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Asymptotique de modèles d'ondes nonlinéaires dans les domaines à bord

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Résumé

Cette thèse est consacrée à l'étude du comportement asymptotique de modèles d'ondes non linéaires, plus précisément l'équation des ondes non linéaire et l'équation de Schrödinger non linéaire, dans des domaines à bord. Nous sommes principalement intéressés par le comportement en temps long (existence globale et comportement de type scattering) de solutions à ces équations dans des domaines extérieurs d'un obstacle non-captant qui n'est pas étoilé, comme le cas dit illuminé depuis l'intérieur ou depuis l'extérieur et le cas dit presque étoilé... Ces obstacles sont des généralisations naturelles de la forme étoilée et ils étaient largement étudiés dans les années 1960 et 1970, après les travaux pionniers de Morawetz pour le cas étoilé dans le cadre de la décroissance de l'énergie locale pour l'équation des ondes linéaire.

Pour l'équation des ondes non linéaire défocalisante critique en énergie en 3 dimensions, le comportement de type scattering n'est connu que pour le cas étoilé. Dans ce travail, nous étendons le scattering pour des obstacles illuminés depuis l'extérieur en utilisant la méthode de multiplicateurs avec des poids qui généralisent le multiplicateur de Morawetz pour être adapté à la géométrie de l'obstacle.

Pour l'équation de Schrödinger non linéaire défocalisante en 2 dimensions, l'existence globale et le scattering sont connus pour le cas étoilé et des non linéarités puissances qui croissent au moins comme la puissance quintique. Dans cette thèse, nous étendons le résultat d'existence globale pour tous obstacles non-captants et pour des non linéarités avec une puissance strictement supérieure à quartique. Pour tels non linéarités, nous montrons aussi le scattering pour les obstacles étoilés et pour une classe d'obstacles presque étoilés.

Mots clés : Équations des ondes et de Schrödinger, comportement de type scattering, illuminé depuis l'intérieur, illuminé depuis l'extérieur, presque étoilé.

Abstract

This thesis is devoted to the study of asymptotics of nonlinear wave models, more specifically the nonlinear wave equation and the nonlinear Schrödinger equation, in domains with boundary. We are mainly interested in the long time behavior (global existence and scattering) of solutions to these equations in domains exterior to non trapping obstacles that are not star-shaped, like the so-called illuminated from interior or from exterior and the so-called almost star-shaped... These obstacles are natural generalizations of the star-shaped and they were extensively studied in the 1960's and 1970's after the pioneering work of Morawetz for the star-shaped case in the setting of local energy decay for the linear wave equation.

For the energy critical nonlinear defocusing wave equation in 3 dimensions, scattering is only known for the star-shaped case. In this work, we extend the scattering for illuminated from exterior obstacles using the method of multipliers with weights that generalize the Morawetz multiplier to suit the geometry of the obstacle.

For the nonlinear defocusing Schrödinger equation in 2 dimensions, global existence and scattering are known for the star-shaped case and for nonlinearities that grow at least as the quintic power. In this thesis, we extend the global existence result for all non trapping obstacles and for nonlinearities with power strictly greater than quartic. For such nonlinearities, we also prove scattering for star-shaped obstacles and for a class of almost star-shaped obstacles.

Keywords : Wave and Schrödinger equations, scattering, illuminated from interior, illuminated from exterior, almost star-shaped.

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Chapitre I

Introduction

1 Les domaines à bord. Diverses géométries

Dans le cadre de cette thèse, on étudie des modèles d'ondes non linéaires dans les domaines à bord. Plus précisément, on s'intéresse au comportement en temps long (existence globale, comportement de type scattering) de l'équation des ondes et l'équation de Schrödinger non linéaire dans des domaines Ω extérieurs d'un obstacle non-captant V. Un obstacle V est dit non-captant si tout rayon lumineux qui se propage dans $\Omega = \mathbb{R}^n \setminus V$ selon les lois de l'optique géométrique ne reste pas dans un compact de $\overline{\Omega}$ un temps infiniment long. Dans cette première section, nous allons présenter différentes géométries non-captantes avec lesquelles nous allons travailler et qui remontent aux années 1960 et 1970 dans le cadre de l'équation des ondes linéaire. Des résultats de décroissance de l'énergie locale ont été obtenus pour des domaines extérieurs à ces obstacles pendant cette période.

1.1 Les obstacles étoilés

Un exemple d'un obstacle non-captant qui a toujours été le plus adapté aux méthodes souvent utilisées dans le traitement de l'équation des ondes et l'équation de Schrödinger (ça sera détaillé plus tard dans la thèse) est l'obstacle étoilé.

Définition 1.1. *Nous appelons un obstacle V ($0 \in V$) étoilé par rapport à l'origine si $n(x) \cdot x \geq 0$ pour $x \in \partial V$ avec n la normale unitaire extérieure à ∂V (intérieure au domaine Ω). Si $n(x) \cdot x > 0$, l'obstacle est appelé strictement étoilé.*

Dans le début des années 1960, et particulièrement dans les travaux de Morawetz ([25],[27]) dans le cadre de la décroissance de l'énergie locale de l'équation des ondes linéaire, la condition de la géométrie étoilée a dû être imposée pour que le terme à bord généré par la méthode utilisée (voir la section 2.3) ait le bon signe. Cependant, des nombreuses généralisations de la géométrie étoilée ont été par la suite introduites et étudiées dans le même cadre.

1.2 Les obstacles illuminés

L'un des obstacles qui généralisent le cas strictement étoilé est le cas dit illuminé depuis l'intérieur (Bloom et Kazarinoff [7]). L'idée de ces obstacles vient du fait que, pour un obstacle strictement étoilé V, une petite boule centrée à l'origine est contenue dans l'intérieur de V et chaque rayon commençant à l'origine recoupe ∂V exactement une fois et est perpendiculaire à la surface de cette boule, cela nous permet de voir la forme étoilée comme illuminée par cette boule plutôt que par une source de lumière située à l'origine (ce qui est impliqué par la condition $x \cdot n(x) > 0$). Figure I.1 ci-dessous représente cette interprétation.

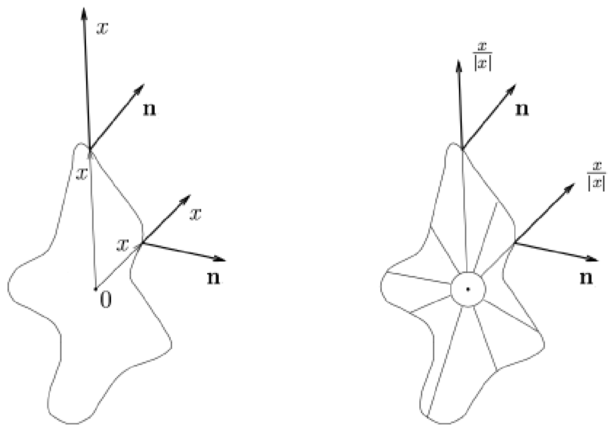

Figure I.1 – Obstacle étoilé

Le concept des obstacles illuminés depuis l'intérieur est de remplacer la boule par un convexe C. Ils sont définis comme suite :

Définition 1.2. *(Bloom et Kazarinoff, [7]) On dit qu'un corps V peut être illuminé depuis l'intérieur si et seulement si il existe un corps convexe régulier C à l'intérieur de V tel que $extC = \mathbb{R}^n \setminus C$ est rempli par une famille de rayons qui ne se croisent pas et qui sont normaux à ∂C et tel que chaque rayon coupe ∂V exactement une fois.*

Un exemple d'un obstacle non étoilé qui peut être illuminé depuis l'intérieur est celui ci-dessous (Figure I.2) qui ressemble à "un os de chien".

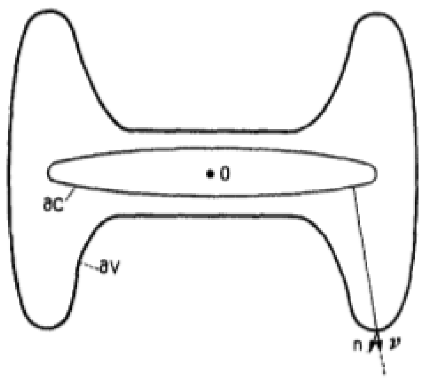

Figure I.2 – Illuminé depuis l'intérieur

Une autre généralisation est le cas dit illuminé depuis l'extérieur (Bloom et Kazarinoff [8] et Liu [24]). Ces obstacles sont une généralisation de l'illuminé depuis l'intérieur et ils ont une géométrie similaire mais légèrement plus complexe, l'obstacle étant contenu dans le corps convexe.

Définition 1.3. *((Liu, [24])) On dit que le bord d'un domaine extérieur Ω (ou l'obstacle V) peut être illuminé depuis l'extérieur si et seulement si il existe un corps convexe C contenant ∂V avec un bord régulier ∂C tel que ∂V est recouvert par une famille de rayons qui ne se croisent pas et qui sont normaux à ∂C. Chaque rayon est entièrement contenu dans Ω dans le sens suivant : pour chaque $x_0 \in \partial V$ il existe $x_1 \in \partial C$ unique et un nombre $s_0(x_1) \leq 0$ tel que*

$$x_0 = s_0(x_1)\nu(x_1) + x_1,$$

où ν est la normale unitaire extérieure à ∂C à x_1, et

$$x = t\nu(x_1) + x_1 \in \Omega, \qquad t \in [s_0, \infty).$$

Chaque obstacle qui peut être illuminé depuis l'intérieur peut aussi être illuminé depuis l'extérieur par l'élargissement du corps convexe original. Un obstacle en forme de serpent (Figure I.3) est un exemple d'un obstacle qui ne peut pas être illuminé depuis son intérieur, mais il peut être illuminé depuis l'extérieur.

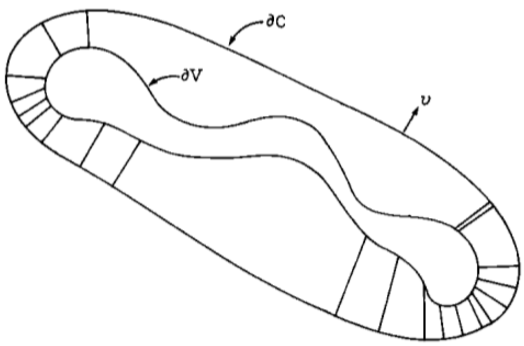

Figure I.3 – Illuminé depuis l'extérieur

De plus, pour ces obstacles (illuminés depuis l'intérieur ou extérieur), la condition de la géométrie étoilée ($x \cdot n(x) \geq 0$) est généralisée par

$$\nu \cdot n \geq 0 \quad \text{sur } \partial V, \tag{I.1}$$

où ν est la normale unitaire extérieure au bord du corps illuminant.

1.3 Les champs de vecteurs de Strauss

Une large généralisation et qui comprend toutes les géométries ci-dessus a été introduite par Strauss en 1975 ([35]). Ce dernier a montré la décroissance uniforme de l'énergie locale de l'équation des ondes linéaire homogène dans des domaines extérieurs dans $\mathbb{R}^n$ pour $n \geq 3$, à condition qu'il existe un champ de vecteurs (maintenant appelé le champ de vecteurs de Strauss) qui satisfait les conditions suivantes :

- $l(x).n(x) > 0, \qquad x \in \partial\Omega,$
- $\partial_i l_j(x)\xi_i\xi_j \geq c|\xi|^2, \quad x \in \overline{\Omega}, \xi \in \mathbb{R}^n.$

En même temps Morawetz a démontré dans [26] un résultat similaire pour des domaines dans $\mathbb{R}^2$ et $\mathbb{R}^3$, en supposant l'existence d'une fonction convexe χ vérifiant certaines propriétés. Mais, en fait, $\nabla\chi$ fournit exactement un champ de vecteurs de Strauss l.

Un exemple d'obstacle qui ne peut pas être illuminé mais pour lequel il existe un champ de vecteurs de Strauss est celui dans la Figure I.4 ci-dessous.

Comme nous l'avons dit au début, l'existence du champ de vecteurs de Strauss généra-

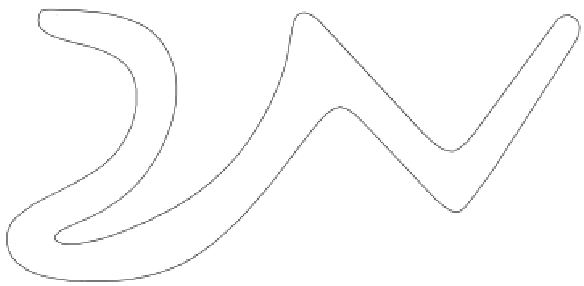

Figure I.4 – Existence d'un champ de vecteurs de Strauss

lise toutes les géométries que nous avons introduites jusqu'à présent (étoilé, illuminé depuis l'intérieur et illuminé depuis l'extérieur). De plus, il faut remarquer qu'avec une restriction (relativement naturelle) sur le comportement de la courbure du bord, l'existence d'un champ de vecteurs de Strauss est une condition optimale sur des obstacles dans $\mathbb{R}^2$ pour avoir la décroissance de l'énergie locale (montré dans [28]). Cependant, ce n'est pas le cas pour les dimensions supérieures à 2, ce qui a motivé la généralisation de ces champs de vecteurs par

Morawetz, Ralston et Strauss [28] en utilisant des méthodes microlocales impliquant des opérateurs pseudo-différentiels pour montrer des résultats de décroissance de l'énergie locale pour les dimensions $n \geq 3$.

1.4 Les obstacles presque étoilés

En 1969, Ivrii a introduit dans [21] la notion d'obstacles presque étoilés, encore dans le cadre de l'équation des ondes linéaire. Il a montré des résultats de décroissance de l'énergie locale pour des domaines extérieurs à ces obstacles dans les dimensions impaires $n > 1$. Un obstacle presque étoilé V $(\Omega = \mathbb{R}^n \setminus V)$ est défini comme suit :

Définition 1.4. *Une région bornée ouverte V avec un bord de classe C^1 est dite presque étoilée s'il existe un voisinage ouvert borné D de $\overline{V}$, une fonction à valeur réelle $\phi \in C^2(\overline{D} \cap \Omega)$ et une constante c_0 telles que :*

- $\phi(x) < c_0$, $x \in D \cap \Omega$, $\phi(x) = c_0$, $x \in \partial D$.
- $|\nabla\phi(x)| \geq const > 0$, $x \in \overline{D} \cap \Omega$.
- *Les surfaces de niveau $\phi(x) = c$ sont fortement convexes ; le rayon de courbure dans toutes les directions à tous les points de $\Omega \cap \overline{D}$ est uniformément majoré.*
- *Aux points d'intersection des surfaces de niveau avec ∂V leurs normales extérieures et la normale extérieure à ∂V forment un angle qui n'est pas supérieur à un angle droit.*

Ces obstacles sont une généralisation naturelle des obstacles étoilés. En fait, si les surfaces de niveau sont des sphères avec un centre commun, alors V est étoilé et inversement.

Ces obstacles correspondent au cas du champ de vecteurs de Strauss et au travail de Morawetz [26], qui sont des travaux effectués en 1975 indépendamment du travail d'Ivrii. En fait, $\nabla\phi$ dans la définition d'Ivrii est un champ de vecteurs de Strauss.

2 L'équation des ondes

On considère dans cette partie l'équation des ondes quintique en dimension 3+1 avec des conditions aux bords de type Dirichlet

$$\begin{aligned}
&\Box u = (\partial_t^2 - \Delta)u = -u^5 \ \text{ dans } \ \mathbb{R} \times \Omega \\
&u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1 \\
&u|_{\mathbb{R}\times\partial\Omega} = 0
\end{aligned} \tag{I.2}$$

où $\Omega \subset \mathbb{R}^3$. Les solutions de cette équation ont une énergie conservée donnée par :

$$E = E(t) = \int_\Omega \frac{|\partial_t u|^2}{2} + \frac{|\nabla u|^2}{2} + \frac{|u|^6}{6} dx. \tag{I.3}$$

et l'invariance d'échelle :

$$u(t,x) \mapsto \frac{1}{\lambda^{1/2}} u(\frac{t}{\lambda}, \frac{x}{\lambda}). \tag{I.4}$$

Ce problème est d'énergie critique car pour des données initiales (u_0, u_1) dans l'espace de l'énergie $\dot{H}^1 \times L^1$, la norme est invariante sous (I.4).

2.1 Existence globale de l'équation des ondes non linéaire critique

L'existence globale de l'équation des ondes non linéaire critique en 3+1 dimension est bien connue. Pour le cas d'espace entier ($\Omega = \mathbb{R}^3$), les premiers résultats ont été obtenus par Grillakis ([16],[17]). Ce dernier a montré qu'il existe des solutions globales régulières de l'équation des ondes critique, si les données sont lisses. Puis Shatah et Struwe ([32],[33]) ont montré l'existence et l'unicité des solutions au problème de Cauchy dans une classe du type

$$u \in C(\mathbb{R}; \dot{H}^1(\mathbb{R}^3)) \cap L^5_{loc}(\mathbb{R}; L^{10}(\mathbb{R}^3)), \ \partial_t u \in C(\mathbb{R}; L^2(\mathbb{R}^3)), \tag{I.5}$$

pour toutes données initiales dans l'espace d'énergie $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Un résultat similaire a été obtenu indépendamment par Kapitanski [22] à peu près en même temps. En outre, Bahouri et Shatah ont fourni dans [3] des informations supplémentaires globales pour les solutions satisfaisant (I.5), c'est-à-dire $u \in L^5(\mathbb{R}; L^{10}(\mathbb{R}^3))$. Ils ont montré cette information globale comme une conséquence de l'estimation de décroissance suivante :

$$\lim_{|t|\to+\infty} \frac{1}{6} \int_\Omega |u(t,x)|^6 dx = 0. \tag{I.6}$$

Pour le cas de domaines, nous sommes intéressés par les domaines Ω extérieurs d'un obstacle non-captant V. Les premiers résultats d'existence globale étaient dus à Smith et Sogge en 1995 ; ils ont étendu dans [34] le théorème de Grillakis sur l'existence globale de solutions dans l'espace entier $\mathbb{R}^3$. Ils ont montré que si Ω est l'extérieur d'un obstacle régulier, compact, et strictement convexe $V \subset \mathbb{R}^3$, alors il existe une solution unique, globale et régulière de l'équation des ondes critique dans $\mathbb{R}_+ \times \Omega$ en supposant que les données initiales $(u_0, u_1) \in C^\infty(\Omega)$. En 2008, Burq, Lebeau et Planchon [9] ont étendu l'existence globale pour tout domaine $\Omega \subset \mathbb{R}^3$; ils ont montré le théorème suivant :

Théorème 2.1. *(Burq, Lebeau, Planchon [9]) Pour toutes $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ il existe une solution unique (globale en temps) u de (I.2) dans l'espace*

$$C^0(\mathbb{R}; H_0^1(\Omega)) \cap C^1(\mathbb{R}; L^2(\Omega)) \cap L^5_{loc}(\mathbb{R}; L^{10}(\mathbb{R}^3)).$$

2.2 L'asymptotique de l'équation des ondes non linéaire critique

L'intérêt principal de cette thèse est l'asymptotique, plus particulièrement le comportement de type scattering de solutions de l'équation non linéaire :

$$\begin{aligned} &\Box u = (\partial_t^2 - \Delta) u = -u^5 \;\; \text{dans} \;\; \mathbb{R} \times \Omega \\ &u|_{\mathbb{R}\times\partial\Omega} = 0. \\ &(\nabla u(t,\cdot), \partial_t u(t,\cdot)) \in L^2(\Omega) \quad t \in \mathbb{R} \end{aligned} \tag{I.7}$$

vers une solution de l'équation homogène :

$$\begin{aligned} &\Box v = 0 \;\; \text{dans} \;\; \mathbb{R} \times \Omega \\ &v|_{\mathbb{R}\times\partial\Omega} = 0. \\ &(\nabla v(t,\cdot), \partial_t v(t,\cdot)) \in L^2(\Omega) \quad t \in \mathbb{R} \end{aligned} \tag{I.8}$$

Rappelons que la fonctionnelle d'énergie est définie par

$$E_0(v;t) = \frac{1}{2} \int_\Omega |\nabla v(t,x)|^2 + |\partial_t v(t,x)|^2 dx$$

et $t \mapsto E_0(v;t)$ est conservée quand v est une solution de l'équation homogène (I.8).

Dans le cas $\Omega = \mathbb{R}^3$, l'estimation de décroissance (I.6) montrée par Bahouri et Shatah

[3] a été utilisée par Bahouri et Gérard dans [2] pour montrer le scattering. De plus, dans [2], Bahouri et Gérard ont utilisé la décomposition de profil pour démontrer qu'il existe une fonction $A : [0, \infty) \to [0, \infty)$ telle que pour chaque solution u de (I.7) satisfaisant (I.5), on a

$$\|u\|_{L^5(\mathbb{R};L^{10}(\mathbb{R}^3))} \leq A(E). \tag{I.9}$$

Remarquons que (I.9) a été dérivée par contradiction, donc Bahouri et Gérard n'ont pas décrit la fonction A. Néanmoins, dans [36] Tao a redémontré (I.9) en donnant une expression explicite de A.

L'asymptotique de solutions de l'équation des ondes critique dans le cas de domaines est plus compliquée que le cas de l'espace entier et moins traité. En effet, le scattering est connu juste pour les domaines extérieurs d'un obstacle étoilé. Ce résultat est dû à Blair, Smith et Sogge, ils ont montré dans [6] la proposition suivante :

Proposition 2.2. *(Blair, Smith, Sogge [6]) Supposons que u résout le problème non-linéaire* (I.7) *et V est un obstacle étoilé par rapport à l'origine. Alors, il existe des solutions uniques $v_\pm$ de* (I.8) *telles que*

$$\lim_{t\to\infty\pm} E_0(u - v_\pm; t) = 0.$$

En outre, u satisfait

$$\|u\|_{L^5(\mathbb{R};L^{10}(\Omega))} + \|u\|_{L^4(\mathbb{R};L^{12}(\Omega))} < \infty.$$

Pour montrer le scattering, Blair, Smith et Sogge ont utilisé la même estimation de décroissance (I.6) montrée par Bahouri et Shatah dans [3] après l'avoir étendue à leur cas d'obstacles en faisant des légères modifications sur la preuve pour traiter le terme à bord. Ils ont utilisé aussi l'estimation de Strichartz (I.10) ci-dessous obtenue dans leur article pour tout obstacle non-captant V sur les fonctions $w(t, x)$ satisfaisant des conditions de Dirichlet homogènes

$$\begin{aligned} &\|w\|_{L^5(\mathbb{R};L^{10}(\Omega))} + \|w\|_{L^4(\mathbb{R};L^{12}(\Omega))} \\ &\qquad \leq c\left(\|(\nabla w(0,\cdot), \partial_t w(0,\cdot))\|_{L^2(\Omega)} + \|\Box w\|_{L^1(\mathbb{R};L^2(\Omega))}\right). \end{aligned} \tag{I.10}$$

2.3 La méthode de multiplicateurs

L'estimation de décroissance (I.6) qui était la clé principale pour montrer le scattering pour le cas d'espace entier $\mathbb{R}^3$ (Bahouri-Shatah [3]) ainsi que pour le cas d'obstacles étoilés (Blair, Smith et Sogge [6]) a été prouvée en utilisant la méthode de multiplicateurs. Cette méthode remonte aux années 1950 (Friedrichs), mais Morawetz était la première à l'utiliser dans le début des années 1960 pour obtenir des résultats de décroissance pour l'équation des ondes linéaire ($\Box u = 0$) dans le cas d'obstacles étoilés ([25] et [27]). L'idée de cette méthode est de multiplier l'équation par un facteur Nu définie comme suit

$$Nu = Au + B \cdot \nabla u + C\partial_t u$$

puis exprimer le produit comme une identité de divergence ou d'énergie de la forme

$$\mathrm{div}_{t,x}(\cdots) + \textit{termes restants} = 0$$

et enfin intégrer cette identité sur un domaine de $\mathbb{R}^{n+1}$ pour obtenir les estimations requises. Le seul cas où le multiplicateur différentiel est adapté à la fois à l'équation des ondes en termes de commutation (éviter les termes restants) et la géométrie de l'obstacle en terme de signe du terme à bord, c'est le cas d'un obstacle étoilé. Par conséquent, Blair, Smith et Sogge ont utilisé dans [6] le multiplicateur de Morawetz

$$u + x \cdot \nabla u + t\partial_t u$$

pour obtenir l'estimation de décroissance pour l'équation non linéaire critique.

Même si cette méthode est parfaitement adaptée uniquement à la géométrie étoilée, des résultats remarquables ont été obtenus dans le cas linéaire en l'utilisant pour une classe plus générale d'obstacles tels que l'illuminé depuis l'intérieur (Bloom et Kazarinoff [7]) et l'illuminé depuis l'extérieur (Bloom et Kazarinoff [8] et Liu [24]). Depuis ces résultats, la théorie linéaire a subi diverses améliorations, notamment par des méthodes microlocales en utilisant des opérateurs pseudo-différentiels (Morawetz, Ralston et Strauss [28]), mais ces méthodes ne sont pas adaptées au cas non linéaire. Donc, c'est le type d'obstacles qui ont été introduits par Bloom, Kazarinoff ([7], [8]) et Liu [24] et qui ont une géométrie précise qui généralise la forme étoilée qu'on trouve le plus susceptible d'être étendu à l'équation non linéaire critique. Cependant, le prix à payer pour la généralisation de la méthode de multiplicateurs pour ces obstacles est qu'on est obligé d'adapter le multiplicateur à la géométrie de l'obstacle pour avoir un bon signe sur le terme à bord, en perdant dans le processus la commutation avec l'opérateur des ondes. Cela signifie qu'on se retrouve avec des intégrales en espace-temps qui doivent être traitées.

2.4 Résultats pour des obstacles illuminés

Un résultat principal de cette thèse est que nous prouvons l'estimation de décroissance L^6 pour des obstacles illuminés depuis l'extérieur. Nous obtenons le théorème suivant :

Théorème 2.3. *Supposons que u résout l'équation des ondes non linéaire (I.7) et V est un obstacle non captant avec un bord régulier qui peut être illuminé depuis l'extérieur par un corps strictement convexe C satisfaisant la condition géométrique suivante :*

$$\min_{\partial V}(s_0 + \rho_1 - 2(\rho_{2M} - \rho_1)) > 0 \tag{I.11}$$

où

- *s_0 est la distance algébrique de ∂C au $x \in \partial V$ le long de la normale extérieure au ∂C.*
- *$\rho_{2M} = \max_{\partial V} \rho_2$ où ρ_i $(i = 1, 2)$ sont les rayons de courbure de ∂C (avec la convention $\rho_2 \geq \rho_1$).*

alors

$$\lim_{t \longrightarrow \infty} \int_{\Omega} |u(t, x)|^6 dx = 0$$

Remarque 2.4. *On peut construire des obstacles qui sont illuminés depuis l'extérieur ou l'intérieur qui satisfont la condition (I.11). En particulier, cette condition est moins restrictive pour des obstacles illuminés depuis l'intérieur, où le corps illuminant est à l'intérieur de l'obstacle et donc $s_0 > 0$. Par exemple, une petite perturbation du cas étoilé, c'est-à-dire un obstacle illuminé par une petite perturbation d'une boule (pas une grande différence entre les rayons de courbures), doit facilement satisfait la condition (I.11).*

Nous obtenons ce décroissance L^6 comme une conséquence de la proposition ci-dessous (Proposition 2.5) et un argument de type Gronwall.

Proposition 2.5. *Supposons que u résout l'équation des ondes non linéaire (I.7) et V est un obstacle non captant avec un bord régulier qui peut être illuminé depuis l'extérieur par un corps strictement convexe C satisfaisant la condition géométrique :*

$$\min_{\partial V}(s_0 + \rho_1 - 2(\rho_{2M} - \rho_1)) > 0$$

alors, on a l'inégalité différentielle suivante :

$$\sqrt{\eta_0}\int_{s+\rho_{2M}\leq \epsilon T+M}\left(|\nabla^* u|^2+\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2\right)dx+\int_{s+\rho_{2M}\leq T+M}\frac{u^6(T,x)}{3}dx$$
$$\leq 2c_1\beta E+\frac{1}{T}\left(C_0E+C_2E\ln(1+T)+2(c_2+c_3T)flux(0,T)\right)$$
$$+\frac{\eta_0}{T}\int_0^T\int_{s+\rho_{2M}\leq \epsilon t+M}\left(|\nabla^* u|^2+\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2\right)dxdt$$

pour un $0<\beta<1$ arbitraire, et où $0<\eta_0,\epsilon<1$ et toutes les constantes dépendent de la géométrie de l'obstacle. Et avec s la distance algébrique de ∂C au $x\in\overline{\Omega}$ le long de la normale extérieure au ∂C, et $\nabla^* u$ est la partie angulaire du gradient ($|\nabla^* u|^2=|\nabla u|^2-|\partial_s u|^2$).

Idée de la preuve : Nous prouvons l'inégalité différentielle dans la proposition 2.5 en utilisant la méthode de multiplicateurs avec le multiplicateur généralisé suivant :

$$u+\alpha\cdot\nabla u+(t+M)\partial_t u$$

où M est une constante positive et α est un champs de vecteurs défini comme suit :

$$\alpha=(s+\rho_{2M})\nu,$$

où ν est la normale unitaire extérieure au bord du corps illuminant.

Nous obtenons une identité de divergence que nous intégrons sur le cône tronqué

$$K_{T_1}^{T_2}=\{(x,t);s+\rho_{2M}\leq t+M,T_1\leq t\leq T_2\},\quad 0<T_1<T_2,$$

et nous utilisons des intégrations par parties, l'inégalité de Hardy et un système de coordonnées adapté à la géométrie de l'obstacle pour obtenir l'inégalité différentielle voulue.

Enfin, comme dans le cas d'un obstacle étoilé, l'estimation de décroissance L^6 ainsi que les estimations de Strichartz globales (I.10) obtenues par Blair, Smith et Sogge, donnent le résultat de scattering :

Corollaire 2.6. *Supposons que u résout l'équation des ondes non linéaire (I.7) et V est un obstacle non captant avec un bord régulier qui peut être illuminé depuis l'extérieur par un corps* *strictement convexe C satisfaisant la condition géométrique :*

$$\min_{\partial V}(s_0 + \rho_1 - 2(\rho_{2M} - \rho_1)) > 0$$

alors il existe des solutions uniques $v_\pm$ pour le problème linéaire homogène

$$\begin{cases} \Box v = 0 \;\; dans \;\; \mathbb{R} \times \Omega \\ v|_{\mathbb{R}\times\partial\Omega = 0} \end{cases}$$

telles que

$$\lim_{t\to\infty\pm} E_0(u - v_\pm; t) = 0.$$

De plus, u satisfait :

$$\|u\|_{L^5(\mathbb{R};L^{10}(\Omega))} + \|u\|_{L^4(\mathbb{R};L^{12}(\Omega))} < \infty.$$

3 L'équation de Schrödinger

On considère dans cette partie l'équation de Schrödinger non linéaire dans un domaine extérieur $\Omega = \mathbb{R}^n \setminus V$, où V est un obstacle non-captant avec un bord régulier, et avec des conditions aux bords de type Dirichlet

$$\begin{aligned} & i\partial_t u + \Delta u = \alpha |u|^{p-1} u \ \text{ dans } \ \Omega = \mathbb{R}^n \setminus V, \ \ p \geq 1 \\ & u|_{\mathbb{R}\times\partial\Omega} = 0 \\ & u(0,x) = u_0(x) \end{aligned} \tag{I.12}$$

avec $\alpha = \{-1, 0, 1\}$. $\alpha = 0$ correspond à l'équation linéaire, $\alpha = 1$ à l'équation défocalisante, et $\alpha = -1$ à l'équation focalisante.
Les solutions à (I.12) sont invariantes par l'invariance d'échelle :

$$u(t,x) \longrightarrow \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x) \tag{I.13}$$

Cette invariance d'échelle définit une notion de criticité, en particulier, pour un espace de Banach donné de donnée initiale u_0, le problème est appelé critique si la norme est invariante par (I.13). Le problème est appelé sous-critique si la norme de la solution rééchelonnée diverge lorsque $\lambda \to \infty$; si la norme réduit à zéro, alors le problème est supercritique. En outre, considérant le problème (I.12) pour $u_0 \in \dot{H}^s(\mathbb{R}^n)$, le problème est critique lorsque

$$s = s_c := \frac{n}{2} - \frac{2}{p-1},$$

sous-critique lorsque $s > s_c$, et supercritique lorsque $s < s_c$.
Maintenant, on note

$$M(u) = \int_\Omega |u|^2 dx \text{ et } E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\alpha}{p+1}\int_\Omega |u|^{p+1} dx, \tag{I.14}$$

la masse et l'énergie qui sont conservées.

3.1 L'obstruction de petites dimensions. L'équation dans des domaines extérieurs en 3D.

Nous sommes intéressés par des domaines extérieurs en 2 dimensions. En fait, les faibles dimensions (inférieures à 3) sont connues pour être les plus délicates, même dans le cas de l'espace entier $\mathbb{R}^n$ en raison de la décroissance faible. Mais, après que le problème de scattering pour le cas sous-critique H^1 ($p < 1+4/(n-2)$) a été réglé par Ginibre et Velo [15] pour $\mathbb{R}^n$ avec $n \geq 3$, l'obstruction de la dimension a été résolue par Nakanishi [29] en utilisant une technique qui n'est pas bien adaptée aux domaines 2D, notre sujet d'intérêt. Cependant, l'introduction des inégalités interactives de Morawetz par Colliander, Keel, Staffilani, Takaoka, et Tao ([12],[13]) a apporté une contribution fondamentale concernant l'existence globale et le scattering dans le cas d'espace entier et aussi une technique qui sera utile dans le cas de domaines. Encore une fois, une obstruction apparaît en petites dimensions et se manifeste dans le mauvais signe du terme bilaplacien qui vient de l'utilisation du poids convexe lié à la distance euclidienne. Cette obstruction a été surmontée simultanément et indépendamment par Colliander, Grillakis et Tzirakis dans [10] ainsi que par Planchon et Vega dans [30].

Dans [30] les auteurs ont également utilisé la technique du multiplicateur bilinéaire pour obtenir des résultats pour les domaines extérieurs en 3D. Ils ont montré dans [30] une estimation de Strichartz $L^4_{t,x}$ et ils l'ont utilisée avec des estimations régularisantes près du bord pour montrer l'existence locale pour la famille d'équations non linéaires (I.12) pour $1 < p < 5$ et $u_0 \in H^1_0(\Omega)$, et l'existence globale pour le cas défocalisant. Ils ont également montré le scattering pour l'équation cubique défocalisante extérieure d'un obstacle étoilé pour donnée initiale dans H^1_0. Un élément crucial dans la preuve de ces résultats a été l'obtention du contrôle suivant au bord :

Proposition 3.1. *(Planchon et Vega, [30]) Soit $\Omega = \mathbb{R}^n \setminus V$, où V est un obstacle étoilé et $V \subset\subset K$, K compact. Supposons que u est une solution de (I.12) avec $\alpha = 0, 1$ (linéaire ou défocalisante) et $n \geq 3$. Alors,*

$$\int_0^T \int_{\partial\Omega} |\partial_n u|^2 dS_x dt + \int_0^T \int_{K\setminus V} (|\nabla u|^2 + |u|^2) dx dt \lesssim \sup_{t\in[0,T]} \|u\|^2_{\dot{H}_0^{\frac{1}{2}}} \lesssim (ME)^{\frac{1}{2}}.$$

Idée de la preuve : Cette proposition a été montrée en utilisant l'identité du viriel :

$$M_h(t) = \int_\Omega |u|^2(x,t) h(x) dx,$$

où h est une fonction régulière quelconque à valeur réelle sur Ω, et en calculant la double dérivée en temps pour obtenir :

$$\begin{aligned}\frac{d^2}{dt^2}M_h(t) = &- \int |u|^2\Delta^2 h + 2\int_{\partial V} |\partial_n u|^2 \partial_n h + 4\int Hessh(\nabla u, \nabla \overline{u}) \\ &+ \frac{2(p-1)}{p+1}\alpha \int |u|^{p+1}\Delta h\end{aligned} \tag{I.15}$$

où n est la normale unitaire extérieure à l'obstacle. Pour obtenir le contrôle désiré, on doit choisir $h = \sqrt{1+|x|^2}$ et se limiter au cas étoilé qui est nécessaire pour assurer un terme au bord avec un bon signe. Une autre restriction est (comme dans le cas d'espace entier) la dimension, qui ne peut pas être moins de 3 sinon le terme bilaplacien dans (I.15) aura un mauvais signe.

D'autres résultats pour les domaines extérieurs en 3D sont les suivants : Dans le cas d'énergie critique $p = 5$, Ivanovici a montré dans [18] que le problème est localement bien posé pour donnée initiale dans H^1 et globalement bien posé pour des petites données, à l'extérieur d'obstacles strictement convexes, en utilisant le paramétrix de Melrose-Taylor pour obtenir des estimations de Strichartz. Ensuite, Ivanovici et Planchon ont étendu ce résultat dans [19] pour tout domaine non-captant dans $\mathbb{R}^3$ en utilisant l'effet régularisant dans $L^5_x(L^2_t)$ pour l'équation linéaire. Leur résultat local en temps est valable aussi pour la condition au bord de type Neumann. Ils ont également étendu le scattering pour l'équation non linéaire défocalisante à l'extérieur d'obstacles étoilés avec donnée initiale dans H^1_0 pour $3 \leq p < 5$. On notera aussi le résultat très récent de Killip, Visan, et Zhang [23] pour l'équation non linéaire quintique (H^1 critique) défocalisante extérieure d'un strict convexe avec la condition au bord de type Dirichlet. Ils ont montré l'existence globale et le scattering pour toute donnée initiale dans l'espace de l'énergie.

3.2 L'équation de Schrödinger dans des domaines extérieurs en 2D

Comme nous l'avons vu dans la section précédente, l'utilisation de la technique de multiplicateur bilinéaire dans le cadre de domaines extérieurs ([30]) impose deux restrictions. La première est l'hypothèse de la géométrie étoilée et la deuxième est la nécessité d'avoir la dimension $n \geq 3$. Cependant, Planchon et Vega ont récemment résolu le problème de restriction de la dimension dans [31] et ils ont obtenu des résultats d'existence globale et du scattering dans des domaines 2D extérieurs à des obstacles étoilés pour le problème non linéaire défocalisant avec donnée initiale dans H^1_0 et pour $p \geq 5$. Ils ont montré le théorème suivant :

Théorème 3.2. *(Planchon et Vega, [31]) Soit $\Omega = \mathbb{R}^2 \setminus V$, où V est un domaine étoilé et* *borné, et* $u_0 \in H_0^1(\Omega)$*. Alors, il existe une solution unique à*

$$\begin{cases} i\partial_t u + \Delta u = |u|^{p-1}u \quad x \in \Omega,\ t \in \mathbb{R},\ p \geq 5 \\ u|_{\mathbb{R}\times\partial\Omega} = 0 \\ u(0,x) = u_0(x), \end{cases}$$

telle que

$$u \in C(\mathbb{R}; H_0^1(\Omega)) \cap L_t^{p-1}L_x^\infty.$$

De plus, il existe $u_+, u_- \in H_0^1(\Omega)$ *uniques telles que*

$$\|u(\cdot,t) - e^{it\Delta_\Omega}u_\pm\| = o(1) \quad t \to \pm\infty.$$

Idée de la preuve : Une idée clé dans la preuve de ce résultat était la technique de produit tensoriel (développée par exemple dans [11] pour obtenir une inégalité de Morawetz d'interaction sur $\mathbb{R}$) en construisant $v(x,y) = u(x)u(y)$ solution de l'équation de Schrödinger dans $\Omega \times \Omega$, puis en utilisant le contrôle au bord obtenu par les multiplicateurs de Morawetz en dimension $n = 4$ ce qui résout le problème du mauvais signe du bilaplacien en dimension 2. Plus précisément, ils ont montré pour u solution du problème linéaire ou défocalisant extérieur d'un étoilé le contrôle suivant

$$\int_{\mathbb{R}}\int_{\partial\Omega\times\Omega} |\partial_n u(x)|^2|u(y)|^2 \frac{dS_x dy}{\sqrt{1+|y|^2}} dt \leq CM^{3/2}E^{1/2} \tag{I.16}$$

en faisant le même calcul que dans la preuve de la Proposition 3.1 mais avec

$$M_h(t) = \int_{\Omega\times\Omega} |v|^2(x,y,t)h(x,y)dxdy,$$

et $h(x,y) = \sqrt{|x|^2+|y|^2}$. Ensuite, ils ont utilisé une autre idée clé associée au théorème suivant :

Théorème 3.3. *(Planchon and Vega, [31]) Soit* $\omega \in \mathbb{R}^n$*,* $n > 1$*, avec* $|\omega| = 1$ *et* $\rho_\omega(z) = |z \cdot \omega|$*,* u_j *solution de*

$$i\partial_t u_j + \Delta u_j = \alpha|u_j|^{p-1}u_j \qquad u_j|_{\partial\Omega_j} = 0$$

avec $\Omega_j \subset \mathbb{R}^n$*,* $p > 1$*,* $\alpha \in \{-1,0,1\}$ *et* $j = 1,2$*. Alors, si* $x = x^\perp + s\omega$ *et* $x^\perp \cdot \omega = 0$*, on a pour*

$$I_\rho(t) = \int_{\Omega_1\times\Omega_2} \rho(x-y)|u_1(x)|^2|u_2(y)|^2 dxdy$$

$$\begin{aligned}\frac{d^2}{dt^2} I_{\rho_\omega} &= \int_s |\partial_s(R(u_1\overline{u_2}))|^2 ds \\ &+ \alpha\frac{p-1}{p+1}\Big(\int_s R(|u_1|^2)R(|u_2|^{p+1})ds + \int_s R(|u_2|^2)R(|u_1|^{p+1})ds\Big) \\ &+ \int_s\int_{x\cdot\omega=s}\int_{y\cdot\omega=s}\Big|u_1(x^\perp+s\omega)\partial_s u_2(y^\perp+s\omega) \\ &- u_2(y^\perp+s\omega)\partial_s u_1(x^\perp+s\omega)\Big|^2 dx^\perp dy^\perp ds \\ &- \int_{\partial\Omega_1\times\Omega_2}|u_2|^2(y)\partial_n\rho_\omega(x-y)|\partial_n u_1|^2(x)dS_x dy \\ &- \int_{\Omega_1\times\partial\Omega_2}|u_1|^2(x)\partial_n\rho_\omega(x-y)|\partial_n u_2|^2(y)dx dS_y\end{aligned}$$

où ∂_n est la normale extérieure à $\partial\Omega$ et $R(f)(s,w) = \int_{\{x\cdot\omega=s\}\cap\Omega} f d\mu_{s,\omega}$ et la transformée de Radon.

La nouvelle idée est que les auteurs ont utilisé Théorème 3.3 avec $\Omega = \Omega_1 = -\Omega_2$, $u_1 = u$ et $u_2(x) = u(-x)$, puis à nouveau avec $\Omega = \Omega_1 = \Omega_2$ et $u_1 = u_2 = u$, ce qui permet certaines éliminations sur addition (il n'a été utilisé que pour $\Omega = \Omega_1 = \Omega_2$ et $u_1 = u_2 = u$ dans le cas de 3D : Théorème 2.5 dans [30]). Ceci avec le contrôle au bord (I.16) donne

$$\|D^{1/2}(|u|^2)\|_{L^2_{t,x}} \le CM^{3/4}E^{1/4} \tag{I.17}$$

où $D^s = (-\Delta)^{s/2}$, pour la solution de l'équation linéaire ainsi que pour celle de l'équation non linéaire. Cette estimation permet d'obtenir l'estimation de Strichartz $L_t^{p-1}L_x^\infty$ globale en temps (pour les obstacles étoilés) qui est l'ingrédient clé pour montrer l'existence locale. Elle implique également le contrôle de la solution dans la norme $L_t^4 L_x^8$, ce qui est utile pour démontrer l'existence globale et le scattering.

Remarque 3.4. *Les auteurs de [31] ont opté pour l'utilisation de Théorème 3.3 et le poids directionnel qui est plus naturel dans un sens géométrique ; ils moyennent ensuite sur toutes les directions. Cependant, leur calcul peut être fait en utilisant directement le poids "moyenné" $\rho(x,y) = |x-y| + |x+y|$: on remarque que pour $x \in \partial\Omega$ et y grand,*

$$\nabla_x\rho(x,y) \sim \frac{2x}{|y|},$$

ce qui explique le poids présent dans (I.16) et la condition d'étoilement. Cette approche non directionnelle sera plus appropriée dans notre contexte, où l'on effectue la technique de produit tensoriel plus d'une fois (voir la section suivante).

3.3 Résultats pour des domaines extérieurs en 2D

Le résultat de Planchon et Vega pour les non-linéarités avec $p \geq 5$ (Théorème 3.2) est donné pour les obstacles étoilés, même pour l'existence. Cela est dû au fait que l'estimation de Strichartz $L_t^{p-1}L_x^\infty$ a été obtenue en utilisant le mécanisme du multiplicateur bilinéaire qui a nécessité l'hypothèse de la géométrie étoilée. Dans ce travail, nous étendons le résultat de l'existence pour tout obstacle non-captant et pour $p > 4$ en utilisant les estimations de Strichartz obtenues par Blair, Smith et Sogge dans [5] :

Théorème 3.5. *(Blair, Smith, and Sogge, [5]) Soit $\Omega = \mathbb{R}^n \setminus V$ un domaine extérieur d'un obstacle non-captant compact avec un bord régulier, et Δ l'opérateur Laplacien standard sur Ω, soumis à : soit des conditions au bord de type Dirichlet soit de type Neumann. Supposons que $p > 2$ et $q < \infty$ satisfont*

$$\begin{cases} \dfrac{3}{p} + \dfrac{2}{q} \leq 1, & n = 2, \\ \dfrac{1}{p} + \dfrac{1}{q} \leq \dfrac{1}{2}, & n \geq 3. \end{cases}$$

Alors, pour $e^{it\Delta} f$ solution de l'équation de Schrödinger linéaire avec donnée initiale f, on a les estimations suivantes

$$\|e^{it\Delta} f\|_{L^p([0,T];L^q(\Omega))} \leq C\|f\|_{H^s(\Omega)},$$

à condition que

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2} - s.$$

Pour des conditions au bord de type Dirichlet, les estimations sont vraies avec $T = \infty$.

Plus précisément, nous considérons le problème suivant :

$$\begin{aligned} &i\partial_t u + \Delta u = \pm|u|^{p-1}u \ \text{ dans } \ \Omega = \mathbb{R}^2 \setminus V, \ \ p > 4 \\ &u|_{\mathbb{R}\times\partial\Omega} = 0 \\ &u(0,x) = u_0(x) \in H_0^1(\Omega) \end{aligned} \tag{I.18}$$

et on met $p = \frac{3}{1-\epsilon_0} + 1$ avec $0 < \epsilon_0 < 1$.

En utilisant l'estimation de Blair, Smith et Sogge (Théorème 3.5) en 2D, on obtient une autre estimation linéaire dans l'espace de Besov $\dot{B}_2^{s_c,1}$ donnée par la proposition suivante.

Proposition 3.6. *Soit $\Omega = \mathbb{R}^2 \setminus V$, où V est un obstacle non-captant avec un bord régulier.*

Pour $e^{it\Delta}f$ solution de l'équation de Schrödinger linéaire avec donnée initiale f, on a

$$\|e^{it\Delta}f\|_{L^{\frac{3}{1-\epsilon_0}}([0,+\infty);L^\infty(\Omega))} \lesssim \|f\|_{\dot{B}_2^{s_c,1}(\Omega)}$$

Remarque 3.7. *Les espaces de Besov sont définis ici en utilisant la localisation spectrale associée au domaine (voir [20]).*

La Proposition 3.6 nous permet de mettre en place l'argument de point fixe et de résoudre localement en temps dans l'espace fonctionnel E_T donné, pour $T > 0$, par :

$$E_T = C([0,T]; \dot{B}_2^{s_c,1}(\Omega)) \cap L^{\frac{3}{1-\epsilon_0}}([0,T]; L^\infty(\Omega)).$$

L'argument de point fixe nous dit aussi que si la donnée initiale est dans H^1 alors le temps de vie est comme une puissance inverse de cette norme qui est contrôlée pour le cas défocalisant. Cela nous permet de prolonger au delà de l'intervalle d'existence locale. On obtient alors le théorème suivant :

Théorème 3.8. *Soit $\Omega = \mathbb{R}^2 \setminus V$, où V est un obstacle non-captant, et $u_0 \in \dot{B}_2^{s_c,1}$. Alors, il existe $T(u_0)$ telle que l'équation non linéaire :*

$$\begin{cases} i\partial_t u + \Delta u = \pm|u|^{p-1}u \quad x \in \Omega,\ t \in \mathbb{R},\ p > 4 \\ u|_{\mathbb{R}\times\partial\Omega} = 0 \\ u(0,x) = u_0(x), \end{cases}$$

admet une solution unique u dans l'espace

$$C([0,T]; \dot{B}_2^{s_c,1}(\Omega)) \cap L^{p-1}([0,T]; L^\infty(\Omega)).$$

De plus, si $u_0 \in H_0^1(\Omega)$, alors la solution reste dans $H_0^1(\Omega)$ et elle est globale en temps pour l'équation défocalisante.

On montre ensuite le scattering pour la solution de l'équation défocalisante avec donnée initiale dans $H_0^1(\Omega)$ pour le cas d'obstacles étoilés en utilisant les lois de conservation de la masse et de l'énergie ainsi que le contrôle en $L_t^4 L_x^8$ obtenu par Planchon et Vega comme une conséquence de (I.17). Nous montrons aussi le scattering pour une classe d'obstacles presque étoilés qui satisfont la condition géométrique suivante : $0 < \epsilon < 1$ étant donné

$$(x_1, \epsilon x_2) \cdot n_x > 0 \ \text{ pour } \ x = (x_1, x_2) \in \partial V \tag{I.19}$$

où n_x est la normale unitaire extérieure à ∂V.
Selon la définition d'Ivrii (Définition 1.4), un obstacle presque étoilé avec des ellipses comme surfaces de niveau satisfait la condition (I.19), où l'inégalité stricte correspond à un angle strictement inférieur à un angle droit dans la 4ème condition de la Définition 1.4. Plus explicitement, la fonction ϕ est donnée par $\phi(x) = \sqrt{x_1^2 + \epsilon x_2^2}$ et ça correspond à la fonction jauge du corps convexe délimité par l'ellipse donnée par l'équation $x_1^2 + \epsilon x_2^2 = c^2$.

Remarque 3.9. *Nous avons choisi de travailler avec des obstacles presque étoilés et d'utiliser la fonction jauge de l'ellipse plutôt qu'avec des obstacles illuminés (par une ellipse) qui impose l'utilisation de la distance au corps illuminant. La raison principale est que le calcul est beaucoup plus facile avec la fonction jauge. Sur l'exemple de l'obstacle en forme d'os de chien, la condition géométrique* (I.19) *est plus faible que la condition d'illumination* (I.1)*, comme on peut s'en convaincre sur la Figure I.2, en dessinant les surfaces de niveau respectivement associées aux dilatations de l'ellipse ou à la distance euclidienne à celle-là.*

Pour obtenir le scattering pour tels obstacles, nous montrons que la solution u est contrôlée dans une certaine norme de type $L_t^a L_x^b$ par une constante dépendant de la masse et de l'énergie. Cela va jouer le rôle du contrôle $L_t^4 L_x^8$ dans le cas étoilé, et il est une conséquence de la proposition suivante :

Proposition 3.10. *Soit $\Omega = \mathbb{R}^2 \setminus V$, avec V un obstacle satisfaisant la condition* (I.19)*. Supposons u est une solution de*

$$\begin{aligned} &i\partial_t u + \Delta u = \alpha |u|^{p-1} u \quad \text{dans } \Omega, \quad p > 1 \\ &u|_{\mathbb{R}\times\partial\Omega} = 0, \end{aligned}$$

avec $\alpha = \{0, 1\}$. Alors on a

$$\|D^{-1/2}(|v|^2)\|_{L^2_{t,X}} \lesssim M^{7/4} E^{1/4}$$

où M et E sont la masse et l'énergie de u et $v(X) = v(x,y) = u(x)u(y)$ est la solution de

$$\begin{aligned} &i\partial_t v + \Delta v = \alpha(|u|^{p-1}(x) + |u|^{p-1}(y))v \quad \text{dans } \Omega \times \Omega \\ &v|_{\partial(\Omega\times\Omega)} = 0. \end{aligned}$$

Idée de la preuve : Cette proposition est obtenue en utilisant la technique de multiplicateur bilinéaire et en effectuant le produit tensoriel deux fois. D'abord, nous avons obtenu des estimations de contrôle au bord par le calcul de la double dérivée en temps de quantités de la forme

$$M_\rho(t) = \int_{\Omega\times\Omega\times\Omega\times\Omega} |U|^2(x,y,z,w,t)\rho(x,y,z,w)dxdydzdw$$

où ρ est une fonction réelle convexe en 8D choisie de manière adaptée à la condition géométrique (I.19) et $U(x,y,z,w) = v(x,y)v(z,w) = u(x)u(y)u(z)u(w)$ est la solution du problème suivant en 8D :

$$\begin{aligned}
&i\partial_t U + \Delta U = \alpha N(u)U \quad \text{in } \Omega\times\Omega\times\Omega\times\Omega \\
&U|_{\partial(\Omega\times\Omega\times\Omega\times\Omega)} = 0,
\end{aligned}$$

avec

$$N(u) = |u|^{p-1}(x) + |u|^{p-1}(y) + |u|^{p-1}(z) + |u|^{p-1}(w).$$

Faire la technique de produit tensoriel deux fois (et donc augmenter la dimension à 8) est nécessaire pour assurer que les termes bilaplaciens générés ont le bon signe pour tout $0 < \epsilon < 1$ donné. Une des estimations de contrôle qu'on obtient au bord et qui sert comme une alternative de (I.16) (du cas étoilé) est

$$\int\int_{\partial\Omega\times\Omega\times\Omega\times\Omega} \frac{|\partial_{n_x}u(x)|^2}{\rho_1(x,y,z,w)}|u(y)|^2|u(z)|^2|u(w)|^2 d\sigma_x dydzdwdt \lesssim M^{7/2}E^{1/2}$$

où

$$\rho_1 = \sqrt{x_1^2 + y_1^2 + z_1^2 + w_1^2 + \epsilon(x_2^2 + y_2^2 + z_2^2 + w_2^2)}.$$

Une autre estimation au bord qu'on obtient en utilisant le poids

$$\begin{aligned}
\rho_2 = &\sqrt{x_1^2 + \epsilon x_2^2 + z_1^2 + \epsilon z_2^2 + \left(\frac{y_1 - w_1}{\sqrt{2}}\right)^2 + \epsilon\left(\frac{y_2 - w_2}{\sqrt{2}}\right)^2} \\
&+ \sqrt{x_1^2 + \epsilon x_2^2 + z_1^2 + \epsilon z_2^2 + \left(\frac{y_1 + w_1}{\sqrt{2}}\right)^2 + \epsilon\left(\frac{y_2 + w_2}{\sqrt{2}}\right)^2}
\end{aligned}$$

est :

$$\int\int_{\partial\Omega\times\Omega\times\Omega\times\Omega} \frac{|\partial_{n_x}u(x)|^2|u(y)|^2|u(z)|^2|u(w)|^2}{\sqrt{|x|^2 + |z|^2 + |y \pm w|^2}} d\sigma_x dydzdwdt \lesssim M^{7/2}E^{1/2}. \tag{I.20}$$

Après avoir obtenu les contrôles nécessaires au bord, nous faisons à nouveau le même calcul sur

M_ρ avec le poids

$$\begin{aligned}\rho(x,y,z,w) =& \sqrt{|x-z|^2+|y-w|^2}+\sqrt{|x+z|^2+|y-w|^2}\\ &+\sqrt{|x-z|^2+|y+w|^2}+\sqrt{|x+z|^2+|y+w|^2}\end{aligned}$$

pour obtenir l'estimation dans Proposition 3.10 : par exemple, si $x \in \partial\Omega$ et y, z, w sont grands,

$$|\nabla_x \rho| \lesssim \frac{1}{\lambda_-} + \frac{1}{\lambda_+},$$

avec $\lambda_\pm^2 = |x|^2 + |z|^2 + |y \pm w|^2$, ce qui explique le poids présent dans (I.20).

Enfin, nous obtenons le théorème suivant :

Théorème 3.11. *Soit $\Omega = \mathbb{R}^2 \setminus V$, où V est un obstacle étoilé ou presque étoilé satisfaisant la condition (I.19), et $u_0 \in H^1_0(\Omega)$. Alors, la solution globale de l'équation défocalisante :*

$$\begin{cases} i\partial_t u + \Delta u = |u|^{p-1}u \quad x \in \Omega,\ t \in \mathbb{R},\ p > 4 \\ u|_{\mathbb{R}\times\partial\Omega} = 0 \\ u(0,x) = u_0(x), \end{cases}$$

a un comportement de type scattering dans H^1_0.

4 Conclusions et Perspectives

Dans cette thèse, nous avons obtenu des nouveaux résultats d'asymptotique concernant l'équation des ondes critique 3D et de l'équation de Schrödinger 2D. La principale nouveauté est que nous avons surmonté la restriction au cas étoilé et nous avons montré des résultats de scattering pour des obstacles non-étoilés qui avaient eté étudiés auparavant dans le cadre de la décroissance locale de l'énergie pour l'équation des ondes linéaire. Voici quelques problèmes ouverts :

- Il serait bon d'étendre l'argument de l'équation des ondes à tout obstacle pour lequel un champ de vecteurs de Strauss existe, plutôt que juste un cas perturbatif de la forme étoilée. En outre, pour l'équation des ondes critique dans des dimensions supérieures, on peut effectuer un argument direct (similaire à celui pour Schrödinger) avec la dérivée en temps du moment cinétique (c'est-à-dire en évitant l'intégration sur le cône, qui induit naturellement la condition d'illumination) et un poids similaire à celui du cas presque étoilé avec des ellipsoïdes pour obtenir un contrôle du terme à bord (et puis la décroissance de la norme $L^{2n/(n-2)}$ par l'argument classique de $\mathbb{R}^n$), mais ce qui manque là est d'avoir ensuite des bonnes estimations de Strichartz pour utiliser cette décroissance.
- Symétriquement, il serait bon d'étendre le cas de Schrödinger à tout obstacle non-captant en 2D. Le fait que le bilaplacien joue un grand rôle pour Schrödinger, mais peu ou pas de rôle pour les ondes est un peu déroutant.
- Il serait bien de faire des décompositions en profils sur des intervalles de temps arbitrairement grands pour l'équation des ondes (par opposition au cadre d'un intervalle fixe de Gallagher et Gérard [14]). Cependant, il existe de graves obstructions à surmonter afin de pouvoir traiter quelque chose d'autre qu'un strictement convexe. Par exemple, dans le travail récent de Killip, Visan et Zhang [23], les auteurs traitent l'équation de Schrödinger sur des grands intervalles de temps, mais encore à l'extérieur d'un convexe.
- Il serait bien d'obtenir des résultats pour des conditions aux bords autres que Dirichlet. Actuellement Neumann semble inaccessible, mais il y a des travaux récents pour des données non homogènes aux bords, par exemple, $u = g$, où g est une fonction spécifiée du temps et de l'espace (voir [1]).
- Il serait également intéressant de traiter des situations où le bord 2D n'est pas lisse, comme le suggère [4], parce que l'essence des arguments de produit tensoriel est juste intégration par parties (par exemple, formule de Stokes), et donc il ne nécessite que le bord soit C^1 par morceaux.

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Chapitre II

Asymptotics of the critical non-linear wave equation for a class of non star-shaped obstacles

1 Introduction

In this paper we are working on the energy critical nonlinear wave equation in 3+1 dimension in a domain $\Omega = \mathbb{R}^3 \setminus V$ where V is a non-trapping obstacle with smooth boundary

$$\begin{aligned} &\Box u = (\partial_t^2 - \Delta)u = -u^5 \;\; in \;\; \mathbb{R} \times \Omega \\ &u|_{\mathbb{R}\times\partial\Omega} = 0 \\ &(\nabla u(t,\cdot), \partial_t u(t,\cdot)) \in L^2(\Omega) \quad t \in \mathbb{R} \end{aligned} \tag{II.1}$$

which enjoys the conservation of energy

$$E = E(t) = \int_\Omega \frac{|\partial_t u|^2}{2} + \frac{|\nabla u|^2}{2} + \frac{|u|^6}{6} dx.$$

In the boundaryless case ($\Omega = \mathbb{R}^3$), the first results for the global existence were obtained by Grillakis ([8], [9]). He showed that there are global smooth solutions of the critical wave equation, if the data is smooth. Shatah and Struwe ([17], [18]) extended this theorem by showing that there are global solutions for the data lying in the energy space $H^1 \times L^2$. They also obtained results for critical wave equation in higher dimensions.
For the case of obstacles, the first results were due to Smith and Sogge [19]. They showed that Grillakis theorem extends to the case where Ω is the complement of a smooth, compact, strictly convex obstacle with Dirichlet boundary conditions. This result was later extended to the case of arbitrary domains in $\mathbb{R}^3$ and data in the energy space by Burq, Lebeau and Planchon [6]. The case of the nonlinear critical Neumann wave equation in 3-dimensions was subsequently handled by Burq and Planchon [7].

More specifically, in this paper we are interested in asymptotics, i.e. how solutions to the nonlinear equation scatter to a solution to the homogeneous linear equation. In the boundaryless case ($\Omega = \mathbb{R}^3$), first results were obtained by Bahouri-Gérard in [1] ; in their paper they used the following decay estimate proved by Bahouri-Shatah [2]

$$\lim_{|t|\to+\infty} \frac{1}{6} \int_\Omega |u(t,x)|^6 dx = 0.$$

to get

$$\|u\|_{L^5(\mathbb{R};L^{10}(\Omega))} + \|u\|_{L^4(\mathbb{R};L^{12}(\Omega))} < \infty.$$

and thus scattering. Moreover, in [1] Bahouri-Gérard used profile decomposition to show that $\|u\|_{L^5(\mathbb{R};L^{10}(\Omega))}$ is also controlled by a universal function of the energy $f(E)$. Then the scattering result was extended to the case of star-shaped obstacles ($x \cdot n \geq 0$ for $x \in \partial V$ with n the outward pointing unit normal vector to ∂V) by Blair, Smith, and Sogge in [3]. They used the same L^6 decay estimate proved by Bahouri-Shatah in [2] after they extended it to their case of obstacles making slight modifications on the proof to handle the boundary term.

In the papers by Bahouri-Shatah [2] and Blair, Smith, and Sogge [3], the L^6 decay estimate which is the main key to prove scattering was proved using the method of multipliers. The method of multipliers is also called Friedrichs' ABC method as it dates back to Kurt O. Friedrichs in the 1950's. The idea of this method is to multiply the equation with a factor Nu, with N is a linear first-order differential operator, defined as

$$Nu = Au + B \cdot \nabla u + C\partial_t u$$

and then to express the product as a divergence or energy identity of the form

$$\mathrm{div}_{t,x}(\cdots) + \textit{Remaining terms} = 0$$

and finally to integrate this divergence identity over a domain in $\mathbb{R}^{n+1}$ and subsequently derive the required estimates. The only case where the differential multiplier is adapted to both the wave equation in terms of commutation (avoiding remaining terms) and the geometry of the obstacle in terms of the sign of the boundary term, is the star-shaped case. The method of multipliers was used in the 1960's and 1970's to prove uniform decay results for the homogeneous linear wave equation ($\Box u = 0$) outside obstacles. Cathleen S. Morawetz was the first to succeed in proving uniform local energy decay for star-shaped obstacles with Dirichlet boundary condition using this method ([13] and [15]). Since then, the results of Morawetz have been considerably improved. Better decay rates have been achieved (as in odd dimensions $n \geq 3$, Huygen's principle has been shown to imply an exponential rate of decay whenever there is some sort of decay [11], [14]). Moreover, the class of obstacles under consideration has been enlarged ; decay results have been derived for a special case of non trapping obstacles referred to as "almost star-shaped regions"(Ivrii [10]) and for non trapping obstacles with simple and direct geometrical generalizations to the star-shaped such as the "illuminated from interior"(Bloom and Kazarinoff [4]) and the "illuminated from exterior"(Bloom and Kazarinoff [5], Liu [12]). For these cases, decay results have been proved using the method of multipliers after generalizing the multipliers to suit the geometry of the obstacle, and although these generalized multipliers lead to volume integrals that were avoided before, it turned out that these integrals were actually useful in the estimates. Other wider generalizations that include all the above geometries later followed, Strauss [20] proved uniform local energy decay for the homogeneous linear wave equation in exterior domains in $\mathbb{R}^n$ $n \geq 3$, provided a strictly expansive vector field (now called the Straussian vector field) exists, that leaves $\overline{\Omega}$ strictly invariant, then these Straussian vector fields were generalized by Morawetz, Ralston, and Strauss [16] by introducing escape functions, using them to construct a pseudo-differential operator $P(x, D)$, and finally setting Pu as a multiplier.

Though the microlocal methods of Morawetz, Ralston, and Strauss provided general results for the linear case, they cannot be easily extended to the nonlinear wave equation for which the star-shapedness has been so far a restriction to obtain decay results. Thus, we consider in this paper obstacles with explicit geometry that is a direct generalization of the star-shapedness, and we prove the L^6 decay estimate and thus the scattering for such obstacles using Friedrichs' ABC method after generalizing the multipliers to suit our case. The price to pay for such a generalization is that the our multiplier, which is adapted to the geometry of the obstacle, no longer has exact commutation properties with the wave operator. Therefore, unlike the star-shaped case we do get volume integrals that we deal with using a Gronwall-like argument.

Our main results are :

Theorem 1.1. *Suppose u solves the nonlinear wave equation (II.1) and that V is a non-trapping obstacle with regular boundary that can be illuminated from its exterior by a strictly convex body C satisfying the geometric condition :*

$$\min_{\partial V}(s_0 + \rho_1 - 2(\rho_{2M} - \rho_1)) > 0 \tag{II.2}$$

where

- *s_0 is the algebraic distance from ∂C to $x \in \partial V$ along the exterior normal to ∂C.*
- *$\rho_{2M} = \max\limits_{(\sigma_1, \sigma_1)} \rho_2$ where ρ_i $(i = 1, 2)$ are the radii of curvature of ∂C $(\rho_2 \geq \rho_1)$.*

then

$$\lim_{t \longrightarrow \infty} \int_\Omega |u(t,x)|^6 dx = 0$$

Remark 1.2. *We can construct obstacles that are illuminated from the exterior or from the interior that satisfy the condition (II.2). In particular, it would be easy to see this for illuminated from interior obstacles, where the illuminating body is inside the obstacle and thus $s_0 > 0$, by considering a dog bone like obstacle (Figure II.1) that is a small perturbation of the star-shaped.*

Remark 1.3. *Remark that if the data has compact support, the computation that proves the above result provides an explicit decay rate for the local energy. In particular, it recovers Bloom-Kazarinoff for the linear equation ([4]) without using the fact that $\Box \partial_t u = 0$.*

As a result of this decay estimate, we get scattering

Corollary 1.4. *Suppose u solves the nonlinear wave equation (II.1) and that V is a non-trapping obstacle with regular boundary that can be illuminated from its exterior by a strictly convex body satisfying the geometric condition :*

$$\min_{\partial V}(s_0 + \rho_1 - 2(\rho_{2M} - \rho_1)) > 0$$

then there exists unique solutions $v_\pm$ to the homogeneous linear problem

$$\begin{cases} \Box v = 0 \ \ in \ \ \mathbb{R} \times \Omega \\ v|_{\mathbb{R}\times\partial\Omega = 0} \end{cases} \tag{II.3}$$

such that

$$\lim_{t\to\infty\pm} E_0(u - v_\pm; t) = 0. \tag{II.4}$$

Moreover, u satisfies :

$$\|u\|_{L^5(\mathbb{R};L^{10}(\Omega))} + \|u\|_{L^4(\mathbb{R};L^{12}(\Omega))} < \infty. \tag{II.5}$$

2 The geometry of the obstacle

We consider in this paper illuminated from exterior obstacles, defined as such (Liu [12]) :

Definition 2.1. *We say that the boundary of an exterior domain $\Omega = \mathbb{R}^3 \setminus V$ (or the obstacle V) can be illuminated from the exterior if and only if there exists a convex body C containing ∂V with smooth boundary ∂C such that ∂V is filled by a family of non-intersecting rays normal to ∂C. Each ray is completely contained in Ω in the following sense : for each $x_0 \in \partial V$ there exists a unique $x_1 \in \partial C$ and a number $s_0(x_1) \leq 0$ such that*

$$x_0 = s_0(x_1)\nu(x_1) + x_1,$$

where ν is the outward unit normal to ∂C at x_1, and

$$x = t\nu(x_1) + x_1 \in \Omega, \qquad t \in [s_0, \infty).$$

This definition actually generalizes the following definition of illuminated from interior obstacles introduced by Bloom and Kazarinoff [4] :

Definition 2.2. *We say that a body V can be illuminated from the interior if and only if there exists a smooth convex body C inside V such that $extC$ is filled by a family of non-intersecting rays normal to ∂C and such that each ray intersects ∂V exactly once.*

In fact, every body that can be illuminated from the interior can also be illuminated from its exterior by enlarging the original convex body (Liu [12]) ; thus our results proven for illuminated from exterior obstacles also hold for illuminated from interior. Furthermore, as we mentioned in the introduction, these geometries are direct generalizations of the star-shaped. More precisely, the illuminated from interior is a generalization of the strict star-shapedness ($x \cdot n > 0$). The condition $x \cdot n > 0$ implies that each ray beginning at the origin intersects ∂V exactly once, which means that the interior of a strictly star-shaped obstacle can be illuminated by a source of light situated at the origin. A small ball centered at the origin is contained in the interior of V and the above light rays are perpendicular to the surface of this ball, hence our strictly star-shaped is illuminated from its interior by this ball.
An example of a non star-shaped body that can be illuminated from interior is a “dog bone”(Figure II.1), and a “snake-shaped”body (Figure II.2) is an example of an obstacle that cannot be illuminated from its interior but can be illuminated from the exterior.

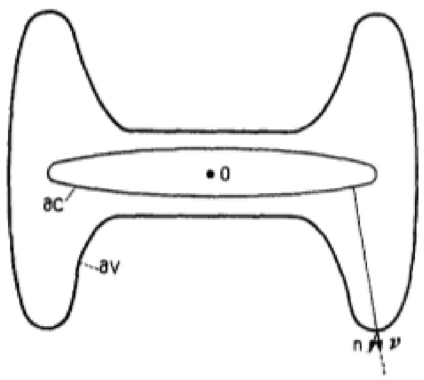

Figure II.1 – dog bone

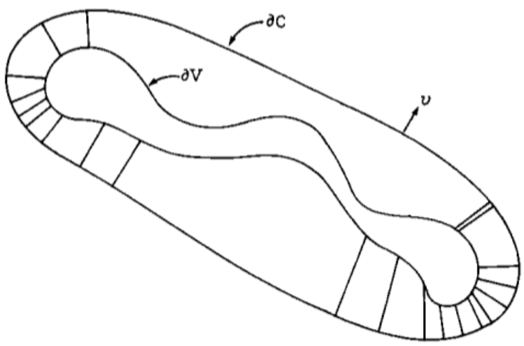

Figure II.2 – snake

2.1 The illuminating coordinate system

In this section we will introduce the coordinate system that we are going to use which is the one in the paper by Liu [12] in which he was dealing with similar obstacle for the linear equation. We again denote by ∂C the smooth and convex surface of the illuminating body. Let $x = (x_1, x_2, x_3)$ be Cartesian coordinates in $\mathbb{R}^3$ with the origin inside C and V. If X^0 is on ∂C, then in a neighborhood of X^0 we choose the parametric curves to be the two principal curves on ∂C. If the neighborhood of X^0 is an all-umbilic surface, then we still can choose the parametric curves to be orthogonal to each other. Furthermore, we let the parameters be the arc-length parameters. Thus, if $X^0 \in \partial C$, then X^0 is given in local coordinates by

$$X^0 = X^0(\sigma_1, \sigma_2) = \left(X^{01}(\sigma_1, \sigma_2), X^{02}(\sigma_1, \sigma_2), X^{03}(\sigma_1, \sigma_2)\right),$$

where $\sigma_1 = const.$ and $\sigma_2 = const.$ are the parameterizations of the arc-length of the principal curves near X^0. A finite number of (σ_1, σ_2) coordinate patches cover ∂C. Next, corresponding to each point $X^0(\sigma_1, \sigma_2)$ on ∂C, we make the choice $x = X(s, \sigma_1, \sigma_2)$, where

$$x = X(s, \sigma_1, \sigma_2) = s\nu(\sigma_1, \sigma_2) + X^0(\sigma_1, \sigma_2) = sX_s + X^0(\sigma_1, \sigma_2), \tag{II.6}$$

where

$$\nu(\sigma_1, \sigma_2) = \left(\frac{X^0_{\sigma_1}}{|X^0_{\sigma_1}|} \times \frac{X^0_{\sigma_2}}{|X^0_{\sigma_2}|}\right)$$

is the unit exterior normal to ∂C, with $X^0_{\sigma_i} \equiv \partial X^0/\partial\sigma_i,\ i = 1, 2$. By Definition 2.1, for each $X^1 \in \partial V$, there is a unique triple $(s_0, \sigma_1, \sigma_2)$ with $s_0 \leq 0$ such that

$$X^1 = s_0\nu(\sigma_1, \sigma_2) + X^0(\sigma_1, \sigma_2).$$

We denote by κ_1 and κ_2 the principal curvatures at $X^0(\sigma_1, \sigma_2)$ and $\rho_i = \frac{1}{\kappa_i}$ $(i = 1, 2)$ the principal radii of curvature of ∂C. We assume $0 < \kappa_2 \leq \kappa_1$ $(\rho_2 \geq \rho_1 > 0)$. Furthermore, we always assume that

$$\min_{\partial V}(s_0 + \rho_1) > 0 \tag{II.7}$$

This condition implies that for every $x \in \Omega$, we have $s + \rho_i > 0$ $(i = 1, 2)$ since $x = s\nu(\sigma_1, \sigma_2) + X^0(\sigma_1, \sigma_2)$ with $s_0 \leq s < \infty$, where s_0 corresponds to the point on ∂V associated with X^0.

Remark 2.3. *Generically, Definition 2.1 implies the condition* (II.7) *([5] page 26 Lemma 2.1, [12] page 316 Remark after Lemma 1). Moreover, note that for a star-shaped obstacle, which is in fact an obstacle that is illuminated from the exterior by some ball $B(0, R_0)$, $s + \rho_i$ is nothing but $r = |x|$. This explains the significance of this value and makes the need of such an assumption in a computation that is a generalization of the star-shaped totally logical.*

Now, we state the following geometrical lemma that we will use later :

Lemma 2.4. *There exist a constant $a_0 > 0$ such that $s + \rho_{2M} \geq a_0 r$.*

Proof. The existence of a_0 is due to the boundedness of the the obstacle V and the illuminating body C. In fact, if $s \geq 0$ ($x \in \overline{extC}$) then $r - r_0(\sigma_1, \sigma_2) \leq s < r$ where $r_0(\sigma_1, \sigma_2) = |X^0(\sigma_1, \sigma_2)|$ and if $s < 0$ ($x \in \overset{\circ}{C} \cap \overline{\Omega}$) then $s + \rho_{2M} \geq c$ for some positive constant c.

We also recall the following lemmas about the coordinate system, these lemmas were originally stated and proved by Bloom and Kazarinoff [4] for illuminated from interior obstacles, and then they were extended by Liu [12] for illuminated from exterior obstacles.

Lemma 2.5. *The level surfaces $s = const.$, $\sigma_i = const.$ $(i = 1, 2)$ define a set of local coordinate systems in $\overline{\Omega}$ with each ray $\{x : \sigma_1 = const., \sigma_2 = const., \text{ and } s_0 \leq s < \infty\}$ normally incident on ∂C and*

$$\left| \frac{D(x_1, x_2, x_3)}{D(s, \sigma_1, \sigma_2)} \right| = \Lambda(\kappa_1 s + 1)(\kappa_2 s + 1) > 0$$

where $\Lambda = |X^0_{\sigma_1}||X^0_{\sigma_2}|$.

Lemma 2.6. *$\nu \cdot n \geq 0$ on ∂V.*

Remark 2.7. *Note that ν is actually defined on ∂C, however, by the construction of the illuminating geometry there is a one to one map between any point outside (or inside) of C and the boundary point. Thus $\nu(x)$ where x is not on the boundary of C really means $\nu(\pi(x))$ where $\pi(x)$ is the projection on the boundary along the ray.*

Remark 2.8. *We recall the following calculus formulas that we will use in our computation $(i = 1, 2)$:*

$$\partial_{\sigma_i} \nu \equiv \frac{\partial \nu}{\partial \sigma_i} = \kappa_i X^0_{\sigma_i}$$

and thus

$$X_{\sigma_i} = (\kappa_i s + 1) X^0_{\sigma_i}.$$

Moreover, remark that for any scalar function $f = f(s, \sigma_1, \sigma_2)$ *and every vector field written in the new coordinate system*

$$F = F^0 \nu + F^1 \frac{X^0_{\sigma_1}}{\left|X^0_{\sigma_1}\right|} + F^2 \frac{X^0_{\sigma_2}}{\left|X^0_{\sigma_2}\right|},$$

we can express the gradient and the divergence as follows (see section 1 of the appendix for further details) :

$$\nabla f = \partial_s f \nu + \frac{1}{|X_{\sigma_1}|} \partial_{\sigma_1} f \frac{X^0_{\sigma_1}}{\left|X^0_{\sigma_1}\right|} + \frac{1}{|X_{\sigma_2}|} \partial_{\sigma_2} f \frac{X^0_{\sigma_2}}{\left|X^0_{\sigma_2}\right|}$$

and

$$divF = \frac{1}{|X_{\sigma_1}| \, |X_{\sigma_2}|} \left[\partial_s \left(|X_{\sigma_1}| \, |X_{\sigma_2}| \, F^0 \right) + \partial_{\sigma_1} \left(|X_{\sigma_2}| \, F^1 \right) + \partial_{\sigma_2} \left(|X_{\sigma_1}| \, F^2 \right) \right]$$

In particular, we have

$$\nabla s = \nu$$

$$\nabla f \cdot \nu = \partial_s f$$

$$|\nabla f|^2 = (\partial_s f)^2 + \frac{1}{(\kappa_1 s + 1)^2 |X^0_{\sigma_1}|^2} (\partial_{\sigma_1} f)^2 + \frac{1}{(\kappa_2 s + 1)^2 |X^0_{\sigma_2}|^2} (\partial_{\sigma_2} f)^2$$

Denote by

$$|\nabla^*_i f|^2 = \frac{1}{(\kappa_i s + 1)^2 |X^0_{\sigma_i}|^2} (\partial_{\sigma_i} f)^2, \ i = 1, 2$$

and

$$|\nabla^* f|^2 = |\nabla^*_1 f|^2 + |\nabla^*_2 f|^2$$

thus

$$|\nabla f|^2 = |\partial_s f|^2 + |\nabla^* f|^2.$$

3 Proof of the L^6 decay estimate (Theorem 1.1)

We must show that for any $\epsilon_0 > 0$, there exists T_0 such that whenever $t \geq T_0$,

$$\frac{1}{6}\int_{\Omega} |u(t,x)|^6 dx \leq \epsilon_0.$$

First, we begin by multiplying the wave equation $\Box u + u^5 = 0$ by $\partial_t u$, we get the following divergence or energy identity

$$\partial_t(e(u)) - \operatorname{div}(\nabla u \partial_t u) = 0$$

where

$$e(u) = \frac{1}{2}(|\partial_t u|^2 + |\nabla u|^2) + \frac{1}{6}|u|^6$$

denotes the energy density. Integrating over the region $\{(x,t); s + \rho_{2M} > t + M, 0 \leq t \leq T\}$, where $\rho_{2M} = \max\limits_{(\sigma_1,\sigma_2)} \rho_2$ and M is a positive constant chosen such that $M \geq \rho_{2M}$ and the illuminating body $C \subset \{x; s + \rho_{2M} \leq M\}$, and using the divergence theorem and the Dirichlet boundary condition, we get

$$\int_{s+\rho_{2M} > T+M} e(u)(T,x)dx + \frac{1}{\sqrt{2}} flux(0,T) \leq \int_{s+\rho_{2M}>M} e(u)(0,x)dx \tag{II.8}$$

where the flux on the mantle is defined by :

$$flux(a,b) = \int_{M_a^b} \left(\frac{1}{2}|\nu\partial_t u + \nabla u|^2 + \frac{u^6}{6}\right) d\sigma$$

with

$$M_a^b = \{(x,t); s + \rho_{2M} = t + M, a \leq t \leq b\}$$

Since the solution has finite energy, we may select M large so that the right hand side of (II.8) is less than $\frac{\epsilon_0}{2}$. Hence, it will suffice to show the existence of T_0 such that whenever $T > T_0$ we have

$$\frac{1}{6}\int_{s+\rho_{2M} \leq T+M} |u(T,x)|^6 dx \leq \frac{\epsilon_0}{2}$$

This is a consequence of the following proposition :

Proposition 3.1. *Suppose u solves the nonlinear wave equation (II.1) and that V is a non-trapping obstacle with regular boundary that can be illuminated from its exterior by a strictly* *convex body satisfying the geometric condition :*

$$\min_{\partial V}(s_0+\rho_1-2(\rho_{2M}-\rho_1))>0$$

then

$$\begin{aligned}&\sqrt{\eta_0}\int_{s+\rho_{2M}\leq\epsilon T+M}\left(|\nabla^* u|^2+\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2\right)dx+\int_{s+\rho_{2M}\leq T+M}\frac{u^6(T,x)}{3}dx\\&\leq 2c_1\beta E+\frac{1}{T}\left(C_0E+C_2E\ln(1+T)+2(c_2+c_3T)flux(0,T)\right)\\&+\frac{\eta_0}{T}\int_0^T\int_{s+\rho_{2M}\leq\epsilon t+M}\left(|\nabla^* u|^2+\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2\right)dxdt\end{aligned}$$

for some arbitrary $0<\beta<1$ *and where* $0<\eta_0,\epsilon<1$ *and all the other constants depend on the geometry of the obstacle.*

As a result of Proposition 3.1, we will show that given $\epsilon_0>0$, $\exists T_0$ such that $\forall T\geq T_0$ we have

$$\frac{1}{T}\int_0^T\int_{s+\rho_{2M}\leq\epsilon T+M}\left(|\nabla^* u|^2+\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2\right)dxdt<\frac{\epsilon_0}{2}. \tag{II.9}$$

For simplicity, let :

$$\phi(t)=\int_{s+\rho_{2M}\leq\epsilon T+M}\left(|\nabla^* u|^2+\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2\right)dx,$$

$$\psi(T)=\frac{1}{T}\int_0^T\phi(t)dt,$$

$$\gamma(t)=\frac{2c_1\beta E}{\eta}+\frac{1}{t\eta}\left(C_0E+C_2E\ln(1+t)+2(c_2+c_3t)flux(0,t)\right).$$

with $\eta=\sqrt{\eta_0}$.
Thus we want to show the existence of T_0 such that $\forall T\geq T_0$ we have $\psi(T)<\frac{\epsilon_0}{2}$.

$$\psi(T)+T\frac{d\psi}{dt}=\phi(T)$$

From the differential inequality in Proposition 3.1 we have

$$\phi(T)\leq\gamma(T)+\eta\psi(T)$$

so

$$\frac{d\psi}{dt} + (1-\eta)\frac{\psi(T)}{T} \leq \frac{\gamma(T)}{T}$$

$$\frac{1}{T^{1-\eta}}\frac{d(T^{1-\eta}\psi)}{dt} \leq \frac{\gamma(T)}{T}$$

$$\frac{d(T^{1-\eta}\psi)}{dt} \leq \frac{\gamma(T)}{T^{\eta}}$$

$$T^{1-\eta}\psi(T) \leq \psi(1) + \int_1^T \frac{\gamma(t)}{t^{\eta}}dt$$

Hence,

$$\psi(T) \leq J(T) + \frac{\psi(1)}{T^{1-\eta}} \tag{II.10}$$

with

$$J(T) = \frac{1}{T^{1-\eta}}\int_1^T \frac{\gamma(t)}{t^{\eta}}dt.$$

But, note that $flux(0,t) \xrightarrow[T\to\infty]{} 0$ by the classical energy-conservation law on the exterior of a truncated cone stated above. Thus, $\exists t_0$ such that $\forall t \geq t_0$ we have

$$\frac{1}{t\eta}\left(C_0E + C_2E\ln(1+t) + 2(c_2+c_3T)flux(0,t)\right) < \frac{\epsilon_0(1-\eta)}{12},$$

and choose β such that

$$\frac{2c_1\beta E}{\eta} = \frac{\epsilon_0(1-\eta)}{12}.$$

Hence,

$$\exists t_0 \text{ such that } \forall t \geq t_0 \text{ we have } \gamma(t) < \frac{\epsilon_0(1-\eta)}{6}$$

and γ is bounded :

$$\gamma(t) \leq M,\ \forall t.$$

Moreover,

$$\exists T_0 > t_0 \text{ such that } \forall t > T_0, \text{ we have } \frac{1}{t^{1-\eta}}\frac{M}{1-\eta}(t_0^{1-\eta}-1) < \frac{\epsilon_0}{6} \text{ and } \frac{\psi(1)}{t^{1-\eta}} < \frac{\epsilon_0}{6}.$$

Thus, for all $T > T_0$,

$$\begin{aligned} J(T) &= \frac{1}{T^{1-\eta}} \int_1^{t_0} \frac{\gamma(t)}{t^\eta} dt + \frac{1}{T^{1-\eta}} \int_{t_0}^{T} \frac{\gamma(t)}{t^\eta} dt \\ &\leq \frac{1}{T^{1-\eta}} \frac{M}{1-\eta} (t_0^{1-\eta} - 1) + \frac{1}{T^{1-\eta}} \frac{\epsilon_0}{6} (T^{1-\eta} - t_0^{1-\eta}) \\ &< \frac{\epsilon_0}{3} \end{aligned}$$

and by (II.10), $\psi(T) < \frac{\epsilon_0}{2}$ which is (II.9).

Now, from Proposition 3.1, we have

$$\int_{s+\rho_{2M} \leq T+M} \frac{u^6(T,x)}{3} dx \leq \eta\gamma(T) + \eta^2\psi(T) < \gamma(T) + \psi(T) < \epsilon_0$$

Hence, for all $T \geq T_0$, we have

$$\int_{s+\rho_{2M} \leq T+M} \frac{u^6(T,x)}{3} dx < \epsilon_0$$

which ends the proof of Theorem 1.1.

4 Proof of the differential inequality (Proposition 3.1)

The method we are going to use to prove our result is the method of multipliers and we will generalize the Morawetz multipliers that were used for star-shaped obstacles in way that suits our obstacle.

4.1 Divergence Identity and Integral Equality

We multiply the wave equation by

$$(u + \alpha \cdot \nabla u + (t+M)\partial_t u),$$

where $M \geq \rho_{2M}$ is the positive constant chosen in the previous section and α is a vector field defined as follows :

$$\alpha = (s + \rho_{2M})\nu \tag{II.11}$$

and we get the following divergence identity :

$$\partial_t Q + \mathrm{div} P + R = 0$$

where

$$\begin{cases} Q = (t+M)\dfrac{|\nabla u|^2}{2} + (t+M)\dfrac{|u|^6}{6} + (t+M)\dfrac{|\partial_t u|^2}{2} + \partial_t u(\alpha \cdot \nabla u) + (\partial_t u)u \\ P = \left(\dfrac{|\nabla u|^2}{2} + \dfrac{|u|^6}{6} - \dfrac{|\partial_t u|^2}{2}\right)\alpha - \left((t+M)\partial_t u + \alpha \cdot \nabla u + u\right)\nabla u \\ R = (\mathrm{div}\alpha - 3)\dfrac{|\partial_t u|^2}{2} + (1 - \mathrm{div}\alpha)\dfrac{|\nabla u|^2}{2} + (5 - \mathrm{div}\alpha)\dfrac{|u|^6}{6} + H_\alpha(\nabla u, \nabla u) \end{cases}$$

where $H_\alpha(\nabla u, \nabla u) = \sum_{i,j=1}^{3} \partial_i \alpha_j \partial_i u \partial_j u = \nabla u \cdot ((\nabla u \cdot \nabla)\,\alpha)$.

Integrating the divergence identity over the truncated cone

$$K_{T_1}^{T_2} = \{(x,t); s + \rho_{2M} \leq t + M, T_1 \leq t \leq T_2\}\,, \quad 0 < T_1 < T_2,$$

and applying the divergence theorem, we get

$$\int_{D(T_2)} Q(T_2,x)dx - \int_{D(T_1)} Q(T_1,x)dx - \frac{1}{\sqrt{2}}\int_{M_{T_1}^{T_2}} (Q - P\cdot\nu)\, d\sigma$$
$$- \int_{T_1}^{T_2}\int_{\partial V} P\cdot n d\sigma dt + \int_{K_{T_1}^{T_2}} R dx dt = 0 \tag{II.12}$$

where $d\sigma$ denotes the Lebesgue measure on the corresponding surface and n is the outward pointing unit normal vector to ∂V ; and where

$$D(T_i) = \{x \in \Omega; s + \rho_{2M} \le T_i + M\}$$

and

$$M_{T_1}^{T_2} = \{(x,t); s + \rho_{2M} = t + M, T_1 \le t \le T_2\}\,.$$

4.2 The differential inequality

Now, we deal with the terms of the integral equality (II.12) in order to get the desired differential inequality.

The boundary term

By the Dirichlet boundary condition, we get :

$$-\int_{T_1}^{T_2}\int_{\partial V} P\cdot n d\sigma dt = -\int_{T_1}^{T_2}\int_{\partial V} \frac{|\nabla u|^2}{2}\alpha\cdot n - (\alpha\cdot\nabla u)(\nabla u\cdot n) d\sigma dt$$
$$= \int_{T_1}^{T_2}\int_{\partial V} \frac{1}{2}|\nabla u\cdot n|^2(\alpha\cdot n) d\sigma dt \ge 0 \tag{II.13}$$

since $\alpha\cdot n = (s+\rho_{2M})\nu\cdot n$ with $s+\rho_{2M} > 0$ (by our assumption) and $\nu\cdot n \ge 0$ (Lemma 2.6).

The terms on the time slices

We have

$$Q(T_2,x) = (T_2+M)\frac{|\nabla u|^2}{2} + (T_2+M)\frac{|u|^6}{6} + (T_2+M)\frac{|\partial_t u|^2}{2} + \partial_t u(\alpha\cdot\nabla u) + (\partial_t u)u$$

Introduce

$$
\begin{aligned}
I(T_2) &= \frac{1}{4}(T_2 + M + (s + \rho_{2M})) \left[\partial_t u + \frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right]^2 \\
&\quad + \frac{1}{4}(T_2 + M - (s + \rho_{2M})) \left[\partial_t u - \frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right]^2 \\
&\quad + (T_2 + M)\frac{u^6}{6} \\
&= (T_2 + M)\left(\frac{|\partial_t u|^2}{2} + \frac{|\partial_s u|^2}{2} + \frac{u^6}{6}\right) + (\partial_t u)(s+\rho_{2M})\partial_s u + (\partial_t u)u \\
&\quad + \frac{T_2+M}{2}\left(\frac{u^2}{(s+\rho_{2M})^2} + \frac{2u\partial_s u}{s+\rho_{2M}}\right)
\end{aligned}
$$

by Remark 2.8, we have

$$|\nabla u|^2 = |\partial_s u|^2 + |\nabla^* u|^2$$

where

$$|\nabla^* u|^2 = |\nabla_1^* u|^2 + |\nabla_2^* u|^2$$

and

$$|\nabla_i^* u|^2 = \frac{1}{(\kappa_i s + 1)^2 |X^0_{\sigma_i}|^2}(\partial_{\sigma_i} u)^2,\ i = 1, 2$$

so

$$I(T_2) + (T_2+M)\frac{|\nabla^* u|^2}{2} = Q(T_2, \cdot) + \frac{T_2+M}{2}\left(\frac{u^2}{(s+\rho_{2M})^2} + \frac{2u\partial_s u}{s+\rho_{2M}}\right)$$

Thus,

$$
\begin{aligned}
\int_{D(T_2)} Q(T_2, x)dx &= \int_{D(T_2)} I(T_2) + (T_2+M)\frac{|\nabla^* u|^2}{2}dx - \frac{T_2+M}{2}\int_{D(T_2)} \frac{2u\partial_s u}{s+\rho_{2M}} + \frac{u^2}{(s+\rho_{2M})^2}dx \\
&= \int_{D(T_2)} I(T_2) + (T_2+M)\frac{|\nabla^* u|^2}{2}dx - \frac{T_2+M}{2}\int_{D(T_2)} \frac{\partial_s((s+\rho_{2M})u^2)}{(s+\rho_{2M})^2}dx
\end{aligned}
$$

Now, integrating by parts and using Dirichlet boundary condition, we get

$$
\begin{aligned}
-\int_{D(T_2)} \frac{\partial_s((s+\rho_{2M})u^2)}{(s+\rho_{2M})^2}dx &= \int_{D(T_2)} (s+\rho_{2M})u^2 \partial_s\left(\frac{(s+\rho_1)(s+\rho_2)}{(s+\rho_{2M})^2}\right)\frac{\Lambda}{\rho_1\rho_2}dsd\sigma_1 d\sigma_2 \\
&\quad - \int_{\partial D(T_2)} \frac{u^2}{T_2+M}dS_2
\end{aligned}
$$

where dS_2 is the measure on $\partial D(T_2)$ and

$$\partial D(T_2) = \{x, s+\rho_{2M} = T_2 + M\}.$$

But

$$\begin{aligned}
\partial_s\left(\frac{(s+\rho_1)(s+\rho_2)}{(s+\rho_{2M})^2}\right) &= \frac{(2s+\rho_1+\rho_2)(s+\rho_{2M}) - 2(s+\rho_1)(s+\rho_2)}{(s+\rho_{2M})^3} \\
&= \frac{1}{(s+\rho_{2M})^3}\left(\sum_{i,j=1,i\neq j}^{2}(\rho_{2M}-\rho_i)(s+\rho_j)\right) \geq 0
\end{aligned}$$

so

$$\begin{aligned}
\int_{D(T_2)} Q(T_2,x)dx &= \int_{D(T_2)} I(T_2) + (T_2+M)\frac{|\nabla^* u|^2}{2}dx - \frac{T_2+M}{2}\int_{\partial D(T_2)}\frac{u^2}{T_2+M}dS_2 \\
&\quad + \frac{T_2+M}{2}\int_{D(T_2)}\frac{u^2}{(s+\rho_{2M})^2}\left(\sum_{i,j=1,i\neq j}^{2}(\rho_{2M}-\rho_i)(s+\rho_j)\right)\frac{\Lambda}{\rho_1\rho_2}dsd\sigma_1 d\sigma_2 \\
&= \int_{D(T_2)} I(T_2) + (T_2+M)\frac{|\nabla^* u|^2}{2}dx - \frac{1}{2}\int_{\partial D(T_2)} u^2 dS_2 \\
&\quad + \frac{T_2+M}{2}\int_{D(T_2)}\left(\sum_{i=1}^{2}\frac{\rho_{2M}-\rho_i}{s+\rho_i}\right)\frac{u^2}{(s+\rho_{2M})^2}dx
\end{aligned} \tag{II.14}$$

Similarly we get

$$\begin{aligned}
\int_{D(T_1)} Q(T_1,x)dx &= \int_{D(T_1)} I(T_1) + (T_1+M)\frac{|\nabla^* u|^2}{2}dx - \frac{1}{2}\int_{\partial D(T_1)} u^2 dS_1 \\
&\quad + \frac{T_1+M}{2}\int_{D(T_1)}\left(\sum_{i=1}^{2}\frac{\rho_{2M}-\rho_i}{s+\rho_i}\right)\frac{u^2}{(s+\rho_{2M})^2}dx
\end{aligned} \tag{II.15}$$

The term on the mantle

On the mantle $M_{T_1}^{T_2}$, we have $s+\rho_{2M} = t+M$, and recall that $\nabla u\cdot\nu = \partial_s u$ (Remark 2.8), thus we get :

$$\begin{aligned}
Q - P\cdot\nu &= (s+\rho_{2M})(\partial_t u + \partial_s u)^2 + u(\partial_t u + \partial_s u) \\
&= (s+\rho_{2M})\left(\partial_t u + \partial_s u + \frac{u}{s+\rho_{2M}}\right)^2 - u(\partial_t u + \partial_s u) - \frac{u^2}{s+\rho_{2M}}
\end{aligned}$$

so

$$-\frac{1}{\sqrt{2}}\int_{M_{T_1}^{T_2}}(Q-P\cdot\nu)d\sigma = -\frac{1}{\sqrt{2}}\int_{M_{T_1}^{T_2}}(s+\rho_{2M})\left(\partial_t u+\partial_s u+\frac{u}{s+\rho_{2M}}\right)^2 d\sigma$$
$$+\frac{1}{\sqrt{2}}\int_{M_{T_1}^{T_2}} u(\partial_t u+\partial_s u)+\frac{u^2}{s+\rho_{2M}}d\sigma \tag{II.16}$$

Now setting $\overline{u}(y)=u(s+\rho_{2M}-M,y)$ on the mantle, we have

$$\nabla\overline{u}=\nu\partial_t u+\nabla u \text{ and } \partial_s\overline{u}=\partial_t u+\partial_s u,$$

the second term in (II.16) can be written as such

$$\frac{1}{\sqrt{2}}\int_{M_{T_1}^{T_2}} u(\partial_t u+\partial_s u)+\frac{u^2}{s+\rho_{2M}}d\sigma=\int_{\overline{M}_{T_1}^{T_2}}\left(\overline{u}\partial_s\overline{u}+\frac{\overline{u}^2}{s+\rho_{2M}}\right)dy=\int_{\overline{M}_{T_1}^{T_2}}\frac{\partial_s((s+\rho_{2M})^2\overline{u}^2)}{2(s+\rho_{2M})^2}dy$$

where

$$\overline{M}_{T_1}^{T_2}=\{x, T_1+M\leq s+\rho_{2M}\leq T_2+M\}$$

Integrating by parts and using the Dirichlet boundary condition, we get

$$\begin{aligned}\int_{\overline{M}_{T_1}^{T_2}}\frac{\partial_s((s+\rho_{2M})^2\overline{u}^2)}{2(s+\rho_{2M})^2}dy &= \int_{\overline{M}_{T_1}^{T_2}}\frac{\partial_s((s+\rho_{2M})^2\overline{u}^2)}{2(s+\rho_{2M})^2}\frac{\Lambda}{\rho_1\rho_2}(s+\rho_1)(s+\rho_2)dsd\sigma_1 d\sigma_2\\
&=-\int_{\overline{M}_{T_1}^{T_2}}\frac{1}{2}(s+\rho_{2M})^2\overline{u}^2\partial_s\left(\frac{(s+\rho_1)(s+\rho_2)}{(s+\rho_{2M})^2}\right)\frac{\Lambda}{\rho_1\rho_2}dsd\sigma_1 d\sigma_2\\
&\quad+\frac{1}{2}\int_{\partial D(T_2)}\overline{u}^2 dS_2-\frac{1}{2}\int_{\partial D(T_1)}\overline{u}^2 dS_1\\
&=-\frac{1}{2}\int_{\overline{M}_{T_1}^{T_2}}\left(\sum_{i,j=1,i\neq j}^{2}(\rho_{2M}-\rho_i)(s+\rho_j)\right)\frac{\overline{u}^2}{s+\rho_{2M}}\frac{\Lambda}{\rho_1\rho_2}dsd\sigma_1 d\sigma_2\\
&\quad+\frac{1}{2}\int_{\partial D(T_2)}\overline{u}^2 dS_2-\frac{1}{2}\int_{\partial D(T_1)}\overline{u}^2 dS_1\end{aligned}$$

Thus the term on the mantle (II.16) becomes

$$-\frac{1}{\sqrt{2}}\int_{M_{T_1}^{T_2}}(Q-P\cdot\nu)d\sigma = -\frac{1}{\sqrt{2}}\int_{M_{T_1}^{T_2}}(s+\rho_{2M})(\partial_t u+\partial_s u+\frac{u}{s+\rho_{2M}})^2 d\sigma$$
$$-\frac{1}{2}\int_{\overline{M}_{T_1}^{T_2}}\left(\sum_{i,j=1,i\neq j}^{2}(\rho_{2M}-\rho_i)(s+\rho_j)\right)\frac{\overline{u}^2}{s+\rho_{2M}}\frac{\Lambda}{\rho_1\rho_2}ds d\sigma_1 d\sigma_2$$
$$+\frac{1}{2}\int_{\partial D(T_2)}u^2 dS_2-\frac{1}{2}\int_{\partial D(T_1)}u^2 dS_1 \tag{II.17}$$

The remainder term

We have $\alpha=(s+\rho_{2M})\nu$ thus by Remark 2.8, we get

$$\begin{aligned}\mathrm{div}\alpha &= \frac{1}{(\kappa_1 s+1)(\kappa_2 s+1)}\partial_s((\kappa_1 s+1)(\kappa_2 s+1)(s+\rho_{2M}))\\ &= 1+\frac{\kappa_1(s+\rho_{2M})}{\kappa_1 s+1}+\frac{\kappa_2(s+\rho_{2M})}{\kappa_2 s+1}\\ &= 3+\frac{\rho_{2M}-\rho_1}{s+\rho_1}+\frac{\rho_{2M}-\rho_2}{s+\rho_2}\end{aligned}$$

since $s+\rho_i>0$ and $\rho_{2M}\geq\rho_2\geq\rho_1$ then $\mathrm{div}\alpha-3\geq 0$ and

$$\int_{K_{T_1}^{T_2}}(\mathrm{div}\alpha-3)\frac{|\partial_t u|^2}{2}dxdt\geq 0 \tag{II.18}$$

Now, in the following remainder term

$$\int_{K_{T_1}^{T_2}}(5-\mathrm{div}\alpha)\frac{|u|^6}{6}dxdt$$

we have

$$5-\mathrm{div}\alpha=2-\frac{\rho_{2M}-\rho_1}{s+\rho_1}-\frac{\rho_{2M}-\rho_2}{s+\rho_2}$$

Imposing the following geometric condition

$$\min_{\partial V}(s_0+\rho_1-(\rho_{2M}-\rho_1))>0$$

we get that (recall that $s \geq s_0$)

$$\frac{\rho_{2M} - \rho_1}{s + \rho_1} + \frac{\rho_{2M} - \rho_2}{s + \rho_2} < 2$$

and thus

$$\int_{K_{T_1}^{T_2}} (5 - \text{div}\alpha) \frac{|u|^6}{6} dxdt \geq 0 \tag{II.19}$$

We still have to deal with the following term in R :

$$H_\alpha(\nabla u, \nabla u) + (1 - \text{div}\alpha) \frac{|\nabla u|^2}{2}$$

We have $H_\alpha(\nabla u, \nabla u) = \nabla u \cdot ((\nabla u \cdot \nabla)\, \alpha)$; and by Remark 2.8, we have

$$\nabla u = \partial_s u \nu + \frac{1}{(\kappa_1 s + 1) \left| X^0_{\sigma_1} \right|^2} \partial_{\sigma_1} u X^0_{\sigma_1} + \frac{1}{(\kappa_2 s + 1) \left| X^0_{\sigma_2} \right|^2} \partial_{\sigma_2} u X^0_{\sigma_2}$$

$$\nabla u \cdot \nabla = \partial_s u \partial_s + \frac{1}{(\kappa_1 s + 1)^2 |X^0_{\sigma_1}|^2} \partial_{\sigma_1} u \partial_{\sigma_1} + \frac{1}{(\kappa_2 s + 1)^2 |X^0_{\sigma_2}|^2} \partial_{\sigma_2} u \partial_{\sigma_2}$$

$$\begin{aligned}(\nabla u \cdot \nabla)\alpha =& ((\nabla u \cdot \nabla)\, (s + \rho_{2M}))\, \nu + (s + \rho_{2M})\, ((\nabla u \cdot \nabla)\, \nu) \\ =& \partial_s u \nu + (s + \rho_{2M}) \left(\frac{\kappa_1}{(\kappa_1 s + 1)^2 |X^0_{\sigma_1}|^2} \partial_{\sigma_1} u X^0_{\sigma_1} + \frac{\kappa_2}{(\kappa_2 s + 1)^2 |X^0_{\sigma_2}|^2} \partial_{\sigma_2} u X^0_{\sigma_2} \right)\end{aligned}$$

Hence

$$H_\alpha(\nabla u, \nabla u) = \nabla u \cdot ((\nabla u \cdot \nabla)\, \alpha) = (\partial_s u)^2 + \frac{\kappa_1 (s + \rho_{2M})}{(\kappa_1 s + 1)^3 |X^0_{\sigma_1}|^2} (\partial_{\sigma_1} u)^2 + \frac{\kappa_2 (s + \rho_{2M})}{(\kappa_2 s + 1)^3 |X^0_{\sigma_2}|^2} (\partial_{\sigma_2} u)^2$$

Using

$$\begin{aligned}|\nabla u|^2 &= (\partial_s u)^2 + \frac{1}{(\kappa_1 s + 1)^2 |X^0_{\sigma_1}|^2} (\partial_{\sigma_1} u)^2 + \frac{1}{(\kappa_2 s + 1)^2 |X^0_{\sigma_2}|^2} (\partial_{\sigma_2} u)^2 \\ &= (\partial_s u)^2 + |\nabla^*_1 u|^2 + |\nabla^*_2 u|^2\end{aligned}$$

we get

$$H_\alpha(\nabla u, \nabla u) = |\nabla u|^2 + \frac{\rho_{2M} - \rho_1}{s + \rho_1} |\nabla^*_1 u|^2 + \frac{\rho_{2M} - \rho_2}{s + \rho_2} |\nabla^*_2 u|^2 \geq 0$$

Moreover, we have

$$(1-\mathrm{div}\alpha)\,\frac{|\nabla u|^2}{2}=-|\nabla u|^2-\left(\frac{\rho_{2M}-\rho_1}{s+\rho_1}+\frac{\rho_{2M}-\rho_2}{s+\rho_2}\right)\frac{|\nabla u|^2}{2}$$

Hence

$$\int_{K_{T_1}^{T_2}} H_\alpha(\nabla u,\nabla u)+(1-\mathrm{div}\alpha)\frac{|\nabla u|^2}{2}dxdt \tag{II.20}$$
$$=\int_{K_{T_1}^{T_2}}\left(\sum_{i=1}^{2}\frac{\rho_{2M}-\rho_i}{s+\rho_i}|\nabla_i^* u|^2\right)dxdt-\int_{K_{T_1}^{T_2}}\left(\sum_{i=1}^{2}\frac{\rho_{2M}-\rho_i}{s+\rho_i}\right)\frac{|\nabla u|^2}{2}dxdt$$

Recall that

$$|\nabla u|^2=|\partial_s u|^2+|\nabla^* u|^2$$

and note that

$$\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}=\partial_s u+\frac{u}{s+\rho_{2M}},$$

hence

$$|\nabla u|^2=|\nabla^* u|^2+\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2-\left|\frac{u}{s+\rho_{2M}}\right|^2-\frac{2u\partial_s u}{s+\rho_{2M}}$$

Substituting this in (II.20), we get :

$$\begin{aligned}
&\int_{K_{T_1}^{T_2}} H_\alpha(\nabla u,\nabla u)+(1-\mathrm{div}\alpha)\frac{|\nabla u|^2}{2}dxdt\\
&=\int_{K_{T_1}^{T_2}}\left(\sum_{i=1}^{2}\frac{\rho_{2M}-\rho_i}{s+\rho_i}|\nabla_i^* u|^2-\frac{1}{2}\left(\sum_{i=1}^{2}\frac{\rho_{2M}-\rho_i}{s+\rho_i}\right)|\nabla^* u|^2\right)dxdt\\
&-\frac{1}{2}\int_{K_{T_1}^{T_2}}\left(\sum_{i=1}^{2}\frac{\rho_{2M}-\rho_i}{s+\rho_i}\right)\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2dxdt\\
&+\frac{1}{2}\int_{K_{T_1}^{T_2}}\left(\sum_{i=1}^{2}\frac{\rho_{2M}-\rho_i}{s+\rho_i}\right)\left|\frac{u}{s+\rho_{2M}}\right|^2dxdt\\
&+\frac{1}{2}\int_{K_{T_1}^{T_2}}\left(\sum_{i=1}^{2}\frac{\rho_{2M}-\rho_i}{s+\rho_i}\right)\frac{2u\partial_s u}{s+\rho_{2M}}dxdt\\
&=I+II+III+IV
\end{aligned} \tag{II.21}$$

$$I=\frac{1}{2}\int_{K_{T_1}^{T_2}}\left(\frac{\rho_{2M}-\rho_1}{s+\rho_1}-\frac{\rho_{2M}-\rho_2}{s+\rho_2}\right)|\nabla_1^* u|^2+\left(\frac{\rho_{2M}-\rho_2}{s+\rho_2}-\frac{\rho_{2M}-\rho_1}{s+\rho_1}\right)|\nabla_2^* u|^2dxdt \tag{II.22}$$

Integrating by parts the last term and using the Dirichlet boundary condition, we get :

$$\begin{aligned}
IV &= \sum_{i=1}^{2} \frac{1}{2} \int_{K_{T_1}^{T_2}} \frac{\rho_{2M}-\rho_i}{s+\rho_i} \frac{2u\partial_s u}{s+\rho_{2M}} dxdt \\
&= \sum_{i,j=1,i\neq j}^{2} \frac{1}{2} \int_{T_1}^{T_2} \int_{s+\rho_{2M}\leq t+M} (\rho_{2M}-\rho_i)\partial_s(u^2) \frac{s+\rho_j}{s+\rho_{2M}} \frac{\Lambda}{\rho_1\rho_2} ds d\sigma_1 d\sigma_2 dt \\
&= -\sum_{i,j=1,i\neq j}^{2} \frac{1}{2} \int_{T_1}^{T_2} \int_{s+\rho_{2M}\leq t+M} (\rho_{2M}-\rho_i) u^2 \partial_s \left(\frac{s+\rho_j}{s+\rho_{2M}} \right) \frac{\Lambda}{\rho_1\rho_2} ds d\sigma_1 d\sigma_2 dt \\
&\quad + \sum_{i,j=1,i\neq j}^{2} \frac{1}{2} \int_{\overline{M}_{T_1}^{T_2}} (\rho_{2M}-\rho_i) u^2 \frac{s+\rho_j}{s+\rho_{2M}} \frac{\Lambda}{\rho_1\rho_2} ds d\sigma_1 d\sigma_2 \\
&= -\sum_{i,j=1,i\neq j}^{2} \frac{1}{2} \int_{T_1}^{T_2} \int_{s+\rho_{2M}\leq t+M} (\rho_{2M}-\rho_i)(\rho_{2M}-\rho_j) \frac{u^2}{(s+\rho_{2M})^2} \frac{\Lambda}{\rho_1\rho_2} ds d\sigma_1 d\sigma_2 dt \\
&\quad + \sum_{i,j=1,i\neq j}^{2} \frac{1}{2} \int_{\overline{M}_{T_1}^{T_2}} (\rho_{2M}-\rho_i)(s+\rho_j) \frac{u^2}{s+\rho_{2M}} \frac{\Lambda}{\rho_1\rho_2} ds d\sigma_1 d\sigma_2
\end{aligned}$$

and we have

$$III = \sum_{i,j=1,i\neq j}^{2} \frac{1}{2} \int_{T_1}^{T_2} \int_{s+\rho_{2M}\leq t+M} (\rho_{2M}-\rho_i)(s+\rho_j) \frac{u^2}{(s+\rho_{2M})^2} \frac{\Lambda}{\rho_1\rho_2} ds d\sigma_1 d\sigma_2 dt$$

Hence,

$$\begin{aligned}
III + IV &= \frac{1}{2} \int_{\overline{M}_{T_1}^{T_2}} \sum_{i,j=1,i\neq j}^{2} (\rho_{2M}-\rho_i)(s+\rho_j) \frac{u^2}{s+\rho_{2M}} \frac{\Lambda}{\rho_1\rho_2} ds d\sigma_1 d\sigma_2 \\
&\quad + \frac{1}{2} \int_{T_1}^{T_2} \int_{s+\rho_{2M}\leq t+M} \sum_{i,j=1,i\neq j}^{2} (\rho_{2M}-\rho_i)(s+2\rho_j-\rho_{2M}) \frac{u^2}{(s+\rho_{2M})^2} \frac{\Lambda}{\rho_1\rho_2} ds d\sigma_1 d\sigma_2 dt
\end{aligned}$$

Now substituting in (II.21) we get

$$
\begin{aligned}
&\int_{K_{T_1}^{T_2}} H_\alpha(\nabla u,\nabla u)+(1-\mathrm{div}\alpha)\frac{|\nabla u|^2}{2}dxdt\\
&\quad=\frac{1}{2}\int_{K_{T_1}^{T_2}}\left(\frac{\rho_{2M}-\rho_1}{s+\rho_1}-\frac{\rho_{2M}-\rho_2}{s+\rho_2}\right)|\nabla_1^* u|^2+\left(\frac{\rho_{2M}-\rho_2}{s+\rho_2}-\frac{\rho_{2M}-\rho_1}{s+\rho_1}\right)|\nabla_2^* u|^2dxdt\\
&\quad-\frac{1}{2}\int_{K_{T_1}^{T_2}}\left(\sum_{i=1}^{2}\frac{\rho_{2M}-\rho_i}{s+\rho_i}\right)\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2 dxdt\\
&\quad+\frac{1}{2}\int_{\overline{M}_{T_1}^{T_2}}\sum_{i,j=1,i\neq j}^{2}(\rho_{2M}-\rho_i)(s+\rho_j)\frac{u^2}{s+\rho_{2M}}\frac{\Lambda}{\rho_1\rho_2}dsd\sigma_1d\sigma_2\\
&\quad+\frac{1}{2}\int_{T_1}^{T_2}\int_{s+\rho_{2M}\leq t+M}\sum_{i,j=1,i\neq j}^{2}(\rho_{2M}-\rho_i)(s+2\rho_j-\rho_{2M})\frac{u^2}{(s+\rho_{2M})^2}\frac{\Lambda}{\rho_1\rho_2}dsd\sigma_1d\sigma_2dt
\end{aligned}
\tag{II.23}
$$

and note that since we imposed the geometric condition

$$\min_{\partial V}(s_0+\rho_1-(\rho_{2M}-\rho_1))>0$$

we have $s+2\rho_j-\rho_{2M}>0$ and thus the last term in (II.23) is nonnegative.

The differential inequality

Now, summing up all the terms ((II.13), (II.14), (II.15), (II.17), (II.18), (II.19), and (II.23)) in the integral equality (II.12) and dropping the nonnegative terms, we get the following differential inequality :

$$\begin{aligned}
&\int_{D(T_2)} I(T_2) + (T_2+M)\frac{|\nabla^* u|^2}{2}dx \\
&\quad \leq \int_{D(T_1)} I(T_1) + (T_1+M)\frac{|\nabla^* u|^2}{2}dx \\
&\quad + \frac{T_1+M}{2}\int_{D(T_1)} \left(\sum_{i=1}^{2}\frac{\rho_{2M}-\rho_i}{s+\rho_i}\right)\frac{u^2}{(s+\rho_{2M})^2}dx \\
&\quad + \frac{1}{\sqrt{2}}\int_{M_{T_1}^{T_2}} (s+\rho_{2M})\left(\partial_t u + \partial_s u + \frac{u}{s+\rho_{2M}}\right)^2 d\sigma \\
&\quad + \frac{1}{2}\int_{K_{T_1}^{T_2}} \left(\frac{\rho_{2M}-\rho_2}{s+\rho_2} - \frac{\rho_{2M}-\rho_1}{s+\rho_1}\right)|\nabla_1^* u|^2 + \left(\frac{\rho_{2M}-\rho_1}{s+\rho_1} - \frac{\rho_{2M}-\rho_2}{s+\rho_2}\right)|\nabla_2^* u|^2 dxdt \\
&\quad + \frac{1}{2}\int_{K_{T_1}^{T_2}} \left(\sum_{i=1}^{2}\frac{\rho_{2M}-\rho_i}{s+\rho_i}\right)\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2 dxdt
\end{aligned}$$

Recall that for $i=1,2$ we have :

$$\begin{aligned}
I(T_i) &= \frac{1}{4}(T_i+M+(s+\rho_{2M}))\left[\partial_t u + \frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right]^2 \\
&+ \frac{1}{4}(T_i+M-(s+\rho_{2M}))\left[\partial_t u - \frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right]^2 \\
&+ (T_i+M)\frac{u^6}{6}
\end{aligned}$$

Thus

$$I(T_2) \geq \frac{1}{2}(T_2+M-(s+\rho_{2M}))\left((\partial_t u)^2 + \left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2\right) + (T_2+M)\frac{u^6}{6}$$

we also have,

$$I(T_1) \leq \frac{1}{2}(T_1+M+(s+\rho_{2M}))\left((\partial_t u)^2 + \left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2\right) + (T_1+M)\frac{u^6}{6}$$

so we get

$$\begin{aligned}
&\int_{D(T_2)} \frac{1}{2}(T_2+M-(s+\rho_{2M}))\left((\partial_t u)^2+\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2\right)+(T_2+M)\frac{u^6}{6}+(T_2+M)\frac{|\nabla^* u|^2}{2}dx\\
&\leq \int_{D(T_1)} \frac{1}{2}(T_1+M+(s+\rho_{2M}))\left((\partial_t u)^2+\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2+\frac{u^6}{3}+|\nabla^* u|^2\right)dx\\
&+\frac{T_1+M}{2}\int_{D(T_1)}\left(\sum_{i=1}^{2}\frac{\rho_{2M}-\rho_i}{s+\rho_i}\right)\frac{u^2}{(s+\rho_{2M})^2}dx+\frac{1}{\sqrt{2}}\int_{M_{T_1}^{T_2}}(s+\rho_{2M})\left(\partial_t u+\partial_s u+\frac{u}{s+\rho_{2M}}\right)^2 d\sigma\\
&+\frac{1}{2}\int_{K_{T_1}^{T_2}}\left(\frac{\rho_{2M}-\rho_2}{s+\rho_2}-\frac{\rho_{2M}-\rho_1}{s+\rho_1}\right)|\nabla_1^* u|^2+\left(\frac{\rho_{2M}-\rho_1}{s+\rho_1}-\frac{\rho_{2M}-\rho_2}{s+\rho_2}\right)|\nabla_2^* u|^2 dxdt\\
&+\frac{1}{2}\int_{K_{T_1}^{T_2}}\left(\sum_{i=1}^{2}\frac{\rho_{2M}-\rho_i}{s+\rho_i}\right)\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2 dxdt \qquad \text{(II.24)}\\
&\leq A_1+A_2+A_3+A_4+A_5
\end{aligned}$$

We have

$$\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2=\left|\frac{u}{s+\rho_{2M}}+\partial_s u\right|^2\leq 2\left|\frac{u}{s+\rho_{2M}}\right|^2+2|\partial_s u|^2\leq 2\left|\frac{u}{s+\rho_{2M}}\right|^2+2|\nabla u|^2$$

and on $D(T_1)$ we have : $s+\rho_{2M}\leq T_1+M$, thus

$$A_1\leq (T_1+M)\int_{D(T_1)}\left((\partial_t u)^2+2\left|\frac{u}{s+\rho_{2M}}\right|^2+3|\nabla u|^2+\frac{u^6}{3}\right)dx$$

Moreover, by the geometric condition we imposed

$$\min_{\partial V}(s_0+\rho_1-(\rho_{2M}-\rho_1))>0$$

we have

$$\sum_{i=1}^{2}\frac{\rho_{2M}-\rho_i}{s+\rho_i}<2$$

Thus

$$A_2\leq (T_1+M)\int_{D(T_1)}\left|\frac{u}{s+\rho_{2M}}\right|^2 dx$$

so

$$A_1+A_2\leq (T_1+M)\int_{D(T_1)}\left((\partial_t u)^2+3\left|\frac{u}{s+\rho_{2M}}\right|^2+3|\nabla u|^2+\frac{u^6}{3}\right)dx$$

Recall that there exist a constant $a_0 > 0$ such that $s + \rho_{2M} \geq a_0 r$ (Lemma 2.4). Thus,

$$\int \left| \frac{u}{s+\rho_{2M}} \right|^2 dx \leq \frac{1}{a_0^2} \int \frac{u^2}{r^2} dx$$

and by Hardy's inequality :

$$\int \frac{u^2}{r^2} dx \leq C \int |\nabla u|^2 dx$$

we get that

$$\int \left| \frac{u}{s+\rho_{2M}} \right|^2 dx \lesssim \int |\nabla u|^2 dx \tag{II.25}$$

so

$$A_1 + A_2 \leq (c_0 + c_1 T_1) E$$

where c_0 and c_1 are constants that depend on the geometry of the obstacle and E is the conserved energy. Now, the term on the mantle A_3 can be written as follows :

$$\begin{aligned} A_3 &= \int_{\overline{M}_{T_1}^{T_2}} (s+\rho_{2M}) \left(\partial_s \overline{u} + \frac{\overline{u}}{s+\rho_{2M}} \right)^2 dy \\ &\leq (T_2 + M) \int_{\overline{M}_{T_1}^{T_2}} \left(\partial_s \overline{u} + \frac{\overline{u}}{s+\rho_{2M}} \right)^2 dy \\ &\leq 2(T_2 + M) \int_{\overline{M}_{T_1}^{T_2}} \left(|\partial_s \overline{u}|^2 + \left| \frac{\overline{u}}{s+\rho_{2M}} \right|^2 \right) dy \end{aligned}$$

Similarly to (II.25), we have

$$\int \left| \frac{\overline{u}}{s+\rho_{2M}} \right|^2 dy \lesssim \int |\nabla \overline{u}|^2 dy$$

hence

$$A_3 \lesssim (T_2 + M) \int_{\overline{M}_{T_1}^{T_2}} |\nabla \overline{u}|^2 dy \leq (c_2 + c_3 T_2) flux(T_1, T_2)$$

where c_2 and c_3 are constants that depend on the geometry of the obstacle, and where

$$flux(T_1, T_2) = \int_{M_{T_1}^{T_2}} \left(\frac{1}{2} |\nu \partial_t u + \nabla u|^2 + \frac{u^6}{6} \right) d\sigma = \sqrt{2} \int_{\overline{M}_{T_1}^{T_2}} \left(\frac{|\nabla \overline{u}|^2}{2} + \frac{\overline{u}^6}{6} \right) dy.$$

$$
\begin{aligned}
A_4 &\leq \frac{1}{2}\int_{K_{T_1}^{T_2}} \left| \frac{\rho_{2M}-\rho_2}{s+\rho_2} - \frac{\rho_{2M}-\rho_1}{s+\rho_1} \right| |\nabla^* u|^2 dxdt \\
&\leq \frac{1}{2}\int_{K_{T_1}^{T_2}} \left(\frac{\rho_{2M}-\rho_2}{s+\rho_2} + \frac{\rho_{2M}-\rho_1}{s+\rho_1} \right) |\nabla^* u|^2 dxdt
\end{aligned}
$$

thus,

$$
A_4 + A_5 \leq \frac{1}{2}\int_{K_{T_1}^{T_2}} \left(\frac{\rho_{2M}-\rho_2}{s+\rho_2} + \frac{\rho_{2M}-\rho_1}{s+\rho_1} \right) \left(|\nabla^* u|^2 + \left| \frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}} \right|^2 \right) dxdt
$$

Thus (II.24) becomes

$$
\begin{aligned}
&\int_{D(T_2)} (T_2 + M - (s+\rho_{2M})) \left(|\nabla^* u|^2 + \left| \frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}} \right|^2 \right) + T_2 \frac{u^6}{3} dx \\
&\leq 2(c_0 + c_1 T_1)E + 2(c_2 + c_3 T_2) flux(T_1, T_2) \qquad \text{(II.26)} \\
&+ \sum_{i=1}^{2} \int_{T_1}^{T_2} \int_{\Omega} \frac{\rho_{2M}-\rho_i}{s+\rho_i} \left(|\nabla^* u|^2 + \left| \frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}} \right|^2 \right) dxdt
\end{aligned}
$$

Split Ω into $\{s+\rho_{2M} \leq \epsilon t + M\}$ and $\{s+\rho_{2M} > \epsilon t + M\}$, where $0 < \epsilon < 1$ is a constant that depends on the geometry of the obstacle to be later specified, then the last term in (II.26) becomes

$$
\begin{aligned}
J &= \sum_{i=1}^{2} \int_{T_1}^{T_2} \int_{\Omega} \frac{\rho_{2M}-\rho_i}{s+\rho_i} \left(|\nabla^* u|^2 + \left| \frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}} \right|^2 \right) dxdt \\
&\leq \sum_{i=1}^{2} \int_{T_1}^{T_2} \int_{s+\rho_{2M}>\epsilon t+M} \frac{\rho_{2M}-\rho_i}{s+\rho_{2M}-(\rho_{2M}-\rho_i)} \left(|\nabla^* u|^2 + \left| \frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}} \right|^2 \right) dxdt \\
&+ \max_{\partial V} \left(\frac{\rho_{2M}-\rho_1}{s_0+\rho_1} + \frac{\rho_{2M}-\rho_2}{s_0+\rho_2} \right) \int_{T_1}^{T_2} \int_{s+\rho_{2M}\leq\epsilon t+M} \left(|\nabla^* u|^2 + \left| \frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}} \right|^2 \right) dxdt \\
&= J_1 + J_2
\end{aligned}
$$

$$
\begin{aligned}
J_1 &\leq \sum_{i=1}^{2} \int_{T_1}^{T_2} \frac{\rho_{2M}-\rho_i}{\epsilon t + N} \left(\int_{s+\rho_{2M}>\epsilon t+M} \left(|\nabla^* u|^2 + \left| \frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}} \right|^2 \right) dx \right) dt \\
&\leq 2\int_{T_1}^{T_2} \frac{\rho_{2M}-\rho_{1m}}{\epsilon t + N} \left(\int_{s+\rho_{2M}>\epsilon t+M} \left(|\nabla^* u|^2 + \left| \frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}} \right|^2 \right) dx \right) dt
\end{aligned}
$$

with $N = M - \rho_{2M} \geq 0$. Since $s + \rho_{2M} \geq a_0 r$ and using Hardy's inequality, we get

$$\int_{s+\rho_{2M} > \epsilon t + M} \left(|\nabla^* u|^2 + \left| \frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}} \right|^2 \right) dx \lesssim E$$

hence,

$$J_1 \lesssim \frac{2(\rho_{2M} - \rho_{1m})}{\epsilon} \ln \left(\frac{\epsilon T_2 + N}{\epsilon T_1 + N} \right) E \leq C_1 E + C_2 E \ln(1 + T_2)$$

where C_1 and C_2 are constants that depend on the geometry of the obstacle. Thus

$$J \leq C_1 E + C_2 E \ln(1+T_2) + \eta_0 \int_{T_1}^{T_2} \int_{s+\rho_{2M} \leq \epsilon t + M} \left(|\nabla^* u|^2 + \left| \frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}} \right|^2 \right) dxdt \quad \text{(II.27)}$$

with

$$\eta_0 = \max_{\partial V} \left(\frac{\rho_{2M} - \rho_1}{s_0 + \rho_1} + \frac{\rho_{2M} - \rho_2}{s_0 + \rho_2} \right)$$

The geometric condition

$$\min_{\partial V} (s_0 + \rho_1 - (\rho_{2M} - \rho_1)) > 0$$

which we assumed so far implies that $\eta_0 < 2$ which is not enough as we want $0 < \eta_0 < 1$ for the proof of the L^6 decay estimate (Theorem 1.1). Thus, we impose a stronger condition

$$\min_{\partial V} (s_0 + \rho_1 - 2(\rho_{2M} - \rho_1)) > 0$$

On the other hand,

$$\int_{D(T_2)} (T_2 + M - (s + \rho_{2M})) \left(|\nabla^* u|^2 + \left| \frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}} \right|^2 \right) + T_2 \frac{u^6}{3} dx \quad \text{(II.28)}$$
$$\geq T_2(1-\epsilon) \int_{s+\rho_{2M} \leq \epsilon T_2 + M} \left(|\nabla^* u|^2 + \left| \frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}} \right|^2 \right) dx + T_2 \int_{D(T_2)} \frac{u^6}{3} dx$$

Using (II.27) and (II.28), (II.26) becomes

$$
\begin{aligned}
&T_2(1-\epsilon)\int_{s+\rho_{2M}\leq \epsilon T_2+M}\left(|\nabla^* u|^2+\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2\right)dx+T_2\int_{D(T_2)}\frac{u^6}{3}dx\\
&\leq C_0E+2c_1T_1E+C_2E\ln(1+T_2)+2(c_2+c_3T_2)flux(T_1,T_2)\\
&+\eta_0\int_{T_1}^{T_2}\int_{s+\rho_{2M}\leq \epsilon t+M}\left(|\nabla^* u|^2+\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2\right)dxdt
\end{aligned}
\tag{II.29}
$$

Now, setting $T_2 = T$ and $T_1 = \beta T$ for some $0 < \beta < 1$ and choosing $0 < \epsilon < 1$ such that $\epsilon = 1-\sqrt{\eta_0}$, (II.29) yields

$$
\begin{aligned}
&T\sqrt{\eta_0}\int_{s+\rho_{2M}\leq \epsilon T+M}\left(|\nabla^* u|^2+\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2\right)dx+T\int_{s+\rho_{2M}\leq T+M}\frac{u^6(T,x)}{3}dx\\
&\leq C_0E+2c_1\beta TE+C_2E\ln(1+T)+2(c_2+c_3T)flux(0,T)\\
&+\eta_0\int_{0}^{T}\int_{s+\rho_{2M}\leq \epsilon t+M}\left(|\nabla^* u|^2+\left|\frac{\partial_s((s+\rho_{2M})u)}{s+\rho_{2M}}\right|^2\right)dxdt
\end{aligned}
\tag{II.30}
$$

which ends the proof of Proposition 3.1.

5 Proof of the scattering (Corollary 1.4)

As a result of the L^6 decay estimate we get the scattering result of Corollary 1.4. The proof of this corollary was done in the paper of Blair, Smith, and Sogge [3] and we replicate it here for the sake of completeness.
We have the following Strichartz estimate on functions $w(t,x)$ satisfying homogeneous Dirichlet boundary condition on non-trapping obstacles

$$\|w\|_{L^5(\mathbb{R};L^{10}(\Omega))} + \|w\|_{L^4(\mathbb{R};L^{12}(\Omega))} \leq c\left(\|(\nabla w(0,\cdot),\partial_t w(0,\cdot))\|_{L^2(\Omega)} + \|\Box w\|_{L^1(\mathbb{R};L^2(\Omega))}\right) \quad \text{(II.31)}$$

and we define the conserved energy of the linear equation (II.3)

$$E_0(v;t) = \frac{1}{2}\int_\Omega |\nabla v|^2 + |\partial_t v|^2 dx$$

Attention was restricted to the v_+ function, as symmetric arguments will yield the existence of a v_- asymptotic to u at $-\infty$. As remarked in [3], it is enough to prove (II.5) as (II.4) follows as a consequence. They first established the existence of the wave operator, that is for any solution v to the linear equation (II.3), there exists a unique solution u to the non linear equation (II.1) such that

$$\lim_{t\longrightarrow\infty} E_0(u-v;t) = 0$$

Given (II.31), for any $\delta > 0$ one may select T large so that $\|v\|_{L^5([T,\infty);L^{10}(\Omega))} \leq \delta$. Given any $w(t,x)$ satisfying $\|w\|_{L^5([T,\infty);L^{10}(\Omega))} \leq \delta$, we have a unique solution to the linear problem

$$\Box\widetilde{w} = -(v+w)^5$$

$$\lim_{t\longrightarrow\infty} E_0(\widetilde{w};t) = 0$$

as the right hand side is in $L^1([T,\infty);L^2(\Omega))$. The estimate (II.31) then also ensures that

$$\|\widetilde{w}\|_{L^5([T,\infty);L^{10}(\Omega))} \leq c\,\|v+w\|^5_{L^5([T,\infty);L^{10}(\Omega))} \leq 32c\delta^5$$

Hence for δ sufficiently small, the map $w \longrightarrow \widetilde{w}$ is seen to be a contraction on the ball of radius δ in $L^5\left([T,\infty)\,;L^{10}(\Omega)\right)$. The unique fixed point w can be uniquely extended over all of $\mathbb{R}\times\Omega$. Hence taking $u = v + w$ shows the existence of the wave operator.

To see that the wave operator is surjective, they used the L^6 decay estimate which we pro-

ved in Theorem 1.1 for our obstacle. This decay estimate establishes that the non linear effects of the solution map for the non linear equation (II.1) diminish as time evolves.
By the result of Theorem 1.1, given any $\epsilon > 0$, there exists T sufficiently large such that

$$\sup_{t \geq T} \|u(t, \cdot)\|_{L^6} < \epsilon$$

Hence for any $S > T$ we obtain the following for any solution u to (II.1)

$$\begin{aligned} \|u\|_{L^5([T,S];L^{10}(\Omega))} + \|u\|_{L^4([T,S];L^{12}(\Omega))} &\leq c\left(E + \left\|u^5\right\|_{L^1([T,S];L^2(\Omega))}\right) \\ &\leq cE + c\epsilon \|u\|^4_{L^4([T,S];L^{12}(\Omega))} \end{aligned}$$

A continuity argument now yields $\|u\|_{L^5([T,\infty);L^{10}(\Omega))} + \|u\|_{L^4([T,\infty);L^{12}(\Omega))} \leq 2cE$ and by time reflection argument, (II.5) follows. However, this implies that the linear problem

$$\Box w = -u^5$$

$$\lim_{t \longrightarrow \infty} E_0(w; t) = 0$$

admits a solution, showing that the wave operator is indeed surjective as $v = u - w$ is the desired solution to (II.3).

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Chapitre III

The NLS in the exterior of 2D star-shaped and almost star-shaped obstacles

1 Introduction and Background

We are interested in this paper in the nonlinear Schrödinger equation in exterior domains $\Omega = \mathbb{R}^n \setminus V$ where V is a non-trapping obstacle with smooth boundary with Dirichlet boundary condition

$$\begin{aligned} &i\partial_t u + \Delta u = \pm |u|^{p-1}u \;\; in \;\; \Omega = \mathbb{R}^n \setminus V, \;\; p \geq 1 \\ &u|_{\mathbb{R}\times\partial\Omega} = 0 \\ &u(0,x) = u_0(x) \end{aligned} \tag{III.1}$$

The class of solutions to (III.1) is invariant by the scaling

$$u(t,x) \longrightarrow \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x) \tag{III.2}$$

This scaling defines a notion of criticality, specifically, for a given Banach space of initial data u_0, the problem is called critical if the norm is invariant under (III.2). The problem is called subcritical if the norm of the rescaled solution diverges as $\lambda \to \infty$; if the norm shrinks to zero, then the problem is supercritical. Moreover, considering the initial value problem (III.1) for $u_0 \in \dot{H}^s(\mathbb{R}^n)$, the problem is critical when $s = s_c := \frac{d}{2} - \frac{2}{p-1}$, subcritical when $s > s_c$, and supercritical when $s < s_c$.
Now, denote by

$$M(u) = \int_\Omega |u|^2 dx \text{ and } E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx \pm \frac{1}{p+1}\int_\Omega |u|^{p+1} dx, \tag{III.3}$$

the mass and the energy which are conserved.

For the case of 3D exterior domains, Planchon and Vega obtained in [18] an $L^4_{t,x}$ Strichartz estimate and they used it along with local smoothing estimates near the boundary to prove the local well-posedness of the family of nonlinear equations (III.1) for $1 < p < 5$ and $u_0 \in H^1_0(\Omega)$, and that the solution is global for the defocusing case (+ sign in (III.1)). They also proved scattering for the cubic defocusing nonlinear equation outside star-shaped obstacles for initial data in H^1_0. For the energy critical case $p = 5$, Ivanovici proved in [10] local well-posedness for solutions with initial data in H^1 and global well-posedness for small data, outside strictly convex obstacles using the Melrose-Taylor parametrix. Scattering results were also obtained for all subquintic defocusing nonlinearities. Ivanovici and Planchon then extended in [11] the local well posedness (and global for small energy data) to the quintic nonlinear Schrödinger equation for any non-trapping domain in $\mathbb{R}^3$ using the smoothing effect in $L^5_x(L^2_t)$ for the linear equation. Their local result also holds for the Neumann boundary condition. They also extended the scattering of solutions to the defocusing nonlinear equation outside star-shaped obstacles with initial data in H^1_0 for $3 \leq p < 5$. A very recent result was obtained by Killip, Visan, and Zhang in [14] for the quintic defocusing NLS in the exterior of strictly convex 3D obstacles with the Dirichlet boundary condition, where they proved global well-posedness and scattering for all initial data in the energy space.

Our main interest here is exterior domains in 2 dimensions which is known to be the most difficult one regarding scattering questions even in the case of the full space $\mathbb{R}^n$. In fact, after the results of Ginibre and Velo [9] for $\mathbb{R}^n$ $(n \geq 3)$ for the H^1 subcritical case that corresponds to the case $0 < s_c < 1$, the obstruction of the dimension was removed by Nakanishi [17] (in dimensions 1 and 2, all powers p have an s_c that is less than 1), but his techniques are not well suited for the domains case. However, a fundamental contribution to the existence and scattering theory in the whole space and that turned out later [18] to be suitable for the case of exterior domains, was by Colliander, Keel, Staffilani, Takaoka, and Tao ([7], [8]) through introducing the Morawetz interactive inequalities. Similar problem with low dimensions appears due to the sign of the bilaplacian term that comes from the use of a convex weight which is the euclidean distance. The sign turns out to be wrong for dimensions less than 3. This obstruction was then overcome simultaneously and independently by Colliander, Grillakis, and Tzirakis in [5] as well as by Planchon and Vega in [18].

In [18] the authors also used the bilinear multiplier technique to obtain their results for exterior domains in 3D. Again, just like in the whole space, the obstruction of the dimension appears as a result of the sign of the bilaplacian. That is why the local smoothing (Prop. 2.7 in [18]), which is a key ingredient in the proof of existence and scattering, was given in dimension 3 and higher. However, Planchon and Vega recently removed this restriction in [19] and they obtained global existence and scattering results in 2D domains exterior to star-shaped obstacles to the nonlinear defocusing problem with initial data in H^1_0 and for $p \geq 5$.

The main idea in [19] was using the tensor product technique (as developed e.g. in [6] to obtain a quadrilinear Morawetz interaction estimate in $\mathbb{R}$) by constructing $v(x, y) = u(x)u(y)$ solution of the nonlinear Schrödinger in $\Omega \times \Omega$, and then using the local smoothing inequality obtained from Morawetz's multipliers in dimension $n = 4$ thus resolving the issue of the wrong sign of the bilaplacian in dimension 2. Their local smoothing estimate is a key step to get that $D^{1/2}(|u|^2)$ is in $L^2_{t,x}$ for both the nonlinear and linear solutions which leads to obtain the global in time Strichartz estimate $L^{p-1}_t L^\infty_x$ (for the case of star-shaped obstacles) which is the key factor to get their result.

In this paper we extend the result of Planchon and Vega in two directions, the range of nonlinearities and the class of obstacles under consideration. First, we extend the local existence for $p > 4$ and for any non-trapping obstacle by using the following set of Strichartz estimates obtained by Blair, Smith, and Sogge in [1] :

Theorem 1.1. *(Theorem 1.1, [1]) Let $\Omega = \mathbb{R}^n \setminus V$ be the exterior domain to a compact non-trapping obstacle with smooth boundary, and Δ the standard Laplace operator on Ω, subject to either Dirichlet or Neumann conditions. Suppose that $p > 2$ and $q < \infty$ satisfy*

$$\begin{cases} \dfrac{3}{p} + \dfrac{2}{q} \leq 1, & n = 2, \\ \dfrac{1}{p} + \dfrac{1}{q} \leq \dfrac{1}{2}, & n \geq 3. \end{cases}$$

Then for $e^{it\Delta} f$ solution to the linear Schrödinger equation with initial data f, the following estimates hold

$$\|e^{it\Delta} f\|_{L^p([0,T];L^q(\Omega))} \leq C\|f\|_{H^s(\Omega)},$$

provided that

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2} - s.$$

For Dirichlet boundary conditions, the estimates hold with $T = \infty$.

Remark 1.2. *Remark that as an application to the nonlinear Schrödinger equation in 3D exterior domains, the authors used their above result and interpolation to establish the $L^4_t L^\infty_x$ Strichartz estimate and present a simple proof to the well-posedness result for small energy data to the quintic nonlinear Schrödinger equation, a result first obtained by Ivanovici and Planchon [11].*

We will use in our work the Besov spaces which are defined here using the spectral localization associated to the domain. We refer to [12] and section 2 of our appendix for a detailed discussion and references, and we provide only basic definitions here. Let $\psi(\cdot) \in C_0^\infty(\mathbb{R} \setminus \{0\})$ and $\psi_j(\cdot) = \psi(2^{-2j}\cdot)$. On the domain Ω, one has the spectral resolution of the Dirichlet Laplacian, and we may define smooth spectral projections $\Delta_j = \psi_j(-\Delta_D)$ as continuous operators on L^2 (they are also continuous on L^p for all p). Moreover, just like the whole space case, these projections obey Bernstein estimates.

Definition 1.3. *Let $f \in S^{'}(\Omega)$ and let $\Delta_j = \psi(-2^{-2j}\Delta_D)$ be a spectral localization with respect to the Dirichlet Laplacian Δ_D such that $\sum_j \Delta_j = Id$. We say f belongs to $\dot{B}^{s,q}_p(\Omega)$ ($s \in \mathbb{R}$,*

$1 \leq p, q \leq +\infty$*) if*

$$\left(2^{js}\|\Delta_j f\|_{L^p}\right) \in l^q,$$

and $\sum_j \Delta_j f$ *converges to* f *in* S'.

Note that $\dot{B}_2^{1,2} = \dot{H}_0^1$ and by analogy we set $\dot{H}^s$ to be just $\dot{B}_2^{s,2}$. The Banach space $\dot{B}_p^{s,q}$ is equipped with following norm :

$$\|f\|_{\dot{B}_p^{s,q}} := \left(\sum_{j\in\mathbb{Z}} \|2^{js}\Delta_j f\|_{L^p}^q\right)^{\frac{1}{q}}.$$

Remark 1.4. *In our range of interest, this intrinsic definition may be proved to be equivalent with the more well-known definition using the restriction to the domain* Ω *of functions in* $\dot{B}_p^{s,q}(\mathbb{R}^n)$*. However, we will not need this equivalence.*

We first obtain the following result :

Theorem 1.5. *Let* Ω *be* $\mathbb{R}^2 \setminus V$*, where* V *is a non-trapping obstacle, and* $u_0 \in \dot{B}_2^{s_c,1}(\Omega)$*. Then, there exists* $T(u_0)$ *such that the nonlinear equation :*

$$\begin{cases} i\partial_t u + \Delta u = \pm|u|^{p-1}u \quad x \in \Omega,\ t \in \mathbb{R},\ p > 4 \\ u|_{\mathbb{R}\times\partial\Omega} = 0 \\ u(0,x) = u_0(x), \end{cases}$$

admits a unique solution u *in the function space*

$$C([0,T]; \dot{B}_2^{s_c,1}(\Omega)) \cap L^{p-1}([0,T]; L^\infty(\Omega)).$$

Moreover, if $u_0 \in H_0^1(\Omega)$*, then the solution stays in* $H_0^1(\Omega)$ *and it is global in time for the defocusing equation.*

Then, we prove the scattering for the defocusing equation with initial data in $H_0^1(\Omega)$ for star-shaped obstacles as well as for a class of almost star-shaped obstacles satisfying the following geometric condition : Given $0 < \epsilon < 1$

$$(x_1, \epsilon x_2) \cdot n_x > 0 \quad \text{for} \quad x = (x_1, x_2) \in \partial V \tag{III.4}$$

where n_x is the exterior unit normal to ∂V.

Almost star-shaped obstacles that are a natural generalization of the star-shaped were introduced by Ivrii in [13] in the setting of local energy decay for the linear wave equation. In section 3.2.1, we provide an explicit definition for such obstacles as well as an interpretation of the geometric condition (III.4).

We obtain the following theorem :

Theorem 1.6. *Let Ω be $\mathbb{R}^2 \setminus V$, where V is star-shaped or almost star-shaped satisfying the condition* (III.4)*, and $u_0 \in H_0^1(\Omega)$. Then the global solution for the defocusing equation*

$$\begin{cases} i\partial_t u + \Delta u = |u|^{p-1}u \;\; x \in \Omega, \; t \in \mathbb{R}, \; p > 4 \\ u|_{\mathbb{R}\times\partial\Omega} = 0 \\ u(0,x) = u_0(x), \end{cases}$$

scatters in H_0^1.

2 Proof of the local and global existence (Theorem 1.5)

We want to solve

$$\begin{aligned} &i\partial_t u + \Delta u = \pm |u|^{p-1}u \ \ in \ \ \Omega = \mathbb{R}^2 \setminus V, \ \ p > 4 \\ &u|_{\mathbb{R}\times\partial\Omega} = 0 \\ &u(0,x) = u_0(x) \end{aligned} \tag{III.5}$$

We will set $p = \frac{3}{1-\epsilon_0} + 1$ with $0 < \epsilon_0 < 1$.

Note that the Sobolev space with the invariant norm under the scaling (III.2) is $\dot{H}^{s_c}$ with $s_c = \frac{1}{3} + \frac{2\epsilon_0}{3}$.
Using the estimate obtained by Blair, Smith, and Sogge (Theorem 1.1), we can obtain another linear estimate in the Besov space $\dot{B}_2^{s_c,1}$. This is stated in the following proposition :

Proposition 2.1. *Let $\Omega = \mathbb{R}^2 \setminus V$, where V is a non-trapping obstacle with smooth boundary, and Δ is the Dirichlet Laplacian. Then for $e^{it\Delta}f$ solution to the linear Schrödinger equation with initial data f, we have*

$$\|e^{it\Delta}f\|_{L^{\frac{3}{1-\epsilon_0}}([0,+\infty];L^\infty(\Omega)}) \lesssim \|f\|_{\dot{B}_2^{s_c,1}(\Omega)} \tag{III.6}$$

Proof. For exterior domains in $\mathbb{R}^2$ and given any $0 < \epsilon < 1$, we have the following Strichartz estimate obtained by Blair, Smith, and Sogge

$$\|e^{it\Delta}f\|_{L_t^{\frac{3}{1-\epsilon}} L_x^{\frac{2}{\epsilon}}} \leq C(\epsilon)\|f\|_{\dot{H}^{\frac{1}{3}(1-\epsilon)}} \tag{III.7}$$

On a dyadic block $\Delta_j f$, where Δ_j is defined via the Dirichlet Laplacian Δ, the Blair-Smith-Sogge estimate is written as follows

$$\|\Delta_j(e^{it\Delta}f)\|_{L_t^{\frac{3}{1-\epsilon}} L_x^{\frac{2}{\epsilon}}} \lesssim 2^{j\frac{1-\epsilon}{3}}\|\Delta_j f\|_{L^2} \tag{III.8}$$

for any $0 < \epsilon < 1$. This can be easily obtained using (III.7) and the fact that Δ_j commutes with $e^{it\Delta}$ as well as a Bernstein's inequality.
Now, we choose $\epsilon = \epsilon_0$, we have

$$2^{\epsilon_0 j}\|\Delta_j(e^{it\Delta}f)\|_{L_t^{\frac{3}{1-\epsilon_0}} L_x^{\frac{2}{\epsilon_0}}} \lesssim 2^{j\frac{1+2\epsilon_0}{3}}\|\Delta_j f\|_{L^2}$$

But by Bernstein we have,

$$\|\Delta_j(e^{it\Delta}f)\|_{L^\infty_x} \lesssim 2^{j\epsilon_0}\|\Delta_j(e^{it\Delta}f)\|_{L_x^{\frac{2}{\epsilon_0}}}$$

hence

$$\begin{aligned}\|e^{it\Delta}f\|_{L_t^{\frac{3}{1-\epsilon_0}}L^\infty_x} &\leq \sum_j \|\Delta_j(e^{it\Delta}f)\|_{L_t^{\frac{3}{1-\epsilon_0}}L^\infty_x}\\ &\lesssim \sum_j 2^{j\epsilon_0}\|\Delta_j(e^{it\Delta}f)\|_{L_t^{\frac{3}{1-\epsilon_0}}L_x^{\frac{2}{\epsilon_0}}}\\ &\lesssim \sum_j 2^{j\frac{1+2\epsilon_0}{3}}\|\Delta_j f\|_{L^2}\left(=\|f\|_{\dot{B}_2^{s_c,1}}\right)\end{aligned}$$

Hence we get the following linear estimate

$$\left\|e^{it\Delta}f\right\|_{L_t^{\frac{3}{1-\epsilon_0}}L^\infty_x} \lesssim \|f\|_{\dot{B}_2^{s_c,1}}$$

which ends the proof of Proposition 2.1. □

Now, using the estimate (III.6), we can solve the nonlinear equation (III.5) with initial data in $\dot{B}_2^{s_c,1}$ locally in time in the function space E_T given by : for $T>0$

$$E_T = C([0,T];\dot{B}_2^{s_c,1}(\Omega)) \cap L^{\frac{3}{1-\epsilon_0}}([0,T];L^\infty(\Omega)).$$

Set $F(x) = |x|^{\frac{3}{1-\epsilon_0}}x$ (or $-|x|^{\frac{3}{1-\epsilon_0}}x$ in the focusing case) and choose T small enough so that $\|e^{it\Delta}u_0\|_{L_{[0,T]}^{\frac{3}{1-\epsilon_0}}L^\infty_x} < c$ for a small constant c to be determined and which is linked to the size of the Besov norm of u_0. The larger the latter is, the smaller the former will have to be.

Remark 2.2. *Remark that the smallness of this quantity can be made explicit if u_0 is in H^1 (not just $\dot{B}_2^{s_c,1}$), and then T will be like an inverse power of the norm $\dot{H}^1$ of u_0 (see for example page 22 of [4] for a similar reasoning). Moreover, for the defocusing case, the H^1 norm is controlled and thus the local time of existence is uniform and one can consequently iterate the local existence result to a global result.*

We define the following mapping for $w \in E_T$

$$\phi(w)(t) := \int_{s<t} e^{i(t-s)\Delta}F(e^{is\Delta}u_0 + w(s))ds$$

then we have

$$\|\phi(w)\|_{E_T} \lesssim \|F(e^{it\Delta}u_0+w)\|_{L^1([0,T];\dot{B}^{s_c,1}_2)} \lesssim \|e^{it\Delta}u_0+w\|_{L^\infty_T\dot{B}^{s_c,1}_2}\|e^{it\Delta}u_0+w\|^{\frac{3}{1-\epsilon_0}}_{L_T^{\frac{3}{1-\epsilon_0}}L^\infty_x} \tag{III.9}$$

The first part can be shown using the linear estimate (III.6), as for the second part, it is due to the following lemma (for the special case $f = e^{it\Delta}u_0 + w$ and $g = 0$) :

Lemma 2.3. *Consider $f, g \in L^\infty_T \dot{B}^{s,q}_p \cap L^{\alpha-1}_T L^\infty_x$ with $0 < s < 2$, then if $F(x) = |x|^{\alpha-1}x$ (or $|x|^\alpha$) and $\alpha \geq 3$ we have*

$$\begin{aligned}\|F(f)-F(g)\|_{L^1_T\dot{B}^{s,q}_p} &\lesssim \|f-g\|_{L^\infty_T\dot{B}^{s,q}_p}(\|f\|^{\alpha-1}_{L^{\alpha-1}_TL^\infty_x}+\|g\|^{\alpha-1}_{L^{\alpha-1}_TL^\infty_x})\\ &+\|f-g\|_{L^{\alpha-1}_TL^\infty_x}(\|f\|_{L^\infty_T\dot{B}^{s,q}_p}\|f\|^{\alpha-2}_{L^{\alpha-1}_TL^\infty_x}+\|g\|_{L^\infty_T\dot{B}^{s,q}_p}\|g\|^{\alpha-2}_{L^{\alpha-1}_TL^\infty_x})\end{aligned}$$

Proof. This lemma can be proved by writing

$$F(f) - F(g) = (f-g)\int_0^1 F'(\theta f + (1-\theta)g)d\theta,$$

and splitting this difference into two paraproducts. For a detailed proof, we refer to Lemma 4.10 in [11] (Lemma 2.5 of our appendix) which is given for functions in $\dot{B}^{s,q}_p \cap L^r$. In fact, we are considering a special case of that lemma with $r = \infty$, the time norms are harmless and can be easily inserted using Hölder. Note that such a result is by now classical if the domain is just $\mathbb{R}^n$, and where the easiest path to prove it is to use the characterization of Besov spaces using finite differences. By contrast, on domains, [11] provides a direct proof using paraproducts which are based on the spectral localization. □

Choosing the small constant c such that $c^{\frac{3}{1-\epsilon_0}}\|u_0\|_{\dot{B}^{s_c,1}_2} << 1$, the estimate (III.9) shows that one can have a small ball in w of E_T that maps into itself. A similar argument on $\|\phi(w) - \phi(w')\|_{E_T}$ for $w' \in E_T$ shows that ϕ is a contraction on the small ball : by Lemma 2.3 (with $\alpha = \frac{3}{1-\epsilon_0} + 1$), if $u = e^{it\Delta}u_0 + w$ and $v = e^{it\Delta}u_0 + w'$

$$\begin{aligned}\|\phi(w)-\phi(w')\|_{E_T} \lesssim \|F(u)-F(v)\|_{L^1([0,T];\dot{B}^{s_c,1}_2)} &\lesssim \|w-w'\|_{L^\infty_T\dot{B}^{s_c,1}_2}(\|u\|^{\alpha-1}_{L^{\alpha-1}_TL^\infty_x}+\|v\|^{\alpha-1}_{L^{\alpha-1}_TL^\infty_x})\\ &+\|w-w'\|_{L^{\alpha-1}_TL^\infty_x}(\|u\|_{L^\infty_T\dot{B}^{s_c,1}_2}\|u\|^{\alpha-2}_{L^{\alpha-1}_TL^\infty_x}+\|v\|_{L^\infty_T\dot{B}^{s_c,1}_2}\|v\|^{\alpha-2}_{L^{\alpha-1}_TL^\infty_x})\end{aligned}$$

Note that the smallness comes from the $||\cdot||^{\alpha-k}$ factors, with $k = 1, 2$. Hence, by the fixed point theorem, there exists a unique w in the small ball such that $\phi(w) = w$ and thus u set as

$u = e^{it\Delta}u_0 + w$ is a solution to the nonlinear Schrödinger equation (III.5) that satisfies

$$u = e^{it\Delta}u_0 + \int_{s<t} e^{i(t-s)\Delta}F(u(s))ds. \tag{III.10}$$

Now, we will show that if the initial data $u_0 \in H_0^1$, then the solution u remains in H_0^1. In fact, if $u_0 \in H_0^1$ then $u_0 \in L^2 = \dot{B}_2^{0,2}$ and $u_0 \in \dot{H}^1 = \dot{B}_2^{1,2}$ (from now on $\dot{H}^1$ will always correspond to $\dot{H}_0^1$). Using the following interpolation inequality

$$\|u_0\|_{\dot{B}_2^{s_c,1}} \lesssim \|u_0\|_{\dot{B}_2^{1,\infty}}^{s_c} \|u_0\|_{\dot{B}_2^{0,\infty}}^{1-s_c}$$

and the fact that

$$\|u_0\|_{\dot{B}_2^{1,\infty}} \leq \|u_0\|_{\dot{B}_2^{1,2}}$$

and

$$\|u_0\|_{\dot{B}_2^{0,\infty}} \leq \|u_0\|_{\dot{B}_2^{0,2}}$$

we get that

$$\|u_0\|_{\dot{B}_2^{s_c,1}} \lesssim \|u_0\|_{\dot{H}^1}^{s_c} \|u_0\|_{L^2}^{1-s_c}$$

Thus $u_0 \in \dot{B}_2^{s_c,1}$ and the nonlinear equation (III.5) with initial data $u_0 \in H_0^1(\Omega)$ has a local solution in E_T given by the Duhamel formula (III.10). Hence, we have

$$\|u\|_{C_T\dot{H}^1} \leq \|u_0\|_{\dot{H}^1} + \||u|^{\frac{3}{1-\epsilon_0}}u\|_{L_T^1\dot{H}^1} \leq \|u_0\|_{\dot{H}^1} + \|u\|^{\frac{3}{1-\epsilon_0}}_{L_T^{\frac{3}{1-\epsilon_0}}L_x^\infty} \|u\|_{L_T^\infty \dot{H}^1}$$

where the nonlinearity is again dealt with by Lemma 2.3 (with $s = 1$, $p = q = 2$). We also have

$$\|u\|_{C_T L_x^2} \leq \|u_0\|_{L_x^2} + \|u\|^{\frac{3}{1-\epsilon_0}}_{L_T^{\frac{3}{1-\epsilon_0}}L_x^\infty} \|u\|_{L_T^\infty L_x^2}.$$

As the solution u is constructed such that its $L_T^{\frac{3}{1-\epsilon_0}}L_x^\infty$ norm is sufficiently small, the above inequalities yield that $u \in C([0,T]; H_0^1)$.

3 Scattering for the defocusing equation (Proof of Theorem 1.6)

In this section, we will show that for the defocusing case with initial data in H_0^1 and for domains Ω exterior to star-shaped obstacles as well as for a class of almost star-shaped obstacles (see section 3.2.1), the solution to the nonlinear equation scatters in H_0^1. To prove that is suffices to show that given any interval I of time where the solution exists the $L_I^{\frac{3}{1-\epsilon_0}} L_x^\infty$ norm is controlled by a universal constant that is independent I. To achieve this we will use the conservation laws of the mass and energy (III.3), as well as additional space-time control of the solution.

3.1 The case of star-shaped obstacles

For star-shaped obstacles, in addition to the conservation laws of the mass and energy, we will use the fact that the $L_t^4 L_x^8$ norm is controlled, which is a consequence of the following result by Planchon and Vega [19] :

Proposition 3.1. *(Planchon-Vega, [19]) Let Ω be $\mathbb{R}^2 \backslash V$, where V is a star-shaped and bounded domain. Then u the solution of*

$$\begin{cases} i\partial_t u + \Delta u = |u|^{p-1} u \;\; p \geq 1 \\ u|_{\mathbb{R}\times\partial\Omega} = 0 \end{cases}$$

satisfies

$$\|D^{1/2}(|u|^2)\|_{L_t^2 L_x^2} \lesssim M^{3/4} E^{1/4},$$

where u is extended by zero for $x \notin \Omega$ to make sense of the half-derivative operator.

Remark 3.2. *Remark that this result is also true for the linear equation, and it plays the key role in proving the $L_t^{p-1} L_x^\infty$ (with $p-1 \geq 4$) Strichartz estimate for star-shaped obstacles in their paper. This is what restricted the range of p in [19], whereas the result of Blair, Smith, and Sogge (Theorem 1.1) allows us to get that estimate with a $p > 4$.*

This proposition combined with a Sobolev embedding yields that

$$\|u\|_{L_I^4 L_x^8} \lesssim M^{3/8} E^{1/8}.$$

Hence we now know that the solution u to the defocusing equation exterior to star-shaped obstacles is such that

$$u \in L^4_I L^8_x \cap L^\infty_I \dot{H}^1$$

But, using the fact that L^8 is continuously included in $\dot{B}^{0,8}_8$ and $\dot{H}^1 = \dot{B}^{1,2}_2$, as well as the following inequalities for Besov spaces :

$$q_1 \leq q_2 \Rightarrow \|u\|_{\dot{B}^{s,q_2}_p} \leq \|u\|_{\dot{B}^{s,q_1}_p}$$

$$p_1 \leq p_2 \Rightarrow \|u\|_{\dot{B}^{s-n(\frac{1}{p_1}-\frac{1}{p_2}),q}_{p_2}} \lesssim \|u\|_{\dot{B}^{s,q}_{p_1}}$$

we get the following continuous embeddings :

$$L^8_x \subset \dot{B}^{-1/4,\infty}_\infty \text{ and } \dot{H}^1 \subset \dot{B}^{0,\infty}_\infty$$

So, the solution u is such that

$$u \in L^4_I(\dot{B}^{-1/4,\infty}_\infty) \cap L^\infty_I(\dot{B}^{0,\infty}_\infty)$$

thus using the well known interpolation inequalities for Lebesgue and Besov spaces, we get that

$$u \in L^q_I(\dot{B}^{\gamma,\infty}_\infty)$$

with

$$\frac{1}{q} = \frac{\alpha}{4} + \frac{1-\alpha}{\infty} = \frac{\alpha}{4}$$

and

$$\gamma = \frac{-\alpha}{4} + 0 \times (1-\alpha) = \frac{-\alpha}{4}$$

for any $\alpha \in]0,1[$. We conveniently choose $\alpha = \frac{8}{9}(1-\epsilon_0)$ (based on the scaling of the space $L_T^{\frac{3}{1-\epsilon_0}} L^\infty_x$), and get that $u \in L_I^{\frac{9}{2(1-\epsilon_0)}}\left(\dot{B}^{-\frac{2}{9}(1-\epsilon_0),\infty}_\infty\right)$, and

$$\begin{aligned}\|u\|_{L_I^{\frac{9}{2(1-\epsilon_0)}} \dot{B}^{-\frac{2}{9}(1-\epsilon_0),\infty}_\infty} &\lesssim \|u\|^\alpha_{L^4_I(\dot{B}^{-1/4,\infty}_\infty)} \|u\|^{1-\alpha}_{L^\infty_I(\dot{B}^{0,\infty}_\infty)} \\ &\lesssim \|u\|^\alpha_{L^4_I L^8_x} \|u\|^{1-\alpha}_{L^\infty_I \dot{H}^1} \leq C(M,E) \qquad \text{(III.11)}\end{aligned}$$

Now, given any interval I of time where the solution exists, and given any $\eta > 0$ there is a finite number of disjoint intervals $I_1, \cdots I_N$ such that $\bigcup_{j=1}^{N} I_j = I$ with $N = N(\eta)$ and

$$\|u\|_{L_{I_j}^{\frac{9}{2(1-\epsilon_0)}} \dot{B}_{\infty}^{-\frac{2}{9}(1-\epsilon_0),\infty}} = \eta, \quad j < N(\eta),$$

$$\|u\|_{L_{I_j}^{\frac{9}{2(1-\epsilon_0)}} \dot{B}_{\infty}^{-\frac{2}{9}(1-\epsilon_0),\infty}} \leq \eta, \quad j = N(\eta).$$

Hence, due to (III.11),

$$N(\eta) \lesssim C(M, E)\eta^{-1}.$$

Now, we fix an ϵ (to be chosen later) such that $0 < \epsilon < \epsilon_0$ and we introduce the following lemma :

Lemma 3.3. *Let $\Omega = \mathbb{R}^2 \setminus V$, where V is a non-trapping obstacle with smooth boundary, and Δ is the Dirichlet Laplacian. Then for $e^{it\Delta} f$ solution to the linear Schrödinger equation with initial data f, we have*

$$\|e^{it\Delta} f\|_{L_t^{\frac{3}{1-\epsilon}} \dot{B}_{\infty}^{\frac{2(\epsilon_0-\epsilon)}{3},1}} \lesssim \|f\|_{\dot{B}_2^{s_c,1}} \tag{III.12}$$

Proof. To prove this we will use again the Blair-Smith-Sogge estimate on a dyadic block $\Delta_j f$:

$$\|\Delta_j(e^{it\Delta} f)\|_{L_t^{\frac{3}{1-\epsilon}} L_x^{\frac{2}{\epsilon}}} \lesssim 2^{j\frac{1-\epsilon}{3}} \|\Delta_j f\|_{L^2}$$

thus

$$2^{\frac{2\epsilon_0+\epsilon}{3} j} \|\Delta_j(e^{it\Delta} f)\|_{L_t^{\frac{3}{1-\epsilon}} L_x^{\frac{2}{\epsilon}}} \lesssim 2^{j\frac{1+2\epsilon_0}{3}} \|\Delta_j f\|_{L^2}$$

But by Bernstein we have,

$$\|\Delta_j(e^{it\Delta} f)\|_{L_x^\infty} \lesssim 2^{j\epsilon} \|\Delta_j(e^{it\Delta} f)\|_{L_x^{\frac{2}{\epsilon}}}$$

hence

$$2^{\frac{2(\epsilon_0-\epsilon)}{3} j} \|\Delta_j(e^{it\Delta} f)\|_{L_t^{\frac{3}{1-\epsilon}} L_x^\infty} \lesssim 2^{\frac{2\epsilon_0+\epsilon}{3} j} \|\Delta_j(e^{it\Delta} f)\|_{L_t^{\frac{3}{1-\epsilon}} L_x^{\frac{2}{\epsilon}}} \lesssim 2^{\frac{1+2\epsilon_0}{3} j} \|\Delta_j f\|_{L_x^2}$$

and thus get

$$\|e^{it\Delta} f\|_{L_t^{\frac{3}{1-\epsilon}} \dot{B}_{\infty}^{\frac{2(\epsilon_0-\epsilon)}{3},1}} \leq \sum_j 2^{\frac{2(\epsilon_0-\epsilon)}{3} j} \|\Delta_j(e^{it\Delta} f)\|_{L_t^{\frac{3}{1-\epsilon}} L_x^\infty} \lesssim \|f\|_{\dot{B}_2^{s_c,1}}$$

□

Using the Duhamel formula (III.10) and the above estimate (III.12) shows that the solution we constructed locally is also in $L_I^{\frac{3}{1-\epsilon}}\dot{B}_\infty^{\frac{2(\epsilon_0-\epsilon)}{3},1}$, and in particular we have by Duhamel on I_j :

$$\|u\|_{L_{I_j}^{\frac{3}{1-\epsilon}}\dot{B}_\infty^{\frac{2(\epsilon_0-\epsilon)}{3},1}} \lesssim \|u(t_j)\|_{\dot{B}_2^{s_c,1}} + \|u\|^{\frac{3}{1-\epsilon_0}}_{L_{I_j}^{\frac{3}{1-\epsilon_0}}L_x^\infty}\|u\|_{L_{I_j}^\infty\dot{B}_2^{s_c,1}} \tag{III.13}$$

On the other hand, we have the following interpolation inequality

$$\|u\|_{\dot{B}_\infty^{0,1}} \lesssim \|u\|^\beta_{\dot{B}_\infty^{-\frac{2}{9}(1-\epsilon_0),\infty}}\|u\|^{1-\beta}_{\dot{B}_\infty^{\frac{2}{3}(\epsilon_0-\epsilon),\infty}}$$

with $0 = -\frac{2}{9}\beta(1-\epsilon_0) + \frac{2}{3}(1-\beta)(\epsilon_0-\epsilon)$. For simplicity, we choose $\epsilon = \frac{2\epsilon_0}{3}$, and thus $\beta = \epsilon_0$. Using the fact that $\dot{B}_\infty^{0,1}$ is continuously included in L^∞ and that

$$\|u\|_{\dot{B}_\infty^{\frac{2}{9}\epsilon_0,\infty}} \le \|u\|_{\dot{B}_\infty^{\frac{2}{9}\epsilon_0,1}}$$

we get that

$$\|u\|_{L_x^\infty} \lesssim \|u\|^{\epsilon_0}_{\dot{B}_\infty^{-\frac{2}{9}(1-\epsilon_0),\infty}}\|u\|^{1-\epsilon_0}_{\dot{B}_\infty^{\frac{2}{9}\epsilon_0,1}}$$

hence,

$$\|u\|_{L_{I_j}^{\frac{3}{1-\epsilon_0}}L_x^\infty} \lesssim \|u\|^{\epsilon_0}_{L_{I_j}^{\frac{9}{2(1-\epsilon_0)}}\dot{B}_\infty^{-\frac{2}{9}(1-\epsilon_0),\infty}}\|u\|^{1-\epsilon_0}_{L_{I_j}^{\frac{3}{1-\epsilon}}\dot{B}_\infty^{\frac{2}{9}\epsilon_0,1}} \tag{III.14}$$

Now, since

$$\|u\|_{\dot{B}_2^{s_c,1}} \lesssim \|u\|^{s_c}_{\dot{H}^1}\|u\|^{1-s_c}_{L^2} \le K$$

where K is a constant that depends on the conserved mass and energy, (III.13) and (III.14) yield

$$\begin{aligned}\|u\|_{L_{I_j}^{\frac{3}{1-\epsilon}}\dot{B}_\infty^{\frac{2}{9}\epsilon_0,1}} &\lesssim K + \|u\|^{\frac{3\epsilon_0}{1-\epsilon_0}}_{L_{I_j}^{\frac{9}{2(1-\epsilon_0)}}\dot{B}_\infty^{-\frac{2}{9}(1-\epsilon_0),\infty}}\|u\|^3_{L_{I_j}^{\frac{3}{1-\epsilon}}\dot{B}_\infty^{\frac{2}{9}\epsilon_0,1}}K \\ &\lesssim K + \eta^{\frac{3\epsilon_0}{1-\epsilon_0}}K\|u\|^3_{L_{I_j}^{\frac{3}{1-\epsilon}}\dot{B}_\infty^{\frac{2}{9}\epsilon_0,1}}\end{aligned}$$

choosing η small enough, we conclude that $\|u\|_{L_{I_j}^{\frac{3}{1-\epsilon}}\dot{B}_\infty^{\frac{2}{9}\epsilon_0,1}}$ is bounded and consequently (by (III.14)) $\|u\|_{L_{I_j}^{\frac{3}{1-\epsilon_0}}L_x^\infty}$ remains bounded by a universal constant C_1 independent of the time interval of existence I. Therefore,

$$\|u\|_{L_I^{\frac{3}{1-\epsilon_0}}L_x^\infty} \le C_1 N \lesssim C(M,E).$$

Hence, our global solution satisfies

$$\|u\|_{L_{\mathbb{R}}^{\frac{3}{1-\epsilon_0}} L_x^\infty} \leq C(M,E).$$

Finally, defining $u_+ \in H_0^1$ as

$$u_+ = u_0 + \int_0^\infty e^{i\tau\Delta}|u|^{p-1}u(\tau)d\tau$$

and similarly for u_-, we get the scattering

$$\|u(\cdot,t) - e^{it\Delta}u_\pm\| = o(1) \quad t \to \pm\infty.$$

3.2 The case of almost star-shaped obstacles

In this section, we will prove the scattering for the defocusing equation for almost star-shaped obstacles V satisfying the following geometric condition : Given an ϵ such that $0 < \epsilon < 1$,

$$(x_1, \epsilon x_2) \cdot n_x > 0 \text{ for } x = (x_1, x_2) \in \partial V \tag{III.15}$$

where n_x is the exterior unit normal to ∂V.

In this case, we lost the $L_t^4 L_x^8$ control which was obtained under the star-shaped assumption. However, we will establish a similar control in some $L_t^a L_x^b$ norm that will play the same role in proving the scattering.

3.2.1 Geometry of the obstacle

In 1969, Ivrii introduced the notion of almost star-shaped obstacles in the setting of the linear wave equation. He proved in [13] local energy decay results for domains exterior to such obstacles in odd dimensions $n > 1$. An almost star-shaped obstacle V ($\Omega = \mathbb{R}^n \setminus V$) is defined as follows :

Definition 3.4. *A bounded open region V with a boundary in class C^1 is said to be almost star-shaped if there exists a D bounded open neighborhood of $\overline{V}$, a real-valued function $\phi \in C^2(\overline{D} \cap \Omega)$ and a constant c_0 such that :*

- $\phi(x) < c_0$, $x \in D \cap \Omega$, $\phi(x) = c_0$, $x \in \partial D$.
- $|\nabla \phi(x)| \geq const > 0$, $x \in \overline{D} \cap \Omega$.

– *The level surfaces $\phi(x) = c$ are strongly convex; the radius of curvature in all directions at all points of $\Omega \cap \overline{D}$ is uniformly bounded from above.*
– *At points of intersection of the level surfaces with ∂V their outer normals and the outer normal to ∂V form an angle which is not greater than a right angle.*

These obstacles are a natural generalization of the star-shaped obstacles. If the level surfaces are spheres with a common center, then V is star-shaped and conversely. According to the above definition, an almost star-shaped obstacle with ellipses as level surfaces satisfies the geometric condition (III.15), where the strict inequality corresponds to an angle strictly less than a right angle in the 4th condition of the above definition. More explicitly, the function ϕ is given by $\phi(x) = \sqrt{x_1^2 + \epsilon x_2^2}$ and this corresponds to what is called the gauge function of the convex body delimited by the ellipse given by the equation $x_1^2 + \epsilon x_2^2 = c^2$.

We also remark that the case of almost star-shaped obstacles corresponds to the works of Strauss [20] and Morawetz [16] that followed in 1975 (independently of Ivrii's work which was unknown to them at that time) on local energy decay for the linear wave equation. Moreover, in the same setting and around the same time in the 70's, another generalization to the star-shaped case was introduced which is the illuminating geometry. Decay results were obtained for the so-called illuminated from interior and illuminated from exterior obstacles (see [2], [3], [15]). We opted to work here with almost star-shaped obstacles and use the gauge function of the ellipse rather than the illuminating geometry (that would impose using the distance to the ellipse) mainly because the computation is much easier with the gauge function. The dog bone like obstacle in Figure III.1 below is an almost star-shaped obstacle (and also illuminated from interior).

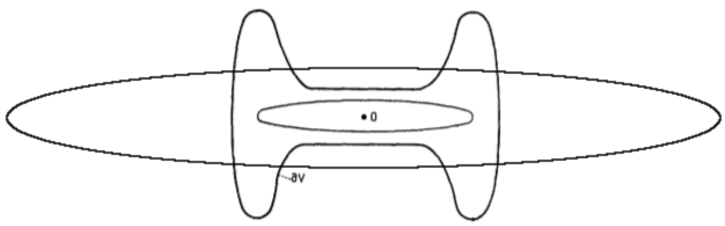

Figure III.1 – dog bone

3.2.2 Space-time control of the solution

In this part, we will prove that the norm of u in some $L_t^a L_x^b$ is controlled by a constant depending on the mass and the energy. This will be a consequence of the following proposition which is an alternative to Proposition 3.1 that is restricted to the star-shaped case :

Proposition 3.5. *Let Ω be $\mathbb{R}^2 \setminus V$, with V is an obstacle satisfying condition (III.15). Assume u is a solution to*

$$\begin{aligned} &i\partial_t u + \Delta u = \alpha |u|^{p-1} u \quad \textit{in } \Omega, \;\; p > 1 \\ &u|_{\mathbb{R}\times\partial\Omega} = 0, \end{aligned}$$

with $\alpha = \{0, 1\}$. Then we have

$$\|D^{-1/2}(|v|^2)\|_{L^2_{t,X}} \lesssim M^{7/4} E^{1/4} \tag{III.16}$$

where $v(X) = v(x, y) = u(x)u(y)$ is the solution to

$$\begin{aligned} &i\partial_t v + \Delta v = \alpha(|u|^{p-1}(x) + |u|^{p-1}(y))v \quad \textit{in } \Omega \times \Omega \\ &v|_{\partial(\Omega\times\Omega)} = 0, \end{aligned}$$

and where we extend $v(\cdot)$ by zero for $x \notin \Omega$ or $y \notin \Omega$, so that (III.16) makes sense for $x \in \mathbb{R}^4$.

This proposition means that (the extension to $\mathbb{R}^4$ of) $|v|^2 \in L^2_t \dot{H}^{-1/2}_X$ and its norm is controlled by a constant depending on the mass and the energy of the solution u.
From now on we will use the notation $C(M, E)$ to denote a constant that depends on the conserved mass and energy of u. This constant may vary from line to line. Moreover, all implicit constants are allowed to depend on the geometry of the obstacle (in particular, they may and will depend on ε appearing in (III.15)). Finally, we also have :

Lemma 3.6. *Let v be again the extension by zero of our solution v to the whole space $\mathbb{R}^4$. Then $|v|^2 \in L^\infty_t H^s_X$, $\forall 0 < s < 1$ and its norm is controlled by $C(M, E)$.*

Proof. We have $u \in H^1$ thus, $\forall 0 < s < 1$, $u \in H^s$ and consequently (by Sobolev embedding), $u \in L^m$ for all $m < \infty$. Now, given any $2 < p < \infty$, we can easily prove that $|u|^2 \in L^p(\mathbb{R}^2)$ and $|u|^2 \in W^{1,q}(\mathbb{R}^2)$ with $1/q = 1/2 + 1/p$. Hence, by Sobolev interpolation inequality, $|u|^2 \in H^{1-2/p}(\mathbb{R}^2)$. So, for any $0 < s < 1$, we have $|u|^2 \in H^s$ and its norm is controlled by $C(M, E)$.

Now, we have

$$\begin{aligned}
\||v|^2\|_{\dot{H}^s}^2 &= \int_{\mathbb{R}^4}(|\xi|^2+|\zeta|^2)^s|\widehat{|v|^2}(\xi,\zeta)|^2 d\xi d\zeta \\
&\leq C_s\int_{\mathbb{R}^4}(|\xi|^{2s}+|\zeta|^{2s})|\widehat{|u|^2}(\xi)|^2|\widehat{|u|^2}(\zeta)|^2 d\xi d\zeta \\
&\leq 2C_s\||u|^2\|_{\dot{H}^s(\mathbb{R}^2)}^2\|u\|_{L^4(\mathbb{R}^2)}^4 \leq C(M,E)
\end{aligned}$$

and it is easy to see that

$$\||v|^2\|_{L^2(\mathbb{R}^4)} = \|u\|_{L^4(\mathbb{R}^2)}^4 \leq C(M,E)$$

$\square$

Fix $0<s<1$ to be chosen later, we have

$$\||v|^2\|_{L_t^{\frac{1+2s}{s}}L_X^2} \lesssim \||v|^2\|_{L_t^2\dot{H}_X^{-1/2}}^{\frac{2s}{1+2s}}\||v|^2\|_{L_t^\infty H_X^s}^{1-\frac{2s}{1+2s}} \leq C(M,E)$$

Consequently, we get our desired control (which now makes sense irrespective of $x\in\mathbb{R}^2$ or $x\in\Omega$)

$$\|u\|_{L_t^{\frac{4(1+2s)}{s}}L_x^4} \leq C(M,E).$$

Now, we are ready to continue the proof which is practically the same as in section 3.1. The solution u is such that

$$u\in L_I^{\frac{4(1+2s)}{s}}L_x^4\cap L_I^\infty\dot{H}^1$$

So,

$$u\in L_I^{\frac{4(1+2s)}{s}}(\dot{B}_\infty^{-1/2,\infty})\cap L_I^\infty(\dot{B}_\infty^{0,\infty})$$

Using the well known interpolation inequalities for Lebesgue and Besov spaces, we get that

$$\|u\|_{L_I^q(\dot{B}_\infty^{\gamma,\infty})} \leq C(M,E)$$

with $q=\frac{4(1+2s)}{s\alpha}$ and $\gamma=\frac{-\alpha}{2}$ for any $\alpha\in]0,1[$. Here, the convenient choice is $\alpha=\frac{4}{3}\frac{(1-\epsilon_0)(1+2s)}{1+3s}$ based on the scaling of the space $L_T^{\frac{3}{1-\epsilon_0}}L_x^\infty$. However, to assure that $0<\alpha<1$, we need to choose s such that $\frac{1-4\epsilon_0}{1+8\epsilon_0}<s<1$. Note that when $1/4\leq\epsilon_0<1$ ($p\geq 5$) any $0<s<1$ will do, but for $0<\epsilon_0<1/4$ ($4<p<5$), we have a restriction on the choice of s. Now, as in section 3.1, we decompose any given interval I of time where the solution exists : Given any $\eta>0$

there is a finite number of disjoint intervals $I_1, \cdots I_N$ such that $\bigcup_{j=1}^{N} I_j = I$ with $N = N(\eta)$ and

$$\|u\|_{L^q_{I_j}\dot{B}^{\gamma,\infty}_\infty} = \eta, \quad j < N(\eta), \text{ and } \|u\|_{L^q_{I_j}\dot{B}^{\gamma,\infty}_\infty} \leq \eta, \quad j = N(\eta).$$

On the other hand, (III.13) still holds

$$\|u\|_{L^{\frac{3}{1-\epsilon}}_{I_j}\dot{B}^{\frac{2(\epsilon_0-\epsilon)}{3},1}_\infty} \lesssim \|u(t_j)\|_{\dot{B}^{s_c,1}_2} + \|u\|^{\frac{3}{1-\epsilon_0}}_{L^{\frac{3}{1-\epsilon_0}}_{I_j}L^\infty_x}\|u\|_{L^\infty_{I_j}\dot{B}^{s_c,1}_2}$$

We choose $\epsilon = \frac{s\epsilon_0}{1+3s} < \epsilon_0$ and we get the following inequality

$$\|u\|_{L^\infty} \lesssim \|u\|^{\epsilon_0}_{\dot{B}^{\gamma,\infty}_\infty}\|u\|^{1-\epsilon_0}_{\dot{B}^{\frac{2}{3}(\epsilon_0-\epsilon),1}_\infty}$$

hence

$$\|u\|_{L^{\frac{3}{1-\epsilon_0}}_{I_j}L^\infty_x} \lesssim \|u\|^{\epsilon_0}_{L^q_{I_j}\dot{B}^{\gamma,\infty}_\infty}\|u\|^{1-\epsilon_0}_{L^{\frac{3}{1-\epsilon}}_{I_j}\dot{B}^{\frac{2}{3}(\epsilon_0-\epsilon),1}_\infty}$$

and the rest follows exactly as in section 3.1.

3.2.3 Proof of Proposition 3.5

In this section we will provide the proof of Proposition 3.5 following an approach similar to one used by Planchon and Vega in [19] to prove Proposition 3.1. First, we will state the following remark that will be useful in our computations :

Remark 3.7. *If H is a function in $\mathbb{R}^{2n}$ of the form*

$$H(x) = \sqrt{x_1^2 + \cdots + x_n^2 + \epsilon(x_{n+1}^2 + \cdots + x_{2n}^2)}$$

with $0 < \epsilon < 1$. Then,

$$\Delta^2 H = \frac{A}{H^3} + \frac{B(x_{n+1}^2 + \cdots + x_{2n}^2)}{H^5} + \frac{C(x_{n+1}^2 + \cdots + x_{2n}^2)^2}{H^7}$$

with

$$A = -n(n+2)\epsilon^2 - 2n(n-3)\epsilon - n^2 + 4n - 3$$
$$B = 2\epsilon(\epsilon-1)(3(\epsilon+1)(n+2) - 15)$$
$$C = -15\epsilon^2(\epsilon-1)^2 < 0$$

Moreover, when $n \geq 3$ *then* $A, B < 0$ $\forall 0 < \epsilon < 1$, *and hence* $\Delta^2 H < 0$.

Now, we have the following proposition :

Proposition 3.8. *Let* Ω *be* $\mathbb{R}^2 \setminus V$, *with* V *is an obstacle satisfying condition* (III.15). *Assume* u *is a solution to*

$$\begin{aligned} &i\partial_t u + \Delta u = \alpha |u|^{p-1} u \quad \text{in } \Omega, \quad p > 1 \\ &u|_{\mathbb{R}\times\partial\Omega} = 0, \end{aligned}$$

with $\alpha = \{0, 1\}$. *Then we have the following estimate*

$$\int\int_{\partial\Omega\times\Omega\times\Omega\times\Omega} \frac{|\partial_{n_{\partial V}} u(x)|^2}{\rho_1(x,y,z,w)} |u(y)|^2 |u(z)|^2 |u(w)|^2 d\sigma_x dy dz dw dt \lesssim M^{7/2} E^{1/2} \tag{III.17}$$

where

$$\rho_1 = \sqrt{x_1^2 + y_1^2 + z_1^2 + w_1^2 + \epsilon(x_2^2 + y_2^2 + z_2^2 + w_2^2)}$$

and M *and* E *are the conserved mass and energy.*

Proof. First, define $v(x, y) = u(x)u(y)$ solution to the problem

$$\begin{aligned} &i\partial_t v + \Delta v = \alpha(|u|^{p-1}(x) + |u|^{p-1}(y))v \quad \text{in } \Omega \times \Omega \\ &v|_{\partial(\Omega\times\Omega)} = 0, \end{aligned}$$

For star-shaped obstacles, in order to obtain local smoothing near the boundary, Planchon and Vega ([19]) considered

$$\int_{\Omega\times\Omega} |v|^2(x, y, t) h(x, y) dx dy$$

with $h(x, y) = \sqrt{|x|^2 + |y|^2}$ and computed the double derivative with respect to time of the 4D integral thus overcoming the problem of the wrong sign of the bilaplacian in 2D. To generalize their procedure to obstacles satisfying condition (III.15), we should take a weight of the form $\sqrt{x_1^2 + \epsilon x_2^2 + y_1^2 + \epsilon y_2^2}$ to ensure that the boundary term has a right sign. However, this will not be enough to cover all epsilons with $0 < \epsilon < 1$ since the bilaplacian will not always have the right sign (see Remark 3.7). This problem can be solved by increasing the dimension through applying the tensor product technique again. Remark that to ensure a right sign of the bilaplacian it is enough to be in 6D ; but to preserve the symmetry of the computations (which is essential in Proposition 3.5), we will apply the tensor product technique again for v. Thus we define $U(x,y,z,w) = v(x,y)v(z,w) = u(x)u(y)u(z)u(w)$ solution to the 8D problem

$$\begin{aligned} &i\partial_t U + \Delta U = \alpha N(u)U \quad \text{in } \Omega\times\Omega\times\Omega\times\Omega \\ &U|_{\partial(\Omega\times\Omega\times\Omega\times\Omega)} = 0, \end{aligned}$$

with

$$N(u) = |u|^{p-1}(x) + |u|^{p-1}(y) + |u|^{p-1}(z) + |u|^{p-1}(w).$$

Now, we consider

$$M_{\rho_1}(t) = \int_{\Omega\times\Omega\times\Omega\times\Omega} |U|^2(x,y,z,w,t)\rho_1(x,y,z,w)dxdydzdw$$

for

$$\rho_1 = \sqrt{x_1^2 + y_1^2 + z_1^2 + w_1^2 + \epsilon(x_2^2 + y_2^2 + z_2^2 + w_2^2)}$$

with $x = (x_1, x_2),\ y = (y_1, y_2),\ z = (z_1, z_2),\ w = (w_1, w_2)$.

and we compute $\frac{d^2}{dt^2}M_{\rho_1}(t)$. This is a standard computation and similar to the one [18] and [19], up to slight modifications to the nonlinear term. We replicate this computation here so that the argument will be self-contained : We have

$$i\partial_t(|U|^2) = U\Delta\overline{U} - \overline{U}\Delta U = \mathrm{div}(U\nabla\overline{U} - \overline{U}\nabla U) = -2i\,\mathrm{div}(Im\overline{U}\nabla U)$$

hence, by integration by parts and using the Dirichlet boundary condition we get

$$\frac{d}{dt}M_{\rho_1}(t) = -2Im\int \rho_1 \mathrm{div}(\overline{U}\nabla U) = 2Im\int \overline{U}\nabla U\cdot\nabla\rho_1$$

Now,

$$\begin{aligned} \frac{d^2}{dt^2}M_{\rho_1}(t) &= 2Im\int(\partial_t\overline{U}\nabla U + \overline{U}\nabla\partial_t U)\cdot\nabla\rho_1 = -2Im\int \partial_t U(2\nabla\overline{U}\cdot\nabla\rho_1 + \overline{U}\Delta\rho_1) \\ &= -2Re\int(\Delta U - \alpha N(u)U)(2\nabla\overline{U}\cdot\nabla\rho_1 + \overline{U}\Delta\rho_1) \\ &= -4Re\int \Delta U\nabla\overline{U}\cdot\nabla\rho_1 + 2\int|\nabla U|^2\Delta\rho_1 + 2Re\int\overline{U}\nabla U\cdot\nabla(\Delta\rho_1) \\ &\quad + 2\alpha\int N(u)\nabla(|U|^2)\nabla\rho_1 + 2\alpha\int N(u)|U|^2\Delta\rho_1 \\ &= -4Re\int \Delta U\nabla\overline{U}\cdot\nabla\rho_1 + 2\int|\nabla U|^2\Delta\rho_1 - \int|U|^2\Delta^2\rho_1 - 2\alpha\int|U|^2\nabla N\cdot\nabla\rho_1. \end{aligned}$$

Integrating by parts again,

$$\int \Delta U \nabla \overline{U} \cdot \nabla \rho_1 = -\int |\partial_n U|^2 \partial_n \rho_1 - \int \nabla U \cdot \nabla(\nabla \overline{U} \cdot \nabla \rho_1)$$

where n is the normal pointing into the domain. and thus

$$\begin{aligned} 2Re \int \Delta U \nabla \overline{U} \cdot \nabla \rho_1 &= -2\int |\partial_n U|^2 \partial_n \rho_1 - \int \nabla(|\nabla U|^2) \cdot \nabla \rho_1 - 2\int Hess\rho_1(\nabla U, \nabla \overline{U}) \\ &= -2\int |\partial_n U|^2 \partial_n \rho_1 + \int |\nabla U|^2 \Delta \rho_1 - 2\int Hess\rho_1(\nabla U, \nabla \overline{U}) \end{aligned}$$

Moreover, by integrating by parts we have

$$\begin{aligned} &-2\alpha \int |U|^2 \nabla N \cdot \nabla \rho_1 \\ &= \frac{2(p-1)}{p+1}\alpha \int |U|^2(|u|^{p-1}(x)\Delta_x \rho_1 + |u|^{p-1}(y)\Delta_y \rho_1 + |u|^{p-1}(z)\Delta_z \rho_1 + |u|^{p-1}(w)\Delta_w \rho_1) \end{aligned}$$

and we finally obtain

$$\begin{aligned} \frac{d^2}{dt^2}M_{\rho_1}(t) &= -\int |U|^2 \Delta^2 \rho_1 + 2\int |\partial_n U|^2 \partial_n \rho_1 + 4\int Hess\rho_1(\nabla U, \nabla \overline{U}) \\ &+ \frac{2(p-1)}{p+1}\alpha \int |U|^2(|u|^{p-1}(x)\Delta_x \rho_1 + |u|^{p-1}(y)\Delta_y \rho_1 + |u|^{p-1}(z)\Delta_z \rho_1 + |u|^{p-1}(w)\Delta_w \rho_1) \end{aligned} \tag{III.18}$$

From our choice of the convex function ρ_1 we have that the terms with the Hessian as well as those with the Laplacian are positive.
We also have from Remark 3.7 that 8D bilaplacian $(n = 4)$ $\Delta^2 \rho_1$ is negative $\forall 0 < \epsilon < 1$. Now, we deal with boundary term. First, we look at the term $\partial_n \rho_1$ with n the normal pointing into $\Omega \times \Omega \times \Omega \times \Omega$, we have

$$\begin{aligned} n &= (n_x, 0, 0, 0) \quad \text{if } x \in \partial\Omega,\ y, z, w \in \Omega \\ n &= (0, n_y, 0, 0) \quad \text{if } y \in \partial\Omega,\ x, z, w \in \Omega \\ n &= (0, 0, n_z, 0) \quad \text{if } z \in \partial\Omega,\ x, y, w \in \Omega \\ n &= (0, 0, 0, n_w) \quad \text{if } w \in \partial\Omega,\ x, y, z \in \Omega \end{aligned}$$

Hence, if $x \in \partial\Omega$

$$\partial_n \rho_1 = \frac{(x_1, \epsilon x_2) \cdot n_x}{\rho_1}$$

which is strictly positive by the geometric condition we imposed (III.15). Moreover,

$$\partial_n \rho_1 \geq \frac{C}{\rho_1}$$

and we also have

$$|\partial_n U|^2 = |\partial_{n_x} u(x)|^2 |u(y)|^2 |u(z)|^2 |u(w)|^2,$$

and we deal similarly when $y, z,$ or $w \in \partial\Omega$. Hence, (III.18) yields

$$\int\int_{\partial\Omega\times\Omega\times\Omega\times\Omega} \frac{|\partial_{n_x} u(x)|^2}{\rho_1} |u(y)|^2 |u(z)|^2 |u(w)|^2 d\sigma dt \lesssim M^{7/2} E^{1/2}$$

which ends the proof of Proposition 3.8. □

Due to the fact that we are doing the tensor product technique more than once, and we are dealing now with four 2D variables, we will need extra estimates on the boundary. We have the following proposition :

Proposition 3.9. *Under the conditions of Proposition 3.8, we have the following estimate*

$$\int\int_{\partial\Omega\times\Omega\times\Omega\times\Omega} \frac{|\partial_{n_x} u(x)|^2 |u(y)|^2 |u(z)|^2 |u(w)|^2}{\sqrt{|x|^2 + |z|^2 + |y \pm w|^2}} d\sigma_x dy dz dw dt \lesssim M^{7/2} E^{1/2} \tag{III.19}$$

Remark 3.10. *Remark that Proposition 3.9 is obviously improving over Proposition 3.8, as the new weight has less decay in some directions (actually, no decay in direction $y - w$ or $y + w$ for example !), whereas ρ_1 is uniformly decaying in all directions.*

Proof. To prove the estimates (III.19), we do the same standard procedure as in Proposition 3.8 with the weight ρ_2 defined as

$$\begin{aligned}\rho_2 = &\sqrt{x_1^2 + \epsilon x_2^2 + z_1^2 + \epsilon z_2^2 + \left(\frac{y_1 - w_1}{\sqrt{2}}\right)^2 + \epsilon\left(\frac{y_2 - w_2}{\sqrt{2}}\right)^2} \\ &+ \sqrt{x_1^2 + \epsilon x_2^2 + z_1^2 + \epsilon z_2^2 + \left(\frac{y_1 + w_1}{\sqrt{2}}\right)^2 + \epsilon\left(\frac{y_2 + w_2}{\sqrt{2}}\right)^2} \\ &:= \rho_2^- + \rho_2^+\end{aligned}$$

Again, we consider

$$M_{\rho_2}(t) = \int_{\Omega\times\Omega\times\Omega\times\Omega} |U|^2(x, y, z, w, t) \rho_2(x, y, z, w) dx dy dz dw$$

and we compute $\frac{d^2}{dt^2}M_{\rho_2}(t)$ to get

$$\frac{d^2}{dt^2}M_{\rho_2}(t) = -\int |U|^2\Delta^2\rho_2 + 2\int |\partial_n U|^2\partial_n\rho_2 + 4\int Hess\rho_2(\nabla U, \nabla\overline{U}) \tag{III.20}$$
$$+\frac{2(p-1)}{p+1}\alpha\int |U|^2(|u|^{p-1}(x)\Delta_x\rho_2 + |u|^{p-1}(y)\Delta_y\rho_2 + |u|^{p-1}(z)\Delta_z\rho_2 + |u|^{p-1}(w)\Delta_w\rho_2)$$

Note that ρ_2 is convex thus the Hessian is positive, and the terms with the Laplacian are positive as well. As for the term of the bilaplacian, note that the functions ρ_2^- and ρ_2^+ of (x,y,z,w) can be also viewed as functions of

$$(x, z, \frac{y-w}{\sqrt{2}}, \frac{y+w}{\sqrt{2}}) := (\xi_1, \xi_2, \xi_3, \xi_4)$$

with $\nabla_{\xi_3}\rho_2^+ = 0$ and $\nabla_{\xi_4}\rho_2^- = 0$. Since the bilaplacian in invariant under rotation, we have

$$\Delta^2_{x,y,z,w}\rho_2^- = \Delta^2_{\xi_1,\xi_2,\xi_3,\xi_4}\rho_2^- = \Delta^2_{\xi_1,\xi_2,\xi_3}\left(\sqrt{\xi_{11}^2+\xi_{21}^2+\xi_{31}^2+\epsilon(\xi_{12}^2+\xi_{22}^2+\xi_{32}^2)}\right)$$

and by Remark 3.7, this 6D bilaplacian ($n=3$) is negative. Similarly, $\Delta^2\rho_2^+ < 0$, hence we have $\Delta^2\rho_2 < 0$.
Now, we deal the boundary term in (III.20). First, we want to control the terms we get on the boundary when $(y,w) \in \partial(\Omega\times\Omega)$. If $y \in \partial\Omega$ then

$$\nabla_y\rho_2 = \frac{1}{2\rho_2^-}(y_1 - w_1, \epsilon(y_2 - w_2)) + \frac{1}{2\rho_2^+}(y_1 + w_1, \epsilon(y_2 + w_2))$$

Introduce

$$\gamma = \sqrt{x_1^2 + \epsilon x_2^2 + z_1^2 + \epsilon z_2^2 + \frac{1}{2}(y_1^2 + \epsilon y_2^2 + w_1^2 + \epsilon w_2^2)}$$

Thus,

$$\rho_2^- = \gamma\sqrt{1 - \frac{y_1w_1 + \epsilon y_2w_2}{\gamma^2}}$$

and

$$\rho_2^+ = \gamma\sqrt{1 + \frac{y_1w_1 + \epsilon y_2w_2}{\gamma^2}}$$

now, we write

$$\frac{1}{\rho_2^\pm} = \frac{1}{\gamma} + \frac{1}{\gamma}\left(\frac{1}{\sqrt{1 \pm \frac{y_1w_1+\epsilon y_2w_2}{\gamma^2}}} - 1\right)$$

and substitute in $\nabla_y \rho_2$ to get

$$\nabla_y \rho_2 = \frac{(y_1, \epsilon y_2)}{\gamma} + \frac{(y_1, \epsilon y_2)}{2} \left[\frac{1}{\gamma} \left(\frac{1}{\sqrt{1 - \frac{y_1 w_1 + \epsilon y_2 w_2}{\gamma^2}}} - 1 \right) + \frac{1}{\gamma} \left(\frac{1}{\sqrt{1 + \frac{y_1 w_1 + \epsilon y_2 w_2}{\gamma^2}}} - 1 \right) \right]$$
$$+ \frac{(w_1, \epsilon w_2)}{2} \left[\frac{1}{\gamma} \left(\frac{1}{\sqrt{1 + \frac{y_1 w_1 + \epsilon y_2 w_2}{\gamma^2}}} - 1 \right) - \frac{1}{\gamma} \left(\frac{1}{\sqrt{1 - \frac{y_1 w_1 + \epsilon y_2 w_2}{\gamma^2}}} - 1 \right) \right]$$

Using the fact the y is bounded and γ is large enough, there exists a positive constant c such that

$$\frac{|y_1 w_1 + \epsilon y_2 w_2|}{\gamma^2} \leq \frac{c}{\gamma} < 1$$

and thus

$$\left| \frac{1}{\gamma} \left(\frac{1}{\sqrt{1 \pm \frac{y_1 w_1 + \epsilon y_2 w_2}{\gamma^2}}} - 1 \right) \right| \leq \left| \frac{1}{\gamma} \left(\frac{1}{\sqrt{1 - \frac{c}{\gamma}}} - 1 \right) \right| \lesssim \frac{1}{\gamma^2}$$

this implies that

$$|\nabla_y \rho_2| \lesssim \frac{1}{\gamma} \leq \frac{\sqrt{2}}{\rho_1}$$

so, the boundary term obtained when $y \in \partial\Omega$ is controlled by Proposition 3.8 :

$$\int \int_{\Omega \times \partial\Omega \times \Omega \times \Omega} |u(x)|^2 |\partial_{n_y} u(y)|^2 |u(z)|^2 |u(w)|^2 |\partial_n \rho_2| dx d\sigma_y dz dw dt$$
$$\lesssim \int \int_{\Omega \times \partial\Omega \times \Omega \times \Omega} \frac{|u(x)|^2 |\partial_{n_y} u(y)|^2 |u(z)|^2 |u(w)|^2}{\rho_1} dx d\sigma_y dz dw dt \lesssim M^{7/2} E^{1/2}$$

similarly for the boundary term generated when $w \in \partial\Omega$.
Now, when $x \in \partial\Omega$, then

$$\partial_n \rho_2 = \nabla_x \rho_2 \cdot n_x = \left(\frac{1}{\rho_2^-} + \frac{1}{\rho_2^+} \right) (x_1, \epsilon x_2) \cdot n_x$$

Again by the geometry of the obstacle, we have $(x_1, \epsilon x_2) \cdot n_x > 0$ and thus

$$\partial_n \rho_2 \gtrsim \frac{1}{\rho_2^-} + \frac{1}{\rho_2^+} \gtrsim_\varepsilon \frac{1}{\sqrt{|x|^2 + |z|^2 + |y - w|^2}} + \frac{1}{\sqrt{|x|^2 + |z|^2 + |y + w|^2}}$$

and

$$|\partial_n U|^2 = |\partial_{n_x} u(x)|^2 |u(y)|^2 |u(z)|^2 |u(w)|^2.$$

and we deal similarly when $z \in \partial\Omega$. So, finally (III.20) yields

$$\int\int_{\partial\Omega\times\Omega\times\Omega\times\Omega} \frac{|\partial_{n_x}u(x)|^2|u(y)|^2|u(z)|^2|u(w)|^2}{\sqrt{|x|^2+|z|^2+|y-w|^2}} d\sigma_x dydzdwdt$$
$$+\int\int_{\partial\Omega\times\Omega\times\Omega\times\Omega} \frac{|\partial_{n_x}u(x)|^2|u(y)|^2|u(z)|^2|u(w)|^2}{\sqrt{|x|^2+|z|^2+|y+w|^2}} d\sigma_x dydzdwdt \lesssim M^{7/2}E^{1/2}.$$

□

Now, we are ready to prove Proposition 3.5. Again, we proceed in a similar argument to that in previous propositions, we consider

$$M_\rho(t) = \int_{(\Omega\times\Omega)\times(\Omega\times\Omega)} |U|^2(X,Y,t)\rho(X,Y)dXdY$$

where $U(X,Y) = v(X)v(Y) = u(x)u(y)u(z)u(w)$, with $X = (x,y),\ Y = (z,w) \in \Omega\times\Omega$, is the solution to the problem

$$\begin{aligned} &i\partial_t U + \Delta U = \alpha N(u)U \quad \text{in } \Omega\times\Omega\times\Omega\times\Omega \\ &U|_{\partial(\Omega\times\Omega\times\Omega\times\Omega)} = 0, \end{aligned}$$

with

$$N(u) = |u|^{p-1}(x) + |u|^{p-1}(y) + |u|^{p-1}(z) + |u|^{p-1}(w)$$

for

$$\begin{aligned} \rho(X,Y) =& |X-Y| + |X'+Y| + |X'-Y| + |X+Y| \\ =& \sqrt{|x-z|^2+|y-w|^2} + \sqrt{|x+z|^2+|y-w|^2} \\ &+ \sqrt{|x-z|^2+|y+w|^2} + \sqrt{|x+z|^2+|y+w|^2} \end{aligned}$$

where $X' = (x,-y)$. Doing the same standard computation, we get

$$\begin{aligned} \frac{d^2}{dt^2}M_\rho(t) =& -\int |U|^2\Delta^2\rho + 2\int |\partial_n U|^2\partial_n\rho + 4\int Hess\rho(\nabla U, \nabla\overline{U}) \\ &+ \frac{2(p-1)}{p+1}\alpha\int |U|^2(|u|^{p-1}(x)\Delta_x\rho + |u|^{p-1}(y)\Delta_y\rho + |u|^{p-1}(z)\Delta_z\rho + |u|^{p-1}(w)\Delta_w\rho) \end{aligned} \tag{III.21}$$

The weight ρ is convex and thus the Hessian term and the Laplacian terms are positive. Moreover, we have

$$-\Delta^2\rho = 12\left(\frac{1}{|X-Y|^3}+\frac{1}{|X'+Y|^3}+\frac{1}{|X'-Y|^3}+\frac{1}{|X+Y|^3}\right)$$

Now, we control the boundary term. If $x \in \partial\Omega$ and $y, z, w \in \Omega$ then $\partial_n\rho = \nabla_x\rho \cdot n_x$ and we have

$$\nabla_x\rho = \frac{x-z}{|X-Y|}+\frac{x+z}{|X'+Y|}+\frac{x-z}{|X'-Y|}+\frac{x+z}{|X+Y|}$$

Setting $\lambda_-^2 = |x|^2+|z|^2+|y-w|^2$ and $\lambda_+^2 = |x|^2+|z|^2+|y+w|^2$, we have :

$$|X-Y|^2 = \lambda_-^2\left(1-\frac{2x\cdot z}{\lambda_-^2}\right), \quad |X'+Y|^2 = \lambda_-^2\left(1+\frac{2x\cdot z}{\lambda_-^2}\right)$$
$$|X'-Y|^2 = \lambda_+^2\left(1-\frac{2x\cdot z}{\lambda_+^2}\right), \quad |X+Y|^2 = \lambda_+^2\left(1+\frac{2x\cdot z}{\lambda_+^2}\right)$$

Reasoning as in Proposition 3.9, we write

$$\frac{1}{|X\pm Y|} = \frac{1}{\lambda_\pm}+\frac{1}{\lambda_\pm}\left(\frac{1}{\sqrt{1\pm\frac{2x\cdot z}{\lambda_\pm^2}}}-1\right)$$

and

$$\frac{1}{|X'\pm Y|} = \frac{1}{\lambda_\mp}+\frac{1}{\lambda_\mp}\left(\frac{1}{\sqrt{1\pm\frac{2x\cdot z}{\lambda_\mp^2}}}-1\right)$$

and we substitute in $\nabla_x\rho$ which yields some convenient cancellations in the z terms. Then, using the fact that $|x|$ is under control and $\lambda_\pm$ are large enough, there exists a positive constant c' such that

$$\frac{|2x\cdot z|}{\lambda_\pm^2} \leq \frac{c'}{\lambda_\pm} < 1$$

and thus

$$\left|\frac{1}{\lambda_-}\left(\frac{1}{\sqrt{1\pm\frac{2x\cdot z}{\lambda_-^2}}}-1\right)\right| \leq \left|\frac{1}{\lambda_-}\left(\frac{1}{\sqrt{1-\frac{c'}{\lambda_-}}}-1\right)\right| \lesssim \frac{1}{\lambda_-^2}$$

and similarly for λ_+. This yields that

$$|\nabla_x \rho| \lesssim \frac{1}{\lambda_-} + \frac{1}{\lambda_+},$$

and thus the boundary term generated when $x \in \partial\Omega$ is controlled by (III.19) of Proposition 3.9, and similarly when $z \in \partial\Omega$.
Now, when $y \in \partial\Omega$ or $w \in \partial\Omega$, we do a similar procedure but with

$$\widetilde{\lambda}_\pm^2 = |y|^2 + |w|^2 + |x \pm z|^2,$$

and we get the same control on the boundary terms by Proposition 3.9. So, finally, (III.21) yields

$$\int \int_{(\Omega\times\Omega)\times(\Omega\times\Omega)} \frac{|v(X)|^2|v(Y)|^2}{|X-Y|^3} dX dY dt \lesssim M^{7/2}E^{1/2}$$

which actually holds on $\mathbb{R}^4 \times \mathbb{R}^4$ provided we extend v by zero inside the obstacle. Then, by Plancherel's theorem we get

$$\|D^{-1/2}(|v|^2)\|^2_{L^2_{t,X}}$$

which ends the proof of Proposition 3.5.

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Appendices

1 Geometric Preliminaries

In this appendix we will we will recall some basic definitions and properties of geometry and differential geometry of surfaces in three dimensions that we used in section 2.1. We start first with some basic geometric definitions :

- Normal curvature : Given a regular surface and a curve within that surface, the normal curvature at a point is the amount of the curve's curvature in the direction of the surface normal.
- Radius of curvature : The radius of curvature is the radius of a circle that mathematically best fits the curve at a given point. It is also simply the reciprocal of the curvature of a curve at that point.
- Principal curvatures : A normal plane at a given point P is one that contains the normal, and will therefore also contain a unique direction tangent to the surface and cut the surface in a plane curve. This curve will in general have different curvatures for different normal planes at P. The principal curvatures at P, denoted by κ_1 and κ_2, are the maximum and minimum values of this curvature. The principal curvatures thus measure the maximum and minimum bending of a regular surface at each point.
- Principal radii of curvature : The minimum and maximum of the radius of curvature at a given point are known as the principal radii of curvature denoted by ρ_1 and ρ_2 which are the reciprocal of κ_1 and κ_2 respectively.
- Principal curves : A curve on a regular surface is a principal curve if and only if the velocity always points in a principal direction (i.e. the directions in which the principal curvatures occur).
- Umbilics : Umbilics or umbilical points are points on a surface that are locally spherical. At such points the normal curvature is the same in all directions, hence, both principal curvatures are equal.

The illuminating geometry. A Curvilinear coordinate system :

We described in section 2.1 the illuminating coordinate system, such that corresponding to each point $X^0(\sigma_1, \sigma_2)$ on the boundary of the illuminating body ∂C (with $\sigma_1 = const.$ and $\sigma_2 = const.$ being the parameterizations of the arc-length of the principal curves near X^0), we have

$$x = X(s, \sigma_1, \sigma_2) = s\nu(\sigma_1, \sigma_2) + X^0(\sigma_1, \sigma_2) \ \in \Omega,$$

where ν is the exterior unit normal to ∂C and s is the algebraic distance from ∂C to $x \in \Omega = \mathbb{R}^3 \setminus V$.

Remark : There is no problem dealing with umbilic points basically because one can assume without loss of generality that the set of umbilics on ∂C is an isolated set, and thus ∂C can be subdivided into a finite number of regions with boundaries that contain all the umbilics. The fact that this is not a restrictive assumption is due to the theorem that an immersion close to an embedding is an embedding [7, Theorem 8.8] and the theorem stating that if X is a compact orientable 2-manifold, then there exists a dense open set of immersions of X into $\mathbb{R}^3$ such that if f is any one of them, then $f(X)$ has only a finite set of umbilics [3, Theorem 6.5].

This illuminating coordinate system is a 3D orthogonal curvilinear coordinate system. Consider a point P in 3D space defined in a curvilinear coordinate system (q_1, q_2, q_3), then the gradient and divergence can be expressed with respect to these coordinates. In fact, if f is a scalar function and $A = A_1e_1 + A_2e_2 + A_3e_3$ is a vector field of the orthogonal curvilinear coordinates, where e_1, e_2 and e_3 are the unit tangent vectors to the q_1, q_2 and q_3 curves at P respectively (the coordinate curves of a curvilinear coordinate system are the analogues of the coordinate axes in the Cartesian system), then we have

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}e_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}e_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}e_3$$

and

$$\text{div}A = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3A_1) + \frac{\partial}{\partial q_2}(h_3h_1A_2) + \frac{\partial}{\partial q_3}(h_1h_2A_3)\right]$$

where h_1, h_2 and h_3 are called the factors of proportionality and they are given by $h_i = \left|\frac{\partial x}{\partial q_i}\right|$ $(i = 1, 2, 3)$, thus $e_i = \frac{\partial x}{\partial q_i}/h_i$.

Now, if we take q_1, q_2 and q_3 to be the illuminating coordinates s, σ_1 and σ_2, then we have

$$x = X(s, \sigma_1, \sigma_2) = s\nu(\sigma_1, \sigma_2) + X^0(\sigma_1, \sigma_2),$$

and

$$\partial_{\sigma_i}\nu \equiv \frac{\partial \nu}{\partial \sigma_i} = \kappa_i X^0_{\sigma_i}$$

and thus

$$X_{\sigma_i} = (\kappa_i s + 1)X^0_{\sigma_i}.$$

Then the factors of proportionality are given by

$$1, \quad |X_{\sigma_1}| = (\kappa_1 s + 1)|X^0_{\sigma_1}|, \quad |X_{\sigma_2}| = (\kappa_2 s + 1)|X^0_{\sigma_2}|$$

and the unit tangent vectors are

$$\nu, \quad \frac{X^0_{\sigma_1}}{|X^0_{\sigma_1}|}, \quad \frac{X^0_{\sigma_2}}{|X^0_{\sigma_2}|}$$

Substituting these in the above expressions of the divergence and the gradient, we get the expressions stated in Remark 2.8.

2 Spectral Projectors and the Besov Spaces

In our work the Besov spaces are defined using the spectral localization associated to the domain. First we recall the definition of the Besov spaces :

Definition 2.1. *Let $f \in S^{'}(\Omega)$ and let $\Delta_j = \psi(-2^{-2j}\Delta_D)$ be a spectral localization with respect to the Dirichlet Laplacian Δ_D such that $\sum_j \Delta_j = Id$. We say f belongs to $\dot{B}^{s,q}_p(\Omega)$ ($s \in \mathbb{R}$, $1 \leq p, q \leq +\infty$) if*

$$\left(2^{js}\|\Delta_j f\|_{L^p}\right) \in l^q,$$

and $\sum_j \Delta_j f$ converges to f in $S^{'}$. The operator $\psi(-2^{-2j}\Delta_D)$ is defined by (22) below.

Note that $\dot{B}^{1,2}_2 = \dot{H}^1_0$ and by analogy we set $\dot{H}^s$ to be just $\dot{B}^{s,2}_2$. The space $\dot{B}^{s,q}_p$ is equipped with following norm :

$$\|f\|_{\dot{B}^{s,q}_p} := \left(\sum_{j\in\mathbb{Z}} \|2^{js}\Delta_j f\|^q_{L^p}\right)^{\frac{1}{q}}.$$

Now, we recall the Dynkin-Helffer-Sjöstrand formula ([2, 4]) and we refer to the appendix of [8] for a nice presentation of the use of almost-analytic extensions in the context of functional calculus.

Definition 2.2. *(see [8, Lemma A.1]) Let $\psi \in C_0^\infty(\mathbb{R})$, possibly complex valued. We assume that there exists $\tilde{\psi} \in C_0^\infty(\mathbb{C})$ such that $|\bar{\partial}\tilde{\psi}(z)| \leq C|Imz|$ and $\tilde{\psi}|_{\mathbb{R}} = \psi$. Then we have (as a bounded operator in $L^2(\Omega)$)*

$$\psi(-h^2\Delta_D) = \frac{i}{2\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{\psi}(z)(z + h^2\Delta_D)^{-1} d\bar{z} \wedge dz. \tag{22}$$

We set

$$\|\psi\|_N := \sum_{m=0}^{N} \int_{\mathbb{R}} |\partial^m \psi(x)|\langle x\rangle^{m-1} dx.$$

We recall the following lemma :

Lemma 2.3. *(see [1, Proposition 6.2]) Let $z \notin \mathbb{R}$ and $|Imz| \lesssim |Rez|$, then Δ_D satisfies*

$$\|(z - \Delta_D)^{-1}\|_{L^p(\Omega)\to L^p(\Omega)} \leq \frac{c}{|Imz|}\left(\frac{|z|}{|Imz|}\right)^\alpha, \quad \forall z \notin \mathbb{R} \tag{23}$$

for $1 \leq p \leq +\infty$, with a constant $c = c(p) > 0$ and $\alpha = \alpha(n,p) > n|\frac{1}{2} - \frac{1}{p}|$.

Remark that, for all $h \in (0, 1]$, the operator $h^2\Delta_D$ satisfies (23) with the same constants c and α (this is nothing but scale invariance).

This lemma yields the L^p boundedness of the operator $\psi(-h^2\Delta_D)$:

Corollary 2.4. *(see [1]) For $N \geq \alpha+1$ the integral (22) is norm convergent and $\forall h \in (0,1]$*

$$\|\psi(-h^2\Delta_D)\|_{L^p(\Omega)\to L^p(\Omega)} \leq c\|\psi\|_{N+1}, \tag{24}$$

for some constant c independent of h.

In the rest of the appendix we will provide a proof to Lemma 2.3, which is a consequence of the following lemma :

Lemma 2.5. *([5, Lemma 4.10]) Consider $\alpha \geq 3$, $u, v \in \dot{B}^{s,q}_p \cap L^r$, with $0 < s < 2$, $\frac{1}{m} = \frac{\alpha-1}{r} + \frac{1}{p}$: Then, if $F(x) = |x|^{\alpha-1}x$ or $F(x) = |x|^\alpha$,*

$$\begin{aligned}\|F(u)-F(v)\|_{\dot{B}^{s,q}_m} &\lesssim \|u-v\|_{\dot{B}^{s,q}_p}(\|u\|^{\alpha-1}_{L^r} + \|v\|^{\alpha-1}_{L^r})\\ &\quad + \|u-v\|_{L^r}(\|u\|_{\dot{B}^{s,q}_p}\|u\|^{\alpha-2}_{L^r} + \|v\|_{\dot{B}^{s,q}_p}\|v\|^{\alpha-2}_{L^r})\end{aligned}$$

Remark 2.6. *We note that all the computations we perform here were done by Ivanovici and Planchon in [5] and we replicate them for self containment. In our work, we are considering $\dot{B}^{s,q}_p \cap L^\infty$. We can easily reinstate the time norms using Hölder to get the result of Lemma 2.3.*

Before starting to prove this lemma, we state some results that will be used in the proof.

Lemma 2.7. *Let f_j be such that $S_j f_j = f_j$, and $\|f_j\|_{L^p} \lesssim 2^{-js}\eta_j$, with $s > 0$ and $(\eta_j)_j \in l^q$. Then $g = \sum_j f_j \in \dot{B}^{s,q}_p$.*

We have, by support conditions,

$$g = \sum_k \Delta_k \sum_{k<j} S_j f_j.$$

Now,

$$\|\Delta_k(\sum_{k<j} S_j f_j)\|_p \lesssim 2^{-ks}\sum_{k<j} 2^{-s(j-k)}\eta_j,$$

which by an $l^1 - l^q$ convolution provides the result.

Lemma 2.8. *Let f_j be such that $(I-S_j)f_j = f_j$, and $\|f_j\|_{L^p} \lesssim 2^{-js}\eta_j$, with $s < 0$ and $(\eta_j)_j \in l^q$. Then $g = \sum_j f_j \in \dot{B}^{s,q}_p$.*

We have, by support conditions,

$$g = \sum_k \Delta_k \sum_{k>j} (I - S_j) f_j.$$

Now,

$$\|\Delta_k (\sum_{k>j} (I - S_j) f_j)\|_p \lesssim 2^{-ks} \sum_{k<j} 2^{-s(j-k)} \eta_j,$$

which by an $l^1 - l^q$ convolution provides the result.

Lemma 2.9. *Consider $f \in \dot{B}_p^{s,q}$ and $g \in L^r$, with $0 < s < 2$, $\frac{1}{m} = \frac{1}{r} + \frac{1}{p}$: let*

$$T_g f = \sum_j S_j g \Delta_j f.$$

Then

$$T_g f \in \dot{B}_m^{s,q}.$$

Proof. We split the paraproduct $T_g f$:

$$T_g f = \sum_j S_j (S_j g \Delta_j f) + \sum_j (I - S_j)(S_j g \Delta_j f);$$

using only Hölder one can easily see by Lemma 2.7 that the first part is in $\dot{B}_m^{s,q}$. For the second one, $K_g f$, taking once again advantage of the spectral supports we have

$$\Delta_k K_g f = \Delta_k \sum_{j<k} (I - S_j)((S_j g)^\alpha \Delta_j f).$$

Notice that here we can't apply Lemma 2.8 because we don't have a negative regularity ($0 < s < 2$). In order to lose regularity, one can derive $K_g f$ and apply to it Lemma 2.8 with regularity $s - 2 < 0$. We have

$$\begin{aligned}
\Delta_D K_g f &= \sum_{j<k} (I - S_j) \Delta_D (S_j g \Delta_j f) \\
&= \sum_{j<k} (I - S_j) \left(\Delta_D S_j g \Delta_j f + (\Delta_D \Delta_j f) S_j g + 2 \nabla S_j g \cdot \nabla \Delta_j f \right)
\end{aligned}$$

The first two pieces can be easily dealt with by Lemma 2.8 (we only need Hölder and the derivative terms are handled by Bernstein) ; the resulting function is in $\dot{B}_m^{s-2,q}$. The remaining cross term is handled using the following (from [6]) :

$$\nabla \Delta_j f = \nabla \exp(4^{-j}\Delta_D)\tilde{\Delta}_j f,$$

where the new dyadic block $\tilde{\Delta}_j$ is built on the function $\tilde{\psi}(\xi) = \exp(|\xi|^2)\psi(\xi)$. From the continuity properties of $\sqrt{s}\nabla \exp(s\Delta_D)$ on L^p, $1 < p < +\infty$, we immediately deduce

$$\|\nabla \Delta_j f\|_p \lesssim 2^j \|\tilde{\Delta}_j f\|_p. \tag{25}$$

The same argument holds for the term $\nabla S_j g$ in L^r, and we can easily sum and conclude. $\square$

Now we are ready to prove Lemma 2.5. In order to obtain a factor $u - v$, we write

$$F(u) - F(v) = (u - v)\int_0^1 F'(\theta u + (1-\theta)v)d\theta. \tag{26}$$

We need to efficiently split this difference into two paraproducts involving $u-v$ and $F'(w)$ with $w = \theta u + (1-\theta)v$, and this requires an estimate on $F'(w)$: write another telescopic series

$$\begin{aligned}
F'(w) =& \sum_j F'(S_{j+1}w) - F'(S_j w)\\
=& \sum_j S_j(F'(S_{j+1}w) - F'(S_j w)) + \sum_j (I - S_j)(F'(S_{j+1}w) - F'(S_j w))\\
=& S_1 + S_2.
\end{aligned}$$

Applying Lemma 2.7, we get that the first sum $S_1 \in \dot{B}^{s,q}_\lambda$ with $\frac{1}{\lambda} = \frac{\alpha-2}{r} + \frac{1}{p}$, as

$$|F'(S_{j+1}w) - F'(S_j w)| \lesssim |\Delta_j w|(|S_{j+1}w|^{\alpha-2} + |S_j w|^{\alpha-2}).$$

The second sum S_2 requires again a trick to be able to be able to apply Lemma 2.8 ; to avoid unnecessary cluttering, we set $F(x) = x^\alpha$, ignoring the sign issue (recall that $\alpha \geq 3$, hence $F'''(x)$ is well-defined as a function) : we apply Δ_D, let $\beta = \alpha - 1 \geq 2$

$$\begin{aligned}
\Delta_D S_2 =& \alpha \sum_j (I - S_j)\Delta_D((S_{j+1}w)^{\alpha-1} - (S_j w)^{\alpha-1})\\
=& \alpha \sum_j (I - S_j)\left(\beta(S_{j+1}w)^{\beta-1}\Delta_D S_{j+1}w - \beta(S_j w)^{\beta-1}\Delta_D S_j w\right.\\
&\left. + \beta(\beta-1)(S_{j+1}w)^{\beta-2}(\nabla S_{j+1}w)^2 - \beta(\beta-1)(S_j w)^{\beta-2}(\nabla S_j w)^2\right).
\end{aligned}$$

Now, we estimate the terms on the right as follows :

$$\begin{aligned}
|((S_{j+1}w)^{\beta-1} - (S_jw)^{\beta-1})\Delta_D S_{j+1}w| &\lesssim C_\beta |\Delta_j w||\Delta_D S_{j+1}w|(|S_{j+1}w|^{\beta-2} + |S_jw|^{\beta-2}) \\
|(S_jw)^{\beta-1}\Delta_D\Delta_j w| &\leq |\Delta_D\Delta_j w||S_{j+1}w|^{\beta-1} \\
|((S_{j+1}w)^{\beta-2} - (S_jw)^{\beta-2})(\nabla S_{j+1}w)^2| &\lesssim \tilde{C}_\beta |\Delta_j w|^{\beta-2}|\nabla S_{j+1}w|^2 \\
|(S_jw)^{\beta-2}((\nabla S_jw)^2 - (\nabla S_{j+1}w)^2)| &\leq |\nabla\Delta_j w|(|\nabla S_jw| + |\nabla S_{j+1}w||S_jw|^{\beta-2}
\end{aligned}$$

The third line was written in this way to deal with the worst case, namely $2 \leq \beta < 3$ (otherwise the power of $\Delta_j w$ in the third bound will be replaced by $|\Delta_j w|(|S_jw|^{\beta-3} + |S_{j+1}w|^{\beta-3})$). Note that all that was done above is decomposing conveniently the terms so that one can just integrate, apply Hölder, use Bernstein to deal with the derivatives, and use (25) to eliminate the ∇ operator. Then one can apply Lemma 2.8 and thus obtain an intermediary result

$$F'(w) \in \dot{B}^{s,q}_\lambda, \quad \text{with } \frac{1}{\lambda} = \frac{\alpha-2}{r} + \frac{1}{p}.$$

Now going back to the difference $F(u) - F(v)$ as expressed in (26), we perform a simple paraproduct decomposition in two terms to which Lemma 2.9 may be applied :

$$\sum_j S_{j+2}(\int_0^1 F'(w)d\theta)\Delta_j(u-v) + \sum_j S_j(u-v)\Delta_j(\int_0^1 F'(w)d\theta)$$

Observe that the integration in θ is irrelevant. For the first sum, we apply Lemma 2.9 for $F'(w) \in L^{r/(\alpha-1)}$ and $u - v \in \dot{B}^{s,q}_p$. As for the second sum, we use $u - v \in L^r$ and $F'(w) \in \dot{B}^{s,q}_\lambda$. This completes the proof.

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Asymptotique de modèles d’ondes nonlinéaires dans les domaines à bord

Résumé. Cette thèse est consacrée à l’étude du comportement asymptotique de modèles d’ondes non linéaires, plus précisément l’équation des ondes non linéaire et l’équation de Schrödinger non linéaire, dans des domaines à bord. Nous sommes principalement intéressés par le comportement en temps long (existence globale et comportement de type scattering) de solutions à ces équations dans des domaines extérieurs d’un obstacle non-captant qui n’est pas étoilé, comme le cas dit illuminé depuis l’intérieur ou depuis l’extérieur et le cas dit presque étoilé... Ces obstacles sont des généralisations naturelles de la forme étoilée et ils étaient largement étudiés dans les années 1960 et 1970, après les travaux pionniers de Morawetz pour le cas étoilé dans le cadre de la décroissance de l’énergie locale pour l’équation des ondes linéaire.
Pour l’équation des ondes non linéaire défocalisante critique en énergie en 3 dimensions, le comportement de type scattering n’est connu que pour le cas étoilé. Dans ce travail, nous étendons le scattering pour des obstacles illuminés depuis l’extérieur en utilisant la méthode de multiplicateurs avec des poids qui généralisent le multiplicateur de Morawetz pour être adapté à la géométrie de l’obstacle.
Pour l’équation de Schrödinger non linéaire défocalisante en 2 dimensions, l’existence globale et le scattering sont connus pour le cas étoilé et des non linéarités puissances qui croissent au moins comme la puissance quintique. Dans cette thèse, nous étendons le résultat d’existence globale pour tous obstacles non-captants et pour des non linéarités avec une puissance strictement supérieure à quartique. Pour tels non linéarités, nous montrons aussi le scattering pour les obstacles étoilés et pour une classe d’obstacles presque étoilés.

Mots clés : Équations des ondes et de Schrödinger, comportement de type scattering, illuminé depuis l’intérieur, illuminé depuis l’extérieur, presque étoilé.

Asymptotics of nonlinear wave models in domains with boundary

Abstract. This thesis is devoted to the study of asymptotics of nonlinear wave models, more specifically the nonlinear wave equation and the nonlinear Schrödinger equation, in domains with boundary. We are mainly interested in the long time behavior (global existence and scattering) of solutions to these equations in domains exterior to non trapping obstacles that are not star-shaped, like the so-called illuminated from interior or from exterior and the so-called almost star-shaped... These obstacles are natural generalizations of the star-shaped and they were extensively studied in the 1960’s and 1970’s after the pioneering work of Morawetz for the star-shaped case in the setting of local energy decay for the linear wave equation.
For the energy critical nonlinear defocusing wave equation in 3 dimensions, scattering is only known for the star-shaped case. In this work, we extend the scattering for illuminated from exterior obstacles using the method of multipliers with weights that generalize the Morawetz multiplier to suit the geometry of the obstacle.
For the nonlinear defocusing Schrödinger equation in 2 dimensions, global existence and scattering are known for the star-shaped case and for nonlinearities that grow at least as the quintic power. In this thesis, we extend the global existence result for all non trapping obstacles and for nonlinearities with power strictly greater than quartic. For such nonlinearities, we also prove scattering for star-shaped obstacles and for a class of almost star-shaped obstacles.

Keywords : Wave and Schrödinger equations, scattering, illuminated from interior, illuminated from exterior, almost star-shaped.