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Nicolas Dub

Enumeration of triangulations modulo symmetries and of rooted triangulations counted by their number of $(d-2)$-simplices in dimension $d \geq 2$

Devant le jury composé de :

Frédérique BASSINO	Professeure, LIPN, Villetaneuse	Directrice de thèse
Valentin BONZOM	Maître de conférences, LIPN, Villetaneuse	Co-directeur de thèse
Eric FUSY	Directeur de recherches, IGM, Marne-la-allée	Rapporteur
Thomas KRAJEWSKI	Maître de conférences, CPT, Marseille	Examinateur
Adrian TANASA	Professeur, LABRI, Talence	Rapporteur

Abstract

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by Nicolas Dub

$O(N)$ invariants are the observables of real tensor models. We represent them by regular colored graphs dual to d-dimensional triangulations. We enumerate the invariants using permutation group techniques and reveal that the algebraic structure organizing them differs from that of the unitary invariants. At fixed rank and fixed number of vertices, an associative semi-simple algebra with dimension the number of invariants naturally emerges from the formulation. Using the representation theory of the symmetric group, we enlighten a few crucial facts: the enumeration of $O(N)$ invariants gives a sum of constrained Kronecker coefficients, there is a representation theoretic orthogonal basis of the algebra that reflects its dimension; normal ordered 2-point correlators of the Gaussian model evaluate using permutation group language, these functions provide other representation theoretic orthogonal bases of the algebra.

Tensor models are furthermore generalizations of matrix models and as such, it is natural to ask whether they satisfy some form of the topological recursion. The world of unitary-invariant observables is howbeit much richer in tensor models. It is therefore *a priori* unclear which set of observables could satisfy the topological recursion. Here we show that some set of observables is present in arbitrary tensor models which have non-vanishing couplings for the quartic melonic interactions. It satisfies the blobbed topological recursion in a universal way. The spectral curve is a disjoint union of Gaussian spectral curves, with the cylinder function receiving an additional holomorphic part. This result is achieved *via* a perturbative rewriting of tensor models as multi-matrix models due to Bonzom, Lionni and Rivasseau. It is then possible to formally integrate all degrees of freedom except those which enter the recursion, meaning interpreting the Feynman graphs as stuffed maps. We further provide new expressions to relate the expectations of $U(N)^d$-invariant observables on the tensor and matrix sides.

Résumé

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Enumeration of triangulations modulo symmetries and of rooted triangulations counted by their number of $(d-2)$-simplices in dimension $d \geq 2$

par Nicolas Dub

Les invariants orthogonaux sont les observables des modèles de tenseurs réels. Nous les représentons au travers de graphes colorés et réguliers qui sont duaux à des triangulations de dimension d. Nous énumérons ces invariants à l'aide de méthodes empruntées au groupe symétrique et montrons que la structure algébrique qui les régit diffère du cas unitaire. À rang et nombre de sommets fixés, une algèbre associative et semi-simple de dimension le nombre d'invariants émerge naturellement de notre formulation. À l'aide de la théorie des représentations du groupe symétrique nous prouvons notamment que l'énumération des invariants orthogonaux se traduit par une somme de coefficients de Kronecker contraints et qu'il existe une base de Fourier orthogonale de l'algèbre qui reflète sa dimension.
Les modèles de tenseurs généralisent les modèles de matrices, l'on est de fait en droit de se demander s'ils satisfont une certaine forme de récurrence topologique. Le monde des observables unitaires étant néanmoins bien plus riche pour les tenseurs, il est difficile *a priori* de savoir quel ensemble d'observables est en mesure de satisfaire cette récurrence. Un de ces ensembles est cependant présent dans tout modèle de tenseurs dont les constantes de couplage des interactions quartiques meloniques sont non nulles. La courbe spectrale est une union disjointe de courbes spectrales gaussiennes, et l'amplitude du cylindre se dote d'une part holomorphe. Ce résultat est obtenu par une réécriture perturbative des modèles de tenseurs en modèles dits multi-matrices, et due à Bonzom, Lionni et Rivasseau. Il est ainsi possible d'intégrer, du moins formellement, tous les degrés de liberté, sauf ceux entrant dans la récurrence. Les graphes de Feynman s'interprètent alors comme des cartes farcies. Finalement, nous donnons de nouvelles relations liant valeurs moyennes des observables tensorielles et matricielles.

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S_{EH}	the Einstein-Hilbert action of pure gravity
σ_k	a k-simplex
n_k	the number of k-simplices of a simplicial complex
$h(\Sigma)$	the genus of the surface Σ
$\chi(\Sigma)$	the Euler characteristic of the surface Σ
$\omega(x)$	the trace of the resolvent of a matrix
$W_{n,g}(x_1, \ldots, x_n)$	the g^{th} term in the genus expansion of the n-point correlation function
$W_{n,g}(x_1, \ldots, x_n \vert t_1, \ldots, t_d \vert t)$	the generating function of maps with n boundaries, maximal degree of faces d and genus g
$\hat{\mathbb{C}}$	the Riemann sphere
$K(z, z_1)$	the topological recursion kernel
$d\mu_C$	the Gaussian measure of covariance C
$B(T, \overline{T})$	a bubble invariant
$\omega(\mathcal{G})$	the Gurau degree of a graph $\mathcal{G}$
S_k	the symmetric group of order $k!$
$Z_C^{H\to G}$	the number of elements of $H \leq G$ in the conjugacy class C of G
z_C	the number of elements in a group commuting with any element in the conjugacy class C
$\mathcal{Z}^{S_\infty[S_d]}(t, \vec{x})$	the generating function of the number of wreath product elements
$Z_d(n)$	the number of rank-d $U(N)$ tensor invariants of order n
$Z_d(2n)$	the number of rank-d $O(N)$ tensor invariants of order n
$\mathbb{C}[S_n]$	the group algebra of the symmetric group
$\mathbf{C}(R_1, R_2, R_3)$	the Kronecker coefficients
$\mathcal{K}_d(2n)$	the double coset algebra associated to the $O(N)$ counting
$Q^{R,S,T,\tau}$	the Fourier basis of $\mathcal{K}_3(2n)$

$\mathcal{H}_p(x)$	:	the Hermite polynomial of order p
$\omega_{n,g}(z_1, \dots, z_n)$	:	the invariants of the spectral curve of genus g
$S_k[S_2]$	:	the hyperoctaedral group of order $2^k k!$
$\mathbb{T}^2$	:	the 2-torus
$\mathbb{S}^k$	:	the k-sphere
χ^R	:	the character in the representation R of the symmetric group
$B^{R;\,r,\nu_r}_{i;\,m_r}$	:	the branching coefficients of an irrep r when decomposed in an orthonormal basis of the irreps R
$O_{\sigma_1,\sigma_2,\sigma_3}(T)$	:	an observable of a rank-3 $O(N)$ tensor model
$\mathcal{B}$	:	a set of bubbles
$A_{N,\mathcal{B}}(G)$	:	the amplitude of a Feynman graph G
$\bigsqcup_\alpha I_\alpha$	:	the disjoint union of the sets I_α
Γ_c	:	a cut of colour c

"If I have seen further it is by standing on the shoulders of giants."

Introduction

Matrix and tensor models – Random tensor models are a generalization of matrix models. Since their inception [1, 2, 3], random tensors offer a framework for studying random discrete geometries as they aim at extending the successful relationship between random matrices [4] and two-dimensional quantum gravity to higher dimensions. The main goal of this approach is to devise a transition from discrete geometries to continuum geometries in any dimension. Howbeit, the original proposals for tensor models were plagued with difficulties and no significant development occurred in the field for twenty years. Only recently have random tensor models witnessed major progress [5] with, for instance, the advent of a new large N expansion generalizing 't Hooft genus expansion [6] for higher dimensional (pseudo-)manifolds. Moreover, recall that matrix models are intimately connected to combinatorial maps, the latter being generated by the Feynman expansion of the former [4, 7]. For instance, in the 1-Hermitian matrix model,

$$\int dM\ e^{-\frac{N}{2t}\mathrm{tr}M^2+N\sum_{k\geq 1}\frac{t_k}{k}\mathrm{tr}M^k} = \sum_{\text{maps}} \frac{t^n}{n!} N^{2-2h} \prod_{k\geq 1} t_k^{n_k}, \tag{1}$$

where the sum is over maps of genus h with n labeled edges and n_k faces of degree $k \geq 1$. The quantity in the exponential on the left-hand side is called the action, or the potential, and the t_ks are called the coupling constants. In tensor models, this relationship is also generalized, meaning that the Feynman expansion of tensor models generate piecewise-linear d-dimensional (pseudo-)manifolds [5, 8, 9, 10]. This is why tensor models were already proposed as candidate for quantum gravity in the early 90s, long before a large N limit was found [11]. The existence of such a limit for tensors naturally unveiled several analytical results, among which the discovery of their critical behavior (branched polymers [12, 13]), the universal property of random tensors [14], and the discovery of new families of renormalizable non-local quantum field theories with interesting UV [15, 16, 17] and nonperturbative behaviors supporting the discovery of new universality classes for gravity [18, 19, 20].

More recently, and quite unexpectedly, tensor models have been shown to provide the same large N limit as the Sachdev-Ye-Kitaev (SYK) model [21, 22] of condensed matter physics (a model which is exactly solvable at large N in the IR and exhibits maximal chaos, it is dual to the Jackiw-Teitelboim 2D gravity [23, 24]). For its deep connections with black hole physics and AdS/CFT correspondence, the SYK model embodies a vibrant topic of research. This has driven the development of tensor models in the last few years. Indeed, new models have been introduced and their large N limit explored [25, 26]. Some models could also be explored beyond leading order using combinatorial techniques [27].

A key feature of tensor models is that the set of observables and interactions is quite larger than in matrix models [28], and grows with d. In $U(N)$-invariant matrix models, observables are products of traces $\operatorname{tr} M^n$, for M Hermitian. In $U(N)^2$-invariant models, they are products of traces $\operatorname{tr}(MM^\dagger)^n$, for M a complex matrix. In both cases, there is a single invariant at fixed degree in M (in addition to products of invariants of smaller degree). More generally, there is a set of generators of the ring of $U(N)^d$-invariant polynomials, called the set of bubbles. They are characterized by a d-tuple of permutations, up to a left and a right action on the tuple. There is a graphical representation as d-regular bipartite graphs with edges labeled by a color from $[\![1, d]\!]$ such that all colors are incident on every vertex. They have been studied in [29, 30, 31], where a relation to Kronecker coefficients was found. Enforcing other sets of symmetries leads to other sets of observables, like using $O(N)^d$ instead of $U(N)^d$ relaxes the bipartiteness of the bubbles [33, 32] as will be seen later on.
This enlarged set of observables in tensor compared to matrix models is the source of various universality classes found in the large N and continuum limit. Indeed, it is well-known in 2D that models built with interaction $\operatorname{tr} M^k$, generating k-angulations, have all the same universality class (that of pure 2D quantum gravity). However, for $d > 2$, there are more possible bubbles, *i.e.* more interactions at fixed order k in $T, \overline{T}$, which correspond to different d-dimensional building blocks. Choosing different bubbles as interactions can then lead to different universality classes [9]. This however does not seem to be the case in 3D, where all planar bubbles (dual to building blocks homeomorphic to the ball) used as interactions always lead to the universality class of random trees (*i.e.* branched polymers) [10].

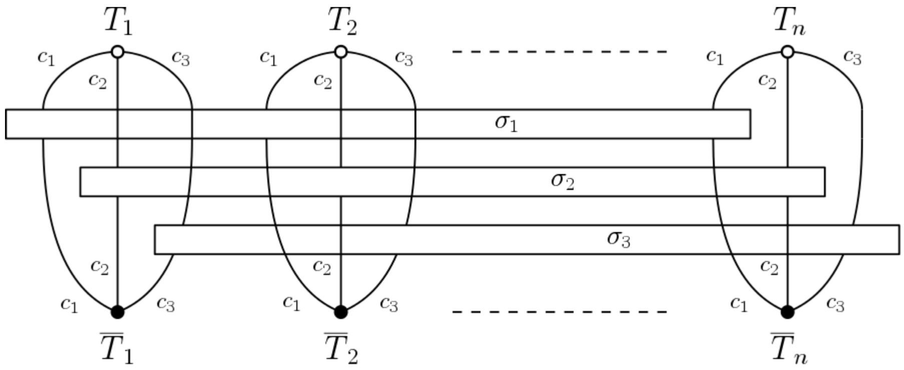


FIGURE 1 – Symmetric group view of unitary invariants.

Orthogonal invariants – As already mentioned, several, if not all, of the previous studies heavily rely on the understanding of the combinatorics of the Feynman graphs and observables of the considered tensor model. Let us now introduce two particular contributions on tensor model graphs that the present work extends.

In [29], the authors worked out the enumeration of the unitary invariants, as observables in complex tensor models. One way of comprehending the theory space of rank-d complex tensor models is to specify its set of observables. The latter are merely $U(N)^d$-invariants (at time, we simply call them $U(N)$, complex or unitary tensor invariants). We know that a convenient manner to represent $U(N)$ invariants defines as a canonical mapping to d-regular bipartite colored graphs [34]. Stated in this way, the inventory of tensor invariants formulates by uniquely using permutation groups (see Figure 1). One should record that these symmetry group techniques and its representation theory have been developed during the last years [35]–[46]. They turned out to be powerful, flexible and versatile enough to address diverse enumeration problems from scalar field theory and matrix models, to gauge (QED, 2D and 4D Yang-Mills) and string theories. In physics, for instance, they brighten the half-BPS sector of $\mathcal{N} = 4$ SYM [35]–[40]. Moreover, unforeseen correspondances arise from these studies, for instance, counting Feynman graphs in ϕ^4 scalar field theory relates to string theory on a cylinder or listing Feynman graphs of QED relates to the counting of ribbon graphs [39]. These correspondances emerge from another interface playing a hinge role between enumeration problems: *via* the Burnside lemma, with each enumeration problem using the symmetric group (and its subgroups), we can associate a Topological Field Theory on a 2-complex (named TFT_2) with gauge group given by the symmetric group (and its subgroups). Such a formulation also unfolds multiple interpretations of the counting formulae with links with the theory of covering spaces in algebraic and complex geometry (see references in [39]). The reference [29] establishes several enumeration formulae pertaining to observables of complex tensor models. Using the Burnside lemma, one recasts the enumeration of $U(N)$ invariants into a partition function of a permutation lattice gauge field theory, a TFT_2. It is *via* this mapping that one elucidates that counting unitary invariants corresponds to counting branched covers of the 2-sphere. Branched covers are well known objects in algebraic and complex geometry [47], in topological string theory, and in dimension 2, they correspond to complex maps [38]. Thus, there is an underlying geometry inherited by tensor models from the TFT_2 formulation that still needs to be understood. There is however a proviso: the counting formulae are only valid when the size N of the tensor is larger than the number of tensors convoluted. More generally, one should resort to a more careful study [45, 46]. The study of tensor invariants has a follow-up in [30]. Their equivalent classes are viewed as the basis elements of a vector space $\mathcal{K}_d(n)$, a subspace of $\mathbb{C}[S_n]^{\otimes d}$, the rank-$d$ group algebra of the symmetric group S_n. $\mathcal{K}_d(n)$ shows stability under an associative product, and it is endowed with a non-degenerate pairing. Therefore, at a fixed rank d and fixed number of vertices n, tensor invariants span a semi-simple algebra. (Note that, importantly, other algebraic structures could be set up on tensor invariants [48, 49, 50]. The above structure is however unique, up to isomorphism.) As a consequence of the Wedderburn-Artin theorem, any semi-simple algebra decomposes as a sum of irreducible matrix subalgebras. The representation theory of the symmetric group sheds more light on the remaining analysis as it enables to reach the Wedderburn-Artin matrix decomposition of the algebra of tensor observables: the dimension of the algebra is a sum of squares of the Kronecker coefficients (these are multiplicity dimensions in the decomposition of a tensor product of representations in irreps; Kronecker coefficients are still under active investigation in Combinatorics and Computational Complexity Theory, see, for instance, [51, 52] and more references therein), each square matching exactly the dimension of a matrix subalgebra. The orthogonal bases of the algebra and its matrix subalgebras have been worked out, meanwhile the Gaussian 2-point correlators also provide new representation theoretic orthogonal bases.

In this work, we consider $O(N)$ tensor models and their observables and investigate if they support the same previous enumeration and algebraic analysis. Fleshed out the first time in [33], such models extended the large N expansion to real tensors. The graphs that determine the $O(N)$ invariants keep the edge coloring but are not bipartite. This naturally leads to a class of observables wider than that of the $U(N)$ tensor models. To enumerate $O(N)$ invariants, we use a standard counting recipe: we use tuples

of permutations on which act permutation (sub)groups that define equivalence classes. We then count the points in the resulting double coset space. The equivalence relation in the present setting is radically different from the $U(N)$ situation and requires more work to obtain a valuable counting formula. With their generating functions in hand, we provide software (Mathematica, Sage) codes to achieve the counting of $O(N)$ observables for any tensor rank. We emphasize that our results match the seminal work of Read in [55] that dealt with the enumeration of k-regular graphs with $2n$ vertices with k-edge-coloring. However, Read's formula was only evaluated for the $k = 3$ regular graphs with $2n = 2, 4, 6$ vertices with edges of three different colors. Our code extends this counting for any k and any n. We produce integer sequences that are new (un-reported as of yet) to the On-Line Encyclopedia of Integer Sequences [56].
Moreover, seeking other correspondances, we address the TFT formulation of our counting and show that to count $O(N)$ observables amounts to count covers of glued cylinders with defects (the rank of the tensors relates to the number of cylinders and defects). After introducing the algebra of $O(N)$ invariants, we show that it is semi-simple, and as such, admits a Wedderburn-Artin decomposition. An invariant orthogonal basis of the algebra transpires in our analysis but it does not yield the decomposition of the algebra in matrix subalgebras. We proceed to the representation theoretic formulation of the counting and its consequences. As to be distinguished from the $U(N)$ case, the dimension of the algebra is a sum of constrained Kronecker coefficients restricted to partitions with all even length rows. The representation theoretic tools exhibit a basis of the algebra, the dimension of which directly reflects the sum of constrained Kroneckers. The Gaussian 2- and 1-point correlators also compute in terms of permutation group formulae. A corollary of that analysis is that 2-point functions, in the normal order, select a representation theoretic orthogonal basis of the algebra. In that sense, the Gaussian integration in the representation Fourier space performs as a pairing of observables.

Blobbed topological recursion – A natural question for tensor models is to go beyond the large N limit. In particular, it is natural to ask whether the methods used for this purpose for matrix models still work for tensor models and whether it depends on the set of chosen interactions. In this work we will focus on the topological recursion of Eynard-Orantin [57, 58, 59, 60]. Let us nevertheless mention previous works on tensor models beyond the large N limit. A standard combinatorial analysis of maps was applied to the Feynman graphs of the so-called colored tensor models by Gurau and Schaeffer [61], and extended to the set of Feynman graphs of the multiorientable model (which has

$U(N)^2 \times O(N)$ symmetry, at $d = 3$) by Fusy and Tanasa [62]. They classify the Feynman graphs appearing at a given order of the $1/N$ expansion of their respective models. They also identify those which are the most singular in the continuum limit at each order of the $1/N$ expansion, thereby allowing for a double-scaling limit, whose 2-point function was calculated.
In parallel, interest grew around the so-called quartic melonic model. It is a tensor model with up to d quartic interactions having a special structure called melonic. This interest in the quartic model comes from the existence of the Hubbard-Stratonovich technique which transforms this tensor model into a multi-matrix model. It opened up a new way of analyzing tensor models through matrix models. The double-scaling limit for this model was done in [63] (for a result similar to [61]). It was also realized in [64] that in the large N limit, the eigenvalues do not spread because the Coulomb repulsion is subdominant. Instead, they all fall in the potential well, as anticipated in [65]. One can then study the fluctuations around the saddle point, an analysis started in [64] where the leading order fluctuations were shown to obey Wigner's semi-circle law.

In [66], the first instance of topological recursion in the context of tensor models was established. Recall that in the ordinary Hermitian 1-matrix model, the topological recursion applies to the calculation of the n-point, genus g, correlation functions $W_{n,g}(x_1, \dots, x_n)$ which appear in the expansion of connected n-point functions

$$\langle \mathrm{tr} \frac{1}{x_1 - M} \cdots \mathrm{tr} \frac{1}{x_n - M} \rangle_c = \sum_{g \geq 0} N^{2-n-2g} W_{n,g}(x_1, \dots, x_n). \tag{2}$$

In terms of maps, it is a recursion on the generating functions of maps of genus g, with n marked faces whose perimeters are tracked by the variables $x_1, \dots, x_n$. The topological recursion takes a universal form, and uses a spectral curve as initial data. The spectral curve is determined by the disc and cylinder functions $W_{1,0}(x)$ and $W_{2,0}(x_1, x_2)$.
In [66], the matrix model is the one obtained in [64] for the fluctuations of the eigenvalues around the saddle point. It has d Hermitian matrices $M_1, \dots, M_d$ where M_c is said to be of color c and the correlations now need to have the colors of their variables specified,

$$W_n(x_1, c_1; \dots, x_n, c_n) = \langle \mathrm{tr} \frac{1}{x_1 - M_{c_1}} \cdots \mathrm{tr} \frac{1}{x_n - M_{c_n}} \rangle_c. \tag{3}$$

As it turns out, the coupling between the colors is not too strong and a topological recursion can be derived where the spectral curve is a disjoint union of d spectral curves for

Gaussian matrix models, with an additional holomorphic term for the cylinder function. This is due to

condition 1 the $U(N)^d$ symmetry. It implies that the matrices of different colors can only interact through products of traces of different colors. The action is of the form

$$S_N(M_1, \ldots, M_d) = \sum_{p_1,\ldots,p_d \geq 0} t_N(p_1, \ldots, p_d)\ \mathrm{tr}M_1^{p_1} \ldots \mathrm{tr}M_d^{p_d}, \tag{4}$$

condition 2 the $1/N$ expansion. It is such that only the quadratic terms of the action survive the large N limit,

$$S_N(M_1, \ldots, M_d) \underset{N\to\infty}{\sim} N \sum_{c=1}^{d} a_c \mathrm{tr}M_c^2 + \sum_{c,c'=1}^{d} b_{cc'} \mathrm{tr}M_c \mathrm{tr}M_{c'}, \tag{5}$$

(in which sense is explained in the text).

Those two conditions guarantee that an extension of the topological recursion, called the blobbed topological recursion, or rather a multi-colored extension of the latter, holds with the spectral curve being a disjoint union of Gaussian spectral curves, except for $W_{2,0}(x_1, c_1; x_2, c_2)$ which has an additional holomorphic part compared to its usual form. The blobbed topological recursion was introduced by Borot [67] and further formalized by Borot and Shadrin [68]. In our context, it applies to matrix models with multi-trace interactions having a topological expansion, *i.e.* of the form

$$S_N(M) = \sum_{n,h\geq 0} \sum_{p_1,\ldots,p_n \geq 0} N^{2-n-2h}\ t^{(h)}(p_1, \ldots, p_n)\ \mathrm{tr}M^{p_1} \ldots \mathrm{tr}M^{p_n}. \tag{6}$$

Combinatorially, those types of models generate stuffed maps, defined in [67]. They are maps which are not built by the gluings of disks but as gluings of surfaces of genus h with n boundary components of perimeters $p_1, \ldots, p_n$. In [66] this interpretation survives with an additional coloring of the boundary components.

In the blobbed topological recursion, the recursion for correlation functions still has the same universal term as the ordinary topological recursion, which calculates the singular parts of the correlation functions. In addition, there are now holomorphic contributions [67, 68]. It is also important to keep in mind that the action (4) is in fact topological only for $d = 4d' + 2$, for $d' \in \mathbb{N}$ [66], meaning that the couplings take the form $t_N(p_1, \ldots, p_d) = \sum_{h\geq 0} N^{2-d-2h} t^{(h)}(p_1, \ldots, p_d)$. For other values of d, one can do

as if the action were topological by absorbing some N-dependence into the couplings $t^{(h)}(p_1, \ldots, p_d)$, and then apply the blobbed topological recursion.
Here we show how to apply this approach to arbitrary $U(N)^d$-invariant models, provided that there are quartic melonic interactions (among others) and some invertibility condition of a quadratic form at large N. This revolves around the fact that after some intermediate field techniques and formal integration, such tensor models can always be rewritten as matrix models with a set of d Hermitian matrices satisfying the conditions 1 and 2. Therefore, the correlation functions of these matrices satisfy the blobbed topological recursion, with the same spectral curve as in the quartic melonic model of [66]. Remarkably (and evidently from [67]), the specifics of the model, *i.e.* the choice of interactions, only contribute to some effective action and not to the form of the blobbed topological recursion. In combinatorial terms, the specifics only contribute to the generating functions of the stuffings of the maps. Proving the blobbed topological recursion does not require knowing the explicit effective action, but only that the conditions 1 and 2 are satisfied. In this sense, the blobbed topological recursion is universal in our framework. The only difference between our formulas and those of [66] is that the generating functions of the stuffing were explicitly known in [66] while their explicit dependence on the coupling constants will be left unknown here (their N-dependence is however important).

Method – There are however some technical obstacles to overcome. Arbitrary tensor models cannot be directly transformed into matrix models using the Hubbard-Stratonovich transformation as the latter only works for quartic interactions. This first obstacle was overcome in [70] where it was shown that there are still matrix models rewritings. This was proven using a bijection between the Feynman graphs of the tensor model and those of the corresponding matrix model. A second proof was also provided by manipulations of formal integrals (integrals which are only defined as their Feynman series). Here we will repeat this proof, adapting it to go through the second obstacle, which we explain now.
The method of [70] turns a tensor model into a matrix model with complex matrices M_C labeled by subsets of $[\![1, d]\!]$, *i.e.* $C \subset [\![1, d]\!]$. However, applying the same recipe as in [66] requires to have d Hermitian matrices $M_1, \ldots, M_d$ instead. This is remedied in two steps. We first show that it is possible to replace the complex matrices M_C with pairs of Hermitian matrices $(Y_C, \Phi_C)_{C\subset[\![1,d]\!]}$. Then, provided that the quartic melonic interactions are turned on, it is possible to integrate formally over all matrices except $Y_1, \ldots, Y_d$. In terms of Feynman graphs, this means that one has combinatorial maps with colored edges

corresponding to the matrices $Y_1, \dots, Y_d$, and everything else is packed in some stuffing of the maps. Keeping in mind the goal of the topological recursion, it is necessary to control the N-dependence of the stuffings in terms of their boundary components.
The next step is to observe that all the eigenvalues of $Y_1, \dots, Y_d$ fall into some potential well at large N, and move on to the study of their fluctuations. This is where one observes that the conditions 1 and 2 are still satisfied and lead to the blobbed topological recursion. It would be interesting to know whether condition 2 could be removed in general. It is known to be possible in the case of a single matrix model as originally done in [67]. However, in the multi-colored case, it would require a 1-cut (Brown's) lemma for a system of coupled equations with catalytic variables, thus extending the framework of [71], which is outside the reach of this manuscript.

Expectations – When discussing the topological recursion in the context of tensor models, there is another natural question to address, which is how to relate the expectations of generic observables on the tensor side to the quantities evaluated *via* the topological recursion on the matrix side. In [64] for the quartic melonic model, it was shown that the expectations of $\mathrm{tr}\, M_c^n$ are expectations of Hermite polynomials of some melonic cyclic bubbles on the tensor side. This relation can also be inverted *via* Hermite polynomials. In [66], the expectations of arbitrary tensor observables (bubbles) were expressed in terms of quantities evaluated by the matrix models, but it involved summing over Wick contractions.
In the present work, we generalize the Hermite polynomial relationship of [64] to arbitrary observables on both the tensor and matrix sides. To express the expectation of a matrix observable in terms of tensorial observables, one has to take derivatives of the potential (which in the case of the quartic melonic model is quadratic, therefore leading to Hermite polynomials). The other way around, *i.e.* to express the expectation of a tensorial observable in terms of matrix expectations, one has to take derivatives of some effective potential for the matrices Y_Cs (in the quartic melonic model, it reduces again to a quadratic potential, hence Hermite polynomials), which comes from integrating all the matrices Φ_Cs.

Plan – The manuscript is structured as follows. Chapter 1 starts by briefly introducing the problem of quantum gravity in any dimension, and then gives the recipe for working it out in $d = 2$ with the help of random matrices. It ends by an elementary example of the topological recursion formalism applied to Hermitian matrices. In Chapter 2 random tensor models are defined and some of their properties and features are given. The symmetric group techniques for counting tensor invariants are also discussed. In Chapter 3 things get more involved. A particularly important type of tensor model is introduced, *viz.* the quartic melonic model, and the heavy machinery of the topological recursion formalism applied to it, it serves as a reference point for our study in Part III. Chapter 4 sets up our notations for real tensor models and their $O(N)$ invariants. We then develop the double coset counting using permutation group formalism. We also discuss therein the TFT formulation of the counting and its consequences, introduce the basics of the representation theory of the symmetric group, and re-interpret the counting in that language. Chapter 5 discusses the double coset algebra built out of the $O(N)$ invariants and lists its properties. Next, Chapter 6 details the 1- and 2-point correlators of the Gaussian tensor models and their representation theoretic consequences. Chapter 7 briefly lists a few remarks on the counting of invariants of the real symplectic group $Sp(2N)$. The counting principle here is similar to that of the $O(N)$ model, but with subtleties that one should pay heed to. In Chapter 8 we define the tensor models of interest and their multi-matrix equivalent. Theorems 8.3.2 and 8.3.3 give some of the relationships between the expectations of observables on the tensor and matrix sides. Chapter 9 explains how to formally integrate all matrices except $Y_1, \ldots, Y_d$, leading to an effective matrix model in Theorem 9.1.1. We use the same technique of formal integration to express the expectations of tensorial observables in terms of matrix expectations in Theorem 9.2.1. The large N limit of the effective model is discussed and leads to a matrix model for the fluctuations. We study the latter in Chapter 10, by describing the Schwinger-Dyson equations, which can be analyzed along the lines of [66, 67]. We only present some key aspects which are needed to state Theorem 10.4.1, about the blobbed topological recursion, since everything works as in [66]. We then summarize our work and draw some of its perspectives. Finally, the manuscript closes with an appendix that divides into two main parts: the first collects identities of the representation theory of the symmetric group that are useful in the text, while the other details the software codes that generate the sequences of numbers of invariants at sundry tensor ranks $d = 3, 4, \ldots$.

Introduction en Français

Matrices et tenseurs aléatoires – Les modèles de tenseurs aléatoires sont une généralisation des modèles de matrices. Depuis leur création [1, 2, 3], ils offrent un cadre dans lequel étudier la géométrie aléatoire et visent à étendre le succès des modèles de matrices [4] quant à la gravité quantique $2D$ aux dimensions supérieures. Cette approche a pour but d'opérer une transition entre géométries discrètes et continues en dimension arbitraire. Cependant, les premiers modèles de tenseurs proposés souffraient de multiples difficultés et aucun développement significatif ne vit le jour pendant une vingtaine d'années. Ce n'est que récemment que d'importants progrès [5] ont été réalisés, comme par exemple la découverte d'un nouveau développement en $1/N$ généralisant le développement en genre de 't Hooft [6] aux (pseudo-)variétés de dimension supérieure. Il convient également de rappeler que les modèles de matrices sont intimement liés aux cartes combinatoires, ces dernières étant générées par le développement en graphes de Feynman de ceux-ci [4, 7]. Par exemple, dans le cas du modèle à une matrice hermitienne, on a :

$$\int dM \; e^{-\frac{N}{2t}\mathrm{tr}M^2+N\sum_{k\geq 1}\frac{t_k}{k}\mathrm{tr}M^k} = \sum_{\text{cartes}} \frac{t^n}{n!} N^{2-2h} \prod_{k\geq 1} t_k^{n_k}, \tag{7}$$

où la somme porte sur les cartes de genre h avec n arêtes étiquetées et n_k faces de degré $k \geq 1$. La grandeur à l'intérieur de l'exponentielle dans le terme de gauche est appelée action, ou potentiel, et les t_k sont les constantes de couplage. Les modèles de tenseurs généralisent également cette relation. En effet, le développement en graphes de Feynman des modèles de tenseurs génère des (pseudo-)variétés de dimension d, linéaires par morceaux [5, 8, 9, 10]. C'est pourquoi les modèles de tenseurs avaient déjà été proposés comme solution au problème de la gravité quantique au début des années 1990, bien avant qu'une limite large N ne soit découverte [11]. L'existence d'une telle limite pour les tenseurs a naturellement contribué à dévoiler plusieurs résultats analytiques, parmi lesquels on peut citer la découverte de leur comportement critique (polymères ramifiés [12, 13]), la propriété d'universalité des tenseurs aléatoires [14] et la découverte de nouvelles familles de théories quantiques des champs renormalisables et non-locales, avec des comportements non-perturbatifs et UV intéressants [15, 16, 17], appuyant ainsi la découverte d'une nouvelle classe d'universalité pour la gravité [18, 19, 20].

Plus récemment, et de manière plutôt étonnante, il a été montré que les modèles de tenseurs admettaient la même limite large N que le modèle Sachdev-Ye-Kitaev (SYK) [21, 22] de la matière condensée. Ce dernier est exactement résoluble à large N dans l'IR, il manifeste la propriété de chaos maximal et est dual à la gravité $2D$ de Jackiw-Teitelboim [23, 24]. En partie pour ses liens avec la physique des trous noirs et la correspondance AdS/CFT, le modèle SYK est un sujet de recherche brûlant qui a porté le développement des modèles de tenseurs ces dernières années. En effet, de nouveaux modèles ont été introduits et leur limite large N étudiée [25, 26]. Certains d'entre eux ont même pu être explorés au-delà de l'ordre dominant en usant de techniques combinatoires [27].

Un trait caractéristique des modèles de tenseurs est le fait que leurs ensembles d'observables et d'interactions sont bien plus larges que dans les modèles de matrices [28], et croissent avec d. Dans les modèles de matrices dits $U(N)$-invariants, les observables sont des produits de traces $\operatorname{tr} M^n$, où M est hermitienne. Dans les modèles $U(N)^2$-invariants, ce sont des produits de traces du type $\operatorname{tr}(MM^\dagger)^n$, où M est une matrice complexe. Dans les deux cas, il n'y a qu'un seul invariant à degré en M fixé (en plus de produits d'invariants de degrés inférieurs). Plus généralement, il existe un ensemble de générateurs de l'anneau des polynômes $U(N)^d$-invariants appelés bulles. Celles-ci se caractérisent par un d-uplet de permutations, aux actions à gauche et à droite près sur ce dernier. L'on peut les représenter graphiquement comme des graphes bipartis d-réguliers dont les arêtes portent une couleur dans $[\![1, d]\!]$ de telle sorte que toutes les couleurs soient incidentes à chaque sommet. Ces graphes ont été étudiés dans [29, 30, 31], où une relation avec les coefficients de Kronecker a été mise à jour. Enfin, imposer d'autres symétries amène d'autres ensembles d'observables, comme le fait d'utiliser $O(N)^d$ à la place de $U(N)^d$ qui assouplit la condition de bipartisme des bulles [33, 32] comme nous allons le voir.
Cet ensemble élargi d'observables tensorielles est source de nombreuses classes d'universalité qui apparaissent dans les limites large N et continue. En effet, il est bien connu que les modèles $2D$, construits avec une interaction de type $\operatorname{tr} M^k$ et générant les k-angulations, ont tous la même classe d'universalité, à savoir celle de la gravité quantique $2D$ dans le vide. Néanmoins, pour $d > 2$, il existe potentiellement plus de bulles, donc plus d'interactions à ordre k fixé en T, $\overline{T}$, qui correspondent à différents composants élémentaires de dimension d. Un choix différent de bulles comme interaction peut alors conduire à

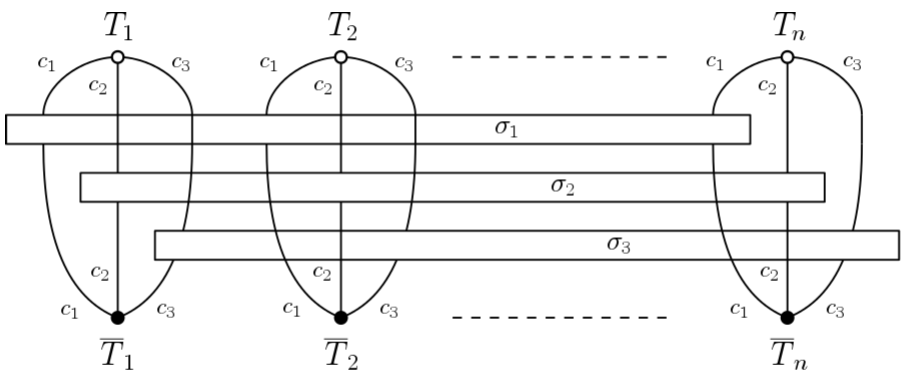


FIGURE 2 – Invariants unitaires représentés au travers du groupe symétrique.

différentes classes d'universalité [9]. Cela ne semble cependant pas être le cas en $3D$, où toutes les bulles planaires (duales à des composants élémentaires homéomorphes à la sphère) utilisées comme interaction mènent toujours à la classe d'universalité des arbres aléatoires (polymères ramifiés) [10].

Invariants orthogonaux – Comme mentionné précédemment, de nombreuses, si ce n'est toutes les études sus-citées reposent sur la compréhension de la combinatoire des graphes de Feynman et observables du modèle de tenseurs considéré. Ce travail s'appuie sur et étend deux principaux résultats sur les graphes des modèles de tenseurs.

Dans [29], les auteurs ont énuméré les invariants unitaires comme des observables dans des modèles de tenseurs complexes. En effet, une façon de décrire l'espace des tenseurs complexes de rang d est d'en donner les observables. Ces dernières ne sont autres que les invariants $U(N)^d$ (nous les appellerons indifféremment invariants $U(N)$, complexes ou unitaires). Nous savons que ces invariants sont idéalement représentés par des graphes colorés bipartis d-réguliers [34]. Leur inventaire est alors uniquement déterminé par des groupes de permutations (voir Figure 2). Il est important de noter que ces techniques de groupes symétriques ainsi que leurs théories des représentations ont été développées ces dernières années [35]–[46]. Elles se sont révélées puissantes, flexibles et suffisamment versatiles pour s'attaquer à divers problèmes d'énumération tant en théorie des champs scalaires et modèles de matrices qu'en théories de jauge (QED, Yang-Mills en $2D$ et $4D$) et théories des cordes. En physique, par exemple, elles jettent un éclairage nouveau sur le secteur demi-BPS de la théorie de Yang-Mills Supersymétrique $\mathcal{N} = 4$ [35]–[40]. De plus, de nouvelles correspondances inattendues se dégagent de ces études, comme par exemple le lien entre graphes de Feynman de la théorie ϕ^4 et théorie des cordes sur un cylindre, ou encore entre graphes de Feynman de l'électrodynamique quantique et le comptage de graphes à rubans [39]. Ces dernières émergent d'une autre interface occupant un rôle charnière entre problèmes d'énumération. En effet, *via* le lemme de Burnside, à tout problème d'énumération impliquant le groupe symétrique, on peut associer une théorie des champs topologique sur un 2-complexe (appelé TFT_2) dont le groupe de jauge est donné par le groupe symétrique. Une telle formulation amène à de nouvelles interprétations des formules de comptage avec notamment des liens avec la théorie des revêtements en géométrie algébrique et complexe (voir les références dans [39]). La référence [29] établit de nombreuses formules se rapportant à l'énumération des observables des modèles de tenseurs complexes. À l'aide du lemme de Burnside, l'énumération des invariants unitaires est refondue en une fonction de partition d'une théorie de jauge sur réseau, une TFT_2. Ainsi, compter les invariants unitaires revient à compter les revêtements ramifiés de la 2-sphère. Les revêtements ramifiés sont des objets bien connus en géométrie algébrique et complexe [47], en théorie des cordes topologique et correspondent, en dimension 2, à des applications complexes [38]. Il existe donc une géométrie sous-jacente pour les tenseurs, héritée de la formulation en termes de TFT_2 et qui reste à être élucidée. Il est cependant important de souligner que les formules de comptage ne sont valides que lorsque la taille N des tenseurs est supérieure à leur nombre. Plus généralement, une étude plus approfondie est nécessaire [45, 46].

L'étude des invariants tensoriels se poursuit dans [30]. Leurs classes d'équivalences sont vues comme les éléments d'une base d'un espace vectoriel $\mathcal{K}_d(n)$, un sous-espace de $\mathbb{C}[S_n]^{\otimes d}$, l'algèbre de groupe de rang d du groupe symétrique S_n. L'espace $\mathcal{K}_d(n)$ est stable par produit associatif et est muni d'un produit scalaire. C'est pourquoi, à rang d et nombre de sommets n fixés, les invariants tensoriels engendrent une algèbre semi-simple. (Il est néanmoins important de noter que d'autres structures algébriques peuvent être construites à partir de ces invariants [48, 49, 50]. La précédente est en revanche unique, à isomorphismes près.) Par le théorème de Wedderburn-Artin, toute algèbre semi-simple se décompose en somme de sous-algèbres de matrices irréductibles. La théorie des représentations du groupe symétrique permet ensuite d'expliciter la décomposition en matrices de Wedderburn-Artin de l'algèbre : la dimension de l'algèbre n'est autre qu'une somme de carrés de coefficients de Kronecker (ces derniers peuvent être vus comme les multiplicités dans la décomposition d'un produit tensoriel de représentations en représentations irréductibles ; ils sont toujours activement étudiés en combinatoire et théorie de la complexité computationnelle, voir par exemple [51, 52] et les références qui y sont contenues), chaque carré correspondant exactement à une des sous-algèbres de matrices. Les bases orthogonales de l'algèbre et de ses sous-algèbres ont été explicitées, d'autres encore proviennent des corrélateurs gaussiens à 2 points.

Dans ces travaux, nous nous intéressons aux modèles de tenseurs $O(N)$ (appelés aussi orthogonaux, ou réels), ainsi qu'à leurs observables et cherchons à déterminer s'ils suivent le même schéma d'énumération que précédemment ou ont le même comportement algébrique. Etudiés avec précision pour la première fois dans [33], ces modèles étendent le développement large N aux tenseurs réels. Les graphes correspondant aux invariants orthogonaux conservent la coloration de leurs arêtes mais perdent leur caractère biparti. Cela amène tout naturellement une classe d'observables plus large que dans le cas unitaire. Pour énumérer les invariants orthogonaux, nous utilisons des n-uplets de permutations sur lesquels agissent des (sous-)groupes de permutations, définissant ainsi des classes d'équivalences. Nous comptons ensuite les points dans la classe double ainsi formée. La relation d'équivalence est ici radicalement différente du cas unitaire et nécessite plus de labeur pour obtenir une formule de comptage satisfaisante. Equipés de leurs fonctions génératrices, nous donnons quelques codes informatiques (Mathematica, Sage) pour compter les observables orthogonales à n'importe quel rang. Nous soulignons que nos résultats sont en accord avec les travaux fondateurs de Read [55] qui traitent de l'énumération des graphes à $2n$ sommets k-réguliers à arêtes k-colorées. Cependant, Read n'appliqua sa formule qu'aux cas $k = 3$ pour $2n = 2, 4, 6$, alors que notre code étend ce comptage à tous k et n. Nous produisons des suites nouvelles (non encore rapportées) qui n'apparaissent pas dans l'OEIS [56].
De plus, la formulation en termes de TFT de notre comptage montre que compter les invariants orthogonaux revient à compter des revêtements de cylindres accolés présentant des lacunes (le rang des tenseurs est en lien direct avec le nombre de ces cylindres et lacunes). Après avoir introduit l'algèbre des invariants orthogonaux, nous montrons qu'elle est semi-simple et en tant que telle admet une décomposition de Wedderburn-Artin. Notre analyse révèle l'existence d'une base orthogonale invariante de l'algèbre, mais qui ne fournit pas la décomposition de l'algèbre en sous-algèbres matricielles. Par opposition au cas unitaire, la dimension de l'algèbre est une somme de coefficients de Kronecker contraints et restreints aux partitions paires. Les outils de la théorie des représentations fournissent une base à l'algèbre, dont la dimension dépend directement de la somme des Kronecker contraints. Les corrélateurs gaussiens à 1 et 2 points s'expriment également dans le langage du groupe symétrique. Comme corollaire à cette analyse, les fonctions

à 2 points normalement ordonnées sélectionnent une base orthogonale de représentation de l'algèbre. En ce sens, l'intégration se traduit par un couplage des observables dans l'espace de Fourier.

Récurrence topologique *à blobs* – Une question qui se pose pour les modèles de tenseurs est la possibilité d'aller au-delà de la limite large N. Tout particulièrement, il est naturel de se demander si les méthodes utilisées à cette fin dans les modèles de matrices sont encore applicables aux tenseurs et si elles dépendent des interactions choisies. Dans ces travaux nous nous focaliserons sur la récurrence topologique d'Eynard-Orantin [57, 58, 59, 60]. Citons néanmoins quelques travaux précédents menés sur les modèles de tenseurs au-delà de la limite large N. Une analyse combinatoire standard a été appliquée aux graphes des modèles dits colorés par Gurau et Schaeffer [61], et étendue à l'ensemble des graphes de Feynman du modèle multi-orientable (qui a pour symétrie $U(N)^2 \times O(N)$ en dimension $d = 3$) par Fusy et Tanasa [62]. Les graphes de Feynman apparaissant à un ordre donné en $1/N$ ont été classés dans chaque modèle. Les auteurs ont également identifié les graphes les plus singuliers dans la limite continue à chaque ordre en $1/N$, prouvant ainsi l'existence d'une double limite d'échelle dont la fonction à 2 points a été calculée.
En parallèle, un modèle particulier, dit quartique melonique, commença à attirer l'attention. C'est un modèle possédant jusqu'à d interactions quartiques d'une structure particulière appelée melonique. L'intérêt porté au modèle quartique melonique provient essentiellement de l'existence de la technique d'Hubbard-Stratonovich qui transforme ce modèle de tenseurs en un modèle multi-matrices, ce qui ouvrit ainsi la voie à une nouvelle méthode d'analyse des modèles de tenseurs, à savoir au travers des modèles de matrices. La double limite d'échelle de ce modèle fut explicitée dans [63] (un résultat similaire à [61]). Dans [64], les auteurs ont également réalisé que dans la limite large N, les valeurs propres ne s'étalent pas car la répulsion coulombienne est sous-dominante. En revanche, elles tombent toutes au fond du puits de potentiel, comme anticipé dans [65]. L'on peut alors étudier leurs fluctuations autour du point col, une analyse débutée dans [64], où il a été montré que l'ordre dominant des fluctuations obéit à la loi du demi-cercle de Wigner.

Dans [66], une première instance de la récurrence topologique dans le contexte des modèles de tenseurs a été établie. On rappelle que dans le cas du modèle à une matrice hermitienne, la récurrence topologique s'applique au calcul des fonctions de corrélation $W_{n,g}(x_1, \ldots, x_n)$ à n points, de genre g qui apparaissent dans le développement des fonctions à n points connexes

$$\langle \mathrm{tr}\frac{1}{x_1 - M} \cdots \mathrm{tr}\frac{1}{x_n - M}\rangle_c = \sum_{g\geq 0} N^{2-n-2g} W_{n,g}(x_1,\dots,x_n). \tag{8}$$

Dans le langage des cartes, c'est une récurrence sur les fonctions génératrices des cartes de genre g avec n faces marquées dont les périmètres sont reliés aux variables $x_1,\dots,x_n$. La récurrence topologique revêt une forme universelle et repose sur la donnée initiale d'une courbe spectrale. La courbe spectrale est quant à elle déterminée par les fonctions du disque et du cylindre $W_{1,0}(x)$ et $W_{2,0}(x_1,x_2)$.
Dans [66], le modèle de matrices obtenu est celui trouvé en [64] pour les fluctuations des valeurs propres autour du point col. Il se compose de d matrices hermitiennes $M_1,\dots,M_d$ où M_c est dite de couleur c et où les fonctions de corrélation doivent refléter la couleur des variables,

$$W_n(x_1,c_1;\dots,x_n,c_n) = \langle \mathrm{tr}\frac{1}{x_1 - M_{c_1}} \cdots \mathrm{tr}\frac{1}{x_n - M_{c_n}}\rangle_c. \tag{9}$$

Il se trouve que le couplage entre les couleurs est relativement faible et l'on peut écrire une récurrence topologique où la courbe spectrale est une union disjointe de d courbes spectrales de modèles de matrices gaussiens, à ceci près que la fonction du cylindre acquiert un terme holomorphe. Ceci est dû aux conditions suivantes :

condition 1 la symétrie $U(N)^d$. Celle-ci implique que les matrices de différentes couleurs ne peuvent interagir qu'au travers de produits de traces de différentes couleurs. L'action prend la forme

$$S_N(M_1,\dots,M_d) = \sum_{p_1,\dots,p_d\geq 0} t_N(p_1,\dots,p_d)\ \mathrm{tr}M_1^{p_1}\dots\mathrm{tr}M_d^{p_d}, \tag{10}$$

condition 2 le développement en $1/N$. Il est tel que seuls les termes quadratiques de l'action survivent dans la limite large N,

$$S_N(M_1,\dots,M_d) \underset{N\to\infty}{\sim} N\sum_{c=1}^{d} a_c \mathrm{tr}M_c^2 + \sum_{c,c'=1}^{d} b_{cc'}\mathrm{tr}M_c\mathrm{tr}M_{c'}. \tag{11}$$

Ces deux conditions garantissent qu'une extension de la récurrence topologique, dite *à blobs* (ou multi-colorée), existe et dont la courbe spectrale est une union disjointe de courbes spectrales gaussiennes, excepté pour $W_{2,0}(x_1,c_1;x_2,c_2)$ qui se dote d'une partie holomorphe supplémentaire. La récurrence topologique *à blobs* a été introduite par Borot

[67] et plus tard formalisée par Borot et Shadrin [68]. Dans notre contexte, elle s'applique aux modèles de matrices avec des interactions multi-traces admettant un développement topologique, à savoir de la forme

$$S_N(M) = \sum_{n,h\geq 0} \sum_{p_1,\ldots,p_n\geq 0} N^{2-n-2h}\, t^{(h)}(p_1,\ldots,p_n)\, \mathrm{tr}M^{p_1}\ldots\mathrm{tr}M^{p_n}. \tag{12}$$

D'un point de vue combinatoire, ces types de modèles génèrent des cartes farcies, définies en [67]. Ce sont des cartes qui ne sont pas construites en recollant des disques, mais des surfaces de genre h à n composantes du bord de périmètres $p_1,\ldots,p_n$. Dans [66], cette interprétation survit avec néanmoins une coloration supplémentaire des composantes du bord.

Dans la version *à blobs*, la récurrence sur les fonctions de corrélation possède toujours le même terme universel que dans le cas ordinaire, à savoir celui qui calcule les parties singulières de ces fonctions. Qui plus est, des contributions holomorphes s'ajoutent [67, 68]. Il reste néanmoins important de garder à l'esprit que l'action (10) n'est en fait topologique que pour $d = 4d' + 2$, pour $d' \in \mathbb{N}$ [66], ce qui signifie que les constantes de couplage prennent la forme $t_N(p_1,\ldots,p_d) = \sum_{h\geq 0} N^{2-d-2h} t^{(h)}(p_1,\ldots,p_d)$. Pour d'autres valeurs de d, on peut faire comme si l'action était topologique en réabsorbant une partie de la dépendance en N dans les constantes de couplage $t^{(h)}(p_1,\ldots,p_d)$, puis en appliquant la récurrence topologique.

Ici nous montrons comment mettre cette approche en œuvre pour des modèles $U(N)^d$-invariants arbitraires, sous réserve qu'il existe des interactions quartiques meloniques (entre autres) ainsi qu'une certaine condition d'inversibilité d'une forme quadratique à large N. Tout cela repose sur le fait qu'après emploi de certaines techniques de champ intermédiaire et d'intégrations formelles, de tels modèles de tenseurs peuvent toujours être réécrits comme des modèles de matrices à d matrices hermitiennes satisfaisant aux conditions 1 et 2. C'est pourquoi les fonctions de corrélation de ces matrices satisfont la récurrence topologique *à blobs* et ce avec la même courbe spectrale que le modèle quartique melonique de [66]. De manière remarquable (et évidente d'après [67]), les détails du modèle, à savoir le choix des interactions, ne contribuent qu'à une action effective et non pas à la forme de la récurrence topologique *à blobs*. En termes combinatoires, les détails ne contribuent qu'aux fonctions génératrices de la farce des cartes. Démontrer la validité de la récurrence *à blobs* ne nécessite pas de connaître explicitement l'action effective, mais seulement que les conditions 1 et 2 soient satisfaites. En ce sens, la récurrence topologique *à blobs* est universelle dans notre cadre. La seule différence entre nos formules et celles de [66] repose sur le fait que les fonctions génératrices de la farce étaient connues explicitement dans [66], alors que leur dépendance en les constantes de couplage demeure inconnue ici (leur dépendance en N reste néanmoins importante).

Méthode – Il existe cependant quelques obstacles techniques à surmonter. Tout modèle de tenseurs ne peut être directement transformé en un modèle de matrices grâce à la transformation d'Hubbard-Stratonovich étant donné que cette dernière ne fonctionne qu'avec des interactions quartiques. Ce premier obstacle a été surmonté dans [70] où il a été montré qu'il était toujours possible de réécrire le modèle avec des matrices. Ceci fut prouvé en utilisant une bijection entre les graphes de Feynman du modèle de tenseurs et ceux du modèle de matrices correspondant. Une seconde preuve fut également donnée par des manipulations d'intégrales formelles (qui ne sont définies que *via* leurs séries de Feynman). Ici nous répéterons cette preuve en l'adaptant afin de surmonter le second obstacle que nous explicitons maintenant.
La méthode employée dans [70] transforme un modèle de tenseurs en un modèle de matrices avec des matrices complexes M_C étiquetées par des sous-ensembles de $[\![1, d]\!]$, à savoir $C \subset [\![1, d]\!]$. Néanmoins, appliquer la même recette que dans [66] nécessite à la place d'avoir d matrices hermitiennes $M_1, \ldots, M_d$. Ce problème est résolu en deux étapes. Nous montrons d'abord qu'il est possible de remplacer chaque matrice complexe M_C par une paire de matrices hermitiennes $(Y_C, \Phi_C)_{C \subset [\![1,d]\!]}$. Ensuite, et sous réserve que les interactions quartiques meloniques soient allumées, il est possible d'intégrer formellement sur toutes les matrices hormis $Y_1, \ldots, Y_d$. En termes de graphes de Feynman, cela signifie que l'on est en présence de cartes combinatoires à arêtes colorées correspondant aux matrices $Y_1, \ldots, Y_d$, tout le reste étant entassé dans une quelconque farce de la carte. Gardant en mémoire le but de la récurrence topologique, il est nécessaire de contrôler la dépendance en N des différentes farces en termes de leurs composantes du bord.
La prochaine étape consiste à observer que toutes les valeurs propres de $Y_1, \ldots, Y_d$ tombent dans un puits de potentiel à large N et de poursuivre par l'étude de leurs fluctuations. C'est ici que l'on observe que les conditions 1 et 2 sont toujours satisfaites et conduisent à la récurrence topologique *à blobs*. Il serait intéressant de savoir si la condition 2 peut être supprimée en général. L'on sait cela possible dans le cas d'un modèle mono-matrice [67], cependant, dans le cas multi-coloré, cela nécessiterait un lemme à une coupure (ou lemme de Brown) pour un système d'équations couplées possédant des variables catalytiques, élargissant partant le cadre de [71], ce qui dépasse les limites de ce manuscrit.

Attentes – En traitant de la récurrence topologique dans le contexte des modèles de tenseurs, il est une autre question qui vient naturellement à l'esprit, à savoir comment relier les valeurs moyennes d'observables du côté des tenseurs aux quantités évaluées *via* la récurrence topologique du côté des matrices. Dans [64] pour le modèle quartique melonique, il a été montré que les valeurs moyennes de tr M_c^n sont des valeurs moyennes de polynômes d'Hermite en des bulles cycliques melonique du côté des tenseurs. Cette relation peut être inversée grâce aux polynômes d'Hermite. Dans [66], les valeurs moyennes d'observables tensorielles (bulles) arbitraires ont été exprimées en termes de quantités évaluées par les modèles de matrices, mais cela impliquait de sommer sur des contractions de Wick.
Dans ce manuscrit, nous généralisons la relation sur les polynômes d'Hermite de [64] à des observables arbitraires à la fois du côté des tenseurs et des matrices. Afin d'exprimer la valeur moyenne d'une observable matricielle à l'aide d'observables tensorielles, l'on est contraint de dériver le potentiel (qui dans le cas quartique melonique est quadratique, ce qui conduit aux polynômes d'Hermite). Dans le cas contraire, c'est-à-dire exprimer la valeur moyenne d'une observable tensorielle en termes de valeurs moyennes matricielles, il faut prendre les dérivées d'un potentiel effectif en les matrices Y_C (dans le modèle quartique melonique, il s'agit encore d'un potentiel quadratique, d'où les polynômes d'Hermite), qui vient par intégration de toutes les matrices Φ_C.

Plan – Ce travail est structuré de la manière suivante. Le Chapitre 1 commence par introduire le problème de la gravité quantique en dimension arbitraire, puis en donne une solution en dimension $d = 2$ à l'aide des modèles de matrices, qu'il introduit. Il se termine par un exemple élémentaire d'application du formalisme de la récurrence topologique au modèle hermitien mono-matrice. Dans le Chapitre 2, on définit les modèles de tenseurs aléatoires et on en donne les principales propriétés et caractéristiques. L'on introduit également les techniques issues de la théorie du groupe symétrique pour compter les invariants tensoriels. Dans le Chapitre 3 on rentre dans le vif du sujet. Un modèle de tenseurs aléatoires particulièrement important est introduit, à savoir le modèle quartique melonique, et le formalisme de la récurrence topologique lui est appliqué ; il sert de référence pour l'analyse de la Partie III. Le Chapitre 4 présente nos notations pour les tenseurs réels et leurs invariants orthogonaux. L'on développe ensuite le comptage double classe en utilisant le formalisme du groupe des permutations. L'on discute également dans ce chapitre de la formulation TFT du comptage et de ses conséquences, puis on introduit le b.a.-ba de la théorie des représentations du groupe symétrique avant de réinterpréter le comptage dans ce langage. Le Chapitre 5 traite de l'algèbre double classe construite à partir des invariants $O(N)$ et liste ses propriétés. Ensuite, le Chapitre 6 détaille les corrélateurs gaussiens à 1 et 2 points et leurs conséquences en termes de théorie des représentations. Le Chapitre 7 liste brièvement quelques remarques concernant le comptage d'invariants du groupe symplectique réel $Sp(2N)$. Le principe est similaire au cas orthogonal, moyennant quelques subtilités. Dans le Chapitre 8, on définit les modèles de tenseurs qui nous intéressent, ainsi que leurs équivalents multi-matriciels. Les théorèmes 8.3.2 et 8.3.3 révèlent quelques relations entre valeurs moyennes d'observables du côté tensoriel et matriciel. Le Chapitre 9 explique comment formellement intégrer toutes les matrices à l'exception de $Y_1, \ldots, Y_d$, ce qui conduit à un modèle de matrices effectif dans le théorème 9.1.1. Les mêmes techniques d'intégration formelle sont utilisées pour exprimer les valeurs moyennes d'observables tensorielles en termes de valeurs moyennes matricielles dans le théorème 9.2.1. La limite large N du modèle effectif est discutée et conduit à un modèle de matrices pour les fluctuations. Ce dernier est étudié dans le Chapitre 10 à l'aide des équations de Schwinger-Dyson suivant l'analyse de [66, 67]. Seuls certains aspects essentiels à l'établissement du théorème 10.4.1 sur la récurrence topologique *à blobs* sont traités, étant donné que tout fonctionne comme dans [66]. Nous résumons ensuite nos travaux et en tirons les perspectives. Finalement, le manuscrit se clôt sur un appendice qui se divise en deux grandes parties. La première recueille des identités de la théorie des représentations du groupe symétrique qui nous sont utiles dans le texte, alors que la seconde détaille les codes informatiques qui génèrent les suites du nombre d'invariants à divers rangs $d = 3, 4, \ldots$

Part I

State of the art: of Quantum Gravity and Random Tensors

Chapter 1

Quantum gravity as random geometry

Quantum gravity (QG) has been evading physicists for quite some time now. From as early as the 1960s, Feynman rules for the gravitational field are established and an attempt to combine the mathematics of quantum mechanics and General Relativity (GR) results in the Wheeler-DeWitt field equation [72]. However, the latter remains ill-defined as standard quantum field theory methods fail because of the nonrenormalizability of GR. Hawking's "wave function of the universe" approach [73] is similarly flawed and one has to wait until the 1980s for a renaissance of the genre with string theory and later Loop Quantum Gravity and Tensorial Field Theories. For further accounts on the history of QG, see for instance [74].

1.1 Looking for a quantum theory of gravity

Let $d > 2$ be an integer. In d-dimensional space, pure gravity is described by the partition function

$$Z_{EH} = \int D[g] e^{-S_{EH}[g]}, \tag{1.1}$$

where the integration is performed over the space of all (semi-)Riemannian metrics of a given smooth manifold Σ. Denoting by R the (Ricci) scalar curvature of g and Λ the cosmological constant, the Einstein-Hilbert (EH) action reads in local coordinates

$$S_{EH}[g] = \frac{1}{16\pi G} \int_{\Sigma} d^d x \sqrt{|g|}(R - 2\Lambda), \tag{1.2}$$

in natural units where $c = 1$, G is Newton's fundamental constant of gravity and $|g|$ the determinant of the metric. In the classical theory, one then recovers the vacuum Einstein field equations by applying the least action principal to S_{EH}. Namely, in local coordinates, for μ, ν ranging from 1 to d and $R_{\mu\nu}$ being the Ricci tensor ($g^{\mu\nu}R_{\mu\nu} = R$):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{1.3}$$

From there one can obtain some rough insight into QG by "linearizing" Einstein's equations on Minkowski space, *i.e.* by taking the metric to be a small perturbation of the flat Minkowski metric $\eta_{\mu\nu}$: $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$ and demanding that this metric be a solution of (1.3) up to first order in ε. The quantum theory thus obtained turns out to be the theory of a massless spin-2 particle called the graviton. The standard viewpoint of quantum field theory would then be first to study the linearized equations and then turning on the interactions by incorporating the non-linear terms. This approach however violates the "background independance" spirit of GR as it privileges the particular solution being perturbed about (the Minkowski metric). This plagues the theory with infinities: it is not renormalizable. As it is now thought that for a theory to accurately describe our universe, it should be renormalizable, this failure calls for a different approach. A common one - mainly developed by Regge [75] - consists in discretizing the manifold in the following way. Consider a triangulation[1] $\mathcal{T}$ of Σ consisting of blocks of d-simplices σ_d. Historically, the discretized EH action treats all the lengths of the triangulation as many geometric degrees of freedom, yielding a classical approximation of gravity on the triangulation. Assume now that all edges have the same length, say l. The theory now becomes an intrinsically quantum one, as the degrees of freedom are the triangulation itself. Such a triangulation can be seen as piecewise flat, the d-simplices are flat and the curvature is concentrated around the $(d-2)$-simplices. The latter being given by the so-called deficit angle

$$\delta(\sigma_{d-2}) = 2\pi - \sum_{\sigma_{d-1} \supset \sigma_{d-2}} \theta(\sigma_{d-1}, \sigma_{d-2}), \tag{1.4}$$

where the $\theta(\sigma_{d-1}, \sigma_{d-2})$ are the dihedral angles hinged on σ_{d-2} and the sum is taken over pairs of $(d-1)$-simplices sharing the same $(d-2)$-simplex. For instance in dimension $d = 2$, the curvature is concentrated over the vertices where arrangements of triangles meet. A vertex with positive (resp. negative) deficit angle thus represents a concentration of negative (resp. positive) curvature.

The EH action then takes the discrete form of the Regge action, where the first term encodes curvature and the second being the total volume of Σ, measured by the cosmological

[1] Such a triangulation may not exist in general when Σ is only topological [76]. However, we know from [77, 78] that every smooth manifold admits a (piecewise-linear) triangulation.

constant:

$$S_{Regge} = \frac{1}{8\pi G}\Big(\sum_{\sigma_{d-2}} \text{vol}(\sigma_{d-2})\delta(\sigma_{d-2}) - 2\Lambda \sum_{\sigma_d} \text{vol}(\sigma_d)\Big). \tag{1.5}$$

Denoting by $v_k = \frac{l^k}{k!}\sqrt{\frac{k+1}{2^k}}$ the volume of a k-simplex and by n_k the number thereof, it can be shown [12] that the action reduces to

$$S_{Regge} = \frac{1}{8\pi G}(v_{d-2}\Delta_{d-2} - 2\Lambda n_d v_d), \tag{1.6}$$

where $\Delta_{d-2} = 2\pi\, n_{d-2} - \frac{d(d+1)}{2} \arccos\left(\frac{1}{d}\right) n_d$ is the sum of the deficit angles. Indeed, the deficit is linked to the number $\frac{d(d+1)}{2}$ of σ_{d-1} simplices around each σ_{d-2} simplex, the dihedral angle being $\arccos\left(\frac{1}{d}\right)$[2]. We then substitute the integral over the metrics of Σ with a sum over homogeneous triangulations of the manifold

$$\int D[g] \quad \longleftrightarrow \quad \sum_{\mathcal{T} \text{ triangulation of } \Sigma} . \tag{1.7}$$

A complete account of this substitution in d dimensions can be found in [12, 79], here we only cite the result for $d = 2$:

$$Z(\Lambda) = \left[e^{-\frac{1}{4G}}\right]^{2-2h(\Sigma)} \sum_{\mathcal{T} \text{ triangulation of } \Sigma} \left[e^{\frac{\Lambda}{8\pi G}}\right]^{\#\text{triangles}}, \tag{1.8}$$

where $h(\Sigma)$ is the genus - the number of handles - of the surface Σ.

Let us now turn our focus to the two dimensional case, where the problem of QG can be solved explicitly. This will allow us to introduce the central notion of random matrices.

1.2 Discretizing surfaces

Take $d = 2$ and suppose Σ is an orientable compact manifold without boundary. In this setting, classical gravity is trivial. Indeed, from the Gauss-Bonnet theorem, the scalar

[2] It is worth noticing that in dimension $d = 3$, as $\arccos\left(\frac{1}{3}\right) \neq \frac{p}{q}\pi$, the sum of deficit angles cannot be zero, hence the question of simulating flat three dimensional spaces with tetrahedra.

curvature[3] term is topological, thus non-dynamical:

$$\int_\Sigma d^2x\,\sqrt{|g|}R = 4\pi\chi(\Sigma), \tag{1.9}$$

where[4] $\chi(\Sigma) = 2 - 2h(\Sigma)$ is the Euler characteristic of the surface. This, however, does no longer hold in the quantum case, the reason of which is twofold. On the one hand, large quantum fluctuations may change the genus of the surface, the partition function should hence involve a sum over surfaces of all genera h. We write

$$\begin{aligned} Z(\Lambda) &= \sum_h \int D[g] e^{-\frac{1}{16\pi G}\left(\int d^2x\,\sqrt{|g|}R - 2\Lambda\int d^2x\,\sqrt{|g|}\right)} \\ &= \sum_h \int D[g] e^{-\frac{1-h}{2G}+\frac{\Lambda}{8\pi G}A}, \end{aligned} \tag{1.10}$$

with $A = \int d^2x\sqrt{|g|}$ being the area of the surface. On the other, at fixed genus, the random contributions stem from local (quantum) fluctuations of the metric field that modify locally the area of the surface - it can be seen by factoring out the Gauss-Bonnet term in the path integral (1.10). An illustration of this phenomenon is best given by the Brownian map, where "random fluctuations" of the graph distance produce highly fractal patterns as seen in Figure 1.1.

Note that even in two dimensions, the integral over the (infinite dimensional) space of metrics is hard to perform in general, especially if Σ is not compact. Nevertheless, a way was developed by Polyakov [81] in the 1980s as he was working on the quantization of the bosonic string and which is nowadays known as Liouville's approach to QG (see [4] for a quick review). We, however, will not be taking this route. So let us once again discretize the surface with triangles. The triangles are chosen to be equilateral so that the triangulation is locally flat when there are exactly six triangles incident to a vertex v and has positive (resp. negative) curvature when there are less (resp. more). More precisely, the discrete equivalent of the Ricci scalar for each vertex is $R_v = 2\pi(6 - \deg v)/\deg v$, so that

$$\frac{1}{4\pi}\int_\Sigma d^2x\,\sqrt{|g|}R \quad \to \quad \sum_{v\in V(\mathcal{T})}\left(1 - \frac{\deg\, v}{6}\right), \tag{1.11}$$

where $\deg\, v$ is the degree of the vertex $v \in V(\mathcal{T})$, *i.e.* the number of triangles incident

[3] Note that in dimension $d = 2$, the Gaussian curvature is simply given by half the Ricci scalar.

[4] When Σ is non-orientable the Euler characteristic reads $\chi = 2-k$, where k, the non-orientable genus, is the number of real projective planes appearing in a connected sum decomposition of the surface.

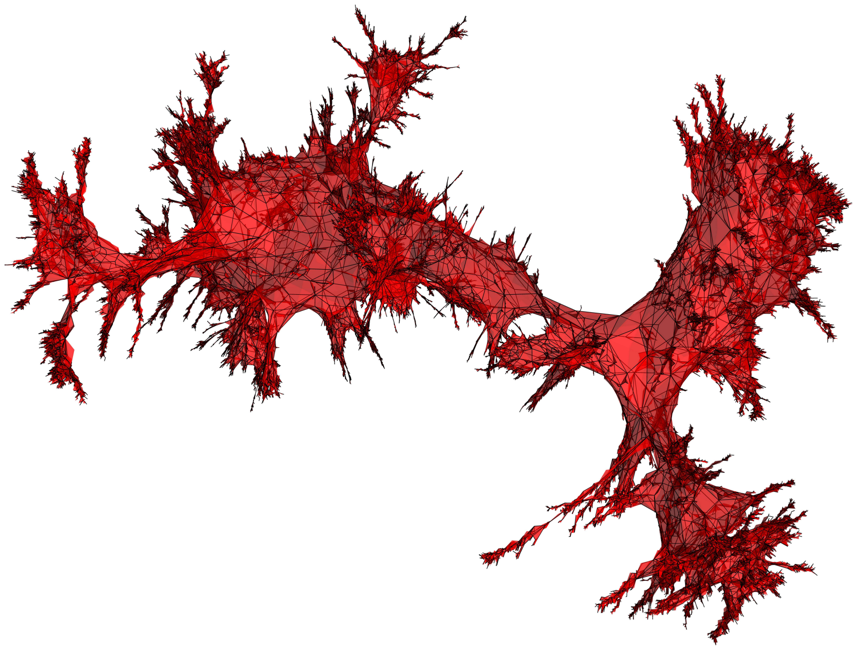


FIGURE 1.1 – A Brownian map of genus 0, or more precisely a very large random bipartite quadrangulation of the sphere. Image by Jérémie Bettinelli [80].

to it. Writing V, E and F respectively the total number of vertices, edges and faces (triangles) in $\mathcal{T}$, the following relations hold: $\sum_{v\in V(\mathcal{T})} \deg\, v = 3F$ (as there are three corners per triangle) and $3F = 2E$ (since each face has three edges, each of which shared by two faces). From (1.6) it now follows that the discretized action takes the form

$$S_{Regge}(\Lambda) = \frac{1}{4G}(V - \frac{1}{2}F)v_0 - \frac{\Lambda}{8\pi G}v_2 F. \tag{1.12}$$

But from the definition of the Euler characteristic,

$$2 - 2h = V - E + F \overset{3F=2E}{=} V - \frac{1}{2}F, \tag{1.13}$$

the action reduces to the expression

$$S_{Regge}(\Lambda) = \frac{1}{4G}(2 - 2h) - \frac{\Lambda}{8\pi G}v_2 F, \tag{1.14}$$

given at the end of the previous section.
A way of implementing such triangulations of surfaces uses the theory of random matrices, that we now introduce.

1.3 Random matrices

Broadly speaking, random matrix theories deal with probability laws on spaces of matrices. Originally introduced by Wishart [82] in multivariate statistics, they made their first appearances in physics with Wigner in the 1950s [83] through the study of the energy spectrum of large nuclei. Since that time, random matrices have spread to almost every field of mathematics and physics. They have been used in chaotic quantum theory to compute for instance the Rydberg levels of hydrogen atoms in a strong magnetic field [84], by 't Hooft [6] in the large N limit of $U(N)$ Quantum Chromodynamics, in string theory and to count various maps and knots [85].
A random matrix theory is defined by the choice of both a matrix ensemble and probability measure on that ensemble. Depending on the symmetries of the system, one finds that one has to consider three sets of matrices, invariant respectively under the orthogonal $O(N)$, unitary $U(N)$ and symplectic group $Sp(2N)$. It is conventional to refer to them by the integer $\beta = 1, 2, 4$ that counts the number of real parameters of the off-diagonal entries. One can furthermore assume that the matrices follow a Gaussian distribution:

$$\rho(M)dM \propto \exp\left(-\frac{1}{2}\beta M^2\right) dM. \tag{1.15}$$

For $\beta = 2$ this defines the Gaussian Unitary Ensemble (GUE) of random matrices. The Lebesgue measure is then expressed as the product of the Lebesgue measures on the real components of the matrix:

$$dM = \prod_i dM_{ii} \prod_{i<j} d\,\mathrm{Re}\, M_{ij}\, d\,\mathrm{Im}\, M_{ij}. \tag{1.16}$$

However, measures that we will encounter are all constructed *via* a potential V:

$$d\mu(M) = e^{-\mathrm{tr}\, V(M)} dM, \tag{1.17}$$

often chosen to be a polynomial function, but more involved potentials, sometimes called semi-classical, can also be considered. A matrix of the GUE has real eigenvalues and can be diagonalized as $M = U\Lambda U^\dagger$, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ and U is a unitary matrix. The measure (1.16) can then be rewritten in terms of the measures on Λ and U as

$$dM = \Delta^2(\lambda) d\Lambda\, dU, \tag{1.18}$$

where $d\Lambda = \prod_i d\lambda_i$ is the Lebesgue measure on $\mathbb{R}^N$, dU is the Haar measure on $U(N)$ and $\Delta(\lambda)$ is the Vandermonde determinant

$$\Delta(\lambda) := \det_{i,j} \lambda_i^{j-1} = \prod_{j<i} (\lambda_i - \lambda_j). \tag{1.19}$$

To show this, notice that (1.16) is invariant under $U(N)$-conjugation, thus one has for the differential of M:

$$\begin{aligned} dM_{ij} &= (d\Lambda + [dU, \Lambda])_{ij} \\ &= \delta_{ij} d\lambda_i + (\lambda_i - \lambda_j) dU_{ij}. \end{aligned} \tag{1.20}$$

Computing the Jacobian then yields

$$\frac{dM}{d\Lambda dU} = \det_{i,i'} \left(\frac{dM_{ii}}{d\lambda_{i'}} \right) \det_{i\neq j, i'\neq j'} \left(\frac{dM_{ij}}{dU_{i'j'}} \right) = \Delta^2(\lambda). \tag{1.21}$$

The partition function associated to the measure $d\mu(M)$ (1.17) then becomes

$$Z = \frac{1}{Z_0} \int_{\mathbb{R}^N} \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda) e^{-\sum_i V(\lambda_i)}, \tag{1.22}$$

where Z_0 is a normalizing factor related to the volume of the GUE. Explicitly[5], it reads [101]

$$Z_0^{-1} = \frac{1}{(2\pi)^N N!} \operatorname{vol}(U(N)) = \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{j=1}^N j!}. \tag{1.23}$$

It will henceforth be omitted.

Introductory example – Following what was said in the previous section, we consider a particular random triangulation of a surface as examplified in Figure 1.2. Consider $N \times N$ Hermitian matrices, the partition function of such a model is given by

$$Z(t_3, N) = \int dM \; e^{-N\left(\frac{1}{2t}\mathrm{tr}M^2 - \frac{t_3}{3}\mathrm{tr}M^3\right)}, \tag{1.24}$$

where the measure of integration is (1.16). This integral may be computed by formally expanding the t_3-term of the exponential, and by explicitly performing the term-by-term

[5] The prefactor in the intermediate term stems from the liberty in choosing U in the diagonalization of M. Indeed, the set of eigenvalues remains unchanged if one multiplies U on the right by elements of the sets $U(1)^N$ of unitary diagonal matrices and S_N of permutation matrices.

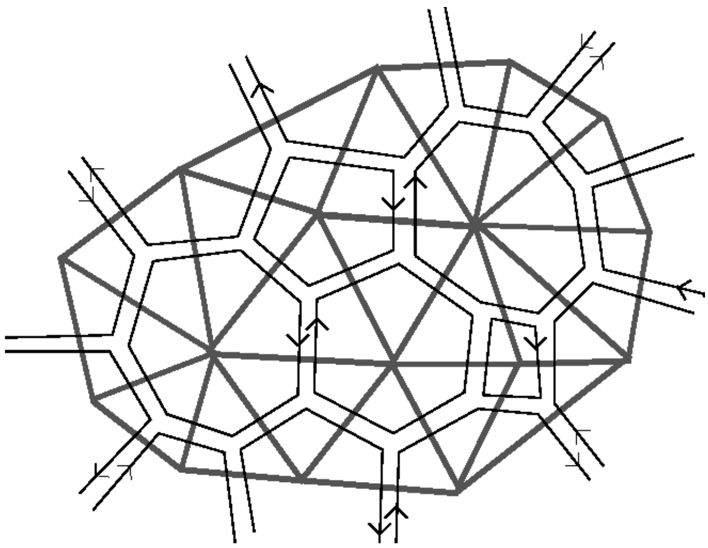


FIGURE 1.2 – An example of a random triangulation of a surface. Each triangle is dual to a three point vertex of a matrix model.

Gaussian integration:

$$\begin{aligned} Z(t_3, N) &= \sum_{k=0}^{\infty} N^k \frac{t_3^k}{k!} \int dM \frac{(\mathrm{tr} M^3)^k}{3^k} e^{-\frac{N}{2t}\mathrm{tr} M^2} \\ &= \sum_{k=0}^{\infty} N^k \frac{t_3^k}{3^k k!} \langle (\mathrm{tr} M^3)^k \rangle . \end{aligned} \tag{1.25}$$

These integrations can be carried out by applying Wick's theorem:

$$\langle M_{i_1 j_1} \dots M_{i_{2n} j_{2n}} \rangle = N^{-n} \sum_{\{p_a, q_a\}} \prod_{m=1}^{n} \delta_{i_{p_m} j_{q_m}} \delta_{i_{q_m} j_{p_m}}, \tag{1.26}$$

where the sum is taken over all the pairings of indices such that $\bigcup_{a=1}^{n} \{p_a, q_a\} = [\![1, 2n]\!]$. As an example let us work out the case $k = 2$:

$$\begin{aligned} \left\langle \mathrm{tr} M^3 \, \mathrm{tr} M^3 \right\rangle = & \, MMMMMM + MMMMMM + MMMMMM \\ & + MMMMMM + MMMMMM + MMMMMM \\ & + MMMMMM + MMMMMM + MMMMMM \\ & + MMMMMM + MMMMMM + MMMMMM \end{aligned} \tag{1.27}$$

$$+ MMMMMM + MMMMMM + MMMMMM,$$

where each matrix pair contraction is given by the propagator

$$M_{ij}M_{kl} = \langle M_{ij}M_{kl}\rangle = \frac{t}{N}\delta_{il}\delta_{jk}. \tag{1.28}$$

Hence we obtain (repeated indices are summed):

$$\begin{aligned} \langle N^2\,\mathrm{tr}M^3\,\mathrm{tr}M^3\rangle &= N^2\langle M_{ij}M_{jk}M_{ki}M_{lm}M_{mn}M_{nl}\rangle \\ &= \frac{1}{N}(\delta_{ik}\delta_{jj}\delta_{km}\delta_{il}\delta_{ml}\delta_{nn} + \delta_{ik}\delta_{jj}\delta_{kn}\delta_{in}\delta_{ln}\delta_{mm} + \dots) \\ &= 12N^2 + 3. \end{aligned} \tag{1.29}$$

This computation can be given the following graphical interpretation. Consider the general matrix model (7). To each matrix element M_{ij} one associates an oriented double half-edge, where each line carries an index of the matrix. Each factor of the form $\mathrm{tr}\,M^k$ is pictured as a k-valent vertex with k outgoing double half-edges, each vertex carrying a weight Nt_k. By virtue of Wick's theorem, the result of the Gaussian integration is obtained by summing over all possible ways of connecting the half-edges of the integrand into pairs so as to form a closed graph, called fat or ribbon graph. Each vertex now counts for a power of N, the sum over all indices results in a weight N per loop (face) of the graph, and each propagator, or edge, comes with a t/N factor. We obtain the total dependence on N of a ribbon graph as

$$N^{V-E+F} = N^{\chi} = N^{2-2h}. \tag{1.30}$$

We conclude that in our example, the dominant graphs $O(N^2)$ are the (twelve) planar ones, that can be drawn on the sphere, there are also three non planar $O(1)$ graphs, that can only be drawn on the torus. This result can be generalized to arbitrary n-point correlation functions of traces of powers of M:

Theorem 1.3.1 ('t Hooft [86], Brézin–Itzykson–Parisi–Zuber [87])**.**

$$\left\langle \prod_{k=1}^{n} N\frac{\mathrm{tr}M^{p_k}}{p_k}\right\rangle_c = \sum_{\Gamma}\frac{N^{\chi(\Gamma)}}{|\,\mathrm{Aut}(\Gamma)|}, \tag{1.31}$$

where the sum is taken over all connected graphs Γ with n vertices of valence p_k, $k \in [\![1, n]\!]$.

Furthermore, on taking the logarithm of the partition function, one is left with only connected graphs. In our example, we get the following expansion for what is called the free energy

$$\ln Z(t_3, N) = F(t_3, N) = \sum_{h\geq 0} N^{2-2h} F_h(t_3), \tag{1.32}$$

where F_h is the generating function of triangulations of genus h. In the large N limit, only the leading order F_0 survives, it counts the *planar* triangulations, *i.e.* triangulations of the sphere $\mathbb{S}^2$.

1.3.1 The steepest way to the continuum limit

One way of computing this large N limit is through the so-called steepest descent method - or stationary phase approximation - that we now introduce. Consider the general matrix model given by the partition function

$$\begin{aligned} Z(N) &= \int dM e^{-N \,\mathrm{tr} V(M)} \\ &= \int \prod_i d\lambda_i \Delta^2(\lambda) e^{-N\sum_i V(\lambda_i)}, \end{aligned} \tag{1.33}$$

where $\Delta(\lambda)$ is again the Vandermonde determinant, the λ_is are the eigenvalues of M and we consider a polynomial potential. Let us first rewrite this integral as:

$$Z(N) = \int \prod_{i=1}^{N} d\lambda_i e^{-NS(\lambda)}, \tag{1.34}$$

the action being:

$$S(\lambda) = \sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N} \sum_{j<i} \log |\lambda_i - \lambda_j|. \tag{1.35}$$

This action exhibits a remarkable feature. Because of the presence of the Vandermonde determinant, the eigenvalues cannot all fall in the potential well created by V, it is as if they were experiencing a (2D logaritmic) Coulomb repulsion[6]. We are now interested in finding the stationary points of the action, *i.e.* in solutions $\{\lambda_i\}$ satisfying $\dfrac{\partial S}{\partial \lambda_i} = 0$, for

[6] This is known in the litterature as the Coulomb gas picture.

all $i \in [\![1, N]\!]$. We obtain the following equations for all $i \in [\![1, N]\!]$,

$$\frac{2}{N}\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j} = V'(\lambda_i). \tag{1.36}$$

They can be solved by introducing the (trace of the) resolvent of the matrix M:

$$\omega(x) = \frac{1}{N}\mathrm{tr}\frac{1}{x - M} = \frac{1}{N}\sum_i \frac{1}{x - \lambda_i}. \tag{1.37}$$

Multiplying now (1.36) by $1/(x - \lambda_i)$ and summing over i, one gets successively

$$\begin{aligned}
\frac{2}{N}\sum_i\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j}\frac{1}{x - \lambda_i} &= \sum_i \frac{V'(\lambda_i)}{x - \lambda_i},\\
\frac{1}{N}\sum_i\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j}\left(\frac{1}{x - \lambda_i} - \frac{1}{x - \lambda_j}\right) &= \sum_i \frac{V'(\lambda_i) - V'(x)}{x - \lambda_i} + \sum_i \frac{V'(x)}{x - \lambda_i}\\
\frac{1}{N}\sum_{i,j}\frac{1}{x - \lambda_i}\frac{1}{x - \lambda_j} - \frac{1}{N}\sum_i \frac{1}{(x - \lambda_i)^2} &= \sum_i \frac{V'(\lambda_i) - V'(x)}{x - \lambda_i} + NV'(x)\omega(x)\\
\omega^2(x) + \frac{1}{N}\omega'(x) - V'(x)\omega(x) &= \frac{1}{N}\sum_i \frac{V'(\lambda_i) - V'(x)}{x - \lambda_i},
\end{aligned} \tag{1.38}$$

where we have divided the whole expression by N. We denote $p(x) = \frac{1}{N}\sum_{i=1}^N \frac{V'(x) - V'(\lambda_i)}{x - \lambda_i}$, it is a polynomial in x of degree $l - 2$ for $\deg V = l$. In the large N limit, we can neglect the ω'/N term, we arrive at the equation:

$$\omega^2(x) - V'(x)\omega(x) + p(x) = 0. \tag{1.39}$$

Notice that in this limit, the distribution of eigenvalues $\rho(\lambda) = \frac{1}{N}\sum_i \delta(\lambda - \lambda_i)$ becomes continuous and ω is given by the Stieltjes transform of ρ:

$$\omega(x) = \int d\lambda \frac{\rho(\lambda)}{x - \lambda}. \tag{1.40}$$

Eq. (1.36) can then be rewritten

$$\int d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'} = V'(\lambda). \tag{1.41}$$

The eigenvalue density is extracted from $\omega(x)$ *via*

$$\rho(x) = \frac{1}{2i\pi} \lim_{\epsilon \to 0} \left(\omega(x + i\epsilon) - \omega(x - i\epsilon) \right). \tag{1.42}$$

A general treatment of (1.39) can be found in [4], here we will focus on the Gaussian case, in order to introduce the topological recursion later on. Let us write

$$\begin{aligned} W_1(x) &= \langle \mathrm{tr} \, \frac{1}{x - M} \rangle = \sum_{n \geq 0} x^{-n-1} \langle \mathrm{tr} \, M^n \rangle, \\ P_1(x) &= \langle \mathrm{tr} \frac{V'(x) - V'(M)}{x - M} \rangle \quad \text{and set} \quad V(x) = \frac{1}{2t} x^2. \end{aligned} \tag{1.43}$$

Note that in the saddle-point approximation, $W_1(x) \sim N\omega(x)$, such that the 1-point correlation function W_1 satisfies again (1.39) with $P_1 = \lim_{N\infty} p$. The solution reads

$$W_1(x) = \frac{V'(x)}{2} - \frac{\sqrt{V'(x)^2 - 4P_1(x)}}{2}, \tag{1.44}$$

where we chose the negative branch as W_1 scales like $1/x$ for $|x|$ large and with $P_1(x) = \frac{1}{t}$. In the general case, W_1 has $2(l-1)$ branch points corresponding to the roots of the polynomial $V'^2 - 4P_1$. The support of ρ is then composed of $l-1$ disconnected pieces. In the simplest case, which interests us here, the potential has only one minimum, there is thus just one connected support, thus only two branch points (with opposite values as V is even). Then:

$$\begin{aligned} V'(x)^2 - 4P_1(x) &= \frac{x^2}{t^2} - \frac{4}{t} \\ &= \frac{1}{t^2}(x - 2\sqrt{t})(x + 2\sqrt{t}), \end{aligned} \tag{1.45}$$

which was to be expected according to Brown's lemma (see next section). Such that

$$W_1(x) = \frac{1}{2t} \left(x - \sqrt{x^2 - 4t} \right), \tag{1.46}$$

from which we get back Wigner's celebrated semi-circle law: for $\lambda \in [-2\sqrt{t}, 2\sqrt{t}]$,

$$\rho(\lambda) = \frac{1}{2\pi t} \sqrt{4t - \lambda^2}. \tag{1.47}$$

1.3.2 The double scaling limit

Consider now the following quartic model,

$$Z(t_4, N) = \int dM\, e^{-N\left(\frac{1}{2}\mathrm{tr}M^2 + \frac{t_4}{4}\mathrm{tr}M^4\right)}. \tag{1.48}$$

The free energy $F(t_4, N) = \log Z(t_4, N)$ admits an expansion over Feynman graphs. Graphs contributing to $F(t_4, N)$ are ribbon graphs with 4-valent vertices and no external leg, while those contributing to the 2-point function $G_2(t_4, N) = \langle \frac{1}{N}\mathrm{tr}M^2 \rangle$ are graphs with quartic vertices and a marked edge. Let us now use the apparatus developed in the previous paragraphs to compute the 2-point function in the large N limit. The potential is now the following

$$V(x) = \frac{1}{2}x^2 + \frac{t_4}{4}x^4, \tag{1.49}$$

and the correlation function becomes

$$W_1(x) = \frac{1}{2}(x + t_4x^3) - \frac{1}{2}(1 + t_4x^2 + 2t_4a^2)\sqrt{x^2 - 4a^2}, \tag{1.50}$$

with

$$a^2 = \frac{1}{6t_4}(\sqrt{1 + 12t_4} - 1). \tag{1.51}$$

From (1.42) together with (1.43) we find that

$$\rho(\lambda) = \frac{1}{2\pi}(1 + t_4\lambda^2 + 2t_4a^2)\sqrt{4a^2 - \lambda^2}. \tag{1.52}$$

Notice that as $t_4 \to 0$, $a \to 1$ and we recover Wigner's semi-circle law (for $t = 1$). We may now compute the 2-point function from

$$G_2(t_4) = \int d\lambda \lambda^2 \rho(\lambda). \tag{1.53}$$

This leads to (the sum is taken over the unmarked faces):

$$G_2(t_4) = \frac{(1 + 12t_4)^{3/2} - 18t_4 - 1}{54t_4^2} = \sum_{n \geq 1} \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} (-t_4)^{n-1}, \tag{1.54}$$

a result already obtained by Tutte in the 1960s [88] for counting rooted planar quadrangulations. From Theorem 1.3.1, it follows that $G_2(t_4, N)$ can be expanded over powers of

N, indexed by the genus h of the surface:

$$G_2(t_4, N) = \sum_{h\geq 0} N^{2-2h} G_{2,h}(t_4). \tag{1.55}$$

From Eq. (1.54) we see that the planar contribution of the 2-point function exhibits a critical behavior (square root branch point) at $t_c = -\frac{1}{12}$ and can thus be expanded as

$$G_2(t_4) \underset{t\to t_4}{\sim} \sum_n n^{\gamma-3} \left(\frac{t_4}{t_c}\right)^n \underset{t\to t_4}{\sim} |t_c - t_4|^{2-\gamma}, \tag{1.56}$$

which is known as the *critical limit*. In our quartic potential example, $\gamma = -\frac{1}{2}$. This is generalized for higher order genera to[7] [8]

$$G_{2,h}(t_4) \underset{t\to t_4}{\sim} \sum_n n^{(\gamma_{\text{str}}-2)(1-h)-1} \left(\frac{t_4}{t_c}\right)^n \underset{t\to t_4}{\sim} a_h |t_c - t_4|^{(2-\gamma_{\text{str}})(1-h)}. \tag{1.57}$$

We see that the higher genus contributions are enhanced when t_4 approaches t_c, this means that if we take simultaneously the limit $N \to \infty$ and $t_4 \to t_c$, the large N genus suppression would be compensated by the enhanced $t_4 \to t_c$ one, enabling every genus surfaces to be treated on equal footing. To see this, define

$$\kappa^{-1} = N(t_4 - t_c)^{(2-\gamma_{\text{str}})/2} \tag{1.58}$$

and take the limits $N \to \infty$ and $t_4 \to t_c$ by keeping fixed the coupling κ, yielding

$$G_2(\kappa) = \sum_{h\geq 0} a_h \kappa^{2h-2}. \tag{1.59}$$

This is known as the *double scaling limit*.

[7] The critical exponent γ_{str} is called the string susceptibility exponent, it characterizes asymptotically the number of planar graphs with a fixed number of vertices. Note that in the planar case, γ_{str} coincides with γ of (1.56).

[8] This is quite a striking result, it holds for every genus and while the exponent could *a priori* be any complicated function of h, it is merely linear.

1.3.3 Loop equations and topological expansion

We now turn our attention to the loop equations. In essence, loop equations are relations between correlation functions obtained by integrating by parts in the matrix integral. Equivalently, loop equations follow from the invariance of the matrix integral under changes of integration variables – in this sense, loop equations are Schwinger–Dyson equations and we will use the two names interchangeably. Consider thus the following vanishing integrals:

$$\frac{1}{Z}\int dM \sum_{a,b} \frac{\partial}{\partial M_{ab}} \left((M^n)_{ab} e^{-N\,\mathrm{tr}\,V(M)}\right) = 0, \tag{1.60}$$

and let us define the connected correlation functions of order n:

$$W_n(x_1,\dots,x_n) = \sum_{k_1,\dots,k_n} \prod_{i=1}^{n} x^{-k_i-1} \left\langle \prod_{i=1}^{n} \mathrm{tr} M^{k_i} \right\rangle_c = \left\langle \prod_{i=1}^{n} \mathrm{tr} \frac{1}{x_i - M} \right\rangle_c, \tag{1.61}$$

where the subscript c of course stands for "connected" and the connected expectation value is defined by restricting its Feynman expansion to connected graphs only. For our model, the loop equations yield

$$\sum_{k=0}^{n-1} \left(\langle \mathrm{tr} M^k\rangle\langle \mathrm{tr} M^{n-1-k}\rangle + \langle \mathrm{tr} M^k \mathrm{tr} M^{n-1-k}\rangle_c\right) - N\langle \mathrm{tr}(M^n V'(M))\rangle = 0. \tag{1.62}$$

To get the equation satisfied by the generating functions, multiply this equation by x^{-n-1} and sum over n. The first term becomes $W_1(x)^2 + W_2(x,x)$. For the second term use the following trick

$$\begin{aligned}\sum_{n\geq 0}\langle \mathrm{tr}(M^n V'(M))\rangle x^{-n-1} &= \langle \mathrm{tr}\frac{V'(x)}{x-M}\rangle - \langle \mathrm{tr}\frac{V'(x)-V'(M)}{x-M}\rangle \\ &= V'(x)W_1(x) - P_1(x),\end{aligned} \tag{1.63}$$

yielding the 1-point equation

$$W_1(x)^2 + W_2(x,x) - NV'(x)W_1(x) + NP_1(x) = 0. \tag{1.64}$$

We will now act repeatedly on (1.64) with the loop insertion operator defined as follows. If we write the potential as $V(M) = \sum_n t_n M^n$, then for all $x \in \mathbb{C}$, $\frac{\delta}{\delta V(x)} = \sum_n x^{-n-1}\frac{\partial}{\partial t_n}$

is such that

$$\frac{\delta}{\delta V(x)} W_n(x_1, \ldots, x_n) = W_{n+1}(x, x_1, \ldots, x_n). \tag{1.65}$$

Before giving the n-point equation, define the sets $I_1, I_2 \subseteq [\![2, n]\!]$ that are such that $I_1 \sqcup I_2 = [\![2, n]\!]$. We furthermore write x_{I_α} the $|I_\alpha|$-tuple of x values. We then have

$$\begin{aligned}\sum_{I_1, I_2} & W_{|I_1|+1}(x_1, x_{I_1}) W_{|I_2|+1}(x_1, x_{I_2}) + W_{n+1}(x_1, x_1, \ldots, x_n) \\ & + \sum_{j=2}^{n} \frac{\partial}{\partial x_j} \frac{W_{n-1}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) - W_{n-1}(x_2, \ldots, x_n)}{x_1 - x_j} \\ & \quad - NV'(x_1) W_n(x_1, \ldots, x_n) + NP_n(x_1, \ldots, x_n) = 0,\end{aligned} \tag{1.66}$$

where we defined

$$P_n(x_1, \ldots, x_n) = \left\langle \mathrm{tr} \frac{V'(x_1) - V'(M)}{x_1 - M} \prod_{i=2}^{n} \mathrm{tr} \frac{1}{x_i - M} \right\rangle_c, \tag{1.67}$$

it is a polynomial function in its first argument x_1.

Finally, we know by Theorem 1.3.1 that all correlation functions admit the following topological expansion

$$W_n(x_1, \ldots, x_n) = \sum_{g \geq 0} N^{2-2g-n} W_{n,g}(x_1, \ldots, x_n). \tag{1.68}$$

Plugging this into (1.66) then yields

$$\begin{aligned}\sum_{g_1+g_2=g} \sum_{I_1, I_2} & W_{|I_1|+1, g_1}(x_1, x_{I_1}) W_{|I_2|+1, g_2}(x_1, x_{I_2}) + W_{n+1, g-1}(x_1, x_1, \ldots, x_n) \\ & + \sum_{j=2}^{n} \frac{\partial}{\partial x_j} \frac{W_{n-1,g}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) - W_{n-1,g}(x_2, \ldots, x_n)}{x_1 - x_j} \\ & \quad - NV'(x_1) W_{n,g}(x_1, \ldots, x_n) + NP_{n,g}(x_1, \ldots, x_n) = 0.\end{aligned} \tag{1.69}$$

1.4 A story of maps

There are various ways to define combinatorial maps, that will not be listed here. Instead we will focus on two particular definitions that will be of interest for us, namely the topological one and through Feynman graphs.

Definition 1.4.1 (Cellular embedded graph). *A cellular embedded graph is the datum* (S, G, f) *where*

- S *is a closed (connected, oriented) topological surface,*
- G *is a connected graph,*
- $f : G \to S$ *is such that*
  - *(i)* $f(G)$ *is a union of Jordan arcs,*
  - *(ii)* $S \backslash f(G)$ *is a union of simply connected open subsets of* S*.*

Two cellular embedded graphs (S, G, f) and (S', G', f') are said to be isomorphic if and only if there exist a surface homeomorphism $\psi : S \to S'$ and a graph homeomorphism $\phi : G \to G'$ such that the following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{f} & S \\ \phi \downarrow & & \downarrow \psi \\ G' & \xrightarrow[f']{} & S' \end{array} . \tag{1.70}$$

We are now equipped to give our first definition of a map.

Definition 1.4.2 (Combinatorial map [60]). *A map is an equivalence class of cellular embedded graphs modulo graph isomorphisms.*

We call *face* a connected component of $S \backslash f(G)$, its degree is defined as the number of arcs making up its boundary. A face can be marked by marking an edge, which is oriented so that the face is on its right. A map with n boundaries is then a map with n marked faces such that each face has only one marked edge oriented such that it is on its right. Let us define the set of maps with n boundaries, v vertices and genus g as $\mathcal{M}_{n,g}^{v}$ and write for $m \in \mathcal{M}_{n,g}^{v}$, $n_i(m)$ the number of unmarked faces of m of degree $1 \leq i \leq l$ and $k_j(m)$ the degree of the j^{th} marked face. We furthermore allow marked faces to have degree at least one[9], except for the atomic map for which $k_1(\bullet) = 0$. The generating function of these maps is given by:

$$\begin{aligned} W_{n,g}(x_1,\dots,x_n|t_1,\dots,t_l|t) &\equiv \sum_v^{\infty} t^v \sum_{m\in\mathcal{M}_{n,g}^v} \prod_{i=1}^{l} \frac{t_i^{n_i(m)}}{|\operatorname{Aut}(m)|} \prod_{j=1}^{n} x_j^{-k_j(m)-1} \\ &\in \mathbb{Q}[x_1^{-1},\dots,x_n^{-1},t_1,\dots,t_l][[t]], \end{aligned} \tag{1.71}$$

where $|\operatorname{Aut}(m)| = 1$ for all $n \geq 1$, as marking a face kills the symmetries of the map. Notice also that the number of vertices is redundant in the previous formula. Indeed, from the definition of the Euler characteristic of a map with boundaries $m \in \mathcal{M}_{n,g}^v$,

$$\chi(m) = 2 - 2g - n = v - e + \sum_{i=1}^{l} n_i(m), \tag{1.72}$$

where e is the number of edges of the map, that expresses as twice the number of half-edges:

$$2e = \sum_{j=1}^{n} k_j(m) + \sum_{i=1}^{l} i n_i(m), \tag{1.73}$$

yielding overall

$$v = \frac{1}{2}\sum_{j=1}^{n} k_j + \frac{1}{2}\sum_{i=1}^{l}(i-2)n_i + \chi. \tag{1.74}$$

$W_{n,g}$ can thus be recast into a formal power series in $\mathbb{Q}[[x_1^{-1},\dots,x_n^{-1},t_1,\dots,t_l]]$ by making the following substitutions

$$t_i \to t_i t^{\frac{i}{2}-1}, \qquad x_j \to \frac{x_j}{\sqrt{t}}, \qquad t \to 1, \tag{1.75}$$

and working with $t = 1$. We will use the two definitions interchangeably and merely write $W_{n,g}(x_1,\dots,x_n)$ the generating function. It relates to the matrix correlation functions by setting the potential to

$$V(x) = \frac{x^2}{2} + \sum_{i=1}^{l} \frac{t_i}{i} x^i. \tag{1.76}$$

[9] This means that our graphs may contain tadpoles and multi-edges.

1.5 Topological recursion for the 1-Hermitian matrix model

The topological recursion (short for Eynard-Orentin topological recursion [57, 58, 59, 60]) is in layman's terms, a way of counting maps of a given topology, using only planar maps. Recall Equation (1.44) of the previous section where we solved for $W_1(x)$ in the Gaussian case. Let us now get more involved. First define

$$f(x, y) = y^2 - V'(x)y + P_1(x). \tag{1.77}$$

Notice that W_1 satisfies $f(x, W_1(x)) = 0$. We know that W_1 is defined near infinity $\left(\sim \frac{1}{x}\right)$ but the form of the polynomial

$$V'(x)^2 - 4P_1(x) \propto \prod_{i=1}^{2l-2}(x - a_i) \tag{1.78}$$

tells us it cannot be analytically continued on the whole complex plane, but instead on

$$\hat{\mathbb{C}} \setminus \bigcup_{i=1}^{l-1}[a_{2i-1}, a_{2i}], \tag{1.79}$$

where $\hat{\mathbb{C}}$ is the Alexandroff compactification of $\mathbb{C}$ obtained by adding a point at infinity, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \simeq \mathbb{CP}^1$ and is called the Riemann sphere. Also the intervals $[a_{2i-1}, a_{2i}] := \Gamma_i$ are the branch cuts and we write $\Gamma = \bigcup_{i=1}^{l-1} \Gamma_i$. One can thus define W_1 either on $\hat{\mathbb{C}} \setminus \Gamma$ or on the whole Riemann sphere, rendering it multivalued. In the case where W_1 has only one branch cut $[a_-, a_+]$, one has the following useful lemma, referred to as Brown's or 1-cut lemma:

Lemma 1.5.1 (Brown). *The singular part of W_1 can be written*

$$V'(x)^2 - 4P_1(x) = M(x)^2(x - a_+)(x - a_-), \quad a_\pm = \alpha \pm 2\gamma, \tag{1.80}$$

where
$$\begin{cases} \alpha \in \mathbb{C}[t_1, \dots, t_l][[t]], & \alpha = O(t) \\ \gamma^2 \in \mathbb{C}[t_1, \dots, t_l][[t]], & \gamma^2 = t + O(t^2) \\ M(x) \in \mathbb{C}[t_1, \dots, t_l][[t]][x], & M(x) = \dfrac{V'(x)}{x} + O(t). \end{cases}$$

So that the solution to $f(x,y)=0$ is given by

$$W_1(x) = \frac{V'(x)}{2} \pm \frac{1}{2}M(x)\sqrt{(x-a_+)(x-a_-)}, \tag{1.81}$$

which is of course (1.44) when we choose the negative branch. At this point however, W_1 remains multivalued, it is then convenient to introduce a two-sheeted cover of the Riemann sphere. This is done by parametrizing the algebraic curve $f(x,y)=0$ through the so-called Zhukowsky transformation given for $z \in \hat{\mathbb{C}}$ by

$$\begin{aligned} x(z) &= \frac{a_+ + a_-}{2} + \frac{a_+ - a_-}{4}(z + z^{-1}), \\ y(z) &= \frac{4}{a_+ - a_-} z^{-1}. \end{aligned} \tag{1.82}$$

The exterior of the unit disc $|z| > 1$ then corresponds to the physical sheet, *i.e.* Equation (1.81) with the negative sign, while the interior $|z| < 1$ is mapped to the non-physical sheet, *i.e.* Equation (1.81) with the positive sign. The two sheets are exchanged by the (local Galois) involution that leaves the covering map x invariant:

$$\iota : z \mapsto 1/z. \tag{1.83}$$

Note that the fixed points of ι are the zeros of $x'(z)$, namely the branch points $z = \pm 1$, and their images by x are the extremities of the cut a_- and a_+.

1.5.1 Disc and cylinder functions

From (1.68) we immediately see that in the large N limit the only surviving terms are the planar ones, *i.e* those for which $g = 0$. The planar correlation functions then satisfy (1.66). There are two of them of importance for us here:

- the *disc function* $W_{1,0}(x) = \lim_{N\to\infty} \frac{1}{N} W_1(x)$, which is the generating function for rooted planar maps with one marked face, and
- the *cylinder function* $W_{2,0}(x_1, x_2) = \lim_{N\to\infty} W_2(x_1, x_2)$, which is the generating function for planar maps with two boundaries.

Disc function – We already computed the disc amplitude in the previous section, it is given by

$$W_{1,0}(x) = \frac{1}{2t}\left(x - \sqrt{x^2 - 4t}\right), \tag{1.84}$$

it has a cut along $\Gamma = [-2\sqrt{t}, 2\sqrt{t}]$.

Cylinder function in the Gaussian case – By setting $n = 2$ in (1.66), one gets

$$(V'(x_1) - 2W_{1,0}(x_1))W_{2,0}(x_1, x_2) = \frac{\partial}{\partial x_2}\frac{W_{1,0}(x_1) - W_{1,0}(x_2)}{x_1 - x_2} + P_{2,0}(x_1, x_2). \tag{1.85}$$

We now restrict ourselves to the Gaussian case again, then by denoting the singular part $\sigma(x) = \sqrt{x^2 - 4t}$, we have

$$\frac{\partial W_{1,0}(x)}{\partial x} = -\frac{W_{1,0}(x)}{\sigma(x)}, \tag{1.86}$$

which leads to the following expression for the Gaussian cylinder function

$$W_{2,0}(x_1, x_2) = \frac{x_1 x_2 - \sigma(x_1)\sigma(x_2) - 4t}{2(x_1 - x_2)^2\sigma(x_1)\sigma(x_2)}. \tag{1.87}$$

1.5.2 Topological recursion formula

Our correlation functions are singular on the cut $\Gamma = [-2\sqrt{t}, 2\sqrt{t}]$ except for $(n, g) \in \{(1, 0), (2, 0)\}$, we thus perform a Zhukowsky tranformation (1.82)

$$x(z) = \sqrt{t}(z + z^{-1}), \tag{1.88}$$

that allows us to turn our correlation functions into differential forms by defining

$$\omega_{n,g}(z_1, \dots, z_n) := W_{n,g}(x_1, \dots, x_n)dx_1 \dots dx_n + \delta_{(n,g),(2,0)}\frac{dx_1 dx_2}{(x_1 - x_2)^2}, \tag{1.89}$$

where for all $i \in [\![1, n]\!]$,

$$x_i = x(z_i) \quad \text{and} \quad dx_i = x'(z_i)dz_i. \tag{1.90}$$

They have the following properties [60] that we do not prove here:

1. $\omega_{n,g}(z_1, \dots, z_n)$ is a meromorphic form in every variable,
2. $\omega_{n,g}$ is ι-antisymmetric, *i.e.* for all $i \in [\![1, n]\!]$, $(n, g) \neq (1, 0)$, $\omega_{n,g}(z_1, \dots, z_n) = -\omega_{n,g}(z_1, \dots, \iota(z_i), \dots, z_n)$, and
3. for $2g - 2 + n > 0$ and $(n, g) \notin \{(1, 0), (2, 0)\}$, $\omega_{n,g}(z_1, \dots, z_n)$ has poles in z_1 only at $z_1 \to \pm 1$.

One can then compute the disc and cylinder forms of the GUE:

$$\begin{aligned}\omega_{1,0}(z) &= \frac{1-z^{-2}}{z}dz \\ \omega_{2,0}(z_1,z_2) &= \frac{dz_1 dz_2}{(z_1-z_2)^2}.\end{aligned} \tag{1.91}$$

Finally, we will need the following notations. For a differential form φ denote

$$\Delta\varphi = \varphi - \iota^*\varphi, \tag{1.92}$$

where the * denotes the pullback on differential forms, also write

$$G(z,z_1) = \int_{\iota(z)}^{z} \omega_{2,0}(\cdot, z_1), \tag{1.93}$$

and define the recursion kernel

$$K(z,z_1) = \frac{\Delta G(z,z_1)}{2\Delta\omega_{1,0}(z)}. \tag{1.94}$$

This allows us to write the following theorem:

Theorem 1.5.2 (Eynard '04)**.** *For all* $2g-2+n>0$*, one has*

$$\begin{aligned}\omega_{n,g}(z_1,\dots,z_n) = \operatorname*{Res}_{z\to\pm1} K(z,z_1)\Bigg[&\omega_{n+1,g-1}(z,\iota(z),z_2,\dots,z_n) \\ &+ \sideset{}{'}\sum_{\substack{I_1\sqcup I_2=[\![2,n]\!] \\ g_1+g_2=g}} \omega_{|I_1|+1,g_1}(z,z_{I_1})\,\omega_{|I_2|+1,g_2}(\iota(z),z_{I_2})\Bigg],\end{aligned} \tag{1.95}$$

where the primed summation $\sum'$ is to be understood as the exclusion of the terms $(I_1,g_1)=(\emptyset,0)$ and $(I_1,g_1)=([\![2,n]\!],g)$. This formula expresses $\omega_{n,g}$ in terms of invariants with larger Euler characteristic $\chi = 2-2g-n$, hence the name topological recursion. It stops after $|\chi|$ steps, when it reaches $\chi=0$ and hence $\omega_{2,0}$.

Chapter 2

Random tensor models

Tensor models are a natural generalization in dimension $d > 2$ of matrix models. Introduced almost thirty years ago, their aim was to reproduce the successes of matrix models in providing a theory of random geometries, especially in ways of quantizing QG. The basic idea is that a certain integral over a space of rank-d tensors is the partition function of a theory of d-dimensional triangulations.
Let V be a vector space over a field $\mathbb{K}$ of dimension $\dim_{\mathbb{K}} V = N$. A tensor T of type (p, q) can be seen as an element of $V^{\otimes p} \otimes V^{*\otimes q}$, where $V^{\otimes p}$ is to be understood as the tensor product of p copies of V, and V^* is the corresponding dual space. If we equip V and V^* with the canonically associated bases $\{e_a\}$ and $\{\varepsilon^b\}$ ($\varepsilon^b(e_a) = \delta^b_a$) then the components of T are defined as

$$T = T^{a_1 \dots a_p}{}_{b_1 \dots b_q}\, e_{a_1} \otimes \cdots \otimes e_{a_p} \otimes \varepsilon^{b_1} \otimes \cdots \otimes \varepsilon^{b_q}, \tag{2.1}$$

where repeated indices are summed. In what follows we will often identify the tensor with its components in a given basis (that will not be specified). Also we do not distinguish between covariant (down) and contravariant (up) indices. Hence, for us a tensor of rank $d \in \mathbb{N}_{>2}$ will be written $T_{a_1 \dots a_d}$, where of course $a_c \in [\![1, N]\!]$ for all $c \in [\![1, d]\!]$.
Consider now such a tensor and its complex conjugate, treated as independent variables. We further assume that they obey no symmetry relations upon permutation of their indices. In general, the main object of interest in any quantum theory is the expectation value of a certain observable O, taken to be a function $O(T, \overline{T})$ of the components of the tensor, we define it as

$$\langle O(T, \overline{T}) \rangle = \frac{1}{Z(N)} \int_{V^{\otimes d}} d\mu_C(T, \overline{T}) O(T, \overline{T}) e^{-N^{d-1} V(T, \overline{T})}, \tag{2.2}$$

where $Z(N) = \int d\mu_C(T,\overline{T})e^{-N^{d-1}V(T,\overline{T})}$ is the partition function and $d\mu_C(T,\overline{T})$ the Gaussian measure of covariance C defined as

$$d\mu_C(T,\overline{T}) = \det C^{-1} e^{-N^{d-1}\sum_{a_p,b_p} T_{a_1\dots a_d}(C^{-1})^{a_1\dots a_d}_{b_1\dots b_d}\overline{T}_{b_1\dots b_d}} \prod_{a_p} dT_{a_1\dots a_d} d\overline{T}_{a_1\dots a_d}. \tag{2.3}$$

Standard (invariant) tensor models are built from the standard Gaussian measure with identity covariance, reducing the Gaussian measure to

$$d\mu_0(T,\overline{T}) := e^{-N^{d-1}T\cdot\overline{T}} \prod_{a_p} dT_{a_1\dots a_d} d\overline{T}_{a_1\dots a_d}. \tag{2.4}$$

Non invariant tensor models, such as those arising from group field theories require more involved covariances [89], such as projectors over gauge invariant states or inverse Laplacians. Finally, a typical interaction term $V(T,\overline{T})$ may be written in the form $V(T,\overline{T}) = \sum_{i\in I} N^{\delta_i}\lambda_i B_i(T,\overline{T})$, for some finite set I, scaling coefficients $\{\delta_i\}$ and coupling constants $\{\lambda_i\}$. The B_is are polynomials in the tensor entries and will be introduced in the next section. We will get back to them in many more details in Part III. An example of such an interaction term is given by the so-called multi-orientable model [90] where $V_{\text{mo}}(T,\overline{T}) = \frac{\lambda}{4}\sum_{i,j,k,l,m,n} T_{ijk}\overline{T}_{imn}T_{njl}\overline{T}_{lmk}$.

2.1 Uncolored tensor models

There is a natural transformation of a complex covariant rank $d > 2$ tensor T under the action of the tensor product of representations of $\bigotimes_{c=1}^{d} U(N_c)$ where each factor acts on a tensor index independently. The complex conjugate of T is a contravariant tensor of the same rank and denoted $\overline{T}$. Here T is considered an object in $W = \bigotimes_{c=1}^{d} V_c$, where V_c for all $c \in [\![1,d]\!]$ is some Hermitian space of dimension $\dim_{\mathbb{C}} V_c = N_c$.
This defines *uncolored* tensors[1], they transform in a given basis as

$$\begin{aligned} T^U_{a_1\dots a_d} &= \sum_{b_1,\dots,b_d} U^{(1)}_{a_1b_1}\dots U^{(d)}_{a_db_d} T_{b_1\dots b_d} \\ \overline{T}^U_{a_1\dots a_d} &= \sum_{b_1,\dots,b_d} \overline{U}^{(1)}_{a_1b_1}\dots \overline{U}^{(d)}_{a_db_d} \overline{T}_{b_1\dots b_d}, \end{aligned} \tag{2.5}$$

[1] The name is historical. The first models [11] to be proven to admit a large N expansion required $d+1$ pairs of *colored* conjugate tensors T^c, $\bar{T}^c$. It was then shown that integrating out all tensors but one in the initial colored model leads to an action for a single uncolored tensor, hence effectively uncoloring the colored model [28].

where for all $c \in [\![1, d]\!]$ the $U^{(c)}s$ are in fact the matrices of the group representations $\rho_c \in \mathrm{Hom}(U(N_c), GL(V_c))$. One can of course construct other such models by taking a different Lie group in lieu of each $U(N_c)$, *e.g.* in the multi-orientable model [90], W carries a representation of $U(N_1) \times O(N_2) \times U(N_3)$, while the main focus of Part II will be put on real tensors transforming under tensor products of representations of the orthogonal group $O(N)$.

We are now interested in constructing polynomial invariants of the components of the tensors, called - by analogy with matrix models - trace invariants. Those can be obtained by contracting, in all ways possible, pairs of covariant and contravariant tensors. As it turns out, these contractions are in bijection with k-edge-colored bipartite graphs called bubbles that we now introduce.

Definition 2.1.1 (**k-bubble**)**.** *A bipartite closed k-colored graph or k-bubble is a graph $B = (V(B), E(B))$ that is a collection $V(B)$ of vertices of fixed valence k and set $E(B)$ of edges, such that*

- *$V(B)$ can be partitioned into two disjoint sets $V^\bullet$ and V° of equal size, such that each edge e may only connect a vertex $v^\bullet \in V^\bullet$ called black to a vertex $v^\circ \in V^\circ$ called white;*
- *the graph has a k-line coloring τ, that is an assignment of a color to each edge, $\tau : E(B) \to [\![1, k]\!]$, such that the colors of the k edges incident to any vertex are all different. Note that $\tau^{-1}(c)$ is the subset of lines of color c.*

The trace invariant associated to the k-bubble B is denoted

$$\mathrm{Tr}_B(T, \overline{T}) \text{ or } B(T, \overline{T}) := \sum_{i,j} \delta^B_{ij} \prod_{v,v' \in V(B)} T_{i_v} \overline{T}_{j_{v'}}\,, \qquad \delta^B_{ij} = \prod_{c=1}^{k} \prod_{e^c \in \tau^{-1}(c)} \delta_{i^c_{v^\bullet(e^c)}\, j^c_{v^\circ(e^c)}} \quad (2.6)$$

Following the Italian school of Pezzana, we know that every regular bipartite $(d+1)$-edge-colored graph $\mathcal{G}$ represents a d-dimensional colored triangulation of an orientable pseudo-manifold. Besides, from our previous definition, a k-bubble of $\mathcal{G}$ is a maximally connected subgraph of $\mathcal{G}$ comprising only edges with k fixed colors. We then get the colored triangulation by duality, where every k-bubble B_k represents a $(d-k)$-subsimplex σ_{d-k} and the $(k+1)$-bubbles containing B_k represent the faces of σ_{d-k}. In particular, $(d-2)$-subsimplices correspond to cycles in the graph alternating two different colors. In general we have the following correspondences:

dual triangulation		**colored graph**
d-simplex	$\leftrightarrow$	vertex
$(d-1)$-simplex	$\leftrightarrow$	edge
$(d-2)$-simplex	$\leftrightarrow$	bicolored cycle
$(d-k)$-simplex	$\leftrightarrow$	k-bubble

A colored d-simplex is a simplex, the boundary $(d-1)$-subsimplices of which have a color in $[\![0, d]\!]$ such that each color appears exactly once. Through the colors, we can thus define a canonical attaching rule between colored simplices as follows.
Notice that in a colored d-simplex, every $(d-2)$-subsimplex is shared by exactly two $(d-1)$-subsimplices, say of colors c, c', and can thus be labeled by the pair of colors $\{c, c'\}$. Similarly, a $(d-k)$-subsimplex is identified by a k-bubble of colors in $[\![1, d]\!]$. The attaching rule is then the following. Take two different $(d-1)$-subsimplices σ_c and σ'_c of the same color $c \in [\![0, d]\!]$ living in two different d-simplices and attach them in the only way that identifies all the subsimplices of σ_c and σ'_c which have the same color labels. In other words, it is the only attaching map which preserves all induced colorings of their k-subsimplices for $k \in [\![0, d-1]\!]$. Let us give two lower-dimensional examples for the sake of clarity.
In two dimensions, a colored triangle has edges of colors 0, 1, 2, and vertices ($(d-2)$-simplices) labeled by the pairs of colors $\{0, 1\}$, $\{1, 2\}$, $\{0, 2\}$ where the vertex with colors $\{c, c'\}$ is the one shared by the edges of colors c and c'. Two triangles can be glued along an edge of say color 0 by identifying the vertices of colors $\{0, 1\}$ on both triangles, and similarly identifying the vertices of colors $\{0, 2\}$, see Figure 2.1.

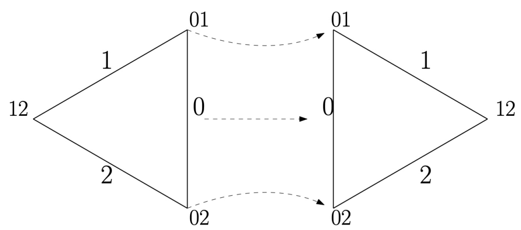


FIGURE 2.1 – Unique gluing of triangles of colors $\{0, 1, 2\}$ which respects all subcolorings.

In three dimensions, a colored tetrahedron has four triangles (faces) of colors 0, 1, 2, 3, six edges ($(d-2)$-simplices) colored $\{c, c'\}_{0 \leq c < c' \leq 3}$ where $\{c, c'\}$ labels the edge shared by the triangles of colors c and c', and four vertices ($(d-3)$-simplices) with labels $\{0, 1, 2\}$, $\{0, 1, 3\}$, $\{0, 2, 3\}$, $\{1, 2, 3\}$ where the vertex with label $\{c, c', c''\}$ is the one shared by the three triangles of colors c, c' and c''. Two tetrahedra can be glued along a triangle of color say 0 by identifying pairwise the edges which have 0 in their labels, *i.e.* both edges of colors $\{0, c\}$ for $c \neq 0$ in the two tetrahedra are identified, and further identifying pairwise the vertices which have 0 in their labels, *i.e.* both vertices with colors $\{0, c, c'\}$ in the two tetrahedra are identified for all $1 \leq c < c' \leq 3$, as shown in Figure 2.2.

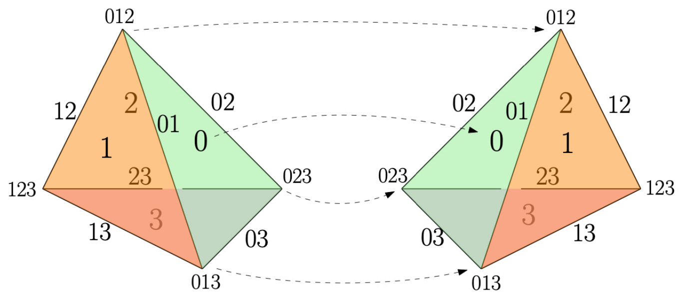


FIGURE 2.2 – Unique gluing of tetrahedra of colors $\{0, 1, 2, 3\}$ which respects all subcolorings.

2.2 Gurau degree and melons

Let us now introduce some definitions.

Definition 2.2.1 (Jacket). *Let $\mathcal{G}$ be a $(d+1)$-colored graph and τ be a cycle on $[\![0, d]\!]$. A colored jacket J of $\mathcal{G}$ is an edge-colored ribbon graph having as 1-skeleton the graph $\mathcal{G}$ and with faces made of graph cylces of colors $(\tau^q(0), \tau^{q+1}(0))$, for $q \in [\![0, d]\!]$, modulo the orientation of the cycle.*

There are $d!/2$ such jackets for every $(d+1)$-colored graph, and being ribbon graphs, they are completely classified by their genus g_J.

Definition 2.2.2 (Gurau degree [5]). *The degree $\omega(\mathcal{G})$ of a colored graph $\mathcal{G}$ is the sum of the genera of its jackets*

$$\omega(\mathcal{G}) = \sum_J g_J \geq 0. \tag{2.7}$$

The main characteristic of the degree is that it allows to count the number of faces of a graph. Indeed, for a $(d+1)$-colored graph with $2p$ vertices, the total number of faces is given by

$$F = \frac{d(d-1)}{2}p + d - \frac{2}{(d-1)!}\omega(\mathcal{G}). \tag{2.8}$$

The (reduced) degree can also be defined for the dual d-dimensional triangulation $\mathcal{T}$ and reads

$$\omega(\mathcal{T}) = d + \frac{d(d-1)}{4} n_d - n_{d-2}. \tag{2.9}$$

Notice that for $d = 2$, the degree corresponds to (twice) the genus of the surface. In higher dimensions, it provides a generalization of the latter. It is *not* a topological invariant, but it combines topological and combinatorial information about the graph.
There is a single quadratic trace invariant up to a factor, it is associated to the dipole, a bubble made of two vertices and connected by d edges. Its trace invariant reads

$$\mathrm{Tr}_D(T, \overline{T}) = \sum_{a_p} T_{a_1 \dots a_d} \overline{T}_{a_1 \dots a_d} = T \cdot \overline{T}, \tag{2.10}$$

and is used to construct the normalized Gaussian measure (2.4). Let now I be a finite set, $\{B_i\}_{i \in I}$ a set of bubbles and $\{t_i\}$ their coupling constants. This allows us to define the most general $(d+1)$-dimensional generic tensor model *via* the following partition function:

$$Z(\{t_i\}, N) = \int d\mu_0(T, \overline{T}) \exp - N^d \sum_{i \in I} N^{-\frac{2}{(d-1)!}\omega(B_i)} t_i B_i(T, \overline{T}). \tag{2.11}$$

Remark 2.2.3. *A large N expansion exists if and only if ω is bounded from below.*

Let us now focus on a specific family of graphs called *melonic*, defined by inserting recursively dipoles on the fundamental melon, himself a d-dipole.

Definition 2.2.4 (Melon). *The melonic graphs or melons are the graphs generated by melonic insertions on the elementary melon.*

This process is illustrated in Figures 2.3 and 2.4 hereunder.

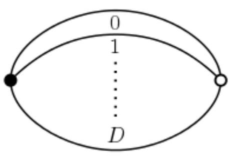


FIGURE 2.3 – The D-dipole or elementary melon.

Theorem 2.2.5 (Bonzom, Gurau, Riello, Rivasseau, 2011 [12])**.** *The degree of a graph vanishes if and only if the graph is a melon.*

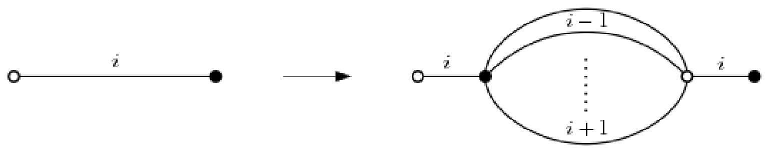


FIGURE 2.4 – A melonic insertion on an edge of color i.

Proof. Let us give the proof for the necessary condition, the reverse being trivial. We start by computing the degree for the elementary melon, from which we can generate every melonic graph by repeated melonic insertions. For the elementary melon M, $F = \frac{d(d+1)}{2}$ and $p = 1$, hence from (2.8) $\omega(M) = 0$.

Lemma 2.2.6. *Melonic insertion leaves the degree unchanged.*

Proof. By melonic insertion,

$$\begin{cases} p & \to \quad p+1 \\ F & \to \quad F + \frac{d(d-1)}{2}, \end{cases} \tag{2.12}$$

such that again by (2.8), the degree remains the same. □

Which completes the proof. □

Remark 2.2.7. *Melonic graphs form an infinite family of diagrams with vanishing degree.*

From (2.7) and Theorem 2.2.5 one can then conclude that the generic tensor model (2.11) has a well defined large N expansion dominated by melonic interactions Feynman graphs.

2.3 Counting $U(N)$ invariants

In this section we restrict our discussion to $d = 3$, generalization to arbitrary dimension is straightforward.

2.3.1 Symmetric group enumeration

Recall that invariants are generated by all the possible ways to contract pairs of covariant and contravariant tensors. Diagrammatically, one can view these contractions as all possible pairings between, say, n white and n black vertices, as depicted in Figure 2.5[2].

[2] In Figure 2.5 the color labels have been made explicit to emphasize the fact that each white vertex connects to a black one through edges of the same color. This is the most general 3-bubble with invariant given by Equation (2.6).

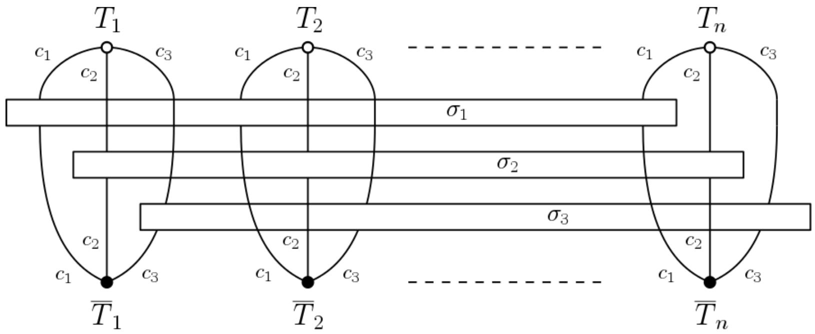


FIGURE 2.5 – Diagrammatic contraction of pairs of rank-3 tensors defining the triple of permutations $(\sigma_1, \sigma_2, \sigma_3)$.

The different connections can be parametrized by a permutation $\sigma \in S_n$. Different permutations can give the same graph if they are related by an equation of the form $\sigma' = \gamma_1\sigma\gamma_2$, where γ_1, γ_2 live in subgroups H_1, H_2 of S_n related to the symmetries of the white and black vertices respectively. This allows to count invariants as points in double cosets of permutation groups. The enumeration of possible graphs is then recast into counting triples of permutations

$$(\sigma_1, \sigma_2, \sigma_3) \in (S_n \times S_n \times S_n), \tag{2.13}$$

under the equivalence relation

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1\sigma_1\gamma_2, \gamma_1\sigma_2\gamma_2, \gamma_1\sigma_3\gamma_2), \tag{2.14}$$

where γ_1, $\gamma_2 \in S_n$. We are thus counting points in the double coset

$$\text{Diag}(S_n)\backslash(S_n \times S_n \times S_n)/\text{Diag}(S_n). \tag{2.15}$$

We denote the number of points in this double coset as $Z_3(n)$. For more general subgroups $H_1 \leq G$, $H_2 \leq G$, the cardinality of such a coset is given by (see below)

$$|H_1\backslash G/H_2| = \frac{1}{|H_1||H_2|}\sum_C Z_C^{H_1\to G} Z_C^{H_2\to G} z_C, \tag{2.16}$$

where the sum runs over conjugacy classes C of G, $Z_C^{H\to G}$ is the number of elements of H in C and z_C stands for the size of the centralizer of C. The conjugacy classes of S_n^3 are entirely determined by a triple of partitions of n, namely (p_1, p_2, p_3). This correspondence holds because each conjugacy class is determined by a cycle structure.

The diagonal subgroup now produces conjugacy classes of the form (p,p,p), and applying (2.16), one gets

$$Z_3(n) = \frac{1}{(n!)^2} \sum_{p \vdash n} \left(\frac{n!}{z_p}\right)^2 z_p^3 = \sum_{p\vdash n} \prod_{i=1}^{n} i^{p_i} p_i! \,, \tag{2.17}$$

where the sum over $p = (p_\ell)_\ell$ is performed over all partitions of $n = \sum_i ip_i$ and

$$z_p = \prod_i i^{p_i} p_i! \tag{2.18}$$

denotes the number of elements in S_n commuting with any permutation of cycle type p. This sequence can be generated [29] and shows

$$1, 4, 11, 43, 161, 901, 5579, 43206, 378360, 3742738, \dots \tag{2.19}$$

which corresponds to the series A110143 of the OEIS website.
Note that the number $Z_3(n)$ counts both connected and disconnected invariants. Figure 2.6 on the next page shows the graphical representation of the connected invariants up to $n = 3$. The first order terms are (the capital letters refer to Figure 2.6):

- $Z_3(1) = 1$ consists in the single connected dipole A,
- $Z_3(2) = 4$ consists in three connected invariants, B_1, B_2 and B_3, one of which being given by

$$\sum_{i_a, i'_a} T_{i_{c_1} i_{c_2} i_{c_3}} \overline{T}_{i'_{c_1} i_{c_2} i_{c_3}} T_{i'_{c_1} i'_{c_2} i'_{c_3}} \overline{T}_{i_{c_1} i'_{c_2} i'_{c_3}} \tag{2.20}$$

  while the two others are obtained by a color permutation, plus one disconnected invariant of the form

$$\left(\sum_{i_a} T_{i_{c_1} i_{c_2} i_{c_3}} \overline{T}_{i_{c_1} i_{c_2} i_{c_3}} \right)^2 , \tag{2.21}$$

  which is nothing but twice the dipole term (A, A),
- $Z_3(3) = 11$ consists in seven connected invariants, *viz.* C_1, C_2, C_3, D_1, D_2, D_3 and E, plus four disconnected ones given by the combinations (A, A, A), (A, B_1), (A, B_2) and (A, B_3).

To get the number of connected invariants $Z_3^c(n)$ one can use the plethystic logarithm function (see Chapter 4) and finds

$$1, 3, 7, 26, 97, 624, 4163, 34470, 314493, 3202839, \dots \tag{2.22}$$

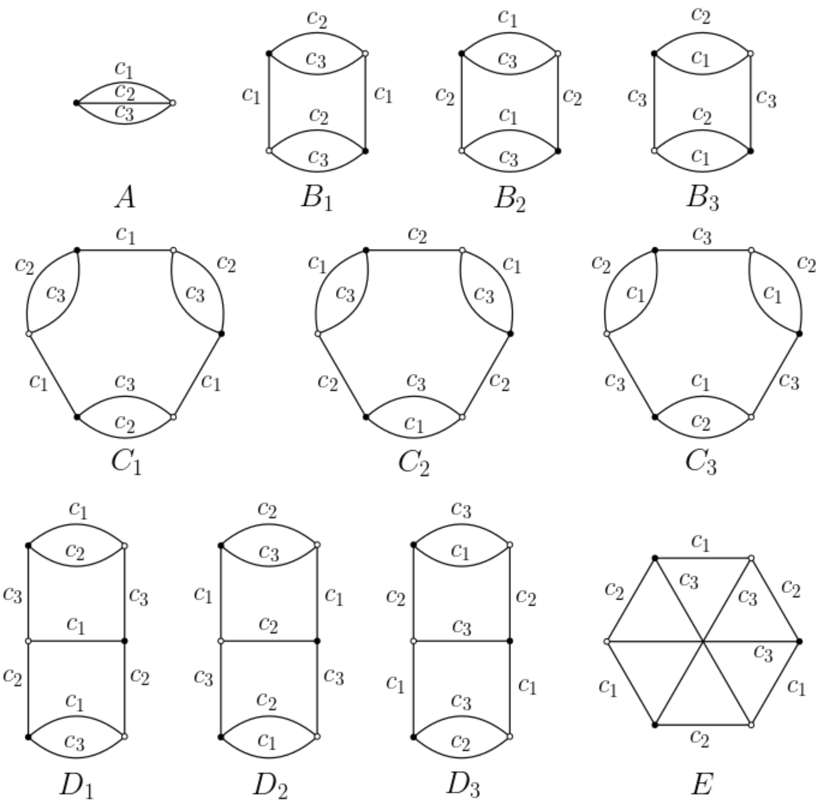


FIGURE 2.6 – Colored graphs associated to connected contractions of pairs of rank-3 tensors defining the triple of permutations $(\sigma_1, \sigma_2, \sigma_3)$.

Note that Equation (2.17) easily generalizes to arbitrary dimension d:

$$Z_d(n) = \sum_{p \vdash n} z_p^{d-2}. \tag{2.23}$$

Let us now revisit this counting once more in the Topological Field Theory framework. Equation (2.16) is in fact a consequence of Burnside's lemma for counting the number of orbits of a group action, we state it without proof:

Proposition 2.3.1. ***(Burnside's lemma)***
Consider a finite set X and a finite group H acting on it by multiplication. The number of orbits of the H-action on X, denoted $|X/H|$ is given by the average number of fixed points of the group action. More explicitly,

$$|X/H| = \frac{1}{|H|} \sum_{h \in H} X^h, \tag{2.24}$$

where $X^h = \{x \in X \,|\, h \cdot x = x\}$ is the set of fixed points of $h \in H$.

Consider thus the double coset as the orbit of the $H_1 \times H_2$ - action on G. The fixed-point counting formula for the number of orbits is given by

$$|H_1 \backslash G / H_2| = \frac{1}{|H_1||H_2|} \sum_{h_1 \in H_1} \sum_{h_2 \in H_2} \sum_{g \in G} \delta(h_1 g h_2 g^{-1}), \tag{2.25}$$

where δ is the delta function over the group G, equal to 1 if the argument is the identity element and 0 otherwise. From this definition of the delta function, one sees that h_1 and h_2 have to be in the same conjugacy class of G. Next, organise the sums according to the conjugacy classes C of G and note the number of elements in the conjugacy class C from H_1 (resp. H_2) $Z_C^{H_1 \to G}$ (resp. $Z_C^{H_2 \to G}$). So the counting picks up a $Z_C^{H_1 \to G} Z_C^{H_2 \to G}$ term from the h_1, h_2 sums and for each such pair, there are z_C possible gs, hence (2.16). Particularizing to our double coset (2.15), this becomes - using (A.5) -,

$$\begin{aligned} Z_3(n) &= \frac{1}{(n!)^2} \sum_{\gamma_i \in S_n} \sum_{\sigma_i \in S_n} \delta(\gamma_1 \sigma_1 \gamma_2 \sigma_1^{-1}) \delta(\gamma_1 \sigma_2 \gamma_2 \sigma_2^{-1}) \delta(\gamma_1 \sigma_3 \gamma_2 \sigma_3^{-1}) \\ &= \frac{1}{(n!)^2} \sum_{\gamma_i \in S_n} \sum_{R_l \vdash n} \chi^{R_1}(\gamma_1) \chi^{R_1}(\gamma_2) \chi^{R_2}(\gamma_1) \chi^{R_2}(\gamma_2) \chi^{R_3}(\gamma_1) \chi^{R_3}(\gamma_2) \\ &= \sum_{R_l \vdash n} \mathbf{C}(R_1, R_2, R_3)^2, \end{aligned} \tag{2.26}$$

where the symbol

$$\mathbf{C}(R_1, R_2, R_3) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^{R_1}(\sigma) \chi^{R_2}(\sigma) \chi^{R_3}(\sigma), \tag{2.27}$$

stands for the Kronecker coefficient that can be defined as the multiplicity of the irreducible representation R_3 in the tensor product of the irreps R_1 and R_2, or equivalently as the multiplicity of the one-dimensional representation in $R_1 \otimes R_2 \otimes R_3$. We refer the reader to Appendix A for a precise definition of representations and characters of the symmetric group. Hence, counting observables of tensor models of rank 3 coincides with a sum of squares of Kronecker coefficients. As it turns out, this sum is also the dimension of an algebra:

$$\dim \mathcal{K}_3(n) = \sum_{R_l \vdash n} \mathbf{C}(R_1, R_2, R_3)^2. \tag{2.28}$$

2.3.2 Double coset algebra

In what follows we only state the main results pertaining to this algebra, a detailed account thereof can be found in [30], while the orthogonal case will be addressed in Chapter 5. We consider $\mathbb{C}[S_n] := \text{Span}_{\mathbb{C}}\{\sigma,\ \sigma \in S_n\}$ the group algebra over S_n.

$\mathcal{K}_3(n)$ **as a double coset algebra in** $\mathbb{C}[S_n]^{\otimes 3}$ – Consider elements $\sigma_1 \otimes \sigma_2 \otimes \sigma_3 \in \mathbb{C}[S_n]^{\otimes 3}$ and the left and right diagonal action of $\text{Diag}(\mathbb{C}[S_n])$ on the triple as:

$$\sigma_1 \otimes \sigma_2 \otimes \sigma_3 \quad \to \quad \sum_{\gamma_i \in S_n} \gamma_1 \sigma_1 \gamma_2 \otimes \gamma_1 \sigma_2 \gamma_2 \otimes \gamma_1 \sigma_3 \gamma_2, \tag{2.29}$$

$\mathcal{K}_3(n)$ is the vector subspace of $\mathbb{C}[S_n]^{\otimes 3}$ which is left invariant by this group action:

$$\mathcal{K}_3(n) = \text{Span}_{\mathbb{C}} \left\{ \sum_{\gamma_i \in S_n} \gamma_1 \sigma_1 \gamma_2 \otimes \gamma_1 \sigma_2 \gamma_2 \otimes \gamma_1 \sigma_3 \gamma_2, \quad \sigma_1, \sigma_2, \sigma_3 \in S_n \right\}. \tag{2.30}$$

The equivalence classes defining $\mathcal{K}_3(n)$ are the double cosets (2.15). Noticing that $id \otimes id \otimes id$ is the unit of $\mathcal{K}_3(n)$, one easily verifies the following proposition.

Proposition 2.3.2. *$\mathcal{K}_3(n)$ is an associative unital subalgebra of $\mathbb{C}[S_n]^{\otimes 3}$.*

A Fourier basis of invariants – The Fourier transform of the basis (2.29) gives another basis of invariants labeled by the tuple $(R_1, R_2, R_3, \tau_1, \tau_2)$ and given by

$$Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} = \kappa \sum_{\sigma_l \in S_n} \sum_{i_a, j_a} C^{R_1,R_2;R_3,\tau_1}_{i_1,i_2;i_3} C^{R_1,R_2;R_3,\tau_2}_{j_1,j_2;j_3} D^{R_1}_{i_1 j_1}(\sigma_1) D^{R_2}_{i_2 j_2}(\sigma_2) D^{R_3}_{i_3 j_3}(\sigma_3) \sigma_1 \otimes \sigma_2 \otimes \sigma_3, \tag{2.31}$$

with $\kappa = \frac{d(R_1)d(R_2)d(R_3)}{(n!)^3}$ and $i_a, j_a \in [\![1, d(R_a)]\!]$. Besides, $C^{R_1,R_2;R_3,\tau_1}_{i_1,i_2;i_3}$ are Clebsch-Gordan coefficients involved in the tensor product representations of S_n with multiplicities $\tau_1, \tau_2 \in [\![1, \mathbf{C}(R_1, R_2, R_3)]\!]$. These basis elements are invariant under left and right diagonal action and multiply like matrices

$$Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} Q^{R'_1,R'_2,R'_3}_{\tau_2,\tau_3} = \delta_{\vec{R},\vec{R}'} Q^{R_1,R_2,R_3}_{\tau_1,\tau_3}, \tag{2.32}$$

where we denoted $\vec{R} = (R_1, R_2, R_3)$.

Orthogonality of the Q-basis – Consider the pairing $\boldsymbol{\delta}_3 : \mathbb{C}[S_n]^{\otimes 3} \times \mathbb{C}[S_n]^{\otimes 3} \to \mathbb{C}$

$$\boldsymbol{\delta}_3(\sigma_1 \otimes \sigma_2 \otimes \sigma_3,\, \sigma'_1 \otimes \sigma'_2 \otimes \sigma'_3) = \delta(\sigma_1 \sigma_1'^{-1})\delta(\sigma_2 \sigma_2'^{-1})\delta(\sigma_3 \sigma_3'^{-1}), \tag{2.33}$$

which extends linearly to the Q-basis:

$$\boldsymbol{\delta}_3(Q^{R_1,R_2,R_3}_{\tau_1,\tau_2},\, Q^{R'_1,R'_2,R'_3}_{\tau'_1,\tau'_2}) = \kappa d(R_3)^2 \delta_{\vec{R},\vec{R}'} \delta_{\tau_1,\tau'_1} \delta_{\tau_2,\tau'_2}. \tag{2.34}$$

Hence the Q-basis is orthogonal and since the pairing is bilinear non-degenerate, the following theorem holds.

Theorem 2.3.3. *$\mathcal{K}_3(n)$ is an associative unital semi-simple*[3] *algebra.*

[3] A (finite-dimensional) algebra is said to be *semi-simple* if it can be expressed as a Cartesian product of simple algebras, *i.e.* algebras having no non-trivial two-sided ideals.

Chapter 3

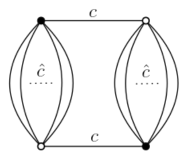
The quartic melonic tensor model

In this chapter we will cite results pertaining to the so-called quartic melonic model. This model is of significance as it is the first tensor model on which the topological recursion formalism has successfully been applied, yielding a blobbed topological recursion [66].
As before, we denote by $T_{a_1\dots a_d}$ the components of the tensor T of rank $d > 2$ and size N and $\overline{T}_{a_1\dots a_d}$ those of $\overline{T}$. Let $E_c \simeq \mathbb{C}^N$ be the vector space of color c. Then T lives in $\bigotimes_{c=1}^d E_c$ and transforms as

$$T \to \bigotimes_{c=1}^{d} U^{(c)} T. \tag{3.1}$$

For all $c \in [\![1, d]\!]$ and $\hat{c}$ its complement, we denote the quartic melonic bubble invariant of color c as

$$B_c(T,\overline{T}) = \sum_{a_p,\, b_p} T_{a_1\dots a_c \dots a_d} \overline{T}_{a_1\dots a_{c-1} b_c a_{c+1}\dots a_d} T_{b_1\dots b_c \dots b_d} \overline{T}_{b_1\dots b_{c-1} a_c b_{c+1}\dots b_d} = \tag{3.2}$$



We furthermore define $H_c(T,\overline{T})$ as the matrix obtained by contracting all the elements of T with those of $\overline{T}$ except for the ones in position c. It can be written out in components as

$$H_c(T,\overline{T})_{ab} = \sum_{a_1,\dots,a_{c-1},a_{c+1},\dots,a_d} T_{a_1\dots a_{c-1} a a_{c+1}\dots a_d} \overline{T}_{a_1\dots a_{c-1} b a_{c+1}\dots a_d}. \tag{3.3}$$

Our bubble invariant then rewrites $B_c(T,\overline{T}) = \mathrm{tr}_{E_c} H_c H_c^\dagger$ and the partition function of the model reads

$$Z_{\text{Tensor}}(N, \{g_c\}) = \int_{(\mathbb{C}^N)^{\otimes d}} d\mu_0(T,\overline{T}) \exp -N^{d-1}\frac{1}{2}\sum_{c=1}^{d} g_c^2 B_c(T,\overline{T}), \tag{3.4}$$

where the g_cs are the bubbles' coupling constants. We now introduce an intermediate Hermitian $N \times N$ matrix field X_c in order to split the interaction term; this is known in the litterature as the Hubbard-Stratonovich transformation and expresses as

$$\exp -N^{d-1}\frac{1}{2}g_c^2 B_c(T,\overline{T}) = \int dX_c\ \exp -N^{d-1}\left(\frac{1}{2}\mathrm{tr}_{E_c} X_c^2 - ig_c \mathrm{tr}_{E_c}\left(H_c(T,\overline{T})X_c\right)\right). \tag{3.5}$$

The integral over T and $\overline{T}$ is now a Gaussian integral and can be computed explicitly. Introducing the notation $\tilde{X}_c = \mathbb{1}^{\otimes(c-1)} \otimes X_c \otimes \mathbb{1}^{\otimes(d-c)}$, we rewrite the partition function using the previous representation as

$$Z_{\text{Tensor}}(N,\{g_c\}) = \int \prod_{c=1}^{d} dX_c \exp\Big[-\frac{1}{2}\sum_{c=1}^{d}\mathrm{tr}_{\otimes_c E_c}\tilde{X}_c^2 - \mathrm{tr}_{\otimes_c E_c}\ln\Big(\mathbb{1}^{\otimes d} + i\sum_{c=1}^{d} g_c \tilde{X}_c\Big)\Big]. \tag{3.6}$$

If $P(T,\overline{T})$ is a polynomial in the tensor entries and f a polynomial function that takes as argument an $N \times N$ matrix, we have the corresponding expectation values:

$$\begin{aligned}\langle P(T,\overline{T})\rangle_T &= \frac{1}{Z_{\text{Tensor}}}\int d\mu_0(T,\overline{T})\ P(T,\overline{T})\ e^{-N^{d-1}\frac{1}{2}\sum_{c=1}^{d} g_c^2 B_c(T,\overline{T})},\\ \langle f(X_c)\rangle_M &= \frac{1}{Z_{\text{Tensor}}}\int \prod_{c=1}^{d} dX_c\ f(X_c)\ e^{-\frac{1}{2}\sum_{c=1}^{d}\mathrm{tr}_{\otimes_c E_c}\tilde{X}_c^2 - \mathrm{tr}_{\otimes_c E_c}\ln\left(\mathbb{1}^{\otimes d} + i\sum_{c=1}^{d} g_c\tilde{X}_c\right)}.\end{aligned} \tag{3.7}$$

In particular, in the symmetric case where $g_c^2 = \frac{\lambda}{2}$, $\forall c \in [\![1,d]\!]$, we have the following proposition [64]

Proposition 3.0.1. *For $p \in \mathbb{N}^*$ we have:*

$$\langle \mathrm{tr}\, H_c'(T,\overline{T})^p\rangle_T = \langle \mathrm{tr}\,\mathcal{H}_p(X_c)\rangle_M, \tag{3.8}$$

and

$$\langle \mathrm{tr}\, X_c^p\rangle_M = \langle \mathrm{tr}\,\mathcal{H}_p(H_c'(T,\overline{T}))\rangle_T, \tag{3.9}$$

where $H_c' = \frac{\sqrt{\lambda}}{2i\sqrt{2}}H_c$ is a rescaling of our matrix (3.3), and $\mathcal{H}_p$ is the Hermite polynomial of order p defined as $\mathcal{H}_p(x) = e^{-\frac{1}{2}\frac{d^2}{dx^2}}x^p$.

This result will be generalized in Theorems 8.3.4 and 9.2.1 of Part III to arbitrary tensor models with arbitrary bubble invariants.

3.1 Large N limit and fluctuations

The large N limit of general quartic models is addressed in [9, 104]. To put it in a nutshell, general quartic interactions possess two pairs of vertices linked together by $k \geq 1$ colors and two other pairs by $d-k$ colors. The large N limit always yields branched polymers as long as k is different from $d/2$. For $k = d/2$, the model possesses a branched polymer phase and a planar map one, with a transition between the two. The melonic models correspond to $k = 1$ (or $k = d-1$ by symmetry). Let us now turn our attention to the study of the eigenvalue fluctuations for the latter.

In [64] the saddle point analysis revealed that all the eigenvalues collapse into the potential well by following a Wigner semi-circle law of width $1/(1-\alpha^2)$. The usual spreading of the eigenvalue density is here absent due to the fact that the Vandermonde is negligible at large N. The extremum of the potential found by the authors is

$$\alpha = \frac{\sqrt{1+2d\lambda}-1}{2id\sqrt{\lambda/2}}. \tag{3.10}$$

This result is further generalized in [66] to the non symmetric eigenvalue distribution of the model (3.6), and reads

$$\alpha_c = g_c \frac{\sqrt{1+4\sum_p g_p^2}-1}{2i\sum_p g_p^2}. \tag{3.11}$$

In order to study the fluctuations around these values, we follow again [64, 66] and make the change of variables

$$X_c = \alpha_c \mathbb{1}_{E_c} + \frac{1}{N^{\frac{d-2}{2}}} M_c, \tag{3.12}$$

where the scaling of the fluctuations is chosen such that the leading terms in the action for the X_cs scale like the Vandermonde contribution, *i.e.* in N^2. The partition function then rewrites

$$\begin{aligned} Z_{\text{Tensor}}(N,\{g_c\}) &= \frac{e^{-\frac{N^d}{2}\sum_c \alpha_c^2}}{(1+i\sum_c g_c\alpha_c)^{N^d}} Z_{\text{Fluct}}(N,\{\alpha_c\}), \quad \text{for} \\ Z_{\text{Fluct}}(N,\{\alpha_c\}) &= \int \prod_{c=1}^d dM_c \exp\Big(-\frac{N}{2}\sum_c \text{tr}_{E_c} M_c^2 + \sum_{p\geq 2} \frac{N^{\frac{2-d}{2}p}}{p} \text{tr}_{\otimes_c E_c}\Big(\sum_c \alpha_c \tilde{M}_c\Big)^p\Big), \end{aligned} \tag{3.13}$$

with the same notational convention as earlier $\tilde{M}_c = \mathbb{1}^{\otimes(c-1)} \otimes M_c \otimes \mathbb{1}^{\otimes(d-c)}$. Notice that the second term of Z_{Fluct} consists of multi-trace interactions. In order to derive the loop equations for this model, we follow [66] and write them for the general d-matrix model with interactions $\prod_{c=1}^{d} \operatorname{tr} M_c^{q_c}$ with generic action

$$S = N \sum_{c=1}^{d} \operatorname{tr} V_c(M_c) + N^{2-d} \sum_{a_1,\dots,a_d \geq 0} t_{a_1\dots a_d} \prod_{c=1}^{d} \operatorname{tr} M_c^{a_c}. \tag{3.14}$$

3.2 Correlation functions

We introduce the following generating functions for products of n traces of colors $c_1, \dots, c_n$,

$$\overline{W}_n(x_1, c_1; \dots; x_n, c_n) = \Big\langle \prod_{i=1}^{n} \operatorname{tr}_{E_{c_i}} \frac{1}{x_i - M_{c_i}} \Big\rangle = \sum_{k_1,\dots,k_n \geq 0} \overline{W}_n^{(k_1,c_1;\dots;k_n,c_n)} \prod_{i=1}^{n} x_i^{-k_i-1}, \tag{3.15}$$

i.e.

$$\overline{W}_n^{(k_1,c_1;\dots;k_n,c_n)} = \Big[\prod_{i=1}^{n} x^{-k_i-1}\Big] \overline{W}_n(x_1, c_1; \dots; x_n, c_n) = \Big\langle \prod_{i=1}^{n} \operatorname{tr}_{E_{c_i}} M_{c_i}^{k_i} \Big\rangle, \tag{3.16}$$

and their connected counterparts

$$W_n(x_1, c_1; \dots; x_n, c_n) = \Big\langle \prod_{i=1}^{n} \operatorname{tr}_{E_{c_i}} \frac{1}{x_i - M_{c_i}} \Big\rangle_c = \sum_{k_1,\dots,k_n \geq 0} W_n^{(k_1,c_1;\dots;k_n,c_n)} \prod_{i=1}^{n} x_i^{-k_i-1}, \tag{3.17}$$

i.e.

$$W_n^{(k_1,c_1;\dots;k_n,c_n)} = \Big[\prod_{i=1}^{n} x^{-k_i-1}\Big] W_n(x_1, c_1; \dots; x_n, c_n) = \Big\langle \prod_{i=1}^{n} \operatorname{tr}_{E_{c_i}} M_{c_i}^{k_i} \Big\rangle_c. \tag{3.18}$$

The variable x_i is said to be of color c_i when it is the generating parameter for $\operatorname{tr}_{E_{c_i}} M_{c_i}^{k_i}$ expanded around infinity. We denote $\mathbb{C}_c$ the copy of $\mathbb{C}$ of color c, so that $x_i \in U_{c_i}$ for some open subset of $\mathbb{C}_{c_i}$. We will also need the functions

$$W_n^{(k_1,c'_1;\dots;k_l,c'_l)}(x_1, c_1; \dots; x_{n-l}, c_{n-l}) = \Big\langle \prod_{i=1}^{l} \operatorname{tr}_{E_{c'_i}} M_{c'_i}^{k_i} \prod_{j=1}^{n-l} \operatorname{tr}_{E_{c_j}} \frac{1}{x_j - M_{c_j}} \Big\rangle_c, \tag{3.19}$$

which are obtained from $W_n(x_1, c_1; \ldots; x_{n-l}, c_{n-l}; x'_1, c'_1; \ldots; x'_l, c'_l)$ by extracting some series coefficients, *viz.*

$$W_n^{(k_1,c'_1;\ldots;k_l,c'_l)}(x_1, c_1; \ldots; x_{n-l}, c_{n-l}) = \Big[\prod_{i=1}^{l} x_i'^{-k_i-1}\Big] W_n(x_1, c_1; \ldots; x_{n-l}, c_{n-l}; x'_1, c'_1; \ldots; x'_l, c'_l). \tag{3.20}$$

Furthermore we write $\overline{(k_i, c_i)} := (k_1, c_1; \ldots; k_{i-1}, c_{i-1}\,;\, k_{i+1}, c_{i+1}; \ldots; k_n, c_n),\ \forall i \in [\![1, n]\!]$.
It will also appear natural to introduce *global* correlation functions which are defined on (an open subset of)

$$E_n = \Big(\bigcup_{c=1}^{d} \mathbb{C}_c \setminus \Gamma_c\Big)^n, \tag{3.21}$$

so that each x_i can be evaluated on any color. These correlation functions are

$$W_n(x_1, \ldots, x_n) = \sum_{c_1,\ldots,c_n=1}^{d} W_n(x_1, c_1; \ldots; x_n, c_n) \prod_{i=1}^{n} \mathbb{1}(x_i, c_i). \tag{3.22}$$

where $\mathbb{1}(x, c)$ is 1 if $x \in \mathbb{C}_c$ and 0 otherwise. In terms of components,

$$W_n(x_1, \ldots, x_n) = \sum_{\substack{k_1,\ldots,k_n \geq 0 \\ (c_1,\ldots,c_n)\in[\![1,d]\!]^n}} W_n^{(k_1,c_1;\ldots;k_n,c_n)} \prod_{i=1}^{n} x_i^{-k_i-1}\mathbb{1}(x_i, c_i). \tag{3.23}$$

The correlation functions $W_n(x_1, c_1; \ldots; x_n, c_n)$ are said to be the *local expressions* of $W_n(x_1, \ldots, x_n)$, since each variable is assigned a fixed color. We are now ready to derive the loop equations of the quartic melonic model.

3.3 Exact resolvent equations

3.3.1 1-point equation

The Schwinger-Dyson equations are obtain from the family of equations

$$\forall c \in [\![1, d]\!], \qquad \frac{1}{Z}\int \prod_{i=1}^{d} dM_i \sum_{a,b} \frac{\partial}{\partial (M_c)_{ab}} \big((M_c^n)_{ab} e^{-S}\big) = 0. \tag{3.24}$$

Computing the derivatives explicitly and summing over $n \geq 0$ with x^{-n-1} yields

$$\begin{aligned}&\overline{W}_2(x,c;x,c) - N\sum_{n\geq 0}\langle \operatorname{tr}(M_c^n V_c'(M_c))\rangle x^{-n-1} \\ &- N^{2-d}\sum_{\substack{a_c\geq 1\\ a_{c'}\geq 0,\ \forall c'\neq c}} a_c t_{a_1\dots a_d}\sum_{n\geq 0}\langle \operatorname{tr} M_c^{n+a_c-1}\prod_{c'\neq c}\operatorname{tr} M_{c'}^{a_{c'}}\rangle x^{-n-1} = 0,\end{aligned} \tag{3.25}$$

for all $c \in [\![1,d]\!]$. The second and third terms are split using the standard trick

$$\sum_{n\geq 0}\operatorname{tr} M^{n+a}x^{-n-1} = x^a \operatorname{tr}\frac{1}{x-M} - \operatorname{tr}\frac{x^a - M^a}{x-M}, \tag{3.26}$$

yielding for the second contribution

$$\sum_{n\geq 0}\langle \operatorname{tr}(M_c^n V_c'(M_c))\rangle x^{-n-1} = V_c'(x)\overline{W}_1(x,c) - P_1(x,c), \tag{3.27}$$

where $P_1(x,c) = \langle \operatorname{tr}\frac{V_c'(x)-V_c'(M_c)}{x-M_c}\rangle$. The last contribution is similar and reads

$$\begin{aligned}&\sum_{n\geq 0}\langle \operatorname{tr} M_c^{n+a_c-1}\prod_{c'\neq c}\operatorname{tr} M_{c'}^{a_{c'}}\rangle x^{-n-1} \\ &= x^{a_c-1}\langle \operatorname{tr}\frac{1}{x-M_c}\prod_{c'\neq c}\operatorname{tr} M_{c'}^{a_{c'}}\rangle - \langle \operatorname{tr}\frac{x^{a_c-1}-M_c^{a_c-1}}{x-M_c}\prod_{c'\neq c}\operatorname{tr} M_{c'}^{a_{c'}}\rangle.\end{aligned} \tag{3.28}$$

The quantity $\langle \operatorname{tr}\frac{x^{a_c-1}-M_c^{a_c-1}}{x-M_c}\prod_{c'\neq c}\operatorname{tr} M_{c'}^{a_{c'}}\rangle$ must be further split by

$$\frac{x^{a_c-1}-M_c^{a_c-1}}{x-M_c} = -\sum_{q=0}^{a_c-2} x^{a_c-2-q}M_c^q, \tag{3.29}$$

yielding overall, for all $c \in [\![1,d]\!]$ and $x \in \mathbb{C}_c$,

$$\begin{aligned}&\overline{W}_2(x,c;x,c) - NV_c'(x)\overline{W}_1(x,c) + NP_1(x,c) \\ &- N^{2-d}\sum_{\substack{a_c\geq 1\\ a_{c'}\geq 0,\ \forall c'\neq c}} a_c t_{a_1\dots a_d}\left(x^{a_c-1}\overline{W}_d^{(a_1,c_1';\dots;a_{d-1},c_{d-1}')}(x,c) + \sum_{q=0}^{a_c-2} x^{a_c-2-q}\overline{W}_d^{(\overline{(a_c,c)}\,;\,q,c)}\right) = 0.\end{aligned} \tag{3.30}$$

We now would like to rewrite (3.30) in terms of connected correlation functions. For this purpose denote $R = \{R_1, \dots, R_{\ell(R)}\}$ a set-partition of $[\![1, n]\!]$, then

$$\overline{W}_n(x_1, c_1; \dots; x_n, c_n) = \sum_{R \vdash [\![1,n]\!]} \prod_{\alpha=1}^{\ell(R)} W_{|R_\alpha|}(\{x_{R_\alpha}, c_{R_\alpha}\}), \tag{3.31}$$

with the short-hand notation $\{x_{R_\alpha}, c_{R_\alpha}\} = \{x_r, c_r\}_{r \in R_\alpha}$. This first gives

$$\overline{W}_2(x, c; x, c) = W_1(x, c)^2 + W_2(x, c\,;\, x, c). \tag{3.32}$$

Now take $R = \{R_1, \dots, R_{\ell(R)}\}$ to be as set-partition of $[\![1, d]\!]$ and for $c \in [\![1, d]\!]$ denote $R_c = \{c\} \cup R'_c$ the part that contains c, where R'_c can be empty. Then,

$$\overline{W}_d^{(a_1, c'_1; \dots; a_{d-1}, c'_{d-1})}(x, c) = \sum_{R \vdash [\![1,d]\!]} W_{|R_c|}^{(a_{R'_c}, c_{R'_c})}(x, c) \prod_{\substack{\alpha \\ R_\alpha \neq R_c}} W_{|R_\alpha|}^{(a_{R_\alpha}, c_{R_\alpha})}, \tag{3.33}$$

and

$$\overline{W}_d^{(\overline{(a_c, c)}\,;\, q, c)} = \sum_{R \vdash [\![1,d]\!]} W_{|R_c|}^{(a_{R'_c}, c_{R'_c}\,;\, q, c)} \prod_{\substack{\alpha \\ R_\alpha \neq R_c}} W_{|R_\alpha|}^{(a_{R_\alpha}, c_{R_\alpha})}. \tag{3.34}$$

The loop equation now reads in terms of connected components

$$\begin{aligned} W_1(x, c)^2 &+ W_2(x, c\,;\, x, c) - NV'_c(x) W_1(x, c) + NP_1(x, c) \\ &- N^{2-d} \sum_{R \vdash [\![1,d]\!]} \sum_{\substack{a_c \geq 1 \\ a_{c'} \geq 0,\, \forall c' \neq c}} a_c t_{a_1 \dots a_d} \prod_{\substack{\alpha \\ R_\alpha \neq R_c}} W_{|R_\alpha|}^{(a_{R_\alpha}, c_{R_\alpha})} \\ &\times \left(x^{a_c - 1} W_{|R_c|}^{(a_{R'_c}, c_{R'_c})}(x, c) + \sum_{q=0}^{a_c - 2} x^{a_c - 2 - q} W_{|R_c|}^{(a_{R'_c}, c_{R'_c}\,;\, q, c)} \right) = 0. \end{aligned} \tag{3.35}$$

3.3.2 n-point equations

For a potential of the form $V_c(x) = \sum_n t_n^{(c)} x^n$ define the colored loop insertion operator as $\frac{\delta}{\delta V_c(x)} = \sum_n x^{-n-1} \frac{\partial}{\partial t_n^{(c)}}$ and for integers A, n denote $\mathcal{I}_A(n)$ the set of lists of the form $I = (I_1, \dots, I_A)$ such that $I_\alpha \subseteq [\![2, n]\!]$ and $\bigsqcup_{\alpha=1}^{A} I_\alpha = [\![2, n]\!]$. Repeated action of the loop insertion operator on (3.35) then gives for the global correlation functions

$$\begin{aligned}
&\sum_{(I_1,I_2)\in\mathcal{I}_2(n)} W_{|I_1|+1}(x_1,x_{I_1})W_{|I_2|+2}(x_1,x_{I_2}) + W_{n+1}(x_1,x_1,\dots,x_n)\\
&- N\sum_{c=1}^{d}\mathbb{1}(x_1,c)\left(V_c'(x_1)W_n(x_1,\dots,x_n) - P_n(x_1,\dots,x_n)\right)\\
&+\sum_{j=2}^{n}\mathbb{1}(x_1,x_j)\frac{\partial}{\partial x_j}\frac{W_{n-1}(x_2,\dots,x_n)-W_{n-1}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)}{x_j-x_1}\\
&- N^{2-d}\sum_{R\vdash[\![1,d]\!]}\ \sum_{(I_1,\dots,I_{\ell(R)})\in\mathcal{I}_{\ell(R)}(d)}\ \sum_{c=1}^{d}\mathbb{1}(x_1,c)\sum_{\substack{a_c\geq 1\\ a_{c'}\geq 0,\,\forall c'\neq c}} a_c t_{a_1\dots a_d}\prod_{\substack{\alpha\\ R_\alpha\neq R_c}} W^{(a_{R_\alpha},c_{R_\alpha})}_{|R_\alpha|+|I_\alpha|}(x_{I_\alpha})\\
&\quad\times\left(x_1^{a_c-1}W^{(a_{R_c'},c_{R_c'})}_{|R_c|+|I_c|}(x_1,x_{I_c}) + \sum_{q=0}^{a_c-2}x_1^{a_c-2-q}W^{(a_{R_c'},c_{R_c'};q,c)}_{|R_c|+|I_c|}(x_{I_c})\right) = 0.
\end{aligned}\tag{3.36}$$

I_c is defined as I_{α^*} where α^* is the index such that $R_{\alpha^*} = R_c$. Moreover, we defined $\mathbb{1}(x,y) = \sum_{c=1}^{d}\mathbb{1}(x,c)\mathbb{1}(y,c)$, it is 1 if and only if x and y are of the same color.

3.4 Disc and cylinder function at genus zero

Having treated the general potential case, we come back to the quartic melonic model, where we have the following.

- The potentials are $V_c(x) = x^2/2$ for all $c\in[\![1,d]\!]$ and $x\in\mathbb{C}_c$;
- The color couplings are [66]

$$t_{a_1\dots a_d} = -\frac{1}{\sum_c a_c}\binom{\sum_c a_c}{a_1,\dots,a_d}\prod_{c=1}^{d}\alpha_c^{a_c}N^{d-2-\frac{d-2}{2}\sum_c a_c},\qquad \sum_c a_c\geq 2.\tag{3.37}$$

Crucially the $t_{a_1\dots a_d}$s depend on N and we read from (3.37) that the dominant ones are given by $\sum_c a_c = 2$. As the $a_c\geq 0$, the latter reduce to the two only couplings $t_{0\dots2\dots0}$ and $t_{0\dots1\dots1\dots0}$, where all dots represent zeros. Thus at large N the action behaves as

$$S_{N_\infty} = -\frac{N}{2}\sum_{c=1}^{d}(1-\alpha_c^2)\mathrm{tr}\,M_c^2 + \sum_{c\neq c'}\alpha_c\alpha_{c'}\mathrm{tr}\,M_c\mathrm{tr}\,M_{c'}.\tag{3.38}$$

We will now cite, without re-deriving them, the expressions for the disc and cylinder functions.

3.4.1 Colored disc functions

With the previous large N action, the disc function of color c satisfies the equation [66]

$$W_{1,0}(x,c)^2 - (1-\alpha_c^2)xW_{1,0}(x,c) + (1-\alpha_c^2) = 0, \qquad \forall c \in \mathbb{C}_c, \tag{3.39}$$

the solution of which being

$$W_{1,0}(x,c) = \frac{1-\alpha_c^2}{2}\left(x - \sqrt{x^2 - \frac{4}{1-\alpha_c^2}}\right). \tag{3.40}$$

This is the disc function of the GUE, it has a cut along $\Gamma_c = \left[-\frac{2}{\sqrt{1-\alpha_c^2}}, \frac{2}{\sqrt{1-\alpha_c^2}}\right]$ which is said to be the cut of color c.

3.4.2 Colored cylinder functions

We introduce the notation

$$\sigma(x,c) = \sqrt{x^2 - \frac{4}{1-\alpha_c^2}}, \qquad \forall x \in \mathbb{C}_c \backslash \Gamma_c. \tag{3.41}$$

Then it can be shown that the colored cylinder functions take the form

$$\begin{aligned} W_{2,0}(x_1,c;x_2,c) = &\frac{x_1x_2 - \sigma(x_1,c)\sigma(x_2,c) - 4/(1-\alpha_c^2)}{2(x_1-x_2)^2\sigma(x_1,c)\sigma(x_2,c)} \\ &- \frac{\sum_{p\neq c}(\alpha_c\alpha_p)^2}{\sum_{p=1}^d \alpha_p^2 - 1}\frac{W_{1,0}(x_1,c)W_{1,0}(x_2,c)}{(1-\alpha_c^2)\sigma(x_1,c)\sigma(x_2,c)}, \quad \text{for } (x_1,x_2)\in\mathbb{C}_c^2 \end{aligned} \tag{3.42}$$

and

$$W_{2,0}(x_1,c_1;x_2,c_2) = -\frac{\alpha_{c_1}\alpha_{c_2}}{\sum_{p=1}^d \alpha_p^2 - 1}\frac{W_{1,0}(x_1,c_1)W_{1,0}(x_2,c_2)}{\sigma(x_1,c_1)\sigma(x_2,c_2)}, \quad \text{for } (x_1,x_2)\in\mathbb{C}_{c_1}\times\mathbb{C}_{c_2}. \tag{3.43}$$

3.5 Blobbed topological recursion

3.5.1 Spectral curve

As before, we denote $\hat{\mathbb{C}}_c$ the copy of color $c \in [\![1, d]\!]$ of the Riemann sphere. For each color, define

$$f_c(x, y) = y^2 - (1 - \alpha_c^2)xy + (1 - \alpha_c^2). \tag{3.44}$$

These polynomials vanish along a curve $\mathcal{C} \subset \bigcup_{c=1}^d \hat{\mathbb{C}}_c$ defined by

$$f(x, y) = \sum_{c=1}^{d} \mathbb{1}(x, c)\mathbb{1}(y, c) f_c(x, y) = 0. \tag{3.45}$$

We then introduce the following Zhukovski parametrization

$$x(z) = \sum_{c=1}^{d} \mathbb{1}(x, c)\mathbb{1}(z, c)\frac{1}{\sqrt{1 - \alpha_c^2}}(z + z^{-1}). \tag{3.46}$$

In order to write the topological recursion, we will pull back the correlation functions on the z-planes using the Zhukowski transformation, this will allow us to turn them into differential forms:

$$\begin{aligned}\omega_n(z_1, \dots, z_n) = W_n(x(z_1), \dots, x(z_n))dx(z_1)\dots dx(z_n)& \\ - \delta_{n,2}\sum_{c=1}^{d} \mathbb{1}(z_1, c)\mathbb{1}(z_2, c)\frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2}&,\end{aligned} \tag{3.47}$$

and give the expressions for the disc and cylindar differentials. The (global) disc function has simple zeroes at $z = \pm 1$, those are the zeroes of $dx(z)$:

$$\omega_{1,0}(z) = \sum_{c=1}^{d} \mathbb{1}(z, c)\frac{1 - z^{-2}}{z}dz. \tag{3.48}$$

While the cylinder forms are

$$\begin{aligned}\omega_{2,0}(z_1, c; z_2, c) &= \frac{dz_1 dz_2}{(z_1 - z_2)^2} - \frac{\sum_{p\neq c}(\alpha_c\alpha_p)^2}{\sum_{p=1}^d \alpha_p^2 - 1}\frac{dz_1 dz_2}{z_1^2 z_2^2}, \\ \omega_{2,0}(z_1, c_1; z_2, c_2) &= -\frac{\alpha_{c_1}\alpha_{c_2}\sqrt{(1 - \alpha_{c_1})(1 - \alpha_{c_2})}}{\sum_{p=1}^d \alpha_p^2 - 1}\frac{dz_1 dz_2}{z_1^2 z_2^2}.\end{aligned} \tag{3.49}$$

Notice that $\omega_{2,0}(z_1, c; z_2, c)$ has a double pole on the diagonal $z_1 = z_2$ and that both $\omega_{2,0}(z_1, c; z_2, c)$ and $\omega_{2,0}(z_1, c_1; z_2, c_2)$ have poles on $z_1 = 0$ and $z_2 = 0$ on each color. The points $z = \pm 1$ in each color are called ramification points and we write

$$\mathcal{R} = \{z = \pm 1 | \forall c \in [\![1, d]\!], z \in \mathbb{C}_c\}. \tag{3.50}$$

Finally we define a global bi-differential by writing

$$\omega_{2,0}(z_1, z_2) = \sum_{c_1, c_2 = 1}^{d} \mathbb{1}(z_1, c_1)\mathbb{1}(z_2, c_2)\left(\delta_{c_1 c_2} \frac{dz_1 dz_2}{(z_1 - z_2)^2} - \Omega_{c_1 c_2} \frac{dz_1 dz_2}{z_1^2 z_2^2}\right), \tag{3.51}$$

where

$$\Omega_{cc'} = \frac{\alpha_c \alpha_{c'} \sqrt{(1 - \alpha_c)(1 - \alpha_{c'})} + \alpha_c^2 \big(\sum_{p \neq c} \alpha_p^2 + \alpha_c - 1\big)\delta_{cc'}}{\sum_p \alpha_p^2 - 1}. \tag{3.52}$$

3.5.2 Blobbed topological recursion formula

We know that correlation functions admit a topological expansion, this is true also for their associated differential forms, thus we write

$$W_n(x_1, \dots, x_n) = \sum_{g \geq 0} N^{2-2g-n} W_{n,g}(x_1, \dots, x_n), \tag{3.53}$$

$$\omega_n(z_1, \dots, z_n) = \sum_{g \geq 0} N^{2-2g-n} \omega_{n,g}(z_1, \dots, z_n). \tag{3.54}$$

With the same notations as in section 1.5.2, the kernel of the topological recursion is again given by

$$K(z, z_1) = \frac{\Delta G(z, z_1)}{2\Delta\omega_{1,0}(z)}, \tag{3.55}$$

and finally define the polar $P\omega_{n,g}$ and holomorphic $H\omega_{n,g}$ parts of $\omega_{n,g}(z_1, \dots, z_n)$ by

$$\begin{aligned} P\omega_{n,g}(z_1, \dots, z_n) &= \sum_{z \in \mathcal{R}} \mathrm{Res}_z\, G(z, z_1)\omega_{n,g}(z, z_2, \dots, z_n) \\ H\omega_{n,g}(z_1, \dots, z_n) &= \omega_{n,g}(z_1, \dots, z_n) - P\omega_{n,g}(z_1, \dots, z_n). \end{aligned} \tag{3.56}$$

We can then give topological recursion formulae for $P\omega_{n,g}$ and $H\omega_{n,g}$, *viz.*

$$\begin{aligned} P\omega_{n,g}(z_1,\ldots,z_n) &= \sum_{z\in\mathcal{R}} \mathrm{Res}_z\, K(z,z_1)\Bigg(\omega_{n+1,g-1}(z,\iota(z),z_2,\ldots,z_n) \\ &\qquad + \sideset{}{'}\sum_{\substack{(I_1,I_2)\in\mathcal{I}_2(n)\\ g_1+g_2=g}} \omega_{|I_1|+1,g_1}(z,z_{I_1})\omega_{|I_2|+1,g_2}(\iota(z),z_{I_2})\Bigg), \\ H\omega_{n,g}(z_1,\ldots,z_n) &= \frac{1}{2\pi i}\oint_{\bigcup_{c=1}^{d} U(1)_c} \omega_{2,0}(z_1,z)\nu_{n,g}(z,z_2,\ldots,z_n), \end{aligned} \tag{3.57}$$

where $U(1)_c$ is the copy of the unit circle of color c, and

$$\nu_{n,g}(z,z_2,\ldots,z_n) = V_{n,g}(x(z),x(z_2),\ldots,x(z_n))dx(z_2)\ldots dx(z_n), \tag{3.58}$$

for

$$\begin{aligned} V_{n,g}(x,x_2,\ldots,x_n) = \sum_{c=1}^{d} \sum_{\substack{R\vdash[\![1,d]\!]\\ R_c=\{c\}}} \sum_{(I_1,\ldots,I_{\ell(R)})\in\mathcal{I}_{\ell(R)}(d)} \mathbb{1}(x,c) \\ \sum_{\substack{h_1,\ldots,h_{\ell(R)}\geq 0\\ d-\ell(R)+\sum_{\alpha=1}^{\ell(R)} h_\alpha=g}} \sum_{\substack{a_c\geq 1\\ a_{c'}\geq 0,\forall c'\neq c}} a_c t_{a_1\ldots a_d} x^{a_c} \prod_{R_\alpha\neq\{c\}} W^{(a_{R_\alpha},c_{R_\alpha})}_{|R_\alpha|+|I_\alpha|,h_\alpha}(x_{I_\alpha}). \end{aligned} \tag{3.59}$$

Part II

On the counting of orthogonal tensor invariants

Chapter 4

Counting $O(N)$ invariants

We now turn our attention to real tensor models and begin by setting up our notations. Consider $d \geq 2$ real vector spaces V_c, $c \in [\![1, d]\!]$, of respective dimensions N_c, and the action of $\bigotimes_{c=1}^{d} O(N_c)$ on $\bigotimes_{c=1}^{d} V_c$. Let T be a tensor of rank d with components $T_{i_1 \dots i_d}$ transforming under the tensor product of d fundamental representations of the groups $O(N_c)$. Each group $O(N_c)$ acts independently on a tensor index i_c and we can write

$$T^{O}_{i_1 \dots i_d} = \sum_{j_1, \dots, j_d} O^{(1)}_{i_1 j_1} \dots O^{(d)}_{i_d j_d} \, T_{j_1 \dots j_d} \, . \tag{4.1}$$

The observables in this model are the contractions of an even number, say $2n$ with $n \in \mathbb{N}$, of tensors T which are obviously invariant under $\bigotimes_{c=1}^{d} O(N_c)$ transformations. We simply name them $O(N)$ invariants. Such invariants generalize real matrix traces and will be denoted:

$$O_K(T) = \sum_{j_c^{(k)}} K(\{j_c^{(k)}\}_{1 \leq c \leq d}^{1 \leq k \leq 2n}) \, T_{j_1^{(1)} \dots j_d^{(1)}} \dots T_{j_1^{(2n)} \dots j_d^{(2n)}} \, , \tag{4.2}$$

where the kernel $K(\cdot)$ factors in Kronecker deltas and identifies the indices of the tensors in a particular pattern; the sole contractions permitted involve the tensor indices with identical color labels $c \in [\![1, d]\!]$. An elegant way of encoding the contraction pattern of tensors consists in a d-regular graph with edge coloring with d different colors, and one of each color at every vertex (representing each tensor). Those graphs are defined as the bubbles of Chapter 2 albeit for the bipartiteness condition that is relaxed. We hence slightly shift the notation and write b the colored graph, the invariant denotes equivalently $O_K(T)$ or $O_b(T)$. We will detail this in the next chapter.
We build a physical model by introducing a partition function

$$Z = \int d\nu(T) \exp(-S_N(T)) \, , \tag{4.3}$$

where the action $S_N(T) = \sum_b \lambda_b N^{-\rho(b)} O_b(T)$ is defined as a finite sum over some $O(N)$ tensor invariants representing the model interactions each with coupling λ_b and scaling parameter $\rho(b)$; $d\nu(T)$ is a tensor field measure. In this part, we will consider only correlators that are Gaussian. This means that the field measure will be Gaussian and of the form

$$d\nu(T) = \prod_{j_l} dT_{j_1 \dots j_d}\, e^{-O_2(T)} , \qquad O_2(T) = \sum_{j_k} (T_{j_1 \dots j_d})^2 . \tag{4.4}$$

In other terms, $O_2(T)$ plays the role of a quadratic mass term. The free propagator of the Gaussian measure is given by

$$\langle T_{i_1 \dots i_d} T_{j_1 \dots j_d} \rangle = \int d\nu(T)\, T_{i_1 \dots i_d} T_{j_1 \dots j_d} = \delta_{i_1 j_1} \dots \delta_{i_d j_d}, \tag{4.5}$$

and will be used in the Wick theorem for computing Gaussian correlators. We will be interested in the mean values of observables that are defined by

$$\begin{aligned} \langle O_b(T) \rangle &= \frac{1}{\int d\nu(T)} \int d\nu(T) O_b(T) , \\ \langle O_b(T) O_{b'}(T) \rangle &= \frac{1}{\int d\nu(T)} \int d\nu(T) O_b(T) O_{b'}(T) . \end{aligned} \tag{4.6}$$

The second correlator will be restricted to normal order allowing only Wick contractions from $O_b(T)$ to $O_{b'}(T)$. In Chapter 6, enlightened by the symmetric group formulation of the $O(N)$ invariants, we will reformulate (4.6) and analyse the representation algebraic structure brought by the 2-point correlator. The first correlator is sketched as it evaluates by modifying the previous calculation method.

We now proceed to the counting *per se*. Counting the number of invariants based on the contractions of $2n$ copies of real tensors $T_{i_1 \dots i_d}$ starts by a symmetric group construction. Actually, this enumeration problem expresses as a permutation-TFT that we also discuss. Finally, switching to representation theory, we derive the same counting formula in terms of the Kronecker coefficients.

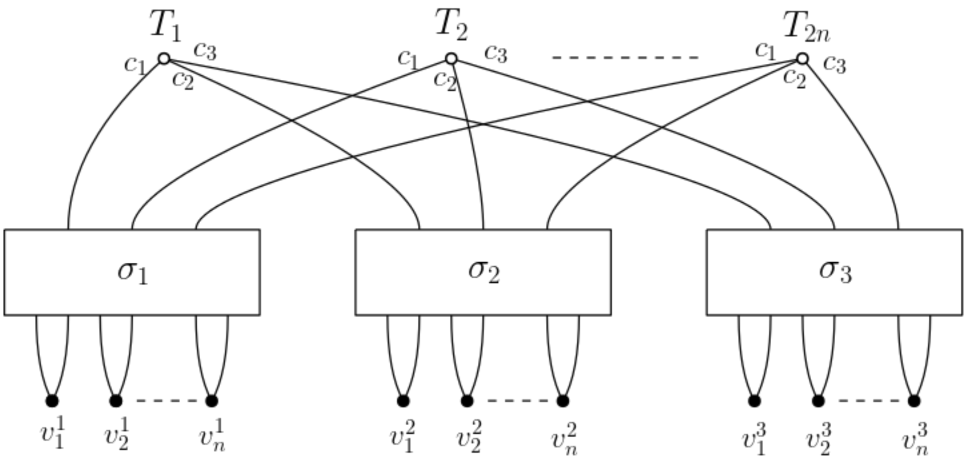


FIGURE 4.1 – Diagrammatic contraction of rank-3 orthogonal tensors defining the triple of permutations $(\sigma_1, \sigma_2, \sigma_3)$.

4.1 Enumeration of rank $d \geq 3$ tensor invariants

Orthogonal invariants are in one-to-one correspondence with d-regular colored graphs (see for instance [33]). We emphasize again that contrary to the graphs corresponding to unitary invariants [11, 29], the present graphs are not bipartite and, so, their dual triangulations might be non-orientable. It is however always possible to make a graph bipartite by inserting another type of vertex of valence 2 called "black" (henceforth the initial vertices are called "white") on each edge of the graph. We therefore perform that transformation and denote the new vertices v_i^c, $i \in [\![1, n]\!]$ (recall that $2n$ is the number of tensors) and $c \in [\![1, d]\!]$. The resulting graph is neither regular, nor properly edge-colored, as opposed to the unitary case. It is however bipartite as illustrated in Figure 4.1, that is to be compared with Figure 2.5 of the previous part. This property concedes a description of a colored graph in symmetric group language. We shall focus on $d = 3$ as the general case will follow from this one.

We denote S_{2n} the symmetric group of order $(2n)!$. Counting possible graphs consists in enumerating triples

$$(\sigma_1, \sigma_2, \sigma_3) \in S_{2n} \times S_{2n} \times S_{2n}, \tag{4.7}$$

subjected to the equivalence

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1 \sigma_1 \gamma, \gamma_2 \sigma_2 \gamma, \gamma_3 \sigma_3 \gamma)\,, \tag{4.8}$$

where $\gamma \in S_{2n}$ and the γ_is belong to the wreath product[1] subgroup $S_n[S_2] \subset S_{2n}$. We intend to count the points in the double coset

$$(S_n[S_2] \times S_n[S_2] \times S_n[S_2]) \backslash (S_{2n} \times S_{2n} \times S_{2n}) / \mathrm{Diag}(S_{2n})\,. \tag{4.9}$$

Let us denote $Z_3(2n)$ the cardinality of this double coset. Recall from Chapter 2 that in a broader setting, for two subgroups $H_1 \leq G$ and $H_2 \leq G$, the cardinality of the double coset $|H_1 \backslash G / H_2|$ is given by

$$|H_1 \backslash G / H_2| = \frac{1}{|H_1||H_2|} \sum_C Z_C^{H_1 \to G} Z_C^{H_2 \to G} z_C\,, \tag{4.10}$$

where z_C stands for the number of elements of G commuting with any element in the conjugacy class C. The sum is over conjugacy classes of G, and $Z_C^{H \to G}$ is the number of elements of H in the conjugacy class C of G. The conjugacy classes of $S_{2n} \times S_{2n} \times S_{2n}$ are determined by triples (p_1, p_2, p_3), where each p_i is a partition of $2n$. The presence of the subgroup $\mathrm{Diag}(S_{2n})$ implies that only conjugacy classes determined by a triple (p, p, p) should be conserved in the above sum. Applying (4.10), we get

$$\begin{aligned} Z_3(2n) &= \frac{1}{[n!2^n]^3 (2n)!} \sum_{p \vdash 2n} Z_{(p,p,p)}^{S_n[S_2]^3 \to S_{2n}^3} \frac{(2n)!}{z_p} z_p^3 \\ &= \frac{1}{[n!2^n]^3} \sum_{p \vdash 2n} Z_{(p,p,p)}^{S_n[S_2]^3 \to S_{2n}^3} z_p^2, \end{aligned} \tag{4.11}$$

with $z_p = \prod_i i^{p_i} p_i!$ and where the sum over $p = (p_\ell)_\ell$ is performed over all partitions of $2n = \sum_i i p_i$. The cardinality of a conjugacy class C_p of S_{2n} with cycle structure[2] determined by a partition p is given by $|C_p| = (2n)!/z_p$. Next, we must determine the size of $Z_{(p,p,p)}^{S_n[S_2]^3 \to S_{2n}^3}$ which factors as

$$Z_{(p,p,p)}^{S_n[S_2]^3 \to S_{2n}^3} = (Z_p^{S_n[S_2] \to S_{2n}})^3\,. \tag{4.12}$$

[1] The wreath product $S_n[S_2] = S_2 \wr S_n$ is defined as the semi-direct product $S_2^n \rtimes S_n$ and is known in the literature as the hyperoctahedral group. Seen as a permutation group, it is the signed symmetric group of permutations of the set $\{-n, -n+1, \dots, -1, 1, \dots, n-1, n\}$.

[2] Two permutations have the same cycle structure or are of the same cycle type if the unordered list of sizes of their cycles coincide. The cycle type of a permutation in S_{2n} determines a list $p = (p_1, \dots, p_{2n})$ of numbers $p_i \geq 0$ of cycles of length i. The list p is a partition of $2n$.

We can get a single factor in this product from

$$\frac{1}{n!2^n} Z_p^{S_n[S_2]\to S_{2n}} = [t^n x^p] \mathcal{Z}^{S_\infty[S_2]}(t,\vec{x}), \tag{4.13}$$

where appears the generating function of the number of wreath product elements in a certain conjugacy class $p \vdash 2n$, namely

$$\mathcal{Z}^{S_\infty[S_d]}(t,\vec{x}) = \sum_n t^n Z^{S_n[S_d]}(\vec{x}) = \exp \sum_{i=1}^{\infty} \frac{t^i}{i} \sum_{q\vdash d} \frac{1}{z_q} \prod_{\ell=1}^{d} x_{\ell i}^{\nu_\ell}, \tag{4.14}$$

with $\vec{x} = (x_1, x_2, \dots)$, $q = (\nu_\ell)_\ell$ a partition of d such that $\sum_\ell \ell\nu_\ell = d$ and in multi-index notation $x^p := x_1^{p_1} \dots x_{2n}^{p_{2n}}$. We adopt the "combinatorists'" notation and write as $[x^\alpha]Z(x)$ the coefficient of x^α in the series expansion of Z. To understand where this comes from, define the *cycle index polynomial* of $H \leq S_n$ as

$$Z^H(\vec{x}) = \frac{1}{|H|} \sum_{p\vdash n} Z_p^{H\to S_n} \prod_i x_i^{p_i}. \tag{4.15}$$

The generating function for cycle index polynomials of S_n is (see Chapter 15.13 in [53])

$$\mathcal{Z}^{S_\infty}(t,\vec{x}) = \sum_{n=0}^{\infty} t^n Z^{S_n}(\vec{x}) = \exp \sum_{i=1}^{\infty} \frac{t^i x_i}{i}. \tag{4.16}$$

The cycle index polynomial of a wreath product is given for $H \leq G$ by (see Chapter 15.5 in [54])

$$Z^{G[H]}(\vec{x}) = Z^G(\vec{r}), \quad \text{with } r_i = Z^H(x_i, x_{2i}, x_{3i}, \dots), \tag{4.17}$$

so that finally one can write the generating function of cycle index polynomials of $S_n[S_d]$[3]:

$$\begin{aligned} \mathcal{Z}^{S_\infty[S_d]}(t,\vec{x}) &= \sum_{n=0}^{\infty} t^n Z^{S_n[S_d]}(\vec{x}) \\ &= \sum_{n=0}^{\infty} t^n \sum_{p\vdash 2n} \frac{1}{d^n n!} Z_p^{S_n[S_d]\to S_{2n}} \prod_i x_i^{p_i} \\ &= \exp \sum_{i=1}^{\infty} \frac{t^i}{i} Z^{S_d}(x_i, \dots, x_{di}), \end{aligned} \tag{4.18}$$

which is indeed (4.14).

[3] For $d = 2$, $\mathcal{Z}^{S_\infty[S_2]}(t,\vec{x}) = \exp \sum_{i=1}^{\infty} \frac{t^i}{2i}(x_i^2 + x_{2i})$.

The expression (4.11) finally computes to

$$Z_3(2n) = \sum_{p \vdash 2n} \left([t^n x^p]\mathcal{Z}^{S_\infty[S_2]}(t, \vec{x})\right)^3 z_p^2 \,. \tag{4.19}$$

In general, for arbitrary d, the above calculation is straightforward and yields, for any $d \geq 2$,

$$Z_d(2n) = \sum_{p \vdash 2n} \left([t^n x^p]\mathcal{Z}^{S_\infty[S_2]}(t, \vec{x})\right)^d z_p^{d-1} \,. \tag{4.20}$$

We can generate the sequences $Z_3(2n)$ and $Z_4(2n)$ (both with $n \in [\![1, 10]\!]$) using a Mathematica program shown in Appendix B.1 and obtain, respectively,

$$1, 5, 16, 86, 448, 3580, 34981, 448628, 6854130, 121173330 \tag{4.21}$$

and

$$\begin{gathered} 1, 14, 132, 4154, 234004, 24791668, 3844630928, 809199787472, \\ 220685007519070, 75649235368772418 \,. \end{gathered} \tag{4.22}$$

Following Read [55], the number $Z_d(2n)$ of d-regular colored graphs made with $2n$ vertices is the coefficient of t^n in $\prod_m \Phi_m(t)$, where

$$\Phi_m(t) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} \frac{A_{m/2}(j)^d}{j!\, m^j} t^{mj/2} & \text{if } m \text{ is even,} \\ \displaystyle\sum_{j=0}^{\infty} \frac{((2j)!)^{d-1}}{(j!)^d} \left(\frac{m^{d-2}}{2^d}\right)^j t^{mj} & \text{if } m \text{ is odd,} \end{cases} \tag{4.23}$$

and the function $A_k(j)$ relates to the j^{th} Hermite polynomial by $A_k(j) = (i\sqrt{k})^j \mathcal{H}_j(\frac{1}{2i\sqrt{k}})$. We generate the corresponding sequences $Z_3(2n)$ and $Z_4(2n)$, $n \in [\![1, 10]\!]$, using a Mathematica program (in Appendix B.2) and the results match with (4.21) and (4.22), respectively. Hence, both methods yield the same results. The sequence (4.21) naturally corresponds to the OEIS sequence A002830 (number of 3-regular edge-colored graphs with $2n$ nodes) [56]. The sequence (4.22) however is not yet reported on the OEIS. Hence, the formula (4.20) is likely to generate arbitrary new sequences for each $d > 3$. We must underline that the above counting of observables concerns connected and disconnected graphs (generalized multi-matrix invariants). To obtain only connected invariants, we use the plethystic logarithm (for recent applications of this function in supersymmetric

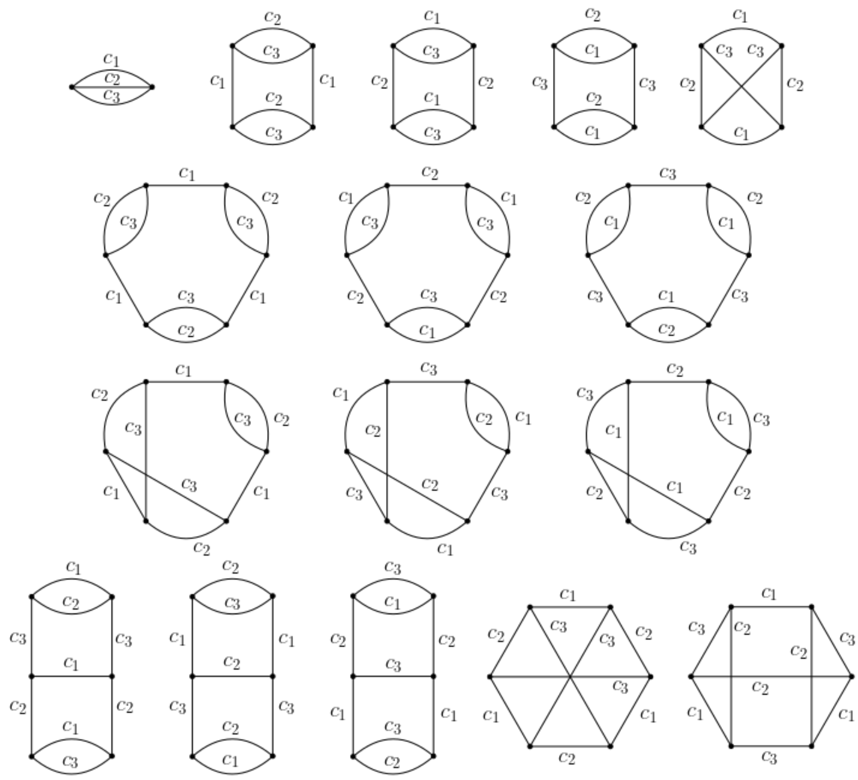


FIGURE 4.2 – Connected colored graphs associated with rank-3 orthogonal tensor invariants with up to 6 vertices.

gauge theory and further references, see [99]) transform on the generating series of the disconnected invariants. This is achieved in the following manner. Define the generating function of disconnected invariants as

$$Z_3(x) = \sum_{n=0}^{\infty} Z_3(n) x^n, \tag{4.24}$$

the plethystic logarithm is defined as

$$\text{Plog}\, Z_3(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log Z_3(x^k), \tag{4.25}$$

where $\mu(k)$ is the Möbius function given by

$$\mu(k) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k \text{ has repeating prime factors,} \\ (-1)^n & \text{if } k \text{ is a product of } n \text{ distinct primes.} \end{cases} \tag{4.26}$$

We obtain the enumeration of connected invariants (see Appendix B.2) as the coefficients in the previous series' expansion. For rank $d = 3$ and 4, respectively, up to order $n = 10$ they read,

$$1, 4, 11, 60, 318, 2806, 29359, 396196, 6231794, 112137138 \tag{4.27}$$

and

$$\begin{gathered} 1, 13, 118, 3931, 228316, 24499085, 3816396556, 805001547991, \\ 219822379032704, 75417509926065404\,. \end{gathered} \tag{4.28}$$

As an illustration, Figure 4.2 depicts the rank-3 connected orthogonal invariants up to order 3.

4.2 Topological Field Theory formulation

From the above symmetric group formulation of the counting of tensor invariants, one extracts more information *via* other correspondences. In particular, the enumeration reformulates as a partition function of a Topological Field Theory on a 2-complex (in short TFT_2) with S_{2n} and its subgroup $S_n[S_2]$ as gauge groups. For a review of TFTs, see [91, 92] and, in notation closer to what we aim at, [38, 39]. Let us however recall some key features here. Consider a 2-dimensional cellular complex (a Hausdorff space together with a cellular structure) X. We call vertices the 0-cells, edges the 1-cells and plaquettes the 2-cells. Then one can define a partition function for a finite group G by assigning a group element g_e to each edge e and a weight $w(g_P)$ to each plaquette P, where $g_P = \prod_{e \in P} g_e$. A natural choice independent of the size of the plaquette is given by

$$w(g_P) = \delta(g_P) = \begin{cases} 1 & \text{if } g_P = id, \\ 0 & \text{otherwise.} \end{cases} \tag{4.29}$$

The partition function of the model then writes

$$Z[X; G] = \frac{1}{|G|^V} \sum_{g_e} \prod_{P} w(g_P), \tag{4.30}$$

where V is the number of vertices in the cell decomposition. We will be interested in cases where G is taken to be the groups S_{2n} and $S_n[S_2]$, but let us exemplify it with S_n. Take as an elementary example the torus realized as a rectangle, with opposite sides identified. This is a cell decomposition with a single vertex, two edges a, c and a single plaquette. Assign to each edge a group element in S_n:

$$a \longrightarrow \sigma\,, \quad c \longrightarrow \gamma. \tag{4.31}$$

Thus the weight for the single plaquette is

$$w(g_P) = \delta(\gamma\sigma\gamma^{-1}\sigma^{-1}), \tag{4.32}$$

and the partition function is given by

$$Z[\mathbb{T}^2; S_n] = \frac{1}{n!} \sum_{\sigma,\gamma \in S_n} \delta(\gamma\sigma\gamma^{-1}\sigma^{-1}). \tag{4.33}$$

This partition function counts equivalence classes of homomorphisms from the fundamental group of the torus $\pi_1(\mathbb{T}^2) \simeq \langle a, c \mid cac^{-1}a^{-1} = 1\rangle$ to S_n (weighted by the number of elements of S_n which fix the homomorphism under conjugation) where the equivalence relation identifies a homomorphism with each of its conjugates by elements in S_n. By Riemann's existence theorem, this is equivalent to counting equivalence classes of covering spaces of $\mathbb{T}^2$ of degree n (see for instance [100]), counted with weight equal to the inverse of the order of the automorphism group of the cover. The partition function (4.33) thus counts n-fold covers of the torus and can be interpreted as a partition function of a TFT_2 on a cellular complex having the topology of a cylinder, as shown in Figure 4.3 below.

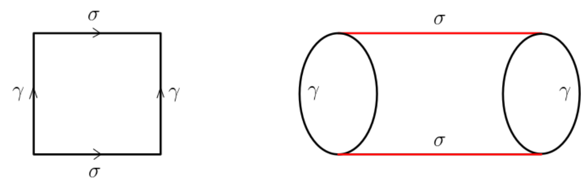


FIGURE 4.3 – TFT_2 associated with the counting of n-fold covers of the torus.

Consider now the counting of classes in the double coset (4.9), denote it as $Z_3(2n)$, and then consider the relation (4.10). Using Burnside's lemma, we have in standard notations:

$$Z_3(2n) = \frac{1}{[n!2^n]^3(2n)!} \sum_{\gamma_i \in S_n[S_2]} \sum_{\sigma_i \in S_{2n}} \sum_{\gamma \in S_{2n}} \delta(\gamma_1\sigma_1\gamma\sigma_1^{-1})\delta(\gamma_2\sigma_2\gamma\sigma_2^{-1})\delta(\gamma_3\sigma_3\gamma\sigma_3^{-1})\,, \tag{4.34}$$

where δ is the Kronecker symbol on S_{2n}. This counting interprets as a partition function of a TFT_2 on a cellular complex given by Figure 4.4. On that lattice, we use two gauge groups S_{2n} and $S_n[S_2]$. The topology of that 2-complex is that of three cylinders sharing the same end circle. Thus, enumerating orthogonal invariant corresponds to a S_{2n}–TFT$_2$ on three cylinders glued along one circle, with a restriction of the gauge group to be $S_n[S_2]$ at the opposite boundary circle. This TFT$_2$ has boundary holonomies endowed with $S_n[S_2]$ group elements.

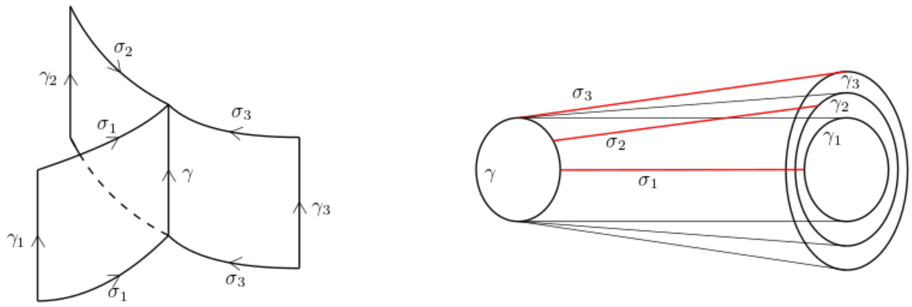


FIGURE 4.4 – S_{2n}-TFT$_2$ associated with the counting of orthogonal invariants.

By successively integrating some delta functions, the TFT$_2$ formulation produces alternative interpretations of the same counting. We extract γ from (4.34) and get $\gamma = \sigma_3^{-1}\gamma_3^{-1}\sigma_3$ such that

$$Z_3(2n) = \frac{1}{[n!2^n]^3(2n)!} \sum_{\gamma_i \in S_n[S_2]} \sum_{\sigma_i \in S_{2n}} \delta(\gamma_1\sigma_1(\sigma_3^{-1}\gamma_3^{-1}\sigma_3)\sigma_1^{-1})\delta(\gamma_2\sigma_2(\sigma_3^{-1}\gamma_3^{-1}\sigma_3)\sigma_2^{-1}) . \tag{4.35}$$

A change of variables $\sigma_{1,2} \leftarrow \sigma_{1,2}\sigma_3^{-1}$ leads us to

$$Z_3(2n) = \frac{1}{[n!2^n]^3} \sum_{\gamma_i \in S_n[S_2]} \sum_{\sigma_{1,2} \in S_{2n}} \delta(\gamma_1\sigma_1\gamma_3\sigma_1^{-1})\delta(\gamma_2\sigma_2\gamma_3\sigma_2^{-1}) . \tag{4.36}$$

This integration illustrates, in Figure 4.5, as the removal of a 1-cell associated with the variable γ in the 2-complex. The partition function therefore shows two types of invariances: the extraction of γ corresponds to one type of topological invariance, and then, it is followed by the change of variables $\sigma_{1,2} \to \sigma_{1,2}\sigma_3^{-1}$ corresponding to a topological invariance of a second kind.

Thus, the partition function (4.36) can also be written as

$$Z_3(2n) = Z(\mathbb{S}^1 \times I; D_{S_n[S_2]}{}^{\times 3}) , \tag{4.37}$$

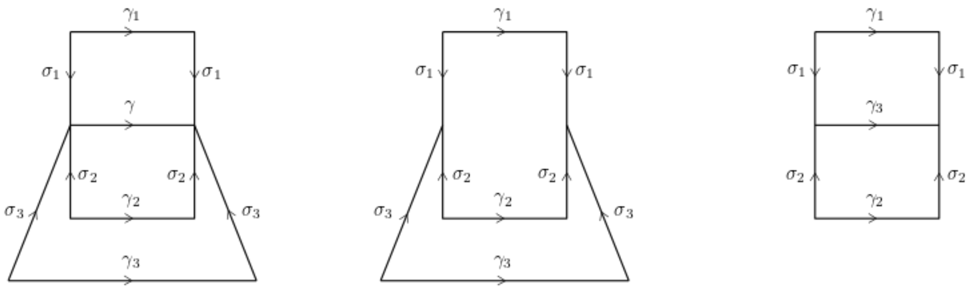


FIGURE 4.5 – Topological transformations of the 2-complex leaving the partition function stable.

where $Z(\mathbb{S}^1 \times I; D_{S_n[S_2]}{}^{\times 3})$ is the partition function obtained by inserting three $S_n[S_2]$-defects - written $D_{S_n[S_2]}$ -, one at each end of the cylinder $\mathbb{S}^1 \times I$, and another one at finite time $t_0 \in I$, for some finite interval I, see Figure 4.6. A defect is defined as a closed non-intersecting loop with a marked point. The relation (4.37) shows that orthogonal invariants are in one-to-one correspondence with n-fold covers of the cylinder with three defects, up to a (symmetry) factor, *viz.* the stabilizer subgroup of the graph that we denote $\mathrm{Aut}(G_{\sigma_1,\sigma_2,\sigma_3})$.

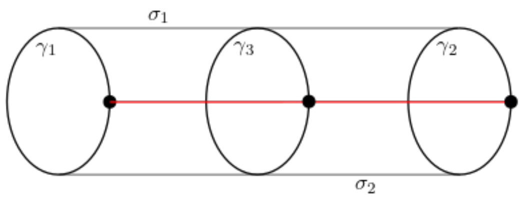


FIGURE 4.6 – Cylinder with three defects.

The order of the stabilizer infers from

$$\mathrm{Sym}(\sigma_1, \sigma_2) = \sum_{\gamma_i \in S_n[S_2]} \delta(\gamma_1 \sigma_1 \gamma_3 \sigma_1^{-1}) \delta(\gamma_2 \sigma_2 \gamma_3 \sigma_2^{-1}) = \mathrm{Aut}(G_{\sigma_1,\sigma_2,\sigma_3}), \tag{4.38}$$

which also relates to the number of equivalences $(S_n[S_2] \times S_n[S_2]) \backslash (S_n \times S_n)/\mathrm{Diag}(S_n[S_2])$ corresponding to a fixed (σ_1, σ_2).

The TFT formulation of the counting enriches it with a geometrical picture. Most of the time, the base space of the TFT is viewed as a string worldsheet. The counting now becomes a counting of worldsheet maps over a cylinder with defects. As noticed elsewhere [29, 30], this once again shows that a link may exist between tensor models and string theory, which could be elucidated *via* the TFT formalism. Such a link may be worth investigating in the future.

Rank-d counting and TFT$_2$ – More generally, for rank $d \geq 3$, the counting $Z_d(2n)$ has a TFT$_2$ formulation that generalizes what we discussed above in a straightforward manner:

$$\begin{aligned} Z_d(2n) &= \frac{1}{[n!2^n]^d(2n)!} \sum_{\gamma_i \in S_n[S_2]} \sum_{\sigma_i \in S_{2n}} \sum_{\gamma \in S_{2n}} \prod_{i=1}^{d} \delta(\gamma_i \sigma_i \gamma \sigma_i^{-1}) \\ &= \frac{1}{[n!2^n]^d} \sum_{\gamma_i \in S_n[S_2]} \sum_{\sigma_i \in S_{2n}} \prod_{i=1}^{d-1} \delta(\gamma_i \sigma_i \gamma_d \sigma_i^{-1}) , \end{aligned} \tag{4.39}$$

where we extracted γ as previously: $\gamma = \sigma_d^{-1}\gamma_d^{-1}\sigma_d$.
The first equation of (4.39) shows that, in rank d, the TFT$_2$-formulation of the counting extends Figure 4.4 as the gluing of d cylinders along one circle. After integration, the second equation reveals that the counting of orthogonal invariants therefore amounts to the counting of weighted covers of $d-1$ cylinders with d defects, one of the defects being shared by all cylinders. In formula, denoting C_i the i^{th} cylinder with base circle $\mathbb{S}^1_i$ and C^{d-1} the quotient space $\bigsqcup_{i=1}^{d-1} C_i \,/\, \sim$, with the identification $\mathbb{S}^1_i \sim \mathbb{S}^1_j$, we have

$$Z_d(2n) = Z(C^{d-1}; D_{S_n[S_2]}{}^{\times d}).$$

4.3 The counting as a Kronecker sum

We now revisit the counting (4.34) under a different light, that of the representation theory of the symmetric group (Appendix A reviews the main identities used in this chapter and the following). Irreducible representations (irreps) of the symmetric group S_{2n} are labeled by partitions $R \vdash 2n$, that are also Young diagrams.
Starting from the Burnside lemma formulation of (4.34), consider the following expansion of the counting of rank-3 invariants using the representation theory of S_{2n}:

$$\begin{aligned} Z_3(2n) &= \frac{1}{[n!2^n]^3(2n)!} \sum_{\gamma_l \in S_n[S_2]} \sum_{\sigma_l \in S_{2n}} \sum_{\gamma \in S_{2n}} \delta(\gamma_1\sigma_1\gamma\sigma_1^{-1})\delta(\gamma_2\sigma_2\gamma\sigma_2^{-1})\delta(\gamma_3\sigma_3\gamma\sigma_3^{-1}) \\ &= \frac{1}{[n!2^n]^3(2n)!} \sum_{\gamma_l \in S_n[S_2]} \sum_{\gamma \in S_{2n}} \sum_{R_l \vdash 2n} \chi^{R_1}(\gamma_1)\chi^{R_1}(\gamma)\chi^{R_2}(\gamma_2)\chi^{R_2}(\gamma)\chi^{R_3}(\gamma_3)\chi^{R_3}(\gamma) \\ &= \frac{1}{[n!2^n]^3} \sum_{R_l \vdash 2n} \mathbf{C}(R_1,R_2,R_3) \sum_{\gamma_1 \in S_n[S_2]} \chi^{R_1}(\gamma_1) \sum_{\gamma_2 \in S_n[S_2]} \chi^{R_2}(\gamma_2) \sum_{\gamma_3 \in S_n[S_2]} \chi^{R_3}(\gamma_3), \end{aligned} \tag{4.40}$$

where χ^R denotes the character in the representation R, we used the identity (A.5) in Appendix A.1 to compute the deltas, and the Kronecker coefficient is defined as

$$\mathbf{C}(R_1, R_2, R_3) = \frac{1}{(2n)!} \sum_{\gamma \in S_{2n}} \chi^{R_1}(\gamma)\chi^{R_2}(\gamma)\chi^{R_3}(\gamma)\,. \tag{4.41}$$

Recall that the Kronecker defines the multiplicity of the representation R_3 in the tensor product $R_1 \otimes R_2$, or the multiplicity of the trivial representation in $R_1 \otimes R_2 \otimes R_3$ when expanded back in irreps.

Above, the sums over the subgroup $S_n[S_2]$ have not yet been performed. To proceed with these sums, we will use a useful result by Howe [93] (see also [95, 96] or a more recent use of it in [43]):

$$\frac{1}{|S_n[S_2]|} \sum_{\gamma \in S_n[S_2]} \chi^R(\gamma) = \delta_{R,\text{even}}, \tag{4.42}$$

where $\delta_{R,\text{even}} = 1$ if R is an "even" partition, that is, all its row lengths are even - we denote it naturally $2R$ - and $\delta_{R,\text{even}} = 0$ otherwise. This result is actually a consequence of what is called Littlewood's formula and Frobenius's reciprocity, which we now recall. Written out in terms of characters, Littlwood's formula (see Proposition 4.1 in [94]) reads, for all $\sigma \in S_{2n}$,

$$\mathbb{1}_{S_n[S_2]} \uparrow^{S_{2n}} (\sigma) = \sum_{S \vdash 2n} \chi^{2S}(\sigma), \tag{4.43}$$

where $\mathbb{1}_{S_n[S_2]} \uparrow^{S_{2n}}$ is the character of the trivial representation of $S_n[S_2]$ induced on S_{2n}. Now denoting $\chi^R \downarrow_{S_n[S_2]}$ the restriction of a character of S_{2n} to $S_n[S_2]$, the following equality, known as Frobenius' reciprocity theorem[4], holds:

$$\left\langle \mathbb{1}_{S_n[S_2]}, \chi^R \downarrow_{S_n[S_2]} \right\rangle_{S_n[S_2]} = \left\langle \mathbb{1}_{S_n[S_2]} \uparrow^{S_{2n}}, \chi^R \right\rangle_{S_{2n}}. \tag{4.44}$$

By expliciting the scalar products, the previous equation becomes successively

$$\frac{1}{n!2^n} \sum_{\gamma \in S_n[S_2]} \mathbb{1}_{S_n[S_2]}(\gamma)\chi^R \downarrow_{S_n[S_2]} (\gamma) = \frac{1}{(2n)!} \sum_{\sigma \in S_{2n}} \mathbb{1}_{S_n[S_2]} \uparrow^{S_{2n}} (\sigma)\chi^R(\sigma), \tag{4.45}$$

$$\frac{1}{n!2^n} \sum_{\gamma \in S_n[S_2]} \chi^R(\gamma) = \frac{1}{(2n)!} \sum_{S \vdash 2n} \sum_{\sigma \in S_{2n}} \chi^{2S}(\sigma)\chi^R(\sigma), \tag{4.46}$$

$$\frac{1}{n!2^n} \sum_{\gamma \in S_n[S_2]} \chi^R(\gamma) = \sum_{S \vdash 2n} \delta_{R,2S}, \tag{4.47}$$

[4] This is a special case. In all generality, for G a finite group with subgroup H and class functions $\varphi : G \to \mathbb{C}$ and $\psi : H \to \mathbb{C}$, the theorem states $\langle \psi, \mathrm{Res}^G_H\, \varphi \rangle_H = \langle \mathrm{Ind}^G_H\, \psi, \varphi \rangle_G$.

which gives the desired result. Note that in the second line we used Littlewood's formula and the fact that by definition for $\gamma \in S_n[S_2]$, $\mathbb{1}_{S_n[S_2]}(\gamma) = 1$ and $\chi^R \downarrow_{S_n[S_2]} (\gamma) = \chi^R(\gamma)$, while in the third, the characters' orthogonality relation (A.6) was used.
Thus, we obtain, inserting this in (4.40)

$$Z_3(2n) = \sum_{R_l \vdash 2n} \mathbf{C}(2R_1, 2R_2, 2R_3)\,. \tag{4.48}$$

Comparing this sequence and (4.21), we produce a Sage code (see Appendix B.3) showing that the numbers generated by (4.48) match with (4.21).
In the next chapter, we will show that this number is also the dimension of an algebra $\mathcal{K}_3(2n)$. It is an interesting problem to investigate how the counting of colored graphs could contribute to the famous problem of giving a combinatorial interpretation to the Kronecker coefficients [51, 52] (in the same way that Littlewood-Richardson coefficients have found a combinatorial description). From previous work [30], we know that the sum of squares of Kronecker coefficients associated with S_n equals the number of d-regular bipartite colored graphs made with n black and n white vertices. Here the interpretation is the following, the number of d-regular colored graphs (not necessarily bipartite) equals the sum of all Kroneckers precluded those that are defined with partitions with odd rows. An idea to contribute to the above problem is to refine the counting of graphs in a way to boil down to a single Kronecker coefficient. In other words, given a non vanishing Kronecker coefficient is it possible to list all graphs contributing to that Kronecker coefficient ? This is certainly a difficult problem that will require new tools in representation theory.

Counting rank-d tensor invariants – The above counting generalizes quite naturally at any rank d as

$$\begin{aligned} Z_d(2n) &= \frac{1}{[n!2^n]^d (2n)!} \sum_{\gamma_l \in S_n[S_2]} \sum_{\sigma_l \in S_{2n}} \sum_{\gamma \in S_{2n}} \prod_{i=1}^{d} \delta(\gamma_i \sigma_i \gamma \sigma_i^{-1}) \\ &= \frac{1}{[n!2^n]^d} \sum_{R_l \vdash 2n} \sum_{\gamma_l \in S_n[S_2]} \mathbf{C}_d(R_1, \dots, R_d) \chi^{R_1}(\gamma_1) \dots \chi^{R_d}(\gamma_d) \\ &= \sum_{R_l \vdash 2n} \mathbf{C}_d(2R_1, \dots, 2R_d), \end{aligned} \tag{4.49}$$

where we introduced the notation

$$\mathbf{C}_k(R_1, \dots, R_k) = \frac{1}{(2n)!} \sum_{\gamma \in S_{2n}} \chi^{R_1}(\gamma) \dots \chi^{R_k}(\gamma)\,. \tag{4.50}$$

This counts the multiplicity of the one dimensional trivial S_{2n} irrep in the tensor product of irreps $R_1 \otimes \ldots \otimes R_k$, $k \geq 4$. It expresses as a convoluted product of Kronecker coefficients as

$$\mathbf{C}_k(R_1, \ldots, R_k) = \sum_{S_l \vdash 2n} \mathbf{C}(R_1, R_2, S_1) \left[\prod_{i=1}^{k-4} \mathbf{C}(S_i, R_{i+2}, S_{i+1}) \right] \mathbf{C}(S_{k-3}, R_{k-1}, R_k) \,. \quad (4.51)$$

Chapter 5

Double coset algebra

We now discuss the underlying structure, an algebra, determined by the counting of the $O(N)$ invariants. The rank-3 case is first addressed for the sake of simplicity, and from that, we will infer the general rank-d case whenever possible.
Consider $\mathbb{C}[S_{2n}]$, the group algebra of S_{2n}. Our construction depends on tensor products of that space.

5.1 $\mathcal{K}_d(2n)$ as a double coset algebra in $\mathbb{C}[S_{2n}]^{\otimes d}$

We fix $d = 3$. Consider $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$ as an element of the group algebra $\mathbb{C}[S_{2n}]^{\otimes 3}$, and three left actions of the subgroup $S_n[S_2]$ and the diagonal right action of $\mathrm{Diag}(\mathbb{C}[S_{2n}])$ on this triple as:

$$\sigma_1 \otimes \sigma_2 \otimes \sigma_3 \to \sum_{\gamma_i \in S_n[S_2]} \sum_{\gamma \in S_{2n}} \gamma_1 \sigma_1 \gamma \otimes \gamma_2 \sigma_2 \gamma \otimes \gamma_3 \sigma_3 \gamma \,. \tag{5.1}$$

$\mathcal{K}_3(2n)$ is the vector subspace of $\mathbb{C}[S_{2n}]^{\otimes 3}$ which is invariant under these subgroup actions:

$$\mathcal{K}_3(2n) = \mathrm{Span}_{\mathbb{C}} \left\{ \sum_{\gamma_i \in S_n[S_2]} \sum_{\gamma \in S_{2n}} \gamma_1 \sigma_1 \gamma \otimes \gamma_2 \sigma_2 \gamma \otimes \gamma_3 \sigma_3 \gamma, \quad (\sigma_1, \sigma_2, \sigma_3) \in S_{2n}^3 \right\} . \tag{5.2}$$

It is obvious that $\dim \mathcal{K}_3(2n) = Z_3(2n)$, since each basis element represents the graph equivalent class counted once in $Z_3(2n)$. Pick two basis elements, called henceforth graph basis elements, and consider their product

$$\Big[\sum_{\gamma_i \in S_n[S_2]} \sum_{\gamma \in S_{2n}} \gamma_1 \sigma_1 \gamma \otimes \gamma_2 \sigma_2 \gamma \otimes \gamma_3 \sigma_3 \gamma \Big] \Big[\sum_{\tau_i \in S_n[S_2]} \sum_{\tau \in S_{2n}} \tau_1 \sigma_1' \tau \otimes \tau_2 \sigma_2' \tau \otimes \tau_3 \sigma_3' \tau \Big]$$

$$= \sum_{\gamma_i,\tau_i \in S_n[S_2]} \sum_{\gamma,\tau \in S_{2n}} \gamma_1\sigma_1\gamma\tau_1\sigma_1'\tau \otimes \gamma_2\sigma_2\gamma\tau_2\sigma_2'\tau \otimes \gamma_3\sigma_3\gamma\tau_3\sigma_3'\tau$$
$$= \sum_{\tau_i \in S_n[S_2]} \sum_{\gamma \in S_{2n}} \Big[\sum_{\gamma_i \in S_n[S_2]} \sum_{\tau \in S_{2n}} \gamma_1(\sigma_1\gamma\tau_1\sigma_1')\tau \otimes \gamma_2(\sigma_2\gamma\tau_2\sigma_2')\tau \otimes \gamma_3(\sigma_3\gamma\tau_3\sigma_3')\tau \Big]. \quad (5.3)$$

This shows that the multiplication remains in the vector space. Hence, $\mathcal{K}_3(2n)$ is an algebra and (5.3) defines a graph multiplication. The proof is totally similar for $\mathcal{K}_d(2n)$ (considering d factors in the tensor product) which is thus an algebra of dimension $Z_d(2n)$.

The product of graphs in the algebra $\mathcal{K}_3(2n)$ illustrates as in Figure 5.1 below.

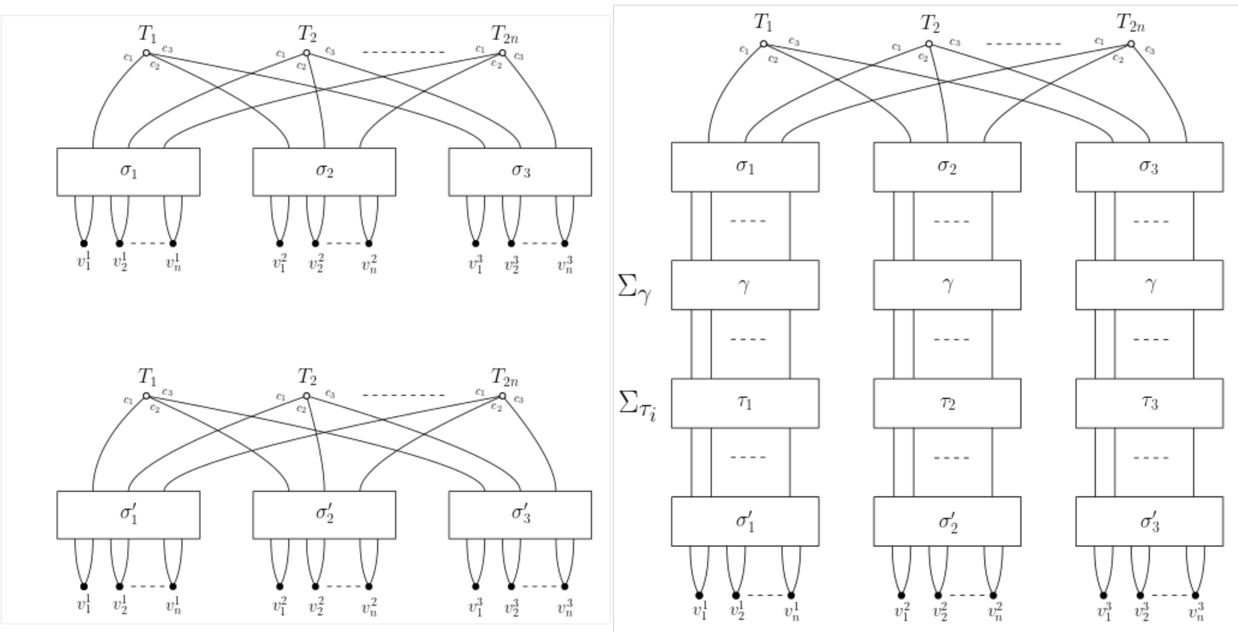


FIGURE 5.1 – Product of two graph basis elements (on the left) gives a sum of graphs (on the right).

Gauge fixing – There is a gauge fixing procedure in the construction of orthogonal invariants. One initially fixes a permutation σ_i but is still able to generate all invariants. Consider $\xi = (12)(34)\dots(2n-1,2n)$, and fix σ_1 to belong to the stabilizer of ξ, *i.e.* $\sigma_1^{-1}\xi\sigma_1 = \xi$. Since $\mathrm{Stab}_\xi = S_n[S_2]$, we simply mean that we choose σ_1 to be in that subgroup. We already observe a difference with the unitary case [30]. Indeed, while the gauge fixing in the unitary case leads to the definition of a permutation centralizer algebra, the gauge fixing here will not bring such an algebra. The main difference with the unitary case also rests on the fact that the left and right invariances on the triple $(\sigma_1, \sigma_2, \sigma_3)$ in this case are radically different.

Associativity – In the graph basis, we can check the associativity of the product of elements of $\mathcal{K}_3(2n)$:

$$\begin{aligned}
&\Bigg(\Big[\sum_{\gamma_i\in S_n[S_2]}\sum_{\gamma\in S_{2n}}\gamma_1\sigma_1\gamma\otimes\gamma_2\sigma_2\gamma\otimes\gamma_3\sigma_3\gamma\Big]\Big[\sum_{\tau_i\in S_n[S_2]}\sum_{\tau\in S_{2n}}\tau_1\sigma_1'\tau\otimes\tau_2\sigma_2'\tau\otimes\tau_3\sigma_3'\tau\Big]\Bigg)\\
&\times\Big[\sum_{\alpha_i\in S_n[S_2]}\sum_{\alpha\in S_{2n}}\alpha_1\sigma_1''\alpha\otimes\alpha_2\sigma_2''\alpha\otimes\alpha_3\sigma_3''\alpha\Big]\\
&=\sum_{\tau_i,\,\alpha_i}\sum_{\gamma,\,\tau}\Big[\sum_{\gamma,\alpha}\gamma_1\sigma_1\gamma\tau_1\sigma_1'\tau\alpha_1\sigma_1''\alpha\otimes\gamma_2\sigma_2\gamma\tau_2\sigma_2'\tau\alpha_2\sigma_2''\alpha\otimes\gamma_3\sigma_3\gamma\tau_3\sigma_3'\tau\alpha_3\sigma_3''\alpha\Big]\\
&=\Big[\sum_{\gamma_i\in S_n[S_2]}\sum_{\gamma\in S_{2n}}\gamma_1\sigma_1\gamma\otimes\gamma_2\sigma_2\gamma\otimes\gamma_3\sigma_3\gamma\Big]\\
&\times\Bigg(\Big[\sum_{\tau_i\in S_n[S_2]}\sum_{\tau\in S_{2n}}\tau_1\sigma_1'\tau\otimes\tau_2\sigma_2'\tau\otimes\tau_3\sigma_3'\tau\Big]\Big[\sum_{\alpha_i\in S_n[S_2]}\sum_{\alpha\in S_{2n}}\alpha_1\sigma_1''\alpha\otimes\alpha_2\sigma_2''\alpha\otimes\alpha_3\sigma_3''\alpha\Big]\Bigg)\,.
\end{aligned} \tag{5.4}$$

The proof easily extends to any d, and we therefore claim the following:

Proposition 5.1.1. *$\mathcal{K}_3(2n)$ is an associative unital sub-algebra of $\mathbb{C}[S_{2n}]^{\otimes 3}$.*

The unit is given by the equivalence class of (id, id, id). Such element corresponds to the disconnected graph made with n connected components with full contraction of n pairs of tensors (*i.e.* dipole graphs).

Pairing – There is an inner product (that we will call pairing) on $\mathcal{K}_d(2n)$ defined from the linear extension of the delta function from the symmetric group to the tensor product group algebra (see (A.24) in Appendix A.3 for details pertaining to the following notation). Take two basis elements (in obvious notation) and evaluate using proper change of variables:

$$\begin{aligned}
\boldsymbol{\delta}\Big(\sum_{\gamma_i,\gamma}\bigotimes_i^d\gamma_i\sigma_i\gamma\,;\sum_{\tau_i,\tau}\bigotimes_i^d\tau_i\sigma_i'\tau\Big) &= \sum_{\gamma_i,\gamma}\sum_{\tau_i,\tau}\prod_i^d\delta(\gamma_i\sigma_i\gamma(\tau_i\sigma_i'\tau)^{-1})\\
&=(2n)!(n!2^n)\sum_{\gamma_i,\gamma}\prod_i^d\delta(\gamma_i\sigma_i\gamma(\sigma_i')^{-1})\,.
\end{aligned} \tag{5.5}$$

Thus, either the tuples $(\sigma_1,\dots,\sigma_d)$ and $(\sigma_1',\dots,\sigma_d')$ define equivalent graphs $G_{\sigma_1\dots\sigma_d}$ and $G_{\sigma_1'\dots\sigma_d'}$, respectively, or the result is 0. This precisely tells us that the graph basis forms an orthogonal system. The above computes further using the order of the automorphism group of the graph

$$\boldsymbol{\delta}\Big(\sum_{\gamma_i,\gamma}\bigotimes_i^d \gamma_i\sigma_i\gamma\,;\sum_{\tau_i,\tau}\bigotimes_i^d \tau_i\sigma_i'\tau\Big) = (2n)!(n!2^n)\delta(G_{\sigma_1\ldots\sigma_d};G_{\sigma_1'\ldots\sigma_d'})\mathrm{Aut}(G_{\sigma_1\ldots\sigma_d})\,. \qquad (5.6)$$

Therefore, there exists a non degenerate bilinear pairing on $\mathcal{K}_d(2n)$ and the following holds:

Theorem 5.1.2. *$\mathcal{K}_d(2n)$ is an associative unital semi-simple algebra.*

As a corollary of Theorem 5.1.2, the Wedderburn-Artin[1] theorem guarantees that $\mathcal{K}_d(2n)$ decomposes into matrix subalgebras. It might be interesting to investigate a basis of such a decomposition of $\mathcal{K}_d(2n)$ in irreducible matrix subalgebras. One could be tempted to think that, at $d = 3$, restricting to $\mathcal{K}_3(2n)$, the Kronecker coefficients for even partitions could be themselves squares, and therefore define the dimensions of the irreducible subalgebras. This is not the case as can easily be shown using the same Sage code given in Appendix B.3 (by printing the Kronecker). This point is postponed for future investigations. In the meantime, it is legitimate to ask a representation basis with labels that reflect the dimension (4.48). This is the purpose of the next section.

5.2 Constructing a representation theoretic basis of $\mathcal{K}_3(2n)$

Let us introduce the representation basis of $\mathbb{C}[S_{2n}]$ given by the elements

$$Q^R_{ij} = \frac{\kappa_R}{(2n)!}\sum_{\sigma\in S_{2n}} D^R_{ij}(\sigma)\sigma\,, \qquad \text{with } \kappa_R^2 = (2n)!\,d(R)\,, \qquad (5.7)$$

that obey the orthogonality relation $\delta(Q^R_{ij}\,;Q^{R'}_{i'j'}) = \delta_{RR'}\delta_{ii'}\delta_{jj'}$. The basis $\{Q^R_{ij}\}$ counts $\sum_{R\vdash 2n} d(R)^2 = (2n)!$ elements and forms the Fourier theoretic basis of $\mathbb{C}[S_{2n}]$. Appendix A.3 collects a few other properties of this basis for a general permutation group.

We fix $d = 3$ and build now the invariant representation theoretic (Fourier for short) basis of the algebra $\mathcal{K}_3(2n)$ (5.2). Consider the right diagonal action $\rho_R(\cdot)$ and the three

[1] The Wedderburn-Artin theorem states that any finite dimensional semi-simple algebra is isomorphic to a finite product of matrix algebras over division algebras.

left actions $\varrho_i(\cdot)$ on the tensor product $\mathbb{C}[S_{2n}]^{\otimes 3}$. Then we write:

$$\begin{aligned}
&\sum_{\gamma_1,\gamma_2,\gamma_3\in S_n[S_2]}\sum_{\gamma\in S_{2n}} \varrho_1(\gamma_1)\varrho_2(\gamma_2)\varrho_3(\gamma_3)\rho_R(\gamma)\, Q^{R_1}_{i_1j_1}\otimes Q^{R_2}_{i_2j_2}\otimes Q^{R_3}_{i_3j_3}\\
&=\sum_{\gamma_a}\sum_{\gamma}\gamma_1\, Q^{R_1}_{i_1j_1}\gamma\otimes\gamma_2\, Q^{R_2}_{i_2j_2}\,\gamma\otimes\gamma_3\, Q^{R_3}_{i_3j_3}\,\gamma\\
&=\sum_{\gamma_a}\sum_{\gamma}\sum_{p_l,q_l} D^{R_1}_{p_1i_1}(\gamma_1)Q^{R_1}_{p_1q_1}D^{R_1}_{j_1q_1}(\gamma)\otimes D^{R_2}_{p_2i_2}(\gamma_2)Q^{R_2}_{p_2q_2}D^{R_2}_{j_2q_2}(\gamma)\otimes D^{R_3}_{p_3i_3}(\gamma_3)Q^{R_3}_{p_3q_3}D^{R_3}_{j_3q_3}(\gamma)\\
&=\frac{(2n)!}{d(R_3)}\sum_{\gamma_a}\sum_{p_l,q_l}\sum_{\tau} C^{R_1,R_2;R_3,\tau}_{j_1,j_2;j_3}C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3}D^{R_1}_{p_1i_1}(\gamma_1)D^{R_2}_{p_2i_2}(\gamma_2)D^{R_3}_{p_3i_3}(\gamma_3)Q^{R_1}_{p_1q_1}\otimes Q^{R_2}_{p_2q_2}\otimes Q^{R_3}_{p_3q_3}.
\end{aligned} \tag{5.8}$$

We used (A.22) to multiply group elements with the Q-basis, see Appendix A.3; then use (A.18) to sum over γ the three representation matrices, see Appendix A.2.
We couple this last result with a Clebsch-Gordan coefficient, in order to get, using (A.16):

$$\begin{aligned}
&\sum_{j_l} C^{R_1,R_2;R_3,\tau}_{j_1,j_2;j_3}\sum_{\gamma_a}\sum_{\gamma}\varrho_1(\gamma_1)\varrho_2(\gamma_2)\varrho_3(\gamma_3)\rho_R(\gamma)\, Q^{R_1}_{i_1j_1}\otimes Q^{R_2}_{i_2j_2}\otimes Q^{R_3}_{i_3j_3}\\
&=(2n)!\sum_{p_l,q_l} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3}\sum_{\gamma_1} D^{R_1}_{p_1i_1}(\gamma_1)\sum_{\gamma_2} D^{R_2}_{p_2i_2}(\gamma_2)\sum_{\gamma_3} D^{R_3}_{p_3i_3}(\gamma_3)Q^{R_1}_{p_1q_1}\otimes Q^{R_2}_{p_2q_2}\otimes Q^{R_3}_{p_3q_3}\,.
\end{aligned} \tag{5.9}$$

Once again, we should stress that $\sum_{\gamma\in S_n[S_2]} D^R_{pq}(\gamma)\neq 0$ if and only if R is a partition of $2n$ with even rows. This condition will be always assumed in the next calculations. Now, we can split the Wigner matrix element using branching coefficients of $S_n[S_2]$ in S_{2n}. Consider V^R an irrep of S_{2n} (see Appendix A listing a few basic facts on representation theory of S_n and our notations), and the subgroup inclusion $S_n[S_2]\subset S_{2n}$, we can decompose V^R in irreps V^r of $S_n[S_2]$ as

$$V^R=\bigoplus_r V^r\otimes V_{R,r}, \tag{5.10}$$

where $V_{R,r}$ is a vector space of dimension the multiplicity of the irreducible representations r in R. A state in this decomposition denotes $|r,m_r,\nu_r\rangle$, where m_r labels the states of V^r and $\nu_r\in[\![1,\dim V_{R,r}]\!]$.
The branching coefficients that are of interest are the coefficients of $|r,m_r,\nu_r\rangle$ when decomposed in an orthonormal basis of the irreps R:

$$B^{R;\,r,\nu_r}_{i;\,m_r}=\langle R,i\,|r,m_r,\nu_r\rangle=\langle r,m_r,\nu_r\,|R,i\rangle\ . \tag{5.11}$$

The last relation is deduced from the fact that we use real representations. Using the decomposition of the identity, the branching coefficients satisfy the following identities

$$\sum_i B^{R;\,r,\nu_r}_{i;\,m_r} B^{R;\,s,\nu_s}_{i;\,m_s} = \delta_{rs}\delta_{\nu_r\nu_s}\delta_{m_r m_s} \tag{5.12}$$

$$\sum_{r,m_r,\nu_r} B^{R;\,r,\nu_r}_{i;\,m_r} B^{R';\,r,\nu_r}_{i';\,m_r} = \delta_{RR'}\delta_{ii'}\,. \tag{5.13}$$

We have the following useful relation, for $\sigma \in S_n[S_2]$,

$$\sum_j D^R_{ij}(\sigma)\, B^{R;\,r,\nu_r}_{j;m_r} = \sum_{m'_r} D^r_{m_r m'_r}(\sigma)\, B^{R;\,r,\nu_r}_{i;m'_r}\,, \tag{5.14}$$

where $D^r_{m_r m'_r}(\sigma)$ is the representation matrix of σ as an element of $S_n[S_2]$. Restricting this to $r = [2n]$, the one-dimensional trivial representation of $S_n[S_2]$, we obtain:

$$\sum_j D^R_{ij}(\sigma)\, B^{R;[2n],1}_{j;1} = D^{[2n]}_{11}(\sigma)\, B^{R;[2n],1}_{i;1} = B^{R;[2n],1}_{i;1}\,. \tag{5.15}$$

We now treat the sum over the representation matrices in (5.9). Inserting twice a complete set of states therein, we get

$$\sum_{\sigma\in S_n[S_2]} D^R_{ij}(\sigma) = \sum_{\sigma\in S_n[S_2]} \sum_{\substack{r,\nu_r,m_r\\ s,\nu_s,m_s}} B^{R;\,r,\nu_r}_{i;\,m_r} B^{R;\,s,\nu_s}_{j;\,m_s} \langle r,\nu_r,m_r|\sigma|s,\nu_s,m_s\rangle\,. \tag{5.16}$$

Noting that $\sum_{\sigma\in S_n[S_2]}\sigma = \sum_{\sigma\in S_n[S_2]} \mathbb{1}_{S_n[S_2]}(\sigma)\sigma$ is, up to the factor $1/[n!2^n]$, nothing but the projector onto the trivial representation of $S_n[S_2]$, the overlap computes to

$$\sum_{\sigma\in S_n[S_2]} \langle r,\nu_r,m_r|\sigma|s,\nu_s,m_s\rangle = (2^n n!)\delta_{r,[2n]}\delta_{s,[2n]}\delta_{1m_r}\delta_{1m_s}\delta_{1\nu_s}\delta_{1\nu_r}\,, \tag{5.17}$$

since we have

$$\begin{aligned}
&\sum_{\sigma\in S_n[S_2]} \sigma|s,\nu_s,m_s\rangle = \sum_{\sigma\in S_n[S_2]} \mathbb{1}_{S_n[S_2]}(\sigma)\sum_k D^s_{m_s k}(\sigma)|s,\nu_s,k\rangle\\
&= \sum_{\sigma\in S_n[S_2]} D^{[2n]}_{11}(\sigma)\sum_k D^s_{m_s k}(\sigma)|s,\nu_s,k\rangle = \frac{2^n n!}{d([2n])}\sum_k \delta_{s,[2n]}\delta_{1m_s}\delta_{1\nu_s}\delta_{1k}|s,\nu_s,k\rangle\\
&= (2^n n!)\delta_{s,[2n]}\delta_{1m_s}\delta_{1\nu_s}|[2n],1,1\rangle\,.
\end{aligned} \tag{5.18}$$

Hence,

$$\sum_{\sigma\in S_n[S_2]} D^R_{ij}(\sigma) = 2^n n!\, B^R_i B^R_j\,, \tag{5.19}$$

where we have defined $B^R_i = \langle R, i\,|[2n], 1, 1\rangle$.
From the above calculation, we finally get from (5.9):

$$\begin{aligned} &\sum_{j_l} C^{R_1,R_2;R_3,\tau}_{j_1,j_2;j_3} \sum_{\gamma_a}\sum_{\gamma} \varrho_1(\gamma_1)\varrho_2(\gamma_2)\varrho_3(\gamma_3)\rho_R(\gamma)\, Q^{R_1}_{i_1j_1}\otimes Q^{R_2}_{i_2j_2}\otimes Q^{R_3}_{i_3j_3} \\ &= (2n)!(n!2^n)^3 B^{R_1}_{i_1}B^{R_2}_{i_2}B^{R_3}_{i_3}\sum_{p_l,q_l} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3} B^{R_1}_{p_1}B^{R_2}_{p_2}B^{R_3}_{p_3} Q^{R_1}_{p_1q_1}\otimes Q^{R_2}_{p_2q_2}\otimes Q^{R_3}_{p_3q_3}. \end{aligned} \tag{5.20}$$

We now define an element

$$\begin{aligned} Q^{R_1,R_2,R_3,\tau} &= \kappa_{\vec{R}} \sum_{p_l,q_l} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3} B^{R_1}_{p_1}B^{R_2}_{p_2}B^{R_3}_{p_3} Q^{R_1}_{p_1q_1}\otimes Q^{R_2}_{p_2q_2}\otimes Q^{R_3}_{p_3q_3} \\ &= \kappa_{\vec{R}} \frac{\kappa_{R_1}\kappa_{R_2}\kappa_{R_3}}{((2n)!)^3} \sum_{\sigma_i}\sum_{p_l,q_l} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3}\Big[\prod_{i=1}^{3} B^{R_i}_{p_i} D^{R_i}_{p_iq_i}(\sigma_i)\Big]\sigma_1\otimes\sigma_2\otimes\sigma_3, \end{aligned} \tag{5.21}$$

where $\kappa_{\vec{R}}$ is a normalization constant to be fixed later and the notation $\vec{R}$ stands for (R_1, R_2, R_3). The set $\{Q^{R_1,R_2,R_3,\tau}\}$ is of cardinality the counting of orthogonal invariants given by (4.48).

Invariance – Let us check that the element $Q^{R_1,R_2,R_3,\tau}$ is invariant under left multiplication on each factor and diagonal right multiplication:

$$\begin{aligned} &(\gamma_1\otimes\gamma_2\otimes\gamma_3)Q^{R_1,R_2,R_3,\tau}(\gamma\otimes\gamma\otimes\gamma) = \kappa_{\vec{R}}\sum_{p_l,q_l} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3} B^{R_1}_{p_1}B^{R_2}_{p_2}B^{R_3}_{p_3} \\ &\times \sum_{\ell_1,j_1} D^{R_1}_{\ell_1p_1}(\gamma_1)Q^{R_1}_{\ell_1j_1}D^{R_1}_{q_1j_1}(\gamma)\otimes \sum_{\ell_2,j_2} D^{R_2}_{\ell_2p_2}(\gamma_2)Q^{R_2}_{\ell_2j_2}D^{R_2}_{q_2j_2}(\gamma)\otimes \sum_{\ell_3,j_3} D^{R_3}_{\ell_3p_3}(\gamma_3)Q^{R_3}_{\ell_3j_3}D^{R_3}_{q_3j_3}(\gamma) \\ &= \kappa_{\vec{R}}\sum_{j_l} C^{R_1,R_2;R_3,\tau}_{j_1,j_2;j_3}\sum_{p_l,\ell_l} D^{R_1}_{\ell_1p_1}(\gamma_1)B^{R_1}_{p_1}D^{R_2}_{\ell_2p_2}(\gamma_2)B^{R_2}_{p_2}D^{R_3}_{\ell_3p_3}(\gamma_3)B^{R_3}_{p_3}\, Q^{R_1}_{\ell_1j_1}\otimes Q^{R_2}_{\ell_2j_2}\otimes Q^{R_3}_{\ell_3j_3} \\ &= \kappa_{\vec{R}}\sum_{j_l,\ell_l} C^{R_1,R_2;R_3,\tau}_{j_1,j_2;j_3} B^{R_1}_{\ell_1}B^{R_2}_{\ell_2}B^{R_3}_{\ell_3}\, Q^{R_1}_{\ell_1j_1}\otimes Q^{R_2}_{\ell_2j_2}\otimes Q^{R_3}_{\ell_3j_3} \\ &= Q^{R_1,R_2,R_3,\tau}\,, \end{aligned} \tag{5.22}$$

where we used once again (A.22) and (A.16) as intermediate steps and the identity (5.15) to get the last line. We now check a few properties of the product of elements of $\mathcal{K}_3(2n)$.

Product – The elements (5.7) of the Fourier basis of $\mathbb{C}[S_{2n}]$ multiply as follows (see Appendix A.3.)

$$Q^R_{ij} Q^{R'}_{kl} = \frac{\kappa_R}{d(R)} \delta_{RR'} \delta_{jk} Q^{R'}_{il} \,. \tag{5.23}$$

The definition (5.21) and relation (5.23) allow us to compute the product

$$\begin{aligned} & Q^{R_1,R_2,R_3,\tau} Q^{R'_1,R'_2,R'_3,\tau'} \\ &= \frac{\kappa_{\vec{R}}\kappa_{\vec{R}'}\kappa_{R_1}\kappa_{R_2}\kappa_{R_3}}{d(R_1)d(R_2)d(R_3)} \delta_{\vec{R}\vec{R}'} \sum_{p_l\, q_l\, a_l\, b_l} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3} C^{R'_1,R'_2;R'_3,\tau'}_{b_1,b_2;b_3} \\ &\quad \times B^{R_1}_{p_1} B^{R_2}_{p_2} B^{R_3}_{p_3} B^{R'_1}_{a_1} B^{R'_2}_{a_2} B^{R'_3}_{a_3} Q^{R'_1}_{p_1 b_1} \otimes Q^{R'_2}_{p_2 b_2} \otimes Q^{R'_3}_{p_3 b_3} \delta_{q_1 a_1} \delta_{q_2 a_2} \delta_{q_3 a_3} \\ &= \frac{\kappa_{\vec{R}}\kappa_{R_1}\kappa_{R_2}\kappa_{R_3}}{d(R_1)d(R_2)d(R_3)} \delta_{\vec{R}\vec{R}'} \Big[\sum_{q_l} C^{R'_1,R'_2;R'_3,\tau}_{q_1,q_2;q_3} B^{R'_1}_{q_1} B^{R'_2}_{q_2} B^{R'_3}_{q_3} \Big] Q^{R'_1,R'_2,R'_3,\tau'} . \end{aligned} \tag{5.24}$$

Hence, the product of two basis elements expands in terms of $Q^{R_1,R_2,R_3,\tau}$. In a compact notation, we write

$$Q^{R_1,R_2,R_3,\tau} Q^{R'_1,R'_2,R'_3,\tau'} = \delta_{\vec{R}\vec{R}'} k(\vec{R}',\tau) Q^{R'_1,R'_2,R'_3,\tau'} , \tag{5.25}$$

which shows that the product is almost orthogonal. Still it cannot represent the basis of the Wedderburn-Artin matrix decomposition. The basis $\{Q^{R_1,R_2,R_3,\tau}\}$ therefore decomposes $\mathcal{K}_3(2n)$ into blocks mutually orthogonal in the labels R_1, R_2, R_3. Still in each block the decomposition remains unachieved.

Associativity – We check the associativity of the product in the Q-basis. On the one hand, we have

$$\begin{aligned} & \Big(Q^{R_1,R_2,R_3,\tau} Q^{R'_1,R'_2,R'_3,\tau'} \Big) Q^{R''_1,R''_2,R''_3,\tau''} \\ &= \frac{\kappa_{\vec{R}}\kappa_{R_1}\kappa_{R_2}\kappa_{R_3}}{d(R_1)d(R_2)d(R_3)} \delta_{\vec{R}\vec{R}'} \Big[\sum_{q_l} C^{R'_1,R'_2;R'_3,\tau}_{q_1,q_2;q_3} B^{R'_1}_{q_1} B^{R'_2}_{q_2} B^{R'_3}_{q_3} \Big] \\ &\times \frac{\kappa_{\vec{R}'}\kappa_{R'_1}\kappa_{R'_2}\kappa_{R'_3}}{d(R'_1)d(R'_2)d(R'_3)} \delta_{\vec{R}'\vec{R}''} \Big[\sum_{q_l} C^{R''_1,R''_2;R''_3,\tau'}_{q_1,q_2;q_3} B^{R''_1}_{q_1} B^{R''_2}_{q_2} B^{R''_3}_{q_3} \Big] Q^{R''_1,R''_2,R''_3,\tau''} , \end{aligned} \tag{5.26}$$

while on the other,

$$
\begin{aligned}
&Q^{R_1,R_2,R_3,\tau}\left(Q^{R_1',R_2',R_3',\tau'}Q^{R_1'',R_2'',R_3'',\tau''}\right)\\
&=\frac{\kappa_{\vec{R}'}\kappa_{R_1'}\kappa_{R_2'}\kappa_{R_3'}}{d(R_1')d(R_2')d(R_3')}\delta_{\vec{R}'\vec{R}''}\Big[\sum_{q_l}C^{R_1'',R_2'';R_3'',\tau'}_{q_1,q_2;q_3}B^{R_1''}_{q_1}B^{R_2''}_{q_2}B^{R_3''}_{q_3}\Big]\\
&\times\frac{\kappa_{\vec{R}}\kappa_{R_1}\kappa_{R_2}\kappa_{R_3}}{d(R_1)d(R_2)d(R_3)}\delta_{\vec{R}\vec{R}''}\Big[\sum_{q_l}C^{R_1'',R_2'';R_3'',\tau}_{q_1,q_2;q_3}B^{R_1''}_{q_1}B^{R_2''}_{q_2}B^{R_3''}_{q_3}\Big]Q^{R_1'',R_2'',R_3'',\tau''}.
\end{aligned}
\tag{5.27}
$$

The two expressions are identical.

Pairing – We use the pairing on $\mathbb{C}[S_{2n}]^{\otimes 3}$ along the lines of (A.26) and evaluate:

$$
\begin{aligned}
&\boldsymbol{\delta}(Q^{R_1,R_2,R_3,\tau};Q^{R_1',R_2',R_3',\tau'})\\
&=\kappa_{\vec{R}}\kappa_{\vec{R}'}\sum_{p_l,q_l,a_l,b_l}C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3}C^{R_1',R_2';R_3',\tau'}_{b_1,b_2;b_3}B^{R_1}_{p_1}B^{R_2}_{p_2}B^{R_3}_{p_3}B^{R_1'}_{a_1}B^{R_2'}_{a_2}B^{R_3'}_{a_3}\\
&\qquad\times\boldsymbol{\delta}(Q^{R_1}_{p_1q_1}\otimes Q^{R_2}_{p_2q_2}\otimes Q^{R_3}_{p_3q_3};Q^{R_1'}_{a_1b_1}\otimes Q^{R_2'}_{a_2b_2}\otimes Q^{R_3'}_{a_3b_3})\\
&=\kappa_{\vec{R}}\kappa_{\vec{R}'}\sum_{p_l,q_l,a_l,b_l}C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3}C^{R_1',R_2';R_3',\tau'}_{b_1,b_2;b_3}B^{R_1}_{p_1}B^{R_2}_{p_2}B^{R_3}_{p_3}B^{R_1'}_{a_1}B^{R_2'}_{a_2}B^{R_3'}_{a_3}\\
&\qquad\times\delta_{\vec{R}\vec{R}'}\delta_{p_1a_1}\delta_{p_2a_2}\delta_{p_3a_3}\delta_{q_1b_1}\delta_{q_2b_2}\delta_{q_3b_3}\\
&=\kappa^2_{\vec{R}}\,d(R_3)\sum_{p_l}\Big[\prod_{i=1}^{3}B^{R_i}_{p_i}\Big]^2\delta_{\vec{R}\vec{R}'}\delta_{\tau\tau'}\\
&=\kappa^2_{\vec{R}}\,d(R_3)\delta_{\vec{R}\vec{R}'}\delta_{\tau\tau'}\,,
\end{aligned}
\tag{5.28}
$$

where, in the first line, we used (A.26), in the last, (A.13), and the fact that, by (5.12), the following holds $\sum_p(B^R_p)^2=\sum_p\langle[2n],1,1|\,R,p\rangle\langle R,p\,|[2n],1,1\rangle=1$, for all $R\vdash 2n$. We could therefore fix the normalization $\kappa^2_{\vec{R}}=1/d(R_3)$.
The following statement holds:

Proposition 5.2.1. $\{Q^{R_1,R_2,R_3,\tau}\}$ *is an invariant orthonormal basis of* $\mathcal{K}_3(2n)$.

Proof. It is sufficient to show that the graph basis expands in terms of the Q-basis. We express any graph basis element $G_{\sigma_1,\sigma_2,\sigma_3}=\sum_{\gamma_i\in S_n[S_2]}\sum_{\gamma\in S_{2n}}\gamma_1\sigma_1\gamma\otimes\gamma_2\sigma_2\gamma\otimes\gamma_3\sigma_3\gamma$ as

$$
G_{\sigma_1,\sigma_2,\sigma_3}=\sum_{R_l,\tau}\boldsymbol{\delta}(Q^{R_1,R_2,R_3,\tau};G_{\sigma_1,\sigma_2,\sigma_3})Q^{R_1,R_2,R_3,\tau}.
\tag{5.29}
$$

The definition of $Q^{R_1,R_2,R_3,\tau}$ calls a linear combination of triples $\tau_1\otimes\tau_2\otimes\tau_3$ that must have a non trivial overlap with $G_{\sigma_1,\sigma_2,\sigma_3}$. Let us compute the overlap between the bases.

Start with (5.21) and then write (using (A.16) and then (5.15))

$$\boldsymbol{\delta}(Q^{R_1,R_2,R_3,\tau};G_{\sigma_1,\sigma_2,\sigma_3}) = \kappa_{\vec{R}}\frac{\kappa_{R_1}\kappa_{R_2}\kappa_{R_3}}{((2n)!)^3}((2^n n!))^3(2n!)\sum_{a_l,b_l} C^{R_1,R_2;R_3,\tau}_{b_1,b_2;b_3}\Big[\prod_{i=1}^{3} B^{R_i}_{a_i} D^{R_i}_{a_i b_i}(\sigma_i)\Big]. \tag{5.30}$$

This number is, up to the normalization factor $((2^n n!))^3(2n!)$, the coefficient of the triple $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$ in $Q^{R_1,R_2,R_3,\tau}$. □

We note that the basis $\{Q^{R_1,R_2,R_3,\tau}\}$ is of the correct cardinality, that of $Z_3(2n)$ as we sought. Finding the Wedderburn-Artin matrix basis of $\mathcal{K}_3(2n)$ would mean that $Z_3(2n)$ can be written as a sum of squares. Interestingly, within the TFT$_2$ formulation of the counting, we note that the partition function (4.36) computes further using (A.5) as

$$\begin{aligned} Z_3(2n) &= \frac{1}{[n!2^n]^3}\sum_{R_l\vdash 2n}\Big(\sum_{\gamma_1}\chi^{R_1}(\gamma_1)\Big)\Big(\sum_{\gamma_2}\chi^{R_2}(\gamma_2)\Big)\Big(\sum_{\gamma_3}\chi^{R_1}(\gamma_3)\chi^{R_2}(\gamma_3)\Big) \\ &= \frac{1}{n!2^n}\sum_{R_l\vdash 2n}\sum_{\gamma_3}\chi^{2R_1}(\gamma_3)\chi^{2R_2}(\gamma_3) \\ &= \frac{1}{n!2^n}\sum_{\gamma_3}\Bigg(\sum_{R\vdash 2n}\chi^{2R}(\gamma_3)\Bigg)^2, \end{aligned} \tag{5.31}$$

thus, as a normalized sum of squares. This shows that $Z_3(2n)$ could admit several decompositions into squares. If $\left(\sum_{R\vdash 2n}\chi^{2R}(\gamma_3)\right)^2$ is the dimension of a subalgebra (given that the characters are integers via the Murnaghan-Nakayama rule), this would mean that this decomposition into subalgebras would be labeled by γ_3 and will be even different from the Wedderburn-Artin decomposition. This decomposition deserves further clarification in the present $O(N)$ setting.

About projectors – Let us conclude this chapter by defining the normalized projectors

$$\begin{aligned} P_i^{S_n[S_2]} &= \frac{1}{n!2^n}\sum_{\gamma_i\in S_n[S_2]}\varrho_i(\gamma_i), \\ P_R^{S_{2n}} &= \frac{1}{(2n)!}\sum_{\gamma\in S_{2n}}\rho_R(\gamma), \end{aligned} \tag{5.32}$$

and by checking that the trace of their product indeed yields the dimension of the algebra $\mathcal{K}_3(2n)$:

$$\dim \mathcal{K}_3(2n) = \mathrm{tr}_{\mathbb{C}[S_{2n}]^{\otimes 3}}(P_1^{S_n[S_2]} P_2^{S_n[S_2]} P_3^{S_n[S_2]} P_R^{S_{2n}}) = \mathrm{tr}_{\mathcal{K}_3(2n)}(\mathbb{1})\,. \tag{5.33}$$

We have

$$\begin{aligned}
&\sum_{\gamma_a \in S_n[S_2]} \sum_{\gamma \in S_{2n}} \varrho_1(\gamma_1)\varrho_2(\gamma_2)\varrho_3(\gamma_3)\rho_R(\gamma)\, Q^{R_1}_{i_1 j_1} \otimes Q^{R_2}_{i_2 j_2} \otimes Q^{R_3}_{i_3 j_3} \\
&= \sum_{\gamma_a} \sum_{\gamma} \sum_{p_l, q_l} D^{R_1}_{p_1 i_1}(\gamma_1) D^{R_1}_{j_1 q_1}(\gamma) D^{R_2}_{p_2 i_2}(\gamma_2) D^{R_2}_{j_2 q_2}(\gamma) D^{R_3}_{p_3 i_3}(\gamma_3) D^{R_3}_{j_3 q_3}(\gamma) \\
&\times Q^{R_1}_{p_1 q_1} \otimes Q^{R_2}_{p_2 q_2} \otimes Q^{R_3}_{p_3 q_3}\,.
\end{aligned} \tag{5.34}$$

To compute the trace, pair this with $Q^{R_1}_{i_1 j_1} \otimes Q^{R_2}_{i_2 j_2} \otimes Q^{R_3}_{i_3 j_3}$ using the orthonormality property $\delta(Q^R_{ij}; Q^S_{kl}) = \delta_{RS}\delta_{ik}\delta_{jl}$ and sum over R_l, i_l, j_l yielding

$$\begin{aligned}
&\sum_{R_l \vdash 2n} \sum_{\gamma_a} \sum_{\gamma} \sum_{p_l, q_l, i_l, j_l} D^{R_1}_{p_1 i_1}(\gamma_1) D^{R_1}_{j_1 q_1}(\gamma) D^{R_2}_{p_2 i_2}(\gamma_2) D^{R_2}_{j_2 q_2}(\gamma) D^{R_3}_{p_3 i_3}(\gamma_3) D^{R_3}_{j_3 q_3}(\gamma) \\
&\times \delta_{i_1 p_1}\delta_{j_1 q_1}\delta_{i_2 p_2}\delta_{j_2 q_2}\delta_{i_3 p_3}\delta_{j_3 q_3} \\
&= \sum_{R_l \vdash 2n} \sum_{\gamma_a} \sum_{\gamma} \sum_{i_l, j_l} D^{R_1}_{i_1 i_1}(\gamma_1) D^{R_1}_{j_1 j_1}(\gamma) D^{R_2}_{i_2 i_2}(\gamma_2) D^{R_2}_{j_2 j_2}(\gamma) D^{R_3}_{i_3 i_3}(\gamma_3) D^{R_3}_{j_3 j_3}(\gamma) \\
&= (2n)! \sum_{R_l \vdash 2n} \sum_{\gamma_a} \mathbf{C}(R_1, R_2, R_3) \chi^{R_1}(\gamma_1) \chi^{R_2}(\gamma_2) \chi^{R_3}(\gamma_3)\,.
\end{aligned} \tag{5.35}$$

Hence we find (4.40) using Burnside's lemma, and we have $Z_3(n) = \dim \mathcal{K}_3(2n)$.

Chapter 6

Correlators

Let us now analyze Gaussian correlators, starting with $d = 3$ we then extend the result to any d. We consider the normal ordered correlator of two observables $O_b(T)O_{b'}(T)$ in the Gaussian measure $d\nu(T)$ (4.4). Normal order means that we only allow contractions from $O_b(T)$ to $O_{b'}(T)$.

6.1 Rank $d = 3$ correlator

Before computing the correlators, a few remarks are in order. A 3-tuple of permutations labels the observables: $O_b(T) = O_{\sigma_1,\sigma_2,\sigma_3}(T)$ and $O_{b'}(T) = O_{\tau_1,\tau_2,\tau_3}(T)$. Recall that an observable $O_{\sigma_1,\sigma_2,\sigma_3}(T)$ is in fact defined by a contraction of tensor indices. This contraction pattern, that gives in return the colored edges of the graph associated with the observable, is not defined by the triple $(\sigma_1, \sigma_2, \sigma_3)$ but instead by the following triple

$$(\widetilde{\sigma}_1, \widetilde{\sigma}_2, \widetilde{\sigma}_3) = (\sigma_1^{-1}\xi\sigma_1, \sigma_2^{-1}\xi\sigma_2, \sigma_3^{-1}\xi\sigma_3), \tag{6.1}$$

where we recall that ξ is the fixed permutation $(12)(34)\dots(2n-1, 2n)$. The justification of this is immediate: each swap in ξ corresponds to a label of the half-lines of the vertex v_i^c, see Figure 4.1. Consider the l^{th} edge of color c from the l^{th} tensor. The vertex links v_i^c the image of $\sigma_c(l)$ and the pre-image through σ_c of $\xi(\sigma_c(l))$. We will need the following convenient notation for tensors: $T_{a_{i1}a_{i2}a_{i3}}$, where the index $i \in [\![1, 2n]\!]$ stands for the label of the tensor which at the end will not matter in the definition of the observable. Using this, an observable made of the contraction of $2n$ tensors can be expressed, for $a_{ic} \in [\![1, N]\!]$, as:

$$O_{\sigma_1,\sigma_2,\sigma_3}(T) = \sum_{a_{ic}} \prod_{i=1}^{2n} \prod_{c=1}^{3} \delta^{a_{ic}}_{a_{\widetilde{\sigma}_c(i)c}} \prod_{i=1}^{2n} T_{a_{i1}a_{i2}a_{i3}}. \tag{6.2}$$

There are many redundant Kroneckers δs in the previous expression. However, the calculus here is discrete and so there are no particular issues. When we will compute the correlator using Wick's theorem, it is the triple $(\widetilde{\sigma}_1, \widetilde{\sigma}_2, \widetilde{\sigma}_3)$ that will be concerned.
The Wick contraction between two observables, in the normal order, introduces a permutation $\mu \in S_{2n}$. A correlator simply counts cycles of a convolution of permutations. Let us determine which convolution that is, using twice (6.2) and the free propagator (4.5):

$$\begin{aligned}\langle O_{\sigma_1,\sigma_2,\sigma_3}(T) O_{\tau_1,\tau_2,\tau_3}(T)\rangle &= \frac{1}{\int d\nu(T)} \int d\nu(T) O_{\sigma_1,\sigma_2,\sigma_3}(T) O_{\tau_1,\tau_2,\tau_3}(T) \\ &= \sum_{\mu} \sum_{a_{ic}, b_{kl}} \Big[\prod_{i=1}^{2n} \prod_{c=1}^{3} \delta^{a_{ic}}_{a_{\widetilde{\sigma}_c(i)c}} \delta^{b_{ic}}_{b_{\widetilde{\tau}_c(i)c}} \Big] \Big[\prod_{i=1}^{2n} \prod_{c=1}^{3} \delta^{a_{ic}}_{b_{\mu(i)c}} \Big] .\end{aligned} \tag{6.3}$$

Summing over the b_{kl} variables and using a change of variables, $b_{ic} = a_{\mu^{-1}(i)c}$, lead us to

$$\begin{aligned}\langle O_{\sigma_1,\sigma_2,\sigma_3}(T) O_{\tau_1,\tau_2,\tau_3}(T)\rangle &= \sum_{\mu} \sum_{a_{ic}} \Big[\prod_{i=1}^{2n} \prod_{c=1}^{3} \delta^{a_{ic}}_{a_{\widetilde{\sigma}_c(i)c}} \delta^{a_{\mu^{-1}(i)c}}_{a_{\mu^{-1}\widetilde{\tau}_c(i)c}} \Big] \\ &= \sum_{\mu} \sum_{a_{ic}} \Big[\prod_{i=1}^{2n} \prod_{c=1}^{3} \delta^{a_{ic}}_{a_{\widetilde{\sigma}_c(i)c}} \delta^{a_{ic}}_{a_{\mu^{-1}\widetilde{\tau}_c\mu\widetilde{\sigma}_c(i)c}} \Big],\end{aligned} \tag{6.4}$$

where we also used $\widetilde{\sigma}_c^{-1} = \widetilde{\sigma}_c$. We can already guess that the correlator expresses as a power of N in a number of cycles of $\mu^{-1}\widetilde{\tau}_c\mu\widetilde{\sigma}_c$. However, the proof is not obvious because of the redundancy of the δs introduced in the definition of the observable, see (6.2).
The following statement holds

Lemma 6.1.1. *Let a_i be an integer, $a_i \in [\![1, N]\!]$, for $i \in [\![1, 2n]\!]$. Then (at fixed color c that we will omit in the ensuing notation),*

$$\sum_{a_i} \Big[\prod_{i=1}^{2n} \delta^{a_i}_{a_{\widetilde{\sigma}(i)}} \delta^{a_i}_{a_{\mu^{-1}\widetilde{\tau}\mu\widetilde{\sigma}(i)}} \Big] = N^{\mathbf{c}(\mu^{-1}\widetilde{\tau}\mu\widetilde{\sigma})} , \tag{6.5}$$

where $\mathbf{c}(\sigma)$ is the number of cycles of the permutation σ.

Proof. The sole issue here is the redundancy of the Kroneckers. In fact, there is enough information in the above sum to withdraw the correct number of cycles. Call "vertex δs" those appearing in the product $\prod_{i=1}^{2n} \delta^{a_i}_{a_{\widetilde{\sigma}(i)}}$, and (Wick) "contraction δs" the remaining ones coming from the resolution of the Wick contraction. Note that there are redundancies in each product of δs.

Consider a fixed index i: to make things easier, we start by the simple case given by $\mu^{-1}\widetilde{\tau}\mu\widetilde{\sigma}(i) = i$. If $\mu^{-1}\widetilde{\tau}\mu\widetilde{\sigma}^{-1}(i) = i$, then (i) is a 1-cycle of $\mu^{-1}\widetilde{\tau}\mu\widetilde{\sigma}$ and we also have $\widetilde{\sigma}(i) = \mu^{-1}\widetilde{\tau}\mu(i)$. Thus we have, among the contraction δs, two distinct δs which become trivial, *viz.* $\delta^{a_i}_{a_i}$ and $\delta^{a_{\widetilde{\sigma}(i)}}_{a_{\widetilde{\sigma}(i)}}$. The sums over a_i and $a_{\widetilde{\sigma}(i)}$ boil down to a single sum precisely because of the vertex $\delta^{a_i}_{a_{\widetilde{\sigma}(i)}}$. Hence that cycle is counted once.
Let us inspect the general case. For an arbitrary i, call $q_i \geq 1$ the smallest integer such that $(\mu^{-1}\widetilde{\tau}\mu\widetilde{\sigma})^{q_i}(i) = i$, and which defines a q_i-cycle of $\mu^{-1}\widetilde{\tau}\mu\widetilde{\sigma}$ (the case $q_i = 1$ has been dealt with above). In the product (6.5), we collect all contraction δs involved in the cycle starting at some fixed i

$$\prod_{l=1}^{q_i} \delta^{a_{(\mu^{-1}\widetilde{\tau}\mu\widetilde{\sigma})^{l-1}(i)}}_{a_{(\mu^{-1}\widetilde{\tau}\mu\widetilde{\sigma})^{l}(i)}} \,. \tag{6.6}$$

Since this product is at arbitrary i, we have a companion and distinct product of contraction δs that starts at $\widetilde{\sigma}(i)$: $\prod_{l=1}^{q_i} \delta^{a_{(\mu^{-1}\widetilde{\tau}\mu\widetilde{\sigma})^{l-1}(\widetilde{\sigma}(i))}}_{a_{(\mu^{-1}\widetilde{\tau}\mu\widetilde{\sigma})^{l}(\widetilde{\sigma}(i))}}$. Hence, we combine both products and multiply by one vertex δ

$$\delta^{a_i}_{a_{\widetilde{\sigma}(i)}} \prod_{l=1}^{q_i} \delta^{a_{(\mu^{-1}\widetilde{\tau}\mu\widetilde{\sigma})^{l-1}(i)}}_{a_{(\mu^{-1}\widetilde{\tau}\mu\widetilde{\sigma})^{l}(i)}} \, \delta^{a_{(\mu^{-1}\widetilde{\tau}\mu\widetilde{\sigma})^{l-1}(\widetilde{\sigma}(i))}}_{a_{(\mu^{-1}\widetilde{\tau}\mu\widetilde{\sigma})^{l}(\widetilde{\sigma}(i))}} \,, \tag{6.7}$$

which evaluates to N after performing the sum over the corresponding a_js. Again, the q_i-cycle is counted once. It just remains to observe that the cycles, each defined by a subset of indices a_j, define partitions of the entire set of indices a_i (once an index is used in a cycle it cannot reappear in another). Thus, the sum over a_i factorizes along cycles, which completes the proof. □

Note that there may be alternative ways of defining real tensor observables using pairings and without introducing the gauge redundancy. In any case, we could work in this setting, keeping track of the necessary information.
From Lemma 6.1.1 applied to each color $c = 1, 2, 3$, we finally come to

$$\langle O_{\sigma_1,\sigma_2,\sigma_3}(T) O_{\tau_1,\tau_2,\tau_3}(T)\rangle = \sum_{\mu} N^{\sum_{c=1}^{3} \mathbf{c}(\mu^{-1}\widetilde{\tau}_c\mu\widetilde{\sigma}_c)} \,. \tag{6.8}$$

The 1-point correlator can be recovered from the above discussion. First, the 1-point correlator cannot be normal ordered. Introduce the Wick contraction μ that belongs to S^*_{2n} the subset defined by the pairings of S_{2n} (a permutation pairing is made only of transpositions). Then, we obtain

$$\langle O_{\sigma_1,\sigma_2,\sigma_3}(T)\rangle = \sum_{\mu\in S_{2n}^*}\sum_{a_{ic}}\Big[\prod_{i=1}^{2n}\prod_{c=1}^{3}\delta^{a_{ic}}_{a_{\widetilde{\sigma}_c(i)c}}\Big]\Big[\prod_{i=1}^{2n}\prod_{c=1}^{3}\delta^{a_{ic}}_{a_{\mu(i)c}}\Big]. \tag{6.9}$$

Next, we adapt Lemma 6.1.1 to $\sum_{a_i}\Big[\prod_{i=1}^{2n}\delta^{a_i}_{a_{\widetilde{\sigma}(i)}}\delta^{a_i}_{a_{\widetilde{\sigma}\mu(i)}}\Big] = N^{\mathbf{c}(\widetilde{\sigma}\mu)}$ and get

$$\langle O_{\sigma_1,\sigma_2,\sigma_3}(T)\rangle = \sum_{\mu\in S_{2n}^*} N^{\sum_{c=1}^3\mathbf{c}(\mu\widetilde{\sigma}_c)}. \tag{6.10}$$

6.2 Representation theoretic basis and orthogonality

We re-express the 2-point function in order to make explicit some of its properties. Inserting three auxiliary permutations $\alpha_c\in S_{2n}$, the above sum (6.8) reads

$$\langle O_{\sigma_1,\sigma_2,\sigma_3}(T)O_{\tau_1,\tau_2,\tau_3}(T)\rangle = \sum_{\mu,\,\alpha_c} N^{\sum_{c=1c}^3\mathbf{c}(\alpha_c)}\prod_{c=1}^{3}\delta(\mu^{-1}\widetilde{\tau}_c\mu\widetilde{\sigma}_c\alpha_c) = N^{6n}\sum_{\mu}\prod_{c=1}^{3}\delta(\mu^{-1}\widetilde{\tau}_c\mu\widetilde{\sigma}_c\Omega_c), \tag{6.11}$$

where we introduced the central element $\Omega_c = \sum_{\alpha_c\in S_{2n}} N^{\mathbf{c}(\alpha_c)-2n}\alpha_c$. The proof rests on the equality $\mathbf{c}(\alpha_c^{-1}) = \mathbf{c}(\alpha_c)$, which holds because each cycle has an inverse, a cycle of the same length. Then, we can rewrite (6.11) as

$$\begin{aligned}
&\langle O_{\sigma_1,\sigma_2,\sigma_3}(T)O_{\tau_1,\tau_2,\tau_3}(T)\rangle\\
&= N^{6n}\sum_{\mu}\boldsymbol{\delta}[(\mu^{-1})^{\otimes 3}(\widetilde{\tau}_1\otimes\widetilde{\tau}_2\otimes\widetilde{\tau}_3)\mu^{\otimes 3}(\widetilde{\sigma}_1\otimes\widetilde{\sigma}_2\otimes\widetilde{\sigma}_3)(\Omega_1\otimes\Omega_2\otimes\Omega_3)]\\
&= N^{6n}\sum_{\mu}\boldsymbol{\delta}[(\widetilde{\tau}_1\otimes\widetilde{\tau}_2\otimes\widetilde{\tau}_3)\mu^{\otimes 3}(\widetilde{\sigma}_1\otimes\widetilde{\sigma}_2\otimes\widetilde{\sigma}_3)(\mu^{-1})^{\otimes 3}(\Omega_1\otimes\Omega_2\otimes\Omega_3)]\,,
\end{aligned} \tag{6.12}$$

where in the last equation we used the fact the Ω_cs are central. We introduce the representation theoretic element by pairing a basis element $Q^{R_1,R_2,R_3,\tau}$ (5.21) and an observable $O_{\sigma_1,\sigma_2,\sigma_3}$ as

$$\begin{aligned}
O^{R_1,R_2,R_3,\tau} &= \sum_{\sigma_l}\boldsymbol{\delta}(Q^{R_1,R_2,R_3,\tau}\sigma_1^{-1}\otimes\sigma_2^{-1}\otimes\sigma_3^{-1})O_{\sigma_1,\sigma_2,\sigma_3}\\
&= \kappa_{\vec{R}}\prod_{i=1}^{3}\frac{\kappa_{R_i}}{(2n)!}\sum_{\sigma_l}\sum_{p_l,q_l}C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3}\Big[\prod_{i=1}^{3}B^{R_i}_{p_i}D^{R_i}_{p_iq_i}(\sigma_i)\Big]O_{\sigma_1,\sigma_2,\sigma_3}.
\end{aligned} \tag{6.13}$$

As a linear combination of observables, we can calculate their correlators:

$$
\begin{aligned}
\langle O^{R_1,R_2,R_3,\tau}\, O^{R_1',R_2',R_3',\tau'}\rangle &= N^{6n}\kappa_{\vec{R}}\kappa_{\vec{R}'}\Big[\prod_{i=1}^{3}\frac{\kappa_{R_i}}{(2n)!}\frac{\kappa_{R_i'}}{(2n)!}\Big] \\
&\quad\times\sum_{\mu}\boldsymbol{\delta}\Bigg[\sum_{\sigma_l,\sigma_l'}\sum_{p_l,q_l,p_l',q_l'} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3}C^{R_1',R_2';R_3',\tau'}_{q_1',q_2';q_3'}\Big[\prod_{i=1}^{3}B^{R_i}_{p_i}D^{R_i}_{p_iq_i}(\sigma_i)B^{R_i'}_{p_i'}D^{R_i'}_{p_i'q_i'}(\sigma_i')\Big] \\
&\quad\times(\widetilde{\sigma}_1'\otimes\widetilde{\sigma}_2'\otimes\widetilde{\sigma}_3')\mu^{\otimes 3}(\widetilde{\sigma}_1\otimes\widetilde{\sigma}_2\otimes\widetilde{\sigma}_3)(\mu^{-1})^{\otimes 3}(\Omega_1\otimes\Omega_2\otimes\Omega_3)\Bigg] \\
&= N^{6n}\kappa_{\vec{R}}\kappa_{\vec{R}'}\Big[\prod_{i=1}^{3}\frac{\kappa_{R_i}}{(2n)!}\frac{\kappa_{R_i'}}{(2n)!}\Big]\sum_{\mu}\boldsymbol{\delta}\Bigg[\sum_{p_l,q_l,p_l',q_l'} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3}C^{R_1',R_2';R_3',\tau'}_{q_1',q_2';q_3'} \\
&\quad\times\Big[\bigotimes_{i=1}^{3}B^{R_i'}_{p_i'}\sum_{\sigma_i'}(\sigma_i')^{-1}\xi D^{R_i'}_{p_i'q_i'}(\sigma_i')\sigma_i'\Big]\mu^{\otimes 3}\Big[\bigotimes_{i=1}^{3}B^{R_i}_{p_i}\sum_{\sigma_i}(\sigma_i)^{-1}\xi D^{R_i}_{p_iq_i}(\sigma_i)\sigma_i\Big](\mu^{-1})^{\otimes 3} \\
&\quad\times(\Omega_1\otimes\Omega_2\otimes\Omega_3)\Bigg]\,.
\end{aligned}
\tag{6.14}
$$

Next, we introduce the operator $T_\xi : S_{2n}\to S_{2n}$ that acts on S_{2n} as $T_\xi(\sigma)=\sigma^{-1}\xi\sigma=\widetilde{\sigma}$ and extends by linearity on $\mathbb{C}[S_{2n}]$. The operator T_ξ actually maps any permutation to a pairing. Its image in $\mathbb{C}[S_{2n}]$ is the vector subspace generated by all pairings (more properties are derived in Appendix A.3). We re-express the above correlator as

$$
\begin{aligned}
&\langle O^{R_1,R_2,R_3,\tau}\, O^{R_1',R_2',R_3',\tau'}\rangle \\
&= N^{6n}\kappa_{\vec{R}}\kappa_{\vec{R}'}\sum_{\mu}\boldsymbol{\delta}\Bigg[\sum_{p_l,q_l,p_l',q_l'} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3}C^{R_1',R_2';R_3',\tau'}_{q_1',q_2';q_3'} \\
&\quad\times\Big[\bigotimes_{i=1}^{3}B^{R_i'}_{p_i'}T_\xi Q^{R_i'}_{p_i'q_i'}\Big]\mu^{\otimes 3}\Big[\bigotimes_{i=1}^{3}B^{R_i}_{p_i}T_\xi Q^{R_i}_{p_iq_i}\Big](\mu^{-1})^{\otimes 3}(\Omega_1\otimes\Omega_2\otimes\Omega_3)\Bigg] \\
&= N^{6n}\sum_{\mu}\boldsymbol{\delta}\Bigg[\Big[T_\xi^{\otimes 3}\sum_{p_l',q_l'}C^{R_1',R_2';R_3',\tau'}_{q_1',q_2';q_3'}\bigotimes_{i=1}^{3}B^{R_i'}_{p_i'}Q^{R_i'}_{p_i'q_i'}\Big]\mu^{\otimes 3} \\
&\quad\times\Big[T_\xi^{\otimes 3}\sum_{p_l,q_l}C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3}\bigotimes_{i=1}^{3}B^{R_i}_{p_i}Q^{R_i}_{p_iq_i}\Big](\mu^{-1})^{\otimes 3}(\Omega_1\otimes\Omega_2\otimes\Omega_3)\Bigg] \\
&= N^{6n}\sum_{\mu}\boldsymbol{\delta}\Big[(T_\xi^{\otimes 3}Q^{R_1',R_2',R_3',\tau'})\mu^{\otimes 3}(T_\xi^{\otimes 3}Q^{R_1,R_2,R_3,\tau})(\mu^{-1})^{\otimes 3}(\Omega_1\otimes\Omega_2\otimes\Omega_3)\Big] \\
&= N^{6n}(2n)!\,\boldsymbol{\delta}\Big[(T_\xi^{\otimes 3}Q^{R_1',R_2',R_3',\tau'})(T_\xi^{\otimes 3}Q^{R_1,R_2,R_3,\tau})(\Omega_1\otimes\Omega_2\otimes\Omega_3)\Big]\,,
\end{aligned}
\tag{6.15}
$$

where we used the right diagonal invariance of the basis $Q^{R'_1,R'_2,R'_3,\tau'}$ to achieve the last stage of the calculation. Hence, this correlator computed with the Gaussian measure of $O(N)$ tensor models in the normal order, regarded as an inner product on the space of observables, corresponds to the group theoretic inner product of the algebra $\mathcal{K}_3(2n)$ calculated on a product of the transformed basis $T_\xi^{\otimes 3} Q^{R_1,R_2,R_3,\tau}$ with an insertion of the factor $\Omega_1 \otimes \Omega_2 \otimes \Omega_3$. The action $T_\xi^{\otimes 3}$ on $Q^{R_1,R_2,R_3,\tau}$ reflects the fact that it is the triple $(\widetilde{\sigma}_1, \widetilde{\sigma}_2, \widetilde{\sigma}_3)$ which plays a major role for computing the cycles associated with Feynman amplitudes in this theory (meanwhile the triple $(\sigma_1, \sigma_2, \sigma_3)$ was associated with the class counting of the double coset space and its resulting algebra). In $U(N)$ models [29], there is a correspondence between Gaussian 2-point correlators in normal order and the inner product on the algebra of observables but without the presence of the operator $T_\xi^{\otimes 3}$. The presence of $T_\xi^{\otimes 3}$ determines therefore a feature proper to $O(N)$ tensor models.
We can further evaluate the above inner product as in Appendix A.4 and find:

$$\langle O^{R_1,R_2,R_3,\tau} \, O^{R'_1,R'_2,R'_3,\tau'} \rangle = \delta_{\vec{R}'\vec{R}} \delta_{\tau'\tau} \sum_{S_i,\tau_i} \prod_{i=1}^{3} \mathrm{Dim}_N(S_i) \left(\sum_{b_i,c_i,p_i} D^{S_i}_{b_i c_i}(\xi) C^{S_i,S_i;R_i,\tau_i}_{b_i,c_i;p_i} B^{R_i}_{p_i} \right)^2 , \tag{6.16}$$

which expresses the orthogonality of the representation theoretic basis $\{O^{R_1,R_2,R_3,\tau}\}$ (corresponding to normal ordered Gaussian correlators) of $\mathcal{K}_3(2n)$. Note also that the pairing between basis elements is a representation translation of the Gaussian integration.

Rank-d 2-point correlator – We obtain the 2-point correlator at rank d in a straightforward manner from the above derivation. We generalize (6.2) and (6.3) by extending the product over c up to $d \geq 3$ and considering a tensor $T_{a_{i1}a_{i2}\dots a_{id}}$. The calculations are direct: we get (6.8) and (6.10) by changing the sum over c running over the colored cycles up to d. Meanwhile, the orthogonality of the 2-point function is a property specific to the rank 3 and cannot be reproduced easily at any rank.

Chapter 7

On $Sp(2N)$ tensor invariants

We provide a few remarks on the counting of real $Sp(2N)$ tensor invariants. Carrozza and Pozsgay recently addressed symplectic complex tensor models in the context of tensor-like SYK models [26]. The authors focused on the complex group $U(N) \cap Sp(2N, \mathbb{C})$ (its quantum mechanical tensor model admits a large N expansion and shares similar properties with the SYK model) and, at the combinatorial level, on the improvement of the numerical computations of the number of its singlets in rank 3. We could ask, in the same vein as discussed above using symmetric group formulae, how to enumerate real symplectic invariants in the pure tensor model setting, *i.e.* with no spacetime attached to the tensor. We stress that, unlike in [26], we are interested in real and Bosonic fields and address in the following the symplectic group itself $Sp(2N, \mathbb{R}) = Sp(2N)$ and its (symplectic) invariants in any rank. We show below that they follow an enumeration principle with the same diagrammatics than that of the $O(N)$ invariants albeit with some changes occuring at the level of the coset equivalence relation. Interestingly in this $Sp(2N)$ setting, the "virtual" vertices v_i^c, in Figure 4.1, find an interpretation: they correspond precisely to symplectic matrix J insertions in the $Sp(2N)$ invariants.
Let us recall the usual notation and introduce the real $2N \times 2N$ symplectic matrix J which writes in blocks

$$J = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix} , \qquad J^2 = -\mathbb{1}_{2N} , \tag{7.1}$$

where $\mathbb{1}_N$, for all N, is the identity matrix of $M_N(\mathbb{R})$. A matrix $K \in Sp(2N)$ obeys $K^T J K = J$.

A rank-d real tensor T, with components $T_{p_1\dots p_d}$, $p_c \in [\![1, 2N]\!]$, transforms under the fundamental representation of $\bigotimes_{c=1}^{d} Sp(2N_c)$ for fixed N_c, if each group $Sp(2N_c)$ acts on the index p_c such that the transformed tensor satisfies:

$$T^K_{q_1\dots q_d} = \sum_{p_1,\dots,p_d} K^{(1)}_{q_1p_1} \dots K^{(d)}_{q_1p_1} \, T_{p_1\dots p_d} \,, \tag{7.2}$$

where $K^{(c)} \in Sp(2N_c)$, $c \in [\![1, d]\!]$.

Observables in $Sp(2N)$ tensor models are the contractions of an even number of tensors T. They are invariant under $\bigotimes_{c=1}^{d} Sp(2N_c)$ transformations and we call them $Sp(2N)$ invariants.

In understood notation, we define a new trace on two rank-d tensors as

$$\mathrm{Tr}(T\, J^d\, T) = \sum_{p_i,q_i} J^{(1)}_{p_1q_1} \dots J^{(d)}_{p_dq_d} \; T_{p_1\dots p_d}T_{q_1\dots q_d} \,. \tag{7.3}$$

Thus, the tensor indices that are contracted couple with J. This is the generalization of the symplectic form over matrices which is defined as $\omega_J(M, W) = \mathrm{tr}(M^T JW)$, and that is invariant under symplectomorphisms.

We check that $\mathrm{Tr}(T\, J^d\, T)$ is invariant under symplectic transformations:

$$\begin{aligned}\mathrm{Tr}(T^K\, J^d\, T^K) &= \sum_{r_i,s_i} \sum_{p_i,q_i} \big(K_{p_1r_1}K_{q_1s_1}J^{(1)}_{p_1q_1}\big) \dots \big(K_{p_dr_d}K_{q_ds_d}J^{(d)}_{p_dq_d}\big) T_{r_1\dots r_d}T_{s_1\dots s_d} \\ &= \mathrm{Tr}(TJ^dT) \,.\end{aligned} \tag{7.4}$$

Now, we extend the trace (7.3) to arbitrary number of tensors. Still the contraction obtained is an $Sp(2N)$ invariant. We can easily observe that the $Sp(2N)$ invariants can be viewed once again in terms of d-regular colored graphs with a decoration on each edge. The decoration seals the symplectic matrix J on each pair of contracted tensor indices. Therefore, J can be represented by a new vertex on each edge which plays precisely the same role as a black vertex v_i^c in Figure 4.1.

The counting of $Sp(2N)$ invariants is more subtle than that of $O(N)$ invariants. Indeed, for simplicity, let us consider in rank 3 (generalizing the following argument at any rank d is straightforward), $2n$ tensors and count the possible triples $(\sigma_1, \sigma_2, \sigma_3) \in S_{2n} \times S_{2n} \times S_{2n}$

subjected to the following invariance:

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1 \sigma_1 \gamma, \gamma_2 \sigma_2 \gamma, \gamma_3 \sigma_3 \gamma), \tag{7.5}$$

where, on the right, we have the ordinary diagonal action of $\mathrm{Diag}(S_{2n})$ on the triple. Meanwhile, on the left, the γ_is belong to an identical subgroup $G_i = G'$ but that is not any more $S_n[S_2]$. Switching the half-edges of the vertices v_i^c produces a sign. This hints to the fact that we should switch to the group algebra $\mathbb{C}[S_{2n}] \times \mathbb{C}[S_{2n}] \times \mathbb{C}[S_{2n}]$ to perform the coset. At this point, note that nothing excludes that the number of $Sp(2N)$ invariants matches the number of orthogonal invariants. Such interesting questions require much more work and is left for future investigations.

Part III

Blobbed topological recursion for correlation functions in tensor models

Chapter 8

Definition of the tensor and matrix models

8.1 Bubbles and partition function

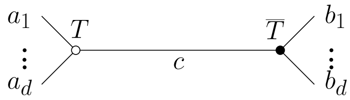
Let us begin this chapter by refreshing the reader's memory about (un)colored tensor models and expanding upon it.

Let $d > 2$ be an integer. For $c \in [\![1, d]\!]$, we call $E_c \simeq \mathbb{C}^N$ the space of color c. Recall from Chapter 2 that a tensor T of rank d is an object in $\bigotimes_{c=1}^{d} E_c$ and its elements are denoted $T_{a_1 \dots a_d}$, with $a_c \in [\![1, N]\!]$ for all $c \in [\![1, d]\!]$. Also, in (un)colored tensor models, one is interested in polynomials in the tensor entries which are invariant under the natural action of (a representation of) $U(N)^d$ on T and $\overline{T}$. This group acts as a different copy of $U(N)$ on each color index,

$$T \to \bigotimes_{c \in [\![1,d]\!]} U^{(c)}\, T. \tag{8.1}$$

The only way to realize this invariance is to identify the index of a T and a $\overline{T}$ which are in the same position, *i.e.* have the same color, and sum over the values of that index. This is represented graphically as follows

$$\sum_{a_c=1}^{N} T_{\dots a_c \dots}\overline{T}_{\dots a_c \dots} =$$



. (8.2)

A *bubble* is a connected, bipartite graph whose edges are labeled by a color in $[\![1, d]\!]$, and such that each vertex has degree d and all colors are incident to each of them. If B is a bubble, the above rule associates to it a polynomial which is invariant under $U(N)^d$ and denoted $B(T, \overline{T})$. These polynomials generate the ring of $U(N)^d$-invariant polynomials. If B has $2n$ vertices and one labels the white vertices from 1 to n and similarly for black vertices, then B can be described as a d-tuple $(\tau^{(1)}, \dots, \tau^{(d)})$ of permutations on $[\![1, n]\!]$. Set $\tau^{(c)}(v) = v'$ if there is an edge of color c connecting the white vertex v to the black vertex v'. The associated polynomial is

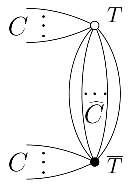
$$B(T, \overline{T}) = \sum_{\substack{(i_1^{(c)}, \dots, i_n^{(c)}) \\ (j_1^{(c)}, \dots, j_n^{(c)})}} \delta^{\tau^{(1)} \dots \tau^{(d)}}_{(i_1^{(c)}, \dots, i_n^{(c)}), (j_1^{(c)}, \dots, j_n^{(c)})} \prod_{v=1}^{n} T_{i_v^{(1)} \dots i_v^{(d)}} \overline{T}_{j_v^{(1)} \dots j_v^{(d)}}, \tag{8.3}$$

with by definition

$$\delta^{\tau^{(1)} \dots \tau^{(d)}}_{(i_1^{(c)}, \dots, i_n^{(c)}), (j_1^{(c)}, \dots, j_n^{(c)})} = \prod_{v=1}^{n} \prod_{c=1}^{d} \delta_{i_v^{(c)}, j^{(c)}_{\tau^{(c)}(v)}}. \tag{8.4}$$

Invariance under relabeling of the white and black vertices implies invariance of $B(T, \overline{T})$ under left product of $\tau^{(1)}, \dots, \tau^{(d)}$ by σ_L and right product by σ_R, two permutations on $[\![1, n]\!]$. If $C \subset [\![1, d]\!]$ and $\hat{C}$ is its complement, then denote

$$E_C = \bigotimes_{c \in C} E_c \qquad \text{and} \qquad H_C(T, \bar{T}) =$$



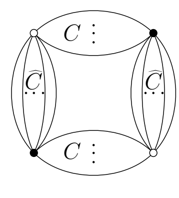
$$\in E_C \otimes E_C^*, \tag{8.5}$$

the matrix obtained by contracting all the colors from $\widehat{C}$ between T and $\overline{T}$. We will write $\big(H_C(T, \overline{T})\big)_{(i^{(c)}), (j^{(c)})}$ the matrix elements. There is a single quadratic invariant (up to a factor), given by the contraction of T with $\overline{T}$ along all colors, namely our dipole (2.10),

$$T \cdot \overline{T} = \sum_{a_1, \dots, a_d = 1}^{N} T_{a_1 \cdots a_d} \overline{T}_{a_1 \cdots a_d} = H_{\emptyset}(T, \overline{T}). \tag{8.6}$$

For quartic invariants, we choose a color subset $C \subset [\![1, d]\!]$ and connect the indices of T with colors in C with a $\overline{T}$ and those with colors in $\widehat{C}$ with another $\overline{T}$,

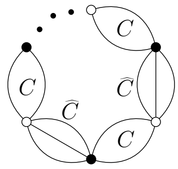
$$Q_C(T, \overline{T}) = \mathrm{tr}_{E_C}\Big(H_C(T, \overline{T})^2\Big) =$$



$$, \tag{8.7}$$

where the notation tr_{E_C} indicates that the trace is taken in the spaces with colors in C. It is invariant under $C \to \widehat{C}$. Note that for $C = \{c\}$ for some $c \in [\![1, d]\!]$ we get back the quartic melonic model of Chapter 3. In this part, we will furthermore consider cyclic interactions, labeled by a color set C and an integer $n \geq 2$

$$B_{C,n}(T,\overline{T}) = \mathrm{tr}_{E_C}\Big(H_C(T,\overline{T})^n\Big) = \tag{8.8}$$



It is again symmetric under the exchange of C and $\widehat{C}$. We say that the cyclic interaction is *melonic* if $|C| = 1$, meaning that in $H_C(T,\overline{T})$, T and $\overline{T}$ are contracted along all colors except one. Let I be a finite set and $\{B_i\}_{i\in I}$ a finite set of bubbles, $\{t_i\}$ their coupling constants and $\{s_i\}$ some scaling coefficients. Denote $\mathcal{B} = \{(B_i, t_i, s_i)\}_{i\in I}$. Then the partition function is

$$\begin{aligned} Z_{\text{Tensor}}(N,\mathcal{B}) &= \int_{(\mathbb{C}^N)^{\otimes d}} dTd\overline{T}\ \exp -N^{d-1}T\cdot\overline{T} + V_{N,\mathcal{B}}(T,\overline{T}), \\ \text{with} \quad V_{N,\mathcal{B}}(T,\overline{T}) &= \sum_{i\in I} N^{s_i} t_i\, B_i(T,\overline{T}), \end{aligned} \tag{8.9}$$

and the free energy is given by $F(N,\mathcal{B}) = \ln Z_{\text{Tensor}}(N,\mathcal{B})$. Here the measure $dTd\overline{T}$ is proportional to the product of the Lebesgue measures over the tensor entries, normalized so that

$$Z_{\text{Tensor}}(N,\emptyset) = \int_{(\mathbb{C}^N)^{\otimes d}} dTd\overline{T}\ \exp -N^{d-1}T\cdot\overline{T} = 1. \tag{8.10}$$

Moreover, we only consider (8.9) to make sense by expanding $e^{V_{N,\mathcal{B}}(T,\overline{T})}$ as a series in $T, \overline{T}$ and integrating each term with the Gaussian weight $e^{-N^{d-1}T\cdot\overline{T}}$, *i.e.* we write

$$e^{V_{N,\mathcal{B}}(T,\overline{T})} = \sum_{\{n_i\geq 0\}_{i\in I}} \prod_{i\in I} \frac{1}{n_i!}\big(N^{s_i} t_i B_i(T,\overline{T})\big)^{n_i}, \tag{8.11}$$

and perform the integral at fixed $\{n_i\}$ using Wick's theorem. More precisely, we expand $B_i(T,\overline{T})^{n_i}$ as a polynomial in the tensor entries

$$\prod_{i\in I} B_i(T,\overline{T})^{n_i} = \sum_{\{a_q^{(c)},b_q^{(c)}\}} \delta^{(\{n_i\})}\big(\{a_q^{(c)}, b_q^{(c)}\}\big) T_{a_1^{(1)}\cdots a_1^{(d)}} \overline{T}_{b_1^{(1)}\cdots b_1^{(d)}} \dots T_{a_p^{(1)}\cdots a_p^{(d)}} \overline{T}_{b_p^{(1)}\cdots b_p^{(d)}}, \tag{8.12}$$

by taking a product of (8.3). The tensor $\delta^{(\{n_i\})}$ is thus a product of Kroneckers. Here p is the total degree, *i.e.* if B_i is of degree p_i in $T, \overline{T}$, then $p = \sum_{i\in I} n_i p_i$. Then Wick's theorem is applied, and leads to an expansion onto pairings, which are here simply permutations $\sigma \in S_p$ on the set of p elements,

$$\begin{aligned}&\int dT d\overline{T}\ e^{-N^{d-1}T\cdot\overline{T}} T_{a_1^{(1)}\dots a_1^{(d)}} \overline{T}_{b_1^{(1)}\dots b_1^{(d)}} \dots T_{a_p^{(1)}\dots a_p^{(d)}} \overline{T}_{b_p^{(1)}\dots b_p^{(d)}} \\ &= N^{-(d-1)p} \sum_{\sigma\in S_p} \delta^{(\sigma)}\big(\{a_q^{(c)}, b_q^{(c)}\}\big),\end{aligned} \tag{8.13}$$

with

$$\delta^{(\sigma)}\big(\{a_q^{(c)}, b_q^{(c)}\}\big) = \prod_{q=1}^{p}\prod_{c=1}^{d} \delta_{a_q^{(c)}, b_{\sigma(q)}^{(c)}}. \tag{8.14}$$

A Feynman graph denoted $G = (\{n_i\}, \sigma)$ has amplitude

$$A_{N,\mathcal{B}}(G) = N^{-(d-1)p} \prod_{i\in I} \frac{(N^{s_i} t_i)^{n_i}}{n_i!} \sum_{\{a_q^{(c)}, b_q^{(c)}\}} \delta^{(\{n_i\})}\big(\{a_q^{(c)}, b_q^{(c)}\}\big)\delta^{(\sigma)}\big(\{a_q^{(c)}, b_q^{(c)}\}\big). \tag{8.15}$$

Since the tensors $\delta^{(\{n_i\})}$ and $\delta^{(\sigma)}$ are products of Kroneckers, and the sums range from 1 to N, those sums give $N^{F(G)}$ for some function $F(G)$ which can be given a simple graphical interpretation. Draw n_i copies of $B_i(T, \overline{T})$ and use σ to connect each white vertex (labeled with q) to a black vertex (labeled $\sigma(q)$) with an edge. It is customary to give the color 0 to those edges. A bicolored cycle of colors $\{0, c\}$ is a closed path alternating an edge of color 0 and an edge of color c. Denote $F_c(G)$ the number of bicolored cycles of colors $\{0, c\}$. Then, by tracking the sequence of index identification along the Kroneckers in the above calculation, it comes that $F(G)$ is the total number of such bicolored cycles, $F(G) = \sum_{c=1}^{d} F_c(G)$. Therefore

$$A_{N,\mathcal{B}}(G) = N^{F(G)-(d-1)p+\sum_{i\in I} s_i n_i} \prod_{i\in I} \frac{t_i^{n_i}}{n_i!}, \tag{8.16}$$

and the free energy reads

$$F(N, \mathcal{B}) = \sum_{\text{connected } G} A_{N,\mathcal{B}}(G). \tag{8.17}$$

The parameters s_i are necessary so that the model has a large N limit which is non-trivial. A non-trivial large N limit is such that $F(N, \mathcal{B})$ appropriately rescaled[1] is a non-trivial[2]

[1] The usual rescaling is N^{-d}.

[2] In fact, one usually requires a condition which is a bit stronger: that an infinite number of graphs from the Feynman expansion contributes to $\lim_{N\to\infty} F(N, \mathcal{B})/N^d$. It could be that only a finite number

function of the coupling constants. *A priori*, those parameters depend on the whole set $\{B_i\}$. However, all models solved so far are such that s_i is determined by B_i solely. For instance, if $B_i = B_{C,n}$ then $s_i = (|C|-1)n + d - |C|$. If $P(T,\overline{T})$ is a polynomial in the tensor entries, its expectation is

$$\langle P(T,\overline{T})\rangle_{\mathcal{B}} = \frac{1}{Z_{\text{Tensor}}(N,\mathcal{B})}\int dTd\overline{T}\ P(T,\overline{T})\ \exp -N^{d-1}T\cdot\overline{T} + V_{N,\mathcal{B}}(T,\overline{T}). \tag{8.18}$$

8.2 Contracted bubbles

We introduce another representation of $U(N)^d$, this time on matrices. Let $(\Phi_C)_{C\subset[\![1,d]\!]}$ be a set of Hermitian, or complex, matrices labeled by all color subsets (except the empty set), such that $\Phi_C \in E_C \otimes E_C^*$. The action of $U(N)^d$ is

$$\Phi_C \to \bigotimes_{c\in C} U^{(c)}\ \Phi_C \bigotimes_{c\in C} U^{(c)\dagger}. \tag{8.19}$$

Bubbles are graphs which can be associated to polynomials which generate the ring of $U(N)^d$-invariant polynomials in $T,\overline{T}$. In this representation, the same role is played by *contracted bubbles.*

Definition 8.2.1. *A contracted bubble $P=(B,\pi)$ is obtain from a bubble B by*

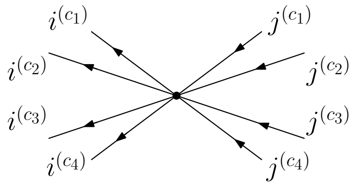
- *orienting the edges of B from white to black vertices,*
- *choosing a pairing π of the vertices of B into pairs $\{v,\pi(v)\}$ where v is white and $\pi(v)$ is black,*
- *identifying the vertices v and $\pi(v)$ of each pair and removing the loops.*

Equivalently, *i.e.* by a trivial bijection, P is a connected graph with oriented edges, each carrying a color $c\in[\![1,d]\!]$, and such that the sub-graph P_c, for c in $[\![1,d]\!]$, made of the edges of color c only, is a disjoint union of oriented cycles. As a remark, we recall that P can further be transformed into a map with colored edges, as shown in [70]. This will not be necessary here. There is a bijection between the vertices of $P=(B,\pi)$ and the white vertices of B. For this reason we will identify them and use the same notations. From the definition, we see that every vertex v of P carries a color set $C_v \subset [\![1,d]\!]$, with

of graphs contribute at large N so that the free energy is a polynomial in the coupling constants. In practice, these two conditions have always been equivalent and we will not discuss these subtleties further.

exactly one incoming and one outgoing edge of every color $c \in C_v$. To obtain an invariant polynomial $P(\{\Phi_C\})$ from the contracted bubble P, one associates to every vertex v_C of color set C a matrix Φ_C, to every incoming incident edge of color c a right index $j_v^{(c)}$ and to every outgoing incident edge a left index $i_v^{(c)}$, *e.g.*

$$\left(\Phi_{\{c_1,c_2,c_3,c_4\}}\right)_{(i^{(c_1)}i^{(c_2)}i^{(c_3)}i^{(c_4)}),(j^{(c_1)}j^{(c_2)}j^{(c_3)}j^{(c_4)})} = \tag{8.20}$$



A left index of color c of a Φ_C at vertex v_C is identified with the right index of the same color of a $\Phi_{C'}$ at vertex $v_{C'}$ if there is an edge of color c from v_C to $v_{C'}$.

Proposition 8.2.2. *The polynomial associated to the contracted bubble $P = (B, \pi)$ is related to the bubble polynomial of B as follows*

$$P(\{H_C(T, \overline{T})\}) = B_P(T, \overline{T}). \tag{8.21}$$

Proof. We simply rewrite $B(T, \overline{T})$ in terms of the matrices $H_C(T, \overline{T})$ given the choice of pairing π. Indeed, each white vertex carries a T and each black vertex carries a $\overline{T}$. If v and $\pi(v)$ are connected by edges with colors in $\widehat{C}$, then the sums over the indices of this T and $\overline{T}$ with colors in $\widehat{C}$ form the matrix $H_C(T, \overline{T})$. Therefore

$$\begin{aligned} B(T, \overline{T}) &= \sum_{\substack{(i_v^{(c)})_{c \in C_v} \\ (j_v^{(c)})_{c \in C_v}}} \prod_{v=1}^{n} \left[H_{C_v}(T, \overline{T})_{(i_v^{(c)}),(j_{\pi(v)}^{(c)})} \prod_{c \in C_v} \delta_{i_v^{(c)}, j_{\tau^{(c)}(v)}^{(c)}} \right] \\ &= \sum_{\substack{(i_v^{(c)})_{c \in C_v} \\ (j_v^{(c)})_{c \in C_v}}} \delta^{\tau^{(1)} \dots \tau^{(d)}, \pi}_{(i_v^{(c)})_{c \in C_v}, (j_v^{(c)})_{c \in C_v}} \prod_{v=1}^{n} H_{C_v}(T, \overline{T})_{(i_v^{(c)}),(j_v^{(c)})}, \end{aligned} \tag{8.22}$$

with

$$\delta^{\tau^{(1)} \dots \tau^{(d)}, \pi}_{(i_v^{(c)})_{c \in C_v}, (j_v^{(c)})_{c \in C_v}} = \prod_{v=1}^{n} \prod_{c \in C_v} \delta_{i_v^{(c)}, j^{(c)}_{\pi^{-1} \circ \tau^{(c)}(v)}}. \tag{8.23}$$

Using the bijection between the white vertices of B and the vertices of $P = (B, \pi)$, we recognize the dependence on H_C as the function $P(\{H_C\})$,

$$P(\{\Phi_C\}) = \sum_{\substack{(i_v^{(c)})_{c\in C_v}\\ (j_v^{(c)})_{c\in C_v}}} \delta^{\tau^{(1)}\dots\tau^{(d)},\pi}_{(i_v^{(c)})_{c\in C_v},(j_v^{(c)})_{c\in C_v}} \prod_{v=1}^{n} \bigl(\Phi_{C_v}\bigr)_{(i_v^{(c)}),(j_v^{(c)})}, \tag{8.24}$$

which concludes the proof. □

8.3 Multi-matrix model and expectation values

Let $\{P_i\}_{i\in I}$ be a finite set of contracted bubbles and denote $\mathcal{P} = \{(P_i, t_i, s_i)\}_{i\in I}$ and

$$V_{N,\mathcal{P}}(\{\Phi_C\}) = \sum_{i\in I} N^{s_i} t_i\, P_i(\{\Phi_C\}). \tag{8.25}$$

We then define the partition function, for pairs of matrices $\{X_C, \Phi_C\}_{C\subset[\![1,d]\!]}$,

$$\begin{aligned} Z_{\mathrm{MM}}(N,\mathcal{P}) = \int \prod_{C\subset[\![1,d]\!]} dX_C d\Phi_C\ \exp - \sum_{C\subset[\![1,d]\!]} \mathrm{tr}_{E_C}\bigl(X_C\Phi_C\bigr) + V_{N,\mathcal{P}}(\{\Phi_C\})\\ - \mathrm{tr}_{\otimes_c E_c} \ln\Bigl(\mathbb{1} - N^{-(d-1)} \sum_C \tilde{X}_C\Bigr), \end{aligned} \tag{8.26}$$

where $\tilde{X}_C = \mathbb{1}_{E_{\widehat{C}}} \otimes X_C$ is the lift of X_C to $\bigotimes_{c=1}^d E_c$ by adding the identity to the colors $c \notin C$. Here MM stands for "multi-matrix". In order to proceed to the Feynman expansion, the above logarithm has to be expanded as

$$\begin{aligned} -\mathrm{tr}_{\otimes_c E_c} \ln\Bigl(\mathbb{1} - N^{-(d-1)} \sum_C \tilde{X}_C\Bigr) &= \sum_{n\geq 1} \frac{N^{-(d-1)n}}{n}\ \mathrm{tr}_{\otimes_c E_c}\Bigl(\sum_C \tilde{X}_C\Bigr)^n \\ &= \sum_{\text{words } w = C_1\cdots C_n} \frac{N^{-(d-1)n}}{n} \mathrm{tr}_{\otimes_c E_c} \tilde{X}_{C_1}\cdots\tilde{X}_{C_n}. \end{aligned} \tag{8.27}$$

There are two possibilities for the integral over X_C, Φ_C, for each $C \subset [\![1,d]\!]$, and the precise form of the Feynman expansion depends on those choices:

- $X_C = \Phi_C^\dagger$ are complex matrices, adjoint to each other;

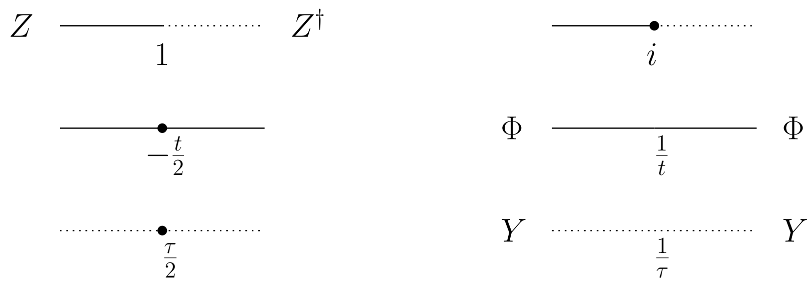


FIGURE 8.1 – On the left hand side are the Feynman rules for the complex model, with one propagator and two types of bivalent vertices. On the right hand side are the Feynman rules for the Hermitian model, with two types of propagators and one type of bivalent vertices.

- Φ_C is Hermitian and $X_C = -iY_C$ where Y_C is Hermitian. In this case, one needs the coupling constant of the quartic bubble Q_C to be negative. We write it $-t_C/2$ with $t_C > 0$.

The equivalence between those two choices is not a given *a priori* because they require different Feynman expansions. In the first case, one uses the quadratic term $\mathrm{tr}_{E_C} X_C \Phi_C$ to define the propagator. In the second case however, one cannot use this term since it reads $-i\mathrm{tr}_{E_C} Y_C \Phi_C$ in terms of Hermitian matrices, and this is not positive-definite. This is the reason why we need to add the condition $t_C > 0$. The following lemma proves the equivalence we need.

Lemma 8.3.1. *For positive coupling constants t, τ, and a potential U which is a series in two variables, the following equality holds formally*

$$\int_{\mathbb{M}_N(\mathbb{C})} dZdZ^\dagger \; e^{-\mathrm{tr}ZZ^\dagger - \frac{t}{2}\mathrm{tr}Z^2 - \frac{\tau}{2}\mathrm{tr}Z^{\dagger 2} + U(Z,Z^\dagger)} = \int_{H_N^2} dYd\Phi \; e^{i\mathrm{tr}Y\Phi - \frac{t}{2}\mathrm{tr}\Phi^2 - \frac{\tau}{2}\mathrm{tr}Y^2 + U(\Phi,-iY)}. \tag{8.28}$$

Here $\mathbb{M}_N(\mathbb{C})$ is the set of complex $N \times N$ matrices and H_N the set of $N \times N$ Hermitian matrices.

Proof. The Feynman rules of the left hand side and right hand side, for propagators and bivalent vertices, are in Figure 8.1. The Feynman expansion of the left hand side is obtained from the expansion

$$\sum_{l,n,m} \frac{(-t/2)^n(-\tau/2)^m}{l!n!m!} \int_{\mathbb{M}_N(\mathbb{C})} dZdZ^\dagger \; e^{-\mathrm{tr}ZZ^\dagger} \left(\mathrm{tr}Z^2\right)^n \left(\mathrm{tr}Z^{\dagger 2}\right)^m U(Z,Z^\dagger)^l, \tag{8.29}$$

and performing Wick contractions between Zs and $Z^\dagger$s. The Feynman rules are thus

- solid half-edges corresponding to the matrix Z and dotted half-edges corresponding to $Z^\dagger$;
- the propagator, coming from the quadratic term $-\mathrm{tr}ZZ^\dagger$, gives rise to edges which have a solid half and a dotted half, with weight 1;
- special vertices of degree 2 with weight $-\frac{t}{2}$ with two incident solid half-edges;
- special vertices of degree 2 with weight $-\frac{\tau}{2}$ with two incident dotted half-edges;
- other vertices coming from the series expansion of $U(Z, Z^\dagger)$.

We call the set of graphs from this expansion $\mathcal{G}_{\text{complex}}$. The Feynman expansion of the right hand side is obtained from the expansion

$$\sum_{l,p} \frac{i^p}{l!p!} \int_{H_N^2} dY d\Phi \; e^{-\frac{t}{2}\mathrm{tr}\Phi^2 - \frac{\tau}{2}\mathrm{tr}Y^2} \Big(\mathrm{tr}\,\Phi Y\Big)^p U(\Phi, -iY)^l, \tag{8.30}$$

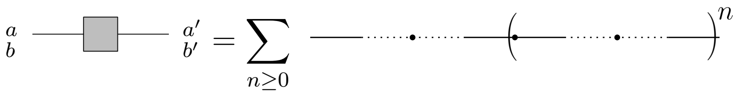
and performing independent Wick contractions between pairs of Φs and between pairs of Ys.

- Solid half-edges corresponding to the matrix Φ and dotted half-edges corresponding to X;
- propagators, coming from the quadratic terms $-\frac{t}{2}\mathrm{tr}\Phi^2 - \frac{\tau}{2}\mathrm{tr}Y^2$, give rise to two types of edges: either two solid half-edges, with weight $1/t$, or two dotted half-edges, with weight $1/\tau$;
- special vertices of degree 2 with weight i with an incident solid half-edge and an incident dotted half-edge;
- other vertices coming from the series expansion of $U(\Phi, -iY)$.

We call the set of graphs from this expansion $\mathcal{G}_{\text{Hermitian}}$. We show that summing the chains of bivalent vertices in both $\mathcal{G}_{\text{complex}}$ and $\mathcal{G}_{\text{Hermitian}}$ leads to the same new set of rules, for a set of graphs we denote $\mathcal{G}_{\text{summed}}$. These graphs are defined as follows.

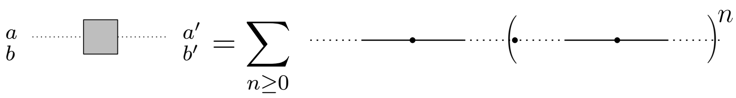
- They have solid and dotted half-edges, and three types of edges: fully solid edges with weight $\tau/(t\tau + 1)$, fully dotted edges with weight $-t/(t\tau + 1)$ and edges made of a solid and a dotted half-edge with weight $1/(t\tau + 1)$.
- Other vertices coming from the series expansion of U.

In the expansion of U, the solid half-edges are associated to the first variable (Z or Φ) and the dotted half-edges to the second variable ($Z^\dagger$ or $-iY$). The sum of bivalent chains starting and ending on solid half-edges in $\mathcal{G}_{\text{complex}}$ gives



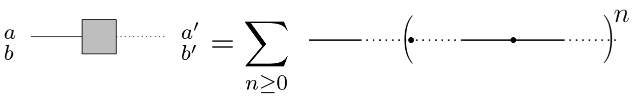
$$= \frac{\tau}{t\tau+1}\delta_{aa'}\delta_{bb'}. \tag{8.31}$$

Here the indices a, b, a', b' are the matrix indices which are identified along Wick contractions. Notice that each vertex contributes to either $2\times(-t/2)$ or $2\times\tau/2$ where the extra factors of 2 come from the two possibilities to add the bivalent vertices, since they are symmetric under the exchange of their incident half-edges. The sum of bivalent chains starting and ending on dotted half-edges is obtained by exchanging τ with $-t$,



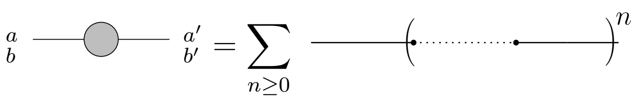
$$= \frac{-t}{t\tau+1}\delta_{aa'}\delta_{bb'}. \tag{8.32}$$

The last sum of bivalent chains is between a solid half-edge and a dotted half-edge

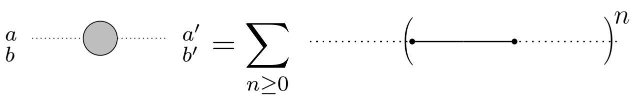


$$= \frac{1}{t\tau+1}\delta_{aa'}\delta_{bb'}. \tag{8.33}$$

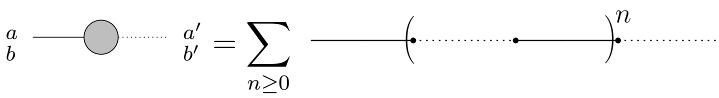
These are indeed the rules for $\mathcal{G}_{\text{summed}}$. Performing the same operation in $\mathcal{G}_{\text{Hermitian}}$, one gets



$$= \frac{1}{t}\sum_{n\geq 0}\left(\frac{i^2}{t\tau}\right)^n = \frac{\tau}{t\tau+1}\delta_{aa'}\delta_{bb'} \tag{8.34}$$



$$= \frac{1}{\tau}\sum_{n\geq 0}\left(\frac{i^2}{t\tau}\right)^n = \frac{t}{t\tau+1}\delta_{aa'}\delta_{bb'} \tag{8.35}$$



$$= \frac{i}{t\tau}\sum_{n\geq 0}\left(\frac{i^2}{t\tau}\right)^n = \frac{i}{t\tau+1}\delta_{aa'}\delta_{bb'}. \tag{8.36}$$

These are not exactly the expected rules, but the difference is compensated by the other vertices. Indeed, in $\mathcal{G}_{\text{Hermitian}}$, the other vertices come from $U(\Phi, -iY)$, *i.e.* there is a factor $-i$ on each dotted half-edge incident to such a vertex. Those factors can be reabsorbed so that the weight of the vertices really comes from $U(\Phi, Y)$, by multiplying each dotted half-edge in (8.34), (8.35) and (8.36) by $-i$. This turns those rules into those of $\mathcal{G}_{\text{summed}}$. □

Let $O(\{X_C, \Phi_C\})$ a $U(N)^d$-invariant function (for the simultaneous action (8.19) on X_Cs and Φ_Cs). Its expectation is

$$\langle O(\{X_C, \Phi_C\})\rangle_{\mathcal{P}} = \frac{1}{Z_{\text{MM}}(N,\mathcal{P})} \int \prod_{C\subset[\![1,d]\!]} dX_C d\Phi_C \; O(\{X_C, \Phi_C\})$$
$$\times \exp - \sum_{C\subset[\![1,d]\!]} \mathrm{tr}_{E_C}\big(X_C\Phi_C\big) + V_{N,\mathcal{P}}(\{\Phi_C\}) - \mathrm{tr}_{\otimes_c E_c}\ln\Big(\mathbb{1} - N^{-(d-1)}\sum_C \tilde{X}_C\Big). \quad (8.37)$$

Lemma 8.3.2. *Let f be a series which takes as arguments $N\times N$ matrices labeled by the subsets of $[\![1,d]\!]$. Then*

$$\langle f(\{H_C(T,\overline{T})\})\rangle_{V_{N,\mathcal{B}}=0} = \langle f(\{\Phi_C\})\rangle_{V_{N,\mathcal{P}}=0}. \quad (8.38)$$

It was proven in [70], both using a bijection between their Feynman expansions, and using formal integrals in the case of complex variables. Here we briefly reproduce the calculation using formal integrals, in order to later relate the expectations of observables on the tensor and matrix sides using the same technique.

Proof. Let us first focus on the case where the expectation on the right hand side of (8.38) is evaluated using complex variables only, $\Phi_C^\dagger = X_C$. Then (8.38) comes from

$$f(h_1,\dots,h_m) = \int_{\mathbb{C}^m} \prod_{l=1}^m dx_l d\phi_l \; e^{\sum_{l=1}^m(-x_l\phi_l + x_l h_l)} \; f(\phi_1,\dots,\phi_m), \quad (8.39)$$

where $\overline{x}_l = \phi_l$ for each l. It holds *via* Wick's theorem, order by order in its series expansion. Making use of (8.39) with every matrix elements of H_C, Φ_C as variables, one gets

$$f(\{H_C(T,\overline{T})\}) = \int \prod_C d\Phi_C dX_C \; e^{\sum_C \mathrm{tr}_{E_C}(-X_C\Phi_C + X_C H_C(T,\overline{T}))} f(\{\Phi_C\}). \quad (8.40)$$

It is now possible to directly integrate the above equation over $T, \overline{T}$ with a Gaussian distribution, leading to

$$\int dT d\overline{T} \; f(\{H_C(T,\overline{T})\}) \; e^{-N^{d-1}T\cdot\overline{T}}$$
$$= \int \prod_C d\Phi_C dX_C \; f(\{\Phi_C\}) \; e^{-\sum_C \mathrm{tr}_{E_C} X_C\Phi_C - \mathrm{tr}_{\otimes_c E_c}\ln\left(\mathbb{1} - N^{-(d-1)}\sum_C \tilde{X}_C\right)}. \quad (8.41)$$

This equality holds up to irrelevant constants. Moreover, the measure on the tensor side has been normalized. On the matrix side, the normalization is trivial when $V_{N,\mathcal{P}} = 0$. This proves (8.38). In the case one wishes to use the Hermitian Φ_C, Y_C with $X_C = -iY_C$, it is necessary to have, instead of vanishing potentials, $V_{N,\mathcal{P}}(\{\Phi_C\}) = -N^{d-1}t_C \mathrm{tr}_{E_C}\Phi_C^2/2$ (and in turn to have on the tensor side a quartic interaction $V_{N,\mathcal{B}}(T,\overline{T}) = -N^{d-1}t_C Q_C(T,\overline{T})/2$). Then Lemma 8.3.1 can be applied to turn the integrals over the complex matrix elements to real matrix elements. The coefficient τ needed in Lemma 8.3.1 comes from the expansion of the logarithm in the definition (8.37) of the expectation. □

Theorem 8.3.3. *Let $\mathcal{B} = \{(B_i, t_i, s_i)\}_{i\in I}$ as in Section 8.1 and $\mathcal{P} = \{(P_i, t_i, s_i)\}_{i\in I}$ such that $P_i = (B_i, \pi_i)$. Then,*

$$Z_{Tensor}(N,\mathcal{B}) = Z_{MM}(N,\mathcal{P}). \tag{8.42}$$

Let f be a series which takes as arguments $N \times N$ matrices labeled by the subsets of $[\![1,d]\!]$. Then

$$\langle f(\{H_C(T,\overline{T})\})\rangle_{\mathcal{B}} = \langle f(\{\Phi_C\})\rangle_{\mathcal{P}}. \tag{8.43}$$

Again, (8.42) was proved in [70], both using a bijection, and using formal integrals.

Proof. The equality between the partition functions simply derives from Lemma 8.3.2 for $f(\{\Phi_C\}) = e^{V_{N,\mathcal{P}}(\{\Phi_C\})}$ and the fact that, from Proposition 8.2.2, $V_{N,\mathcal{P}}(\{H_C(T,\overline{T})\}) = V_{N,\mathcal{B}}(T,\overline{T})$. This takes care of the denominators in (8.43), and the latter is then equivalent to

$$\langle f(\{H_C(T,\overline{T})\})e^{V_{N,\mathcal{P}}(\{H_C(T,\overline{T})\})}\rangle_{V_{N,\mathcal{B}}=0} = \langle f(\{\Phi_C\})e^{V_{N,\mathcal{P}}(\{\Phi_C\})}\rangle_{V_{N,\mathcal{P}}=0}, \tag{8.44}$$

which follows from Lemma 8.3.2. □

Theorem 8.3.4. *With the same notations as previously,*

$$\langle P(\{X_C\})\rangle_{\mathcal{P}} = \sum_{\substack{(i_v^{(c)})_{c\in C_v}\\ (j_v^{(c)})_{c\in C_v}}} \delta^{\tau^{(1)}\dots\tau^{(d)},\pi}_{(i_v^{(c)})_{c\in C_v},(j_v^{(c)})_{c\in C_v}} \left\langle e^{-V_{N,\mathcal{B}}(T,\overline{T})} \prod_{v=1}^{n} \frac{-\partial}{\partial\big(\Phi_{C_v}\big)_{(j_v^{(c)}),(i_v^{(c)})}} \, e^{V_{N,\mathcal{P}}}\big|_{\Phi_C = H_C(T,\overline{T})} \right\rangle_{\mathcal{B}}. \tag{8.45}$$

This theorem generalizes Proposition 1 of [64] in two ways.

- First, [64] is focused on the quartic melonic model, $V_{N,\mathcal{B}} = \sum_c \frac{-t_c}{2} N^{d-1}\mathrm{tr} H_c(T,\overline{T})^2$. In this case, $V_{N,\mathcal{P}}(\{\Phi_C\}) = \sum_c \frac{-t_c}{2} N^{d-1}\mathrm{tr}\,\Phi_{\{c\}}^2$ is quadratic. The theorem thus explains the appearance of Hermite polynomials in [64].

- Second, Theorem 8.3.4 presents the expectation of an arbitrary contracted bubble, while [64] only considered $\langle \mathrm{tr}_{E_c} X^n_{\{c\}} \rangle$.

The reciprocal theorem is Theorem 9.2.1. It is presented later because it uses the technique of partial integration of Section 9.1.

Proof. To prove (8.45), we apply Lemma 8.3.2 to the expectation on the right hand side,

$$\begin{aligned} &\Big\langle e^{-V_{N,\mathcal{B}}(T,\overline{T})} \prod_{v=1}^{n} \frac{-\partial}{\partial\big(\Phi_{C_v}\big)_{(j_v^{(c)}),(i_v^{(c)})}} \, e^{V_{N,\mathcal{P}}}\big|_{\Phi_C = H_C(T,\overline{T})} \Big\rangle_{\mathcal{B}} \\ &\qquad = \Big\langle e^{-V_{N,\mathcal{P}}(\{\Phi_C\})} \prod_{v=1}^{n} \frac{-\partial}{\partial\big(\Phi_{C_v}\big)_{(j_v^{(c)}),(i_v^{(c)})}} e^{V_{N,\mathcal{P}}} \Big\rangle_{\mathcal{P}}, \end{aligned} \tag{8.46}$$

which can then be rewritten as

$$\begin{aligned} &\Big\langle e^{-V_{N,\mathcal{P}}(\{\Phi_C\})} \prod_{v=1}^{n} \frac{-\partial}{\partial\big(\Phi_{C_v}\big)_{(j_v^{(c)}),(i_v^{(c)})}} e^{V_{N,\mathcal{P}}} \Big\rangle_{\mathcal{P}} \\ &= \frac{1}{Z_{\mathrm{MM}}(N,\mathcal{P})} \int \prod_C dX_C d\Phi_C \prod_{v=1}^{n} \frac{-\partial}{\partial\big(\Phi_{C_v}\big)_{(j_v^{(c)}),(i_v^{(c)})}} e^{V_{N,\mathcal{P}}(\{\Phi_C\})} \\ &\quad \times \exp - \sum_C \mathrm{tr}_{E_C}\big(X_C \Phi_C\big) - \mathrm{tr}_{\otimes_c E_c} \ln\big(\mathbb{1} - N^{-(d-1)} \sum_C \tilde{X}_C\big) \\ &= \frac{1}{Z_{\mathrm{MM}}(N,\mathcal{P})} \int \prod_C dX_C d\Phi_C \prod_{v=1}^{n} \frac{\partial}{\partial\big(\Phi_{C_v}\big)_{(j_v^{(c)}),(i_v^{(c)})}} e^{-\sum_C \mathrm{tr}_{E_C}(X_C \Phi_C)} \\ &\quad \times \exp V_{N,\mathcal{P}}(\{\Phi_C\}) - \mathrm{tr}_{\otimes_c E_c} \ln\big(\mathbb{1} - N^{-(d-1)} \sum_C \tilde{X}_C\big) \\ &= \frac{1}{Z_{\mathrm{MM}}(N,\mathcal{P})} \int \prod_C dX_C d\Phi_C \prod_{v=1}^{n} \big(X_{C_v}\big)_{(i_v^{(c)}),(j_v^{(c)})} \\ &\quad \times \exp - \sum_C \mathrm{tr}_{E_C}\big(X_C \Phi_C\big) + V_{N,\mathcal{P}}(\{\Phi_C\}) - \mathrm{tr}_{\otimes_c E_c} \ln\big(\mathbb{1} - N^{-(d-1)} \sum_C \tilde{X}_C\big) \\ &= \big\langle \prod_{v=1}^{n} \big(X_{C_v}\big)_{(i_v^{(c)}),(j_v^{(c)})} \big\rangle. \end{aligned} \tag{8.47}$$

In the second equality, we have used integration by parts. Summing over all indices with the tensor $\delta^{\tau^{(1)} \dots \tau^{(d)}, \pi}$, one recognizes the expansion (8.24) of $P(\{X_C\})$. □

Chapter 9

Effective matrix model

Denote $X_c = X_{\{c\}}$ for $c \in [\![1, d]\!]$ to make the notation lighter. We will show that correlators of the form

$$\left\langle \mathrm{tr}_{V_{c_1}} \frac{1}{x_{c_1} - \delta X_{c_1}} \cdots \mathrm{tr}_{V_{c_n}} \frac{1}{x_{c_n} - \delta X_{c_n}} \right\rangle_c, \tag{9.1}$$

where δX_c is some fluctuation of X_c around its saddle point value, satisfy the blobbed topological recursion. To this aim, we will integrate over all matrices except for $X_1, \ldots, X_d$ to get an effective matrix model for them. This effective matrix model has multi-trace interactions. It is convenient to write them using partitions. Given a set of integers $\lambda_1 \geq \cdots \geq \lambda_l > 0$, we say that $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition of length $\ell(\lambda) = l$ and size $|\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i$. We also allow the special case $\lambda = (0)$ with $\ell(\lambda) = 1$. If M is a matrix on V, we denote the multi-trace (also known as power sums)

$$I_\lambda(M) = \prod_{i=1}^{\ell(\lambda)} \mathrm{tr}_V M^{\lambda_i}. \tag{9.2}$$

Furthermore, we denote $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(d)})$ a vector of partitions, one for every color, and

$$I_{\boldsymbol{\lambda}}(\{M_c\}) = \prod_{c=1}^{d} I_{\lambda^{(c)}}(M_c). \tag{9.3}$$

It contains $\ell(\lambda^{(c)})$ traces of color c, for a total of $\ell(\boldsymbol{\lambda}) = \sum_{c=1}^{d} \ell(\lambda^{(c)})$ traces. It is of total degree $|\boldsymbol{\lambda}| = \sum_{c=1}^{d} |\lambda^{(c)}|$ in the matrices.

Proposition 9.0.1. *A series F in the matrices $X_1, \ldots, X_d$ which is invariant under $U(N)^d$ has an expansion*

$$F(\{X_c\}) = \sum_{\boldsymbol{\lambda}} F(\boldsymbol{\lambda}) I_{\boldsymbol{\lambda}}(\{X_c\}). \tag{9.4}$$

Proof. F has an expansion onto contracted bubbles. The latter are easily seen to be disjoint union of unicycles for all colors, and their corresponding polynomials are $I_{\boldsymbol{\lambda}}$. □

9.1 Partial integrals

We will assume

- the free energy $F(N,\mathcal{B})$ has a $1/N$ expansion

$$F(N,\mathcal{B}) = N^{d'} \sum_{i\geq 0} N^{-\delta_i} F_i(\mathcal{B}), \tag{9.5}$$

  where $(\delta_i)_{i\geq 0}$ is an increasing, positive sequence and $d' > 2$. As far as we know, all tensor models for which the scaling of the free energy is known satisfy $d' = d$;

- I is a finite set and s_i is rational for all $i \in I$. This implies rationality of the δ_is *via* Equation (8.16);

- each $F_i(\mathcal{B})$ exists in a neighborhood of the origin in the space of coupling constants;

- denoting $-t_c/2$ the coupling constant of the quartic bubble $Q_{\{c\}}$ for $c \in [\![1,d]\!]$, we will need $t_c > 0$ in order to choose $\Phi_{\{c\}}$ Hermitian and $X_c = -iY_c$ with Y_c Hermitian.

The results of this section are a bit simplified by rescaling the matrices X_c in the definition (8.26) by N^{d-1}, so that up to irrelevant constants,

$$\begin{aligned} Z_{\mathrm{MM}}(N,\mathcal{P}) = \int \prod_{C\subset[\![1,d]\!]} dX_C d\Phi_C \ \exp - \sum_{C\subset[\![1,d]\!]} N^{d-1}\mathrm{tr}_{E_C}\big(X_C\Phi_C\big) + V_{N,\mathcal{P}}(\{\Phi_C\}) \\ - \mathrm{tr}_{\otimes_c E_c}\ln\Big(\mathbb{1} - \sum_C \tilde{X}_C\Big). \end{aligned} \tag{9.6}$$

Denote

$$L(\{X_C\}) = -\mathrm{tr}_{\otimes_c E_c}\ln\Big(\mathbb{1} - \sum_C \tilde{X}_C\Big) - \frac{1}{2}N^{d-1}\sum_{c=1}^{d}\mathrm{tr}_{E_c}X_c^2, \tag{9.7}$$

which just removes some quadratic terms from the expansion of the logarithm (recall $X_c = X_{\{c\}}$).

Theorem 9.1.1. *Define the partial free energy*

$$\exp F_{N,\mathcal{P}}(\{X_c\}) = \int \prod_{\substack{C\subset[\![1,d]\!]\\|C|\geq 2}} dX_C d\Phi_C \prod_{c=1}^{d} d\Phi_{\{c\}} e^{-\sum_C N^{d-1}\mathrm{tr}(X_C\Phi_C)+V_{N,\mathcal{P}}(\{\Phi_C\})+L(\{X_C\})}, \tag{9.8}$$

where $X_c = -iY_c$, Y_c Hermitian for $c\in[\![1,d]\!]$. In the integral $\Phi_{\{c\}}$ are Hermitian, while the pairs $\{\Phi_C, X_C\}$ can be chosen complex, $\Phi_C^\dagger = X_C$, or with $X_C = -iY_C$ with Φ_C, Y_C Hermitian (provided $t_C > 0$) for all $|C| > 1$. Then

$$F_{N,\mathcal{P}}(\{X_c\}) = \sum_{\boldsymbol{\lambda}} N^{d'-\ell(\boldsymbol{\lambda})}\ t_{N,\mathcal{P}}(\boldsymbol{\lambda})\ I_{\boldsymbol{\lambda}}(\{X_c\}), \tag{9.9}$$

where $t_{N,\mathcal{P}}(\boldsymbol{\lambda})$ has a $1/N$ expansion which starts at order $\mathcal{O}(1)$

$$t_{N,\mathcal{P}}(\boldsymbol{\lambda}) = \sum_{i\geq 0} N^{-\delta_i(\boldsymbol{\lambda})} t_{\mathcal{P}}^{(i)}(\boldsymbol{\lambda}), \tag{9.10}$$

and $(\delta_i(\boldsymbol{\lambda}))_{i\geq 0}$ are positive, increasing sequences of rationals. Furthermore

$$Z_{Tensor}(N,\mathcal{B}) = \int \prod_{c=1}^{d} dX_c \exp\Big(\frac{1}{2}N^{d-1}\sum_{c=1}^{d}\mathrm{tr}_{E_c}X_c^2 + F_{N,\mathcal{P}}(\{X_c\})\Big), \tag{9.11}$$

with the same relation between $\mathcal{B}$ and $\mathcal{P}$ as in Theorem 8.3.4.

Proof. The proof first establishes (9.11), by showing that the Feynman expansion of $Z_{\text{Tensor}}(N,\mathcal{B})$ can be obtained from that of $F_{N,\mathcal{P}}(\{X_c\})$. Then, for each Feynman graph of the expansion of $F_{N,\mathcal{P}}(\{X_c\})$, we build a special Feynman graph for $Z_{\text{Tensor}}(N,\mathcal{B})$ in a way such that the N-dependence of this construction is controlled. Thus, the $1/N$ expansion of the $Z_{\text{Tensor}}(N,\mathcal{B})$ implies (9.9), (9.10).

Feynman rules – Theorem 8.3.4 allows for studying the matrix model instead of the tensor model. Then proving (9.11) amounts to show that one can write the Feynman rules so as to integrate first over all matrices Φ_C, X_C, except $X_1,\ldots,X_d$, and then integrate the latter. This will be clear once we have described how we use the Feynman expansion on $Z_{\text{MM}}(N,\mathcal{P})$. We choose for the Feynman expansion of $Z_{\text{MM}}(N,\mathcal{P})$ to use $\Phi_{\{c\}}$ Hermitian and $X_c = -iY_c$ with Y_c Hermitian for $c\in[\![1,d]\!]$. As for the matrices Φ_C, X_C for $|C|>1$, we can use the complex matrices or the Hermitian matrices either way. Let us choose the latter, since it will let us have a unified description of the Feynman rules, and it is the setup for Theorem 9.2.1. This however requires $t_C > 0$. We thus write integrals over Hermitian matrices,

$$Z_{\rm MM}(N,\mathcal{P}) = \int \prod_C d\Phi_C dY_C \; e^{\sum_C \mathrm{tr}_{E_C}\left(iN^{d-1}\Phi_C Y_C - \frac{t_C}{2}\Phi_C^2 - \frac{N^{d-|C|}}{2}Y_C^2\right) + V_{N,\mathcal{P}}(\{\Phi_C\}) + K(\{Y_C\})}, \tag{9.12}$$

with

$$\begin{aligned} K(\{Y_C\}) &= -\mathrm{tr}_{\bigotimes_{c=1}^d E_c} \ln\Big(\mathbb{1} + i\sum_C \tilde{Y}_C\Big) + \frac{1}{2}\sum_C N^{d-|C|}\mathrm{tr}_{E_C} Y_C^2 \\ &= \sum_{\text{words } w\in W} \frac{(-i)^{|w|}}{|w|}\mathrm{tr}_{\bigotimes_{c=1}^d E_c} \tilde{Y}_{C_1}\dots\tilde{Y}_{C_{|w|}}, \end{aligned} \tag{9.13}$$

which is the series expansion of the logarithm minus its single-trace, quadratic terms (since we have isolated them to be used for the propagators). The set W is a set of words

$$W = \{w = C_1\cdots C_{|w|} \,|\, \forall q \in [\![1,|w|]\!],\; C_q \subset [\![1,d]\!],\; |w| \neq 2 \text{ or } w = C_1C_2 \text{ with } C_1 \neq C_2\}. \tag{9.14}$$

We have also separated from $V_{N,\mathcal{P}}$ its quadratic terms but we retain the notation. To perform the Feynman expansion, we first expand

$$\begin{aligned} &e^{\sum_C \mathrm{tr}_{E_C} iN^{d-1}\Phi_C Y_C + V_{N,\mathcal{P}}(\{\Phi_C\}) + K(\{Y_C\})} \\ &= \sum_{\{n_C\},\{n_i\},\{n_w\}} \frac{i^{\sum_C n_C}(-i)^{\sum_w n_w|w|}\left(N^{s_i}t_i\right)^{n_i}}{\prod_C n_C!\prod_i n_i!\prod_w n_w!|w|} \\ &\qquad \times \prod_C \Big(\mathrm{tr}_{E_C}\Phi_C Y_C\Big)^{n_C} \prod_i \Big(P_i(\{\Phi_C\})\Big)^{n_i} \prod_w \Big(\mathrm{tr}_{\bigotimes_c E_c}\tilde{Y}_{C_1}\dots\tilde{Y}_{C_{|w|}}\Big)^{n_w}. \end{aligned} \tag{9.15}$$

Here the indices C span the subsets of $[\![1,d]\!]$, and $i \in I$, and $w \in W$. Wick's theorem can then be applied and, at fixed $\{n_C, n_i, n_w\}$, expresses the Gaussian moments as sums over pairings $\{\sigma_C, \rho_C\}$, the first identifying pairs of Φ_Cs and the second identifying pairs of Y_Cs,

$$\int e^{-\frac{t_C}{2}\mathrm{tr}_{E_C}\Phi_C^2}\Phi_{Ca_1b_1}\dots\Phi_{Ca_{2n}b_{2n}} = \frac{1}{t_C^n}\sum_{\sigma_C}\prod_{\{i,j\}\in\sigma_C}\delta_{a_i,a_j}\delta_{b_i,b_j}, \tag{9.16}$$

where a pairing σ_C is a way to partition $[\![1,2n]\!]$ into disjoint pairs $\{i,j\}$, and

$$\int e^{-\frac{N^{d-|C|}}{2}\mathrm{tr}_{E_C}Y_C^2}Y_{Ca_1b_1}\dots Y_{Ca_{2n}b_{2n}} = N^{-(d-|C|)n}\sum_{\rho_C}\prod_{\{i,j\}\in\rho_C}\delta_{a_i,a_j}\delta_{b_i,b_j}. \tag{9.17}$$

We call a Feynman graph $G = (\{n_C, \sigma_C, \rho_C\}_{C \subset [\![1,d]\!]}, \{n_i\}_{i \in I}, \{n_w\}_{w \in W})$. The Feynman rules are as follows.

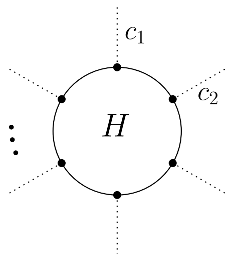
- Solid half-edges of color type C correspond to the matrix Φ_C, and dotted half-edges of color type C correspond to Y_C.

- Propagators come from the quadratic terms $-\frac{1}{2}\sum_C (t_C \mathrm{tr}_{E_C} \Phi_C^2 + N^{d-|C|} \mathrm{tr}_{E_C} Y_C^2)$ and give rise to two types of edges: fully solid edges of color type C with weight $1/t_C$ and fully dotted edges of color type C with weight $N^{|C|-d}$.

- A special bivalent vertex of weight iN^{d-1} with an incident solid half-edge and an incident dotted half-edge.

- Vertices coming from the expansion of K have dotted incident half-edges, and vertices coming from the expansion of $V_{N,\mathcal{P}}$ have solid incident half-edges.

The sums over all the indices, identified along interactions and Wick contractions will be described below. Denote $\mathcal{G}$ the set of connected Feynman graphs for this Feynman expansion. It is clear with those rules that one can first perform the integrals over the matrices Φ_C, Y_C for $|C| > 1$ and $\Phi_{\{1\}}, \ldots, \Phi_{\{d\}}$ by summing over the corresponding pairings, and then integrate over $Y_1, \ldots, Y_d$ by summing over $\rho_1, \ldots, \rho_d$. This proves (9.11). In fact, one can integrate the matrices Φ_C, Y_C in any particular order. This means that partial integrals can be performed as one wishes. However, only in our case will we be able to describe the effective action $F_{N,\mathcal{P}}(\{Y_c\})$. We denote $\mathcal{G}(\{Y_c\})$ the set of connected Feynman graphs for $F_{N,\mathcal{P}}(\{Y_c\})$. They are constructed using the same set of Feynman rules except that fully dotted edges of color $c \in [\![1, d]\!]$ are not allowed anymore. This means that all dotted half-edges of color $c \in [\![1, d]\!]$ are left hanging, *viz.*



. (9.18)

For instance, in the expansion of $K(\{Y_C\})$, one finds terms with $w = c_1 \cdots c_n \in [\![1, d]\!]^n$. They contribute to $H_w \in \mathcal{G}(\{Y_c\})$ as graphs with a single vertex and n hanging dotted half-edges of colors $c_1, \ldots, c_n$ (respecting the cyclic order). Clearly there is a bijection between $\mathcal{G}$ and collections of graphs from $\mathcal{G}(\{Y_c\})$ connected by Wick pairings $\rho_1, \dots, \rho_d$. Graphically, one obtains from $G \in \mathcal{G}$ the collections of graphs from $\mathcal{G}(\{Y_c\})$ by cutting all dotted edges into half-edges. The other way around, performing the Wick pairings $\rho_1, \dots, \rho_d$ means connecting the corresponding dotted half-edges. From the above Feynman rules we have found

$$F_{N,\mathcal{P}}(\{Y_c\}) = \sum_{G \in \mathcal{G}(\{Y_c\})} A_G(N, \mathcal{P}; \{Y_c\}), \tag{9.19}$$

where $A_G(N, \mathcal{P}; \{Y_c\})$ is the amplitude of G. It is a polynomial in the matrices Y_cs (since it is a finite object). From the invariance under unitary transformations and Proposition 9.0.1, we find that

$$F_{N,\mathcal{P}}(\{Y_c\}) = \sum_{\boldsymbol{\lambda}} F_{N,\mathcal{P}}(\boldsymbol{\lambda}) I_{\boldsymbol{\lambda}}(\{Y_c\}). \tag{9.20}$$

To establish the $1/N$ expansion of $F_{N,\mathcal{P}}(\boldsymbol{\lambda})$, we need to look into the structure of the Feynman graphs and their N-dependence. The reader already familiar with faces as the origin of the N-dependence can skip this discussion.

Faces of Feynman graphs – As usual in matrix models, a factor N comes from each face, but we need to explain what faces are in this multi-matrix, multi-size context. In ordinary single-trace matrix models, it corresponds to a sequence of identifications of matrix indices *via* propagators and interactions on a Wick contraction. In terms of Feynman graphs, there is a cyclic order of the half-edges incident to each vertex. This means that Feynman graphs are in fact combinatorial maps. A face is then a closed path which follows an edge to a vertex, then uses the cyclic order around that vertex to move to another half-edge (*e.g.* counter-clockwise), then follows that edge, etc. Consider an index of color c of a Φ_C in $V_{N,\mathcal{P}}$, or of a X_C in $K(\{X_C\})$, for $c \in C$. In a Feynman graph $G \in \mathcal{G}$, the propagator identifies it with an index of another interaction which has the same color, then because the interaction are unitary invariant, it is further identified with an index of the same color of another matrix, which is then identified to another index of the same color by a propagator, and so on. One ends up with a free sum from 1 to N for each such cycle, hence a power of N. We thus see that there could be a notion of faces, but it is color by color, and it requires to track the index identifications in $V_{N,\mathcal{P}}(\{\Phi_C\})$. One of the key results of [70] is that the interaction $V_{N,\mathcal{P}}(\{\Phi_C\})$ can be given the structure of a map with edges labeled by color type $C \subset [\![1, d]\!]$. Consider $G \in \mathcal{G}$ and denote $G_c \subset G$ for $c \in [\![1, d]\!]$ the sub-graph whose edges have color set $C \ni c$. It is a disjoint union of ordinary combinatorial maps and the *faces of color c* are defined as the faces of those maps. The weight of a graph then goes like $N^{F(G)}$ (and other factors of N).

Feynman graphs for $F_{N,\mathcal{P}}(\{Y_c\})$ – Let $H \in \mathcal{G}(\{Y_c\})$. It has faces of color $c \in [\![1,d]\!]$ as defined above, but some faces go around some hanging dotted half-edges. We call them external faces, while the others are internal faces. Every internal face contributes with a factor N. However, since dotted half-edges of color c correspond to the matrix Y_c which is not integrated over, the Feynman amplitude receives a matrix Y_c every time one goes around a face and meets a hanging dotted half-edge. The matrix indices of Y_c are then identified along the face. An external face thus receives $\mathrm{tr}_{E_c} Y_c^{l_f}$ where l_f is the number of such bivalent interactions around the face. This shows that for every $H \in \mathcal{G}(\{Y_c\})$, there exists a unique $\boldsymbol{\lambda}$ such that

$$A_H(N,\mathcal{P};\{Y_c\}) = \tilde{A}_H N^{\eta(G)} I_{\boldsymbol{\lambda}}(\{Y_c\}), \tag{9.21}$$

where $\tilde{A}_H$ is independent of N and of the matrices Y_cs. For $H \in \mathcal{G}(\{Y_c\})$, we denote n_i the number of interactions P_i, and b_C the number of bivalent interactions $\mathrm{tr}(X_C\Phi_C)$ for $C \subset [\![1,d]\!]$, and $F_{\text{int}}(G)$ the number of internal faces. It comes

$$\eta(H) = F_{\text{int}}(H) + \sum_{i\in I} n_i s_i + \sum_{C} (d-1) b_C. \tag{9.22}$$

We can thus write $\mathcal{G}(\{Y_c\}) = \bigcup_{\boldsymbol{\lambda}} \mathcal{G}_{\boldsymbol{\lambda}}(\{Y_c\})$, where the amplitude of $H \in \mathcal{G}_{\boldsymbol{\lambda}}(\{Y_c\})$ is proportional to $I_{\boldsymbol{\lambda}}(\{Y_c\})$, and

$$F_{N,\mathcal{P}}(\boldsymbol{\lambda}) = \sum_{H\in\mathcal{G}_{\boldsymbol{\lambda}}(\{Y_c\})} \tilde{A}_H N^{\eta(H)}. \tag{9.23}$$

Denote

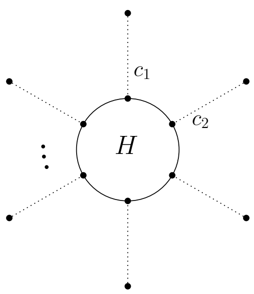
$$d_{\boldsymbol{\lambda}} = \sup_{H\in\mathcal{G}_{\boldsymbol{\lambda}}(\{Y_c\})} \eta(H). \tag{9.24}$$

If $d_{\boldsymbol{\lambda}} < \infty$, then $d_{\boldsymbol{\lambda}}$ is actually a maximum. Indeed, we see in Equation (8.16) that the exponent of N is a finite sum of integers - except for the s_i, but there are a finite number of them and they are rationals and the same for all graphs. Therefore one can write the exponents of N with the same denominator for all graphs, while the numerators consist in a sequence of integers. If the latter has a finite supremum, it is obviously a maximum.

$1/N$ **expansion** – A graph $G \in \mathcal{G}$ is made out of sub-graphs $H_{\boldsymbol{\lambda}_1}, \dots, H_{\boldsymbol{\lambda}_R} \in \mathcal{G}(\{Y_c\})$ connected by dotted edges of colors in $[\![1, d]\!]$. We denote E' the number of those edges and further denote F'_c the number of faces of color c which go along them, and $F' = \sum_{c=1}^{d} F'_c$. The sub-graphs $H_{\boldsymbol{\lambda}_1}, \dots, H_{\boldsymbol{\lambda}_R} \in \mathcal{G}(\{X_c\})$ come with powers of N, $\eta(H_{\boldsymbol{\lambda}_1}), \dots, \eta(H_{\boldsymbol{\lambda}_R})$ and amplitudes $\tilde{A}_{H_{\boldsymbol{\lambda}_1}}, \dots, \tilde{A}_{H_{\boldsymbol{\lambda}_R}}$. Altogether, the amplitude of G is

$$A_{N,\mathcal{P}}(G) = N^{F'-(d-1)E'+\sum_{r=1}^{R} \eta(H_{\boldsymbol{\lambda}_r})} \prod_{r=1}^{R} \tilde{A}_{H_{\boldsymbol{\lambda}_r}}. \tag{9.25}$$

Consider $H \in \mathcal{G}_{\boldsymbol{\lambda}}(\{Y_c\})$. We build a graph $G(H) \in \mathcal{G}$ as follows. Let $H_c \in \mathcal{G}(\{Y_c\})$ be the graph which consists in a single vertex and a single dotted half-edge of color c. Then we connect each hanging dotted half-edge of color c of H to H_c,

$G(H) =$  , (9.26)

H corresponds to the interaction $I_{\boldsymbol{\lambda}}$ and therefore has $\ell(\boldsymbol{\lambda})$ external faces, and $|\boldsymbol{\lambda}|$ hanging dotted half-edges. In G, each external face of H becomes an internal face, adding a factor of N to the amplitude. Each univalent vertex H_c is also a connected component for the $d-1$ sub-graphs $\mathcal{G}_{c'}$, $c' \neq c$. The total number F' of faces which go along the newly added, fully dotted edges of color $c \in [\![1, d]\!]$ is thus $\ell(\boldsymbol{\lambda}) + (d-1)|\boldsymbol{\lambda}|$. The number E' of those new edges is equal to $|\boldsymbol{\lambda}|$. Formula (9.25) thus gives

$$A_{N,\mathcal{P}}(G(H)) = N^{\ell(\boldsymbol{\lambda})+\eta(H)}\, A(G(H)). \tag{9.27}$$

By assumption, this is bounded by $N^{d'}$ for all H. Taking the supremum over all $H \in \mathcal{G}_{\boldsymbol{\lambda}}(\{Y_c\})$, one finds

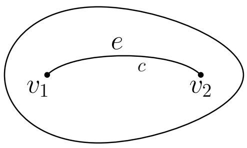
$$d_{\boldsymbol{\lambda}} \leq d' - \ell(\boldsymbol{\lambda}), \tag{9.28}$$

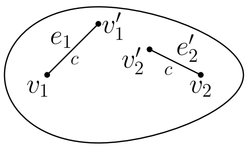
further implying that $d_{\boldsymbol{\lambda}}$ is a maximum. This shows that $F_N(\boldsymbol{\lambda}) \leq N^{d'-\ell(\boldsymbol{\lambda})}$ and it is obvious that the sub-leading orders are rational powers of N. □

While it is not necessary for our purposes, the leading order of $F(N, \mathcal{P})$ can be described as follows.

Proposition 9.1.2. *The graphs contributing to the leading order of $F(N, \mathcal{P})$ are trees over sub-graphs $H_{\boldsymbol{\lambda}_1}, \ldots, H_{\boldsymbol{\lambda}_R}$, with $H_{\boldsymbol{\lambda}_r} \in \mathcal{G}_{\boldsymbol{\lambda}_r}(\{Y_c\})$, satisfying $\eta(H_{\boldsymbol{\lambda}_r}) = d' - \ell(\boldsymbol{\lambda})$.*

Proof. We start with the following lemma: an edge e in $G \in \mathcal{G}$ is not a bridge, then G is not leading order. Indeed, if e is of color c and not a bridge, it can be cut into edges of color c as follows without disconnecting G,

$G =$ 

$\to \quad G' =$ 

. (9.29)

The exponent of N for G' is calculated from the Feynman rules as follows

$$A_{N,\mathcal{P}}(H) = N^{F + \sum_{i \in I} n_i s_i + \sum_C (d-1) b_C - (d - |C|) E_C} A(H), \tag{9.30}$$

for any $H \in \mathcal{G}$, where $A(H)$ is independent of N. We now look at the variations of the quantities appearing in this formula. From G to G', the number of edges changes by 1, the number of contracted bubbles b_C of each type is unchanged. The number F_c of faces of color c can increase or decrease by one. Moreover, $\{v'_1\}$ and $\{v'_2\}$ are new connected components of the sub-graphs $G_{c'}$ for $c' \neq c$, implying that the number of faces $F_{c'}$ varies by exactly two. Therefore the total variation of the exponent of N is

$$\Delta\big(F + \sum_{i \in I} n_i s_i - (d-1)(E - b)\big) = \pm 1 + 2(d-1) - (d-1) \geq d - 2 > 0, \tag{9.31}$$

which proves the lemma. As a consequence, a leading order graph is a tree over sub-graphs $H_{\boldsymbol{\lambda}_1}, \ldots, H_{\boldsymbol{\lambda}_R} \in \mathcal{G}(\{X_c\})$. To maximize the exponent of N, we need to maximize each $\eta(H_{\boldsymbol{\lambda}_r})$. As seen in the above proof, $H_{\boldsymbol{\lambda}_r}$ therefore contributes to the leading order if and only if $\eta(H_{\boldsymbol{\lambda}_r}) = d' - \ell(\boldsymbol{\lambda})$. It is then easy to check that every graph obtained this way behaves as $N^{d'}$ with respect to N and is thus a leading order graph. □

9.2 Bubble observables

In Theorem 8.3.4, we wrote the expectations of the multi-matrix model in terms of expectations of the tensor model. Here we provide the reverse theorem.

Theorem 9.2.1. *Define the effective action*

$$e^{W_{N,\mathcal{P}}(\{X_C\})} = \int \prod_{C\subset[\![1,d]\!]} d\Phi_C \; e^{-\sum_C \mathrm{tr}_{E_C}(X_C\Phi_C)+V_{N,\mathcal{P}}(\{\Phi_C\})}, \tag{9.32}$$

with the same techniques as in the proof of Theorem 9.1.1 to define $F_{N,\mathcal{P}}(\{X_c\})$. Then for any contracted bubble $P=(B,\pi)$,

$$\langle B(T,\overline{T})\rangle_{\mathcal{B}} = \sum_{\substack{(i_v^{(c)})_{c\in C_v}\\ (j_v^{(c)})_{c\in C_v}}} \delta^{\tau^{(1)}\dots\tau^{(d)},\pi}_{(i_v^{(c)})_{c\in C_v},(j_v^{(c)})_{c\in C_v}} \left\langle e^{-W_{N,\mathcal{P}}(\{X_C\})} \prod_v \frac{-\partial}{\partial\big(X_{C_v}\big)_{(j_v^{(c)}),(i_v^{(c)})}} e^{W_{N,\mathcal{P}}(\{X_C\})}\right\rangle_{\mathcal{P}}. \tag{9.33}$$

It is the reciprocal of Theorem 8.3.4. It also generalizes [64] in the same two ways: to arbitrary bubbles $B(T,\overline{T})$ instead of melonic cycles $\mathrm{tr}H_{\{c\}}(T,\overline{T})^n$, and to an arbitrary model instead of the quartic melonic one. There, since $V_{N,\mathcal{P}}(\{\Phi_C\}) = \sum_{c=1}^d -\frac{t_c}{2}\mathrm{tr}_{E_c}\Phi_{\{c\}}^2$, the integral defining $W_{N,\mathcal{P}}$ can be performed to find $W_{N,\mathcal{P}}(\{X_c\}) = \frac{1}{2t_c}\mathrm{tr}_{E_c}X_c^2$. This is how we recover the Hermite polynomials found in [64]. Notice that the equivalent of our theorems 8.3.4 and 9.2.1 in [64] both feature Hermite polynomials. Here we see that in general it is not the same object in both theorems, since one has derivatives with respect to $V_{N,\mathcal{P}}(\{\Phi_C\})$, while the other has derivatives with respect to $W_{N,\mathcal{P}}(\{X_C\})$.

Proof. We start with the following equalities

$$\langle B(T,\overline{T})\rangle_{\mathcal{B}} = \langle P(\{H_C(T,\overline{T})\})\rangle_{\mathcal{B}} = \langle P(\{\Phi_C\})\rangle_{\mathcal{P}}, \tag{9.34}$$

the first one being Proposition 8.2.2 and the second Theorem 8.3.3. Then we expand P as a polynomial (8.24) and use the matrices X_Cs as sources,

$$\begin{aligned}\langle\prod_{v=1}^n \big(\Phi_{C_v}\big)_{(i_v^{(c)}),(j_v^{(c)})}\rangle_{\mathcal{P}} = \frac{1}{Z_{\mathrm{MM}}(N,\mathcal{P})}\int \prod_{C\subset[\![1,d]\!]} dX_C d\Phi_C \; e^{V_{N,\mathcal{P}}(\{\Phi_C\})-\mathrm{tr}_{\otimes_c E_c}\ln\left(\mathbb{1}-N^{-(d-1)}\sum_C \tilde{X}_C\right)}\\ \times\prod_{v=1}^n \frac{-\partial}{\partial\big(X_{C_v}\big)_{(j_v^{(c)}),(i_v^{(c)})}} e^{-\sum_C \mathrm{tr}_{E_C}(X_C\Phi_C)},\end{aligned} \tag{9.35}$$

then integrate by parts

$$\langle\prod_{v=1}^{n}(\Phi_{C_v})_{(i_v^{(c)}),(j_v^{(c)})}\rangle_{\mathcal{P}} = \frac{1}{Z_{\mathrm{MM}}(N,\mathcal{P})}\int\prod_{C\subset[\![1,d]\!]} dX_C d\Phi_C\ e^{-\sum_C \mathrm{tr}_{E_C}(X_C\Phi_C)+V_{N,\mathcal{P}}(\{\Phi_C\})}$$
$$\times\prod_{v=1}^{n}\frac{\partial}{\partial(X_{C_v})_{(j_v^{(c)}),(i_v^{(c)})}}e^{-\mathrm{tr}_{\otimes_c E_c}\ln\left(\mathbb{1}-N^{-(d-1)}\sum_C\tilde{X}_C\right)}. \tag{9.36}$$

At this stage, one performs the integrals over all Φ_Cs,

$$\langle\prod_{v=1}^{n}(\Phi_{C_v})_{(i_v^{(c)}),(j_v^{(c)})}\rangle_{\mathcal{P}} = \frac{1}{Z_{\mathrm{MM}}(N,\mathcal{P})}\int\prod_{C\subset[\![1,d]\!]} dX_C\ e^{W_{N,\mathcal{P}}(\{X_C\})}$$
$$\times\prod_{v=1}^{n}\frac{\partial}{\partial(X_{C_v})_{(j_v^{(c)}),(i_v^{(c)})}}e^{-\mathrm{tr}_{\otimes_c E_c}\ln\left(\mathbb{1}-N^{-(d-1)}\sum_C\tilde{X}_C\right)}, \tag{9.37}$$

and integrate by parts again to find the Theorem. □

9.3 Comparison with ordinary multi-trace matrix models

The model (9.11) with action (9.9) has a natural interpretation in terms of stuffed maps as introduced in [67], with additional colors on their boundary components, and a non-topological expansion. Recall that a map can be seen as a gluing of polygons along their boundaries. Here a polygon is simply a 2-cell homeomorphic to a disc, with k boundary edges. We then call k the perimeter of the boundary. In stuffed maps, polygons are replaced with *elementary 2-cells of topology* (h,λ), where $\lambda=(\lambda_1,\dots,\lambda_{\ell(\lambda)})$ is a partition. Such a 2-cell is homeomorphic to a surface of genus h with $\ell(\lambda)$ boundary components of perimeters $\lambda_1,\dots,\lambda_{\ell(\lambda)}$. A *stuffed map* is a gluing of elementary 2-cells along the edges of their boundary components. In matrix models, a polygon of perimeter k corresponds to an interaction $\mathrm{tr}X^k$ in Feynman graphs. In matrix models with multi-trace interactions, whose partition functions are of the form

$$\int dX\ \exp\sum_{\lambda,h}N^{2-\ell(\lambda)-2h}t^{(h)}(\lambda)I_\lambda(X), \tag{9.38}$$

the interaction $N^{2-\ell(\lambda)-2h}I_\lambda(X)$ is naturally interpreted as an elementary 2-cell of topology (h,λ). Each trace in $I_\lambda(X)=\prod_{i=1}^{\ell(\lambda)}\mathrm{tr}X^{\lambda_i}$ gives rise to a boundary component, and the exponent λ_i gives its perimeter. When interpreting the Feynman expansion of a matrix model with multi-trace interactions in terms of stuffed maps, notice that the only way to associate the genus h to an elementary 2-cell is that the exponent of N in front of $I_\lambda(X)$ is $N^{2-\ell(\lambda)-2h}$ and the coupling constant $t^{(h)}(\lambda)$ is independent of N. In fact, a matrix model with an interaction like

$$\sum_\lambda N^{2-\ell(\lambda)}t_N(\lambda)I_\lambda(X), \tag{9.39}$$

has a well-defined large N limit when $t_N(\lambda)$ itself admits a $1/N$ expansion starting at order $\mathcal{O}(1)$, *i.e.* $t_N(\lambda)=\sum_{i\geq 0}t^{(i)}(\lambda)N^{-\delta_i(\lambda)}$ where $(\delta_i(\lambda))_{i\geq 0}$ is an increasing sequence of non-negative numbers. However, for the expansions of the free energy and of the correlation functions to be called topological, they must be series in $1/N^2$. It is the case when the coupling constants $t_N(\lambda)$ are themselves series in $1/N^2$, *i.e.* $\delta_i(\lambda)\in 2\mathbb{N}$ but not in general. The model (9.46) fits into this framework, with the following amendments.

- It has d matrices $X_1,\ldots,X_d$ and interaction in the form $I_{\boldsymbol{\lambda}}(\{X_c\})$. In terms of stuffed maps, it simply means that the boundary components of elementary 2-cells are now colored and a partition $\lambda^{(c)}$ is needed to describe the perimeters of the boundary components of each color $c\in[\![1,d]\!]$. An elementary 2-cell with boundary profile $\boldsymbol{\lambda}$ moreover comes with the weight $N^{d'-\ell(\boldsymbol{\lambda})}t_{N,\mathcal{P}}(\boldsymbol{\lambda})$.

- Obvious from the above discussion, the $1/N$ expansion is not topological, since $d'\neq 2$, and the sequence $(\delta_i(\boldsymbol{\lambda}))_{i\geq 0}$ in (9.10) may not consist of even integers.

In the following section we first focus on the consequence of $d'>2$ for the large N limit.

9.4 Large N limit and fluctuations

Theorem 9.1.1 provides the form we are looking for to be able to apply the blobbed topological recursion, as in [66]. In this section, we thus follow the first step of [66] which is to subtract the leading contribution at large N.

9.4.1 Subtracting the leading order

Changing variables from $X_c = U_c D_c U_c^\dagger$ to unitary matrices $(U_1, \dots, U_d)$ and eigenvalues, $D_c = \mathrm{diag}(x_1^{(c)}, \dots, x_N^{(c)})$ for all $c \in [\![1, d]\!]$, in the matrix formulation of Theorem 9.1.1, the angular parts are trivially integrated out. The change of variables also produces a squared Vandermonde determinant for all $c \in [\![1, d]\!]$. This gives

$$Z_{\mathrm{MM}}(N, \mathcal{P}) = \int \prod_{c=1}^{d} \prod_{i_c=1}^{N} dx_{i_c}^{(c)} \, \exp\Big(\frac{1}{2} N^{d-1} \sum_{c=1}^{d} \mathrm{tr}_{E_c} D_c^2 + F_{N,\mathcal{P}}(\{D_c\}) + 2 \sum_{c=1}^{d} \sum_{i_c < j_c} \ln |x_{i_c}^{(c)} - x_{j_c}^{(c)}|\Big). \tag{9.40}$$

If one looks for saddle-points such that the eigenvalues do not scale with N, then one sees that

- the quadratic terms scale like N^d,
- all terms from $F_{N,\mathcal{P}}(\{X_c\})$ scale like $N^{d'}$,
- all the terms from Vandermonde determinants scale like N^2.

This means that we can look for a solution without repulsion between eigenvalues, meaning the eigenvalues $x_i^{(c)}$ can simply fall onto their preferred value for all c. We moreover set $d' = d$ so the two first types of contributions have the same scale. Consider

$$X_c = \alpha_c \mathbb{1}_{E_c} + \frac{1}{N^{\frac{d-2}{2}}} M_c. \tag{9.41}$$

The α_cs are set on a saddle point of the action for $\{X_c\}$. Moreover, the scaling $1/N^{\frac{d-2}{2}}$ of the fluctuations is chosen so that the leading terms of the action in the X_cs scale like N^2, *i.e.* the same as the Vandermonde contributions.

9.4.2 Matrix model for the fluctuations

Before plugging (9.41) into (9.11), let us see its effect on a multi-trace interaction for a single color (which we do not write explicitly),

$$I_\lambda\Big(\alpha \mathbb{1} + \frac{1}{N^{\frac{d-2}{2}}} M\Big) = \prod_{i=1}^{\ell(\lambda)} \mathrm{tr}\Big(\alpha \mathbb{1} + \frac{1}{N^{\frac{d-2}{2}}} M\Big)^{\lambda_i} = \sum_{\mu_1, \dots, \mu_{\ell(\lambda)}=0}^{\lambda_1, \dots, \lambda_{\ell(\lambda)}} \prod_{i=1}^{\ell(\lambda)} \binom{\lambda_i}{\mu_i} \alpha^{\lambda_i - \mu_i} N^{-\frac{d-2}{2}\mu_i} \mathrm{tr} M^{\mu_i}. \tag{9.42}$$

It is easily rewritten as a sum over partitions $\mu \subset \lambda$. We denote the skew-partition $\lambda - \mu = (\lambda_1 - \mu_1, \dots, \lambda_{\ell(\lambda)} - \mu_{\ell(\lambda)})$ (we can complete μ with zeros if needed for $\mu_{i \geq \ell(\mu)}$) and $|\lambda - \mu| = \sum_{i\geq 1} \lambda_i - \mu_i$. Also use the notation $\binom{\lambda}{\mu} = \prod_{i=1}^{\ell(\mu)} \binom{\lambda_i}{\mu_i}$. Then

$$I_\lambda\Big(\alpha \mathbb{1} + \frac{1}{N^{\frac{d-2}{2}}} M\Big) = \sum_{\mu \subset \lambda} \binom{\lambda}{\mu} \alpha^{|\lambda-\mu|} \, N^{\ell(\lambda)-\ell(\mu)-\frac{d-2}{2}|\mu|} \, I_\mu(M). \tag{9.43}$$

We thus get the same expansion for $I_{\boldsymbol{\lambda}}(\{\alpha_c \mathbb{1}_{E_c} + M_c/N^{\frac{d-2}{2}}\})$ by taking a product over the colors. Overall,

$$F_{N,\mathcal{P}}\Big(\{\alpha_c \mathbb{1}_{E_c} + \frac{1}{N^{\frac{d-2}{2}}} M_c\}\Big) = \sum_{\boldsymbol{\mu}} N^{2-\ell(\boldsymbol{\mu})-\frac{d-2}{2}(|\boldsymbol{\mu}|-2)} \, s_{N,\mathcal{P}}(\boldsymbol{\mu}) \, I_{\boldsymbol{\mu}}(\{M_c\}), \tag{9.44}$$

where the sum runs over all d-tuples of partitions $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(d)})$ and $\ell(\boldsymbol{\mu}) = \sum_{c=1}^d \ell(\mu^{(c)})$ and $|\boldsymbol{\mu}| = \sum_{c=1}^d |\mu^{(c)}|$. The coefficients are

$$s_{N,\mathcal{P}}(\boldsymbol{\mu}) = \sum_{\boldsymbol{\lambda} \supset \boldsymbol{\mu}} t_{N,\mathcal{P}}(\boldsymbol{\lambda}) \prod_{c=1}^d \binom{\lambda^{(c)}}{\mu^{(c)}} \alpha_c^{|\lambda^{(c)}-\mu^{(c)}|}. \tag{9.45}$$

They all have $1/N$ expansions starting at order $\mathcal{O}(1)$, simply obtained by using the $1/N$ expansion of the coefficients $t_{N,\mathcal{P}}(\boldsymbol{\lambda}) = \sum_{i\geq 0} N^{-\delta_i(\boldsymbol{\lambda})} t_{\mathcal{P}}^{(i)}(\boldsymbol{\lambda})$. As a result, the matrix model from Theorem 9.1.1 becomes

$$\begin{aligned} &Z_{\text{MM}}(N,\mathcal{P}) = e^{\frac{1}{2}N^2 \sum_c \alpha_c^2 + F_{N,\mathcal{P}}(\{\alpha_c \mathbb{1}_{E_c}\})} \, Z_{\text{Fluct}}(N,\mathcal{P}), \\ \text{with} \qquad &Z_{\text{Fluct}}(N,\mathcal{P}) = \int \prod_{c=1}^d dM_c \, \exp\Big(\frac{1}{2} N \sum_c \operatorname{tr}_{E_c} M_c^2 + S_{N,\mathcal{P}}(\{M_c\})\Big), \end{aligned} \tag{9.46}$$

where $S_{N,\mathcal{P}}(\{M_c\})$ is

$$S_{N,\mathcal{P}}(\{M_c\}) = \sum_{\boldsymbol{\mu}, |\boldsymbol{\mu}|\geq 2} N^{2-\ell(\boldsymbol{\mu})-\frac{d-2}{2}(|\boldsymbol{\mu}|-2)} s_{N,\mathcal{P}}(\boldsymbol{\mu}) \, I_{\boldsymbol{\mu}}(\{M_c\}), \tag{9.47}$$

with

$$s_{N,\mathcal{P}}(\boldsymbol{\mu}) = \sum_{i\geq 0} N^{-\eta_i(\boldsymbol{\mu})} s_{\mathcal{P}}^{(i)}(\boldsymbol{\mu}), \tag{9.48}$$

and $(\eta_i(\boldsymbol{\mu}))_{i\geq 0}$ is an increasing sequence of non-negative rationals. The reason there is no linear term in (9.46) (and no $|\boldsymbol{\mu}| = 1$ term above) is that the set $\{\alpha_c\}$ is a saddle point.

Chapter 10

Blobbed topological recursion for colored, multi-trace matrix models

Going back to the discussion of Section 9.3, but replacing the original model $Z_{\text{MM}}(N, \mathcal{P})$ with the one for the fluctuations, $Z_{\text{Fluct}}(N, \mathcal{P})$ and action (9.47), we see that the latter differs from the multi-trace models of [67] by

- the fact that it has d matrices,
- the fact that the $1/N$ expansions of the coupling constants in (9.47) are not topological.

To remedy the first difference, consider the following model, which is a generalization of [67] to colored matrices. For a vector $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_d)$ of non-negative integers, denote $|\boldsymbol{\ell}| = \sum_{c=1}^{d} \ell_c$. Also denote

$$E_{\boldsymbol{\ell}} = \bigotimes_{c=1}^{d} E_c^{\otimes \ell_c}, \tag{10.1}$$

and $M_c^{(i)} = \mathbb{1} \otimes \cdots \otimes M_c \otimes \cdots \otimes \mathbb{1}$ the matrix acting on $E_c^{\otimes \ell_c}$ by M_c on the i^{th} factor and the identity everywhere else. Consider the model with partition function

$$Z_{\text{Top}}(N, S) = \int \prod_{c=1}^{d} dM_c \ \exp \sum_{\boldsymbol{\ell}} N^{2-|\boldsymbol{\ell}|} \mathrm{tr}_{E_{\boldsymbol{\ell}}} S_{N,\boldsymbol{\ell}}(M_1^{(1)}, \ldots, M_1^{(\ell_1)}; \ldots; M_d^{(1)}, \ldots, M_d^{(\ell_d)}). \tag{10.2}$$

When the action has the following expansion

$$\begin{aligned} & S_{N,\boldsymbol{\ell}}(M_1^{(1)}, \ldots, M_1^{(\ell_1)}; \ldots; M_d^{(1)}, \ldots, M_d^{(\ell_d)}) \\ &= \sum_{h \geq 0} N^{-2h} S_{\boldsymbol{\ell}}^{(h)}(M_1^{(1)}, \ldots, M_1^{(\ell_1)}; \ldots; M_d^{(1)}, \ldots, M_d^{(\ell_d)}), \end{aligned} \tag{10.3}$$

it is said to be topological (because it leads to an expansion in $1/N^2$ for the free energy, like the genus expansion of single-trace matrix models). When it has an expansion onto products of traces of the matrices M_cs, *i.e.*

$$\mathrm{tr}_{E_{\boldsymbol{\ell}}} S_{N,\boldsymbol{\ell}}(M_1^{(1)},\ldots,M_1^{(\ell_1)};\ldots;M_d^{(1)},\ldots,M_d^{(\ell_d)}) = \sum_{\substack{\boldsymbol{\lambda} \\ \ell(\lambda^{(c)})=\ell_c}} S_N(\boldsymbol{\lambda}) I_{\boldsymbol{\lambda}}(\{M_c\}), \tag{10.4}$$

and an invertible quadratic form, then the Feynman expansion corresponds to an expansion onto stuffed maps (non-necessarily topological), as described in Section 9.3. Finally, a topological expansion onto stuffed maps is obtained from

$$\mathrm{tr}_{E_{\boldsymbol{\ell}}} S_{N,\boldsymbol{\ell}}(M_1^{(1)},\ldots,M_1^{(\ell_1)};\ldots;M_d^{(1)},\ldots,M_d^{(\ell_d)}) = \sum_{\substack{h\geq 0 \\ \ell(\lambda^{(c)})=\ell_c}} N^{-2h} S^{(h)}(\boldsymbol{\lambda}) I_{\boldsymbol{\lambda}}(\{M_c\}). \tag{10.5}$$

We will present the loop equations of this model. However, we will calculate the disc and cylinder function only in the case where a special property of (9.47) is satisfied, namely

$$S^{(0)}(\boldsymbol{\lambda}) = \mathcal{O}(N^{\frac{d-2}{2}}), \qquad \text{for } |\boldsymbol{\lambda}| \geq 3, \tag{10.6}$$

which in fact will imply that the large N limit is that of a Gaussian model. The second key difference with [67] is that (9.47) is in general non-topological. Since the leading order of the coupling constants is nevertheless $N^{2-\ell(\boldsymbol{\lambda})}$, as in (10.2). Therefore, we can define new coupling constants as follows

$$N^{-2h} S^{(h)}_{N,\mathcal{P}}(\boldsymbol{\lambda}) = \sum_{\substack{i\geq 0 \\ 2h=\lfloor \frac{d-2}{2}(|\boldsymbol{\lambda}|-2)+\eta_i(\boldsymbol{\lambda})\rfloor}} N^{-\frac{d-2}{2}(|\boldsymbol{\lambda}|-2)-\eta_i(\boldsymbol{\lambda})}\, s^{(i)}_{\mathcal{P}}(\boldsymbol{\lambda}), \tag{10.7}$$

i.e. we absorb $s^{(i)}_{\mathcal{P}}(\boldsymbol{\lambda})$ and its N-dependent prefactor into the coefficients $S^{(h)}_{N,\mathcal{P}}(\boldsymbol{\lambda})$ by rounding down its order to the closest $2h$. Here we will follow the same route as in [66, 67] and do everything as if $S^{(h)}_{N,\mathcal{P}}(\boldsymbol{\lambda})$ were independent of N, except for the explicit evaluation of the disc function and the cylinder function using (10.6). In principle, one also has to eventually re-expand the coefficients $S^{(h)}_{N,\mathcal{P}}(\boldsymbol{\lambda})$ as in (10.7).

10.1 Correlation functions

If $P(\{M_c\})$ is an observable, its expectation is

$$\langle P(\{M_c\})\rangle = \frac{1}{Z_{\mathrm{Top}}(N,S)}\int \prod_{c=1}^{d} dM_c\ P(\{M_c\}) \\ \times \exp\sum_{\boldsymbol{\ell}} N^{2-|\boldsymbol{\ell}|}\mathrm{tr}_{E_{\boldsymbol{\ell}}} S_{N,\boldsymbol{\ell}}(M_1^{(1)},\ldots,M_1^{(\ell_1)};\ldots;M_d^{(1)},\ldots,M_d^{(\ell_d)}). \tag{10.8}$$

The natural set of observables are the expectations of products of traces of the matrices $\{M_c\}$, *i.e.* expectations of $I_{\boldsymbol{\lambda}}(\{M_c\})$. We recall here for the reader's sake the definitions of the correlation functions introduced in section 3.2

$$\overline{W}_n(x_1,c_1;\ldots;x_n,c_n) = \Big\langle \prod_{i=1}^{n} \mathrm{tr}_{E_{c_i}} \frac{1}{x_i - M_{c_i}}\Big\rangle = \sum_{k_1,\ldots,k_n\geq 0} \overline{W}_n^{(k_1,c_1;\ldots;k_n,c_n)} \prod_{i=1}^{n} x_i^{-k_i-1}, \tag{10.9}$$

i.e.

$$\overline{W}_n^{(k_1,c_1;\ldots;k_n,c_n)} = \Big[\prod_{i=1}^{n} x^{-k_i-1}\Big]\overline{W}_n(x_1,c_1;\ldots;x_n,c_n) = \Big\langle \prod_{i=1}^{n}\mathrm{tr}_{E_{c_i}} M_{c_i}^{k_i}\Big\rangle, \tag{10.10}$$

and their connected counterparts

$$W_n(x_1,c_1;\ldots;x_n,c_n) = \Big\langle \prod_{i=1}^{n} \mathrm{tr}_{V_{c_i}} \frac{1}{x_i - M_{c_i}}\Big\rangle_c = \sum_{k_1,\ldots,k_n\geq 0} W_n^{(k_1,c_1;\ldots;k_n,c_n)} \prod_{i=1}^{n} x_i^{-k_i-1}, \tag{10.11}$$

i.e.

$$W_n^{(k_1,c_1;\ldots;k_n,c_n)} = \Big[\prod_{i=1}^{n} x^{-k_i-1}\Big]W_n(x_1,c_1;\ldots;x_n,c_n) = \Big\langle \prod_{i=1}^{n}\mathrm{tr}_{E_{c_i}} M_{c_i}^{k_i}\Big\rangle_c. \tag{10.12}$$

The variable x_i is said to be of color c_i when it is the generating parameter for $\mathrm{tr}_{E_{c_i}} M_{c_i}^{k_i}$ expanded around infinity. We denote $\mathbb{C}_c$ the copy of $\mathbb{C}$ of color c, so that $x_i \in U_{c_i}$ for some open subset of $\mathbb{C}_{c_i}$. We will also need the functions

$$W_n^{(k_1,c'_1;\ldots;k_l,c'_l)}(x_1,c_1;\ldots;x_{n-l},c_{n-l}) = \Big\langle \prod_{i=1}^{l}\mathrm{tr}_{E_{c'_i}} M_{c'_i}^{k_i} \prod_{j=1}^{n-l}\mathrm{tr}_{E_{c_j}}\frac{1}{x_j - M_{c_j}}\Big\rangle_c, \tag{10.13}$$

which are obtained from $W_n(x_1, c_1; \dots; x_{n-l}, c_{n-l}; x'_1, c'_1; \dots; x'_l, c'_l)$ by extracting some series coefficients

$$W_n^{(k_1,c'_1;\dots;k_l,c'_l)}(x_1, c_1; \dots; x_{n-l}, c_{n-l}) = \Big[\prod_{i=1}^{l} {x'}_i^{-k_i-1}\Big] W_n(x_1, c_1; \dots; x_{n-l}, c_{n-l}; x'_1, c'_1; \dots; x'_l, c'_l). \tag{10.14}$$

As a special case of such functions, when $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(d)})$ is a d-tuple of partitions, and $c \in [\![1, d]\!]$ and $j \in [\![1, \ell(\lambda^{(c)})]\!]$, we denote $\boldsymbol{\lambda}_{(c,j)} = (\lambda'^{(1)}, \dots, \lambda'^{(d)})$ the d-tuple of partitions with

$$\lambda'^{(c')} = \lambda^{(c')} \quad \text{for } c' \neq c \quad \text{and} \quad \lambda'^{(c)} = \big(\lambda^{(c)}_1, \dots, \lambda^{(c)}_{j-1}, \lambda^{(c)}_{j+1}, \dots, \lambda^{(c)}_{\ell(\lambda^{(c)})}\big), \tag{10.15}$$

i.e. the j^{th} row of $\lambda^{(c)}$ is removed. Then we denote

$$W_n^{(\boldsymbol{\lambda}_{(c,j)})}(x_1, c_1; \dots; x_p, c_p) = W_n^{(\lambda_i^{c'}, c')^{(c',i)\neq(c,j)}_{(c',i)\in[\![1,d]\!]\times[\![1,\ell(\lambda^{(c')})]\!]}}(x_1, c_1; \dots; x_p, c_p), \tag{10.16}$$

with $n = p + \ell(\boldsymbol{\lambda}) - 1$. It will also appear natural to introduce *global* correlation functions which are defined on (an open subset of)

$$E_n = \Big(\bigcup_{c=1}^{d} \mathbb{C}_c \setminus \Gamma_c\Big)^n, \tag{10.17}$$

so that each x_i can be evaluated on any color. These correlation functions are

$$W_n(x_1, \dots, x_n) = \sum_{c_1,\dots,c_n \in [\![1,d]\!]} W_n(x_1, c_1; \dots; x_n, c_n) \prod_{i=1}^{n} \mathbb{1}(x_i, c_i), \tag{10.18}$$

where $\mathbb{1}(x, c)$ is 1 if $x \in \mathbb{C}_c$ and 0 otherwise. In terms of components

$$W_n(x_1, \dots, x_n) = \sum_{\substack{k_1,\dots,k_n \geq 0 \\ c_1,\dots,c_n \in [\![1,d]\!]}} W_n^{(k_1,c_1;\dots;k_n,c_n)} \prod_{i=1}^{n} x_i^{-k_i-1} \mathbb{1}(x_i, c_i). \tag{10.19}$$

The correlation functions $W_n(x_1, c_1; \dots; x_n, c_n)$ are said to be the *local expressions* of $W_n(x_1, \dots, x_n)$, since each variables is assigned a fixed color.

10.2 Loop equations

In this section we use the form (10.4) of the action.

10.2.1 1-point equation

The Schwinger-Dyson equations are obtained from

$$\frac{1}{Z_{\mathrm{Top}}(N,S)}\int\prod_{c=1}^{d}dM_c\sum_{a,b=1}^{N}\frac{\partial}{\partial(M_c)_{ab}}\Big(\big(M_c^n\big)_{ab}\,e^{\sum_{\boldsymbol{\lambda}}N^{2-\ell(\boldsymbol{\lambda})}S_N(\boldsymbol{\lambda})I_{\boldsymbol{\lambda}}(\{M_c\})}\Big)=0, \tag{10.20}$$

by making the action of the derivative above explicit on each term of the integrand, and summing over $n\geq 0$ with x^{-n-1}. One gets

$$\overline{W}_2(x,c;x,c)+\sum_{\boldsymbol{\lambda}}N^{2-\ell(\boldsymbol{\lambda})}S_N(\boldsymbol{\lambda})\sum_{j=1}^{\ell(\lambda^{(c)})}\lambda_j^{(c)}\Big\langle\mathrm{tr}_{E_c}\frac{M_c^{\lambda_j^{(c)}-1}}{x-M_c}\prod_{i\neq j}\mathrm{tr}_{E_c}M_c^{\lambda_i^{(c)}}\prod_{c'\neq c}I_{\lambda^{(c')}}(M_{c'})\Big\rangle=0. \tag{10.21}$$

We now work towards rewriting (10.21) in terms of connected correlation functions. Denote $R=\{R_1,\ldots,R_{\ell(R)}\}$ a set-partition of $[\![1,n]\!]$, then

$$\overline{W}_n(x_1,c_1;\ldots;x_n,c_n)=\sum_{R\vdash[\![1,n]\!]}\prod_{\alpha}W_{|R_\alpha|}(\{x_{R_\alpha},c_{R_\alpha}\}), \tag{10.22}$$

with the short-hand notation $\{x_{R_\alpha},c_{R_\alpha}\}=\{x_r,c_r\}_{r\in R_\alpha}$. This first gives

$$\overline{W}_2(x,c;x,c)=W_1(x,c)^2+W_2(x,c;x,c). \tag{10.23}$$

The contribution of the interaction is split in the usual way using

$$\frac{M_c^{\lambda_j^{(c)}-1}}{x-M_c}=\frac{x^{\lambda_j^{(c)}-1}}{x-M_c}+\sum_{q=0}^{\lambda_j^{(c)}-2}x^{\lambda_j^{(c)}-2-q}M_c^q, \tag{10.24}$$

which leads to

$$\begin{aligned}&\left\langle \mathrm{tr}_{E_c}\frac{M_c^{\lambda_j^{(c)}-1}}{x-M_c}\prod_{i\neq j}\mathrm{tr}_{E_c}M_c^{\lambda_i^{(c)}}\prod_{c'\neq c}I_{\lambda^{(c')}}(M_{c'})\right\rangle\\&=x^{\lambda_j^{(c)}-1}\overline{W}^{(\boldsymbol{\lambda}_{(c,j)})}_{\ell(\boldsymbol{\lambda})}(x,c)-\sum_{q=0}^{\lambda_j^{(c)}-2}x^{\lambda_j^{(c)}-2-q}\overline{W}^{(\boldsymbol{\lambda}_{(c,j)};q,c)}_{\ell(\boldsymbol{\lambda})}.\end{aligned} \tag{10.25}$$

Then rewrite each of the two terms using connected correlations. To do so, denote

$$L(\boldsymbol{\lambda})=\{(c,i)|(c,i)\in[\![1,d]\!]\times[\![1,\ell(\lambda^{(c)})]\!]\}, \tag{10.26}$$

and $\mathcal{P}(L(\boldsymbol{\lambda}))$ the set of partitions of $L(\boldsymbol{\lambda})$, *i.e.* $R=\{R_1,\ldots,R_{\ell(R)}\}\in\mathcal{P}(L(\boldsymbol{\lambda}))$ if the R_αs are non-empty, disjoint, and $\bigsqcup_\alpha R_\alpha=L(\boldsymbol{\lambda})$. Moreover, for a fixed pair $(c,j)\in L(\boldsymbol{\lambda})$, we denote $R(c,j)$ the part which contains (c,j), and

$$R(c,j)=\{(c,j)\}\cup R'(c,j), \tag{10.27}$$

where $R'(c,j)$ can be empty. Then,

$$\overline{W}^{(\boldsymbol{\lambda}_{(c,j)})}_{\ell(\boldsymbol{\lambda})}(x,c)=\sum_{R\in\mathcal{P}(L(\boldsymbol{\lambda}))}W^{(\lambda_{R'(c,j)},c_{R'(c,j)})}_{|R(c,j)|}(x,c)\prod_{\substack{\alpha\\R_\alpha\neq R(c,j)}}W^{(\lambda_{R_\alpha},c_{R_\alpha})}_{|R_\alpha|}. \tag{10.28}$$

Here we use the short-hand notation $(\lambda_{R_\alpha},c_{R_\alpha})=(\lambda_i^{(c')},c')_{(c',i)\in R_\alpha}$. Furthermore

$$\overline{W}^{(\boldsymbol{\lambda}_{(c,j)};q,c)}_{\ell(\boldsymbol{\lambda})}=\sum_{R\in\mathcal{P}(L(\boldsymbol{\lambda}))}W^{(\lambda_{R'(c,j)},c_{R'(c,j)};q,c)}_{|R(c,j)|}\prod_{\substack{\alpha\\R_\alpha\neq R(c,j)}}W^{(\lambda_{R_\alpha},c_{R_\alpha})}_{|R_\alpha|}. \tag{10.29}$$

The loop equation (10.21) then reads

$$\begin{aligned}&W_1(x,c)^2+W_2(x,c;x,c)+\sum_{\substack{\boldsymbol{\lambda}\\R\in\mathcal{P}(L(\boldsymbol{\lambda}))}}\sum_{j=1}^{\ell(\lambda^{(c)})}N^{2-\ell(\boldsymbol{\lambda})}S_N(\boldsymbol{\lambda})\prod_{\substack{\alpha\\R_\alpha\neq R(c,j)}}W^{(\lambda_{R_\alpha},c_{R_\alpha})}_{|R_\alpha|}\\&\times\lambda_j^{(c)}\Big(x^{\lambda_j^{(c)}-1}W^{(\lambda_{R'(c,j)},c_{R'(c,j)})}_{|R(c,j)|}(x,c)-\sum_{q=0}^{\lambda_j^{(c)}-2}x^{\lambda_j^{(c)}-2-q}W^{(\lambda_{R'(c,j)},c_{R'(c,j)};q,c)}_{|R(c,j)|}\Big)=0,\end{aligned} \tag{10.30}$$

which can also be written as a global equation

$$W_1(x)^2 + W_2(x,x) + \sum_{\substack{\boldsymbol{\lambda} \\ R \in \mathcal{P}(L(\boldsymbol{\lambda}))}} \sum_{c=1}^{d} \mathbb{1}(x,c) \sum_{j=1}^{\ell(\lambda^{(c)})} N^{2-\ell(\boldsymbol{\lambda})} S_N(\boldsymbol{\lambda}) \prod_{\substack{\alpha \\ R_\alpha \neq R(c,j)}} W_{|R_\alpha|}^{(\lambda_{R_\alpha}, c_{R_\alpha})}$$
$$\times \lambda_j^{(c)} \Big(x^{\lambda_j^{(c)}-1} W_{|R(c,j)|}^{(\lambda_{R'(c,j)}, c_{R'(c,j)})}(x,c) - \sum_{q=0}^{\lambda_j^{(c)}-2} x^{\lambda_j^{(c)}-2-q} W_{|R(c,j)|}^{(\lambda_{R'(c,j)}, c_{R'(c,j)}; q, c)} \Big) = 0. \quad (10.31)$$

10.2.2 n-point equations

The single-trace terms of the potential are

$$\sum_{c=1}^{d} \sum_{\lambda \geq 1} N \big(s^{(0)}(\lambda, c) + o(1) \big) \mathrm{tr}_{E_c} M_c^{\lambda}. \quad (10.32)$$

Then the loop insertion operator with respect to color $c \in [\![1, d]\!]$ is

$$\delta_x = \sum_{c=1}^{d} \sum_{\lambda \geq 0} \mathbb{1}(x,c) x^{-\lambda-1} \frac{\partial}{\partial s^{(0)}(\lambda, c)}. \quad (10.33)$$

With the notations of paragraph 3.3.2, repeated actions of the loop insertion operator on (10.31) gives

$$\sum_{(I_1,I_2) \in \mathcal{I}_2(n)} W_{|I_1|+1}(x_1, x_{I_1}) W_{|I_2|+1}(x_1, x_{I_2}) + W_{n+1}(x_1, x_1, \ldots, x_n)$$
$$+ \sum_{j=2}^{n} \mathbb{1}(x_1, x_j) \frac{\partial}{\partial x_j} \frac{W_{n-1}(x_2, \ldots, x_n) - W_{n-1}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)}{x_j - x_1}$$
$$+ \sum_{\substack{\boldsymbol{\lambda} \\ R \in \mathcal{P}(L(\boldsymbol{\lambda}))}} \sum_{(I_1, \ldots, I_{\ell(R)}) \in \mathcal{I}_{\ell(R)}(n)} \sum_{c=1}^{d} \mathbb{1}(x_1, c) \sum_{j=1}^{\ell(\lambda^{(c)})} N^{2-\ell(\boldsymbol{\lambda})} S_N(\boldsymbol{\lambda}) \prod_{\substack{\alpha \\ R_\alpha \neq R(c,j)}} W_{|R_\alpha|+|I_\alpha|}^{(\lambda_{R_\alpha}, c_{R_\alpha})}(x_{I_\alpha})$$
$$\times \lambda_j^{(c)} \Big(x_1^{\lambda_j^{(c)}-1} W_{|R(c,j)|+|I(c,j)|}^{(\lambda_{R'(c,j)}, c_{R'(c,j)})}(x_1, x_{I(c,j)}) - \sum_{q=0}^{\lambda_j^{(c)}-2} x_1^{\lambda_j^{(c)}-2-q} W_{|R(c,j)|+|I(c,j)|}^{(\lambda_{R'(c,j)}, c_{R'(c,j)}; q, c)}(x_{I(c,j)}) \Big) = 0.$$
$$(10.34)$$

$I(c,j)$ is defined as I_{α^*} where α^* is the index such that $R_{\alpha^*} = R(c,j)$. Moreover, $\mathbb{1}(x,y) = \sum_{c=1}^{d} \mathbb{1}(x,c)\mathbb{1}(y,c)$ is 1 if and only if x and y are variables of the same color.

10.2.3 Topological expansion

All correlation functions admit the usual topological expansion

$$W_{n+p}^{(k_1,c_1;\dots;k_p,c_p)}(x_1,\dots,x_n) = \sum_{h\geq 0} N^{2-n+p-2h}\, W_{n+p,h}^{(k_1,c_1;\dots,k_p,c_p)}(x_1,\dots,x_n). \tag{10.35}$$

Plugging it into (10.34) leads to

$$\begin{aligned}
&\sum_{\substack{(I_1,I_2)\in\mathcal{I}_2(n)\\ h=0,\dots,g}} W_{|I_1|+1,h}(x_1,x_{I_1})W_{|I_2|+1,g-h}(x_1,x_{I_2}) + W_{n+1,g-1}(x_1,x_1,\dots,x_n)\\
&\qquad + \sum_{j=2}^{n} \mathbb{1}(x_1,x_j)\frac{\partial}{\partial x_j}\frac{W_{n-1,g}(x_2,\dots,x_n)-W_{n-1,g}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)}{x_j-x_1}\\
&+ \sum_{\substack{\boldsymbol{\lambda}\\ R\in\mathcal{P}(L(\boldsymbol{\lambda}))\\ (I_1,\dots,I_{\ell(R)})\in\mathcal{I}_{\ell(R)}(n)}} \sum_{\substack{h,h_1,\dots,h_{\ell(R)}\geq 0\\ \ell(\boldsymbol{\lambda})-\ell(R)+h+\sum_{\alpha=1}^{\ell(R)} h_\alpha=g}} \sum_{c=1}^{d}\mathbb{1}(x_1,c)\sum_{j=1}^{\ell(\lambda^{(c)})} S^{(h)}(\boldsymbol{\lambda}) \prod_{\substack{\alpha\\ R_\alpha\neq R(c,j)}} W_{|R_\alpha|+|I_\alpha|,h_\alpha}^{(\lambda_{R_\alpha},c_{R_\alpha})}(x_{I_\alpha})\\
&\times\lambda_j^{(c)}\Big(x_1^{\lambda_j^{(c)}-1}W_{|R(c,j)|+|I(c,j)|,h(c,j)}^{(\lambda_{R'(c,j)},c_{R'(c,j)})}(x_1,x_{I(c,j)}) - \sum_{q=0}^{\lambda_j^{(c)}-2} x_1^{\lambda_j^{(c)}-2-q} W_{|R(c,j)|+|I(c,j)|,h(c,j)}^{(\lambda_{R'(c,j)},c_{R'(c,j)};q,c)}(x_{I(c,j)})\Big) = 0,
\end{aligned} \tag{10.36}$$

where $h(c,j)$ is the h_{α^*} where α^* is such that $R_{\alpha^*} = R(c,j)$.

10.3 Large N limit

Restricting (10.36) to $g=0$ gives the constraint $h = h_1 = \cdots = h_{\ell(R)} = 0$, which in turn implies $\ell(R) = \ell(\boldsymbol{\lambda})$. This reduces the sum over partitions of $L(\boldsymbol{\lambda})$ to a single one for each $\boldsymbol{\lambda}$, *i.e.* the partition into singletons, $R = \{\{c',i\}\}$, for $c' \in [\![1,d]\!]$ and $i \in [\![1,\ell(\lambda^{(c')})]\!]$. For $(I_1,\dots,I_{\ell(\boldsymbol{\lambda})}) \in \mathcal{I}_{\ell(\boldsymbol{\lambda})}(n)$, we denote $I(c',i) = I_\alpha$ when $R_\alpha = \{c',i\}$. This gives, for a generic potential,

$$\begin{aligned}
&\sum_{(I_1,I_2)\in\mathcal{I}_2(n)} W_{|I_1|+1,0}(x_1,x_{I_1})W_{|I_2|+1,0}(x_1,x_{I_2})\\
&\qquad + \sum_{j=2}^{n} \mathbb{1}(x_1,x_j)\frac{\partial}{\partial x_j}\frac{W_{n-1,0}(x_2,\dots,x_n)-W_{n-1,0}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)}{x_j-x_1}
\end{aligned}$$

$$+ \sum_{\substack{\boldsymbol{\lambda} \\ (I_1,\dots,I_{\ell(\boldsymbol{\lambda})})\in\mathcal{I}_{\ell(\boldsymbol{\lambda})}(n)}} \sum_{c=1}^{d} \mathbb{1}(x_1,c) \sum_{j=1}^{\ell(\lambda^{(c)})} S^{(0)}(\boldsymbol{\lambda}) \prod_{(c',i)\neq(c,j)} W^{(\lambda_i^{(c')},c')}_{|I(c',i)|+1,0}(x_{I(c',i)})$$

$$\times \lambda_j^{(c)} \Big(x_1^{\lambda_j^{(c)}-1} W_{|I(c,j)|+1,0}(x_1, x_{I(c,j)}) - \sum_{q=0}^{\lambda_j^{(c)}-2} x_1^{\lambda_j^{(c)}-2-q} W^{(q,c)}_{|I(c,j)|+1,0}(x_{I(c,j)}) \Big) = 0. \quad (10.37)$$

In the case (9.46) of $Z_{\mathrm{Fluct}}(N,\mathcal{P})$, which we are interested in, the leading order coefficients $S^{(0)}(\boldsymbol{\lambda})$ actually have an extra N-dependence and satisfy (10.6). It implies that the partitions $\boldsymbol{\lambda}$ appearing in (10.37) must be of size 2, *viz.* $|\boldsymbol{\lambda}| = 2$. These partitions are of the form $\boldsymbol{\lambda} = (\lambda^{(1)},\dots,\lambda^{(d)})$ with

- either $\lambda^{(c)} = (2)$ for some c. We write the corresponding part of the action $\frac{1}{2}\sum_{c=1}^{d} a_c \mathrm{tr}_{E_c} M_c^2$;
- or $\lambda^{(c)} = (1,1)$ for some c. We write the corresponding part of the action $\frac{1}{2}\sum_{c=1}^{d} b_{cc} \big(\mathrm{tr}_{E_c} M_c\big)^2$;
- or $\lambda^{(c)} = (1)$ and $\lambda^{(c')} = (1)$ for some $c \neq c'$. We write the corresponding part of the action $\frac{1}{2}\sum_{c\neq c'}^{d} b_{cc'} \mathrm{tr}_{E_c} M_c \, \mathrm{tr}_{E_{c'}} M_{c'}$, with $b_{cc'} = b_{c'c}$.

This means that at large N, the correlation functions behave as if the action simply was

$$-\frac{N}{2}\sum_c \mathrm{tr}_{E_c} M_c^2 - \sum_{|\boldsymbol{\lambda}|=2} N^{2-\ell(\boldsymbol{\lambda})} s_{\mathcal{P}}^{(0)}(\boldsymbol{\lambda})\, I_{\boldsymbol{\lambda}}(\{M_c\}) = \frac{N}{2}\sum_c a_c \mathrm{tr}_{E_c} M_c^2 + \frac{1}{2}\sum_{c,c'} b_{c,c'} \mathrm{tr}_{E_c} M_c \, \mathrm{tr}_{E_{c'}} M_{c'}. \quad (10.38)$$

Then (10.37) becomes

$$\sum_{(I_1,I_2)\in\mathcal{I}_2(n)} W_{|I_1|+1,0}(x_1,x_{I_1}) W_{|I_2|+1,0}(x_1,x_{I_2})$$
$$+ \sum_{j=2}^{n} \mathbb{1}(x_1,x_j) \frac{\partial}{\partial x_j} \frac{W_{n-1,0}(x_2,\dots,x_n) - W_{n-1,0}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)}{x_j - x_1}$$
$$- \sum_{c=1}^{d} \mathbb{1}(x_1,c) \Big(a_c x_1 W_n(x_1,\dots,x_n) + \sum_{(I_1,I_2)\in\mathcal{I}_2(n)} W_{|I_1|+1,0}(x_1,x_{I_1}) \sum_{c'=1}^{d} b_{cc'} W^{(1,c')}_{|I_2|+1,0}(x_{I_2}) \Big) = 0. \quad (10.39)$$

10.3.1 Disc function

The disc function of color c is $W_{1,0}(x,c) = \lim_{N\to\infty} \frac{1}{N} W_1(x,c)$, *i.e.* the generating function of planar stuffed maps with a single boundary of arbitrary perimeter. The global disc function is $W_{1,0}(x) = \sum_{c=1}^{d} \mathbb{1}(x,c) W_{1,0}(x,c)$. Equation (10.39) can be directly applied. Before doing so, however, let us briefly mention the case of a generic potential, by setting $n = 1$ in (10.37). For a fixed color c,

$$W_{1,0}(x,c)^2 + \sum_{\boldsymbol{\lambda}} \sum_{j=1}^{\ell(\lambda^{(c)})} S^{(0)}(\boldsymbol{\lambda}) \prod_{(c',i)\neq(c,j)} W_{1,0}^{(\lambda_i^{(c')},c')} \lambda_j^{(c)} \Big(x^{\lambda_j^{(c)}-1} W_{1,0}(x,c) - \sum_{q=0}^{\lambda_j^{(c)}-2} x^{\lambda_j^{(c)}-2-q} W_{1,0}^{(q,c)} \Big) = 0. \tag{10.40}$$

This is thus a set of d equations on d functions $W_{1,0}(x,c)$ with a single catalytic variable, and some explicit dependence on coefficients of the unknown series. This generalizes the classical equation of the 1-matrix, multi-trace model [67]. The analytic properties of its disc function, described in [69], derive from an extension of [71], which applies to stuffed maps with unbounded face degrees. Here, we would require a further extension, to a system of equations. Instead of pursuing the generic route, we focus on the specific model $Z_{\text{Fluct}}(N,\mathcal{P})$. Setting $n = 1$ in (10.39), one finds

$$W_{1,0}(x,c)^2 - a_c\big(xW_{1,0}(x,c) - 1\big) = 0, \tag{10.41}$$

where we have used $W_{1,0}^{(1,c')} = 0$ for all c', since $W_{1,0}^{(1,c')} = \lim_{N\to\infty} \frac{1}{N}\langle \mathrm{tr}_{E_{c'}} M_{c'}\rangle$ and the model is invariant under $\{M_c\} \to \{-M_c\}$ at large N. The disc function of color c is thus

$$W_{1,0}(x,c) = \frac{a_c}{2}\left(x - \sqrt{x^2 - \frac{4}{a_c}} \right), \tag{10.42}$$

as in [66]. It has a cut along $\Gamma_c = [-\frac{2}{\sqrt{a_c}}, \frac{2}{\sqrt{a_c}}]$ (if $a_c > 0$), which is said to be the cut of color c. The global disc function $W_{1,0}(x) = \sum_{c=1}^{d} W_{1,0}(x,c)\mathbb{1}(x,c)$ thus has d cuts, along $\bigcup_{c=1}^{d} \Gamma_c$.

10.3.2 Cylinder function

The cylinder function is the leading order of the two-point function. The local expression is

$$W_{2,0}(x_1,c_1;x_2,c_2) = \lim_{N\to\infty} W_2(x_1,c_1;x_2,c_2) = \lim_{N\to\infty} \langle \mathrm{tr}_{E_{c_1}} \frac{1}{x_1 - M_{c_1}} \mathrm{tr}_{E_{c_2}} \frac{1}{x_2 - M_{c_2}} \rangle_c. \tag{10.43}$$

By setting $n = 2$ in (10.39), one finds

$$\big(2W_{1,0}(x_1,c_1) - a_{c_1}x_1\big) W_{2,0}(x_1,c_1;x_2,c_2) + \delta_{c_1,c_2} \frac{\partial}{\partial x_2} \frac{W_{1,0}(x_1,c_1) - W_{1,0}(x_2,c_2)}{x_1 - x_2} - W_{1,0}(x_1,c_1) \sum_{c'=1}^{d} b_{c_1c'} W_{2,0}^{(1,c')}(x_2,c_2) = 0, \tag{10.44}$$

where a term $-W_{2,0}(x_1,c_1;x_2,c_2)\sum_{c'=1}^{d} b_{c_1c'}W_{1,0}^{(1,c')}$ has been removed because $W_{1,0}^{(1,c')} = 0$ in this model. It generalizes the equations found for $W_{2,0}(x_1,c_1;x_2,c_2)$ in [66]. The method used to solve them still works here. It processes by first finding the values of $W_{2,0}^{(1,c')}(x_2,c_2)$. To do so, we extract the coefficient of the equations at order $1/x_1$. It gives

$$-a_{c_1}W_{2,0}^{(1,c_1)}(x_2,c_2) - \sum_{c'=1}^{d} b_{c_1,c'} W_{2,0}^{(1,c')}(x_2,c_2) = \delta_{c_1,c_2} \frac{\partial W_{1,0}(x_2,c_2)}{\partial x_2}, \tag{10.45}$$

which can be given a matrix form. Introduce the following $d\times d$ matrices $A = \mathrm{diag}(a_1,\ldots,a_d)$, $B = (b_{c,c'})_{1\le c,c'\le d}$ and

$$W_{2,0}^{(1)}(x) = \big(W_{2,0}^{(1,c)}(x,c')\big)_{1\le c,c'\le d}, \qquad \partial W_{1,0}(x) = \mathrm{diag}\Big(\frac{\partial W_{1,0}(x,c)}{\partial x},\ldots,\frac{\partial W_{1,0}(x,c)}{\partial x}\Big). \tag{10.46}$$

It comes

$$W_{2,0}^{(1)}(x) = -(A+B)^{-1}\partial W_{1,0}(x), \tag{10.47}$$

provided $A+B$ is invertible. Denoting $\sigma(x,c) = \sqrt{x^2 - 4/a_c}$, we have

$$\frac{\partial W_{1,0}(x,c)}{\partial x} = -\frac{W_{1,0}(x,c)}{\sigma(x,c)}, \tag{10.48}$$

and it comes

$$W_{2,0}(x_1, c_1; x_2, c_2) = \delta_{c_1 c_2} \frac{x_1 x_2 - \sigma(x_1, c_1)\sigma(x_2, c_2) - 4/a_{c_1}}{2(x_1 - x_2)^2 \sigma(x_1, c_1)\sigma(x_2, c_2)} - \frac{1}{a_{c_1}} \Big(B \frac{1}{A + B}\Big)_{c_1 c_2} \frac{W_{1,0}(x_1, c_1) W_{1,0}(x_2, c_2)}{\sigma(x_1, c_1)\sigma(x_2, c_2)}. \tag{10.49}$$

The first part of this formula is the cylinder function for the GUE, while the other term is due to the multi-trace interaction. Notice that the latter is not manifestly invariant under the exchange symmetry $(x_1, c_1) \leftrightarrow (x_2, c_2)$. The cylinder function is thus basically the same as in the quartic melonic model [66]. The difference is that in that case, the matrices A, B were specific functions of the coupling constants $t_1, \dots, t_d$, while they are here, in general, functions of all the coupling constants t_i, $i \in I$.

10.4 Blobbed topological recursion

The remaining is exactly similar to [66], with the replacement $a_c = 1 - \alpha_c^2$, which already followed the theorems of [67, 68].

10.4.1 Spectral curve

Denote the Riemann sphere $\hat{\mathbb{C}}$, and $\hat{\mathbb{C}}_c$ its copy of color $c \in [\![1, d]\!]$. For each color, define

$$f_c(x, y) = y^2 - a_c x y + a_c. \tag{10.50}$$

The Gaussian spectral curve is defined as $\mathcal{C}_{\text{Gaussian}} \subset \bigcup_{c=1}^{d} \hat{\mathbb{C}}_c^2$ by

$$f(x, y) = \sum_{c=1}^{d} \mathbb{1}(x, c)\mathbb{1}(y, c) f_c(x, y) = 0. \tag{10.51}$$

Recall that $\Gamma_c = [-\frac{2}{\sqrt{a_c}}, \frac{2}{\sqrt{a_c}}]$ (for $a_c > 0$) and denote $\Gamma = \bigcup_{c=1}^{d} \Gamma_c$. It can be checked as in [67] that our correlation functions $W_{n,g}(x_1, \dots, x_n)$ are only singular along Γ (with respect to each variable), except for $(n, g) \in \{(1, 0), (2, 0)\}$. We therefore introduce a Zhukovski parametrization

$$x(z) = \sum_{c=1}^{d} \mathbb{1}(x, c)\mathbb{1}(z, c) \frac{1}{\sqrt{a_c}} (z + z^{-1}), \tag{10.52}$$

for $|z| \geq 1$ in each color. As shown in [67], the correlation functions for $(n,g) \notin \{(1,0),(2,0)\}$ are holomorphic for $|z| \geq 1$ except at $z = \pm 1$. Moreover, they can be analytically continued to the interior of a neighborhood of the unit circle, except at $z = \pm 1$. In our case (see below), this analytic continuation can be performed for all $0 < |z| < 1$. The correlation functions can thus be turned into differential forms

$$\begin{aligned}\omega_{n,g}(z_1,\dots,z_n) = W_{n,g}(x(z_1),\dots,x(z_n))dx(z_1)\dots dx(z_n) \\ + \delta_{(n,g),(2,0)} \sum_{c=1}^{d} \mathbb{1}(z_1,c)\mathbb{1}(z_2,c)\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}.\end{aligned} \tag{10.53}$$

For $(n,g) \notin \{(1,0),(2,0)\}$, they are holomorphic on $\mathcal{C}^n_{\text{Gaussian}}$ except at $z_i = 0, \pm 1$. In the framework of the blobbed topological recursion, the singularities at 0 and at ± 1 play different roles. This is because the disc function has (simple) zeroes at $z = \pm 1$,

$$W_{1,0}(x(z)) = \sum_{c=1}^{d} \mathbb{1}(z,c)\frac{\sqrt{a_c}}{z} \quad \Rightarrow \quad \omega_{1,0}(z) = \sum_{c=1}^{d} \mathbb{1}(z,c)\frac{1-z^{-2}}{z}dz, \tag{10.54}$$

because they are the zeroes of $dx(z)$. As for the cylinder function, it becomes

$$\begin{aligned}\omega_{2,0}(z_1,z_2) =& \frac{dz_1\,dz_2}{(z_1-z_2)^2}\sum_{c=1}^{d}\mathbb{1}(z_1,c)\mathbb{1}(z_2,c) \\ &- \frac{dz_1dz_2}{z_1^2z_2^2}\sum_{c_1,c_2=1}^{d}\frac{\mathbb{1}(z_1,c_1)\mathbb{1}(z_2,c_2)}{\sqrt{a_{c_1}a_{c_2}}}\frac{1}{a_{c_1}}\Big(B\frac{1}{A+B}\Big)_{c_1c_2},\end{aligned} \tag{10.55}$$

which is singular along the diagonal $z_1 = z_2$ as expected, but also has poles at $z_1 = 0$ and $z_2 = 0$ on each color. The points $z = \pm 1$ in each color are called the ramification points and we denote

$$\mathcal{R} = \{z = \pm 1 \in \mathbb{C}_c, c \in [\![1,d]\!]\}. \tag{10.56}$$

The spectral curve is also supplemented with the canonical involution $\iota(z) = 1/z$ which preserves the ramification points.

10.4.2 Topological recursion formula

Recall the notations $G(z,z_1) = \int_{\iota(z)}^{z}\omega_{2,0}(\cdot,z_1)$, and $\Delta\varphi = \varphi - \iota^*\varphi$ for a differential form φ. The kernel of the topological recursion is $K(z,z_1) = \frac{\Delta G(z,z_1)}{2\Delta\omega_{1,0}(z)}$. We further define the polar and holomorphic part of $\omega_{n,g}(z_1,\dots,z_n)$ as follows (terminology justified below)

$$\begin{aligned} P\omega_{n,g}(z_1,\dots,z_n) &= \sum_{z\in\mathcal{R}} \mathrm{Res}_z\, G(z,z_1)\omega_{n,g}(z,\dots,z_n), \\ H\omega_{n,g}(z_1,\dots,z_n) &= \omega_{n,g}(z_1,\dots,z_n) - P\omega_{n,g}(z_1,\dots,z_n). \end{aligned} \tag{10.57}$$

Theorem 10.4.1. *Assume that $A+B$ is invertible, so that $\omega_{2,0}$ is given by (10.55). For all $(n,g)\notin\{(1,0),(2,0)\}$, the holomorphic part is holomorphic while the polar part has poles on $\mathcal{R}$. They are given by*

$$\begin{aligned} P\omega_{n,g}(z_1,\dots,z_n) = &\sum_{z\in\mathcal{R}} \mathrm{Res}_z\, K(z,z_1)\Big(\omega_{n+1,g-1}(z,\iota(z),z_2,\dots,z_n) \\ &+ \sideset{}{'}\sum_{\substack{(I_1,I_2)\in\mathcal{I}_2(n)\\ g_1+g_2=g}} \omega_{|I_1|+1,g_1}(z,z_{I_1})\omega_{|I_2|+1,g_2}(\iota(z),z_{I_2})\Big), \end{aligned} \tag{10.58}$$

and

$$H\omega_{n,g}(z_1,\dots,z_n) = \frac{1}{2i\pi}\oint_{z\in\bigcup_{c=1}^d U(1)_c} \omega_{2,0}(z_1,z)\nu_{n,g}(z,z_2,\dots,z_n), \tag{10.59}$$

where $U(1)_c$ is the copy of the unit circle of color c, and

$$\nu_{n,g}(z,z_2,\dots,z_n) = V_{n,g}(x(z),x(z_2),\dots,x(z_n))\ dx(z_2)\dots dx(z_n) \tag{10.60}$$

for

$$\begin{aligned} V_{n,g}(x,x_2,\dots,x_n) = &\sum_{c=1}^{d}\sum_{\boldsymbol{\lambda}}\sum_{j=1}^{\ell(\lambda^{(c)})}\sum_{\substack{R\in\mathcal{P}(L(\boldsymbol{\lambda}))\\ R'(c,j)=\emptyset}}\sum_{(I_1,\dots,I_{\ell(R)})\in\mathcal{I}_{\ell(R)}(n)} \mathbb{1}(x,c) \\ &\sum_{\substack{h,h_1,\dots,h_{\ell(R)}\geq 0\\ \ell(\boldsymbol{\lambda})-\ell(R)+h+\sum_{\alpha=1}^{\ell(R)}h_\alpha=g}} S^{(h)}(\boldsymbol{\lambda})x^{\lambda_j^{(c)}}\prod_{R_\alpha\neq\{(c,j)\}} W^{(\lambda_{R_\alpha},c_{R_\alpha})}_{|R_\alpha|+|I_\alpha|,h_\alpha}(x_{I_\alpha}). \end{aligned} \tag{10.61}$$

Proof. The loop equations have the same form as in [66] and all the arguments, which were borrowed from [67, 68] apply. ∎

Summary and outlook

First of all, this work paves the way for a new formulation of real tensor models, their observables and correlators in terms of symmetric groups and their representation theory. The formulation is particularly convenient for implementing heavy computations using software resources, thus, leading to a gain of confidence in the computational process. Furthermore, with its multiple facets, the formalism elaborated here may shed a new light on some results since it bridges theories - combinatorics, TFT and physics through observables and correlators - which from the outset may look rather different.
We have enumerated $O(N)$ or rank-d real tensor invariants as d-regular colored graphs using a permutation group formalism. These invariants define the points of a double coset of S_{2n}^d. We used Mathematica and Sage codes to generate the sequences associated with the number of these invariants from their generating functions. The sequences obtained at $d \geq 4$ are new according to the OEIS. Translated into the TFT_2 formulation, the same counting delivers the number of covers of gluing of cylinders with defects. Such covers have been also observed while counting Feynman graphs of scalar field theory [39] and relate to a string theory on cylinders. Thus, there should be an equivalent way of describing tensor observables in purely string theoretic language. Moreover, this link with covers must be made precise: covers in 2D are related to holomorphic maps and may, in return, give a geometry to the space of orthogonal invariants. This point fully deserves further investigation.
Another piece of information reveals itself with the representation theoretic formulation of the counting: the number of orthogonal invariants is a sum of constrained Kronecker coefficients. The Kronecker coefficient is a core object in Computational Complexity theory: either finding a combinatorial rule describing it (finding which combinatorial objects it counts), or its vanishing property or otherwise remains under active investigation (see references in [51, 52]). It concentrates a lot of research efforts since one expects that, roughly speaking, understanding that object could lead to a separation of complexity classes P vs NP. In our present work (as is similarly done in [30]), we show that the

number of tensor model observables - represented by colored graphs and thus combinatorial structures - links to a sum of Kronecker coefficients (in [30], it is a sum of square of these coefficients). It remains of course to be seen how this would help with one of the famous problems stated above. Perhaps a refined counting of colored graphs (endowed with specific properties) could boil down the sum to a single Kronecker element. Such a study may bring some progress in the field.

The equivalence classes associated with the colored graphs are mapped in the tensor product of the group algebra $\mathbb{C}[S_{2n}]^{\otimes d}$. They form the basis vectors of a subspace, namely $\mathcal{K}_d(2n)$, that is in fact a semi-simple algebra. We call it a double coset algebra. Note also that, as elements of an algebra, d-regular colored graphs multiply in a specific way, and yield back a combination of d-regular colored graphs. In rank 3, we have found a "natural" representation theoretic basis $\{Q^{R,S,T,\tau}\}$, of $\mathcal{K}_3(2n)$, *i.e.* invariant and orthonormal. Unlike the unitary case [30], this basis decomposes the algebra into blocks but does not provide its Wedderburn-Artin (WA) decomposition in matrix subalgebras. This raises other questions: in which basis can the WA decomposition be made explicit? Is there a simple enough combination starting from $Q^{R,S,T,\tau}$ that produces the WA decomposition? A starting point of that analysis might be given by the work of Bremner [97] that constructs the WA basis of a finite dimensional unital algebra over rationals. Finally, is there a way to understand why the sum of constrained Kronecker coefficients is actually a sum of squares (each of which being the dimension of a matrix subalgebra entering in the WA decomposition)? Such points deserve future clarifications.

We also addressed normal ordered Gaussian 2-point correlators in this work and showed that they formulate completely as a function of the size N of the tensor indices and permutation cycles. We generated an orthogonal representation basis from these correlators. This result is similar to what is observed in the unitary case, with the following distinction: there is an operator acting on the triple defining the observables. We showed that computing Gaussian correlators in representation theory space actually translates to computing an inner product. Finally, we briefly sketched the main feature of $Sp(2N)$ invariants: although they obey the same diagrammatics than the $O(N)$ invariants, they satisfy a different rule concerning their equivalence classes. Thus, for the symplectic group and its invariants, the story could be radically different from the orthogonal case and will require more work.

On a different note, we have shown that, as long as there are quartic melonic interactions, one can find in arbitrary tensor models a set of correlation functions which satisfy the blobbed topological recursion in a universal way. The spectral curve is then a disjoint union of Gaussian spectral curves, with an additional holomorphic part to the cylinder function which always has the same form.

Those results rely on the conditions 1 and 2 presented in the introduction and detailed throughout Part III. In particular, the specifics of the model, *i.e.* the choice of interactions, do not matter as long as the effective action obtained after the formal integration of all the matrices except $Y_1, \ldots, Y_d$ has a well-defined $1/N$ expansion as we have described in Theorem 9.1.1. This is why our formulae all have the same structure as in the case of the quartic melonic model in [66].

We have also provided theorems 8.3.3, 8.3.4 and 9.2.1 to relate the expectations of the $U(N)^d$-invariant observables on the tensor and matrix sides.

There are still many interesting questions about the topological recursion for tensor models. Are there other sets of correlation functions satisfying the topological recursion? Is it always a blobbed recursion? Can condition 2 from the introduction be removed? Is it possible to proceed without going to matrix models and derive topological recursions directly in the tensor formulation? There have been some efforts to use directly the Schwinger-Dyson equations of tensor models, in [65] and [102], thus extracting the double scaling limit of tensor models with melonic interactions for instance, but this is still far from the topological recursion. We hope some of those questions can be tackled in the near future.

Appendix A

The symmetric group and its representation theory

This appendix gathers useful identities and notations about the symmetric group S_n and its representation theory. The presentation here is a summary of Appendix A, withdrawn from [30], and the textbook by Hammermesh [98].

A.1 Representation theory of the symmetric group

Let n be a positive integer and S_n, the group of permutation of n elements. The Young diagrams or partitions R of n, denoted $R \vdash n$, label the irreducible representations (irreps) of S_n. Consider V_R a space of dimension $d(R)$ (that will be made explicit below). An irrep $\varrho_R : S_n \to \mathrm{End}(V_R)$ is given by a matrix D^R with entries $\varrho_R(\sigma)|R,i\rangle = \sum_{l=1}^{d(R)} D^R_{li}(\sigma)|R,l\rangle$ with $\sigma \in S_n$ and with $|R,i\rangle$, $i \in [\![1, d(R)]\!]$, an orthogonal basis of states for V_R (this basis obeys $\langle R,j|R,i\rangle = \delta_{ij}$).
We write in short $\varrho_R(\sigma) = \sigma$ and define the matrix elements as $\langle R,j|\sigma|R,i\rangle = D^R_{ji}(\sigma)$. It is common to assimilate the irreducible representation ϱ_R and the carrier space V_R with their label R.
From the commuting action of the unitary group $U(N)$ and S_n on a tensor product space $V^{\otimes n}$, the Schur-Weyl duality teaches us that we associate an irrep R of S_n with an irrep of $U(N)$, provided N bounds the length $l(R)$ of the first column of R, in symbol $l(R) \leq N$. Let us denote $d(R)$ the dimension of R and $\mathrm{Dim}_N(R)$ the dimension of an irrep of $U(N)$, then those are given by

$$d(R) = n!/h(R)\,, \qquad \mathrm{Dim}_N(R) = f_N(R)/h(R)\,, \tag{A.1}$$

where $h(R)$ is the product of the hook lengths and $f_N(R)$ is the product of box weights (the content shifted by N) given respectively by $h(R) = \prod_{i,j}(c_j - j + r_i - i + 1)$ and $f_N(R) = \prod_{i,j}(N - i + j)$; the pairs (i, j) label the boxes of the Young diagram with i the row label and j the column label. The i^{th} row length is r_i and c_j is the column length of the j^{th} column.

We now restrict to real representations and so the $D^R_{ij}(\sigma)$ must be real matrices. These satisfy the following properties:

$$\sum_i D^R_{ai}(\sigma)D^R_{ib}(\sigma') = D^R_{ab}(\sigma\sigma') \,, \quad D^R_{ab}(id) = \delta_{ab} \,, \quad D^R_{ij}(\sigma^{-1}) = D^R_{ji}(\sigma) \,, \tag{A.2}$$

$$\sum_{\sigma\in S_n} D^R_{ij}(\sigma)D^S_{kl}(\sigma) = \frac{n!}{d(R)} \delta_{RS}\, \delta_{ik}\delta_{jl} \quad \text{(orthogonality)} \,. \tag{A.3}$$

The character of a given irrep R is simply the trace of $D^R(\sigma)$, $\chi^R(\sigma) = \mathrm{tr}(D^R(\sigma)) = \sum_i D^R_{ii}(\sigma)$. The Kronecker delta $\delta(\sigma)$ of the symmetric group (defined to be equal to 1 when $\sigma = id$ and 0 otherwise) decomposes as

$$\delta(\sigma) = \sum_{R\vdash n} \frac{d(R)}{n!}\, \chi^R(\sigma). \tag{A.4}$$

The following identities are easily proven using the orthogonality relations of the representation matrices:

$$\sum_{\gamma\in S_n} \delta(\gamma\sigma\gamma^{-1}\tau^{-1}) = \sum_{R\vdash n} \chi^R(\sigma)\chi^R(\tau) \tag{A.5}$$

$$\sum_{\sigma\in S_n} \chi^R(\sigma)\chi^S(\sigma) = n!\, \delta_{RS} \tag{A.6}$$

$$\sum_{\gamma\in S_n} \chi^R(A\gamma B\gamma^{-1}) = \frac{n!}{d(R)}\chi^R(A)\chi^R(B) \underset{\text{central element}}{\overset{\text{If } B \text{ is a}}{=}} n!\, \chi^R(AB) \tag{A.7}$$

Another useful identity expresses as

$$\frac{1}{n!} \sum_{\sigma\in S_n} \chi^R(\sigma)N^{\mathbf{c}(\sigma)} = \mathrm{Dim}_N(R) \,, \qquad \sum_{\sigma\in S_n} D^R_{ij}(\sigma)N^{\mathbf{c}(\sigma)} = \delta_{ij} f_N(R) \,, \tag{A.8}$$

where $\mathbf{c}(\sigma)$ is the number of cycles of σ.

Defining the central element $\Omega \in \mathbb{C}[S_n]$, as $\Omega = \sum_{\sigma\in S_n} N^{n-\mathbf{c}(\sigma)}\sigma$, the first relation in (A.8) can also be written as

$$\frac{N^n}{n!}\chi^R(\Omega) = \mathrm{Dim}_N(R)\,. \tag{A.9}$$

A.2 Clebsch-Gordan coefficients

Consider two carrier spaces V_{R_1} and V_{R_2} of two irreps of S_n labeled by two Young diagrams R_1 and R_2, respectively. The tensor product representation $V_{R_1}\otimes V_{R_2}$ can be decomposed into a direct sum of irreps V_{R_3} with multiplicities

$$V_{R_1}\otimes V_{R_2} = \bigoplus_{R_3\vdash n} V_{R_3}\otimes V^{\mathrm{m}}_{R_3}\,. \tag{A.10}$$

The tensor product space is spanned by a tensor product of the basis $|R_1,i_1\rangle\otimes|R_2,i_2\rangle := |R_1,i_1;R_2,i_2\rangle$. On the right hand side, the direct sum corresponds to a basis set $|R_3,i_3,\tau_{R_3}\rangle$. The label i_3 runs over states of R_3, and τ_{R_3}, the so-called multiplicity, runs over an orthogonal basis in the multiplicity space $V^{\mathrm{m}}_{R_3}$.
The Clebsch-Gordan coefficients are the branching coefficients between these bases:

$$C^{R_1,R_2;\,R_3,\,\tau_{R_3}}_{i_1,i_2;\,i_3} := \langle R_1,i_1;R_2,i_2|R_3,\tau_{R_3},i_3\rangle = \langle R_3,\tau_{R_3},i_3|R_1,i_1;R_2,i_2\rangle \tag{A.11}$$

Note that they are real.
The following relations are detailed in Appendix A.2 in [30]:

$$\sum_{j_1,j_2} D^{R_1}_{i_1j_1}(\gamma)D^{R_2}_{i_2j_2}(\gamma)C^{R_1,R_2;\,R_3,\,\tau}_{j_1,j_2;\,j_3} = \sum_{i_3} C^{R_1,R_2;\,R_3,\,\tau}_{i_1,i_2;\,i_3}\,D^{R}_{i_3j_3}(\gamma)\,; \tag{A.12}$$

$$\sum_{i_1,i_2} C^{R_1,R_2;\,R_3,\,\tau}_{i_1,i_2;\,i_3}C^{R_1,R_2;\,R_3',\,\tau'}_{i_1,i_2;\,j_3} = \delta_{R_3R_3'}\,\delta_{\tau\tau'}\,\delta_{i_3j_3}\,; \tag{A.13}$$

$$\sum_{R_3,i_3,\tau} C^{R_1,R_2;\,R_3,\,\tau}_{i_1,i_2;\,i_3}C^{R_1,R_2;\,R_3,\,\tau}_{j_1,j_2;\,i_3} = \delta_{i_1j_1}\,\delta_{i_2j_2}\,; \tag{A.14}$$

$$\sum_{R_3,\tau;\,i_3,j_3} C^{R_1,R_2;\,R_3,\,\tau}_{i_1,i_2;\,i_3}D^{R_3}_{i_3j_3}(\gamma)C^{R_1,R_2;\,R_3,\,\tau}_{j_1,j_2;\,j_3} = D^{R_1}_{i_1j_1}(\gamma)D^{R_2}_{i_2j_2}(\gamma)\,; \tag{A.15}$$

$$\sum_{j_1,j_2,j_3} D^{R_1}_{i_1j_1}(\gamma)D^{R_2}_{i_2j_2}(\gamma)D^{R_3}_{i_3j_3}(\gamma)C^{R_1,R_2;\,R_3,\,\tau}_{j_1,j_2;\,j_3} = C^{R_1,R_2;\,R_3,\,\tau}_{i_1,i_2;i_3}\,; \tag{A.16}$$

$$\sum_{i_l,j_l} C^{R_1,R_2;R_3,\tau_1}_{i_1,i_2;i_3}C^{R_1,R_2;R_3,\tau_2}_{j_1,j_2;j_3}D^{R_1}_{i_1j_1}(\gamma_1\sigma_1\gamma_2)D^{R_2}_{i_2j_2}(\gamma_1\sigma_2\gamma_2)D^{R_3}_{i_3j_3}(\gamma_1\sigma_3\gamma_2)$$

$$= \sum_{i_l, j_l} C^{R_1,R_2;R_3,\tau_1}_{i_1,i_2;i_3} C^{R_1,R_2;R_3,\tau_2}_{j_1,j_2;j_3} D^{R_1}_{i_1 j_1}(\sigma_1) D^{R_2}_{i_2 j_2}(\sigma_2) D^{R_3}_{i_3 j_3}(\sigma_3)\,; \tag{A.17}$$

$$\sum_{\sigma\in S_n} D^{R_1}_{i_1 j_1}(\sigma) D^{R_2}_{i_2 j_2}(\sigma) D^{R_3}_{i_3 j_3}(\sigma) = \frac{n!}{d(R_3)} \sum_{\tau} C^{R_1,R_2;R_3,\tau}_{i_1,i_2;i_3} C^{R_1,R_2;R_3,\tau}_{j_1,j_2;j_3}\,. \tag{A.18}$$

Furthermore, we can generalize the second relation (A.8) as follows: given two permutations A and B, we have

$$\begin{aligned}
&\sum_{\sigma\in S_n} D^R_{ij}(\sigma) N^{\mathbf{c}(\sigma^{-1}A\sigma B)} \\
&= \sum_{\gamma,\sigma\in S_n} D^R_{ij}(\sigma)\delta(\gamma^{-1}\sigma^{-1}A\sigma B) N^{\mathbf{c}(\gamma)} \\
&\overset{(A.4)}{=} \sum_{S,a} \frac{d(S)}{n!} \sum_{\gamma,\sigma} D^R_{ij}(\sigma) D^S_{aa}(\gamma^{-1}\sigma^{-1}A\sigma B) N^{\mathbf{c}(\gamma)} \\
&= \sum_{S,a} \frac{d(S)}{n!} \sum_{m,n,o,p} \Big[\sum_{\gamma} D^S_{ma}(\gamma) N^{\mathbf{c}(\gamma)}\Big]\Big[\sum_{\sigma} D^S_{nm}(\sigma) D^S_{op}(\sigma) D^R_{ij}(\sigma)\Big] D^S_{no}(A) D^S_{pa}(B)\,,
\end{aligned} \tag{A.19}$$

with the property $\mathbf{c}(\gamma) = \mathbf{c}(\gamma^{-1})$. We now use (A.8) and (A.18) to write

$$\begin{aligned}
&\sum_{\sigma\in S_n} D^R_{ij}(\sigma) N^{\mathbf{c}(\sigma^{-1}A\sigma B)} \\
&= \sum_{S,a} \frac{d(S)}{n!} \sum_{m,n,o,p} \delta_{ma} f_N(S) \Big(\frac{n!}{d(R)} \sum_{\tau} C^{S,S;R,\tau}_{n,o;i} C^{S,S;R,\tau}_{m,p;j}\Big) D^S_{no}(A) D^S_{pa}(B) \\
&= \sum_{S,\tau} \frac{d(S)}{d(R)} f_N(S) \Big(\sum_{n,o} C^{S,S;R,\tau}_{n,o;i} D^S_{no}(A)\Big)\Big(\sum_{a,p} C^{S,S;R,\tau}_{a,p;j} D^S_{pa}(B)\Big)\,.
\end{aligned} \tag{A.20}$$

A.3 Basis of the group algebra $\mathbb{C}[S_n]$

The matrix basis of the group algebra $\mathbb{C}[S_n]$ is defined by the elements

$$Q^R_{ij} = \frac{\kappa_R}{n!} \sum_{\sigma\in S_n} D^R_{ij}(\sigma)\sigma\,, \tag{A.21}$$

where the constant $\kappa_R^2 = n! d(R)$ is fixed by a normalization. The basis set $\{Q^R_{ij}\}$ is of cardinality $\sum_{R\vdash n}(d(R))^2 = n!$. The elements Q^R_{ij} form a representation theoretic Fourier basis for $\mathbb{C}[S_n]$.

The left and right multiplication by group elements on Q^R_{ij} expand as

$$\tau\, Q^R_{ij} = \sum_l D^R_{li}(\tau)\, Q^R_{lj}\,, \qquad Q^R_{ij}\,\tau = \sum_l Q^R_{il}\, D^R_{jl}(\tau)\,. \tag{A.22}$$

Using the definition of the basis and (A.22), one gets

$$\begin{aligned} Q^R_{ij} Q^{R'}_{kl} &= \frac{\kappa_R \kappa_{R'}}{(n!)^2} \sum_{\sigma\in S_n}\sum_{\tau\in S_n} D^R_{ij}(\sigma)\sigma D^{R'}_{kl}(\tau)\tau = \frac{\kappa_R}{n!}\sum_{\sigma\in S_n} D^R_{ij}(\sigma)\sigma Q^{R'}_{kl} \\ &= \frac{\kappa_R}{n!}\sum_{\sigma\in S_n} D^R_{ij}(\sigma)\sum_m D^{R'}_{mk}(\sigma) Q^{R'}_{ml} = \frac{\kappa_R}{n!}\sum_m \frac{n!}{d(R)}\delta_{RR'}\delta_{im}\delta_{jk}Q^{R'}_{ml} \\ &= \frac{\kappa_R}{d(R)}\delta_{RR'}\delta_{jk}Q^{R'}_{il}\,. \end{aligned} \tag{A.23}$$

We consider the Kronecker δ on S_n, and extend it (by linearity) as a pairing denoted again δ on $\mathbb{C}[S_n]$, and then once again extend the result to $\mathbb{C}[S_n]^{\otimes d}$, $d > 1$, such that

$$\boldsymbol{\delta}(\sigma_1 \otimes \ldots \otimes \sigma_d; \sigma'_1 \otimes \ldots \otimes \sigma'_d) = \delta(\sigma_1 \sigma'^{-1}_1)\ldots\delta(\sigma_d^{-1}\sigma'^{-1}_d)\,. \tag{A.24}$$

Calculating the inner product $\delta(Q^R_{ij}; Q^{R'}_{i'j'})$, we obtain

$$\delta(Q^R_{ij}; Q^{R'}_{i'j'}) = \frac{\kappa_R^2}{n!d(R)}\delta_{RR'}\delta_{ii'}\delta_{jj'} = \delta_{RR'}\delta_{ii'}\delta_{jj'}\,. \tag{A.25}$$

Then, for multiple tensor factors, we obtain

$$\boldsymbol{\delta}(Q^{R_1}_{i_1j_1} \otimes \ldots \otimes Q^{R_d}_{i_dj_d}\,;\, Q^{R'_1}_{i'_1j'_1} \otimes \ldots \otimes Q^{R'_d}_{i'_dj'_d}) = \prod_{a=1}^d \delta_{R_aR'_a}\delta_{i_ai'_a}\delta_{j_aj'_a} \tag{A.26}$$

Hence, the basis $\{Q^{R_1}_{i_1j_1}\otimes\ldots\otimes Q^{R_d}_{i_dj_d}\}$ is a Fourier theoretic orthonormal basis for $\mathbb{C}[S_n]^{\otimes d}$. In the text, we focus on S_{2n} and we introduce the operator $T_\xi : S_{2n} \to S_{2n}$ that acts on S_{2n} as $T_\xi(\sigma) = \sigma^{-1}\xi\sigma$. In a natural way, T_ξ extends by linearity on $\mathbb{C}[S_{2n}]$. Then, without any possible confusion with the tensor notation T itself, $T_\xi \in \mathrm{End}(\mathbb{C}[S_{2n}])$ is the image of the mapping $T : S_{2n} \to \mathrm{End}(\mathbb{C}[S_{2n}])$, $\xi \mapsto T_\xi$. We then extend T by linearity to $T : \mathbb{C}[S_{2n}] \to \mathrm{End}(\mathbb{C}[S_{2n}])$, $\lambda\xi + \rho \mapsto T_{\lambda\xi+\rho} = \lambda T_\xi + T_\rho$, $\lambda \in \mathbb{C}$.

We are interested in the properties of the transformed basis $T_\xi Q^R_{ij}$ which is nothing but the Fourier transform of the pairing $\sigma^{-1}\xi\sigma$. First, let us see how they multiply:

$$(T_\xi Q^R_{ij})\,(T_\xi Q^{R'}_{i'j'}) = \frac{\kappa_R\kappa_{R'}}{(2n!)^2}\sum_{\sigma,\rho\in S_{2n}} D^R_{ij}(\sigma)D^{R'}_{i'j'}(\rho)\sigma^{-1}\xi\sigma\rho^{-1}\xi\rho\,. \tag{A.27}$$

Note that the group order is now $(2n)!$. Introduce a change of variable $\sigma\to\sigma\rho^{-1}$, and

$$\begin{aligned}
&(T_\xi Q^R_{ij})\,(T_\xi Q^{R'}_{i'j'}) = \frac{\kappa_R\kappa_{R'}}{(2n!)^2}\sum_{\sigma,\rho\in S_{2n}}\sum_k D^R_{ik}(\sigma)D^R_{kj}(\rho)D^{R'}_{i'j'}(\rho)\rho^{-1}\sigma^{-1}\xi\sigma\xi\rho \\
&= \frac{\kappa_{R'}}{(2n!)}\sum_{\rho\in S_{2n}} D^{R'}_{i'j'}(\rho)\sum_k D^R_{kj}(\rho)T_{(T_\xi Q^R_{ik})\xi}(\rho) \\
&= \frac{\kappa_{R'}}{(2n!)}\sum_{\rho\in S_{2n}} D^{R'}_{i'j'}(\rho)T_{\sum_k D^R_{kj}(\rho)(T_\xi Q^R_{ik})\xi}(\rho)\,.
\end{aligned} \tag{A.28}$$

Thus, the product of the transformed basis elements does not re-express easily in terms of the transformed elements themselves. The left and right multiplications of fixed permutations on the elements $T_\xi Q^R_{ij}$, counterparts of (A.22), are given by:

$$\tau(T_\xi Q^R_{ij}) = \sum_a (T_\xi Q^R_{ia})D^R_{aj}(\tau)\tau\,,\qquad (T_\xi Q^R_{ij})\tau = \sum_a (T_\xi Q^R_{ia})D^R_{ja}(\tau)\tau\,. \tag{A.29}$$

The inner product of these elements expresses as:

$$\delta(T_\xi Q^R_{ij},\, T_\xi Q^{R'}_{i'j'}) = \frac{\kappa_R\kappa_{R'}}{(2n!)^2}\sum_{\sigma,\rho\in S_{2n}} D^R_{ij}(\sigma)D^{R'}_{i'j'}(\rho)\,\delta(T_\xi(\sigma),T_\xi(\rho))\,. \tag{A.30}$$

This is simply the Fourier transform of the delta $\delta(\sigma^{-1}\xi\sigma\rho^{-1}\xi\rho)$ which tells us that the sole terms remaining in this sum are those which define the same pairing. A closer look shows that $\delta(\sigma^{-1}\xi\sigma\rho^{-1}\xi\rho) = \delta(\xi\sigma\rho^{-1}\xi\rho\sigma^{-1})$. Then, this means that the elements that contribute to the sum are those $\sigma\rho^{-1}$ that belong to the stabilizer of ξ, that is $\sigma\rho^{-1}\in S_n[S_2]$. Hence, we change variable as $\sigma\to\bar\sigma=\sigma\rho^{-1}$, rename again $\bar\sigma$ as σ and then rewrite, using the orthogonality of the representation matrices:

$$\begin{aligned}\delta(T_\xi Q^R_{ij},\, T_\xi Q^{R'}_{i'j'}) &= \frac{\kappa_R\kappa_{R'}}{(2n!)^2}\sum_{\rho\in S_{2n}}\sum_{\sigma\in S_n[S_2]} D^R_{ij}(\sigma\rho)D^{R'}_{i'j'}(\rho)\\ &= \frac{\kappa_R\kappa_{R'}}{(2n!)^2}\sum_a\sum_{\sigma\in S_n[S_2]} D^R_{ia}(\sigma)\sum_{\rho\in S_{2n}} D^R_{aj}(\rho)D^{R'}_{i'j'}(\rho)\\ &= \delta_{RR'}\delta_{jj'}\frac{\kappa_R^2}{(2n!)^2}\frac{2n!}{d(R)}\sum_a\sum_{\sigma\in S_n[S_2]} D^R_{ia}(\sigma)\delta_{ai'}\\ &= \delta_{RR'}\delta_{jj'}\sum_{\sigma\in S_n[S_2]} D^R_{ii'}(\sigma)\,.\end{aligned} \tag{A.31}$$

In the text, we compute a formula for that sum in terms of branching coefficients, see (5.19). It turns out that the sum is non vanishing only if the partition R is even, meaning that the length of each of its rows is even. Hence, from the above relation, (A.31), the set of the transformed basis elements does not form an orthogonal system.

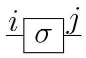
It is instructive to perform the same evaluation in an alternative way to discover new identities satisfied by the Clebsch-Gordan coefficients. Consider the expansion of the above inner product as follows:

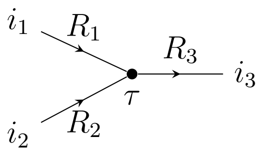
$$\begin{aligned}&\delta(T_\xi Q^R_{ij},\, T_\xi Q^{R'}_{i'j'})\\ &= \frac{\kappa_R\kappa_{R'}}{(2n!)^2}\sum_S\frac{d(S)}{2n!}\sum_{\sigma,\rho\in S_n} D^R_{ij}(\sigma)D^{R'}_{i'j'}(\rho)\chi^S(\sigma^{-1}\xi\sigma\rho^{-1}\xi\rho)\\ &= \frac{\kappa_R\kappa_{R'}}{(2n!)^2}\sum_S\frac{d(S)}{2n!}\sum_{a,b,c,d,e,f} D^S_{bc}(\xi)D^S_{ef}(\xi)\sum_{\sigma,\rho} D^S_{ba}(\sigma)D^S_{cd}(\sigma)D^R_{ij}(\sigma)D^S_{fa}(\rho)D^S_{ed}(\rho)D^{R'}_{i'j'}(\rho)\\ &= \frac{\kappa_R\kappa_{R'}}{(2n!)^2}\sum_S\frac{d(S)}{2n!}\sum_{b,c,e,f} D^S_{bc}(\xi)D^S_{ef}(\xi)\frac{(2n!)^2}{d(R)d(R')}\sum_{\tau,\tau'} C^{S,S;R,\tau}_{b,c;i}C^{S,S;R',\tau'}_{f,e;i'}\sum_{a,d} C^{S,S;R,\tau}_{a,d;j}C^{S,S;R',\tau'}_{a,d;j'}\\ &= \frac{\kappa_R\kappa_{R'}}{d(R)d(R')}\sum_S\frac{d(S)}{2n!}\sum_{b,c,e,f} D^S_{bc}(\xi)D^S_{ef}(\xi)\sum_{\tau,\tau'} C^{S,S;R,\tau}_{b,c;i}C^{S,S;R',\tau'}_{f,e;i'}\delta_{RR'}\delta_{\tau\tau'}\delta_{jj'}\\ &= \delta_{RR'}\delta_{jj'}\frac{\kappa_R^2}{d(R)^2}\sum_{S,\tau}\frac{d(S)}{2n!}\sum_{b,c,e,f} D^S_{bc}(\xi)D^S_{ef}(\xi)C^{S,S;R,\tau}_{b,c;i}C^{S,S;R,\tau}_{f,e;i'}\\ &= \delta_{RR'}\delta_{jj'}\frac{1}{d(R)}\sum_{S,\tau} d(S)F(S,R,\tau;i)F(S,R,\tau;i')\,,\end{aligned} \tag{A.32}$$

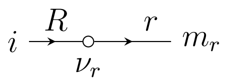
where, at some intermediate steps, we used successively (A.18) and (A.13), and where $F(S,R,\tau;i) = \sum_{b,c} D^S_{bc}(\xi)C^{S,S;R,\tau}_{b,c;i}$. Using $\sum_{\sigma\in S_n[S_2]} D^R_{ij}(\sigma) = (2^n n!)B^R_i B^R_j$ (see (5.19)), we arrive to a new identity:

$$\sum_{S,\tau} d(S) \big(\sum_{b,c} D^S_{bc}(\xi) C^{S,S;R,\tau}_{b,c;i} \big) \big(\sum_{e,f} D^S_{ef}(\xi) C^{S,S;R,\tau}_{e,f;j} \big) = \frac{(2^n n!)}{d(R)} B^R_i B^R_j \,. \tag{A.33}$$

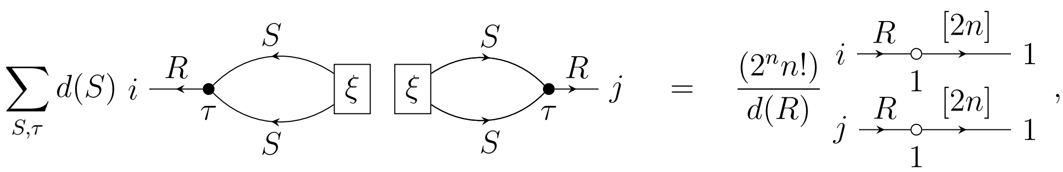
Note the similarity of the left-hand-side member with (A.20) (adjusted for the symmetric group S_{2n}).

There exist graphical ways of representing identities in representation theory in general. For the permutation group, Appendix A2 of [30] lists such graphical representations for most of the identities given above. For instance, we use the graphical representation of the representation matrix $D^R_{ij}(\sigma)$ as 

, the Clebsch-Gordan coefficient $C^{R_2,R_2;R_3,\tau}_{i_1,i_2;i_3}$ represents as follows 

and the branching coefficient $B^{R;\,r,\nu_r}_{i;\,m_r}$ looks like 

. Then the convolution given by (A.33) translates as the factorization:



(A.34)

hence, a new identity satisfied by the Clebsch-Gordan of the symmetric group.

A.4 2-point correlator evaluation

We prove in this part (6.16). To proceed, we will make use of (A.8), (A.13) and (A.18), or alternatively (A.20), of Appendix A.2. Introducing $k_{\vec R} = \kappa_{\vec R} \frac{\kappa_{R_1}\kappa_{R_2}\kappa_{R_3}}{((2n)!)^3}$, then from (6.15), we focus on the $\boldsymbol{\delta}$ function:

$$\begin{aligned}
&\boldsymbol{\delta}\Big[(T^{\otimes 3}_\xi Q^{R'_1,R'_2,R'_3,\tau'})(T^{\otimes 3}_\xi Q^{R_1,R_2,R_3,\tau})(\Omega_1 \otimes \Omega_2 \otimes \Omega_3)\Big] \\
&= k_{\vec R} k'_{\vec R} \sum_{p_l,q_l,p'_l,q'_l} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3} C^{R'_1,R'_2;R'_3,\tau'}_{q'_1,q'_2;q'_3} \Big[\prod_{i=1}^3 B^{R'_i}_{p'_i} B^{R_i}_{p_i}\Big] \\
&\quad \times \sum_{\sigma'_i,\sigma_i} \sum_{\alpha_i} \Big[\prod_{i=1}^3 N^{\mathbf{c}(\alpha_i)-2n}\Big] \Big[\prod_{i=1}^3 D^{R'_i}_{p'_i q'_i}(\sigma'_i) D^{R_i}_{p_i q_i}(\sigma_i) \delta((\sigma'_i)^{-1}\xi\sigma'_i(\sigma_i)^{-1}\xi\sigma_i\alpha_i)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= k_{\vec{R}} k'_{\vec{R}} \sum_{p_l,q_l,p'_l,q'_l} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3} C^{R'_1,R'_2;R'_3,\tau'}_{q'_1,q'_2;q'_3} \left[\prod_{i=1}^{3} B^{R'_i}_{p'_i} B^{R_i}_{p_i} \right] \sum_{\sigma'_i,\sigma_i} \left[\prod_{i=1}^{3} D^{R'_i}_{p'_i q'_i}(\sigma'_i) D^{R_i}_{p_i q_i}(\sigma_i) \right] \\
&\quad \times \sum_{S_i,a_i,g_i} \sum_{\alpha_i} \left[\prod_{i=1}^{3} N^{\mathbf{c}(\alpha_i)-2n} D^{S_i}_{g_i a_i}(\alpha_i) \right] \\
&\quad \times \sum_{b_i,c_i,d_i,e_i,f_i} \prod_{i=1}^{3} \frac{d(S_i)}{2n!} D^{S_i}_{a_i b_i}((\sigma'_i)^{-1}) D^{S_i}_{b_i c_i}(\xi) D^{S_i}_{c_i d_i}(\sigma'_i) D^{S_i}_{d_i e_i}((\sigma_i)^{-1}) D^{S_i}_{e_i f_i}(\xi) D^{S_i}_{f_i g_i}(\sigma_i) \\
&= k_{\vec{R}} k'_{\vec{R}} \frac{N^{-6n}}{(2n!)^3} \sum_{p_l,q_l,p'_l,q'_l} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3} C^{R'_1,R'_2;R'_3,\tau'}_{q'_1,q'_2;q'_3} \left[\prod_{i=1}^{3} B^{R'_i}_{p'_i} B^{R_i}_{p_i} \right] \sum_{S_i,a_i,g_i} \left[\prod_{i=1}^{3} \delta_{g_i a_i} f_N(S_i) d(S_i) \right] \\
&\quad \times \sum_{b_i,c_i,d_i,e_i,f_i} \sum_{\sigma'_i,\sigma_i} \left[\prod_{i=1}^{3} D^{S_i}_{b_i a_i}(\sigma'_i) D^{S_i}_{c_i d_i}(\sigma'_i) D^{R'_i}_{p'_i q'_i}(\sigma'_i) D^{S_i}_{f_i g_i}(\sigma_i) D^{S_i}_{e_i d_i}(\sigma_i) D^{R_i}_{p_i q_i}(\sigma_i) \right] \\
&\quad \times \left[\prod_{i=1}^{3} D^{S}_{b_i c_i}(\xi) D^{S}_{e_i f_i}(\xi) \right] .
\end{aligned} \tag{A.35}
$$

It is the moment to use (A.18) to integrate the representation matrices and get:

$$
\begin{aligned}
& k_{\vec{R}} k'_{\vec{R}} \frac{N^{-6n}}{(2n!)^3} \sum_{p_l,q_l,p'_l,q'_l} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3} C^{R'_1,R'_2;R'_3,\tau'}_{q'_1,q'_2;q'_3} \left[\prod_{i=1}^{3} B^{R'_i}_{p'_i} B^{R_i}_{p_i} \right] \sum_{S_i} \left[\prod_{i=1}^{3} (2n!) \mathrm{Dim}_N(S_i) \right] \\
&\quad \times \sum_{a_i,b_i,c_i,d_i,e_i,f_i} \prod_{i=1}^{3} \left[\frac{2n!}{d(R_i) d(R'_i)} \sum_{\tau'_i,\tau_i} C^{S_i,S_i;R'_i,\tau'_i}_{b_i,c_i;p'_i} C^{S_i,S_i;R'_i,\tau'_i}_{a_i,d_i;q'_i} C^{S_i,S_i;R_i,\tau_i}_{f_i,e_i;p_i} C^{S_i,S_i;R_i,\tau_i}_{a_i,d_i;q_i} \right] \\
&\quad \times \left[\prod_{i=1}^{3} D^{S_i}_{b_i c_i}(\xi) D^{S_i}_{e_i f_i}(\xi) \right] \\
&= k^2_{\vec{R}} N^{-6n} \prod_{i=1}^{3} \left[\frac{2n!}{d(R_i) d(R'_i)} \right] \sum_{p_l,q_l,p'_l,q'_l} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3} C^{R'_1,R'_2;R'_3,\tau'}_{q'_1,q'_2;q'_3} \left[\prod_{i=1}^{3} B^{R'_i}_{p'_i} B^{R_i}_{p_i} \right] \\
&\quad \times \sum_{S_i} \left[\prod_{i=1}^{3} \mathrm{Dim}_N(S_i) \right] \\
&\quad \times \sum_{b_i,c_i,e_i,f_i} \prod_{i=1}^{3} \left[\sum_{\tau'_i,\tau_i} C^{S_i,S_i;R'_i,\tau'_i}_{b_i,c_i;p'_i} C^{S_i,S_i;R_i,\tau_i}_{f_i,e_i;p_i} \delta_{R'_i R_i} \delta_{\tau'_i \tau_i} \delta_{q'_i q_i} \right] \left[\prod_{i=1}^{3} D^{S_i}_{b_i c_i}(\xi) D^{S_i}_{e_i f_i}(\xi) \right] \\
&= \frac{(2n!)^3 k^2_{\vec{R}} N^{-6n}}{\prod_{i=1}^{3} d(R_i)^2} \left[\prod_{i=1}^{3} \delta_{R'_i R_i} \right] \sum_{p_l,q_l,p'_l} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3} C^{R_1,R_2;R_3,\tau'}_{q_1,q_2;q_3} \left[\prod_{i=1}^{3} B^{R'_i}_{p'_i} B^{R_i}_{p_i} \right] \\
&\quad \times \sum_{S_i} \left[\prod_{i=1}^{3} \mathrm{Dim}_N(S_i) \right]
\end{aligned}
$$

$$
\begin{aligned}
&\times \sum_{b_i,c_i,e_i,f_i} \prod_{i=1}^{3} \Big[\sum_{\tau_i} C^{S_i,S_i;R_i,\tau_i}_{b_i,c_i;p'_i} C^{S_i,S_i;R_i,\tau_i}_{f_i,e_i;p_i} \Big] \Big[\prod_{i=1}^{3} D^{S_i}_{b_ic_i}(\xi) D^{S_i}_{e_if_i}(\xi) \Big] \\
&= \kappa^2_{\vec{R}} N^{-6n} \Big[\prod_{i=1}^{3} \delta_{R'_iR_i} \Big] \delta_{\tau'\tau} \sum_{q_3} \delta_{q_3q_3} \sum_{p_l,p'_l} \Big[\prod_{i=1}^{3} B^{R'_i}_{p'_i} B^{R_i}_{p_i} \Big] \sum_{S_i} \Big[\prod_{i=1}^{3} \mathrm{Dim}_N(S_i) \Big] \\
&\times \sum_{b_i,c_i,e_i,f_i} \prod_{i=1}^{3} \Big[\sum_{\tau_i} C^{S_i,S_i;R_i,\tau_i}_{b_i,c_i;p'_i} C^{S_i,S_i;R_i,\tau_i}_{f_i,e_i;p_i} \Big] \Big[\prod_{i=1}^{3} D^{S_i}_{b_ic_i}(\xi) D^{S_i}_{e_if_i}(\xi) \Big] \\
&= \kappa^2_{\vec{R}} d(R_3) N^{-6n} \Big[\prod_{i=1}^{3} \delta_{R'_iR_i} \Big] \delta_{\tau'\tau} \sum_{S_i,\tau_i} \Big[\prod_{i=1}^{3} \mathrm{Dim}_N(S_i) \Big] \\
&\times \prod_{i=1}^{3} \Big[\sum_{b_i,c_i,p'_i} D^{S_i}_{b_ic_i}(\xi) C^{S_i,S_i;R_i,\tau_i}_{b_i,c_i;p'_i} B^{R_i}_{p'_i} \Big] \Big[\sum_{p_i,e_i,f_i} D^{S_i}_{e_if_i}(\xi) C^{S_i,S_i;R_i,\tau_i}_{f_i,e_i;p_i} B^{R_i}_{p_i} \Big] \\
&= N^{-6n} \delta_{\vec{R}'\vec{R}} \delta_{\tau'\tau} \sum_{S_i,\tau_i} \Big[\prod_{i=1}^{3} \mathrm{Dim}_N(S_i) \Big] \Big[\sum_{b_i,c_i,p_i} D^{S_i}_{b_ic_i}(\xi) C^{S_i,S_i;R_i,\tau_i}_{b_i,c_i;p_i} B^{R_i}_{p_i} \Big]^2 . \qquad \text{(A.36)}
\end{aligned}
$$

The evaluation finally yields

$$
\begin{aligned}
&\langle O^{R_1,R_2,R_3,\tau}\, O^{R'_1,R'_2,R'_3,\tau'} \rangle = \delta_{\vec{R}'\vec{R}} \delta_{\tau'\tau} F(R_1,R_2,R_3,\tau) \\
&F(R_1,R_2,R_3,\tau) = \sum_{S_i,\tau_i} \Big[\prod_{i=1}^{3} \mathrm{Dim}_N(S_i) \Big] \Big[\sum_{b_i,c_i,p_i} D^{S_i}_{b_ic_i}(\xi) C^{S_i,S_i;R_i,\tau_i}_{b_i,c_i;p_i} B^{R_i}_{p_i} \Big]^2 . \qquad \text{(A.37)}
\end{aligned}
$$

This is (6.16) and implies the orthogonality of the representation theoretic basis $\{O^{R_1,R_2,R_3,\tau}\}$.

Appendix B

Codes

We list here some algorithms which count the number of orthogonal invariants as given in Part II. We use Mathematica and Sage softwares in the following.

B.1 Mathematica code for $Z_d(t)$

We wish to compute the number $Z_d(2n)$ of rank-d orthogonal invariants made with $2n$ tensors. In order to obtain that number, we first code the generating function, denoted `Z[X,t]`, of the counting of the number of elements of the wreath product $S_n[S_2]$ in a certain conjugacy class of S_{2n}. Doing this, we use the built-in function `Count[list, pattern]` which counts the number of elements in a `list` matching a `pattern`. Then, we extract a coefficient of t^n in `Z[X,t]` that is involved in `Zd[X,n,d]` that encodes $Z_d(2n)$. We finally give the counting for ranks 3 and 4, successively, for $n \in [\![1, 10]\!]$.

```
X = Array[x, 20];
PP[n_] := IntegerPartitions[n]
Sym[q_, n_] := Product[i^(Count[q, i]) Count[q, i]!, {i, 1, n}]
Symd[X_, k_, q_] := Product[(X[[k*l]]/l)^(Count[q, l])/(Count[q, l]!), {l, 1, 2}]

Z[X_, t_] := Product[Exp[(t^i/i)*Sum[Symd[X, i, PP[2][[j]]], {j, 1, Length[PP[2]]}]],
            {i, 1, 15}]
Zprim[X_, n_] := Coefficient[Series[Z[X, t], {t, 0, n}], t^n]
CC[X_, n_, q_] := Coefficient[Zprim[X, n], Product[X[[i]]^(Count[q, i]), {i, 1, 2*n}]]
Zd[X_, n_, d_] := Sum[(CC[X, n, PP[2*n][[i]]])^d*(Sym[PP[2*n][[i]], 2*n])^(d - 1),
                {i, 1, Length[PP[2*n]]}]

Table[Zd[X, i, 3], {i, 1, 10}]

(out) {1, 5, 16, 86, 448, 3580, 34981, 448628, 6854130, 121173330}
```

```
Table[Zd[X, i, 4], {i, 1, 10}]

(out) {1, 14, 132, 4154, 234004, 24791668, 3844630928, 809199787472, 220685007519070,
75649235368772418}
```

B.2 Mathematica code for counting with Hermite polynomials

This part is dedicated to the implementation of an algorithm realizing Read's enumeration of k-regular graphs on $2n$ vertices with edges of k different colors where one of each color is at every vertex. We want to compare Read's results with the previous sequences. Read's generating function that encodes the above enumeration denotes `ZR[t,d,n]`, in the following program. Then, `ZR[d,n]` yields the counting at rank d with $2n$ vertices and that is given by the coefficient of t^n in `ZR[t,d,n]`. We evaluate $Z_3(2n)$ and $Z_4(2n)$ for the ranks 3 and 4, respectively, and confirm that the results of Read match with the previous results.

Next, the number of connected rank-d tensor invariants made with $2n$ tensors, written below `ZRc[d,n]`, can be obtained using the plethystic logarithm (Plog) function. The Plog function $\text{Plog} Z_d(t)$, denoted `Plog[ZR,t,d,n]`, is defined with the `MoebiusMu` implementing the Möbius function.

```
A[p_, v_] := (I Sqrt[p])^v HermiteH[v, 1/(2 I Sqrt[p])]
ZR[t_, d_, n_] = 1;

For[m = 0, m <= 20, m++
     {If[OddQ[m],
      Phi[m, t_, d_, n_] := (Sum[((2 v)!)^(d - 1)/(v!)^(d)*(m^(d - 2)/2^d)^
         v t^(m v), {v, 0, n}]),
      Phi[m, t_, d_, n_] := (Sum[(A[m/2, v])^d/(v! m^v) t^(m v/2), {v, 0, n}])]
      };
    ZR[t_, d_, n_] = ZR[t, d, n]*Phi[m, t, d, n]
]

ZR[d_, n_] := Coefficient[Series[ZR[t, d, n], {t, 0, n}], t^n]
Plog[F_, t_, d_, n_] := Sum[MoebiusMu[i]/i Log[F[t^i, d, n]], {i, 1, n}]
ZRc[d_, n_] := Coefficient[Series[Plog[ZR, t, d, n], {t, 0, n}], t^n]

Table[ZR[3, i], {i, 1, 10}]
```

```
(Out) {1, 5, 16, 86, 448, 3580, 34981, 448628, 6854130, 121173330}

Table[ZR[4, i], {i, 1, 10}]

(Out) {1, 14, 132, 4154, 234004, 24791668, 3844630928, 809199787472, 220685007519070,
75649235368772418}

Table[ZRc[3, i], {i, 1, 10}]

(Out) {1, 4, 11, 60, 318, 2806, 29359, 396196, 6231794, 112137138}

Table[ZRc[4, i], {i, 1, 10}]

(Out) {1, 13, 118, 3931, 228316, 24499085, 3816396556, 805001547991, 219822379032704,
75417509926065404}
```

B.3 Sage code: Counting from the sum of Kroneckers in rank $d = 3$

We provide here a Sage code that recovers the same counting through the sum of constrained Kronecker coefficients with even partitions (4.48).
We need the library `SymmetricFunctions(QQ)` which - as its name suggests - introduces symmetric functions. The Kronecker coefficient associated with three partitions R, S and T deduces as the usual Hall scalar product of Schur symmetric functions. In the following, $s(S)$ is the Schur function associated with the partition S.

```
s = SymmetricFunctions(QQ).s()
for n in range(1,4) :
    Total=0
    for R in Partitions(2*n) :
        i=0
        rep=0
        while ( (i < R.length()) & (rep==0) ):
            if ( (R.get_part(i)%2) !=0 ):
                rep = 1
            i=i+1
        if (rep ==0) :
            for S in Partitions (2*n) :
```

```
                j=0
                rep2=0
                while ( (j < S.length()) & (rep2==0) ):
                    if ( (S.get_part(j)%2) !=0 ):
                        rep2 = 1
                    j=j+1
                if (rep2 ==0) :
                    for T in Partitions (2*n) :
                        k=0
                        rep3=0
                        while ( (k < T.length()) & (rep3==0) ):
                            if ( (T.get_part(k)%2) !=0 ):
                                rep3 = 1
                            k=k+1
                        if (rep3 ==0) :
                            a = ( s(S).itensor(s(T)) ).scalar ( s(R) )
                            Total =Total+a

    print "Number of invariants at 2n =", 2*n, "is", Total

(out) Number of invariants at 2n = 2 is 1
Number of invariants at 2n = 4 is 5
Number of invariants at 2n = 6 is 16
Number of invariants at 2n = 8 is 86
```

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