Université Paris 13 Nord **Institut Galilée**

Laboratoire Analyse Géométrie et Applications **UMR 7539**

THÈSE

pour obtenir le grade de

DOCTEUR de l'UNIVERSITÉ PARIS 13

Discipline : Mathématique

présentée et soutenue publiquement par

Quoc Hung Phan

le 25 mars 2013

Titre :

Analyse qualitative des solutions de systèmes de réaction-diffusion et théorèmes de type Liouville.

Directeur de thèse :

Philippe Souplet

JURY

Président :	Mme. Marie-François Bidaut-Véron	Université François Rabelais
Rapporteurs :	M. Frank Pacard	Ecole Polytechnique
	M. Tobias Weth	Universität Frankfurt
Examinateurs :	M. Scott Armstrong	CEREMADE Université Paris-Dauphine
	M. Philippe Souplet	Université Paris 13 Nord
	M. Frederic Weissler	Université Paris 13 Nord
	M. Hatem Zaag	Université Paris 13 Nord

Mis en page avec la classe thloria.

Remerciements

Je tiens à exprimer mes profonds remerciements et ma gratitude à Philippe Souplet qui a dirigé mes travaux avec beaucoup de disponibilité et de gentillesse. C'est grâce à ses précieux conseils, sa rigueur scientifique et ses talents pédagogiques que j'ai pu surmonter les moments difficiles pendant ma thèse.

Je suis très reconnaissant envers Frank Pacard et Tobias Weth qui m'ont fait l'immense honneur de rapporter cette thèse.

Je voudrais remercier vivement Scott Armstrong, Marie-François Bidaut-Véron, Frederic Weissler et Hatem Zaag pour avoir accepté de participer à ma soutenance de thèse en tant que membres du jury. Je remercie sincèrement Marie-François Bidaut-Véron pour ses remarques et suggestions pertinentes.

Je souhaite profiter de cette occasion pour exprimer ma reconnaissance à Frédéric Klopp qui a encadré mon stage de Master 2 et qui m'a encouragé à poursuivre mes études supérieures. Je remercie particulièrement Lionel Schwartz qui m'a beaucoup aidé pendant les premiers jours où j'étais en France.

Mes remerciements vont aussi à tous les membre du LAGA, que j'ai eu de la chance de côtoyer durant toutes ces années dans un excellent environnement scientifique et humain. Je suis redevable envers Isabelle, Yolande, Jean-Philippe et Philippe pour leur aide de problèmes administratifs et leur gentillesse.

Merci à mes amis vietnamiens pour l'ambiance agréable et familiale.

Enfin, je remercie du fond du coeur ma famile. Merci à mes parents et ma soeur qui m'ont constamment soutenu et encouragé. Un immense merci à ma femme et mon fils qui sont toujours une source formidable de motivation. Cette thèse leur est dédiée.

Résumé

Cette thèse est consacrée à l'étude des propriétés qualitatives des solutions d'équations et de systèmes elliptiques et paraboliques non-linéaires.

Dans la première partie de la thèse, nous nous intéressons aux équations et aux systèmes elliptiques à coefficients singuliers ou dégénérés de type Hardy-Hénon, à l'équation parabolique de même type, ainsi qu'à un système parabolique à coefficients constants mais non coopératif. Nous obtenons des théorèmes de type Liouville elliptiques et paraboliques et nous développons leurs applications : estimations a priori, estimations des singularités en temps ou en espace, estimations de la décroissance à l'infini.

Dans une deuxième partie, nous prouvons l'existence globale et le caractère borné des solutions pour un système parabolique, fortement couplé, de type Keller-Segel issu de la criminologie.

Mots-clés: Hardy-Hénon ; Théorème de type Liouville ; Borne universelle, Nonexistence ; Estimation des singularités ; Estimation de la décroissance ; Estimation a priori ; Explosion ; Système parabolique ; Keller-Segel ; Modélisation de la criminologie.

Abstract

This dissertation is devoted to the study of qualitative properties of solutions for some nonlinear elliptic and parabolic equations and systems.

In the first part of the dissertation, we are interested in elliptic equations and systems with singular or degenerate coefficients of Hardy-Hénon type, in parabolic equations of the same type, and in a noncooperative parabolic system with constant coefficients. We obtain elliptic and parabolic Liouville-type theorems and we develop their applications : a priori estimates, singularity estimates in space or in time, decay estimates.

In the second part, we prove the global existence and a priori bound of solutions of a Keller-Segel type, strongly coupled, parabolic system arising in crime modelling.

Keywords: Hardy-Hénon ; Liouville type theorem ; Universal bound ; Non-existence ; Singularity estimate ; Decay estimate ; A priori estimate ; Blow-up ; Parabolic system ; Keller-Segel ; Crime modelling.

Table des matières

Liste des travaux rassemblés dans la thèse

1. Quoc Hung Phan and Philippe Souplet. Liouville-type theorems and bounds of solutions of Hardy-Hénon equations. *J. Differential Equations*, 252(3) :2544-2562, 2012 (cf. Chapitre 2 de la thèse).
2. Quoc Hung Phan. Liouville-type theorems and bounds of solutions of Hardy-Hénon systems. *Adv. Differential Equations*, 17(7-8) :605-634, 2012 (cf. Chapitre 3 de la thèse).
3. Quoc Hung Phan. Singularity and blow-up estimates via Liouville-type theorems for Hardy-Hénon parabolic equations. *J. Evol. Equ.*, accepté (cf. Chapitre 4 de la thèse).
4. Quoc Hung Phan. Optimal Liouville-type theorem for a parabolic system, à soumettre (cf. Chapitre 5 de la thèse).
5. Raul Manasevich, Quoc Hung Phan, Philippe Souplet. Global existence of solutions for a chemotaxis-type system arising in crime modelling. *European J. Appl. Math.*, 24(2) :273-296, 2013 (cf. Chapitre 6 de la thèse).

Chapitre 1

Introduction générale

L'objet principal de cette thèse est l'obtention de théorèmes de type Liouville et l'étude de leur connexion avec les propriétés locales des solutions d'équations et de systèmes de réaction-diffusion elliptiques et paraboliques. En outre, nous nous intéressons dans cette thèse à l'étude de l'existence globale de solutions d'un modèle chimiotaxique issu de la criminologie.

On rappelle qu'un théorème de type Liouville est un résultat de non-existence de solutions (non-triviales) dans l'espace entier (ou demi-espace). Au cours des trois dernières décennies, les propriétés de ce type pour les problèmes elliptiques et paraboliques ont été largement étudiées (voir les travaux [GS81a, CL91, BM02, SZ96, Ser64, RZ00, Mit96, BVV91, MP01, PQS07a, AS11, dFF94, dFS05, QS12b] et [BV98, PQ06, PQS07b, BPQ11, AS10, MZ00, MZ98], resp., pour les problèmes elliptiques et paraboliques). Pour le modèle elliptique scalaire, de très nombreux travaux sont consacrés à établir des théorèmes de type Liouville pour l'équation

$$-\Delta u = f(u) \quad \text{dans } \mathbb{R}^N,$$

et ses généralisations. En particulier, Gidas et Spruck ont donné dans leur travail célèbre [GS81a] en 1981 le résultat de Liouville pour le modèle typique

$$-\Delta u = u^p \quad \text{dans } \mathbb{R}^N, \tag{1.1}$$

avec $p > 1$. Ils ont montré que (1.1) n'a aucune solution classique positive si $p < p_S$ où p_S est l'exposant de Sobolev,

$$p_S := \begin{cases} \frac{N+2}{N-2} & \text{si} \quad N \geq 3, \\ \infty & \text{si} \quad N = 1, 2. \end{cases}$$

Ce résultat est optimal en raison de l'existence de solutions positives de (1.1) si $p \geq p_S$.

La même question s'est posée pour le système analogue suivant (à savoir le système de Lane-Emden)

$$\begin{cases} -\Delta u = v^p, \\ -\Delta v = u^q, \end{cases} \tag{1.2}$$

avec $p, q > 0$. La conjecture de Lane-Emden affirme qu'il n'existe aucune solution classique positive de (1.2) dans $\mathbb{R}^N$ si et seulement si (p, q) est en-dessous de l'hyperbole

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}.$$

Cette conjecture a été démontrée en dimension $N \leq 4$ (voir [Mit96, MP01, SZ96, PQS07a, Sou09b]).

D'autre part, le modèle parabolique

$$u_t - \Delta u = |u|^{p-1}u \tag{1.3}$$

a été étudié par beaucoup d'auteurs. L'obtention d'un théorème de type Liouville pour (1.3) est une question dui est loin d'être complètement résolue. Jusqu'à présent, on dispose d'un résultat optimal pour les solutions radiales [PQ05, BPQ11], et d'un résultat partiel pour les solutions positives non-radiales [BV98]. Rappelons également, lorsque $p < p_S$, le résultat de Merle et Zaag [MZ98] de nonexistence pour les solutions définies sur $\mathbb{R}^N \times (-\infty, 0)$ qui décroissent au moins comme $|t|^{-1/(p-1)}$ lorsque $t \to -\infty$. Les résultats de Liouville elliptiques et paraboliques jouent un rôle particulier dans la théorie qualitative des solutions, car ils ont des applications très importantes concernant les propriétés locales et globales des solutions :

- l'existence de solutions du problème de Dirichlet dans les domaines bornés pour des équations et systèmes elliptiques non-linéaires, en particulier pour ceux qui n'ont pas de structure variationnelle,
- l'estimation a priori des solutions du problème de Dirichlet dans les domaines bornés,
- l'estimation des singularités et l'estimation de décroissance à l'infini,
- l'estimation universelle, l'inégalité de Harnack,
- le taux d'explosion initial pour les solutions locales,
- le taux de décroissance pour les solutions globales.

Il y a plusieurs méthodes de démonstration du théorème de type Liouville, notamment pour le modèle typique (1.1), (1.2) et son analogue parabolique (1.3). Nous rappelons ici quelques méthodes usuelles :

Pour le modèle elliptique (scalaire et système)

- la méthode de fonction-test ([MP01]), qui donne le résultat de type Liouville non-optimal pour les solutions, mais donne le résultat optimal pour les sur-solutions ;
- la formule de Bochner combinée avec les arguments de multiplicateurs non-linéaires [GS81a, BVV91, SZ02]. Cette technique est délicate et donne le résultat optimal dans le cas scalaire ;
- l'argument de "moving plane" [BM98, BM02, CL91, CL09, Dan92, RZ00]. Cet argument donne le résultat optimal dans le cas scalaire en combinant avec la transformation de Kelvin. Cependant, la transformation de Kelvin n'est en général applicable dans le cas du système que si toutes les non-linéarités sont sous-critiques ou critiques ;
- l'identité de Pohozaev combinée avec des arguments supplémentaires ("feedback", mesure, l'interpolation...) [SZ96, Sou09b, QS12a]. Cette méthode est optimale en basse dimension ($N \leq 4$) ;
- le principe du maximum et comparaison [QS12b, AS11], qui peut donner des résultats optimaux pour certains types de systèmes, ou pour les sur-solutions.

Pour le modèle parabolique (scalaire et système), quelques méthodes utilisées dans le cas elliptique peuvent être appliquées mais le résultat est loin d'être optimal.

- la méthode de fonction-test, qui donne le résultat de type Fujita ;
- la technique de Gidas-Spruck modifiée par Bidaut-Véron [BV98], qui se limite à l'exposant $p < p_B$;
- la méthode de "moving plane" [PQ05] ;
- l'argument d'intersection-comparaison [PQ06, BPQ11, Sou09a], qui donne le résultat optimal dans la classe des solutions radiales ;
- l'argument d'énergie combiné avec l'estimation de décroissance à l'infini et le théorème de type Liouville elliptique [BPQ11, AS10, QS12a].

En 1981, Gidas et Spruck ont utilisé dans [GS81b] le résultat de type Liouville pour (1.1), combiné avec un argument de changements d'échelles (appelé encore méthode d'explosion) pour démontrer l'estimation a priori des solutions des problèmes aux limites correspondants. L'idée de cette méthode est la suivante : s'il y a une suite non-bornée de solutions explosives, par des changements d'échelles au voisinage des points de maximum, on en déduit une suite bornée de solutions qui convergent vers une solution dans l'espace entier ou demi-espace, en contradiction avec le théorème de Liouville dans l'espace entier ou le demi-espace. Plus récemment, Polacik, Quittner et Souplet ont découvert de nouvelles connexions entre les théorèmes de type Liouville non-linéaires et les propriétés locales des solutions des problèmes elliptiques et paraboliques [PQS07a, PQS07b]. Plus précisément, ils ont développé une méthode générale pour déduire des estimations universelles des solutions locales à partir de théorèmes de type Liouville. Cette méthode est basée sur des arguments de changements d'échelles et sur une propriété cruciale de "doublement". L'argument de doublement, qui est une extension d'une idée de Hu [Hu96], est différent de la méthode classique de changement d'échelle de Gidas-Spruck, qui s'applique seulement aux problèmes aux limites. Cette méthode permet d'obtenir de nouveaux résultats sur les singularités d'équations et de systèmes elliptiques et paraboliques. De plus, cela met aussi en évidence que, pour le modèle typique, le théorème de type Liouville et l'estimation universelle des singularités sont finalement équivalents.

Dans cette thèse, nous nous intéressons aux problèmes elliptiques et paraboliques de type Hardy-Hénon suivants

- $$-\Delta u = |x|^a u^p, \tag{1.4}$$
- $$\begin{cases} -\Delta u = |x|^a v^p \\ -\Delta v = |x|^b u^q, \end{cases} \tag{1.5}$$
- $$u_t - \Delta u = |x|^a |u|^{p-1} u. \tag{1.6}$$

Le modèle (1.4) est appelé l'équation de Hénon (resp. Hardy, Lane-Emden) si $a > 0$ (resp. $a < 0$, $a = 0$). Le cas $a < 0$ est appelé l'équation de Hardy car cela correspond à l'inégalité de Hardy-Sobolev [GY00]. Le cas $a > 0$ a été introduit par Hénon [H́73] comme un modèle pour la distribution de masse dans des grappes sphériquement symétriques d'étoiles. L'équation (1.4) a été largement étudiée, notamment pour l'existence et la non-existence dans un domaine borné, la multiplicité et les propriétés de symétrie des solutions ([SWS02, SW03, Ser05]). Récemment, le théorème de type Liouville pour les solutions (de signe variable) d'indice de Morse fini et pour les solutions stables a été démontré dans [Faz11, DDG11, Esp08, WY12].

Le modèle (1.5) est la généralisation du système de Lane-Emden. Pour ce modèle, l'existence et la non-existence de la solution dans les domaines bornés sont été étudiées dans [CR10, dFPR08], et des résultats sur le comportement asymptotique des solutions pour l'exposant presque critique ont été obtenus dans [HY08]. Récemment, le théorème de type Liouville pour les solutions d'indice de Morse fini et pour les solutions stables a été démontré dans [Cow11, FG11].

Concernant le modèle parabolique (1.6), le résultat de type Fujita a été montré dans [Pin97]. Le problème de Cauchy correspondant a été largement étudié, et l'existence et la non-existence de solutions ont été établies dans [Wan93, Pin97, Hir08, FT00]. Les questions de comportement asymptotique et de stabilisation ont été considérées dans [Wan93, DLY06, LL08] et le phénomène d'explosion pour le problème de Cauchy et aux limites a été étudié dans [AT05, Wan93, GS11, GLS10].

Notre premier but est d'obtenir des théorèmes de type Liouville et d'étudier leurs applications pour l'analyse qualitative des solutions locales. Nous utilisons essentiellement les méthodes ci-dessus, combinées avec quelques nouvelles idées pour déduire les théorèmes de type Liouville. Pour des applications sur les propriétés des solutions de ces problèmes, nous employons essentiellement l'argument de changement d'échelles et de doublement issu des travaux de Polacik, Quittner et Souplet [PQS07a, PQS07b], et nous introduisons certaines nouvelles idées concernant le changement d'échelles et l'estimation des singularités et de la décroissance à l'infini, afin de surmonter les difficultés provenant de la dégénérescence et de la singularité du terme $|x|^a$.

Dans une deuxième direction, nous nous intéressons à l'existence globale des solutions d'un système de type chimiotaxique issu de la criminologie. La premier modèle mathématique de cambriolages dans les zones d'habitat résidentiel a été introduit dans [SDP$^+$08] en prenant la limite continue de deux équations aux différences, l'une représente l'attractivité des maisons individuelles au cambriolage, et l'autre représente le mouvement des cambrioleurs. La limite continue donne un système aux dérivées partielles du type chimiotaxique. Après ce travail, plusieurs articles ont été consacrés à l'étude de modèles variés provenant de la criminologie (voir [BN10, CCM12, Pit10, RB10, SBB10, SDP+08]). Nous considérons dans cette thèse un modèle dans [Pit10] et étudions l'existence globale et la borne a priori des solutions.

Cette thèse est composée de 6 chapitres. Le premier chapitre, introductif, présente le sujet et résume les résultats obtenus. Les chapitres suivants correspondent chacun à un article. Nous allons présenter ici d'une manière détaillée le contenu de chacun d'eux.

- Le chapitre 2 présente de nouveaux théorèmes de type Louville ainsi que leurs applications pour le modèle elliptique (1.4).
- Dans le même axe, un autre problème étudié dans le chapitre 3 est le cas du système de Hardy-Hénon (1.5). Pour la preuve du théorème du type Liouville, nous utilisons la technique introduite par Serrin-Zou [SZ96] et développée par Souplet [Sou09b]. La preuve est très délicate, basée sur une combinaison d'identité de type Rellich-Pohozaev, d'inégalités de Sobolev et d'interpolation sur S^{N-1}, et d'arguments de "feedback" et de mesure.
- Le chapitre 4 est consacré à l'étude de la propriété de type Liouville et des singularités pour l'équation parabolique de Hardy-Hénon

$$u_t - \Delta u = |x|^a |u|^{p-1} u, \quad (x,t) \in \Omega \times I \tag{1.7}$$

  où Ω est un domaine de $\mathbb{R}^N$, $p > 1$, $a > \max\{-2, -N\}$, et I est un intervalle de $\mathbb{R}$.
- Dans le chapitre 5, nous démontrons un résultat optimal de type Liouville pour le système parabolique non coopératif

$$\begin{cases} u_t - \Delta u = u^p - \beta u^r v^{r+1}, \\ v_t - \Delta v = v^p - \beta u^{r+1} v^r. \end{cases} \tag{1.8}$$

  où $p = 2r + 1, r > 0$, $\beta \in \mathbb{R}$.
- Le chapitre 6 concerne l'existence globale de solutions d'un modèle de type Keller-Segel issu de la criminologie.

1.1 L'équation elliptique de Hardy-Hénon (chapitre 2)

Nous nous intéressons à l'équation de Hardy-Hénon suivante

$$-\Delta u = |x|^a u^p, \quad x \in \Omega, \tag{1.9}$$

avec $p > 1$, $a \in \mathbb{R}$, Ω un domaine de $\mathbb{R}^N$, $N \geq 2$. L'équation (1.9) est appelée l'équation de Hénon (resp. Hardy, Lane-Emden) si $a > 0$ (resp. $a < 0$, $a = 0$). Nous considérons les solutions dans l'espace

$$\begin{cases} C^2(\Omega), & \text{si } a \geq 0, \\ C^2(\Omega \setminus \{0\}) \cap C(\Omega), & \text{si } a < 0. \end{cases} \tag{1.10}$$

Pour le cas $a = 0$ (l'équation de Lane-Emden), le théorème de type Liouville a été complètement résolu dans l'article célèbre de Gidas et Spruck [GS81a]. Le cas $a < 0$ est démontré dans [BVG10, Theorem 1.7]. Cependant, le résultat optimal de Liouville pour le cas $a > 0$ reste ouvert et semble difficile. Comme $p_S(a) > p_S$, les techniques classiques de [BVV91, GS81a] (la formule de Bochner combinée avec des arguments délicats de multiplicateurs non-linéaires) et de [CL91] (la transformation de Kelvin combinée avec le "moving plane") ne marchent plus pour $p > p_S$. Dans la classe des solutions radiales, le théorème de type Liouville reste vrai si et seulement si $p < p_S(a)$, où $p_S(a) = (N + 2 + 2a)/(N - 2)$ est l'exposant de Hardy-Sobolev. Il est donc naturel de conjecturer qu'il n'existe aucune solution positive de (1.9) si et seulement si $p < p_S(a)$. Nous avons réussi à démontrer la conjecture en dimension $N = 3$ dans le cas des solutions bornées.

Théorème 1.1.1. *Soient $a > 0$, $p > 1$ et $N = 3$. Si $p < p_S(a)$, l'équation (1.9) n'a aucune solution bornée positive dans $\Omega = \mathbb{R}^N$.*

Nous établissons ensuite des estimations au voisinage de la singularité isolée et de la décroissance à l'infini. Ce résultat suivant est aussi une extension de [BVV91, GS81a].

Théorème 1.1.2. *Soient $a > -2$ et $1 < p < p_S$. Il existe une constante $C = C(N, p, a) > 0$ telle que :*

(i) Toutes les solutions non-négatives de l'équation (1.9) dans $\Omega = \{x \in \mathbb{R}^N ; 0 < |x| < \rho\}$ ($\rho > 0$) satisfont

$$u(x) \le C|x|^{-\frac{2+a}{p-1}} \quad et \quad |\nabla u(x)| \le C|x|^{-\frac{p+1+a}{p-1}}, \qquad 0 < |x| < \rho/2. \tag{1.11}$$

(ii) Toutes les solutions non-négatives de l'équation (1.9) dans $\Omega = \{x \in \mathbb{R}^N ; |x| > \rho\}$ ($\rho \ge 0$) satisfont

$$u(x) \le C|x|^{-\frac{2+a}{p-1}} \quad et \quad |\nabla u(x)| \le C|x|^{-\frac{p+1+a}{p-1}}, \qquad |x| > 2\rho. \tag{1.12}$$

On note que l'estimation de décroissance (1.12), combinée avec l'identité de Rellich-Pohozaev, donne une démonstration beaucoup plus simple de la nonexistence de la solution positive de (1.9) dans $\mathbb{R}^N$ pour $p < \min\{p_S, p_S(a)\}$. Notre démonstration du Théorème 1.1.2 est basée sur l'observation que les estimations (1.11) et (1.12) pour p, a peuvent être réduites à la propriété de Liouville pour le même p **mais avec a remplacé par** 0. Cette réduction s'appuie sur deux ingrédients :

(i) un changement de variable, qui permet de remplacer le coefficient $|x|^a$ par une fonction régulière qui est bornée dans un domaine approprié ;

(ii) une généralisation d'un argument de doublement et de changement d'échelles de [PQS07a].

Pour les applications du résultat de type Liouville, nous considérons le problème aux limites suivant :

$$\begin{cases} -\Delta u = |x|^a u^p, & x \in \Omega, \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{1.13}$$

avec la fonction positive ou nulle $\varphi \in C(\partial\Omega)$.

Théorème 1.1.3. *Soient $N \ge 2$, $a > -2$, $p > 1$*

(i) Supposons $p < \min(p_S, p_S(a))$. Soient $M > 0$ et $0 \le \varphi \in C(\partial\Omega)$ avec $\|\varphi\|_\infty \le M$. Alors, toutes les solutions positives $u \in C^2(\Omega \setminus \{0\}) \cap C(\overline{\Omega})$ du problème (1.13) satisfont

$$\|u\|_{L^\infty(\Omega)} \le C,$$

où la constante $C > 0$ ne dépend que de Ω, a, p, M.

(ii) Supposons $p \ge p_S(a)$, alors l'affirmation (i) est fausse. Plus précisément, il existe une suite bornée de nombres réels $b_k > 0$ et une suite de solutions u_k de (1.13) avec $\Omega = B_1$ et $\varphi_k \equiv b_k$, telles que $u_k(0) \to \infty$ lorsque $k \to \infty$.

En particulier, pour $a \le 0$, il s'en déduit que l'hypothèse $p < p_S(a)$ dans l'assertion (i) est optimale.

Enfin, nous clarifions aussi certains résultats précédents sur les estimations a priori des solutions pour le problème de Dirichlet correspondant.

1.2 Le système elliptique de Hardy-Hénon (chapitre 3)

Nous nous intéressons au système elliptique semi-linéaire de type Hardy-Hénon

$$\begin{cases} -\Delta u = |x|^a v^p, & x \in \Omega, \\ -\Delta v = |x|^b u^q, & x \in \Omega, \end{cases} \tag{1.14}$$

où $p, q > 0$, $a, b \in \mathbb{R}$ et Ω est un domaine de $\mathbb{R}^N$, $N \geq 3$. Nous considérons les solutions dans la classe suivante

$$C^2(\Omega \setminus \{0\}) \cap C(\Omega). \tag{1.15}$$

Le cas particulier $a = b = 0$ (le système de Lane-Emden) a été étudié par beaucoup de mathématiciens. La conjecture de Lane-Emden dit qu'il n'existe aucune solution positive dans $\Omega = \mathbb{R}^N$ si et seulement si

$$\frac{N}{p+1} + \frac{N}{q+1} > N - 2. \tag{1.16}$$

La conjecture a été démontrée pour les solutions radiales en toute dimension [Mit96, SZ98]. Pour les solutions non radiales en dimension $N \leq 2$, la conjecture est une conséquence d'un résultat de Mitidieri et Pohozaev [MP01] qui est valable pour les sur-solutions. En dimension $N = 3$, elle a été démontrée par Serrin et Zou [SZ96] sous une condition supplémentaire que (u, v) a au plus une croissance polynômiale à l'infini. Cette restriction a ensuite été enlevée par Polacik, Quittner et Souplet [PQS07a] et donc la conjecture est vraie pour $N = 3$. Récemment, cette conjecture a été démontrée pour $N = 4$ par Souplet [Sou09b], et quelques résultats partiels ont aussi été établis pour $N \geq 5$ ([Sou09b, BM02, Lin98, CL09]).

Pour le système général avec $a \neq 0$ ou $b \neq 0$, la propriété de type Liouville a été moins étudiée. La non-existence des sur-solutions a été étudiée dans [AS11, MP01]. Pour les solutions radiales, le résultat optimal suivant est connu [BVG10].

Proposition A. *Soient $a, b > -2$. Alors le système (1.14) n'a aucune solution positive radiale dans $\Omega = \mathbb{R}^N$ si et seulement si*

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} > N - 2. \tag{1.17}$$

L'hyperbole

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} = N - 2 \tag{1.18}$$

joue donc le rôle critique dans le cas radial. Ceci conduit à la conjecture suivante.

Conjecture B. *Soient $a, b > -2$. Alors le système (1.14) n'a aucune solution positive dans $\Omega = \mathbb{R}^N$ si et seulement si (p, q) satisfait (1.17).*

Cette conjecture semble difficile, même pour le cas $a = b = 0$. De plus, dans le cas scalaire $-\Delta u = |x[^a u^p$, la condition optimale pour la non-existence de solution positive dans l'espace entier $\mathbb{R}^N$ n'est pas connue pour $a > 0$ (voir [PS12, Section 1.1]). Cependant, la non-existence des sur-solutions dans $\mathbb{R}^N$ (ou un domaine extérieur) a été complètement établie :

Proposition 1.2.1. *Soient $a, b > -2$ et $N \geq 3$. Si $pq \leq 1$, ou si $pq > 1$ et*

$$\max\left\{\frac{2(p+1) + a + bp}{pq-1}, \frac{2(q+1) + b + aq}{pq-1}\right\} \geq N - 2, \tag{1.19}$$

alors le système (1.14) n'a aucune sur-solution positive dans $\Omega = \mathbb{R}^N$.

La Proposition 1.2.1 a été démontrée par Miditieri et Pohozaev [MP01, Section 18] via la méthode des fonctions-test pour $p, q > 1$. Par la technique d'EDO de Serrin et Zou dans [SZ96], on peut démontrer cette proposition pour tous $p, q > 0$. Récemment, Armstrong et Sirakov [AS11] ont développé un nouveau type de principe du maximum qui donne une démonstration très simple de ce rśultat pour tous $p, q > 0$.

Nous obtenons les résultats de type Liouville suivants :

Théorème 1.2.1. *Soient $a, b > -2$ et $N \geq 3$. Supposons $pq > 1$ et (1.17). Si $N \geq 4$, on suppose que*

$$0 \leq a - b \leq (N-2)(p-q), \tag{1.20}$$

$$\max\{\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\} > N-3. \tag{1.21}$$

Alors le système (1.14) n'a aucune solution bornée positive dans $\Omega = \mathbb{R}^N$.

Théorème 1.2.2. *Soient $a, b > -2$ et $N \geq 3$. Supposons que $pq > 1$, et que (1.16), (1.17) et (1.21) qont vérifiées. Alors le système (1.14) n'a aucune solution positive dans $\Omega = \mathbb{R}^N$.*

En particulier ceci démontre la Conjecture B en dimension $N = 3$ dans la classe des solutions bornées et en dimension $N \leq 4$ avec $a, b \leq 0$, sans aucune hypothèse sur la croissance des solutions :

Corollaire 1.2.1. *(i) Soit $N = 3$. Alors la Conjecture B est vraie dans la classe des solutions bornées.*

(ii) Soit $N = 3$ ou 4 et soient $a, b \leq 0$. Alors la Conjecture B est vraie.

La preuve du Théorème 1.2.1 emploie la technique introduite par Serrin et Zou dans [SZ96] et développée par Souplet dans [Sou09b], elle est basée sur une combinaison d'identités de type Rellich-Pohozaev, une propriété de comparaison unilatérale entre composantes via le principe du maximum, des inégalités de Sobolev et d'interpolation sur S^{N-1}, et des arguments de "feedback" et de mesure. La preuve du Théorème 1.2.2 est basée sur l'estimation de la décroissance à l'infini des solutions et l'identité de type Rellich-Pohozaev.

1.3 L'équation parabolique de Hardy-Hénon (chapitre 4)

Nous étudions l'équation parabolique semi-linéaire de la forme

$$u_t - \Delta u = |x|^a |u|^{p-1} u, \quad (x,t) \in \Omega \times I \tag{1.22}$$

où Ω est un domaine de $\mathbb{R}^N$, $p > 1$, $a > \max\{-2, -N\}$, et I est un intervalle de $\mathbb{R}$.

Comme le modèle type (1.3), cette équation a des propriétés remarquables : la structure de gradient, l'invariance par changement d'échelles, l'existence d'une fonctionnelle d'énergie.

Nous considérons les solutions dans la classe

$$\begin{cases} C^{2,1}(\Omega \times I), & \text{si } a \geq 0, \\ C^{2,1}(\Omega \setminus \{0\} \times I) \cap C^{0,0}(\Omega \times I), & \text{si } a < 0. \end{cases} \tag{1.23}$$

La restriction $a > -2$ est naturelle en raison de la régularité des solutions stationnaires à l'origine pour $N \geq 2$ (cf. [BV00, Lemma 6.2], [DP04, GS81a]). En dimension $N = 1$, il faut supposer en plus que u est solution au sens des distributions (si $a < 0$, ceci ne découle de (1.23) que si $N \geq 2$).

On introduit les exposants suivants :

$$p_S(a) := \begin{cases} \frac{N+2+2a}{N-2} & \text{si} \quad N \geq 3, \\ \infty & \text{si} \quad N = 1, 2, \end{cases} \tag{1.24}$$

et

$$p_B := \begin{cases} \frac{N(N+2)}{(N-1)^2} & \text{si} \quad N \geq 2, \\ \infty & \text{si} \quad N = 1. \end{cases} \tag{1.25}$$

1.3.1 Théorèmes de type Liouville

Dans un premier temps, on s'intéresse à la propriété de type Liouville – i.e. la non-existence de solutions du problème (1.22) dans l'espace entier $\mathbb{R}^N \times \mathbb{R}$. On rappelle que son analogue elliptique est

$$-\Delta u = |x|^a |u|^{p-1} u, \quad x \in \mathbb{R}^N. \tag{1.26}$$

Le résultat de type Liouville pour (1.26) joue un rôle important dans le problème parabolique mais il n'est pas complètement résolu. Il y a une conjecture prédisant que la non-existence de solution positive reste vraie sous la condition $p < p_S(a)$. Cependant, le résultat de type Liouville pour (1.26) a été seulement démontré sous la condition plus forte, à savoir $p < \min\{p_S, p_S(a)\}$, qui n'est pas optimale quand $a > 0$. Récemment, la conjecture a été résolue dans [PS12] pour la solution bornée en dimension $N = 3$.

Pour l'équation parabolique correspondante, la propriété de type Liouville a été étudiée dans le cas particulier $a = 0$ (voir [BPQ11, BV98, PQ06, PQS07b]). Les résultats suivants sont connus.

Théorème A.
(i) Soient $a = 0$ et $1 < p < p_S$. Alors, l'équation (1.22) n'a aucune solution radiale non-triviale non-négative dans $\mathbb{R}^N \times \mathbb{R}$.

(ii) Soient $a = 0$ et $1 < p < p_B$. Alors, l'équation (1.22) n'a aucune solution non-triviale non-négative dans $\mathbb{R}^N \times \mathbb{R}$.

Pour une solution nodale radiale, on note $z_I(u)$ le nombre de changements de signe de fonction radiale $u = u(r)$ dans l'intervalle I.

Théorème B.
(i) Soient $a = 0$, $1 < p < p_S$ et soit $u = u(r,t)$ une solution radiale de (1.22) dans $\mathbb{R}^N \times \mathbb{R}$, telle que le nombre de changements de signe satisfait

$$z_{(0,\infty)}(u(t)) \leq M, \quad \forall t \in \mathbb{R}.$$

Alors $u \equiv 0$.

(ii) Soient $a = 0$, $N = 1$ et soit $u = u(x,t)$ une fonction radiale de (1.22) dans $\mathbb{R} \times \mathbb{R}$, telle que le nombre de changements de signe satisfait

$$z_{\mathbb{R}}(u(t)) \leq M, \quad \forall t \in \mathbb{R}.$$

Alors $u \equiv 0$.

Le Théorème A a été démontré dans [BV98, PQS07b, PQ06], et le Théorème B a été récemment démontré dans [BPQ11]. La borne supérieure sur l'exposant p dans le Théorème A(i) et dans le Théorème B(i) est optimale en raison de l'existence de solutions (bornées) radiales positives de $-\Delta u = |u|^{p-1}u$ dans $\mathbb{R}^N$ pour $p \geq p_S$.

Dans le cas où $a \neq 0$, la propriété de type Liouville a été moins étudiée, même pour les solutions radiales. Jusqu'à présent, le seul résultat de ce type disponible est un théorème de type Fujita (voir [Pin97]), qui dit qu'il n'existe aucune solution positive dans $\mathbb{R}^N \times \mathbb{R}_+$ si et seulement si $1 < p \leq 1 + \frac{2+a}{N}$. Nous établissons les théorèmes de type Liouville dans le cas $a \neq 0$ pour une plage plus grande de valeurs de p. Nous avons les résultats suivants.

Théorème 1.3.1. *(i) Soit $1 < p < \min\{p_B, p_S(a)\}$ et soit u une solution bornée non-négative de l'équation (1.22) dans $\mathbb{R}^N \times \mathbb{R}$. Alors $u \equiv 0$.*

(ii) Soit $1 < p < p_S(a)$ et soit u une solution bornée radiale non-négative de l'équation (1.22) dans $\mathbb{R}^N \times \mathbb{R}$. Alors $u \equiv 0$.

Pour les solutions de signe variable, nous avons le résultat suivant.

Théorème 1.3.2. *Soit $1 < p < p_S(a)$ et $u = u(r,t)$ est une solution radiale de (1.22) dans $\mathbb{R}^N \times \mathbb{R}$, telle que le nombre de changements de signe satisfait*

$$z_{(0,\infty)}(u(t)) \leq M, \quad \forall t \in \mathbb{R}.$$

Alors $u \equiv 0$.

Les preuves des Théorèmes 1.3.1 et 1.3.2 suivent les idées de [BPQ11, AS10, QS11], qui se décomposent en trois étapes :

1. montrer l'estimation de la décroissance à l'infini de la solution (voir Théorème 1.3.3(ii) et Théorème 1.3.4(ii) ci-dessous) ;
2. utiliser la fonctionnelle de Lyapunov et l'estimation de la décroissance à l'infini de la solution pour déduire que les ensembles de α- et ω-limite d'une solution sont non-vides et sont constitués d'états équilibrés ;
3. combiner ce résultat avec le théorème elliptique de type Liouville pour obtenir une contradiction.

Un outil important dans ces preuves est la propriété de la fonctionnelle d'énergie. Après avoir montré l'estimation de la décroissance à l'infini de la solution, on peut définir la fonctionnelle d'énergie par

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(t)|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |x|^a u^{p+1}(t) dx.$$

Et nous avons la propriété suivante :

Quels que soient $t_1 < t_2$,

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N} u_t^2(t) dxdt \leq E(t_1) - E(t_2).$$

On note que la condition $p < p_S(a)$ dans les Théorème 1.3.1(ii) et 1.3.2 est optimale, en raison de l'existence de solutions radiales, bornées positives de $-\Delta u = |x|^a u^p$ dans $\mathbb{R}^N$ pour $p \geq p_S(a)$.

Le Théorème 1.3.1(ii) pour $a > 0$ peut être démontré par une autre méthode, complètement différente, à savoir l'argument d'intersection-comparaison (voir [PQ06]). Pour ce cas, la preuve est totalement similaire à celle dans [PQ06].

Concernant le Théorème 1.3.1, il y a une conjecture prédisant que la non-existence de solution non-négative nontriviale dans $\mathbb{R}^N \times \mathbb{R}$ reste vraie si $p < p_S(a)$. Cependant, cette conjecture semble difficile, même pour le cas particulier $a = 0$.

1.3.2 Les estimations des singularités et de la décroissance à l'infini

Pour le sujet suivant, nous obtenons les estimations des singularités et de la décroissance à l'infini de solutions de l'équation (1.22). Le théorème suivant est un analogue parabolique du Théorème 1.1.2. Les résultats similaires pour $a = 0$ ont été démontrés dans [PQS07b, Theorem 3.1].

Théorème 1.3.3. *(i) Soit u une solution non-négative de (1.22) dans $\Omega \times (0,T)$ où $\Omega = \{0 < |x| < \rho\}$. Supposons que*

$$p < p_B, \qquad \text{ou bien que } u \text{ est radiale.} \tag{1.27}$$

Alors, pour tout $0 < |x| < \rho/2$ et $t \in (0,T)$, on a

$$|x|^{a/(p-1)}u(x,t) + \left||x|^{a/(p-1)}\nabla u(x,t)\right|^{2/(p+1)} \leq C\left(t^{-1/(p-1)} + (T-t)^{-1/(p-1)} + |x|^{-2/(p-1)}\right), \tag{1.28}$$

où $C = C(N,p,a)$.

(ii) Soit u une solution non-négative de (1.22) dans $\Omega \times (0,T)$ où $\Omega = \{|x| > \rho\}$. Supposons que

$$p < p_B, \qquad \text{ou bien que } u \text{ est radiale.} \tag{1.29}$$

Alors, pour tout $|x| > 2\rho$ et $t \in (0,T)$, on a

$$|x|^{a/(p-1)}u(x,t) + \left||x|^{a/(p-1)}\nabla u(x,t)\right|^{2/(p+1)} \leq C\left(t^{-1/(p-1)} + (T-t)^{-1/(p-1)} + |x|^{-2/(p-1)}\right) \tag{1.30}$$

où $C = C(N,p,a)$.

Pour les solutions de signe variable, nous avons le résultat ci-dessous. Nous soulignons qu'il n'y a aucune restriction sur la borne supérieure de l'exposant p.

Théorème 1.3.4. *(i) Soit $u = u(r,t)$ une solution radiale de (1.22) dans $\Omega \times (0,T)$ où $\Omega = \{0 < |x| < \rho\}$, telle que le nombre de changements de signe satisfait*

$$z_{(0,\rho)}(u(t)) \leq M, \quad \forall t \in (0,T).$$

Alors, pour tout $0 < |x| < \rho/2$ et $t \in (0,T)$, on a

$$|x|^{a/(p-1)}|u(x,t)| + \left||x|^{a/(p-1)}\nabla u(x,t)\right|^{\frac{2}{p+1}} \leq C\left(t^{-1/(p-1)} + (T-t)^{-1/(p-1)} + |x|^{-2/(p-1)}\right)$$

où $C = C(N,p,a,M)$.

(ii) Soit $u = u(r,t)$ une solution radiale de (1.22) dans $\Omega \times (0,T)$ où $\Omega = \{|x| > \rho\}$, telle que le nombre de changements de signe satisfait

$$z_{(\rho,\infty)}(u(t)) \leq M, \quad \forall t \in (0,T).$$

Alors, pour tout $|x| > 2\rho$ et $t \in (0,T)$, on a

$$|x|^{a/(p-1)}|u(x,t)| + \left||x|^{a/(p-1)}\nabla u(x,t)\right|^{\frac{2}{p+1}} \leq C\left(t^{-1/(p-1)} + (T-t)^{-1/(p-1)} + |x|^{-2/(p-1)}\right)$$

où $C = C(N,p,a,M)$.

Les preuves des Théorème 1.3.3 et 1.3.4 sont basées sur :

1. un changement de variable, qui permet de remplacer le coefficient $|x|^a$ par une fonction régulière qui est bornée et bornée loin 0 dans un domaine spatial approprié ;
2. une généralisation de l'argument de changement d'échelles et de doublement de [PQS07a].
3. Le théorème de type Liouville correspondant pour l'équation (1.22) avec $a = 0$.

Les estimations du Théorème 1.3.4 dans le cas $a = 0$ donnent une forme similaire à celle dans [MM04, Corollary 3.2] et [MM09, Proposition 2.5 et 2.7] pour les solutions radiales de l'équation de la chaleur non linéaire sur-critique. À la différence de ces travaux, les constantes C sont ici universelles, mais au détriment d'une restriction supplémentaire sur le nombre fini de changements de signe des solutions. Notre argument est basé sur un changement d'échelles et un argument de doublement tandis que celui de [MM04, MM09] est basé sur des estimations d'énergie.

Si l'on remplace l'intervalle $(0, T)$ par $\mathbb{R}$ dans les Théorème 1.3.3(ii) et 1.3.4(ii), alors on obtient une estimation de décroissance spatiale

$$|u(x,t)| \le C|x|^{-(2+a)/(p-1)}, \quad |\nabla u(x,t)| \le C|x|^{-(p+1+a)/(p-1)}, \quad |x| > 0,\ t \in \mathbb{R}.$$

Ceci est une caractéristique importante qui sera utilisée dans la preuve du Théorème 1.3.1 et du Théorème 1.3.2.

1.3.3 L'estimation a priori des solutions globales et l'estimation à l'explosion

Comme application des résutats de type Liouville, on considère le problème aux limites correspondant :

$$\begin{cases} u_t - \Delta u = |x|^a u^p, & x \in \Omega,\ 0 < t < T, \\ u = 0, & x \in \partial\Omega,\ 0 < t < T, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.31}$$

où Ω est un domaine borné régulier de $\mathbb{R}^N$ qui contient l'origine, et $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$. On s'intéresse aux solutions dont l'existence locale et l'unicité sont obtenues par l'argument de point-fixe. D'après la régularité parabolique, les solutions du problème (1.31) sont dans $C^{2,1}(\Omega \setminus \{0\} \times (0,T)) \cap C^{0,0}(\bar{\Omega} \times (0,T))$. Des résultats concernant l'existence locale et globale des solutions de (1.31) ont été obtenus dans [Wan93]). On désigne par u la solution maximale dans $L^\infty_{loc}([0,T); L^\infty(\Omega))$ et $T = T_{max}(u_0)$ son temps maximal d'existence. Nous nous intéressons ici à l'estimation a priori des solutions. Nous avons le résultat suivant.

Théorème 1.3.5. *Soit $1 < p < \min(p_S, p_S(a))$. Supposons que u est une solution globale du problème (1.31) avec donnée initiale $u_0 \ge 0$. Alors,*

$$\sup_{t \ge 0} \|u(t)\|_\infty \le C(\|u_0\|_\infty). \tag{1.32}$$

De plus, si $\Omega = B_R$ et u_0 est radiale, alors (1.32) reste vraie pour $1 < p < p_S(a)$.

On rappelle que la borne a priori des solutions non-négatives du problème elliptique $-\Delta u = |x|^a u^p$ a été obtenue sous la condition $p < \min(p_S, p_S(a))$ (voir le Théorème 1.1.3). Théorème 1.3.5 nous assure que la borne a priori (1.32) pour l'analogue parabolique reste encore vraie sous cette condition. Dans le cas particulier $a = 0$, cette borne a priori a été établie par Giga [Gig86] pour les solutions non négatives, puis par Quittner [Qui99] sans hypothèse de signe.

Nous étudions ensuite les taux d'explosion initiaux et finaux en temps. Le résultat pour $a = 0$ a été établi dans [PQS07b]. L'estimation d'explosion finale en temps pour le problème (1.31) a été démontrée dans [AT05, Theorem 1.2 and 1.3], sous une condition plus forte $1 < p < 1 + \min\{2/N, (2+a)/N\}$.

Théorème 1.3.6. *Soit u une solution positive de (1.31) dans $\Omega \times (0, T)$. Supposons*

$$p < \min\{p_B, p_S(a)\}, \qquad \text{ou bien } p < p_S(a),\ \Omega \text{ est une boule } B_R \text{ et } u \text{ est radiale.}$$

(i) Si $T < \infty$, alors on a

$$u(x,t) \leq C\left(1 + t^{-1/(p-1)} + (T-t)^{-1/(p-1)}\right), \quad x \in \Omega,\ 0 < t < T,$$

où $C = C(\Omega, p, a)$.

(ii) Si u est globale, alors on a

$$u(x,t) \leq C\left(1 + t^{-1/(p-1)}\right), \quad x \in \Omega,\ t > 0,$$

où $C = C(\Omega, p, a)$.

Nous nous intéressons finalement à la borne universelle des solutions non-négatives globales du problème (1.31). Cette borne universelle donne une conclusion moins précise que le Théorème 1.3.6, mais elle peut être appliquée dans une autre zone de paramètres, en raison de la méthode complètement différente. La démonstration de ce théorème est essentiellement basée sur les idées de [Qui01, QSW04].

Théorème 1.3.7. *Soient $a \geq 0$, $N \leq 4$, et $1 < p < \frac{N+2+a}{N-2+a}$. Pour tout $\tau > 0$, il existe $C = C(\Omega, p, a, \tau) > 0$ telle que toute solution globale non-négative du problème (1.31) satisfait*

$$\sup_{t \geq \tau} \|u(t)\|_\infty \leq C(\Omega, p, a, \tau).$$

1.4 Un système parabolique non coopératif (chapitre 5)

Nous étudions le système parabolique non coopératif, semi-linéaire de la forme

$$\begin{cases} u_t - \Delta u = u^p - \beta u^r v^{r+1}, & (x,t) \in \Omega \times I \\ v_t - \Delta v = v^p - \beta u^{r+1} v^r, & (x,t) \in \Omega \times I, \end{cases} \tag{1.33}$$

où $p = 2r+1, r > 0$, $\beta \in \mathbb{R}$, Ω est un domaine de $\mathbb{R}^N$, et I est un intervalle de $\mathbb{R}$.

La partie stationnaire du système 1.33) correspond à un système de type Schrödinger ou Gross-Pitaevskii qui intervient en optique non linéaire ou dans des modèles de condensats de Bose-Einstein (voir [BDW10, DWW10, WW08]). De plus il a été montré dans [WW08] que les estimations a priori pour le système parabolique (1.33) peuvent fournir des informations utiles pour l'étude des solutions stationnaires. Nous nous intéressons à la propriété de type Liouville – i.e. la non-existence de solutions du problème (1.33) dans l'espace entier $\mathbb{R}^N \times \mathbb{R}$. On rappelle tout d'abord son analogue elliptique

$$\begin{cases} -\Delta u = u^p - \beta u^r v^{r+1}, & x \in \mathbb{R}^N \\ -\Delta v = v^p - \beta u^{r+1} v^r, & x \in \mathbb{R}^N. \end{cases} \tag{1.34}$$

Le résultat de type Liouville pour (1.34) joue un rôle important dans le problème parabolique. Pour les solutions radiales, il a été montré dans [QS12a] que le problème (1.34) n'a pas de solution radiale positive si $\beta < 1$ et $p < p_S$, et la condition sur β ou p est optimale. Pour les solutions générales, le problème (1.34) n'a aucune solution positive si $\beta < 1$ et $p < p_S^*$, où $p_S^* = p_S$ si $N \leq 4$, $p_S^* = (N-1)/(N-3)$ si $N \geq 5$ (voir [QS12a]). La condition sur p en dimension $N \leq 4$ est optimale en raison de l'existence de solutions bornées radiales positives de (1.34) si $p \geq p_S$ (et $\beta < 1$).

Pour le problème parabolique correspondant, la propriété de type Liouville est moins bien comprise, même pour les solutions radiales. Récemment, Quittner et Souplet ont prouvé dans [QS11] un théorème de type Liouville pour les solutions radiales dans le cas spécial $p = 3$ et $N \leq 3$. Nous allons établir un théorème optimal de type Liouville du problème (1.34) pour la dimension $N = 1$ et pour les solutions radiales en toute dimension dans la zone sous-critique $p < p_S$.

Rappelons $p = 2r + 1$ et $r > 0$, notre résultat principal est le suivant :

Théorème 1.4.1. *Soient $N = 1$ et $\beta < \frac{r}{3r+2}$. Alors le système (1.33) n'a aucune solution positive dans $\mathbb{R} \times \mathbb{R}$.*

Théorème 1.4.2. *Soient $p < p_S$ et $\beta < \frac{r}{3r+2}$. Alors le système (1.33) n'a aucune solution radiale positive dans $\mathbb{R}^N \times \mathbb{R}$.*

La preuve du Théorème 1.4.1 est basée sur des estimations intégrales combinées avec la formule de Böchner-Wietzenbök [BV98], et celle du Théorème 1.4.2 suit l'idée dans [BPQ11, AS10, QS11]. Notre résultat semble être le premier exemple de théorème de type Liouville dans la zone sous-critique de Sobolev pour un système parabolique [1] (même si l'on se restreint aux solutions radiales). De plus, cela semble également être la première application de la technique de Gidas-Spruck à un système parabolique.

1.5 Un système de type Keller-Segel parabolique issu de la criminologie (chapitre 6)

Nous considérons un système parabolique non-linéaire, fortement couplé provenant d'un modèle issu de la criminologie. Ce système modélise les cambriolages dans les zones résidentielles [Pit10]. Le système est de type chimiotaxique et comporte une fonction logarithmique de sensitivité et des termes d'interaction spécifique et de relaxation.

$$\begin{cases} \dfrac{\partial A}{\partial t} = \eta \Delta A + \psi N A(1-A) + \tilde{A} - A, & x \in \Omega,\ t > 0, \\ \dfrac{\partial N}{\partial t} = \nabla \cdot \Big[\nabla N - N \nabla \vartheta(A) \Big] + \omega - \omega N, & x \in \Omega,\ t > 0, \\ \dfrac{\partial A}{\partial \nu} = \dfrac{\partial N}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ A(x,0) = A_0(x), \quad N(x,0) = N_0(x), & x \in \Omega. \end{cases} \tag{1.35}$$

On note ici

$$\vartheta(A) = \chi \log A, \quad \text{avec } \chi > 0 \text{ une constante.} \tag{1.36}$$

La valeur de χ donnée dans [Pit10] est $\chi = 2$. En plus, on suppose que

$$\Omega \text{ est un domaine borné et suffisamment régulier de } \mathbb{R}^2, \text{ ou bien un carré,} \tag{1.37}$$

$$\eta, \psi, \omega, \tilde{A} > 0 \text{ sont des constantes} \tag{1.38}$$

et que la donnée initiale satisfait

$$(A_0, N_0) \in H^{1+\beta}(\Omega) \times L^2(\Omega) \text{ pour } \beta > 0, \text{ avec } A_0 > 0 \text{ sur } \overline{\Omega} \text{ et } N_0 \geq 0 \text{ dans } \Omega. \tag{1.39}$$

Le vecteur normal extérieur sur $\partial\Omega$ est noté par ν. On peut montrer par des arguments classiques que le problème (1.35) est localement bien posé.

Dans [Pit10], l'auteur obtient des résultats concernant la stabilité linéarisée ou l'instabilité de l'état stable homogène et présente des simulations numériques qui suggèrent l'existence de "hotspots". Cependant, l'existence (locale et) globale de solutions avait été laissée ouverte.

Sous des hypothèses convenables sur les données initiales du problème, nous donnons une preuve rigoureuse de l'existence globale et une borne des solutions du système (1.35), et résolvons ainsi un problème laissé ouvert sur ce modèle.

1. avec l'exception du résultat de [MZ00] qui, pour une classe différente de systèmes, classifie les solutions sur $\mathbb{R}^N \times (-\infty, 0)$ qui décroissent au moins comme $|t|^{-1/(p-1)}$, en vue de l'étude du comportement auto-similaire à l'explosion.

On note

$$\varepsilon_0 = \varepsilon_0(\Omega) = \mu^{-2}K^{-1/2},$$

où μ est la meilleure constante dans l'inégalité de Poincaré-Sobolev

$$\|u - |\Omega|^{-1}\int_\Omega u\,dx\|_2 \le \mu\|\nabla u\|_1, \quad u \in W^{1,1}(\Omega), \tag{1.40}$$

et K est la meilleure constante dans l'estmation d'interpolation

$$\int_\Omega |\nabla u|^4 dx \le K\,\mathrm{osc}^2(u)\int_\Omega |\Delta u|^2 dx. \tag{1.41}$$

Aussi, l'injection de Sobolev entraine que $A_0 \in C(\overline{\Omega})$, et on pose

$$A_{\max} = \max\{1, \tilde{A}, \max_{\overline{\Omega}} A_0\} \ge A_{\min} = \min\{1, \tilde{A}, \min_{\overline{\Omega}} A_0\} > 0.$$

Notre résultat principal est le suivant :

Théorème 1.5.1. *Supposons (1.36), (1.37)-(1.39) et*

$$\Big(\frac{A_{\max}}{A_{\min}}\Big)^2 (A_{\max} - A_{\min}) \max\{\|N_0\|_1, |\Omega|\} < \varepsilon_0 \eta \psi^{-1}\chi^{-2}. \tag{1.42}$$

Alors la solution du problème (1.35) est globale et satisfait la borne uniforme

$$\sup_{t\ge 0} \|A(t)\|_\infty < \infty \tag{1.43}$$

et

$$\sup_{t\ge \tau} \|N(t)\|_\infty < \infty, \qquad \textit{pour tout } \tau > 0. \tag{1.44}$$

La preuve du Théorème 1.5.1 est basée sur la construction d'entropies approchées et sur l'utilisation de diverses inégalités fonctionnelles, combinée avec les propriétés du semi-groupe de la chaleur.

D'autre part, lorsque $\chi \le 1$, en utilisant un argument de [Bil99], nous montrons que la conclusion du Théorème 1.5.1 reste vraie sous une hypothèse simple sur A_0 et $\tilde{A}$ et sans aucune restriction de taille sur N_0. Toutefois, ces techniques ne s'appliquent pas au modèle criminologique donné dans [Pit10] pour lequel $\chi = 2$.

Théorème 1.5.2. *Supposons que (1.36), (1.37)-(1.39) sont vérifiées, avec* $0 < \chi \le 1$, $\tilde{A} \le 1$ *et* $\max_{\overline{\Omega}} A_0 \le 1$. *Alors la solution du problème (1.35) est globale et satisfait la borne uniforme (1.43)-(1.44).*

Pour des applications pratiques, nous fournissons également des conditions explicites numériques pour l'existence globale lorsque le domaine est un carré. La valeur de ε_0 dans le Théorème 1.5.1 peut être estimée de manière explicite.

Théorème 1.5.3. *Soit* $\Omega = (0, L)^2$, *avec* $L > 0$. *Alors le résutat du Théorème 1.5.1 est vrai avec*

$$\varepsilon_0 = \frac{1}{3\sqrt{3}} \approx 0.19.$$

Bibliographie

[AS10] Kaouther Ammar and Philippe Souplet. Liouville-type theorems and universal bounds for nonnegative solutions of the porous medium equation with source. *Discrete Contin. Dyn. Syst.*, 26(2) :665–689, 2010.

[AS11] Scott N. Armstrong and Boyan Sirakov. Nonexistence of positive supersolutions of elliptic equations via the maximum principle. *Comm. Partial Differential Equations*, 36(11) :2011–2047, 2011.

[AT05] Daniele Andreucci and Anatoli F. Tedeev. Universal bounds at the blow-up time for nonlinear parabolic equations. *Adv. Differential Equations*, 10(1) :89–120, 2005.

[BDW10] Thomas Bartsch, Norman Dancer, and Zhi-Qiang Wang. A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system. *Calc. Var. Partial Differential Equations*, 37(3-4) :345–361, 2010.

[Bil99] Piotr Biler. Global solutions to some parabolic-elliptic systems of chemotaxis. *Adv. Math. Sci. Appl*, 20 :347–359, 1999.

[BM98] Isabeau Birindelli and Enzo Mitidieri. Liouville theorems for elliptic inequalities and applications. *Proc. Roy. Soc. Edinburgh Sect. A*, 128(6) :1217–1247, 1998.

[BM02] Jérôme Busca and Raul Manásevich. A Liouville-type theorem for Lane-Emden systems. *Indiana Univ. Math. J.*, 51(1) :37–51, 2002.

[BN10] Henri Berestycki and Jean-Pierre Nadal. Self-organised critical hot spots of criminal activity. *European J. Appl. Math.*, 21(4-5) :371–399, 2010.

[BPQ11] Thomas Bartsch, Peter Poláčik, and Pavol Quittner. Liouville-type theorems and asymptotic behavior of nodal radial solutions of semilinear heat equations. *J. Eur. Math. Soc. (JEMS)*, 13(1) :219–247, 2011.

[BV98] Marie-Françoise Bidaut-Véron. Initial blow-up for the solutions of a semilinear parabolic equation with source term. In *Équations aux dérivées partielles et applications*, pages 189–198. Gauthier-Villars, Éd. Sci. Méd. Elsevier, Paris, 1998.

[BV00] Marie-Francoise Bidaut-Veron. Local behaviour of the solutions of a class of nonlinear elliptic systems. *Adv. Differential Equations*, 5(1-3) :147–192, 2000.

[BVG10] Marie Françoise Bidaut-Veron and Hector Giacomini. A new dynamical approach of Emden-Fowler equations and systems. *Adv. Differential Equations*, 15(11-12) :1033–1082, 2010.

[BVV91] Marie-Françoise Bidaut-Véron and Laurent Véron. Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations. *Invent. Math.*, 106(3) :489–539, 1991.

[CCM12] Robert Stephen Cantrell, Chris Cosner, and Raul Manasevich. Global bifurcation of solutions for crime modeling equations. *SIAM J. Math. Anal.*, (to appear), 2012.

[CL91] Wenxiong Chen and Congming Li. Classification of solutions of some nonlinear elliptic equations. *Duke Math. J.*, 63(3) :615–622, 1991.

[CL09] Wenxiong Chen and Congming Li. An integral system and the Lane-Emden conjecture. *Discrete Contin. Dyn. Syst.*, 24(4) :1167–1184, 2009.

[Cow11] Craig Cowan. A Liouville theorem for a fourth order Hénon equation. *arXiv :1110.2246*, 2011.

[CR10] Marta Calanchi and Bernhard Ruf. Radial and non radial solutions for Hardy-Hénon type elliptic systems. *Calc. Var. Partial Differential Equations*, 38(1-2) :111–133, 2010.

[Dan92] E. Norman Dancer. Some notes on the method of moving planes. *Bull. Austral. Math. Soc.*, 46(3) :425–434, 1992.

[DDG11] E. Norman Dancer, Yihong Du, and Zongming Guo. Finite Morse index solutions of an elliptic equation with supercritical exponent. *J. Differential Equations*, 250(8) :3281–3310, 2011.

[dFF94] Djairo G. de Figueiredo and Patricio L. Felmer. A Liouville-type theorem for elliptic systems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 21(3) :387–397, 1994.

[dFPR08] Djairo G. de Figueiredo, Ireneo Peral, and Julio D. Rossi. The critical hyperbola for a Hamiltonian elliptic system with weights. *Ann. Mat. Pura Appl. (4)*, 187(3) :531–545, 2008.

[dFS05] Djairo G. de Figueiredo and Boyan Sirakov. Liouville type theorems, monotonicity results and a priori bounds for positive solutions of elliptic systems. *Math. Ann.*, 333(2) :231–260, 2005.

[DLY06] Yinbin Deng, Yi Li, and Fen Yang. On the stability of the positive steady states for a nonhomogeneous semilinear Cauchy problem. *J. Differential Equations*, 228(2) :507–529, 2006.

[DP04] Louis Dupaigne and Augusto C. Ponce. Singularities of positive supersolutions in elliptic PDEs. *Selecta Math. (N.S.)*, 10(3) :341–358, 2004.

[DWW10] E. Norman Dancer, Juncheng Wei, and Tobias Weth. A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27(3) :953–969, 2010.

[Esp08] Pierpaolo Esposito. Linear instability of entire solutions for a class of non-autonomous elliptic equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 138(5) :1005–1018, 2008.

[Faz11] Mostafa Fazly. Liouville-type theorems for stable and finite Morse index solutions of a quasi-linear elliptic equation. *arXiv :1109.5142*, 2011.

[FG11] Mostafa Fazly and Nassif Ghoussoub. On the Hénon-Lane-Emden conjecture. *arXiv :1107.5611*, 2011.

[FT00] Stathis Filippas and Achilles Tertikas. On similarity solutions of a heat equation with a nonhomogeneous nonlinearity. *J. Differential Equations*, 165(2) :468–492, 2000.

[Gig86] Yoshikazu Giga. A bound for global solutions of semilinear heat equations. *Comm. Math. Phys.*, 103(3) :415–421, 1986.

[GLS10] Jong-Shenq Guo, Chang-Shou Lin, and Masahiko Shimojo. Blow-up behavior for a parabolic equation with spatially dependent coefficient. *Dynam. Systems Appl.*, 19(3-4) :415–433, 2010.

[GS81a] Basilis Gidas and Joel Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.*, 34(4) :525–598, 1981.

[GS81b] Basilis Gidas and Joel Spruck. A priori bounds for positive solutions of nonlinear elliptic equations. *Comm. Partial Differential Equations*, 6(8) :883–901, 1981.

[GS11] Jong-Shenq Guo and Masahiko Shimojo. Blowing up at zero points of potential for an initial boundary value problem. *Commun. Pure Appl. Anal.*, 10(1) :161–177, 2011.

[GY00] N. Ghoussoub and C. Yuan. Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents. *Trans. Amer. Math. Soc.*, 352(12) :5703–5743, 2000.

[H́73] M Hénon. Numerical experiments on the stability of spherical stellar systems. *Astronom. Astrophys.*, 24 :229–238, 1973.

[Hir08] Munemitsu Hirose. Existence of global solutions for a semilinear parabolic Cauchy problem. *Differential Integral Equations*, 21(7-8) :623–652, 2008.

[Hu96] Bei Hu. Remarks on the blowup estimate for solution of the heat equation with a nonlinear boundary condition. *Differential Integral Equations*, 9(5) :891–901, 1996.

[HY08] Haiyang He and Jianfu Yang. Asymptotic behavior of solutions for Hénon systems with nearly critical exponent. *J. Math. Anal. Appl.*, 347(2) :459–471, 2008.

[Lin98] Chang-Shou Lin. A classification of solutions of a conformally invariant fourth order equation in $\mathbf{R}^n$. *Comment. Math. Helv.*, 73(2) :206–231, 1998.

[LL08] BaiShun Lai and Yi Li. On the semistability of the minimal positive steady state for a nonhomogeneous semilinear Cauchy problem. *Math. Res. Lett.*, 15(5) :923–939, 2008.

[Mit96] Enzo Mitidieri. Nonexistence of positive solutions of semilinear elliptic systems in $\mathbf{R}^N$. *Differential Integral Equations*, 9(3) :465–479, 1996.

[MM04] Hiroshi Matano and Frank Merle. On nonexistence of type II blowup for a supercritical nonlinear heat equation. *Comm. Pure Appl. Math.*, 57(11) :1494–1541, 2004.

[MM09] Hiroshi Matano and Frank Merle. Classification of type I and type II behaviors for a supercritical nonlinear heat equation. *J. Funct. Anal.*, 256(4) :992–1064, 2009.

[MP01] Enzo Mitidieri and Stanislav I. Pohozaev. A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities. *Tr. Mat. Inst. Steklova*, 234 :1–384, 2001.

[MZ98] Frank Merle and Hatem Zaag. Optimal estimates for blowup rate and behavior for nonlinear heat equations. *Comm. Pure Appl. Math.*, 51(2) :139–196, 1998.

[MZ00] Frank Merle and Hatem Zaag. A Liouville theorem for vector-valued nonlinear heat equations and applications. *Math. Ann.*, 316(1) :103–137, 2000.

[Pin97] Ross G. Pinsky. Existence and nonexistence of global solutions for $u_t = \Delta u + a(x)u^p$ in $\mathbf{R}^d$. *J. Differential Equations*, 133(1) :152–177, 1997.

[Pit10] Ashley B. Pitcher. Adding police to a mathematical model of burglary. *European J. Appl. Math.*, 21(4-5) :401–419, 2010.

[PQ05] Peter Poláčik and Pavol Quittner. Liouville type theorems and complete blow-up for indefinite superlinear parabolic equations. In *Nonlinear elliptic and parabolic problems*, volume 64 of *Progr. Nonlinear Differential Equations Appl.*, pages 391–402. Birkhäuser, Basel, 2005.

[PQ06] Peter Poláčik and Pavol Quittner. A Liouville-type theorem and the decay of radial solutions of a semilinear heat equation. *Nonlinear Anal.*, 64(8) :1679–1689, 2006.

[PQS07a] Peter Poláčik, Pavol Quittner, and Philippe Souplet. Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems. *Duke Math. J.*, 139(3) :555–579, 2007.

[PQS07b] Peter Poláčik, Pavol Quittner, and Philippe Souplet. Singularity and decay estimates in superlinear problems via Liouville-type theorems. II. Parabolic equations. *Indiana Univ. Math. J.*, 56(2) :879–908, 2007.

[PS12] Quoc Hung Phan and Philippe Souplet. Liouville-type theorems and bounds of solutions of Hardy-Hénon equations. *J. Differential Equations*, 252(3) :2544–2562, 2012.

[QS11] Pavol Quittner and Philippe Souplet. Parabolic Liouville-type theorems via their elliptic counterparts. *Discrete Contin. Dyn. Syst.*, suppl. :1206–1213, 2011.

[QS12a] Pavol Quittner and Philippe Souplet. Optimal Liouville-type theorems for noncooperative elliptic Schrödinger systems and applications. *Comm. Math. Phys.*, 311(1) :1–19, 2012.

[QS12b] Pavol Quittner and Philippe Souplet. Symmetry of components for semilinear elliptic systems. *SIAM J. Math. Anal.*, 44(4) :2545–2559, 2012.

[QSW04] Pavol Quittner, Philippe Souplet, and Michael Winkler. Initial blow-up rates and universal bounds for nonlinear heat equations. *J. Differential Equations*, 196(2) :316–339, 2004.

[Qui99] Pavol Quittner. A priori bounds for global solutions of a semilinear parabolic problem. *Acta Math. Univ. Comenian. (N.S.)*, 68(2) :195–203, 1999.

[Qui01] Pavol Quittner. Universal bound for global positive solutions of a superlinear parabolic problem. *Math. Ann.*, 320(2) :299–305, 2001.

[RB10] Nancy. Rodriguez and Andrea L. Bertozzi. Local existence and uniqueness of solutions to a pde model for criminal behavior. *Math. Models Methods Appl. Sci.*, 20(suppl.) :1425–1457, 2010.

[RZ00] Wolfgang Reichel and Henghui Zou. Non-existence results for semilinear cooperative elliptic systems via moving spheres. *J. Differential Equations*, 161(1) :219–243, 2000.

[SBB10] Martin B. Short, Andrea L. Bertozzi, and P. Jeffrey Brantingham. Nonlinear patterns in urban crime : hotspots, bifurcations, and suppression. *SIAM J. Appl. Dyn. Syst.*, 9(2) :462–483, 2010.

[SDP+08] Martin B. Short, Maria R. D'Orsogna, Virginia B. Pasour, George E. Tita, P. Jeffrey Brantingham, Andrea L. Bertozzi, and L. B. Chayes. A statistical model of criminal behavior. *Math. Models Methods Appl. Sci.*, 18(suppl.) :1249–1267, 2008.

[Ser64] James Serrin. Local behavior of solutions of quasi-linear equations. *Acta Math.*, 111 :247–302, 1964.

[Ser05] Enrico Serra. Non radial positive solutions for the Hénon equation with critical growth. *Calc. Var. Partial Differential Equations*, 23(3) :301–326, 2005.

[Sou09a] Philippe Souplet. An optimal Liouville-type theorem for radial entire solutions of the porous medium equation with source. *J. Differential Equations*, 246(10) :3980–4005, 2009.

[Sou09b] Philippe Souplet. The proof of the Lane-Emden conjecture in four space dimensions. *Adv. Math.*, 221(5) :1409–1427, 2009.

[SW03] Didier Smets and Michel Willem. Partial symmetry and asymptotic behavior for some elliptic variational problems. *Calc. Var. Partial Differential Equations*, 18(1) :57–75, 2003.

[SWS02] Didier Smets, Michel Willem, and Jiabao Su. Non-radial ground states for the Hénon equation. *Commun. Contemp. Math.*, 4(3) :467–480, 2002.

[SZ96] James Serrin and Henghui Zou. Non-existence of positive solutions of Lane-Emden systems. *Differential Integral Equations*, 9(4) :635–653, 1996.

[SZ98] James Serrin and Henghui Zou. Existence of positive solutions of the Lane-Emden system. *Atti Sem. Mat. Fis. Univ. Modena*, 46(suppl.) :369–380, 1998. Dedicated to Prof. C. Vinti (Italian) (Perugia, 1996).

[SZ02] James Serrin and Henghui Zou. Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities. *Acta Math.*, 189(1) :79–142, 2002.

[Wan93] Xuefeng Wang. On the Cauchy problem for reaction-diffusion equations. *Trans. Amer. Math. Soc.*, 337(2) :549–590, 1993.

[WW08] Juncheng Wei and Tobias Weth. Radial solutions and phase separation in a system of two coupled Schrödinger equations. *Arch. Ration. Mech. Anal.*, 190(1) :83–106, 2008.

[WY12] Chao Wang and Dong Ye. Some Liouville theorems for Hénon type elliptic equations. *J. Funct. Anal.*, 262(4) :1705–1727, 2012.

Chapitre 2

Liouville-type theorems and bounds of solutions of Hardy-Hénon equations

Liouville-type theorems and bounds of solutions of Hardy-Hénon equations [2]

Quoc Hung Phan and Philippe Souplet

Abstract. We consider the Hardy-Hénon equation $-\Delta u = |x|^a u^p$ with $p > 1$ and $a \in \mathbb{R}$ and we are concerned in particular with the Liouville property, i.e. the nonexistence of positive solutions in the whole space $\mathbb{R}^N$. It has been conjectured that this property is true if (and only if) $p < p_S(a)$, where $p_S(a)$ is the Hardy-Sobolev exponent, given by $(N+2+2a)/(N-2)$. However, when $N \geq 3$, the conjecture had up to now been proved only for $a \leq 0$. Indeed the case $a > 0$ seems more difficult, due to $p_S(a) > (N+2)/(N-2)$.

In this paper, we prove the conjecture for $a > 0$ in dimension $N = 3$, in the case of bounded solutions. Next, for the conjecture in the case $a < 0$, and for related estimates near isolated singularities and at infinity, we give new proofs – based in particular on doubling-rescaling arguments – and we provide some extensions of these estimates. These proofs are significantly simpler than the previously known ones. Finally, we clarify some of the previous results on a priori estimates for the related Dirichlet problem.

2.1 Introduction

This article is devoted to the study of positive solutions of the following elliptic equation

$$-\Delta u = |x|^a u^p, \quad x \in \Omega, \tag{2.1}$$

where $p > 1$, $a \in \mathbb{R}$ and Ω is a domain of $\mathbb{R}^N$ with $N \geq 2$. (For the case $N = 1$, see Proposition 2.5.1 and Remark 2.5.2 in Appendix.) Equation (2.1) is traditionally called the Hénon (resp., Hardy, or Lane-Emden) equation for $a > 0$ (resp., $a < 0$, $a = 0$). Throughout this paper, unless otherwise specified, solutions are considered in the class

$$\begin{cases} C^2(\Omega), & \text{if } a \geq 0, \\ C^2(\Omega \setminus \{0\}) \cap C(\Omega), & \text{if } a < 0, \end{cases} \tag{2.2}$$

and are assumed to satisfy the equation pointwise, except at $x = 0$ if $a < 0$ and $0 \in \Omega$.

Our primary interest is in the Liouville property – i.e. the nonexistence of positive solution in the entire space $\Omega = \mathbb{R}^N$ – and on singularity and decay estimates of solutions. The case $a = 0$ has been widely studied by many authors. Here, the optimal Liouville-type result has been established by Gidas and Spruck in their celebrated article [15]. Namely, equation (2.1) has no positive solution if and only if

$$p < p_S := \frac{N+2}{N-2} \qquad (= \infty \text{ if } N \leq 2).$$

The case $a \neq 0$ is less completely understood. Let us first recall that if $a \leq -2$, then (2.1) has no positive solution in any domain Ω containing the origin (cf. [15], [1, Lemma 6.2] and [13]). We therefore restrict ourselves to the case $a > -2$ in the rest of this article. Let us introduce the Hardy-Sobolev exponent

$$p_S(a) := \frac{N+2+2a}{N-2} \qquad (= \infty \text{ if } N = 2).$$

In the case of radial solutions, we have the following complete result (stated in [15]; see [2] for a detailed proof).

Proposition A. *Let $N \geq 2$, $a > -2$ and $p > 1$.*

2. J. Differential Equations, 252(3) : 2544-2562, 2012

(i) If $p < p_S(a)$, then equation (2.1) has no positive radial solution in $\Omega = \mathbb{R}^N$.
(ii) If $p \geq p_S(a)$, then equation (2.1) possesses bounded, positive radial solution in $\Omega = \mathbb{R}^N$.

The Hardy-Sobolev exponent $p_S(a)$ thus plays a critical role in the radial case and this, in addition to the above mentioned result for $a = 0$, supports the following natural conjecture :

Conjecture B. *If $N \geq 2$, $a > -2$ and $1 < p < p_S(a)$, then equation (2.1) has no positive solutions in $\Omega = \mathbb{R}^N$.*

The condition $p < p_S(a)$ is the best possible due to Proposition A(ii). However, apart from the radial case, the best available nonexistence result up to now is the following.

Theorem C. *Let $N \geq 2$, $a > -2$ and $p > 1$.*

(i) If

$$p < \min(p_S, p_S(a)), \tag{2.3}$$

then equation (2.1) has no positive solution in $\Omega = \mathbb{R}^N$.

(ii) The conclusion of part (i) remains true if

$$p \leq \frac{N+a}{N-2}. \tag{2.4}$$

Theorem C(i) was proved in [2, Theorem 1.7]. A more general class of elliptic systems was actually treated in [2] and the result was already contained in [3], although not explicitly stated there. See Remark 2.1.1(c),(d) for a discussion of earlier results in this direction. As for Theorem C(ii), it can be found in e.g. [18, Example 3.2] (see also Remark 2.5.1 below). The proof, based on the rescaled test-function method, is relatively easier than that of part (i). We note that condition (2.4) in Theorem C(ii) becomes better than (2.3) when $a \geq 2$.

Theorem C(i) in particular implies Conjecture B for $a < 0$, since $p_S(a) < p_S$ in this case. **However, the conjecture is still an open problem for $a > 0$.** And indeed, the case $a > 0$ seems more difficult, since then $p_S(a) > p_S$ and classical techniques from [15, 4] (Bochner formula combined with delicate nonlinear multiplier arguments) and from [12] (Kelvin transform combined with moving planes) fail for $p > p_S$.

The first aim of this paper is to give a contribution in this direction. Namely we shall prove Conjecture B for dimension $N = 3$ and $a > 0$ in the class of bounded solutions.

Theorem 2.1.1. *Let $N \geq 2$, $a > 0$, $p > 1$ and $N = 3$. If $p < p_S(a)$, then equation (2.1) has no positive bounded solution in $\Omega = \mathbb{R}^N$.*

Remarks 2.1.1. (a) The proof of Theorem 2.1.1 uses the technique introduced by Serrin and Zou in [23] and further developed by the second author in [26], which is based on a combination of Pohozaev identity, Sobolev inequality on S^{N-1} and a measure argument. By using additional interpolation and feedback arguments from [26], one could extend the result to higher dimensions $N \geq 4$, but at the expense of the further restriction $p < (N-1)/(N-3) \leq p_S$. Therefore, for $N \geq 4$, these techniques do not seem to lead to any improvement of Theorem C(i).

(b) Theorem 2.1.1 is still true for polynomially bounded solutions, i.e. if $u(x) \leq C|x|^q$ for x large, with some $q > 0$ (see after the end of the proof). We note that, although it would be desirable to show Theorem 2.1.1 without any growth restriction on the solutions, Liouville type theorems for bounded solutions are usually sufficient for applications such as a priori estimates and universal bounds, obtained by rescaling arguments (see [16, 20]).

On the other hand, the Liouville property is not true in general if the continuity assumption in (2.2) (at $x = 0$) is relaxed. For instance, (2.1) admits a distributional solution of the form $u(x) = C|x|^{-\alpha}$, $\alpha = (2+a)/(p-1)$, whenever $N \geq 3$, $p > (N+a)/(N-2)$ and $a > -2$. However, Theorem C(ii) (for $p \leq (N+a)/(N-2)$) remains true for distributional supersolutions (see [18] and Remark 2.5.1 below).

(c) Prior to [2], Theorem C(i) had been proved in the special case $a \geq 2$ (with $p < p_S$) in [15, Theorem 4.1]. The restriction $a \geq 2$ comes from the assumption that the x-depending coefficient be a C^2 function.

(d) It is claimed in [16] that the Liouville property is true for

$$a > -2, \quad 1 < p < p_S, \quad p \neq p_S(a) \tag{2.5}$$

(cf. Theorem [16, Theorem 4.2], which is not proved there but attributed for $a < 0$ to ref. 3 in the bibliography of [16], a work which doesn't seem to have actually appeared). However, solutions in Theorem [16, Theorem 4.2] are assumed to be in $C^2(\mathbb{R}^N)$, which is not relevant for $a < 0$. Nevertheless, for solutions in the regularity class (2.2), the Liouville property is false in part of the range (2.5), namely for $-2 < a < 0$, $p_S(a) \leq p < p_S$, as shown by Proposition A(ii).

We now turn to the second topic of this paper, which concerns the closely connected subjet of singularity and decay estimates. Namely, we will present a simpler proof, as well as an extension, of results from [15, 4] for $p < p_S$. By the same token, we will obtain a new and much simpler proof of Theorem C(i), hence in particular of Conjecture B for $a < 0$. Concerning singularity and decay estimates, we have the following :

Theorem 2.1.2. *Let $N \geq 2$, $a > -2$ and $1 < p < p_S$. There exists a constant $C = C(N, p, a) > 0$ such that the following holds.*

(i) Any nonnegative solution of equation (2.1) in $\Omega = \{x \in \mathbb{R}^N;\, 0 < |x| < \rho\}$ ($\rho > 0$) satisfies

$$u(x) \leq C|x|^{-\frac{2+a}{p-1}} \quad \text{and} \quad |\nabla u(x)| \leq C|x|^{-\frac{p+1+a}{p-1}}, \qquad 0 < |x| < \rho/2. \tag{2.6}$$

(ii) Any nonnegative solution of equation (2.1) in $\Omega = \{x \in \mathbb{R}^N;\, |x| > \rho\}$ ($\rho \geq 0$) satisfies

$$u(x) \leq C|x|^{-\frac{2+a}{p-1}} \quad \text{and} \quad |\nabla u(x)| \leq C|x|^{-\frac{p+1+a}{p-1}}, \qquad |x| > 2\rho. \tag{2.7}$$

The first part of estimate (2.6) was proved in [15, Theorem 3.1] and [4, Theorem 6.3] (cf. also [3, Corollary 6.4] for the exterior domain case (ii)). In addition, we also estimate the gradient – a feature that will be used for our proof of Theorem C(i).

Our proof of Theorem 2.1.2 is based on the observation that estimates (2.6) and (2.7) for given p, a can be rather easily reduced to the Liouville property for the same p **but with** a **replaced by** 0.[3] This reduction relies on two ingredients :

(i) a change of variable, that allows to replace the coefficient $|x|^a$ with a smooth function which is bounded and bounded away from 0 in a suitable spatial domain;

(ii) a generalization of a doubling-rescaling argument from [20] (see Lemma 2.2.1 below).

We can then obtain an easy derivation of Theorem C(i) from Theorem 2.1.2, by combining the Pohozaev identity with the decay estimate (2.7). We note that the gradient part of estimate (2.7) is crucial for the proof in order to estimate some of the terms appearing in the Pohozaev identity.

As the third topic of this paper, let us finally consider the associated boundary value problem :

$$\begin{cases} -\Delta u = |x|^a u^p, & x \in \Omega, \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{2.8}$$

Here we assume that

$$\Omega \subset \mathbb{R}^N \text{ is a smoothly bounded domain containing the origin} \tag{2.9}$$

3. Of course, the Liouville property for $p < p_S$ and $a = 0$ is a deep result, but its proof (see [15, 4] and also [21, Chapter 8]) is easier than in the case $a \neq 0$ and, furthermore, an alternative proof [12] by the method of moving planes is known.

and that $\varphi \in C(\partial\Omega)$ is a nonnegative function. It is well-known that Liouville-type results enable one to derive a priori bounds for positive solutions of elliptic Dirichlet problems, via the blow-up method of [16]. In the case of (2.8), this was actually done in [16, Theorem 4.1]. Unfortunately, that statement suffers from shortcomings similar to those mentioned in Remark 2.1.1(d) above. Namely, it is claimed in [16, Theorem 4.1] that, under assumption (2.5), positive C^2 solutions of (2.8) satisfy a uniform a priori bound. However, no such solutions obviously exist when $a < 0$, so that one probably has to interpret this as a statement about positive solutions in the natural class $C^2(\Omega \setminus \{0\}) \cap C(\overline{\Omega})$. But it turns out (see Theorem 2.1.3(ii) below) that such an a priori bound is not true for $-2 < a < 0$, $p_S(a) \le p < p_S$. We thus provide the following corrected version of [16, Theorem 4.1].

Theorem 2.1.3. *Let $N \ge 2$, $a > -2$, $p > 1$ and assume (2.9).*

(i) Assume (2.3). Let $M > 0$ and $0 \le \varphi \in C(\partial\Omega)$ with $\|\varphi\|_\infty \le M$. Then all positive solutions $u \in C^2(\Omega \setminus \{0\}) \cap C(\overline{\Omega})$ of problem (2.8) satisfy

$$\|u\|_{L^\infty(\Omega)} \le C,$$

where the constant $C > 0$ depends only on Ω, a, p, M.

(ii) Assume $p \ge p_S(a)$. Then assertion (i) fails. More precisely, there exists a bounded sequence of real numbers $b_k > 0$ and a sequence of solutions u_k of (2.8) with $\Omega = B_1$ and $\varphi_k \equiv b_k$, such that $u_k(0) \to \infty$ as $k \to \infty$.

In particular for $a \le 0$, it follows that the assumption $p < p_S(a)$ in assertion (i) is optimal.

We close this introduction by mentioning other work related to the boundary value problem (2.8). The existence and non-existence of positive solutions of (2.8), especially for the case $\varphi = 0$, have been studied (see for instance [14, 19, 22], and the references therein). More precisely, if $a < 0$, one obtains the existence of a positive solution in $H_0^1(\Omega)$ provided that $1 < p < p_S(a)$, by using variational methods and Caffarelli-Kohn-Nirenberg estimates (see [8]) ; if $p \ge p_S(a)$, one proves non-existence of nontrivial solutions in starshaped domains as a consequence of a generalized Pohozaev-type identity. If $a \ge 0$, one obtains the existence of a solution for $1 < p < p_S$ by standard variational argument. On the other hand, if Ω is a ball, W.-M. Ni [19] proved the existence of a radial solution in a larger range, namely for $1 < p < p_S(a)$, by using the Mountain Pass Lemma in a space of radial functions. Recently, the question of multiplicity and qualitative properties of solutions for the Hénon equation, such as the symmetry-breaking, have been widely studied. If Ω is a ball and $a > 0$, numerical computation (see [11]) suggested that for some values of the parameter $a > 0$, the ground state solutions (i.e. solutions with minimal energy) are nonradial. It was then confirmed by Smets, Su and Willem (see [25]) that, if $1 < p < p_S$, there exists $a^* > 0$ such that for $a > a^*$, Ni's radial solution is not the ground state solution. Further results on the subcritical Hénon equation such as symmetry properties of solutions and blowup profile of ground states as $a \to \infty$ or $p \to p_S$ can be found in [5, 6, 7, 9, 10, 24].

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 2.1.2 by doubling-rescaling arguments. In Section 3 we provide a simple proof of Theorem C(i), based on the Pohozaev-type identity and on Theorem 2.1.2. Section 4 is devoted to the more delicate proof of Theorem 2.1.1. Finally, in Appendix, we collect the proofs of some results which we use and are more or less known, but whose proofs we prefer to provide, either for completeness, or because we couldn't find a satisfactory proof in the literature. This includes a Pohozaev-type identity and an interpolation lemma. The proof of Theorem 2.1.3, along the lines of [16], is also given there.

Notation. For $R > 0$, we set $B_R = \{x \in \mathbb{R}^N;\ |x| < R\}$. We shall use spherical coordinates $r = |x|$, $\theta = x/|x| \in S^{N-1}$ and write $u = u(r, \theta)$. The derivative $\partial/\partial r = \frac{x}{|x|} \cdot \nabla$ will be denoted by $'$. The surface measures on S^{N-1} and on the sphere $\{x \in \mathbb{R}^N;\ |x| = R\}$, $R > 0$, will be denoted respectively by $d\theta$ and by $d\sigma_R$. For given function $w = w(\theta)$ on S^{N-1} and $1 \le k \le \infty$, we set $\|w\|_k = \|w\|_{L^k(S^{N-1})}$. When no confusion is likely, we shall denote $\|u\|_k = \|u(r, \cdot)\|_k$.

2.2 Singularity and decay estimates

In this Section, we give a relatively simple proof of Theorem 2.1.2. We need the following lemma, which is an extension of Theorem 6.1 in [20]. The main difference with that result is that the estimate is uniform with respect with the (Hölder bounded) coefficient $c(x)$.

Lemma 2.2.1. *Let $N \geq 1$, $1 < p < p_S$ and $\alpha \in (0, 1]$. Let $c \in C^\alpha(\overline{B}_1)$ satisfy*

$$\|c\|_{C^\alpha(\overline{B}_1)} \leq C_1 \qquad \text{and} \qquad c(x) \geq C_2, \quad x \in \overline{B}_1, \tag{2.10}$$

for some constants $C_1, C_2 > 0$. There exists a constant C, depending only on α, C_1, C_2, p, N, such that, for any nonnegative classical solution u of

$$-\Delta u = c(x)u^p, \quad x \in B_1, \tag{2.11}$$

u satisfies

$$|u(x)|^{(p-1)/2} + |\nabla u(x)|^{(p-1)/(p+1)} \leq C\big(1 + \operatorname{dist}^{-1}(x, \partial B_1)\big), \quad x \in B_1.$$

Proof. Arguing by contradiction, we suppose that there exist sequences c_k, u_k verifying (2.10), (2.11) and points y_k, such that the functions

$$M_k = |u_k|^{(p-1)/2} + |\nabla u_k|^{(p-1)/(p+1)}$$

satisfy

$$M_k(y_k) > 2k\big(1 + \operatorname{dist}^{-1}(y_k, \partial B_1)\big) \geq 2k \operatorname{dist}^{-1}(y_k, \partial B_1).$$

By the doubling lemma in [20, Lemma 5.1], there exists x_k such that

$$M_k(x_k) \geq M_k(y_k), \quad M_k(x_k) > 2k \operatorname{dist}^{-1}(x_k, \partial B_1),$$

and

$$M_k(z) \leq 2M_k(x_k), \quad \text{for all } z \text{ such that } |z - x_k| \leq kM_k^{-1}(x_k). \tag{2.12}$$

We have

$$\lambda_k := M_k^{-1}(x_k) \to 0, \quad k \to \infty, \tag{2.13}$$

due to $M_k(x_k) \geq M_k(y_k) > 2k$.

Next we let

$$v_k = \lambda_k^{2/(p-1)} u_k(x_k + \lambda_k y), \qquad \tilde{c}_k(y) = c_k(x_k + \lambda_k y).$$

We note that $|v_k|^{(p-1)/2}(0) + |\nabla v_k|^{(p-1)/(p+1)}(0) = 1$,

$$\Big[|v_k|^{(p-1)/2} + |\nabla v_k|^{(p-1)/(p+1)}\Big](y) \leq 2, \quad |y| \leq k, \tag{2.14}$$

due to (2.12), and we see that v_k satisfies

$$-\Delta v_k = \tilde{c}_k(y)v_k^p, \quad |y| \leq k. \tag{2.15}$$

On the other hand, due to (2.10), we have $C_2 \leq \tilde{c}_k \leq C_1$ and, for each $R > 0$ and $k \geq k_0(R)$ large enough,

$$|\tilde{c}_k(y) - \tilde{c}_k(z)| \leq C_1|\lambda_k(y - z)|^\alpha \leq C_1|y - z|^\alpha, \quad |y|, |z| \leq R. \tag{2.16}$$

Therefore, by Ascoli's theorem, there exists $\tilde{c}$ in $C(\mathbb{R}^N)$, with $\tilde{c} \geq C_2$ such that, after extracting a subsequence, $\tilde{c}_k \to \tilde{c}$ in $C_{loc}(\mathbb{R}^N)$. Moreover, (2.16) and (2.13) imply that $|\tilde{c}_k(y) - \tilde{c}_k(z)| \to 0$ as $k \to \infty$, so that the function $\tilde{c}$ is actually a constant $C > 0$.

Now, for each $R > 0$ and $1 < q < \infty$, by (2.15), (2.14) and interior elliptic L^q estimates, the sequence v_k is uniformly bounded in $W^{2,q}(B_R)$. Using standard imbeddings and interior elliptic

Schauder estimates, after extracting a subsequence, we may assume that $v_k \to v$ in $C^2_{loc}(\mathbb{R}^N)$. It follows that $v \geq 0$ is a classical solution of

$$-\Delta v = Cv^p, \quad y \in \mathbb{R}^N,$$

and $|v|^{(p-1)/2}(0) + |\nabla v|^{(p-1)/(p+1)}(0) = 1$. Since $p < p_S$, this contradicts the Liouville-type result [15, Theorem 1.1] and concludes the proof. □

Proof of Theorem 2.1.2. Assume either $\Omega = \{x \in \mathbb{R}^N;\, 0 < |x| < \rho\}$ and $0 < |x_0| < \rho/2$, or $\Omega = \{x \in \mathbb{R}^N;\, |x| > \rho\}$ and $|x_0| > 2\rho$. We denote

$$R = \tfrac{1}{2}\,|x_0|$$

and observe that, for all $y \in B_1$, $\frac{|x_0|}{2} < |x_0 + Ry| < \frac{3|x_0|}{2}$, so that $x_0 + Ry \in \Omega$ in either case. Let us thus define

$$U(y) = R^{\frac{2+a}{p-1}} u(x_0 + Ry).$$

Then U is a solution of

$$-\Delta U = c(y)U^p, \quad y \in B_1, \qquad \text{with } c(y) = \left|y + \frac{x_0}{R}\right|^a.$$

Notice that $|y + \frac{x_0}{R}| \in [1,3]$ for all $y \in \overline{B}_1$. Moreover $\|c\|_{C^1(\overline{B}_1)} \leq C(a)$. Then applying Lemma 2.2.1, we have $U(0) + |\nabla U(0)| \leq C$, hence

$$u(x_0) \leq CR^{-\frac{2+a}{p-1}}, \quad |\nabla u(x_0)| \leq CR^{-\frac{p+1+a}{p-1}},$$

which yields the desired conclusion. □

Remarks 2.2.1. Lemma 2.2.1 does not hold any longer if the Hölder norm in (2.10) is replaced with the uniform norm, as shown by the following counter-example. Let $N \geq 3$, $N/(N-2) < p < (N+2)/(N-2)$ and $u(x) = (1 + |x|^2)^{-1/(p-1)}$ then

$$-\Delta u = a(x)u^p, \quad x \in \mathbb{R}^N, \qquad \text{with } a(x) = \left(\frac{2N}{p-1} - \frac{4p}{(p-1)^2}\right) + \frac{4p}{(p-1)^2}(1 + |x|^2)^{-1}.$$

Since $p > N/(N-2)$ then $\frac{2N}{p-1} - \frac{4p}{(p-1)^2} > 0$. Thus, $0 < C_2 \leq a(x) \leq C_1$ for all $x \in \mathbb{R}^N$. Let $u_\lambda(y) = \lambda^{2/(p-1)} u(\lambda y)$. Then

$$-\Delta u_\lambda = a_\lambda(y) u_\lambda^p, \quad y \in B(0,1), \qquad \text{with } a_\lambda(y) = a(\lambda y),$$

whereas $u_\lambda(0) = \lambda^{2/(p-1)} \to \infty$ as $\lambda \to \infty$. Therefore, the conclusion of Lemma 2.2.1 fails. In fact, we see that $a_\lambda(y) - a_\lambda(0) = C > 0$ for $|y| = \lambda^{-1}$; consequently, the modulus of continuity of a_λ near 0 is not uniform w.r.t. $\lambda \to \infty$, and in particular assumption (2.10) of Lemma 2.2.1 is not satisfied.

2.3 A simple proof of Theorem C(i)

A basic ingredient to the proof of both Theorems C(i) and 2.1.1 is the following Pohozaev-type identity. It is more or less known, but we give a proof in Appendix for completeness, especially since there is a slight technical difficulty when $a < 0$.

Lemma 2.3.1 (Rellich-Pohozaev identity). *Let $p > 1$, $N \geq 2$, $a > -2$ and let u be a positive solution of (2.1) in $\mathbb{R}^N$. For all $R > 0$, there holds*

$$\left(2\frac{N+a}{p+1} - N + 2\right) \int_{B_R} |x|^a u^{p+1}\, dx$$
$$= \int_{|x|=R} \left(2R^{1+a}\frac{u^{p+1}}{p+1} + 2R^{-1}|x.\nabla u|^2 - R|\nabla u|^2 + (N-2)uu'\right) d\sigma_R. \tag{2.17}$$

Proof of Theorem C(i). Let u be a positive solution of (2.1) and define

$$F(R) = \int_{B_R} |x|^a u^{p+1}\, dx. \tag{2.18}$$

By Rellich-Pohozaev indentity, we have

$$F(R) \leq C\big(G_1(R) + G_2(R)\big), \tag{2.19}$$

where

$$G_1(R) = R^{N+a} \int_{S^{N-1}} u^{p+1}(R,\theta)\, d\theta \tag{2.20}$$

and

$$G_2(R) = R^N \int_{S^{N-1}} \big(|D_x u(R,\theta)|^2 + R^{-2}u^2(R,\theta)\big)\, d\theta. \tag{2.21}$$

Now, by (2.7) in Theorem 2.1.2, we have

$$u(x) \leq C|x|^{-\frac{2+a}{p-1}} \text{ and } |\nabla u(x)| \leq C|x|^{-\frac{p+1+a}{p-1}}, \qquad x \neq 0.$$

Due to $p < p_S(a)$, it follows that

$$G_1(R) + G_2(R) \leq CR^{N-\frac{2(p+1+a)}{p-1}} \to 0, \text{ as } R \to \infty.$$

Therefore, $u \equiv 0$ by (2.19). □

2.4 Proof of Theorem 2.1.1

2.4.1 Functional inequalities and basic estimates

Lemma 2.4.1 (Sobolev inequalities on S^{N-1}). *Let $N \geq 2$, let $j \geq 1$ be integer and $1 < k < \lambda \leq \infty$ satisfy $k \neq (N-1)/j$. For $w = w(\theta) \in W^{j,k}(S^{N-1})$, we have*

$$\|w\|_\lambda \leq C(\|D_\theta^j w\|_k + \|w\|_1),$$

where $C = C(j,k,N) > 0$ and

$$\begin{cases} \frac{1}{k} - \frac{1}{\lambda} = \frac{j}{N-1}, & \text{if } k < (N-1)/j, \\ \lambda = \infty, & \text{if } k > (N-1)/j. \end{cases}$$

See e.g. [23].

Lemma 2.4.2 (Elliptic L^k-estimates on an annulus)**.** *Let $N \geq 2$ and $1 < k < \infty$. For $z = z(x) \in W^{2,k}(B_{2R} \setminus B_{R/4})$ and $R > 0$, we have*

$$\int_{B_R \setminus B_{R/2}} |D_x^2 z|^k \, dx \leq C\Big(\int_{B_{2R} \setminus B_{R/4}} |\Delta z|^k \, dx + R^{-2k} \int_{B_{2R} \setminus B_{R/4}} |z|^k \, dx \Big), \tag{2.22}$$

with $C = C(N, k) > 0$.

Lemma 2.4.3 (An interpolation inequality on an annulus)**.** *Let $N \geq 2$. For $z = z(x) \in W^{2,1}(B_{2R})$ and $R > 0$, we have*

$$\int_{B_R \setminus B_{R/2}} |D_x z| \, dx \leq CR \int_{B_{2R} \setminus B_{R/4}} |\Delta z| \, dx + CR^{-1} \int_{B_{2R} \setminus B_{R/4}} |z| \, dx, \tag{2.23}$$

with $C = C(N) > 0$.

Lemmas 2.4.2 and 2.4.3 follow from the case $R = 1$ and an obvious dilation argument. For $R = 1$, (2.22) is just the standard interior elliptic estimate. As for (2.23) with $R = 1$, this is a variant of an estimate stated without proof in [26, Lemma 2.3]. If the L^1 norms in (2.23) are replaced with L^k norms with $1 < k < \infty$, then it follows from standard elliptic and interpolation inequalities. However for $k = 1$, we could not find a reference in the literature and we therefore provide a proof in Appendix. Note that [26, Lemma 2.3] can be proved by a very similar argument.

The following basic integral estimates for solutions of (2.1) follows from the rescaled test-function method (see [18, Section I.3]). We give a proof in Appendix for completeness.

Lemma 2.4.4. *Let $N \geq 2$, $a > -2$ and u be a positive solution of (2.1) with $\Omega = \mathbb{R}^N$. Then there holds*

$$\int_{B_R} |x|^a u^p \, dx \leq CR^{N-2-\frac{2+a}{p-1}}, \quad R > 0, \tag{2.24}$$

with $C = C(N, p, a) > 0$.

We now deduce the following lemma.

Lemma 2.4.5. *Let $N \geq 2$, $a > -2$ and u be a positive solution of (2.1) with $\Omega = \mathbb{R}^N$. Then, for all $R > 0$, there hold*

$$\int_{B_R \setminus B_{R/2}} u \, dx \leq CR^{N-\frac{2+a}{p-1}}, \tag{2.25}$$

$$\int_{B_R \setminus B_{R/2}} |D_x u| \, dx \leq CR^{N-1-\frac{2+a}{p-1}}, \tag{2.26}$$

$$\int_{B_R} |\Delta u| \, dx \leq CR^{N-2-\frac{2+a}{p-1}}. \tag{2.27}$$

Proof. Estimate (2.27) is just (2.24). Next, by Hölder's inequality and (2.24), we obtain

$$\int_{B_R \setminus B_{R/2}} u \, dx \leq CR^{\frac{N(p-1)}{p}} \Big(\int_{B_R \setminus B_{R/2}} u^p \, dx \Big)^{\frac{1}{p}} \leq CR^{-\frac{Na}{p} + \frac{N(p-1)}{p}} \Big(\int_{B_R} |x|^a u^p \, dx \Big)^{1/p} \leq CR^{N-\frac{2+a}{p-1}},$$

hence (2.25). Finally, adding up estimates (2.25) for $R/2, R$ and $2R$, we obtain (2.25) on $B_{2R} \setminus B_{R/4}$ and this, along with (2.27) and Lemma 2.4.3 yields (2.26). □

2.4.2 Proof of Theorem 2.1.1

The proof consists of 4 steps. Starting from the Pohozaev inequality, which yields formulas (2.18)-(2.21), we shall control the terms $G_1(R), G_2(R)$ suitably for appropriate values of R. For sake of clarity, although here $N = 3$, we shall keep the letter N in the proof. We fix a number $\varepsilon > 0$, which will be ultimately chosen small. In what follows, C denotes any positive constant independent of R (but possibly depending on ε).

Step 1 : *Estimation of $G_1(R)$ and $G_2(R)$ in terms of suitable norms.* Recall that $\|u\|_k$ denotes $\|u(R,\cdot)\|_{L^k(S^{N-1})}$. By Lemma 2.4.1, since $N = 3$, we have

$$\|u\|_{p+1} \le \|u\|_\infty \le C\left(\|D^2_\theta u\|_{1+\varepsilon} + \|u\|_1\right) \le C\left(R^2\|D^2_x u\|_{1+\varepsilon} + \|u\|_1\right)$$

and

$$\|D_x u\|_2 \le C\left(\|D_\theta D_x u\|_{1+\varepsilon} + \|D_x u\|_1\right) \le C\left(R\|D^2_x u\|_{1+\varepsilon} + \|D_x u\|_1\right),$$
$$\|u\|_2 \le \|u\|_\infty \le C\left(R^2\|D^2_x u\|_{1+\varepsilon} + \|u\|_1\right).$$

Therefore,

$$G_1(R) \le CR^{N+a+2(p+1)}\left(\|D^2_x u\|_{1+\varepsilon} + R^{-2}\|u\|_1\right)^{p+1} \tag{2.28}$$

and

$$G_2(R) \le CR^{N+2}(\|D^2_x u\|_{1+\varepsilon} + R^{-1}\|D_x u\|_1 + R^{-2}\|u\|_1)^2. \tag{2.29}$$

Step 2 : *Control of the averages.* For any $R > 1$, we claim that

$$\int_{R/2}^{R} \|u(r)\|_1 r^{N-1}\,dr \le CR^{N-\frac{2+a}{p-1}}, \tag{2.30}$$

$$\int_{R/2}^{R} \|D_x u(r)\|_1 r^{N-1}\,dr \le CR^{N-1-\frac{2+a}{p-1}} \tag{2.31}$$

and

$$\int_{R/2}^{R} \|D^2_x u(r)\|^{1+\varepsilon}_{1+\varepsilon} r^{N-1}\,dr \le CR^{N-2-\frac{2+a}{p-1}+a\varepsilon}. \tag{2.32}$$

Estimates (2.30)-(2.31) follow from (2.25)-(2.26) in Lemma 2.4.5. Next, by using Lemma 2.4.2, equation (2.1), the boundedness of u and (2.27), we obtain

$$\begin{aligned}
\int_{R/2}^{R} \|D^2_x u(r)\|^{1+\varepsilon}_{1+\varepsilon} r^{N-1}\,dr &= \int_{B_R\setminus B_{R/2}} |D^2_x u|^{1+\varepsilon}\,dx \\
&\le C\int_{B_{2R}\setminus B_{R/4}} |\Delta u|^{1+\varepsilon}\,dx + CR^{-2(1+\varepsilon)}\int_{B_{2R}\setminus B_{R/4}} u^{1+\varepsilon}\,dx \\
&\le C\int_{B_{2R}\setminus B_{R/4}} |x|^{a\varepsilon}u^{p\varepsilon}|\Delta u|\,dx + CR^{-2(1+\varepsilon)}\int_{B_{2R}\setminus B_{R/4}} u^{1+\varepsilon}\,dx \\
&\le CR^{a\varepsilon}\int_{B_{2R}\setminus B_{R/4}} |\Delta u| + CR^{-2(1+\varepsilon)}\int_{B_{2R}\setminus B_{R/4}} u\,dx \\
&\le CR^{N-2-\frac{2+a}{p-1}+a\varepsilon} + R^{N-2-\frac{2+a}{p-1}-2\varepsilon} \le CR^{N-2-\frac{2+a}{p-1}+a\varepsilon}.
\end{aligned}$$

Hence (2.32) holds.

Step 3 : *Measure argument.* For a given $K > 0$, let us define the sets

$$\Gamma_1(R) = \{r \in (R, 2R); \|u(r)\|_1 > KR^{-\frac{2+a}{p-1}}\},$$
$$\Gamma_2(R) = \{r \in (R, 2R); \|D_x u(r)\|_1 > KR^{-1-\frac{2+a}{p-1}}\},$$
$$\Gamma_3(R) = \{r \in (R, 2R); \|D_x^2 u(r)\|_{1+\varepsilon}^{1+\varepsilon} > KR^{-2-\frac{2+a}{p-1}+a\varepsilon}\}.$$

By estimate (2.30), for $R > 1$, we have

$$C \geq R^{-N+\frac{2+a}{p-1}} \int_R^{2R} \|u(r)\|_1 r^{N-1}\, dr \geq R^{-N+\frac{2+a}{p-1}} |\Gamma_1(R)| R^{N-1} K R^{-\frac{2+a}{p-1}} = K|\Gamma_1(R)|R^{-1}.$$

Consequently, $|\Gamma_1| \leq R/4$ for $K \geq 4C$. Similarly, from estimates (2.31) and (2.32), we obtain $|\Gamma_2|, |\Gamma_3| \leq R/4$. Therefore, for each $R \geq 1$, we can assert the existence of

$$\tilde{R} \in (R, 2R) \setminus \bigcup_{i=1}^{3} \Gamma_i(R) \neq \emptyset. \tag{2.33}$$

Step 4 : *Conclusion.* If follows from (2.28)-(2.29) in Step 1 and (2.33) in Step 3 that

$$G_1(\tilde{R}) \leq CR^{N+a+2(p+1)} \left(R^{(-2-\frac{2+a}{p-1}+a\varepsilon)/(1+\varepsilon)} + R^{-2-\frac{2+a}{p-1}}\right)^{p+1} \leq C\left(R^{-a_1(\varepsilon)} + R^{-a_1(0)}\right), \tag{2.34}$$

where

$$a_1(\varepsilon) = (p+1)\left[\left(2 + \frac{2+a}{p-1} - a\varepsilon\right)\frac{1}{1+\varepsilon} - 2 - \frac{N+a}{p+1}\right],$$

and

$$G_2(\tilde{R}) \leq CR^{N+2} \left(R^{(-2-\frac{2+a}{p-1}+a\varepsilon)/(1+\varepsilon)} + R^{-2-\frac{2+a}{p-1}}\right)^{2} \leq C\left(R^{-a_2(\varepsilon)} + R^{-a_2(0)}\right), \tag{2.35}$$

where

$$a_2(\varepsilon) = -N - 2 + \frac{2}{1+\varepsilon}\left(2 + \frac{2+a}{p-1} - a\varepsilon\right).$$

Let $\tilde{a} = \min\left(a_1(\varepsilon), a_1(0), a_2(\varepsilon), a_2(0)\right)$. Combining (2.34) and (2.35), we obtain

$$F(R) \leq F(\tilde{R}) \leq CR^{-\tilde{a}}, \quad R \geq 1.$$

By straightforward computation, we see that

$$a_1(0) = a_2(0) = \frac{N + 2 + 2a - (N-2)p}{p-1} > 0,$$

due to $p < p_S(a)$. Therefore, for $\varepsilon > 0$ small enough, we have $\tilde{a} > 0$, so that $\int_{\mathbb{R}^N} |x|^a u^{p+1} = 0$, hence $u \equiv 0$: a contradiction. The proof is complete. □

Finally we note that the above proof still works if, instead of assuming u bounded, one assumes that $u(x) \leq C|x|^q$ for x large, with some $q > 0$. Indeed, estimate (2.32) above can be replaced with

$$\int_{R/2}^{R} \|D_x^2 u(r)\|_{1+\varepsilon}^{1+\varepsilon} r^{N-1}\, dr \leq CR^{N-2-\frac{2+a}{p-1}+(qp+a)\varepsilon}$$

and the rest of proof is similar.

2.5 Appendix : Proof of Lemmas 2.3.1, 2.4.3, 2.4.4, Theorem 2.1.3, and the case $N = 1$.

We start with the following simple Lemma.

Lemma 2.5.1. *Let $N \geq 2$, $a > -2$, $p > 1$, $0 \in \Omega$ and u be a positive solution of (2.1). For any $R > 0$ such that $B_R \subset\subset \Omega$, we have*

$$\int_{B_R} |\nabla u|^2\, dx = \int_{B_R} |x|^a u^{p+1}\, dx + \int_{|x|=R} uu'\, d\sigma_R < \infty. \tag{2.36}$$

In particular,

$$\text{there exists a sequence } \varepsilon_i \to 0^+ \text{ such that } \quad \varepsilon_i \int_{|x|=\varepsilon_i} |\nabla u|^2\, d\sigma_{\varepsilon_i} \to 0. \tag{2.37}$$

Moreover, u is a distributional solution of (2.1).

Proof. If $a \geq 0$, the result is immediate, so we may assume $a < 0$. Recall that solutions are assumed to belong to the class (2.2). For $0 < \rho < R$ such that $B_R \subset\subset \Omega$, we have

$$\begin{aligned}\int_{B_R\setminus B_\rho} |\nabla u|^2\, dx &= -\int_{B_R\setminus B_\rho} u\Delta u\, dx + \int_{|x|=R} uu'\, d\sigma_R - \int_{|x|=\rho} uu'\, d\sigma_\rho \\ &= \int_{B_R\setminus B_\rho} |x|^a u^{p+1}\, dx + \int_{|x|=R} uu'\, d\sigma_R - \int_{|x|=\rho} uu'\, d\sigma_\rho. \end{aligned}\tag{2.38}$$

On the other hand, we have

$$\int_{|x|=\rho} uu'\, d\sigma_\rho = \rho^{N-1} f'(\rho), \qquad \text{where } f(\rho) := \frac{1}{2}\int_{S^{N-1}} u^2(\rho,\theta)\, d\theta.$$

Since $f \in C^1((0,R]) \cap C([0,R])$ due to (2.2), we infer the existence of a sequence $\rho_i \to 0^+$ such that $\lim_{i\to\infty} \rho_i f'(\rho_i) = 0$. Since $N \geq 2$, passing to the limit in (2.38) with $\rho = \rho_i$, we obtain (2.36), where the RHS is finite due to $a > -2 \geq -N$ and (2.2). Since

$$\int_{\varepsilon=0}^{R} \int_{|x|=\varepsilon} |\nabla u|^2\, d\sigma_\varepsilon\, d\varepsilon = \int_{B_R} |\nabla u|^2\, dx,$$

assertion (2.37) follows.

Let now $\varphi \in C_0^\infty(\Omega)$ and denote $\Omega_\varepsilon = \Omega \cap \{|x| > \varepsilon\}$ for $\varepsilon > 0$ small. From (2.1), using Green's formula, we obtain

$$\Big|\int_{\Omega_\varepsilon} |x|^a u^p \varphi dx + \int_{\Omega_\varepsilon} u\Delta\varphi dx\Big| = \Big|-\int_{\Omega_\varepsilon} \varphi\Delta u dx + \int_{\Omega_\varepsilon} u\Delta\varphi dx\Big| = \Big|\int_{|x|=\varepsilon} \varphi' u d\sigma_\varepsilon - \int_{|x|=\varepsilon} u'\varphi d\sigma_\varepsilon\Big|. \tag{2.39}$$

We note that, by (2.37),

$$\int_{|x|=\varepsilon_i} |\nabla u|\, d\sigma_{\varepsilon_i} \leq \left(\varepsilon_i^{N-1} \int_{|x|=\varepsilon_i} |\nabla u|^2\, d\sigma_{\varepsilon_i}\right)^{1/2} \to 0, \quad \text{as } i \to \infty. \tag{2.40}$$

Passing to the limit in (2.39) with $\varepsilon = \varepsilon_i$ and using (2.40) and the continuity of u at 0, we obtain

$$\int_{\Omega} |x|^a u^p \varphi\, dx + \int_{\Omega} u\Delta\varphi\, dx = 0,$$

so that u is a distributional solution of (2.1). □

Proof of Lemma 2.3.1. Since u is a solution of (2.1) then

$$(x.\nabla u)\Delta u = -(x.\nabla u)|x|^a u^p = -\mathrm{div}\left(x|x|^a \frac{u^{p+1}}{p+1}\right) + \frac{N+a}{p+1}|x|^a u^{p+1}.$$

Thus, for $0 < \varepsilon < R$, we have

$$\int\limits_{B_R\setminus B_\varepsilon} (x.\nabla u)\Delta u dx = \frac{N+a}{p+1} \int\limits_{B_R\setminus B_\varepsilon} |x|^a u^{p+1} dx - R^{1+a} \int\limits_{|x|=R} \frac{u^{p+1}}{p+1} d\sigma_R + \varepsilon^{1+a} \int\limits_{|x|=\varepsilon} \frac{u^{p+1}}{p+1} d\sigma_\varepsilon.$$

Letting $\varepsilon \to 0$, using the continuity of u, we obtain

$$\int\limits_{B_R} (x.\nabla u)\Delta u\, dx = \frac{N+a}{p+1} \int\limits_{B_R} |x|^a u^{p+1}\, dx - \int\limits_{|x|=R} R^{1+a} \frac{u^{p+1}}{p+1}\, d\sigma_R. \tag{2.41}$$

Next, by direct computation, we have the following identity

$$\mathrm{div}\left(2(x.\nabla u)\nabla u - x|\nabla u|^2\right) = 2(x.\nabla u)\Delta u - (N-2)|\nabla u|^2. \tag{2.42}$$

It follows that, for $0 < \varepsilon < R$,

$$\int\limits_{B_R\setminus B_\varepsilon} \left(2(x.\nabla u)\Delta u - (N-2)|\nabla u|^2\right) dx$$
$$= \int\limits_{|x|=R} \left(2(x.\nabla u)\nabla u - x|\nabla u|^2\right) \frac{x}{|x|}\, d\sigma_R - \int\limits_{|x|=\varepsilon} \left(2(x.\nabla u)\nabla u - x|\nabla u|^2\right) \frac{x}{|x|}\, d\sigma_\varepsilon.$$

Letting $\varepsilon = \varepsilon_i \to 0$, where ε_i is given by Lemma 2.5.1, we obtain

$$\int\limits_{B_R} \left(2(x.\nabla u)\Delta u - (N-2)|\nabla u|^2\right) dx = \int\limits_{|x|=R} \left(2(x.\nabla u)\nabla u - x|\nabla u|^2\right) \frac{x}{|x|}\, d\sigma_R. \tag{2.43}$$

From (2.41), (2.36) and (2.43) we deduce (2.17). □

Proof of Lemma 2.4.3. As mentioned before, it suffices to consider the case $R = 1$, and we can also assume that u is smooth. For $r > 0$, set $A_r := \{r/4 < |x| < 3r/2\}$ and let ν_r and dS_r respectively denote the outer unit normal and surface measure on ∂A_r. Next we denote by $G_r(x; y)$ the Green kernel of the $-\Delta$ in with Dirichlet boundary conditions. By a simple rescaling argument, we see that $G_r(x; y) = r^{2-N} G_1(r^{-1}x; r^{-1}y)$. Also, we shall denote by $\tilde{x}, \tilde{y}$ the variables for $G_1 = G_1(\tilde{x}, \tilde{y})$.

Let $1/2 < |x| < 1$ and $1 < r < 4/3$. It follows from the Green representation formula that

$$\begin{aligned} u(x) &= -\int_{A_r} \Delta u(y) G_r(x; y)\, dy - \int_{\partial A_r} u(y)\partial_{\nu_r} G_r(x; y)\, dS_r(y), \\ &= -r^{2-N} \int_{A_r} \Delta u(y) G_1(r^{-1}x; r^{-1}y)\, dy - r^{1-N} \int_{\partial A_r} u(y)\, \nu_r \cdot \nabla_{\tilde{y}} G_1(r^{-1}x; r^{-1}y)\, dS_r(y), \end{aligned}$$

hence

$$\begin{aligned} \nabla u(x) = &- r^{1-N} \int_{A_r} \Delta u(y) \nabla_{\tilde{x}} G_1(r^{-1}x; r^{-1}y)\, dy \\ &- r^{-N} \int_{\partial A_r} u(y)\, \nu_r \cdot \nabla_{\tilde{x}} \nabla_{\tilde{y}} G_1(r^{-1}x; r^{-1}y)\, dS_r(y). \end{aligned} \tag{2.44}$$

We now use the estimates $|\nabla_{\tilde{x}} G_1(\tilde{x}, \tilde{y})| \le C|\tilde{x} - \tilde{y}|^{1-N}$ and $|\nabla_{\tilde{x}} \nabla_{\tilde{y}} G_1(\tilde{x}, \tilde{y})| \le C|\tilde{x} - \tilde{y}|^{-N}$ (see e.g. [17]). It follows that

$$|\nabla u(x)| \le C \int_{A_r} |\Delta u(y)|\, |x-y|^{1-N}\, dy + \int_{\partial A_r} |u(y)|\, |x-y|^{-N}\, dS_r(y). \tag{2.45}$$

Now, for $|y| \le 2$, we note that $\int_{1/2<|x|<1} |x-y|^{1-N}\, dx \le \int_{B_3} |z|^{1-N}\, dz < \infty$. Moreover we have $|x-y| > 1/6$ for any $y \in \partial A_r$ (recalling that $1/2 < |x| < 1$ and $1 < r < 4/3$). Combining this with (2.45) and using Fubini's Theorem, we thus obtain, for $1 < r < 4/3$,

$$\begin{aligned}\int_{1/2<|x|<1} |\nabla u(x)|\, dx &\le C \int_{A_r} |\Delta u(y)| \Big(\int_{1/2<|x|<1} |x-y|^{1-N}\, dx \Big)\, dy \\ &\quad + C \int_{\partial A_r} |u(y)| \Big(\int_{1/2<|x|<1} |x-y|^{-N}\, dx \Big)\, dS_r(y) \\ &\le C \int_{A_r} |\Delta u(y)|\, dy + C \int_{\partial A_r} |u(y)|\, dS_r(y).\end{aligned}$$

Integrating over $r \in (1, 4/3)$, we obtain

$$\begin{aligned}\frac{1}{3} \int_{1/2<|x|<1} |\nabla u(x)|\, dx &\le C \int_{r=1}^{4/3} \int_{A_r} |\Delta u(y)|\, dy\, dr + C \int_{r=1}^{4/3} \int_{\partial A_r} |u(y)|\, dS_r(y)\, dr \\ &\le C \int_{1/4<|x|<2} |\Delta u(y)|\, dy + C \int_{1/4<|x|<2} |u(y)|\, dy\end{aligned}$$

and the lemma is proved. □

Proof of Lemma 2.4.4. We use the rescaled test-function method (see e.g. [18]). Fix $\phi \in \mathcal{D}(\mathbb{R}^N)$, $0 \le \phi \le 1$, such that $\phi(x) = 1$ for $|x| \le 1$ and $\phi(x) = 0$ for $|x| > 2$. For each $R > 0$, put $\phi_R(x) = \phi(x/R)$. Let $m = 2p/(p-1) > 2$. We have

$$|\Delta \phi_R^m(x)| = |m\phi_R^{m-1} \Delta \phi_R + m(m-1)\phi_R^{m-2} |\nabla \phi_R|^2| \le CR^{-2} \phi_R^{m-2}.$$

By Lemma 2.5.1 in Appendix, we know that u is a distributional solution. We thus have

$$\int_{\mathbb{R}^N} |x|^a u^p \phi_R^m\, dx = -\int_{\mathbb{R}^N} \Delta u \phi_R^m\, dx = -\int_{\mathbb{R}^N} u \Delta(\phi_R^m)\, dx \le CR^{-2} \int_{R<|x|<2R} u \phi_R^{m-2}\, dx.$$

Now applying the Hölder's inequality, it follows that

$$\int_{\mathbb{R}^N} |x|^a u^p \phi_R^m\, dx \le CR^{\frac{N}{p'}-2} \left(\int_{R<|x|<2R} u^p \phi_R^{p(m-2)}\, dx \right)^{1/p} = CR^{\frac{N}{p'}-2} \left(\int_{R<|x|<2R} u^p \phi_R^m\, dx \right)^{1/p}.$$

Therefore,

$$\int_{\mathbb{R}^N} |x|^a u^p \phi_R^m\, dx \le CR^{\theta} \left(\int_{R<|x|<2R} |x|^a u^p \phi_R^m\, dx \right)^{1/p}, \tag{2.46}$$

with $\theta = \frac{(N-2)(p-1)-(2+a)}{p}$, hence $\int_{\mathbb{R}^N} |x|^a u^p \phi_R^m\, dx \le CR^{p\theta/(p-1)}$ and (2.24) follows. □

Remarks 2.5.1. Recall from [18] that Theorem C(ii) for $N-2-\frac{2+a}{p-1} < 0$ is a direct consequence of estimate (2.24) ; whereas, in case $N-2-\frac{2+a}{p-1} = 0$, (2.24) implies $\int_{\mathbb{R}^N} |x|^a u^p\, dx < \infty$ and, letting $R \to \infty$ in (2.46), we then obtain $\int_{\mathbb{R}^N} |x|^a u^p\, dx = 0$, hence $u \equiv 0$. Note that, by the same token, Theorem C(ii) remains in fact true for distributional supersolutions.

We finally prove Theorem 2.1.3. The proof of assertion (i) is similar to that of [16, Theorem 4.1]. However, due to the problem with that result, mentioned in the paragraph preceding Theorem 2.1.3, we prefer to sketch the proof. Moreover, the proof is facilitated by the availability of the universal bounds in Theorem 2.1.2 (cf. the case $P = 0$ below).

Proof of Theorem 2.1.3. (i) Suppose that assertion (i) is false. Then there exists a sequence of solutions u_k and a sequence of points $P_k \in \Omega$ such that

$$M_k = \sup_{x\in\Omega} u_k(x) = u_k(P_k) \to \infty \text{ as } k \to \infty.$$

We may assume that $P_k \to P \in \overline{\Omega}$ as $k \to \infty$.

Case 1 : $P \in \Omega \setminus \{0\}$ or $P \in \partial\Omega$. We rescale the solution according to

$$U_k(y) = \lambda_k^{\frac{2}{p-1}} u_k(P_k + \lambda_k y), \quad \lambda_k = M_k^{-\frac{p-1}{2}}.$$

Then U_k is a solution of

$$-\Delta U_k = \left|P_k + \lambda_k y\right|^a U_k^p$$

in a rescaled domain, with $0 \le U_k \le 1$ and $U_k(0) = 1$. Using elliptic estimates and standard imbeddings similarly as in [16], we deduce that some subsequence of U_k converges to a solution $v > 0$ of the equation $-\Delta v = \ell v^p$, for some $\ell > 0$, either in $\mathbb{R}^N$, or in a half-space with 0 boundary conditions. Since $p < p_S$, this contradicts one of the Liouville-type results [15, Theorem 1.1] or [16, Theorem 1.3].

Case 2 : $P = 0$. We now rescale the solution according to

$$U_k(y) = \lambda_k^{\frac{2+a}{p-1}} u_k(P_k + \lambda_k y), \quad \lambda_k = M_k^{-\frac{p-1}{2+a}}.$$

Then U_k is a solution of

$$-\Delta U_k = \left|y + \lambda_k^{-1} P_k\right|^a U_k^p$$

in a rescaled domain containing $B(0, \rho\lambda_k^{-1})$ for some $\rho > 0$. Moreover, it follows from estimate (2.6) in Theorem 2.1.2 that the sequence $\lambda_k^{-1}|P_k| = |P_k|\, u_k^{\frac{p-1}{2+a}}(P_k)$ is bounded. We may thus assume that $\lambda_k^{-1} P_k \to x_0 \in \mathbb{R}^N$ as $k \to \infty$. A similar limiting procedure as in Case 1 then produces a positive solution v of

$$-\Delta v = |y + x_0|^a v^p, \quad y \in \mathbb{R}^N. \tag{2.47}$$

More precisely, in the case $-2 < a < 0$, by elliptic regularity (which is applicable since the u_k are distributional solutions in virtue of Lemma 2.5.1), the u_k satisfy a local $W^{2,m}$ bound for $N/2 < m < N/|a|$, hence a local Hölder bound, and this is sufficient to pass to the limit to obtain a solution of (2.47), with $v(\cdot - x_0)$ in the class (2.2).

Since we assumed (2.3), after a space shift, this gives a contradiction with Theorem C(i).

(ii) Assume $p \ge p_S(a)$. Then we know that (2.1) has a bounded, positive radial solution U in $\mathbb{R}^N$ (see [15, Appendix A] and [2]). Moreover, as $r \to \infty$, we have

$$U(r) \sim \frac{C_0}{r^{(2+a)/(p-1)}}, \qquad \text{if } p > p_S(a),$$
$$U(r) \sim C_0 r^{-N+2}, \qquad \text{if } p = p_S(a).$$

For $\lambda > 0$, let

$$U_\lambda(y) = \lambda^{(2+a)/(p-1)} U(\lambda|y|),$$

then $-\Delta U_\lambda = |y|^a U_\lambda^p$ on B_1 and $U_{\lambda|\partial B_1} = \lambda^{(2+a)/(p-1)} U(\lambda) \to C_0$ (resp. 0), as $\lambda \to \infty$, if $p > p_S(a)$ (resp., $p = p_S(a)$). The assertion follows by observing that

$$U_\lambda(0) = \lambda^{(2+a)/(p-1)} \to \infty \text{ as } \lambda \to \infty.$$

□

In the following Proposition, we briefly comment on the very particular case $N = 1$, where somewhat stronger conclusions can be obtained. Namely, we have a nonexistence result which is stronger than Theorem C and Theorem 1.2(ii) becomes void for $N = 1$. Also we get universal bounds which are stronger than in Theorem 1.2(i) and this in turn imposes an a priori restriction on the size of the boundary data $\varphi(\rho)$ in Theorem 1.3.

Proposition 2.5.1. *Assume $N = 1$, $a > -2$ and $p > 1$.*

(i) For any $b \geq 0$, there exist no nontrivial nonnegative C^2 solution of $-u'' \geq |x|^a u^p$ in (b, ∞).

(ii) For any $\rho > 0$, there exists a constant $C = C(\rho, p, a) > 0$ such that any nonnegative C^2 solution of $-u'' = |x|^a u^p$ in $(0, \rho)$ satisfies $u \leq C$ in $(0, \rho)$.

Proof. (i) Assume the contrary. Since $u'' \leq 0$, there exists $\ell = \lim_{x\to\infty} u'(x) \in [-\infty, \infty)$, and necessarily $\ell \geq 0$ due to $u \geq 0$. Therefore, $u' \geq 0$ in $(0, \infty)$, hence $u \geq c > 0$ for $x \geq x_0 > 0$ large enough. For each $R > 1$, define $v_R(x) := u(x_0 + 2R + x)$ for $x \in [-R, R]$. The function v_R satisfies $-v_R'' \geq \tilde{c} R^a v_R$ in $[-R, R]$, with $\tilde{c} > 0$ independent of R. Multiplying this inequality with $\varphi_R(x) := \cos(\pi x/(2R))$, which satisfies $-\varphi_R'' = 2^{-2}\pi^2 R^{-2}\varphi_R$ in $[-R, R]$, $\varphi_R(\pm R) = 0$, $\varphi_R'(R) \leq 0$ and $\varphi_R'(-R) \geq 0$, we obtain

$$\begin{aligned}\tilde{c} R^a \int_{-R}^{R} v_R \varphi_R \, dx &\leq 2^{-2}\pi^2 R^{-2} \int_{-R}^{R} v_R \varphi_R \, dx + \big(v_R \varphi_R'\big)(R) - \big(v_R \varphi_R'\big)(-R) \\ &\leq 2^{-2}\pi^2 R^{-2} \int_{-R}^{R} v_R \varphi_R \, dx,\end{aligned}$$

hence $4\tilde{c} R^{a+2} \leq \pi^2$, which is a contradiction with $a > -2$ for R large.

(ii) First note that, by the proof of Theorem 1.2(i), we have $u(\rho/2) + |u'(\rho/2)| \leq K = K(\rho, p, a)$. Since u is concave, we deduce that

$$\frac{u(x) - u(\rho/2)}{x - (\rho/2)} \leq u'(\rho/2) \leq K, \qquad x \in (\rho/2, \rho)$$

and

$$\frac{u(x) - u(\rho/2)}{x - (\rho/2)} \geq u'(\rho/2) \geq -K, \qquad x \in (0, \rho/2).$$

It follows that $u(x) \leq C := K + K\rho/2$ for $x \in (0, \rho)$. □

Remarks 2.5.2. In the case $a < -2$, there is an important difference between the cases $N = 1$ and $N \geq 2$, since there exist local near 0 – and even global – solutions $u \in C(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$, of the form $u(x) = c|x|^\gamma$, where $\gamma = -(a + 2)/(p - 1) \in (0, 1)$ for $1 < p < -a - 1$. We note that such solutions are not distributional solutions near 0.

Acknowledgement. We thank M.-F. Bidaut-Véron and Dong Ye for interesting discussion and suggestions concerning this work, and the anonymous referee for useful comments.

Bibliographie

[1] Marie-Francoise Bidaut-Veron. Local behaviour of the solutions of a class of nonlinear elliptic systems. *Adv. Differential Equations*, 5(1-3) :147–192, 2000.

[2] Marie Françoise Bidaut-Veron and Hector Giacomini. A new dynamical approach of Emden-Fowler equations and systems. *Adv. Differential Equations*, 15(11-12) :1033–1082, 2010.

[3] Marie-Françoise Bidaut-Véron and Thierry Raoux. Asymptotics of solutions of some nonlinear elliptic systems. *Comm. Partial Differential Equations*, 21(7-8) :1035–1086, 1996.

[4] Marie-Françoise Bidaut-Véron and Laurent Véron. Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations. *Invent. Math.*, 106(3) :489–539, 1991.

[5] Jaeyoung Byeon, Sungwon Cho, and Junsang Park. On the location of a peak point of a least energy solution for Hénon equation. *Discrete Contin. Dyn. Syst.*, 30(4) :1055–1081, 2011.

[6] Jaeyoung Byeon and Zhi-Qiang Wang. On the Hénon equation : asymptotic profile of ground states. II. *J. Differential Equations*, 216(1) :78–108, 2005.

[7] Jaeyoung Byeon and Zhi-Qiang Wang. On the Hénon equation : asymptotic profile of ground states. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23(6) :803–828, 2006.

[8] Luis Caffarelli, Robert V. Kohn, and Louis Nirenberg. First order interpolation inequalities with weights. *Compositio Math.*, 53(3) :259–275, 1984.

[9] Daomin Cao and Shuangjie Peng. The asymptotic behaviour of the ground state solutions for Hénon equation. *J. Math. Anal. Appl.*, 278(1) :1–17, 2003.

[10] Daomin Cao, Shuangjie Peng, and Shusen Yan. Asymptotic behaviour of ground state solutions for the Hénon equation. *IMA J. Appl. Math.*, 74(3) :468–480, 2009.

[11] Goong Chen, Jianxin Zhou, and Wei-Ming Ni. Algorithms and visualization for solutions of nonlinear elliptic equations. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 10(7) :1565–1612, 2000.

[12] Wenxiong Chen and Congming Li. Classification of solutions of some nonlinear elliptic equations. *Duke Math. J.*, 63(3) :615–622, 1991.

[13] Louis Dupaigne and Augusto C. Ponce. Singularities of positive supersolutions in elliptic PDEs. *Selecta Math. (N.S.)*, 10(3) :341–358, 2004.

[14] N. Ghoussoub and C. Yuan. Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents. *Trans. Amer. Math. Soc.*, 352(12) :5703–5743, 2000.

[15] Basilis Gidas and Joel Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.*, 34(4) :525–598, 1981.

[16] Basilis Gidas and Joel Spruck. A priori bounds for positive solutions of nonlinear elliptic equations. *Comm. Partial Differential Equations*, 6(8) :883–901, 1981.

[17] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[18] Enzo Mitidieri and Stanislav I. Pohozaev. A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities. *Tr. Mat. Inst. Steklova*, 234 :1–384, 2001.

[19] Wei Ming Ni. A nonlinear Dirichlet problem on the unit ball and its applications. *Indiana Univ. Math. J.*, 31(6) :801–807, 1982.

[20] Peter Poláčik, Pavol Quittner, and Philippe Souplet. Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems. *Duke Math. J.*, 139(3) :555–579, 2007.

[21] Pavol Quittner and Philippe Souplet. *Superlinear parabolic problems.* Birkhäuser Advanced Texts : Basler Lehrbücher. [Birkhäuser Advanced Texts : Basel Textbooks]. Birkhäuser Verlag, Basel, 2007. Blow-up, global existence and steady states.

[22] Enrico Serra. Non radial positive solutions for the Hénon equation with critical growth. *Calc. Var. Partial Differential Equations*, 23(3) :301–326, 2005.

[23] James Serrin and Henghui Zou. Non-existence of positive solutions of Lane-Emden systems. *Differential Integral Equations*, 9(4) :635–653, 1996.

[24] Didier Smets and Michel Willem. Partial symmetry and asymptotic behavior for some elliptic variational problems. *Calc. Var. Partial Differential Equations*, 18(1) :57–75, 2003.

[25] Didier Smets, Michel Willem, and Jiabao Su. Non-radial ground states for the Hénon equation. *Commun. Contemp. Math.*, 4(3) :467–480, 2002.

[26] Philippe Souplet. The proof of the Lane-Emden conjecture in four space dimensions. *Adv. Math.*, 221(5) :1409–1427, 2009.

Chapitre 3

Liouville-type theorems and bounds of solutions for Hardy-Hénon elliptic systems

Liouville-type theorems and bounds of solutions for Hardy-Hénon elliptic systems [4]

Quoc Hung Phan

Abstract. We consider the Hardy-Hénon system $-\Delta u = |x|^a v^p$, $-\Delta v = |x|^b u^q$ with $p, q > 0$ and $a, b \in \mathbb{R}$ and we are concerned in particular with the Liouville property, i.e. the nonexistence of positive solutions in the whole space $\mathbb{R}^N$. In view of known results, it is a natural conjecture that this property should be true if and only if $(N + a)/(p + 1)+$ $(N + b)/(q + 1) > N - 2$. In this paper, we prove the conjecture for dimension $N = 3$ in the case of bounded solutions and in dimensions $N \leq 4$ when $a, b \leq 0$, among other partial nonexistence results. As far as we know, this is the first optimal Liouville type result for the Hardy-Hénon system. Next, as applications, we give results on singularity and decay estimates as well as a priori bounds of positive solutions.

3.1 Introduction

We study the semilinear elliptic systems of Hardy-Hénon type

$$\begin{cases} -\Delta u = |x|^a v^p, & x \in \Omega, \\ -\Delta v = |x|^b u^q, & x \in \Omega, \end{cases} \tag{3.1}$$

where $p, q > 0$, $a, b \in \mathbb{R}$ and Ω is a domain of $\mathbb{R}^N$, $N \geq 3$.

Throughout this paper, unless otherwise specified, solutions are considered in the class

$$C^2(\Omega \setminus \{0\}) \cap C(\Omega). \tag{3.2}$$

Let us first note that, if $\min\{a, b\} \leq -2$, then (3.1) has no positive solution in class (3.2) in any domain Ω containing the origin [2, Proposition 2.1]. We therefore restrict ourselves to the case $\min\{a, b\} > -2$.

We are interested in the Liouville type theorem- i.e. the nonexistence of positive solution in the entire space $\Omega = \mathbb{R}^N$- and its applications such as a priori bounds and singularity and decay estimates of solutions.

We recall the case $a = b = 0$ of (3.1), the so-called Lane-Emden system, which has been widely studied by many authors. Here, the Lane-Emden conjecture states that there is no positive classical solution in $\Omega = \mathbb{R}^N$ if and only if

$$\frac{N}{p+1} + \frac{N}{q+1} > N - 2. \tag{3.3}$$

This conjecture is known to be true for radial solutions in all dimensions [14, 19]. For non-radial solutions, in dimension $N \leq 2$, the conjecture is a consequence of a result of Mitidieri and Pohozaev [15]. In dimension $N = 3$, it was proved by Serrin and Zou [18] under the additional assumption that (u, v) has at most polynomial growth at ∞. This assumption was then removed by Polacik, Quittner and Souplet [17] and hence the conjecture is true for $N = 3$. Recently, the conjecture was proved for $N = 4$ by Souplet [20], and some partial results were also established for $N \geq 5$ (see [20, 5, 13, 7]).

For the general system with $a \neq 0$ or $b \neq 0$, the Liouville property is less understood. In fact, the nonexistence of **supersolution** has been studied in [1, 15]. The following result is essentially known.

4. Adv. Differential Equations, 17(8-9) : 605-634, 2012

Theorem A. *Let $a, b > -2$ and $N \geq 3$. If $pq \leq 1$, or if $pq > 1$ and*

$$\max\left\{\frac{2(p+1)+a+bp}{pq-1}, \frac{2(q+1)+b+aq}{pq-1}\right\} \geq N-2, \tag{3.4}$$

then system (3.1) has no positive supersolution in $\Omega = \mathbb{R}^N$.

Moreover, it is not difficult to check that condition (3.4) is optimal for supersolutions (consider functions of the form $u(x) = (c_1 + c_2|x|^2)^{-\gamma_1}$, $v(x) = (c_3 + c_4|x|^2)^{-\gamma_2}$). Miditieri and Pohozaev proved Theorem A for $p, q > 1$ by rescaled test-function method (see [15, Section 18]). Theorem A for all $p, q > 0$ can be proved by an argument totally similar to that of Serrin and Zou in [18]. There, the authors treated the special case $a = b = 0$, but this argument still works for the general case $a, b > -2$. However, their proof is rather involved, especially for $p < 1$ or $q < 1$. Very recently, Amstrong and Sirakov [1] developed a new maximum principle type argument which, among other things, allows for a simpler proof of Theorem A for all $p, q > 0$. It follows from the arguments in [1, Section 6]. Also, Theorem A remains true if Ω is an exterior domain.

As usual, it is expected that the optimal range of nonexistence for solutions should be larger than for supersolutions. However, this question seems still difficult, even in the special case $a = b = 0$. Furthermore, even for the scalar equation $-\Delta u = |x|^a u^p$, the optimal condition for nonexistence of positive solution on the whole of $\mathbb{R}^N$ has not been completely settled yet when $a > 0$ (see the recent paper [16] and cf. [10, 3]). Concerning system (3.1), the following optimal result regarding radial solutions is known [3].

Proposition B. *Let $a, b > -2$. Then system (3.1) has no positive radial solution in $\Omega = \mathbb{R}^N$ if and only if*

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} > N-2. \tag{3.5}$$

The hyperbola

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} = N-2. \tag{3.6}$$

thus plays a critical role in the radial case and this, combined with the case of Lane-Emden system, leads to the following conjecture.

Conjecture C. *Let $a, b > -2$. Then system (3.1) has no positive solution in $\Omega = \mathbb{R}^N$ if and only if (p, q) satisfies (3.5).*

In this paper, we prove the conjecture for dimension $N = 3$ in the class of bounded solutions, and for dimensions $N \leq 4$ when $a, b \leq 0$, without any growth assumption (among other partial results in higher dimensions).

From now on, we restrict ourselves to the case $pq > 1$ (cf. Theorem A), and without loss of generality, it will be assumed that

$$p \geq q.$$

Let us denote

$$\alpha = \frac{2(p+1)}{pq-1}, \qquad \beta = \frac{2(q+1)}{pq-1}. \tag{3.7}$$

Then (3.5) is equivalent to

$$\alpha\left(1+\frac{b}{2}\right) + \beta\left(1+\frac{a}{2}\right) > N-2. \tag{3.8}$$

We have obtained the following Liouville type results.

Theorem 3.1.1. *Let $a, b > -2$ and $N \geq 3$. Assume $pq > 1$ and (3.5). If $N \geq 4$, assume in addition that*

$$0 \leq a - b \leq (N-2)(p-q), \tag{3.9}$$

$$\alpha = \max\{\alpha, \beta\} > N - 3. \tag{3.10}$$

Then system (3.1) has no positive bounded solution in $\Omega = \mathbb{R}^N$.

Theorem 3.1.2. *Let $a, b > -2$ and $N \geq 3$. Assume $pq > 1$, $p \geq q$, (3.3), (3.5) and (3.10). Then system (3.1) has no positive solution in $\Omega = \mathbb{R}^N$.*

As an immediate consequence, we obtain Conjecture C in the following special cases.

Corollary 3.1.1. *(i) If $N = 3$, then Conjecture C is true for bounded solutions.* [5]

(ii) If $N = 3$ or 4 and $a, b \leq 0$, then Conjecture C is true.

Remarks 3.1.1. (a) The proof of Theorem 3.1.1 uses the technique introduced by Serrin and Zou in [18] and further developed by Souplet in [20], which is based on a combination of Rellich-Pohozaev identity, a comparison property between components via the maximum principle, Sobolev and interpolation inequality on S^{N-1} and feedback and measure arguments. As for the idea of the proof of Theorem 3.1.2, see after Theorem 3.1.3 below.

(b) Theorem 3.1.1 is still true for polynomially bounded solutions, i.e. if $u(x) \leq C|x|^q$ for x large, with some $q > 0$. This follows from easy modifications of the proof. Let us recall that Liouville type theorems for bounded solutions are usually sufficient for applications such as a priori estimates and universal bounds, obtained by rescaling arguments (see [11, 17]).

(c) For dimension $N \geq 4$, the conclusion of Theorem 3.1.1 still holds if we replace conditions (3.9) and (3.10) by $\min\{\alpha, \beta\} > N - 3$.

(d) If $a + b \leq 2(4-N)/(N-3)$ then condition (3.10) is a consequence of (3.5). Then it follows from Theorem 3.1.2 that Conjecture C is true if $a, b \leq 0$ and $a + b \leq 2(4-N)/(N-3)$.

We next study the strongly related question of singularity and decay estimates for solutions of system (3.1). We have the following theorem.

Theorem 3.1.3. *Let $a, b > -2$ and $N \geq 3$. Assume $pq > 1$, $p \geq q$, (3.3) and (3.10). Then there exists a constant $C = C(N, p, q, a, b) > 0$ such that the following holds.*

(i) Any nonnegative solution of system (3.1) in $\Omega = \{x \in \mathbb{R}^N;\, 0 < |x| < \rho\}$ ($\rho > 0$) satisfies

$$u(x) \leq C|x|^{-\alpha - \frac{a+bp}{pq-1}}, \quad v(x) \leq C|x|^{-\beta - \frac{b+aq}{pq-1}}, \quad 0 < |x| < \rho/2. \tag{3.11}$$

(ii) Any nonnegative solution of system (3.1) in $\Omega = \{x \in \mathbb{R}^N;\, |x| > \rho\}$ ($\rho \geq 0$) satisfies

$$u(x) \leq C|x|^{-\alpha - \frac{a+bp}{pq-1}}, \quad v(x) \leq C|x|^{-\beta - \frac{b+aq}{pq-1}}, \quad |x| > 2\rho. \tag{3.12}$$

The proof of Theorem 3.1.3 is based on :

- a change of variable, that allows to replace the coefficients $|x|^a, |x|^b$ with smooth functions which are bounded and bounded away from 0 in a suitable spatial domain ;

- a generalization of a doubling-rescaling argument from [17] (see Lemma 3.4.1 below) ;

- a known Liouville theorem for the Lane-Emden system [20].

With Theorem 3.1.3 at hand (along with the corresponding decay estimates for the gradients – cf. Proposition 3.4.1 below), one can then deduce Theorem 3.1.2 from the Rellich-Pohozaev identity.

5. After the completion of the present work, we received a preprint [9] by M.Fazly and N.Ghoussoub where they obtain Theorem 3.1.1 in the special case $N = 3$ with $a, b \geq 0$. They also prove interesting result about *solutions with finite Morse index* in the scalar case.

Finally, as an application of our Liouville theorems, we derive a priori bounds of solutions of the following boundary value problem

$$\begin{cases} -\Delta u = |x|^a v^p, & x \in \Omega, \\ -\Delta v = |x|^b u^q, & x \in \Omega, \\ (u,v) = (\varphi, \psi), & x \in \partial\Omega, \end{cases} \tag{3.13}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain containing the origin, $\varphi, \psi \in C(\partial\Omega)$ are nonnegative. For this, we essentially follow the classical blow-up method of Gidas and Spruck [11]. We have the following.

Theorem 3.1.4. *Let φ, ψ be nonegative functions in $C(\partial\Omega)$. Under the assumptions of Theorem 3.1.2, all nonnegative solutions of (3.13) in $C^2(\Omega \setminus \{0\}) \cap C(\overline{\Omega})$ are uniformly bounded.*

The boundary value problem (3.13) has been investigated, especially for the case $\varphi = \psi = 0$, and the existence and non-existence of nonnegative solutions have been established [6, 8]. More precisely, the existence of a nonnegative solution is obtained via variational methods and non-existence of nontrivial solutions in starshaped domains is a consequence of a generalized Pohozaev-type identity. Further results on the asymptotic behavior of solutions for Hénon systems with nearly critical exponent can be found in [12].

Remarks 3.1.2. The conclusions of Theorem 3.1.3 remain true under the assumption that system (3.1) with $a = b = 0$ does not admit positive bounded solution in $\Omega = \mathbb{R}^N$ (instead of (3.3) and (3.10)). As for Theorem 3.1.4, it remains true under the assumptions that both system (3.1) and system (3.1) with $a = b = 0$ do not admit positive bounded solution in $\Omega = \mathbb{R}^N$ (instead of (3.3), (3.5) and (3.10)).

The rest of paper is organized as follows. In Section 2, we recall some functional inequalities, Rellich-Pohozaev identity and prove a comparison property between the two components. Section 3 is devoted to the proof of Theorem 3.1.1. The proof is quite long and involved, and for the sake of clarity, we separate it in two cases : $N \geq 4$ and $N = 3$. Section 4 is devoted to applications of Liouville property, we establish the singularity and decay estimates as well as a priori bound of solutions. The proof of Theorem 3.1.2 is then given in Section 5. Finally, for completeness, we collect in Appendix the proofs of some results which are more or less known.

3.2 Preliminaries

For $R > 0$, we set $B_R = \{x \in \mathbb{R}^N;\ |x| < R\}$. We shall use spherical coordinates $r = |x|$, $\theta = x/|x| \in S^{N-1}$ and write $u = u(r, \theta)$. The surface measures on S^{N-1} and on the sphere $\{x \in \mathbb{R}^N;\ |x| = R\}$, $R > 0$, will be denoted respectively by $d\theta$ and by $d\sigma_R$. For given function $w = w(\theta)$ on S^{N-1} and $1 \leq k \leq \infty$, we set $\|w\|_k = \|w\|_{L^k(S^{N-1})}$. When no confusion is likely, we shall denote $\|u\|_k = \|u(r, \cdot)\|_k$.

3.2.1 Some functional inequalities

Lemma 3.2.1 (Sobolev inequalities on S^{N-1})**.** *Let $N \geq 2, j \geq 1$ is integer and $1 < k < \lambda \leq \infty$, $k \neq (N-1)/j$. For $w = w(\theta) \in W^{j,k}(S^{N-1})$, we have*

$$\|w\|_\lambda \leq C(\|D_\theta^j w\|_k + \|w\|_1)$$

where

$$\begin{cases} \frac{1}{k} - \frac{1}{\lambda} = \frac{j}{N-1}, & \text{if } k < (N-1)/j, \\ \lambda = \infty & \text{if } k > (N-1)/j. \end{cases}$$

and $C = C(j, k, N) > 0$.

See e.g [18].

Lemma 3.2.2 (Elliptic L^p- estimates on an annulus)**.** *Let $1 < k < \infty$. For $R > 0$ and $z = z(x) \in W^{2,k}(B_{2R} \setminus B_{R/4})$, we have*

$$\int_{B_R \backslash B_{R/2}} |D_x^2 z|^k dx \leq C\Big(\int_{B_{2R} \backslash B_{R/4}} |\Delta z|^k dx + R^{-2k} \int_{B_{2R} \backslash B_{R/4}} |z|^k dx\Big),$$

with $C = C(n,k) > 0$.

Lemma 3.2.3 (An interpolation inequality on an annulus)**.** *For $R > 0$ and $z = z(x) \in W^{2,1}(B_{2R} \setminus B_{R/4})$, we have*

$$\int_{B_R \backslash B_{R/2}} |D_x z| dx \leq CR \int_{B_{2R} \backslash B_{R/4}} |\Delta z| dx + CR^{-1} \int_{B_{2R} \backslash B_{R/4}} |z| dx,$$

with $C = C(n) > 0$.

Lemmas 3.2.2 and 3.2.3 follow from the case $R = 1$ and an obvious dilation argument. Lemma 3.2.2 with $R = 1$ is just the standard elliptic estimate. For Lemma 3.2.3 with $R = 1$, see e.g. [16].

3.2.2 Basic estimates, identities and comparison properties

We have the following basic integral estimates for solutions of (3.1).

Lemma 3.2.4. *Let $pq > 1$, $a, b > -2$, $N \geq 3$ and (u,v) be a positive solution of (3.1) in $\Omega = \mathbb{R}^N$. Then there holds*

$$\int_{B_R \backslash B_{R/2}} |x|^a v^p \, dx \leq CR^{N-2-\alpha-\frac{a+bp}{pq-1}}, \qquad \int_{B_R \backslash B_{R/2}} |x|^b u^q dx \leq CR^{N-2-\beta-\frac{b+aq}{pq-1}}, \quad R > 0, \tag{3.14}$$

with $C = C(N,p,q,a,b) > 0$.

A simple proof of Lemma 3.2.4 is given in Appendix, based on ideas from [1]. We now deduce the following lemma.

Lemma 3.2.5. *Let $pq > 1$ $a, b > -2$, $N \geq 3$ and (u,v) solution of (3.1), there hold*

$$\int_{B_R \backslash B_{R/2}} u dx \leq CR^{N-\alpha-\frac{a+bp}{pq-1}}, \qquad \int_{B_R \backslash B_{R/2}} v dx \leq CR^{N-\beta-\frac{b+aq}{pq-1}}, \tag{3.15}$$

$$\int_{B_R \backslash B_{R/2}} |D_x u| dx \leq CR^{N-1-\alpha-\frac{a+bp}{pq-1}}, \qquad \int_{B_R \backslash B_{R/2}} |D_x v| dx \leq CR^{N-1-\beta-\frac{b+aq}{pq-1}}, \tag{3.16}$$

$$\int_{B_R \backslash B_{R/2}} |\Delta u| dx \leq CR^{N-2-\alpha-\frac{a+bp}{pq-1}}, \qquad \int_{B_R \backslash B_{R/2}} |\Delta v| dx \leq CR^{N-2-\beta-\frac{b+aq}{pq-1}}, \tag{3.17}$$

with $C = C(N,p,q,a,b) > 0$.

Proof. Estimates (3.17) is just (3.14). We next prove (3.15). Since $pq > 1$ then we assume $p > 1$. By Hölder's inequality and (3.14), we have

$$\begin{aligned}\int_{B_R\setminus B_{R/2}} v dx &\le CR^{\frac{N(p-1)}{p}}\Big(\int_{B_R\setminus B_{R/2}} v^p dx\Big)^{\frac{1}{p}} \le CR^{\frac{N(p-1)}{p}-\frac{a}{p}}\Big(\int_{B_R\setminus B_{R/2}} |x|^a v^p dx\Big)^{\frac{1}{p}}\\ &\le CR^{\frac{N(p-1)}{p}-\frac{a}{p}+\frac{1}{p}(N-2-\alpha-\frac{a+bp}{pq-1})} = CR^{\frac{1}{p}\left(N(p-1)-a+N-2-\alpha-\frac{a+bp}{pq-1})\right)}\\ &= CR^{N-\beta-\frac{b+aq}{pq-1}}.\end{aligned}$$

If $q \ge 1$ then similarly, we have

$$\int_{B_R\setminus B_{R/2}} u dx \le CR^{N-\alpha-\frac{a+bp}{pq-1}}.$$

If $q < 1$, we follow the idea in [18]. For $\gamma \in (0,1]$, we denote

$$\overline{u^\gamma}(r) = \int_{S^{N-1}} u^\gamma(r,\theta) d\theta.$$

Since $q \in (0,1)$ then $\overline{u^q}(r)$ is non-increasing function of r (see [18, Lemma 2.3]). Hence

$$\begin{aligned}\overline{u^q}(r) &\le 2r^{-1}\int_{r/2}^{r} \overline{u^q}(s) ds \le Cr^{-N-b}\int_{r/2}^{r} s^{N-1+b}\overline{u^q}(s) ds = Cr^{-N-b}\int_{B_r\setminus B_{r/2}} |x|^b u^q dx\\ &\le Cr^{-N-b+N-2-\beta-\frac{b+aq}{pq-1}} = Cr^{-q\left(\alpha+\frac{a+bp}{pq-1}\right)}.\end{aligned}$$

We choose a positive number $\gamma_0 \le q$ and an integer $m_0 \ge 1$ such that

$$\Big(\frac{N}{N-2}\Big)^{m_0}\gamma_0 = 1.$$

Define

$$\gamma_i = \Big(\frac{N}{N-2}\Big)^{i}\gamma_0, \quad i = 0,1,2,...,m_0.$$

By Hölder's inequality, we have

$$\overline{u^{\gamma_0}}(r) \le C\Big(\overline{u^q}(r)\Big)^{\gamma_0/q} \le Cr^{-\gamma_0\left(\alpha+\frac{a+bp}{pq-1}\right)}.$$

Using [18, Lemma 2.5 (ii)] with iteration, we obtain

$$\overline{u^{\gamma_i}}(r) \le Cr^{-\gamma_i\left(\alpha+\frac{a+bp}{pq-1}\right)}, \quad i = 1,2,...,m_0.$$

Taking $i = m_0$ then

$$\overline{u}(r) \le Cr^{-\left(\alpha+\frac{a+bp}{pq-1}\right)},$$

and therefore

$$\int_{B_R\setminus B_{R/2}} u dx \le CR^{N-\alpha-\frac{a+bp}{pq-1}}.$$

Finally, (3.16) is a consequence of (3.15), (3.17) and Lemma 3.2.3. □

The following Rellich-Pohozaev identity plays a key role in the proof of Theorem 3.1.1. It is probably known (see e.g [6]), but we give a proof in Appendix for completeness, especially since there is a slight technical difficulty when $a < 0$ or $b < 0$.

Lemma 3.2.6 (Rellich-Pohozaev identity)**.** *Let $a_1, a_2 \in \mathbb{R}$ satisfy $a_1 + a_2 = N - 2$ and (u, v) solution of (3.1), there holds*

$$\begin{aligned}
&\left(\frac{N+a}{p+1} - a_1\right) \int_{B_R} |x|^a v^{p+1} dx + \left(\frac{N+b}{q+1} - a_2\right) \int_{B_R} |x|^b u^{q+1} dx \\
&= R^{1+b} \int_{|x|=R} \frac{u^{q+1}}{q+1} d\sigma_R + R^{1+a} \int_{|x|=R} \frac{v^{p+1}}{q+1} d\sigma_R \\
&\quad + R \int_{|x|=R} \left(u'v' - \nabla u.\nabla v\right) d\sigma_R + \int_{|x|=R} \left(a_1 u'v + a_2 uv'\right) d\sigma_R
\end{aligned}$$

where $' = \frac{1}{|x|} x.\nabla = \partial/\partial r$.

We next prove an important comparison property for system (3.1) under condition on the difference $a - b$. We follow the ideas of Bidaut-Véron in [2] and Souplet in [20].

Lemma 3.2.7 (Comparison property)**.** *Let $pq > 1$, $N \geq 3$ and (u, v) be a positive solution of (3.1). Assume (3.9). Then*

$$|x|^a \frac{v^{p+1}}{p+1} \leq |x|^b \frac{u^{q+1}}{q+1}. \tag{3.18}$$

Proof. Let $\sigma := (q+1)/(p+1) \in (0, 1]$, $l := \sigma^{-1/(p+1)}$, $h := (a-b)/(p+1)$ and $w := v - l|x|^{-h} u^\sigma$. For all $x \neq 0$, we have

$$\Delta w = \Delta v - l\sigma |x|^{-h} u^{\sigma-1} \Delta u + l|x|^{-h} u^\sigma K,$$

where

$$K = \frac{h(N-2-h)}{|x|^2} + \sigma(1-\sigma)\frac{|\nabla u|^2}{u^2} + 2h\sigma \frac{x}{|x|^2} \cdot \frac{\nabla u}{u}.$$

If $\sigma = 1$ then $h = 0$, thus $K = 0$.

If $\sigma \in (0, 1)$ then it follows from (3.9) that

$$K = h\Big(N - 2 - \frac{h}{1-\sigma}\Big)\frac{1}{|x|^2} + \sigma(1-\sigma)\Big|\frac{\nabla u}{u} + \frac{h}{1-\sigma}\frac{x}{|x|^2}\Big|^2 \geq 0.$$

Hence,

$$\begin{aligned}
\Delta w &\geq \Delta v - l\sigma |x|^{-h} u^{\sigma-1} \Delta u \\
&= |x|^{a-h} u^{\sigma-1} \Big((v/l)^p - \Big(|x|^{-h} u^\sigma\Big)^p \Big).
\end{aligned}$$

It follows that

$$\Delta w \geq 0 \text{ in the set } \{x \in \mathbb{R}^N \setminus \{0\};\ w(x) \geq 0\}. \tag{3.19}$$

If $p \geq 2$, then for any $R > 0$ and $\varepsilon \in (0, R)$, we have

$$\int_{B_R \setminus B_\varepsilon} |\nabla w_+|^2 dx = -\int_{B_R \setminus B_\varepsilon} w_+ \Delta w dx + \int_{|x|=R} w_+ \partial_\nu w \, d\sigma_R + \int_{|x|=\varepsilon} w_+ \partial_\nu w \, d\sigma_\varepsilon$$

Using (3.19), the boundedness of w_+ near $x = 0$, and passing to the limit with $\varepsilon = \varepsilon_i \to 0$, where ε_i is given by Lemma 3.6.1, we deduce that

$$\int_{B_R} |\nabla w_+|^2 dx \leq R^{N-1} \int_{S^{N-1}} w_+(R) w_r(R) d\theta \leq \frac{R^{N-1}}{2} f_1'(R), \tag{3.20}$$

where $f_1(R) := \int_{S^{N-1}} (w_+)^2(R)d\theta$.

On the other hand, let $g(R) = \int_{S^{N-1}} v^p(R)d\theta$ and note that $f_1 \le Cg^{2/p}$. Lemma 3.2.5 guarantees that

$$\int_{R/2}^{R} g(r)r^{N-1+a}dr \le CR^{N-2-\alpha-\frac{a+bp}{pq-1}}.$$

Therefore $g(R_i) \to 0$ for some sequence $R_i \to \infty$. Consequently, $f_1(R_i) \to 0$ and there exists a sequence $\tilde{R}_i \to \infty$ such that $f_1'(\tilde{R}_i) \le 0$. Letting $i \to \infty$ in (3.20) with $R = \tilde{R}_i$, we conclude that w_+ is constant in $\mathbb{R}^N$. If $w = C > 0$ then $v \ge C > 0$ in $\mathbb{R}^N$, contradicting Lemma 3.2.5. Hence, $w_+ = 0$.

If $1 < p < 2$, then for any $R > 0$, $\varepsilon \in (0, R)$ and $\eta > 0$, we have

$$(p-1)\int_{B_R\setminus B_\varepsilon} (w_+ + \eta)^{p-2}|\nabla w_+|^2 dx = -\int_{B_R\setminus B_\varepsilon} (w_+ + \eta)^{p-1}\Delta w dx$$
$$+ \int_{|x|=R} (w_+ + \eta)^{p-1}\partial_\nu w\, d\sigma_R + \int_{|x|=\varepsilon} (w_+ + \eta)^{p-1}\partial_\nu w\, d\sigma_\varepsilon.$$

Letting $\eta \to 0$ (passing to the limit in the LHS via monotone convergence) and using (3.19), it follows that

$$(p-1)\int_{B_R\setminus B_\varepsilon} w_+^{p-2}|\nabla w_+|^2 dx \le \int_{|x|=R} w_+^{p-1}\partial_\nu w\, d\sigma_R + \int_{|x|=\varepsilon} w_+^{p-1}\partial_\nu w\, d\sigma_\varepsilon.$$

Next passing to the limit with $\varepsilon = \varepsilon_i \to 0$, where ε_i is given by Lemma 3.6.1, we deduce that

$$(p-1)\int_{B_R} w_+^{p-2}|\nabla w_+|^2 dx \le R^{N-1}\int_{S^{N-1}} w_+^{p-1}(R)w_r(R)d\theta \le \frac{R^{N-1}}{p} f_2'(R), \tag{3.21}$$

where $f_2(R) := \int_{S^{N-1}} (w_+)^p(R)d\theta$. Using $f_2 \le g$ and arguing as above, we have $w_+ = 0$. □

3.3 Proof of Theorem 3.1.1

We first prove the theorem for dimension $N \ge 4$. The proof consists of 6 steps similar to those in [20]. We repeat these steps in detail for completeness and because of the additional technicalities introduced by the coefficient $|x|^a$, $|x|^b$. Suppose that there exists a positive solution (u, v) of (3.1) in $\mathbb{R}^N$.

Step 1 : *Preparations.* Let us choose a_1, a_2 such that $a_1 + a_2 = N - 2$ and

$$\frac{N+a}{p+1} > a_1, \ \frac{N+b}{q+1} > a_2 \tag{3.22}$$

and set $F(R) = \int_{B_R} |x|^b u^{q+1}$. By the Rellich-Pohozaev identity (Lemma 3.2.6) and the comparison property (3.18), we have

$$F(R) \le C\big(G_1(R) + G_2(R)\big),$$

where

$$G_1(R) = R^{N+b}\int_{S^{N-1}} u^{q+1}(R)d\theta, \tag{3.23}$$

$$G_2(R) = R^N \int_{S^{N-1}} \big(|D_x u(R)| + R^{-1}u(R)\big)\big(|D_x v(R)| + R^{-1}v(R)\big)\, d\theta. \tag{3.24}$$

We may assume that

$$p \geq \frac{N+2}{N-2}. \tag{3.25}$$

In fact, if $q \leq p < \frac{N+2}{N-2}$, then we may apply Theorem 3.1.2 (which will be proved independently of Theorem 3.1.1 in Section 5).

Step 2 : *Estimation of* $G_1(R)$. Let

$$\lambda = \frac{N-1}{N-3},\ k = \frac{p+1}{p} \quad \text{and} \quad \varepsilon > 0. \tag{3.26}$$

(The number ε will be ultimately chosen small; in what follows, the constant C may depend on ε.) By the Lemma 3.2.1, we have

$$\|u\|_\lambda \leq C\left(\|D_\theta^2 u\|_{1+\varepsilon} + \|u\|_1\right) \leq C\left(R^2\|D_x^2 u\|_{1+\varepsilon} + \|u\|_1\right).$$

We show that

$$\frac{1}{k} - \frac{1}{q+1} < \frac{2}{N-1}. \tag{3.27}$$

Indeed

$$\frac{1}{k} - \frac{1}{q+1} = \frac{pq-1}{(p+1)(q+1)} = \frac{2}{(p+1)\beta} = \frac{2}{\alpha+\beta+2} < \frac{2}{N-1}.$$

On the other hand, from (3.25), there exists $\mu > 0$ such that

$$\frac{1}{k} - \frac{1}{\mu} = \frac{2}{N-1}.$$

It follows from (3.27) that $\mu > q+1$. By Lemma 3.2.1, we have

$$\|u\|_\mu \leq C\left(\|D_\theta^2 u\|_k + \|u\|_1\right) \leq C\left(R^2\|D_x^2 u\|_k + \|u\|_1\right).$$

If $\lambda < q+1$ then

$$\|u\|_{q+1} \leq \|u\|_\lambda^\nu \|u\|_\mu^{1-\nu} \leq C\left(R^2\|D_x^2 u\|_{1+\varepsilon} + \|u\|_1\right)^\nu \left(R^2\|D_x^2 u\|_k + \|u\|_1\right)^{1-\nu}. \tag{3.28}$$

If $\lambda \geq q+1$ then (3.28) is still valid with $\nu = 1$. In both cases, we see that ν is given by

$$\nu = 1-(p+1)A,\ \text{with } A = \left(\frac{N-3}{N-1} - \frac{1}{q+1}\right)_+. \tag{3.29}$$

Therefore,

$$\left[R^{-N-b}G_1(R)\right]^{1/q+1} \leq CR^2\left(\|D_x^2 u\|_{1+\varepsilon} + R^{-2}\|u\|_1\right)^\nu \left(\|D_x^2 u\|_k + R^{-2}\|u\|_1\right)^{1-\nu}. \tag{3.30}$$

Step 3 : *Estimation of* $G_2(R)$. Let

$$m = \frac{q+1}{q}, \quad \rho = \frac{N-1}{N-2}.$$

By Lemma 3.2.1, we have

$$\|D_x u\|_\rho \leq C\left(\|D_\theta D_x u\|_{1+\varepsilon} + \|D_x u\|_1\right) \leq C\left(R\|D_x^2 u\|_{1+\varepsilon} + \|D_x u\|_1\right). \tag{3.31}$$

Case 1. $q > 1/(N-2)$. Let γ_1, γ_2 be defined by

$$\frac{1}{\gamma_1} = \frac{p}{p+1} - \frac{1}{N-1}, \quad \frac{1}{\gamma_2} = \frac{q}{q+1} - \frac{1}{N-1}.$$

Then we have

$$k < \gamma_1 < \infty, \quad m < \gamma_2 < \infty.$$

Assume that we can find $z \in (1, \infty)$ such that

$$\frac{1}{k} - \frac{1}{N-1} \le \frac{1}{z} \le 1 - \frac{1}{N-1} \tag{3.32}$$

and

$$\frac{1}{m} - \frac{1}{N-1} \le 1 - \frac{1}{z} \le 1 - \frac{1}{N-1}. \tag{3.33}$$

By the same estimate as in [20], we have

$$\begin{aligned} G_2(R) \le CR^{N+2} &\left(\|D_x^2 u\|_{1+\varepsilon} + R^{-1}\|D_x u\|_1 + R^{-2}\|u\|_1\right)^{\tau_1} \\ &\times \left(\|D_x^2 u\|_k + R^{-1}\|D_x u\|_1 + R^{-2}\|u\|_1\right)^{1-\tau_1} \\ &\times \left(\|D_x^2 v\|_{1+\varepsilon} + R^{-1}\|D_x v\|_1 + R^{-2}\|v\|_1\right)^{\tau_2} \\ &\times \left(\|D_x^2 v\|_m + R^{-1}\|D_x v\|_1 + R^{-2}\|v\|_1\right)^{1-\tau_2}. \end{aligned} \tag{3.34}$$

where

$$\tau_1 = 1 - (p+1)A_1, \quad A_1 = \frac{N-2}{N-1} - \frac{1}{z},$$
$$\tau_2 = 1 - (q+1)A_2, \quad A_2 = \frac{1}{z} - \frac{1}{N-1}.$$

Case 2. $q \le 1/(N-2)$. Then (3.34) remains true with $\tau_1 = 1, \tau_2 = 0$.

Step 4 : *Control the averages.* For any $R > 1$ we claim that

$$\begin{cases} \int_{R/2}^R \|u(r)\|_1 r^{N-1} dr \le CR^{N-\alpha-\frac{a+bp}{pq-1}}, \\ \int_{R/2}^R \|v(r)\|_1 r^{N-1} dr \le CR^{N-\beta-\frac{b+aq}{pq-1}}, \end{cases} \tag{3.35}$$

$$\begin{cases} \int_{R/2}^R \|D_x u(r)\|_1 r^{N-1} dr \le CR^{N-1-\alpha-\frac{a+bp}{pq-1}}, \\ \int_{R/2}^R \|D_x v(r)\|_1 r^{N-1} dr \le CR^{N-1-\beta-\frac{b+aq}{pq-1}}, \end{cases} \tag{3.36}$$

$$\int_{R/2}^R \|D_x^2 u(r)\|_k^k r^{N-1} dr \le CR^{\frac{a}{p}} F(2R), \tag{3.37}$$

$$\int_{R/2}^R \|D_x^2 v(r)\|_m^m r^{N-1} dr \le CR^{\frac{b}{q}} F(2R), \tag{3.38}$$

$$\begin{cases} \int_{R/2}^R \|D_x^2 u(r)\|_{1+\varepsilon}^{1+\varepsilon} r^{N-1} dr \le CR^{N-2-\alpha-\frac{a+bp}{pq-1}+a\varepsilon}, \\ \int_{R/2}^R \|D_x^2 v(r)\|_{1+\varepsilon}^{1+\varepsilon} r^{N-1} dr \le CR^{N-2-\beta-\frac{b+aq}{pq-1}+b\varepsilon}, \end{cases} \tag{3.39}$$

Estimates (3.35) and (3.36) follow from Lemma 3.2.5. Let us next prove (3.37), (3.38) and (3.39). Indeed,

$$\begin{aligned}\int_{R/2}^{R} \|D_x^2 u(r)\|_k^k r^{N-1} dr &= \int_{B_R\setminus B_{R/2}} |D_x^2 u|^k dx \\ &\leq C\Big(\int_{B_{2R}\setminus B_{R/4}} |\Delta u|^k dx + R^{-2k}\int_{B_{2R}\setminus B_{R/4}} u^k dx\Big) \\ &= C\Big(\int_{B_{2R}\setminus B_{R/4}} |x|^{ka} v^{p+1} dx + R^{-2k}\int_{B_{2R}\setminus B_{R/4}} u^k dx\Big) \\ &\leq C\Big(R^{a/p}F(2R) + R^{-2k}\int_{B_{2R}\setminus B_{R/4}} u^k dx\Big).\end{aligned}$$

By Hölder's inequality, for $R>1$, we have

$$\begin{aligned}A_1 = R^{-2k}\int_{B_{2R}\setminus B_{R/4}} u^k dx &\leq CR^{-2k}R^{N(pq-1)/p(q+1)}\Big(\int_{B_{2R}\setminus B_{R/4}} u^{q+1}dx\Big)^{(p+1)/p(q+1)} \\ &\leq CR^{\eta_1/p}F(2R),\end{aligned}$$

with $\eta_1 = -2(p+1) + N(pq-1)/(q+1) - b(p+1)/(q+1)$, where we used $(p+1)/p(q+1) < 1$, along with

$$F(R) \geq F(1) > 0, \quad R > 1.$$

We show that $\eta_1 < a$. Indeed

$$\begin{aligned}a - \eta_1 &= 2(p+1) - N\frac{pq-1}{q+1} + b\frac{p+1}{q+1} + a \\ &= 2(p+1) - N\frac{pq-1}{q+1} + b\frac{p(q+1)-(pq-1)}{q+1} + a \\ &= \frac{2}{\beta}\left((p+1)\beta - N + \frac{b}{2}p\beta - b + \frac{a}{2}\beta\right) \\ &= \frac{2}{\beta}\left(2+\alpha+\beta - N + \frac{b}{2}\alpha + \frac{a}{2}\beta\right).\end{aligned}$$

Hence (3.37) holds.

Also, using Hölder's inequality, Lemma 3.2.7 and Lemma 3.2.2, we have

$$\begin{aligned}\int_{R/2}^{R} \|D_x^2 v(r)\|_m^m r^{N-1} dr &= \int_{B_R\setminus B_{R/2}} |D_x^2 v|^m dx \\ &\leq C\Big(\int_{B_{2R}\setminus B_{R/4}} |\Delta v|^m dx + R^{-2m}\int_{B_{2R}\setminus B_{R/4}} v^m dx\Big) \\ &= C\Big(\int_{B_{2R}\setminus B_{R/4}} |x|^{mb} u^{q+1} dx + R^{-2m}\int_{B_{2R}\setminus B_{R/4}} v^m dx\Big) \\ &\leq C\Big(R^{\frac{b}{q}}F(2R) + R^{-2m+\frac{(b-a)m}{p+1}}\int_{B_{2R}\setminus B_{R/4}} u^{\frac{m(q+1)}{p+1}} dx\Big).\end{aligned}$$

By Hölder's inequality, for $R > 1$, we have

$$\begin{aligned} A_2 &= R^{-2m+\frac{(b-a)m}{p+1}} \int_{B_{2R}\setminus B_{R/4}} u^{\frac{m(q+1)}{p+1}} dx \\ &\le CR^{-2m+\frac{(b-a)m}{p+1}+\frac{N(pq-1)}{q(p+1)}} \left(\int_{B_{2R}\setminus B_{R/4}} u^{q+1} dx \right)^{(q+1)/q(p+1)} \\ &\le CR^{-2m+\frac{N(pq-1)}{q(p+1)}-\frac{am}{p+1}} \left(\int_{B_{2R}\setminus B_{R/4}} |x|^b u^{q+1} dx \right)^{(q+1)/q(p+1)} \\ &\le CR^{\eta_2/q} F(2R), \end{aligned}$$

with $\eta_2 = -2(q+1) + N(pq-1)/(p+1) - a(q+1)/(p+1)$, where we used $(q+1)/q(p+1) < 1$. The similar computation gives

$$\eta_2 - b = \frac{2}{\alpha}\left(2 + \alpha + \beta - N + \frac{b}{2}\alpha + \frac{a}{2}\beta\right).$$

Hence, $\eta_2 < b$ and then (3.38) holds.

On the other hand, by using Lemma 3.2.2, 3.2.4, equation (3.1) and the boundedness of u, we obtain

$$\begin{aligned} \int_{R/2}^{R} \|D_x^2 u(r)\|_{1+\varepsilon}^{1+\varepsilon} r^{N-1}\, dr &= \int_{B_R\setminus B_{R/2}} |D_x^2 u|^{1+\varepsilon}\, dx \\ &\le C \int_{B_{2R}\setminus B_{R/4}} |\Delta u|^{1+\varepsilon}\, dx + CR^{-2(1+\varepsilon)} \int_{B_{2R}\setminus B_{R/4}} u^{1+\varepsilon}\, dx \\ &\le C \int_{B_{2R}\setminus B_{R/4}} |x|^{a\varepsilon} v^{p\varepsilon} |x|^a v^p\, dx + CR^{-2(1+\varepsilon)} \int_{B_{2R}\setminus B_{R/4}} u^{1+\varepsilon}\, dx \\ &\le CR^{a\varepsilon} \int_{B_{2R}\setminus B_{R/4}} |x|^a v^p dx + CR^{-2(1+\varepsilon)} \int_{B_{2R}\setminus B_{R/4}} u\, dx \\ &\le CR^{N-2-\alpha-\frac{a+bp}{pq-1}+a\varepsilon} + CR^{N-2-\alpha-\frac{a+bp}{pq-1}-2\varepsilon} \\ &\le CR^{N-2-\alpha-\frac{a+bp}{pq-1}+a\varepsilon}. \end{aligned}$$

By the similar calculation for v, (3.39) holds.

Step 5 : *measure and feedback argument.* For a given $K > 0$, let us define the sets

$$\begin{aligned} \Gamma_1(R) &= \{r \in (R, 2R); \|D_x^2 u(r)\|_k^k > KR^{-N+\frac{a}{p}} F(4R)\}, \\ \Gamma_2(R) &= \{r \in (R, 2R); \|D_x^2 v(r)\|_m^m > KR^{-N+\frac{b}{q}} F(4R)\}, \\ \Gamma_3(R) &= \{r \in (R, 2R); \|D_x^2 u(r)\|_{1+\varepsilon}^{1+\varepsilon} > KR^{-2-\alpha-\frac{a+bp}{pq-1}+a\varepsilon}\}, \\ \Gamma_4(R) &= \{r \in (R, 2R); \|D_x^2 v(r)\|_{1+\varepsilon}^{1+\varepsilon} > KR^{-2-\beta-\frac{b+aq}{pq-1}+b\varepsilon}\}, \\ \Gamma_5(R) &= \{r \in (R, 2R); \|u(r)\|_1 > KR^{-\alpha-\frac{a+bp}{pq-1}}\}, \\ \Gamma_6(R) &= \{r \in (R, 2R); \|v(r)\|_1 > KR^{-\beta-\frac{b+aq}{pq-1}}\}, \\ \Gamma_7(R) &= \{r \in (R, 2R); \|D_x u(r)\|_1 > KR^{-1-\alpha-\frac{a+bp}{pq-1}}\}, \\ \Gamma_8(R) &= \{r \in (R, 2R); \|D_x v(r)\|_1 > KR^{-1-\beta-\frac{b+aq}{pq-1}}\}. \end{aligned}$$

By estimate (3.37) and (3.35), for $R > 1$ we have

$$\begin{aligned} CR^{a/p}F(4R) &\geq \int_R^{2R} \|D_x^2 u(r)\|_k^k r^{N-1} dr \\ &\geq |\Gamma_1(R)| R^{N-1} K R^{-N+\frac{a}{p}} F(4R) = |\Gamma_1(R)| K R^{-1+\frac{a}{p}} F(4R) \end{aligned}$$

and

$$\begin{aligned} C &\geq R^{-N+\alpha+\frac{a+bp}{pq-1}} \int_R^{2R} \|u(r)\|_1 r^{N-1} dr \\ &\geq R^{-N+\alpha+\frac{a+bp}{pq-1}} |\Gamma_5(R)| R^{N-1} K R^{-\alpha-\frac{a+bp}{pq-1}} = |\Gamma_5(R)| K R^{-1}. \end{aligned}$$

Consequently, $|\Gamma_1| \leq R/10$ and $|\Gamma_5| \leq R/10$ for $K > 10C$. Similarly, $|\Gamma_i| \leq R/10,\ i = 1, ..., 8$. Therefore, for each $R \geq 1$, we can find

$$\tilde{R} \in (R, 2R) \setminus \bigcup_{i=1}^{8} \Gamma_i(R) \neq \emptyset. \tag{3.40}$$

Let us check that

$$2 + \alpha + \frac{a+bp}{pq-1} > \frac{N}{k} - \frac{a}{pk}, \tag{3.41}$$

$$2 + \beta + \frac{b+aq}{pq-1} > \frac{N}{m} - \frac{b}{qm}. \tag{3.42}$$

Indeed, by computation

$$\begin{aligned} M &= (2+\alpha)k + \frac{a+bp}{pq-1}k - N + \frac{a}{p} \\ &= p\beta k + \frac{(a+bp)(p+1)}{p(pq-1)} - N + \frac{a}{p} \\ &= \beta(p+1) + \frac{(a+bp)}{2p}\alpha - N + \frac{a}{p} \\ &= p\beta + \beta + \frac{a}{2p}\alpha + \frac{b}{2}\alpha - N + \frac{a}{p} \\ &= \alpha + \beta + 2 - N + \frac{b}{2}\alpha + \frac{a}{2p}(\alpha+2) \\ &= \alpha + \beta + 2 - N + \frac{b}{2}\alpha + \frac{a}{2}\beta > 0. \end{aligned}$$

Thus, (3.41) holds. Similarly for (3.42). Therefore, for $\varepsilon > 0$ small enough, we have

$$\frac{1}{1+\varepsilon}(2 + \alpha + \frac{a+bp}{pq-1} - a\varepsilon) > \frac{N}{k} - \frac{a}{pk}, \tag{3.43}$$

$$\frac{1}{1+\varepsilon}(2 + \beta + \frac{b+aq}{pq-1} - b\varepsilon) > \frac{N}{m} - \frac{b}{qm}. \tag{3.44}$$

By (3.30) and the definition of the sets Γ_i, we may now control $G_1(\tilde{R})$ as follows

$$\begin{aligned} \left[R^{-N-b} G_1(\tilde{R})\right]^{1/q+1} &\leq CR^2 \left(R^{-2-\alpha-\frac{a+bp}{pq-1}} + R^{(-2-\alpha-\frac{a+bp}{pq-1}+a\varepsilon)/(1+\varepsilon)}\right)^{\nu} \\ &\quad \times \left(R^{-\frac{N}{k}+\frac{a}{pk}} F^{1/k}(4R) + R^{-2-\alpha-\frac{a+bp}{pq-1}}\right)^{1-\nu}. \end{aligned}$$

Using (3.41) and (3.43), we obtain

$$G_1(\tilde{R}) \le C\left(R^{-a_1(0)} + R^{-a_1(\varepsilon)}\right) F^{b_1}(4R) \tag{3.45}$$

where

$$a_1(\varepsilon) = (q+1)\left[\left(\alpha + 2 + \frac{a+bp}{pq-1} - a\varepsilon\right)\frac{\nu}{1+\varepsilon} + \left(\frac{N}{k} - \frac{a}{pk}\right)(1-\nu) - 2 - \frac{N+b}{q+1}\right]$$
$$b_1 = \frac{(1-\nu)}{k}(q+1).$$

On the other hand, it follows from (3.34), (3.41)-(3.44) that

$$\begin{aligned} G_2(\tilde{R}) \le\ & CR^{N+2}\left(R^{-\alpha-2-\frac{a+bp}{pq-1}} + R^{(-2-\alpha-\frac{a+bp}{pq-1}+a\varepsilon)/(1+\varepsilon)}\right)^{\tau_1} \\ & \times \left(R^{-\beta-2-\frac{b+aq}{pq-1}} + R^{(-2-\beta-\frac{b+aq}{pq-1}+b\varepsilon)/(1+\varepsilon)}\right)^{\tau_2} \\ & \times \left(R^{-\frac{N}{k}+\frac{a}{pk}} F^{1/k}(4R) + R^{-\alpha-2-\frac{a+bq}{pq-1}}\right)^{1-\tau_1} \\ & \times \left(R^{-\frac{N}{m}+\frac{b}{qm}} F^{1/m}(4R) + R^{-\beta-2-\frac{b+aq}{pq-1}}\right)^{1-\tau_2} \\ \le\ & C\left(R^{-a_2(0)} + R^{-a_2(\varepsilon)} + R^{-a_3(\varepsilon)} + R^{-a_4(\varepsilon)}\right) F^{b_2}(4R). \end{aligned} \tag{3.46}$$

where

$$\begin{aligned} a_2(\varepsilon) =& -N - 2 + \frac{\tau_1}{1+\varepsilon}\left(\alpha + 2 + \frac{a+bp}{pq-1} - a\varepsilon\right) + \tau_2\left(\beta + 2 + \frac{b+aq}{pq-1}\right) \\ & + \left(N - \frac{a}{p}\right)\frac{(1-\tau_1)}{k} + \left(N - \frac{b}{q}\right)\frac{(1-\tau_2)}{m}, \end{aligned}$$

$$\begin{aligned} a_3(\varepsilon) =& -N - 2 + \tau_1\left(\alpha + 2 + \frac{a+bp}{pq-1}\right) + \frac{\tau_2}{1+\varepsilon}\left(\beta + 2 + \frac{b+aq}{pq-1} - b\varepsilon\right) \\ & + \left(N - \frac{a}{p}\right)\frac{(1-\tau_1)}{k} + \left(N - \frac{b}{q}\right)\frac{(1-\tau_2)}{m}, \end{aligned}$$

$$\begin{aligned} a_4(\varepsilon) =& -N - 2 + \frac{\tau_1}{1+\varepsilon}\left(\alpha + 2 + \frac{a+bp}{pq-1} - a\varepsilon\right) + \frac{\tau_2}{1+\varepsilon}\left(\beta + 2 + \frac{b+aq}{pq-1} - b\varepsilon\right) \\ & + \left(N - \frac{a}{p}\right)\frac{(1-\tau_1)}{k} + \left(N - \frac{b}{q}\right)\frac{(1-\tau_2)}{m}, \end{aligned}$$

$$b_2 = \frac{1-\tau_1}{k} + \frac{1-\tau_2}{m}.$$

Let $\tilde{a} = \min\left(a_i(0), a_j(\varepsilon);\quad i = 1,2;\quad j = 1,2,3,4\right)$ and $\tilde{b} = \max(b_1, b_2)$. Combining (3.45) and (3.46), we obtain

$$F(R) \le CR^{-\tilde{a}} F^{\tilde{b}}(4R), R \ge 1. \tag{3.47}$$

We claim that there exist a constant $M_0 > 0$ and a sequence $R_i \to \infty$ such that

$$F(4R_i) \le M_0 F(R_i).$$

Assume that the claim is false. Then, for any $M_0 > 0$, there exists $R_0 > 0$ such that $F(4R) \geq M_0 F(R)$ for all $R \geq R_0$. But since u is bounded, we have $F(R) \leq CR^{N+b}$. Thus

$$M_0^i F(R_0) \leq F(4^i R_0) \leq C(4^i R_0)^{N+b} = CR_0^{N+b} 4^{i(N+b)}, \forall i \geq 0.$$

This is a contradiction for i large if we choose $M_0 > 4^{N+b}$.

Now we assume we have proved that $\tilde{a} > 0$ and $\tilde{b} < 1$, then from (3.47) we have

$$F(4R_i) \leq CR_i^{-\tilde{a}/(1-\tilde{b})}.$$

Letting $i \to \infty$, we obtain $\int_{\mathbb{R}^N} |x|^b u^{q+1} = 0$, hence $u \equiv 0 \equiv v$: contradiction.

Step 6 : *Fulfillment of the conditions $\tilde{a} > 0$ and $\tilde{b} < 1$*

Verification of $b_1 < 1$. If $q \leq 2/(N-3)$ then $b_1 = 0$. If $q > 2/(N-3)$ then

$$\begin{aligned} 1 - b_1 = 1 - p(q+1)A = 1 - p\left((q+1)\frac{N-3}{N-1} - 1\right) &= \frac{(N-1)(p+1) - p(q+1)(N-3)}{N-1} \\ = \frac{2(p+1) - (N-3)(pq-1)}{N-1} &= \frac{pq-1}{N-1}(\alpha + 3 - N). \end{aligned}$$

Thus, $0 \leq b_1 < 1$.

Verification of $a_1(0) > 0$.

$$\begin{aligned} a_1(0) &= (q+1)\left[\alpha + \frac{a+bp}{pq-1} - \frac{N+b}{q+1} - (1-\nu)\frac{1}{k}\left((2+\alpha)k + \frac{a+bp}{pq-1}k - N - \frac{a}{p}\right)\right] \\ &= (q+1)\alpha + \frac{(q+1)(a+bp)}{pq-1} - N - b - b_1 M \\ &= \alpha + \beta + 2 + \frac{(q+1)(a+bp)}{pq-1} - N - b - b_1 M \\ &= M - b_1 M = (1-b_1)M. \end{aligned}$$

Hence $a_1(0) > 0$.

Verification of $a_2(0) > 0$ and $b_2 < 1$.

Case $q > 1/(N-2)$. Here we must ensure the existence of $z \in (1, \infty)$ satisfying (3.32) and (3.33), that is

$$\max\left(\frac{1}{k} - \frac{1}{N-1}, \frac{1}{N-1}\right) \leq \frac{1}{z} \leq \min\left(1 - \frac{1}{N-1}, \frac{1}{q+1} + \frac{1}{N-1}\right). \tag{3.48}$$

We have

$$b_2 = pA_1 + qA_2 = p\left(\frac{N-2}{N-1} - \frac{1}{z}\right) + q\left(\frac{1}{z} - \frac{1}{N-1}\right) = \frac{p(N-2) - q}{N-1} - \frac{p-q}{z}.$$

Hence, there exists $z \in (1, \infty)$ satisfying (3.48) and such that $b_2 < 1$, if the following hold

$$\frac{1}{k} - \frac{1}{N-1} \leq \frac{1}{q+1} + \frac{1}{N-1}, \tag{3.49}$$

$$\frac{p(N-2) - q}{N-1} - 1 < \frac{(N-2)(p-q)}{N-1}, \tag{3.50}$$

$$\frac{p(N-2) - q}{N-1} - 1 < (p-q)\left(\frac{1}{q+1} + \frac{1}{N-1}\right). \tag{3.51}$$

Inequality (3.49) is true by (3.27). Inequality (3.50) is equivalent to $q < (N-1)/(N-3)$, which is true due to $q \leq p(q+1)/(p+1) = 1 + (2/\alpha) < (N-1)/(N-3)$. Inequality (3.51) is also true due to $\alpha > N - 3$.

We have

$$\begin{aligned} a_2(0) =& -N-2+\tau_1\left(\alpha+2+\frac{a+bp}{pq-1}\right)+\tau_2\left(\beta+2+\frac{b+aq}{pq-1}\right)\\ &+\left((2+\alpha)k+\frac{a+bp}{pq-1}k-M\right)\frac{(1-\tau_1)}{k}\\ &+\left((2+\beta)m+\frac{b+aq}{pq-1}m-M\right)\frac{(1-\tau_2)}{m}\\ =& -N-2+2+\alpha+\frac{a+bp}{pq-1}+2+\beta+\frac{b+aq}{pq-1}-M\left(\frac{1-\tau_1}{k}+\frac{1-\tau_2}{m}\right)\\ =&M-Mb_2=M(1-b_2)>0. \end{aligned}$$

Case $q\le 1/(N-2)$**.** Since $\tau_1=1,\tau_2=0$, we deduce

$$\begin{aligned} a_2(0) =& -N-2+\alpha+2+\frac{a+bp}{pq-1}+\frac{N}{m}-\frac{b}{qm}\\ =&\frac{1}{q+1}\left(-N+(q+1)\alpha+\frac{(q+1)(a+bp)}{pq-1}-\frac{b(pq-1)}{pq-1}\right)\\ =&\frac{1}{q+1}\left(\alpha+\beta+\frac{b}{2}\alpha+\frac{a}{2}\beta-N+2\right)>0 \end{aligned}$$

and also $b_2=1/m<1$.

Note that $a_2(0)=a_3(0)=a_4(0)$. Thus $a_i(\varepsilon)>0,\ i=1,...,4$ for ε small enough. Theorem is proved for $N\ge 4$.

For $N=3$, conditions (3.9) and (3.10) are not necessary and the proof becomes much less complicated due to the Sobolev imbedding $W^{2,1+\varepsilon}\subset L^\infty$ on S^2. For sake of clarity, although here $N=3$, we shall keep the letter N in the proof.

Step 1 : *Preparations.* Let us choose a_1,a_2 satisfying (3.22) and set

$$F(R)=\int_{B_R}|x|^b u^{q+1}dx+\int_{B_R}|x|^a v^{p+1}dx.$$

By the Rellich-Pohozaev identity (Lemma 3.2.6), we have

$$F(R)\le C\big(G_{11}(R)+G_{12}(R)+G_2(R)\big),$$

where

$$\begin{aligned} G_{11}(R)&=R^{N+b}\int_{S^{N-1}}u^{q+1}(R)d\theta,\\ G_{12}(R)&=R^{N+a}\int_{S^{N-1}}v^{p+1}(R)d\theta,\\ G_2(R)&=R^N\int_{S^{N-1}}\left(|D_xu(R)|+R^{-1}u(R)\right)\left(|D_xv(R)|+R^{-1}v(R)\right)d\theta. \end{aligned}$$

Step 2 : *Estimations of* $G_{11}(R),G_{12}(R)$ *and* $G_2(R)$.

By Lemma 3.2.1, since $N=3$, we have

$$\|u\|_{q+1}\le C\|u\|_\infty\le C\left(\|D^2_\theta u\|_{1+\varepsilon}+\|u\|_1\right)\le C\left(R^2\|D^2_x u\|_{1+\varepsilon}+\|u\|_1\right)$$

and

$$\|D_xu\|_2\le C\left(\|D_\theta D_xu\|_{1+\varepsilon}+\|D_xu\|_1\right)\le C\left(R\|D^2_xu\|_{1+\varepsilon}+\|D_xu\|_1\right).$$

Similarly,

$$\|v\|_{p+1} \le C\|v\|_\infty \le C\left(\|D_\theta^2 v\|_{1+\varepsilon} + \|v\|_1\right) \le C\left(R^2\|D_x^2 v\|_{1+\varepsilon} + \|v\|_1\right)$$

and

$$\|D_x v\|_2 \le C\left(\|D_\theta D_x v\|_{1+\varepsilon} + \|D_x v\|_1\right) \le C\left(R\|D_x^2 v\|_{1+\varepsilon} + \|D_x v\|_1\right).$$

Therefore,

$$G_{11}(R) \le CR^{N+b+2(q+1)}\left(\|D_x^2 u\|_{1+\varepsilon} + R^{-2}\|u\|_1\right)^{q+1}, \tag{3.52}$$

$$G_{12}(R) \le CR^{N+a+2(p+1)}\left(\|D_x^2 v\|_{1+\varepsilon} + R^{-2}\|v\|_1\right)^{p+1} \tag{3.53}$$

and

$$G_2(R) \le CR^{N+2}(\|D_x^2 u\|_{1+\varepsilon} + R^{-1}\|D_x u\|_1 + R^{-2}\|u\|_1)(\|D_x^2 v\|_{1+\varepsilon} + R^{-1}\|D_x v\|_1 + R^{-2}\|v\|_1) \tag{3.54}$$

Step 3 : *Conclusion.* We can find

$$\tilde{R} \in (R, 2R) \setminus \bigcup_{i=3}^{8} \Gamma_i(R) \neq \emptyset, \tag{3.55}$$

where the sets Γ_i are defined in Step 4 of the proof of the case $N \ge 4$. If follows from (3.52)-(3.54) in Step 2 and (3.55) in Step 3 that

$$\begin{aligned} G_{11}(\tilde{R}) &\le CR^{N+b+2(q+1)}\left(R^{(-2-\alpha-\frac{a+bp}{pq-1}+a\varepsilon)/(1+\varepsilon)} + R^{-2-\alpha-\frac{a+bp}{pq-1}}\right)^{q+1} \\ &\le C\left(R^{-c_1(\varepsilon)} + R^{-c_1(0)}\right), \end{aligned}$$

where

$$c_1(\varepsilon) = (q+1)\left[\left(2+\alpha+\frac{a+bp}{pq-1} - a\varepsilon\right)\frac{1}{1+\varepsilon} - 2 - \frac{N+b}{q+1}\right].$$

Similarly

$$\begin{aligned} G_{12}(\tilde{R}) &\le CR^{N+a+2(p+1)}\left(R^{(-2-\beta-\frac{b+aq}{pq-1}+b\varepsilon)/(1+\varepsilon)} + R^{-2-\beta-\frac{b+aq}{pq-1}}\right)^{p+1} \\ &\le C\left(R^{-c_2(\varepsilon)} + R^{-c_2(0)}\right), \end{aligned}$$

where

$$c_2(\varepsilon) = (p+1)\left[\left(2+\beta+\frac{b+aq}{pq-1} - b\varepsilon\right)\frac{1}{1+\varepsilon} - 2 - \frac{N+a}{p+1}\right],$$

and

$$\begin{aligned} G_2(\tilde{R}) &\le CR^{N+2}\left(R^{(-2-\alpha-\frac{a+bp}{pq-1}+a\varepsilon)/(1+\varepsilon)} + R^{-2-\alpha-\frac{a+bp}{pq-1}}\right) \\ &\quad\times\left(R^{(-2-\beta-\frac{b+aq}{pq-1}+b\varepsilon)/(1+\varepsilon)} + R^{-2-\beta-\frac{b+aq}{pq-1}}\right) \\ &\le C\left(R^{-c_3(\varepsilon)} + R^{-c_4(\varepsilon)} + R^{-c_5(\varepsilon)} + R^{-c_3(0)}\right), \end{aligned}$$

where

$$\begin{aligned} c_3(\varepsilon) &= -N-2+\frac{1}{1+\varepsilon}\left(2+\alpha+\frac{a+bp}{pq-1} - a\varepsilon\right) + \frac{1}{1+\varepsilon}\left(2+\beta+\frac{b+aq}{pq-1} - b\varepsilon\right), \\ c_4(\varepsilon) &= -N-2+\frac{1}{1+\varepsilon}\left(2+\alpha+\frac{a+bp}{pq-1} - a\varepsilon\right) + 2+\beta+\frac{b+aq}{pq-1}, \\ c_5(\varepsilon) &= -N-2+\frac{1}{1+\varepsilon}\left(2+\beta+\frac{b+aq}{pq-1} - b\varepsilon\right) + 2+\alpha+\frac{a+bp}{pq-1}. \end{aligned}$$

Letting $\tilde{c} = \min\left(c_i(\varepsilon), c_j(0); i = 1, ..., 5, j = 1, ..., 3\right)$, we obtain

$$F(R) \leq F(\tilde{R}) \leq CR^{-\tilde{c}}, \quad R \geq 1.$$

By straightforward computation, we see that

$$c_i(0) > 0, \; i = 1, ..., 5.$$

Therefore, for $\varepsilon > 0$ small enough, we have $\tilde{c} > 0$, so that $\int_{\mathbb{R}^N} \left(|x|^a v^{p+1} + |x|^b u^{q+1}\right) dx = 0$, hence $u \equiv v \equiv 0$: a contradiction. The proof is complete. □

3.4 Applications : Singularity and decay estimates and a priori bound

3.4.1 Singularity and decay estimates

We now prove Theorem 3.1.3. We need the following lemma which is analogous to [16, Lemma 2.1]. We give here the proof for completeness.

Lemma 3.4.1. *Assume $pq > 1$, $p \geq q$, (3.3) and (3.10). Assume in addition that $c, d \in C^\gamma(\overline{B}_1)$ for some $\gamma \in (0,1]$ and*

$$\|c\|_{C^\gamma(\overline{B}_1)} \leq C_1, \; \|d\|_{C^\gamma(\overline{B}_1)} \leq C_1 \;\; \text{and} \;\; c(x) \geq C_2, \; d(x) \geq C_2, \; x \in \overline{B}_1, \tag{3.56}$$

for some constants $C_1, C_2 > 0$. There exists a constant C, depending only on $\gamma, C_1, C_2, p, q, N$, such that, for any nonnegative classical solution (u, v) of

$$\begin{cases} -\Delta u = c(x) v^p, & x \in B_1 \\ -\Delta v = d(x) u^q, & x \in B_1 \end{cases} \tag{3.57}$$

(u, v) satisfies

$$|u(x)|^{\frac{1}{\alpha}} + |v(x)|^{\frac{1}{\beta}} + |\nabla u(x)|^{\frac{1}{\alpha+1}} + |\nabla v(x)|^{\frac{1}{\beta+1}} \leq C\big(1 + \mathrm{dist}^{-1}(x, \partial B_1)\big), \quad x \in B_1. \tag{3.58}$$

Proof. Arguing by contradiction, we suppose that there exist sequences c_k, d_k, u_k, v_k verifying (3.56), (3.57) and points y_k, such that the functions

$$M_k = |u|^{\frac{1}{\alpha}} + |v|^{\frac{1}{\beta}} + |\nabla u|^{\frac{1}{\alpha+1}} + |\nabla v|^{\frac{1}{\beta+1}}$$

satisfy

$$M_k(y_k) > 2k\big(1 + \mathrm{dist}^{-1}(y_k, \partial B_1)\big) \geq 2k \, \mathrm{dist}^{-1}(y_k, \partial B_1).$$

By the Doubling Lemma in [17, Lemma 5.1], there exists x_k such that

$$M_k(x_k) \geq M_k(y_k), \quad M_k(x_k) > 2k \, \mathrm{dist}^{-1}(x_k, \partial B_1),$$

and

$$M_k(z) \leq 2M_k(x_k), \quad \text{for all } z \text{ such that } |z - x_k| \leq k M_k^{-1}(x_k). \tag{3.59}$$

We have

$$\lambda_k := M_k^{-1}(x_k) \to 0, \quad k \to \infty, \tag{3.60}$$

due to $M_k(x_k) \geq M_k(y_k) > 2k$.

Next we let

$$\tilde{u}_k = \lambda_k^\alpha u_k(x_k + \lambda_k y), \; \tilde{v}_k = \lambda_k^\beta v_k(x_k + \lambda_k y), \quad \tilde{c}_k(y) = c_k(x_k + \lambda_k y), \; \tilde{d}_k(y) = d_k(x_k + \lambda_k y).$$

We note that $|\tilde{u}_k(0)|^{\frac{1}{\alpha}} + |\tilde{v}_k(0)|^{\frac{1}{\beta}} + |\nabla \tilde{u}_k(0)|^{\frac{1}{\alpha+1}} + |\nabla \tilde{v}_k(0)|^{\frac{1}{\beta+1}} = 1$,

$$\left[|\tilde{u}_k|^{\frac{1}{\alpha}} + |\tilde{v}_k|^{\frac{1}{\beta}}\right](y) \leq 2, \quad |y| \leq k, \tag{3.61}$$

due to (3.59), and we see that $(\tilde{u}_k, \tilde{v}_k)$ satisfies

$$\begin{cases} -\Delta \tilde{u}_k = \tilde{c}_k(y)\tilde{v}_k^p, & |y| \leq k, \\ -\Delta \tilde{v}_k = \tilde{d}_k(y)\tilde{u}_k^q, & |y| \leq k. \end{cases} \tag{3.62}$$

On the other hand, due to (3.56), we have $C_2 \leq \tilde{c}_k, \tilde{d}_k \leq C_1$ and, for each $R > 0$ and $k \geq k_0(R)$ large enough,

$$\begin{cases} |\tilde{c}_k(y) - \tilde{c}_k(z)| \leq C_1|\lambda_k(y-z)|^\gamma \leq C_1|y-z|^\gamma, & |y|,|z| \leq R, \\ |\tilde{d}_k(y) - \tilde{d}_k(z)| \leq C_1|\lambda_k(y-z)|^\gamma \leq C_1|y-z|^\gamma, & |y|,|z| \leq R. \end{cases} \tag{3.63}$$

Therefore, by Ascoli's theorem, there exist $\tilde{c}, \tilde{d}$ in $C(\mathbb{R}^N)$ such that, after extracting a subsequence, $(\tilde{c}_k, \tilde{d}_k) \to (\tilde{c}, \tilde{d})$ in $C_{loc}(\mathbb{R}^N)$. Moreover, (3.63) and (3.60) imply that $|\tilde{c}_k(y) - \tilde{c}_k(z)| \to 0$ as $k \to \infty$, so that the function $\tilde{c}$ is actually a constant $C \geq C_2$. Similarly, $\tilde{d}$ is actually a constant $D \geq C_2$.

Now, for each $R > 0$ and $1 < q < \infty$, by (3.62), (3.61) and interior elliptic L^q estimates, the sequence $(\tilde{u}_k, \tilde{v}_k)$ is uniformly bounded in $W^{2+\gamma,q}(B_R)$. Using standard imbeddings, after extracting a subsequence, we may assume that $(\tilde{u}_k, \tilde{v}_k) \to (\tilde{u}, \tilde{v})$ in $C^2_{loc}(\mathbb{R}^N)$. It follows that $(\tilde{u}, \tilde{v})$ is a nonnegative classical solution of

$$\begin{cases} -\Delta \tilde{u} = C\tilde{v}^p, & y \in \mathbb{R}^N, \\ -\Delta \tilde{v} = D\tilde{u}^q, & y \in \mathbb{R}^N, \end{cases}$$

and $\tilde{u}^{\frac{1}{\alpha}}(0) + \tilde{v}^{\frac{1}{\beta}}(0) + |\nabla \tilde{u}(0)|^{\frac{1}{\alpha+1}} + |\nabla \tilde{v}(0)|^{\frac{1}{\beta+1}} = 1$. This contradicts the Liouville-type result of Lane-Emden system in [20] and concludes the proof. □

In addition to Theorem 3.1.3, we shall at the same time prove the following, corresponding gradient estimates, which will be useful in the proof of Theorem 3.1.2.

Proposition 3.4.1. *Under the assumptions of Theorem 3.1.3, (u, v) also satisfies the estimates*

$$|\nabla u(x)| \leq C|x|^{-\alpha-1-\frac{a+bp}{pq-1}}, \quad |\nabla v(x)| \leq C|x|^{-\beta-1-\frac{b+aq}{pq-1}}, \tag{3.64}$$

for $0 < |x| < \rho/2$ (resp. $|x| > 2\rho$).

Proof of Theorem 3.1.3 and Proposition 3.4.1. Assume either $\Omega = \{x \in \mathbb{R}^N;\, 0 < |x| < \rho\}$ and $0 < |x_0| < \rho/2$, or $\Omega = \{x \in \mathbb{R}^N;\, |x| > \rho\}$ and $|x_0| > 2\rho$. Let $R_0 = |x_0|/2 > 0$. We rescale (u, v) by setting

$$U(y) = R_0^{\alpha+\frac{a+bp}{pq-1}} u(x_0 + R_0 y), V(y) = R_0^{\beta+\frac{b+aq}{pq-1}} v(x_0 + R_0 y).$$

Then (U, V) is solution of

$$\begin{cases} -\Delta U = c(y)V^p, & y \in B(0,1) \\ -\Delta V = d(y)U^q, & y \in B(0,1). \end{cases}$$

where

$$c(y) = |y + \frac{x_0}{R_0}|^a, \quad d(y) = |y + \frac{x_0}{R_0}|^b.$$

Notice that $|y+\frac{x_0}{R_0}|\in[1,3]$, $\forall y\in\overline{B}(0,1)$. Moreover $\|c\|_{C^1(\overline{B}_1)}\le C(a)$ and $\|d\|_{C^1(\overline{B}_1)}\le C(b)$, then applying Lemma 3.4.1 we have $U(0)+V(0)+|\nabla U(0)|+|\nabla V(0)|\le C$. Hence

$$u(x_0)\le CR_0^{-\alpha-\frac{a+bp}{pq-1}},\qquad v(x_0)\le CR_0^{-\beta-\frac{b+aq}{pq-1}}$$

and

$$|\nabla u(x_0)|\le CR_0^{-\alpha-1-\frac{a+bp}{pq-1}},\qquad |\nabla v(x_0)|\le CR_0^{-\beta-1-\frac{b+aq}{pq-1}}.$$

The announced results are proved. □

3.4.2 A priori bound

Proof of Theorem 3.1.4. Suppose that Theorem 3.1.4 is false. Let $d=\mathrm{dist}(0,\partial\Omega)>0$. Due to estimate (3.11) in Theorem 3.1.3, all solutions of (3.13) are uniformly bounded, away from $\{0\}\cup\partial\Omega$. Then there are only two following possibilities.

Case 1 : There exist sequence of solutions (u_k,v_k) and a sequence of points $P_k\to P\in\partial\Omega$ such that

$$N_k=\sup_{x\in\Omega:\mathrm{dist}(x,\partial\Omega)<d/2}\left(u_k^{\frac{1}{\alpha}}(x)+v_k^{\frac{1}{\beta}}(x)\right)=u_k^{\frac{1}{\alpha}}(P_k)+v_k^{\frac{1}{\beta}}(P_k)\to\infty \text{ as } k\to\infty. \tag{3.65}$$

Let $d_k=\mathrm{dist}(P_k,\partial\Omega)$, we rescale solution according to

$$U_k(y)=\lambda_k^{\alpha}u_k(P_k+\lambda_k y),\quad V_k(y)=\lambda_k^{\beta}v_k(P_k+\lambda_k y);\quad \lambda_k=N_k^{-1}$$

then

$$\begin{cases}-\Delta U_k=|P_k+\lambda_k y|^a V_k^p, & |y|<\lambda_k^{-1}d_k,\\ -\Delta V_k=|P_k+\lambda_k y|^b U_k^q, & |y|<\lambda_k^{-1}d_k.\end{cases}$$

If there exists a subsequence $\lambda_k^{-1}d_k\to\infty$ then by using the elliptic estimates and standard imbeddings, we deduce that some subsequence of (U_k,V_k) converges in $C_{\mathrm{loc}}(\mathbb{R}^N)$ to a solution (U,V) in $\mathbb{R}^N$ of the following system

$$\begin{cases}-\Delta U=|P|^aV^p, & y\in\mathbb{R}^N\\ -\Delta V=|P|^bU^q, & y\in\mathbb{R}^N.\end{cases}$$

This contradicts Liouville-type result of Lane-Emden system (see [20]).

If $\lambda_k^{-1}d_k$ is bounded then by the argument similar to that in [11], there exist $\ell_1,\ell_2>0$ and functions U,V solving the following problem in the half-space

$$\begin{cases}-\Delta U=\ell_1V^p, & x\in H_s^N\\ -\Delta V=\ell_2U^q, & x\in H_s^N\\ U(x)=V(x)=0, & x\in\partial H_s^N\\ U^{\frac{1}{\alpha}}(0)+V^{\frac{1}{\beta}}(0)=1,\end{cases}$$

where $H_s^N:=\{y\in\mathbb{R}^N:y_1>-s\}$ for some $s>0$. In view of assumption (3.3), this contradicts the Liouville-type result of [17, Theorem 4.2] for the Lane-Emden system in a half-space.

Case 2 : There exists a sequence of solutions (u_k,v_k) and a sequence of points $P_k\to 0\in\Omega$ such that

$$M_k=\sup_{|x|<d/2}\left(u_k^{\frac{1}{\alpha+\frac{a+bp}{pq-1}}}(x)+v_k^{\frac{1}{\beta+\frac{b+aq}{pq-1}}}(x)\right)=u_k^{\frac{1}{\alpha+\frac{a+bp}{pq-1}}}(P_k)+v_k^{\frac{1}{\beta+\frac{b+aq}{pq-1}}}(P_k)\to\infty \text{ as } k\to\infty.$$

We denote by

$$U_k(y) = \lambda_k^{\alpha+\frac{a+bp}{pq-1}} u_k(P_k+\lambda_k y), V_k(y) = \lambda_k^{\beta+\frac{b+aq}{pq-1}} v_k(P_k+\lambda_k y), \quad \lambda_k = M_k^{-1}.$$

Then (U_k, V_k) is solution to

$$\begin{cases} -\Delta U_k = |y+\frac{P_k}{\lambda_k}|^a V_k^p, y \in B(0, \frac{d}{2\lambda_k}) \\ -\Delta V_k = |y+\frac{P_k}{\lambda_k}|^b U_k^q, y \in B(0, \frac{d}{2\lambda_k}). \end{cases} \tag{3.66}$$

Moreover, it follows from estimate (3.11) in Theorem 3.1.3. that the sequence $\lambda_k^{-1}|P_k| = |P_k|M_k$ is bounded. We may thus assume that $\lambda_k^{-1}P_k \to x_0$ as $k \to \infty$.

From (3.66), by using the elliptic estimates and standard imbeddings, we deduce that some subsequence of (U_k, V_k) converges in $C_{\text{loc}}(\mathbb{R}^N)$ to a solution (U, V) in $\mathbb{R}^N$ of the following system

$$\begin{cases} -\Delta U = |y+x_0|^a V^p, \quad y \in \mathbb{R}^N \\ -\Delta V = |y+x_0|^b U^q, \quad y \in \mathbb{R}^N. \end{cases}$$

with

$$U^{\frac{1}{\alpha+\frac{a+bp}{pq-1}}}(0) + V^{\frac{1}{\beta+\frac{b+aq}{pq-1}}}(0) = 1.$$

After a space shift, this gives a contradiction with Theorem 3.1.2. □

3.5 Proof of Theorem 3.1.2.

Let (u, v) be a positive solution of system (3.1). By the Rellich-Pohozaev identity (Lemma 3.2.6) with (3.22), we have

$$\begin{aligned} &\int_{B_R} |x|^a v^{p+1}\,dx + \int_{B_R} |x|^b u^{q+1}\,dx \\ &\qquad \le H(R) := CR^{N+a} \int_{S^{N-1}} v^{p+1}(R,\theta)\,d\theta + CR^{N+b} \int_{S^{N-1}} u^{q+1}(R,\theta)\,d\theta \\ &\qquad + CR^N \int_{S^{N-1}} \left(|D_x u(R,\theta)| + R^{-1}u(R,\theta)\right)\left(|D_x v(R,\theta)| + R^{-1}v(R,\theta)\right)\,d\theta. \end{aligned}$$

Now, by Theorem 3.1.3 and Proposition 3.4.1, for $x \neq 0$, we have

$$u(x) \le C|x|^{-\alpha-\frac{a+bp}{pq-1}}, \qquad v(x) \le C|x|^{-\beta-\frac{b+aq}{pq-1}}$$

and

$$|\nabla u(x)| \le C|x|^{-\alpha-1-\frac{a+bp}{pq-1}}, \qquad |\nabla v(x)| \le C|x|^{-\beta-1-\frac{b+aq}{pq-1}}.$$

By straightforward calculations, it follows that

$$H(R) \le CR^{N-2-(1+\frac{b}{2})\alpha-(1+\frac{a}{2})\beta} \to 0, \text{ as } R \to \infty,$$

due to (3.5) (which is equivalent to (3.8)). Therefore, $u \equiv v \equiv 0$. □

3.6 Appendix

We start with the following simple Lemma.

Lemma 3.6.1. *Let $a, b > -2$, $pq > 1$, $N \geq 3$, $0 \in \Omega$ and (u, v) be positive solution of (3.1). Then :*

$$\text{There exists a sequence } \varepsilon = \varepsilon_i \to 0 \text{ such that } \int_{|x|=\varepsilon_i} \left(|\nabla u|^2 + |\nabla v|^2\right) d\sigma_{\varepsilon_i} \to 0. \tag{3.67}$$

Moreover, (u, v) is a distributional solution of (3.1).

Proof. Since solution is considered in $C(\mathbb{R}^N)$, then $|x|^a v^p, |x|^b u^q \in L^k_{loc}(\mathbb{R}^N)$ with $1 < k \leq N/2$ due to $a, b > -2$. Hence, $u, v \in W^{2,k}_{loc}(\mathbb{R}^N)$ with $1 < k \leq N/2$ by elliptic regularity. By Sobolev embedding, it follows that

$$|\nabla u|, |\nabla u| \in L^N_{loc}(\mathbb{R}^N). \tag{3.68}$$

Consequently,

$$\begin{aligned} \int_{\rho=0}^{\varepsilon} \int_{|x|=\rho} \left(|\nabla u|^2 + |\nabla v|^2\right) d\sigma_\varepsilon \, d\rho &= \int_{|x|<\varepsilon} \left(|\nabla u|^2 + |\nabla v|^2\right) dx \\ &\leq C\varepsilon^{N-2} \left(\|\nabla u\|^2_{L^N(B_\varepsilon)} + \|\nabla u\|^2_{L^N(B_\varepsilon)}\right). \end{aligned}$$

Therefore, there exists $\varepsilon_i \in (0, \varepsilon)$ such that

$$\int_{|x|=\varepsilon_i} \left(|\nabla u|^2 + |\nabla v|^2\right) d\sigma_{\varepsilon_i} \leq C\varepsilon^{N-3} \left(\|\nabla u\|^2_{L^N(B_\varepsilon)} + \|\nabla u\|^2_{L^N(B_\varepsilon)}\right),$$

and assertion (3.67) follows by letting $\varepsilon \to 0$.

Let now $\varphi \in C_0^\infty(\Omega)$ and denote $\Omega_\varepsilon = \Omega \cap \{|x| > \varepsilon\}$ for $\varepsilon > 0$ small. From (3.1), using Green's formula, we obtain

$$\Big|\int_{\Omega_\varepsilon} (|x|^a v^p \varphi + u\Delta\varphi) \, dx\Big| = \Big|-\int_{\Omega_\varepsilon} \varphi \Delta u \, dx + \int_{\Omega_\varepsilon} u \Delta \varphi \, dx\Big| = \Big|\int_{|x|=\varepsilon} \varphi \frac{\partial u}{\partial r} \, d\sigma_\varepsilon - \int_{|x|=\varepsilon} u \frac{\partial \varphi}{\partial r} \, d\sigma_\varepsilon\Big|.$$

Similarly,

$$\Big|\int_{\Omega_\varepsilon} |x|^b u^q \varphi \, dx + \int_{\Omega_\varepsilon} v \Delta \varphi \, dx\Big| = \Big|\int_{|x|=\varepsilon} \varphi \frac{\partial v}{\partial r} \, d\sigma_\varepsilon - \int_{|x|=\varepsilon} v \frac{\partial \varphi}{\partial r} \, d\sigma_\varepsilon\Big|.$$

Passing to the limit with $\varepsilon = \varepsilon_i$, we conclude that (u, v) is a distributional solution of (3.1). □

Proof of Lemma 3.2.6. Since u is a solution of (3.1) then

$$\begin{aligned} &(x.\nabla u)\Delta v + (x.\nabla v)\Delta u = -(x.\nabla u)|x|^b u^q - (x.\nabla v)|x|^a v^p \\ &= -\text{div}\left(x|x|^b \frac{u^{q+1}}{q+1} + x|x|^a \frac{v^{p+1}}{p+1}\right) + \frac{N+b}{q+1}|x|^b u^{q+1} + \frac{N+a}{p+1}|x|^a v^{p+1}. \end{aligned} \tag{3.69}$$

Integrating (3.69) on $B_R \setminus B_\varepsilon$ and letting $\varepsilon \to 0$, we have

$$\begin{aligned} \int_{B_R} (x.\nabla u)\Delta v + (x.\nabla v)\Delta u \, dx &= \int_{B_R} \left(\frac{N+b}{q+1}|x|^b u^{q+1} + \frac{N+a}{p+1}|x|^a v^{p+1}\right) dx \\ &- \int_{|x|=R} \left(R^{1+b} \frac{u^{q+1}}{q+1} + R^{1+a} \frac{v^{p+1}}{p+1}\right) d\sigma_R. \end{aligned} \tag{3.70}$$

On the other hand, we have

$$\begin{aligned}\int\limits_{B_R\setminus B_\varepsilon} \nabla u.\nabla v\,dx &= -\int\limits_{B_R\setminus B_\varepsilon} u\Delta v\,dx + \int\limits_{|x|=R} uv'\,d\sigma_R - \int\limits_{|x|=\varepsilon} uv'\,d\sigma_\varepsilon \\ &= \int\limits_{B_R\setminus B_\varepsilon} |x|^b u^{q+1}\,dx + \int\limits_{|x|=R} uv'\,d\sigma_R - \int\limits_{|x|=\varepsilon} uv'\,d\sigma_\varepsilon. \end{aligned} \tag{3.71}$$

Letting $\varepsilon = \varepsilon_i \to 0$ in (3.71), where ε_i is given by Lemma 3.6.1, we obtain

$$\int\limits_{B_R} \nabla u.\nabla v\,dx = \int\limits_{B_R} |x|^b u^{q+1}\,dx + \int\limits_{|x|=R} uv'\,d\sigma_R.$$

Similarly,

$$\int\limits_{B_R} \nabla u.\nabla v\,dx = \int\limits_{B_R} |x|^a v^{p+1}\,dx + \int\limits_{|x|=R} u'v\,d\sigma_R.$$

Hence, for $a_1 + a_2 = N - 2$, we have

$$\int\limits_{B_R} (N-2)\nabla u.\nabla v\,dx = \int\limits_{B_R} \left(a_1|x|^a v^{p+1} + a_2|x|^b u^{q+1}\right)dx + \int\limits_{|x|=R} (a_1u'v + a_2uv')\,d\sigma_R. \tag{3.72}$$

By direct computation, we have the following identity

$$(x.\nabla u)\Delta v + (x.\nabla v)\Delta u - (N-2)\nabla u.\nabla v = \operatorname{div}\left[(x.\nabla u)\nabla v + (x.\nabla v)\nabla u - x\nabla u.\nabla v\right]. \tag{3.73}$$

Integrating (3.73) on $B_R \setminus B_\varepsilon$ and letting $\varepsilon = \varepsilon_i \to 0$, where ε_i is given by Lemma 3.6.1, we have

$$\int\limits_{B_R} \left[(x.\nabla u)\Delta v + (x.\nabla v)\Delta u - (N-2)\nabla u.\nabla v\right]dx = \int\limits_{|x|=R} R\left(2u'v' - \nabla u.\nabla v\right)d\sigma_R. \tag{3.74}$$

The Rellich-Pohozaev identity follows from (3.70), (3.72), and (3.74). □

For the proof of Lemma 3.2.4, we need the following lemma (see [4, Lemma 3.2] and [1]).

Lemma 3.6.2. *Assume $h \in L^\infty(B_3 \setminus B_{1/2})$ is nonnegative, and $u \geq 0$ satisfies*

$$-\Delta u \geq h(x) \quad \text{in } B_3 \setminus B_{1/2}.$$

There exists a constant $C = C(N)$ such that

$$\inf_{B_2\setminus B_1} u \geq C\int_{B_2\setminus B_1} h(x)dx. \tag{3.75}$$

Proof of Lemma 3.2.4. Let $m_1(R) = \inf_{B_{2R}\setminus B_R} u$, $m_2(R) = \inf_{B_{2R}\setminus B_R} v$. It follows from Lemma 3.6.2 that

$$m_1(R) \geq CR^{2-N} \int\limits_{B_{2R}\setminus B_R} |x|^a v^p dx \geq CR^{2+a} m_2^p(R), \quad R > \rho, \tag{3.76}$$

$$m_2(R) \geq CR^{2-N} \int\limits_{B_{2R}\setminus B_R} |x|^b u^q dx \geq CR^{2+b} m_1^q(R), \quad R > \rho. \tag{3.77}$$

Therefore,

$$m_1(R) \geq CR^{2+a+p(2+b)} m_1^{pq}(R), \quad m_2(R) \geq CR^{2+b+q(2+a)} m_2^{pq}(R), \quad R > \rho, \tag{3.78}$$

hence

$$m_1(R) \leq CR^{-\alpha-\frac{a+bp}{pq-1}}, \quad m_2(R) \leq CR^{-\beta-\frac{b+aq}{pq-1}}.$$

Combining this with (3.76) and (3.77), we have the desired estimates in Lemma 3.2.4. □

Acknowledgement. The author would like to thank Professor Philippe Souplet for valuable suggestions, and the referee for a careful reading and useful comments.

Bibliographie

[1] Scott N. Armstrong and Boyan Sirakov. Nonexistence of positive supersolutions of elliptic equations via the maximum principle. *Comm. Partial Differential Equations*, 36(11) :2011–2047, 2011.

[2] Marie-Francoise Bidaut-Veron. Local behaviour of the solutions of a class of nonlinear elliptic systems. *Adv. Differential Equations*, 5(1-3) :147–192, 2000.

[3] Marie Françoise Bidaut-Veron and Hector Giacomini. A new dynamical approach of Emden-Fowler equations and systems. *Adv. Differential Equations*, 15(11-12) :1033–1082, 2010.

[4] Haïm Brezis and Xavier Cabré. Some simple nonlinear PDE's without solutions. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)*, 1(2) :223–262, 1998.

[5] Jérôme Busca and Raul Manásevich. A Liouville-type theorem for Lane-Emden systems. *Indiana Univ. Math. J.*, 51(1) :37–51, 2002.

[6] Marta Calanchi and Bernhard Ruf. Radial and non radial solutions for Hardy-Hénon type elliptic systems. *Calc. Var. Partial Differential Equations*, 38(1-2) :111–133, 2010.

[7] Wenxiong Chen and Congming Li. An integral system and the Lane-Emden conjecture. *Discrete Contin. Dyn. Syst.*, 24(4) :1167–1184, 2009.

[8] Djairo G. de Figueiredo, Ireneo Peral, and Julio D. Rossi. The critical hyperbola for a Hamiltonian elliptic system with weights. *Ann. Mat. Pura Appl. (4)*, 187(3) :531–545, 2008.

[9] Mostafa Fazly and Nassif Ghoussoub. On the Hénon-Lane-Emden conjecture. *arXiv :1107.5611.*

[10] Basilis Gidas and Joel Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.*, 34(4) :525–598, 1981.

[11] Basilis Gidas and Joel Spruck. A priori bounds for positive solutions of nonlinear elliptic equations. *Comm. Partial Differential Equations*, 6(8) :883–901, 1981.

[12] Haiyang He and Jianfu Yang. Asymptotic behavior of solutions for Hénon systems with nearly critical exponent. *J. Math. Anal. Appl.*, 347(2) :459–471, 2008.

[13] Chang-Shou Lin. A classification of solutions of a conformally invariant fourth order equation in $\mathbf{R}^n$. *Comment. Math. Helv.*, 73(2) :206–231, 1998.

[14] Enzo Mitidieri. Nonexistence of positive solutions of semilinear elliptic systems in $\mathbf{R}^N$. *Differential Integral Equations*, 9(3) :465–479, 1996.

[15] Enzo Mitidieri and Stanislav I. Pohozaev. A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities. *Tr. Mat. Inst. Steklova*, 234 :1–384, 2001.

[16] Quoc Hung Phan and Philippe Souplet. Liouville-type theorems and bounds of solutions of Hardy-Hénon equations. *J. Differential Equations*, 252(3) :2544–2562, 2012.

[17] Peter Poláčik, Pavol Quittner, and Philippe Souplet. Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems. *Duke Math. J.*, 139(3) :555–579, 2007.

[18] James Serrin and Henghui Zou. Non-existence of positive solutions of Lane-Emden systems. *Differential Integral Equations*, 9(4) :635–653, 1996.

[19] James Serrin and Henghui Zou. Existence of positive solutions of the Lane-Emden system. *Atti Sem. Mat. Fis. Univ. Modena*, 46(suppl.) :369–380, 1998. Dedicated to Prof. C. Vinti (Italian) (Perugia, 1996).

[20] Philippe Souplet. The proof of the Lane-Emden conjecture in four space dimensions. *Adv. Math.*, 221(5) :1409–1427, 2009.

Chapitre 4

Singularity and blow-up estimates via Liouville-type theorems for Hardy-Hénon parabolic equations

Singularity and blow-up estimates via Liouville-type theorems for Hardy-Hénon parabolic equations [6]

Quoc Hung Phan

Abstract. We consider the Hardy-Hénon parabolic equation $u_t - \Delta u = |x|^a |u|^{p-1}u$ with $p > 1$ and $a \in \mathbb{R}$. We establish the space-time singularity and decay estimates, and Liouville-type theorems for radial and nonradial solutions. As applications, we study universal and a priori bound of global solutions as well as the blow-up estimates for the corresponding initial boundary value problem.

4.1 Introduction

In this paper, we study the semilinear parabolic equation of the form

$$u_t - \Delta u = |x|^a |u|^{p-1}u, \quad (x,t) \in \Omega \times I \tag{4.1}$$

where Ω is a domain of $\mathbb{R}^N$, $p > 1$, and I is an interval of $\mathbb{R}$. We assume throughout that $a > -2$ when $N \geq 2$, and $a > -1$ when $N = 1$.

Throughout this paper, unless otherwise specified, solutions are considered in the class

$$\begin{cases} C^{2,1}(\Omega \times I), & \text{if } a \geq 0, \\ C^{2,1}(\Omega \setminus \{0\} \times I) \cap C^{0,0}(\Omega \times I), & \text{if } a < 0, \end{cases} \tag{4.2}$$

and are assumed to satisfy the equation pointwise, except at $x = 0$ if $a < 0$ and $0 \in \Omega$. This choice is natural since we are primarily interested in classical solutions, except for possible singularity at the origin if $a < 0$ and $0 \in \Omega$. For $N = 1$ (and $-1 < a < 0$ and $0 \in \Omega$), we instead consider distributional solutions which belong to $C^{0,0}(\Omega \times I)$.

The restriction $a > -2$ when $N \geq 2$ is reasonable due to the regularity at the origin of stationary solutions (cf. [5, Lemma 6.2], [8, 10]). In this case, it turns out that any (classical) solution in the sense (4.2) is also a distributional solution (see Lemma 4.5.1 in Appendix). The case $N = 1$ is more peculiar – see Proposition 4.5.1 and the preceding paragraph.

For the statement of main results, let us introduce the following exponents :

$$p_S(a) := \begin{cases} \frac{N+2+2a}{N-2} & \text{if} \quad N \geq 3, \\ \infty & \text{if} \quad N = 1, 2, \end{cases} \tag{4.3}$$

$p_S := p_S(0)$ and

$$p_B := \begin{cases} \frac{N(N+2)}{(N-1)^2} & \text{if} \quad N \geq 2, \\ \infty & \text{if} \quad N = 1. \end{cases} \tag{4.4}$$

4.1.1 Liouville-type theorems

As the first topic, we are interested in the Liouville property – i.e. the nonexistence of solution of problem (4.1) in the entire space $\mathbb{R}^N \times \mathbb{R}$. We first recall its elliptic counterpart

$$-\Delta u = |x|^a |u|^{p-1}u, \quad x \in \mathbb{R}^N. \tag{4.5}$$

The Liouville-type result for (4.5) plays an important role in the parabolic problem but it is not completely solved. For radial solutions, the problem (4.5) has no positive radial solution if

6. to appear in J. Evol. Equ.

and only if $p < p_S(a)$ and it has been conjectured that the nonexistence of positive solution holds under that condition. However, the Liouville-type result for (4.5) was only proved under stronger assumption, namely $p < \min\{p_S, p_S(a)\}$, which is not optimal when $a > 0$. Recently, the conjecture was shown in [22] for bounded positive solution in dimension $N = 3$.

For corresponding parabolic equation, the Liouville property has been studied in special case $a = 0$ for nonnegative and nodal radial solutions (see [3, 4, 24, 26]). The following results are known to be true.

Theorem A.

(i) Let $a = 0$ and $1 < p < p_S$. Then equation (4.1) has no nontrivial nonnegative radial solution in $\mathbb{R}^N \times \mathbb{R}$.

(ii) Let $a = 0$ and $1 < p < p_B$. Then equation (4.1) has no nontrivial nonnegative solution in $\mathbb{R}^N \times \mathbb{R}$.

Theorem B.

(i) Let $a = 0$, $1 < p < p_S$ and let $u = u(r,t)$ be a classical radial solution of (4.1) in $\mathbb{R}^N \times \mathbb{R}$ with the number of sign-changes satisfying

$$z_{(0,\infty)}(u(t)) \leq M, \quad \forall t \in \mathbb{R}.$$

Then $u \equiv 0$.

(ii) Let $a = 0$, $N = 1$ and let $u = u(x,t)$ be a classical solution of (4.1) in $\mathbb{R} \times \mathbb{R}$ with the number of sign-changes satisfying

$$z_{\mathbb{R}}(u(t)) \leq M, \quad \forall t \in \mathbb{R}.$$

Then $u \equiv 0$.

Theorem A was shown in [4, 26, 24], and Theorem B is recently proved in [3]. The upper bound of exponent p in Theorem A(i) and in Theorem B(i) is optimal due to the existence of positive (bounded) radial solution of $-\Delta u = |u|^{p-1}u$ in $\mathbb{R}^N$ for $p \geq p_S$.

For case $a \neq 0$, the Liouville property is much less understood even for radial solution. Up to now, the only available result of this kind is the Fujita-type (see [23], or [21, section 26]), which states there is no positive solution in $\mathbb{R}^N \times \mathbb{R}_+$ if and only if $1 < p \leq 1 + \frac{2+a}{N}$. In this paper, we will establish Liouville-type theorems in case $a \neq 0$ for a larger range of p. We have the following results.

Theorem 4.1.1. *(i) Let $1 < p < \min\{p_B, p_S(a)\}$ and u be bounded nonnegative solution of equation (4.1) in $\mathbb{R}^N \times \mathbb{R}$. Then $u \equiv 0$.*

(ii) Let $1 < p < p_S(a)$ and u be bounded nonnegative radial solution of equation (4.1) in $\mathbb{R}^N \times \mathbb{R}$. Then $u \equiv 0$.

For sign-changing solution, let us recall the definition of zero number. Given an open interval $I \subset \mathbb{R}$ and $v \in C(I)$, then the zero number of v in I is defined by

$$z_I(v) := \sup\{j : \exists x_1, ..., x_{j+1} \in I, x_1 < x_2 < ... < x_{j+1}, v(x_i)v(x_{i+1}) < 0, \text{ for } i = 1, ..., j\}.$$

We have the following result.

Theorem 4.1.2. *Let $1 < p < p_S(a)$ and let $u = u(r,t)$ be a radial solution of (4.1) in $\mathbb{R}^N \times \mathbb{R}$ with the number of sign-changes satisfying*

$$z_{(0,\infty)}(u(t)) \leq M, \quad \forall t \in \mathbb{R}.$$

Then $u \equiv 0$.

The proofs of Theorem 4.1.1 and Theorem 4.1.2 follow the idea as in [3, 1, 30], which consists of three steps :

1. Showing spatial decay of solutions (see Theorem 4.1.3(ii) and Theorem 4.1.4(ii) below).
2. Using the Lyapunov functional and decay estimate of solutions to show that both α- and ω-limit sets of any solution are nonempty and consist of equilibria.
3. Combining with the nonexistence of nontrivial equilibria to have the contradiction.

Remarks 4.1.1. (a) We note that the condition $p < p_S(a)$ in Theorem 4.1.1 (ii) and Theorem 4.1.2 is optimal, due to the existence of bounded positive radial solution of $-\Delta u = |x|^a u^p$ in $\mathbb{R}^N$ for $p \geq p_S(a)$.

(b) Theorem 4.1.1(ii) for $a > 0$ can be proved by another, completely different method , namely intersection-comparison argument (see [24]). For this case, the proof is totally similar to that in [24].

(c) Related to Theorem 4.1.1, it is a natural conjecture that the nonexistence of entire nonnegative nontrivial solution holds for $p < p_S(a)$. However, it seems still difficult, even for special case $a = 0$.

4.1.2 Singularity and decay estimates

As the next topic, we establish the space-time singularity and decay estimates of solutions of equation (4.1). The following theorem is a parabolic counterpart of [22, Theorem 1.2]. The similar results for $a = 0$ have been proved in [26, Theorem 3.1].

Theorem 4.1.3. *(i) Let u be a nonnegative solution of (4.1) on $\Omega \times (0,T)$ where $\Omega = \{0 < |x| < \rho\}$. Assume that either*

$$p < p_B, \qquad \text{or} \quad u \text{ is radial.} \tag{4.6}$$

Then for all $0 < |x| < \rho/2$ and $t \in (0,T)$, there holds

$$|x|^{a/(p-1)}u(x,t) + \left||x|^{a/(p-1)}\nabla u(x,t)\right|^{\frac{2}{p+1}} \leq C\left(t^{-1/(p-1)} + (T-t)^{-1/(p-1)} + |x|^{-2/(p-1)}\right), \tag{4.7}$$

where $C = C(N,p,a)$.

(ii) Let u be a nonnegative solution of (4.1) in $\Omega \times (0,T)$ where $\Omega = \{|x| > \rho\}$. Assume that either

$$p < p_B, \qquad \text{or} \quad u \text{ is radial.}$$

Then for all $|x| > 2\rho$ and $t \in (0,T)$, there holds

$$|x|^{a/(p-1)}u(x,t) + \left||x|^{a/(p-1)}\nabla u(x,t)\right|^{\frac{2}{p+1}} \leq C\left(t^{-1/(p-1)} + (T-t)^{-1/(p-1)} + |x|^{-2/(p-1)}\right), \tag{4.8}$$

where $C = C(N,p,a)$.

For sign-changing solution, we have the following. We stress that there is no restriction on the upper bound of exponent p.

Theorem 4.1.4. *(i) Let $u = u(r,t)$ be a radial solution of (4.1) on $\Omega \times (0,T)$ where $\Omega = \{0 < |x| = r < \rho\}$ with the number of sign-changes satisfying*

$$z_{(0,\rho)}(u(t)) \leq M, \quad \forall t \in (0,T).$$

Then for all $0 < r < \rho/2$ *and* $t \in (0, T)$*, there holds*

$$r^{a/(p-1)}|u(r,t)| + \left|r^{a/(p-1)}u_r(r,t)\right|^{2/(p+1)} \leq C\left(t^{-1/(p-1)} + (T-t)^{-1/(p-1)} + r^{-2/(p-1)}\right)$$

where $C = C(N, p, a, M)$.

(ii) Let $u = u(r,t)$ *be a radial solution in* $\Omega \times (0,T)$ *where* $\Omega = \{|x| = r > \rho\}$ *with the number of sign-changes satisfying*

$$z_{(\rho,\infty)}(u(t)) \leq M, \quad \forall t \in (0,T).$$

Then for all $r > 2\rho$ *and* $t \in (0, T)$*, there holds*

$$r^{a/(p-1)}|u(r,t)| + \left|r^{a/(p-1)}u_r(r,t)\right|^{2/(p+1)} \leq C\left(t^{-1/(p-1)} + (T-t)^{-1/(p-1)} + r^{-2/(p-1)}\right)$$

where $C = C(N, p, a, M)$.

The proofs of Theorem 4.1.3 and Theorem 4.1.4 rely on :

1. a change of variable, that allows to replace the coefficient $|x|^a$ with a smooth function which is bounded and bounded away from 0 in a suitable spatial domain ;
2. a generalization of a doubling-rescaling argument from [25] (see Lemma 4.2.1 below).
3. The corresponding Liouville-type theorem for equation (1) with $a = 0$.

Remarks 4.1.2. (a) The estimates in Theorem 4.1.4 in case $a = 0$ give a similar form as in [19, Corollary 3.2] and [20, Proposition 2.5 and 2.7] for radial solutions of supercritical nonlinear heat equation. As an improvement, the constants C are here universal, but at expense of further restriction on finite number of sign-changes of solutions. Our argument is based on rescaling and doubling property while that one in [19, 20] is based on energy estimates.

(b) If we replace the interval $(0, T)$ by $\mathbb{R}$ in Theorem 4.1.3(ii) and in Theorem 4.1.4(ii), then we have the spatial decay estimate

$$|u(x,t)| \leq C|x|^{-(2+a)/(p-1)}, \quad |\nabla u(x,t)| \leq C|x|^{-(p+1+a)/(p-1)}, \quad |x| > 0, \ t \in \mathbb{R}.$$

This is an important feature that will be used in proof of Theorem 4.1.1 and Theorem 4.1.2.

4.1.3 A priori bound of global solutions and blow-up estimates

As applications of Liouville-type results, let us consider the corresponding initial-boundary value problem :

$$\begin{cases} u_t - \Delta u = |x|^a u^p, & x \in \Omega, \ 0 < t < T, \\ u = 0, & x \in \partial\Omega, \ 0 < t < T, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{4.9}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and contains the origin and initial data $u_0 \in L^\infty(\Omega)$. The well-posedness and the global existence of solutions of problem (4.9) were proved in [32]. We denote by u the maximal mild solution in $L^\infty_{loc}([0,T); L^\infty(\Omega))$ and by $T = T_{max}(u_0)$ its existence time. We are here interested in a priori bound of nonnegative solutions. Our result is as follows.

Theorem 4.1.5. *Let* $1 < p < \min(p_S, p_S(a))$*. Assume* u *is any global solution of problem (4.9) with initial data* $u_0 \geq 0$*. Then*

$$\sup_{t \geq 0} \|u(t)\|_\infty \leq C(\|u_0\|_\infty). \tag{4.10}$$

Moreover, if $\Omega = B_R$ *and* u_0 *is radial then (4.10) still holds whenever* $1 < p < p_S(a)$.

Remarks 4.1.3. We recall that a priori bound of nonnegative solutions of elliptic problem $-\Delta u = |x|^a u^p$ has been proved under the condition $p < \min(p_S, p_S(a))$ (see [22, Theorem 1.3]). Theorem 4.1.5 say that a priori bound (4.10) for parabolic counterpart also holds under this condition. In special case $a = 0$, such a priori bound was proved by Giga [12] for nonnegative solutions, and by Quittner [27] for sign-changing solutions.

We next give results of universal initial and final time blow-up rates. The similar result for case $a = 0$ has been proved in [26]. The final time blow-up estimate of problem (4.9) was established in [2, Theorem 1.2 and 1.3], under a stronger condition $1 < p < 1 + \min\{2/N, (2+a)/N\}$.

Theorem 4.1.6. *Let u be a positive solution of (4.9). Assume that either*

$$p < \min\{p_B, p_S(a)\}, \qquad \text{or } p < p_S(a),\ \Omega \text{ is a ball } B_R \text{ and } u \text{ is radial.} \tag{4.11}$$

(i) If $T < \infty$ then there holds

$$u(x,t) \le C\left(1 + t^{-1/(p-1)} + (T-t)^{-1/(p-1)}\right), \quad x \in \Omega,\ 0 < t < T, \tag{4.12}$$

where $C = C(\Omega, p, a)$.

(ii) If u is global then there holds

$$u(x,t) \le C\left(1 + t^{-1/(p-1)}\right), \quad x \in \Omega,\ t > 0, \tag{4.13}$$

where $C = C(\Omega, p, a)$.

Theorem 4.1.6(ii) in particular implies universal bounds, away from $t = 0$, for all global solutions of problem (4.9). In last result, we provide such bounds under different assumptions on p, a and N. This result gives a less precise conclusion than that in Theorem 4.1.6(ii) but it can be applied in a different range of parameters, due to a completely different method. Whereas Theorem 4.1.6 relies on Liouville theorems and doubling arguments, the method of proof of Theorem 4.1.7 is different, based on a combination of energy and rescaling arguments (see [28, 31]).

Theorem 4.1.7. *Let $a > 0$, $N \le 4$, and $1 < p < \frac{N+2+a}{N-2+a}$ ($1 < p < \infty$ when $N = 1$). Then for all $\tau > 0$, there exists $C = C(\Omega, p, a, \tau)$ such that any nonnegative global solution of problem (4.9) satisfies*

$$\sup_{t \ge \tau} \|u(t)\|_\infty \le C(\Omega, p, a, \tau). \tag{4.14}$$

In this paper, the proofs of Theorem 4.1.3-4.1.7 all make use of rescaling techniques, combined with some additional arguments, such as, doubling properties, parabolic Liouville-type theorems (for both case $a = 0$ and $a \ne 0$), or energy arguments. The classical rescaling argument was first introduced by Gidas and Spruck ([11]) for elliptic problem, it was then significantly improved in [12, 16, 25, 26] for elliptic and parabolic problems. In particular, the authors in [25, 26] have developed the doubling property (which is an extension of an idea of [16]) that enables one to obtain a variety of important results such as : singularity and decay estimates, a priori bound and universal bounds of solutions, etc.... We essentially employ this powerful idea and introduce some new rescalings to deal with some new difficulties arising due to the degeneracy and singularity of the term $|x|^a$. We intend to provide the details of various rescaling arguments in order to make precise the differences among cases.

We close the introduction by mentioning other work related to problem (4.9). The Cauchy problem corresponding to problem (4.9) (i.e. $\Omega = \mathbb{R}^n$) has been widely studied, and the existence and nonexistence of global solution were established [23, 15, 9]. The asymptotics, stabilization and blow-up phenomenon of the Cauchy problem are considered in [32, 7, 17]. The blow-up phenomenon for initial-boundary value problem (4.9) can be found in [14, 13], where the authors constructed a special solution that blows up at the origin, and also gave some sufficient conditions that ensure the origin is not a blow-up point.

The organization of this paper is as follows. Section 2 contains the proof of the singularity and decay estimates (Theorem 4.1.3 and 4.1.4). Section 3 contains the proof of Liouville-type theorems. In Section 4, we give the proofs of Theorems 4.1.5-4.1.7.

4.2 Singularity and decay estimates

In this Section, we give a relatively simple proof of Theorem 4.1.3. Theorem 4.1.4 can be proved by the same argument. We need the following lemma.

Lemma 4.2.1. *Let $\mathcal{C} = \{x \in \mathbb{R}^N : 1 < |x| < 2\}$, $\alpha \in (0,1]$ and $c \in C^\alpha(\overline{\mathcal{C}})$ be a function satisfying*

$$\|c\|_{C^\alpha(\overline{\mathcal{C}})} \leq C_1 \qquad \text{and} \qquad c(x) \geq C_2, \quad x \in \overline{\mathcal{C}}, \tag{4.15}$$

for some constants $C_1, C_2 > 0$. Let u be positive classical solution of

$$u_t - \Delta u = c(x)u^p, \quad (x,t) \in \mathcal{C} \times \mathbb{R}. \tag{4.16}$$

Assume that either

$$p < p_B, \qquad \text{or} \;\; c, u \text{ are radial.} \tag{4.17}$$

Then there exists a constant $C = C(\alpha, C_1, C_2, p, N)$, such that, for all $(x,t) \in \mathcal{C} \times (0,T)$, there holds

$$|u(x,t)| + |\nabla u(x,t)|^{2/(p+1)} \leq C\big(1 + t^{-1/(p-1)} + (T-t)^{-1/(p-1)} + \mathrm{dist}^{-2/(p-1)}(x, \partial\mathcal{C})\big). \tag{4.18}$$

Proof. We follow the argument in [26], we denote the parabolic distance

$$d_P((x,t),(y,s)) := |x-y| + |t-s|^{1/2}. \tag{4.19}$$

Let $D = \mathcal{C} \times (0,T) \in \mathbb{R}^{N+1}$ then the estimate (4.18) can be written as

$$u(x,t) + |\nabla u(x,t)|^{2/(p+1)} \leq C\big(1 + d^{-2/(p-1)}((x,t), \partial D)\big), \; (x,t) \in D \tag{4.20}$$

Arguing by contradiction, we suppose that there exist sequences c_k, u_k, T_k verifying (4.15), (4.16) and points (y_k, τ_k), such that the functions

$$M_k = |u_k|^{(p-1)/2} + |\nabla u_k|^{(p-1)/(p+1)}$$

satisfy

$$M_k(y_k, \tau_k) > 2k\big(1 + d_P^{-1}((y_k,\tau_k), \partial D_k)\big) > 2k\, d_P^{-1}((y_k,\tau_k), \partial D_k\big), \quad D_k = \mathcal{C} \times (0, T_k).$$

By the doubling lemma in [25, Lemma 5.1] with $X = \mathbb{R}^{N+1}$, equipped with parabolic distance d_P, there exists $(x_k, t_k) \in D_k$ such that

$$M_k(x_k,t_k) \geq M_k(y_k,\tau_k), \quad M_k(x_k,t_k) > 2k\, d_P^{-1}((x_k,t_k), \partial D_k),$$

and

$$M_k(x,t) \leq 2M_k(x_k,t_k), \qquad \text{for all } (x,t) \text{ such that } d_P((x,t),(x_k,t_k)) \leq kM_k^{-1}(x_k,t_k). \tag{4.21}$$

We note that (x,t) satisfying (4.21) is automatically contained in D_k.

We have

$$\lambda_k := M_k^{-1}(x_k,t_k) \to 0 \quad \text{as } k\to\infty, \tag{4.22}$$

due to $M_k(x_k,t_k) \geq M_k(y_k,\tau_k) > 2k$.

We now consider the nonradial and radial cases separately.

A. The nonradial case. Let

$$v_k(y,s) = \lambda_k^{2/(p-1)} u_k(x_k+\lambda_k y, t_k+\lambda_k^2 s), \qquad \tilde c_k(y) = c_k(x_k+\lambda_k y), \quad (y,s)\in \tilde D_k,$$

where

$$\tilde D_k = \{|y|<k/2\}\times\{|s|<k^2/4\}.$$

We note that $|v_k|^{(p-1)/2}(0,0)+|\nabla v_k|^{(p-1)/(p+1)}(0,0)=1$,

$$\Big[|v_k|^{(p-1)/2}+|\nabla v_k|^{(p-1)/(p+1)}\Big](y,s)\leq 2, \quad (y,s)\in\tilde D_k, \tag{4.23}$$

due to (4.21), and we see that v_k satisfies

$$\partial_s v_k - \Delta v_k = \tilde c_k(y) v_k^p, \quad (y,s)\in \tilde D_k. \tag{4.24}$$

On the other hand, due to (4.15), we have $C_2\leq \tilde c_k\leq C_1$ and, for each $R>0$ and $k\geq k_0(R)$ large enough,

$$|\tilde c_k(y)-\tilde c_k(z)| \leq C_1|\lambda_k(y-z)|^\alpha \leq C_1|y-z|^\alpha, \quad |y|,|z|\leq R. \tag{4.25}$$

Therefore, by Ascoli's theorem, there exists $\tilde c$ in $C(\mathbb{R}^N)$, with $\tilde c\geq C_2$ such that, after extracting a subsequence, $\tilde c_k\to\tilde c$ in $C_{loc}(\mathbb{R}^N)$. Moreover, (4.25) and (4.22) imply that $|\tilde c_k(y)-\tilde c_k(z)|\to 0$ as $k\to\infty$, so that the function $\tilde c$ is actually a constant $C>0$.

Now, for each $R>0$ and $1<q<\infty$, by (4.24), (4.23) and interior parabolic L^q estimates, the sequence v_k is uniformly bounded in $W_q^{2,1}(B_R\times(-R,R))$. Using standard embeddings and interior parabolic L^q estimates, after extracting a subsequence, we may assume that $v_k\to v$ in $C^{2,1}_{loc}(\mathbb{R}^N\times\mathbb{R})$. It follows that $v\geq 0$ is a classical solution of

$$v_s - \Delta v = Cv^p, \quad (y,s)\in\mathbb{R}^N\times\mathbb{R},$$

and $|v|^{(p-1)/2}(0,0)+|\nabla v|^{(p-1)/(p+1)}(0,0)=1$. Since $p<p_B$, this contradicts Theorem A(ii).

B. The radial case. Since u_k, c are radial, we write $u_k=u_k(r,t)$, $c_k=c_k(r)$ and $M_k=M(r,t)$, where $r=|x|$. Then u_k solves the equation

$$u_t - u_{rr} - \frac{N-1}{r}u_r = c_k(r)u^p.$$

Assume that $|x_k|=r_k$, it follows from (4.21) that

$$M_k(r,t)\leq 2M_k(r_k,t_k), \qquad \text{for all } (r,t) \text{ such that } |r-r_k|+|t-t_k|^{1/2}\leq k\lambda_k := kM_k^{-1}(r_k,t_k).$$

We rescale by

$$v_k(\rho,s) := \lambda_k^{2/(p-1)}u_k(r_k+\lambda_k\rho, t_k+\lambda_k^2 s), \quad (\rho,s)\in\tilde D_k,$$

where

$$\tilde D_k = \big(-\min(r_k/\lambda_k,\, k/2),\ k/2\big)\times(-k^2/4, k^2/4).$$

Then v_k solves the equation

$$v_s - v_{\rho\rho} - \frac{N-1}{\rho+r_k/\lambda_k}v_\rho = \tilde c_k v^p, \quad \tilde c_k(\rho)=c_k(r_k+\lambda_k\rho), \quad (\rho,s)\in\tilde D_k,$$

and we note that $|v_k|^{(p-1)/2}(0,0) + |\nabla v_k|^{(p-1)/(p+1)}(0,0) = 1$,

$$\Big[|v_k|^{(p-1)/2} + |\nabla v_k|^{(p-1)/(p+1)}\Big](\rho,s) \le 2, \quad (\rho,s) \in \tilde{D}_k.$$

Similar to the nonradial case, after extracting a subsequence, we may assume that $\tilde{c}_k(\rho) \to C$ in $C_{loc}(\mathbb{R})$. Since $r_k \in (1,2)$ then $r_k/\lambda_k \to \infty$. Passing to the limit, we obtain a nonnegative bounded solution v of the equation

$$v_s - v_{\rho\rho} = Cv^p \quad \text{in} \quad \mathbb{R}\times\mathbb{R}$$

and

$$|v|^{(p-1)/2}(0,0) + |\nabla v|^{(p-1)/(p+1)}(0,0) = 1.$$

This contradicts Theorem A(ii) for $N = 1$ and concludes the proof. □

Proof of Theorem 4.1.3. Assume either $\Omega = \{x \in \mathbb{R}^N;\ 0 < |x| < \rho\}$ and $0 < |x_0| < \rho/2$, or $\Omega = \{x \in \mathbb{R}^N;\ |x| > \rho\}$ and $|x_0| > 2\rho$. Let $R = \frac{2}{3}|x_0|$ and we denote

$$U(y,s) = R^{\frac{2+a}{p-1}} u(Ry, R^2 s).$$

Then U is a solution of

$$U_s - \Delta U = c(y)U^p, \quad (y,s) \in \mathcal{C}\times(0, R^{-2}T), \qquad \text{with } c(y) = |y|^a, \ \ \mathcal{C} = \{y \in \mathbb{R}^N : 1 < |y| < 2\}.$$

Notice that $|y| \in [1,2]$ for all $y \in \overline{\mathcal{C}}$. Moreover $\|c\|_{C^1(\overline{\mathcal{C}})} \le C(a)$. Then applying Lemma 4.2.1, we have

$$U(R^{-1}x_0, R^{-2}t) + |\nabla U(R^{-1}x_0, R^{-2}t)|^{2/(p+1)} \le C\big(1 + (R^{-2}t)^{-1/(p-1)} + (R^{-2}T - R^{-2}t)^{-1/(p-1)}\big),$$

hence

$$R^{a/(p-1)}u(x_0,t) + \big|R^{a/(p-1)}\nabla u(x_0,t)\big|^{2/(p+1)} \le C\big(t^{-1/(p-1)} + (T-t)^{-1/(p-1)} + R^{-2/(p-1)}\big),$$

which yields the desired conclusion. □

Proof of Theorem 4.1.4. By the similar argument, we have the same estimate (4.18) for radial solution with finite number of sign-changes. The only thing taken into consideration is that, in the last step of proof of Lemma 4.2.1, we have a contradiction with Liouville-type theorem for nodal solution (see [3, Theorem 1.4]). The rest of proof is similar. □

4.3 Liouville-type theorem

In this section, we will only prove Theorem 4.1.1. And by this method, one can prove Theorem 4.1.2 similarly. The proof is based on properties of energy functional. We note that the solutions in the case of the whole space $\mathbb{R}^N$ need not a priori belong to the energy space. However, as shown in the following lemma, this will turn out to be true thanks to the spatial decay estimates in (4.28). Moreover the case $a < 0$ is more delicate and requires additional arguments.

For u solution of equation (4.1) in $\mathbb{R}^N \times \mathbb{R}$, we denote (formally) the energy functional

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(t)|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |x|^a u^{p+1}(t)dx. \tag{4.26}$$

Lemma 4.3.1. *Assume $p < p_S(a)$ and (4.6). For any solution u of equation (4.1) in $\mathbb{R}^N\times\mathbb{R}$, the energy functional (4.26) is well defined for any $t \in \mathbb{R}$. Moreover, for any $t_1 < t_2$, we have*

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N} u_t^2(t)dxdt \le -E(t_2) + E(t_1). \tag{4.27}$$

Proof. By Theorem 4.1.3 (see Remark 4.1.2(b)), u has spatial decay estimates

$$u(x,t) \le C|x|^{-\frac{2+a}{p-1}}, \qquad |\nabla u(x,t)| \le C|x|^{-1-\frac{2+a}{p-1}}, \quad |x|>1. \tag{4.28}$$

We first show that

$$|u_t(x,t)| \le C|x|^{-2-\frac{2+a}{p-1}}, \quad |x|>2. \tag{4.29}$$

Indeed, for any $R>2$, let $U(y,s) = R^{(2+a)/(p-1)}u(Ry,R^2s)$ then $U(y,s) \le C$ in $(B_4 \setminus B_{1/2}) \times \mathbb{R}$, where $C = C(N,p,a)$, and

$$U_s - \Delta U = |y|^a U^p, \quad (y,s) \in (B_4 \setminus B_{1/2}) \times \mathbb{R}$$

It follows from bootstrap argument for parabolic regularity that $|U_s(y,s)| \le C$ for all $(y,s) \in (B_2 \setminus B_1) \times \mathbb{R}$, where C does not depend on R. Hence,

$$|u_t(x,t)| \le CR^{-2-(2+a)/(p-1)} \le C|x|^{-2-(2+a)/(p-1)}, \quad (x,t) \in (B_{2R} \setminus B_R) \times \mathbb{R},$$

and (4.29) follows.

Combining these decay estimates with $p < p_S(a)$, we have for any $t \in \mathbb{R}$,

$$|\nabla u(t)|^2 \in L^1(\mathbb{R}^N \setminus B_1), \;\; u_t^2(t) \in L^1(\mathbb{R}^N \setminus B_1), \qquad |x|^a u^{p+1}(t) \in L^1(\mathbb{R}^N).$$

Hence, if $a \ge 0$, since there is no singularity at $x=0$, the energy functional (4.26) is well defined and (4.27) holds since

$$\frac{dE(t)}{dt} = -\|u_t(t)\|_2^2.$$

We now consider case $a<0$. Since the term $f = |x|^a u^p \in L^\infty_{loc}(\mathbb{R}; L^{\tilde q}(B_2))$ for any $1 < \tilde q < N/|a|$, using the cut-off function and variation-of-constants formula (obtained by Lemma 4.5.2 in Appendix), we have

$$u \in L^\infty_{loc}(\mathbb{R}; W^{2-\delta,\tilde q}(B_1)), \quad 1 < \tilde q < N/|a|.$$

Choose $\delta > 0$ small enough such that $W^{2-\delta,\tilde q}(B_1)$ is continuously embedded into $W^{1,2}(B_1)$. Hence, $|\nabla u(t)|^2 \in L^1(\mathbb{R}^N)$ and the energy functional (4.26) is well defined. To prove (4.27), we may assume that $t_1 = 0, t_2 > 0$, we consider the following problem

$$\begin{cases} \partial_t v_\varepsilon - \Delta v_\varepsilon = (|x|+\varepsilon)^a v_\varepsilon^p, & (x,t) \in \mathbb{R}^N \times [0,t_2], \\ v(x,0) = u(x,0). \end{cases}$$

Note that $a<0$, by comparison property we have v_ε is increasing and $0 < v_\varepsilon \le u$. This implies in particular that v_ε satisfies the first part of spatial estimate (4.28). Let us show spatial decay of ∇v_ε. For any $R>2$, let $V(y,s) = R^{(2+a)/(p-1)}v_\varepsilon(Ry,R^2s)$, then $V(y,s) \le C$ in $(B_4 \setminus B_{1/2}) \times \mathbb{R}$, where $C = C(N,p,a)$, and

$$V_s - \Delta V = (|y| + \varepsilon/R)^a V^p, \quad (y,s) \in (B_4 \setminus B_{1/2}) \times \mathbb{R}$$

For any $\varepsilon \le 1/2$, we have $1/4 < |y| + (\varepsilon/R) < 5$ for all $1/2 < |y| < 4$. The parabolic estimates imply $|\nabla V(y,s)| \le C$ for all $(y,s) \in (B_2 \setminus B_1) \times \mathbb{R}$, where C does not depend on R. Hence, $|\nabla v_\varepsilon(x,t)| \le CR^{-1-(2+a)/(p-1)}$ for all $(x,t) \in (B_{2R} \setminus B_R) \times \mathbb{R}$. Therefore, $|\nabla v_\varepsilon(x,t)| \le C|x|^{-1-(2+a)/(p-1)}$ for any $\varepsilon < 1/2$ and $|x|>2$.

Let $v = \lim v_\varepsilon$ and $e^{t\Delta}$ denote the heat semigroup in $\mathbb{R}^N$, we show that $v = u$. Indeed, by the variation-of-constants formula, we deduce that

$$u(t) - v(t) = \int_0^t e^{(t-s)\Delta}(|.|^a(u^p - v^p))ds = \int_0^t e^{(t-s)\Delta}(|.|^a H(u,v)(u-v))ds,$$

where $0 \le H(u,v) \le pu^{p-1}$. Let $w = u - v$ then $w(0) = 0$ and

$$\|w(t)\|_\infty \le \int_0^t (t-s)^{-N/(2q)} \|w(s)\|_\infty \||x|^a H(u,v)\|_q ds.$$

We choose $q = 1$ when $N = 1$, and $q = N/(|a|+\gamma)$ when $N \ge 2$, where $\gamma > 0$ satisfies $|a|+\gamma < 2$. Then $N/(2q) < 1$ and

$$\||x|^a H(u,v)\|_q \le \|p|x|^a u^{p-1}\|_q \le C.$$

Hence,

$$\|w(t)\|_\infty \le C \int_0^t (t-s)^{-N/(2q)} \|w(s)\|_\infty ds.$$

Consequently, $w \equiv 0$, or $u \equiv v$.

Let us denote by $E_\varepsilon(t)$ the energy functional with respect to v_ε, which is well-defined due to the spatial decay of v_ε and ∇v_ε. Then we have

$$\int_0^{t_2} \int_{\mathbb{R}^N} |\partial_t v_\varepsilon|^2(t) dxdt = -E_\varepsilon(t_2) + E(0). \tag{4.30}$$

Hence

$$\begin{aligned}\int_0^{t_2} \int_{\mathbb{R}^N} |\partial_t v|^2(t) dxdt &\le \liminf_{\varepsilon \to 0} \int_0^{t_2} \int_{\mathbb{R}^N} |\partial_t v_\varepsilon|^2(t) dxdt \\ &\le \liminf_{\varepsilon \to 0} \Big(-\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_\varepsilon(t_2)|^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^N} |x|^a v_\varepsilon^{p+1}(t_2) dx \Big) + E(0)\end{aligned}$$

By monotone convergence, we have

$$\int_{\mathbb{R}^N} |x|^a v_\varepsilon^{p+1}(t_2) dx \to \int_{\mathbb{R}^N} |x|^a u^{p+1}(t_2) dx.$$

On the other hand,

$$\liminf_{\varepsilon \to 0} \Big(-\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_\varepsilon(t_2)|^2 dx \Big) \le -\liminf_{\varepsilon \to 0} \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_\varepsilon(t_2)|^2 dx \le -\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v(t_2)|^2 dx$$

Therefore, (4.27) follows. Lemma is proved. □

Remarks 4.3.1. If we assume in addition that $a > -N/2$ then by parabolic regularity, one can see that $u_t \in L^2_{loc}(\mathbb{R}^N)$ for any $t \in \mathbb{R}$. This combined with spatial decay estimates implies

$$\frac{dE(t)}{dt} = -\|u_t(t)\|^2_{L^2(\mathbb{R}^N)},$$

and Lemma 4.3.1 is then straightforward.

Proof of Theorem 4.1.1. We shall prove (i) and (ii) at the same time. Assume that u is a bounded nontrivial nonnegative solution of (4.1). Then u satisfies spatial decay estimates (4.28). Combining with the boundedness of u and $p < p_S(a)$ we have

$$|E(u(t))| \le C, \quad \forall t \in R.$$

It follows from Lemma 4.3.1 that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^N} u_t^2(x,t) dxdt < \infty.$$

Consequently, there exists $t_k \to \infty$ such that

$$u_t(t_k) \to 0 \quad \text{in} \quad L^2(\mathbb{R}^N). \tag{4.31}$$

We now show that

$$\|u(t_k)\|_{L^\infty(\mathbb{R}^N)} \to 0. \tag{4.32}$$

Indeed, if not then there exists x_k such that $u(x_k, t_k) \geq C > 0$. It follows from spatial decay estimates of u that x_k is bounded. We may assume that $x_k \to x_\infty$. Let $v_k(x) := u(x, t_k)$, then there exists a subsequence which converges in $C_{loc}(\mathbb{R}^N)$ to a function v satisfying

$$-\Delta v = |x|^a v^p,$$

and $v(x_\infty) \geq C$. We note also that if u is radial then so is v. This contradicts Liouville-type theorem for Hardy-Hénon equations (see [6, 22]). Hence (4.32) is true.

Let $\alpha_k = \int_{\mathbb{R}^N} |x|^a u^{p+1}(t_k)dx$, for any $R > 0$ we have

$$\begin{aligned} \alpha_k &= \int_{|x|<R} |x|^a u^{p+1}(t_k)dx + \int_{|x|>R} |x|^a u^{p+1}(t_k)dx \\ &\leq CR^{N+a}\|u(t_k)\|^{p+1}_{L^\infty(\mathbb{R}^N)} + CR^{-[(2+a)(p+1)/(p-1)-N-a]}. \end{aligned}$$

Hence, $\limsup_{k\to\infty} \alpha_k \leq CR^{-[(2+a)(p+1)/(p-1)-N-a]}$. Letting $R \to \infty$, we obtain

$$\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \int_{\mathbb{R}^N} |x|^a u^{p+1}(t_k)dx = 0. \tag{4.33}$$

We next show that $\|\nabla u(t_k)\|_{L^2(\mathbb{R}^N)} \to 0$. Let φ be a smooth function in $\mathbb{R}^N$, $0 \leq \varphi \leq 1$, $\varphi(x) = 0$ in B_1, $\varphi(x) = 0$ if $|x| \geq 2$ and $|\nabla\varphi| \leq C\varphi^{1/2}$. For any $R > 0$, let $\varphi_R(x) = \varphi(x/R)$, then we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u(t_k)|^2 \varphi_R dx &= -\int_{\mathbb{R}^N} u(t_k)\nabla u(t_k).\nabla\varphi_R dx + \int_{\mathbb{R}^N} \big(|x|^a u^{p+1}(t_k) - u_t(t_k)u(t_k)\big)\varphi_R dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(t_k)|^2\varphi_R dx + \frac{1}{2}\int_{\mathbb{R}^N} u^2(t_k)|\nabla\varphi_R|^2\varphi_R^{-1/2}dx \\ &\quad + \int_{\mathbb{R}^N} \big(|x|^a u^{p+1}(t_k) - u_t(t_k)u(t_k)\big)\varphi_R dx. \end{aligned}$$

Using (4.31)-(4.33) and the compact support of φ_R, we deduce that

$$\lim_{k\to\infty} \int_{B_R} |\nabla u(t_k)|^2 dx = 0,$$

for any $R > 0$. On the other hand, if follows from spatial decay estimate of ∇u that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u(t_k)|^2 dx &\leq \int_{B_R} |\nabla u(t_k)|^2 dx + \int_{|x|>R} |\nabla u(t_k)|^2 dx \\ &\leq \int_{B_R} |\nabla u(t_k)|^2 dx + CR^{N-2-\frac{4+2a}{p-1}}. \end{aligned} \tag{4.34}$$

By letting $k \to \infty$ and then $R \to \infty$ in (4.34), we obtain

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} |\nabla u(t_k)|^2 dx = 0. \tag{4.35}$$

Combining this with (4.33) we obtain $E(t_k) \to 0$ as $k \to \infty$.

Similarly, there exist $s_k \to -\infty$ such that $u_t(s_k) \to 0$ in $L^2(\mathbb{R}^N)$ and we deduce that $E(s_k) \to 0$ as $k \to \infty$. It follows from Lemma 4.3.1 that

$$\int_{s_k}^{t_k} \int_{\mathbb{R}^N} u_t^2(t)dxdt \le E(s_k) - E(t_k).$$

Let $k \to \infty$ we obtain $u_t \equiv 0$. This contradicts Liouville-type theorem for Hardy-Hénon elliptic equations (see [6, 22]). □

The proof of Theorem 4.1.2 follows the same steps as in that of Theorem 4.1.1. The only one thing taken into consideration is the Liouville-type elliptic theorem of (4.5) for nodal radial solutions with finite zero number. This is guaranteed by the following lemma.

Lemma 4.3.2. *Assume that $p < p_S(a)$ and u is nodal radial solution of (4.5) with s finite zero number, then $u \equiv 0$.*

Proof. By Theorem 4.1.4, we deduce that u has decay estimates

$$|u(r)| \le Cr^{-(2+a)/(p-1)}, \quad |u_r(r)| \le Cr^{-(p+1+a)/p-1)}.$$

Combining these with Rellich-Pohozaev identity and the argument in [22, Theorem 1.1], we deduce that $u \equiv 0$. Lemma is proved. □

4.4 Problems with boundary condition

In this section, we will prove Theorems 4.1.5-4.1.7. Let u be a nonnegative solution of problem (4.9), and as in the previous section, we denote

$$E(u(t)) = E(t) := \frac{1}{2}\int_\Omega |\nabla u(t)|^2 dx - \frac{1}{p+1}\int_\Omega |x|^a u^{p+1}(t)dx. \tag{4.36}$$

Then $E(u(t))$ is well defined and similar to Lemma 4.3.1, for $t_1 < t_2$, we have

$$\int_{t_1}^{t_2} \int_\Omega u_t^2(t)dxdt \le -E(t_2) + E(t_1). \tag{4.37}$$

We need the following lemma.

Lemma 4.4.1. *Consider problem (4.9) with nonnegative initial data $u_0 \in L^\infty \cap H_0^1(\Omega)$. If $E(u_0) < 0$ then $T_{\max}(u_0) < \infty$.*

Proof. We follow the concavity method in [18] (see also [29, Theorem 17.6]). Assume that $T_{\max}(u_0) = \infty$. Let $M(t) = \frac{1}{2}\int_0^t \|u(s)\|_2^2 ds$ then

$$\begin{aligned} M''(t) &= \int_\Omega uu_t(t)dx = -\int_\Omega |\nabla u(t)|^2 dx + \int_\Omega |x|^a u^{p+1}(t)dx \\ &= -(p+1)E(u(t)) + \frac{p-1}{2}\int_\Omega |\nabla u(t)|^2 dx \\ &\ge -(p+1)E(u_0) > 0. \end{aligned}$$

Consequently, $M'(t) \to \infty$ and $M(t) \to \infty$ as $t \to \infty$. Moreover,

$$M''(t) \ge -(p+1)E(u(t)) \ge -(p+1)E(u(t)) + (p+1)E(u_0) \ge (p+1)\int_0^t \|u_t(s)\|_2^2 ds,$$

hence

$$\begin{aligned} M(t)M''(t) &\geq \frac{p+1}{2}\left(\int_0^t \|u_t(s)\|_2^2 ds\right)\left(\int_0^t \|u(s)\|_2^2 ds\right) \\ &\geq \frac{p+1}{2}\left(\int_0^t \int_\Omega u(x,s)u_t(x,s)dxds\right)^2 \\ &= \frac{p+1}{2}\left(M'(t)-M'(0)\right)^2. \end{aligned}$$

Since $M'(t) \to \infty$ as $t \to \infty$, there exist $\alpha, t_0 > 0$ such that

$$M(t)M''(t) \geq (1+\alpha)(M'(t))^2, \quad t \geq t_0.$$

This guarantees that the nonincreasing function $t \mapsto M^{-\alpha}(t)$ is concave on $[t_0, \infty)$ which contradicts the fact that $M^{-\alpha}(t) \to 0$ as $t \to \infty$. □

Proof of Theorem 4.1.5. Assume that the bound (4.10) does not hold for global nonnegative solutions. Then there exist $t_k > 0$ and $u_{0,k} \geq 0$ such that $\|u_{0,k}\|_\infty \leq C_0$ and the solutions u_k with initial data $u_{0,k}$ satisfying

$$M_k := u_k(x_k, t_k) = \sup\{u_k(x,t) : x \in \Omega,\ t \in [0, t_k]\} \to \infty \quad \text{as} \quad k \to \infty. \tag{4.38}$$

By the point fixed argument, there exists $\delta > 0$ such that

$$u_k(x,t) \leq 2C_0, \quad \forall (x,t) \in \Omega \times [0, \delta].$$

Hence $t_k \geq \delta$ for k large enough. We will show by variation-of-constants formula that

$$\sup_k \|u_k(\delta/2)\|_{H^1(\Omega)} \leq C. \tag{4.39}$$

Indeed, (4.39) is straightforward if $a \geq 0$. If $a < 0$ then

$$\|\nabla u_k(\delta/2\|_2 \leq C\|u_{0,k}\|_2 + C\int_0^{\delta/2} s^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q}-\frac{1}{2})}\||x|^a\|_q \, ds.$$

We now choose $q = 1$ when $N = 1$, and $q = \frac{N}{\gamma+|a|}$ when $N \geq 2$, where $\gamma > 0$ is small such that $\gamma + |a| < 2$. Then $|x|^a \in L^q(\Omega)$ and $\frac{1}{2} + \frac{N}{2}(\frac{1}{q} - \frac{1}{2}) < 1$. Hence (4.39).

Combining (4.39) with Lemma 4.4.1, we obtain

$$0 \leq E(u_k(\delta/2) \leq C. \tag{4.40}$$

We may assume that $x_k \to x_0 \in \bar{\Omega}$. We denote $d_k = \mathrm{dist}(x_k, \partial\Omega)$.

A. Nonradial case.

Case A_1 : $x_0 \in \Omega \setminus \{0\}$. Let

$$v_k(y,s) = \lambda_k^{2/(p-1)} u_k(x_k + \lambda_k y, t_k + \lambda_k^2 s), \quad (y,s) \in Q_k, \tag{4.41}$$

where $\lambda_k = M_k^{-(p-1)/2}$ and $Q_k = \{(y,s) : (x_k + \lambda_k y, t_k + \lambda_k^2 s) \in \Omega \times (0, t_k)\}$. It follows that $0 \leq v_k \leq 1 = v_k(0,0)$ and v_k solves the problem

$$\partial_s v_k - \Delta v_k = |x_k + \lambda_k y|^a v_k^p, \qquad (y,s) \in \tilde{Q}_k := \{(y,s) : |y| < \frac{d_k}{\lambda_k}, -\frac{t_k}{2\lambda_k^2} < s < 0\}.$$

Using parabolic L^p-estimates together with standard embedding theorems, we may assume $v_k \to v$ in $C^{0,0}_{loc}(\mathbb{R}^N \times (-\infty, 0))$, and

$$v_s - \Delta v = |x_0|^a v^p, \quad \text{in } \mathbb{R}^N \times (-\infty, 0).$$

with $0 \le v \le v(0,0) = 1$. Using (4.40), we have

$$\int_{\tilde{Q}_k} |\partial_s v_k|^2 dyds = \lambda_k^{\frac{4}{p-1}-N+2} \int_{t_k/2}^{t_k} \int_{|x-x_k|<d_k} |\partial_t u_k|^2 dxdt \le \lambda_k^{\frac{4}{p-1}-N+2} \int_{\delta/2}^{\infty} \int_{\Omega} |\partial_t u_k|^2 dxdt$$
$$\le \lambda_k^{\frac{4}{p-1}-N+2} \Big[E(u_k(\delta/2)) - \lim_{t\to\infty} E(u_k(t))\Big] \to 0 \text{ as } k \to \infty.$$

Since $\partial_s v_k \to v_s$ in $\mathcal{D}'(\mathbb{R}^N \times (-\infty,0)$ then $v_s \equiv 0$. This contradicts the Liouville-type theorem for Lane-Emden equation (see [10]).

Case $A_2 : x_0 \in \partial\Omega$. We rescaling u as in (4.41).

If $d_k/\lambda_k \to \infty$ then we have the same contradiction as in the first case. If $d_k/\lambda_k \to c > 0$ then similarly, we have a function v solving the problem

$$\begin{cases} v_s - \Delta v = lv^p & \text{in } H_c^N \times (-\infty,0), \\ v = 0 & \text{on } \partial H_c^N \times (-\infty,0), \end{cases}$$

and satisfying $0 \le v \le v(0,0) = 1$, where $H_c^N := \{y \in \mathbb{R}^n : y_1 > -c\}$. As in case A_1, we have $v_s = 0$, hence contradiction.

Case $A_3 : x_0 = 0$. We have the following two possibilities :

(i) If $M_k|x_k|^{(2+a)/(p-1)} \le C$, then let $\lambda_k = M_k^{-(p-1)/(2+a)}$ we have $\lambda_k^{-1}x_k$ is bounded. We may assume that $\lambda_k^{-1}x_k \to P$. Let

$$w_k(y,s) = \lambda_k^{(2+a)/(p-1)} u_k(x_k + \lambda_k y, t_k + \lambda_k^2 s),$$

Then w_k solves

$$\partial_s w_k - \Delta w_k = |\lambda_k^{-1}x_k + y|^a w_k^p, \quad (y,s) \in \tilde{Q}_k := \{(y,s) : |y| < \frac{d_k}{\lambda_k}, -\frac{t_k}{2\lambda_k^2} < s < 0\}.$$

A similar limiting procedure as in Case A_1 then produces a solution w of

$$w_s - \Delta w = |y + P|^a w^p, \quad (y,s) \in \mathbb{R}^N \times (-\infty,0).$$

with $0 \le w \le w(0,0) = 1$. Using (4.40) and $p < p_S(a)$, we have

$$\begin{aligned}\int_{\tilde{Q}_k} |\partial_s w_k|^2 dyds &= \lambda_k^{(4+2a)/(p-1)-N+2} \int_{t_k/2}^{t_k} \int_{|x-x_k|<d_k} |\partial_t u_k|^2 dxdt \\ &\le \lambda_k^{(4+2a)/(p-1)-N+2} \int_{\delta/2}^{\infty} \int_{\Omega} |\partial_t u_k|^2 dxdt \\ &\le \lambda_k^{(4+2a)/(p-1)-N+2} \Big[E(u_k(\delta/2)) - \lim_{t\to\infty} E(u_k(t))\Big] \to 0 \text{ as } k \to \infty.\end{aligned}$$

Since $\partial_s w_k \to w_s$ in $\mathcal{D}'(\mathbb{R}^N \times (-\infty,0)$ then $w_s \equiv 0$. Hence $-\Delta w = |y+P|^a w^p$ in $\mathbb{R}^N$ with $w(0,0) = 1$. After a spatial shift, we have a contradiction with Liouville-type theorem for Hardy-Hénon elliptic equation (see [22, 6]).

(ii) If there exists a subsequence of k, still denoted by k, such that $M_k|x_k|^{(2+a)/(p-1)} \to \infty$, then we can choose $m_k > 1$ such that

$$M_k|x_k|^{(2m_k+a)/(p-1)} = 1.$$

Let

$$w_k(y,s) = \lambda_k^{\frac{2m_k+a}{m_k(p-1)}} u_k(x_k + \lambda_k y, t_k + \lambda_k^2 s), \quad (y,s) \in Q_k,$$

where $\lambda_k = M_k^{-m_k(p-1)/(2m_k+a)}$, then w_k solves the problem

$$\partial_s w_k - \Delta w_k = |\lambda_k^{-1/m_k} x_k + \lambda_k^{1-1/m_k} y|^a w_k^p, (y,s) \in \tilde{Q}_k = \{(y,s) : |y| < \frac{d_k}{\lambda_k}, -\frac{t_k}{2\lambda_k^2} < s < 0\}.$$

Since $\lambda_k^{-1/m_k}|x_k| = 1$ and $0 < \lambda_k^{1-1/m_k} < 1$, we may assume that $\lambda_k^{-1/m_k} x_k \to P$ with $|P| = 1$ and $\lambda_k^{1-1/m_k} \to l \in [0,1]$. A similar limiting procedure as in Case A_1 then produces a solution w of

$$w_s - \Delta w = |P + ly|^a w^p, \quad (y,s) \in \mathbb{R}^N \times (-\infty, 0).$$

We will show that $w_s \equiv 0$, indeed,

$$\begin{aligned}
\int_{\tilde{Q}_k} |\partial_s w_k|^2 dyds &= \lambda_k^{(4m_k+2a)/m_k(p-1)-N+2} \int_{t_k/2}^{t_k} \int_{|x-x_k|<d_k} |\partial_t u_k|^2 dxdt \\
&\leq \lambda_k^{(4m_k+2a)/m_k(p-1)-N+2} \int_{\delta/2}^{\infty} \int_{\Omega} |\partial_t u_k|^2 dxdt \\
&\leq \lambda_k^{(4m_k+2a)/m_k(p-1)-N+2} \Big[E(u_k(\delta/2)) - \lim_{t\to\infty} E(u_k(t)) \Big] \to 0.
\end{aligned}$$

since

$$\frac{4m_k + 2a}{m_k(p-1)} - N + 2 \geq \min\{4/(p-1) - N + 2, (4+2a)/(p-1) - N + 2\} > 0.$$

Therefore $w_s \equiv 0$, and we have the contradiction.

B. Radial case. Assume $\Omega = B_R$, we will write $u_k = u_k(r,t)$, $r = |x| \in (0,R)$, $M_k = M_k(r_k, t_k)$. Then u_k solves the equation

$$u_t - u_{rr} - \frac{N-1}{r} u_r = r^a u^p.$$

We have 3 subcases :

Case B_1 : $r_k \to r_0 \in (0,R)$. Let $\lambda_k = M_k^{-(p-1)/2}$, we rescale by

$$v_k(\rho, s) := \lambda_k^{2/(p-1)} u_k(r_k + \lambda_k \rho, t_k + \lambda_k^2 s), \quad (\rho, s) \in \big(0, \frac{R - r_k}{\lambda_k}\big) \times (-t_k/\lambda_k^2, 0)$$

Then v_k solves the equation

$$v_s - v_{\rho\rho} - \frac{N-1}{\rho + r_k/\lambda_k} v_\rho = |r_k + \lambda_k \rho|^a v^p,$$

we note that $v_k(0,0) = 1$, after extracting a subsequence, we can assume that $v_k \to v$ that satisfies

$$v_s - v_{\rho\rho} = r_0^a v^p \quad \text{in} \quad \mathbb{R} \times (-\infty, 0)$$

and

$$v(0,0) = 1.$$

By the argument similar to the case A_1, we have $v_s = 0$ and a contradiction with Liouville-type theorem for Lane-Emden equation with $N = 1$.

Case B_2 : $r_k \to R$. We have the following two possibilities :

(i) If $\frac{R-r_k}{\lambda_k} \to \infty$. The same rescaling as in case B_1 leads to a contradiction as in case B_1.

(ii) If $\frac{R-r_k}{\lambda_k} \to c$. The same rescaling as in case B_1 leads to a contradiction with the Liouville-type theorem in half space for Lane-Emden equation with $N = 1$.

Case B_3 : $r_k \to 0$. It follows from the singularity estimate in Theorem 4.1.3(i) and $t_k \geq \delta$ that

$$M_k r_k^{(2+a)/(p-1)} \leq C.$$

Let $\lambda_k = M_k^{-(p-1)/(2+a)}$ then $\lambda_k^{-1} r_k$ is bounded. We may assume that $\lambda_k^{-1} r_k \to P$. Let

$$w_k(\rho, s) = \lambda_k^{(2+a)/(p-1)} u_k(\lambda_k \rho, t_k + \lambda_k^2 s), \quad 0 < \rho < R/\lambda_k, \quad -t_k/\lambda_k^2 < s < 0.$$

Then w_k solves

$$\partial_s w - w_{\rho\rho} - \frac{N-1}{\rho} w_\rho = |\rho|^a w^p.$$

After extracting a subsequence, we can assume that $w_k \to w$ in $C_{loc}^{0,0}(\mathbb{R} \times (-\infty, 0))$ and

$$\partial_s w - w_{\rho\rho} - \frac{N-1}{\rho} w_\rho = |\rho|^a w^p.$$

By similar argument as in case A_3, we have $w_s = 0$. We therefore have a contradiction with Liouville-type theorem for radial solutions of Hardy-Hénon elliptic equation. Theorem is proved. □

We now turn to prove Theorem 4.1.6.

Proof of Theorem 4.1.6. It suffices to prove assertion (i).

Suppose that estimate (4.12) is false. Then there exist sequences $T_k \in (0, \infty)$, u_k, $y_k \in \Omega$, $s_k \in (0, T_k)$, such that u_k solves problem (4.9) (with T replaced by T_k) and the functions

$$M_k := u_k^{(p-1)/2}$$

satisfy

$$M_k(y_k, s_k) > 2k(1 + d_k^{-1}(s_k)),$$

where $d_k := (\min(t, T_k - t))^{1/2}$. We apply Doubling Lemma in [25, Lemma 5.1], with $X = \mathbb{R}^{N+1}$, equipped with parabolic distance (4.19), $\Sigma = \Sigma_k = \bar{\Omega} \times [0, T_k]$, $D = D_k = \bar{\Omega} \times (0, T_k)$, and $\Gamma = \Gamma_k = \bar{\Omega} \times \{0, T_k\}$. Notice that

$$d_k(t) = d_P((x,t), \Gamma_k), \quad (x,t) \in \Sigma_k.$$

Then there exist $x_k \in \Omega$, $t_k \in (0, T_k)$ such that

$$\begin{aligned} &M_k(x_k, t_k) > 2k d_k^{-1}(t_k), \\ &M_k(x_k, t_k) \geq M_k(y_k, s_k) > 2k, \end{aligned}$$

and

$$M_k(x,t) \leq 2M_k(x_k, t_k), \quad (x,t) \in D_k \cap \tilde{B}_k, \tag{4.42}$$

where

$$\tilde{B}_k = \{(x,t) \in \mathbb{R}^{N+1} : |x - x_k| + |t - t_k|^{1/2} \leq k M_k^{-1}\}.$$

We may assume that $x_k \to x_0 \in \bar{\Omega}$. Let $d_k = \operatorname{dist}(x_k, \partial\Omega)$.

A. The nonradial case. We have 3 subcases.

Case A_1 : $x_0 \in \Omega \setminus \{0\}$. Let

$$v_k(y,s) = \lambda_k^{2/(p-1)} u_k(x_k + \lambda_k y, t_k + \lambda_k^2 s), \quad (y,s) \in \tilde{D}_k,$$

where $\lambda_k = M_k^{-1}(x_k, t_k)$ and $\tilde{D}_k = \big(\lambda_k^{-1}(\Omega - x_k) \cap \{|y| < \frac{k}{2}\}\big) \times \Big(-\frac{k^2}{4}, \frac{k^2}{4}\Big)$.

We have $v_k(0,0) = 1$, and it follows from (4.42) that $v_k \leq 2^{2/(p-1)}$, and v_k solves the problem

$$\partial_s v_k - \Delta v_k = |x_k + \lambda_k y|^a v_k^p, \qquad (y,s) \in \tilde{D}_k.$$

Using parabolic L^p-estimates together with standard embedding theorems, we may assume $v_k \to v$ in $C^{0,0}_{loc}(\mathbb{R}^N \times \mathbb{R})$, and

$$v_s - \Delta v = |x_0|^a v^p, \quad \text{in } \mathbb{R}^N \times \mathbb{R}.$$

with $v(0,0) = 1$. This contradicts Theorem A(ii).

Case A_2 : $x_0 \in \partial\Omega$. We rescale u as in case A_1. Let

$$v_k(y,s) = \lambda_k^{2/(p-1)} u_k(x_k + \lambda_k y, t_k + \lambda_k^2 s), \quad \lambda_k = M_k^{-1}(x_k, t_k).$$

If $d_k/\lambda_k \to \infty$ then we have the same contradiction as in case A_1. If $d_k/\lambda_k \to c > 0$ then similarly, we have a function v solving the problem

$$\begin{cases} v_s - \Delta v = l v^p & \text{in } H_c^N \times \mathbb{R}, \\ v = 0 & \text{on } \partial H_c^N \times \mathbb{R}, \end{cases}$$

and satisfying $v(0,0) = 1$, where $H_c^N := \{y \in \mathbb{R}^n : y_1 > -c\}$. This contradicts [26, Theorem 2.19(ii)].

Case A_3 : $x_0 = 0$. We have two possibilities :

(i) If $M_k^{2/(2+a)}(x_k, t_k)|x_k| \leq C$, then letting $\lambda_k = M_k^{-2/(2+a)}(x_k, t_k)$, we have $\lambda_k^{-1} x_k$ is bounded. We may assume that $\lambda_k^{-1} x_k \to P$. Let

$$w_k(y,s) = \lambda_k^{(2+a)/(p-1)} u_k(x_k + \lambda_k y, t_k + \lambda_k^2 s),$$

Then w_k solves

$$\partial_s w_k - \Delta w_k = |\lambda_k^{-1} x_k + y|^a w_k^p, \quad (y,s) \in \tilde{D}_k.$$

A similar limiting procedure as in Case A_1 then produces a solution w of

$$w_s - \Delta w = |y + P|^a w^p, \quad (y,s) \in \mathbb{R}^N \times \mathbb{R}.$$

After a spatial shift, we have a contradiction with Theorem 4.1.1(i).

(ii) If there exists a subsequence of k, still denoted by k, such that $M_k^{2/(2+a)}(x_k, t_k)|x_k| \to \infty$. We can choose $m_k > 1$ such that

$$M_k^{2/(2m_k+a)}(x_k, t_k)|x_k| = 1.$$

Let

$$w_k(y,s) = \lambda_k^{\frac{2m_k+a}{m_k(p-1)}} u_k(x_k + \lambda_k y, t_k + \lambda_k^2 s), \quad (y,s) \in \tilde{D}_k,$$

where $\lambda_k = M_k^{-2m_k/(2m_k+a)}(x_k, t_k)$, then w_k solves the problem

$$\partial_s w_k - \Delta w_k = |\lambda_k^{-1/m_k} x_k + \lambda_k^{1-1/m_k}|^a w_k^p, \quad (y,s) \in \tilde{D}_k.$$

Since $\lambda_k^{-1/m_k}|x_k| = 1$ and $0 < \lambda_k^{1-1/m_k} < 1$ then we may assume that $\lambda_k^{-1/m_k} x_k \to P$ with $|P| = 1$ and $\lambda_k^{1-1/m_k} \to l \in [0,1]$. A similar limiting procedure as in case A_1 then produces a solution w of

$$w_s - \Delta w = |P + ly|^a w^p, \quad (y,s) \in \mathbb{R}^N \times \mathbb{R}.$$

We have a contradiction with Theorem 4.1.1(i) if $l \neq 0$, or with Theorem A(ii) if $l = 0$.

B. The radial case. Assume $\Omega = B_R$, we will write $u_k = u_k(r,t)$, $r \in (0,R)$, $M_k = M_k(r,t)$, where $r = |x|$. Then u_k solves the equation

$$u_t - u_{rr} - \frac{N-1}{r}u_r = r^a u^p.$$

Denote $r_k = |x_k|$, we have 3 cases :

Case B_1 : $r_k \to r_0 \in (0,R)$. Let $\lambda_k = M_k^{-1}(r_k,t_k)$, we rescale by

$$v_k(\rho,s) := \lambda_k^{2/(p-1)} u_k(r_k + \lambda_k\rho, t_k + \lambda_k^2 s), \quad \rho < \min\{\frac{k}{2}, \frac{r_0}{\lambda_k}, \frac{R-r_0}{\lambda_k}\},\ |s| < \frac{k^2}{4}.$$

Then v_k solves the equation

$$v_s - v_{\rho\rho} - \frac{N-1}{\rho + r_k/\lambda_k}v_\rho = |r_k + \lambda_k\rho|^a v^p,$$

we note that $v_k(0,0) = 1$, after extracting a subsequence, we can assume that $v_k \to v$ that satisfies

$$v_s - v_{\rho\rho} = r_0^a v^p \quad \text{in} \quad \mathbb{R}\times\mathbb{R}$$

and

$$v(0,0) = 1.$$

This contradicts Theorem A(ii) for $N = 1$.

Case B_2 : $r_k \to R$. We have the following two possibilities :

(i) $\frac{R-r_k}{\lambda_k} \to \infty$. The same rescaling as in case B_1 lead to a contradiction with Theorem A(ii) for $N = 1$.

(ii) $\frac{R-r_k}{\lambda_k} \to c$ The same rescaling as in case B_1 lead to a contradiction with the Liouville-type theorem in half space with $N = 1$ (see [26, Theorem 2.19]).

Case B_3 : $r_k \to 0$. We have the following two possibilities :

(i) $M_k^{2/(2+a)}(r_k,t_k)r_k \leq C$. Let $\lambda_k = M_k^{-2/(2+a)}(r_k,t_k)$ then $\lambda_k^{-1}r_k$ is bounded. We may assume that $\lambda_k^{-1}r_k \to P$. Let

$$w_k(\rho,s) = \lambda_k^{(2+a)/(p-1)} u_k(r_k + \lambda_k\rho, t_k + \lambda_k^2 s), \quad \rho < \min\{\frac{k}{2}, \frac{R}{2\lambda_k}\},\ |s| < \frac{k^2}{4}.$$

Then w_k solves

$$\partial_s w - w_{\rho\rho} - \frac{N-1}{\rho}w_\rho = |\rho|^a w^p.$$

After extracting a subsequence, we can assume that $w_k \to w$ in $C^{0,0}_{loc}(\mathbb{R}\times\mathbb{R})$ and

$$\partial_s w - w_{\rho\rho} - \frac{N-1}{\rho}w_\rho = |\rho|^a w^p.$$

So we have a contradiction with Theorem 4.1.1(ii)

(ii) There exists a subsequence of k, still denoted by k such that $M_k^{2/(2+a)}(r_k,t_k)r_k \to \infty$. We can choose $m_k > 1$ such that

$$M_k^{2/(2m_k+a)}(r_k,t_k)r_k = 1.$$

Let $\lambda_k = M_k^{-2m_k/(2m_k+a)}(r_k,t_k)$ then $\lambda_k^{-1/m_k}r_k = 1$ and $0 < \lambda_k^{1-1/m_k} < 1$, we may assume that $\lambda_k^{1-1/m_k} \to l \in [0,1]$.

If $l = 0$ then we rescale

$$w_k(\rho, s) = \lambda_k^{\frac{2m_k+a}{m_k(p-1)}} u_k(r_k + \lambda_k\rho, t_k + \lambda_k^2 s), \quad \rho < \min\{\frac{k}{2}, \frac{R}{2\lambda_k}\},\ |s| < \frac{k^2}{4}.$$

It follows that w_k solves the problem

$$\partial_s w - w_{\rho\rho} - \frac{N-1}{\rho + \lambda_k^{-1} r_k} w_\rho = |\lambda_k^{-1/m_k} r_k + \lambda_k^{1-1/m_k} \rho|^a w_k^p.$$

A similar limiting procedure as in Case A_1 then produces a solution w of

$$w_s - w_{\rho\rho} = w^p, \quad (\rho, s) \in \mathbb{R} \times \mathbb{R},$$

with $w(0,0) = 1$, and we have a contradiction with Theorem A(ii) for $N = 1$.

If $l \neq 0$ then we rescale

$$w_k(\rho, s) = \lambda_k^{\frac{2m_k+a}{m_k(p-1)}} u_k(\lambda_k\rho, t_k + \lambda_k^2 s), \quad \rho < \min\{\frac{k}{2}, \frac{R}{\lambda_k}\},\ |s| < \frac{k^2}{4},$$

It follows that w_k solves the problem

$$\partial_s w - w_{\rho\rho} - \frac{N-1}{\rho} w_\rho = |\lambda_k^{1-1/m_k} \rho|^a w^p.$$

Passing to the limit, we obtain w solutions to

$$w_s - w_{\rho\rho} - \frac{N-1}{\rho} w_\rho = l^a |\rho|^a w^p, \quad (\rho, s) \in \mathbb{R} \times \mathbb{R}.$$

with $w(0,0) = 1$. This contradicts Theorem 4.1.1(i). □

We now give proof of Theorem 4.1.7. We first need the following lemma, which is proved by the same argument as in [31, Lemma 4.1].

Lemma 4.4.2. *Let $a > 0$, $p > 1 + \frac{a}{N}$ and $\varepsilon \in (0, (p+1)/2)$. Then there exists $a_\varepsilon \in (0, a)$ such that, for any nonnegative solution u of (4.9) and $t \in (0, T/2)$, it holds*

$$\int_0^t \int_\Omega |x|^{a_\varepsilon} u^{\frac{p+1}{2} - \varepsilon} dxdt \le C(p, a, \Omega, \varepsilon)(1+t)\left(1 + T^{-1/(p-1)}\right).$$

Proof. The proof is nearly the same as in [31, Lemma 4.1]. Only one thing we take into consideration is the condition $p > 1 + \frac{a}{N}$, which implies the Hölder 's inequality

$$\int_\Omega |x|^a u^p(t) \varphi_1 dx \ge C\bigg(\int_\Omega u(t)\varphi_1 dx\bigg)^p.$$

□

Proof of Theorem 4.1.7. If $p < p_B$ then the estimate (4.14) is a consequence of Theorem 4.1.6(ii). We may assume that $p_B \le p < \frac{N+2+a}{N-2+a}$ (this in particular implies $a < \frac{N+2}{4N-1}$ and $N \ge 2$). We shall follow the steps similar to those in [31]. In order not to repeat the same things, we only precise the modifications and the differences coming from the weight term $|x|^a$.

By Theorem 4.1.5, we know that global solutions of problem (4.9) satisfy the a priori estimate

$$\|u(t)\|_\infty \le C(\Omega, p, a, \|u(t_0)\|_\infty), \quad t \ge t_0 \ge 0,$$

where C remains bounded for $\|u(t_0)\|_\infty$ bounded. Therefore, it is sufficient to show the existence of $C(\Omega, p, a, \tau) > 0$ such that any global solution u of problem (4.9) satisfies

$$\inf_{t \in (0,\tau)} \|u(t)\|_\infty \le C(\Omega, p, a, \tau). \tag{4.43}$$

Using the boundedness of $|x|^a$ in Ω, by the well-known estimate (see [34]), we have

$$\|u(t)\|_\infty \le C\|u(t_0)\|_q (t-t_0)^{-N/(2q)}, \quad 0 \le t-t_0 \le T(\|u(t_0)\|_q) \tag{4.44}$$

where $r \in [q,\infty]$ and $q > q_c := N(p-1)/2$. We note that $p \ge p_B > 1 + \frac{2a}{N}$, hence $q_c > 1$. Since $p < \frac{N+2+a}{N-2+a}$, we can choose $q \in (q_c, p+1)$ such that $aq/(p+1-q) < N$. It follows from (4.44) and Hölder inequality that

$$\|u(t)\|_\infty \le C\|u(t_0)\|_q (t-t_0)^{-N/(2q)} \le C\||x|^{a/(p+1)}u(t_0)\|_{p+1}(t-t_0)^{-N/(2q)}.$$

Therefore, (4.14) will follow if we can show that

$$\inf_{t\in(0,\tau)} \||x|^{a/(p+1)}u(t)\|_{p+1} \le C(\Omega,p,a,\tau).$$

We argue by contradiction, assume that for each k, there exists a global solution $u_k \ge 0$ such that

$$\int_\Omega |x|^a u_k^{p+1}(t)dx > k, \quad \forall t \in (0,\tau/2). \tag{4.45}$$

Denote

$$E_k(t) = E(u_k(t)) = \frac{1}{2}\int_\Omega |\nabla u_k(t)|^2 dx - \frac{1}{p+1}\int_\Omega |x|^a u_k^{p+1}(t)dx.$$

Then $E_k'(t) = -\|\partial_t u_k(t)\|_2^2 \le 0$ and u_k satisfies the identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_k^2(t)dx = -\int_\Omega \|\nabla u_k\|^2(t)dx + \int_\Omega |x|^a u_k^{p+1}(t)dx \tag{4.46}$$

$$= -2E_k(t) + \frac{p-1}{p+1}\int_\Omega |x|^a u_k^{p+1}(t)dx. \tag{4.47}$$

Step 1. We claim that

$$E_k(\tau/4) \ge k^{1/2} \tag{4.48}$$

for all $k \ge k_0(\Omega,p,a)$ large enough.

Assume that (4.48) is false. Since $p > 1 + \frac{2a}{N}$, we have

$$\|u_k\|_2 \le C\||x|^{a/p+1}u_k\|_{p+1}. \tag{4.49}$$

Using (4.47) and (4.49), for all $t \ge \tau/4$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_k^2(t)dx \ge -2k^{1/2} + \frac{p-1}{p+1}\int_\Omega |x|^a u_k^{p+1}(t)dx \ge -2k^{1/2} + C\left(\int_\Omega u_k^2(t)dx\right)^{(p+1)/2}. \tag{4.50}$$

This implies

$$\int_\Omega u_k^2(t)dx \le Ck^{\frac{1}{p+1}}, \quad t \ge \tau/4. \tag{4.51}$$

Combining (4.45) with (4.50), we deduce

$$\frac{1}{4}k\tau \le \int_{\tau/4}^{\tau/2}\int_\Omega |x|^a u_k^{p+1}(t)dxdt \le C(k^{\frac{1}{p+1}} + k^{1/2}\tau),$$

which gives a contradiction for k large enough.

Step 2. Let $\alpha > 0$ to be fixed later and $F_k = \{t \in (0, \tau/4] : -E_k'(t) \geq E_k^{1+1/\alpha}(t)\}$. By the same argument as in [31] we have, $|F_k| < \tau/8$ for all $k \geq k_0$ large enough.

Step 3. Choose $\alpha \geq (p+1)/(p-1)$. We claim that for all $k \geq k_0$ large,

$$\|\partial_t u_k(t)\|_2^2 \leq C \left(\int_\Omega |x|^a u_k^{p+1}(t) dx \right)^{(\alpha+1)/\alpha}, \quad \text{for } t \in (0, \tau/4] \setminus F_k. \tag{4.52}$$

For all $t \in (0, \tau/4] \setminus F_k$, we have

$$\|\partial_t u_k(t)\|_2^2 = -E_k' \leq E_k^{1+1/\alpha}(t) \leq \|\nabla u_k(t)\|_2^{2(1+1/\alpha)}. \tag{4.53}$$

This along with (4.46) and (4.49) implies

$$\begin{aligned}
\|\nabla u_k(t)\|_2^2 &\leq \int_\Omega |x|^a u_k^{p+1}(t) dx + \|u_k(t)\|_2 \|\partial_t u_k(t)\|_2 \\
&\leq \int_\Omega |x|^a u_k^{p+1}(t) dx + \||x|^{a/p+1} u_k(t)\|_{p+1} \|\nabla u_k(t)\|_2^{1+1/\alpha} \\
&\leq \int_\Omega |x|^a u_k^{p+1}(t) dx + C\||x|^{a/p+1} u_k(t)\|_{p+1}^{2\alpha/(\alpha-1} + \frac{1}{2}\|\nabla u_k(t)\|_2^2 \\
&\leq C \int_\Omega |x|^a u_k^{p+1}(t) dx + \frac{1}{2}\|\nabla u_k(t)\|_2^2
\end{aligned}$$

where we have used $\alpha \geq (p+1)/(p-1)$ and (4.45). Consequently,

$$\|\nabla u_k(t)\|_2^2 \leq C \int_\Omega |x|^a u_k^{p+1}(t) dx.$$

Combing this with (4.53), we have (4.52).

Step 4. Let $0 < q < (p+1)/2$, $b = (p+1-q)(\alpha+1)/\alpha$ and

$$G_k = \{t \in (0, \tau/4] : \|\partial_t u_k(t)\|_2^2 \leq C\|u_k\|_\infty^b\}.$$

We claim that $|G_k| > 0$.

Due to $a > a_\varepsilon$ and Lemma 4.4.2, for $A = A(p, q, a, \Omega, \tau)$ large enough, the set

$$\tilde{G}_k := \{t \in (0, \tau/4] : \int_\Omega |x|^a u_k^q \geq A\}$$

satisfies

$$|\tilde{G}_k| < \tau/8. \tag{4.54}$$

It follows from (4.45) that, for all $t \in (0, \tau/4] \setminus \tilde{G}_k$,

$$\int_\Omega |x|^a u_k^{p+1}(t) dx \leq C\|u_k(t)\|_\infty^{p+1-q} \int_\Omega |x|^a u_k^q(t) dx \leq C\|u_k(t)\|_\infty^{p+1-q}.$$

Therefore, $G_k \supset (0, \tau/4] \setminus (F_k \cup \tilde{G}_k)$. The claim follows from Step 2 and (4.54).

Step 5. Construction of a sequence of rescaling times.

If $N \leq 3$, for each k large, we just pick any $t_k \in G_k$.

If $N > 3$, we follows the argument in Step 5 of [31], (since $|x|^a$ is bounded in Ω, Lemma 5.2 and 5.3 in [31] still hold), and there exists $t_k \in (0, \tau)$ such that

$$\|\partial_t u_k(t_k)\|_r \leq \|u_k(t_k)\|_\infty^{\frac{b}{2}+\beta(p-1)}, \quad 2 \leq r \leq \infty. \tag{4.55}$$

where $\beta = \frac{N}{2}(\frac{1}{2} - \frac{1}{r})$. (Note that when $N \leq 3$, (4.55) holds for $r = 2$, $\beta = 0$.)

Step 6. We will now obtain a contradiction by using rescaling argument. By (4.45), we have $M_k = \|u_k(t_k)\|_\infty \to \infty$. Let $x_k \in \bar{\Omega}$ be such that $M_k = u_k(x_k, t_k)$. We may assume that $x_k \to x_0 \in \bar{\Omega}$. Similar to the proof of Theorem 4.1.6, we have the following three cases.

Case 1 : $x_0 \in \Omega \setminus \{0\}$. Let $\nu_k = M_k^{-(p-1)/2}$ and

$$w_k(y) = M_k^{-1} u_k(x_k + \nu_k y, t_k), \tag{4.56}$$

$$\tilde{w}_k(y) = M_k^{-p} \partial_t u_k(x_k + \nu_k y, t_k). \tag{4.57}$$

Then we have

$$\begin{cases} \Delta w_k + |x_k + \nu_k y|^a w_k^p = \tilde{w}_k & \text{in } \Omega_k \\ w_k = 0 & \text{on } \partial\Omega_k \end{cases} \tag{4.58}$$

where $\Omega_k = \nu_k^{-1}(\Omega - x_k)$. Moreover, $0 \le w_k \le 1 = w_k(0)$. Now passing to the limit we obtain a contradiction with elliptic Liouville-type theorem for Lane-Emden equation [11]. We only have to show that the functions w_k are locally uniformly Hölder continuous and $\tilde{w}_k \to 0$ in $L^r_{loc}(\mathbb{R}^N)$ with some $r > N/2$.

Let $R > 0$. Using (4.55) we obtain for k large enough,

$$\begin{aligned} \left(\int_{B_R} |\tilde{w}_k(y)|^r dy\right)^{1/r} &= M_k^{-p} \left(\int_{B_R} |\partial_t u_k(x_k + \nu_k y, t_k)|^r dy\right)^{1/r} \\ &= M_k^{-p} \nu_k^{-N/r} \left(\int_{\Omega} |\partial_t u_k(x, t_k)|^r dx\right)^{1/r} \\ &\le M_k^{-p} M_k^{N(p-1)/(2r)} M_k^{\frac{b}{2}+\beta(p-1)} = C M_k^{\gamma_1}. \end{aligned} \tag{4.59}$$

where $\gamma_1 = -p + \frac{N(p-1)}{4} + \frac{\alpha+1}{2\alpha}(p+1-q)$ and $r \in [2, \infty)$.

By taking q close to $(p+1)/2$ and α sufficiently large, γ_1 will be negative provided $(N-3)p < N-1$, which is always true since $N \le 4$ and $p \le \frac{N+2+a}{N-2+a}$.

Case 2 : $x_0 \in \partial\Omega$. By the same argument as in the Case 1, we have the contradiction with Liouville-type theorem for Lane-Emden equation if $d(x_k, \partial\Omega)\nu_k^{-1} \to \infty$, or with Liouville-type theorem for Lane-Emden equation in half-space if $d(x_k, \partial\Omega)\nu_k^{-1}$ is bounded.

Case 3 : $x_0 = 0$. We have the following two possibilities :

(i) If $M_k^{(p-1)/(2+a)}|x_k| \le C$, let $\nu_k = M_k^{-(p-1)/(2+a)}$ then $\nu_k^{-1} x_k$ is bounded. We may assume that $\nu_k^{-1} x_k \to P$. Let $w_k, \tilde{w}_k$ defined as in (4.56) and (4.57), by the same procedures, it is sufficient to show that functions w_k are locally uniformly Hölder continuous and $\tilde{w}_k \to 0$ in $L^r_{loc}(\mathbb{R}^N)$ with some $r > N/2$.

Similarly as in (4.59),

$$\left(\int_{B_R} |\tilde{w}_k(y)|^r dy\right)^{1/r} \le M_k^{-p} M_k^{N(p-1)/(2r+ar)} M_k^{\frac{b}{2}+\beta(p-1)} \le C M_k^{\gamma_2}.$$

where $r \in [2, \infty)$ and $\gamma_2 = -p + \frac{N(p-1)}{4} + \frac{\alpha+1}{2\alpha}(p+1-q) + \frac{N(p-1)}{r}(\frac{1}{2+a} - \frac{1}{2}) \le \gamma_1$. By taking q close to $(p+1)/2$ and α sufficiently large, γ_2 will be negative provided $(N-3)p < N-1$.

(ii) If there exists a subsequence of k, still denoted by k, such that $M_k^{(p-1)/(2+a)}|x_k| \to \infty$. We can choose $m_k > 1$ such that

$$M_k^{(p-1)/(2m_k+a)}|x_k| = 1.$$

Let $\nu_k = M_k^{-m_k(p-1)/(2m_k+a)}$, we may assume that $\nu_k^{-1/m_k} x_k \to P$ with $|P| = 1$, and $\nu_k^{1-1/m_k} \to l \in [0,1]$. Let $w_k, \tilde{w}_k$ defined as in (4.56) and (4.57), by the same procedures, it is sufficient to show that functions w_k are locally uniformly Hölder continuous and $\tilde{w}_k \to 0$ in $L^r_{loc}(\mathbb{R}^N)$ with some $r > N/2$.

Similarly as in (4.59),

$$\left(\int_{B_R} |\tilde{w}_k(y)|^r dy\right)^{1/r} \leq M_k^{-p} M_k^{\frac{N}{r}\frac{m_k(p-1)}{2m_k+a})} M_k^{\frac{b}{2}+\beta(p-1)} \leq C M_k^{\gamma_3}.$$

where $r \in [2,\infty)$ and $\gamma_3 = -p + \frac{N(p-1)}{4} + \frac{\alpha+1}{2\alpha}(p+1-q) + \frac{N(p-1)}{r}(\frac{m_k}{2m_k+a} - \frac{1}{2}) \leq \gamma_1$. By taking q close to $(p+1)/2$ and α sufficiently large, γ_3 will be negative provided $(N-3)p < N-1$. Theorem is proved. □

4.5 Appendix

Lemma 4.5.1. *Assume that $0 \in \Omega$, $a > -2$, $N \geq 2$ and u is solution of (4.1) in $\Omega \times (0,T)$ in the sense of (4.2). Then u is distributional solution in the sense*

$$-\int_0^T \int_\Omega (u(\varphi_t + \Delta\varphi)dxdt = \int_0^T \int_\Omega |x|^a u^p \varphi dxdt \tag{4.60}$$

for all $\varphi \in C_0^\infty(\Omega \times (0,T))$.

Proof. We follow the argument in [22]. If $a \geq 0$ then the result is immediate, so we may assume $-2 < a < 0$.

Denote $d\sigma_\rho$ the surface measure on the sphere $\{x \in \mathbb{R}^N : |x| = \rho\}$. For $0 < \varepsilon < R$ such that $B_R \subset\subset \Omega$, for any $\tau > 0$, we have

$$\begin{aligned}\int_\tau^{T-\tau}\int_{B_R\setminus B_\varepsilon} |\nabla u|^2 dxdt &= -\int_\tau^{T-\tau}\int_{B_R\setminus B_\varepsilon} u\Delta u dxdt + \int_\tau^{T-\tau}\int_{|x|=R} uu' d\sigma_R dt - \int_\tau^{T-\tau}\int_{|x|=\varepsilon} uu' d\sigma_\varepsilon dt \\ &= \int_\tau^{T-\tau}\int_{B_R\setminus B_\varepsilon} |x|^a u^{p+1} dxdt + \int_{B_R\setminus B_\varepsilon} (u(\tau) - u(T-\tau))dxdt \\ &\quad + \int_\tau^{T-\tau}\int_{|x|=R} uu' d\sigma_R dt - \int_\tau^{T-\tau}\int_{|x|=\varepsilon} uu' d\sigma_\varepsilon dt. \end{aligned}\tag{4.61}$$

On the other hand, we have

$$\int_\tau^{T-\tau}\int_{|x|=\varepsilon} uu'\, d\sigma_\varepsilon = \varepsilon^{N-1} f'(\varepsilon), \qquad \text{where } f(r) := \frac{1}{2}\int_\tau^{T-\tau}\int_{S^{N-1}} u^2(r,\theta)\, d\theta.$$

Since $f \in C^1((0,R]) \cap C([0,R])$ due to our regularity assumption (4.2), we infer the existence of a sequence $\varepsilon_i \to 0$ such that $\lim_{i\to\infty} \varepsilon_i f'(\varepsilon_i) = 0$. Passing to the limit in (4.61) with $\varepsilon = \varepsilon_i$ and noting $a > -N$, we have

$$\int_\tau^{T-\tau}\int_{B_R} |\nabla u|^2\, dx < \infty.$$

Hence, there exist $\rho_i \to 0^+$ (depending on τ) such that

$$\int_\tau^{T-\tau}\int_{|x|=\rho_i} \rho_i |\nabla u|^2\, d\sigma_{\rho_i} dt \to 0.$$

Consequently,

$$\int_{\tau}^{T-\tau} \int_{|x|=\rho_i} |\nabla u| \, d\sigma_{\rho_i} \le C \left((T-2\tau)\rho_i^{N-1} \int_{|x|=\rho_i} |\nabla u|^2 \, d\sigma_{\rho_i} \right)^{1/2} \to 0. \tag{4.62}$$

Let now $\varphi \in C_0^\infty(\Omega \times (0,T))$ and denote $\Omega_\varepsilon = \Omega \cap \{|x| > \rho\}$ for $\rho > 0$ small. From (4.1), using Green's formula, we obtain

$$\begin{aligned} \Big| \int_0^T \int_{\Omega_\rho} |x|^a u^p \varphi \, dx + \int_0^T \int_{\Omega_\rho} u(\varphi_t + \Delta\varphi) \, dx \Big| &= \Big| - \int_0^T \int_{\Omega_\rho} \varphi \Delta u \, dx + \int_0^T \int_{\Omega_\rho} u \Delta\varphi \, dx \Big| \\ &= \Big| \int_0^T \int_{|x|=\rho} \varphi \frac{\partial u}{\partial r} \, d\sigma_\rho - \int_0^T \int_{|x|=\varepsilon} u \frac{\partial \varphi}{\partial r} \, d\sigma_\rho \Big|. \end{aligned} \tag{4.63}$$

Passing (4.63) to the limit with $\rho = \rho_i$, we conclude that u is a distributional solution of (4.1). $\square$

Lemma 4.5.2. *Assume that u is bounded solution of (4.1) in $\mathbb{R}^N \times [0,T)$ in the sense of (4.2). Assume in addition that u is distributional solution. Then u is integral solution in the sense that*

$$u(t) = e^{t\Delta} u(0) + \int_0^t e^{(t-s)\Delta} (|.| u^p(s)) ds \tag{4.64}$$

Proof. Lemma is standard for $a \ge 0$, so we only need to treat the case $a < 0$. For any $\varepsilon > 0$, let us consider following problem

$$\begin{cases} \partial_t v_\varepsilon - \Delta v_\varepsilon = (|x| + \varepsilon)^a u^p, \\ v_\varepsilon(0) = u(0). \end{cases}$$

Then $v_\varepsilon \le u$ by comparison property. Since v_ε is increasing as $\varepsilon \to 0^+$, setting that $v_\varepsilon \to v$, by monotone convergence, we have

$$v(t) = e^{t\Delta} u(0) + \int_0^t e^{(t-s)\Delta} (|.|^a u^p(s)) ds.$$

It suffices to show that $u = v$.

Let $z = u - v$. Then z is a bounded, nonnegative distributional solution of $z_t - \Delta z = 0$ in $Q_T := R^N \times (0,T)$. By parabolic regularity (see e.g. [29, Remark 48.3], we deduce that $z \in C^{2,1}(Q_T)$. Moreover, since $u, v \in C(\bar{Q}_T)$, it follows that $z \in C(\bar{Q}_T)$ with $z \equiv 0$ at $t = 0$. By standard uniqueness properties (see e.g. [33, Theorem 2.4], we conclude that $z = 0$ in Q_T.

$\square$

In dimension $N = 1$, we have assumed $a > -1$ in order to make sense of distributional solutions. Actually, the definition (4.2) is no longer consistent for $a < 0$ and $N = 1$ since $\Omega \setminus 0$ is no longer connected and the problem should require boundary conditions at $x = 0$. The following result shows that, for $N = 1$ and $a \in (-1, 0)$, there even exist solutions in the sense (4.2) which are not distributional solutions.

Proposition 4.5.1. *Let $N = 1$ and $a \in (-1, 0)$, then there exists solution u of (4.1) in $(-1,1) \times (0,1)$, but u is not distributional solution.*

Proof. Let B be unit ball in $\mathbb{R}^3$ and $v = v(r)$ be an regular positive radial solution in B of the following Hardy elliptic equation

$$-\Delta v = |x|^{p-1+a} v^p, \quad v(0) > 0,\ v(1) = 0. \tag{4.65}$$

Since $p < p_S(p-1+a) = 5 + 2(p-1+a)$, the existence of such function v with homogeneous Dirichlet boundary condition was shown in [6, Theorem 1.6 (iii)].

Let $w(r) = rv(r)$ then $w'' = r^a w^p$, $r \in (0,1)$ and $w(0) = 0$, $w'(0) = v(0) > 0$. We set

$$u(x,t) = w(|x|), \quad (x,t) \in (-1,1) \times (0,1).$$

Then u is solution of (4.1) in the sense of (4.2) in $(-1,1)\times(0,1)$ with $u_t \equiv 0$. On the other hand, $u_x(0^+,) = v(0)$, $u_x(0^-,t) = -v(0)$. This implies $u_{xx}(0,t)$ has a Dirac $2v(0)\delta_0$. Therefore, u is no longer distributional solution. □

Acknowledgement. The author would like to thank Professor Philippe Souplet for valuable suggestions and comments.

Bibliographie

[1] Kaouther Ammar and Philippe Souplet. Liouville-type theorems and universal bounds for nonnegative solutions of the porous medium equation with source. *Discrete Contin. Dyn. Syst.*, 26(2) :665–689, 2010.

[2] Daniele Andreucci and Anatoli F. Tedeev. Universal bounds at the blow-up time for nonlinear parabolic equations. *Adv. Differential Equations*, 10(1) :89–120, 2005.

[3] Thomas Bartsch, Peter Poláčik, and Pavol Quittner. Liouville-type theorems and asymptotic behavior of nodal radial solutions of semilinear heat equations. *J. Eur. Math. Soc. (JEMS)*, 13(1) :219–247, 2011.

[4] Marie-Françoise Bidaut-Véron. Initial blow-up for the solutions of a semilinear parabolic equation with source term. In *Équations aux dérivées partielles et applications*, pages 189–198. Gauthier-Villars, Éd. Sci. Méd. Elsevier, Paris, 1998.

[5] Marie-Francoise Bidaut-Veron. Local behaviour of the solutions of a class of nonlinear elliptic systems. *Adv. Differential Equations*, 5(1-3) :147–192, 2000.

[6] Marie Françoise Bidaut-Veron and Hector Giacomini. A new dynamical approach of Emden-Fowler equations and systems. *Adv. Differential Equations*, 15(11-12) :1033–1082, 2010.

[7] Yinbin Deng, Yi Li, and Fen Yang. On the stability of the positive steady states for a nonhomogeneous semilinear Cauchy problem. *J. Differential Equations*, 228(2) :507–529, 2006.

[8] Louis Dupaigne and Augusto C. Ponce. Singularities of positive supersolutions in elliptic PDEs. *Selecta Math. (N.S.)*, 10(3) :341–358, 2004.

[9] Stathis Filippas and Achilles Tertikas. On similarity solutions of a heat equation with a nonhomogeneous nonlinearity. *J. Differential Equations*, 165(2) :468–492, 2000.

[10] B. Gidas and J. Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.*, 34(4) :525–598, 1981.

[11] B. Gidas and J. Spruck. A priori bounds for positive solutions of nonlinear elliptic equations. *Comm. Partial Differential Equations*, 6(8) :883–901, 1981.

[12] Yoshikazu Giga. A bound for global solutions of semilinear heat equations. *Comm. Math. Phys.*, 103(3) :415–421, 1986.

[13] Jong-Shenq Guo, Chang-Shou Lin, and Masahiko Shimojo. Blow-up behavior for a parabolic equation with spatially dependent coefficient. *Dynam. Systems Appl.*, 19(3-4) :415–433, 2010.

[14] Jong-Shenq Guo and Masahiko Shimojo. Blowing up at zero points of potential for an initial boundary value problem. *Commun. Pure Appl. Anal.*, 10(1) :161–177, 2011.

[15] Munemitsu Hirose. Existence of global solutions for a semilinear parabolic Cauchy problem. *Differential Integral Equations*, 21(7-8) :623–652, 2008.

[16] Bei Hu. Remarks on the blowup estimate for solution of the heat equation with a nonlinear boundary condition. *Differential Integral Equations*, 9(5) :891–901, 1996.

[17] BaiShun Lai and Yi Li. On the semistability of the minimal positive steady state for a nonhomogeneous semilinear Cauchy problem. *Math. Res. Lett.*, 15(5) :923–939, 2008.

[18] Howard A. Levine. Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$. *Arch. Rational Mech. Anal.*, 51 :371–386, 1973.

[19] Hiroshi Matano and Frank Merle. On nonexistence of type II blowup for a supercritical nonlinear heat equation. *Comm. Pure Appl. Math.*, 57(11) :1494–1541, 2004.

[20] Hiroshi Matano and Frank Merle. Classification of type I and type II behaviors for a supercritical nonlinear heat equation. *J. Funct. Anal.*, 256(4) :992–1064, 2009.

[21] E. Mitidieri and S. I. Pokhozhaev. A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities. *Tr. Mat. Inst. Steklova*, 234 :1–384, 2001.

[22] Quoc Hung Phan and Philippe Souplet. Liouville-type theorems and bounds of solutions of Hardy-Hénon equations. *J. Differential Equations*, 252(3) :2544–2562, 2012.

[23] Ross G. Pinsky. Existence and nonexistence of global solutions for $u_t = \Delta u + a(x)u^p$ in $\mathbf{R}^d$. *J. Differential Equations*, 133(1) :152–177, 1997.

[24] Peter Poláčik and Pavol Quittner. A Liouville-type theorem and the decay of radial solutions of a semilinear heat equation. *Nonlinear Anal.*, 64(8) :1679–1689, 2006.

[25] Peter Poláčik, Pavol Quittner, and Philippe Souplet. Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems. *Duke Math. J.*, 139(3) :555–579, 2007.

[26] Peter Poláčik, Pavol Quittner, and Philippe Souplet. Singularity and decay estimates in superlinear problems via Liouville-type theorems. II. Parabolic equations. *Indiana Univ. Math. J.*, 56(2) :879–908, 2007.

[27] P. Quittner. A priori bounds for global solutions of a semilinear parabolic problem. *Acta Math. Univ. Comenian. (N.S.)*, 68(2) :195–203, 1999.

[28] Pavol Quittner. Universal bound for global positive solutions of a superlinear parabolic problem. *Math. Ann.*, 320(2) :299–305, 2001.

[29] Pavol Quittner and Philippe Souplet. *Superlinear parabolic problems.* Birkhäuser Advanced Texts : Basler Lehrbücher. [Birkhäuser Advanced Texts : Basel Textbooks]. Birkhäuser Verlag, Basel, 2007. Blow-up, global existence and steady states.

[30] Pavol Quittner and Philippe Souplet. Parabolic Liouville-type theorems via their elliptic counterparts. *Discrete Contin. Dyn. Syst.*, (suppl.) :1206–1213, 2011.

[31] Pavol Quittner, Philippe Souplet, and Michael Winkler. Initial blow-up rates and universal bounds for nonlinear heat equations. *J. Differential Equations*, 196(2) :316–339, 2004.

[32] Xuefeng Wang. On the Cauchy problem for reaction-diffusion equations. *Trans. Amer. Math. Soc.*, 337(2) :549–590, 1993.

[33] N. A. Watson. *Parabolic equations on an infinite strip*, volume 127 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1989.

[34] Fred B. Weissler. Local existence and nonexistence for semilinear parabolic equations in L^p. *Indiana Univ. Math. J.*, 29(1) :79–102, 1980.

Chapitre 5

Optimal Liouville-type theorem for a parabolic system

Optimal Liouville-type theorem for a parabolic system

Quoc Hung Phan

Abstract. We prove a Liouville-type theorem for a parabolic system in dimension $N = 1$ and for radial solutions in all dimensions under an optimal Sobolev growth restriction on the nonlinearities. This seems to be the first example of a Liouville-type theorem in the whole Sobolev subcritical range for a parabolic system (even for radial solutions). Moreover, this also seems to be the first application of the Gidas-Spruck technique to a parabolic system.

5.1 Introduction

In this note, we study the semilinear parabolic system of the form

$$\begin{cases} u_t - \Delta u = u^p - \beta u^r v^{r+1}, & (x,t) \in \Omega \times I \\ v_t - \Delta v = v^p - \beta u^{r+1} v^r, & (x,t) \in \Omega \times I, \end{cases} \tag{5.1}$$

where $p = 2r+1, r > 0$, $\beta \in \mathbb{R}$, Ω is a domain of $\mathbb{R}^N$, and I is an interval of $\mathbb{R}$.

For the statement of main results, let us introduce the Sobolev exponent :

$$p_S := \begin{cases} \frac{N+2}{N-2} & \text{if} \quad N \geq 3, \\ \infty & \text{if} \quad N = 1, 2. \end{cases} \tag{5.2}$$

The main goal of this note is to prove the Liouville property – i.e. the nonexistence of solution of problem (5.1) in the entire space $\mathbb{R}^N \times \mathbb{R}$. As applications, we give some consequences of Liouville type theorems on the qualitative analysis of solutions of related problem on bounded or unbounded domains, such as a priori estimate, singularity estimate, blow-up estimate, etc. Let us first recall its elliptic counterpart

$$\begin{cases} -\Delta u = u^p - \beta u^r v^{r+1}, & x \in \mathbb{R}^N \\ -\Delta v = v^p - \beta u^{r+1} v^r, & x \in \mathbb{R}^N. \end{cases} \tag{5.3}$$

System (5.3) arises in mathematical models for various phenomena in physics, such as nonlinear optics and Bose-Einstein condensation (see [2, 5, 13]). It has been recently studied in various mathematical directions such as Liouville-type results, a priori estimates, regularity, symmetry property, existence of entire nonradial solutions...(see [11, 7, 12, 6, 5]). It is well known that the Liouville-type result for (5.3) plays an important role in both elliptic and parabolic problems. For radial solutions, it has been showed in [11] that the problem (5.3) has no positive radial solution if for $\beta < 1$ and $p < p_S$, and condition on β or p is optimal. For general solutions, problem (5.3) has no positive solution if $\beta < 1$ and $p < p_S^*$, where $p_S^* = p_S$ if $N \leq 4$, $p_S^* = (N-1)/(N-3)$ if $N \geq 5$ (see [11]). The condition of p for Liouville-type results in dimension $N \leq 4$ is optimal due to the existence of bounded positive radial solution of (5.3) if $p \geq p_S$ (and $\beta < 1$).

For the corresponding parabolic problem (5.1), the Liouville property is much less understood even for radial solutions. By the test-function argument, one can easily deduce the Fujita-type result of problem (5.1), namely there is no positive solution in $\mathbb{R}^N \times \mathbb{R}_+$ if $1 < p \leq 1 + \frac{2}{N}$ and $\beta < 1$. Recently, Quittner and Souplet have proved in [10] the Liouville type theorem for radial solution for special case $p = 3$ and $N \leq 3$. In this paper, we will establish a Liouville-type theorem for problem (5.3) in dimension $N = 1$ and for radial solutions in all dimensions in full subcritical range $p < p_S$.

Our main result is the following

Theorem 5.1.1. *Let $N = 1$ and $\beta < \frac{r}{3r+2}$ then system (5.1) does not possess any positive classical solution in $\mathbb{R} \times \mathbb{R}$.*

Theorem 5.1.2. *Let $p < p_S$ and $\beta < \frac{r}{3r+2}$, then system (5.1) does not possess any radial, positive, classical solution in $\mathbb{R}^N \times \mathbb{R}$.*

The assumption $p < p_S$ in Theorem 5.1.2 is optimal due to the existence of bounded radial solution when $p \geq p_S$ and $\beta < 1$, but there is further restriction on β, namely $\beta < \frac{r}{3r+2}$. We stress that our result is not a perturbation one. Although it is not known if condition on β is optimal, we have an explicit value. Our tools for proof of Theorem 5.1.1 are based on integral estimate combined with Bochner formula [4]. The proof of Theorem 5.1.2 follows the idea in [3, 1, 10], which consists of three steps :

1. Showing spatial decay of solutions
2. Using the Lyapunov functional and decay estimate of solutions to show that both α- and ω-limit sets of any solution are nonempty and consist of equilibria.
3. Combining with the nonexistence of nontrivial equilibria to have the contradiction.

5.2 Applications of Liouville-type theorems

From the Liouville-type theorems, one can deduce the singularity estimates by rescaling and doubling arguments. We mean a symmetric domain Ω either the whole space $\mathbb{R}^N$, a ball $B_R = B(0, R)$, an annulus $\{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$, or an exterior domain $\{x \in \mathbb{R}^N : |x| > R\}$. We just give here the result without proof since it is totally similar to that in [8, Theorem 3.1].

Proposition 5.2.1. *Let $\beta < \frac{r}{3r+2}$, Ω be a domain of $\mathbb{R}^N$. Let (u, v) be a positive solution of (5.1) in $\Omega \times (0, T)$. Assume either*

$$N = 1, \quad \text{or } p < p_S, \Omega \text{ is symmetric and } u, v \text{ are radial}$$

Then for all $(x, t) \in \Omega \times (0, T)$, there holds

$$u(x,t) + v(x,t) \leq C(N,p,\beta)\left(t^{-1/(p-1)} + (T-t)^{-1/(p-1)} + \mathrm{dist}^{-2/(p-1)}(x, \partial\Omega)\right). \tag{5.4}$$

Let us next consider the corresponding boundary value problem :

$$\begin{cases} u_t - \Delta u = u^p - \beta u^r v^{r+1}, & (x,t) \in \Omega \times (0,T) \\ v_t - \Delta v = v^p - \beta u^{r+1} v^r, & (x,t) \in \Omega \times (0,T), \\ (u,v) = 0, & x \in \partial\Omega,\ 0 < t < T, \\ (u(x,0), v(x,0)) = (u_0, v_0), & x \in \Omega. \end{cases} \tag{5.5}$$

where u_0, v_0 are non-negative radial in $C_0(\Omega)$. We only state the results (without proof) of universal initial and final time blow-up rates, this result also implies a priori bound of nonnegative solutions.

Proposition 5.2.2. *Let $\beta < \frac{r}{3r+2}$ and (u, v) be a positive radial solution of (5.5) in $\Omega \times (0, T)$. Assume either*

$$N = 1, \quad \text{or } p < p_S, \Omega \text{ is symmetric and } u, v \text{ are radial.}$$

(i) If $T < \infty$ then there holds

$$u(x,t) + v(x,t) \leq C\left(1 + t^{-1/(p-1)} + (T-t)^{-1/(p-1)}\right), \quad x \in \Omega,\ 0 < t < T, \tag{5.6}$$

where $C = C(\Omega, p, \beta)$.

(ii) If (u, v) is global then there holds

$$u(x,t) + v(x,t) \leq C\left(1 + t^{-1/(p-1)}\right), \quad x \in \Omega,\ t > 0, \tag{5.7}$$

where $C = C(\Omega, p, \beta)$.

We note that the proof of Proposition 5.2.2 closely follows the idea of [8, Theorem 4.1], which is based on rescaling and doubling arguments, Liouville-type theorem in whole space and on half-line. Here, the relevant result on half-line is given by the following lemma.

Lemma 5.2.1. *Assume that* $\beta < \frac{r}{3r+2}$ *and* (u, v) *be a positive solution of (5.1) on* $(0, \infty) \times \mathbb{R}$, *with the zero boundary condition* $u(0, t) = v(0, t) = 0$. *Then* $u = v = 0$.

The proof of Lemma 5.2.1 is as follows. From Proposition 5.2.1, we have the decay estimate of (u,v). Then using $\alpha-$ and $\omega-$ limits arguments, one can deduce the Lemma for bounded solutions. The boundedness of solutions is then removed by doubling argument.

5.3 Liouville-type theorems in one dimension

For the sake of simplicity, we denote by $\int$ the integral $\int_{(-1,1)\times(-1,1)} dxdt$.

Lemma 5.3.1. *Assume* $N = 1$. *Let* $0 \le \varphi \in \mathcal{D}((-1, 1) \times (-1, 1))$ *and* (u, v) *be positive solution of (5.1) in* $(-1, 1) \times (-1, 1)$. *Denote*

$$I_1 = \int \varphi u^{-2}|u_x|^4, \quad I_2 = \int \varphi v^{-2}|v_x|^4, \quad I := I_1 + I_2,$$
$$L_1 = \int \varphi u^{2p}, \quad L_2 = \int \varphi v^{2p}, \quad L = L_1 + L_2.$$

Then there holds

$$\begin{aligned} I + L \le& C \int \big(\varphi|u_t|u^{-1}|u_x|^2 + \varphi|v_t|v^{-1}|v_x|^2 + |\varphi_x u_x|(u^p + u^{-1}|u_x|^2) + |\varphi_x v_x|(v^p + v^{-1}|v_x|^2\big) \\ &+ C \int (|\varphi_t| + |\varphi_{xx}|)(u^{p+1} + v^{q+1}), \end{aligned} \tag{5.8}$$

where $C = C(N, p, \beta)$.

Proof. We first prove the lemma for $\beta \ge 0$. Denoting

$$J_1 := \int \varphi u^{-1}|u_x|^2 u_{xx}, \quad J_2 := \int \varphi v^{-1}|v_x|^2 v_{xx}, \quad J = J_1 + J_2.$$

Applying [9, Lemma 8.9] with $q = 0$, $k = -1/2$,

$$\frac{1}{2}I_1 - \frac{3}{2}J_1 \le \frac{1}{2}\int |u_x|^2\varphi_{xx} + \int \big(u_{xx} + \frac{1}{2}u^{-1}|u_x|^2\big)u_x\varphi_x = \frac{1}{2}\int u^{-1}|u_x|^2 u_x\varphi_x. \tag{5.9}$$

Therefore,

$$\frac{1}{2}I - \frac{3}{2}J \le \frac{1}{2}\int u^{-1}|u_x|^2 u_x\varphi_x + \frac{1}{2}\int v^{-1}|v_x|^2 v_x\varphi_x. \tag{5.10}$$

We now calculate J_1,

$$\begin{aligned} -J_1 &= \int \varphi u^{-1}|u_x|^2(u^p - \beta u^r v^{r+1} - u_t) \\ &= \int \varphi u^{p-1}|u_x|^2 - \beta \int \varphi u^{r-1}v^{r+1}|u_x|^2 - \int \varphi u_t u^{-1}|u_x|^2. \end{aligned} \tag{5.11}$$

On the other hand,

$$\begin{aligned} \int \varphi u^{r-1}v^{r+1}|u_x|^2 &= \int \varphi u_x\partial_x(\frac{1}{r}u^r v^{r+1}) - \frac{r+1}{r}\int \varphi u^r v^r u_x v_x \\ &= -\frac{1}{r}\int \varphi u^r v^{r+1}u_{xx} - \frac{1}{r}\int \varphi_x u^r v^{r+1}u_x - \frac{r+1}{r}\int \varphi u^r v^r u_x v_x. \end{aligned}$$

Hence,

$$-J_1 = \int \varphi u^{p-1}|u_x|^2 - \int \varphi u_t u^{-1}|u_x|^2 + \frac{\beta}{r}\int \varphi u^r v^{r+1} u_{xx} + \frac{\beta}{r}\int \varphi_x u^r v^{r+1} u_x + \frac{\beta(r+1)}{r}\int \varphi u^r v^r u_x v_x.$$

Similarly,

$$-J_2 = \int \varphi v^{p-1}|v_x|^2 - \int \varphi v_t v^{-1}|v_x|^2 + \frac{\beta}{r}\int \varphi u^{r+1} v^r v_{xx} + \frac{\beta}{r}\int \varphi_x u^{r+1} v^r v_x + \frac{\beta(r+1)}{r}\int \varphi u^r v^r u_x v_x.$$

Consequently,

$$-J = \int \varphi(u^{p-1}|u_x|^2 + (v^{p-1}|v_x|^2) + \frac{2\beta(r+1)}{r}\int \varphi u^r v^r u_x v_x + \frac{\beta}{r(r+1)}\int \varphi_x \partial_x(u^{r+1}v^{r+1}) + \frac{\beta}{r}\int \varphi u^r v^{r+1} u_{xx} + \frac{\beta}{r}\int \varphi u^{r+1} v^r v_{xx} - \int \varphi u_t u^{-1}|u_x|^2 - \int \varphi v_t v^{-1}|v_x|^2.$$

Using the Cauchy-Schwarz inequality and noting $\beta \geq 0$, we have

$$\int \varphi(u^{p-1}|u_x|^2 + v^{p-1}|v_x|^2) + \frac{2\beta(r+1)}{r}\int \varphi u^r v^r u_x v_x \geq \Big(1 - \frac{\beta(r+1)}{r}\Big)\int \varphi(u^{p-1}|u_x|^2 + v^{p-1}|v_x|^2).$$

Hence,

$$-J \geq \Big(1 - \frac{\beta(r+1)}{r}\Big)\int \varphi(u^{p-1}|u_x|^2 + v^{p-1}|v_x|^2) - \frac{\beta}{r(r+1)}\int \varphi_{xx} u^{r+1} v^{r+1} + \frac{\beta}{r}\int \varphi u^r v^{r+1} u_{xx} + \frac{\beta}{r}\int \varphi u^{r+1} v^r v_{xx} - \int \varphi u_t u^{-1}|u_x|^2 - \int \varphi v_t v^{-1}|v_x|^2.$$

We calculate last two terms

$$\int \varphi u^{p-1}|u_x|^2 = \int \varphi u_x \partial_x(\frac{u^p}{p}) = -\frac{1}{p}\int \varphi u^p u_{xx} - \frac{1}{p}\int \varphi_x u_x u^p, \tag{5.12}$$

$$\int \varphi v^{p-1}|v_x|^2 = \int \varphi v_x \partial_x(\frac{v^p}{p}) = -\frac{1}{p}\int \varphi v^p v_{xx} - \frac{1}{p}\int \varphi_x v_x v^p. \tag{5.13}$$

Substituting $u_{xx} = u_t - u^p + \beta u^r v^{r+1}$ and $v_{xx} = v_t - v^p + \beta u^{r+1} v^r$, we have

$$-J \geq a_1 \int \varphi(u^{2p} + v^{2p}) + a_2 \int \varphi(u^{p+r}v^{r+1} + u^{r+1}v^{p+r}) + a_3 \int \varphi(u^{2r}v^{2r+2} + u^{2r+2}v^{2r}) - C\int \Big(\varphi|u_t|u^{-1}|u_x|^2 + \varphi|v_t|v^{-1}|v_x|^2 + |\varphi_x u_x|u^p + |\varphi_x v_x|v^p\Big) - C\int (|\varphi_t| + |\varphi_{xx}|)(u^{p+1} + v^{p+1}),$$

where

$$a_1 = \frac{1}{p} - \frac{\beta(r+1)}{pr} > 0, \tag{5.14}$$

$$a_2 = -\beta\Big(\frac{1}{p} - \frac{\beta(r+1)}{pr} + \frac{1}{r}\Big), \tag{5.15}$$

$$a_3 = \frac{\beta^2}{r} > 0. \tag{5.16}$$

We now show that $a_1 + a_2 + a_3 > 0$. Indeed,

$$\begin{aligned} a_1 + a_2 + a_3 &= \frac{1}{p} - \beta\Big(\frac{r+1}{pr} + \frac{1}{p} + \frac{1}{r}\Big) + \beta^2\Big(\frac{1}{r} + \frac{r+1}{pr}\Big) \\ &= \frac{1}{p} - \beta\frac{2}{r} + \beta^2\frac{3r+2}{pr} < 0 \\ &= \frac{1}{p}(1-\beta)\Big(1 - \beta\frac{3r+2}{r}\Big) > 0. \end{aligned}$$

The assumption on β also implies $a_1 > a_3$. Let $\eta > 0$ be small enough such that $(a_1-\eta)+a_2+a_3 \geq 0$ and $a_1 - \eta - a_3 > 0$. We have for all u, v positive that

$$(u^{2p} + v^{2p}) - 2(u^{p+r}v^{r+1} + v^{p+r}u^{r+1}) + (u^{2r}v^{2r+2} + v^{2r}u^{2r+2}) \geq 0, \tag{5.17}$$

$$(u^{2p} + v^{2p}) - (u^{p+r}v^{r+1} + v^{p+r}u^{r+1}) \geq 0. \tag{5.18}$$

Taking $(5.17) * a_3 + (5.18) * (a_1 - \eta - a_3)$, we get

$$(a_1 - \eta)(u^{2p} + v^{2p}) - (a_1 - \eta + a_3)(u^{p+r}v^{r+1} + v^{p+r}u^{r+1}) + a_3(u^{2r}v^{2r+2} + v^{2r}u^{2r+2}) \geq 0. \tag{5.19}$$

By noting that $a_1 - \eta + a_3 > -a_2$, from (5.19), we have

$$(a_1 - \eta)(u^{2p} + v^{2p}) + a_2(u^{p+r}v^{r+1} + u^{r+1}v^{p+r}) + a_3(u^{2r}v^{2r+2} + u^{2r+2}v^{2r}) \geq 0.$$

This implies

$$\begin{aligned} -J \geq &\eta \int \varphi(u^{2p} + v^{2p}) - C\int \Big(\varphi|u_t||u^{-1}|u_x|^2\varphi|v_t|v^{-1}|v_x|^2 + |\varphi_x u_x|u^p + |\varphi_x v_x|v^p\Big) \\ &- C\int (|\varphi_t| + |\varphi_{xx}|)(u^{p+1} + v^{p+1}). \end{aligned}$$

Combining this with (5.10), we have desired estimate (5.8).

For the case $\beta < 0$, from (5.11) we have

$$-J_1 \geq \int \varphi u^{p-1}|u_x|^2 - \int \varphi u_t u^{-1}|u_x|^2.$$

Hence,

$$-J \geq \int \varphi u^{p-1}|u_x|^2 + \int \varphi v^{p-1}|v_x|^2 - \int \varphi u_t u^{-1}|u_x|^2 - \int \varphi v_t v^{-1}|v_x|^2.$$

Using (5.12), (5.13) and substituting $u_{xx} = u_t - u^p + \beta u^r v^{r+1}$ and $v_{xx} = v_t - v^p + \beta u^{r+1}v^r$, we have

$$\begin{aligned} -J \geq &\frac{1}{p}\int \varphi(u^{2p} + v^{2p}) - \frac{\beta}{p}\int \varphi(u^{p+r}v^{r+1} + u^{r+1}v^{p+r}) \\ &- C\int \Big(\varphi|u_t|u^{-1}|u_x|^2 + \varphi|v_t|v^{-1}|v_x|^2 + |\varphi_x u_x|u^p + |\varphi_x v_x|v^p + |\varphi_t|(u^{p+1} + v^{p+1})\Big). \end{aligned}$$

Noting $\beta < 0$ and combing with (5.10), we have estimate (5.8). Lemma is proved. □

Lemma 5.3.2. *Assume $N = 1$. Let (u,v) be positive solution of (5.1) in $(-1, 1) \times (-1, 1)$, then*

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (u^{2p} + v^{2p})dxdt \leq C(N, p, \beta). \tag{5.20}$$

Proof. We follow the argument as in proof of [9, Proposition 21.5]. One can choose the test-function φ such that $\varphi = 1$ in $(-1/2, 1/2) \times (-1/2, 1/2)$, $0 \le \varphi \le 1$ and

$$|\varphi_x| \le C\varphi^{(3p+1)/4p}, \quad |\varphi_{xx}| \le C\varphi^{(p+1)/2p}, \quad |\varphi_t| \le C\varphi^{(3p+1)/4p} \le C\varphi^{(p+1)/2p}.$$

The same argument as in proof of [9, Proposition 21.5] yields

$$\begin{aligned}
&\int |u_x|^2\varphi^{-1}|\varphi_x|^2 \le \varepsilon(I+L) + C(\varepsilon),\\
&\int |v_x|^2\varphi^{-1}|\varphi_x|^2 \le \varepsilon(I+L) + C(\varepsilon),\\
&\int \big(\varphi(|u_t|u^{-1}|u_x|^2 + (u^p + u^{-1}|u_x|^2)|u_x\varphi_x| + u^{p+1}(|\varphi_t| + |\varphi_{xx}|)\big)\\
&\qquad \le \varepsilon(I+L) + C(\varepsilon)\Big(1 + \int \varphi|u_t|^2\Big)\\
&\int \big(\varphi(|v_t|v^{-1}|v_x|^2 + (v^p + v^{-1}|v_x|^2)|v_x\varphi_x| + v^{p+1}(|\varphi_t| + |\varphi_{xx}|)\big)\\
&\qquad \le \varepsilon(I+L) + C(\varepsilon)\Big(1 + \int \varphi|v_t|^2\Big).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int \varphi(|u_t|^2 + |v_t|^2) &= \int \varphi u_t(u_{xx} + u^p - \beta u^r v^{r+1}) + \int \varphi v_t(v_{xx} + v^p - \beta u^{r+1}v^r)\\
&= \int \varphi\partial_t\Big(\frac{u^{p+1}+v^{p+1}}{p+1} - \frac{|u_x|^2+|v_x|^2}{2} - \frac{\beta u^{r+1}v^{r+1}}{r+1}\Big) - \int(\varphi_x u_x u_t + \varphi_x v_x v_t)\\
&= \int \varphi_t\Big(-\frac{u^{p+1}+v^{p+1}}{p+1} + \frac{|u_x|^2+|v_x|^2}{2} + \frac{\beta u^{r+1}v^{r+1}}{r+1}\Big) - \int(\varphi_x u_x u_t + \varphi_x v_x v_t)\\
&\le C\int |\varphi_t|\big(u^{p+1} + v^{p+1} + |u_x|^2 + |v_x|^2\big) + \frac{1}{2}\int \varphi(|u_t|^2 + |v_t|^2) + \frac{1}{2}\int \varphi^{-1}\varphi_x^2(u_x^2 + v_x^2).
\end{aligned}$$

Hence,

$$\int \varphi(|u_t|^2 + |v_t|^2 \le C\int |\varphi_t|\big(u^{p+1} + v^{p+1} + |u_x|^2 + |v_x|^2\big) + |v_t|^2) + \frac{1}{2}\int \varphi^{-1}\varphi_x^2(u_x^2 + v_x^2).$$

The same argument as in proof of [9, Proposition 21.5] then

$$\int \varphi(|u_t|^2 + |v_t|^2) \le 2\varepsilon(I+L) + C(\varepsilon).$$

Therefore,

$$I + L \le C(\varepsilon) + C(N, p, \beta)\varepsilon(I+L).$$

By choosing ε sufficiently small, we obtain $I, L \le C$. □

Proof of Theorem 5.1.1. Let (u, v) be a solution of (5.1) in $\mathbb{R} \times \mathbb{R}$. For any $R > 0$, we rescale

$$u_R(x,t) = R^{2/(p-1)}u(Rx, R^2t), \quad v_R(x,t) = R^{2/(p-1)}v(Rx, R^2t).$$

Then (u_R, v_R) is also solution. By Lemma 5.3.2, we have

$$\begin{aligned}
\int_{-R^2/2}^{R^2/2}\int_{-R/2}^{R/2}(u^{2p} + v^{2p})(y,s)\,dyds &= R^{3-4p/(p-1)}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(u_R^{2p} + v_R^{2p})(x,t)\,dxdt\\
&\le CR^{3-4p/(p-1)}.
\end{aligned}$$

Letting $R \to \infty$ then $u \equiv v \equiv 0$. □

5.4 Liouville-type theorem for radial solutions

By the rescaling-doubling arguments as in proof of [10, Theorem 4.1], one can show the following lemma.

Lemma 5.4.1. *Assume that $p < p_S$ and $\beta < \frac{r}{3r+2}$. Let (u, v) be a positive radial solution of (5.1) in $\mathbb{R}^N \times \mathbb{R}$. Then there exists constant $C = C(N, p, \beta)$ such that*

$$(u(x,t) + v(x,t)|x|^{\frac{2}{p-1}} + (|\nabla u(x,t)| + |\nabla v(x,t)|)|x|^{\frac{p+1}{p-1}} \leq C. \tag{5.21}$$

With the decay estimate (5.21) at hand and taking into account $p < p_S$, one can define the energy functional for (u, v) solution of equation (5.1) in $\mathbb{R}^N \times \mathbb{R}$ as follows

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2)dx - \frac{1}{p+1}\int_{\mathbb{R}^N}(u^{p+1} + v^{p+1})dx + \frac{\beta}{r+1}\int_{\mathbb{R}^N} u^{r+1}v^{r+1}dx. \tag{5.22}$$

And we have

$$\frac{dE(t)}{dt} = -\|u_t(t)\|_2^2 - \|v_t(t)\|_2^2. \tag{5.23}$$

Proof of Theorem 5.1.2. We first prove the theorem for bounded solutions. Assume that (u, v) is a bounded nontrivial nonnegative solution of (5.1). By Lemma 5.4.1, the boundedness of (u, v) and $p < p_S$, we have

$$|E(t)| \leq C, \quad \forall t \in R.$$

Hence,

$$\int_{\mathbb{R}}(\|u_t\|_2^2 + \|v_t\|_2^2)dt < \infty.$$

Consequently, there exists $t_k \to \infty$ such that

$$|u_t|(t_k) + |v_t|(t_k) \to 0 \quad \text{in} \quad L^2(\mathbb{R}^N). \tag{5.24}$$

We now show that

$$\|u(t_k) + v(t_k)\|_{L^\infty} \to 0. \tag{5.25}$$

Indeed, if not then there exists x_k such that $u(x_k, t_k) + v(x_k, t - k) \geq C > 0$. It follows from the decay estimate of u, v (see Lemma 5.4.1) that x_k is bounded. We may assume that $x_k \to x_\infty$. Let $(U_k, V_k)(x) := (u(x, t_k), v(x, t_k))$ then there exists a subsequence which converges in $C_{loc}(\mathbb{R}^N)$ to a function (U, V) satisfying

$$\begin{cases} -\Delta U = U^p - \beta U^r V^{r+1}, & x \in \mathbb{R}^N \\ -\Delta V = V^p - \beta U^{r+1} V^r, & x \in \mathbb{R}^N \end{cases}$$

and $(U+V)(x_\infty) \geq C$. We note also that if U, V is radial. This contradicts Liouville-type theorem for radial solution (see [11, Proposition 5]). Hence (5.25) is true.

Let $\alpha_k = \int_{\mathbb{R}^N} u^{p+1}(t_k)dx$, for any $R > 0$ we have

$$\begin{aligned} \alpha_k &= \int_{|x|<R} u^{p+1}(t_k)dx + \int_{|x|>R} u^{p+1}(t_k)dx \\ &\leq CR^N \|u(t_k)\|_{L^\infty(\mathbb{R}^N)}^{p+1} + CR^{-[2(p+1)/(p-1)-N]}. \end{aligned}$$

Hence, $\limsup_{k\to\infty} \alpha_k \le CR^{-[2(p+1)/(p-1)-N]}$. Letting $R \to \infty$, we obtain

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} u^{p+1}(t_k)dx = 0. \tag{5.26}$$

Similarly, we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} v^{p+1}(t_k)dx = \lim_{k\to\infty} \int_{\mathbb{R}^N} u^{r+1}v^{r+1}(t_k)dx = 0. \tag{5.27}$$

We next show that $\|\nabla u(t_k)\|_{L^2(\mathbb{R}^N)} \to 0$. Let φ be a smooth function in $\mathbb{R}^N$, $0 \le \varphi \le 1$, $\varphi(x) = 0$ in B_1, $\varphi(x) = 0$ if $|x| \ge 2$ and $|\nabla\varphi| \le C\varphi^{1/2}$. For any $R > 0$, letting $\varphi_R(x) = \varphi(x/R)$, then we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u(t_k)|^2\varphi_R dx &= -\int_{\mathbb{R}^N} u(t_k)\nabla u(t_k).\nabla\varphi_R dx + \int_{\mathbb{R}^N} (u^{p+1} - \beta u^{r+1}v^{r+1} - u_t u)(t_k)\varphi_R dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(t_k)|^2\varphi_R dx + \frac{1}{2}\int_{\mathbb{R}^N} u^2(t_k)|\nabla\varphi_R|^2\varphi_R^{-1/2}dx \\ &\quad + \int_{\mathbb{R}^N} (u^{p+1} - \beta u^{r+1}v^{r+1} - u_t u)(t_k)\varphi_R dx.\end{aligned}$$

Using (5.24)-(5.27) and the compact support of φ_R, we deduce that

$$\lim_{k\to\infty} \int_{B_R} |\nabla u(t_k)|^2 dx = 0,$$

for any $R > 0$. On the other hand, if follows from spatial decay estimate of ∇u that

$$\int_{\mathbb{R}^N} |\nabla u(t_k)|^2 dx \le \int_{B_R} |\nabla u(t_k)|^2 dx + \int_{|x|>R} |\nabla u(t_k)|^2 dx \le \int_{B_R} |\nabla u(t_k)|^2 dx + CR^{N-2-\frac{4}{p-1}}. \tag{5.28}$$

By letting $k \to \infty$ and then $R \to \infty$ in (5.28), we obtain

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} |\nabla u(t_k)|^2 dx = 0. \tag{5.29}$$

Similarly,

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} |\nabla v(t_k)|^2 dx = 0. \tag{5.30}$$

We deduce from (5.26), (5.27), (5.29) and (5.30) that $E(t_k) \to 0$ as $k \to \infty$.

Similarly, there exist $s_k \to -\infty$ such that $|u_t|(s_k) + |v_t|(s_k) \to 0$ in $L^2(\mathbb{R}^N)$ and we deduce that $E(s_k) \to 0$ as $k \to \infty$. It follows from Lemma 5.23 that

$$\int_{s_k}^{t_k} \int_{\mathbb{R}^N} (u_t^2 + v_t^2)dxdt = E(s_k) - E(t_k).$$

Letting $k \to \infty$, we obtain $u_t \equiv v_t \equiv 0$. This contradicts Liouville-type theorem for radial solution (see [11]).

So far we proved our Liouville-type theorem for bounded solutions. In fact, Liouville-type theorem for bounded solutions is sufficient for the proof of universal singularity and decay estimates in Proposition 5.2.1 (by using doubling argument as in [8, Theorem 3.1]), which in particular implies universal bound of all entire positive solutions. Therefore, the boundedness assumption can be removed.

□

Bibliographie

[1] Kaouther Ammar and Philippe Souplet. Liouville-type theorems and universal bounds for nonnegative solutions of the porous medium equation with source. *Discrete Contin. Dyn. Syst.*, 26(2) :665–689, 2010.

[2] Thomas Bartsch, Norman Dancer, and Zhi-Qiang Wang. A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system. *Calc. Var. Partial Differential Equations*, 37(3-4) :345–361, 2010.

[3] Thomas Bartsch, Peter Poláčik, and Pavol Quittner. Liouville-type theorems and asymptotic behavior of nodal radial solutions of semilinear heat equations. *J. Eur. Math. Soc. (JEMS)*, 13(1) :219–247, 2011.

[4] Marie-Françoise Bidaut-Véron. Initial blow-up for the solutions of a semilinear parabolic equation with source term. In *Équations aux dérivées partielles et applications*, pages 189–198. Gauthier-Villars, Éd. Sci. Méd. Elsevier, Paris, 1998.

[5] E. Norman Dancer, Juncheng Wei, and Tobias Weth. A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27(3) :953–969, 2010.

[6] E. Norman Dancer and Tobias Weth. Liouville-type results for non-cooperative elliptic systems in a half-space. *J. Lond. Math. Soc. (2)*, 86(1) :111–128, 2012.

[7] Yuxia Guo, Bo Li, and Juncheng Wei. Entire nonradial solutions for non-cooperative coupled elliptic system with critical exponents. *preprint*, 2012.

[8] Peter Poláčik, Pavol Quittner, and Philippe Souplet. Singularity and decay estimates in superlinear problems via Liouville-type theorems. II. Parabolic equations. *Indiana Univ. Math. J.*, 56(2) :879–908, 2007.

[9] Pavol Quittner and Philippe Souplet. *Superlinear parabolic problems.* Birkhäuser Advanced Texts : Basler Lehrbücher. [Birkhäuser Advanced Texts : Basel Textbooks]. Birkhäuser Verlag, Basel, 2007. Blow-up, global existence and steady states.

[10] Pavol Quittner and Philippe Souplet. Parabolic Liouville-type theorems via their elliptic counterparts. *Discrete Contin. Dyn. Syst.*, suppl. :1206–1213, 2011.

[11] Pavol Quittner and Philippe Souplet. Optimal Liouville-type theorems for noncooperative elliptic Schrödinger systems and applications. *Comm. Math. Phys.*, 311(1) :1–19, 2012.

[12] Hugo Tavares, Susanna Terracini, Gianmaria Verzini, and Tobias Weth. Existence and nonexistence of entire solutions for non-cooperative cubic elliptic systems. *Comm. Partial Differential Equations*, 36(11) :1988–2010, 2011.

[13] Juncheng Wei and Tobias Weth. Radial solutions and phase separation in a system of two coupled Schrödinger equations. *Arch. Ration. Mech. Anal.*, 190(1) :83–106, 2008.

Chapitre 6

Global existence of solutions for a chemotaxis-type system arising in crime modelling

Global existence of solutions for a chemotaxis-type system arising in crime modelling [7]

Raul Manasevich, Quoc Hung Phan and Philippe Souplet

Abstract. We consider a nonlinear, strongly coupled, parabolic system arising in the modelling of burglary in residential areas. This model appeared in [19], as a more realistic version of Short et al. [24] model. The system under consideration is of chemotaxis-type and involves a logarithmic sensitivity function and specific interaction and relaxation terms. Under suitable assumptions on the data of the problem, we give a rigorous proof of the existence of a global and bounded, classical solution, thereby solving a problem left open in previous work on this model. Our proofs are based on the construction of approximate entropies and on the use of various functional inequalities. We also provide explicit numerical conditions for global existence when the domain is a square, including concrete cases involving values of the parameters which are expected to be physically relevant.

6.1 Introduction

6.1.1 Model and main results

In a series of recent papers, models based on partial differential equations have been derived to study crime, see [3], [8, 19, 20, 23, 24]. Many of these papers are most related to modelling burglary of houses. A basic issue here is to obtain patterns that describe the location of hotspots, and study phenomena such as appearance and disappearance of them, their stability and movement.

The primary interest in considering continuum systems to study crime comes from two sides. On the one hand we want to gain information about this social phenomenon that is so important to the whole society. On the other hand the models that have emerged so far are very interesting since they give rise to challenging mathematical questions.

In this paper we will provide rigorous answers to some important issues that have arisen in one of the proposed models for burglary of houses. We begin with a short description of the main ideas and models that have originated our paper.

A very successful model was obtained recently by Short et al. in [24]. They first derived an agent-based statistical model to study the dynamics of hotspots, taking two sociological effects into account : the 'broken window effect' and the 'repeat near-repeat effect'. The first effect refers to the observation that crime in an area leads to more crime, and the second one to the observation that houses burglarized at some moment have an increased probability of being burglarized again for some period of time after that moment.

The agent-based model considered relies on the assumption that criminal agents are walking randomly on a two-dimensional lattice and are committing burglaries when encountering an attractive opportunity. An attractiveness value is assigned to every house, which measures how easily the house can be burgled without consequences for the criminal agent. In addition to walking randomly, the criminal agents move toward areas of high attractiveness values. In turn, when a burglary occurs, it increases the attractiveness of the house that was burglarized and of those nearby. If no additional burglaries occur, then the local attractiveness decays toward a constant value.

In a second step, by taking a suitable limit of the equations for the discrete model, the authors in [24] obtained the following system of parabolic differential equations (in non dimensional

7. Europ. J. Appl. Math., 24(2) :273-296, 2013

form) :

$$\begin{cases} \dfrac{\partial A}{\partial t} = \eta \triangle A + NA + A^0 - A, \\ \\ \dfrac{\partial N}{\partial t} = \nabla \cdot \Big[\nabla N - N \nabla \vartheta(A) \Big] - NA + \overline{A} - A^0. \end{cases} \tag{6.1}$$

Here and throughout the article, we denote

$$\vartheta(A) = \chi \log A, \quad \text{with } \chi > 0 \text{ constant.} \tag{6.2}$$

The model described in [24] involves the specific value $\chi = 2$ (we note that the parameter χ cannot be scaled out by a linear change of dependent or independent variables). The functions $A(x;t)$ and $N(x,t)$ respectively represent the attractiveness value and the criminal density at position x and time t. The first equation describes the evolution of the attractiveness of individual houses to burglary, and the second describes the burglar movement. Here Ω is a bounded domain in $\mathbb{R}^2$, $\eta > 0$ is the diffusion rate of attractiveness, A^0 is the intrinsic attractiveness, $\overline{A}$ is a constant that in the equilibrium case represents the average attractiveness.

A first effort to study the dynamics of the model represented by problem (6.1) was done in [20], where a corresponding initial-boundary value problem with no flux boundary conditions is considered. To deal with this problem, the authors in [20] assume, for simplicity, that Ω is a square and under some symmetry conditions are able to map this problem into one with periodic problem boundary conditions. Their main result is local existence of a solution, but global existence is left open. In addition, and as a simplification of the model in [24], they considered a generalized version of a Keller-Segel chemotaxis model with the goal of understanding possible conditions for global existence vs blow-up of solutions in finite time for the original model.

After this work was submitted, we were informed of a very recent work [21] where the author proves global existence of solutions for a parabolic system modelling crime behavior different from the one in [24] and different from the one we will consider in this paper.

In [24], an important hypothesis is that burglars are generated in the model at a constant rate and leave the lattice immediately after they have committed a burglary. In [19], a modified agent-based model for burglary was obtained, a new condition is introduced to modify these effects. This new condition, that is refer as burglar fatigue, models the effect that if burglars are sufficiently deterred, they will eventually get tired and will stop looking for houses to burglar. This consideration introduces changes to the original equations of Short et al. Some additional changes to the original equations come in by considering in the model that, if too many burglaries occur at some location, then burglars are likely to assume that most of the high-value is gone or that the owners of the place have implemented more strict security measures.

As in the original model, by taking the continuous limit to this new agent-based model, a new system of parabolic equations is obtained in [19]. This system, complemented with initial data and no-flux boundary conditions, leads us to study the following initial-boundary value problem (again in non dimensional form) :

$$\begin{cases} \dfrac{\partial A}{\partial t} = \eta \Delta A + \psi NA(1-A) + \tilde{A} - A, & x \in \Omega,\ t > 0, \\ \\ \dfrac{\partial N}{\partial t} = \nabla \cdot \Big[\nabla N - N \nabla \vartheta(A) \Big] + \omega - \omega N, & x \in \Omega,\ t > 0, \\ \\ \dfrac{\partial A}{\partial \nu} = \dfrac{\partial N}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ \\ A(x,0) = A_0(x), \quad N(x,0) = N_0(x), & x \in \Omega. \end{cases} \tag{6.3}$$

Here again the function ϑ is defined by (6.2) with $\chi = 2$ in [19]. Moreover, it is assumed – as we will do throughout this paper – that

$$\Omega \text{ is a sufficiently smooth bounded domain of } \mathbb{R}^2, \text{ or a square,} \tag{6.4}$$

$$\eta, \psi, \omega, \tilde{A} > 0 \text{ are constants} \tag{6.5}$$

and that the initial data satisfy

$$(A_0, N_0) \in H^{1+\beta}(\Omega) \times L^2(\Omega) \text{ for some } \beta > 0, \text{ with } A_0 > 0 \text{ in } \overline{\Omega} \text{ and } N_0 \geq 0 \text{ a.e. in } \Omega. \tag{6.6}$$

The outward normal vector on $\partial\Omega$ is denoted by ν. (We point out that all the functional analytic properties needed for the proofs remain true in a square, although the latter is not smooth at the corner points.) Problem (6.3) is locally well posed, the positivity of A being of course understood as part of the definition of solution (see Section 2 for details).

In [19], some results about linearized stability/instability of the homogeneous steady-states and some numerical simulations suggesting the existence of hotspots were given for system (6.3). However, the (local and) global existence of solutions was left open.

In this paper, our main goal is to give a rigorous proof of the global existence and boundedness of solutions of system (6.3) under suitable assumptions. In what follows, we denote

$$\varepsilon_0 = \varepsilon_0(\Omega) = \mu^{-2} K^{-1/2},$$

where μ is the best constant in the Poincaré-Sobolev inequality

$$\|u - |\Omega|^{-1} \int_\Omega u\, dx\|_2 \leq \mu \|\nabla u\|_1, \quad u \in W^{1,1}(\Omega),$$

and K is the best constant in the interpolation estimate

$$\int_\Omega |\nabla u|^4 dx \leq K \operatorname{osc}^2(u) \int_\Omega |\Delta u|^2 dx, \quad u \in H^2(\Omega) \text{ with } \tfrac{\partial u}{\partial \nu} = 0$$

(cf. Lemmas 6.2.3 and 6.2.4).

Also, recalling that $A_0 \in C(\overline{\Omega})$ by Sobolev imbedding, we set

$$A_{\max} = \max\{1, \tilde{A}, \max_{\overline{\Omega}} A_0\} \geq A_{\min} = \min\{1, \tilde{A}, \min_{\overline{\Omega}} A_0\} > 0.$$

Our main result is the following :

Theorem 6.1.1. *Assume (6.2), (6.4)-(6.6) and*

$$\Big(\frac{A_{\max}}{A_{\min}}\Big)^2 (A_{\max} - A_{\min}) \max\{\|N_0\|_1, |\Omega|\} < \varepsilon_0 \eta \psi^{-1} \chi^{-2}. \tag{6.7}$$

Then the solution of problem (6.3) is global and satisfies the uniform bounds

$$\sup_{t \geq 0} \|A(t)\|_\infty < \infty \tag{6.8}$$

and

$$\sup_{t \geq \tau} \|N(t)\|_\infty < \infty, \qquad \textit{for all } \tau > 0. \tag{6.9}$$

On the other hand, when $\chi \leq 1$, by using an argument from [5], we can show that the conclusion of Theorem 6.1.1 remains true under a simple assumption on A_0 and $\tilde{A}$ and without any size restriction on N_0. However, this does not apply to the criminological model in [19] where $\chi = 2$.

Theorem 6.1.2. *Assume (6.2), (6.4)-(6.6), with* $0 < \chi \leq 1$, $\tilde{A} \leq 1$ *and* $\max_{\overline{\Omega}} A_0 \leq 1$. *Then the solution of problem (6.3) is global and satisfies the uniform bounds (6.8)-(6.9).*

6.1.2 Explicit global existence conditions and discussion

Interestingly, in view of practical applications, the constant appearing in Theorem 6.1.1 can be estimated explicitly in the case of a square.

Theorem 6.1.3. *Let $\Omega = (0, L)^2$, with $L > 0$. Then the result of Theorem 6.1.1 is true with*

$$\varepsilon_0 = \frac{1}{3\sqrt{3}} \approx 0.19.$$

With Theorem 6.1.3 at hand, we shall now explicitly compute our sufficient condition for global existence and boundedness in concrete cases involving values of the parameters which are expected to be physically relevant (cf. [24, 19]). The parameters of system (6.3) are expressed in [19] in terms of parameters of the dimensional form of the model. To avoid confusion, dimensional parameters are marked with a hat in what follows.

The domain is taken to be a square lattice where exactly one house is located at each lattice site. Distances are thus measured in units of house separations and the adimensional side length of the domain is thus essentially equal to $\sqrt{p}$, where $p \gg 1$ is the total number of houses. After a suitable renormalization (involving the diffusion and mean lifetime parameters of the burglars – see table 1 and formula (3.3) in [19, p.406]), the measure of the adimensionalized spatial domain Ω is given by $|\Omega| = p/3500$. We chose $p = 3500$, hence $|\Omega| = 1$. Next, the parameter ω is given by $\omega = \hat{\omega}_2/\hat{\omega}_1$, where $\hat{\omega}_1$ and $\hat{\omega}_2$ are the mean lifetimes of the attractiveness and of the active burglars, respectively. The values chosen in [19] are $\hat{\omega}_1 = (1/14)\,\text{day}^{-1}$ and $\hat{\omega}_2 = 6\,\text{day}^{-1}$, hence $\omega = 84$. The parameter ψ is given by $\psi = \hat{\theta}\hat{\Gamma}/\hat{\omega}_1\hat{\omega}_2$, where $\hat{\Gamma}$, describing the source term, stands for the number of burglars becoming active per time and surface unit and $\hat{\theta}$ is the factor that boosts attractiveness due to a burglary. We take $\hat{\Gamma} = 0.002$ burglars $\cdot$ day^{-1} $\cdot$ house separation^{-2}, which is the value taken in [24] for the analogous parameter (the value $\hat{\Gamma} = 0.5$ used in [19] seems too high, apparently due to a lack of spatial normalization). For the domain that we consider, this means a total source term of 7 burglars $\cdot$ day^{-1}. Concerning $\hat{\theta}$, following [19], we take $\hat{\theta} = 1$ burglars^{-1} $\cdot$ day^{-1}, which leads to $\psi = (14/3) \times 10^{-3} \approx 0.0047$. As for the attractiveness diffusion parameter η, which models neighboring effects, smaller values mean that repeat victimization is more and more likely than near-repeat victimization. The values taken in [24] range from 0.01 to 0.2, whereas [19] takes $\eta = 0.001$.

Last, let us consider the initial data. Concerning the (adimensional) attractiveness A, recall that the constant $\tilde{A}$ is assumed to represent its static component, while $A - \tilde{A}$ represents the dynamic component associated with the boost hypothesis. However it is not made completely clear in [24, 19] how A should be evaluated or measured. Here we shall assume that

$$\tilde{A} \le A_0(x) \le 1, \tag{6.10}$$

which implies that $A_{\min} = \tilde{A}$ and $A_{\max} = 1$. Consequently (cf. Lemma 6.2.1 below), the attractiveness remains for all times larger or equal to $\tilde{A}$, and at most one. Note that in the numerical simulations from [19], the initial data are assumed to be close to the homogeneous steady-state :

$$(A^*, N^*) = \left(\frac{\psi - 1 + \sqrt{(\psi-1)^2 + 4\psi\tilde{A}}}{2\psi}, 1 \right),$$

which in particular satisfies (6.10). As for the initial burglar density N_0, we assume for simplicity that its average value $\overline{N_0}$ satisfies

$$\overline{N_0} := |\Omega|^{-1} \|N_0\|_1 \le 1 = N^*.$$

In dimensional units (cf. [19]), in view of the above choice of $\hat{\omega}_2$ and $\hat{\Gamma}$, this corresponds to an initial total population less than ≈ 1.2 burglars (although this may seem small, recall that we have at the same time a total source term of 7 burglars $\cdot$ day^{-1}).

Now, for the above-chosen numerical values of the parameters, the global existence and boundedness condition (6.7) in Theorems 6.1.1 and 6.1.3 becomes

$$\tilde{A}^{-2}(1-\tilde{A}) < \gamma\,\eta, \qquad \text{where } \gamma = \frac{1}{12\sqrt{3}}(|\Omega|\psi)^{-1} = \frac{125\sqrt{3}}{21} \approx 10.31. \tag{6.11}$$

This is equivalent to

$$\tilde{A} > \tilde{A}_- := \frac{2}{1+\sqrt{1+4\gamma\,\eta}}.$$

The values of $\tilde{A}_-$ as a function of η are given in the following table :

η	$\tilde{A}_-$
0.01	0.91
0.05	0.73
0.1	0.61
0.2	0.49

Therefore, our global existence conditions are compatible with attractiveness values which are up to about twice those of their static component in the range of η used by [24]. We note that it is suggested in [19] that a ratio of order 10 (instead of 2) might be desirable, even for smaller values of η that the ones we consider here. However, this is still beyond the range in which we can rigorously prove global existence (and actually, global existence for such systems need not be taken for granted – see next subsection).

6.1.3 Related results and comments

As noted in [24], problems (6.1) and (6.3) belong to the family of chemotaxis-type systems. A more general class of such systems takes the form

$$\begin{cases} \dfrac{\partial A}{\partial t} = \eta\Delta A - A + f(A,N), \\[2ex] \dfrac{\partial N}{\partial t} = \Delta N - \nabla(N\nabla h(A)) - \omega N + g(A,N). \end{cases} \tag{6.12}$$

The typical feature of chemotaxis-type systems is the presence of the anti-diffusion term, here $-\nabla(N\nabla h(A))$, by which the population, of density N, tends to move towards higher concentrations of A. The best-known among this class of problems is the Keller-Segel model [15], corresponding to $f = N$, $h = A$ and $g = 0$.

In more general systems (6.12), the nonlinear sensitivity function h, or an even more general anti-diffusion term of the form $-\nabla(N\kappa(A,N)\nabla A)$, is designed to incorporate additional features such as the prevention of overcrowding, sometimes also called volume-filling effect. Other related models involve nonlinear diffusion terms $\Delta\phi(A)$ instead of ΔA. There is a very large literature on such systems (see [12, 13] for recent surveys and, for instance, [9] and the references therein).

System (6.12) with a logarithmic sensitivity function $h = \chi\log(A)$ was considered in [4, 25] for $f = N$ and $g = \omega = 0$. Without any size restriction on the initial data, existence of a global classical solution was proved in [4] under the restriction $\chi \le 1$, whereas existence of a (possibly singular) global weak solution was proved in [25] for any $\chi > 0$. System (6.12) with a logarithmic sensitivity function was also considered in [2] (see also [18]), in conjunction with a quadratic absorption term, namely $g = -cN^2$, and $f = N$. In this case, global classical existence was proved for any $\chi > 0$ without size restriction on the initial data, but the proof made crucial use of the quadratic absorption term.

Here, with only a linear absorption $-\omega N$ and the specific f under consideration, a size condition on the initial data is required to prove existence of global classical solution, but we do not need absolute restrictions on the parameter χ. Although we do not know what happens for larger initial data, let us recall that in the case of the Keller-Segel model in two space dimensions, global existence is true only for small initial mass and that blow-up may occur for large mass. However proving blow-up for 2-dimensional parabolic chemotaxis models is a very challenging task. The only known blow-up result for the $2d$ Keller-Segel system is that of Herrero and Velázquez [11], and its proof, based on matched asymptotics, is highly technical. In simplified parabolic-elliptic chemotaxis models, where the term $\partial A/\partial t$ in (6.12) is replaced by 0, blow-up proofs are easier and there are more results available. In particular, when $h = \chi \log A$, $f = N$, $g = \omega = 0$ and the space dimension n is at least 3, finite time blow-up is known [16] to occur for $\chi > 2n/(n-2)$. For blow-up results concerning the parabolic-elliptic Keller-Segel system, see e.g. the survey [6].

Our global existence proofs, as is customary in chemotaxis-type problems, rely on a priori estimates obtained by energy and entropy arguments. However, in the case of Theorem 6.1.1, some care is needed to properly take into account the size condition on the initial data. In the case of the Keller-Segel system, an exact entropy functional was found and used to prove global existence for suitably small mass in [17, 10, 4]. Although system (6.3) is not known to possess an exact entropy functional, *assuming condition (6.7) on the data of the problem,* we can nevertheless construct an *approximate* entropy functional (with the typical $N \log N$ growth – cf. Lemma 6.3.1), which is the key to our a priori estimates. We also stress that, in spite of uniform lower and upper bounds for A which easily follow from the maximum principle, the proof of the estimates of N is far from being immediate. As for Theorem 6.1.2, under the assumptions $\chi \leq 1$, $\tilde{A} \leq 1$ and $\max\limits_{\overline{\Omega}} A_0 \leq 1$, we can use a modified entropy functional from [5].

Remarks 6.1.1. By straightforward modifications of the method (see Remark 6.4.1 for more details), one can show that a result similar to Theorem 6.1.1 remains true for the initial-boundary value problem associated with the more general system (6.12) where f, g, h are (sufficiently smooth) functions satisfying the following conditions

- $(\mathcal{H}_1)$ $\quad g(A, 0) \geq 0, \quad \forall A \geq 0.$

- $(\mathcal{H}_2)$ There exist two positive constants $A_{\min} < A_{\max}$ such that

$$
\begin{aligned}
-A_{\min} + f(A_{\min}, N) \geq 0, \quad \forall N \geq 0,\\
-A_{\max} + f(A_{\max}, N) \leq 0, \quad \forall N \geq 0.
\end{aligned}
$$

- $(\mathcal{H}_3)$

$$|g(A, N)| \leq g_1(A) N^{1-\delta} + g_2(A),$$

  where $\delta \in (0, 1)$ and g_1, g_2 are bounded in $[A_{\min}, A_{\max}]$.

- $(\mathcal{H}_4)$

$$|f(A, N)| \leq f_1(A) N + f_2(A),$$

  where f_1, f_2 are bounded in $[A_{\min}, A_{\max}]$.

- $(\mathcal{H}_5)$

$$\sup_{[A_{\min}, A_{\max}]} \left| \frac{d^i h(A)}{dA^i} \right| < \infty, \quad i = 1, 2.$$

The outline of the rest of the article is as follows. Subsection 2.1 is devoted to local existence and uniqueness, whereas Subsection 2.2 contains basic estimates, energy identities and functional inequalities which will needed for proving global existence. Sections 3–5 are then devoted to the proofs of Theorems 1.1, 1.3 and 1.2, respectively. Finally, our main conclusions are summarized in Section 6.

6.2 Preliminaries

6.2.1 Local existence and uniqueness

We have the following local existence and uniqueness result.

Proposition 6.2.1. *Assume (6.2), (6.4)-(6.6) and fix any $\beta \in (0, \frac{1}{2})$. Then problem (6.3) admits a unique, classical, maximal in time solution (A, N) with $A > 0$. Moreover, if its maximal existence time T^* is finite, then* $\lim_{t \to T^*} \|A(t)\|_{H^{1+\beta}} + \|N(t)\|_{L^2} = \infty$.

Note that the solution given by Proposition 6.2.1 is of course independent of the choice of $\beta \in (0, \frac{1}{2})$ (due to local uniqueness and the fact that $H^{1+\beta}(\Omega) \subset H^{1+\gamma}(\Omega)$ for $\beta > \gamma$). The procedure for proving Proposition 6.2.1 is rather standard (see, e.g., [18, 2, 14] for more details on similar problems). Although the function ϑ is singular at $A = 0$, this actually causes no difficulty, because a positive lower bound for A can be deduced from the maximum principle (first working with a smooth replacement of ϑ). We sketch the proof for completeness.

We first recall some properties of theory of abstract evolution equation. Let $\mathcal{H}, \mathcal{V}$ be two separable Hilbert spaces with dense and compact embedding $\mathcal{V} \subset \mathcal{H}$. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ and denoting the dual space of $\mathcal{V}$ by $\mathcal{V}'$.

Consider the initial value problem

$$\begin{cases} \dfrac{dU}{dt} + \mathcal{A}U = F(U), \quad 0 < t < \infty, \\ U(0) = U_0. \end{cases} \tag{6.13}$$

Here, $\mathcal{A}$ is bounded linear operator from $\mathcal{V}$ to $\mathcal{V}'$ which is defined by a symmetric bilinear form $a(.,.)$ on $\mathcal{V}$ satisfying

$$\begin{aligned} &|a(U, V)| \leq M\|U\|_{\mathcal{V}}\|V\|_{\mathcal{V}}, \quad U, V \in \mathcal{V}, \\ &a(U, U) \geq c\|U\|_{\mathcal{V}}^2, \quad U \in \mathcal{V}, \end{aligned} \tag{6.14}$$

with some positive constants c, M. $F : \mathcal{V} \to \mathcal{V}'$ is a continuous mapping such that, for each $\theta > 0$, there exist nondecreasing functions ϕ_θ, ψ_θ such that

$$\|F(U)\|_{\mathcal{V}'} \leq \theta\|U\|_{\mathcal{V}} + \phi_\theta(\|U\|_{\mathcal{H}}), \tag{6.15}$$

$$\|F(U) - F(V)\|_{\mathcal{V}'} \leq \theta\|U - V\|_{\mathcal{V}} + (\|U\|_{\mathcal{V}} + \|V\|_{\mathcal{V}} + 1)\psi_\theta(\|U\|_{\mathcal{H}} + \|V\|_{\mathcal{H}})\|U - V\|_{\mathcal{H}}, \tag{6.16}$$

for $U, V \in \mathcal{V}$. Then by standard argument (see [22]), the following holds :

Proposition 6.2.2. *Assume (6.14)-(6.16) and let $M > 0$ and $U_0 \in \mathcal{H}$ with $\|U_0\|_{\mathcal{H}} \leq M$. Then there exists $T = T(M) > 0$ and a unique local-in-time solution U to (6.13) such that*

$$U \in C([0, T]; \mathcal{H}) \cap H^1([0, T]; \mathcal{V}') \cap L^2([0, T]; \mathcal{V}). \tag{6.17}$$

To deduce Proposition 6.2.1 from Proposition 6.2.2, we fix β in $(0, \frac{1}{2})$ and we set

$$\mathcal{V} := H_N^{2+\beta}(\Omega) \times H^1(\Omega), \quad \mathcal{H} := H^{1+\beta}(\Omega) \times L^2(\Omega).$$

Then

$$\mathcal{V}' = H^\beta(\Omega) \times (H^1(\Omega))'.$$

The linear operator $\mathcal{A}$ is defined by

$$\mathcal{A} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where $T_1 = -\Delta + 1$ is regarded as an operator from $H_N^{2+\beta}(\Omega)$ to $H^\beta(\Omega)$ and $T_2 = -\Delta + \omega$ is the Laplace operator equipped with the Neumann boundary condition in $H^1(\Omega)'$. Then the bilinear form $a(.,.)$ is defined by

$$a(U,V) = \left(T_1^{(1+\beta)/2}A_1, T_1^{(1+\beta)/2}A_2\right)_{L^2} + \int_\Omega (\nabla N_1 \nabla N_2 + \omega N_1 N_2)dx,$$

where $U = (A_1, N_1)$ and $V = (A_2, N_2)$.

For any fixed $\alpha \in (0, A_{\min})$, we pick a function $\vartheta_\alpha \in C^\infty(\mathbb{R})$ such that $\vartheta_\alpha(s) = \chi \log s$ for $s \geq A_{\min}$. Define $F_\alpha(U)$ by

$$F_\alpha(U) = \begin{pmatrix} \psi NA(1-A) + \tilde{A} \\ -\nabla(N\nabla\vartheta_\alpha(A)) + \omega \end{pmatrix}, \qquad U = (A, N).$$

and denote by (P_α) the modified problem (6.3).

As a consequence of Proposition 6.2.2, problem (P_α) admits a unique, maximal in time solution $(\hat{A}, \hat{N})$, defined in an interval $[0, T^*)$. Moreover, if $T^* < \infty$ then $\lim_{t\to T^*} \|\hat{A}(t)\|_{H^{1+\beta}} + \|\hat{N}(t)\|_{L^2} = \infty$. Furthermore, it is classical for $t > 0$. This follows from a standard bootstrap argument based on parabolic regularity and imbedding theorems.

Observing that $A_{\min}$ is a subsolution of the equation for $\hat{A}$, we deduce that $\hat{A} \geq A_{\min}$ in $(0, T^*)$, so that $(\hat{A}, \hat{N})$ actually solves the original problem (6.3).

As for local uniqueness for (6.3), the notion of solution to (6.3) implies that A is uniformly positive on $\overline{\Omega} \times [0, T]$ for any $0 < T < T^*$ (since $A \in C([0, T^*); H^{1+\beta}(\Omega))$. Since α can be arbitrary close to 0 in problem (P_α), the uniqueness for the original problem follows from the uniqueness for (P_α).

6.2.2 Basic estimates, energy identities and functional inequalities

Our first lemma provides the primary L^1 control of N and uniform lower and upper bounds for A. The latter can be easily obtained from the maximum principle, owing to the special form of the nonlinear term in the equation for A.

Lemma 6.2.1 (A priori estimates for A and N)**.** *For all $t \in (0, T^*)$, we have*

$$A(x,t) \geq \min\{1, \tilde{A}, \inf A_0(x)\} := A_{\min}, \tag{6.18}$$

$$A(x,t) \leq \max\{1, \tilde{A}, \sup A_0(x)\} := A_{\max}, \tag{6.19}$$

$$N(x,t) \geq 1 - e^{-\omega t} > 0, \tag{6.20}$$

$$\|N(t)\|_1 = e^{-\omega t}\|N_0\|_1 + |\Omega|(1 - e^{-\omega t}), \tag{6.21}$$

hence in particular

$$\|N(t)\|_1 \leq \max\{\|N_0\|_1, |\Omega|\} := N_{1,\max}. \tag{6.22}$$

Proof. To check (6.18) and (6.19), it suffices to observe that $A_{\min}, A_{\max}$ are, respectively, sub-/supersolution of the equation for A, with Neumann boundary conditions, and to apply the maximum principle. The proof of (6.20) is similar, in view of the regularity of the solution. As for (6.21), it follows by integrating the equation for N in space. □

We next state the basic energy identities.

Lemma 6.2.2 (Energy identities)**.** *For all $t \in (0, T^*)$, we have*

$$\frac{1}{2}\frac{d}{dt}\|A(t)\|_2^2 + \|A(t)\|_2^2 + \eta\|\nabla A(t)\|_2^2 = \psi \int_\Omega NA^2(1-A)dx + \int_\Omega \tilde{A}A\,dx, \tag{6.23}$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla A(t)\|_2^2 + \|\nabla A(t)\|_2^2 + \eta\|\Delta A(t)\|_2^2 = -\psi \int_\Omega NA(1-A)\Delta A\,dx, \tag{6.24}$$

$$\frac{d}{dt}\int_\Omega (N\log N - N + 1)dx + \omega\int_\Omega (N\log N - N + 1)dx + \int_\Omega \frac{|\nabla N|^2}{N}dx - \int_\Omega \nabla N.\nabla\vartheta(A)dx$$
$$= \omega\int_\Omega (\log N - N + 1)dx \leq 0, \tag{6.25}$$

$$\frac{1}{2}\frac{d}{dt}\|N(t)\|_2^2 + \omega\|N(t)\|_2^2 + \|\nabla N(t)\|_2^2 = \int_\Omega N\nabla N.\nabla\vartheta(A)dx + \omega\int_\Omega N dx. \tag{6.26}$$

Proof. Formulae (6.23)-(6.26) follow by integration by parts after multiplying, respectively, the first equation in (6.3) by A and $-\Delta A$, and the second equation in (6.3) by $\log N - 1$ and N. Note that these manipulations are licit in view of (6.20), of the classical regularity of N and of the higher parabolic regularity applied to the equation for A. □

Lemma 6.2.3 (Poincaré's inequality)**.** *For any uniformly positive function $u \in H^1(\Omega)$, there holds*

$$\|u\|_2^2 \leq \mu^2\|u\|_1 \int_\Omega \frac{|\nabla u|^2}{u}dx + |\Omega|^{-1}\|u\|_1^2, \tag{6.27}$$

where μ is the best constant of the following Poincaré-Sobolev inequality :

$$\|u - |\Omega|^{-1}\int_\Omega u\,dx\|_2 \leq \mu\|\nabla u\|_1, \quad u \in W^{1,1}(\Omega). \tag{6.28}$$

Proof. Denote $\overline{u} = |\Omega|^{-1}\int_\Omega u\,dx$. Using the orthogonality of $u - \overline{u}$ and $\overline{u}$ in L^2 and (6.28), we have

$$\|u\|_2^2 = \|u - \overline{u}\|_2^2 + |\Omega|\overline{u}^2$$
$$\leq \mu^2\|\nabla u\|_1^2 + |\Omega|\overline{u}^2. \tag{6.29}$$

Since $\|\nabla u\|_1^2 \leq \|u\|_1 \int_\Omega \frac{|\nabla u|^2}{u}dx$ by the Cauchy-Schwarz inequality, we deduce (6.27). □

A key ingredient in the proof of our main result is the following interpolation estimate. We recall that $H^2(\Omega) \subset C(\overline{\Omega})$, since we are in two space dimensions ; however Lemma 6.2.4 remains true in any dimension if we assume $u \in H^2(\Omega) \cap C(\overline{\Omega})$.

Lemma 6.2.4. *Assume that $u \in H^2(\Omega)$ satisfies $\partial u/\partial\nu = 0$ on $\partial\Omega$ (in the sense of traces). Then*

$$\int_\Omega |\nabla u|^4 dx \leq K\,\mathrm{osc}^2(u)\int_\Omega |\Delta u|^2 dx, \tag{6.30}$$

where $K = K(\Omega) > 0$.

Proof. **Step 1.** We first give a homogeneous version of the standard elliptic L^2-estimate, namely :

$$\|D^2u\|_2 := \Big(\sum_{ij}\|u_{x_ix_j}\|_2^2\Big)^{1/2} \leq C(\Omega)\|\Delta u\|_2, \tag{6.31}$$

for any $u \in H^2(\Omega)$ with Neumann boundary condition.

To verify (6.31), we start from $\|D^2u\|_2 \leq C(\Omega)\| - \Delta u + u\|_2$, which is well known (see e.g. [7, Chapter 9]), hence

$$\|D^2u\|_2 \leq C(\Omega)(\|\Delta u\|_2 + \|u\|_2). \tag{6.32}$$

Let $(e_k)_{k\geq 0}$ be a Hilbert basis of L^2 made of eigenfunctions of $-\Delta$ with domain H^2 equipped with Neumann conditions. Denote the eigenvalues by $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and observe that $e_0 = C = Const$. Set $f = -\Delta u$, decompose $f = \sum_{k\geq 0} c_k e_k$, and note that

$$c_0 = (f, e_0) = C \int_\Omega f\,dx = -C \int_{\partial\Omega} \partial_\nu u\,d\sigma = 0.$$

Let $v := \sum_{k\geq 1} \lambda_k^{-1} c_k e_k$. Then $-\Delta v = f$ with $\partial_\nu v = 0$ and v satisfies

$$\|v\|_2^2 = \sum_{k\geq 1} \lambda_k^{-2} |c_k|^2 \leq \lambda_1^{-2} \|\Delta u\|_2^2. \tag{6.33}$$

Since the difference $z = v - u$ satisfies $\Delta z = 0$, $\partial_\nu z = 0$, it follows that $\int_\Omega |\nabla z|^2\,dx = 0$, so that

$$u = v + Const.$$

Combining this with (6.32) and (6.33), we obtain

$$\|D^2 u\|_2 = \|D^2 v\|_2 \leq C(\Omega)(\|\Delta v\|_2 + \|v\|_2),$$

and

$$\|D^2 u\|_2 \leq C(\Omega)(1 + \lambda_1^{-1})\|\Delta u\|_2.$$

Hence (6.31).

Step 2. By density, it suffices to prove the Lemma for $u \in C^2(\overline{\Omega})$. We may assume without loss of generality that $\min_\Omega u = 0$ and $\mathrm{osc}(u) = \|u\|_\infty$. For a matrix $M = (m_{ij})$, we denote $|M|_1 = \max_i\{\sum_j |m_{ij}|\}$. Observing that

$$\nabla \cdot (|\nabla u|^2 \nabla u) = |\nabla u|^2 \Delta u + 2\,{}^t\nabla u(D^2 u)\nabla u = {}^t\nabla u\big(2D^2 u + (\Delta u)I\big)\nabla u,$$

where I is the identity matrix, we have

$$\big|\nabla \cdot (|\nabla u|^2 \nabla u)\big| \leq \big|2D^2 u + (\Delta u)I\big|_1 |\nabla u|^2.$$

Using the divergence theorem, it follows that

$$\begin{aligned}\int_\Omega |\nabla u|^4 dx &= \int_\Omega \nabla u \cdot |\nabla u|^2 \nabla u\,dx = -\int_\Omega u \nabla(|\nabla u|^2 \nabla u) dx \\ &\leq \int_\Omega u\big|2D^2 u + (\Delta u)I\big|_1 |\nabla u|^2 dx \leq \frac{1}{2}\int_\Omega |\nabla u|^4 dx + \frac{1}{2}\int_\Omega u^2 \big|2D^2 u + (\Delta u)I\big|_1^2 dx.\end{aligned}$$

Consequently,

$$\int_\Omega |\nabla u|^4 dx \leq \|u\|_\infty^2 \int_\Omega \big|2D^2 u + (\Delta u)I\big|_1^2 dx. \tag{6.34}$$

Since $\int_\Omega \big|2D^2 u + (\Delta u)I\big|_1^2 dx \leq C\|D^2 u\|_2^2$, it follows from (6.31) that

$$\int_\Omega \big|2D^2 u + (\Delta u)I\big|_1^2 dx \leq K(\Omega)\|\Delta u\|_2^2 \tag{6.35}$$

which, along with (6.34), implies the conclusion. □

We shall also need the following classical smoothing properties of the Neumann heat semigroup. For $d, \lambda > 0$, we define the operator $\mathcal{A} = \mathcal{A}_{d,\lambda} = -d\Delta + \lambda$ on $L^2(\Omega)$, with domain $D(\mathcal{A}) = \{v \in H^2(\Omega);\ \partial A/\partial\nu = 0 \text{ on } \partial\Omega \text{ (in the sense of traces)}\}$. We denote by $T(t) = T_{d,\lambda}(t)$ the semigroup generated by $\mathcal{A}$. It is well known that, for any $0 \leq m \leq 2$, $1 \leq p \leq q \leq \infty$ and any $\phi \in L^2(\Omega) \cap L^p(\Omega)$, we have

$$\|T(t)\phi\|_{W^{m,q}(\Omega)} \leq Ce^{-\lambda t}\big(1 + t^{-\frac{m}{2} - \frac{1}{p} + \frac{1}{q}}\big)\|\phi\|_{L^p(\Omega)}, \qquad t \geq 0, \tag{6.36}$$

with $C = C(\Omega, m, p, q, \eta) > 0$.

6.3 Global existence and boundedness : Proof of Theorem 6.1.1

The following lemma is the key to the proof of our main result. It provides an approximate entropy functional ϕ, which is available whenever condition (6.7) is satisfied. This function enables us to get an H^1-bound for A and a time-averaged H^2-bound.

Lemma 6.3.1 (Approximate entropy functional). *Assume (6.7) and let*

$$c_1 := \frac{\eta}{2}\Big(1 - K\chi^4\eta^{-2}\mu^4\psi^2 N_{1,\max}^2 A_{\min}^{-4} A_{\max}^4 (A_{\max} - A_{\min})^2\Big) > 0. \tag{6.37}$$

(i) Then the function

$$\phi(t) := \sigma \int_\Omega (N\log N - N + 1)dx + \frac{1}{2}\|\nabla A(t)\|_2^2, \qquad \text{with } \sigma = \frac{2\psi^2}{\eta} A_{\max}^4 \mu^2 N_{1,\max},$$

satisfies the differential inequality

$$\phi' + \tilde{\omega}\phi + c_1\|\Delta A\|_2^2 \le c_2, \quad 0 < t < T^*, \tag{6.38}$$

where $\tilde{\omega} = \min(\omega, 2)$ *and* $c_2 = c_2(K, \chi, A_{\min}, A_{\max}, \psi, N_{1,\max}, \eta, |\Omega|, \mu) > 0$.

(ii) We have

$$\sup_{t\in(0,T^*)} \|\nabla A(t)\|_2^2 \le c_3, \tag{6.39}$$

with $c_3 := 2\max(\phi(0), c_2\tilde{\omega}^{-1})$, *and*

$$\int_s^t \|\Delta A(\tau)\|_2^2\, d\tau \le c_4(1 + t - s), \quad 0 < s < t < T^*, \tag{6.40}$$

with $c_4 = c_1^{-1}\max(c_2, c_3/2)$.

Proof. Set

$$\phi_1(t) := \|\nabla A(t)\|_2^2, \qquad \phi_2(t) := \int_\Omega (N\log N - N + 1)dx \ge 0 \tag{6.41}$$

(due to $s\log s - s + 1 \ge 0$, $s > 0$). In this proof, C will denote a generic positive constant depending only on $K, \chi, A_{\min}, A_{\max}, \psi$, $N_{1,\max}, \eta, |\Omega|, \mu$ and on $\varepsilon_1, \varepsilon_2, \varepsilon_3$ below.

On the one hand, it follows from (6.24) that, for any $\varepsilon_1 > 0$,

$$\frac{1}{2}\phi_1' + \phi_1 + \eta\|\Delta A\|_2^2 \le \varepsilon_1\|\Delta A\|_2^2 + \frac{\psi^2}{4\varepsilon_1}\int_\Omega N^2A^2(1-A)^2dx. \tag{6.42}$$

Since

$$\|N\|_2^2 \le \mu^2 N_{1,\max}\int_\Omega \frac{|\nabla N|^2}{N}dx + C \tag{6.43}$$

due to Lemma 6.2.3, we deduce from (6.42) that

$$\frac{1}{2}\phi_1' + \phi_1 \le a_1\int_\Omega \frac{|\nabla N|^2}{N}dx + a_2\|\Delta A\|_2^2 + C \tag{6.44}$$

where

$$a_1 = \frac{\psi^2}{4\varepsilon_1}A_{\max}^4\mu^2 N_{1,\max} > 0, \qquad a_2 = \varepsilon_1 - \eta. \tag{6.45}$$

On the other hand, it follows from (6.25) that, for any $\varepsilon_2, \varepsilon_3 > 0$,

$$\begin{aligned}\phi_2' + \omega\phi_2 + \int_\Omega \frac{|\nabla N|^2}{N}dx &\le \int_\Omega \nabla N\nabla\vartheta(A)dx \le \varepsilon_2\int_\Omega \frac{|\nabla N|^2}{N}dx + \frac{1}{4\varepsilon_2}\int_\Omega N|\nabla\vartheta(A)|^2dx\\ &\le \varepsilon_2\int_\Omega \frac{|\nabla N|^2}{N}dx + \frac{1}{4\varepsilon_2}\Big(\varepsilon_3\|N\|_2^2 + \frac{1}{4\varepsilon_3}\int_\Omega |\nabla\vartheta(A)|^4dx\Big).\end{aligned}$$

Using

$$\int_\Omega |\nabla\vartheta(A)|^4 dx \le K\chi^4 A_{\min}^{-4}(A_{\max}-A_{\min})^2\|\Delta A\|_2^2$$

due to Lemma 6.2.4, and (6.43), we deduce that

$$\phi_2' + \omega\phi_2 \le a_3\int_\Omega \frac{|\nabla N|^2}{N}dx + a_4\|\Delta A\|_2^2 + C, \tag{6.46}$$

where

$$a_3 = \varepsilon_2 + \frac{\varepsilon_3\mu^2 N_{1,\max}}{4\varepsilon_2} - 1, \qquad a_4 = \frac{K\chi^4}{16\varepsilon_2\varepsilon_3}A_{\min}^{-4}(A_{\max}-A_{\min})^2 > 0. \tag{6.47}$$

Now setting $\tilde\omega = \min(\omega, 2)$ and combining (6.44) and (6.46), we see that $\phi = \frac{1}{2}\phi_1 + \sigma\phi_2$ satisfies

$$\phi' + \tilde\omega\phi \le (a_1+\sigma a_3)\int_\Omega \frac{|\nabla N|^2}{N}dx + \big(a_2+\sigma a_4\big)\|\Delta A\|_2^2 + C. \tag{6.48}$$

Assume $a_3 < 0$ and choose $\sigma = -a_1/a_3 > 0$. Then we have $a_2 + \sigma a_4 < 0$ provided $a_1 a_4 < a_2 a_3$, that is

$$K\chi^4 A_{\min}^{-4}A_{\max}^4(A_{\max}-A_{\min})^2\psi^2\mu^2 N_{1,\max} < 16\varepsilon_1(\eta-\varepsilon_1)\big(4(1-\varepsilon_2)\varepsilon_2 - \varepsilon_3\mu^2 N_{1,\max}\big)\varepsilon_3. \tag{6.49}$$

The best condition, maximizing the RHS in (6.49), is obtained by choosing $\varepsilon_1 = \eta/2$, $\varepsilon_2 = 1/2$ and then $\varepsilon_3 = (2\mu^2 N_{1,\max})^{-1}$, which in turn implies $a_3 = -1/4 < 0$. Inequality (6.49) is then equivalent to

$$K\chi^4 A_{\min}^{-4}A_{\max}^4(A_{\max}-A_{\min})^2\psi^2\mu^2 N_{1,\max} < 4\eta^2(1-\varepsilon_3\mu^2 N_{1,\max})\varepsilon_3 = \eta^2(\mu^2 N_{1,\max})^{-1},$$

which is true, due to (6.37). Then we have $a_2 = -\eta/2$,

$$a_1 = \frac{\psi^2}{2\eta}A_{\max}^4\mu^2 N_{1,\max}, \quad a_4 = \frac{K\chi^4}{4}\mu^2 N_{1,\max}\,A_{\min}^{-4}(A_{\max}-A_{\min})^2, \quad \sigma = \frac{2\psi^2}{\eta}A_{\max}^4\mu^2 N_{1,\max}.$$

We conclude from (6.48) that (6.38) holds with $c_1 = -a_2 - \sigma a_4$, which yields the value given in (6.37).

(ii) Multiplying (6.38) with $e^{\tilde\omega t}$ and integrating between 0 and t, we obtain

$$\phi(t) \le \max(\phi(0), c_2\tilde\omega^{-1}), \quad 0 < t < T^*. \tag{6.50}$$

This guarantees (6.39), in view of (6.41). Inequality (6.40) then follows after integrating (6.38) over (s,t), taking (6.50) into account. □

Building on estimate (6.40) from the previous lemma, we shall now derive uniform estimates for $\|N(t)\|_2$ and $\|A(t)\|_{H^m(\Omega)}$, which in turn will guarantee the global existence of the solution.

Lemma 6.3.2. *Assume that (6.40) is satisfied for some $c_4 > 0$. Then we have*

$$\sup_{t\in(0,T^*)} \|N(t)\|_2 < \infty \tag{6.51}$$

and, for each $m \in (0,2)$ and $\tau \in (0,T^*)$,

$$\sup_{t\in(\tau,T^*)} \|A(t)\|_{H^m(\Omega)} < \infty. \tag{6.52}$$

Proof. Let us first establish (6.51). It follows from (6.26) that

$$\begin{aligned}\frac{d}{dt}\|N(t)\|_2^2 + 2\omega\|N\|_2^2 + 2\|\nabla N(t)\|_2^2 &\leq 2\,\omega N_{1;\max} + 2\int_\Omega N\nabla N.\nabla\vartheta(A)dxdt \\ &\leq 2\,\omega N_{1;\max} + \int_\Omega \nabla N^2.\nabla\vartheta(A)dx \\ &\leq 2\,\omega N_{1;\max} - \int_\Omega N^2.\Delta\vartheta(A)dx \\ &\leq 2\,\omega N_{1;\max} + \|N\|_4^2\|\Delta\vartheta(A)\|_2.\end{aligned}$$

By the Sobolev embedding of $W^{1,1}(\Omega)$ into $L^2(\Omega)$ with constant C_S, we have, for any $\varepsilon_1 > 0$,

$$\begin{aligned}\|N\|_4^2 = \|N^2\|_2 &\leq C_S(\|\nabla N^2\|_1 + \|N^2\|_1) = C_S(2\|N\nabla N\|_1 + \|N\|_2^2) \\ &\leq C_S(2\|N\|_2\|\nabla N\|_2 + \|N\|_2^2) \leq \varepsilon_1\|\nabla N\|_2^2 + (C_S + C_S^2/\varepsilon_1)\|N\|_2^2.\end{aligned}$$

Choosing $\varepsilon_1 = \|\Delta\vartheta(A)\|_2^{-1}$, we get

$$\frac{d}{dt}\|N(t)\|_2^2 + 2\omega\|N\|_2^2 + \|\nabla N(t)\|_2^2 \leq 2\,\omega N_{1;\max} + \big(C_S\|\Delta\vartheta(A)\|_2 + C_S^2\|\Delta\vartheta(A)\|_2^2\big)\|N\|_2^2. \quad (6.53)$$

Pick now $\varepsilon > 0$, and denote by $C(\varepsilon)$ a generic positive constant depending on the solution and on ε, but independent of $t \in (0, T^*)$. By the Gagliardo-Nirenberg inequality

$$\|N\|_2 \leq C_G(\|\nabla N\|_2^{1/2}\|N\|_1^{1/2} + \|N\|_1),$$

we have

$$\|N\|_2^2 \leq 2C_G^2(\|\nabla N\|_2\|N\|_1 + \|N\|_1^2) \leq \varepsilon\|\nabla N\|_2^2 + C(\varepsilon)N_{1;\max}^2.$$

Therefore, using (6.22), we have

$$\|\nabla N\|_2^2 \geq \frac{1}{\varepsilon}\|N\|_2^2 - C(\varepsilon).$$

Combining this with (6.53), we obtain

$$\frac{d}{dt}\|N(t)\|_2^2 + \frac{1}{\varepsilon}\|N\|_2^2 \leq C(\varepsilon) + \big(1 + 2C_S^2\|\Delta\vartheta(A)\|_2^2\big)\|N\|_2^2.$$

Now setting

$$\varphi(t) = \|N(t)\|_2^2, \qquad \rho(t) = \int_0^t \Big(\frac{1}{\varepsilon} - 1 - 2C_S^2\|\Delta\vartheta(A)\|_2^2\Big)ds,$$

we are thus left with the differential inequality

$$\varphi'(t) + \rho'(t)\varphi(t) \leq C(\varepsilon).$$

On the other hand, we have $\Delta\vartheta(A) = \chi(A^{-1}\Delta A - A^{-2}|\nabla A|^2)$, hence

$$\|\Delta\vartheta(A)\|_2^2 \leq 2\chi^2\big(A^{-2}\|\Delta A\|_2^2 + A^{-4}\|\nabla A\|_4^4\big).$$

It then follows from (6.40), (6.18) and Lemma 6.2.4 that

$$\int_s^t \|\Delta\vartheta(A)\|_2^2 \leq c_5(1 + t - s), \quad 0 < s < t < T^*.$$

By choosing $\varepsilon = (2 + 2C_S^2c_5)^{-1}$ and letting $c_6 = 2C_S^2c_5$, we have

$$\rho(t) - \rho(s) \geq t - s - c_6, \quad 0 < s < t < T^*.$$

By integration, we get

$$\varphi(t) \leq \varphi(0)e^{-\rho(t)} + C\int_0^t e^{\rho(s)-\rho(t)}ds \leq \varphi(0)e^{c_6-t} + C\int_0^t e^{c_6+s-t}ds \leq e^{c_6}(\varphi(0)+C)$$

and (6.51) follows.

To prove (6.52), we rewrite the first equation via the variation-of-constants formula

$$A(t) = T_{\eta,1}(t)A_0 + \int_0^t T_{\eta,1}(t-s)(\psi NA(1-A)+\tilde{A})(s)ds$$

(where $T_{\eta,1}$ is defined at the end of Section 2). By (6.51), we have

$$M := \sup_{t\in(0,T^*)} \|\big(\psi NA(1-A)+\tilde{A}\big)(t)\|_2 < \infty.$$

Fix $m \in (0,2)$. It follows from (6.36) that, for any $0 < \tau \leq t < T^*$,

$$\begin{aligned}\|A(t)\|_{H^m(\Omega)} &\leq Ce^{-t}(1+t^{-m/2})\|A_0\|_2 \\ &\quad + C\int_0^t e^{-(t-s)}(1+(t-s)^{-m/2})\|\big(\psi NA(1-A)+\tilde{A}\big)(s)\|_2 ds \\ &\leq C(1+\tau^{-m/2})\|A_0\|_2 + CM\int_0^\infty e^{-s}(1+s^{-m/2})ds =: C(\tau),\end{aligned}$$

which proves (6.52). □

Proof of Theorem 6.1.1.

Step 1. *Global existence.* In view of the local theory stated in Section 2, this is a direct consequence of estimates (6.51) and (6.52) in Lemma 6.3.2.

Step 2. *Boundedness.* Estimate (6.8) was already obtained in Lemma 6.3.2. Starting from the global estimates obtained in Lemma 6.3.2, we shall use a standard bootstrap argument (see e.g., [17]) to prove (6.9).

As a consequence of (6.52) and Sobolev imbeddings, we have, for each $\tau > 0$,

$$\sup_{t\in(\tau,\infty)} \|\nabla A(t)\|_p < \infty, \quad 1 \leq p < \infty. \tag{6.54}$$

Using (6.18) and (6.51), it follows that

$$M_q := \sup_{t\in(\tau,\infty)} \|(NA^{-1}\nabla A)(t)\|_q < \infty, \quad 1 \leq q < 2. \tag{6.55}$$

Set $T(t) = T_{1,\omega}(t)$. From the second equation, using $T(t)1 = e^{-\omega t}$, we have

$$\begin{aligned}N(t) &= T(t)N_0 + \int_0^t T(t-s)\big(-\nabla(N\nabla\vartheta(A))\big)ds + \int_0^t T(t-s)\omega ds \\ &= T(t)N_0 + 1 - e^{-\omega t} + \int_0^t \nabla T(t-s)\cdot NA^{-1}\nabla A ds.\end{aligned}$$

After a time-shift, combining this with (6.36), (6.51) and (6.55), we obtain, for any $q \in [1,2)$, $m \in [0,1)$ and $t \geq \tau/2$,

$$\begin{aligned}\|N(\tfrac{\tau}{2}+t)\|_{W^{m,q}(\Omega)} &\leq Ce^{-t}(1+t^{-m/2})\|N(\tfrac{\tau}{2})\|_q + C \\ &\quad + C\int_0^t e^{-(t-s)}\big(1+(t-s)^{-\frac{m+1}{2}}\big)\|\big(NA^{-1}\nabla A\big)(\tfrac{\tau}{2}+s)\|_q ds \\ &\leq C(1+\tau^{-m/2}) + CM_q\int_0^\infty e^{-s}\big(1+s^{-(m+1)/2}\big)ds =: \tilde{C}(\tau).\end{aligned} \tag{6.56}$$

Using Sobolev's imbedding again, taking q close to 2^- and m close to 1^-, we deduce

$$\sup_{t\in(\tau,\infty)} \|N(t)\|_p < \infty, \quad 1 \le p < \infty.$$

Due to (6.54), this implies that (6.55) is true for any $q \in [1,\infty)$. The argument in (6.56) then yields

$$\sup_{t\in(\tau,\infty)} \|N(t)\|_{W^{m,q}(\Omega)} < \infty, \quad 0 < m < 1,\ 1 \le q < \infty$$

and (6.9) follows by a further application of Sobolev imbeddings. □

6.4 The case of a square domain : Proof of Theorem 6.1.3

Let us consider the square $\Omega = (0,L)^2$. Theorem 6.1.3 is a consequence of Theorem 6.1.1 and of the following two lemmas, where we estimate the constants K and μ in inequalities (6.30) and (6.28)

Lemma 6.4.1. *For $\Omega = (0,L)^2$, the constant K in estimate (6.30) satisfies $K \le 12$.*

Proof. Recall that the constant $K = K(\Omega)$ is determined from inequality (6.35) in the proof of Lemma 6.2.4. Repeatedly using the Cauchy-Schwarz inequality, we first compute

$$\begin{aligned}\big|2D^2u + (\Delta u)I\big|_1 &= \max\big\{|3u_{xx} + u_{yy}| + 2|u_{xy}|, |u_{xx} + 3u_{yy}| + 2|u_{xy}|\big\}\\ &= 2|u_{xy}| + \max\big\{|3u_{xx} + u_{yy}|, |u_{xx} + 3u_{yy}|\big\}\\ &\le 2|u_{xy}| + \big(10(u_{xx}^2 + u_{yy}^2)\big)^{1/2}\end{aligned}$$

and then

$$\big|2D^2u + (\Delta u)I\big|_1^2 \le 12\big(u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2\big).$$

Consequently,

$$\int_\Omega \big|2D^2u + (\Delta u)I\big|_1^2 dx \le 12\big(\|u_{xx}\|_2^2 + \|u_{yy}\|_2^2 + 2\|u_{xy}\|_2^2\big). \tag{6.57}$$

We next claim that

$$\|u_{xx}\|_2^2 + \|u_{yy}\|_2^2 + 2\|u_{xy}\|_2^2 = \|\Delta u\|_2^2. \tag{6.58}$$

Indeed, using the Fourier expansion $u = \sum_{j,k\ge 0} a_{jk} \cos\frac{j\pi x}{L} \cos\frac{k\pi y}{L}$, we have

$$\begin{aligned}\|u_{xx}\|_2^2 + \|u_{yy}\|_2^2 &= \frac{\pi^4}{2L^2}\sum_{j\ge1} j^4 a_{j0}^2 + \frac{\pi^4}{2L^2}\sum_{k\ge1} k^4 a_{0k}^2 + \frac{\pi^4}{4L^2}\sum_{j,k\ge1}(j^4+k^4)a_{jk}^2,\\ \|u_{xy}\|_2^2 &= \frac{\pi^4}{4L^2}\sum_{j,k\ge1} j^2k^2 a_{jk}^2,\\ \|u_{xx} + u_{yy}\|_2^2 &= \frac{\pi^4}{2L^2}\sum_{j\ge1} j^4 a_{j0}^2 + \frac{\pi^4}{2L^2}\sum_{k\ge1} k^4 a_{0k}^2 + \frac{\pi^4}{4L^2}\sum_{j,k\ge1}(j^2+k^2)^2 a_{jk}^2,\end{aligned}$$

which implies (6.58).

The conclusion follows by combining (6.35), (6.57) and (6.58). □

Lemma 6.4.2. *For $\Omega = (0,L)^2$, the constant μ in estimate (6.28) satisfies $\mu \le \sqrt{\frac{3}{2}}$.*

Proof. Let u be a smooth function with zero average. There exist mean values $\zeta(x), \xi(y)$ in $[0, L]$ such that

$$\frac{1}{L}\int_0^L u(s,y)ds = u(\xi(y),y), \quad \frac{1}{L}\int_0^L u(x,t)dt = u(x,\zeta(x)).$$

It follows that

$$\begin{aligned} u(x,y) &= \frac{1}{L}\int_0^L u(s,y)ds + \int_{\xi(y)}^x u_x(s,y)ds \\ &= \frac{1}{L}\int_0^L u(x,t)dt + \int_{\zeta(x)}^y u_y(x,t)dt. \end{aligned}$$

Consequently,

$$\begin{aligned} \int_\Omega u^2(x,y)dxdy =& \frac{1}{L^2}\Big(\int_\Omega u(x,y)dxdy\Big)^2 \\ &+ \int_\Omega \Big(\frac{1}{L}\int_0^L u(s,y)ds \int_{\zeta(x)}^y u_y(x,t)dt\Big)dxdy \\ &+ \int_\Omega \Big(\frac{1}{L}\int_0^L u(x,t)dt \int_{\xi(y)}^x u_x(s,y)ds\Big)dxdy \\ &+ \int_\Omega \Big(\int_{\xi(y)}^x u_x(s,y)ds \int_{\zeta(x)}^y u_y(x,t)dt\Big)dxdy. \end{aligned}$$

Since u has zero average, it follows that

$$\|u\|_2^2 \le \frac{1}{L}\|u\|_1(\|u_x\|_1 + \|u_y\|_1) + \|u_x\|_1\|u_y\|_1.$$

Moreover, by [1, Theorem 3.2], we have $\|u\|_1 \le (L/\sqrt{2})\|\nabla u\|_1$, hence

$$\|u\|_2^2 \le \frac{1}{\sqrt{2}}\|\nabla u\|_1(\|u_x\|_1 + \|u_y\|_1) + \frac{1}{4}(\|u_x\|_1 + \|u_y\|_1)^2.$$

Using the elementary inequality $\|u_x\|_1 + \|u_y\|_1 \le \sqrt{2}\|\nabla u\|_1$, we deduce that

$$\|u\|_2 \le \sqrt{\frac{3}{2}}\|\nabla u\|_1.$$

Since $C^1(\bar\Omega)$ is dense in $W^{1,1}(\Omega)$, Lemma 6.4.2 follows by letting $u = N - \frac{1}{|\Omega|}\int_\Omega N(x,y)dxdy$. □

Remarks 6.4.1. *Let us briefly justify the generalization in Remark 1.2. Conditions* $(\mathcal{H}_1), (\mathcal{H}_2), (\mathcal{H}_5)$ *are natural. By condition* $(\mathcal{H}_3)$ *and* $N^{1-\delta} \le \varepsilon N + C(\varepsilon)$*, one can show that* $\|N(t)\|_1$ *is uniformly bounded. And then, one has following estimates*

$$|g(A,N)\log N| \le C_1(A)N + C_2(A) \qquad (a)$$

$$|g(A,N)|N \le \varepsilon N^2 + C_3(\varepsilon, A) \qquad (b)$$

Inequality (a) *and boundedness of* $\|N(t)\|_1$ *imply (6.46). Inequality* (b) *implies (6.53). By condition* $(\mathcal{H}_4)$*, one has*

$$|f(A,N)|^2 \le C_4(A)N^2 + C_5(A)$$

which implies inequality (6.44)

6.5 The case $\chi \leq 1$: Proof of Theorem 6.1.2

For $\chi \leq 1$, a basic entropy estimate is given by the following.

Lemma 6.5.1. *Assume* $0 < \chi \leq 1$ *and* $A_{\max} = 1$. *Then*

$$\sup_{0<t<T^*} \int_\Omega N|\log N|\,dx < \infty. \tag{6.59}$$

Proof. Following [5, Section 4], we define

$$Y(t) = \int_\Omega N(\log N - c\log A)\,dx.$$

Letting $F = \psi NA(1-A) + \tilde{A} - A$ and $G = \omega(1-N)$, we have

$$\begin{aligned} Y'(t) &= \int_\Omega N_t(\log N - c\log A)\,dx + \int_\Omega (N_t - cNA^{-1}A_t)\,dx \\ &= Q + \int_\Omega (\log N - c\log A)G\,dx + \int_\Omega G\,dx - c\int_\Omega NA^{-1}F\,dx, \end{aligned} \tag{6.60}$$

where

$$\begin{aligned} Q &= \int_\Omega \Big[\Delta N - \nabla\cdot(N\nabla\vartheta(A))\Big](\log N - c\log A)\,dx - c\eta\int_\Omega NA^{-1}\Delta A\,dx \\ &= -\int_\Omega N^{-1}|\nabla N|^2\,dx + (\chi + c(1+\eta))\int_\Omega A^{-1}\nabla N\cdot\nabla A - c(\chi+\eta)\int_\Omega NA^{-2}|\nabla A|^2\,dx \\ &= -\int_\Omega N^{-1}A^{-2}\Big[A^2|\nabla N|^2 + c(\chi+\eta)N^2|\nabla A|^2 - (\chi + c(1+\eta))NA\nabla N\cdot\nabla A\Big]\,dx. \end{aligned}$$

We see that $Q \leq 0$ provided the discriminant $\delta := (\chi + c(1+\eta))^2 - 4c(\chi+\eta)$ is nonpositive, that is

$$(1+\eta)^2c^2 - 2c(\chi + 2\eta - \chi\eta) + \chi^2 \leq 0. \tag{6.61}$$

There exists a constant c satisfying (6.61) if and only if

$$|\chi + 2\eta - \chi\eta| \geq (1+\eta)\chi,$$

which is equivalent to $\chi \leq 1$.

On the other hand, we have $G \leq \omega$ and, since $A \leq 1$ due to (6.19), $F \geq -A$. Using (6.18), (6.22) and the inequality $\log s \leq s$ $(s > 0)$, we deduce from (6.60) that

$$Y'(t) \leq -\omega Y + \omega\int_\Omega (\log N - c\log A)\,dx + \omega|\Omega| + c\int_\Omega N\,dx \leq -\omega Y + C.$$

We infer that $\sup_{0<t<T^*} Y(t) < \infty$, so that (6.59) follows from (6.19), (6.22). □

With the bound (6.5.1) at hand, we can now use the following better Sobolev type inequality in place of Lemma 6.2.3.

Lemma 6.5.2. *For any* $\varepsilon > 0$, *there holds*

$$\|u\|_2^2 \leq \varepsilon\int_\Omega u|\log u|\,dx \int_\Omega \frac{|\nabla u|^2}{u}dx + C(\Omega, \|u\|_1, \varepsilon) \tag{6.62}$$

for any uniformly positive function $u \in H^1(\Omega)$.

Proof. Fix $k > 1$. Since $\nabla[u-k]_+ = \chi_{\{u>k\}}\nabla u$, we deduce from (6.29) that

$$\int_\Omega [u-k]_+^2 dx \le \mu^2 \left(\int_{u>k} |\nabla u| dx\right)^2 + |\Omega|^{-1}\left(\int_\Omega [u-k]_+ dx\right)^2.$$

On the other hand, we have $u^2 \le [u-k]_+^2 + 2ku$ (just consider separately the cases $u \le$ or $> k$). Therefore, by integration, we obtain

$$\begin{aligned}\int_\Omega u^2 dx &\le \mu^2 \left(\int_{u>k} |\nabla u| dx\right)^2 + |\Omega|^{-1}\|u\|_1^2 + 2k\|u\|_1 \\ &\le \mu^2 \int_{u>k} \frac{|\nabla u|^2}{u} dx \cdot \int_{u>k} u dx + |\Omega|^{-1}\|u\|_1^2 + 2k\|u\|_1 \\ &\le \mu^2 \int_\Omega \frac{|\nabla u|^2}{u} dx \cdot \frac{\int_\Omega u|\log u| dx}{\log k} + |\Omega|^{-1}\|u\|_1^2 + 2k\|u\|_1\end{aligned} \tag{6.63}$$

and the Lemma follows by taking $k = \exp(\mu^2/\varepsilon)$. □

Proof of Theorem 6.1.2. We claim that

$$\sup_{t\in(0,T^*)} \|\nabla A(t)\|_2^2 < \infty \tag{6.64}$$

and

$$\int_s^t \|\Delta A(\tau)\|_2^2 d\tau \le C(1+t-s), \quad 0 < s < t < T^*. \tag{6.65}$$

Fix $\varepsilon > 0$. By Lemmas 6.5.1 and 6.5.2, we have

$$\|N\|_2^2 \le \varepsilon \int_\Omega \frac{|\nabla N|^2}{N} dx + C(\varepsilon), \quad 0 < t < T^*. \tag{6.66}$$

Using estimate (6.66) instead of (6.27) in the proof of Lemma 6.3.1(i), we obtain the differential inequality (6.48) where the constants a_i are defined by formulae (6.45) and (6.47) with $\mu^2 N_{1,\max}$ replaced by ε. Like before, we assume $a_2, a_3 < 0$ and choose $\sigma = -a_1/a_3 > 0$. Then we have $a_2 + \sigma a_4 < 0$ provided $a_1 a_4 < a_2 a_3$, which is now equivalent to

$$K\chi^4 A_{\min}^{-4} A_{\max}^4 (A_{\max} - A_{\min})^2 \psi^2 \varepsilon < 16\varepsilon_1(\eta - \varepsilon_1)\big(4(1-\varepsilon_2)\varepsilon_2 - \varepsilon_3\varepsilon\big)\varepsilon_3.$$

Choosing $\varepsilon_1 = \eta/2$, $\varepsilon_2 = 1/2$, $\varepsilon_3 = 1/(2\varepsilon)$, the condition becomes $K\chi^4 A_{\min}^{-4} A_{\max}^4 (A_{\max} - A_{\min})^2 \psi^2 \varepsilon < \eta^2 \varepsilon^{-1}$ which is verified for ε suitably small. Estimates (6.64)-(6.65) then follow as in the proof of Lemma 6.3.1(ii).

The rest of the proof of the theorem then relies on Lemma 6.3.2 similarly as in the proof of Theorem 6.1.1. □

6.6 Conclusion

In this article, we have considered a nonlinear, strongly coupled, parabolic system arising in the modelling of burglary in residential areas. The system involves two spatio-temporal unknowns : the attractiveness value of the property and the criminal density. The system is of chemotaxis-type and involves a logarithmic sensitivity function and specific interaction and relaxation terms.

This model appeared in [19], as a modification of the model of Short et al. [24]. In this last paper an important hypothesis is that burglars are generated at a constant rate and leave the lattice immediately after they have committed a burglary. This is modify in [19] by adding a new condition referred as burglar fatigue that models the effect that if burglars are sufficiently deterred, they will eventually get tired and will stop looking for houses. Some additional changes to the original equations come in by considering that if too many burglaries occur at some location, then burglars are likely to assume that most of the high-value is gone or that the owners of the place have implemented more strict security measures.

In [19] some results about linearized stability/instability of the homogeneous steady-states and some numerical simulations suggesting the existence of hotspots were given. However, the (local and) global existence of solutions was left open.

In this article, under suitable assumptions on the data of the problem, we have given a rigorous proof of the existence of a global and bounded, classical solution, thereby solving the problem left open in [19]. In the range of anti-diffusion parameter χ relevant for the criminological model, our sufficient condition for global existence roughly says that, at the initial time, the product of the oscillation of the attractiveness value and of the total criminal population should not be two large. Our proofs are based on the construction of approximate entropies and on the use of various functional inequalities.

We have also provided explicit numerical conditions for global existence when the domain is a square, including concrete cases involving values of the parameters which are expected to be physically relevant. In such cases, in the range of diffusion parameter η used by [24], our global existence conditions are compatible with magnitudes of attractivity which are up to about twice those of their static component. It is suggested in [19] that a ratio of order 10 (instead of 2) might be desirable, even for smaller values of η that the ones we consider here. However, this is still beyond the range in which we can rigorously prove global existence (and actually, global existence for such systems need not be taken for granted, as shown by the existing chemotaxis literature).

Bibliographie

[1] G. Acosta, R.G. Durán. An optimal Poincaré inequality in L^1 for convex domains. *Proc. Amer. Math. Soc.* 132 (2004), 195-202.

[2] M. Aida, K. Osaki, T. Tsujikawa, A. Yagi, M. Mimura. Chemotaxis and growth system with singular sensitivity function. *Nonlinear Anal, Real World Appl.* 6 (2005), 323-336.

[3] H. Berestycki, J.-P. Nadal. Self-organised critical hot spots of criminal activity. *European J. Appl. Math.*, 21 (2010), 371-399.

[4] P. Biler. Local and global solvability of some parabolic systems modeling chemotaxis. *Adv. Math. Sci. Appl.* 8 (1998), 715-743.

[5] P. Biler. Global solutions to some parabolic-elliptic systems of chemotaxis. *Adv. Math. Sci. Appl.* 9 (1999), 347-359.

[6] A. Blanchet. On the parabolic-elliptic Patlak-Keller-Segel system in dimension 2 and higher. Preprint ArXiV 1109.1543.

[7] H. Brezis. *Analyse fonctionnelle.* Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[8] S. Cantrell, C. Cosner, R. Manasevich. Global bifurcation of solutions for crime modeling equations. *SIAM J. Math. Anal.* 44 (2012) 1340-1358.

[9] M. DiFrancesco, J. Rosado. Fully parabolic Keller-Segel model for chemotaxis with prevention of overcrowding. *Nonlinearity* 21 (2008), 2715-2730.

[10] H. Gajewski, K. Zacharias. Global behaviour of a reaction-diffusion system modelling chemotaxis. *Math. Nachr.* 195 (1998), 77-114.

[11] M.A. Herrero, J.J.L. Velázquez. A blow-up mechanism for a chemotaxis model. *Ann. Scuola Norm. Super. Pisa Cl. Sci.* 24 (1997), 633-683.

[12] T. Hillen, K.J. Painter. A user's guide to PDE models for chemotaxis. *J. Math. Biol.* 58 (2009), 83-217.

[13] D. Horstmann. From 1970 until present : the Keller-Segel model in chemotaxis and its consequences. *Jahresberichte der DMV* 105 (2003), 103-165.

[14] D. Horstmann, M. Winkler. Boundedness vs. blow-up in a chemotaxis system, *J. Differential Equations* 215 (2005) 52-107.

[15] E.F. Keller, L.A. Segel. Initiation of slime mold aggregation viewed as an instability. *J. Theoret. Biol.* 26 (1970) 399-415.

[16] T. Nagai, T. Senba, Behavior of radially symmetric solutions of a system related to chemotaxis. *Nonlinear Anal.* 30 (1997) 3837-3842.

[17] T. Nagai, T. Senba, K. Yoshida. Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis. *Funkcial. Ekvac.*, 40 (1997), 411-433.

[18] K. Osaki, T. Tsujikawa, A. Yagi, M. Mimura. Exponential attractor for a chemotaxis-growth system of equations. *Nonlinear Anal. TMA* 51 (2002), 119-144.

[19] A.B. Pitcher. Adding police to a mathematical model of burglary. *European J. Appl. Math.*, 21 (2010), 401-419.

[20] N. Rodriguez A.L,. Bertozzi. Local existence and uniqueness of solutions to a PDE model for criminal behavior. *Math. Models Methods Appl. Sci.*, 20 (2010), 1425-1457.

[21] N. Rodriguez. On the Global Well-Posedness Theory for a class of PDE Models for Criminal Activity. Physica D (to appear).

[22] S.-U. Ryu, A. Yagi. Optimal control of Keller-Segel equations. *J. Math. Anal. Appl.*, 256 (2001), 45-66.

[23] M.B. Short, A.L. Bertozzi, P.J. Brantingham. Nonlinear patterns in urban crime : hotspots, bifurcations, and suppression. *SIAM J. Appl. Dyn. Syst.* 9 (2010), 462-483.

[24] M.B. Short, M.R. D'Orsogna, V.B. Pasour, G.E. Tita, P.J. Brantingham, A.L. Bertozzi, L.B. Chayes. A statistical model of criminal behavior. *Math. Models Methods Appl. Sci.* 18 (2008), 1249-1267.

[25] C. Stinner, M. Winkler. Global weak solutions in a chemotaxis system with large singular sensitivity. *Nonlinear Anal., Real World Appl.* 12 (2011), 3727-3740.