Université Paris 13
et l'Institut de Mathématique Simion Stoilow de l'Académie Roumaine

THÈSE

Pour obtenir le grade de

Docteur de l'Université Paris 13

Discipline: Mathématiques

Présentée et soutenue publiquement par

Oana LUPAȘCU

le 29 novembre 2013

Modélisation probabiliste et déterministe de la rupture

Thèse dirigée par **M. Lucian BEZNEA** et **M. Ioan R. IONESCU**

JURY

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Résumé et remerciements/ Abstract

Résumé. Cette thèse présente des modèles probabilistes et déterministes de rupture et des phénomènes de branchement, on étudie : les processus de branchement à valeurs mesures et leur EDP non linéaires, les processus de Markov de la subordination au sens de Bochner sur les espaces L^p, et les EDP non linéaires liées au déclenchement des avalanches.

La première partie présente les aspects stochastiques. On utilise plusieurs outils théoriques, analytiques et probabilistes de la théorie du potentiel. D'abord, on construit des processus de branchement (de Markov) sur l'ensemble des configurations finies de l'espace d'état d'un processus standard, contrôlés par un noyau de branchement et un noyau tuant. On établit des connexions avec les équations différentielles partielles liées aux fonctions de transition d'un processus de branchement. Si le processus de base est le mouvement brownien, on a une équation d'évolution non linéaire avec le gradient au carré. Si on part d'un super-processus, on obtiendra un processus de branchement ayant l'espace d'état des configurations finies de mesures positives finies sur un espace topologique. L'outil principal pour démontrer la régularité des trajectoires d'un processus de branchement est l'existence des fonctions surharmoniques convenables, ayant les niveaux compacts. En suite, on démontre que la subordination induite par un semi-groupe de convolution (la subordination au sens de Bochner) d'un C_0 -semi-groupe d'opérateurs sous-markoviens sur l'espace L^p est associée à la subordination de processus droit de Markov. En conséquence, on résout le problème des martingales associé au L^p -générateur infinitésimal d'un semi-groupe subordonné. Il s'avère qu'un élargissement de l'espace de base est nécessaire. La principale étape de la preuve est la préservation sous une subordination de la propriété d'un processus de Markov d'être un processus droit borélien.

La deuxième partie de la thèse est consacrée à la modélisation du déclenchement d'une avalanche d'un matériau visco-plastique de faible épaisseur (sols, neige ou autre géo-matériaux) sur une surface avec topographie (montagnes, vallées). On part d'un modèle d'un fluide visco-plastique avec topographie et on introduit un critère simple, capable de distinguer si une avalanche se produit ou pas. Ce critère est déduit d'un problème d'optimisation (analyse de la charge limite). Comme la fonctionnelle de dissipation plastique est non régulière et non coercive dans les espaces de Sobolev classiques, on utilise l'espace des fonctions à déformation tangentielle bornée, pour prouver l'existence d'un champ de vitesse optimal, associé au déclenchement d'une avalanche. On propose aussi une stratégie numérique, sans maillage, pour résoudre le problème de la charge limite et pour obtenir la fracture de déclenchement. La fracture du matériau pendant la phase de déclenchement est modélisée par une discontinuité de ce champ de vitesse. Enfin, l'approche numérique proposée est illustrée par la résolution de certains problèmes modélisant le déclenchement des avalanches.

Mots-clefs. Fonction excessive, processus de Markov, fonction compacte de Lyapunov, branchement discret, processus à valeurs mesures, noyau de branchement, subordination au sens de Bochner, L^p semi-groupe, déclenchement d'avalanche, fluide visco-plastique, topographie, déformation tangentielle bornée, méthode sans maillage, analyse de charge limite.

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Abstract. The thesis presents probabilistic and deterministic models for rupture and branching type phenomena, by studying: measure-valued discrete branching processes and their nonlinear PDEs, the Markov processes of the Bochner subordinations on L^p-spaces, and the nonlinear PDEs associated to the flow onset of dense avalanches.

The first part presents the stochastic aspects. Several analytic and probabilistic potential theoretical tools are used. First, it is given a construction for the branching Markov processes on the space of finite configurations of the state space of a given standard process (called base process), controlled by a branching kernel and a killing one. There are established connections with the nonlinear partial differential equations associated with the transition functions of the branching processes. When the base process is the Brownian motion, then a nonlinear evolution equation involving the square of the gradient occurs. Starting with a superprocess as base process, the result is a branching process with state space the finite configurations of positive finite measures on a topological space. A key tool in proving the path regularity of the branching process is the existence of a convenient superharmonic function having compact level sets. Second, it is shown that the subordination induced by a convolution semigroup (the subordination in the sense of Bochner) of a C_0-semigroup of sub-Markovian operators on an L^p space is actually associated to the subordination of a right (Markov) process. As a consequence, it is solved the martingale problem associate with the L^p-infinitesimal generator of the subordinate semigroup. It turns out that an enlargement of the base space is necessary. A main step in the proof is the preservation under such a subordination of the property of a Markov process to be a Borel right process.

The second part of the thesis deals with the modeling of the onset of a shallow avalanche (soils, snow or other geomaterials) over various bottom topologies (mountains, valleys). Starting from a shallow visco-plastic model with topography, a simple criterion able to distinguish if an avalanche occurs or not, is introduced. This criterion is deduced from an optimization problem, called limit load analysis. The plastic dissipation functional involved is non-smooth, and non coercive in the classical Sobolev spaces. The appropriate functional space is the space of bounded tangential deformation functions and the existence of an onset velocity field (collapse flow) is proved. To propose a numerical strategy, a mesh free method is used to reduce the limit load problem to the minimization of a shape dependent functional. The collapse flow velocity field, which is discontinuous, is associated to an optimum sub-domain and to a rigid flow. The description of the sub-domains is given through a level set of a Fourier function while genetic algorithms are used to solve the resulted non convex and non-smooth global optimization problem. Finally, the proposed numerical approach is illustrated by solving some safety factor problems associated to avalanche onset.

Keywords. Excessive function, Markov process, compact Lyapunov function, discrete branching, measure-valued process, branching kernel, subordination in the sense of Bochner, L^p-semigroup, avalanche onset, visco-plastic fluid, topography, bounded deformation functions, mesh free method, limit load problem.

Introduction

The branching phenomena arise in applicative fields such as mechanics (avalanches, disintegration of snow blocs) or astrophysics (formation and fragmentation of planets). This thesis is related to the present scientific effort in developing the connection between branching processes and the fragmentation phenomenon with applications to material ruptures models, as the avalanche phenomena. We construct and study measure-valued branching processes and their nonlinear partial differential equations, we associate Markov processes to the subordination in the sense of Bochner of the L^p-semigroups, and we investigate the nonlinear partial differential equations modeling the flow onset of dense avalanches (visco-plastic Saint-Venant model with topography), we emphasize a numerical approach.

The branching processes describe the time evolution of a system of particles, located in an Euclidean domain. A branching process is interpreted in a suggestive way as describing the evolution of a random cloud. It turns out that, in order to describe such Markovian time evolution phenomena, the natural state space to be considered is a set of positive finite measures on that domain. Therefore, the methods of constructing measure-valued Markov processes are of particular interest. We establish natural connexions with nonlinear partial differential equations of the type $\Delta u - u = -\sum_{k=1}^{\infty} q_k u^k$, where the coefficients q_k are positive and $\sum_{k=1}^{\infty} q_k = 1$, associated with the transition functions of the branching processes.

The subordination in the sense of Bochner is a convenient way of transforming the semigroups of operators and their infinitesimal generators. Recall that in particular, this is a method of studying the fractional powers of the Laplace operator. The second objective from the stochastic processes part of the thesis is to show that any subordination in the sense of Bochner of a sub-Markovian L^p-semigroups is actually produced by the subordination of a Markov process. It turns out that an enlargement of the base space is necessary. Our approach to the branching processes and Bochner subordination uses analytic and probabilistic tools from potential theory.

Modeling avalanche formation of soils, snow, granular materials or other geomaterials, is a complex task. The main method in modeling the shallow avalanche onset is the study of a global non smooth optimization problem, called the safety factor (of limit load) problem. This optimization problem is reconsidered in the space of bounded deformations functions, a suitable space to capture the discontinuities of the onset velocity field and the velocity boundary conditions are relaxed. We prove that the initial optimization problem is not changed and the reformulated safety factor problem has at least one solution, modeling the onset flow field of the avalanche. We also develop a Discontinuous Velocity Domain Splitting method (DVDS method), a numerical technique to solve the safety factor problem through a shape optimization problem. Our numerical approach is illustrated by numerical simulations of some limit load problems.

The thesis is the result of a PhD program which started in September 2010, in the frame of a cotutelle agreement between Paris 13 University, France, and Simion Stoilow Institute of Mathematics of the Romanian Academy (IMAR) in Bucharest, Romania. It was funded jointly by the French Institute in Bucharest and by IMAR (POSDRU project 82514).

Part I

Branching and subordinate Markov processes

Introduction to Part I

The organization of this part of the thesis is the following. In Chapter 1 we collect some preliminaries on the resolvents of kernels and basic notions of potential theory. We present in (1.3.2) and Proposition 1.3.4 a suitable result on the existence of a càdlàg, quasi-left continuous strong Markov processes, given a resolvent of kernels, imposing conditions on the resolvent.

The description of a discrete branching process is as follows (cf. e.g., [92], page 235, and [34]). An initial particle starts at a point of a set E and moves according to a standard Markov process with state space E (called base process) until a random terminal time when it is destroyed and replaced by a finite number of new particles, its direct descendants. Each direct descendant moves according to the same right standard process until its own terminal time when it too is destroyed and replaced by second generation particles and the process continues in this manner. N. Ikeda, M. Nagasawa, S. Watanabe, and M. L. Silverstein (cf. [53], [54], and [92]) indicated the natural connection between discrete branching processes and nonlinear partial differential operators Λ of the type

$$\Lambda u := \mathcal{L}u + \sum_{k=1}^{\infty} q_k u^k,$$

where $\mathcal{L}$ is the infinitesimal generator of the given base process and the coefficients q_k are positive, Borelian functions with $\sum_{k=1}^{\infty} q_k = 1$. In the description of a branching process this particular case means that each direct descendant starts at the terminal position of the parent particle and $q_k(x)$ is the probability that a particle destroyed at x has precisely k descendants. It is possible to consider a more general nonlinear part for the above operator, generated by a branching kernel B; the descendants start to move from their birthplaces which have been distributed according to B. Thus, these processes are also called "non-local branching" (cf. [34]); from the literature about branching processes we indicate the classical monographs [47], [5], [4], the recent one [67], and the lecture notes [66] and [35].

In Chapter 2 we construct discrete branching Markov processes associated to operators of the type Λ, using several analytic and probabilistic potential theoretical methods. The base space of the process is the set S of all finite configurations of E. The branching kernels on the space of finite configurations are introduced in Section 2.1.

Section 2.5 is devoted to the construction of the measure-valued discrete branching processes. The first step is to solve the nonlinear evolution equation induced by Λ (see Proposition 2.2.1 and Remark 2.2.2 (*ii*) below). Then, using a technique of absolutely monotonic operators developed in [92], it is possible to construct (cf. Theorem 2.3.1) a Markovian transition function on S, formed by branching kernels. We follow the classical approach from [53] and [92], but we consider a more general frame, the given topological space E being a Lusin one and not more compact (see Ch. 5 in [4] for the locally compact space situation).

The second step is to show that the transition function we constructed on S is indeed associated to a standard process with state space S. The main result of this Chapter is Theorem 2.5.2, its proof involves the entrance space S_1, an extension of S constructed by using a Ray type compactification method. We apply the mentioned results from Section 1.3 and Section 1.2, showing that the required imposed conditions from (1.3.2) are satisfied by the resolvent of kernels on S associated with the branching semigroup constructed in the previous step. In Proposition 2.4.2 (ii) we emphasize relations between a class of excessive functions with respect to the base process X and two classes of excessive functions (defined on S) with respect to the forthcoming branching process: the linear and the exponential type excessive functions. A particular linear excessive function for the branching process becomes a function having compact level sets (called compact Lyapunov function) and will lead to the tightness property of the capacity on S (see Proposition 2.4.2 (iii)). It turns out that it is necessary to make a perturbation of $\mathcal{L}$ with a kernel induced by the given branching kernel, and we present it in Proposition 1.4.2.

The above mentioned tools were useful in the case of the continuous branching processes too (cf. [12] and [19]), e.g., for the superprocesses in the sense of E. B. Dynkin (cf. [40]; see Section 3.1 for the basic definitions), like the super-Brownian motion, processes on the space of all finite measures on E induced by operators of the form $\mathcal{L}u - u^\alpha$ with $1 < \alpha \leqslant 2$. We establish in Remark 2.3.2 several links with the nonlinear partial differential equations associated with the branching semigroups and we point out connections between the continuous and discrete branching processes. Note that a cumulant semigroup (similar to the continuous branching case; see (2.6.2) below) is introduced in Theorem 2.3.1 for the discrete branching processes. In particular, when the base process X is the Brownian motion, then the cumulant semigroup of the induced discrete branching process formally satisfies a nonlinear evolution equation involving the square of the gradient. Finally, recall that the method of finding a convenient compact Lyapunov function was originally applied in order to obtain martingale solutions for singular stochastic differential equations on Hilbert spaces (cf. [15] and [22]) and for proving the standardness property of infinite dimensional Lévy processes on Hilbert spaces (see [17]).

We complete the main result of Chapter 2 with an application as suggested in [47], page 50, where T. E. Harris emphasizes the interest for branching processes for which "each object is a very complicated entity; e.g., an object may itself be a population". More precisely, because we may consider base processes with general state space, it might be a continuous branching process playing this role. In Section 2.6, Corollary 2.6.1, we obtain in this way a branching Markov process, having the space of finite configurations of positive finite measures on E as base space; an additional suggestive interpretation of this branching process is exposed in Remark 2.6.2. Note that in [11] new branching processes are generated starting with a superprocess and using an appropriate subordination theory.

In Chapter 3 we study the subordination in the sense of Bochner for L^p-semigroups and the associated Markov processes. Let $(P_t)_{t\geq 0}$ be a C_0-semigroup on a Banach space B and $\mu = (\mu_t)_{t\geq 0}$ be a convolution semigroup on $\mathbb{R}_+$. Recall that the subordinate of $(P_t)_{t\geq 0}$ in the sense of Bochner is the C_0-semigroup $(P_t^\mu)_{t\geq 0}$ on B defined as

$$P_t^\mu u := \int_0^\infty P_s u \; \mu_t(\, \mathrm{d}s), \; t \geq 0, \, u \in B.$$

The probabilistic counterpart of the Bochner subordination is a procedure of introducing jumps in the evolution of a given Markov process, by means of a positive real-valued stationary stochastic process $(\xi_t)_{t\geqslant 0}$, with independent nonnegative increments (called subordinator), induced by $\mu = (\mu_t)_{t\geqslant 0}$. More precisely, if $X = (X_t)_{t\geqslant 0}$, is a (Borel) right Markov process with state space E, then define the subordinate process $X^\xi = (X_t^\xi)_{t\geqslant 0}$ as

$$X_t^\xi := X_{\xi_t}, \;\; t \geqslant 0.$$

A specific problem is to show that regularity properties are transferred from the given process X to the subordinate one X^ξ; for example the Feller or strongly Feller properties of the corresponding transition functions (see, e.g., [61] and Section 3 from Ch. V in [27]). Applications to the Dirichlet forms are developed in [63] and [1], to the pseudo differential operators in [61] and [62], while to establish Harnack inequalities in [46]. Recall that the classical example of such an operator obtained by Bochner subordination is the square root of the Laplace operator; see (3.2.1) below for some details. For other various developments and applications see [90], [50], [31], [36], [91], and [28].

Recall that if $(P_t)_{t\geqslant 0}$ is the C_0-semigroup on $L^p(E,m)$ induced by the transition function of $X = (X_t)_{t\geqslant 0}$ (where m is a σ-finite P_t-*subinvariant measure*, i.e., $\int_E P_t f \,\mathrm{d}m \leqslant \int_E f \,\mathrm{d}m$ for all $f \in L^p(E,m)$, $f \geqslant 0$, and $t \geqslant 0$), then the transition function of the subordinate process $X^\xi = (X_t^\xi)_{t\geqslant 0}$ is $(P_t^\mu)_{t\geqslant 0}$. A converse of this statement is the main result of Chapter 3 and it is given in Section 3.4 below, Theorem 3.4.1.

Consequently, one can apply results on the subordination (in the sense of Bochner) of the Markov processes for the subordination of the L^p-semigroups on arbitrary Lusin measurable state spaces. As an example, the process X^ξ from Theorem 3.4.1 may be regarded as the solution of the martingale problem associate with the infinitesimal generator $\mathcal{L}^\mu$ of the subordinate semigroup $(P_t^\mu)_{t\geqslant 0}$. The precise result is given in Corollary 3.4.3.

A second consequence of Theorem 3.4.1 is the validity of the quasi continuity property for the subordinate semigroup $(P_t^\mu)_{t\geqslant 0}$, with respect to the Choquet capacity associated with the given C_0-semigroup $(P_t)_{t\geqslant 0}$. Recall that this property is analogous to the quasi-regularity condition from the Dirichlet forms theory (cf. [73]); the role of the capacity induced by the energy is played in this L^p frame by the capacity associated to the process. We give more details in Section 3.4, Corollary 3.4.4.

The crucial argument in proving Theorem 3.4.1 is the association of a right process to a C_0-resolvent of sub-Markovian contractions on an L^p-space, proved in [16], where the necessity of the enlargement of the space E is also discussed; for the reader's convenience we present in Section 3.4, Remark 3.4.2, some details about the construction of the larger space E_1. A second main argument is to show that if X is a transient (Borel) right process with state space a Lusin topological space then X^ξ is a (Borel) right process with the same state space and topology, in particular, it is strong Markov. Results of this type were obtained by Bouleau in [31]. We present here a different approach (cf. Corollary 3.3.2 below), based on a characterization of the property of a resolvent of kernels to be associated to a right process, in terms of excessive measures (due to Steffens, see [93]), combined with a result of Sharpe on the preservation of the properties of a process under change of realization (cf. Theorem (19.3) from [91]). Note that according with [16] (Theorem 1.3 and the comment before it) there are some difficulties in applying Steffens' result in the non-transient case, therefore we present it in Section 1.1. Note also that the paper [51] contains a related result, namely, it is shown that the resolvent of a semigroup of kernels obtained by subordination from the transition function of a transient right process is the resolvent of a right process, but possible in some different topology and assuming in addition that it is proper; however, no information about the subordinate process is given and no process is associated to a given transition function (semigroup of kernels), as we are doing in Corollary 3.3.2. The above mentioned result of Steffens is an essential argument in [51] too.

A preliminary result is to show that the min-stability property of the excessive functions is preserved under the subordination by a convolution semigroup (cf. Proposition 3.2.3). We complete in this way results from [22], where this property was supposed to be satisfied by the subordinate resolvent, in order to associate to it a càdlàg Markov process; see the Example following Corollary 5.4 from [22]. Recall that from the probabilistic point of view the stability to the point-wise infimum of the convex cone of all excessive functions is precisely the property that all the points of the state space of the process are nonbranch points.

This first part of the thesis is essentially based on results from the papers [18] and [70].

Chapter 1

Analytic and probabilistic potential theory for resolvents of kernels

1.1 Excessive functions and measures, energy functional

Let $(E, \mathcal{B})$ be a measurable space. We denote by $p\mathcal{B}$ the set of all numerical, positive $\mathcal{B}$-measurable functions on E.

For a familiy $\mathcal{G}$ of real-valued functions on E we denote by $\sigma(\mathcal{G})$ (resp. by $\mathcal{T}(\mathcal{G})$) the $\sigma-$algebra (resp. the topology) generated by $\mathcal{G}$ and by $b\mathcal{G}$ (resp. $[\mathcal{G}]$, $\overline{\mathcal{G}}$) the subfamily of bounded functions from $\mathcal{G}$ (resp. the linear space spanned by $\mathcal{G}$, the closure in the supremum norm of $\mathcal{G}$).

Kernels. A *kernel* on $(E, \mathcal{B})$ is a map $N : p\mathcal{B} \longrightarrow p\mathcal{B}$ such that $N0 = 0$ and for each sequence $(f_n)_n \subset p\mathcal{B}$ we have $N(\sum_n f_n) = \sum_n Nf_n$. The kernel N is called *Markovian* (respectively *sub-Markovian, bounded*) provided that $N1 = 1$ (resp. $N1 \leqslant 1$, $N1$ is a bounded function).

(i) If N is a kernel on $(E, \mathcal{B})$ then for each $x \in E$ the map $f \longmapsto Nf(x)$ is a measure on $(E, \mathcal{B})$ denoted by N_x or $N(x, \cdot)$, i.e, : $Nf(x) = \int f \, \mathrm{d}N_x$, $N(1_A)(x) = N(x, A) = N_x(A)$ for all $A \in \mathcal{B}$.

(ii) If $(N_k)_k$ is a sequence of kernels then $\sum\limits_k N_k$ is a kernel and $N_1 \circ N_2$ is also a kernel on $(E, \mathcal{B})$.

Example. If $g : E \times E \longrightarrow \overline{\mathbb{R}}_+$ is $\mathcal{B} \times \mathcal{B}$-measurable and λ is a σ-finite measure on $(E, \mathcal{B})$, then we define $Gf(x) := \int_E g(x, y) f(y) \lambda(\,\mathrm{d}y)$ for all $x \in E, f \in p\mathcal{B}$. By Fubini theorem it follows that G is a kernel on $(E, \mathcal{B})$.

Let $E = \mathbb{R}^d$, λ be the Lebesgue measure on $\mathbb{R}^d$, $v : \mathbb{R}^d \longrightarrow \overline{\mathbb{R}}_+$ a Borel measurable function, and set $g(x, y) := v(x - y)$. Then

$$Gf(x) = \int_{\mathbb{R}^d} f(y)v(x-y)\lambda(\,\mathrm{d}y) = v * f(x),\ x \in \mathbb{R}^d.$$

G is called *convolution kernel* with density with respect to the d-dimensional Lebesgue measure.

Resolvent of kernels. A family $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is called *resolvent of kernels* on $(E, \mathcal{B})$ provide that U_α is a bounded kernel on $(E, \mathcal{B})$ for each $\alpha > 0$ and the resolvent equation holds, that is,

$$U_\alpha f = U_\beta f + (\beta - \alpha)U_\alpha U_\beta f \;\text{ for all } \alpha, \beta > 0 \text{ and } f \in bp\mathcal{B}.$$

Note that in particular we have $U_\alpha U_\beta = U_\beta U_\alpha$ for all $\alpha, \beta > 0$ and $U_\alpha \leqslant U_\beta$ provided that $\alpha \geqslant \beta$. In particular, we may consider the kernel $U := \sup_\alpha U_\alpha$,

$$Uf(x) := \sup_\alpha U_\alpha f(x) = \lim_{\alpha \to 0} U_\alpha f(x) \text{ for all } f \in p\mathcal{B} \text{ and } x \in E.$$

The kernel U is called the *initial kernel* of the resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$.

The resolvent $\mathcal{U}$ is called *proper* provide there exists $f \in p\mathcal{B}$, $f > 0$, such that Uf is a bounded function.

A resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is called *Markovian* (resp. *sub-Markovian*) provided that each kernel αU_α, $\alpha > 0$, is Markovian (resp. sub-Markovian), i.e., $\alpha U_\alpha 1 = 1$ (resp. $\alpha U_\alpha 1 \leqslant 1$).

If $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ and $q > 0$, then the family of kernels $\mathcal{U}_q = (U_{q+\alpha})_{\alpha>0}$ is also a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ and its initial kernel is U_q. In particular, the resolvent $\mathcal{U}_q = (U_{q+\alpha})_{\alpha>0}$ is always proper, since U_q is a bounded kernel.

Transition function. A family of kernels $\mathbb{T} = (T_t)_{t\geqslant 0}$ on $(E, \mathcal{B})$ is called (time homogenous) *transition function* provided that T_t is a sub-Markovian kernel on $(E, \mathcal{B})$ for each $t \geqslant 0$ and $T_t \circ T_s = T_{t+s}$ for all $t, s \geqslant 0$ i.e.,

$$T_{t+s}(x, A) = \int_E T_s(y, A)T_t(x,\, \mathrm{d}y) \text{ for all } A \in \mathcal{B} \text{ and } x \in E$$

(the Chapman-Kolmogorov equation). We assume that T_0 is the identity operator and for each $f \in bp\mathcal{B}$ the real-valued function $\mathbb{R}_+ \times E \ni (t, x) \longmapsto T_t f(x)$ is measurable.

The transition function $\mathbb{T} = (T_t)_{t\geqslant 0}$ on $(E, \mathcal{B})$ is called *Markovian* provided that $T_t 1 = 1$, $t > 0$.

The *resolvent of kernels associated with the transition function* $\mathbb{T} = (T_t)_{t\geqslant 0}$ is the family $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $(E, \mathcal{B})$ defined by

$$U_\alpha f := \int_0^\infty e^{-\alpha t}\, T_t f \,\mathrm{d}t, \quad f \in p\mathcal{B}.$$

Note that the resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ associated with the transition function $\mathbb{T} = (T_t)_{t\geqslant 0}$ is sub-Markovian. Indeed, we have $U_\alpha 1 = \int_0^\infty e^{-\alpha t} T_t 1 \,\mathrm{d}t \leqslant \int_0^\infty e^{-\alpha t}\,\mathrm{d}t = \frac{1}{\alpha}$. Analogously, if $\mathbb{T} = (T_t)_{t\geqslant 0}$ is Markovian, then the associated resolvent has the same property.

Examples

(1.1.1) *Convolution semigroups on* $\mathbb{R}^d$*.* Let $(\mu_t)_{t\geqslant 0}$ be a (vaguely continuous) *convolution semigroup of probability measures* on $\mathbb{R}^d$, that is, $(\mu_t)_{t\geqslant 0}$ is a family of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $\mu_0 = \delta_0$, $\mu_t * \mu_s = \mu_{t+s}$ for all $s, t \geqslant 0$, and $\int f \,\mathrm{d}\mu_t \to f(0)$ (as t tends to zero) for every $f \in C_0(\mathbb{R}^d)$. Define then for $f \in bp\mathcal{B}$ and $t \geqslant 0$

$$T_t f(x) := f * \mu_t(x) = \int_{\mathbb{R}^d} f(x+y)\mu_t(\,\mathrm{d}y).$$

Then the family $\mathbb{T} = (T_t)_{t\geqslant 0}$ is a Markovian transition function on $\mathbb{R}^d$.

(1.1.2) *The Gaussian semigroup on* $\mathbb{R}^d$. Recall that *the density of the Gaussian kernel* on $\mathbb{R}^d$ is the function $g_t : \mathbb{R}^d \longrightarrow \mathbb{R}$, $t > 0$, defined as

$$g_t(x) := \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2t}},\ x \in \mathbb{R}^d.$$

One can check (see e.g., Ch. 0 in [27]) that the family $(g_t)_{t>0}$ has the following convolution property: $g_s * g_t = g_{s+t}$ for all $s, t > 0$. Consequently, the family of probabilities $(\nu_t)_{t\geqslant 0}$, where $\nu_t := g_t \cdot \lambda$ if $t > 0$ and $\nu_0 := \delta_0$, is a convolution semigroup on $\mathbb{R}^d$ as described in (1.1.1); here we have denoted by λ the d-dimensional Lebesgue measure. The *Gaussian kernel* P_t, $t > 0$, is the (Markovian) convolution kernel (defined above) induced by the density g_t, $P_t f := g_t * f$ for all $f \in p\mathcal{B}$, that is

$$P_t f(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2t}} f(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d.$$

The Gaussian semigroup on $\mathbb{R}^d$ is the family of kernels $\mathbb{P} = (P_t)_{t\geqslant 0}$, where P_0 is the identity operator, that is, $P_0 f := f$ for all f. Clearly, the Gaussian semigroup is a Markovian transition function on $\mathbb{R}^d$, it is induced by the convolution semigroup $(\nu_t)_{t\geqslant 0}$.

(1.1.3) *The translation to the right semigroup on the line.* For each $t \geqslant 0$ let $\mu_t := \delta_t$. Then the family $(\mu_t)_{t\geqslant 0}$ is a convolution semigroup of probability measures on $\mathbb{R}$, which induces (by convolution, cf. (1.1.1)) the transition function of the uniform motion to the right; see, e.g., (3.7) in Ch. I of [28].

For the rest of this chapter $\mathcal{U} = (U_\alpha)_{\alpha>0}$ will be a fixed sub-Markovian resolvent of kernels on the measurable space $(E, \mathcal{B})$. As basic references for the presentation below we indicate the monographs [91], [36], and [13].

Excessive function. A $\mathcal{B}$-universally measurable function $v : E \longrightarrow \overline{\mathbb{R}}_+$ is called *$\mathcal{U}$-supermedian* if $\alpha U_\alpha v \leqslant v$ for all $\alpha > 0$.

Clearly, the property of $\mathcal{U}$ to be sub-Markovian means that the constant function one is $\mathcal{U}$-supermedian.

A $\mathcal{U}$-supermedian function v is named *$\mathcal{U}$-excessive* if in addition $\sup_{\alpha>0} \alpha U_\alpha v = v$.

Observe that for each v is $\mathcal{U}$-supermedian and $x \in E$, the function $\alpha \longmapsto \alpha U_\alpha v(x)$ is increasing and therefore we may consider the function $\widehat{v} : E \longrightarrow \overline{\mathbb{R}}_+$ defined as

$$\widehat{v}(x) := \sup_{\alpha>0} \alpha U_\alpha v(x) = \lim_{\alpha\to\infty} \alpha U_\alpha v(x), \quad x \in E.$$

The function $\widehat{v}$ is called the *$\mathcal{U}$-excessive regularization* of v.

We denote by $\mathcal{E}(\mathcal{U})$ (resp. $\mathcal{S}(\mathcal{U})$) the set of all real-valued $\mathcal{B}$-measurable $\mathcal{U}$-excessive functions (resp. $\mathcal{U}$-supermedian functions).

(1.1.4) *Basic properties of the $\mathcal{U}$ -supermedian and $\mathcal{U}$-excessive functions*

• The sets $\mathcal{S}(\mathcal{U})$ and $\mathcal{E}(\mathcal{U})$ are ordered convex cones and $\mathcal{S}(\mathcal{U})$ is min-stable (i.e., if $u, v \in \mathcal{S}(\mathcal{U})$ then $\inf(u, v) \in \mathcal{S}(\mathcal{U})$).

• If $(v_n)_n$ is a sequence of $\mathcal{U}$-supermedian functions, then $\inf_n v_n$ and $\liminf_n v_n$ are also $\mathcal{U}$-supermedian functions. More generally, if $\mathcal{A}$ is a family of $\mathcal{U}$-supermedian functions and we know that the function $\inf \mathcal{A}$ is $\mathcal{B}$-universally measurable, then it is easy to see that $\inf \mathcal{A}$ is also $\mathcal{U}$-supermedian. An example of main interest is the reduced function $R_q^A u$ of an $\mathcal{U}_q$-excessive function u on a set $A \in \mathcal{B}$, which is a $\mathcal{U}_q$-supermedian function; see Section 1.2 below.

• If v is a $\mathcal{U}$-supermedian function then $\widehat{v}$ (its $\mathcal{U}$-excessive regularization) is an $\mathcal{U}$-excessive function, $\widehat{v} \leqslant v$, and the set $[v \neq \widehat{v}]$ is $\mathcal{U}$-negligible, that is $U_\beta(1_{[v\neq\widehat{v}]}) = 0$ for some (and therefore for all) $\beta > 0$.

(1.1.5) • Let $v \in \mathcal{S}(\mathcal{U})$. Then $\widehat{v} \in \mathcal{E}(\mathcal{U})$, it is the biggest $\mathcal{U}$ -excessive function point-wise dominated by v, and it coincides with the $\mathcal{U}_q$-excessive regularization of v.

Consequently, the following assertions are equivalent:

(a) We have $v = \widehat{v}$.

(b) The function v is $\mathcal{U}$-excessive.

(c) The function v is $\mathcal{U}_q$-excessive for some $q > 0$.

• If $u, v \in \mathcal{E}(\mathcal{U})$ then there exists the greatest lower bound of u and v in $\mathcal{E}(\mathcal{U})$ with respect to the point-wise order relation, denoted $u \wedge v$. More precisely, we have

$$u \wedge v = \widehat{\inf(u, v)}.$$

• If $f \in p\mathcal{B}$ then the function Uf is $\mathcal{U}$-excessive. If $q > 0$ and $v \in \mathcal{S}(\mathcal{U}_q)$ the $U_\beta v$ is $\mathcal{U}_q$-excessive for all $\beta > 0$.

• If $(v_n)_n$ is a sequence of $\mathcal{U}$-supermedian functions which is point-wise increasing to v, then the function v is also $\mathcal{U}$-supermedian and the sequence $(\widehat{v_n})_n$ increases to $\widehat{v}$.

• Let $(v_n)_n$ be a sequence of $\mathcal{U}$-excessive functions. Then there exists the greatest lower bound of $(v_n)_n$ in $\mathcal{E}(\mathcal{U})$ with respect to the point-wise order relation, denoted $\bigwedge\limits_n v_n$, and we have

$$\bigwedge_n v_n = \widehat{\inf_n v_n}.$$

If in addition the sequence $(v_n)_n$ is point-wise increasing to v, then the function v is also $\mathcal{U}$-excessive.

• (*Hunt's Approximation Theorem*) Assume that the resolvent $\mathcal{U}$ is proper. Then for each $v \in \mathcal{E}(\mathcal{U})$ there exists a sequence $(f_n)_n$ in $bp\mathcal{B}$ such that Uf_n is bounded for all n and the sequence $(Uf_n)_n$ is point-wise increasing to v. (For the proof see, e.g., Proposition (2.6) in Ch. II from [28]).

• If $q < q'$ then $\mathcal{S}(\mathcal{U}_q) \subset \mathcal{S}(\mathcal{U}_{q'})$, $\mathcal{E}(\mathcal{U}_q) \subset \mathcal{E}(\mathcal{U}_{q'})$, and $\mathcal{S}(\mathcal{U}) = \bigcap_{q>0} \mathcal{S}(\mathcal{U}_q)$, $\mathcal{E}(\mathcal{U}) = \bigcap_{q>0} \mathcal{E}(\mathcal{U}_q)$.

(1.1.6) • Assume that the resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is associated with a transition function $\mathbb{T} = (T_t)_{t\geqslant 0}$ and consider the set $\mathcal{E}(\mathbb{T})$ defined as

$$\mathcal{E}(\mathbb{T}) := \{v \in p\mathcal{B} : T_t v \leqslant v \text{ for all } t > 0 \text{ and } \lim_{t\to 0} T_t v = v\}.$$

Then $\mathcal{E}(\mathcal{U}) = \mathcal{E}(\mathbb{T})$. To prove that, let $u \in \mathcal{E}(\mathbb{T})$. From $T_t u \leqslant u$ for all $t > 0$ we obtain $U_\alpha u = \int_0^\infty e^{-\alpha t} T_t u \, \mathrm{d}t \leqslant \int_0^\infty e^{-\alpha t} u \, \mathrm{d}t = u \int_0^\infty e^{-\alpha t} \, \mathrm{d}t = \dfrac{u}{\alpha}$ and so, $u \in \mathcal{S}(\mathcal{U})$. Because $u \in \mathcal{E}(\mathbb{T})$, the map $t \longmapsto T_t u$ is decreasing and there exists the point-wise limit $\lim_{t\searrow 0} T_t u = \sup_{t>0} T_t u = \lim_{n\to\infty} T_{t_n} u = u$, where $(t_n)_n$ is a sequence of positive numbers decreasing to zero. We have $\alpha U_\alpha u = \alpha \int_0^\infty e^{-\alpha t} T_t u \,\mathrm{d}t = \int_0^\infty e^{-s} T_{s/\alpha} u \,\mathrm{d}s$. For each fixed $s > 0$ we have $T_{s/\alpha} u \longrightarrow u$ as $\alpha \to \infty$, therefore by dominated convergence we get $\lim_{\alpha\to\infty} \int_0^\infty e^{-s} T_{s/\alpha} u \,\mathrm{d}s = \int_0^\infty e^{-s} u \,\mathrm{d}s = u \int_0^\infty e^{-s} \,\mathrm{d}s = u$. It follows that $\widehat{u} = \lim_{\alpha\to\infty} \alpha U_\alpha u = \lim_{\alpha\to\infty} \int_0^\infty e^{-s} T_{s/\alpha} u \,\mathrm{d}s = u$. Hence $u \in \mathcal{E}(\mathcal{U})$ and we conclude that $\mathcal{E}(\mathbb{T}) \subset \mathcal{E}(\mathcal{U})$. Let now $u \in \mathcal{E}(\mathcal{U})$. Passing to the level $q > 0$ of the resolvent we may assume that the initial kernel U is proper. Indeed, it is easy to see that, as in the case of the $\mathcal{U}$-excessive functions (cf. 1.1.4), we have also $\mathcal{E}(\mathbb{T}) = \bigcap_{q>0} \mathcal{E}(\mathbb{T}_q)$, where $\mathbb{T}_q$ is the transition function defined as $\mathbb{T}_q := (e^{-qt} T_t)_{t \geqslant 0}$. Note also that $\mathcal{U}_q$ is precisely the resolvent family associated with $\mathbb{T}_q$. By Hunt's Approximation Theorem (cf. 1.1.4) there exists a sequence $(f_n)_n \in bp\mathcal{B}$ such that $Uf_n \nearrow u$. Consequently, to prove that the function v belongs to $\mathcal{E}(\mathbb{T})$, it is enough to show that $T_t Uf \leqslant Uf$ for all $t > 0$ and that $\lim_{t\to 0} T_t Uf = Uf$. We have $T_t Uf = \int_0^\infty T_{t+s} f \,\mathrm{d}s = \int_t^\infty T_s f \,\mathrm{d}s \leqslant \int_0^\infty T_s f \,\mathrm{d}s = Uf$ and $\lim_{t\to 0} T_t Uf = \lim_{t\to 0} \int_t^\infty T_s f \,\mathrm{d}s = \int_0^\infty T_s f \,\mathrm{d}s = Uf$. □

Examples. (i) Suppose that $v \in C^2(\mathbb{R}^d)$. Then v is excessive with respect to the Gaussian semigroup $\mathbb{P} = (P_t)_{t \geqslant 0}$ on $\mathbb{R}^d$ introduced in the example (1.1.2) (i.e., v belongs to $\mathcal{E}(\mathbb{P})$) if and only if v is a classical superharmonic function on $\mathbb{R}^d$, that is, $\Delta v \leqslant 0$.

(ii) A positive real-valued function defined on $\mathbb{R}$ is excessive with respect to the translation to the right semigroup on the line (see the example (1.1.3) if and only if it is decreasing and lower semicontinuous.

(iii) A typical example of an $\mathcal{U}$-supermedian function which in general is not $\mathcal{U}$-excessive is the point-wise infimum of a sequence of $\mathcal{U}$-excessive functions; see also (1.1.4).

Excessive measure. Let $\mathsf{Exc}(\mathcal{U})$ be the set of all *$\mathcal{U}$-excessive measures* on E: $\xi \in \mathsf{Exc}(\mathcal{U})$ if and only if it is a σ-finite measure on $(E, \mathcal{B})$ such that $\xi \circ \alpha U_\alpha \leqslant \xi$ for all $\alpha > 0$. Recall that if $\xi \in \mathsf{Exc}(\mathcal{U})$ then actually $\xi \circ \alpha U_\alpha \nearrow \xi$ as $\alpha \to \infty$.

We denote by $\mathsf{Pot}(\mathcal{U})$ the set of all *potential* $\mathcal{U}$-excessive measures: if $\xi \in \mathsf{Exc}(\mathcal{U})$ then $\xi \in \mathsf{Pot}(\mathcal{U})$ if $\xi = \mu \circ U$, where μ is a σ-finite on $(E, \mathcal{B})$.

Energy functional. If $q > 0$ then the *energy functional* $L_q : \mathsf{Exc}(\mathcal{U}_q) \times \mathcal{E}(\mathcal{U}_q) \longrightarrow \overline{\mathbb{R}}_+$ is defined as

$$L_q(\xi, u) := \sup\{\nu(u) : \mathsf{Pot}(\mathcal{U}_q) \ni \nu \circ U_q \leqslant \xi\}.$$

By Theorem 1.4.5 from [13] for all $\xi \in \mathsf{Exc}(\mathcal{U}_q)$ and $u \in \mathcal{E}(\mathcal{U}_q)$

$$L_q(\xi, u) = \sup\{\xi(f) : f \in p\mathcal{B}, U_q f \leqslant u\}$$

and as a consequence if $f \in p\mathcal{B}$ and $u = U_q f$ then

$$L_q(\xi, u) = \xi(f).$$

Also, if $\xi = \mu \circ U_q \in \mathsf{Pot}(\mathcal{U}_q)$ and $u \in \mathcal{E}(\mathcal{U}_q)$ then

(1.1.7) $$L_q(\xi, u) = \int u \, \mathrm{d}\mu.$$

If the resolvent $\mathcal{U}$ is proper then we also consider the case $q = 0$. The 0-level energy functional is denoted by L and all the above properties remain true.

From now on in this section we assume that:

(1.1.8) *$(E, \mathcal{B})$ is a Lusin measurable space (i.e., it is measurable isomorphic to a Borel subset of a metrizable compact space endowed with the Borel σ-algebra) and $\mathcal{B}$ is generated by $b\mathcal{E}(\mathcal{U}_q)$ for some $q > 0$.*

Non-branch points for a resolvent of kernels. Suppose that the resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is proper. We denote by $D_{\mathcal{U}}$ the set of all *non-branch points* with respect to $\mathcal{U}$,

$$D_{\mathcal{U}} := \{x \in E : \inf(u, v)(x) = \widehat{\inf(u, v)}(x) \text{ for all } u, v \in \mathcal{E}(\mathcal{U}) \text{ and } \widehat{1}(x) = 1\}.$$

Since $\mathcal{B}$ is countably generated, we have $D_{\mathcal{U}} \in \mathcal{B}$ and the set $E \setminus D_{\mathcal{U}}$ is $\mathcal{U}$-negligible.

The next result is essentially from [93] (see also Corollary 2.3 from [12]) and it turns out that it will be a main tool in the next chapters. Therefore, for the reader convenience we present its proof here.

(1.1.9) Let $q \geqslant 0$. Then the following three assertions are equivalent for the proper resolvent $\mathcal{U}$.

(i) All the points of E are non-branch points with respect to $\mathcal{U}$.

(ii) We have $1 \in \mathcal{E}(\mathcal{U}_q)$ and the following two properties hold:
(UC) **Uniqueness of charges** for $\mathcal{U}_q$: If two potential $\mathcal{U}_q$-excessive measures $\mu \circ U_q, \nu \circ U_q$ are equal then $\mu = \nu$.

(SSP) **Specific solidity of potentials** for $\mathcal{U}_q$: If $\eta, \eta', \mu \circ U_q \in \mathsf{Exc}(\mathcal{U}_q)$ are such that $\eta + \eta' = \mu \circ U_q$, then there exists a measure ν on E such that $\eta = \nu \circ U_q$.

(iii) The linear space $[b\mathcal{E}(\mathcal{U}_q)]$ spanned by $b\mathcal{E}(\mathcal{U}_q)$ is an unitary algebra.

Proof of (1.1.9).

We show first a property of the $\mathcal{U}_q$-supermedian functions, $q \geqslant 0$; we use the notation $\mathcal{U}_0 = \mathcal{U}$.

(1.1.10) If $v : E \longrightarrow \mathbb{R}_+$ and $\varphi : I \longrightarrow \mathbb{R}_+$ is an increasing concave function, where $I = [0, a)$, $a > 0$, $Im(v) \subset I$, and if $v \in \mathcal{S}(\mathcal{U}_q)$, then $\varphi \circ v \in \mathcal{S}(\mathcal{U}_q)$. Particularly, the vector space $[b\mathcal{S}(\mathcal{U}_q)]$ spanned by $\mathcal{S}(\mathcal{U}_q)$ is an algebra.

The first assertion follows by Jensen's inequality, applied to the sub-probability measure $\mu_x(\mathrm{d}y) := \alpha U_{q+\alpha}(x, \mathrm{d}y)$ for all $x \in E$. Indeed, we may assume that $\varphi(0) = 0$ and for all $x \in E$ we have

$$\alpha U_{q+\alpha}(\varphi \circ v)(x) = \int \varphi \circ v \; \mathrm{d}\mu_x \leqslant \varphi(\mu_x(v)) = \varphi(\alpha U_{q+\alpha} v(x)) \leqslant \varphi(v(x)),$$

where the last inequality holds because φ is increasing and note that $\mu_x(v) \in I$ because $0 \leqslant \mu_x(v) \leqslant v(x)$.

To prove that $[b\mathcal{S}(\mathcal{U}_q)]$ is an algebra, it is sufficient to show that $v^2 \in [b\mathcal{S}(\mathcal{U}_q)]$ for every $v \in b\mathcal{S}(\mathcal{U}_q)$. We may assume that $v \leqslant 1$ and let $\varphi : [0, 1] \longrightarrow \mathbb{R}_+$ defined by $\varphi(x) = 2x - x^2$. Then φ is concave and increasing, so, by the above consideration we have $\varphi \circ v \in b\mathcal{S}(\mathcal{U}_q)$ and therefore $v^2 = 2v - \varphi \circ v \in [b\mathcal{S}(\mathcal{U}_q)]$. Hence (1.1.10) holds.

(i) $\Longrightarrow$ *(iii)*. As before, to prove that $[b\mathcal{E}(\mathcal{U}_q)]$, $q \geqslant 0$, is an algebra, it is sufficient to show that $v^2 \in [b\mathcal{E}(\mathcal{U}_q)]$ for every $v \in b\mathcal{E}(\mathcal{U}_q)$. We may assume that $v \leqslant 1$. By (1.1.10) v^2 belongs to $[b\mathcal{S}(\mathcal{U}_q)]$, $v^2 = 2v - w$ with $w := 2v - v^2 \in b\mathcal{S}(\mathcal{U}_q)$. It remains to show that $w \in \mathcal{E}(\mathcal{U}_q)$. But w is a finely continuous $\mathcal{U}_q$-supermedian function, hence it is $\mathcal{U}_q$-excessive; see e.g. Corollary 1.3.4 from [13].

(iii) $\Longrightarrow$ *(i)*. Let $\mathcal{A}$ be the closure of $[b\mathcal{E}(\mathcal{U}_q)]$ in the supremum norm, it is a Banach algebra and therefore a lattice with respect to the point-wise order relation. Since $\lim_{\alpha\to\infty} \alpha U_{q+\alpha} v = v$, point-wise for all $v \in \mathcal{E}(\mathcal{U}_q)$, it follows that the same property holds for all $v \in \mathcal{A}$. Consequently, since $1 \in \mathcal{A}$, we have $\widehat{1} = 1$ and if $v_1, v_2 \in \mathcal{E}(\mathcal{U}_q)$ then the $\mathcal{U}_q$-supermedian function $v := \inf(v_1, v_2)$ belongs to $\mathcal{A}$, therefore $\widehat{v} = v$ and we conclude that $D_{\mathcal{U}_q} = E$.

(i) $\Longrightarrow$ *(ii)*. We show that the uniqueness of charges property (UC) holds for $\mathcal{U}_q$, $q \geqslant 0$, that is: if μ, ν are two measures on $(E, \mathcal{B})$ such that their potentials $\mu \circ U_q$ and $\nu \circ U_q$ are σ-finite and

$$\mu \circ U_q = \nu \circ U_q,$$

then $\mu = \nu$.

Indeed, the resolvent equation implies that if $\beta > 0$ then the measures $\mu \circ U_{q+\beta}$ and $\mu \circ U_q U_{q+\beta}$ are σ-finite, hence

$$\mu \circ U_{q+\beta} = \nu \circ U_{q+\beta} \text{ for all } \beta > 0. \tag{1.1.11}$$

Let further $g \in bp\mathcal{B}$, $g > 0$, be such that $\mu \circ U_q(g) = \nu \circ U_q(g) < \infty$ and $U_q g$ is bounded (in the case $q = 0$). Set $h := U_q g$, so $0 < h \in L^1(E, \mu + \nu) \cap b\mathcal{E}(\mathcal{U}_q)$. If $f \in [b\mathcal{E}(\mathcal{U}_q)]$, $0 \leqslant f \leqslant 1$, then fh belongs to $[b\mathcal{E}(\mathcal{U}_q)]$ (because it is an algebra by the already proved implication $(i) \Longrightarrow (iii)$) and therefore $\lim_n nU_{q+n}(fh) = fh$. Since $nU_{q+n}(fh) \leqslant nU_{q+n}h \leqslant h \in L^1(E, \mu + \nu)$, by (1.1.11) and the dominated convergence, we obtain that $\mu(fh) = \nu(fh)$ for all $f \in [b\mathcal{E}(\mathcal{U}_q)]$, which is an algebra of bounded functions generating the σ-algebra $\mathcal{B}$. By the monotone class theorem we conclude that $\mu = \nu$. Hence the uniqueness of charges property (UC) holds for the resolvent $\mathcal{U}_q$.

We prove now that the specific solidity of potentials property (SSP) also holds for $\mathcal{U}_q$. Let $\xi, \eta, \mu \circ U_q \in \mathsf{Exc}(\mathcal{U}_q)$ such that $\xi + \eta = \mu \circ U_q$. We may assume that the measure μ is finite. Consider the functional $\varphi_\xi : b\mathcal{E}(\mathcal{U}_q) \longrightarrow \mathbb{R}_+$ defined as

$$\varphi_\xi(v) := L_q(\xi, v) \text{ for all } v \in b\mathcal{E}(\mathcal{U}_q).$$

Note that $L_q(\xi, v) \leqslant L_q(\mu \circ U_q, v) = \mu(v) < \infty$. We may extand φ_ξ to a real valued linear functional on $[b\mathcal{E}(\mathcal{U}_q)]$ and we get

$$\varphi_\xi(f) + \varphi_\eta(f) = \mu(f) \text{ for all } f \in [b\mathcal{E}(\mathcal{U}_q)] \tag{1.1.12}$$

Observe that φ_ξ is positive, i.e.,

$$\varphi_\xi(f) \geqslant 0 \text{ provided that } f \in [b\mathcal{E}(\mathcal{U}_q)] \text{ is positive.} \tag{1.1.13}$$

This follows because (by the properties of the energy functional L_q) φ_ξ is increasing as a functional on $\mathcal{E}(\mathcal{U}_q)$: if $u, v \in \mathcal{E}(\mathcal{U}_q)$ and $u \leqslant v$, then $\varphi_\xi(u) \leqslant \varphi_\eta(v)$. We claim that if $(f_n)_n \subset [b\mathcal{E}(\mathcal{U}_q)]$ is decreasing poinwise to zero then the sequence $(\varphi_\xi(f_n))_n$ also decreases to zero. Note first that by monotone convergence we have $\lim_n \mu(f_n) = 0$. From (1.1.12) and (1.1.13) it follows that $0 \leqslant \varphi_\xi(f_n) \leqslant \mu(f_n)$ for all n and thus

$$0 \leqslant \lim_n \varphi_\xi(f_n) \leqslant \lim_n \mu(f_n) = 0.$$

We can apply now Daniell's theorem on the vector lattice $[b\mathcal{E}(\mathcal{U}_q)]$, for the functional φ_ξ. Hence there exists a positive measure ν on $\mathcal{B} = \sigma([b\mathcal{E}(\mathcal{U}_q)])$ such that

$$\varphi_\xi(f) = \nu(f) \text{ for all } f \in bp\mathcal{B}.$$

Taking $f = U_q g$ with $g \in bp\mathcal{B}$, we get $\xi(g) = L_q(\xi, U_q g) = \varphi_\xi(f) = \nu(U_q g)$ for all $g \in bp\mathcal{B}$, so $\xi = \nu \circ U_q$.

(ii) $\Longrightarrow$ *(iii)*. Let $q \geqslant 0$. We prove first the following assertion:

(1.1.14) If m_1, m_2 are two $\mathcal{U}_q$-excessive measures such that $m_1 + m_2 = \delta_x \circ U_q$ for some $x \in E$, then there exist two positive constants c_1, c_2 with $c_1 + c_2 = 1$, such that $m_i = c_i\, \delta_x$, $i = 1, 2$.

Indeed, by the specific solidity of potentials property (SSP) for the resolvent $\mathcal{U}_q$ there exists $\mu_1 \circ U_q, \mu_2 \circ U_q \in \mathsf{Exc}(\mathcal{U}_q)$ such that $m = \mu_1 \circ U_q$ and $m_2 = \mu_2 \circ U_q$. It follows that $\delta_x \circ U_q = (\mu_1 + \mu_2) \circ U_q$ and by the uniqueness of charges property (UC) for the resolvent $\mathcal{U}_q$ we get $\mu_1 + \mu_2 = \delta_x$, $m_i = c_i\, \delta_x$, $i = 1, 2$, as claimed.

We extend by linearity the $\mathcal{U}_q$-excessive regularization operator $\widehat{\ }$ from $b\mathcal{S}(\mathcal{U}_q)$ to $[b\mathcal{S}(\mathcal{U}_q)]$. We have $\widehat{g}(x) = \lim_\alpha \alpha U_{q+\alpha} g(x)$ for any $g \in [b\mathcal{S}(\mathcal{U}_q)]$ and it follows that $\widehat{\ }$ is monotone on $[b\mathcal{S}(\mathcal{U}_q)]$ and that the mapping $g \longmapsto \widehat{g}(x)$ is continuous in the uniform norm on $[b\mathcal{S}(\mathcal{U}_q)]$.

Recall that by (1.1.10) $[b\mathcal{S}(\mathcal{U}_q)]$ is an algebra, therefore, to show that $[b\mathcal{E}(\mathcal{U}_q)]$ is also an algebra we have to prove that

(1.1.15) $$\text{if } g, h \in [b\mathcal{E}(\mathcal{U}_q)] \text{ then } \widehat{gh} = gh.$$

Fix $g \in [b\mathcal{S}(\mathcal{U}_q)]$ with $0 \leq g \leq 1$, and define for all $f \in bp\mathcal{B}$

$$m_1(f) := \widehat{gU_q f}(x), \quad m_2(f) := (\widehat{1 - g)U_q} f(x).$$

Each m_i is additive and $m_1 + m_2 = \delta_x \circ U_q$. Thus, by Daniell's Theorem, each m_i defines a measure on $(E, \mathcal{B})$. We claim that moreover m_1 and m_2 are $\mathcal{U}_q$-excessive. Indeed, using the monotonicity of $\widehat{\ }$ we have $m_1(\alpha U_{q+\alpha} f) = g(\alpha \widehat{U_{q+\alpha}} U_q f)(x) \leqslant \widehat{gU_q f}(x) = m_1(f)$. From (1.1.14) we get now $m_i = c_i\, \delta_x \circ U_q$, $i = 1, 2$, with $c_1 + c_2 = 1$.

We choose a sequence $U_q f_n$ of potentials increasing to the constant $\mathcal{U}_q$-excessive function 1. It follows that $c_1 = \lim_n c_1 U_q f_n(x) = \lim_n m_1(f_n)(x) = \lim_n \widehat{gU_q f_n}(x) \leqslant \widehat{g}(x)$, hence $c_1 \leq \widehat{g}(x)$ and analogously, $c_2 \leqslant 1 - \widehat{g}(x)$. Since $1 = c_1 + c_2 \leq \widehat{g}(x) + (1 - \widehat{g}(x)) = 1$ we get $c_1 = \widehat{g}(x)$, $c_2 = 1 - \widehat{g}(x)$, and we conclude that

$$\widehat{gU_q f}(x) = \widehat{g}(x) U_q f(x) \text{ for all } g \in [b\mathcal{S}(\mathcal{U}_q)] \text{ and } f \in bp\mathcal{B}.$$

Let now $h \in b\mathcal{E}(\mathcal{U}_q)$ and consider an increasing sequence of (bounded) potentials $(U_q f_n)_n$, converging point-wise to h. Reasoning as before and using the last equality we obtain

$$\widehat{gh}(x) = \lim_n \widehat{gU_q} f_n(x) \leq \lim_n \widehat{gU_q f_n}(x) = \lim_n \widehat{g}(x) U_q f_n(x) = \widehat{g}(x) h(x)$$

and analogously $\widehat{(1-g)}h(x) \leq (1-\widehat{g}(x))h(x)$, which implies

$$\widehat{gh}(x) = \widehat{g}(x)h(x) \text{ for all } g \in [b\mathcal{S}(\mathcal{U}_q)] \text{ and } h \in [b\mathcal{E}(\mathcal{U}_q)].$$

This, in particular, yields

$$\widehat{gh} = gh \text{ and any } g, h \in [b\mathcal{E}(\mathcal{U}_q)],$$

hence (1.1.15) holds and therefore $[b\mathcal{E}(\mathcal{U}_q)]$ is an unitary algebra. □

Further we also consider the following condition:

(1.1.16)
All the points in E are non-branch points with respect to $\mathcal{U}_q$ for some $q > 0$.

Remark 1.1.1. (i) *Conditions (1.1.16) and (i) of (1.1.9) are equivalent provided that the resolvent $\mathcal{U}$ is proper.* Indeed, since $1 \in \mathcal{S}(\mathcal{U})$, by (1.1.5) we have: $1 \in \mathcal{E}(\mathcal{U})$ if and only if $1 \in \mathcal{E}(\mathcal{U}_q)$. Also, by the equivalence $(b) \Longleftrightarrow (c)$ from (1.1.5) we get the inclusion $\mathcal{D}_{\mathcal{U}_q} \subset \mathcal{D}_{\mathcal{U}}$. It remains to show that if $\mathcal{D}_{\mathcal{U}} = E$ and $u, v \in \mathcal{E}(\mathcal{U}_q)$ then $\inf(u,v) \in \mathcal{E}(\mathcal{U}_q)$. By Hunt's Approximation Theorem (see (1.1.4)) there exist two sequences $(f_n)_n$, $(g_n)_n \subset bp\mathcal{B}$ such that $U_q f_n \nearrow u$, $U_q g_n \nearrow v$ and since $\mathcal{U}$ is proper we may assume that $U(f_n + g_n)$ is a bounded function for all n. Because $\inf(U_q f_n, U_q g_n) \nearrow \inf(u,v)$, it is sufficient to show that for each n the function $w := \inf(U_q f_n, U_q g_n)$ is $\mathcal{U}_q$-excessive. For, observe that the hypothesis $\mathcal{D}_{\mathcal{U}} = E$ implies that $[b\mathcal{E}(\mathcal{U})]$ is a vector lattice with respect to the point-wise infimum and supremum. By the resolvent equation we have $U_q f_n, U_q g_n \in [b\mathcal{E}(\mathcal{U})]$ and therefore w also belongs to $[b\mathcal{E}(\mathcal{U})]$. Since for every $f \in [b\mathcal{E}(\mathcal{U})]$ we have $\lim_{\alpha\to\infty} \alpha U_\alpha f = f$ point-wise, we conclude that $\lim_{\alpha\to\infty} \alpha U_{q+\alpha} w = \lim_{\alpha\to\infty} \alpha U_\alpha w = w$, hence $w \in \mathcal{E}(\mathcal{U}_q)$.

(ii) *We shall prove in Section 1.4, Proposition 1.4.1, that the linear space $[b\mathcal{E}(\mathcal{U}_q)]$ spanned by $b\mathcal{E}(\mathcal{U}_q)$ does not depend on $q > 0$, even in the non-transient case. Consequently, conditions (1.1.8) and (1.1.16) do not depend on $q > 0$ (use also the equivalence $(i) \Longleftrightarrow (iii)$ from (1.1.9)). Note that the fact that (1.1.16) does not depend on $q > 0$ is also a consequence of the above assertion (i).*

(iii) *If conditions (1.1.8) and (1.1.16) are satisfied then, using again (1.1.9), the following assertions hold:* if μ and ν are two finite measures on $(E, \mathcal{B})$ such that $\int v \, \mathrm{d}\mu = \int v \, \mathrm{d}\nu$ for all $v \in b\mathcal{E}(\mathcal{U}_q)$ then $\mu = \nu$.

1.2 Completion of the space, fine and natural topologies, induced capacities

In this section we assume that conditions (1.1.8) and (1.1.16) hold. Our presentation follows that from [13], Ch. 1.

Completion of the base space E. There exists a second Lusin measurable space $(E_1, \mathcal{B}_1)$ such that $E \subset E_1$, $E \in \mathcal{B}_1$, $\mathcal{B} = \mathcal{B}_1|_E$, and a resolvent of kernels $\mathcal{U}^1 = (U^1_\alpha)_{\alpha>0}$ on $(E_1, \mathcal{B}_1)$ such that for some $q > 1$ $\sigma(\mathcal{E}(\mathcal{U}^1_q)) = \mathcal{B}_1$, every point of E_1 is a non-branch point with respect to $\mathcal{U}^1_q$, $U^1_q(1_{E_1\setminus E}) = 0$, and $\mathcal{U}$ is the restriction of $\mathcal{U}^1$ to E (i.e., $U_\alpha g = U^1_\alpha g^1$ for all $\alpha > 0$, where $g^1 \in \mathrm{p}\mathcal{B}_1$ and $g^1|_E = g$).

We clearly have $\mathsf{Exc}(\mathcal{U}_q) = \mathsf{Exc}(\mathcal{U}^1_q)$ and the following property holds (for one and therefore for all $q > 0$):

(1.2.1) every $\xi \in \mathsf{Exc}(\mathcal{U}^1_q)$ with $L_q(\xi, 1) < \infty$ is a potential on E_1 (with respect to $\mathcal{U}^1_q$).

One can take for E_1 the set of all extreme points of the set $\{\xi \in \mathsf{Exc}(\mathcal{U}_q) : L_q(\xi, 1) = 1\}$, endowed with the σ-algebra $\mathcal{B}_1$ generated by the functionals $\widetilde{u}$ defined as

$$\widetilde{u}(\xi) := L_q(\xi, u) \text{ for all } \xi \in E_1 \text{ and } u \in \mathcal{E}(\mathcal{U}_q).$$

Let $(E', \mathcal{B}')$ be a Lusin measurable space such that $E \subset E'$, $E \in \mathcal{B}'$, $\mathcal{B} = \mathcal{B}'|_E$, and there exists a proper sub-Markovian resolvent of kernels $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ on $(E', \mathcal{B}')$ with $D_{\mathcal{U}'_q} = E'$, $\sigma(\mathcal{E}(\mathcal{U}'_q)) = \mathcal{B}'$, $U'_q(1_{E'\setminus E}) = 0$, E' satisfies (1.2.1) with respect to $\mathcal{U}'$, and $\mathcal{U}$ is the restriction of $\mathcal{U}'$ to E. Then the map $x \longmapsto \delta_x \circ U'_q$ is a measurable isomorphism between $(E', \mathcal{B}')$ and the measurable space $(E_1, \mathcal{B}_1)$.

Fine topology. The *fine topology* on E (associated with $\mathcal{U}$) is the coarsest topology on E such that every $\mathcal{U}_q$-excessive function is continuous for some $q > 0$. Note that by the assertion (i) of Remark 1.1.1 the fine topology does not depend on $q > 0$.

Extension of excessive functions from E **to** E_1**.** For every $u \in \mathcal{E}(\mathcal{U}_q)$ we consider the function $\widetilde{u} : E_1 \longrightarrow \overline{\mathbb{R}}_+$ defined above,

$$\widetilde{u}(\xi) := L_q(\xi, u), \quad \xi \in E_1. \tag{1.2.2}$$

Then by (1.1.7) we have $\widetilde{u}(\varepsilon_x \circ U_\beta) = u(x)$ for all $x \in E$ and therefore, by the embedding of E in E_1,

$$\widetilde{u}|_E = u.$$

In addition, $\widetilde{u}$ is $\mathcal{U}^1_q$-excessive and it is the (unique) extension by fine continuity of u from E to E_1. Consequently, the fine topology on E is the trace of the fine topology on E_1 (generated by the $\mathcal{U}_q^1$-excessive functions) and E is a finely dense subset of E_1.

Ray cone. If $q > 0$ then a *Ray cone* associated with $\mathcal{U}_q$ is a cone $\mathcal{R}$ of bounded $\mathcal{U}_q$-excessive functions such that: $U_\alpha(\mathcal{R}) \subset \mathcal{R}$ for all $\alpha > 0$, $U_q\big((\mathcal{R} - \mathcal{R})_+\big) \subset \mathcal{R}$, $\sigma(\mathcal{R}) = \mathcal{B}$, it is min-stable, separable in the supremum norm and $1 \in \mathcal{R}$. Using a slightly modified version of Proposition 1.5.1 from [13], one can show that such a Ray cone always exists.

If the initial kernel of $\mathcal{U}$ is bounded, then one can take $q = 0$, that is, there is a Ray cone of $\mathcal{U}$-excessive functions.

Below, if we say Ray cone it is always meant to be associated with one fixed resolvent $\mathcal{U}_q$.

A *Ray topology* on E is a topology generated by a Ray cone. A Ray topology is Lusin and its Borel σ-algebra is $\mathcal{B}$.

A metrizable Lusin topology on E is called *natural* provide that its Borel σ-algebra is precisely $\mathcal{B}$ and it is smaller than the fine topology on E.

Since a Ray cone is a subset of $\mathcal{U}_q$, we clearly have that any Ray topology on E is natural.

Reduced function and the induced Choquet capacities. If $q > 0$, then for all $u \in \mathcal{E}(\mathcal{U}_q)$ and every subset A of E we consider the function

$$R_q^A u := \inf\{v \in \mathcal{E}(\mathcal{U}_q) : \, v \geqslant u \text{ on } A\},$$

called the q-order *reduced function* of u on A.

It is known that if $A \in \mathcal{B}$ then $R_q^A u$ is universally $\mathcal{B}$-measurable and in addition it is a $\mathcal{U}_q$-supermedian function; cf. (1.1.4). If the set $A \in \mathcal{B}$ is finely open and $u \in p\mathcal{B}$ then $R_q^A u \in p\mathcal{B}$.

Let λ be a finite measure on $(E, \mathcal{B})$. We also fix a natural topology $\mathcal{T}$ on E and let $u_o := U_q 1$. It turns out that the functional $M \longmapsto c_\lambda^q(M)$, $M \subset E$, defined as

$$c_\lambda^q(M) := \inf\{\int R_q^G u_o \, \mathrm{d}\lambda : \, G \in \mathcal{T}, \; M \subset G\}$$

is a Choquet capacity on the topological space E (endowed with the topology $\mathcal{T}$).

Tightness of the capacity induced by a sub-Markovian resolvent of kernels. The capacity c_λ^q on $(E, \mathcal{T})$ is named *tight* provided that there exists an increasing sequence $(K_n)_n$ of $\mathcal{T}$-compact sets such that

$$\inf_n c_\lambda^q(E \setminus K_n) = 0$$

or equivalently $\inf_n R_q^{E\setminus K_n} u_o = 0$ λ-a.e.

(1.2.3) The following assertions are equivalent (see Proposition 4.1 in [22] and Proposition 2.1.1 in [21]):

(*i*) The capacity c_λ^q is *tight*.

(*ii*) There exist a $\mathcal{U}_q$-excessive function v which is finite λ-a.e. and a bounded strictly positive $\mathcal{U}_q$-excessive function u such that the set $[\frac{v}{u} \leqslant \alpha]$ is relatively compact for all $\alpha > 0$.

(*iii*) There exists a $\mathcal{U}_q$-excessive function v which is λ-integrable and such that the set $[\frac{v}{u_o} \leqslant \alpha]$ is relatively compact for all $\alpha > 0$; recall that $u_o := U_q 1$.

Remark 1.2.1. *If there exists a strictly positive constant k such that $k \leqslant u_o$ (in particular, this happens if the resolvent $\mathcal{U}$ is Markovian), then in the above assertion (ii) one can take $u = 1$. In this case we say that v has compact level sets.*

1.3 Markov processes associated with a resolvent of kernels, path regularity

Remark 1.3.1. *Suppose that $\mathcal{U}$ is the resolvent of a Borel right Markov process with state space E, a metrizable Lusin topological space (i.e., E is homeomorphic to a Borel subset of a compact metrizable space) and $\mathcal{B}$ is the Borel σ-algebra on E. Then the topology of E is natural, conditions (1.1.8) and (1.1.16) are satisfied and the following property is verified provided that $\mathcal{U}$ is proper:*

(NSP) **Natural solidity of potentials**: *Every $\mathcal{U}$-excessive measure dominated by a potential is also a potential.*

Proposition 1.3.2. *The following assertions hold for the sub-Markovian resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$.*

(i) *Assume that conditions (1.1.8) and (1.1.16) are satisfied and that $\mathcal{U}$ is the resolvent of a transition function $\mathbb{T} = (T_t)_{t\geqslant 0}$ on $(E, \mathcal{B})$, i.e.,*

$$U_\alpha f = \int_0^\infty e^{-\alpha t}\, T_t f \,\mathrm{d}t \qquad \text{for all } f \in p\mathcal{B}.$$

Then $\mathbb{T} = (T_t)_{t\geqslant 0}$ is uniquely determined by $\mathcal{U} = (U_\alpha)_{\alpha>0}$.

(ii) Suppose that $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is the resolvent family of a (Borel) right process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ with state space E, that is

$$U_\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t)\,\mathrm{d}t \text{ for all } f \in bp\mathcal{B} \text{ and } x \in E.$$

If in addition $\mathcal{U}$ is the resolvent of a transition function $\mathbb{T} = (T_t)_{t\geqslant 0}$ on $(E, \mathcal{B})$, then $\mathbb{T} = (T_t)_{t\geqslant 0}$ is precisely the transition function of the process X, i.e.,

$$T_t f(x) = E^x(f(X_t)),\ f \in p\mathcal{B} \text{ and } x \in E.$$

Proof. Assertion (ii) is clearly a consequence of (i) since by Remark 1.3.1 conditions (1.1.8) and (1.1.16) are satisfied provided that $\mathcal{U}$ is the resolvent of a (Borel) right process.

To prove (i), let $\mathbb{T}' = (T'_t)_{t\geqslant 0}$ be a second transition function having the same resolvent $\mathcal{U}$. If $q > 0$, $v \in b\mathcal{E}(\mathcal{U}_q)$, and $x \in E$, then using (1.1.6) we deduce that both functions $t \longmapsto T_t v(x)$ and $t \longmapsto T'_t v(x)$ are right continuous on $[0, \infty)$ and by hypothesis

$$\int_0^\infty e^{-\alpha t} T_t v(x)\,\mathrm{d}t = U_\alpha v(x) = \int_0^\infty e^{-\alpha t} T'_t v(x)\,\mathrm{d}t \text{ for all } \alpha > 0.$$

The uniqueness property of the Laplace transform implies now that $T_t v(x) = T'_t v(x)$ for all $v \in b\mathcal{E}(\mathcal{U}_q)$, $x \in E$, and $t > 0$. By assertion (iii) of Remark 1.1.1 we conclude that the kernels T_t and T'_t coincide, so, assertion (i) holds. □

We present now a result related to the existence of a right process having $\mathcal{U}$ as associated resolvent, showing essentially that the converse of the assertion from Remark 1.3.1 is true.

(1.3.1) The following assertions are equivalent for a sub-Markovian resolvent of kernels $\mathcal{U}$ on the Lusin measurable space $(E, \mathcal{B})$.

(1.3.1a) All the points of E are nonbranch points with respect to $\mathcal{U}_q$, $\mathcal{B}$ is generated by $b\mathcal{E}(\mathcal{U}_q)$, and the natural density of potentials property hold for some $\mathcal{U}_q$, $q > 0$.

(1.3.1b) There exists a Lusin topology on E such that $\mathcal{B}$ is the σ-algebra of all Borel sets of E, and there exists a right process with state space E, having $\mathcal{U}$ as the associated resolvent.

(1.3.1c) For every natural topology on E there exists a right process with state space E, having $\mathcal{U}$ as the associated resolvent.

The equivalence between (1.3.1a) and (1.3.1b) is essentially due to J. Steffens [93] in the transient case; see also Sections 1.7 and 1.8 in [13]. For the non-transient case see Theorem 1.3 from [16]. The implication (1.3.1a) $\Longrightarrow$ (1.3.1c) follows from [13], Corollary 1.8.12, using Proposition 3.5.3.

Observe that condition (1.3.1a) means precisely that conditions (1.1.8), and (1.1.16) for some $q > 0$ are satisfied.

Remark. (i) If $\mathcal{U}$ is the resolvent of a right process X, then the following fundamental result of G. A. Hunt holds for all $A \in \mathcal{B}$, $x \in E$, and $u \in \mathcal{E}(\mathcal{U}_q)$:

$$R_q^A u(x) = E^x(e^{-qD_A} u(X_{D_A})),$$

where D_A is the *entry time* of A,

$$D_A := \inf\{t \geqslant 0 : \ X_t \in A\};$$

see e.g. [36].

(ii) If λ is a finite measure on $(E, \mathcal{B})$ then the tightness of the capacity c_λ^q is equivalent to

$$P^\lambda(\lim_n D_{E \setminus K_n} < \zeta) = 0.$$

In particular, if the capacity c_λ^q on $(E, \mathcal{T})$ is tight for one $q > 0$ then this happens for all $q > 0$.

Path regularity: the càdlàg and the quasi-left continuity properties of the trajectories. Recall that a right (Markov) process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ with state space E is called *standard* if for every finite measure μ on $(E, \mathcal{B})$ X has *càdlàg trajectories* under P^μ, i.e., it possesses left limits in E P^μ-a.e. on $[0, \zeta)$ and X is *quasi-left continuous up to* ζ P^μ-a.e., that is, for every increasing sequence $(T_n)_n$ of stopping times with $T_n \nearrow T$ we have $X_{T_n} \longrightarrow X_T$ P^μ-a.e. on $[T < \zeta]$, ζ being the life time of X; a *stopping time* is a map $T : \Omega \longrightarrow \overline{\mathbb{R}}_+$ such that the set $[T \leqslant t]$ belongs to $\mathcal{F}_t$ for all $t \geqslant 0$.

The next result is the convenient one for the construction of the discrete branching measure-valued processes we give in Section 2.5 below, it follows from [19], Theorem 2.1, and it is a consequence of [22], Theorem 5.2, Corollary 5.3 (ii), and Theorem 5.5 (i).

(1.3.2) Suppose that the following three conditions are satisfied by the sub-Markovian resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha > 0}$ on $(E, \mathcal{B})$:

($h1$) Conditions (1.1.8) and (1.1.16) are satisfied, that is: $\sigma(\mathcal{E}(\mathcal{U}_q)) = \mathcal{B}$ and all the points of E are non-branch points with respect to $\mathcal{U}_q$ for some $q > 0$.

$(h2)$ For every $x \in E$ there exists $v_x \in \mathcal{E}(\mathcal{U}_q)$ such that $v_x(x) < \infty$ and the set $[v_x \leqslant n]$ is relatively compact for all n; such a function v_x is called *compact Lyapunov function.*

$(h3)$ There exists a countable subset $\mathcal{F}$ of $[b\mathcal{E}(\mathcal{U}_q)]$ generating the topology of E, $1 \in \mathcal{F}$, and there exists $u_o \in \mathcal{E}(\mathcal{U}_q)$, $u_o < \infty$, such if ξ, η are two finite $\mathcal{U}_q$-excessive measures with $L_q(\xi + \eta, u_o) < \infty$ and such that $L_q(\xi, \varphi) = L_q(\eta, \varphi)$ for all $\varphi \in \mathcal{F}$, then $\xi = \eta$; here recall that L_q denotes the energy functional associated with $\mathcal{U}_q$, defined in Section 1.1.

Then there exists a càdlàg process X with state space E such that $\mathcal{U}$ is the resolvent of X, i.e., for all $\alpha > 0$, $f \in bp\mathcal{B}$, and $x \in E$ we have $U_\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t)\,\mathrm{d}t$.

Remark 1.3.3. (i) *As we already noted in Remark 1.3.1 and (1.3.1), condition* $(h1)$ *is necessary in order to deduce that* $\mathcal{U}$ *is the resolvent family of a (Borel) right process; see [91], [93], [13], and [16].*

(ii) *According to [71] and [14], condition* $(h2)$ *is necessary for proving that the process* X *has càdlàg trajectories. It is related to the tightness of the associated capacity, we gave some details in (1.2.3).*

(iii) *The quasi-left continuity of the forthcoming measure-valued branching process will be deduced from the next proposition. Some arguments in its proof are classical, e.g., similar to the Ray resolvent case (see for example Theorem (9.21) from [91] , the proof of Lemma IV.3.21 from [73], and the proof of Theorem 3.7.7 from [13]). However, none of the existing results covers our context, therefore we present here a complete proof of it.*

Proposition 1.3.4. *Let* X *be a right process with state space* E *and càdlàg trajectories. Let* $\mathbb{T} = (T_t)_{t \geqslant 0}$ *be its transition function and* $\mathcal{U} = (U_\alpha)_{\alpha > 0}$ *its resolvent. Assume that there exists a countable subset* $\mathcal{F}$ *of* $[b\mathcal{E}(\mathcal{U}_q)]$ *generating the topology of* E *and a family* $\mathcal{K} \subset \overline{[\mathcal{F}]}$ *which is multiplicative (i.e., if* $f, g \in \mathcal{K}$ *then* $fg \in \mathcal{K}$*) and separating the points of* E*. Then the process* X *is quasi-left continuous, hence standard, provided that* $T_t f$ *belongs to* $\overline{[\mathcal{F}]}$ *for all* $f \in \mathcal{K}$ *and* $t > 0$*. If* $(T_t)_{t \geqslant 0}$ *is Markovian then it is enough to assume that* $\mathcal{K}$ *is a family of bounded, continuous, real-valued functions on* E *which is multiplicative, separating the points of* E*, and* $T_t f$ *is a continuous function for all* $f \in \mathcal{K}$ *and* $t > 0$*.*

Proof. As we already mentioned, we follow the classical approach, cf., e.g., page 48 from [91], page 115 in [73], see also pages 133-134 in [13] and the proof of Theorem 5.5 (ii) from [22].

We start with the construction of a convenient compactification of E, as in the proof of Theorem 5.2 from [22].

Let K be the compactification of E with respect to $\mathcal{F}$. Since for every real-valued function $u \in \mathcal{E}(\mathcal{U}_q^o)$ the real-valued process $(e^{-qt} u \circ X_t)_{t \geqslant 0}$ is a right continuous (P^x-integrable) supermartingale under P^x for all $x \in E$, it follows that this process has left limits P^x-a.s. and we conclude that X has left limits in K a.s.

Let $(T_n)_n$ be an increasing sequence of stopping times and $T = \lim_n T_n$. It is no loss of generality to assume that T is bounded. From the above considerations the limit $Z := \lim_n X_{T_n}$ exists in K a.s. and $Z(\omega) \in E$ if $T(\omega) < \zeta(\omega)$.

In order to prove that $Z = X_T$ a.s. on $[T < \zeta]$, it is enough to show that for every $x \in E$ and $G \in pb\mathcal{B}(K \times K)$,

$$E^x(G 1_{E \times E}(Z, X_T)) = E^x(G 1_{E \times E}(Z, Z)). \tag{1.3.3}$$

Indeed, taking as G the indicator function of the diagonal of $K \times K$, from (1.3.3) we get $P^x([Z \in E, Z \neq X_T]) = 0$.

Note that every function f from $\overline{[\mathcal{F}]}$ has an extension by continuity from E to K, denoted by $\overline{f}$. Since $[b\mathcal{E}(\mathcal{U}_q)]$ is an algebra, we may assume that $\mathcal{F}$ is multiplicative. In order to prove (1.3.3) we first use the strong Markov property (clearly, $\overline{f}(Z) \in \mathcal{F}_T$) and then the P^x-a.s. equality $\lim_n f(X_{T_n}) T_t g(X_{T_n}) = \overline{f}(Z)\overline{T_t g}(Z)$ (because we take $f \in [\mathcal{F}]$ and $T_t g$ belongs to $\overline{[\mathcal{F}]}$ provided that $g \in \mathcal{K}$):

$$E^x(\overline{f}(Z) U_\alpha g(X_T)) = E^x(\overline{f}(Z) E^{X_T}\int_0^\infty e^{-\alpha t} g(X_t)\,\mathrm{d}t) = E^x(\overline{f}(Z) e^{\alpha T} \int_T^\infty e^{-\alpha t} g(X_t)\,\mathrm{d}t) =$$

$$\lim_n E^x(f(X_{T_n}) e^{\alpha T_n} \int_{T_n}^\infty e^{-\alpha t} g(X_t)\,\mathrm{d}t) = \lim_n E^x(f(X_{T_n}) U_\alpha g(X_{T_n})) =$$

$$\lim_n E^x(f(X_{T_n}) \int_0^\infty e^{-\alpha t} T_t g(X_{T_n})\,\mathrm{d}t) = E^x(\overline{f}(Z) \int_0^\infty e^{-\alpha t} \overline{T_t g}(Z)\,\mathrm{d}t).$$

By a monotone class argument we have for all $h \in bp\mathcal{B}(K)$

$$E^x(h 1_E(Z) U_\alpha g(X_T)) = E^x(h 1_E(Z) \int_0^\infty e^{-\alpha t} \overline{T_t g}(Z)\,\mathrm{d}t)$$

and therefore

$$E^x(h 1_E(Z) U_\alpha g(X_T)) = E^x(h(Z) \int_0^\infty e^{-\alpha t} T_t g(Z)\,\mathrm{d}t; Z \in E) = E^x(h(Z) U_\alpha g(Z); Z \in E).$$

Because $\lim_{\alpha \to \infty} \alpha U_\alpha g = g$ (since g is continuous), multiplying by α and letting α tend to infinity we get

$$E^x(h 1_E(Z) g(X_T)) = E^x((h g 1_E)(Z)).$$

Using again monotone class arguments we obtain first

$$E^x(h1_E(Z) \cdot k1_E(X_T)) = E^x(h1_E(Z) \cdot k1_E(Z)) \quad \text{for all } h, k \in pb\mathcal{B}(K),$$

and then (1.3.3).

If the transition function $(T_t)_{t \geqslant 0}$ is Markovian then the limit $Z = \lim_n X_{T_n}$ exists in E a.s. Therefore, in this case it is enough to show that (1.3.3) holds for every $G \in pb\mathcal{B}(E \times E)$. Note that the extensions by continuity of f and $T_t g$ from E to K are not longer necessary, in particular, $\lim_n f(X_{T_n}) T_t g(X_{T_n}) = f(Z) T_t g(Z)$ P^x-a.s. □

1.4 Perturbation with kernels of the sub-Markovian semigroups

The first result of this section is a direct consequence of Hunt's Approximation Theorem, applied to the level $q > 0$ of a resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha > 0}$ on a measurable space $(E, \mathcal{B})$.

Proposition 1.4.1. *The vector space* $[b\mathcal{E}(\mathcal{U}_q)]$ *does not depend on* $q > 0$. *If in addition* $\mathcal{U}$ *is bounded then* $[b\mathcal{E}(\mathcal{U})] = [b\mathcal{E}(\mathcal{U}_q)]$ *for all* $q > 0$.

Proof. Observe first that if $\mathcal{U}$ is proper then there exists a second resolvent of kernel $\mathcal{U}' = (U'_\alpha)_{\alpha > 0}$ on $(E, \mathcal{B})$ such its initial kernel is bounded and $\mathcal{E}(\mathcal{U}') = \mathcal{E}(\mathcal{U})$. Therefore, in this case (replacing $\mathcal{U}$ by $\mathcal{U}'$) we may suppose that the kernel $U = U_0 := \sup U_\alpha$ is bounded.

If $0 \leqslant \alpha < q$ then because $\mathcal{E}(\mathcal{U}_\alpha) \subset \mathcal{E}(\mathcal{U}_q)$ it is sufficient to prove that $b\mathcal{E}(\mathcal{U}_q) \subset [b\mathcal{E}(\mathcal{U}_\alpha)]$. For, if $v \in b\mathcal{E}(\mathcal{U}_q)$ then clearly $U_\alpha v \in b\mathcal{E}(\mathcal{U}_\alpha)$ and by Hunt's Approximation Theorem (see (1.1.4)) we also have that $v + (q - \alpha) U_\alpha v$ belongs to $b\mathcal{E}(\mathcal{U}_\alpha)$. So, $v = (v + (q - \alpha) U_\alpha v) - (q - \alpha) U_\alpha v) \in [b\mathcal{E}(\mathcal{U}_\alpha)]$. □

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a fixed right (Markov) process with state space E. Let further $\mathbb{T} = (T_t)_{t \geqslant 0}$ be its transition function and assume that $\mathcal{U} = (U_\alpha)_{\alpha > 0}$ is the associated resolvent of kernels.

We fix a function $c \in bp\mathcal{B}$ and we denote by $\mathbb{T}^c = (T_t^c)_{t \geqslant 0}$ the transition function of the process obtained by killing X with the multiplicative functional $(e^{-\int_0^t c(X_s)\,\mathrm{d}s})_{t \geqslant 0}$. It is given by the Feynman-Kac formula:

$$T_t^c f(x) = E^x(e^{-\int_0^t c(X_s)\,\mathrm{d}s} f(X_t)),\ t > 0, f \in bp\mathcal{B},\, x \in E.$$

Note that if $\mathcal{L}$ is the infinitesimal generator of X, then the above killed process has the generator $\mathcal{L} - c$.

We denote by $\mathcal{U}^c = (U^c_\alpha)_{\alpha>0}$ the resolvent of kernels induced by $\mathbb{T}^c = (T^c_t)_{t\geqslant 0}$, i.e., the resolvent family of the process killed with c,

$$U^c_\alpha f(x) = E^x\int_0^\infty e^{-\alpha - \int_0^t c(X_s)\,\mathrm{d}s} f(X_t)\,\mathrm{d}t,\ \alpha>0, f\in bp\mathcal{B},\ x\in E.$$

Proposition 1.4.2. *Let K be a sub-Markovian kernel on $(E,\mathcal{B})$. Then for any $f\in bp\mathcal{B}$ the equation*

$$(1.4.1)\qquad r_t(x) = T^c_t f(x) + \int_0^t T^c_{t-u}(cKr_u)(x)\,\mathrm{d}u,\ t\geqslant 0,\ x\in E,$$

has a unique solution $Q_t f\in bp\mathcal{B}$, the function $(t,x)\longmapsto Q_t f(x)$ is measurable and the following assertions hold.

(i) The family $\mathbb{Q}=(Q_t)_{t\geqslant 0}$ is a semigroup of sub-Markovian kernels on $(E,\mathcal{B})$ and it is the transition function of a Borel right process with state space E.

(ii) The function $t\longmapsto Q_t f(x)$ is right continuous on $[0,\infty)$ for each $x\in E$ if and only if the function $t\longmapsto T^c_t f(x)$ has the same property.

(iii) Let $\mathcal{U}^o=(U^o_\alpha)_{\alpha>0}$ be the resolvent of kernels on $(E,\mathcal{B})$ induced by $\mathbb{Q}=(Q_t)_{t\geqslant 0}$,

$$U^o_\alpha = \int_0^\infty e^{-\alpha t} Q_t\,\mathrm{d}t,\ \alpha>0.$$

Then for all $\beta>0$ we have

$$U^o_\beta = U^c_\beta + U^c_\beta cKU^o_\beta = U^c_\beta + G_\beta U^c_\beta,$$

where G_β is the bounded kernel defined as

$$G_\beta := \sum_{k\geqslant 1} (U^c_\beta cK)^k.$$

(iv) We have

$$\mathcal{E}(\mathcal{U}^o_\beta)\subset\mathcal{E}(\mathcal{U}^c_\beta),\quad [b\mathcal{E}(\mathcal{U}^o_\beta)] = [b\mathcal{E}(\mathcal{U}^c_\beta)] = [b\mathcal{E}(\mathcal{U}_\beta)],\ \textit{and}\ G_\beta(\mathcal{E}(\mathcal{U}^c_\beta))\subset\mathcal{E}(\mathcal{U}^o_\beta).$$

Proof. The uniqueness is a straitforward consequence of Gronwall's Lemma: If r_t and r'_t are two solutions of (1.4.1) and we set $v_t := \|r_t - r'_t\|_\infty$ then $v_t \leqslant \int_0^t \|T^c_{t-u}cK(r_u - r'_u)\|_\infty du \leqslant \beta_0\int_0^t v_u du$ for all $t\geqslant 0$, where $\beta_0 = \|cK1\|_\infty$, hence $v_t = 0$. To prove the existence, define inductively the kernels Q^n_t, $n\geqslant 0$, as $Q^0_t f := T^c_t f$,

$$(1.4.2)\qquad Q^{n+1}_t f := T^c_t f + \int_0^t T^c_{t-u} cKQ^n_u f du, f\in bp\mathcal{B}.$$

Clearly the function $(t,x) \longmapsto Q_t^n f(x)$ is measurable. We claim that the sequence of kernels $(Q_t^n)_n$ is increasing. Indeed, $Q_t^1 f = T_t^c f + \int_0^t T_{t-u}^c cKQ_u^0 f du \geqslant Q_t^0 f$. If we suppose that $Q_t^{n-1} f \leqslant Q_t^n f$ then $Q_t^{n+1} f = T_t^c f + \int_0^t T_{t-u}^c cKQ_u^n f du \geqslant T_t^c f + \int_0^t T_{t-u}^c cKQ_u^{n-1} du = Q_t^n f$. We can now define the kernel Q_t as $Q_t := \sup_n Q_t^n$. The function $(t,x) \longmapsto Q_t f(x)$ is measurable and we claim that Q_t is sub-Markovian. Indeed, it is sufficient to prove by induction that $Q_t^n 1 \leqslant 1$ for all $n \geqslant 0$. The inequality holds for $n = 1$ because $(T_t^c)_{t\geqslant 0}$ is sub-Markovian and $Q_t^0 1 = T_t^c 1$. If we assume that $Q_t^n 1 \leqslant 1$ then $Q_t^{n+1} 1 = T_t^c 1 + \int_0^t T_u^c (cKQ_t^n 1) du \leqslant T_t^c 1 + \int_0^t T_u^c c\, du = E^x(e^{-\int_0^t c(X_u)\, du} + \int_0^t e^{-\int_0^s c(X_u)\, du} c(X_s)\, ds) = 1$. Passing to the limit (pointwise) in (1.4.2) we get that $Q_t f$ is indeed a solution of (1.4.1).

The semigroup property of $(Q_t)_{t\geqslant 0}$ is a consequence of the uniqueness. We have to show that $Q_{t'+t} f = Q_t(Q_{t'} f)$, so, it is enough to prove that the mapping $t \longmapsto Q_{t'+t} f$ verifies (1.4.1) with $Q_{t'} f$ instead of f. We have $Q_{t'+t} f = T_t^c T_{t'}^c f + \int_0^{t'} T_t^c (T_{t'-u}^c cKQ_u f du) + \int_{t'}^{t'+t} T_{t'+t-u}^c cKQ_u f du = T_t^c (T_{t'}^c f + \int_0^{t'} T_{t'-u}^c cKQ_u f du) + \int_0^t T_{t-s}^c cKQ_{t'+s} f ds = T_t^c Q_{t'} f + \int_0^t T_{t-s}^c cKQ_{t'+s} f ds$.

To prove (ii) observe first that because the family $(Q_t)_{t\geqslant 0}$ is a semigroup, it is enough to verify the right continuity in $t = 0$. The assertion follows by dominate convergence since $Q_t f(x)$ is a solution of (1.4.1) and the function $u \longmapsto T_{t-u}^c cKr_u(x)$ is bounded on $[0,\infty)$: $\lim_{t\searrow 0} \int_0^t T_{t-u}^c cKr_u(x)\, du = 0$.

(iii) and (iv). The equality $U_\beta^o = U_\beta^c + U_\beta^c cKU_\beta^o$ follows from (1.4.1) by a straightforward calculation. Then by induction

$$U_\beta^o = U_\beta^c + (U_\beta^c cK)U_\beta^c + \ldots + (U_\beta^c cK)^n U_\beta^c + (U_\beta^c cK)^{n+1} U_\beta^o$$

and letting n tends to infinity we have $U_\beta^o = U_\beta^c + G_\beta U_\beta^c$. The kernel G_β is bounded because

$$U_\beta^c cK1 \leqslant ||\frac{c}{c+\beta}||_\infty \lim_{t\to\infty} \int_0^t T_u^{c+\beta}(c+\beta)\, \mathrm{d}u \leqslant \frac{c_o}{c_o+\beta},$$

where $c_o := ||c||_\infty$. If $u \in b\mathcal{E}(\mathcal{U}_\beta^o)$ then clearly $\alpha U_{\beta+\alpha}^c u \leqslant u$ for all $\alpha > 0$ because $U_\alpha^c \leqslant U_\alpha^o$. From $\lim_{t\to 0} Q_t u = u$ we get by (ii) that $\lim_{t\to 0} T_t^c u = u$, hence $u \in \mathcal{E}(\mathcal{U}_\beta^c)$, $b\mathcal{E}(\mathcal{U}_\beta^o) \subset \mathcal{E}(\mathcal{U}_\beta^c)$. The inequality $U_\beta^c \leqslant U_\beta^o$ for all $\beta > 0$ implies that the function $G_\beta U_\beta^c f = U_\beta^o f - U_\beta^c f$ is $\mathcal{U}_\beta^o$-excessive for every $f \in bp\mathcal{B}$. If $v \in b\mathcal{E}(\mathcal{U}_\beta^c)$ then we take a sequence $(f_n)_n \subset bp\mathcal{B}$ such that $U_\beta^c f_n \nearrow v$ and therefore $G_\beta U_\beta^c f_n \nearrow G_\beta v \in \mathcal{E}(\mathcal{U}_\beta^o)$, $U_\beta^o f_n \nearrow v + G_\beta v \in b\mathcal{E}(\mathcal{U}_\beta^o)$, so $v \in [b\mathcal{E}(\mathcal{U}_\beta^o)]$. We clearly have $\mathcal{E}(\mathcal{U}_\beta) \subset \mathcal{E}(\mathcal{U}_\beta^c) \subset \mathcal{E}(\mathcal{U}_{c_o+\beta})$ and by Proposition 1.4.1 we get $[b\mathcal{E}(\mathcal{U}_\beta)] = [b\mathcal{E}(\mathcal{U}_{c_o+\beta})] = [b\mathcal{E}(\mathcal{U}_\beta^c)]$.

To complete the proof of (i) we check now that (1.1.8) and (1.1.16) hold for $\mathcal{U}^o$. From $[b\mathcal{E}(\mathcal{U}_\beta^o)] = [b\mathcal{E}(\mathcal{U}_\beta^c)]$ and since $\mathcal{U}^c$ verifies (1.1.8) and (1.1.16) we get $\mathcal{B} = \sigma(\mathcal{E}(\mathcal{U}_\beta^c)) = \sigma(\mathcal{E}(\mathcal{U}_\beta^o))$ and by (1.1.9) we conclude that (1.1.16) holds for $\mathcal{U}^o$ too. The constant function 1 is $\mathcal{U}^o$-supermedian and it belongs to $[b\mathcal{E}(\mathcal{U}_\beta^o)]$ because $1 \in \mathcal{E}(\mathcal{U}_\beta) \subset [b\mathcal{E}(\mathcal{U}_\beta^o)]$. Therefore, using also (1.1.6), we get $\lim_{t\searrow 0} Q_t 1 = \lim_{t\searrow 0} e^{-\beta t} Q_t 1 = 1$, hence $1 \in \mathcal{E}(\mathcal{U}^o)$.

It remains to show that $(Q_t)_{t\geqslant 0}$ is the transition function of a right process with state space E. By Proposition 5.2.4 from [13] it follows that the natural solidity of potentials property (NSP) holds for $\mathcal{U}^o_\beta$. Because $\mathcal{U}$ is the resolvent of a right process with state space E and we proved that $[b\mathcal{E}(\mathcal{U}^o_\beta)] = [b\mathcal{E}(\mathcal{U}_\beta)]$, it follows that the topology of E is natural for $\mathcal{U}^o$ too. From the property (1.1.16) we get that all the points of E are nonbranch points with respect to $\mathcal{U}^o_q$. By (1.3.1) (the implication (1.3.1a) $\Longrightarrow$ (1.3.1c)) we conclude that there exists a right process with state space E, having $\mathcal{U}^o$ as associated resolvent. From assertion (ii) of Proposition 1.3.2 we conclude now that the transition function of this right process is precisely $(Q_t)_{t\geqslant 0}$, completing the proof. □

Remark. (i) *Proposition 1.4.2 is a result on the perturbation with kernels of the sub-Markovian semigroups and resolvents. In fact, the existence of the solution of the equation* (1.4.1) *is a linear version of Proposition 2.2.1 below.*

(ii) *In the particular case of a Banach space, if A is the generator of the semigroup $(T^c_t)_{t\geq 0}$ then the solution of the equation* (1.4.1) *has as generator $A - c + cK$, where K is a bounded operator (see Lema 1.4 from [23] for a complete proof).*

If $M \in \mathcal{B}$, $\beta > 0$, and $u \in \mathcal{E}(\mathcal{U}^c_\beta)$ (resp. $u \in \mathcal{E}(\mathcal{U}^o_\beta)$), let ${}^c\!R^M_\beta u$ (resp. ${}^o\!R^M_\beta u$) be the reduced function of u on M with respect to $\mathcal{U}^c_\beta$ (resp. $\mathcal{U}^o_\beta$). Let further $v_o := U^o_\beta 1 = U^c_\beta f_o$, where $f_o := 1 + cKU^o_\beta 1$, and fix a finite measure λ on $(E, \mathcal{B})$. We denote by c^c_λ (resp. c^o_λ) the induced capacity:

$$c^c_\lambda(M) := \inf\{\int {}^c\!R^D_\beta v_o \,\mathrm{d}\lambda : D \text{ open}, M \subset D\}$$

(resp. $c^o_\lambda(M) := \inf\{\int {}^o\!R^D_\beta v_o \,\mathrm{d}\lambda : D \text{ open}, M \subset D\}$).

Corollary 1.4.3. *We have $c^o_\lambda \leqslant c^c_{\lambda'} \leqslant c_{\lambda'}$, where $\lambda' := \lambda + \lambda \circ G_\beta$. In particular, if the capacity $c_{\lambda'}$ is tight then the capacities $c^c_{\lambda'}$ and c^o_λ are also tight.*

Proof. Because $\mathcal{E}(\mathcal{U}_\beta) \subset \mathcal{E}(\mathcal{U}^c_\beta)$ we get ${}^c\!R^M_\beta v_o \leqslant {}^c\!R^M_\beta u_o \leqslant R^M_\beta u_o$ for every $M \in \mathcal{B}$ (where recall that $u_o = U_\beta f_o \geqslant v_o$) and therefore $c^c_{\lambda'} \leqslant c_{\lambda'}$. By assertion (iii) of Proposition 1.4.2 we may apply the result from [13], Proposition 5.2.5, to obtain the inequality of kernels ${}^o\!R^M_\beta \leqslant {}^c\!R^M_\beta + G_\beta {}^c\!R^M_\beta$ which leads to $c^o_\lambda(M) \leqslant c^c_{\lambda'}(M)$. □

Remark. A result related to Corollary 1.4.3 was stated in Proposition 3.7 from [24].

Chapter 2

Discrete branching Markov processes on the finite configurations

2.1 Spaces of measures, branching kernels, branching processes

Let $M(E)$ be the set of all positive finite measures on E.

For a function $f \in p\mathcal{B}$ we shall consider the mappings $l_f : M(E) \longrightarrow \overline{\mathbb{R}}_+$ and $e_f : M(E) \longrightarrow [0,1]$, defined by

$$l_f(\mu) := \langle \mu, f \rangle := \int f d\mu,\ \mu \in M(E), \quad e_f := e^{-l_f}.$$

$M(E)$ is endowed with the weak topology and note that the Borel σ-algebra $\mathcal{M}(E)$ on $M(E)$ is generated by $\{l_f : f \in bp\mathcal{B}\}$.

A second set of measures as state space for a forthcoming discrete branching process will be the set S of all positive measures μ on E which are finite sums of Dirac measures: $\mu = \sum_{k=1}^{m} \delta_{x_k}$, where $x_1, \ldots, x_m \in E$, called the space of *finite configurations* of E (cf. [92]). The space S of all finite configurations of E is identified with the union of all symmetric m-th powers $E^{(m)}$ of E, hence

$$S = \bigcup_{m \geq 1} E^{(m)};$$

$E^{(m)}$ is the factorization of the Cartesian product E^m by the equivalence relation induced by the permutation group σ^m; for details see, e.g., [53] and [20]. The space $E^{(m)}$ is endowed with the quotient topology, where E^m is equipped with the product topology. The space S is equipped with the canonical topological structure and we denote by $\mathcal{B}(S)$ the Borel σ-algebra on S.

Branching kernels. Recall that if p_1, p_2 are two finite measures on $M(E)$, then their convolution $p_1 * p_2$ is the finite measure on $M(E)$ defined for every $F \in bp\mathcal{M}(E)$ by

$$\int p_1 * p_2(d\nu)F(\nu) := \int p_1(d\nu_1) \int p_2(d\nu_2)F(\nu_1 + \nu_2).$$

In particular, if $f \in p\mathcal{B}$ then

(2.1.1) $$p_1 * p_2(e_f) = p_1(e_f) \cdot p_2(e_f).$$

Note that if p_1 and p_2 are concentrated on S then $p_1 * p_2$ has the same property.

According with [92], a kernel N on $(S, \mathcal{B}(S))$ (resp. on $(M(E), \mathcal{M}(E))$) which is sub-Markovian (i.e., $N1 \leq 1$) is called *branching kernel* provided that for all $\mu, \nu \in S$ (resp. for all $\mu, \nu \in M(E)$) we have

$$N_{\mu+\nu} = N_\mu * N_\nu,$$

where N_μ denotes the measure on $(S, \mathcal{B}(S))$ (resp. on $(M(E), \mathcal{M}(E))$) such that $Ng(\mu) = \int g\, dN_\mu$ for all $g \in bp\mathcal{B}(S)$ (resp. $g \in bp\mathcal{M}(E)$).

A right (Markov) process with state space $M(E)$ or S is called *branching process* provided that its transition function is formed by branching kernels. For the probabilistic interpretation of this analytic branching property see e.g., [43], page 337.

Example of branching kernels on $M(E)$. We set $\mathcal{S} := bp\mathcal{B}$. Recall that a function $\varphi : \mathcal{S} \longrightarrow \mathbb{R}$ is called *negative definite* provided that for all $n \geq 2$, $\{f_1, f_2, ..., f_n\} \subset \mathcal{S}$, and $\{a_1, a_2, ..., a_n\} \subset \mathbb{R}$ with $\sum_{i=1}^n a_i = 0$ we have

$$\sum_{i,j} a_i a_j \varphi(f_i + f_j) \leq 0.$$

(2.1.2) Let $\varphi : \mathcal{S} \longrightarrow \mathbb{R}_+$ be a negative definite function such that if $(f_n)_n$ is pointwise decreasing to f then $(\varphi(f_n))_n$ is decreasing to $\varphi(f)$. Then there exists a unique finite measure $\bar{P}$ on $(M(E), \mathcal{M}(E))$ such that

$$\bar{P}(e_f) = e^{-\varphi(f)} \text{ for all } f \in \mathcal{S}.$$

Let now $V : \mathcal{S} \longrightarrow \mathcal{S}$ be a map such that for every $x \in E$ the function $f \longmapsto Vf(x)$ is negative definite. Applying (2.1.2), there exists a unique kernel Q on $(M(E), \mathcal{M}(E))$ such that

(2.1.3) $$Q(e_f) = e_{Vf} \quad \text{for all } f \in \mathcal{S}$$

(see (A.6) from [43] and Proposition 4.1 in [12]). We claim that Q is a branching kernel on $M(E)$. Indeed, we have to prove that for all $\mu, \nu \in M(E)$

$$Q_{\mu+\nu} = Q_\mu * Q_\nu.$$

By a monotone class argument it is sufficient to check the above equality for functions on $M(E)$ of the form e_f, $f \in \mathcal{S}$. Using (2.1.1) and (2.1.3) then we have

$$Q_\mu * Q_\nu(e_f) = Q_\mu(e_f) \cdot Q_\nu(e_f) = e_{Vf}(\mu) \cdot e_{Vf}(\nu) = Q_{\mu+\nu}(e_f).$$

Multiplicative functions and branching kernels on S. For every real-valued $\mathcal{B}$-measurable function φ consider the function $\widehat{\varphi} : \bigcup_{m\geq 1} E^m \longrightarrow \mathbb{R}$ defined as

$$\widehat{\varphi}(x) := \varphi(x_1) \cdot \ldots \cdot \varphi(x_m) \text{ for } x = (x_1, \ldots, x_m) \in E^m.$$

Note that the restriction of $\widehat{\varphi}$ to each E^m is invariant under the action of σ^m. Therefor $\widehat{\varphi}$ may be considered as a real-valued function defined on S. Such a function $\widehat{\varphi} : S \longrightarrow \mathbb{R}$ is called *multiplicative* (cf. [92]; see also [20]).

With this notation a multiplicative function $\widehat{\varphi}$, $\varphi \in p\mathcal{B}$, $\varphi \leq 1$, is the restriction to S of an exponential function on $M(E)$,

$$\widehat{\varphi} = e_{-ln\varphi}.$$

(2.1.4) For every sub-Markovian kernel $B : p\mathcal{B}(S) \longrightarrow p\mathcal{B}$ there exists a branching kernel $\widehat{B}$ on $(S, \mathcal{B}(S))$ such that for every $\mathcal{B}$-measurable function φ, $|\varphi| \leq 1$, we have

$$\widehat{B}\widehat{\varphi} = \widehat{B\widehat{\varphi}}.$$

Conversely, if H is a branching kernel on $(S, \mathcal{B}(S))$ then there exists a unique sub-Markovian kernel $B : p\mathcal{B}(S) \longrightarrow p\mathcal{B}$ such that $H = \widehat{B}$ (see Proposition 3.2 in [20]).

Example of branching kernels on S. Let $q_k \in p\mathcal{B}$, $k \geq 1$, satisfying $\sum_{k\geq 1} q_k \leq 1$. Consider the kernel $B : p\mathcal{B}(S) \longrightarrow p\mathcal{B}$ defined as

(2.1.5) $$Bg(x) := \sum_{k\geq 1} q_k(x) g_k(x, \ldots, x), \quad g \in bp\mathcal{B}(S),\ x \in E,$$

with $g_k := g|_{E^{(k)}}$ for all $k \geq 1$. Using (2.1.4) there exists a branching kernel $\widehat{B}$ on S such that for all $\varphi \in p\mathcal{B}$, $\varphi \leq 1$, we have

$$\widehat{B}\widehat{\varphi}|_E = \sum_{k\geq 1} q_k \varphi^k.$$

2.2 Nonlinear evolution equations

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a fixed Borel right process with space E and suppose that its transition function $(T_t)_{t \geqslant 0}$ of X is Markovian.

Let $B : bp\mathcal{B}(S) \longrightarrow bp\mathcal{B}$ be a sub-Markovian kernel such that

(2.2.1) $$\sup_{x \in E} Bl_1(x) < \infty.$$

Note that (2.2.1) is precisely condition 4.1.2 from [92], (2.1) in [34], or (4.3.1) from [67] and if B is given by (2.1.5) then (2.2.1) is equivalent with

$$\sup_{x \in E} \sum_{k \geqslant 1} k q_k(x) < \infty.$$

In Section 1.4 we fixed a function $c \in bp\mathcal{B}$ and we denoted by $(T_t^c)_{t \geqslant 0}$ the transition function of the process obtained by killing X with the multiplicative functional $(e^{-\int_0^t c(X_s)ds})_{t \geqslant 0}$. Recall that $\mathcal{U} = (U_\alpha)_{\alpha > 0}$ is the resolvent of the process X and $\mathcal{U}^c = (U_\alpha^c)_{\alpha > 0}$ the resolvent of kernels induced by $(T_t^c)_{t \geqslant 0}$, i.e., the resolvent family of the above mentioned killed process (by means of the function c).

Denote by $\mathcal{B}_{\mathsf{u}}$ the set of all functions $\varphi \in p\mathcal{B}$ such that $\varphi \leqslant 1$. Recall that a map $H : \mathcal{B}_{\mathsf{u}} \longrightarrow \mathcal{B}_{\mathsf{u}}$ is called *absolutely monotonic* provided that there exists a sub-Markovian kernel $\mathbf{H} : bp\mathcal{B}(S) \longrightarrow bp\mathcal{B}$ such that $H\varphi = \mathbf{H}\widehat{\varphi}$ for all $\varphi \in \mathcal{B}_{\mathsf{u}}$. By (2.1.4) we have:

(2.2.2) *A map $H : \mathcal{B}_{\mathsf{u}} \longrightarrow \mathcal{B}_{\mathsf{u}}$ is absolutely monotonic if and only if there exists a branching kernel $\widehat{\boldsymbol{H}}$ on S such that $\widehat{\boldsymbol{H}}\widehat{\varphi} = \widehat{H\varphi}$ for all $\varphi \in \mathcal{B}_{\mathsf{u}}$.*

We also have (cf. Lemma 2.2 and Theorem 1 from [92]):

(2.2.3) *If H, K are absolutely monotonic then their composition HK is also absolutely monotonic and $\widehat{\boldsymbol{HK}} = \widehat{\boldsymbol{H}}\widehat{\boldsymbol{K}}$. The map $H \longmapsto \widehat{\boldsymbol{H}}$ is a bijection between the set of all absolutely monotonic mappings and the set of all branching kernels on S.*

In the next proposition we solve an appropriate integral equation (following the approach of [92], see also [20]).

Proposition 2.2.1. *For any $\varphi \in \mathcal{B}_{\mathsf{u}}$ the equation*

(2.2.4) $$h_t(x) = T_t^c\varphi(x) + \int_0^t T_{t-u}^c(cB\widehat{h_u})(x)du,\ t \geqslant 0,\ x \in E,$$

has a unique solution $(t,x) \longmapsto H_t\varphi(x)$ *jointly measurable in* (t,x)*, such that* $H_t\varphi \in \mathcal{B}_{\mathsf{U}}$ *for all* $t > 0$ *and the following assertions hold.*

(i) For each $t > 0$ *the mapping* $\varphi \longmapsto H_t\varphi$ *is absolutely monotonic and it is Lipschitz with the constant* $\beta_o t$*, where*

$$\beta_o := \|c\|_\infty \|Bl_1\|_\infty.$$

(ii) The familiy $(H_t)_{t\geqslant 0}$ *is a semigroup of nonlinear operators on* $\mathcal{B}_{\mathsf{U}}$*. If* $B1 = 1$ *then* $H_t 1 = 1$ *for all* $t \geqslant 0$*.*

(iii) For each $x \in E$ *the function* $t \longmapsto H_t\varphi(x)$ *is right continuous on* $[0,\infty)$*, provided that* $t \longmapsto T_t^c\varphi(x)$ *is right continuous.*

Proof. Let

$$K\varphi := B\widehat{\varphi}.$$

With this notation (2.2.4) becomes

(2.2.5) $$h_t(x) = T_t^c\varphi(x) + \int_0^t T_{t-u} cKh_u(x)\, du, \quad t \geqslant 0,\ x \in E.$$

We prove first the uniqueness. As in [92], the inequality (4.11), one can see that if $\varphi, \psi \in \mathcal{B}_{\mathsf{U}}$ and $\mu \in S$ then

(2.2.6) $$|\widehat{\varphi}(\mu) - \widehat{\psi}(\mu)| \leqslant l_1(\mu) ||\varphi - \psi||_\infty.$$

From (2.2.1) and the (2.2.6) we conclude that

(2.2.7) the mapping $\varphi \longmapsto cK\varphi$ is Lipschitz with the constant β_0.

If h_t and h'_t are two solutions of (2.2.4) then for all $t \geqslant 0$

$$\|h_t - h'_t\|_\infty \leqslant \int_0^t \|T_{t-u}(|\, cKh_u - cKh'_u\,|)\|_\infty du \leqslant \beta_o \int_0^t \|\, h_u - h'_u\, \|_\infty\, du.$$

It follows by Gronwall's Lemma that $\|\, h_t - h'_t\, \|_\infty = 0$.

To prove the existence, define inductively the operators H_t^n, $n \geqslant 0$, as $H_t^0\varphi := T_t^c\varphi$,

(2.2.8) $$H_t^{n+1}\varphi := T_t^c\varphi + \int_0^t T_{t-u} cKH_u^n\varphi\, du,\ \varphi \in \mathcal{B}_{\mathsf{U}}.$$

Clearly the function $(t,x) \longmapsto H_t^n\varphi(x)$ is measurable. We claim that the sequence $(H_t^n\varphi)_n$ is increasing. Indeed, $H_t^1\varphi = T_t^c\varphi + \int_0^t T_{t-u}cKH_u^0\varphi\ du \geqslant H_t^0\varphi$. If we suppose that $H_t^{n-1}\varphi \leqslant H_t^n\varphi$ then $H_t^{n+1}\varphi = T_t^c\varphi + \int_0^t T_{t-u}cKH_u^n\varphi\ du \geqslant T_t^c\varphi +$

$\int_0^t T_{t-u} cKH_u^{n-1}\varphi \; du = H_t^n\varphi$. The last inequality holds because if $\varphi \leqslant \psi$ then $cK\varphi \leqslant cK\psi$ and in addition one can prove inductively that $H_t^n\varphi \leqslant H_t^n\psi$ for all n.

We claim now that

$$(2.2.9) \qquad H_t^n 1 \leqslant 1 \quad \text{for all } n \geqslant 0.$$

We proceed again by induction. The inequality holds for $n = 1$ because $(T_t^c)_{t\geqslant 0}$ is sub-Markovian and $H_t^0 1 = T_t^c 1$. If we assume that $H_t^n 1 \leqslant 1$ then $\widehat{H_t^n 1} \leqslant 1$ and therefore

$$H_t^{n+1}1 = T_t^c 1 + \int_0^t T_u^c(cB\widehat{H_t^n 1})du \leqslant T_t^c 1 + \int_0^t T_u^c c \, du =$$

$$E^x(e^{-\int_0^t c(X_u)\,du} + \int_0^t e^{-\int_0^s c(X_u)\,du} c(X_s)\,ds) = 1.$$

If $\varphi \in \mathcal{B}_{\mathsf{u}}$ then by (2.2.9) $H_t^n\varphi \in \mathcal{B}_{\mathsf{u}}$ for all $n \geqslant 0$. For $x \in E$, $t \geqslant 0$, and $\varphi \in \mathcal{B}_{\mathsf{u}}$ we set

$$H_t\varphi(x) := \sup_n H_t^n\varphi(x).$$

The function $(t, x) \longmapsto H_t\varphi(x)$ is measurable, by (2.2.9) we have $H_t 1 \leqslant 1$, $H_t(\mathcal{B}_{\mathsf{u}}) \subset \mathcal{B}_{\mathsf{u}}$, and passing to the pointwise limit in (2.2.8) it follows that $(H_t\varphi)_{t\geqslant 0}$ verifies (2.2.5).

(i) We show inductively that for all n the operator H_t^n is absolutely monotonic. If $n = 1$ then $H_t^1\varphi = T_t^c\varphi = \mathbf{T}_t\widehat{\varphi}$, where $\mathbf{T}_t : bp\mathcal{B}(S) \longrightarrow bp\mathcal{B}$ is the kernel defined by $\mathbf{T}_t g = T_t^c(g|_E)$ for all $g \in bp\mathcal{B}(S)$. Hence $H_t^1\varphi = \mathbf{T}_t\widehat{\varphi}$ for all $\varphi \in \mathcal{B}_{\mathsf{u}}$ and therefore H_t^1 is absolutely monotonic. Suppose now that H_t^n is absolutely monotonic, $H_t^n\varphi = \mathbf{H}_t^n\widehat{\varphi}$. We have

$$H_t^{n+1}\varphi = \mathbf{T}_t\widehat{\varphi} + \int_0^t T_{t-u}cB\widehat{H_u^n\varphi}\; du = (\mathbf{T}_t + \int_0^t T_{t-u}cB\widehat{\mathbf{H}_u^n}\; du)\widehat{\varphi},$$

where $\widehat{\mathbf{H}_u^n}$ is the branching kernel on S associated by (2.2.2) with $\mathbf{H}_u^n$. Taking

$$(2.2.10) \qquad \mathbf{H}_t^{n+1} := \mathbf{T}_t + \int_0^t T_{t-u}cB\widehat{\mathbf{H}_u^n}\; du,$$

it follows that H_t^{n+1} is also absolutely monotonic. One can deduce from (2.2.10) that for all $t \geqslant 0$ the sequence of kernels $(\mathbf{H}_t^n)_{n\geqslant 0}$ is increasing and therefore we may consider the kernel $\mathbf{H}_t$ defined as $\mathbf{H}_t := \sup_n \mathbf{H}_t^n$. From the above considerations for all $\varphi \in \mathcal{B}_{\mathsf{u}}$ we have $H_t\varphi = \sup_n H_t^n\varphi = \sup_n \mathbf{H}_t^n\widehat{\varphi} = \mathbf{H}_t\widehat{\varphi}$ and we conclude that H_t is absolutely monotonic.

We prove now the Lipschitz property of the mapping $\varphi \longmapsto H_t\varphi$. For, if $\varphi, \psi \in \mathcal{B}_{\mathsf{u}}$ and $t \geqslant 0$ then by (2.2.5) and (2.2.7)

$$||H_t\varphi - H_t\psi||_\infty \leqslant ||\varphi - \psi||_\infty + \beta_0 \int_0^t ||H_u\varphi - H_u\psi||_\infty du$$

and by Gronwall's Lemma we conclude that $||H_t\varphi - H_t\psi||_\infty \leqslant \beta_0 t||\varphi - \psi||_\infty$.

(ii) The semigroup property of $(H_t)_{t\geqslant 0}$ is a consequence of the uniqueness. Indeed, we have to show that $H_{t'+t}\varphi = H_t(H_{t'}\varphi)$, so, it is enough to prove that the mapping $t \longmapsto H_{t'+t}\varphi$ verifies (2.2.5) with $H_{t'}\varphi$ instead of φ. We have

$$H_{t'+t}\varphi = T_t^c T_{t'}\varphi + \int_0^{t'} T_t^c(T_{t'-u}^c cKH_u\varphi\, du) + \int_{t'}^{t'+t} T_{t'+t-u}^c cKH_u\varphi\, du =$$

$$T_t^c(T_{t'}\varphi + \int_0^{t'} T_{t'-u}^c cKH_u\varphi\ du) + \int_0^t T_{t-s}^c cKH_{t'+s}\varphi\, ds = T_t^c H_{t'}\varphi + \int_0^t T_{t-s}^c cKH_{t'+s}\varphi\, ds.$$

Suppose now that $B1 = 1$ and define inductively the operators $H_t'^n$, $n \geqslant 0$, as $H_t'^0\varphi := T_t^c\varphi + \int_0^t T_u^c cK\varphi du$,

$$H_t'^{n+1}\varphi := T_t^c\varphi + \int_0^t T_u^c cKH_{t-u}'^n\varphi\, du,\ \varphi \in \mathcal{B}_{\mathsf{u}}. \tag{2.2.11}$$

We already observed that $T_t^c 1 + \int_0^t T_u^c c\, du = 1$, therefore $H_t'^0 1 = 1$ and by induction we get that $H_t'^n 1 = 1$ for all $n \in \mathbb{N}$. Using (2.2.7) as before we obtain

$$||H_t'^{n+1}\varphi - H_t'^n\varphi||_\infty \leqslant \beta_o \int_0^t ||H_u'^n\varphi - H_u'^{n-1}\varphi||_\infty du$$

and because $||H_t'^1\varphi - H_t'^0\varphi||_\infty \leqslant \beta_o||\varphi||_\infty \int_0^t (2 + \beta_o u) du = ||\varphi||_\infty(2\beta_o t + \frac{(\beta_o t)^2}{2})$ again by induction

$$||H_t'^{n+1}\varphi - H_t'^n\varphi||_\infty \leqslant ||\varphi||_\infty \left(2\frac{(\beta_o t)^{n+1}}{(n+1)!} + \frac{(\beta_o t)^{n+2}}{(n+2)!}\right).$$

Consequently, if $t_o > 0$ is fixed then

$$\sup_{\substack{x\in E\\ t\leqslant t_o}} \left|H_t'^{n+1}\varphi(x) - H_t'^n\varphi(x)\right| \leqslant \left(2\frac{(\beta_o t_o)^{n+1}}{(n+1)!} + \frac{(\beta_o t_o)^{n+2}}{(n+2)!}\right).$$

It follows that the sequence $(H_t'^n\varphi)_n$ is Cauchy in the supremum norm and passing to the limit in (2.2.11), we deduce that the point-wise limit of this sequence verifies (2.2.5), hence it is $H_t\varphi$ by the uniqueness of the solution. In particular, $H_t 1 = \lim_n H_t'^n 1 = 1$.

(iii) Because the family $(H_t)_{t\geqslant 0}$ is a semigroup, it is enough to prove the right continuity in $t = 0$. Since $H_t\varphi(x)$ is a solution of (2.2.5) and the function $u \longmapsto T_{t-u}^c cKh_u(x)$ is bounded on $[0, \infty)$, by dominate convergence we get $\lim_{t\searrow 0} \int_0^t T_{t-u}^c cKh_u(x)\, du = 0$, hence $t \longmapsto H_t\varphi(x)$ is right continuous in $t = 0$. □

Remark 2.2.2. (i) *If $\varphi \in \mathcal{B}_{\mathsf{u}}$ and $t \geqslant 0$ then the sequence $(H_t^n \varphi)_{n \geqslant 0}$ defined by (2.2.8) converges uniformly to the solution $H_t\varphi$ of (2.2.4). The assertion is a consequence of the following inequality which may be proved by induction:*

$$\|H_t^{n+1}\varphi - H_t^n \varphi\|_\infty \leqslant \frac{(\beta_o t)^n}{n!}\|\varphi\|_\infty \quad \textit{for all } n \geqslant 0.$$

(ii) *Note that if $\mathcal{L}$ is the infinitesimal generator of the base process X, then (2.2.4) is formally equivalent to*

$$\frac{d}{dt}h_t = (\mathcal{L} - c)h_t + cB\widehat{h_t},\ t \geqslant 0.$$

(iii) *If B is given by (2.1.5) then the condition $B1 = 1$ is equivalent with*

$$\sum_{k \geqslant 1} q_k(x) = 1 \ \textit{ for all } x \in E.$$

2.3 Semigroups of branching kernels on the finite configurations, the cumulant semigroup

Let further $(H_t)_{t \geqslant 0}$ be the semigroup of nonlinear operators on $\mathcal{B}_{\mathsf{u}}$ given by assertion (ii) of Proposition 2.2.1.

Theorem 2.3.1. *For each $t \geqslant 0$ let $\widehat{\boldsymbol{H}_t}$ be the branching kernel on S associated by (2.2.2) with the absolutely monotonic operator H_t from Proposition 2.2.1, $H_t\varphi = \widehat{\boldsymbol{H}_t}\widehat{\varphi}|_E$ for all $\varphi \in \mathcal{B}_{\mathsf{u}}$. Then the following assertions hold.*

(i) *The family $(\widehat{\boldsymbol{H}_t})_{t \geqslant 0}$ is a sub-Markovian semigroup of branching kernels on $(S, \mathcal{B}(S))$.*

(ii) *For each $t \geqslant 0$ and $f \in bp\mathcal{B}$ define the function $V_t f \in p\mathcal{B}$ as*

$$V_t f := -ln H_t(e^{-f}).$$

Then the family $(V_t)_{t \geqslant 0}$ is a semigroup of (nonlinear) operators on $bp\mathcal{B}$ and

$$\widehat{\boldsymbol{H}_t}(e_f) = e_{V_t f} \ \textit{ for all } f \in bp\mathcal{B}. \tag{2.3.1}$$

Proof. Assertion (i) follows from (2.2.3) and Proposition 2.2.1. To prove assertion (ii) it is enough to show that if $f \in bp\mathcal{B}$ then $V_t f \in bp\mathcal{B}$. If $f \leqslant M$, because $H_t(e^{-M}) \geqslant T_t^c(e^{-M})$ and since the transition function of X is Markovian, we have $V_t f \leqslant V_t M \leqslant -ln(e^{-M}E^x(e^{-\int_0^t c(X_s)ds})) \leqslant M + t\|c\|_\infty$. □

Remark 2.3.2. (i) *If $v \in \mathcal{B}_{\boldsymbol{u}}$ is such that $\widehat{v}$ is an invariant function with respect to the branching semigroup $(\widehat{\boldsymbol{H}}_t)_{t\geqslant 0}$ on S, then v belongs to the domain of $\mathcal{L}$ (the infinitesimal generator of the base process X) and*

$$(\mathcal{L} - c)v + cB\widehat{v} = 0. \tag{2.3.2}$$

In particular, if B is given by (2.1.5), then v is the solution of the nonlinear equation

$$(\mathcal{L} - c)v + c\sum_{k\geqslant 1} q_k v^k = 0.$$

It turns out that $(\widehat{\boldsymbol{H}}_t)_{t\geqslant 0}$ is the main tool in the treatment of the (nonlinear) Dirichlet problem associated with the equation (2.3.2); see Proposition 6.1 and Subsection 6.2 in [20].

(ii) *The equation (2.2.4) and the equality (2.3.1) are analogous to* (2.2) *and respectively* (2.4) *from [34], where, however, the forthcoming branching process having $(\widehat{\boldsymbol{H}}_t)_{t\geqslant 0}$ as transition function is used. Assertion* (ii) *of Theorem 2.3.1 shows that the semigroup approach for the continuous branching (developed by E.B. Dynkin [40] and P.J. Fitzsimmons [43]; see also [12] and [67]) is analogue to the above construction of the branching semigroup in the discrete branching case, due to N. Ikeda, M. Nagasawa, S. Watanabe, and M.L. Silverstein. Recall that in the case of the continuous branching the family $(V_t)_{t\geqslant 0}$ is the so called "cumulant semigroup"; for more details see Section 2.6 below.*

(iii) *Assume that E is an Euclidean open set, X is the Brownian motion on E, and B is given by (2.1.5). Then by (2.3.1) and Remark 2.2.2* (ii) *one can see that the cumulant semigroup $(V_t)_{t\geqslant 0}$ is formally the solution of the following evolution equation*

$$\frac{d}{dt}V_t f = \Delta V_t f - |\nabla V_t f|^2 + c(1 - \sum_{k\geqslant 1} q_k e^{(1-k)V_t f}),\ t \geqslant 0.$$

This should be compare with the equation satisfied by the cumulant semigroup of a mesure-valued continuous branching process (cf. $(2.2)'$ *from [43] and (2.6.1) below), in particular, in the case of the super-Brownian motion: $\frac{d}{dt}V_t f = \Delta V_t f - (V_t f)^2$, $t \geqslant 0$.*

(iv) *We refer to the survey article [19] for a version of assertion* (ii) *of Theorem 2.3.1 and for further connections between the continuous and discrete measure-valued processes.*

2.4 Linear and exponential excessive functions

Let further

$$\beta_1 := ||Bl_1||_\infty.$$

and assume that $B1 = 1$, hence $\beta_1 \geqslant 1$. We suppose that $\beta_1 > 1$ and that the function c is such that $c < \frac{\beta_1}{\beta_1 - 1}$. Let $(Q_t)_{t\geqslant 0}$ be the semigroup given by Proposition 1.4.2, with the sub-Markovian kernel K on $(E, \mathcal{B})$ defined as $Kf := \frac{c}{c+\beta_1} B(l_f)$ and with $c + \beta_1$ instead of c.

Lemma 2.4.1. *If B is given by (2.1.5) and c does not depend on $x \in E$, then*

$$Q_t f(x) = e^{-(c+\beta_1)t} E^x (e^{\int_0^t c q_o(X_s)\, ds} f(X_t)), \quad f \in bp\mathcal{B}, x \in E, t > 0,$$

where $q_o := \sum_{k\geqslant 1} k q_k$, and we have $[b\mathcal{E}(\mathcal{U}^o)] = [b\mathcal{E}(\mathcal{U}_\beta)]$ for all $\beta > 0$.

Proof. Observe first that in this case

$$B(l_f) = q_o f,\ f \in bp\mathcal{B},$$

and equation (1.4.1) with $c + \beta_1$ instead of c and $Kf = \frac{c}{c+\beta_1} B(l_f)$ becomes

$$k_t(x) = T_t f(x) - \int_0^t T_{t-u}(b k_u)(x) du,\ t \geqslant 0,\ x \in E,$$

where $k_t := e^{(c+\beta_1)t} r_t$ and $b := -c q_o$. By Proposition 3.3 (i) from [12] the above equation has a unique solution $T_t^b f$ and consequently $Q_t = T_t^{c+\beta_1 - c q_o}$ for all $t \geqslant 0$. The claimed expression of Q_t follows now also from [12], the equality (3.3). Let $\beta_2 := c + \beta_1 - c\beta_1$. From Proposition 3.3 (iii) from [12], since $0 < \beta_2 \leqslant c + \beta_1 - c q_o \leqslant \beta_1$, we have $T_t^{\beta_1} \leqslant Q_t \leqslant T_t^{\beta_2}$, $\mathcal{E}(\mathcal{U}_{\beta_2}) \subset \mathcal{E}(\mathcal{U}^o) \subset \mathcal{E}(\mathcal{U}_{\beta_1})$. By Proposition 1.4.1 we conclude that $[b\mathcal{E}(\mathcal{U}^o)] = [b\mathcal{E}(\mathcal{U}_{\beta_1})] = [b\mathcal{E}(\mathcal{U}_\beta)]$ for all $\beta > 0$. □

In the next proposition we prove relations between a set of excessive functions with respect to the base process X and excessive functions with respect to the forthcoming branching process on S. The key tool is the equality from the assertion (i) below. Note that a similar equality was obtained in the case of continuous branching processes in [43], Proposition 2.7 (see also [12], Proposition 4.2). It turns out that a perturbation of the generator of the base process was necessary in that case too, however a simpler one, the perturbed semigroup $(Q_t)_{t\geqslant 0}$ being expressed with a Feynman-Kac formula, as in the particular case discussed in the above Lemma 2.4.1.

Proposition 2.4.2. *The following assertions hold.*

(i) *If $f \in bp\mathcal{B}$ and $t > 0$ then*

$$e^{-\beta_1 t} \widehat{\boldsymbol{H}}_t(l_f) = l_{Q_t f}.$$

(ii) *If $\beta > 0$ and $\beta' := \beta_1 + \beta$ then the following assertions are equivalent for every $u \in bp\mathcal{B}$.*

(ii.a) $u \in b\mathcal{E}(\mathcal{U}^o_\beta)$.

(ii.b) $l_u \in \mathcal{E}(\widehat{\mathcal{U}}_{\beta'})$.

(ii.c) For every $\alpha > 0$ *we have* $1 - e_{\alpha u} \in \mathcal{E}(\widehat{\mathcal{U}}_{\beta'})$.

(iii) If $u \in \mathcal{E}(\mathcal{U}^o_\beta)$ *is a compact Lyapunov function on* E *then* $l_{1+u} \in \mathcal{E}(\widehat{\mathcal{U}}_{\beta'})$ *is a compact Lyapunov function on* S.

Proof. (i) We show first that if $f \in bp\mathcal{B}$ and N is a kernel on S or from S to E, such that $N(l_1)$ is a bounded function then

$$N(e_{\lambda f})'_{\lambda=0} := \lim_{\lambda \to 0} \frac{N(e_{\lambda f}) - N1}{\lambda} = -N(l_f). \tag{2.4.1}$$

Indeed, the assertion follows since $\frac{1-e_{\lambda f}}{\lambda} \nearrow l_f$ pointwise on S. The Lipschitz property of H_t (Proposition 2.2.1 (i)) implies

$$||H_t(e^{-f}) - 1||_\infty \leqslant \beta_o t ||1 - e^{-f}||_\infty \leqslant \beta_o t ||f||_\infty, \quad \left| \frac{H_t(e^{-\lambda f}) - 1}{\lambda} \right| \leqslant \beta_o t ||f||_\infty.$$

Applying (2.4.1) with $N = \widehat{\mathbf{H}_t}$ and since $\widehat{\mathbf{H}_t}(e_{\lambda f})|_E = H_t(e^{-\lambda f})$ we deduce that $\widehat{\mathbf{H}_t}(l_f)|_E \leqslant \beta_o t ||f||_\infty$ and we claim that

$$\widehat{\mathbf{H}_t}(l_f) = -\widehat{\mathbf{H}_t}(e_{\lambda f})'_{\lambda=0} = l_{\widehat{\mathbf{H}}_{t(l_f)|_E}}. \tag{2.4.2}$$

By (2.4.1) it is sufficient to show the second equality. Let $\mu \in E^{(m)}$, $\mu = \sum_{k=1}^m \delta_{x_k}$. Using again (2.4.1) for the kernel $\widehat{\mathbf{H}_t}|_E$ and since $\widehat{\mathrm{H}}_t 1 = 1$ we get

$$\widehat{\mathbf{H}_t}(e_{\lambda f})'_{\lambda=0}(\mu) = (\prod_{k=1}^m \widehat{\mathbf{H}_t}(e_{\lambda f})|_E(x_k))'_{\lambda=0} =$$

$$(\widehat{\mathbf{H}_t}(e_{\lambda f})|_E)'_{\lambda=0}(x_1) \cdot \widehat{\mathbf{H}_t}(e_0)(x_2) \cdot \ldots \cdot \widehat{\mathbf{H}_t}(e_0)(x_m) + \ldots =$$
$$-[\widehat{\mathbf{H}_t}(l_f)(x_1) + \ldots + \widehat{\mathbf{H}_t}(l_f)(x_m)] = -l_{\widehat{\mathbf{H}}_{t(l_f)|_E}}(\mu).$$

For each $t \geqslant 0$ define the function $\varphi_t : \mathbb{R}_+ \longrightarrow bp\mathcal{B}$ by $\varphi_t(\lambda) := V_t(\lambda f)$. We clearly have $H_t(e^{-\lambda f}) = e^{-\varphi_t(\lambda)}$ and from Proposition 2.2.1 we obtain

$$e^{-\varphi_t(\lambda)} = T^c_t(e^{-\lambda f}) + \int_0^t T^c_{t-u} cB\widehat{\mathbf{H}_t}(e_{\lambda f})du, \quad t \geqslant 0.$$

We have $\varphi_t(0) = 0$, and using (2.4.2) we get $\varphi'_t(0) = \widehat{\mathbf{H}_t}(l_f)|_E$. By derivation of the above equation in $\lambda = 0$ and multiplying with $e^{-\beta_1 t}$, applying (2.4.1) for $N = T^c_{t-u}cB\widehat{\mathbf{H}_t}$, and again from (2.4.2) we conclude that

$$e^{-\beta_1 t}\varphi'_t(0) = T^{c+\beta_1}_t f + \int_0^t T^{c+\beta_1}_{t-u}(c + \beta_1)K(e^{-\beta_1 u}\varphi'_u(0))du, \quad t \geqslant 0,$$

where $Kf := \frac{c}{c+\beta_1}B(l_f)$ and $(T_t^{c+\beta_1})_{t\geqslant 0}$ is the transition function of the process obtained by killing X with the multiplicative functional $(e^{-\beta_1 t-\int_0^t c(X_s)ds})_{t\geqslant 0}$, $T_t^{c+\beta_1} = e^{-\beta_1 t}T_t^c$. Hence $e^{-\beta_1 t}\varphi_t'(0)$ verifies (1.4.1) with $c+\beta_1$ instead of c and the kernel K. Proposition 1.4.2 implies $e^{-\beta_1 t}\varphi_t'(0) = Q_t f$ and by (2.4.2) $e^{-\beta_1 t}\widehat{\mathbf{H}}_t(l_f) = l_{e^{-\beta_1 t}\widehat{\mathbf{H}}_t(l_f)|_E} = l_{Q_t f}$.

The proof of (ii) is similar to that of Corollary 4.3 from [12], using the above assertion (i).

(iii) Let $u \in \mathcal{E}(\mathcal{U}_\beta^o)$ be a Lyapunov function on E and for each $n \in \mathbb{N}^*$ consider the compact set K_n such that $[u \leqslant n] \subset K_n$. Since $l_1 = m$ on $E^{(m)}$, $m \geqslant 1$, we conclude that $[l_{1+u} \leqslant n]$ is included in the compact set $K_n \cup (K_n)^{(2)} \cup \ldots \cup (K_n)^{(n)}$ of S. □

2.5 Construction of branching processes on the finite configurations

Let $\mathcal{A} := \overline{[b\mathcal{E}(\mathcal{U}_\beta)]}$ (= the closure in the supremum norm of $[b\mathcal{E}(\mathcal{U}_\beta)]$). Note that $\mathcal{A}$ is an algebra; see, e.g. Corollary 2.3 from [12]. By Proposition 1.4.1 $\mathcal{A}$ does depend on $\beta > 0$ and using also Proposition 1.4.2 (iv) we get $\mathcal{A} = \overline{[b\mathcal{E}(\mathcal{U}_\beta^c)]} = \overline{[b\mathcal{E}(\mathcal{U}_\beta^o)]}$. We need a supplementary hypothesis:

$(*)$ *There exist a countable subset $\mathcal{F}_o$ of $b\mathcal{E}(\mathcal{U}_\beta^o)$ which is additive, $0 \in \mathcal{F}_o$, and separates the finite measures on E and a separable vector lattice $\mathcal{C} \subset \mathcal{A}$ such that $\{e^{-u} : u \in \mathcal{F}_o\} \subset \mathcal{C}$ and $T_t^c\varphi, T_t^c(cB\widehat{\varphi}) \in \mathcal{C}$ for all $\varphi \in \mathcal{C} \cap \mathcal{B}_{\mathsf{U}}$ and $t > 0$.*

Remark. Recall that $1 - e^{-u} \in \mathcal{E}(\mathcal{U}_\beta^o)$ provided that $u \in \mathcal{E}(\mathcal{U}_\beta^o)$ (cf. Proposition 2.4.2 (ii)) and therefore $\{e^{-u} : u \in \mathcal{F}_o\} \subset \mathcal{A} \cap \mathcal{B}_{\mathsf{U}}$.

Proposition 2.5.1. *The following assertions hold.*

(i) If c does not depend on $x \in E$ and B is given by (2.1.5) with $\sum_{k\geqslant 1} ||q_k||_\infty < \infty$, then $(*)$ is verified taking any countable subset $\mathcal{F}_o$ of $b\mathcal{E}(\mathcal{U}_\beta^o)$ which is additive, $0 \in \mathcal{F}_o$, and separates the finite measures on E, and as $\mathcal{C}$ the closure in the supremum norm of a separable vector lattice $\mathcal{C}_o \subset \mathcal{A}$ such that $\{e^{-u} : u \in \mathcal{F}_o\} \subset \mathcal{C}_o$, $(q_k)_{k\geqslant 1} \subset \mathcal{C}_o$ and $T_t(\mathcal{C}_o) \subset \mathcal{C}_o$ for all $t > 0$.

(ii) Assume that E is a locally compact space, $(T_t^c)_{t\geqslant 0}$ a C_0-semigroup on $C_0(E)$, $c \in bC(E)$, and $B\widehat{\varphi}$ and $B(l_\varphi)$ also belong to $C_0(E)$ for all $\varphi \in C_0(E) \cap \mathcal{B}_{\mathsf{U}}$. Then $(*)$ holds taking $\mathcal{C} = C_0(E) \oplus \mathbb{R}$ and for any countable subset $\mathcal{F}_o$ of $C_0(E) \cap b\mathcal{E}(\mathcal{U}_\beta^o)$ which is additive, $0 \in \mathcal{F}_o$, and separates the finite measures on E.

(iii) If condition $(*)$ holds then $V_t(\mathcal{F}_o) \subset \overline{\mathcal{C}}$ (the closure in the supremum norm of $\mathcal{C}$) for every $t \geqslant 0$.

Proof. By (2.1.5) $B\widehat{\varphi} = \sum_{k\geqslant 1} q_k \varphi^k$ and $\mathcal{C}$ is an algebra. Therefore $B\widehat{\varphi} \in \mathcal{C} \cap \mathcal{B}_{\mathsf{u}}$ provided that $\varphi \in \mathcal{C} \cap \mathcal{B}_{\mathsf{u}}$, so, assertion (i) holds.

Assertion (ii) is clear, observing that $C_0(E) \oplus \mathbb{R} \subset \mathcal{A}$. Note that by Remark 2.2.2 (i) we have $Q_t(C_0(E)) \subset C_0(E)$ for all $t > 0$ and using (1.4.1) one can see that $(Q_t)_{t\geqslant 0}$ is also a C_0-semigroup on $C_0(E)$.

(iii) Using condition $(*)$ it follows that $H_t^n(e^{-u}) \in \overline{\mathcal{C}} \cap \mathcal{B}_{\mathsf{u}}$ for all $n \geqslant 0$ and $u \in \mathcal{F}_o$, where H_t^n is given by (2.2.8). Since the sequence $(H_t^n(e^{-u}))_n$ converges uniformly (see Remark 2.2.2, $H_t(e^{-u})$ also belongs to $\overline{\mathcal{C}}$ which is an algebra and we conclude that $V_t u = -ln H_t(e^{-u}) \in \overline{\mathcal{C}}$. □

We state now the main result of this chapter, the existence of a discrete branching process associated with the base process X, the branching kernel B an the killing kernel c.

Theorem 2.5.2. *If the base process X is standard and condition $(*)$ holds then there exists a branching standard process with state space S, having $(\widehat{\boldsymbol{H}}_t)_{t\geqslant 0}$ as transition function.*

Proof. According to (1.3.2), in order to show that $(\widehat{\mathbf{H}_t})_{t\geqslant 0}$ is the transition function of a càdlàg process with state space S, we have to verify conditions $(h1)$-$(h3)$ for $\widehat{\mathcal{U}}_{\beta'}$.

We show first that $(h1)$ is satisfied by $\widehat{\mathcal{U}}_{\beta'}$, in particular, all the points of S are non-branch points for $\widehat{\mathcal{U}}_{\beta'}$. We proceed as in the proof of Proposition 4.5 from [12]. According to Corollary 3.6 from [93], it will be sufficient to prove that the uniqueness of charges and the specific solidity of potentials properties hold for $\widehat{\mathcal{U}}_{\beta'} = (\widehat{U}_{\beta'+\alpha})_{\alpha>0}$, where recall that $\widehat{U}_\alpha = \int_0^\infty e^{-\alpha t}\widehat{\mathbf{H}_t}\, dt$.

The uniqueness of charges property. We have to show that if μ, ν are two finite measures on S such that $\mu \circ \widehat{U}_{\beta'} = \nu \circ \widehat{U}_{\beta'}$ then $\mu = \nu$. We get $\mu(1) = \nu(1)$ and by Hunt's approximation theorem $\mu(F) = \nu(F)$ for every $F \in [b\mathcal{E}(\widehat{\mathcal{U}}_{\beta'})]$. We already observed that the multiplicative family of functions

$$\widehat{\mathcal{F}}_o := \{e_u : u \in \mathcal{F}_o\}$$

is a subset of $[b\mathcal{E}(\widehat{\mathcal{U}}_{\beta'})]$. Therefore $\mu(e_u) = \nu(e_u)$ for every $u \in \mathcal{F}_o$ and $\mathcal{B}(S) = \sigma(\widehat{\mathcal{F}}_o) = \sigma(\widehat{\mathcal{E}}(\mathcal{U}_{\beta'}))$. By a monotone class argument we conclude that $\mu = \nu$.

The specific solidity of potentials. We have to show that if $\xi, \mu \circ \widehat{U}_{\beta'} \in \mathsf{Exc}(\widehat{\mathcal{U}}_{\beta'})$ and $\xi \prec \mu \circ \widehat{U}_{\beta'}$, then ξ is a potential. Here $\prec$ denotes the specific order relation on the convex cone $\mathsf{Exc}(\widehat{\mathcal{U}}_{\beta'})$ of all $\widehat{\mathcal{U}}_{\beta'}$-excessive measures: if $\xi, \xi' \in \mathsf{Exc}(\widehat{\mathcal{U}}_{\beta'})$ then $\xi \prec \xi'$ means that there exists $\eta \in \mathsf{Exc}(\widehat{\mathcal{U}}_{\beta'})$ such that $\xi + \eta = \xi'$.

Let $\mathcal{A}_o$ be the additive semigroup generated by $\{V_t u : u \in \mathcal{F}_o, t \geqslant 0\}$ and $[\widehat{\mathcal{A}_o}]$ the vector space spanned by $\{e_v : v \in \mathcal{A}_o\}$. Then $[\widehat{\mathcal{A}_o}]$ is an algebra of functions on S, $1 \in [\widehat{\mathcal{A}_o}]$, and since $\widehat{\mathcal{F}}_o \subset [\widehat{\mathcal{A}}]$ we have $\sigma([\widehat{\mathcal{A}}]) = \mathcal{B}(S)$. We prove now that

$$(2.5.1) \qquad [\widehat{\mathcal{A}_o}] \subset [b\mathcal{E}(\widehat{\mathcal{U}}_{\beta'})].$$

Since $\widehat{\mathcal{F}}_o \subset [b\mathcal{E}(\widehat{\mathcal{U}}_{\beta'})]$ we get from (2.3.1) that $e_{V_t u} \in [b\mathcal{E}(\widehat{\mathcal{U}}_{\beta'})]$ for all $u \in \mathcal{F}_o$. By Corollary 2.3 from [12] the vector space $[b\mathcal{S}(\widehat{\mathcal{U}}_{\beta'})]$ is an algebra and therefore $e_v \in [b\mathcal{S}(\widehat{\mathcal{U}}_{\beta'})]$ for all $v \in \mathcal{A}_o$. It remains to prove that the map $s \longmapsto \widehat{\mathbf{H}_s}(e_v)(\mu)$ is right continuous on $[0,\infty)$ for every $v \in \mathcal{A}_o$ and $\mu \in S$. Because $\widehat{\mathbf{H}_s}(e_v) = \widehat{H_s(e^{-v})}$, we have to prove the right continuity of the mapping $s \longmapsto H_s(e^{-v})(x)$, $x \in E$. According to Proposition 2.2.1 (iii) it will be sufficient to show that the map $s \longmapsto T_s^c(e^{-v})(x)$ is right continuous for every $v \in \mathcal{A}_o$ and $x \in E$. This last right continuity property is satisfied since by Proposition 2.5.1 (iii) the function e^{-v} belongs to the algebra $\mathcal{A}$.

Let $\xi, \mu \circ \widehat{U}_{\beta'} \in \mathsf{Exc}(\widehat{\mathcal{U}}_{\beta'})$, $\xi \prec \mu \circ \widehat{U}_{\beta'}$. We may suppose that $\mu(1) \leqslant 1$; if it is not the case, then $\mu = \sum_n \mu_n$ with $\mu_n(1) \leqslant 1$ for all n and by Ch. 2 in [13] there exists a sequence $(\xi_n)_n \subset \mathsf{Exc}(\widehat{\mathcal{U}}_{\beta'})$ such that $\xi = \sum_n \xi_n$ and $\xi_n \prec \mu_n \circ \widehat{U}_\beta$ for every n. Let $\varphi_\xi : \mathcal{E}(\widehat{\mathcal{U}}_{\beta'}) \longrightarrow \overline{\mathbb{R}}_+$ the functional defined by $\varphi_\xi(F) := \widehat{L}_{\beta'}(\xi, F)$, $F \in \mathcal{E}(\widehat{\mathcal{U}}_{\beta'})$, where $\widehat{L}_{\beta'}$ denotes the energy functional associated with $\widehat{\mathcal{U}}_{\beta'}$. By (2.5.1) we may extend φ_ξ to an increasing linear functional on $[\widehat{\mathcal{A}_o}]$. Let $\mathcal{M}$ be the closure of $[\widehat{\mathcal{A}_o}]$ with respect to the supremum norm. Clearly, $\mathcal{M}$ is a vector lattice and we claim that φ_ξ extends to a positive linear functional on $\mathcal{M}$. Indeed, if $(F_n)_n \subset [\widehat{\mathcal{A}_o}]$ is a sequence converging uniformly to zero and we consider a sequence $(\nu_k \circ \widehat{U}_{\beta'})_k \subset \mathsf{Exc}(\widehat{\mathcal{U}}_{\beta'})$, $\nu_k \circ \widehat{U}_{\beta'} \nearrow \xi$, then we have $|\varphi_\xi(F_n)| = \sup_k |\nu_k(F_n)| \leqslant \liminf_k \nu_k(|F_n|) \leqslant \varepsilon \liminf_k \nu_k(1) = \varepsilon \widehat{L}_{\beta'}(\xi, 1) \leqslant \varepsilon\mu(1) \leqslant \varepsilon$, provided that $n \geqslant n_0$ and $||F_n||_\infty < \varepsilon$ for all $n \geqslant n_0$. Since $\xi \prec \mu \circ \widehat{U}_{\beta'}$ we have $\varphi_\xi(F) \leqslant \mu(F)$ for every $F \in \mathcal{M}_+$. Consequently, if $(F_n)_n \subset \mathcal{M}_+$ is a sequence decreasing pointwise to zero, then $\varphi_\xi(F_n) \searrow 0$. By Daniell's Theorem there exists a measure ν on $(S, \mathcal{B}(S))$ such that $\varphi_\xi(F) = \nu(F)$ for all $F \in \mathcal{M}$. In particular, if $u \in \mathcal{F}_o$ then $\widehat{L}_{\beta'}(\xi, \widehat{\mathbf{H}}_t(e_u)) = \varphi_\xi(e_{V_t u}) = \nu(\widehat{\mathbf{H}}_t(e_u))$ and therefore $\widehat{L}_{\beta'}(\xi, \widehat{U}_{\beta'}(e_u)) = \lim_k \nu_k(\widehat{U}_{\beta'}(e_u)) = \int_0^\infty e^{-\beta' t} \lim_k \nu_k(\widehat{\mathbf{H}}_t(e_u))dt = \int_0^\infty e^{-\beta' t} \widehat{L}_{\beta'}(\xi, \widehat{\mathbf{H}}_t(e_u))dt = \nu(\widehat{U}_{\beta'}(e_u))$. We conclude that $\xi = \nu \circ \widehat{U}_{\beta'}$. Observe that actually we proved he following assertion:

(2.5.2) *If ξ, η are two finite measures from $\mathsf{Exc}(\widehat{\mathcal{U}}_{\beta'})$ and $\widehat{L}_{\beta'}(\xi, \widehat{\boldsymbol{H}}_t(e_u)) = \widehat{L}_{\beta'}(\eta, \widehat{\boldsymbol{H}}_t(e_u))$ for all $u \in \mathcal{F}_o$ and $t \geqslant 0$, then $\xi = \eta$.*

Because X is a Hunt process, Theorem (47.10) in [91] implies that X has càdlàg trajectories in any Ray topology (see Section 1.2). Consider a Ray topology $\mathcal{T}(\mathcal{R})$ with respect to $\mathcal{U}^o$, which is finer than the original topology and it is generated by a

Ray cone $\mathcal{R} \subset b\mathcal{E}(\mathcal{U}^o_\beta)$ such that $\mathcal{F}_o \subset \mathcal{R}$. So, without loosing the generality, we may assume in the sequel that the original topology of E is a Ray one.

We check now condition $(h2)$. Let $\lambda \in S$ and set as before $\lambda' = \lambda + \lambda \circ G_\beta$. Since the base process X on E has càdlàg trajectories the capacity $c_{\lambda'}$ is tight (see Remark 1.3.3) and by Corollary 1.4.3 the capacity c^o_λ is also tight. According to (1.2.3) and Remark 1.2.1 there exists a compact Lyapunov function $u \in \mathcal{E}(\mathcal{U}^o_\beta) \cap L^1(\lambda)$. Proposition 2.4.2 (ii) implies that $l_u \in \mathcal{E}(\widehat{\mathcal{U}}_{\beta'})$ is a compact Lyapunov function on S and $l_u(\lambda) < \infty$, hence $(h2)$ holds.

We show that $(h3)$ also holds for $\widehat{\mathcal{U}}'_\beta$. We take l_1 as the function u_o; observe that by Proposition 2.4.2 (ii) we have $l_1 \in \mathcal{E}(\widehat{\mathcal{U}}_{\beta'})$ and clearly l_1 is a real-valued function. Let $\mathcal{C}_o$ be a countable subset of $b\mathcal{E}(\mathcal{U}^o_\beta)$ such that $\mathcal{F}_o \subset \mathcal{C}_o$, $\mathcal{C}_o$ is additive, and $p\mathcal{C}$ is included in the closure in the supremum norm of $(\mathcal{C}_o - \mathcal{C}_o)_+$. Let further $\mathcal{R}_o$ be a countable dense subset of the Ray cone $\mathcal{R}$ such that $\mathcal{C}_o \subset \mathcal{R}_o$. We may consider $\widehat{\mathcal{R}}_o := \{e_u : u \in \mathcal{R}_o\}$ as the countable set $\mathcal{F}$ from $(h3)$. Note that since $\mathcal{R}_o$ generates the (Ray) topology on E, by Lemma 0.2 from [53] (see also the proof of Lemma 2.4 from [20]), one can see that $\widehat{\mathcal{R}}_o$ generates the topology of S. Let further ξ, η be two finite $\mathcal{E}(\widehat{\mathcal{U}}_{\beta'})$-excessive measures such that $\widehat{L}_{\beta'}(\xi, e_u) = \widehat{L}_{\beta'}(\eta, e_u)$ for all $u \in \mathcal{R}_o$ and

$$\widehat{L}_{\beta'}(\xi + \eta, l_1) < \infty. \tag{2.5.3}$$

To show that $\xi = \eta$ we can now proceed as in the proof of Theorem 4.9 from [12], Step I, page 699; this procedure was also used in the proof of Theorem 3.5 from [19]. Because the σ-algebra $\mathcal{B}(S)$ is generated by the multiplicative family $\widehat{\mathcal{F}}_o$, a monotone class argument implies that $\xi = \eta$ provided that

$$\xi(e_u) = \eta(e_u) \text{ for all } u \in \mathcal{F}_o.$$

By (2.5.2) the above equality holds if

$$\widehat{L}_{\beta'}(\xi, \widehat{\mathbf{H}_t}(e_u)) = \widehat{L}_{\beta'}(\eta, \widehat{\mathbf{H}_t}(e_u)) \text{ for all } u \in \mathcal{F}_o \text{ and } t \geqslant 0. \tag{2.5.4}$$

From (2.5.3) and (1.2.1) there exist two measures μ and ν on S_1 such that $\xi = \mu \circ \widehat{U}^1_{\beta'}$ and $\eta = \nu \circ \widehat{U}^1_{\beta'}$. Let further $\widetilde{\mathcal{C}}_o := \{\widetilde{e}_u : u \in \mathcal{C}_o\}$. Because $\widetilde{\mathcal{C}}_o$ is a multiplicative class of functions on S_1 and $\mu(\widetilde{e}_u) = \widehat{L}_{\beta'}(\xi, e_u) = \widehat{L}_{\beta'}(\eta, e_u) = \nu(\widetilde{e}_u)$ for every $u \in \mathcal{C}_o$, by the monotone class theorem we have

$$\mu(F) = \nu(F) \text{ for all } F \in \sigma(\widetilde{\mathcal{C}}_o). \tag{2.5.5}$$

If $u \in \mathcal{F}_o$ then by Proposition 2.5.1 (iii) there exists a sequence $(f_n)_n \subset (\mathcal{C}_o - \mathcal{C}_o)_+$ converging uniformly to $V_t u$. Note that if $f \in (\mathcal{C}_o - \mathcal{C}_o)_+$ then e_f has a finely continuous extension $\widetilde{e}_f$ from S to S_1. Since $e_u \in [b\mathcal{E}(\widehat{\mathcal{U}}_{\beta'})]$, using by (2.3.1) we get that $e_{V_t u}$ belongs to $[b\mathcal{E}(\widehat{\mathcal{U}}_{\beta'})]$ and by (1.2.2) it has a unique extension $\widetilde{e}_f$ from S to S_1 (by fine continuity too). As a consequence, and using (2.2.6), for every $\lambda \in S$ we have $|e_{f_n}(\lambda) - e_{V_t u}(\lambda)| \leqslant ||f_n - V_t u||_\infty \cdot l_1(\lambda)$, hence $|\widetilde{e}_{f_n} - \widetilde{e}_{V_t u}| \leqslant ||f_n - V_t u||_\infty \cdot \widetilde{l}_1$ on S_1. It follows that $(\widetilde{e}_{f_n})_n$ converges pointwise to $\widetilde{e}_{V_t u}$ on the set $[\widetilde{l}_1 < \infty] \in \sigma(\widetilde{\mathcal{C}}_o)$. From (2.5.3) we get $(\mu + \nu)(\widetilde{l}_1) = \widehat{L}_\beta(\xi + \eta, l_1) < \infty$, hence $\widetilde{l}_1 < \infty$ $(\mu + \nu)$-a.e. Therefore, $1_{[\widetilde{l}_1 < \infty]} \cdot \widetilde{e}_{V_t u}$ is $\sigma(\widetilde{\mathcal{C}}_o)$-measurable and by (2.5.5) we now deduce that $\mu(\widetilde{e}_{V_t u}) = \nu(\widetilde{e}_{V_t u})$ for all $u \in \mathcal{F}_o$. We conclude that (2.5.4) holds, so $\xi = \eta$. Applying (1.3.2), $\widehat{\mathcal{U}}$ is the resolvent of a standard process with state space S.

The quasi-left continuity follows by Proposition 1.3.4, taking $\widehat{\mathcal{F}}_o$ as the multiplicative set $\mathcal{K}$, since by Proposition 2.2.1 (ii) the semigroup $(\widehat{\mathbf{H}}_t)_{t \geqslant 0}$ is Markovian. □

2.6 Continuous branching as base process

In this section we give an example of a branching Markov process, having as base space the set of all finite configurations of positive finite measures on a topological space. Note that an example of branching type process on this space was given in [20], obtained by perturbing a diagonal semigroup with a branching kernel.

Let Y be a standard (Markov) process with state space a Lusin topological space F, called *spatial motion*. We fix a *branching mechanism*, that is, a function $\Phi : F \times [0, \infty) \longrightarrow \mathbb{R}$ of the form

$$\Phi(x, \lambda) = -b(x)\lambda - a(x)\lambda^2 + \int_0^\infty (1 - e^{-\lambda s} - \lambda s) N(x, ds)$$

where $a \geqslant 0$ and b are bounded $\mathcal{B}$-measurable functions and $N : p\mathcal{B}((0, \infty)) \longrightarrow p\mathcal{B}(F)$ is a kernel such that $N(u \wedge u^2) \in bp\mathcal{B}(F)$. Examples of branching mechanisms are $\Phi(\lambda) = -\lambda^\alpha$ for $1 < \alpha \leq 2$.

We first present the measure-valued branching Markov process associated with the spatial motion Y and the branching mechanism Φ, the (Y, Φ)-superprocess, a standard process with state space $M(F)$, the space of all positive finite measures on $(F, \mathcal{B}(F))$, endowed with the weak topology (cf. [43] and [40], see also [12]). For each $f \in bp\mathcal{B}(F)$ the equation

$$v_t(x) = P_t f(x) + \int_0^t P_s(x, \Phi(\cdot, v_{t-s})) ds, \quad t \geqslant 0, \quad x \in F,$$

has a unique solution $(t, x) \longmapsto N_t f(x)$ jointly measurable in (t, x) such that $\sup\limits_{0 \leq s \leq t} ||v_s||_\infty < \infty$ for all $t > 0$; we have denoted by $(P_t)_{t \geqslant 0}$ the transition function of the spatial motion

Y. Assume that Y is conservative, that is $P_t 1 = 1$. The mappings $f \longmapsto N_t f$, $t \geqslant 0$, form a nonlinear semigroup of operators on $bp\mathcal{B}(F)$ and the above equation is formally equivalent with

$$\begin{cases} \frac{d}{dt} v_t(x) = \mathsf{L} v_t(x) + \Phi(x, v_t(x)) \\ v_0 = f, \end{cases} \tag{2.6.1}$$

where L is the infinitesimal generator of the spatial motion Y. For every $t \geqslant 0$ there exists a unique kernel T_t on $(M(F), \mathcal{M}(F))$ such that

$$T_t(e_f) = e_{N_t f}, \quad f \in bp\mathcal{B}(F), \tag{2.6.2}$$

where for a function $g \in bp\mathcal{B}(F)$ the exponential function e_g is defined on $M(F)$ as in Section 2.1. Since the family $(N_t)_{t\geqslant 0}$ is a (nonlinear) semigroup on $bp\mathcal{B}(F)$, $(T_t)_{t\geqslant 0}$ is a linear semigroup of kernels on $M(F)$. Suppose that F is a locally compact space, $(P_t)_{t\geqslant 0}$ is a C_0-semigroup on $C_0(F)$, and a, b, and N do not depend on $x \in F$. We may assume that $b \geqslant 0$. Arguing as in the proof of Proposition 4.8 from [12] one can see that $N_t(C_0(F)) \subset C_0(F)$ and that $N_t(b\mathcal{E}(\mathcal{U}_{b'})) \subset b\mathcal{E}(\mathcal{U}_{b'})$ for every $t \geqslant 0$, where $b' := b + \beta$, with $\beta > 0$. Then $(T_t)_{t\geqslant 0}$ is the transition function of a standard process with state space $M(F)$, called (Y, Φ)-*superprocess*; see, e.g., [43], [12], and [19]. In addition, the (Y, Φ)-superprocess is a branching process on $M(F)$, i.e., T_t is a branching kernel on $M(F)$ for all $t > 0$. Recall that the nonlinear semigroup $(N_t)_{t\geq 0}$ is called the *cumulant semigroup* of this branching process.

We can apply now the results from Section 2.5, starting with the (Y, Φ)-superprocess as base process with state space $E := M(F)$.

Corollary 2.6.1. *Let c and $(q_k)_{k\geqslant 1}$ be positive real numbers such that $\sum_{k\geqslant 1} q_k = 1$, $\sum_{k\geqslant 1} k q_k =: q_o < \infty$, and $0 < \beta < c + q_o - c q_o$. Then there exists a discrete branching process with state space the set of all finite configurations of positive finite measures on F, associated to c and $(q_k)_{k\geqslant 1}$, and with base process the (Y, Φ)-superprocess.*

Proof. We apply Thoerem 2.5.2, so, we have to check condition $(*)$. Let $\mathcal{R}$ be a Ray cone with respect to the resolvent $\mathcal{W} = (W_\alpha)_{\alpha>0}$ of the process Y on F, constructed as in the proof of Proposition 4.8 from [12], $\mathcal{R} \subset b\mathcal{E}(\mathcal{W}_{b'})$, such that $[\mathcal{R} \cap C_0(F)]$ is dense in $C_0(F)$. Let $\mathcal{R}_o$ be a countable, additive, dense subset of $\mathcal{R}$. Then $\{e_r : r \in \mathcal{R}_o\}$ is a multiplicative set of functions on E and separates the measures on E. Let further $\mathcal{C}$ be the closure in the supremum norm of the vector space spanned by $\{e_w : w \in b\mathcal{E}(\mathcal{W}_{b'}) \cap C_0(F)\}$ and denote by $\mathcal{U} = (U_\alpha)_{\alpha>0}$ the resolvent of the (Y, Φ)-superprocess on E. By Corollary 4.4 from [12] $1 - e_w \in \mathcal{E}(\mathcal{U}_\beta)$ for all $w \in b\mathcal{E}(\mathcal{W}_{b'})$. Therefore $\mathcal{C} \subset \mathcal{A}$

and we may take as $\mathcal{F}_o$ the additive semigroup generated by the set $\{1-e_w : w \in \mathcal{R}_o\}$. From (2.6.2) and the above considerations $T_t(\mathcal{C}) \subset \mathcal{C}$ and since $\mathcal{C}$ is a Banach algebra we clearly have $\{e^{-u} : u \in \mathcal{F}_o\} \subset \mathcal{C}$, hence condition $(*)$ holds. □

The following comment completes the Final Remark from [20], page 27.

Remark 2.6.2. *Recall first a suggestive description of a superprocess, stated in [41], page 55: "A measure-valued Markov process describes the evolution of a random cloud. The branching property means that any parts of the cloud at time t do not interact after t." Consequently, taking into account the interpretation of a branching process given in Introduction to Part I, one can think that the process constructed in Corollary 2.6.1 describes the evolution of a random cloud not only controlled by a branching mechanism Φ but also driven by a discrete branching process, in a new dimension, along which the splitting into a random number of clouds takes place, commanded by a branching kernel B.*

Chapter 3

Subordination in the sense of Bochner and associated Markov processes

3.1 Convolution semigroups on the real line, subordinators

Consider a convolution semigroups $(\mu_t)_{t\geqslant 0}$ on the real line, as introduced in the example (1.1.3).

Bernstein function of a convolution semigroup $(\mu_t)_{t\geqslant 0}$**.** An arbitrarily often differentiable function $f : (0,\infty) \to \mathbb{R}$ is called *Bernstein function* if $f \geqslant 0$ and $(-1)^n f^{(n)} \leqslant 0$ hold for $n \in \mathbb{N}$. Bernstein functions can be fully characterized by a *Lévy-Khinchin formula*,

$$f(x) = a + bx + \int_0^\infty (1 - e^{-sx})\mu(\mathrm{d}s),$$

with $a, b \geqslant 0$ and a non-negative measure μ on $(0,\infty)$ such that $\int_0^\infty s(s+1)^{-1}\mu(\mathrm{d}s) < \infty$. These and the following result can be found in the monograph [10].

(3.1.1) Every convolution semigroup $(\mu_t)_{t\geqslant 0}$ of sub-probability measures on $[0,\infty)$ is uniquely characterized by some Bernstein function f, and vice versa. This correspondence is given by

$$\int_0^\infty e^{-sx}\mu_t(\mathrm{d}s) = e^{-tf(x)}.$$

Examples. The function $x \longmapsto a, a \geqslant 0$, is a Bernstein function with corresponding convolution semigroup $(e^{-at}\delta_0)_{t\geqslant 0}$. Further, the function $x \longmapsto bx, b \geqslant 0$, is a Bernstein function with associated convolution semigroup $(\delta_{bt})_{t\geqslant 0}$. For $b = 1$, then the transition function define as in exemple (1.1.1) is $T_t f(x) = f(x+t)$.

(3.1.2) Bernstein functions of special interest are the *fractional powers*, defined for any $\alpha \in [0,1]$ by $f_\alpha(x) = x^\alpha$, that is

$$x^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-xs}) s^{-\alpha-1}\,\mathrm{d}s, x \geqslant 0.$$

The corresponding convolution semigroup of the fractional powers is called *the one-sided stable semigroup of order* α, denoted by $(\eta_t^\alpha)_{t>0}$. For $\alpha = 0$ we find $\eta_t^0 = e^{-\alpha t}\delta_0$, and for $\alpha = 1$ is follows that $\eta_t^1 = \delta_t$.

Subordinator associated with a convolution semigroup on the real line. We may interpret the convolution semigroup $(\mu_t)_{t\geqslant 0}$ as the transition function of a positive real-valued stationary stochastic process $(\xi_t)_{t\geqslant 0}$ with independent nonnegative increments. Since $\mu_0 = \delta_0$ and since the measures μ_t are supported on $[0,\infty)$, we have almost surely $\xi_0 = 0$ and almost surely increasing paths $t \longmapsto \xi_t$. The converse assertion is also true: every such process defines (uniquely) a convolution semigroup of sub-probability measures on $[0,\infty]$. We call $(\xi_t)_{t\geqslant 0}$ the *subordinator* induced by $(\mu_t)_{t\geqslant 0}$.

3.2 Subordination in the sense of Bochner of the C_0-semigroups and resolvents of kernels

In the sequel we fix a transition function $\mathbb{P} = (P_t)_{t\geqslant 0}$ on $(E,\mathcal{B})$ and a convolution semigroup $(\mu_t)_{t\geqslant 0}$ on $\mathbb{R}_+^*$.

For each $t \geqslant 0$ we define the kernel P_t^μ on $(E,\mathcal{B})$ by

$$P_t^\mu f := \int_0^\infty P_s f \mu_t(\mathrm{d}s) \quad \text{for all } f \in bp\mathcal{B}.$$

The next proposition collects some basic properties of the subordination of the semigroups of kernels; we follow the approach of [27], Ch. V, Section 3, and for the reader convenience we also present the proof.

Proposition 3.2.1. *The family* $\mathbb{P}^\mu = (P_t^\mu)_{t\geqslant 0}$ *is a sub-Markovian semigroup of kernels on* $(E,\mathcal{B})$ *and* $\mathcal{E}(\mathbb{P}) \subset \mathcal{E}(\mathbb{P}^\mu)$.

Proof. Since for all $t_1, t_2 > 0$, and $f \in bp\mathcal{B}$ we have

$$P^{\mu}_{t_1} P^{\mu}_{t_2} f = \int_0^{\infty} P_{s_1}(P^{\mu}_{t_2} f)\mu_{t_1}(\mathrm{d}s_1) = \int_0^{\infty} \left(\int_0^{\infty} P_{s_1}(P_{s_2} f)\mu_{t_2}(\mathrm{d}s_2) \right) \mu_{t_1}(\mathrm{d}s_1) =$$

$$\int_0^{\infty}\int_0^{\infty} P_{s_1+s_2} f \mu_{t_2}(\mathrm{d}s_2)\mu_{t_1}(\mathrm{d}s_1) = \int_0^{\infty} P_s f(\mu_{t_1} * \mu_{t_2})(\mathrm{d}s) = \int_0^{\infty} P_s f \mu_{t_1+t_2}(\mathrm{d}s) = P^{\mu}_{t_1+t_2} f,$$

it follows that the family of kernels $\mathbb{P}^{\mu} = (P^{\mu}_t)_{t \geqslant 0}$ is indeed a semigroup.

If $u \in p\mathcal{B}$ is $\mathbb{P}$-supermedian (i.e., $P_s u \leqslant u$ for each $s \geqslant 0$ then $P^{\mu}_t u = \int_0^{\infty} P_s u\, \mu_t(\mathrm{d}s) \leq \int_0^{\infty} u \mu_t(\mathrm{d}s) = u\, \mu_t(1) = u$. Hence the function u is $\mathbb{P}^{\mu}$-supermedian too. In particular, taking $u = 1$, we get that P^{μ}_t is a sub-Markovian kernel for each $t > 0$.

We prove now that

$$\mathcal{E}(\mathbb{P}) \subset \mathcal{E}(\mathbb{P}^{\mu}).$$

Fix $u \in \mathcal{E}(\mathbb{P})$. Then we already observed that $P^{\mu}_t u \leqslant u$ for every $t > 0$. Let $x \in E$, $a < u(x)$ and $0 < b < 1$. Then there exist $s_0 > 0$ and $t_0 > 0$ such that $P_s u(x) > a$ for every $0 < s < s_0$ and $\mu_{t_0}(]0, s_0[) > b$, hence $P^{\mu}_{t_0} u(x) \geqslant \int_0^{s_0} P_s u(x)\mu_{t_0}(\,\mathrm{d}s) \geqslant ab$. This implies that $u \in \mathcal{E}(\mathbb{P}^{\mu})$. □

The semigroup $\mathbb{P}^{\mu} = (P^{\mu}_t)_{t>0}$ is called the sub-Markovian *semigroup subordinated to* $\mathbb{P} = (P_t)_{t>0}$ *by means of* $(\mu_t)_{t>0}$.

Subordination of infinitesimal operators. Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of a C_0-contraction semigroup $(P_t)_{t \geq 0}$ on a Banach space B; formally we have $P_t = e^{tA}$. Let further $(P^{\mu}_t)_{t \geq 0}$ be the *subordinate of* $(P_t)_{t \geq 0}$ *in the sense of Bochner* (by means of $(\mu_t)_{t>0}$), and let $(A^f, \mathcal{D}(A^f))$ be the infinitesimal generator of $(P^{\mu}_t)_{t \geq 0}$, called the *subordinate generator*. Hence again formally $e^{tA^f} = P^{\mu}_t = \int_0^{\infty} e^{sA}\mu_t(\,\mathrm{d}s)$ and comparing with (3.1.1) we deduce that

$$-A^f = f(-A),$$

(see also [90]).

(3.2.1) If we take $A = \Delta$ (i.e., A is the generator of the Gaussian semigroup) and the Bernstein function $f(x) = x^{1/2}$, $x \in \mathbb{R}$ (this is possible, according with (3.1.2)), then $-(-\Delta)^{1/2}$ is obtain from the Δ by subordination. So, the subordination in the sense of Bochner is a convenient way to obtain the fractional powers of the Laplace operator.

Let $\mathbb{P} = (P_t)_{t>0}$ be a measurable sub-markovian semigroup on $(E, \mathcal{B})$. If $\mu_t = e^{-\alpha t}\delta_t, t > 0$, for some $\alpha \geqslant 0$, then $\mathbb{P}^{\mu} = (P_t)^{\alpha}_{t>0}$, where $P^{\alpha}_t = e^{-\alpha t} P_t, t > 0$.

We recall that if $(P_t)_{t\geqslant 0}$ is the C_0-semigroup on $L^p(E,m)$ induced by the transition function of $X = (X_t)_{t\geqslant 0}$ (where m is a σ-finite P_t-*subinvariant measure*, i.e., $\int_E P_t f \,\mathrm{d}m \leqslant \int_E f \,\mathrm{d}m$ for all $f \in L^p_+(E,m)$ and $t \geqslant 0$), then the transition function of the subordinate process $X^\xi = (X^\xi_t)_{t\geqslant 0}$ is $(P^\mu_t)_{t\geqslant 0}$. A converse of this statement is the main result of this chapter.

Let $(V_\alpha)_{\alpha>0}$ be the (sub-Markovian) strongly continuous resolvent of contractions on $L^p(E,m)$ induced by $(P_t)_{t\geqslant 0}$,

$$V_\alpha = \int_0^\infty e^{-\alpha t} P_t \,\mathrm{d}t, \ \alpha > 0.$$

Subordination by convolution semigroups. For a familiy $\mathcal{G}$ of real valued functions on E we denote by $b\mathcal{G}$ the subfamily of bounded functions from $\mathcal{G}$.

Recall that a family $\mu = (\mu_t)_{t\geqslant 0}$ of measures on $\mathbb{R}_+$ is called a (vaguely continuous) *convolution semigroup* on $\mathbb{R}_+$ if for all $s, t \geqslant 0$ one has

$$\mu_s * \mu_t = \mu_{s+t}\,, \ \mu_t(\mathbb{R}_+) \leqslant 1, \ \text{ and } \lim_{t\to 0} \mu_t = \mu_0 := \delta_0 \ \text{(vaguely)}\ .$$

Let $\mathcal{U}^\mu = (U^\mu_\alpha)_{\alpha>0}$ be the resolvent of the subordinate semigroup $\mathbb{T}^\mu = (T^\mu_t)_{t\geqslant 0}$.

Let $\mathsf{L} : \ \mathrm{Exc}(\mathcal{U}) \times \mathcal{E}(\mathcal{U}) \longrightarrow \overline{\mathbb{R}}_+$ be the *energy functional* (associated with $\mathcal{U}$) defined as

$$\mathsf{L}(\eta, v) := \sup\{\mu(v) : \ \mathrm{Pot}(\mathcal{U}) \ni \mu \circ U \leqslant \eta\}$$

for all $\eta \in \mathrm{Exc}(\mathcal{U})$ and $v \in \mathcal{E}(\mathcal{U})$; where $\mu(v) := \int_E v \ \mathrm{d}\mu$. The energy functional associated with $\mathcal{U}^\mu_q$, $q > 0$, will be denoted by L^μ_q.

Lemma 3.2.2. *Let $\eta, \nu \circ U^\mu_q$ be two $\mathcal{U}^\mu_q$-excessive measures with $\eta \leqslant \nu \circ U^\mu_q$ and suppose that the measure $\nu \circ U$ is σ-finite. Define the positive measure η' on $(E, \mathcal{B})$ as*

$$\eta'(f) := \mathsf{L}^\mu_q(\eta, Uf), \ f \in bp\mathcal{B}.$$

Then the measure η' is $\mathcal{U}$-excessive.

Proof. If $\alpha > 0$ then $\eta' \circ \alpha U_\alpha(f) = \mathsf{L}^\mu_q(\eta, \alpha U_\alpha U f) \leqslant \mathsf{L}^\mu_q(\eta, Uf) = \eta'(f)$. We show now that η' is a σ-finite measure. If $f_0 \in bp\mathcal{B}$, $f_0 > 0$, is such that $\nu(Uf_0) < \infty$, then $\eta'(f_0) = \mathsf{L}^\mu_q(\eta, Uf_0) \leqslant \mathsf{L}^\mu_q(\nu \circ U^\mu_q, Uf_0) = \nu(Uf_0) < \infty$. Hence the measure η' is σ-finite and we conclude that it is $\mathcal{U}$-excessive. □

Proposition 3.2.3. *Assume that the resolvent $\mathcal{U}$ is proper and that all the points of E are nonbranch points with respect to $\mathcal{U}$. Then the same property holds for the resolvent $\mathcal{U}^\mu_q$, $q > 0$.*

Proof. By (1.1.9) we have to show that conditions (UC) and (SSP) are verified by the resolvent $\mathcal{U}_q^\mu$. Let ν_1 and ν_2 be two positive finite measures on E such that $\nu_1 \circ U_q^\mu = \nu_2 \circ U_q^\mu$. Using Hunt's approximation theorem (see e.g. Theorem 1.2.8 from [13]), we have $\nu_1(v) = \nu_2(v)$ for all $v \in \mathcal{E}(\mathcal{U}_q^\mu)$ and because $\mathcal{E}(\mathcal{U}) \subset \mathcal{E}(\mathcal{U}^\mu) \subset \mathcal{E}(\mathcal{U}_q^\mu)$, we get $\nu_1(v) = \nu_2(v)$ for all $v \in \mathcal{E}(\mathcal{U})$. In particular, we have $\nu_1(Uf) = \nu_2(Uf)$ for all $f \in p\mathcal{B}$, hence $\nu_1 \circ U = \nu_2 \circ U$. Since by hypothesis all the points of E are nonbranch points with respect to $\mathcal{U}$, by (1.1.9) it follows that the uniqueness of charges property (UC) holds for $\mathcal{U}$, hence $\nu_1 = \nu_2$ and we conclude that (UC) also holds for $\mathcal{U}_q^\mu$.

We check now that the specific solidity of potentials property (SSP) holds with respect to $\mathcal{U}_q^\mu$. Let η_1, η_2, and $\nu \circ U_q^\mu$ be $\mathcal{U}_q^\mu$-excessive measures such that

$$\eta_1 + \eta_2 = \nu \circ U_q^\mu. \tag{3.2.2}$$

We may assume that ν is a finite measure, consequently the measures η_1 and η_2 are also finite. We define the positive measures η_1' and η_2' on E as

$$\eta_1'(f) := \mathsf{L}_q^\mu(\eta_1, Uf), \quad \eta_2'(f) := \mathsf{L}_q^\mu(\eta_2, Uf) \quad \text{for all } f \in bp\mathcal{B}.$$

By Lemma 3.2.2 the measures η_1' and η_2' are $\mathcal{U}$-excessive and using (3.2.2) we have for every $f \in bp\mathcal{B}$

$$\eta_1'(f) + \eta_2'(f) = \mathsf{L}_q^\mu(\eta_1, Uf) + \mathsf{L}_q^\mu(\eta_2, Uf) = \mathsf{L}_q^\mu(\nu \circ U_q^\mu, Uf) = \nu(Uf).$$

We obtain the following equality of $\mathcal{U}$-excessive measures: $\eta_1' + \eta_2' = \nu \circ U$. Since by hypothesis the property (SSP) holds for $\mathcal{U}$, we deduce from the last equality that there exists a measure λ on E such that $\eta_1' = \lambda \circ U$. Hence for all $f \in bp\mathcal{B}$

$$\mathsf{L}_q^\mu(\eta_1, Uf) = \eta_1'(f) = \lambda(Uf) = \mathsf{L}_q^\mu(\lambda \circ U_q^\mu, Uf).$$

In particular, taking $f = U_q^\mu g$, with $g \in bp\mathcal{B}$, and since $UU_q^\mu = U_q^\mu U$, it follows that

$$\eta_1(Ug) = \mathsf{L}_q^\mu(\eta_1, UU_q^\mu g) = \mathsf{L}_q^\mu(\lambda \circ U_q^\mu, U_q^\mu Ug) = \lambda \circ U_q^\mu(Ug) \quad \text{for all } g \in bp\mathcal{B}.$$

Note that in addition we have $\eta_1(Ug) \leqslant \nu(U_q^\mu Ug) \leqslant \frac{1}{q}\nu(Ug)$. In particular, the measures $\eta_1 \circ U$ and $(\lambda \circ U_q^\mu) \circ U$ are σ-finite and equal. By the uniqueness of charges property (UC) for the resolvent $\mathcal{U}$ we conclude that $\eta_1 = \lambda \circ U_q^\mu$, so, the property (SSP) holds with respect to $\mathcal{U}_q^\mu$. □

3.3 Subordination of right processes

Theorem 3.3.1. *Assume that E is a Lusin topological space with $\mathcal{B}$ as Borel σ-algebra and let $\mathbb{T} = (T_t)_{t \geqslant 0}$ be the transition function of a transient (Borel) right process with* *state space E. Then the subordinate semigroup $\mathbb{T}^\mu = (T_t^\mu)_{t\geqslant 0}$ is also the transition function of a (Borel) right process with state space the topological space E.*

Proof. By Proposition 3.2.3 all the points of E are nonbranch points with respect to $\mathcal{U}_q^\mu$. Since $\mathcal{E}(\mathcal{U}) \subset \mathcal{E}(\mathcal{U}^\mu) \subset \mathcal{E}(\mathcal{U}_q^\mu)$ we get that any natural topology with respect to $\mathcal{U}$ is also natural with respect to $\mathcal{U}^\mu$.

We show that (NSP) holds for $\mathcal{U}_q^\mu$, i.e., if $\eta, \nu \circ U_q^\mu \in \mathsf{Exc}(\mathcal{U}_q^\mu)$ and $\eta \leqslant \nu \circ U_q^\mu$ then there exists a measure λ such that $\eta = \lambda \circ U_q^\mu$. Indeed, we may suppose that the measure ν is finite and by Lemma 3.2.2 the measure η' belongs to $\mathsf{Exc}(\mathcal{U})$ and we have for all $f \in bp\mathcal{B}$

$$\eta'(f) = \mathsf{L}_q^\mu(\eta, Uf) \leqslant \mathsf{L}_q^\mu(\nu \circ U_q, Uf) = \nu(Uf).$$

Since $\mathcal{U}$ is the resolvent of a right process and $\eta' \leqslant \nu \circ U$, by Remark 1.3.1 there exists a measure λ such that $\eta' = \lambda \circ U$. Reasoning as in the last part of the proof of Proposition 3.2.3, we conclude that $\eta = \lambda \circ U_q^\mu$.

By Proposition 3.2.3 and the implication $(1.3.1a) \Longrightarrow (1.3.1c)$ from (1.3.1) applied to the resolvent $\mathcal{U}^\mu$, there exists a right process with state space E, having $\mathcal{U}^\mu$ as associated resolvent. From Proposition 1.3.2 we conclude that the transition function of this process is precisely $\mathbb{T}^\mu = (T_t^\mu)_{t\geqslant 0}$. □

Remark. In [22] it is derived the existence of a right process with càdlàg trajectories, having $\mathbb{T}^\mu$ as transition function, as an application of a result about the preservation of the path regularity of a process by certain transformations. A main argument in that approach is the relation between the càdlàg property of the trajectories and the existence of a nest of compact sets (cf. [14] and [13]), or equivalently, the existence of an excessive functions having compact level sets (a so called *compact Lyapunov function*); see [15] and [22] for further developments and applications.

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a (Borel) right process with state space E and $(\xi_t)_{t\geqslant 0}$ be the subordinator with path space Ω' (and state space $\mathbb{R}_+$) induced by $\mu = (\mu_t)_{t\geqslant 0}$. For each $t \geqslant 0$ define the *subordinate process* $X^\xi = (X_t^\xi)_{t\geq 0}$ as

$$X_t^\xi(\omega, \omega') := X_{\xi_t(\omega')}(\omega) \text{ for all } (\omega, \omega') \in \Omega \times \Omega' \text{ and } t \geqslant 0.$$

For a detailed discussion about the the appropriate probability space structure and filtration of $\Omega \times \Omega'$ see Section 8 in the pioneering article of E. Nelson [79]; see also [31] for a different approach.

Corollary 3.3.2. *Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a (Borel) right process with state space the Lusin topological space E and transition function $\mathbb{T} = (T_t)_{t\geqslant 0}$ and let $\xi =$*

$(\xi_t)_{t\geqslant 0}$ *be the subordinator induced by* $\mu=(\mu_t)_{t\geqslant 0}$. *Then the subordinate process* $X^\xi = (X_t^\xi)_{t\geq 0}$ *is a right process with state space* E *and transition function* $\mathbb{T}^\mu=(T_t^\mu)_{t\geqslant 0}$.

Proof. By Theorem 5 from [79] it follows that $X^\xi=(X_t^\xi)_{t\geq 0}$ is a Markov process with transition function $\mathbb{T}^\mu=(T_t^\mu)_{t\geqslant 0}$. For the reader's convenience we check here that the transition function of X^ξ is indeed $\mathbb{T}^\mu$: if P'^0 is the probability on Ω' corresponding to $\xi_0=0$, with the notations from [79] we have for all $t>0$ and $f\in bp\mathcal{B}$

$$T_t^\mu f(x)=\int_0^\infty T_s f(x)\mu_t(\,\mathrm{d}s)=\int_0^\infty E^x[f(X_s)]\mu_t(\,\mathrm{d}s)$$

$$\int_{\Omega'}\left(\int_\Omega f(X_{\xi_t(\omega')}(\omega))P^x(\,\mathrm{d}\omega)\right)P'^0(\,\mathrm{d}\omega')=P^x\cdot P'^0(f(X_t^\xi)).$$

Clearly $t\longmapsto X_t^\xi$ is right continuous and therefore X^ξ is a right continuous realization of the semigroup $\mathbb{T}^\mu$. On the other hand Theorem 3.3.1 implies that $\mathbb{T}^\mu$ has a right continuous realization which is a right process and note that by [13], Section 1.7, the $\mathcal{U}_q^\mu$-excessive functions are nearly Borel. From Theorem (19.3) in [91] we conclude that $X^\xi=(X_t^\xi)_{t\geq 0}$ is also a right process. □

3.4 Markov processes associated with subordinate L^p-semigroups

Let $(V_\alpha)_{\alpha>0}$ be the (sub-Markovian) strongly continuous resolvent of contractions on $L^p(E,m)$ induced by $(P_t)_{t\geqslant 0}$,

$$V_\alpha=\int_0^\infty e^{-\alpha t}P_t\,\mathrm{d}t,\ \alpha>0.$$

Theorem 3.4.1. *Let* $p\in[1,\infty)$ *and* $(P_t)_{t\geqslant 0}$ *be a* C_0*-semigroup of sub-Markovian contractions on* $L^p(E,m)$, *where* $(E,\mathcal{B})$ *is a Lusin measurable space and* m *is a* σ*-finite measure on* $(E,\mathcal{B})$. *Assume that* m *is a* P_t*-subinvariant measure and*

$(*)$ *there exists* $f\in L^p(E,m)$, $f>0$, *such that* $V_\alpha f\leqslant 1$ *for all* $\alpha>0$.

Let further $\mu=(\mu_t)_{t\geqslant 0}$ *be a convolution semigroup on* $\mathbb{R}_+$.

Then there exist a Lusin topological space E_1 *with* $E\subset E_1$, $E\in\mathcal{B}_1$ *(the* σ*-algebra of all Borel subsets of* E_1*),* $\mathcal{B}=\mathcal{B}_1|_E$, *and a (Borel) right process* X *with state space* E_1 *such that the transition functions of* X *and of the subordinate process* X^ξ, *regarded as families of operators on* $L^p(E_1,m_1)$, *coincide with* $(P_t)_{t\geqslant 0}$ *and the subordinate semigroup of operators* $(P_t^\mu)_{t\geqslant 0}$ *respectively,*

$$P_t^\mu u=E^\cdot[u(X_t^\xi)],\ t\geqslant 0,\, u\in L^p(E,m),$$

where m_1 is the measure on $(E_1, \mathcal{B}_1)$ extending m with zero on $E_1 \setminus E$.

Proof. By Theorem 2.2 from [16] (see also Remark 3.4.2 below) there exist a Lusin topological space E_1 with $E \subset E_1$, $E \in \mathcal{B}_1$, $\mathcal{B} = \mathcal{B}_1|_E$, and a right process X with state space E_1, such that its resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$, regarded on $L^p(E_1, m_1)$, coincides with $(V_\alpha)_{\alpha>0}$ (the resolvent of operators induced by $(P_t)_{t\geqslant o}$). Since by hypothesis m is P_t-subinvariant, the measure m_1 on E_1 is $\mathcal{U}$-excessive; recall that m_1 is the measure on $(E_1, \mathcal{B}_1)$ extending m with zero on $E_1 \setminus E$. In addition, from $(*)$ there exists $f \in p\mathcal{B}_1$, $f > 0$, such that $Uf < \infty$ m-a.e., where U is the initial kernel of $\mathcal{U}$. By [16], Proposition A1 (the implication 4) $\Longrightarrow$ 1)), the measure m is dissipative. Therefore we can apply Proposition 1.4 from [16] to deduce that the process X is transient.

Let $\mathbb{T} = (T_t)_{t\geqslant 0}$ be the transition function of the right process X. If $f \in p\mathcal{B}_1$ then $\int T_t f \, \mathrm{d}m_1 \leqslant \int f \, \mathrm{d}m_1$ and so, $T_t f = 0$ m_1-a.e., provided that $f = 0$ m_1-a.e. It follows that T_t becomes a linear contraction on both spaces $L^\infty(E_1, m_1)$ and $L^1(E_1, m_1)$. Consequently, T_t induces a contraction on $L^p(E_1, m_1)$ for all $p \in [1, \infty)$ and one can see that $(T_t)_{t\geqslant 0}$ is a C_0-semigroup of sub-Markovian contractions on $L^p(E_1, m_1)$. We derive that, regarded as a family of operators on $L^p(E_1, m_1)$, $(T_t)_{t\geqslant 0}$ coincides with $(P_t)_{t\geqslant 0}$ (having the same resolvent).

By Corollary 3.3.2 the subordinate process X^ξ is also a right process with state space E_1 and $\mathbb{T}^\mu = (T_t^\mu)_{t\geqslant 0}$ is its transition function. Note that m_1 is an excessive measure with respect to $\mathbb{T}^\mu = (T_t^\mu)_{t\geqslant 0}$. As before, $(T_t^\mu)_{t\geqslant 0}$ becomes a C_0-semigroup of sub-Markovian contractions on $L^p(E_1, m_1)$. Let $(P_t^\mu)_{t\geqslant 0}$ be the semigroup of operators on $L^p(E, m)$ obtained from $(P_t)_{t\geqslant 0}$ by subordination with μ. If $f \in L^p(E_1, m_1)$ and $g \in L^{p'}(E_1, m_1)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, then by the properties of the Bochner integral and using Fubini Theorem,

$$\int_{E_1} g P_t^\mu f \, \mathrm{d}m_1 = \int_{E_1} g(x) \int_0^\infty P_s f(x) \mu_t(\,\mathrm{d}s)\ m_1(\,\mathrm{d}x) =$$

$$\int_0^\infty \mu_t(\,\mathrm{d}s) \int_{E_1} g(x) T_s f(x)\ m_1(\,\mathrm{d}x) =$$

$$\int_{E_1} g(x) \int_0^\infty T_s f(x) \mu_t(\,\mathrm{d}s)\, m_1(\,\mathrm{d}x) = \int_{E_1} g T_t^\mu f \, \mathrm{d}m_1.$$

We conclude that T_t^μ and P_t^μ coincide for each $t \geqslant 0$, regarded as operators on $L^p(E_1, m_1)$ and the proof is complete. □

Remark 3.4.2. *We present some details about the construction of the larger space E_1 from Theorem 3.4.1; we follow the proof of Theorem 2.2 from [16].*
(i) For each $\alpha > 0$ we consider a kernel $\overline{V}_\alpha$ on $(E, \mathcal{B})$ such that V_α and $\overline{V}_\alpha$ coincide as

operators on $L^p(E,m)$.
(ii) Using a "trivial modification" procedure (see also Subsection 3.2 from [22]), there exists a sub-Markovian resolvent of kernels $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ on $(E,\mathcal{B})$ such that (1.1.8) holds for $\mathcal{U}'_q$, all the points of E are nonbranch points with respect to $\mathcal{U}'_q$ for some $q>0$, and $\overline{V}_\alpha f = U'_\alpha f$ m-a.e. for all $f \in p\mathcal{B}$ and $\alpha > 0$.
(iii) Let E_1 be the set of all extreme points of the set $\{\eta \in \mathsf{Exc}(\mathcal{U}'_q) : \mathsf{L}_q(\eta,1)=1\}$, endowed with the σ-algebra $\mathcal{B}_1$ generated by the functionals $\widetilde{u}$, $\widetilde{u}(\eta) := \mathsf{L}_q(\eta,u)$ for all $\eta \in \mathsf{Exc}(\mathcal{U}'_q)$ and $u \in \mathcal{E}(\mathcal{U}'_q)$.; here L_q denotes the energy functional associated with $\mathcal{U}'_q$.
(iv) It turns out that $(E_1,\mathcal{B}_1)$ is a Lusin measurable space, the map $x \longmapsto \varepsilon_x \circ U'_\beta$ identifies E with a subset of E_1, $E \in \mathcal{B}_1$, $\mathcal{B} = \mathcal{B}_1|_E$ and there exists a sub-Markovian resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $(E_1,\mathcal{B}_1)$ such that (1.3.1a) holds for $\mathcal{U}_q$ on E_1 and $\mathcal{U}'$ is the restriction of $\mathcal{U}$ to E, i.e., $U_\alpha f|_E = U'_\alpha(f|_E)$ for all $f \in p\mathcal{B}_1$ and $\alpha > 0$. Note that by (1.3.1) $\mathcal{U}$ is the resolvent of a right process with state space E_1.

The martingale problem

Corollary 3.4.3. *Under the assumptions of Theorem 3.4.1, let $(\mathcal{L}^\mu, D(\mathcal{L}^\mu))$ be the infinitesimal generator of the semigroup $(P_t^\mu)_{t\geq 0}$, the subordinate in the sense of Bochner on $L^p(E,m)$ of the C_0-semigroup $(P_t)_{t\geq 0}$. Let further η be a probability measure on $(E,\mathcal{B})$ having density with respect to m, $\frac{\mathrm{d}\eta}{\mathrm{d}m} \in L_+^{p'}(E,m)$, where $\frac{1}{p}+\frac{1}{p'}=1$, and consider the probability P^η of the subordinate process X^ξ with the initial distribution η.*

Then for every $u \in D(\mathcal{L}^\mu)$ the process

$$u(X_t^\xi) - \int_0^t \mathcal{L}^\mu u(X_s^\xi)\,\mathrm{d}s, \quad t \geq 0,$$

is a martingale under P^η with respect to the filtration of X^ξ.

Proof. The assertion follows from [15], Proposition 1.4, applied on the larger space E_1, the state space of the subordinate process X^ξ given by Theorem 3.4.1. □

A second consequence of Theorem 3.4.1 is the validity of the quasi continuity property for the elements of a dense subspace of the domain $D(\mathcal{L}^\mu)$ of the subordinate semigroup $(P_t^\mu)_{t\geqslant 0}$, with respect to the capacity on E_1 associated with the process X. Recall that this property is analogous to the quasi-regularity condition from the Dirichlet forms theory (cf. [73]); the role of the capacity induced by the energy is played in this L^p frame by the capacity associated to the process. The precise result is the following corollary.

Corollary 3.4.4. *There exists a subspace $\mathcal{D}$ of the L^p domain $D(\mathcal{L}^\mu)$ of the infinitesimal generator of the subordinate semigroup $(P_t^\mu)_{t\geq 0}$, which is dense in $L^p(E,m)$ and* *such that every element of $\mathcal{D}$ possesses a quasi continuous version with respect to the capacity associated with the right process X on E_1 and any finite measure on E.*

Proof. We present first the suitable capacity of the transient Markov process X on E_1. Let $M \in \mathcal{B}_1$ and P_{T_M} be the associated *hitting kernel*,

$$P_{T_M} f(x) = E^x(f \circ X_{T_M}\,;\, T_M < \infty)\,, \quad x \in E,\ f \in p\mathcal{B}_1,$$

where $T_M(\omega) := \inf\{t > 0 : X_t(\omega) \in M\}$, $\omega \in \Omega$. Fix $p := Uf_0$, with $0 < f_0 \leqslant 1$, $f_0 \in p\mathcal{B}_1$, and let λ be a finite measure on $(E, \mathcal{B})$. Then the functional $M \longmapsto c_\lambda(M)$, $M \subset E_1$, defined as

$$c_\lambda(M) := \inf\Big\{\int P_{T_G} p \,\mathrm{d}\lambda \,:\, M \subset G \text{ open}\Big\},$$

is a Choquet capacity on E_1 (see e.g. [13]).

Let $\beta > 0$ and set $\mathcal{D} := U^\mu_\beta U_\beta(L^p(E_1, m_1))$. Since $U^\mu_\beta U_\alpha = U_\alpha U^\mu_\beta$ for all $\alpha, \beta > 0$, we get $\mathcal{D} \subset D(\mathcal{L}) \cap D(\mathcal{L}^\mu)$. Because $(U_\alpha)_{\alpha>0}$ and $(U^\mu_\alpha)_{\alpha>0}$ are C_0-resolvents of contractions on $L^p(E_1, m_1)$, we deduce that $\mathcal{D}$ is dense in $L^p(E_1, m_1)$. The claimed assertion follows now because by Proposition 3.2.6 from [13] if $f \in p\mathcal{B}_1 \cap L^p(E_1, m_1)$, then $U_\beta f$ is c_λ-quasi continuous. □

Remark. *The subordinate of a Lévy process on $\mathbb{R}^d$ is a Lévy proces too. Consequently, the methods developed by [26] for the simulation of the Lévy processes using Monte Carlo method is also applicable to the process obtained after introducing jumps by the Bochner subordination in the evolution of a given Lévy process.*

Part II

Deterministic approach of the shallow avalanche onset

Introduction to Part II

An important issue in geophysics is the understanding of the phenomena related to shallow avalanche of soils, snow or other geomaterials ([2, 85]). The real problem is three dimensional and the mathematical and numerical modeling is very complex. For that, reduced 2-D models (called also Saint-Venant models) are introduced (see [6, 8, 9, 88, 89, 97, 74, 75]) to capture the principal features of the flow.

Figure 3.1: A snow avalanche flow.

Natural avalanches and debris flows are often associated with complicated mountain topologies, which makes the prediction very difficult (see Figure 3.1). For that, a lot of studies include the bottom curvature effect into the classical Saint-Venant equations to describe channelized flows along talwegs [45, 98, 86, 97] (see also the review [87]) or flows on more general basal geometries [64, 29, 30]. Very recently, the model obtained in [55] for plane slopes, was extended in [56] to the case of a general basal topography by using a local base given by the bottom geometry and the associated differential operators.

The first goal of the second part of the thesis is to show that the model obtained in [56] can describe also the avalanche onset of a flow. We introduce here a simple criterion, which relates the yield limit (material resistance) to the external forces distribution, able to distinguish if an avalanche occurs or not. This criterion is related with the maximum of the loading parameter such that the fluid/solid can withstand without collapsing. The safety factor is the solution of an global optimization problem, called limit load analysis. The velocity field, solution of this optimization problem, is called the collapse flow or onset velocity field.

Figure 3.2: The onset of a snow avalanche as a fracture process.

In many applications, the strains are localized on some surfaces where the velocity of the collapse flow exhibits discontinuities (see Figure 3.2). From mathematical and numerical points of view, the avalanche onset modeling was and remains a difficult problem. The second objective is to prove the existence of an onset velocity field in an appropriate functional space. Since the functional involved in the global optimization is non-smooth, and non coercive in classical Sobolev spaces, we have to consider it in the space of bounded tangential deformation functions (i.e., the space of velocities which have their tangential rate of deformation in the space of bounded measures), similar to the space introduced in [94, 96].

The third objective is to propose a numerical strategy to solve the limit load problem and to get the onset flow field of the avalanche. The numerical solutions methods in limit analysis are based on the discretization of the kinematic or static variational principles (established in [38]) using the finite element method technics and the convex and linear programming. Despite great progress in the last decades (X-FEM, re-meshing techniques), the finite element method remains associated to continous fields and it is not so well adapted for modeling strain localization and velocities discontinuities on unknown surfaces. For that, we will adapt here the discontinuous velocity domain splitting (DVDS) method, introduced in [60]. DVDS is a mesh free method which does not use a finite element discretization of the solid. It focuses on the strain localization and completely neglect the bulk deformations. The limit load problem is thus reduced to the minimization of a shape-dependent functional (plastic dissipation power). The avalanche collapse flow velocity field, which is discontinuous, is associated to an optimum sub-domain and a rigid flow. It has localized deformations only, at the boundary of the sub-domain.

The main novelty of this part consists in finding the appropriate functional space of the limit load problem, in obtaining an existence result for the onset velocity field. The specific Stokes formula are proved, the variational formulation of the velocity field by using the tangential plane Stokes formula associated to these operators, and the set of tangential rigid velocities are deduced. As far as we know, the use of a mesh free technique (DVDS) for a numerical approach of the shallow avalanche onset problem is also new. All the new results of this part can be found in [58, 59].

This part is organized as follows: In Chapter 4 we present the 3-D dimensional mechanical problem and we discuss the choice of the visco-plastic model adopted here.

In Chapter 5 we introduce the shallow flow problem. Firstly, we give a geometrical description of the bottom surface and the expressions of the differential operators acting in the tangential plane. We prove here specific Stokes formula and we deduce the set of tangential rigid velocities. We recall from [56] the boundary-value problem for the visco-plastic Saint-Venant model with topography formulated on the local base associated to the bottom surface. Finaly, we give the variational formulation of the velocity field by using the tangential plane Stokes formula associated to these operators.

Chapter 6 is devoted to the mathematical approach of the limit load problem obtained from the variational formulation, described before. We introduce a global optimization problem (called the limit load analysis or safety factor problem) on classical Sobolev spaces to study the link between the yield limit, the external forces and the thickness distributions for which the shallow flow of a visco-plastic fluid does, or does not occur. Then, we consider the same optimization problem in the space of bounded tangential deformation functions. The boundary conditions, expressed for smooth functions have to be relaxed for non-smooth velocity fields considered in this new functional framework. For that, we have to add some additional boundary integrals on the plastic dissipation functionals and external forces power. In these integrals, which are modeling a discontinuity of a non-smooth velocity field located at boundary, we have to define the tangential normal on a bottom boundary. We prove that the above relaxation of the boundary conditions does not change the initial optimization problem. At the end of this chapter we prove that the reformulated safety factor problem has at least a solution, modeling the avalanche onset. For that, we have to study coercivity properties of the plastic dissipation functional and to describe the kernel of the tangential deformation operator by using the space of tangential rigid velocities introduced before.

In Chapter 7 we adapted the DVDS numerical technique ([60]) to solve the safety factor problem. This last problem is reduced to a shape optimization problem. The description of the subset shape is given by a level set of a Fourier function and we use genetic algorithms to solve the resulted non convex and non-smooth global optimization problem. We illustrate the proposed numerical approach in solving some safety factor problems. First, we consider the case of a plane slope with a non-uniform thickness distribution. For a circular dome geometry of a Bingham (Von-Mises plasticity) fluid we give a comparison between our results and a dynamic finite element/finite volume method. Then, we analyze the avalanche of a square dome of a Drucker-Prager fluid, and the last example concerns the avalanche of a thick Bingham fluid over an obstacle. Finally, we illustrate our technique in the case of a complex basal topography. For a half-sphere et quarter of a sphere covered with a Bingham fluid/solid with constant thickness distribution we compute the safety factor and the avalanche onset. In the last example we analyze the case of a quarter of a ellipsoid filled with a Drucker-Prager fluid/solid with a non-uniform thickness distribution.

Chapter 4

3-D mechanical modeling

In last few years, a lot of efforts in geophysics and engineering have been devoted to the study of the physics of avalanche formation and to the flow of soils, snow or other geomaterials. It has been recognized that the real problem is 3-D and that the behavior of the material is best represented by visco-plastic fluid type models.

4.1 Field equations and boundary conditions

We consider here the evolution, on the time interval $(0, T)$, of a visco-plastic fluid/solid occupying a domain $\mathcal{D}(t) \subset \mathbb{R}^3$. In what follows the space and time coordinates, as well as all mechanical fields, are non dimensional. The boundary $\partial\mathcal{D}(t)$ is divided into three disjoint parts $\partial\mathcal{D}(t) = \Gamma_b(t) \cup \Gamma_s(t) \cup \Gamma_l(t)$, the bottom, the free and the lateral boundaries (see Figure 4.1).

The notation $\boldsymbol{u}$ stands for the velocity field, $\boldsymbol{\sigma}$ for the (non dimensional) Cauchy stress tensor field, $p = -\operatorname{trace}(\boldsymbol{\sigma})/3$ for the (non dimensional) pressure and $\boldsymbol{\sigma'} = \boldsymbol{\sigma} + pI_3$ the (non dimensional) stress deviator tensor. The momentum balance law in the Eulerian coordinates reads

$$\rho\Big(\mathrm{St}\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\Big) - \mathbf{div}\,\boldsymbol{\sigma'} + \nabla p = \frac{1}{\mathrm{Fr}^2}\rho \boldsymbol{f} \quad \text{in } \mathcal{D}(t), \tag{4.1.1}$$

where $\rho > 0$ is the (non dimensional) mass density and $\boldsymbol{f}$ denotes the (non dimensional) body forces. We have denoted by

$$\mathrm{St} = \frac{L_c}{V_c T_c}, \quad \mathrm{Fr}^2 = \frac{V_c^2}{L_c f_c}$$

the Strouhal and Froude numbers, where $\rho_c, V_c, L_c, f_c, T_c$ are the characteristic density, velocity, length, body forces and time respectively and we have chosen the characteristic yields stress $\kappa_c = \rho_c V_c^2$. Since we deal with an incompressible fluid, we have

$$\operatorname{div}\,\boldsymbol{u} = 0 \quad \text{in } \mathcal{D}(t). \tag{4.1.2}$$

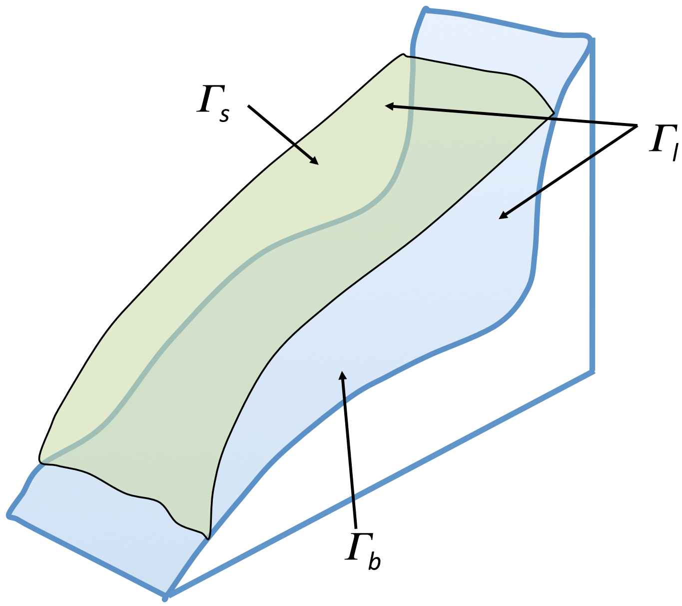

Figure 4.1: 3-D representation of the domain $\mathcal{D}(t)$ and the partition of the boundary $\partial\mathcal{D}(t)$.

The fact that the fluid region is advected by the flow is expressed by

$$\frac{d}{dt}1_{\mathcal{D}(t)} = \mathrm{St}\frac{\partial 1_{\mathcal{D}(t)}}{\partial t} + \boldsymbol{u}\cdot\nabla 1_{\mathcal{D}(t)} = 0, \tag{4.1.3}$$

where $1_{\mathcal{D}(t)}$ is the characteristic function of the domain $\mathcal{D}(t)$.

To complete equations (4.1.1–4.1.3) with the boundary conditions let $\boldsymbol{n}$ be the outward unit normal on $\partial\mathcal{D}(t)$. We denote by $\sigma_n = \boldsymbol{\sigma n}\cdot\boldsymbol{n}$, $u_n = \boldsymbol{u}\cdot\boldsymbol{n}$ and by $\boldsymbol{\sigma}_T = \boldsymbol{\sigma n} - \sigma_n\boldsymbol{n}$, $\boldsymbol{u}_T = \boldsymbol{u} - u_n\boldsymbol{n}$ the normal and tangential parts of surface forces and of the velocity on the boundary. On $\Gamma_b(t)$, which corresponds to the bottom part, the visco-plastic fluid is in contact with Coulomb friction with a rigid structure, described by:

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0,\quad \boldsymbol{\sigma}_T = C_f\sigma_n\boldsymbol{\Lambda}^f(\boldsymbol{u}_T),\quad \begin{cases} \boldsymbol{\Lambda}^f(\boldsymbol{u}_T) = \dfrac{\boldsymbol{u}_T}{|\boldsymbol{u}_T|} & \text{if } |\boldsymbol{u}_T| \neq 0, \\ |\boldsymbol{\Lambda}_f(\boldsymbol{u}_T)| \leq 1 & \text{if } |\boldsymbol{u}_T| = 0, \end{cases}$$

where C_f is the friction coefficient. The (unknown) boundary $\Gamma_s(t)$ is a free surface, i.e., we assume a stress free condition $\boldsymbol{\sigma n} = 0$. We will suppose that the lateral boundary

$\Gamma_l(t)$, is splinted into two parts, $\Gamma_l^0(t)$ and $\Gamma_l^1(t)$, and we will consider two kinds of boundary conditions: adherence $\boldsymbol{u} = 0$ on $\Gamma_l^0(t)$ and a stress free condition $\boldsymbol{\sigma n} = 0$ on $\Gamma_l^1(t)$. Finally, the initial conditions are given by

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \mathcal{D}(0) = \mathcal{D}_0.$$

4.2 The visco-plastic model

Let us begin by describing the visco-plastic fluid model used. In contrast with a classical viscous fluid, which cannot sustain a shear stress at rest, the Cauchy stress tensor $\boldsymbol{\sigma}$ of a visco-plastic fluid belongs to an admissible convex set $K = \{\boldsymbol{\sigma} \in \mathbb{R}_S^{3\times 3}\ ;\ ||\boldsymbol{\sigma}'|| \leq \kappa(p)\}$, where $\kappa = \kappa(p)$ is the yield limit. The boundary of K stands for the flow/no flow condition. Conversely, if the stress is in K then the rate of deformation tensor $\boldsymbol{D}(\boldsymbol{u}) = \dfrac{1}{2}(\boldsymbol{\nabla u} + \boldsymbol{\nabla}^T\boldsymbol{u})$ vanishes. If the stress tensor is not in K then we deal with an incompressible viscous flow described by the following constitutive equation:

$$\text{(4.2.1)} \qquad \begin{cases} \boldsymbol{\sigma}' = \dfrac{2}{\mathrm{Re}}\eta(\|\boldsymbol{D}(\boldsymbol{u})\|, p)\boldsymbol{D}(\boldsymbol{u}) + \kappa(p)\dfrac{\boldsymbol{D}(\boldsymbol{u})}{\|\boldsymbol{D}(\boldsymbol{u})\|} & \text{if } \boldsymbol{D}(\boldsymbol{u}) \neq 0, \\ \|\boldsymbol{\sigma}'\| \leq \kappa(p) & \text{if } \boldsymbol{D}(\boldsymbol{u}) = 0, \end{cases}$$

where $\eta = \eta(\|\boldsymbol{D}(\boldsymbol{u})\|, p) > 0$ is the (non dimensional) viscosity ($\|\boldsymbol{A}\| =: |\boldsymbol{A}|/\sqrt{2} = \sqrt{\boldsymbol{A} : \boldsymbol{A}/2}$ denotes the second invariant of the stress deviator tensor) and $\mathrm{Re} = \rho_c V_c L/\eta_c$ is the Reynolds number (with η_c is the characteristic viscosity). We can recast the above equation in a different form by writing the rate of deformation $\boldsymbol{D}(\boldsymbol{u})$ as a function of the stress deviator

$$\text{(4.2.2)} \qquad \boldsymbol{D}(\boldsymbol{u}) = \frac{\mathrm{Re}}{2\eta}\left[\mathcal{F}(\boldsymbol{\sigma})\right]_+ \frac{\boldsymbol{\sigma}'}{\|\boldsymbol{\sigma}'\|},$$

where $[\ x\ ]_+ = (|x| + x)/2$ is the positive part, and $\mathcal{F}$ is the yield function:

$$\mathcal{F}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma}'\| - \kappa(p).$$

The expression (4.2.2) of the visco-plastic constitutive law (4.2.1) was used by Perzyna [84] and Duvaut Lions [39] in extending inviscid plastic models to account for rate effects (visco-plastic regularization method).

Note that the state of stress, $\boldsymbol{\sigma}'$, is represented as the sum of a viscous contribution $\boldsymbol{\sigma}^V = 2\eta(\|\boldsymbol{D}\|, p)\boldsymbol{D}$ (rate dependent) and a contribution $\boldsymbol{S} = \kappa(p)\dfrac{\boldsymbol{D}}{\|\boldsymbol{D}\|}$, related to plastic effects (rate independent). The viscous part of the stress $\boldsymbol{\sigma}^V$, as for a classical viscous fluid, is continuous in $\boldsymbol{D}$ and vanishes for $\boldsymbol{D} = 0$, i.e.,

$$\|\boldsymbol{D}\|\eta(\|\boldsymbol{D}\|, p) \to 0, \quad \text{for} \quad \|\boldsymbol{D}\| \to 0. \tag{4.2.3}$$

At difference with the viscous contribution $\boldsymbol{\sigma}^V$, the plastic part $\boldsymbol{S}$ is not continuous in $\boldsymbol{D}$ and $\boldsymbol{S}$ does not vanish for $\boldsymbol{D} = 0$. For $\boldsymbol{D} \neq 0$ we get $\|\boldsymbol{\sigma}'\| = \|\boldsymbol{D}\|\eta(\|\boldsymbol{D}\|, p) + \kappa(p) > \kappa(p)$ and since (4.2.3) holds we obtain a continuous transition between flow and no-flow states (i.e., the flow rule and the non flow condition are compatible).

For $\kappa(p) \equiv 0$ the plastic effects are vanishing and (4.2.1) reduces to a viscous fluid model. If η is independent of $\|\boldsymbol{D}\|$ and p, (4.2.1) reduces to the incompressible Navier-Stokes model but other choices can also be considered (Prandtl-Eyring, Norton, etc).

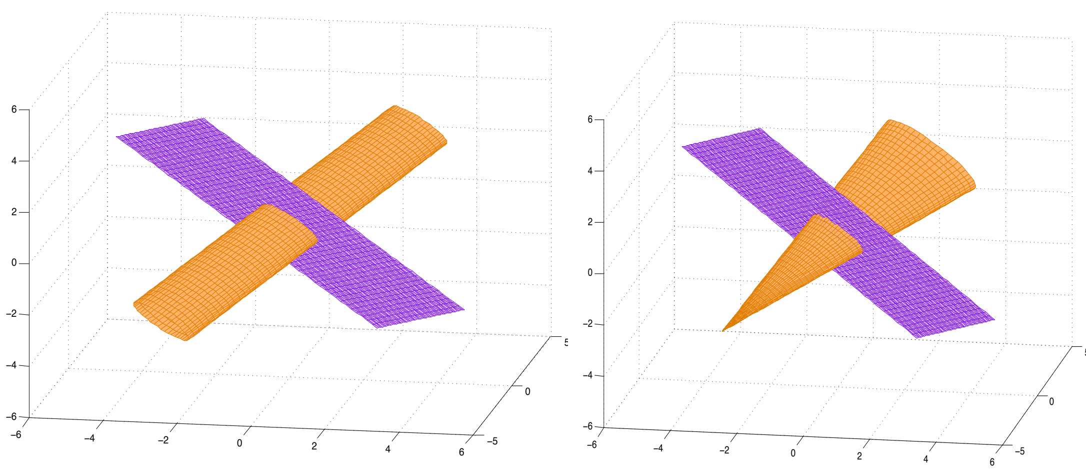

Figure 4.2: The 3-D flow/no flow condition ($||\boldsymbol{\sigma}'|| = \kappa(p)$) represented in the space of principal stresses $O(-\sigma_1)(-\sigma_2)(-\sigma_3)$ and the intersection with the deviatoric plane $\sigma_1 + \sigma_2 + \sigma_3 = 0$. Left: Von-Mises model ($\kappa^0 = 1, \mu = 0$). Right: Drucker-Prager model ($\kappa^0 = 1, \mu = 0.2$).

The Von-Mises plasticity criterion (see Figure 4.2 left):

$$\kappa(p) \equiv \kappa^0 > 0$$

was introduced to describe the plasticity of metals. If η is independent of $\|\boldsymbol{D}\|$ and p the constitutive equation (4.2.1) recover the classical *Bingham fluid* model (see [25]), used for many fluids with a solid like behavior (for instance soils or sediments in oil drilling processes). This model, also denominated "Bingham solid" (see for instance [80]) was also considered to describe the (high rate) deformation of many solid materials having a fluid like behavior.

The plasticity (flow/no flow) criterion

$$\kappa(p) = \kappa^0 + \mu p$$

(see Figure 4.2 right) is called the Drucker-Prager model (see [37]). This yielding criterion was constructed as a simplification of Mohr-Coulomb plasticity: $\tau_{max}(\boldsymbol{\sigma'}) \leq C + p\tan(\phi)$ where $\tau_{max}(\boldsymbol{\sigma'})$ is the Mohr-Coulomb tangential stress, C is the cohesion and ϕ is the angle of internal friction. The fact that the Mohr-Coulomb criterion is expressed in terms of the eigenvalues $\sigma_3 \leq \sigma_2 \leq \sigma_1$ of the stress tensor $\boldsymbol{\sigma}$ through $\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin(\phi) \leq 2C\cos(\phi)$, makes it very difficult to handle numerically. This is not the case for Drucker-Prager yield condition: $\|\boldsymbol{\sigma'}\| \leq \kappa^0 + \mu p$ which involves the norm of the stress deviator tensor. The correspondence between the constitutive coefficients κ^0 and μ of the Drucker-Prager model and the coefficients of the Mohr-Coulomb model is not simple to establish. The usual choice is $\kappa^0 = C\cos(\phi), \quad \mu = \tan(\phi)$, but other choices can be found if one choose to reproduce different experimental settings. For the non-associate incompressible flow we can consider $\mu = \sin(\phi)$ in the in-plane case and $\mu = 6\sin(\phi)/(\sqrt{3}(3 - \sin(\phi))$ for the triaxial compression (see for instance in [77]). For constant viscosity η (not depending on $\|\boldsymbol{D}\|$ and p) we refer to the model as the "Drucker-Prager fluid". With an appropriate choice of the viscosity η the constitutive law (4.2.1) recovers the model proposed by Jop, Forterre and Pouliquen [44] for granular materials.

Chapter 5

The shallow flow problem

Since the numerical integration of the three dimensional equations of visco-plastic fluids is very complex and poses many challenges, reduced 2-D models, called also Saint-Venant models, are generally considered. Such models are able to capture the principal features of the flow: onset, dynamic propagation and arrest.

When the fluid is relatively shallow and spreads slowly, lubrication-style asymptotic approximations can be used to build reduced models for the spreading dynamics of visco-plastic fluids. For two-dimensional (sheet) flow lubrication models were introduced by Liu and Mei [68, 69] and applied to problems of mud flow, while Balmforth et al. [6] considered the axisymmetric version of the problem to model the extrusion of lava domes. The lubrication model has been successfully extended to three dimensions in [7] and used thereafter in [8, 9]. Other model, which considers the same adherence conditions on the bottom as the lubrication models, was recently obtained in [42].

When the movement is faster, shallow water theory for non-viscous flows may be used in conjunction with Coulomb frictional type boundary condition at the bottom. A depth integrated theory, obeying a Mohr-Coulomb type yield criterion, was introduced by Savage and Hutter [88, 89] and developed thereafter by many authors (see for instance [97, 74, 75]). These models takes into account the frictional dissipation between the flowing layers parallel to the basal plane through an anisotropy factor which depends on the friction angles. The importance of this anisotropy factor is still an open question and an accurate derivation of the equations is still lacking (see e.g. [83]).

A large number of complex fluids exhibit an effective slip between the fluid and the wall. As it has been noted in [9], if slip becomes sufficiently severe, the flow can become relatively plug-like across the film thickness. Then, the shear stresses, which are dominant in the lubrication models become small, while the extensional and in-plane shear stress becomes important. This is the case when dealing with free liquid threads and films [81], ice shelves and streams [72] or snow avalanches. This situation demands a different depth integrated theory, obtained in [55] for fluids with a Drucker-Prager type yield criterion (which includes the Bingham model) flowing down inclined planes and extended in [56] to the case of a general basal topography.

5.1 Geometrical description of the bottom surface

Coordinates associated to the bottom surface. To describe the shallow flow of a visco-plastic fluid/solid we shall use a system of coordinates adapted to the geometry of the flow. Let us describe first the bottom surface $\mathcal{S}_b$ (see Figure 5.1), given through a general parametric representation by $\boldsymbol{r}_b(x_1, x_2) = B_1(x)\boldsymbol{c}_1 + B_2(x)\boldsymbol{c}_2 + B_3(x)\boldsymbol{c}_3$, where $x = (x_1, x_2)$ are the parametric coordinates belonging to a two dimensional domain $\Omega \subset \mathbb{R}^2$ and $\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\}$ is the Cartesian basis with the vertical in the $\boldsymbol{c}_3$ direction. Note that, in general, $x = (x_1, x_2)$ are not physical coordinates.

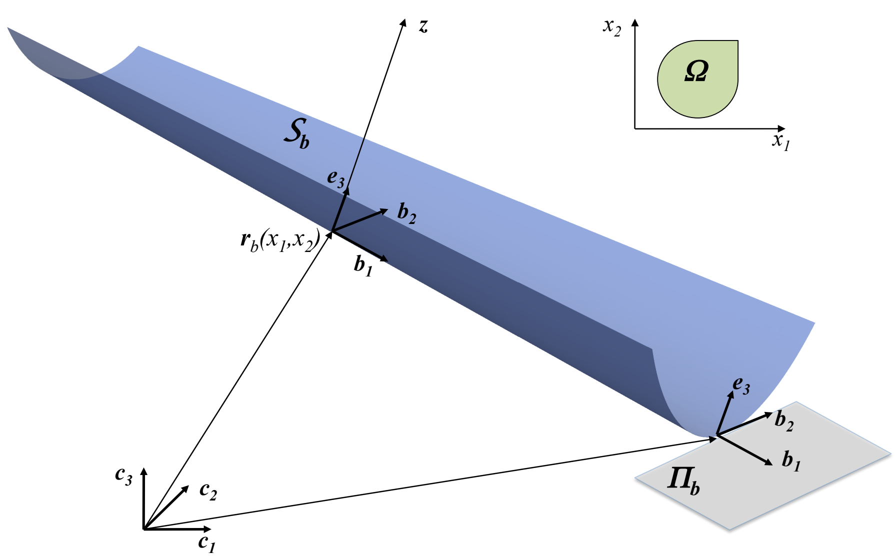

Figure 5.1: 3-D representation bottom surface $\mathcal{S}_b$, described through a parametric representation by $\boldsymbol{r}_b(x_1, x_2)$.

Let $\Pi_b = \Pi_b(x)$ be the two dimensional vectorial space tangent to the bottom surface $\mathcal{S}_b$. We denote by

$$\boldsymbol{b}_1(x) = \frac{\partial \boldsymbol{r}_b}{\partial x_1}(x), \quad \boldsymbol{b}_2(x) = \frac{\partial \boldsymbol{r}_b}{\partial x_2}(x)$$

the covariant basic vectors and by $g_{11} = |\boldsymbol{b}_1|^2, g_{22} = |\boldsymbol{b}_2|^2, g_{12} = \boldsymbol{b}_1 \cdot \boldsymbol{b}_2$, the covariant fundamental magnitudes of the first order. Le $\Pi_b = \Pi_b(x)$ be the two dimensional vectorial space tangent to the bottom surface $\mathcal{S}_b$ (i.e., $\Pi_b(x) := Sp\{\boldsymbol{b}_1(x), \boldsymbol{b}_2(x)\}$). We denote by L_1, L_2 the Lamé coefficients and by g the element of area in the tangent plane

$$L_1(x) = \sqrt{g_{11}}, \quad L_2(x) = \sqrt{g_{22}}, \quad g(x) = \sqrt{g_{11}g_{22} - g_{12}^2}.$$

We denote also by $\boldsymbol{e}_1, \boldsymbol{e}_2$ the covariant physical basis and by $\boldsymbol{e}_3$ the unit normal vector on $\mathcal{S}_b$:

$$\boldsymbol{e}_1 = \frac{1}{L_1}\boldsymbol{b}_1, \quad \boldsymbol{e}_2 = \frac{1}{L_2}\boldsymbol{b}_2, \quad \boldsymbol{e}_3 = \frac{\boldsymbol{b}_1 \wedge \boldsymbol{b}_2}{g}.$$

To introduce the contravariant tangent basis, denoted by $\boldsymbol{b}^1, \boldsymbol{b}^2$, and the contravariant fundamental magnitudes of the first order

$$g^{11} = |\boldsymbol{b}^1|^2 = \frac{g_{22}}{g^2}, \quad g^{22} = |\boldsymbol{b}^2|^2 = \frac{g_{11}}{g^2}, \quad g^{12} = \boldsymbol{b}^1 \cdot \boldsymbol{b}^2 = -\frac{g_{12}}{g^2}.$$

The fundamental magnitudes of the second order are given by

$$k_{11} = \frac{\partial^2 \boldsymbol{r}_b}{\partial x_1^2} \cdot \boldsymbol{e}_3, \quad k_{22} = \frac{\partial^2 \boldsymbol{r}_b}{\partial x_2^2} \cdot \boldsymbol{e}_3, \quad k_{12} = k_{21} = \frac{\partial^2 \boldsymbol{r}_b}{\partial x_1 \partial x_2} \cdot \boldsymbol{e}_3.$$

which define the curvature tensor $\boldsymbol{k}$:

$$\boldsymbol{k} = k_{ij}\boldsymbol{b}^i \otimes \boldsymbol{b}^j = k^i_j \boldsymbol{b}_i \otimes \boldsymbol{b}^j = k^{ij}\boldsymbol{b}_i \otimes \boldsymbol{b}_j,$$

with summation on i and j from 1 to 2.

Differential operators in the bottom tangent plane. We recall the formula associated to the differential operators in the tangent plane.

For a scalar field acting on $\mathcal{S}_b$ and given through $\phi : \Omega \to \mathbb{R}$ the tangential gradient $\nabla_T \phi$ is given by

$$\nabla_T \phi = \frac{\partial \phi}{\partial x_k}\boldsymbol{b}^k = \frac{\partial \phi}{\partial x_k} g^{ki}\boldsymbol{b}_i,$$

where the summation is done from 1 to 2. For a vector field acting on $\mathcal{S}_b$ and given through $\boldsymbol{\Psi} : \Omega \to \Pi_b$, with $\boldsymbol{\Psi}(x) = \Psi^i(x)\boldsymbol{b}_i(x) = \Psi_i(x)\boldsymbol{b}^i(x)$ we have

(5.1.1) $$\boldsymbol{\nabla}_T \boldsymbol{\Psi} = (\frac{\partial \Psi^i}{\partial x_k} + \Gamma^i_{jk}\Psi^j)\boldsymbol{b}^k \otimes \boldsymbol{b}_i = (\frac{\partial \Psi_i}{\partial x_k} - \Gamma^j_{ki}\Psi_j)\boldsymbol{b}^k \otimes \boldsymbol{b}^i = g^{kl}(\frac{\partial \Psi^i}{\partial x_l} + \Gamma^i_{jl}\Psi^j)\boldsymbol{b}_k \otimes \boldsymbol{b}_i$$

(5.1.2) $$D_T(\boldsymbol{\Psi}) = \frac{1}{2}(\boldsymbol{\nabla}_T \boldsymbol{\Psi} + \boldsymbol{\nabla}^t_T \boldsymbol{\Psi}),$$

$$\mathrm{div}_T \boldsymbol{\Psi} = \frac{\partial \Psi^k}{\partial x_k} + \Psi^i \Gamma^k_{ik} = g^{ki}(\frac{\partial \Psi_i}{\partial x_k} - \Psi_j \Gamma^j_{ki}),$$

where the Christoffel symbols are given by $\Gamma^k_{ij} = \dfrac{\partial^2 \boldsymbol{r}_b}{\partial x_i \partial x_j} \cdot \boldsymbol{b}^k$. Let us also note that for a tensor $\boldsymbol{T} = T^{ij} \boldsymbol{b}_i \otimes \boldsymbol{b}_j$ the trace and the norm are given by

$$\mathrm{trace}(\boldsymbol{T}) = g_{ij} T^{ij}, \quad |\boldsymbol{T}|^2 = g_{ik} g_{jl} T^{lk} T^{ij}.$$

Tangential normal and Stokes formula. We give here the definition of the tangential normal to a boundary of a domain laying in the bottom surface and Stokes formula associated to the tangential plane operators.

Let define the tangential normal $\boldsymbol{n}_T$ of a subdomain ω_b of $\mathcal{S}_b$. For that, let ω be a subdomain of Ω such that $\omega_b = \{\boldsymbol{r}_b(x); x \in \omega\}$. The boundary $\partial\omega_b$ of ω_b is defined through a curve $\mathcal{C} = \{s \to \boldsymbol{r}_b(x_1(s), x_2(s))\} := \partial\omega_b$, where $s \to (x_1(s), x_2(s))$ is the parametric representation of the curve $C := \partial\omega \subset \bar{\Omega}$. Here s is the curvilinear length, i.e., the unitar tangent and normal vectors are

$$(\tau_1, \tau_2) = \left(\frac{dx_1}{ds}, \frac{dx_2}{ds}\right), \qquad (n_1, n_2) = (-\tau_2, \tau_1).$$

Let $\boldsymbol{n}_T$ be the intersection of $\mathcal{N}_\mathcal{C}$, the normal plane on $\mathcal{C}$, with the tangent plane Π_b on $\mathcal{S}_b$, i.e., $Sp\{\boldsymbol{n}_T\} = \Pi_b \cap \mathcal{N}_\mathcal{C}$ (see Figure 5.2). More precisely, we define

$$\boldsymbol{n}_T = n_i \boldsymbol{b}^i = -\tau_2 \boldsymbol{b}^1 + \tau_1 \boldsymbol{b}^2.$$

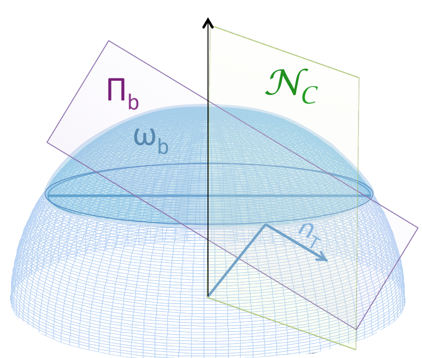

Figure 5.2: Reprezentation of the tangential normal $\boldsymbol{n}_T$ on a curve $\mathcal{C} = \partial\omega_b \subset \mathcal{S}_b$.

For a smooth function $\boldsymbol{\Psi}$ we have the following formula

$$\text{(5.1.3)} \qquad \int_\omega \mathbf{div}_T(\boldsymbol{\Psi}) g dx = \int_{\partial\omega} \boldsymbol{\Psi} \cdot \boldsymbol{n}_T g dS, \quad \int_\omega \boldsymbol{\nabla}_T \boldsymbol{\Psi} g dx = \int_{\partial\omega} \boldsymbol{\Psi} \otimes \boldsymbol{n}_T g dS,$$

$$\int_{\omega} D_T(\boldsymbol{\Psi}) g dx = \int_{\partial\omega} E_T(\boldsymbol{\Psi}) g dS, \tag{5.1.4}$$

where E_T is given in (6.2.3). To prove these, we remark that

$$\frac{\partial g^2}{\partial x_1} = 2g^2 \left(\frac{\partial^2 \boldsymbol{r}_b}{\partial x_1^2} \cdot \boldsymbol{b}^1 + \frac{\partial^2 \boldsymbol{r}_b}{\partial x_1 \partial x_2} \cdot \boldsymbol{b}^2 \right) \text{ and } \frac{\partial g^2}{\partial x_2} = 2g^2 \left(\frac{\partial^2 \boldsymbol{r}_b}{\partial x_1 \partial x_2} \cdot \boldsymbol{b}^1 + \frac{\partial^2 \boldsymbol{r}_b}{\partial x_2^2} \cdot \boldsymbol{b}^2 \right).$$

Using the Christoffel symbols, we obtain:

$$\frac{\partial g^2}{\partial x_1} = 2g^2 \Gamma^k_{1k}, \quad \frac{\partial g^2}{\partial x_2} = 2g^2 \Gamma^k_{2k}, \text{ and } \frac{\partial g}{\partial x_i} = g \Gamma^k_{ik} \text{ for } i = 1, 2.$$

Taking $\boldsymbol{\Psi} = \Psi^i \boldsymbol{b}_i$ we have $g \,\mathbf{div}_T(\boldsymbol{\Psi}) = \dfrac{\partial}{\partial x_1}(g\Psi^1) + \dfrac{\partial}{\partial x_2}(g\Psi^2)$, and the first formula in (5.1.3) yields.

To prove the second formula in (5.1.3) let us remark that

$$\begin{aligned}\int_{\Omega} \boldsymbol{\nabla}_T \boldsymbol{\Psi} g dx &= \int_{\Omega} g \frac{\partial \Psi^i}{\partial x_k} \boldsymbol{b}^k \otimes \boldsymbol{b}_i + g \Psi_j \Gamma^i_{jk} \boldsymbol{b}^k \otimes \boldsymbol{b}_i dx \\ &= \int_{\Omega} \frac{\partial}{\partial x_k}(g \Psi_i \boldsymbol{b}^k \otimes \boldsymbol{b}_i) - \Psi_i \frac{\partial}{\partial x_k}(g \boldsymbol{b}^k \otimes \boldsymbol{b}_i) + g \Psi_j \Gamma^i_{jk} \boldsymbol{b}^k \otimes \boldsymbol{b}_i dx.\end{aligned}$$

We multiply the last two terms of the integral with $\boldsymbol{b}_n$ to the left and $\boldsymbol{b}^m$ to the right and using the Christoffel symbols we have

$$\Psi_i \frac{\partial}{\partial x_k}(g \boldsymbol{b}^k \otimes \boldsymbol{b}_i) - g \Psi_j \Gamma^i_{jk} \boldsymbol{b}^k \otimes \boldsymbol{b}_i = g \left\{ \left[\Psi_m \boldsymbol{b}_n \cdot \frac{\partial \boldsymbol{b}^k}{\partial x_k} + \Psi_i \frac{\partial \boldsymbol{b}_i}{\partial x_n} \cdot \boldsymbol{b}^m \right] + \Psi_m \Gamma^l_{nl} - \Psi_j \Gamma^m_{jn} \right\}.$$

Because $\dfrac{\partial \boldsymbol{b}^k}{\partial x_k} \cdot \boldsymbol{b}_n = -\boldsymbol{b}^k \cdot \dfrac{\partial \boldsymbol{b}_n}{\partial x_k} = -\Gamma^k_{nk}$, we obtain:

$$\int_{\Omega} \boldsymbol{\nabla}_T \boldsymbol{\Psi} g dx = \int_{\partial\Omega} g \Psi^i \boldsymbol{b}^k \otimes \boldsymbol{b}_i n_k dS = \int_{\partial\Omega} g \boldsymbol{n}_T \otimes \boldsymbol{\Psi} dS.$$

Since $g D_T(\boldsymbol{\Psi}) = D_T(g\boldsymbol{\Psi}) - (\boldsymbol{\nabla}_T g \otimes \boldsymbol{\Psi} + \boldsymbol{\Psi} \otimes \boldsymbol{\nabla}_T g)$ from (5.1.3) we get (5.1.4).

5.2 Geometrical description of the shallow flow

The system of parallel surfaces. We introduce the 3-D system of coordinates by using a system of parallel surfaces, defined as the loci of points at constant distances along the normals of the bottom surface $\mathcal{S}_b$. The position vector $\boldsymbol{r}$ is expressed from the curvilinear coordinates (x, z) (here z is the coordinate representing the distance to $\mathcal{S}_b$) through

$$\boldsymbol{r} = \boldsymbol{r}(x, z) = \boldsymbol{r}_b(x) + z\boldsymbol{e}_3(x).$$

We use now the 3-D system of coordinates, introduced above, to describe the domain $\mathcal{D}(t) \subset \mathbb{R}^3$ (see Figure 5.3) occupied by the visco-plastic fluid/solid on the time interval $(0, T)$ through a single scalar variable $h(t, x) \geq 0$, the thickness of the fluid/solid:

$$\mathcal{D}(t) = \{\boldsymbol{r}(x, z)\ ;\ x \in \Omega,\ 0 < z < h(t, x)\}.$$

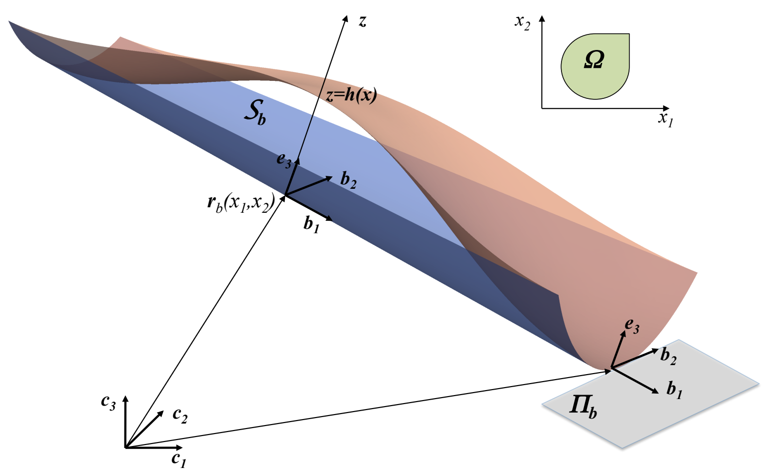

Figure 5.3: 3-D representation bottom surface $\mathcal{S}_b$, described through a parametric representation by $\boldsymbol{r}_b(x_1, x_2)$, and of the fluid domain $\mathcal{D}(t)$, described through the thickness function $z = h(t, x_1, x_2)$.

Normal and tangential decomposition. We shall use the following unique decompositions into tangential $\boldsymbol{v}$ and normal w components of the velocity field,

$$\boldsymbol{u} = \boldsymbol{v} + w\boldsymbol{e}_3, \quad \boldsymbol{v} \cdot \boldsymbol{e}_3 = 0.$$

$\sigma_{33} \in \mathbb{R}$ and $\boldsymbol{\sigma}_{3T} \in \Pi_b$ are the normal and tangent components of the stress vector acting on the surfaces $z = const$ and $\boldsymbol{\sigma}_T \in \Pi_b \otimes \Pi_b$ is the tangential part of the Cauchy stress tensor acting from Π_b into Π_b, defined through the unique decomposition

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_T + \boldsymbol{\sigma}_{3T} \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{\sigma}_{3T} + \sigma_{33}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3.$$

The body forces $\boldsymbol{f}$ are decomposed into the tangent and normal parts with respect to the bottom surface:

$$\boldsymbol{f} = \boldsymbol{f}_T + f_N\boldsymbol{e}_3, \quad \boldsymbol{f}_T \cdot \boldsymbol{e}_3 = 0.$$

The fact that region $\mathcal{D}(t)$ is advected by the flow (see (4.1.3)) can be written now as

$$\text{St}\frac{\partial h}{\partial t} + \boldsymbol{v} \cdot \nabla_T h - w = 0 \quad \text{for } z = h(t, x). \tag{5.2.1}$$

Boundary conditions. The partition of $\partial\mathcal{D}(t)$ could be described as $\Gamma_s(t) = \{\boldsymbol{r}(x, z) \; ; \; x \in \Omega, \; z = h(t, x) > 0\}$, $\Gamma_b(t) = \{\boldsymbol{r}(x, z) \; ; \; x \in \Omega, \; z = 0, \; h(t, x) > 0\}$, $\Gamma_l(t) = \{\boldsymbol{r}(x, z) \; ; \; x \in \partial\Omega, \; h(t, x) > z > 0\}$. For the sake of simplicity we will suppose that $h(t, x) > 0$ for all $x \in \partial\Omega$ and the lateral boundary $\Gamma_l(t)$ could be split into two parts $\Gamma_l^0(t) = \{\boldsymbol{r}(x, z) \; ; \; x \in \Gamma_0, \; h(t, x) > z > 0\}$ and $\Gamma_l^1(t) = \{\boldsymbol{r}(x, z) \; ; \; x \in \Gamma_1, \; 0 < z < h(t, x)\}$ following a partition of $\partial\Omega = \Gamma_0 \cup \Gamma_1$. We will consider two kinds of boundary conditions: adherence on $\Gamma_l^0(t)$ and non stress on $\Gamma_l^1(t)$.

5.3 Assumptions and conservation equations

The shallow flow assumptions. In the shallow flow approximation, $\varepsilon \ll 1$ will be a small parameter representing the aspect ratio of the thickness. The shallow model presented in this section was derived in [56] under the following asymptotic assumptions: the normal components of the velocity as well as the tangential stresses are of order of ε, i.e., :

$$h = \mathcal{O}(\varepsilon), \quad w = \mathcal{O}(\varepsilon), \quad \boldsymbol{\sigma}_{3T} = \mathcal{O}(\varepsilon).$$

As a consequence of the above assumptions, the tangential component $\boldsymbol{v}$ has a small variation with respect to the thickness variable z, i.e., $\boldsymbol{v} = \boldsymbol{v}(t, x) + \mathcal{O}(\varepsilon^2)$. Other consequences of the above scalings are the expression of the normal stress acting on the bottom and its average on the thickness

$$\sigma_{33}|_{z=0} = -\rho h(\boldsymbol{k}\boldsymbol{v} \cdot \boldsymbol{v} - \frac{f_N}{\text{Fr}^2}) + \mathcal{O}(\varepsilon^2), \qquad \overline{\sigma_{33}} = -\rho h(\frac{1}{2}\boldsymbol{k}\boldsymbol{v} \cdot \boldsymbol{v} - \frac{1}{\text{Fr}^2}\frac{f_N}{2}) + \mathcal{O}(\varepsilon^2),$$

as a function of the tangential velocity, the bottom curvature and the normal body forces. We denote with $\overline{a}$ the average on the thickness of any function a:

$$\overline{a}(x) := \frac{1}{h(x)} \int_0^{h(x)} a(x, z) \, dz.$$

The shallow problem consists in finding the thickness $h : [0, T] \times \Omega \to \mathbb{R}_+$, the tangential velocity $\boldsymbol{v} : [0, T] \times \Omega \to \Pi_b$, and the tangential averaged stress $\boldsymbol{\tau} : [0, T] \times \Omega \to$

$\Pi_b \otimes \Pi_b$ defined bellow. The tangential averaged stress $\boldsymbol{\tau}$ is related to the average of the tangential part of the Cauchy stress through

$$\boldsymbol{\tau} = \overline{\boldsymbol{\sigma}_T} - \overline{\sigma_{33}}\boldsymbol{I}_2,$$

where $\boldsymbol{I}_2$ is the identity tensor in the tangent plane.

The thickness equation. The fact that the fluid region is advected by the flow (see (5.2.1)) is expressed by an evolution equation for the thickness function h:

$$\mathrm{St}\frac{\partial h}{\partial t} + \mathrm{div}_T(h\boldsymbol{v}) = 0. \tag{5.3.1}$$

The shallow momentum balance equation. The tangential momentum balance equation reads (see (4.1.1))

$$h\rho\Big(\mathrm{St}\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla_T)\boldsymbol{v}\Big) - \mathrm{div}_T(h\boldsymbol{\tau}) + h\rho\left[-\frac{1}{\mathrm{Fr}^2}f_N + \boldsymbol{k}\boldsymbol{v}\cdot\boldsymbol{v}\right]_+ C_f\boldsymbol{\Lambda}^f(\boldsymbol{v}) = \frac{h}{\mathrm{Fr}^2}\mathbf{F}, \tag{5.3.2}$$

where $\mathbf{F}$ are the "shallow external forces"

$$\mathbf{F} = \rho\boldsymbol{f}_T + \rho\frac{1}{2h}\nabla_T(h^2 f_N),$$

with $f_N, \boldsymbol{f}_T$ computed for $z = 0$.

5.4 The shallow constitutive law

The shallow constitutive (visco-plastic), see (4.2.1), which relates the averaged stress $\boldsymbol{\tau}$ to the rate of deformation $D_T(\boldsymbol{v}) = \frac{1}{2}(\boldsymbol{\nabla}_T\boldsymbol{v} + \boldsymbol{\nabla}_T^t\boldsymbol{v})$ acting in the tangential plane Π_b, is

$$\begin{cases} \boldsymbol{\tau} = \dfrac{2\eta^{shallow}}{\mathrm{Re}}[\mathrm{div}_T(\boldsymbol{v})\boldsymbol{I}_2 + D_T(\boldsymbol{v})] + \kappa\dfrac{D_T(\boldsymbol{v}) + \mathrm{div}_T(\boldsymbol{v})\boldsymbol{I}_2}{\sqrt{\frac{1}{2}\left[|D_T(\boldsymbol{v})|^2 + (\mathrm{div}_T\boldsymbol{v})^2\right]}} & \text{if } |D_T(\boldsymbol{v})| \neq 0, \\ \sqrt{\dfrac{1}{2}\left[|\boldsymbol{\tau}|^2 - 3r^2\right]} \leq \kappa & \text{if } |D_T(\boldsymbol{v})| = 0, \end{cases} \tag{5.4.1}$$

where $\eta^{shallow}$ is the shallow viscosity deduced form the expression of the 3-D viscosity η, which may depend on the strain rate and on the pressure.

The above equation can also be written in an inverse form. To state it, we have to introduce the shallow yield function $\mathcal{F}^{shallow} : \mathbb{R}_S^{2\times 2} \to \mathbb{R}$, defined through:

$$\mathcal{F}^{shallow}(\boldsymbol{\tau}) := \sqrt{\frac{1}{2}\left[|\boldsymbol{\tau}|^2 - 3r^2\right]} - \kappa(r), \quad \text{with } r = -\frac{1}{3}\,\mathrm{trace}(\boldsymbol{\tau}),$$

while the yield limit κ is the same as for the three dimensional yielding (flow/no flow) criterion.

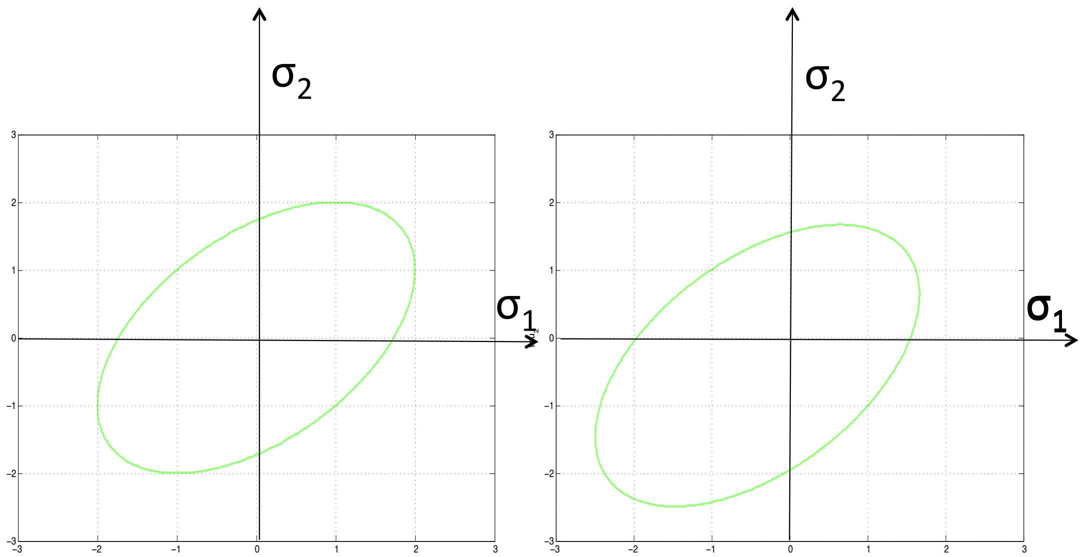

Figure 5.4: The shallow flow/no flow condition ($\mathcal{F}^{shallow}(\boldsymbol{\tau}) = 0$) represented in the plane of principal plane stresses $O(-\tau_1)(-\tau_2)$. Left: the shallow Von-Mises model ($\bar{\kappa}^0 = 1, \mu = 0$). Right: the shallow Drucker-Prager model ($\bar{\kappa}^0 = 1, \mu = 0.2$).

In Figure 5.4 it is plotted the shallow flow/no flow condition ($\mathcal{F}^{shallow}(\boldsymbol{\tau}) = 0$) in the plane of the averaged principal plane stresses. The shallow yield conditions correspond to the 3-D flow/no flow condition plotted in Figure 4.2. On the left hand side it is plotted the shallow Von-Mises model ($\bar{\kappa}^0 = 1, \mu = 0$) while the shallow Drucker-Prager model ($\bar{\kappa}^0 = 1, \mu = 0.2$) is depicted on the right hand side. Remark the fact that the Drucker-Prager model is not symmetric with respect to the origin. Indeed, this model have a different behavior for "traction" or "compression" (consolidation) processes.

The constitutive equation of the shallow rigid visco-plastic fluid (5.4.1) have now the following expression:

$$D_T(\boldsymbol{v}) = \frac{\mathrm{Re}}{2\eta^{shallow}} \left[\mathcal{F}^{shallow}(\boldsymbol{\tau})\right]_+ \frac{\boldsymbol{\tau} + r\boldsymbol{I}_2}{\sqrt{\frac{1}{2}\left[|\boldsymbol{\tau}|^2 - 3r^2\right]}}, \tag{5.4.2}$$

Notice that the 3-D model and the shallow model have the same structure. To see the link between these two models one has to consider the plain stress imbedding of the

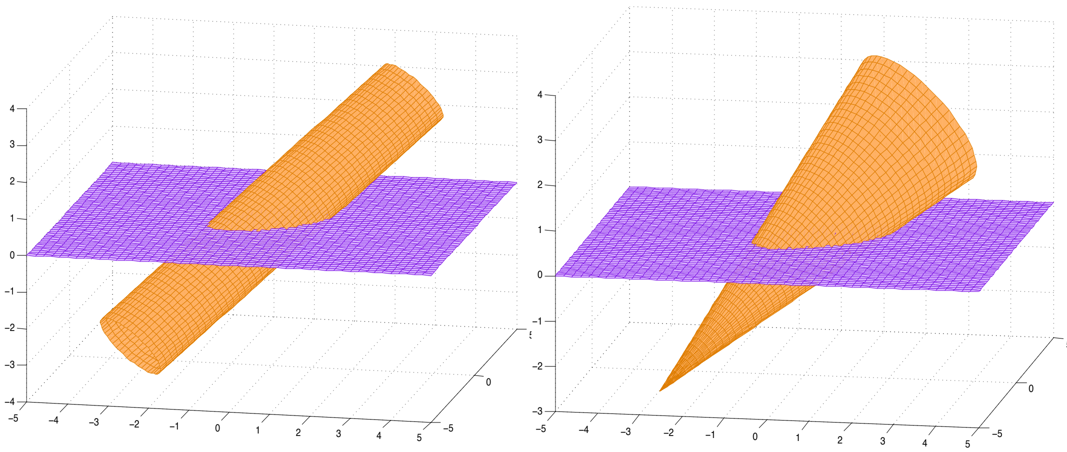

Figure 5.5: The 3-D flow/no flow condition ($||\boldsymbol{\sigma}'|| = \kappa(p)$) represented in the space of principal stresses $O(-\sigma_1)(-\sigma_2)(-\sigma_3)$ and the intersection with the plane $\sigma_3 = 0$. Left: Von-Mises model ($\kappa^0 = 1, \mu = 0$). Right: Drucker-Prager model ($\kappa^0 = 1, \mu = 0.2$).

2-D stress tensors space into a 3-D stress tensors space. More precisely, $\boldsymbol{\tau}$ and $D_T(\boldsymbol{v})$ satisfy the shallow flow constitutive law (5.4.1) (or equivalently (5.4.2)) if and only if $\boldsymbol{\sigma}$ and $\boldsymbol{D}(\boldsymbol{u})$ satisfy (4.2.1) (or equivalently (4.2.2)). That means that the shallow constitutive equation is the plane stress projection of the initial (3-D) visco-plastic model (see Figure 5.5).

5.5 Shallow rigid velocities

We study here the kernel of the tangential rate of deformation operator D_T and we introduce the set of tangential rigid velocities $\mathcal{R}_T$, defined as:

$$\mathcal{R}_T =: \{\boldsymbol{r}_T = r_{Ti}\boldsymbol{b}^i \ ; \ r_{Ti}(x) = \boldsymbol{a}\cdot\boldsymbol{b}_i(x) + (\boldsymbol{r}_{\boldsymbol{b}}(x)\wedge\boldsymbol{b}_i(x))\cdot\boldsymbol{w}, \ \boldsymbol{a},\boldsymbol{w}\in\mathbb{R}^3\}. \tag{5.5.1}$$

Let us prove that the tangential rigid motions defined above by (5.5.1) is the kernel of the tangential strain rate operator D_T. This space is the projection on the tangential plane Π_b of the 3-D rigid motions $\mathcal{R} =: \{\boldsymbol{r} = \boldsymbol{a} + \mathbf{x}\wedge\boldsymbol{w}, \ ; \ \boldsymbol{a},\boldsymbol{w}\in\mathbb{R}^3\}$.

For that, let $\boldsymbol{r}_T\in\mathcal{R}_T$ and from (5.1.1) and (5.1.2) we have $D_T(\boldsymbol{r}_T) = [D_T(\boldsymbol{r}_T)]_{ij}\boldsymbol{b}^i\otimes\boldsymbol{b}^j$.

$$\begin{aligned}[D_T(\boldsymbol{r}_T)]_{ij} &= \frac{1}{2}\left(\frac{\partial r_{Ti}}{\partial x_j} + \frac{\partial r_{Tj}}{\partial x_i}\right) - \Gamma^l_{ij}r_{Tl} \\ &= (\boldsymbol{a} + \boldsymbol{w}\wedge\boldsymbol{r}_b)\cdot\left[\frac{1}{2}\left(\frac{\partial\boldsymbol{b}_i}{\partial x_j} + \frac{\partial\boldsymbol{b}_j}{\partial x_i}\right) - \Gamma^l_{ij}\boldsymbol{b}_l\right] + \frac{1}{2}[(\boldsymbol{w}\wedge\boldsymbol{b}_i)\cdot\boldsymbol{b}_j + (\boldsymbol{w}\wedge\boldsymbol{b}_j)\cdot\boldsymbol{b}_i].\end{aligned}$$

But $(\boldsymbol{w} \wedge \boldsymbol{b}_i) \cdot \boldsymbol{b}_j = -(\boldsymbol{w} \wedge \boldsymbol{b}_j) \cdot \boldsymbol{b}_i$ if $i \neq j$ and $\left(\frac{\partial \boldsymbol{b}_i}{\partial x_j} + \frac{\partial \boldsymbol{b}_j}{\partial x_i} \right) - \Gamma_{ij}^l \boldsymbol{b}_l = 0$, hence

$$D_T(\boldsymbol{r}_T) = 0.$$

5.6 Statement of the shallow flow problem

We can formulate now the shallow flow problem of a rigid visco-plastic fluid: find the thickness $h : [0,T] \times \Omega \to \mathbb{R}_+$, the horizontal velocity $\boldsymbol{v} : [0,T] \times \Omega \to \Pi_b$, and the averaged stress $\boldsymbol{\tau} : [0,T] \times \Omega \to \Pi_b \otimes \Pi_b$ which satisfies the equations (5.3.1), (5.3.2), (5.4.2), the boundary conditions

$$\boldsymbol{v} = 0 \quad \text{on } \Gamma_0, \quad \text{and} \quad \boldsymbol{\tau n} = 0 \quad \text{on } \Gamma_1,$$

and the initial conditions:

$$h(0) = h_0, \quad \boldsymbol{v}(0) = \boldsymbol{v}_0.$$

Comparison with Savage and Hutter model. We summarize here the principal features of the visco-plastic shallow flow model by comparison with Savage and Hutter model [88, 89]. The constitutive model of Savage and Hutter deals with an incompressible, inviscid fluid/solid with a cohesion-less Mohr-Coulomb plasticity (flow/no-flow) condition, which implies a linear dependence of the yield limit with respect to the pressure. The model used here deals with an incompressible, viscous fluid/solid with a rather general plasticity condition for which the yield limit could have a general dependence on the pressure. It can include Drucker-Prager and Von-Mises /Bingham plasticity models excluded by Savage and Hutter model. Morever, with an appropriate choice of the viscosity, the model described in this section recovers the visco-plastic model proposed by Jop, Forterre and Pouliquen [44]. Concerning the bottom boundary, both models use Coulomb frictional conditions. The bottom surface is described by its elevation in the Savage and Hutter model, while a general parametric description is given here. The thickness evolution equation (5.3.1) has the same form for both models while the shallow momentum equation is different. In the Savage and Hutter model the frictional terms and the plastic terms of the shallow model are introduced as external forces through "net driving acceleration" terms and "pressure coefficients" terms, respectively. In contrast, in our model the resulting shallow equations have the same structure as the three dimensional ones: the 2-D (tangent) momentum balance law (5.3.2) is completed with a "shallow constitutive equation" (5.4.2) which links the projection of the averaged stresses on the tangent plane to the rate of deformations (expressed through the tangent differential operators). Even if the shallow flow/no flow (yield) condition and viscosity are not the same as in the three dimensional case, the shallow constitutive law (5.4.1), which has the same structure, can be derived from the 3-D model (4.2.1).

Shallow flow variational formulation. In what follows we need to characterize the rest configuration, i.e., to see when $h(t) \equiv h_0, \boldsymbol{v}(t) \equiv 0$ is a solution of the shallow problem defined above. Since (5.3.1) is trivially verified we need to see when (5.3.2) and (5.4.2) are satisfied for $\boldsymbol{v}(t) \equiv 0$. This cannot be done directly (the constitutive equation (5.4.2) and the friction law are not invertible for $\boldsymbol{v}(t) \equiv 0$). That is why, we need a variational formulation in terms of velocities which include (5.3.2) and (5.4.2).

From the tangential momentum balance equation (5.3.2) and from the shallow rigid visco-plastic law (5.4.2) we have deduced the following variational inequality in terms of velocities $\boldsymbol{v}(t) \in V =: \{\boldsymbol{\Psi} \in H^1(\Omega)^2 \ ; \ \boldsymbol{\Psi}(x) \in \Pi_b(x) \ ; \ \boldsymbol{\Psi} = 0 \text{ on } \Gamma_0 \ \}$:

$$
\begin{aligned}
&\frac{1}{\mathrm{St}} \int_\Omega h\rho \frac{\partial \boldsymbol{v}}{\partial t} \cdot (\boldsymbol{\Psi} - \boldsymbol{v}) g \, dx + \int_\Omega h\rho (\boldsymbol{v} \cdot \nabla_T) \boldsymbol{v} \cdot (\boldsymbol{\Psi} - \boldsymbol{v}) g \, dx + \\
&\frac{1}{\mathrm{Re}} \int_\Omega 2\eta^{shallow} h \big[D_T(\boldsymbol{v}) : D_T(\boldsymbol{\Psi} - \boldsymbol{v}) + \mathrm{div}_T \boldsymbol{v} \, \mathrm{div}_T(\boldsymbol{\Psi} - \boldsymbol{v}) \big] g \, dx + \\
&\int_\Omega h\rho \left[-\frac{1}{\mathrm{Fr}^2} f_N + \boldsymbol{k}\boldsymbol{v} \cdot \boldsymbol{v} \right]_+ C_f (|\boldsymbol{\Psi}| - |\boldsymbol{v}|) g \, dx \\
&+ \int_\Omega h\kappa \Big[\sqrt{\frac{1}{2} [|D_T(\boldsymbol{\Psi})|^2 + (\mathrm{div}_T \boldsymbol{\Psi})^2]} - \sqrt{\frac{1}{2} [|D_T(\boldsymbol{v})|^2 + (\mathrm{div}_T \boldsymbol{v})^2]} \Big] g \, dx \\
&\geq \frac{1}{\mathrm{Fr}^2} \int_\Omega h\rho \big[\boldsymbol{f}_T \cdot (\boldsymbol{\Psi} - \boldsymbol{v}) - h \frac{f_N}{2} \mathrm{div}_T (\boldsymbol{\Psi} - \boldsymbol{v}) \big] g \, dx,
\end{aligned} \tag{5.6.1}
$$

for all $\boldsymbol{\Psi} \in V$. To obtain the above variational formulation we have used appropriate Stokes formula for integrals over curved surfaces.

Chapter 6

The limit load problem and the flow onset

The onset of the flow (or collapse flow field) is best studied through an idealized mechanical model (perfectly rigid-plastic material) subjected to a slowly increasing load, called the "limit analysis problem" (see [33] for a complete description). The main problem is to find the maximum multiple of the force distribution, that the solid/fluid can be withstand without flowing (collapsing), and the associated (collapse, onset) flow field. The final result of such an analysis is a non-dimensional number called "safety factor" (or "limit load").

6.1 Safety factor analysis problem

When modelling landslides, or snow avalanches, the fluid/solid is totally at rest (blocked) in its natural configuration and the beginning of a flow can be seen as a "disaster". In order to get the characterization of the fact the fluid is totally at rest in its initial configuration we have to check whenever $h(t) \equiv h_0, \boldsymbol{v}(t) \equiv 0$ is a solution of (5.6.1) and (5.3.1). If we look for $\boldsymbol{v} = 0$ in (5.6.1) then we get the following variational inequality

$$\begin{aligned}(6.1.1)\quad & \int_\Omega h_0\kappa_0\sqrt{\frac{1}{2}\left[|D_T(\boldsymbol{\Psi})|^2 + (\mathrm{div}_T\boldsymbol{\Psi})^2\right]}g\,dx + \frac{1}{\mathrm{Fr}^2}\int_\Omega \rho h_0[-f_N]_+ C_f|\boldsymbol{\Psi}|g\,dx \\ & \geq \frac{1}{\mathrm{Fr}^2}\int_\Omega h_0\rho\big[\boldsymbol{f}_T\cdot\boldsymbol{\Psi} - \frac{f_N}{2}h_0\mathrm{div}_T\boldsymbol{\Psi}\big]g\,dx,\end{aligned}$$

for all $\boldsymbol{\Psi} \in V$, where κ_0 is the distribution of the shallow yield limit in the rest configuration.

Note that the onset variational inequality (6.1.1) has no time dependency, while the Saint-Venant visco-plastic model described through the variational problem (5.6.1) is time dependent. Moreover, the problem (6.1.1) is rate independent which means that the viscous effects, related to the rate dependency, will disappear in the optimization problem modeling the avalanche onset. This fact has important mathematical consequences for the regularity of the solution. As we can see in the next, the solutions of the rigid-plastic shallow flow problem (6.1.1) have spatial discontinuities.

The above variational inequality can be written in terms of safety factor (or limit load). To do that, we denote by $\phi(x,\cdot):\Pi_b(x)\otimes\Pi_b(x)\to\mathbb{R}_+$

$$\phi(x,D_T) =: \sqrt{\frac{1}{2}\,[|D_T|^2+(\mathrm{trace}\,D_T)^2]},$$

the shallow plastic strain rate potential and by

$$G(\boldsymbol{\Psi}) =: \int_\Omega a\phi(D_T(\boldsymbol{\Psi}))\; gdx + \int_\Omega q|\boldsymbol{\Psi}|\; gdx, \quad L(\boldsymbol{\Psi}) =: \int_\Omega (\mathbf{c}\cdot\boldsymbol{\Psi}\; + b\mathrm{div}_T(\boldsymbol{\Psi}))\; gdx.$$

the total dissipation power (plastic and frictional dissipation) and the external forces dissipation power, where

$$a =: h_0\kappa_0, \quad q =: \frac{\rho}{\mathrm{Fr}^2}[-f_N]_+h_0C_f, \quad \mathbf{c} =: \frac{\rho}{\mathrm{Fr}^2}h_0\boldsymbol{f}_T, \quad b =: -\frac{\rho}{2\mathrm{Fr}^2}h_0^2f_N.$$

We can define the *safety factor (or limit load)* λ^* as

$$\lambda^* = \inf_{\boldsymbol{\Psi}\in V, L(\boldsymbol{\Psi})=1} G(\boldsymbol{\Psi}). \tag{6.1.2}$$

and we remark that (6.1.1) is verified (i.e., the solid/fluid is at rest) if and only if the safety factor $\lambda^*\geq 1$. We can formulate now the following flow/no flow criterion:

The avalanche shallow flow of the visco-plastic fluid/solid starts if and only if $\lambda^*<1$.

6.2 Functional framework

The plastic dissipation functional G involved in (6.1.2) is non-smooth, and non coercive in the classical Sobolev spaces. Moreover, its expression is valid only for smooth velocity fields from the Sobolev space V. On the other hand, fractures are modeled by velocity fields with discontinuities and the gradient operator involved in the definition of the rate of deformation tensor has to be understood in the sense of distributions. The space of bounded deformation (i.e., the space of velocities which have their rate of deformation in the space of bounded measures) was introduced in [94, 96] to handle non-smooth velocity fields in plasticity.

Let define now $BD_T(\Omega)$, the space of bounded tangential deformations functions $\mathbf{\Psi}$, for which the tangential strain rate $D_T(\mathbf{\Psi})$ belongs to the space of bounded measures $M^1(\Omega)$:

$$BD_T(\Omega) =: \{\mathbf{\Psi} : \Omega \to \mathbb{R}^2 \ ; \ \mathbf{\Psi} \in L^1(\Omega)^2, D_T(\mathbf{\Psi}) \in M^1(\Omega)^{2\times 2}\}.$$

We assume the following regularity conditions:

$$\boldsymbol{r}_b \in [C^2(\bar{\Omega})]^3, \quad a, b \in C^0(\bar{\Omega}),\ a(x) \geq a_0 > 0, \quad \boldsymbol{c} \in [L^\infty(\Omega)]^2, \quad q \in L^\infty(\Omega),\ q \geq 0.$$

Since the fundamental magnitudes $g_{ij} \in C^1(\bar{\Omega})$ and the Christoffel symbols $\Gamma^k_{ij} \in C^0(\bar{\Omega})$, we get that $D(\mathbf{\Psi}) \in M^1(\Omega)^{2\times 2}$ if and only if $D_T(\mathbf{\Psi}) \in M^1(\Omega)^{2\times 2}$. Hence we can identify $BD(\Omega)$ (introduced and discussed in [76, 94, 95, 96]) with $BD_T(\Omega)$, defined above.

Since $\mathbf{\Psi} \to \phi(\mathbf{\Psi})$ is an equivalent norm on $\mathbb{R}^{2\times 2}$ and satisfies the conditions of theorem 4.1, chapter 2 from [95], $\phi(\mathbf{\Psi})$ is a bounded positive measure on Ω. We can use this to extend the functionals G and L for all $\mathbf{\Psi} \in BD(\Omega)$ through the formula

$$G(\mathbf{\Psi}) = \int_\Omega a\ g d\phi(\mathbf{\Psi}) + \int_\Omega q|\mathbf{\Psi}|\ gdx, \quad L(\mathbf{\Psi}) = \int_\Omega \mathbf{c} \cdot \mathbf{\Psi}\ gdx + \int_\Omega b\ gd\mathrm{div}_T(\mathbf{\Psi}).$$

In order to handle the velocity boundary conditions on Γ_0 for non-smooth velocity fields, we have to add some additional boundary integrals in functionals G and L. These integrals are modeling a discontinuity surface of a non-smooth velocity field located at the boundary Γ_0. To do that we introduce

$$G_0(\mathbf{\Psi}) =: \int_{\Gamma_0} \frac{1}{2} a \sqrt{[|\mathbf{\Psi}|^2 \cdot |\boldsymbol{n}_T|^2 + 3(\mathbf{\Psi} \cdot \boldsymbol{n}_T)^2]}\ gdS, \quad \tilde{G}(\mathbf{\Psi}) =: G(\mathbf{\Psi}) + G_0(\mathbf{\Psi}),$$

$$L_0(\mathbf{\Psi}) =: -\int_{\Gamma_0} b\mathbf{\Psi} \cdot \boldsymbol{n}_T\ gdS, \quad \tilde{L}(\mathbf{\Psi}) =: L(\mathbf{\Psi}) + L_0(\mathbf{\Psi}),$$

recall that $\boldsymbol{n}_T$ is the normal on the boundary Γ_0 laying in the tangent plan Π_b.

If we reformulate the optimization problem (6.1.2) as the relaxed safety factor (or limit load) problem through

$$\tilde{\lambda}^* = \inf_{\mathbf{\Psi} \in BD_T(\Omega), \tilde{L}(\mathbf{\Psi})=1} \tilde{G}(\mathbf{\Psi}), \tag{6.2.1}$$

and since $G_0(\mathbf{\Psi}) = 0$ and $L_0(\mathbf{\Psi}) = 0$ for all $\mathbf{\Psi} \in V$, we get that

$$\tilde{\lambda}^* = \inf_{\mathbf{\Psi} \in BD_T(\Omega), \tilde{L}(\mathbf{\Psi})=1} \tilde{G}(\mathbf{\Psi}) \leq \lambda^* = \inf_{\mathbf{\Psi} \in V, L(\mathbf{\Psi})=1} G(\mathbf{\Psi}).$$

The following theorem states that the above relaxation of the boundary conditions does not change the initial optimization problem.

Theorem 6.2.1. *The safety factor for the initial optimization problem is equal with the safety factor for the relaxed optimization problem, i.e.,*

$$\tilde{\lambda}^* = \inf_{\mathbf{\Psi}\in BD(\Omega), \tilde{L}(\mathbf{\Psi})=1} \tilde{G}(\mathbf{\Psi}) = \lambda^* = \inf_{\mathbf{\Psi}\in V, L(\mathbf{\Psi})=1} G(\mathbf{\Psi}).$$

Proof. We shall prove that for all $\mathbf{\Psi} \in BD(\Omega)$ there exists a sequence $\mathbf{\Psi}_n \in V$ such that $L(\mathbf{\Psi}_n) \to \tilde{L}(\mathbf{\Psi})$ and $G(\mathbf{\Psi}_n) \to \tilde{G}(\mathbf{\Psi})$. It is sufficient to give the proof for $\mathbf{\Psi} \in C^\infty(\bar{\Omega})$ only. Indeed, if $\mathbf{\Psi} \in BD(\Omega)$ then there exist $\mathbf{\Phi}_n \in C^\infty(\bar{\Omega})$ such that $\mathbf{\Phi}_n \to \mathbf{\Psi}$ strongly in $L^1(\Omega)^2$, $D_T(\mathbf{\Phi}_n) \to D_T(\mathbf{\Psi})$ narrowly on Ω (i.e., $D_T(\mathbf{\Phi}_n) \to D_T(\mathbf{\Psi})$ weak* in $M^1(\Omega)^{2\times 2}$, $||D_T(\mathbf{\Phi}_n)||_{M^1(\Omega)} \to ||D_T(\mathbf{\Psi})||_{M^1(\Omega)}$) and $G(\mathbf{\Phi}_n) \to G(\mathbf{\Psi})$ (see Theorem 5.2, chapter 2 from [95]). Using the continuity of the trace map with respect to the above intermediate topology we obtain $G_0(\mathbf{\Phi}_n) \to G_0(\mathbf{\Psi})$ and $L_0(\mathbf{\Phi}_n) \to L_0(\mathbf{\Psi})$.

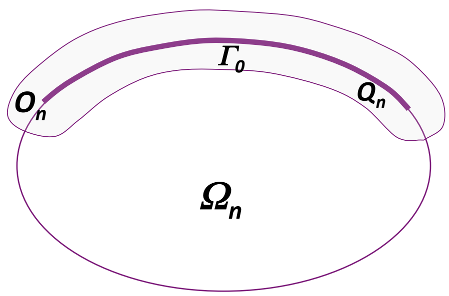

Figure 6.1: The decomposition of the domain Ω.

Let $\mathbf{\Psi} \in C^\infty(\bar{\Omega})$. For all $n \in \mathbb{N}$ we define $O_n = \{x \in \mathbb{R}^N;\ \mathrm{dist}(x, \Gamma_0) < 1/n\}$, $\Omega_n = \Omega \setminus \bar{O}_n$ and $Q_n = \Omega \cap O_n$ (see Figure 6.1). Let $u_n \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le u_n \le 1$, $u_n = 0$ on Γ_0 and $u_n = 1$ on Ω_n. If we take $\mathbf{\Psi}_n = \mathbf{\Psi} u_n$ then $\mathbf{\Psi}_n \in C^\infty(\bar{\Omega}) \cap V$ and for all $v \in C^\infty(\bar{\Omega})$ we have $\displaystyle\int_\Omega v \frac{\partial}{\partial x_i} \mathbf{\Psi}_n \; gdx = \int_{Q_n} \frac{\partial}{\partial x_i}(v \mathbf{\Psi} u_n)\; gdx - \int_{Q_n} u_n \mathbf{\Psi} \frac{\partial}{\partial x_i}(v)\; gdx + \int_{\Omega_n} v \frac{\partial}{\partial x_i}(u_n \mathbf{\Psi})\; gdx$. Using (5.1.3) in the first integral, denoted by I_1, we get $I_1 = \int_{\partial Q_n} v u_n \mathbf{\Psi} \cdot \boldsymbol{n}_T \; gdS$, and having in mind that $u_n = 0$ on Γ_0 we deduce $I_1 \to -\int_{\Gamma_0} v \mathbf{\Psi} \cdot \boldsymbol{n}_T \; gdS$. Since the measure of Q_n is vanishing the second integral converges to 0 and from $u_n = 1$ on Ω_n we get that the third integral converges to $\displaystyle\int_\Omega v \frac{\partial}{\partial x_i} \mathbf{\Psi}\; gdx$. We have just obtained that

$$\int_\Omega v \frac{\partial}{\partial x_i} \mathbf{\Psi}_n \; gdx \to \int_\Omega v \frac{\partial}{\partial x_i} \mathbf{\Psi}\; gdx - \int_{\Gamma_0} v \mathbf{\Psi} \cdot \boldsymbol{n}_T \; gdS. \tag{6.2.2}$$

Denoting by

$$E_T(\mathbf{\Psi}) =: \frac{1}{2}(\mathbf{\Psi} \otimes \boldsymbol{n}_T + \boldsymbol{n}_T \otimes \mathbf{\Psi}) \tag{6.2.3}$$

and using again (5.1.3) we get $\displaystyle\int_\Omega vD_T(\boldsymbol{\Psi}_n)\ gdx \to \int_\Omega vD_T(\boldsymbol{\Psi})\ gdx - \int_{\Gamma_0} vE_T(\boldsymbol{\Psi})\ gdS.$ From the convergence (6.2.2) we get $\int_\Omega b\ \mathrm{div}_T(\boldsymbol{\Psi}_n)\ gdx \to$ $\int_\Omega b\ \mathrm{div}_T(\boldsymbol{\Psi})\ gdx - \int_{\Gamma_0} b\boldsymbol{\Psi}\cdot\boldsymbol{n}_T\ gdS$, which means that $L(\boldsymbol{\Psi}_n) \to \tilde{L}(\boldsymbol{\Psi})$.

Since $D_T(\boldsymbol{\Psi}_n) = u_n D_T(\boldsymbol{\Psi}) + \dfrac{1}{2}(\nabla_T u_n \otimes \boldsymbol{\Psi} + \boldsymbol{\Psi} \otimes \nabla_T u_n)$, from (5.1.4) we get $\int_{\Omega_n} g\phi(D_T(\boldsymbol{\Psi}_n)) \to \int_\Omega g\phi(D_T(\boldsymbol{\Psi}))$ and $\int_{O_n} g\phi(D_T(\boldsymbol{\Psi}_n)) \to \int_{\Gamma_0} g\phi(-E_T(\boldsymbol{\Psi}))$. Bearing in mind that $\mathrm{trace}(E_T(\boldsymbol{\Psi})) = \boldsymbol{\Psi}\cdot\boldsymbol{n}_T$ and $|E_T(\boldsymbol{\Psi})|^2 = \frac{1}{2}(|\boldsymbol{\Psi}|^2 \cdot |\boldsymbol{n}_T|^2 + (\boldsymbol{\Psi}\cdot\boldsymbol{n}_T)^2)$, we get $G_0(\boldsymbol{\Psi}) = \int_{\Gamma_0} \phi(-E_T(\boldsymbol{\Psi}))\ gdS$, hence $G(\boldsymbol{\Psi}_n) = \int_{\Omega_n} g\phi(\boldsymbol{\Psi}_n) + \int_{O_n} g\phi(\boldsymbol{\Psi}_n) \to$ $\int_\Omega g\phi(\boldsymbol{\Psi}) + \int_{\Gamma_0} g\phi(-E_T(\boldsymbol{\Psi}))g = \tilde{G}(\boldsymbol{\Psi})$.

□

6.3 Existence of an onset velocity field

We prove here the existence solution in the space $BD_T(\Omega)$ of the relaxed optimization problem (6.2.1).

Theorem 6.3.1. *We assume that $meas(\Gamma_0) > 0$ or $q > 0$. Then there exist an onset velocity field $\boldsymbol{v}^* \in BD_T(\Omega)$, with $\tilde{L}(\boldsymbol{v}^*) = 1$, solution of the relaxed optimization problem*

$$\tilde{G}(\boldsymbol{v}^*) = \tilde{\lambda}^* = \min_{\boldsymbol{\Psi}\in BD_T(\Omega), \tilde{L}(\boldsymbol{\Psi})=1} \tilde{G}(\boldsymbol{\Psi}).$$

Proof. Let $\boldsymbol{v}_n \in BD(\Omega)$, with $\tilde{L}(\boldsymbol{v}_n) = 1$ be a minimizing sequence. Because $\tilde{G}(\boldsymbol{v}_n) \to \tilde{\lambda}^*$, the sequence $\tilde{G}(\boldsymbol{v}_n)$ is bounded. Since $meas(\Gamma_0) > 0$ we deduce that the continuous semi-norm $\boldsymbol{\Psi} \to G_0(\boldsymbol{\Psi}) + \int_\Omega q|\boldsymbol{\Psi}|\ gdx$ is a semi-norm on the space of tangential rigid motions $\mathcal{R}_\mathcal{T}$.

Now, we can use Proposition 2.4 of [95] to obtain that $\boldsymbol{\Psi} \to \tilde{G}(\boldsymbol{\Psi})$ is an equivalent norm on $BD(\Omega)$. This means that $(\boldsymbol{v}_n)$ is bounded in $BD(\Omega)$, hence it contains a subsequence, denoted again by $(\boldsymbol{v}_n)$, and there exists $\boldsymbol{v}^* \in BD(\Omega)$ such that $\boldsymbol{v}_n \to v^*$ weakly in $BD(\Omega)$. Since $\tilde{L}$ is a linear continuous functional we get $\tilde{L}(\boldsymbol{v}_n) \to \tilde{L}(v^*)$, which means that $\tilde{L}(v^*) = 1$. Using the weak lower semicontinuity of $\tilde{G}$ to get that $\tilde{G}(v^*) \leq \liminf \tilde{G}(\boldsymbol{v}_n) = \tilde{\lambda}^*$. □

Chapter 7

Numerical approach and simulations

The classical numerical methods used in solving the limit load extremal problems, as (6.1.2), are based on the finite element discretization and the convex and linear programming (first results were obtained in [49, 52], but see also [3, 33, 32, 65, 78]).

Even if the finite element method have been intensively developed in the last years (X-FEM, re-meshing techniques), it is associated to continuous fields and it is not adapted for capturing discontinuities on unknown surfaces. But, almost all nontrivial known solutions of the limit load problems have spatial discontinuities. This is not so surprising if we have in mind that the extremal problem (6.1.2) models phenomena as ductile fracture or strain localization. For that, to solve the limit load extremal problem (6.2.1), involving $\tilde{G}$, we make use here of a mesh free method which does not use a finite element discretization of the solid. This new limit analysis method is called *discontinuous velocity domain splitting* (DVDS). Even if a detailed description of DVDS can be found in [60], we shall briefly recall it in the next section.

7.1 Discontinuous velocity domain splitting method

The mathematical foundation of the DVDS method is given in [57] (see also [48]), while a detailed description of this limit analysis method, can be found in [60]. For in-plane or 3-D problems the DVDS method considers a special class of velocities fields constructed as follows. First, the body is split into two sub-domains: on one sub-domain the velocity is vanishing and on the other one the velocity corresponds to a rigid motion. Thus, the discontinuous velocity field is determined only by the shape of one sub-domain (more precisely by its boundary) and by a rigid motion. In this way, the deformation is concentrated (localized) at the boundary of a sub-domain and the plastic dissipation power depends only on the shape of the sub-domain and on the rigid velocity. The problem is thus reduced to the minimization of a shape dependent functional.

Let describe here the DVDS test function set. We consider as test functions the velocity fields of the form $\boldsymbol{r}_T 1_\omega$, where $\boldsymbol{r}_T$ belongs to the set of tangential rigid motions (5.5.1) and 1_ω is the characteristic function of a subdomain $\omega \subset \Omega$ (i.e., $1_\omega(x) = 1$ if $x \in \omega$ and $1_\omega(x) = 0$ if $x \notin \omega$) (see Figure 7.1). To be more precisely let us define $\mathcal{V}$ the set of DVDS velocity fields

$$\mathcal{V} := \{\boldsymbol{r}_T 1_\omega \; ; \; \omega \subset \Omega, \; \boldsymbol{r}_T \in \mathcal{R}_T\}.$$

Since the space of bounded deformation $BD_T(\Omega)$ include functions with spatial discontinuities we have $\boldsymbol{r}_T 1_\omega \in BD_T(\Omega)$ and $\mathcal{V} \subset BD_T(\Omega)$. We can take now test functions from $\mathcal{V}$ in $\tilde{G}$ to get an upper-bound λ_1^* of λ^*:

$$\lambda_1^* = \inf_{\boldsymbol{r}_T 1_\omega \in \mathcal{V}, \tilde{L}(\boldsymbol{r}_T 1_\omega)=1} \tilde{G}(\boldsymbol{\Psi}) \geq \lambda^* = \inf_{\boldsymbol{\Psi} \in BD_T(\Omega), \tilde{L}(\boldsymbol{\Psi})=1} \tilde{G}(\boldsymbol{\Psi}).$$

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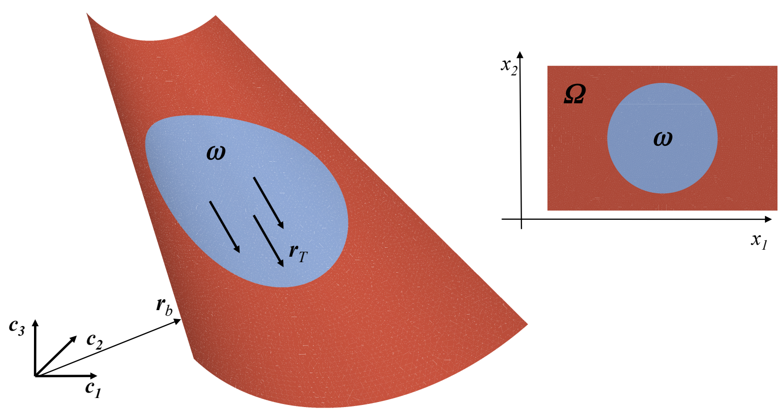

Figure 7.1: The DVDS special class of velocities.

Having in mind that $D_T(\boldsymbol{r}_T) = 0$, for all DVDS velocity field the plastic dissipation involved in (6.1.2) can be computed explicitly as a function of ω and $\boldsymbol{r}_T$ from ω and $\boldsymbol{r}_T$. Using the formula of $\tilde{G}(\boldsymbol{r}_T 1_\omega), \tilde{L}(\boldsymbol{r}_T 1_\omega)$, which could be obtained in a similar way as in the proof of Theorem 6.2.1, we introduce the following shape dependent functional $J_T(\omega, \boldsymbol{r}_T)$:

$$J_T(\omega, \boldsymbol{r}_T) := \frac{\tilde{G}(\boldsymbol{r}_T 1_\omega)}{\left[\tilde{L}(\boldsymbol{r}_T 1_\omega)\right]_+} = \frac{\int_{\partial\omega\setminus\Gamma_1} \frac{1}{2} a \sqrt{[|\boldsymbol{r}_T|^2 |\boldsymbol{n}_T|^2 + 3(\boldsymbol{r}_T \cdot \boldsymbol{n}_T)^2]} \, gdS + \int_\omega q |\boldsymbol{r}_T| \, gdx}{\left[\int_\omega \mathbf{c} \cdot \boldsymbol{r}_T \, gdx - \int_{\partial\omega\setminus\Gamma_1} b \boldsymbol{r}_T \cdot \boldsymbol{n}_T \, gdS\right]_+},$$

where $[s]_+ = (s + |s|)/2$ is the positive part and $\boldsymbol{n}_T$ is the normal to $\partial\omega$ in the plane Π_b. Then we have

$$\lambda_1^* =: \inf_{\omega\subset\Omega, \boldsymbol{r}_T\in\mathcal{R}_T} J_T(\omega, \boldsymbol{r}_T) \geq \lambda^*.$$

The optimal value λ_1^* is an upper-bound for the safety factor (limit load) λ^*. For the anti-plane flow DVDS gives an exact evaluation of the safety factor, i.e., $\lambda_1^* = \lambda^*$ (see [57] for a rigorous proof). Moreover, as it was founded in [60], in all in-plane flows problems for which we have an estimation of the safety factor λ^*, the DVDS upper-bound λ_1^* is very close (less than 2-5 %) to the global minimum λ^*.

From the optimal set ω^*, which is the solution of the above shape optimization problem, and the optimal rigid flow $\boldsymbol{r}_T^*$, i.e.,

$$J_T(\omega^*, \boldsymbol{r}_T^*) =: \min_{\omega\subset\Omega, \boldsymbol{r}_T\in\mathcal{R}_T} J_T(\omega, \boldsymbol{r}_T),$$

one can construct the DVDS onset velocity filed $\boldsymbol{v}^* =: \boldsymbol{r}_T^* 1_{\omega^*}$. The boundary of ω^*, delimiting the flow zone from the non-flow zone, represents the collapse fracture surface. The existence of an the optimal set ω^* and of an optimal rigid flow $\boldsymbol{r}_T^*$ could be proved using similar arguments as developed in [57] for the anti-plane problem, while the uniqueness results cannot be expected in general (see again [57] for details). The study of the existence and of the uniqueness of the optimal set ω^* and the optimal rigid flow $\boldsymbol{r}_T^*$ is beyond of the scope of the present work.

7.2 Level set approach

We use a level set approach to represent the sub-domain ω involved in the minimization of the functional J_T. This is an useful tool in representation of the optimal set when the topology is not known. Since the global minimization technique makes use of genetic algorithm the description of the level set function have to be done through a few number of parameters. Thus, the main features of our approach consists in the following principal ingredients: the description of the sub-domain ω as a level set of a function described by a small number of parameters, the description of the vector field $\boldsymbol{r}_T$, the reconstruction of the topology of ω and the computation of the cost function J_T.

The parameterization of the level set function, which describe the sub-domain ω is based on Fourier's series. To illustrate this approach we consider $\Omega = (0, 1) \times (0, 1)$ and let $a_0, (a_{i,j}) \subset [-1, 1]$ $(0 \leq i, j \leq m)$ be a family of $(m + 1)^2$ parameters. At each choice of parameters $(a_{i,j})$ we consider the function $\phi(a_{i,j}) : [0, 1] \times [0, 1] \to \mathbb{R}$ as

$$\phi(x_1, x_2) := \sum_{i,j=1}^{m} \frac{1}{ij} a_{i,j} \sin(\pi i x_1/2) \sin(\pi j x_2/2) + \sum_{i=1}^{m} \frac{1}{i} a_{i,0} \sin(\pi i x_1/2) + \sum_{j=1}^{m} \frac{1}{j} a_{0,j} \sin(\pi j x_2/2),$$

and after the normalization of the coefficients $(a_{i,j})$ we define the function $\Phi(a_0, a_{ij})$: $[0,1] \times [0,1] \to \mathbb{R}$

$$\Phi(x_1, x_2) =: 2\frac{\phi(x_1, x_2) - m_\phi}{M_\phi - m_\phi} + a_0 - 1,$$

where $m_\phi = \min \phi(x_1, x_2), M_\phi = \max \phi(x_1, x_2)$. The function $\Phi(a_0, a_{ij})$ defines the subset ω through its level set

$$\omega = \omega(a_0, a_{ij}) = \{(x_1, x_2) \in [0,1] \times [0,1] : \ \Phi(x_1, x_2) > 0\}.$$

The boundary of ω, defined as

$$\partial\omega = \{(x_1, x_2) \in [0,1] \times [0,1] : \ \Phi(x_1, x_2) = 0\} \cup (\partial\Omega \cap \bar{\omega}).$$

is a union of no overlapping simple closed curves.

Because the shape depend functional J_T is homogenous of degree 0 in $\boldsymbol{r}_T$, it is sufficient to consider normalized rigid motions belonging to the set of normalized rigid motions $\mathcal{R}_{T,1}$:

$$\mathcal{R}_{T,1} =: \{\boldsymbol{r}_T \ ; \ r_{Ti} = \boldsymbol{a}' \cdot \boldsymbol{b}_i + (\boldsymbol{r_b} \wedge \boldsymbol{b}_i) \cdot \boldsymbol{w}', \ \boldsymbol{a}', \boldsymbol{w}' \in \mathbb{R}^3, \ |\boldsymbol{a}'|^2 + |\boldsymbol{w}'|^2 = 1\}.$$

Indeed, for all $\boldsymbol{r}_T \in \mathcal{R}_T$ we have $\boldsymbol{r}_T / \sqrt{|\boldsymbol{a}|^2 + |\boldsymbol{w}|^2} \in \mathcal{R}_{T,1}$ and

$$J_T(\omega, \boldsymbol{r}_T) = J_T(\omega, \boldsymbol{r}_T / \sqrt{|\boldsymbol{a}|^2 + |\boldsymbol{w}|^2}).$$

We can use now the angles $\psi \in \left[0, \frac{\pi}{2}\right]$, $\theta, \ \bar{\theta} \in (0, 2\pi)$ and $\varphi, \ \bar{\varphi} \in (0, \pi)$ to describe $\boldsymbol{a}'$ and $\boldsymbol{w}'$ involved in the definition of $\boldsymbol{r}_T \in \mathcal{R}_{T,1}$:

$$\boldsymbol{a}' = \cos(\psi)(\cos(\theta)\sin(\varphi)\boldsymbol{c}_1 + \sin(\theta)\sin(\varphi)\boldsymbol{c}_2 + \cos(\varphi)\boldsymbol{c}_3),$$

$$\boldsymbol{w}' = \sin(\psi)(\cos(\bar{\theta})\sin(\bar{\varphi})\boldsymbol{c}_1 + \sin(\bar{\theta})\sin(\bar{\varphi})\boldsymbol{c}_2 + \cos(\bar{\varphi})\boldsymbol{c}_3),$$

which means that $\boldsymbol{r}_T = \boldsymbol{r}_T(\psi, \theta, \bar{\theta}, \varphi, \bar{\varphi})$. Using this simplification we define the cost function $\mathcal{J}_T : \mathbb{R}^{(m+1)^2+4} \to \mathbb{R}_+$ by

$$\mathcal{J}_T(a_0, a_{i,j}, \psi, \theta, \bar{\theta}, \varphi, \bar{\varphi}) =: J_T(\omega(a_0, a_{ij}), \boldsymbol{r}_T(\psi, \theta, \bar{\theta}, \varphi, \bar{\varphi})),$$

as a discretization of the DVDS total dissipation functional J_T on the set of subsets $\omega(a_0, a_{ij})$ defined as level sets of Fourier's series. We can see now that the cost function can be computed from integrals on Ω and on the level set of $\{\Phi = 0\}$:

$$\mathcal{J}_T = \frac{\int_{\{\Phi=0\}\setminus\Gamma_1} \frac{1}{2} a \sqrt{[|\boldsymbol{r}_T|^2 |\boldsymbol{n}_T|^2 + 3(\boldsymbol{r}_T \cdot \boldsymbol{n}_T)^2]} \ gdS + \int_\Omega H(\Phi) q |\boldsymbol{r}_T| \ gdx}{\left[\int_\Omega H(\Phi) \boldsymbol{c} \cdot r_T \ gdx - \int_{\{\Phi=0\}\setminus\Gamma_1} b \boldsymbol{r}_T \cdot \boldsymbol{n}_T \ gdS\right]_+},$$

where we have denoted by $H(x) = (x/|x| + 1)/2$ the Heaviside step function. To compute the above expresion of the functional $\mathcal{J}_T(\omega, \boldsymbol{r}_T)$ we use the following formula:

$$\boldsymbol{r}_T \cdot \boldsymbol{n}_T = r_T^i \cdot n_i, \quad |\boldsymbol{r}_T|^2 = r_{Ti} \cdot r_T^i, \quad |\boldsymbol{n}_T|^2 = g^{ij} n_i n_j,$$

where $r_T^i(x) = \boldsymbol{a} \cdot \boldsymbol{b}^i(x) + (\boldsymbol{r_b}(x) \wedge \boldsymbol{b}^i(x)) \cdot \boldsymbol{w},\ \boldsymbol{a}, \boldsymbol{w} \in \mathbb{R}^3$.

To compute the boundary integrals on every connected sub-domains of ω, we use the rectangle method, while for the integrals over Ω we use the Simpson method.

Finally, for the global minimization of the cost functional $\mathcal{J}_T$ over $[-1,1]^{(m+1)^2} \times \left(0, \dfrac{\pi}{2}\right) \times (0, 2\pi)^2 \times (0, \pi)^2$ we use standard genetic algorithm techniques (see [82] for details on stochastic optimization methods).

7.3 Plane slope computations

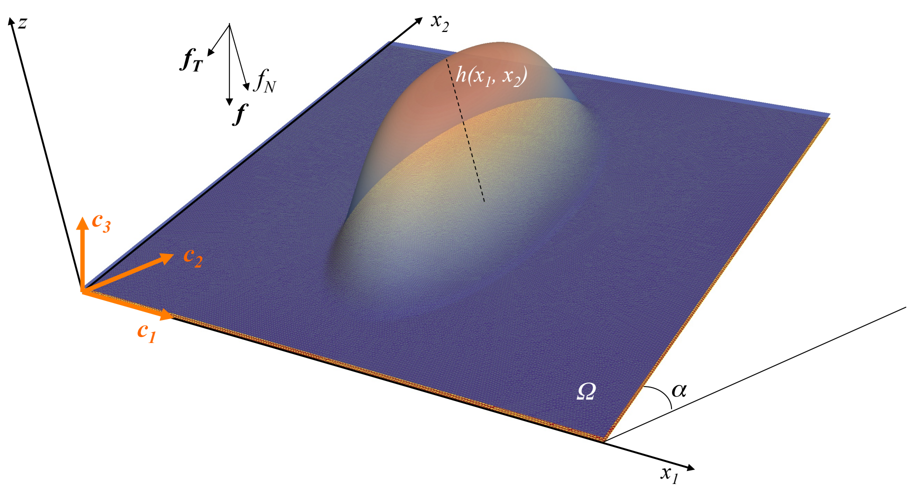

Figure 7.2: 3-D representation of the fluid domain $\mathcal{D}_0$ flowing on a plane slope of angle α, the bottom part Γ_b and the Ox_1z section described through the thickness function $(x_1, x_2) \to z = h(t, x_1, x_2)$.

We consider here a plane slope of angle α (see Figure 7.2), which can be written in the general framework of a system of parallel surfaces through

$$\boldsymbol{r}_b(x_1, x_2) = x_1 \boldsymbol{c}_1 + \cos(\alpha) x_2 \boldsymbol{c}_2 + \sin(\alpha) x_2 \boldsymbol{c}_3$$

where $x = (x_1, x_2)$ are the coordinates in the slope plane. For all numerical simulations presented in this section we choose the domain $\Omega = [0,1] \times [0,1]$, the slope angle

$\alpha = 45°$, the density $\rho = 1$ and the gravitational acceleration $\gamma = 10$. All the integrals involved in the cost function $\mathcal{J}$ were done on 50×50 points grid.

Circular dome. In the first example we consider a circular dome (see Figure 7.3), given by the following thickness distribution:

$$h_0(x_1, x_2) = \begin{cases} h_D \left(1 + \cos \frac{\pi}{\delta} \sqrt{(x_1 - x_{01})^2 + (x_2 - x_{02})^2}\right) + h_e \text{ if } (x_1, x_2) \in B(x_0, \delta), \\ h_e \quad \text{else.} \end{cases}$$

where $x_0 = (0.5, 0.5), \delta = 0.25$ and $h_D = 0.125, h_e = 0.01$.

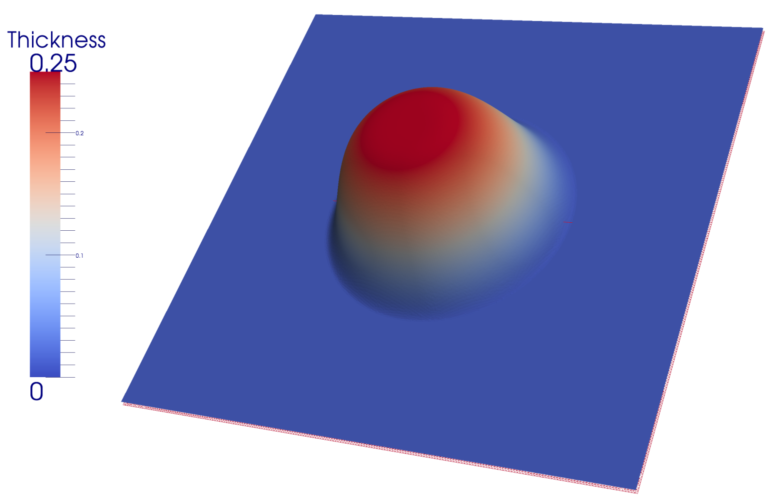

Figure 7.3: The distribution of the thickness function and the circular dome geometry of the shallow fluid.

For a Bingham fluid ($\mu = 0, \kappa_0 = 10$) with no friction ($C_f = 0$) we obtain the safety factor $\lambda^* = 0.6464$. The avalanche onset velocity $\boldsymbol{v}^*$, plotted in Figure 7.4 left, shows that the fracture is circular and occurs at the base of the dome. In order to compare this result with another method we compute the dynamic onset of the flow for a visco-plastic fluid using the finite element/finite volume numerical method described in [55]. In Figure 7.4 right, we plot the norm of the velocity field computed at the beginning of the flow. We can see that the onset region described by the our approach is the same as for the dynamic computations.

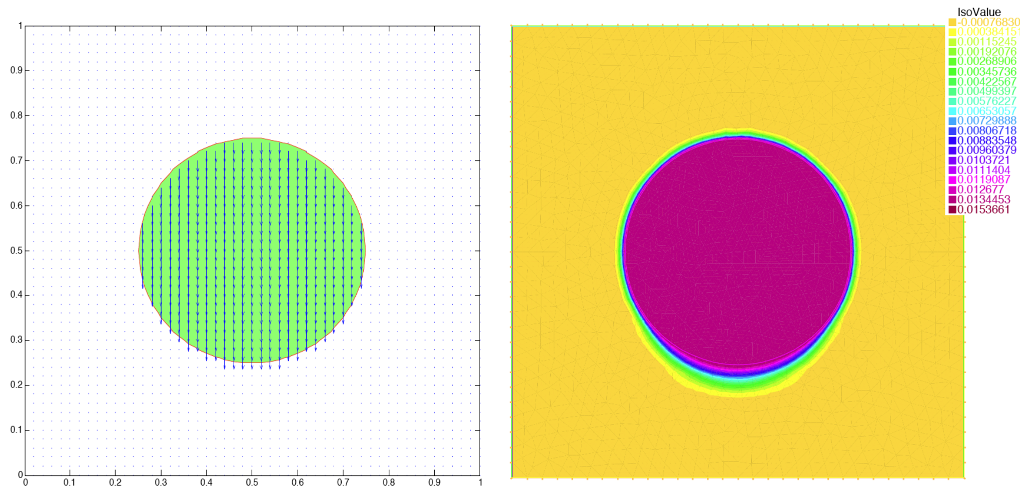

Figure 7.4: Left: the fracture and the velocity onset obtained with our approach for the circular dome geometry. Right: the norm of the velocity field at the beginning of the flow obtained with dynamic finite element/finite computations .

Square dome. In the second simulation we consider a square dome (see Figure 7.5 left), given by:

$$h_0(x_1,x_2) = \begin{cases} h_D(1+\cos(\frac{\pi}{\delta}(x_1-x_{01})))(1+\cos(\frac{\pi}{\delta}(x_2-x_{02}))) + h_e, \\ \qquad\qquad \text{if } |x_1-x_{01}|, |x_2-x_{02}| < \delta, \\ h_e \quad \text{else}, \end{cases}$$

for a Drucker-Prager fluid ($\mu = \tan(30^0), \kappa_0 = 0.1$) with no friction ($C_f = 0$). We obtain the safety factor $\lambda^* = 0.3053$ and the onset velocity is plotted is Figure 7.5 right. We remark that the avalanche fracture, which is a rounded rectangle, occurs at the base of the dome.

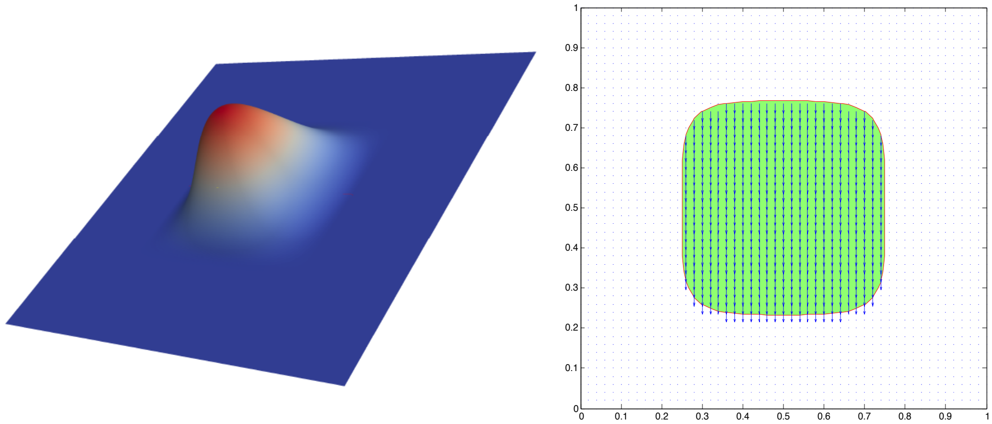

Figure 7.5: Left: the distribution of the thickness function for the square dome geometry. Right: the velocity onset obtained with our approach.

Circular dome with obstacles. In the third simulation we consider a circular dome over a thick uniform fluid in the presence of a circular obstacle $B = B(x_0^o, \delta^o)$ (see Figure 7.6 left). The obstacle is located at $x_0^o = (0.5, 0)$ and has a radius of $\delta^o = 0.2$, while the thickness function h_0 is the same as in the first simulation, but with $h_e = 0.1$, ten times larger. It is more suitable to model the obstacle by the penalization of the yield limit κ_0. For that, we consider the same domain $\Omega = [0, 1] \times [0, 1]$, as before, but with $\kappa_0 = 1$ on $\Omega \setminus B$ outside the obstacle and $\kappa_0 = 50$ inside the obstacle $\Omega \cap B$. We find the safety factor $\lambda^* = 3.4713$ and the onset velocity $\boldsymbol{v}^*$ is plotted is Figure 7.5 right. We remark that the avalanche fracture is rather different form the dome over a thin film, computed in the first simulation. The fracture is close to the corners of the domain and avoids the circular obstacle B.

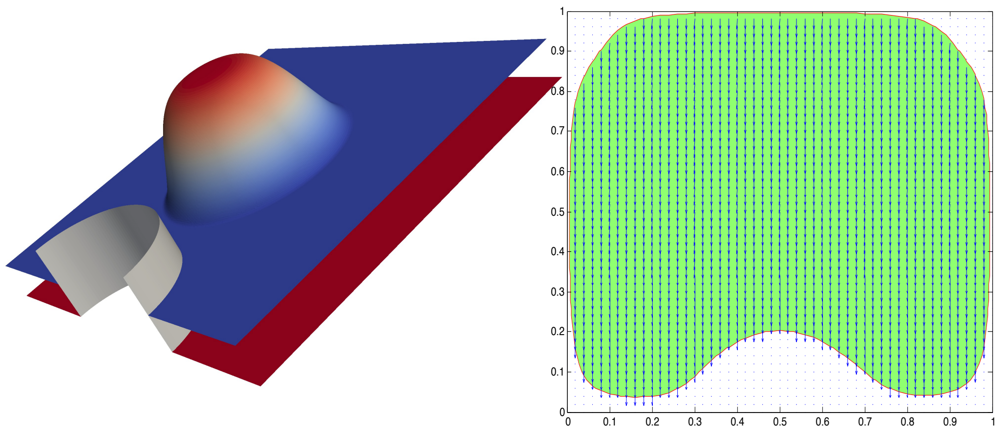

Figure 7.6: Left: the distribution of the thickness function for a thick fluid with an obstacle. Right: the velocity onset obtained with our approach.

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7.4 Complex basal topography computations

In this section we give some numerical computations of the safety factor λ^* and of the onset velocity $\boldsymbol{v}^*$ for complex basal topography. For all numerical simulations we choose the density $\rho = 1$ and the gravitational acceleration $\gamma = 10$. All the integrals involved in the cost function $\mathcal{J}_T$ were done on 50×50 points grid.

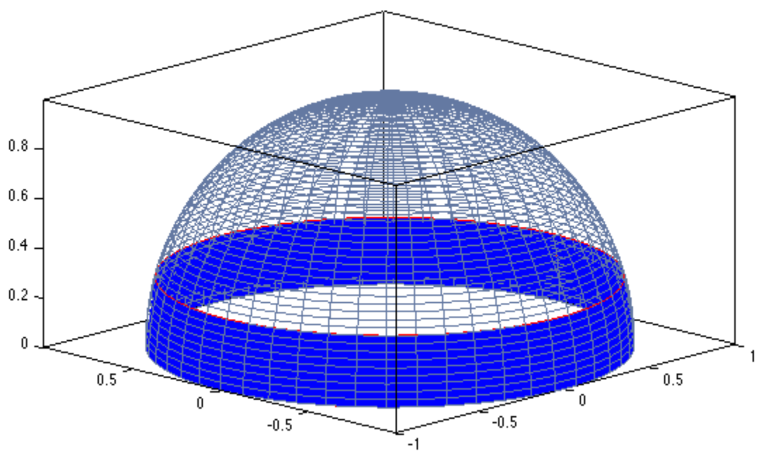

Figure 7.7: The half sphere covered with an uniform thickness Bingham fluid, computed with a DVDS approach. The fracture $\partial\omega^*$ associated to the velocity onset $\boldsymbol{v}^* =: \boldsymbol{r}_T^* 1_{\omega^*}$ occurs in the inferior part of the half sphere.

Half sphere with uniform thickness. In the first example, which is an academic problem, we have considered a half sphere with a constant thickness $h_0(x) = 0.1$ (see Figure 7.7). The parametric representation of the bottom surface $\mathcal{S}_b$ is given by:

$$\boldsymbol{r}_b(\theta, \varphi) = (\cos(\theta)\sin(\varphi), \sin(\theta)\sin(\varphi), \cos(\varphi)).$$

where $x_1 = \theta \in [0, 2\pi)$, $x_2 = \varphi \in [0, \frac{\pi}{2}), \Omega = [0, 2\pi) \times [0, \frac{\pi}{2})$. The half sphere is covered uniformly (h_0 =0.1) with a Bingham fluid ($\mu = 0, \kappa_0 = 10$) which is in contact with friction ($C_f = 0.5$) with the bottom surface. The half sphere has stress free boundary conditions on the horizontal-plane, i.e., $\Gamma_0 = \emptyset, \Gamma_1 = \{\varphi = \pi/2\}$. The computed safety factor was founded to be $\lambda^* = 0.10$ and the onset velocity $\boldsymbol{v}^* =: \boldsymbol{r}_T^* 1_{\omega^*}$, plotted in Figure 7.7, shows that the fracture occurs at the inferior part of the half sphere. In this region, the tangential gravitational forces are more important, while the frictional resistance, proportional to the normal gravitational forces is very reduced.

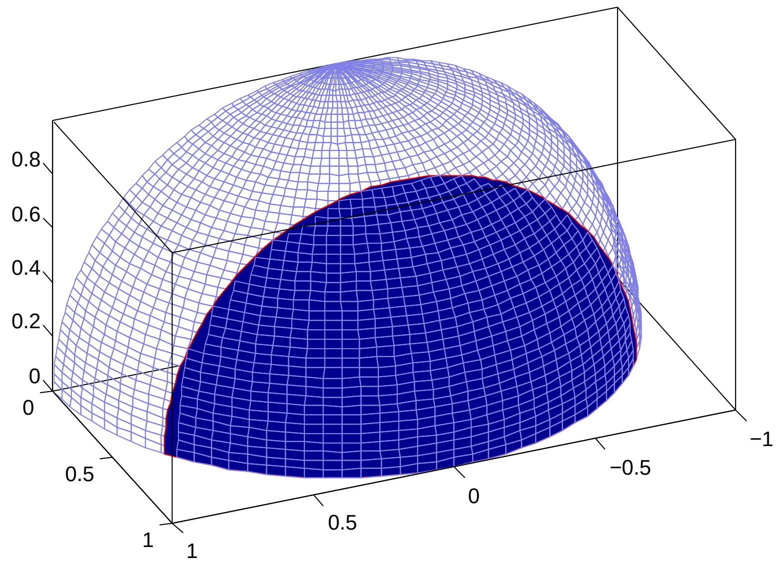

Figure 7.8: The quarter of a sphere covered with an uniform thickness Bingham fluid, computed with a DVDS approach. The fracture associated to the velocity onset $\boldsymbol{v}^* =: \boldsymbol{r}_T^* 1_{\omega^*}$ occurs at the boundary of the optimal sub-domain $\partial\omega^*$.

The rigid motion $\boldsymbol{r}^*$ associated to the onset velocity is given by a vertical translation $(\boldsymbol{a}' = (0, 0, -1), \boldsymbol{w}' = (0, 0, 0))$.

Quarter of a sphere with uniform thickness. In the second example we consider a quarter of a sphere with Dirichlet boundary conditions (see Figure 7.8) covered by the same material (Bingham), with the same thickness and the same friction coefficient as in the previous example. For that we put $x_1 = \theta \in (0, \pi),\ x_2 = \varphi \in [0, \frac{\pi}{2}),\ \Omega = (0, \pi) \times [0, \frac{\pi}{2})$ and $\Gamma_0 = \{\varphi = \pi/2\} \cup \{\theta = 0\} \cup \{\theta = \pi\}, \Gamma_1 = \emptyset$. The computed safety factor is $\lambda^* = 0.1324$ and the DVDS onset velocity $\boldsymbol{v}^* =: \boldsymbol{r}_T^* 1_{\omega^*}$ is plotted in Figure 7.8. As before in the optimal subdomain ω^*, included in the inferior part of the sphere, the tangential gravitational forces are important, while the frictional resistance is reduced. We remark the influence of the boundary conditions on $\{\theta = 0\} \cup \{\theta = \pi\}$ in the shape of the optimal set ω^*. Moreover, we can see here that the fracture occurs on the horizontal boundary $\{\varphi = \pi/2\}$, where the shallow material is fixed. This example illustrate the importance of the relaxation of the boundary conditions.

Quarter of an ellipsoid with nonuniform thickness. In the last simulation we consider a quart of an ellipsoid (see Figure 7.9), which is described by the following parametric representation of the bottom surface $\mathcal{S}_b$:

$$\boldsymbol{r}_b(\theta,\varphi) = (a\cos(\theta)\sin(\varphi), b\sin(\theta)\sin(\varphi), c\cos(\varphi)).$$

where $x_1 = \theta \in (0,\pi),\ x_2 = \varphi \in \left(\frac{\pi}{2},\pi\right]$ and $\Omega = (0,\pi)\times\left(\frac{\pi}{2},\pi\right]$.

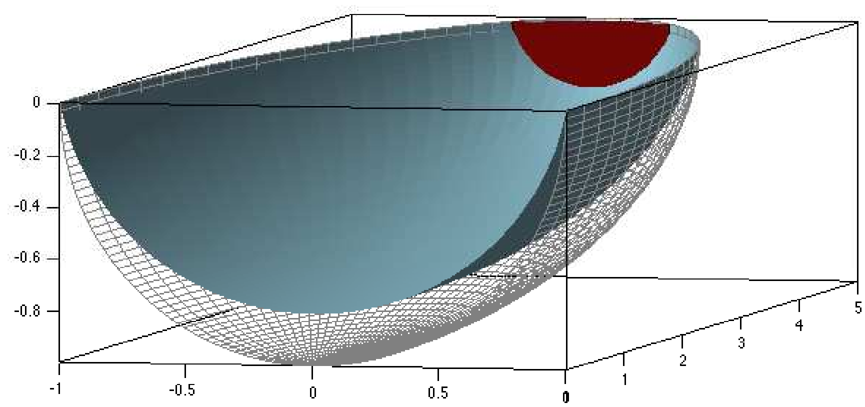

Figure 7.9: The quarter of an ellipsoid filled with an non uniform thickness Drucker-Prager fluid, computed with a DVDS approach. The fracture $\partial\omega^*$ associated to the velocity onset $\boldsymbol{v}^* =: \boldsymbol{r}_T^* 1_{\omega^*}$ occurs in the upper part of the ellipsoid.

We consider here a Drucker-Prager fluid/solid ($\mu = \tan(30^o), \kappa_0 = 0.1$) with a non-uniform thickness (given by $h_0(\theta,\varphi) = -0.2\cos(\theta)$) which is in frictionless contact ($C_f = 0$) with the bottom surface. The quarter of ellipsoid has stress free boundary conditions, i.e., $\Gamma_0 = \{\theta = 0\}\cup\{\theta = \pi\}, \Gamma_1 = \{\varphi = \pi/2\}$. The computed safety factor is $\lambda^* = 0.1590$ and from the onset velocity $\boldsymbol{v}^* =: \boldsymbol{r}_T^* 1_{\omega^*}$, we deduce that the fracture occurs (see Figure 7.9) at the upper part of the ellipsoid. As in above example in the onset region, the gravitational forces are very close to the tangential plane.

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Résumé. Cette thèse présente des modèles probabilistes et déterministes de rupture et des phénomènes de branchement, on étudie : les processus de branchement à valeurs mesures et leur EDP non linéaires, les processus de Markov de la subordination au sens de Bochner sur les espaces L^p, et les EDP non linéaires liées au déclenchement des avalanches. La première partie présente les aspects stochastiques. On utilise plusieurs outils théoriques, analytiques et probabilistes de la théorie du potentiel. D'abord, on construit des processus de branchement (de Markov) sur l'ensemble des configurations finies de l'espace d'état d'un processus standard, contrôlés par un noyau de branchement et un noyau tuant. On établit des connexions avec les équations différentielles partielles liées aux fonctions de transition d'un processus de branchement. Si on part d'un super-processus, on obtiendra un processus de branchement ayant l'espace d'état des configurations finies de mesures positives finies sur un espace topologique. L'outil principal pour démontrer la régularité des trajectoires d'un processus de branchement est l'existence des fonctions surharmoniques convenables, ayant les niveaux compacts. En suite, on démontre que la subordination induite par un semi-groupe de convolution (la subordination au sens de Bochner) d'un C_0 -semi-groupe d'opérateurs sous-markoviens sur l'espace L^p est associée à la subordination de processus droit de Markov. En conséquence, on résout le problème des martingales associé au L^p -générateur infinitésimal d'un semi-groupe subordonné. Il s'avère qu'un élargissement de l'espace de base est nécessaire. La principale étape de la preuve est la préservation sous une subordination de la propriété d'un processus de Markov d'être un processus droit borélien. La deuxième partie de la thèse est consacrée à la modélisation du déclenchement d'une avalanche d'un matériau visco-plastique de faible épaisseur (sols, neige ou autre géo-matériaux) sur une surface avec topographie (montagnes, vallées). On introduit un critére simple, déduit d'un problème d'optimisation (analyse de la charge limite), capable de distinguer si une avalanche se produit ou pas. Comme la fonctionnelle de dissipation plastique est non régulière et non coercive dans les espaces de Sobolev classiques, on utilise l'espace des fonctions à déformation tangentielle bornée, pour prouver l'existence d'un champ de vitesse optimal, associé au déclenchement d'une avalanche. La fracture du matériau pendant la phase de déclenchement est modélisée par une discontinuité de ce champ de vitesse. On propose aussi une stratégie numérique, sans maillage, pour résoudre le problème de charge limite et pour obtenir la fracture de déclenchement. Enfin, l'approche numérique proposée est illustrée par la résolution de certains problèmes modélisant le déclenchement des avalanches.

Probabilistic and deterministic models for fracture type phenomena

Abstract. The thesis presents probabilistic and deterministic models for rupture and branching type phenomena, by studying: measure-valued discrete branching processes and their nonlinear PDEs, the Markov processes of the Bochner subordinations on L^p spaces, and the nonlinear PDEs associated to the flow onset of dense avalanches. The first part presents the stochastic aspects. Several analytic and probabilistic potential theoretical tools are used. First, it is given a construction for the branching Markov processes on the space of finite configurations of the state space of a given standard process (called base process), controlled by a branching kernel and a killing one. There are established connections with the nonlinear partial differential equations associated with the transition functions of the branching processes.When the base process is the Brownian motion, then a nonlinear evolution equation involving the square of the gradient occurs. Starting with a superprocess as base process, the result is a branching process with state space the finite configurations of positive finite measures on a topological space. A key tool in proving the path regularity of the branching process is the existence of a convenient superharmonic function having compact level sets. Second, it is shown that the subordination induced by a convolution semigroup (the subordination in the sense of Bochner) of a C_0 -semigroup of sub-Markovian operators on an L^p space is actually associated to the subordination of a right (Markov) process. As a consequence, it is solved the martingale problem associate with the L^p -infinitesimal generator of the subordinate semigroup. It turns out that an enlargement of the base space is necessary. A main step in the proof is the preservation under such a subordination of the property of a Markov process to be a Borel right process. The second part of the thesis deals with the modeling of the onset of a shallow avalanche (soils, snow or other geomaterials) over various bottom topologies (mountains, valleys). Starting from a shallow visco-plastic model with topography, a simple criterion able to distinguish if an avalanche occurs or not, is introduced. This criterion is deduced from an optimization problem, called limit load analysis. The plastic dissipation functional involved is non-smooth, and non coercive in the classical Sobolev spaces. The appropriate functional space is the space of bounded tangential deformation functions and the existence of an onset velocity field (collapse flow) is proved. To propose a numerical strategy, a mesh free method is used to reduce the limit load problem to the minimization of a shape dependent functional. The collapse flow velocity field, which is discontinuous, is associated to an optimum sub-domain and to a rigid flow. Finally, the proposed numerical approach is illustrated by solving some safety factor problems associated to avalanche onset.

Discipline: Mathématiques

Mots-clefs. Fonction excessive, processus de Markov, fonction compacte de Lyapunov, branchement discret, processus à valeurs mesures, noyau de branchement, subordination au sens de Bochner, L^p semi-groupe, délenchement d'avalanche, fluide visco-plastique, topographie, déformation tangentielle bornée, méthode sans maillage, analyse de charge limite.

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