

with $\log^+(x) = \max(0, \log(x))$, for all $x \geq 0$. Under this assumption together with (H_{norm}) and $(H_{\text{derivative}})$, Biggins' theorem (apply [40] to $\alpha = -1$ and $X_i = \log \mathsf{A}(i)$) implies that $M_\infty(T)$ is not degenerated.

Fact 6.2. *Under the hypotheses (H_{norm}), $(H_{\text{derivative}})$ and $(H_{X\log X})$, the random variable $M_\infty(T)$ has expectation 1 and is positive on the event of non-extinction, which has positive probability.*

Now notice that in each case, the hypothesis (H_κ) supersedes the hypothesis $(H_{X\log X})$. The tail behavior of M_∞ plays a crucial role in this work.

We owe to [37, Theorem 2.1, Theorem 2.2] the following fact.

Fact 6.3. *Under the hypotheses (H_{norm}), $(H_{\text{derivative}})$ and (H_κ), the random variable M_∞ has finite moments of order p for all p in $[1, \kappa)$ if $\kappa \leq 2$ and for all p in $[1, 2]$ if $\kappa > 2$.*

If $\kappa \leq 2$, the asymptotic tail probability of M_∞ satisfies

$$\mathbf{P}(M_\infty > s) \asymp_{s\to\infty} s^{-\kappa}. \tag{6.5}$$

Now, we introduce for $p \geq 1$, the function φ_p defined on $(0, \infty)$ by

$$\varphi_p(a) = \mathbf{E}\left[\left(\frac{M_\infty^2}{a + M_\infty}\right)^p\right]. \tag{6.6}$$

Its behavior at infinity will play a very important role.

Lemma 6.4. *With the previous notations and hypotheses, assume that κ is in $(1, 2]$ and that p is a real number in $[\kappa/2, \kappa)$.*

- *If $1 < \kappa \leq 2$ and $\kappa/2 < p < \kappa$, then $\varphi_p(a) \asymp_{a\to\infty} a^{p-\kappa}$;*
- *if $1 < \kappa \leq 2$ and $p = \kappa/2$, then $\varphi_p(a) \asymp_{a\to\infty} a^{p-\kappa}\log(a)$;*
- *if $\kappa > 2$ and $1 < p < 2$, then we may find $C > 0$ such that, for all large enough a, $\varphi_p(a) \leq Ca^{p-2}$;*
- *if $\kappa > 2$, then $\varphi_1(a) \sim_{a\to\infty} \mathbf{E}[M_\infty^2]a^{-1}$.*

Proof. Write $\mathbf{P}_{M_\infty}$ for the distribution of M_∞ . Differentiating the function $x \mapsto \frac{x^2}{a+x^2}$, we obtain

$$\varphi_p(a) = \int_0^\infty \left(\int_0^s p \frac{x^2 + 2ax}{(a+x)^2} \left(\frac{x^2}{a+x} \right)^{p-1} \mathrm{d}x \right) \mathbf{P}_{M_\infty}(\mathrm{d}s).$$

Using Tonelli's theorem together with the change of variable $y = x/a$ yields

$$\varphi_p(a) = pa^{p-\kappa} \int_0^\infty \left(\frac{y^2}{1+y} \right)^{p-1} y^{-\kappa} \left(1 - \frac{1}{(1+y)^2} \right) [(ay)^\kappa \mathbf{P}(M_\infty > ay)] \,\mathrm{d}y. \tag{6.7}$$

Let $f(y)$ be the integrand in the last equation.

Now assume that $1 < \kappa \leq 2$ and write $\underline{\ell}$ (respectively $\overline{\ell}$) for the inferior (respectively superior) limit of $s^\kappa \mathbf{P}(M_\infty > s)$, as s goes to infinity Consider $\varepsilon > 0$ so small that $\underline{\ell} - \varepsilon > 0$. Let $N > 0$ be large enough that

$$\forall s \geq N, \qquad s^\kappa \mathbf{P}(M_\infty > s) \in (\underline{\ell} - \varepsilon, \overline{\ell} + \varepsilon).$$

Assume that $a > N$. On the interval $(0, N/a)$, dominating $\mathbf{P}(M_\infty > ay)$ by 1 yields

$$f(y) \leq a^\kappa y^{2p-1} \max_{0 \leq y \leq 1} \frac{2+y}{(1+y)^{p+1}},$$

so that in any case,

$$pa^{p-\kappa} \int_0^{N/a} f(y) \,\mathrm{d}y \leq \left[pN^{2p} \max_{0 \leq y \leq 1} \frac{2+y}{(1+y)^{p+1}} \right] a^{-p},$$

which will be negligible. On the other hand, if y is in the interval $[N/a, \infty)$, then

$$f(y) \leq (\overline{\ell} + \varepsilon) \left(1 - \frac{1}{(1+y)^2} \right) \left(\frac{y^2}{1+y} \right)^{p-1} y^{-\kappa}, \tag{6.8}$$

and

$$f(y) \geq (\underline{\ell} - \varepsilon) \left(1 - \frac{1}{(1+y)^2} \right) \left(\frac{y^2}{1+y} \right)^{p-1} y^{-\kappa}. \tag{6.9}$$

Those bounds are integrable on $(0, \infty)$ if $p > \kappa/2$ and in this case, we may conclude by applying the monotone convergence theorem.

Now assume that $p = \kappa/2$. The bounds above are still integrable at the neighborhood of ∞, but not at the neighborhood of 0. As a consequence, the main contribution in the integral comes from the term

$$\int_{N/a}^1 f(y) \,\mathrm{d}y \leq (\overline{\ell} + \varepsilon) \int_{N/a}^1 y^{-1} \frac{2+y}{(1+y)^{p+1}} \,\mathrm{d}y \asymp_{a \to \infty} \log(a),$$

and similarly for the lower bound.

Finally, assume that $\kappa > 2$ and recall that in this case, by our hypotheses, $\mathbf{E}[M_\infty^2]$ is finite, thus by Markov's inequality, for all $r > 0$, $\mathbf{P}(M_\infty > r) \leq \mathbf{E}[M_\infty^2]/r^2$. Now, if

$1 < p < 2$, the rest of the computations is exactly the same as in the first point, whereas if $p = 1$, by dominated convergence,

$$a\varphi_1(a) = \mathbf{E}\left[\frac{M_\infty^2}{1 + M_\infty/a}\right] \xrightarrow[a\to\infty]{} \mathbf{E}\Big[M_\infty^2\Big]. \qquad \square$$

6.2.3 Subdiffusive random walk on a weighted Galton-Watson tree

Using Section 6.2.1, we may define, on the weighted Galton-Watson tree T, a probability kernel P^T and an irreducible, nearest-neighbor random walk associated to it. A recurrence-transience criterion is given in [41], in the case of i.i.d. weights and in [21], in our setting.

Fact 6.5 (Null recurrence)**.** *Assume that* (H_{norm})*,* $(H_{\mathrm{derivative}})$ *and* $(H_{X\log X})$ *hold. Then, for* $\mathbf{GW}$*-almost every infinite tree* t*, the random walk on* t *of probability kernel* P^t *is null recurrent.*

We give a short proof if this fact, for the reader's convenience. It is slightly different than the original proof in [21].

Proof. For a weighted tree t, let $\beta(t) = P^t_\varnothing(\tau_{\varnothing_*} = \infty)$ and $\mathscr{C}(t) = P_\varnothing(\tau^+_\varnothing = \infty)/P^t_\varnothing(\varnothing, \varnothing_*)$. These are the conductances between, respectively, $\varnothing_*$ and infinity, and $\varnothing$ and infinity. It is well-known that $\beta(t) > 0$ if and only if the random walk on the weighted tree t is transient. Moreover, by the Markov property (or electrical network considerations)

$$\beta(t) = \frac{\mathscr{C}(t)}{1 + \mathscr{C}(t)} \quad \text{and} \quad \mathscr{C}(t) = \sum_{i=1}^{\nu_t(\varnothing)} \mathsf{A}_t(i)\beta(t[i]).$$

Now if T is a weighted Galton-Watson tree and $\mathbf{E}[\sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i)] = 1$, taking the expectation in the previous identities leads to

$$\mathbf{E}[\mathscr{C}(T)] = \mathbf{E}[\beta(T)] = \mathbf{E}\left[\frac{\mathscr{C}(T)}{1 + \mathscr{C}(T)}\right],$$

which implies that, almost surely, $\mathscr{C}(T) = 0$, and the recurrence is proved.

To prove that it is null-recurrent, consider, for any recurrent weighted tree t, $\alpha(t) = E^t_\varnothing[\tau_{\varnothing_*}]$. We want to show that, almost surely on the event of non-extinction, $\alpha(T) = \infty$. The function α satisfies, by the Markov property,

$$\alpha(t) = P^t_\varnothing(\varnothing, \varnothing_*) + \sum_{i=1}^{\nu_t(\varnothing)} P^t_\varnothing(\varnothing, i)(1 + \alpha(t[i]) + \alpha(t)).$$

Thus we see that, if $\alpha(t)$ is finite, so are $\alpha(t[x])$ for x in t. In this case, one has

$$\alpha(t) = 1 + \sum_{i=1}^{\nu_t(\varnothing)} \mathsf{A}_t(i) + \sum_{i=1}^{\nu_t(\varnothing)} \mathsf{A}_t(i)\alpha(t[i]),$$

and iterating the previous identity, for all $n \geq 1$,

$$\alpha(t) = 1 + 2\sum_{k=1}^{n} M_k(t) + \sum_{|x|=n} \mathsf{c}_t(x)\alpha(t[x]) \geq \sum_{k=1}^{n} M_k(t).$$

This show that

$$\mathbf{P}(\alpha(T) < \infty) \leq \mathbf{P}(\sum_{k=1}^{\infty} M_k(T) < \infty),$$

but our assumptions and Biggins' theorem imply that, almost surely on the event of non-extinction, $M_n(T) \to M_\infty(T) > 0$, thus $\mathbf{P}(\alpha(T) < \infty)$ is the probability that T is finite. □

6.3 Useful inequalities

6.3.1 Elementary analysis lemmas

Lemma 6.6. *Let $(u_n)_{n\geq 0}$ be a non-increasing positive sequence converging to 0, $\alpha > 1$ and $C > 0$.*

1. If for n large enough, $u_n - u_{n+1} \leq Cu_n^\alpha$, then

$$\liminf_{n\to\infty} n^{1/(\alpha-1)} u_n \geq [C(\alpha-1)]^{-1/(\alpha-1)}.$$

2. If for n large enough, $u_n - u_{n+1} \geq Cu_n^\alpha$, then

$$\limsup_{n\to\infty} n^{1/(\alpha-1)} u_n \leq [C(\alpha-1)]^{-1/(\alpha-1)}.$$

Proof. Let n be a positive integer. By the mean value theorem, there exists a number ξ_n in the interval $(1/u_n, 1/u_{n+1})$ such that

$$\frac{1}{u_{n+1}^{\alpha-1}} - \frac{1}{u_n^{\alpha-1}} = \frac{u_n - u_{n+1}}{u_n u_{n+1}}(\alpha-1)\xi_n^{\alpha-2}.$$

Now, if we assume that, eventually, $u_n - u_{n+1} \leq Cu_n^\alpha$, then we have that $u_n \sim u_{n+1}$ and $\xi_n \sim u_n^{2-\alpha}$. Let $\varepsilon > 0$. Eventually,

$$\frac{1}{u_{n+1}^{\alpha-1}} - \frac{1}{u_n^{\alpha-1}} \leq C(1+\varepsilon)(\alpha-1)\frac{u_n}{u_{n+1}} \leq C(1+\varepsilon)^2(\alpha-1).$$

Summing this inequality and letting ε go to 0 yields

$$\limsup_{n\to\infty} \frac{1}{nu_n^{\alpha-1}} \leq C(\alpha-1),$$

which proves the first point.

Now assume that, eventually, $u_n - u_{n+1} \geq C u_n^\alpha$. In this case,

$$\begin{aligned}
\frac{1}{u_{n+1}^{\alpha-1}} - \frac{1}{u_n^{\alpha-1}} &\geq C(\alpha-1)\frac{u_n^{\alpha-1}}{u_{n+1}}\xi_n^{2-\alpha} \\
&\geq C(\alpha-1)\begin{cases} \dfrac{u_n}{u_{n+1}} & \text{if } \alpha \geq 2, \\ \dfrac{u_n^{\alpha-1}}{u_{n+1}^{\alpha-1}}, & \text{if } 1 < \alpha < 2. \end{cases} \\
&\geq C(1-\alpha),
\end{aligned}$$

by the assumption that the sequence (u_n) is non-increasing. Summing this inequality yields

$$\liminf_{n\to\infty} \frac{1}{n u_n^{\alpha-1}} \geq C(1-\alpha),$$

which proves the second point. ☐

Lemma 6.7. *Let $(u_n)_{n\geq 0}$ be a non-increasing positive sequence converging to* 0 *and* $C > 0$*.*

1. If for n large enough, $u_n - u_{n+1} \leq C u_n^2 \log(1/u_n)$, then $\liminf u_n n \log n \geq C^{-1}$*.*

2. If for n large enough, $u_n - u_{n+1} \geq C u_n^2 \log(1/u_n)$, then $\limsup u_n n \log n \leq C^{-1}$*.*

Proof. Consider, for $x > e$, $f(x) = x/\log(x)$. Then, $f'(x) = \frac{1}{\log(x)}\left(1 - \frac{1}{\log(x)}\right)$. For any large enough n, by the mean value theorem, there exists ξ_n in $(1/u_n, 1/u_{n+1})$ such that

$$f\left(\frac{1}{u_{n+1}}\right) - f\left(\frac{1}{u_n}\right) = \left(\frac{1}{u_{n+1}} - \frac{1}{u_n}\right) f'(\xi_n) = \frac{u_n - u_{n+1}}{u_n u_{n+1}} f'(\xi_n).$$

The function f' is decreasing on $[e^2, \infty)$, so for any large enough integer n,

$$f'\left(\frac{1}{u_{n+1}}\right) < f'(\xi_n) < f'\left(\frac{1}{u_n}\right).$$

Now the first assumption implies

$$0 \leq 1 - \frac{u_{n+1}}{u_n} \leq C u_n \log(1/u_n) \to 0,$$

therefore, for any $\varepsilon > 0$, eventually,

$$f\left(\frac{1}{u_{n+1}}\right) - f\left(\frac{1}{u_n}\right) \leq C\frac{u_n}{u_{n+1}}\left(1 + \frac{1}{\log(u_n)}\right) \leq C(1+\varepsilon).$$

Summing this inequality and letting ε go to 0 implies

$$\limsup \frac{1}{n u_n \log(1/u_n)} \leq C. \tag{6.10}$$

Applying the log function, we see that

$$\log(1/u_n) \leq \log n + o\left(\log(1/u_n)\right),$$

thus, for n large enough,

$$\log(1/u_n) \geq (1-\varepsilon)\log n.$$

Plugging this in (6.10) and letting ε go to 0 yields the first point.

Now, the assumption of the second point implies that, eventually,

$$f\left(\frac{1}{u_{n+1}}\right) - f\left(\frac{1}{u_n}\right) \geq C\frac{u_n \log(1/u_n)}{u_{n+1}\log(1/u_{n+1})}\left(1+\frac{1}{\log(u_{n+1})}\right) \geq C(1-\varepsilon),$$

because the function $x \mapsto x\log(1/x)$ is increasing on $[0, e^{-1}]$. We conclude in the same way as in the first case. □

6.3.2 Moments of a sum of independent random variables

For later use, we collect here two inequalities regarding moments of a finite sum of independent random variables. The first point is taken from [51] while the second may be found in [54, p. 82].

Fact 6.8. *Let p be a real number in $[1,2]$ and assume that $\xi_1, \dots, \xi_k$ are independent real-valued random variables such that for all $1 \leq i \leq k$, $\mathbb{E}[|\xi_i|^p] < \infty$.*

1. If $\xi_1, \dots, \xi_k$ are non-negative, then

$$\mathbb{E}[(\xi_1 + \cdots + \xi_k)^p] \leq \sum_{i=1}^{k}\mathbb{E}[\xi_i^p] + \left(\sum_{i=1}^{k}\mathbb{E}\xi_i\right)^p. \tag{6.11}$$

2. If $\xi_1, \dots, \xi_k$ are centered, then

$$\mathbb{E}[|\xi_1 + \cdots + \xi_k|^p] \leq 2\sum_{i=1}^{k}\mathbb{E}[|\xi_i|^p]. \tag{6.12}$$

6.3.3 Renormalized positive random variables

For any non-negative random variable ξ such that $\mathbb{E}\xi \in (0,\infty)$, write

$$\langle\xi\rangle = \frac{\xi}{\mathbb{E}[\xi]}. \tag{6.13}$$

The following FKG-type inequality is taken from [27, Formula 3.3].

Lemma 6.9. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a convex differentiable function and ξ be a random variable with values in a Borel subset J of $\mathbb{R}$. Let x_0 be a non-negative real number and I be an open sub-interval of $\mathbb{R}_+$ containing x_0. Assume that $h : I \times J \to (0,\infty)$ is a Borel function such that $\partial h/\partial x$ exists on $I \times J$. Assume that*

$$\mathbb{E}[h(x_0,\xi)] < \infty, \quad \mathbb{E}|\phi\langle h(x_0,\xi)\rangle| < \infty \quad \textit{and} \quad \mathbb{E}\left[\sup_{x\in I}\left|\frac{\partial h}{\partial x}(x,\xi)\right|\right] < \infty.$$

For x in I, write

$$\psi(x,\xi) = \frac{1}{(\mathbb{E}h(x,\xi))^2}\left(\frac{\partial h}{\partial x}(x,\xi)\mathbb{E}[h(x,\xi)] - h(x,\xi)\mathbb{E}\left[\frac{\partial h}{\partial x}(x,\xi)\right]\right),$$

and further assume that

$$\mathbb{E}[\sup_{x\in I}|\phi'\langle h(x,\xi)\rangle\psi(x,\xi)|] < \infty.$$

Then, the function

$$f : x \mapsto \mathbb{E}[\phi\langle h(x,\xi)\rangle] \tag{6.14}$$

is well-defined and differentiable on I. Finally, if we assume that the functions

$$y \mapsto h(x_0,y) \text{ and } y \mapsto \frac{\partial}{\partial x}\log(x_0,y) \text{ are monotonic on } J,$$

then, if they have the same monotonicity, $f'(x_0) \geq 0$ and $f'(x_0) \leq 0$ otherwise.

As a particular case, the following inequality will play a key role. It is already stated in [26, proof of lemma 3.1]. Nevertheless, we give a detailed proof for the reader's convenience.

Lemma 6.10. *Let ξ be a non-negative random variable such that $\mathbb{E}[\xi]$ is in $(0,\infty)$. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuously differentiable, regularly varying at infinity, convex function. Then,*

$$\mathbb{E}\left[\phi\left\langle\frac{\xi}{1+\xi}\right\rangle\right] \leq \mathbb{E}[\phi\langle\xi\rangle]. \tag{6.15}$$

Proof. For x in $[0,1]$ and y in $[0,\infty)$, let $h(x,y) = y/(1+xy)$. Notice that

$$\frac{y}{1+y} \leq h(x,y) \leq y.$$

Together with the assumption that $\xi \geq 0$ and $\mathbb{E}[\xi] \in (0,\infty)$, this implies that

$$0 < \mathbb{E}\left[\frac{\xi}{1+\xi}\right] \leq \sup_{x\in[0,1]} \mathbb{E}[h(x,\xi)] \leq \mathbb{E}[\xi] < \infty. \tag{6.16}$$

On the other hand, for x in $(0,1]$ and $y \geq 0$, one has

$$\frac{\partial h}{\partial x} = -\frac{y^2}{(1+xy)} = -h(x,y)^2, \tag{6.17}$$

thus $|(\partial h/\partial x)(x,y)| \leq x^{-2}$. Now, by continuity of ϕ and ϕ', the preceding inequalities show that the first conditions of the previous lemma are satisfied whenever I is a compact sub-interval of $(0,1)$ and thus that the function

$$f : x \longmapsto \mathbb{E}\left[\phi\left\langle\frac{\xi}{1+x\xi}\right\rangle\right]$$

is well-defined and differentiable on $(0,1)$. It is non-increasing on $(0,1)$ because, for any x in $(0,1)$, the function $y \mapsto h(x,y)$ is increasing on $\mathbb{R}_+$, and by (6.17), for all $y \geq 0$,

$$\frac{\partial}{\partial x}\log(x,y) = -h(x,y).$$

We now want to study the limits of $f(x)$ as x goes to 0 and as x goes to 1. For x in $[1/2,1]$, we have

$$\frac{\frac{\xi}{1+\xi}}{\mathbb{E}\Big[\frac{\xi}{1+\xi/2}\Big]} \leq \Big\langle \frac{\xi}{1+x\xi}\Big\rangle \leq \frac{\frac{\xi}{1+\xi/2}}{\mathbb{E}\Big[\frac{\xi}{1+\xi}\Big]}, \tag{6.18}$$

so by convexity,

$$\phi\bigg(\Big\langle\frac{\xi}{1+x\xi}\Big\rangle\bigg) \leq \phi\bigg(\mathbb{E}\Big[\frac{\xi}{1+\xi/2}\Big]^{-1}\bigg) + \phi\bigg(2\mathbb{E}\Big[\frac{\xi}{1+\xi}\Big]^{-1}\bigg), \tag{6.19}$$

and by the dominated convergence theorem, f is continuous at 1.

For the limit at 0, first remark that we may assume that $\mathbb{E}[\phi\langle\xi\rangle]$ is finite, otherwise the inequality (6.15) is trivially true. The continuity of ϕ on $\mathbb{R}_+$ together with the assumption that ϕ is regularly varying at infinity imply that in fact, for any $a > 0$, $\mathbb{E}[\phi(a\xi)] < \infty$. Since, for any x in $[0,1]$,

$$\Big\langle\frac{\xi}{1+x\xi}\Big\rangle \leq \frac{\xi}{\mathbb{E}\Big[\frac{\xi}{1+\xi}\Big]},$$

we may once again apply the dominated convergence theorem, and we have $f(0+) = f(0) = \mathbb{E}[\phi\langle\xi\rangle]$ and the proof is complete. $\square$

We extend the previous lemma to random sums.

Lemma 6.11. *Let* $\mathbf{A} = (A_1, A_2, ..., A_N)$ *and* $\mathbf{X} = (X_1, X_2, \ldots, X_N)$ *be random non-negative tuples with the same (possibly random) number of elements and let* B *be a non-negative random variable. Assume that, given* N, $\mathbf{A}$ *and* $\mathbf{X}$ *are independent,* B *is independent of* $\mathbf{X}$ *and* $X_1, \ldots, X_N$ *are independent. Assume also that the components of* $\mathbf{X}$ *have a finite, non-null expectation. Let* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a continuously differentiable, regularly varying at infinity, convex function. Then,*

$$\mathbb{E}\bigg[\phi\bigg(\sum_{i=1}^N A_i\Big\langle\frac{X_i}{1+X_i}\Big\rangle + B\bigg)\bigg] \leq \mathbb{E}\bigg[\phi\bigg(\sum_{i=1}^N A_i\langle X_i\rangle + B\bigg)\bigg]. \tag{6.20}$$

Proof. We first deal with the case where N is deterministic equal to $k \geq 0$. If $k = 0$, the result is trivial. If $k = 1$, write

$$\mathbb{E}\Big[\phi\Big(A_1\Big\langle\frac{X_1}{1+X_1}\Big\rangle + B\Big)\Big] = \mathbb{E}\mathbb{E}\Big[\phi\Big(A_1\Big\langle\frac{X_1}{1+X_1}\Big\rangle + B\Big)\Big| A_1, B\Big].$$

Notice that, for any $a, b \geq 0$, the function $x \mapsto \phi(ax+b)$ still satisfies the hypotheses of the previous lemma, therefore,

$$\mathbb{E}\left[\phi\left(A_1\left\langle\frac{X_1}{1+X_1}\right\rangle + B\right)\,\middle|\, A_1, B\right] \leq \mathbb{E}[\phi(A_1\langle X_1\rangle + B) \,|\, A_1, B]. \tag{6.21}$$

Now, for $k \geq 2$,

$$\begin{aligned}\mathbb{E}\left[\phi\left(\sum_{i=1}^k A_i\left\langle\frac{X_i}{1+X_i}\right\rangle + B\right)\right] &= \mathbb{E}\left[\phi\left(A_1\left\langle\frac{X_1}{1+X_1}\right\rangle + \sum_{i=2}^k A_i\left\langle\frac{X_i}{1+X_i}\right\rangle + B\right)\right] \\ &= \mathbb{E}\left[\phi\left(A_1\left\langle\frac{X_1}{1+X_1}\right\rangle + \widetilde{B}\right)\right],\end{aligned}$$

with $\widetilde{B} = \sum_{i=2}^k A_i\left\langle\frac{X_i}{1+X_i}\right\rangle + B$. Applying (6.21) allows us to conclude by induction. The case where N is random derives from the deterministic case by conditionning. $\square$

6.4 Lower bound

From now on, we work under the hypotheses (H_{norm}), $(H_{\text{derivative}})$ and (H_κ). Let, for $n \geq 1$,

$$u_n = \mathbf{E}[\mathscr{C}_n(T)] \qquad \text{and} \qquad a_n = \frac{1}{u_n}. \tag{6.22}$$

We start with an easy *a priori* upper bound.

Lemma 6.12. *In any case, one has*

$$\limsup_{n\to\infty} n u_n \leq 1. \tag{6.23}$$

Proof. Let $n \geq 2$. Recall that, by (6.2),

$$\mathscr{C}_n(T) = \sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i)\frac{\mathscr{C}_{n-1}(T[i])}{1+\mathscr{C}_{n-1}(T[i])}.$$

Conditionally on $\nu_T(\varnothing)$, the random variables $\mathscr{C}_{n-1}(T[1]), \ldots, \mathscr{C}_{n-1}(T[\nu_T(\varnothing)])$ are i.i.d. and are independent of $\mathsf{A}_T(1), \ldots, \mathsf{A}_T(\nu(\varnothing))$. By the hypothesis (H_{norm}),

$$\mathbf{E}[\mathscr{C}_n(T)] = \mathbf{E}\left[\frac{\mathscr{C}_{n-1}(T)}{1+\mathscr{C}_{n-1}(T)}\right]. \tag{6.24}$$

We may also write (6.24) as

$$\mathbf{E}\mathscr{C}_n = \mathbf{E}\mathscr{C}_{n-1} - \mathbf{E}\left[\frac{\mathscr{C}_{n-1}^2}{1+\mathscr{C}_{n-1}}\right]. \tag{6.25}$$

Now notice that

$$\mathbf{E}\left[\frac{\mathscr{C}_{n-1}^2}{1+\mathscr{C}_{n-1}}\right] \geq \mathbf{E}\left[\frac{\mathscr{C}_{n-1}^2}{(1+\mathscr{C}_{n-1})^2}\right] \geq \left(\mathbf{E}\left[\frac{\mathscr{C}_{n-1}}{1+\mathscr{C}_{n-1}}\right]\right)^2 = \mathbf{E}[\mathscr{C}_n]^2.$$

Hence, we have

$$u_{n-1} - u_n \geq u_n^2,$$

which implies that

$$u_n^{-1}(1+u_n)^{-1} \geq u_{n-1}^{-1}.$$

Now, let $\varepsilon > 0$. Since, as n goes to infinity, u_n goes to 0, we have, when n is large enough,

$$(1-u_n)^{-1} \geq 1 - u_n - \varepsilon u_n,$$

thus

$$u_n^{-1} - u_{n-1}^{-1} \geq 1 - \varepsilon.$$

Summing this inequality yields

$$\liminf_{n\to\infty} \frac{u_n^{-1}}{n} \geq 1 - \varepsilon. \qquad \square$$

Now we go back to the relation (6.2). Together with (6.24), this implies that

$$\langle \mathscr{C}_n(T) \rangle = \sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i) \left\langle \frac{\mathscr{C}_{n-1}(T)}{1+\mathscr{C}_{n-1}(T)} \right\rangle, \tag{6.26}$$

so we may apply Lemma 6.13 to obtain the following inequality.

Lemma 6.13. *Let ϕ be a non-negative convex, continuously differentiable function, regularly varying at infinity. For any integer $n \geq 1$,*

$$\mathbf{E}[\phi(\langle \mathscr{C}_n(T) \rangle)] \leq \mathbf{E}[\phi(M_n(T))] \leq \mathbf{E}[\phi(M_\infty(T))]. \tag{6.27}$$

Proof. Recall that, by definition, $\mathscr{C}_1(T) = \sum_{i=1}^{\nu_T(\varnothing)} \mathsf{A}_T(i)$ and iterate the recursive equation (6.26) together with Lemma 6.11. The last inequality comes from the fact that, by Jensen's inequality, $\phi(M_n(T))$ is a sub-martingale. $\square$

The order of magnitude in Lemma 6.12 is only sharp in the case $\kappa > 2$. To obtain more refined bounds, let, for $a > 0$,

$$\phi_a(x) = \frac{x^2}{a+x}. \tag{6.28}$$

The function ϕ_a is convex. By (6.25), we have

$$\mathbf{E}[\mathscr{C}_n] = \mathbf{E}[\mathscr{C}_{n-1}] - \mathbf{E}[\mathscr{C}_{n-1}]\mathbf{E}\left[\frac{\langle \mathscr{C}_{n-1} \rangle^2}{\frac{1}{\mathbf{E}\mathscr{C}_{n-1}} + \langle \mathscr{C}_{n-1} \rangle} \right]. \tag{6.29}$$

$$= \mathbf{E}[\mathscr{C}_{n-1}] - \mathbf{E}[\mathscr{C}_{n-1}]\mathbf{E}\Big[\phi_{1/\mathbf{E}\mathscr{C}_{n-1}}(\langle \mathscr{C}_{n-1} \rangle)\Big] \tag{6.30}$$

$$\geq \mathbf{E}[\mathscr{C}_{n-1}] \left(1 - \mathbf{E}[\phi_{a_{n-1}}(M_\infty)]\right), \tag{6.31}$$

where, for the last inequality, we have used Lemma 6.13. We are now ready to give a lower estimate of $\mathbf{E}[\mathscr{C}_n]$, as n goes to infinity.

Proposition 6.14. *Under the hypotheses* (H_{norm}), ($H_{\text{derivative}}$) *and* (H_κ),

1. $\liminf_{n\to\infty} n^{1/(\kappa-1)}\mathbf{E}[\mathscr{C}_n(T)] > 0$, *if* $1 < \kappa < 2$;
2. $\liminf_{n\to\infty} n \log n\mathbf{E}[\mathscr{C}_n(T)] > 0$, *if* $\kappa = 2$ *and*
3. $\liminf_{n\to\infty} n\mathbf{E}[\mathscr{C}_n(T)] \geq \mathbf{E}[M_\infty^2]$, *if* $\kappa > 2$.

Proof. Recall the notation

$$\varphi_p(a) = \mathbf{E}\left[\left(\frac{M_\infty^2}{a + M_\infty}\right)^p\right].$$

By (6.29), we have

$$u_n \geq u_{n-1}(1 - \varphi_1(a_{n-1})). \tag{6.32}$$

Now assume that $1 < \kappa < 2$. By Lemma 6.4, there exists $C > 0$ such that, for n large enough,

$$\varphi_1(a_{n-1}) \leq C a_{n-1}^{1-\kappa}.$$

Hence, for n large enough,

$$u_{n-1} - u_n \leq C u_{n-1}^{\kappa},$$

so we may conclude by Lemma 6.6.

If we assume that $\kappa = 2$, by Lemma 6.4 there exists $C > 0$ such that, for n large enough,

$$\varphi_1(a_{n-1}) \leq C a_{n-1}^{-1} \log(a_{n-1}).$$

Applying this time Lemma 6.7, yields the result.

Finally, if $\kappa > 2$, then by our hypotheses, $\mathbf{E}[M_\infty^2]$ is finite and by dominated convergence,

$$\varphi_1(a_{n-1}) \sim_{n\to\infty} \mathbf{E}[M_\infty^2] a_{n-1}^{-1}.$$

Using again Lemma 6.6 concludes the proof. □

Remark 6.1. If we assume that we are in the "non-lattice case" (see [37]) we may also give explicit lower bounds (depending on the law of M_∞) in the cases $1 < \kappa < 2$ and $\kappa = 2$. However, since our method does not provide explicit upper bounds, we chose not to make this additional assumption.

6.5 Upper bound and almost-sure convergence

Iterating (6.2), we obtain, for all $1 \leq k \leq n$,

$$\langle \mathscr{C}_n(T) \rangle = \frac{\mathbf{E}[\mathscr{C}_{n-k}(T)]}{\mathbf{E}[\mathscr{C}_n(T)]} \sum_{|x|=k} \mathsf{c}_T(x) \langle \mathscr{C}_{n-k}(T[x]) \rangle - \frac{1}{\mathbf{E}[\mathscr{C}_n(T)]} \sum_{|x|\leq k} \mathsf{c}_T(x) \phi_1\Big(\mathscr{C}_{n-|x|}(T[x])\Big), \tag{6.33}$$

where $\phi_1 : x \mapsto x^2/(1+x)$.

$$\langle \mathscr{C}_n(T) \rangle = \frac{a_n}{a_{n-k}} M_\infty(T) + \frac{a_n}{a_{n-k}} X_{k,n}(T) - \sum_{j=1}^{k} \frac{a_n}{a_{n-j}} Y_{j,n}(T), \tag{6.34}$$

where

$$X_{k,n}(T) = \sum_{|x|=k} \mathsf{c}_T(x) \left(\mathscr{C}_{n-k}(T[x]) - M_\infty(T[x]) \right), \tag{6.35}$$

and

$$Y_{j,n}(T) = \sum_{|x|=j} \mathsf{c}_T(x) \phi_{a_{n-j}} \langle \mathscr{C}_{n-j} \rangle (T[x]). \tag{6.36}$$

For later use, we also introduce

$$\xi_{k,n}(T) = \sum_{j=1}^{k} Y_{j,n}(T). \tag{6.37}$$

We want to let $k = k_n$ in the previous decomposition, but first we need to know when the sequence (a_{n-k_n}) is equivalent to (a_n), which is the purpose of the two following lemmas.

Lemma 6.15. *In any case, we may find $C > 0$ such that, for n large enough,*

$$\varphi_1(a_n) \leq \frac{C}{n}.$$

Proof. By the lower bounds (Proposition 6.14), we may find $C_1 > 0$ such that for large enough n,

$$a_n \leq C_1 \times \begin{cases} n^{1/(\kappa-1)} & \text{if } 1 < \kappa < 2; \\ n \log n & \text{if } \kappa = 2 \text{ and} \\ n & \text{if } \kappa > 2. \end{cases}$$

Combine this with the fact that (by Lemma 6.4 and the end of the proof of Proposition 6.14) there exists $C_2 > 0$ such that, for a large enough,

$$\varphi_1(a) \leq C_2 \times \begin{cases} a^{1-\kappa} & \text{if } 1 < \kappa < 2; \\ a^{-1} \log(a) & \text{if } \kappa = 2 \text{ and} \\ a^{-1} & \text{if } \kappa > 2. \end{cases} \qquad \square$$

Lemma 6.16. *Let $(k_n)_{n\geq 1}$ be a sequence of non-negative integers. If $k_n = o(n)$, then $a_{n-k_n} \sim a_n$.*

Proof. Iterating the inequality (6.32), recalling that the function φ_1 is non-increasing, we obtain that, for any large enough integer n,

$$1 \geq \frac{u_n}{u_{n-k_n}} \geq \prod_{i=1}^{k_n} (1 - \varphi_1(a_{n-i})) \geq (1 - \varphi_1(a_{n-k_n}))^{k_n}.$$

By the previous lemma, we may find $C > 0$ such that, eventually,

$$(1 - \varphi_1(a_{n-k_n}))^{k_n} \geq \left(1 - \frac{C}{n - k_n}\right)^{k_n} \xrightarrow[n\to\infty]{} 1. \qquad \square$$

We are now ready to prove the convergence of $\langle \mathscr{C}_n \rangle$ towards M_∞ in L^p, for $p \in (1, \kappa \wedge 2)$.

Lemma 6.17. *Let p be a real number in $(1, \kappa \wedge 2)$. Let $1 \leq k \leq n$, then we have the following inequalities, in any case,*

$$\|X_{k,n}\|_p \leq 2^{1+1/p} \|M_\infty\|_p e^{k\psi(p)/p}; \tag{6.38}$$

$$\|Y_{k,n}\|_p \leq e^{j\psi(p)/p} \varphi_p^{1/p}(a_{n-j}) + \|M_\infty\|_p \varphi_1(a_{n-j}). \tag{6.39}$$

As a consequence, the sequence $\langle \mathscr{C}_n \rangle$ converges, in L^p, towards M_∞, for any $p \in (1, \kappa \wedge 2)$.

Proof. For $X_{k,n}$, notice that the random variables

$$\left(\langle \mathscr{C}_{n-k} \rangle (T[x]) - M_\infty(T[x])\right), \quad \text{for } |x| = n$$

are i.i.d., centered, and independent of $\mathcal{F}_k$. Thus we may apply, conditionally on $\mathcal{F}_k$, the inequality (6.12), to obtain

$$\mathbf{E}[|X_{k,n}|^p \,|\, \mathcal{F}_k] \leq 2 \sum_{|x|=k} \mathsf{c}_T(x)^p \mathbf{E}[|\langle \mathscr{C}_{n-k} \rangle - M_\infty|^p].$$

Taking the expecation on both sides, we obtain

$$\mathbf{E}[|X_{k,n}|^p] \leq 2 e^{k\psi(p)} \mathbf{E}[|\langle \mathscr{C}_{n-k} \rangle - M_\infty|^p]$$

Now recall that, by convexity, $\mathbf{E}[\langle \mathscr{C}_n \rangle^p] \leq \mathbf{E}[M_\infty^p]$, therefore,

$$\|X_{k,n}\|_p \leq 2^{1/p} \|\mathscr{C}_{n-k} - M_\infty\|_p e^{k\psi(p)/p} \leq 2^{1+1/p} \|M_\infty\|_p e^{k\psi(p)/p}.$$

For $Y_{j,n}$, conditionally on $\mathcal{F}_j$, we may use the inequality (6.11):

$$\begin{aligned}\mathbf{E}[Y_{j,n}{}^p] &\leq \mathbf{E}\Big[\sum_{|x|=j} \mathsf{c}_T(x)^p\Big] \mathbf{E}\Big[\phi^p_{a_{n-j}}(\langle \mathscr{C}_{n-j} \rangle)\Big] + \mathbf{E}\Big[\phi_{a_{n-j}}(\mathscr{C}_{n-j})\Big]^p \mathbf{E}\Big[\Big(\sum_{|x|=j} \mathsf{c}_T(x)\Big)^p\Big] \\ &\leq e^{j\psi(p)} \varphi_p(a_{n-j}) + \mathbf{E}(M_\infty^p) \varphi(a_{n-j})^p.\end{aligned}$$

Now, let $(k_n)_{n\geq 1}$ be any sequence of non-negative integers that go to infinity as n goes to infinity and such that $k_n = o(n)$ (*e.g.* $k_n = \lfloor \log(n) \rfloor$). By Minkowski's inequality,

$$\|\langle \mathscr{C}_n \rangle - M_\infty\|_p \leq \left(\frac{a_n}{a_{n-k_n}} - 1\right) \|M_\infty\|_p + \frac{a_n}{a_{n-k_n}} \|X_{k_n,n}\|_p + a_n \Big\| \sum_{j=1}^{k_n} \frac{1}{a_{n-j}} Y_{j,n} \Big\|_p.$$

By Lemma 6.16, the sequence (a_{n-k_n}) is equivalent to (a_n). By the inequality (6.38) and the fact that, by hypothesis, $\psi(p) < 0$, the term $\|X_{n,k_n}\|_p$ goes to 0 as n goes to infinity.

Since (a_n) is increasing, by Minkowski's inequality,

$$\begin{aligned}\Big\|\sum_{i=1}^{k_n}\frac{a_n}{a_{n-j}}Y_{j,n}\Big\|_p &\le \frac{a_n}{a_{n-k_n}}\sum_{j=1}^{k_n}\|Y_{j,n}\|_p \\ &\le \frac{a_n}{a_{n-k_n}}\left(\frac{e^{\psi(p)/p}}{1-e^{\psi(p)/p}}\varphi_p^{1/p}(a_{n-k_n}) + \|M_\infty\|_p k_n\varphi_1(a_{n-k_n})\right).\end{aligned}$$

By Lemma 6.4, the first term in the above sum goes to 0 as n goes to infinity, and by Lemma 6.15 and our choice of k_n, so does the second term $k_n\varphi_1(a_{n-k_n})$ and we have proved the convergence in L^p. □

The almost-sure convergence will be obtained by Borel-Cantelli's lemma together with a monotony argument. To this end, however, we need to know the order of magnitude of $\mathbf{E}[\mathscr{C}_n]$.

Lemma 6.18. *Assume that* $1 < p < \kappa \wedge 2$. *Let, for* $n \ge 1$,

$$k_n = \left\lfloor \frac{-2}{\psi(p)}\log(a_n)\right\rfloor.$$

Then, there exists $C < \infty$ *such that, in any case, for all* $n \ge 1$,

$$\mathbf{E}\Big[\xi^p_{k_n,n} + |X_{k_n,n}|^p\Big] \le C a_n^{p-\kappa\wedge 2}. \tag{6.40}$$

Proof. Our choice of (k_n) is such that $a_{n-k_n} \sim_{n\to\infty} a_n$. Moreover, by (6.38),

$$\mathbf{E}|X_{k_n,n}|^p \le 2^{p+1}\mathbf{E}[M_\infty^p]e^{k_n\psi(p)} \le C_1 a_n^{-2},$$

for some finite constant C_1. By (6.39), there exists $C_2 < \infty$ such that

$$\mathbf{E}[\xi^p_{k_n,n}] \le C_2\left(\varphi_p(a_{n-k_n}) + k_n^p\varphi_1^p(a_{n-k_n})\right).$$

Now Lemma 6.4 shows that there exist finite constants C_3 and C_4 such that

$$\varphi_p(a_{n-k_n}) \le C_3 a_n^{p-\kappa} \qquad \text{and}$$

$$k_n^p\varphi_1^p(a_{n-k_n}) \le C_4\log(a_n)^p\begin{cases} a_n^{(1-\kappa)p} & \text{if } 1<\kappa<2,\\ a_n^{-p}(\log(a_n))^p & \text{if } \kappa = 2,\\ a_n^{-p} & \text{if } \kappa > 2.\end{cases}$$

Since $p > 1$, we have $p - \kappa\wedge 2 > (1-\kappa\wedge 2)p$, so that, in any case,

$$\limsup_{n\to\infty} a_n^{\kappa\wedge 2-p}\mathbf{E}\Big[\xi^p_{k_n,n} + |X_{k_n,n}|^p\Big] \le C_2C_3 < \infty. \qquad □$$

Lemma 6.19. *If* $\kappa \in (1,2]$, *we may find* $\delta_0 > 0$, $c_0 > 0$ *and* $n_0 \ge 1$ *such that*

$$\mathbf{P}(\langle\mathscr{C}_n\rangle > r) \ge c_0 r^{-\kappa}, \qquad \forall r \in [1, \delta_0 a_n],\ \forall n \ge n_0.$$

Proof. Let $\delta > 0$ and $r \in [1, \delta a_n]$. Let (k_n) be defined as in the previous lemma. By the union bound,

$$\mathbf{P}(\langle \mathscr{C}_n \rangle > r) \geq \mathbf{P}\Big(\frac{a_n}{a_{n-k_n}} M_\infty > 3r\Big) - \mathbf{P}\Big(\frac{a_n}{a_{n-k_n}} |X_{k_n,n}| > r\Big) - \mathbf{P}\Big(\frac{a_n}{a_{n-k_n}} \xi_{k_n,n} > r\Big).$$

Assume that n is so large that

$$1 \leq \frac{a_n}{a_{n-k_n}} \leq 2.$$

Then,

$$\mathbf{P}(\langle \mathscr{C}_n \rangle > r) \geq \mathbf{P}(M_\infty > 3r) - \mathbf{P}\Big(|X_{k_n,n}| > \frac{r}{2}\Big) - \mathbf{P}\Big(\xi_{k_n,n} > \frac{r}{2}\Big).$$

By Markov's inequality,

$$\begin{aligned} r^\kappa \left(\mathbf{P}\Big(|X_{k_n,n}| > \frac{r}{2}\Big) + \mathbf{P}(\xi_{k_n,n} > r)\right) &\leq r^{\kappa-p} 2^p \mathbf{E}\Big[|X_{k_n,n}|^p + \xi_{k_n,n}^p\Big] \\ &\leq C_1 a_n^{p-\kappa} \delta^{\kappa-p} a_n^{\kappa-p} = C_1 \delta^{\kappa-p}, \end{aligned}$$

for some finite constant C_1, where, for the last inequality we have used the previous lemma, together with the assumption that $r \in [1, \delta a_n]$.

On the other hand, by Fact 6.3,

$$\inf_{r\geq 1} r^\kappa \mathbf{P}(M_\infty > 3r) =: C_2 > 0.$$

This implies that, for all r in $[1, \delta a_n]$,

$$r^\kappa \mathbf{P}(\langle \mathscr{C}_n \rangle > r) \geq C_2 - \delta^{\kappa-p} C_1,$$

which is positive as soon as δ is small enough. □

Proposition 6.20. *Under the hypotheses* (H_{norm}), $(H_{\text{derivative}})$ *and* (H_κ),

1. $\limsup_{n\to\infty} n^{1/(\kappa-1)} \mathbf{E}[\mathscr{C}_n(T)] < \infty$, *if* $1 < \kappa < 2$;
2. $\limsup_{n\to\infty} n \log n \mathbf{E}[\mathscr{C}_n(T)] < \infty$, *if* $\kappa = 2$ *and*
3. $\limsup_{n\to\infty} n \mathbf{E}[\mathscr{C}_n(T)] < \infty$, *if* $\kappa > 2$.

Proof. The last point was already stated as the *a priori* bound (6.23).

Assume that $1 < \kappa < 2$. Let $n \geq n_0$, $\delta_0 > 0$ and $c_0 > 0$ be such that the conclusion of the previous lemma holds. By Markov's inequality,

$$\mathbf{E}\left[\frac{\langle \mathscr{C}_{n-1}\rangle^2}{a_n + \langle \mathscr{C}_{n-1}\rangle}\right] \geq \mathbf{P}(\langle \mathscr{C}_{n-1} > \delta_0 a_n \rangle) \frac{\delta_0^2 a_n^2}{a_n + \delta_0 a_n} \geq c_0 \frac{\delta_0^2}{1+\delta_0} a_n^{1-\kappa},$$

where, for the last inequality, we have used the previous lemma. Therefore, by (6.25), with $u_n := \mathbf{E}[\mathscr{C}_n]$,

$$u_{n-1} - u_n = u_{n-1} \mathbf{E}\left[\frac{\langle \mathscr{C}_{n-1}\rangle^2}{a_n + \langle \mathscr{C}_{n-1}\rangle}\right] \geq C u_n^\kappa,$$

for some finite constant C. By Lemma 6.6, this implies that

$$\limsup_{n\to\infty} n^{1/(\kappa-1)} u_n \leq C < \infty.$$

Now assume that $\kappa = 2$. As before, let n_0, δ_0 and c_0 be chosen as in the previous lemma. Let n be greater than n_0 and write $\mathbf{P}_{\langle\mathscr{C}_n\rangle}$ for the distribution of $\langle\mathscr{C}_n\rangle$. By Tonelli's theorem,

$$\begin{aligned}\mathbf{E}\left[\frac{\langle\mathscr{C}_{n-1}\rangle^2}{a_n + \langle\mathscr{C}_{n-1}\rangle}\right] &= \int_0^\infty \frac{x^2}{a_n + x}\mathbf{P}_{\langle\mathscr{C}_n\rangle}(\mathrm{d}x)\\ &= \int_0^\infty \frac{x^2 + 2a_n x}{(a_n + x)^2}\mathbf{P}(\langle\mathscr{C}_n\rangle > x)\,\mathrm{d}x\\ &\geq \int_1^{\delta_0 a_n} \frac{x^2 + 2a_n x}{(a_n + x)^2} c_0 x^{-2}\,\mathrm{d}x,\end{aligned}$$

by the previous lemma. The change of variable $y = x/a_n$ leads to

$$\mathbf{E}\left[\frac{\langle\mathscr{C}_{n-1}\rangle^2}{a_n + \langle\mathscr{C}_{n-1}\rangle}\right] \geq c_0 a_n^{-1} \int_{1/a_n}^{\delta_0} \frac{1 + 2/y}{(1+y)^2}\,\mathrm{d}y \geq C a_n^{-1}\log(a_n),$$

for some constant $C > 0$. Together with (6.25), this implies that, for all large enough n,

$$u_n - u_{n-1} \geq C u_n^2 \log(1/u_n),$$

and we may conclude by Lemma 6.7. $\square$

Proposition 6.21. *In any case, the sequence $\langle\mathscr{C}_n\rangle$ converges to M_∞, almost surely.*

Proof. Fix p in $(1, \kappa \wedge 2)$. Let (k_n) be as in Lemma 6.18. Write, for $n \geq 1$,

$$\eta_n = \frac{a_{n-k_n}}{a_n}\langle\mathscr{C}_n\rangle = M_\infty + X_{k_n,n} - a_{n-k_n}\sum_{j=1}^{k_n}\frac{1}{a_{n-j}}Y_{j,n}.$$

Then, using first Lemma 6.18 and then Proposition 6.20,

$$\mathbf{E}[|\eta_n - M_\infty|^p] \leq C_1 a_n^{p-\kappa\wedge 2} \leq C_2 \begin{cases} n^{-\frac{\kappa-p}{p(\kappa-1)}} & \text{if } 1 < \kappa < 2\\ (n\log(n))^{-(2/p-1)} & \text{if } \kappa = 2\\ n^{-(2/p-1)} & \text{if } \kappa > 2.\end{cases}$$

In any case, we may find an integer $\alpha \geq 2$ such that

$$\sum_{n\geq 1}\mathbf{E}[|\eta_{n^\alpha} - M_\infty|^p] < \infty.$$

By Markov's inequality and Borel-Cantelli's lemma, this implies that the sequence (η_{n^α}) converges almost surely to M_∞ and so does $(\langle\mathscr{C}_{n^\alpha}\rangle)$.

Now, let, for $n \geq 1$, $r_n = \lceil n^{1/\alpha} \rceil$. Then, for all $n \geq 1$,

$$(r_n - 1)^\alpha \leq n \leq r_n^\alpha,$$

and by the fact that the sequence $(\mathscr{C}_n)$ is decreasing,

$$\mathscr{C}_{r_n^\alpha} \leq \mathscr{C}_n \leq \mathscr{C}_{(r_n-1)^\alpha}.$$

This implies that

$$\frac{u_{r_n^\alpha}}{u_{(r_n-1)^\alpha}} \langle \mathscr{C}_{r_n^\alpha} \rangle \leq \langle \mathscr{C}_n \rangle \leq \frac{u_{(r_n-1)^\alpha}}{u_{r_n^\alpha}} \langle \mathscr{C}_{(r_n-1)^\alpha} \rangle.$$

Now, write $(r_n - 1)^\alpha = r_n^\alpha - s_n$. Since $s_n = o(r_n^\alpha)$, we may use Lemma 6.16 to see that

$$\frac{u_{r_n^\alpha}}{u_{(r_n-1)^\alpha}} \xrightarrow[n\to\infty]{} 1,$$

which concludes the proof. □

Proposition 6.22. *If $\kappa > 2$, the convergence of $\langle \mathscr{C}_n \rangle$ towards M_∞ also holds in L^2. Moreover, the expectation of $\mathscr{C}_n$ satisfies*

$$n\mathbf{E}[\mathscr{C}_n] \xrightarrow[n\to\infty]{} \|M_\infty\|_2.$$

Proof. We already know that, for all $n \geq 1$, $\mathbf{E}[\langle \mathscr{C}_n \rangle^2] \leq \mathbf{E}[M_\infty^2]$. Now, by the almost-sure convergence of $\langle \mathscr{C}_n \rangle$ to M_∞ and Fatou's lemma, $\mathbf{E}[M_\infty^2] \leq \liminf [\mathscr{C}_n^2]$, thus $\mathbf{E}[\langle \mathscr{C}_n \rangle^2] \to \mathbf{E}[M_\infty^2]$.

Finally, by dominated convergence,

$$\mathbf{E}\left[\frac{\langle \mathscr{C}_{n-1}^2 \rangle}{1 + \mathscr{C}_{n-1}}\right] \xrightarrow[n\to\infty]{} \mathbf{E}\left[M_\infty^2\right],$$

so

$$\mathbf{E}\left[\frac{\langle \mathscr{C}_{n-1}^2 \rangle}{a_n + \langle \mathscr{C}_{n-1} \rangle}\right] \sim u_n \mathbf{E}\left[M_\infty^2\right],$$

and by the identity (6.25) and Lemma 6.6, $u_n \sim \frac{\mathbf{E} M_\infty^2}{n}$. □

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Résumé

Cette thèse a pour objet d'étude divers modèles de marches aléatoires sur les arbres aléatoires. Nous nous sommes consacrés principalement aux aspects qui relevaient à la fois de la théorie des probabilités et de la théorie ergodique.

Notre premier modèle est celui des marches aléatoires sur les arbres à longueurs récursives (qui généralise un modèle apparaissant dans un travail récent de Curien et Le Gall). Nous montrons pour ce modèle sous des conditions très générales qu'un phénomène appelé « chute de dimension » se produit pour la mesure harmonique et donnons une formule assez explicite permettant de calculer cette dimension.

En utilisant les outils développés pour ce dernier modèle, nous nous intéressons à la marche aléatoire lambda-biaisée sur un arbre de Galton-Watson infini, pour lequel de nombreuses conjectures sont toujours ouvertes. Notre approche nous permet de calculer la dimension de la mesure harmonique en fonction de la loi de la conductance de l'arbre. C'est un résultat nouveau qui nous permet de vérifier numériquement certaines de ces conjectures ouvertes.

Le reste de la thèse porte sur un modèle très riche appelé marche aléatoire sur un arbre pondéré aléatoire. D'abord dans le cas transient, où nous montrons par une approche différente de celle des parties précédentes que le phénomène de chute de dimension se produit. Puis sur un cas récurrent appelé sous-diffusif, où nous nous intéressons à la vitesse de convergence vers 0 de la conductance entre la racine et le niveau n de l'arbre lorsque n tend vers l'infini. Nous montrons que la loi limite de cette conductance renormalisée par son espérance est la limite de la martingale de Mandelbrot.

Mots-clés : probabilités ; théorie ergodique ; arbres aléatoires ; marches aléatoires.

Abstract

The subject of this thesis is the study of various models of random walks on random trees, with an emphasis on the aspects that fall at the intersection of probability theory and ergodic theory.

We called our first model "random walks on Galton-Watson trees with recursive lengths". It generalizes a model appearing in a recent work by Curien and Le Gall. We show that under fairly general assumptions, a phenomenon called "dimension drop" holds for this model and we give a formula for this dimension.

Using the tools developed for the study of the previous model, we turn to the case of transient lambda-biased random walks on infinite Galton-Watson trees, for which many famous problems are still open. Our approach allows us to compute the dimension of the harmonic measure as a function of the law of the conductance of the tree. With this new result, we check numerically the validity of some twenty-year-old conjectures.

The remainder of this thesis is about a very rich model called random walk on a random weighted Galton-Watson tree. First, we study the transient case, where we show with a different method than in the previous parts, that the dimension drop phenomenon occurs. Then we turn to a recurrent case called subdiffusive and we investigate the rate of decay of the conductance between the root and the n-th level of the tree, as n goes to infinity. We prove that this conductance, suitably renormalized converges to the limit of the Mandelbrot martingale.

Keywords : probability theory; ergodic theory; random trees; random walks.