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Théories spectrale et de résonances pour l'opérateur de Schrödinger avec champ magnétique.

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Résumé

Cette thèse traite de certaines propriétés spectrales de deux classes spécifiques des opérateurs de Schrödinger avec champs électromagnétiques en dimension deux. Nous nous intéressons tout d'abord à l'hamiltonien de Landau perturbé par un potentiel dépendant d'un petit paramètre semi-classique ou d'une grande constante de couplage. Nous obtenons alors le comportement asymptotique de la fonction de comptage des valeurs propres dans les trous spectraux avec une estimation optimale du reste.

Le second modèle étudié dans cette thèse est un hamiltonien quadratique avec champ magnétique fort. Nous donnons également la description de la fonction de comptage des valeurs propres lorsque l'intensité du champ magnétique tend vers l'infini. Nous montrons de plus que près des niveaux de Landau, il existe des résonances dont la largeur est polynomialement petite par rapport à l'intensité du champ magnétique.

Mots-clés: Opérateurs de Schrödinger, champs magnétiques, distribution des valeurs propres, résonances, largeur de résonances.

Abstract

This Ph.D thesis deals with some spectral properties of two specific classes of two-dimensional Schrödinger operators with electromagnetic fields. We are firstly interested in the Landau Hamiltonian perturbed by a potential depending on a small semi-classical constant or on a large coupling constant. We obtain an asymptotic behavior of the eigenvalue counting function with sharp remainder estimate.

The second model studied in this thesis is a quadratic Hamiltonian with strong magnetic field. We also give the description of the counting function of eigenvalues when the strength of magnetic field tends to infinity. We show in addition that near Landau levels there exist resonances whose width is polynomially small with respect to the strength of magnetic field.

Keywords: Schrödinger operators, magnetic fields, distribution of eigenvalues, resonances, resonance width.

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Notations

Dans ce manuscrit, nous utilisons les notations et les conventions suivantes :
• $\sigma(P)$ - le spectre de l'opérateur P.
• $\sigma_{\mathrm{d}}(P)$ - le spectre discret de l'opérateur P.
• $\sigma_{\mathrm{ess}}(P)$ - le spectre essentiel de l'opérateur P.
• $\rho(P)$ - l'ensemble résolvant de l'opérateur P.
• $\mathcal{S}(\mathbb{R}^n)$ - l'espace de Schwartz.
• $S^k_\delta(m,\mathbb{R}^{2n})$ - l'espace des symboles.
• $S^0(\mathbb{R}^{2n})$ - l'espace des symboles bornés.
• $M_n(\mathbb{C})$ -l'ensemble des matrices carrées de taille n.
• $S^k_\delta(\mathbb{R}^{2n};M_n(\mathbb{C}))$ - l'espace des symboles à valeurs matricielles où les coefficients appartiennent à $S^k_\delta(\mathbb{R}^{2n})$.
• $a^w(x,hD_x)$ (ou $Op^w_h(a)$) - opérateur $h-$pseudo-différentiel associé au symbole $a(x,\xi)$.
• $\mathrm{tr}(P)$ - la trace de l'opérateur P.
• Soit g une fonction dépendant de h. On écrit $g(h)\sim\sum\limits_{j\geq 0}\alpha_j h^j$, $h\to 0$ si pour tout $N\in\mathbb{N}$, on a $\lim\limits_{h\to 0}\left(g(h)-\sum\limits_{j=0}^{N}\alpha_j h^j\right)h^{-N}=0$.
• Soit $a(x,\xi;h)$ un symbole dépendant de h. On écrit $a(x,\xi;h)\sim\sum\limits_{j\geq 0}a_j(x,\xi)h^j$ dans $S^k_\delta(m,\mathbb{R}^{2n})$ si pour tout $N\in\mathbb{N}$, on a $a(x,\xi;h)-\sum\limits_{j=0}^{N}a_j(x,\xi)h^j\in S^{k-N-1}_\delta(m,\mathbb{R}^{2n})$.
• $\mathcal{L}(L^2(\mathbb{R}^n))$ - l'espace des opérateurs linéaires bornés de $L^2(\mathbb{R}^n)$ dans $L^2(\mathbb{R}^n)$.
• Pour $x\in\mathbb{R}^n$, $\langle x\rangle:=(1+|x|^2)^{\frac{1}{2}}$ et $D_x=\frac{1}{i}\partial_x$.
• On dit que $\lambda\in\mathbb{R}$ est une valeur critique d'une fonction $V\in C^1(\mathbb{R}^n;\mathbb{R})$ si et seulement s'il existe $x\in\{x\in\mathbb{R}^n;\ V(x)=\lambda\}$ tel que $\nabla_x V(x)=0$.
• $\mathcal{F}_h$ - la transformé de Fourier semi-classique et son inverse $\mathcal{F}_h^{-1}$:
pour $\theta\in\mathcal{S}(\mathbb{R}^n)$, $\mathcal{F}_h\theta(\xi)=\int e^{-i(x,\xi)/h}\theta(x)dx$ et $\mathcal{F}_h^{-1}\theta(x)=\frac{1}{(2\pi h)^n}\int e^{i(x,\xi)/h}\theta(\xi)d\xi$.
• $\|.\|_{\mathrm{tr}}$ (resp. $\|.\|_{\mathrm{HS}}$) - la norme trace (resp. Hilbert-Schmidt) des opérateurs.

Chapitre 1

Introduction

1.1 Motivation

Après une série de résultats intéressants concernant les opérateurs de Schrödinger avec champs magnétiques de J. Avron, I. Herbst et B. Simon publiée dans les années soixante-dix du siècle dernier (voir [2, 1, 3]), les propriétés spectrales des opérateurs de ce type ont été intensivement étudiées. Dans la littérature, il existe de nombreux modèles des opérateurs de Schrödinger dépendant des champs magnétiques et électriques. Le changement de champs magnétiques et électriques entraîne effectivement la modification des propriétés spectrales. Le but de cette thèse est principalement d'étudier quelques problèmes spectraux pour deux classes spécifiques d'opérateurs de Schrödinger avec champs magnétiques en dimension deux.

Le premier problème spectral qui a attiré l'attention des mathématiciens est de savoir comment décrire le spectre discret de l'opérateur de Schrödinger avec champ magnétique. Plus précisément, le problème porte sur le comportement du nombre de valeurs propres discrètes par rapport à certains paramètres physiques (par exemple la constante de Plank, l'intensité du champ magnétique, la constante de couplage, etc). Afin de traiter cette question, nous mentionnons ici trois méthodes utilisées dans la littérature.

La première est une approche variationnelle (voir [2, 46, 48, 51, 52]). Cette méthode nous permet de calculer le terme principal dans l'asymptotique de la fonction de comptage des valeurs propres dans des situations générales, notamment dans le cas où le potentiel électrique est peu régulier. Néanmoins, elle n'est pas vraiment efficace pour optimiser le reste. En général, elle exige en plus que le potentiel soit de signe constant.

La deuxième est une approche dûe à V. Ivrii basée sur l'analyse microlocale et la technique multi-échelle (voir [34, 35, 36]). Elle nous permet d'estimer localement le noyau de Schwartz du projecteur spectral de l'opérateur de Schrödinger magnétique avec estimation optimale du reste. Ensuite, une formule reliant le noyau de Schwartz et la fonction de comptage des valeurs propres est utilisée pour trouver l'asymptotique du nombre de valeurs propres avec estimation optimale du reste. Cependant, pour construire le noyau de Schwartz, on utilise toujours une équation d'évolution. Dans plusieurs cas, il est difficile de résoudre une telle équation (voir [14, 15]). Pour cette raison, M. Dimassi et J. Sjöstrand [22] ont développé une méthode stationnaire basée sur la formule dite de Helffer-Sjöstrand.

Cette approche nous permettra de traiter des situations où non seulement le paramètre spectral est implicite, mais aussi quand il n'y a pas vraiment d'équation d'évolution associée (voir [14, 15, 19, 21, 22]). Nous signalons ici que cette méthode exige quelques conditions sur la régularité du potentiel électrique.

Dans ce travail, on reprendra l'approche stationnaire développée par M.Dimassi et J.Sjöstrand pour étudier des formules de trace et la répartition des valeurs propres de deux classes des opérateurs de Schrödinger bidimensionnels suivants :

• L'opérateur de Schrödinger avec champ magnétique constant et la perturbation dépendant du petit paramètre semi-classique $h \searrow 0$, (désigné $H(h)$) ou de la grande constante de couplage $\lambda \nearrow +\infty$, (désigné H_λ).

• L'opérateur de Schrödinger avec champ magnétique fort ($B \nearrow +\infty$) et un potentiel additionnel. Cet opérateur peut être considéré comme un hamiltonien quadratique perturbé.

Pour le premier modèle dans le cas semi-classique, nous obtenons des développements asymptotiques complets de la trace de $f(H(h))$ avec $f \in C_0^\infty(\mathbb{R})$. Notre méthode se généralise facilement pour le cas de dimension $2d$ (avec $d > 1$). Dans le cas des grandes constantes de couplage, nous donnons aussi un développement asymptotique complet en puissances de λ^{-1} de la trace de $f(H_\lambda)$, où $f \in C_0^\infty(\mathbb{R})$. En corollaire, nous obtenons la formule de trace du type Weyl avec reste optimal. Ce résultat généralise celui de G. Raikov qui a obtenu seulement le terme principal dans l'asymptotique de la fonction de comptage des valeurs propres sans aucun contrôle du reste (voir [50] et aussi [47]).
À notre connaissance, il n'y a pas beaucoup de travaux traitant des problèmes spectraux du deuxième modèle. Dans l'article de H. Matsumoto et N. Ueki [45], on a étudié les spectres des hamiltoniens quadratiques non-perturbés en dimension deux. D'autre part, le spectre absolument continu du deuxième modèle a été abordé par P. Exner, A.Joye et H. Kovařík dans [26]. La répartition des valeurs propres n'était cependant pas encore traitée. Nous nous proposons donc d'établir des formules de trace et l'asymptotique du nombre de valeurs propres au dessous du spectre essentiel du deuxième modèle lorsque l'intensité du champ magnétique tend vers l'infini. Ici, nos résultats sont nouveaux et n'ont pas été étudiés dans les travaux [7, 12, 26, 40, 44, 45].

Le deuxième problème spectral est de chercher l'existence des résonances et leur localisation pour les opérateurs de Schrödinger avec champs magnétiques constants. La résonance peut être définie par dilatation analytique (voir [58, 59]), par transition complexe (voir [19, 27]) ou par prolongement méromorphe de la résolvante à partir du demi-plan complexe supérieur. Pour l'opérateur de Schrödinger tridimensionnel avec champ magnétique constant, J. F. Bony, V. Bruneau et G. Raikov ont obtenu une borne supérieure et une borne inférieure du nombre de résonances près des niveaux de Landau (voir aussi [38]). Ils ont de plus établi la relation entre la fonction de décalage spectral et les résonances (appelée approximation de Breit-Wigner). Lorsque l'intensité du champ magnétique tend vers l'infini, X. P. Wang a montré l'existence des résonances près des niveaux de Landau et il a aussi obtenu des développements asymptotiques complets pour l'énergie ainsi que pour la largeur des résonances dans le cas où le potentiel électrique a une haute barrière (voir [58, 59] et aussi [28]).

Comme dans le résultat de X. P. Wang [59], nous prouvons l'existence des résonances près de niveaux de Landau pour notre deuxième modèle et nous donnons un développement asymptotique complet en puissances de B^{-1} de ces résonances. Ici B est l'intensité du champ magnétique. On en déduit que les parties imaginaires de ces résonances sont de l'ordre de $\mathcal{O}(B^{-\infty})$. Le problème consistant à prouver l'absence de valeurs propres plongées est encore ouvert. Rappelons que ce problème est encore ouvert même pour l'opérateur de Stark (voir [19]).

1.2 Opérateur de Schrödinger quadratique avec champ magnétique fort

Introduisons l'opérateur de Schrödinger non-perturbé bidimensionnel avec champ magnétique fort $B \nearrow +\infty$

$$P_0(B,\omega) = (D_x - By)^2 + D_y^2 + \omega^2 x^2,$$

où $\omega \neq 0$ est une constante fixe. Il est bien connu que l'opérateur $P_0(B,\omega)$ est essentiellement auto-adjoint sur $C_0^\infty(\mathbb{R}^2)$ (voir [26, 45]) et son spectre est absolument continu et donné par

$$\sigma(P_0(B,\omega)) = \sigma_{\text{ess}}(P_0(B,\omega)) = \left[\sqrt{B^2+\omega^2}, \infty\right).$$

L'opérateur perturbé est maintenant défini par

$$P(B,\omega) := P_0(B,\omega) + V(x,y).$$

Supposons que le potentiel V est à valeurs réelles, indéfiniment dérivable et tend vers zéro à l'infini. On peut alors montrer que $V(P_0(B,\omega)+i)^{-1}$ est un opérateur compact. Grâce aux théorèmes de Kato-Rellich et de Weyl, l'opérateur $P(B,\omega)$ est aussi essentiellement auto-adjoint sur $C_0^\infty(\mathbb{R}^2)$ et les spectres essentiels de $P(B,\omega)$ et $P_0(B,\omega)$ coïncident, c.-à-.d.

$$\sigma_{\text{ess}}(P(B,\omega)) = \sigma_{\text{ess}}(P_0(B,\omega)) = \left[\sqrt{B^2+\omega^2}, \infty\right).$$

Par conséquent, le spectre discret de $P(B,\omega)$ est inclus dans l'intervalle $\left(-\infty, \sqrt{B^2+\omega^2}\right)$. Comme mentionné dans l'introduction, il y a peu de travaux concernant l'opérateur $P(B,\omega)$. On peut citer les travaux [7, 12, 40] où la répartition des valeurs propres a été réalisée pour le cas où on remplace $\omega^2 x^2$ par un potentiel $W(x)$ borné.

Pour l'opérateur $P(B,\omega)$, on étudiera dans ce travail lorsque $B \nearrow +\infty$

- des formules de trace,
- la répartition des valeurs propres,
- l'existence de résonances.

Posons $\alpha := \sqrt{B^2+\omega^2}$ et notons que $\frac{B}{\alpha} \to 1$ quand $B \nearrow +\infty$.

1.2.1 Fonction de comptage des valeurs propres de $P(B,\omega)$

On commence par annoncer les principaux résultats.

Théorème 1.2.1. (cf. théorème 3.3.1) *Soit $f \in C_0^\infty((-\infty,0);\mathbb{R})$ dont le support ne dépend pas de B. Supposons que $V \in S^0(\mathbb{R}^2)$ tend vers zéro à l'infini. On a alors*

$$\operatorname{tr}\left(f\left(P(B,\omega)-\sqrt{B^2+\omega^2}\right)\right) \sim B\sum_{j=0}^{\infty} A_j(f)B^{-j}, B \nearrow +\infty. \tag{1.1}$$

où

$$\begin{aligned} A_0(f) &= \frac{1}{2\pi}\iint f\left(\omega^2x^2+V(x,y)\right)dxdy,\\ A_1(f) &= \frac{1}{8\pi}\iint \left(\partial_x^2V(x,y)+\partial_y^2V(x,y)\right)f'\left(\omega^2x^2+V(x,y)\right)dxdy. \end{aligned} \tag{1.2}$$

Théorème 1.2.2. (cf. théorème 3.3.2) *Soit λ un nombre réel négatif, fixé. En plus de l'hypothèse du théorème 1.2.1, supposons que λ n'est pas une valeur critique de $\omega^2x^2+V(x,y)$. Il existe alors $\sigma>0$ assez petit et C assez grand tels que, pour tout $f \in C_0^\infty((\lambda-\sigma,\lambda+\sigma);\mathbb{R})$ et $\Psi \in C_0^\infty((-\frac{1}{C},\frac{1}{C});\mathbb{R})$, il existe une suite de fonctions $c_j \in C^\infty(\mathbb{R})$, $j\in\mathbb{N}$, $\forall M,N\in\mathbb{N}$, on a*

$$\begin{aligned} &\operatorname{tr}\left(f\left(P(B,\omega)-\sqrt{B^2+\omega^2}\right)\mathcal{F}_{\frac{1}{\alpha}}^{-1}\Psi\left(\tau-P(B,\omega)+\sqrt{B^2+\omega^2}\right)\right)\\ &= B\left(\sum_{j=0}^{M}c_j(\tau)B^{-j}+\mathcal{O}\left(\frac{B^{-M-1}}{\langle\tau\rangle^N}\right)\right), \quad \text{quand } B\nearrow+\infty, \end{aligned} \tag{1.3}$$

uniformément en $\tau\in\mathbb{R}$, où

$$c_0(\tau)=\frac{1}{2\pi}f(\tau)\Psi(0)\int_{S_\tau}\frac{dS_\tau}{|\nabla_{x,y}(\omega^2x^2+V(x,y))|}. \tag{1.4}$$

Ici dS_τ désigne la densité riemannienne sur $S_\lambda := \{(x,y)\in\mathbb{R}^2|\ \omega^2x^2+V(x,y)=\lambda\}$.

Théorème 1.2.3. (cf. théorème 3.3.3) *Soit $\lambda<0$ fixé. On désigne par $N_B(\lambda)$ le nombre de valeurs propres de $P(B,\omega)-\sqrt{B^2+\omega^2}$ dans $(-\infty,\lambda]$ comptées avec leur multiplicité. Sous les hypothèses du théorème 1.2.2, on a*

$$N_B(\lambda)=B\Big(C_0+\mathcal{O}(B^{-1})\Big), \tag{1.5}$$

où

$$C_0=\frac{1}{2\pi}\iint_{\{(x,y)\in\mathbb{R}^2|\ \omega^2x^2+V(x,y)\le\lambda\}}dxdy.$$

Dans la suite, nous présentons brièvement l'idée de la démonstration.

Idée de la démonstration du théorème 1.2.1

La démonstration du théorème 1.2.1 se décompose essentiellement en trois étapes. On commence par construire un problème de Grushin approprié afin de réduire l'étude du spectre de $P(B,\omega)$ près de l'énergie $\alpha+z$ à celle d'un opérateur $h:=\frac{1}{\alpha}$−pseudo-différentiel $E_{-+}(z)$ près de zéro, $E_{-+}(z): L^2(\mathbb{R})\to L^2(\mathbb{R})$. Ici

$$E_{-+}(z)=z-\omega^2x^2+a^w(x,hD_x,z,h),$$

avec $a(x,\xi,z,h) \in S^0(\mathbb{R}^2)$. Remarquons que, dans notre cas, la présence du potentiel de bord non borné $\omega^2 x^2$ conduit au fait que l'hamiltonien effectif n'est plus un opérateur borné de $L^2(\mathbb{R})$ dans lui-même. Ceci est effectivement la différence principale entre ce travail et celui de M. Dimassi [15]. Notons que M. Dimassi a établi les théorèmes 1.2.1, 1.2.2 et 1.2.3 dans le cas où $\omega = 0$ (voir [15]).

Pour surmonter cette difficulté, par micro-localisation appropriée, modulo un terme négligeable, nous pouvons restreindre l'étude sur les valeurs bornées de x car pour B grand, près de $\lambda < 0$, on est près de la surface $\{(x,\xi) \in \mathbb{R}^2 | \; \omega^2 x^2 + V(x,\xi) = \lambda\}$. En fait, on peut montrer, à l'aide de la formule dite de Helffer et Sjöstrand et du problème de Grushin, que

$$\begin{aligned} \mathrm{tr}(f(P(B,\omega)-\alpha)) &= \mathrm{tr}(f(P(B,\omega)-\alpha)\chi^w(x,hD_x)) \\ &= \mathrm{tr}\left(-\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)(E_{-+}(z))^{-1}\partial_z E_{-+}(z)L(dz)\chi^w(x,hD_x)\right) + \mathcal{O}(h^\infty), \end{aligned} \tag{1.6}$$

où $\tilde{f} \in C_0^\infty(\mathbb{C})$ étant une extension presque analytique de f et χ étant une fonction à support compact qui dépend de V. Grâce à l'estimation (1.6), on n'aura besoin que d'étudier l'hamiltonien effectif $E_{-+}(z)$ dans le support de χ. En utilisant le calcul symbolique, on peut effectivement remplacer $E_{-+}(z)$ dans (1.6) par un opérateur $h-$pseudo-différentiel borné $\tilde{E}(z)$, c.-à-.d.

$$\mathrm{tr}(f(P(B,\omega)-\alpha)) = \mathrm{tr}\left(-\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)(\tilde{E}(z))^{-1}\partial_z \tilde{E}(z)L(dz)\chi^w(x,hD_x)\right) + \mathcal{O}(h^\infty). \tag{1.7}$$

On démontre ensuite que la contribution du membre de droite de (1.7) est négligeable quand on restreint l'intégrale sur le domaine $\{z \in \mathrm{supp}\tilde{f} | \; |\mathrm{Im} z| < h^\delta\}$ pour $\delta \in (0, \frac{1}{2})$. Dans la zone $K_\delta = \{z \in \mathbb{C} | \; |\mathrm{Im} z| \geq h^\delta\}$, on prouve que $(\tilde{E}(z))^{-1}\partial_z \tilde{E}(z)$ est un opérateur $h-$pseudo-différentiel ayant un développement asymptotique en puissances de h. En combinant ceci avec (1.7), on obtient le développement de $\mathrm{tr}(f(P(B,\omega)-\alpha)$ en puissances de h. Ceci termine la preuve du théorème 1.2.1.

Idée de la démonstration du théorème 1.2.2

La démonstration du théorème 1.2.2 est similaire à celle du théorème 1.2.1. Plus précisément, on utilisera les propriétés de l'hamiltonien effectif et un résultat dû à M. Dimassi [14, Théorème 1.8] pour déduire la preuve.

Idée de la démonstration du théorème 1.2.3

Le théorème 1.2.3 peut être considéré comme un corollaire des théorèmes 1.2.1, 1.2.2 et de l'argument taubérien (voir aussi [56, Chapitre 5]). En effet, l'étude de la fonction de comptage des valeurs propres $N_B(\lambda)$ se ramènera à celle d'une fonction

$$M(t,h) := \sum_{t \leq \gamma_j(h)} \phi(\gamma_j(h))$$

où $\gamma_j(h)$ est valeur propre de $P(B,\omega)-\alpha$ et le support de ϕ est contenu dans un petit voisinage de λ. L'argument taubérien permet ensuite d'établir une relation entre la fonction $M(t,h)$ et la trace d'un opérateur du type (1.3) :

$$M(t,h) = \int_{-\infty}^{t} \mathrm{tr}\left(\phi(P(B,\omega)-\alpha)\mathcal{F}_h^{-1}\Psi(\tau-(P(B,\omega)-\alpha))\right)d\tau + \mathcal{O}(1). \tag{1.8}$$

Par conséquent, l'estimation de $N_B(\lambda)$ découle de (1.8), du théorème 1.2.2 et de quelques calculs élémentaires.

1.2.2 Existence des résonances de $P(B,\omega)$

On s'intéresse maintenant à l'existence des résonances de l'opérateur $P(B,\omega)$. Avant de donner la définition spécifique des résonances, nous ferons l'hypothèse suivante sur le potentiel V :

($\mathbf{H_1}$) V est une fonction à valeurs réelles. Il existe de plus des constantes $\alpha_1,\alpha_2,\alpha_3>0$ et $\delta>0$ telles que V admet un prolongement analytique dans le domaine

$$\mathcal{A}=\{(z_1,z_2)\in\mathbb{C}^2;\ |\mathrm{Im}(z_1,z_2)|\le\alpha_1|\mathrm{Re}(z_1,z_2)|+\alpha_2\},$$

et pour tout $(z_1,z_2)\in\mathcal{A}$

$$|V(z_1,z_2)|\le\alpha_3\langle\mathrm{Re}(z_1,z_2)\rangle^{-\delta}.$$

L'hypothèse ($\mathbf{H_1}$) nous permet de définir les résonances de l'opérateur $P(B,\omega)$ en effectuant une dilatation analytique. En effet, on note d'abord (voir l'appendice 3.6) que l'opérateur $P(B,\omega)$ est unitairement équivalent à

$$\begin{aligned}&P_1(B,\omega):=\sqrt{B^2+\omega^2}(D_y^2+y^2)+\omega^2x^2\\&+V^w\left((B^2+\omega^2)^{-\frac14}D_y+(1+\frac{\omega^2}{B^2})^{-\frac12}x,(B^2+\omega^2)^{-\frac12}D_x+B^{-\frac12}(1+\frac{\omega^2}{B^2})^{-\frac34}y\right).\end{aligned}$$

Soit θ réel. Considérons l'opérateur unitaire $U_\theta:L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$, $u(x,y)\mapsto u(e^\theta x,e^{-\theta}y)$. Un calcul simple donne

$$\begin{aligned}&P_{1,\theta}(B,\omega):=U_\theta P_1(B,\omega)U_\theta^{-1}=\sqrt{B^2+\omega^2}(e^{2\theta}D_y^2+e^{-2\theta}y^2)+e^{2\theta}\omega^2x^2\\&+V^w\Big(e^\theta((B^2+\omega^2)^{-\frac14}D_y+(1+\frac{\omega^2}{B^2})^{-\frac12}x),e^{-\theta}((B^2+\omega^2)^{-\frac12}D_x+B^{-\frac12}(1+\frac{\omega^2}{B^2})^{-\frac34}y)\Big).\end{aligned}$$

On simplifie les notations en posant

$$P_{0,\theta}(B,\omega):=\sqrt{B^2+\omega^2}(e^{2\theta}D_y^2+e^{-2\theta}y^2)+e^{2\theta}\omega^2x^2, \tag{1.9}$$

et

$$\begin{aligned}&V_\theta^w(B,\omega)\\&=V^w\Big(e^\theta((B^2+\omega^2)^{-\frac14}D_y+(1+\frac{\omega^2}{B^2})^{-\frac12}x),e^{-\theta}((B^2+\omega^2)^{-\frac12}D_x+B^{-\frac12}(1+\frac{\omega^2}{B^2})^{-\frac34}y)\Big).\end{aligned}$$

On a donc

$$P_{1,\theta}(B,\omega)=P_{0,\theta}(B,\omega)+V_\theta^w(B,\omega).$$

Grâce à l'analyticité du potentiel, on peut étendre la formule de $P_{1,\theta}(B,\omega)$ pour θ dans un petit voisinage complexe de 0. Il résulte alors de [10, Théorème 1] (voir aussi [32]) que l'on peut définir les résonances de l'opérateur $P(B,\omega)$ comme les valeurs propres discrètes de l'opérateur non auto-adjoint $P_{1,\theta}(B,\omega)$. Notons que ces valeurs propres et leurs multiplicités sont indépendantes de θ (voir [10]).

On fixe désormais $\theta \in \mathbb{C}$ tel quel $\mathrm{Im}\theta < 0$ et $|\theta|$ suffisamment petit. Il sera montré ci-dessous que les spectres essentiels de $P_{0,\theta}(B,\omega)$ et $P_{1,\theta}(B,\omega)$ sont les mêmes et égaux à $\bigcup_{n\in\mathbb{N}} \{(2n+1)\sqrt{B^2+\omega^2}+e^{2\theta}\lambda,\ \lambda \geq 0\}$. Les résonances sont donc réparties en dehors de ces demi-droites. Comme mentionné ci-dessus, on étudie l'existence et la localisation des résonances près des niveaux de Landau. Pour faire cela, nous suivons la stratégie utilisée par X.P. Wang dans [58, 59].

Fixons $n \in \mathbb{N}$. On pose $\nu_n := (2n+1)\sqrt{B^2+\omega^2}$ et $h := \frac{1}{\sqrt{B^2+\omega^2}}$. Soit $E \in \mathbb{R}\setminus\{0\}$. En construisant à nouveau un problème de Grushin, nous démontrons que z est une valeur propre de $P_{1,\theta}(B,\omega) - \nu_n$ près de E si et seulement si 0 est valeur propre d'un opérateur $h-$pseudo-différentiel (ce qu'on a appelé l'hamiltonien effectif). Ici, l'hamiltonien effectif se détermine par

$$E_{-+}(z) = z - A_\theta(h) + h^2 G_\theta(z;h), \tag{1.10}$$

où $G_\theta(z;h)$ est holomorphe en z dans un ensemble borné dépendant de B et $A_\theta(h)$ est indépendant de z. L'opérateur $A_\theta(h)$ est de plus un opérateur $h-$pseudo-différentiel avec symbole $a(e^\theta x, e^{-\theta}\xi;h)$ qui admet un développement asymptotique complet en puissances de h :

$$a(x,\xi;h) - a_0(x,\xi) \sim \sum_{j\geq 1} h^j a_j(x,\xi),$$

où

$$a_0(x,\xi) = \omega^2 x^2 + V(x,\xi) \ \text{ et } \ a_1(x,\xi) = \frac{(2n+1)}{4}\Delta V(x,\xi). \tag{1.11}$$

Par conséquent, l'existence des résonances de $P(B,\omega)$ se ramène à l'étude du spectre discret de $A_\theta(h)$. Les étapes cruciales pour démontrer l'existence des résonances sont les suivantes

- Montrer la décroissance exponentielle des fonctions propres de $A_\theta(h)$ associées aux valeurs propres près de E.
- Établir une estimation de la résolvante dans le cas non auto-adjoint.

Afin d'effectuer ces étapes, on a besoin des hypothèses suivantes :

($\mathbf{H_2}$) Soit $E \in \mathbb{R}\setminus\{0\}$. Supposons que $a_0(x_0,y_0) = E$ et E est un extremum local (maximum ou minimum) non dégénéré de a_0, c.-à-.d. la Hessienne $a_0''(x_0,y_0) < 0$ (ou $a_0''(x_0,y_0) > 0$).

Par une translation, on peut supposer que $(x_0,y_0) = (0,0)$. Posons

$$\Omega_E = \{(x,y) \in \mathbb{R}^2 |\ a_0(x,y) = E\}.$$

($\mathbf{H_3}$) (La condition de non-capture) Supposons que $\Omega_E = \{(0,0)\} \cup \Gamma$, où Γ est une courbe connexe de $\mathbb{R}^2$ et $(0,0)$ est un point isolé et que l'hamiltonien classique $a_0(x,\xi)$ est non captif sur Γ :

$$\begin{aligned}\{a_0(x,\xi), G_0(x,\xi)\} &= \partial_\xi a_0 \partial_x G_0 - \partial_x a_0 \partial_\xi G_0 \\ &= \xi\partial_\xi a_0(x,\xi) - x\partial_x a_0(x,\xi) \neq 0,\ \forall (x,\xi) \in \Gamma,\end{aligned} \tag{1.12}$$

où $G_0(x,\xi) = x.\xi$, $\forall(x,\xi) \in \mathbb{R}^2$.

Pour obtenir la largeur des résonances, comme dans [41, 59], on utilise la méthode BKW pour construire une solution approchée du problème $E_{-+}(z)u = \mathcal{O}(h^\infty)$. Puis, en étudiant un problème de Grushin approprié, on obtient un développement asymptotique complet à coefficients réels des résonances en puissances de h. Par conséquent, la largeur des résonances est au moins de taille $\mathcal{O}(h^\infty)$.

Le résultat principal dans cette direction se présente comme suit

Théorème 1.2.4. (Le théorème 4.2.1) *Pour $n \in \mathbb{N}$, on définit*

$$U_n = \left\{ z \in \mathbb{C};\ \mathrm{Re} z \in \left](2n+1)B + E - \frac{C_0}{B}, (2n+1)B + E + \frac{C_0}{B}\right[, \mathrm{Im} z \in \left] -\frac{1}{C_0 B}, 0 \right] \right\}, \tag{1.13}$$

où $C_0 > 1$ peut être arbitrairement grand et en dehors d'un ensemble discret de $\mathbb{R}$. Sous les hypothèses $(\mathbf{H_1})$, $(\mathbf{H_2})$ et $(\mathbf{H_3})$, il existe des résonances de $P(B,\omega)$ dans U_n qui sont toutes données par des développements asymptotiques complets en puissances de B^{-1} lorsque $B \nearrow +\infty$:

$$E_{n,j}(B,\omega) \sim (2n+1)B + E + \frac{1}{2}\Big(\pm (2j+1)(\lambda\mu)^{\frac{1}{2}} + (2n+1)\frac{\lambda+\mu}{2}\Big)B^{-1} + \sum_{k\geq 2} c_{\pm;n,j}^{(k)} B^{-k}, \tag{1.14}$$

où $c_{\pm;n,j}^{(k)} \in \mathbb{R}$, λ et μ sont les valeurs propres de la Hessienne $a_0''(0,0)$, et le signe $+(-)$ correspond à un minimum (respectivement un maximum) local.

De plus les résonances de $P(B,\omega)$ dans U_n sont algébriquement simples et la largeur des résonances est d'ordre de $\mathcal{O}(B^{-\infty})$.

1.3 Hamiltonien de Landau avec champ magnétique constant.

Désignons par B la constante magnétique dont le signe est fixé $B > 0$. On considère l'opérateur de Schrödinger non-perturbé

$$H_0 = (D_x - By)^2 + D_y^2. \tag{1.15}$$

Il est bien connu que l'opérateur H_0 est essentiellement auto-adjoint sur $C_0^\infty(\mathbb{R}^2)$ et son spectre est constitué par les valeurs propres de multiplicité infinie $(2n+1)B, n \in \mathbb{N}$ (voir [2]). Ces valeurs propres sont traditionnellement appelées niveaux de Landau. On perturbe l'opérateur H_0 par un potentiel électrique $\lambda V(hx,hy)$ dépendant de deux paramètres $\lambda > 0$ grand et $h > 0$ petit. Supposons que la fonction V soit de classe C^∞, bornée ainsi que toutes ses dérivées et

$$\lim_{|(x,y)|\to\infty} V(x,y) = 0. \tag{1.16}$$

L'opérateur de multiplication $\lambda V(hx,hy)$ est donc relativement H_0- compact. D'après le théorème de Kato-Rellich et le théorème de Weyl, on déduit que l'opérateur de Schrödinger perturbé $H(V) := H_0 + \lambda V(hx, hy)$ est essentiellement auto-adjoint sur $C_0^\infty(\mathbb{R}^2)$ et que

$$\sigma_{\text{ess}}(H(V)) = \sigma_{\text{ess}}(H_0) = \bigcup_{n\in\mathbb{N}}\{(2n+1)B\}.$$

On s'intéresse alors aux formules de trace et à la répartition des valeurs propres de l'opérateur $H(V)$ dans les trous spectraux selon deux cas : le régime semi-classique ($\lambda = 1, h \searrow 0$) ou la grande constante de couplage ($h = 1, \lambda \nearrow +\infty$).

1.3.1 Cas semi-classique

Considérons l'opérateur de Schrödinger semi-classique

$$H(h) = H_0 + V(hx, hy),$$

où le potentiel V vérifie (1.16). En choisissant $B =$ constante, on peut supposer que $B = 1$.

Soient a et b deux réels fixés tels que $[a, b] \subset \mathbb{R} \setminus \sigma_{\text{ess}}(H(h))$. On définit

$$\begin{aligned} l_0 &:= \min\{q \in \mathbb{N}; V^{-1}([a-(2q+1), b-(2q+1)]) \neq \emptyset\}, \\ l &:= \sup\{q \in \mathbb{N}; V^{-1}([a-(2q+1), b-(2q+1)]) \neq \emptyset\}. \end{aligned} \tag{1.17}$$

On établira un développement asymptotique en puissances de h^2 de $\operatorname{tr}(f(H(h), h))$ dans les deux cas suivants :
a) $f(x, h) = f(x)$, avec $f \in C_0^\infty((a, b)\,; \mathbb{R})$.
b) $f(x, h) = f(x)\mathcal{F}_{h^2}^{-1}\theta(\tau - x)$, où $f, \theta \in C_0^\infty(\mathbb{R}; \mathbb{R})$, $\tau \in \mathbb{R}$.
Par conséquent, on obtiendra le comportement asymptotique du nombre de valeurs propres de $H(h)$ avec l'estimation optimale du reste lorsque $h \searrow 0$.

Présentons maintenant les résultats principaux :

Théorème 1.3.1. (cf. théorème 5.2.1) *Soit $f \in C_0^\infty((a, b); \mathbb{R})$. Supposons que $V \in S^0(\mathbb{R}^2)$ tendant vers zéro à l'infini. Il existe alors une suite de nombres réels $(a_k(f))_{k\in\mathbb{N}}$ telle que*

$$\operatorname{tr}(f(H(h)) \sim \sum_{k=0}^{\infty} a_k(f) h^{2(k-1)}, \tag{1.18}$$

où

$$a_0(f) = \sum_{j=l_0}^{l} \frac{1}{2\pi} \iint f(2j + 1 + V(x, y))dxdy. \tag{1.19}$$

Théorème 1.3.2. (cf. théorème 5.2.2) *Fixons $\mu \in \mathbb{R} \setminus \sigma_{\text{ess}}(H(h))$ qui n'est pas valeur critique de $(2j + 1 + V)$, pour $j = l_0, ..., l$. On suppose que $f \in C_0^\infty((\mu - \epsilon, \mu + \epsilon)\,; \mathbb{R})$ et $\theta \in C_0^\infty(\left(-\frac{1}{C}, \frac{1}{C}\right); \mathbb{R})$, avec $\theta = 1$ dans un petit voisinage de zéro. Il existe alors $\epsilon > 0$, $C > 0$ et une suite de fonctions $c_j \in C^\infty(\mathbb{R}; \mathbb{R})$, $j \in \mathbb{N}$, tels que pour tout $M, N \in \mathbb{N}$, on a*

$$\operatorname{tr}\left(f(H(h))\mathcal{F}_{h^2}^{-1}\theta(t - H(h))\right) = \sum_{k=0}^{M} c_k(t) h^{2(k-1)} + \mathcal{O}\left(\frac{h^{2M}}{\langle t\rangle^N}\right) \tag{1.20}$$

uniformément en $t \in \mathbb{R}$, où

$$c_0(t) = \frac{1}{2\pi} f(t) \sum_{j=l_0}^{l} \int_{\{(x,y)\in\mathbb{R}^2 \mid 2j+1+V(x,y)=t\}} \frac{dS_t}{|\nabla V(x,y)|}. \tag{1.21}$$

Théorème 1.3.3. (cf. corollaire 5.2.3) *On garde les hypothèses du théorème 1.3.1 et on suppose de plus que a et b ne sont pas des valeurs critiques des fonctions $(2j+1+V)$, $j = l_0, ..., l$. Soit $N_h([a,b])$ la fonction de comptage des valeurs propres de $H(h)$ dans l'intervalle $[a,b]$ comptées avec leur multiplicité. Alors on a*

$$N_h([a,b]) = h^{-2} C_0 + \mathcal{O}(1),\ h \searrow 0, \tag{1.22}$$

où

$$C_0 = \frac{1}{2\pi} \sum_{j=l_0}^{l} \mathrm{Vol}\left(V^{-1}([a-(2j+1), b-(2j+1)])\right). \tag{1.23}$$

Remarque 1.3.4. Notons que l'opérateur $H(h)$ est unitairement équivalent à l'opérateur de Schrödinger semi-classique avec champ magnétique fort $P(h) := (hD_x - \frac{1}{h}y)^2 + D_y^2 + V(x,y)$, et que le théorème 1.3.3 a été démontré par V. Ivrii (voir [36]) en se basant sur une réduction de l'opérateur $P(h)$ en une forme canonique du type "forme normale de Birkhoff". Dans ce travail, on donne une autre démonstration en exploitant la méthode de l'hamiltonien effectif et l'approche stationnaire.

Idée de la démonstration

Pour démonstrer les résultats ci-dessus, on utilise à nouveaux l'approche stationnaire et la méthode de l'hamiltonien effectif comme expliqué dans la section précédent. En fait, l'hamiltonien effectif dans ce cas est un opérateur h^2-pseudo-différentiel borné, à valeurs matricielles

$$E_{-+}(z) : L^2(\mathbb{R}; \mathbb{C}^{l-l_0+1}) \to L^2(\mathbb{R}; \mathbb{C}^{l-l_0+1}),$$

dont le symbole est donné par

$$e_{-+}(x,\xi,z,h) \sim \sum_{j=0}^{\infty} e_{-+}^{j}(x,\xi,z) h^j, \quad \text{dans } S^0(\mathbb{R}^2; M_{l-l_0+1}(\mathbb{C})),$$

avec

$$e_{-+}^{0}(x,\xi,z) = \left((z-(2j+1)-V(x,\xi))\delta_{i,j}\right)_{l_0 \le i,j \le l}.$$

En suivant la stratégie de la méthode de l'hamiltonien effectif, on obtient le développement de $\mathrm{tr}(f(H(h))$ en puissances de h.

Remarquons d'autre part que par un changement de variables $(x,y) \mapsto (x,-y))$, les opérateurs $H(h)$ et $H(-h)$ sont unitairement équivalents. Donc $\mathrm{tr}(f(H(h)) = \mathrm{tr}(f(H(-h))$. Ceci montre que $\mathrm{tr}(f(H(h))$ a un développement asymptotique en puissances de h^2. Et les théorèmes 1.3.2 et 1.3.3 sont prouvés de la même manière que les théorèmes 1.2.2 et 1.2.3.

1.3.2 Asymptotiques dans la limite de grande constante de couplage

Appliquons les résultats ci-dessus à l'étude de l'opérateur de Schrödinger avec champ magnétique constant suivant :

$$H_\lambda = (D_x - y)^2 + D_y^2 + \lambda V(x, y).$$

On désigne les points dans le plan $\mathbb{R}^2$ par $X := (x, y)$. Introduisons l'hypothèse suivante sur le potentiel V :

($\mathbf{H_0}$) Supposons que V est strictement positive et que pour tout $N \in \mathbb{N}$,

$$V(X) = \sum_{j=0}^{N-1} \omega_{2j}\left(\frac{X}{|X|}\right)|X|^{-\delta-2j} + r_{2N}(X), \text{ pour } |X| \geq 1, \tag{1.24}$$

où

- $\omega_0 \in C^\infty(\mathbb{S}^1; (0, +\infty))$, $\omega_{2j} \in C^\infty(\mathbb{S}^1; \mathbb{R})$, $j \geq 1$. Ici $\mathbb{S}^1$ désigne le cercle unité.
- δ est une constante positive,
- $|\partial_X^\beta r_{2N}(X)| \leq C_\beta(1 + |X|)^{-|\beta|-\delta-2N}$, $\forall \beta \in \mathbb{N}^2$.

On a $\sigma(H_\lambda) \subset [1, +\infty)$, puisque V est positive. Soit $a > 1$ et $[a, b] \subset \mathbb{R} \setminus \sigma_{\text{ess}}(H_\lambda)$. On s'intéresse à nouveau à la formule de trace et à la répartition des valeurs propres de l'opérateur H_λ dans l'intervalle $[a, b]$ lorsque λ tend vers l'infini. Remarquons que $\sigma_{\text{ess}}(H_\lambda) = \cup_{j=0}^{\infty}\{(2j + 1)\}$, donc il existe $q \in \mathbb{N}$ tel que $2q + 1 < a < b < 2q + 3$.

Théorème 1.3.5. (cf. théorème 5.2.4) *Soit $f \in C_0^\infty((a, b); \mathbb{R})$. Sous l'hypothèse ($\mathbf{H_0}$), il existe une suite de nombres réels $(b_k(f))_{k\geq 0}$ telle que*

$$\operatorname{tr}(f(H_\lambda)) \sim \lambda^{\frac{2}{\delta}} \sum_{k=0}^{\infty} b_k(f)\lambda^{-\frac{2k}{\delta}}, \ \lambda \nearrow +\infty \tag{1.25}$$

où

$$b_0(f) = \frac{1}{2\pi\delta}\int_0^{2\pi}(\omega_0(\cos\theta, \sin\theta))^{\frac{2}{\delta}}d\theta \sum_{j=0}^{q}\int f(u)(u - (2j + 1))^{-1-\frac{2}{\delta}}du. \tag{1.26}$$

Théorème 1.3.6. (cf. théorème 5.2.5) *Soient $f \in C_0^\infty((a - \epsilon, b + \epsilon); \mathbb{R})$ et $\theta \in C_0^\infty((-\frac{1}{C}, \frac{1}{C}); \mathbb{R})$, avec $\theta = 1$ près de 0. Il existe alors $\epsilon > 0$, $C > 0$ et une suite de fonctions $c_j \in C^\infty(\mathbb{R}; \mathbb{R})$, $j \in \mathbb{N}$, tels que pour tout $M, N \in \mathbb{N}$, on a*

$$\operatorname{tr}\left(f(H_\lambda)\mathcal{F}^{-1}_{\lambda^{-\frac{2}{\delta}}}\theta(t - H_\lambda)\right) = \lambda^{\frac{2}{\delta}} \sum_{k=0}^{M} c_k(t)\lambda^{-\frac{2k}{\delta}} + \mathcal{O}\left(\frac{\lambda^{-\frac{2M}{\delta}}}{\langle t\rangle^N}\right), \ \lambda \nearrow +\infty \tag{1.27}$$

uniformément en $t \in \mathbb{R}$, où

$$c_0(t) = \frac{1}{2\pi}f(t)\sum_{j=0}^{q}\int_{\{X\in\mathbb{R}^2 |\ 2j+1+W(X)=t\}} \frac{dS_t}{|\nabla_X W(X)|}. \tag{1.28}$$

Ici $W(X) = \omega_0(\frac{X}{|X|})|X|^{-\delta}$.

Théorème 1.3.7. (cf. corollaire 5.2.6) *Supposons que l'hypothèse* ($\mathbf{H_0}$) *est vraie. Soit* $N_\lambda([a,b])$ *la fonction de comptage des valeurs propres de* H_λ *dans* $[a,b]$ *comptées avec leur multiplicité. Alors on a quand* $\lambda \nearrow +\infty$

$$N_\lambda([a,b]) = \lambda^{\frac{2}{\delta}} D_0 + \mathcal{O}(1),$$

où

$$D_0 = \frac{1}{4\pi} \sum_{j=0}^{q} \left((a-2j-1)^{-\frac{2}{\delta}} - (b-2j-1)^{-\frac{2}{\delta}} \right) \int_0^{2\pi} (\omega_0(\cos\theta, \sin\theta))^{\frac{2}{\delta}}\, d\theta.$$

Remarque 1.3.8. Sous l'hypothèse que $V(X) = \omega_0(\frac{X}{|X|})|X|^{-\delta}(1+o(1))$, G. Raikov a montré que $N_\lambda([a,b]) = \lambda^{\frac{2}{\delta}}(D_0 + o(1))$ (voir [49, 50]). Le théorème 1.3.7 donne une estimation optimale du reste. Cette estimation est meilleure que dans le cas standard (l'opérateur de Schrödinger sans champ magnétique). Nous rappelons que dans ce cas-là, le reste est de l'ordre $\mathcal{O}(\lambda^{\frac{1}{\delta}})$ (voir [16]).

Idée de la démonstration

Pour démontrer les théorèmes 1.3.5, 1.3.6 et 1.3.7, nous établirons une réduction dont le but est de ramener l'étude dans la limite de grande constante de couplage $\lambda \nearrow +\infty$ à celle dans le cas semi-classique $h = \lambda^{-\frac{1}{\delta}} \searrow 0$. En effet, étant donné l'intervalle $[a,b]$ comme ci-dessus, on peut voir que pour tout $C > 0$, l'ensemble $\{(x,y,\xi,\eta) \in \mathbb{R}^4 |\ |(x,y)| < C$ et $(\xi - y)^2 + \eta^2 + \lambda V(x,y) \in [a,b]\}$ est vide pour λ assez grand. Alors modulo une quantité $\mathcal{O}(\lambda^{-\infty})$, la trace des opérateurs $f(H_\lambda)$ et $f(H_\lambda)\mathcal{F}^{-1}_{\lambda^{-\frac{2}{\delta}}}\theta(t - H_\lambda)$ ne dépendent que du comportement de V à l'infini. L'hypothèse (1.24) montre d'autre part que l'on peut construire une fonction lisse φ dépendant de h telle que

- $\varphi(X;h) = \varphi_0(X) + \varphi_2(X)h^2 + \varphi_4(X)h^4 + ... + \varphi_{2j}(X)h^{2j} + ...,$
- $\lambda V(X) = \varphi(hX;h)$ pour $|X|$ grand,

où les fonctions φ_{2j} sont uniformément bornées ainsi que leur dérivées. Posons $Q = H_0 + \varphi(hX;h)$. Il n'est pas difficile de voir que les théorèmes 1.3.1, 1.3.2 et 1.3.3 restent encore valables pour l'opérateur Q. Les théorèmes 1.3.5, 1.3.6 et 1.3.7 sont donc les corolaires des assertions suivantes :

$$\mathrm{tr}(f(H_\lambda)) = \mathrm{tr}(f(Q)) + \mathcal{O}(h^\infty), \tag{1.29}$$

et

$$\mathrm{tr}\left(f(H_\lambda)\mathcal{F}^{-1}_{\lambda^{-\frac{2}{\delta}}}\theta(t - H_\lambda)\right) = \mathrm{tr}\left(f(Q)\mathcal{F}^{-1}_{\lambda^{-\frac{2}{\delta}}}\theta(t - Q)\right) + \mathcal{O}(h^\infty) \tag{1.30}$$

Maintenant, nous expliquons comment traiter (1.29). On utilise d'abord la formule de Helffer-Sjöstrand

$$\begin{aligned} &f(H_\lambda - f(Q) \\ &= -\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)[(z - H_\lambda)^{-1} - (z-Q)^{-1}]L(dz). \end{aligned} \tag{1.31}$$

Ensuite, à l'aide de la formule de résolvante et la construction de $\varphi(hX;h)$, on peut montrer l'identité cruciale suivante

$$\begin{aligned} [(z - H_\lambda)^{-1} - (z-Q)^{-1}] = A_1(z) + A_2(z)(z-Q)^{-1} + (z - H_\lambda)^{-1}A_3(z) \\ + (z - H_\lambda)^{-1}A_4(z)(z-Q)^{-1},\ \mathrm{Im} z \neq 0, \end{aligned} \tag{1.32}$$

où $z \mapsto A_j(z)$ $j = 1, 2, 3, 4$ sont holomorphes dans un voisinage complexe $\mathcal{U}$ de $[a, b]$. De plus, les opérateurs $A_2(z), A_3(z)$ et $A_4(z)$ sont de classe trace et $\|A_j(z)\|_{\text{tr}} = \mathcal{O}(h^\infty)$, $j = 2, 3, 4$.

Finalement, en substituant (1.32) dans (1.31) et en utilisant le fait que $\bar{\partial}_z \tilde{f} = \mathcal{O}(|\text{Im}z|^\infty)$, $\|(z - H_\lambda)^{-1}\| = \mathcal{O}(|\text{Im}z|^{-1})$ et $\|(z - Q)^{-1}\| = \mathcal{O}(|\text{Im}z|^{-1})$, on obtient (1.29). De la même manière, on obtient (1.30).

Chapitre 2

Rappels sur les opérateurs $h-$pseudo-différentiels.

L'objectif de ce chapitre est de rappeler quelques résultats élémentaires sur les opérateurs $h-$pseudo-différentiels (voir [22, 25, 56]) qui seront utilisés dans la suite de cette thèse.

2.1 Opérateurs $h-$pseudo-différentiels

On commence par donner les définitions des fonctions d'ordre, des symboles et des opérateurs $h-$pseudo-différentiels.

Définition 2.1.1. *Une fonction $m : \mathbb{R}^n \to (0, +\infty)$ est appelée fonction d'ordre s'il existe des constantes $C, N > 0$ telles que*

$$m(w) \leq C\langle z - w\rangle^N m(z),$$

pour tout $z, w \in \mathbb{R}^n$.

Remarquons que si m_1 et m_2 sont deux fonctions d'ordre, alors m_1m_2 est aussi une fonction d'ordre.

Définition 2.1.2. *Soit m une fonction d'ordre sur $\mathbb{R}^{2n}$. Pour $k \in \mathbb{R}$, $\delta \in [0, \frac{1}{2}]$, on désigne par $S^k_\delta(m, \mathbb{R}^{2n})$ l'ensemble des fonctions $a(x, \xi, h)$ définies sur $\mathbb{R}^{2n} \times (0, h_0]$ telles que*

- *$a(x, \xi, h)$ est une fonction lisse par rapport à (x, ξ),*
- *Pour tout multi-indice $\alpha, \beta \in \mathbb{N}^n$, il existe $C_{\alpha,\beta} > 0$ tel que*

$$|\partial^\alpha_x \partial^\beta_\xi a(x, \xi, h)| \leq C_{\alpha,\beta} h^{-k-\delta(|\alpha|+|\beta|)} m(x, \xi),$$

  pour tout $(x, \xi) \in \mathbb{R}^{2n}$.

Si $k = 0, \delta = 0$, on écrit $S^0(m; \mathbb{R}^{2n})$ au lieu de $S^0_0(m; \mathbb{R}^{2n})$. Définissons aussi

$$S^{-\infty}(m, \mathbb{R}^{2n}) := \bigcap_{N\geq 0} S^{-N}(m, \mathbb{R}^{2n}).$$

Lorsque la fonction d'ordre m vaut 1, on omettra m et on notera $S^k_\delta(\mathbb{R}^{2n}) := S^k_\delta(1, \mathbb{R}^{2n})$.

Définition 2.1.3. *À un symbole $a \in S^k_\delta(m, \mathbb{R}^{2n})$, on associe un opérateur $h-$pseudo-différentiel de $\mathcal{S}(\mathbb{R}^n)$ dans $\mathcal{S}(\mathbb{R}^n)$ défini par*

$$a^w(x, hD_x)u(x) = \frac{1}{(2\pi h)^n} \iint e^{i(x-y).\xi/h} a(\frac{x+y}{2}, \xi)u(y)dyd\xi,\ u \in \mathcal{S}(\mathbb{R}^n)$$

Théorème 2.1.4. (Composition des opérateurs)

1. *Soient $a \in S^0_\delta(m_1, \mathbb{R}^{2n})$ et $b \in S^0_\delta(m_2, \mathbb{R}^{2n})$, alors il existe un symbole $a\#b \in S^0_\delta(m_1 m_2, \mathbb{R}^{2n})$ tel que $a^w(x, hD_x)b^w(x, hD_x) = (a\#b)^w(x, hD_x)$ au sens des opérateurs de $\mathcal{S}(\mathbb{R}^n)$ à $\mathcal{S}(\mathbb{R}^n)$. Ici $a\#b$ est déterminé par la formule suivante*

$$a\#b = e^{\frac{ih}{2}\sigma(D_x, D_\xi, D_y, D_\eta)}\Big(a(x,\xi)b(y,\eta)\Big)\Big|_{(x,\xi)=(y,\eta)},$$

où σ est le produit symplectique défini par $\sigma(D_x, D_\xi, D_y, D_\eta) = D_y.D_\xi - D_x.D_\eta$.

2. *En particulier, $a\#b - ab \in S^{2\delta-1}_\delta(m_1 m_2, \mathbb{R}^{2n})$.*

Théorème 2.1.5. (Continuité sur L^2) *Supposons que a appartient à $S^0_\delta(\mathbb{R}^{2n})$, alors*

$$a^w(x, hD_x) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

est un opérateur borné et il existe une constante $C > 0$ indépendant de h telle que $\|a^w(x, hD_x)\| \leq C$.

Théorème 2.1.6. (Supports disjoints) *Soient $a, b \in S^0_\delta(\mathbb{R}^{2n})$. Supposons qu'il existe $\gamma > 0$ indépendant de h tel que*

$$\mathrm{dist}\big(\mathrm{supp}(a), \mathrm{supp}(b)\big) \geq \gamma > 0.$$

Pour $\delta \in [0, \frac{1}{2})$, on a alors

$$\|a^w(x, hD_x)b^w(x, hD_x)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} = \mathcal{O}(h^\infty).$$

Théorème 2.1.7. (Inversibilité) *Soit $\delta \in [0, \frac{1}{2})$. Supposons que $a \in S^0_\delta(m, \mathbb{R}^{2n})$ et $|a(x,\xi)| \geq \gamma m(x,\xi)$ $(\gamma > 0)$ pour tout $(x,\xi) \in \mathbb{R}^{2n}$. Pour h suffisamment petit, l'opérateur $a^w(x, hD_x)^{-1}$ est un opérateur $h-$pseudo-différentiel avec symbole appartenant à $S^0_\delta(\frac{1}{m}, \mathbb{R}^{2n})$.*

Théorème 2.1.8. (Inégalité de Gårding, version faible) *Soit a un symbole à valeurs réelles dans $S^0(\mathbb{R}^{2n})$ et*

$$a(x,\xi) \geq \gamma > 0,\ \text{pour tout } (x,\xi) \in \mathbb{R}^{2n}.$$

Alors, pour $\varepsilon > 0$, il existe $h_0 = h_0(\varepsilon) > 0$ telle que

$$\langle a^w(x, hD_x)u, u\rangle \geq (\gamma - \varepsilon)\|u\|,$$

pour tout $h \in (0, h_0]$ et $u \in L^2(\mathbb{R}^n)$.

Théorème 2.1.9. (Inégalité de Gårding) *Soit a un symbole dans $S^0(\mathbb{R}^{2n})$. On suppose que $a \geq 0$ sur $\mathbb{R}^{2n}$. Il existe alors $h_0 > 0$ et $C \geq 0$ tels que*

$$\langle a^w(x, hD_x)u, u\rangle \geq -Ch\|u\|,$$

pour tout $h \in (0, h_0]$ et $u \in L^2(\mathbb{R}^n)$.

Théorème 2.1.10. *Soient $a \in S^0(m, \mathbb{R}^{2n})$ et $\kappa : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ une transformation canonique. Alors il existe un opérateur unitaire U tel que $a^w(x, hD_x)U = U(a \circ k)^w(x, hD_x)$.*

2.2 Calcul fonctionnel par la formule de Helffer-Sjöstrand.

On commence par introduire la formule de Helffer-Sjöstrand.

Proposition 2.2.1. *Soit $f \in C_0^\infty(\mathbb{R})$, alors il existe une fonction $\tilde{f} \in C_0^\infty(\mathbb{C})$ telle que pour tout $N \in \mathbb{N}$*

$$|\bar{\partial}_z \tilde{f}(z)| \leq C_N |\mathrm{Im} z|^N,$$

$$\tilde{f}\Big|_{\mathbb{R}} = f.$$

Ici $\bar{\partial}_z := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. On dit que $\tilde{f}$ est une extension presque analytique de f.

Théorème 2.2.2. (Formule de Helffer- Sjöstrand) *Soit A un opérateur auto-adjoint sur un espace de Hilbert $\mathcal{H}$. Soit $f \in C_0^2(\mathbb{R})$ et $\tilde{f} \in C_0^1(\mathbb{C})$ l'extension presque analytique de f avec $|\bar{\partial}_z \tilde{f}(z)| = \mathcal{O}(|\mathrm{Im} z|)$. Alors*

$$f(A) = -\frac{1}{\pi} \int \bar{\partial}_z \tilde{f}(z)(z - A)^{-1} L(dz),$$

où $L(dz) = dxdy$ est la mesure de Lebesgue sur $\mathbb{C} \sim \mathbb{R}^2_{x,y}$.

Théorème 2.2.3. (Caractérisation de Beal) *Soient $A = A_h : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n), 0 < h \leq 1$. Alors les assertions suivantes sont équivalentes :*

1. *$A = a^w(x, hD_x, h)$ pour certain $a \in S^0(\mathbb{R}^{2n})$.*
2. *Pour tout $N \in \mathbb{N}$ et pour toutes les formes linéaires $l_1(x, \xi), ..., l_N(x, \xi)$ sur $\mathbb{R}^{2n}$, l'opérateur $\mathrm{ad}_{l_1^w(x,hD_x)} \circ ... \circ \mathrm{ad}_{l_N^w(x,hD_x)} A_h$ appartient à $\mathcal{L}(L^2(\mathbb{R}^n))$ et est de norme $\mathcal{O}(h^N)$ dans cet espace.*

Ici $\mathrm{ad}_A B$ est le commutateur de A et B donné par $\mathrm{ad}_A B := [A, B] = AB - BA$.

Théorème 2.2.4. *Soient $a \in S^0(m, \mathbb{R}^{2n})$ et $f \in C_0^\infty(\mathbb{R})$. Alors $f(a^w(x, hD_x))$ est un opérateur $h-$pseudo-différentiel dont le symbole appartient à $S^0(m^{-k}, \mathbb{R}^{2n})$ pour tout $k \in \mathbb{N}$.*

2.3 Opérateurs à trace.

On rappelle quelques critères pour qu'un opérateur pseudo-différentiel soit de classe trace ou Hilbert-Schmidt.

Théorème 2.3.1. *Soit $a \in \mathcal{S}'(\mathbb{R}^{2n})$. Alors $a^w(x, hD_x)$ est Hilbert-Schmidt si et seulement si $a \in L^2(\mathbb{R}^{2n})$. On a de plus*

$$\|a^w(x, hD_x)\|^2_{\mathrm{HS}} = \frac{1}{(2\pi h)^n} \iint |a(x, \xi)|^2 dx d\xi.$$

Théorème 2.3.2. *Soit a un symbole dans $S^0(\mathbb{R}^{2n})$. On suppose que $\partial^\alpha_{x,\xi} a(x,\xi) \in L^1(\mathbb{R}^{2n})$ pour tout $|\alpha| \leq 2n+1$. Alors $a^w(x,hD_x)$ est de classe trace et on a*

$$\operatorname{tr}(a^w(x,hD_x)) = \frac{1}{(2\pi h)^n} \iint a(x,\xi)dxd\xi,$$

$$\|a^w(x,hD_x)\|_{\operatorname{tr}} \leq C_n h^{-n} \sum_{|\alpha|\leq 2n+1} \|\partial^\alpha_{x,\xi} a(x,\xi)\|_{L^1(\mathbb{R}^{2n})}.$$

Théorème 2.3.3. *Soit $a \in S^0(\mathbb{R}^{2n})$. Alors, pour tout symbole $b \in C_0^\infty(\mathbb{R}^{2n})$, on a*

$$\operatorname{tr}(a^w(x,hD_x) \circ b^w(x,hD_x)) = \frac{1}{(2\pi h)^n} \iint a(x,\xi)b(x,\xi)dxd\xi.$$

Chapitre 3

Spectral asymptotics in the strong magnetic field

Dans ce chapitre, nous présentons l'article [24].

SPECTRAL ASYMPTOTICS FOR TWO-DIMENSIONAL SCHRÖDINGER OPERATORS WITH STRONG MAGNETIC FIELDS.

ANH TUAN DUONG

ABSTRACT. In this paper we study the perturbed quadratic Hamiltonian in two-dimensional case, $P(B,\omega) = (D_x - By)^2 + D_y^2 + \omega^2 x^2 - \sqrt{B^2+\omega^2} + V(x,y)$. Here, B is the strong constant magnetic field, $\omega \neq 0$ is a fixed constant, and the potential V vanishes at infinity. For $f \in C_0^\infty((-\infty,0);\mathbb{R})$ and B large enough, we give a full asymptotic expansion in powers of B^{-1} of the trace of $f(P(B,\omega))$. Moreover, we also obtain a Weyl formula with optimal remainder estimate of the counting function of eigenvalues of $P(B,\omega)$ as $B \to \infty$.

3.1 Introduction

We consider the two-dimensional Schrödinger operator with constant magnetic field

$$\begin{aligned} P(B,\omega) &= P_0(B,\omega) + V(x,y) \\ &= (D_x - By)^2 + D_y^2 + \omega^2 x^2 - \sqrt{B^2+\omega^2} + V(x,y), \quad (x,y) \in \mathbb{R}^2, \end{aligned} \tag{3.1}$$

where $D_\nu = \frac{1}{i}\partial_\nu$, B is the strong constant magnetic field and $\omega \neq 0$ is a fixed constant.

Throughout this paper, we always assume that the electric potential V satisfies the following hypothesis :

(H) The potential V is a real-valued smooth function, bounded with all its derivatives, and tends to zero at infinity, i.e., $\forall \beta, \gamma \in \mathbb{N}$

$$|\partial_x^\beta \partial_y^\gamma V(x,y)| \leq C_{\beta,\gamma},\ \forall (x,y) \in \mathbb{R}^2.$$

It is well known that the operator $P(B,\omega)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ (see [26, 44, 45]). For $V \equiv 0$, it was shown that the spectrum of the unperturbed operator $P_0(B,\omega)$ is absolutely continuous and equal to the interval $[0,\infty)$ (see [45]). Since V tends to zero at infinity, the operator $V(P_0(B,\omega)+i)^{-1}$ is compact (see [2]). According to the Weyl theorem (see [54]), the operators $P(B,\omega)$ and $P_0(B,\omega)$ have the same essential spectrum, i.e., $\sigma_{\text{ess}}(P(B,\omega)) = \sigma_{\text{ess}}(P_0(B,\omega)) = [0,\infty)$. Thus the spectrum of $P(B,\omega)$ in the interval $(-\infty,0)$ is discrete.

Recently a substantial progress has been made in the analysis of the magnetic Schrödinger operators with long-range perturbations going to 0 as $|x| \to +\infty$ and the works around the trace formulae have generated many results on the distribution of eigenvalues near Landau levels and Weyl's formula with sharp remainder estimate of the counting function of eigenvalues (see [2, 11, 15, 18, 20, 26, 31, 36, 39, 46, 48, 51, 52, 57] and the references given there).

To our best knowledge, there are only a few works concerning the model (3.1) (see [26, 44, 45]). In [45], the authors studied a quadratic Hamiltonian without perturbation by using the theory of metapletic representations. In [26], the authors investigated the absolutely continuous spectrum of $P(B,\omega)$. By applying the Mourre theory, they proved that a part of the absolutely continuous spectrum of $P(B,\omega)$ persists. On the other hand, we can consider the model (3.1) as the quantum hall system Hamiltonian with the unbounded edge potential $W(x) = \omega^2 x^2$ (see [7, 12, 40]).

In this work, we give a complete asymptotic expansion in powers of B^{-1} of the trace of the operators $f(P(B,\omega))$ and $f(P(B,\omega))\mathcal{F}_{\frac{1}{\alpha}}^{-1}\Psi(\lambda - P(B,\omega))$, where $f \in C_0^\infty((-\infty,0);\mathbb{R})$, $\Psi \in C_0^\infty((-\frac{1}{C},\frac{1}{C});\mathbb{R})$ with C large enough. Here $\alpha := \sqrt{B^2+\omega^2}$ and the semiclassical Fourier transformation $\mathcal{F}_{\frac{1}{\alpha}}^{-1}\Psi(x) = \frac{\alpha}{2\pi}\int_{\mathbb{R}} e^{i\alpha t x}\Psi(t)dt$. The proof is based on the effective Hamiltonian method and the stationary techniques developed by M. Dimassi [14] (see also M. Dimassi-J. Sjöstrand [22]). More precisely, we reduce the spectral study of the operator $P(B,\omega)$ near a fixed energy z to the study of a $\frac{1}{\alpha}-$ pseudodifferential operator $E_{-+}(z)$ called the effective Hamiltonian. Then we apply to the operator $E_{-+}(z)$ the time independent approach. Furthermore, thanks to a Tauberian theorem (see [15, 56]), we deduce Weyl's formula with optimal remainder estimate of the counting function of isolated eigenvalues of $P(B,\omega)$ in $(-\infty,\lambda]$, where $\lambda < 0$ is a fixed constant.

The paper is organized as follows. In Section 3.2 we recall some notations and definitions for symbols and pseudodifferential operators. Our main results are announced in Section 3.3. In Subsection 3.4.1, we establish and study the effective Hamiltonian. Then we prove some trace formulae in Subsection 3.4.2. The proofs of our main results are given in Section 3.5 and Section 3.6. Finally, we construct some linear canonical transformations in Appendix 3.6.

3.2 Notations and Definitions.

In this paper, we will use the notations in [22, 25] for symbols and h - pseudodifferential operators. Here h is the semiclassical parameter.

Definition 3.2.1. *A function $m : \mathbb{R}^{2n} \to (0, \infty)$ is called an order function if there exist constants C, N such that*

$$m(w) \leq C\langle \tilde{w} - w\rangle^N m(\tilde{w}),$$

for all $\tilde{w}, w \in \mathbb{R}^{2n}$.

Definition 3.2.2. *Let m be an order function on $\mathbb{R}^{2n}$. For $k \in \mathbb{R}$, $0 \leq l < \frac{1}{2}$, we define the class of symbols :*

$$S_l^k(m, \mathbb{R}^{2n}) := \left\{ a \in C^\infty(\mathbb{R}^{2n}) |\ \forall \beta \in \mathbb{N}^{2n}, \exists C_\beta > 0 \text{ s.t. } |\partial^\beta a| \leq C_\beta h^{-k-l|\beta|} m \right\}.$$

If $m = 1$ (resp. $l = 0$), we write $S_l^k(\mathbb{R}^{2n}) := S_l^k(1, \mathbb{R}^{2n})$ (resp. $S^k(m, \mathbb{R}^{2n}) := S_0^k(m, \mathbb{R}^{2n})$).

Definition 3.2.3. *Let $a(.; h) \in S_l^{k_0}(m, \mathbb{R}^{2n})$ depend on h. We say that $a(.; h)$ admits an asymptotic expansion in powers of h, if there exists a sequence of symbols of $S_l^{k_0}(m, \mathbb{R}^{2n})$, $(a_j)_{j\in\mathbb{N}}$, such that for any $N \in \mathbb{N}$, and for any $\beta \in \mathbb{N}^{2n}$, there exists $C_{N,\beta} > 0$, s.t.*

$$\left| \partial^\beta \left(a(.; h) - \sum_{j=0}^N h^j a_j \right) \right| \leq C_{N,\beta} h^{N+1-k_0-l|\beta|} m.$$

We write $a(.; h) \sim \sum_{j=0}^\infty h^j a_j$ in $S_l^{k_0}(m, \mathbb{R}^{2n})$.

Definition 3.2.4. *(see [4]) We denote by $S^0(\mathbb{R}^{2n}; \mathcal{L}(L^2(\mathbb{R}^n)))$ the set of operator-valued functions $a \in C^\infty(\mathbb{R}^{2n}; \mathcal{L}(L^2(\mathbb{R}^n)))$ satisfying :*
For $\beta \in \mathbb{N}^{2n}$, there exists $C_\beta > 0$ such that

$$\|\partial^\beta a(x, \xi)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C_\beta,\ \forall (x, \xi) \in \mathbb{R}^{2n}.$$

We will use the standard Weyl quantization of symbols. More precisely, if $a(x, \xi)$, $(x, \xi) \in \mathbb{R}^{2n}$, is a symbol in $S_l^k(m, \mathbb{R}^{2n})$ (resp. $S^0(\mathbb{R}^{2n}; \mathcal{L}(L^2(\mathbb{R}^n)))$), then $a^w(x, hD_x)$ is the operator defined by

$$a^w(x, hD_x)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(\frac{x+y}{2}, \xi) u(y) dy d\xi,$$

for $u \in \mathcal{S}(\mathbb{R}^n)$ (resp. $u \in \mathcal{S}(\mathbb{R}^n; L^2(\mathbb{R}^n))$). Sometimes, we write $Op_h^w(a)$ instead of $a^w(x, hD_x)$. For the theory of $h-$ pseudodifferential operators with operator-valued symbols, we refer to [4, 6, 29].

Let $g(\alpha)$ be a function depending on a large parameter α. We say that g has a complete asymptotic expansion in powers of α^{-1}, and we write

$$g(\alpha) \sim \sum_{j=0}^\infty c_j \alpha^{-j}, \text{ as } \alpha \to \infty,$$

if and only if, for all $N \in \mathbb{N}$,

$$\lim_{\alpha\to\infty}\Big[g(\alpha)-\sum_{j=0}^{N}c_j\alpha^{-j}\Big]\alpha^N=0.$$

We write $g(\alpha)=\mathcal{O}(\alpha^{-\infty})$ if for all $N\in\mathbb{N}$, $g(\alpha)=\mathcal{O}_N(\alpha^{-N})$.

3.3 Main results.

In this section, we state our main results :

Theorem 3.3.1. *Let $f \in C_0^\infty((-\infty,0);\mathbb{R})$ and assume that V satisfies the assumption $(\mathbf{H})$. Then, the following full expansion holds :*

$$\operatorname{tr}\Big(f(P(B,\omega))\Big)\sim B\sum_{j=0}^{\infty}B_j(f)B^{-j},\ \text{as } B\to\infty. \tag{3.2}$$

Moreover,

$$B_0(f)=\frac{1}{2\pi}\iint f\big(\omega^2x^2+V(x,y)\big)dxdy, \tag{3.3}$$

$$B_1(f)=\frac{1}{8\pi}\iint\big[\partial_x^2V(x,y)+\partial_y^2V(x,y)\big]f'\big(\omega^2x^2+V(x,y)\big)dxdy. \tag{3.4}$$

Theorem 3.3.2. *Fix $\lambda<0$. In addition to the assumption $(\mathbf{H})$ suppose that $\nabla_{x,y}(\omega^2x^2+V(x,y))\neq 0$ on the surface*

$$\{(x,y)\in\mathbb{R}^2|\ \omega^2x^2+V(x,y)=\lambda\}.$$

Then, there exist $\sigma>0$ small enough and $C>0$ large enough such that, for $f\in C_0^\infty((\lambda-\sigma,\lambda+\sigma);\mathbb{R})$ and $\Psi\in C_0^\infty((-\frac{1}{C},\frac{1}{C});\mathbb{R})$, there exist functions $C_j(\tau)\in C^\infty(\mathbb{R})$, $\forall M,N\in\mathbb{N}$, we have

$$\operatorname{tr}\Big(f(P(B,\omega))\mathcal{F}^{-1}_{\frac{1}{\alpha}}\Psi(\tau-P(B,\omega))\Big)=B\Big(\sum_{j=0}^{M}C_j(\tau)B^{-j}+\mathcal{O}\big(\frac{B^{-M-1}}{\langle\tau\rangle^N}\big)\Big),\ \text{as } B\to\infty, \tag{3.5}$$

uniformly in $\tau\in\mathbb{R}$. Here $\langle\tau\rangle=(1+\tau^2)^{\frac{1}{2}}$,

$$C_0(\tau)=\frac{1}{2\pi}f(\tau)\Psi(0)\int_{\{(x,y)\in\mathbb{R}^2|\ \omega^2x^2+V(x,y)=\lambda\}}\frac{dS_\tau}{|\nabla_{x,y}(\omega^2x^2+V(x,y))|}, \tag{3.6}$$

and S_τ is the surface measure on $\{(x,y)\in\mathbb{R}^2|\ \omega^2x^2+V(x,y)=\lambda\}$.

Theorem 3.3.3. *Fix $\lambda<0$, and let $N_\alpha(\lambda)$ be the number of eigenvalues of $P(B,\omega)$ in $(-\infty,\lambda]$ counted with their multiplicities. Under the assumptions of Theorem 3.3.2, we have*

$$N_\alpha(\lambda)=B\Big(M_0+\mathcal{O}(B^{-1})\Big),\ \text{as } B\to\infty, \tag{3.7}$$

where

$$M_0 = \frac{1}{2\pi} \iint_{\{(x,y)\in\mathbb{R}^2 |\ \omega^2x^2+V(x,y)\le\lambda\}} dxdy. \tag{3.8}$$

Remark 3.3.4. Since $\alpha = \sqrt{B^2+\omega^2}$, and $\omega \neq 0$ is fixed, it follows that

$$\alpha^{-1} \sim \sum_{j=1}^{\infty} c_j(\omega)B^{-j} \text{ and } B^{-1} \sim \sum_{j=1}^{\infty} d_j(\omega)\alpha^{-j}.$$

Thus, we only need to prove that the left hand sides of (3.2), (3.5) and (3.7) have asymptotic expansions in powers of α^{-1}.

Outline of the proof

Using linear changes of variables, we prove in Appendix 3.6 that the operator $P(b,\omega)$ is unitarily equivalent to the operator $\tilde{P}(\alpha) := \tilde{P}_0(\alpha) + V^w(\alpha)$, where $\tilde{P}_0(\alpha) := \alpha(D_y^2 + y^2) + \omega^2x^2 - \alpha$ and $V^w(\alpha) := V^w(\frac{1}{\sqrt{\alpha}}D_y + \frac{B}{\alpha}x, \frac{1}{\alpha}D_x + \frac{B\sqrt{\alpha}}{\alpha^2}y)$.

By constructing a suitable Grushin problem (see Lemma 3.4.3) we reduce the spectral study of $(\tilde{P}(\alpha) - z)$ for $z \in (\underline{a}, \bar{a}) \subset (-\infty, 0)$ to the study of a $\frac{1}{\alpha}-$ pseudodifferential operator $E_{+-}(z)$ called effective Hamiltonian. Here we notice that, in our case ($\omega \neq 0$), the effective Hamiltonian is no longer a bounded operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, because of the presence of the edge potential $W(x) = \omega^2x^2$ (see Lemma 3.4.3). This is the main difference from the work of M. Dimassi [15] where ω is assumed to be equal to zero. To overcome this difficulty, we show that if $f \in C_0^\infty((-\infty,0);\mathbb{R})$, we have $\mathrm{tr}(f(\tilde{P}(\alpha))) = \mathrm{tr}(f(\tilde{P}(\alpha))\chi^w) + \mathcal{O}(\alpha^{-\infty})$ for some compactly supported function χ (see formula (3.57)). Thus, by using the Helffer-Sjöstrand formula, we prove that (see formula (3.57))

$$\mathrm{tr}(f(\tilde{P}(\alpha))\chi^w) = \mathrm{tr}\Big(-\frac{1}{\pi}\int \bar{\partial}_z\tilde{f}(z)E_{-+}(z)^{-1}\partial_z E_{-+}(z)\chi^w L(dz)\Big). \tag{3.9}$$

Here, $\tilde{f} \in C_0^\infty(\mathbb{C})$ is an almost analytic extension of f. Then, we only need to study $E_{-+}(z)$ in the right hand side of (3.9) on the support of χ. Next, by applying the $\frac{1}{\alpha}-$ pseudodifferential calculus, modulo $\mathcal{O}(\alpha^{-\infty})$ we can replace $E_{-+}(z)$ in (3.9) by a bounded operator $\tilde{E}(z)$ (see Proposition 3.4.7) such that

$$\mathrm{tr}(f(\tilde{P}(\alpha))\chi^w) = \mathrm{tr}\Big(-\frac{1}{\pi}\int \bar{\partial}_z\tilde{f}(z)\tilde{E}(z)^{-1}\partial_z \tilde{E}(z)\chi^w L(dz)\Big) + \mathcal{O}(\alpha^{-\infty}). \tag{3.10}$$

Moreover, we show that $\tilde{E}(z)$ has a complete asymptotic expansion in powers of $\frac{1}{\alpha}$ (see Proposition 3.4.5). Then by applying the time independent method used by M. Dimassi [14] (see also [22]), we prove Theorem 3.3.1 and Theorem 3.3.2.

By combining Theorems 3.3.1 and 3.3.2 with Tauberian arguments, we obtain the proof of Theorem 3.3.3.

3.4 The effective Hamiltonian and trace formulae

In this section, we construct the effective Hamiltonian $E_{-+}(z)$ and we give a trace formula linking the operators $\tilde{P}(\alpha)$ and $E_{-+}(z)$. Without loss of generality we may assume that $\omega = 1$.

3.4.1 The effective Hamiltonian.

Let g be a positive function satisfying $g(\alpha) \to \infty$ as $\alpha \to \infty$. Set

$$\Omega_\alpha := \{z \in \mathbb{C}| \ \mathrm{Re} z \leq 2\alpha - g(\alpha)\}. \tag{3.11}$$

We denote by $\phi(y) := \pi^{-\frac{1}{4}} e^{-\frac{y^2}{2}}$ the normalized eigenfunction of the one-dimensional harmonic operator corresponding to the ground state energy $E = 1$ (i.e., $(D_y^2 + y^2)\phi(y) = \phi(y)$ and $\|\phi\|_{L^2(\mathbb{R})} = 1$). The following lemma is well known :

Lemma 3.4.1. *The following statements hold :*

1. *ϕ is an even function.*
2. *$\langle D_y^k \phi(y), y^l \phi(y)\rangle = 0$, when $k + l$ is an odd number.*
3. *$\|\phi'(y)\|^2_{L^2(\mathbb{R})} = \|y\phi(y)\|^2_{L^2(\mathbb{R})} = \frac{1}{2}$.*

We now consider the following operators

$$\begin{aligned} R_- \ :\ & L^2(\mathbb{R}) \to L^2(\mathbb{R}^2), & R_+ \ :\ & L^2(\mathbb{R}^2) \to L^2(\mathbb{R}), \\ & u(x) \mapsto u(x)\phi(y) & & v(x,y) \mapsto \textstyle\int v(x,y)\phi(y)dy \end{aligned}$$

$$\begin{aligned} \Pi := R_- R_+ \ & : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \\ & v(x,y) \mapsto \int v(x,t)\phi(t)dt\phi(y). \end{aligned}$$

From the definition of R_- and R_+, we have

$$R_+ R_- u(x) = R_+(u(x)\phi(y)) = \int u(x)\phi(y)\phi(y)dy = u(x).$$

Lemma 3.4.2. *There exists $\alpha_0 > 0$ large enough such that, for $\alpha \geq \alpha_0$, and $z \in \Omega_\alpha$, the operator*

$$(I - \Pi)\tilde{P}(\alpha)(I - \Pi) - z \ :\ (I - \Pi)L^2(\mathbb{R}^2) \to (I - \Pi)L^2(\mathbb{R}^2)$$

is invertible. In addition, there exists $C > 0$ such that

$$\|R(z)\| \leq \frac{C}{g(\alpha)} \ \textit{uniformly in } z \in \Omega_\alpha, \tag{3.12}$$

where $R(z) := \Big((I - \Pi)\tilde{P}(\alpha)(I - \Pi) - z\Big)^{-1}(I - \Pi) :\ L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$.

Proof. In the following, we denote by $\sigma(A)$ (resp. $\rho(A)$) the spectrum (resp. resolvent set) of the operator A. It is clear that $\sigma((I-\Pi)\tilde{P}_0(\alpha)(I-\Pi)) = [2\alpha, +\infty)$ (see [32, Proposition 6.9]). Hence, for $z \in \Omega_\alpha$, we have

$$\mathrm{dist}\Big(z, \sigma((I-\Pi)\tilde{P}_0(\alpha)(I-\Pi))\Big)\|(I-\Pi)u\| \leq \|[(I-\Pi)\tilde{P}_0(\alpha)(I-\Pi)-z](I-\Pi)u\|. \tag{3.13}$$

On the other hand, for $z \in \Omega_\alpha$, one has $g(\alpha) \le \operatorname{dist}\Big(z, \sigma((I-\Pi)\tilde{P}_0(\alpha)(I-\Pi))\Big)$. Thus,

$$g(\alpha)\|(I-\Pi)u\| \le \|[(I-\Pi)\tilde{P}_0(\alpha)(I-\Pi) - z](I-\Pi)u\|. \tag{3.14}$$

Moreover, since V is bounded simultaneously with all its derivatives, it follows from the Calderón-Vaillancourt theorem (see [22, Theorem 7.11]) that there exists $C_1 > 0$ such that

$$\|V^w(\alpha)(I-\Pi)u\| \le C_1\|(I-\Pi)u\|.$$

Combining this with (3.14), we have

$$(g(\alpha) - C_1)\|(I-\Pi)u\| \le \|[(I-\Pi)\tilde{P}(\alpha)(I-\Pi) - z](I-\Pi)u\| \tag{3.15}$$

uniformly in $z \in \Omega_\alpha$, which yields (3.12). We recall that $\lim\limits_{\alpha\to\infty} g(\alpha) = +\infty$. □

Next, we construct a Grushin problem for the perturbed Hamiltonian :

$$\mathcal{P}(z) = \begin{pmatrix} \tilde{P}(\alpha) - z & R_- \\ R_+ & 0 \end{pmatrix} \; : \; \mathcal{D} \times L^2(\mathbb{R}) \to L^2(\mathbb{R}^2) \times L^2(\mathbb{R}),$$

where $\mathcal{D}$ is the domain of $\tilde{P}(\alpha)$.

Lemma 3.4.3. *For α large enough, the operator $\mathcal{P}(z)$ is uniformly invertible for $z \in \Omega_\alpha$. The inverse of $\mathcal{P}(z)$ is holomorphic in z and given by*

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}, \tag{3.16}$$

where

$$E_{-+}(z) = z - x^2 - R_+V^w(\alpha)R_- - R_+a(z)[V^w(\alpha), \Pi]R_-, \tag{3.17}$$

with $a(z) = (I + [\Pi, V^w(\alpha)]R(z))^{-1}$. Moreover,

$$E(z) = R(z)a(z) \; ; \; E_-(z) = R_+a(z), \tag{3.18}$$

$$E_+(z) = R_- - R(z)a(z)[V^w(\alpha), \Pi]R_-. \tag{3.19}$$

Here $[A, B] = AB - BA$ is the commutator of A and B.

Proof. We claim that $[V^w(\alpha), \Pi] = \mathcal{O}(\alpha^{-1/2})$ in $\mathcal{L}(L^2(\mathbb{R}^2))$. Indeed, the natural restriction of Π on $L^2(\mathbb{R}_y)$ is the projection onto the eigenspace associated to the eigenvalue 1 of the operator $D_y^2 + y^2 \; : \; L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Let $f \in C_0^\infty((0,2);\mathbb{R})$, $f(y) = 1$ for y near 1. Then one has $\Pi = f(D_y^2 + y^2)$.

According to [22, Theorem 8.7], Π is a pseudodifferential operator whose symbol belongs to $S^0(\langle y\rangle^{-\infty}\langle\eta\rangle^{-\infty}, \mathbb{R}^4)$. By using the pseudodifferential calculus, it follows that $[V^w(\alpha), \Pi]$ has an asymptotic expansion of the form

$$[V^w(\alpha), \Pi] = \sum_{j=1}^{N} a_j^w(\frac{B}{\alpha}x, \frac{1}{\alpha}D_x)b_j^w(y, D_y)\alpha^{-\frac{j}{2}} + \mathcal{O}(\alpha^{-\frac{N+1}{2}}), \; \forall N \in \mathbb{N}, \tag{3.20}$$

where $a_j, b_j \in S^0(\mathbb{R}^2)$. Consequently, one has

$$[V^w(\alpha), \Pi] = \mathcal{O}(\frac{1}{\sqrt{\alpha}}), \text{ in } \mathcal{L}(L^2(\mathbb{R}^2)), \tag{3.21}$$

which yields the claim.

Next we construct an approximate inverse of $\mathcal{P}(z)$ as follows :

$$\tilde{\mathcal{E}}(z) = \begin{pmatrix} R(z) & R_- \\ R_+ & z - x^2 - R_+ V^w(\alpha) R_- \end{pmatrix}.$$

A straightforward computation gives

$$\mathcal{P}(z)\tilde{\mathcal{E}}(z) = \begin{pmatrix} \tilde{P}(\alpha) - z & R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} R(z) & R_- \\ R_+ & z - x^2 - R_+ V^w(\alpha) R_- \end{pmatrix} =: \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \tag{3.22}$$

with

$$\begin{aligned} &A_4 = R_+ R_- = I_{L^2(\mathbb{R})}, \\ &A_3 = 0 \text{ (here we used } \ R_+(I - \Pi) = R_+ - R_+ R_- R_+ = 0), \\ &A_2 = (\tilde{P}(\alpha) - z) R_- + R_-(z - x^2 - R_+ V^w(\alpha) R_-), \\ &A_1 = (\tilde{P}(\alpha) - z) R(z) + R_- R_+. \end{aligned}$$

Using the fact that $\tilde{P}_0(\alpha)$ commutes with Π and $\Pi R(z) = 0$, one has $(\tilde{P}(\alpha) - z)R(z) = I - \Pi + \Pi V^w(\alpha) R(z)$. Thus,

$$A_1 = I - \Pi + \Pi V^w(\alpha) R(z) + \Pi = I + [\Pi, V^w(\alpha)] R(z).$$

On the other hand, since $(\alpha(D_y^2 + y^2) - \alpha) R_- = 0$ and $V^w(\alpha) R_- - R_- R_+ V^w(\alpha) R_- = [V^w(\alpha), \Pi] R_-$, it follows that $A_2 = [V^w(\alpha), \Pi] R_-$. Consequently,

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = I_2 + \mathcal{O}(\frac{1}{\sqrt{\alpha}}),$$

where I_2 is the identity matrix from $L^2(\mathbb{R}^2) \times L^2(\mathbb{R})$ to $L^2(\mathbb{R}^2) \times L^2(\mathbb{R})$. Therefore, for α large enough, $\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ is uniformly invertible for $z \in \Omega_\alpha$ and

$$\mathcal{E}_1(z) := \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}^{-1} = \begin{pmatrix} a(z) & -a(z)[V^w(\alpha), \Pi] R_- \\ 0 & I_{L^2(\mathbb{R})} \end{pmatrix}, \tag{3.23}$$

where $a(z) = (I + [\Pi, V^w(\alpha)] R(z))^{-1}$.

Hence, for α large enough, $\mathcal{P}(z)$ is uniformly invertible for $z \in \Omega_\alpha$ with inverse $\mathcal{E}(z) = \tilde{\mathcal{E}}(z)\mathcal{E}_1(z)$. Using the explicit expressions of $\tilde{\mathcal{E}}(z)$ and $\mathcal{E}_1(z)$, we obtain (3.16). □

Remark 3.4.4. The following standard properties are well known (see [15, 19, 25, 31])

1. $z \in \sigma(\tilde{P}(\alpha)) \Leftrightarrow 0 \in \sigma(E_{-+}(z))$.

2. $(z-\tilde{P}(\alpha))^{-1}=-E(z)+E_{+}(z)(E_{-+}(z))^{-1}E_{-}(z)$, for $z\in\rho(\tilde{P}(\alpha))$.
3. $(E_{-+}(z))^{-1}=-R_{+}(z-\tilde{P}(\alpha))^{-1}R_{-}$, for $z\in\rho(\tilde{P}(\alpha))$.
4. $\partial_z E_{-+}(z)=E_{-}(z)E_{+}(z)$.

We denote $h=\alpha^{-1}$, which plays the role of semiclassical parameter. From now on, we write $\tilde{P}(h)$ (resp. $V^w(h^{\frac{1}{2}})$) instead of $\tilde{P}(\alpha)$ (resp. $V^w(\alpha)$). The symbol of $V^w(h^{\frac{1}{2}})$ is denoted by $V(h^{\frac{1}{2}})$. In the next proposition, we obtain the main properties of $E_{-+}(z)$.

Proposition 3.4.5. *Let Ω be a bounded open set in $\mathbb{C}$ (independent of h). Then for $h\in(0,h_0)$ ($h_0>0$ small enough), the following assertions hold :*
i)

$$E_{-+}(z)-\Big(z-x^2-V^w((1-h^2)^{\frac{1}{2}}x,hD_x)\Big)\in \mathrm{Op}_h^w(S^{-1}(\mathbb{R}^2)). \tag{3.24}$$

ii) For $\psi\in C_0^\infty(\mathbb{R})$, the symbol of the operator $\psi(x)\Big(E_{-+}(z)-(z-x^2)\Big)$ has a complete asymptotic expansion in powers of h in $S^0(\mathbb{R}^2)$. More precisely, there exists $a(x,\xi,z;h)\sim\sum_{j=0}^{\infty}a_j(x,\xi,z)h^j$ in $S^0(\mathbb{R}^2)$ such that

$$\psi(x)\Big(E_{-+}(z)-(z-x^2)\Big)=a^w(x,hD_x,z;h), \tag{3.25}$$

with

$$a_0(x,\xi,z)=a_0(x,\xi):=-\psi(x)V(x,\xi), \tag{3.26}$$

$$a_1(x,\xi,z)=a_1(x,\xi):=-\frac{1}{4}\psi(x)\Big(\partial_x^2V(x,\xi)+\partial_\xi^2V(x,\xi)\Big)+\frac{1}{2i}\partial_x\psi(x)\partial_\xi V(x,\xi). \tag{3.27}$$

Proof. i) According to Lemma 3.4.3, one has

$$E_{-+}(z)-(z-x^2)=-R_{+}V^w(h^{\frac{1}{2}})R_{-}-R_{+}a(z)[V^w(h^{\frac{1}{2}}),\Pi]R_{-}. \tag{3.28}$$

We first demonstrate that $E_{-+}(z)-(z-x^2)$ is an h- pseudodifferential operator with bounded symbol. To do this we use the Beal's characterization of $h-$ pseudodifferential operators (see [22, Chapter 8] or [25, Chapter 8]).

Let $l(x,\xi)$ be a linear form on $\mathbb{R}^2$ and let $l^w(x,hD_x)$ be the corresponding $h-$ pseudodifferential operator. Using the fact that R_+,R_- commute with $l^w(x,hD_x)$ and that $V^w(h^{\frac{1}{2}})$ is an $h-$ pseudodifferential operator in x whose symbol is bounded operator in y, we obtain

$$\begin{aligned}[l^w(x,hD_x),R_{+}V^w(h^{\frac{1}{2}})R_{-}]&=R_{+}[l^w(x,hD_x),V^w(h^{\frac{1}{2}})]R_{-}\\&=\mathcal{O}(h),\end{aligned} \tag{3.29}$$

where in the last equality, we have used $\|R_+\|=\mathcal{O}(1)$ and $\|R_-\|=\mathcal{O}(1)$.

Similarly, for $N\in\mathbb{N}$ and linear forms $l_1(x,\xi),...,l_N(x,\xi)$ on $\mathbb{R}^2$, we also obtain

$$[l_1^w(x,hD_x),[...[l_N^w(x,hD_x),R_{+}V^w(h^{\frac{1}{2}})R_{-}]...]]=\mathcal{O}(h^N). \tag{3.30}$$

According to [22, Proposition 8.3], $R_+V^w(h^{\frac{1}{2}})R_-$ is an h- pseudodifferential operator with bounded symbol.

Using the same arguments, we also obtain that the operator $R_+a(z)[V^w(h^{\frac{1}{2}}),\Pi]R_-$ is an $h-$ pseudodifferential operator with bounded symbol. Therefore, $E_{-+}(z)-(z-x^2)$ is an h- pseudodifferential operator with bounded symbol denoted by $\tilde{a}(x,\xi,z;h)$.

Now we will prove that $\tilde{a}(x,\xi,z;h)$ has an asymptotic expansion like (3.44). First, we consider the operator

$$R_+V^w(h^{\frac{1}{2}})R_-:L^2(\mathbb{R})\to L^2(\mathbb{R}).$$

Recall that $V(h^{\frac{1}{2}})=V(h^{\frac{1}{2}}\eta+(1-h^2)^{\frac{1}{2}}x,h\xi+h^{\frac{1}{2}}(1-h^2)^{\frac{1}{2}}y)$. Applying Taylor's formula to $V(h^{\frac{1}{2}})$ at $X:=((1-h^2)^{\frac{1}{2}}x,h\xi)$, we get

$$\begin{aligned}V(h^{\frac{1}{2}})&=V(X)+\Big[\partial_1V(X)\eta+\partial_2V(X)(1-h^2)^{\frac{1}{2}}y\Big]h^{\frac{1}{2}}\\&+\frac{1}{2}\Big[\partial_1^2V(X)\eta^2+\partial_2^2V(X)((1-h^2)^{\frac{1}{2}}y)^2+2\partial_{1,2}^2V(X)\eta(1-h^2)^{\frac{1}{2}}y\Big]h+\dots\\&+\mathcal{O}_{N,X}\Big(\big(\eta(1-h^2)^{\frac{1}{2}}y\big)^N\Big)h^{\frac{N}{2}},\end{aligned}\tag{3.31}$$

where ∂_j is the partial derivative with respect to the jth-variable, $j=1,2$, and $\mathcal{O}_{N,X}$ depends on N,X.

Notice that

$$\begin{aligned}&R_+\Big[\partial_1^k\partial_2^lV(X)\eta^ky^l\Big]^w(x,y,D_x,D_y)R_-u(x)\\&=\partial_1^k\partial_2^lV^w((1-h^2)^{\frac{1}{2}}x,hD_x)u(x)\langle(\eta^ky^l)^w(y,D_y)\phi(y),\phi(y)\rangle.\end{aligned}\tag{3.32}$$

A simple computation shows that

$$(\eta^ky^l)^w(y,D_y)\phi(y)=\frac{1}{i^k}\sum_{j=0}^{\min(k,l)}C_k^j\frac{1}{2^j}y^{l-j}\phi^{(k-j)}(y),\tag{3.33}$$

where $\phi^{(k-j)}(y)$ is the derivative of order $k-j$ of $\phi(y)$. When $k+l$ is an odd number, it follows from Lemma 3.4.1 that $\langle\phi^{(k-j)}(y),y^{l-j}\phi(y)\rangle=0$ for all $0\le j\le\min(k,l)$. Therefore, for $k+l$ odd,

$$\langle(\eta^ky^l)^w(y,D_y)\phi(y),\phi(y)\rangle=0,\ \text{for all } 0\le j\le\min(k,l).\tag{3.34}$$

Since $\phi'(y)=-y\phi(y)$, one has $(\eta y)^w(y,D_y)\phi(y)=\frac{1}{2}\phi(y)+y\phi'(y)=\frac{1}{2}\phi(y)-y^2\phi(y)$. Combining this with Lemma 3.4.1, we obtain

$$\langle(\eta y)^w(y,D_y)\phi(y),\phi(y)\rangle=0.\tag{3.35}$$

Putting together (3.31), (3.32), (3.34) and (3.35), we get $\forall N\in\mathbb{N}$,

$$R_+V^w(h^{\frac{1}{2}})R_-=\sum_{j=0}^{N}r_j^w((1-h^2)^{\frac{1}{2}}x,hD_x)h^j+\mathcal{O}(h^{N+1}),\tag{3.36}$$

where $r_j \in S^0(\mathbb{R}^2)$. In particular, $r_0((1-h^2)^{\frac{1}{2}}x,\xi) = V((1-h^2)^{\frac{1}{2}}x,\xi)$ and $r_1((1-h^2)^{\frac{1}{2}}x,\xi) = \frac{1}{4}\Big(\partial_1^2 V((1-h^2)^{\frac{1}{2}}x,\xi) + \partial_2^2 V((1-h^2)^{\frac{1}{2}}x,\xi)\Big)$.

Next we study the operator $R_+a(z)[V^w(h^{\frac{1}{2}}),\Pi]R_-$. From the definition of $a(z)$ (see Lemma 3.4.3), one has

$$\begin{aligned} R_+a(z)[V^w(h^{\frac{1}{2}}),\Pi]R_- &= R_+(I+[\Pi,V^w(h^{\frac{1}{2}})]R(z))^{-1}[V^w(h^{\frac{1}{2}}),\Pi]R_- \\ &= R_+\sum_{j\geq 0}(-1)^j([\Pi,V^w(h^{\frac{1}{2}})]R(z))^j[V^w(h^{\frac{1}{2}}),\Pi]R_- \qquad (3.37)\\ &= R_+[V^w(h^{\frac{1}{2}}),\Pi]R_- + R_+\sum_{j\geq 1}(-1)^j([\Pi,V^w(h^{\frac{1}{2}})]R(z))^j[V^w(h^{\frac{1}{2}}),\Pi]R_- \qquad (3.38)\\ &= R_+\sum_{j\geq 1}(-1)^j([\Pi,V^w(h^{\frac{1}{2}})]R(z))^j[V^w(h^{\frac{1}{2}}),\Pi]R_-, \qquad (3.39)\end{aligned}$$

where we have used $R_+\Pi = R_+$ and $\Pi R_- = R_-$ to deduce $R_+[V^w(h^{\frac{1}{2}}),\Pi]R_- = 0$.

We write

$$\begin{aligned} R(z) &= \Big((I-\Pi)[h^{-1}(D_y^2+y^2-1)+x^2+V^w(h^{\frac{1}{2}})](I-\Pi)-z\Big)^{-1}(I-\Pi)\\ &= h\Big((I-\Pi)[(D_y^2+y^2-1)+hx^2+hV^w(h^{\frac{1}{2}})](I-\Pi)-hz\Big)^{-1}(I-\Pi) \qquad (3.40)\\ &= hH_1\big[I+h(I-\Pi)(V^w(h^{\frac{1}{2}})-z)(I-\Pi)H_1\big]^{-1}(I-\Pi) \qquad (3.41)\\ &= H_1\sum_{j\geq 0}(-1)^j[(I-\Pi)(V^w(h^{\frac{1}{2}})-z)H_1]^j(I-\Pi)h^{j+1}, \qquad (3.42)\end{aligned}$$

where $H_1 = \Big((I-\Pi)(D_y^2+y^2-1+hx^2)(I-\Pi)\Big)^{-1}(I-\Pi)$ is a bounded operator since $(I-\Pi)((D_y^2+y^2-1)+hx^2)(I-\Pi) \geq (I-\Pi)$. Moreover, if we consider H_1 as an operator from $L^2(\mathbb{R}_y)$ to $L^2(\mathbb{R}_y)$ and x as a parameter, we have

$$\|H_1\|_{\mathcal{L}(L^2(\mathbb{R}_y))} \leq C$$

for some constant C independent of both x and h. Since $\partial_x H_1 = -H_1(2hx)H_1$, there also exists $C_1 > 0$ such that

$$\|\partial_x H_1\|_{\mathcal{L}(L^2(\mathbb{R}_y))} \leq C_1.$$

Similarly, we can show that

$$\|\partial_x^\beta H_1\|_{\mathcal{L}(L^2(\mathbb{R}_y))} \leq C_\beta$$

for all $\beta \in \mathbb{N}$ and C_β is also independent of x. Combining this with the fact that H_1 does not depend on the dual variable ξ of x, we conclude that H_1 is an $h-$ pseudodifferential operator in the $x-$ variable whose symbol is bounded operator in the $y-$ variable.

Making use of the symbolic calculus of pseudodifferential operators with operator-valued symbols (see [4]), and using (3.20), (3.39) and (3.42), we deduce that, $\forall N \in \mathbb{N}$,

$$R_+a(z)[V^w(h^{\frac{1}{2}}),\Pi]R_- = \sum_{j=0}^{N}\tilde{r}_j^w(x,\xi,z;hx^2;(1-h^2)^{\frac{1}{2}}x)h^{\frac{j}{2}+2} + \mathcal{O}(h^{\frac{N+1}{2}+2}), \qquad (3.43)$$

where $\tilde{r}_j \in S^0(\mathbb{R}^2)$.

As a consequence of (3.36) and (3.43), the operator $E_{-+}(z) - \Big(z - x^2 - V^w((1-h^2)^{\frac{1}{2}}x, hD_x)\Big)$ is an $h-$ pseudodifferential operator with symbol belonging to $S^{-1}(\mathbb{R}^2)$.

ii) It follows from (3.36) and (3.43) that the symbol $\tilde{a}(x,\xi,z;h)$ of $E_{-+}(z) - (z-x^2)$ can be written as follows :

$$\tilde{a}(x,\xi,z;h) \sim \sum_{j\geq 0} \tilde{a}_j(x,\xi,z;h)h^{\frac{j}{2}}, \tag{3.44}$$

where $\tilde{a}_j(x,\xi,z;h)$ is of the form $r_j(x,\xi,z;hx^2;(1-h^2)^{\frac{1}{2}}x)$ with $r_j \in S^0(\mathbb{R}^2)$. In particular, $\tilde{a}_0(x,\xi,z;h) = V((1-h^2)^{\frac{1}{2}}x, h\xi)$, $\tilde{a}_1(x,\xi,z;h) = 0$ and $\tilde{a}_2(x,\xi,z;h) = \frac{1}{4}\Big(\partial_1^2 V((1-h^2)^{\frac{1}{2}}x,\xi) + \partial_\xi^2 V((1-h^2)^{\frac{1}{2}}x,\xi)\Big)$.

Now we prove (3.25). Let $\psi \in C_0^\infty(\mathbb{R})$, we claim that the symbol $a(x,\xi,z;h)$ of the operator $\psi(x)(E_{-+}(z) - (z-x^2))$ has an asymptotic expansion in powers of $h^{\frac{1}{2}}$ in $S^0(\mathbb{R}^2)$

$$a(x,\xi,z;h) \sim \sum_{j\geq 0} \tilde{\tilde{a}}_j(x,\xi,z)h^{\frac{j}{2}}. \tag{3.45}$$

Indeed, let $\tilde{\psi} \in C_0^\infty(\mathbb{R})$ such that $\tilde{\psi} = 1$ on the support of ψ. We set $\tilde{V}(h^{\frac{1}{2}}) := \tilde{\psi}(x)V(h^{\frac{1}{2}})$ and $\tilde{R}(z;h^{\frac{1}{2}}) := \Big((I-\Pi)[h^{-1}(D_y^2+y^2-1) + x^2\tilde{\psi}(x) + \tilde{V}^w(h^{\frac{1}{2}})](I-\Pi) - z\Big)^{-1}(I-\Pi)$. Applying the $h-$ pseudodifferential calculus, one obtains

$$\begin{aligned} &a^w(x,hD_x,z;h) \\ &= \psi(x)\Big(- R_+\tilde{V}^w(h^{\frac{1}{2}})R_- - R_+(I + [\Pi,\tilde{V}^w(h^{\frac{1}{2}})]\tilde{R}(z;h^{\frac{1}{2}}))^{-1}[\tilde{V}^w(h^{\frac{1}{2}}),\Pi]R_-\Big) + \mathcal{O}(h^\infty). \end{aligned} \tag{3.46}$$

We apply Taylor's formula to $V(h^{\frac{1}{2}})$ at $(h^{\frac{1}{2}}\eta + x, h\xi + h^{\frac{1}{2}}(1-h^2)^{\frac{1}{2}}y)$ and obtain

$$\tilde{V}(h^{\frac{1}{2}}) = \sum_{j=0}^{N} V_j(h^{\frac{1}{2}}\eta + x, h\xi + h^{\frac{1}{2}}(1-h^2)^{\frac{1}{2}}y)\psi_j(x)h^2 + \mathcal{O}(h^{2(N+1)}), \tag{3.47}$$

where $V_j \in S^0(\mathbb{R}^2)$ and $\psi_j \in C_0^\infty(\mathbb{R})$. We next use the same arguments as (3.31), (3.36) where $V(h^{\frac{1}{2}})$ is replaced by the right hand side of (3.47). Then we obtain, for all $N \in \mathbb{N}$,

$$- R_+\tilde{V}^w(h^{\frac{1}{2}})R_- = \sum_{j=0}^{N} v_j^w(x,hD_x)h^j + \mathcal{O}(h^{N+1}), \tag{3.48}$$

where $v_j \in S^0(\mathbb{R}^2)$ is independent of h. In particular,

$$v_0(x,\xi) = -\tilde{\psi}(x)V(x,\xi);\ v_1(x,\xi) = -\frac{1}{4}\Big(\partial_x^2(\tilde{\psi}(x)V(x,\xi)) + \partial_\xi^2(\tilde{\psi}(x)V(x,\xi))\Big). \tag{3.49}$$

As (3.42), it is easy to see that

$$\tilde{R}(z;h^{\frac{1}{2}}) = \tilde{H}_1 \sum_{j\geq 0}(-1)^j[(I-\Pi)(\tilde{\psi}(x)x^2 + \tilde{V}^w(h^{\frac{1}{2}}) - z)\tilde{H}_1]^j(I-\Pi)h^{j+1}, \tag{3.50}$$

where $\tilde{H}_1 = ((I-\Pi)(D_y^2+y^2-1)(I-\Pi))^{-1}(I-\Pi)$. The operator $\tilde{H}_1$ is independent of both x and h. Then making use of the symbolic calculus of pseudodifferential operators with operator-valued symbols, we obtain for all $N \in \mathbb{N}$,

$$-R_+(I+[\Pi,\tilde{V}^w(h^{\frac{1}{2}})]\tilde{R}(z;h^{\frac{1}{2}}))^{-1}[\tilde{V}^w(h^{\frac{1}{2}}),\Pi]R_- = \sum_{j=0}^{N} \tilde{v}_j^w(x,hD_x,z)h^{\frac{j}{2}+2} + \mathcal{O}(h^{\frac{N+1}{2}+2}), \tag{3.51}$$

where $\tilde{v}_j \in S^0(\mathbb{R}^2)$ is also independent of h. Thus, (3.45) follows from (3.46), (3.48) and (3.51).

Put $J(h^{\frac{1}{2}}) := R_+(I+[\Pi,\tilde{V}^w(h^{\frac{1}{2}})]\tilde{R}(z;h^{\frac{1}{2}}))^{-1}[\tilde{V}^w(h^{\frac{1}{2}}),\Pi]R_-$. We construct the following bounded operator, $\Sigma : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \Sigma u(x,y) = u(x,-y)$. Obviously, one has $\Sigma^2 = I, \Sigma^* = \Sigma$. The following lemma shows that $J(h^{\frac{1}{2}}) = J(-h^{\frac{1}{2}})$ which completes the proof of (3.25) :

Lemma 3.4.6. *The following assertions hold :*

1. $\Sigma\tilde{V}^w(h^{\frac{1}{2}})\Sigma = \tilde{V}^w(-h^{\frac{1}{2}})$.
2. $[\Sigma,\Pi] = 0$.
3. $R_+\Sigma = R_+$ *and* $\Sigma R_- = R_-$.
4. $\Sigma\tilde{R}(z;h^{\frac{1}{2}})\Sigma = \tilde{R}(z;-h^{\frac{1}{2}})$.

Proof. i) Let $a(x,y,\xi,\eta) \in S^0(\mathbb{R}^4)$ arbitrarily. By a change of variables, one has

$$\begin{aligned}
&\Sigma a^w(x,y,D_x,D_y)\Sigma u(x,y)\\
&= \frac{1}{(2\pi)^2}\iiiint e^{i((x-x')\xi+(-y-y')\eta)}a(\frac{x+x'}{2},\frac{-y+y'}{2},\xi,\eta)u(x',-y')dx'dy'd\xi d\eta\\
&= \frac{1}{(2\pi)^2}\iiiint e^{i((x-x')\xi+(y-y'')\eta')}a(\frac{x+x'}{2},\frac{-y-y''}{2},\xi,-\eta')u(x',y'')dx'dy''d\xi d\eta'\\
&= a^w(x,-y,D_x,-D_y)u(x,y).
\end{aligned}$$

Applying this to $\tilde{V}(h^{\frac{1}{2}})$, we obtain $\Sigma\tilde{V}^w(h^{\frac{1}{2}})\Sigma = \tilde{V}^w(-h^{\frac{1}{2}})$.

ii) By using the definition of Σ and Π, one has

$$\begin{aligned}
[\Sigma,\Pi]u(x,y) &= \Sigma\Pi u(x,y) - \Pi u(x,-y)\\
&= \int u(x,t)\phi(t)dt\phi(-y) - \int u(x,-t)\phi(t)dt\phi(y) \qquad (3.52)\\
&= \int (u(x,t)-u(x,-t))\phi(t)dt\phi(y). \qquad (3.53)
\end{aligned}$$

However, since the integrand in the right hand side of (3.53) is an odd function with respect to t, we obtain $[\Sigma,\Pi]u(x,y) = 0$ for all $u \in L^2(\mathbb{R}^2)$. Therefore, $[\Sigma,\Pi] = 0$.

iii) The same arguments give (iii).

iv) The assertion (iv) is an easy consequence of (i) and (ii). □

□

3.4.2 Trace formulae.

Assume that the hypotheses in Theorem 3.3.1 hold and let $\underline{a} < \bar{a} < 0$ such that $\mathrm{supp} f \subset (\underline{a}, \bar{a})$. Then we can find an almost analytic extension $\tilde{f}$ of f satisfying $\tilde{f} \in C_0^\infty(\Omega), \tilde{f}|_{\mathbb{R}} = f, \bar{\partial}_z \tilde{f}(z) = \mathcal{O}(|\mathrm{Im} z|^\infty)$, where $\Omega := (\underline{a}, \bar{a}) + i(-1, 1)$ and $\bar{\partial}_z = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ for $z = x + iy$ (see [22]). Notice that we can take $\tilde{f}$ with support in an arbitrarily small neighbourhood of the support of f. Since the potential V vanishes at infinity, one has that

$$\Sigma_{[\underline{a},\bar{a}]} := \Big\{(x, \xi) \in \mathbb{R}^2 |\ x^2 + V(x, \xi) \in [\underline{a}, \bar{a}]\Big\} \tag{3.54}$$

is a compact set in $\mathbb{R}^2$.

Next we are going to prove that $f(\tilde{P}(h))$ is of trace class and give *a priori* estimate of $\mathrm{tr}\Big(f(\tilde{P}(h))\Big)$.

Proposition 3.4.7. *The operator $f(\tilde{P}(h))$ is of trace class. Moreover, for all functions $\chi \in C_0^\infty(\mathbb{R}^2; \mathbb{R})$ such that $\chi = 1$ near $\Sigma_{[\underline{a},\bar{a}]}$, we have*

$$\mathrm{tr}\Big(f(\tilde{P}(h))\Big) = \mathrm{tr}\left(-\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)(\tilde{E}(z))^{-1}\partial_z \tilde{E}(z) L(dz)\chi^w(x, hD_x)\right) + \mathcal{O}(h^\infty). \tag{3.55}$$

Here, $\tilde{f}$ is an almost analytic extension of f and $\tilde{E}(z) = z - x^2\psi(x) + \psi(x)(E_{-+}(z) - (z - x^2))$ for some $\psi \in C_0^\infty(\mathbb{R}; [0, 1])$ such that $\psi(x)\chi(x, \xi) = \chi(x, \xi),\ \forall (x, \xi) \in \mathbb{R}^2$.

Proof. Suppose that $\chi \in C_0^\infty(\mathbb{R}^2; \mathbb{R})$, $\chi = 1$ near $\Sigma_{[\underline{a},\bar{a}]}$. Then we can choose a function $\tilde{V} \in S^0(\mathbb{R}^2)$ such that

- $\tilde{V}(x, y) = V(x, y)$ outside a small neighbourhood of $\Sigma_{[\underline{a},\bar{a}]}$,
- $\chi = 1$ on the support of $(\tilde{V} - V)$,
- $|z - x^2 - \tilde{V}((1 - h^2)^{\frac{1}{2}}x, y)| \geq \varepsilon_0\langle x\rangle^2$ for all $(x, y) \in \mathbb{R}^2$, $z \in \Omega$ and h small enough.

Indeed, let U be a neighbourhood of $\Sigma_{[\underline{a},\bar{a}]}$ such that $\Sigma_{[\underline{a},\bar{a}]} \subset U$ and $\chi = 1$ on U. Now for $(x, y) \in \mathbb{R}^2 \setminus U$, one has either $x^2 + V(x, y) < \underline{a}$ or $x^2 + V(x, y) > \bar{a}$. However, since $x^2+V(x, y)$ is a continuous function and $\lim\limits_{|(x,y)|\to\infty} V(x, y) = 0$, one obtains $x^2+V(x, y) > \bar{a}$ for all $(x, y) \in \mathbb{R}^2 \setminus U$. From this, there also exists $\varepsilon > 0$ such that $x^2 + V(x, y) \geq \bar{a} + \varepsilon$ (or in other words $V(x, y) \geq \bar{a} + \varepsilon - x^2$) for all $(x, y) \in \mathbb{R}^2 \setminus U$.

Hence, it suffices to choose $\tilde{V}$ such that $\tilde{V}(x, y) \geq \bar{a} + \varepsilon - x^2$ for $(x, y) \in \Sigma_{[\underline{a},\bar{a}]}$ and $\tilde{V}(x, y) = V(x, y)$ for $(x, y) \in \mathbb{R}^2 \setminus U$. Then we obtain $\tilde{V}(x, y) \geq \bar{a} + \varepsilon - x^2$ for $(x, y) \in \mathbb{R}^2$ and $\mathrm{supp}(V - \tilde{V}) \subset U$. In particular, $\tilde{V}((1 - h^2)^{\frac{1}{2}}x, y) \geq \bar{a} + \varepsilon - (1 - h^2)x^2 > \bar{a} + \varepsilon - x^2$ uniformly for $(x, y) \in \mathbb{R}^2$ and h small. This shows that $|z - x^2 - \tilde{V}((1 - h^2)^{\frac{1}{2}}x, y)| \geq \varepsilon$ for $(x, y) \in \mathbb{R}^2$. Combining this with the fact that $\tilde{V}$ vanishes at infinity, one obtains that there exists $\varepsilon_0 > 0$ such that $|z - x^2 - \tilde{V}((1 - h^2)^{\frac{1}{2}}x, y)| \geq \varepsilon_0\langle x\rangle^2$ for $(x, y) \in \mathbb{R}^2$. The we have constructed the function $\tilde{V}$ satisfying the above conditions.

Set $\tilde{E}_{-+}(z) = E_{-+}(z)+V^w((1-h^2)^{\frac{1}{2}}x, hD_x)-\tilde{V}^w((1-h^2)^{\frac{1}{2}}x, hD_x)$. Then the principal symbol of the symbol of $\tilde{E}_{-+}(z)$ is $z - x^2 - \tilde{V}((1 - h^2)^{\frac{1}{2}}x, \xi)$ satisfying $|z - x^2 - \tilde{V}((1 - h^2)^{\frac{1}{2}}x, \xi)| \geq \varepsilon_0\langle x\rangle^2$ uniformly in $z \in \Omega$ and $(x, \xi) \in \mathbb{R}^2$. Therefore, for h small enough, the symbol of $\tilde{E}_{-+}(z)$ is elliptic and then $\tilde{E}_{-+}(z)$ is invertible. Moreover, $z \mapsto \tilde{E}_{-+}(z)^{-1}$ is holomorphic on Ω.

Applying the so-called Helffer-Sjötrand formula and Remark 3.4.4, one has

$$\begin{aligned} f(\tilde{P}(h)) &= -\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)(z-\tilde{P}(h))^{-1}L(dz) \\ &= -\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)\Big(-E(z)+E_+(z)(E_{-+}(z))^{-1}E_-(z)\Big)L(dz) \\ &= -\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)E_+(z)(E_{-+}(z))^{-1}E_-(z)L(dz). \end{aligned}$$

Here we have employed the fact that $E(z)$ is analytic to get $-\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)E(z)L(dz)=0$.

On the other hand, the following decomposition

$$E_{-+}(z)^{-1} = \tilde{E}_{-+}(z)^{-1} + E_{-+}(z)^{-1}(\tilde{E}_{-+}(z)-E_{-+}(z))\tilde{E}_{-+}(z)^{-1}$$

yields

$$\begin{aligned} &-\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)E_+(z)(E_{-+}(z))^{-1}E_-(z)L(dz) = -\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)E_+(z)\tilde{E}_{-+}(z)^{-1}E_-(z)L(dz) \\ &-\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)E_+(z)E_{-+}(z)^{-1}(\tilde{E}_{-+}(z)-E_{-+}(z))\tilde{E}_{-+}(z)^{-1}E_-(z)L(dz) \\ =&-\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)E_+(z)E_{-+}(z)^{-1}(\tilde{E}_{-+}(z)-E_{-+}(z))\tilde{E}_{-+}(z)^{-1}E_-(z)L(dz). \end{aligned}$$

In the last equality, we have employed the analyticity of $E_+(z)\tilde{E}_{-+}(z)^{-1}E_-(z)$ on the support of $\tilde{f}$ to deduce $\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)E_+(z)\tilde{E}_{-+}(z)^{-1}E_-(z)L(dz)=0$.

By the construction of $\tilde{E}_{-+}(z)$, the support of symbol of $\tilde{E}_{-+}(z)-E_{-+}(z)$ is compact. Hence, $\tilde{E}_{-+}(z)-E_{-+}(z)$ is a trace operator (see [22, 56]). Thus, $f(\tilde{P}(h))$ is of trace class. Furthermore, making use of the cyclicity of the trace, one has

$$\mathrm{tr}\big(f(\tilde{P}(h))\big) = \mathrm{tr}\Big(-\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)E_{-+}(z)^{-1}(\tilde{E}_{-+}(z)-E_{-+}(z))\tilde{E}_{-+}(z)^{-1}\partial_z E_{-+}(z)L(dz)\Big).$$

Now, by choosing h small enough, we have $\chi=1$ near support of $V((1-h^2)^{\frac{1}{2}}x,\xi)-\tilde{V}((1-h^2)^{\frac{1}{2}}x,\xi)$ which is the symbol of $\big(\tilde{E}_{-+}(z)-E_{-+}(z)\big)$. It follows from [22, Proposition 9.5] that

$$\|f(\tilde{P}(h))(1-\chi^w(x,hD_x))\|_{\mathrm{tr}} = \mathcal{O}(h^\infty). \tag{3.56}$$

Consequently,

$$\begin{aligned} \mathrm{tr}(f(\tilde{P}(h))) &= \mathrm{tr}(f(\tilde{P}(h))\chi^w(x,hD_x)) + \mathcal{O}(h^\infty) \\ &= \mathrm{tr}\Big(-\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)E_{-+}(z)^{-1}\partial_z E_{-+}(z)\chi^w(x,hD_x)L(dz)\Big) + \mathcal{O}(h^\infty). \end{aligned} \tag{3.57}$$

Let ψ be in $C_0^\infty(\mathbb{R};[0,1])$ such that $\psi(x)\chi(x,\xi)=\chi(x,\xi)$. If we consider the function ψ as a constant function with respect to ξ, then $\mathrm{supp}(1-\psi)\cap\mathrm{supp}\chi=\emptyset$. Let us put

$$\tilde{E}(z) = z - x^2\psi(x) + \psi(x)(E_{-+}(z)-(z-x^2)). \tag{3.58}$$

According to [14, Lemma 1.1], there exists $\epsilon>0$ such that

- $\tilde{E}(z)^{-1}$ exists and is holomorphic on $\{z \in \tilde{\Omega} | \, |\mathrm{Im} z| \neq 0\}$, where $\tilde{\Omega} := (\underline{a}, \bar{a}) + i(-\epsilon, \epsilon)$,
- $\|\tilde{E}(z)^{-1}\| = \mathcal{O}(|\mathrm{Im} z|^{-1})$ for $z \in \tilde{\Omega}$, $|\mathrm{Im} z| \neq 0$.

Choosing $\tilde{f}$ with support in $\tilde{\Omega}$, we get that $\tilde{E}(z)^{-1}$ exists and is holomorphic on $\{z \in \mathrm{supp}\tilde{f} | \, |\mathrm{Im} z| \neq 0\}$.

It follows from the resolvent identity that

$$\begin{aligned} E_{-+}(z)^{-1} - \tilde{E}(z)^{-1} &= E_{-+}(z)^{-1}(\tilde{E}(z) - E_{-+}(z))\tilde{E}(z)^{-1} \\ &= E_{-+}(z)^{-1}((1-\psi(x))x^2 + (\psi(x)-1)(E_{-+}(z) - (z - x^2)))\tilde{E}(z)^{-1}. \end{aligned}$$

Following the proof of [22, Proposition 8.6], we obtain for $z \in \mathrm{supp}\tilde{f}$, $\mathrm{Im} z \neq 0$, that $\tilde{E}(z)^{-1}$ is an $h-$pseudodifferential operator with symbol $e(x, \xi, z; h)$ satisfying, for all $\beta \in \mathbb{N}^2$,

$$|\partial^\beta e| \leq C_\beta \max(1, \frac{h^{\frac{1}{2}}}{|\mathrm{Im} z|})^5 |\mathrm{Im} z|^{-|\beta|-1}.$$

Moreover $\bar{\partial}_z \tilde{f}(z) = \mathcal{O}(|\mathrm{Im} z|^\infty)$, then we obtain the symbol of $|\mathrm{Im} z|^{-1}\bar{\partial}_z \tilde{f}(z)\tilde{E}(z)^{-1}$ belongs to $S^0(\mathbb{R}^2)$. Since $\mathrm{supp}(1-\psi) \cap \mathrm{supp}\chi = \emptyset$, it follows that

$$\|((\psi(x)-1)x^2 + (1-\psi(x))(E_{-+}(z) - (z-x^2)))|\mathrm{Im} z|^{-1}\bar{\partial}_z \tilde{f} \tilde{E}(z)^{-1}\partial_z E_{-+}(z)\chi^w\|_{\mathrm{tr}} = \mathcal{O}(h^\infty).$$

From this and the fact that $\|E_{-+}(z)^{-1}\| = \|R_+(z - \tilde{P}(h))^{-1}R_-\| = \mathcal{O}(\frac{1}{|\mathrm{Im} z|})$, we get

$$\mathrm{tr}\Big(-\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)(E_{-+}(z)^{-1} - \tilde{E}(z)^{-1})\partial_z E_{-+}(z)\chi^w(x, hD_x)L(dz)\Big) = \mathcal{O}(h^\infty). \tag{3.59}$$

Thus, (3.57) and (3.59) imply

$$\begin{aligned} \mathrm{tr}(f(\tilde{P}(h))) &= \mathrm{tr}\Big(-\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)\tilde{E}(z)^{-1}\partial_z E_{-+}(z)\chi^w(x, hD_x)L(dz)\Big) + \mathcal{O}(h^\infty) \\ &= \mathrm{tr}\Big(-\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)\tilde{E}(z)^{-1}\partial_z \tilde{E}(z)\chi^w(x, hD_x)L(dz)\Big) + \mathcal{O}(h^\infty). \end{aligned} \tag{3.60}$$

In the last equality of (3.60), we have used $\partial_z E_{-+}(z)\chi^w(x, hD_x) = \partial_z \tilde{E}(z)\chi^w(x, hD_x) + \mathcal{O}(h^\infty)$ which follows from the fact that $\psi(x)\chi(x, \xi) = 1$. This ends the proof of Proposition 3.4.7. □

By combining the arguments in the proof of Proposition 3.4.7 and the techniques in [22, Chapter 12] (see also [14]), one obtains the following result :

Lemma 3.4.8. *Under assumptions of Theorem 3.3.1, we have*

$$\begin{aligned} \mathrm{tr}\Big(f(\tilde{P}(h))\mathcal{F}_h^{-1}\Psi(\tau - \tilde{P}(h))\Big) = \mathrm{tr}\Big(-\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)\mathcal{F}_h^{-1}\Psi(\tau - z)(\tilde{E}(z))^{-1}\partial_z \tilde{E}(z)L(dz)\chi^w\Big) \\ + \mathcal{O}(h^\infty), \end{aligned} \tag{3.61}$$

where χ, $\tilde{E}(z)$ are constructed in the proposition 3.4.7

We recall that $z-\psi(x)(x^2+V(x,\xi))$ is the principal symbol of the symbol of $\tilde{E}(z)$. It is a linear function with respect to z. Furthermore, V tends to zero at infinity. Then for z in some compact set of $(-\infty,0)$ and $|(x,\xi)|$ large enough, $|z-\psi(x)(x^2+V(x,\xi))| \geq \text{const} > 0$. Thus, the following proposition is a consequence of [14, Lemma 1.2] (see also [22, Chapter 8]) :

Proposition 3.4.9. *Let f be as in Proposition 3.4.7. Let $a_f(x,\xi;h)$ be the symbol of the operator*

$$A_f = -\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)\tilde{E}(z)^{-1}\partial_z \tilde{E}(z)L(dz). \tag{3.62}$$

Then there exists a sequence of symbols $(c_j(x,\xi))_{j\in\mathbb{N}}$ in $S^0(\mathbb{R}^2)$ such that

$$a_f(x,\xi;h) \sim \sum_{j=0}^{\infty} c_j(x,\xi)h^j. \tag{3.63}$$

Moreover,

$$c_0(x,\xi) = f(\psi(x)(x^2+V(x,\xi))), \tag{3.64}$$
$$c_1(x,\xi) = -a_1(x,\xi)f'(\psi(x)(x^2+V(x,\xi))). \tag{3.65}$$

Here $a_1(x,\xi)$ is given in Proposition 3.4.5.

3.5 Proofs of Theorem 3.3.1 and Theorem 3.3.2

We start by proving Theorem 3.3.1. It results from Propositions 3.4.7 and 3.4.9 that for all $N \in \mathbb{N}$,

$$\mathrm{tr}(f(\tilde{P}(h))) = \sum_{j=1}^{N} \mathrm{tr}(c_j^w(x,hD_x)\chi^w(x,hD_x))h^j + \mathcal{O}(h^{N+1}). \tag{3.66}$$

Since the support of χ is compact, one has (see [56, Proposition II-56])

$$\mathrm{tr}(c_j^w(x,hD_x)\chi^w(x,hD_x)) = \frac{1}{2\pi h}\iint c_j(x,\xi)\chi(x,\xi)dxd\xi. \tag{3.67}$$

Thus, for all $N \in \mathbb{N}$

$$\mathrm{tr}(f(\tilde{P}(h))) = \sum_{j=1}^{N} A_j(f)h^j + \mathcal{O}(h^{N+1}), \tag{3.68}$$

where $A_j(f) = \frac{1}{2\pi}\iint c_j(x,\xi)\chi(x,\xi)dxd\xi$.

Remark that $\psi = 1$ on $\mathrm{supp}\chi$, and $\chi = 1$ on the set $\{(x,\xi) \in \mathbb{R}^2 |\ x^2+V(x,\xi) \in \mathrm{supp} f\}$. Therefore, $c_0(x,\xi)\chi(x,\xi) = f(x^2+V(x,\xi))$ and

$$A_0(f) = \frac{1}{2\pi}\iint f\left(x^2+V(x,\xi)\right)dxd\xi. \tag{3.69}$$

On the other hand $\psi'(x)\chi(x,\xi) = 0$ for all $(x,\xi) \in \mathbb{R}^2$. Consequently, $c_1(x,\xi)\chi(x,\xi) = \frac{1}{4}(\partial_x^2 V(x,\xi) + \partial_\xi^2 V(x,\xi))f'(x^2 + V(x,\xi))$ and

$$A_1(f) = \frac{1}{8\pi}\iint \left(\partial_x^2 V(x,\xi) + \partial_\xi^2 V(x,\xi)\right) f'\left(x^2 + V(x,\xi)\right) dx d\xi. \tag{3.70}$$

Theorem 3.3.1 is proved.

The following corollary is a simple consequence of Theorem 3.3.1. We recall that $\omega = 1$.

Corollary 3.5.1. *Let λ be a negative real number. Let $N_h(\lambda)$ be the number of eigenvalues below λ of $\tilde{P}(h)$. Assume that the hypotheses in Theorem 3.3.1 hold and that*

$$\mathrm{Vol}\left(\{(x,y) \in \mathbb{R}^2 |\ x^2 + V(x,y) = \lambda\}\right) = 0. \tag{3.71}$$

Then,

$$\lim_{h\to 0} hN_h(\lambda) = M_0, \tag{3.72}$$

where

$$M_0 = \frac{1}{2\pi}\iint_{\{(x,y)\in\mathbb{R}^2|\ x^2+V(x,y)\le\lambda\}} dx dy. \tag{3.73}$$

Proof. It is clear that the operator $\tilde{P}(h)$ is bounded from below. Then there exists $M > 0$ such that $\tilde{P}(h) > -\frac{M}{2}$. For $\varepsilon > 0$ small enough such that $\lambda + \varepsilon < 0$, we put $I(\pm\varepsilon) = [-M \mp \varepsilon, \lambda \pm \varepsilon]$. Let us choose $f_{\pm\varepsilon} \in C_0^\infty(\mathbb{R};[0,1])$, where $f_{+\varepsilon} = 1$ on $I(+\frac{\varepsilon}{2})$, $f_{+\varepsilon} = 0$ outside $I(+\varepsilon)$ and $f_{-\varepsilon} = 1$ on $I(-\varepsilon)$, $f_{-\varepsilon} = 0$ outside $I(-\frac{\varepsilon}{2})$. It is easy to see that

$$f_{-\varepsilon}(\tilde{P}(h)) \le 1_{[-M,\lambda]}(\tilde{P}(h)) \le f_{+\varepsilon}(\tilde{P}(h)). \tag{3.74}$$

Here $1_{[-M,\lambda]}$ is the characteristic function of the interval $[-M,\lambda]$ and $A \ge B \Leftrightarrow A - B$ is a positive operator.

From Theorem 3.3.1, one has

$$\lim_{h\to 0} h\mathrm{tr}\Big(f_{+\varepsilon}(\tilde{P}(h))\Big) = \frac{1}{2\pi}\iint f_{+\varepsilon}(x^2 + V(x,y))dxdy$$

and

$$\lim_{h\to 0} h\mathrm{tr}\Big(f_{-\varepsilon}(\tilde{P}(h))\Big) = \frac{1}{2\pi}\iint f_{-\varepsilon}(x^2 + V(x,y))dxdy.$$

Combining this with (3.74), one gets

$$\frac{1}{2\pi}\iint f_{-\varepsilon}(x^2 + V(x,y))dxdy \le \lim_{h\to 0} hN_h(\lambda) \le \frac{1}{2\pi}\iint f_{+\varepsilon}(x^2 + V(x,y))dxdy. \tag{3.75}$$

On the other hand

$$\begin{aligned}
&\Big|\iint \Big(f_{+\varepsilon}(x^2 + V(x,y)) - 1_{[-M,\lambda]}(x^2 + V(x,y))\Big) dxdy\Big| \\
&\le \iint 1_{[\lambda,\lambda+\varepsilon]}(x^2 + V(x,y))dxdy + \iint 1_{[-M-\varepsilon,M]}(x^2 + V(x,y))dxdy \\
&= \iint 1_{[\lambda,\lambda+\varepsilon]}(x^2 + V(x,y))dxdy \overset{\varepsilon\to 0}{\longrightarrow} \mathrm{Vol}(\{(x,y) \in \mathbb{R}^2 |\ x^2 + V(x,y) = \lambda\}) = 0.
\end{aligned}$$

Here we used the fact that $x^2+V(x,y) > M$ for all $(x,y) \in \mathbb{R}^2$ to deduce $\iint 1_{[-M-\varepsilon,M]}(x^2+V(x,y))dxdy = 0$. Similarly, one also obtains

$$\Big|\iint \Big(f_{+\varepsilon}(x^2+V(x,y)) - 1_{[-M,\lambda]}(x^2+V(x,y))\Big)dxdy\Big| \overset{\varepsilon\to 0}{\longrightarrow} 0.$$

Now let ε tend to zero in (3.75) one obtains (3.72). □

Remark 3.5.2. It follows from the corollary 3.5.1 that $N_h(\lambda) = \frac{1}{h}(M_0 + o(1))$. Naturally we want to give a more precise estimate for the term $o(1)$. The optimal estimate is actually given in Theorem 3.3.3.

Now, we are going to prove Theorem 3.3.2. We choose χ in Lemma 3.4.8 such that $\{(x,\xi) \in \mathbb{R}^2 |\ x^2+V(x,\xi) \le \lambda+\sigma_0 < 0\}$ is a subset of suppχ, for some $\sigma_0 > 0$, and choose $\psi \in C_0^\infty(\mathbb{R};[0,1])$, $\psi(x)\chi(x,\xi) = \chi(x,\xi)$. Then, for $\tau \le \lambda+\sigma_0$, we have

$$\psi(x)(x^2+V(x,\xi)) = \tau \Leftrightarrow x^2+V(x,\xi) = \tau. \tag{3.76}$$

On the other hand, under the hypotheses of Theorem 3.3.2, there exists $0 < \sigma < \sigma_0$, such that

$$\nabla_{x,\xi}(x^2+V(x,\xi)) \neq 0 \text{ for all } (x,\xi) \in \{(x,\xi) \in \mathbb{R}^2 |\ \lambda-\sigma < x^2+V(x,\xi) < \lambda+\sigma\}. \tag{3.77}$$

It follows that the principal symbol of $\tilde{E}(z)$, $z - \psi(x)(x^2+V(x,\xi))$, is strictly microhyperbolic (in the sense of [14]) in $(\lambda-\sigma, \lambda+\sigma)$. Let $f \in C_0^\infty((\lambda-\sigma,\lambda+\sigma);\mathbb{R})$, we apply [14, Theorem 1.8] to obtain : For $M \ge 1$, uniformly in $\tau \in (\lambda-\sigma,\lambda+\sigma)$

$$\operatorname{tr}\Big(-\frac{1}{\pi}\int \bar{\partial}_z \tilde{f}(z)\mathcal{F}_h^{-1}\Psi(\tau-z)(\tilde{E}(z))^{-1}\partial_z\tilde{E}(z)L(dz)\chi^w\Big) = \sum_{j=0}^{M} C_j(\tau)h^{j-1} + \mathcal{O}(h^M), \tag{3.78}$$

where C_j is a smooth function. In particular,

$$C_0(\tau) = \frac{1}{2\pi}f(\tau)\Psi(0)\int_{\{(x,\xi)\in\mathbb{R}^2|\ \psi(x)(x^2+V(x,\xi))=\tau\}} \frac{dS_\tau}{|\nabla_{x,\xi}(\psi(x)(x^2+V(x,\xi)))|}.$$

Combining this with (3.76), one has

$$C_0(\tau) = \frac{1}{2\pi}f(\tau)\Psi(0)\int_{\{(x,\xi)\in\mathbb{R}^2|\ x^2+V(x,\xi)=\tau\}} \frac{dS_\tau}{|\nabla_{x,\xi}(x^2+V(x,\xi))|}.$$

Let us denote by $\mu_j(h), j = 1, ..., m_0$ the eigenvalues of $\tilde{P}(h)$ in $(\lambda-\sigma,\lambda+\sigma)$. Then there exists a positive constant ε_0 such that $\min\limits_{0\le j\le m_0} |\tau-\mu_j(h)| \ge \varepsilon_0 \langle\tau\rangle$ for all $\tau \in \mathbb{R}\backslash(\lambda-\sigma,\lambda+\sigma)$. From the spectral theorem and Theorem 3.3.1, we have

$$\mathcal{O}\Big(\frac{1}{h}\Big) = \operatorname{tr}(f(\tilde{P}(h))) = \sum_{j=1}^{m_0} f(\mu_j(h)).$$

Since $\mathcal{F}^{-1}\Psi \in \mathcal{S}(\mathbb{R})$ ($\mathcal{F}^{-1}$ is the classical inverse Fourier transformation), we get, for all $N \geq 1$,

$$\begin{aligned}\operatorname{tr}\Big(f(\tilde{P}(h))\mathcal{F}_h^{-1}\Psi(\tau - \tilde{P}(h))\Big) &= \sum_{j=1}^{m_0} f(\mu_j(h))\frac{1}{h}\mathcal{F}^{-1}\Psi(\frac{\tau-\mu_j(h)}{h}) \\ &= \mathcal{O}\Big(\frac{1}{h^2}\Big)\max_{0\leq j\leq m_0}\Big|\frac{\tau-\mu_j(h)}{h}\Big|^{-N} \\ &= \mathcal{O}\Big(h^{N-2}/\langle\tau\rangle^N\Big),\end{aligned} \tag{3.79}$$

uniformly in $\tau \in \mathbb{R}\setminus(\lambda-\sigma, \lambda+\sigma)$. Therefore, it results from (3.78) and (3.79) that, uniformly in $\tau \in \mathbb{R}$,

$$\operatorname{tr}\Big(f(\tilde{P}(h))\mathcal{F}_h^{-1}\Psi(\tau - \tilde{P}(h))\Big) = \sum_{j=0}^{M} C_j(\tau)h^{j-1} + \mathcal{O}\Big(h^M/\langle\tau\rangle^N\Big).$$

This ends the proof of Theorem 3.3.2.

3.6 Proof of Theorem 3.3.3.

In this section, we give the proof of Theorem 3.3.3 which is based on Theorems 3.3.1, 3.3.2 and some Tauberian arguments.

Let σ be given in (3.77). Choosing $f_1 \in C_0^\infty((-\infty, \lambda-\sigma/2); [0,1])$ and $f_2 \in C_0^\infty((\lambda-\sigma, \lambda+\sigma); [0,1])$ such that $f_1+f_2 = 1$ on $[-M, \lambda+\sigma/2]$, where $M > \sup_{(x,y)\in\mathbb{R}^2}|V(x,y)|$. Let $\mu_0(h) \leq \mu_1(h) \leq ...$ be the eigenvalues of $\tilde{P}(h)$ in $(-\infty, 0)$. Then,

$$N_h(\lambda) = \sum_{\mu_j(h)\leq\lambda}(f_1+f_2)(\mu_j(h)) = \sum_{\mu_j(h)} f_1(\mu_j(h)) + \sum_{\mu_j(h)\leq\lambda} f_2(\mu_j(h)). \tag{3.80}$$

It follows from Theorem 3.3.1 that

$$\sum_{\mu_j(h)} f_1(\mu_j(h)) = \operatorname{tr}(f_1(\tilde{P}(h))) = \frac{1}{2\pi h}\iint f_1(x^2+V(x,y))dxdy + \mathcal{O}(1). \tag{3.81}$$

Set

$$M(\tau,h) = \sum_{\mu_j(h)\leq\tau} f_2(\mu_j(h)). \tag{3.82}$$

Replacing Ψ by $\Psi * \Psi$ if necessary, for $C > 0$ large enough, we can choose a function $\Psi \in C_0^\infty((-\frac{1}{C}, \frac{1}{C}); \mathbb{R})$ such that :

- $\Psi(0) = 1$,
- $\mathcal{F}^{-1}\Psi \geq 0$ on $\mathbb{R}$,
- $\mathcal{F}^{-1}\Psi \geq \varepsilon_0$ on $[-\varepsilon_1, \varepsilon_1]$ for some small constants $\varepsilon_0, \varepsilon_1$.

Since $M(\tau,h)$ is monotone with respect to τ, we use the Tauberian arguments (see [56, Theorem V-13]) to obtain the following result :

Proposition 3.6.1. *Uniformly in h small and $\lambda \in \mathbb{R}$ one has*

$$M(\lambda, h) = \int_{-\infty}^{\lambda} \operatorname{tr}\Big(f_2(\tilde{P}(h))\mathcal{F}_h^{-1}\Psi(\tau - \tilde{P}(h))\Big)d\tau + \mathcal{O}(1), \tag{3.83}$$

where Ψ is constructed as above.

Applying Proposition 3.6.1 and Theorem 3.3.2, we get

$$M(\lambda, h) = h^{-1}\int_{-\infty}^{\lambda} C_0(\tau)d\tau + \mathcal{O}(1), \tag{3.84}$$

where

$$\int_{-\infty}^{\lambda} C_0(\tau)d\tau = \int_{\lambda-\sigma}^{\lambda} \frac{1}{2\pi} f_2(\tau)d\tau \int_{\{(x,y)\in\mathbb{R}^2|\ x^2+V(x,y)=\tau\}} \frac{dS_\tau}{|\nabla_{x,y}(x^2+V(x,y))|}. \tag{3.85}$$

The condition (3.77) allows us to apply the coarea formula (see, e.g., [56, Lemma V-9])

$$\begin{aligned} &f_2(\tau)\int_{\{(x,y)\in\mathbb{R}^2|\ x^2+V(x,y)=\tau\}} \frac{dS_\tau}{|\nabla_{x,y}(x^2+V(x,y))|} \\ &= \partial_\tau\Big(\iint_{\{(x,y)\in\mathbb{R}^2|\ x^2+V(x,y)\leq\tau\}} f_2(x^2+V(x,y))dxdy\Big),\ \forall \tau \in (\lambda-\sigma, \lambda). \end{aligned}$$

Thus,

$$\int_{-\infty}^{\lambda} C_0(\tau)d\tau = \frac{1}{2\pi}\iint_{\{(x,y)\in\mathbb{R}^2|\ x^2+V(x,y)\leq\lambda\}} f_2(x^2+V(x,y))dxdy.$$

Finally, one obtains from (3.81), (3.82) and (3.84)

$$\begin{aligned} N_h(\lambda) &= \frac{1}{2\pi h}\iint_{\{(x,y)\in\mathbb{R}^2|\ x^2+V(x,y)\leq\lambda\}} (f_1+f_2)(x^2+V(x,y))dxdy + \mathcal{O}(1) \\ &= \frac{1}{2\pi h}\iint_{\{(x,y)\in\mathbb{R}^2|\ x^2+V(x,y)\leq\lambda\}} dxdy + \mathcal{O}(1). \end{aligned}$$

Theorem 3.3.3 is now proved.

Appendix A

Let $a \in S^0(m, \mathbb{R}^4)$. Suppose that $\kappa : \mathbb{R}^4 \to \mathbb{R}^4$ is a linear canonical transformation. According to [22, Theorem A_2, Chapter 7], there exists an associated unitary operator U_κ such that $U_\kappa^{-1} a^w U_\kappa = (a \circ \kappa)^w$.

As mentioned in the last of section 3.3, we construct some linear canonical transformations to prove the unitary equivalence of $P(B,\omega) = (D_x - By)^2 + D_y^2 - \sqrt{B^2+\omega^2} + \omega^2 x^2 + V(x,y)$ and $\tilde{P}(\alpha) = \alpha(D_y^2 + y^2) - \alpha + \omega^2 x^2 + V^w(\frac{1}{\sqrt{\alpha}} D_y + \frac{B}{\alpha} x, \frac{1}{\alpha} D_x + \frac{B\sqrt{\alpha}}{\alpha^2} y)$,

Let $p(x,y,\xi,\eta) = (\xi - By)^2 + \eta^2 - \alpha + \omega^2 x^2 + V(x,y)$ which is the symbol of $P(B,\omega)$, we construct the following linear canonical transformations $\kappa_0, \kappa_1, \kappa_2$:

The first one is

$$\kappa_0 : (x,y,\xi,\eta) \mapsto (x + \frac{1}{B}\eta, y + \frac{1}{B}\xi, \xi, \eta).$$

Then the new symbol is given by $p_1(x,y,\xi,\eta) = p \circ \kappa_0(x,y,\xi,\eta) = (By)^2 + \eta^2 - \alpha + \omega^2(x + \frac{1}{B}\eta)^2 + V(x + \frac{1}{B}\eta, y + \frac{1}{B}\xi)$.

The second one is given by

$$\kappa_1 : (x,y,\xi,\eta) \mapsto (x, y, \xi - \frac{\omega^2 B}{\alpha^2} y, \eta - \frac{\omega^2 B}{\alpha^2} x).$$

The new symbol is $p_2(x,y,\xi,\eta) = p_1 \circ \kappa_1(x,y,\xi,\eta) = (By)^2 + \frac{\alpha^2}{B^2}\eta^2 - \alpha + \frac{\omega^2 B^2}{\alpha^2} x^2 + V(\frac{B^2}{\alpha^2} x + \frac{1}{B}\eta, \frac{B^2}{\alpha^2} y + \frac{1}{B}\xi)$.

The last one is constructed as follows

$$\kappa_2 : (x,y,\xi,\eta) \mapsto (\frac{\alpha}{B} x, \frac{\sqrt{\alpha}}{B} y, \frac{B}{\alpha}\xi, \frac{B}{\sqrt{\alpha}}\eta),$$

then the new symbol is $\tilde{p}(x,y,\xi,\eta) = p_2 \circ \kappa_2(x,y,\xi,\eta) = \alpha(\eta^2 + y^2) - \alpha + \omega^2 x^2 + V(\frac{1}{\sqrt{\alpha}}\eta + \frac{B}{\alpha} x, \frac{1}{\alpha}\xi + \frac{B\sqrt{\alpha}}{\alpha^2} y)$. Therefore, $P(B,\omega) = p^w(x,y,D_x,D_y)$ is unitarily equivalent to $\tilde{P}(\alpha) = \tilde{p}^w(x,y,D_x,D_y)$.

Chapitre 4

Resonances in the strong magnetic field

Dans ce chapitre, nous présentons l'article [23].

RESONANCES OF TWO-DIMENSIONAL SCHRÖDINGER OPERATORS WITH STRONG MAGNETIC FIELDS.

ANH TUAN DUONG

ABSTRACT. The purpose of this paper is to study the Schrödinger operator $P(B,\omega) = (D_x - By)^2 + D_y^2 + \omega^2 x^2 + V(x,y),\ (x,y) \in \mathbb{R}^2$, with the magnetic field B large enough and the constant $\omega \neq 0$ is fixed and proportional to the strength of the electric field. Under certain assumptions on the potential V, we prove the existence of resonances near Landau levels as $B \to \infty$. Moreover, we show that the width of resonances is of size $\mathcal{O}\Big(B^{-\infty}\Big)$.

4.1 Introduction.

The model of Schrödinger operator studied in this article is the following

$$P(B,\omega) = P_0(B,\omega) + V(x,y) = (D_x - By)^2 + D_y^2 + \omega^2 x^2 + V(x,y), \tag{4.1}$$

defined on $L^2(\mathbb{R}^2)$, where $D_\nu = \frac{1}{i}\partial_\nu$, B is the strong magnetic field, $\omega \neq 0$ is a fixed constant and the potential V is a real smooth function decreasing at infinity.

It is well-known that the operator $P(B,\omega)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$, (see [26, 45]). For $V \equiv 0$, $\omega = 0$, it was shown that the spectrum of the unperturbed Hamiltonian $P_0(B,0) := (D_x - By)^2 + D_y^2$ consists of eigenvalues $\lambda_n = (2n+1)B,\ n \in \mathbb{N}$ with infinite multiplicities called Landau levels (see [2, 15, 46, 52]). However, in the case $\omega \neq 0$, the essential spectrum of $P_0(B,\omega)$ is absolutely continuous and equal to the semi-axis $[\sqrt{B^2+\omega^2}, +\infty)$, (see [45]). On the other hand, whenever the potential V vanishes at the infinity, one can show as in [2] that $V(P_0(B,\omega)+i)^{-1}$ is a compact operator. By applying the Weyl theorem (see [32]) the essential spectrum of $P(B,\omega)$ is equal to that of $P_0(B,\omega)$. Further, the absolutely continuous spectrum of $P(B,\omega)$ was investigated in [26]. Recently, the counting function of discrete eigenvalues of $P(B,\omega)$ in $(-\infty,\sqrt{B^2+\omega^2})$ has been studied in [24].

Until now, there has been little discussion about the spectral problem of $P(B,\omega)$ which can be also regarded as the quantum hall system Hamiltonian with the unbounded edge potential (see [12, 40] and also [7]). In this work, we propose to study the existence of resonances of $P(B,\omega)$ near Landau levels when the strength of magnetic field tends to infinity. Roughly speaking, the resonances of $P(B,\omega)$ are defined by eigenvalues of some dilated operator (see below).

The past thirty years have seen increasingly rapid advances in the study of resonances of Schrödinger operators with magnetic fields (see [5, 19, 38, 58, 59] and references therein). For the two-dimensional Stark Hamiltonian with strong magnetic field, X.P.Wang proved that there exist resonances near Landau levels, (see [59]). Moreover, M.Dimassi and V.Petkov showed that there does not exist resonances in the upper-half complex plane, (see [19]). However, we notice here that the definitions of resonances in these articles are a little bit different. The resonances are defined by the complex dilation in [59] and by the complex transition in the $x-$variable in [19]. For three-dimensional Schrödinger operators without Stark effect, one of the results of J.F.Bony *et al.* showed that there exist infinitely many resonances in a vicinity of each Landau level (see [5]).

By using the arguments in [22, Chapter 7], we obtain that $P(B,\omega)$ is unitarily equivalent to

$$\begin{aligned}P_1(B,\omega) &:= \sqrt{B^2+\omega^2}(D_y^2+y^2)+\omega^2x^2\\ &+V^w\Big((B^2+\omega^2)^{-\frac14}D_y+(1+\frac{\omega^2}{B^2})^{-\frac12}x,(B^2+\omega^2)^{-\frac12}D_x+B^{-\frac12}(1+\frac{\omega^2}{B^2})^{-\frac34}y\Big).\end{aligned}$$

Here we use the Weyl quantization (see [30, 33]).

Let θ be real. Consider the unitary operator $U_\theta : L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$, $u(x,y)\mapsto u(e^\theta x, e^{-\theta}y)$. One has

$$\begin{aligned}P_{1,\theta}(B,\omega) &:= U_\theta P_1(B,\omega)U_\theta^{-1} = \sqrt{B^2+\omega^2}(e^{2\theta}D_y^2+e^{-2\theta}y^2)+e^{2\theta}\omega^2x^2\\ &+V^w\Big(e^\theta((B^2+\omega^2)^{-\frac14}D_y+(1+\frac{\omega^2}{B^2})^{-\frac12}x),e^{-\theta}((B^2+\omega^2)^{-\frac12}D_x+B^{-\frac12}(1+\frac{\omega^2}{B^2})^{-\frac34}y)\Big).\end{aligned}$$

We set

$$P_{0,\theta}(B,\omega) := \sqrt{B^2+\omega^2}(e^{2\theta}D_y^2+e^{-2\theta}y^2)+e^{2\theta}\omega^2x^2, \tag{4.2}$$

and

$$V_\theta^w(B,\omega)=V^w\Big(e^\theta((B^2+\omega^2)^{-\frac14}D_y+(1+\frac{\omega^2}{B^2})^{-\frac12}x),e^{-\theta}((B^2+\omega^2)^{-\frac12}D_x+B^{-\frac12}(1+\frac{\omega^2}{B^2})^{-\frac34}y)\Big),$$

then we have

$$P_{1,\theta}(B,\omega)=P_{0,\theta}(B,\omega)+V_\theta^w(B,\omega).$$

By using the analytic extension of the potential V (see $(\mathbf{H_1})$), we can extend the formula of $P_{1,\theta}(B,\omega)$ from the real axis to a small complex neighbourhood of 0 with respect to θ. In this paper we define the resonances of $P(B,\omega)$ as the eigenvalues of the non-selfadjoint operator $P_{1,\theta}(B,\omega)$ for $\theta \in \mathbb{C}$, $|\theta|$ small and $\mathrm{Im}\theta < 0$. Moreover the eigenvalues and their multiplicities are independent of θ (see [10, 32]).

From now on, we fix $\theta \in \mathbb{C}$ satisfying $\mathrm{Im}\theta < 0$ and $|\theta|$ small enough. Note that the essential spectra of $P_{0,\theta}(B,\omega)$ and $P_{1,\theta}(B,\omega)$ are coincident and given by $\bigcup_{n\in\mathbb{N}}\{(2n+1)\sqrt{B^2+\omega^2}+e^{2\theta}\lambda,\ \lambda \geq 0\}$ (see Lemma 4.3.1). Then the resonances are distributed outside these semi-lines. As is mentioned above, we are interested in localizing the resonances near Landau levels. To do this, we follow the strategy used by X.P.Wang in [58, 59] and the recent progress in the analysis of two-dimensional Schrödinger operators with magnetic fields (see [14, 15, 19]).

We fix $n \in \mathbb{N}$. Then we set $\mu_n := (2n+1)\sqrt{B^2+\omega^2}$ and $h := \frac{1}{\sqrt{B^2+\omega^2}}$. Let $E \in \mathbb{R}\setminus\{0\}$. In Section 4.3 we prove that z is an eigenvalue of $P_{1,\theta}(B,\omega) - \mu_n$ near E if and only if 0 is an eigenvalue of an $h-$ pseudodifferential operator (called the effective Hamiltonian). Here the effective Hamiltonian is given by

$$E_{-+}(z) = z - A_\theta(h) + h^2 G_\theta(z;h), \tag{4.3}$$

where $G_\theta(z;h)$ is holomorphic for z in some large, bounded set T_n (see (4.22)) and $A_\theta(h)$ does not depend on z (see Theorem 4.3.5). Moreover $A_\theta(h)$ is also an $h-$ pseudodifferential operator with symbol $a(e^\theta x, e^{-\theta}\xi; h)$ which admits a complete expansion in powers of h (see identity (4.33)) :

$$a(x,\xi;h) - a_0(x,\xi) \sim \sum_{j\geq 1} h^j a_j(x,\xi),$$

where

$$a_0(x,\xi) = \omega^2 x^2 + V(x,\xi) \ \text{ and } \ a_1(x,\xi) = \frac{(2n+1)}{4}\Delta V(x,\xi). \tag{4.4}$$

For the $h-$ pseudodifferential operators, we refer the readers to [22, 25].

Therefore, the localization of resonances of $P(B,\omega)$ can be deduced from studying the spectrum of $A_\theta(h)$. In fact, the crucial steps to prove the existence of resonances are the following :

• Prove the exponential decay of the eigenfunctions of $A_\theta(h)$ associated to the eigenvalues near E (see Theorem 4.4.4).

• Establish a resolvent estimate in the non-selfadjoint case (see Proposition 4.4.6).

For these two points, we need a non-trapping condition (see $(\mathbf{H_3})$).

For the width of resonances, as in [41] we use the WKB method to construct an approximate solution of the problem $E_{-+}(z)u = \mathcal{O}(h^\infty)$. Thus by studying a suitable Grushin problem, we obtain the expansion of each resonance in powers of h with real coefficients. This means that the width of resonances is at least of size $\mathcal{O}(h^\infty)$.

The rest of paper is organized as follows. In Section 4.2 we give our assumptions and results. The essential spectrum of $P_{1,\theta}(B,\omega)$ is computed in Subsection 4.3.1. Next the Grushin problem is constructed in Subsection 4.3.2 to establish a reduction to the effective Hamiltonian. In Section 4.4, we study the spectral property of the leading term of the effective Hamiltonian. The existence of resonances of $P_{1,\theta}(B,\omega)$ is proved in Subsection 4.5.1 and the width of resonances is showed in Subsection 4.5.2.

4.2 Assumptions and Results.

In this section, we will present our hypotheses and our main result. We recall the operator

$$P(B,\omega) = (D_x - By)^2 + D_y^2 + \omega^2 x^2 + V(x,y),\ (x,y) \in \mathbb{R}^2,$$

where the potential V satisfies the following hypothesis :

($\mathbf{H_1}$) There exist constants $\alpha_1, \alpha_2, \alpha_3 > 0$ and $\delta > 0$ such that V admits an analytic extension on the domain

$$\mathcal{A} = \{(z_1,z_2) \in \mathbb{C}^2;\ |\mathrm{Im}(z_1,z_2)| \le \alpha_1 |\mathrm{Re}(z_1,z_2)| + \alpha_2\},$$

and for all $(z_1,z_2) \in \mathcal{A}$

$$|V(z_1,z_2)| \le \alpha_3 \langle \mathrm{Re}(z_1,z_2)\rangle^{-\delta}.$$

Here we denote $\langle (t,s)\rangle = (1+t^2+s^2)^{\frac{1}{2}},\ (t,s) \in \mathbb{R}^2$.

We recall that the total electric potential $a_0(x,y) = \omega^2 x^2 + V(x,y)$ (see (4.4)). We introduce the following assumption :

($\mathbf{H_2}$) Let $E \in \mathbb{R} \setminus \{0\}$. Suppose that a_0 has a local non-degenerate maximum (or minimum) point E at (x_0,y_0), i.e., the definite Hessian $a_0''(x_0,y_0) < 0$ (or $a_0''(x_0,y_0) > 0$).

By the translation, we can always assume that $(x_0,y_0) = (0,0)$. Set

$$\Omega_E = \{(x,y) \in \mathbb{R}^2;\ a_0(x,y) = E\}.$$

($\mathbf{H_3}$) (On the non-trapping condition) Assume that $\Omega_E = \{(0,0)\} \cup \Gamma$, where Γ is a connected curve and $(0,0)$ is an isolated point, and that the classical Hamiltonian $a_0(x,\xi)$ is non-trapping on Γ :

$$\begin{aligned}\{a_0(x,\xi), G_0(x,\xi)\} &= \partial_\xi a_0 \partial_x G_0 - \partial_x a_0 \partial_\xi G_0 \\ &= \xi \partial_\xi a_0(x,\xi) - x \partial_x a_0(x,\xi) \neq 0,\ \forall (x,\xi) \in \Gamma,\end{aligned} \tag{4.5}$$

where $G_0(x,\xi) = x.\xi,\ \forall (x,\xi) \in \mathbb{R}^2$.

Our main result is the following :

Theorem 4.2.1. *Assume that the assumptions* ($\mathbf{H_1}$), ($\mathbf{H_2}$) *and* ($\mathbf{H_3}$) *hold. For each* n, *we define*

$$U_n = \Big\{ z \in \mathbb{C};\ \mathrm{Re} z \in \Big](2n+1)B + E - \frac{C_0}{B}, (2n+1)B + E + \frac{C_0}{B}\Big[, \mathrm{Im} z \in \Big] -\frac{1}{C_0 B}, 0\Big]\Big\}, \tag{4.6}$$

where $C_0 > 1$ can be arbitrarily large outside a discrete set in $\mathbb{R}$. Then for B large enough, the resonances of $P(B,\omega)$ in U_n exist and are all given by complete expansions in powers of B^{-1} :

$$E_{n,j}(B,\omega) \sim (2n+1)B + E + \frac{1}{2}\Big(\pm(2j+1)(\lambda\mu)^{\frac{1}{2}} + (2n+1)\frac{\lambda+\mu}{2}\Big)B^{-1} + \sum_{k\ge 2} c_{\pm;n,j}^{(k)} B^{-k}, \tag{4.7}$$

where $c_{\pm;n,j}^{(k)} \in \mathbb{R}$, λ and μ are eigenvalues of the Hessian $a_0''(0,0)$, and the sign $+(-)$ corresponds to a local minimum, maximum respectively.

Moreover, the resonances of $P(B,\omega)$ in U_n are all algebraically simple and the width of resonances is of order $\mathcal{O}(B^{-\infty})$.

Remark 4.2.2. Notice that in the half-plane $\{z \in \mathbb{C};\ \mathrm{Re}z < \sqrt{B^2+\omega^2}\}$, the poles of the meromorphic extension of the resolvent from $\{z \in \mathbb{C};\ \mathrm{Im}z > 0\}$ to $\{z \in \mathbb{C};\ \mathrm{Im}z < 0\}$ are all given by the discrete eigenvalues of $P(B,\omega)$. Then the resonances in this half-plane are identically equal to the set of discrete eigenvalues of $P(B,\omega)$. Therefore, let $E < 0$, the width of $E_{0,k}(B,\omega)$ is equal to 0.

We want to give an example to illustrate our main result.

Consider $V(x,y) = -\frac{c_1}{x^4+1} - \frac{c_2}{y^2+1}$, $c_1, c_2 > 0$. Then $a_0(x,y) = \omega^2 x^2 - \frac{c_1}{x^4+1} - \frac{c_2}{y^2+1}$ and one has

$$\begin{cases} \partial_x a_0(x,y) = 0 \\ \partial_y a_0(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} 2\omega^2 x + \frac{4c_1 x^3}{(x^4+1)^2} = 0 \\ \frac{2c_2 y}{(y^2+1)^2} = 0. \end{cases} \tag{4.8}$$

The system (4.8) has only one solution $(x,y) = (0,0)$ and $a_0(0,0) = -c_1 - c_2$. It is easy to compute the Hessian at $(0,0)$ of a_0 :

$$a_0''(0,0) = \begin{pmatrix} 2\omega^2 & 0 \\ 0 & 2c_2 \end{pmatrix} > 0.$$

It shows that a_0 has a local minimum point at $(0,0)$. On the other hand, $a_0^{-1}(-c_1 - c_2) = \{(0,0)\}$. Therefore we do not need to verify the non-trapping condition as in this case $\Gamma = \emptyset$. Our main result shows that there exist resonances of $P(B,\omega)$ near $(2n+1)B - c_1 - c_2$ for B large enough and for all $n \in \mathbb{N}$.

4.3 Reduction to the semiclassical effective Hamiltonian.

In this section, we reduce the study of resonances of $P(B,\omega)$ to the spectral study of an $h-$ pseudodifferential operator.

4.3.1 Spectral properties of $P_{1,\theta}(B,\omega)$

In this subsection, we compute the essential spectrum of $P_{1,\theta}(B,\omega)$ and we give some resolvent estimates.

Lemma 4.3.1. *Let θ be in a small complex neighbourhood of 0. Then,*

$$\sigma_{\mathrm{ess}}(P_{1,\theta}(B,\omega)) = \sigma_{\mathrm{ess}}(P_{0,\theta}(B,\omega)) = \bigcup_{n\in\mathbb{N}} \{\mu_n + e^{2\theta}\lambda,\ \lambda \geq 0\}, \tag{4.9}$$

where $\mu_n = (2n+1)\sqrt{B^2+\omega^2}$ and $P_{0,\theta}(B,\omega)$ given by (4.2).

Proof. Recall that $P_{0,\theta}(B,\omega) = \sqrt{B^2+\omega^2}(e^{2\theta}D_y^2 + e^{-2\theta}y^2) + e^{2\theta}\omega^2 x^2$. Then when $\omega = 0$, one has $P_{0,\theta}(B,0) = \sqrt{B^2+\omega^2}(e^{2\theta}D_y^2 + e^{-2\theta}y^2)$. We are going to determine the spectrum of $P_{0,\theta}(B,0) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$.

Denote by $P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}$ the restriction of $P_{0,\theta}(B,0)$ on $L^2(\mathbb{R}_y)$. Let us consider

$$\mathfrak{A} = \{\psi_n(y);\ \psi_n(y) = H_n(y)e^{-\frac{y^2}{2}} \text{ where } H_n(\cdot) \text{ is the nth Hermite polynomial, } n \in \mathbb{N}\}$$

the set of normalized eigenfunctions of one-dimensional harmonic operator, i.e.,

$$(D_y^2 + y^2)(\psi_n(y)) = (2n+1)\psi_n(y).$$

For $\theta \in \mathbb{C}$ near 0, we set $\psi_{n,\theta}(y) = e^{-\frac{\theta}{2}}\psi_n(e^{-\theta}y)$, $n \in \mathbb{N}$. We put $\mathfrak{A}_\theta = \{\psi_{n,\theta};\ n \in \mathbb{N}\}$. Then one has $\mathfrak{A}_\theta \subset L^2(\mathbb{R})$ and $P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}(\psi_{n,\theta}(y)) = \mu_n\psi_{n,\theta}(y)$, $n \in \mathbb{N}$. It shows that $\bigcup_{n\in\mathbb{N}}\{\mu_n\} \subset \sigma\Big(P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}\Big)$. On the other hand, since $P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}$ is elliptic for $|\theta|$ small, then its spectrum is discrete. In addition, $\mathfrak{A}_\theta$ is a dense set in $L^2(\mathbb{R})$ (see [32, Chapter 16]). Then if $\lambda \notin \bigcup_{n\in\mathbb{N}}\{\mu_n\}$ is an eigenvalue of $P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}$ and the corresponding eigenfunction f, one has $f \in \mathfrak{A}_\theta^\perp = \{0\}$. Therefore,

$$\sigma\Big(P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}\Big) = \bigcup_{n\in\mathbb{N}}\{\mu_n\}. \tag{4.10}$$

In fact, we can write $P_{0,\theta}(B,0) = \mathrm{Id}_{L^2(\mathbb{R}_x)} \otimes P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}$ and the multiplication operator $e^{2\theta}\omega^2x^2 = e^{2\theta}\omega^2x^2\Big|_{L^2(\mathbb{R}_x)} \otimes \mathrm{Id}_{L^2(\mathbb{R}_y)}$. Here $e^{2\theta}\omega^2x^2\Big|_{L^2(\mathbb{R}_x)}$ is the natural restriction of the multiplication operator $e^{2\theta}\omega^2x^2$ on $L^2(\mathbb{R}_x)$. Then it is easy to verify that the operator $P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}$ and the multiplication operator $e^{2\theta}\omega^2x^2\Big|_{L^2(\mathbb{R}_x)}$ satisfy [55, Theorem XIII.35]. It enables us to obtain :

$$\sigma(P_{0,\theta}(B,\omega)) = \sigma\Big(P_{0,\theta}(B,0)\Big|_{L^2(\mathbb{R}_y)}\Big) + \sigma\Big(e^{2\theta}\omega^2x^2\Big|_{L^2(\mathbb{R}_x)}\Big). \tag{4.11}$$

Moreover, $\sigma\Big(e^{2\theta}\omega^2x^2\Big|_{L^2(\mathbb{R}_x)}\Big) = \{e^{2\theta}\lambda;\ \lambda \geq 0\}$. Combining this with (4.10) and (4.11), one obtains

$$\sigma(P_{0,\theta}(B,\omega)) = \bigcup_{n\in\mathbb{N}}\{\mu_n + e^{2\theta}\lambda; \lambda \geq 0\}, \tag{4.12}$$

and then the discrete spectrum of $P_{0,\theta}(B,\omega)$ is empty.

Now we prove that the essential spectrum of $P_{1,\theta}(B,\omega)$ is equal to that of $P_{0,\theta}(B,\omega)$. Firstly we can show as in [2] that $V(P_0(B,\omega) - z)^{-1}$ is a compact operator for $z \notin \sigma(P_0(B,\omega))$. By the unitary equivalence, $T(\theta) := V_\theta^w(B,\omega)(P_{0,\theta}(B,\omega) - z)^{-1}$ is also a compact operator for θ real and $z \notin \sigma(P_{0,\theta}(B,\omega))$. Further, since $T(\theta)$ is an analytic bounded operator-valued function in θ near 0, it is compact for all θ near 0 (see [55, page 126, Lemma 5]). From this we can apply [37, page 244, Theorem 5.35] and achieve

$$\sigma_{\mathrm{ess}}(P_{1,\theta}(B,\omega)) = \sigma_{\mathrm{ess}}(P_{0,\theta}(B,\omega)) = \bigcup_{n\in\mathbb{N}}\{\mu_n + e^{2\theta}\lambda; \lambda \geq 0\}, \tag{4.13}$$

where $\mu_n = (2n+1)\sqrt{B^2+\omega^2}$. □

For each fixed $n \in \mathbb{N}$, we denote by ψ_n the normalized eigenfunction of the harmonic oscillator corresponding to the eigenvalue $(2n+1)$ (i.e., $(D_y^2+y^2)\psi_n(y) = (2n+1)\psi_n(y)$ and $\|\psi_n\|_{L^2(\mathbb{R})} = 1$). Put $\psi_{n,\theta}(y) = e^{-\frac{\theta}{2}}\psi_n(e^{-\theta}y)$. Since $\psi_n(y)$ is of the form $H_n(y)e^{-y^2/2}$, where H_n is the nth - Hermite polynomial, we have $\overline{\psi_{n,\theta}} = \psi_{n,\bar{\theta}}$ and $\langle \psi_{n,\theta}, \psi_{n,\bar{\theta}}\rangle = \|\psi_n\|^2 = 1$. We next consider the following operators

$$\begin{aligned} R_- : L^2(\mathbb{R}) &\to L^2(\mathbb{R}^2), & R_+ : L^2(\mathbb{R}^2) &\to L^2(\mathbb{R}) \\ u(x) &\mapsto u(x)\psi_{n,\theta}(y) & u(x,y) &\mapsto \langle u, \psi_{n,\bar{\theta}}\rangle_{L^2(\mathbb{R}_y)} \end{aligned}$$

and

$$\begin{aligned} \Pi_n \ : L^2(\mathbb{R}^2) &\to L^2(\mathbb{R}^2) \\ u(x,y) &\mapsto \langle u, \psi_{n,\bar{\theta}}\rangle_{L^2(\mathbb{R}_y)}\psi_{n,\theta}(y), \end{aligned}$$

here the scalar product $\langle .,.\rangle_{L^2(\mathbb{R}_y)}$ denotes the integration in the y variable. The natural restriction of Π_n on $L^2(\mathbb{R}_y)$ we also denote by Π_n. From the definition, one has

$$\begin{aligned} R_+R_-u(x) &= R_+(u(x)\psi_{n,\theta}(y)) = u(x) \\ R_-R_+v(x,y) &= \langle u, \psi_{n,\bar{\theta}}\rangle_{L^2(\mathbb{R}_y)}\psi_{n,\theta}(y) = \Pi_n v(x,y). \end{aligned}$$

Lemma 4.3.2. *Let θ be in a small complex neighbourhood of 0. Then we have*

$$\sigma(\Pi_n P_{0,\theta}(B,\omega)\Pi_n) = \{\mu_n + e^{2\theta}\lambda;\ \lambda \geq 0\}, \tag{4.14}$$

and

$$\sigma((1-\Pi_n)P_{0,\theta}(B,\omega)(1-\Pi_n)) = \bigcup_{k\in\mathbb{N}\setminus\{n\}} \{\mu_k + e^{2\theta}\lambda;\ \lambda \geq 0\}. \tag{4.15}$$

Here $\Pi_n P_{0,\theta}(B,\omega)\Pi_n : \Pi_n L^2(\mathbb{R}^2) \to \Pi_n L^2(\mathbb{R}^2)$ and $(1-\Pi_n)P_{0,\theta}(B,\omega)(1-\Pi_n) : (1-\Pi_n)L^2(\mathbb{R}^2) \to (1-\Pi_n)L^2(\mathbb{R}^2)$.

Proof. First we demonstrate (4.14).

We observe that the Hilbert space $\Pi_n L^2(\mathbb{R}_y)$ is generated by $\psi_{n,\theta}$. Then it can be readily verified that

$$\sigma\Big(\Pi_n P_{0,\theta}(B,0)\Pi_n\Big|_{\Pi_n L^2(\mathbb{R}_y)}\Big) = \{\mu_n\}. \tag{4.16}$$

We recall that $\sigma\Big(e^{2\theta}\omega^2x^2\Big|_{L^2(\mathbb{R}_x)}\Big) = \{e^{2\theta}\lambda;\ \lambda \geq 0\}$. Then as in (4.11), it follows from [55, Theorem XIII.35] that :

$$\begin{aligned} \sigma(\Pi_n P_{0,\theta}(B,\omega)\Pi_n) &= \sigma\Big(\Pi_n P_{0,\theta}(B,0)\Pi_n\Big|_{\Pi_n L^2(\mathbb{R}_y)}\Big) + \sigma\Big(e^{2\theta}\omega^2x^2\Big|_{L^2(\mathbb{R}_x)}\Big) && (4.17) \\ &= \{\mu_n + e^{2\theta}\lambda;\ \lambda \geq 0\}. && (4.18) \end{aligned}$$

Secondly, we prove (4.15) in the same way as above.

By applying [32, Proposition 6.9], one has

$$\sigma\Big((1-\Pi_n)P_{0,\theta}(B,0)(1-\Pi_n)\Big|_{L^2(\mathbb{R}_y)}\Big) = \bigcup_{k\in\mathbb{N}\setminus\{n\}} \{\mu_k\}. \tag{4.19}$$

Then we use again [55, Theorem XIII.35] and derive

$$\sigma((1-\Pi_n)P_{0,\theta}(B,\omega)(1-\Pi_n)) = \sigma\Big((1-\Pi_n)P_{0,\theta}(B,0)(1-\Pi_n)\Big|_{L^2(\mathbb{R}_y)}\Big) + \sigma\Big(e^{2\theta}\omega^2x^2\Big|_{L^2(\mathbb{R}_x)}\Big) \tag{4.20}$$

$$= \bigcup_{k\in\mathbb{N}\setminus\{n\}} \{\mu_k + e^{2\theta}\lambda;\ \lambda \geq 0\}. \tag{4.21}$$

□

From now on, we fix $n \in \mathbb{N}$. We put

$$T_n = \Big\{z \in \mathbb{C};\ |\mathrm{Re}z| \leq 2\beta\sqrt{B^2+\omega^2}, |\mathrm{Im}z| \leq 2|\mathrm{Im}\theta|(1-\beta)\sqrt{B^2+\omega^2}\Big\}, \tag{4.22}$$

where $0 < \beta < 1$.

Proposition 4.3.3. *Let $\theta \in \mathbb{C}$ with $|\theta|$ small and $\mathrm{Im}\theta < 0$. Then for $z \in T_n$ the operator $R_{0,\theta}(B,\omega,z) := \Big((1-\Pi_n)P_{0,\theta}(B,\omega)(1-\Pi_n) - \mu_n - z\Big)^{-1}(1-\Pi_n)$ exists and the following estimate holds :*

There exists $C_1 > 0$ independent of B such that

$$\|R_{0,\theta}(B,\omega,z)\|_{L^2(\mathbb{R}^2)} \leq \frac{C_1}{\sqrt{B^2+\omega^2}},\ \text{uniformly in } z \in T_n. \tag{4.23}$$

Moreover, for B large enough, the operator $R_{1,\theta}(B,\omega,z) = \Big((1-\Pi_n)P_{1,\theta}(B,\omega)(1-\Pi_n) - \mu_n - z\Big)^{-1}(1-\Pi_n)$ exists for all $z \in T_n$ and the following estimate holds :

There exists $C_2 > 0$ independent of B such that

$$\|R_{1,\theta}(B,\omega,z)\|_{L^2(\mathbb{R}^2)} \leq \frac{C_3}{\sqrt{B^2+\omega^2}},\ \text{uniformly in } z \in T_n. \tag{4.24}$$

Proof. From (4.15) and the definition of T_n, one has $\sigma((1-\Pi_n)P_{0,\theta}(B,\omega)(1-\Pi_n)-\mu_n) \cap T_n = \emptyset$. It implies the existence of $R_{0,\theta}(B,\omega,z)$, $z \in T_n$. For $z \in T_n$, we put

$$C(z) = \Big((1-\Pi_n)(e^{2\theta}D_y^2 + e^{-2\theta}y^2 + e^{2\theta}\omega^2x^2)(1-\Pi_n) - (2n+1) - \frac{z}{\sqrt{B^2+\omega^2}}\Big)^{-1}(1-\Pi_n).$$

Since $\frac{1}{\sqrt{B^2+\omega^2}}T_n = \{\frac{1}{\sqrt{B^2+\omega^2}}z;\ z \in T_n\}$ is a compact set independent of both B and ω, then there exists $z_0 \in T_n$ such that $\|C(z_0)\| = \sup_{z\in T_n}\|C(z)\|$. Remark that $\|C(z_0)\|$ does not depend on both B and ω. On the other hand, by a change of variables $x \mapsto (B^2+\omega^2)^{\frac{1}{4}}x$, we have $R_{0,\theta}(B,\omega,z)$ is unitarily equivalent to $\frac{1}{\sqrt{B^2+\omega^2}}C(z)$. Then

$$\|R_{0,\theta}(B,\omega,z)\| \leq \frac{1}{\sqrt{B^2+\omega^2}}\|C(z_0)\|,$$

uniformly in $z \in T_n$. Note that $\|C(z_0)\|$ is finite for $\mathrm{Im}\theta < 0$. And this proves (4.23).

As a consequence of (4.23), we prove (4.24). Indeed, for $z \in T_n$,

$$\begin{aligned}
&(1-\Pi_n)P_{1,\theta}(B,\omega)(1-\Pi_n)-\mu_n-z\\
&=(1-\Pi_n)P_{0,\theta}(B,\omega)(1-\Pi_n)-\mu_n-z+(1-\Pi_n)V_\theta^w(B,\omega)(1-\Pi_n)\\
&=\Big((1-\Pi_n)P_{0,\theta}(B,\omega)(1-\Pi_n)-\mu_n-z\Big)\Big(1+R_{0,\theta}(B,\omega,z)(1-\Pi_n)V_\theta^w(B,\omega)(1-\Pi_n)\Big).
\end{aligned}$$

Since V is bounded together with all its derivatives, there exists $C>0$ such that $\|R_{0,\theta}(B,\omega,z)(1-\Pi_n)V_\theta^w(B,\omega)(1-\Pi_n)\| \le \frac{C}{\sqrt{B^2+\omega^2}}$ uniformly in $z \in T_n$. Then for B large enough, $\|R_{0,\theta}(B,\omega,z)(1-\Pi_n)V_\theta^w(B,\omega)(1-\Pi_n)\| \le \frac{1}{2}$ and then $R_{1,\theta}(B,\omega,z)$ exists. Moreover, for B sufficiently large, there exists $C_2>0$ such that

$$\begin{aligned}
\|R_{1,\theta}(B,\omega,z)\| &= \left\|\Big(1+R_{0,\theta}(B,\omega,z)(1-\Pi_n)V_\theta^w(B,\omega)(1-\Pi_n)\Big)^{-1}R_{0,\theta}(B,\omega,z)\right\| \quad (4.25)\\
&\le \frac{C_2}{\sqrt{B^2+\omega^2}},
\end{aligned}$$

uniformly in $z \in T_n$. □

Remark 4.3.4. Let Q be equal to $R_{0,\theta}(B,\omega,z)$ or $R_{1,\theta}(B,\omega,z)$. Let K be a compact set in $\mathbb{R}$. Using the theory of $h-$ pseudodifferential operators of operator-valued symbols, we can view Q as an $h-$ pseudodifferential operator in the $x-$ variable whose symbol $q(x,\xi,\theta;h)$ is bounded operator in the $y-$ variable. In particular, the proof of Proposition 4.3.3 shows that $q(x,\xi,\theta;h)$ is well-defined on $\mathbb{R}^2_{x,\xi}$ (resp. $K\times\mathbb{R}$) for $\mathrm{Im}\theta<0$ (resp. $\mathrm{Im}\theta=0$).

4.3.2 Grushin problem

From now on, we use $h=\frac{1}{\sqrt{B^2+\omega^2}}$. To indicate that the operators depend on h, we replace the indices (B,ω) by h. For example, we write $P_{1,\theta}(h)$ (resp. $V_\theta^w(h)$) instead of $P_{1,\theta}(B,\omega)$ (resp. $V_\theta^w(B,\omega)$).

We now study the Grushin problem for $P_{1,\theta}(h)-\mu_n$: Set

$$\mathcal{P}(z)=\begin{pmatrix} P_{1,\theta}(h)-\mu_n-z & R_- \\ R_+ & 0\end{pmatrix} : \mathcal{D}\times L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)\times L^2(\mathbb{R}), \quad (4.26)$$

where $\mathcal{D}\subset L^2(\mathbb{R}^2)$ is the domain of $P_{1,\theta}(h)$.

Fix $\theta\in\mathbb{C}$ with $|\theta|$ small enough and $\mathrm{Im}\theta<0$.

Theorem 4.3.5. *For B large enough, the operator $\mathcal{P}(z)$ is invertible uniformly for $z\in T_n$. Moreover, the inverse of $\mathcal{P}(z)$ is holomorphic in $z\in T_n$ and given by*

$$\mathcal{E}(z)=\begin{pmatrix} E(z) & E_+(z)\\ E_-(z) & E_{-+}(z)\end{pmatrix},$$

where

$$E_{-+}(z)=z-e^{2\theta}\omega^2x^2-R_+V_\theta^w(h)R_- - R_+b(z)[V_\theta^w(h),\Pi_n]R_- \quad (4.27)$$

with $b(z) = (I + [\Pi_n, V_\theta^w(h)]R_{1,\theta}(h,z))^{-1}$.

Here the operators $E(z)$, $E_-(z)$ *and* $E_+(z)$ *are given by*

$$\begin{aligned} E(z) &= R_{1,\theta}(h,z)b(z) \; ; \; E_-(z) = R_+ b(z) \\ E_+(z) &= R_- - R_{1,\theta}(h,z)b(z)[V_\theta^w(h), \Pi_n]R_-. \end{aligned} \tag{4.28}$$

In addition, z *is an eigenvalue of* $P_{1,\theta}(h) - \mu_n$ *if and only if* 0 *is an eigenvalue of* $E_{-+}(z)$. *Here the notation* $[.,.]$ *is the commutator which is defined by* $[A, C] = AC - CA$.

Proof. The proof follows from a simple modification of [58, Theorem 2.2] (see also [19, Section 6]). So we omit the details. □

Now we are interested in studying the operator $E_{-+}(z)$. In fact, for $z \in T_n$ (T_n is defined in (4.22)) and h sufficiently small, we prove that $E_{-+}(z) - (z - e^{2\theta}\omega^2 x^2)$ is an $h-$ pseudodifferential operator with bounded symbol.

By applying the Beal's characterization of pseudodifferential operators (see [22]), one easily sees that Π_n is a pseudodifferential operator with bounded symbol $\pi_n(y, \eta)$. Then making use of the pseudodifferential calculus (see [22, Chapter 7]), one obtains that the symbol of the commutator $[V_\theta^w(h), \Pi_n]$ has the asymptotics $\sum_{j\geq 1} b_j h^{\frac{j}{2}}$ in $S^0(\mathbb{R}^4)$. It follows that

$$[V_\theta^w(h), \Pi_n] = \mathcal{O}(h^{\frac{1}{2}}) \tag{4.29}$$

in $\mathcal{L}(L^2(\mathbb{R}^2))-$ the space of bounded operators from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$.

Now for $z \in T_n$ and h small enough,

$$\begin{aligned} E_{-+}(z) &= z - e^{2\theta}\omega^2 x^2 - R_+ V_\theta^w(h) R_- - R_+ b(z)[V_\theta^w(h), \Pi_n]R_- \\ &= z - e^{2\theta}\omega^2 x^2 - R_+ V_\theta^w(h) R_- - R_+ \sum_{j\geq 0} \left([V_\theta^w(h), \Pi_n]R_{1,\theta}(h,z)\right)^j [V_\theta^w(h), \Pi_n]R_-. \end{aligned} \tag{4.30}$$

Here we used (4.24), (4.29) and the Neumann series. It follows from this and the fact that $R_+[V_\theta^w(h), \Pi_n]R_- = 0$ that

$$\begin{aligned} E_{-+}(z) &= z - e^{2\theta}\omega^2 x^2 - R_+ V_\theta^w(h) R_- - R_+ \sum_{j\geq 1} \left([V_\theta^w(h), \Pi_n]R_{1,\theta}(h,z)\right)^j [V_\theta^w(h), \Pi_n]R_- \\ &= z - e^{2\theta}\omega^2 x^2 - R_+ V_\theta^w(h) R_- + h^2 G_\theta(z;h), \end{aligned} \tag{4.31}$$

where $G_\theta(z;h)$ is holomorphic for $z \in T_n$. We set

$$A_\theta(h) = e^{2\theta}\omega^2 x^2 + R_+ V_\theta^w(h) R_-. \tag{4.32}$$

As in [15], we can prove that $R_+ V_\theta^w(h) R_-$ is an $h-$ pseudodifferential operator with bounded symbol which belongs to $S^0((x^2+\xi^2)^{-\frac{\delta}{2}})$, where δ is given in Assumption ($\mathbf{H_1}$). Here $S^0((x^2+\xi^2)^{-\frac{\delta}{2}})$ is the class of symbols with order function $(x^2+\xi^2)^{-\frac{\delta}{2}}$ (see [22, Chapter 7]). Therefore,

$$A_\theta(h) = a^w(e^\theta x, e^{-\theta} h D_x; h), \tag{4.33}$$

where $a(.,.;h)$ is holomorphic in some conic neighbourhood of $\mathbb{R}^2$ and we have the following complete expansion in powers of h :

$$a(x,\xi;h) - a_0(x,\xi) \sim \sum_{j\geq 1} h^j a_j(x,\xi) \tag{4.34}$$

with

$$a_0(x,\xi) = \omega^2 x^2 + V(x,\xi) \text{ and } a_1(x,\xi) = \frac{(2n+1)}{4}\Delta V(x,\xi). \tag{4.35}$$

By using the arguments in [58], we can prove that $E_{-+}(z) - (z - e^{2\theta}\omega^2 x^2)$ is an $h-$ pseudodifferential operator with bounded symbol. Moreover the symbol also admits a complete expansion in powers of h.

Fix $\theta \in \mathbb{C}$ with $|\theta|$ small enough and $\text{Im}\theta < 0$. We have obtained the following :

Proposition 4.3.6. *For $z \in T_n$ and h sufficiently small, the operator $E_{-+}(z) - (z - e^{2\theta}\omega^2 x^2)$ is an $h-$ pseudodifferential operator with bounded symbol. Moreover, the symbol admits a complete expansion in powers of h in $S^0(\mathbb{R}^2)$:*

$$E_{-+}(z) - (z - e^{2\theta}\omega^2 x^2) = a_0^w(e^\theta x, e^{-\theta} hD_x) + a_1^w(e^\theta x, e^{-\theta} hD_x)h + \mathcal{O}(h^2), \tag{4.36}$$

where a_0, a_1 are given in (4.35).

Remark 4.3.7. It follows from the theory of $h-$ pseudodifferential operators of operator-valued symbols, formula (4.31) and Remark 4.3.4 that the symbol corresponding to $E_{-+}(z)$ is well-defined for x in a compact set and $\text{Im}\theta = 0$.

Thanks to Theorem 4.3.5, our purpose is now to study the spectrum of the effective Hamiltonian $E_{-+}(z)$.

4.4 Spectral properties of the leading term of $E_{-+}(z)$.

In this section, we investigate the spectrum of $A_\theta(h)$ near E . We recall that $A_\theta(h) = e^{2\theta}\omega^2 x^2 + R_+ V_\theta^w(h) R_-$ which satisfies (4.33). For $\theta \in \mathbb{C}$ with $|\theta|$ small enough and $\text{Im}\theta < 0$, we have the following :

Lemma 4.4.1. *The essential spectrum of $A_\theta(h)$ is equal to the set $\{e^{2\theta}\lambda;\ \lambda \geq 0\}$.*

Proof. We regard $R_+ V_\theta^w(h) R_-$ as the perturbation of the multiplication operator $e^{2\theta}\omega^2 x^2$. Recall that the decay of V at infinity implies that the symbol of $R_+ V_\theta^w(h) R_-$ belongs to $S^0((x^2+\xi^2)^{-\frac{\delta}{2}})$ (see Assumption $(\mathbf{H_1})$). Therefore it follows that $R_+ V_\theta^w(h) R_-(e^{2\theta}\omega^2 x^2 - z)^{-1}$ is a compact operator, for $z \notin \sigma(e^{2\theta}\omega^2 x^2)$ (see [43, page 62]). Thus, we apply [37, Theorem 52.35] to obtain the essential spectrum of $A_\theta(h)$ is equal to that of $e^{2\theta}\omega^2 x^2$. In addition, the essential spectrum of the multiplication operator $e^{2\theta}\omega^2 x^2$ is nothing but $\{e^{2\theta}\lambda;\ \lambda \geq 0\}$. The lemma is proved. ∎

Without loss of generality, we may next assume that the total electric potential $a_0(x,\xi)$ has a local minimum at $(0,0)$ (otherwise we study $-A_\theta(h)$). Moreover, the real part of θ can be ignored by a unitary transformation, it is then sufficient to consider $\mathrm{Re}\theta = 0$.

To study the spectrum of $A_\theta(h)$ near E, it is very important to know some properties of the principal symbol. We will see below that $a_0(e^\theta x, e^{-\theta}\xi) - E$ is elliptic outside $(0,0)$.

So far, we want to show the exponential decay of eigenfunctions of $A_\theta(h)$ corresponding to eigenvalues near E. Then it is essential to study an operator of the form $e^{\frac{f(x)}{h}} A_\theta(h) e^{-\frac{f(x)}{h}}$, of which the principal symbol is $a_0(e^\theta x, e^{-\theta}(\xi + if'(x)))$. So by choosing a suitable function f, we show below that $a_0(e^\theta x, e^{-\theta}(\xi+if'(x)))-E$ has the same properties as $a_0(e^\theta x, e^{-\theta}\xi) - E$.

From now on we fix $\theta = i\gamma$ with $\gamma < 0$. Let $\beta > 0$, we set $B(\beta) = \{(x,\xi) \in \mathbb{R}^2;\ |x| + |\xi| < \beta\}$.

Lemma 4.4.2. *For $\beta > 0$ sufficiently small and $|\gamma|$ small enough, there exists a smooth function $f(x)$ such that*

$$f(x) > 0 \ \text{for} \ x \in \mathbb{R} \setminus \{0\}, \tag{4.37}$$

$$f(x) = c_1 x^2 \ \text{for} \ x \text{ near } 0, \tag{4.38}$$

where c_1 is a small positive constant, and the following lower bounds hold :

There exists $C > 0$ large enough such that

$$\mathrm{Re}\Big(a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E\Big) \geq \frac{1}{C}(x^2 + \xi^2) \ \text{for} \ (x,\xi) \in B(\beta), \tag{4.39}$$

$$\Big|a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E\Big| \geq \frac{|\gamma|}{C} \ \text{for} \ (x,\xi) \in \mathbb{R}^2 \setminus B(\beta). \tag{4.40}$$

Proof. Put $\Gamma_1 := \{x \in \mathbb{R};\ \exists \xi \in \mathbb{R}, (x,\xi) \in \Gamma\}$. Remember that $\Gamma = \{(x,\xi) \in \mathbb{R}^2;\ \omega^2 x^2 + V(x,\xi) = E\}$ and $\lim\limits_{|(x,\xi)| \to \infty} V(x,\xi) = 0$, then Γ_1 is a bounded set.

For $\beta, |\gamma|$ small enough chosen later on, we construct a real smooth function f depending on these constants :

$$f(x) = c_1 x^2 \chi_1(x) + c_2 \chi_2(x) + c_3 \chi_3(x),$$

where $\chi_1 \in C_0^\infty(\mathbb{R};[0,1])$, $\chi_3, \chi_2 \in C^\infty(\mathbb{R};[0,1])$ satisfy :

• $\chi_1 = 1$ on $\{x \in \mathbb{R};\ \exists \xi \in \mathbb{R} \text{ s.t. } (x,\xi) \in B(\beta)\}$, the support of χ_1 lies in some small neighbourhood of 0 and $\chi_1 + \chi_2 = 1$.

• The support of χ_3 lies outside a neighbourhood of $\Gamma_1 \cup \{0\}$ and $\chi_3(x) = 1$ for $|x|$ large,

and positive constants $c_1, c_2, c_3 > 0$ small enough (to be chosen later on). Remark that c_1, c_2 depend on γ, c_3 is independent of γ.

Then it is easy to see that (4.37), (4.38) are verified. By using a symplectic change of coordinates if necessary, one can assume that Hessian of a_0 at $(0,0)$ is given by

$$a_0''(0,0) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

We start by proving (4.39). Notice that the constant C may change from line to line in what follows. For $\beta > 0$ small enough, by applying Taylor formula of order three to a_0 at $(0,0)$, one obtains :

$$a_0(e^{i\gamma}x, e^{-i\gamma}\xi) = E + \frac{1}{2}(\lambda e^{i2\gamma}x^2 + \mu e^{-i2\gamma}\xi^2) + \mathcal{O}((x,\xi)^3) \tag{4.41}$$

for $(x,\xi) \in B(\beta)$. We replace ξ by $\xi + if'(x)$ in (4.41). Since $f'(x) = 2c_1 x$ for $x \in \{y \in \mathbb{R};\ \exists \eta \in \mathbb{R} \text{ s.t. } (y,\eta) \in B(\beta)\}$, we can choose c_1 small enough such that there exists $C > 0$ large :

$$\mathrm{Re}\Big(a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E\Big) \geq \frac{1}{C}(x^2 + \xi^2),$$

for $(x,\xi) \in B(\beta)$. Thus the lower bound estimate (4.39) is proved.

Now we demonstrate the estimate (4.40). The proof is divided into two cases according to the sign of E.

Case $E < 0$.

First remark that Ω_E is a compact set in this case. Since $(0,0)$ is an isolated point, then for β small enough, $\Gamma = \Gamma \cap \mathbb{R}^2 \setminus B(\beta) = \Omega_E \cap \mathbb{R}^2 \setminus B(\beta)$. It implies that Γ is also a compact set. We choose a neighbourhood of Γ as follows :

The non-trapping condition on Γ (see Assumption ($\mathbf{H}_3$)) implies that, for each $(x_0,\xi_0) \in \Gamma$, there exists $\varepsilon(x_0,\xi_0) > 0$ such that

$$|x\partial_x a_0(x,\xi) - \xi\partial_\xi a_0(x,\xi)| \geq \frac{1}{C(x_0,\xi_0)} > 0, \tag{4.42}$$

for all $(x,\xi) \in D\big((x_0,\xi_0), \varepsilon(x_0,\xi_0)\big)$. Here $C(x_0,\xi_0)$ is a large constant depending on (x_0,ξ_0) and $D\big((x_0,\xi_0), \varepsilon(x_0,\xi_0)\big) = \{(x,\xi) \in \mathbb{R}^2;\ |x - x_0|^2 + |\xi - \xi_0|^2 < \varepsilon(x_0,\xi_0)^2\}$. The compactness of Γ gives : there exists a finite number of such discs such that

$$\Gamma \subset \bigcup_{j=1}^{k} D\big((x_j,\xi_j), \varepsilon(x_j,\xi_j)\big) =: \mathcal{V}(\Gamma).$$

a) For $(x,\xi) \in \mathcal{V}(\Gamma)$:

$$|x\partial_x a_0(x,\xi) - \xi\partial_\xi a_0(x,\xi)| \geq \frac{1}{\max\limits_{1\leq j\leq k} C(x_j,\xi_j)} > 0, \tag{4.43}$$

where $C(x_j,\xi_j), j = 1,...,k$, are given in (4.42).

Since $\mathrm{Im}(e^{i\gamma}x, e^{-i\gamma}\xi) = \sin(\gamma)(x,-\xi)$ and $\mathrm{Re}(e^{i\gamma}x, e^{-i\gamma}\xi) = \cos(\gamma)(x,\xi)$, then

$$\mathrm{Im}(e^{i\gamma}x, e^{-i\gamma}\xi).\nabla a_0(\mathrm{Re}(e^{i\gamma}x, e^{-i\gamma}\xi)) = \sin(\gamma)\Big(x\partial_x a_0(\cos(\gamma)(x,\xi)) - \xi\partial_\xi a_0(\cos(\gamma)(x,\xi))\Big)$$

Combining this with (4.43) we have, for $|\gamma|$ small enough,

$$\Big|\mathrm{Im}(e^{i\gamma}x, e^{-i\gamma}\xi).\nabla a_0(\mathrm{Re}(e^{i\gamma}x, e^{-i\gamma}\xi))\Big| \geq \frac{|\gamma|}{C},\ \forall (x,\xi) \in \mathcal{V}(\Gamma), \tag{4.44}$$

where C is a large constant.

Notice that (x,ξ) near Γ corresponds to x near Γ_1. Then we can choose c_1, c_2 small depending on γ such that $|f'(x)| \leq c'|\gamma|$ for x near Γ_1. Here the constant c' is small enough. Thus the inequality (4.44) remains true when we replace $(e^{i\gamma}x, e^{-i\gamma}\xi)$ by $w := (e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x)))$, i.e., there exists C sufficiently large such that

$$|\mathrm{Im}w.\nabla a_0(\mathrm{Re}w)| \geq \frac{|\gamma|}{C}. \tag{4.45}$$

Remark that the Hessian of a_0 is given by

$$a_0''(x,\xi) = \begin{pmatrix} 2\omega^2 + \partial^2_{xx}V(x,\xi) & \partial^2_{x\xi}V(x,\xi) \\ \partial^2_{\xi x}V(x,\xi) & \partial^2_{\xi\xi}V(x,\xi) \end{pmatrix}$$

and $\partial^\alpha_{x\xi}a_0(x,\xi) = \partial^\alpha_{x\xi}V(x,\xi)$ for all $\alpha \in \mathbb{N}^2$, $|\alpha| \geq 3$. Then we apply the Taylor formula of order two to $a_0(w)$ at $\mathrm{Re}w$:

$$a_0(w) = a_0(\mathrm{Re}w) + i\mathrm{Im}w\nabla a_0(\mathrm{Re}w) + \omega^2 x^2 \sin^2(\gamma) + r(w), \tag{4.46}$$

where $|r(w)| \leq C\sin^2(|\gamma|)$. Combine this with (4.45), we have for C large enough,

$$|a_0(w) - E| \geq |\mathrm{Im}(a_0(w) - E)| \geq \frac{|\gamma|}{C}, \ \forall (x,\xi) \in \mathcal{V}(\Gamma). \tag{4.47}$$

b) For $(x,\xi) \in \mathbb{R}^2 \setminus B(\beta)$ and $(x,\xi) \notin \mathcal{V}(\Gamma)$: From now on, we set $\tilde{\mathcal{V}}(\Gamma) := \mathbb{R}^2 \setminus (B(\beta) \cup \mathcal{V}(\Gamma))$.

Choose $R > 0$ sufficiently large such that : $\omega^2 R^2 > \sup_{\mathbb{R}^2} |V|$ and $\sup_{\{(x,\xi)\in\mathbb{R}^2;\ |\xi|\geq R\}} |V(x,\xi)| < -\frac{E}{2}$. Then

$$\omega^2 x^2 + V(x,\xi) - E \geq -\frac{E}{2} > 0 \tag{4.48}$$

for all $(x,\xi) \in \{(x,\xi) \in \tilde{\mathcal{V}}(\Gamma);\ |x| \geq R \text{ or } |\xi| \geq R\}$. In fact for $|x| \geq R$ we use $\omega^2 x^2 + V(x,\xi) - E \geq \omega^2 R^2 - \sup_{\mathbb{R}^2} |V| - E > -E$ and for $|\xi| \geq R$ we use $\omega^2 x^2 + V(x,\xi) - E \geq -\sup_{\{(x,\xi)\in\mathbb{R}^2;\ |\xi|\geq R\}} |V(x,\xi)| - E > -\frac{E}{2}$. Then for $|\gamma|$ small and c_3 small (c_3 is independent of γ),

$$\mathrm{Re}\Big(a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E\Big) \geq -\frac{E}{4} > 0 \tag{4.49}$$

on the set $\{(x,\xi) \in \tilde{\mathcal{V}}(\Gamma);\ |x| \geq R \text{ or } |\xi| \geq R\}$.

Since $\Gamma \cap \tilde{\mathcal{V}}(\Gamma) = \emptyset$ and $a_0(x,\xi) - E = 0$ if and only if $(x,\xi) \in \Gamma \cup \{(0,0)\}$, one has $a_0(x,\xi) - E \neq 0$ on $\tilde{\mathcal{V}}(\Gamma)$. Thus, on the compact set $\{(x,\xi) \in \tilde{\mathcal{V}}(\Gamma);\ |x| \leq R, |\xi| \leq R\}$,

$$|a_0(x,\xi) - E| \geq \mathrm{const} > 0.$$

By a perturbation argument, for $|\gamma|$ and c_1, c_2, c_3 small enough, one obtains

$$\Big|a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E\Big| \geq \mathrm{const} > 0, \tag{4.50}$$

for $(x,\xi) \in \{(x,\xi) \in \tilde{\mathcal{V}}(\Gamma);\ |x| \leq R, |\xi| \leq R\}$.

Therefore, from (4.49) and (4.50), one has

$$\Big|a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E\Big| \geq \text{const} > 0, \tag{4.51}$$

for all $(x,\xi) \in \tilde{\mathcal{V}}(\Gamma)$.

It follows from (4.47), (4.51) that, for $|\gamma|$ and c_1, c_2, c_3 small enough, there exists C large enough such that

$$\Big|a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E\Big| \geq \frac{|\gamma|}{C}, \tag{4.52}$$

for all $(x,\xi) \in \mathbb{R}^2 \setminus B(\beta)$.

Case $E > 0$.

Note that Ω_E is no longer a compact set. In fact, there are two asymptotes of Γ : $x = \pm\frac{\sqrt{E}}{\omega}$ since V vanishes at infinity. Let $\varepsilon > 0$ small, we can choose $R > 0$ large enough such that $\Gamma \cap \mathbb{R}^2 \setminus D((0,0),R) \subset \{(x,\xi) \in \mathbb{R}^2 \setminus D((0,0),R);\ |\omega^2x^2 - E| < \varepsilon\}$. Here $R >> \beta$.

First, for $(x,\xi) \in D((0,0),R) \setminus B(\beta)$, by using the same arguments as in case $E < 0$, i.e., the non-trapping condition on Γ and the compactness of $\overline{D((0,0),R)}$, we obtain (4.40).

Next we divide the set $\mathbb{R}^2 \setminus D((0,0),R)$ into two sets

$$\mathbb{R}^2 \setminus D((0,0),R) = R(\varepsilon) \cup R(\varepsilon)^c,$$

where

$$R(\varepsilon) := \{(x,\xi) \in \mathbb{R}^2 \setminus D((0,0),R);\ |\omega^2x^2 - E| < \varepsilon\}.$$

- In $R(\varepsilon)$, when $\varepsilon \to 0$ one has

$$V(x,\xi),\ x\partial_x V(x,\xi),\ \xi\partial_\xi V(x,\xi) = o(1),$$

and

$$\omega^2x^2 - E = o(1).$$

Therefore, for ε sufficiently small and $(x,\xi) \in R(\varepsilon)$

$$\Big|\{x.\xi, a_0(x,\xi)\}\Big| = \Big|x(2\omega^2x + \partial_x V(x,\xi)) - \xi\partial_\xi V(x,\xi)\Big| \geq |2E + o(1)| \geq E. \tag{4.53}$$

Apply again the perturbation argument, one gets

$$\Big|\mathrm{Im}(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))).\nabla a_0(\mathrm{Re}(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))))\Big| \geq \frac{E|\gamma|}{2} > 0. \tag{4.54}$$

The same arguments as in (4.47), one obtains

$$\Big|a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E\Big| \geq \frac{E|\gamma|}{2} > 0. \tag{4.55}$$

• In $R(\varepsilon)^c$, one has $|\omega^2 x^2 - E| \geq \varepsilon$. Since $\lim_{|(x,\xi)|\to\infty} V(x,\xi) = 0$, we choose R large enough such that

$$|\omega^2 x^2 + V(x,\xi) - E| \geq \frac{\varepsilon}{2},$$

for all $(x,\xi) \in R(\varepsilon)^c$. Now we apply again the perturbation argument to obtain, for $|\gamma|$ and c_1, c_2, c_3 small enough,

$$\Big| a_0(e^{i\gamma}x, e^{-i\gamma}(\xi + if'(x))) - E \Big| \geq \frac{\varepsilon}{4} > 0. \tag{4.56}$$

The proof of the lemma is thus complete. □

Since x and ξ play the same role, the following lemma can be proved by using the same arguments as above :

Lemma 4.4.3. *For β sufficiently small and $|\gamma|$ small enough, there exists a smooth function $g(\xi)$ such that*

$$g(\xi) > 0 \ \textit{for} \ \xi \in \mathbb{R}\setminus\{0\} \ \textit{and} \ g(\xi) = c_2\xi^2 \ \textit{for} \ \xi \textit{ near } 0, \tag{4.57}$$

where c_2 is a small positive constant, and the following lower bounds hold :
There exists $C > 0$ sufficiently large such that

$$\mathrm{Re}\Big(a_0(e^{i\gamma}(x + ig'(\xi)), e^{-i\gamma}\xi) - E \Big) \geq \frac{1}{C}(x^2 + \xi^2) \textit{ for } (x,\xi) \in B(\beta), \tag{4.58}$$

$$\Big| a_0(e^{i\gamma}(x + ig'(\xi)), e^{-i\gamma}\xi) - E \Big| \geq \frac{|\gamma|}{C} \textit{ for } (x,\xi) \in \mathbb{R}^2 \setminus B(\beta). \tag{4.59}$$

Recall that $\gamma = \mathrm{Im}\theta < 0$.

By relying on Lemma 4.4.2 and Lemma 4.4.3, we prove the exponential decay of eigenfunctions corresponding to eigenvalues of $A_{i\gamma}(h)$ near E :

Theorem 4.4.4. *Let f be constructed in Lemma 4.4.2. Let $C_0 > 0$ be a large and fixed constant, we define a neighbourhood of E,*

$$D = \{z \in \mathbb{C};\ |z - E| < C_0 h\}.$$

Suppose that $\lambda(h) \in D$ is an eigenvalue of $A_{i\gamma}(h)$ and $u(h)$ is a normalized eigenfunction associated to $\lambda(h)$, then there exists $C > 0$ such that

$$\|e^{f(x)/h}u(h)\| \leq C, \tag{4.60}$$

Proof. Let $\tilde{a}_{i\gamma}$ be the symbol of $A_{i\gamma}(h) - e^{2i\gamma}\omega^2x^2$. We recall that $(x,\xi) \mapsto \tilde{a}_{i\gamma}(x,\xi;h)$ is holomorphic in some conic neighbourhood of $\mathbb{R}^2$. Let us put $A_f(h) = e^{\frac{f}{h}}A_{i\gamma}(h)e^{-\frac{f}{h}}$. Then by using the contour integration in the ξ variable, one has for $u \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned}
\Big(A_f(h) - e^{2i\gamma}\omega^2x^2\Big)u(x) &= \frac{1}{2\pi h}\iint e^{(i(x-y)\xi + f(x) - f(y))/h}\tilde{a}_{i\gamma}\Big(\frac{x+y}{2}, \xi; h\Big)u(y)dyd\xi \\
&= \frac{1}{2\pi h}\iint e^{i(x-y)(\xi - if'(x,y))/h}\tilde{a}_{i\gamma}\Big(\frac{x+y}{2}, \xi; h\Big)u(y)dyd\xi \\
&= \frac{1}{2\pi h}\iint e^{i(x-y)\xi/h}\tilde{a}_{i\gamma}\Big(\frac{x+y}{2}, \xi + if'(x,y); h\Big)dyd\xi,
\end{aligned}$$

where $f'(x,y)$ is determined by $f(x)-f(y)=(x-y)f'(x,y)$.

Using the analyticity of $\tilde{a}_{i\gamma}$, it follows from Cauchy formula that $A_f(h)-e^{2i\gamma}\omega^2x^2$ is also an $h-$ pseudodifferential operator with bounded symbol. Moreover, the symbol of $A_f(h)-e^{2i\gamma}\omega^2x^2$ can also be expanded in powers of h in $S^0(\mathbb{R}^2)$ (the set of bounded symbols) with the principal symbol is $V(e^{i\gamma}x, e^{-i\gamma}(\xi+if'(x)))$. Then the principal symbol of $A_f(h)$ is $a_0(e^{i\gamma}x, e^{-i\gamma}(\xi+if'(x)))$.

Let us put $u_f(h):=e^{\frac{f}{h}}u(h)$ which belongs to $L^2(\mathbb{R})$ since f is bounded. Let $B(\beta)$ be as in Lemma 4.4.2, we choose a partition of unity $\chi_1+\chi_2=1$, $\mathrm{supp}\chi_1\subset B(\beta)$, $\chi_1=1$ near $(0,0)$. The idea of the proof is to estimate separately $\chi_1^w(x,hD_x)u_f(h)$ and $\chi_2^w(x,hD_x)u_f(h)$.

We first evaluate $\chi_2^w(x,hD_x)u_f(h)$. From (4.40), the symbol $\chi_2(x,\xi)\big(a_0(e^{i\gamma}x, e^{-i\gamma}(\xi+if'(x)))-\lambda(h)\big)^{-1}$ exists. Moreover since $(A_f(h)-\lambda(h))u_f(h)=0$, one has

$$\Big(\chi_2(x,\xi)\big(a_0(e^{i\gamma}x, e^{-i\gamma}(\xi+if'(x)))-\lambda(h)\big)^{-1}\Big)^w(x,hD_x)(A_f(h)-\lambda(h))u_f(h)=0.$$

By applying the $h-$ pseudodifferential calculus in the right hand side of this equality, one obtains

$$\Big(\chi_2^w(x,hD_x)+\mathcal{O}(h)\Big)u_f(h)=0. \tag{4.61}$$

Secondly, we estimate $u_f^1(h):=\chi_1^w(x,hD_x)u_f(h)$. In fact, it results from the compactness of the support of χ_1 and $\big(A_f(h)-\lambda(h)\big)u_f(h)=0$ that

$$\mathrm{Re}\big\langle(A_f(h)-\lambda(h))u_f^1(h),u_f^1(h)\big\rangle=\mathrm{Re}\big\langle[A_f(h),\chi_1^w(x,hD_x)]u_f(h),u_f^1(h)\big\rangle \tag{4.62}$$

$$=\mathcal{O}(h)\Big\langle u_f(h),u_f^1(h)\Big\rangle. \tag{4.63}$$

In addition, by using (4.39) and the Gårding inequality, one obtains

$$\mathrm{Re}\big\langle(A_f(h)-\lambda(h))u_f^1(h),u_f^1(h)\big\rangle=\Big\langle\mathrm{Re}\big(a_0(e^{i\gamma}x,e^{-i\gamma}(hD_x+if'(x)))-E\big)u_f^1(h),u_f^1(h)\Big\rangle$$
$$+\mathcal{O}(h)\|u_f^1(h)\|^2\geq\frac{1}{C}\Big\langle(h^2D_x^2+x^2-C_1h)u_f^1(h),u_f^1(h)\Big\rangle, \tag{4.64}$$

for some large constants C, C_1.

Since $h^2D_x^2$ is a positive operator, then

$$\Big\langle(h^2D_x^2+x^2-C_1h)u_f^1(h),u_f^1(h)\Big\rangle\geq\Big\langle(x^2-C_1h)u_f^1(h),u_f^1(h)\Big\rangle. \tag{4.65}$$

Let M be a large constant. We decompose the scalar product in (4.65) into two parts according to $|x|>Mh$ and $|x|\leq Mh$. Then one has

$$\Big\langle(x^2-C_1h)u_f^1(h),u_f^1(h)\Big\rangle=\Big\langle(x^2-C_1h)u_f^1(h),u_f^1(h)\Big\rangle_{L^2(\{x\in\mathbb{R};\ |x|>Mh^{1/2}\})} \tag{4.66}$$
$$+\Big\langle(x^2-C_1h)u_f^1(h),u_f^1(h)\Big\rangle_{L^2(\{x\in\mathbb{R};\ |x|\leq Mh^{1/2}\})}.$$

By using $x^2 \geq 0$, one obtains

$$\begin{aligned}\Big\langle (x^2 - C_1 h) u_f^1(h), u_f^1(h) \Big\rangle &\geq \Big\langle (M^2 - C_1) h u_f^1(h), u_f^1(h) \Big\rangle_{L^2(\{x\in\mathbb{R};\ |x|>Mh^{1/2}\})} \\ &\quad - C_1 h \Big\langle u_f^1(h), u_f^1(h) \Big\rangle_{L^2(\{x\in\mathbb{R};\ |x|\leq Mh^{1/2}\})} \\ &= \Big\langle (M^2 - C_1) h u_f^1(h), u_f^1(h) \Big\rangle \\ &\quad - M^2 h \Big\langle u_f^1(h), u_f^1(h) \Big\rangle_{L^2(\{x\in\mathbb{R};\ |x|\leq Mh^{1/2}\})}.\end{aligned} \tag{4.67}$$

For $|x| \leq Mh^{1/2}$, one has $\frac{f(x)}{h} = \frac{c_1 x^2}{h} \leq c_1 M^2$. Combining this with the fact that $\|u(h)\| = 1$, one derives : There exists $C(M) > 0$ such that

$$\Big\langle u_f^1(h), u_f^1(h) \Big\rangle_{L^2(\{x\in\mathbb{R};\ |x|\leq Mh^{1/2}\})} < C(M). \tag{4.68}$$

Then, combining (4.65), (4.67) and (4.68) one has

$$\Big\langle (h^2 D_x^2 + x^2 - C_1 h) u_f^1(h), u_f^1(h) \Big\rangle \geq (M^2 - C_1) h \|u_f^1(h)\|^2 - M^2 C(M) h. \tag{4.69}$$

From (4.62), (4.64) and (4.69), there exists $C_2 > 0$ such that

$$(M^2 - C_1) h \|u_f^1(h)\|^2 - M^2 C(M) h \leq C_2 h \|u_f^1(h)\| \|u_f(h)\|, \tag{4.70}$$

which implies that

$$\|u_f^1(h)\|^2 \leq \frac{M^2 C(M)}{(M^2 - C_1)} + \frac{C_2}{(M^2 - C_1)} \|u_f(h)\|^2.$$

From (4.61), (4.70) and the fact that $\chi_1^w(x, hD_x) + \chi_2^w(x, hD_x) = 1$ we have

$$\begin{aligned}\|u_f(h)\|^2 &= \|(\chi_1^w(x, hD_x) + \chi_2^w(x, hD_x)) u_f(h)\|^2 \\ &\leq \frac{M^2 C(M)}{(M^2 - C_1)} + \frac{C_2}{(M^2 - C_1)} \|u_f(h)\|^2 + C_3 h \|u_f(h)\|^2,\end{aligned}$$

for some $C_3 > 0$. The proof follows by choosing M sufficiently large, h small enough. □

Since x and ξ play the same role, we also obtain the following :

Theorem 4.4.5. *Let g and D be as in Lemma 4.4.3 and Theorem 4.4.4. Suppose that $\lambda(h) \in D$ is an eigenvalue of $A_{i\gamma}(h)$ and $u(h)$ is a normalized eigenfunction associated to $\lambda(h)$, then there exists $C > 0$ such that*

$$\|e^{g(hD_x)/h} u(h)\| \leq C. \tag{4.71}$$

Thanks to the theorems 4.4.4 and 4.4.5, we need only study the symbol of the operator $A_{i\gamma}(h)$ near $(0,0)$. We also have assumed near $(0,0)$ that $a_0(x,\xi) = E + \frac{1}{2}(\lambda x^2 + \mu \xi^2) + \mathcal{O}((x,\xi)^3)$ (i.e., the matrix $a_0''(0,0)$ is diagonal). Putting

$$A_{i\gamma}^0(h) = \frac{1}{2}(\lambda e^{2i\gamma} x^2 + \mu e^{-2i\gamma} h^2 D_x^2) + \frac{2n+1}{4} \Delta V(0,0) h. \tag{4.72}$$

Since $\Delta V(0,0) = \lambda + \mu - \omega^2$, then it is clear that the spectrum of $A^0_{i\gamma}(h), \sigma(A^0_{i\gamma}(h)) = \{he_k;\ k \in \mathbb{N}\}$ where

$$e_k = \frac{(2k+1)\sqrt{\lambda\mu}}{2} + \frac{(2n+1)(\lambda+\mu-2\omega^2)}{4},\ k \in \mathbb{N}. \tag{4.73}$$

Notice here that in the work of X.P.Wang (see [59]) the total electric potential is $W(x,y) = \omega x + V(x,y)$ and then $\Delta W(x,y) = \Delta V(x,y)$. But in our case, the total electric potential is $a_0(x,y) = \omega^2 x^2 + V(x,y)$. Then $\Delta a_0(x,y) = 2\omega^2 + \Delta V(x,y)$. This explains why there is $-2\omega^2$ in the formula of e_k.

Let C_0 be a large fixed constant such that $C_0 \neq e_k$, $k \in \mathbb{N}$. Henceforth we denote the neighbourhood of E,

$$D = \{z \in \mathbb{C};\ |z - E| \leq C_0 h\}. \tag{4.74}$$

For $\beta > 0$ small and $j \in \mathbb{N}$, let $D_j = \{z \in D;\ |z - E - he_j| \leq \beta h\}$. Then we prove that the spectrum of $A_{i\gamma}(h)$ in $D \setminus \bigcup_{j\in\mathbb{N}} D_j$ is empty.

Proposition 4.4.6. *Let z be in $D \setminus \bigcup_{j\in\mathbb{N}} D_j$. For h small enough, the following resolvent estimate holds :*

$$\|(A_{i\gamma}(h) - z)^{-1}\| \leq Ch^{-1},$$

for some $C > 0$.

Proof. Choosing $\{\chi_1, \chi_2\}$ a partition of unity on $\mathbb{R}^2$, $\chi_1 = 1$ near $(0,0)$ and $\chi_1 = 0$ outside an $\beta-$ neighbourhood of $(0,0)$.

From the proof of the inequality (4.40), it is easy to see that for $z \in D$, one has $|a_0(e^{i\gamma}x, e^{-i\gamma}\xi) - z| \geq \frac{|\gamma|}{C}$ for $(x,\xi) \in \mathrm{supp}\chi_2$. Denote by $B_1(z)$ the $h-$ pseudodifferential operator with symbol $\chi_2(x,\xi)(a_0(e^{i\gamma}x, e^{-i\gamma}\xi) - z)^{-1}$. Then one has

$$B_1(z)(A_{i\gamma}(h) - z) = \chi_2^w(x, hD_x) + \mathcal{O}(h). \tag{4.75}$$

For $z \in D \setminus \bigcup_{j\in\mathbb{N}} D_j$, then $z - E \in \rho(A^0_{i\gamma}(h))$ which shows the existence of $B_2(z) := (A^0_{i\gamma}(h) + E - z)^{-1}$. Our purpose is to study $\chi_1^w(x, hD_x)B_2(z)(A_{i\gamma}(h) - z)$. It can be readily verified that $x^j(hD_x)^k B_2(z)$ is unitarily equivalent to

$$h^{\frac{k+j}{2}-1} x^j D_x^k \Big(\frac{1}{2}(\lambda e^{2i\gamma}x^2 + \mu e^{-2i\gamma}D_x^2) + \frac{2n+1}{4}\Delta V(0,0) + \frac{E-z}{h}\Big)^{-1}.$$

Then,

$$\|x^j(hD_x)^k B_2(z)\| \leq Ch^{\frac{k+j}{2}-1},\ 0 \leq k+j \leq 2. \tag{4.76}$$

We have

$$\begin{aligned}\chi_1^w(x, hD_x)B_2(z)(A_{i\gamma}(h) - z) &= \chi_1^w(x, hD_x)\Big(1 + B_2(z)(A_{i\gamma}(h) - A^0_{i\gamma}(h) - E)\Big)\\ &= \chi_1^w(x, hD_x)\Big(1 + B_2(z)(A_{i\gamma}(h) - A^0_{i\gamma}(h) - E)\Big)\Big(\chi_3^w(x, hD_x) + 1 - \chi_3^w(x, hD_x)\Big),\end{aligned}$$

where $\chi_3 = 1$ near suppχ_1 and $\chi_3 = 0$ outside $2\beta-$ neighbourhood of $(0,0)$. From (4.76) and supp$\chi_1 \cap$ supp$\chi_3 = \emptyset$, we obtain by inductive arguments that

$$\chi_1^w(x, hD_x)B_2(z)(A_{i\gamma}(h) - A_{i\gamma}^0(h) - E)(1 - \chi_3^w(x, hD_x)) = \mathcal{O}(h^\infty). \tag{4.77}$$

One has the symbol of $A_{i\gamma}(h) - A_{i\gamma}^0(h) - E$ on the support of χ_3 is $\mathcal{O}((x, \xi))h + \mathcal{O}(h^2)$. Thus we use again (4.76) to obtain

$$\|\chi_1^w(x, hD_x)B_2(z)(A_{i\gamma}(h) - A_{i\gamma}^0(h) - E)\chi_3^w(x, hD_x)\| \leq C\beta + Ch, \tag{4.78}$$

for some C large.

These arguments give

$$\chi_1^w(x, hD_x)B_2(z)(A_{i\gamma}(h) - z) = \chi_1^w(x, hD_x) + \mathcal{O}(\beta) + \mathcal{O}(h). \tag{4.79}$$

From (4.75) and (4.79), there exist $C_1, C_2, C_3, C_4 > 0$ such that

$$\begin{aligned}(1 - C\beta - C_1 h)\|u\| &\leq \|\chi_1^w(x, hD_x)B_2(z)(A_{i\gamma}(h) - z)u\| + \|B_1(z)(A_{i\gamma}(h) - z)u\| \\ &\leq C_2 h^{-1}\|(A_{i\gamma}(h) - z)u\| + C_3\|(A_{i\gamma}(h) - z)u\| \\ &\leq C_4 h^{-1}\|(A_{i\gamma}(h) - z)u\|, \ \forall u \in C_0^\infty(\mathbb{R}).\end{aligned}$$

We choose β and h small such that $(1 - C\beta - C_1 h) > \frac{1}{2}$. It implies that $A_{i\gamma}(h)$ does not have any eigenvalue in $D \setminus \bigcup_{j\in\mathbb{N}} D_j$. On the other hand Lemma 4.4.1 asserts that $\sigma_{\text{ess}}(A_{i\gamma}(h)) \cap D \setminus \bigcup_{j\in\mathbb{N}} D_j = \emptyset$. Thus $D \setminus \bigcup_{j\in\mathbb{N}} D_j \subset \rho(A_{i\gamma}(h))$ and $\|(A_{i\gamma}(h) - z)^{-1}\| \leq Ch^{-1}$ for all $z \in D \setminus \bigcup_{j\in\mathbb{N}} D_j$. □

From Proposition 4.4.6, the spectrum of $A_{i\gamma}(h)$ near E is contained in $\bigcup_{j\in\mathbb{N}} D_j$. Further, one obtains the following :

Theorem 4.4.7. *Let D be the set defined in (4.74). Then for h sufficiently small, there is one to one correspondence between the eigenvalues of $A_{i\gamma}(h)$ in D and the set $\{e_j;\ e_j < C_0,\ j \in \mathbb{N}\}$. Moreover, we can rearrange the eigenvalues of $A_{i\gamma}(h)$ in D such that the jth eigenvalue is*

$$E_j(h) = E + he_j + \mathcal{O}(h^{\frac{3}{2}}). \tag{4.80}$$

Proof. According to Proposition 4.4.6, it suffices to show that for each j such that $e_j < C_0$, in $D_j = \{z \in D;\ |z - E - he_j| \leq \beta h\}$ there exists uniquely an eigenvalue of $A_{i\gamma}(h)$. Denote $\Gamma_j = \{z \in \mathbb{C};\ |z - E - he_j| = 2\beta h\} \subset D \setminus \bigcup_{j\in\mathbb{N}} D_j$. For $z \in \Gamma_j$, one has $\|(z - A_{i\gamma}(h))^{-1}\| \leq Ch^{-1}$. Let us define for each $j \in \mathbb{N}$

$$\Xi_j(h) = \frac{1}{2\pi i}\int_{\Gamma_j}(z - A_{i\gamma}(h))^{-1}dz. \tag{4.81}$$

In the same notations as in Proposition 4.4.6, one has

$$B_1(z)(A_{i\gamma}(h) - z) = \chi_2^w(x, hD_x) + \mathcal{O}(h)$$

and

$$\chi_1^w(x,hD_x)B_2(z)(A_{i\gamma}(h)-z)=\chi_1^w(x,hD_x)(1+B_2(z)(A_{i\gamma}(h)-A_{i\gamma}^0(h)-E)).$$

Then,

$$\begin{aligned}&(B_1(z)+\chi_1^w(x,hD_x)B_2(z))(A_{i\gamma}(h)-z)\\&=1+\mathcal{O}(h)+\chi_1^w(x,hD_x)B_2(z)(A_{i\gamma}(h)-A_{i\gamma}^0(h)-E)\\&=1+\mathcal{O}(h)+\chi_1^w(x,hD_x)B_2(z)(A_{i\gamma}(h)-A_{i\gamma}^0(h)-E)\chi_3^w(x,hD_x),\end{aligned}$$

which shows that

$$\begin{aligned}&(A_{i\gamma}(h)-z)^{-1}=B_1(z)+\chi_1^w(x,hD_x)B_2(z) \quad (4.82)\\&-\Big(\mathcal{O}(h)+\chi_1^w(x,hD_x)B_2(z)(A_{i\gamma}(h)-A_{i\gamma}^0(h)-E)\chi_3^w(x,hD_x)\Big)(A_{i\gamma}(h)-z)^{-1}.\end{aligned}$$

We notice here that $B_1(z)$ is holomorphic in $z\in D$. Then, by inserting (4.82) into (4.81), one obtains

$$\begin{aligned}\Xi_j(h)&=\chi_1^w(x,hD_x)\Pi^0(h)+\int_{\Gamma_j}(\mathcal{O}(1)+\mathcal{O}(\beta)h^{-1})dz\\&=\chi_1^w(x,hD_x)\Pi^0(h)+B_3(h,\beta), \quad (4.83)\end{aligned}$$

where $\Pi^0(h)$ is the spectral projector associated to e_j of $A_{i\gamma}^0(h)$ and $\|B_3(h,\beta)\|\to 0$ as $\beta,h\to 0$. It implies that, for $j\in\mathbb{N}$ such that $e_j<C_0$, $\mathrm{rank}\Xi_j(h)=\mathrm{rank}\chi_1\Pi^0(h)=1$. According to [55, Theorem XII.6] there exists uniquely a simple eigenvalue of $A_{i\gamma}(h)$ inside Γ_j denoted by $E_j(h)$.

Now we prove the estimate (4.80). Let $\phi_{j,i\gamma}(x;h)$ be the unique eigenvalue associated to he_j of $A_{i\gamma}^0(h)$. Since $\phi_{j,i\gamma}(x;h)$ is of the form $p_j(\frac{e^{i\gamma}x}{\sqrt{h}})h^{-\frac14}e^{-c\frac{e^{i2\gamma}x^2}{2h}}$ where p_j is a polynomial, by a scaling argument, one gets

$$\|x^j(hD_x)^k(\chi_1^w(x,hD_x)\phi_{j,i\gamma}(x;h))\|=\mathcal{O}(h^{\frac{k+j}{2}}),\ 0\le k+j\le 3.$$

It follows from this that

$$(A_{i\gamma}(h)-E-A_{i\gamma}^0(h))\chi_1^w(x,hD_x)\phi_{j,i\gamma}(x;h)=\mathcal{O}(h^{\frac32}). \quad (4.84)$$

A direct computation gives

$$\begin{aligned}&(A_{i\gamma}(h)-E-he_j)\chi_1^w\phi_{j,i\gamma}(x;h)\\&=[A_{i\gamma}(h),\chi_1^w]\phi_{j,i\gamma}(x;h)+\chi_1^w(A_{i\gamma}(h)-E-A_{i\gamma}^0(h))(\phi_{j,i\gamma}(x;h))\\&=[A_{i\gamma}(h),\chi_1^w]\phi_{j,i\gamma}(x;h)+\chi_1^w(A_{i\gamma}(h)-E-A_{i\gamma}^0(h))(\chi_3^w+1-\chi_3^w)(\phi_{j,i\gamma}(x;h)).\end{aligned}$$

Combining this with (4.77) and (4.84), one has

$$(A_{i\gamma}(h)-E-he_j)\chi_1^w\phi_{j,i\gamma}(x;h)=[A_{i\gamma}(h),\chi_1^w]\phi_{j,i\gamma}(x;h)+\mathcal{O}(h^{\frac32}). \quad (4.85)$$

Let $\tilde{\chi} \in C_0^\infty(\mathbb{R}^2)$ such that $\tilde{\chi} = 1$ near $(0,0)$ and $\chi_1\tilde{\chi} = \tilde{\chi}$. It follows from the $h-$ pseudo-differential calculus that $[A_{i\gamma}(h), \chi_1^w]\tilde{\chi}^w = \mathcal{O}(h^\infty)$. Combining this with the exponential decay of $\phi_{j,i\gamma}(x;h)$, one obtains

$$\begin{aligned}[A_{i\gamma}(h), \chi_1^w]\phi_{j,i\gamma}(x;h) &= [A_{i\gamma}(h), \chi_1^w]\tilde{\chi}^w\phi_{j,i\gamma}(x;h) + [A_{i\gamma}(h), \chi_1^w](1-\tilde{\chi}^w)\phi_{j,i\gamma}(x;h) \\ &= \mathcal{O}(h^\infty). \end{aligned} \tag{4.86}$$

From (4.85) and (4.86), one obtains

$$(A_{i\gamma}(h) - E - he_j)\chi_1^w(x, hD_x)\phi_{j,i\gamma}(x;h) = \mathcal{O}(h^{\frac{3}{2}}). \tag{4.87}$$

Let $u_{j,i\gamma}(h)$ be a normalized eigenfunction of $A_{i\gamma}(h)$ associated to $E_j(h)$. We denote by $A_{i\gamma}(h)^*$ the adjoint of $A_{i\gamma}(h)$. Now we take $u_{j,i\gamma}(h)^*$ an eigenfunction of $A_{i\gamma}(h)^*$ such that $A_{i\gamma}(h)^*u_{j,i\gamma}(h)^* = \bar{E}_j(h)u_{j,i\gamma}(h)^*$ and $\langle u_{j,i\gamma}(h), u_{j,i\gamma}(h)^*\rangle = 1$.

It follows from (4.83), (4.87) that

$$\Big\langle (A_{i\gamma}(h) - E - he_j)(u_{j,i\gamma}(h) - B_3(h,\beta)u_{j,i\gamma}(h)), u_{j,i\gamma}(h)^* \Big\rangle = \mathcal{O}(h^{\frac{3}{2}}). \tag{4.88}$$

Remark that $u_{j,i\gamma}(h)$ (resp. $u_{j,i\gamma}(h)^*$) is the eigenfunction of $A_{i\gamma}(h)$ (resp. $A_{i\gamma}(h)^*$). Then (4.88) follows that

$$(E_j(h) - E - he_j)(1 - \langle B_3(h,\beta)u_{j,i\gamma}(h), u_{j,i\gamma}(h)^*\rangle) = \mathcal{O}(h^{\frac{3}{2}}). \tag{4.89}$$

Recall that $\|B_3(h,\beta)\| \to 0$ as $h, \beta \to 0$. Then for h, β sufficiently small, it results from (4.89) that $E_j(h) = E + he_j + \mathcal{O}(h^{\frac{3}{2}})$. □

By using the fact that $E_j(h)$ is a simple eigenvalue and repeating the same arguments as in [41] (see also [31]), we obtain :

Theorem 4.4.8. *The eigenvalue $E_j(h)$ of $A_{i\gamma}(h)$ can be expanded asymptotically in powers of h, i.e.,*

$$E_j(h) \sim \sum_{k\geq 0} \lambda_{j,k}h^k,$$

where $\lambda_{j,0} = E, \lambda_{j,1} = e_j$.

4.5 Proof of the main theorem

In this section, we prove the existence of resonances and show the width of resonances is $\mathcal{O}(h^\infty)$.

4.5.1 The existence of resonances.

In this subsection, we prove the existence of resonances of $P(B,\omega)$ in each set U_n, $n \in \mathbb{N}$. For fix $j \in \mathbb{N}$, let us recall some notations used in the proof of Theorem 4.4.7. We consider $u_{j,i\gamma}(h)$ be a normalized eigenfunction of $A_{i\gamma}(h)$ associated to $E_j(h)$. We denote by $A_{i\gamma}(h)^*$ the adjoint of $A_{i\gamma}(h)$. Take $u_{j,i\gamma}(h)^*$ an eigenfunction of $A_{i\gamma}(h)^*$ such that $A_{i\gamma}(h)^*u_{j,i\gamma}(h)^* = \bar{E}_j(h)u_{j,i\gamma}(h)^*$, $\langle u_{j,i\gamma}(h), u_{j,i\gamma}(h)^*\rangle = 1$. Let $\tilde{\Pi}_j(h)$ be the spectral projection associated to $E_j(h)$ of $A_{i\gamma}(h)$ defined by $\tilde{\Pi}_j(h)(u) = \langle u, u_{j,i\gamma}(h)^*\rangle u_{j,i\gamma}(h)$.

Lemma 4.5.1. *For ε small enough, we put $\Omega_j := \{z \in \mathbb{C};\ |z - E_j(h)| < \varepsilon h\}$. Let $z \in \Omega_j$, then for h sufficiently small, one has*

$$\tilde{R}(z) = ((1 - \tilde{\Pi}_j(h))A_{i\gamma}(h)(1 - \tilde{\Pi}_j(h)) - z)^{-1}(1 - \tilde{\Pi}_j(h)) \tag{4.90}$$

exists. Moreover, $\tilde{R}(z)$ is holomorphic in $z \in \Omega_j$ and $\|\tilde{R}(z)\| \le Ch^{-1}$.

Proof. By Theorem 4.4.7, there is only a simple eigenvalue $E_j(h)$ of $A_{i\gamma}(h)$ in Ω_j. This shows that

$$\sigma\Big((1 - \tilde{\Pi}_j(h))A_{i\gamma}(h)(1 - \tilde{\Pi}_j(h))\Big) = \sigma(A_{i\gamma}(h)) \setminus \{E_j(h)\}.$$

Then,

$$\Omega_j \subset \rho\Big((1 - \tilde{\Pi}_j(h))A_{i\gamma}(h)(1 - \tilde{\Pi}_j(h))\Big),$$

which gives the existence of $\tilde{R}(z)$. The estimate of $\tilde{R}(z)$ can be followed immediately by imitating the proof of Proposition 4.4.6. Then we omit the details. □

Theorem 4.5.2. *For each $n \in \mathbb{N}$ fixed above, and let h be small enough, there exists only one resonance $E_{n,j}(h)$ of $P(B,\omega)$ in $\{z \in \mathbb{C};\ |z - (2n+1)h^{-1} - E_j(h)| < \varepsilon h\}$,*

$$E_{n,j}(h) = (2n+1)h^{-1} + E + he_j + \mathcal{O}(h^2) \tag{4.91}$$

which is algebraically simple. In particular, for all j such that $e_j < C_0$ (C_0 is defined in (4.6)), $E_{n,j}(h)$ is a resonance of $P(B,\omega)$ in U_n. Remark that U_n is defined by (4.6).

Proof. Let

$$R_-^1 : \mathbb{C} \to L^2(\mathbb{R}),\ \lambda \mapsto \lambda u_{j,i\gamma}(h)$$
$$R_+^1 : L^2(\mathbb{R}) \to \mathbb{C},\ v \mapsto \langle v, u_{j,i\gamma}(h)^* \rangle.$$

Then $R_-^1 R_+^1 = \tilde{\Pi}_j(h)$ and $R_+^1 R_-^1 = 1$. Let us consider the following Grushin problem for $E_{-+}(z)$:

$$\mathcal{P}_1(z) = \begin{pmatrix} E_{-+}(z) & R_-^1 \\ R_+^1 & 0 \end{pmatrix} : L^2(\mathbb{R}) \times \mathbb{C} \to L^2(\mathbb{R}) \times \mathbb{C}.$$

We treat this problem in the same way as Theorem 4.3.5. In the same notations as in Lemma 4.5.1, we put

$$\tilde{\mathcal{E}}_1(z) = \begin{pmatrix} -\tilde{R}(z) & R_-^1 \\ R_+^1 & E_j(h) - z \end{pmatrix}.$$

By a simple computation, we also get $\mathcal{P}_1(z)\tilde{\mathcal{E}}_1(z) = I + \mathcal{O}(h)$ and $\tilde{\mathcal{E}}_1(z)\mathcal{P}_1(z) = I + \mathcal{O}(h)$ uniformly in $z \in \Omega_j$. So $\mathcal{P}_1(z)$ is invertible, whose inverse is

$$\begin{aligned}\mathcal{E}_1(z) &= \tilde{\mathcal{E}}_1(z)\begin{pmatrix} (1 - h^2 G_{i\gamma}(z;h)\tilde{R}(z))^{-1} & -(1 - h^2 G_{i\gamma}(z;h)\tilde{R}(z))^{-1} h^2 G_{i\gamma}(z;h)u_{j,i\gamma}(h) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e(z) & e_+(z) \\ e_-(z) & e_{-+}(z) \end{pmatrix},\end{aligned}$$

where $h^2 G_{i\gamma}(z;h) = (E_{-+}(z) - z + A_{i\gamma}(h)) = \mathcal{O}(h^2)$ (see (4.31)). The right lower corner element of $\mathcal{E}_1(z)$ is

$$\begin{aligned} e_{-+}(z) &= E_j(h) - z - \left\langle (1 - h^2 G_{i\gamma}(z;h)\tilde{R}(z))^{-1} h^2 G_{i\gamma}(z;h) u_{j,i\gamma}(h), u_{j,i\gamma}(h)^* \right\rangle \\ &= E_j(h) - z + \mathcal{O}(h^2). \end{aligned}$$

We have $e_{-+}(z) : \mathbb{C} \to \mathbb{C}$ and $0 \in \sigma(E_{-+}(z))$ if and only if $z \in \sigma(e_{-+}(z))$. Combining this with (4.26), $z \in \sigma(e_{-+}(z))$ if and only if $(2n+1)h^{-1} + z \in \sigma(P_{1,i\gamma}(h))$.

By applying the Rouché theorem, $e_{-+}(z) = 0$ has a unique simple solution in Ω_j, $z = E_j(h) + \mathcal{O}(h^2)$. Therefore $E_{n,j}(h) := (2n+1)h^{-1} + E_j(h) + \mathcal{O}(h^2)$ is an unique resonance of $P(B,\omega)$ in $\{z \in \mathbb{C};\ |z - (2n+1)h^{-1} - E_j(h)| < \varepsilon h\}$. □

4.5.2 The width of resonances.

In this subsection, we use the same notations as in the preceding sections. Let $j \in \mathbb{N}$ and $\Omega_j := \{z \in \mathbb{C};\ |z - E_j(h)| < \varepsilon h\}$ be fixed as above (here $\varepsilon > 0$ small enough). We want to construct an approximate eigenvalue $\tilde{z}_j(h) \in \Omega_j$ and an approximate eigenfunction $\tilde{u}_{j,\theta}(h)$ such that

$$E_{-+}(\tilde{z}_j(h))\tilde{u}_{j,\theta}(h) = \mathcal{O}(h^\infty), \tag{4.92}$$

where we recall that

$$E_{-+}(z) = z - A_\theta(h) + h^2 G_\theta(z;h)$$

and $\theta \in \mathbb{C}$ with $|\theta|$ small enough, $\mathrm{Im}\theta < 0$ (see (4.31)).

Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 1$ near $[-R, R]$ $(R > 0)$. It follows from Remark 4.3.7 that $\tilde{G}_\theta(z;h) := \chi(x) G_\theta(z;h)$ is well-defined and holomorphic in $\theta \in \mathbb{C}$, $|\theta|$ small enough. In addition, $\tilde{G}_\theta(z;h)$ is self-adjoint if z real and $\mathrm{Im}\theta = 0$.

Let $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ such that $\mathrm{supp}\tilde{\chi} \subset (-R, R)$. Using the $h-$ pseudodifferential calculus, one has

$$\|\tilde{\chi}(x)[\tilde{G}_\theta(z;h) - G_\theta(z;h)]\| = \mathcal{O}(h^\infty). \tag{4.93}$$

Then we are led to construct an approximate solution for the following problem

$$\Big(A_0(h) - h^2 \tilde{G}_0(z;h)\Big) u(x) = z u(x),\ z \in \Omega_j. \tag{4.94}$$

In fact, the same problem was studied in [41, 42, 59] by using the WKB method, so we only want to recall main steps. First of all, with the help of Theorem 4.4.4, one can construct an approximate solution near 0 to

$$(A_0(h) - z^{(j)}(h)) u^{(j)}(x;h) = \mathcal{O}(e^{\frac{-d(x)}{h}} h^\infty) \tag{4.95}$$

in the form

$$z^{(j)}(h) \sim \sum_{l \geq 0} \lambda_l^{(j)} h^l$$

and

$$u^{(j)}(x;h) \sim e^{\frac{-d(x)}{h}} \sum_{l \geq 0} u_l^{(j)}(x) h^l,$$

where $d(x)$ is some phase function holomorphic in x near 0 and $\mathrm{Re}(d(x)) > 0$ for $x \in \mathbb{R} \setminus \{0\}$, $\lambda_0^{(j)} = E$, $\lambda_1^{(j)} = e_j$, $\lambda_l^{(j)} \in \mathbb{R}$ for all $l, j \in \mathbb{N}$. Here $z^{(j)}(h) - E_j(h) = \mathcal{O}(h^\infty)$ and $\|u_0^{(j)}\| > \mathrm{const} > 0$.

After that, one solves the following problem by using inductive arguments in $k \in \mathbb{N}$:

$$\Big(A_0(h) - h^2\tilde{G}_0(t_{k-1}^{(j)}(h); h) - t_k^{(j)}(h)\Big)v_k^{(j)}(x; h) = \mathcal{O}(e^{-\frac{d(x)}{h}}h^\infty), \tag{4.96}$$

where $v_0^{(j)}(x; h) = u^{(j)}(x; h)$, $t_0^{(j)}(h) = z^{(j)}(h)$. The solution of (4.96) is of the form

$$t_k^{(j)}(h) \sim \sum_{l\geq 0} \lambda_{l,k}^{(j)} h^l \text{ and } v_k^{(j)}(x; h) \sim e^{-\frac{d(x)}{h}} \sum_{l\geq 0} v_{l,k}^{(j)}(x) h^l,$$

where $t_{k+1}^{(j)}(h) - t_k^{(j)}(h) = \mathcal{O}(h^{k+2})$ and $v_{k+1}^{(j)}(x; h) - v_k^{(j)}(x; h) = \mathcal{O}(h^{k+2})$, $\lambda_{l,k}^{(j)}$ are real and $v_{l,k}^{(j)}$ are holomorphic near 0. Here $\lambda_{0,k}^{(j)} = E$ and $\lambda_{1,k}^{(j)} = e_j$. Taking the diagonal series, we get an approximate solution of (4.94) :

$$\tilde{z}_j(h) \sim \sum_{l\geq 0} \lambda_{l,l}^{(j)} h^l,$$

and

$$\tilde{u}_j(x; h) \sim e^{-\frac{d(x)}{h}} \sum_{l\geq 0} v_{l,l}^{(j)}(x) h^l,$$

where $\lambda_{0,0}^{(j)} = E$ and $\lambda_{1,1}^{(j)} = e_j$.

Let $\chi_1 \in C_0^\infty(\mathbb{R})$, $\chi_1 = 1$ near 0. Let us choose $\tilde{u}_{j,\theta}(x; h) = \chi_1(x)\tilde{u}_j(e^\theta x; h)$. From the analyticity of $\tilde{G}_\theta(z; h)$ with respect to θ near 0 and (4.93), we obtain an approximate eigenvalue $\tilde{z}_j(h)$ and an approximate eigenfunction $\tilde{u}_{j,\theta}(x; h)$ of (4.92).

Thus, we have proved the following theorem :

Theorem 4.5.3. *Let $\Omega_j := \{z \in \mathbb{C};\ |z - E_j(h)| < \varepsilon h\}$, for some small constant $\varepsilon > 0$, $j \in \mathbb{N}$. There exist $\tilde{z}_j(h) \in \Omega_j$ and $\tilde{u}_{j,\theta}(\cdot; h) \in L^2(\mathbb{R})$ verifying $\|\tilde{u}_{j,\theta}(\cdot; h)\| > \mathrm{const} > 0$, such that*

$$E_{-+}(\tilde{z}_j(h))\tilde{u}_{j,\theta}(x; h) = \mathcal{O}(h^\infty), \tag{4.97}$$

where

$$\tilde{z}_j(h) \sim \sum_{l\geq 0} \lambda_{l,l}^{(j)} h^l,$$

$\lambda_{0,0}^{(j)} = E$, $\lambda_{1,1}^{(j)} = e_j$ and $\lambda_{l,l}^{(j)} \in \mathbb{R}$, $\forall j, l \in \mathbb{N}$.

Now we carry out again subsection 4.5.1 in which $\theta = i\gamma$, the eigenfunction $u_{j,i\gamma}(h)$ of $A_{i\gamma}(h)$ is replaced by $\tilde{u}_{j,i\gamma}(\cdot; h)$ and $u_{j,i\gamma}(h)^*$ is replaced by $\frac{1}{\langle \tilde{u}_{j,i\gamma}(\cdot;h), \tilde{u}_{j,-i\gamma}(\cdot;h)\rangle}\tilde{u}_{j,-i\gamma}(\cdot; h)$ and $E_j(h)$ is replaced by $\tilde{z}_j(h)$. We then obtain

$$E_{n,j}(h) \sim (2n+1)h^{-1} + \sum_{l\geq 0} \lambda_{l,l}^{(j)} h^l. \tag{4.98}$$

4.5.3 End of the proof of Theorem 4.2.1

We end the proof of our main result in this subsection by proving the following

Proposition 4.5.4. *The resonance $E_{n,j}(B,\omega)$ has an asymptotic expansion in powers of B^{-1} as $B \to \infty$:*

$$E_{n,j}(B,\omega) \sim (2n+1)B+E+\frac{1}{2}\Big((2j+1)\sqrt{\lambda\mu}+\frac{(2n+1)(\lambda+\mu)}{2}\Big)B^{-1}+\sum_{k\geq 2} c_{n,j}^{(k)}B^{-k}, \quad (4.99)$$

where $c_{n,j}^{(k)} \in \mathbb{R}$ and λ, μ are two eigenvalues of $a_0''(0,0)$. In particular, the width of resonance $E_{n,j}(B,\omega)$ is of order $\mathcal{O}(B^{-\infty})$.

Proof. For B large enough, one has $|\frac{\omega}{B}| < 1$. Thus, for all $N \in \mathbb{N}$,

$$\sqrt{B^2+\omega^2} = B\sqrt{1+\frac{\omega^2}{B^2}} = B\Big(1+\frac{1}{2}\frac{\omega^2}{B^2}+\sum_{k\geq 2} a_k B^{-2k} + \mathcal{O}(B^{-2(N+1)})\Big), \quad (4.100)$$

$$\frac{1}{\sqrt{B^2+\omega^2}} = \frac{1}{B}\frac{1}{\sqrt{1+\frac{\omega^2}{B^2}}} = \frac{1}{B}\Big(1-\frac{1}{2}\frac{\omega^2}{B^2}+\sum_{k\geq 2} b_k B^{-2k} + \mathcal{O}(B^{-2(N+1)})\Big), \quad (4.101)$$

where $a_k, b_k \in \mathbb{R}$.

Replacing h by $\frac{1}{\sqrt{B^2+\omega^2}}$ in (4.98) and taking into account (4.100), (4.101), we obtain, for all $N \in \mathbb{N}$,

$$E_{n,j}(B,\omega) = (2n+1)B + \sum_{k=0}^{N} c_{n,j}^{(k)} B^{-k} + \mathcal{O}(B^{-N-1}), \quad (4.102)$$

where $c_{n,j}^{(0)} = E$, $c_{n,j}^{(1)} = e_j + \frac{2n+1}{2}\omega^2 = \frac{1}{2}\Big((2j+1)\sqrt{\lambda\mu}+\frac{(2n+1)(\lambda+\mu)}{2}\Big)$, $c_{n,j}^{(k)} \in \mathbb{R}$. In particular, the imaginary part of $E_{n,j}(B,\omega)$ is of order $\mathcal{O}(B^{-\infty})$. This ends the proof of Proposition 4.5.4. □

Chapitre 5

Spectral asymptotics for the Landau hamiltonian.

Dans ce chapitre, nous présentons un travail en collaboration avec M. Dimassi [17]

TRACE ASYMPTOTIC FORMULA FOR THE SCHÖDINGER OPERATORS OPERATORS WITH CONSTANT MAGNETIC FIELDS.

MOUEZ DIMASSI AND ANH TUAN DUONG

ABSTRACT. In this paper, we consider the $2D$- Schrödinger operator with constant magnetic field $H(V)=(D_x-By)^2+D_y^2+V_h(x,y)$, where V tends to zero at infinity and h is a small positive parameter. We will be concerned with two cases : the semi-classical limit regime $V_h(x,y)=V(hx,hy)$, and the large coupling constant limit case $V_h(x,y)=h^{-\delta}V(x,y)$. We obtain a complete asymptotic expansion in powers of h^2 of $\mathrm{tr}(\Phi(H(V),h))$, where $\Phi(\cdot,h)\in C_0^\infty(\mathbb{R};\mathbb{R})$. We also give a Weyl type asymptotics formula with optimal remainder estimate of the counting function of eigenvalues of $H(V)$.

5.1 Introduction

Let $H_0=(D_x-By)^2+D_y^2$ be the $2D-$ Schrödinger operator with constant magnetic field $B>0$. Here $D_\nu=\frac{1}{i}\partial_\nu$. It is well known that the operator H_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ and its spectrum consists of eigenvalues of infinite multiplicity (called Landau levels, see, e.g., [2])

$$\sigma(H_0)=\sigma_{\mathrm{ess}}(H_0)=\bigcup_{n=0}^{\infty}\{(2n+1)B\}.$$

Here $\sigma(H_0)$ (resp. $\sigma_{\rm ess}(H_0)$) denotes the spectrum (resp. the essential spectrum) of the operator H_0.

Let $V \in C^\infty(\mathbb{R}^2;\mathbb{R})$ and assume that V is bounded with all its derivatives and satisfies

$$\lim_{|(x,y)|\to\infty} V(x,y) = 0. \tag{5.1}$$

We now consider the perturbed Schrödinger operator

$$H(V) = H_0 + \ V_h(x,y), \tag{5.2}$$

where V_h is a potential depending on a semi-classical parameter $h>0$, and is of the form $V_h(x,y) = V(hx,hy)$ or $V_h(x,y) = h^{-\delta}V(x,y)$, $(\delta > 0)$. The Kato-Rellich theorem and the Weyl criterion show that $H(V)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ and

$$\sigma_{\rm ess}(H(V)) = \sigma_{\rm ess}(H_0) = \bigcup_{n=0}^{\infty}\{(2n+1)B\}.$$

The spectral properties of the $2D$-Schrödinger operator with constant magnetic field $H(V)$ have been intensively studied in the last ten years. In the case of perturbations, the Landau levels $\Lambda_n = (2n+1)B$ become accumulation points of the eigenvalues and the asymptotics of the function counting the number of the eigenvalues lying in a neighborhood of Λ_n have been examined by many authors in different aspects. For recent results, the reader may consult [51, 15, 52, 46, 39, 20, 7] and the references therein.

The asymptotics with precise remainder estimate for the counting spectral function of the operator $H(h) := H_0 + V(hx,hy)$ have been obtained by V. Ivrii [36]. In fact, he constructs a micro-local canonical form for $H(h)$, which leads to the sharp remainder estimates.

However, there are only a few works treating the case of the large coupling constant limit (i.e., $V_h(x,y) = h^{-\delta}V(x,y)$) (see [47, 49, 50]). In this case, the asymptotic behavior of the counting spectral function depends both on the sign of the perturbation and on its decay properties at infinity. In [50], G. Raikov obtained only the main asymptotic term of the counting spectral function as $h \searrow 0$.

The method used in [50] is of variational nature. By this method one can find the main term in the asymptotics of the counting spectral function with a weaker assumption on the perturbation V. However, it is quite difficult to establish with these techniques an asymptotic formula involving sharp remainder estimates.

For both the semi-classical and large coupling constant limit, we give a complete asymptotic expansion of the trace of $\Phi(H(V),h)$ in powers of h^2. We also establish a Weyl-type asymptotic formula with optimal remainder estimate for the counting function of eigenvalues of $H(V)$. The remainder estimate in Corollary 5.2.4 and Corollary 5.2.6 is $\mathcal{O}(1)$, so it is better than in the standard case (without magnetic field, see e.g., [16]). To prove our results, we show that the spectral study of $H(h)$ near some energy level z can be reduced to the study of an h^2-ΨDO $E_{-+}(z)$ called *the effective Hamiltonian*. Our results are still true for the case of dimension $2d$ with $d \geq 1$. For the transparency of the presentation, we shall mainly be concerned with the two-dimensional case.

The paper is organized as follows : In the next section we state the assumptions and the results precisely, and we give an outline of the proofs. In Section 5.3 we reduce the spectral study of $H(V)$ to the one of a system of h^2-pseudo-differential operators $E_{-+}(z)$. In Section 5.4, we establish a trace formula involving the effective Hamiltonian $E_{-+}(z)$, and we prove the results concerning the semi-classical case. Finally, Section 5.5 is devoted to the proofs of the results concerning the large coupling constant limit case.

5.2 Formulations of main results

5.2.1 Semi-classical case

In this section we will be concerned with the semi-classical magnetic Schrödinger operator

$$H(h) = H_0 + V(hx, hy),$$

where V satisfies (5.1). By choosing $B =$ constant, we may actually assume that $B = 1$.

Fix two real numbers a and b such that $[a, b] \subset \mathbb{R} \setminus \sigma_{\text{ess}}(H(h))$. We define

$$\begin{aligned} l_0 &:= \min \left\{ q \in \mathbb{N}; V^{-1}([a - (2q + 1), b - (2q + 1)]) \neq \emptyset \right\}, \\ l &:= \sup \left\{ q \in \mathbb{N}; V^{-1}([a - (2q + 1), b - (2q + 1)]) \neq \emptyset \right\}. \end{aligned} \tag{5.1}$$

We will give an asymptotic expansion in powers of h^2 of $\text{tr}(f(H(h), h))$ in the two following cases :

a) $f(x, h) = f(x)$, where $f \in C_0^\infty((a, b)\,; \mathbb{R})$.

b) $f(x, h) = f(x)\widehat{\theta}(\frac{x-\tau}{h^2})$, where $f, \theta \in C_0^\infty(\mathbb{R}; \mathbb{R})$, $\tau \in \mathbb{R}$, and $\widehat{\theta}$ is the Fourier transform of θ.

As a consequence, we get a sharp remainder estimate for the counting spectral function of $H(h)$ when $h \searrow 0$. Let us state the results precisely.

Theorem 5.2.1. *Assume* (5.1)*, and let* $f \in C_0^\infty((a, b); \mathbb{R})$*. There exists a sequence of real numbers* $(\alpha_j(f))_{j \in \mathbb{N}}$*, such that*

$$\text{tr}(f(H(h))) \sim \sum_{k=0}^{\infty} \alpha_k(f) h^{2(k-1)}, \tag{5.2}$$

where

$$\alpha_0(f) = \sum_{j=l_0}^{l} \frac{1}{2\pi} \iint f((2j + 1) + V(x, y))dxdy. \tag{5.3}$$

Let $\theta \in C_0^\infty(\mathbb{R})$, and let ϵ be a positive constant. Set

$$\breve{\theta}(\tau) = \frac{1}{2\pi} \int e^{it\tau} \theta(t)dt, \quad \breve{\theta}_\epsilon(t) = \frac{1}{\epsilon}\breve{\theta}(\frac{t}{\epsilon}).$$

In the sequel we shall say that λ is not a critical value of V if and only if $V(X) = \lambda$ for some $X \in \mathbb{R}^2$ implies $\nabla_X V(X) \neq 0$.

Theorem 5.2.2. *Fix $\mu \in \mathbb{R} \setminus \sigma_{\text{ess}}(H(h))$ which is not a critical value of $(2j+1+V)$, for $j = l_0, ..., l$. Let $f \in C_0^\infty((\mu-\epsilon, \mu+\epsilon);\mathbb{R})$ and $\theta \in C_0^\infty((-\frac{1}{C}, \frac{1}{C});\mathbb{R})$, with $\theta = 1$ near 0. Then there exist $\epsilon > 0$, $C > 0$ and a functional sequence $c_j \in C^\infty(\mathbb{R};\mathbb{R})$, $j \in \mathbb{N}$, such that for all $M, N \in \mathbb{N}$, we have*

$$\operatorname{tr}\left(f(H(h))\breve{\theta}_{h^2}(t - H(h))\right) = \sum_{k=0}^{M} c_k(t) h^{2(k-1)} + \mathcal{O}\left(\frac{h^{2M}}{\langle t\rangle^N}\right) \tag{5.4}$$

uniformly in $t \in \mathbb{R}$, where

$$c_0(t) = \frac{1}{2\pi} f(t) \sum_{j=l_0}^{l} \int_{\{(x,y)\in\mathbb{R}^2 |\ 2j+1+V(x,y)=t\}} \frac{dS_t}{|\nabla V(x,y)|}. \tag{5.5}$$

Corollary 5.2.3. *In addition to the hypotheses of Theorem 5.2.1 suppose that a and b are not critical values of $((2j+1)+V)$ for all $j = l_0, ..., l$. Let $\mathcal{N}_h([a,b])$ be the number of eigenvalues of $H(h)$ in the interval $[a,b]$ counted with their multiplicities. Then we have*

$$\mathcal{N}_h([a,b]) = h^{-2} C_0 + \mathcal{O}(1), h \searrow 0, \tag{5.6}$$

where

$$C_0 = \frac{1}{2\pi} \sum_{j=l_0}^{l} \operatorname{Vol}\left(V^{-1}([a-(2j+1), b-(2j+1)])\right). \tag{5.7}$$

5.2.2 Large coupling constant limit case.

We apply the above results to the Schrödinger operator with constant magnetic field in the large coupling constant limit case. More precisely, consider

$$H_\lambda = (D_x - y)^2 + D_y^2 + \lambda V(x,y). \tag{5.8}$$

Here λ is a large constant, and the electric potential V is assumed to be strictly positive. Let $X := (x,y) \in \mathbb{R}^2$. We suppose in addition that for all $N \in \mathbb{N}$,

$$V(X) = \sum_{j=0}^{N-1} \omega_{2j}\left(\frac{X}{|X|}\right) |X|^{-\delta-2j} + r_{2N}(X), \text{ for } |X| \geq 1, \tag{5.9}$$

where

- $\omega_0 \in C^\infty(\mathbb{S}^1; (0,+\infty))$, $\omega_{2j} \in C^\infty(\mathbb{S}^1;\mathbb{R})$, $j \geq 1$. Here $\mathbb{S}^1$ denotes the unit circle.
- δ is some positive constant,
- $|\partial_X^\beta r_{2N}(X)| \leq C_\beta(1+|X|)^{-|\beta|-\delta-2N}$, $\forall \beta \in \mathbb{N}^2$.

Since V is positive, it follows that $\sigma(H_\lambda) \subset [1,+\infty)$. Fix two real numbers a and b such that $a > 1$ and $[a,b] \subset \mathbb{R} \setminus \sigma_{\text{ess}}(H_\lambda)$. Since $\sigma_{\text{ess}}(H_\lambda) = \cup_{j=0}^{\infty}\{(2j+1)\}$, there exists $q \in \mathbb{N}$ such that $2q+1 < a < b < 2q+3$. The following results are consequences of Theorem 5.2.1, Theorem 5.2.2 and Corollary 5.2.3.

Theorem 5.2.4. *Assume (5.9), and let $f \in C_0^\infty((a,b);\mathbb{R})$. There exists a sequence of real numbers $(b_j(f))_{j\in\mathbb{N}}$, such that*

$$\operatorname{tr}(f(H_\lambda)) \sim \lambda^{\frac{2}{\delta}} \sum_{k=0}^{\infty} b_k(f)\lambda^{-\frac{2k}{\delta}}, \ \lambda \nearrow +\infty \tag{5.10}$$

where

$$b_0(f) = \frac{1}{2\pi\delta}\int_0^{2\pi} (\omega_0(\cos\theta,\sin\theta))^{\frac{2}{\delta}} d\theta \sum_{j=0}^{q} \int f(u)(u-(2j+1))^{-1-\frac{2}{\delta}} du. \tag{5.11}$$

Theorem 5.2.5. *Let $f \in C_0^\infty((a-\epsilon, b+\epsilon);\mathbb{R})$ and $\theta \in C_0^\infty\left(\left(-\frac{1}{C},\frac{1}{C}\right);\mathbb{R}\right)$, with $\theta = 1$ near 0. Then there exist $\epsilon > 0$, $C > 0$ and a functional sequence $c_j \in C^\infty(\mathbb{R};\mathbb{R})$, $j \in \mathbb{N}$, such that for all $M, N \in \mathbb{N}$, we have*

$$\operatorname{tr}\left(f(H_\lambda)\breve{\theta}_{\lambda^{-\frac{2}{\delta}}}(t-H_\lambda)\right) = \lambda^{\frac{2}{\delta}} \sum_{k=0}^{M} c_k(t)\lambda^{-\frac{2k}{\delta}} + \mathcal{O}\left(\frac{\lambda^{-\frac{2M}{\delta}}}{\langle t\rangle^N}\right) \tag{5.12}$$

uniformly in $t \in \mathbb{R}$, where

$$c_0(t) = \frac{1}{2\pi} f(t) \sum_{j=0}^{q} \int_{\{X\in\mathbb{R}^2 |\ 2j+1+W(X)=t\}} \frac{dS_t}{|\nabla_X W(X)|}. \tag{5.13}$$

Here $W(X) = \omega_0(\frac{X}{|X|})|X|^{-\delta}$.

Corollary 5.2.6. *Let $\mathcal{N}_\lambda([a,b])$ be the number of eigenvalues of H_λ in the interval $[a,b]$ counted with their multiplicities. We have*

$$\mathcal{N}_\lambda([a,b]) = \lambda^{\frac{2}{\delta}} D_0 + \mathcal{O}(1), \ \lambda \to +\infty,$$

where

$$D_0 = \frac{1}{4\pi} \sum_{j=0}^{q} \left((a-2j-1)^{-\frac{2}{\delta}} - (b-2j-1)^{-\frac{2}{\delta}}\right) \int_0^{2\pi} (\omega_0(\cos\theta,\sin\theta))^{\frac{2}{\delta}} d\theta.$$

5.2.3 Outline of the proofs

The purpose of this subsection is to provide a broad outline of the proofs. By a change of variable on the phase space, the operator $H(h)$ is unitarily equivalent to

$$P(h) := P_0 + V^w(h) = -\frac{\partial^2}{\partial y^2} + y^2 + V^w(x + hD_y, hy + h^2 D_x), X = (x,y) \in \mathbb{R}^2.$$

Let $\Pi = 1_{[c,d]}(P_0)$ be the spectral projector of the harmonic oscillator on the interval $[c,d] = \left[a - \|V\|_{L^\infty(\mathbb{R}^2)}, b + \|V\|_{L^\infty(\mathbb{R}^2)}\right]$. Using the explicit expression of Π we will reduce the spectral study of $(P-z)$ for $z \in [a,b] + i[-1,1]$ to the study of a system of h^2-pseudo-differential operator, $E_{-+}(z)$ depending only on x (see Remark 5.3.7 and Corollary 5.3.9).

In particular, modulo $\mathcal{O}(h^\infty)$, we are reduced to proving Theorem 5.2.1 and Theorem 5.2.2 for a system of h^2-pseudo-differential operator (see Proposition 5.4.1). Thus, (5.2) and (5.4) follows easily from Theorem 1.8 in [14] (see also [15]). Corollary 5.2.3 is a simple consequence of Theorem 5.2.1, Theorem 5.2.2 and a Tauberian-argument.

To deal with the large coupling constant limit case, we note that for all $M > 0$ and λ large enough, we have

$$\big\{(x,y,\eta,\xi)\in\mathbb{R}^4; |(x,y)|<M,\ (\xi-y)^2+\eta^2+\lambda V(x,y)\in[a,b]\big\}=\emptyset.$$

Thus, on the symbolic level, only the behavior of $V(x,y)$ at infinity contributes to the asymptotic behavior of the left hand sides of (5.10) and (5.12). Since, for $|X|$ large enough, $\lambda V(X) = \varphi_0(hX)+\varphi_2(hX)h^2+\cdots+\varphi_{2j}(hX)h^{2j}+\cdots$ with $h=\lambda^{-\frac{1}{\delta}}$ and $\varphi_0(X) = \omega_0(\frac{X}{|X|})|X|^{-\delta}$, Theorem 5.2.4 (resp. Theorem 5.2.5) follows from Theorem 5.2.1 (resp. Theorem 5.2.2).

5.3 The effective Hamiltonian

5.3.1 Classes of symbols

Let $M_n(\mathbb{C})$ be the space of complex square matrices of order n. We recall the standard class of semi-classical matrix-valued symbols on $T^*\mathbb{R}^d=\mathbb{R}^{2d}$:

$$S^m(\mathbb{R}^{2d};M_n(\mathbb{C}))=\{a\in C^\infty(\mathbb{R}^{2d}\times(0,1];M_n(\mathbb{C})):\ \|\partial_x^\alpha\partial_\xi^\beta a\|_{\mathcal{L}(M_n(\mathbb{C}))}\le C_{\alpha,\beta}h^{-m}\}.$$

We note that the symbols are tempered as $h\searrow 0$. The more general class $S^m_\delta(\mathbb{R}^{2d};M_n(\mathbb{C}))$, where the right hand side in the above estimate is replaced by $C_{\alpha,\beta}h^{-m-\delta(|\alpha|+|\beta|)}$, has nice quantization properties as long as $0\le\delta\le 1/2$ (we refer to [22, Chapter 7]).

For h-dependent symbol $a\in S^m(\mathbb{R}^{2d};M_n(\mathbb{C}))$, we say that a has an asymptotic expansion in powers of h and we write

$$a\sim\sum_{j\ge0}a_jh^j,\text{ in } S^m_\delta(\mathbb{R}^{2d};M_n(\mathbb{C})),$$

if there exists a sequence of symbols $a_j(x,\xi)\in S^m_\delta(\mathbb{R}^{2d};M_n(\mathbb{C}))$ such that for all $N\in\mathbb{N}$, we have

$$a-\sum_{j=0}^{N}a_jh^j\in S^{m-N-1}_\delta(\mathbb{R}^{2d};M_n(\mathbb{C})).$$

In the special case when $m=\delta=0$ (resp. $m=\delta=0, n=1$), we will write $S^0(\mathbb{R}^{2d};M_n(\mathbb{C}))$ (resp. $S^0(\mathbb{R}^{2d})$) instead of $S^0_0(\mathbb{R}^{2d};M_n(\mathbb{C}))$ (resp. $S^0_0(\mathbb{R}^{2d};M_1(\mathbb{C}))$.

We will use the standard Weyl quantization of symbols. More precisely, if $a\in S^m_\delta(\mathbb{R}^{2d};M_n(\mathbb{C}))$, then $a^w(x,hD_x;h)$ is the operator defined by

$$a^w(x,hD_x;h)u(x)=(2\pi h)^{-d}\iint e^{\frac{i(x-y,\xi)}{h}}a\left(\frac{x+y}{2},\xi;h\right)u(y)dyd\xi,\ u\in\mathcal{S}(\mathbb{R}^d;\mathbb{C}^n).$$

In order to prove our main results, we shall recall some well-known results

Proposition 5.3.1. (Composition formula) *Let* $a_i \in S_\delta^{m_i}(\mathbb{R}^{2d}; M_n(\mathbb{C}))$, $i = 1, 2$, $\delta \in [0, \frac{1}{2})$. *Then* $b^w(y, hD_y; h) = a_1^w(y, hD_y) \circ a_2^w(y, hD_y)$ *is an h-pseudo-differential operator, and*

$$b(y, \eta; h) \sim \sum_{j=0}^{\infty} b_j(y, \eta) h^j, \text{ in } S_\delta^{m_1+m_2}(\mathbb{R}^{2d}; M_n(\mathbb{C})).$$

Proposition 5.3.2. (Beals characterization) *Let* $A = A_h : \mathcal{S}(\mathbb{R}^d; \mathbb{C}^n) \to \mathcal{S}'(\mathbb{R}^d; \mathbb{C}^n)$, $0 < h \leq 1$. *The following two statements are equivalent :*
(1) $A = a^w(x, hD_x; h)$, *for some* $a = a(x, \xi; h) \in S^0(\mathbb{R}^{2d}; M_n(\mathbb{C}))$.
(2) For every $N \in \mathbb{N}$ *and for every sequence* $l_1(x, \xi), \ldots, l_N(x, \xi)$ *of linear forms on* $\mathbb{R}^{2d}$, *the operator* $\mathrm{ad}_{l_1^w(x,hD_x)} \circ \cdots \circ \mathrm{ad}_{l_N^w(x,hD_x)} A_h$ *belongs to* $\mathcal{L}(L^2, L^2)$ *and is of norm* $\mathcal{O}(h^N)$ *in that space. Here,* $\mathrm{ad}_A B := [A, B] = AB - BA$.

Proposition 5.3.3. (L^2- boundedness) *Let* $a = a(x, \xi; h) \in S_\delta^0(\mathbb{R}^{2d}; M_n(\mathbb{C}))$, $0 \leq \delta \leq 1/2$. *Then* $a^w(x, hD_x; h)$ *is bounded :* $L^2(\mathbb{R}^d; \mathbb{C}^n) \to L^2(\mathbb{R}^d; \mathbb{C}^n)$, *and there is a constant* C *independent of* h *such that for* $0 < h \leq 1$ *;*

$$\|a^w(x, hD_x; h)\| \leq C.$$

5.3.2 Reduction to a semi-classical problem

Here, we shall make use of a strong field reduction onto the jth eigenfunction of the harmonic oscillator, $j = l_0 \cdots l$, and a well-posed Grushin problem for $H(h)$. We show that the spectral study of $H(h)$ near some energy level z can be reduced to the study of an h^2-ΨDO $E_{-+}(z)$ called *the effective Hamiltonian.* Without any loss of generality we may assume that $l_0 = 1$.

Lemma 5.3.4. *There exists a unitary operator* $\widetilde{W} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ *such that*

$$P(h) = \widetilde{\mathcal{W}} H(h) \widetilde{\mathcal{W}}^*$$

where $P(h) := P_0 + V^w(h)$, $P_0 := -\frac{\partial^2}{\partial y^2} + y^2$ *and* $V^w(h) := V^w(x + hD_y, hy + h^2 D_x)$.

Proof. The linear symplectic mapping

$$\widetilde{S} : \mathbb{R}^4 \to \mathbb{R}^4 \text{ given by } (x, y, \xi, \eta) \mapsto \left(\frac{1}{h}x + \eta, y + h\xi, h\xi, \eta\right),$$

maps the Weyl symbol of the operator $H(h)$ into the Weyl symbol of the operator $P(h)$. By Theorem A.2 in [22, Chapter 7], there exists a unitary operator $\widetilde{W} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ associated to $\widetilde{S}$ such that $P(h) = \widetilde{\mathcal{W}} H(h) \widetilde{\mathcal{W}}^*$. □

Introduce the operator $R_j^- : L^2(\mathbb{R}_x) \to L^2(\mathbb{R}_{x,y}^2)$ by

$$(R_j^- v)(x, y) = \phi_j(y) v(x),$$

where ϕ_j is the jth normalized eigenfunction of the harmonic oscillator. Further, the operator $R_j^+ : L^2(\mathbb{R}^2_{x,y}) \to L^2(\mathbb{R}_x)$ is defined by

$$(R_j^+ u)(x) = \int \phi_j(y) u(x,y) dy.$$

Notice that R_j^+ is the adjoint of R_j^-. An easy computation shows that $R_j^+ R_j^- = I_{L^2(\mathbb{R}_x)}$ and $R_j^- R_j^+ = \Pi_j$, where

$$\Pi_j : \ L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \ v(x,y) \mapsto \int v(x,t)\phi_j(t) dt \phi_j(y).$$

Define $\Pi = \sum_{j=1}^{l} \Pi_j$.

Lemma 5.3.5. *Let* $\Omega := \{z \in \mathbb{C} | \ \mathrm{Re} z \in [a,b], \ |\mathrm{Im} z| < 1\}$. *The operator*

$$(I-\Pi)P(h)(I-\Pi) - z : \ (I-\Pi)L^2(\mathbb{R}^2) \to (I-\Pi)L^2(\mathbb{R}^2)$$

is uniformly invertible for $z \in \Omega$.

Proof. It follows from the definition of Π that $\sigma((I-\Pi)P_0(I-\Pi)) = \{2k+1 | \ k \neq 1, ..., l\}$. Hence

$$\sigma((I-\Pi)P(h)(I-\Pi)) \subset \bigcup_{k \neq 1,\dots,l} \left[2k+1 - \|V\|_{L^\infty(\mathbb{R}^2)}, 2k+1 + \|V\|_{L^\infty(\mathbb{R}^2)}\right],$$

which implies

$$\sigma((I-\Pi)P(h)(I-\Pi)) \cap [a,b] = \emptyset.$$

Consequently,

$$\|(I-\Pi)P(h)(I-\Pi) - z\| \geq \mathrm{dist}\left([a,b], \sigma((I-\Pi)P(h)(I-\Pi))\right) > 0$$

uniformly for $z \in \Omega$. Thus, we obtain

$$(I-\Pi)P(h)(I-\Pi) - z : \ (I-\Pi)L^2(\mathbb{R}^2) \to (I-\Pi)L^2(\mathbb{R}^2)$$

is uniformly invertible for $z \in \Omega$. $\square$

For $z \in \Omega$, we put

$$\mathcal{P}(z) = \begin{pmatrix} (P(h)-z) & R_1^- & ... & R_l^- \\ R_1^+ & 0 & ... & 0 \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ R_l^+ & 0 & ... & 0 \end{pmatrix} \text{ and } \widetilde{\mathcal{E}}(z) = \begin{pmatrix} R(z) & R_1^- & ... & R_l^- \\ R_1^+ & A_1 & ... & 0 \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ R_l^+ & 0 & ... & A_l \end{pmatrix},$$

where $A_j = z - (2j+1) - R_j^+ V^w(h) R_j^-, j = 1, ..., l$ and $R(z) = ((I-\Pi)P(h)(I-\Pi) - z)^{-1}(I-\Pi)$.

Let $\mathcal{E}_1(z) := \mathcal{P}(z)\widetilde{\mathcal{E}}(z) = (a_{k,j})_{k,j=1}^{l+1}$. In the next step we will compute explicitly $a_{k,j}$.

Using the fact that $\Pi R(z) = 0$ as well as the fact that Π commutes with P_0, we deduce that $(P(h) - z)R(z) = (I - \Pi) + [\Pi, V^w(h)]R(z)$. Consequently,

$$a_{1,1} = (P(h) - z)R(z) + \sum_{j=1}^{l} R_j^- R_j^+ = I + [\Pi, V^w(h)]R(z). \tag{5.1}$$

Next, from the definition of A_1 and the fact that $P_0R_1^- = 3R_1^-$ (we recall that $l_0 = 1$), one has

$$\begin{aligned} a_{1,2} &= (P(h) - z)R_1^- + R_1^- A_1 \\ &= -(z-3)R_1^- + V^w(h)R_1^- + R_1^-(z-3) - \Pi_1 V^w(h)R_1^- \\ &= V^w(h)R_1^- - \Pi_1 V^w(h)R_1^- \\ &= [V^w(h), \Pi_1]R_1^-. \end{aligned}$$

Similarly, $a_{1,j} = [V^w(h), \Pi_{j-1}]R_{j-1}^-, j \geq 3$.

Since $R_1^+(1 - \Pi_1) = R_1^+ - R_1^+R_1^-R_1^+ = 0$ and $R_1^+\Pi_j = R_1^+R_j^-R_j^+ = 0$ for $j \neq 1$, it follows that $a_{2,1} = R_1^+R(z) = 0$. Evidently, $a_{2,2} = R_1^+R_1^- = I_{L^2(\mathbb{R})}$ and $a_{2,j} = R_1^+R_j^- = 0$ for $j \geq 3$. The same arguments as above show that $a_{k,j} = \delta_{j,k}I_{L^2(\mathbb{R})}$ for all $k \geq 3$. Summing up we have proved

$$\mathcal{E}_1(z) = \mathcal{P}(z)\widetilde{\mathcal{E}}(z) = \begin{pmatrix} I + [\Pi, V^w(h)]R(z) & [V^w(h), \Pi_1]R_1^- & \dots & [V^w(h), \Pi_l]R_l^- \\ 0 & I_{L^2(\mathbb{R})} & \dots & 0 \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ 0 & 0 & \dots & I_{L^2(\mathbb{R})} \end{pmatrix},$$

Let $f_j \in C_0^\infty(\mathbb{R})$, $f_j = 1$ near $2j+1$ and $\mathrm{supp} f_j \subset [2j, 2j+2]$. By the spectral theorem we have $\Pi_j = f_j(D_y^2 + y^2)$. On the other hand, the functional calculus of pseudo-differential operators shows that $\Pi_j = f_j(D_y^2 + y^2) = B^w(y, D_y)$ with $B(y, \eta) = \mathcal{O}(\langle y \rangle^{-\infty}\langle \eta \rangle^{-\infty})$.

The composition formula of pseudo-differential operators (Proposition 5.3.1) gives

$$[V^w(h), \Pi_j] = \sum_{k=1}^{N} b_{k,j}^w(x, h^2D_x)c_{k,j}^w(y, D_y)h^k + \mathcal{O}(h^{N+1}),\ \forall N \in \mathbb{N}, \tag{5.2}$$

where $b_{k,j}, c_{k,j} \in S^0(\mathbb{R}^2)$. This together with the Calderon-Vaillancourt theorem (Proposition 5.3.3) yields $[V^w(h), \Pi_j] = \mathcal{O}(h)$ in $\mathcal{L}(L^2(\mathbb{R}^2))$. Therefore, for h is sufficiently small, $\mathcal{E}_1(z)$ is uniformly invertible for $z \in \Omega$, and

$$\mathcal{E}_1(z)^{-1} = \begin{pmatrix} a(z) & -a(z)[V^w(h), \Pi_1]R_1^- & \dots & -a(z)[V^w(h), \Pi_l]R_l^- \\ 0 & I_{L^2(\mathbb{R})} & \dots & 0 \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ 0 & 0 & \dots & I_{L^2(\mathbb{R})} \end{pmatrix},$$

where $a(z) = (I + [\Pi, V^w(h)]R(z))^{-1}$. Using the explicit expressions of $\widetilde{\mathcal{E}}(z)$ and $\mathcal{E}_1(z)^{-1}$, we get

$$\mathcal{E}(z) := \widetilde{\mathcal{E}}(z)\mathcal{E}_1(z)^{-1}$$

$$= \begin{pmatrix} R(z)a(z) & -R(z)a(z)[V^w(h),\Pi_1]R_1^- + R_1^- & \dots & -R(z)a(z)[V^w(h),\Pi_l]R_l^- + R_l^- \\ R_1^+a(z) & A_1 - R_1^+a(z)[V^w(h),\Pi_1]R_1^- & \dots & -R_1^+a(z)[V^w(h),\Pi_l]R_l^- \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ R_l^+a(z) & -R_l^+a(z)[V^w(h),\Pi_1]R_1^- & \dots & A_l - R_l^+a(z)[V^w(h),\Pi_l]R_l^- \end{pmatrix}.$$

Thus, we have proved the following theorem.

Theorem 5.3.6. *Let Ω be as in Lemma 5.3.5. Then $\mathcal{P}(z)$ is uniformly invertible for $z \in \Omega$ with inverse $\mathcal{E}(z)$. In addition, $\mathcal{E}(z)$ is holomorphic in $z \in \Omega$.*

From now on, we write $\mathcal{E}(z) = (B_{k,j})_{k,j=1}^{l+1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}$, where $E_{-+}(z) = (B_{k,j})_{k,j=2}^{l+1}$, $E(z) = R(z)a(z)$, $E_-(z) = \begin{pmatrix} R_1^+a(z) \\ . \\ . \\ . \\ R_l^+a(z) \end{pmatrix}$, and

$$E_+(z) = \left(-R(z)a(z)[V^w(h),\Pi_1]R_1^- + R_1^- \quad \dots \quad -R(z)a(z)[V^w(h),\Pi_l]R_l^- + R_l^-\right).$$

Remark 5.3.7. *The following formulas are consequences of the fact that $\mathcal{E}(z)$ is the inverse of $\mathcal{P}(z)$ as well as the fact that $R_j^\pm$ are independent of z (see [15, 31]) :*

$$(z - P(h))^{-1} = -E(z) + E_+(z)(E_{-+}(z))^{-1}E_-(z), \ z \in \rho(P(h)), \tag{5.3}$$

$$\partial_z E_{-+}(z) = E_-(z)E_+(z). \tag{5.4}$$

In what follows, the explicit formulae for $E(z)$ and $E_\pm(z)$ are not needed. We just indicate that they are holomorphic in z. In the remainder of this section, we will prove that the symbol of the operator $E_{-+}(z)$ is in $S^0(\mathbb{R}^2; M_l(\mathbb{C}))$, and has a complete asymptotic expansion in powers of h. Moreover, we will give explicitly the principal term.

Proposition 5.3.8. *For $1 \le k, j \le l$, the operators $R_j^+V^w(h)R_j^-$ and $R_k^+a(z)[V^w(h),\Pi_j]R_j^-$ are h^2-pseudo-differential operators with bounded symbols. Moreover, there exist $v_{j,n}, b_{k,j,n} \in S^0(\mathbb{R}^2)$, $n = 1, 2, ..,$ such that*

$$R_j^+V^w(h)R_j^- = \sum_{n=0}^{N} h^{2n} v_{j,n}^w(x, h^2D_x) + \mathcal{O}(h^{2(N+1)}), \tag{5.5}$$

$$R_k^+a(z)[V^w(h),\Pi_j]R_j^- = \sum_{n=1}^{N} b_{k,j,n}^w(x, h^2D_x, z)h^n + \mathcal{O}(h^{N+1}), \ \textit{for } k \neq j, \tag{5.6}$$

$$R_j^+a(z)[V^w(h),\Pi_j]R_j^- = \sum_{n=1}^{N} b_{j,j,2n}^w(x, h^2D_x, z)h^{2n} + \mathcal{O}(h^{2(N+1)}), \ \forall N \in \mathbb{N}, \tag{5.7}$$

Here

$$v_{j,0}(x,\xi) = V(x,\xi), \quad j = 1, ..., l.$$

Proof. The proofs of (5.5), (5.6) and (5.7) are quite similar, and are based on the Beal's characterization of h^2-pseudo-differential operators (see Proposition 5.3.2). We give only the main ideas of the proof of (5.5) and we refer to [15, 22, 31] for more details. Let Q denote the left hand side of (5.5). Let $l^w(x, h^2D_x)$ be as in Proposition 5.3.2. Using the fact that $R_j^{\pm}$ commutes with $l^w(x, h^2D_x)$ as well as the fact that $V^w(h)$ is an h^2-pseudo-differential operator on x, we deduce from Proposition 5.3.2 that $Q = q^w(x, h^2D_x; h)$, with $q \in S^0(\mathbb{R}^2)$. On the other hand, writing

$$V^w(h) = V^w(x, h^2D_x) + hD_y\left(\frac{\partial V}{\partial x}\right)^w(x, h^2D_x) + hy\left(\frac{\partial V}{\partial y}\right)^w(x, h^2D_x) + \cdots, \quad (5.8)$$

and using Proposition 5.3.2, we see that $q(x,\xi;h)$ has an asymptotic expansion in powers of h.

Notice that the odd powers of h in (5.5) and (5.7) disappear, due to the special properties of the eigenfunctions of the harmonic oscillator (i.e., $\int_{\mathbb{R}} y^{2j+1}|\phi_j(y)|^2 dy = \int_{\mathbb{R}} \phi_j(y)\, \partial_y^{2j+1}\phi_j(y) dy = 0$). Finally, since $R_j^+R_j^- = I_{L^2(\mathbb{R})}$, it follows from (5.8) that $v_{j,0}(x,\xi) = V(x,\xi)$.

□

Let $e_{-+}(x,\xi,z,h)$ denote the symbol of $E_{-+}(z)$. The following corollary follows from the above proposition and the definition of $E_{-+}(z)$.

Corollary 5.3.9. *We have*

$$e_{-+}(x,\xi,z,h) \sim \sum_{j=0}^{\infty} e_{-+}^j(x,\xi,z)h^j, \quad \text{in } S^0(\mathbb{R}^2; M_l(\mathbb{C})),$$

with

$$e_{-+}^0(x,\xi,z) = \Big((z-(2j+1)-V(x,\xi))\delta_{i,j}\Big)_{1\le i,j\le l}.$$

5.4 Proof of Theorem 5.2.1 and Theorem 5.2.3

5.4.1 Trace formulae

Let $f \in C_0^\infty((a,b);\mathbb{R})$, where $(a,b) \subset \mathbb{R}\setminus\sigma_{\text{ess}}(P(h))$, and let $\theta \in C_0^\infty(\mathbb{R};\mathbb{R})$. Set

$$\Sigma_j([a,b]) = \left\{(x,\xi)\in\mathbb{R}^2 |\ 2j+1+V(x,\xi)\in[a,b]\right\}, j = 1, ..., l$$

and

$$\Sigma_{[a,b]} = \bigcup_{j=1}^{l} \Sigma_j([a,b]). \quad (5.9)$$

Let $\widetilde{f} \in C_0^\infty((a,b)+i[-1,1])$ be an almost analytic extension of f, i.e., $\widetilde{f} = f$ on $\mathbb{R}$ and $\overline{\partial}_z \widetilde{f}$ vanishes on $\mathbb{R}$ to infinite order, i.e. $\overline{\partial}_z \widetilde{f}(z) = \mathcal{O}_N(|\mathrm{Im}\, z|^N)$ for all $N \in \mathbb{N}$. Then the functional calculus due to Helffer-Sjöstrand (see e.g. [22, Chapter 8]) yields

$$f(P(h)) = -\frac{1}{\pi}\int \overline{\partial}_z \widetilde{f}(z)(z-P(h))^{-1}L(dz), \tag{5.10}$$

$$f(P(h))\breve{\theta}_{h^2}(t-P(h)) = -\frac{1}{\pi}\int \overline{\partial}_z \widetilde{f}(z)\breve{\theta}_{h^2}(t-z)(z-P(h))^{-1}L(dz). \tag{5.11}$$

Here $L(dz) = dxdy$ is the Lebesgue measure on the complex plane $\mathbb{C} \sim \mathbb{R}^2_{x,y}$. In the last equality we have used the fact that $\widetilde{f}(z)\breve{\theta}_{h^2}(t-z)$ is an almost analytic extension of $f(x)\breve{\theta}_{h^2}(t-x)$, since $z \mapsto \breve{\theta}_{h^2}(t-z)$ is analytic.

Proposition 5.4.1. *For h small enough, we have*

$$\mathrm{tr}(f(P(h))) = \mathrm{tr}\left(-\frac{1}{\pi}\int \overline{\partial}_z \widetilde{f}(z)(E_{-+}(z))^{-1}\partial_z E_{-+}(z)L(dz)\chi^w(x,h^2D_x)\right) + \mathcal{O}(h^\infty), \tag{5.12}$$

$$\mathrm{tr}\left(f(P(h))\breve{\theta}_{h^2}(t-P(h))\right) = \tag{5.13}$$

$$\mathrm{tr}\left(-\frac{1}{\pi}\int \overline{\partial}_z \widetilde{f}(z)\breve{\theta}_{h^2}(t-z)(E_{-+}(z))^{-1}\partial_z E_{-+}(z)L(dz)\chi^w(x,h^2D_x)\right) + \mathcal{O}(h^\infty),$$

where $\chi \in C_0^\infty(\mathbb{R}^2;\mathbb{R})$ is equal to one in a neighbourhood of $\Sigma_{[a,b]}$.

Proof. Replacing $(z-P(h))^{-1}$ in (5.10) by the right hand side of (5.3), and using the fact that $E(z)$ is holomorphic in z, we obtain

$$f(P(h)) = -\frac{1}{\pi}\int \overline{\partial}_z \widetilde{f}(z)E_+(z)(E_{-+}(z))^{-1}E_-(z)L(dz). \tag{5.14}$$

Let $\widetilde{V} \in S^0(\mathbb{R}^2)$ be a real-valued function coinciding with V for large (x,y), and having the property that

$$|z-(2j+1)-\widetilde{V}(x,y)| > c > 0, \;\; j = 1,2,...,l, \tag{5.15}$$

uniformly in $z \in \mathrm{supp}\, \widetilde{f}$, and $(x,y) \in \mathbb{R}^2$. We recall that for $z \in \mathrm{supp}\, \widetilde{f}$, $\mathrm{Re} z \in (a,b) \subset \mathbb{R} \setminus \sigma_{\mathrm{ess}}(H(h)) = \mathbb{R} \setminus \cup_{k=0}^{\infty}\{(2k+1)\}$. Then (5.15) holds for $\widetilde{V} \in S^0(\mathbb{R}^2)$ with $\|\widetilde{V}\|$ small enough.

Set $\widetilde{E}_{-+}(z) := E_{-+}(z) + \left(V^w(x,h^2D_x) - \widetilde{V}^w(x,h^2D_x)\right) I_l$, and let $\widetilde{e}(x,\xi,z)$ be the principal symbol of $\widetilde{E}_{-+}(z)$. Here I_l denotes the unit matrix of order l. It follows from (5.15) that $|\det e(x,\xi,z)| > c^l$. Then for sufficiently small $h>0$, the operator $\widetilde{E}_{-+}(z)$ is elliptic, and $\widetilde{E}_{-+}(z)^{-1}$ is well defined and holomorphic for z in some fixed complex neighbourhood of supp$\widetilde{f}$, (see chapter 7 of [22]). Hence, by an integration by parts, we get

$$-\frac{1}{\pi}\int \overline{\partial}_z \widetilde{f}(z)E_+(z)\widetilde{E}_{-+}(z)^{-1}E_-(z)L(dz) = 0.$$

Combining this with (5.14) and using the resolvent identity for $\mathrm{Im}\, z \neq 0$

$$E_{-+}(z)^{-1} = \widetilde{E}_{-+}(z)^{-1} + E_{-+}(z)^{-1}(\widetilde{E}_{-+}(z) - E_{-+}(z))\widetilde{E}_{-+}(z)^{-1},$$

we obtain

$$\mathrm{tr}\,(f(P(h))) = -\frac{1}{\pi}\mathrm{tr}\left(\int \overline{\partial}_z\widetilde{f}(z)E_+(z)E_{-+}(z)^{-1}(\widetilde{E}_{-+}(z) - E_{-+}(z))\widetilde{E}_{-+}(z)^{-1}E_-(z)L(dz)\right). \tag{5.16}$$

Since the symbol of $E_{-+}(z) - \widetilde{E}_{-+}(z)$ is $(\widetilde{V} - V)I_l$ belonging to $C_0^\infty(\mathbb{R}^2; M_l(\mathbb{C}))$, we have $E_{-+}(z) - \widetilde{E}_{-+}(z)$ is a trace class operator. It is then clear that we can permute integration and the operator "tr" in the right hand side of (5.16).

Using the property of cyclic invariance of the trace, and applying (5.4) we get

$$\mathrm{tr}\left(E_+(z)E_{-+}(z)^{-1}(\widetilde{E}_{-+}(z) - E_{-+}(z))\widetilde{E}_{-+}(z)^{-1}E_-(z)\right) =$$

$$\mathrm{tr}\left(E_{-+}(z)^{-1}(\widetilde{E}_{-+}(z) - E_{-+}(z))\widetilde{E}_{-+}(z)^{-1}\partial_z E_{-+}(z)\right).$$

Let $\chi \in C_0^\infty(\mathbb{R}^2)$ be equal to 1 in a neighbourhood of supp $(\widetilde{V} - V)$. From the composition formula for two $h^2 - \Psi$DOs with Weyl symbols (see Proposition 5.3.1), we see that all the derivatives of the symbol of the operator $(E_{-+}(z) - \widetilde{E}_{-+}(z))(\widetilde{E}_{-+}(z))^{-1}\partial_z E_{-+}(z)(1 - \chi^w(x, h^2D_x))$ are $\mathcal{O}(h^{2N}\langle(x,\xi)\rangle^{-N})$ for every $N \in \mathbb{N}$. The trace class-norm of this expression is therefore $\mathcal{O}(h^\infty)$, and consequently

$$\mathrm{tr}(E_+(z)E_{-+}(z)^{-1}(\widetilde{E}_{-+}(z) - E_{-+}(z))\widetilde{E}_{-+}(z)^{-1}E_-(z)) = \tag{5.17}$$

$$\mathrm{tr}(E_{-+}(z)^{-1}(\widetilde{E}_{-+}(z) - E_{-+}(z))\widetilde{E}_{-+}(z)^{-1}\partial_z E_{-+}(z)\chi^w(x, h^2D_x)) + \mathcal{O}(h^\infty|\mathrm{Im}z|^{-1}).$$

Here we recall from (5.3) that $E_{-+}(z)^{-1} = \mathcal{O}(|\mathrm{Im}z|^{-1})$.

Inserting (5.17) into (5.16), and using the fact that $\widetilde{E}_{-+}(z)^{-1}\partial_z E_{-+}(z)$ is holomorphic in z we obtain (5.12). The proof of (5.13) is similar.

□

Trace formulas involving effective Hamiltonian like (5.12) and (5.13) were studied in [14, 15]. Applying Theorem 1.8 in [14] to the left hand side of (5.12), we obtain

$$\mathrm{tr}(f(P(h))) \sim \sum_{j=0}^{\infty} \beta_j h^{j-2}, \ (h \searrow 0). \tag{5.18}$$

To use Theorem 1.8 in [14] we make the following definition.

Definition 5.4.2. We say that $p(x,\xi) \in S^0(\mathbb{R}^2; M_l(\mathbb{C}))$, is micro-hyperbolic at (x_0, ξ_0) in the direction $T \in \mathbb{R}^2$, if there are constants $C_0, C_1, C_2 > 0$ such that

$$(\langle dp(x,\xi), T\rangle\omega, \omega) \geq \frac{1}{C_0}\|\omega\|^2 - C_1\|p(x,\xi)\omega\|^2.$$

for all $(x,\xi) \in \mathbb{R}^2$ with $\|(x,\xi) - (x_0,\xi_0)\| \leq \frac{1}{C_2}$ and all $\omega \in \mathbb{C}^l$.

The assumption of Theorem 5.2.2 implies that the principal symbol $e^0_{-+}(x,\xi,z)$ of $E_{-+}(z)$ is micro-hyperbolic at every point $(x_0,\xi_0)\in\Sigma_\mu:=\{(x,\xi)\in\mathbb{R}^2;\ \det(e^0_{-+}(x,\xi,\mu))=0\}$. Thus, according to Theorem 1.8 in [14] there exists $C>0$ large enough and $\epsilon>0$ small such that for $f\in C_0^\infty(]\mu-\epsilon,\mu+\epsilon[;\mathbb{R})$, $\theta\in C_0^\infty(]-\frac{1}{C},\frac{1}{C}[;\mathbb{R})$, we have :

$$\operatorname{tr}\Big(f(P(h))\breve{\theta}_{h^2}(t-P(h))\Big)\sim\sum_{j=0}^{\infty}\gamma_j(t)h^{j-2},\ (h\searrow 0), \tag{5.19}$$

with $\gamma_0(t)=c_0(t)$.

By observing that the h-pseudo-differential calculus can be extended to $h<0$, we have

$$\left|h^2\operatorname{tr}\Big(f(P(h))\breve{\theta}_{h^2}(t-P(h))\Big)-\sum_{0\leq j\leq N}\gamma_j(t)h^j\right|\leq C_N|h|^{N+1},\ \ h\in]-h_N,h_N[\backslash\{0\}.$$

$$\left|h^2\operatorname{tr}(f(P(h))-\sum_{0\leq j\leq N}\beta_jh^j\right|\leq C_N|h|^{N+1},\ \ h\in]-h_N,h_N[\backslash\{0\}.$$

By the change of variable $(x,y)\to(x,-y)$, we see that $P(h)$ is unitarily equivalent to $P(-h)$. From this we deduce that $h^2\operatorname{tr}(f(P(h)))$ and $h^2\operatorname{tr}\Big(f(P(h))\breve{\theta}_{h^2}(t-P(h))\Big)$ are unchanged when we replace h by $-h$. We recall that if A and B are unitarily equivalent trace class operators then $\operatorname{tr}(A)=\operatorname{tr}(B)$. Consequently, $\gamma_{2j+1}=\beta_{2j+1}=0$. This ends the proof of Theorem 5.2.1 and Theorem 5.2.2.

5.4.2 Proof of Corollary 5.2.3.

Pick $\sigma>0$ small enough. Let $\phi_1\in C_0^\infty((a-\sigma,a+\sigma);[0,1])$, $\phi_2\in C_0^\infty(\big(a+\frac{\sigma}{2},b-\frac{\sigma}{2}\big);[0,1])$, $\phi_3\in C_0^\infty((b-\sigma,b+\sigma);[0,1])$ satisfy $\phi_1+\phi_2+\phi_3=1$ on $\big(a-\frac{\sigma}{2},b+\frac{\sigma}{2}\big)$. Let $\gamma_0(h)\leq\gamma_1(h)\leq\cdots\leq\gamma_N(h)$ be the eigenvalues of $H(h)$ counted with their multiplicity and lying in the interval $(a-\sigma,b+\sigma)$. We have

$$\begin{aligned}\mathcal{N}_h(a,b)&=\sum_{a\leq\gamma_j(h)\leq b}(\phi_1+\phi_2+\phi_3)(\gamma_j(h))\\&=\sum_{a\leq\gamma_j(h)}\phi_1(\gamma_j(h))+\sum\phi_2(\gamma_j(h))+\sum_{\gamma_j(h)\leq\lambda_2}\phi_3(\gamma_j(h))\\&=\sum_{a\leq\gamma_j(h)}\phi_1(\gamma_j(h))+\operatorname{tr}(\phi_2(H(h)))+\sum_{\gamma_j(h)\leq b}\phi_3(\gamma_j(h)).\end{aligned}\tag{5.20}$$

According to Theorem 5.2.1, we have

$$\operatorname{tr}\left(\phi_m(H(h))\right)=\frac{1}{2\pi h^2}\sum_{j=1}^{l}\int_{\mathbb{R}^2}\phi_m((2j+1)+V(X))dX+\mathcal{O}(1),\quad m=1,2,3. \tag{5.21}$$

Set $M(\tau,h) := \sum_{\gamma_j(h)\le\tau} \phi_3(\gamma_j(h))$. Evidently, in the sense of distribution, we have

$$\mathcal{M}(\tau) := M'(\tau,h) = \sum_j \delta(\tau-\gamma_j(h))\phi_3(\gamma_j(h)). \tag{5.22}$$

In what follows, we choose $\theta \in C_0^\infty((-\frac{1}{C},\frac{1}{C});[0,1])$, ($C>0$ large enough) such that $\theta(0)=1,\ \breve{\theta}(t)\ge 0, t\in\mathbb{R},\ \breve{\theta}(t)\ge\epsilon_0, t\in[-\delta_0,\delta_0]$ for some $\delta_0>0,\epsilon_0>0$.

Corollary 5.4.3. *There is $C_0>0$, such that, for all $(\lambda,h)\in\mathbb{R}\times(0,h_0)$, we have :*

$$|M(\lambda+\delta_0h^2,h)-M(\lambda-\delta_0h^2,h)|\le C_0.$$

Proof. Since $\phi_3\ge 0$, it follows from the construction of θ that

$$\frac{\epsilon_0}{h^2}\sum_{\lambda-\delta_0h^2\le\gamma_j(h)\le\lambda+\delta_0h^2}\phi_3(\gamma_j(h)) \le \sum_{|\lambda-\gamma_j(h)|<\delta_0h^2}\breve{\theta}_{h^2}(\lambda-\gamma_j(h))\phi_3(\gamma_j(h)) \le$$

$$\sum_j \breve{\theta}_{h^2}(\lambda-\gamma_j(h))\phi_3(\gamma_j(h)) = \breve{\theta}_{h^2}\star\mathcal{M}(\lambda) = \operatorname{tr}\Big(\phi_3(H(h))\breve{\theta}_{h^2}(\lambda-H(h))\Big).$$

Now Corollary 5.4.3 follows from (5.4). □

According to Corollary 5.4.3, we have

$$\int\left\langle\frac{\tau-\lambda}{h^2}\right\rangle^{-2}\mathcal{M}(\tau)d\tau = \sum_{k\in\mathbb{Z}}\int_{\{\delta_0k\le\frac{\tau-\lambda}{h^2}\le\delta_0(k+1)\}}\left\langle\frac{\tau-\lambda}{h^2}\right\rangle^{-2}\mathcal{M}(\tau)d\tau \le C_0\left(\sum_{k\in\mathbb{Z}}\langle\delta_0k\rangle^{-2}\right). \tag{5.23}$$

On the other hand, since $\breve{\theta}\in\mathcal{S}(\mathbb{R})$ and $\theta(0)=1$, there exists $C_1>0$ such that :

$$\left|\int_{-\infty}^{\lambda}\breve{\theta}_{h^2}(\tau-y)dy - 1_{(-\infty,\lambda)}(\tau)\right| = \left|\int_{\frac{\tau-\lambda}{h^2}}^{+\infty}\breve{\theta}(y)dy - 1_{(-\infty,\lambda)}(\tau)\right| \le C_1\left\langle\frac{\tau-\lambda}{h^2}\right\rangle^{-2},$$

uniformly in $\tau\in\mathbb{R}$ and $h\in(0,h_0)$. Consequently,

$$\left|\int_{-\infty}^{\lambda}\breve{\theta}_{h^2}\star\mathcal{M}(\tau)d\tau - \int_{-\infty}^{\lambda}\mathcal{M}(\tau)d\tau\right| \le C_1\int\left\langle\frac{\tau-\lambda}{h^2}\right\rangle^{-2}\mathcal{M}(\tau)d\tau. \tag{5.24}$$

Putting together (5.22), (5.23) and (5.24), we get

$$\int_{-\infty}^{\lambda}\breve{\theta}_{h^2}\star\mathcal{M}(\tau)d\tau = M(\lambda,h)+\mathcal{O}(1). \tag{5.25}$$

Note that $\breve{\theta}_{h^2}\star\mathcal{M}(\tau) = \operatorname{tr}\Big(\phi_3(H(h))\breve{\theta}_{h^2}(\tau-H(h))\Big)$. As a consequence of (5.4), (5.5) and (5.25) we obtain

$$M(\lambda,h) = h^{-2}m(\lambda)+\mathcal{O}(1), \tag{5.26}$$

where

$$m(\lambda) = \int_{-\infty}^{\lambda} c_0(\tau)d\tau = \frac{1}{2\pi}\sum_{j=1}^{l}\int_{\{X\in\mathbb{R}^2|(2j+1)+V(X)\leq\lambda\}} \phi_3((2j+1)+V(X))dX. \quad (5.27)$$

Here we have used the fact that if E is not a critical value of $V(X)$, then

$$\frac{\partial}{\partial E}\left(\int_{\{X\in\mathbb{R}^2|V(X)\leq E\}} \phi(V(X))dX\right) = \phi(E)\int_{S_E}\frac{dS_E}{|\nabla_X V|},$$

where $S_E = V^{-1}(E)$ (see [56, Lemma V-9]).
Applying (5.2), (5.26) and (5.27) to ϕ_1 and writing :

$$\sum_{a\leq\gamma_j(h)} \phi_1(\gamma_j(h)) = \sum_j \phi_1(\gamma_j(h)) - \sum_{\gamma_j(h)<a} \phi_1(\gamma_j(h)),$$

we get

$$\sum_{a\leq\gamma_j(h)} \phi_1(\mu_j(h)) = h^{-2}m_1(a) + \mathcal{O}(1), \quad (5.28)$$

with

$$m_1(a) = \frac{1}{2\pi}\sum_{j=1}^{l}\int_{\{X\in\mathbb{R}^2|(2j+1)+V(X)\geq a\}} \phi_1((2j+1)+V(X))dX. \quad (5.29)$$

Now Corollary 5.2.3 results from (5.21), (5.22), (5.26), (5.27), (5.28) and (5.29).

5.5 Proof of Theorem 5.2.4 and Theorem 5.2.6

As we have noticed in the outline of the proofs, we will construct a potential

$$\varphi(X;h) = \varphi_0(X) + \varphi_2(X)h^2 + \cdots + \varphi_{2j}(X)h^{2j} + \cdots,$$

such that for all $f \in C_0^\infty((a,b);\mathbb{R})$ and $\theta \in C_0^\infty(\mathbb{R};\mathbb{R})$, we have

$$\mathrm{tr}(f(H_\lambda)) = \mathrm{tr}(f(Q)) + \mathcal{O}(h^\infty), \quad (5.30)$$

$$\mathrm{tr}\left(f(H_\lambda)\breve{\theta}_{\lambda^{-\frac{2}{\delta}}}(t-H_\lambda)\right) = \mathrm{tr}\left(f(Q)\breve{\theta}_{h^2}(t-Q)\right) + \mathcal{O}(h^\infty), \quad (5.31)$$

where $Q := H_0 + \varphi(hX;h)$ and $h = \lambda^{-\frac{1}{\delta}}$. By observing that Theorem 5.2.1, Theorem 5.2.2 and Corollary 5.2.3 remain true when we replace $H(h) = H_0 + V(hX)$ by Q, Theorem 5.2.4, Theorem 5.2.5 and Corollary 5.2.6 follow from (5.30) and (5.31). The remainder of this paper is devoted to the proof of (5.30) and (5.31).

5.5.1 Construction of reference operator Q

Set $h = \lambda^{-\frac{1}{\delta}}$. For $M > 0$, put

$$\Omega_M(h) = \{X \in \mathbb{R}^2 |\ h^{-\delta} V(X) > M\}. \tag{5.32}$$

Since $\omega_0 > 0$ and continuous on the unit circle, there exist two positive constants C_1 and C_2 such that $C_1 < (\min_{\mathbb{S}^1} \omega_0)^{1/\delta} \leq (\max_{\mathbb{S}^1} \omega_0)^{1/\delta} < C_2$.

According to the hypothesis (5.9), there exists $h_0 > 0$ such that

$$B(0, C_1 M^{-1/\delta} h^{-1}) \subset \Omega_M(h) \subset B(0, C_2 M^{-1/\delta} h^{-1}), \quad \text{for all } 0 < h \leq h_0.$$

Here $B(0, r)$ denotes the ball of center 0 and radius r.

Let $\chi \in C_0^\infty(B(0, C_1 M^{-1/\delta}); [0, 1])$ satisfying $\chi = 1$ near zero. Set

– $\varphi(X; h) := (1 - \chi(X)) h^{-\delta} V(\frac{X}{h}) + M\chi(X)$,

– $W_h(X) := h^{-\delta} V(X) - \varphi(hX; h) = \chi(hX)(h^{-\delta} V(X) - M)$.

By the construction of $\varphi(\cdot; h)$ and W_h, we have

$$|\partial_X^\alpha \varphi(X; h)| \leq C_\alpha, \text{ uniformly for } h \in (0, h_0], \tag{5.33}$$

$$\varphi(hX; h) > \frac{M}{2} \text{ for } X \in \Omega_{\frac{M}{2}}(h), \tag{5.34}$$

$$\text{supp} W_h \subset B(0, C_1 M^{-1/\delta} h^{-1}) \subset \Omega_M(h). \tag{5.35}$$

On the other hand, it follows from (5.9) that for all $N \in \mathbb{N}$, there exist $\varphi_0, ..., \varphi_{2N}, K_{2N+2}(\cdot; h) \in C^\infty(\mathbb{R}^2; \mathbb{R})$, uniformly bounded with respect to $h \in (0, h_0]$ together with their derivatives such that :

$$\varphi(X; h) = \sum_{j=0}^{N} \varphi_{2j}(X) h^{2j} + h^{2N+2} K_{2N+2}(X; h) \tag{5.36}$$

with

$$\varphi_0(X) = (1 - \chi(X)) \omega_0 \left(\frac{X}{|X|} \right) |X|^{-\delta} + M\chi(X).$$

In fact, if $X \in \text{supp}\chi$ then $\omega_0 \left(\frac{X}{|X|} \right) |X|^{-\delta} > C_1^\delta |X|^{-\delta} > M$, which implies that $\varphi_0(X) \geq (1 - \chi(X))M + M\chi(X) = M$ for all $X \in \text{supp}\chi$. Consequently, we have

Lemma 5.5.1. *If $\varphi_0(X) < M$ then $\varphi_0(X) = \omega_0 \left(\frac{X}{|X|} \right) |X|^{-\delta}$.*

Let $\psi \in C^\infty(\mathbb{R}; [\frac{M}{3}, +\infty))$ satisfying $\psi(t) = t$ for all $t \geq \frac{M}{2}$. We define

$$F_1(X; h) := \psi(\varphi(hX; h)) \text{ and } F_2(X; h) := \psi(h^{-\delta} V(X)).$$

Let $\mathcal{U}$ be a small complex neighborhood of $[a, b]$. From now on, we choose $M > a + b$ large enough such that

$$F_j(X; h) - \text{Re} z \geq \frac{M}{4}, \ j = 1, 2, \tag{5.37}$$

uniformly for $z \in \mathcal{U}$. This choice of M implies that :

- If $2j+1+\varphi(X;h) \in [a,b]$ then $\varphi_0(X) < M$ for all $h \in (0, h_0]$,
- The function defined by $z \mapsto (z - H_{F_j})^{-1}$ is holomorphic from $\mathcal{U}$ to $\mathcal{L}(L^2(\mathbb{R}^2))$, where $H_{F_j} := H_0 + F_j(X;h),\ j = 1,2$.

Moreover, it follows from (5.33) that $\partial_X^\alpha F_j(X;h) = \mathcal{O}_\alpha(h^{-\delta})$.

Finally, (5.34) shows that

$$\begin{aligned} &\operatorname{dist}\left(\operatorname{supp} W_h, \operatorname{supp}[\varphi(h\cdot;h) - F_1(\cdot;h)]\right) \geq \frac{a_1(M)}{h}, \\ &\operatorname{dist}\left(\operatorname{supp} W_h, \operatorname{supp}[h^{-\delta}V(\cdot) - F_2(\cdot;h)]\right) \geq \frac{a_2(M)}{h}, \end{aligned} \tag{5.38}$$

with $a_1(M), a_2(M) > 0$ independent of h.

Lemma 5.5.2. *Let $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^2)$. For $z \in \mathcal{U}$, the operators $\widetilde{\chi}(hX)(z - H_{F_j})^{-1}$, $j = 1,2$, belong to the class of Hilbert-Schmidt operators. Moreover*

$$\|\widetilde{\chi}(hX)(z - H_{F_j})^{-1}\|_{\mathrm{HS}} = \mathcal{O}(h^{-3-\delta}). \tag{5.39}$$

Here we denote by $\|.\|_{\mathrm{HS}}$ the Hilbert-Schmidt norm of operators.

Proof. We prove (5.39) for $j = 1$. The case $j = 2$ is treated in the same way.

Using the resolvent equation, one has

$$(z - H_{F_1})^{-1} = \left(z - \frac{M}{6} - H_0\right)^{-1} + \left(z - \frac{M}{6} - H_0\right)^{-1}\left(F_1(X;h) - \frac{M}{6}\right)(z - H_{F_1})^{-1}. \tag{5.40}$$

On the other hand, the operator $\left(z - \frac{M}{6} - H_0\right)^{-1}$ was shown to be an integral operator with integral kernel $K_0(X,Y,z)$ satisfying $|K_0(X,Y,z)| \leq Ce^{-\frac{1}{8}|X-Y|^2}$ uniformly for $z \in \mathcal{U}$ (see [9, Formula 2.17]). Let $K_1(X,Y,z)$ be the integral kernel of $\widetilde{\chi}(hX)(z - \frac{M}{6} - H_0)^{-1}$. Then $K_1(X,Y,z) = \widetilde{\chi}(hX)K_0(X,Y,z)$.

Let $\langle X\rangle = (1+|X|^2)^{\frac{1}{2}}$, $X \in \mathbb{R}^2$. Since $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^2)$, one has $\widetilde{\chi}(hX)h^3\langle X\rangle^3$ is uniformly bounded for $h > 0$. Combining this with the fact that $\frac{1}{\langle X\rangle^3}e^{-\frac{1}{8}|X-Y|^2} \in L^2(\mathbb{R}^4)$, we obtain

$$\|K_1(X,Y,z)\|_{L^2(\mathbb{R}^4)} = \left\|\widetilde{\chi}(hX)h^3\langle X\rangle^3 \frac{1}{h^3\langle X\rangle^3}K_0(X,Y,z)\right\|_{L^2(\mathbb{R}^4)} = \mathcal{O}(h^{-3}). \tag{5.41}$$

It shows that $\widetilde{\chi}(hX)(z - \frac{M}{6} - H_0)^{-1}$ is a Hilbert-Schmidt operator and

$$\left\|\widetilde{\chi}(hX)\left(z - \frac{M}{6} - H_0\right)^{-1}\right\|_{\mathrm{HS}} = \|K_1(X,Y,z)\|_{L^2(\mathbb{R}^4)} = \mathcal{O}(h^{-3}). \tag{5.42}$$

Consequently, (5.40) and (5.42) imply that

$$\begin{aligned} \|\widetilde{\chi}(hX)(z - H_{F_1})^{-1}\|_{\mathrm{HS}} &\leq \left\|\widetilde{\chi}(hX)\left(z - \frac{M}{6} - H_0\right)^{-1}\right\|_{\mathrm{HS}} \\ &+ \left\|\widetilde{\chi}(hX)\left(z - \frac{M}{6} - H_0\right)^{-1}\right\|_{\mathrm{HS}} \left\|F_1(X;h) - \frac{M}{6}\right\|_{L^\infty(\mathbb{R}^2)} \|(z - H_{F_1})^{-1}\| \\ &= \mathcal{O}(h^{-3-\delta}), \end{aligned}$$

where we have used $F_1(X;h) = \mathcal{O}(h^{-\delta})$. □

Lemma 5.5.3. *For $z \in \mathcal{U}$, the operator*

$$W_h(X)(H_{F_1} - z)^{-1}(\varphi(hX;h) - F_1(X;h))$$

belongs to the class of Hilbert-Schmidt operators. Moreover,

$$\|W_h(X)(H_{F_1} - z)^{-1}(\varphi(hX;h) - F_1(X;h))\|_{\mathrm{HS}} = \mathcal{O}(h^\infty). \tag{5.43}$$

Proof. Let $H^0_{F_1} := -\Delta + F_1(X;h)$. We denote by $G(X,Y;z)$ (resp. $G_0(X,Y;\mathrm{Re}z)$) the Green function of $(H_{F_1} - z)^{-1}$ (resp. $(H^0_{F_1} - \mathrm{Re}z)^{-1}$).

From the functional calculus, one has

$$\begin{aligned} (H_{F_1} - z)^{-1} &= \int_0^\infty e^{tz} e^{-tH_{F_1}}\, dt, \\ (H^0_{F_1} - \mathrm{Re}z)^{-1} &= \int_0^\infty e^{t\mathrm{Re}z} e^{-tH^0_{F_1}}\, dt. \end{aligned} \tag{5.44}$$

For $t \geq 0$, the Kato inequality (see [11, Formula 1.8]) implies that

$$|e^{-tH_{F_1}}u| \leq e^{-tH^0_{F_1}}|u| \text{ (pointwise)}, \ u \in L^2(\mathbb{R}^2). \tag{5.45}$$

Then (5.44) and (5.45) yield

$$|(H_{F_1} - z)^{-1}u| \leq (H^0_{F_1} - \mathrm{Re}z)^{-1}|u| \text{ (pointwise)}, \ u \in L^2(\mathbb{R}^2). \tag{5.46}$$

Consequently, applying [8, Theorem 10] we have $|G(X,Y;z)| \leq G_0(X,Y;\mathrm{Re}z)$ for a.e. $X, Y \in \mathbb{R}^2$. From this, one obtains

$$|W_h(X)G(X,Y;z)(\varphi(hY;h) - F_1(Y;h))| \leq |W_h(X)G_0(X,Y;\mathrm{Re}z)(\varphi(hY;h) - F_1(Y;h))| \tag{5.47}$$

for a.e. $X, Y \in \mathbb{R}^2$.

On the other hand, using (5.38) M. Dimassi proved that (see [13, Proposition 3.3])

$$\|W_h(X)G_0(X,Y;\mathrm{Re}z)(\varphi(hY;h) - F_1(Y;h))\|_{L^4(\mathbb{R}^4)} = \mathcal{O}(h^\infty). \tag{5.48}$$

Thus, (5.47) and (5.48) give

$$\|W_h(X)G(X,Y;z)(\varphi(hY;h) - F_1(Y;h))\|_{L^4(\mathbb{R}^4)} = \mathcal{O}(h^\infty). \tag{5.49}$$

The estimate (5.49) shows that the operator $W_h(X)(H_{F_1} - z)^{-1}(\varphi(hX;h) - F_1(X;h))$ is Hilbert-Schmidt and

$$\|W_h(X)(H_{F_1} - z)^{-1}(\varphi(hX;h) - F_1(X;h))\|_{\mathrm{HS}} = \mathcal{O}(h^\infty). \tag{5.50}$$

□

By using the same arguments as in Lemma 5.5.3, we also obtain

Lemma 5.5.4. *For $z \in \mathcal{U}$, the operator*

$$W_h(X)(H_{F_2} - z)^{-1}(h^{-\delta}V(X) - F_2(X;h))$$

belongs to the class of Hilbert-Schmidt operators and

$$\|W_h(X)(H_{F_2} - z)^{-1}(h^{-\delta}V(X) - F_2(X;h))\|_{\mathrm{HS}} = \mathcal{O}(h^\infty).$$

Let $Q := H_0 + \varphi(hX;h)$. For $z \in \mathcal{U}$, $\mathrm{Im} z \neq 0$, put

$$G(z) = (z - H_\lambda)^{-1} - (z - Q)^{-1} - (z - H_{F_2})^{-1}W_h(z - H_{F_1})^{-1}. \tag{5.51}$$

Proposition 5.5.5. *The operator $G(z)$ is of trace class and satisfies the following estimate :*

$$\|G(z)\|_{\mathrm{tr}} = \mathcal{O}(h^\infty|\mathrm{Im} z|^{-2}), \tag{5.52}$$

uniformly for $z \in \mathcal{U}$ with $\mathrm{Im} z \neq 0$.

Proof. It follows from the resolvent equation that

$$(z - H_\lambda)^{-1} - (z - Q)^{-1} = (z - H_\lambda)^{-1}W_h(z - Q)^{-1}. \tag{5.53}$$

On the other hand, one has

$$\begin{aligned}(z - H_\lambda)^{-1} &= (z - H_{F_2})^{-1} \\ &+ (z - H_\lambda)^{-1}(h^{-\delta}V(X) - F_2(X;h))(z - H_{F_2})^{-1}\end{aligned} \tag{5.54}$$

and

$$\begin{aligned}(z - Q)^{-1} &= (z - H_{F_1})^{-1} \\ &+ (z - H_{F_1})^{-1}(\varphi(hX;h) - F_1(X;h))(z - Q)^{-1}.\end{aligned} \tag{5.55}$$

Substituting (5.54) and (5.55) into the right hand side of (5.53), one gets

$$\begin{aligned}G(z) &= (z - H_{F_2})^{-1}W_h(z - H_{F_1})^{-1}(\varphi(hX;h) - F_1(X;h))(z - Q)^{-1} \\ &+ (z - H_\lambda)^{-1}(h^{-\delta}V(X) - F_2(X;h))(z - H_{F_2})^{-1}W_h(z - H_{F_1})^{-1} \\ &+ (z - H_\lambda)^{-1}(h^{-\delta}V(X) - F_2(X;h))(z - H_{F_2})^{-1}W_h(z - H_{F_1})^{-1}\times \\ &\times (\varphi(hX;h) - F_1(X;h))(z - Q)^{-1} =: A(z) + B(z) + C(z).\end{aligned}$$

Next we choose $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^2)$ such that $\widetilde{\chi}(hX)W_h(X) = W_h(X)$. It follows from Lemma 5.5.2 and Lemma 5.5.3 that

$$\begin{aligned}&\|A(z)\|_{\mathrm{tr}} \leq \|(z - H_{F_2})^{-1}\widetilde{\chi}(hX)\|_{\mathrm{HS}}\|W_h(z - H_{F_1})^{-1}(\varphi(hX,h) - F_1(X;h))\|_{\mathrm{HS}}\|(z - Q)^{-1}\| \\ &= \mathcal{O}(h^\infty|\mathrm{Im} z|^{-1}).\end{aligned}$$

Here we have used the fact that $\|(z - Q)^{-1}\| = \mathcal{O}(|\mathrm{Im} z|^{-1})$. Similarly, we also obtain $\|B(z)\|_{\mathrm{tr}} = \mathcal{O}(h^\infty|\mathrm{Im} z|^{-1})$ and $\|C(z)\|_{\mathrm{tr}} = \mathcal{O}(h^\infty|\mathrm{Im} z|^{-2})$. Thus,

$$\|G(z)\|_{\mathrm{tr}} \leq \|A(z)\|_{\mathrm{tr}} + \|B(z)\|_{\mathrm{tr}} + \|C(z)\|_{\mathrm{tr}} = \mathcal{O}(h^\infty|\mathrm{Im} z|^{-2}).$$

□

5.5.2 Proof of (5.30) and Theorem 5.2.4

Let $f \in C_0^\infty((a,b);\mathbb{R})$ and let $\widetilde{f} \in C_0^\infty(\mathcal{U})$ be an almost analytic extension of f. From the Helffer- Sjötrand formula and (5.51), we get

$$\begin{aligned} &f(H_\lambda) - f(Q) \\ &= -\frac{1}{\pi}\int \overline{\partial}_z \widetilde{f}(z)[(z-H_\lambda)^{-1} - (z-Q)^{-1}]L(dz) \\ &= -\frac{1}{\pi}\int \overline{\partial}_z \widetilde{f}(z)\left[(z-H_{F_2})^{-1}W_h(z-H_{F_1})^{-1} + G(z)\right]L(dz). \end{aligned} \tag{5.56}$$

Notice that $(z-H_{F_2})^{-1}W_h(z-H_{F_1})^{-1}$ is holomorphic in $z \in \mathcal{U}$, then

$$-\frac{1}{\pi}\int \overline{\partial}_z \widetilde{f}(z)(z-H_{F_2})^{-1}W_h(z-H_{F_1})^{-1}L(dz) = 0. \tag{5.57}$$

Thus, (5.56) and (5.57) follow that

$$f(H_\lambda) - f(Q) = -\frac{1}{\pi}\int \overline{\partial}_z \widetilde{f}(z)G(z)L(dz), \tag{5.58}$$

which together with (5.52) yields (5.30).

Applying Theorem 5.2.1 to the operator Q and using (5.30) we obtain (5.10) with

$$b_0(f) = \sum_{j=0}^{q} \frac{1}{2\pi}\iint f(2j+1+\varphi_0(X))dX$$

According to Lemma 5.5.1, one has

$$2j+1+\varphi_0(X) \in [a,b] \Longleftrightarrow \varphi_0(X) = \omega_0\left(\frac{X}{|X|}\right)|X|^{-\delta}, j = 0,...,q.$$

Thus, after a change of variable in the integral we get

$$b_0(f) = \frac{1}{2\pi\delta}\int_0^{2\pi}(\omega_0(\cos\theta,\sin\theta))^{\frac{2}{\delta}}d\theta\sum_{j=0}^{q}\int f(u)(u-(2j+1))^{-1-\frac{2}{\delta}}du.$$

We recall that supp$f \subset]a,b[$, with $2q+1 < a < b < 2q+3$. This ends the proof of Theorem 5.2.4.

5.5.3 Proof of (5.31) and Theorem 5.2.5

The proof of (5.31) is a slight modification of (5.30). For that, let $\phi \in C_0^\infty((-2,2);[0,1])$ such that $\phi = 1$ on $[-1,1]$. Put $\phi_h(z) = \phi(\frac{\mathrm{Im}z}{h^2})$, then $\widetilde{f}(z)\phi_h(z)$ is also an almost analytic extension of f. Applying again the Helffer-Sjöstrand formula, we get

$$\begin{aligned} & f(H_\lambda)\breve{\theta}_{\lambda^{-\frac{2}{\delta}}}(t-H_\lambda) - f(Q)\breve{\theta}_{h^2}(t-Q) \\ & = -\frac{1}{\pi}\int \overline{\partial}_z(\widetilde{f}\phi_h)(z)\breve{\theta}_{h^2}(t-z)[(z-H_\lambda)^{-1} - (z-Q))^{-1}]L(dz) \\ & = -\frac{1}{\pi}\int \overline{\partial}_z(\widetilde{f}\phi_h)(z)\breve{\theta}_{h^2}(t-z)\left[(z-H_{F_2})^{-1}W_h(z-H_{F_2})^{-1} + G(z)\right]L(dz) \\ & = -\frac{1}{\pi}\int \overline{\partial}_z(\widetilde{f}\phi_h)(z)\breve{\theta}_{h^2}(t-z)G(z)L(dz), \end{aligned} \tag{5.59}$$

where in the last equality we have used the fact that $(z-H_{F_2})^{-1}W_h(z-H_{F_1})^{-1}$ is holomorphic in $z \in \mathcal{U}$.

According to the Paley-Wiener theorem (see e.g. [53, Theorem IX.11]) the function $\breve{\theta}_{h^2}(t-z)$ is analytic with respect to z and satisfies the following estimate

$$\breve{\theta}_{h^2}(t-z) = \mathcal{O}\left(\frac{1}{h^2}\exp\left(\frac{|\mathrm{Im}z|}{Ch^2}\right)\right). \tag{5.60}$$

Combining this with the fact that $\overline{\partial}_z(\widetilde{f}\phi_h)(z) = \mathcal{O}(|\mathrm{Im}z|^\infty)\phi_h(z) + \mathcal{O}\left(\frac{1}{h^2}\right)1_{[h^2,2h^2]}(|\mathrm{Im}z|)$, and using Proposition 5.5.5 we get

$$\|\overline{\partial}_z(\widetilde{f}\phi_h)(z)\breve{\theta}_{h^2}(t-z)G(z)\|_{\mathrm{tr}} = \mathcal{O}(h^\infty).$$

This together with (5.59) ends the proof of (5.31).

By observing that $X.\nabla_X\left(\omega_0\left(\frac{X}{|X|}\right)\right) = 0$, we get

$$X.\nabla_X\left(\omega_0\left(\frac{X}{|X|}\right)|X|^{-\delta}\right) = -\delta\omega_0\left(\frac{X}{|X|}\right)|X|^{-\delta}. \tag{5.61}$$

Then, since $\omega_0 > 0$, we have $\nabla_X(\omega_0(\frac{X}{|X|})|X|^{-\delta}) \neq 0$ for $X \in \mathbb{R}^2 \setminus \{0\}$. It implies that the functions $2j+1+\varphi_0(X)$, $j = 1, ..., q$, do not have any critical values in the interval $[a,b]$. Consequently, Theorem 5.2.5 follows from (5.31) and Theorem 5.2.2.

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