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Combinatorial Hopf algebras based on the selection/quotient rule

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Chapter 1

Introduction

Major advances in combinatorics during the last decades rely upon the study of algebraic structures associated to various combinatorial objects. Hopf algebra based on partitions, graphs, permutations and tableaux can be listed here. These objects are endowed with a product and a coproduct that encode certain combinatorial properties. Studying these algebraic structures, we obtain new insights on the combinatorics and, conversely, combinatorial properties allow us to better understand the algebra behind it. Hopf algebras based on trees are known nowadays.

A first type of combinatorial Hopf algebra (CHA) structure is constructed using the selection/quotient principle. This simply means that the comultiplication is of the form

$$\Delta(S) = \sum_{\substack{A\subseteq S \\ +Conditions}} S[A] \otimes S/A, \tag{1.1}$$

where $S[A]$ is a substructure of S and $S/_A$ is the corresponding quotient. We call these structures **CHAs of type I**.

Examples of such Hopf algebras are the Connes-Kreimer Hopf algebra of Feynman graphs, underlying the combinatorics of perturbative renormalization in quantum field theory [CK00] or in non-commutative Moyal quantum field theory [TK13], [TVT08] (the interested reader may also refer to [Tan10b], [Tan12] for some short reviews on these algebras). For the sake of completeness, let us also mention that similar Hopf algebraic structures have been proposed [Mar03], [Tan10a] for quantum gravity spin-foam models.

A second type of combinatorial Hopf algebra structure relies on the selection/complement principle. This means that the comultiplication is of the form:

$$\Delta(S) = \sum_{\substack{A\subseteq S \\ +Conditions}} S[A] \otimes [S - A]. \tag{1.2}$$

Examples of such Hopf algebras are the (commutative and noncommutative) polynomial Hopf algebras and the Loday-Ronco Hopf algebra of planar binary trees [LR98] or the Hopf algebra of matrix quasi-symmetric functions **MQSym**, the Hopf algebra of free quasi-symmetric function [MR95], [GKL+95], [DKKT97] and [DHT02]. We call these structures **CHAs of type II**.

In this thesis, we first define a non-commutative, non-cocommutative CHA of type I on packed words (see Chapter 3). Moreover, we implement a non-commutative, non-cocommutative CHA of type I on certain type of labelled graphs (that we call totally assigned graphs) - see Chapter 5.

On the other hand, matroid theory was first formalized in 1935 by Whitney [Whi35] who introduced the notion as an attempt to study the properties of vector spaces in an abstract manner. A matroid is an abstraction of the notion of linear independence in a vector space. Matroids arise naturally in combinatorial optimization and can be used as a framework for approaching a considerable variety of combinatorial problems. An important and active research direction in matroid theory involves the numerous links between matroids and graphs. In [Tut79], Tutte said "If a theorem about graphs can be expressed in terms of edges and circuits only it probably exemplifies a more general theorem about matroid". The development of matroids as a tool to be applied to graphs was mostly fostered in the area of combinatorics. Another very active and rich part of matroid theory centers on the Tutte polynomial, its properties and evaluations throughout combinatorics.

The Tutte polynomial [Tut54] is a well known invariant of graphs and matroids, important in combinatorics, knot theory and combinatorial physics. Two properties of the Tutte polynomial of graphs are of particular interest: the contraction-deletion rule and the duality. Thus, the Tutte polynomial plays an important role in the field of statistical physics where it appears as the partition function of the $q-$state Potts models $Z_G(q,v)$ (see [Sok05]). In fact, if G is a graph on n-vertices then

$$T = (x-1)(y-1)Z_G$$

and so the partition function of the $q-$state Potts model is simply the Tutte polynomial expressed in different variables.
These contraction and deletion operations are natural reductions for many network models arising from a wide range of problems at the heart of computer science, engineering, optimization, physics.
The Tutte polynomial can be evaluated at particular points (x, y) to give numerical graphical invariants, including the number of spanning trees, the number of forests, the number of connected spanning subgraphs and many more.
A Tutte-Grothendick invariant of matroids is a function F from (finite) matroids to a domain of scalars, which satisfies the multiplicative and invariances laws

$$F(M_1 \oplus M_2) = F(M_1)F(M_2), \tag{1.3}$$

$$F(M) = F(M/e) + F(M\backslash e), \tag{1.4}$$

if e is neither loop nor coloop of M (see Definition 2.2.17 and respective 2.2.18), and

$$F(M_1) = F(M_2), \tag{1.5}$$

if M_1 and M_2 are two isomorphic matroids (see Definition 2.2.10).
We refer the reader to [Bol98, Oxl92, Tut84, Wel93, Whi86, Whi92] for background on graphs, matroids and the Tutte polynomial.

Finally, let us mention here that the matroid Hopf algebra introduced in [Sch94] is a Hopf algebra that may be associated to any family of matroids that is closed under formation of minors and direct sums. This Hopf algebra has as basis the set of isomorphism classes of matroids belonging to the given family, with product induced by the direct sum operation, and coproduct of a matroid is of type (1.1). This Hopf algebra was also briefly considered in connection with the characteristic and Tutte polynomials of matroids in [KRS99] and [Kun04].

In Chapter 4 below, we investigate the characters of this matroid Hopf algebra and we give a new proof of the universality property of the Tutte polynomial.

Chapter 2

Background and general results

Contents

In this chapter we present some general notions of Hopf algebras and matroid theory that will be useful in the following chapters.

2.1 Hopf algebras

In this section, we introduce the axioms of algebra, coalgebra, bialgebra and Hopf algebra. The classical references on this subject are [Abe80, Swe69]. The notion of algebras over a commutative ring A will be recalled. An algebra over A is defined by giving an A-module morphism which is called its structure map. Coalgebras are naturally in a dual relationship with algebras.

Definition 2.1.1. *Let k be a commutative ring. A k-module $\mathcal{A}$ is called* **a k-associative algebra with unit (k-AAU)** *or, for short, a* **k-algebra** *if the following diagrams commute.*

$$\begin{array}{ccc} \mathcal{A}\otimes_k\mathcal{A}\otimes_k\mathcal{A} & \xrightarrow{Id\otimes m} & \mathcal{A}\otimes_k\mathcal{A} \\ {\scriptstyle m\otimes Id}\downarrow & & \downarrow{\scriptstyle m} \\ \mathcal{A}\otimes_k\mathcal{A} & \xrightarrow[m]{} & \mathcal{A} \end{array} .$$

$$\begin{array}{ccccc} k\otimes_k\mathcal{A} & \xrightarrow{\eta\otimes Id} & \mathcal{A}\otimes_k\mathcal{A} & \xleftarrow{Id\otimes\eta} & \mathcal{A}\otimes k \\ & {\scriptstyle Id}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle Id} & \\ & & \mathcal{A} & & \end{array} .$$

The commutation of these diagrams is equivalent to the following compositional equations. If we identify $\mathcal{A}\otimes_k k$ and $k\otimes_k\mathcal{A}$ with $\mathcal{A}$, if the following equations holds, then $\mathcal{A}$ is called an k-algebra.

$$m\circ(Id\otimes m)=m\circ(m\otimes Id),\quad m\circ(Id\otimes\eta)=m\circ(\eta\otimes Id)=Id_{\mathcal{A}}. \tag{2.1}$$

m is said to be the ***multiplication map*** *of $\mathcal{A}$, η the* ***unit map*** *of $\mathcal{A}$.*

Definition 2.1.2. *Let $\mathcal{A}$, $\mathcal{B}$ be k-algebras. The linear mapping $\varphi:\mathcal{A}\to\mathcal{B}$ is a ring morphism as well as a k-module morphism, then we call φ a k-algebra morphism.*

Proposition 2.1.3. *Let $(\mathcal{A},m_{\mathcal{A}},\eta_{\mathcal{A}})$, $(\mathcal{B},m_{\mathcal{B}},\eta_{\mathcal{B}})$ be k-algebras. The k-module morphism $\varphi:\mathcal{A}\to\mathcal{B}$ is a k-algebra morphism if and only if the following diagrams commute.*

$$\begin{array}{ccc} \mathcal{A}\otimes_k\mathcal{A} & \xrightarrow{\varphi\otimes\varphi} & \mathcal{B}\otimes_k\mathcal{B} \\ {\scriptstyle m_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle m_{\mathcal{B}}} \\ \mathcal{A} & \xrightarrow[\varphi]{} & \mathcal{B} \end{array} .$$

$$\begin{array}{ccc} k & \xrightarrow{\eta_{\mathcal{A}}} & \mathcal{A} \\ & {\scriptstyle \eta_{\mathcal{B}}}\searrow & \downarrow{\scriptstyle\varphi} \\ & & \mathcal{B} \end{array} .$$

In other words, it is necessary and sufficient that

$$m_{\mathcal{B}} \circ (\varphi \otimes \varphi) = \varphi \circ m_{\mathcal{A}}, \quad \varphi \circ \eta_{\mathcal{A}} = \eta_{\mathcal{B}}. \tag{2.2}$$

The map τ denotes the k-module isomorphism $\mathcal{A} \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ defined by $a \otimes b \mapsto b \otimes a$.

Definition 2.1.4. *The algebra $\mathcal{A}$ is commutative if and only if $m_{\mathcal{A}} \circ \tau = m_{\mathcal{A}}$.*

The map τ_{23} denotes the isomorphism $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D} \to \mathcal{A} \otimes \mathcal{C} \otimes \mathcal{B} \otimes \mathcal{D}$ defined by $a \otimes b \otimes c \otimes d \mapsto a \otimes c \otimes b \otimes d$.

Proposition 2.1.5. *Let $(\mathcal{A}, m_{\mathcal{A}}, \eta_{\mathcal{A}})$ and $(\mathcal{B}, m_{\mathcal{B}}, \eta_{\mathcal{B}})$ be two k-algebras. The k-module $\mathcal{A} \otimes_k \mathcal{B}$ is an algebra with the following multiplication map and unit map given by*

$$m_{\mathcal{A} \otimes \mathcal{B}} := (m_{\mathcal{A}} \otimes m_{\mathcal{B}}) \circ \tau_{23}, \tag{2.3}$$

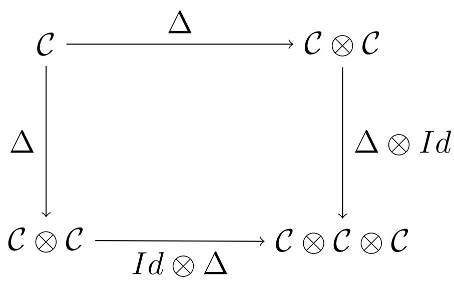
$$\eta_{\mathcal{A} \otimes \mathcal{B}} := \eta_{\mathcal{A}} \otimes \eta_{\mathcal{B}}. \tag{2.4}$$

Proof. One can directly check that $m_{\mathcal{A} \otimes \mathcal{B}}$ and $\eta_{\mathcal{A} \otimes \mathcal{B}}$ satisfy conditions (2.1). ☐

We define a k-coalgebra dually to a k-algebra [Bou06a].

Definition 2.1.6. *A coalgebra over a commutative ring k is a vector space $\mathcal{C}$ over k together with k-linear maps $\Delta : \mathcal{C} \to \mathcal{C} \otimes_k \mathcal{C}$ and $\epsilon : \mathcal{C} \to k$ such that*
1. $(Id_{\mathcal{C}} \otimes \Delta) \circ \Delta = (\Delta \otimes Id_{\mathcal{C}}) \circ \Delta$
2. $(Id_{\mathcal{C}} \otimes \epsilon) \circ \Delta = Id_{\mathcal{C}} = (\epsilon \otimes Id_{\mathcal{C}}) \circ \Delta$.
We call Δ the coproduct and ϵ the counit.

Equivalently, the following diagrams commute.



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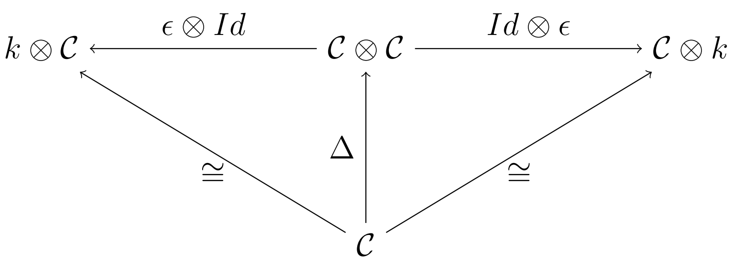

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In the first diagram we identify $\mathcal{C} \otimes (\mathcal{C} \otimes \mathcal{C})$ with $(\mathcal{C} \otimes \mathcal{C}) \otimes \mathcal{C}$, which are naturally isomorphic modules. Similarly, in the second diagram the naturally isomorphic spaces $\mathcal{C}$, $\mathcal{C} \otimes k$ and $k \otimes \mathcal{C}$ are identified.
The first diagram is the dual of the one expressing the associativity of algebra multiplication (called the coassociativity of the comultiplication); the second diagram is the dual of the one expressing the existence of a multiplicative identity.

Sweedler's notations: A coproduct is a sum of tensors. For simplifying various types of operations, we use the following notations. Given a k-coalgebra $(\mathcal{C}, \Delta, \epsilon)$ and $x \in \mathcal{C}$, we can write

$$\Delta(x) = \sum_{(1)(2)} x^{(1)} \otimes x^{(2)}. \tag{2.5}$$

For k-linear maps f, g from $\mathcal{C}$ to $\mathcal{C}$ or k, we write

$$(f \otimes g) \circ \Delta(x) = \sum_{(1)(2)} f(x^{(1)}) \otimes g(x^{(2)}). \tag{2.6}$$

The coassociative law can be rewritten as follows.

$$\begin{aligned} \sum_{(1)(2)} \left(\sum_{x^{(1)}} \left(x^{(1)}\right)^{(1)} \otimes \left(x^{(1)}\right)^{(2)} \right) \otimes x^{(2)} &= \sum_{(1)(2)} x^{(1)} \otimes \left(\sum_{x^{(2)}} \left(x^{(2)}\right)^{(1)} \otimes \left(x^{(2)}\right)^{(2)} \right) \\ &:= \sum_{(1)(2)(3)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)}. \end{aligned} \tag{2.7}$$

In general, we define

$$\Delta^{(1)} = \Delta, \quad \Delta^{(n)} = (\Delta \otimes Id^{\otimes(n-1)}) \circ (\Delta \otimes Id^{\otimes(n-2)}) \circ \cdots \circ \Delta, \ (n > 1). \tag{2.8}$$

For all $x \in \mathcal{C}$, we have

$$\Delta^{(n)}(x) = \sum_{(1)\ldots(n+1)} x^{(1)} \otimes \cdots \otimes x^{(n+1)}. \tag{2.9}$$

The counitary property may be expressed as following: for all $x \in \mathcal{C}$

$$\sum_{(1)(2)} \epsilon(x^{(1)})x^{(2)} = \sum_{(1)(2)} x^{(1)}\epsilon(x^{(2)}) = x. \tag{2.10}$$

Example 2.1.7. *Consider the polynomial ring $k[X]$. It has as unit the monomial 1. It is easy to see that $k[X]$ is a commutative algebra.*
This becomes a coalgebra if we define

$$\Delta(X^n) = \sum_{k=0}^{n} \binom{n}{k} X^k \otimes X^{n-k}$$

and

$$\epsilon(X^n) = \begin{cases} 1 & if \quad n = 0 \\ 0 & if \quad n > 0 \end{cases}$$

for all $n \geq 0$. Now $k[X]$ is both a unital associative algebra and a coalgebra, and the two structures are compatible.

Definition 2.1.8. *A coalgebra $(\mathcal{C}, \Delta, \epsilon)$ is called cocommutative if $\tau \circ \Delta = \Delta$.*

Example 2.1.9. *Let A be an alphabet. Let $k\langle A \rangle$ be the non-commutative polynomials in the variables A. The multiplication is the usual concatenation of words. The unit is the empty word. The shuffle coproduct is given by*

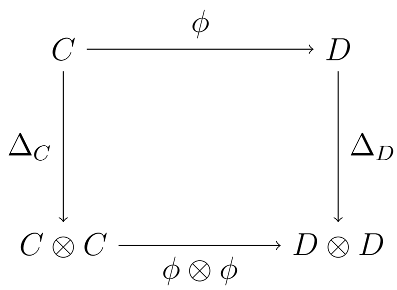
$$\Delta(w) = \sum_{I+J=[1\ldots|w|]} w[I] \otimes w[J], \tag{2.11}$$

and the counit is given by

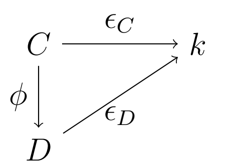
$$\epsilon(w) = \begin{cases} 1 & \text{if } w = 1_{A^*} \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

Definition 2.1.10. *Let C and D be two coalgebras and $\phi : C \to D$ be a linear map. The map ϕ is called a coalgebra morphism if*

$$\Delta_D \circ \phi = (\phi \otimes \phi) \circ \Delta_C;$$



and $\epsilon_D \circ \phi = \epsilon_C$



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Proposition 2.1.11. *Let $\mathcal{C}$, $\mathcal{D}$ be two coalgebras. Then $\mathcal{C} \otimes \mathcal{D}$ is a coalgebra with the coproduct given by*

$$\Delta_{\mathcal{C}\otimes\mathcal{D}} := \tau_{23} \circ (\Delta_{\mathcal{C}} \otimes \Delta_{\mathcal{D}}). \tag{2.13}$$

The counit is given by

$$\epsilon_{\mathcal{C}} \otimes \epsilon_{\mathcal{D}} : x \otimes y \longmapsto \epsilon_{\mathcal{C}}(x)\epsilon_{\mathcal{D}}(y), \tag{2.14}$$

for all $x \in \mathcal{C}$ and $y \in \mathcal{D}$.

Proof. One must check that for all $x \in \mathcal{C}$ and $y \in \mathcal{D}$,

$$(\Delta_{\mathcal{C}\otimes\mathcal{D}} \otimes Id_{\mathcal{C}\otimes\mathcal{D}}) \circ \Delta_{\mathcal{C}\otimes\mathcal{D}}(x \otimes y) = (Id_{\mathcal{C}\otimes\mathcal{D}} \otimes \Delta_{\mathcal{C}\otimes\mathcal{D}}) \circ \Delta_{\mathcal{C}\otimes\mathcal{D}}(x \otimes y). \tag{2.15}$$

Using Sweedler's notation, one has

$$\Delta_{\mathcal{C}}(x) = \sum_{(1)(2)} x^{(1)} \otimes x^{(2)}. \tag{2.16}$$

$$\Delta_{\mathcal{D}}(y) = \sum_{(3)(4)} y^{(3)} \otimes y^{(4)}. \tag{2.17}$$

Then, the left-hand-side (LHS) of Equation (2.15) can be rewritten as follows.

$$\begin{aligned}
(\Delta_{\mathcal{C}\otimes\mathcal{D}} \otimes Id_{\mathcal{C}\otimes\mathcal{D}}) \circ \Delta_{\mathcal{C}\otimes\mathcal{D}}(x \otimes y) &= (\Delta_{\mathcal{C}\otimes\mathcal{D}} \otimes Id_{\mathcal{C}\otimes\mathcal{D}}) \circ (Id_{\mathcal{C}} \otimes \tau_{23} \otimes Id_{\mathcal{D}}) \circ (\Delta_{\mathcal{C}} \otimes \Delta_{\mathcal{D}})(x \otimes y) \\
&= (\Delta_{\mathcal{C}\otimes\mathcal{D}} \otimes Id_{\mathcal{C}\otimes\mathcal{D}}) \circ (Id_{\mathcal{C}} \otimes \tau_{23} \otimes Id_{\mathcal{D}}) \circ (\Delta_{\mathcal{C}}(x) \otimes \Delta_{\mathcal{D}}(y)) \\
&= (\Delta_{\mathcal{C}\otimes\mathcal{D}} \otimes Id_{\mathcal{C}\otimes\mathcal{D}}) \circ (Id_{\mathcal{C}} \otimes \tau_{23} \otimes Id_{\mathcal{D}}) \circ \\
&\quad \left(\sum_{(1)(2)} x^{(1)} \otimes x^{(2)} \otimes \sum_{(3)(4)} y^{(3)} \otimes y^{(4)} \right) \\
&= (\Delta_{\mathcal{C}\otimes\mathcal{D}} \otimes Id_{\mathcal{C}\otimes\mathcal{D}}) \left(\sum_{(1)(2)(3)(4)} x^{(1)} \otimes y^{(3)} \otimes x^{(2)} \otimes y^{(4)} \right) \\
&= \sum_{(1)(2)(3)(4)} \Delta_{\mathcal{C}\otimes\mathcal{D}}(x^{(1)} \otimes y^{(3)}) \otimes x^{(2)} \otimes y^{(4)} \\
&= \sum_{(1)(2)(3)(4)} \left(\sum_{x^{(1)},y^{(3)}} \left(x^{(1)}\right)^{(1)} \otimes \left(y^{(3)}\right)^{(1)} \otimes \left(x^{(1)}\right)^{(2)} \otimes \left(y^{(3)}\right)^{(2)} \right) \\
&\quad \otimes x^{(2)} \otimes y^{(4)} \\
&= \sum_{(1)(2)(3)} x^{(1)} \otimes y^{(1)} \otimes x^{(2)} \otimes y^{(2)} \otimes x^{(3)} \otimes y^{(3)}.
\end{aligned} \tag{2.18}$$

On the other hand, the right-hand-side (RHS) of Equation (2.15) can be rewritten as follows.

$$(Id_{\mathcal{C}\otimes\mathcal{D}} \otimes \Delta_{\mathcal{C}\otimes\mathcal{D}}) \circ \Delta_{\mathcal{C}\otimes\mathcal{D}}(x \otimes y) = \sum_{(1)(2)(3)} x^{(1)} \otimes y^{(1)} \otimes x^{(2)} \otimes y^{(2)} \otimes x^{(3)} \otimes y^{(3)}. \tag{2.19}$$

From two Equations (2.18) and (2.19), one gets the coassociativity of $\Delta_{\mathcal{C}\otimes\mathcal{D}}$.
For $x \in \mathcal{C}$ and $y \in \mathcal{D}$, one has

$$\begin{aligned}
(\epsilon_{\mathcal{C}} \otimes \epsilon_{\mathcal{D}} \otimes Id_{\mathcal{C}\otimes\mathcal{D}}) \circ \Delta_{\mathcal{C}\otimes\mathcal{D}}(x \otimes y) &= (\epsilon_{\mathcal{C}} \otimes \epsilon_{\mathcal{D}} \otimes Id_{\mathcal{C}\otimes\mathcal{D}})(\sum_{(1)(2)(3)(4)} x^{(1)} \otimes y^{(3)} \otimes x^{(2)} \otimes y^{(4)}) \\
&= \sum_{(1)(2)(3)(4)} \epsilon_{\mathcal{C}}(x^{(1)}) \otimes \epsilon_{\mathcal{D}}(y^{(3)}) \otimes x^{(2)} \otimes y^{(4)} \\
&= \sum_{(1)(2)(3)(4)} \epsilon_{\mathcal{C}}(x^{(1)})x^{(2)} \otimes \epsilon_{\mathcal{D}}(y^{(3)})y^{(4)} \\
&= \left(\sum_{(1)(2)} \epsilon_{\mathcal{C}}(x^{(1)})x^{(2)} \right) \otimes \left(\sum_{(3)(4)} \epsilon_{\mathcal{D}}(y^{(3)})y^{(4)} \right) \\
&= x \otimes y.
\end{aligned} \tag{2.20}$$

Similarly, one has
$$(Id_{\mathcal{C}\otimes\mathcal{D}} \otimes \epsilon_{\mathcal{C}} \otimes \epsilon_{\mathcal{D}}) \circ \Delta_{\mathcal{C}\otimes\mathcal{D}}(x \otimes y) = x \otimes y. \tag{2.21}$$
Then, the unitary property holds. One can get the conclusion. □

A bialgebra is both an algebra and a coalgebra with some compatibility between these two structures.

Theorem 2.1.12. *Let H be a space with an algebra structure (H, m, η) and a coalgebra structure (H, Δ, ϵ). The following are equivalent:*

(i) *Δ and ϵ are algebra morphisms.*

(ii) *m and η are coalgebra morphisms.*

(iii) *For all $x, y \in H$:*
$$\Delta(xy) = \sum_x \sum_y x^{(1)}y^{(2)} \otimes x^{(2)}y^{(2)}, \quad \Delta(1) = 1 \otimes 1, \tag{2.22}$$
$$\epsilon(xy) = \epsilon(x)\epsilon(y), \quad \epsilon(1_H) = 1_k. \tag{2.23}$$

Proof. (i) ⇔ (ii). The conditions under which Δ is a k-algebra morphism are

(1) $\Delta \circ m = (m \otimes m) \circ \tau_{23} \circ (\Delta \otimes \Delta)$,

(2) $\Delta \circ \eta = \eta \otimes \eta$,

and the condition under which ϵ is a k-algebra morphism are

(3) $\epsilon \circ m = \epsilon \otimes \epsilon$,

(4) $\epsilon \circ \eta = 1_k$.

On the other hand, m is a k-coalgebra morphism if it satisfies conditions (1), (3); η is a k-coalgebra morphism if it satisfies conditions (2), (4). This allows to conclude that (i) ⇔ (ii).
(i) ⇔ (iii) is clear from the definition. □

Definition 2.1.13 ([Abe80]). *A bialgebra is a 4-tuple $(H, m, \eta, \Delta, \epsilon)$ which satisfies one of the equivalent conditions of Theorem 2.1.12.*

Definition 2.1.14. *Let H and H' be two bialgebras and $\phi : H \to H'$. We call ϕ a bialgebra morphism if ϕ is an algebra morphism and a coalgebra morphism.*

Definition 2.1.15. *(Convolution) Let $\mathcal{C} = (\mathcal{C}, \Delta, \epsilon)$ be a coalgebra and $\mathcal{A} = (\mathcal{A}, m, \eta)$ be an algebra. If $f, g \in Hom(\mathcal{C}, \mathcal{A})$,*
$$f * g := m \circ (f \otimes g) \circ \Delta,$$
i.e, for all $x \in \mathcal{C}$, one has
$$(f * g)(x) := \sum_x f(x^{(1)})g(x^{(2)}). \tag{2.24}$$
This product is called convolution.

Proposition 2.1.16. *The vector space $Hom(\mathcal{C},\mathcal{A})$ endowed with the convolution product is an algebra where the unit is given by*

$$e(x) := \eta \circ \epsilon(x) = \epsilon(x)1_{\mathcal{A}}. \tag{2.25}$$

Proof. Let f, g and h be in $Hom(\mathcal{C},\mathcal{A})$. For $x \in \mathcal{C}$, one has

$$\begin{aligned}(f*g)*h(x) &= \sum_x (f*g)(x^{(1)})h(x^{(2)})\\ &= \sum_x f(\left(x^{(1)}\right)^{(1)})gf(\left(x^{(1)}\right)^{(2)})h(x^{(2)}) = \sum_x f(x^{(1)})g(x^{(2)})h(x^{(3)}).\end{aligned} \tag{2.26}$$

Similarly, one has

$$f*(g*h) = \sum_x f(x^{(1)})g(x^{(2)})h(x^{(3)}). \tag{2.27}$$

Then, one has

$$(f*g)*h = f*(g*h). \tag{2.28}$$

In the other hand, for $f \in Hom(\mathcal{C},\mathcal{A})$, $x \in \mathcal{C}$, one has

$$\begin{aligned}e*f(x) &= \sum_x e(x^{(1)})f(x^{(2)}) = \sum_x \epsilon(x^{(1)})f(x^{(2)}) = \sum_x f(\epsilon(x^{(1)})x^{(2)})\\ &= f\left(\sum_x \epsilon(x^{(1)})x^{(2)}\right) = f(x).\end{aligned} \tag{2.29}$$

Similarly, $f*e = f$.
Thus, one gets the conclusion. □

Example 2.1.17. *Let H be a bialgebra. We can then take $\mathcal{A} = H$ and $\mathcal{C} = H$. Then $Hom(H,H)$ is equipped with a product $*$, the unit is $x \longrightarrow \epsilon(x)1_H$.*

Definition 2.1.18. *Let H be a bialgebra. H is called a Hopf algebra if Id_H has an inverse in the algebra $(Hom(H,H),*)$. The unique inverse of Id_H (if it exists) is called the antipode of H and is noted S.*

Example 2.1.19. *Let $(k[x],\times,1)$ be the polynomial algebra (one variable). We define the morphism $\Delta_\odot : k[x] \longleftrightarrow k[x]\otimes k[x]$ by $\Delta_\odot(x) = x\otimes x$and $\epsilon(x) = 1$. One can check that $(k[x],\times,1,\Delta_\odot,\epsilon)$ is a bialgebra which admits no antipode.*

Theorem 2.1.20. *The following properties hold for the antipode S of k-Hopf algebra H.*

(i) $S(gh) = S(h)S(g)$, *for all* $g,h \in H$.

(ii) $S(1_H) = 1_H$, *i.e.* $S\circ\eta = \eta$.

(iii) $\epsilon\circ S = \epsilon$.

(iv) $\tau\circ(S\otimes S)\circ\Delta = \Delta\circ S$.

(v) If H is commutative or cocommutative, then $S^2 = Id_H$.

Proof. (i) From Proposition 2.1.11, $H \otimes H, \Delta_{H\otimes H}, \epsilon_{H\otimes H})$ is a coalgebra. Then, $Hom(H \otimes H, H)$ is an algebra where the convolution product, denoted $\star$, is given as following. For f, $g \in Hom(H \otimes H, H)$ and x, $y \in H$, one has

$$f \star g(x \otimes y) = m \circ (f \otimes g) \circ \Delta_{H\otimes H}(x \otimes y) = \sum_{x,y} f(x^{(1)} \otimes y^{(1)}) g(x^{(2)} \otimes y^{(2)}). \tag{2.30}$$

The unit e is given by

$$e(x \otimes y) = \epsilon(x)\epsilon(y)1_H. \tag{2.31}$$

Note that m and $S\circ m$ are in $Hom(H\otimes H, H)$. Now if $(S\circ m)\star m = \eta\circ\epsilon = m\star(m\circ(S\otimes S)\circ\tau)$ holds, then it implies (i) $S \circ m = m \circ (S \otimes S) \circ \tau$.

$$\begin{aligned}(S \circ m) \star m(g \otimes h) &= \sum_{g,h} S(g^{(1)}h^{(1)})g^{(2)}h^{(2)} = \sum_{gh} S((gh)^{(1)})(gh)^{(2)} \\ &= (S * Id_H)(gh) = \epsilon(gh)1_H = \epsilon(g)\epsilon(h)1_H \\ &= e(g \otimes h).\end{aligned} \tag{2.32}$$

In the other hand, one has

$$\begin{aligned}m \star (m \circ (S \otimes S) \circ \tau)(g \otimes h) &= \sum_{g,h} g^{(1)}h^{(1)}S(h^{(2)})S(g^{(2)}) \\ &= \sum_g g^{(1)} \left(\sum_h h^{(1)}S(h^{(2)})\right) S(g^{(2)}) = \epsilon(h) \sum_g g^{(1)}S(g^{(2)}) \\ &= \epsilon(g)\epsilon(h)1_H = e(g \otimes h).\end{aligned} \tag{2.33}$$

(ii) From the fact that $\epsilon(1_H) = 1_k$ and $\Delta(1_H) = 1_H \otimes 1_H$,one has

$$S(1_H) = S(1_H)1_H = S * Id_H(1_H) = \eta \circ \epsilon(1_H) = 1_H. \tag{2.34}$$

(iii) One has

$$\begin{aligned}\epsilon(h) &= \epsilon(\epsilon(h)1_H) = \epsilon \circ \eta \circ \epsilon(h) \\ &= \epsilon\left(\sum_h h^{(1)}S(h^{(2)})\right) = \sum_h \epsilon(h^{(1)})\epsilon(S(h^{(2)})) \\ &= \epsilon(S(h)1_H) \\ &= \epsilon(S(x)).\end{aligned} \tag{2.35}$$

(iv) From Proposition 2.1.5, $H \otimes H$ is an algebra; Then, $(Hom(H, H \otimes H), m_{H\otimes H}, \eta_{H\otimes H})$ is an algebra where the convolution product, denote $\star_1$, is given as following. For f, $g \in Hom(H, H \otimes H)$ and $x \in H$, one has

$$f \star_1 g(x) = m_{H\otimes H} \circ (f \otimes g) \circ \Delta(x). \tag{2.36}$$

Using Proposition 2.1.12, one has $\Delta \circ m = (m \otimes m) \circ \tau_{23} \circ (\Delta \otimes \Delta)$. Thus, for $h \in H$,

$$\begin{aligned}(\Delta \circ S) \star_1 \Delta(h) &= m_{H \otimes H} \circ (\Delta \circ S \otimes \Delta) \circ \Delta(h) \\ &= (m \otimes m) \circ \tau_{23} \circ (\Delta \otimes \Delta) \circ (S \otimes Id) \circ \Delta(h) \\ &= \Delta \circ m \circ (S \otimes Id) \circ \Delta(h) \\ &= \Delta(S * Id)(h) \\ &= \Delta(\epsilon(h) 1_H) \\ &= \epsilon(h) 1_H \otimes 1_H \\ &= e_1(h). \end{aligned} \tag{2.37}$$

Thus, $\Delta \circ S \star_1 \Delta = e_1$.
In the other hand, for $h \in H$, one has

$$\begin{aligned}\Delta \star_1 (\tau \circ (S \otimes S) \circ \Delta)(h) &= \sum_h h^{(1)} S(h^{(4)}) \otimes h^{(2)} S(h^{(3)}) \\ &= \sum_x h^{(1)} S(h^{(3)}) \otimes \epsilon(h^{(2)}) 1_H \\ &= \sum_x h^{(1)} S(h^{(3)}) \epsilon(h^{(2)}) \otimes 1_H \\ &= \sum_x h^{(1)} S(h^{(2)}) \otimes 1_H \\ &= \epsilon(h) 1_H \otimes 1_H \\ &= e_1(h). \end{aligned} \tag{2.38}$$

Then, $\Delta \star_1 (\tau \circ (S \otimes S) \circ \Delta) = e_1$.
Using the associativity of $\star_1$, one has

$$\begin{aligned}\Delta \circ S &= (\Delta \circ S) \star_1 (\Delta \star_1 (\tau \circ (S \otimes S) \circ \Delta)) \\ &= ((\Delta \circ S) \star_1 \Delta) \star_1 (\tau \circ (S \otimes S) \circ \Delta) \\ &= \tau \circ (S \otimes S) \circ \Delta. \end{aligned} \tag{2.39}$$

(v) If H is commutative, then one has

$$\begin{aligned} S^2 * S(x) &= \sum_x S^2(x^{(1)}) S(x^{(2)}) = S\left(\sum_x x^{(2)} S(x^{(1)})\right) = S\left(\sum_x S(x^{(1)}) x^{(2)}\right) \\ &= S(\eta \circ \epsilon(x)) = \eta \circ \epsilon(x). \end{aligned} \tag{2.40}$$

If H is cocommutative, then one has

$$S^2 * S(x) = \sum_x S^2(x^{(2)}) S(x^{(1)}) = S\left(\sum_x x^{(1)} S(x^{(2)})\right) = S(\eta \circ \epsilon(x)) = \eta \circ \epsilon(x). \tag{2.41}$$

Thus, one has $S^2 * S = e$. It implies that S^2 is inverse of S, i.e $S^2 = Id_H$. □

Remark 2.1.21. *It exists a Hopf algebra where the antipode map S is not bijective ([Tak71]). Let $M_m(k)$ be the $m \times m$ matrix algebra over k, then the duality $C = M_m(k)^*$ is a coalgebra. Let $H(M_m(k)^*)$ be a Hopf algebra generated by the coalgebra C. Following Theorem 11 in [Tak71], with $m > 1$, the antipode of $H(M_m(k)^*)$ is not bijective.*

Because $H = H^+ \oplus k.1_H$, one has

$$\Delta(x) = a(1 \otimes 1) + b \otimes 1 + 1 \otimes c + \sum_{x^{(i)} \in H^+} x_{(1)} \otimes x_{(2)}. \tag{2.42}$$

Using the property of the counit, one gets

$$\Delta(x) = x \otimes 1 + 1 \otimes x - \epsilon(x) 1 \otimes 1 + \sum_{(1)(2)} x_{(1)} \otimes x_{(2)} \tag{2.43}$$

We will note Δ_+ the last sum.

$$\Delta_+ = (I^+ \otimes I^+) \circ \Delta, \tag{2.44}$$

where I^+ is a projector to $H_0 = k1_H$.

Proposition 2.1.22. *Let H be a bialgebra. Then the following are equivalent*

i) H admits an antipode;

ii) There exists $S : H \longrightarrow H$ such that

$$\begin{aligned} S(1_{\mathcal{H}}) &= 1_{\mathcal{H}}; & (2.45)\\ S(x) &= -x - \sum_{x^{(i)} \in H^+} S(x^{(1)}) x^{(2)} \quad \textit{for all } x \in H_+. & (2.46) \end{aligned}$$

Proof. $(i) \Rightarrow (ii)$ H admits an antipode S. From Theorem 2.1.20 (ii), one has $S(1_H) = 1_H$. For $x \in H^+$, one has

$$\begin{aligned} \epsilon(x) 1_H = S * Id_H(x) &= m \circ (S \otimes Id_H) \left(\sum_x x^{(1)} \otimes x^{(2)} \right) \\ &= S(1)x + S(x) 1_H + \sum_{x^{(i)} \in H^+} S(x^{(1)}) x^{(2)}. \end{aligned} \tag{2.47}$$

Thus, one has

$$S(x) = -x - \sum_{x^{(i)} \in H^+} S(x^{(1)}) x^{(2)}. \tag{2.48}$$

$(ii) \Rightarrow (i)$ For $x = \lambda 1_H$, one has

$$S * Id_H(x) = \lambda S(1_H) Id_H(1_H) = \lambda 1_H = e(x). \tag{2.49}$$

Thus, one has, in this case $Id_H * S(x) = e(x)$.
For $x \in H_+$, one has $\Delta_+(x) = \sum_{x^{(i)} \in H^+} x^{(1)} \otimes x^{(2)}$. Then, one has

$$\begin{aligned} S * Id_H(x) &= S(x) 1_H + S(1_H) x + \sum_{x^{(i)} \in H^+} S(x^{(1)}) x^{(2)} = S(x) + x + \sum_{x^{(i)} \in H^+} S(x^{(1)}) x^{(2)} \\ &= 0 = e(x). \end{aligned} \tag{2.50}$$

One gets the conclusion. ◻

Remark 2.1.23. *(i) When* Δ_+ *is nilpotent, formula* (2.46) *gives a recursive computation of* S*. This means that the successive replacement of* $S(x)$ *(left hand side of* (2.46) *by its value (right hand side of* (2.46)*) comes to an end (i.e. this algorithm converges).)*
(ii) When Δ_+ *is not nilpotent, formula* (2.46) *is exact (as shows the proof) but it does not need to be convergent. For example, let* $x \in H$ *be a group-like element, then one has* $x - 1 \in H_+$*.*

$$\Delta_+(x-1) = (x-1) \otimes (x-1). \tag{2.51}$$

Then, Equation (2.46) *reads, in this case,*

$$\begin{aligned} &S(x-1) = -(x-1) - S(x-1)(x-1),\\ &S(x) - 1 = -x + 1 - xS(x) + S(x) + x - 1,\\ &xS(x) = 1. \end{aligned}$$

Definition 2.1.24. *Let* H *be a Hopf algebra and* I *a subspace of* H*.*

1. *I is called a Hopf subalgebra of H if I is a subbialgebra and $S(I) \subseteq I$.*

2. *I is called a Hopf ideal of H if it is a biideal and $S(I) \subseteq I$.*

Definition 2.1.25. *Let* H *and* H' *be two Hopf algebras. The map* $\phi : H \to H'$ *is called a Hopf algebra morphism if* ϕ *is a bialgebra morphism and if* $\phi \circ S_H = S_{H'} \circ \phi$*.*

Proposition 2.1.26. *Let* H_1*,* H_2 *be two Hopf algebras. If* $\phi : H_1 \to H_2$ *is a bialgebra morphism, then one has*

$$\phi \circ S_1 = S_2 \circ \phi. \tag{2.52}$$

Proof. Let us prove that $\phi \circ S_1$ is the inverse of ϕ. For $x \in H_1$, one has

$$\begin{aligned} \phi \circ S_1 * \phi(x) &= m_2 \circ (\phi \circ S_1 \otimes \phi) \circ \Delta_1(x)\\ &= m_2 \circ (\phi \otimes \phi) \circ (S_1 \otimes Id_1) \circ \Delta_1(x)\\ &= \phi \circ m_1 \circ (S_1 \otimes Id_1) \circ \Delta_1(x)\\ &= \phi \circ \eta_1 \circ \epsilon_1(x)\\ &= \eta_2 \circ \epsilon_1(x)\\ &= e(x). \end{aligned} \tag{2.53}$$

Similarly, one has $\phi * \phi \circ S_1 = e$. Thus, $\phi \circ S_1$ is the inverse of ϕ.
Let now us prove that $\phi \circ S_2$ is the inverse of ϕ. Indeed, for $x \in H_1$, one has

$$\begin{aligned} S_2 \circ \phi * \phi(x) &= m_2 \circ (S_2 \circ \phi \otimes \phi) \circ \Delta_1(x)\\ &= m_2 \circ (S_2 \otimes Id_2) \circ (\phi \otimes \phi) \circ \Delta_1(x)\\ &= m_2 \circ (S_2 \otimes Id_2) \circ \Delta_2 \circ \phi(x)\\ &= \epsilon_2(\phi(x))1_{H_2}\\ &= \epsilon_1(x)1_{H_2}\\ &= e(x). \end{aligned} \tag{2.54}$$

In the same way, one has that $\phi * S_2 \circ \phi = e$.
Thus, one gets the conclusion. ☐

Definition 2.1.27. *Let H a Hopf algebra. Let $x \in H$.*

1. *x is called a group-like element if $x \neq 0$, $\Delta(x) = x \otimes x$ and $\epsilon(x) = 1$.*
2. *x is called a primitive element if $\Delta(x) = x \otimes 1 + 1 \otimes x$.*

The set of group-like elements of H is denoted $G(H)$ and the subspace of primitive elements of H is denoted $P(H)$.

Remark 2.1.28. *If x is a group-like element, then $S(x) = x^{-1}$. If x is a primitive element, then $S(x) = -x$.*

Let us mention that in most of the studied cases, the spaces have finite graded dimensions. Thus, in order to prove the Hopf algebra structures, it sufficies to show the compatibility of the product and the coproduct.

Proposition 2.1.29. *Let $(\mathcal{H}, m, 1, \Delta, \epsilon)$ be a $\mathbb{N}$-graded bialgebra such that $H_0 = k1_{\mathcal{H}}$. Then $\mathcal{H}$ is a Hopf algebra.*

Let us recall the definition of codendriform bialgebras [Agu04, Lod01, LR02, Foi07].

Definition 2.1.30. *A dendriform algebra is a family $(A, \prec, \succ)$ such that:*
1. A is a k-vector space and:

$$\prec: \begin{cases} A \otimes A & \longrightarrow A \\ a \otimes b & \longrightarrow a \prec b, \end{cases} \quad | \succ: \begin{cases} A \otimes A \longrightarrow A \\ a \otimes b \longrightarrow a \succ b. \end{cases}$$

2. For all $a, b, c \in A$:

$$\begin{aligned} (a \prec b) \prec c &= a \prec (b \prec c + b \succ c), & (2.55) \\ (a \succ b) \prec c &= a \succ (b \prec c), & (2.56) \\ (a \prec b + a \succ b) \succ c &= a \succ (b \succ c). & (2.57) \end{aligned}$$

Definition 2.1.31. *A dendriform coalgebra is a family $(C, \Delta_\prec, \Delta_\succ)$ such that:*
1. C is a k-vector space and:

$$\Delta_\prec : \begin{cases} C & \longrightarrow C \otimes C \\ a & \longrightarrow \Delta_\prec(a) = a'_\prec \otimes a''_\prec, \end{cases} \quad |\Delta_\succ : \begin{cases} C \longrightarrow C \otimes C \\ a \longrightarrow \Delta_\succ(a) = a'_\succ \otimes a''_\succ. \end{cases}$$

2. For all $a \in C$:

$$\begin{aligned} (\Delta_\prec \otimes Id) \circ \Delta_\prec(a) &= (Id \otimes \Delta_\prec + Id \otimes \Delta_\succ) \circ \Delta_\prec(a), & (2.58) \\ (\Delta_\succ \otimes Id) \circ \Delta_\succ(a) &= (Id \otimes \Delta_\prec) \circ \Delta_\succ(a), & (2.59) \\ (\Delta_\prec \otimes Id + \Delta_\succ \otimes Id) \circ \Delta_\succ(a) &= (Id \otimes \Delta_\succ) \circ \Delta_\succ(a). & (2.60) \end{aligned}$$

Definition 2.1.32. *A codendriform bialgebra is a family* $(A, m, \Delta_{\prec}, \Delta_{\succ})$ *such that:*

1. $(A, \Delta_{\prec}, \Delta_{\succ})$ *is a dendriform coalgebra.*
2. (A, m) *is an associative (non unitary) algebra.*
3. *The following compatibilities are satisfied: for all* $a, b \in A$,

$$\tilde{\Delta}(a \prec b) = a'b' \otimes a'' \prec b'' + a' \otimes a'' \prec b + a'b \otimes a'' + b' \otimes a \prec b'' + b \otimes a, \quad (2.61)$$
$$\tilde{\Delta}(a \succ b) = a'b' \otimes a'' \succ b'' + a' \otimes a'' \succ b + ab' \otimes b'' + b' \otimes a \succ b'' + a \otimes b. \quad (2.62)$$

2.2 Matroids and the Tutte polynomial for matroids

In this section we briefly recall some matroid theory notions. We give the definition of matroids, of the associated Tutte polynomial as well as some further properties which will be useful to prove the results of Chapter 4. Finally, the Hopf algebra of isomorphic classes of matroids is given.

2.2.1 Matroid theory

In this subsection we recall the definition and some properties of the Tutte polynomial for matroids as well as of the matroid Hopf algebra defined in [Sch94].

Let us first define a graph in the following way:

Definition 2.2.1. *A graph* Γ *is defined as a set of vertices* V *and of edges* E *together with an incidence relation between them.*

After the book [Oxl92], one gives the following definitions:

Definition 2.2.2. *A* **matroid** M *is a pair* $(E, \mathcal{I})$ *consisting of a finite set* E *and a collection of subsets of* E *satisfying the following set of axioms:*

(I1) $\mathcal{I}$ *is non-empty.*

(I2) Every subset of every member of $\mathcal{I}$ *is also in* $\mathcal{I}$.

(I3) If X *and* Y *are in* $\mathcal{I}$ *and* $|X| = |Y| + 1$, *then there is at least an element* x *in* $X \setminus Y$ *such that* $Y \cup \{x\}$ *is in* $\mathcal{I}$.

M is called a matroid on E. The set E above is called the **ground set** of the matroid M. The members of $\mathcal{I}$ are called **independent sets** of the matroid M. A subset of E which is not in $\mathcal{I}$ is called **dependent**.

Proposition 2.2.3 (Proposition 1.1.1 of [Oxl92]). *Let* E *be the set of column labels of an* $m \times n$ *matrix* A *over a field* F, *and let* $\mathcal{I}$ *be the set of subsets* X *of* E *for which the multiset*[1] *of columns labelled by* X *is linearly independent in the vector space* $V(m, F)$. *Then* $(E, \mathcal{I})$ *is a matroid.*

[1] In general, taking a set of labels give a multiset, but here (in the case of independent) there is no multiplicity.

This matroid is called the **vector matroid**.

Example 2.2.4. *Let A be a matrix*

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

over field $\mathbb{R}$. The ground set is given by $E = \{1,2,3,4,5,6\}$ and the collection of subset $\mathcal{I}$ consists of all subsets of $E - \{6\}$ with at most three elements except for any subset containing $\{1,2\}$. In details, one has $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}\}$.

Let E be an $n-$element set and let $\mathcal{I}$ be the collection of subsets of E with at most r elements, $0 \le r \le n$. One can check that $(E, \mathcal{I})$ is a matroid; it is called the **uniform matroid** $U_{r,n}$. If the ground set is an empty set, there is one matroid, namely $U_{0,0}$. This matroid, denoted by **1**, is called the **empty matroid**.

A forest of graph is a subgraph without cycles [Ber73].

Definition 2.2.5. *Let G be a graph. Let E be the set of edges of graph G and $\mathcal{I}$ be the set of forests of G. Then $(E, \mathcal{I})$ is a matroid. This matroid is called the* **cycle matroid** *of G. It is denoted by $M(G)$.*

Note that not every matroid is a graphic matroid. All matroids on the ground set which contains three or less elements, are graphic (see Table 2.1). The smallest example of a non-graphic matroid is the uniform matroid $U_{2,4}$.

Remark 2.2.6. *The matroids representations of two different graphs might be the same. For example, for the graphs G and H in Figure 2.1a and Figure 2.1b respectively, $M(G) = M(H)$.*

A cycle matroid of G is has the ground set $E(G) = \{1,2,3,4,5,6\}$ and the set $\mathcal{I}$ consists of all subsets of $E(G) \setminus \{6\}$ with at most three elements except for any subset containing $\{1,2\}$.

Definition 2.2.7. *The maximal independent sets of a matroid is called* **a base** *or* **a basis***. The minimal dependent sets of a matroid are called circuits.*

Example 2.2.8. *(i) The bases of the cycle matroid $M(G)$ of the graph G in Figure 2.1a are all subsets of $E(G) \setminus \{6\}$ with at most three elements except for any subset containing $\{1,2\}$.*
(ii) The bases of the uniform matroid $U_{k,n}$ are all k-element subsets of the ground set.

Let $M = (E, \mathcal{I})$ be a matroid and let $\mathcal{B} = \{B\}$ be the collection of bases of M.

A spanning tree of a graph on n vertices is a subset of $n-1$ edges that form a tree [Ber73].

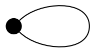
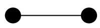
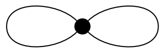
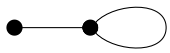
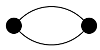
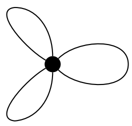
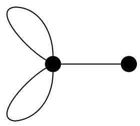
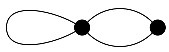
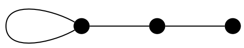
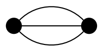
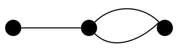
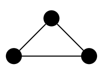
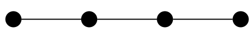
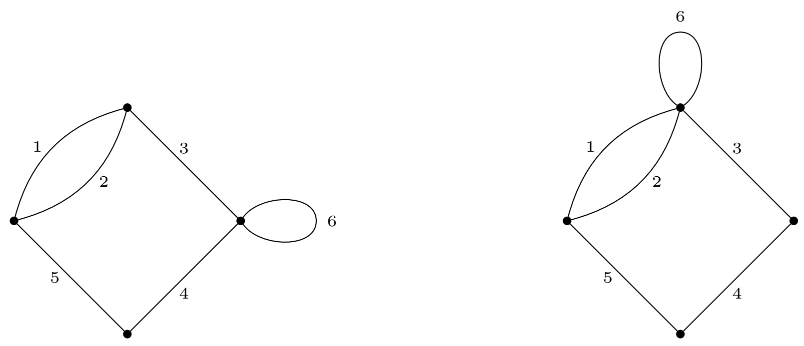
Number of elements	A corresponding graph	non-isomorphic matroids
0		$(\emptyset,\ \{\emptyset\})$
1		$(\{1\},\ \{\emptyset\})$
1		$(\{1\},\ \{\emptyset, \{1\}\})$
2		$(\{1,2\},\ \{\emptyset\})$
2		$(\{1,2\},\ \{\emptyset, \{1\}\})$
2		$(\{1,2\},\ \{\emptyset, \{1\}, \{2\}\})$
2		$(\{1,2\},\ \{\emptyset, \{1\}, \{2\}, \{1,2\}\})$
3		$(\{1,2,3\},\ \{\emptyset\})$
3		$(\{1,2,3\},\ \{\emptyset, \{1\}\})$
3		$(\{1,2,3\},\ \{\emptyset, \{1\}, \{2\}\})$
3		$(\{1,2,3\},\ \{\emptyset, \{1\}, \{2\}, \{1,2\}\})$
3		$(\{1,2,3\},\ \{\emptyset, \{1\}, \{2\}, \{3\}\})$
3		$(\{1,2,3\},$ $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\})$
3		$(\{1,2,3\},$ $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\})$
3		$(\{1,2,3\},$ $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\},$ $\{1,2,3\}\})$

Table 2.1: The matroids with ground set of cardinal at most 3.



(a) The graph G. (b) The graph H.

Figure 2.1: Two different graphs have the same representation matroid.

Example 2.2.9. *Let G be a connected graph. The collection of bases of a graphic matroid $M(G)$ is the set of spanning trees.*

Definition 2.2.10. *Two matroids M_1 and M_2 are said* **isomorphic** *(written $M_1 \cong M_2$), if there is a bijection ϕ from $E(M_1)$ to $E(M_2)$ such that, for all $X \subseteq E(M_1)$, $\phi(X)$ is independent in M_2 if and only if X is independent in M_1.*

A matroid that is isomorphic to the cycle matroid of graph is called **graphic**.
A matroid M that is isomorphic to a vector matroid of a matrix D over a field F is said to be **representable over** F **or** F**-representable** for M. In fact, every graphic matroid is representable over every field.

Remark 2.2.11. *If one takes $n = 1$, there are only two matroids, namely $U_{0,1}$ and $U_{1,1}$ and both of these matroids are graphic matroids. The graphs of these two matroids correspond to the graphs with one edge of Fig.2.2 and Fig. 2.3. In the first case, the edge is a loop*

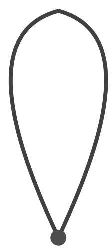


Figure 2.2: The graph corresponding to the matroid $U_{0,1}$.

Figure 2.3: The graph corresponding to the matroid $U_{1,1}$.

(in graph theoretical terminology) or a tadpole (in quantum field theory (QFT) language).

In the second case, the edge represents a bridge (in graph theoretical terminology) or a 1*-particle-reducible line (in QFT terminology) - the number of connected components of the graphs increases by* 1 *if one deletes the respective edge.*

Theorem 2.2.12 (Theorem 2.1.1 in [Oxl92]). *Let M be a matroid and $\mathcal{B}^\star(M)$ be $\{E(M)\setminus B : B \in \mathcal{B}(M)\}$. Then $\mathcal{B}^\star(M)$ is the set of bases of a matroid on $E(M)$, denoted M^*.*

The matroid in the last theorem, whose ground set is $E(M)$ and whose set of bases is $\mathcal{B}^\star(M)$, is called the **dual** of M and denoted by $M^\star$. Thus, $\mathcal{B}(M^\star) = \mathcal{B}^\star(M)$. One has $(M^\star)^\star = M$. The bases of $M^\star$ are called **cobases** of M.

Example 2.2.13. *Consider the uniform matroid $U_{k,n}$. Its bases are all of the k-element subsets of E. Hence all bases of its dual are all the $(n-k)$-element subsets of E. One has $U^\star_{k,n} = U_{n-k,n}$.*

Example 2.2.14. *Table 2.2 gives the dual of all matroids in Table 2.1.*

Definition 2.2.15. *Let $M = (E, \mathcal{I})$ be a matroid. The* **rank** *$r(A)$ of $A \subset E$ is given by the following formula:*

$$r(A) := max\{|B| \ s.t. \ B \in \mathcal{I}, B \subset A\} \ . \tag{2.63}$$

Definition 2.2.16. *Let $M = (E, \mathcal{I})$ be a matroid with a ground set E. For a subset $A \subset E$, the* **nullity** *function is given by*

$$n(A) := |A| - r(A). \tag{2.64}$$

Definition 2.2.17. *Let $M = (E, \mathcal{I})$ be a matroid. The element $e \in E$ is a* ***loop*** *iff $\{e\}$ is a circuit.*

Definition 2.2.18. *Let $M = (E, \mathcal{I})$ be a matroid. The element $e \in E$ is a* ***coloop*** *iff, for any basis B, $e \in B$.*

Let us remark that e is a loop iff e is a coloop in M^*.
Note that in a graphic matroid, a loop is an edge which is a loop in the graph-theoretic sense, while a coloop is an edge which is a **bridge** or **isthmus** in the graph (see again Figures 2.2 and 2.3).
Let M be a matroid $(E, \mathcal{I})$ and T be a subset of E. Let $\mathcal{I}|_T$ be the set $\{I \subseteq T \text{ s. t. } I \in \mathcal{I}\}$. One can then check that the pair $(T, \mathcal{I}|_T)$ is a matroid, which is denoted by $M|_T$ and is called the **restriction** of M to T.
Let us now define two more basic operations on matroids. Let $\mathcal{I}' = \{I \subseteq E - T \text{ s.t. } I \in \mathcal{I}\}$. One can check that $(E-T, \mathcal{I}')$ is a matroid. We denote this matroid by $M\backslash T$ - the **deletion** of T from M. The **contraction** of T from M, M/T, is given by the formula: $M/T = (M^\star \backslash T)^\star$.

Example 2.2.19. *Let M be a cycle matroid of graph G in Figure 2.1a. Let $T = \{1, 3\}$. One then has*

$$\begin{aligned} M\backslash T &= (\{2,4,5,6\}, \{\emptyset, \{2\}, \{4\}, \{5\}, \{2,4\}, \{2,5\}, \{4,5\}, \{2,4,5\}\}) \\ M/T &= (\{2,4,5,6\}, \{\emptyset, \{4\}, \{5\}\}). \end{aligned}$$

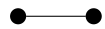
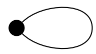
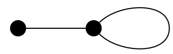
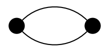
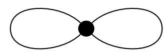
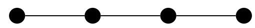
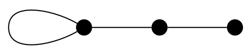
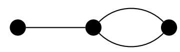
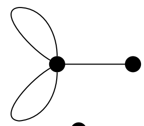
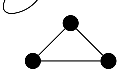
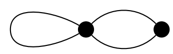
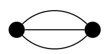
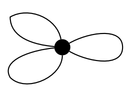
Matroids	Dual matroids	A corresponding graph
$(\emptyset, \{\emptyset\})$	$(\emptyset, \{\emptyset\})$	
$(\{1\}, \{\emptyset\})$	$(\{1\}, \{\emptyset, \{1\}\})$	
$(\{1\}, \{\emptyset, \{1\}\})$	$(\{1\}, \{\emptyset\})$	
$(\{1,2\}, \{\emptyset\})$	$(\{1,2\},$ $\{\emptyset, \{1\}, \{2\}, \{1,2\}\})$	
$(\{1,2\}, \{\emptyset, \{1\}\})$	$(\{1,2\}, \{\emptyset, \{2\}\})$	
$(\{1,2\}, \{\emptyset, \{1\}, \{2\}\})$	$(\{1,2\}, \{\emptyset, \{1\}, \{2\}\})$	
$(\{1,2\},$ $\{\emptyset, \{1\}, \{2\}, \{1,2\}\})$	$(\{1,2\}, \{\emptyset\})$	
$(\{1,2,3\}, \{\emptyset\})$	$(\{1,2,3\},$ $\{\emptyset, \{1\}, \{2\}, \{3\},$ $\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\})$	
$(\{1,2,3\}, \{\emptyset, \{1\}\})$	$(\{1,2,3\}, \{\emptyset, \{2\}, \{3\},$ $\{2,3\}\})$	
$(\{1,2,3\}, \{\emptyset, \{1\}, \{2\}\})$	$(\{1,2,3\},$ $\{\emptyset, \{1\}, \{2\}, \{3\},$ $\{1,3\}, \{2,3\}\})$	
$(\{1,2,3\},$ $\{\emptyset, \{1\}, \{2\}, \{1,2\}\})$	$(\{1,2,3\}, \{\emptyset, \{3\}\})$	
$(\{1,2,3\},$ $\{\emptyset, \{1\}, \{2\}, \{3\}\})$	$(\{1,2,3\},$ $\{\emptyset, \{1\}, \{2\}, \{3\},$ $\{1,2\}, \{1,3\}, \{2,3\}\})$	
$(\{1,2,3\},$ $\{\emptyset, \{1\}, \{2\}, \{3\}$ $\{1,2\}, \{1,3\}\})$	$(\{1,2,3\}, \{\emptyset, \{2\}, \{3\}\})$	
$(\{1,2,3\},$ $\{\emptyset, \{1\}, \{2\}, \{3\})$ $\{1,2\}, \{1,3\}, \{2,3\}\}$	$(\{1,2,3\},$ $\{\emptyset, \{1\}, \{2\}, \{3\}\})$	
$(\{1,2,3\},$ $\{\emptyset, \{1\}, \{2\}, \{3\}$ $\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\})$	$(\{1,2,3\}, \{\emptyset\})$	

Table 2.2: Dual matroids with ground set of cardinal at most 3.

Let us give the following definition:

Definition 2.2.20. *Let M_1 and M_2 be the matroids $(E_1, \mathcal{I}_1)$ and $(E_2, \mathcal{I}_2)$ where E_1 and E_2 are disjoint. Let*

$$M_1 \oplus M_2 := (E_1 \cup E_2, \{I_1 \cup I_2 \ s.t.\ I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}) \ .$$

Then $M_1 \oplus M_2$ is a matroid. This matroid is called the direct sum of M_1 and M_2.

A matroid N is a **minor** of the matroid M if it is obtained from M by any combination of restrictions and contractions. We further call a family of matroids to be **minor-closed** if it closed under formation of minors and direct sums.

Example 2.2.21 (Corollary 3.2.2). *Every minor of a graphic matroid is graphic.*

Let us also recall the following results:

Lemma 2.2.22 (). *Let M be a matroid $(E, \mathcal{I})$ and T be a subset of E. One has:*

$$M|_T = M\backslash_{E-T} \ . \tag{2.65}$$

Proof. It is easy to see that $E(M\backslash_{E-T}) = T = E(M|_T)$.
Moreover, from the definition, the collection of independent subsets of $M\backslash_{E-T}$ is given by

$$\mathcal{I}(M\backslash_{E-T}) = \{I \subset E - (E - T) \text{ s.t. } I \in \mathcal{I}\} = \{I \subset T \text{ s.t. } I \in \mathcal{I}\} = \mathcal{I}(M|_T).$$

One then gets the conclusion. □

Let us recall some basic results (see [Oxl92]).

Lemma 2.2.23 (Corollary 3.1.25 [Oxl92]). *If e is either a coloop or a loop of a matroid $M = (E, \mathcal{I})$, then $M/e = M\backslash e$.*

Lemma 2.2.24 (Proposition 3.1.6 [Oxl92]). *Let $M = (E, \mathcal{I})$ be a matroid and $T \subseteq E$, then, for all $X \subseteq E - T$,*

$$r_{M/T}(X) = r_M(X \cup T) - r_M(T) \ . \tag{2.66}$$

Let us now define the Tutte polynomial for matroids [Tut54]:

Definition 2.2.25. *Let $M = (E, \mathcal{I})$ be a matroid. The* **Tutte polynomial** *is given by the following formula:*

$$T_M(x, y) := \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{n(A)}. \tag{2.67}$$

Example 2.2.26. *1) Let $U_{k,n}$ be a uniform matroid, $0 \le k \le n$. The Tutte polynomial of this matroid is given by the following formula:*

$$T_{U_{k,n}}(x, y) = \sum_{i=0}^{k} \binom{n}{i} (x-1)^{k-i} + \sum_{i=k+1}^{n} \binom{n}{i} (y-1)^{i-k}. \tag{2.68}$$

2) Let M be a cycle matroid of graph G in Figure 2.1a. One has

$$\begin{aligned} T_M(x,y) &= (x-1)^3 + 5(x-1)^2 + (x-1)^3(y-1) + 9(x-1) + 6(x-1)^2(y-1) \\ &\quad +7 + 12(x-1)(y-1) + (x-1)^2(y-1)^2 + 12(y-1) + 3(x-1)(y-1)^2 \\ &\quad +6(y-1)^2 + (y-1)^3 \\ &= x^3y + x^2y + xy + xy^2 + x^2y^2 + y^2 + y^3. \end{aligned} \tag{2.69}$$

Let us recall, from [BO92] that

$$T_M(x,y) = T_{M^\star}(y,x). \tag{2.70}$$

Proposition 2.2.27 (Lemma 6.2.1 of [BO92]). *The Tutte polynomial is a Tutte-Grothendieck invariant* (1.3) - (1.5) *for the class of all matroids. This means that*

(i) $T_M(x,y) = T_{M'}(x,y)$ *where* $M \cong M'$*;*

(ii) $T_M(x,y) = T_{M/e}(x,y) + T_{M\setminus e}(x,y)$ *if e is neither a loop nor coloop;*

(iii) $T_M(x,y) = T_{M|e}(x,y)T_{M\setminus e}(x,y)$ *if e is either a loop or a coloop.*

The Tutte polynomial for matroids also satisfies a multiplicative law.

Proposition 2.2.28 (Proposition 6.2.5 of [BO92]). *Let the two matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$, where E_1 and E_2 are disjoint. One has*

$$T_{M_1 \oplus M_2}(x,y) = T_{M_1}(x,y)T_{M_2}(x,y). \tag{2.71}$$

2.2.2 Matroid Hopf algebra

The **matroid-minor Hopf algebra**, introduced in [Sch94], has as canonical basis the set of all isomorphism classes of matroids. This Hopf algebra is briefly considered in connection with the characteristic and Tutte polynomials of matroids in [KRS99] and [Kun04].
Let us denote by $\mathcal{M}$ a minor-closed family of matroids. We further note by $\widetilde{\mathcal{M}}$ the set of isomorphic classes of matroids belonging to $\mathcal{M}$. As already mentioned in [Sch94], direct sums induce a product on $\widetilde{\mathcal{M}}$. Let $k(\widetilde{\mathcal{M}})$ finally denote the monoid algebra of $\widetilde{\mathcal{M}}$ over some commutative ring k with unit.
In [Sch94], as a particularization of a more general construction of incidence Hopf algebras, the following result was proved:

Proposition 2.2.29 (Proposition 2.1 **of [CS05]).** *If $\mathcal{M}$ is a minor-closed family of matroids then $k(\widetilde{\mathcal{M}})$ is a coalgebra, with coproduct Δ and counit ϵ determined by*

$$\Delta(M) := \sum_{A \subseteq E} M|A \otimes M/A \tag{2.72}$$

and respectively by $\epsilon(M) = \begin{cases} 1, & \text{if } E = \emptyset, \\ 0 & \text{otherwise}, \end{cases}$ *for all $M = (E, \mathcal{I}) \in \mathcal{M}$. If, furthermore, the family $\mathcal{M}$ is closed under formation of direct sums, then $k(\widetilde{\mathcal{M}})$ is a Hopf algebra, with product induced by direct sum.*

We refer to this Hopf algebra as to the matroid Hopf algebra. We follow [CS05] and, by a slight abuse of notation, we denote in the same way a matroid and its isomorphic class, since the distinction will be clear from the context (as it is already in Proposition 2.2.29).
We denote the unit of this Hopf algebra by **1** (the empty matroid, or $U_{0,0}$).

Example 2.2.30. *(Example 2.4 of [CS05]) The class $\mathcal{U}$ of all uniform matroids is minor-closed, and the coproduct of the matroids $U_{k,n}$ is given by:*

$$\Delta(U_{k,n}) = \sum_{i=0}^{k} \binom{n}{i} U_{i,i} \otimes U_{k-i,n-i} + \sum_{i=k+1}^{n} \binom{n}{i} U_{k,i} \otimes U_{0,n-i}, \forall k, n.$$

2.3 Structure of cocommutative Hopf algebra

The results presented in this section follow [DMT$^+$].
In order to extend Schützenberger's factorization to general perturbations, the combinatorial aspects of the Hopf algebra of a deformed shuffle product is developed systematically in a parallel way with those of the shuffle product, with an emphasis on the Lie elements as studied by Ree. In particular, we will give an effective construction of pair of bases in duality.
Many algebras of functions [DDMS11] and many special sums [MJOP00, MJOP01] are governed by shuffle products, their perturbations (adding a "superposition term" [DTPK10]) or deformations [TU96].

In order to better understand the mechanisms of these products, we wish here to examine, with full generality the products which are defined by a recursion of the type

$$au \star bv = a\,(u \star bv) + b\,(au \star v) + \phi(a, b)\, u \star v \ , \tag{2.73}$$

the empty word being the neutral of this new product.

Moreover, we present a version of the Cartier-Quillen-Milnor and Moore (CQMM in the sequel) without any use of the Poincaré-Birkhoff-Witt (PBW) construction. We are obliged to restate the CQMM theorem without supposing any basis because we aim at "varying the scalars" in forthcoming papers (germs of functions, arithmetic functions, etc.) and, in order to do this at ease, we must cope safely with cases where torsion (non-zero annihilators) may appear (and then, one cannot have any basis). See (counter) examples in the subsection 2.3.3.

2.3.1 General results on summability and duality

2.3.1.1 Total algebras and duality

Series and infinite sums
We here recall the results used to handle infinite sums in the sequel. The underlying topology is that of the pointwise convergence (the target being endowed with the discrete topology).

In the sequel, we will need to construct spaces of functions on different monoids (mainly direct products of free monoids). We set, once for all the general construction of the corresponding convolution algebra.

Let A be a unitary commutative ring and M a monoid. Let us denote A^M the set[2] of all (graphs of) mappings[3] $M \to A$. This set is endowed with its classical structure of module. In order to extend the product defined in $A[M]$ (the algebra of the monoid M), it is essential that, in the sums

$$f * g = \sum_{m\in M} \Big(\sum_{uv=m} f(u)g(v) \Big) m \tag{2.74}$$

the inner sums $\sum_{uv=m} f(u)g(v)$ make sense. For that, we suppose that the monoid M fulfills condition "D" (i.e. M is of finite decomposition type [Bou06a] Ch III.10). Formally, we say that M satisfies condition "D" iff, for all $m \in M$, the set

$$\{(u, v) \in M \times M \text{ s.t. } uv = m\} \tag{2.75}$$

is finite. In this case, Equation (2.74) endows A^M with the structure of an associative algebra with unit (AAU). This algebra is traditionally called the total algebra of M (see [Bou06a] Ch III.10)[4]. Here, it will be denoted, with an unambiguous abuse of denotation, by $A\langle\langle M\rangle\rangle$. The pairing

$$A\langle\langle M\rangle\rangle \otimes A[M] \quad\longrightarrow\quad A \tag{2.76}$$

defined by[5]

$$\langle f \mid g\rangle \quad := \quad \sum_{m\in M} f(m)g(m) \tag{2.77}$$

allows to consider the total algebra as the dual of the module $A[M]$ i.e., through this pairing

$$A\langle\langle M\rangle\rangle \simeq (A[M])^* \ .$$

One says that a family $(f_i)_{i\in I}$ of $A\langle\langle M\rangle\rangle$ is summable [BR88] iff, for every $m \in M$, the mapping $i \mapsto \langle f_i \mid m\rangle$ is finitely supported. In this case, the sum $\sum_{i\in I} f_i$ is exactly the mapping $m \longmapsto \sum_{i\in I} \langle f_i \mid m\rangle$ so that, one has by definition

$$\Big\langle \sum_{i\in I} f_i \mid m \Big\rangle = \sum_{i\in I} \langle f_i \mid m\rangle \ . \tag{2.78}$$

Finally, let us remark that the set $M_1 \otimes M_2 = \{u \otimes v\}_{(u,v)\in M_1\times M_2}$ is a (monoidal) basis of $A[M_1] \otimes A[M_2]$ and $M_1 \otimes M_2$ is a monoid (in the product algebra $A[M_1] \otimes A[M_2]$) isomorphic to the direct product $M_1 \times M_2$.

[2] In general Y^X is the set of all (total) mappings $X \to Y$ [Bou06d] Ch 2.5.2.

[3] According to [Bou06d], Y^X is the set of all "graphs" of $\Gamma \subset X \times Y$ which are functional in X such that $dom(\Gamma) = X$.

[4] Actually, the algebra of commutative (resp. noncommutative) series on an alphabet X is the total algebra of the free commutative (resp. free) monoid on X

[5] Here $A[M]$ is identified with the submodule of finitely supported functions $M \to A$.

Summable families in Hom **spaces.**
In fact, $A\langle\langle M\rangle\rangle \simeq (A[M])^* = \mathrm{Hom}(A[M], A)$ and the notion of summability developed above can be seen as a particular case of that of a family of endomorphisms $f_i \in \mathrm{Hom}(V, W)$ for which $\mathrm{Hom}(V, W)$ appears as a complete space[6].

The definition extends that of a summable family of series (2.78).

Definition 2.3.1. *i) A family $(f_i)_{i\in I}$ of elements in* $\mathrm{Hom}(V, W)$ *is said to be* summable *iff for all $x \in V$, the map $i \mapsto f_i(x)$ has finite support. As a quantized criterium it reads*

$$(\forall x \in V)(\exists F \subset I,\ F \text{ finite})(\forall i \notin F)(f_i(x) = 0). \tag{2.79}$$

ii) If the family $(f_i)_{i\in I} \in \mathrm{Hom}(V, W)^I$ fulfils the condition (2.79) *above, its sum is given by*

$$(\sum_{i\in I} f_i)(x) := \sum_{i\in I} f_i(x). \tag{2.80}$$

It is an easy exercise to show that the mapping $V \to W$ defined by the equation (2.80) is in fact in $\mathrm{Hom}(V, W)$.

Remark 2.3.2. *(i) As the limiting process is defined by linear conditions, if a family $(f_i)_{i\in I}$ is summable, so is*

$$(a_i f_i)_{i\in I} \tag{2.81}$$

for an arbitrary family of coefficients $(a_i)_{i\in I} \in A^I$.
(ii) With $V = A[M]$ and $W = A$, one has $Hom(V, W) \cong A\langle\langle M\rangle\rangle$ and it is easy to check that with Definition 2.3.1 one recovers the previous notion of summability of Equation (2.80).

This tool will be used in subsection 2.3.2 to give an analytic presentation of the theorem of CQMM in the case when $V = W = \mathcal{B}$ is a bialgebra.
The most interesting feature of this operation is the interchange of sums. Let us state it formally as a proposition.

Proposition 2.3.3. *Let $(f_i)_{i\in I}$ be a family of elements in* $\mathrm{Hom}(V, W)$ *and $(I_j)_{j\in J}$ be a partition of I ([Bou06d] ch II §4 n^o 7 Def. 6), then, the following statements are equivalent*
i) $(f_i)_{i\in I}$ is summable
ii) for all $j \in J$, $(f_i)_{i\in I_j}$ is summable and the family $(\sum_{i\in I_j} f_i)_{j\in J}$ is summable.
In these conditions, one has

$$\sum_{i\in I} f_i = \sum_{j\in J}(\sum_{i\in I_j} f_i). \tag{2.82}$$

We derive at once from this the following practical criterium for double sums.

Corollary 2.3.4. *Let $(f_{\alpha,\beta})_{(\alpha,\beta)\in X\times Y}$ be a doubly indexed summable family in $Hom(V, W)$, then, for fixed α (resp. β) the "row-families" $(f_{\alpha,\beta})_{\beta\in Y}$ (resp. the "column-families" $(f_{\alpha,\beta})_{\alpha\in X}$) are summable and their sums are summable. Moreover*

$$\sum_{(\alpha,\beta)\in X\times Y} f_{\alpha,\beta} = \sum_{\alpha\in X}\sum_{\beta\in Y} f_{\alpha,\beta} = \sum_{\beta\in Y}\sum_{\alpha\in X} f_{\alpha,\beta}\ . \tag{2.83}$$

[6]Complete is in the sense of [Bou06b] Ch3 §1 n$^{\mathrm{O}}$ 1. Every summable (infinite sum) has a limit.

2.3.1.2 Substitutions

Let $\mathcal{A}$ be an AAU and $f \in \mathcal{A}$. For every polynomial $P \in A\langle z \rangle$ $(= A[z^*])$, one can compute $P(f)$ by

$$P(f) = \sum_{n\geq 0} \langle P \mid X^n \rangle f^n \ . \tag{2.84}$$

One checks at once that $P \mapsto P(f)$ is a morphism[7] of AAU's between $A\langle X \rangle$ and $\mathcal{A}$. Moreover, this morphism is compatible with the substitutions as one checks easily that, for $Q \in A[X]$

$$P(Q)(f) = P(Q(f)) \tag{2.85}$$

(it suffices to check that $P \mapsto P(Q)(f)$ and $P \mapsto P(Q(f))$ are two morphisms which coincide on $P = X$).
In order to substitute within series, one needs some limiting process. The framework of $\mathcal{A} = \mathrm{Hom}(V, W)$ and summable families will be here sufficient (see subsection 2.3.1.1). We suppose that $(V, \delta_V, \epsilon_V)$ is a co-AAU and that $(W, \mu_W, 1_W)$ is an AAU. Then $(\mathrm{Hom}(V, W), *, e)$ is an AAU (with $e = 1_W \circ \epsilon_V$). A series $S \in A[[X]]$ and $f \in \mathrm{Hom}(V, W)$ being given, we say that $f \in Dom(S)$ iff the family $(\langle S \mid X^n \rangle f^{*n})_{n\geq 0}$ is summable[8]. We have the following properties

Proposition 2.3.5. *Let S and T be two series in $A[[X]]$. If $f \in Dom(S) \cap Dom(T)$ and $\alpha \in A$, one has*

$$(\alpha S)(f) = \alpha S(f) \ ; \ (S + T)(f) = S(f) + T(f) \tag{2.86}$$

and

$$(TS)(f) = T(f) * S(f) \ . \tag{2.87}$$

If $((f)^{*n})_{n\geq 0}$ is summable and $S(0) = 0$ then

$$f \in Dom(S) \cap Dom(T(S)) \ ; \ S(f) \in Dom(T) \tag{2.88}$$

and

$$T(S)(f) = T(S(f)). \tag{2.89}$$

Proof. Let us first prove (2.87). As $f \in Dom(S) \cap Dom(T)$, the families $(\langle S \mid X^n \rangle f^{*n})_{n\geq 0}$ and $(\langle T \mid X^m \rangle f^{*m})_{m\geq 0}$ are summable, then so is

$$(\langle T \mid X^m \rangle f^{*m} * \langle S \mid X^n \rangle f^{*n})_{n,m\geq 0} \tag{2.90}$$

as, for every $x \in V$, $\delta_V(x) = \sum_{i=1}^{N} x_i^{(1)} \otimes x_i^{(2)}$ and for every $i \in I$,

$$supp_{w.r.t.\ m}(\langle T \mid X^m \rangle f^{*m}(x_i^{(1)})) \ ; \ supp_{w.r.t.\ n}(\langle S \mid X^n \rangle f^{*n}(x_i^{(2)}))$$

are finite[9]. Then outside of the Cartesian product of the (finite) union of these supports, the product

$$(\langle T \mid X^m \rangle f^{*m} * \langle S \mid X^n \rangle f^{*n})(x) = \mu_W((\langle T \mid X^m \rangle f^{*m} \otimes \langle S \mid X^n \rangle f^{*n})(\delta_V(x))) \tag{2.91}$$

[7]In case $\mathcal{A}$ is a geometric space, this morphism is called "evaluation at f" and corresponds to a Dirac measure.

[8]Where f^{*n} denotes straightforwardly the n-th power of f w.r.t. the convolution product.

[9]Let $(X_{i,j,k,\ldots,m})$ be a multiindexed family. The $supp_{w.r.t.\ k}$ is the support of mapping $k \longrightarrow X_{i,j,k,\ldots,m}$.

is zero. Hence the summability.
Now

$$\begin{aligned} T(f) * S(f) &= \left(\sum_{m=0}^{\infty} \langle T \mid X^m \rangle f^{*m}\right) * \left(\sum_{n=0}^{\infty} (\langle S \mid X^n \rangle f^{*n}\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle T \mid X^m \rangle \langle S \mid X^n \rangle f^{*(n+m)} \\ &= \sum_{s=0}^{\infty} \left(\sum_{n+m=s}^{\infty} \langle T \mid X^m \rangle \langle S \mid X^n \rangle \right) f^{*s} \\ &= \sum_{s=0}^{\infty} (\langle TS \mid X^s \rangle) f^{*s} = (TS)(f). \end{aligned} \tag{2.92}$$

We now prove the statements (2.88) and (2.89). If $((f)^{*n})_{n \geq 0}$ is summable then f belongs to all domains (i.e. is universally substitutable) by virtue of (2.81). For all $x \in V$, there exists $N_x \in \mathbb{N}$ such that

$$n > N_x \Longrightarrow (f)^{*n}(x) = 0 \ .$$

Now, for S such that $S(0) = 0$, one has $S = \sum_{n=1}^{\infty} \langle S \mid X^n \rangle X^n$ and then

$$S^k = \sum_{n=k}^{\infty} \left\langle S^k \mid X^n \right\rangle X^n.$$

Now, in view of (2.87), one has

$$S(f)^{*n}(x) = S^n(f)(x) = \sum_{m=n}^{\infty} \langle S^n \mid X^m \rangle (f)^{*m}(x) \tag{2.93}$$

which is zero for $n > N_x$. Hence the summability of $(S(f)^{*n})_{n \geq 0}$ which implies that $S(f) \in Dom(T)$. The family $(\langle T \mid X^n \rangle \langle S^n \mid X^m \rangle (f)^{*m})_{(n,m) \in \mathbb{N}^2}$ is summable because, if $x \in V$ and if n or m is greater than N_x then

$$\langle T \mid X^n \rangle \langle S^n \mid X^m \rangle (f)^{*m}(x) = 0. \tag{2.94}$$

Thus $T(S(f))$ is the sum

$$\begin{aligned} T(S(f)) &= \sum_{n=0}^{\infty} \langle T \mid X^n \rangle S(f)^{*n} \\ &= \sum_{n=0}^{\infty} \langle T \mid X^n \rangle \sum_{m=n}^{\infty} \langle S^n \mid X^m \rangle (f)^{*m} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \langle T \mid X^n \rangle \langle S^n \mid X^m \rangle (f)^{*m} \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \langle T \mid X^n \rangle \langle S^n \mid X^m \rangle \right) (f)^{*m} \\ &= \sum_{m=0}^{\infty} \langle T(S) \mid X^m \rangle (f)^{*m} = T(S)(f). \end{aligned} \tag{2.95}$$

□

In the free case (i.e. $V = W$ are the bialgebra $(A\langle X\rangle, \texttt{conc}, 1_{X^*}, \Delta_{⧢}, \epsilon)$), one has a very useful representation of the convolution algebra $\mathrm{Hom}(V, W)$ through images of the diagonal series. This representation will provide us with the key lemma 2.3.10. Let

$$\mathcal{D}_X := \sum_{w \in X^*} w \otimes w. \tag{2.96}$$

be the diagonal series attached to X.

Proposition 2.3.6. *Let A be a commutative unitary ring and X an alphabet. Then*

i) For every $f \in \mathrm{End}(A\langle X\rangle)$, the family $(u \otimes f(u))_{u \in X^*}$ is summable in $A\langle\langle X^* \otimes X^*\rangle\rangle$ and its sum is

$$\sum_{v,w \in X^*} \langle f(v) \mid w\rangle\, v \otimes w, \tag{2.97}$$

where $\langle|\rangle$ is the pairing given by $(\forall u, v \in X^*)(\langle u \mid v\rangle = \delta_{u,v})$.

ii) The map

$$f \mapsto \rho(f) := \sum_{u \in X^*} u \otimes f(u) \tag{2.98}$$

is faithful representation from $(\mathrm{End}(A\langle X\rangle), *)$ to $(A\langle\langle X^* \otimes X^*\rangle\rangle, ⧢ \otimes \texttt{conc})$. In particular, for $f \in \mathrm{End}(A\langle X\rangle)$ and $P \in A[z]$, one has

$$\rho(P(f)) = P(\rho(f)). \tag{2.99}$$

iii) If $f(1_{X^*}) = 0$ and $S \in A[[z]]$ is a series, then $(\rho(f)^n)_{n \geq 0}$ is summable in $(A\langle\langle X^* \otimes X^*\rangle\rangle,$ $⧢ \otimes \texttt{conc})$ and

$$\rho(S(f)) = S(\rho(f)). \tag{2.100}$$

Proof. i) One easily checks that for every $v \otimes w \in X^* \otimes X^*$, the mapping $\langle u \otimes f(u) \mid v \otimes w\rangle^{\otimes 2}$ is finitely supported. Hence the family $(u \otimes f(u))_{u \in X^*}$ is summable. One has

$$\begin{aligned}
\sum_{u \in X^*} u \otimes f(u) &= \sum_{v,w \in X^*} \sum_{u \in X^*} \langle u \otimes f(u) \mid v \otimes w\rangle^{\otimes 2}\, v \otimes w \\
&= \sum_{v,w \in X^*} \langle v \otimes f(v) \mid v \otimes w\rangle^{\otimes 2}\, v \otimes w \\
&= \sum_{v,w \in X^*} \langle f(v) \mid w\rangle\, v \otimes w.
\end{aligned} \tag{2.101}$$

ii) Let us prove that ρ is a representation

$$\begin{aligned}
&\rho(f)(⧢ \otimes \texttt{conc})\rho(g) = \sum_{u,v \in X^*} (u \otimes f(u)(⧢ \otimes \texttt{conc})(v \otimes g(v)) \\
&= \sum_{u,v \in X^*} (u ⧢ v \otimes (\texttt{conc}(f(u) \otimes g(v)))) \\
&= \sum_{u,v \in X^*} \sum_{w \in X^*} (\langle u ⧢ v \mid w\rangle\, w \otimes \texttt{conc}(f(u)g(v))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{w\in X^*} w\otimes\left(\sum_{u,v\in X^*}(\langle u \sqcup\!\sqcup\, v \mid w\rangle\, \mathtt{conc}(f(u)g(v))\right)\\
&= \sum_{w\in X^*} w\otimes\left(\sum_{u,v\in X^*}(\langle u \otimes v \mid \Delta(w)\rangle\, \mathtt{conc}(f(u)g(v))\right)\\
&= \sum_{w\in X^*} w\otimes(\mathtt{conc}(f\otimes g)\Delta(w)) = \sum_{w\in X^*} w\otimes(f*g(w)). \qquad (2.102)
\end{aligned}
$$

Moreover, this representation is faithful because $\rho(f)=0$ then $f=0$.
iii) It is easy to check. □

2.3.2 Theorem of Cartier-Quillen-Milnor-Moore (analytic form)

2.3.2.1 General properties of bialgebras

From now on, we suppose that A is a unitary commutative $\mathbb{Q}$-algebra (i.e. $\mathbb{Q}\subset A$). The aim of CQMM theorem is to provide necessary and sufficient conditions for a bialgebra $\mathcal{B}$ to be an enveloping algebra, we will discuss this condition in detail in the sequel.

Let $(\mathcal{B},\mu,1_{\mathcal{B}},\Delta,\epsilon)$ be a (general) A-bialgebra. From Theorem 2.1.3 in [Abe80], one can always consider the Lie algebra of primitive elements $\mathrm{Prim}(\mathcal{B})$. Let $\mathcal{U}(\mathrm{Prim}(\mathcal{B}))$ be the universal enveloping A-algebra of $\mathrm{Prim}(\mathcal{B})$. By universal property, there exists an algebra morphism $i_{\mathcal{B},\mathcal{U}}$ from $\mathcal{U}(\mathrm{Prim}(\mathcal{B}))$ to $\mathcal{B}$ which fix all primitive elements of $\mathcal{B}$.

$$i_{\mathcal{B},\mathcal{U}}:\mathcal{U}(\mathrm{Prim}(\mathcal{B}))\to\mathcal{B}\ .$$

Then, $\mathcal{A}=i_{\mathcal{B},\mathcal{U}}(\mathcal{U}(\mathrm{Prim}(\mathcal{B})))$ is the subalgebra generated by the primitive elements.
Let $i_{\mathcal{A},P}$ be a natural map from $\mathrm{Prim}(\mathcal{B})$ to $\mathcal{A}$.
Let $i_{\mathcal{B},P}$ be a canonical map from $\mathrm{Prim}(\mathcal{B})$ to $\mathcal{B}$. One then has a factorization $i_{\mathcal{B},P}=i_{\mathcal{A},P}\circ i_{\mathcal{B},\mathcal{A}}$.
Let $i_{\mathcal{U},P}$ be a natural map from a Lie algebra $\mathrm{Prim}(\mathcal{B})$ into its enveloping algebra $\mathcal{U}(\mathrm{Prim}(\mathcal{B}))$.
As the image of $i_{\mathcal{B},\mathcal{U}}$ is a subspace of $\mathcal{A}$, there is a unique factorization $i_{\mathcal{B},\mathcal{U}}=i_{\mathcal{B},\mathcal{A}}\circ i_{\mathcal{A},\mathcal{U}}$.

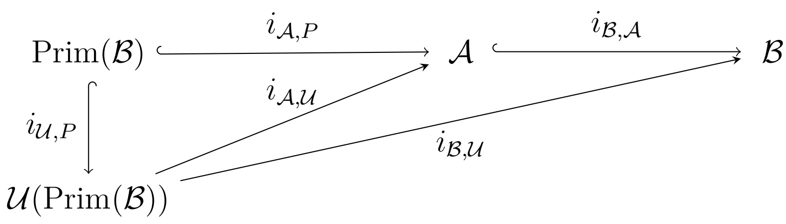


Figure 2.4: The sub-algebra $\mathcal{A}$ generated by primitive elements.

Remark 2.3.7. *The mapping $i_{\mathcal{B},\mathcal{A}}$ is into but $i_{\mathcal{B},\mathcal{A}}\otimes i_{\mathcal{B},\mathcal{A}}$ may not be so.*
This is the case for $\mathcal{B}=(\mathbb{Q}[\epsilon][x],.,1_{\mathbb{Q}[\epsilon][x]},\Delta,c)$ where $(\mathbb{Q}[\epsilon][x],.,1_{\mathcal{B}})$ is the usual polynomial algebra with coefficients in the algebra of dual numbers $\mathbb{Q}[\epsilon]$ (with $\epsilon^2=0$) and

$$\Delta(x)=x\otimes 1+1\otimes x+\epsilon x\otimes x,\ \ c(x)=0$$

(see details and proofs below, in section 2.3.3).

In general, one has (only) $\Delta(\mathcal{A}) \subset Im(i_{\mathcal{B},\mathcal{A}} \otimes i_{\mathcal{B},\mathcal{A}})$, this can be simply seen from the following combinatorial argument.
For any list of primitive elements $L = [g_1, g_2, \cdots, g_n]$ and $I = \{i_1 < i_2 < .. < i_k\} \subset \{1, 2, .., n\}$, put $L[I] = g_{i_1} g_{i_1} \cdots g_{i_k}$, the product of the sublist.
One has

$$\Delta(g_1 g_2 \cdots g_n) = \Delta(L[1, 2, .., n]) = \sum_{I+J=\{1,2,..,n\}} L[I] \otimes L[J] \ . \tag{2.103}$$

From (2.103) one gets also that $i_{\mathcal{B},\mathcal{U}}$ is a morphism of bialgebras. If for any reason, there exists a lifting[10] of $\Delta \circ i_{\mathcal{B},\mathcal{A}}$ as a comultiplication of $\mathcal{A}$, then $i_{\mathcal{B},\mathcal{U}}$ is into (see the statement and the proof below). Formula (2.103) proves that we have the maps (save the – hypothetical – dotted one) in Figure 2.5.
$G \subset \mathrm{Prim}(\mathcal{B})$ is any generating set of the AAU $\mathcal{A}$. Let $P \in A\langle G\rangle$, one has

$$s_G(P) = \sum_{w \in G^*} \langle P \mid w\rangle s_G(w). \tag{2.104}$$

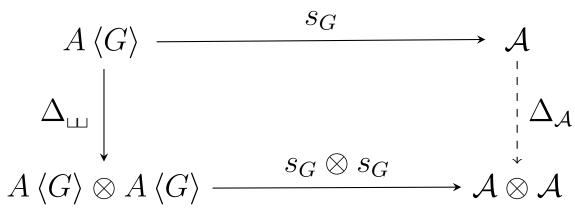


Figure 2.5: The unique lifting $\Delta_{\mathcal{A}}$ (when it exists).

We emphasize the fact that, in the diagram above, G must be understood set-theoretically (i.e. with no relation between the elements[11]).
In fact, one has the following proposition

Proposition 2.3.8. *Let $\mathcal{B}$ be a bialgebra over a (commutative) $\mathbb{Q}$-algebra A, the notations being those of figures 2.4 and 2.5, then the following statements are equivalent*

i) For a generating set $G \subset \mathrm{Prim}(\mathcal{B})$, $\ker(s_G) \subset \ker((s_G \otimes s_G) \circ \Delta_{\sqcup\sqcup})$.

ii) For any generating set $G \subset \mathrm{Prim}(\mathcal{B})$, $\ker(s_G) \subset \ker((s_G \otimes s_G) \circ \Delta_{\sqcup\sqcup})$.

iii) $i_{\mathcal{B},\mathcal{U}}$ is into.

Proof. *i)* $\Longrightarrow$ *iii)* In order to prove this, we need to construct the arrows σ, τ which are a decomposition of a section[12] of $i_{\mathcal{B},\mathcal{U}}$.

[10] This means that a true comultiplication $\mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ such that $\Delta \circ i_{\mathcal{B},\mathcal{A}} = (i_{\mathcal{B},\mathcal{A}} \otimes i_{\mathcal{B},\mathcal{A}}) \circ \Delta_{\mathcal{A}}$.
[11] We will see, below and in paragraph 2.3.3 how it is crucial to consider that $[\lambda x]$ and $\lambda[x]$ are not necessarily equal, when $\lambda x \in G$ (for clarity, $[y] \in A\langle G\rangle$ is the image of $y \in G$).
[12] Let s be a surjective map $X \longrightarrow Y$. σ is a section of s iff $\sigma \circ s = Id_Y$.

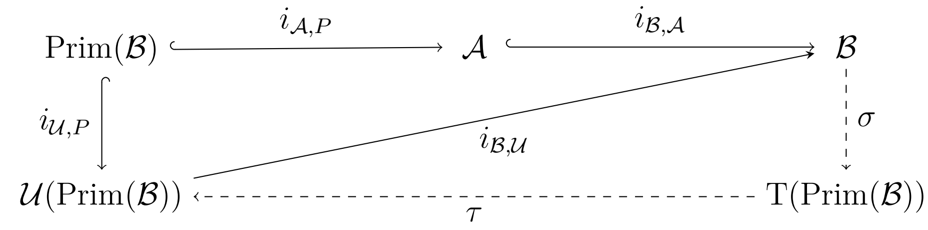


Figure 2.6: The sub-bialgebra $\mathcal{A}$ generated by primitive elements.

Let us remark that, when $\mathrm{Prim}(\mathcal{B})$ is free as an A-module, the proof of this fact is a consequence of the PBW theorem[13]. But, here, we will construct the section in the general case using projectors which are now classical for the free case but which still can be computed analytically [Reu93] as they lie in $\mathbb{Q}[[X]]$ and still converge in $\mathcal{A}$.
(Injectivity of $i_{\mathcal{B},\mathcal{U}}$, construction of the section $\tau \circ \sigma$)
As $\mathcal{A}$ is the subalgebra of $\mathcal{B}$ generated by $\mathrm{Prim}(\mathcal{B})$, one has $Im(i_{\mathcal{B},\mathcal{U}}) = \mathcal{A}$.
Let I be the identity from $\mathcal{B}$ to $\mathcal{B}$. One has $I = I^+ + e$, where e is a unit of convolution product on $\mathrm{Hom}(\mathcal{B}, \mathcal{B})$.
Remark that all series $\sum_{n\geq 0} a_n (I^+)^{*n}$ are summable on $\mathcal{A}$ (not in general on $\mathcal{B}$ for example in case $\mathcal{B}$ contains non-trivial group-like elements).

We define

$$\pi_{1,\mathcal{A}} := \log_*(I) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} (I^+)^{*n} \tag{2.105}$$

and remark that, in view of Proposition (2.3.6), in the case when $\mathcal{B} = A\langle X\rangle$ one has $\mathcal{A} = \mathcal{B}$ and, with $S(X) = \log(1 + X)$

$$\begin{aligned}&\sum_{w\in X^*} w \otimes \pi_{1,\mathcal{A}}(w) = \rho(\log(I)) = \rho(S(I^+)) = S(\rho(I^+)) = \\ &S(\sum_{\substack{w\in X^* \\ w\neq 1_{X^*}}} w \otimes w) = S(\mathcal{D}_X - 1_{X^*} \otimes 1_{X^*}) = \log(\mathcal{D}_X).\end{aligned} \tag{2.106}$$

We first prove that $\pi_{1,\mathcal{A}}$ is a projector $\mathcal{A} \to \mathrm{Prim}(\mathcal{B})$. The key point is that $\Delta_{\mathcal{A}}$ (the restriction of the comultiplication to $\mathcal{A}$) is a morphism of bialgebras [14] $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. We first prove that $\Delta_{\mathcal{A}}$ "commutes" with the convolution. This is a consequence of the following property

Lemma 2.3.9. *Let $f_i \in \mathrm{End}(\mathcal{B}_i)$, be such that $\varphi f_1 = f_2\varphi$.*
i) Then, if $P \in A[z]$, one has

$$\varphi P(f_1) = P(f_2)\varphi \ . \tag{2.107}$$

ii) If the series $\sum_{n\geq 0} (I^+_{(i)})^{*n}$, $i = 1, 2$ are summable and, if $f_1(1) = 0$ (which implies $f_2(1) = 0$) and $S \in A[[z]]$, the families $(\langle S \mid X^n\rangle\, f_i^{*n})_{n\in\mathbb{N}}$ are summable, we denote by $S(f_i)$ their

[13]See [Bou06c] Ch2 §1 n$^{\mathrm{o}}$ 6 th 1 for a field of characteristic zero and §1 Ex. 10 for the free case (over a ring A with $\mathbb{Q} \subset A$).
[14]In fact it is the case for any cocommutative bialgebra, be it generated by its primitive elements or not.

$$\begin{array}{ccc} \mathcal{B}_1 & \xrightarrow{\ \varphi\ } & \mathcal{B}_2 \\ f_1 \downarrow & & \downarrow f_2 \\ \mathcal{B}_1 & \xrightarrow[\ \varphi\ ]{} & \mathcal{B}_2 \end{array}$$

Figure 2.7: Intertwining with a morphism of bialgebras (the functions of f_i below will be computed with the respective convolution products).

sums (note that this definition is coherent with the previous ones when S is a polynomial). One has, for the convolution product,

$$\varphi S(f_1) = S(f_2)\varphi \ . \tag{2.108}$$

Proof. The only delicate part is (ii). First, one remarks that, if φ is a morphism of bialgebras, one has

$$(\varphi \otimes \varphi) \circ \Delta_1^+ = \Delta_2^+ \circ \varphi \tag{2.109}$$

then, the image by φ of an element of order less than N (i.e. such that $\Delta_1^{+(N)}(x) = 0$) is of order less than N. Let now S be a univariate series $S = \sum_{k=0}^{\infty} a_k z^k$. For every element x of order less than N and $f_i \in End(\mathcal{B}_i)$, $i = 1, 2$, one has

$$\begin{aligned} S(f)(x) &= \sum_{k=0}^{\infty} a_k f^{*k}(x) = \sum_{k=0}^{\infty} a_k \mu^{(k-1)} \circ f^{\otimes k} \circ \Delta^{(k-1)}(x) \\ &= \sum_{k=0}^{\infty} a_k \mu^{(k-1)} \circ (f^{\otimes k}) \circ (I^{+\otimes k}) \circ \Delta^{(k-1)}(x) \\ &= \sum_{k=0}^{N} a_k \mu^{(k-1)} \circ (f^{\otimes k}) \circ \Delta_+^{(k-1)}(x) \ . \end{aligned} \tag{2.110}$$

Since the sum is finite, we use (i) to show that $\varphi \circ S(f_1) = S(f_2) \circ \varphi$. □

Thanks to Lemma 2.3.9, we can now prove that $\pi_{1,\mathcal{A}}$ is a projector $\mathcal{A} \to \mathrm{Prim}(\mathcal{B})$.

In case $\mathcal{B}$ is cocommutative, the comultiplication Δ is a morphism of bialgebras. Using Lemma 2.3.9 (ii), one has

$$\Delta \circ \log_*(I) = \log_*(I \otimes I) \circ \Delta. \tag{2.111}$$

But

$$\begin{aligned} \log_*(I \otimes I) &= \log_*((I \otimes e) * (e \otimes I)) \\ &= \log_*(I \otimes e) + \log_*(e \otimes I) \\ &= \log_*(I) \otimes e + e \otimes \log_*(I). \end{aligned} \tag{2.112}$$

Then

$$\Delta(\log_*(I)) = (\log_*(I) \otimes e + e \otimes \log_*(I)) \circ \Delta \tag{2.113}$$

which implies that $\log_*(I)(\mathcal{B}) \subset \mathrm{Prim}(\mathcal{B})$. To finish the proof that π_1 is a projector onto $\mathrm{Prim}(\mathcal{B})$, it suffices to remark that, for $x \in \mathrm{Prim}(\mathcal{B})$ and $n \geq 2$, $(\mathrm{I}^+)^{*n}(x) = 0$ then

$$\log_*(I)(x) = \mathrm{I}^+(x) = x \ . \tag{2.114}$$

Now, we consider

$$I_{\mathcal{A}} = \exp_*(\log_*(I_{\mathcal{A}})) = \sum_{n\geq 0} \frac{1}{n!}\pi_{1,\mathcal{A}}^{*n} \ , \tag{2.115}$$

where $\pi_{1,\mathcal{A}} = \log_*(I_{\mathcal{A}})$.
Let us prove that the summands of (2.115) form a resolution of unity.
Note that, in any $\mathbb{Q}-$algebra, for $(P_i)_{i\in I}$ being a family of elements, the space generated by $\{P_i^n\}_{i\in I}$ is exactly the space generated by the symmetrized products.
First, one defines $\mathcal{A}_{[n]}$ as the linear span of the powers $\{P^n\}_{P\in \mathrm{Prim}(\mathcal{B})}$ or, equivalently, of the symmetrized products

$$\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} P_{\sigma(1)}P_{\sigma(2)}\cdots P_{\sigma(n)} \ . \tag{2.116}$$

It is obvious that $Im((\pi_{1,\mathcal{A}})^{*n}) \subset \mathcal{A}_{[n]}$ (since $\pi_{1,\mathcal{A}}$ is a projector onto $Prim(\mathcal{B})$). We remark that

$$\pi_{1,\mathcal{A}}^{*n} = \mu_{\mathcal{B}}^{(n-1)}\pi_{1,\mathcal{A}}^{\otimes n}\Delta^{(n-1)} = \mu_{\mathcal{B}}^{(n-1)}\pi_{1,\mathcal{A}}^{\otimes n}I^{+\otimes n}\Delta^{(n-1)} = \mu_{\mathcal{B}}^{(n-1)}\pi_{1,\mathcal{A}}^{\otimes n}\Delta_+^{(n-1)} \tag{2.117}$$

as $\pi_{1,\mathcal{A}}I^+ = \pi_{1,\mathcal{A}}$. Now, let $P \in \mathrm{Prim}(\mathcal{A})$. We compute $\pi_{1,\mathcal{A}}^{*n}(P^m)$. Indeed, if $m < n$, one has

$$\pi_{1,\mathcal{A}}^{*n}(P^m) = \mu_{\mathcal{B}}^{n-1}\Delta_+^{n-1}(P^m) = 0 \ . \tag{2.118}$$

If $n = m$, one has, from Equation (2.103)

$$\Delta_+^{n-1}(P^n) = n!P^{\otimes n} \tag{2.119}$$

and hence $\pi_{1,\mathcal{A}}^{*n}$ is the identity on $\mathcal{A}_{[n]}$. If $m > n$, the nullity of $\pi_{1,\mathcal{A}}^{*n}(P^m)$ is a consequence of the following lemma.

Lemma 2.3.10. *Let $\mathcal{B}$ be a bialgebra and P a primitive element of $\mathcal{B}$. Then*
i) The series $\log_*(I)$ is summable on each power P^m.
ii) $\log_*(I)(P^m) = 0$ for $m > 2$.

Proof. i) As $\Delta_+^{*N}(P^m) = 0$ for $N > m$, one has $I^{+*N}(P^m) = 0$ for these values.
ii) Let a be a letter, the morphism of AAU $\varphi_P : A[a] \to \mathcal{B}$, defined by

$$\varphi_P(a) = P \tag{2.120}$$

is, in fact, a morphism of bialgebras.
One denotes that $\pi_{1,[A[a]]} = \log(I_{A[a]})$.
One checks easily that $\pi_{1,[A[a]]}(a^m) = 0$ for $m > 2$ which is a consequence of the general equality (see (2.106))

$$\sum_{w\in X^*}(w\otimes \pi_1(w)) = \log(\sum_{w\in X^*} w\otimes w) \tag{2.121}$$

because, for $Y = \{a\}$ (and then $A\langle X\rangle = A[a]$) one has

$$\log(\sum_{w\in X^*} w\otimes w) = \log(\sum_{n\geq 0} a^n\otimes a^n) \quad =$$

$$\begin{array}{ccc} A[a] & \xrightarrow{\varphi_P} & \mathcal{B} \\ {\scriptstyle I^+_{A[a]}}\downarrow & & \downarrow{\scriptstyle I^+_{\mathcal{B}}} \\ A[a] & \xrightarrow[\varphi_P]{} & \mathcal{B} \end{array}$$

Figure 2.8: Intertwining with one primitive element.

$$\log(\sum_{n\geq 0}\frac{1}{n!}(a\otimes a)^{(\sqcup\!\sqcup\,\otimes conc)\,n}) = \log(\exp(a\otimes a)) \quad = \quad a\otimes a. \tag{2.122}$$

This proves that $\pi_{1,\mathcal{A}}^{*n}(\mathcal{A}_{[m]}) = 0$ for $m \neq n$ and hence the summands of the sum

$$I_{\mathcal{A}} = \exp_*(\log_*(I_{\mathcal{A}})) = \sum_{n\geq 0}\frac{1}{n!}\pi_{1,\mathcal{A}}^{*n} \ . \tag{2.123}$$

are pairwise orthogonal projectors with $Im(\pi_{1,\mathcal{A}}^{*n}) = \mathcal{A}_{[n]}$ and then

$$\mathcal{A} = \oplus_{n\geq 0}\, \mathcal{A}_{[n]} \ . \tag{2.124}$$

This decomposition enables to construct σ by

$$\sigma(P^n) = \frac{1}{n!}\Delta_+^{(n-1)}(P^n) \in T_n(\mathrm{Prim}(\mathcal{B})) \tag{2.125}$$

for $n \geq 1$ and, one sets $\sigma(1_{\mathcal{B}}) = 1_{T(\mathrm{Prim}(\mathcal{B}))}$.
It is easy to check that $i_{\mathcal{B},\mathcal{U}} \circ \tau \circ \sigma = I_{\mathcal{A}}$ as $\mathcal{A}$ is (linearly) generated by the powers $(P^m)_{P\in\mathrm{Prim}(\mathcal{B}),m\geq 0}$. □

End of the proof of proposition 2.3.8
iii) $\Longrightarrow$ *ii)* If $i_{\mathcal{B},\mathcal{U}}$ is into, then $i_{\mathcal{A},\mathcal{U}}$ is one-to-one and one gets a comultiplication

$$\Delta_{\mathcal{A}} \ : \ \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$$

such that, for any list of primitive elements $L = [g_1, g_2, \cdots g_n]$ (the denotations are the same as previously)

$$\Delta_{\mathcal{A}}(g_1 g_2\cdots g_n) = \Delta(L[\{1,2,..,n\}]) = \sum_{I+J=\{1,2,..,n\}} L[I]\otimes_{\mathcal{A}} L[J] \tag{2.126}$$

but, this time, the tensor product $\otimes_{\mathcal{A}}$ is understood as being in $\mathcal{A}\otimes\mathcal{A}$. This guarantees that the diagram Fig. 2.5 commutes for any G.
ii) $\Longrightarrow$ *i)* Obvious. □

2.3.2.2 The theorem from the point of view of summability

From now on, the morphism $i_{\mathcal{B},\mathcal{U}}$ is supposed into.
The bialgebra $\mathcal{B}$ being supposed cocommutative, we discuss the equivalent conditions under which we are in the presence of an enveloping algebra i.e.

$$\mathcal{B} \cong_{A-bialg} \mathcal{U}(\mathrm{Prim}(\mathcal{B})) \tag{2.127}$$

from the point of view of the convergence of the series $\log_*(I)$[15]. These conditions are known as the theorem of CQMM.

Theorem 2.3.11. *[Bou06c] Let $\mathcal{B}$ be a A-cocommutative bialgebra (A is a $\mathbb{Q}$-AAU) and $\mathcal{A}$, as above, the subalgebra generated by* $\mathrm{Prim}(\mathcal{B})$*. Then, the following conditions are equivalent.*

i) $\mathcal{B}$ admits an increasing filtration

$$\mathcal{B}_0 = A.1_{\mathcal{B}} \subset \mathcal{B}_1 \subset \cdots \subset \mathcal{B}_n \subset \mathcal{B}_{n+1} \cdots$$

compatible with the structures of algebra (i.e. for all $p, q \in \mathbb{N}$, one has $\mathcal{B}_p\mathcal{B}_q \subset \mathcal{B}_{p+q}$) and coalgebra :

$$\forall n \in \mathbb{N}, \qquad \Delta(\mathcal{B}_n) \subset \sum_{p+q=n} \mathcal{B}_p \otimes \mathcal{B}_q.$$

ii) $((\mathrm{Id}^+)^{*n})_{n\in\mathbb{N}}$ is summable in $\mathrm{End}(\mathcal{B})$.

iii) $\mathcal{B} = \mathcal{A}$.

Proof. We prove

$$\text{(ii)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(i)} \Longrightarrow \text{(ii)}. \tag{2.129}$$

(ii) $\Longrightarrow$ (iii)
The image of $i_{\mathcal{B},\mathcal{U}}$ it is the subalgebra generated by the primitive elements. Let us prove that, when $((\mathrm{Id}^+)^{*n})_{n\in\mathbb{N}}$ is summable, one has $Im(i_{\mathcal{B},\mathcal{U}}) = \mathcal{B}$. The series $\log(1+z)$ is without constant term so, in virtue of (2.89) and the summability of $((\mathrm{Id}^+)^{*n})_{n\in\mathbb{N}}$, one has

$$\exp(\log(e + \mathrm{Id}^+)) = \exp(\log(1+z))(Id^+) = 1_{\mathrm{End}(\mathcal{B})} + Id^+ = e + Id^+ = I. \tag{2.130}$$

Set $\pi_1 = \log(e + \mathrm{Id}^+)$.
To end this part, let us compute, for $x \in \mathcal{B}$

$$x = \exp(\pi_1)(x) = (\sum_{n\geq 0} \frac{1}{n!}\pi_1^{*n})(x) = (\sum_{n=0}^{N} \frac{1}{n!}\mu^{(n-1)}\pi_1^{\otimes n})\Delta^{(n-1)}(x) \tag{2.131}$$

[15] In a A-bialgebra, one can always consider the series of endomorphisms

$$\sum_{n\geq 1} \frac{(-1)^{n-1}}{n}(I^+)^{*n} . \tag{2.128}$$

The family $(\frac{(-1)^{n-1}}{n}(I^+)^{*n})_{n\geq 0}$ is summable iff $((I^+)^{*n})_{n\geq 0}$ is (use (2.81)).

where N is the first order for which $\Delta^{+(n-1)}(x) = 0$ (as $\pi_1 \circ \mathrm{Id}^+ = \pi_1$). This proves that $\mathcal{B}$ is generated by its primitive elements.
The implications (iii) $\Longrightarrow$ (i) is obvious.
(i) $\Longrightarrow$ (ii)
As $\mathcal{B}$ is a graded bialgebra, one then has that Δ_+ is a locally nilpotent. It implies (ii). □

Remark 2.3.12. *i) The equivalence (i) $\Longleftrightarrow$ (iii) is the classical CQMM theorem (see [Bou06c]). The equivalence with (ii) could be called the "Convolutional CQMM theorem". The combinatorial aspects of this last one will be the subject of a forthcoming paper.*
ii) When $\mathrm{Prim}(\mathcal{B})$ *is free, we have* $\mathcal{B} \cong_{k-bialg} \mathcal{U}(\mathrm{Prim}(\mathcal{B}))$ *and* $\mathcal{B}$ *is an enveloping algebra.*
iii) The (counter) example is the following with $A = k[x]$ *(k is a field of characteristic zero). Let* Y *be an alphabet and* $A\langle Y\rangle$ *be the usual free algebra (the space of non-commutative polynomials over* Y*) and* ϵ*, the "constant term" linear form. Let conc be the concatenation and* Δ *the dual law of the shuffle product (supra).*
Then the bialgebra $(A\langle Y\rangle, conc, 1_{Y^*}, \Delta, \epsilon)$ *is a Hopf algebra (it is the enveloping algebra of the Lie polynomials). Let* $A_+\langle Y\rangle = \ker(\epsilon)$ *and, for* $N \geq 2$ $J_N = x^N.A_+\langle Y\rangle$ *then,* J_N *is a Hopf ideal and* $\mathrm{Prim}(A\langle Y\rangle/(J_N))$ *is never free (no basis).*

2.3.3 Examples and counterexamples

2.3.3.1 Lamperti product

Let Y be a totally ordered alphabet and A, a unitary ring. The free monoid and free algebra, over Y, are denoted respectively by Y^* and $A\langle Y\rangle$. The neutral of Y^* (and then of $A\langle Y\rangle$) is denoted by 1_{Y^*}.

Definition 2.3.13 (The Lamperti product ([Lam65, Fli74])). *Let* f *and* g *be words. The Lamperti product of* f *and* g *is given by*

$$f\mathcal{L}_{\alpha,\beta,\gamma}\, g := \sum \alpha^{n_1}\beta^{n_2}\gamma^{n_3} f_1g_1h_1f_2g_2h_2\dots f_kg_kh_k, \tag{2.132}$$

where $f = f_1h_1f_2h_2\dots f_kh_k$*,* $g = g_1h_1g_2h_2\dots g_kh_k$*,* $n_1 = \mid f_1\dots f_k\mid$*,* $n_2 = \mid g_1\dots g_k\mid$*,* $n_3 = \mid h_1\dots h_k\mid$*,* f_i*,* g_i*,* h_i *are factors.*

Let $w = f_1g_1h_1f_2g_2h_2\dots f_kg_kh_k$, then, one can rephrase (2.132) in terms of subwords as

$$f\mathcal{L}_{\alpha,\beta,\gamma}\, g = \sum_{\substack{w\in Y^* \\ I+J+K=[1\dots|w|] \\ w[I+K]=f \\ w[J+K]=g}} \alpha^{|I|}\beta^{|J|}\gamma^{|K|}w. \tag{2.133}$$

Example 2.3.14. *Let* a*,* b *and* c *be three distinct letters in* Y*. One has*

$$a\mathcal{L}_{\alpha,\beta,\gamma}\, b \quad = \quad \alpha\beta(ab+ba). \tag{2.134}$$

Moreover, one has

$$ab\mathcal{L}_{\alpha,\beta,\gamma}\, c \quad = \quad \alpha^2\beta(abc+acb+cab), \tag{2.135}$$

$$\tag{2.136}$$

Remark 2.3.15. *The Lamperti product $\mathcal{L}_{\alpha,\beta,\gamma}$ is not associative in general. Indeed, one can check*

$$\begin{aligned}(a\mathcal{L}_{\alpha,\beta,\gamma}\ b)\mathcal{L}_{\alpha,\beta,\gamma}\ c &= \alpha\beta(ab+ba)\mathcal{L}_{\alpha,\beta,\gamma}\ c\\ &= \alpha\beta[\alpha^2\beta(abc+acb+cab)+\alpha^2\beta(bac+bca+cba)]\\ &= \alpha^3\beta^2(abc+acb+cab+bac+bca+cba),\end{aligned} \tag{2.137}$$

whereas

$$\begin{aligned}a\mathcal{L}_{\alpha,\beta,\gamma}\ (b\mathcal{L}_{\alpha,\beta,\gamma}\ c) &= a\mathcal{L}_{\alpha,\beta,\gamma}\ [\alpha\beta(bc+cb)]\\ &= \alpha\beta[\alpha\beta^2(abc+bac+bca)+\alpha\beta^2(acb+cab+cba)]\\ &= \alpha^2\beta^3(abc+bac+bca+acb+cab+cba).\end{aligned} \tag{2.138}$$

Then, one can see that the Lamperti product is associative if $\alpha=\beta\in\{0,1\}$. This condition is sufficient [Luq99, DFLL01].
Let us give some cases where the Lamperti product is associative (see Table 2.3).

α	β	γ	Nature of $\mathcal{L}_{\alpha,\beta,\gamma}$
0	0	0	0
0	0	1	Hadamard product
1	1	0	Shuffle product
1	1	1	Infiltration
1	1	q	q-infiltration
0	0	q	q-Hadamard

Table 2.3: The cases of coefficients α, β and γ in which the Lamperti product is associative.

From (2.133), one has

$$\langle f\mathcal{L}_{\alpha,\beta,\gamma}\ g \mid w\rangle = \sum_{I+J+K=[1\ldots|w|]} \alpha^{|I|}\beta^{|J|}\gamma^{|K|}\,\langle f \mid w[I+K]\rangle\,\langle g \mid w[J+K]\rangle\,. \tag{2.139}$$

So, the coproduct dual of the Lamperti product (2.132) can be given as follows.

$$\sum_{f,g}\langle f\mathcal{L}_{\alpha,\beta,\gamma}\ g \mid w\rangle\, f\otimes g = \sum_{I+J+K=[1\ldots|w|]} \alpha^{|I|}\beta^{|J|}\gamma^{|K|} w[I+K]\otimes w[J+K] := \Delta_{\mathcal{L}}(w). \tag{2.140}$$

We will prove that it is sufficient to know it on the letters.

Proposition 2.3.16. *$\Delta_{\mathcal{L}}$ is morphism $A\langle X\rangle \to A\langle X\rangle\otimes A\langle X\rangle$.*

Proof. Let us start with a letter x of Y^*. One has

$$\Delta_{\mathcal{L}}(x) = \underbrace{\alpha x\otimes 1_{Y^*}}_{I=\{1\},J=K=\emptyset} + \underbrace{\beta 1_{Y^*}\otimes x}_{J=\{1\},I=K=\emptyset} + \underbrace{\gamma x\otimes x}_{K=\{1\},I=J=\emptyset}\ . \tag{2.141}$$

Let w be a word in Y^*. One has

$$\begin{aligned}
\Delta_{\mathcal{L}}(w) &= \sum_{I+J+K=[1\ldots|w|]} \alpha^{|I|}\beta^{|J|}\gamma^{|K|} w[I+K]\otimes w[J+K]\\
&= \sum_{\substack{I+J+K=[1\ldots|w|]\\ 1\in I}} \alpha^{|I|}\beta^{|J|}\gamma^{|K|} w[I+K]\otimes w[J+K]\\
&\quad+ \sum_{\substack{I+J+K=[1\ldots|w|]\\ 1\in J}} \alpha^{|I|}\beta^{|J|}\gamma^{|K|} w[I+K]\otimes w[J+K]\\
&\quad+ \sum_{\substack{I+J+K=[1\ldots|w|]\\ 1\in K}} \alpha^{|I|}\beta^{|J|}\gamma^{|K|} w[I+K]\otimes w[J+K]\\
&= \sum_{I'+J+K=[2\ldots|w|]} \alpha^{|I'|+1}\beta^{|J|}\gamma^{|K|} w[1]w[I'+K]\otimes w[J+K]\\
&\quad+ \sum_{I+J'+K=[2\ldots|w|]} \alpha^{|I|}\beta^{|J'|+1}\gamma^{|K|} w[I+K]\otimes w[1]w[J'+K]\\
&\quad+ \sum_{I+J+K'=[2\ldots|w|]} \alpha^{|I|}\beta^{|J|}\gamma^{|K|} w[1]w[I+K']\otimes w[1]w[J+K']\\
&= \alpha(w[1]\otimes 1)\mathtt{conc}^{\otimes 2}\sum_{I'+J+K=[2\ldots|w|]} \alpha^{|I'|}\beta^{|J|}\gamma^{|K|} w[I'+K]\otimes w[J+K]\\
&\quad+\beta(1\otimes w[1])\mathtt{conc}^{\otimes 2}\sum_{I+J'+K=[2\ldots|w|]} \alpha^{|I|}\beta^{|J'|}\gamma^{|K|} w[I+K]\otimes w[J'+K]\\
&\quad+\gamma(w[1]\otimes w[1])\mathtt{conc}^{\otimes 2}\sum_{I+J+K'=[2\ldots|w|]} \alpha^{|I|}\beta^{|J|}\gamma^{|K'|} w[I'+K']\otimes w[J+K']\\
&= [\alpha(w[1]\otimes 1_{Y^*})+\beta(1_{Y^*}\otimes w[1])+\gamma(w[1]\otimes w[1])]\,\mathtt{conc}^{\otimes 2}\Delta_{\mathcal{L}}(w[2\ldots|w|])\\
&= \Delta_{\mathcal{L}}(w[1])\mathtt{conc}^{\otimes 2}\Delta_{\mathcal{L}}(w[2\ldots|w|]). \qquad (2.142)
\end{aligned}$$

From this, one can get the conclusion. □

From this proof, one can get the recursion of this product (letter-by-letter) as follows.

$$\Delta_{\mathcal{L}}(w) = \begin{cases} \alpha w\otimes 1_{Y^*} + \beta 1_{Y^*}\otimes w + \gamma w\otimes w & \text{if } w \text{ is a letter;}\\ \Delta_{\mathcal{L}}(w[1])\mathtt{conc}^{\otimes 2}\Delta_{\mathcal{L}}(u) & \text{if } w\in Y^* \text{ and } w = w[1]u. \end{cases} \qquad (2.143)$$

Proposition 2.3.17. *The dual product of $\Delta_{\mathcal{L}}$ is $\mathcal{L}_{\alpha,\beta,\gamma}$. This product is given by the recursive formula. Let u be a word in Y^*. Let u and v be two words in Y^*, a and b two letters in Y, one has*

$$\begin{cases} u\mathcal{L}_{\alpha,\beta,\gamma}1_{Y^*} = 1_{Y^*}\mathcal{L}_{\alpha,\beta,\gamma}u = u,\\ au\mathcal{L}_{\alpha,\beta,\gamma}bv = \alpha a(u\mathcal{L}_{\alpha,\beta,\gamma}bv) + \beta b(au\mathcal{L}_{\alpha,\beta,\gamma}v) + \gamma\delta_{a,b}a(u\mathcal{L}_{\alpha,\beta,\gamma}v). \end{cases} \qquad (2.144)$$

Proof. One has

$$\begin{aligned}
au\mathcal{L}_{\alpha,\beta,\gamma}bv &= \sum_{w\in Y^*}\langle au\mathcal{L}_{\alpha,\beta,\gamma}bv \mid w\rangle w\\
&= \sum_{w\in Y^*}\langle au\otimes bv \mid \Delta_{\mathcal{L}}(w)\rangle w
\end{aligned}$$

$$
\begin{aligned}
&= \langle au \otimes bv \mid \Delta_{\mathcal{L}}(1_{Y^*})\rangle + \sum_{w \in Y^+} \langle au \otimes bv \mid \Delta_{\mathcal{L}}(w)\rangle w \\
&= \sum_{\substack{x \in Y \\ w' \in Y^*}} \langle au \otimes bv \mid \Delta_{\mathcal{L}}(xw')\rangle xw' \\
&= \sum_{\substack{x \in Y \\ w' \in Y^*}} \langle au \otimes bv \mid \Delta_{\mathcal{L}}(x)\Delta_{\mathcal{L}}(w')\rangle xw' \\
&= \sum_{\substack{x \in Y \\ w' \in Y^*}} \langle au \otimes bv \mid (\alpha x \otimes 1_{Y^*} + \beta 1_{Y*} \otimes x + \gamma x \otimes x)\Delta_{\mathcal{L}}(w')\rangle xw' \\
&= \sum_{\substack{x = a \\ w' \in Y^*}} \langle au \otimes bv \mid \alpha(a \otimes 1_{Y^*})\Delta_{\mathcal{L}}(w')\rangle aw' \\
&\quad + \sum_{\substack{x = b \\ w' \in Y^*}} \langle au \otimes bv \mid \beta(1_{Y^*} \otimes a)\Delta_{\mathcal{L}}(w')\rangle bw' \\
&\quad + \delta_{a,b} \sum_{\substack{x = a \\ w' \in Y^*}} \langle au \otimes bv \mid \gamma(a \otimes a)\Delta_{\mathcal{L}}(w')\rangle aw' \\
&= \alpha a(u \mathcal{L}_{\alpha,\beta,\gamma} bv) + \beta b(au \mathcal{L}_{\alpha,\beta,\gamma} v) + \gamma \delta_{a,b}(u \mathcal{L}_{\alpha,\beta,\gamma} v). \qquad (2.145)
\end{aligned}
$$

Thus, one gets the conclusion. □

From Proposition 2.3.17, one gets the recursive formula of the Lamperti product.

Remark 2.3.18. *As $\Delta_{\mathcal{L}}$ is a morphism, we can isolate the place "$|w|$" instead of the place "1". Doing this we get the equivalent recursion.*

$$
\begin{cases}
u \mathcal{L}_{\alpha,\beta,\gamma} 1_{Y^*} = 1_{Y^*} \mathcal{L}_{\alpha,\beta,\gamma} u = u, \\
ua \mathcal{L}_{\alpha,\beta,\gamma} vb = \alpha(u \mathcal{L}_{\alpha,\beta,\gamma} bv)a + \beta(au \mathcal{L}_{\alpha,\beta,\gamma} v)b + \gamma \delta_{a,b}(u \mathcal{L}_{\alpha,\beta,\gamma} v)a,
\end{cases} \qquad (2.146)
$$

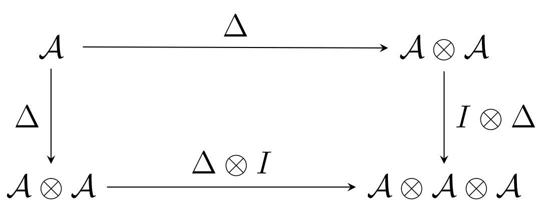
with $\mathcal{L}_{\alpha,\beta,\gamma}$ as it appears in [Lot97] for infiltration ($\alpha = \beta = \gamma = 1$) and in [Fli74] for the shuffle[16] *($\alpha = \beta = 1$, $\gamma = 0$).*

Let us discuss about the coassociativity of the Lamperti coproduct.

Lemma 2.3.19. *Let $\mathcal{A}$ be an algebra and $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a morphism. Let G be a generating subset of $\mathcal{A}$. G satisfies*

$$(\forall g \in G)((\Delta \otimes Id) \circ \Delta(g) = (Id \otimes \Delta) \circ \Delta(g)), \qquad (2.147)$$

iff Δ is coassociative.



[16] This is called Hurwitz product.

From Proposition 2.3.16 and Lemma 2.3.19, it is sufficient to check that $\Delta_{\mathcal{L}}$ is coassociative on letters. Let x be a letter in Y. One has

$$\begin{aligned}(\Delta_{\mathcal{L}} \otimes I) \circ \Delta_{\mathcal{L}}(x) &= \alpha^2 x \otimes 1_{Y^*} \otimes 1_{Y^*} + \alpha\beta 1_{Y^*} \otimes x \otimes 1_{Y^*} + \alpha\gamma x \otimes x \otimes 1_{Y^*} \\ &+ \beta 1_{Y^*} \otimes 1_{Y^*} \otimes x + \alpha\gamma x \otimes 1_{Y^*} \otimes x + \beta\gamma 1_{Y^*} \otimes x \otimes x + \gamma^2 x \otimes x \otimes x. \end{aligned} \tag{2.148}$$

$$\begin{aligned}(I \otimes \Delta_{\mathcal{L}}) \circ \Delta_{\mathcal{L}}(x) &= \alpha x \otimes 1_{Y^*} \otimes 1_{Y^*} + \alpha\beta 1_{Y^*} \otimes x \otimes 1_{Y^*} + \beta^2 1_{Y^*} \otimes 1_{Y^*} \otimes x \\ &+ \beta\gamma 1_{Y^*} \otimes x \otimes x + \alpha\gamma x \otimes x \otimes 1_{Y^*} + \beta\gamma x \otimes 1_{Y^*} \otimes x + \gamma^2 x \otimes x \otimes x. \end{aligned} \tag{2.149}$$

From Equation (2.148) and (2.149), if $\Delta_{\mathcal{L}}$ is coassociative on letter, then one gets

$$\begin{cases} \alpha^2 = \alpha, \\ \beta^2 = \beta, \\ \alpha\gamma = \beta\gamma. \end{cases} \tag{2.150}$$

Proposition 2.3.20. *$\Delta_{\mathcal{L}}$ is coassociative if and only if*

1) $\alpha = \beta \in \{0, 1\}$;

2) $\alpha \in \{0, 1\}$, $\beta = 1 - \alpha$ and $\gamma = 0$.

Proof. From (2.150), it implies that there are 4 cases for the coassociativity of $\Delta_{\mathcal{L}}$.

1) $\alpha = \beta = 0$, $\gamma = q$ arbitrary. The Lamperti product has the formula

$$\Delta_{\mathcal{L}_0}(x) = qx \otimes x. \tag{2.151}$$

This is the q-Hadamard product.

2) $\alpha = \beta = 1$, $\gamma = q$ arbitrary. The Lamperti product has the formula

$$\Delta_{\mathcal{L}_1}(x) = x \otimes 1 + 1 \otimes x + qx \otimes x. \tag{2.152}$$

This is the q-infiltration product.

3) $\alpha = 0$, $\beta = 1$, $\gamma = 0$. The Lamperti product has the formula

$$\Delta_{\mathcal{L}_{0,1,0}}(x) = 1 \otimes x. \tag{2.153}$$

4) $\alpha = 1$, $\beta = 0$, $\gamma = 0$. The Lamperti product has the formula

$$\Delta_{\mathcal{L}_{1,0,0}}(x) = x \otimes 1. \tag{2.154}$$

□

2.3.3.2 ϕ-deformation shuffle

Let $Y=\{y_i\}_{i\in I}$ be still a totally ordered alphabet and $A\langle Y\rangle$ be equipped with the ϕ-deformed stuffle defined by [EM12].

i) for any $w\in Y^*$, $1_{Y^*}\uplus_\phi w = w\uplus_\phi 1_{Y^*}=w$,

ii) for any $y_i,y_j\in Y$ and $u,v\in Y^*$,

$$y_iu\uplus_\phi y_jv \quad=\quad y_j(y_iu\uplus_\phi v)+y_i(u\uplus_\phi y_jv)+\phi(y_i,y_j)u\uplus_\phi v, \tag{2.155}$$

where ϕ is an arbitrary mapping

$$\phi:Y\times Y \quad\longrightarrow\quad AY,$$

where AY is the free A-module on Y.

Definition 2.3.21. *Let*

$$\phi:Y\times Y \quad\longrightarrow\quad AY$$

be defined by its structure constants

$$(y_i,y_j) \quad\longmapsto\quad \phi(y_i,y_j)=\sum_{k\in I}\gamma_{i,j}^k\,y_k.$$

Proposition 2.3.22. *The recursion (2.155) defines a unique mapping*

$$\uplus_\phi\ :Y^*\times Y^* \quad\longrightarrow\quad A\langle Y\rangle.$$

Proof. Let us denote $(Y^*\times Y^*)_{\le n}$ the set of words $(u,v)\in Y^*\times Y^*$ such that $|u|+|v|\le n$. We construct a sequence of mappings

$$\uplus_{\phi_{\le n}}:(Y^*\times Y^*)_{\le n}\longrightarrow A\langle Y\rangle.$$

which satisfy the recursion of (2.155). For $n=0$, we have only a pre-image and $\uplus_{\phi_{\le 0}}(1_{Y^*})=1_{Y^*}\otimes 1_{Y^*}$. Suppose $\uplus_{\phi_{\le n}}$ already constructed and let
$(u,v)\in(Y^*\times Y^*)_{\le n+1}\setminus(Y^*\times Y^*)_{\le n}$, i.e. $|u|+|v|=n+1$.
One has three cases : $u=1_{Y^*},\ v=1_{Y^*}$ and $(u,v)\in Y^+\times Y^+$. For the first two, one uses the initialization of the recursion, thus

$$\uplus_{\phi_{\le n+1}}(w,1_{Y^*})=\uplus_{\phi_{\le n+1}}(1_{Y^*},w)=w\ .$$

For the last case, write $u=y_iu',\ v=y_jv'$ and use, to get

$$\uplus_{\phi_{\le n+1}}(y_iu',y_jv')=y_i\uplus_{\phi_{\le n}}(u',y_jv')+y_j\uplus_{\phi_{\le n}}(y_iu',v')+y_{i+j}\uplus_{\phi_{\le n}}(u',v')$$

this proves the existence of the sequence $(\uplus_{\phi_{\le n}})_{n\ge 0}$. Every $\uplus_{\phi_{\le n+1}}$ extends the preceding so there is a mapping

$$\uplus_\phi:Y^*\times Y^*\longrightarrow A\langle Y\rangle.$$

which extends all the $\sqcup\!\!\!\sqcup_{\phi_{\leq n+1}}$ (the graph of which is the union of the graphs of the $\sqcup\!\!\!\sqcup_{\phi_{\leq n}}$). This proves the existence. For unicity, just remark that, if there were two mappings $\sqcup\!\!\!\sqcup_\phi, \sqcup\!\!\!\sqcup'_\phi$, the fact that they must fulfil the recursion (2.155) implies that $\sqcup\!\!\!\sqcup_\phi = \sqcup\!\!\!\sqcup'_\phi$. □

We still denote by ϕ and $\sqcup\!\!\!\sqcup_\phi$ the linear extension of ϕ and $\sqcup\!\!\!\sqcup_\phi$ to $AY \otimes AY$ and $A\langle Y\rangle \otimes A\langle Y\rangle$ respectively.

Then $\sqcup\!\!\!\sqcup_\phi$ is a law of algebra (with 1_{Y^*} as unit) on $A\langle Y\rangle$.

Lemma 2.3.23. *Let Δ be the morphism $A\langle Y\rangle \to A\langle\langle Y^* \otimes Y^*\rangle\rangle$ defined on the letters by*

$$\Delta(y_s) = y_s \otimes 1 + 1 \otimes y_s + \sum_{n,m\in I} \gamma^s_{n,m}\, y_n \otimes y_m \ . \tag{2.156}$$

Then

i) for all $w \in Y^+$ we have

$$\Delta(w) = w \otimes 1 + 1 \otimes w + \sum_{u,v\in Y^+} \langle \Delta(w) \mid u \otimes v\rangle\, u \otimes v \tag{2.157}$$

ii) for all $u, v, w \in Y^*$, one has

$$\langle u \sqcup\!\!\!\sqcup_\phi v \mid w\rangle = \langle u \otimes v \mid \Delta(w)\rangle^{\otimes 2} \ . \tag{2.158}$$

Proof.

i) By recurrence on $|w|$. If $w = y_s$ is of length one, it is obvious from the definition. If $w = y_s w'$, we have, from the fact that Δ is a morphism

$$\begin{aligned}\Delta(w) \quad = \quad & \Big(y_s \otimes 1 + 1 \otimes w + \sum_{i,j\in I} \gamma^s_{i,j} y_i \otimes y_j\Big) \\ & \Big(w' \otimes 1 + 1 \otimes w' + \sum_{u,v\in Y^+} \langle u \otimes v \mid \Delta(w')\rangle\Big)\end{aligned} \tag{2.159}$$

the development of which proves that $\Delta(w)$ is of the desired form.

ii) Let $S(u, v) := \sum_{w\in Y^*} \langle u \otimes v \mid \Delta(w)\rangle\, w$. It is easy to check (and left to the reader) that, for all $u \in Y^*$, $S(u, 1) = S(1, u) = u$. Let us now prove that, for all $y_i, y_j \in Y$ and $u, v \in Y^*$

$$S(y_i u, y_j v) = y_i S(u, y_j v) + y_j S(y_i u, v) + \phi(y_i, y_j) S(u, v) \ . \tag{2.160}$$

Indeed, noticing that $\Delta(1) = 1 \otimes 1$, one has

$$\begin{aligned} S(y_i u, y_j v) \quad &= \sum_{w\in Y^*} \langle y_i u \otimes y_j v \mid \Delta(w)\rangle w = \sum_{w\in Y^+} \langle y_i u \otimes y_j v \mid \Delta(w)\rangle w \\ &= \sum_{y_s\in Y,\ w'\in Y^*} \langle y_i u \otimes y_j v \mid \Delta(y_s w')\rangle\, y_s w' \\ &= \sum_{y_s\in Y,\ w'\in Y^*} \langle y_i u \otimes y_j v \mid \Big(y_s \otimes 1 + 1 \otimes y_s + \sum_{n,m\in I} \gamma^s_{n,m}\, y_n \otimes y_m\Big)\Delta(w')\rangle\, y_s w' \end{aligned}$$

$$\begin{aligned}
&= \sum_{y_s\in Y,\ w'\in Y^*} \langle y_iu\otimes y_jv(y_s\otimes 1)\Delta(w')\rangle\, y_sw' \\
&\quad + \sum_{y_s\in Y,\ w'\in Y^*} \langle y_iu\otimes y_jv \mid (1\otimes y_s)\Delta(w')\rangle\, y_sw' \\
&\quad + \sum_{y_s\in Y,\ w'\in Y^*} \langle y_iu\otimes y_jv \mid (\sum_{n,m\in I} \gamma^s_{n,m}\, y_n\otimes y_m)\Delta(w')\rangle\, y_sw' \\
&= \sum_{w'\in Y^*} \langle u\otimes y_jv \mid \Delta(w')\rangle\, y_iw' + \sum_{w'\in Y^*} \langle y_iu\otimes v \mid \Delta(w')\rangle\, y_jw' \\
&\quad + \sum_{y_s\in Y,\ w'\in Y^*} \langle u\otimes v \mid \gamma^s_{i,j}\Delta(w')\rangle\, y_sw' \\
&= y_i \sum_{w'\in Y^*} \langle u\otimes y_jv \mid \Delta(w')\rangle\, w' + y_j \sum_{w'\in Y^*} \langle y_iu\otimes v \mid \Delta(w')\rangle w' \\
&\quad + \sum_{y_s\in Y} \gamma^s_{i,j}\, y_s \sum_{w'\in Y^*} \langle u\otimes v \mid \Delta(w')\rangle\, w' \\
&= y_iS(u,y_jv) + y_jS(y_iu,v) + \phi(y_i,y_j)S(u,v)
\end{aligned} \tag{2.161}$$

then the computation of S shows that, for all $u,v\in Y^*$, $S(u,v) = u ⧢_\phi v$ as S is bilinear, so $S = ⧢_\phi$.

□

Theorem 2.3.24. *i) The law $⧢_\phi$ is commutative if and only if the extension*

$$\phi : AY\otimes AY \longrightarrow AY$$

is so.

ii) The law $⧢_\phi$ is associative if and only if the extension

$$\phi : AY\otimes AY \longrightarrow AY$$

is so.

iii) Let $\gamma^z_{x,y} := \langle \phi(x,y)|z\rangle$ be the structure constants of ϕ (w.r.t. the basis Y), then $⧢_\phi$ is dualizable[17] if and only if $(\gamma^z_{x,y})_{x,y,z\in X}$ has the following decomposition property[18]

$$(\forall z\in X)(\#\{(x,y)\in X^2 | \gamma^z_{x,y}\neq 0\} < +\infty)\ . \tag{2.162}$$

Proof. (i) First, let us suppose ϕ commutative and consider T, the twist, i.e. the operator in $A\langle\langle Y^*\otimes Y^*\rangle\rangle$ defined by

$$\langle T(S) \mid u\otimes v\rangle = \langle S \mid v\otimes u\rangle\ . \tag{2.163}$$

It is an easy check to prove that T is a morphism of algebras. If ϕ is commutative, then so is the following diagram.

[17] In this context, it means that $⧢_\phi$ admits a dual comultiplication that is $\Delta : k\langle Y\rangle \longrightarrow k\langle Y\rangle\otimes k\langle Y\rangle$ such that for all P, Q, $R\in k\langle Y\rangle$,

$$\langle \Delta(P) \mid Q\otimes R\rangle = \langle P \mid Q ⧢_\phi R\rangle.$$

[18] One can prove that, in case Y is a semigroup, the associated ϕ fulfils (2.162) iff Y fulfils "condition D" of Bourbaki (see [Bou06a])

$$\begin{array}{ccc} Y & \xrightarrow{\ \Delta_{\uplus_\phi}\ } & A\langle\!\langle Y^*\otimes Y^*\rangle\!\rangle \\ & \searrow^{\Delta_{\uplus_\phi}} & \downarrow T \\ & & A\langle\!\langle Y^*\otimes Y^*\rangle\!\rangle \end{array}$$

and, then, the two morphisms $\Delta_{\uplus_\phi}$ and $T\circ\Delta_{\uplus_\phi}$ coincide on the generators Y of the algebra $A\langle Y\rangle$ and hence over $A\langle Y\rangle$ itself. Now for all $u,v,w\in Y^*$, one has

$$\begin{aligned} & \langle v\uplus_\phi u \mid w\rangle = \langle v\otimes u \mid \Delta_{\uplus_\phi}(w)\rangle = \langle u\otimes v \mid T\circ\Delta_{\uplus_\phi}(w)\rangle \\ = & \ \langle u\otimes v \mid \Delta_{\uplus_\phi}(w)\rangle = \langle u\uplus_\phi v \mid w\rangle \end{aligned} \tag{2.164}$$

which proves that $v\uplus_\phi u=u\uplus_\phi v$. Conversely, if $\uplus_\phi$ is commutative, one has, for $i,j\in I$

$$\phi(y_j,y_i)=y_j\uplus_\phi y_i-(y_j\sqcup\!\sqcup\, y_i)=y_i\uplus_\phi y_j-(y_i\sqcup\!\sqcup\, y_j)=\phi(y_i,y_j)\ . \tag{2.165}$$

(ii) Likewise, if ϕ is associative, let us define the operators

$$\overline{\Delta_{\uplus_\phi}\otimes I} : A\langle\!\langle Y^*\otimes Y^*\rangle\!\rangle \to A\langle\!\langle Y^*\otimes Y^*\otimes Y^*\rangle\!\rangle \tag{2.166}$$

by

$$\langle\overline{\Delta_{\uplus_\phi}\otimes I}(S) \mid u\otimes v\otimes w\rangle = \langle S \mid (u\uplus_\phi v)\otimes w\rangle \tag{2.167}$$

and, similarly,

$$\overline{I\otimes\Delta_{\uplus_\phi}} : A\langle\!\langle Y^*\otimes Y^*\rangle\!\rangle \to A\langle\!\langle Y^*\otimes Y^*\otimes Y^*\rangle\!\rangle \tag{2.168}$$

by

$$\langle\overline{I\otimes\Delta_{\uplus_\phi}}(S) \mid u\otimes v\otimes w\rangle = \langle S \mid u\otimes(v\uplus_\phi w)\rangle \tag{2.169}$$

it is easy to check by direct calculation that they are well defined morphisms and that the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{\ \Delta_{\uplus_\phi}\ } & A\langle\!\langle Y^*\otimes Y^*\rangle\!\rangle \\ \Delta_{\uplus_\phi}\downarrow & & \downarrow\overline{I\otimes\Delta_{\uplus_\phi}} \\ A\langle\!\langle Y^*\otimes Y^*\rangle\!\rangle & \xrightarrow[\ \overline{\Delta_{\uplus_\phi}\otimes I}\ ]{} & A\langle\!\langle Y^*\otimes Y^*\otimes Y^*\rangle\!\rangle \end{array}$$

is commutative. This proves that the two composite morphisms

$$\overline{\Delta_{\uplus_\phi}\otimes I}\circ\Delta_{\uplus_\phi}$$

and

$$\overline{I\otimes\Delta_{\uplus_\phi}}\circ\Delta_{\uplus_\phi}$$

coincide on Y and then on $A\langle Y\rangle$. Now, for $u, v, w, t \in Y^*$, one has

$$\begin{aligned}
& \langle (u \uplus_\phi v) \uplus_\phi w \mid t\rangle = \langle (u \uplus_\phi v) \otimes w \mid \Delta_{\uplus_\phi}(t)\rangle \\
= \;& \langle u \otimes v \otimes w \mid (\overline{\Delta_{\uplus_\phi} \otimes I})\Delta_{\uplus_\phi}(t)\rangle \\
= \;& \langle u \otimes v \otimes w \mid (\overline{I \otimes \Delta_{\uplus_\phi}})\Delta_{\uplus_\phi}(t)\rangle \\
= \;& \langle u \otimes (v \uplus_\phi w) \mid \Delta_{\uplus_\phi}(t)\rangle = \langle u \uplus_\phi (v \uplus_\phi w) \mid t\rangle
\end{aligned}$$

which proves the associativity of the law $\uplus_\phi$. Conversely, if $\uplus_\phi$ is associative, the direct expansion of the right hand side of

$$0 = (y_i \uplus_\phi y_j) \uplus_\phi y_k - y_i \uplus_\phi (y_j \uplus_\phi y_k) \tag{2.170}$$

proves the associativity of ϕ.
iii) We suppose that $(\gamma_{x,y}^z)_{x,y,z\in X}$ satisfies (2.162). In this case $\Delta_{\uplus_\phi}$ takes its values in $A\langle Y\rangle \otimes A\langle Y\rangle$ so its dual, the law $\uplus_\phi$ is dualizable[19]. Conversely, if $Im(\Delta_{\uplus_\phi}) \subset A\langle Y\rangle \otimes A\langle Y\rangle$, one has, for every $s \in I$

$$\sum_{n,m\in I} \gamma_{n,m}^s y_n \otimes y_m = \Delta(y_s) - (y_s \otimes 1 + 1 \otimes y_s) \in A\langle Y\rangle \otimes A\langle Y\rangle$$

which proves the claim. □

From now on, we suppose that $\phi : AY \otimes AY \longrightarrow AY$ is an associative and commutative law (of algebra) on AY.

Theorem 2.3.25. *Let A be a $\mathbb{Q}$-algebra. Then if ϕ is dualizable [20], let $\Delta_{\uplus_\phi} : A\langle Y\rangle \longrightarrow A\langle Y\rangle \otimes A\langle Y\rangle$ denote its dual comultiplication, then*

a) $\mathcal{B}_\phi = (A\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\uplus_\phi}, \varepsilon)$ is a bialgebra.

b) If A is a $\mathbb{Q}$-algebra then, the following conditions are equivalent

i) $\mathcal{B}_\phi$ is an enveloping bialgebra

ii) the algebra AX admits an increasing filtration $\Big((AY)_n\Big)_{n\in\mathbb{N}}$

$$(AY)_0 = \{0\} \subset (AY)_1 \subset \cdots \subset (AY)_n \subset (AY)_{n+1} \subset \cdots$$

compatible with both the multiplication and the comultiplication $\Delta_{\uplus_\phi}$ i.e.

$$\begin{aligned}
(AY)_p(AY)_q &\subset (AY)_{p+q} \\
\Delta_{\uplus_\phi}((AY)_n) &\subset \sum_{p+q=n} (AY)_p \otimes (AY)_q \ .
\end{aligned}$$

[19] In this context, it means that $\uplus_\phi$ admits a dual comultiplication that is $\Delta : k\langle Y\rangle \longrightarrow k\langle Y\rangle \otimes k\langle Y\rangle$ such that for all $P, Q, R \in k\langle Y\rangle$, $\langle \Delta(P) \mid Q \otimes R\rangle = \langle P \mid Q \uplus_\phi R\rangle$.

[20] For the pairing defined by

$$(\forall x, y \in Y)(\langle x \mid y\rangle = \delta_{x,y}) \ .$$

iii) $\mathcal{B}_\phi$ *is isomorphic to* $(A\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\sqcup\sqcup}, \epsilon)$ *as a bialgebra.*

iv) I^+ *is* $\star$*-nilpotent.*

Proof. We only prove the following implication (the other ones are easy)
$iv) \Longrightarrow iii)$ It suffices to prove that the morphism Φ given on each letter by $\Phi(y) = \pi_1(y)$ is an isomorphism

$$\Phi : \mathcal{B}_\phi \longrightarrow (A\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\sqcup\sqcup}, \epsilon)\ . \tag{2.171}$$

As I^+ is $\star$-nilpotent, the sum

$$I = \sum_{k\geq 0} \frac{1}{k!} \pi_{1,\mathcal{B}_\phi}^{*k} \tag{2.172}$$

is in fact finitely supported and can be rewritten As I^+ is $\star$-nilpotent, the sum

$$I = \sum_{k\geq 0}^{N} \frac{1}{k!} \pi_{1,\mathcal{B}_\phi}^{*k} = e + \pi_{1,\mathcal{B}_\phi} + \sum_{k\geq 2}^{N} \frac{1}{k!} \pi_{1,\mathcal{B}_\phi}^{*k} \tag{2.173}$$

for $N \geq 2$ large enough and $e = \pi_{1,\mathcal{B}_\phi}^{*0} = 1_{\mathcal{B}_\phi} \circ \epsilon$. As ϕ is associative and dualizable, let $\gamma_y^{y_1 y_2 \cdots y_k}$ be the structure constants of $\Delta_{\uplus_\phi}^{k-1}$ restricted to AY, or equivalently, for all $y \in Y$

$$\Delta_{\uplus_\phi}^{k-1}(y) = \sum_{y_1 y_2 \cdots y_k \in Y} \gamma_y^{y_1 y_2 \cdots y_k} y_1 \otimes y_2 \otimes \cdots \otimes y_k \tag{2.174}$$

then using a rearrangement of the star-log of the diagonal series, in virtue of (2.173) and using $e(y) = 0$, we get, for $y \in Y$

$$y = \pi_{1,\mathcal{B}_\phi}(y) + \sum_{k\geq 2}^{N} \frac{1}{k!} \sum_{y_1 y_2 \cdots y_k \in Y} \gamma_y^{y_1 y_2 \cdots y_k} \pi_{1,\mathcal{B}_\phi}(y_1) \pi_{1,\mathcal{B}_\phi}(y_2) \cdots \pi_{1,\mathcal{B}_\phi}(y_k) \tag{2.175}$$

This proves that the morphism (of AAU) given by $\Phi(y) = \pi_{1,\mathcal{B}_\phi}(y_1)$ is onto. The fact that it is into derives from the following lemma

Lemma 2.3.26. *Let* Φ *be a morphism* $A\langle X\rangle \to A\langle X\rangle$ *such that* $\Phi(x) \equiv x \mod 2$*; explicitly*

$$\Phi(x) = x + \sum_{|w|\geq 2} \langle \Phi(x) \mid w\rangle\, w \tag{2.176}$$

then, Φ *is into.*

Proof. (of lemma 2.3.26) One has, for all $w \in X^*$, $\Phi(w) = w + \sum_{|u|>|w|} \langle \Phi(w) \mid u\rangle\, u$. Then Φ can be written $\Phi = Id + \Phi^+$, continued by (uniform) continuity as a morphism $\Phi : A\langle\langle X\rangle\rangle \to A\langle\langle X\rangle\rangle$, space in which the series

$$\sum_{k\geq 0} (-1)^k (\Phi^+)^{\circ k} \tag{2.177}$$

converges and sums up to $\Phi^{-1} \in Aut(A\langle\langle X\rangle\rangle)$ which proves the injectivity of Φ. □
Now Theorem 2.3.25 is completely proved. □

Remark 2.3.27. *i) Theorem 2.3.25 a) holds for general (dualizable, coassociative) ϕ be it commutative of not.*
ii) It can happen that there is no antipode (and then, I^+ cannot be $\star$-nilpotent) as the following example shows.
Let $Y = \{y_0, y_1\}$ and $\phi(y_i, y_j) = y_{(i+j \ mod \ 2)}$, then

$$\begin{aligned}\Delta(y_0) &= y_0 \otimes 1 + 1 \otimes y_0 + y_0 \otimes y_0 + y_1 \otimes y_1 \\ \Delta(y_1) &= y_1 \otimes 1 + 1 \otimes y_1 + y_0 \otimes y_1 + y_1 \otimes y_0\end{aligned} \tag{2.178}$$

then, from Equations 2.178, one derives that $1 + y_0 + y_1$ is group-like. As this element has no inverse in $K\langle Y\rangle$. Thus, the bialgebra $\mathcal{B}_\phi$ cannot be a Hopf algebra.
iii) When I^+ is nilpotent, the antipode exists and is computed by

$$a_{\sqcup\!\sqcup_\phi} = (I)^{*-1} = (e + I^+)^{*-1} = \sum_{n\geq 0} (-1)^k (I^+)^{*k} \tag{2.179}$$

(see section (2.3.2)).
iv) In QFT, the antipode of a vector $h \in \mathcal{B}$ is computed by

$$S(1) = 1,\ S(h) = -h + \sum_{(1)(2)} S(h_{(1)})h_{(2)} \tag{2.180}$$

and by using the fact that S is an anti-morphism. This formula is used in contexts where I^+ is $\star$-nilpotent (although the concerned bialgebras are often not cocommutative). Here, one can prove this recursion from (2.179)*.*
v) (on lemma 2.3.26) Such a morphism (with $\Phi(x) \equiv x \mod 2$) can be seen as "tangent to identity", it is automatically into (as proves the lemma), but - by no means - onto, as shows the example of $\Phi(x) = x + x^2$ with $X = x$ (just notice that, with $\Phi^{-1} \in A\langle\langle X\rangle\rangle$, computation ((2.177)*) shows that $\Phi^{-1}(x) \notin A\langle X\rangle$).*

We have depicted the framework which is common to different kinds of shuffles. For all these, provided that I^+ be $*$-nilpotent, the bialgebra

$$(A\langle Y\rangle, \texttt{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup_\phi}, \varepsilon)$$

is isomorphic to

$$(A\langle Y\rangle, \texttt{conc}, 1_{Y^*}, \Delta_{\sqcup\!\sqcup}, \varepsilon)$$

and the straightening algorithm is simply the morphism which sends each $y_s \in Y$ to $\pi_1(y_s) = \log(I)(y_s)$ (this bialgebra is then a Hopf algebra). In other cases, such as the infiltration given by

$$\Delta(y_s) = y_s \otimes 1 + 1 \otimes y_s + y_s \otimes y_s$$

group-like elements without inverse may appear (and therefore no Hopf structure can be hoped).

2.3.3.3 Miscellaneous examples

First example
The coproduct is given on the generators as follows

$$\Delta_q(y_s) = y_s \otimes 1_{Y^*} + 1_{Y^*} \otimes y_s + q \left(\sum_{i+j=s} y_i \otimes y_j \right). \tag{2.181}$$

It is easy to check that $\mathcal{B}_q = (A\langle Y \rangle, \texttt{conc}, 1_{Y^*}, \Delta_q, \epsilon)$ is a bialgebra.
One has that I^+ is $\star$-nilpotent. Then, from Lemma 1, one gets

$$\mathrm{Prim}(\mathcal{B}_q) = Im(\pi_1(\mathcal{B}_q)). \tag{2.182}$$

Now, we prove that $\mathcal{B}_q$ is generated by its primitive elements.
Let $y'_s = \pi_1(y_s)$. One then has

$$y_s = \sum_{k\geq 1} \frac{q^{k-1}}{k!} \sum_{s_1+\cdots+s_k=s} y'_{s_1} \ldots y'_{s_k}. \tag{2.183}$$

For example, one has some first primitive elements of $\mathcal{B}_q$:

$$y_1 \tag{2.184}$$
$$y_2 - \frac{q}{2} y_1 y_1 \tag{2.185}$$
$$y_3 - \frac{q}{2}(y_1y_2 + y_2y_1) + \frac{q^2}{3} y_1y_1y_1 \tag{2.186}$$
$$y_4 - \frac{q}{2}(y_1y_3 + y_2y_2 + y_3y_1) + \frac{q^2}{3}(y_1y_1y_2 + y_1y_2y_1 + y_2y_1y_1) - \frac{1}{4}q^4 y_1y_1y_1y_1 \tag{2.187}$$

...

Using Lemma 2, $\pi_1 = \log_*(I)$ is a projector. In the case above, $i_{\mathcal{B},\mathcal{U}}$ is into. From Proposition 2.3.8, this implies that $\mathcal{B}_q$ is isomorphic to the algebra of the primitive of $\mathcal{B}_q$.
Second example
The coproduct is given on the generators as follows, with $\chi(y_i, y_j) = q^{ij}$.

$$\Delta_q^\chi(y_s) = y_s \otimes 1_{Y^*} + 1_{Y^*} \otimes y_s + \sum_{i+j=s} q^{ij} y_i \otimes y_j. \tag{2.188}$$

In the same manner, one has that $\mathcal{B}_q^\chi = (A\langle Y \rangle, \texttt{conc}, 1_{Y^*}, \Delta_q^\chi, \epsilon)$ is a bialgebra. Moreover, $\mathcal{B}_q^\chi$ is generated by its primitive elements.
Let $y'_s = \pi_1(y_s)$. One then has

$$y_s = \sum_{k\geq 1} \frac{1}{k!} \sum_{s_1+\cdots+s_k=s} q^{\prod_{j=1}^k s_j} y'_{s_1} \ldots y'_{s_1}. \tag{2.189}$$

For example, one has some first primitive elements of $\mathcal{B}_q^\chi$:

$$y_1 \tag{2.190}$$

$$y_2 - \frac{q}{2}y_1y_1 \tag{2.191}$$

$$y_3 - \frac{q^2}{2}(y_1y_2 + y_2y_1) + \frac{q^2}{3}y_1y_1y_1 \tag{2.192}$$

$$y_4 - \frac{1}{2}(q^3y_1y_3 + q^4y_2y_2 + q^3y_3y_1) + \frac{1}{3}(q^5y_1y_1y_2 + q^5y_1y_2y_1 + q^4y_2y_1y_1) - \frac{1}{4}q^5y_1y_1y_1y_1 \tag{2.193}$$

$\dots$

Using Lemma 2, $\pi_1 = \log_*(I)$ is a projector. In the cases above, $j_{\mathcal{B},\mathcal{U}}$ is into. From Proposition 2.3.8, this implies that $\mathcal{B}_q^X$ is isomorphic to the algebra of primitive of $\mathcal{B}_q^X$.

2.3.3.4 Counterexamples

It has been said that, with $\mathcal{B} = (\mathbb{Q}[\epsilon][x], ., 1_{\mathbb{Q}[\epsilon][x]}, \Delta, c)$ (notations as above), $i_{\mathcal{B},\mathcal{U}}$ is not into. Let us show this statement.

The q-infiltration coproduct [DFLL01] Δ_q is defined on the free algebra $K\langle X\rangle$ (K is a unitary ring), by its values on the letters

$$\Delta_q(x) = x \otimes 1 + 1 \otimes x + q(x \otimes x) \tag{2.194}$$

where $q \in K$. One can show easily that, for a word $w \in X^*$,

$$\Delta_q(w) = \sum_{I\cup J=[1..|w|]} q^{|I\cap J|} w[I] \otimes w[J] \tag{2.195}$$

with, as above (for $I = \{i_1 < i_2 < .. < i_k\} \subset \{1, 2, .., n\}$ and $w = a_1a_2\cdots a_n$), $w[I] = a_{i_1}a_{i_2}\cdots a_{i_k}$.

Then, with $K = \mathbb{Q}[\epsilon],\ q = \epsilon,\ X = \{x\}$, one has (as a direct application of Equation (2.195))

$$\Delta_\epsilon(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k} + \epsilon \sum_{k=1}^{n} k \binom{n}{k} x^k \otimes x^{n-k+1}\ . \tag{2.196}$$

This proves that, here, the space of primitive elements is a submodule of $K.x$ and solving $\Delta_\epsilon(\lambda x) = (\lambda x) \otimes 1 + 1 \otimes (\lambda x)$, one finds $\lambda = \lambda_1\epsilon$. Together with $\epsilon\, x \in \mathrm{Prim}(\mathcal{B})$ this proves that $\mathrm{Prim}(\mathcal{B})$ is of $\mathbb{Q}$-dimension one (in fact equal to $\mathbb{Q}.(\epsilon\, x)$). Now, the consideration of the morphism of Lie algebras $\mathrm{Prim}(\mathcal{B}) \to K[x]/(\epsilon K[x])$ which sends $\epsilon\, x$ to x proves that, in $\mathcal{U}(\mathrm{Prim}(\mathcal{B}))$, we have $(\epsilon\, x)(\epsilon\, x) \neq 0$ and $i_{\mathcal{B},\mathcal{U}}$ cannot be into.

For a graded counterexample[21], one can see that, with $K = \mathbb{Q}[\epsilon]$, $X = \{x, y, z\}$, $\mathcal{B} = K\langle X\rangle$ and

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \epsilon\,(y \otimes z),\ \Delta(y) = y \otimes 1 + 1 \otimes y,\ \Delta(z) = z \otimes 1 + 1 \otimes z \tag{2.197}$$

the same phenomenon occurs (for the gradation, one takes
$\deg(y) = \deg(z) = 1,\ \deg(x) = 2$).

[21] This example is due to Darij Grinberg.

Chapter 3

A word Hopf algebra based on the selection/quotient principle

Contents

The results presented in this chapter follow [DHNT13b] (see also [DHNT13a] for an extended abstract).

3.1 Algebra structure

3.1.1 Definitions

Let X be an infinite totally ordered alphabet $\{x_i\}_{i\geq 0}$ and X^* be the set of words with letters in the alphabet X.
A word w of length $n = |w|$ is a mapping $i \mapsto w[i]$ from $[1..|w|]$ to X. For a letter $x \in X$, the partial degree $|w|_x$ is the number of times the letter x occurs in the word w, i.e.

$$|w|_x := \sum_{j=1}^{|w|} \delta_{w[j],x}. \tag{3.1}$$

For a word $w \in X^*$, one defines the alphabet $\mathrm{Alph}(w)$ as the set of its letters, while $\mathrm{IAlph}(w)$ is the set of indices in $\mathrm{Alph}(w)$:

$$\mathrm{Alph}(w) := \{x \in X \text{ s.t. } \ |w|_x \neq 0\} \ ; \ \mathrm{IAlph}(w) := \{i \in \mathbb{N} \text{ s.t. } |w|_{x_i} \neq 0\}. \tag{3.2}$$

The upper bound $\sup(w)$ is the supremum of $IAlph(w)$, i. e.

$$\sup(w) := \sup_{\mathbb{N}}(\mathrm{IAlph}(w)). \tag{3.3}$$

Note that $\sup(1_{X^*}) = 0$.
Let us define the substitution operators. Let $w = x_{i_1} \dots x_{i_m}$ and $\phi : \mathrm{IAlph}(w) \longrightarrow \mathbb{N}$, with $\phi(0) = 0$. We set:

$$S_\phi(x_{i_1} \dots x_{i_m}) := x_{\phi(i_1)} \dots x_{\phi(i_m)}. \tag{3.4}$$

Let us define the pack operator of a word w. Let $\{j_1, \dots, j_k\} = \mathrm{IAlph}(w) \setminus \{0\}$ with $j_1 < j_2 < \dots < j_k$ and define ϕ_w as

$$\phi_w(i) := \begin{cases} m & \text{if } i = j_m \\ 0 & \text{if } i = 0 \end{cases}. \tag{3.5}$$

The corresponding packed word, denoted by $\mathrm{pack}(w)$, is $S_{\phi_w}(w)$. A word $w \in X^*$ is said to be *packed* if $w = \mathrm{pack}(w)$.

Example 3.1.1. *1) Let $w = x_1x_1x_5x_0x_4$. One then has* $\mathrm{pack}(w) = x_1x_1x_3x_0x_2$.
2) Let $w = x_1x_6x_2x_4$. One then has $\mathrm{pack}(w) = x_1x_4x_2x_3$.

Remark 3.1.2. *The presence of the letter x_0 dramatically influences the picture since one has an infinite number of distinct packed words of weight m (the weight is, here, the sum of the indices), which are obtained by inserting multiple copies of the letter x_0.*

Example 3.1.3. *The packed words of weight 2 are of the form: $x_0^{k_1}x_1x_0^{k_2}x_1x_0^{k_3}$, with $k_1, k_2, k_3 \geq 0$. This set is $x_0^*x_1x_0^*x_1x_0^*$.*

The operator $\mathrm{pack} : X^* \longrightarrow X^*$ is idempotent ($\mathrm{pack} \circ \mathrm{pack} = \mathrm{pack}$). It defines, by linear extension, a projector. The image, $\mathrm{pack}(X^*)$, is the set of packed words.

Let u, v be two words; one defines the shifted concatenation $*$ by

$$u * v := uT_{\sup(u)}(v), \tag{3.6}$$

where, for $t \in \mathbb{N}$, $T_t(w)$ denotes the "vertical shift by t", i.e., the image of w by S_ϕ for $\phi(n) = n + t$ if $n > 0$ and $\phi(0) = 0$ (in general, all letters can be reindexed except x_0). It is straightforward to check that, in case the words are packed, the result of a shifted concatenation is a packed word.

Definition 3.1.4. *Let k be a field. One defines a vector space $\mathcal{H} = span_k(\mathrm{pack}(X^*))$. One can endow this space with a product (on the words) given by*

$$\begin{aligned} \mu :\ \mathcal{H} \otimes \mathcal{H} &\longrightarrow \mathcal{H}, \\ u \otimes v &\longmapsto u * v. \end{aligned} \tag{3.7}$$

Remark 3.1.5. *The product above is similar to the shifted concatenation for permutations (see [DHT02]). Moreover, if u, v are two words in X^*, then* $\sup(u * v) = \sup(u) + \sup(v)$.

Proposition 3.1.6. *The triplet* $(\mathcal{H}, \mu, 1_{X^*})$ *is an AAU.*

Proof. Let u, v, w be three words in $\mathcal{H}$. One then has:

$$\begin{aligned}(u * v) * w &= (uT_{\sup(u)}(v))T_{\sup(u*v)}(w) = u(T_{\sup(u)}(v)(T_{\sup(u)+\sup(v)})(w)) \\ &= uT_{\sup(u)}(vT_{\sup(v)}(w)) = u * (v * w).\end{aligned} \tag{3.8}$$

Thus, μ is associative. On the other hand, for all $u \in \text{pack}(X^*)$, one has:

$$u * 1_{X^*} = uT_{\sup(u)}(1_{X^*}) = u1_{X^*} = u \ ,$$

and

$$1_{X^*} * u = (1_{X^*})T_{\sup(1_{X^*})}(u) = (1_{X^*})u = u \ .$$

Now remark that $\text{pack}(1_{X^*}) = 1_{X^*}$. This is clear from the fact that $1_{X^*} = 1_{\mathcal{H}}$. One concludes that $(\mathcal{H}, \mu, 1_{X^*})$ is an AAU. □

We call this algebra WMat.

Remark 3.1.7. *(i) The product* (3.7) *is non-commutative. For example:* $x_1 * x_1x_1 \neq x_1x_1 * x_1$.
(ii) $(k\langle X\rangle, \mu, 1_{X^*})$ *is also an AAU. Moreover,* $\mathcal{H}$ *is a subalgebra of* $k\langle X\rangle$.

Let $w = x_{k_1} \dots x_{k_n}$ be a word and $I \subseteq [1 \dots n]$. A sub-word $w[I]$ is defined as $x_{k_{i_1}} \dots x_{k_{i_l}}$, where $i_j \in I$ and $i_1 < \cdots < i_\ell$.

Proposition 3.1.8. *(i) Let* u *and* v *be two words such that, for* $x_i \in \text{Alph(u)}$ *and* $x_j \in \text{Alph}$*,* $i < j$ *or* $j = 0$*. One has*

$$pack(uv) = pack(u) * pack(v). \tag{3.9}$$

(ii) Let w *be a word. For* $t \geq 0$*, one has*

$$pack(T_t(w)) = pack(w). \tag{3.10}$$

(iii) A map $\text{pack} : k\langle X\rangle \longrightarrow \mathcal{H}$ *is a AAU morphism.*

Proof.
(i) Let $v := x_{j_1} \cdots x_{j_k}$ and $IAlph(v) \setminus \{0\} = \{s_1 < \cdots < s_m\}$. Since $j_i > sup(u)$ when $j_i \neq 0$, then

$$pack(uv) = pack(u)x_{j'_1} \cdots x_{j'_k}, \text{ where } \begin{cases} x_{j'_i} = x_0 \text{ if } j_i = 0, \\ x_{j'_i} = x_{\ell+sup(pack(u))} \text{ if } j_i = s_\ell. \end{cases} \tag{3.11}$$

One then has

$$\text{pack}(uv) = \text{pack}(u)T_{sup(\text{pack}(u))}(\text{pack}(v)) = \text{pack}(u) * \text{pack}(v). \tag{3.12}$$

(ii) It is easy to see this.
(iii) From (i), it follows that $pack(u * v) = pack(u) * pack(v)$ for two words u, v. □

3.1.2 WMat **is a free algebra**

As our context is that of Hopf algebras, the expression "free algebra" stands for "free k-associative unital algebra". It is known that it is the algebra of noncommutative polynomials $k\langle Z\rangle$ over some alphabet Z. By construction, one has $k\langle Z\rangle = k[Z^*]$, where Z^* is the free monoid of alphabet Z [Lot97].
WMat is, by construction, the algebra of the monoid pack(X^*). In order to check to check that WMat is a free algebra, it is sufficient to show that pack(X^*) is a free monoid on its irreducible words, which is shown below. In the following diagram, a free monoid is a pair $(F(X), j_X)$ where $F(X)$ is a monoid, $j_X : X \to F(X)$ is a mapping such that for all $M \in Mon$ and all $f : X \to M$ there exists a unique $f_X \in Hom(F(X), M)$ such that $f = f_x \circ j_X$.

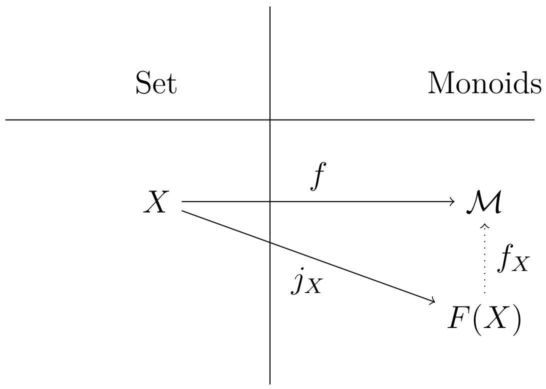


Figure 3.1: The universal process of a free algebra.

Here we give an "internal" characterization of free monoids in terms of irreducible elements.

Definition 3.1.9 ([DTPK10]). *Let $(M, *)$ be a monoid. A element $p \in M$ is called an irreducible if and only if it cannot be written in the form $p = q * r$, where $q, r \neq 1_M$.*

The following lemma is standard

Lemma 3.1.10 ([DTPK10]). *Let $(M, *)$ be a monoid $Z = irr(M)$ its set of irreducibles, $j_M : Z \hookrightarrow M$ the canonical inclusion mapping and $\mu_M : Z^* \to M$ the morphism deduced from j_M by the universal process (Figure 3.1). Then M is a free monoid iff μ_M is one-to-one or, equivalently, all $m \in M$ can be decomposed uniquely as a product of irreducibles.*

For the monoid pack(X^*), we will speak of *irreducible words* and denote by Irr(pack(X^*)) the set of its irreducible elements.

Example 3.1.11. *The word $x_1x_1x_1$ is an irreducible word. The word $x_1x_1x_2$ is a reducible word because it can be written as $x_1x_1x_2 = x_1x_1 * x_1$.*

Proposition 3.1.12. *If w is a packed word, then w can be written uniquely as $w = v_1 * v_2 * \cdots * v_n$, where v_i are non-trivial irreducible words, $1 \leq i \leq n$.*

Proof. The i^{th} position of word w is called an admissible cut if $\sup(w[1\dots i]) = inf(w[i+1\dots|w|]) - 1$ or $\sup(w[i+1\dots|w|]) = 0$, where $inf(w)$ is the infimum of $\mathrm{IAlph}(w)$.
Since the length of word is finite, we may write $w = v_1 * v_2 * \cdots * v_n$, with n maximal and v_i non-trivial, $1 \le i \le n$.
We assume that one word can be written in two ways

$$w = v_1 * v_2 * \cdots * v_n \tag{3.13}$$

and

$$w = v'_1 * v'_2 * \cdots * v'_m. \tag{3.14}$$

Denoting by k the first index such that $v_k \neq v'_k$, without loss of generality we may suppose that $|v_k| < |v'_k|$. From Equation (3.13), the k^{th} position is an admissible cut of w. From Equation (3.14), the k^{th} position is not an admissible cut of w. Thus, we obtain a contradiction. Hence, we have $n = m$ and $v_i = v'_i$ for all $1 \le i \le n$. □

This proves that $\mathrm{pack}(X^*)$ is free on its irreducibles. Let us emphasize that $\mathrm{pack}(X^*)$ is isomorphic to $(\mathrm{Irr}(\mathrm{pack}(X^*)))^*$ and thus WMat to $k\langle \mathrm{Irr}(\mathrm{pack}(X^*))\rangle$.

3.2 Bialgebra structure

Let us give the definition of the coproduct and prove that the coassociativity property holds.

Definition 3.2.1. *Let $A \subset X$, one defines $w/A := S_{\phi_A}(w)$ with $\phi_A(i) = \begin{cases} i \text{ if } x_i \notin A, \\ 0 \text{ if } x_i \in A \end{cases}$. Let u be a word. One defines ${}^{w}/_{u} :={}^{w} /_{\mathrm{Alph}(u)}$.*

Definition 3.2.2. *The coproduct of $\mathcal{H}$ is given by*

$$\Delta(w) := \sum_{I+J=[1\dots|w|]} \mathrm{pack}(w[I]) \otimes \mathrm{pack}(w[J]/w[I]), \forall w \in \mathcal{H}, \tag{3.15}$$

where this sum runs over all partitions of $[1\dots|w|]$ divided into two blocks, $I \cup J = [1\dots|w|]$ and $I \cap J = \emptyset$.

Example 3.2.3. *One has:*

$$\begin{aligned}\Delta(x_1x_2x_1) = x_1x_2x_1 \otimes 1_{X^*} + x_1 \otimes x_1x_0 + x_1 \otimes x_1^2 + x_1 \otimes x_0x_1 + x_1x_2 \otimes x_0 + x_1^2 \otimes x_1 \\ + x_2x_1 \otimes x_0 + 1_{X^*} \otimes x_1x_2x_1.\end{aligned} \tag{3.16}$$

Let us now prove the coassociativity. Let $I = [i_1, \dots, i_n]$.Let α be a mapping:

$$\begin{aligned}\alpha : I &\longrightarrow [1\dots n], \\ i_s &\longmapsto s.\end{aligned} \tag{3.17}$$

Lemma 3.2.4 (Transitivity of selection). *Let $w \in X^*$ be a word, I be a subset of $[1 \dots |w|]$ and $I_1 \subset [1 \dots |I|]$. One then has*

$$\mathrm{pack}(w[I])[I_1] = S_{\phi_{w[I]}}(w[I_1']), \tag{3.18}$$

where I_1' is $\alpha^{-1}(I_1)$ and $\phi_{w[I]}$ is the packing map of $w[I]$ that is given in (3.5) *.*

Proof. Using the definition of packing map $\phi_{w[I]}$, one can directly check that equation (3.18) holds. □

Example 3.2.5. *Let $w = x_1x_4x_3x_7x_6x_0$. Let $I = \{1,3,4,6\}$ and $I_1 = \{1,3\}$. One has $I_1' = \{1,4\}$. One has $\mathrm{pack}(w[I])[I_1] = x_1x_3$. On other hand, one has $w[I_1'] = x_1x_7$ and $S_{\phi_{w[I]}}(w[I_1']) = x_1x_3$.*

Lemma 3.2.6. *Let $w \in X^*$ be a word and ϕ be a strictly increasing map from $\mathrm{IAlph}(w)$ to $\mathbb{N}$. One then has:*

1)

$$\mathrm{pack}(S_\phi(w)) = \mathrm{pack}(w). \tag{3.19}$$

2)

$$S_\phi(w_1/w_2) = S_\phi(w_1)/S_\phi(w_2). \tag{3.20}$$

Proof.
1) One has

$$\mathrm{pack}(S_\phi(w)) = S_{\phi_w}(S_\phi(w)) = S_{\phi_w\circ\phi}(w),$$

where ϕ_w is the packing map which is given in (3.5). Note that both ϕ and ϕ_w are strictly increasing maps. Let $I = \mathrm{IAlph}(w) = \{j_1, j_2, \dots, j_k\}$, $j_1 < j_2 < \cdots < j_k$, the image set $\phi(I) = \{j_i', j_i' = \phi(j_i), i = 1 \cdots k\}$ one has $j_1' < j_2' < \cdots < j_k'$. From the definition of ϕ_w, one has: $\phi_w(j_i') = i = \phi_w(j_i)$. This leads to:

$$S_{\phi_w\circ\phi}(w) = S_{\phi_w}(w) = \mathrm{pack}(w). \tag{3.21}$$

2) Let $X = \mathrm{Alph}(w_2)$ and $Y = \mathrm{Alph}(S_\phi(w_1))$.
Let us rewrite the two sides of equation (3.20), the LHS and the RHS:

$$LHS = S_\phi(S_{\phi_X}(w_1)) = S_{\phi\circ\phi_X}(w_1), \tag{3.22}$$

$$RHS = S_{\phi_Y}(S_\phi(w_1)) = S_{\phi_Y\circ\phi}(w_1). \tag{3.23}$$

With $x_i \in \mathrm{Alph}(w_1)$, one has two cases:

1. If $x_i \in X$, then $\phi_X(i) = 0$ and $\phi \circ \phi_X(i) = 0$ because ϕ is a strictly increasing map.

   On the other hand, $\phi(i) \in Y$ and this implies $\phi_Y \circ \phi(i) = \phi_Y(\phi(i)) = 0$.

2. If $x_i \notin X$, then $\phi_X(i) = i$ and $\phi \circ \phi_X(i) = \phi(i)$.

   On the other hand, because ϕ is a strictly increasing map, then $\phi(i) \notin Y$, and $\phi_Y \circ \phi(i) = \phi(i)$.

One thus has $\phi \circ \phi_X(i) = \phi_Y \circ \phi(i)$. Using this result and the two equations above (3.22) and (3.23), one concludes the proof. □

Lemma 3.2.7 (Transitivity of quotients). *Let w be a word in $\mathcal{H}$, and I, J, K be three disjoint subsets of $\{1 \dots |w|\}$. One then has:*

$$(w[K]/w[I])/(w[J]/w[I]) = w[K]/w[I+J]. \tag{3.24}$$

Proof.
Using Lemma 3.2.6, one has:

$$\begin{aligned}(w[K]/w[I])/(w[J]/w[I]) &= S_{\phi_I}(w[K])/S_{\phi_I}(w[K]) = S_{\phi_I}(w[K]/w[J]) = S_{\phi_I}(S_{\phi_J}(w[K])) \\ &= S_{\phi_I \circ \phi_J}(w[K]) = w[K]/w[I+J]. \end{aligned} \tag{3.25}$$

□

Proposition 3.2.8. *The vector space $\mathcal{H}$ endowed with the coproduct* (3.15) *is a coassociative coalgebra with counit (c-AAU). The co-unit is given by:*

$$\epsilon(w) := \begin{cases} 1 & \textit{if } w = 1_{\mathcal{H}}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof.
Let us first prove the coassociativity of the coproduct (3.15), namely

$$(\Delta \otimes Id) \circ \Delta(w) = (Id \otimes \Delta) \circ \Delta(w). \tag{3.26}$$

The LHS of the coassociativity condition (3.26) can be written:

$$\begin{aligned}(\Delta \otimes Id) \circ \Delta(w) = &\sum_{I+J=[1\dots|w|]} \left(\sum_{I_1+I_2=[1\dots|I|]} \mathrm{pack}(\mathrm{pack}(w[I])[I_1]) \otimes \mathrm{pack}({}^{\mathrm{pack}(w[I][I_2])}/_{\mathrm{pack}(w[I][I_1])}) \right) \\ \otimes \mathrm{pack}({}^{w[J]}/_{w[I]}) = &\sum_{I+J=[1\dots|w|]} \left(\sum_{I'_1+I'_2=I} \mathrm{pack}(S_\phi(w[I'_1])) \otimes \mathrm{pack}({}^{S_\phi(w[I'_2])}/_{S_\phi(w[I'_1])}) \right) \\ \otimes \mathrm{pack}({}^{w[J]}/_{w[I]}) = &\sum_{I'_1+I'_2+J=[1\dots|w|]} \mathrm{pack}(w[I'_1]) \otimes \mathrm{pack}({}^{w[I_2]}/_{w[I'_1]}) \otimes \mathrm{pack}({}^{w[J]}/_{w[I_1+I_2]}). \end{aligned} \tag{3.27}$$

The RHS of the coassociativity condition (3.26) can be written as:

$$(Id \otimes \Delta) \circ \Delta(w) = \sum_{I+J=[1...|w|]} \mathrm{pack}(w[I]) \otimes \left(\sum_{J_1+J_2=[1...|J|]} \mathrm{pack}(\mathrm{pack}({}^{w[J]}/_{w[I]}))[J_1]) \right.$$
$$\left. \otimes \mathrm{pack}({}^{\mathrm{pack}({}^{w[J]}/_{w[I]})[J_2]}/_{\mathrm{pack}({}^{w[J]}/_{w[I]})[J_1]}))\right) = \sum_{I+J=[1...|w|]} \mathrm{pack}(w[I]) \otimes \left(\sum_{J'_1+J'_2=J} \mathrm{pack}({}^{w[J'_1]}/_{w[I]}) \right.$$
$$\left. \otimes \mathrm{pack}(S_\phi({}^{{}^{w[J'_2]}/_{w[I]}}/_{{}^{w[J'_1]}/_{w[I]}}))\right) = \sum_{I+J=[1...|w|]} \mathrm{pack}(w[I]) \otimes \left(\sum_{J'_1+J'_2=J} \mathrm{pack}({}^{w[J'_1]}/_{w[I]}) \right.$$
$$\left. \otimes \mathrm{pack}({}^{w[J'_2]}/_{w_{[I+J'_1]}})\right) = \sum_{I+J'_1+J'_2=[1...|w|]} \mathrm{pack}(w[I]) \otimes \mathrm{pack}({}^{w[J'_1]}/_{w[I]}) \otimes \mathrm{pack}({}^{w[J'_2]}/_{w_{[I+J'_1]}}). \tag{3.28}$$

Using the equations (3.27) and (3.28), one concludes that the coproduct (3.15) is coassociative.
Let us now prove the following

$$(\epsilon \otimes Id) \circ \Delta(w) = (Id \otimes \epsilon) \circ \Delta(w), \tag{3.29}$$

for all word $w \in \mathcal{H}$.
Let us rewrite the LHS and the RHS of the equation (3.29):

$$\begin{aligned} LHS &= (\epsilon \otimes Id) \left(\sum_{I+J=[1...|w|]} \mathrm{pack}(w[I]) \otimes \mathrm{pack}(w[J]/w[I]) \right) \\ &= \sum_{I+J=[1...|w|]} \epsilon(\mathrm{pack}(w[I])) \otimes \mathrm{pack}(w[J]/w[I]) = 1_{\mathcal{H}} \otimes \mathrm{pack}(w) = \mathrm{pack}(w). \end{aligned} \tag{3.30}$$

$$\begin{aligned} RHS &= (Id \otimes \epsilon) \left(\sum_{I+J=[1...|w|]} \mathrm{pack}(w[I]) \otimes \mathrm{pack}(w[J]/w[I]) \right) \\ &= \sum_{I+J=[1...|w|]} \mathrm{pack}(w[I]) \otimes \epsilon(\mathrm{pack}(w[J]/w[I])) = \mathrm{pack}(w) \otimes 1_{\mathcal{H}} = \mathrm{pack}(w). \end{aligned} \tag{3.31}$$

One thus concludes that $(\mathcal{H}, \Delta, \epsilon)$ is a c-AAU. □

Remark 3.2.9. *This coalgebra is not cocommutative. For example, one has*

$$\begin{aligned} T_{12} \circ \Delta(x_1^2) &= T_{12}(x_1^2 \otimes 1_{\mathcal{H}} + 2x_1 \otimes x_0 + 1_{\mathcal{H}} \otimes x_1^2) \\ &= x_1^2 \otimes 1_{\mathcal{H}} + 2x_0 \otimes x_1 + 1_{\mathcal{H}} \otimes x_1^2 \neq \Delta(x_1^2), \end{aligned}$$

where the operator T_{12} is given by $T_{12}(u \otimes v) = v \otimes u$.

Lemma 3.2.10. *Let u, v be two words. Let $I_1 + J_1 = [1 \dots |u|]$ and $I_2 + J_2 = [|u|+1 \dots |u| + |v|]$. One then has*

$$\mathrm{pack}({}^{u*v[J_1+J_2]}/_{u*v[I_1+I_2]}) = \mathrm{pack}({}^{u[J_1]}/_{u[I_1]}) * \mathrm{pack}({}^{v[J_2']}/_{v[I_2']}), \tag{3.32}$$

where I_2' is the set $\{k - |u|, k \in I_2\}$ and J_2' is the set $\{k - |u|, k \in J_2\}$.

Proof. One has:

$$\begin{aligned}
&\mathrm{pack}({}^{u*v[J_1+J_2]}/_{u*v[I_1+I_2]}) = \mathrm{pack}(S_{\phi_{I_1+I_2}}(u * v[J_1 + J_2])) = \mathrm{pack}(S_{\phi_{I_1+I_2}}(uT_{\sup(u)}(v)[J_1 + J_2]))\\
=&\mathrm{pack}(S_{\phi_{I_1}+\phi_{I_2}}(u[J_1]T_{\sup(u)}(v)[J_2])) = \mathrm{pack}(S_{\phi_{I_1}}S_{\phi_{I_2}}(u[J_1]T_{\sup(u)}(v[J_2'])))\\
=&\mathrm{pack}(S_{\phi_{I_1}}(u[J_1])S_{\phi_{I_2}}(T_{\sup(u)}(v[J_2']))) = \mathrm{pack}({}^{u[J_1]}/_{u[I_1]})T_{\sup({}^{u[J_1]}/_{u[I_1]})}\mathrm{pack}(S_{\phi_{I_2'}}(v[J_2']))\\
=&\mathrm{pack}(({}^{u[J_1]}/_{u[I_1]}) * \mathrm{pack}({}^{u[J_2']}/_{u[I_2']}).
\end{aligned} \tag{3.33}$$

□

Proposition 3.2.11. *Let u, v be two words in $\mathcal{H}$. One has:*

$$\Delta(u * v) = \Delta(u) *^{\otimes 2} \Delta(v), \tag{3.34}$$

where $*^{\otimes 2} := (* \otimes *) \circ \tau_{23}$.

Proof.
One has:

$$\begin{aligned}
\Delta(u * v) =& \sum_{\substack{I_1+I_2=I\\ J_1+J_2=J\\ I_1,J_1\subset[1\dots|u|]\\ I_2,J_2\subset[|u|+1\dots|u|+|v|]}} (\mathrm{pack}(u * v[I_1 + I_2])) \otimes \Big(\mathrm{pack}({}^{(u*v[J_1+J_2]}/_{(u*v[I_1+I_2]})\Big)\\
=& \sum_{\substack{I_1+J_1=[1\dots|u|]\\ I_2'+J_2'=[1\dots|v|]}} \Big(\mathrm{pack}(u[I_1]) \otimes \mathrm{pack}({}^{u[J_1]}/_{u[I_1]})\Big) * \Big(\mathrm{pack}(v[I_2]) \otimes \mathrm{pack}({}^{u[J_2']}/_{u[I_2']})\Big)\\
=& \left(\sum_{I_1+J_1=[1\dots|u|]} \mathrm{pack}(u[I_1]) \otimes \mathrm{pack}({}^{u[J_1]}/_{u[I_1]})\right) * \left(\sum_{I_2'+J_2'=[1\dots|v|]} \mathrm{pack}(v[I_2]) \otimes \mathrm{pack}({}^{u[J_2']}/_{u[I_2']})\right)\\
=&\Delta(u) *^{\otimes 2} \Delta(v).
\end{aligned} \tag{3.35}$$

□

It is easy to check that $\mathcal{H}$ is graded by the word's length.
Using Proposition 2.1.22, the antipode map is given by the recursion:

$$\begin{aligned}
&S(1_{\mathcal{H}}) = 1_{\mathcal{H}}\\
&S(w) = -w - \sum_{I+J=[1\dots|w|],I,J\neq\emptyset} S(\mathrm{pack}(w[I])) * \mathrm{pack}({}^{w[J]}/_{w[I]}),\ \forall w \neq 1_{\mathcal{H}}.
\end{aligned} \tag{3.36}$$

One easily now concludes:

Theorem 3.2.12. *The triplet $(\mathcal{H}, \Delta, *)$ is a Hopf algebra.*

Proof. The conclusion follows from the above results. □

Let us end this subsection by mentioning that we have checked the coassociativity of the WMat coproduct (3.15) with Maple (see Appendix A).
The coassociativity of a similar coproduct, on commutative alphabets, has been also tested in a analogous way with Maple (see Appendix B).

Remark 3.2.13. *Note that $\mathcal{H}$ is a free associative CHA in the sense of Loday and Ronco [LR10]. A CHA is a free (or cofree) Hopf algebra which is free (or cofree) and equipped with a given isomorphism to the free algebra over the indecomposables (respective the cofree coalgebra over the primitives).*

3.3 The Hilbert series of the Hopf algebra Wmat

In this section, we compute the number of packed words with length n and supremum k. It is the same as the number of cyclically ordered partitions of an n-element set. Using the formula of Stirling numbers of the second kind (see [Com74]), one can get the explicit formula for the number of packed words with length n, number which we denote by d_n.

Definition 3.3.1. *The Stirling numbers of the second kind count the number of set partitions of an n-element set into precisely k non-void parts. The Stirling numbers, denoted by $S(n,k)$ are given by the following recursive relations:*

1. $S(n,n) = 1 (n \geq 0)$,
2. $S(n,0) = 0 (n > 0)$,
3. $S(n+1,k) = S(n,k-1) + kS(n,k)$, *for* $0 < k \leq n$.

One can define a word without x_0 by its positions, this means that if a word $w = x_{i_1}x_{i_2}\dots x_{i_n}$ has length n and alphabet $\mathrm{IAlph}(w) = \{1,2,\dots,k\}$, then this word can be determined from the list $[S_1, S_2, \dots, S_k]$, where S_i is the set of positions of x_i in the word w, with $1 \leq i \leq k$. It is straightforward to check that $(S_i)_{0 \leq i \leq k}$ is a partition of $[1 \dots n]$.
The set of packed words with length n and supremum k splits into two subsets: *"pure"* packed words (which have no x_0 in their alphabet), denote $pack^+_{n,s}(X)$ and packed words which have x_0 in their alphabet, denote $\mathrm{pack}^0_{n,k}(X)$. It is clear that:

$$d(n,k) = \#\mathrm{pack}^+_{n,k}(X) + \#\mathrm{pack}^0_{n,k}(X). \tag{3.37}$$

Let us now compute the cardinal of these two sets $\mathrm{pack}^+_{n,k}(X)$ and $\mathrm{pack}^0_{n,k}(X)$.
Consider a word $w \in \mathrm{pack}^+_{n,k}(X)$, then $\mathrm{IAlph}(w) = \{1,2,\dots,k\}$. This word is determined by $[S_1, S_2, \dots, S_k]$, in which S_i is a set of positions of x_i, for $1 \leq i \leq k$. One can see that:

1. $S_i \neq \emptyset$, $\forall i \in [1,k]$;

2. $\sqcup_{1\le i\le k} S_i = \{1, 2, \dots, n\}$.

Note that 1-2 hold even with $w = 1_{\mathcal{H}}$. Thus, one has the cardinal of packed words with length n and supremum k:

$$d^+(n,k) = \#\mathrm{pack}^+_{n,k}(X) = S(n,k)k!. \tag{3.38}$$

Similarly, a word $w \in \mathrm{pack}^0_{n,k}(X)$ can be determined by $[S_0, S_1, S_2, \dots, S_k]$ where S_i is the set of positions of x_i, for all $0 \le i \le k$. After computation, one has:

$$d^0(n,k) = \#\mathrm{pack}^0_{n,k}(X) = S(n,k+1)(k+1)!. \tag{3.39}$$

From the two equations above, one can get the number of packed words with length n and supremum k:

$$d(n,k) = d^+(n,k) + d^0(n,k) = S(n,k)k! + S(n,k+1)(k+1)! = S(n+1,k+1)k!. \tag{3.40}$$

From this formula, using Maple, one can get some values of $d(n,k)$. We give in the Table 3.1 the first values.

		k								
		0	1	2	3	4	5	6	7	8
	0	1	0	0	0	0	0	0	0	0
	1	1	1	0	0	0	0	0	0	0
	2	1	3	2	0	0	0	0	0	0
	3	1	7	12	6	0	0	0	0	0
n	4	1	15	50	60	24	0	0	0	0
	5	1	31	180	390	360	120	0	0	0
	6	1	63	602	2100	3360	2520	720	0	0
	7	1	127	1932	10206	25200	31920	20160	5040	0
	8	1	255	6050	46620	166824	317520	332640	181440	40320

Table 3.1: Values of $d(n,k)$ given by the explicit formula (3.40) and computed with Maple.

Note that the values of Table 3.1 correspond to those of the triangular array $A028246$ of Sloane [Slo]. The exponential generating function is given in [Slo] by the formula $-log(1 - y(e^x - 1))$.

Remark 3.3.2. *Formulas* (3.38) *and* (3.39) *imply that the packed words of length n and supremum k without, and respectively with, x_0 are in bijection with the ordered partitions of $[n]$ in k parts and respectively in $k+1$ parts. Therefore, formula* (3.40) *implies that the set of packed words of length n with supremum k is in bijection with the circularly ordered partitions of $n+1$ elements in $k+1$ parts.*

The formula for the number of packed words of length n, d_n ($n \ge 1$), is then given by

$$d_n = \sum_{k=0}^{n} d(n,k) = \sum_{k=0}^{n} S(n+1,k+1)k!. \tag{3.41}$$

n	0	1	2	3	4	5	6	7	8	9	10
d_n	1	2	6	26	150	1082	9366	94586	1091670	14174522	204495126

Table 3.2: Some value of d_n by the formula (3.41).

Using again Maple, one can get the values listed in Table 3.2.
The number of packed words is the sequence $A000629$ of Sloane [Slo], where it is also mentioned that this sequence corresponds to the ordered Bell numbers sequence times two (except for the 0th order term).
The ordinary and exponential generating function of our sequence are also given in [Slo]. The ordinary one is given by the formula: $\sum_{n\geq 0} \frac{2^n n! x^n}{\prod_{k=0}^{n}(1+kx)}$. The exponential one is given by: $\frac{e^x}{2-e^x}$. Let us give the proof of this.
Firstly, recall that the exponential generating function of the ordered Bell numbers (see, for example, page 109 of Philippe Flajolet's book [FS08]) is:

$$\frac{1}{2-e^x} = \sum_{n\geq 0}\sum_{k=0}^{n} S(n,k)k!\frac{x^n}{n!}. \tag{3.42}$$

By deriving both side of equation (3.42) with respect to x , one obtains:

$$\frac{e^x}{2-e^x} = \sum_{n\geq 1}\sum_{k=1}^{n} S(n,k)k!\frac{x^{n-1}}{(n-1)!}. \tag{3.43}$$

From equations (3.41) and (3.43), one gets the exponential generating function of our sequence:

$$\frac{e^x}{2-e^x} = \sum_{n\geq 0}\sum_{k=0}^{n} S(n+1,k+1)k!\frac{x^n}{n!} = \sum_{n\geq 0} d_n\frac{x^n}{n!}. \tag{3.44}$$

Let us now investigate the combinatorics of irreducible packed words.
Firstly, we notice that one still has an infinity of irreducible packed words of weight m, which are again obtained by adding multiple copies of the letter x_0.

Example 3.3.3. *The word $x_1x_0^kx_1x_0^kx_1$ (with k an arbitrary integer) is an irreducible packed word of weight* 3.

Let us denote by i_n the number of irreducible packed words of length n. Then one has:

$$i_n = \sum_{\substack{j_1+\cdots+j_k=n\\ j_l\neq 0}} (-1)^{k+1} d_{j_1}\dots d_{j_k}. \tag{3.45}$$

Using Maple, one can get the values of i_n, which we give in Table 3.3 below. Note that this sequence does not appear in Sloane's On-Line Encyclopedia of Integer Sequences [Slo].

n	0	1	2	3	4	5	6	7	8	9	10
i_n	1	2	2	10	66	538	5170	59906	704226	9671930	145992338

Table 3.3: Ten first values of the number of irreducible packed words.

Chapter 4

Recipe theorem for the Tutte polynomial for matroids, renormalization group-like approach

Contents

The results presented in this chapter follow [DHNKT13a] (see also [DHNKT13b] for an extended abstract).

4.1 Quantum field theory, the renormalization group

Since our proof of the universality of the matroid Tutte polynomial is based on an equation analogue to the QFT renormalization group equation, let us first introduce this equation.
A QFT model (for a general introduction to QFT see for example the books [ZJ02] or [KSF01]) is defined by means of a functional integral of the exponential of an *action* S which, from a mathematical point of view, is a functional of the *fields* of the model.
For the Φ^4 scalar model - the simplest QFT model - there is only one type of field, which we denote by $\Phi(x)$. From a mathematical point of view, for an Euclidean QFT scalar model, one has $\Phi : \mathbb{R}^D \to \mathbb{K}$, where D is usually taken equal to 4 (the dimension of the space) and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ (real, respectively complex fields).

The principal aim of QFT in the functional integral formulation is to define and compute expectation values of observables of the type

$$\langle \mathcal{O} \rangle = \frac{\int D\Phi \mathcal{O}[\Phi] \exp(-S[\Phi])}{\int D\Phi \exp(-S[\Phi])}. \tag{4.1}$$

In this setting, an observable is a functional of the fields, a simple example being a product of n fields evaluated at different space-times points, leading to the n-point correlation functions

$$G(x_1, \dots, x_n) = \frac{\int D\Phi \Phi(x_1) \cdots \Phi(x_n) \exp(-S[\Phi])}{\int D\Phi \exp(-S[\Phi])}. \tag{4.2}$$

This functional integral is usually defined in perturbation theory by separating the quadratic part of the action which is used to define a Gaussian measure on the space of fields. Then, non quadratic terms are treated as a perturbation and the functional integral is expand over Feynman graphs.
The quantities computed in QFT are generally divergent. One thus has to consider a real, positive, *cut-off* Λ - the flowing parameter. In the Wilsonian approach to renormalization, the functional integral is performed in a step by step procedure by successive integrations over fields with decreasing momenta. Let us denote by $\widetilde{\Phi}(p)$ the Fourier transform of $\Phi(x)$, define the Wilsonian effective action as

$$S_\Lambda[\Phi] = \int [D\Phi'] \exp\left\{ -\int_{\mathbb{R}^{2D}} \frac{1}{2} d^D p d^D q \, \widetilde{\Phi'}(p) C^{-1}_{\Lambda,\Lambda_0}(p,q) \widetilde{\Phi'}(q) + S_0[\Phi + \Phi'] \right\}. \tag{4.3}$$

The quadratic part is written using the covariance

$$C_{\Lambda,\Lambda_0}(p,q) = \delta(p-q) \int_{\frac{1}{\Lambda_0}}^{\frac{1}{\Lambda}} d\alpha e^{-\alpha p^2}. \tag{4.4}$$

where δ is the 4−dimension function which is given as $\delta(p-q) = \begin{cases} 1 \text{ if } p = q, \\ 0 \text{ otherwise.} \end{cases}$

It involves an ultraviolet cut-off Λ_0 and a floating infrared cut-off Λ. Its role is to enforce the integration over fields with momenta between Λ and Λ_0 for the fluctuating field Φ'.
The effective action obeys the *renormalization group equation* (see [Pol84])

$$\Lambda \frac{\partial S_\Lambda}{\partial \Lambda} = \int_{\mathbb{R}^{2D}} \frac{1}{2} d^D p d^D q \Lambda \frac{\partial C_{\Lambda,\Lambda_0}}{\partial \Lambda} \left(\frac{\delta^2 S}{\delta \tilde{\Phi}(p) \delta \tilde{\Phi}(q)} - \frac{\delta S}{\delta \tilde{\Phi}(p)} \frac{\delta S}{\delta \tilde{\Phi}(q)} \right), \tag{4.5}$$

where $\tilde{\Phi}$ represents the Fourier transform of the function Φ. The first term in the RHS of the equation above corresponds to the derivation of a propagator associated to a bridge in the respective Feynman graph. The second term corresponds to an edge which is not a bridge and is part of some circuit in the graph. This is diagrammatically represented in Fig. 4.1.

This equation can then be used to prove perturbative renormalizability in QFT. In our context, it is useful to notice that a perturbative solution to this equation generates a sum over all connected graphs. To this aim, it is not necessary to include all the field theoretical

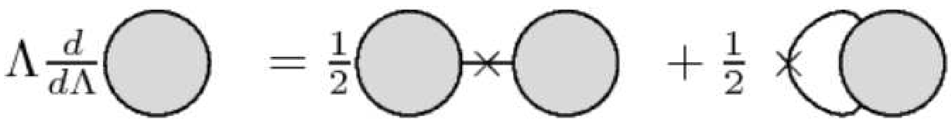


Figure 4.1: Diagrammatic representation of the flow equation.

structure. Instead, we consider $\phi \in \mathbb{R}$ as a single real variable so that the functional integral defining the effective action (4.3) reduces to a perturbed Gaussian integral

$$S_t(\phi) = \log \int_{\mathbb{R}} \frac{d\phi'}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2t}\phi'^2 + S_0(\phi + \phi')\right\}. \tag{4.6}$$

S_0 is a formal power series in ϕ and extra variables λ_n

$$S_0(\phi) = \sum_n \frac{\lambda_n}{n!}\phi^n. \tag{4.7}$$

Performing the Gaussian integration over ϕ, the effective action turns out to be a sum over all connected graphs. The weight $\lambda_\Gamma(t)$ of a graph Γ in this expansion obeys the differential equation

$$\frac{d\lambda_\Gamma}{dt} = \sum_{\substack{e \text{ edge of } \Gamma \\ \Gamma - e \text{ connected}}} \lambda_{\Gamma - e} \quad + \sum_{\substack{e \text{ edge of } \Gamma \\ \Gamma - e \text{ disconnected}}} \lambda_{\Gamma'}\lambda_{\Gamma''} \tag{4.8}$$

where $\Gamma - e$ is the graph obtained by removing an edge in Γ and Γ' and Γ" are the two connected components of $\Gamma - e$ when it is disconnected.

Let us also stress here, that an equation of this type is also used to prove a result of E. M. Wright which expresses the generating function of connected graphs under certain conditions (fixed excess). To get this generating functional (see, for example, Proposition $II.6$ in the book [FS08]), one needs to consider contributions of two types of edges (first contribution when the edge is a bridge and a second one when not - see again Fig. 4.1).
In the sequel, we generalize such an equation to matroids using a Hopf algebra formulation. This generalization is made possible by the fact that equation (4.8) above has two types of terms. These terms correspond to removing an edge firstly by leaving the graph connected and secondly by increasing the number of connected components of the graph. These two types of terms correspond to the terms containing δ_{loop} and respectively δ_{coloop} in the differential equations (4.37) and (4.43) (see section 4.4).

4.2 Matroid Hopf algebra characters

In this section we define two infinitesimal characters which are then exponentiated in an appropriate way such that a Hopf algebra character is obtained.

Let us first recall the following definitions:

Definition 4.2.1. *Let $\mathcal{M}$ be a minor-closed family of matroids and let f, g be two mappings in $Hom(k(\widetilde{\mathcal{M}}), k(\widetilde{\mathcal{M}}))$. The convolution product of f and g is given by the following formula*

$$f * g = m \circ (f \otimes g) \circ \Delta, \tag{4.9}$$

where we have denoted by m the multiplication law, given here by the matroid direct sum.

Definition 4.2.2. *A matroid Hopf algebra* **character** *f is an algebra morphism from the matroid Hopf algebra into a fixed commutative ring $\mathbb{K}$. This means that one has:*

$$f(M_1 \oplus M_2) = f(M_1)f(M_2), \quad f(\mathbf{1}) = 1_{\mathbb{K}}. \tag{4.10}$$

Definition 4.2.3. *A matroid Hopf algebra* **infinitesimal character** *g is a linear morphism from the matroid Hopf algebra into a fixed commutative ring $\mathbb{K}$, such that*

$$g(M_1 \oplus M_2) = g(M_1)\epsilon(M_2) + \epsilon(M_1)g(M_2). \tag{4.11}$$

Since we work in a Hopf algebra where the non-trivial part of the coproduct is nilpotent, we can also define an exponential map by the following expression

$$\exp_*(\delta) := \epsilon + \delta + \frac{1}{2}\delta * \delta + \ldots = \sum_{k\geq 0} \frac{1}{k!}\delta^{*k} \tag{4.12}$$

where δ is an infinitesimal character.

Lemma 4.2.4. *If δ is an infinitesimal character, then $\exp_\star(\delta)$ is a character*

Proof. One can use induction to prove that

$$\delta^{*k}(xy) = \sum_{i=0}^{k} \binom{k}{i} \delta^{*i}(x)\delta^{*(k-i)}(y) \tag{4.13}$$

One then has

$$\begin{aligned} \exp_*(\delta)(x)\exp_*(\delta)(y) &= \left(\sum_{i\geq 0}\frac{1}{i!}\delta^{*i}(x)\right)\left(\sum_{j\geq 0}\frac{1}{j!}\delta^{*j}(j)\right) = \sum_{k\geq 0}\sum_{i+j=k}\frac{1}{i!j!}\delta^{*i}(x)\delta^{*j}(y) \\ &= \sum_{k\geq 0}\frac{1}{k!}\delta^{*k}(xy) = \exp_*(\delta)(xy). \end{aligned} \tag{4.14}$$

□

As already stated above (see Remark 2.2.11), there are only two matroids with unit cardinal ground set, $U_{0,1}$ and $U_{1,1}$. We now define two maps δ_{loop} and δ_{coloop}.

$$\delta_{\text{loop}}(M) := \begin{cases} 1_{\mathbb{K}} \text{ if } M = U_{0,1}, \\ 0_{\mathbb{K}} \text{ otherwise }. \end{cases} \tag{4.15}$$

$$\delta_{\text{coloop}}(M) := \begin{cases} 1_{\mathbb{K}} \text{ if } M = U_{1,1}, \\ 0_{\mathbb{K}} \text{ otherwise }. \end{cases} \tag{4.16}$$

One can directly check that these maps are **infinitesimal characters** of the matroid Hopf algebra defined in Proposition 2.2.29.

We now define the following map:

$$\alpha_{x,y,s}(M) := \exp_* s\{\delta_{\text{coloop}} + (y-1)\delta_{\text{loop}}\} * \exp_* s\{(x-1)\delta_{\text{coloop}} + \delta_{\text{loop}}\}(M). \tag{4.17}$$

Example 4.2.5. *One has*

$$\begin{aligned} \alpha(x,y,s,U_{2,4}) &= (exp_* s\{\delta_{tree} + (y-1)\delta_{loop}\} * exp_* s\{(x-1)\delta_{tree} + \delta_{loop}\})(U_{2,4}) \\ &= (exp_* s\{\delta_{tree} + (y-1)\delta_{loop}\} \otimes exp_* s\{(x-1)\delta_{tree} + \delta_{loop}\})(\Delta(U_{2,4})) \\ &= (exp_* s\{\delta_{tree} + (y-1)\delta_{loop}\} \otimes exp_* s\{(x-1)\delta_{tree} + \delta_{loop}\})(1 \otimes U_{2,4} \\ &\quad +4U_{1,1} \otimes U_{1,3} + 6U_{2,2} \otimes U_{0,2} + 4U_{2,3} \otimes U_{0,1} + U_{2,4} \otimes 1) \\ &= s^4(x-1)^2 + 4s^4(x-1) + 6s^4 + 4s^4(y-1) + s^4(y-1)^2 \\ &= s^4[x^2 + y^2 + 2x + 2y]. \end{aligned} \tag{4.18}$$

One then has:

Proposition 4.2.6. *The map* (4.17) *is a character.*

Proof. The proof can be done by a direct check. On a more general basis, this is a consequence of the fact that δ_{loop} and δ_{coloop} are infinitesimal characters and the space of infinitesimal characters is a vector space; thus $s\{\delta_{\text{coloop}} + (y-1)\delta_{\text{loop}}\}$ and $s\{(x-1)\delta_{\text{coloop}} + \delta_{\text{loop}}\}$ are infinitesimal characters.
Using Lemma 4.2.4 and since the convolution of two characters is a character, one gets that α is a character. □

4.3 Proof of a Tutte polynomial convolution formula

Let $M = (E, \mathcal{I})$ be a matroid. One then has the following result:

Lemma 4.3.1. *One has:*

$$\exp_*\{a\delta_{\text{coloop}} + b\delta_{\text{loop}}\}(M) = a^{r(M)} b^{n(M)}. \tag{4.19}$$

Proof. Using the definition (4.12), the LHS of the identity (4.19) above reads:

$$\left(\sum_{k=0}^{\infty} \frac{(a\delta_{\text{coloop}} + b\delta_{\text{loop}})^{*k}}{k!} \right)(M). \tag{4.20}$$

All the terms in the sum above vanish, except the one for whom k is equal to $|E|$. Using the definition (4.9) of the convolution product, this term writes

$$\frac{1}{k!}\left(\sum_{i=0}^{k} a^{k-i}b^{i} \sum_{\substack{i_1+\cdots+i_n=k-i\\ j_1+\cdots+j_m=i}} \delta_{\text{coloop}}^{\otimes(i_1)}\otimes\delta_{\text{loop}}^{\otimes(j_1)}\otimes\cdots\otimes\delta_{\text{coloop}}^{\otimes(i_n)}\otimes\delta_{\text{loop}}^{\otimes(j_m)}\right) \left(\sum_{(i)} M^{(1)}\otimes\cdots\otimes M^{(k)}\right), \tag{4.21}$$

where we have used the notation $\Delta^{(k-1)}(M)=\sum_{(i)} M^{(1)}\otimes\cdots\otimes M^{(k)}$. Using the definitions (4.15) and respectively (4.16) of the infinitesimal characters δ_{loop} and respectively δ_{coloop}, implies that the submatroids $M^{(j)}$ $(j=1,\ldots,k)$ are equal to $U_{0,1}$ or $U_{1,1}$.
Using the definition of the rank and of the nullity of a matroid, the number of δ_{coloop} and δ_{loop} appearing in Equation (4.21) equal to the rank and the nullity of the matroid M, respectively. One then gets that

$$\exp_*\{a\delta_{\text{coloop}}+b\delta_{\text{loop}}\}(M) = \frac{1}{|E|!}a^{r(M)}b^{n(M)}\left(\sum_{\substack{i_1+\cdots+i_n=r(M)\\ j_1+\cdots+j_m=n(M)}} \delta_{\text{coloop}}^{\otimes(i_1)}\otimes\delta_{\text{loop}}^{\otimes(j_1)}\otimes\cdots \otimes\delta_{\text{coloop}}^{\otimes(i_n)}\otimes\delta_{\text{loop}}^{\otimes(j_m)}\right)\left(\sum_{(i)} M^{(1)}\otimes\cdots\otimes M^{(|E|)}\right). \tag{4.22}$$

It is easy to check that

$$\sum_{\substack{i_1+\cdots+i_n=r(M)\\ j_1+\cdots+j_m=n(M)}} \delta_{\text{coloop}}^{\otimes(i_1)}\otimes\delta_{\text{loop}}^{\otimes(j_1)}\otimes\cdots\otimes\delta_{\text{coloop}}^{\otimes(i_n)}\otimes\delta_{\text{loop}}^{\otimes(j_m)}\left(\sum_{(i)} M^{(1)}\otimes\cdots\otimes M^{(|E|)}\right) = \binom{|E|}{r(M)}. \tag{4.23}$$

From Equations (4.22) and (4.23), one gets the conclusion. □

Example 4.3.2. *Let us illustrate Lemma 4.3.1 for the uniform matroid $U_{k,n}$. One has $r(U_{k,n})=k$ and $n(U_{n,k})=n-k$. We now use the definitions* (4.15) *and respectively* (4.16) *of δ_{loop} and respectively δ_{coloop} to work out the LHS of identity* (4.19). *One has:*

$$\begin{aligned} exp_*\{a\delta_{\text{coloop}}+b\delta_{\text{loop}}\}(U_{k,n}) &= \frac{1}{n!}a^k b^{n-k}\delta_{\text{coloop}}^{\otimes k}\otimes\delta_{\text{loop}}^{\otimes(n-k)}\left(\binom{n}{n-1}\cdots\binom{2}{1}U_{1,1}^{\otimes k}\otimes U_{0,1}^{\otimes(n-k)}\right)\\ &= a^k b^{n-k}. \end{aligned} \tag{4.24}$$

Working out the definition formula (4.17) of the character α, one gets the following equivalent expression:

$$\begin{aligned} \alpha_{x,y,s}(M) = {} & \exp_*\left(s(\delta_{\text{coloop}}+(y-1)\delta_{\text{loop}})\right)*\exp_*\left(s(-\delta_{\text{coloop}}+\delta_{\text{loop}})\right)\\ & *\exp_*\left(s(\delta_{\text{coloop}}-\delta_{\text{loop}})\right)*\exp_*\left(s((x-1)\delta_{\text{coloop}}+\delta_{\text{loop}})\right). \end{aligned} \tag{4.25}$$

One then has:

Proposition 4.3.3. *The character* α *is related to the Tutte polynomial for matroids by the following identity:*

$$\alpha_{x,y,s}(M) = s^{|E|} T_M(x,y). \tag{4.26}$$

Proof. Using the definition (4.9) of the convolution product in the definition (4.17) of the character α, one has the following identity:

$$\alpha_{x,y,s}(M) = \sum_{A\subseteq E} \exp_* s\{\delta_{\text{coloop}} + (y-1)\delta_{\text{loop}}\}(M|_A)\ \exp_* s\{(x-1)\delta_{\text{coloop}} + \delta_{\text{loop}}\}(M/A). \tag{4.27}$$

We can now apply Lemma 4.3.1 on each of the two terms in the RHS of equation (4.27) above. One has

$$\begin{aligned}
\alpha_{x,y,s}(M) &= \sum_{A\subseteq E} s^{r(M|_A)} \left(s(y-1)\right)^{n(M|_A)} \left(s(x-1)\right)^{r(M/A)} s^{n(M/A)} \\
&= \sum_{A\subseteq E} s^{r(M|_A)+n(M|_A)+r(M/A)+n(M/A)} (x-1)^{r(M/A)} (y-1)^{n(M|_A)} \\
&= \sum_{A\subseteq E} s^{|E|} (x-1)^{r(M)-r(A)} (y-1)^{n(A)} \\
&= s^{|E|} T_M(x,y).
\end{aligned} \tag{4.28}$$

□

Example 4.3.4. *Let* $U_{k,n}$ *be a uniform matroid,* $0 \leq k \leq n$*. One has*

$$\alpha(x,y,s,U_{k,n}) = \sum_{i=0}^{k} \binom{n}{i} s^n (x-1)^{k-i} + \sum_{i=k+1}^{n} \binom{n}{i} s^n (y-1)^{i-k} = s^n T_{U_{k,n}}(x,y). \tag{4.29}$$

Using (2.70) and Proposition 4.3.3, one has the following consequence:

Corollary 4.3.5. *One has:*

$$\alpha_{x,y,s}(M) = \alpha_{y,x,s}(M^\star). \tag{4.30}$$

Proposition 4.3.3 allows to give a different proof of a matroid Tutte polynomial convolution identity, shown in [KRS99] and in [EL98]. One has:

Corollary 4.3.6. *(Theorem* 1 *of [KRS99]) The Tutte polynomial satisfies:*

$$T_M(x,y) = \sum_{A\subset E} T_{M|A}(0,y) T_{M/A}(x,0). \tag{4.31}$$

Proof. Taking $s = 1$, this is as a direct consequence of identity (4.25), and of Proposition 4.3.3. □

4.4 The recipe theorem

Let us define a map $\varphi_{a,b} : k(\widetilde{\mathcal{M}}) \to k(\widetilde{\mathcal{M}})$,

$$M \longmapsto p^{r(M)} q^{n(M)} M, \tag{4.32}$$

where p and q are scalars belonging to the ring k^* (the group of invertible elements of k).

Lemma 4.4.1. *The map $\varphi_{p,q}$ is a bialgebra automorphism.*

Proof. One can directly check that the map $\varphi_{p,q}$ is an algebra automorphism. Let us now check that this map is also a coalgebra automorphism. Using Lemma 2.2.22 and Lemma 2.2.24,

$$r(M|_T) + r(M/T) = r(M). \tag{4.33}$$

Thus, using the definitions of the map $\varphi_{p,q}$ and of the matroid coproduct, one has:

$$\Delta \circ \varphi_{p,q}(M) = \sum_{T \subseteq E} (p^{r(M|_T)} q^{n(M|_T)} M|_T) \otimes (p^{r(M/_T)} q^{n(M/_T)} M/_T). \tag{4.34}$$

Using again the definition of the map $\varphi_{p,q}$ leads to

$$\Delta \circ \varphi_{p,q}(M) = (\varphi_{p,q} \otimes \varphi_{p,q}) \circ \Delta(M), \tag{4.35}$$

which concludes the proof. □

Let us now define:

$$[f, g]_* := f * g - g * f. \tag{4.36}$$

Using the definition (4.17) of the Hopf algebra character α, one can directly prove the following result:

Proposition 4.4.2. *The character α is the solution of the differential equation:*

$$\frac{d\alpha}{ds} = x\alpha * \delta_{\text{coloop}} + y\delta_{\text{loop}} * \alpha + [\delta_{\text{coloop}}, \alpha]_* - [\delta_{\text{loop}}, \alpha]_* . \tag{4.37}$$

It is the fact that the matroid Tutte polynomial is a solution of the differential equation (4.37) that will be used now to prove the universality of the matroid Tutte polynomial. In order to do that, we take a four-variable matroid polynomial $Q_M(x, y, a, b)$ satisfying a multiplicative law

$$Q_{M_1 \oplus M_2}(x, y, a, b) = Q_{M_1}(x, y, a, b) Q_{M_2}(x, y, a, b), \forall M_1, M_2 \text{ matroids} \tag{4.38}$$

and which has the following properties:

- if e is a coloop, then
$$Q_M(x, y, a, b) = x Q_{M \setminus e}(x, y, a, b) \ , \tag{4.39}$$
- if e is a loop, then
$$Q_M(x, y, a, b) = y Q_{M/e}(x, y, a, b) \tag{4.40}$$

- if e is neither a loop nor coloop, then

$$Q_M(x,y,a,b) = aQ_{M\backslash e}(x,y,a,b) + bQ_{M/e}(x,y,a,b). \tag{4.41}$$

Remark 4.4.3. *Note that, when one deals with the same problem in the case of graphs, a supplementary multiplicative condition for the case of one-vertex joint of two graphs (i. e. identifying a vertex of the first graph and a vertex of the second graph into a single vertex of the resulting graph) is required (see, for example, [EMM11] or [Sok05]).*

We now define the map:

$$\beta(x,y,a,b,s,M) := s^{|E|}Q_M(x,y,a,b). \tag{4.42}$$

One then directly checks (using the definition (4.42) above and the multiplicative property of the polynomial Q) that this map is again a matroid Hopf algebra character.

Proposition 4.4.4. *The character* (4.42) *satisfies the following differential equation:*

$$\frac{d\beta}{ds}(M) = \left(x\beta * \delta_{\text{coloop}} + y\delta_{\text{loop}} * \beta + b[\delta_{\text{coloop}},\beta]_* - a[\delta_{\text{loop}},\beta]_*\right)(M). \tag{4.43}$$

Proof. Applying the definition (4.9) of the convolution product, the RHS of equation (4.43) above writes

$$\begin{aligned}&(x-b)\sum_{A\subseteq E}\beta(M|_A)\delta_{\text{coloop}}(M/_A) + (y-a)\sum_{A\subseteq E}\delta_{\text{loop}}(M|_A)\beta(M/_A)\\&+b\sum_{A\subseteq E}\delta_{\text{coloop}}(M|_A)\beta(M/_A) + a\sum_{A\subseteq E}\beta(M|_A)\delta_{\text{loop}}(M/_A).\end{aligned} \tag{4.44}$$

Using the definitions (4.15) and respectively (4.16) of the infinitesimal characters δ_{loop} and respectively δ_{coloop}, constraints the sums on the subsets A above. The RHS of (4.43) becomes:

$$\begin{aligned}&(x-b)\textstyle\sum_{A,M/_A=U_{1,1}}\beta(M|_A) + (y-a)\sum_{A,M|_A=U_{0,1}}\beta(M/_A)\\&+b\textstyle\sum_{A,M|_A=U_{1,1}}\beta(M/_A) + a\sum_{A,M/_A=U_{0,1}}\beta(M|_A).\end{aligned} \tag{4.45}$$

We now apply the definition of the Hopf algebra character β; one obtains:

$$\begin{aligned}&s^{|E|-1}[(x-b)\textstyle\sum_{A,M/_A=U_{1,1}}Q(x,y,a,b,M|_A) + (y-a)\sum_{A,M|_A=U_{0,1}}Q_{M/_A}(x,y,a,b)\\&+b\textstyle\sum_{A,M|_A=U_{1,1}}Q_{M/_A}(x,y,a,b) + a\sum_{A,M/_A=U_{0,1}}Q_{M|_A}(x,y,a,b)].\end{aligned} \tag{4.46}$$

We can now directly analyze the four particular cases $M/_A = U_{1,1}$, $M/_A = U_{0,1}$, $M|_A = U_{1,1}$ and $M|_A = U_{0,1}$:

- If $M/_A = U_{1,1}$, we can denote the ground set of $M/_A$ by $\{e\}$. Note that e is a coloop. From Lemma 2.2.22, one has $M|_A = M\backslash_{E-A} = M\backslash e$. One then has $Q_M(x,y,a,b) = xQ_{M|_A}(x,y,a,b)$.
- If $M|_A = U_{0,1}$, then $A = \{e\}$ and e is a loop of M. Thus, one has $Q_M(x,y,a,b) = yQ_{M/_A}(x,y,a,b)$

- If $M|_A = U_{1,1}$, then $A = \{e\}$. One has to distinguish between two subcases:
  - e is a coloop of M. Then, by Lemma 2.2.23, $M/e = M\backslash e$. Thus, one has $Q_M(x,y,a,b) = xQ_{M|_A}(x,y,a,b)$.
  - e is neither a loop nor coloop of M.
- If $M/_A = U_{0,1}$, one can denote the ground set of $M/_A$ by $\{e\}$. There are again two subcases to be considered:
  - e is a loop of M, one has that $M|_A = M\backslash_{(E-A)} = M\backslash_{\{e\}} = M/e$. Then one has $Q_M(x,y,a,b) = yQ_{M|_A}(x,y,a,b)$.
  - e is a nonseparating point of M, then one has $M|_A = M\backslash_{(E-A)} = M\backslash_{\{e\}}$

We now insert all of this in equation (4.46); this leads to three types of sums over some element e of the ground set E, e being a loop, a coloop or a nonseparating point:

$$s^{|E|-1}[\sum_{e\in E: e \text{ is a coloop}} Q_M(x,y,a,b) + \sum_{e\in E: e \text{ is a loop}} Q_M(x,y,a,b) + \sum_{e\in E: e \text{ is a regular element}} Q_M(x,y,a,b)] \tag{4.47}$$

This rewrites as

$$|E|s^{|E|-1}Q_M(x,y,a,b) = \frac{d\beta}{ds}(M), \tag{4.48}$$

which completes the proof. □

We can now state the **main result** of this paper, the recipe theorem specifying how to recover the general matroid polynomial Q_M as an evaluation of the Tutte polynomial T_M:

Theorem 4.4.5. *If one has a four-variable matroid polynomial $Q_M(x,y,a,b)$ satisfying the multiplicative law* (4.38) *and the conditions* (4.39), (4.40) *and* (4.41), *then one has:*

$$Q_M(x,y,a,b) = a^{n(M)}b^{r(M)}T_M(\frac{x}{b},\frac{y}{a}). \tag{4.49}$$

Proof. The proof is a direct consequence of Propositions 4.3.3, 4.4.2 and 4.4.4 and of Lemma 4.4.1. This comes from the fact that one can apply the automorphism $\varphi_{a,b}$ defined in (4.32) to the differential equation (4.43). One then obtains the differential equation (4.37) with modified parameters x/b and y/a. Finally, the solution of this differential equation is (trivially) related to the matroid Tutte polynomial T_M (see Proposition 4.3.3) and this concludes the proof. □

As already announced above, this is a new proof of the universality property of the Tutte polynomial for matroids (the interested reader may refer for example to T. Brylawski and J. Oxley's article [BO92]).

Let us end this section by stating that all the results obtained in this paper naturally hold for graphs (instead of matroids), since graphs are a particular class of matroids (the graphic matroids, see subsection 2.2). We have thus given here the proofs of the graph results conjectured in [KM11].

Chapter 5

A combinatorial non-commutative Hopf algebra of graphs

Contents

The results presented in this Chapter follow [DFHN$^+$].

5.1 Why discrete scales?

As already announced above, the idea of decorating the edges of a graph with discrete scales comes from quantum field theory, or more precisely from the multi-scale analysis technique used in perturbative and in constructive renormalization (see Vincent Rivasseau's book [Riv91]).

In quantum field theory each edge of a graph is associated to a *propagator* $C = 1/H$ (which, in elementary particle physics represents a particle). Introducing discrete scales comes to a "slicing" of the propagator

$$C = \int_0^\infty e^{-\alpha H} d\alpha \, , \; \sum_{i=0}^{\infty} C^i \tag{5.1}$$

$$C^a = \int_{M^{-2a}}^{M^{-2(a-1)}} e^{-\alpha H} d\alpha \, , \; C^0 = \int_1^\infty e^{-\alpha H} d\alpha. \tag{5.2}$$

When some discrete integer a is associated to a given edge, this means that the propagator assigned to this edge lies within a given energy scale. One thus introduces more information

(replacing graphs by "assigned graphs") which yields in turn some refinement of the analysis, as we will explain here.
When integrating over the energy scales of the internal propagators in a Feynman graph in quantum field theory, one obtains the *Feynman integral* associated to the respective graphs. Usually, these integrals are divergent. This is when *renormalization* comes in, subtracting (when possible) the divergent parts of these Feynman integrals, in a self-consistent way (see again Vincent Rivasseau's book [Riv91] or any other textbook on renormalization). Nevertheless, these divergences only appear for high energies (the so-called *ultraviolet regime*)[1], which corresponds, within the multi-scale formalism, to the case when all the integer scales associated to the internal edges are higher then the edges associated to the external edges (see again Vincent Rivasseau's book [Riv91] for details).

When dealing with this divergence subtraction (the subtraction of the so-called "counterterms"), an important "technical" complication is given by the issue of "overlapping divergences", which is given by overlapping subgraphs which lead, independently, to divergences. This problem is solved in an elegant way within the multi-scale analysis, where all subgraphs leading to divergences are either disjoint or nested.

Let us also emphasize that the multi-scale renormalization technique splits the counterterms into two categories: "useful" and "useless" counterterms (the useful ones being the ones corresponding to subgraphs where all the integer scales associated to the internal edges are higher then the edges associated to the external edges). This refining is not possible without the scale decoration of edges; furthermore, it also solves another major problem of renormalization, the so-called "renormalon problem" (which is an issue when one wants to sum over the contribution of each term in perturbation theory).

This versatile technique of multi-scale analysis was successfully applied for scalar quantum field theory renormalization (see again [Riv91]), the condensed matter case [BG90], [FT90], [Riv12], renormalization of scalar quantum field theory on the non-commutative Moyal space (see [GMRT09], [GMRVT06], [GRVT06], [VFR06] and [VT07]) and recently to the renormalization of quantum gravity tensor models [GR13],[COR].

The combinatorics of the multi-scale renormalization was encoded in a Hopf algebraic framework in [KRT]. As already announced above, the Hopf algebraic setting of [KRT] is commutative, and the assigned graphs designed there can have equal scale integers for several edges of the same graph.

In this chapter, we allow graphs with self-loops (*i. e.* edges which have both ends hooked to the same vertex) and multi-edges. However, graphs made of a single vertex and empty edge set are not allowed.

Let us also mention that a commutative and a non-commutative CHA of graphs **of type II**, were defined in [Sch94].

[1]Divergences for low energies (the *infrared regime*) can also appear in quantum field theory, but one can deal with this type of divergences in a different way. This lies outside the purpose of this section.

5.2 Non-commutative graph algebra structure

In this section we define the space of totally assigned graphs (TAG) and a non-commutative algebra structure on this space.

Definition 5.2.1. *A* **totally ordered scale assignment** μ *for a graph* Γ *is a total order on the set* $E(\Gamma)$ *of edges of* Γ.

It will be convenient to visualize the total order μ by choosing a compatible labelling, i.e. an injective increasing map from $\big(E(\Gamma),\mu\big)$ into $\mathbb{N}^* = \{1,2,3,\ldots\}$. There is of course an infinite number of possible labellings. The unique such map with values in $\{1,\ldots,|E(\Gamma)|\}$ will be called the *standard labelling* associated with μ.

Example 5.2.2. *An example of a totally ordered scale assignment with nonstandard labelling is given in Fig. 5.1.*

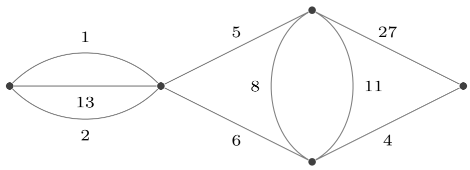


Figure 5.1: A graph with a totally ordered scale assignment.

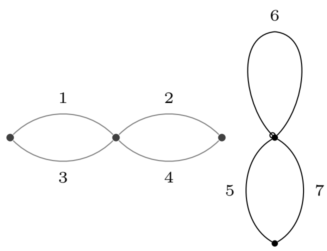
Definition 5.2.3. *A* **totally assigned graph** *(TAG) is a pair* (Γ,μ) *formed by a graph* Γ *(not necessarily connected), together with a totally ordered scale assignment* μ.

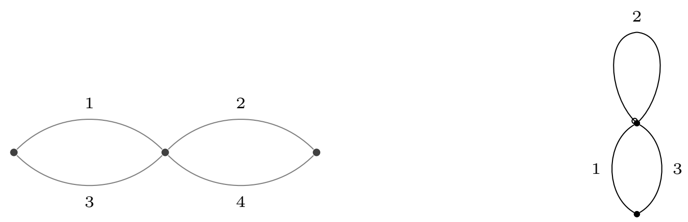
Consider now a field $\mathbb{K}$ of characteristic 0, and let $\mathcal{H}$ be the $\mathbb{K}$- vector space freely spanned by TAGs. The **product** m on $\mathcal{H}$ is given by:

$$m\Big((\Gamma_1,\mu),(\Gamma_2,\nu)\Big) := (\Gamma_1,\mu)\cdot(\Gamma_2,\nu) := (\Gamma_1\sqcup\Gamma_2,\mu\sqcup\nu), \tag{5.3}$$

where $\Gamma_1\sqcup\Gamma_2$ is the disjoint union of the two graphs, and where $\mu\sqcup\nu$ is the *ordinal sum order*, i.e. the unique total order on $E(\Gamma_1)\sqcup E(\Gamma_2)$ which coincides with μ (resp. ν) on Γ_1 (resp. Γ_2), and such that $e_1 < e_2$ for any $e_1\in\Gamma_1$ and $e_2\in\Gamma_2$. Although the disjoint union of graphs is commutative, the product is not because the total orders $\mu\sqcup\nu$ and $\nu\sqcup\mu$ are different (see also Remark 5.2.7 below). Associativity is however obvious. The empty TAG is the empty graph, denoted by $1_{\mathcal{H}}$.

Example 5.2.4. *Let* (Γ_1,μ_1) *and* (Γ_2,μ_2) *be the two graphs in Fig. 5.2. One has*

$m\Big((\Gamma_1,\mu_1),(\Gamma_2,\mu_2)\Big) =$  .



(a) The TAG (Γ_1, μ_1). (b) The TAG (Γ_2, μ_2).

Figure 5.2: Two examples of TAGs.

Let $G = (\Gamma, \mu)$ be a non-empty TAG. The set $E(\Gamma)$ of its edges is endowed with the total order μ. Say that G is *decomposable* if it can be split into two non-empty components $G_1 = (\Gamma_1, \mu_1)$ and $G_2 = (\Gamma_2, \mu_2)$ such that:

1. μ_i is the restriction of the total order μ to $E(\Gamma_i)$, $i = 1, 2$.
2. For any $e_1 \in E(\Gamma_1)$ and $e_2 \in E(\Gamma_2)$ we have $e_1 < e_2$.
3. The two components are disconnected, i.e. no edge of Γ_1 hook to any edge of Γ_2.

In that case we obviously have $G = G_1 \cdot G_2$ for the product just defined above. Otherwise the TAG G is called *indecomposable*.

Summing up:

Proposition 5.2.5. *$(\mathcal{H}, m, 1_{\mathcal{H}})$ is a free associative unitary algebra.*

Proof. The freeness of the algebra remains to be proved. Let Ind denote the set of indecomposables TAGs.
Iterating this decomposition process, we clearly can obtain any TAG G as a finite product of indecomposable TAGs:

$$G_1 \cdots G_k$$

with $G_j = (\Gamma_j, \mu_j)$, $j = 1, \ldots, k$, and where μ_j is the total order μ restricted to $E(\Gamma_j)$. The set $E(\Gamma_k)$ is the smallest terminal segment of $E(\Gamma)$ such that no edge in it hook to other edges in $E(\Gamma)$. Hence the last component G_k is uniquely defined, and the whole decomposition as well by iterating this argument.
One then has that $\mathcal{H}$ is isomorphic to $\mathbb{K}\langle Ind \rangle$.
Hence, the set of TAGs endowed with the product defined above is the free monoid generated by the indecomposable TAGs, which proves Proposition 5.2.5. □

Remark 5.2.6. *An indecomposable TAG is not necessarily connected (see Fig. 5.3).*

Remark 5.2.7. *The standard labelling of the product $(\Gamma_1, \mu_1) \cdot (\Gamma_2, \mu_2)$ is obtained by keeping the standard labelling for $E(\Gamma_1)$ and shifting the standard labelling of $E(\Gamma_2)$ by $|E(\Gamma_1)|$.*

Let us end this section by the following example illustrating the non-commutativity of our product:

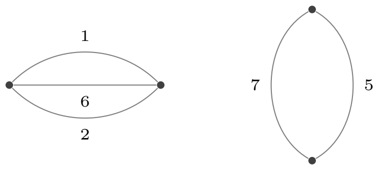
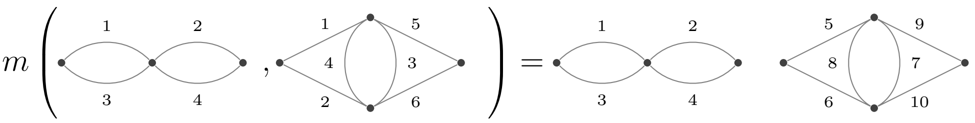


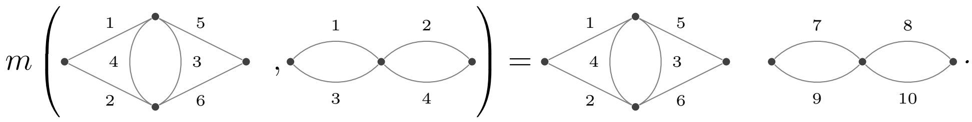
Figure 5.3: A non connected indecomposable TAG.

Example 5.2.8. *One has*



(5.4)

and



(5.5)

5.3 Hopf algebra structure

Let us first give the following definitions:

Definition 5.3.1. *A subgraph γ of a graph Γ is the graph formed by a given subset of edges e of the set of edges of the graph Γ together with the vertices that the edges of e hook to in Γ.*

Let us notice that a subgraph is not necessary connected nor spanning.

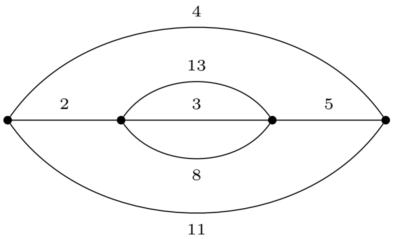
Definition 5.3.2. *A* **totally assigned subgraph** (γ, ν) *of a given TAG (Γ, μ) is a subgraph γ of Γin the sense of Definition 5.3.1, together with the total order ν on $E(\gamma)$ induced by μ. The* **shrinking** $(\Gamma, \mu)/(\gamma, \nu)$ *of a given TAG (Γ, μ) by a totally assigned subgraph (γ, ν) is defined as follows: the cograph Γ/γ is obtained by shrinking each connected component of γ on a point (i.e. to shrink a subgraph means to erase erase its edges and identify its vertices), and the totally ordered scale assignment μ/ν of the cograph Γ/γ is given by restricting the total order μ on the edges of the cograph, i.e. the edges of Γ which are not internal to γ. The TAG $(\Gamma/\gamma, \mu/\nu)$ is called a totally assigned cograph.*

Example 5.3.3. *1) Let us give an example, see Figure 5.5 of the shrinking of a totally assigned subgraph.*
2) Let us give another example, see Figure 5.7 of the shrinking of a totally assigned subgraph.

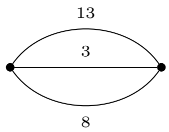
Let us now define the coproduct $\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ as

$$\Delta\Big((\Gamma, \mu)\Big) := \sum_{\emptyset \subseteq (\gamma,\nu) \subseteq (\Gamma,\mu)} (\gamma, \nu) \otimes (\Gamma/\gamma, \mu/\nu) \tag{5.6}$$

for any TAG (Γ, μ).



(a) The TAG (Γ, μ).



(b) The totally assigned subgraph (γ, ν).

Figure 5.4: An example of a totally assigned subgraph.

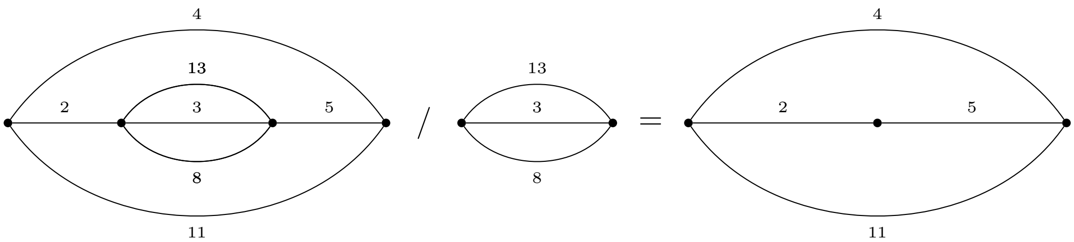
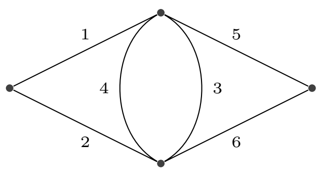
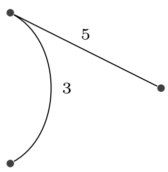


Figure 5.5: The cograph $(\Gamma/\gamma, \mu/\nu)$ is obtained by shrinking of (γ, ν), in Fig. 5.4b "inside" (Γ, μ), Fig. 5.4a.



(a) The TAG (Γ_1, μ_1).



(b) The totally assigned subgraph (γ_1, ν_1).

Figure 5.6: An example of a totally assigned subgraph.

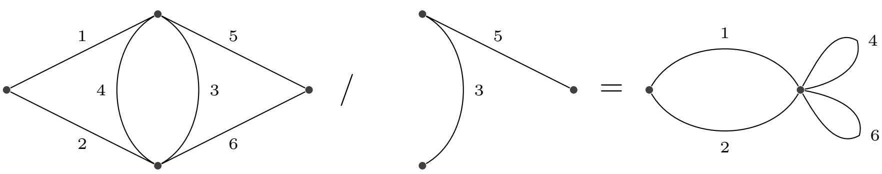
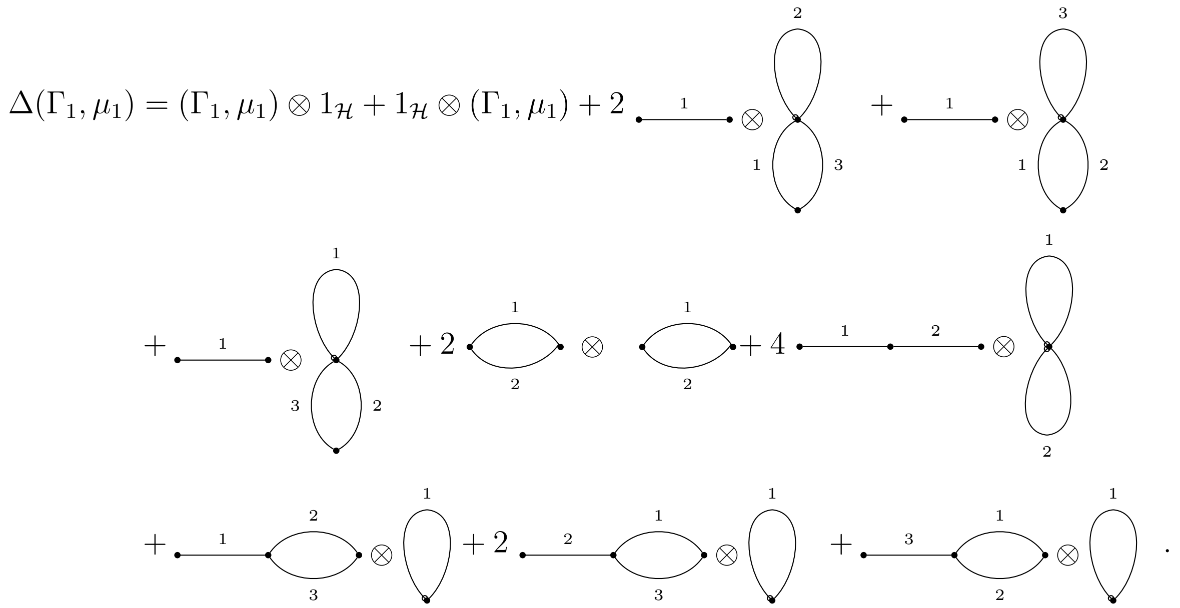


Figure 5.7: The cograph $(\Gamma_1/\gamma_1, \mu_1/\nu_1)$ is obtained by shrinking of (γ_1, ν_1), in Fig. 5.4b "inside" of (Γ_1, μ_1), Fig. 5.4a.

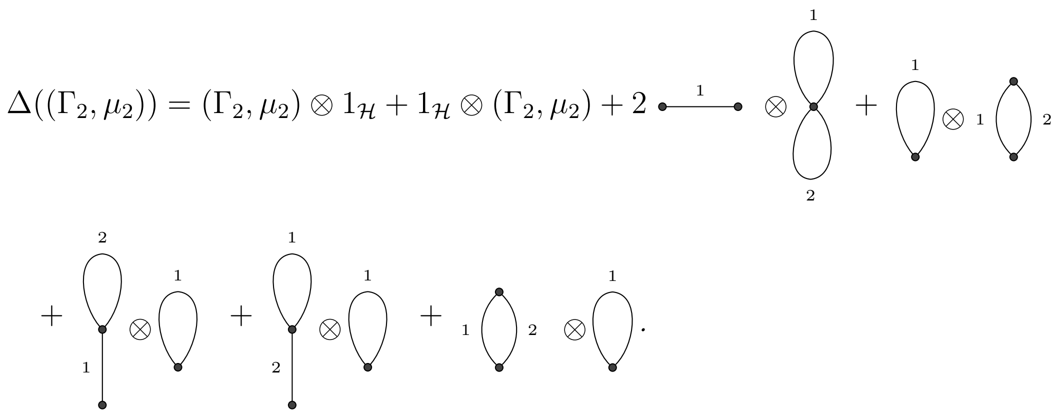
Example 5.3.4. *1) Let* (Γ_1, μ_1) *be the TAG in Fig. 5.2a.*

One has the coproduct:



(5.7)

2) Let (Γ_2,μ_2) *be the TAG given in Figure 5.2b.*



Lemma 5.3.5. *Let* (Γ,μ) *be a TAG in* $\mathcal{H}$*. Let* (γ,ν) *and* (δ,ν') *be two totally assigned subgraphs such that* $(\delta,\nu')\subseteq(\gamma,\nu)\subseteq(\Gamma,\mu)$*. Then, one has*

$$(\Gamma/\gamma,\mu/\nu) = \Big((\Gamma/\delta)/(\gamma/\delta),(\mu/\nu')/(\nu/\nu')\Big). \tag{5.8}$$

Proof. Since, $\delta\subseteq\gamma\subseteq\Gamma$, then one has $\gamma/\delta\subseteq\Gamma/\delta$. One has $\Gamma/\gamma = (\Gamma/\delta)/(\gamma/\delta)$. Moreover, the total order ν (resp. ν') is induced by restriction of μ (resp. ν or μ) to the set of edges of δ. Then $\mu/\nu = (\mu/\nu')/(\nu/\nu')$, which concludes the proof.

□

Proposition 5.3.6. *The coproduct defined in* (5.6) *is coassociative.*

Proof. Let $(\Gamma, \mu) \in \mathcal{H}$. Then, one has:

$$\begin{aligned}(\Delta \otimes Id) \circ \Delta\Big((\Gamma, \mu)\Big) &= (\Delta \otimes Id) \left(\sum_{(\gamma,\nu)\subset(\Gamma,\mu)} (\gamma, \nu) \otimes (\Gamma/\gamma, \mu/\nu) \right) \\ &= \sum_{(\gamma,\nu)\subset(\Gamma,\mu)} \left(\sum_{(\gamma',\nu')\subset(\gamma,\nu)} (\gamma', \nu') \otimes (\gamma/\gamma', \nu/\nu') \right) \otimes (\Gamma/\gamma, \mu/\nu) \\ &= \sum_{\substack{(\gamma,\nu)\subset(\Gamma,\mu) \\ (\gamma',\nu')\subset(\gamma,\nu)}} (\gamma', \nu') \otimes (\gamma/\gamma', \nu/\nu') \otimes (\Gamma/\gamma, \mu/\nu). \end{aligned} \tag{5.9}$$

and

$$\begin{aligned}(Id \otimes \Delta) \circ \Delta((\Gamma, \mu)) &= (Id \otimes \Delta) \left(\sum_{(\gamma,\nu))\subset(\Gamma,\mu)} (\gamma, \nu) \otimes (\Gamma/\gamma, \mu/\nu) \right) \\ &= \sum_{(\gamma,\nu)\subset(\Gamma,\mu)} (\gamma, \nu) \otimes \left(\sum_{(\gamma',\nu')\subset(\Gamma/\gamma,\mu/\nu)} (\gamma', \nu') \otimes \Big((\Gamma/\gamma)/\gamma', (\mu/\nu)/\nu'\Big) \right). \end{aligned} \tag{5.10}$$

There is a one-to-one correspondence between the assigned subgraphs $(\gamma', \nu') \subseteq (\Gamma/\gamma, \mu/\nu)$ and the assigned subgraphs $(\gamma_1, \nu_1) \subseteq (\Gamma, \mu)$ such that $(\gamma, \nu) \subseteq (\gamma_1, \nu_1)$. Indeed, starting from an assigned subgraph $(\gamma_1, \nu_1) \subseteq (\Gamma, \mu)$ such that $(\gamma, \nu) \subseteq (\gamma_1, \nu_1)$, we find an assigned subgraph $(\gamma', \nu') \subseteq (\Gamma/\gamma, \mu/\nu)$ by restricting the total order ν_1 to the edges of γ_1 which are not internal to γ, and the inverse operation consists in extending the total order ν' to all edges of γ_1 in the unique way compatible with the total order μ on $E(\Gamma)$.
Applying Lemma 5.3.5, one has $\Big((\Gamma/\gamma)/\gamma', (\mu/\nu)/\nu'\Big) = (\Gamma/\gamma_1, \mu/\nu_1)$. Equation (5.10) can then be rewritten as follows:

$$\begin{aligned}&(Id \otimes \Delta) \circ \Delta\Big((\Gamma, \mu)\Big) \\ &= \sum_{\substack{(\gamma_1,\nu_1)\subset(\Gamma,\mu) \\ (\gamma,\nu)\subset(\gamma_1,\nu_1)}} (\gamma, \nu) \otimes (\gamma_1/\gamma, \nu_1/\nu) \otimes (\Gamma/\gamma_1, \mu/\nu_1). \end{aligned} \tag{5.11}$$

Using equations (5.9) and (5.11), one concludes the proof. □

Furthermore, we define the counit $\epsilon : \mathcal{H} \longrightarrow \mathbb{K}$ by

$$\epsilon\Big((\Gamma, \mu)\Big) := \begin{cases} 1 \text{ if } (\Gamma, \mu) = 1_{\mathcal{H}}; \\ 0 \text{ otherwise.} \end{cases} \tag{5.12}$$

Theorem 5.3.7. *The triple $(\mathcal{H}, \Delta, \epsilon)$ is a coassociative coalgebra with counit.*

Proof. Let us show that ϵ is a counit of the coalgebra. For any TAG (Γ, μ), one has

$$(\epsilon \otimes Id) \circ \Delta\Big((\Gamma, \mu)\Big) = \epsilon\Big((\Gamma, \mu)\Big)1 + \epsilon(1_{\mathcal{H}})(\Gamma, \mu) + \sum_{(\gamma,\nu)\subsetneq(\Gamma,\mu)} \epsilon(\gamma, \nu)(\Gamma/\gamma, \mu/\nu) = (\Gamma, \mu).$$

Analogously, one has: $(Id \otimes \epsilon) \circ \Delta\Big((\Gamma, \mu)\Big) = (\Gamma, \mu)$. One thus concludes that the maps Id, $(\epsilon \otimes Id) \circ \Delta$ and $(Id \otimes \epsilon) \circ \Delta$ coincide on TAGs, thus proving that ϵ is a counit of Δ. Using now Proposition 5.3.6, one concludes the proof. □

Example 5.3.8. *Let us illustrate the coassociativity of our coproduct on the example of the standard labeled TAG of Fig. 5.8. When acting with the coproduct on this standard labeled*

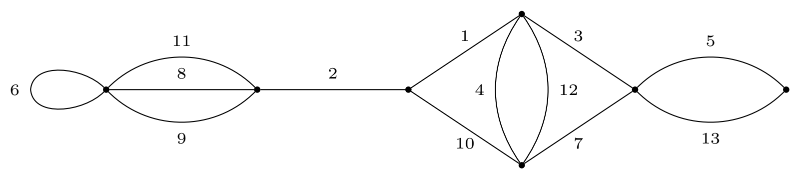


Figure 5.8: A standard labeled TAG.

TAG, one gets the term of Fig. 5.9, which is one of the terms appearing in the sum. Another

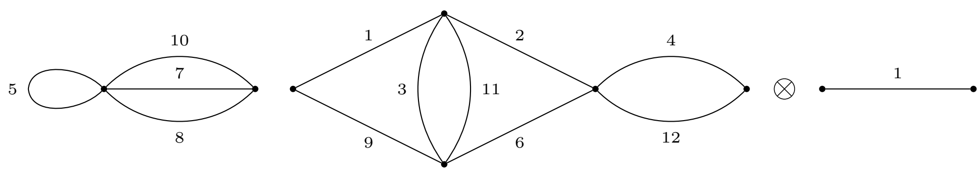


Figure 5.9: One of the terms obtained by acting with the coproduct on the standard labeled TAG of Fig 5.8.

type of term is the one of Fig. 5.10 (which again adds up to the rest of the coproduct terms). Acting now on these terms with $(\Delta \otimes Id)$ and respectively with $(Id \otimes \Delta)$ leads to the same

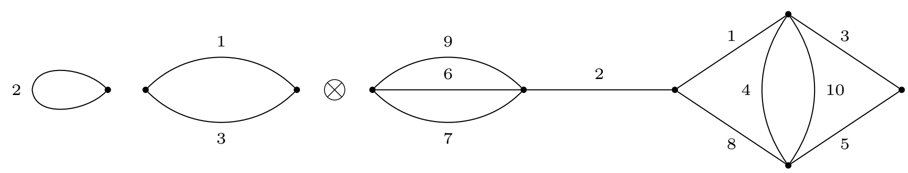


Figure 5.10: Another terms obtained by acting with the coproduct on the standard labeled TAG of Fig 5.8.

term represented in Fig. 5.11 with standard labelling. Let us also emphasize that this term cannot be obtained from other terms of Δ because of the diagrammatic difference of the various disconnected components of the graphs.

Example 5.3.9. *Let us illustrate the coassociativity of our coproduct on the example of the standard labeled TAG of Fig. 5.12.*

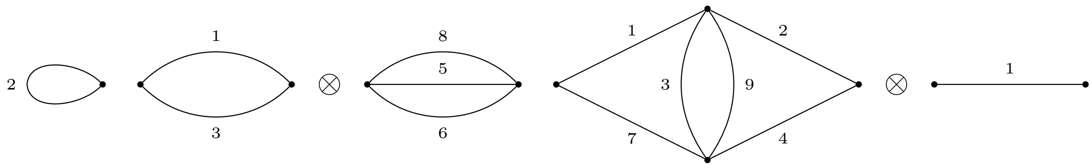


Figure 5.11: The resulting in the LHS and RHS of the coassociativity identity.

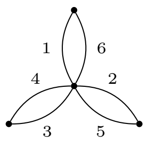


Figure 5.12: A standard labeled TAG.

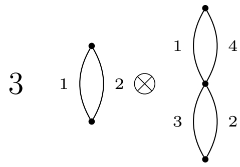


Figure 5.13: One of the terms obtained by acting with the coproduct on the standard labeled TAG of Fig 5.8.

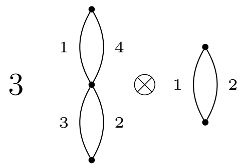


Figure 5.14: Another terms obtained by acting with the coproduct on the standard labeled TAG of Fig 5.12.

When acting with the coproduct on this standard labeled TAG, one gets the term of Fig. 5.13, which is one of the terms appearing in the sum. Another type of term is the one of Fig. 5.14 (which again adds up to the rest of the coproduct terms). Acting now on these terms with $(\Delta \otimes Id)$ *and respectively with* $(Id \otimes \Delta)$ *leads to the same term represented in Fig. 5.15 with standard labelling.*

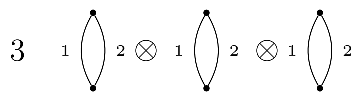


Figure 5.15: The resulting in the LHS and RHS of the coassociativity identity.

Let us also emphasize that this term cannot be obtained from other terms of Δ *because of the* *diagrammatic difference of the various disconnected components of the graphs.*

One has:

Proposition 5.3.10. *Let* (Γ_1, μ_1) *and* (Γ_2, μ_2) *be two TAGs in* $\mathcal{H}$*. One has*

$$\Delta\Big(m\big((\Gamma_1, \mu_1), (\Gamma_2, \mu_2)\big)\Big) = m^{\otimes 2} \circ \tau_{23}\big(\Delta(\Gamma_1, \mu_1), \Delta(\Gamma_2, \mu_2)\big) \tag{5.13}$$

where τ_{23} *is the flip of the two middle factors in* $\mathcal{H}^{\otimes 4}$*.*

Proof. One has

$$\begin{aligned}
\Delta\Big(m\big((\Gamma_1, \mu_1), (\Gamma_2, \mu_2)\big)\Big) &= \Delta(\Gamma_1 \sqcup \Gamma_2, \mu_1 \sqcup \mu_2) &(5.14)\\
&= \sum_{\emptyset \subseteq (\gamma,\nu) \subseteq (\Gamma_1 \sqcup \Gamma_2, \mu_1 \sqcup \mu_2)} (\gamma, \nu) \otimes \big((\Gamma_1 \sqcup \Gamma_2)/\gamma, (\mu_1 \sqcup \mu_2)/\nu\big)\\
&= \sum_{\substack{(\gamma_1,\nu_1) \subseteq (\Gamma_1, \mu_1)\\ (\gamma_2,\nu_2) \subseteq (\Gamma_2, \mu_2)}} (\gamma_1 \sqcup \gamma_2, \nu_1 \sqcup \nu_2) \otimes (\Gamma_1/\gamma_1 \sqcup \Gamma_2/\gamma_2, \mu_1/\nu_1 \sqcup \mu_2/\nu_2)\\
&= m^{\otimes 2} \circ \tau_{23}(\Delta(\Gamma_1, \mu_1), \Delta(\Gamma_2, \mu_2)). &(5.15)
\end{aligned}$$

□

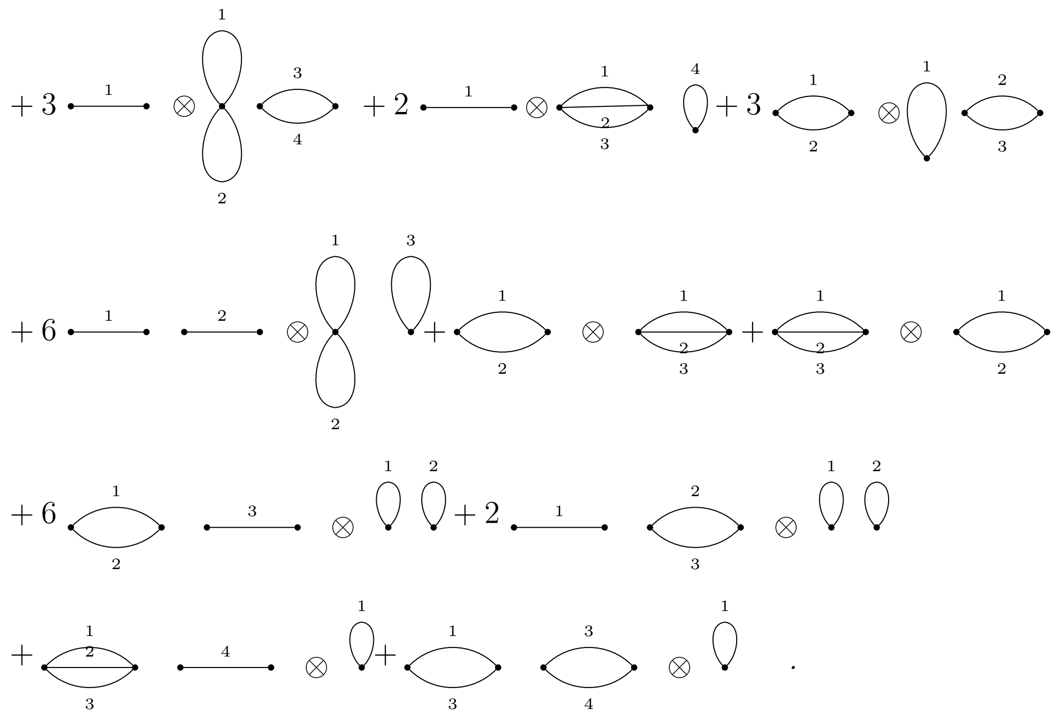
Example 5.3.11. *Let* (Γ_1, μ_1) *and* (Γ_2, μ_2) *be the graph in Fig. 5.16.*

(a) A TAG (Γ_1, μ_1). (b) A TAG (Γ_2, μ_2).

Figure 5.16: Two TAGs.

One has:

$$\Delta(m((\Gamma_1,\mu_1)\otimes(\Gamma_2,\mu_2))) = (\Gamma_1\sqcup\Gamma_2,\mu_1\sqcup\mu_2)\otimes 1_{\mathcal{H}} + 1_{\mathcal{H}}\otimes(\Gamma_1\sqcup\Gamma_2,\mu_1\sqcup\mu_2)$$



(5.16)

One has:

Theorem 5.3.12. *$(\mathcal{H}, m, 1_{\mathcal{H}}, \Delta, \epsilon)$ is a bialgebra.*

Proof. Using Proposition 5.3.10, it follows that Δ is a morphism of algebras. Moreover, it is easy to check that the counit ϵ is also a morphism of algebras. One thus concludes the proof. □

For all $n \in \mathbb{N}$, one calls $\mathcal{H}(n)$ the vector space generated by the TAGs with n edges. Then one has $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}(n)$. Moreover, one has:

1. For all $m, n \in \mathbb{N}$, $\mathcal{H}(m)\mathcal{H}(n) \subseteq \mathcal{H}(m+n)$.
2. For all $n \in \mathbb{N}$, $\Delta\Big(\mathcal{H}(n)\Big) \subseteq \sum_{i+j=n} \mathcal{H}(i)\otimes\mathcal{H}(j)$.

One thus concludes that $\mathcal{H}$ is a **graded bialgebra**. Note that $\mathcal{H}$ is **connected**.
We can now state the main result of this section:

Theorem 5.3.13. *The bialgebra $(\mathcal{H}, m, 1_{\mathcal{H}}, \Delta, \epsilon)$ is a Hopf algebra.*

Proof. The bialgebra $(\mathcal{H}, m, 1_{\mathcal{H}}, \Delta, \epsilon)$ is connected and graded. The conclusion follows. The antipode $S : \mathcal{H} \longrightarrow \mathcal{H}$ verifies $S(1_{\mathcal{H}}) = 1_{\mathcal{H}}$, and is given on a non-empty TAG (Γ, μ) by one of the two following recursive formulas:

$$S(\Gamma,\mu) = -(\Gamma,\mu) - \sum_{\emptyset\subsetneq(\gamma,\nu)\subsetneq(\Gamma,\mu)} S(\gamma,\nu)\cdot(\Gamma/\gamma,\mu/\nu) \tag{5.17}$$

$$= -(\Gamma,\mu) - \sum_{\emptyset\subsetneq(\gamma,\nu)\subsetneq(\Gamma,\mu)} (\gamma,\nu)\cdot S(\Gamma/\gamma,\mu/\nu). \tag{5.18}$$

□

Note that if one considers now the core Hopf algebra $\mathcal{H}^c$ of graphs (without any edge scale decoration [Kre10, KM11]), one has:

Proposition 5.3.14. *The map π from $\mathcal{H}$ to $\mathcal{H}^c$ defined on the TAGs by $\pi\big((\Gamma,\mu)\big) = \Gamma$ is a Hopf algebra morphism.*

Proof. This statement directly follows from the definitions. □

A further non-commutative Hopf algebra of TAGs can be defined when considering only graphs of a given quantum field theoretical model and defining the coproduct as the appropriate sum on the class of superficially divergent graphs (see for example [KRT], where such a Hopf algebra was defined, in a commutative setting).

5.4 Quadri-coalgebra structure

A quadri-algebra ([AL04]) is a family $(A, \nwarrow, \swarrow, \searrow, \nearrow)$, such that A is a vector space, and $\nwarrow$, $\swarrow$, $\searrow$, $\nearrow$ are four products on A, satisfying nine axioms below. Putting $\leftarrow = \nwarrow + \swarrow$, $\rightarrow = \nearrow + \searrow$, $\uparrow = \nwarrow + \nearrow$, $\downarrow = \swarrow + \searrow$, and $\star = \nwarrow + \swarrow + \searrow + \nearrow = \leftarrow + \rightarrow = \uparrow + \downarrow$, these axioms imply that $(A, \uparrow, \downarrow)$ and $(A, \leftarrow, \rightarrow)$ are dendriform algebras ([Lod01, LR98]), and $\star$ is an associative (non unitary) product.

$$\begin{array}{lll}
(x \nwarrow y) \nwarrow z = x \nwarrow (y \star z), & (x \nearrow y) \nwarrow z = x \nearrow (y \leftarrow z), & (x \uparrow y) \nearrow z = x \nearrow (y \rightarrow z) \\
(x \swarrow y) \nwarrow z = x \swarrow (y \uparrow z), & (x \searrow y) \nwarrow z = x \searrow (y \nwarrow z), & (x \downarrow y) \nearrow z = x \searrow (y \nearrow z) \\
(x \leftarrow y) \swarrow z = x \swarrow (y \downarrow z), & (x \rightarrow y) \swarrow z = x \searrow (y \swarrow z), & (x \star y) \searrow z = x \searrow (y \searrow z).
\end{array}$$

Dually, a quadri-coalgebra is a family $(C, \Delta_\nwarrow, \Delta_\swarrow, \Delta_\searrow, \Delta_\nearrow)$, where C is a vector space, $\Delta_\nwarrow$, $\Delta_\swarrow$, $\Delta_\searrow$, $\Delta_\nearrow : C \longrightarrow C \otimes C$, such that:

$$\begin{array}{rclrcl}
(\Delta_\nwarrow \otimes Id) \circ \Delta_\nwarrow &=& (Id \otimes \Delta_\star) \circ \Delta_\nwarrow, & (\Delta_\nearrow \otimes Id) \circ \Delta_\nwarrow &=& (Id \otimes \Delta_\leftarrow) \circ \Delta_\nearrow, \\
(\Delta_\uparrow \otimes Id) \circ \Delta_\nearrow &=& (Id \otimes \Delta_\rightarrow) \circ \Delta_\nearrow, & (\Delta_\swarrow \otimes Id) \circ \Delta_\nwarrow &=& (Id \otimes \Delta_\uparrow) \circ \Delta_\swarrow, \\
(\Delta_\searrow \otimes Id) \circ \Delta_\nwarrow &=& (Id \otimes \Delta_\nwarrow) \circ \Delta_\searrow, & (\Delta_\downarrow \otimes Id) \circ \Delta_\nearrow &=& (Id \otimes \Delta_\nearrow) \circ \Delta_\searrow, \\
(\Delta_\leftarrow \otimes Id) \circ \Delta_\swarrow &=& (Id \otimes \Delta_\downarrow) \circ \Delta_\swarrow, & (\Delta_\rightarrow \otimes Id) \circ \Delta_\swarrow &=& (Id \otimes \Delta_\swarrow) \circ \Delta_\searrow, \\
(\Delta_\star \otimes Id) \circ \Delta_\searrow &=& (Id \otimes \Delta_\searrow) \circ \Delta_\searrow, &&&
\end{array}$$

where $\Delta_\leftarrow = \Delta_\nwarrow + \Delta_\swarrow$, $\Delta_\rightarrow = \Delta_\searrow + \Delta_\nearrow$, $\Delta_\uparrow = \Delta_\nwarrow + \Delta_\nearrow$, $\Delta_\downarrow = \Delta_\swarrow + \Delta_\searrow$ and $\Delta_\star = \Delta_\nwarrow + \Delta_\swarrow + \Delta_\searrow + \Delta_\nearrow$. This implies that $(C, \Delta_\leftarrow, \Delta_\rightarrow)$ and $(C, \Delta_\uparrow, \Delta_\downarrow)$ are dendriform coalgebras; moreover, $\Delta_\star$ is coassociative (non counitary): it is called the coassociative coproduct induced by the quadri-coalgebra structure.

Definition 5.4.1. *1. Let $\mathcal{H}_+$ be the augmentation ideal of $\mathcal{H}$. It is given a coassociative, non counitary coprodut $\Delta_\star$ defined for all nonempty TAG (Γ,μ) by:*

$$\Delta_\star\big((\Gamma,\mu)\big) = \Delta\big((\Gamma,\mu)\big) - (\Gamma,\mu) \otimes 1 - 1 \otimes (\Gamma,\mu) = \sum_{\emptyset \subsetneq (\gamma,\nu) \subsetneq (\Gamma,\mu)} (\gamma,\nu) \otimes (\Gamma/\gamma, \mu/\nu),$$

2. *Let (Γ,μ) be a nonempty TAG. We denote by $a(\Gamma,\mu)$ the smallest edge of (Γ,μ) and $b(\Gamma,\mu)$ the greatest edge of (Γ,μ) for the scale assignment μ.*

Proposition 5.4.2. *We define four coproducts on $\mathcal{H}_+$ by:*

$$\begin{array}{rcl}
\Delta_{\nwarrow}\Big((G,\mu)\Big) & := & \displaystyle\sum_{\substack{\emptyset\subsetneq(\gamma,\nu)\subsetneq(\Gamma,\mu),\\ a(G)\in\gamma,b(G)\in\gamma}}(\gamma,\nu)\otimes(\Gamma/\gamma,\mu/\nu),\\
\Delta_{\swarrow}\Big((G,\mu)\Big) & := & \displaystyle\sum_{\substack{\emptyset\subsetneq(\gamma,\nu)\subsetneq(\Gamma,\mu),\\ a(G)\in\gamma,b(G)\notin\gamma}}(\gamma,\nu)\otimes(\Gamma/\gamma,\mu/\nu),\\
\Delta_{\searrow}\Big((G,\mu)\Big) & := & \displaystyle\sum_{\substack{\emptyset\subsetneq(\gamma,\nu)\subsetneq(\Gamma,\mu),\\ a(G)\notin\gamma,b(G)\notin\gamma}}(\gamma,\nu)\otimes(\Gamma/\gamma,\mu/\nu),\\
\Delta_{\nearrow}\Big((G,\mu)\Big) & := & \displaystyle\sum_{\substack{\emptyset\subsetneq(\gamma,\nu)\subsetneq(\Gamma,\mu),\\ a(G)\notin\gamma,b(G)\in\gamma}}(\gamma,\nu)\otimes(\Gamma/\gamma,\mu/\nu).
\end{array}$$

Then $(\mathcal{H}_+,\Delta_{\nwarrow},\Delta_{\swarrow},\Delta_{\searrow},\Delta_{\nearrow})$ is a quadri-coalgebra, and the induced coassociative coproduct is $\Delta_\star$. Moreover, for all $x,y\in\mathcal{H}_+$:

$$\begin{array}{rcl}
\Delta_{\nwarrow}(xy) & = & xy'_{\uparrow}\otimes y''_{\uparrow}+x'_{\leftarrow}y\otimes x''_{\leftarrow}+x'_{\leftarrow}y'_{\uparrow}\otimes x''_{\leftarrow}y''_{\uparrow}\\
\Delta_{\swarrow}(xy) & = & x\otimes y+x'_{\leftarrow}\otimes x''_{\leftarrow}y+xy'_{\downarrow}\otimes y''_{\downarrow}+x'_{\leftarrow}y'_{\downarrow}\otimes x''_{\leftarrow}y''_{\downarrow}\\
\Delta_{\searrow}(xy) & = & x'_{\rightarrow}\otimes x''_{\rightarrow}y+y'_{\downarrow}\otimes xy''_{\downarrow}+x'_{\rightarrow}y'_{\downarrow}\otimes x''_{\rightarrow}y''_{\downarrow}\\
\Delta_{\nearrow}(xy) & = & y\otimes x+y'_{\uparrow}\otimes xy''_{\uparrow}+x'_{\rightarrow}y\otimes x''_{\rightarrow}+x'_{\rightarrow}y'_{\downarrow}\otimes x''_{\rightarrow}y''_{\downarrow},
\end{array}$$

where $\Delta_{\leftarrow}(x)=x'_{\leftarrow}\otimes x''_{\leftarrow}$, $\Delta_{\rightarrow}(x)=x'_{\rightarrow}\otimes x''_{\rightarrow}$, $\Delta_{\uparrow}(y)=y'_{\uparrow}\otimes y''_{\uparrow}$, and $\Delta_{\downarrow}(y)=y'_{\downarrow}\otimes y''_{\downarrow}$.

Proof. Obviously, $\Delta_{\nwarrow}+\Delta_{\swarrow}+\Delta_{\searrow}+\Delta_{\nearrow}=\Delta_\star$. Let (G,μ) be a nonempty TAG. We write, using the coassociativity of $\Delta_\star$:

$$\begin{array}{rl}
& (\Delta_\star\otimes Id)\circ\Delta_\star\Big((\Gamma,\mu)\Big)=(Id\otimes\Delta_\star)\circ\Delta_\star\Big((\Gamma,\mu)\Big)\\
= & \displaystyle\sum_{\emptyset\subsetneq(\gamma',\nu')\subsetneq(\gamma,\nu)\subsetneq(\Gamma,\mu)}(\gamma',\nu')\otimes(\gamma/\gamma',\nu/\nu')\otimes(\Gamma/\gamma,\mu/\nu).
\end{array}$$

Then each relation defining quadri-coalgebras corresponds to the terms in this sum such that:

$$\begin{array}{cc}
a\in\gamma',b\in\gamma', & a\in\gamma/\gamma',b\in\gamma',\\
a\in\Gamma/\gamma,b\in\gamma', & a\in\gamma',b\in\gamma/\gamma',\\
a\in\gamma/\gamma',b\in\gamma/\gamma', & a\in\Gamma/\gamma,b\in\gamma/\gamma',\\
a\in\gamma',b\in\Gamma/\gamma, & a\in\gamma/\gamma',b\in\Gamma/\gamma,\\
a\in\Gamma/\gamma,b\in\Gamma/\gamma. &
\end{array}$$

Let x,y be nonempty TAGs. By definition of the product, $a(xy)=a(x)$ and $b(xy)=b(y)$. Let $\emptyset\subsetneq(\gamma,\nu)\subsetneq xy$. We put $(\gamma_x,\nu_x)=x\cap(\gamma,\nu)$ and $(\gamma_y,\nu_y)=y\cap(\gamma,\nu)$. Then $(\gamma,\nu)=(\gamma_x,\nu_x)(\gamma_y,\nu_y)$ and $xy/(\gamma,\nu)=x/(\gamma_x,\nu_x)y/(\gamma_y,\nu_y)$. If $a(x,y)\in\gamma$, $b(xy)\notin\gamma$, then $a(x)\in\gamma_x$ and $a(y)\notin\gamma_y$. Four cases are possible:

- $x = \gamma$. This gives the term $x \otimes y$.
- $\gamma \subsetneq x$. This gives the term $x'_{\leftarrow} \otimes x''_{\leftarrow} y$.
- $x \subsetneq \gamma$. This gives the term $xy'_{\downarrow} \otimes y''_{\downarrow}$.
- $\gamma_x, \gamma_y \neq \emptyset$ and $\gamma_x \subsetneq x$, $\gamma_y \subsetneq y$. This gives the term $x'_{\leftarrow} y'_{\downarrow} \otimes x''_{\leftarrow} y''_{\downarrow}$.

This proves the compatibility between $\Delta_{\swarrow}$ and the product. The three other compatibilities are proved in the same way. □

Summing, we obtain the following compatibilities:

$$\begin{aligned}
\Delta_{\leftarrow}(xy) &= x \otimes y + xy' \otimes y'' + x'_{\leftarrow} y \otimes x''_{\leftarrow} + x'_{\leftarrow} \otimes x''_{\leftarrow} y + x'_{\leftarrow} y' \otimes x''_{\leftarrow} y'' \\
\Delta_{\rightarrow}(xy) &= y \otimes x + y' \otimes xy'' + x'_{\rightarrow} \otimes x''_{\rightarrow} y + x'_{\rightarrow} y \otimes x''_{\rightarrow} + x'_{\rightarrow} y' \otimes x''_{\rightarrow} y'' \\
\Delta_{\uparrow}(xy) &= y \otimes x + x' y \otimes x'' + xy'_{\uparrow} \otimes y''_{\uparrow} + y'_{\uparrow} \otimes xy''_{\uparrow} + x' y'_{\uparrow} \otimes x'' y''_{\uparrow} \\
\Delta_{\downarrow}(xy) &= x \otimes y + x' \otimes x'' y + xy'_{\downarrow} \otimes y''_{\downarrow} + y'_{\downarrow} \otimes xy''_{\downarrow} + x' y'_{\downarrow} \otimes x'' y''_{\downarrow}.
\end{aligned}$$

Hence:

Corollary 5.4.3. *$(\mathcal{H}_+, m^{op}, \Delta_{\leftarrow}, \Delta_{\rightarrow})$ and $(\mathcal{H}_+, m, \Delta_{\uparrow}, \Delta_{\downarrow})$ are codendriform Hopf algebras.*

Chapter 6

Concluding remarks

In this dissertation we focus on the study of Hopf algebras of type I, namely the *selection/quotient* one. These Hopf algebra are graded and non-cocommutative and sometimes, even non-commutative.

In Chapter 2, we recall all the algebraic structures which are used in the following chapters. The commutative Hopf algebra structure is studied at the end of chapter. We give the analytic form of CQMM theorem as well as its applications to the ϕ-deformed shuffle.

In Chapter 3, the main result is the study of a new Hopf algebra on packed words. Our Hopf algebra is free and non-cocommutative. The number of packed words is twice of the ordered Bell number. We can thus investigate the ordinary generating function of packed words as well as the Hilbert series of WMat.

In Chapter 4, the main result is obtained from using a quantum field theory renormalization-group-like equation to prove the universality of the Tutte polynomial for matroids. This equation derived from the study of appropriate characters of the Hopf algebra of isomorphic classes of matroids.

In Chapter 5, we define a CHA of *totally assigned graphs*. This Hopf algebra is free, non-commutative and non-cocommutative.

Let us now mention a few concrete perspectives for future work. One question that has not been settled is the cofreeness of WMat. Another perspective is to find an explicit polynomial realization of WMat (if any). This appears as particularly interesting because polynomial realizations of Hopf algebras substantially simplify the coproduct coassociativity proof [Thi12].
An example of polynomial realizations for the Hopf algebra of trees [CK98] was given in [FNT10]. The same question can also be asked in the case of the Hopf algebra of totally assigned graphs. Another interesting question is whether these Hopf algebra are self-dual.

The work of Connes and Kreimer [CK00] explores and settles the Hopf algebraic formulation of renormalization for general perturbative quantum field theory. The work of Broadhurst and Kreimer (see [BK99] and references within) develops many computational aspects. They show how to use the coproduct structure of the Hopf algebra to use Hochschild cohomology to resum the perturbative series appearing in quantum field theory.

On the other hand, generalizing graphs to matroids suggests a hierarchy of difficulty. As announced in Chapter 2, one knows that not every matroid is a graphic matroid.
An interesting subject for the future seems to us to be the study the cohomology theory for the matroid Hopf algebra [Sch94]. The task is to translate from the language of graph theory to the language of matroid theory the results of [Kre06] or [TK13].

Chapter A

Maple codes for WMat

To test our results with Maple, we implement a random word in the following way. To each word we associate a certain monomial which encodes, using a given alphabet the position of any letter and its value. We associate the monomial whose powers correspond to the values of the letters and its indices correspond to the positions of the respective letters.

$$x_2x_2x_3x_1 \longrightarrow a_1^2a_2^2a_2^3a_4. \tag{A.1}$$

One has to keep in mind that the letter x_0 can also be present in the words, which is encoded with a supplementary word length variable. This information is encoded with the help of a supplementary variable, associated to the length of the word.
We now implement the generator of random words. It takes as an input: first the length of the word; second, the maximum value of the powers to be generated; third is the alphabet the word will be using.

```
randmon := proc(l,h,a) local i,mon; mon:=1 : for i to l do mon:=mon *
    cat(a,i)^(rand(0..h)()) od: mon*cat(q,a)^l end;
```

We thus obtained a Maple function which takes as arguments the length of the word and the maximal power and returns a random word (by random generation of numbers between 1 and the maximal power).
Using this idea we can then implement, in Maple, packed words (obtained with a Maple function taking as an argument a general word).
The argument of the function *packword* is a word; the function then returns the corresponding packed word.

```
pack_J := proc (J, k)
        local x, res;
        res := 1;
        for x in J while x <> k do res := res+1 end do;
        res
end proc
```

```
packword := proc (m)
       local res, i, j, m1, res1;
       res := m;
       j := 1;
       m1 := min(op(`minus`({op(m)}, {0})));
       res1 := [0];
       while j < max(op(m)) do
              for i to nops(res) do
                     if op(i, res) = m1 then res := subs(op(i, res) = j, res)
                        end if
              end do;
              j := j+1;
              res1 := [op(res1), m1];
              m1 := min(op(`minus`({op(m)}, {op(res1)})))
       end do;
       res
end proc
```

Now, we implement the product, namely the shifted concatenation Equation (3.6).
The following function takes as input: two words; the alphabet of the first word. This function shifts the second word by the maximum index of the first word and converts it into the first word's alphabet.

```
shift2nd := proc (m1, m2, a)
       local res, mon, x, i;
       mon := 1;
       if m1 = 1 or m1 = q0 then
              mon := mon*m2
       else
              if m2 = q0 or m2 = 1 then
                     mon := mon*q0
              else
                     for x in `minus`(indets(m2), {cat(q, a)}) do mon := mon*x
                        end do;
                     res := m2*mon^mmax(m1, a);
                     subs({seq(cat(a, i) = cat(a, i+degree(m1, cat(q, a))), i =
                        1 .. degree(m2, cat(q, a)))}, res)
              end if
       end if
end proc
```

We then can implement the product of two monomials, i.e. two words, with unit coefficients.

```
prod := proc (m1, m2, a, b)
       if m1 = q0 or m2 = q0 then
              m1*m2/q0
       else
```

```
                sort(m1*shift2nd(m1, m2, a))
        end if
end proc
```

We then implement the product of two monomials with non-unit coefficients.

```
mprod := proc (m1, m2, a)
        if type(op(1, m1), integer) and type(op(1, m2), integer) then
                op(1, m1)*op(1, m2)*prod(m1/op(1, m1), m2/op(1, m2), a)
        elif type(op(1, m2), integer) then
                op(1, m2)*prod(m1, m2/op(1, m2), a)
        elif type(op(1, m1), integer) then
                op(1, m1)*prod(m1/op(1, m1), m2, a)
        else
                prod(m1, m2, a)
        end if
end proc
```

In order to implement the coproduct, one must first define the quotient function.

```
quot := proc (m, A, a)
        local i, res;
        res := m;
        for i to degree(m, cat(q, a)) do
                if `in`(degree(m, cat(a, i)), A) then
                                res := res*cat(a, i)^(-degree(m, cat(a, i)))
                end if
        end do;
        res
end proc
```

The following function represents the implementation of the coproduct; the input is a word in the alphabet a and the output is in the alphabet b (the LHS term of the coproduct) and in the alphabet c (the RHS term of the coproduct).

```
coprod := proc (m, a, b, c)
        local i, j, A, S, C, res;
        res := 0;
        A := {seq(i, i = 1 .. degree(m, cat(q, a)))};
        if m = cat(q, 0) then
                res := res+cat(q, 0)*cat(q, 0)
        else
                for S in allsubset(A) do
                        C := `minus`(A, S);
                        res := res+packword(subsword(m, S, a), a,
                            b)*packword(quot(subsword(m, C, a), {seq(degree(m,
                            cat(a, i)), `in`(i, S))}, a), a, c)
                end do
```

```
        end if;
        res
end proc
```

We now implement the LHS and the RHS of the coproduct formula. For this purpose, one needs to refine the above function by considering two distinct alphabets to "build up" the words, such that one can easily separate - as function of the different alphabets - the LHS from the RHS.

We now define a function separator which gives us the part of the word written in one given alphabet; this allows us to separate the LHS and the RHS of the coproduct terms.

```
seperator := proc (m, a)
        local i, res;
        res := 1;
        for i to degree(m, cat(q, a)) do
                res := res*cat(a, i)^degree(m, cat(a, i))
        end do;
        res*cat(q, a)^degree(m, cat(q, a))
end proc
```

We now check the coassociativity. We define $LHS = (\Delta \otimes Id) \circ \Delta$ and $RHS = (Id \otimes \Delta) \circ \Delta$.

```
LHS := proc (m, a, b, c)
        local res, A, S, C;
        res := 0;
        A := {seq(i, i = 1 .. degree(m, qa))};
        for S in allsubset(A) do
                C := `minus`(A, S);
                res := res+coprod(packword(subsword(m, S, a), a, b), b, a,
                    b)*packword(quot(subsword(m, C, a), {seq(degree(m, cat(a, i)),
                    `in`(i, S))}, a), a, c)
        end do;
        res
end proc

RHS := proc (m, a, b, c)
        local res, A, S, C;
        res := 0;
        A := {seq(i, i = 1 .. degree(m, qa))};
        for S in allsubset(A) do
                C := `minus`(A, S);
                res := res+packword(subsword(m, S, a), a,
                    a)*coprod(packword(quot(subsword(m, C, a), {seq(degree(m,
                    cat(a, i)), `in`(i, S))}, a), a, b), b, b, c)
        end do;
        res
end proc
```

Finally, using all of the above, we check the coassociativity condition for random words up to length 10, with maximal power 10.

```
st := time(); T := randmon(10, 10, a); simplify(LHS(T, a, b, c)-RHS(T, a, b,
    c)); time()-st;
```

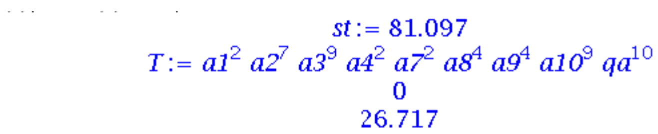


Figure A.1: The result of the test of the coassociativity for random word T with length 10 with maximal power 10.

Chapter B

Maple codes for WMat on commutative alphabet

Let us check the coassociativity of coproduct of packed words on commutative alphabet. For example, one has

$$x_2x_1x_3x_1 = x_1^2x_2x_3. \tag{B.1}$$

The vector space spanning on these packed words is endowed with the similar product and coproduct with WMat.
To test the coassociativity of the coproduct with Maple, we implement a random word in the following way. Each word is associated to a list.
Let us implement the function which generates the random words. It takes as an input: first the length of word, second the maximum value of the alphabet of words.

```
randword := proc(l,h)
        Generate(list(posint(range=h),l)):
end proc
```

The argument of the function *packword* is the random list. It returns the corresponding packed list.

```
packword:= proc(m)
        local res,i,j,m1,res1:
        res := m;
        j := 1;
        m1 := min(op({op(m)} minus {0}));
        res1 := [0];
        while j < max(op(m)) do
                for i from 1 to nops(res) do
                        if op(i,res) = m1 then res:=subs(op(i,res) = j,res) fi:
                od:
                j := j+1:
```

```
            res1 := [op(res1),m1]:
            m1 := min(op({op(m)} minus {op(res1)})):
            od:
       res
end proc
```

We now implement the product, namely shifted concatenation. The function takes as input: two lists. This function shifts the second list by the maximum of the first list and returns the list that results from concatenating two list.

```
prod := proc(m1,m2):
       local i,res:
       res := m1;
       for i from 1 to nops(m2) do
              if op(i,m2) = 0 then
                     res := [op(res),op(i,m2)]
              else
                     res := [op(res),op(i,m2) + max(op(m1))]
              fi:
       od:
       res:
end proc
```

We now implement the quotient function. The input of the quotient function are a list and a set A. It transforms the values of list at position i^{th} as following: if $i \in A$ then the values becomes 0 and otherwise.

```
quotient := proc(m,A)
       local res,i,j:
       res := m:
       for i in A do
              for j from 1 to nops(m) do
                     if op(j,m) = i then res := subs(op(j,m)=0,res): fi:
              od:
       od:
       res:
end proc
```

We can then implement the coproduct.

```
coprod := proc(m)
       local res,i,A:
       A := NULL:
       res := 0;
       for i from 1 to nops(m) do
              A := A,i:
       od:
       for i in allsubsets({A}) do
```

```
            res := res +
                sort(packword(subsword(m,i)))*packword(quotient(subsword(m,{A}
                minus i), subsword(m,i))):
        od:
        sort(res):
end proc
```

We now implement the LHS and the RHS of the coproduct formula.

```
LHS := proc(m)
        local res,A,i:
        res :=0:
        sort(m):
        A := NULL:
        for i to nops(m) do A := A,i od:
        for i in allsubsets({A}) do
            res := res +
                sort(coprod(subsword(m,i)))*packword(quotient(subsword(m,{A}
                minus i), subsword(m,i)))
        od:
        sort(res):
end proc

RHS := proc(m)
        local res,A,i:
        res :=0:
        sort(m):
        A := NULL:
        for i to nops(m) do A := A,i od:
        for i in allsubsets({A}) do
            res := res + sort(packword(subsword(m,i)))*coprod(subsword(m, {A}
                minus i))
        od:
        sort(res):
end proc
```

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Algèbres de Hopf combinatoire basées sur la règle de sélection/quotient

Résume

Dans cette thèse, nous nous concentrons sur l'étude des algèbres de Hopf de type I, à savoir de type *sélection/quotient*.
Nous présentons une structure d'algèbre de Hopf sur l'espace vectoriel engendré par de les mots tassés avec coproduit sélection/quotient. C'est un algèbre libre sur ses mots irreducible. Nous montrons que la serie de Hilbert de cette algèbre de Hopf.
Nous donnons une nouvelle preuve de l'universalité du polynôme de Tutte pour les matroïdes. Cette preuve utilise des caractères appropriés de l'algèbre de Hopf des matroïdes introduite par Schmitt (1994). Nous montrons que ces caractères sont des solutions des équations différentielles du même type que les équations différentielles utilisées pour décrire le flux du groupe de renormalisation en théorie quantique de champs. Cette approche nous permet aussi de démontrer, d'une manière différente, une formule de convolution du polynôme de Tutte des matroïdes, formule publiée par Kook, Reiner et Stanton (1999) et par Etienne et Las Vergnas (1998).
Dans la dernière partie, nous définissons une algèbre de Hopf non-commutative de graphes. La non-commutativité du produit est obtenue grâce à des étiquettes entières distinctes associées aux arrêtes du graphe. Cette idée est inspirée de certaines techniques analytiques utilisées en renormalisation en théories quantiques des champs. Nous définissons ensuite une structure d'algèbre de Hopf, avec un coproduit basé sur une règle de type sélection/quotient, et nous demontrons la coassociativité de ce coproduit. Nous analysons finalement la structure de quadri-cogèbre et les structures codendriformes associées.

Mots-clefs : Combinatoire algébrique, Algèbre de Hopf, Graphes, Matroïds, Polynôme de Tutte

---

Combinatorial Hopf algebras based on the selection/quotient rule

Abstract

In this thesis, we focus on the study of Hopf algebras of type I, namely the *selection/quotient*.
We study the new Hopf algebra structure on the vector space spanned by packed words. We show that this algebra is free on its irreducible packed words. We also compute the Hilbert series of this Hopf algebra.
We provide a new way to obtain the universality of the Tutte polynomial for matroids. This proof uses appropriate characters of Hopf algebra of matroids, algebra introduced by Schmitt (1994). We show that these Hopf algebra characters are solutions of some differential equations which are of the same type as the differential equations used to describe the renormalization group flow in quantum field theory. This approach allows us to also prove, in a different way, a matroid Tutte polynomial convolution formula published by Kook, Reiner and Stanton (1999) and by Etienne and Las Vergnas (1998).
We define a non-commutative Hopf algebra of graphs. The non-commutativity of the product is obtained thanks to some discrete labels associated to the graph edges. This idea is inspired from certain analytic techniques used in quantum field theory renormalization. We then define a Hopf algebra structure, with a coproduct based on a selection/quotient rule, and prove the coassociativity of this coproduct. We analyze the associated quadri-coalgebra and codendrifrom structures.

Keywords: Algebraic combinatorics, Hopf algebra, Graphs, Matroids, Tutte polynomial