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Titre :

Etude qualitative d'un système parabolique-elliptique de type Keller-Segel et de systèmes elliptiques non coopératifs

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Jury

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Cette thèse est dédiée à
la mémoire de mes grand-parents,
paysans,
les uns à Saint-Calais dans la Sarthe,
les autres à Tourailles dans le Loir-et-Cher.

Un voyage de mille lis commence par un pas.

Lao Tseu

Admirant les gestes harmonieux de son boucher et le rythme musical de son couteau dans les carcasses qu'il dépeçait, le prince Mai loua l'habileté de son art. Le boucher lui répondit :

- Mon habileté vient du fait que je suis le Tao.

Au tout début de ma carrière je ne voyais que le boeuf. Après des années de pratique je ne considère plus l'animal entier et travaille en me laissant guider par mon esprit plutôt que par mes yeux.

Je m'accorde à la constitution naturelle de l'animal et le fil de mon couteau suit les interstices et s'engage dans les cavités. Je ne coupe ni muscles ni nerfs et encore moins les os. Un bon boucher change son couteau tous les ans parce qu'il tranche, un boucher ordinaire tous les mois parce qu'il hache. Moi, je me sers du même couteau depuis dix-neuf ans et bien qu'il ait découpé des milliers de boeufs, on dirait que son tranchant vient d'être aiguisé.

La finesse de la lame s'introduit dans les espaces des articulations et des fibres et je manie mon couteau avec aisance dans les espaces vides que j'élargis ainsi.

Je pose mon attention sur les difficultés particulières rencontrées à chaque fois, agis lentement, et les parties se séparent d'elles-mêmes, comme si j'émiettais une poignée de terre.

Alors je retire mon couteau, me relève, souffle et le range.

- Bien, dit le prince, les paroles de ce boucher m'apprennent l'art de diriger ma vie.

Tchouang Tseu

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A propos de celle-ci, j'en profite pour dire qu'elle a été pour moi un cadre de travail agréable, bien plus en tout cas que l'image grise qui en est souvent donnée d'une université coincée entre les tours de « cités ». J'y ai vu aussi de la verdure, des arbres et des fleurs. De plus, j'ai souvent été reçu avec le sourire dans ses différents services administratifs ou à la cantine. Concernant le LAGA, merci notamment à Isabelle et Yolande pour leur bonne humeur et leur gentillesse, qui rendent les démarches administratives tellement plus agréables.

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Résumé

Cette thèse est consacrée à l'étude de deux problèmes :

D'une part, nous considérons un système parabolique-elliptique de type Patlak-Keller-Segel avec sensitivité de type puissance et exposant critique. Nous étudions les solutions radiales de ce système dans une boule de l'espace euclidien et obtenons des résultats d'existence-unicité, de régularité ainsi qu'une alternative d'explosion. Concernant le comportement qualitatif en temps long des solutions radiales, pour toute dimension d'espace supérieure ou égale à trois, nous montrons un phénomène de masse critique qui généralise le cas déjà connu de la dimension deux mais présente par rapport à celui-ci un comportement très différent dans le cas de la masse critique. Dans le cas d'une masse sous-critique, nous montrons de plus que les densités de cellule convergent uniformément à vitesse exponentielle vers l'unique solution stationnaire. Ce dernier résultat est valable pour toute dimension d'espace supérieure ou égale à deux et n'était, à notre connaissance, pas connu même pour le cas très étudié de la dimension deux.

D'autre part, nous étudions des systèmes elliptiques (semi-linéaires et complètement non-linéaires) non coopératifs. Dans le cas de l'espace ou d'un demi-espace (ou même d'un cône), sous une hypothèse de structure naturelle sur les non-linéarités, nous donnons des conditions suffisantes pour avoir la proportionnalité des composantes, ce qui permet de ramener l'étude à celle d'une équation scalaire et ainsi d'obtenir des résultats de classification et de type Liouville pour le système. Dans le cas d'un domaine borné, grâce aux théorèmes de type Liouville obtenus, la méthode de renormalisation de Gidas et Spruck permet d'obtenir une estimation a priori des solutions bornées et finalement de déduire l'existence d'une solution non triviale, via une méthode topologique utilisant la théorie du degré.

Abstract

This thesis is concerned with the study of two problems :

On the one hand, we consider a parabolic-elliptic system of Patlak-Keller-Segel type with a critical power type sensitivity. We study the radially symmetric solutions of this system on a ball of the euclidean space and obtain wellposedness and regularity results together with a blow-up alternative. As for the long time qualitative behaviour of the radial solutions, for any space dimension greater or equal to three, we show that a critical mass phenomenon occurs, which generalizes the well-known case of dimension two but, with respect to the latter, with a very different qualitative behaviour in the case of the critical mass. When the mass is subcritical, we moreover show that the cell density converges uniformly with exponential speed toward the unique steady state. This result is valid for any space dimension greater or equal to two, which was, to our knowledge, not known even for the most studied case of dimension two.

On the other hand, we study noncooperative (semilinear and fully nonlinear) elliptic systems. In the case of the whole space or of a half-space (or even for a cone), under a natural structure condition on the nonlinearities, we give sufficient conditions to have proportionnality of the components, which allows to reduce the system to a scalar equation and then to get classification and Liouville type results. In the case of a bounded domain, thanks to the obtained Liouville type theorems, the blow-up method of Gidas and Spruck then allows to get an a priori estimate on the bounded solutions and eventually to deduce the existence of a non trivial solution by a topological method using the degree theory.

Liste de publications liées à la thèse

[**P1**] Wellposedness and regularity for a degenerate parabolic equation arising in a model of chemotaxis with nonlinear sensitivity. Discrete and Continuous Dynamical Systems - Series B Volume 19, Issue 1, Pages : 231 - 256, 2013

http://aimsciences.org/journals/displayArticlesnew.jsp?paperID=9508

[**P2**] A semilinear parabolic-elliptic chemotaxis system with critical mass in any space dimension. Nonlinearity 26 (2013), no. 9, 2669–2701.

http://iopscience.iop.org/0951-7715/26/9/2669?fromSearchPage=true

[**P3**] Exponential speed of uniform convergence of the cell density toward equilibrium for subcritical mass in a Patlak-Keller-Segel model. Journal of Differential Equations, accepté pour publication.

[**P4**] Symmetry of components and Liouville theorems for noncooperative elliptic systems on the half-space (**avec Philippe Souplet**). C. R. Math. Acad. Sci. Paris 352 (2014), no. 4, 321–325.

www.em-consulte.com/article/884832/symmetry-of-components-and-liouville-theorems-for-

[**P5**] Proportionality of Components, Liouville Theorems and a Priori Estimates for Noncooperative Elliptic Systems (**avec Philippe Souplet et Boyan Sirakov**). Arch. Ration. Mech. Anal. 213 (2014), no. 1, 129–169.

http://link.springer.com/article/10.1007%2Fs00205-013-0719-4

[**P6**] Proportionality of Components, Liouville Theorems and a Priori Estimates for Noncooperative Fully Nonlinear Elliptic Systems (**avec Boyan Sirakov**). A soumettre prochainement.

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Introduction générale

Dans cette introduction générale, nous souhaiterions donner un aperçu des différents résultats obtenus pendant notre thèse, effectuée sous la direction de Philippe Souplet.
Nos travaux ont porté sur deux problèmes différents :

– Un modèle de chimiotaxie avec masse critique en toute dimension.
(Voir Chapitres 1,2,3.)
– Classification et résultats d'existence ou de non-existence pour des systèmes elliptiques semi-linéaires et complètement non-linéaires.
(Voir Chapitres 4,5,6.)

Ils seront par conséquent présentés dans deux sections distinctes.

1 Un modèle de chimiotaxie avec masse critique en toute dimension

1.1 Origine du problème

On parle de phénomène de chimiotaxie lorsque le mouvement d'organismes (cellules, bactéries) est affecté voire dirigé par la présence d'une substance chimique. On peut assister à des phénomènes de répulsion comme d'attraction, et dans ce dernier cas, la substance chimique est appelée chimioattractant. Par exemple, des cellules peuvent être attirées par des nutriments ou bien repoussées en présence d'une substance qui leur est toxique.
Un exemple plus intéressant est celui des amibes *Dyctyostelium discoideum* qui, en cas de manque de nutriments, se mettent à secréter de l'adénosine monophosphate cyclique (AMPc) qui attire les autres amibes. La chimiotaxie se révèle alors être un puissant moyen de communication entre les amibes qui induit un mouvement collectif. Il a été observé des phénomènes d'agrégation où les amibes, initialement monocellulaires, forment finalement une société, i.e. un organisme pluricellulaire. Celui-ci peut ensuite se déplacer pour aller chercher de la nourriture ou bien former comme une tige au bout de laquelle sont créées des spores. Celles-ci sont alors projetées au loin dans l'espoir d'un environnement plus clément, les cellules formant la tige se sacrifiant pour la survie de la société. Pour en savoir davantage sur la vie sociale des amibes *Dyctyostelium discoideum*, nous renvoyons le lecteur à l'article

[21] de M.A. Herrero et L. Sastre.

Pour modéliser ce phénomène d'agrégation, C.S. Patlak en 1953 puis E.F. Keller et L.A. Segel en 1970 ont proposé le système suivant (cf. [29, 28]) :

$$\rho_t = D_1 \Delta\rho - \nabla[\chi \nabla c] \quad (1)$$
$$c_t = D_2 \Delta c + \mu \rho \quad (2)$$

pour tout $t > 0$ et $y \in \Omega \subset \mathbb{R}^N$, où $\rho = \rho(t, y) \geq 0$ désigne la densité de cellules (amibes), $c = c(t, y) \geq 0$ la concentration du chimioattractant (AMPc), $\chi = \chi(\rho, c) \geq 0$ la sensitivité des cellules au chimioattractant, $\mu \geq 0$ le taux de création de chimioattractant par cellule et par unité de temps et $D_1, D_2 > 0$ sont des coefficients de diffusion.

Examinons plus précisément les hypothèses qui sous-tendent ce modèle :

– Il est fait une hypothèse de diffusion aussi bien pour les cellules que pour le produit chimique.
– Le flux de cellules dû au chimioattractant est supposé proportionnel au gradient de la concentration en chimioattractant, ce qui est un fait observé expérimentalement par les biologistes.

Par ailleurs, les cellules ainsi que le chimioattractant sont supposés se trouver dans un domaine borné Ω de $\mathbb{R}^N$ avec $N \geq 2$ ($N = 2$ et $N = 3$ étant les cas pertinents d'un point de vue physique). Nous devons donc préciser les conditions aux bord faites pour ρ et c :

– On fait une hypothèse de flux nul au bord de Ω pour les cellules

$$D_1 \frac{\partial \rho}{\partial \nu} - \chi(\rho, c) \frac{\partial c}{\partial \nu} = 0 \quad \text{sur } \partial\Omega, \quad (3)$$

ce qui entraine la conservation de la masse des cellules au cours du temps.
– On fait une hypothèse de type Dirichlet au bord de Ω pour le chimioattractant :

$$c = 0 \quad \text{sur } \partial\Omega. \quad (4)$$

Dans certains cas, comme par exemple pour les amibes *Dyctyostelium discoideum* et l'AMPc, il se trouve que le chimioattractant diffuse beaucoup plus vite que les cellules. Le système présente alors deux échelles de temps, la concentration en produit chimique dans la deuxième équation évoluant beaucoup plus vite que la densité de cellules dans la première équation. A la limite, on peut faire l'hypothèse que la concentration en chimioattractant atteint instantanément son équilibre.

Ceci conduit, après renormalisation, au système parabolique-elliptique suivant :

$$\rho_t = \Delta\rho - \nabla[\chi \nabla c] \quad (5)$$
$$-\Delta c = \rho \quad (6)$$

avec les mêmes conditions au bord que précédemment, qui deviennent :

$$\frac{\partial \rho}{\partial \nu} - \chi(\rho, c)\,\frac{\partial c}{\partial \nu} = 0 \quad \text{sur } \partial\Omega, \tag{7}$$

$$c = 0 \quad \text{sur } \partial\Omega. \tag{8}$$

Pour un compte-rendu général sur les mathématiques de la chimiotaxie, le lecteur peut consulter le chapitre écrit par M.A. Herrero dans [20] ainsi que l'article [23] de T. Hillen et K. J. Painter. Pour un compte-rendu sur les modèles de type Patlak-Keller-Segel, voir les deux articles de D. Horstmann [24, 25].

Le fait que la sensitivité χ dépende de ρ et de c est une observation expérimentale. Mais ce cas général est actuellement hors de portée mathématiquement. Cependant, dans [26], D. Horstmann et M. Winkler se sont penchés sur le cas où la sensitivité χ ne dépend que de ρ et ont montré le résultat suivant :

– S'il existe $C > 0$ tel que $\chi(\rho) \leq C\rho^q$ pour $\rho \geq 1$ et si $q < \frac{2}{N}$, alors la densité de cellules ρ existe globalement et est même uniformément bornée en temps.
– S'il existe $C > 0$ tel que $\chi(\rho) \geq C\rho^q$ pour $\rho \geq 1$ et si $q > \frac{2}{N}$, alors ρ peut exploser en temps fini.

On constate donc que l'exposant

$$\boxed{q = \frac{2}{N}}$$

est critique pour ce système.

Ceci nous conduit à nous intéresser au système (PKS_q) suivant pour $q = \frac{2}{N}$:

$$\rho_t = \Delta\rho - \nabla[\rho^q\,\nabla c] \tag{9}$$

$$-\Delta c = \rho \tag{10}$$

avec pour conditions au bord

$$\frac{\partial \rho}{\partial \nu} - \rho^q\,\frac{\partial c}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{11}$$

$$c = 0 \quad \text{on } \partial\Omega. \tag{12}$$

De plus, si on considère les solutions renormalisées par changement d'échelle de ce système, définies pour tout $\lambda > 0$ par

$$\rho_\lambda(t,x) = \lambda^{\frac{2}{q}}\,\rho(\lambda^2\,t, \lambda\,x) \tag{13}$$

$$c_\lambda(t,x) = \lambda^{\frac{2}{q}-2}\,c(\lambda^2\,t, \lambda\,x) \tag{14}$$

pour tout $t > 0$ et $x \in \mathbb{R}^N$, on constate que la norme L^1 (correspondant à la masse totale des cellules) est invariante par le changement d'échelle exactement pour la valeur $q = \frac{2}{N}$. Ce fait est important car il ouvre la porte à la possibilité d'un phénomène de masse critique.

Remarques :
- Les modèles de type Patlak-Keller-Segel interviennent aussi pour décrire les effondrements stellaires, ρ désignant alors la masse volumique et c le potentiel gravitationnel. Voir le livre de S. Chandrasekhar [11].
- Le système (PKS_q) intervient aussi en thermodynamique généralisée et peut être vu comme la description macroscopique d'une collection de n particules suivant une équation stochastique de Langevin généralisée. En fait, il peut être vu comme une équation de Fokker-Planck non-linéaire obtenue par une limite thermodynamique propre $n \to \infty$. Pour plus de détails, voir l'article de P.H. Chavanis [12, section 3.4].

1.2 Résultats connus pour $N = 2$ et $q = 1$

Nous voudrions rappeler certains résultats connus sur le système (PKS_q) pour $N = 2$ et l'exposant critique $q = 1$. Celui-ci s'écrit alors :

$$\rho_t = \Delta\rho - \nabla[\rho\,\nabla c] \tag{15}$$
$$-\Delta c = \rho. \tag{16}$$

Considérant les solutions radiales de (PKS_1) dans $\Omega = D$ où $D \subset \mathbb{R}^2$ est la boule unité ouverte centrée à l'origine, il a été montré que ce système admet 8π pour masse critique. En effet, si on note $\overline{m}$ la masse totale des cellules dans D, il a été démontré que :
- Si $\overline{m} < 8\pi$, alors $\rho(t)$ existe globalement en temps et converge vers une solution stationnaire lorsque le temps $t \to +\infty$.
  (voir [5] par P. Biler, G. Karch, P. Laurençot et T. Nadzieja).
- Si $\overline{m} = 8\pi$, alors $\rho(t)$ existe globalement en temps mais explose en temps infini vers une masse de Dirac centrée à l'origine.
  (voir à nouveau [5] et aussi [27] par N.I. Kavallaris et P. Souplet pour la vitesse d'explosion asymptotique).
- Si $\overline{m} > 8\pi$, alors $\rho(t)$ explose en temps fini vers une masse de Dirac centrée à l'origine.
  (voir [22] par M.A. Herrero et J.L. Velazquez).

Des résultats sont également connus sur ce système lorsque Ω est autre que la boule unité.
- Lorque $\Omega = \mathbb{R}^2$, il a été prouvé que l'on a la même dichotomie entre existence globale et explosion en temps fini.
  Voir les travaux de J. Dolbeault et B. Perthame dans [15], A. Blanchet, J.A. Carrillo et N. Masmoudi dans [7], P. Biler, G. Karch, P. Laurençot et T. Nadzieja dans [6] et A. Blanchet, J. Dolbeault et B. Perthame dans [8].

– Mais si on considère des solutions générales, i.e. sans hypothèse de symétrie radiale, dans un domaine borné régulier Ω de $\mathbb{R}^N$, les résultats diffèrent puisqu'il peut exister des points d'explosion de masse 4π au bord de Ω.
Voir le livre de T. Suzuki [37].

Le système parabolique-parabolique dans $\mathbb{R}^2$ a également été étudié et révèle un comportement plus complexe. Voir par exemple les articles [3] de P. Biler, L. Corrias et J. Dolbeault et [10] de V. Calvez et L. Corrias.

1.3 Réduction du cas radial à un problème scalaire

A partir de maintenant, nous allons nous intéresser aux solutions radiales du système (PKS_q) définies dans $\Omega = D$ (où $D \subset \mathbb{R}^N$ est la boule ouverte supposée centrée à l'origine de rayon un, sans perte de généralité) pour

$$N \geq 2$$

et pour l'exposant critique

$$q = \frac{2}{N}.$$

En adaptant la procédure décrite dans [4] de P. Biler, D. Hilhorst et T. Nadzieja (voir aussi [5, 27]), on peut tirer profit de l'hypothèse de symétrie radiale des solutions et ramener l'étude du système (PKS_q) à celle d'une seule équation parabolique.
En effet, notant V_N le volume de D pour la mesure de Lebesgue dy, si on pose pour $t \geq 0$ et $x \in [0,1]$,

$$u(t,x) = \frac{1}{N^{\frac{2}{q}} V_N} \int_{B(0,x^{1/N})} \rho(\frac{t}{N^2}, y)dy,$$

alors on vérifie formellement (voir les détails dans [P2]) que u est solution du problème suivant, noté (PDE_m) :

$$u_t = x^{2-\frac{2}{N}} u_{xx} + u\, {u_x}^q \qquad t > 0 \quad 0 < x \leq 1 \tag{17}$$
$$u(t,0) = 0 \qquad t \geq 0 \tag{18}$$
$$u(t,1) = m \qquad t \geq 0 \tag{19}$$
$$u_x(t,x) \geq 0 \qquad t > 0 \quad 0 \leq x \leq 1, \tag{20}$$

où

$$m = \frac{\overline{m}}{N^{\frac{2}{q}} V_N} \geq 0 \tag{21}$$

et, pour mémoire,

$$q = \frac{2}{N}\ .$$

Il est important de remarquer que la quantité ayant un sens physique est u_x, la dérivée de u, et non pas u. En effet, si on note $\rho(t,y)=\tilde{\rho}(t,|y|)$ pour $t\geq 0$ et $y\in\overline{D}$, on a la formule suivante :

$$\tilde{\rho}(t,x)=N^{\frac{2}{q}}\,u_x(N^2\,t,x^N)\qquad\text{for all } x\in[0,1]. \tag{22}$$

Ceci explique également la condition (20).

1.4 Principaux résultats obtenus

1.4.1 Théorie locale en temps

L'équation parabolique (17) présente plusieurs difficultés :

– D'une part, la diffusion est dégénérée puisque $x^{2-\frac{2}{N}}\to 0$ lorsque $x\to 0$.
– D'autre part, la non-linéarité fait intervenir le gradient et est de plus non-Lipschitz pour $N\geq 3$ puisqu'alors $0<q<1$.

On peut néanmoins obtenir un résultat d'existence et unicité de solutions classiques ainsi que des résultats de régularité pour le problème précédent. Pour ce faire, nous introduisons le cadre fonctionnel suivant. On définit pour $m\geq 0$ l'espace

$$\boxed{Y_m=\{u\in C([0;1]),\ u \text{ croissante },\ u'(0)\text{ existe},\ u(0)=0,\ u(1)=m\}}$$

qui sera l'espace des conditions initiales ainsi que l'espace

$$Y_m^{1,\gamma}=\{u\in Y_m\cap C^1([0,1]),\ \sup_{x\in(0,1]}\frac{|u'(x)-u'(0)|}{x^\gamma}<\infty\}$$

pour $m\geq 0$ et $0<\gamma\leq 1$ qui nous servira à préciser la régularité des solutions.

Maintenant, si on fixe une condition initiale $u_0\in Y_m$, le problème (17)-(20) s'écrit alors, de manière équivalente,

$$\boxed{u_t=x^{2-\frac{2}{N}}u_{xx}+u\,u_x{}^q}\qquad\text{pour } (t,x)\in(0,T]\times(0,1] \tag{23}$$

$$u(0)=u_0 \tag{24}$$

$$u(t)\in Y_m\qquad\text{pour } t\in[0,T]. \tag{25}$$

Nous introduisons également pour toute fonction $f:]0,1]\to\mathbb{R}$ la quantité

$$\mathcal{N}[f]=\sup_{x\in(0,1]}\frac{f(x)}{x},$$

qui nous servira à contrôler l'explosion de la solution.

Enfin, pour $T > 0$, nous appellerons solution classique du problème (PDE_m) sur $[0, T]$ avec condition initiale $u_0 \in Y_m$ une fonction

$$u \in C([0,T] \times [0,1]) \bigcap C^1((0,T] \times [0,1]) \bigcap C^{1,2}((0,T] \times (0,1])$$

telle que (23)(24)(25) sont satisfaites. Une solution classique du problème (PDE_m) sur $[0, T)$ est définie de manière analogue.

Théorème 1.1. *Soit* $N \geq 3$, $q \in (0,1)$ *et* $m \geq 0$.
Soit $K > 0$ *et* $u_0 \in Y_m$ *tel que* $\mathcal{N}[u_0] \leq K$.

i) Il existe $T_{max} = T_{max}(u_0) > 0$ *et une unique solution classique maximale* u *du problème* (PDE_m) *avec condition initiale* u_0. *De plus,* u *vérifie la condition suivante :*

$$\sup_{t \in (0,T]} \sqrt{t}\, \|u(t)\|_{C^1([0,1])} < \infty \textit{ pour tout } T \in (0, T_{max}).$$

ii) Il existe $\tau = \tau(K) > 0$ *tel que* $T_{max} \geq \tau$.
iii) Alternative d'explosion : $T_{max} = +\infty$ *ou* $\lim_{t \to T_{max}} \mathcal{N}[u(t)] = +\infty$
iv) $u_x(t,0) > 0$ *pour tout* $t \in (0, T_{max})$.
v) Si $0 < t_0 < T < T_{max}$ *et* $x_0 \in (0,1)$, *alors pour tout* $\gamma \in (0,q)$,

$$u \in C^{1+\frac{\gamma}{2}, 2+\gamma}([t_0, T] \times [x_0, 1])$$

vi) Pour tout $t \in (0, T_{max})$, $u(t) \in Y_m^{1,\frac{2}{N}}$.

La preuve de ce résultat repose sur un changement d'inconnue qui donne un problème auxiliaire avec une diffusion standard de type Laplacien dans la boule unité de $\mathbb{R}^{N+2}$, ce qui permet d'éliminer une des difficultés de l'équation. Une procédure d'approximation de la non-linéarité est ensuite nécessaire puisque celle-ci est non-Lipschitz. Voir [P1] pour plus de détails.
Il est à noter que le problème auxiliaire joue aussi un rôle important pour obtenir diverses estimations dont découlent en particulier des résultats de compacité et de régularité.
Quant à la "norme" $\mathcal{N}$, elle se révèle bien adaptée aux arguments de comparaison en vue des résultats d'existence globale ou d'explosion en temps fini obtenus. Pour plus de précisions sur ces deux dernières remarques, voir [P2] .

Remarque 1.1.

i) Le théorème précédent, traitant le cas $N \geq 3$, *n'est pas valable uniquement pour* $q = \frac{2}{N}$ *mais pour tout* $q \in (0,1)$. *Il a été démontré dans [P1] .*
ii) Ce théorème est vrai également pour $N = 2$ *et* $q = 1$. *La condition iv) peut alors être améliorée en* $u_x > 0$ *sur* $[0,1]$ *pour tout* $t \in (0, T_{max})$. *Ceci a été montré dans [P3] .*

1.4.2 Phénomène de masse critique

Nous nous intéressons maintenant au comportement asymptotique des solutions dans le cas où $N \geq 3$. Dans un premier temps, il est naturel de s'intéresser aux solutions stationnaires du problème (PDE_m).
Il a été montré dans [P2] que celles-ci sont les restrictions à $[0,1]$ d'une famille $(U_a)_{a\geq 0}$ de fonctions définies sur $[0,+\infty)$ ayant la structure simple suivante :

- $U_1 \in C^1([0,1]) \cap C^2((0,1])$, $U_1(0) = 0$, $\dot{U_1}(0) = 1$, U_1 est strictement croissante sur $[0, A]$ pour un certain $A > 0$, atteint son maximum M pour $x = A$ puis devient plate.
- Toutes les fonctions $(U_a)_{a\geq 0}$ sont obtenues par dilatation de U_1, i.e. pour tout $a \geq 0$,
$$U_a(x) = U_1(ax)$$
pour tout $x \geq 0$.

Remarque 1.2. *Pour $N = 2$, les solutions stationnaires sont les restrictions à $[0,1]$ de la famille $(U_a|_{[0,1]})_{a\geq 0}$ où*
$$U_1(x) = \frac{x}{1+\frac{x}{2}}$$
et
$$U_a(x) = U_1(ax)$$
pour tout $x \in [0,1]$ et $a \geq 0$. Le rôle des dilatations est donc le même pour $N = 2$. En revanche, pour $a > 0$, U_a n'atteint pas son supremum sur $[0,+\infty)$ qui est 2. Ce fait a de profondes conséquences sur le comportement asymptotique des solutions du problème (PDE_m), notamment pour la masse critique, définie ci-après.

Le prochain théorème justifiera que nous posions la définition suivante :

Définition 1.1. *On appelle **masse critique** du problème (PDE_m) la quantité M suivante :*

i) Pour $N = 2$, $M = 2$ est le supremum de U_1 sur $[0,+\infty)$ (qui correspond, d'après (21), à la masse critique 8π du problème (PKS_1)).
i) Pour $N \geq 3$, M est le maximum de U_1 sur $[0,+\infty)$.

La description précédente des solutions stationnaires a pour conséquence simple le résultat suivant, valable pour $N \geq 3$:

- Si $0 \leq m < M$, il y a une unique solution stationnaire, donnée par $U_a|_{[0,1]}$, où $a = a(m) \in [0, A)$ est complètement déterminé par m.
- Si $m = M$, il existe un continuum de solutions stationnaires : $\left(U_a|_{[0,1]}\right)_{a\geq A}$.
Il est remarquable que les densités de cellules correspondantes aient leur support strictement inclus dans D lorsque $a > A$.
- Si $m > M$, il n'y a pas de solution stationnaire.

L'appellation de **masse critique** pour M est justifiée par le théorème suivant, prouvé dans les Théorèmes 1.2 et 1.3 de [P2].

Théorème 1.2. *Soit* $N \geq 3$ *et* $u_0 \in Y_m$.

i) Si $m \leq M$, *alors*

$$T_{max}(u_0) = +\infty$$

et il existe $a \geq 0$ *tel que*

$$u(t) \underset{t\to+\infty}{\longrightarrow} U_a \quad \text{dans} \quad C^1([0,1]).$$

Plus précisément, $a = a(m) \in [0, A)$ *si* $0 \leq m < M$ *et* $a \geq A$ *si* $m = M$.

ii) Si $m > M$, *alors*

$$T_{max}(u_0) < \infty.$$

Remarque 1.3.

i) Le comportement asymptotique pour la masse critique est très différent pour $N = 2$ *et* $N \geq 3$. *En effet, dans le premier cas, il y a explosion en temps infini alors que dans le second, il y convergence vers une solution stationnaire en temps infini.*

ii) Il est remarquable d'obtenir la convergence vers une solution stationnaire même dans le cas critique où il existe un continuum d'états d'équilibre, car la solution $u(t)$ *pourrait par exemple osciller le long d'un sous-ensemble de ce continuum.*

La preuve de i) repose sur le fait que

$$\boxed{\mathcal{F}[u] = \int_0^1 \frac{\dot{u}^{2-q}}{(2-q)(1-q)} - \frac{u^2}{2x^{2-q}}} \tag{26}$$

est une fonctionnelle de Lyapunov stricte pour le système dynamique engendré par l'équation d'évolution (PDE_m) *ainsi que sur la structure particulière de l'ensemble des solutions stationnaires.*

iii) La preuve de ii) repose sur la construction d'une sous-solution explosive en temps fini. En l'occurrence, celle-ci se trouve devenir une solution auto-similaire au bout d'un certain temps et son profil V *est obtenu comme solution pour* $\varepsilon > 0$ *assez petit de l'équation*

$$\begin{aligned} x^{2-q}\ddot{V} + V\dot{V}^q &= \varepsilon x\dot{V} \qquad x > 0 \\ V(0) &= 0 \\ \dot{V}(0) &= 1, \end{aligned}$$

qui est une perturbation de l'équation stationnaire.

iv) Dans [P3] , il a été montré que le même résultat reste vrai pour $N = 2$ *dans le cas sous-critique, i.e.* $m < 2$. *Plus précisément, si* $m < 2$ *et* $u_0 \in Y_m$, *alors*

$$u(t) \underset{t\to+\infty}{\longrightarrow} U_a \quad \text{dans} \quad C^1([0,1])$$

où U_a est l'unique solution stationnaire. A notre connaissance, la convergence en norme C^1 n'était pas connue. On montrera en fait un résultat plus fort, à savoir que la convergence a lieu à vitesse exponentielle (pour $N \geq 2$).

1.4.3 Convergence uniforme à vitesse exponentielle dans le cas sous-critique

Nous nous intéressons maintenant au cas sous-critique $m < M$ lorsque $N \geq 2$. Nous faisons donc désormais les hypothèses suivantes

$$\boxed{N \geq 2}. \tag{27}$$

$$\boxed{0 < m < M} \tag{28}$$

et

$$\boxed{u_0 \in Y_m}. \tag{29}$$

Nous désignons par u l'unique solution du problème (PDE_m) avec u_0 pour condition initiale. D'après les résultats précédents, on sait déjà que celle-ci existe globalement en temps et que

$$u(t) \underset{t \to +\infty}{\longrightarrow} U_a \quad \text{dans} \quad C^1([0,1])$$

où U_a est l'unique solution stationnaire. Il est en fait possible de montrer que la convergence a lieu à vitesse exponentielle, comme précisé ci-dessous.

Théorème 1.3. *Supposons (27)(28)(29).*
Soit $U_a = U_{a(m)}$ l'unique solution stationnaire du problème (PDE_m), i.e. du problème (23)-(25).
Soit $\lambda_1 = \lambda_1(a) > 1$ la meilleure constante dans l'inégalité de type Hardy de la Proposition 1.1 ci-après. Fixons $\lambda \in (0, \lambda_1 - 1)$.
Alors il existe $C = C(u_0, \lambda) > 0$ tel que pour tout $t \geq 1$,

$$\|u(t) - U_a\|_{C^1([0,1])} \leq C \exp(-\lambda \dot{U}_a(1)^q \, t).$$

Remarque 1.4.

i) Rappelons que, d'après (22), la densité de cellules ρ est donnée, à constante multiplicative et changement de variable près, par la dérivée de u, si bien que ce théorème montre en fait la convergence uniforme à vitesse exponentielle de la densité de cellules vers l'équilibre.

ii) Ce théorème couvre le cas $N = 2$ et le résultat est, à notre connaissance, nouveau même dans ce cas très étudié.

iii) La preuve de ce résultat se fait en deux étapes : on montre dans un premier temps la convergence à vitesse exponentielle dans l'espace L^2 à poids $L^2\left((0,1), \frac{dx}{x^{2-q}}\right)$, ceci en linéarisant et en utilisant l'inégalité de type Hardy de la Proposition 1.1. Puis, on montre que l'équation (23) a un effet régularisant, ce qui permet d'obtenir la convergence C^1.

iv) L'espace $L^2\left((0,1),\frac{dx}{x^{2-q}}\right)$ apparaît très naturellement dans ce problème. En effet, l'équation (23) peut être vue comme un flot gradient

$$u_t = -\nabla\mathcal{F}[u(t)] \tag{30}$$

sur une "variété riemannienne de dimension infinie $(\mathcal{M},g)$", le poids $\frac{1}{x^{2-q}}$ intervenant dans la métrique g.
Rappelons que pour $N \geq 3$, $\mathcal{F}$ est définie en (26). Il existe une formule analogue lorsque $N = 2$ (voir [P3]).

Nous voudrions préciser l'origine de la constante $\lambda_1 > 1$ intervenant dans le théorème précédent. Pour cela, nous avons besoin de définir l'espace de fonctions suivant :

$$\boxed{H = \left\{h \in L^2\left((0,1),\frac{dx}{x^{2-q}}\right), \int_0^1 \dot{h}^2 < \infty,\ h(0) = h(1) = 0\right\}}.$$

On montre en fait que la norme naturelle associée à H est équivalente à la norme H_0^1 et que par conséquent $H = H_0^1(0,1)$.

On peut alors montrer l'inégalité de type Hardy suivante :

Proposition 1.1. *Soit $a \in (0,A)$.*
Il existe $\lambda_1 = \lambda_1(a) > 1$ tel que pour tout $h \in H$,

$$\int_0^1 \frac{\dot{h}^2}{\dot{U_a}^q} \geq \lambda_1 \int_0^1 \frac{h^2}{x^{2-q}}. \tag{31}$$

De plus, il existe $\phi_1 \in H$ tel qu'il y a égalité dans (31) si et seulement si $h = c\,\phi_1$, où $c \in \mathbb{R}$.

Remarque 1.5.
L'inégalité de Hardy ci-dessus correspond à une forme de stricte convexité de $\mathcal{F}$ en U_a. Ceci explique donc intuitivement, grâce à (30), que la convergence ait lieu à vitesse exponentielle.

1.5 Questions ouvertes

Un certain nombre de questions apparaissent naturellement :

a) Peut-on déterminer le bassin d'attraction de chacune des solutions stationnaires dans le cas critique ?

Dans le cas critique, il y a un continuum d'états d'équilibre et on a vu que toute solution converge vers l'un d'entre eux. Mais est-il possible de préciser lequel ? Et comment se transmet l'invariance par dilatation des états d'équilibre (pour le problème (PDE_m)) aux bassins d'attraction ?

b) La vitesse exponentielle de convergence uniforme dégénère-t-elle ou non dans le cas critique ?

On sait que pour $N \geq 2$, la densité de cellules converge uniformément vers l'unique solution stationnaire dans le cas d'une masse sous-critique, et ceci à vitesse exponentielle. Mais cette vitesse dégénère-t-elle dans le cas de la masse critique ? C'est le cas pour $N = 2$, comme cela a été montré dans [27]. Qu'en est-il pour $N \geq 3$? Sachant que la variété centrale semble être celle des états stationnaires, il semble raisonnable de penser que la vitesse de convergence uniforme doit rester exponentielle dans le cas critique mais cela reste à prouver.

c) Le phénomène de masse critique persiste-t-il pour les solutions de (PKS_q) sans symétrie radiale ?

On peut penser que non et plutôt s'attendre à l'existence de certaines solutions explosant au bord de la boule unité D pour une masse égale à la moitié de la masse critique, comme c'est le cas pour $N = 2$ et $q = 1$.

d) Dans le cas de l'espace entier $\mathbb{R}^n$, que se passe-t-il pour les solutions de (PKS_q) sans symétrie radiale ?

Il devrait a priori y avoir un phénomène de masse critique dans ce cas, avec la même masse critique que pour les solutions radiales dans le cas de la boule unité D.
Le cas d'une masse sous-critique nous paraît très intéressant.
En effet, jusqu'à présent, les solutions radiales du problème (PKS_q) ont été étudiées via le problème (PDE_m) en utilisant en particulier la structure de flot gradient de ce dernier. Or, il se trouve que le problème original (PKS_q) a aussi une structure de flot gradient et que la métrique Riemannienne sous-jacente engendre une distance qui généralise celle de Wasserstein $\mathcal{W}_2$ déjà bien utilisée dans le cas $N = 2$ et $q = 1$. Une discussion avec Jose Antonio Carrillo laisse entrevoir la possibilité de montrer au moins l'existence globale de solutions pour (PKS_q) via un schéma minimisant. Nous espérons aussi pouvoir montrer la convergence à vitesse exponentielle par la même méthode de flot gradient sur une "variété Riemannienne de dimension infinie" utilisée dans le cas radial.

2 Classification et résultats d'existence pour des systèmes elliptiques

2.1 Origine du problème

Dans cette partie, nous nous intéressons dans un premier temps aux systèmes elliptiques semi-linéaires du type

$$\begin{cases} -\Delta u = f(x,u,v) & x \in \Omega, \\ -\Delta v = g(x,u,v) & x \in \Omega, \end{cases} \tag{32}$$

où Ω est un domaine de $\mathbb{R}^n$, puis dans un second temps, aux systèmes plus généraux du type

$$\begin{cases} -F(D^2u) = f(x,u,v) & x \in \Omega, \\ -F(D^2v) = g(x,u,v) & x \in \Omega, \end{cases} \tag{33}$$

où F est un opérateur de Isaacs, i.e. un opérateur uniformément elliptique positivement homogène (la définition sera rappelée plus loin). F peut donc être complètement non-linéaire mais les résultats sont nouveaux même dans le cas linéaire.

Sans pour l'instant entrer trop dans les détails, nous souhaiterions présenter les grandes lignes de ce travail. Notre objectif est en particulier d'obtenir des résultats de classification ou de type Liouville dans le cas où Ω est l'espace entier $\mathbb{R}^n$ ou bien le demi-espace $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \ : \ x_n > 0\}$. Par la méthode de renormalisation de Gidas-Spruck (voir [19]), les théorèmes de Liouville obtenus permettent (à une complication près due à l'éventuelle existence de solutions semi-triviales) d'obtenir une borne a priori sur les solutions. Via le degré topologique, il est ensuite possible d'obtenir l'existence d'une solution classique strictement positive lorsque Ω est un domaine borné régulier.

La plupart des théorèmes de Liouville connus sur les systèmes elliptiques dans $\Omega = \mathbb{R}^n$ ou $\Omega = \mathbb{R}^n_+$ ont été obtenus en utilisant l'une des deux méthodes suivantes :

- Par l'utilisation de plans (ou sphères) mobiles et de la transformée de Kelvin, ce qui requiert une hypothèse de coopérativité du système (voir [32, 18, 17, 38]).
- Par l'utilisation d'identités intégrales comme celle de Pohozaev, ce qui n'est possible que si le système a une structure variationnelle (voir [33, 34, 35, 31, 36, 14]).

La stratégie adoptée ici est différente : on cherche à se ramener à un problème scalaire en établissant, pour toute solution (classique ou de viscosité) de (32) et (33) dans $\mathbb{R}^n$ ou $\mathbb{R}^n_+$, la proportionnalité des composantes

$$u = Kv$$

où $K > 0$ est une constante indépendante de la solution. Ceci fait, on applique ensuite bien sûr les résultats connus pour le problème scalaire obtenu en remplaçant u par Kv dans le système.
Pour cela, nous allons faire une hypothèse de structure naturelle sur les non-linéarités f et g pour espérer avoir la proportionnalité des composantes $u = Kv$, à savoir :

$$\exists K > 0 : \quad [f(x,u,v) - Kg(x,u,v)][u - Kv] \leq 0 \quad \text{pour tout } (u,v) \in \mathbb{R}^2 \text{ et } x \in \Omega, \tag{34}$$

Cette condition, qui a tout d'abord été mise à profit dans [30] dans le cas où $\Omega = \mathbb{R}^n$, ne demande pas que le système soit coopératif ou variationnel.

Elle est naturelle pour plusieurs raisons :

– D'un point de vue mathématique :

Lorsque Ω est borné, alors, si on pose

$$w = (u - Kv)^2,$$

on remarque que (en se restreignant au système (32) par simplicité)

$$\Delta w \geq 2(u - Kv)(Kg - f) \geq 0,$$

si bien que si $u = Kv$ sur $\partial\Omega$, alors $w = 0$ par le principe du maximum, i.e.

$$u = Kv.$$

Précisons tout de suite que cet argument simple est insuffisant dans le cas d'un domaine non borné puisque nous ne faisons aucune hypothèse sur (u, v) à l'infini. On sait d'ailleurs que la condition (34) n'est pas suffisante en général pour garantir la proportionnalité des composantes. Par exemple, considérant le cas où $\Omega = \mathbb{R}^n$, il a été montré dans [30, Theorème 1.4 ii)] que, si on suppose $p \geq p_S = \frac{n+2}{n-2}$ et $r \geq 0$, le système suivant

$$\begin{cases} -\Delta u = u^r \ v^p & x \in \mathbb{R}^n, \\ -\Delta v = u^r \ v^p & x \in \mathbb{R}^n \end{cases} \tag{35}$$

admet une solution classique positive (u, v) avec $u = v + 1$, donc sans égalité des composantes (dans ce cas, $K = 1$).

La condition (34) se révèle néanmoins très naturelle a posteriori de par la généralité du Théorème 2.3 concernant le cas du demi-espace $\Omega = \mathbb{R}^n_+$, qui montre la proportionnalité des composantes, en supposant de plus que u et v sont à croissance sous-linéaire, ce qui est donc en particulier valable pour les solutions bornées.

Une autre raison, beaucoup plus heuristique, est la suivante : si on considère le système parabolique associé à (32), on obtient formellement

$$\frac{d}{dt}\int_\Omega [u - Kv]^2(t) \leq 0,$$

ce qui laisse espérer que $\|[u-Kv](t)\|_2 \to 0$ lorsque $t \to +\infty$. Ceci impliquerait alors que les solutions stationnaires satisfont $u(\infty) = Kv(\infty)$.
Un autre indice en faveur de l'attractivité de la droite $u = Kv$ provient du fait que la condition (34) signifie que le champ de vecteurs (f, g) du système différentiel sous-jacent pointe en direction de la droite $u = Kv$.

– D'un point de vue physique :

Il se trouve qu'un certain nombre de systèmes rencontrés dans les applications satisfont la condition (34). C'est par exemple le cas pour le système

$$\begin{cases} -\Delta u = u^r v^p[av^q - cu^q] \\ -\Delta v = v^r u^p[bu^q - dv^q]. \end{cases} \tag{36}$$

lorsqu'on suppose que les paramètres réels a, b, c, d, p, q, r vérifient

$$a, b > 0, \quad c, d \geq 0, \qquad p, r \geq 0, \quad q > 0, \quad q \geq |p - r|. \tag{37}$$

Ceci a été établi dans [30] et est rappelé ci-dessous :

Proposition 2.1. *Supposons (37).*
(i) Alors les non-linéarités du système (36) *satisfont* (34)*.*
(ii) Supposons que $ab \geq cd$. Alors le nombre K est unique.
De plus, $K = 1$ si et seulement si $a + d = b + c$ et $K > 1$ si et seulement si $a + d > b + c$.
En outre, si $ab > cd$ (resp. $ab = cd$), alors $a - cK^q > 0$ (resp. $= 0$) et $bK^q - d > 0$ (resp. $= 0$).
Or, ce système recouvre au moins les deux situations suivantes en biologie et en physique.

$$(LV)\begin{cases} -\Delta u = u\Big[av - cu\Big] \\ -\Delta v = v\Big[bu - dv\Big], \end{cases} \qquad (BE)\begin{cases} -\Delta u = u\Big[av^2 - cu^2\Big] \\ -\Delta v = v\Big[bu^2 - dv^2\Big]. \end{cases}$$

En effet, le premier système (LV) est le système de Lotka-Volterra intervenant en dynamique des populations pour décrire une interaction symbiotique de deux espèces, le second est le modèle de Bose-Einstein qui sert à modéliser les condensats éponymes mais intervient aussi en optique non-linéaire. Dans le cas présent, l'interaction entre les deux états quantiques est attractive alors que l'auto-interaction est répulsive ou neutre.

2.2 Résultats obtenus sur les systèmes du type (32)

Les résultats présentés dans cette sous-section correspondent à la Note aux Compte-Rendus de l'Académie des Sciences [P4], écrite avec Philippe Souplet, ainsi qu'à l'article plus développé [P5], fruit d'une collaboration avec Philippe Souplet et Boyan Sirakov.

2.2.1 Cas où $\Omega = \mathbb{R}^n$

Nous étudions le système (36) et supposons toujours que l'hypothèse (37) est vérifiée, ce qui assure que la condition de structure (34) est satisfaite, d'après la

Proposition 2.1.

Le résultat suivant donne des conditions suffisantes pour avoir la proportionnalité des composantes $u = Kv$.

Théorème 2.1. *Supposons (37) et $ab \geq cd$.*
Soit $K > 0$ la constante définie dans la Proposition 2.1 et (u, v) une solution classique strictement positive de (36) dans $\mathbb{R}^n$.

(i) Supposons que

$$r \leq \frac{n}{(n-2)_+}. \tag{38}$$

Si $p + q < 1$, supposons de plus que (u, v) est bornée. Alors $u \equiv Kv$.

(ii) Supposons que

$$p \leq \frac{2}{(n-2)_+} \quad et \quad c, d > 0. \tag{39}$$

Si $q + r \leq 1$, supposons de plus que (u, v) est bornée. Alors $u \equiv Kv$.

Ce théorème, lorsqu'il s'applique, permet de ramener l'étude du système (36) à celui de l'équation

$$-\Delta v = C\, v^\sigma$$

où $C = K^p(bK^q - d) \geq 0$ et

$$\sigma = p + q + r > 0.$$

Or, on sait par exemple, d'après la Proposition 2.1, que si $ab > cd$ alors $C > 0$ et que l'équation précédente admet des solutions si et seulement si $n \geq 3$ et $\sigma \geq \frac{n+2}{n-2}$. Lorsque $\sigma = \frac{n+2}{n-2}$, on sait même que cette solution est unique à translation et changement d'échelle près, ce qui permet alors de classifier de manière précise les solutions du système (36).

De même, en utilisant le Théorème de type Liouville bien connu pour cette équation, nous obtenons le résultat suivant :

Théorème 2.2. *Supposons (37), $ab > cd$, et*

$$\sigma := p + q + r < \frac{n+2}{(n-2)_+}.$$

(i) Alors le système (36) n'a pas de solution classique bornée strictement positive.

(ii) Supposons en outre que

$$p + q \geq 1, \qquad ou \qquad p \leq \frac{2}{(n-2)_+}, \qquad ou \qquad \sigma \leq \frac{n}{(n-2)_+}$$

(Remarquons que cette hypothèse est vérifiée dans chacun des "cas physiques" $q \geq 1$ ou $n \leq 4$). Alors le système (36) n'a pas de solution classique (bornée ou non) strictement positive.

Maintenant que le cas des solutions strictement positives a été traité, il est naturel de considérer la question des solutions positives au sens large, ce qui est l'objet du résultat suivant :

Corollaire 2.1. *Sous les hypothèses du Théorème 2.2(i) (resp., 2.2(ii)), supposant de plus que*

$$q + r > 1 \qquad ou \qquad (q + r = 1 \ \ et \ p > 0),$$

alors toute solution positive bornée (resp., positive) de (36) est de la forme

$$(C_1, 0) \qquad ou \qquad (0, C_2),$$

où C_1, C_2 sont des constantes positives ou nulles.

De plus, si en outre $p = 0$, $r > 0$ et $c > 0$ (resp., $d > 0$), alors $C_1 = 0$ (resp., $C_2 = 0$), alors que si $r = 0$, alors $C_1 = C_2 = 0$.

2.2.2 Cas où $\Omega = \mathbb{R}^n_+$

Dans ce cas, grâce au principe du maximum de Phragmèn-Lindelöf, nous obtenons un théorème général sous la seule l'hypothèse (34) sur les non-linéarités (ce qui montre comme celle-ci est naturelle).
Rappelons auparavant que $u : \mathbb{R}^n_+ \to \mathbb{R}$ est dite à croisance sous-linéaire si $u(x) = o(|x|)$ lorsque $|x| \to \infty$, $x \in \mathbb{R}^n_+$.

Théorème 2.3. *Supposons (34).*
Soit (u, v) une solution classique de (32) dans $\mathbb{R}^n_+$ telle que $u = Kv$ sur $\partial\mathbb{R}^n_+$.
Si u et v sont à croissance sous-linéaire, alors

$$u \equiv Kv \quad \text{dans } \mathbb{R}^n_+.$$

Remarque 2.1.

- *i) Ce résultat s'applique notamment aux solutions classiques bornées vérifant la condition de Dirichlet $u = v = 0$ sur $\partial\mathbb{R}^n_+$.*
- *ii) Il est possible d'affaiblir l'hypothèse de sous-linéarité des solutions en utilisant les moyennes semi-sphériques (voir [P4] et [P5]).*

En utilisant le résultat récent de Z. Chen, C-S. Lin and W. Zou dans [13] établissant que pour $p > 1$, l'équation $-\Delta u = u^p$ n'a pas de solution classique positive bornée non triviale sur $\mathbb{R}^n_+$ s'annulant sur $\partial\mathbb{R}^n_+$, nous pouvons alors déduire le théorème de Liouville suivant :

Corollaire 2.2. *Supposons que (34) est vraie pour un certain $K > 0$ et qu'il existe des nombres $c > 0$ et $p > 1$ telles que*

$$f(x, Ks, s) = cs^p, \quad s \geq 0,\, x \in \mathbb{R}^n_+.$$

Alors le système (32) n'a pas de solution classique positive bornée non triviale sur $\mathbb{R}^n_+$ telle que $u = v = 0$ sur la frontière $\partial\mathbb{R}^n_+$.

Si on fait des hypothèses plus restrictives sur les non-linéarités, on peut obtenir un résultat de classification des solutions classiques dans $\mathbb{R}^n_+$ sans hypothèse de croissance à l'infini. On appelle solution **semi-triviale** une solution (u, v) telle que $u = 0$ ou $v = 0$.

Théorème 2.4. *Soit $p, q, r, s \geq 0$.*
Supposons que f, g satisfont la condition (34) pour une constante $K > 0$ et que, pour un $c > 0$,

$$f(x, u, v) \geq c\, u^r\, v^p \quad \textit{and} \quad g(x, u, v) \geq c\, u^q\, v^s \quad \textit{pour tout } u, v \geq 0 \textit{ et } x \in \mathbb{R}^n_+. \tag{40}$$

Soit (u, v) une solution classique positive de (32) dans $\mathbb{R}^n_+$, telle que $u = Kv$ sur $\partial\mathbb{R}^n_+$.

(i) Ou bien $u \leq Kv$ ou bien $u \geq Kv$ dans $\mathbb{R}^n_+$.

(ii) Si

$$r \leq \frac{n+1+p}{n-1} \quad \textit{ou} \quad q \leq \frac{1+s}{n-1}, \tag{41}$$

et

$$s \leq \frac{n+1+q}{n-1} \quad \textit{ou} \quad p \leq \frac{1+r}{n-1}, \tag{42}$$

alors ou bien $u \equiv Kv$ ou bien (u, v) est semi-triviale.

(iii) Si (41)-(42) *sont vérifiées et si* $\min(p + r, q + s) \leq (n + 1)/(n - 1)$, *alors (u, v) est semi-triviale.*

2.2.3 Estimation a priori et existence dans un domaine borné

Nous considérons maintenant le problème de Dirichlet suivant :

$$\begin{cases} -\Delta u = u^r v^p \Big[a(x)v^q - c(x)u^q\Big] + \mu(x)u, & x \in \Omega, \\ -\Delta v = v^r u^p \Big[b(x)u^q - d(x)v^q\Big] + \nu(x)v, & x \in \Omega, \\ \quad u = v = 0, & x \in \partial\Omega, \end{cases} \tag{43}$$

où Ω est un domaine borné régulier de $\mathbb{R}^n$.

Par simplicité, nous nous sommes restreints ici à un terme de plus bas degré linéaire. Remarquons également que la dépendance en espace des coefficients a, b, c, d et la présence d'un terme de plus bas degré entrainent que les non-linéarités ne satisfont pas la condition (34). C'est pourquoi il n'est pas possible de déduire simplement ici que $u = Kv$ et de ramener le système à une équation scalaire.

Théorème 2.5. *Soit $p, r \geq 0$, $q > 0$ tels que*

$$q \geq |p-r|, \qquad q+r > 1 \text{ ou } (q+r = 1 \text{ et } p > 0), \qquad r \leq 1, \qquad 1 < p+q+r < \frac{n+2}{(n-2)_+}. \tag{44}$$

Soit $a, b, c, d, \mu, \nu \in C(\overline{\Omega})$ satisfaisant $a, b > 0$, $c, d \geq 0$ dans $\overline{\Omega}$ et

$$\inf_{x \in \Omega} \left[a(x)b(x) - c(x)d(x)\right] > 0. \tag{45}$$

(i) Alors il existe $M > 0$, dépendant seulement de p, q, r, Ω et des bornes uniformes de a, b, c, d, μ, ν telle que toute solution classique strictement positive (u, v) de (43) satisfait

$$\sup_{\Omega} u \leq M, \quad \sup_{\Omega} v \leq M.$$

(ii) Supposons de plus que a, b, c, d, μ, ν sont Höldériennes et que $\mu, \nu < \lambda_1(-\Delta, \Omega)$ dans $\overline{\Omega}$. Alors il existe au moins une solution classique strictement positive de (43).

2.3 Résultats obtenus sur les systèmes de type (33)

Le travail présenté ci-après est le fruit d'une visite de trois semaines à Boyan Sirakov à la PUC de Rio de Janeiro en Avril 2014. Nous souhaitons préciser que celle-ci a été financée par le Réseau Franco-Brésilien de Mathématiques.

Nous nous penchons désormais sur le cas du système (33), où nous supposons que F est un opérateur de Isaacs, i.e. que F satisfait les deux hypothèses suivantes, où on a noté $\mathcal{S}_n$ l'ensemble des matrices symétriques réelles de taille n :

– F est uniformément elliptique :
  il existe $\Lambda > \lambda > 0$ tel que pour tout $(M, N) \in {\mathcal{S}_n}^2$ avec $N \geq 0$,

$$(H_1) \qquad \lambda\, tr(N) \leq F(M+N) - F(M) \leq \Lambda\, tr(N).$$

– F est positivement 1-homogène :
  pour tout $t \geq 0$ et $M \in \mathcal{S}_n$, on a

$$(H_2). \qquad F(t\, M) = t\, F(M).$$

A titre d'exemples d'opérateurs de Isaacs, le cas où F est la trace est le plus simple puisque l'opérateur obtenu est le Laplacien. Deux exemples très importants sont aussi les opérateurs extrémaux de Pucci définis par

$$\mathcal{M}^+(M) = \Lambda \sum_{\mu_i > 0} \mu_i + \lambda \sum_{\mu_i < 0} \mu_i$$

et

$$\mathcal{M}^-(M) = \lambda \sum_{\mu_i > 0} \mu_i + \Lambda \sum_{\mu_i < 0} \mu_i,$$

pour tout $M \in \mathcal{S}_n$, où $(\mu_i)_{i=1..n})$ sont les valeurs propres de M.
Un autre exemple est celui de l'opérateur de Barenblatt défini par

$$F(M) = \max(tr(M),\ 2tr(M))$$

pour tout $M \in \mathcal{S}_n$, qui intervient en théorie de l'élastoplasticité.

Puisque nous traitons ici des opérateurs elliptiques complètement non-linéaires, lorsque nous parlerons de solutions, ce sera bien sûr au sens de solutions de viscosité. Pour plus de détails sur ces différentes notions, voir le livre de L.A. Caffarelli et X. Cabre [9].

Le principe employé pour attaquer le système (33) est, dans les grandes lignes, similaire à celui utilisé dans le cas du laplacien. En revanche, il y a une perte d'efficacité due, non pas à la méthode elle-même ou au fait qu'il s'agisse d'un système, mais à la compréhension à ce jour incomplète de l'équation

$$-F(D^2u) = u^p$$

pour un opérateur de Isaacs quelconque. En revanche, l'inéquation

$$-F(D^2u) \geq u^p$$

est très bien comprise, au moins dans le cas d'un cône ou de l'espace tout entier, dans le sens où on connait l'exposant optimal de non-existence d'une solution non triviale. Tout progrès futur dans la compréhension de l'équation

$$-F(D^2u) = u^p,$$

ne serait-ce que pour un opérateur ou une classe d'opérateurs particulière, aura donc des conséquences bénéfiques immédiates sur nos résultats.

Pour exprimer ceux-ci, nous avons besoin de définir des exposants correspondant, au signe près, au degré d'homogénéité des solutions fondamentales dans le domaine considéré.

Plus précisément, dans le cas où $\Omega = \mathbb{R}^n$, on sait d'après l'article de S. Armstrong, B. Sirakov et C.K. Smart [2] qu'il existe une solution Φ orientée vers le haut, non nulle et unique à constante près, de l'équation

$$-F(D^2\Phi) = 0 \qquad \text{dans } \mathbb{R}^n \setminus \{0\}.$$

Cette solution Φ est homogène de degré $-\alpha^*$, i.e. si $\alpha^* \neq 0$, alors

$$\Phi(x) = t^{\alpha^*}\Phi(tx) \qquad \text{pour tout } t > 0,\ x \in \mathbb{R}^n \setminus \{0\}$$

et si $\alpha^*(F) = 0$, alors

$$\Phi(x) = \Phi(tx) + \log(t) \qquad \text{pour tout } t > 0,\ x \in \mathbb{R}^n \setminus \{0\},$$

où $\alpha^* = \alpha^*(F, n) > -1$ est déterminé de manière unique par F et la dimension n. Pour plus de détails voir [2, Théorème 1.3].
A titre d'exemples, notons que dans le cas du Laplacien, i.e. lorsque F est la trace, on sait que pour $n \geq 1$

$$\alpha^*(F) = n - 2.$$

Dans le cas des opérateurs extrémaux de Pucci, on a

$$\alpha^*(\mathcal{M}^+) = \frac{\lambda}{\Lambda}(n-1) - 1$$

et

$$\alpha^*(\mathcal{M}^-) = \frac{\Lambda}{\lambda}(n-1) - 1.$$

De même, il existe une notion de solution fondamentale dans le cas d'un cône

$$\mathcal{C}_\omega = \{tx, t > 0, x \in \omega\}$$

où ω est un sous-domaine (strict) C^2 de la sphère unité de $\mathbb{R}^n$.
En effet, il existe une solution singulière $\Psi^+ \in C\left(\overline{\mathcal{C}_\omega} \setminus \{0\}\right)$

$$\begin{cases} -F(D^2\Psi^+) = 0, & \text{dans } \mathcal{C}_\omega \\ \Psi^+ = 0, & \text{sur } \partial\mathcal{C}_\omega \setminus \{0\} \end{cases} \tag{46}$$

telle que

$$\Psi^+(x) > 0 \text{ sur } \mathcal{C}_\omega \qquad \text{et} \qquad \Psi^+(x) = t^{\alpha^+}\Psi^+(tx) \qquad \text{pour tout } t > 0,\ x \in \mathcal{C}_\omega.$$

avec

$$\alpha^+ = \alpha^+(F, n, \mathcal{C}_\omega) > 0$$

déterminé de façon unique par F, n et $\mathcal{C}_\omega$.
Pour plus de détails, voir le résultat [1, Théorème 1.1] de S. Armstrong, B. Sirakov et C.K. Smart.

Nous pouvons désormais énoncer les différents résultats.

2.3.1 Cas où $\Omega = \mathcal{C}_\omega$

Commençons par le cas où Ω est un cône $\mathcal{C}_\omega$, ω étant un sous-domaine (strict) C^2 de la sphère unité de $\mathbb{R}^n$.

Théorème 2.6. *Supposons que f et g satisfont la condition (34) pour un certain $K > 0$. Soit (u, v) une solution de viscosité bornée sur $\mathcal{C}_\omega$ de*

$$\begin{cases} -F(D^2u) = f(x, u, v), & x \in \mathcal{C}_\omega \\ -F(D^2v) = g(x, u, v), & x \in \mathcal{C}_\omega \\ u = K\,v & sur\ \partial\mathcal{C}_\omega. \end{cases} \tag{47}$$

Alors

$$u = K\,v.$$

Ce résultat s'applique notamment aux solutions de viscosité bornées vérifiant la condition de Dirichlet $u = v = 0$ sur $\partial\mathcal{C}_\omega$.

Nous pouvoir déduire du Théorème 2.6 le résultat de type Liouville suivant :

Corollary 2.1. *Sous les hypothèses du théorème précédent et s'il existe $C > 0$ et*

$$\sigma \in \left(0, \frac{\alpha^+(F, \mathcal{C}_\omega) + 2}{\alpha^+(F, \mathcal{C}_\omega)}\right)$$

tels que pour tout $x \in \mathbb{R}^n_+$ et $y \geq 0$,

$$f(x, Ky, y) = C\,y^\sigma,$$

alors l'unique solution de viscosité positive au sens large et bornée de (47) est la solution nulle.

2.3.2 Cas où $\Omega = \mathbb{R}^n$

Dans le cas où $\Omega = \mathbb{R}^n$, nous considérons le système suivant :

$$\begin{cases} -F(D^2u) = u^r v^p[av^q - cu^q] & \text{dans } \mathbb{R}^n \\ -F(D^2v) = v^r u^p[bu^q - dv^q] & \text{dans } \mathbb{R}^n. \end{cases} \tag{48}$$

Nous supposons toujours que les paramètres réels a, b, c, d, p, q, r satisfont

$$a, b > 0, \quad c, d \geq 0, \qquad p, r \geq 0, \quad q > 0, \quad q \geq |p - r|. \tag{49}$$

Nous donnons maintenant des conditions suffisantes pour avoir la proportionnalité des composantes $u = Kv$.

Théorème 2.7. *Soit F un opérateur de Isaacs.*
Supposons (49) et

$$ab \geq cd.$$

Soit $K > 0$ la constante donnée dans la Proposition 2.1.
Soit (u, v) une solution de viscosité strictement positive de (48) dans $\mathbb{R}^n$.

i) Supposons que

$$\alpha^*(F) \leq 0 \ ou \ \left(\alpha^*(F) > 0 \ et \ 0 \leq r \leq \frac{\alpha^*(F)+2}{\alpha^*(F)}\right).$$

Si $p+q<1$, nous supposons de plus que u et v sont bornées. Alors

$$u = Kv.$$

ii) Supposons que

$$p \leq \frac{2}{\alpha^*(F)} \qquad et \qquad c, d > 0.$$

Si $q+r \leq 1$, nous supposons de plus que u et v sont bornées. Alors

$$u = Kv.$$

Ceci a pour conséquence le théorème de Liouville suivant :

Théorème 2.8. *Soit F un opérateur de Isaacs.*
Supposons (49) et

$$ab > cd.$$

Nous notons

$$\sigma = p+q+r > 0$$

et faisons l'hypothèse que

$$\alpha^*(F) \leq 0 \quad ou \quad 0 \leq r \leq \frac{\alpha^*(F)+2}{\alpha^*(F)} \quad ou \quad 0 \leq p \leq \frac{2}{\alpha^*(F)}.$$

Supposons de plus que l'équation

$$-F(D^2u) = u^\sigma \text{ n'a pas de solution de viscosité strictement positive bornée dans } \mathbb{R}^n. \tag{50}$$

Alors (48) n'a pas de solution de viscosité strictement positive bornée dans $\mathbb{R}^n$.

Théorème 2.9. *Sous les hypothèses du Théorème 2.8, si nous supposons de plus que*

$$q+r > 1 \ ou \ (q+r = 1 \ et \ p > 0)$$

alors toute solution de viscosité positive (36) est semi-triviale, i.e.

$$(u,v) = (C_1, 0) \qquad ou \qquad (u,v) = (0, C_2)$$

avec $C_1, C_2 \geq 0$. De plus :
- Si $r=0$, alors $(u,v)=(0,0)$.
- Si $r>0$, $p=0$ et $c>0$ (resp. $d>0$), alors $C_1=0$ (resp. $C_2=0$).

Remarque 2.2. *Nous nous heurtons donc ici à la compréhension encore fort sommaire de l'équation*

$$-F(D^2u) = u^\sigma$$

pour un opérateur de Isaacs quelconque. Néanmoins, nous pouvons observer que :

i) La condition de type Liouville (54) *est vérifiée lorsque l'une des deux conditions suivantes est satisfaite :*

a) $\alpha^*(F) \leq 0$

b) $\alpha^*(F) > 0$ *et* $\sigma \leq \frac{\alpha^*(F)+2}{\alpha^*(F)}$

ii) Dans le cas où F *est l'un des opérateurs extrémaux de Pucci, alors on sait d'après [16] que si* $\alpha^*(F) > 0$*, il existe*

$$\overline{\sigma} > \frac{\alpha^*(F)+2}{\alpha^*(F)}$$

tel que (54) *est satisfaite pour les solutions radiales si et seulement si* $\sigma < \overline{\sigma}$*,* $\overline{\sigma}$ *n'étant pas connu explicitement en fonction de* n, λ, Λ.

iii) Si on considère, l'opérateur de Barenblatt défini par

$$F(M) = \max(tr(M),\ 2tr(M))$$

pour tout $M \in \mathcal{S}_n$*, alors toute solution de*

$$-F(D^2u) = u^\sigma$$

est surharmonique donc les solutions de cette dernière équation coïncident avec celles de

$$-\Delta u = u^\sigma.$$

Ainsi, (54) *est vérifiée si*

$$\sigma < \frac{n+2}{n-2}.$$

Ce raisonnement évident sur l'équation donne cependant un résultat non trivial sur le système (48), grâce au Théorème 2.8, bien que les seconds membres du système n'aient pas a priori de signe défini.

2.3.3 Estimation a priori et existence dans un domaine borné

Nous considérons désormais le système suivant :

$$\begin{cases} -F(D^2u) = u^r v^p\Big[a(x)v^q - c(x)u^q\Big] + h_1(x,u,v), & x \in \Omega, \\ -F(D^2v) = v^r u^p\Big[b(x)u^q - d(x)v^q\Big] + h_2(x,u,v), & x \in \Omega, \\ \qquad u = v = 0, & x \in \partial\Omega, \end{cases} \tag{51}$$

où Ω est un domaine borné régulier de $\mathbb{R}^n$.

Le résultat suivant fournit une borne a priori sur les solutions de viscosité strictement positives de (51). Nous notons

$$\alpha^+ = \alpha^+(F, \mathbb{R}^n_+).$$

Théorème 2.10. *Soit F un opérateur de Isaacs.*
Soit $p, r \geq 0$, $q > 0$, $q \geq |p - r|$ tels que

$$q + r > 1 \text{ ou } (q + r = 1 \text{ et } p > 0) \tag{52}$$

Nous posons

$$\sigma = p + q + r > 1.$$

Faisons l'hypothèse que

$$\alpha^*(F) \leq 0 \quad \text{or} \quad 0 \leq r \leq \frac{\alpha^*(F) + 2}{\alpha^*(F)} \quad \text{or} \quad 0 \leq p \leq \frac{2}{\alpha^*(F)}. \tag{53}$$

Supposons de plus que

$$-F(D^2 u) = u^\sigma \text{ n'a pas de solution de viscosité strictement positive bornée,} \tag{54}$$

qu'elle soit considérée dans $\mathbb{R}^n$ ou dans $\mathbb{R}^n_+$ avec condition de Dirichlet au bord.

Soit $a, b, c, d \in C(\overline{\Omega})$ satisfaisant $a, b > 0$, $c, d \geq 0$ dans $\overline{\Omega}$ et

$$\inf_{x \in \Omega} \left[a(x)b(x) - c(x)d(x)\right] > 0. \tag{55}$$

Soit $h_1, h_2 \in C(\overline{\Omega} \times [0, \infty)^2)$ satisfaisant

$$\lim_{u+v \to \infty} \frac{h_i(x, u, v)}{(u + v)^\sigma} = 0, \quad i = 1, 2, \tag{56}$$

et supposons que l'un des deux ensembles suivants d'hypothèses est vérifié :

$$\begin{cases} r \leq 1, \quad \text{et, notant } \bar{m} := \min\{\inf_{x \in \Omega} a(x), \inf_{x \in \Omega} b(x)\} > 0, \\ \liminf_{v \to \infty,\ u/v \to 0} \dfrac{h_1(x, u, v)}{u^r v^{p+q}} > -\bar{m}, \quad \liminf_{u \to \infty,\ v/u \to 0} \dfrac{h_2(x, u, v)}{v^r u^{p+q}} > -\bar{m}, \end{cases} \tag{57}$$

ou

$$\begin{cases} m := \min\{\inf_{x \in \Omega} c(x), \inf_{x \in \Omega} d(x)\} > 0, \quad \text{et} \\ \limsup_{u \to \infty,\ v/u \to 0} \dfrac{h_1(x, u, v)}{u^{r+q} v^p} < m, \quad \limsup_{v \to \infty,\ u/v \to 0} \dfrac{h_2(x, u, v)}{v^{r+q} u^p} < m \end{cases} \tag{58}$$

(avec des limites uniformes par rapport à $x \in \overline{\Omega}$ in (56)– (58)).
Alors il existe $M > 0$ telle que toute solution de viscosité strictement positive (u, v) de (51) satisfait

$$\sup_\Omega u \leq M, \qquad \sup_\Omega v \leq M. \tag{59}$$

Remarque 2.3. *Si*

$$1 < \sigma < \frac{\alpha^+ + 2}{\alpha^+},$$

alors les conditions (53) *et* (54) *sont vérifiées. Voir la Remarque 1.3 du Chapitre 5 pour plus de détails.*

Pour obtenir un résultat d'existence, nous devons supposer de plus que F est sous-additif, i.e.

$$F(M+N) \leq F(M) + F(N) \text{ for all } (M,N) \in {\mathcal{S}_n}^2,$$

ce qui est équivalent à F convexe pour un opérateur de Isaacs (car il est alors continu et positivement 1-homogène, i.e. vérifie (H_2)).

Théorème 2.11. *Soit F un opérateur de Isaacs sous-additif.*
Supposons (52)–(57) satisfaites. Supposons de plus que a, b, c, d, h_1, h_2 sont Höldériennes et que pour un $\epsilon > 0$

$$\inf_{x\in\Omega,\ u,v>0} u^{-1}\, h_1(x,u,v) > -\infty, \qquad \inf_{x\in\Omega,\ u,v>0} v^{-1}\, h_2(x,u,v) > -\infty, \tag{60}$$

$$\sup_{x\in\Omega,\ u>0} u^{-1}\, h_1(x,u,0) < \lambda_1^+(-F,\Omega), \qquad \sup_{x\in\Omega,\ v>0} v^{-1}\, h_2(x,0,v) < \lambda_1^+(-F,\Omega), \tag{61}$$

$$\sup_{x\in\Omega,\ u,v\in(0,\epsilon)^2} (u+v)^{-1}\left[h_1(x,u,v) + h_2(x,u,v)\right] < \lambda_1^+(-F,\Omega). \tag{62}$$

Alors il existe une solution de viscosité strictement positive bornée de (51).

2.4 Questions ouvertes

Deux problèmes intéressants se présentent à l'esprit dans le prolongement de ce travail :

a) Extension à plus de 2 équations.

La méthode employée utilisait le fait que le système comportait deux équations. Est-il possible de la généraliser pour plus de deux équations ? Il y a en particulier des problèmes ouverts importants pour les systèmes de type Bose-Einstein qui permettraient d'avoir des résultats lorsqu'il y a plus de deux états quantiques.

b) Etude de systèmes mixtes (dissipatifs-répulsifs), de la forme

$$\Delta U = U^r V^p, \quad -\Delta V = V^s U^q.$$

Certaines propriétés de ces systèmes jouent un rôle essentiel au niveau technique dans l'analyse réalisée dans [P5] , mais de nombreuses questions intéressantes restent ouvertes. Par exemple, est-il possible de déterminer ou au moins d'améliorer les conditions sur les paramètres p, q, r, s de non-existence d'une solution positive ?

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Chapitre 1

Un système de type Patlak-Keller-Segel avec masse critique en toute dimension [1]

Dans ce chapitre, nous étudions les solutions radiales dans une boule D de $\mathbb{R}^N$ d'un système parabolique-elliptique de type Patlak-Keller-Segel avec une sensitivité non-linéaire de type puissance, l'exposant étant critique. Pour $N = 2$, ce système est bien connu pour sa masse critique 8π. Nous montrons qu'un phénomène de masse critique se produit également pour $N \geq 3$, mais avec un comportement qualitatif très différent dans le cas de la masse critique. Plus précisément, si la masse totale des cellules est inférieure ou égale à la masse critique $\overline{M}$, alors la densité de cellules converge vers une solution stationnaire. Dans le cas de la masse critique, ce résultat n'est pas évident a priori car il existe un continuum d'états d'équilibre. De plus, il contraste fortement avec le cas $N = 2$ où la solution explose en temps infini. En revanche, si la masse totale des cellules est strictement supérieure à $\overline{M}$, alors la densité de cellules explose en temps fini. Ceci provient de l'existence (contrairement au cas $N = 2$) d'une famille de solutions autosimilaires explosant en temps fini. Notons enfin que, contrairement au cas $N = 2$, la densité de cellules peut être concentrée dans une boule strictement incluse dans D.

1 Introduction

1.1 Origin of the problem

Chemotaxis is the biological phenomenon whereby some cells or bacteria direct their movement according to some chemical present in their environment which can be attractive or repulsive. We shall focus on the case where the chemical is attractive

1. Ce chapitre est tiré de l'article [P2].

(then called chemoattractant) and self-emitted by cells. For instance, in case of starvation, amoebas *Dyctyostelium discoideum* emit cyclic adenosine monophosphate (cAMP) which attract themselves. Chemotaxis is thus a strong mean of communication for cells and leads to collective motion.
For more details on the social life of amoebas *Dyctyostelium discoideum*, see the article [18] of M.A. Herrero and L. Sastre.

1.1.1 Mathematical formulation

Assuming that cells and chemoattractant are diffusing and that cells are sensitive to the chemical's concentration gradient (a fact experimentally observed), Patlak in 1953 (cf. [37]) and Keller and Segel in 1970 (cf. [27]) have proposed the following mathematical model, a parabolic-parabolic system known as Patlak-Keller-Segel system :

$$\rho_t = D_1\,\Delta\rho - \nabla[\chi\,\nabla c] \tag{1.1}$$

$$c_t = D_2\,\Delta c + \mu\,\rho \tag{1.2}$$

where ρ is the cell density, c the chemoattractant concentration, D_1 and D_2 are diffusion coefficients, $\chi = \chi(\rho, c)$ is the sensitivity of cells to the chemoattractant and μ the creation rate of chemical by cells.
Cells and chemoattractant are assumed to lie in a bounded domain Ω of $\mathbb{R}^N$ with $N \geq 2$ ($\mathbb{R}^2$ or $\mathbb{R}^3$ physically speaking) so we have to specify the boundary conditions. For the cell density ρ, it is natural to impose a no flux boundary condition

$$D_1\,\frac{\partial\rho}{\partial\nu} - \chi(\rho, c)\,\frac{\partial c}{\partial\nu} = 0 \quad \text{ on } \partial\Omega, \tag{1.3}$$

where ν denotes the outward unit normal vector to the boundary $\partial\Omega$.
For the chemoattractant concentration c, Dirichlet boundary conditions are assumed :

$$c = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

Some cells diffuse much slower than the chemoattractant and we will make this assumption. In this case, two timescales appear in the system and to the limit, we can assume that the chemical concentration c reaches instantaneously its stationary state. After renormalization, these considerations lead to the simplified parabolic-elliptic Patlak-Keller-Segel system :

$$\rho_t = \Delta\rho - \nabla[\chi\,\nabla c] \tag{1.5}$$

$$-\Delta c = \rho \tag{1.6}$$

with the same boundary conditions as above, which then become :

$$\frac{\partial\rho}{\partial\nu} - \chi(\rho, c)\,\frac{\partial c}{\partial\nu} = 0 \quad \text{ on } \partial\Omega. \tag{1.7}$$

$$c = 0 \quad \text{on } \partial\Omega. \tag{1.8}$$

We would like to add that there also exists cells which have a velocity comparable to that of the chemoattractant. This is for instance the case of Escherichia coli which moreover has a 'run and tumble' motion. Hence, in this case, diffusion does not seem to be the most suitable modeling. On that subject, see the article of B. Perthame [34] for a kinetic approach which takes into account these characteristics and allows to recover the Patlak-Keller-Segel model in a diffusion limit.

For a review on mathematics of chemotaxis, see the chapter written by M.A. Herrero in [17] and the article [23] of T. Hillen and K. J. Painter. For a review on the Patlak-Keller-Segel model, see both articles of D. Horstmann [20, 21].

In [22], D. Horstmann and M. Winkler have studied the case where the sensitivity χ depends only on ρ and shown that :

- If $\chi(\rho) \leq C\rho^q$ for $\rho \geq 1$ and $q < \frac{2}{N}$, then the cell density ρ exists globally and is even uniformly bounded in time.
- If $\chi(\rho) \geq C\rho^q$ for $\rho \geq 1$ and $q > \frac{2}{N}$, then ρ can blow up.

See also [13, 28, 31, 32, 38] for related results.

Thus, the power $q = \frac{2}{N}$ of the nonlinearity $\chi(\rho)$ is critical for that system. This is why we are going to focus on the following problem, noted (PKS_q), with a special interest to the case $q = \frac{2}{N}$:

$$(PKS_q) \qquad \begin{cases} \rho_t = \Delta\rho - \nabla[\rho^q \nabla c] \\ -\Delta c = \rho \end{cases} \tag{1.9}$$

where the boundary conditions become

$$\frac{\partial\rho}{\partial\nu} - \rho^q \frac{\partial c}{\partial\nu} = 0 \quad \text{on } \partial\Omega, \tag{1.10}$$

$$c = 0 \quad \text{on } \partial\Omega. \tag{1.11}$$

We would like to stress that $q = \frac{2}{N}$ is exactly the exponent for which the mass, i.e. the L^1 norm of ρ, is invariant by the rescaling of this system, given by

$$\rho_\lambda(t,y) = \lambda^{\frac{2}{q}} \rho(\lambda^2 t, \lambda y) \tag{1.12}$$

$$c_\lambda(t,y) = \lambda^{\frac{2}{q}-2} c(\lambda^2 t, \lambda y) \tag{1.13}$$

for all $t > 0$, $y \in \mathbb{R}^N$ and $\lambda > 0$. This fact opens the door to the possibility of a critical mass phenomenon.

Remark 1.1. *System (PKS_q) can also be seen as the macroscopic description of a collection of n particles following a generalized stochastic Langevin equation. Making a mean field approximation, it is actually obtained as a nonlinear Fokker-Planck equation in a proper thermodynamic limit $n \to \infty$. For more details, see the article of P.H. Chavanis [10, section 3.4].*

1.1.2 Radial setting

In this paper, we restrict our study to the case of radially symmetric solutions of (PKS_q) where Ω is the open unit ball $B \subset \mathbb{R}^N$ centered at the origin. Note that by using the scaling of the system and its invariance by translation, we can of course cover the case of any open ball of $\mathbb{R}^N$.

We would like to point out that for $N = 2$, the critical exponent is $q = 1$, so the latter system reduces to the most studied Keller-Segel parabolic-elliptic type system :

$$\rho_t = \Delta\rho - \nabla[\rho\,\nabla c] \tag{1.14}$$

$$-\Delta c = \rho \tag{1.15}$$

It is a well-known fact that this system exhibits a critical mass phenomenon. More precisely, denoting $\overline{m}$ the total mass of the cells, it has been shown for radially symmetric solutions that :

- If $\overline{m} < 8\pi$, then $\rho(t)$ is global and converges to a steady state as t goes to infinity.
  (see [3] by P. Biler, G. Karch, P. Laurençot and T. Nadzieja).
- If $\overline{m} = 8\pi$, then $\rho(t)$ blows up in infinite time to a Dirac mass centered at the origin.
  (see again [3] and [25] by N.I. Kavallaris and P. Souplet for refined asymptotics).
- If $\overline{m} > 8\pi$, then $\rho(t)$ blows up in finite time to a Dirac mass.
  (see [19] by M.A. Herrero and J.L. Velazquez).

Moreover, this system exhibits a similar phenomenon in the case of the whole plane $\mathbb{R}^2$. See the work of [4, 6, 7, 14]. In the nonradial case in a bounded domain, results are slightly different (see the book [39] of T. Suzuki). The behaviour of the parabolic-parabolic system in $\mathbb{R}^2$ seems more intricate. See [1, 11].

From now on, we consider the case $N \geq 3$.

Adapting the procedure described in the article [2] of P. Biler, D. Hilhorst and T. Nadzieja (or also in [3, 25]), we can reduce the system $(PKS)_q$ to a single one-dimensional equation.
Indeed, denoting $Q(t,r) = \int_{B(0,r)} \rho(t,y)\,dy$ the total mass of the cells in $B(0,r)$ at time t for $0 \leq r \leq 1$, we can make the following formal computations :

$$\begin{aligned} Q_t &= \int_{S(0,r)} \left[\frac{\partial\rho}{\partial\nu} - \rho^q\,\frac{\partial c}{\partial\nu}\right] d\sigma_N \\ &= \int_{S(0,r)} \tilde{\rho}_r\,d\sigma_N - \tilde{\rho}^q \int_{B(0,r)} \Delta c\,dy \\ &= \sigma_N\,r^{N-1}\tilde{\rho}_r + \tilde{\rho}^q\,Q \end{aligned}$$

where σ_N denotes the surface area of the unit sphere in $\mathbb{R}^N$ and

$$\rho(t,y) = \tilde{\rho}(t,|y|)$$

for any $y \in \overline{B}$. Since we can write

$$Q(t,r) = \int_0^r \sigma_N\, s^{N-1}\tilde{\rho}(t,s)ds,$$

we have both following formulas

$$Q_r = \sigma_N\, r^{N-1}\,\tilde{\rho}$$

and

$$\sigma_N\, r^{N-1}\tilde{\rho}_r = Q_{rr} - \frac{N-1}{r}Q_r,$$

which imply :

$$Q_t = Q_{rr} - \frac{N-1}{r}\,Q_r + \left[\frac{Q_r}{\sigma_N\, r^{N-1}}\right]^q\, Q. \tag{1.16}$$

Then, setting $P(t,x) = Q(t, x^{\frac{1}{N}})$, we obtain :

$$P_t = N^2\, x^{2-\frac{2}{N}}\, P_{xx} + \left[\frac{N}{\sigma_N}\right]^q\, {P_x}^q\, P. \tag{1.17}$$

Finally, setting $u(t,x) = \frac{1}{N^{\frac{2}{q}}\, V_N} P(\frac{t}{N^2}, x)$ where $V_N = \frac{\sigma_N}{N}$ is the volume of B, we get :

$$u_t = x^{2-\frac{2}{N}}\, u_{xx} + u\, {u_x}^q. \tag{1.18}$$

Moreover, by the no flux boundary condition on the cell density, it is formally clear that the total cell mass $\overline{m}$ is constant in time. Hence, setting

$$m = \frac{\overline{m}}{N^{\frac{2}{q}} V_N},$$

we also have the boundary conditions that for all $t \geq 0$,

$$u(t,0) = 0$$

$$u(t,1) = m.$$

A simple calculation also shows that, for $r \geq 0$,

$$\tilde{\rho}(t,r) = N^{\frac{2}{q}}\, u_x(N^2\, t, r^N). \tag{1.19}$$

Hence, ρ is simply proportional to u_x, up to a time rescaling and a change of variable. It means that the derivative of u is the quantity with physical meaning and should then be nonnegative.

Finally, we shall focus on the following problem, noted (PDE_m) :

$$(PDE_m) \qquad \begin{cases} u_t = x^{2-\frac{2}{N}}\, u_{xx} + u\, {u_x}^q & t>0 \quad 0<x\le 1 \\ u(t,0)=0 & t\ge 0 \\ u(t,1)=m & t\ge 0 \\ u_x(t,x)\ge 0 & t>0 \quad 0\le x\le 1. \end{cases} \tag{1.20}$$

Conversely, starting from a solution u of (PDE_m) we would like to show (at least formally) how to get a solution of (PKS_q).
First, it is easy to check that

$$\tilde{\rho}_r(t,r) = N^{\frac{2}{q}+1} r^{N-1} u_{xx}(N^2\,t, r^N)$$

and, denoting $\tilde{c}$ the radial profile of c, since $-\Delta c = \rho$, we have

$$\tilde{c}_r(t,r) = -\frac{N^{\frac{2}{q}-1}}{r^{N-1}}\, u(N^2\,t, r^N)$$

so that, by (1.11), we obtain

$$\tilde{c}(t,r) = \int_r^1 \frac{N^{\frac{2}{q}-1}}{s^{N-1}}\, u(N^2\,t, s^N). \tag{1.21}$$

Now, we define (ρ, c) by their profiles given in formulas (1.19) and (1.21).
If we denote $r=|y|$ for any $y \in \overline{B}$ and

$$\alpha(r) = \tilde{\rho}_r - \tilde{\rho}^q\, \tilde{c}_r,$$

then, by the following general fact

$$r^{N-1}\ \mathrm{div}[\alpha(r)\vec{e_r}] = \frac{d}{dr}[r^{N-1}\alpha],$$

we obtain

$$\begin{aligned} r^{n-1}\ \mathrm{div}[\nabla\rho - \rho^q\nabla c] &= \frac{d}{dr}[r^{N-1}\tilde{\rho}_r - \tilde{\rho}^q r^{N-1}\tilde{c}_r] \\ &= N^{\frac{2}{q}+1}\frac{d}{dr}\left[r^{2N-2}u_{xx}(N^2t, r^N) + (u\, {u_x}^q)(N^2t, r^N)\right] \\ &= N^{\frac{2}{q}+1}\frac{d}{dr}u_t(N^2t, r^N) && \text{by (1.20)} \\ &= r^{N-1}\ N^{\frac{2}{q}+2}u_{xt}(N^2t, r^N) \\ &= r^{N-1}\rho_t && \text{by (1.19).} \end{aligned}$$

Hence, (ρ, c) is a solution of (PKS_q). We just have to check that ρ also satisfies the no flux boundary conditions (1.10) which are equivalent to

$$\tilde{\rho}_r - \tilde{\rho}^q\, \tilde{c}_r = 0 \qquad \text{for } r=1. \tag{1.22}$$

Thanks to the previous formulas on $\tilde{\rho}_r$ and $\tilde{c}_r$, (1.22) becomes

$$[u_{xx} + {u_x}^q\, u]\,(N^2\, t, 1) = 0,$$

which can be obtained from (1.20) since $u_t(N^2\, t, 1) = 0$.

Now, it seems reasonable to consider problem (PDE_m) as our model for chemotaxis. Equation (1.20) presents two difficulties since the diffusion is degenerate at $x = 0$ and the nonlinearity is not Lipschitz continuous. We shall assume that the initial data u_0 belongs to the class

$$Y_m = \{u \in C([0;1]),\ u \text{ nondecreasing },\ u'(0) \text{ exists},\ u(0) = 0,\ u(1) = m\}.$$

For such u_0, we have established in [P1] the existence and uniqueness of a maximal classical solution u such that $u(t) \in Y_m$ for all $t \in [0, T_{max}(u_0))$, where $T_{max}(u_0)$ is the maximal existence time. See Subsection 3.1 below for precise definitions and more details.

1.2 Main results

We now focus on the case of the critical exponent $q = \frac{2}{N}$ with $N \geq 3$.

The set of stationary solutions can be precisely described.
We shall prove that the stationary solutions of (PDE_m) are the restrictions to $[0, 1]$ of a family of functions $(U_a)_{a\geq 0}$ with the following properties :

- $U_1(0) = 0$, U_1 is nondecreasing and reaches its maximum M at $x = A$ from which U_1 is flat.
- All $(U_a)_{a\geq 0}$ are obtained by dilation of U_1, i.e. $U_a(x) = U_1(ax)$ for all $x \geq 0$.

Using this, we can then prove :

Theorem 1.1. *Let $m \geq 0$. Considering problem (PDE_m) with $q = \frac{2}{N}$:*
i) If $0 \leq m < M$, then there exists a unique stationary solution.
ii) If $m = M$, there exists a continuum of steady states : $\left(U_a|_{[0,1]}\right)_{a\geq A}$.
Note that the corresponding cell densities have their support strictly inside B.
iii) If $m > M$, there is no stationary solution.

The previous theorem leads us to set the following definition.

Definition 1.1.
We call M the critical mass of problem (PDE_m) for $q = \frac{2}{N}$ and

$$\overline{M} = N^N V_N \times M$$

the critical mass of system (PKS_q).

We would like to stress that the system (PKS_q) exhibits two levels of criticality. We have already seen the first level which consists in choosing the right exponent $q = \frac{2}{N}$ in order to balance the diffusion and aggregation forces in the system. Once this exponent is chosen, a second level of criticality arises with the choice of the mass. The following two theorems state that a critical mass phenomenon indeed occurs.

Theorem 1.2. *Let $N \geq 3$ and $q = \frac{2}{N}$.*
If $m \leq M$ and $u_0 \in Y_m$, then

$$u(t) \underset{t\to+\infty}{\longrightarrow} U_a \quad in \quad C^1([0,1])$$

for some $a \geq 0$.

Theorem 1.3. *Let $N \geq 3$ and $q = \frac{2}{N}$.*
If $m > M$, then for all $u_0 \in Y_m$,

$$T_{max}(u_0) < \infty.$$

Moreover, denoting $\mathcal{N}[u] = \sup\limits_{x\in(0,1]} \frac{u(x)}{x}$, we have :

$$\lim_{t\to T_{max}} \mathcal{N}[u(t)] = \infty.$$

In addition, for slightly supercritical mass, we can show the existence of blowing-up self-similar solutions.

Theorem 1.4. *There exists $M^+ > M$ such that for all $m \in (M, M^+]$ there exists a family of blowing-up self-similar solutions of problem (PDE_m).*
Moreover, the corresponding cell densities have their support strictly inside B.

1.3 Comments and related results

1.3.1 Description of the ideas of the proofs

The global existence part of Theorem 1.2 for subcritical or critical mass is based on comparison with suitable supersolutions, combined with some continuation results obtained in [P1]. Our convergence statements heavily rely on Lyapunov functional type arguments.

More precisely, we show that the evolution problem (PDE_m) induces a gradient type dynamical system on $Y_m^1 = Y_m \cap C^1([0,1])$, with global relatively compact trajectories. Moreover, we exhibit a strict Lyapunov functional :

$$\mathcal{F}[u] = \int_0^1 \frac{\dot{u}^{2-q}}{(2-q)(1-q)} - \frac{u^2}{2x^{2-q}} dx.$$

Indeed, formally, it is easy to check that

$$\frac{d}{dt}\mathcal{F}[u(t)] = \int_0^1 u_{tx}\frac{\dot{u}^{1-q}}{1-q} - \frac{u_t\, u}{x^{2-q}}dx = \left[u_t\frac{\dot{u}^{1-q}}{1-q}\right]_0^1 - \int_0^1 u_t\left[\frac{d}{dx}\frac{\dot{u}^{1-q}}{1-q} + \frac{u}{x^{2-q}}\right]dx.$$

Thanks to the boundary conditions and to (1.20), we then have

$$\frac{d}{dt}\mathcal{F}[u(t)] = -\int_0^1 \frac{(u_t)^2}{\dot{u}^q\, x^{2-q}}dx.$$

However, this computation is not rigorously valid, since u_x can vanish on a whole interval for instance. Nevertheless, we can overcome this difficulty and prove that $\mathcal{F}$ is indeed a strict Lyapunov functional by expressing it as the limit as ϵ goes to zero of a family of strict Lyapunov functionals $\mathcal{F}_\epsilon$ for suitable approximate problems (cf. problem (PDE_m^ϵ) introduced in Subsection 3.2.2). We note that the proof of the compactness of the trajectories relies on a different transformation, leading to another auxiliary problem $(tPDE_m)$ (cf. Subsection 3.2.1 below). In the subcritical case, since there is a single steady state, this immediately implies the convergence of the trajectory. But in the critical case, the situation is more delicate, since there exists a continuum of steady states and the solution could oscillate without stabilizing. However, thanks to a good relation between order and topology of the set of stationary solutions, we can prove stabilization by arguments in the spirit of (though simpler than) those in the articles [30, 41] of H. Matano and T.I. Zelenyak.

As for our blow-up results (Theorems 1.3 and 1.4), their proofs rely on the construction of a subsolution which becomes a self-similar solution after some time. The latter's profile is solution of an appropriate auxiliary ordinary differential equation which is a perturbation of the stationary solution's equation. The construction, as well as the study of the steady states (cf. Theorem 1.1), requires some rather delicate ODE arguments.

1.3.2 Open problems

i) A natural and very interesting question would be to determine the basin of attraction of a given steady state U_a with $a \geq A$.

ii) For the self-similar solutions in Theorem 1.4, it is easy to see that the blow-up rate of the central density of cells (proportional to $u_x(t,0)$) behaves like $(T_{max} - t)^{-\frac{N}{2}}$. It would be interesting to know if all solutions of problem (PDE_m) blow up at the self-similar rate or if there also exists blow-up of type II, i.e. if there exists solutions with blow-up speed faster than that of the self-similar solutions.

1.3.3 Comparison with the case $N = 2$

It is instructive to compare the cases $N \geq 3$ and $N = 2$ for problem (PDE_m) with $q = \frac{2}{N}$. The behaviour is the same for both in the subcritical case since solutions

converge to a unique steady state and also in the supercritical case since solutions blow up in finite time. But for the critical case, the qualitative behaviour differs strongly. Indeed, for $N = 2$, blow-up occurs in infinite time whereas for $N \geq 3$, there is still convergence to a regular steady state whose corresponding cell density has support strictly inside B, a phenomenon which never occurs for $N = 2$. We would like to suggest an "explanation" for this.
Denoting S_N the set of stationary solutions for $N \geq 2$, we can see that we could as well define the critical mass as

$$M = \sup_{U \in S_N} \|U\|_{\infty,[0,1]}.$$

The main difference is that this supremum is not reached for $N = 2$ whereas it is for $N \geq 3$, which allows us in the latter case to find a supersolution that prevents blow-up. Thus, convergence or infinite-time blow-up seems to be determined by whether or not the critical mass is reached by stationary solutions.

1.3.4 Related literature for porous medium type diffusion

Finally, we would like to make the link between our work and the article [5] of A. Blanchet, J.A. Carrillo and P. Laurençot (see also the articles [26] of I. Kim and Y. Yao and [9] of J. Bedrossian, N. Rodriguez and A.L. Bertozzi for further results in this direction). It will allow to identify a formula for the critical mass M.
The authors there study in the whole space $\mathbb{R}^N$ for $N \geq 3$ the following Patlak-Keller-Segel system (PKS^p) with porous-medium like nonlinear diffusion :

$$\mu_t = \operatorname{div}[\nabla \mu^p - \mu \nabla c] \quad t > 0 \quad x \in \mathbb{R}^N \tag{1.23}$$

$$-\Delta c = \mu \quad t > 0 \quad x \in \mathbb{R}^N \tag{1.24}$$

where μ is the cell density and c the concentration of the chemoattractant.
They could show that for the critical exponent $p = 2 - \frac{2}{N}$, the system (PKS^p) exhibits a critical mass phenomenon. See also [12] for a explanation of this exponent for parabolic-elliptic Patlak-Keller-Segel systems with general nonlinear diffusion.
More precisely, denoting $\overline{m}$ the total mass of the cells, they have shown the existence of M_c such that :

- If $\overline{m} < M_c$, solutions exist globally.
- If $\overline{m} = M_c$, solutions exist globally in time. Moreover, there are infinitely many compactly supported stationary solutions.
- If $\overline{m} > M_c$, there are solutions which blow up in finite time.

A. Blanchet and P. Laurençot also proved in [8] the existence of self-similar compactly supported blowing-up solutions for $\overline{m} \in (M_c, M_c^+]$ where $M_c^+ > M_c$. See also [40] by Y. Yao and A.L. Bertozzi for recent formal and numerical results on self-similar and non self-similar blow-up for a generalization of system (PKS^p) with kernel of power-law type.

We can then observe similarities between both problem $(PKS_{\frac{2}{N}})$ and $(PKS^{2-\frac{2}{N}})$ for $N \geq 3$. Here, we would like to thank P. Laurençot for suggesting us that both problems should share the same stationary solutions, as we can indeed verify, at least formally : denoting K_Ω the Dirichlet kernel of the Laplacian in a bounded domain Ω, it is easy to see that the steady states of (PKS_q) and (PKS^p) are respectively the solutions of

$$\frac{\rho^{1-q}}{1-q} - K_\Omega * \rho = C \qquad \text{and} \qquad p\,\frac{\mu^{p-1}}{p-1} - K_\Omega * \mu = C',$$

where C and C' are any real constants. Hence, the map $\rho \mapsto \mu := (2-q)^{\frac{1}{q}}\,\rho$ defines a correspondence between steady states of (PKS_q) with constant C and steady states of (PKS^{2-q}) with constant $C' = (2-q)^{\frac{1}{q}}\,C$.
Then the formula for M_c given in [5] also gives a formula for the critical cell mass $\overline{M}$ in our case :

$$\overline{M} = \left[\frac{N^2\sigma_N}{C_*(N-1)}\right]^{\frac{N}{2}} \quad \text{or equivalently} \quad M = \frac{N\sigma_N{}^{\frac{N}{2}-1}}{[C_*(N-1)]^{\frac{N}{2}}},$$

where C_* is the optimal constant in the following variant of the Hardy-Littlewood-Sobolev inequality : for all $h \in L^1(\mathbb{R}^N) \cap L^{2-\frac{2}{N}}(\mathbb{R}^N)$,

$$\left| \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{h(x)h(y)}{|x-y|^{N-2}} dx\,dy \right| \leq C_* \|h\|_{2-\frac{2}{N}}^{2-\frac{2}{N}} \|h\|_1^{2/N}.$$

We would like to make the heuristic remark that if we roughly put $N = 2$ in the above formula, then we recover the well-known critical mass $\overline{M} = 8\pi$ since in this case $C_* = 1$ and $\sigma_2 = 2\pi$.

It is interesting that, in spite of similarities in the qualitative behaviour, the two problems $(PKS_{\frac{2}{N}})$ and $(PKS^{2-\frac{2}{N}})$ seem to require different techniques. We would like to point out that our results are restricted to the radial setting, but on the other hand, they give a fairly more precise asymptotic description.

Throughout the rest of the paper, we assume :

$$\boxed{m \geq 0,\ N \geq 3 \ \text{and}\ q = \frac{2}{N}.}$$

2 The set of stationary solutions of problem (PDE_m)

We begin by studying the steady states of problem (PDE_m), i.e. the solutions of the following problem :

$$x^{2-q}u'' + uu'^q = 0 \qquad x \in (0,1] \tag{2.1}$$
$$u(0) = 0 \tag{2.2}$$
$$u(1) = m \tag{2.3}$$

As is customary in evolution problems, this is essential in order to understand the large-time asymptotics of solutions of problem (PDE_m).

Note that, even if we will use the results of this section only for $q = \frac{2}{N}$, they are all valid for any $q \in (0, 1)$.

2.1 Existence of a steady state depending on $m \geq 0$

In view of proving Theorem 1.1, we first need to study the following Cauchy problem.

Definition 2.1. *For $a \geq 0$, we define the problem (P_a) by :*

$$x^{2-q}u'' + uu'^q = 0 \tag{2.4}$$

$$u(0) = 0 \tag{2.5}$$

$$u'(0) = a \tag{2.6}$$

Definition 2.2. *Let $R > 0$.*
We say that u is a solution of problem (P_a) on $[0; R[$ if :
- *$u \in C^1([0; R[) \cap C^2(]0; R[)$.*
- *u is nondecreasing.*
- *u satisfies (2.4) on $]0; R[$ and also conditions (2.5) and (2.6).*

This definition can obviously be adapted for the case of a closed interval $[0; R]$ or for $R = +\infty$.

We first state some a priori properties of the solutions of (P_a).

Lemma 2.1. *Let $a \geq 0$ and $R > 0$.*
Let u be a solution of (P_a) on $[0; R[$. Then :
- *i) If there exists $x_0 \in [0; R[$ such that $u'(x_0) = 0$ then $u(x) = u(x_0)$ for all $x \in [x_0; R[$.*
- *ii) For all $x \in [0; R[$, $0 \leq u(x) \leq 2$ and $0 \leq u'(x) \leq a$.*

Proof : i) We first note that since $u' \geq 0$, then, thanks to (2.4), u is concave on $]0; R[$. If $x_0 > 0$ then for all $x \in [x_0; R[$, $0 \leq u'(x) \leq u'(x_0) = 0$. If $x_0 = 0$, we have to use in addition the continuity of u' at $x = 0$.
ii) If $a = 0$ then by i), $u = 0$, hence the result.
Now, let us suppose $a > 0$ and fix $x \in [0; R[$. u' is nonincreasing on $]0; R]$ and continuous at $x = 0$, so $0 \leq u'(x) \leq a$. Hence, we just have to prove that $u(x) \leq 2$.
Let us denote $x_0 = \sup\{x \in [0; R[, u'(x) > 0\} > 0$. Let $x \in [0; x_0[$.
First note that by i), $u'(x) > 0$. Since u is concave, then $u'(x) \leq \dfrac{u(x)}{x}$ so $-\dfrac{u''(x)}{u'(x)^{q+1}} \geq \dfrac{1}{x^{1-q}}$. Hence, by integration, we get

$$\frac{1}{u'(x)^q} \geq \frac{1}{u'(x)^q} - \frac{1}{u'(0)^q} \geq x^q$$

and finally $xu'(x) \leq 1$. Since $u''(x)$ in nonpositive and

$$uu'(x) = -xu''(x)(xu'(x))^{1-q},$$

then $uu'(x) \leq -xu''(x) = (u - xu')'(x)$. Finally, by integration,

$$\frac{u^2(x)}{2} \leq u(x) - xu'(x) \leq u(x)$$

so $u(x) \leq 2$ for all $x \in [0; x_0[$.
If $x_0 = R$, all is done. Else, $u'(x_0) = 0$ and $u(x_0) \leq 2$ by continuity, so by i), $u(x) = u(x_0) \leq 2$ for all $x \in [x_0; R[$. □

We will prove that solutions of problem (P_a) exist on $[0, \infty)$. We begin by showing the local existence.

Lemma 2.2. *Let $a \geq 0$ and $\tau > 0$.*
If τ is small enough, there exists a unique solution of (P_a) on $[0; \tau]$.

Proof : If $a = 0$, then from the previous lemma i), it is clear that 0 is the unique solution of the problem (P_0) on $[0, \infty)$.
If $a > 0$, let us define

$$E_a = \{u \in C^1([0, \tau]),\ u(0) = 0,\ u'(0) = a,\ \|u' - a\|_\infty \leq \frac{a}{2}\}.$$

E_a equipped with the metric induced by the norm $\|u\|_{E_a} = \|u'\|_\infty$ is a complete metric space. Any $u \in E_a$ is nondecreasing on $[0, \tau]$ since $u' \geq \frac{a}{2}$.
It is clear that the following function F is well defined :

$$F : E_a \to C^1([0; \tau])$$

$$F(u)(x) = ax - \int_0^x \int_0^y \frac{u(s)}{s} \frac{u'(s)^q}{s^{1-q}} dsdy.$$

Since for all $u \in E_a$, $\|u'\|_\infty \leq \dfrac{3}{2}a$, we easily get that

$$\|F(u)' - a\|_\infty \leq (\frac{3}{2}a)^{q+1} \frac{\tau^q}{q} \leq \frac{a}{2}$$

if τ is chosen small enough. Hence, F sends E_a into E_a.
By the mean value theorem, since for all $u \in E_a$, $\frac{a}{2} \leq u' \leq \frac{3}{2}a$,

$$\|F(u) - F(v)\|_{E_a} \leq \left(\frac{(\frac{3}{2}a)^q}{q} + \frac{\frac{3}{2}a}{(\frac{a}{2})^{1-q}} \right) \tau^q \|v - u\|_{E_a}$$

Hence, if τ is small enough, F is a contraction so there exists a fixed point of F. Since $F(u) \in C^2((0, \tau])$ when $u \in C^1([0; \tau])$ then u is a solution of (P_a).
Finally, it is easy to check that a solution is necessarily a fixed point of F, which proves the uniqueness. □

Remark 2.1. *Let u be the local solution of (P_a) on $[0,\tau]$. We would like to stress that u is not C^2 up to $x=0$. Indeed, one can see that*

$$u'(x) = a - \frac{a^{q+1}}{q}x^q + o(x^q).$$

Moreover, one can obtain an expansion of u at any order in powers of x^q. We proved in [P1] that solutions of problem (PDE_m) share these properties with the stationary solutions.

Theorem 2.1. *Let $a \geq 0$.*
There exists a unique maximal solution of (P_a).
Moreover, it is globally defined on $[0,\infty)$.

Proof : For the sake of completeness, we prefer to give a (standard) proof.
Existence : Leaving aside the obvious case $a=0$, let $a>0$.
By Lemma 2.2, for a given τ small enough, we have a unique classical solution u_τ of (P_a) on $[0;\tau]$. Setting $W=(u,u')$, we can now consider the following ordinary differential equation on the interval $[\frac{\tau}{2},+\infty[$:

$$\begin{aligned} W' &= F(x,W) \\ W\left(\frac{\tau}{2}\right) &= \left(u_\tau\left(\frac{\tau}{2}\right), u'_\tau\left(\frac{\tau}{2}\right)\right) \end{aligned} \tag{2.7}$$

where $F(x,u,v) = \left(v, \dfrac{-uv^q}{x^{2-q}}\right)$. Let us denote $\Omega = \mathbb{R}\times]0;+\infty[$. Since F is locally Lipschitz continuous with respect to W in Ω, by classical Cauchy-Lipschitz theory, there exists a maximal solution $u \in C^2([\frac{\tau}{2};X^*))$ of problem (2.7) such that

$$(u(x),u'(x)) \in \Omega \text{ for all } x \in [\frac{\tau}{2};X^*).$$

By local uniqueness in the classical Cauchy-Lipschitz theory, $u_\tau = u$ around $x=\frac{\tau}{2}$, so we can glue u_τ and u and get a solution of problem (P_a) on $[0;X^*)$.
If $X^*=+\infty$, all is done. So we suppose that $X^*<\infty$.
Since u is nondecreasing and bounded above by Lemma 2.1 ii), then

$$l = \lim_{x\to X^*} u(x) \text{ exists.}$$

Hence, we can extend continuously the function u by setting $u(x)=l$ for $x \geq X^*$. But $(u(x),u'(x))$ must leave any compact of Ω as x goes to X^* so, by Lemma 2.1) ii), the only possibility is that $\lim_{x\to X^*} u'(x)=0$. So $u\in C^1([0;+\infty))$.
And now, thanks to (2.4), $\lim_{x\to X^*} u''(x)=0$, so $u\in C^2((0;+\infty)$. Moreover, u clearly satisfies (2.4) on $(0,+\infty)$ and is then a global solution of (P_a).
Uniqueness : Let v another global solution of (P_a). By the result of uniqueness around $x=0$ and the uniqueness due to classical Cauchy-Lipschitz theory in Ω,

u and v coincide for all $x \in [0, X^*)$. If $X^* = \infty$, all is done. Now, assume that $X^* < +\infty$. $v(X^*) = u(X^*)$ by continuity. As $v'(X^*) = u'(X^*) = 0$, then by Lemma 2.1) i), $v(x) = v(X^*) = u(X^*)$ for all $x \geq X^*$. Hence, $v = u$. □

Notation 2.1. *Let $a \geq 0$.*
We denote U_a the unique solution of (P_a) on $[0, +\infty)$.

Lemma 2.3. *Let $a \geq 0$.*
There exists $x_0 \geq 0$ such that $U_a(x) = U_a(x_0)$ for all $x \geq x_0$.

Proof : Suppose the contrary. By Lemma 2.1)i), it implies that $U_a'(x) > 0$ for all $x \geq 0$. Since U_a is nondecreasing and has 2 as an upper bound, there exists $l \leq 2$ such that $U_a(x)$ tends to l as x goes to infinity.
As U_a' is nonnegative and nonincreasing, then $U_a'(x)$ has a nonnegative limit as x goes to $+\infty$, but this limit has to be 0 since U_a is bounded from above.
Moreover, for all $x > 0$,

$$\frac{d}{dx} U_a'(x)^{1-q} = U_a(x) \frac{d}{dx} \frac{1}{x^{1-q}}.$$

Let $x_0 > 0$ and $x \geq x_0$. By monotonicity, $U_a(x_0) \leq U_a(x) \leq l$ so by integration on $[x_0; x]$,

$$l \left(\frac{1}{x^{1-q}} - \frac{1}{x_0^{1-q}} \right) \leq U_a'(x)^{1-q} - U_a'(x_0)^{1-q} \leq U_a(x_0) \left(\frac{1}{x^{1-q}} - \frac{1}{x_0^{1-q}} \right).$$

Finally, we let x go to $+\infty$ and then x_0 go to $+\infty$ to obtain that

$$U_a'(x_0) \underset{x_0 \to +\infty}{\sim} \frac{l^{\frac{1}{1-q}}}{x_0}.$$

Then, $U_a(x)$ goes to infinity as x goes to ∞, which is a contradiction. □

Proposition 2.1. *$U_a(x) = U_1(ax)$ for all $x \geq 0$ and all $a \geq 0$.*

Proof : Let $V(x) = U_1(ax)$. Clearly, $V \in C^1([0, +\infty)) \cap C^2((0; +\infty))$, $V(0) = 0$, $V'(0) = a$ and for $x > 0$,

$$x^{2-q} V''(x) = (ax)^{2-q} U_1''(ax) a^q = U_1(ax) U_1'(ax)^q a^q = V(x) V'(x)^q.$$

The result then follows from the uniqueness of the solution of problem (P_a). □

Remark 2.2. *Behind this proof is the fact that L_a and D commute, where for any $u \in C^1([0, \infty))$, $a \geq 0$ and $x \geq 0$, $L_a(u)(x) = u(ax)$ and $Du = x\, u_x$.*

This proposition drives us to the natural following definition.

Definition 2.3. *The number $M = \max\limits_{x \geq 0} U_1(x)$ will be called the critical mass.*
Note that M also is the maximal value of each U_a, for all $a > 0$.

We also will use the following notation.

Notation 2.2. *We denote by A the first $x \geq 0$ such that $U_1(x) = M$.*

Proposition 2.2.
i) If $m \in [0; M)$, there exists a unique $a \in [0; A)$ such that $U_a(1) = m$.
ii) $U_a(1) = M$ if and only if $a \geq A$.

Proof : i) $U_a(1) = U_1(a)$ and U_1 is a bijection from $[0; A)$ to $[0; M)$.
ii) $U_a(1) = M$ if and only if $U_1(a) = M$, which is equivalent to $a \geq A$. □

From this follows Theorem 1.1 announced in the introduction.

2.2 Order and topological properties of the set of stationary solutions

Now, we shall describe two simple but very important properties of the set of stationary solutions : it has a total order and its topology behaves well with respect to that order.

Notation 2.3. *Let $m \geq 0$.*
- *$Y^1_m = \{u \in C^1([0; 1]),\ u \text{ nondecreasing},\ u(0) = 0,\ u(1) = m\}$ is the complete metric space equipped with the distance induced by the C^1 norm.*
- *$V_m(u, \epsilon) \subset Y^1_m$ is the open ball of center $u \in Y^1_m$ and radius $\epsilon > 0$.*
- *We say that two functions $u, v \in Y^1_m$ satisfy $u \prec v$ if $u \leq v$ and $u \neq v$.*
  If $u \in Y^1_m$ and $V \subset Y^1_m$, we say that $u \prec V$ if for all $v \in V$, $u \prec v$.
  Similarly, we define $V \prec u$.

Proposition 2.3.
i) Suppose $0 \leq a < b$. Then $U_a \prec U_b$.
The set of stationary solutions is a totally ordered set.
ii) Suppose $A \leq a < b$.
α) There exists $\epsilon > 0$ such that $V_M(U_a, \epsilon) \prec U_b$.
β) There exists $\epsilon > 0$ such that $\{u \in Y^1_M,\ u \leq U_a\} \cap V_M(U_b, \epsilon) = \emptyset$.

Proof : i) Let $x \geq 0$. $U_a(x) = U_1(ax) \leq U_1(bx) = U_b(x)$, so $U_a \leq U_b$.
Since, $U'_a(0) < U'_b(0)$, then $U_a \prec U_b$.
ii) α) There exists $\gamma > 0$ such that for all $x \in [0; \gamma]$, $\frac{a+b}{2}x \leq U_b(x)$.
Let us set $\epsilon_1 = \frac{b-a}{2}$, $\epsilon_2 = \frac{1}{2} \min\limits_{x \in [\gamma; \frac{A}{b}]} [U_b(x) - U_a(x)]$ and $\epsilon = \min(\epsilon_1, \epsilon_2)$.
Note that $\epsilon > 0$ since for all $x \in [\gamma; \frac{A}{b}]$, we have $ax < bx \leq A$, hence

$$U_a(x) = U_1(ax) < U_1(bx) = U_b(x)$$

because U_1 is increasing on $[0; A]$.
Let $u \in V_M(U_a, \epsilon)$. Since $u \in V_M(U_a, \epsilon_1)$, then

$$u' \leq U'_a + \frac{b-a}{2} \leq a + \frac{b-a}{2} = \frac{a+b}{2}.$$

Hence, for all $x \in [0;\gamma]$,

$$u(x) \leq \frac{a+b}{2}x \leq U_b(x).$$

Since $u \in V_M(U_a, \epsilon_2)$, it is clear that for all $x \in [\gamma; \frac{A}{b}]$, $u(x) < U_b(x)$. Moreover, if $x \geq \frac{A}{b}$ then $u(x) \leq M = U_b(x)$. Hence,

$$u \prec U_b.$$

β) Let $u \in Y^1_M$ such that $u \leq U_a$. Then $u'(0) \leq a$. Let us set $\epsilon = \frac{b-a}{2} > 0$. Since $\|U_b - u\|_{C^1} \geq U_b'(0) - u'(0) \geq b - a$ then $u \notin V_M(U_b, \epsilon)$. □

3 Summary of local in time results

In this section, we give some useful results on problem (PDE_m) and two other auxiliary parabolic problems. For proofs, see [P1].

3.1 Wellposedness of problem (PDE_m)

Before stating the existence and uniqueness of classical solutions for problem (PDE_m), we need to fix some definitions.

Definition 3.1.

$$Y_m = \{u \in C([0;1]),\ u \textit{ nondecreasing },\ u'(0) \textit{ exists, } u(0) = 0,\ u(1) = m\}.$$

Definition 3.2. *Let $T > 0$.*
We say that u is a classical solution of (PDE_m) (see (1.20)) on $[0,T)$ with initial condition $u_0 \in Y_m$ if :

- $u \in C([0,T) \times [0,1]) \bigcap C^1((0,T) \times [0,1]) \bigcap C^{1,2}((0,T) \times (0,1])$.
- $u(0) = u_0$.
- $u(t) \in Y_m$ *for* $t \in [0,T)$.
- u *satisfies* (1.20) *on* $(0,T) \times (0,1]$.

Definition 3.3. *For any real function defined on $(0,1]$, we set :*

$$\mathcal{N}[u] = \sup_{x \in (0,1]} \frac{u(x)}{x}.$$

Theorem 3.1. *Let $q \in (0,1)$, $m \geq 0$ and $u_0 \in Y_m$.*

i) There exists $T_{max} = T_{max}(u_0) > 0$ and a unique maximal classical solution u of problem (PDE_m) with initial condition u_0.
Moreover, u satisfies the following condition :

$$\sup_{t\in(0,T]} \sqrt{t}\, \|u(t)\|_{C^1([0,1])} < \infty \textit{ for any } T \in (0, T_{max}). \tag{3.1}$$

ii) Blow-up alternative : $T_{max} = +\infty$ *or* $\lim_{t \to T_{max}} \mathcal{N}[u(t)] = +\infty$.

iii) $u_x(t,0) > 0$ *for all* $t \in (0, T_{max})$.

Moreover, a classical comparison principle is available for problem (PDE_m).

Lemma 3.1. *Let* $T > 0$. *Assume that :*

– $u_1, u_2 \in C([0,T] \times [0,1]) \bigcap C^1((0,T] \times [0,1]) \bigcap C^{1,2}((0,T] \times (0,1))$.

– *For all* $t \in (0,T]$, $u_1(t)$ *and* $u_2(t)$ *are nondecreasing.*

– *There exists* $i_0 \in \{1,2\}$ *and some* $\gamma < \frac{1}{q}$ *such that*

$$\sup_{t \in (0,T]} t^\gamma \left\| u_{i_0}(t) \right\|_{C^1([0,1])} < \infty.$$

Suppose moreover that :

$$(u_1)_t \leq x^{2-\frac{2}{N}}(u_1)_{xx} + u_1(u_1)_x^q \quad \textit{for all } (t,x) \in (0,T] \times (0,1) \tag{3.2}$$

$$(u_2)_t \geq x^{2-\frac{2}{N}}(u_2)_{xx} + u_2(u_2)_x^q \quad \textit{for all } (t,x) \in (0,T] \times (0,1) \tag{3.3}$$

$$u_1(0,x) \leq u_2(0,x) \quad \textit{for all } x \in [0,1] \tag{3.4}$$

$$u_1(t,0) \leq u_2(t,0) \quad \textit{for } t \geq 0 \tag{3.5}$$

$$u_1(t,1) \leq u_2(t,1) \quad \textit{for } t \geq 0 \tag{3.6}$$

Then $u_1 \leq u_2$ *on* $[0,T] \times [0,1]$.

3.2 Two auxiliary parabolic problems

3.2.1 Problem $(tPDE_m)$

We now introduce an auxiliary transformed problem $(tPDE_m)$ which will be helpful in order to get some estimates implying the compactness of the trajectories in the subcritical and critical case $m \leq M$. This transformation was also important in [P1] in order to establish the blow-up alternative (see Theorem 3.2 ii) below), a property which will be used in the global existence part of the proof of Theorem 1.2.

Denoting B the open unit ball in $\mathbb{R}^{N+2}$ and $Z_m = \{w \in C(\overline{B}),\ w|_{\partial B} = m\}$, we define $\begin{array}{rcl} \theta_0 & : & Y_m \longrightarrow Z_m \\ & & u_0 \longmapsto w_0 \end{array}$ where $\left\{ \begin{array}{lll} w_0(y) & = \frac{u_0(|y|^N)}{|y|^N} & \text{if } y \in \overline{B} \backslash \{0\} \\ & = u_0'(0) & \text{if } y = 0 \end{array} \right.$.

We now make the following change of unknown

$$w(t,y) = \frac{u(N^2 t, |y|^N)}{|y|^N} \quad \text{if } y \in \overline{B} \backslash \{0\} \tag{3.7}$$

in problem (PDE_m) with initial condition $u_0 \in Y_m$ and obtain the following problem $(tPDE_m)$ with initial condition $w_0 = \theta_0(u_0)$.

Definition 3.4. *Let* $m \geq 0$ *and* $T > 0$.
Let $u_0 \in Y_m$ *and* $w_0 = \theta_0(u_0)$.
We define problem $(tPDE_m)$ *with initial condition* w_0 *by :*

$$(tPDE_m)\begin{cases} w_t = \Delta w + N^2 w(w + \frac{y.\nabla w}{N})^q & on\ (0,T] \times \overline{B} \\ w(0) = w_0 & \\ w + \frac{y.\nabla w}{N} \geq 0 & on\ (0,T] \times \overline{B} \\ w = m & on\ [0,T] \times \partial B \end{cases} \tag{3.8}$$

A classical solution on $[0,T]$ *for problem* $(tPDE_m)$ *with initial condition* w_0 *is a function*

$$w \in C([0,T] \times \overline{B}) \bigcap C^{1,2}((0,T] \times \overline{B})$$

such that all conditions of (3.8) *are satisfied.*
We define analogously a classical solution on $[0,T)$.

Let us give now the corresponding wellposedness result.

Theorem 3.2. *Let* $u_0 \in Y_m$ *and* $w_0 = \theta_0(u_0)$.

i) There exists $T^* = T^*(w_0) > 0$ *and a unique maximal classical radially symmetric solution* w *of problem* $(tPDE_m)$ *with initial condition* w_0.
Moreover, w *satisfies the following condition :*

$$\sup_{t \in (0,T]} \sqrt{t}\, \|w(t)\|_{C^1(\overline{B})} < \infty \text{ for any } T \in (0,T^*). \tag{3.9}$$

ii) Blow-up alternative : $T^* = +\infty$ *or* $\lim_{t \to T^*} \|w(t)\|_{\infty,\overline{B}} = +\infty$.

Connection between problems (PDE_m) **and** $(tPDE_m)$:
Let $w_0 = \theta_0(u_0)$ with $u_0 \in Y_m$. Then,

$$T_{max}(u_0) = N^2 T^*(w_0).$$

Moreover, for all $(t,x) \in [0,T_{max}) \times [0,1]$,

$$u(t,x) = x\ \tilde{w}(\frac{t}{N^2}, x^{\frac{1}{N}}), \tag{3.10}$$

where we write $f = \tilde{f}(|\cdot|)$ for any radial function f on B.

3.2.2 Problem (PDE_m^ϵ)

We define an approximate problem (PDE_m^ϵ) of problem (PDE_m). It will be useful since we can easily find a strict Lyapunov functional $\mathcal{F}_\epsilon$ for (PDE_m^ϵ) and then prove by this way that $\mathcal{F} = \lim_{\epsilon \to 0} \mathcal{F}_\epsilon$ is a strict Lyapunov functional for the dynamical system induced by problem (PDE_m) in the subcritical and critical case $m \leq M$.

Definition 3.5. *Let $\epsilon > 0$. We set :*

$$f_\epsilon(x) = (x+\epsilon)^q - \epsilon^q \quad \text{for } x \geq 0$$

Observe in particular that $0 \leq f_\epsilon(x) \leq x^q$ for all $x \in [0, +\infty)$.

Definition 3.6. *Let $\epsilon > 0$, $m \geq 0$ and $T > 0$.*
We define problem (PDE_m^ϵ) with initial condition $u_0 \in Y_m$ by :

$$(PDE_m^\epsilon) \begin{cases} u_t = x^{2-\frac{2}{N}} u_{xx} + u f_\epsilon(u_x) & \text{on } (0,T] \times (0,1] \\ u(0) = u_0 & \\ u(t,0) = 0 & \text{for all } t \in [0,T] \\ u(t,1) = m & \text{for all } t \in [0,T] \end{cases} \tag{3.11}$$

A classical solution on $[0,T]$ of problem (PDE_m^ϵ) with initial condition $u_0 \in Y_m$ is a function

$$u^\epsilon \in C([0,T] \times [0,1]) \bigcap C^1((0,T] \times [0,1]) \bigcap C^{1,2}((0,T] \times (0,1])$$

such that all conditions of (3.11) are satisfied.
A classical solution of problem (PDE_m^ϵ) on $[0,T)$ is defined analogously.

Theorem 3.3. *Let $m \geq 0$, $\epsilon > 0$, $K > 0$ and $u_0 \in Y_m$ with $\mathcal{N}[u_0] \leq K$.*

i) There exists $T^\epsilon_{max} = T^\epsilon_{max}(u_0) > 0$ and a unique maximal classical solution u^ϵ on $[0, T^\epsilon_{max})$ of problem (PDE_m^ϵ) with initial condition u_0.
Moreover, u^ϵ satisfies the following condition :

$$\sup_{t\in(0,T]} \sqrt{t}\, \|u^\epsilon(t)\|_{C^1([0,1])} < \infty \text{ for all } T \in (0, T^\epsilon_{max}). \tag{3.12}$$

ii) Blow-up alternative : $T^\epsilon_{max} = \infty$ or $\lim\limits_{t\to T^\epsilon_{max}} \mathcal{N}[u^\epsilon(t)] = +\infty$.

iii) $(u^\epsilon)_x > 0$ on $t \in (0, T^\epsilon_{max}) \times [0,1]$.

iv) $u^\epsilon \in C^2\left((0, T^\epsilon_{max}) \times (0,1]\right)$. (not optimal)

Connection with problem (PDE_m) :
Let us fix an initial condition $u_0 \in Y_m$.
The next lemma shows the convergence of maximal classical solutions u^ϵ of (PDE_m^ϵ) to the maximal classical solution of (PDE_m) in various spaces.
These results are essential in our proof that $\mathcal{F} = \lim\limits_{\epsilon\to 0} \mathcal{F}_\epsilon$ is a strict Lyapunov functional in the case $m \leq M$.

Lemma 3.2. *Let $u_0 \in Y_m$.*

i) $T_{max}(u_0) \leq T^\epsilon_{max}(u_0)$ for any $\epsilon > 0$.

ii) Let $[t_0, T] \subset (0, T_{max}(u_0))$.

α) $u^\epsilon \underset{\epsilon\to 0}{\longrightarrow} u$ in $C^{1,2}([t_0,T] \times (0,1])$.
Moreover, there exists $K > 0$ independent of ϵ such that for all $(t,x) \in [t_0,T] \times (0,1]$, $|u^\epsilon_{xx}| \leq \frac{K}{x^{1-q}}$.

β) $(u^\epsilon)_x \underset{\epsilon\to 0}{\longrightarrow} u_x$ in $C([t_0,T] \times [0,1])$.

γ) $(u^\epsilon)_t \underset{\epsilon\to 0}{\longrightarrow} u_t$ in $C([t_0,T] \times [0,1])$.

4 Convergence to a stationary state in critical and subcritical case $m \leq M$

All this section only concerns the case $m \leq M$.

We shall prove that problem (PDE_m) defines a continuous dynamical system on Y_m^1 which admits a strict Lyapunov functional. We shall be able to prove that classical solutions of (PDE_m) converge to a stationary state as times goes to infinity, even in the case $m = M$ where there is a continuum of steady states.

4.1 Estimates

Lemma 4.1. *Let $m \leq M$ and $u_0 \in Y_m$. Then,*

$$T_{max}(u_0) = \infty.$$

Moreover, for each $K > 0$, for any $u_0 \in Y_m$ with $\mathcal{N}[u_0] \leq K$, we have

$$\sup_{t\in[0,\infty)} \mathcal{N}[u(t)] \leq C_K$$

where $C_K = \frac{A}{M} \max(K, M)$.

Proof : Let $T_{max} = T_{max}(u_0)$.
From Theorem 3.1, it is sufficient to prove that

$$\sup_{t\in[0,T_{max})} \mathcal{N}[u(t)] < \infty.$$

This fact easily follows from a comparison with a supersolution of problem (PDE_m).
The main idea is that since $m \leq M$, if a is large enough then $u_0 \leq U_a$ and U_a is then a supersolution so $0 \leq u(t) \leq U_a$ for all $t \in [0, T_{max})$.
More precisely, since u_0 is differentiable at $x = 0$, $x \mapsto \frac{u_0(x)}{x}$ can be extended continuously to $[0; 1]$, so there exists $K \geq M$ such that $\mathcal{N}[u_0] \leq K$.
Let us set $a = \frac{K}{M}A$.
For $x \in [0; \frac{M}{K}]$, $u_0(x) \leq Kx \leq U_a(x)$ since by concavity, U_a is above its chord between $x = 0$ and $x = \frac{M}{K}$. On $[\frac{M}{K}, 1]$, $u_0 \leq m \leq M = U_a$.
Hence, $u_0 \leq U_a$ on $[0, 1]$. Finally, since $M \geq m$, U_a is a supersolution of the PDE, so $u(t) \leq U_a$ for all $t \in [0, T_{max})$. By concavity of U_a, we see that $\mathcal{N}[U_a] = a$, so $\sup_{t\in[0,T_{max})} \mathcal{N}[u(t)] \leq a < \infty$. Then $T_{max} = \infty$.
We notice that the choice of a depends only on K, whence the second part of the lemma. □

Before going further, we would like to recall some notation and properties of the heat semigroup. For reference, see for instance the book [29] of A. Lunardi.

Notation 4.1.

- *B denotes the open unit ball in $\mathbb{R}^{N+2}$.*
- *$X_0 = \{W \in C(\overline{B}),\ W|_{\partial B} = 0\}$.*
- *$(S(t))_{t\geq 0}$ denotes the heat semigroup on X_0. It is the restriction on X_0 of the Dirichlet heat semigroup on $L^2(B)$.*
- *$(X_\theta)_{\theta\in[0,1]}$ denotes the scale of interpolation spaces for $(S(t))_{t\geq 0}$, where $X_0 = L^2(B)$, $X_1 = D(-\Delta)$ and $X_\alpha \hookrightarrow X_\beta$ with dense continuous injection for any $\alpha > \beta$, $(\alpha,\beta) \in [0,1]^2$.*

Properties 4.1.

- *$X_{\frac{1}{2}} = \{W \in C^1(\overline{B}),\ W|_{\partial B} = 0\}$.*
- *Let $\gamma_0 \in (0;\frac{1}{2}]$. For any $\gamma \in [0, 2\gamma_0)$,*
$$X_{\frac{1}{2}+\gamma_0} \subset C^{1,\gamma}(\overline{B})$$
  with continuous embedding.
- *There exists $C_D \geq 1$ such that for any $\theta \in [0;1]$, $W \in C(\overline{B})$ and $t > 0$,*
$$\|S(t)W\|_{X_\theta} \leq \frac{C_D}{t^\theta}\|W\|_\infty.$$

We just want to introduce a specific notation we are going to use.

Notation 4.2. *Let $(a,b) \in (0,1)^2$. We denote $I(a,b) = \int_0^1 \frac{ds}{(1-s)^a s^b}$.*
For all $t \geq 0$, $\int_0^t \frac{ds}{(t-s)^a s^b} = t^{1-a-b} I(a,b)$.

We will now give an estimate from which follows a compactness result.

Lemma 4.2. *Let $m \leq M$, $\gamma \in [0;1)$, $t_0 > 0$ and $K > 0$.*
There exists $D_K > 0$ such that for any $u_0 \in Y_m$ with $\mathcal{N}[u_0] \leq K$, then
$$\sup_{t\geq t_0} \|u(t)\|_{C^{1,\frac{\gamma}{N}}} \leq D_K.$$
As a consequence, $\{u(t),\ \mathcal{N}[u_0] \leq K,\ t \geq t_0\}$ is relatively compact in Y_m^1.

Proof : Let $u_0 \in Y_m$ such that $\mathcal{N}[u_0] \leq K$. Let $w_0 = \theta_0(u_0)$.
First step : from Lemma 4.1, there exists $C_K > 0$ such that
$$\sup_{t\in[0,\infty)} \mathcal{N}[u(t)] \leq C_K.$$
Since for $t \geq 0$, $\|w(t)\|_{\infty,\overline{B}} = \mathcal{N}[u(\frac{t}{N^2})]$, we deduce that w is global and that
$$\sup_{t\in[0,\infty)} \|w(t)\|_{\infty,\overline{B}} = \sup_{t\in[0,\infty)} \mathcal{N}[u(t)] \leq C_K.$$

Second step : Let $\tau = \frac{t_0}{N^2}$ and $t \in [0,\tau]$.
Denoting $W_0 = w_0 - m$, then
$$w(t) - m = S(t)W_0 + \int_0^t S(t-s)N^2 w\left(w + \frac{x.\nabla w}{N}\right)^q ds, \tag{4.1}$$

so

$$\|w(t)\|_{C^1} \leq m + \frac{C_D}{\sqrt{t}}(C_K + m) + N^2 \int_0^t \frac{C_D}{\sqrt{t-s}} C_K \|w(s)\|_{C^1}^q ds.$$

Setting $h(t) = \sup\limits_{s \in (0,t]} \sqrt{s}\|w(s)\|_{C^1}$, we have $h(t) < \infty$ by (3.9) and

$$\sqrt{t}\|w(t)\|_{C^1} \leq m\sqrt{\tau} + C_D(C_K + m) + N^2 C_K C_D \sqrt{t} \int_0^t \frac{1}{s^{\frac{q}{2}}\sqrt{t-s}} h(s)^q ds,$$

$$\sqrt{t}\|w(t)\|_{C^1} \leq m\sqrt{\tau} + C_D(m + C_K) + N^2 C_K C_D I\left(\frac{1}{2}, \frac{q}{2}\right) t^{1-\frac{q}{2}} h(t)^q.$$

Let $T \in (0, \tau]$. Then,

$$h(T) \leq m\sqrt{\tau} + C_D(m + C_K) + N^2 C_K C_D I\left(\frac{1}{2}, \frac{q}{2}\right) T^{1-\frac{q}{2}} h(T)^q. \tag{4.2}$$

Setting $A = m\sqrt{\tau} + C_D(m + C_K)$ and $B = N^2 C_K C_D I\left(\frac{1}{2}, \frac{q}{2}\right) 2^q$, assume that there exists $T \in [0, \tau]$ such that $h(T) = 2A$. Then,

$$A^{1-q} \leq B T^{1-\frac{q}{2}} \text{ which implies } T \geq \left(\frac{A^{1-q}}{B}\right)^{\frac{1}{1-q}}.$$

Let us set $\tau' = \min\left(\tau, \frac{1}{2}\left(\frac{A^{1-q}}{B}\right)^{\frac{1}{1-q}}\right)$.
Since $h \geq 0$ is nondecreasing, $h_0 = \lim\limits_{t \to 0^+} h(t)$ exists and $h_0 \leq A$ by (4.2). So by continuity of h on $(0, \tau']$, $h(t) \leq 2A$ for all $t \in (0, \tau']$, that is to say :

$$\|w(t)\|_{C^1} \leq \frac{2A}{\sqrt{t}} \text{ for all } t \in (0, \tau'],$$

where A and τ' only depend on K. Then, setting $A_K = 2A$, we have

$$\sup_{t \in [0,\tau']} \sqrt{t}\|w(t)\|_{C^1} \leq A_K.$$

Third step : Let $\gamma_0 \in (\frac{\gamma}{2}, \frac{1}{2})$ and $t \in [0, \tau']$.
Setting $W = w - m$ and $W_0 = w_0 - m$, then for $t \geq 0$, due to (4.1), we get

$$\|W(t)\|_{X_{\frac{1}{2}+\gamma_0}} \leq \frac{C_D}{t^{\frac{1}{2}+\gamma_0}}(C_K + m) + N^2 \int_0^t \frac{C_D}{(t-s)^{\frac{1}{2}+\gamma_0}} C_K \frac{(A_K)^q}{s^{\frac{q}{2}}} ds.$$

Then we deduce that :

$$\begin{aligned} t^{\frac{1}{2}+\gamma_0}\|W(t)\|_{X_{\frac{1}{2}+\gamma_0}} &\leq C_D(C_K + m) + N^2 C_K C_D (A_K)^q t^{\frac{1}{2}+\gamma_0} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\gamma_0} s^{\frac{q}{2}}} ds \\ &\leq C_D(m + C_K) + N^2 C_K C_D (A_K)^q I(\frac{1}{2} + \gamma_0, \frac{q}{2}) \tau'^{(1-\frac{q}{2})}. \end{aligned}$$

Hence, since $X_{\frac{1}{2}+\gamma_0} \subset C^{1,\gamma}(\overline{B})$, we deduce that there exists $A'_K > 0$ depending only on K such that $\sup_{t\in[0,\tau']} t^{\frac{1}{2}+\gamma}\|w(t)\|_{C^{1,\gamma}(\overline{B})} \leq A'_K$. Then,

$$\|w(\tau')\|_{C^{1,\gamma}(\overline{B})} \leq \frac{A'_K}{\tau'^{(\frac{1}{2}+\gamma)}} =: A''_K.$$

Last step : Let $t' \geq \frac{t_0}{N^2}$. Since $\tau' \leq \frac{t_0}{N^2}$, we can apply the same arguments by taking $w_0(t'-\tau)$ as initial data instead of w_0, so we obtain

$$\text{for all } t' \geq \frac{t_0}{N^2},\ \|w(t')\|_{C^{1,\gamma}(\overline{B})} \leq A''_K.$$

Finally, coming back to $u(t)$, thanks to formula (3.10), we get an upper bound D_K for $\|u(t)\|_{C^{1,\frac{\gamma}{N}}}$ valid for any $u_0 \in Y_m$ such that $\mathcal{N}[u_0] \leq K$. □

4.2 A continuous dynamical system $(T(t))_{t\geq 0}$

We recall the definition of a continuous dynamical system on Y^1_m.
For reference, see [16, chap. 9, p.142].

Definition 4.1. *A continuous dynamical system on Y^1_m is a one-parameter family of mappings $(T(t))_{t\geq 0}$ from Y^1_m to Y^1_m such that :*
i) $T(0) = Id$.
ii) $T(t+s) = T(t)T(s)$ for any $t, s \geq 0$.
iii) For any $t \geq 0$, $T(t) \in C(Y^1_m, Y^1_m)$.
iv) For any $u_0 \in Y_m$, $t \mapsto T(t)u_0 \in C((0,\infty), Y^1_m)$.

Remark 4.1. *Continuity at $t = 0$ is sometimes included in the definition, but it is not required for our needs.*

Definition 4.2. *Let $u_0 \in Y^1_m$ and $t \geq 0$.*
We define $T(t)u_0 = u(t)$ where we recall that u is the classical solution of problem (PDE_m) with initial condition u_0.

Proposition 4.1. *$(T(t))_{t\geq 0}$ is a continuous dynamical system on Y^1_m.*

Proof : Thanks to Lemma 4.1, we know that $T(t)$ is well defined for all $t \geq 0$ and by definition of a classical solution, $T(t)$ maps Y^1_m into Y^1_m.
ii) is clear by uniqueness of the global classical solution.
iv) comes from the fact that $u \in C((0,\infty), C^1([0,1]))$.
iii) Let $t > 0$, $u_0 \in Y^1_m$ and $(u_n)_{n\geq 1} \in Y^1_m$.
Assume that $u_n \xrightarrow[n\to\infty]{C^1} u_0$. Let us show that $u_n(t) \xrightarrow[n\to\infty]{C^1} u(t)$.
We proceed in two steps.
First step : We show that if $u_n \xrightarrow[n\to\infty]{C^0} u_0$, then $u_n(t) \xrightarrow[n\to\infty]{C^0} u(t)$. Let $\eta > 0$.

By (3.1), there exists $C > 0$ such that for all $s \in (0,t]$, $\|u_x(s)\|_\infty \leq \frac{C}{\sqrt{s}}$.
So we can choose $\eta' > 0$ such that

$$\eta' e^{\int_0^t [\|u_x(s)\|_\infty^q + 1] \, ds} \leq \eta.$$

Let $n_0 \geq 1$ such that for all $n \geq n_0$, $\|u_n - u_0\|_{\infty,[0,1]} \leq \eta'$.
Let $n \geq n_0$ and $s \in [0,t]$. We denote $u_n(s)$ the solution at time s of problem (PDE_m) with initial condition u_n and set :

$$z(s) = [u_n(s) - u(s)] e^{-\int_0^s [\|u_x(s')\|_\infty^q + 1] \, ds'}.$$

We see that z satisfies

$$z_s = x^{2-\frac{2}{N}} z_{xx} + b\, z_x + c\, z \tag{4.3}$$

where

$$b(s,x) = u_n(s) \frac{(u_n)_x(s,x)^q - u_x(s,x)^q}{(u_n)_x(s,x) - u_x(s,x)} \text{ if } (u_n)_x(s,x) \neq u_x(s,x) \text{ and } 0 \text{ else}$$

and

$$c = [{u_x}^q - \|u_x\|_\infty^q - 1] < 0.$$

Since $z \in C([0,t] \times [0,1])$, z reaches its maximum and its minimum.
Assume that this maximum is greater than η'. Since $z = 0$ for $x = 0$ and $x = 1$ and $z \leq \eta'$ for $s = 0$, it can be reached only in $(0,t] \times (0,1)$ but this is impossible because $c < 0$ and (4.3). We make a similar reasoning for the minimum. Hence, $|z| \leq \eta'$ on $[0,t] \times [0,1]$.
Finally, $\|u_n - u\|_{\infty,[0,1]\times[0,t]} \leq \eta' e^{\int_0^t [\|(u_x(s))\|_\infty^q + 1] \, ds} \leq \eta$ for all $n \geq n_0$. Whence the result.

Second step : since $u_n \xrightarrow[n\to\infty]{C^1} u_0$, $\|u_n\|_{C^1}$ is bounded so there exists $K > 0$ such that for all $n \geq 1$, $\mathcal{N}[u_n] \leq K$. Then, from Lemma 4.2, since $t > 0$, $\{u_n(t),\ n \geq 1\}$ is relatively compact in Y_m^1 and has a single accumulation point $u(t)$ from first step. Whence the result. □

4.3 A strict Lyapunov functional for $(T(t))_{t\geq 0}$

4.3.1 Reminder on strict Lyapunov functionals

We recall some definitions in the context of a continuous dynamical system $(T(t))_{t\geq 0}$ on Y_m^1, including strict Lyapunov functional and Lasalle's invariance principle.

Definition 4.3. *Let* $u_0 \in Y_m^1$.
- $\gamma_1(u_0) = \{T(t)u_0,\ t \geq 1\}$ *is the trajectory of* u_0 *from* $t = 1$.
- $\omega(u_0) = \{v \in Y_m^1,\ \exists t_n \to +\infty,\ t_n \geq 1,\ T(t_n)u_0 \underset{n\to+\infty}{\longrightarrow} v \text{ in } Y_m^1\}$ *is the* ω*-limit set of* u_0.

Definition 4.4.
i) $\mathcal{F} \in C(Y^1_m, \mathbb{R})$ *is a Lyapunov functional if for all* $u_0 \in Y^1_m$,

$$t \mapsto \mathcal{F}[T(t)u_0] \text{ is nonincreasing on } [1, +\infty).$$

ii) A Lyapunov functional $\mathcal{F}$ *is a strict Lyapunov functional if*

$$\mathcal{F}[T(t)u_0] = \mathcal{F}[u_0] \text{ for all } t \geq 0 \text{ implies that } u_0 \text{ is an equilibrium point.}$$

Proposition 4.2. *Lasalle's invariance principle.*
Let $u_0 \in Y^1_m$. *Assume that the dynamical system* $(T(t))_{t\geq 0}$ *admits a strict Lyapunov functional and that* $\gamma_1(u_0)$ *is relatively compact in* Y^1_m.
Then the ω*-limit set* $\omega(u_0)$ *is nonempty and consists of equilibria of the dynamical system.*

See [16, p. 143] for a proof.

4.3.2 Approximate Lyapunov functionals

We recall that for all $\epsilon > 0$ and $x > 0$, $f_\epsilon(x) = (x+\epsilon)^q - \epsilon^q$. In order to introduce the approximate Lyapunov functional, we first need a double primitive H_ϵ of $\frac{1}{f_\epsilon}$ that converges uniformly to $H(x) = \dfrac{x^{2-q}}{(2-q)(1-q)}$ on compacts of $\mathbb{R}^+$. The next lemma provides it.

Lemma 4.3. *Let* $q \in (0,1)$ *and* $\epsilon \in (0,1]$.
For $x \geq 0$, *we set* $H_\epsilon(x) = \int_0^x \int_1^y \frac{dt}{f_\epsilon(t)} dy + \frac{x}{1-q}$ *and* $H(x) = \dfrac{x^{2-q}}{(2-q)(1-q)}$.
i) H_ϵ *is continuous on* $[0; +\infty)$, *twice differentiable on* $(0; +\infty)$.
$H''_\epsilon = \frac{1}{f_\epsilon}$ *on* $(0, +\infty)$.
ii) H_ϵ *converges uniformly to* H *on* $[0; R]$ *as* ϵ *tends to* 0, *for any* $R > 0$.

Proof : Let $R > 0$ and $x \in [0, R]$. We begin with two remarks :
- Since $f_\epsilon \geq f_1$ and f_1 is concave, then, denoting $K = \frac{f_1(R)}{R}$, we have

$$f_\epsilon(t) \geq Kt \qquad \text{for any } t > 0.$$

- We also note that for any $t > 0$,

$$0 \leq t^q - f_\epsilon(t) \leq \epsilon^q.$$

Indeed, setting $g(\epsilon) = (t+\epsilon)^q - \epsilon^q$, we have

$$t^q - f_\epsilon(t) = g(0) - g(\epsilon) = -\epsilon \int_0^1 g'(\epsilon s) ds = \epsilon \int_0^1 \frac{q}{(s\epsilon)^{1-q}} - \frac{q}{(t+\epsilon s)^{1-q}} ds \leq \epsilon^q.$$

i) Let us set for $y > 0$, $\gamma_\epsilon(y) = \int_1^y \frac{dt}{f_\epsilon(t)}$. Since for all $t > 0$, $f_\epsilon(t) \geq K\,t$, then $\gamma_\epsilon(y) = O(|\log(y)|)$. Hence γ_ϵ is integrable on $(0, R]$. Then, H_ϵ is continuous on

$[0, R]$. The other facts are obvious.
ii) We can write

$$H_\epsilon(x) - H(x) = \int_0^x \int_1^y \frac{1}{f_\epsilon(t)} - \frac{1}{t^q}\, dt\, dy = \int_0^x \int_1^y \frac{t^q - f_\epsilon(t)}{t^q f_\epsilon(t)} dt dy.$$

Using the first two remarks, we get

$$|H_\epsilon(x) - H(x)| \leq \frac{\epsilon^q}{K} \int_0^R |\int_1^y \frac{dt}{t^{1+q}}| \leq \frac{\epsilon^q}{Kq}[\frac{R^{1-q}}{1-q} + R].$$

Whence the result. □

Definition 4.5. *Let $\epsilon > 0$. For $u \in Y_m^1$, we define*

$$\mathcal{F}_\epsilon(u) = \int_0^1 H_\epsilon(u_x) - \frac{u^2}{2x^{2-q}} dx.$$

We would like to remind to the reader that, if $u_0 \in Y_m$ is given, u^ϵ denotes the solution of problem (PDE_m^ϵ) (see (3.11)) with initial condition u_0.

Lemma 4.4. *Let $u_0 \in Y_m^1$. For all $0 < t < s < T_{max}^\epsilon$,*

$$\mathcal{F}_\epsilon[u^\epsilon(s)] \leq \mathcal{F}_\epsilon[u^\epsilon(t)].$$

More precisely, for all $t > 0$,

$$\frac{d}{dt}\mathcal{F}_\epsilon[u^\epsilon(t)] = -\int_0^1 \frac{(u^\epsilon)_t^2}{x^{2-q} f_\epsilon((u^\epsilon)_x)} dx.$$

Proof : Let $t > 0$ and $\eta > 0$ such that $I = [t-\eta, t+\eta] \subset (0, T_{max}^\epsilon)$.
By Theorem 3.3 iii), there exists $\mu > 0$ such that $(u^\epsilon)_x \geq \mu$ on $I \times [0;1]$ so it will allow us to use that $H_\epsilon \in C^2([\mu, +\infty))$.
By Lemma 3.2, $(u^\epsilon)_{tx}$, $(u^\epsilon)_t$, $(u^\epsilon)_x$ are bounded on $I \times [0;1]$ and there exists $K > 0$ such that $|(u^\epsilon)_{xx}| \leq \frac{K}{x^{1-q}}$ on $I \times [0;1]$. Moreover, $0 \leq \frac{u^\epsilon}{x^{2-q}} \leq \frac{\|(u^\epsilon)_x\|_{C^1}}{x^{1-q}} \leq \frac{K'}{x^{1-q}}$ on $I \times [0;1]$ for some $K' > 0$. Note also that $u_{tx}^\epsilon = u_{xt}^\epsilon$ since $u^\epsilon \in C^2((0,T) \times (0,1])$. All these facts allow us to differentiate $\mathcal{F}_\epsilon[u^\epsilon(t)]$ and then to integrate by parts :

$$\frac{d}{dt}\mathcal{F}_\epsilon(u^\epsilon(t)) = \int_0^1 (u^\epsilon)_{tx} H'_\epsilon((u^\epsilon)_x) - \frac{(u^\epsilon)_t u^\epsilon}{x^{2-q}} dx$$

$$= [(u^\epsilon)_t H'_\epsilon((u^\epsilon)_x)]_0^1 - \int_0^1 (u^\epsilon)_t \left[\frac{(u^\epsilon)_{xx}}{f_\epsilon((u^\epsilon)_x)} + \frac{u^\epsilon}{x^{2-q}}\right].$$

Hence, $\frac{d}{dt}\mathcal{F}_\epsilon(u^\epsilon(t)) = -\int_0^1 \frac{(u^\epsilon)_t^2}{x^{2-q} f_\epsilon((u^\epsilon)_x)}$ since $(u^\epsilon)_t(t,0) = (u^\epsilon)_t(t,1) = 0$. □

4.3.3 A strict Lyapunov functional

Definition 4.6. *For $u \in Y_m^1$, we define*

$$\mathcal{F}(u) = \int_0^1 H(u_x) - \frac{u^2}{2x^{2-q}} dx$$

where $H(v) = \dfrac{v^{2-q}}{(2-q)(1-q)}$ *for all* $v \in \mathbb{R}$.

Theorem 4.1.
$\mathcal{F}$ is a strict Lyapunov functional for the dynamical system $T(t)_{t\geq 0}$ on Y_m^1.

This fact cannot be obtained directly by the formal computation shown in the introduction since u_x can vanish on a whole interval. Indeed, consider for instance an initial condition $u_0 \in Y_M$ such that $u_0 \geq U_a$ where $a > A$. By comparison principle, for all $t \geq 0$, $M \geq u(t) \geq U_a$ so $u_x(t) = 0$ at least on $[\frac{A}{a}, 1]$.

Proof : First, by application of the dominated convergence theorem, since H is continuous on $[0, \infty)$, it is clear that $\mathcal{F}$ is continuous on Y_m^1.
Secondly, let $u_0 \in Y_m^1$ and $0 < t \leq s$. Let us show that $\mathcal{F}(u(t)) \geq \mathcal{F}(u(s))$.
Let $t' > 0$. We first need to show that $\mathcal{F}_\epsilon(u^\epsilon(t')) \underset{\epsilon\to 0}{\longrightarrow} \mathcal{F}(u(t'))$.
Lemma 3.2 tells us that $u^\epsilon(t') \underset{\epsilon\to 0}{\longrightarrow} u(t')$ in $C^1([0,1])$. In particular, $(u^\epsilon(t'))_{\epsilon\in(0;1)}$ is bounded in $C^1([0;1])$ so we have a domination independent of ϵ. Since H_ϵ converges uniformly to H on compact subsets of $[0; +\infty)$ and H is continuous on $[0; +\infty)$, by the dominated convergence theorem, we obtain easily that $\mathcal{F}_\epsilon(u^\epsilon(t')) \underset{\epsilon\to 0}{\longrightarrow} \mathcal{F}(u_0(t'))$.
Then, as we know from Lemma 4.4 that $\mathcal{F}_\epsilon[u^\epsilon(t)] \geq \mathcal{F}_\epsilon[u^\epsilon(s)]$, the result follows by letting ϵ go to zero.
Thirdly, denoting $R = \sup\limits_{t'\in[t;s]} \|u_0(t')\|_{C^1}$, we want to show that

$$\mathcal{F}[u(t)] - \mathcal{F}[u_0(s)] \geq \frac{1}{(R+1)^q} \iint_{[t;s]\times[0;1]} (u_0)_t^2. \tag{4.4}$$

By Lemma 4.4,

$$\mathcal{F}_\epsilon(u^\epsilon(t)) - \mathcal{F}_\epsilon(u^\epsilon(s)) = \iint_{[t;s]\times[0;1]} \frac{(u^\epsilon)_t^2}{x^{2-q} f_\epsilon((u^\epsilon)_x)}.$$

By Lemma 3.2, $(u^\epsilon)_x$ tends to u_x in $C([t,s]\times[0;1])$, so there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0;\epsilon_0)$, $\sup\limits_{t'\in[t;s]} \|u^\epsilon(t')\|_{C^1} \leq R+1$.
Note that $\frac{1}{x^{2-q}} \geq 1$ for $0 < x \leq 1$ and that $0 < f_\epsilon(u_x^\epsilon) \leq (u_x^\epsilon)^q \leq (R+1)^q$ on $[t,s]\times[0,1]$. So,

$$\mathcal{F}_\epsilon(u^\epsilon(t)) - \mathcal{F}_\epsilon(u^\epsilon(s)) \geq \frac{1}{(R+1)^q} \iint_{[t;s]\times[0;1]} (u_t^\epsilon)^2.$$

By Lemma 3.2, $(u^\epsilon)_t$ tends to $(u_0)_t$ in $C([t;s]\times[0,1])$, hence by taking the limit as ϵ goes to 0, we obtain the result.
Finally, assume that $\mathcal{F}[u(t)] = \mathcal{F}[u_0]$ for all $t \geq 0$.
Let $[t,T] \subset (0,\infty)$. Formula (4.4) shows that $u_t = 0$ on $[t,T]$. So $u_t = 0$ on $(0,\infty)\times[0,1]$. Then, by continuity of u on $[0,\infty)\times[0,1]$, we get that $T(t)u_0 = u_0$ for all $t \geq 0$, i.e. u_0 is an equilibrium of the dynamical system $(T(t))_{t\geq 0}$. □

4.4 Convergence to a stationary state for $m \leq M$: proof of Theorem 1.2

In the case of $m < M$, there is a unique stationary solution for problem (PDE_m) so the convergence is not really surprising.
But for $m = M$ there is a continuum of stationary solutions (all $U_a|_{[0,1]}$ for $a \geq A$) and the behaviour could be much more complicated. However, thanks to the good properties of the set of steady states (see Proposition 2.3) and since the problem is one-dimensional, convergence can be shown by arguments in the spirit of [30] or [41].

Proof of Theorem 1.2 : Let $u_0 \in Y_m$. Let us set $u_1 = u_0(1) \in Y_m^1$.
To get the result, we just have to study $\lim\limits_{t\to+\infty} T(t)u_1$.
Thanks to Lemma 4.2, $\gamma_1(u_1)$ is relatively compact in Y_m^1 and since $\mathcal{F}$ is a strict Lyapunov functional for $(T(t))_{t\geq 0}$, we know by Lasalle's invariance principle (Proposition 4.2) that the ω-limit set $\omega(u_1)$ is non empty and contains only stationary solutions.
First case : $m < M$. Then from Theorem 1.1, there exists a unique stationary solution U_a with $a < M$. Hence, $\omega(u_1) = \{U_a\}$ so $T(t)u_1 \underset{t\to+\infty}{\longrightarrow} U_a$.
Second case : $m = M$. Assume by contradiction that $\omega(u_1)$ contains two different stationary solutions U_a and U_b with $A \leq a < b$. Then we chose any $c \in (a,b)$. From Proposition 2.3) ii), there exists $\epsilon > 0$ such that $V_M(U_a,\epsilon) \prec U_c$ and $\{u \in Y_M^1,\ u \leq U_c\} \cap V_M(U_b,\epsilon) = \emptyset$.
Since $U_a \in \omega(u_1)$, there exists t_a such that $T(t_a)u_1 \in V_M(U_a,\epsilon)$. Hence, $T(t_a)u_1 \leq U_c$ and then by comparison principle, for all $t \geq t_a$, $T(t)u_1 \leq U_c$. But, since $U_b \in \omega(u_1)$, there exists $t_b \geq t_a$ such that $T(t_b)u_1 \in V_M(U_b,\epsilon)$, and this is a contradiction because $\{u \in Y_M^1,\ u \leq U_c\} \cap V_M(U_b,\epsilon) = \emptyset$. Hence, $\omega(u_1)$ is a singleton $\{U_a\}$ with $a \geq A$ so $T(t)u_1 \underset{t\to+\infty}{\longrightarrow} U_a$. □

5 Finite time blow-up and self-similar solutions in supercritical case $m > M$

In this section, we only consider the supercritical case, i.e. when $m > M$.
We shall prove that classical solutions of problem (PDE_m) blow up in finite time. The idea of the proof is to exhibit a subsolution $\underline{u}(t,x) = V(a(t)x)$ which turns out

to be a self-similar solution after some time.
This is why we are interested in the following ordinary differential equation.

5.1 An auxiliary ordinary differential equation

Definition 5.1. *Let $\epsilon > 0$.*
We define the problem (Q_ϵ) by :

$$\begin{aligned} x^{2-q}\ddot{V} + V\dot{V}^q &= \epsilon x\dot{V} \qquad x > 0 && (5.1)\\ V(0) &= 0 && (5.2)\\ \dot{V}(0) &= 1 && (5.3) \end{aligned}$$

Definition 5.2. *Let $\epsilon > 0$ and $R > 0$.*
We say that V is a solution of problem (Q_ϵ) on $[0; R[$ if :
- *$V \in C^1([0; R[) \cap C^2(]0; R[)$.*
- *V is nondecreasing.*
- *V satisfies (27) on $]0; R[$ and the conditions (27) and (27).*

This definition can obviously be adapted for the case of a closed interval $[0; R]$ or for $R = +\infty$.

We summarize in the following theorem some very helpful results about solutions of problem (Q_ϵ).
Recall that U_1 is the solution of problem (P_1) and that A is the first point from which U_1 is constant.

Theorem 5.1. *There exists a unique solution of problem (Q_ϵ) on $[0, \infty)$.*
If $\epsilon > 0$ is small enough, V_ϵ is concave and there is a first point $A_\epsilon < \infty$ from which V_ϵ is constant with value M_ϵ greater than M.
Moreover :

i) $\|V_\epsilon - U_1\|_{C^1([0,\infty))} \underset{\epsilon\to 0}{\longrightarrow} 0$.
ii) $A_\epsilon \underset{\epsilon\to 0}{\longrightarrow} A$.
iii) $V_\epsilon(A+1) \underset{\epsilon\to 0}{\longrightarrow} M$.
That is to say that the constant reached by V_ϵ can be chosen as close to M as we wish provided that ϵ is small enough.

Remark : The fact that $M_\epsilon > M$ and that V_ϵ is concave for small $\epsilon > 0$ follow from the proof of Theorem 1.4.

The proof of Theorem 5.1 follows from the following successive lemmas. We begin by giving an a priori property of the solutions of problem (Q_ϵ).

Lemma 5.1. *Let $\epsilon > 0$ and V a solution of problem (Q_ϵ) on $[0; +\infty[$.*
If for some $x_0 \geq 0$, $\dot{V}(x_0) = 0$, then for all $x \geq x_0$, $V(x) = V(x_0)$.

Proof : First, $x_0 > 0$ since $\dot{V}(0) = 1$, then $V(x_0) > 0$ since V is nondecreasing. Let $x_1 = \sup\{r \geq x_0,\ s.t.\ V \text{ constant on } [x_0, r]\}$.
Assume by contradiction that $x_1 < \infty$. Then, by continuity, $V(x_1) = V(x_0) > 0$ and $\dot{V}(x_1) = 0$. Writing equation (5.1) as

$$x^{2-q}\ddot{V} = \dot{V}^q(-V + \epsilon x\dot{V}^{1-q}),$$

we see that the first factor of the RHS is nonnegative and that the second one keeps negative for x close enough to x_1 by continuity since $-V(x_1) + \epsilon x_1 \dot{V}(x_1)^{1-q} < 0$. Hence $\ddot{V}$ is nonpositive for x close enough to x_1. But $\dot{V}(x_1) = 0$ and $\dot{V} \geq 0$, then $\dot{V} = 0$ near of x_1, which contradicts the definition of x_1. So, $x_1 = \infty$. □

We now prove the local existence of a solution of problem (Q_ϵ).

Lemma 5.2. *Let $\epsilon \in (0, 1]$.*
There exists $\delta > 0$ independent of ϵ such that the problem (Q_ϵ) admits a unique solution on $[0, \delta]$.

Proof : The method used is a fixed point argument, as for the local existence of the solutions of problem (P_a).
Let us define

$$E = \{V \in C^1([0, \delta]),\ \dot{V} \geq 0,\ V(0) = 0,\ \dot{V}(0) = 1,\ \|\dot{V} - 1\|_\infty \leq \frac{1}{2}\}.$$

E equipped with the metric induced by the norm $\|V\|_E = \|\dot{V}\|_\infty$ is a complete metric space. We define F by :

$$F : E \to C^1([0; \delta])$$

$$F(V)(x) = x - \int_0^x \int_0^y \frac{V(s)}{s} \frac{\dot{V}(s)^q}{s^{1-q}} ds\, dy + \epsilon \int_0^x \int_0^y \frac{\dot{V}}{s^{1-q}} ds\, dy.$$

Since for all $V \in E$, $\|\dot{V}\|_\infty \leq \frac{3}{2}$ and $0 < \epsilon \leq 1$, we easily get that

$$\|F(V)' - 1\|_\infty \leq \left((\frac{3}{2})^{q+1} + \frac{3}{2}\right) \frac{\delta^q}{q} \leq \frac{1}{2},$$

provided that δ is chosen small enough. Hence, F sends E into E.
We can apply the mean value theorem to function $z \mapsto z^q$, since for all $V \in E$, $\frac{1}{2} \leq \dot{V} \leq \frac{3}{2}$. Finally, we obtain for all $(V_1, V_2) \in E^2$:

$$\|F(V_2) - F(V_1)\|_E \leq \left(\frac{(\frac{3}{2})^q}{q} + \frac{\frac{3}{2}}{(\frac{1}{2})^{1-q}} + \frac{1}{q}\right) \delta^q \|V_2 - V_1\|_E.$$

Hence, if δ is small enough, F is a contraction so there exists a fixed point V of F. Since $F(V) \in C^2((0, \delta])$ when $V \in C^1([0; \tau])$ then V is a solution of (Q_ϵ).
Finally, it is easy to check that a solution of (Q_ϵ) is necessarily a fixed point of F, which proves the uniqueness. □

Lemma 5.3. *Let $\epsilon \in (0,1]$.*
There exists a unique $A_\epsilon \geq \delta$ and a unique maximal solution V_ϵ on $[0, A_\epsilon)$ of (Q_ϵ) such that $\dot{V}_\epsilon > 0$ on $[0, A_\epsilon)$.

Proof : the proof follows from Lemma 5.2, similarly to that of Theorem 2.1.

Lemma 5.4. *Let $\epsilon \in (0,1]$.*
i) We have the following formula : for $x \in [0, A_\epsilon)$:

$$\dot{V}_\epsilon(x) = \exp(\frac{\epsilon(1-q)x^q}{q}) \left(1 - (1-q)\int_0^x \frac{V_\epsilon(s)}{s^{2-q}} \exp(-\frac{\epsilon(1-q)s^q}{q})ds\right)^{\frac{1}{1-q}}.$$

ii) For all $x \in [0, A_\epsilon)$,

$$0 < \dot{V}_\epsilon(x) \leq \exp(\frac{\epsilon(1-q)x^q}{q}),$$

$$0 \leq V_\epsilon(x) \leq \int_0^x \exp(\frac{\epsilon(1-q)s^q}{q})ds.$$

Proof : $i)$ Let us set $w = \dot{V}_\epsilon$. On $(0, A_\epsilon)$, w satisfies

$$\dot{w} = -\frac{V_\epsilon(s)}{s^{2-q}} w^q + \frac{\epsilon}{x^{1-q}} w.$$

We recognize a Bernoulli type ordinary differential equation. Since $w > 0$ on $[0, A_\epsilon)$, then we can divide by w^q. Setting $z = w^{1-q}$, we obtain

$$\dot{z} = \frac{\epsilon(1-q)}{x^{1-q}} z - \frac{V_\epsilon(s)(1-q)}{s^{2-q}}$$

which can be easily integrated. Whence the formula.
$ii)$ V_ϵ is increasing and $V_\epsilon(0) = 0$ so $V_\epsilon \geq 0$ on $[0, A_\epsilon)$ whence the results. □

Remark 5.1. *A consequence of last lemma is that*
if $1 - (1-q)\int_0^x \frac{V_\epsilon(s)}{s^{2-q}} \exp(-\frac{\epsilon(1-q)s^q}{q})ds \leq 0$, then $A_\epsilon \leq x$.

Lemma 5.5. *Let $\epsilon \in (0,1]$.*
There exists a unique solution V_ϵ of problem (Q_ϵ) defined on $[0, +\infty)$.

Proof : If $A_\epsilon = \infty$ then all is already done.
Else, if $A_\epsilon < \infty$ then $\dot{V}_\epsilon(x)$ has to go to zero as x goes to A_ϵ. Indeed, $(V_\epsilon(x), \dot{V}_\epsilon(x))$ must go out of any compact of $\mathbb{R} \times (0, +\infty)$ as x goes to A_ϵ but by $ii)$ of the above lemma $V_\epsilon(x)$ and $\dot{V}_\epsilon(x)$ keep bounded for x bounded. So $\dot{V}_\epsilon(x) \underset{x \to A_\epsilon}{\longrightarrow} 0$.
Hence, by Cauchy criterion, $V_\epsilon(x)$ has a limit L_ϵ as x goes to A_ϵ. Moreover, by the equation (5.1), $\ddot{V}_\epsilon(x)$ goes to zero as x goes to A_ϵ. Since the constants are solutions of (5.1), then V_ϵ can be extended by the constant L_ϵ on $[A_\epsilon, +\infty)$ to a C^2 function

on $(0,+\infty)$ which is a solution of problem (5.1).
For proving the uniqueness, let V a solution on $[0,+\infty)$. By uniqueness of the solution on $[0,A_\epsilon)$ then $V=V_\epsilon$ on $[0,A_\epsilon)$. Then, by continuity, $V(A_\epsilon)=V_\epsilon(A_\epsilon)$ and $\dot V(A_\epsilon)=0$. But now by Lemma 5.1, V is constant on $[A_\epsilon,\infty)$ so $V=V_\epsilon$ on $[0,\infty)$. □

Lemma 5.6.
i) $A_\epsilon<\infty$ *for* ϵ *small enough.*
Moreover $A_\epsilon \underset{\epsilon\to 0}{\longrightarrow} A$.
ii) $\|V_\epsilon-U_1\|_{C^1([0,\infty))} \underset{\epsilon\to 0}{\longrightarrow} 0$.
As a consequence, $V_\epsilon(A+1) \underset{\epsilon\to 0}{\longrightarrow} M$.

Proof :
First step : We show that there exists $\gamma\in(0,\delta]$ independent of ϵ such that

$$\|V_\epsilon-U_1\|_{C^1([0,\gamma])} \underset{\epsilon\to 0}{\longrightarrow} 0.$$

Let us set $\gamma=\min(\delta,\delta',A)$ where δ is a short existence time for all V_ϵ and $\delta'=(\frac{q}{2})^{\frac{1}{q}}$.
$\dot U_1>0$ on $[0,A)$ so equation (2.4) can be written on $[0,A)$ as

$$\frac{d}{dx}\frac{\dot{U_1}^{1-q}}{1-q}=-\frac{U_1}{x^{2-q}}.$$

Hence, we have the following formula

$$\dot U_1(x)=\left(1-(1-q)\int_0^x\frac{U_1(s)}{s^{2-q}}ds\right)^{\frac{1}{1-q}}, \tag{5.4}$$

which is valid for all $x\in[0,A]$, by continuity.
Let $x\in[0,\gamma]$.

$$\begin{aligned}\dot V_\epsilon(x)-\dot U_1(x)=&\\ \left(e^{\epsilon\frac{(1-q)x^q}{q}}-1\right)&\left(1-(1-q)\int_0^x\frac{V_\epsilon(s)}{s^{2-q}}\exp(-\frac{\epsilon(1-q)s^q}{q})ds\right)^{\frac{1}{1-q}}\\ &+\left(1-(1-q)\int_0^x\frac{V_\epsilon(s)}{s^{2-q}}\exp(-\frac{\epsilon(1-q)s^q}{q})ds\right)^{\frac{1}{1-q}}\\ &-\left(1-(1-q)\int_0^x\frac{U_1(s)}{s^{2-q}}ds\right)^{\frac{1}{1-q}}.\end{aligned}$$

Then,

$$|\dot V_\epsilon(x)-\dot U_1(x)|\le(e^{\epsilon\frac{(1-q)x^q}{q}}-1)+\int_0^x\left|\frac{V_\epsilon(s)}{s^{2-q}}e^{-\frac{\epsilon(1-q)s^q}{q}}-\frac{U_1(s)}{s^{2-q}}\right|ds$$

because $|a^{\frac{1}{1-q}} - b^{\frac{1}{1-q}}| \leq \frac{1}{1-q}|a-b|$ for all $(a,b) \in [0,1]^2$ since $\frac{1}{1-q} > 1$.

$$|\dot{V}_\epsilon(x) - \dot{U}_1(x)| \leq (e^{\epsilon\frac{(1-q)\gamma^q}{q}} - 1) + \int_0^x \frac{|V_\epsilon(s) - U_1(s)|}{s^{2-q}} e^{-\frac{\epsilon(1-q)s^q}{q}} ds$$
$$+ \int_0^x \frac{U_1(s)}{s^{2-q}}(1 - e^{-\frac{\epsilon(1-q)s^q}{q}})ds.$$

Then, we get

$$\|\dot{V}_\epsilon - \dot{U}_1\|_{\infty,[0,\gamma]} \leq (e^{\epsilon\frac{(1-q)\gamma^q}{q}} - 1)$$
$$+\|\dot{V}_\epsilon - \dot{U}_1\|_{\infty,[0,\gamma]}\frac{\gamma^q}{q} + (1 - e^{-\frac{\epsilon(1-q)\gamma^q}{q}})\int_0^\gamma \frac{U_1(s)}{s^{2-q}}ds.$$

Since $\frac{\gamma^q}{q} \leq \frac{1}{2}$ and $\int_0^\gamma \frac{U_1(s)}{s^{2-q}}ds \leq \frac{1}{1-q}$, then

$$\|\dot{V}_\epsilon - \dot{U}_1\|_{\infty,[0,\gamma]} \leq 2\left[e^{\epsilon\frac{(1-q)\gamma^q}{q}} - 1 + \frac{1 - e^{-\frac{\epsilon(1-q)\gamma^q}{q}}}{1-q}\right].$$

Hence, $\|V_\epsilon - U_1\|_{C^1([0,\gamma])} \underset{\epsilon\to 0}{\longrightarrow} 0$.

Second step : Let $A' < A$.
Let us show that for ϵ small enough $A_\epsilon \geq A'$ and that

$$\|V_\epsilon - U_1\|_{C^1([0,A'])} \underset{\epsilon\to 0}{\longrightarrow} 0.$$

Let us denote $V_{a,b}$ the solution of (5.1) such that $V_{a,b}(\gamma) = a$ and $\dot{V}_{a,b}(\gamma) = b$. U_1 is the solution of equation (5.1) for $\epsilon = 0$ and initial condition $(a,b) = (U_1(\gamma), \dot{U}_1(\gamma))$. Since $\dot{U}_1 > 0$ on $[\gamma, A']$, the classical Cauchy-Lipschitz theory is here available, and by continuity of the solutions on $[\gamma, A']$ with respect to the parameter ϵ and the initial condition (a,b), we know that if ϵ is small enough and if the initial condition (a,b) is close enough to $(U_1(\gamma), \dot{U}_1(\gamma))$ then $\|V_{a,b} - U_1\|_{C^1([\gamma,A'])}$ is as small as we wish. But, thanks to the first step, taking ϵ even smaller, $(V_\epsilon(\gamma), \dot{V}_\epsilon(\gamma))$ can be made as close as necessary of $(U_1(\gamma), \dot{U}_1(\gamma))$. Finally, $\|V_\epsilon - U_1\|_{C^1([0,A'])}$ is as small as necessary when ϵ is small enough. This implies of course that $A_\epsilon \geq A'$ for ϵ small enough. Whence the results.

Third step : Let us show that $\|V_\epsilon - U_1\|_{C^1([0,A+1])} \underset{\epsilon\to 0}{\longrightarrow} 0$.
Let $\alpha > 0$ and $\eta = (\frac{\alpha}{4})^{1-q}$.
By formula (5.4), we have $(1-q)\int_0^A \frac{U_1(s)}{s^{2-q}}ds = 1$ and since $\dot{U}_1 = 0$ on $[A,\infty)$, there exists $A' < A$ such that $1 - (1-q)\int_0^{A'} \frac{U_1(s)}{s^{2-q}}ds \leq \eta$ and $\|\dot{U}_1\|_{\infty,[A',\infty]} \leq \frac{\alpha}{2}$.
Let $x \in [A', \min(A_\epsilon, A+1)]$. We know that
$\dot{V}_\epsilon(x) = \exp(\frac{\epsilon(1-q)x^q}{q})\left(1 - (1-q)\int_0^x \frac{V_\epsilon(s)}{s^{2-q}}\exp(-\frac{\epsilon(1-q)s^q}{q})ds\right)^{\frac{1}{1-q}}$
then $\dot{V}_\epsilon(x) \leq \exp(\frac{\epsilon(1-q)(A+1)^q}{q})\left(1 - (1-q)\int_0^{A'} \frac{V_\epsilon(s)}{s^{2-q}}ds\right)^{\frac{1}{1-q}}$ since $V_\epsilon \geq 0$.
There exists $\epsilon_1 > 0$ such that if $0 < \epsilon \leq \epsilon_1$, $\exp(\frac{\epsilon(1-q)(A+1)^q}{q}) \leq 2$, so $\dot{V}_\epsilon(x) \leq$

$2\eta^{\frac{1}{1-q}} = \frac{\alpha}{2}$. Then $\|\dot{V}_\epsilon\|_{\infty,[A',A+1]} \leq \frac{\alpha}{2}$ since $\dot{V}_\epsilon = 0$ in $[A_\epsilon, \infty]$. Finally, if $0 < \epsilon \leq \epsilon_1$, then $\|\dot{V}_\epsilon - \dot{U}_1\|_{\infty,[A',A+1]} \leq \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$. From the second step, let $\epsilon_2 > 0$ such that $\|\dot{V}_\epsilon - \dot{U}_1\|_{\infty,[0,A']} \leq \alpha$ for $0 < \epsilon \leq \epsilon_2$. Let us set $\epsilon_0 = \min(\epsilon_1, \epsilon_2)$. Then, if if $0 < \epsilon \leq \epsilon_0$, then $\|\dot{V}_\epsilon - \dot{U}_1\|_{\infty,[0,A+1]} \leq \alpha$.
Fourth step : Let us show that for ϵ small enough, $A_\epsilon < \infty$ and even that $A_\epsilon \underset{\epsilon\to 0}{\longrightarrow} A$. Let $A' \in (A, A+1]$. Since, by formula (5.4), we have

$$1 - (1-q)\int_0^{A'} \frac{U_1(s)}{s^{2-q}} ds = -(1-q)\int_A^{A'} \frac{U_1(s)}{s^{2-q}} < 0$$

and since

$$\|V_\epsilon - U_1\|_{C^1([0,A'])} \underset{\epsilon\to 0}{\longrightarrow} 0,$$

then there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$,

$$1 - (1-q)\int_0^{A'} \frac{V_\epsilon(s)}{s^{2-q}} \exp\left(-\frac{\epsilon(1-q)s^q}{q}\right) ds < 0.$$

Let $0 < \epsilon < \epsilon_0$. From remark 5.1, then $A_\epsilon \leq A' < \infty$. Combined with step 2, this proves that $A_\epsilon \underset{\epsilon\to 0}{\longrightarrow} A$
Last step : Let $\epsilon > 0$ small enough so that $A_\epsilon \leq A+1$. Then, $\dot{V}_\epsilon = \dot{U}_1 = 0$ on $[A+1, \infty)$ thanks to Lemma 5.1. Moreover, thanks to step 3, $V_\epsilon(A+1) \underset{\epsilon\to 0}{\longrightarrow} U_1(A+1) = M$. Whence $\|V_\epsilon - U_1\|_{C^1([0,\infty))} \underset{\epsilon\to 0}{\longrightarrow} 0$. □

5.2 Blow-up and existence of self-similar solutions : proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3 : Let $m > M$ and $u_0 \in Y_m$.
From Theorem 5.1, we can fix $\epsilon > 0$ such that V_ϵ is constant equal to L on $[A_\epsilon, \infty)$ with $L < m$.
Pick $t_0 \in (0, T_{max}(u_0))$. From Theorem 3.1 iii), we know that $u_x(t_0, 0) > 0$, then $\alpha = \min_{x\in(0,1]} \frac{u(t_0,x)}{x} > 0$. We now set

$$a_0 = \frac{\alpha}{\|\dot{V}_\epsilon\|_{\infty,[0,+\infty)}} > 0.$$

Then for all $x \in [0,1]$,

$$V_\epsilon(a_0\, x) \leq \|\dot{V}_\epsilon\|_{\infty,[0,+\infty)}\, a_0\, x \leq \frac{a_0\, \|\dot{V}_\epsilon\|_{\infty,[0,+\infty)}}{\alpha} u(t_0, x).$$

Hence, a_0 is small enough so that

$$V_\epsilon(a_0 x) \leq u(t_0, x) \ \text{ for all } x \in [0,1].$$

Let us set

$$a(t) = \frac{a_0}{(1 - \epsilon\, a_0^q\, qt)^{\frac{1}{q}}}$$

and

$$T^* = \frac{1}{\epsilon\, a_0^q\, q}.$$

Remark that $a(0) = a_0$ and

$$\dot{a}(t) = \epsilon\, a(t)^{1+q} \text{ for } t \in [0, T^*).$$

We shall show that

$$\underline{u}(t,x) = V_\epsilon(a(t)x)$$

is a subsolution for $(PDE)_m$ with initial condition $u_0(t_0)$.
Let us set $T = \min(T_{max}(u_0) - t_0, T^*)$.
Indeed, $\underline{u}(0,x) = V_\epsilon(a_0 x) \leq u_0(t_0, x)$ for all $x \in [0,1]$. For all $t \in [0,T)$, we see that $\underline{u}(t,0) = 0 = u_0(t_0+t, 0)$ and $\underline{u}(t,1) \leq L \leq m = u_0(t_0+t, 1)$.
Moreover, a straightforward calculation shows that

$$\underline{u}_t - x^{2-q}\underline{u}_{xx} - \underline{u}\ \underline{u}_x{}^q = a^q \left[\frac{\dot{a}}{a^{1+q}} y \dot{V}_\epsilon(y) - y^{2-q}\ddot{V}_\epsilon(y) - V_\epsilon(y)\dot{V}_\epsilon(y)^q \right] = 0$$

where $y = a(t)x$.

From the comparison principle (cf Lemma 3.1), $\underline{u}(t) \leq u_0(t_0+t)$ for all $t \in [0,T)$. Now, if we assume that $T_{max}(u_0) > t_0 + T^*$, then $T = T^*$ and by letting t go to T^*, since $a(t) \underset{t \to T^*}{\longrightarrow} +\infty$, we obtain

$$L \leq u_0(t_0 + T^*, x) \text{ for all } x \in (0,1].$$

Since $u_0(t_0 + T^*, 0) = 0$, this contradicts the continuity of $u_0(t_0 + T^*)$ at $x = 0$. Hence, $T_{max}(u_0) \leq t_0 + T^* < \infty$. □

Proof of Theorem 1.4 : From Theorem 5.1, we know the existence of $\epsilon_1 > 0$ such that for $\epsilon \in (0, \epsilon_1]$, $A_\epsilon \leq A+1$ and V_ϵ is flat from $x = A_\epsilon$. We set $u_{\epsilon,a_0}(t,x) = V_\epsilon(a(t)x)$ where $a(t) = \dfrac{a_0}{(1 - \epsilon\, a_0^q\, qt)^{\frac{1}{q}}}$ and $a_0 \geq A_\epsilon$. The calculation in the proof of the previous theorem and the fact that V_ϵ is constant from the point $x = A_\epsilon$ prove that $(u_{\epsilon,a_0})_{a_0 \geq A_\epsilon}$ is a family of solutions of problem (PDE_{M_ϵ}) where $M_\epsilon = V_\epsilon(A+1)$. Now, by using the same methods as in the proof of Lemma 5.6, we can prove that

$$\epsilon \mapsto V_\epsilon \text{ is continuous from } [0, \epsilon_1] \text{ to } C^1([0, +\infty)). \tag{5.5}$$

Then, $\epsilon \mapsto M_\epsilon = V_\epsilon(A+1)$ is continuous in $[0, \epsilon_1]$ so its image is a compact interval I. Since u_{ϵ,a_0} blows up, we necessarily have $M_\epsilon > M$ if $\epsilon \in (0, \epsilon_1]$ so $I = [M, M^+]$ with $M^+ > 0$.

Finally, denoting $K_\epsilon = \sup\limits_{x\in(0,\infty)} \frac{V_\epsilon(x)}{x}$, it is clear that $K_\epsilon = \sup\limits_{x\in(0,A_\epsilon]} \frac{V_\epsilon(x)}{x}$ since V_ϵ is flat from $x = A_\epsilon$. Then, we have

$$\mathcal{N}[u_{\epsilon,a_0}(t)] = K_\epsilon\, a(t) = \frac{K_\epsilon}{q^{\frac{1}{q}}\epsilon^{\frac{1}{q}}(T_{max}-t)^{\frac{1}{q}}},$$

where

$$T_{max} = \frac{1}{q\epsilon a_0^q}.$$

We now prove that V_ϵ is concave for small ϵ, which implies that $K_\epsilon = 1$ so that the blow-up speed is known explicitly. Let $\epsilon \in [0, \epsilon_1]$.
By Lemma 5.2, there exists $\delta > 0$ independent of ϵ such that $\dot{V}_\epsilon \geq \frac{1}{2}$ on $[0, \delta]$. Since V_ϵ is moreover nondecreasing on $[0, A+1]$, then $\min\limits_{x\in(0,A+1]} \frac{V_\epsilon}{x} \geq \frac{\delta}{2}$. By (5.5), there exists $C > 0$ such that $\|\dot{V}_\epsilon\|_\infty \leq C$ for all $\epsilon \in [0, \epsilon_1]$.
Let $\epsilon_2 = \min(\epsilon_1, \frac{\delta}{2\,C^{1-q}})$, $\epsilon \in (0, \epsilon_2]$ and $x \in [0, A+1]$.
We have $\epsilon\, x\, \dot{V}_\epsilon(x)^{1-q} \leq \epsilon_2\, x\, C^{1-q} \leq \frac{\delta}{2} x \leq V_\epsilon(x)$ and since

$$x^{2-q}\ddot{V}_\epsilon(x) = -\dot{V}_\epsilon(x)^q \left[V_\epsilon(x) - \epsilon x \dot{V}_\epsilon(x)^{1-q}\right],$$

we deduce that V_ϵ is concave and $K_\epsilon = \dot{V}_\epsilon(0) = 1$ for $\epsilon \in [0, \epsilon_2]$. □

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Chapitre 2

Théorie locale en temps, régularité et alternative d'explosion [1]

Dans ce chapitre, nous étudions un problème parabolique dégénéré unidimensionnel avec une non-linéarité faisant intervenir le gradient de façon non Lipschitz. Cette équation d'évolution intervient lorsque qu'on s'intéresse aux solutions radiales du système de type Patlak-Keller-Segel étudié dans le chapitre 1. Nous montrons que ce problème est bien posé dans un espace fonctionnel approprié et obtenons également des résultats de régularité ainsi qu'une alternative d'explosion. Un problème transformé ainsi qu'un problème approximé apparaissent naturellement dans la preuve. Notons que ceux-ci s'étaient déjà révélés essentiels lorsque nous avons étudié le comportement qualitatif en temps long dans le chapitre 1 et montré l'existence d'une masse critique.

1 Introduction

In this paper, we are mainly interested in studying the local in time wellposedness of the following problem (PDE_m) :

$$\begin{aligned} u_t = x^{2-\frac{2}{N}}\, u_{xx} + u\, {u_x}^q \qquad & t > 0 \quad 0 < x \leq 1 && (1.1)\\ u(t,0) = 0 \qquad & t \geq 0 && (1.2)\\ u(t,1) = m \qquad & t \geq 0 && (1.3)\\ u_x(t,x) \geq 0 \qquad & t > 0 \quad 0 \leq x \leq 1, && (1.4) \end{aligned}$$

where N is an integer greater or equal to 2, $m \geq 0$ and $0 < q < 1$.

This problem follows from a chemotaxis model being aimed at describing a collection of cells diffusing and emitting a chemical which attracts themselves. These

1. Ce chapitre est tiré de l'article [P1].

cells are assumed to lie in a physical domain corresponding to the open unit ball $D \subset \mathbb{R}^N$ ($N = 2$ or $N = 3$ being the most relevant cases) and if we suppose moreover that cells diffuse much more slowly than the chemoattractant, we get the following parabolic-elliptic Patlak-Keller-Segel system (PKS_q) :

$$\rho_t = \Delta\rho - \nabla[\rho^q\,\nabla c] \qquad t > 0 \quad \text{on } D \tag{1.5}$$

$$-\Delta c = \rho \qquad t > 0 \quad \text{on } D \tag{1.6}$$

with the following boundary conditions :

$$\frac{\partial\rho}{\partial\nu} - \rho^q\,\frac{\partial c}{\partial\nu} = 0 \quad \text{ on } \partial D \tag{1.7}$$

$$c = 0 \quad \text{on } \partial D \tag{1.8}$$

where ρ is the cell density and c the chemoattractant concentration. Note that on the boundary ∂D are imposed a natural no flux condition for ρ and Dirichlet conditions for c.
Problem (PDE_m) follows from (PKS_q) when considering radially symmetric solutions and after having made some transformations and a renormalization.
What is essential to know is that :

- m is proportional to the cells mass $\int_B \rho$.
- The derivative of u is the quantity with physical interest since u_x is proportional to the cells density ρ, up to a rescaling in time and a change of variable. More precisely, denoting $\rho(t, y) = \tilde{\rho}(t, |y|)$ for $t \geq 0$ and $y \in \overline{D}$,

$$\tilde{\rho}(t, x) = N^{\frac{2}{q}}\,u_x(N^2\,t, x^N) \qquad \text{for all } x \in [0, 1].$$

- The power $q = \frac{2}{N}$ is critical.

Much more detail about problem (PKS_q) and its link with (PDE_m) are given in the introduction of [P2]. See [22, 20, 12] for references concerning the biological background and [14, 15, 11, 17, 16, 19, 1, 2, 13, 8, 24, 3, 4, 5, 6] for related mathematical results.

The critical case $N = 2$, $q = 1$ is already well-known for its critical mass 8π. See [1, 13]. Our aim is to provide a rigorous framework in view of the study that we have carried out in [P2] on the global behaviour of solutions of problem (PDE_m) in the case $N \geq 3$ and $q = \frac{2}{N} \in (0, 1)$. In particular, we will prove the local in time existence and uniqueness of a maximal classical solution u for problem (PDE_m) with initial condition $u_0 \in Y_m$ where Y_m is a space of functions which will be made explicit in the next section. Moreover, we have a blow-up alternative, regularity results and a description giving an idea of the shape of solutions.
Let us point out that solutions of (PDE_m) are uniformly bounded in view of the maximum principle and that possible finite singularities are thus of gradient blow-up type. However, we shall show (see Theorem 3.1)iii)) that the solution can be

continued as long as the slopes with respect to the origin are controlled, which is a crucial fact for the analysis in [P2].

In the way to prove these results, we will need some related problems, in particular a transformed problem ($tPDE_m$) and an approximated problem (PDE_m^ϵ) for $\epsilon > 0$. We also would like to point out the role played by both problems when proving in [P2] that problem (PDE_m) exhibits a critical mass phenomenon. More precisely, we showed there the existence of $M > 0$ such that :

- If $m \leq M$, then u is global and

  $$u(t) \underset{t\to\infty}{\longrightarrow} U \qquad \text{in } C^1([0,1])$$

  where U is a steady state of (PDE_m).
- If $m > M$ then $u(t)$ blows up in finite time $T_{max} < \infty$.
  Moreover,

  $$\lim_{t\to T_{max}} \mathcal{N}[u(t)] = +\infty$$

  where $\mathcal{N}[f] = \sup\limits_{x\in(0,1]} \frac{f(x)}{x}$ for any real function f defined on $(0,1]$.

We precisely described the set of steady states and in particular proved that there exists only one stationary solution for $m < M$, none for $m > M$ but a whole continuum for $m = M$ (in which case u_x has support strictly inside $[0,1)$). The critical case $m = M$ could then be much more intricate since the solution could for instance oscillate between various stationary solutions. In order to treat the case $m \leq M$, we used some dynamical systems methods and proved (with help of ($tPDE_m$)) that all trajectories are relatively compact and (with help of (PDE_m^ϵ)) the existence of a strict Lyapunov functional $\mathcal{F} = \lim\limits_{\epsilon\to 0} \mathcal{F}_\epsilon$ where $\mathcal{F}_\epsilon$ is a strict Lyapunov functional for (PDE_m^ϵ).

Eventually, we would like to stress that problem (PDE_m) is not standard since it presents two difficulties :

- The diffusion is degenerate since $x^{2-\frac{2}{N}}$ goes to 0 as x goes to 0.
- The nonlinearity, which involves a gradient term, is not Lipschitz since $q \in (0,1)$.

The outline of the rest of the paper is as follows :

2 Notation and strategy

We give the definition of Y_m, a space of functions appropriate for our study.

Definition 2.1. *Let $m \geq 0$.*
$Y_m = \{u \in C([0;1]),\ u \textit{ nondecreasing },\ u'(0) \textit{ exists, } u(0) = 0,\ u(1) = m\}$

We are interested in the following evolution equation called (PDE_m) with

$$N \geq 2,\ q \in (0,1) \text{ and } m \geq 0.$$

Definition 2.2. *Let $T > 0$.*
We define problem (PDE_m) with initial condition $u_0 \in Y_m$ by :

$$u_t = x^{2-\frac{2}{N}} u_{xx} + u\, {u_x}^q \quad \text{on } (0,T] \times (0,1] \tag{2.1}$$
$$u(0) = u_0 \tag{2.2}$$
$$u(t) \in Y_m \quad \text{for } t \in [0,T] \tag{2.3}$$

A classical solution of problem (PDE_m) with initial condition $u_0 \in Y_m$ on $[0,T]$ is a function

$$u \in C([0,T] \times [0,1]) \bigcap C^1((0,T] \times [0,1]) \bigcap C^{1,2}((0,T] \times (0,1])$$

such that (2.1)(2.2)(2.3) are satisfied.
A classical solution of problem (PDE_m) on $[0,T)$ is defined similarly.

We would like to briefly describe the strategy used to obtain a maximal classical solution of problem (PDE_m), as well as approximate solutions of it that turned out to be very helpful in [P2]. At the same time, we introduce the notation used throughout this paper.

First step : we introduce the change of unknown, denoted θ_0, in order to get rid of the degenerate diffusion. It turns out (see formulae (5.11)(5.12)(5.13)(5.14)) that the transformed equation becomes nondegenerate and involves the radial heat operator, but in N+2 space dimensions.

Definition 2.3. *Let B denote the open unit ball in $\mathbb{R}^{N+2}$.*
We define the transformation
$\theta_0 : \ Y_m \longrightarrow Z_m$
$u \longrightarrow w$ *where* $w(y) = \frac{u(|y|^N)}{|y|^N}$ *for all* $y \in \overline{B} \backslash \{0\}$

where

$$Z_m = \{w \in C(\overline{B}),\ w|_{\partial B} = m\}.$$

Remark 2.1. *To avoid any confusion, we would like to stress that the physical domain D (where the cells live) lies in $\mathbb{R}^N$ but that the ball B (where the transformed problem is posed) lies in $\mathbb{R}^{N+2}$.*

Setting $w_0 = \theta_0(u_0) \in Z_m$ and $w(t,y) = \frac{u(N^2 t, |y|^N)}{|y|^N}$ for all $y \in \overline{B}$, we obtain a transformed problem called $(tPDE_m)$ with simple Laplacian diffusion which will allow us to use the heat semigroup.

Definition 2.4. *Let $m \geq 0$, and $T > 0$.*
Let $w_0 = \theta_0(u_0)$ where $u_0 \in Y_m$.
We define problem $(tPDE_m)$ with initial condition w_0 by :

$$w_t = \Delta w + N^2 w \left(w + \frac{y.\nabla w}{N}\right)^q \quad \text{on } (0,T] \times \overline{B} \tag{2.4}$$

$$w(0) = w_0 \tag{2.5}$$

$$w + \frac{y.\nabla w}{N} \geq 0 \quad \text{on } (0,T] \times \overline{B} \tag{2.6}$$

$$w = m \quad \text{on } [0,T] \times \partial B \tag{2.7}$$

A classical solution of problem $(tPDE_m)$ with initial condition w_0 on $[0,T]$ is a function

$$w \in C([0,T] \times \overline{B}) \bigcap C^{1,2}((0,T] \times \overline{B})$$

such that (2.4)(2.5)(2.6)(2.7) are satisfied.
We define analogously a classical solution on $[0,T)$.

Second step : since equation $(tPDE_m)$ still has a non Lipschitz nonlinearity, we want to define an approximate problem $(tPDE_m^\epsilon)$ for $\epsilon > 0$ to get rid of it.
This is why we introduce the following function :

Definition 2.5. *Let $\epsilon > 0$. We set :*

$$f_\epsilon(x) = (x+\epsilon)^q - \epsilon^q \quad \text{if } x \geq 0$$

and f_ϵ can be extended to $\mathbb{R}$ so that it satisfies both following conditions :

$$f_\epsilon \in C^3(\mathbb{R})$$

$$-|x|^q \leq f_\epsilon(x) < 0 \text{ for all } x \in (-\infty, 0)$$

Observe in particular that $|f_\epsilon(x)| \leq |x|^q$ for all $x \in \mathbb{R}$.

Remark 2.2. *Note that the conditions on f_ϵ on $(-\infty,0)$ are purely technical. Indeed, the choice of the extension does not play any role since we will prove that actually $(u^\epsilon)_x > 0$ on $[0,1]$, where u^ϵ is the maximal classical solution of problem (PDE_m^ϵ) with initial condition $u_0 \in Y_m$ as defined below.*

Definition 2.6. *Let $\epsilon > 0$, $m \geq 0$ and $T > 0$.*
Let $w_0 = \theta_0(u_0)$ where $u_0 \in Y_m$.
We define problem $(tPDE_m^\epsilon)$ with initial condition w_0 by :

$$w_t = \Delta w + N^2 w \; f_\epsilon\left(w + \frac{y.\nabla w}{N}\right) \quad \text{on } (0,T] \times \overline{B} \tag{2.8}$$

$$w(0) = w_0 \tag{2.9}$$

$$w = m \quad \text{on } [0,T] \times \partial B \tag{2.10}$$

A classical solution for problem $(tPDE_m^\epsilon)$ with initial condition w_0 on $[0,T]$ is a function

$$w^\epsilon \in C([0,T] \times \overline{B}) \bigcap C^{1,2}((0,T] \times \overline{B})$$

such that (2.8)(2.9)(2.10) are satisfied.
We define similarly a classical solution on $[0,T)$.

The setting of problem $(tPDE_m^\epsilon)$ is standard and allows to find a unique classical maximal solution w^ϵ on $[0, T_\epsilon^*)$ with initial condition $w_0 = \theta(u_0)$ for any $u_0 \in Y_m$. Then, a compactness property and the monotonicity of the family $(w^\epsilon)_{\epsilon>0}$ allows to get a local solution of $(tPDE_m)$ by letting ϵ go to 0. Eventually, since a comparison principle is available, we obtain a unique maximal classical solution w for problem $(tPDE_m)$. Since w_0 is radial, so is $w(t)$ which can then be written $w(t,y) = \tilde{w}(t, |y|)$ for all $y \in \overline{B}$. Eventually, setting

$$u(t,x) = \tilde{w}(\frac{t}{N^2}, x^{\frac{1}{N}}) \qquad \text{for all } x \in [0,1],$$

we get a classical solution for problem (PDE_m) that will be proved to be actually maximal.

As explained before, we will also need solutions of (PDE_m^ϵ), an approximate version of problem (PDE_m).

Definition 2.7. *Let $\epsilon > 0$, $m \geq 0$ and $T > 0$.*
We define problem (PDE_m^ϵ) with initial condition $u_0 \in Y_m$ by :

$$u_t = x^{2-\frac{2}{N}} u_{xx} + u f_\epsilon(u_x) \quad \textit{on } (0,T] \times (0,1) \tag{2.11}$$

$$u(0) = u_0 \tag{2.12}$$

$$u(t,0) = 0 \qquad \textit{for all } t \in [0,T] \tag{2.13}$$

$$u(t,1) = m \qquad \textit{for all } t \in [0,T] \tag{2.14}$$

A classical solution of problem (PDE_m^ϵ) with initial condition $u_0 \in Y_m$ on $[0,T]$ is a function

$$u^\epsilon \in C([0,T] \times [0,1]) \bigcap C^1((0,T] \times [0,1]) \bigcap C^{1,2}((0,T] \times (0,1])$$

such that (2.11)(2.12)(2.13)(2.14) *are satisfied.*
A classical solution of problem (PDE_m^ϵ) on $[0,T)$ is defined similarly.

We will see that each of the four problems we have described admits a unique maximal classical solution and we would like to fix now the notation we will use throughout this paper for these solutions.

Notation 2.1.

- *Let $u_0 \in Y_m$.*
  We denote u [resp. u^ϵ] the maximal classical solution of problem (PDE_m) [resp. (PDE_m^ϵ)] with initial condition u_0.
- *Let $w_0 = \theta_0(u_0)$ where $u_0 \in Y_m$.*
  We denote w [resp. w^ϵ] the maximal classical solution of problem $(tPDE_m)$ [resp. $(tPDE_m^\epsilon)$] with initial condition w_0.

3 Main results : local wellposedness, regularity and blow-up alternative for problem (PDE_m)

Definition 3.1. *For any real function f defined on $(0,1]$, we set*

$$\mathcal{N}[f] = \sup_{x\in(0,1]} \frac{f(x)}{x}.$$

Theorem 3.1. *Let $N \geq 2$, $q \in (0,1)$ and $m \geq 0$.*
Let $K > 0$ and $u_0 \in Y_m$ with $\mathcal{N}[u_0] \leq K$.

i) There exists $T_{max} = T_{max}(u_0) > 0$ and a unique maximal classical solution u of problem (PDE_m) with initial condition u_0.
Moreover, u satisfies the following condition :

$$\sup_{t\in(0,T]} \sqrt{t}\, \|u(t)\|_{C^1([0,1])} < \infty \text{ for any } T \in (0, T_{max})$$

ii) There exists $\tau = \tau(K) > 0$ such that $T_{max} \geq \tau$.
iii) Blow up alternative : $T_{max} = +\infty$ or $\lim_{t\to T_{max}} \mathcal{N}[u(t)] = +\infty$
iv) $u_x(t,0) > 0$ for all $t \in (0, T_{max})$.
v) If $0 < t_0 < T < T_{max}$ and $x_0 \in (0,1)$, then for any $\gamma \in (0,q)$,

$$u \in C^{1+\frac{\gamma}{2},2+\gamma}([t_0,T] \times [x_0,1])$$

vi) For all $t \in (0, T_{max})$, $u(t) \in Y_m^{1,\frac{2}{N}}$ where for any $\gamma > 0$,

$$Y_m^{1,\gamma} = \{u \in Y_m \cap C^1([0,1]),\ \sup_{x\in(0,1]} \frac{|u'(x) - u'(0)|}{x^\gamma} < \infty\}.$$

Remember that the radially symmetric cells density ρ is related to the derivative of u by :

$$\tilde{\rho}(t,x) = N^{\frac{2}{q}}\, u_x(N^2\, t, x^N) \qquad \text{for all } x \in [0,1].$$

We can have an idea of the shape of u_x, especially near the origin since we can show :

Proposition 3.1. *Let $u_0 \in Y_m$.*

i) For all $(t,x) \in (0, T_{max}(u_0)) \times [0,1]$,

$$u_x(t,x) = h(t, x^{\frac{1}{N}})$$

with $h \in C^{1,1}((0, T_{max}(u_0)) \times [0,1])$.

ii) Let $[t_0, T] \subset (0, T_{max}(u_0))$.
There exists $\delta > 0$ such that for all $(t,x) \in [t_0,T] \times [0,\delta]$,

$$u_x(t,x) = h(t, x^{\frac{1}{N}})$$

with $h \in C^{1,\infty}([t_0,T] \times [0,1])$ such that for any $t \in [t_0,T]$, $h(t,\cdot)$ has odd derivatives vanishing at $x = 0$.

iii) Let $t \in (0, T_{max}(u_0))$.
$u_x(t,x)$ admits an expansion of any order in powers of $x^{\frac{2}{N}}$ at $x = 0$.
For instance, $u_x(t,x) = a(t) + b(t)x^{\frac{2}{N}} + o(x^{\frac{2}{N}})$.

4 Additional results

4.1 Problem $(tPDE_m)$

Theorem 4.1. *Let $u_0 \in Y_m$ and $w_0 = \theta_0(u_0)$.*

i) There exists $T^* = T^*(w_0) > 0$ and a unique maximal classical solution w of problem $(tPDE)$ with initial condition w_0.
Moreover, w satisfies the following condition :

$$\sup_{t\in(0,T]} \sqrt{t}\, \|w(t)\|_{C^1(\overline{B})} < \infty \text{ for any } T \in (0, T^*).$$

ii) Blow-up alternative : $T^* = +\infty$ or $\lim_{t\to T^*} \|w(t)\|_{\infty,\overline{B}} = +\infty$.

iii) $w > 0$ on $(0, T^*) \times \overline{B}$.

iv) $w \in C^{1+\frac{\gamma}{2},2+\gamma}([t_0, T] \times \overline{B})$ for all $\gamma \in (0, q)$ and all $[t_0, T] \subset (0, T^*)$.

Connection with problem (PDE_m) :

$$T_{max}(u_0) = N^2 T^*(w_0)$$

and for all $(t, x) \in [0, T_{max}) \times [0, 1]$,

$$u(t, x) = x\, \tilde{w}(\frac{t}{N^2}, x^{\frac{1}{N}}) \tag{4.1}$$

where for any radially symmetric function f on B, we will denote $f(y) = \tilde{f}(|y|)$ for all $y \in B$.

4.2 Problem (PDE_m^ϵ)

Theorem 4.2. *Let $m \geq 0$, $\epsilon > 0$ and $K > 0$.*
Let $u_0 \in Y_m$ with $\mathcal{N}[u_0] \leq K$.

i) There exists $T_{max}^\epsilon = T_{max}^\epsilon(u_0) > 0$ and a unique maximal classical solution u^ϵ on $[0, T_{max}^\epsilon)$ of problem (PDE_m^ϵ) with initial condition u_0.
Moreover, u^ϵ satisfies the following condition :

$$\sup_{t\in(0,T]} \sqrt{t}\, \|u^\epsilon(t)\|_{C^1([0,1])} < \infty \text{ for all } T \in (0, T_{max}^\epsilon). \tag{4.2}$$

ii) There exists $\tau = \tau(K) > 0$ such that for all $\epsilon > 0$, $T_{max}^\epsilon \geq \tau$.
Moreover, there exists $C = C(K) > 0$ independent of ϵ such that

$$\sup_{t\in[0,\tau]} \mathcal{N}[u^\epsilon(t)] \leq C. \tag{4.3}$$

iii) Blow up alternative : $T_{max}^\epsilon = \infty$ or $\lim_{t\to T_{max}^\epsilon} \mathcal{N}[u^\epsilon(t)] = \infty$.

iv) $(u^\epsilon)_x > 0$ on $(0, T_{max}^\epsilon) \times [0, 1]$.

v) If $0 < t_0 < T < T^{\epsilon}_{max}$ *and* $x_0 \in (0,1)$*, then for any* $\gamma \in (0,1)$*,*

$$u^{\epsilon} \in C^{1+\frac{\gamma}{2},2+\gamma}([t_0,T] \times [x_0,1]).$$

vi) If $u_0 \in Y^{1,\gamma}_m$ *with* $\gamma > \frac{1}{N}$ *then* $u^{\epsilon} \in C([0,T^{\epsilon}_{max}), C^1([0,1]))$*.*

Connection with problem (PDE_m) **:**
Fixing an initial condition $u_0 \in Y_m$, the next lemma shows the convergence of maximal classical solutions u^{ϵ} of (PDE^{ϵ}_m) to the maximal classical solution of (PDE_m) in various spaces.
These results turned out to be essential in [P2] since, starting from a strict Lyapounov functional $\mathcal{F}_{\epsilon}$ for (PDE^{ϵ}_m) in the subcritical case (m less or equal to the critical mass M), we obtained a strict Lyapounov functional $\mathcal{F}$ for (PDE_m) by setting $\mathcal{F} = \lim\limits_{\epsilon \to 0} \mathcal{F}_{\epsilon}$. We point out that it does not seem possible to construct a Lyapunov functional for (PDE_m) by a direct approach (cf. p.7 in [P2]).

Lemma 4.1. *Let* $u_0 \in Y_m$*.*
i) $T_{max}(u_0) \leq T^{\epsilon}_{max}(u_0)$ *for any* $\epsilon > 0$*.*
ii) Let $[t_0,T] \subset (0, T_{max}(u_0))$*.*
$\alpha)$ $u^{\epsilon} \underset{\epsilon \to 0}{\longrightarrow} u$ *in* $C^{1,2}([t_0,T] \times (0,1])$*.*
Moreover, there exists $K > 0$ *independent of* ϵ *such that for all* $(t,x) \in [t_0,T] \times (0,1]$*,* $|u^{\epsilon}_{xx}| \leq \frac{K}{x^{1-q}}$*.*
$\beta)$ $(u^{\epsilon})_x \underset{\epsilon \to 0}{\longrightarrow} u_x$ *in* $C([t_0,T] \times [0,1])$*.*
$\gamma)$ $(u^{\epsilon})_t \underset{\epsilon \to 0}{\longrightarrow} u_t$ *in* $C([t_0,T] \times [0,1])$*.*

Connection with problem $(tPDE^{\epsilon}_m)$**.**
Let $u_0 \in Y_m$ and $w_0 = \theta_0(u_0)$. Then

$$T^{\epsilon}_{max}(u_0) = N^2 T^{*}_{\epsilon}(w_0).$$

Moreover, for all $(t,x) \in [0, T^{\epsilon}_{max}) \times [0,1]$,

$$u^{\epsilon}(t,x) = x\ \tilde{w}^{\epsilon}(\frac{t}{N^2}, x^{\frac{1}{N}}).$$

5 Proofs

5.1 Comparison principles

The four problems we have defined each admit a comparison principle which is in particular available for classical solutions.
Whence the uniqueness of the maximal classical solution in each case.

Lemma 5.1. ***Comparison principle for problem*** (PDE_m)
Let $T > 0$*. Assume that :*

- $u_1, u_2 \in C([0,T]\times[0,1]) \bigcap C^1((0,T]\times[0,1]) \bigcap C^{1,2}((0,T]\times(0,1))$.
- *For all* $t \in (0,T]$, $u_1(t)$ *and* $u_2(t)$ *are nondecreasing.*
- *There exists* $i_0 \in \{1,2\}$ *and some* $\gamma < \frac{1}{q}$ *such that :*

$$\sup_{t\in(0,T]} t^{\gamma} \, \|u_{i_0}(t)\|_{C^1([0,1])} < \infty.$$

Suppose moreover that :

$$(u_1)_t \le x^{2-\frac{2}{N}}(u_1)_{xx} + u_1 (u_1)_x^q \quad \textit{for all } (t,x) \in (0,T]\times(0,1). \tag{5.1}$$
$$(u_2)_t \ge x^{2-\frac{2}{N}}(u_2)_{xx} + u_2 (u_2)_x^q \quad \textit{for all } (t,x) \in (0,T]\times(0,1). \tag{5.2}$$
$$u_1(0,x) \le u_2(0,x) \quad \textit{for all } x \in [0,1]. \tag{5.3}$$
$$u_1(t,0) \le u_2(t,0) \quad \textit{for } t \ge 0. \tag{5.4}$$
$$u_1(t,1) \le u_2(t,1) \quad \textit{for } t \ge 0. \tag{5.5}$$

Then $u_1 \le u_2$ *on* $[0,T]\times[0,1]$.

Proof : Let us set $z = (u_1 - u_2)e^{-\int_0^t (\|u_{i_0}(s)\|_{C^1}^q + 1)ds}$. The hypotheses made show that $z \in C([0;T]\times[0;1]) \bigcap C^1((0,T]\times[0,1]) \bigcap C^{1,2}((0,T]\times(0,1))$.
Assume now by contradiction that $\max\limits_{[0;T]\times[0;1]} z > 0$.
By assumption, $z \le 0$ on the parabolic boundary of $[0,T]\times[0,1]$.
Hence, $\max\limits_{[0;T]\times[0;1]} z$ is reached at a point $(t_0,x_0) \in (0;T]\times(0;1)$.
Then $z_x(t_0,x_0) = 0$ so $(u_1)_x(t_0,x_0) = (u_2)_x(t_0,x_0)$.
Moreover, $z_{xx}(t_0,x_0) \le 0$ and $z_t(t_0,x_0) \ge 0$.
We have $z_t(t_0,x_0) \le x^{2-\frac{2}{N}} z_{xx}(t_0,x_0) + \left[(u_{i_0})_x(t_0,x_0)^q - \|u_{i_0}(t_0)\|_{C^1}^q - 1\right] z(t_0,x_0)$. The LHS of the inequality is nonnegative and the RHS is negative, whence the contradiction.

Remark 5.1. ***Comparison principle for problem*** (PDE_m^{ϵ})
Under the same assumptions (except the monotonicity of $u_1(t)$ *and* $u_2(t)$*), an analogous comparison principle is available for problem* (PDE_m^{ϵ}) *for any* $\epsilon > 0$.

Lemma 5.2. ***Comparison principle for problem*** $(tPDE_m)$
Let $T > 0$. *Assume that :*

- $w_1, w_2 \in C([0,T]\times[0,1]) \bigcap C^{1,2}((0,T]\times\overline{B})$.
- *For* $i = 1,2$, *for all* $(t,y) \in (0,T]\times\overline{B}$, $w_i(t,y) = \tilde{w}_i(t,|y|)$.
- *For* $i = 1,2$, *for all* $(t,y) \in (0,T]\times\overline{B}$, $w_i(t,y) + \frac{y.\nabla w_i(t,y)}{N} \ge 0$.
- *There exists* $i_0 \in \{1,2\}$ *and some* $\gamma < \frac{1}{q}$ *such that :*

$$\sup_{t\in(0,T]} t^{\gamma} \, \|\tilde{w}_{i_0}(t)\|_{C^1([0,1])} < \infty.$$

Suppose moreover that :

$$(w_1)_t \leq \Delta w_1 + N^2 w_1 (w_1 + \tfrac{y.\nabla w_1}{N})^q \quad \text{on } (0,T] \times \overline{B}. \tag{5.6}$$
$$(w_2)_t \geq \Delta w_2 + N^2 w_2 (w_2 + \tfrac{y.\nabla w_2}{N})^q \quad \text{on } (0,T] \times \overline{B}. \tag{5.7}$$
$$w_1(0,y) \leq w_2(0,y) \quad \text{for all } y \in \overline{B}. \tag{5.8}$$
$$w_1(t,y) \leq w_2(t,y) \quad \text{for all } (t,y) \in [0,T] \times \partial B. \tag{5.9}$$
$$\tag{5.10}$$

Then $w_1 \leq w_2$ *on* $[0,T] \times \overline{B}$.

Proof : For $i = 1, 2$, let us set

$$u_i(t,x) = x\ \tilde{w}_i(\frac{t}{N^2}, x^{\frac{1}{N}}). \tag{5.11}$$

Calculations show that, for $0 < t \leq T$ and $0 < x \leq 1$:

$$(u_i)_t(t,x) = \frac{x}{N^2}\ (\tilde{w}_i)_t(\frac{t}{N^2}, x^{\frac{1}{N}}). \tag{5.12}$$
$$\begin{aligned}(u_i)_x(t,x) &= \left[\tilde{w}_i + \frac{r(\tilde{w}_i)_r}{N}\right](\frac{t}{N^2}, x^{\frac{1}{N}}) \\ &= \left[w_i + \frac{y.\nabla w_i)}{N}\right](\frac{t}{N^2}, x^{\frac{1}{N}}).\end{aligned} \tag{5.13}$$
$$\begin{aligned}x^{2-\frac{2}{N}}(u_i)_{xx}(t,x) &= \frac{x}{N^2}\left[(\tilde{w}_i)_{rr} + \frac{N+1}{r}(\tilde{w}_i)_r\right](\frac{t}{N^2}, x^{\frac{1}{N}}) \\ &= \frac{x}{N^2}\ \Delta w_i(\frac{t}{N^2}, x^{\frac{1}{N}}).\end{aligned} \tag{5.14}$$

It is easy to check that

$$u_i \in C([0,N^2T] \times [0,1]) \bigcap C^1((0,N^2T] \times [0,1]) \bigcap C^{1,2}((0,N^2T] \times (0,1]).$$

Special attention has to be paid to the fact that u_i is C^1 up to $x = 0$ but this is clear because of (5.11) and (5.13).
Clearly, u_1 and u_2 satisfy all assumptions of Lemma 5.1, so $u_1 \leq u_2$ on $[0,T] \times [0,1]$. Then $w_1 \leq w_2$ on $[0,T] \times \overline{B}\backslash\{0\}$. But by continuity of w_1 and w_2, we get $w_1 \leq w_2$ on $[0,T] \times \overline{B}$.

Remark 5.2. *A similar comparison principle is available for problem* $(tPDE_m^\epsilon)$ *for any* $\epsilon > 0$ *(except that we do not have to suppose* $w_i(t,y) + \frac{y.\nabla w_i(t,y)}{N} \geq 0$ *for* $i = 1, 2$*).*

5.2 Preliminaries to local existence results

First, we would like to recall some notation and properties of the heat semigroup. For reference, see for instance the book [21] of A. Lunardi.

Notation 5.1.

- *B denotes the open unit ball in $\mathbb{R}^{N+2}$.*
- *$X_0 = \{W \in C(\overline{B}),\ W|_{\partial B} = 0\}$.*
- *$(S(t))_{t\geq 0}$ denotes the heat semigroup on X_0. It is the restriction on X_0 of the Dirichlet heat semigroup on $L^2(B)$.*
- *$(X_\theta)_{\theta\in[0;1]}$ denotes the scale of interpolation spaces for $(S(t))_{t\geq 0}$.*

Properties 5.1.

- *$X_{\frac{1}{2}} = \{W \in C^1(\overline{B}),\ W|_{\partial B} = 0\}$.*
- *Let $\gamma_0 \in (0; \frac{1}{2}]$. For any $\gamma \in [0, 2\gamma_0)$,*

$$X_{\frac{1}{2}+\gamma_0} \subset C^{1,\gamma}(\overline{B})$$

  with continuous embedding.
- *There exists $C_D \geq 1$ such that for any $\theta \in [0; 1]$, $W \in C(\overline{B})$ and $t > 0$,*

$$\|S(t)W\|_{X_\theta} \leq \frac{C_D}{t^\theta}\|W\|_\infty.$$

For reference, we recall some notation and then introduce two spaces of functions more in order to state a useful lemma on θ_0.

Notation 5.2. *Let $m \geq 0$ and $\gamma > 0$.*

- *For $W \in C^1(\overline{B})$, the C^1 norm of W is $\|W\|_{C^1} = \|W\|_{\infty,\overline{B}} + \|\nabla W\|_{\infty,\overline{B}}$.*
- *$Y_m = \{u \in C([0; 1])$ nondecreasing, $u'(0)$ exists, $u(0) = 0, u(1) = m\}$.*
- *$Z_m = \{w \in C(\overline{B}),\ w|_{\partial B} = m\}$.*
- *$Y_m^{1,\gamma} = \{u \in Y_m \cap C^1([0, 1]),\ \sup\limits_{x\in(0,1]} \frac{|u'(x)-u'(0)|}{x^\gamma} < \infty\}$.*
- *$Z_m^{1,\gamma} = \{w \in Z_m \cap C^1(\overline{B}),\ \sup\limits_{y\in\overline{B}\backslash\{0\}} \frac{|\nabla w(y)|}{|y|^\gamma} < \infty\}$.*
- $$\begin{array}{rl}\theta_0 : & Y_m \longrightarrow Z_m \\ & u \longrightarrow w \text{ where } w(y) = \frac{u(|y|^N)}{|y|^N} \text{ for all } y \in \overline{B}\backslash\{0\},\ w \text{ continuous on } \overline{B}.\end{array}$$
- *Let $(a, b) \in (0, 1)^2$. We denote $I(a, b) = \int_0^1 \frac{ds}{(1-s)^a s^b}$.*
  For all $t \geq 0$, $\int_0^t \frac{ds}{(t-s)^a s^b} = t^{1-a-b} I(a, b)$.

Lemma 5.3. *Let $m \geq 0$.*

i) θ_0 sends Y_m into Z_m.

ii) Let $\gamma > \frac{1}{N}$. θ_0 sends $Y_m^{1,\gamma}$ into $Z_m^{1,N\gamma-1}$.

Proof : i) Let $u \in Y_m$ and $w = \theta_0(u)$. Clearly, w can be extended in a continuous function on $\overline{B}$ by setting $w(0) = u'(0)$.
ii) Let $u \in Y_m^{1,\gamma}$. It is clear that $w \in C^1(\overline{B}\backslash\{0\})$.
Let $y \in \overline{B}\backslash\{0\}$. $w(y) = \int_0^1 u'(t|y|^N)dt = w(0) + \int_0^1 [u'(t|y|^N) - u'(0)]dt$.
Since $u \in Y_m^{1,\gamma}$, there exists $K > 0$ such that $|w(y) - w(0)| \leq K|y|^{N\gamma}$. Since $N\gamma > 1$, w is differentiable at $y = 0$ and $\nabla w(0) = 0$.

$\nabla w(y) = N\frac{y}{|y|^2}[u'(|y|^N) - w(y)] = N\frac{y}{|y|^2}[u'(|y|^N) - u'(0) + w(0) - w(y)]$ So $|\nabla w(y)| \leq 2NK|y|^{N\gamma-1}$.
Then $w \in C^1(\overline{B})$ and $\sup\limits_{y\in\overline{B}\setminus\{0\}} \frac{|\nabla w(y)|}{|y|^{N\gamma-1}} < \infty$, ie $w \in Z_m^{1,N\gamma-1}$.

Lemma 5.4. ***A density lemma.***
Let $u_0 \in Y_m$. There exists a sequence $(u_n) \in Y_m^{1,1}$ such that

$$\|u_n - u_0\|_{\infty,[0,1]} \underset{n\to\infty}{\longrightarrow} 0$$

and

$$\mathcal{N}(u_n) \leq \mathcal{N}(u_0).$$

Proof : Let $\epsilon > 0$. Let $\mathcal{T} = \{(x,y) \in \mathbb{R}^2,\ 0 \leq x \leq 1,\ 0 \leq y \leq \mathcal{N}[u_0]\, x\}$. The graph of u_0 lies inside $\mathcal{T}$. Since u_0 is uniformly continuous on $[0,1]$, for n_0 large enough, we can construct a nondecreasing piecewise affine function on $[0,1]$ $v \in Y_m$ such that $\|u_0 - v\|_{\infty,[0,1]} \leq \frac{\epsilon}{2}$ and for all $k = 0 \dots n_0$, $v(x_k) = u_0(x_k)$ where $x_k = \frac{k}{n_0}$ and v is affine between the successive points $P_k = (x_k, u(x_k))$.
Since $\mathcal{T}$ is convex and all points P_k are in $\mathcal{T}$ then the graph of v lies also inside $\mathcal{T}$.
We now just have to find a function $w \in C^2([0,1]) \bigcap Y_m$ whose graph is in $\mathcal{T}$ and such that $\|w - v\|_{\infty,[0,1]} \leq \frac{\epsilon}{2}$.
In order to do that, we extend v to a nondecreasing function $\overline{v}$ on $\mathbb{R}$: we simply extend the first and last segments $[P_0, P_1]$ and $[P_{n_0-1}, P_{n_0}]$ to a straight line, so that $\overline{v}$ is in particular affine on $(-\infty, \frac{1}{n_0}]$ and $[1 - \frac{1}{n_0}, +\infty)$.
Let $(\rho_\alpha)_{\alpha>0}$ a mollifiers family such that $\int_{\mathbb{R}} \rho_\alpha = 1$, $supp(\rho_\alpha) \subset [-\alpha, \alpha]$ and ρ_α is even.
Since $\overline{v}$ is Lipschitz continuous, there exists $\alpha_0 > 0$ such for all $(x,y) \in \mathbb{R}^2$, if $|x - y| \leq \alpha_0$ then $|\overline{v}(x) - \overline{v}(y)| \leq \frac{\epsilon}{2}$. Let $\alpha_1 = \min(\alpha_0, \frac{1}{2n_0})$ and $w = \rho_{\alpha_1} * \overline{v}$. Remark that since v is nondecreasing, so is w and $\|w - v\|_{\infty,[0,1]} \leq \frac{\epsilon}{2}$.
Note that, since $\int_{\mathbb{R}} y\rho_{\alpha_1}(y)dy = 0$, if for all $y \in [x - \alpha_1, x + \alpha_1]$, $\overline{v}(y) = ay + b$ (resp. $\overline{v}(y) \leq ay + b$) then $w(x) = ax + b$ (resp. $w(x) \leq ax + b$).
Since the graph of v lies inside $\mathcal{T}$ on $[0,1]$, this implies that the graph of w lies inside $\mathcal{T}$ on $[\dfrac{1}{2n_0}, 1 - \dfrac{1}{2n_0}]$.
Moreover, since $\overline{v}$ is affine on $(-\infty, \frac{1}{n_0}]$ and on $[1 - \frac{1}{n_0}, +\infty)$, then w is affine and coincides with v on $[0, \frac{1}{2n_0}]$ and on $[1 - \frac{1}{2n_0}, 1]$.
So $w \in Y_m$ and the graph of w on $[0,1]$ lies inside $\mathcal{T}$.
Finally, $\|w - u_0\|_{\infty,[0,1]} \leq \epsilon$ and $w \in C^2([0,1]) \bigcap Y_m \subset Y_m^{1,1}$.

5.3 Solutions of problem $(tPDE_m^\epsilon)$

Theorem 5.1. ***Wellposedness of problem*** $(tPDE_m^\epsilon)$***.***
Let $\epsilon > 0$ and $K > 0$.
Let $w_0 \in Z_m$ with $\|w_0\|_{\infty,\overline{B}} \leq K$.

i) There exists $T^*_\epsilon = T^*_\epsilon(w_0) > 0$ and a unique maximal classical solution w^ϵ of problem $(tPDE^\epsilon_m)$ on $[0, T^*_\epsilon)$ with initial condition w_0.
Moreover, w^ϵ satisfies the following condition :

$$\sup_{t\in(0,T]} \sqrt{t}\, \|w^\epsilon(t)\|_{C^1(\overline{B})} < \infty \text{ for all } T \in (0, T^*_\epsilon). \tag{5.15}$$

ii) We have the following blow-up alternative :

$$T^*_\epsilon = +\infty \quad \text{or} \quad \lim_{t\to T^*_\epsilon} \|w^\epsilon(t)\|_{\infty,\overline{B}} = +\infty.$$

iii) There exists $\tau = \tau(K) > 0$ such that for all $\epsilon > 0$, $T^*_\epsilon \geq \tau$.
Moreover, there exists $C = C(K) > 0$ independent of ϵ such that

$$\sup_{t\in[0,\tau]} \|w^\epsilon(t)\|_{\infty,\overline{B}} \leq C.$$

iv) There exist $\tau' = \tau'(K) > 0$ and $C' = C'(K) > 0$ both independent of ϵ such that

$$\sup_{t\in(0,\tau']} \sqrt{t}\|w^\epsilon(t)\|_{C^1(\overline{B})} \leq C'.$$

v) If $0 < t_0 < T < T^*_\epsilon$, then for any $\gamma \in (0, 1)$,

$$w^\epsilon \in C^{1+\frac{\gamma}{2},2+\gamma}([t_0, T] \times \overline{B}).$$

vi) $w^\epsilon > 0$ on $(0, T^*_{\epsilon'}) \times \overline{B}$.
vii) If $w_0 \in Z_m \bigcap C^1(\overline{B})$, then $w^\epsilon \in C([0, T^*_\epsilon), C^1(\overline{B}))$.

The proof of this theorem is based on a series of lemmas. We start with the following small time existence result for the auxiliary problem obtained by setting $W = w - m$ in $(tPDE^\epsilon_m)$.

Lemma 5.5. *Let $m \geq 0$, $\epsilon > 0$ and $W_0 \in X_0$.*
There exists $\tau = \tau(\epsilon) > 0$ and a unique mild solution

$$W^\epsilon \in C([0;\tau], X_0) \bigcap \{W \in L^\infty_{loc}((0;\tau], X_{\frac{1}{2}}),\ \sup_{t\in(0;\tau]} \sqrt{t}\, \|W(t)\|_{C^1} < \infty\}$$

of the following problem :

$$W_t = \Delta W + N^2(m + W)f_\epsilon(m + W + \frac{y.\nabla W}{N}) \quad \text{on} \quad (0, \tau] \times \overline{B} \tag{5.16}$$

$$W = 0 \quad \text{on} \quad [0, \tau] \times \partial B \tag{5.17}$$

$$W(0) = W_0 \tag{5.18}$$

More precisely, $W^\epsilon \in C((0, \tau], X_\gamma)$ for any $\gamma \in [\frac{1}{2}, 1)$.

Proof : Note that the initial data is singular with respect to the nonlinearity since the latter needs a first derivative but $W_0 \in X_0$. Although the argument is relatively well known, we give the proof for completeness. We shall adapt an argument given for instance in [26, theorem 51.25, p.495].
We define $E = E_1 \cap E_2$, where $E_1 = C([0;\tau]; X_0)$,
$E_2 = \{W \in L^\infty_{loc}((0;\tau], X_{\frac{1}{2}}),\ \sup_{t\in(0;\tau]} \sqrt{t}\, \|W(t)\|_{C^1} < \infty\}$
and τ will be made precise later. For $W \in E$, we define its norm :

$$\|W\|_E = \max\left[\sup_{t\in[0;\tau]} \|W(t)\|_\infty,\ \sup_{t\in(0;\tau]} \sqrt{t}\, \|W(t)\|_{C^1}\right]$$

and for $K \geq 0$ to be made precise later, we set $E_K = \{W \in E,\ \|W\|_E \leq K\}$.
E_K equipped with the metric induced by $\| \ \|_E$ is a complete space.
We now define $\Phi : E_K \longrightarrow E$ by

$$\Phi(W)(t) = S(t)W_0 + \int_0^t S(t-s)F_\epsilon(W(s))ds$$

where

$$F_\epsilon(W) = N^2(m+W)f_\epsilon(m+W+\frac{y.\nabla W}{N})$$

For the proof that $\Phi(W) \in C([0;\tau]; X_0) \bigcap C((0,\tau], X_\gamma)$ for any $\gamma \in [\frac{1}{2}, 1)$ when $W \in E$, we refer to [26], p.496 since the proof is similar.
Next, by properties of analytics semigroups and due to $0 < q < 2$, we get that for $t \in (0,\tau]$ and $W \in E_K$,

$$\|\Phi(W(t))\|_\infty \leq C_D\|W_0\|_\infty + \frac{\tau^{1-\frac{q}{2}}}{1-\frac{q}{2}} C_D N^2(m+K)(m\sqrt{\tau}+K)^q$$

$$\sqrt{t}\|\Phi(W(t))\|_{C^1} \leq C_D\|W_0\|_\infty + \tau^{1-\frac{q}{2}} C_D N^2(m+K)(m\sqrt{\tau}+K)^q I\left(\frac{1}{2}, \frac{q}{2}\right)$$

It is now obvious that Φ sends E_K into E_K provided that $K \geq 2C_D\|W_0\|_\infty$ and τ is small enough.
Let $(W_1, W_2) \in (E_K)^2$. We have

$$\begin{aligned} F_\epsilon(W_1) - F_\epsilon(W_2) = \quad & N^2 f_\epsilon(m+W_1+\tfrac{y.\nabla W_1}{N})[W_1 - W_2] \\ +N^2(W_2+m) \quad & \left[f_\epsilon(m+W_1+\tfrac{y.\nabla W_1}{N}) - f_\epsilon(m+W_2+\tfrac{y.\nabla W_2}{N})\right] \end{aligned}$$

Now, since $f_\epsilon \in C^1(\mathbb{R})$ and $|f'_\epsilon| \leq L_\epsilon$, we see that for any $s \in (0,\tau]$,

$$\begin{aligned} \|F_\epsilon(W_1(s)) - F_\epsilon(W_2(s))\|_\infty \leq \quad & N^2\frac{(m\sqrt{s}+K)^q}{s^{\frac{q}{2}}}\|(W_1-W_2)(s)\|_\infty \\ & +N^2(m+K)L_\epsilon\|(W_1-W_2)(s)\|_{C^1} \\ & \leq \beta_1(s)\|W_1-W_2\|_E \end{aligned}$$

where $\beta_1(s) = N^2 \left[\frac{(m\sqrt{s}+K)^q}{s^{\frac{q}{2}}} + (m+K)\frac{L_\epsilon}{\sqrt{s}} \right]$.

Let $t \in (0, \tau]$.

Since $\Phi(W_1)(t) - \Phi(W_2)(t) = \int_0^t S(t-s)[F_\epsilon(W_1(s)) - F_\epsilon(W_2(s))]ds$, we have

$$\|\Phi(W_1)(t) - \Phi(W_2)(t)\|_\infty \leq \beta_2 \|W_1 - W_2\|_E$$

$$\sqrt{t}\|\Phi(W_1)(t) - \Phi(W_2)(t)\|_{C^1} \leq \beta_3 \|W_1 - W_2\|_E$$

where

$$\beta_2 = C_D N^2 \left[\frac{\tau^{1-\frac{q}{2}}}{1-\frac{q}{2}}(m\sqrt{\tau}+K)^q + 2\sqrt{\tau}(m+K)L_\epsilon \right]$$

and

$$\beta_3 = C_D N^2 \left[\tau^{1-\frac{q}{2}}(m\sqrt{\tau}+K)^q I(\frac{1}{2}, \frac{q}{2}) + \sqrt{\tau}(m+K)L_\epsilon I(\frac{1}{2}, \frac{1}{2}) \right].$$

So, since $0 < q < 2$, Φ is a contraction for τ small enough. Hence, there exists a fixed point of Φ, that is to say a mild solution.

The uniqueness of the mild solution comes from the uniqueness of the fixed point given by the contraction mapping theorem.

Remark 5.3. *If $W_0 \in X_{\frac{1}{2}} = \{W \in C^1(\overline{B}),\ W|_{\partial B} = 0\}$, then a slight modification of the proof shows that $W^\epsilon \in C([0,\tau], C^1(\overline{B}))$. Indeed, we just have to replace the space E in the proof by $E = C([0,\tau], X_{\frac{1}{2}})$. Or, we also can refer to [26, theorem 51.7, p.470]. This remark will be helpful later for a density argument.*

Lemma 5.6. *Let $\epsilon > 0$ and $W_0 \in X_0$.*

i) There exists $T_\epsilon^* = T_\epsilon^*(W_0) > 0$ and a unique maximal

$$W^\epsilon \in C([0, T_\epsilon^*), X_0) \bigcap C^1((0, T_\epsilon^*), X_0) \bigcap C((0, T_\epsilon^*), X_1) \tag{5.19}$$

such that $W^\epsilon(0) = W_0$ and for all $t \in (0, T_\epsilon^*)$,

$$\frac{d}{dt}W^\epsilon(t) = \Delta W^\epsilon(t) + N^2(m + W^\epsilon(t))f_\epsilon \left[m + W^\epsilon(t) + \frac{y.\nabla W^\epsilon(t)}{N} \right] \tag{5.20}$$

$$\sup_{t\in(0,T]} \sqrt{t}\, \|W^\epsilon(t)\|_{C^1(\overline{B})} < \infty \text{ for all } T \in (0, T_\epsilon^*) \tag{5.21}$$

ii) Moreover, if $0 < t_0 < T < T_\epsilon^*$, then for any $\gamma \in (0,1)$,

$$W^\epsilon \in C^{\frac{1+\gamma}{2}, 2+\gamma}([t_0, T] \times \overline{B})$$

iii) In particular, $W^\epsilon \in C([0, T_\epsilon^*) \times \overline{B}) \bigcap C^{1,2}((0, T_\epsilon^*) \times \overline{B})$ is the unique maximal classical solution of the following problem :

$$W_t = \Delta W + N^2(m+W)f_\epsilon[m + W + \frac{y.\nabla W}{N}] \quad \text{on } (0, T_\epsilon^*) \times \overline{B} \tag{5.22}$$

$$W = 0 \quad \text{on } [0, T_\epsilon^*) \times \partial B \tag{5.23}$$

$$W(0) = W_0 \tag{5.24}$$

Moreover, W^ϵ satisfies

$$\sup_{t\in(0,T]} \sqrt{t}\, \|W^\epsilon(t)\|_{C^1(\overline{B})} < \infty \text{ for all } T \in (0, T^*_\epsilon) \tag{5.25}$$

and we have the following blow-up alternative :

$$T^*_\epsilon = +\infty \quad \text{or} \quad \lim_{t\to T^*_\epsilon} \|W^\epsilon(t)\|_\infty = +\infty$$

Proof : i) In the proof of Lemma 5.5, we notice that for fixed $\epsilon > 0$, the minimal existence time $\tau = \tau(\epsilon)$ is uniform for all $W_0 \in X_0$ such that $\|W_0\|_\infty \leq r$, where $r > 0$. Then a standard argument shows that there exists a unique maximal mild solution W^ϵ with existence time $T^*_\epsilon > 0$ of problem (5.16)(5.17)(5.18). It also gives the following blow-up alternative :

$$T^*_\epsilon = +\infty \quad \text{or} \quad \lim_{t\to T^*_\epsilon} \|W^\epsilon(t)\|_\infty = +\infty$$

For reference, see for instance [26, Proposition 16.1, p. 87-88].
Clearly, W^ϵ satisfies (5.21). Let us show that W^ϵ satisfies (5.19)(5.20).
Let $t_0 \in (0, T^*_\epsilon)$, $T \in (0, T^*_\epsilon - t_0)$ and $t \in [0, T]$. Then,

$$W^\epsilon(t_0 + t) = S(t)W^\epsilon(t_0) + \int_0^t S(t-s)F_\epsilon(W^\epsilon(t_0 + s))ds$$

Since $W^\epsilon \in C((0, T^*_\epsilon), X_{\frac{1}{2}})$, we have $\sup_{0\leq s\leq T} \|W^\epsilon(t_0 + s)\|_{C^1} < \infty$. Then,

$$F_\epsilon(W^\epsilon(t_0 + \cdot)) \in L^\infty((0, T), X_0)$$

We now apply [21, proposition 4.2.1, p.129] to get $W^\epsilon(t_0+\cdot) \in C^{\frac{1}{2}}([0, T], X_{\frac{1}{2}})$. Then,

$$F_\epsilon(W^\epsilon(t_0 + \cdot)) \in C^{\frac{1}{2}}([0, T], X_0)$$

We eventually apply [23, theorem 3.2, p.111] to conclude that W^ϵ satisfies (5.19)(5.20)(5.21) on any segment $[t_0, T] \subset (0, T^*_\epsilon)$, hence on $(0, T^*_\epsilon)$.
Conversely, since a solution of (5.19)(5.20)(5.21) is a mild solution, this proves the maximality and the uniqueness.
ii) Let $t_0 \in (0, T^*_\epsilon)$, $T \in (0, T^*_\epsilon - t_0)$ and $t \in [0, T]$. Since in particular $F_\epsilon(W^\epsilon(t_0 + \cdot)) \in C([0, T], X_0)$, then $F_\epsilon(W^\epsilon(t_0 + \cdot)) \in L^p([t_0, T] \times \overline{B})$ for any $p \geq 1$. Hence, since W^ϵ satisfies (5.20), by interior boundary L^p-estimates, we obtain that $W^\epsilon \in W^{1,2}_p([t_0, T] \times \overline{B})$ for any $1 \leq p < \infty$. Hence, by Sobolev's embedding theorem, (see for instance [18, p.26]) we see that

$$W^\epsilon \in C^{\frac{1+\gamma}{2}, 1+\gamma}([t_0, T] \times \overline{B}) \text{ for any } \gamma \in (0, 1)$$

Eventually, since $F_\epsilon(W^\epsilon(t_0 + \cdot)) \in C^{\frac{\gamma}{2},\gamma}([t_0, T] \times \overline{B})$ then by Schauder interior-boundary parabolic estimates,

$$W^\epsilon \in C^{1+\frac{\gamma}{2}, 2+\gamma}([t_0, T] \times \overline{B}) \text{ for any } \gamma \in (0, 1)$$

iii) Allmost all is obvious now. Since a classical solution is mild and a mild solution is classical as seen in i) and ii), then W^ϵ is also maximal in the sense of the classical solutions of (5.22)(5.23) and (5.24).
The uniqueness of the maximal classical solution comes from the uniqueness of the maximal mild solution.

Proof of theorem 5.1 : i)ii)v) The correspondence between the solutions of problem (5.22)(5.23)(5.24) and problem $(tPDE^\epsilon_m)$ is given by $w^\epsilon = W^\epsilon + m$. The previous lemma then gives the result. Note that the existence time is of course the same for both problems.
iii) Let us set $L = \max(K, m)$ and for $(t, x) \in [0, \frac{1}{qL^qN^2}) \times \overline{B}$,

$$\overline{w}(t, x) = \frac{L}{(1 - qL^qN^2t)^{\frac{1}{q}}}.$$

Obviously, $\overline{w}(t)|_{\partial B} \geq L \geq m$ for all $t \geq 0$ and $\overline{w}(0) = L \geq w_0$.
Moreover, $\overline{w}_t = N^2\overline{w}^{q+1} \geq \Delta\overline{w} + N^2\overline{w}f_\epsilon(\overline{w})$ since for all $x \in \mathbb{R}$, $|f_\epsilon(x)| \leq |x|^q$.
Then $\overline{w}$ is a supersolution for problem $(tPDE^\epsilon_m)$, so if $0 \leq t < \min(T^*_\epsilon, \frac{1}{2qL^qN^2})$, then $0 \leq w^\epsilon(t) \leq \overline{w}(t) \leq 2^{\frac{1}{q}}L$.
We set $\tau = \frac{1}{2qN^2L^q}$ and $C = 2^{\frac{1}{q}}L$. By blow-up alternative ii), we get

$$T^*_\epsilon \geq \tau \qquad \text{and} \qquad \sup_{t\in[0,\tau]} \|w^\epsilon(t)\|_{\infty,\overline{B}} \leq C$$

Note that τ and C depend on K, but is independent of ϵ.
iv) Noting $W_0 = w_0 - m$, then for $t \in [0, \tau]$,

$$w^\epsilon(t) = m + S(t)W_0 + \int_0^t S(t-s)N^2w^\epsilon f_\epsilon\left(w^\epsilon + \frac{x.\nabla w^\epsilon}{N}\right)ds$$

so

$$\|w^\epsilon(t)\|_{C^1} \leq m + \frac{C_D}{\sqrt{t}}(C + m) + N^2\int_0^t \frac{C_D}{\sqrt{t-s}}C\|w^\epsilon(s)\|^q_{C^1}ds$$

Setting $h(t) = \sup_{s\in(0,t]} \sqrt{s}\|w^\epsilon(s)\|_{C^1}$, we have $h(t) < \infty$ by (5.15) and

$$\sqrt{t}\|w^\epsilon(t)\|_{C^1} \leq m\sqrt{\tau} + C_D(C + m) + N^2CC_D\sqrt{t}\int_0^t \frac{1}{s^{\frac{q}{2}}\sqrt{t-s}}h(s)^q ds$$

$$\sqrt{t}\|w^\epsilon(t)\|_{C^1} \leq m\sqrt{\tau} + C_D(m + C) + N^2CC_DI\left(\frac{1}{2}, \frac{q}{2}\right)t^{1-\frac{q}{2}}h(t)^q$$

Let $T \in (0, \tau]$. Then,

$$h(T) \leq m\sqrt{\tau} + C_D(m + C) + N^2CC_DI\left(\frac{1}{2}, \frac{q}{2}\right)T^{1-\frac{q}{2}}h(T)^q \tag{5.26}$$

Setting $A = m\sqrt{\tau} + C_D(m+C)$ and $B = N^2 C C_D I\left(\frac{1}{2}, \frac{q}{2}\right) 2^q$, assume that there exists $T \in [0,\tau]$ such that $h(T) = 2A$. Then,

$$A^{1-q} \leq BT^{1-\frac{q}{2}} \text{ which implies } T \geq \left(\frac{A^{1-q}}{B}\right)^{\frac{1}{1-q}}$$

Let us set $\tau' = \min\left(\tau, \frac{1}{2}\left(\frac{A^{1-q}}{B}\right)^{\frac{1}{1-q}}\right)$.
Since $h \geq 0$ is nondecreasing, $h_0 = \lim_{t\to 0^+} h(t)$ exists and $h_0 \leq A$ by (5.26). So by continuity of h on $(0,\tau']$, $h(t) \leq 2A$ for all $t \in (0,\tau']$, that is to say :

$$\|w^\epsilon(t)\|_{C^1} \leq \frac{2A}{\sqrt{t}} \text{ for all } t \in (0,\tau']$$

where A and τ' only depend on K.
vi) 0 is a subsolution of problem $(tPDE^\epsilon_m)$. Then by comparison principle $w^\epsilon \geq 0$. The strong maximum principle implies that $w^\epsilon > 0$ on $(0, T^*_{\epsilon'}) \times \overline{B}$ (see [9, theorem 5, p.39]).
vii) This fact is a consequence of remark 5.3.

5.4 Solutions of problem (PDE^ϵ_m)

Using the connection with problem $(tPDE^\epsilon_m)$ through the transformation θ_0, we shall now provide the proof of Theorem 4.2.

Proof : i) The uniqueness of the classical maximal solution for problem (PDE^ϵ_m) comes from the comparison principle for this problem.
We shall now exhibit a classical solution of problem (PDE^ϵ_m) satisfying (4.2) and will prove in i)bis) that it is maximal.
Let us set $w_0 = \theta_0(u_0)$.
Remind that $w^\epsilon \in C([0,T^*_\epsilon) \times \overline{B}) \bigcap C^{1,2}((0,T^*_\epsilon) \times \overline{B})$.
Remark that a classical solution of $(tPDE^\epsilon_m)$ composed with a rotation is still a classical solution. Then by uniqueness, since w_0 is radial, so is $w^\epsilon(t)$ for all $t \in [0, T^*_\epsilon)$.
Hence, $w^\epsilon(t,y) = \tilde{w}^\epsilon(t, \|y\|)$ for all $(t,y) \in [0,T^*_\epsilon) \times \overline{B}$. Let us define :

$$u^\epsilon(t,x) = x\, \tilde{w}^\epsilon\left(\frac{t}{N^2}, x^{\frac{1}{N}}\right) \text{ for } (t,x) \in [0, N^2 T^*_\epsilon) \times [0,1] \tag{5.27}$$

Since

$$(u^\epsilon)_t(t,x) = \frac{x}{N^2}\, (\tilde{w}^\epsilon)_t \left(\frac{t}{N^2}, x^{\frac{1}{N}}\right) \tag{5.28}$$

$$\begin{aligned} (u^\epsilon)_x(t,x) &= \left[\tilde{w}^\epsilon + \frac{r(\tilde{w}^\epsilon)_r}{N}\right]\left(\frac{t}{N^2}, x^{\frac{1}{N}}\right) \\ &= \left[\tilde{w}^\epsilon + \frac{y.\nabla \tilde{w}^\epsilon)}{N}\right]\left(\frac{t}{N^2}, x^{\frac{1}{N}}\right) \end{aligned} \tag{5.29}$$

$$\begin{aligned} x^{2-\frac{2}{N}}(u^\epsilon)_{xx}(t,x) &= \frac{x}{N^2}\left[(\tilde{w}^\epsilon)_{rr} + \frac{N+1}{r}(\tilde{w}^\epsilon)_r\right]\left(\frac{t}{N^2}, x^{\frac{1}{N}}\right) \\ &= \frac{x}{N^2}\,\Delta w^\epsilon\left(\frac{t}{N^2}, x^{\frac{1}{N}}\right) \end{aligned} \tag{5.30}$$

it is easy to check that

$$u^\epsilon \in C([0, N^2T^*_\epsilon) \times [0,1]) \bigcap C^1((0, N^2T^*_\epsilon) \times [0,1]) \bigcap C^{1,2}((0, N^2T^*_\epsilon) \times (0,1])$$

is a classical solution of problem (PDE^ϵ_m) on $[0, N^2T^*_\epsilon)$.
Special attention has to be paid to the fact that u^ϵ is C^1 up to $x = 0$ but this is clear because of (5.27) and (5.29).
Since $\|w_0\|_{\infty,\overline{B}} = \mathcal{N}[u_0] \le K$, formula (5.29) and theorem 5.1 iii)iv) imply that there exist $\tau' = \tau'(K) \in (0,1]$ and $C = C(K) > 0$ independent of ϵ such that $\sup_{t\in(0,\tau']} \|w^\epsilon(t)\|_{\infty,\overline{B}} + \sup_{t\in(0,\tau']} \sqrt{t}\,\|w^\epsilon(t)\|_{C^1,\overline{B}} \le C$. Then,

$$\sup_{t\in(0,\tau']} \sqrt{t}\,\|u^\epsilon(t)\|_{C^1([0,1])} \le C \tag{5.31}$$

It is also clear from formula (5.27) that $T^\epsilon_{max} \ge N^2T^*_\epsilon$.
ii) From theorem 5.1 iv), $T^*_\epsilon \ge \tau'$, then u^ϵ is at least defined on $[0, N^2\tau']$ and can be extended to a maximal solution. This minimal existence time $\tau = N^2\tau'$ only depends on K.
Moreover, by formula (5.27),

$$\sup_{t\in[0,\tau]} \mathcal{N}[u^\epsilon(t)] = \sup_{(t,y)\in[0,\tau']\times\overline{B}\setminus\{0\}} w^\epsilon(t,y) = \sup_{t\in[0,\tau']} \|w^\epsilon(t)\|_{\infty,\overline{B}} \le C$$

i)bis) If $T^*_\epsilon = \infty$, then formula (5.27) gives a global solution u^ϵ then $T^\epsilon_{max} = \infty$.
Suppose $T^*_\epsilon < \infty$.
Assume that $T^\epsilon_{max} > N^2T^*_\epsilon$ with maybe $T^\epsilon_{max} = \infty$. Then, there exists in particular a classical solution u^ϵ of (PDE^ϵ_m) on $[0, N^2T^*_\epsilon]$. By uniqueness, on $[0, N^2T^*_\epsilon)$, u^ϵ coincides with the solution given by (5.27). By blow-up alternative for w^ϵ, $\lim_{t\to T^*_\epsilon} \|w^\epsilon(t)\|_{\infty,\overline{B}} = \infty$ thus $\lim_{t\to N^2T^*_\epsilon} \mathcal{N}[u^\epsilon(t)] = \infty$.
But, since (4.3) and $u^\epsilon \in C([\tau', N^2T^*_\epsilon], C^1([0,1]))$ then

$$\sup_{t\in[0,N^2T^*_\epsilon]} \mathcal{N}[u^\epsilon(t)] < \infty$$

which provides a contradiction. Whence i).
Moreover, this proves that the solution u^ϵ is actually maximal.
iii) The blow-up alternative for problem (PDE^ϵ_m) follows directly from $i)bis)$ and from the blow-up alternative for problem $(tPDE^\epsilon_m)$.
iv) This point needs some work that will be done in the next lemma.
v) This follows from Theorem 5.1 iv) and formulas (5.27)(5.28)(5.29) (5.30).

vi) This follows from Lemma 5.3)i), Theorem 5.1 vii) and formula (5.29).

The next lemma, whose proof is rather technical, is very important since it shows that $(u^\epsilon)_x > 0$ on $(0, T^\epsilon_{max}) \times [0,1]$, which will imply later that solutions of (PDE_m) at time t are nondecreasing. Moreover, this fact is essential in [P2] in order to prove that some functional $\mathcal{F}_\epsilon$ is a strict Lyapunov functional for the dynamiacl system induced by problem (PDE^ϵ_m).

Lemma 5.7. *Let $\epsilon > 0$, $u_0 \in Y_m$ and $w_0 = \theta_0(u_0)$.*
Let us set $T^\epsilon_{max} = T^\epsilon_{max}(u_0)$ and $T^*_\epsilon = T^*_\epsilon(w_0)$.
i) $0 \le u^\epsilon \le m$ on $[0, T^\epsilon_{max}) \times [0,1]$.
ii) $u^\epsilon \in C^{1,3}((0, T^\epsilon_{max}) \times (0,1])$ and $u^\epsilon \in C^2((0, T^\epsilon_{max}) \times (0,1])$ (not optimal).
iii) For all $(t,x) \in (0, T^\epsilon_{max}) \times [0,1]$, $(u^\epsilon)_x(t,x) > 0$.
iv) $w^\epsilon + \frac{y.\nabla w^\epsilon}{N} > 0$ for any $(t,y) \in (0, T^*_\epsilon) \times \overline{B}$.

Proof : *i)* 0 and m are respectively sub- and supersolution for problem (PDE^ϵ_m) satisfied by u^ϵ. Whence the result by comparison principle.

ii) Let $[t_0, T] \subset (0, T^\epsilon_{max})$.
Let $w_0 = \theta_0(u_0)$. We set $t'_0 = \frac{t_0}{N^2}$ and $T' = \frac{T}{N^2}$. We now refer to [9, p.72, Theorem 10] and apply it to $D = (t'_0, T') \times \overline{B}$.
We recall that w^ϵ satisfies on D

$$w_t = \Delta w + c\, w \tag{5.32}$$

with $c = N^2 f_\epsilon(w + \frac{y.\nabla w}{N})$. Let $\gamma \in (0,1)$. ∇c is Hölder continuous with exponent γ in D because $w^\epsilon \in C^{1+\frac{\gamma}{2}, 2+\gamma}([t'_0, T'] \times \overline{B})$ and f'_ϵ is Lipschitz continuous on compact sets of $\mathbb{R}$. Then $\partial_t \nabla w^\epsilon$ and $\partial_\alpha w^\epsilon$ are Hölder continuous with exponent γ in D for any multi-index $|\alpha| \le 3$. Thus,
$w \in C^{1,3}([t'_0, T'] \times \overline{B})$, so $u^\epsilon \in C^{1,3}([t_0, T] \times (0,1])$ by formula (5.27).
We apply the same theorem again : $\partial_\alpha c$ is Hölder continuous with exponent γ for any $|\alpha| \le 2$ since f''_ϵ is Lipschitz continuous on compact sets of $\mathbb{R}$. So, $\partial_t \partial_\alpha w$ is Hölder continuous with exponent γ for any $|\alpha| \le 2$. Then, c_t and $\partial_t \Delta w$ are continuous so by (5.32), w_{tt} is continuous.
By (5.32) again, it is clear that $\partial_\alpha \partial_t w$ is continuous for $|\alpha| \le 1$ hence $w \in C^2([t'_0, T'] \times \overline{B})$. It follows from formula (5.27) that $u^\epsilon \in C^2([t_0, T] \times (0,1])$.
In particular, $(u^\epsilon)_{t,x} = (u^\epsilon)_{x,t}$.
iii) Let $T \in (0, T^\epsilon_{max})$.
We prove the result in two steps.
First step : We now show that $v^\epsilon := u^\epsilon_x \ge 0$ on $[0,T] \times [0,1]$.
We divide the proof in three parts.

- First part : We show the result for any $u_0 \in Y^{1,\gamma}_m$ where $\gamma > \frac{1}{N}$.
  Since u^ϵ satisfies on $(0,T] \times (0,1]$

$$u^\epsilon_t = x^{2-q} u^\epsilon_{xx} + u^\epsilon f_\epsilon(u^\epsilon_x) \tag{5.33}$$

and thanks to ii), we can now differentiate this equation with respect to x. We denote

$$b = \left[(2 - \frac{2}{N})x^{1-\frac{2}{N}} + u^\epsilon f_\epsilon'(v)\right]$$

and obtain the partial differential equation satisfied by v^ϵ :

$$v_t = x^{2-\frac{2}{N}} v_{xx} + b\, v_x + f_\epsilon(v)v \quad \text{on} \quad (0,T)\times(0,1) \tag{5.34}$$

$$v(0,\cdot\,) = (u_0)' \tag{5.35}$$

$$v(t,0) = u_x^\epsilon(t,0) \quad \text{for} \quad t\in(0,T] \tag{5.36}$$

$$v(t,1) = u_x^\epsilon(t,1) \quad \text{for} \quad t\in(0,T] \tag{5.37}$$

By Theorem 4.2 vii), we know that $u^\epsilon \in C([0,T],C^1([0,1]))$, then $v^\epsilon \in C([0,T]\times[0,1])$ and v^ϵ reaches its minimum on $[0,T]\times[0,1]$.
From $i)$ follows that $u_x^\epsilon(t,0)\geq 0$ and $u_x^\epsilon(t,1)\geq 0$ for all $t\in(0,T]$. Then, from (5.35), (5.36) and (5.37), $v^\epsilon \geq 0$ on the parabolic boundary of $[0,T]\times[0,1]$. From (5.34), we see that v^ϵ cannot reach a negative minimum in $(0,T]\times(0,1)$ since for all $x\neq 0$, $x\, f_\epsilon(x) > 0$. So $v^\epsilon \geq 0$ on $[0,T]\times[0,1]$.

– Second part : We show that if $u_0\in Y_m$, there exists $\tau>0$ such that for all $t\in[0,\tau]$, $u^\epsilon(t)$ is non decreasing on $[0,1]$.
Let $u_0 \in Y_m$. From Lemma 5.4, there exists a sequence $(u_n)_{n\geq 1}$ of $Y_m^{1,1}$ such that $\|u_n - u_0\|_{\infty,[0,1]} \underset{n\to\infty}{\longrightarrow} 0$ and $\mathcal{N}[u_n]\leq \mathcal{N}[u_0]$.
By Theorem 4.2 ii), there exists a common small existence time $\tau>0$ for all solutions $(u_n^\epsilon)_{n\geq 0}$ of problem (PDE_m^ϵ) with initial condition u_n. From first part, we know that for all $t\in[0,\tau]$ $u_n^\epsilon(t)$ is a nondecreasing function since $u_n\in Y_m^{1,1}$.
To prove the result, it is sufficient to show that $\|u_n^\epsilon - u^\epsilon\|_{\infty,[0,1]\times[0,\tau]} \underset{n\to\infty}{\longrightarrow} 0$.
Let $\eta>0$. By (4.2), there exists $C>0$ such that for all $t\in[0,\tau]$, $\|(u^\epsilon(t))_x\|_\infty \leq \frac{C}{\sqrt{t}}$. So we can choose $\eta'>0$ such that

$$\eta' e^{\int_0^\tau [\|(u^\epsilon(t))_x\|_\infty^q + 1]\, dt} \leq \eta$$

Let $n_0\geq 1$ such that for all $n\geq n_0$, $\|u_n - u_0\|_{\infty,[0,1]}\leq \eta'$. Let $n\geq n_0$.
Let us set

$$z(t) = [u_n^\epsilon(t) - u^\epsilon(t)]e^{-\int_0^\tau [\|(u^\epsilon(t))_x\|_\infty^q + 1]\, dt}$$

We see that z satisfies

$$z_t = x^{2-\frac{2}{N}} z_{xx} + b\, z_x + c\, z \tag{5.38}$$

where $b = u_n^\epsilon \dfrac{f_\epsilon((u_n^\epsilon)_x) - f_\epsilon((u^\epsilon)_x)}{(u_n^\epsilon)_x - (u^\epsilon)_x}$ if $(u_n^\epsilon)_x \neq (u^\epsilon)_x$ and 0 else,
$c = [f_\epsilon((u^\epsilon)_x) - \|(u^\epsilon)_x\|_\infty^q - 1] < 0$.
Since $z\in C([0,\tau]\times[0,1])$, z reaches its maximum and its minimum.
Assume that this maximum is greater than η'. Since $z=0$ for $x=0$ and $x=1$ and $z\leq\eta'$ for $t=0$, it can be reached only in $(0,\tau]\times(0,1)$ but this

is impossible because $c < 0$ and (5.38). We make the similar reasoning for the minimum. Hence, $|z| \leq \eta'$ on $[0, \tau] \times [0, 1]$.
Eventually, $\|u^\epsilon_n - u^\epsilon\|_{\infty,[0,1]\times[0,\tau]} \leq \eta' e^{\int_0^\tau [\|(u^\epsilon(t))_x\|^q_\infty + 1]\, dt} \leq \eta$ for all $n \geq n_0$. Whence the result.

– Last part : Let $u_0 \in Y_m$. From the second part, there exists $\tau > 0$ such that that for all $t \in [0, \tau]$, $u^\epsilon(t)$ is nondecreasing. Since $u^\epsilon \in C([\tau, T^\epsilon_{max}), C^1([0, 1]))$ and $u_0(\tau)$ is nondecreasing, we can apply the same argument as in the first part to deduce that for all $t \in [\tau, T^\epsilon_{max})$, $u^\epsilon(t)$ is nondecreasing. That concludes the proof of the second step.

Second step : Let us show that $v^\epsilon > 0$ on $(0, T] \times [0, 1]$.
First, from formula (5.29) and Theorem 5.1 vi) follows that $v^\epsilon(t, 0) = (u^\epsilon)_x(t, 0) > 0$ for $t \in (0, T]$.
Assume by contradiction that v^ϵ is zero at some point in $(0, T) \times (0, 1)$.
Let $z = v^\epsilon e^{-\int_0^t [\|v^\epsilon(s)\|^q_\infty + 1]} ds \geq 0$ by second step. z reaches its minimum and satisfies the following equation :

$$z_t = x^{2-\frac{2}{N}} z_{xx} + [(2 - \frac{2}{N})x^{1-\frac{2}{N}} + u^\epsilon f'_\epsilon(v^\epsilon)]z_x + [f_\epsilon(v^\epsilon) - \|v^\epsilon(s)\|^q_\infty - 1]z \qquad (5.39)$$

where $f_\epsilon(v^\epsilon) - \|v^\epsilon(s)\|^q_\infty - 1 \leq -1$ on $[0, T] \times [0, 1]$.
Then, by the strong minimum principle ([9], p.39, Theorem 5) applied to z, we deduce that $v^\epsilon = 0$ on $(0, T) \times (0, 1)$. Then, by continuity, $v^\epsilon(t, 0) = 0$ for $t \in (0, T)$ which contradicts the previous assertion.
Suppose eventually that $v^\epsilon(t, 1) = 0$ for some $t \in (0, T)$. From (5.33), we deduce that $(u^\epsilon)_{xx}(t, 1) = 0$, ie $v^\epsilon_x(t, 1) = 0$.
Since $f_\epsilon(y)y \geq 0$ for all $y \in \mathbb{R}$, we observe that v^ϵ satisfies :

$$v_t \geq x^{2-\frac{2}{N}} v_{xx} + [(2 - \frac{2}{N})x^{1-\frac{2}{N}} + u^\epsilon f'_\epsilon(v)]v_x \qquad (5.40)$$

Since $v^\epsilon > 0$ on $(0, T) \times [\frac{1}{2}, 1)$ and the underlying operator in the above equation is uniformly parabolic on $(0, T) \times [\frac{1}{2}, 1]$, we can apply Hopf's minimum principle (cf. [25, Theorem 3, p.170]) to deduce that $v^\epsilon_x(t, 1) < 0$ what yields a contradiction. In conclusion, $(u^\epsilon)_x > 0$ on $(0, T] \times [0, 1]$ for all $T < T^\epsilon_{max}$, whence the result.

iv) It is clear from iii) thanks to formula (5.29).

We can now deduce the following monotonicity property which will be useful in order to find a solution of problem (PDE_m) by letting ϵ go to zero.

Lemma 5.8. *Let $u_0 \in Y_m$ and $w_0 = \theta_0(u_0)$.*
i) If $\epsilon' < \epsilon$, then $T^*_{\epsilon'} \leq T^*_\epsilon$ and $w^{\epsilon'} \geq w^\epsilon$ on $[0, T^*_{\epsilon'}) \times \overline{B}$.
ii) If $\epsilon' < \epsilon$, then $T^{\epsilon'}_{max} \leq T^\epsilon_{max}$ and $u^{\epsilon'} \geq u^\epsilon$ on $[0, T^{\epsilon'}_{max}) \times [0, 1]$.

Proof : i) $w^{\epsilon'}$ is a supersolution for $(tPDE^\epsilon_m)$ since $w^{\epsilon'} + \frac{y.\nabla w^{\epsilon'}}{N} \geq 0$ for all $(t, y) \in [0, T^*_{\epsilon'}) \times \overline{B}$ and $f_{\epsilon'} \geq f_\epsilon$ on $[0, +\infty)$ for $\epsilon' < \epsilon$. Using the blow-up alternative

for problem $(tPDE_m^\epsilon)$, we get the result by contradiction.
ii) It is clear from i) using the relation between u^ϵ and w in Theorem 4.2 iii). We could as well use a comparison argument as in i).

Remark 5.4. *$(w^\epsilon)_{\epsilon\in(0,1)}$ (resp. $(u^\epsilon)_{\epsilon\in(0,1)}$) is then a nondecreasing family of functions for ϵ decreasing, with an existence time maybe shorter and shorter but not less than a given $\tau > 0$ depending on $\|w_0\|_\infty$ (resp. $\mathcal{N}[u_0]$).*

5.5 Solutions of problem $(tPDE_m)$ and proof of Theorem 3.1

We shall now prove Theorem 4.1, i.e. the local in time wellposedness of problem $(tPDE_m)$.

The small time existence part is obtained by passing to the limit ϵ to 0 in problem $(tPDE_m^\epsilon)$ via the following lemma :

Lemma 5.9. ***Local existence of a classical solution for problem*** $(tPDE_m)$
Let $w_0 = \theta_0(u_0)$ where $u_0 \in Y_m$ with $\|w_0\|_{\infty,\overline{B}} \leq K$.
There exists $\tau' = \tau'(K) > 0$ and $w \in C([0,\tau'] \times \overline{B}) \bigcap C^{1,2}((0,\tau'] \times \overline{B})$ such that $w^\epsilon \underset{\epsilon\to 0}{\longrightarrow} w$ in $C([0,\tau'] \times \overline{B})$ and in $C^{1,2}((0,\tau'] \times \overline{B})$.
Moreover, w is the unique classical solution of problem $(tPDE_m)$ on $[0,\tau']$ and satisfies the following condition :

$$\sup_{t\in(0,\tau']} \sqrt{t}\, \|w(t)\|_{C^1([0,1])} < \infty$$

Proof :
<u>First step :</u> From Theorem 5.1 $iii)iv)$, there exists $\tau' = \tau'(K) > 0$ and $C = C(K) > 0$ both independent of ϵ such that

$$\sup_{t\in[0,\tau']} \|w^\epsilon(t)\|_{\infty,\overline{B}} \leq C$$

$$\sup_{t\in(0,\tau']} \sqrt{t}\|w^\epsilon(t)\|_{C^1(\overline{B})} \leq C$$

Let $t_0 \in (0,\tau']$. Recall that $F_\epsilon(w) = N^2 w f_\epsilon(w + \frac{x.\nabla w}{N})$.
We see that for all $\epsilon > 0$ and $t \in [t_0,\tau']$, $\|w^\epsilon(t)\|_{C^1} \leq \frac{C}{\sqrt{t_0}}$ where C is independent of ϵ.
If $x \geq 0$, $0 \leq f_\epsilon(x) \leq x^q$, so there exists $C' > 0$ which depends on t_0 but is independent of ϵ such that

$$\|F_\epsilon(w^\epsilon)\|_{\infty,[t_0,\tau']\times\overline{B}} \leq C'$$

then for any $p \geq 1$, $\|F_\epsilon(w^\epsilon)\|_{L^p([t_0,\tau']\times\overline{B})} \leq C''$ where C'' depends on t_0 but is independent of ϵ.

We can now use the L^p estimates, then Sobolev embedding and eventually interior-boundary Schauder estimates to obtain that for any $\gamma \in [0, q)$,

$$\|w^\epsilon\|_{C^{1+\frac{\gamma}{2},2+\gamma}([t_0,\tau']\times\overline{B})} \leq C'''$$

where C''' depnds on t_0 but is independent of ϵ since f_ϵ is Hölder continuous with exponent q on $[0, +\infty)$ and Hölder coefficient less or equal to 1.
We now use a sequence $t_k \underset{k\to\infty}{\longrightarrow} 0$ and the Ascoli's theorem for each k and eventually proceed to a diagonal extraction to get a sequence $\epsilon_n \underset{n\to\infty}{\longrightarrow} 0$ such that

$$w^{\epsilon_n} \underset{n\to\infty}{\longrightarrow} w$$

in $C^{1,2}([t_k, \tau'] \times \overline{B})$ for some function w, for each k. So,

$$w \in C^{1,2}((0, \tau'] \times \overline{B}) \tag{5.41}$$

Since by Lemma 5.8 i), w^ϵ is nondecreasing as ϵ decreases to 0, then $w = \lim\limits_{\epsilon\to 0^+} w^\epsilon$ on $(0, \tau'] \times \overline{B}$. w is then unique. Hence,

$$w^\epsilon \underset{\epsilon\to 0}{\longrightarrow} w \qquad \text{in } C^{1,2}([t_0, \tau'] \times \overline{B}) \text{ for each } t_0 \in (0, \tau']$$

For a fixed $s \in (0, \tau']$, $w^\epsilon(s) \underset{\epsilon\to 0}{\longrightarrow} w(s)$ in $C^1(\overline{B})$, then Lemma 5.7 iv) implies

$$w + \frac{y.\nabla w}{N} \geq 0 \quad \text{ on } (0, \tau'] \times \overline{B} \tag{5.42}$$

Moreover, the both following estimates are clear :

$$\|w(t)\|_{\infty,\overline{B}} \leq C \text{ for all } t \in (0, \tau'] \tag{5.43}$$

$$\|w(t)\|_{C^1} \leq \frac{C}{\sqrt{t}} \text{ for all } t \in (0, \tau'] \tag{5.44}$$

Second step : Let us show that $w \in C([0, \tau'] \times \overline{B})$ and that

$$w^\epsilon \underset{\epsilon\to 0}{\longrightarrow} w \qquad \text{in } C([0, \tau'] \times \overline{B})$$

First, remark that from Dini's theorem, the second part is obvious once the first one is known since w^ϵ is nondecreasing on the compact set $[0, \tau'] \times \overline{B}$ and w^ϵ converges pointwise to the continuous function w.
Let $t \in (0, \tau']$. Let us set $W_0 = w_0 - m$. We have

$$w^\epsilon(t) = m + S(t)W_0 + \int_0^t S(t-s)F_\epsilon(w^\epsilon(s))ds$$

$(w^\epsilon + \frac{y.\nabla w^\epsilon}{N})(t, x) \geq 0$ hence $f_\epsilon(w^\epsilon + \frac{y.\nabla w^\epsilon}{N}) = (w^\epsilon + \frac{y.\nabla w^\epsilon}{N} + \epsilon)^q - \epsilon^q$.

Let $s \in (0,t)$. Clearly, $F_\epsilon(w^\epsilon(s)) \underset{\epsilon\to 0}{\longrightarrow} N^2 w(s)\left(w(s)+\frac{y.\nabla w(s)}{N}\right)^q$ in $C(\overline{B})$.
By continuous dependence of the heat semi-group on $C_0(\overline{B})$ with respect to the initial data, we have

$$S(t-s)F_\epsilon(w^\epsilon(s)) \underset{\epsilon\to 0}{\longrightarrow} S(t-s)N^2 w(s)\left(w(s)+\frac{y.\nabla w(s)}{N}\right)^q \quad \text{in } C(\overline{B})$$

Moreover, we have a uniform domination for all $\epsilon \in (0,1)$ since

$$\|S(t-s)F_\epsilon(w^\epsilon(s))\|_{\infty,\overline{B}} \leq C_D N^2 C(C+\frac{C}{\sqrt{s}})^q$$

and the RHS belongs to $L^1(0,t)$. Hence, since $w^\epsilon(t) \underset{\epsilon\to 0}{\longrightarrow} w(t)$ in $C(\overline{B})$, by the Lebesgue's dominated convergence theorem, we obtain :

$$w(t) = m + S(t)W_0 + \int_0^t S(t-s)N^2 w(s)\left(w(s)+\frac{x.\nabla w(s)}{N}\right)^q ds$$

Then $\|w(t)-w_0\|_{\infty,\overline{B}} \leq \|S(t)W_0 - W_0\|_{\infty,\overline{B}} + \int_0^t C_D N^2 C(C+\frac{C}{\sqrt{s}})^q$.
Hence, by the continuity of the heat semigroup at $t=0$ on $C_0(\overline{B})$,

$$w(t) \underset{t\to 0}{\longrightarrow} w_0 \quad \text{in } C(\overline{B})$$

We can then deduce

$$w \in C([0,\tau'], C(\overline{B})) = C([0,\tau']\times\overline{B}) \tag{5.45}$$

Last step : Passing to the limit, since $w^\epsilon \underset{\epsilon\to 0}{\longrightarrow} w$ in $C^{1,2}([t_0,\tau']\times\overline{B})$ for each $t_0 \in (0,\tau']$, then w satisfies :

$$w_t = \Delta w + N^2\, w\,(w+\frac{y.\nabla w}{N})^q \quad \text{on} \quad (0,\tau']\times\overline{B}.$$

Since, moreover, (5.41)(5.42)(5.44)(5.45) hold, w is thus a classical solution of problem $(tPDE_m)$. The uniqueness comes from the comparison principle.

5.6 Proofs of Theorem 4.1 and Theorem 3.1

Proof of Theorem 4.1 : i) and ii) are standard since the small existence time depends on $\|w_0\|_{\infty,\overline{B}}$. For reference, see [26, Proposition 16.1, p. 87-88] for instance.
iii) By Lemma 5.7 iv), since $f_\epsilon(s) \leq s^q$ for all $s \geq 0$, so w^ϵ is a subsolution of $(tPDE_m)$ so by comparison principle,

$$0 \leq w^\epsilon \leq w \text{ on } (0, \min(T^*_\epsilon, T^*))$$

By blow-up alternative for classical solutions of $(tPDE^\epsilon_m)$, it is easy to see by contradiction that $T^*_\epsilon \geq T^*$. It implies that $w \geq w^\epsilon > 0$ on $(0,T^*)\times\overline{B}$ by Theorem 5.1

vi).
iv) We use interior-boundary Schauder estimates.

Proof of Theorem 3.1 : it follows from Theorem 4.1 by exactly the same way as for passing from Theorem 5.1 to Theorem 4.2.
The part vi) will be proved in subsection 5.8.

Remark 5.5. *We can precisely describe the connection between problems* (PDE_m) *and* $(tPDE_m)$*.*
Let $w_0 = \theta_0(u_0)$ *with* $u_0 \in Y_m$*. Then,*

$$T_{max}(u_0) = N^2 T^*(w_0).$$

Moreover, for all $(t,x) \in [0, T_{max}) \times [0,1]$,

$$u(t,x) = x\ \tilde{w}(\frac{t}{N^2}, x^{\frac{1}{N}}). \tag{5.46}$$

5.7 Convergence of maximal classical solutions of problem (PDE_m^ϵ) to classical solutions of problem (PDE_m) as ϵ goes to 0

Proof of Lemma 4.1 : i) Since $0 \leq f_\epsilon(s) \leq s^q$ for all $s \geq 0$ and $(u^\epsilon)_x \geq 0$, it is easy to check that u^ϵ is a subsolution for problem (PDE_m) with initial condition u_0. Hence, by blow-up alternative, this implies that $T_{max}(u_0) \leq T^\epsilon_{max}(u_0)$.
ii) Let $w_0 = \theta_0(u_0)$.
We know that w^ϵ is a subsolution for problem $(tPDE_m)$ with initial condition w_0 thus, setting $T' = \frac{T}{N^2}$, $t'_0 = \frac{t_0}{N^2}$,

$$\sup_{t\in[0,T']} \|w^\epsilon(t)\|_{\infty,\overline{B}} \leq \sup_{t\in[0,T']} \|w(t)\|_{\infty,\overline{B}} =: K < \infty.$$

Applying Theorem 5.1 iv), we know that there exists $\tau' \in (0, t'_0)$ and $C > 0$ both depending on K such that for all $w_0 \in Z_m$ with $\|w_0\|_{\infty,\overline{B}} \leq K$ we have

$$\sup_{t\in(0,\tau']} \sqrt{t}\|w^\epsilon(t)\|_{C^1(\overline{B})} \leq C.$$

So, for all $w_0 \in Z_m$ with $\|w_0\|_{\infty,\overline{B}} \leq K$,

$$\|w^\epsilon(\tau')\|_{C^1(\overline{B})} \leq \frac{C}{\sqrt{\tau'}} =: C'.$$

where C' depends on K and t_0.
For $t \in [t'_0, T']$, we can use $w^\epsilon(t-\tau')$ as initial data to show that

$$\sup_{t\in[t'_0,T']} \|w^\epsilon(t)\|_{C^1(\overline{B})} \leq C'.$$

We can then proceed as in the proof of Lemma 5.9 and show that

$$w^\epsilon \underset{\epsilon\to 0}{\longrightarrow} w \text{ in } C^{1,2}([t_0', T'] \times \overline{B}).$$

Whence the results thanks to formulas (5.27)(5.28) (5.29)(5.30) and their equivalent for u and $\tilde{w}$.

5.8 Regularity of classical solutions of problem (PDE_m)

We already know that classical solutions verify $u(t) \in C^1([0,1])$ for all $t \in (0, T_{max}(u_0))$ but we can actually be more precise, as stated in the next lemma which corresponds exactly to Theorem 3.1 vi).

Lemma 5.10. *Let $u_0 \in Y_m$.*
For all $t \in (0, T_{max}(u_0))$, $u(t) \in Y_m^{1,\frac{2}{N}}$.

Proof : Let $(t,x) \in (0, T_{max}(u_0)) \times [0,1]$ and $w_0 = \theta_0(u_0)$. We know that w is radial, so for all $(s,y) \in (0, \frac{T_{max}(u_0)}{N^2}) \times \overline{B}$, $w(s,y) = \tilde{w}(s, \|y\|)$ with

$$\tilde{w} \in C^{1,2}((0, \frac{T_{max}(u_0)}{N^2}) \times [0,1]) \tag{5.47}$$

We have shown that $u(t,x) = x\,\tilde{w}(\frac{t}{N^2}, x^{\frac{1}{N}})$ so that

$$u_x(t,x) = \tilde{w}(\frac{t}{N^2}, x^{\frac{1}{N}}) + \frac{x^{\frac{1}{N}}}{N}\tilde{w}_r(\frac{t}{N^2}, x^{\frac{1}{N}}) \tag{5.48}$$

This formula already allowed us to prove that $u(t) \in C^1([0,1])$ with $u_x(t,0) = \tilde{w}(\frac{t}{N^2}, 0)$. Since $w(t)$ is radial, then $\tilde{w}_r(\frac{t}{N^2}, 0) = 0$ so we get that

$$|u_x(t,x) - u_x(t,0)| \leq K\, x^{\frac{2}{N}}$$

with $K = (\frac{1}{2} + \frac{1}{N})\|\tilde{w}(\frac{t}{N^2})_{rr}\|_{\infty,[0,1]}$. Hence, $u(t) \in Y_m^{1,\frac{2}{N}}$.

5.9 Shape of the derivative of classical solutions of problem (PDE_m)

We will prove Proposition 3.1.

Proof : i) We set $h(t,x) = \tilde{w}(\frac{t}{N^2}, x) + \frac{x}{N}\tilde{w}_r(\frac{t}{N^2}, x)$.
The result comes from formula (5.48) and because of (5.47).
ii) Since $\tilde{w}(\frac{t}{N^2}, 0) > 0$ for $t \in [t_0, T]$, then by compactness, there exists $\delta > 0$ such

that $\tilde{w}(\frac{t}{N^2}, x^{\frac{1}{N}}) + \frac{x^{\frac{1}{N}}}{N}\tilde{w}_r(\frac{t}{N^2}, x^{\frac{1}{N}}) > 0$ on $[t_0, T] \times [0, \delta]$.
Since $x \mapsto x^q$ is smooth on $(0, \infty)$ and w satisfies

$$w_t = \Delta w + N^2\, w(w + \frac{y.\nabla w}{N})^q$$

then by classical regularity result, $w \in C^{1,\infty}([t_0, T] \times \overline{B}(0, \delta))$. This gives the regularity of h.
iii) Clear since $\tilde{w}$ has odd order derivatives vanishing at $x = 0$.

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Chapitre 3

Convergence uniforme à vitesse exponentielle dans le cas sous-critique [1]

Dans ce chapitre, nous nous intéressons toujours aux solutions radiales du système de Patlak-Keller-Segel étudié, mais dans le cas d'une masse sous-critique et pour $N \geq 2$. Le résultat principal est la convergence uniforme à vitesse exponentielle de la densité de cellules vers l'unique état d'équilibre. Notons que ceci est, à notre connaissance, nouveau même pour le cas très étudié $N = 2$. La preuve exploite le fait que le problème parabolique dégénéré étudié dans le chapitre 2 possède une structure de flot gradient $u_t = -\nabla\mathcal{F}[u(t)]$ sur une "variété Riemannienne de dimension infinie". Nous montrons en particulier une nouvelle inégalité de type Hardy qui est équivalente à la stricte convexité de $\mathcal{F}$ au voisinage de l'équilibre, ce qui explique intuitivement la convergence à vitesse exponentielle vers celui-ci.

1 Introduction

1.1 Origin of the problem

In this paper, we are interested in the speed of convergence toward steady states of solutions of the following problem, called (PDE_m) :

$$\boxed{u_t = x^{2-\frac{2}{N}}\, u_{xx} + u\, {u_x}^q} \qquad t > 0 \quad 0 < x \leq 1 \tag{1.1}$$

$$u(t,0) = 0 \qquad t \geq 0 \tag{1.2}$$

$$u(t,1) = m \qquad t \geq 0 \tag{1.3}$$

$$u_x(t,x) \geq 0 \qquad t > 0 \quad 0 \leq x \leq 1, \tag{1.4}$$

1. Ce chapitre est tiré de l'article [P3].

where $m \geq 0$, N is an integer greater or equal to 2 and q is the critical exponent, i.e.

$$\boxed{q = \frac{2}{N}}.$$

Note that this parabolic problem has degenerate diffusion since $x^{2-\frac{2}{N}}$ vanishes at $x = 0$ and that its nonlinearity involves the gradient and is moreover non Lipschitz when $N \geq 3$ since $0 < q < 1$.

Problem (PDE_m) arose for $N = 2$ in the articles [3] of P. Biler, G. Karch, P. Laurençot and T. Nadzieja and [23] of N. Kavallaris and P. Souplet and then in [P1,P2] for $N \geq 3$ as a key tool in the study of radial solutions of the following chemotaxis system (PKS_q), supposed to describe a collection of cells diffusing in the open unit ball $D \subset \mathbb{R}^N$ and emitting a chemical which attracts themselves :

$$\rho_t = \Delta\rho - \nabla[\rho^q \nabla c] \qquad t > 0 \quad \text{on } D \tag{1.5}$$

$$-\Delta c = \rho \qquad t > 0 \quad \text{on } D, \tag{1.6}$$

with the following boundary conditions :

$$\frac{\partial \rho}{\partial \nu} - \rho^q \frac{\partial c}{\partial \nu} = 0 \quad \text{on } \partial D \tag{1.7}$$

$$c = 0 \quad \text{on } \partial D, \tag{1.8}$$

where ρ is the cell density and c the chemoattractant concentration.

Note that this model relies on the following assumptions :

- Cells diffuse much more slowly than the chemoattractant.
- The cell flux $\vec{F}$ due to the chemoattractant is here described by $\vec{F} = \chi \nabla c$ where

$$\chi(\rho) = \rho^q$$

  is the sensitivity of cells to the chemoattractant.
- On the boundary ∂D, there is a no flux condition for ρ and a Dirichlet conditions for c.

This system (PKS_q) is a particular case of the Patlak-Keller-Segel model. To know more about the latter, the reader can refer to the original works [27] of C.S. Patlak and [24] of E.F. Keller and L.A. Segel. For a review on mathematics of chemotaxis, see the chapter written by M.A. Herrero in [17] and the article [19] of T. Hillen and K. J. Painter. For a review on the Patlak-Keller-Segel model, see both articles of D. Horstmann [20, 21].
We also would like to very briefly recall some important results for the case $N = 2$ and $q = 1$:
- It is known thanks to the works [18] of M.A. Herrero and J.L. Velazquez and [3] of P. Biler, G. Karch, P. Laurençot and T. Nadzieja that 8π is a critical mass for

radial solutions in a ball.
- In the case of the whole plane $\mathbb{R}^2$, this system has a similar behaviour. See [11] by J. Dolbeault and B. Perthame, [5] by A. Blanchet, J.A. Carrillo and N. Masmoudi, [4] by P. Biler, G. Karch, P. Laurençot and T. Nadzieja and [6] by A. Blanchet, J. Dolbeault and B. Perthame.
- For general solutions in a bounded domain of $\mathbb{R}^2$, the results are slightly different since for a mass 4π blow-up at a point of the boundary of the domain can occur (see the book [30] of T. Suzuki).

We now want to recall what is essential to know about the relation between problems (PKS_q) and (PDE_m) (much more can be found in [P2]) :

- m is proportional to the total mass of cells $\int_B \rho$ which is a conserved quantity in time.
- The derivative of u is the quantity with physical interest since u_x is proportional to the cell density ρ, up to a rescaling in time and a change of variable. More precisely, denoting $\rho(t,y) = \tilde{\rho}(t,|y|)$ for $t \geq 0$ and $y \in \overline{D}$, we have

$$\tilde{\rho}(t,x) = N^{\frac{2}{q}}\, u_x(N^2\, t, x^N) \qquad \text{for all } x \in [0,1].$$

- The power $q = \frac{2}{N}$ is critical. Indeed, as a particular case of [22] by D. Horstmann and M. Winkler, we know that the solutions are global in time when $q < \frac{2}{N}$ and can blow up if $q > \frac{2}{N}$

From now on, we will only focus on problem (PDE_m), which becomes our chemotaxis model. We will now list some facts that we have obtained in [P1,P2] for $N \geq 3$ and will later establish some similar results that we need for the case $N = 2$.

1.1.1 Case of dimension $N \geq 3$

In [P1], we have proved the existence of a unique maximal classical solution u of problem (PDE_m) with initial condition $u_0 \in Y_m$ and existence time $T_{max} = T_{max}(u_0) > 0$, where we denote

$$\boxed{Y_m = \{u \in C([0;1]),\ u \text{ nondecreasing },\ u'(0) \text{ exists},\ u(0) = 0,\ u(1) = m\}}$$

and "classical" means here that

$$u \in C([0,T_{max}) \times [0,1]) \bigcap C^1((0,T_{max}) \times [0,1]) \bigcap C^{1,2}((0,T_{max}) \times (0,1]).$$

Actually, we obtained more information about the regularity of the solutions and will refer to [P1] when necessary.

In [P2], we showed that the stationary solutions of (PDE_m) are the restrictions to $[0,1]$ of a family of functions $(U_a)_{a\geq 0}$ on $[0,+\infty)$ with the following simple structure :

- $U_1 \in C^1([0,1]) \cap C^2((0,1])$, $U_1(0) = 0$, $\dot{U}_1(0) = 1$, U_1 is increasing on $[0, A]$ for some $A > 0$ and reaches its maximum M at $x = A$ after which U_1 is flat.
- All $(U_a)_{a\geq 0}$ are obtained by dilation of U_1, i.e. $U_a(x) = U_1(ax)$ for all $x \geq 0$.

An easy consequence of this description is that

- If $0 \leq m < M$, then there exists a unique stationary solution. The latter is given by $U_a|_{[0,1]}$, where $a = a(m) \in [0, A)$ is uniquely determined by m.
- If $m = M$, there exists a continuum of steady states : $\left(U_a|_{[0,1]}\right)_{a\geq A}$. Note that the corresponding cell densities have their support strictly inside D when $a > A$.
- If $m > M$, there is no stationary solution.

We call M the critical mass of problem (PDE_m), which is justified by the following result proved in Theorems 1.2 and 1.3 in [P2], valid for any $u_0 \in Y_m$:

- If $m \leq M$, then
$$T_{max}(u_0) = +\infty$$
and there exists $a \geq 0$ such that
$$u(t) \underset{t\to+\infty}{\longrightarrow} U_a \quad \text{in} \quad C^1([0,1]).$$
More precisely, $a = a(m) \in [0, A)$ if $0 \leq m < M$ and $a \geq A$ if $m = M$.
- If $m > M$, then
$$T_{max}(u_0) < \infty.$$

1.1.2 Case of dimension $N = 2$

For $N = 2$, there is also such a critical mass phenomenon, well studied, with critical mass $M = 2$ corresponding to 8π in the original Patlak-Keller-Segel model (PKS_1) (see [3, 18]).

Problem (PDE_m) then reads

$$\boxed{u_t = x\,u_{xx} + u\,u_x} \qquad t > 0 \quad 0 < x \leq 1 \tag{1.9}$$
$$u(t,0) = 0 \qquad t \geq 0 \tag{1.10}$$
$$u(t,1) = m \qquad t \geq 0 \tag{1.11}$$
$$u_x(t,x) \geq 0 \qquad t > 0 \quad 0 \leq x \leq 1, \tag{1.12}$$

where $m \geq 0$.

It is easy to see that its stationary solutions are all

$$(U_a|_{[0,1]})_{a\geq 0}$$

where
$$U_a(x) = U_1(ax)$$

and

$$U_1(x) = \frac{x}{1+\frac{x}{2}}$$

for all $x \in [0,1]$, $a \geq 0$.

The description of the set of steady states easily gives :

- If $m < 2$, there exists a unique classical steady state of problem (PDE_m), namely $U_a|_{[0,1]}$ where

$$a = a(m) = \frac{m}{1-\frac{m}{2}} \in [0,+\infty).$$

- If $m \geq 2$, there is no classical stationary solution of problem (PDE_m) but only a singular one $\overline{U} = m$ (singular in the sense that the boundary condition at $x = 0$ is lost).

Remark 1.1. *A deep difference with the case $N \geq 3$ is that the steady states here do not reach their upper bound 2 and that the critical value switches from the regular to the singular regime.*
Actually, for all $a > 0$, $\dot{U}_a > 0$ on $[0,1]$. We will see in Theorem 2.1 vii) that this property is shared with the solution u at any time $t > 0$, which means, coming back to the cell density interpretation, that cells are present in the whole ball D. This is in contrast with the case $N \geq 3$, where, at least in the critical mass case, the cells are sometimes present only in a ball strictly inside D.

It is possible to show a similar result as for $N \geq 3$, i.e. that if $0 \leq m < 2$, for any $u_0 \in Y_m$, then

$$T_{max}(u_0) = +\infty$$

and

$$u(t) \underset{t\to+\infty}{\longrightarrow} U_a \quad \text{in} \quad C^1([0,1])$$

where

$$a = a(m) = \frac{m}{1-\frac{m}{2}} \in [0,+\infty).$$

In [3], for the subcritical case $0 \leq m < 2$, the exponential speed of convergence of $u(t)$ toward the unique stationary solution $U_{a(m)}$ as $t \to +\infty$ was proved for all L^p norms with $1 \leq p < \infty$ when the initial condition u_0 is continuous and nondecreasing with $u_0(0) = 0$ and $u_0(1) = m$ (a larger class than Y_m) and also in L^∞ norm for some initial conditions for which global in time $W^{1,\infty}$ bound is known (the result then following by interpolation between L^1 and $W^{1,\infty}$).
As far as we know, the mere convergence in C^1 norm was unknown, and a stronger result (the exponential convergence in C^1 norm) will actually be obtained below, by a very different technique from that in [3]. See section 2 for more details.

1.2 Main result

The main goal of this paper is to study the speed of convergence of solutions of (PDE_m) toward the unique stationary solution U_a for the subcritical case $0 < m < M$ ($m = 0$ being obvious since $u = 0$ because $u_0 \in Y_0 = \{0\}$) when

$$\boxed{N \geq 2}. \tag{1.13}$$

For related results concerning the parabolic-elliptic Patlak-Keller-Segel system in the case of the plane, see [8, 9, 10, 12].

From now on, we fix

$$\boxed{0 < m < M} \tag{1.14}$$

and

$$\boxed{u_0 \in Y_m}. \tag{1.15}$$

We denote u the global solution of (PDE_m) with initial condition u_0. We know that

$$u(t) \underset{t\to+\infty}{\longrightarrow} U_a \quad \text{in} \quad C^1([0,1]),$$

where $U_a = U_{a(m)}$ is the unique stationary state of problem (PDE_m).

Building on this qualitative information, we shall obtain a stronger quantitative one, namely the exponential speed of convergence in $C^1([0,1])$.

Theorem 1.1. *Assume (1.13)(1.14)(1.15).*
Let $U_a = U_{a(m)}$ be the unique stationary solution of (PDE_m), i.e. problem (1.1)-(1.4), and let $\lambda_1 = \lambda_1(a) > 1$ be the best constant of the Hardy type inequality in Proposition 1.1 below.
Let $\lambda \in (0, \lambda_1 - 1)$.
Then there exists $C = C(u_0, \lambda) > 0$ such that for all $t \geq 1$,

$$\|u(t) - U_a\|_{C^1([0,1])} \leq C \exp(-\lambda \dot{U}_a(1)^q \, t).$$

Remark 1.2. *We recall that the derivative of u is, up to a multiplicative constant and a change of variables, the radial part of the cell density ρ in the original Patlak-Keller-Segel model (PKS_q).*
Hence, this result is equivalent to the exponential speed of uniform convergence of $\rho(t)$ toward ρ_a where ρ_a is the cell density corresponding to U_a.

The proof of Theorem 1.1 consists of two steps :
- We first establish exponential convergence in an appropriate weighted L^2 norm, by means of a linearization procedure and a suitable Hardy type inequality.
- We then deduce exponential C^1 convergence by using a smoothing effect after a suitable tranformation of the equation.

In the next subsection, we describe the first step of the proof.

1.3 A Hardy type inequality and exponential convergence in a L^2 weighted space

The following result, which is a Hardy type inequality, requires as a natural framework the two Hilbert spaces $L \supset Y_m$ and H, where

$$\boxed{L = L^2\left((0,1), \frac{dx}{x^{2-q}}\right)}$$

is equipped with the norm

$$\|h\|_L = \sqrt{\int_0^1 \frac{h^2}{x^{2-q}}}$$

and

$$\boxed{H = \left\{h \in L, \int_0^1 \dot{h}^2 < \infty,\, h(0) = h(1) = 0\right\}}$$

with the norm

$$\|h\|_H = \sqrt{\int_0^1 \frac{h^2}{x^{2-q}} + \int_0^1 \dot{h}^2}.$$

Note that, actually, $H = H_0^1 \subset C^{\frac{1}{2}}([0,1])$ and the norms on H and H_0^1 are equivalent (see Remark 3.1).

Remark 1.3. *It is very natural to introduce L from the viewpoint of the evolution equation (PDE_m). This will be justified in the following heuristics subsection.*

Proposition 1.1. *Let $a \in (0, A)$.*
There exists $\lambda_1 = \lambda_1(a) > 1$ such that for all $h \in H$,

$$\int_0^1 \frac{\dot{h}^2}{\dot{U_a}^q} \geq \lambda_1 \int_0^1 \frac{h^2}{x^{2-q}}. \tag{1.16}$$

Moreover, there exists $\phi_1 \in H$ such that there is equality if and only if $h = c\,\phi_1$ for some $c \in \mathbb{R}$.

As will be explained with much more details in subsection 1.4, the evolution problem (PDE_m) can formally be seen as a gradient flow equation

$$u_t = -\nabla \mathcal{F}[u(t)]$$

on some "infinite dimensional Riemannian manifold" $(\mathcal{M}, g)$ where

$$\mathcal{M} = \{u \in Y_m^1,\ \dot{u} > 0 \text{ on } [0,1]\}$$

is an open set of the affine space

$$Y_m^1 = Y_m \cap C^1([0,1])$$

and the metric g is defined by

$$g_u(h,h) = \int_0^1 \frac{h^2}{x^{2-q}\dot{u}^q}$$

for all $u \in \mathcal{M}$ and $h \in T_u\mathcal{M}$, $T_u\mathcal{M}$ denoting the tangent space to $\mathcal{M}$ at u.
The previous result is actually equivalent to the strict convexity of the Lyapunov functional $\mathcal{F}$ at U_a, which makes us expect an exponential speed of convergence toward U_a, measured with the Riemannian distance $d_{\mathcal{M}}(U_a,\cdot)$ defined by the metric g (which is equivalent to $\|\cdot\|_L$ near U_a).
Its proof relies on the theory of compact self-adjoint operators on a separable Hilbert space and on a technique used in the article [1] of P.R Beesack about extensions of Hardy's inequality.
We enjoy the opportunity to thank Philippe Souplet for suggesting this reading.

The following result shows rigorously the expected exponential speed of convergence in L :

Lemma 1.1. *Under the assumptions of Theorem 1.1, there exists* $C = C(u_0,\lambda) > 0$ *such that*

$$\|u(t) - U_a\|_L \leq C \exp(-\lambda \dot{U}_a(1)^q \, t)$$

for all $t \geq 0$.

This result, though not the strongest, is the core of our paper. Its proof is inspired by both the gradient flow structure of problem (PDE_m) and the fact that $\mathcal{M}$ is an open set of an affine space, which allows us to consider all the situation from the viewpoint of U_a. More precisely, if we define

$$h(t) = u(t) - U_a$$

and consider

$$\gamma(t) = g_{U_a}(h(t),h(t)),$$

we want to get a differential inequality on the latter. Since h satisfies

$$h_t = L_{U_a}h + F(x,h,\dot{h}) \tag{1.17}$$

where

$$L_{U_a} = x^{2-q}\dot{U_a}^q \frac{d}{dx}\left[\frac{\dot{h}}{\dot{U_a}^q}\right] + \dot{U_a}^q \, h$$

is the linearized operator at $u = U_a$ and F is some remainder term, we will have two parts to deal with in the derivative of γ. The first term can be managed thanks to the Hardy type inequality in Proposition 1.1. The second imposes to sacrifice a bit of the first one, but without any serious damage since there was anyhow no hope to reach the limit case $\lambda = \lambda_1$, at least by this way.

Remark 1.4.

i) We can show that the degenerate parabolic equation (1.17) satisfied by h is regularizing in time from L to $C^1([0,1])$, at least for large time (see Lemma 5.1). This will be enough to deduce the exponential speed of convergence toward steady states in $C^1([0,1])$, i.e. Theorem 1.1, as an easy consequence of Lemma 1.1.

ii) The constant C we get is unbounded as $\lambda \longrightarrow \lambda_1$ so that we cannot get the same result with $\lambda = \lambda_1 - 1$.

iii) We think that the upper rate $\lambda_1 \dot{U}_a(1)^q$ is not optimal. We believe λ_1 is but not $\dot{U}_a(1)^q$ because it follows from the following rough inequality : $\int_0^1 \frac{h^2}{x^{2-q}} \geq \dot{U}_a(1)^q \int_0^1 \frac{h^2}{x^{2-q}\dot{U}_a{}^q}$ for any $h \in L$ because U_a is concave.

iv) In dimension $N \geq 3$, an interesting question is to know whether the exponential speed of convergence degenerates or not for $a = A$. Indeed, we can see that $\lambda_1(A) = 1$ since if we set $w_A = \frac{d}{da}\Big|_{a=A} U_a$, we remark that $-\frac{d}{dx}\frac{w_A}{\dot{U}_A{}^q} = \frac{w_A}{x^{2-q}}$. Hence we can guess that $\lambda_1(a) \to 1$ as $a \to A$. But, since the center manifold seems to be made of the steady states $(U_a)_{a \geq A}$, it is not clear that the exponential speed of convergence should disappear.
It would then be very different for the critical mass for $N = 2$ and $q = 1$ since the speed of convergence degenerates and is no longer exponential. This has been done in [23]. It was known that infinite time blow-up of u_x occurs. Of course, uniform convergence toward the constant singular steady state $\overline{U} = 2$ cannot hold in this case since $u(t,0) = 0$. However, the authors proved that $|u(t) - 2|_1 \sim C\sqrt{t}e^{-\sqrt{2t}}$ as $t \to \infty$.

1.4 Heuristics

Although the proof of Lemma 1.1 (cf. sections 3-4) can be read without any reference to the following heuristic arguments, we think that they shed some light on the underlying ideas and on the intuition that led to the rigorous proof. Indeed, the latter is inspired by a gradient flow approach, in the spirit of the seminal work of F. Otto [26], a strategy which has already been used successfully for the Patlak-Keller-Segel model. For instance, applying these ideas to system (PKS_1) in $\mathbb{R}^2$ for the subcritical mass case, A. Blanchet, V. Calvez and J.A. Carrillo recovered in [2] the global in time existence of weak solutions and V. Calvez and J.A. Carrillo proved in [8] the exponential speed of convergence of radial solutions toward equilibrium, but measured with the Wasserstein distance $\mathcal{W}_2$.

First, we would like to recall a basic fact about gradient flows in a Euclidean space which provides a sufficient condition to have a exponential speed of convergence to the stationary point. We will give its rigorous proof, even though it is very simple, because it is the scheme for proofs in a more general infinite dimensional setting, as we will then see on a well-known instance in an infinite dimensional Hilbert space. Finally, we will see, without searching to be rigorous, that these ideas are inspiring in the case of problem (PDE_m) which turns out to define a gradient flow on an "infinite dimensional Riemannian manifold".

For basic knowledge about strict Lyapunov functional and Lasalle's invariance principle, we refer the reader to [13, Chapter 9] or to [29, Appendix G]. We also recall some useful properties in subsection 6.1.

We consider the following differential equation in the Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ having a gradient flow structure, i.e.

$$\dot{x}(t) = -\nabla F(x(t))$$

with $F : \mathbb{R}^n \to \mathbb{R}$ smooth.

Lemma 1.2. *Let $x_0 \in \mathbb{R}^d$.*
If the trajectory starting from x_0 is relatively compact in $\mathbb{R}^d$ (then global), if there exists a unique stationary point x_∞ and if F is strictly convex at $x = x_\infty$, i.e. F satisfies for some $\alpha_1 > 0$,

$$d^2F(x_\infty)(\dot{x}, \dot{x}) \geq \alpha_1 |\dot{x}|^2 \text{ for all } \dot{x} \in \mathbb{R}^n,$$

then for any $\alpha \in (0, \alpha_1)$, there exists $C = C(x_0, \alpha) > 0$ such that for all $t \geq 0$

$$|x(t) - x_\infty| \leq C \exp(-\alpha\, t).$$

Proof of Lemma 1.2. First, we observe that F is a strict Lyapunov function since

$$\frac{d}{dt} F(x(t)) = -|\nabla F(x(t))|^2.$$

Since the trajectory $(x(t))_{t \geq 0}$ starting from x_0 is relatively compact, i.e. bounded in the context of an Euclidean space, then from Lasalle's invariance principle, the ω-limit set is made of stationary points. But since there is only one stationary point x_∞, i.e. verifying

$$\nabla F(x_\infty) = 0,$$

we can deduce the convergence of $x(t)$ toward x_∞.

This implies in particular that x_∞ is the minimum of F, so that moreover

$$d^2F(x_\infty) \geq 0.$$

It is then not surprising that the strict convexity assumption on F will give information about the speed of convergence of $x(t)$ toward x_∞. Indeed, if we denote

$$h(t) = x(t) - x_\infty$$

and

$$\gamma(t) = |h(t)|^2,$$

we have
$$\dot{\gamma}(t) = -2\langle \nabla F(x(t)), h(t)\rangle.$$
But $\nabla F(x_\infty) = 0$, so
$$\nabla F(x(t)) = d(\nabla F)(x_\infty).h(t) + \epsilon(h(t))h(t)$$
where $\epsilon(h) \underset{h\to 0}{\longrightarrow} 0$. Hence,
$$\dot{\gamma}(t) = -2d^2F(x_\infty).(h(t), h(t)) + \epsilon(h(t))|h(t)|^2$$
Now, let $\alpha < \alpha_1$.
Since $h(t) \underset{h\to 0}{\longrightarrow} 0$, there exists $t_0 > 0$ such that for all $t \geq t_0$, $\epsilon(h(t)) \leq 2(\alpha_1 - \alpha)$.
Then, for all $t \geq t_0$,
$$\dot{\gamma}(t) \leq -2\alpha\, \gamma(t),$$
which implies
$$\gamma(t) \leq \gamma(t_0)\exp(-2\alpha\, t)$$
for all $t \geq t_0$ and finally we have for all $t \geq 0$,
$$\gamma(t) \leq C\exp(-2\alpha\, t)$$
where $C = C(x_0, \alpha)$ because γ is bounded. Whence the result. □

As said before, this scheme can also be used in an infinite dimensional setting, like a Hilbert space. For example, let us consider the heat equation with Dirichlet condition on an bounded domain Ω
$$u_t = \Delta u.$$
This equation defines a continuous dynamical system on $L^2(\Omega)$ endowed with its standard scalar product $(\cdot\,,\cdot)$. and is moreover regularizing so that, for $t > 0$, $u(t) \in H_0^1(\Omega)$. If we define
$$F(u) = \int_\Omega \frac{|\nabla u|^2}{2},$$
then for $t > 0$,
$$u_t = -\nabla F(u(t))$$
since
$$(\nabla F(u), h) = dF(u).h = \int_\Omega \nabla u \nabla h = -\int_\Omega \Delta u\; h = (-u_t, h)$$
for all $h \in H_0^1(\Omega)$.
It is easy to see that F is a strict Lyapunov function and that 0 is the only stationary solution since the only harmonic function in Ω vanishing on the boundary is the zero

function.
Moreover, since F is quadratic,

$$d^2F(u).(h,h) = 2F(h) = \int_\Omega |\nabla h|^2 \geq \lambda_1(\Omega)\|h\|^2_{L^2(\Omega)}$$

by Poincaré inequality, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ in Ω with Dirichlet condition.
The same computation as above shows that for any $u_0 \in L^2(\Omega)$, for any $\lambda < \lambda_1(\Omega)$, there exists $C = C(u_0, \lambda) > 0$ such that for all $t \geq 0$,

$$\|u(t)\|_{L^2(\Omega)} \leq C\exp(-\lambda t).$$

Note that, actually, the proof also works for $\lambda = \lambda_1$ in this particular instance because F is quadratic so that $u \mapsto \nabla F(u)$ is linear hence there is no $o(h)$ to deal with.

Another more general setting where this method can be applied is that of "infinite dimensional Riemannian manifolds". This idea has been deeply exploited in the very nice paper [26] concerning the porous medium equation.
It turns out that problem (PDE_m) has this kind of gradient flow structure and we will try to take advantage of it. In what follows, we will consider the case of dimension $N \geq 3$ but all this discussion can be made for the case $N = 2$.

If we denote the "infinite dimensional manifold" (actually an open set of the affine space Y_m^1)

$$\boxed{\mathcal{M} = \{u \in Y_m^1,\ \dot{u} > 0 \text{ on } [0,1]\}}$$

where we recall that

$$Y_m^1 = Y_m \cap C^1([0,1]),$$

we know that for $t > 0$, $u(t) \in Y_m^1$ and then for t large enough,

$$u(t) \in \mathcal{M}$$

since $u(t) \underset{t\to+\infty}{\longrightarrow} U_a$ in $C^1([0,1])$ and $\dot{U}_a > 0$ on $[0,1]$.

We can define the "Riemannian metric" g on $\mathcal{M}$ by

$$\boxed{g_u(h,k) = \int_0^1 \frac{hk}{x^{2-q}\dot{u}^q}} \tag{1.18}$$

for any $u \in \mathcal{M}$ and any $(h,k) \in T_u\mathcal{M}^2$, where actually, for any $u \in \mathcal{M}$

$$T_u\mathcal{M} = \mathcal{T}$$

with

$$\mathcal{T} = \{h \in C^1([0,1]),\ h(0) = h(1) = 0\}$$

since $\mathcal{M}$ is an open set of the affine space Y_m^1 which has $\mathcal{T}$ as direction (actually, $Y_m^1 = m\,Id_{[0,1]} + \mathcal{T}$).

Now, we recall the strict Lyapunov functional $\mathcal{F}$ used in [P2] to prove convergence toward steady states in the critical and subcritical mass cases :

$$\boxed{\mathcal{F}[u] = \int_0^1 \frac{\dot{u}^{2-q}}{(2-q)(1-q)} - \frac{u^2}{2x^{2-q}}}.$$

$\mathcal{F}$ can be guessed by the following equivalent formulation of (1.1)

$$u_t = x^{2-q}\dot{u}^q\left[\frac{d}{dx}\frac{\dot{u}^{1-q}}{1-q} + \frac{u}{x^{2-q}}\right]. \tag{1.19}$$

It is easy to see formally that

$$u_t = -\nabla\mathcal{F}[u(t)],$$

which explains intuitively why $\mathcal{F}$ is a strict Lyapunov functional for (PDE_m). Indeed, for any $h \in T_u\mathcal{M}$, we have by definition

$$g_u(\nabla\mathcal{F}[u], h) = d\mathcal{F}(u).h$$

and moreover, by formal computation and integration by parts, we get

$$d\mathcal{F}(u).h = \int_0^1 \frac{\dot{u}^{1-q}\dot{h}}{1-q} - \frac{u\,h}{x^{2-q}} = -\int_0^1 \left[\frac{\ddot{u}}{\dot{u}^q} + \frac{u}{x^{2-q}}\right] h = -g_u(u_t, h).$$

Since we study the subcritical mass case, there exists a unique steady state U_a so we have to compute the second derivative of $\mathcal{F}$ at this point. Formally, we get

$$\boxed{d^2\mathcal{F}[U_a].(h,h) = \int_0^1 \frac{\dot{h}^2}{\dot{U_a}^q} - \frac{h^2}{x^{2-q}}}.$$

As explained before, since $u(t) \underset{t\to+\infty}{\longrightarrow} U_a$ in $C^1([0,1])$, then U_a is the minimum of $\mathcal{F}$ so that we can naturally expect that, for any $h \in T_{U_a}\mathcal{M}$,

$$d^2F(U_a).(h,h) \geq 0.$$

If we can prove the stronger result that for some $\alpha_1 > 0$, we have for all $h \in T_{U_a}\mathcal{M}$,

$$d^2\mathcal{F}[U_a].(h,h) \geq \alpha_1\, g_{U_a}(h,h)$$

or equivalently that for some $\lambda_1 > 1$, for all $h \in T_{U_a}\mathcal{M}$,

$$\int_0^1 \frac{\dot{h}^2}{\dot{U_a}^q} \geq \lambda_1 \int_0^1 \frac{h^2}{x^{2-q}}, \tag{1.20}$$

then we can hope to prove that the speed of convergence is exponential as before.

Remark 1.5. *We thank Philippe Souplet for pointing out the following intuitive explanation of the fact that $\lambda_1 > 1$ in the present context. Indeed, for the subcritical case, the steady states of (1.19) form an increasing family $(U_a)_{a\in(0,A)}$ of solutions of*

$$\frac{d}{dx}f(\dot{u}) + V(x)u = 0$$

where f is the increasing function on $[0,+\infty)$ defined for all $v \geq 0$ by

$$f(v) = \frac{v^{1-q}}{1-q}$$

and

$$V(x) = \frac{1}{x^{2-q}} > 0.$$

Hence, for any $a \in (0,A)$, $w_a = \frac{d}{da}U_a > 0$ and w_a formally satisfies

$$\frac{d}{dx}[f'(\dot{U}_a)\dot{w}_a] + V(x)w_a = 0.$$

If $\phi_1 > 0$ is an eigenvector for the first eigenvalue λ_1, i.e. satisfies

$$\frac{d}{dx}[f'(\dot{U}_a)\dot{\phi}_1] + \lambda_1\, V(x)\phi_1 = 0,$$

then it is easy to see by integration by parts that

$$(\lambda_1 - 1)\int_0^1 V\, w_a\, \phi_1 = [f'(\dot{U}_a)w_a\dot{\phi}_1]_0^1 > 0$$

by Hopf maximum principle on the boundary. Therefore, $\lambda_1 > 1$.

But here, there is an additional difficulty since we have a "Riemannian structure". Indeed, the metric g here depends on the point u, so that if we set

$$\gamma_0(t) = g_{u(t)}(u(t) - U_a, u(t) - U_a)$$

and differentiate it, there will be an extra term. This strategy is in some sense very natural since it takes into account the gradient flow structure. Nevertheless, because of this extra term, we preferred to also take advantage of the fact that $\mathcal{M}$ is an open set of an affine space by considering

$$\gamma(t) = g_{U_a}(u(t) - U_a, u(t) - U_a),$$

i.e. we fixed the point U_a and consider the difference $u(t) - U_a$ belonging to the tangent space $T_{U_a}\mathcal{M}$. Hence, this strategy of linearization somehow uses both the gradient flow structure via the good relation between g and the flow, and the "affine structure" because we can fix U_a and consider the situation from its viewpoint.

Finally, we also remark that if $U \in \mathcal{M}$ is near of U_a, then all measures by the metrics $g_u(h,h)$ are comparable to

$$\|h\|_L = \sqrt{\int_0^1 \frac{h^2}{x^{2-q}}}.$$

Hence, recalling that the Riemannian metric $d_{\mathcal{M}}$ on $\mathcal{M}$ between U_a and U is defined by

$$d_{\mathcal{M}}(U_a, U)^2 = \inf_{\{u \in C^1([0,1],\mathcal{M}),\, u(0)=U_a,\, u(1)=U\}} \int_0^1 g_{u(t)}(u_t, u_t)\, dt,$$

it is clear that $d_{\mathcal{M}}(U_a, U)$ is equivalent to $\|U - U_a\|_L$ for U near of U_a. This consideration naturally leads us to introduce the Hilbert space $L \supset Y_m$, where

$$\boxed{L = L^2\left((0,1), \frac{dx}{x^{2-q}}\right)}.$$

It is also very natural to make the proof of the Hardy type inequality (1.20) in a larger space than $T_{U_a}\mathcal{M}$, namely for all $h \in H$, where H is the Hilbert space

$$\boxed{H = \left\{h \in L, \int_0^1 \dot{h}^2 < \infty,\, h(0) = h(1) = 0\right\}}$$

equipped with the norm

$$\|h\|_H = \sqrt{\int_0^1 \frac{h^2}{x^{2-q}} + \int_0^1 \dot{h}^2}.$$

Outline of the rest of the paper. In section 2, we state some preliminary results for dimension $N = 2$ which will be proved in the appendix.
The next sections are devoted to proofs. In section 3, we will get the strict convexity of $\mathcal{F}$ (or $\mathcal{G}$ if $N = 2$) at U_a by showing its equivalent form expressed in the Hardy type inequality of Proposition 1.1.
In section 4, we show Lemma 1.1 which establishes the exponential speed of convergence toward the steady state in L.
In section 5, we prove that the degenerate parabolic equation satisfied by $h = u - U_a$ is regularizing for large time from L to $C^1([0,1])$, i.e. Lemma 5.1 which therefore easily implies Theorem 1.1. In the appendix, we also recall some basic facts about continuous dynamical systems and Lyapunov functionals.

2 Preliminary results for dimension $N = 2$

In this section, we focus on the most studied case of dimension 2, well-known for its critical mass 8π if we come back to the original Keller-Segel system (1.5). Our aim is to state the results that lead us to Lemma 2.5, i.e. to the C^1 convergence of $u(t)$ toward the unique steady state U_a that we mentioned in the introduction.

We would like to remark that problem (PDE_m) is simpler for $N = 2$ (see (1.9)) than for $N \geq 3$ (see (1.1)) since its nonlinearity is then locally Lipschitz (even bilinear) in (u, u_x). Accordingly, the convergence results for $N = 2$, as well as the required wellposedness and regularity properties, can be proved by similar ideas as in [P1,P2] which treat the case $N \geq 3$. We point out that some of the wellposedness issues for $N = 2$ have been addressed in [23, 3], but that they do not provide all the necessary properties that we need. Therefore, and also for the sake of completeness, we chose to give all the proofs in Appendix, trying to be reasonably self-contained.

2.1 Local wellposedness and regularity for problem (PDE_m)

We first give a wellposedness and regularity theorem which requires the introduction of the following "norm" $\mathcal{N}$ and some notation.

Definition 2.1. *For any real function u defined on $(0, 1]$, we set*

$$\mathcal{N}[u] = \sup_{x \in (0,1]} \frac{u(x)}{x}.$$

Notation 2.1. *Let $m \geq 0$ and $\gamma > 0$.*

– $Y_m = \{u \in C([0; 1])$ *nondecreasing,* $u'(0)$ *exists,* $u(0) = 0, u(1) = m\}$.

– $Y_m^1 = Y_m \cap C^1([0, 1])$.

– $Y_m^{1,\gamma} = \{u \in Y_m \cap C^1([0, 1]),\ \sup\limits_{x \in (0,1]} \frac{|u'(x) - u'(0)|}{x^\gamma} < \infty\}$.

Theorem 2.1. *Let $K > 0$ and $u_0 \in Y_m$ with $\mathcal{N}[u_0] \leq K$.*

i) There exists $T_{max} = T_{max}(u_0) > 0$ and a unique maximal classical solution of (PDE_m) with initial condition u_0, i.e.

$$u \in C([0, T_{max}) \times [0, 1]) \bigcap C^1((0, T_{max}) \times [0, 1]) \bigcap C^{1,2}((0, T_{max}) \times (0, 1])$$

verifying (1.9)(1.10)(1.11)(1.12) and $u(0) = u_0$.
Moreover, u satisfies the following condition :

$$\sup_{t \in (0,T]} \sqrt{t}\, \|u(t)\|_{C^1([0,1])} < \infty \text{ for any } T \in (0, T_{max}). \tag{2.1}$$

ii) There exists $\tau = \tau(K) > 0$ such that $T_{max} \geq \tau$.
iii) Blow up alternative : $T_{max} = +\infty$ or $\lim\limits_{t \to T_{max}} \mathcal{N}[u(t)] = +\infty$
iv) $u \in C^\infty((0, T_{max}) \times (0, 1])$.
v) If $u_0 \in Y_m^{1,\gamma}$ with $\frac{1}{2} < \gamma \leq 1$ then $u \in C([0, T_{max}), C^1([0, 1]))$.
vi) For all $t \in (0, T_{max})$, $u(t) \in Y_m^{1,1}$.
vii) $u_x(t, x) > 0$ for all $(t, x) \in (0, T_{max}) \times [0, 1]$.

At least the four first points were known explicitly or implicitly (see [23]). Concerning point vii), to our knowledge, it was only proved that for all $t \in (0, T_{max})$,

$$u_x(t, 0) > 0.$$

Although vii) is expected, its proof is rather technical and moreover this fact will turn out to be essential in the proof of Lemma 2.4.

2.2 Subcritical case : Lyapunov functional and convergence in $C^1([0,1])$

From now on, we only focus on the subcritical case

$$\boxed{m < 2}$$

which corresponds to mass lower than 8π for the original Keller-Segel system (1.5).

Then, the classical solutions of (PDE_m) are globally defined. More precisely :

Lemma 2.1. *Let $m < 2$ and $u_0 \in Y_m$. Then*

$$T_{max}(u_0) = +\infty.$$

The next lemma, stating in particular the relative compactness of the trajectory $\{u(t),\, t \geq 1\}$ in Y_m^1 for any initial condition $u_0 \in Y_m^1$, will also be useful to check that $(T(t))_{t\geq 0}$ defined below is a continuous dynamical system on Y_m^1.

Definition 2.2. *Let $u_0 \in Y_m^1$ and $t \geq 0$.*
We define $T(t)u_0 = u(t)$ where u is the classical solution of problem (PDE_m) with initial condition u_0.

Lemma 2.2. *Let $m < 2$, $t_0 > 0$ and $K > 0$.*
Then, $\{T(t)u_0,\ \mathcal{N}[u_0] \leq K,\ t \geq t_0\}$ is relatively compact in Y_m^1.

Lemma 2.3. *$(T(t))_{t\geq 0}$ is a continuous dynamical system on Y_m^1.*

We now introduce a functional which is an analogue of $\mathcal{F}$ in the case $q = 1$.

Definition 2.3. *Let $\mathcal{M} = \{u \in Y_m^1,\ u_x > 0 \text{ on } [0,1]\}$.*
We define for all $u \in \mathcal{M}$,

$$\mathcal{G}[u] = \int_0^1 u_x[\ln u_x - 1] - \frac{u^2}{2x}.$$

Indeed, we have the following result.

Lemma 2.4. *$\mathcal{G}$ is a strict Lyapunov functional for $(T(t))_{t\geq 0}$.*

As a consequence, we finally get :

Lemma 2.5. *Let* $0 \leq m < 2$ *and* $u_0 \in Y_m$. *Then*

$$u(t) \underset{t\to+\infty}{\longrightarrow} U_a \quad in \quad C^1([0,1])$$

where

$$a = \frac{m}{1-\frac{m}{2}}.$$

3 A Hardy type inequality

The aim of this section is to prove Proposition 1.1. First, we will need to establish some intermediate lemmas. For reader's convenience, we recall that

$$L = L^2\left((0,1), \frac{dx}{x^{2-q}}\right) \qquad \text{and} \qquad H = \left\{h \in L, \int_0^1 \dot{h}^2 < \infty,\ h(0) = h(1) = 0\right\}$$

and that L and H are equipped with the following norms

$$\|h\|_L^2 = \int_0^1 \frac{h^2}{x^{2-q}} \qquad \text{and} \qquad \|h\|_H^2 = \int_0^1 \frac{h^2}{x^{2-q}} + \int_0^1 \dot{h}^2.$$

Remark 3.1. *Actually, we can see that*

$$H = H_0^1$$

and that $\|\cdot\|_{H_0^1}$ *and* $\|\cdot\|_H$ *are equivalent.*
Indeed, $H \subset H_0^1$ *with continuous embedding is obvious and the reverse is also true by the standard Hardy inequality*

$$\int_0^1 \frac{h^2}{x^2} \leq 4\int_0^1 \dot{h}^2. \tag{3.1}$$

valid for any $h \in H_0^1$. *Note also that* L *and* H *are separable Hilbert spaces.*

We will need the following compactness result.

Lemma 3.1. *The imbedding* $H \subset L$ *is compact.*

Démonstration. For any $\alpha \in (0,1]$, we denote

$$C_0^\alpha = \{h \in C^\alpha([0,1]),\ h(0) = 0\}$$

the Banach space equipped with the norm

$$\|h\|_{C_0^\alpha} = \sup_{\substack{(x,y)\in[0,1]^2 \\ x\neq y}} \frac{|h(x)-h(y)|}{|x-y|^\alpha}.$$

It is clear that $H \subset C_0^{\frac{1}{2}}$ with continuous imbedding since if $h \in H$,

$$|h(x) - h(y)| = |\int_y^x \dot{h}| \leq \sqrt{|x-y|}\sqrt{\int_0^1 \dot{h}^2}.$$

Now, let $\gamma \in (\frac{1-q}{2}, \frac{1}{2})$. The imbedding $C_0^{\frac{1}{2}} \subset C_0^\gamma$ is compact and the imbedding $C_0^\gamma \subset L$ is continuous since for all $h \in C_0^\gamma$,

$$\|h\|_L^2 = \int_0^1 \frac{h^2}{x^{2-q}} \leq \|h\|_{C_0^\gamma}^2 \int_0^1 \frac{1}{x^{2-q-2\gamma}}$$

with $\int_0^1 \frac{1}{x^{2-q-2\gamma}} < \infty$ since $2 - q - 2\gamma < 1$. □

The following lemma, whose proof relies on a technique used in [1] to get extensions of Hardy's inequality, will be essential in the proof of Proposition 1.1.

Lemma 3.2. *Let $0 < a < A$. Then, for all $h \in H$*

$$\int_0^1 \frac{\dot{h}^2}{\dot{U_a}^q} - \frac{h^2}{x^{2-q}} \geq 0 \tag{3.2}$$

with equality if and only if $h = 0$.

Before giving the proof, we recall some useful properties of U_a. For all $a \geq 0$,

$$x^{2-\frac{2}{N}}\ddot{U}_a + U_a \dot{U_a}^{\frac{2}{N}} = 0 \tag{3.3}$$

and

$$\dot{U}_a(0) = a.$$

This implies the concavity of U_a, so

$$\dot{U}_a(1) \leq \dot{U}_a \leq a \text{ on } [0,1].$$

Moreover, for all $x \in [0,1]$,

$$U_a(x) = U_1(ax).$$

Since U_1 is increasing on $[0, A]$ (and flat after $x = A$) for some $A > 0$, then

$$\text{for } 0 < a < A,\ \dot{U}_a > 0 \text{ on } [0,1].$$

Démonstration. We denote for all $x \in [0,1]$,

$$w_a(x) = \frac{d}{da}U_a(x) = x\, \dot{U}_1(ax).$$

We see that $w_a > 0$ on $(0,1]$ since $0 < a < A$.
Moreover, for all $x \in [0,1]$, noting first that

$$\dot{w}_a(x) = \dot{U}_1(ax) + ax\ddot{U}_1(ax),$$

we have

$$\begin{aligned}\frac{\dot{w}_a(x)}{\dot{U_a}^q(x)} &= \frac{\dot{U}_1(ax)^{1-q}}{a^q} + a^{1-q}x\frac{\ddot{U}_1(ax)}{\dot{U}_1(ax)^q} \\ &= \frac{\dot{U}_1(ax)^{1-q}}{a^q} - \frac{U_1(ax)}{ax^{1-q}} \quad \text{by (3.3)}\end{aligned}$$

then, we obtain

$$\begin{aligned}\frac{d}{dx}\left[\frac{\dot{w}_a}{\dot{U_a}^q}\right] &= (1-q)a^{1-q}\frac{\ddot{U}_1(ax)}{\dot{U}_1(ax)^q} + (1-q)\frac{U_1(ax)}{ax^{2-q}} - \frac{\dot{U}_1(ax)}{x^{1-q}} \\ &= -(1-q)a^{1-q}\frac{U_1(ax)}{(ax)^{2-q}} + (1-q)\frac{U_1(ax)}{ax^{2-q}} - \frac{w_a(x)}{x^{2-q}} = -\frac{w_a(x)}{x^{2-q}}\end{aligned}$$

again by (3.3), so we see that w_a satisfies

$$\frac{d}{dx}\left[\frac{\dot{w}_a}{\dot{U_a}^q}\right] + \frac{w_a}{x^{2-q}} = 0. \tag{3.4}$$

We note that this equation could also be obtained by differentiating (3.3) with respect to a.

The proof of Lemma 3.2 will be made by density. Therefore, we first make the following computation for any $h \in C_c^\infty((0,1))$.

$$\begin{aligned}\int_0^1 \frac{[\dot{h} - \frac{\dot{w}_a}{w_a}h]^2}{\dot{U_a}^q} &= \int_0^1 \frac{\dot{h}^2}{\dot{U_a}^q} + \int_0^1 \left[\frac{\dot{w}_a}{w_a}\right]^2 \frac{h^2}{\dot{U_a}^q} - \int_0^1 2h\dot{h}\left[\frac{\dot{w}_a}{w_a}\frac{1}{\dot{U_a}^q}\right] \\ &= \int_0^1 \frac{\dot{h}^2}{\dot{U_a}^q} + \int_0^1 \left[\frac{\dot{w}_a}{w_a}\right]^2 \frac{h^2}{\dot{U_a}^q} + \int_0^1 h^2\frac{d}{dx}\left[\frac{\dot{w}_a}{w_a}\frac{1}{\dot{U_a}^q}\right] \\ &= \int_0^1 \frac{\dot{h}^2}{\dot{U_a}^q} + \int_0^1 h^2\left(\left[\frac{\dot{w}_a}{w_a}\right]^2\frac{1}{\dot{U_a}^q} + \frac{d}{dx}\left[\frac{\dot{w}_a}{w_a}\frac{1}{\dot{U_a}^q}\right]\right)\end{aligned}$$

where we used $h(0) = h(1) = 0$ in the integration by parts.
From (3.4), we deduce

$$\frac{d}{dx}\left[\frac{1}{w_a}\frac{\dot{w}_a}{\dot{U_a}^q}\right] + \frac{1}{\dot{U_a}^q}\left[\frac{\dot{w}_a}{w_a}\right]^2 = \frac{d}{dx}\left[\frac{\dot{w}_a}{\dot{U_a}^q}\right]\frac{1}{w_a} = -\frac{1}{x^{2-q}}$$

which, coming back to the previous computation, implies

$$\int_0^1 \frac{\dot{h}^2}{\dot{U_a}^q} - \frac{h^2}{x^{2-q}} = \int_0^1 \frac{[\dot{h} - \frac{\dot{w}_a}{w_a}h]^2}{\dot{U_a}^q} \geq 0.$$

Let $h \in H$. Since $H = H_0^1(0,1)$ with equivalent norms, then $C_c^\infty((0,1))$ is dense in H so there exists $h_n \in C_c^\infty((0,1))$ such that $h_n \to h$ in L and $\dot{h}_n \to \dot{h}$ in $L^2((0,1), dx)$ with convergence almost everywhere in $(0,1)$ and domination by two functions respectively in L and $L^2(0,1)$. Hence, by Lebesgue's dominated convergence theorem, the previous equation is also valid for h.

Since $w_a \left(\frac{h}{w_a}\right)' = \dot{h} - \dfrac{\dot{w}_a}{w_a} h$, there is equality if and only if $h = c\, w_a$ for some $c \in \mathbb{R}$ almost everywhere on $(0,1)$, but actually everywhere on $[0,1]$ since h and w_a are both continuous. Now, we note that $h(1) = 0$ and $w_a(1) > 0$ since $a < A$, so $h = c\, w_a$ implies $c = 0$, i.e $h = 0$. ☐

Proof of Proposition 1.1. The following procedure is standard.
Considering the symmetric bilinear form Λ defined on H by

$$\Lambda(h,k) = \int_0^1 \frac{\dot{h}\dot{k}}{\dot{U}_a^{\,q}} \qquad \text{for all } (h,k) \in H^2,$$

it is easy to see that Λ is continuous and coercive. Hence, we can apply the Lax-Milgram theorem and prove that for any $\varphi \in H'$, there exists a unique $h \in H$ such that

$$\Lambda(h, \cdot) = \varphi.$$

Thanks to Lemma 3.2, any $f \in L$ defines $\varphi_f \in H'$ by

$$\varphi_f(k) = \int_0^1 \frac{f\,k}{x^{2-q}} \qquad \text{for all } k \in H.$$

We then define $T : L \to L$ by $Tf = h$ where $h \in H$ is such that $\Lambda(h,\cdot) = \varphi_f$. It is easy to see that T is self-adjoint, continuous (thanks to Lax-Milgram) and even compact, thanks to Lemma 3.1.

The end of the proof, which relies on the theory of compact self-adjoint operators on a separable Hilbert space, is completely similar to that of [14, Theorem 2, p.336]. Moreover, since the infimum in

$$\lambda_1 = \inf_{\substack{h \in H \\ h \neq 0}} \frac{\Lambda(h,h)}{\|h\|_L^2}$$

is reached, then Lemma 3.2 implies $\lambda_1 > 1$.

☐

4 Convergence with exponential speed in $L^2\left((0,1), \frac{dx}{x^{2-q}}\right)$

Proof of Lemma 1.1. We let

$$u = U_a + h.$$

To get the result, it is equivalent to show the existence of $C > 0$ such that

$$\gamma(t) \leq C \exp(-2\,\lambda\,\dot{U}_a(1)^q\, t)$$

where

$$\gamma(t) = g_{U_a}(h(t), h(t)) = \int_0^1 \frac{h(t)^2}{x^{2-q}\dot{U_a}^q}.$$

An easy computation shows that for any $t > 0$ and any $x \in (0,1]$,

$$h_t = L_{U_a} h + F(x, h, \dot{h}) \tag{4.1}$$

where

$$L_{U_a} h = \left[x^{2-q}\right]\ \ddot{h} + \left[q \frac{U_a}{\dot{U_a}^{1-q}}\right]\ \dot{h} + \left[\dot{U_a}^q\right]\ h \tag{4.2}$$

$$= x^{2-q}\dot{U_a}^q\ \frac{d}{dx}\left[\frac{\dot{h}}{\dot{U_a}^q}\right] + \dot{U_a}^q\ h \tag{4.3}$$

and

$$F(x, h, \dot{h}) = \frac{q}{\dot{U_a}^{1-q}}\ h\dot{h} + \left[h\ \dot{U_a}^q + U_a\dot{U_a}^q\right]\left[\left(1 + \frac{\dot{h}}{\dot{U_a}}\right)^q - 1 - q\frac{\dot{h}}{\dot{U_a}}\right]. \tag{4.4}$$

We already know from Proposition 2.1 in [P1] that if $t_0 > 0$,

$$h_x(t, x) = \psi(t, x^{\frac{q}{2}}) \qquad \text{for all } t \geq t_0,\ x \in [0,1] \tag{4.5}$$

where

$$\psi \in C^{1,\infty}([t_0, \infty) \times [0,1]) \tag{4.6}$$

(ψ having odd derivatives vanishing at $x = 0$). Formula (4.5) implies

$$\frac{\partial^2 h}{\partial x^2}(t,x) = \frac{q}{2}\frac{\frac{\partial \psi}{\partial x}(t, x^{\frac{q}{2}})}{x^{1-\frac{q}{2}}} \quad \text{for all } t \geq t_0,\ x \in (0,1]. \tag{4.7}$$

Moreover, from Theorem 1.2 in [P2], we know that

$$\|h(t)\|_{C^1([0,1])} \underset{t\to+\infty}{\longrightarrow} 0. \tag{4.8}$$

We will need the following lemma, whose proof is postponed just after this one :

Lemma 4.1. *Let $t > 0$. Denoting $h = h(t)$, we have :*

$$\frac{h\ L_{U_a} h}{x^{2-q} \dot{U_a}^q} \in L^1(0,1)$$

and

$$\int_0^1 \frac{h\ L_{U_a} h}{x^{2-q} \dot{U_a}^q} = -\int_0^1 \left[\frac{\dot{h}^2}{\dot{U_a}^q} - \frac{h^2}{x^{2-q}}\right]. \tag{4.9}$$

By (4.8), there exists $t_0 > 0$ such that for all $t \geq t_0$,

$$\|\frac{h(t)}{\dot{U_a}}\|_{\infty,[0,1]} \leq \frac{1}{2} \qquad \text{and} \qquad \|\dot{h}(t)\|_{\infty,[0,1]} \leq \min\left(1, \frac{2\epsilon}{Ka^{1+q}}, \frac{\delta}{K}\right). \tag{4.10}$$

Recalling (4.4), there exists $K > 0$ such that

$$\left|\frac{h\ F(x,h,\dot{h})}{\dot{U_a}^q}\right| \leq K\left[h^2|\dot{h}| + h^2\dot{h}^2 + U_a|h|\dot{h}^2\right] \tag{4.11}$$

for all $(x,t) \in [0,1] \times [t_0,\infty)$ (h and $\dot{h}$ depending on t).

Let $\delta > 0$ such that $\lambda + \delta \in (0, \lambda_1 - 1)$ and $\epsilon = \frac{\lambda_1 - 1 - \lambda - \delta}{\lambda_1} > 0$ which satisfies

$$(1-\epsilon)(\lambda_1 - 1) - \epsilon = \lambda + \delta. \tag{4.12}$$

It is easy to see that for any $t \geq t_0 + 1$ there exists $M(t) > 0$ such that

$$\sup_{[t-1,t+1]} \|h_t\|_{\infty,[0,1]} \leq M(t).$$

This follows from (4.1)(4.2)(4.7)(4.4) since for all $t \geq t_0$, $\|\dot{h}(t)\|_{\infty,[0,1]} \leq 1$ by (4.10).

Let $t \geq t_0 + 1$. From now on, we denote $h = h(t)$. We want to differentiate $\gamma(t)$ under the integral sign by applying Lebesgue's dominated theorem. This is allowed since

$$\left|\frac{h_t\ h}{x^{2-q}\dot{U_a}^q}\right| \leq \|h_t\|_{\infty,[0,1]} \|\dot{h}\|_{\infty,[0,1]} \frac{1}{x^{1-q}\ \dot{U_a}^q(1)} \leq \frac{M(t)}{x^{1-q}\ \dot{U_a}^q(1)}.$$

Hence, we have :

$$\begin{aligned}
\dot{\gamma}(t) &= 2\int_0^1 \frac{h_t\ h}{x^{2-q}\dot{U_a}^q} = 2\int_0^1 \frac{L_{U_a} h\ h}{x^{2-q}\dot{U_a}^q} + 2\int_0^1 \frac{F(x,h,\dot{h})\ h}{x^{2-q}\dot{U_a}^q} \\
&= -2\int_0^1 \left[\frac{\dot{h}^2}{\dot{U_a}^q} - \frac{h^2}{x^{2-q}}\right] + 2\int_0^1 \frac{F(x,h,\dot{h})\ h}{x^{2-q}\dot{U_a}^q} \\
&= -2(1-\epsilon)\int_0^1 \left[\frac{\dot{h}^2}{\dot{U_a}^q} - \frac{h^2}{x^{2-q}}\right] - 2\epsilon\int_0^1 \left[\frac{\dot{h}^2}{\dot{U_a}^q} - \frac{h^2}{x^{2-q}}\right] + 2\int_0^1 \frac{F(x,h,\dot{h})\ h}{x^{2-q}\dot{U_a}^q} \\
&\leq -2[(1-\epsilon)(\lambda_1 - 1) - \epsilon]\int_0^1 \frac{h^2}{x^{2-q}} - 2\epsilon\int_0^1 \frac{\dot{h}^2}{\dot{U_a}^q} + 2\int_0^1 \frac{F(x,h,\dot{h})\ h}{x^{2-q}\dot{U_a}^q}
\end{aligned}$$

where we have used Lemma 4.1 and Proposition 1.1. Moreover, by (4.11), it is easy to see that

$$\int_0^1 \frac{F(x,h,\dot h)\,h}{x^{2-q}\dot U_a^{\,q}} \le K\left[\|\dot h\|_{\infty,[0,1]} + \|\dot h\|^2_{\infty,[0,1]}\right]\int_0^1 \frac{h^2}{x^{2-q}} + K\int_0^1 \frac{U_a}{x}\frac{|h|}{x}\,x^q\,\dot h^2$$
$$\le 2K\|\dot h\|_{\infty,[0,1]}\int_0^1 \frac{h^2}{x^{2-q}} + Ka\|\dot h\|_{\infty,[0,1]}\int_0^1 \dot h^2.$$

Hence, coming back to the previous calculation and applying (4.12), we have

$$\begin{aligned}\dot\gamma(t) &\le -2(\lambda+\delta-K\|\dot h\|_{\infty,[0,1]})\int_0^1 \frac{h^2}{x^{2-q}} + \left[Ka\|\dot h\|_{\infty,[0,1]} - \frac{2\epsilon}{a^q}\right]\int_0^1 \dot h^2\\ &\le -2\lambda\int_0^1 \frac{h^2}{x^{2-q}} \qquad \text{because of (4.10)}\\ &\le -2\lambda \dot U_a(1)^q\,\gamma(t)\end{aligned}$$

Then for all $t \ge t_0$,

$$\gamma(t) \le C_1 \exp(-2\lambda \dot U_a(1)^q t)$$

where

$$C_1 = \gamma(t_0)\exp(2\lambda \dot U_a(1)^q t_0)$$

depends on λ and u_0.
Since u_0 has a derivative at $x=0$, it is clear that for some $\overline{a}$ large enough, $U_{\overline{a}}$ is a supersolution (see the proof of Lemma 4.1 in [P2]). Hence by the comparison principle (see Lemma 4.1 in [P1]), we have $u(t,x) \le U_{\overline{a}}(x) \le \overline{a}x$ for all $x \in [0,1]$ and $t \ge 0$, which implies that γ is bounded. So there exists $C_2 = C_2(t_0,u_0)$ such that for all $t \in [0,t_0]$,

$$\gamma(t) \le C_2 \exp(-2\lambda \dot U_a(1)^q t),$$

whence the result with $C = \max(C_1,C_2)$ depending on u_0 and λ. □

Remark : we see that t_0 depends on λ and that $t_0 \to +\infty$ as $\lambda \to \lambda_1$. Hence, since t_0 may possibly go to infinity, then we have no bound on C. So, we cannot get the result for $\lambda = \lambda_1$, at least by this way.

Proof of Lemma 4.1. Fixing $\epsilon \in (0,1)$, since $h \in C^2((0,1])$ and from formula (4.3), we see that :

$$\int_\epsilon^1 \frac{L_{U_a}h\,h}{x^{2-q}\dot U_a^{\,q}} = \int_\epsilon^1 \frac{d}{dx}\left[\frac{\dot h}{\dot U_a^{\,q}}\right]\,h + \frac{h^2}{x^{2-q}} = -\int_\epsilon^1 \left[\frac{\dot h^2}{\dot U_a^{\,q}} - \frac{h^2}{x^{2-q}}\right] - \frac{h(\epsilon)\dot h(\epsilon)}{\dot U_a^{\,q}(\epsilon)}$$

since $h(1)=0$. Then, since $h(0)=0$ and $h \in C^1([0,1])$, we have

$$\frac{h^2}{x^{2-q}} \in L^1(0,1)$$

and

$$\frac{h(\epsilon)\dot{h}(\epsilon)}{\dot{U_a}^q(\epsilon)} \underset{\epsilon\to 0}{\longrightarrow} 0.$$

Moreover,

$$\frac{d}{dx}\left[\frac{\dot{h}}{\dot{U_a}^q}\right] \in L^1(0,1)$$

since

$$\frac{d}{dx}\left[\frac{\dot{h}}{\dot{U_a}^q}\right] = \frac{\ddot{h}}{\dot{U_a}^q} + q\frac{U_a}{x^{2-q}\dot{U_a}}\dot{h},$$

$$\left|\frac{U_a}{x}\right| \leq a$$

and because of (4.7). Finally, we get the result by letting ϵ go to zero since the Lebesgue's dominated theorem can be applied. □

5 Convergence with exponential speed in $C^1([0,1])$

We first give a regularizing estimate from L to $C^1([0,1])$ for problem (1.17).

Lemma 5.1. *Let*

$$N \geq 2,$$

$$0 < m < M,$$

$$U_a = U_{a(m)}$$

the unique stationary solution of (PDE_m) *(equations (1.1)-(1.4)) and*

$$u_0 \in Y_m.$$

Then, there exists $\overline{t} = \overline{t}(u_0) > 0$, $T = T(N, u_0) > 0$, $C = C(N, u_0) > 0$ *such that, for all* $t_0 \geq \overline{t}$ *and* $t \in (0, T]$,

$$\|u(t_0+t) - U_a\|_{C^1([0,1])} \leq \frac{C}{t^\beta}\|u(t_0) - U_a\|_L,$$

where

$$\beta = \beta(N) = 1 + \frac{N}{4}.$$

Before giving the proof of this lemma, we need to recall some notation and well-known properties of the Dirichlet heat semigroup on the open unit ball B of $\mathbb{R}^{N+2}$.

Properties 5.1. *We denote* $(S(t))_{t\geq 0}$ *the Dirichlet heat semigroup on* $B = B(0,1) \subset \mathbb{R}^{N+2}$,

$$C_0(\overline{B}) = \left\{f \in C(\overline{B}),\ f = 0 \text{ on } \partial B\right\}$$

and

$$C_0^1(\overline{B}) = \left\{ f \in C^1(\overline{B}),\ f = 0 \ on\ \partial B \right\}.$$

For $p > 1$,

$$\|S(t)f\|_{W^{1,p}(B)} \leq \frac{C}{\sqrt{t}} \|f\|_{L^p(B)} \text{ for all } f \in L^p(B). \tag{5.1}$$

Moreover, let $p > N + 2$. *Since there exists* $C > 0$ *such that for all* $t > 0$,

$$\|S(t)f\|_{C_0^1(\overline{B})} \leq \frac{C}{\sqrt{t}} \|f\|_{C(\overline{B})} \text{ for all } f \in C_0(\overline{B})$$

and

$$\|S(t)f\|_{C(\overline{B})} \leq \frac{C}{t^{\frac{N+2}{2p}}} \|f\|_{L^p(B)} \text{ for all } f \in L^p(B)$$

then there exists $C > 0$ *such that for all* $t > 0$,

$$\|S(t)f\|_{C^1(\overline{B})} \leq \frac{C}{t^{\gamma}} \|f\|_{L^p(B)} \text{ for all } f \in L^p(B) \tag{5.2}$$

where $\gamma = \frac{1}{2} + \frac{N+2}{2p} < 1$.

Proof of Lemma 5.1. As before, we denote $h(t) = u(t) - U_a$. Now, we set

$$w(t, y) = \frac{u(N^2 t, |y|^N)}{|y|^N}$$

$$W_a(y) = \frac{U_a(|y|^N)}{|y|^N}$$

$$f(t, y) = \frac{h(N^2 t, |y|^N)}{|y|^N}$$

for all $t \geq 0$ and $y \in \overline{B}$ where

$$B = B(0, 1) \subset \mathbb{R}^{N+2}$$

denotes the open unit ball in $\mathbb{R}^{N+2}$. Then w is a radial classical solution of the following transformed problem called $(tPDE_m)$:

$$\begin{aligned} w_t &= \Delta w + N^2 w \left(w + \tfrac{y.\nabla w}{N} \right)^q && \text{on } (0, T] \times \overline{B} \\ w(0) &= w_0 && \\ w + \tfrac{y.\nabla w}{N} &\geq 0 && \text{on } (0, T] \times \overline{B} \\ w &= m && \text{on } [0, T] \times \partial B \end{aligned} \tag{5.3}$$

Here, "classical" means that for any $T > 0$,

$$w \in C([0, T] \times \overline{B}) \bigcap C^{1,2}((0, T] \times \overline{B}).$$

Note also that W_a is a radial stationary solution of $(tPDE_m)$ and that

$$f = w - W_a$$

which implies obviously $f = 0$ on ∂B.

All these facts rely on the following calculations relating h to $\tilde{f}$ (and also u to w and U_a to W_a), where

$$f(t,y) = \tilde{f}(t,|y|) \text{ for all } (t,y) \in [0,+\infty) \times \overline{B}.$$

We have for $0 < t \le T$ and $0 < x \le 1$:

$$\begin{aligned}
h(t,x) &= x\,\tilde{f}\left(\frac{t}{N^2}, x^{\frac{1}{N}}\right). && (5.4)\\
h_t(t,x) &= \frac{x}{N^2}\,\tilde{f}_t\left(\frac{t}{N^2}, x^{\frac{1}{N}}\right).\\
h_x(t,x) &= \left[\tilde{f} + \frac{r\tilde{f}_r}{N}\right]\left(\frac{t}{N^2}, x^{\frac{1}{N}}\right)\\
&= \left[f + \frac{y.\nabla f}{N}\right]\left(\frac{t}{N^2}, x^{\frac{1}{N}}\right). && (5.5)\\
x^{2-\frac{2}{N}} h_{xx}(t,x) &= \frac{x}{N^2}\left[\tilde{f}_{rr} + \frac{N+1}{r}\tilde{f}_r\right]\left(\frac{t}{N^2}, x^{\frac{1}{N}}\right)\\
&= \frac{x}{N^2}\,\Delta f\left(\frac{t}{N^2}, x^{\frac{1}{N}}\right).
\end{aligned}$$

Remark : we would like to mention that problem (5.3), which exhibits simple Laplacian diffusion, was already used in [P1] to prove existence of a solution for (PDE_m) and in [P2] to get some estimates implying relative compactness of the trajectories in $C^1([0,1])$. Actually, the solution u was obtained from w by the formula

$$u(t,x) = x\,w\left(\frac{t}{N^2}, x^{\frac{1}{N}}\right)$$

and w was obtained as a limit of solutions w^ϵ of approximations of (5.3) (because the nonlinearity is non Lipschitz), the regularity of w following from that of w^ϵ since for $\alpha \in [0, \frac{2}{N})$, w^ϵ have a bound in $C^{1+\alpha/2, 2+\alpha}$ uniform in ϵ. See section 4.5 in [P1] for more details.

Since $\dot{U}_a > 0$ and $W_a \in C^1(\overline{B})$, then we have

$$W_a + \frac{y.\nabla W_a}{N} \text{ positive and bounded on } \overline{B}. \qquad (5.6)$$

A simple computation shows that

$$f_t = \Delta f + \Phi(y, f, \nabla f) \qquad (5.7)$$

where

$$\begin{aligned}\Phi(y,f,\nabla f) =& N^2 f\left[W_a+\frac{y.\nabla W_a}{N}+f+\frac{y.\nabla f}{N}\right]^q \\ &+N^2W_a\left(W_a+\frac{y.\nabla W_a}{N}\right)^q\left[\left(1+\frac{f+\frac{y.\nabla f}{N}}{W_a+\frac{y.\nabla W_a}{N}}\right)^q-1\right].\end{aligned} \tag{5.8}$$

We observe that, since $\tilde{f}(r)=\frac{h(r^N)}{r^N}$ and B is the unit ball in $\mathbb{R}^{N+2}$, then

$$\|f\|^2_{L^2(B)}=\int_0^1 |S_{N+1}|r^{N+1}\frac{h(r^N)^2}{r^{2N}}\,dr=\frac{|S_{N+1}|}{N}\int_0^1\frac{h^2}{x^{2-\frac{2}{N}}}\,dx=\frac{|S_{N+1}|}{N}\|h\|^2_L$$

hence

$$\|f\|_{L^2(B)}=\sqrt{\frac{|S_{N+1}|}{N}}\|h\|_L.$$

Other observation : by (4.8), we know that for t large enough $\|h(t)\|_{C^1([0,1])}$ is as small as desired. Hence, since $h(t,0)=0$, we deduce from (5.4) that $\|f(t)\|_{C(\overline{B})}$ can be made as small as we wish for large t and then $\|y.\nabla f(t)\|_{C(\overline{B})}$ also from (5.5). Hence, there exists

$$\bar{t}_0=\bar{t}_0(u_0)>0$$

such that for all $t\geq\bar{t}_0$, for all $y\in\overline{B}$,

$$\left|\frac{f+\frac{y.\nabla f}{N}}{W_a+\frac{y.\nabla W_a}{N}}\right|\leq\frac{1}{2}.$$

This fact, the boundedness of $W_a+\frac{y.\nabla W_a}{N}$ on $\overline{B}$(above and below by a positive constant) and (5.8) imply, for any $p\geq 2$, the existence of $C>0$ such that for all $t\geq\bar{t}_0$,

$$\|\Phi(y,f(t),\nabla f(t))\|_{L^p(B)}\leq C\|f(t)\|_{W^{1,p}(B)} \tag{5.9}$$

and

$$\|\Phi(y,f(t),\nabla f(t))\|_{C(\overline{B})}\leq C\|f(t)\|_{C^1(\overline{B})} \tag{5.10}$$

We will now use the regularizing effect of the Dirichlet heat semigroup recalled in Properties 5.1 to show a similar property for (5.7).

First step. We show that for all $p\in(1,+\infty)$, there exists $T_0>0$ and $C>0$ such that

(A) $\qquad \|f(t_0+t)\|_{W^{1,p}}\leq Ct^{-1/2}\|f(t_0)\|_p,\quad$ for all $t_0\geq\bar{t}_0$ et $t\in(0,T_0]$

and

(A') $\qquad \|f(t_0+t)\|_p\leq C\|f(t_0)\|_p,\quad$ pour tout $t_0\geq\bar{t}_0$ et $t\in(0,T_0]$.

Let $t_0 \geq \bar{t}$ and $t \geq t_0$.
Since w is a classical solution of $(tPDE_m)$ for $t > 0$, then f is a classical solution of (5.7), hence also a mild solution. So,

$$f(t_0 + t) = S(t)f(t_0) + \int_0^t S(t-s)\Phi(y, f(t_0+s), \nabla f(t_0+s))\, ds. \tag{5.11}$$

Then, by (5.9) and (5.1) we obtain : (C_1 being a positive constant which my vary from line to line)

$$\|f(t_0+t)\|_{W^{1,p}(B)} \leq \frac{C_1}{\sqrt{t}}\|f(t_0)\|_{L^p(B)} + \int_0^t \frac{C_1}{\sqrt{t-s}}\|f(t_0+s)\|_{W^{1,p}(B)} ds,$$

from which follows

$$\sqrt{t}\|f(t_0+t)\|_{W^{1,p}(B)} \leq C_1\|f(t_0)\|_{L^p(B)} + \sqrt{t}\int_0^t \frac{C_1}{\sqrt{s(t-s)}}\sqrt{s}\|f(t_0+s)\|_{W^{1,p}(B)} ds.$$

We notice that $\int_0^t \frac{ds}{\sqrt{s(t-s)}} = \int_0^1 \frac{dx}{\sqrt{x(1-x)}}$ by the change of variable $x = \frac{s}{t}$.
Let $T_0 = \frac{1}{4{C_1}^2}$. Denoting

$$a(T_0) = \sup_{t \in (t_0, t_0+T_0]} \sqrt{t}\|f(t)\|_{W^{1,p}(B)},$$

we get

$$a(T_0) \leq C_1\|f(t_0)\|_{L^p(B)} + C_1\sqrt{T_0}\, a_1(T_0)$$

which, by the choice of T_0, gives

$$a(T_0) \leq 2C_1\|f(t_0)\|_{L^p(B)}.$$

Hence, for all $t \in (t_0, T_0]$,

$$\|f(t_0+t)\|_{W^{1,p}(B)} \leq \frac{2C_1}{\sqrt{t}}\|f(t_0)\|_{L^p(B)},$$

which proves (A) and allows thanks to (5.11) and (5.9) again to get (A').

Second step. Let us set $n = N + 2$.
We show by iteration the existence of $p \in (n, +\infty)$ and $C > 0$ independent of the solution f such that

(B) $\|f(t_0+t)\|_{W^{1,p}(B)} \leq Ct^{-1/2-(n/2)(1/2-1/p)}\|f(t_0)\|_2$ for all $t_0 \geq \bar{t}_0$ et $t \in (0, T_0]$.

et

(B') $\|f(t_0+t)\|_{L^p(B)} \leq Ct^{-(n/2)(1/2-1/p)}\|f(t_0)\|_2$ for all $t_0 \geq \bar{t}_0$ et $t \in (0, T_0]$.

Indeed, this is true for $p = 2$ thanks to (A) and (A').
Assume that (B) and (B') are true for some $p \in [2, +\infty)$.
If $p < n$, then we prove that (B) and (B') are true for $p = p^*$, $p^* < +\infty$ beeing the optimal exponent such that we have the following Sobolev imbedding

$$W^{1,p}(B) \subset L^{p^*}(B).$$

Indeed, we have by (A) and Sobolev embedding that

$$\begin{aligned} \|f(t_0+t)\|_{W^{1,p^*}} &\leq C(t/2)^{-1/2}\|f(t_0+(t/2))\|_{p^*} \leq C(t/2)^{-1/2}\|f(t_0+(t/2))\|_{W^{1,p}} \\ &\leq C(t/2)^{-1/2}(t/2)^{-1/2-(n/2)(1/2-1/p)}\|f(t_0)\|_2 \\ &= C(t/2)^{-1/2-(n/2)(1/2-1/p+1/n)}\|f(t_0)\|_2 = C't^{-1/2-(n/2)(1/2-1/p^*)}\|f(t_0)\|_2, \end{aligned}$$

and, by Sobolev embedding and (B), we have

$$\|f(t_0+t)\|_{p^*} \leq C\|f(t_0+t)\|_{W^{1,p}} \leq Ct^{-1/2-(n/2)(1/2-1/p)}\|f(t_0)\|_2 = Ct^{-(n/2)(1/2-1/p^*)}\|f(t_0)\|_2.$$

Iterating this process, we obtain after a finite number of steps some $p \in [n, +\infty)$ such that (B) and (B') are true.

- If $p > n$, this is the result we wanted.
- If $p = n$, since B (resp. (B')) is true for $p = 2$ and $p = n$, we can interpolate between $W^{1,2}(B)$ and $W^{1,n}(B)$ (resp. $L^2(B)$ and $L^n(B)$) and get (B) (resp. (B')) for some $p_0 \in (\frac{n}{2}, n)$, which, by an application of the previous process, shows (B) (resp. (B')) for $p = p_0^* > n$.

Last step. We can now prove the result by making a last iteration.
Coming back to (5.11), by (5.10) and (5.2) we obtain :

$$\|f(t_0+t)\|_{C^1(\overline{B})} \leq \frac{C}{t^\gamma}\|f(t_0)\|_{L^p(B)} + \int_0^t \frac{C}{\sqrt{t-s}}\|f(t_0+s)\|_{C^1(\overline{B})}ds,$$

from which follows

$$t^\gamma\|f(t_0+t)\|_{C^1(\overline{B})} \leq C\|f(t_0)\|_{L^p(B)} + t^\gamma \int_0^t \frac{C}{s^\gamma\sqrt{t-s}} s^\gamma \|f(t_0+s)\|_{C^1(\overline{B})}ds.$$

Note that $t^\gamma \int_0^t \frac{ds}{s^\gamma\sqrt{t-s}} = \int_0^1 \frac{dx}{x^\gamma\sqrt{1-x}}\ \sqrt{t}$ which is well defined since $\gamma = \frac{1}{2} + \frac{n}{2p} < 1$.
Let $T = \frac{1}{4C^2}$. Denoting

$$b(T) = \sup_{t \in (t_0, t_0+T]} t^\gamma \|f(t)\|_{C^1(\overline{B})},$$

we get

$$b(T) \leq C\|f(t_0)\|_{L^p(B)} + C\sqrt{T}\ a(T)$$

which, by the choice of T, gives

$$b(T) \leq 2C\|f(t_0)\|_{L^p(B)}.$$

Hence, for all $t \in (t_0, T]$,

$$\|f(t_0+t)\|_{C^1(\overline{B})} \leq \frac{2C}{t^\gamma}\|f(t_0)\|_{L^p(B)},$$

which implies by (B') that

$$\|f(t_0+t)\|_{C^1(\overline{B})} \leq \frac{C'}{t^{\frac{1}{2}+\frac{n}{2p}}}\|f(t_0+t/2)\|_{L^p(B)} \leq \frac{C'}{t^{\frac{1}{2}+\frac{n}{4}}}\|f(t_0)\|_{L^2(B)}.$$

This implies the result since

$$\|h(t_0+t)\|_{C^1([0,1])} \leq C_1\|f(t_0+t)\|_{C^1(\overline{B})} \leq \frac{C_2}{t^{\frac{1}{2}+\frac{n}{4}}}\|f(t_0)\|_{L^2(B)} = \frac{C_3}{t^{\frac{1}{2}+\frac{n}{4}}}\|h(t_0)\|_L.$$

□

We can now give the proof of Theorem 1.1.

Proof of Theorem 1.1. This follows from Lemma 1.1 and from Lemma 5.1, which is a regularizing in time estimate. Indeed, using notation of Lemma 5.1 (having fixed $p > N$), let $t \geq \overline{t} + T$. Then $t - T \geq \overline{t}$ so we obtain,

$$\|u(t) - U_a\|_{C^1([0,1])} \leq \frac{C}{T^\gamma}\|u(t-T) - U_a\|_L \leq C(u_0, p)\exp(-\lambda \dot{U}_a(1)^q\,(t-T)),$$

which gives the result. □

6 Appendix : proofs of the preliminary results for dimension $N = 2$

In this section, we first recall some basic facts about continuous dynamical systems and Lyapunov functionals. In the next subsections are the proofs of all results of section 2.

6.1 Reminder on continuous dynamical systems and Lyapunov functionals

For reader's convenience, we fast recall some very basic facts on continuous dynamical systems, which are general but will be given in the context of

$$Y_m^1 = Y_m \cap C^1([0,1])$$

endowed with the induced topology of $C^1([0,1])$. For reference, see [13, chap. 9].

Here follow the definitions of a continuous dynamical system, its trajectories, stationary points and ω-limit sets.

Definition 6.1. *A continuous dynamical system on Y_m^1 is a one-parameter family of mappings $(T(t))_{t\geq 0}$ from Y_m^1 to Y_m^1 such that :*
i) $T(0) = Id$.
ii) $T(t+s) = T(t)T(s)$ for any $t, s \geq 0$.
iii) For any $t \geq 0$, $T(t) \in C(Y_m^1, Y_m^1)$.
iv) For any $u_0 \in Y_m$, $t \mapsto T(t)u_0 \in C((0,\infty), Y_m^1)$.

Definition 6.2. *Let $u_0 \in Y_m^1$.*
- *u_0 is a stationary point if for all $t \geq 0$, $T(t)u_0 = u_0$.*
- *$\gamma_1(u_0) = \{T(t)u_0,\ t \geq 1\}$ is the trajectory of u_0 from $t = 1$.*
- *$\omega(u_0) = \{v \in Y_m^1,\ \exists t_n \to +\infty,\ t_n \geq 1,\ T(t_n)u_0 \underset{n\to+\infty}{\longrightarrow} v \text{ in } Y_m^1\}$ is the ω-limit set of u_0.*

Now we give the definition of a strict Lyapunov functional and Lasalle's invariance principle.

Definition 6.3.
i) $\mathcal{F} \in C(Y_m^1, \mathbb{R})$ is a Lyapunov functional if for all $u_0 \in Y_m^1$,

$$t \mapsto \mathcal{F}[T(t)u_0] \text{ is nonincreasing on } [0, +\infty).$$

ii) A Lyapunov functional $\mathcal{F}$ is a strict Lyapunov functional if

$$\mathcal{F}[T(t)u_0] = \mathcal{F}[u_0] \text{ for all } t \geq 0 \text{ implies that } u_0 \text{ is an equilibrium point.}$$

Proposition 6.1. *Lasalle's invariance principle.*
Let $u_0 \in Y_m^1$. Assume that the dynamical system $(T(t))_{t\geq 0}$ admits a strict Lyapunov functional and that $\gamma_1(u_0)$ is relatively compact in Y_m^1.
Then the ω-limit set $\omega(u_0)$ is nonempty and consists of equilibria of the dynamical system.

See [13, p. 143] for a proof.

6.2 Wellposedness and regularity for problem (PDE_m)

We first remark that there is a classical comparison principle available for problem (PDE_m), which will for instance imply the uniqueness of the maximal classical solution in Theorem 2.1.

Lemma 6.1. *Let $T > 0$. Assume that :*
- *$u_1, u_2 \in C([0,T] \times [0,1]) \bigcap C^1((0,T] \times [0,1]) \bigcap C^{1,2}((0,T] \times (0,1))$.*
- *For all $t \in (0,T]$, $u_1(t)$ and $u_2(t)$ are nondecreasing.*
- *There exists $i_0 \in \{1,2\}$ and some $\gamma < 1$ such that*

$$\sup_{t\in(0,T]} t^\gamma \, \|u_{i_0}(t)\|_{C^1([0,1])} < \infty. \tag{6.1}$$

Suppose moreover that :

$$u_{1t} \leq x\, u_{1xx} + u_1 u_{1x} \quad \textit{for all } (t,x) \in (0,T] \times (0,1) \tag{6.2}$$
$$u_{2t} \geq x\, u_{2xx} + u_2 u_{2x} \quad \textit{for all } (t,x) \in (0,T] \times (0,1) \tag{6.3}$$
$$u_1(0,x) \leq u_2(0,x) \quad \textit{for all } x \in [0,1] \tag{6.4}$$
$$u_1(t,0) \leq u_2(t,0) \quad \textit{for } t \geq 0 \tag{6.5}$$
$$u_1(t,1) \leq u_2(t,1) \quad \textit{for } t \geq 0 \tag{6.6}$$

Then $u_1 \leq u_2$ *on* $[0,T] \times [0,1]$.

The proof of this result was given in [23] under weaker assumptions. We give a different one in this simpler context.

Proof of Lemma 6.1. Let us set

$$z = (u_1 - u_2) e^{-\int_0^t (\|u_{i_0}(s)\|_{C^1} + 1) ds},$$

well defined thanks to (6.1). The hypotheses made show that

$$z \in C([0;T] \times [0;1]) \bigcap C^1((0,T] \times [0,1]) \bigcap C^{1,2}((0,T] \times (0,1)).$$

Assume now by contradiction that $\max\limits_{[0;T]\times[0;1]} z > 0$.
By assumption, $z \leq 0$ on the parabolic boundary of $[0,T] \times [0,1]$.
Hence, $\max\limits_{[0;T]\times[0;1]} z$ is reached at a point $(t_0, x_0) \in (0;T] \times (0;1)$.
Then $z_x(t_0,x_0) = 0$ so $(u_1)_x(t_0,x_0) = (u_2)_x(t_0,x_0)$.
Moreover, $z_{xx}(t_0,x_0) \leq 0$ and $z_t(t_0,x_0) \geq 0$. But we have

$$z_t(t_0,x_0) \leq x\, z_{xx}(t_0,x_0) + [(u_{i_0})_x(t_0,x_0) - \|u_{i_0}(t_0)\|_{C^1} - 1]\, z(t_0,x_0).$$

The LHS of the inequality is nonnegative and the RHS is negative, whence the contradiction. ☐

Before coming to the proof of Theorem 2.1, we need to fix some notation and recall some facts about the Dirichlet heat semigroup.

For reference, see for instance the book [25] of A. Lunardi.

Notation 6.1.

- *B denotes the open unit ball in $\mathbb{R}^4$.*
- $Z_0 = \{W \in C(\overline{B}),\ W|_{\partial B} = 0\}$.
- *$(S(t))_{t\geq 0}$ denotes the heat semigroup on Z_0. It is the restriction on Z_0 of the Dirichlet heat semigroup on $L^2(B)$.*
- *$(X_\theta)_{\theta\in[0,1]}$ denotes the scale of interpolation spaces for $(S(t))_{t\geq 0}$, where $X_0 = Z_0$, $X_1 = D(-\Delta)$ and $X_\alpha \hookrightarrow X_\beta$ with dense continuous injection for any $\alpha > \beta$, $(\alpha,\beta) \in [0,1]^2$.*

Properties 6.1.
- $X_{\frac{1}{2}} = \{W \in C^1(\overline{B}),\ W|_{\partial B} = 0\}$.
- *Let* $\gamma_0 \in (0; \frac{1}{2}]$. *For any* $\gamma \in [0, 2\gamma_0)$,

$$X_{\frac{1}{2}+\gamma_0} \subset C^{1,\gamma}(\overline{B})$$

with continuous embedding.
- *There exists* $C_D \geq 1$ *such that for any* $\theta \in [0; 1]$, $W \in X_0$ *and* $t > 0$,

$$\|S(t)W\|_{X_\theta} \leq \frac{C_D}{t^\theta}\|W\|_\infty.$$

We just want to introduce some specific notation we are going to use.

Notation 6.2. *Let* $(a, b) \in (0, 1)^2$. *We denote* $I(a, b) = \int_0^1 \frac{ds}{(1-s)^a s^b}$.
For all $t \geq 0$, $\int_0^t \frac{ds}{(t-s)^a s^b} = t^{1-a-b} I(a, b)$.

Notation 6.3. *Let* $m \geq 0$ *and* $\gamma > 0$.
- $Y_m = \{u \in C([0; 1])$ *nondecreasing,* $u'(0)$ *exists,* $u(0) = 0, u(1) = m\}$.
- $Z_m = \{w \in C(\overline{B}),\ w|_{\partial B} = m\}$.
- $Y_m^{1,\gamma} = \{u \in Y_m \cap C^1([0, 1]),\ \sup\limits_{x \in (0,1]} \frac{|u'(x)-u'(0)|}{x^\gamma} < \infty\}$.
- $Z_m^{1,\gamma} = \{w \in Z_m \cap C^1(\overline{B}),\ \sup\limits_{y \in \overline{B}\backslash\{0\}} \frac{|\nabla w(y)|}{|y|^\gamma} < \infty\}$.

Proof of Theorem 2.1. We begin by giving a short proof of points i)ii)iii)iv).

We define the following transformation θ_0, already remarked in [7, section 2.2] and [16, section 2], and also used in [23] :

$$\begin{aligned} \theta_0 :\ & Y_m \longrightarrow Z_m \\ & u \longrightarrow w \text{ where } w(y) = \frac{u(|y|^2)}{|y|^2} \text{ for all } y \in \overline{B}\backslash\{0\}. \end{aligned}$$

The next lemma has been proved in Lemma 4.3 in [P1].

Lemma 6.2. *Let* $m \geq 0$.
i) θ_0 *sends* Y_m *into* Z_m.
ii) If $\gamma > \frac{1}{2}$, *then* θ_0 *sends* $Y_m^{1,\gamma}$ *into* $Z_m^{1,2\gamma-1}$.

If $u_0 \in Y_m$, we set

$$w_0 = \theta_0(u_0) \in Z_m$$

and

$$w(t, y) = \frac{u(4t, |y|^2)}{|y|^2}$$

for all $y \in \overline{B}$.
Then we obtain a transformed problem called $(tPDE_m)$ with simple Laplacian diffusion in $B \subset \mathbb{R}^4$:

$$w_t = \Delta w + 4w\left(w + \tfrac{y.\nabla w}{2}\right) \quad \text{on } (0,T] \times \overline{B} \tag{6.7}$$
$$w(0) = w_0 \tag{6.8}$$
$$w + \tfrac{y.\nabla w}{N} \geq 0 \quad \text{on } (0,T] \times \overline{B} \tag{6.9}$$
$$w = m \quad \text{on } [0,T] \times \partial B \tag{6.10}$$

This relies on the following calculations relating u to $\tilde{w}$, where we denote

$$w(t,y) = \tilde{w}(t,|y|) \text{ for all } (t,y) \in [0,+\infty) \times \overline{B}.$$

We have for $0 < t \leq T$ and $0 < x \leq 1$:

$$u(t,x) = x\,\tilde{w}\left(\frac{t}{4}, \sqrt{x}\right). \tag{6.11}$$
$$u_t(t,x) = \frac{x}{4}\,\tilde{w}_t\left(\frac{t}{4}, \sqrt{x}\right).$$
$$\begin{aligned} u_x(t,x) &= \left[\tilde{w} + \frac{r\tilde{w}_r}{2}\right]\left(\frac{t}{4}, \sqrt{x}\right) \\ &= \left[w + \frac{y.\nabla w}{2}\right]\left(\frac{t}{4}, \sqrt{x}\right). \end{aligned} \tag{6.12}$$
$$\begin{aligned} x\,u_{xx}(t,x) &= \frac{x}{4}\left[\tilde{w}_{rr} + \frac{3}{r}\tilde{w}_r\right]\left(\frac{t}{4}, \sqrt{x}\right) \\ &= \frac{x}{4}\,\Delta w\left(\frac{t}{4}, \sqrt{x}\right). \end{aligned}$$

The existence of a unique maximal classical solution w on $[0,T^*)$ of problem $(tPDE_m)$ with initial condition $w_0 \in Z_m$, i.e. a function

$$w \in C([0,T^*) \times \overline{B}) \bigcap C^{1,2}((0,T^*) \times \overline{B})$$

satisfying (6.7)(6.8)(6.9)(6.10) is standard.
Indeed, we can set $W = w - m$, get a corresponding equation for W, obtain by a fixed point argument a mild solution W on $[0,\tau^*]$ for some small $\tau^* > 0$ by use of the Dirichlet heat semigroup since the nonlinearity is locally Lipschitz in $(w,\nabla w)$ on $C^1(\overline{B})$, and finally exploit regularity results to prove that the solution is classical.
Moreover,

$$\sup_{t \in (0,\tau^*]} \sqrt{t}\|w(t)\|_{C^1(\overline{B})} < \infty \tag{6.13}$$

Again by iteration of regularity results on (6.7), it can also be proved that

$$w \in C^\infty((0,T^*) \times \overline{B}).$$

Since $\tau^* = \tau^*(\|w_0\|_{\infty,\overline{B}})$, we also get the blow-up alternative

$$T^* = +\infty \quad \text{or} \quad \lim_{t\to T^*} \|w(t)\|_{\infty,\overline{B}} = +\infty.$$

Now, we come back to problem (PDE_m). Let $u_0 \in Y_m$ and $w_0 = \theta_0(u_0)$. By uniqueness, w is radial because w_0 is. Then, if we set

$$T_{max}(u_0) = 4\,T^*(w_0).$$

and for all $(t,x) \in [0, T_{max}) \times [0,1]$,

$$u(t,x) = x\ \tilde{w}(\frac{t}{4}, \sqrt{x}), \tag{6.14}$$

then we can check that u is the unique maximal classical solution of problem (PDE_m) with initial condition u_0. The fact that u is nondecreasing on $[0,1]$ will be shown in vii). It is easy to see that the small existence time $\tau^* = \tau^*(\|w_0\|_{\infty,\overline{B}})$ for w gives a small existence time $\tau = \tau(\mathcal{N}[u_0])$ for u, i.e. for each $K > 0$, there exists $\tau = \tau(K) > 0$ such that if $\mathcal{N}[u_0] \leq K$ then the solution u is at least defined on $[0,\tau]$. Moreover, the regularity of w implies the results of regularity on u. Hence, we have proved i)ii)iii) and iv).

v) If $u_0 \in Y_m^{1,\gamma}$ with $\gamma > 1/2$, then from Lemma 6.2 ii),

$$w_0 = \theta_0(u_0) \in Z_m^{1,2\gamma-1} \subset C^1(\overline{B}).$$

We only have to check the continuity at $t = 0$ of $t \mapsto w(t) \in C^1(\overline{B})$. This is clear by the variation of constants formula since $t \mapsto S(t)\Phi \in C([0,+\infty), X_{\frac{1}{2}})$ for any $\Phi \in X_{\frac{1}{2}}$. Hence, we get a maximal classical solution

$$w \in C([0,T^*), C^1(\overline{B}))$$

which, thanks to formula (6.12), gives a maximal classical solution of (PDE_m)

$$u \in C([0,T_{max}), C^1([0,1])).$$

vi) Let $(t,x) \in (0, T_{max}(u_0)) \times (0,1]$. From formulas (6.11) and (6.12), we have

$$u(t,x) = x\ \tilde{w}\left(\frac{t}{4}, \sqrt{x}\right)$$

and

$$u_x(t,x) = \tilde{w}\left(\frac{t}{4}, \sqrt{x}\right) + \sqrt{x}\ \frac{\tilde{w}_r}{2}\left(\frac{t}{4}, \sqrt{x}\right).$$

These formulas allow to prove that $u(t) \in C^1([0,1])$ with $u_x(t,0) = \tilde{w}(\frac{t}{4}, 0)$. Since $w(\frac{t}{4})$ is radial, then $\tilde{w}_r(\frac{t}{4}, 0) = 0$. This implies that for any $y \in [0,1]$,

$$\left|\tilde{w}\left(\frac{t}{4}, y\right) - \tilde{w}\left(\frac{t}{4}, 0\right)\right| \leq K\ \frac{y^2}{2}$$

and

$$\left|\tilde{w}_r\left(\frac{t}{4}, y\right)\right| \leq K\, y$$

where $K = \|\tilde{w}(\frac{t}{4})_{rr}\|_{\infty,[0,1]}$. So, we obtain

$$|u_x(t,x) - u_x(t,0)| \leq K\, x.$$

Hence, $u(t) \in Y_m^{1,1}$.

vii) Let us now show that $u_x(t,x) > 0$ for all $(t,x) \in (0, T_{max}) \times [0,1]$.

We prove the result in two steps. Let $T \in (0, T_{max})$.
First step : We now show that $v := u_x \geq 0$ on $(0,T] \times [0,1]$.
We divide the proof in three parts.

– First part : We show the result for any $u_0 \in Y_m^{1,\gamma}$ where $\gamma > \frac{1}{2}$.
  Since u satisfies on $(0,T] \times (0,1]$

$$u_t = x\, u_{xx} + u\, u_x \tag{6.15}$$

  and thanks to iv), we can now differentiate this equation with respect to x. We denote

$$b = 1 + u$$

  and obtain the partial differential equation satisfied by v :

$$v_t = x\, v_{xx} + b\, v_x + v^2 \quad \text{on} \quad (0,T) \times (0,1) \tag{6.16}$$
$$v(0,\cdot\,) = (u_0)' \tag{6.17}$$
$$v(t,0) = u_x(t,0) \quad \text{for} \quad t \in (0,T] \tag{6.18}$$
$$v(t,1) = u_x(t,1) \quad \text{for} \quad t \in (0,T] \tag{6.19}$$

  By vi), we know that $u \in C([0,T], C^1([0,1]))$, then $v \in C([0,T] \times [0,1])$ and v reaches its minimum on $[0,T] \times [0,1]$.
  By comparison principle, we have

$$0 \leq u \leq m$$

  so

$$u_x(t,0) \geq 0$$

  and

$$u_x(t,1) \geq 0$$

  for all $t \in (0,T]$. Then, from (6.17), (6.18) and (6.19), $v \geq 0$ on the parabolic boundary of $[0,T] \times [0,1]$. From (6.16), we see that v cannot reach a negative minimum in $(0,T) \times (0,1)$. So $v \geq 0$ on $[0,T] \times [0,1]$.

– Second part : We show that if $u_0 \in Y_m$, there exists $\tau \in (0,T)$ such that for all $t \in [0,\tau]$, $u(t)$ is nondecreasing on $[0,1]$.
Let $u_0 \in Y_m$. By Lemma 4.4 in [P1], there exists a sequence $(u_{0,n})_{n\geq 1}$ of $Y_m^{1,1}$ such that
$$\|u_{0,n} - u_0\|_{\infty,[0,1]} \underset{n\to\infty}{\longrightarrow} 0$$
and
$$\mathcal{N}[u_{0,n}] \leq \mathcal{N}[u_0].$$
Since $\mathcal{N}[u_{0,n}]$ is bounded, we know by ii) that there exists a common small existence time $\tau \in (0,T)$ for all solutions $(u_n(t))_{t\geq 0}$ of problem (PDE_m) with initial condition $u_{0,n}$. From first part, we know that for all $t \in [0,\tau]$ $u_n(t)$ is a nondecreasing function since $u_{0,n} \in Y_m^{1,1}$. To prove the result, it is sufficient to show that
$$\|u_n - u\|_{\infty,[0,1]\times[0,\tau]} \underset{n\to\infty}{\longrightarrow} 0.$$
Let $\eta > 0$. By (2.1), there exists $C > 0$ such that for all $t \in [0,\tau]$, $\|u(t)_x\|_\infty \leq \frac{C}{\sqrt{t}}$. So we can choose $\eta' > 0$ such that
$$\eta' e^{\int_0^\tau [\|u(t)_x\|_\infty + 1]\, dt} \leq \eta$$
Let $n_0 \geq 1$ such that for all $n \geq n_0$, $\|u_{0,n} - u_0\|_{\infty,[0,1]} \leq \eta'$. Let $n \geq n_0$.
Let us set
$$z(t) = [u_n(t) - u(t)] e^{-\int_0^\tau [\|u(t)_x\|_\infty + 1]\, dt}$$
We see that z satisfies
$$z_t = x\, z_{xx} + b\, z_x + c\, z \tag{6.20}$$
where $b = u_n(t)$ and $c = [u_x - \|(u)_x\|_\infty - 1] < 0$.
Since $z \in C([0,\tau]\times[0,1])$, z reaches its maximum and its minimum.
Assume that this maximum is greater than η'. Since $z = 0$ for $x = 0$ and $x = 1$ and $z \leq \eta'$ for $t = 0$, it can be reached only in $(0,\tau]\times(0,1)$ but this is impossible because $c < 0$ and (6.20). We make the similar reasoning for the minimum. Hence, $|z| \leq \eta'$ on $[0,\tau]\times[0,1]$.
Eventually, $\|u_n - u\|_{\infty,[0,1]\times[0,\tau]} \leq \eta' e^{\int_0^\tau [\|u(t)_x\|_\infty + 1]\, dt} \leq \eta$ for all $n \geq n_0$. Whence the result.

– Last part : Let $u_0 \in Y_m$. From the second part, there exists $\tau \in (0,T)$ such that that for all $t \in [0,\tau]$, $u(t)$ is nondecreasing. Since $u \in C([\tau,T], C^1([0,1]))$ and $u(\tau) \in Y_m^{1,1}$, we can apply the same argument as in the first part to deduce that for all $t \in [\tau,T]$, $u(t)$ is nondecreasing. This concludes the proof of the first step.

Second step : Let us show that $v > 0$ on $(0,T]\times[0,1]$.
0 is clearly a subsolution of problem $(tPDE_m)$ so $w \geq 0$ on $\overline{B}$ but by strong maximum principle we even have
$$w > 0$$

on B (see [15, Theorem 5 p.39]). Then, from formula (6.12) it follows that

$$v(t,0) = u_x(t,0) > 0$$

for $t \in (0,T]$.
Assume by contradiction that v is zero at some point in $(0,T) \times (0,1)$.
Since v satisfies (6.16) and the underlying operator is parabolic on $(0,T) \times (0,1]$, by the strong minimum principle (see [15, Theorem 5 p.39]), we deduce that $v = 0$ on $(0,T) \times (0,1)$. Then, by continuity, $v(t,0) = 0$ for $t \in (0,T)$ which contradicts the previous assertion.
Suppose eventually that $v(t,1) = 0$ for some $t \in (0,T)$. From (6.15), we deduce that $u_{xx}(t,1) = 0$, ie

$$v_x(t,1) = 0.$$

Since $v^2 \geq 0$, we observe that v satisfies :

$$v_t \geq x\, v_{xx} + [1+u]v_x \tag{6.21}$$

Since $v > 0$ on $(0,T) \times [\frac{1}{2}, 1)$ and the underlying operator in the above equation is uniformly parabolic on $(0,T) \times [\frac{1}{2}, 1]$, we can apply Hopf's minimum principle (cf. [28, Theorem 3, p.170]) to deduce that $v_x(t,1) < 0$ what yields a contradiction. In conclusion, $u_x > 0$ on $(0,T] \times [0,1]$ for all $T < T_{max}$, whence the result.

□

6.3 Subcritical case : Lyapunov functional and convergence in $C^1([0,1])$

Here are the proofs of results in subsection 2.2.

Proof of Lemma 2.1. $m = 0$ is trivial so we assume $0 < m < 2$.
Let $T_{max} = T_{max}(u_0)$.
From Theorem 2.1, in order to get $T_{max} = +\infty$, it is sufficient to prove that

$$\sup_{t \in [0,T_{max})} \mathcal{N}[u(t)] < \infty.$$

This fact easily follows from a comparison with a supersolution of problem (PDE_m).
The main idea is that since $m < 2$, if a_0 is large enough then

$$u_0 \leq U_{a_0}$$

and U_{a_0} is then a supersolution so for all $t \in [0, T_{max})$, $0 \leq u(t) \leq U_{a_0}$ hence

$$\mathcal{N}[u(t)] \leq a_0$$

since U_{a_0} is concave.
Now we give an explicit formula for a_0 which will end the proof. We denote

$$a = \frac{m}{1 - \frac{m}{2}}$$

which defines the unique steady state, i.e. satisfying $U_a(1) = m$.
First, since u_0 is differentiable at $x = 0$, $x \mapsto \frac{u_0(x)}{x}$ can be extended continuously to $[0;1]$, so $m \leq \mathcal{N}[u_0] < +\infty$.
Let us set $x_0 = \frac{m}{\mathcal{N}[u_0]} \in (0,1]$. We can check that for

$$a_0 = \frac{a}{x_0}$$

we have

$$U_{a_0}(x_0) = U_{a_0 x_0}(1) = m.$$

- For $x \in [0;x_0]$, $u_0(x) \leq \mathcal{N}[u_0]x \leq U_{a_0}(x)$ since by concavity, U_{a_0} is above its chord between $x = 0$ and $x = x_0$.
- For $x \in [x_0, 1]$, $u_0(x) \leq m = U_{a_0}(x_0) \leq U_{a_0}(x)$ since U_{a_0} is increasing.
Hence, $u_0 \leq U_{a_0}$ on $[0,1]$ where

$$a_0 = \frac{\mathcal{N}[u_0]}{1 - \frac{m}{2}}$$

Remark : we actually proved the following stronger result, to be used in the next proof.
For each $K > 0$, for any $u_0 \in Y_m$ with $\mathcal{N}[u_0] \leq K$, we have

$$\sup_{t\in[0,\infty)} \mathcal{N}[u(t)] \leq \frac{K}{1 - \frac{m}{2}}.$$

□

Proof of Lemma 2.2. Actually, we will prove the following stronger result :
If $m < 2$, $\gamma \in [0;1)$, $t_0 > 0$ and $K > 0$, then there exists $D_K > 0$ such that for any $u_0 \in Y_m$ with $\mathcal{N}[u_0] \leq K$, we have

$$\sup_{t\geq t_0} \|u(t)\|_{C^{1,\frac{\gamma}{N}}} \leq D_K.$$

Let $u_0 \in Y_m$ such that $\mathcal{N}[u_0] \leq K$. Let $w_0 = \theta_0(u_0)$.
First step : thanks to the final remark in the proof of Lemma 2.1, we have

$$\sup_{t\in[0,\infty)} \mathcal{N}[u(t)] \leq C_K := \frac{K}{1 - \frac{m}{2}}.$$

Since for $t \geq 0$, $\|w(t)\|_{\infty,\overline{B}} = \mathcal{N}[u(\frac{t}{4})]$, we deduce that w is global and that

$$\sup_{t\in[0,\infty)} \|w(t)\|_{\infty,\overline{B}} = \sup_{t\in[0,\infty)} \mathcal{N}[u(t)] \leq C_K.$$

Second step : Let

$$\tau = \frac{t_0}{4}$$

and $t \in [0, \tau]$.
Denoting $W_0 = w_0 - m$, then

$$w(t) - m = S(t)W_0 + 4 \int_0^t S(t-s) w \left(w + \frac{x.\nabla w}{2} \right) ds, \tag{6.22}$$

so

$$\|w(t)\|_{C^1} \leq m + \frac{C_D}{\sqrt{t}}(C_K + m) + 4 \int_0^t \frac{C_D}{\sqrt{t-s}} C_K \|w(s)\|_{C^1} ds.$$

Setting $h(t) = \sup\limits_{s \in (0,t]} \sqrt{s} \|w(s)\|_{C^1}$, we have $h(t) < \infty$ by (6.13) and

$$\sqrt{t}\|w(t)\|_{C^1} \leq m\sqrt{\tau} + C_D(C_K + m) + 4C_K C_D \sqrt{t} \int_0^t \frac{1}{\sqrt{s}\sqrt{t-s}} h(s) ds,$$

$$\sqrt{t}\|w(t)\|_{C^1} \leq m\sqrt{\tau} + C_D(m + C_K) + 4C_K C_D I\left(\frac{1}{2}, \frac{1}{2}\right) \sqrt{t}\, h(t).$$

Let $T \in (0, \tau]$. Then,

$$h(T) \leq m\sqrt{\tau} + C_D(m + C_K) + 4C_K C_D I\left(\frac{1}{2}, \frac{1}{2}\right) \sqrt{T}\, h(T). \tag{6.23}$$

Setting $A = m\sqrt{\tau} + C_D(m + C_K)$ and $B = 8C_K C_D I\left(\frac{1}{2}, \frac{1}{2}\right)$, assume that there exists $T \in [0, \tau]$ such that

$$h(T) = 2A.$$

Then,

$$2A \leq A + \frac{B}{2}\sqrt{T}\, 2A \text{ which implies } T \geq \frac{1}{B^2}.$$

Let us set

$$\tau' = \min\left(\tau, \frac{1}{2B^2}\right).$$

Since $h \geq 0$ is nondecreasing, $h_0 = \lim\limits_{t \to 0^+} h(t)$ exists and $h_0 \leq A$ by (6.23). So by continuity of h on $(0, \tau']$, $h(t) \leq 2A$ for all $t \in (0, \tau']$, that is to say :

$$\|w(t)\|_{C^1} \leq \frac{2A}{\sqrt{t}} \text{ for all } t \in (0, \tau'],$$

where A and τ' only depend on K. Then, setting $A_K = 2A$, we have

$$\sup_{t \in [0, \tau']} \sqrt{t}\|w(t)\|_{C^1} \leq A_K.$$

Third step : Let $\gamma_0 \in (\frac{\gamma}{2}, \frac{1}{2})$ and $t \in [0, \tau']$.
Setting $W = w - m$ and $W_0 = w_0 - m$, then for $t \geq 0$, due to (6.22), we get

$$\|W(t)\|_{X_{\frac{1}{2}+\gamma_0}} \leq \frac{C_D}{t^{\frac{1}{2}+\gamma_0}}(C_K + m) + 4\int_0^t \frac{C_D}{(t-s)^{\frac{1}{2}+\gamma_0}} C_K \frac{A_K}{\sqrt{s}} ds.$$

Then we deduce that :

$$\begin{aligned} t^{\frac{1}{2}+\gamma_0}\|W(t)\|_{X_{\frac{1}{2}+\gamma_0}} &\leq C_D(C_K + m) + 4C_K C_D A_K t^{\frac{1}{2}+\gamma_0} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\gamma_0}\sqrt{s}} ds \\ &\leq C_D(m + C_K) + 4C_K C_D A_K I(\frac{1}{2} + \gamma_0, \frac{1}{2})\sqrt{\tau'}. \end{aligned}$$

Hence, since $X_{\frac{1}{2}+\gamma_0} \subset C^{1,\gamma}(\overline{B})$, we deduce that there exists $A'_K > 0$ depending only on K such that

$$\|w(\tau')\|_{C^{1,\gamma}(\overline{B})} \leq \frac{A'_K}{\tau'^{(\frac{1}{2}+\gamma_0)}} =: A''_K.$$

Last step : Let $t' \geq \frac{t_0}{4}$. Since $\tau' \leq \frac{t_0}{4}$, we can apply the same arguments by taking $w_0(t' - \tau)$ as initial data instead of w_0, so we obtain

$$\text{for all } t' \geq \frac{t_0}{4},\ \|w(t')\|_{C^{1,\gamma}(\overline{B})} \leq A''_K.$$

Finally, coming back to $u(t)$, thanks to formula (6.14), we get an upper bound D_K for $\|u(t)\|_{C^{1,\frac{\gamma}{N}}}$ valid for any $u_0 \in Y_m$ such that $\mathcal{N}[u_0] \leq K$. □

Proof of Lemma 2.3. Thanks to Lemma 2.1, we know that $T(t)$ is well defined for all $t \geq 0$ and by definition of a classical solution, $T(t)$ maps Y_m^1 into Y_m^1.
ii) is clear by uniqueness of the global classical solution.
iv) comes from the fact that $u \in C((0, \infty), C^1([0, 1]))$.
iii) Let $t > 0$, $u_0 \in Y_m^1$ and $(u_n)_{n\geq 1} \in Y_m^1$.
Assume that $u_n \xrightarrow[n\to\infty]{C^1} u_0$. Let us show that $u_n(t) \xrightarrow[n\to\infty]{C^1} u(t)$.
We proceed in two steps.
First step : We want to show that if $u_n \xrightarrow[n\to\infty]{C^1} u_0$, then $u_n(t) \xrightarrow[n\to\infty]{C^0} u(t)$.

Actually, this has already been done in the proof of Theorem 2.1 vii) (in the first step, second part). Indeed, the argument there shows that if all the u_n exist on a common interval $[0, T_0]$, then we have

$$\|u_n - u\|_{\infty,[0,1]\times[0,T_0]} \xrightarrow[n\to\infty]{} 0.$$

But here, for all n, $T_{max}(u_n) = +\infty$, so this result can be applied to $T_0 = t$, which implies the result.

Second step : since $u_n \xrightarrow[n\to\infty]{C^1} u_0$, $\|u_n\|_{C^1}$ is bounded so there exists $K > 0$ such that for all $n \geq 1$, $\mathcal{N}[u_n] \leq K$. Then, from Lemma 2.2, since $t > 0$, $\{u_n(t),\ n \geq 1\}$

is relatively compact in Y_m^1 and has a single accumulation point $u(t)$ from first step. Whence the result.

□

Proof of Lemma 2.4. Let $t > 0$. We can differentiate the integral by applying Lebesgue's dominated theorem. Indeed, let $\eta > 0$ small enough so that

$$I = [t - \eta, t + \eta] \subset (0, T_{max}).$$

Note : here, for $0 \leq m < 2$, $T_{max} = +\infty$.
Since $u \in C(I, C^1([0,1]))$, then $\frac{u}{x}$ is bounded on $I \times [0,1]$. Since moreover, by Theorem 2.1 vii), for all $t \in I$, $u_x(t) > 0$ on $[0,1]$, then $\ln(u_x)$ is bounded on $I \times [0,1]$.

Let $(t, x) \in (0, T_{max}(u_0)) \times (0, 1]$. We recall that

$$u_x(t,x) = \tilde{w}\left(\frac{t}{4}, \sqrt{x}\right) + \sqrt{x}\,\frac{\tilde{w}_r}{2}\left(\frac{t}{4}, \sqrt{x}\right).$$

Hence,

$$u_{xx}(t,x) = \frac{3}{4\sqrt{x}}\tilde{w}_r\left(\frac{t}{4}, \sqrt{x}\right) + \frac{1}{4}\tilde{w}_{rr}\left(\frac{t}{4}, \sqrt{x}\right)$$

and

$$u_{xxx}(t,x) = -\frac{3}{8x^{\frac{3}{2}}}\tilde{w}_r\left(\frac{t}{4}, \sqrt{x}\right) + \left(\frac{3}{8x} + \frac{1}{4}\right)\tilde{w}_{rr}\left(\frac{t}{4}, \sqrt{x}\right) + \frac{1}{8\sqrt{x}}\tilde{w}_{rrr}\left(\frac{t}{4}, \sqrt{x}\right).$$

Since

$$u_t = x\,u_{xx} + u\,u_x$$

and

$$u_{t,x} = x\,u_{xxx} + [1 + u]u_{xx} + {u_x}^2,$$

it is now easy to see that u_t and $u_{t,x}$ are bounded on $I \times [0,1]$.
Since $u \in C^2((0, T_{max}) \times (0,1])$, then $u_{t,x} = u_{x,t}$.
Finally, $\ln(u_x)\,u_{t,x} - \frac{u\,u_t}{x}$ is bounded on $I \times [0,1]$. Hence, by direct calculation,

$$\begin{aligned}\frac{d}{dt}\mathcal{G}[u(t)] &= \int_0^1 \ln(u_x)\,u_{t,x} - \frac{u\,u_t}{x} = -\int_0^1 [\frac{u_{xx}}{u_x} + \frac{u}{x}]u_t \\ &= -\int_0^1 \frac{{u_t}^2}{x u_x}\end{aligned}$$

where an integration by parts was made, using that $u_t(t,0) = u_t(t,1) = 0$.
It is easy to see that $\mathcal{G}$ is continuous on $\mathcal{M}$, $\mathcal{G}$ is nonincreasing on the trajectories, so we have proved that $\mathcal{G}$ is a Lyapunov function. Now, assume that

$$\mathcal{G}[u(t)] = \mathcal{G}[u_0]$$

for all $t \geq 0$. This implies

$$\int_0^t \int_0^1 \frac{{u_t}^2}{x u_x} = 0$$

so

$$u_t = 0$$

on $[0,1] \times [0,t]$ for all $t \geq 0$. Hence, by continuity, for all $t \geq 0$,

$$u(t) = u_0$$

i.e. u is a steady state of (PDE_m). □

Proof of Lemma 2.5. Let $u_0 \in Y_m$ and u the global solution of (PDE_m) with initial condition u_0. Let us set

$$u_1 = u(1) \in Y_m^1.$$

To get the result, we just have to study $\lim_{t\to+\infty} T(t)u_1$.
Thanks to Lemma 2.2, $\gamma_1(u_1)$ is relatively compact in Y_m^1 and since $\mathcal{G}$ is a strict Lyapunov functional for $(T(t))_{t\geq 0}$, we know by Lasalle's invariance principle (Proposition 6.1) that the ω-limit set $\omega(u_1)$ is non empty and contains only stationary solutions. But since there exists only one steady state U_a where

$$a = \frac{m}{1-\frac{m}{2}},$$

then

$$\omega(u_1) = \{U_a\}$$

so

$$T(t)u_1 \underset{t\to+\infty}{\longrightarrow} U_a \ .$$

□

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Chapitre 4

Proportionnalité des composantes, théorèmes de Liouville et estimations a priori pour des systèmes elliptiques non coopératifs [1]

Dans ce chapitre, nous étudions les propriétés des solutions positives de systèmes elliptiques semi-linéaires non coopératifs, sans structure variationnel en général. Nous obtenons de nouveaux résultats de classification et de non-existence aussi bien dans l'espace entier que dans un demi-espace et en déduisons des estimations a priori et l'existence d'une solution strictement positive pour des problèmes de Dirichlet associés. Nous améliorons de manière significative les résultats connus pour une large classe de systèmes présentant une compétition entre les termes d'attraction et de répulsion. Parmi ceux-ci apparaissent des modèles biologiques de type Lotka-volterra mais aussi des modèles physiques pour les condensats de Bose-Einstein.

1 Introduction

This paper is concerned with existence, non-existence and qualitative properties of classical solutions of nonlinear elliptic systems in the form

$$\begin{cases} -\Delta u = f(x,u,v), \\ -\Delta v = g(x,u,v). \end{cases} \tag{1.1}$$

1. Ce chapitre est tiré de l'article [P5] écrit en collaboration avec **Philippe Souplet et Boyan Sirakov**.

In a nutshell, we will consider noncooperative, possibly nonvariational, systems with nonlinearities which have power growth in u, v, and in which the reaction terms dominate the absorption ones. We will be interested in nonexistence or more general classification results in unbounded domains such as $\mathbb{R}^n$ or a half-space in $\mathbb{R}^n$, as well as in their applications to a priori estimates and existence of positive solutions of Dirichlet problems in bounded domains.

1.1 A model case

We will illustrate our results by applying them to the system

$$\begin{cases} -\Delta u = u v^p \left[a(x)v^q - c(x)u^q\right] + \mu(x)u & \text{in } \Omega, \\ -\Delta v = v u^p \left[b(x)u^q - d(x)v^q\right] + \nu(x)v & \text{in } \Omega, \\ \quad u = 0,\ v = 0 & \text{on } \partial\Omega \text{ (if } \partial\Omega \neq \emptyset), \end{cases} \tag{1.2}$$

where $\Omega \subseteq \mathbb{R}^n$,

$$p \geq 0, \qquad q > 0, \qquad q \geq |1-p|, \tag{1.3}$$

and the coefficients a, b, c, d, μ, ν are Hölder continuous functions in $\overline{\Omega}$, with

$$a, b > 0 \ \text{ in } \ \overline{\Omega}, \qquad c, d \geq 0 \ \text{ in } \ \overline{\Omega}. \tag{1.4}$$

Observe (1.2) already covers a large class of systems satisfied by stationary states of coupled reaction-diffusion equations, or by standing waves of Schrödinger equations in the typical form

$$\mathbf{U}_t - \Delta\mathbf{U} = \mathcal{A}(x, \mathbf{U})\mathbf{U}, \qquad\qquad i\mathbf{U}_t - \Delta\mathbf{U} = \mathcal{A}(x, \mathbf{U})\mathbf{U}, \tag{1.5}$$

where $\mathbf{U} = (u, v)^T$ and $\mathcal{A}$ is a matrix which describes the replication rate of and the interaction between the quantities u and v. Let us mention that for $p = 0$ and $q = 1$ we obtain a Lotka-Volterra system, while for $p = 0$ and $q = 2$ we get a system arising in the theory of Bose-Einstein condensates and nonlinear optics, which has been widely studied in the recent years. Systems like (1.2) with $p > 0$ appear in models of chemical interactions. A more detailed discussion and references will be given in Section 1.3, below.

We will almost always assume that the reaction terms in the system dominate the absorption terms, in the following sense

$$D := ab - cd \geq 0 \qquad \text{in } \ \Omega. \tag{1.6}$$

The following two theorems effectively illustrate the more general results below. Here and in the rest of the article, $\lambda_1(-\Delta, \Omega)$ denotes the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions in Ω.

Theorem 1.1. *Let Ω be a smooth bounded domain. Assume that (1.3)–(1.4) hold,*

$$\inf_{\Omega} D > 0, \qquad p + q < \frac{4}{(n-2)_+} \tag{1.7}$$

and

$$\mu, \nu < \lambda_1(-\Delta, \Omega) \ \text{ in } \ \overline{\Omega}. \tag{1.8}$$

Then the system (1.2) has a classical solution (u, v) in Ω, such that $u, v > 0$ in Ω. All such solutions are uniformly bounded in $L^\infty(\Omega)$.

Theorem 1.2. *Assume that (1.3)–(1.4) hold, $\mu = \nu = 0$, and a, b, c, d are constants. Let (u, v) be a nonnegative classical solution of (1.2).*

1. *If $\Omega = \mathbb{R}^n$ and $D \geq 0$ then either $u \equiv 0$, or $v \equiv 0$, or $u \equiv Kv$ for some unique constant $K > 0$.*
2. *If $\Omega = \mathbb{R}^n$, $D > 0$ and $p + q < \frac{4}{(n-2)_+}$, then for some nonnegative constant $C \geq 0$*

$$(u, v) \equiv (C, 0) \quad \text{or} \quad (u, v) \equiv (0, C).$$

   If $p = 0$ then $C = 0$.
3. *If Ω is a half-space of $\mathbb{R}^n$ and $u, v \in L^\infty(\Omega)$, then $u = v \equiv 0$.*

1.2 A quick overview of our goals and methods

As the previous two theorems show, the two main goals we pursue are :

(a) obtain classification (or non-existence) results for solutions of (1.1) in $\mathbb{R}^n$ or in a half-space of $\mathbb{R}^n$. Naturally, to that goal we need to assume some homogeneity of (1.1) in (u, v).

(b) prove a priori estimates and existence statements for the Dirichlet problem for (1.1) in bounded domains. The "blow-up" method of Gidas and Spruck yields such results for general nonlinearities f and g, whose leading terms are the functions for which the non-existence theorems in (a) are proved.

This scheme is well-known and has been used widely since the pioneering works [25, 26]. As can be expected, the main effort falls on the classification results in (a). It should be stressed that these classification results must not be viewed only as a step to the existence results in (b), but are of considerable importance in themselves.

It appears that for systems in the whole space or in a half-space, most methods to prove Liouville type theorems under optimal growth assumptions are based either on moving planes or spheres and Kelvin transform, and hence require some rather restrictive cooperativity assumptions (cf. [41, 21, 20, 47]) ; or on integral identities such as Pohozaev's identity, and hence require some variational structure (cf. [42, 43, 44, 40, 45, 16]).

However, there are large classes of systems appearing in applications, whose structure is not treatable by these techniques. One of our basic observations is that many such systems have an inherent monotonicity structure expressed by the following hypothesis

$$\exists\, K>0\ :\quad [f(x,u,v)-Kg(x,u,v)][u-Kv]\le 0 \quad \text{for all } (u,v)\in\mathbb{R}^2 \text{ and } x\in\Omega, \tag{1.9}$$

which plays a fundamental role in our nonexistence and classification results.

To fix ideas, we will immediately describe a class of systems that appear in applications, satisfy (1.9), but do not seem to be manageable by the well-known methods for establishing Liouville type results. Our techniques naturally extend to even more general systems that satisfy the condition (1.9) (observe that verifying (1.9) for any given system is a matter of simple analysis).

Consider the system

$$\begin{cases} -\Delta u = u^r v^p[av^q - cu^q] \\ -\Delta v = v^r u^p[bu^q - dv^q]. \end{cases} \tag{1.10}$$

In our study of (1.10) we always assume that the real parameters a,b,c,d,p,q,r satisfy

$$a,b>0,\quad c,d\ge 0,\qquad p,r\ge 0,\quad q>0,\quad q\ge |p-r|. \tag{1.11}$$

Proposition 1.1. *Assume (1.11).*

(i) Then the nonlinearities in system (1.10) *satisfy* (1.9).

(ii) Assume moreover that $ab\ge cd$*. Then the number* K *is unique. We have* $K=1$ *if and only if* $a+d=b+c$ *and* $K>1$ *if and only if* $a+d>b+c$*. In addition, if* $ab>cd$ *(resp.* $ab=cd$*), then* $a-cK^q>0$ *(resp.* $=0$*) and* $bK^q-d>0$ *(resp.* $=0$*).*

The proof of this proposition is of course elementary (though tedious), and will be given in the appendix. There is no explicit formula for K, except in some special cases. For instance, when $p=0$ and $r=1$, one easily computes that $K=(\frac{a+d}{b+c})^{1/q}$. We remark that we do not know whether the last hypothesis in (1.11) is necessary for our classification results. However, it cannot be completely relaxed, at least if we want Theorem 1.2(1) (or the more general Theorem 2.1 below) to be true with a unique value of K, as shown by the counterexample in [39, Theorem 1.4(iii)] (where $r=p+q+1\ge n/(n-2)$, with $a=b=1$, $c=d=0$). On the other hand, many models of which we are aware satisfy this hypothesis.

Let us now discuss the use of (1.9). The point of this hypothesis is that for any solution (u,v) of (1.1), the nonnegative functions $(u-Kv)_+$ and $(Kv-u)_+$ are *subharmonic* in Ω, which allows for applications of various forms of the maximum principle.

In particular, if (1.9) holds, the domain Ω is bounded and $u=Kv$ on $\partial\Omega$, then the classical maximum principle implies the classification property

$$u\equiv Kv \quad \text{in } \Omega, \tag{1.12}$$

which reduces the system to a single elliptic equation. Our basic goal will be to prove (1.12) for unbounded domains like the whole space $\mathbb{R}^n$ or a half-space, where additional work and hypotheses are needed. We comment briefly on these next.

First, when Ω is a half-space and u, v have sublinear growth at infinity, (1.12) is a consequence of the Phragmén-Lindelöf maximum principle (a tool which is not often encountered in the context of Liouville theorems for nonlinear systems). This classifies bounded solutions in a half-space, which in particular is sufficient for the application of the blow-up method. We obtain nonexistence and classification results for general unbounded solutions in a half-space as well - then some supplementary assumptions are unavoidable. The proofs of these more general results use properties of spherical means of functions in a half-space, as well as a general nonexistence result for weighted elliptic inequalities in cones from [3].

Next, proving (1.12) in the whole space is where we encounter most difficulties. Probably the most important and novel observation we make is that under our assumptions the functions $Z = \min(u, Kv)$ and $W = |u - Kv|$ satisfy the inequality

$$-\Delta Z \geq cW^\mu Z^r \qquad \text{in } \mathbb{R}^n, \tag{1.13}$$

and, in some cases,

$$\Delta W \geq cZ^p W^\gamma \qquad \text{in } \mathbb{R}^n, \tag{1.14}$$

for appropriate $\mu, \gamma \geq 1$, $c > 0$. It is worth observing that in (1.13) the *superharmonic* function Z satisfies an anti-coercive elliptic inequality with a *subharmonic* weight W^μ, while in (1.14) the subharmonic function W satisfies a coercive inequality with a weight which is a power of a superharmonic function. We do not know of any other work where such combinations of inequalities and weights appear. By using properties of subharmonic and superharmonic functions and by adapting the methods for proving nonexistence of positive solutions of inequalities from [3] (for (1.13)) and from [31] (for (1.14)), we show that under appropriate restrictions on the exponents p, r, we have $W \equiv 0$.

The idea of showing nonexistence results by first proving the property (1.12) was used earlier in [32, 17] for a particular Lotka-Volterra type system, and more recently in [39, 23, 13], where some partial use of (1.9) with $K = 1$ was also made. To our knowledge the present paper is the first systematic study of systems whose nonlinearities satisfy (1.9). Our results very strongly improve on previous ones, both in the generality of the systems considered, and in the results obtained, even when applied to particular systems. Our methods, in particular the above observations, appear to be new.

Finally, as far as step (b) above is concerned, we recall that uniform a priori estimates and existence of positive solutions of Dirichlet problems associated with asymptotically homogeneous systems in bounded domains can be obtained via the rescaling (or blow-up) method of Gidas and Spruck [25] combined with known topological degree arguments (see for instance [12, 32, 17, 21, 20, 47] for systems). Applying this method requires nonexistence theorems for the limiting "blown-up" system in the whole space and in the half-space. We will follow the same scheme here ; however, as an additional and nontrivial difficulty with respect to the cases treated in [12, 32, 17, 21, 20], we will need to deal with the fact that many of the limiting systems that we obtain admit semi-trivial solutions in the whole space, of the form $u = 0, v = C$ or $u = C, v = 0$, with $C > 0$ (for instance system (1.10) with $p, r > 0$). Additional arguments are thus needed to rule out the occurrence of such limits (see Remark 6.2).

1.3 Some systems that appear in applications, to which our results apply

The results in Section 1.1 apply in particular to the following two systems which we already mentioned

$$(LV)\begin{cases} -\Delta u = u\Big[a(x)v - c(x)u + \mu(x)\Big] \\ -\Delta v = v\Big[b(x)u - d(x)v + \nu(x)\Big], \end{cases} \qquad (BE)\begin{cases} -\Delta u = u\Big[a(x)v^2 - c(x)u^2 + \mu(x)\Big] \\ -\Delta v = v\Big[b(x)u^2 - d(x)v^2 + \nu(x)\Big]. \end{cases}$$

The first of these two systems is of Lotka-Volterra type, and appears as a model of symbiotic interaction of biological species. In (LV) the logistic terms $(\mu - cu)u$ and $(\nu - dv)v$ take into account the reproduction and the limitation of resources within each species, while the uv-terms represent the interaction between the two species. A positive solution then corresponds to a coexistence state – see for instance [28, 32, 17] and the references therein for more details on the biological background.

The system (BE) arises in models of Bose-Einstein condensates which involve two different quantum states, as well as in nonlinear optics. In particular, one gets (BE) when looking for standing waves of an evolutionary cubic Schrödinger system. In the present case, the interspecies interaction is attractive, while the self-interaction is repulsive or neutral, leading to phenomena of symbiotic solitons. We refer to [36, 11] for a description of physical phenomena that lead to such systems. These references include systems with spatially inhomogeneous coefficients.

For (LV) and (BE), we get the following result as a direct consequence of Theorem 1.1.

Corollary 1.1. *Assume Ω is a smooth bounded domain, a, b, c, d, μ, ν are Hölder continuous in $\overline{\Omega}$, and* (1.4), (1.8) *hold. Assume further that $n \le 5$ for (LV), $n \le 3$ for (BE).*

If

$$\inf_{x\in\Omega} \, [a(x)b(x) - c(x)d(x)] > 0, \tag{1.15}$$

then there exists at least one positive classical solution of (LV) or (BE) in Ω, such that $u = v = 0$ on $\partial\Omega$. All such solutions are bounded above by a constant which depends only on Ω, and the uniform norms of a, b, c, d, μ, ν.

Observe that (1.15) cannot be removed, as simple examples show. For instance, if $a = b = c = d$ and $\mu = \nu = 0$ in (LV) or (BE), by adding up the two equations we see that any nonnegative solution of the Dirichlet problem vanishes identically.

In spite of the huge number of works on Lotka-Volterra systems (giving a reasonably complete bibliography is virtually impossible), this corollary represents an improvement on known results for (LV) — see [17, Theorem 7.4], where a more restrictive assumption than (1.15) was made on the functions a, b, c, d.

For the system (BE), most of the previously known statements on a priori estimates and existence concerned the case of reversed interactions (a, b positive and c, d negative ; or a, b, c, d negative) ; see [4, 5, 15, 46, 40, 20]. The self-repulsive case which we consider here was also studied in [30], where positive solutions are constructed by variational methods under the additional hypothesis that $a = b$ and a, b, c, d are large constants. Thus Corollary 1.1 completes these works, providing optimal results for the case of attractive interspecies interaction, and repulsive or neutral intraspecies interaction.

Finally, we point out the following third example, which is a special case of a class of systems arising in the modelling of general chemical reactions

$$\left\{\begin{aligned} u_t - \Delta u &= uv\Big[a(x)v - c(x)u\Big], && t > 0,\ x \in \Omega,\\ \gamma v_t - \Delta v &= uv\Big[b(x)u - d(x)v\Big], && t > 0,\ x \in \Omega,\\ u = v &= 0, && t > 0,\ x \in \partial\Omega, \end{aligned}\right. \tag{1.16}$$

where Ω is a bounded domain and $\gamma > 0$. See for instance equation (1) in [18] and equations (3.1), (3.5) in [37], as well as the other examples and references given in these works (note that more general power-like behaviour in the nonlinearities can be considered as well). Specifically, system (1.16) in the case $a(x) = d(x)$ and $b(x) = c(x)$ describes the evolution of the concentrations of two chemical molecules A and B in the reversible reaction

$$A + 2B \underset{k_2}{\overset{k_1}{\rightleftarrows}} 2A + B,$$

under inhomogeneous catalysis with reaction speeds $k_1 = a(x)$, $k_2 = b(x)$, and absorption on the boundary. (Note that the net result of the reaction is $B \rightleftarrows A$ and that the molecules A, B should thus be isomeric.) In this case, it is easy to see by considering $u + v$ that the only nonnegative equilibrium is $(u, v) = (0, 0)$. Hence, Theorem 1.1 shows the existence of a bifurcation phenomenon for the stationary system associated with (1.16), precisely at $a(x) = d(x)$ and $b(x) = c(x)$. Indeed, assume $n \leq 3$ and let the Hölder continuous functions a, b, c, d satisfy $ab = cd + \varepsilon$, $a, b > 0$ and $c, d \geq 0$ in $\overline{\Omega}$. Then there exists a positive steady state beside the trivial one, for each $\varepsilon > 0$.

It is worth noticing that the discussion in [37] (see eqn. (3.6) in that paper) provides a physical explanation as to why the case $ab > cd$ differs strongly from $ab \leq cd$.

As is pointed out in [37], in the case of constant coefficients, $ab \leq cd$ guarantees that the system (1.16) exhibits control of mass (we refer to [37] for definitions), or, in other words, the absorption in the system controls the reaction. Under this assumption, it can be shown that any global and bounded solution converges uniformly to $(0,0)$ as $t \to \infty$ (however, whether or not some solutions may blow up in finite time is a highly nontrivial question in general – see [37] and the references therein). It should then come as no surprise that the case $ab > cd$, in which no control of mass is available, is delicate to study, even in the stationary (elliptic) case.

2 Main results

We will only consider classical solutions, for simplicity. Observe that under our hypotheses on f, g, any continuous weak-Sobolev solution of (1.1) is actually classical, by standard elliptic regularity.

In what follows, we say that (u, v) is semi-trivial if $u \equiv 0$ or $v \equiv 0$. We say that (u, v) is positive if $u, v > 0$ in the domain where a given system is set.

2.1 Classification results in the whole space

In this section we study the system (1.10) in $\mathbb{R}^n$. The following theorem plays a pivotal role in the paper and is probably its most original result.

Theorem 2.1. *Assume (1.11) holds and $ab \geq cd$. Let $K > 0$ be the constant from Proposition 1.1 and (u, v) be a positive classical solution of (1.10) in $\mathbb{R}^n$.*

(i) Assume that

$$r \leq \frac{n}{(n-2)_+}. \tag{2.1}$$

If $p + q < 1$, assume in addition that (u, v) is bounded. Then $u \equiv Kv$.

(ii) Assume that

$$p \leq \frac{2}{(n-2)_+} \quad \text{and} \quad c, d > 0. \tag{2.2}$$

If $q + r \leq 1$, assume in addition that (u, v) is bounded. Then $u \equiv Kv$.

We stress that, remarkably, Theorem 2.1 includes *critical and supercritical* cases, since no upper bound is imposed on the *total degree* $\sigma := p + q + r$ of the system (1.10).

Theorem 2.1 provides a classification of positive solutions of (1.10) in $\mathbb{R}^n$. Specifically, the set of positive solutions of (1.10) is given by $(u, v) = (KV, V)$, where V is either a positive harmonic function, hence constant (if $ab = cd$) or V is a solution of

$$-\Delta V = c_1 V^\sigma \qquad \text{in } \mathbb{R}^n, \tag{2.3}$$

with $c_1 = K^p(bK^q - d) > 0$ (if $ab > cd$, by Proposition 1.1). It is well known that positive solutions of (2.3) exist precisely if $n \geq 3$ and $\sigma \geq (n+2)/(n-2)$. They are moreover unique up to rescaling and translation, and explicit, if $\sigma = (n+2)/(n-2)$ (see [9]). For some related classification results for cooperative systems with $c = d = 0$ in the critical case, which use the method of moving planes, see [27, 29].

Theorem 2.1 significantly improves the results from [39] concerning system (1.10) (see [39, Theorem 2.3]). There, only the case $a = b$, $c = d$ (hence $K = 1$) was considered and, for that case, much stronger restrictions than (2.1) or (2.2) were imposed.

Combining Theorem 2.1 with known results on scalar equations yields the following striking Liouville type result for the noncooperative system (1.10), with an optimal growth assumption on the nonlinearities.

Theorem 2.2. *Assume (1.11), $ab > cd$, and*

$$\sigma := p + q + r < \frac{n+2}{(n-2)_+}.$$

(i) Then system (1.10) does not admit any positive, classical, bounded solution.

(ii) Assume in addition

$$p + q \geq 1, \qquad or \qquad p \leq \frac{2}{(n-2)_+}, \qquad or \qquad \sigma \leq \frac{n}{(n-2)_+}$$

(note this hypothesis is satisfied in each one of the "physical cases" $q \geq 1$ or $n \leq 4$). Then system (1.10) does not admit any positive classical (bounded or unbounded) solution.

Once positive solutions are ruled out, it is natural to ask about nontrivial nonnegative solutions (and this will be important in view of our applications to a priori estimates, below). The following result is a simple consequence of Theorem 2.2 and the strong maximum principle.

Corollary 2.1. *Under the hypotheses of Theorem 2.2(i) (resp., 2.2(ii)), assuming in addition $q + r \geq 1$, any nonnegative bounded (resp., nonnegative) solution of (1.10) is in the form $(C_1, 0)$ or $(0, C_2)$, where C_1, C_2 are nonnegative constants.*

Moreover, if in addition $p = 0$, $r > 0$ and $c > 0$ (resp., $d > 0$), then $C_1 = 0$ (resp., $C_2 = 0$), whereas, if $r = 0$, then $C_1 = C_2 = 0$.

We end this subsection with several remarks on the hypotheses in the above theorems. It is not known whether or not the restrictions (2.1), (2.2) are optimal for the property $u \equiv Kv$. However, the following result shows that this property may fail if p and r are large enough.

Theorem 2.3. *Let $n \geq 3$ and consider system (1.10) with $p = r > (n+2)/(n-2)$, $q > 0$ and $a = b = c = d = 1$. Then there exists a positive solution such that u/v is not constant.*

We remark that if $c = 0$ or $d = 0$ then we can show that at least one of the components dominates the other, without restrictions on p or r.

Proposition 2.1. *Assume (1.11) and $c = 0$ or $d = 0$. Assume that either (u, v) is bounded or $\max(p + q, q + r) > 1$. Then either $u \geq Kv$ in $\mathbb{R}^n$ or $u \leq Kv$ in $\mathbb{R}^n$.*

Remark 2.1. *If u and v are assumed to be radially symmetric, it is easy to show that we have $u \geq Kv$ or $u \leq Kv$, only under the assumptions (1.11) and $ab \geq cd$ (see the end of section 4.1).*

Next, we recall that the property $u = Kv$ is known to be true for all nonnegative solutions of (1.10) provided $p = 0$ (so that the system is cooperative), and $q \geq r > 0$, $c, d > 0$, $q + r > 1$. The proof of this fact (see [32, 13, 23]) relies on the observation that the function $w = (u - Kv)_+$ satisfies $\Delta w \geq c_1(u^{q+r-1} + v^{q+r-1})w$ for some constant $c_1 > 0$, which leads to the "coercive" elliptic inequality

$$\Delta w \geq c_1 w^{q+r} \quad \text{in } \mathbb{R}^n. \tag{2.4}$$

It then follows from a classical result of Keller and Osserman (see also Brezis [7]) that $w \equiv 0$, hence $u \equiv Kv$ (after exchanging the roles of u, v). The same idea applies in the half-space, under homogeneous Dirichlet boundary conditions. However, this argument fails if $p > 0$, or if c or $d = 0$, since one does not obtain a coercive equation like (2.4). Nevertheless, we will be able to use some more general coercivity properties in the proof of Theorem 2.1 for $p \leq 2/(n-2)$, see (1.14).

Finally, we recall that the case $ab < cd$ in system (1.10) is very different, since the absorption features then become dominant. For instance, if $p = 0$ and $ab < cd$, then any nonnegative solution of (1.10) has to be trivial if $q+r > 1$, in sharp contrast with the case $ab > cd$ (when nontrivial solutions (u, Ku) exist if $q + r \geq (n+2)/(n-2)$). Indeed, by Young's inequality, one easily checks that $w = u + tv$ satisfies (2.4) for suitable $t, c_1 > 0$, hence $w \equiv 0$. An interesting question, though outside the scope of this paper, is to determine the optimal conditions on $p, q, r \geq 0$ under which classification results can be proved, when $ab < cd$.

2.2 Classification results in the half-space

We begin with a rather general classification result for system (1.1) on the half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, under the basic structure assumption (1.9).

A function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is said to have *sublinear growth* if $u(x) = o(|x|)$ as $|x| \to \infty$, $x \in \mathbb{R}^n_+$. The following theorem classifies positive solutions with sublinear growth in $\mathbb{R}^n_+$, and implies nonexistence results by reducing the system to a scalar equation. It will thus be sufficient, along with Liouville type results for bounded solutions in the whole space (stated in Section 2.1), in order to prove a priori estimates via the blow-up method.

Theorem 2.4. *Assume that (1.9) holds. Let (u, v) be a classical solution of (1.1) in $\mathbb{R}^n_+$, such that $u = Kv$ on $\partial\mathbb{R}^n_+$. If u and v have sublinear growth, then*

$$u \equiv Kv \quad \text{in } \mathbb{R}^n_+.$$

This theorem is a consequence of the Phragmèn-Lindelöf maximum principle.

Remark 2.2. *Observe that we did not make any assumption on the sign or on the growth of the nonlinearities f and g. Therefore, supercritical nonlinearities can be allowed.*

Theorem 2.4 can be used to deduce Liouville type theorems for noncooperative systems. We have for instance the following result, which applies to the system (1.10).

Corollary 2.2. *Assume that (1.9) holds for some $K > 0$, and there exist constants $c > 0$ and $p > 1$ such that*

$$f(x, Ks, s) = cs^p, \quad s \geq 0.$$

Then system (1.1) has no nontrivial, bounded, classical nonnegative solution in $\mathbb{R}^n_+$, such that $u = v = 0$ on the boundary $\partial\mathbb{R}^n_+$.

This corollary is obtained by combining Theorem 2.4 with a recent result [10], which guarantees that, for any $p > 1$, the scalar equation $-\Delta u = u^p$ has no positive, bounded, classical solution in the half-space, which vanishes on the boundary (this was known before under additional restriction on p, see [25, 14, 22]).

Under further assumptions on the nonlinearities, namely positivity (one may think of $c = d = 0$ in (1.10)), we obtain classification results in the half-space, without making growth restrictions on the solutions.

Theorem 2.5. *Let $p, q, r, s \geq 0$. We assume that f, g satisfy condition (1.9) for some constant $K > 0$ and that, for some $c > 0$,*

$$f(x, u, v) \geq c\, u^r\, v^p \quad \text{and} \quad g(x, u, v) \geq c\, u^q\, v^s \quad \text{for all } u, v \geq 0 \text{ and } x \in \mathbb{R}^n_+. \tag{2.5}$$

Let (u, v) be a nonnegative classical solution of (1.1) in $\mathbb{R}^n_+$, such that $u = Kv$ on $\partial\mathbb{R}^n_+$.

(i) Either $u \leq Kv$ or $u \geq Kv$ in $\mathbb{R}^n_+$.

(ii) If

$$r \leq \frac{n+1+p}{n-1} \quad \text{or} \quad q \leq \frac{1+s}{n-1}, \tag{2.6}$$

and

$$s \leq \frac{n+1+q}{n-1} \quad \text{or} \quad p \leq \frac{1+r}{n-1}, \tag{2.7}$$

then either $u \equiv Kv$ or (u, v) is semitrivial.

(iii) If (2.6)-(2.7) *hold and* $\min(p+r, q+s) \leq (n+1)/(n-1)$, *then* (u,v) *is semitrivial.*

Theorem 2.5 complements [39, Theorem 1.2], which concerned similar problems in $\mathbb{R}^n$.

Remark 2.3. *The restrictions (2.6)–(2.7) are unlikely to be optimal, since they are strongly related to nonexistence results for inequalities in the half-space. Recall that* $-\Delta v = v^p$ *has no positive solutions vanishing on the boundary for each* $p > 1$*, while the same is valid for* $-\Delta v \geq v^p$ *if and only if* $p \leq (n+1)/(n-1)$.

The proof of Theorem 2.5 makes use of a generalization of Theorem 2.4, which we state next. If w is a continuous function in $\overline{\mathbb{R}^n_+}$, we denote with $[w]$ its half-spherical mean, defined by

$$[w](R) = \frac{1}{|S_R^+|}\int_{S_R^+} \frac{w(x)}{R}\,\frac{x_n}{R}\,d\sigma_R(x),$$

for each $R > 0$, where $S_R^+ = \{x \in \mathbb{R}^n_+,\ |x| = R\}$.

Theorem 2.6. *Assume that (1.9) holds. Let* (u,v) *be a classical solution of (1.1) in* $\mathbb{R}^n_+$*, such that* $u = Kv$ *on* $\partial\mathbb{R}^n_+$*. If*

$$\liminf_{R\to\infty}[(u-Kv)_+](R) = 0 \qquad \textit{and} \qquad \liminf_{R\to\infty}[(Kv-u)_+](R) = 0, \tag{2.8}$$

then $u \equiv Kv$.

Remark 2.4. *If* u, v *have sublinear growth then* $\lim_{R\to\infty}[|u|](R) = \lim_{R\to\infty}[|v|](R) = 0$*, which in turn implies (2.8). Hence Theorem 2.4 is a consequence of Theorem 2.6.*

2.3 A priori estimates and existence in bounded domains

We consider the Dirichlet problem

$$\begin{cases} -\Delta u = u^r v^p\Big[a(x)v^q - c(x)u^q\Big] + \mu(x)u, & x \in \Omega, \\ -\Delta v = v^r u^p\Big[b(x)u^q - d(x)v^q\Big] + \nu(x)v, & x \in \Omega, \\ \quad u = v = 0, & x \in \partial\Omega, \end{cases} \tag{2.9}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$. For simplicity, here we restrict ourselves to linear lower order terms. Further results, for systems with more general lower order terms, will be given in Section 6. Note that, due to the space dependence of the coefficients a, b, c, d and to the presence of the lower order terms, the right-hand side of system (2.9) does not satisfy (1.9), in general. Therefore, system (2.9) cannot be directly reduced to a scalar problem via the property $u \equiv Kv$.

Theorem 2.7. *Let $p, r \geq 0$, $q > 0$, and*

$$q \geq |p-r|, \qquad q+r \geq 1, \qquad r \leq 1, \qquad 1 < p+q+r < \frac{n+2}{(n-2)_+}. \tag{2.10}$$

Let $a, b, c, d, \mu, \nu \in C(\overline{\Omega})$ satisfy $a, b > 0$, $c, d \geq 0$ in $\overline{\Omega}$ and

$$\inf_{x\in\Omega} \left[a(x)b(x) - c(x)d(x)\right] > 0. \tag{2.11}$$

(i) Then there exists $M > 0$, depending only on p, q, r, Ω, and the uniform norms of a, b, c, d, μ, ν, such that any positive classical solution (u, v) of (2.9) satisfies

$$\sup_\Omega u \leq M, \quad \sup_\Omega v \leq M.$$

(ii) Assume in addition that a, b, c, d, μ, ν are Hölder continuous and that $\mu, \nu < \lambda_1(-\Delta, \Omega)$ in $\overline{\Omega}$. Then there exists at least one positive classical solution of (2.9).

As we already observed, Theorem 2.7 seems to be new even for very particular cases of (2.9), for instance the system (BE) from Section 1.3.

The rest of the paper is organized as follows. In the preliminary Section 3 we state some essentially known nonexistence results for scalar inequalities with weights. In Section 4 we prove the main classification and Liouville type results for the repulsive-attractive system (1.10) in the whole space. In Section 5 we introduce the half-spherical means, establish their monotonicity properties and prove Theorems 2.4 and 2.6. Then we prove some further properties of half-spherical means of super-harmonic functions, and deduce Theorem 2.5. Finally, Section 6 is devoted to a priori estimates by the rescaling method and existence by topological degree arguments. In the appendix we gather some elementary computations related to Proposition 1.1.

3 Preliminary results. Liouville theorems for weighted inequalities in unbounded domains.

In this section we state three essentially known nonexistence results for scalar elliptic inequalities. We require such properties both for inequalities with source and for inequalities with absorption.

In the rest of the paper, a *weak* solution of an (in)equality in a given domain $\Omega \subset \mathbb{R}^n$ will mean a function in $H^1_{loc}(\Omega) \cap C(\overline{\Omega})$ which verifies the given (in)equality in the sense of distributions.

We begin with the following Liouville type result for weighted elliptic inequalities with space dependence in an exterior domain of the half-space.

Lemma 3.1. *Let $r \geq 0$ and u be a nonnegative weak solution of :*

$$-\Delta u \geq h(x)\, u^r \qquad \text{on } \mathbb{R}^n_+ \setminus B_1, \tag{3.1}$$

where $h \geq 0$ on $\mathbb{R}^n_+ \setminus B_1$ and there exists $\kappa > -2$ such that $\kappa + r \geq -1$, and

$$h(x) \geq c|x|^\kappa \qquad \text{in the cone } \{x \,:\, x_n \geq \delta|x|\} \setminus B_1,$$

for some constants $c, \delta > 0$.

If

$$0 \leq r \leq \frac{n+1+\kappa}{n-1},$$

then $u = 0$.

Démonstration. This follows from Theorem 5.1 or Corollary 5.6 in [3]. Note that theorem was stated for $h(x) = c|x|^\kappa$ but its proof contains the statement of Lemma 3.1. As is explained in Section 3 of [3], the results in that paper hold for any notion of weak solution, for which the maximum principle and some related properties are valid. □

Remark 3.1. *We will apply Lemma 3.1 with h in the form $h(x) = cx_n^s|x|^{-m}$.*

The next result plays a crucial role in our proofs below.

Lemma 3.2. *Assume $0 \leq r \leq n/(n-2)_+$ and let $V \in C(\mathbb{R}^n)$, $V \geq 0$, $V \not\equiv 0$ be such that*

$$\liminf_{R\to\infty} R^{-n} \int_{B_R \setminus B_{R/2}} V(x)\, dx > 0. \tag{3.2}$$

Let $z \geq 0$ be a weak solution of

$$-\Delta z \geq V(x) z^r \qquad \text{in } \mathbb{R}^n. \tag{3.3}$$

Then $z \equiv 0$.

The point is that in inequality (3.3) the potential $V(x)$ is not assumed to be bounded below by a positive constant (in which case the result is well-known - see for instance [33]), but only in *average on large annuli*.

In particular, Lemma 3.2 applies if $V \gneqq 0$ is a *subharmonic* function. Indeed, the mean-value inequality and the well-known fact that for each subharmonic function V the spherical average $\bar{V}(R) = \oint_{\partial B_R} V$ is nondecreasing in R easily imply that (here $\oint$ stands for the average integral)

$$\oint_{B_R} V(x)\, dx \leq \frac{n}{R}\int_0^R \bar{V}(r)\, dr \leq \frac{2n}{R}\int_{R/2}^R \bar{V}(r)\, dr \leq C(n) \oint_{B_R\setminus B_{R/2}} V(x)\, dx,$$

hence, for each $x_0 \in \mathbb{R}^n$

$$C(n)\liminf_{R\to\infty} \oint_{B_R\setminus B_{R/2}} V(x)\, dx \geq \liminf_{R\to\infty} \oint_{B_R} V(x)\, dx = \liminf_{R\to\infty} \oint_{B_R(x_0)} V(x)\, dx \geq V(x_0),$$

which implies, for each subharmonic $V \gneqq 0$ and some positive constant $c(n)$,

$$\liminf_{R\to\infty} R^{-n} \int_{B_R\setminus B_{R/2}} V(x)\,dx \geq c(n) \sup_{\mathbb{R}^n} V.$$

Lemma 3.2 can be proved through a slight modification of the argument introduced in [3]. We will give a full and simplified proof, for completeness.

We first recall the following "quantitative strong maximum principle".

Lemma 3.3. *Let Ω be a smooth bounded domain and K be a compact subset of Ω. There exists a constant $c > 0$ depending only on n, K, dist$(K, \partial\Omega)$, such that if h is a nonnegative bounded function and u satisfies the inequality*

$$-\Delta u \geq h \quad \text{in } \Omega, \qquad \text{then} \qquad \inf_K u \geq c\int_K h(x)\,dx.$$

For a simple proof of Lemma 3.3 one may consult Lemma 3.2 in [8]. Lemma 3.3 can also be seen as a consequence of the fact that the Green function of the Laplacian in any domain is strictly positive away from the boundary of the domain.

Proof of Lemma 3.2. If $n \leq 2$ Lemma 3.2 is immediate, since every positive superharmonic function in $\mathbb{R}^2$ is constant.

Suppose now $n \geq 3$ and u is a solution of (3.3). Set $u_R(x) := u(Rx)$ and $m(R) := \inf_{\partial B_R} u = \inf_{\partial B_1} u_R$. By the maximum principle $m(R) = \inf_{B_R} u = \inf_{B_1} u_R$ and $m(R)$ is nonincreasing in R.

Observe that (3.2) is equivalent to the existence of $R_0 > 0$ and $c_0 > 0$ such that

$$\int_{B_1\setminus B_{1/2}} V(Rx)\,dx \geq c_0 > 0 \quad \text{for } R \geq R_0.$$

From now on we assume that $R \geq R_0$. Since u_R is a solution in $\mathbb{R}^n$ of the inequality

$$-\Delta u_R \geq R^2 V(Rx) u_R^p,$$

we can apply Lemma 3.3 with $\Omega = B_2$ and $K = \bar{B}_1$ and deduce

$$m(R) \geq cR^2 m(R)^p,$$

for some $c > 0$. If $p \leq 1$ this is a contradiction, since $m(R)$ is nonincreasing in R. If $p > 1$ we get

$$m(R) \leq CR^{-\frac{2}{p-1}}. \tag{3.4}$$

Since u is superharmonic, the maximum principle implies that

$$u(x) \geq m(1)|x|^{2-n} \quad \text{in } \mathbb{R}^n \setminus B_1,$$

and hence

$$m(R) \geq cR^{2-n} \quad \text{for } R \geq 1. \tag{3.5}$$

If $p < n/(n-2)$, combining (3.4) and (3.5), and letting $R \to \infty$ yields a contradiction.

Finally, assume that $p = n/(n-2)$, that is, $2/(p-1) = n-2$. Set $\tilde{u}_R(x) = R^{n-2}u(Rx)$. Then

$$-\Delta \tilde{u}_R \geq V(Rx)\tilde{u}_R^p. \tag{3.6}$$

Observe that

$$\tilde{m}(R) := \inf_{\partial B_1} \tilde{u}_R = \inf_{\partial B_R} \frac{u}{\Phi},$$

where $\Phi(x) = |x|^{2-n}$. We proved in (3.4)–(3.5) that $0 < c \leq \tilde{m}(R) \leq C$, for $R \geq R_0$.

By the maximum principle $u(x) \geq \tilde{m}(R)\Phi(x)$ in $\mathbb{R}^n \setminus B_R$, which is equivalent to $\tilde{u}_R \geq \tilde{m}(R)\Phi$ in $\mathbb{R}^n \setminus B_1$, by the $(2-n)$-homogeneity of Φ. In addition, $\tilde{m}$ is nondecreasing in R.

So (3.6) implies

$$-\Delta(\tilde{u}_R - \tilde{m}(R)\Phi) \geq V(Rx)\tilde{u}_R^p \geq cV(Rx) \quad \text{in } \ B_5 \setminus B_1. \tag{3.7}$$

We apply Lemma 3.3 to this inequality, with $\Omega = B_5 \setminus B_1$ and $K = B_4 \setminus B_{3/2}$, to deduce that

$$\tilde{u}_R \geq \tilde{m}(R)\Phi + c_0 = (\tilde{m}(R) + c_0 2^{n-2})\Phi \qquad \text{on } \ \partial B_2,$$

that is,

$$u \geq (\tilde{m}(R) + c_0 2^{n-2})\Phi \qquad \text{on } \ \partial B_{2R}.$$

Hence

$$\tilde{m}(2R) \geq \tilde{m}(R) + c_0 2^{n-2},$$

which implies $\tilde{m}(R) \to \infty$ as $R \to \infty$, a contradiction. □

The following lemma is a generalization of a classical result of Keller and Osserman to weak solutions of coercive problems with weights.

Lemma 3.4. *Let W be a nonnegative weak solution of*

$$\Delta W \geq \frac{A}{1+|x|^2} W^p \qquad \text{in } \ \mathbb{R}^n, \tag{3.8}$$

where $p \geq 0$ and $A > 0$.

(i) If $W \in L^\infty(\mathbb{R}^n)$, then $W = 0$.

(ii) If $p > 1$, then $W = 0$.

The statement (ii) in this lemma appeared first in [31] (see also [34] for an earlier result for potentials with subquadratic decay). We will provide a full and simplified proof in the case of the Laplacian, for the reader's convenience.

Proof of Lemma 3.4. In what follows we denote

$$V(x) = \frac{A}{1+|x|^2}.$$

Step 1. We assume $p > 1$ or $W \in L^\infty(\mathbb{R}^n)$. Given a solution W of (3.8), we prove the existence of a smooth Z satisfying the same equation, with possibly modified constant A.

We pick a nonnegative $\rho \in C^\infty(\mathbb{R}^n)$ with support inside $B(0,1)$ such that $\int_{\mathbb{R}}^n \rho = 1$ and set $Z = W * \rho \in C^\infty(\mathbb{R}^n)$. It is easy to see that

$$\Delta Z \geq [V\, W^p] * \rho$$

in the classical sense. Note that if $|y| \leq 1$, then $V(x-y) \geq \frac{1}{2}\frac{A}{1+|x|^2}$.

So

$$[V\, W^p] * \rho(x) = \int_{\mathbb{R}}^n V(x-y) W^p(x-y)\rho(y)\, dy \geq \frac{C}{2}\frac{A}{1+|x|^2} Z^p,$$

where $C = 1$ if $p \geq 1$ and $C = \frac{1}{\|W\|_\infty^{1-p}}$ if $0 \leq p < 1$ (if $p > 1$ we use Jensen's inequality).

Hence,

$$\Delta Z \geq \tilde{V} Z^p$$

where $\tilde{V} = \frac{\tilde{A}}{1+|x|^2}$ and $\tilde{A} = \frac{CA}{2}$. Note that if $W \in L^\infty(\mathbb{R}^n)$, then $Z \in L^\infty(\mathbb{R}^n)$.

Step 2. From Step 1, we can assume that W is smooth.

(i) Suppose for contradiction that $W \geq 0$ is bounded on $\mathbb{R}^n$ and does not vanish identically. We can assume without loss of generality that $W(0) > 0$, since the problem is invariant with respect to translations (a translation of V gives a function whose behaviour is the same as V).

For any $R > 0$, we denote the spherical mean of W by

$$\overline{W}(R) = \frac{1}{|S_R|}\int_{S_R} W\, d\sigma_R$$

where S_R is the sphere of center 0 and radius R, σ_R is the Lebesgue's measure on S_R and $|S_R| = \sigma_R(S_R)$. It is clear that $\overline{W}$ is bounded on $(0, +\infty)$.

Since W is subharmonic,

$$W(0) \leq \frac{1}{|B_R|}\int_{B_R} W\, dx.$$

It is easy to see that there exists $C > 0$ independent of R such that

$$(W(0))^{\max(p,1)} \leq C\, \frac{1}{|B_R|}\int_{B_R} W^p\, dx.$$

Indeed, if $p \geq 1$ then this is a consequence of Jensen's inequality (and $C = 1$), whereas if $0 \leq p \leq 1$, we can use the boundedness of W (and $C = \|W\|_\infty^{1-p}$).

We know that

$$\frac{d\overline{W}}{dR} = \frac{1}{|S_R|} \int_{B_R} \Delta W \, d\sigma_R,$$

hence

$$\frac{d\overline{W}}{dR} \geq \frac{A}{n} \frac{R}{1+R^2} \frac{1}{|B_R|} \int_{B_R} W^p \, dx \geq \frac{A}{nC} \frac{R}{1+R^2} (W(0))^{\max(p,1)} = C \frac{R}{1+R^2}.$$

But this implies $\overline{W}(R) \underset{R \to +\infty}{\longrightarrow} +\infty$, which is a contradiction.

(ii) We assume $p > 1$ and will prove that W is bounded, which implies the result, by the statement (i).

Arguing as in [35], we define the function W_R on B_R by

$$W_R(x) = C \frac{R^{2\alpha}}{(R^2 - |x|^2)^\alpha},$$

where $\alpha = \frac{2}{p-1}$. We will see, by direct computation, that if $C > 0$ is large enough, then

$$\Delta W_R \leq \frac{A}{1+|x|^2} {W_R}^p. \tag{3.9}$$

Indeed, denoting $r = |x|$, we have

$$\Delta W_R = 2\alpha C R^{2\alpha} \, \frac{n(R^2 - r^2) + 2(\alpha+1) r^2}{(R^2 - r^2)^{\alpha+2}} \leq 2\alpha C R^{2\alpha+2} \, \frac{n + 2(\alpha+1)}{(R^2 - r^2)^{\alpha+2}}$$

and

$$\frac{A}{1+r^2} {W_R}^p = \frac{A}{1+r^2} \frac{C^p R^{2\alpha p}}{(R^2 - r^2)^{\alpha p}} \geq \frac{A}{1+R^2} \frac{C^p R^{2\alpha p}}{(R^2 - r^2)^{\alpha p}}.$$

We note that $\alpha + 2 = \alpha p$. Hence, a sufficient condition to have (3.9) is

$$C^{p-1} \geq (1+R^2) R^{2\alpha+2-2\alpha p} \frac{2\alpha[n + 2(\alpha+1)]}{A}.$$

Since $2\alpha + 2 - 2\alpha p = -2$, for each $R \geq 1$ a sufficient condition for the last inequality is

$$C^{p-1} \geq \frac{4\alpha[n + 2(\alpha+1)]}{A},$$

and this is how we choose C.

It is now easy to see that $W \leq W_R$ on B_R. Note that $W_R(x) \to \infty$ as $x \to \partial B_R$. If we denote $w = W - W_R$ and if S is a C^2 nondecreasing convex function on $\mathbb{R}$ such that $S = 0$ on $(-\infty, 0]$ and $S > 0$ otherwise, then

$$\Delta S(w) \geq S'(w) \Delta w \geq S'(w) V(x) (W^p - {W_R}^p) \geq 0.$$

Hence $S(w)$ is subharmonic on B_R and can be continuously extended on $\overline{B_R}$ by setting $S(w) = 0$ on S_R, so by the maximum principle $S(w) = 0$, that is, $w \leq 0$.

Finally, for all $R \geq 1$, $W \leq W_R$ on B_R, so by letting $R \to \infty$ we obtain $W(x) \leq \lim_{R\to\infty} W_R(x) = C$, for each $x \in \mathbb{R}^n$. □

4 Proofs of the classification and Liouville theorems in the whole space

4.1 Proof of Theorem 2.1.

The key idea is to use the two auxiliary functions

$$W := |u - Kv|$$

and

$$Z := \min(u, Kv),$$

where K is given by Proposition 1.1. Clearly $u \equiv Kv$ is equivalent to $W \equiv 0$, and $u = v \equiv 0$ is equivalent to $Z \equiv 0$, when $K > 0$.

The following two lemmas assert that the functions Z, W satisfy a suitable system of elliptic inequalities, respectively of the form (3.3) and (3.8).

Lemma 4.1. *We suppose that* (1.11) *holds.*

(i) Assume $ab \geq cd$. Then Z is superharmonic.

If $p+q < 1$, suppose in addition that (u, v) is bounded. Then Z is a weak solution of

$$-\Delta Z \geq CW^{\beta}Z^{r} \qquad \text{in } \mathbb{R}^n, \tag{4.1}$$

where $\beta := \max(p + q, 1)$ and $C > 0$.

(ii) Assume $ab > cd$. Then Z is a weak solution of

$$-\Delta Z \geq CZ^{p+q+r} \qquad \text{in } \mathbb{R}^n.$$

Lemma 4.2. *We suppose that* (1.11) *holds, and $ab \geq cd$.*

(i) Then W is subharmonic.

(ii) Assume $r > p$ and $c, d > 0$. We also suppose that (u, v) is bounded in case $q + r < 1$. Then W is a weak solution of

$$\Delta W \geq CZ^{p}W^{\gamma} \qquad \text{in } \mathbb{R}^n, \tag{4.2}$$

where $\gamma := \max(q + r, 1)$ and $C > 0$.

Proof of Lemma 4.1. Let us recall the Kato inequality (valid in particular for weak solutions) :

$$\Delta z_+ \geq \chi_{\{z>0\}}\Delta z. \tag{4.3}$$

(i) Writing

$$Z = \frac{1}{2}\Big(u + Kv - (u - Kv)_+ - (Kv - u)_+\Big),$$

it follows from (4.3) that

$$-\Delta Z \geq \frac{1}{2}\Big(-\Delta(u + Kv) + \chi_{\{u>Kv\}}\Delta(u - Kv) + \chi_{\{u<Kv\}}\Delta(Kv - u)\Big),$$

hence

$$- \Delta Z \geq -\chi_{\{u<Kv\}}\Delta u - K\chi_{\{u>Kv\}}\Delta v - \frac{1}{2}\chi_{\{u=Kv\}}\Delta(u+Kv). \tag{4.4}$$

Now we make use of the inequality

$$x^q - y^q \geq C_q x^{q-1}(x-y), \quad x > y > 0$$

with $C_q = 1$ if $q \geq 1$, $C_q = q$ if $0 < q < 1$. By Proposition 1.1, we have

$$a - cK^q \geq 0, \quad bK^q - d \geq 0. \tag{4.5}$$

Therefore, on the set $\{u \leq Kv\}$, we obtain

$$\begin{aligned} -\Delta u &= u^r v^p(av^q - cu^q) \geq aK^{-q}u^r v^p((Kv)^q - u^q) \\ &\geq aC_q K^{-1}u^r v^{p+q-1}(Kv-u) \ \geq 0. \end{aligned} \tag{4.6}$$

Similarly, on the set $\{u \geq Kv\}$, we get

$$\begin{aligned} -\Delta v &= v^r u^p(bu^q - dv^q) \geq bv^r u^p(u^q - (Kv)^q) \\ &\geq bC_q v^r u^{p+q-1}(u-Kv) \ \geq 0. \end{aligned} \tag{4.7}$$

In particular, $-\chi_{\{u=Kv\}}\Delta(u+Kv) \geq 0$. Hence, we deduce from (4.4) that

$$- \Delta Z \geq -\chi_{\{u<Kv\}}\Delta u - K\chi_{\{u>Kv\}}\Delta v, \tag{4.8}$$

so Z is superharmonic, by (4.6) and (4.7).

Now assume either that $p+q \geq 1$ or that (u,v) is bounded. By using that

$$v^{p+q-1} \geq C(Kv-u)^{p+q-1} \quad \text{ if } \ p+q \geq 1,$$

and $v^{p+q-1} \geq C > 0$ otherwise (since v is bounded), we infer from (4.6) and (4.7) that

$$-\Delta u \geq Cu^r(Kv-u)^\beta \quad \text{ on } \ \{u \leq Kv\},$$

and

$$-\Delta v \geq C(Kv)^r(u-Kv)^\beta \quad \text{ on } \ \{u \geq Kv\}.$$

We then deduce from (4.8) that

$$-\Delta Z \geq Cu^r(Kv-u)_+^\beta + C(Kv)^r(u-Kv)_+^\beta = C|u-Kv|^\beta Z^r.$$

(ii) If $ab > cd$, then the inequalities in (4.5) are strict, that is $a \geq cK^q + \varepsilon$, $bK^q \geq d + \varepsilon$ for some $\varepsilon > 0$. Then we obtain, as in (4.6) and (4.7), that

$$-\Delta u \geq \varepsilon u^r v^{p+q} \geq \varepsilon K^{-p-q} Z^\sigma \qquad \text{on the set} \quad \{u \leq Kv\},$$

and $-\Delta v \geq \varepsilon u^p v^{q+r} \geq \varepsilon K^{-q-r} Z^\sigma$ on the set $\{u \geq Kv\}$, for some $\varepsilon > 0$. The assertion then follows from (4.8). □

Proof of Lemma 4.2. (i) By using (4.3) and Proposition 1.1, we get

$$\begin{aligned}\Delta W &= \Delta(u-Kv)_+ + \Delta(Kv-u)_+\\ &\geq \chi_{\{u>Kv\}}\Delta(u-Kv) + \chi_{\{u<Kv\}}\Delta(Kv-u)\end{aligned}$$

hence

$$\Delta W \geq \chi_{\{u>Kv\}}(Kg-f) + \chi_{\{u<Kv\}}(f-Kg) \geq 0, \tag{4.9}$$

where we have set $f(u,v) = u^r v^p[av^q - cu^q]$, $g(u,v) = v^r u^p[bu^q - dv^q]$.

(ii) In Lemma 7.1(i) in the appendix we show that

$$(Kg-f)(u-Kv) \geq Cu^p v^p (u+Kv)^{q+r-p-1}(u-Kv)^2,$$

when $r > p$ and $c, d > 0$. Using (4.9), we then get

$$\begin{aligned}\Delta W &\geq \chi_{\{u>Kv\}}(Kg-f) + \chi_{\{u<Kv\}}(f-Kg)\\ &\geq Cu^p v^p (u+Kv)^{q+r-p-1}|u-Kv|\\ &\geq C_1 Z^p (u+Kv)^{q+r-1}|u-Kv|.\end{aligned}$$

If $q+r \geq 1$, we conclude by using $(u+Kv)^{q+r-1} \geq |u-Kv|^{q+r-1}$. If $q+r<1$, we conclude by using $(u+Kv)^{q+r-1} \geq C$, in view of the boundedness of (u,v). □

Proof of Theorem 2.1. (i) Assume for contradiction that $u \not\equiv Kv$. By Lemma 4.2(i), the function $W = |u-Kv|$ is subharmonic, nonnegative and nontrivial. Clearly, so is W^β, for each $\beta \geq 1$. Then Lemma 3.2 applies to the inequality $-\Delta Z \geq W^\beta Z^r$, which we proved in Lemma 4.1 (recall the discussion after the statement of Lemma 3.2). Hence $Z \equiv 0$, a contradiction.

(ii) First, we observe that we may assume $q+r>1$. Indeed, if $q+r \leq 1$, then (u,v) is assumed to be bounded and, since $r \leq q+r \leq 1 < n/(n-2)_+$, the conclusion follows from assertion (i).

Next we claim that we may assume $r > p$. Indeed, if $p+q<1$, then this is true due to $q+r>1>p+q$. If $p+q \geq 1$, then we may assume $r > n/(n-2)$ and $n \geq 3$, since otherwise the result is already known from assertion (i). But then $r > 2/(n-2) \geq p$.

Now, by Lemma 4.1(i), Z is superharmonic and positive, hence

$$Z(x) \geq c_1(1+|x|)^{2-n}, \quad x \in \mathbb{R}^n,$$

for some $c_1 > 0$. Therefore

$$Z^p(x) \geq \tilde{c}_1(1+|x|)^{-(n-2)p} \geq \tilde{c}_1(1+|x|)^{-2}, \quad x \in \mathbb{R}^n.$$

Hence we can apply Lemma 3.4 to the inequality $\Delta W \geq Z^p W^\beta$, which we obtained in Lemma 4.2(ii), and conclude that $W \equiv 0$. □

Remark 4.1. *It does not seem possible to go beyond assumptions (2.1), (2.2) by the sole means of the mixed-type system (4.1)-(4.2). Indeed, if* $n \geq 3$, $r > \frac{2}{n-2}$ *and* $p > \frac{2}{n-2}$, *then this system admits positive solutions of the form*

$$Z = C(1+|x|^2)^{-\alpha}, \quad W = B - A(1+|x|^2)^{-\beta},$$

with suitable $B > A > 0$, $C > 0$, $2/(n-2) < 1/\alpha < \min(p, r-1)$ *and* $0 < \beta < p\alpha - 1$ *(this is easily checked by direct computation). We remark that Keller-Osserman type estimates and Liouville theorems for another mixed-type system, namely*

$$-\Delta Z = W^p, \quad \Delta W = Z^q,$$

were obtained in the recent work [6].

Proof of Theorem 2.2. Assume first that $\sigma \leq n/(n-2)_+$. Then the result is a consequence of Lemma 4.1(ii) and Lemma 3.2.

Assume next that $n \geq 3$ and $\sigma > n/(n-2)$. Suppose for contradiction that a positive bounded solution (u, v) exists. By (1.11), we have $r \leq p+q$, hence $r \leq \sigma/2$. Since $\sigma < (n+2)/(n-2)$, we deduce $r \leq n/(n-2)$. Theorem 2.1 then guarantees that $u = Kv$, where K is given by Proposition 1.1. It follows that

$$-\Delta v = K^{-1}u^r v^p(av^q - cu^q) = Cv^\sigma, \quad x \in \mathbb{R}^n,$$

with $C = K^{r-1}(a - cK^q) > 0$ by Proposition 1.1. But this contradicts a well-known Liouville-type result from [26].

Moreover, if either $p + q \geq 1$, or $p \leq 2/(n-2)$ (hence $q + r > 1$ due to $\sigma > n/(n-2)$), then the boundedness assumption is not necessary when applying Theorem 2.1. Finally, we note that if $n \leq 4$, then we always have either $p + q \geq 1$ or $p < 1 \leq 2/(n-2)$. ☐

Proof of Proposition 2.1. We may assume without loss of generality that $d = 0$. (Indeed the system (1.10) with unknown (u, v), parameters a, b, c, d and exponents p, q, r is equivalent to the system (1.10) with unknown (v, u), parameters b, a, d, c and same exponents.) Also we may assume that $c > 0$ since, in the case $c = d = 0$, the result is already known from Theorems 1.4(i) and 1.2 in [39]. (This is actually proved there in the case $a = b = 1$, but the general case immediately follows by scaling.)

Now assume that $(Kv - u)_+ \not\equiv 0$. Since

$$\Delta(Kv - u)_+ \geq \chi_{\{u<Kv\}}\Delta(Kv - u) \geq \chi_{\{u<Kv\}}(f - Kg) \geq 0,$$

due to Proposition 1.1, the function $(Kv - u)_+$ is subharmonic. It follows (see the discussion after the statement of Lemma 3.2)) that

$$\liminf_{R\to\infty} R^{-n} \int_{B_R} (Kv - u)_+(x)\, dx > 0.$$

Consequently, since $v \geq (1/K)(Kv-u)_+$ we have $\liminf_{R\to\infty} \overline{v}(R) =: L > 0$, where $\overline{v}(R) = |S_R|^{-1} \int_{S_R} v \, d\sigma_R$ denote the spherical means. But, since v is superharmonic due to $d=0$, we deduce from [39, Lemma 3.2] that

$$v \geq L > 0, \quad x \in \mathbb{R}^n. \tag{4.10}$$

On the other hand, by Lemma 7.1(ii), we have

$$\begin{aligned}(Kg-f)(u-Kv) &\geq Cu^r v^{p\wedge r}(u+Kv)^{q-1+(p-r)_+}(u-Kv)^2 \\ &\geq Cv^{p\wedge r}|u-Kv|^{q+1+(p\vee r)}.\end{aligned}$$

Therefore,

$$\Delta(u-Kv)_+ \geq \chi_{\{u>Kv\}}(Kg-f) \geq C(u-Kv)_+^{q+(p\vee r)},$$

owing to (4.10). In view of Lemma 4.2(ii), we conclude that $u \leq Kv$. □

Finally, let us justify the statement in Remark 2.1. Let $W_1 := (u-Kv)_+$ and $W_2 := (Kv-u)_+$. By the proof of Lemma 4.2(i), we know that the radially symmetric functions W_1 and W_2 are subharmonic, hence (radially) nondecreasing. Since $W_1 W_2 \equiv 0$, we necessarily have $\lim_{t\to\infty} W_1(t) = 0$ or $\lim_{t\to\infty} W_2(t) = 0$, hence $W_1 \equiv 0$ or $W_2 \equiv 0$.

4.2 Proof of Theorem 2.3

By adding up the two equations, we see that $u+v$ is harmonic and positive, hence constant. We therefore look for a solution such that $v = 1-u$, with $0<u<1$. The system then becomes equivalent to

$$-\Delta u = u^p(1-u)^p[(1-u)^q - u^q] =: f(u). \tag{4.11}$$

To show the existence of a nonconstant positive solution of (4.11), we argue like in the proof of [39, Theorem 1.4]. Consider the initial value problem for the real function $u = u(t)$

$$-(t^{n-1}u')' = t^{n-1}f(u), \quad t>0, \qquad u(0)=\varepsilon, \quad u'(0)=0, \tag{4.12}$$

with $0<\varepsilon<\frac{1}{2}$. It is standard to check that either $u>0$, $u'\leq 0$ for all $t>0$, or u has a first zero $t=R$. If the latter occurs, then the PDE in (4.11) admits a positive solution u in a finite ball with homogeneous Dirichlet conditions, and also $0<u<\varepsilon$. But this is known to be impossible, owing to the Pohozaev identity, whenever

$$h(X) := Xf(X) - (p_S+1)F(X) \geq 0, \quad 0<X<\varepsilon, \tag{4.13}$$

where $F(X) = \int_0^X f(\tau)\,d\tau$ and $p_S = \frac{n+2}{n-2}$. In the case of (4.11) we have $h(0)=0$ and

$$h'(X) = Xf'(X) - p_S f(X) \sim (p-p_S)X^p > 0, \quad \text{as } X \to 0^+.$$

Therefore (4.13) is true for $\varepsilon>0$ sufficiently small, and the conclusion follows.

5 Properties of half-spherical means. Proofs of the classification results in a half-space.

We start by recalling that the classical Phragmén-Lindelöf maximum principle states that a subharmonic function with sublinear growth in the half-space which is nonpositive on $\partial\mathbb{R}^n_+$ is also nonpositive in $\mathbb{R}^n_+$ (see for instance [38]).

Proof of Theorem 2.4. We set $w = u - Kv$. Since u, v have sublinear growth, so do $|w|$ and w_+. Let $\psi \in C^2(\mathbb{R})$ be convex, nondecreasing and such that $0 \le \psi(t) \le t^+$ for all $t \in \mathbb{R}$ and $\psi(t) > 0$ for $t > 0$. Then $\psi(w)$ has sublinear growth.
Since the nonlinearities satisfy condition (1.9), then $w \ge 0$ implies $\Delta w \ge 0$. Hence, we have

$$\Delta\psi(w) = \psi'(w)\Delta w + \psi''(w)|\nabla w|^2 \ge 0,$$

since $\psi'(w) = 0$ if $w \le 0$ and $\Delta w \ge 0$ otherwise. Hence, $\psi(w)$ is subharmonic. Since by hypothesis $w = 0$ on $\partial\mathbb{R}^n_+$, $\psi(w) = 0$ on $\partial\mathbb{R}^n_+$. By the Phragmén-Lindelöf maximum principle, we get $\psi(w) \le 0$, so $w \le 0$. The same argument applied to $-w$ leads to $-w \le 0$. Finally, we obtain $w = 0$, i.e. $u = Kv$. □

We will use the following notation : for any $R > 0$, and any $y \in \partial\mathbb{R}^n_+$, we set

$$S^+_R(y) = \{x \in \mathbb{R}^n_+,\ |x - y| = R\},$$

$$B^+_R(y) = \{x \in \mathbb{R}^n_+,\ |x - y| \le R\},$$

$$D_R(y) = \{y' \in \partial\mathbb{R}^n_+,\ |y' - y| \le R\},$$

and write S^+_R, B^+_R and D_R respectively for $S^+_R(0)$, $B^+_R(0)$ and $D_R(0)$. We recall the definition of the half-spherical means of a function w, namely

$$[w]_y(R) = \frac{1}{R^2|S^+_R|}\int_{S^+_R(y)} w(x)\, x_n\, d\sigma_R(x), \quad R > 0,\ y \in \partial\mathbb{R}^n_+,$$

and denote $[w] := [w]_0$. Observe that $[x_n]$ is a positive constant (independent of R).

The following lemma provides a basic computation for the derivative of the half-spherical mean with respect to the radius.

Lemma 5.1. *Let $u \in C^2(\overline{\mathbb{R}^n_+})$. For any $y \in \partial\mathbb{R}^n_+$ and $R > 0$, we have :*

$$\frac{d}{dR}[u]_y(R) = \frac{1}{R^2|S^+_R|}\left[\int_{B^+_R(y)} \Delta u\, x_n\, dx - \int_{D_R(y)} u(y')\, dy'\right].$$

Démonstration. We have

$$I = \int_{B^+_R(y)} \Delta u\, x_n\, dx - \int_{D_R(y)} u(y')\, dy' = \int_{B^+_R(y)} div(\nabla u\, x_n - u\, e_N)\, dx - \int_{D_R(y)} u(y')\, dy'$$
$$= \int_{S^+_R(y)} \nabla u \cdot \vec{\nu}\, x_n\, d\sigma_R(x) - \int_{S^+_R(y)} u\, \nu_n\, d\sigma_R(x),$$

since $x_n = 0$ on $\partial\mathbb{R}^n_+$. Then, setting $x = y + Rz$, since $\nu(x) = z$ and $\nu_n(x) = \frac{x_n}{R}$, we have

$$\begin{aligned} I &= R^n \int_{S_1^+} \nabla u(y+Rz)\cdot z\, z_n \, d\sigma_1(z) - \frac{1}{R}\int_{S_R^+(y)} x_n u \, d\sigma_R(x) \\ &= R^n \frac{d}{dR}\int_{S_1^+} u(y+Rz)\, z_n \, d\sigma_1(z) - R|S_R^+|[u]_y(R) \\ &= R^n \frac{d}{dR}\Big(R\,|S_1^+|[u]_y(R)\Big) - R|S_R^+|[u]_y(R) \\ &= R|S_R^+|\Big(\frac{d}{dR}\Big(R[u]_y(R)\Big) - [u]_y(R)\Big) = R^2|S_R^+|\frac{d}{dR}[u]_y(R). \end{aligned}$$

□

Next, we give a the generalization of the Phragmén-Lindelöf maximum principle, based on the above monotonicity property, which will play an important role.

Lemma 5.2. *Let $w \in C^2(\overline{\mathbb{R}^n_+})$ be such that $w \le 0$ on $\partial\mathbb{R}^n_+$ and $\Delta w \ge 0$ on the set $\{w > 0\}$. If we assume*

$$\liminf_{R\to\infty} [w^+](R) = 0, \tag{5.1}$$

then $w \le 0$ in $\mathbb{R}^n_+$.

Proof. Let $\psi \in C^2(\mathbb{R})$ be convex, nondecreasing and such that $0 \le \psi(t) \le t^+$ for all $t \in \mathbb{R}$ and $\psi(t) > 0$ for $t > 0$. Then, for any $R > 0$, $0 \le [\psi(w)](R) \le [w^+](R)$. Therefore,

$$\liminf_{R\to\infty} [\psi(w)](R) = 0. \tag{5.2}$$

We also have

$$\Delta\psi(w) = \psi'(w)\Delta w + \psi''(w)|\nabla w|^2 \ge 0,$$

since $\psi'(w) = 0$ if $w \le 0$ and $\Delta w \ge 0$ otherwise. Since $w \le 0$ on $\partial\mathbb{R}^n_+$, then $\psi(w) = 0$ on $\partial\mathbb{R}^n_+$ so Lemma 5.1 gives that $[\psi(w)](R)$ is nondecreasing. But its limit as $R \to \infty$ is zero by (5.2), so $[\psi(w)](R) = 0$ for all $R > 0$. This implies that $\psi(w) \equiv 0$, hence $w \le 0$. □

The proof of Theorem 2.6 is an easy consequence of Lemma 5.2.

Proof of Theorem 2.6. Let $w = u - Kv$. Observe that $w = 0$ on $\partial\mathbb{R}^n_+$ and $w\,\Delta w \ge 0$ thanks to (1.9). Hence we can apply Lemma 5.2 to w and $-w$ and conclude that $w = 0$. □

We now turn to the proof of Theorem 2.5. In order to treat solutions without growth restrictions at infinity in the case of positive nonlinearities, we will need to exploit some further properties of half-spherical means for superharmonic functions.

The following lemma will permit to us to split the proof of Theorem 2.5 in the following way : either the superharmonic functions u, v grow at infinity at least like

x_n and then we apply the nonexistence result for weighted inequalities in Lemma 3.1, or the half-spherical means of u, v decay at infinity and we can use Theorem 2.6.

Lemma 5.3. *Suppose that $u \in C^2(\overline{\mathbb{R}^n_+})$ is nonnegative and superharmonic in $\mathbb{R}^n_+$.*

(i) For each $y \in \partial\mathbb{R}^n_+$, the function $R \mapsto [u]_y(R)$ is nonincreasing and its limit is independent of y.

(ii) Denote $L(u) := \lim_{R\to\infty} [u](R) \in [0,\infty)$. Then we have

$$u(x) \geq \frac{L(u)}{[x_n]}\, x_n, \quad x \in \mathbb{R}^n_+.$$

Assertion (ii) can be deduced from a more general and rather difficult result from [24] ; see Remark 5.1 below. We will provide a direct, more elementary proof.

Proof. (i) That $[u]_y(R)$ is nonincreasing in R is a direct consequence of Lemma 5.1. Set

$$\mu(y) := \lim_{R\to\infty} R^{-(n+1)} \int_{S^+_R(y)} x_n u \, d\sigma_R = |S^+_1| \lim_{R\to\infty} [u]_y(R). \tag{5.3}$$

By L'Hôpital's rule, (5.3) implies that

$$\lim_{R\to\infty} R^{-(n+2)} \int_{B^+_R(y)} x_n u \, dx = \lim_{R\to\infty} R^{-(n+2)} \int_0^R \int_{S^+_r(y)} x_n u \, d\sigma_r \, dr = \frac{\mu(y)}{n+2}$$

(with nonincreasing limit). Now, for $y_1, y_2 \in \partial\mathbb{R}^n_+$, we have $B^+_R(y_1) \subset B^+_{R+|y_1-y_2|}(y_2)$, hence

$$R^{-(n+2)} \int_{B^+_R(y_1)} x_n u \, dx \leq (1+R^{-1}|y_1-y_2|)^{n+2}(R+|y_1-y_2|)^{-(n+2)} \int_{B^+_{R+|y_1-y_2|}(y_2)} x_n u \, dx.$$

By letting $R \to \infty$, we deduce that $\mu(y_1) \leq \mu(y_2)$, which proves that $\mu(y)$ is independent of y.

(ii) The proof is divided in three steps.

Step 1. We recall several properties of Poisson kernels, that is, normal derivatives of Green functions. For $R > 0$, we denote by $P_R(x; y)$ the Poisson kernel of B^+_R. Then for any $\varphi \in C(\partial B^+_R)$, the unique harmonic function v in B^+_R with boundary value φ is given by

$$v(x) = \int_{\partial B^+_R} P_R(x; y)\varphi(y) \, d\sigma_R(y).$$

A simple rescaling argument shows that

$$P_R(x; y) = R^{1-n} P_1(R^{-1}x; R^{-1}y). \tag{5.4}$$

On the other hand, for each $Y \in \partial B^+_1$, $P_1(\cdot, Y)$ is positive in B^+_1 (by the strong maximum principle, since it is harmonic, nonnegative and nontrivial). For each $X \in B^+_1$, since $Y \mapsto P_1(X; Y)$ is continuous on ∂B^+_1, it follows that

$$c(X) := \inf_{Y\in\partial B^+_1} P_1(X; Y) > 0. \tag{5.5}$$

Step 2. Fix $x \in H$, denote by $\tilde{x} = (x_1, \cdots, x_{n-1}, 0)$ its projection onto $\partial\mathbb{R}^n_+$ and set $R = 2x_n$. Since $u(\tilde{x} + \cdot) \geq 0$ is superharmonic in B^+_R, the maximum principle implies that, for all $z \in B^+_R$,

$$u(\tilde{x} + z) \geq \int_{\partial B^+_R} P_R(z; y) u(\tilde{x} + y)\, d\sigma_R(y) \geq \int_{S^+_R} P_R(z; y) u(\tilde{x} + y)\, d\sigma_R(y),$$

hence

$$u(\tilde{x} + z) \geq R^{1-n} \int_{S^+_R} P_1(R^{-1}z; R^{-1}y) u(\tilde{x} + y)\, d\sigma_R(y),$$

due to (5.4).

Now take $z = (0, \cdots, 0, x_n)$, set $X = (0, \cdots, 0, 1/2)$ and $c_0 = c(X)$ (see (5.5)). Using (5.5) and assertion (i), we obtain

$$\begin{aligned} u(x) &\geq R^{1-n} \int_{S^+_R} P_1(X; R^{-1}y) u(\tilde{x} + y)\, d\sigma_R(y) \geq c_0 R^{1-n} \int_{S^+_R} u(\tilde{x} + y)\, d\sigma_R(y) \\ &= c_0 R^{1-n} \int_{S^+_R(\tilde{x})} u(y)\, d\sigma_R(y) \geq c_0 R^{-n} \int_{S^+_R(\tilde{x})} y_n\, u(y)\, d\sigma_R(y) \\ &\geq c_0 |S^+_1| R\, [u]_{\tilde{x}}(R) \geq 2c_0 |S^+_1| L(u)\, x_n. \end{aligned}$$

Step 3. Define $E = \{c \geq 0;\ u \geq c\, x_n \text{ in } \mathbb{R}^n_+\}$. The set E is closed and nonempty. For any $c \in E$, we have $L(u) \geq c[x_n]$, hence E is bounded and

$$\tilde{c} := \max E \leq c^* := [x_n]^{-1} L(u).$$

Assume for contradiction that $\tilde{c} < c^*$. Setting $z = u - \tilde{c}\, x_n$, we see that z is nonnegative, superharmonic and that $L(z) > 0$. By the result of Step 2 applied to z, it follows that $z \geq \varepsilon x_n$ for some $\varepsilon > 0$. But this contradicts the definition of $\tilde{c}$. Therefore $\tilde{c} = c^*$ and the result is proved. □

Remark 5.1. *(i) For any subharmonic function w on $\mathbb{R}^n_+$, the Corollary to Theorem 1 on page 341 in [24] asserts the following : if w_+ has a harmonic majorant, if $\liminf_{R\to\infty} [w](R) \leq 0$ and if, for all $y \in \partial\mathbb{R}^n_+$, $\liminf_{R\to 0} R\, [w]_y(R) \leq 0$, then $w \leq 0$. To deduce Lemma 5.3(ii) from this, set $L = L(u)$ and $w = \frac{L}{[x_n]}\, x_n - u$. Then w is subharmonic, w_+ has a harmonic majorant $\frac{L}{[x_n]}\, x_n$ and $[w](R) = L - [u](R) \underset{R\to\infty}{\longrightarrow} 0$. Moreover, for all $y \in \partial\mathbb{R}^n_+$, $\liminf_{R\to 0} R[w]_y(R) \leq \liminf_{R\to 0} R \frac{L}{[x_n]}\, [x_n] = 0$. Therefore, $w \leq 0$, i.e., $u \geq \frac{L}{[x_n]}\, x_n$.*

(ii) From Lemmas 5.1 and 5.3(ii), we may retrieve the well-known fact that any positive harmonic function in $\mathbb{R}^n_+$, such that $u \in C^2(\overline{\mathbb{R}^n_+})$ and $u = 0$ on the boundary, is necessarily of the form $u = c\, x_n$ with $c > 0$.

We first claim that $L(u) > 0$. Indeed, $[u](R)$ is independent of R by Lemma 5.1. Therefore $L(u) = 0$ would imply $[u](R) \equiv 0$, from which we readily infer $u \equiv 0$. Let then $z = u - \frac{L(u)}{[x_n]}\, x_n$. Then z is harmonic and Lemma 5.3(ii) guarantees $z \geq 0$. Since $L(z) = 0$, the above argument yields $z \equiv 0$.

Proof of Theorem 2.5. (i) Since the functions f and g are nonnegative, u and v are superharmonic. Therefore, by Lemma 5.1(ii),

$$L(u) := \lim_{R\to\infty}[u](R) \in [0,\infty) \qquad \text{and} \qquad L(v) := \lim_{R\to\infty}[v](R) \in [0,\infty). \tag{5.6}$$

First, we observe that we cannot have simultaneously $L(u) > 0$ and $L(v) > 0$. Indeed, by Lemma 5.3(ii), this would imply that, for some $c > 0$, and all $x \in \mathbb{R}^n_+$

$$u(x) \geq c\, x_n \quad \text{and} \quad v(x) \geq c\, x_n,$$

hence $-\Delta u \geq (c\, x_n)^\sigma$ in $\mathbb{R}^n_+$, but this contradicts Lemma 3.1.

Assume for instance $L(u) = 0$. Setting $w = u - Kv$, we have $w^+ \leq u$, hence

$$\lim_{R\to+\infty}[w^+](R) = 0.$$

By Lemma 5.2, this implies $w \leq 0$, i.e. $u \leq Kv$. If $L(v) = 0$, we similarly obtain $u \geq Kv$.

(ii) By what we just proved, it is enough to show that $L(u) = L(v) = 0$. Assume $L(v) > 0$. Therefore, $v \geq cx_n$ for $c > 0$, and

$$\begin{cases} -\Delta u \geq cx_n^p u^r \\ -\Delta v \geq cx_n^s u^q \qquad \text{in } \mathbb{R}^n_+. \end{cases} \tag{5.7}$$

If the first condition in (2.6) is satisfied, then the first inequality in (5.7) combined with Lemma 3.1 yields $u \equiv 0$.

Hence we can assume that the second condition in (2.6) is satisfied. Set $\Psi := x_n|x|^{-n}$, so that $-\Delta\Psi = 0$ in $\mathbb{R}^n_+ \setminus \{0\}$. Let

$$c_0 := \inf_{\partial B_1 \cap \mathbb{R}^n_+} \frac{u}{\Psi}.$$

Note $c_0 > 0$ (if $u = 0$ on $\partial\mathbb{R}^n_+$, this follows from Hopf's lemma). Since u is superharmonic in $\mathbb{R}^n_+$, $u \geq c\Psi$ on $\partial(\mathbb{R}^n_+ \setminus B_1)$ and $\Psi \to 0$ as $|x| \to \infty$, the maximum principle implies

$$u \geq cx_n|x|^{-n} \qquad \text{in } \mathbb{R}^n_+ \setminus B_1.$$

Plugging this into the second inequality of (5.7) we get

$$-\Delta v \geq c\, {x_n}^{s+q}|x|^{-nq}$$

in $\mathbb{R}^n_+ \setminus B_1$, which contradicts Lemma 3.1, applied with $r = 0$ and $\kappa = s - (n-1)q$, in view of the second condition in (2.6).

In case $L(u) > 0$ we use (2.7) in a similar way, to conclude the proof of (ii).

(iii) By (ii), we know that either (u, v) is semi-trivial or $u = Kv$. In the latter case, since $\min(p + r, q + s) \leq (n + 1)/(n - 1)$, we deduce from Lemma 3.1 that $u = 0$ or $v = 0$. □

6 A priori estimates and existence

We consider the following system with general lower order terms, of which (2.9) is a particular case

$$\begin{cases} -\Delta u = u^r v^p \Big[a(x)v^q - c(x)u^q\Big] + h_1(x,u,v), & x \in \Omega, \\ -\Delta v = v^r u^p \Big[b(x)u^q - d(x)v^q\Big] + h_2(x,u,v), & x \in \Omega, \\ \quad u = v = 0, & x \in \partial\Omega, \end{cases} \tag{6.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$.

Theorem 2.7 is a consequence of the following more general statements on a priori estimates and existence.

Theorem 6.1. *Let $p, r \geq 0$, $q > 0$, $q \geq |p-r|$, and*

$$q + r \geq 1, \qquad 1 < \sigma := p + q + r < \frac{n+2}{(n-2)_+}. \tag{6.2}$$

Let $a, b, c, d \in C(\overline{\Omega})$ satisfy $a, b > 0$, $c, d \geq 0$ in $\overline{\Omega}$ and

$$\inf_{x\in\Omega} \, [a(x)b(x) - c(x)d(x)] \, > 0. \tag{6.3}$$

Let $h_1, h_2 \in C(\overline{\Omega} \times [0,\infty)^2)$ satisfy

$$\lim_{u+v\to\infty} \frac{h_i(x,u,v)}{(u+v)^\sigma} = 0, \quad i = 1, 2, \tag{6.4}$$

and assume that one of the following two sets of assumptions is satisfied :

$$\begin{cases} r \leq 1, \quad \text{and, setting } \bar{m} := \min\{\inf_{x\in\Omega} a(x), \inf_{x\in\Omega} b(x)\} > 0, \\ \liminf_{v\to\infty,\ u/v\to 0} \dfrac{h_1(x,u,v)}{u^r v^{p+q}} > -\bar{m}, \qquad \liminf_{u\to\infty,\ v/u\to 0} \dfrac{h_2(x,u,v)}{v^r u^{p+q}} > -\bar{m}, \end{cases} \tag{6.5}$$

or

$$\begin{cases} m := \min\{\inf_{x\in\Omega} c(x), \inf_{x\in\Omega} d(x)\} > 0, \qquad \text{and} \\ \limsup_{u\to\infty,\ v/u\to 0} \dfrac{h_1(x,u,v)}{u^{r+q} v^p} < m, \qquad \limsup_{v\to\infty,\ u/v\to 0} \dfrac{h_2(x,u,v)}{v^{r+q} u^p} < m \end{cases} \tag{6.6}$$

(with uniform limits with respect to $x \in \overline{\Omega}$ in (6.4)–(6.6)). Then there exists $M > 0$ such that any positive classical solution (u, v) of (6.1) satisfies

$$\sup_\Omega u \leq M, \qquad \sup_\Omega v \leq M. \tag{6.7}$$

Theorem 6.2. *Let (6.2)–(6.5) be satisfied. Assume in addition that* a, b, c, d, h_1, h_2 *are Hölder continuous and that for some* $\varepsilon > 0$

$$\inf_{x\in\Omega,\ u,v>0} u^{-1}\, h_1(x,u,v) > -\infty, \qquad \inf_{x\in\Omega,\ u,v>0} v^{-1}\, h_2(x,u,v) > -\infty, \tag{6.8}$$

$$\sup_{x\in\Omega,\ u>0} u^{-1}\, h_1(x,u,0) < \lambda_1(-\Delta,\Omega), \qquad \sup_{x\in\Omega,\ v>0} v^{-1}\, h_2(x,0,v) < \lambda_1(-\Delta,\Omega), \tag{6.9}$$

$$\sup_{x\in\Omega,\ u,v\in(0,\varepsilon)^2} (u+v)^{-1}\,[h_1(x,u,v) + h_2(x,u,v)] < \lambda_1(-\Delta,\Omega). \tag{6.10}$$

Then there exists a positive classical solution of (6.1).

Remark 6.1. *We will not treat the existence question under the assumption (6.6), which seems to be a delicate problem. The reason is that we prove Theorem 6.2 by using a deformation of the system (6.1) via homotopy, adding positive linear terms (see* (6.14) *below). However, with such terms, assumption (6.6) is no longer satisfied and we cannot use Theorem 6.1.*

Remark 6.2. *Like many previous works, our proof of a priori estimates uses the classical rescaling method of Gidas and Spruck [25]. However, as mentioned in the introduction, arises an additional difficulty : to rule out the possibility of semitrivial rescaling limits, of the form* $(C_1, 0)$ *or* $(0, C_2)$ *(see Step 2 below). Under assumption (6.5), this will be achieved by a suitable eigenfunction argument, while (6.6) will guarantee that in each blowing up solution* (u, v) *of (6.1) the components* u *and* v *explode at a comparable rate. Note that a similar difficulty appears in the work [47], which studied a class of cooperative systems with nonnegative nonlinearities in the form of products. In that case, the problem was dealt with by different techniques, namely moving planes and Harnack inequalities.*

For the reader's convenience, before giving the proofs of Theorems 6.1-6.2 we quickly review the role of the hypotheses in these theorems. The first condition in (6.2) guarantees that the strong maximum principle applies to the system (1.10), while the second condition in (6.2) is a usual superlinearity and subcriticality condition on the nonlinearities at infinity. The hypothesis (6.4) says h_1 and h_2 are indeed of "lower order", and disappear in the blow-up limit, while the assumptions (6.5)-(6.6) are used to exclude semitrivial blow-up limits. The hypothesis (6.8) permits to us to apply the strong maximum principle to (6.1), whereas (6.9) implies that for each nonnegative solution of (6.1) we have $u \equiv 0$ if and only if $v \equiv 0$. Finally, (6.10) is a standard superlinearity condition at zero for (6.1).

Proof of Theorem 6.1. We will consider the following parametrized version of system (6.1) (this will be needed in the proof of Theorem 6.2) :

$$\begin{cases} -\Delta u = F(t,x,u,v), & x \in \Omega, \\ -\Delta v = G(t,x,u,v), & x \in \Omega, \\ \quad u = v = 0, & x \in \partial\Omega, \end{cases} \tag{6.11}$$

where

$$F(t,x,u,v) := u^r v^p\Big[(a(x)+tA)v^q - c(x)u^q\Big] + \hat{h}_1(t,x,u,v), \tag{6.12}$$

$$G(t,x,u,v) := v^r u^p\Big[(b(x)+tA)u^q - d(x)v^q\Big] + \hat{h}_2(t,x,u,v), \tag{6.13}$$

and

$$\hat{h}_1(t,x,u,v) = h_1(x,u,v)+At(1+u), \quad \hat{h}_2(t,x,u,v) = h_2(x,u,v)+At(1+v). \tag{6.14}$$

Here $A>0$ is a constant to be fixed below, and t is a parameter in $[0,1]$.

Note that (6.1) is (6.11) with $t=0$. Under assumption (6.5), we will prove the bound in (6.7) for the positive solutions of (6.11), uniformly for $t\in[0,1]$ (but possibly depending on A), whereas under assumption (6.6) we will restrict ourselves to $t=0$ [2] (see Remark 6.1).

We assume for contradiction that there exists a sequence $\{t_j\}\subset[0,1]$ and a sequence (u_j,v_j) of positive solutions of (6.11) with $t=t_j$, such that $\|u_j\|_\infty + \|v_j\|_\infty \to\infty$. We may assume $\|u_j\|_\infty \geq \|v_j\|_\infty$ without loss of generality. Set $\alpha = 2/(\sigma-1)$. Let $x_j\in\Omega$ be such that $u_j(x_j)=\|u_j\|_\infty$ and set

$$\lambda_j := \Big(\|u_j\|_\infty^{1/\alpha} + \|v_j\|_\infty^{1/\alpha}\Big)^{-1} \to 0, \quad \text{ as } j\to\infty.$$

By passing to a subsequence, we may assume that $x_j\to x_\infty\in\overline{\Omega}$ and $t_j\to t_0\in[0,1]$. Setting $d_j := \mathrm{dist}(x_j,\partial\Omega)$, we then split the proof into two cases, according to whether $d_j/\lambda_j\to\infty$ (along some subsequence) or d_j/λ_j is bounded.

Case A : $d_j/\lambda_j\to\infty$.

This case is treated in two steps.

Step 1 : Convergence of rescaled solutions to a semi-trivial entire solution.

We rescale the solutions around x_j as follows :

$$\tilde{u}_j(y) = \lambda_j^\alpha\, u_j(x_j+\lambda_j y), \quad \tilde{v}_j(y) = \lambda_j^\alpha\, v_j(x_j+\lambda_j y), \qquad y\in\Omega_j, \tag{6.15}$$

where $\Omega_j = \{y\in\mathbb{R}^n : |y| < d_j/\lambda_j\}$. Due to the definition of λ_j, it is clear that

$$\tilde{u}_j(y),\ \tilde{v}_j(y) \leq 1, \qquad y\in\Omega_j. \tag{6.16}$$

Moreover, $\tilde{u}_j^{1/\alpha}(0) = \lambda_j\,\|u_j\|_\infty^{1/\alpha} \geq \lambda_j\,(\|u_j\|_\infty^{1/\alpha} + \|v_j\|_\infty^{1/\alpha}\big)/2 = 1/2$, hence

$$\tilde{u}_j(0) \geq 2^{-\alpha}. \tag{6.17}$$

We see that $(\tilde{u},\tilde{v}) = (\tilde{u}_j,\tilde{v}_j)$ satisfies the system

$$\begin{cases} -\Delta\tilde{u} = \tilde{u}^r\tilde{v}^p\Big[(a(x_j+\lambda_j y)+t_jA)\,\tilde{v}^q - b(x_j+\lambda_j y)\,\tilde{u}^q\Big] + \tilde{h}_{1,j}(y), & y\in\Omega_j, \\ -\Delta\tilde{v} = \tilde{v}^r\tilde{u}^p\Big[(b(x_j+\lambda_j y)+t_jA)\,\tilde{u}^q - d(x_j+\lambda_j y)\tilde{v}^q\Big] + \tilde{h}_{2,j}(y), & y\in\Omega_j, \end{cases} \tag{6.18}$$

2. The restriction $t_j=0$ under assumption (6.6) will be used only in Step 2 to exclude semi-trivial rescaling limits.

where

$$\tilde{h}_{i,j}(y) = \lambda_j^{\alpha+2}\hat{h}_i(t_j, x_j + \lambda_j y, \lambda_j^{-\alpha}\tilde{u}_j(y), \lambda_j^{-\alpha}\tilde{v}_j(y)), \quad i = 1, 2.$$

In view of (6.4), (6.16), $\sigma > 1$, and $\alpha + 2 - \alpha\sigma = 0$ we have

$$\sup_{\Omega_j} (|\tilde{h}_{1,j}| + |\tilde{h}_{2,j}|) \leq \lambda_j^{\alpha+2}(\lambda_j^{-\alpha\sigma} o(1) + 2A(1 + \lambda_j^{-\alpha})) \to 0, \quad \text{as } j \to \infty. \tag{6.19}$$

For each fixed $R > 0$, we have $B_{2R} \subset \Omega_j$ for j sufficiently large, and $|\Delta\tilde{u}_j|,\ |\Delta\tilde{v}_j| \leq C(R)$ in B_{2R}, owing to (6.16), (6.18), (6.19). It follows from interior elliptic estimates that the sequences $\tilde{u}_j,\ \tilde{v}_j$ are bounded in $W^{2,m}(B_R)$ for each $1 < m < \infty$. By embedding theorems, we deduce that they are bounded in $C^{1+\gamma}(\overline{B_R})$ for each $\gamma \in (0, 1)$. It follows that, up to some subsequence,

$$\lim_{j\to\infty} (\tilde{u}_j, \tilde{v}_j) = (U, V), \quad \text{locally uniformly on } \mathbb{R}^n,$$

where (U, V) is a bounded nonnegative classical solution of

$$\begin{cases} -\Delta U = U^r V^p \Big[a_0 V^q - c_0 U^q\Big], & y \in \mathbb{R}^n, \\ -\Delta V = V^r U^p \Big[b_0 U^q - d_0 V^q\Big], & y \in \mathbb{R}^n, \end{cases} \tag{6.20}$$

with

$$a_0 = a(x_\infty) + t_0 A > 0,\ b_0 = b(x_\infty) + t_0 A > 0,\ c_0 = c(x_\infty) \geq 0,\ d_0 = d(x_\infty) \geq 0. \tag{6.21}$$

Moreover,

$$c_0 d_0 < a_0 b_0 \tag{6.22}$$

in view of (6.3). Also, $U(0) \geq 2^{-\alpha}$ due to (6.17). By Theorem 2.2(i) and Corollary 2.1, there exists a constant $\bar{C} > 0$ such that $U \equiv \bar{C}$ and $V \equiv 0$, hence

$$\lim_{j\to\infty} (\tilde{u}_j, \tilde{v}_j) = (\bar{C}, 0), \quad \text{locally uniformly on } \mathbb{R}^n. \tag{6.23}$$

Step 2 : Exclusion of semi-trivial rescaling limits.

Let us first consider the case when assumption (6.5) is satisfied. For some $\delta, M_1 > 0$ we have

$$\hat{h}_2(t, x, u, v) \geq (-\bar{m} + \delta) v^r u^{p+q}, \quad \text{for } u \geq M_1 \max(v, 1),$$

(uniformly in $x \in \Omega$ and $t \in [0, 1]$) and hence

$$\tilde{h}_{i,j} \geq (-\bar{m} + \delta)\tilde{v}_j^r \tilde{u}_j^{p+q}, \quad \text{for } \tilde{u}_j \geq M_1 \max(\tilde{v}_j, \lambda_j^\alpha), \quad i = 1, 2.$$

Fix $\varepsilon \in (0, 1)$ with

$$\varepsilon \leq \min\Big\{\frac{\bar{C}}{2M_1}, \Big(\frac{\delta}{2\|d\|_\infty}\Big)^{1/q}\frac{\bar{C}}{2}\Big\}.$$

Take $R > 0$ to be chosen later. By (6.23), there exists j_0 such that, for all $j \geq j_0$, we have $\tilde{u}_j \geq \bar{C}/2$, $\tilde{v}_j \leq \varepsilon$ on B_R, and $\tilde{u}_j \geq \bar{C}/2 \geq M_1 \max(\tilde{v}_j, \lambda_j^\alpha)$, since $\lambda_j^\alpha \to 0$ as $j \to \infty$. Hence

$$\begin{aligned} -\Delta\tilde{v}_j &\geq \tilde{v}_j^r \tilde{u}_j^p \Big[\Big(b(x_j + \lambda_j y) + t_j A - \bar{m} + \delta\Big)\, \tilde{u}_j^q - d(x_j + \lambda_j y)\tilde{v}_j^q\Big] \\ &\geq \tilde{v}_j^r \tilde{u}_j^p \Big[\delta\, \tilde{u}_j^q - \|d\|_\infty \varepsilon^q\Big] \geq \tfrac{\delta}{2}\Big(\tfrac{\bar{C}}{2}\Big)^{p+q} \tilde{v}_j^r \geq \tfrac{\delta}{2}\Big(\tfrac{\bar{C}}{2}\Big)^{p+q} \tilde{v}_j \qquad \text{in } \ B_R, \end{aligned}$$

(in the last inequality we used $r \leq 1$). This implies that the first eigenvalue of the Laplacian in B_R is larger than $\frac{\delta}{2}\Big(\frac{\bar{C}}{2}\Big)^{p+q}$, which is a contradiction for sufficiently large R. More precisely, denote by $\lambda_1(R)$ and φ_R the first eigenvalue and eigenfunction of $-\Delta$ in B_R with Dirichlet boundary conditions. Since $\lambda_1(R) = \lambda_1(1)R^{-2}$, by multiplying the above inequality with φ_R and by integrating by parts, we get

$$\lambda_1(1)R^{-2} \int_{B_R} \tilde{v}_j \varphi_R \, dx = -\int_{B_R} \tilde{v}_j \Delta\varphi_R \, dx \geq -\int_{B_R} \varphi_R \Delta\tilde{v}_j \, dx \geq \tfrac{\delta}{4}\Big(\tfrac{\bar{C}}{2}\Big)^{p+q} \int_{B_R} \tilde{v}_j \varphi_R \, dx.$$

By taking R sufficiently large (depending only on $\delta, \bar{C}, p, q$), this implies $\tilde{v}_j = 0$ on B_R, a contradiction.

Let us now consider the case when assumption (6.6) is satisfied, and $t_j = 0$. Now there exist $\delta, M_1 > 0$ such that

$$h_1(x, u, v) \leq (m - \delta) u^{r+q} v^p, \qquad \text{if } \ u \geq M_1 \max(v, 1).$$

Therefore, for any positive solution (u, v) of (6.1), if $\|u\|_\infty \geq M_1$ then, at a maximum point x_0 of u, we have either $u(x_0) < M_1 v(x_0)$, or else

$$0 \leq -\Delta u(x_0) \leq u^r v^p [a v^q - (c - m + \delta) u^q](x_0).$$

Since u and v are positive we deduce that

$$v(x_0) \geq \left(\frac{\delta}{a(x_0)}\right)^{1/q} u(x_0) \geq \left(\frac{\delta}{\|a\|_\infty}\right)^{1/q} u(x_0).$$

Hence there exists a constant $\eta > 0$ such that, for any positive solution (u, v) of (6.1),

$$\|u\|_\infty = u(x_0) \geq M_1 \Longrightarrow v(x_0) \geq \eta\, u(x_0).$$

In view of definition (6.15), this implies $\tilde{v}_j(0) \geq \eta \tilde{u}_j(0)$, hence $V(0) \geq \eta U(0) \geq \eta 2^{-\alpha}$, which excludes semitrivial limits and leads to a contradiction with the nonexistence of positive solutions of (6.20).

Case B : d_j/λ_j is bounded. We may assume that $d_j/\lambda_j \to c_0 \geq 0$. Arguing similarly to [25, pp. 891-892] (see also [39, p. 265]), after performing local changes of coordinates which flatten the boundary, we end up with a nontrivial nonnegative (bounded) solution (U, V) of system (6.20) in a half-space, with $U = V = 0$ on the boundary. Moreover, (6.22) is satisfied. By Proposition 1.1 and Theorem 2.4, we deduce $U = KV$, $K > 0$, which in turn implies that $-\Delta U = C_1 U^\sigma$, $-\Delta V = C_2 V^\sigma$ in the half-space, for some $C_1, C_2 > 0$. This yields a contradiction with the Liouville-type theorem in [25] for half-spaces. □

Proof of Theorem 6.2. First, it is important to observe that the assumptions $q+r \geq 1$ and (6.8) guarantee that any nonnegative solution of (6.11) satisfies $u > 0$ and $v > 0$ in Ω, unless $t = 0$ and $(u,v) \equiv (0,0)$. Indeed, if $u \not\equiv 0$, then $u > 0$ by the strong maximum principle – note by (6.12) and (6.8) we have

$$F(t,x,u,v) \geq -Cu,$$

for some $C \geq 0$ (which may depend on t,u,v,c,d,A). On the other hand, assume for instance $u \equiv 0$. Then $0 = F(t,x,0,v) \geq At$ (since (6.8) implies $h_1 \geq -C_1 u$ for some $C_1 \geq 0$), so $t = 0$. Then (6.9) implies that

$$-\Delta v \leq h_2(x,0,v) \leq \Big(\lambda_1(-\Delta,\Omega) - \varepsilon_0\Big)v \quad \text{ in } \ \Omega,$$

for some $\varepsilon_0 > 0$. We then easily deduce $v \equiv 0$, by multiplying with the first Dirichlet eigenfunction of $-\Delta$ and by integrating by parts.

Theorem 6.2 follows from a standard topological degree argument. We recall the following fixed point theorem, due to Krasnoselskii and Benjamin (see Proposition 2.1 and Remark 2.1 in [19]). This type of statements are nowadays standard in proving existence results.

Theorem 6.3. *Let $\mathcal{K}$ be a closed cone in a Banach space E, and let $T : \mathcal{K} \to \mathcal{K}$ be a compact mapping. Suppose $0 < \delta < M < \infty$, are such that*

(i) $\eta Tx \neq x$ *for all* $x \in \mathcal{K}, \|x\| = \delta$, *and all* $\eta \in [0,1]$ *;*

and there exists a compact mapping $H : \mathcal{K} \times [0,1] \to \mathcal{K}$ such that

(ii) $H(x,0) = Tx$ *for all* $x \in \mathcal{K}$ *;*

(iii) $H(x,t) \neq x$ *for all* $x \in \mathcal{K}, \|x\| = M$ *and all* $t \in [0,1]$ *;*

(iv) $H(x,1) \neq x$ *for all* $x \in \mathcal{K}, \|x\| \leq M$.

Then there exists a fixed point x of T (i.e. $Tx = x$), such that $\delta \leq \|x\| \leq M$.

Observe that (i) implies $i(T, B_\delta \cap \mathcal{K}) = i(0, B_\delta \cap \mathcal{K}) = 1$, where i is the (homotopy invariant) fixed point index with respect to the relative topology of $\mathcal{K}$, whereas by (iii)-(iv)

$$i(H(\cdot,0), B_R \cap \mathcal{K}) = i(H(\cdot,1), B_R \cap \mathcal{K}) = 0,$$

and the excision property of the index implies Theorem 6.3.

A little care is needed in defining T and H. Let $\mathcal{K}$ denote the cone of nonnegative functions in $E := C(\overline{\Omega}) \times C(\overline{\Omega})$, and let $\mathcal{T} : E \to \mathcal{K}$ be defined by

$$\mathcal{T}(\phi,\psi) = (u_+, v_+),$$

where (u,v) is the solution of the linear problem

$$-\Delta u = \phi, \ -\Delta v = \psi \text{ in } \Omega,$$
$$u = v = 0 \ \text{ on } \partial\Omega.$$

It is clear that $\mathcal{T}$ is compact, since $(\phi, \psi) \to (u, v)$ is such by elliptic estimates, and $(u, v) \to (u_+, v_+)$ is Lipschitz. We set

$$H((u,v),t) := \mathcal{T}\Big(F(t,x,u(x),v(x)), G(t,x,u(x),v(x))\Big),$$

and $T(u,v) = H((u,v),0)$. Recall F, G are defined in (6.12)-(6.13), so fixed points of $H(\cdot,t)$ are solutions of (6.11).

We still have to choose the constant A in (6.12)–(6.14). We do this in the following way : by (6.8), there exists $C_1 > 0$ such that $h_1 \geq -C_1 u$ and $h_2 \geq -C_1 v$, and we set

$$A = \max\left\{C_1 + \lambda_1(-\Delta, \omega),\ \sup_{x\in\Omega} c(x),\ \sup_{x\in\Omega} d(x)\right\}, \tag{6.24}$$

where ω is some smooth strict subdomain of Ω. Once A is fixed, we know from the proof of Theorem 6.1 that there exists a universal bound for the positive solutions (if they exist) of (6.11) valid for all $t \in [0,1]$, and we chose M larger than this bound.

Theorem 6.2 is proved if we show that T has a nontrivial fixed point in $\mathcal{K}$. So it remains to check that the hypotheses of Theorem 6.3 are satisfied.

• Let us first show that $H(\cdot,1)$ does not possess any fixed point in $\mathcal{K}$, which will verify (iv). Assume such a fixed point (u,v) exists, which is then a solution of (6.11), with $t = 1$. We have $u, v > 0$ in Ω, since $t > 0$. Let $\mathcal{S} = u^{1/2}v^{1/2}$. By using the inequality $2\Delta((uv)^{1/2}) \leq v^{1/2}u^{-1/2}\Delta u + u^{1/2}v^{-1/2}\Delta v$, we get

$$\begin{aligned}
-\Delta\mathcal{S} \geq\ & \frac{u^{-1/2}v^{1/2}}{2}\Big[u^r v^p\Big((a(x)+A)v^q - c(x)u^q\Big) + (A - C_1)u + A\Big] \\
& + \frac{u^{1/2}v^{-1/2}}{2}\Big[v^r u^p\Big((b(x)+A)u^q - d(x)v^q\Big) + (A - C_1)v + A\Big] \\
\geq\ & \frac{v^\sigma X^{-1/2}}{2}\Big[(a(x)+A)X^r + (b(x)+A)X^{p+q+1} - c(x)X^{q+r} - d(x)X^{p+1}\Big] \\
& + (A - C_1)\mathcal{S} + A,
\end{aligned}$$

where $X = u/v$. Using (6.24) and the inequality

$$X^r + X^{p+q+1} - X^{q+r} - X^{p+1} = X^r(1 - X^q)(1 - X^{p+1-r}) \geq 0$$

(note $p + 1 \geq 1 \geq r$), it follows that

$$-\Delta\mathcal{S} \geq (A - C_1)\mathcal{S} \quad \text{in}\ \ \omega.$$

We reach a contradiction by testing this inequality with the first Dirichlet eigenfunction in ω, because of (6.24).

• Hypothesis (iii) in Theorem 6.3 is a consequence of the a priori bound for positive solutions of (6.11) which we obtained in the proof of Theorem 6.1, and the observation we made in the beginning of the proof of Theorem 6.2.

• Finally, assume that hypothesis (i) is not verified, which implies that for any (small) $\delta > 0$ we can find a positive solution (u, v) with $\|(u, v)\| \leq \delta$, of (6.1) with the right-hand side of this system multiplied by some $\eta \in [0, 1]$. By adding up the two equations in the system and using (6.10) we obtain, with $\lambda_1 = \lambda_1(-\Delta, \Omega)$ and for some $\varepsilon_0 > 0$,

$$\begin{aligned} -\Delta(u+v) &\leq C(u^r v^{p+q} + v^r u^{p+q}) + (\lambda_1 - \varepsilon_0)(u+v) \\ &\leq C(u+v)^{\sigma-1}(u+v) + (\lambda_1 - \varepsilon_0)(u+v) \\ &\leq (\lambda_1 - \varepsilon_0/2)(u+v) \end{aligned}$$

(we obtained the last inequality by choosing δ sufficiently small). By testing again with the first Dirichlet eigenfunction we get a contradiction.

Theorem 6.2 is proved. □

Remark 6.3. *By simple modifications of the above proof, one can show that assumption (6.8) can be weakened as follows : for each $R > 0$,*

$$\inf_{\substack{x\in\Omega \\ u,v\in(0,R)}} u^{-1}\, h_1(x,u,v) > -\infty, \qquad \inf_{\substack{x\in\Omega \\ u,v\in(0,R)}} v^{-1}\, h_2(x,u,v) > -\infty$$

(which allows for the application of the strong maximum principle) and, for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that, for all $u, v \geq 0$, $x \in \Omega$,

$$h_1(x,u,v) \geq -\varepsilon u^r v^{p+q} - C_\varepsilon(1+u), \quad h_2(x,u,v) \geq -\varepsilon v^r u^{p+q} - C_\varepsilon(1+v).$$

7 Appendix. Proof of Proposition 1.1

In the appendix we study the proportionality constants of the system (1.10) and give the elementary proof of Proposition 1.1. In the following we set

$$f(u,v) = u^r v^p [a v^q - c u^q], \qquad g(u,v) = v^r u^p [b u^q - d v^q].$$

Proof of Proposition 1.1. We first note that if $c = d = 0$ and $q = r - p$, then $Kg - f = (Kb - a)u^r v^r$ and (1.9) is satisfied if and only if $K = a/b$ (and then actually $Kg - f \equiv 0$). We may thus assume that either

$$c > 0, \quad \text{or } d > 0, \quad \text{or } q \neq r - p. \tag{7.1}$$

Set $X = u/v$. For given $K > 0$, we compute, for $u, v > 0$,

$$\begin{aligned} Kg - f &= K u^p v^r (b u^q - d v^q) - u^r v^p (a v^q - c u^q) \\ &= u^p v^{r+q} [K b X^q - K d - a X^{r-p} + c X^{q+r-p}] \end{aligned}$$

and also, factorizing by X^{r-p},

$$Kg - f = u^r v^{p+q}[KbX^{q+p-r} - KdX^{p-r} - a + cX^q].$$

Set

$$m := |r-p| \le q,$$

and define

$$H_K(X) = AX^{q+m} + BX^q - CX^m - D, \qquad X > 0,$$

where

$$\begin{cases} A = c,\ B = Kb,\ C = a,\ D = Kd, & \text{if } r \ge p, \\ A = Kb,\ B = c,\ C = Kd,\ D = a, & \text{otherwise.} \end{cases}$$

We then see that we may write

$$Kg - f = \begin{cases} u^p v^{r+q} H_K(u/v), & \text{if } r \ge p, \\ u^r v^{p+q} H_K(u/v), & \text{otherwise.} \end{cases} \tag{7.2}$$

We next claim that there exists (at least one) $K > 0$ such that $H_K(K) = 0$. Indeed, setting

$$J(K) := H_K(K) = \begin{cases} cK^{q+m} + bK^{q+1} - aK^m - dK, & \text{if } r \ge p, \\ bK^{q+m+1} + cK^q - dK^{m+1} - a, & \text{otherwise,} \end{cases}$$

we easily see that $\lim_{t\to\infty} J(t) = \infty$ and $J(K) < 0$ for small $K > 0$, and the claim follows.

Now pick any $K > 0$ such that $H_K(K) = 0$. We will prove that

$$[Kg(u,v) - f(u,v)][u - Kv] > 0 \ \text{ for all } u, v > 0 \text{ with } u \ne Kv, \tag{7.3}$$

a slightly stronger property than (1.9), which will in particular establish the existence part of Proposition 1.1.

We first consider the case $m > 0$ and set $\ell = q/m \ge 1$. Let us rewrite

$$H_K(X) = h_K(X^m), \qquad \text{with } h_K(t) = At^{\ell+1} + Bt^\ell - Ct - D, \quad t > 0. \tag{7.4}$$

This function is easier to handle than H_K because its last two terms are affine and h_K is convex. We claim that

$$h_K(t) < 0 \text{ for } t > 0 \text{ small.} \tag{7.5}$$

- If $D > 0$, then $h_K(0) = -D < 0$.
- If $\ell > 1$ and $D = 0$, then $h_K(0) = 0$ and $h'_K(0) = -C = -a < 0$.
- If $\ell = 1$ (hence $q = m$) and $D = 0$ (hence $r \ge p$ and $d = 0$), then we may assume $c > 0$ (see (7.1)). We have $h_K(s) = cs^2 + (Kb - a)s$. Then necessarily $Kb - a < 0$ (since $H_K(K) = 0$). Thus $h_K(0) = 0$ and $h'_K(0) < 0$.

In either case, (7.5) is true. On the other hand, we also have

$$\lim_{t\to\infty} h_K(t) = \infty. \tag{7.6}$$

(This is clear unless $A = 0$ and $\ell = 1$, but in that case $D > 0$ due to (7.1), hence $B > C$ due to $H_K(K) = 0$.) Now, since h_K is convex on $[0, \infty)$, it follows from (7.5), (7.6) that h_K has a unique zero on $(0, \infty)$. Consequently, K is the unique zero of H_K on $(0, \infty)$ and, by (7.2), (7.4) and (7.5), we deduce (7.3).

If $m = 0$, then $H_K(X) = (Kb+c)X^q-(a+Kd)$, which is monotonically increasing in X, and (7.3) is clear. The proof of the existence part is thus complete.

Let us now suppose $ab \geq cd$ and show the uniqueness of K. Assume for contradiction that (1.9) is true for two distinct values of K, say $K_2 > K_1 > 0$. Pick $Y \in (K_1, K_2)$. For $i \in \{1, 2\}$, since $H_{K_i}(K_i) = 0$ due to (7.2), it follows from what we already proved that (7.3) is true for $K = K_i$. In particular, $K_1 g(Y, 1) > f(Y, 1) > K_2 g(Y, 1)$. Therefore $g(Y, 1) < 0$ and $f(Y, 1) < 0$, that is $0 < a < cY^q$ and $0 < bY^q < d$. Consequently $ab < cd$: a contradiction.

Finally, suppose $ab > cd$ and assume for contradiction that $cK^q \geq a$ (hence $c > 0$). Then $bK^{1+q+p-r} \geq (ab/c)K^{1+p-r} > dK^{1+p-r}$. It then follows from (7.3) that

$$0 = Kg(K, 1) - f(K, 1) \geq K^r[bK^{1+q+p-r} - dK^{1+p-r} - a + cK^q] > 0.$$

This contradiction shows that $a - cK^q > 0$. The proofs of $bK^q - d > 0$ and of the equalities are similar. □

In the end, we prove related lower bounds which we use in the proofs of Lemma 4.2 and Proposition 2.1.

Lemma 7.1. *Assume (1.11).*

(i) Assume $r > p$ and $c, d > 0$. Then the nonlinearities in the system (1.10) *satisfy, for some $C > 0$,*

$$(Kg - f)(u - Kv) \geq Cu^p v^p (u + Kv)^{q+r-p-1}(u - Kv)^2.$$

(ii) Assume $d = 0$ and $c > 0$. Then

$$(Kg - f)(u - Kv) \geq Cu^r v^{p\wedge r}(u + Kv)^{q-1+(p-r)_+}(u - Kv)^2. \tag{7.7}$$

Proof. (i) We use the same notation as in the above proof of Proposition 1.1. First note that $h_K{}'(K^m) > 0$ since h_K is negative and convex on $(0, K^m)$. Denoting

$$p(t) = \frac{h_K(t)}{t^{\ell+1} - K^{m(\ell+1)}},$$

we observe that $p(t) > 0$ on $[0, K^m) \cup (K^m, +\infty)$ and that $p(t)$ has positive limits as t goes to K^m or $+\infty$ (using that $h_K{}'(K^m) > 0$). Hence, for all $t \geq 0$, we have $p(t) \geq C$ for some constant $C > 0$. So,

$$\frac{H_K(X)}{X^{m(\ell+1)} - K^{m(\ell+1)}} \geq C$$

and we obtain

$$H_K(X)(X-K) \geq C(X-K)(X^{m(\ell+1)} - K^{m(\ell+1)}).$$

Since $(Kg-f)(u-Kv) = u^p v^{r+q+1} H_K(X)(X-K)$, by using the inequality

$$(x^k - y^k)(x-y) \geq C_k (x+y)^{k-1}(x-y)^2, \quad x, y > 0$$

for $k > 0$ and some $C_k > 0$, we get

$$\begin{aligned}(Kg-f)(u-Kv) &\geq C \frac{u^p v^{r+q+1}}{v^{1+q+r-p}} (u^{q+r-p} - (Kv)^{q+r-p})(u-Kv) \\ &\geq C u^p v^p (u+Kv)^{q+r-p-1}(u-Kv)^2.\end{aligned}$$

(ii) Letting $X = u/v$, we have, for $u, v > 0$,

$$Kg - f = u^r v^{p+q} G(X), \quad \text{where } G(X) = KbX^{q+p-r} + cX^q - a.$$

We know from Proposition 1.1 that G vanishes only at $X = K$. Since $G'(K) > 0$, it is easy to see that

$$\frac{G(X)}{X-K} \geq C(X+1)^\ell, \quad X \in [0, \infty) \setminus \{K\},$$

where $\ell = q - 1 + (p-r)_+$. Therefore

$$(Kg-f)(u-Kv) = u^r v^{p+q-1} \frac{G(X)}{X-K}(u-Kv)^2 \geq C u^r v^{p+q-1-\ell}(u+v)^\ell (u-Kv)^2.$$

The assertion follows. □

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Chapitre 5

Proportionnalité des composantes, théorèmes de Liouville et estimations a priori pour des systèmes elliptiques complètement non-linéaires non coopératifs [1]

Dans ce chapitre, nous cherchons à généraliser les résultats du chapitre 5 pour des systèmes elliptiques complètement non-linéaires, plus précisément lorsque l'opérateur considéré n'est plus le Laplacien mais un opérateur de Isaacs, i.e. un opérateur uniformément elliptique positivement homogène, les solutions étant donc comprises au sens de la viscosité.

1 Introduction

1.1 Origin of the problem

We want to generalize the results obtained in [P5] to fully nonlinear elliptic systems. More precisely, we consider the following kind of systems

$$\begin{cases} -F(D^2u) = f(x,u,v), & x \in \Omega \\ -F(D^2v) = g(x,u,v), & x \in \Omega \\ u = K\,v & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

1. Ce chapitre est le fruit d'une visite de trois semaines à la PUC de Rio de Janeiro, sur invitation de **Boyan Sirakov**. Nous tenons à remercier le Réseau Franco-Brésilien de Mathématiques pour le financement de ce séjour. Une version ultérieure de ce chapitre fera l'objet de l'article [P6] en collaboration avec **Boyan Sirakov**.

where F is an Isaacs operator (see Section 2 for a definition), Ω is a domain of $\mathbb{R}^n$ and solutions are thought in the viscosity sense.

Throughout this chapter, we will assume that the nonlinearities f and g (or their leading order terms) satisfy the following condition

$$\exists\, K > 0 \ : \quad [f(x,u,v) - Kg(x,u,v)][u - Kv] \le 0 \quad \text{for all } (u,v) \in \mathbb{R}^2 \text{ and } x \in \Omega. \tag{1.2}$$

We have explained in the general introduction how natural is this condition to expect proportionality of components

$$u = Kv.$$

1.2 Main results

Here, we present the main results we have obtained for fully nonlinear elliptic systems.

1.2.1 Case $\Omega = \mathcal{C}_\omega$

We begin by the case where Ω is a cone

$$\mathcal{C}_\omega = \{tx, t > 0, x \in \omega\}$$

where ω is a C^2 (strict) subdomain of the unit sphere in $\mathbb{R}^n$.

Theorem 1.1. *Let F be an Isaacs operator.*
Assume that f and g satisfy (1.2) with some $K > 0$.
Let (u,v) be a bounded viscosity solution on $\mathcal{C}_\omega$ of

$$\begin{cases} -F(D^2u) = f(x,u,v), & x \in \mathcal{C}_\omega \\ -F(D^2v) = g(x,u,v), & x \in \mathcal{C}_\omega \\ u = K\,v & \text{on } \partial\mathcal{C}_\omega. \end{cases} \tag{1.3}$$

Then

$$u = K\,v.$$

Remark 1.1. *This result in particular applies to bounded viscosity solutions satisfying the Dirichlet condition $u = v = 0$ on $\partial\mathcal{C}_\omega$.*

This result is a consequence of a Phragmèn-Lindelhöf principle which is a particular case of [2, Theorem 1.7].

Recalling that $\alpha^+(F, \mathcal{C}_\omega)$ was defined in the general introduction of this thesis, we obtain as an obvious consequence of Theorem 1.1 and of a nonexistence result on a cone (see Lemma 4.4) the following Liouville theorem on systems.

Corollary 1.1. *Under the hypothesis of the previous theorem and if there exist $C > 0$ and*

$$\sigma \in \left(0, \frac{\alpha^+(F, \mathcal{C}_\omega) + 2}{\alpha^+(F, \mathcal{C}_\omega)}\right)$$

such that for all $x \in \mathbb{R}^n_+$ and $y \geq 0$,

$$f(x, Ky, y) = C\, y^\sigma,$$

then the only bounded solution of (1.3) is the trivial one.

1.2.2 Case $\Omega = \mathbb{R}^n$

For $\Omega = \mathbb{R}^n$, we focus on the following system

$$\begin{cases} -F(D^2u) = u^r v^p [av^q - cu^q] & \text{on } \mathbb{R}^n \\ -F(D^2v) = v^r u^p [bu^q - dv^q] & \text{on } \mathbb{R}^n, \end{cases} \tag{1.4}$$

where we always assume that the real parameters a, b, c, dp, q, r satisfy

$$a, b > 0, \quad c, d \geq 0, \qquad p, r \geq 0, \quad q > 0, \quad q \geq |p - r|. \tag{1.5}$$

This last condition ensures, thanks to Proposition 5.1, that (1.2) is verified.

Recalling that $\alpha^*(F)$ was defined in the general introduction of this thesis, we have the following result :

Theorem 1.2. *Let F be an Isaacs operator.*
Assume (1.5) holds and

$$ab \geq cd$$

and let $K > 0$ be the constant from Proposition 5.1.
Let (u, v) be a positive viscosity solution of (1.4) in $\mathbb{R}^n$.

i) Assume that

$$\alpha^*(F) \leq 0 \text{ or } \left(\alpha^*(F) > 0 \text{ and } 0 \leq r \leq \frac{\alpha^*(F) + 2}{\alpha^*(F)}\right).$$

If $p + q < 1$, we assume moreover that u and v are bounded. Then

$$u = Kv.$$

ii) Assume that

$$p \leq \frac{2}{\alpha^*(F)} \qquad \text{and} \qquad c, d > 0.$$

If $q + r \leq 1$, we assume moreover that u and v are bounded. Then

$$u = Kv.$$

An easy consequence is then the following Liouville type result for the system :

Theorem 1.3. *Let F be an Isaacs operator.*
Assume (1.5) holds and

$$ab > cd.$$

We denote

$$\sigma = p + q + r > 0$$

and assume that

$$\alpha^*(F) \leq 0 \quad \text{or} \quad 0 \leq r \leq \frac{\alpha^*(F)+2}{\alpha^*(F)} \quad \text{or} \quad 0 \leq p \leq \frac{2}{\alpha^*(F)}. \tag{1.6}$$

Assume moreover that the equation

$$-F(D^2u) = u^\sigma \text{ has no bounded positive viscosity solution on } \mathbb{R}^n, \tag{1.7}$$

then (1.4) has no bounded positive viscosity solution in $\mathbb{R}^n$.

Theorem 1.4. *Under the hypothesis of Theorem 1.3, if we moreover assume that*

$$q + r > 1,$$

then any nonnegative bounded viscosity solution of (1.4) is semitrivial, i.e.

$$(u, v) = (C_1, 0) \qquad \text{or} \qquad (u, v) = (0, C_2)$$

with $C_1, C_2 \geq 0$. Moreover :
- If $r = 0$, then $(u, v) = (0, 0)$.
- If $r > 0$, $p = 0$ and $c > 0$ (resp. $d > 0$), then $C_1 = 0$ (resp. $C_2 = 0$).

Remark 1.2. *We here have to face with the still rough understanding we have about nonexistence results concerning the equation*

$$-F(D^2u) = u^\sigma$$

for a general Isaacs operator. Nevertheless, we can observe that :
i) The condition (1.7) is satisfied if one of the following conditions
a) $\alpha^*(F) \leq 0$
b) $\alpha^*(F) > 0$ and $\sigma \leq \frac{\alpha^*(F)+2}{\alpha^*(F)}$ is verified.
This is a consequence of Lemma 4.2 applied with $V = 1$.
ii) When F is one of the Pucci operators $\mathcal{M}^+$ or $\mathcal{M}^-$, then we know from [7] that if $\alpha^*(F) > 0$, then there exists

$$\overline{\sigma} > \frac{\alpha^*(F)+2}{\alpha^*(F)}$$

such that (1.7) is satisfied for radial solutions if and only if $\sigma < \overline{\sigma}$, $\overline{\sigma}$ being not not known explicitly in function of n, λ, Λ.

iii) If we consider the Barenblatt operator defined by

$$F(M) = \max(tr(M), 2tr(M))$$

for all $M \in \mathcal{S}_n$, then, any solution of

$$-F(D^2u) = u^\sigma$$

being superharmonic, we deduce that (1.7) is satisfied if and only if

$$-\Delta u = u^\sigma$$

and then if and only if

$$\sigma < \frac{n+2}{n-2}.$$

1.3 A priori estimates and existence in a bounded domain

We consider the following system with general lower order terms

$$\begin{cases} -F(D^2u) = u^r v^p \Big[a(x)v^q - c(x)u^q\Big] + h_1(x,u,v), & x \in \Omega, \\ -F(D^2v) = v^r u^p \Big[b(x)u^q - d(x)v^q\Big] + h_2(x,u,v), & x \in \Omega, \\ \qquad u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.8}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$ and F is an Isaacs operator.

We denote $\alpha^+ = \alpha^+(F, \mathbb{R}^n_+)$.

Theorem 1.5. *Let F be an Isaacs operator.*
Let $p, r \geq 0$, $q > 0$, $q \geq |p - r|$ and such that

$$q + r > 1 \tag{1.9}$$

We set

$$\sigma = p + q + r > 1.$$

Assume moreover that

$$\alpha^*(F) \leq 0 \quad \textit{or} \quad 0 \leq r \leq \frac{\alpha^*(F)+2}{\alpha^*(F)} \quad \textit{or} \quad p \leq \frac{2}{\alpha^*(F)}. \tag{1.10}$$

and that

$$\textit{the equation } -F(D^2u) = u^\sigma \textit{ has no bounded positive viscosity solution} \tag{1.11}$$

neither on $\mathbb{R}^n$ nor on $\mathbb{R}^n_+$ with Dirichlet condition.

Let $a, b, c, d \in C(\overline{\Omega})$ satisfying $a, b > 0,\ c, d \geq 0$ in $\overline{\Omega}$ and

$$\inf_{x\in\Omega}\ [a(x)b(x) - c(x)d(x)]\ > 0. \tag{1.12}$$

Let $h_1, h_2 \in C(\overline{\Omega} \times [0,\infty)^2)$ satisfy

$$\lim_{u+v\to\infty} \frac{h_i(x,u,v)}{(u+v)^\sigma} = 0, \quad i = 1, 2, \tag{1.13}$$

and assume that one of the following two sets of assumptions is satisfied :

$$\left\{\begin{array}{l} r \leq 1, \quad \text{and, setting } \bar{m} := \min\{\inf\limits_{x\in\Omega} a(x), \inf\limits_{x\in\Omega} b(x)\} > 0, \\ \liminf\limits_{v\to\infty,\ u/v\to 0} \dfrac{h_1(x,u,v)}{u^r v^{p+q}} > -\bar{m}, \qquad \liminf\limits_{u\to\infty,\ v/u\to 0} \dfrac{h_2(x,u,v)}{v^r u^{p+q}} > -\bar{m}, \end{array}\right. \tag{1.14}$$

or

$$\left\{\begin{array}{l} m := \min\{\inf\limits_{x\in\Omega} c(x), \inf\limits_{x\in\Omega} d(x)\} > 0, \qquad \text{and} \\ \limsup\limits_{u\to\infty,\ v/u\to 0} \dfrac{h_1(x,u,v)}{u^{r+q} v^p} < m, \qquad \limsup\limits_{v\to\infty,\ u/v\to 0} \dfrac{h_2(x,u,v)}{v^{r+q} u^p} < m \end{array}\right. \tag{1.15}$$

(with uniform limits with respect to $x \in \overline{\Omega}$ in (1.13)– (1.15)).
Then there exists $M > 0$ such that any positive classical solution (u, v) of (6.1) satisfies

$$\sup_\Omega u \leq M, \qquad \sup_\Omega v \leq M. \tag{1.16}$$

Remark 1.3. *Note that if*

$$1 < \sigma < \frac{\alpha^+ + 2}{\alpha^+},$$

then (1.10) *and* (1.11) *are satisfied.*
Indeed, $\alpha^+ \geq \alpha^*(F)$ since by [2, 3], we know that

$$\alpha^+ = \sup\{\alpha > 0,\ \exists u \in H_\alpha(\mathbb{R}^n_+),\ -F(D^2u) \geq 0 \text{ and } u > 0 \text{ in } \mathbb{R}^n_+\}$$

and that, if $\alpha^*(F) > 0$, we have

$$\alpha^*(F) = \sup\{\alpha > 0,\ \exists u \in H_\alpha(\mathbb{R}^n \setminus \{0\}),\ -F(D^2u) \geq 0 \text{ and } u > 0 \text{ in } \mathbb{R}^n \setminus \{0\}\},$$

where $H_\alpha(\Omega) = \{u \in C(\Omega),\ u$ is $(-\alpha)$-homogeneous$\}$ for Ω being a cone.
Hence, $\sigma < \frac{\alpha^*+2}{\alpha^*}$, whence (1.10)*. The Liouville results for $-F(D^2u) = u^\sigma$ follow from Lemma 4.4 and Lemma 4.2 (applied to $V = 1$).*

In the following theorem of existence, we need some more assumptions, in particular that F is subadditive, i.e. that

$$F(M+N) \leq F(M) + F(N) \text{ for all } (M,N) \in {\mathcal{S}_n}^2,$$

which is equivalent to the convexity of F since we also assume (H_2) for F.

Theorem 1.6. *Let F be a subadditive Isaacs operator.*
Let (1.9)–(1.14) be satisfied. Assume in addition that a, b, c, d, h_1, h_2 are Hölder continuous and that for some $\epsilon > 0$

$$\inf_{x\in\Omega,\ u,v>0} u^{-1}\, h_1(x,u,v) > -\infty, \qquad \inf_{x\in\Omega,\ u,v>0} v^{-1}\, h_2(x,u,v) > -\infty, \tag{1.17}$$

$$\sup_{x\in\Omega,\ u>0} u^{-1}\, h_1(x,u,0) < \lambda_1^+(-F,\Omega), \qquad \sup_{x\in\Omega,\ v>0} v^{-1}\, h_2(x,0,v) < \lambda_1^+(-F,\Omega), \tag{1.18}$$

$$\sup_{x\in\Omega,\ u,v\in(0,\epsilon)^2} (u+v)^{-1}\,[h_1(x,u,v)+h_2(x,u,v)] < \lambda_1^+(-F,\Omega). \tag{1.19}$$

Then there exists a bounded positive classical solution of (6.1).

2 Reminder on some notation and definitions

For the convenience of the reader, we begin by recalling some definitions and notation.

Definition 2.1. *F is an Isaccs operator if the following conditions are satisfied :*
– *F is uniformly elliptic :*
there exist $\Lambda > \lambda > 0$ such that for all $(M, N) \in \mathcal{S}_n{}^2$ with $N \geq 0$,

$$(H_1) \qquad \lambda\, tr(N) \leq F(M+N) - F(M) \leq \Lambda\, tr(N),$$

– *F is 1-homogeneous :*
for all $t \geq 0$ and $M \in \mathcal{S}_n$, we have

$$(H_2) \qquad F(t\,M) = t\,F(M),$$

where we denote $\mathcal{S}_n$ the set of real symmetric matrices.

For the notions of uniformly elliptic operators and viscosity solutions, we refer the reader to [5].

Notation 2.1. *Let $0 < \lambda < \Lambda$. We define the extremal Pucci operators by*

$$\mathcal{M}^+(M) = \Lambda \sum_{\mu_i>0} \mu_i + \lambda \sum_{\mu_i<0} \mu_i$$

and

$$\mathcal{M}^-(M) = \lambda \sum_{\mu_i>0} \mu_i + \Lambda \sum_{\mu_i<0} \mu_i,$$

for all $M \in \mathcal{S}_n$, where the $(\mu_i)_{i=1..n})$ are the eigenvalues of M.

We now would like to stress that (H_1) is actually equivalent to

$$(H_1') \qquad \mathcal{M}^-(M-N) \leq F(M) - F(N) \leq \mathcal{M}^+(M-N)$$

for all $(M,N) \in {\mathcal{S}_n}^2$.

As a consequence, an Isaacs operator F satisfies

$$\mathcal{M}^-(M) \leq F(M) \leq \mathcal{M}^+(M) \tag{2.1}$$

for all $M \in \mathcal{S}_n$, since (H_2) implies that

$$F(0) = 0.$$

We also recall that :

Definition 2.2. *$F : \mathcal{S}_n \to \mathbb{R}$ is a degenerate elliptic operator if $F(M) \leq F(N)$ for any $(M,N) \in {\mathcal{S}_n}^2$ such that $M \leq N$.*

We will also need the definition of one of the principal half-eigenvalues of an Isaacs operator. See [1] for more details.

Definition 2.3. *Let Ω be a bounded smooth domain of $\mathbb{R}^n$ and F be an Isaccs operator. We define the finite real number*

$$\lambda_1^+(-F,\Omega) = \sup\{r \in \mathbb{R}, \exists u \in C(\Omega), u > 0, -F(D^2u) \geq r\ u\ \textit{in}\ \Omega\}.$$

3 Preliminary results on viscosity solutions of systems

In this section, for the sake of completeness, we present some simple but useful results for viscosity solutions of systems that we will employ later in the proofs. The reader familiar with the subject may skip it.

We first would like to recall a transitivity result, without proof since it is a particular case of [1, Lemma 3.2].

Lemma 3.1. *Assume that F_1, F_2, G are uniformly elliptic operators (i.e. satisfy (H_1)) and verify*

$$G(M+N) \geq F_1(M) + F_2(N)$$

for all $(M,N) \in {\mathcal{S}_n}^2$.
Assume that $u \in C(\overline{\Omega})$, $f \in C(\Omega)$ and u is a viscosity solution in Ω of

$$F_1(D^2u) \geq f$$

and that $v \in C(\overline{\Omega})$, $g \in C(\Omega)$ and v is a viscosity solution in Ω of

$$F_2(D^2v) \geq g.$$

Then $w = u + v \in C(\overline{\Omega})$ is a viscosity solution in Ω of

$$G(D^2w) \geq f + g.$$

Actually, we will use the following result, which is an easy consequence of the previous lemma :

Lemma 3.2. *Let F be a uniformly elliptic operator.*
Assume that $u \in C(\overline{\Omega})$, $f \in C(\Omega)$ and u is a viscosity solution in Ω of

$$F(D^2u) \geq f$$

and that $v \in C(\overline{\Omega})$, $g \in C(\Omega)$ and v is a viscosity solution in Ω of

$$F(D^2v) \leq g.$$

Then $w = u - v$ is a viscosity solution in Ω of

$$\mathcal{M}^+(D^2w) \geq f - g.$$

Démonstration. If we define the uniformly elliptic operator $\tilde{F}$ by

$$\tilde{F}(M) = -F(-M)$$

for all $M \in \mathcal{S}_n$, then (H_1') implies that

$$\mathcal{M}^+(M + N) \geq F(M) + \tilde{F}(N)$$

for all $(M, N) \in {\mathcal{S}_n}^2$. The result then follows from Lemma 3.1 applied to u and $-v$ since $\tilde{v} = -v$ is solution of

$$\tilde{F}(D^2\tilde{v}) \geq -g.$$

□

The next lemma is helpful to exploit condition (1.2) on the nonlinearities f and g of the system, as shown in the subsequent result.

Lemma 3.3. *Let Ω be an open set and assume that F satisfies (H_1) and $F(0) = 0$. If w is a viscosity solution on Ω of*

$$\begin{cases} F(D^2w) \geq h \\ F(D^2[-w]) \geq -h \end{cases} \tag{3.1}$$

where

$$\begin{cases} h \geq 0 & \text{if } w > 0 \\ h \leq 0 & \text{if } w < 0 \\ h = 0 & \text{if } w = 0 \end{cases} \tag{3.2}$$

then $|w|$ is a viscosity solution on Ω of

$$F(D^2|w|) \geq |h|$$

Démonstration. Let $\phi \in C^2(\Omega)$ touching by above $|w|$ at $x_0 \in \Omega$.
If $w(x_0) > 0$, then $h(x_0) \geq 0$ and since ϕ touches w by above at x_0, we have

$$F(D^2\phi(x_0)) \geq h(x_0) = |h(x_0)|.$$

If $w(x_0) < 0$, then $h(x_0) \leq 0$ and since ϕ touches $-w$ by above at x_0, we have

$$F(D^2\phi(x_0)) \geq -h(x_0) = |h(x_0)|.$$

If $w(x_0) = 0$, then $h(x_0) = 0$. Moreover, $\phi(x_0) = 0$ and $\phi \geq |w| \geq 0$ so $x_0 \in \Omega$ is a minimum point of ϕ so $D^2\phi(x_0) \geq 0$, whence $F(D^2\phi(x_0) \geq F(0) = 0 = |h(x_0)|$. □

Lemma 3.4. *Assume that F is an Isaacs operator.*
Let (u, v) be a viscosity solution of (1.1) on an open set Ω and assume that the nonlinearities f, g are continuous and satisfy (1.2). Then

$$\mathcal{M}^+(D^2|u - Kv|) \geq |f - Kg| \qquad \textit{on } \Omega$$

in the viscosity sense.

Démonstration. We are going to apply Lemma 3.3. We set

$$w = u - Kv$$

and

$$h = Kg(\cdot, u, v) - f(\cdot, u, v).$$

Since $F(D^2u) = -f$ and $F(D^2[Kv]) = K\,F(D^2v) = -Kg$, from Lemma 3.2, it is easy to see that

$$\mathcal{M}^+(D^2w) \geq h$$

and

$$\mathcal{M}^+(D^2[-w]) \geq -h.$$

Moreover, condition (1.2) means that

$$h\,w \geq 0.$$

By continuity of f, g and (1.2), it is easy to see that for all $x \in \Omega$ and $y \in \mathbb{R}$, we have $Kg(x, Ky, y) - f(x, Ky, y) = 0$ hence if $w = 0$ then $h = 0$. This implies that (3.2) is fully satisfied. Hence, by Lemma 3.3, we get the result. □

This last lemma will be useful when considering system (1.4) on the whole space since the auxiliary function $Z = \min(u, Kv)$ will play a crucial role in our analysis (where $K > 0$ is given by Proposition 5.1).

Lemma 3.5. *Assume that F is a degenerate elliptic operator.*
Let Ω be an open set and let $(u, v, f, g, h) \in C(\Omega)^5$.
Assume that u and v are respectively viscosity solutions of

$$-F(D^2u) \geq f \qquad \text{and} \qquad -F(D^2v) \geq g$$

and that

$$\begin{cases} f \geq h & \text{on } \{u \leq v\} \\ g \geq h & \text{on } \{u > v\} \end{cases} \tag{3.3}$$

Then

$$w = \min(u, v) \in C(\Omega)$$

is a viscosity solution on Ω of

$$-F(D^2w) \geq h.$$

Démonstration. Let $\phi \in C^2(\Omega)$ touching by below w at $x_0 \in \Omega$. Then $\phi \leq u$ and $\phi \leq v$.
If $u(x_0) \leq v(x_0)$ then $w(x_0) = u(x_0)$ and ϕ touches u by below at x_0, hence

$$-F(D^2\phi(x_0)) \geq f(x_0) \geq h(x_0).$$

Similarly, if $u(x_0) > v(x_0)$ then $w(x_0) = v(x_0)$ and ϕ touches v by below at x_0, hence

$$-F(D^2\phi(x_0)) \geq g(x_0) \geq h(x_0).$$

□

4 Liouville theorems for scalar equations on $\mathbb{R}^n$ or a cone

4.1 Liouville theorems for weighted inequalities on the whole space

In this subsection, we present two Liouville type results on $\mathbb{R}^n$.

Lemma 4.1. *Let $w \geq 0$ be a viscosity solution on $\mathbb{R}^n$ of*

$$F(D^2w) \geq \frac{A}{1 + |x|^2} \, w^p, \tag{4.1}$$

where $p > 0$, $A > 0$ and F is a uniformly elliptic operator.
i) If $w \neq 0$, then w is unbounded.
ii) If $p > 1$, then $w = 0$.

Démonstration. Since $F \le \mathcal{M}^+$, then w is a viscosity solution on $\mathbb{R}^n$ of

$$\mathcal{M}^+(D^2 w) \ge \frac{A}{1+|x|^2}\, w^p.$$

i) Assume $w \ne 0$.
We define

$$M(R) = \sup_{B_R} w$$

and will show the existence of $c > 0$ and $R_0 > 0$ such that for all $R \ge R_0$,

$$M(2R) \ge M(R) + c,$$

which implies the result.
The starting point to prove this is to adapt an idea from Observation c) after Theorem 3.4 in [10].

For all $x \in \mathbb{R}^n$, we define

$$f(x) = \frac{A}{1+|x|^2}\, w(x)^p.$$

Since $f \ge 0$ and $f \in C(\mathbb{R}^n)$, thanks to Lemma 7.6, there exists a unique viscosity solution u_R of

$$\begin{aligned} -\mathcal{M}^-(D^2 u_R) &= f \qquad && \text{on } B_{2R} \\ u_R &= 0 && \text{on } \partial B_{2R}. \end{aligned}$$

We define on $\overline{B_{2R}}$

$$v_R = M(2R) - u_R.$$

Then,

$$\mathcal{M}^+(D^2 v_R) = \mathcal{M}^+(-D^2 u_R) = -\mathcal{M}^-(D^2 u_R) = f$$

and

$$\mathcal{M}^+(D^2 w) \ge f.$$

Moreover,

$$v_R = M(2R) \ge w \qquad \text{on } \partial B_{2R}$$

so by comparison principle, we obtain

$$w \le v_R \qquad \text{on } B_{2R},$$

which implies

$$\inf_{B_R} u_R + M(R) \le M(2R).$$

Now, we define on $\overline{B_2}$

$$\tilde{u}_R(x) = u_R(Rx)$$

which is a viscosity solution of

$$-\mathcal{M}^-(D^2\tilde{u}_R) = \frac{A\ R^2}{1+R^2|x|^2}\, w(Rx)^p \geq \epsilon\, w(Rx)^p \qquad \text{on } B_2 \tag{4.2}$$

for some $\epsilon > 0$.
Note that since $w \neq 0$, then there exists $R_0 > 0$ such that

$$\sup_{B_1} w(R_0 \cdot) > 0.$$

Since $\mathcal{M}^+(D^2 w) \geq 0$, then by local maximum principle applied to $w(R\cdot) \geq 0$ for $R > 0$ (see [5, Theorem 4.8 (2)]), for any $q > 0$, there exists $C = C(q) > 0$ such that for all $R \geq R_0$,

$$\|w(R\cdot)\|_{L^q(B_2)} \geq C \sup_{B_1} w(R\cdot) \geq C \sup_{B_1} w(R_0\cdot) > 0.$$

Since $\tilde{u}_R \geq 0$ satisfies (4.2), then by the quantitative strong maximum principle (see Lemma 7.4), there exists $q_0 > 0$ and $c_0 > 0$ such that

$$\inf_{B_1} \tilde{u}_R \geq c_0\, \epsilon\, \|w(R\cdot)^p\|_{L^{q_0}(B_1)}.$$

We choose $q = q_0\, p$ and obtain a constant $c > 0$ such that for all $R \geq R_0$, we have

$$\inf_{B_1} \tilde{u}_R \geq c.$$

Since $\inf_{B_1} \tilde{u}_R = \inf_{B_R} u_R$, this implies

$$M(2R) \geq M(R) + c.$$

ii) We will show that w is bounded on $\mathbb{R}^n$, which proves that $w = 0$ thank to i).

In the spirit of [12], we define the function $w_R \in C^2(B_R)$ by

$$w_R(x) = C\frac{R^{2\alpha}}{(R^2 - |x|^2)^\alpha}$$

for all $x \in B_R$, where

$$\alpha = \frac{2}{p-1}.$$

It is easy to see, by direct computation, that if $C > 0$ is large enough, then for all $x \in B_R$,

$$\Lambda\ \Delta w_R \leq \frac{A}{1+|x|^2} {w_R}^p \tag{4.3}$$

in the classical sense. Indeed, denoting $r = |x|$, we have

$$\Lambda\ \Delta w_R = 2\Lambda\ \alpha C R^{2\alpha}\ \frac{n(R^2-r^2)+2(\alpha+1)r^2}{(R^2-r^2)^{\alpha+2}} \leq 2\Lambda\ \alpha C R^{2\alpha+2}\ \frac{n+2(\alpha+1)}{(R^2-r^2)^{\alpha+2}}$$

and

$$\frac{A}{1+r^2}{w_R}^p = \frac{A}{1+r^2}\frac{C^p R^{2\alpha p}}{(R^2-r^2)^{\alpha p}} \geq \frac{A}{1+R^2}\frac{C^p R^{2\alpha p}}{(R^2-r^2)^{\alpha p}}.$$

We note that

$$\alpha + 2 = \alpha p.$$

Hence, a sufficient condition to have (4.3) is

$$C^{p-1} \geq (1+R^2) R^{2\alpha+2-2\alpha p}\frac{2\Lambda\ \alpha[n+2(\alpha+1)]}{A}.$$

Since $2\alpha+2-2\alpha p = -2$, for each $R \geq 1$ a sufficient condition for the last inequality is

$$C^{p-1} \geq \frac{4\Lambda\ \alpha[n+2(\alpha+1)]}{A},$$

and we choose such a C.
Since w_R is radial and its first and second radial derivative are nonnegative, then w_R is convex so $\mathcal{M}^+(D^2 w_R) = \Lambda\ \Delta w_R$, hence implying

$$\mathcal{M}^+(D^2 w_R) \leq \frac{A}{1+|x|^2}{w_R}^p. \tag{4.4}$$

Since $w_R(x) \underset{x\to\partial B_R}{\longrightarrow} +\infty$, then there exists $R' < R$ such that $w_R \geq \|w\|_{\infty, B_R}$ on $B_R \setminus B_{R'}$.
Assume that $\sup\limits_{B_{R'}} [w - w_R] > 0$.
Since $w \leq w_R$ on $\partial B_{R'}$, then this supremum is reached at $x_0 \in B_{R'}$. Since $w_R \in C^2(B_{R'})$, then by the definition of a viscosity subsolution,

$$\mathcal{M}^+(D^2 w_R(x_0)) \geq \frac{A}{1+|x_0|^2}\ w(x_0)^p > \frac{A}{1+|x_0|^2}\ w_R(x_0)^p,$$

contradicting (4.4). This implies that $w \leq w_R$ on $B_{R'}$ and then on B_R.
Now, for any $x \in \mathbb{R}^n$, we can let R go to $+\infty$ to obtain $w(x) \leq C$. Hence w is bounded on $\mathbb{R}^n$.

□

Lemma 4.2. *Assume that F is an Isaacs operator.*
Let $V \in C(\mathbb{R}^n)$, $V \geq 0$, $V \neq 0$ such that for all $\gamma > 0$

$$\liminf_{R\to+\infty}\frac{1}{R^n}\int_{B_{2R}\setminus B_R} V^\gamma > 0.$$

Assume that

$$\begin{cases} 0 \le r & \text{if } \alpha^*(F) \le 0 \\ 0 \le r \le \dfrac{\alpha^*(F)+2}{\alpha^*(F)} & \text{if } \alpha^*(F) > 0. \end{cases} \tag{4.5}$$

Let $z \geq 0$ be a viscosity solution of

$$-F(D^2 z) \geq V(x)\, z^r, \qquad x \in \mathbb{R}^n.$$

Then

$$z = 0.$$

Démonstration. The case $\alpha^*(F) \leq 0$ is obvious since the only viscosity solutions of

$$-F(D^2 z) \geq 0$$

are the constants (see Lemma 7.2 i)) and because $V \geq 0,\ V \neq 0$.
Hence we suppose

$$\alpha^*(F) > 0.$$

Assume by contradiction that $z \neq 0$. Since $z \geq 0$ and $-F(D^2 z) \geq 0$, then by the strong maximum principle (see [5, Proposition 4.9]), we have $z > 0$ on $\mathbb{R}^n$.

First, we note that the hypothesis on V implies, for each $\gamma > 0$, the existence of $R_0 = R_0(\gamma) > 0$ and $c_0 = c_0(\gamma) > 0$ such that for all $R \geq R_0$,

$$\left(\int_{B_2 \setminus B_1} V(Rx)^\gamma dx\right)^{\frac{1}{\gamma}} \geq c_0.$$

Let $\gamma = \gamma(F, n) > 0$ given in the quantitative strong maximum principle (see Lemma 7.4) and $R_0 = R_0(\gamma)$. Let $R \geq R_0$. We set

$$z_R = z(R\cdot)$$

and

$$m(R) = \inf_{B_R} z = \inf_{B_1} z_R.$$

It is easy to see that z_R is a viscosity solution of

$$-F(D^2 z_R) \geq R^2 V(Rx) {z_R}^r, \qquad x \in \mathbb{R}^n.$$

By the quantitative strong maximum principle (see Lemma 7.4), there exists $C > 0$ such that

$$\inf_{B_1} z_R \geq C\ \|R^2 V(R\cdot) {z_R}^r\|_{L^\gamma(B_1)} \geq C R^2 m(R)^r \left(\int_{B_1} V(Rx)^\gamma dx\right)^{\frac{1}{\gamma}}.$$

Hence, for all $R \geq R_0$

$$m(R) \geq CR^2 c_0(\gamma)\, m(R)^r.$$

First case : Assume $0 \leq r \leq 1$.
For all $R \geq R_0$, since $m(R) > 0$, we have

$$m(R_0)^{1-r} \geq m(R)^{1-r} \geq CR^2 c_0(\gamma)$$

because $R \mapsto m(R)$ is nonincreasing. We then obtain a contradiction when we let R go to infinity.

Second case : Assume $1 < r < \frac{\alpha^*(F)+2}{\alpha^*(F)}$, which is equivalent to

$$\frac{2}{r-1} > \alpha^*(F).$$

From the argument of the previous case, we deduce that for all $R \geq R_0$,

$$m(R) \leq \frac{C}{R^{\frac{2}{r-1}}}$$

for some $C > 0$ and we will prove that for any $R \geq 1$,

$$m(R) \geq \frac{c}{R^{\alpha^*(F)}}$$

for some $c > 0$, which then gives a contradiction by letting R go to infinity.

First, we note that since

$$-F(D^2 z_R) \geq 0 \text{ on } \mathbb{R}^n,$$

then by the minimum principle,

$$m(R) = \min_{\partial B_R} z.$$

If we set

$$c = \inf_{\partial B_1} \frac{z}{\Phi},$$

then we know by Lemma 7.2 ii) that

$$z \geq c\,\Phi \qquad \text{on } \mathbb{R}^n \setminus B_1$$

where we denote by Φ the upward-pointing fundamental solution of F normalized by

$$\min_{\partial B_1} \Phi = 1. \tag{4.6}$$

Now, we use the $[-\alpha^*(F)]$-homogeneity of Φ. Let $R \geq 1$ and $|x| = R$. Then

$$z(x) \geq c\,\Phi\left(|x|\frac{x}{|x|}\right) = c\,\Phi\left(\frac{x}{|x|}\right)\frac{1}{R^{\alpha^*(F)}} \geq \frac{c}{R^{\alpha^*(F)}}$$

by the normalization (4.6) of Φ. Hence, for all $R \geq 1$,

$$m(R) \geq \frac{c}{R^{\alpha^*(F)}}.$$

Last case : Assume $r = \frac{\alpha^*(F)+2}{\alpha^*(F)}$, which is equivalent to

$$\frac{2}{r-1} = \alpha^*(F).$$

As a consequence of this equality, if we set $R_1 = \max(1, R_0)$, we already know that for all $R \geq R_1$,

$$c \leq R^{\alpha^*(F)} m(R) \leq C.$$

We set

$$\tilde{z}_R = R^{\alpha^*(F)} z_R$$

and

$$\tilde{m}(R) := \inf_{\partial B_R} \frac{z}{\Phi} = \inf_{\partial B_1} \frac{\tilde{z}_R}{\Phi},$$

the last equality being due the $[-\alpha^*(F)]$-homogeneity of Φ. If we denote

$$M_1 = \sup_{\partial B_1} \Phi,$$

then for all $R \geq R_1$, we have

$$\frac{c}{M_1} \leq \frac{R^{\alpha^*(F)} m(R)}{M_1} \leq \tilde{m}(R) \leq R^{\alpha^*(F)} m(R) \leq C$$

hence

$$\frac{c}{M_1} \leq \tilde{m}(R) \leq C. \tag{4.7}$$

It is easy to see that $\tilde{z}_R$ is a viscosity solution of

$$-F(D^2 \tilde{z}_R) \geq V(Rx)\tilde{z}_R^r, \qquad x \in \mathbb{R}^n.$$

Hence $-F(D^2\tilde{z}_R) \geq 0$ so, by Lemma 7.2 ii), we deduce that

$$z_R - \tilde{m}(R)\Phi \geq 0 \qquad \text{on } \mathbb{R}^n \setminus B_1. \tag{4.8}$$

Since

$$F(D^2[\tilde{m}(R)\Phi]) = \tilde{m}(R)\, F(D^2\Phi) = 0, \qquad x \neq 0$$

and

$$F(D^2\tilde{z}_R) \leq -V(Rx)\tilde{z}_R^r$$

then, applying Lemma 3.2, we get

$$-\mathcal{M}^-\left(D^2\left[\tilde{z}_R - \tilde{m}(R)\Phi\right]\right) = \mathcal{M}^+\left(D^2\left[\tilde{m}(R)\Phi - \tilde{z}_R\right]\right) \geq V(Rx)\tilde{z}_R^r.$$

We apply the quantitative strong maximum principle (see Lemma 7.4) to the operator $\mathcal{M}^-$ with $\Omega = B_5 \setminus \overline{B_1}$ and $K = \overline{B_4} \setminus B_2$ so there exist $\gamma^- > 0$ and $c_- > 0$ such that

$$\inf_{\overline{B_4}\setminus B_2} \left[\tilde{z}_R - \tilde{m}(R)\Phi\right] \geq c_-\left(\int_{\overline{B_4}\setminus B_2} V(Rx)^{\gamma^-}\, \tilde{z}_R(x)^{r\gamma^-}\, dx\right)^{\frac{1}{\gamma^-}}.$$

By the $[-\alpha^*(F)]$-homogeneity of Φ and (4.6), we have

$$\Phi \geq \frac{1}{4^{\alpha^*(F)}} \qquad \text{on } \overline{B_4} \setminus B_2,$$

which implies by (4.7) and (4.8) that

$$\tilde{z}_R \geq \frac{c}{M_1\, 4^{\alpha^*(F)}} \qquad \text{on } \overline{B_4} \setminus B_2.$$

Hence, we obtain

$$\inf_{\overline{B_4}\setminus B_2} \left[\tilde{z}_R - \tilde{m}(R)\Phi\right] \geq c_-\left[\frac{c}{M_1\, 4^{\alpha^*(F)}}\right]^r \left(\int_{\overline{B_4}\setminus B_2} V(Rx)^{\gamma^-}\, dx\right)^{\frac{1}{\gamma^-}} \geq c_-\left[\frac{c}{M_1\, 4^{\alpha^*(F)}}\right]^r c_0 =: c_1 > 0$$

where $c_0 = c_0(\gamma^-) > 0$ and R large enough, from the hypothesis on V.
Hence, for R large enough and for all $|x| = 1$, we have

$$\tilde{z}_R(2x) \geq \tilde{m}(R)\Phi(2x) + c_1$$

then, we get

$$\frac{1}{2^{\alpha^*(F)}}(2R)^{\alpha^*(F)} z(2Rx) \geq \frac{\tilde{m}(R)}{2^{\alpha^*(F)}}\Phi(x) + c_1$$

and finally

$$\tilde{z}_{2R}(x) \geq \tilde{m}(R)\Phi(x) + c_1 2^{\alpha^*(F)}.$$

Then, we have for R large enough,

$$\tilde{m}(2R) \geq \tilde{m}(R) + \frac{c_1 2^{\alpha^*(F)}}{M_1},$$

so $\tilde{m}(R)$ is unbounded, a contradiction.

□

The previous theorem can obviously be applied to any constant function V, but also under the more general condition on V given in the following lemma.

Lemma 4.3. *Let F be a uniformly elliptic operator.*
Let $V \in C(\mathbb{R}^n)$ satisfying $V \geq 0$, $V \neq 0$ and

$$F(D^2V) \geq 0.$$

Then, for any $\gamma > 0$,

$$\liminf_{R\to+\infty} \frac{1}{R^n} \int_{B_{2R}\setminus B_R} V^\gamma > 0$$

Démonstration. This is an easy consequence of [5, Theorem 4.8 (2)].
Indeed, if we apply the latter to $V_R = V(R\cdot) \geq 0$ for $R > 0$, then for any $\gamma > 0$, there exists $C = C(\gamma) > 0$ such that for all $R > 0$,

$$\left(\int_{B_2\setminus B_1} V(Rx)^\gamma \, dx\right)^{\frac{1}{\gamma}} \geq C \sup_{B_{\frac{5}{3}}\setminus B_{\frac{4}{3}}} V(R\cdot)$$

hence, for all $R > 0$,

$$\left(\frac{1}{R^n} \int_{B_{2R}\setminus B_R} V^\gamma\right)^{\frac{1}{\gamma}} \geq C \sup_{B_{\frac{5}{3}R}\setminus B_{\frac{4}{3}R}} V \geq C \sup_{\partial B_{\frac{5}{3}R}} V = C \sup_{B_{\frac{5}{3}R}} V,$$

the last equality following the comparison principle since $F(D^2V) \geq 0$.
But since $V \neq 0$, there exists $R_0 > 0$ such that

$$\sup_{B_{\frac{5}{3}R_0}} V > 0.$$

hence, for all $R \geq R_0$,

$$\left(\frac{1}{R^n} \int_{B_{2R}\setminus B_R} V^\gamma\right)^{\frac{1}{\gamma}} \geq C \sup_{B_{\frac{5}{3}R_0}} V > 0,$$

which implies the result. □

4.2 A Liouville theorem on a cone

In this subsection, we fix a cone

$$\mathcal{C}_\omega = \{tx, t > 0, x \in \omega\}$$

where ω is a C^2 subdomain of S_1. We will use the following notation

$$B_R^+ = B_R \cap \mathcal{C}_\omega$$

and

$$S_R^+ = S_R \cap \mathcal{C}_\omega$$

and will denote by $\Psi^+ \in C\left(\overline{\mathcal{C}_\omega} \setminus \{0\}\right)$ a singular solution of

$$\begin{cases} -F(D^2\Psi^+) = 0, & \text{in } \mathcal{C}_\omega \\ \Psi^+ = 0, & \text{on } \partial\mathcal{C}_\omega \setminus \{0\} \end{cases} \tag{4.9}$$

such that

$$\Psi^+(x) > 0 \text{ on } \mathcal{C}_\omega \qquad \text{and} \qquad \Psi^+(x) = t^{\alpha^+(F)}\Psi(tx) \qquad \text{for all } t > 0,\ x \in \mathcal{C}_\omega.$$

where

$$\alpha^+(F) > 0$$

is uniquely determined. For more details, see [2, Theorem 1.1].

Lemma 4.4. *Let F an operator satisfying (H_1) and (H_2).*
Let $u \in C(\overline{\mathcal{C}_\omega})$ be a nonnegative viscosity solution on $\mathcal{C}_\omega$ of

$$-F(D^2u) \geq u^p \tag{4.10}$$

where

$$0 < p < \frac{\alpha^+(F) + 2}{\alpha^+(F)}.$$

We assume moreover that u is bounded if $0 < p < 1$. Then

$$u = 0.$$

Remark 4.1. *The boundedness assumtion for $0 < p < 1$ is necessary. Indeed, considering the equation $-F(D^2u) = u^p$ on the half-space $\mathbb{R}^n_+$, there exists a unbounded solution u of the form*

$$u(x) = C\, x_n{}^{\frac{2}{1-p}}$$

with $C > 0$ a well chosen constant.

Démonstration. Since $-F(D^2u) \geq 0$, then by the strong minimum principle, $u > 0$ in $\mathcal{C}_\omega$. If $x_0 \in \mathcal{C}_\omega$, it is clear that $u(\cdot + x_0)$ is also a viscosity solution of (4.10) on $\mathcal{C}_\omega$ hence we can assume that

$$u > 0 \text{ on } \overline{\mathcal{C}_\omega}.$$

Let $x_0 \in \mathcal{C}_\omega$ such that $B(x_0, 2) \subset \mathcal{C}_\omega$. For any $R > 0$, since $RB(x_0, 2) \subset \mathcal{C}_\omega$, we can define

$$u_R(x) = u(R(x_0 + x)) \qquad \text{for all } x \in B_2.$$

It is clear that u_R satisfies

$$-F(D^2u_R) \geq R^2\, u_R{}^p \text{ on } B_2$$

in the viscosity sense.

First case : $p = 1$.
Since $-F(D^2 u_R) \geq R^2\, u_R$ on B_1, then by definition of the principal half-eigenvalue, we have

$$\lambda_1^+(-F, B_1) \geq R^2$$

for all $R > 0$, which is a contradiction since $\lambda_1^+(-F, B_1) < \infty$.

Second case : $0 < p < 1$.
Since u is assumed to be bounded, there exists $c > 0$ such that $-F(D^2 u_R) \geq c\, R^2\, u_R$ on B_1 which implies that

$$\lambda_1^+(-F, B_1) \geq c\, R^2$$

for all $R > 0$, leading to a contradiction.

Third case : $1 < p < \frac{\alpha^+(F)+2}{\alpha^+(F)}$, which is equivalent to

$$\frac{2}{p-1} > \alpha^+(F).$$

Since $-F(D^2 u) \geq 0$, by Lemma 7.3, we know that there exists $c > 0$ such that

$$u \geq c\; \Psi^+ \text{ on } \mathcal{C}_\omega \setminus B_1^+.$$

Hence, by using the $[-\alpha^+(F)]$-homogeneity of ψ^+, we have for all $R \geq 1$ and all $x \in B_1$,

$$u_R(x) \geq \frac{c}{R^{\alpha^+(F)}} \psi^+(x_0 + x) \geq \frac{\tilde{c}}{R^{\alpha^+(F)}}$$

since $R(x_0 + x) \in \mathcal{C}_\omega \setminus B_1^+$, where $\tilde{c} = c \inf_{B_1(x_0)} \psi^+ > 0$ since $\psi^+ > 0$ on $\overline{B_1(x_0)}$. Therefore,

$$-F(D^2 u_R) \geq \tilde{c}^{p-1} R^{2-(p-1)\alpha^+(F)}\; u_R \text{ on } B_1.$$

Hence, for all $R \geq 1$,

$$\lambda_1^+(-F, B_1) \geq \tilde{c}^{p-1} R^{2-(p-1)\alpha^+(F)}$$

which leads to a contradiction by letting R go to infinity. □

5 Proportionality results for systems on $\mathbb{R}^n$ or a cone

5.1 Case of a cone

In this subsection, we fix a cone $\mathcal{C}_\omega$ where ω is a C^2 subdomain of S_1.
We first would like to recall a Phragmèn-Lindelhöf principle which is a particular case of [2, Theorem 1.7].

Lemma 5.1. *Assume that F satisfy (H_1) and (H_2).*
Let $w \in C(\overline{\mathcal{C}_\omega})$ be a bounded viscosity solution of

$$F(D^2 w) \geq 0 \qquad \text{on } \mathcal{C}_\omega \tag{5.1}$$

and

$$w \leq 0 \qquad \text{on } \partial\mathcal{C}_\omega. \tag{5.2}$$

Then

$$w \leq 0.$$

Démonstration. This is a special case of [2, Theorem 1.7]. Using the notation of the latter, we set $\Omega = \Omega' = \mathcal{C}_\omega$, so $\mathcal{D} = \mathcal{C}_\omega$ and we choose $\mathcal{D}' = \mathcal{C}_\omega$. Since w is bounded and $\alpha^- < 0 < \alpha^+$, then condition (1.12) of [2, Theorem 1.7] is clearly satisfied and since (5.1) and (5.2) are verified, then we obtain

$$w \leq 0.$$

□

An easy consequence of this Phragmèn-Lindelhöf principle, we can give the proof of Theorem 1.1 :

Proof of Theorem 1.1 : We consider

$$w = (u - Kv)_+ := \max(u - Kv, 0)$$

and show that it is a viscosity solution on $\mathbb{R}^n_+$ of

$$\mathcal{M}^+(D^2 w) \geq 0$$

Indeed, let $\phi \in C^2(\mathbb{R}^n_+)$ touching w by above at $x_0 \in \mathbb{R}^n_+$.
If $w(x_0) = 0$, then $\phi(x_0) = 0$ and since $\phi \geq w \geq 0$ then $x_0 \in \mathbb{R}^n_+$ is a minimum point of ϕ so $D^2\phi(x_0) \geq 0$, whence $\mathcal{M}^+(D^2\phi(x_0) \geq 0$.
If $w(x_0) > 0$, then $[u - Kv](x_0) > 0$ so by (1.2) we have $Kg(x_0, u(x_0), v(x_0)) - f(x_0, u(x_0), v(x_0)) \geq 0$. But from Lemma 3.2, it follows that $\tilde{w} = u - Kv$ is a a viscosity solution in $\mathbb{R}^n_+$ of

$$\mathcal{M}^+(D^2\tilde{w}) \geq K\, g(x, u, v) - f(x, u, v).$$

Since ϕ touches $\tilde{w}$ by above at x_0, then

$$\mathcal{M}^+(D^2\phi(x_0)) \geq K\, g(x_0, u(x_0), v(x_0)) - f(x_0, u(x_0), v(x_0)) \geq 0.$$

w is then a bounded viscosity solution of

$$\mathcal{M}^+(D^2 w) \geq 0$$

vanishing on $\partial\mathbb{R}^n_+$, so by Lemma 5.1, we get $w \leq 0$ and hence

$$u \leq Kv.$$

We make the same argument with $w = (Kv - u)_+$ and then obtain

$$u = Kv.$$

□

5.2 Case of the whole space

In this section, we focus on system (1.1) on the whole space $\mathbb{R}^n$ with

$$f(u,v) = u^r v^p[av^q - cu^q]$$

and

$$g(u,v) = v^r u^p[bu^q - dv^q]$$

i.e. we consider

$$\begin{cases} -F(D^2u) = u^r v^p[av^q - cu^q] & \text{on } \mathbb{R}^n \\ -F(D^2v) = v^r u^p[bu^q - dv^q] & \text{on } \mathbb{R}^n. \end{cases} \tag{5.3}$$

In our study of (5.3) we always assume that the real parameters a, b, c, d, p, q, r satisfy

$$a, b > 0, \quad c, d \geq 0, \qquad p, r \geq 0, \quad q > 0, \quad q \geq |p - r|. \tag{5.4}$$

The reason of assuming (5.4) is the following result, proved in the appendix of [P5] :

Proposition 5.1. *Assume (5.4).*
(i) Then the nonlinearities f and g in the system (5.3) *satisfy* (1.2) *for some $K > 0$.*

(ii) Assume moreover that $ab \geq cd$. Then the number $K > 0$ is unique.
We have $K = 1$ if and only if $a + d = b + c$ and $K > 1$ if and only if $a + d > b + c$.
In addition, if $ab > cd$ (resp. $ab = cd$), then $a - cK^q > 0$ (resp. $= 0$) and $bK^q - d > 0$ (resp. $= 0$).

Lemma 5.2. *Let F be an Isaacs operator.*
Assume that (5.4) holds and let (u, v) be a positive viscosity solution of (5.3). We set

$$Z = \min(u, Kv)$$

and

$$W = |u - Kv|.$$

i) Assume $ab \geq cd$.
a) Z is a viscosity solution on $\mathbb{R}^n$ of

$$-F(D^2Z) \geq 0$$

and W is a viscosity solution on $\mathbb{R}^n$ of

$$\mathcal{M}^+(D^2W) \geq 0.$$

b) If $p + q < 1$, suppose in addition that (u, v) is bounded.
Then Z is a viscosity solution of

$$-F(D^2Z) \geq CW^\beta Z^r \qquad \text{in } \mathbb{R}^n, \tag{5.5}$$

where $C > 0$ and

$$\beta := \max(p+q, 1).$$

c) Assume $r > p$ and $c, d > 0$.
If $q + r < 1$, suppose in addition that (u, v) is bounded.
Then W is a viscosity solution of

$$\mathcal{M}^+(D^2W) \geq CZ^pW^\gamma \qquad in \ \mathbb{R}^n, \tag{5.6}$$

where $C > 0$ and

$$\gamma := \max(q+r, 1).$$

ii) Assume $ab > cd$.
Then Z is a viscosity solution of

$$-F(D^2Z) \geq CZ^{p+q+r} \qquad in \ \mathbb{R}^n$$

where $C > 0$.

Démonstration. **i) a)** By Proposition 5.1, we have

$$a \geq cK^q, \quad bK^q \geq d. \tag{5.7}$$

Hence, on the set $\{u \leq Kv\}$, we have

$$f(u,v) = u^r v^p [av^q - cu^q] \geq cu^r v^p [(Kv)^q - u^q] \geq 0$$

and on the set $\{u > Kv\}$, we have

$$g(u,v) = u^p v^r [bu^q - dv^q] \geq b\, u^p v^r [u^q - (Kv)^q] \geq 0.$$

Now, we apply Lemma 3.5 to u and Kv with $h = 0$ and deduce that $Z = \min(u, Kv)$ is a viscosity solution of

$$-F(D^2Z) \geq 0.$$

The fact that W is a viscosity solution of

$$\mathcal{M}^+(D^2W) \geq 0$$

is due to Lemma 3.4.

i) b) We recall the following inequality :

$$x^q - y^q \geq C_q x^{q-1}(x-y), \qquad \text{if } x \geq y \geq 0$$

with

$$\begin{cases} C_q = 1 & \text{if } q \geq 1 \\ C_q = q & \text{if } 0 < q < 1. \end{cases} \tag{5.8}$$

Using (5.7), on the set $\{u \le Kv\}$, we have

$$\begin{aligned} f(u,v) = u^r v^p[av^q - cu^q] &\ge \frac{a}{K^q}\, u^r v^p[(Kv)^q - u^q] \\ &\ge \frac{aC_q}{K} u^r v^{p+q-1}(Kv-u) \\ &\ge \frac{aC_q}{K^{p+q}} u^r (Kv)^{p+q-1}(Kv-u) \\ &\ge C_1 Z^r W^\beta \end{aligned}$$

for some $C_1 > 0$ since if $p+q \ge 1$, we clearly have

$$(Kv)^{p+q-1} \ge (Kv-u)^{p+q-1}$$

and if $p+q<1$, then

$$(Kv)^{p+q-1} \ge C_0$$

for some $C_0 > 0$, v being assumed bounded.

Similarly, on the set $\{u > Kv\}$, we have

$$\begin{aligned} g(u,v) = u^p v^r[bu^q - dv^q] &\ge b\, u^p v^r[u^q - (Kv)^q] \\ &\ge bC_q u^{p+q-1} v^r (u-Kv) \\ &\ge C_2 Z^r W^\beta \end{aligned}$$

for some $C_2 > 0$ since if $p+q \ge 1$, then

$$u^{p+q-1} \ge (u-Kv)^{p+q-1}$$

and if $p+q<1$, we have

$$u^{p+q-1} \ge C_0'$$

for some $C_0' > 0$, u being assumed bounded.

Now, we apply Lemma 3.5 to u and Kv with $h = CZ^rW^\beta$ where $C = \min(C_1, KC_2)$ (since $-F(D^2[Kv]) = Kg(u,v)$) and obtain that $Z = \min(u, Kv)$ is a viscosity solution of

$$-F(D^2 Z) \ge CZ^r W^\beta.$$

ii) Since $ab > cd$, we know from Proposition 5.1 that

$$a \ge cK^q + \epsilon, \quad bK^q \ge d + \epsilon \tag{5.9}$$

for some $\epsilon > 0$ small enough.

Hence, on the set $\{u \le Kv\}$, we have

$$\begin{aligned} f(u,v) = u^r v^p[av^q - cu^q] &\ge \epsilon\, u^r v^{p+q} + c\, u^r v^p[(Kv)^q - u^q] \\ &\ge \epsilon\, u^r v^{p+q} \\ &\ge \epsilon K^{-p-q}\, Z^{r+p+q} \end{aligned}$$

and similarly on the set $\{u > Kv\}$, we have

$$\begin{aligned} g(u,v) &= u^p v^r [bu^q - dv^q] = K^{-q} u^p v^r [bK^q u^q - d(Kv)^q] \\ &\geq dK^{-q} u^p v^r [u^q - (Kv)^q] + \epsilon K^{-q} u^{p+q v^r} \\ &\geq \epsilon K^{-q} u^{p+q} v^r \\ &\geq \epsilon K^{-q-r}\, Z^{r+p+q}. \end{aligned}$$

We again apply Lemma 3.5 to u and Kv with $h = CZ^{r+p+q}$ with $C = \epsilon \min(K^{-p-q}, K^{1-q-r}) > 0$ and obtain that $Z = \min(u, Kv)$ is a viscosity solution of

$$-F(D^2 Z) \geq CZ^{p+q+r}.$$

i) c) Thanks to Lemma 7.1 i) in [P5], we know that, since $r > p$ and $c, d > 0$, we have for some $C_0 > 0$

$$|Kg(u,v) - f(u,v)| \geq C_0\, u^p v^p (u + Kv)^{q+r-p-1} |u - Kv|.$$

We also note that for $x, y \geq 0$ and $x + y > 0$, we have

$$\frac{xy}{x+y} \geq \frac{1}{2} \min(x,y).$$

Hence,

$$\begin{aligned} |Kg(u,v) - f(u,v)| &\geq \frac{C_0}{K^p} \left[\frac{u\,Kv}{u+Kv}\right]^p (u+Kv)^{q+r-1} |u - Kv| \\ &\geq \frac{C_0}{(2K)^p} Z^p (u+Kv)^{q+r-1} |u - Kv| \\ &\geq C\, Z^p W^\gamma \end{aligned}$$

for some $C > 0$ since if $q + r \geq 1$, we have

$$(u+Kv)^{q+r-1} \geq |u - Kv|^{q+r-1}$$

and if $q + r < 1$, we have

$$(u+Kv)^{q+r-1} \geq C_0'$$

for some $C_0' > 0$, u and v being assumed bounded.

Now, thanks to Lemma 3.4, we obtain that

$$\mathcal{M}^+(D^2 W) \geq CZ^p W^\gamma \qquad \text{in } \mathbb{R}^n$$

in the viscosity sense.

□

We now can make the following proof :

Proof of Theorem 1.2 : Let (u, v) be a positive viscosity solution of (5.3) in $\mathbb{R}^n$. We set

$$Z = \min(u, Kv)$$

and

$$W = |u - Kv|.$$

i) Assume that $W \neq 0$.
From Lemma 5.2 i) b), we know that Z is a viscosity solution of

$$-F(D^2Z) \geq C\,V\,Z^r \qquad \text{in } \mathbb{R}^n, \tag{5.10}$$

where $C > 0$ and

$$V = W^\beta$$

with $\beta := \max(p + q, 1)$. From Lemma 5.2 i) a), we know that W is a viscosity solution on $\mathbb{R}^n$ of $\mathcal{M}^+(D^2W) \geq 0$ so, since $\beta \geq 1$, V is a viscosity solution on $\mathbb{R}^n$ of

$$\mathcal{M}^+(D^2V) \geq 0.$$

Moreover, $V \geq 0$ and $V \neq 0$, hence, by Lemma 4.3, V satisfies the conditions to apply Lemma 4.2, therefore $Z = 0$, a contradiction since $u, v > 0$. Then $W = 0$, i.e. $u = Kv$.

ii) We can assume

$$q + r > 1.$$

Indeed, if $q+r \leq 1$, then u, v are bounded and $r \leq q+r \leq 1 < \frac{\alpha^*(F)+2}{\alpha^*(F)}$ so the result follows from ii) a).

We can assume

$$r > p.$$

Indeed, if $p+q < 1$, then $p+q < 1 < q+r$ so $r > p$ and if $p+q \geq 1$, we can assume $r > \frac{\alpha^*(F)+2}{\alpha^*(F)}$ else the result follows from ii) a), so $r > \frac{2}{\alpha^*(F)} \geq p$.

Since Z is a viscosity solution on $\mathbb{R}^n$ of

$$-F(D^2Z) \geq 0,$$

then by Lemma 7.2 ii), there exists $m > 0$ such that for all $|x| \geq 1$,

$$Z \geq \frac{m}{|x|^{\alpha^*(F)}}.$$

Hence,

$$Z^p \geq \frac{m^p}{|x|^{p\,\alpha^*(F)}} \qquad \text{for all } x \in \mathbb{R}^n \setminus B_1$$

and since $p \leq \frac{2}{\alpha^*(F)}$, then there exists $A_1 > 0$ such that

$$Z^p \geq \frac{A_1}{1+|x|^2} \qquad \text{for all } x \in \mathbb{R}^n \setminus B_1.$$

Moreover, $Z^p > 0$ is continuous on B_1 so there exists $A_2 > 0$ such that

$$Z^p \geq \frac{A_2}{1+|x|^2} \qquad \text{for all } x \in B_1.$$

Therefore, since $r > p$ and $c, d > 0$, then by Lemma 5.2 i) c), W is a viscosity solution of

$$\mathcal{M}^+(D^2W) \geq \frac{A}{1+|x|^2} W^\gamma \qquad \text{for all } x \in \mathbb{R}^n$$

for some $A > 0$ and $\gamma = q + r > 1$. Then, by Lemma 4.1 ii), we have

$$W = 0.$$

□

We then deduce the proof of Theorem 1.3 :

Proof of Theorem 1.3 : Let (u, v) be a bounded positive viscosity solution on $\mathbb{R}^n$ of (5.3) .
From the hypothesis (**??**), we necessarily have

$$\alpha^*(F) \leq 0 \text{ or } \left[\alpha^*(F) > 0 \text{ and } \left(r \leq \frac{\alpha^*(F)+2}{\alpha^*(F)} \text{ or } p \leq \frac{2}{\alpha^*(F)}\right)\right].$$

Then, from Theorem 1.2, since u, v are assumed bounded, we know that

$$u = K\, v.$$

Then v is a solution of

$$-F(D^2v) = K^p(bK^q - d)v^\sigma.$$

But since $ab > cd$, we know from Proposition 5.1 that $bK^q - d > 0$, so, by using the scaling of the equation and the hypothesis, we get $v = 0$ and hence $u = 0$.

□

We finally give the proof of Theorem 1.4 :

Proof of Theorem 1.4 : Let (u, v) be a nonnegative bounded viscosity solution of (5.3). First note that u is a viscosity solution of

$$-F(D^2u) + Cu \geq 0$$

where $C = c\, v^p\, u^{q+r-1} \geq 0$. Since $-F + C$ is a proper operator, we can apply the strong minimum principle and deduce that $u = 0$ or $u > 0$. The same is true for v.

Assume for instance that $u = 0$.
Case $p > 0$: then $-F(D^2v) = 0$, so by Lemma 7.1, we have $v = C_2 \geq 0$.
Case $p = 0$: then $F(D^2v) = d\, v^{q+r}$.
If $d = 0$, then $v = C_2$ and if $d > 0$, then $v = 0$ by Lemma 4.1 because $q + r > 1$.
The same analysis can be done if $v = 0$. Hence, in all cases (u, v) is semitrivial.
If $r = 0$, then it is clear that if $u = 0$, then $v = 0$, and reciprocally. The last cases are straightforward.

□

6 A priori estimates and existence result in a bounded domain

We consider the following system with general lower order terms

$$\begin{cases} -F(D^2u) = u^r v^p\Big[a(x)v^q - c(x)u^q\Big] + h_1(x,u,v), & x \in \Omega, \\ -F(D^2v) = v^r u^p\Big[b(x)u^q - d(x)v^q\Big] + h_2(x,u,v), & x \in \Omega, \\ \qquad u = v = 0, & x \in \partial\Omega, \end{cases} \tag{6.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$ and F satisfies $(H_1)(H_2)$.

In this section, $\alpha^+(F)$ corresponds to $\alpha^+(F, \mathbb{R}^n_+)$.

Proof of Theorem 1.5. We consider the following parametrized version of system (6.1), needed in the proof of Theorem 1.6 :

$$\begin{cases} -F(D^2u) = f(t,x,u,v), & x \in \Omega, \\ -F(D^2v) = g(t,x,u,v), & x \in \Omega, \\ \qquad u = v = 0, & x \in \partial\Omega, \end{cases} \tag{6.2}$$

where

$$f(t,x,u,v) := u^r v^p\Big[(a(x) + tA)v^q - c(x)u^q\Big] + \hat{h}_1(t,x,u,v), \tag{6.3}$$

$$g(t,x,u,v) := v^r u^p\Big[(b(x) + tA)u^q - d(x)v^q\Big] + \hat{h}_2(t,x,u,v), \tag{6.4}$$

and

$$\hat{h}_1(t,x,u,v) = h_1(x,u,v) + At(1+u), \quad \hat{h}_2(t,x,u,v) = h_2(x,u,v) + At(1+v). \tag{6.5}$$

Here $A > 0$ is a constant to be fixed below, and t is a parameter in $[0, 1]$.

Note that (6.1) is (6.2) with $t = 0$. Under assumption (1.14), we will prove the bound in (1.16) for the positive solutions of (6.2), uniformly for $t \in [0, 1]$ (but possibly depending on A), whereas under assumption (1.15) we will restrict ourselves to $t = 0$. [2]

We assume for contradiction that there exists a sequence $\{t_j\} \subset [0,1]$ and a sequence (u_j, v_j) of positive viscosity solutions of (6.2) with $t = t_j$, such that $\|u_j\|_\infty + \|v_j\|_\infty \to \infty$. Without loss of generality, we can assume

$$\|u_j\|_\infty \geq \|v_j\|_\infty.$$

We set

$$\alpha = \frac{2}{\sigma - 1}.$$

Let $x_j \in \Omega$ be such that $u_j(x_j) = \|u_j\|_\infty$ and let us set

$$\lambda_j := \frac{1}{\|u_j\|_\infty^{1/\alpha} + \|v_j\|_\infty^{1/\alpha}} \to 0, \qquad \text{as } j \to \infty.$$

By passing to a subsequence, we may assume that $x_j \to x_\infty \in \overline{\Omega}$ and $t_j \to t_0 \in [0,1]$. Setting

$$d_j := \operatorname{dist}(x_j, \partial\Omega),$$

we then split the proof into two cases, according to whether $d_j/\lambda_j \to \infty$ (along some subsequence) or d_j/λ_j is bounded.

First case : $d_j/\lambda_j \to \infty$.

This case is treated in two steps.

Step 1 : Convergence of rescaled solutions to a semi-trivial entire solution.

We rescale the solutions around x_j as follows :

$$\tilde{u}_j(y) = \lambda_j^\alpha\, u_j(x_j + \lambda_j y), \quad \tilde{v}_j(y) = \lambda_j^\alpha\, v_j(x_j + \lambda_j y), \qquad y \in \Omega_j, \tag{6.6}$$

where $\Omega_j = \{y \in \mathbb{R}^n : |y| < d_j/\lambda_j\}$. Due to the definition of λ_j, it is clear that

$$\tilde{u}_j(y),\ \tilde{v}_j(y) \leq 1, \qquad y \in \Omega_j. \tag{6.7}$$

Moreover, $\tilde{u}_j^{1/\alpha}(0) = \lambda_j\, \|u_j\|_\infty^{1/\alpha} \geq \lambda_j \,(\|u_j\|_\infty^{1/\alpha} + \|v_j\|_\infty^{1/\alpha}\big)/2 = 1/2$, hence

$$\tilde{u}_j(0) \geq 2^{-\alpha}. \tag{6.8}$$

We see that $(\tilde{u}_j, \tilde{v}_j)$ is a viscosity solution of the system

$$\begin{cases} -F(D^2\tilde{u}) = \tilde{u}^r\tilde{v}^p\Big[(a(x_j+\lambda_j y) + t_j A)\,\tilde{v}^q - b(x_j+\lambda_j y)\,\tilde{u}^q\Big] + \tilde{h}_{1,j}(y), & y \in \Omega_j, \\ -F(D^2\tilde{v}) = \tilde{v}^r\tilde{u}^p\Big[(b(x_j+\lambda_j y) + t_j A)\,\tilde{u}^q - d(x_j+\lambda_j y)\tilde{v}^q\Big] + \tilde{h}_{2,j}(y), & y \in \Omega_j, \end{cases} \tag{6.9}$$

2. The restriction $t_j = 0$ under assumption (1.15) will be used only in Step 2 to exclude semi-trivial rescaling limits.

where

$$\tilde{h}_{i,j}(y) = \lambda_j^{\alpha+2}\hat{h}_i(t_j, x_j + \lambda_j y, \lambda_j^{-\alpha}\tilde{u}_j(y), \lambda_j^{-\alpha}\tilde{v}_j(y)), \quad i = 1, 2.$$

In view of (1.13), (6.7), $\sigma > 1$, and $\alpha + 2 - \alpha\sigma = 0$ we have

$$\sup_{\Omega_j} (|\tilde{h}_{1,j}| + |\tilde{h}_{2,j}|) \leq \lambda_j^{\alpha+2}(\lambda_j^{-\alpha\sigma}o(1) + 2A(1 + \lambda_j^{-\alpha})) \to 0, \quad \text{as } j \to \infty. \tag{6.10}$$

We fix $R > 0$. For j large enough, we have $B_{2R} \subset \Omega_j$. By (6.7) and the Hölder interior continuity (see Proposition 4.10 in [5]), we deduce a uniform C^α bound on B_R for all j large enough, for some $\alpha > 0$. From Ascoli's theorem, we can extract a subsequence (again denoted $(\tilde{u}_j, \tilde{v}_j)$) uniformly converging toward (U_R, V_R) on B_R. By Cantor's diagonal argument, we eventually get a subsequence $(\tilde{u}_j, \tilde{v}_j)$) uniformly converging toward (U, V) on $\mathbb{R}^n$. Since F is uniformly elliptic, then F is continuous on $\mathcal{S}_n$, so by Lemma 7.5, we deduce that (U, V) is a viscosity solution of

$$\begin{cases} -F(D^2U) = U^r V^p\Big[a_0 V^q - c_0 U^q\Big], & y \in \mathbb{R}^n, \\ -F(D^2V) = V^r U^p\Big[b_0 U^q - d_0 V^q\Big], & y \in \mathbb{R}^n, \end{cases} \tag{6.11}$$

with

$$a_0 = a(x_\infty) + t_0 A > 0, \ b_0 = b(x_\infty) + t_0 A > 0, \ c_0 = c(x_\infty) \geq 0, \ d_0 = d(x_\infty) \geq 0. \tag{6.12}$$

Moreover,

$$c_0 d_0 < a_0 b_0 \tag{6.13}$$

in view of (1.12). Also, $U(0) \geq 2^{-\alpha}$ due to (6.8).
Either $\alpha^*(F) \leq 0$ or $\alpha^*(F) > 0$ and then $\frac{\alpha^*(F)+2}{\alpha^*(F)} \geq \frac{\alpha^+(F)+2}{\alpha^+(F)} > \sigma$ since $\alpha^*(F) \leq \alpha^+(F)$ (by definition of $\alpha^+(F)$) . Hence, we can apply Theorem 1.4, so there exists a constant $\bar{C} > 0$ such that

$$U \equiv \bar{C} \text{ and } V \equiv 0.$$

Hence,

$$\lim_{j\to\infty} (\tilde{u}_j, \tilde{v}_j) = (\bar{C}, 0) \quad \text{locally uniformly on } \mathbb{R}^n. \tag{6.14}$$

Step 2 : Exclusion of semi-trivial rescaling limits.

Let us first consider the case when assumption (1.14) is satisfied. For some $\delta, M_1 > 0$ we have

$$\hat{h}_2(t, x, u, v) \geq (-\bar{m} + \delta)v^r u^{p+q}, \quad \text{for } u \geq M_1 \max(v, 1),$$

(uniformly in $x \in \Omega$ and $t \in [0, 1]$) and hence

$$\tilde{h}_{i,j} \geq (-\bar{m} + \delta)\tilde{v}_j^r \tilde{u}_j^{p+q}, \quad \text{for } \tilde{u}_j \geq M_1 \max(\tilde{v}_j, \lambda_j^\alpha), \quad i = 1, 2.$$

Fix $\epsilon \in (0, 1)$ with

$$\epsilon \leq \min\Big\{\frac{\bar{C}}{2M_1}, \Big(\frac{\delta}{2\|d\|_\infty}\Big)^{1/q}\frac{\bar{C}}{2}\Big\}.$$

Take $R > 0$ to be chosen later. By (6.14), there exists j_0 such that, for all $j \geq j_0$, we have $\tilde{u}_j \geq \bar{C}/2$, $\tilde{v}_j \leq \epsilon$ on B_R, and $\tilde{u}_j \geq \bar{C}/2 \geq M_1 \max(\tilde{v}_j, \lambda_j^\alpha)$, since $\lambda_j^\alpha \to 0$ as $j \to \infty$. Hence

$$\begin{aligned} -F(D^2\tilde{v}_j) &\geq \tilde{v}_j^r \tilde{u}_j^p \Big[\Big(b(x_j + \lambda_j y) + t_j A - \bar{m} + \delta\Big)\, \tilde{u}_j^q - d(x_j + \lambda_j y)\tilde{v}_j^q\Big] \\ &\geq \tilde{v}_j^r \tilde{u}_j^p \Big[\delta\, \tilde{u}_j^q - \|d\|_\infty \epsilon^q\Big] \geq \tfrac{\delta}{2}\Big(\tfrac{\bar{C}}{2}\Big)^{p+q} \tilde{v}_j^r \geq \tfrac{\delta}{2}\Big(\tfrac{\bar{C}}{2}\Big)^{p+q} \tilde{v}_j \qquad \text{in } B_R, \end{aligned}$$

(in the last inequality we used $r \leq 1$). This implies that

$$\lambda_1^+(-F, B_R) \geq \frac{\delta}{2}\left(\frac{\bar{C}}{2}\right)^{p+q}$$

which is a contradiction for R large since (as is easily seen by the definition)

$$\lambda_1^+(-F, B_R) = \frac{\lambda_1^+(-F, B_1)}{R^2}.$$

Let us now consider the case when assumption (1.15) is satisfied, and $t_j = 0$. Now there exist $\delta, M_1 > 0$ such that

$$h_1(x, u, v) \leq (m - \delta) u^{r+q} v^p, \qquad \text{if } u \geq M_1 \max(v, 1).$$

Therefore, for any positive solution (u, v) of (6.1), if $\|u\|_\infty \geq M_1$ then, at a maximum point x_0 of u, we have either $u(x_0) < M_1 v(x_0)$, or else

$$0 \leq u^r v^p [a v^q - (c - m + \delta) u^q](x_0)$$

because $\phi \equiv M_1$ touches u by above at x_0. Since u and v are positive we deduce that

$$v(x_0) \geq \left(\frac{\delta}{a(x_0)}\right)^{1/q} u(x_0) \geq \left(\frac{\delta}{\|a\|_\infty}\right)^{1/q} u(x_0).$$

Hence there exists a constant $\eta > 0$ such that, for any positive solution (u, v) of (6.1),

$$\|u\|_\infty = u(x_0) \geq M_1 \Longrightarrow v(x_0) \geq \eta\, u(x_0).$$

In view of definition (6.6), this implies $\tilde{v}_j(0) \geq \eta \tilde{u}_j(0)$, hence $V(0) \geq \eta U(0) \geq \eta 2^{-\alpha}$, which excludes semitrivial limits and leads to a contradiction with the nonexistence of positive solutions of (6.11).

Second case : d_j/λ_j is bounded.
We may assume that $d_j/\lambda_j \to c_0 \geq 0$. After performing local changes of coordinates which flatten the boundary, we end up with a nontrivial nonnegative (bounded) solution (U, V) of system (6.11) in a half-space, with $U = V = 0$ on the boundary. From Theorem 1.1, we have

$$U = K\, V.$$

Moreover, (6.13) is satisfied so $a_0 - c_0K^q > 0$ by Proposition 5.1 ii). Hence $U > 0$ is a bounded viscosity solution of

$$-F(D^2U) = (a_0 - c_0K^q)U^\sigma$$

which contradicts Lemma 4.4 since $\sigma < \frac{\alpha^+(F)+2}{\alpha^+(F)}$. $\square$

Proof of Theorem 1.6. We first observe that by (1.17) and since $q + r \geq 1$, then any nonnegative solution of (6.2) satisfies

$$u > 0 \text{ and } v > 0 \text{ in } \Omega \qquad \text{or} \qquad t = 0 \text{ and } (u, v) \equiv (0, 0).$$

Indeed, by (6.3) and (1.17) we have

$$f(t, x, u, v) \geq -Cu,$$

for some $C \geq 0$ (which depends on t, u, v, c, d, A). Hence, $-F(D^2u) + Cu \geq 0$ and $F + C$ is a proper operator so the strong minimum principle applies and proves that $u \equiv 0$ or $u > 0$. In the case where $u \equiv 0$, then $0 = f(t, x, 0, v)$ and since (1.17) implies that $h_1(t, x, 0, v) \geq 0$, we have $0 \geq At$ so $t = 0$. Then (1.18) implies that

$$-F(D^2v) \leq h_2(x, 0, v) \leq \Big(\lambda_1^+(-F, \Omega) - \varepsilon_0\Big)v \quad \text{in } \Omega,$$

for some $\varepsilon_0 > 0$. But we know by [1] (see what follows Corollary 3.6) that

$$\lambda_1^+(-F, \Omega) = \sup\{\lambda : -F + \lambda \text{ satisfies the maximum principle}\}. \tag{6.15}$$

Since $v = 0$ on $\partial\Omega$, then $v \leq 0$, so $v = 0$.

Theorem 1.6 follows from a standard topological degree argument. We recall the following fixed point theorem, due to Krasnoselskii and Benjamin (see Proposition 2.1 and Remark 2.1 in [6]). This type of statements are nowadays standard in proving existence results.

Theorem 6.1. *Let $\mathcal{K}$ be a closed cone in a Banach space E, and let $T : \mathcal{K} \to \mathcal{K}$ be a compact mapping. Suppose $0 < \delta < M < \infty$, are such that*

(i) *$\eta Tx \neq x$ for all $x \in \mathcal{K}, \|x\| = \delta$, and all $\eta \in [0, 1]$;*

and there exists a compact mapping $H : \mathcal{K} \times [0, 1] \to \mathcal{K}$ such that

(ii) *$H(x, 0) = Tx$ for all $x \in \mathcal{K}$;*

(iii) *$H(x, t) \neq x$ for all $x \in \mathcal{K}, \|x\| = M$ and all $t \in [0, 1]$;*

(iv) *$H(x, 1) \neq x$ for all $x \in \mathcal{K}, \|x\| \leq M$.*

Then there exists a fixed point x of T (i.e. $Tx = x$), such that $\delta \leq \|x\| \leq M$.

Observe that (i) implies $i(T, B_\delta \cap \mathcal{K}) = i(0, B_\delta \cap \mathcal{K}) = 1$, where i is the (homotopy invariant) fixed point index with respect to the relative topology of $\mathcal{K}$, whereas by (iii)-(iv)

$$i(H(\cdot,0), B_R \cap \mathcal{K}) = i(H(\cdot,1), B_R \cap \mathcal{K}) = 0,$$

and the excision property of the index implies Theorem 6.1.

Let $\mathcal{K}$ denote the cone of nonnegative functions in

$$E := C(\overline{\Omega})$$

and let $\mathcal{T} : E \times E \to \mathcal{K}$ be defined by

$$\mathcal{T}(\phi, \psi) = (u_+, v_+),$$

where (u, v) is the unique viscosity solution (see Lemma 4.9) of the problem

$$\begin{gathered} -F(D^2u) = \phi, \;\; -F(D^2v) = \psi \text{ in } \Omega, \\ u = v = 0 \;\; \text{on } \partial\Omega. \end{gathered}$$

We check that $\mathcal{T}$ is compact. Indeed, by [5, Proposition 4.14], fixing $R > 0$ and considering

$$\begin{aligned} -F(D^2u) &= \phi \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

we get a same modulus of continuity for all u coming from $\phi \in B_E(0, R)$. Moreover, all such u have a uniform L^∞ bound since, by comparison principle, $u_{-R} \le u \le u_R$, where we have denoted u_R (resp. u_{-R}) the solution coming from $\phi = R$ (resp. $\phi = -R$). Hence the compactness of $(\phi, \psi) \mapsto (u, v)$ follows from Ascoli's theorem and since $(u, v) \mapsto (u_+, v_+)$ is Lipschitz, then $\mathcal{T}$ is compact.

We set

$$H((u,v),t) := \mathcal{T}\Big(f(t, x, u(x), v(x)), g(t, x, u(x), v(x))\Big),$$

and $T(u, v) = H((u, v), 0)$. Recall f, g are defined in (6.3)-(6.4), so fixed points of $H(\cdot, t)$ are solutions of (6.2).

We still have to choose the constant A in (6.3)–(6.5). We do this in the following way : by (1.17), there exists $C_1 > 0$ such that $h_1 \ge -C_1 u$ and $h_2 \ge -C_1 v$ for all $u, v > 0$ and $x \in \Omega$, and we choose some

$$A > \max\left\{ C_1 + \lambda_1^+(-\Delta, \omega), \; \sup_{x\in\Omega} c(x), \; \sup_{x\in\Omega} d(x) \right\}, \tag{6.16}$$

where ω is some smooth strict subdomain of Ω. Once A is fixed, we know from the proof of Theorem 1.5 that there exists a universal bound for the positive solutions (if they exist) of (6.2) valid for all $t \in [0, 1]$, and we choose M larger than this bound.

Theorem 1.6 is proved if we show that T has a nontrivial fixed point in $\mathcal{K}$. So it remains to check that the hypotheses of Theorem 6.1 are satisfied.

• Let us first show that $H(\cdot,1)$ does not possess any fixed point in $\mathcal{K}$, which will verify (iv). Assume such a fixed point (u,v) exists, which is then a solution of (6.2), with $t=1$. We have $u,v>0$ in Ω, since $t>0$. Let

$$\mathcal{S}=\sqrt{uv}.$$

Let us show that, in the viscosity sense,

$$-F(D^2\mathcal{S})\geq (A-C_1)\mathcal{S}\quad \text{ in } \omega.$$

This will prove that $\lambda_1^+(-F,\omega)\geq A-C_1$, which is in contradiction with the choice of A in (6.16), whence the result.

We know that, in the viscosity sense,

$$\begin{aligned}-F(D^2u)&\geq u^rv^p\Big[(a(x)+A)v^q-c(x)u^q\Big]+(A-C_1)u+A \quad \text{ in } \Omega,\\ -F(D^2v)&\geq v^ru^p\Big[(b(x)+A)u^q-d(x)v^q\Big]+(A-C_1)v+A \quad \text{ in } \Omega.\end{aligned}$$

Formally, since $(u,v)\mapsto\sqrt{uv}$ is concave on $[0,+\infty)^2$, then

$$D^2\mathcal{S}\leq\frac{\partial\mathcal{S}}{\partial u}D^2u+\frac{\partial\mathcal{S}}{\partial v}D^2v.$$

Hence, by subadditivity and $1-$homogeneity of F, we obtain

$$\begin{aligned}-F(D^2\mathcal{S}\geq{}&\frac{u^{-1/2}v^{1/2}}{2}\Big[u^rv^p\Big((a(x)+A)v^q-c(x)u^q\Big)+(A-C_1)u+A\Big]\\ &+\frac{u^{1/2}v^{-1/2}}{2}\Big[v^ru^p\Big((b(x)+A)u^q-d(x)v^q\Big)+(A-C_1)v+A\Big]\\ \geq{}&\frac{v^\sigma X^{-1/2}}{2}\Big[(a(x)+A)X^r+(b(x)+A)X^{p+q+1}-c(x)X^{q+r}-d(x)X^{p+1}\Big]\\ &+(A-C_1)\mathcal{S}+A,\end{aligned}$$

where $X=u/v$. Using (6.16) and the inequality

$$X^r+X^{p+q+1}-X^{q+r}-X^{p+1}=X^r(1-X^q)(1-X^{p+1-r})\geq 0$$

(note $p+1\geq 1\geq r$), it follows that

$$-F(D^2\mathcal{S})\geq (A-C_1)\mathcal{S}\quad \text{ in } \omega.$$

• Hypothesis (iii) in Theorem 6.1 is a consequence of the a priori bound for positive solutions of (6.2) which we obtained in the proof of Theorem 1.5, and the observation we made in the beginning of the proof of Theorem 1.6.

• Finally, assume that hypothesis (i) is not verified, which implies that for any (small) $\delta > 0$ we can find a positive solution (u, v) with $\|(u, v)\| \leq \delta$, of (6.1) with the right-hand side of this system multiplied by some $\eta \in [0, 1]$. Since F is subadditive, then by adding up the two equations in the system and using (1.19) we obtain, with $\lambda_1 = \lambda_1^+(-F, \Omega)$ and for some $\epsilon_0 > 0$,

$$\begin{aligned} -F(D^2[u+v]) &\leq -F(D^2u) - F(D^2v) \leq C(u^r v^{p+q} + v^r u^{p+q}) + (\lambda_1 - \epsilon_0)(u+v) \\ &\leq 2C(u+v)^{\sigma-1}(u+v) + (\lambda_1 - \epsilon_0)(u+v) \\ &\leq (\lambda_1 - \epsilon_0/2)(u+v) \end{aligned}$$

(we obtained the last inequality by choosing δ sufficiently small). But by (6.15), we get $u + v \leq 0$, a contradiction.

Hence, Theorem 1.6 is proved. □

7 Appendix

For the reader's convenience, we recall in this appendix some known facts to which we refer in the proofs.

Lemma 7.1. *Let F be a uniformly elliptic operator.*
If u is a bounded by below viscosity solution of $F(D^2u) = 0$ on $\mathbb{R}^n$, then u is constant.

The proof is well known and is the same as for the Laplacian. We give it for completeness.

Démonstration. If we set

$$v = u - \inf_{\mathbb{R}^n} u,$$

then $v \geq 0$, $\inf_{\mathbb{R}^n} v = 0$ and $F(D^2v) = 0$. Now, we can apply the Harnack inequality (see [5, theorem 4.3]) to $v_R = v(R\cdot) \geq 0$ for any $R > 0$ since $F(D^2v_R) = 0$ on $\mathbb{R}^n$. Hence, the exists $C > 0$ such that for all $R > 0$,

$$\sup_{B_1} v_R \leq C \inf_{B_1} v_R,$$

which implies

$$0 \leq \sup_{B_R} v \leq C \inf_{B_R} v.$$

If we let R go to $+\infty$, we get $v = 0$ since $\inf_{B_R} v \to 0$ as $R \to +\infty$, so

$$u = \inf_{\mathbb{R}^n} u.$$

□

Lemma 7.2. *Let z a viscosity solution on $\mathbb{R}^n$ of*

$$-F(D^2 z) \geq 0.$$

i) Assume $\alpha^*(F) \leq 0$.
If z is bounded by below, then

$$z \text{ is constant.}$$

ii) Assume $\alpha^*(F) > 0$.
Let Φ be the normalized upward-pointing fundamental solution of F.
If $z > 0$, then

$$z \geq m\ \Phi \qquad \text{on } \mathbb{R}^n \setminus B_1$$

where

$$m = \inf_{\partial B_1} \frac{z}{\Phi}.$$

As a consequence,

$$z \geq \frac{m}{|x|^{\alpha^*(F)}} \quad \text{for all } |x| \geq 1.$$

Remark : when $\alpha^*(F) > 0$, the normalization convention is given by

$$\min_{\partial B_1} \Phi = 1. \tag{7.1}$$

To have more details, see definition 1.5 in [3].

Démonstration. **i)** We may assume $z \geq 0$. We set

$$m(R) = \inf_{\partial B_R} z.$$

Since $-F(D^2 z) \geq 0$, then by the comparison principle, we have

$$m(R) = \inf_{B_R} z$$

so $R \mapsto m(R)$ is nonincreasing.
Let us show the existence of R_0 such that for all $R \geq R_0$,

$$m(R) = \inf_{\mathbb{R}^n \setminus B_R} z.$$

This will show that $R \mapsto m(R)$ is also nondecreasing on $[R_0, +\infty)$, hence constant, and finally, by the strong minimum principle, this will prove that z is constant (since the infimum of z on $\overline{B_{2R_0}}$ is reached on $\overline{B_{R_0}}$).

By Remark 1.3 in [3], since $\alpha^*(F) \leq 0$, we know that

$$\lim_{|x| \to \infty} \Phi(x) = -\infty.$$

Hence, there exists $R_0 > 0$ such that for all $|x| \geq R_0$, $\Phi(x) \leq 0$.
Let $R \geq R_0$ and $\delta > 0$. There exists $R_1 > R$ such that

$$- \Phi(x) \geq \frac{m(R)}{\delta} \qquad \text{for all } |x| \geq R_1. \tag{7.2}$$

Let $R' > R_1$. We apply the minimum principle to

$$z - \delta\Phi$$

on

$$A_{R,R'} = B_{R'} \setminus \overline{B_R}.$$

On ∂B_R,

$$z - \delta\Phi \geq z \geq m(R)$$

since $\Phi \leq 0$ on $\mathbb{R}^n \setminus B_{R_0}$ and on $\partial B_{R'}$, we have

$$z - \delta\Phi \geq -\delta\Phi \geq m(R)$$

since $z \geq 0$ and by (7.2), hence

$$z \geq \delta\Phi + m(R) \text{ on } A_{R,R'}.$$

So, by letting R' go to $+\infty$ and then δ to 0, we obtain $z \geq m(R)$ on $\mathbb{R}^n \setminus B_R$, which implies

$$m(R) = \inf_{\mathbb{R}^n \setminus B_R} z$$

for all $R \geq R_0$.

ii) We recall that we have set

$$m = \inf_{\partial B_1} \frac{z}{\Phi} > 0$$

since $z > 0$ and $\Phi > 0$ on $\mathbb{R}^n \setminus \{0\}$. We want to show that

$$z \geq m\,\Phi \text{ on } \mathbb{R}^n \setminus B_1.$$

This will imply the last result since, by (7.1) and the $[-\alpha^*(F)]$-homogeneity of Φ, we have for all $x \neq 0$,

$$\Phi(x) \geq \frac{1}{|x|^{\alpha^*(F)}}.$$

Let $\delta > 0$. Since $\alpha^*(F) > 0$, we know that (see Remark 1.6 in [3])

$$\Phi(x) \underset{|x|\to\infty}{\longrightarrow} 0$$

so there exists $R_1 \geq 1$ such that

$$\delta \geq m\Phi \qquad \text{on } \mathbb{R}^n \setminus B_{R_1}.$$

Let $R \geq R_1$. We want to apply the comparison principle on $B_R \setminus \overline{B_1}$ to

$$z + \delta \qquad \text{and} \qquad m\Phi.$$

This is possible since

$$F(D^2 m\Phi) = mF(D^2\Phi) = 0$$

on $\mathbb{R}^n \setminus \{0\}$ and

$$-F(D^2[z+\delta]) = -F(D^2 z) \geq 0$$

on $\mathbb{R}^n$ and since moreover

$$z + \delta \geq z \geq m\Phi \qquad \text{on } \partial B_1$$

and

$$z + \delta \geq \delta \geq m\Phi \qquad \text{on } \partial B_R.$$

Hence, we deduce that $z + \delta \geq m\Phi$ on $B_R \setminus B_1$. Eventually, letting R go to infinity and then δ to 0, we obtain

$$z \geq m\Phi \qquad \text{on } \mathbb{R}^n \setminus B_1.$$

□

Lemma 7.3. *Let $u \in C(\overline{\mathcal{C}_\omega})$ such that $u > 0$ on $\overline{\mathcal{C}_\omega}$. Then there exists $c > 0$ such that*

$$u \geq c\, \Psi^+ \text{ on } \mathcal{C}_\omega \setminus B_1^+.$$

Démonstration. We set

$$c = \inf_{S_1^+} \frac{u}{\Psi^+} > 0$$

since $u > 0$ and $\Psi^+ = 0$ on $\partial\mathcal{C}_\omega$. We then employ the same technique as for Lemma 7.2 ii).
Indeed, since $\alpha^+(F) > 0$, then

$$\Psi^+(x) \underset{|x|\to\infty}{\longrightarrow} 0$$

and moreover

$$u \geq 0 = \Psi^+ \text{on } \partial\mathcal{C}_\omega \setminus \{0\}.$$

So, for any $\delta > 0$ and any $R > 1$ large enough, by the comparison principle, we have

$$u + \delta \geq c\Psi^+ \text{ on } \overline{B_R^+ \setminus B_1^+},$$

whence the result by letting R go to infinity and then δ to zero. □

Lemma 7.4. *Let Ω be an domain of $\mathbb{R}^n$ and let F be a uniformly elliptic operator. Let $u \geq 0$ be a viscosity solution on Ω of*

$$-F(D^2u) \geq h.$$

Then, for any compact $K \subset \Omega$, there exist $\gamma = \gamma(F, n) > 0$ and $C = C(K) > 0$ such that

$$\inf_K u \geq C \left(\int_K h^\gamma \right)^{\frac{1}{\gamma}}.$$

It is well known (see for instance [5, Proposition 2.9]) that it is easy to pass to the limit with viscosity solutions, as recalled below :

Lemma 7.5. *Let Ω be a domain of $\mathbb{R}^n$ and F be a continuous and degenerate elliptic operator.*
Let (u_j) a sequence of viscosity solutions of

$$-F(D^2u_j) = f_j(x, u_j) \qquad \text{on } \Omega$$

where $f_j \in C(\Omega \times \mathbb{R})$.
Assume that

$$u_j \to u \text{ locally uniformly in } \Omega$$

and

$$f_j \to f \in C(\Omega \times \mathbb{R}) \text{ locally uniformly in } \Omega \times \mathbb{R}.$$

Then u is a viscosity solution of

$$-F(D^2u) = f(x, u) \qquad \text{on } \Omega.$$

Démonstration. First, it is clear that $u \in C(\Omega)$.
Let $\phi \in C^2(\Omega)$ touching u strictly by above at some point $x_0 \in \Omega$. This means that $\phi(x_0) = u(x_0)$ and $\phi > u$ on $\Omega \setminus \{x_0\}$. We want to show that $-F(D^2\phi(x_0)) \leq f(x_0, u(x_0))$.

Let $\epsilon > 0$ such that $K = \overline{B(x_0, \epsilon)} \subset \Omega$. Since $u - \phi < 0$ on ∂K, we set

$$-2\delta = \max_{\partial K}[u - \phi] < 0.$$

Let j_0 such that for all $j \geq j_0$, $\|u - u_j\|_{\infty, K} < \delta$. Let $j \geq j_0$. Then $\max_{\partial K}[u_j - \phi] < -\delta$ and $[u_j - \phi](x_0) > -\delta$ hence $\max_K[u_j - \phi]$ is reached at some point $x_j \in B(x_0, \epsilon)$. Therefore,

$$-F(D^2\phi(x_j)) \leq f_j(x_j, u_j(x_j)).$$

Taking a subsequence, we may assume that $x_j \to y \in K$. But, by locally uniform convergence, we know that

$$\max_K[u_j - \phi] \to \max_K[u - \phi] = 0$$

But $[u_j - \Phi](x_j) \to [u - \phi](y)$, so $[u - \phi](y) = 0$ thus $y = x_0$ since ϕ is strictly above u except at x_0. Since for all $j \geq j_0$, $(x_j, u_j(x_j))$ is in a compact of $\Omega \times \mathbb{R}$, $(x_j, u_j(x_j)) \to (x_0, u(x_0))$ and $f_j \to f$ locally uniformly in $\Omega \times \mathbb{R}$, then

$$f_j(x_j, u_j(x_j)) \to f(x_0, u(x_0)).$$

Moreover,

$$F(D^2\phi(x_j)) \to F(D^2\phi(x_0))$$

since $\phi \in C^2(\Omega)$ and F is continuous on $\mathcal{S}_n$. Hence,

$$-F(D^2\phi(x_0)) \leq f(x_0, u(x_0))$$

by letting j go to infinity.
We make the same argument with ϕ touching u strictly by below at some point $x_0 \in \Omega$ and get

$$-F(D^2\phi(x_0)) \geq f(x_0, u(x_0)),$$

whence the result. □

Lemma 7.6. *Let F be a uniformly elliptic operator, $R > 0$ and $f \in C(\overline{B_R})$. Then there exists a unique viscosity solution $u \in C(\overline{B_R})$ of problem*

$$\begin{cases} -F(D^2u) = f(x) & x \in B_R, \\ u(x) = 0 & x \in \partial B_R \end{cases} \tag{7.3}$$

Démonstration. We just would like to give an idea of the proof, namely to find a subsolution and a supersolution of (7.3).
First, if we set for all $(x, M) \in B_R \times \mathcal{S}_n$, $G(x, M) = F(M) + f(x)$, we see that G is uniformly elliptic and continuous. The comparison principle then applies to G (see Theorem 3.9 in [9]), ensuring in particular the uniqueness of the viscosity solution.
Moreover,

$$P(x) = a\left[\frac{R^2}{2} - \frac{|x|^2}{2}\right]$$

is a supersolution of $-G(x, D^2u) = 0$ on B_R for $a \geq \frac{\|f\|_{\infty,B_R}}{\Lambda n}$ since

$$-F(D^2P) \geq -\mathcal{M}^+(D^2P) = \Lambda\, n\, a \geq f.$$

Similarly, $-P$ is a subsolution of $-G(x, D^2u) = 0$ on B_R since

$$-F(D^2[-P]) \leq -\mathcal{M}^-(D^2[-P]) = \mathcal{M}^+(D^2P) = -\Lambda\, n\, a \leq -\|f\|_{\infty,B_R} \leq f.$$

In addition, $P = 0$ on ∂B_R. Hence, we can apply Theorem (Ishii) in Section 9 of the first chapter in [4] and obtain the existence of a viscosity solution. □

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Chapitre 6

Proportionnalité des composantes et théorèmes de Liouville pour des systèmes elliptiques non coopératifs dans le demi-espace [1]

Dans ce court chapitre, nous nous intéressons aux solutions classiques de systèmes elliptiques semi-linéaires dans le demi-espace et donnons des conditions suffisantes assurant la proportionnalité des composantes, i.e. $u = K\,v$ avec $K > 0$, ce qui réduit alors le système à une seule équation elliptique. Sous une condition naturelle de structure sur les non-linéarités, nous montrons que les solutions à croissance sous-linéaire, donc en particulier les solutions bornées, ont des composantes proportionnelles. Ce résultat couvre le cas de systèmes non coopératifs, non variationnels et éventuellement sur-critiques. Nous obtenons aussi des résultats de proportionnalité sans hypothèse de croissance sur les solutions. Comme conséquence, nous obtenons de nouveaux théorèmes de type Liouville dans le demi-espace, ainsi que des estimations a priori et des résultats d'existence pour des problèmes de Dirichlet associés. Nos preuves reposent sur un principe du maximum, sur les propriétés de moyennes semi-sphériques, sur un théorème de rigidité pour les fonctions surharmoniques ainsi que sur des résultats de nonexistence pour des inéquations scalaires dans le demi-espace.

1. Ce court chapitre est tiré de la Note aux Compte-Rendus de l'Académie des Sciences [P4], écrite en collaboration avec **Philippe Souplet**. Celle-ci annonçait une partie des résultats désormais contenus et améliorés dans [P5] (voir chapitre 4). Nous l'incluons dans la mesure où elle fait partie de nos travaux de thèse.

Version française abrégée

Nous étudions des systèmes elliptiques semi-linéaires dans le demi-espace H_n, $n \geq 1$, du type

$$\begin{cases} -\Delta u = f(u,v) & \text{sur } H_n \\ -\Delta v = g(u,v) & \text{sur } H_n \\ u = v = 0 & \text{sur } \partial H_n \end{cases} \tag{0.1}$$

et donnons des conditions suffisantes pour avoir la proportionnalité des composantes, i.e. $u = Kv$, ce qui permet alors de se ramener au cas scalaire et de classifier les solutions ou obtenir de nouveaux résultats de type Liouville. La condition de structure (1.1) apparaît naturellement et permet d'obtenir le théorème général suivant, valable pour les solutions à croissance sous-linéaire, donc pour les solutions bornées (ce qui est suffisant pour l'application de la méthode de renormalisation de Gidas et Spruck [7]). Notons que nos résultats peuvent aussi s'appliquer à des non-linéarités sur-critiques.

Théorème 0.1. *Soit $(u, v) \in C^2(\overline{H_n})^2$ une solution de (0.1) où f et g satisfont (1.1). Si u et v sont à* ***croissance sous-linéaire****, i.e. $u(x) = o(|x|)$ et $v(x) = o(|x|)$ quand $x \to \infty$, alors $u = Kv$.*

Ce théorème est optimal comme le montre l'exemple $u = x_n$, $v = 0$, $f(u, v) = g(u, v) = uv$, $K = 1$. Il s'applique au système (2.2) comme au système non coopératif et non variationnel (2.3) et permet par exemple d'obtenir le théorème de Liouville suivant (grâce aux résultats de [7, 8]) :

Théorème 0.2. *On note $\sigma = p + q + r$.*

(i) La seule solution bornée de (2.2) ou (2.3) est la solution triviale.

(ii) Si $\sigma \leq (n + 2)/(n - 2)_+$, alors la seule solution à croissance sous-linéaire est la solution triviale.

On peut également obtenir des résultats de proportionnalité sans faire d'hypothèse de croissance sur les solutions. Par exemple, appelant **semi-triviale** une solution telle que $u = 0$ ou $v = 0$, on obtient, à l'aide d'un résultat de rigidité pour les fonctions surharmoniques dans le demi-espace, le théorème suivant :

Théorème 0.3. *Soit $(u, v) \in C^2(\overline{H_n})^2$ une solution positive ou nulle du système (0.1), où l'on suppose que f et g satisfont la condition (1.1) et que*

$$f(u, v) \geq c\, u^r\, v^{p+q} \qquad \text{et} \qquad g(u, v) \geq c\, u^{r+q}\, v^p \quad \text{pour tout } u, v \geq 0,$$

où $c > 0$ et $p, q, r \geq 0$. On note $\sigma = p + q + r$.

(ii) Si $r \leq (n + 1 + p + q)/(n - 1)$ et $p \leq (n + 1 + r + q)/(n - 1)$, alors $u = Kv$ ou (u, v) est semi-triviale.

(iii) Si on suppose la condition plus forte $\sigma \leq (n + 1)/(n - 1)$, alors (u, v) est semi-triviale.

Comme application, nous obtenons des résultats d'existence et d'estimation a priori pour le problème de Dirichlet (2.4). Celui-ci comprend comme cas particuliers des modèles d'interaction symbiotique d'espèces biologiques (de type Lotka-Volterra), de condensats de Bose-Einstein et de réactions chimiques.

1 Introduction

In order to show existence of a classical solution for semilinear elliptic systems of the form

$$\begin{cases} -\Delta u = & f(x,u,v), \qquad x \in \Omega \\ -\Delta v = & g(x,u,v), \qquad x \in \Omega \end{cases}$$

in a bounded regular domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$ with Dirichlet boundary conditions, a well-known method is to first show an a priori estimate for all solutions and then to apply a topological method via degree theory (see [1] for instance). To prove the former, we can use the rescaling (or blow-up) method introduced in [7] (see e.g. [2] for the case of systems), which requires the knowledge of Liouville-type theorems for bounded solutions in the case of the whole space $\mathbb{R}^n$ and the half-space

$$H_n = \{x \in \mathbb{R}^n,\, x_n > 0\}.$$

To get these nonexistence results, most methods seem to use either moving planes (or spheres) and Kelvin transform (and hence require a rather restrictive assumption of cooperativity of the system) or Pohozaev type identity (and hence require some variational structure).

Another possible method is, under the following natural condition

$$[f(u,v) - Kg(u,v)][u - Kv] \leq 0, \quad (u,v) \in \mathbb{R}^2, \quad \text{for some constant } K > 0 \quad (1.1)$$

to show proportionality of the components, i.e. $u = Kv$, so as to reduce the system to a single equation. This then allows to get new classification and nonexistence results by using known results for the scalar case. It is important to note that, since the rescaling method only "sees the highest order terms", only the latter have to satisfy condition (1.1).

This condition (1.1) is natural a priori since in a bounded domain, it would imply proportionality just by considering $w = u - Kv$ and integrating by parts, and a posteriori because of the generality of Theorem 2.1 in the half-space below. We would like to add that, more heuristically, if we consider the parabolic system associated to (1.1), then we have formally $\frac{d}{dt}\int_\Omega [u - Kv]^2(t) \leq 0$, which gives hope that $\|[u - Kv](t)\|_2 \to 0$ as $t \to +\infty$. This would mean that stationary states satisfy $u(\infty) = Kv(\infty)$. Another clue to the attractivity of the diagonal is that condition (1.1) means that the vector field (f,g) of the underlying differential system is pointing toward the latter.

This method has been employed successfully in [12] for general systems in the whole space (see also [10] for earlier use of this method for a particular system). In this note, we focus on the case of the half-space H_n. Whereas a central ingredient in [12] was the use of spherical means, a key tool will here be the following half-spherical means:

$$[u](R) = \frac{1}{R^2|S_R^+|}\int_{S_R^+} u(x)\, x_N \, d\sigma_R(x), \quad R > 0,$$

where $S_R^+ = \{x \in H_n, |x| = R\}$, σ_R is the Lebesgue's measure on S_R^+ and $|S_R^+| = \sigma_R(S_R^+)$. Complete proofs as well as further results obtained in collaboration with B. Sirakov will be provided in the forthcoming article [P5].

2 Main results

The first result is a rather general theorem concerning classical solutions (u, v) with **sublinear growth**, i.e. such that $u(x) = o(|x|)$ and $v(x) = o(|x|)$ as $x \to \infty$. This case in particular covers the case of bounded solutions, sufficient for the rescaling method.

Theorem 2.1. *Assume f and g satisfy (1.1). Let $(u, v) \in C^2(\overline{H_n})^2$ be a solution of the system*

$$-\Delta u = f(u, v) \quad \text{and} \quad -\Delta v = g(u, v) \quad \text{on } H_n. \tag{2.1}$$

If $u = Kv$ on ∂H_n and u and v have sublinear growth, then $u = Kv$.

Remark 2.1. *(a) This theorem is optimal since if one of the components has linear growth, the result is not valid anymore, as shown by the counterexample $u = x_n$, $v = 0$, $f(u, v) = g(u, v) = uv$, $K = 1$.*
(b) Note that no assumption is made on the sign of the solutions or of the nonlinearities f and g.

This theorem in particular applies to the nonnegative solutions of the following two systems:

$$\begin{cases} -\Delta u = u^r\, v^{p+q} & \text{on } H_n \\ -\Delta v = u^{r+q}\, v^p & \text{on } H_n \\ u = v = 0 & \text{on } \partial H_n \end{cases} \qquad \text{where } p, q, r \geq 0, \tag{2.2}$$

$$\begin{cases} -\Delta u = u^r v^p [a v^q - c u^q] & \text{on } H_n \\ -\Delta v = v^r u^p [b u^q - d v^q] & \text{on } H_n \\ u = v = 0 & \text{on } \partial H_n, \end{cases} \tag{2.3}$$

where

$$p, r \geq 0, q > 0, q \geq |r - p|, a, b > 0, c, d \geq 0, cd < ab.$$

Indeed, system (2.2) satisfies condition (1.1) with $K = 1$ and it can be shown (see [P5]) that (2.3) satisfies (1.1) for some (unique) $K > 0$ (K being equal to 1 if and only if $a+d=b+c$). Hence, thanks to the scalar nonexistence results in [7, 8], we can deduce the following Liouville-type result.

Theorem 2.2. *Denote $\sigma = p+q+r$.*
(i) The only nonnegative bounded solution of (2.2) or (2.3) is the trivial one.
(ii) If $\sigma \leq (n+2)/(n-2)_+$, then the only solution with sublinear growth is the trivial one.

Remark 2.2. *Note that system (2.3) is not cooperative (and generally non variational) for $p>0$. The whole space case for systems (2.2) and (2.3) was considered in [12] and, for system (2.3), significantly improved results are given in [P5].*

Concerning solutions without any growth assumption, we give an instance of our results, relying on a rigidity result for superharmonic functions in the half-space. We will say that a solution (u,v) is **semitrivial** if $u=0$ or $v=0$ and that it is **positive** if $u>0$ and $v>0$.

Theorem 2.3. *Let $(u,v) \in C^2(\overline{H_n})^2$ be a nonnegative solution of system (2.1) with boundary conditions $u=v=0$ on ∂H_n. We assume that f,g satisfy condition (1.1) and that*

$$f(u,v) \geq c\, u^r\, v^{p+q} \qquad \text{and} \qquad g(u,v) \geq c\, u^{r+q}\, v^p \qquad \text{for all } u,v \geq 0,$$

for some $c>0$, where $p,q,r \geq 0$. We denote $\sigma = p+q+r$.
(i) We have $u \leq Kv$ or $u \geq Kv$.
(ii) If $r \leq (n+1+p+q)/(n-1)$ and $p \leq (n+1+r+q)/(n-1)$, then $u=Kv$ or (u,v) is semitrivial.
(iii) If we assume the stronger condition $\sigma \leq (n+1)/(n-1)$, then (u,v) is semitrivial.

Remark 2.3. *(a) The previous theorem allows to address supercritical nonlinearities, as can be seen by taking q large enough in system (2.2) for instance.*
(b) Considering system (2.2), under conditions in (ii), by a well-known Liouville-type result of [7], we can deduce that if $\sigma \leq (n+2)/(n-2)_+$, then the system has no positive solution.

As an application, let us now consider the Dirichlet problem

$$\begin{cases} -\Delta u = u^r v^p \Big[a(x)v^q - c(x)u^q\Big] + \lambda u, & x \in \Omega, \\ -\Delta v = v^r u^p \Big[b(x)u^q - d(x)v^q\Big] + \mu u, & x \in \Omega, \\ \quad u = v = 0, & x \in \partial\Omega, \end{cases} \tag{2.4}$$

where Ω is a smoothly bounded domain of $\mathbb{R}^n$, $p, r \geq 0$, $q > 0$, $a, b, c, d, \lambda, \mu \in C(\overline{\Omega})$ satisfy $a, b > 0$, $c, d \geq 0$ in $\overline{\Omega}$. Note that the special cases (a): $r = q = 1$, $p = 0$; (b): $r = 1$, $p = 0$, $q = 2$; and (c): $r = p = q = 1$ respectively correspond to a Lotka-Volterra type system modeling the symbiotic interaction of biological species, to models of Bose-Einstein condensates and to systems involved in models of chemical reaction. These systems were considered for instance in [10, 4], [9, 3], [5]. The following theorem extends or improves some of the results therein.

Theorem 2.4. *Assume $q \geq |p-r|$, $q+r \geq 1$, $r \leq 1$, $1 < p+q+r < (n+2)/(n-2)_+$ and*

$$\sup_{x\in\Omega} \frac{c(x)d(x)}{a(x)b(x)} < 1. \tag{2.5}$$

(i) There exists $M > 0$ such that any positive classical solution (u, v) of (2.4) satisfies $\|u\|_\infty$, $\|v\|_\infty \leq M$.

(ii) Assume in addition that a, b, c, d, λ, μ are Hölder continuous and that $\lambda, \mu < \lambda_1$ on $\overline{\Omega}$.
Then there exists at least a positive classical solution of (2.4).

3 Sketch of proofs

Proof of Theorem 2.1. We denote $w = u - Kv$. Since u and v have sublinear growth, then $|w|$ also. Hence $[w_+](R) \underset{R\to\infty}{\longrightarrow} 0$ and $[(-w)_+](R) \underset{R\to\infty}{\longrightarrow} 0$. So, thanks to condition (1.1), we can apply the following key lemma to w and to $-w$ to get $w = 0$. □

Lemma 3.1. *Let $w \in C^2(\overline{H_n})$ be such that $w \leq 0$ on ∂H_n and $\Delta w \geq 0$ on the set $\{w > 0\}$. If we assume*

$$\liminf_{R\to\infty} [w_+](R) = 0$$

then $w \leq 0$.

Proof. It follows by applying to (a regularized version of) w_+ the formula

$$\frac{d}{dR}[z](R) = \frac{1}{R^2|S_R^+|}\left[\int_{B_R^+} \Delta z\, x_n\, dx - \int_{B_R\cap\partial H_n} z(y')\, dy'\right], \quad R > 0, \quad z \in C^2(\overline{H_n}), \tag{3.1}$$

which can be obtained by direct computation. □

For the proof of Theorem 2.3, we will need the following two lemmas. The first lemma is concerned with half-spherical means, and its second assertion gives a rigidity result for superharmonic functions on H_n.

Lemma 3.2. *Let $u \in C^2(\overline{H_n})$, with $u \geq 0$. If u is superharmonic on H_n, then $[u]$ is nonincreasing and $\lim_{R\to\infty}[u](R) = \lambda \geq 0$. Moreover, $u(x) \geq \frac{\lambda}{[x_n]}\, x_n$ for all $x \in H_n$. (Note that $[x_n]$ is a constant.)*

Proof. The first assertion follows from (3.1). The second assertion can be shown by representing the solutions via Poisson kernels on half-spheres and using the scaling properties of the kernels. Alternatively, it can be shown by applying to the function $w = \frac{\lambda}{[x_n]}\, x_n - u$ a special kind of maximum principle, namely the corollary of Theorem 1 p.341 in [6]. □

Lemma 3.3. *Assume $r, s \geq 0$ and $c > 0$. Let $u \in C^2(\overline{H_n})$ be a nonnegative solution of $-\Delta u \geq c\, {x_n}^s\, u^r$ on H_n. If $r \leq (n+1+s)/(n-1)$, then $u = 0$.*

Proof. It is based on rescaled test-functions, similarly as that of [11, Theorem 10.1, p.36] which concerns the inequality $-\Delta u \geq |x|^s\, u^r$ in a half-space. □

Proof of Theorem 2.3. (i) The functions f and g are nonnegative, so u and v are superharmonic. Then by Lemma 3.2,

$$\lim_{R\to\infty} [u](R) = \lambda \geq 0 \text{ and } \lim_{R\to\infty} [v](R) = \mu \geq 0.$$

• Assume $\lambda > 0$ and $\mu > 0$. Since $u = 0$ and $v = 0$ on ∂H_n, by Lemma 3.2 (i), we have, for some $c > 0$,

$$u(x) \geq c\, x_n \text{ and } v(x) \geq c\, x_n, \quad x \in H_n.$$

Hence, $-\Delta u \geq c\, {x_n}^\sigma$, so by Lemma 3.3 with $r = 0$ we have a contradiction.
• Assume $\lambda = 0$. Then, setting $w = u - Kv$, we have $w_+ \leq u$ since $u, v \geq 0$ and then $\lim_{R\to\infty}[w_+](R) = 0$. By Lemma 3.1, this implies $w \leq 0$, i.e. $u \leq Kv$.
• If $\mu = 0$, similarly, we get $u \geq Kv$.
(ii) Using the notation of (i), if $\lambda > 0$, then $-\Delta v \geq c\, {x_n}^{r+q} v^p$. Since $p \leq (n+1+r+q)/(n-1)$, we have $v = 0$ by Lemma 3.3. Similarly, if $\mu > 0$, then $u = 0$. If $\lambda = \mu = 0$, then, by the proof of (i), we get $u = Kv$.
(iii) If $\sigma \leq (n+1)/(n-1)$, since $-\Delta u \geq cu^\sigma$ or $-\Delta v \geq cv^\sigma$ due to assertion (i), we deduce from Lemma 3.3 that $u = 0$ or $v = 0$. □

Proof of Theorem 2.4. Assertion (i) is obtained by using the rescaling method of [12], relying on Liouville type theorems in a half-space (Theorem 2.2) and in the whole space [12, **?**]. Some special care is needed in order to rule out semitrivial limits (i.e., solutions of the form $(0, C)$ or $(C, 0)$) in the rescaling procedure (this is based on an eigenfunction argument on large balls and uses the assumption $r \leq 1$). As for assertion (ii), it follows from classical topological degree arguments. □

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RESUME : Cette thèse est consacrée à l'étude de deux problèmes :

D'une part, nous considérons un système parabolique-elliptique de type Patlak-Keller-Segel avec sensitivité de type puissance et exposant critique. Nous étudions les solutions radiales de ce système dans une boule de l'espace euclidien et obtenons des résultats d'existence-unicité, de régularité ainsi qu'une alternative d'explosion. Concernant le comportement qualitatif en temps long des solutions radiales, pour toute dimension d'espace supérieure ou égale à trois, nous montrons un phénomène de masse critique qui généralise le cas déjà connu de la dimension deux mais présente par rapport à celui-ci un comportement très différent dans le cas de la masse critique. Dans le cas d'une masse sous-critique, nous montrons de plus que les densités de cellule convergent uniformément à vitesse exponentielle vers l'unique solution stationnaire. Ce dernier résultat est valable pour toute dimension d'espace supérieure ou égale à deux et n'était, à notre connaissance, pas connu même pour le cas très étudié de la dimension deux.

D'autre part, nous étudions des systèmes elliptiques (semi-linéaires et complètement non-linéaires) non coopératifs. Dans le cas de l'espace ou d'un demi-espace (ou même d'un cône), sous une hypothèse de structure naturelle sur les non-linéarités, nous donnons des conditions suffisantes pour avoir la proportionnalité des composantes, ce qui permet de ramener l'étude à celle d'une équation scalaire et ainsi d'obtenir des résultats de classification et de type Liouville pour le système. Dans le cas d'un domaine borné, grâce aux théorèmes de type Liouville obtenus, la méthode de renormalisation de Gidas et Spruck permet d'obtenir une estimation a priori des solutions bornées et finalement de déduire l'existence d'une solution non triviale, via une méthode topologique utilisant la théorie du degré.

TITLE : Qualitative study of a parabolic-elliptic Keller-Segel type system and noncooperative elliptic systems.

ABSTRACT : This thesis is concerned with the study of two problems :

On the one hand, we consider a parabolic-elliptic system of Patlak-Keller-Segel type with a critical power type sensitivity. We study the radially symmetric solutions of this system on a ball of the euclidean space and obtain wellposedness and regularity results together with a blow-up alternative. As for the long time qualitative behaviour of the radial solutions, for any space dimension greater or equal to three, we show that a critical mass phenomenon occurs, which generalizes the well-known case of dimension two but, with respect to the latter, with a very different qualitative behaviour in the case of the critical mass. When the mass is subcritical, we moreover show that the cell density converges uniformly with exponential speed toward the unique steady state. This result is valid for any space dimension greater or equal to two, which was, to our knowledge, not known even for the most studied case of dimension two.

On the other hand, we study noncooperative (semilinear and fully nonlinear) elliptic systems. In the case of the whole space or of a half-space (or even for a cone), under a natural structure condition on the nonlinearities, we give sufficient conditions to have proportionnality of the components, which allows to reduce the system to a scalar equation and then to get classification and Liouville type results. In the case of a bounded domain, thanks to the obtained Liouville type theorems, the blow-up method of Gidas and Spruck then allows to get an a priori estimate on the bounded solutions and eventually to deduce the existence of a non trivial solution by the degree theory.

DISCIPLINE : Mathématiques

MOTS-CLES : EDP, SYSTEMES, PARABOLIQUE, ELLIPTIQUE, KELLER-SEGEL, MASSE CRITIQUE, CHIMIOTAXIE, NON COOPERATIF

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