Université Paris 13 - Institut Galilée
Laboratoire Analyse, Géométrie et Applications, UMR 7539

Numéro attribué par la bibliothèque
|__|__|__|__|__|__|__|__|__|__|__|

THÈSE

pour obtenir le titre de

DOCTEUR DE L'UNIVERSITÉ PARIS-NORD

Discipline : **Mathématiques**

Présentée et soutenue publiquement par :

Van Tien NGUYEN

le 11 décembre 2014

Sujet de la thèse :

ÉTUDE NUMÉRIQUE ET THÉORIQUE DU PROFIL À L'EXPLOSION DANS LES ÉQUATIONS PARABOLIQUES NON LINÉAIRES

Composition du jury :

Mme. Linda El Alaoui	Université Paris-Nord	Co-directrice de thèse
M. Pierre Raphaël	Université de Nice Antipolis	
M. Julio D. Rossi	Université de Buenos Aires	Rapporteur
M. Jérémie Szeftel	CNRS et Université Pierre et Marie Curie	
M. Slim Tayachi	Faculté des Sciences de Tunis	Rapporteur
M. Fred B. Weissler	Université Paris-Nord	Président
M. Hatem Zaag	CNRS et Université Paris-Nord	Directeur de thèse

Remerciements

Je tiens tout d'abord à exprimer mes profonds remerciements et ma gratitude à mon directeur de thèse, Monsieur Hatem Zaag, et à ma co-directrice de thèse, Madame Linda El Alaoui, pour leurs gentillesses, pour leurs soutiens continus, pour leurs patiences et pour le temps qu'ils m'ont accordé tout au long de ces années. Je n'oublierai jamais les conseils judicieux qu'ils m'ont prodigués.

Je suis très reconnaissant envers Messieurs Julio Daniel Rossi et Slim Tayachi qui m'ont fait l'honneur d'être rapporteurs de cette thèse. Je voudrais remercier vivement Messieurs Pierre Raphaël, Jérémie Szeftel et Frederic Weissler pour avoir accepté de participer à ma soutenance de thèse en tant que membres du jury.

Je souhaite profiter de cette occasion pour exprimer ma reconnaissance à mon professeur à l'Université des Sciences au Vietnam, Monsieur Duong Minh Duc, qui m'a encouragé à poursuivre mes études supérieurs. Je remercie particulièrement Monsieur Omnes Pascal pour ses aides avec beaucoup de disponibilité et de gentillesse pendant les premiers jours où je suis arrivé en France pour mon stage à l'Université Paris-Nord.

Mes remerciements vont également à tous les membre du LAGA pour leur accueil chaleureux durant toutes ces années. Un merci particulier à Mesdames Isabelle Barbotin et Yolande Jimenez, Messieurs Jean-Philippe Dru et Olivier Fernandez pour leurs aides de problèmes administratifs et leurs gentilesses, et à Monsieur Gilles Scarella pour m'avoir aidé à faciliter l'accès au server de calcul GAIA.

Mes longues années à étranger n'auraient pas été si joyeuses sans mes chers amis vietnamiens. Je suis reconnaissant que nos chemins se sont croisés.

Mes plus grands remerciements vont à ma famille pour leur soutien et affection inconditionnel. Enfin, un immense merci à ma femme, Yến, sans toi à mes côtés rien ne serait possible.

Kính tặng Bố Mẹ.

à ma femme Yến.

Résumé

On s'intéresse au phénomène d'explosion en temps fini dans les équations aux dérivées partielles paraboliques non linéaires, particulièrement au profil à l'explosion, des points de vue numérique et théorique.

Dans la partie théorique, on s'intéresse au phénomène d'explosion en temps fini pour une classe d'équations semi-linéaires de la chaleur perturbées fortement avec l'exposant sous-critique de Sobolev :

$$\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u + h(u),$$

où $u : (x,t) \in \mathbb{R}^n \times [0,T) \to \mathbb{R}$, $1 < p$, $p < \frac{n+2}{n-2}$ si $n \geq 3$, et la fonction $h \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}^*)$ satisfait

$$j = 0, 1,\ |h^{(j)}(z)| \leq M \left(\frac{|z|^{p-j}}{\log^a(2+z^2)} + 1 \right), \quad |h''(z)| \leq M \frac{|z|^{p-2}}{\log^a(2+z^2)},$$

avec $M > 0$ et $a > 1$, ou $h(z) = \dfrac{\mu |z|^{p-1} z}{\log^a(2+z^2)}$ avec $\mu \in \mathbb{R}$ et $a > 0$.
Travaillant dans le cadre des *variables auto-similaires*, on obtient d'abord l'existence d'une fonctionnelle de Lyapunov, ce qui constitue une étape cruciale pour établir le taux d'explosion de la solution. Dans une seconde étape, on s'intéresse à la structure de la solution au voisinage du temps et du point d'explosion. On classifie tous les comportements asymptotiques possibles pour la solution quand elle s'approche de la singularité. Ensuite, on décrit les profils à l'explosion correspondant à ces comportements asymptotiques. Dans une troisième étape, on construit pour cette équation une solution qui explose en temps fini en un seul point avec un profil d'explosion prescrit. Cette construction s'appuie sur la réduction en dimension finie du problème et sur l'utilisation du théorème de l'indice pour conclure.

Dans la partie numérique, on se propose de développer des méthodes afin de donner des réponses numériques à la question du profil à l'explosion pour certaines équations paraboliques, y compris le modèle de Ginzburg-Landau. Nous proposons deux méthodes. La première est *l'algorithme de remise à l'échelle* (rescaling) proposé par Berger et Kohn en 1988, appliqué à des équations paraboliques satisfaisant une propriété d'*invariance d'échelle*. Cette propriété nous permet de faire un zoom de la solution quand elle est proche de la singularité, tout en gardant la même équation. Le principal avantage de cette méthode est sa capacité à donner une très bonne approximation numérique qui nous permet d'atteindre numériquement le profil à l'explosion. Le profil à l'explosion que l'on obtient numériquement est en bon accord avec le profil théorique. De plus, en considérant une équation de la chaleur non linéaire critique avec un terme de gradient non linéaire, avec peu de résultats théoriques, nous énonçons une conjecture sur le profil à l'explosion, grâce à nos simulations numériques. La deuxième méthode numérique s'appuie aussi sur un raffinement de maillage, dans l'esprit de *l'algorithme de remise à l'échelle* de Berger et Kohn. Cette méthode est applicable à une plus grande classe d'équations dont les solutions explosent en temps fini sans la propriété d'*invariance d'échelle*.

Mots clés : Équation semi-linéaire de la chaleur, perturbation d'ordre inférieur, singularité, explosion numérique, explosion en temps fini, profil, stabilité, comportement asymptotique, équation complexe de Ginzburg-Landau.

Abstract

We are interested in finite-time blow-up phenomena arising in the study of Nonlinear Parabolic Partial Differential Equations, in particular in the blow-up profile, under the theoretical and numerical aspects.

In the theoretical direction, we are interested in particular in finite-time blow-up phenomena for some class of strongly perturbed semilinear heat equations with Sobolev subcritical power nonlinearity:

$$\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u + h(u),$$

where $u : (x,t) \in \mathbb{R}^n \times [0,T) \to \mathbb{R}$, $p > 1$ and $p < \frac{n+2}{n-2}$ if $n \geq 3$, the function $h \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}^*)$ satisfies

$$j = 0, 1,\ |h^{(j)}(z)| \leq M\left(\frac{|z|^{p-j}}{\log^a(2+z^2)} + 1\right), \quad |h''(z)| \leq M\frac{|z|^{p-2}}{\log^a(2+z^2)},$$

where $M > 0$ and $a > 1$, or $h(z) = \dfrac{\mu|z|^{p-1}z}{\log^a(2+z^2)}$ with $\mu \in \mathbb{R}$ and $a > 0$.
Working in the framework of *similarity variables*, we first derive a Lyapunov functional in similarity variables which is a crucial step to derive the blow-up rate of the solution. In a second step, we are interested in the structure of the solution near blow-up time and point. We classify all possible asymptotic behaviors of the solution when it approaches to the singularity. Then we describe blow-up profiles corresponding to these asymptotic behaviors. In a third step, we construct for this equation a solution which blows up in finite time at only one blow-up point with a prescribed blow-up profile. The construction relies on the reduction of the problem to a finite dimensional one and the use of index theory to conclude.

In the numerical direction, we intend to develop methods in order to give numerical answers to the question of the blow-up profile for some parabolic equations including the Ginzburg-Landau model. We propose two methods. The first one is the *rescaling algorithm* proposed by Berger and Kohn in 1988 applied to parabolic equations which are invariant under a scaling transformation. This scaling property allows us to make a zoom of the solution when it is close to the singularity, still keeping the same equation. The main advantage of this method is its ability to give a very good numerical approximation allowing to attain the numerical blow-up profile. The blow-up profile we obtain numerically is in good accordance with the theoretical one. Moreover, by applying the method to a critical nonlinear heat equation with a nonlinear gradient term, where almost nothing is known, we give a conjecture for its blow-up profile thanks to our numerical simulations. The second one is a new mesh-refinement method inspired by the *rescaling algorithm* of Berger and Kohn, which is applicable to more general equations, in particular those with no scaling invariance.

Keyword: Semilinear heat equations, lower order perturbation, singularity, numerical blow-up, finite-time blow-up, profile, stability, asymptotic behavior, complex Ginzburg-Landau equation.

Contents

[1] submitted, arXiv:1403.7547

[2]submitted, arXiv:1404.4018
[3]submitted, arXiv:1406.5233
[4]submitted, arXiv:1410.4079

List of Figures

List of Tables

Chapter I

Introduction

This thesis addresses the problem of finite-time blow-up for parabolic equations from two different points of view. Accordingly it has two parts:

- a theoretical part which is devoted to the study of finite-time blow-up solutions of some class of strongly perturbed semilinear heat equations with Sobolev subcritical power nonlinearity. We focus on the answer to the questions of *classification* of general solutions and *construction* of solutions with some prescribed behavior, and try to see how the strong perturbation we consider affect the known results in the unperturbed case.

- a numerical part which is devoted to developing numerical methods in order to give numerical answers to the question of the blow-up profile for some parabolic equations including the Ginzburg-Landau model. We propose two methods: the first one is the *rescaling method* of Berger and Kohn [14] applied to parabolic equations which are invariant under a scaling transformation, and the second one is a new mesh-refinement method inspired by the *rescaling algorithm* of [14]. This method is applicable to more general equations, in particular those with no scaling invariance.

1 Theoretical study: Blow-up results for strongly perturbed semilinear heat equations

The theoretical objective of this thesis is devoted to the study of finite-time blow-up solutions for the following semilinear parabolic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} &= \Delta u + |u|^{p-1}u + h(u), & \text{in } \Omega \times \in [0,T) \\ u &= 0, & \text{on } \partial\Omega \times [0,T) \\ u(x,0) &= u_0(x), & \text{in } \Omega \end{cases} \tag{1.1}$$

where $u : (x,t) \in \Omega \times [0,T) \to \mathbb{R}$, $u_0 : x \in \Omega \to \mathbb{R}$, Ω is a bounded convex regular open set of $\mathbb{R}^n$ or $\Omega \equiv \mathbb{R}^n$, $T > 0$, Δ stands for the Laplacian in $\mathbb{R}^n$, $p > 1$ and $p < \frac{n+2}{n-2}$ if

$n \geq 3$, the function $h \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}^*)$ satisfies

$$j = 0, 1,\ |h^{(j)}(z)| \leq M \left(\frac{|z|^{p-j}}{\log^a(2+z^2)} + 1 \right), \quad |h''(z)| \leq M \frac{|z|^{p-2}}{\log^a(2+z^2)}, \tag{1.2}$$

where $M > 0$ and $a > 1$, or $h(z) = \dfrac{\mu |z|^{p-1} z}{\log^a(2+z^2)}$ with $\mu \in \mathbb{R}$ and $a > 0$.
Equation (1.1) arises in various physical problems such as the problem of heat flow, or more generally, the problems involving diffusion. It is a simple model for a large class of nonlinear parabolic equations, which are ubiquitous in mathematics and its applications. The subject of blow-up first arose with singularities in gas dynamics, the intense explosion problem, adiabatic explosion and combustion theory (see Zel′dovich, Barenblatt, Librovich and Makhviladze [111], Gel′fand [41], Barenblatt [9]). The two classical models include $u_t = \Delta u + e^u$ (Frank-Kamenetskii equation [35]) and $u_t = \Delta u + |u|^{p-1}u$ (see Zel′dovich, Barenblatt, Librovich and Makhviladze [111], Bebernes and Eberly [12], Bebernes, Bressan and Eberly [11], Bebernes and Kassoy [13], Kassoy and Poland [61], [62], Peral and Vázquez [88]). The problem (1.1) is also related to the Keller-Segel model for chemotaxis which is one of the most widely studied models in mathematical biology (see Keller and Segel [63], Tindall, Maini, Porter and Armitage [99], [100]). There is a rather extensive bibliography devoted to the subject of blow-up. We mention the surveys by Galaktionov and Vázquez [40], Bandle and Bruner [8], that contain all the main references on this subject as well as to the main applications.

The Cauchy problem for equation (1.1) can be solved in some functional space, $u(t) \in \mathcal{H}$. Frequent instances of $\mathcal{H}$ are $\mathcal{C}(\Omega) \cap L^\infty(\Omega)$, or the Lebesgue space $L^q(\Omega)$, or the Sobolev space $\mathcal{H} = H^1(\Omega)$ (see Henry [52], Pazy [87], Weissler [107], [106], Ribaud [92]). Moreover, on can show that either the solution $u(t)$ exists on $[0, +\infty)$ (global existence), or only on $[0, T)$ with $T < +\infty$ (local existence). In the second case, we say that $u(t)$ blows up in finite time T if $u(t)$ satisfies (1.1) in $\Omega \times [0, T)$ and

$$\|u(t)\|_{\mathcal{H}} \to +\infty \quad when \quad t \to T.$$

We call T the blow-up time of $u(t)$. In such a blow-up case, one can show that there is at least one blow-up point a which is defined as follows (see Friedman and McLeod [37]):

A point $\hat{a} \in \Omega$ is a blow-up point of u if $u(x,t)$ is not locally bounded in the neighborhood of $(\hat{a}, T)$, this means that there exists $(x_n, t_n) \to (\hat{a}, T)$ such that $|u(x_n, t_n)| \to +\infty$ when $n \to +\infty$.

The theoretical part of this thesis is devoted to the study of blow-up for equation (1.1). Two sets of questions arise in the study of blow-up:

- **Classification**: We consider an arbitrary blow-up solution. The first question concerns the rate of blow-up, that is whether we can calculate exactly the norms of the solution and its derivatives as t approaches the blow-up time and x approaches a blow-up point?

The second question is about the asymptotic behavior. Given $\hat{a}$ a blow-up point of $u(x,t)$, how can we find more information about the asymptotic behavior in the neighborhood of (a,T)? Can we define a notion of *blow-up profile*? Another aspect is to see whether $u(t)$ has a universal behavior as $t \to T$ (independent from initial data)? Is this universal behavior stable under perturbations of initial data?

- **Construction**: Are there solutions to equation (1.1) which blow up in finite time? Are there sufficient conditions on the initial data u_0 and the nonlinearity which imply blow-up in finite time? Can we construct a blow-up solution which blows up in finite time and verifies some prescribed asymptotic behavior?

In this thesis, we address these questions and will try to see how the perturbation we consider affects the results known in the unperturbed case $(h \equiv 0)$.

1.1 Blow-up results for the semilinear heat equation

In the study of blow-up phenomena for the problem (1.1), we have currently a fair understanding of the types of blow-up for the following semilinear heat equation with no perturbation ($h \equiv 0$):

$$\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u, \tag{1.3}$$

where $u : (x,t) \in \Omega \times [0,T) \to \mathbb{R}$, $p > 1$ or $p < \frac{n+2}{n-2}$ if $n \geq 3$.

There is a rich literature regarding the blow-up problem for equation (1.3), see for instance, the book by Souplet and Quittner [91] and the references therein. Here we only review some relevant results.

a) Existence of finite-time blow-up solutions:
There are various criteria for blow-up in finite time were derived. The earliest results on the existence of finite-time blow-up solutions appear in the 1960s with the works of Kaplan[60], Fujita [38], Friedman [36]. Some early results are due to Ball [7], Levine [65], Levine and Payne [66], [67], Hayakawa [51], Tsutsumi [101], Weissler [108], Fila and Filo [31], Palais [86], Mizoguchi and Yanagida[76], Mizoguchi, Ninomiya and Yanagida [75] and others. We present here some of theirs results. We start with a very simple criterion based on the Kaplan's first eigenvalue method [60]:

Let Ω bounded and $0 \leq u_0 \in L^\infty(\Omega)$. If $\int_\Omega u_0(x)\varphi_1(s)dx$ is sufficiently large, where φ_1 is the eigenfunction corresponding to the first eigenvalue of $-\Delta$ in $W_0^{1,2}(\Omega)$ and $\int_\Omega \varphi_1(x)dx = 1$, then the solution u of (1.3) *must blow up in finite time.*

The next blow-up criterion, introduced by Levine [65] (see also Levine and Payne [66], [67], Ishii [59] and Ball [7]), uses the concavity of an auxiliary function. This concavity method is actually powerful enough to be applied to many other types of second-order parabolic equations as well as other types of evolution equations. The following result is discussed in [65] and Hu [58], that gives a sufficient condition for the existence of finite-time blow-up solutions for equation (1.3) with the domain Ω either bounded or unbounded:

Let Ω be a smooth domain and $0 \not\equiv u_0 \in L^\infty \cap H_0^1(\Omega)$. If $E(u_0) < 0$, where

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \frac{1}{p+1}\int_\Omega |u(x)|^{p+1} dx \tag{1.4}$$

is the energy associated to (1.3), then the solution u of (1.3) must blow up in finite time.

Let us now present Fujita's result [38] (including Fujita's critical exponent) which is one of very first achievements on the study of blow-up:

Let $\Omega = \mathbb{R}^n$ and $u_0 \geq 0$. If $p > p_F := 1 + \frac{2}{n}$, then the solution of (1.3) is global in time, provided that the initial data u_0 satisfies for some $\epsilon > 0$ small,

$$u_0(x) \leq \epsilon G(x,1) \quad \textit{for} \quad x \in \mathbb{R}^n,$$

where $G(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$ is the Gaussian heat kernel.
If $p \leq p_F$, then all non trivial solutions of (1.3) blow up in finite time.

Other criteria have been obtained for non-negative blow-up solutions for (1.3) by using the comparison principle, since the positivity property is naturally supported by the Maximum Principle for such second-order parabolic equations (see for example, Lions [68], Fila [30]).

b) Blow-up rate:
We consider u a blow-up solution to equation (1.3) which blows up at time T. Here we are interested in the Type I blow-up, which is defined as follow (see Matano and Merle [70] for a statement):

We say that the blow-up is of Type I if the quantity $(T-t)^{\frac{1}{p-1}}\|u(t)\|_{L^\infty}$ remains bounded as $t \to T^-$. The blow-up is called Type II if it is not of Type I.

We plan to describe known results for the question: *Are there positive constants c and C such that*

$$c(T-t)^{-\frac{1}{p-1}} \leq \|u(t)\|_{L^\infty} \leq C(T-t)^{-\frac{1}{p-1}}, \quad \forall 0 < t < T? \tag{1.5}$$

Note that in the case of Type I blow-up, the blow-up rate of $\|u(t)\|_{L^\infty}$ is the same as that of the ODE associated to (1.3), namely

$$\frac{dv}{dt} = |v|^{p-1}v,$$

whose solution is

$$v(t) = \kappa(T-t)^{-\frac{1}{p-1}} \quad \text{where} \quad \kappa = (p-1)^{-\frac{1}{p-1}}, \tag{1.6}$$

and note also that the lower bound in (1.5) is always satisfied, in fact the constant $c = \kappa$ (see Weissler [107], Friedman and McLeod [37], Giga and Kohn [44]).
The upper bound in (1.5) has been completely established even in the nonradial case, at least when the domain Ω is convex. Some earlier results were obtained for special classes of solutions by Weissler [109], [108], Muller and Weissler [77]. Giga and Kohn [43], [44] established (1.5) under the assumption that either $u_0 \geq 0$ (so that the solution is positive everywhere) or $1 < p < \frac{3n+8}{3n-4}$. Later Giga, Matsui and Sasayama [45] ($\Omega = \mathbb{R}^n$), [46] (Ω is a convex domain in $\mathbb{R}^n$) proved (1.5) without such assumptions. All these results are obtained mainly by the energy methods. Galaktionov and Posashkov [39] also obtained the same results in one dimensional via a different method. Merle and Zaag [72], [74] refined the estimate (1.5) by proving that

$$(T-t)^{\frac{1}{p-1}}\|u(t)\|_{L^\infty} \to \kappa \quad \text{as} \quad t \to T.$$

(See also Fila and Souplet [32] and the references therein for similar results in this direction).
Let us mention that the proof of (1.5) by the energy method has been done through the introduction for each $\hat{a} \in \Omega$ ($\hat{a}$ may be a blow-up point of u or not) the following *similarity variables* (see [42], [43], [44]):

$$w_{\hat{a}}(y,s) = (T-t)^{\frac{1}{p-1}}u(x,t), \quad y = \frac{x-\hat{a}}{\sqrt{T-t}}, \quad s = -\log(T-t), \tag{1.7}$$

and $w_{\hat{a}}$ (or w for simplicity) solves a new parabolic equation in (y,s): for all $s \geq -\log T$ and $y \in D_{\hat{a},s}$ with $D_{\hat{a},s} = \{y \in \mathbb{R}^n | \hat{a} + ye^{-s/2} \in \Omega\}$,

$$\partial_s w = \frac{1}{\rho}\nabla \cdot (\rho \nabla w) - \frac{w}{p-1} + |w|^{p-1}w, \tag{1.8}$$

where

$$\rho(y) = \left(\frac{1}{4\pi}\right)^{n/2} e^{-\frac{|y|^2}{4}}. \tag{1.9}$$

In view of (1.7), the proof of the upper bound in (1.5) is equivalent to establishing a uniform bound for the global solution w of (1.8), that is

$$\|w(s)\|_{L^\infty} \leq C \quad \text{for all} \quad s \geq -\log T. \tag{1.10}$$

The proof written in [45] (see also [46]) is strongly based on the existence of the following Lyapunov functional:

$$\mathcal{E}_0[w](s) = \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla w|^2 + \frac{1}{2(p-1)}|w|^2 - \frac{1}{p+1}|w|^{p+1}\right)\rho dy. \tag{1.11}$$

Based on this functional, some energy estimates related to this structure and a bootstrap argument given in [90], the authors in [45] have established the following key integral estimate

$$\forall q \geq 2, \quad \sup_{s \geq s'} \int_s^{s+1} \|w(s)\|_{L^{p+1}(\mathbf{B}_R)}^{(p+1)q} ds \leq C_{q,s'} \quad \text{for} \quad s' > -\log T, \tag{1.12}$$

where $C_{q,s'}$ is independent of $\hat{a}$, and $\mathbf{B}_R$ is the open ball in $\mathbb{R}^n$ centered at zero with radius R.
Using the key estimate (1.12) together with the interpolation by Cazenave and Lions [19] and an interior regularity for linear parabolic equations, we then derive (1.10).

We would like to insist that the blow-up rate (1.5) is a fundamental step opening the doors to the study of the asymptotic behavior of blow-up solutions to problem (1.3), whose analysis has been initiated by Giga and Kohn in [42], Herrero and Velázquez in [53], [55], Velázquez in [103], [105], [104], Filippas and Kohn in [33], Filippas and Liu in [34] and others.

c) Asymptotic behavior:
In this paragraph, we plan to describe known results concerning the asymptotic behavior of blow-up solutions of problem (1.3) near blow-up points as t approaches the blow-up time. A fundamental tool for this study is the *similarity variables* introduced in (1.7). One can see from (1.7) that the study of $u(t)$ in the neighborhood of $(\hat{a}, T)$, where $\hat{a}$ is a blow-up point and T is the blow-up time, is equivalent to the study of the asymptotic behavior of $w_{\hat{a}}(s)$ as $s \to +\infty$. Assume $\Omega = \mathbb{R}^n$ (Ω can be a bounded convex domains, but we restrict ourselves to the case of the whole space for simplicity) and the estimate (1.5) is satisfied. We start with the following earliest result by Giga and Kohn [42], [44] (analysis in L^2_ρ, where L^2_ρ is the weighted L^2 space associated with the weight ρ defined in (1.9)):

If $\hat{a}$ is a blow-up point of u, then

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} u(\hat{a} + y\sqrt{T-t}, t) = \lim_{s \to +\infty} w_{\hat{a}}(y, s) = \pm\kappa, \tag{1.13}$$

uniformly on compact sets $|y| \leq R$, where κ is given in (1.6).

Note that the only bounded solutions of (1.8) which are independent of the time are the constant solutions: κ, $-\kappa$ and 0 (see Giga and Kohn [42]). The estimate (1.13) has been refined until the higher order by Filippas, Kohn and Liu [33], [34], Herrero and Velázquez [53], [55], [103], [105], [104]. More precisely, they classified the behavior of $w_{\hat{a}}(y, s)$ for $|y|$ bounded. They prove that one of the following cases occurs (assuming that $w_{\hat{a}} \to \kappa$, up to replacing u by $-u$):

- Case 1 (non-degenerate rate of blow-up): There exists $k \in \{0, \dots, n-1\}$ and a $n \times n$ orthonormal matrix Q such that

$$\forall R > 0, \quad \sup_{|y| \leq R} \left| w_{\hat{a}}(y,s) - \left[\kappa + \frac{\kappa}{2ps}\left((n-k) - \frac{1}{2} y^T A_k y \right) \right] \right| = \mathcal{O}\left(\frac{1}{s^{1+\delta}} \right) \tag{1.14}$$

as $s \to +\infty$, where $\delta > 0$,

$$A_k = Q \begin{pmatrix} I_{n-k} & 0 \\ 0 & 0 \end{pmatrix} Q^{-1}, \tag{1.15}$$

and I_{n-k} is the $(n-k) \times (n-k)$ identity matrix.

- Case 2 (degenerate rate of blow-up): There exists $\mu > 0$ such that

$$\forall R > 0, \quad \sup_{|y| \leq R} |w_{\hat{a}}(y,s) - \kappa| \leq C(R) e^{-\mu s}, \tag{1.16}$$

(this exponential convergence has been refined up to the order 1 by Herrero and Velázquez but we omit that description since we choose in this thesis to concentrate on the non-degenerate rate of blow-up mentioned in the case 1 above).

Back to the original variables $u(x,t)$, this yields information on the blow-up behavior in a small space range of the form $|x - \hat{a}| \leq R\sqrt{T-t}$. In some sense and from a physical point of view, these results do not show the transition between the singular zone ($w_{\hat{a}} > \alpha$ where $\alpha > 0$) and the regular one ($w_{\hat{a}} \simeq 0$) well. Indeed, the convergence is only uniform in the parabolic domain $|x - \hat{a}| \leq R\sqrt{T-t}$ which does not allow us to derive a blow-up profile of $u(t)$ in the original variable (x,t). Velázquez [103] (see also [55]) extended the $|y|$ bounded convergence in (1.14) to the larger set $|y| \leq R\sqrt{s}$, by estimating the effect of the convective term $\frac{1}{2} y \cdot \nabla w_{\hat{a}}$ in the equation (1.8) in L^p spaces with Gaussian measure (note that $\frac{1}{\rho}\nabla \cdot (\rho \nabla w) = \Delta w - \frac{1}{2} y \cdot \nabla w$). However, the convergence (1.17) that Velázquez obtained depends strongly on the considered blow-up point $\hat{a}$, and it is not uniform with respect to $\hat{a}$. Merle and Zaag [73] obtained a related profile existence result, but their result has been shown to be independent of $\hat{a}$ thanks to a compactness property on $w_{\hat{a}}$ uniformly with respect to $\hat{a} \in \mathbb{R}^n$. Particularly, they established the following blow-up profile in the variable $z = \frac{y}{\sqrt{s}}$ (which is the intermediate scale that separates the regular and singular parts in the non-degenerate case):

There exists $k \in \{0, \dots, n\}$ such that

$$\forall R > 0, \quad \sup_{|z| \leq R} \left| w_{\hat{a}}(z\sqrt{s}, s) - f_k(z) \right| \to 0 \quad \text{as } s \to +\infty, \tag{1.17}$$

where

$$f_k(z) = \kappa \left(1 + \frac{p-1}{4p} z^T A_k z \right)^{-\frac{1}{p-1}}, \tag{1.18}$$

and A_k is defined in (1.15).

Following the *classification* of Herrero and Velázquez, Bricmont and Kupiainen showed in [16] (see also [15] and [17]) the existence of initial data for (1.3) such that

$$\sup_{x\in\mathbb{R}^n}\left|(T-t)^{\frac{1}{p-1}}u(x,t)-f_0\left(\frac{x-\hat{a}}{\sqrt{(T-t)|\log(T-t)|}}\right)\right|\to 0 \quad \text{as } t\to T, \tag{1.19}$$

where f_0 is defined in (1.18).
More specifically, given a small function g, they find constants $d_0\in\mathbb{R}$ and $d_1\in\mathbb{R}^n$ such that for each $\hat{a}\in\mathbb{R}^n$, the solution of (1.3) with the datum

$$u_{0,d_0,d_1}(x)=T^{-\frac{1}{p-1}}\left\{f_0(z)\left(1+\frac{d_0+d_1 z}{p-1+\frac{(p-1)^2}{4p}|z|^2}\right)+g(z)\right\},$$

$$z=(x-\hat{a})\sqrt{T|\log T|},$$

has the convergence (1.19). This result was proved again by Merle and Zaag in [71] by reducing the problem to a finite-dimensional one (see also Zaag [110], Masmoudi and Zaag [69], Ebde and Zaag [24], Nouaili and Zaag [85] for further results in this direction). More importantly, the method of [71] allows to derive the stability of the blow-up behavior (1.19) with respect to perturbations in the initial data or the nonlinearity (see also Fermanian, Merle and Zaag [25], [26] for other proofs of stability). This result also opens to the notion of the limiting profile in the $u(x,t)$ variable, in sense that $u(x,t)\to u^*(x)$ when $t\to T$ if $x\neq\hat{a}$ and x is the neighborhood of $\hat{a}$, with

$$u^*(x)\sim\left[\frac{8p|\log|x-\hat{a}||}{(p-1)^2|x-\hat{a}|^2}\right]^{\frac{1}{p-1}} \quad \text{as } x\to\hat{a}. \tag{1.20}$$

Let us mention the following uniform localization estimate by Merle and Zaag [72], [74]:

$\forall\epsilon>0$, $\exists C_\epsilon>0$ *such that* $\forall t\in\left[\frac{T}{2},T\right)$, $\forall x\in\mathbb{R}^n$,

$$|\Delta u|=\left|\partial_t u-|u|^{p-1}u\right|\leq\epsilon|u|^p+C_\epsilon.$$

This result together with the stability of the profile (1.19) strongly rely on the following Liouville theorem for equation (1.3) proved by Merle and Zaag (see [72] and [74]):

Assume that w is a solution in L^∞ of (1.8) *defined on $\mathbb{R}^n\times\mathbb{R}$. Then $w\equiv 0$ or $w\equiv\pm\kappa$ or $w(y,s)=\pm\theta(s-s_0)$ for some $s_0\in\mathbb{R}$, where $\theta(s)=\kappa(1+e^s)^{-\frac{1}{p-1}}$.*

This result has an equivalent formulation for solution of (1.3) via the transformation (1.7):

Assume that u *is a solution in* L^∞ *of* (1.3) *defined for* $(x,t) \in \mathbb{R}^n \times (-\infty, T)$. *Assume in addition that* $|u(x,t)| \le C(T-t)^{-\frac{1}{p-1}}$. *Then* $u \equiv 0$ *or there exist* $T_0 \ge T$ *and* $\omega \in \{-1, 1\}$ *such that for all* $(x,t) \in \mathbb{R}^n \times (-\infty, T)$, $u(x,t) = \kappa(T_0 - t)^{-\frac{1}{p-1}}\omega$.

1.2 Blow-up results for some class of weak perturbations of the semilinear heat equation

In this section, we aim at recalling some blow-up results of the following semilinear heat equation:

$$\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u + h(u, \nabla u), \tag{1.21}$$

where $u : (x,t) \in \Omega \times [0,T) \to \mathbb{R}$, $p > 1$ or $p < \frac{n+2}{n-2}$ if $n \ge 3$, and the perturbation $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies

$$|h(u,v)| \le M(1 + |u|^q + |v|^r), \quad M > 0,\ 0 \le q < p,\ 0 \le r < \frac{2p}{p+1}.$$

In some sense, the term $h(u, \nabla u)$ has a subcritical size when $q < p$ and $r < \frac{2p}{p+1}$. In the *selfsimilar variables* framework introduced in (1.7), we see that w solves the following equation:

$$\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{p-1} w + |w|^{p-1}w + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}} w, e^{\frac{(p+1)s}{2(p-1)}} \nabla w\right), \tag{1.22}$$

where the perturbation term satisfies

$$\left| e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}} w, e^{\frac{(p+1)s}{2(p-1)}} \nabla w\right)\right| \le Ce^{-\delta s}(|w|^q + |\nabla w|^r + 1),$$

with $\delta = \min\left\{\frac{p-q}{p-1}, \frac{2p - r(p+1)}{2(p-1)}\right\} > 0$. This is the reason why we say that equation (1.21) is a "weak" perturbed version of (1.3), and justify why our perturbation in (1.2) is called "strong".

In [43] and [44], Giga and Kohn generalized the results (1.5) for equation (1.3) to (1.21) (hence (1.13)) in the case where the function $h(u,v) \equiv h(u)$ with $1 \le q < p$ (h does not depend on ∇u), provided that either $u_0 \ge 0$ or $1 < p < \frac{3n+8}{3n-4}$. The proof of (1.5) is based on the energy method where the energy has an extra term due to h:

$$J[w](s) = \mathcal{E}_0[w](s) + e^{-\frac{(p+1)s}{p-1}} \int H\left(e^{\frac{ps}{p-1}} w\right) \rho dy,$$

where $\mathcal{E}_0$ as in (1.11), ρ is defined in (1.9), and $H(z) = \int_0^z h(\xi)d\xi$ (note that J is not a Lyapunov functional for equation (1.22)).
Souplet and Tayachi [97] proved (1.5) by considering radial positive solutions of (1.21) in a ball or in $\mathbb{R}^n$ under certain assumptions on the parameters, especially in the cases where r is subcritial, critical ($r = \frac{2p}{p+1}$) and supercritical.

Ebde and Zaag [24] constructed a blowup solution for equation (1.22) with a prescribed blowup profile by the method of [71] (see also Bricmont and Kupiainen [16]). More precisely, they showed the existence of initial data for (1.22) such that

$$\left\| (T-t)^{\frac{1}{p-1}} u(\cdot,t) - f_0\left(\frac{\cdot - \hat{a}}{\sqrt{(T-t)|\log(T-t)|}}\right)\right\|_{W^{1,\infty}} \leq \frac{C}{\sqrt{|\log(T-t)|}}, \tag{1.23}$$

where f_0 is defined in (1.18).
Since the presence of the perturbation including a nonlinear gradient term, their proof needs some involved arguments to control the $h(u, \nabla u)$ term, and the convergence in $W^{1,\infty}$ come from a parabolic regularity estimate for equation (1.22).

1.3 Blow-up results for some class of strong perturbations of the semilinear heat equation

This section is devoted to the description of the main results in the theoretical direction concerning the study of blow-up solutions of the following nonlinear parabolic equation:

$$\begin{cases} u_t &= \Delta u + |u|^{p-1}u + h(u), \\ u(0) &= u_0 \in L^\infty(\mathbb{R}^n), \end{cases} \tag{1.24}$$

where u is defined for $(x,t) \in \mathbb{R}^n \times [0,T)$, p is a subcritical nonlinearity,

$$p > 1 \text{ and } p < \frac{n+2}{n-2} \text{ if } n \geq 3. \tag{1.25}$$

The function h is in $\mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}^*)$ satisfying

$$j = 0, 1,\ |h^{(j)}(z)| \leq M\left(\frac{|z|^{p-j}}{\log^a(2+z^2)} + 1\right), \quad |h''(z)| \leq M\frac{|z|^{p-2}}{\log^a(2+z^2)}, \tag{1.26}$$

where $a > 1$ and $M > 0$, or,

$$h(z) = \frac{\mu|z|^{p-1}z}{\log^a(2+z^2)} \quad \text{with } \mu \in \mathbb{R},\ a > 0. \tag{1.27}$$

Consider u a solution of (1.24) blowing up in some finite time T. The study of u has been done via the *similarity variables* (1.7). Then the problem is converted to the study of the long-time behavior of $w_{\hat{a}}$ for $\hat{a} \in \mathbb{R}^n$. The equation in $w_{\hat{a}}$ (or w for simplicity) is as follows: for all $(y,s) \in \mathbb{R}^n \times [-\log T, +\infty)$,

$$\partial_s w = \frac{1}{\rho}\nabla \cdot (\rho \nabla w) - \frac{w}{p-1} + |w|^{p-1}w + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}} w\right), \tag{1.28}$$

where ρ is defined in (1.9).
Since the perturbation term in equation (1.28) satisfies

$$e^{-\frac{ps}{p-1}} \left| h\left(e^{\frac{s}{p-1}} z\right)\right| \leq \frac{C_0}{s^a}\left(|z|^p + 1\right), \quad \forall z \in \mathbb{R}, \forall s \geq s_0,$$

for some $C_0 > 0$ and $s_0 > 0$, we wonder whether a perturbation of the methods devoted to the study of blow-up solutions of equation (1.3) would work for our problem. In other words, do the results stated for the unperturbed case ($h \equiv 0$) in the previous subsection hold for our perturbed case? We propose to develop three directions in the first part of this thesis:

- In the first direction, we propose a perturbation of the method of [45] in order to derive the blow-up rate for solutions of (1.24). As in [45] (see also [46]), the crucial step is to derive the existence of a Lyapunov functional for equation (1.28). Following the method introduced by Hamza and Zaag in [50] and [49] for perturbations of the semilinear wave equation, we first prove the existence of a Lyapunov functional for equation (1.24). Thanks to energy estimates based on this Lyapunov functional and a blow-up criterion for equation (1.28), we are able to adapt the analysis in [45] for equation (1.3) to obtain the blow-up rate for solutions of equation (1.24).

- In the second direction, we are interested in the *classification* of all possible asymptotic behaviors of the blow-up solution $u(t)$ of (1.24) near blow-up points as t approaches to the blow-up time. Because of the presence of the strong perturbation, we need new crucial ideas to get rid of the effect of this term on the structure of the solution. A key successful step is the linearization of the bounded solution of (1.28) around an implicit profile function which is the solution of the ODE associated to equation (1.28).

- The third direction is devoted to the *construction* of a solution $u(x,t)$ for equation (1.24) which blows up in finite time T at only one blow-up point $\hat{a} \in \mathbb{R}^n$ and converges to some prescribed blow-up profile as $t \to T$. Since the perturbation term h certainly impacts the construction of solutions of equation (1.24), we need crucial modifications in the method of Merle and Zaag [71] in order to totally control this term. This construction is based on *a priori* estimates' technique presented in [71] and based on the estimates of Bricmont and Kupiainen [16], which reduces the problem to a finite-dimensional one. This method also allows us to derive a stability of the blow-up profile under perturbations of initial data.

a) Existence of a Lyapunov functional and blow-up rate:
Following the method introduced by Hamza and Zaag in [50], [49] for perturbations of the semilinear wave equation (see also Giga and Kohn [43]), we introduce for each $\alpha > 0$ the following functional:

$$\mathcal{J}_\alpha[w](s) = \mathcal{E}[w](s)e^{\frac{\gamma}{\alpha}s^{-\alpha}} + \theta s^{-\alpha}, \tag{1.29}$$

where $\gamma = \gamma(\alpha, p, n, \mu, M) > 0$ and $\theta = \theta(\alpha, p, n, \mu, M) > 0$ are sufficiently large constants,

$$\mathcal{E}[w] = \mathcal{E}_0[w] + \mathcal{I}[w], \quad \mathcal{I}[w](s) = -e^{-\frac{p+1}{p-1}s} \int_{\mathbb{R}^n} H\left(e^{\frac{s}{p-1}} w\right) \rho dy, \tag{1.30}$$

where $\mathcal{E}_0$ is defined in (1.11) and $H(z) = \int_0^z h(\xi) d\xi$.
With this introduction, we derive that the functionals $\mathcal{J}_{a-1}[w](s)$ and $\mathcal{J}_a[w](s)$ are decreasing functions of time for equation (1.28) in the cases (1.26) and (1.27) respectively, provided that s is large enough. More precisely, we have established the following result in [82] and [83] (see Theorem 1.1 page 88 and Theorem 1.1 page 176):

Theorem 1.1 (Existence of a Lyapunov functional for equation (1.28)). *Let a, p, n, μ, M be fixed, consider w a solution of equation (1.28). Then there exist $\hat{s}_0 = \hat{s}_0(a, p, n, \mu, M) \geq s_0$ and $\hat{\theta}_0 = \hat{\theta}_0(a, p, n, \mu, M)$ such that if $\theta \geq \hat{\theta}_0$, then for all $s_2 > s_1 \geq \max\{\hat{s}_0, -\log T\}$,*

$$\mathcal{J}_\alpha[w](s_2) - \mathcal{J}_\alpha[w](s_1) \leq -\frac{1}{2} \int_{s_1}^{s_2} \int_{\mathbb{R}^n} (\partial_s w)^2 \rho dy ds, \tag{1.31}$$

where $\alpha = a - 1 > 0$ in the case (1.26) and $\alpha = a > 0$ in the case (1.27).

Remark 1.2. *The proof of Theorem 1.1 mainly relies on the following observation (see Lemm 2.1 page 93 and Lemm 2.2 page 181):*

$$\frac{d}{ds}\mathcal{E}[w](s) \leq -\frac{1}{2} \int_{\mathbb{R}^n} w_s^2 \rho dy + \gamma s^{-(\alpha+1)} \mathcal{E}[w](s) + C s^{-(\alpha+1)}.$$

While in the more general case (1.26) for the perturbation h, we restrict to the range $a > 1$, taking the more specific form (1.27) allows us to overcome technical difficulties in order to derive (1.31) for any $a > 0$.

As mentioned earlier, the existence of this Lyapunov functional $\mathcal{J}_\alpha$ is a crucial step in the derivation of the blow-up rate for equation (1.24). Indeed, with the Lyapunov functional $\mathcal{J}_\alpha$ and some more work, we are able to adapt the analysis in [45] for equation (1.3) and get the following result in [82] and [83] (see Theorem 1.2 page 88 and i) of Theorem 1.2 page 176):

Theorem 1.3 (Blow-up rate for equation (1.24)). *Let a, p, n, μ, M be fixed, p satisfy (1.25). There exists $\hat{s}_1 = \hat{s}_1(a, p, n, \mu, M) \geq \hat{s}_0$ such that if u is a blow-up solution of equation (1.24) with a blow-up time T, then*
(i) for all $s \geq s' = \max\{\hat{s}_1, -\log T\}$,

$$\|w_{\hat{a}}(s)\|_{L^\infty(\mathbb{R}^n)} \leq C, \tag{1.32}$$

where $w_{\hat{a}}$ is defined in (1.7) and C is a positive constant depending only on n, p, μ, M and a bound of $\|w_{\hat{a}}(\hat{s}_0)\|_{L^\infty}$.
(ii) For all $t \in [t_1, T)$ where $t_1 = T - e^{-s'}$,

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(T-t)^{-\frac{1}{p-1}}. \tag{1.33}$$

The proof of Theorem 1.3 is far from being a straightforward adaptation of [45]. Indeed, three major difficulties arise in our case and make the heart of our contribution:
- the existence of a Lyapunov functional in the *similarity variables* (see Theorem 1.1 above),
- the control of the L^2-norm in terms of the energy (see (ii) of Proposition 2.3 page 97), where we rely on a new blow-up criterion greatly simplifying the approach in [43] (see Lemma 2.2 page 96):
Let w be a solution of equation (1.28)*, if there exists $\tilde{s}_1 \geq \max\{\hat{s}_0, -\log T\}$ such that*

$$-4\mathcal{J}_\alpha[w](\bar{s}) + \frac{p-1}{p+1}\left(\int_{\mathbb{R}^n} |w(y,\bar{s})|^2 \rho dy\right)^{\frac{p+1}{2}} > 0 \quad \textit{for some } \bar{s} \geq \tilde{s}_1, \tag{1.34}$$

then w is not defined for all $(y,s) \in \mathbb{R}^n \times [\bar{s}, +\infty)$.
- the proof of a nonlinear parabolic regularity result (see Proposition 2.7 page 100).

b) Asymptotic behavior:
Thanks to the boundedness of the solution in the *similarity variables* (1.7) and the technique of Giga and Kohn [44], [42], we have derived an analogous result on the behavior of $w_{\hat{a}}(y,s)$ cited in (1.13), where $\hat{a}$ is a blow-up point, as $s \to +\infty$. More precisely (see Theorem 1.4 page 89 and $ii)$ of Theorem 1.2 page 176),

Theorem 1.4 (Behavior of $w_{\hat{a}}$ as $s \to +\infty$). *Let a, p, n, μ, M be fixed, p satisfy* (1.2)*. Consider $u(t)$ a solution of equation* (1.24) *which blows up at time T and $\hat{a}$ a blow-up point. Then*

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}} u(\hat{a} + y\sqrt{T-t}, t) = \lim_{s\to+\infty} w_{\hat{a}}(y,s) = \pm\kappa,$$

holds in L^2_ρ, and uniformly on compact subsets of $\mathbb{R}^n$.

Consider $\hat{a}$ a blow-up point of u, we may assume that $w_{\hat{a}} \to \kappa$ in L^2_ρ as $s \to +\infty$ (by changing u_0 in $-u_0$ and h in $-h$). By linearizing $w_{\hat{a}}$ around ϕ, where ϕ is the positive solution of the following ODE associated to equation (1.28):

$$\phi_s = -\frac{\phi}{p-1} + \phi^p + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}}\phi\right) \tag{1.35}$$

such that

$$\phi(s) \to \kappa \quad \text{as} \quad s \to +\infty, \tag{1.36}$$

(see Lemma A.4 page 125 for the existence of ϕ) we have classified all possible asymptotic behavior of $w_{\hat{a}}(s)$ as $s \to +\infty$ in [82] and [83] (see Theorem 1.5 page 90 and Theorem 1.4 page 178):

Theorem 1.5 (Classification of the behavior of $w_{\hat{a}}$ as $s \to +\infty$). *Consider $u(t)$ a solution of equation* (1.24) *which blows up in the finite time T and $\hat{a}$ a blow-up point. Let $w_{\hat{a}}(y,s)$ be a solution of equation* (1.28)*. Then only one of the following possibilities*

occurs:
$i)$ $w_{\hat{a}}(y,s) \equiv \phi(s)$.
$ii)$ *There exists* $l \in \{1,\dots,n\}$ *such that up to an orthogonal transformation of coordinates,*
- if h *is given by* (1.26),

$$w_{\hat{a}}(y,s) = \kappa - \frac{\kappa}{4ps}\left(\sum_{j=1}^{l} y_j^2 - 2l\right) + \mathcal{O}\left(\frac{1}{s^a}\right) + \mathcal{O}\left(\frac{\log s}{s^2}\right) \quad \textit{as} \quad s \to +\infty, \tag{1.37}$$

- if h *is given by* (1.27),

$$w_{\hat{a}}(y,s) = \phi(s) - \frac{\kappa}{4ps}\left(\sum_{j=1}^{l} y_j^2 - 2l\right) + \mathcal{O}\left(\frac{1}{s^{a+1}}\right) + \mathcal{O}\left(\frac{\log s}{s^2}\right) \quad \textit{as} \quad s \to +\infty. \tag{1.38}$$

$iii)$ *There exist an integer number* $m \geq 3$ *and constants* c_β *not all zero such that*

$$w_{\hat{a}}(y,s) = \phi(s) - e^{-\left(\frac{m}{2}-1\right)s} \sum_{|\beta|=m} c_\beta H_\beta(y) + o\left(e^{-\left(\frac{m}{2}-1\right)s}\right) \quad \textit{as} \quad s \to +\infty, \tag{1.39}$$

where $H_\beta(y) = h_{\beta_1}(y_1)\dots h_{\beta_n}(y_n)$ *with*

$$h_k(y) = \sum_{i=0}^{\left[\frac{k}{2}\right]} \frac{k!}{i!(k-2i)!}(-1)^i y^{k-2i}.$$

The convergence takes place in L^2_ρ *as well as in* $\mathcal{C}^{k,\gamma}_{loc}$ *for any* $k \geq 1$ *and* $\gamma \in (0,1)$.

Remark 1.6. *Unlike in [34] and [104], we do not linearize* $w_{\hat{a}}$ *around* κ*, which is an explicit profile. We instead linearize* $w_{\hat{a}}$ *around* ϕ*, which is the key successful step in deriving this result. Indeed, if we linearize* $w_{\hat{a}}$ *around* κ*, we then fall in logarithmic scales* $\gamma = \frac{1}{|\log \epsilon|}$ *with* $\epsilon = T - t$*. Further refinements in this direction should give an expansion of* $w - \kappa$ *in terms of powers of* γ*, i.e in logarithmic scales of* ϵ*. Therefore, we can not reach significantly small error terms in the expansion of the solution* $w_{\hat{a}}$ *as Theorem 1.5 describes. In order to escape this situation, a relevant approximation is required in order to go beyond all logarithmic scales, i.e approximations up to lower order terms such as* ϵ^α *for some* $\alpha > 0$*. Our idea to capture such relevant terms is to abandon the explicit profile obtained as a first order approximation, namely* κ*, and take an implicit profile function as a first order description of the singular behavior, namely* $\phi(s)$ *introduced in* (1.35) *and* (1.36)*.*

Remark 1.7. *If we linearize* $w_{\hat{a}}$ *around* κ*, then we see from Theorem 1.5 that* $\|w_{\hat{a}}(s) - \kappa\|_{L^2_\rho} \sim \frac{1}{s^{a'}}$ *with* $a' = \min\{a,1\}$ *in the cases* $i), ii)$*, and* $a' = a$ *in the case* $iii)$*. This is contrast with the unperturbed case (*$h \equiv 0$*) where three possibilities arrive (either* $w_{\hat{a}} - \kappa \equiv 0$*, or* $\|w_{\hat{a}}(s) - \kappa\|_{L^2_\rho} \sim \frac{1}{s}$ *or* $\|w_{\hat{a}}(s) - \kappa\|_{L^2_\rho} \leq Ce^{-\lambda s}$ *with* $\lambda > 0$*). This fact makes* *the originality of our approach in this thesis, since the perturbation suppresses two of the behaviors which are available in the non-perturbed case. Of course, as we said in the previous remark, linearizing around κ is not the good choice, and one has to linearize around $\phi(s)$ to capture exponentially small terms.*

Remark 1.8. *Expanding $\phi(s)$ in a power series of $\frac{1}{s^a}$ in (1.39), namely $\phi(s) = \kappa + \sum_{i=1}^{k} \frac{C_i}{s^{ia}} + \mathcal{O}\left(\frac{1}{s^{(k+1)a}}\right)$, shows two relevant scales in the expansion of $w_{\hat{a}}$: a slow scale $\frac{1}{s^a}$ and a fast scale $e^{-(m/2-1)s}$. This also another originality of our approach.*

Using the result obtained in Theorem 1.5 and the method of Velázquez [104], we can extend the asymptotic behavior of $w_{\hat{a}}$ to larger regions. More precisely, the following result was established in [82] and [83] (Theorem 1.10 page 92 and Theorem 1.7 page 178):

Theorem 1.9 (Convergence extension of $w_{\hat{a}}$ to larger regions). *For all $K_0 > 0$,*
$i)$ if $ii)$ of Theorem 1.5 occurs, then

$$\sup_{|\xi|\leq K_0} \left|w_{\hat{a}}(\xi\sqrt{s}, s) - f_l(\xi)\right| = \mathcal{O}\left(\frac{1}{s^{\alpha}}\right) + \mathcal{O}\left(\frac{\log s}{s}\right) \quad \text{as } s \to +\infty, \tag{1.40}$$

where $\alpha = a - 1 > 0$ in the case (1.26), $\alpha = a > 0$ in the case (1.27), and

$$f_l(\xi) = \kappa \left(1 + \frac{p-1}{4p}\sum_{j=1}^{l} \xi_j^2\right)^{-\frac{1}{p-1}}, \quad \forall \xi \in \mathbb{R}^n,$$

with l the same as in $ii)$ of Theorem 1.5.
$ii)$ if $iii)$ of Theorem 1.5 occurs, then $m \geq 4$ is even, and

$$\sup_{|\xi|\leq K_0} \left|w_{\hat{a}}\left(\xi e^{(\frac{1}{2}-\frac{1}{m})s}, s\right) - \psi_m(\xi)\right| \to 0 \quad \text{as } s \to +\infty, \tag{1.41}$$

where

$$\psi_m(\xi) = \kappa \left(1 + \kappa^{-p}\sum_{|\beta|=m} c_\beta \xi^\beta\right)^{-\frac{1}{p-1}}, \quad \forall \xi \in \mathbb{R}^n, \tag{1.42}$$

with c_β the same as in Theorem 1.5.

Remark 1.10. *As in the unperturbed case ($h \equiv 0$), we expect that (1.40) is stable and (1.41) should correspond to unstable behaviors (the unstability of (1.41) was proved only in one space dimension by Herrero and Velázquez in [54] and [56] for $h \equiv 0$). While remarking numerical simulations for equation (1.1) in one space dimension, we see that the numerical solutions exhibit only the behavior (1.40), we could never obtain the behavior (1.41). This is probably due to the fact that the behavior (1.41) is unstable.*

Remark 1.11. *Unlike in Theorem 1.5 where the convergence is uniform only in compact sets, and where two time-scales coexist, in particular in* (1.42) *(see Remark 1.8 above), here, the effect of* $\phi(s)$ *disappears in some sense since we work at the order* $o(1)$ *in* (1.41) *and* $\phi(s) = \kappa + o(1)$. *Thus, in the larger zones covered by Theorem 1.9, we recover the same profiles as in the unperturbed case treated by Herrero and Velázquez in [54] and [103]. However, in the derivation of Theorem 1.9 from Theorem 1.5 by the method of Velázquez [103], we need new ideas to get rid of the term in the scale* $\frac{1}{s}$ *coming from the strong perturbation. The key successful step is to linearize* $w_{\hat{a}}$ *around a sharpest profile which is of the form* $\frac{\phi(s)}{\kappa}\varphi(y,s)$, *where* $\varphi(y,s)$ *is the function used for the unperturbed case treated in [103] (see Section 4 of Chapter III and Section 3 of Chapter V for more details).*

c) Construction of a stable blow-up solution:
We are interested in the construction of a solution for (1.24) which blows up in finite time with a prescribed blow-up profile. In [84], the following result was obtained (Theorem 1.1 page 141):

Theorem 1.12 (Existence of a blow-up solution for equation (1.1) **with the description of its profile).** *There exists* $T > 0$ *such that equation* (1.24) *has a solution* $u(x,t)$ *in* $\mathbb{R}^n \times [0,T)$ *satisfying:*
$i)$ *the solution* u *blows up in finite-time* T *at only one blow-up point* $\hat{a} = 0$,
$ii)$

$$\left\|(T-t)^{\frac{1}{p-1}}u(\cdot\sqrt{T-t},t) - f_0\left(\frac{\cdot}{\sqrt{|\log(T-t)|}}\right)\right\|_{W^{1,\infty}(\mathbb{R}^n)} \leq \frac{C}{|\log(T-t)|^{\varrho}}, \qquad (1.43)$$

for all $\varrho \in (0,\nu)$ *with* $\nu = \min\{a-1, \frac{1}{2}\}$ *in the case* (1.26) *and* $\nu = \min\{a, \frac{1}{2}\}$ *in the case* (1.27), C *is some positive constant and* f_0 *is defined in* (1.18).
$iii)$ *There exists* $u_* \in \mathcal{C}(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ *such that* $u(x,t) \to u_*(x)$ *as* $t \to T$ *uniformly on compact subsets of* $\mathbb{R}^n \setminus \{0\}$, *where*

$$u_*(x) \sim \left(\frac{8p|\log|x||}{(p-1)^2|x|^2}\right)^{\frac{1}{p-1}} \quad \text{as} \quad x \to 0.$$

Remark 1.13. *We do not need the condition on the second derivative of* h *in the case* (1.26) *for the proof of Theorem 1.12.*

The objective of Theorem 1.12 deals with the construction of the initial data u_0 of (1.24) such that (1.43) is satisfied. Its proof is based on techniques developed by Bricmont and Kupiainen in [16] and Merle and Zaag in [71] for equation (1.3). Although our result is analogous as in [71], its proof is far from being a straightforward adaptation of the proof written in [71]. Because the perturbation term h certainly impacts on the construction of solutions of (1.24) satisfying (1.43), we need some crucial modifications

in [71] in order to totally control the term h. Although these modifications do not affect the general framework developed in [71], they lay in 3 crucial places:
- We modify the profile around which we study equation (1.28), so that we go beyond the order $\frac{1}{s^a}$ generated by the perturbation term. Indeed, for small $a > 0$ and with the same profile as in [71], the order $\frac{1}{s^a}$ will become too strong and will not allow us to close our estimates. See Section 2 of Chapter III, particularly, see the definition of φ given in (2.1) page 143, which enables us to reach the order $\frac{1}{s^{a+1}}$.
- In order to handle the order $\frac{1}{s^{a+1}}$, we need to modify the definition of the shrinking set near the profile. See Section 3 of Chapter III, and particularly Proposition 3.1 page 146.
- A sharp understanding of the dynamics of the linearized operator of (1.28) around the profile φ, which allows to handle the new definition of the shrinking set. See Lemma 3.5 page 149.

The proof of Theorem 1.12 is divided into 2 steps:
- In the first step, we reduce the problem to a finite-dimensional problem. We show that it is enough to control a finite-dimensional variable in order to control the solution near the profile.
- In the second step, we proceed by contradiction to solve the finite-dimensional problem and conclude using index theory.

As in [71], the method of reduction to the finite-dimensional problem for the proof of Theorem 1.12 allows us to derive the stability of the profile f_0 (1.43) with respect to perturbations in the initial data. More precisely, we have the following (Theorem 1.4 page 143):

Theorem 1.14 (**Stability of the solution constructed in Theorem 1.1**). *Let us denote by $\hat{u}(x,t)$ the solution constructed in Theorem 1.12 and by $\hat{T}$ its blow-up time. Then, there exists a neighborhood $\mathcal{V}_0$ of $\hat{u}(x,0)$ in $W^{1,\infty}$ such that for any $u_0 \in \mathcal{V}_0$, equation* (1.24) *has a unique solution $u(x,t)$ with initial data u_0, and $u(x,t)$ blows up in finite time $T(u_0)$ at one single blow-up point $\hat{a}(u_0)$. Moreover, estimate* (1.43) *is satisfied by $u(x-\hat{a},t)$ and*

$$T(u_0) \to \hat{T}, \quad \hat{a}(u_0) \to 0 \quad \text{as } u_0 \to \hat{u}_0 \text{ in } W^{1,\infty}(\mathbb{R}^n).$$

Remark 1.15. *Note that from a parabolic regularity, our stability result also holds in the larger space $L^\infty(\mathbb{R}^n)$.*

2 Numerical study of finite-time blow-up arising in models of nonlinear evolution equations

The numerical study of the blow-up phenomenon was not as advanced as the theoretical study, in particular when it comes to deriving the numerical blow-up profile. Many of the theoretical blow-up results were obtained with no numerical results obtained before.

This lack of numerical results is due to the limitations consisting of three parts: the non-linearity, the unboundedness of the solution, and the multidimentionality of the physical domain. The delicate problem is how we can obtain a good numerical solution which attains the blow-up profile? Thanks to the recent theoretical results, we have serious hints towards the simulation of such delicate behaviors. In this thesis, we are expected to develop numerical methods in order to give numerical answers to the question of the blow-up profile for some parabolic equations. We propose two methods:
- the first one is the *rescaling algorithm* of Berger and Kohn [14] applied to parabolic equations which are invariant under a scaling transformation,
- the second one is a new mesh-refinement method inspired by the *rescaling algorithm* of Berger and Kohn [14]. This method is applicable to more general equations, in particular those with no scaling invariance.

2.1 Invariant parabolic equations

In this part, we are interested in the blow-up phenomenon appearing in the study of parabolic problems whose solutions have a common property of blowing up in finite time and where the equations are invariant under the following scaling transformation

$$\forall \lambda > 0, \quad u \mapsto u_\lambda(x,t) := \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t). \tag{2.1}$$

In particular, we study blow-up solutions of the following parabolic problems. The first model is

$$\begin{cases} u_t(x,t) &= u_{xx}(x,t) + g(u,u_x), & \text{in} \quad \Omega \times (0,T), \\ u(x,t) &= 0 & \text{on} \quad \partial\Omega \times [0,T), \\ u(x,0) &= u_0(x), & \text{on} \quad \bar{\Omega}. \end{cases} \tag{2.2}$$

where $u(t) : x \in \Omega \mapsto u(x,t) \in \mathbb{R}$, $p > 1$. The function g is given by

$$g(u,u_x) = |u|^{p-1}u + \beta |u_x|^q, \quad \text{with} \quad q = \frac{2p}{p+1},$$

for some $\beta \in \mathbb{R}$. The equation (2.2) can be viewed as a population dynamic model (see Souplet [96] for an example with $\beta < 0$).
The second model is the following complex Ginzburg-Landau equation,

$$\begin{cases} u_t(x,t) &= (1+\imath\gamma)u_{xx} + (1+\imath\delta)|u|^{p-1}u, & \text{in} \quad \Omega \times (0,T), \\ u(x,t) &= 0 & \text{on} \quad \partial\Omega \times [0,T), \\ u(x,0) &= u_0(x), & \text{on} \quad \bar{\Omega}. \end{cases} \tag{2.3}$$

where $u(t) : x \in \Omega \to u(x,t) \in \mathbb{C}$, $p > 1$ and the constants γ, δ are real. This equation appears in various physical situations. An example is the theory of phase transitions and superconductivity. We refer to Levermore and Oliver [64], Popp et al. [89] and references therein for the physical background.
In both problems, Ω is a bounded interval and $u_0 : \bar{\Omega} \to \mathbb{R}$ is a given initial value that belongs to $\mathcal{H}$ where $\mathcal{H} \equiv W^{1,\infty}(\Omega)$ for equation (2.2) and $\mathcal{H} \equiv L^\infty(\Omega)$ for equation

(2.3). In particular, we consider $\Omega = (-1,1)$ and u_0 is positive, nontrivial, smooth and verifies $u_0(-1) = u_0(1) = 0$; in addition, u_0 is symmetric and nondecreasing on the interval $(-1,0)$. One can see that the maximal solution is either global in time, or exists only for $t \in [0,T)$ for some $T > 0$. In that case, the solution blows up in finite time T, namely,

$$\lim_{t\to T} \|u(t)\|_{\mathcal{H}} = +\infty,$$

and T is called the blow-up time of $u(t)$.

a) The theoretical framework:

For equation (2.2) with $\beta = 0$, the equation (2.2) is the semilinear heat equation (1.3) whose behavior is largely well-understood and which is mentioned in the previous sections. When $\beta \neq 0$ and $q > 0$, less is known about blow-up for equation (2.2). As a matter of fact, we loose the gradient structure, and energy methods break down. We keep however a maximum principal. We have several contributions on the subject by Chipot and Weissler [22], Souplet [96], Souplet, Tayachi and Weissler [98], Snoussi, Tayachi and Weissler [95], Snoussi and Tayachi [94], Ebde and Zaag [24] and others. Note that our choice $q = \frac{2p}{p+1}$ is critical in the sense that it is the only choice that makes equation (2.2) invariant under the dilation given in (2.1). In the case where $\beta \in [-2,0)$, in [98] (see also [22], [96]), the authors proved the existence of a non-trivial backward self-similar solution which blows up in finite time, only at one point and described the asymptotic behavior of its radially symmetric profile. More precisely, they showed the existence of a solution of (2.2) of the form

$$u(x,t) = (T-t)^{-\frac{1}{p-1}} v\left(\frac{x}{\sqrt{T-t}}\right), \tag{2.4}$$

where v satisfies for all $\xi \in \mathbb{R}$,

$$\Delta v(\xi) + \beta|\nabla v(\xi)|^q - \left[\frac{\xi}{2}\cdot\nabla v(\xi) + \frac{1}{p-1}v(\xi)\right] + |v(\xi)|^{p-1}v(\xi) = 0.$$

Note that this type of behavior does not hold when $\beta = 0$. Indeed, from Giga and Kohn [42], we know that the only solutions of the form (2.4) are 0 and $\pm\kappa(T-t)^{-\frac{1}{p-1}}$ with κ given in (1.6).

We wonder however whether equation (2.2) has solutions which behave like the solution of the case $\beta = 0$ (equation (1.3)), namely such that

$$\left\|(T-t)^{1/(p-1)}u(\cdot,t) - \bar{f}_\beta\left(\frac{\cdot - \hat{a}}{\sqrt{(T-t)|\log(T-t)|}}\right)\right\|_{L^\infty} \to 0, \quad \text{as} \quad t \to T, \tag{2.5}$$

where $\hat{a}$ is the blow-up point of u,

$$\bar{f}_\beta(z) = \left(p-1+b(p,\beta)|z|^2\right)^{-\frac{1}{p-1}}, \quad \text{with} \quad b(p,0) = \frac{(p-1)^2}{4p}, \tag{2.6}$$

or in the *similarity variables* defined in (1.7),

$$\left\| w_{\hat{a}}(s) - \bar{f}_\beta\left(\frac{\cdot}{\sqrt{s}}\right)\right\|_{L^\infty} \to 0, \quad \text{as} \quad s \to \infty. \tag{2.7}$$

Up to our knowledge, there is no theoretical answer to this equation. We answer it positively through a numerical method in this thesis (see Section 5.2 of Chapter II).

For equation (2.3), when $\gamma \neq \delta$, we have no gradient structure nor maximum principle. Therefore, classical methods cannot be applied here. Up to our knowledge, there are not many papers on this subject, apart from the paper of Popp et al. [89] and the paper by Masmoudi and Zaag [69] who construct a stable blow-up solution. There are also papers by Snoussi and Tayachi [93], Cazenave, Dickstein and Weissler [18] when $\gamma = \delta$ (note that in this case, there is a Lyapunov functional). In [69], the authors constructed a solution for equation (2.3) which blows up in finite time T only at one blow-up point and gave a sharp description of its blow-up profile. Furthermore, they showed the stability of that solution with respect to perturbations in initial data. Their result extends the previous result of Zaag [110] done in the case $\gamma = 0$. Their main result is the following:

For any $(\delta, \gamma) \in \mathbb{R}^2$ *such that*

$$p - \delta^2 - \gamma\delta(p+1) > 0,$$

equation (1.2) *has a solution* $u(x,t)$ *blowing up in finite time* T *only at one blow-up point* $\hat{a}$*, moreover,*

$$\left\| (T-t)^{\frac{1+\imath\delta}{p-1}} |\log(T-t)|^{-\imath\mu} u(t) - \tilde{f}_{\delta,\gamma}\left(\frac{\cdot - \hat{a}}{\sqrt{(T-t)|\log(T-t)|}}\right)\right\|_{L^\infty} \to 0 \quad \text{as } t \to T, \tag{2.8}$$

where $\mu = -\frac{2\gamma b(p,\delta,\gamma)}{(p-1)^2}(1+\delta^2)$, $b(p,\delta,\gamma) = \frac{(p-1)^2}{4(p-\delta^2-\gamma\delta(p+1))}$ *and*

$$\tilde{f}_{\delta,\gamma}(z) = \left(p - 1 + b(p,\delta,\gamma)|z|^2\right)^{-\frac{1+\imath\delta}{p-1}}. \tag{2.9}$$

Remark 2.1. *We remark that equation* (2.3) *is rotation invariant. Therefore,* $e^{\imath\theta}\tilde{f}_{\delta,\gamma}$ *is also an asymptotic profile of the solution of* (2.3) *with* $\theta \in \mathbb{R}$.

b) The numerical method:

Let us first review some aspects of the numerical study of blow-up problems such as the sufficient blow-up conditions, the blow-up rate, the blow-up time, the blow-up set and the blow-up profile. For the problem (2.2), there are several studies in the case $\beta = 0$. The first works were done by Nakagawa and Ushijima in [78], [79] where the authors used the finite difference and the finite element method on a uniform spatial mesh. Some papers focus on the study of the numerical schemes, in particular, some authors established sufficient blow-up conditions for the numerical schemes, for example, Abia, López-Marcos and Martínez [1], [3], Chen [20], [21], Duran, Etcheverry and Rossi[23], N'gohisse and Boni [80]. For the numerical blow-up rate, there is a series of studies by Acosta, Ferreira, Groisman and Rossi [28], [5], [29], [47], Hirota and Ozawa [57], ... Those papers gave the relation between the discretized problem and the continuous ones. For the numerical convergence of the blow-up time, it was investigated in [1], [2], [102], [4], [57] and [80]. On numerical blow-up sets, we would like to mention the works in [27], [48], [29] and [6]. Up to our knowledge, there are not many papers on the numerical blow-up profile, apart from the paper of Berger and Kohn [14] who already obtained very good numerical results on this subject. There is also the work of Baruch et al. [10] studying standing-ring solutions.

Here we rely on the rescaling method suggested in [14] to obtain a numerical solution for equations (2.2) and (2.3). This algorithm fundamentally relies on the scale invariance of those equations:
If u a solution of (1.1) *(or* (1.2)*), then for all $\lambda > 0$, the function u_λ given by*

$$u_\lambda(\xi,\tau) = \lambda^{\frac{2}{p-1}} u(\lambda\xi, \lambda^2\tau), \tag{2.10}$$

is also a solution of (2.2) *(or* (2.3)*).*
This property allows to make a zoom of the solution when it is close to the singularity, still keeping the same equation (see Section 3 of Chapter II for more details of the *rescaling algorithm*). Our aim is to give a numerical confirmation for the theoretical profile for equation (2.2), and especially for the complex Ginzburg-Landau equation (2.3). In Section 5.2 of Chapter II, we give a numerical answer to the question of the blow-up profile (2.7), where no theoretical is available in the case $\beta \neq 0$ in equation (2.2) (see Figure I.1 for an illustration). This way, we have given a numerical evidence for the following conjecture:

Conjecture 2.2. *Equation* (2.2) *has a solution $u(x,t)$ which blows up in finite time T at one blow-up point $x = 0$, moreover,*

$$\left\| (T-t)^{1/(p-1)} u(t) - \bar{f}_\beta\left(\frac{\cdot}{\sqrt{(T-t)|\log(T-t)|}}\right)\right\|_{L^\infty} \to 0, \quad \text{as} \quad t \to T, \tag{2.11}$$

where

$$\bar{f}_\beta(z) = \left(p - 1 + b(p,\beta)|z|^2\right)^{-\frac{1}{p-1}}, \quad \text{with} \quad b(p,0) = \frac{(p-1)^2}{4p}, \tag{2.12}$$

and $b(p,\beta)$ is represented in Figure I.2 and Figure I.3.

While remarking numerical simulation for equation (2.2) with $\beta \neq 0$, we could never obtain the self-similar behavior (2.4) rigorously proved in [98]. On the contrary, we could exhibit the behavior (2.11), at the heart of our conjecture. In our opinion, this is

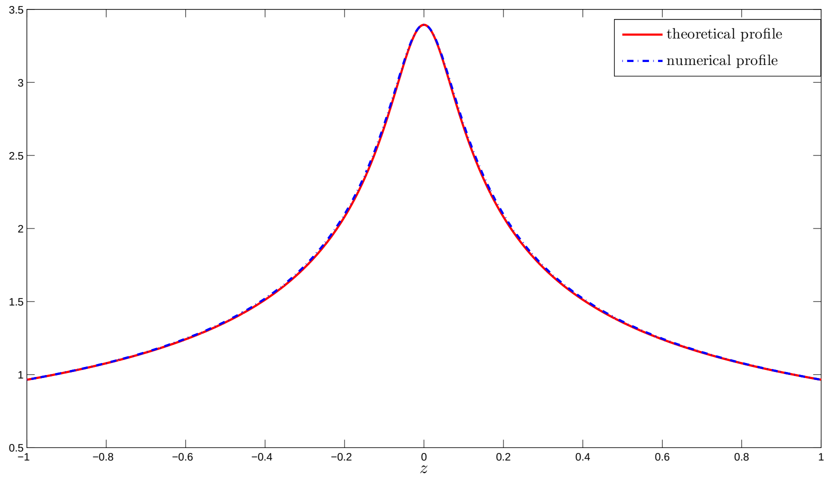

Figure I.1. *The profile of a rescaled version of the solution of* (2.2) *after 80 iterations, for computations with* $\beta = 1$, $p = 5$ *and the initial data* $u_0(x) = 1.2(1 + \cos(\pi x))$. *They coincide within plotting resolution.*

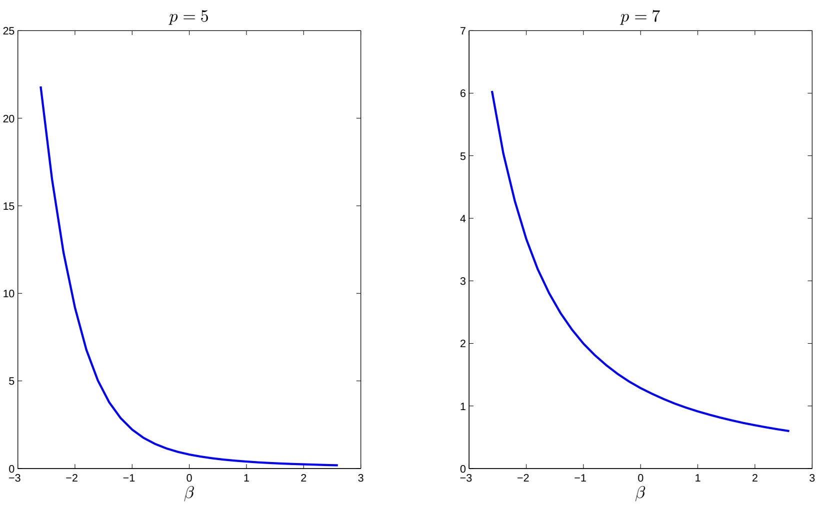

Figure I.2. *The computed values of* $b(5, \beta)$ *(left) and* $b(7, \beta)$ *(right) for various values of* β*, where* $b(p, \beta)$ *is introduced in* (2.12).

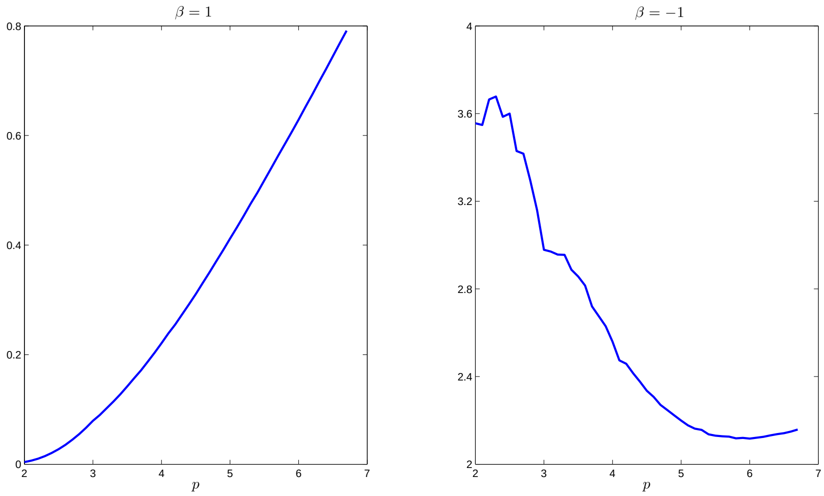

Figure I.3. *The computed values of* $b(p, 1)$ *(left) and* $b(p, -1)$ *(right) for various values of* p*, where* $b(p, \beta)$ *is introduced in* (2.12).

probably due to the fact that the behavior (2.4) is unstable, unlike the behavior (2.11), which we suspect to be stable with respect to perturbations in initial data.

We also proved in Section 4 of Chapter II the convergence of the rescaling method applied to equation (2.2) under some regularity assumptions (see Theorem 4.1 page 56). Furthermore, many numerical simulations were performed to give numerical confirmations for the blow-up profile (2.8) for the complex Ginzburg-Ladau equation (2.3), which has never been done earlier and is quite challenging. In Figure I.4, we give a numerical confirmation for the blow-up profile (2.8). Both the numerical modulus and phase (up to adding a constant $\theta \in \mathbb{R}$) coincide with the theoretical ones within plotting resolution (see Section 5.3 of Chapter II for more numerical simulations).

2.2 Non scale-invariant parabolic equations

The rescaling method of Berger and Kohn [14] is a very powerful algorithm, however, it only applies to some equations satisfying the scale-invariance property (2.1). However, there are many equations whose solutions blow up in finite time but the equation does not satisfy the property (2.1), for example, equation (1.24) with the strong perturbation h. In this thesis, we propose a new mesh-refinement method inspired by the *rescaling algorithm* of Berger and Kohn [14] applied to such problems. Though the method is

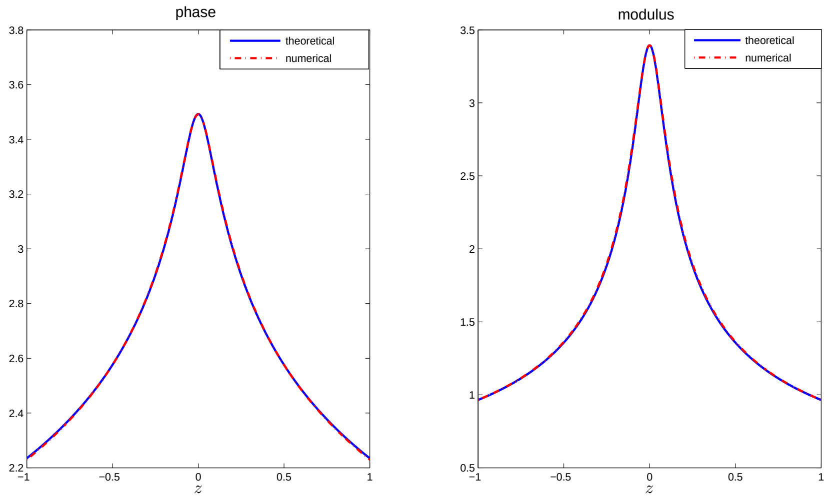

Figure I.4. *The profile of a rescaled version of the solution of the complex Ginzburg-Landau equation* (2.3) *after 80 iterations, for computations with* $\gamma = \delta = 1$, $p = 5$ *and the initial data* $u_0(x) = 1.2(1 + \cos(\pi x))$.

obviously more general, we have applied it to the following problem:

$$\begin{cases} u_t &= u_{xx} + F(u), \quad (x,t) \in (-1,1) \times (0,T), \\ u(1,t) &= u(-1,t) = 0, \quad t \in (0,T), \\ u(x,0) &= \varphi(x), \quad x \in (-1,1), \end{cases} \tag{2.13}$$

where $p > 1$, $\varphi(x) > 0$, $\varphi(x) = \varphi(-x)$, $x\frac{d\varphi(x)}{dx} < 0$ for $x \neq 0$, and

$$F(u) = u^p + \frac{u^p}{\log^a(2 + u^2)} \quad \text{with} \quad a > 0. \tag{2.14}$$

Our method differs from Berger and Kohn's in the following way: we step the solution forward until its maximum value multiplied by a power of its mesh size reaches a preset threshold, where the mesh size and the preset threshold are linked; for the *rescaling algorithm*, the solution is stepped forward until its maximum value reaches a preset threshold, and the mesh size and the preset threshold do not need to be linked. Because we do not need the scale-invariance property (2.1), our method is applicable to a larger class of equations whose solutions blow up in finite time (see Section 4.1 of Chapter V for a full description of this method). Various numerical simulations are performed to illustrate the effectiveness of our method to the problem of the numerical blow-up profile. These numerical results also give confirmations for the theoretical blow-up results of equation (1.24) stated in section 1.3 (see Figure I.5 for an illustration and see Section 4.2 of Chapter V for more numerical results).

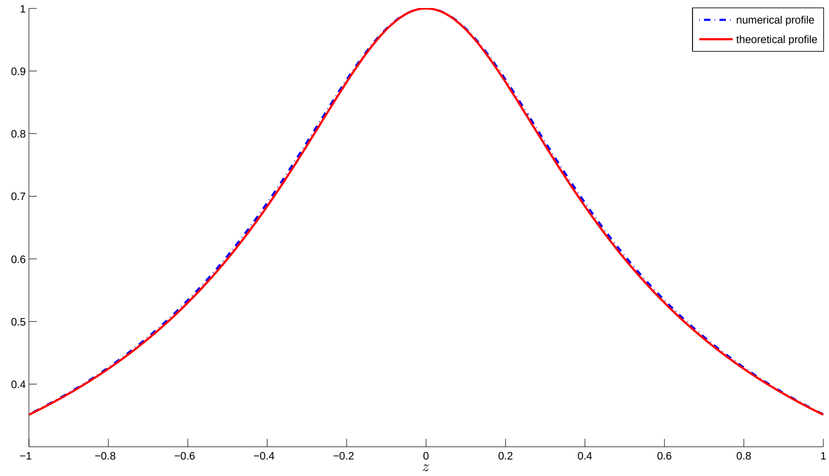

Figure I.5. *The profile of a rescaled version of the solution of* (2.13) *after 40 iterations, for computations with* $a = 0.1$*,* $p = 3$ *and the initial data* $\varphi(x) = 2(1 + \cos(\pi x))$.

This thesis is organized in four chapters which are presented in the papers [81], [82], [84] and [83]:
- Chapter II: Numerical analysis of the rescaling method of [14] for the parabolic equations (2.2) and (2.3) whose solutions blow up in finite time and is invariant under the scaling property.
- Chapter III: Blow-up results for a class of strongly perturbed semilinear heat equations (1.24) in the case $a > 1$. We first derive a Lyapunov functional in the *similarity variables* and then use it to derive the blow-up rate. We also classify all possible asymptotic behaviors and corresponding profiles when the solution approaches to singularity.
- Chapter IV: Construction of a stable blow-up solution for a class of strongly perturbed semilinear heat equations (1.24) and description of it blow-up profile.
- Chapter V: Complement of the blow-up results for a class of strongly perturbed semilinear heat equations (1.24) in the case $a \in (0, 1]$ by taking the particular form (1.27) and description of the mesh-refinement technique for non-invariant parabolic equations.

Bibliography

[1] L. M. Abia, J. C. López-Marcos, and J. Martínez. Blow-up for semidiscretizations of reaction-diffusion equations. *Appl. Numer. Math.*, 20(1-2):145–156, 1996. Workshop on the method of lines for time-dependent problems (Lexington, KY, 1995).

[2] L. M. Abia, J. C. López-Marcos, and J. Martínez. On the blow-up time convergence of semidiscretizations of reaction-diffusion equations. *Appl. Numer. Math.*, 26(4):399–414, 1998.

[3] L. M. Abia, J. C. López-Marcos, and J. Martínez. The Euler method in the numerical integration of reaction-diffusion problems with blow-up. *Appl. Numer. Math.*, 38(3):287–313, 2001.

[4] G. Acosta, R. G. Durán, and J. D. Rossi. An adaptive time step procedure for a parabolic problem with blow-up. *Computing*, 68(4):343–373, 2002.

[5] G. Acosta, J. Fernández Bonder, P. Groisman, and J. D. Rossi. Numerical approximation of a parabolic problem with a nonlinear boundary condition in several space dimensions. *Discrete Contin. Dyn. Syst. Ser. B*, 2(2):279–294, 2002.

[6] L. A. Assalé, Boni T. K., and Nabongo Diabate. Numerical blow-up time for a semilinear parabolic equation with nonlinear boundary conditions. *Journal of Applied Mathematics*, 2008:Article ID 753518, 29 p.–Article ID 753518, 29 p., 2008.

[7] J. M. Ball. Remarks on blow-up and nonexistence theorems for nonlinear evolution equations. *Quart. J. Math. Oxford Ser. (2)*, 28(112):473–486, 1977.

[8] C. Bandle and H. Brunner. Blowup in diffusion equations: a survey. *J. Comput. Appl. Math.*, 97(1-2):3–22, 1998.

[9] G. I. Barenblatt. *Scaling, self-similarity, and intermediate asymptotics*, volume 14 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 1996. With a foreword by Ya. B. Zeldovich.

[10] G. Baruch, G. Fibich, and N. Gavish. Singular standing-ring solutions of nonlinear partial differential equations. *Phys. D*, 239(20-22):1968–1983, 2010.

[11] J. Bebernes, A. Bressan, and D. Eberly. A description of blowup for the solid fuel ignition model. *Indiana Univ. Math. J.*, 36(2):295–305, 1987.

[12] J. Bebernes and D. Eberly. *Mathematical problems from combustion theory*, volume 83 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1989.

[13] J. W. Bebernes and D. R. Kassoy. A mathematical analysis of blowup for thermal reactions—the spatially nonhomogeneous case. *SIAM J. Appl. Math.*, 40(3):476–484, 1981.

[14] M. Berger and R. V. Kohn. A rescaling algorithm for the numerical calculation of blowing-up solutions. *Comm. Pure Appl. Math.*, 41(6):841–863, 1988.

[15] J. Bricmont and A. Kupiainen. Renormalization group and nonlinear PDEs. In *Quantum and non-commutative analysis (Kyoto, 1992)*, volume 16 of *Math. Phys. Stud.*, pages 113–118. Kluwer Acad. Publ., Dordrecht, 1993.

[16] J. Bricmont and A. Kupiainen. Universality in blow-up for nonlinear heat equations. *Nonlinearity*, 7(2):539–575, 1994.

[17] J. Bricmont, A. Kupiainen, and G. Lin. Renormalization group and asymptotics of solutions of nonlinear parabolic equations. *Comm. Pure Appl. Math.*, 47(6):893–922, 1994.

[18] T. Cazenave, F. Dickstein, and F. B. Weissler. Finite-time blowup for a complex ginzburg–landau equation. *SIAM Journal on Mathematical Analysis*, 45(1):244–266, 2013.

[19] T. Cazenave and P. L. Lions. Solutions globales d'équations de la chaleur semi linéaires. *Comm. Partial Differential Equations*, 9(10):955–978, 1984.

[20] Y. G. Chen. Asymptotic behaviours of blowing-up solutions for finite difference analogue of $u_t = u_{xx} + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 33(3):541–574, 1986.

[21] Y. G. Chen. Blow-up solutions to a finite difference analogue of $u_t = \Delta u + u^{1+\alpha}$ in N-dimensional balls. *Hokkaido Math. J.*, 21(3):447–474, 1992.

[22] M. Chipot and F. B. Weissler. Some blowup results for a nonlinear parabolic equation with a gradient term. *SIAM J. Math. Anal.*, 20(4):886–907, 1989.

[23] R. G. Duran, J. I. Etcheverry, and J. D. Rossi. Numerical approximation of a parabolic problem with a nonlinear boundary condition. *Discrete Contin. Dynam. Systems*, 4(3):497–506, 1998.

[24] M. A. Ebde and H. Zaag. Construction and stability of a blow up solution for a nonlinear heat equation with a gradient term. *S$\vec{e}$MA J.*, (55):5–21, 2011.

[25] C. Fermanian Kammerer, F. Merle, and H. Zaag. Stability of the blow-up profile of non-linear heat equations from the dynamical system point of view. *Math. Ann.*, 317(2):347–387, 2000.

[26] C. Fermanian Kammerer and H. Zaag. Boundedness up to blow-up of the difference between two solutions to a semilinear heat equation. *Nonlinearity*, 13(4):1189–1216, 2000.

[27] J. Fernández Bonder, P. Groisman, and J. D. Rossi. On numerical blow-up sets. *Proc. Amer. Math. Soc.*, 130(7):2049–2055, 2002.

[28] R. Ferreira, P. Groisman, and J. D. Rossi. Numerical blow-up for a nonlinear problem with a nonlinear boundary condition. *Math. Models Methods Appl. Sci.*, 12(4):461–483, 2002.

[29] R. Ferreira, P. Groisman, and J. D. Rossi. Numerical blow-up for the porous medium equation with a source. *Numer. Methods Partial Differential Equations*, 20(4):552–575, 2004.

[30] M. Fila. Boundedness of global solutions for the heat equation with nonlinear boundary conditions. *Comment. Math. Univ. Carolin.*, 30(3):479–484, 1989.

[31] M. Fila and J. Filo. A blow-up result for nonlinear diffusion equations. *Math. Slovaca*, 39(3):331–346, 1989.

[32] M. Fila and P. Souplet. The blow-up rate for semilinear parabolic problems on general domains. *NoDEA Nonlinear Differential Equations Appl.*, 8(4):473–480, 2001.

[33] S. Filippas and R. V. Kohn. Refined asymptotics for the blowup of $u_t - \Delta u = u^p$. *Comm. Pure Appl. Math.*, 45(7):821–869, 1992.

[34] S. Filippas and W. X. Liu. On the blowup of multidimensional semilinear heat equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(3):313–344, 1993.

[35] D. A. Frank-Kamenetskii. *Diffusion and Heat Transfer in Chemical Kinetics*. Plenum Press, New York, 1969.

[36] A. Friedman. Remarks on nonlinear parabolic equations. In *Proc. Sympos. Appl. Math., Vol. XVII*, pages 3–23. Amer. Math. Soc., Providence, R.I., 1965.

[37] A. Friedman and B. McLeod. Blow-up of positive solutions of semilinear heat equations. *Indiana Univ. Math. J.*, 34(2):425–447, 1985.

[38] H. Fujita. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I*, 13:109–124 (1966), 1966.

[39] V. A. Galaktionov and S. A. Posashkov. Application of new comparison theorems to the investigation of unbounded solutions of nonlinear parabolic equations. *Differentsial′nye Uravneniya*, 22(7):1165–1173, 1285, 1986.

[40] V. A. Galaktionov and J. L. Vázquez. The problem of blow-up in nonlinear parabolic equations. *Discrete Contin. Dyn. Syst.*, 8(2):399–433, 2002. Current developments in partial differential equations (Temuco, 1999).

[41] I. M. Gel′fand. Some problems in the theory of quasilinear equations. *Amer. Math. Soc. Transl. (2)*, 29:295–381, 1963.

[42] Y. Giga and R. V. Kohn. Asymptotically self-similar blow-up of semilinear heat equations. *Comm. Pure Appl. Math.*, 38(3):297–319, 1985.

[43] Y. Giga and R. V. Kohn. Characterizing blowup using similarity variables. *Indiana Univ. Math. J.*, 36(1):1–40, 1987.

[44] Y. Giga and R. V. Kohn. Nondegeneracy of blowup for semilinear heat equations. *Comm. Pure Appl. Math.*, 42(6):845–884, 1989.

[45] Y. Giga, S. Matsui, and S. Sasayama. Blow up rate for semilinear heat equations with subcritical nonlinearity. *Indiana Univ. Math. J.*, 53(2):483–514, 2004.

[46] Y. Giga, S. Matsui, and S. Sasayama. On blow-up rate for sign-changing solutions in a convex domain. *Math. Methods Appl. Sci.*, 27(15):1771–1782, 2004.

[47] P. Groisman. Totally discrete explicit and semi-implicit Euler methods for a blow-up problem in several space dimensions. *Computing*, 76(3-4):325–352, 2006.

[48] P. Groisman and J. D. Rossi. Asymptotic behaviour for a numerical approximation of a parabolic problem with blowing up solutions. *J. Comput. Appl. Math.*, 135(1):135–155, 2001.

[49] M. A. Hamza and H. Zaag. Lyapunov functional and blow-up results for a class of perturbations of semilinear wave equations in the critical case. *J. Hyperbolic Differ. Equ.*, 9(2):195–221, 2012.

[50] M. A. Hamza and H. Zaag. A Lyapunov functional and blow-up results for a class of perturbed semilinear wave equations. *Nonlinearity*, 25(9):2759–2773, 2012.

[51] K. Hayakawa. On nonexistence of global solutions of some semilinear parabolic differential equations. *Proc. Japan Acad.*, 49:503–505, 1973.

[52] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1981.

[53] M. A. Herrero and J. J. L. Velázquez. Flat blow-up in one-dimensional semilinear heat equations. *Differential Integral Equations*, 5(5):973–997, 1992.

[54] M. A. Herrero and J. J. L. Velázquez. Generic behaviour of one-dimensional blow up patterns. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 19(3):381–450, 1992.

[55] M. A. Herrero and J. J. L. Velázquez. Blow-up behaviour of one-dimensional semilinear parabolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(2):131–189, 1993.

[56] M. A. Herrero and Juan J. L. Velázquez. Comportement générique au voisinage d'un point d'explosion pour des solutions d'équations paraboliques unidimensionnelles. *C. R. Acad. Sci. Paris Sér. I Math.*, 314(3):201–203, 1992.

[57] C. Hirota and K. Ozawa. Numerical method of estimating the blow-up time and rate of the solution of ordinary differential equations—an application to the blow-up problems of partial differential equations. *J. Comput. Appl. Math.*, 193(2):614–637, 2006.

[58] B Hu. *Blow-up theories for semilinear parabolic equations*, volume 2018 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011.

[59] Hitoshi Ishii. Asymptotic stability and blowing up of solutions of some nonlinear equations. *J. Differential Equations*, 26(2):291–319, 1977.

[60] S. Kaplan. On the growth of solutions of quasi-linear parabolic equations. *Comm. Pure Appl. Math.*, 16:305–330, 1963.

[61] D. R. Kassoy and J. Poland. The thermal explosion confined by a constant temperature boundary. I. The induction period solution. *SIAM J. Appl. Math.*, 39(3):412–430, 1980.

[62] D. R. Kassoy and J. Poland. The thermal explosion confined by a constant temperature boundary. II. The extremely rapid transient. *SIAM J. Appl. Math.*, 41(2):231–246, 1981.

[63] E. F. Keller and L. A. Segel. Initiation of slime mold aggregation viewed as an instability. *Journal of Theoretical Biology*, 26(3):399 – 415, 1970.

[64] C. D. Levermore and M. Oliver. The complex Ginzburg-Landau equation as a model problem. In *Dynamical systems and probabilistic methods in partial differential equations (Berkeley, CA, 1994)*, volume 31 of *Lectures in Appl. Math.*, pages 141–190. Amer. Math. Soc., Providence, RI, 1996.

[65] H. A. Levine. Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$. *Arch. Rational Mech. Anal.*, 51:371–386, 1973.

[66] H. A. Levine and L. E. Payne. Nonexistence theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backward in time. *J. Differential Equations*, 16:319–334, 1974.

[67] H. A. Levine and L. E. Payne. Some nonexistence theorems for initial-boundary value problems with nonlinear boundary constraints. *Proc. Amer. Math. Soc.*, 46:277–284, 1974.

[68] P.-L. Lions. Asymptotic behavior of some nonlinear heat equations. *Phys. D*, 5(2-3):293–306, 1982.

[69] N. Masmoudi and H. Zaag. Blow-up profile for the complex Ginzburg-Landau equation. *J. Funct. Anal.*, 255(7):1613–1666, 2008.

[70] H. Matano and F. Merle. On nonexistence of type II blowup for a supercritical nonlinear heat equation. *Comm. Pure Appl. Math.*, 57(11):1494–1541, 2004.

[71] F. Merle and H. Zaag. Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$. *Duke Math. J.*, 86(1):143–195, 1997.

[72] F. Merle and H. Zaag. Optimal estimates for blowup rate and behavior for nonlinear heat equations. *Comm. Pure Appl. Math.*, 51(2):139–196, 1998.

[73] F. Merle and H. Zaag. Refined uniform estimates at blow-up and applications for nonlinear heat equations. *Geom. Funct. Anal.*, 8(6):1043–1085, 1998.

[74] F. Merle and H. Zaag. A Liouville theorem for vector-valued nonlinear heat equations and applications. *Math. Ann.*, 316(1):103–137, 2000.

[75] N Mizoguchi, H Ninomiya, and E Yanagida. Critical exponent for the bipolar blowup in a semilinear parabolic equation. *J. Math. Anal. Appl.*, 218(2):495–518, 1998.

[76] N. Mizoguchi and E. Yanagida. Blowup and life span of solutions for a semilinear parabolic equation. *SIAM J. Math. Anal.*, 29(6):1434–1446 (electronic), 1998.

[77] C. E. Mueller and F. B. Weissler. Single point blow-up for a general semilinear heat equation. *Indiana Univ. Math. J.*, 34(4):881–913, 1985.

[78] T. Nakagawa. Blowing up of a finite difference solution to $u_t = u_{xx} + u^2$. *Appl. Math. Optim.*, 2(4):337–350, 1975/76.

[79] T. Nakagawa and T. Ushijima. Finite element analysis of the semi-linear heat equation of blow-up type. *in Topics in Numerical Analysis, J. J. H. Miller, ed., Academic Press*, 1977.

[80] F. K. N'gohisse and T. K. Boni. Numerical blow-up for a nonlinear heat equation. *Acta Math. Sin. (Engl. Ser.)*, 27(5):845–862, 2011.

[81] V. T. Nguyen. Numerical analysis of the rescaling method for parabolic problems with blow-up in finite time. *submitted, arXiv:1403.7547*.

[82] V. T. Nguyen. On the blow-up results for a class of strongly perturbed semilinear heat equation. *submitted, arXiv:1404.4018*.

[83] V. T. Nguyen and H. Zaag. Blow-up results for a strongly perturbed semilinear heat equation: Theoretical analysis and numerical method. *submitted, arXiv:1410.4079*.

[84] V. T. Nguyen and H. Zaag. Contruction of a stable blow-up solution for a class of strongly perturbed semilinear heat equation. *submitted, arXiv:1406.5233*.

[85] N. Nouaili and H. Zaag. A Liouville theorem for vector valued semilinear heat equations with no gradient structure and applications to blow-up. *Trans. Amer. Math. Soc.*, 362(7):3391–3434, 2010.

[86] B. Palais. Blowup for nonlinear equations using a comparison principle in Fourier space. *Comm. Pure Appl. Math.*, 41(2):165–196, 1988.

[87] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[88] I. Peral and J. L. Vázquez. On the stability or instability of the singular solution of the semilinear heat equation with exponential reaction term. *Arch. Rational Mech. Anal.*, 129(3):201–224, 1995.

[89] S. Popp, O. Stiller, E. Kuznetsov, and L. Kramer. The cubic complex Ginzburg-Landau equation for a backward bifurcation. *Phys. D*, 114(1-2):81–107, 1998.

[90] P. Quittner. A priori bounds for global solutions of a semilinear parabolic problem. *Acta Math. Univ. Comenian. (N.S.)*, 68(2):195–203, 1999.

[91] P. Quittner and P. Souplet. *Superlinear parabolic problems*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2007. Blow-up, global existence and steady states.

[92] F. Ribaud. Cauchy problem for semilinear parabolic equations with initial data in $H_p^s(\mathbf{R}^n)$ spaces. *Rev. Mat. Iberoamericana*, 14(1):1–46, 1998.

[93] S. Snoussi and S. Tayachi. Nonglobal existence of solutions for a generalized Ginzburg-Landau equation coupled with a Poisson equation. *J. Math. Anal. Appl.*, 254(2):558–570, 2001.

[94] S. Snoussi and S. Tayachi. Large time behavior of solutions for parabolic equations with nonlinear gradient terms. *Hokkaido Math. J.*, 36(2):311–344, 2007.

[95] S. Snoussi, S. Tayachi, and F. B. Weissler. Asymptotically self-similar global solutions of a semilinear parabolic equation with a nonlinear gradient term. *Proc. Roy. Soc. Edinburgh Sect. A*, 129(6):1291–1307, 1999.

[96] P. Souplet. Finite time blow-up for a non-linear parabolic equation with a gradient term and applications. *Math. Methods Appl. Sci.*, 19(16):1317–1333, 1996.

[97] P. Souplet and S. Tayachi. Blowup rates for nonlinear heat equations with gradient terms and for parabolic inequalities. *Colloq. Math.*, 88(1):135–154, 2001.

[98] P. Souplet, S. Tayachi, and F. B. Weissler. Exact self-similar blow-up of solutions of a semilinear parabolic equation with a nonlinear gradient term. *Indiana Univ. Math. J.*, 45(3):655–682, 1996.

[99] M. J. Tindall, P. K. Maini, S. L. Porter, and J. P. Armitage. Overview of mathematical approaches used to model bacterial chemotaxis. II. Bacterial populations. *Bull. Math. Biol.*, 70(6):1570–1607, 2008.

[100] M. J. Tindall, S. L. Porter, P. K. Maini, G. Gaglia, and J. P. Armitage. Overview of mathematical approaches used to model bacterial chemotaxis. I. The single cell. *Bull. Math. Biol.*, 70(6):1525–1569, 2008.

[101] M. Tsutsumi. Existence and nonexistence of global solutions for nonlinear parabolic equations. *Publ. Res. Inst. Math. Sci.*, 8:211–229, 1972.

[102] T. K. Ushijima. On the approximation of blow-up time for solutions of nonlinear parabolic equations. *Publ. Res. Inst. Math. Sci.*, 36(5):613–640, 2000.

[103] J. J. L. Velázquez. Higher-dimensional blow up for semilinear parabolic equations. *Comm. Partial Differential Equations*, 17(9-10):1567–1596, 1992.

[104] J. J. L. Velázquez. Classification of singularities for blowing up solutions in higher dimensions. *Trans. Amer. Math. Soc.*, 338(1):441–464, 1993.

[105] J. J. L. Velázquez. Estimates on the $(n-1)$-dimensional Hausdorff measure of the blow-up set for a semilinear heat equation. *Indiana Univ. Math. J.*, 42(2):445–476, 1993.

[106] F. B. Weissler. Local existence and nonexistence for semilinear parabolic equations in L^p. *Indiana Univ. Math. J.*, 29(1):79–102, 1980.

[107] F. B. Weissler. Existence and nonexistence of global solutions for a semilinear heat equation. *Israel J. Math.*, 38(1-2):29–40, 1981.

[108] F. B. Weissler. Single point blow-up for a semilinear initial value problem. *J. Differential Equations*, 55(2):204–224, 1984.

[109] F. B. Weissler. An L^∞ blow-up estimate for a nonlinear heat equation. *Comm. Pure Appl. Math.*, 38(3):291–295, 1985.

[110] H. Zaag. Blow-up results for vector-valued nonlinear heat equations with no gradient structure. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 15(5):581–622, 1998.

[111] Ya. B. Zel′dovich, G. I. Barenblatt, V. B. Librovich, and G. M. Makhviladze. *The mathematical theory of combustion and explosions*. Consultants Bureau [Plenum], New York, 1985. Translated from the Russian by Donald H. McNeill.

Chapter II

Numerical analysis of the rescaling method for parabolic problems with blow-up in finite time [1]

V. T. Nguyen

Université Paris 13, Sorbonne Paris Cité,
LAGA, CNRS (UMR 7539), F-93430, Villetaneuse, France.

Abstract
In this work, we study the numerical solution for parabolic equations whose solutions have a common property of blowing up in finite time and the equations are invariant under the following scaling transformation

$$u \mapsto u_\lambda(x,t) := \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t).$$

For that purpose, we apply the rescaling method proposed by Berger and Kohn [9] to such problems. The convergence of the method is proved under some regularity assumption. Some numerical experiments are given to derive the blow-up profile verifying henceforth the theoretical results.

Keyword: Numerical blow-up, finite-time blow-up, nonlinear parabolic equations.

[1] submitted, arXiv:1403.7547

1 Introduction

We study the solution of the following parabolic problem

$$\begin{cases} u_t(x,t) &= u_{xx}(x,t) + g(u,u_x), & \text{in} \quad \Omega \times (0,T), \\ u(x,t) &= 0 & \text{on} \quad \partial\Omega \times [0,T), \\ u(x,0) &= u_0(x), & \text{on} \quad \bar{\Omega}. \end{cases} \tag{1.1}$$

where $u(t) : x \in \Omega \mapsto u(x,t) \in \mathbb{R}$, $p > 1$. The function g is given by

$$g(u,u_x) = |u|^{p-1}u + \beta|u_x|^q, \quad \text{with} \quad q = \frac{2p}{p+1},$$

for some $\beta \in \mathbb{R}$. This equation can be viewed as a population dynamic model (see [51] for an example with $\beta < 0$).
We also consider the complex Ginzburg-Landau equation,

$$\begin{cases} u_t(x,t) &= (1+\imath\gamma)u_{xx} + (1+\imath\delta)|u|^{p-1}u, & \text{in} \quad \Omega \times (0,T), \\ u(x,t) &= 0 & \text{on} \quad \partial\Omega \times [0,T), \\ u(x,0) &= u_0(x), & \text{on} \quad \bar{\Omega}. \end{cases} \tag{1.2}$$

where $u(t) : x \in \Omega \to u(x,t) \in \mathbb{C}$, $p > 1$ and the constants γ, δ are real. This equation appears in various physical situations. An example is the theory of phase transitions and superconductivity. We refer to Popp et al. [46] and the references therein for the physical background.

In both problems, Ω is a bounded interval and $u_0 : \bar{\Omega} \to \mathbb{R}$ is a given initial value that belongs to H where $H \equiv W^{1,\infty}(\Omega)$ for equation (1.1) and $H \equiv L^\infty(\Omega)$ for equation (1.2). In particular, we consider $\Omega = (-1,1)$ and u_0 is positive, nontrivial, smooth and verifies $u_0(-1) = u_0(1) = 0$; in addition, u_0 is symmetric and nondecreasing on the interval $(-1,0)$. Thanks to a fixed-point argument, the Cauchy problem for equation (1.1) can be solved in $W^{1,\infty}(\Omega)$, locally in time. For equation (1.2), we solve it in $L^\infty(\Omega)$. Then, it is easy to see that the maximal solution is either global in time, or exists only for $t \in [0,T)$ for some $T > 0$. In that case, the solution blows up in finite time T, namely,

$$\lim_{t\to T} \|u(t)\|_H = +\infty,$$

and T is called the blow-up time of $u(t)$.

When $\beta = 0$, the theoretical part for equation (1.1) is largely well-understood. The literature on the subject is huge, so we refer the reader to the book by Souplet and Quittner [47]. When $\beta \neq 0$ and $q > 0$, less is known about blow-up for equation (1.1). As a matter of fact, we loose the gradient structure, and energy methods break down. We keep however a maximum principal. We have several contributions on the subject by [14], [51], [52], [50], [49] and [16]. Note that our choice $q = \frac{2p}{p+1}$ is critical in the sense that it is the only choice that makes equation (1.1) invariant under the dilation given in (1.3) below. As for equation (1.2), when $\gamma \neq \delta$, we have no gradient structure nor maximum principle. Therefore, classical methods cannot be applied here. Up to our knowledge, there are not many papers on this subject, apart from the paper of Popp et al. [46] and the paper by Masmoudi and Zaag [39] who construct a stable blow-up solution. There are also papers by Snoussi and Tayachi [48], Cazenave, Dickstein and Weissler [11] when $\gamma = \delta$ (note that in this case, there is a Lyapunov functional).

In comparison with the theoretical aspects, the numerical analysis of blow-up has received little attention, particularly on the numerical blow-up profile. For other numerical aspects related to sufficient blow-up conditions, the blow-up rate, the blow-up time and the blow-up set, there are several studies for (1.1) in the case $\beta = 0$. The first work on this problem was done in [43], [44] by using the finite difference and finite element method on a uniform spatial mesh. For sufficient blow-up conditions, the solution of semi or full-discretized equation blowing up in finite time was established in [1], [3], [12], [13], [15], [43] and [45]. For the numerical blow-up rate, there is a series of studies by [20], [5], [21], [31], [36] and [45]. Those papers gave the relation between the discretized problem and the continuous ones. For the numerical convergence of the blow-up time, it was investigated in [1], [2], [53], [4], [36] and [45]. On numerical blow-up sets, we would like to mention the works in [19], [32], [21] and [6]. Up to our knowledge, there are not many papers on the numerical blow-up profile, apart from the paper of Berger and Kohn [9] who already obtained very good numerical results on this subject. There is also the work of Baruch et al. [8] studying standing-ring solutions.

For this reason, we will rely on the rescaling method suggested in [9] to obtain a numerical solution for the equations mentioned above. This algorithm fundamentally relies on the scale invariance of equation (1.1) and (1.2): if u a solution of (1.1) (or (1.2)), then for all $\lambda > 0$, the function u_λ given by

$$u_\lambda(\xi, \tau) = \lambda^{\frac{2}{p-1}} u(\lambda\xi, \lambda^2\tau), \tag{1.3}$$

is also a solution of (1.1) (or (1.2)). This property allows to make a zoom of the solution when it is close to the singularity, still keeping the same equation. Our aim is to give a numerical confirmation for the theoretical profile of the semilinear heat equation (1.1) in the case $\beta = 0$ (already done in [9]) and especially the complex Ginzburg-Landau equation (1.2) which has never been done earlier numerically, and is quite challenging. In the case $\beta \neq 0$ in equation (1.1), we give a numerical answer to the question of the blow-up profile, where no theoretical is available. This way, our numerical result gives use to new conjecture.

The paper is organized as follows: In section 2, we give some theoretical framework on the study. Section 3 presents the approximation scheme and the rescaling algorithm. The convergence of the numerical solution for problem (1.1) is proved in section 4. In the last section, we give some numerical experiments to confirm the theoretical results.

Acknowledgement: The author is grateful to L. El Alaoui and H. Zaag for helpful suggestions and remarks during the preparation of this paper.

2 The theoretical framework

Equation (1.1) **in case** $\beta = 0$: The existence of blow-up solution for equation (1.1) has been proved by several authors ([24], [25], [38], [7]). We have lots of results concerning the behavior of the solution u of (1.1) at blow-up time, near blow-up points ([28], [29], [30], [22], [23], [34], [33], [33], [54], [55] and [40], [42]). This study has been done through the introduction for each $a \in \Omega$ (a may be a blow-up point of u or not) the following *similarity variables:*

$$w_{a,T}(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \quad y = \frac{x-a}{\sqrt{T-t}}, \quad s = -\log(T-t), \tag{2.1}$$

and $w_{a,T} = w$ solves a new parabolic equation in (y,s): for all $s \geq -\log T$ and $y \in D_{a,s}$, $D_{a,s} = \{y \in \mathbb{R} | a + ye^{-s/2} \in \Omega\}$,

$$\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1} w. \tag{2.2}$$

Studying solutions of (1.1) near blow-up is therefore equivalent to analyzing large-time asymptotics of solutions of (2.2). Each result for u has an equivalent formulation in terms of w.

One of the main results which is established in [29], [30] is that a is a blow-up point if and only if

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} u(a + y\sqrt{T-t}, t) = \pm\kappa,$$

uniformly in $|y| \leq C$, where $\kappa = (p-1)^{-\frac{1}{p-1}}$.

In [26], [27], the authors used a formal argument adapted from [37] to derive the ansatz

$$u(x,t) \sim (T-t)^{-\frac{1}{p-1}} \left(p - 1 + \frac{(1-p)^2}{4p} \frac{(x-a)^2}{(T-t)|\log(T-t)|} \right)^{-\frac{1}{p-1}}. \tag{2.3}$$

This ansatz has been proved in [54], [10], [42] for some examples of initial data. More precisely, w has a limiting profile in the variable $z = \frac{y}{\sqrt{s}}$ (see [40], [42], [54], [34]), in the sense that

$$\left\| w_{a,T}(s) - f\left(\frac{\cdot}{\sqrt{s}} \right) \right\|_{L^\infty} \to 0 \quad \text{as } s \to +\infty, \tag{2.4}$$

where

$$f(z) = \left(p - 1 + \frac{(p-1)^2}{4p} |z|^2 \right)^{-\frac{1}{p-1}}. \tag{2.5}$$

The profile (2.5) is stable under perturbations of initial data, other profiles are possible but they are suspected to be unstable (see [40], [18], [17]). Note that Herrero and

Velázquez proved the genericity of the behavior (2.3) in [33] and [35] in one space dimension.

Equation (1.1) **in case** $\beta \neq 0$: When $\beta \in (-2, 0)$, in [52] (see also [51], [14], [50], [49]), the authors proved the existence of a non-trivial backward self-similar solution which blows up in finite time, only at one point and described the asymptotic behavior of its radially symmetric profile. More precisely, they showed the existence of a solution of (1.1) of the form

$$u(x,t) = (T-t)^{-\frac{1}{p-1}} v\left(\frac{x}{\sqrt{T-t}}\right), \tag{2.6}$$

where v satisfies for all $\xi \in \mathbb{R}$,

$$\Delta v(\xi) + \beta |\nabla v(\xi)|^q - \left[\frac{\xi}{2} \cdot \nabla v(\xi) + \frac{1}{p-1} v(\xi)\right] + |v(\xi)|^{p-1} v(\xi) = 0.$$

Note that this type of behavior does not hold when $\beta = 0$. Indeed, from Giga and Kohn [28], we know that the only solutions of the form (2.6) are 0 and $\pm\kappa(T-t)^{-\frac{1}{p-1}}$ with $\kappa = (p-1)^{-\frac{1}{p-1}}$.

We wonder however whether equation (1.1) has solutions which behave like the solution of the case $\beta = 0$, namely such that

$$\left\| (T-t)^{1/(p-1)} u(t) - \bar{f}_\beta \left(\frac{\cdot - a}{\sqrt{(T-t)|\log(T-t)|}} \right) \right\|_{L^\infty} \to 0, \quad \text{as} \quad t \to T, \tag{2.7}$$

where a is the blow-up point,

$$\bar{f}_\beta(z) = \left(p - 1 + b(p,\beta)|z|^2\right)^{-\frac{1}{p-1}}, \quad \text{with} \quad b(p,0) = \frac{(p-1)^2}{4p}, \tag{2.8}$$

or in *similarity variables* defined in (2.1),

$$\left\| w(s) - \bar{f}_\beta \left(\frac{\cdot}{\sqrt{s}} \right) \right\|_{L^\infty} \to 0, \quad \text{as} \quad s \to \infty. \tag{2.9}$$

Up to our knowledge, there is no theoretical answer to this equation. We answer it positively through a numerical method in this paper (see Section 5.2 below).

The complex Ginzburg-Landau equation: In [39], Masmoudi and Zaag constructed the first solution to equation (1.2) which blows up in finite time T only at one blow-up point and gave a sharp description of its blow-up profile. Furthermore, they showed the stability of that solution with respect to perturbations in initial data. Their result extends the previous result of Zaag [56] done for $\gamma = 0$. More precisely, they used the following self-similar transformation of equation (1.2):

$$w_{a,T}(y,s) = (T-t)^{\frac{1+\imath\delta}{p-1}} u(x,t), \quad y = \frac{x-a}{\sqrt{T-t}}, \quad s = -\log(T-t), \tag{2.10}$$

and then $w(y,s)$ satisfies the following equation:

$$\partial_s w = (1+\imath\gamma)\Delta w - \frac{1}{2} y\cdot\nabla w - \frac{1+\imath\delta}{p-1} w + (1+\imath\delta)|w|^{p-1}w.$$

Their main result is the following: for any $(\delta,\gamma)\in\mathbb{R}^2$ such that $p-\delta^2-\gamma\delta(p+1)>0$, equation (1.2) has a solution $u(x,t)$ blowing up in finite time T only at one blow-up point $a\in\mathbb{R}$. Moreover,

$$\left\| |s|^{-\imath\mu} w_{a,T}(s) - \tilde{f}_{\delta,\gamma}\left(\frac{\cdot}{\sqrt{s}}\right)\right\|_{L^\infty} \le \frac{C}{1+\sqrt{|s|}}, \tag{2.11}$$

where $\mu = -\frac{2\gamma b(p,\delta,\gamma)}{(p-1)^2}(1+\delta^2),\ b(p,\delta,\gamma) = \frac{(p-1)^2}{4(p-\delta^2-\gamma\delta(p+1))}$ and

$$\tilde{f}_{\delta,\gamma}(z) = \left(p-1+b(p,\delta,\gamma)|z|^2\right)^{-\frac{1+\imath\delta}{p-1}}. \tag{2.12}$$

Remark 2.1. *We remark that equation* (1.2) *is rotation invariant. Therefore, $e^{\imath\theta}\tilde{f}_{\delta,\gamma}$ is also an asymptotic profile of the solution of* (1.2) *with $\theta\in\mathbb{R}$.*

Remark 2.2. *In our paper, we give the first numerical computation of this result. Note that the stability result of [39] concerning that solution makes it visible in numerical simulations.*

3 The numerical method

In this section, we recall the rescaling algorithm introduced in [9].

3.1 The numerical scheme

We first give an Euler approximation of (1.1) and (1.2). Let I be a positive integer and let us discretize the domain $\Omega=(-1,1)$ by the grid $x_i = ih$ where $-I\le i\le I$ and $h=\frac{1}{I}$. Let $\tau>0$ be a time step and $n\ge 0$ be a positive integer. Then, we set $t_n = n\tau$. In what follows, the lowercase letter denotes the exact values, whereas the capital letter denotes its approximation, for example, we write $u_{i,n}\equiv u(x_i,t_n)$ and $U_{i,n}$ the approximation of $u(x_i,t_n)$. In the following, the notation $\mathbf{U}_n$ stands for $(U_{-I},\ldots,U_0,\ldots,U_I)^T$. In addition, we denote

$$\begin{aligned}
\delta_t U_{i,n} &= \frac{U_{i,n+1}-U_{i,n}}{\tau},\\
\delta_x U_{i,n} &= \frac{U_{i+1,n}-U_{i-1,n}}{2h},\\
\delta_x^2 U_{i,n} &= \frac{U_{i-1,n}-2U_{i,n}+U_{i+1,n}}{h^2}.
\end{aligned} \tag{3.1}$$

Discretization of the semilinear heat equation:
The Euler discretization of (1.1) is defined as follows: for $n \geq 0$ and $-I+1 \leq i \leq I-1$,

$$\left\{\begin{array}{rcl} \delta_t U_{i,n} & = & \delta_x^2 U_{i,n} + \left|U_{i,n}^{p-1}\right| U_{i,n} + \beta|\delta_x U_{i,n}|^{\frac{2p}{p+1}}, \\ U_{-I,n} & = & U_{I,n} = 0, \end{array}\right. \tag{3.2}$$

with $U_{i,0} = \phi_i$ where $\phi_i = u_0(x_i)$. Note that $U_{i,n}$ is defined for all $n \geq 0$ and $-I \leq i \leq I$.

Discretization of the Ginzburg-Landau equation:
Let us write the solution of (1.2) as $u = v + \imath w$ and $|u| = \sqrt{v^2+w^2}$. Then (1.2) can be rewritten as follows:

$$\left\{\begin{array}{rcl} v_t & = & v_{xx} - \gamma w_{xx} + \left(v^2+w^2\right)^{\frac{p-1}{2}}(v - \delta w) \\ w_t & = & \gamma v_{xx} + w_{xx} + \left(v^2+w^2\right)^{\frac{p-1}{2}}(\delta v + w) \\ v(x,0) & = & \Re\big(u_0(x)\big), \quad w(x,0) = \Im\big(u_0(x)\big). \end{array}\right. \tag{3.3}$$

Denote by $V_{i,n}$ and $W_{i,n}$ approximations of $v(x_i,t_n)$ and $w(x_i,t_n)$ respectively. On setting $\mathbf{V}_n = (V_{-I,n},\ldots,V_{I,n})^T$, $\mathbf{W}_n = (W_{-I,n},\ldots,W_{I,n})^T$, the Euler scheme approximating the solution of (3.3) is given below: for $n \geq 0$ and $-I+1 \leq i \leq I-1$,

$$\left\{\begin{array}{rcl} \delta_t V_{i,n} & = & \delta_x^2 V_{i,n} - \gamma\delta_x^2 W_{i,n} + R_{i,n}(V_{i,n} - \delta W_{i,n}), \\ \delta_t W_{i,n} & = & \gamma\delta_x^2 V_{i,n} + \delta_x^2 W_{i,n} + R_{i,n}(\delta V_{i,n} + W_{i,n}), \\ V_{-I,n} & = & V_{I,n} = W_{-I,n} = W_{I,n} = 0, \end{array}\right. \tag{3.4}$$

with $V_{i,0} = \Re\big(\phi_i\big), W_{i,0} = \Im\big(\phi_i\big)$ where $\phi_i = u_0(x_i)$ and $R_{i,n} = (V_{i,n}^2 + W_{i,n}^2)^{\frac{p-1}{2}}$.

Remark 3.1. *By Taylor expansion, one can show that the central difference approximation given in* (3.1) *is second-order accurate. Therefore, both difference schemes* (3.2) *and* (3.4) *are first-order accurate in time and second-order in space.*

In what follows, let $\mathbf{a} = (a_{-I},\ldots,a_0,\ldots,a_I)^T$, denote $\|\mathbf{a}\|_\infty = \max_i |a_i|$. We say that $\mathbf{a}$ is positive if each component of $\mathbf{a}$ is positive and write $\mathbf{a} > 0$. Similar notations $\geq, \leq, <$ can be defined.

3.2 The rescaling method

For the sake of clarity, we present the rescaling method in [9], only for the approximation of the semilinear heat equation (3.2). Straightforward adaptations allos to derive it for the Ginzburg-Landau equation approximated in (3.4).
We first introduce some notations:

- $\lambda < 1$ is a scaling factor such that λ^{-1} is a small positive integer.
- M is a maximum amplitude before rescaling.
- α is a parameter controlling the width of the interval to be rescaled.

- ◇ $u^{(k)}(\xi_k, \eta_k)$ is the k-th *rescaled solution* defined in space-time variables (ξ_k, η_k). If $k = 0$, $u^{(0)}(\xi_0, \eta_0) \equiv u(x,t)$, $(\xi_0, \eta_0) \equiv (x,t)$.
- ◇ h_k, τ_k denote the space and time step used to approximate $u^{(k)}$.
- ◇ $U^{(k)}_{i,n}$ is an approximation value of $u^{(k)}(\xi_{k,i}, \eta_{k,n})$ where $\xi_{k,i} = ih_k$ and $\eta_{k,n} = n\tau_k$.

Let $\{(x_i, t_n, F_{i,n}) | -I \leq i \leq I, 0 \leq n \leq N\}$ be a set of data points, we associate the function $F_{h,\tau}$ which is a piecewise linear approximation in both space and time such that $F_{h,\tau}(x_i, t_n) = F_{i,n}$ and for all $(x,t) \in (x_i, x_{i+1}) \times (t_n, t_{n+1})$,

$$\begin{aligned} F_{h,\tau}(x,t) = & \frac{1}{h\tau}\left[F_{i,n}(x_{i+1}-x)(t_{n+1}-t) + F_{i+1,n}(x-x_i)(t_{n+1}-t)\right] \\ & + \frac{1}{h\tau}\left[F_{i,n+1}(x_{i+1}-x)(t-t_n) + F_{i+1,n+1}(x-x_i)(t-t_n)\right]. \end{aligned} \tag{3.5}$$

At some points, we may use the notation $F_{h,n}(x) \equiv F_{h,\tau}(x, t_n)$ for a given t_n and $F_{i,\tau}(t) \equiv F_{h,\tau}(x_i, t)$ for a given x_i.

We now recall the rescaling method introduced in [9].
The solution of (3.2) is integrated until getting the first time step $\mathbf{n}_0$ such that $\|\mathbf{U}_{\mathbf{n}_0}\|_\infty \geq M$. Then we find out a value τ_0^* satisfying

$$(\mathbf{n}_0 - 1)\tau \leq \tau_0^* \leq \mathbf{n}_0\tau \quad \text{and} \quad \|\mathbf{U}_{h,\tau}(\cdot, \tau_0^*)\|_\infty = M,$$

and two grid points $x_{i_0^-}, x_{i_0^+}$, with $i_0^+, i_0^- \in \{-I, \dots, 0, \dots, I\}$, such that

$$\begin{cases} \mathbf{U}_{h,\tau}(x_{i_0^- -1}, \tau_0^*) & < \alpha M & \leq \mathbf{U}_{h,\tau}(x_{i_0^-}, \tau_0^*), \\ \mathbf{U}_{h,\tau}(x_{i_0^+ +1}, \tau_0^*) & < \alpha M & \leq \mathbf{U}_{h,\tau}(x_{i_0^+}, \tau_0^*). \end{cases}$$

On the interval $(x_{i_0^-}, x_{i_0^+})$ and for $t \geq \tau_0^*$, we refine the mesh by a factor λ in space and λ^2 in time. More precisely, we introduce

$$u^{(1)}(\xi_1, \eta_1) = \lambda^{\frac{2}{p-1}} u(\lambda\xi_1, \tau_0^* + \lambda^2\eta_1),$$

which is also a solution of equation (1.1), thanks to the scale invariance property stated after (1.3). From a numerical point of view, it is important to use for $u^{(1)}$ the same discretization as for u. Let h_1 be the space discretization step and τ_1 be the time discretization step, then we need to set $h_1 = h$ and $\tau_1 = \tau$ to use the same scheme (3.2) for approximating $u^{(1)}$. In other words, the approximation of u on the interval $(x_{i_0^-}, x_{i_0^+})$ with the steps $\lambda h, \lambda^2\tau$ is equivalent to the approximation of $u^{(1)}$ on the interval $\lambda^{-1}\left(x_{i_0^-}, x_{i_0^+}\right)$ by using h and τ as discretization parameters.

Let $I_1 = \lambda^{-1} i_0^+$ and $\mathbf{U}_n^{(1)} = \left(U_{-I_1,n}^{(1)}, \ldots, U_{0,n}^{(1)}, \ldots, U_{I_1,n}^{(1)}\right)^T$ be an approximation of $u^{(1)}$ at time $\eta_{1,n}$. Then, $\mathbf{U}_{n+1}^{(1)}$ solves the following equations: for all $n \geq 0$, i between $-I_1+1$ and $I_1 - 1$,

$$\left\{\begin{array}{lcl} \delta_t U_{i,n}^{(1)} & = & \delta_x^2 U_{i,n}^{(1)} + \left|U_{i,n}^{(1)}\right|^{p-1} U_{i,n}^{(1)} + \beta \left|\delta_x U_{i,n}^{(1)}\right|^{\frac{2p}{p+1}}, \\ U_{I_1,n}^{(1)} & = & U_{-I_1,n}^{(1)} = \psi_n^{(1)}, \quad U_{i,0}^{(1)} = \phi_i^{(1)}, \end{array}\right.$$

where

$$\psi_n^{(1)} = \lambda^{\frac{2}{p-1}} \mathbf{U}_{h,\tau}(x_{i_0^+}, \tau_0^* + \lambda^2 n\tau), \quad n \geq 0, \tag{3.6}$$

$$\phi_i^{(1)} = \lambda^{\frac{2}{p-1}} \mathbf{U}_{h,\tau}(\lambda \xi_{1,i}, \tau_0^*), \quad -I_1 \leq i \leq I_1. \tag{3.7}$$

We stop the computation of $\mathbf{U}^{(1)}$ at the first time level $\eta_{1,\mathbf{n}_1}$ ($\mathbf{n}_1 \geq 1$) such that $\left\|\mathbf{U}_{\mathbf{n}_1}^{(1)}\right\|_\infty \geq M$. After that, we determine τ_1^* and two grid points $\xi_{1,i_1^-}, \xi_{1,i_1^+}$ where $i_1^-, i_1^+ \in \{-I_1, \ldots, I_1\}$ by

$$\left\{\begin{array}{l} (\mathbf{n}_1 - 1)\tau_1 \leq \tau_1^* \leq \mathbf{n}_1 \tau_1 \\ \left\|\mathbf{U}_{h,\tau}^{(1)}(\cdot, \tau_1^*)\right\|_\infty = M, \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lll} \mathbf{U}_{h,\tau}^{(1)}(\xi_{1,i_1^- -1}, \tau_1^*) & < \alpha M & \leq \mathbf{U}_{h,\tau}^{(1)}(\xi_{1,i_1^-}, \tau_1^*), \\ \mathbf{U}_{h,\tau}^{(1)}(\xi_{1,i_1^+ +1}, \tau_1^*) & < \alpha M & \leq \mathbf{U}_{h,\tau}^{(1)}(\xi_{1,i_1^+}, \tau_1^*). \end{array}\right.$$

We remark that the computation of $\mathbf{U}^{(1)}$ requires an initial and a boundary conditions. The initial data conditions are already obtained by (3.7). It remains to focus on the boundary condition (3.6). Both $\mathbf{U}$ and $\mathbf{U}^{(1)}$ are stepped forward independently, each on its own grid. A single time step of $\mathbf{U}$ corresponds to λ^{-2} time steps of $\mathbf{U}^{(1)}$. Therefore, the linear interpolation in time of $\mathbf{U}$ is used to find the boundary values of $\mathbf{U}^{(1)}$. After stepping forward $\mathbf{U}^{(1)}$ λ^{-2} times, the values of $\mathbf{U}$ at grid points on the interval $(x_{i_0^-}, x_{i_0^+})$ are modified to better with the fine grid solution $\mathbf{U}^{(1)}$. On the interval where $\mathbf{U}^{(1)} > \alpha M$, the entire procedure is repeated, yielding $\mathbf{U}^{(2)}$, and so forth.

The $(k+1)$-st rescaled solution $u^{(k+1)}$ is introduced when η_k reaches a value τ_k^* satisfying

$$(\mathbf{n}_k - 1)\tau_k \leq \tau_k^* \leq \mathbf{n}_k \tau_k, \quad \mathbf{n}_k > 0 \quad \text{and} \quad \left\|\mathbf{U}^{(k)}(\cdot, \tau_k^*)\right\|_\infty = M. \tag{3.8}$$

The interval $(\xi_{k,i_k^-}, \xi_{k,i_k^+})$ to be rescaled satisfies

$$\left\{\begin{array}{lll} \mathbf{U}_{h,\tau}^{(k)}(\xi_{k,i_k^- -1}, \tau_k^*) & < \alpha M & \leq \mathbf{U}_{h,\tau}^{(k)}(\xi_{k,i_k^-}, \tau_k^*), \\ \mathbf{U}_{h,\tau}^{(k)}(\xi_{k,i_k^+ +1}, \tau_k^*) & < \alpha M & \leq \mathbf{U}_{h,\tau}^{(k)}(\xi_{k,i_k^+}, \tau_k^*). \end{array}\right.$$

The solution $u^{(k+1)}$ is related to $u^{(k)}$ by

$$u^{(k+1)}(\xi_{k+1}, \eta_{k+1}) = \lambda^{\frac{2}{p-1}} u^{(k)}(\lambda \xi_{k+1}, \tau_k^* + \lambda^2 \eta_{k+1}). \tag{3.9}$$

Let $I_{k+1} = \lambda^{-1} i_k^+$ and

$$\mathbf{U}_n^{(k+1)} = \left(U_{-I_{k+1},n}^{(k+1)}, \dots, U_{0,n}^{(k+1)}, \dots, U_{I_{k+1},n}^{(k+1)} \right)^T$$

be an approximation of $u^{(k+1)}$ at time $\eta_{k+1,n}$. Then $\mathbf{U}_{n+1}^{(k+1)}$ is a solution of the following equations: for all $n \geq 0$, i between $-I_{k+1}+1$ and $I_{k+1}-1$,

$$\begin{cases} \delta_t U_{i,n}^{(k+1)} &= \delta_x^2 U_{i,n}^{(k+1)} + \left| U_{i,n}^{(k+1)} \right|^{p-1} U_{i,n}^{(k+1)} + \beta \left| \delta_x U_{i,n}^{(k+1)} \right|^{\frac{2p}{p+1}}, \\ U_{I_k,n}^{(k+1)} &= U_{-I_k,n}^{(k+1)} = \psi_n^{(k+1)}, \quad U_{i,0}^{k+1} = \phi_i^{(k+1)}, \end{cases} \tag{3.10}$$

where

$$\psi_n^{(k+1)} = \lambda^{\frac{2}{p-1}} \mathbf{U}_{h,\tau}^{(k)}(\xi_{k,i_k^+}, \tau_k^* + \lambda^2 n\tau), \quad n \geq 0, \tag{3.11}$$

$$\phi_i^{(k+1)} = \lambda^{\frac{2}{p-1}} \mathbf{U}_{h,\tau}^{(k)}(\lambda \xi_{k+1,i}, \tau_k^*), \quad -I_{k+1} \leq i \leq I_{k+1}. \tag{3.12}$$

We step forward $\mathbf{U}^{(k+1)}$ on the interval $\lambda^{-1}(\xi_{k,i_k^-}, \xi_{k,i_k^+})$ with the space step h_{k+1} and time step τ_{k+1}. Here, we set $h_{k+1} = h_k = \cdots = h$ and $\tau_{k+1} = \tau_k = \cdots = \tau$ to use the same scheme as for $\mathbf{U}^{(k)}, \mathbf{U}^{(k-1)}, \dots, \mathbf{U}$. The initial data of (3.10) is given in (3.12). For the boundary data of (3.10), it is obtained by using the linear interpolation in time of $\mathbf{U}^{(k)}$ given in (3.11). Hence, we step forward independently the previous solutions $\mathbf{U}^{(k)}, \mathbf{U}^{(k-1)}, \dots$ each one on its own grid. Previously, $\mathbf{U}^{(k)}$ is stepped forward once every λ^{-2} time steps of $\mathbf{U}^{(k+1)}$, $\mathbf{U}^{(k-1)}$ once every λ^{-4} time steps of $\mathbf{U}^{(k+1)}, \dots$. After λ^{-2} time steps of $\mathbf{U}^{(k+1)}$, the values of $\mathbf{U}^{(k)}$ on the interval which has been refined need to be updated to fit with the calculation of $\mathbf{U}^{(k+1)}$; this is performed on $\mathbf{U}^{(k-1)}$ after λ^{-4} time steps of $\mathbf{U}^{(k+1)}$ and so forth. We stop the evolution of $\mathbf{U}^{(k+1)}$ when its amplitude reaches the given threshold M and another rescaling can be performed.

To make it clearer, we describe the rescaling method by the following *algorithm*. Assume that we perform up to the K-th *rescaled solution*.

0. Set up parameters: $M, \lambda, \alpha, h, \tau, I$.
1. Initial phase:
   - Forward $\mathbf{U}$ until $\max_i U_i \geq M$.
   - Get the values of τ_0^* and $x_{i_0^-}, x_{i_0+}$.
2. Iterative phase: set $k = 1$, while $k \leq K$ then
   (a) Define a grid for $\mathbf{U}^{(k)}$ on the interval $\lambda^{-1}\left(\xi_{k-1,i_{k-1}^-}, \xi_{k-1,i_{k-1}^+}\right)$.
   (b) Compute the initial data for $\mathbf{U}^{(k)}$ from $\mathbf{U}_{h,\tau}^{(k-1)}(\cdot, \tau_{k-1}^*)$.
   (c) For $i = 0$ to $k-1$: forward $\mathbf{U}^{(i)}$ one step.

(d) Set $n = 1$.

(e) While $\max_i U_i^{(k)} < M$ then

+ Forward $\mathbf{U}^{(k)}$ one step.

+ Compute the boundary values of $\mathbf{U}^{(k)}$ from $\mathbf{U}_{h,\tau}^{(k-1)}(\xi_{k-1,i_{k-1}^+}, \tau_{k-1}^* + \lambda n \tau_k)$.

+ For $j = 0$ to $k-1$: if $\mod\left(n, \lambda^{-2(j+1)}\right) = 0$ then

- Update $\mathbf{U}^{(k-j-1)}$ on the interval to be rescaled.

- Forward $\mathbf{U}^{(k-j-1)}$ one step.

+ Set $n = n + 1$.

(f) Get the values of τ_k^* and $\xi_{k,i_k^-}, \xi_{k,i_k^+}$.

(g) For $i = 1$ to k: update $\mathbf{U}^{k-i}$.

(i) Set $k = k + 1$, $\mathbf{n}_k = n$ and go to step (a).

Remark 3.2. *The value of M should be chosen such that the maximum of the initial data of all rescaled solutions are equal. This means that for all $k \geq 0$,*

$$\lambda^{\frac{2}{p-1}} \|u^{(k)}(\tau_k^*)\|_\infty = \|u_0\|_\infty.$$

Using the fact that $\|u^{(k)}(\tau_k^*)\|_\infty = M$, it yields that $M = \|u_0\|_\infty \lambda^{-\frac{2}{p-1}}$.

To end this section, we want to give a definition of the numerical solution $\mathbf{U}_{h,\tau}(x,t)$ of the rescaling method. Let $\sigma > 0$ small enough, $h > 0$ and $\tau > 0$ be the space and time step, then, for each $(x,t) \in [-1,1] \times [0, T - \sigma]$, we can find an integer $K \geq 0$ such that

$$\mu_{K-1} \leq t < \mu_K \quad \text{and} \quad \left\| \mathbf{U}_{h,\tau}^{(K)}\left(\cdot, \lambda^{-2K}\left(t - \mu_{K-1}\right)\right) \right\|_\infty < M,$$

where $\mu_q := \sum_{i=0}^q \lambda^{2i} \tau_i^*$. Then, $\mathbf{U}_{h,\tau}(x,t)$ is defined as follows:

$$\mathbf{U}_{h,\tau}(x,t) = \begin{cases} \lambda^{-\frac{2K}{p-1}} \mathbf{U}_{h,\tau}^{(K)}\left(\lambda^{-K}x, \lambda^{-2K}\left(t - \mu_{K-1}\right)\right) & \text{if } x \in \Omega_K, \\ \lambda^{-\frac{2(K-1)}{p-1}} \mathbf{U}_{h,\tau}^{(K-1)}\left(\lambda^{-(K-1)}x, \lambda^{-2(K-1)}\left(t - \mu_{K-2}\right)\right) & \text{if } x \in \Omega_{K-1} \backslash \Omega_K, \\ \vdots & \\ \lambda^{-\frac{2}{p-1}} \mathbf{U}_{h,\tau}^{(1)}\left(\lambda^{-1}x, \lambda^{-2}\left(t - \mu_0\right)\right) & \text{if } x \in \Omega_1 \backslash \Omega_2, \\ \mathbf{U}_{h,\tau}^{(0)}(x,t) & \text{if } x \in \Omega \backslash \Omega_1. \end{cases} \tag{3.13}$$

where $\Omega_k = (\lambda^k \xi_{k-1,i_{k-1}^-}, \lambda^k \xi_{k-1,i_{k-1}^+})$ for $k \geq 1$ and $\mathbf{U}_{h,\tau}^{(k)}$ is the linear interpolation defined in (3.5).

One can see that the solution defined in (3.13) tends to infinity when k goes to infinity. We say that the solution defined in (3.13) blows up in a finite time if

$$T_{h,\tau} = \lim_{K \to +\infty} \sum_{k=0}^{K} \lambda^{2k} \tau_k^* < +\infty. \tag{3.14}$$

The time $T_{h,\tau}$ is call the numerical blow-up time.

Remark 3.3. *We can see that $T_{h,\tau}$ defined in* (3.14) *is finite if the solution $\mathbf{U}_{h,\tau}^{(k)}$ (defined from $\mathbf{U}_n^{(k)}$ by* (3.5)*) reaches the given threshold M in a bounded number of time steps, namely when $\bar{\tau} = \sup_{k\geq 0} \tau_k^* < +\infty$. In this case, we see that*

$$T_{h,\tau} \leq \lim_{K\to+\infty} \bar{\tau} \sum_{k=0}^{K} \lambda^{2k} = \frac{\bar{\tau}}{1-\lambda^2} < +\infty.$$

4 Convergence of the rescaling method

This section is devoted to the convergence analysis of the rescaling method for problem (1.1) with $\beta \in \mathbb{R}$ and $q \in [1,2)$ not necessarily $q = \frac{2p}{p+1}$, under some regularity assumptions. Note that the discrete problem (3.2) when $\beta = 0$ has already been treated in [45]. When $\beta \neq 0$, proceeding as for $\beta = 0$, the crucial step is to obtain a comparison principle for the discrete problem (see Lemma 4.10 below). Note that we could not prove analogous results for the equation (1.2), since we already have no comparison principle in the continuous case.

Theorem 4.1. *Consider $h > 0$ sufficiently small and $\tau > 0$ such that $\tau \leq \frac{h^2}{2}$. Let $\sigma > 0$, suppose that the problem* (1.1) *(with $q \in [1,2)$) has a non-negative solution $u(x,t) \in \mathcal{C}^{4,2}([-1,1]\times[0,T-\sigma])$ and the initial data of* (3.2) *satisfies*

$$\sup_{x\in[-1,1]} |\phi_h(x) - u(x,0)| = \mathcal{O}(h^2) \quad as \quad h \to 0.$$

Then the solution $\mathbf{U}_{h,\tau}$ defined in (3.13) *satisfies*

$$\sup_{(x,t)\in[-1,1]\times[0,T-\sigma]} |\mathbf{U}_{h,\tau}(x,t) - u(x,t)| = \mathcal{O}\left(h^2\right) \quad as \quad h \to 0.$$

Remark 4.2. *The convergence of the rescaling method stated in Theorem 4.1 is proved by a recursive application of Proposition 4.5 below. Therefore, it is enough to give the proof of this proposition.*

Remark 4.3. *The choice of the central difference approximation for the gradient in* (3.1) *is crucial to get the $\mathcal{O}(h^2)$ convergence in Theorem 4.1.*

Remark 4.4. *When $\beta \neq 0$ and $q < 2$, we have been unable to show that equation* (1.1) *has a $\mathcal{C}^{4,2}$ solution (this is the case when $q = \frac{2p}{p+1}$). On the contrary, when $\beta = 0$ or $q \geq 2$, we do have $\mathcal{C}^{4,2}$ solutions (just take $p \geq 2$ and $u_0 \in \mathcal{C}^4([-1,1])$, see A for a justification of this fact). Hence, our convergence result (Theorem 4.1) is at least meaningful when $\beta = 0$.*

One can see from the definition of $\mathbf{U}_{h,\tau}$ in (3.13) that $\mathbf{U}_{h,\tau}$ is constructed from $\mathbf{U}_{h,\tau}^{(k)}$ which is the solutions of the problem (3.10). It is reasonable then to consider the following problem with the non-zero Dirichlet condition,

$$\left\{\begin{array}{rcll} v_t(x,t) &=& v_{xx}(x,t) + g(v(x,t), v_x(x,t)) & (x,t) \in (-L,L)\times(0,T), \\ v(-L,t) &=& v(L,t) = v_1(t) & t \in (0,T), \\ v(x,0) &=& v_0(x) & x \in (-L,L), \end{array}\right. \tag{4.1}$$

where $v(t) : x \in (-L, L) \mapsto v(x,t) \in \mathbb{R}$, $p > 1$,

$$g(v, v_x) = |v|^{p-1}v + \beta|v_x|^q, \quad \text{with} \quad q = \frac{2p}{p+1}.$$

Let $I > 0$ and consider the grid $x_i = ih$, $-I \le i \le I$ where $h = \frac{L}{I}$. Let $\tau > 0$ be a time step and denote $t_n = n\tau$. Let

$$\mathbf{V}_n = (V_{-I,n}, \dots, V_{0,n}, \dots, V_{I,n})^T$$

be the approximation of $v(t_n)$ at grid points. Then, $\mathbf{V}_{n+1}$ is a solution of the following equation: for all $n \ge 0$, $i = -I+1, \dots, I-1$,

$$\begin{cases} \delta_t V_{i,n} &= \delta_x^2 V_{i,n} + g(V_{i,n}, \delta_x V_{i,n}) \\ V_{-I,n} &= V_{I,n} = \psi_n, \quad V_{i,0} = \phi_i, \end{cases} \tag{4.2}$$

where ψ_n and ϕ_i stand for $\psi_n^{(k)}$ and $\phi_i^{(k)}$ introduced in (3.7), (3.12), (3.6) and (3.11).

Let $\mathbf{V}_{h,n}(x)$ be the piecewise linear interpolation generated from $\mathbf{V}_n$ by (3.5), then, we get the following results:

Proposition 4.5. *Consider $h > 0$ sufficiently small and $\tau > 0$ such that $\tau \le \frac{h^2}{2}$. Let $\eta \in (0,T)$, suppose that the problem* (4.1) *(with $q \in [1,2)$) has a non-negative solution $v \in \mathcal{C}^{4,2}([-L,L] \times [0, T-\eta])$, the initial data and boundary data of* (4.2) *satisfy*

$$\epsilon_1 = \sup_{x \in [-L,L]} |v(x,0) - \phi_h(x)| = o(h) \quad as \quad h \to 0,$$
$$\epsilon_2 = \sup_{t \in [0,T-\eta]} |v(L,t) - \psi_\tau(t)\| = o(1) \quad as \quad \tau \to 0,$$

where ϕ_h and ψ_τ are the interpolations of ϕ_i and ψ_n defined in (3.5). *Then,*

$$\max_{0 \le n \le N} \|\mathbf{V}_{h,n} - v(t_n)\|_\infty = \mathcal{O}(\epsilon_1 + \epsilon_2 + h^2) \quad as \quad h \to 0,$$

where $N > 0$ is such that $t_N = N\tau \le T - \eta$.

We now state some properties of the discrete scheme (4.2).

Lemma 4.6. *Let $n = 1, 2, \dots, N$, $\mathbf{V}_n$ be the solution of* (4.2) *and $\mathbf{V}_0$ be a symmetric data. Then, $\mathbf{V}_n$ is also symmetric for all $n = 0, 1, \dots N$.*

Proof. It is straightforward from the symmetry of the data and the equation. □

Remark 4.7. *We can consider the problem* (4.2) *on the half interval $[0, L]$ from now on. In particular, we have for $n \ge 0$ and $i = 1, 2, \dots I-1$,*

$$\delta_x^2 V_{0,n} = \frac{2V_{1,n} - 2V_{0,n}}{h^2}, \quad \delta_x^2 V_{i,n} = \frac{V_{i-1,n} - 2V_{i,n} + V_{i+1,n}}{h^2},$$
$$\delta_x V_{0,n} = 0, \quad \delta_x V_{i,n} = \frac{V_{i+1,n} - V_{i-1,n}}{2h}. \tag{4.3}$$

Remark 4.8. *The convergence stated in Proposition 4.5 holds without the symmetric property. However, we handle only symmetric data to simplify the proofs below.*

Lemma 4.9 (Positivity of the discrete solution). *Let $n = 1, 2, \ldots, N$ and $\mathbf{V}_n$ be the solution of* (4.2). *Suppose that $\mathbf{V}_0 \geq 0$ and $V_{I,n} \geq 0$ for $n = 0, 1, \ldots, N$. Assume in addition that $\tau \leq \frac{h^2}{2}$ and $h \leq \left(\frac{2^q}{|\beta| M_0^{q-1}}\right)^{\frac{1}{2-q}}$ if $\beta < 0$, where $M_0 = \max\limits_{0 \leq n \leq N} \|\mathbf{V}_n\|_\infty$. Then, $\mathbf{V}_n \geq 0$ for all n between 0 and N.*

Proof. By induction, we assume that $\mathbf{V}_k \geq 0$ for all $k = 0, 1, \ldots, n$. We need to show that $\mathbf{V}_{n+1} \geq 0$. Using (4.2), we see that

$$V_{0,n+1} = \left(1 - \frac{2\tau}{h^2}\right) V_{0,n} + \frac{2\tau}{h^2} V_{1,n} + \tau V_{0,n}^p,$$

where we used the fact that $\delta_x V_{0,n} = 0$ from Remark 4.7. From the restriction $\tau \leq \frac{h^2}{2}$, we have $V_{0,n+1} \geq 0$.
For i between 1 and $I-1$, we have

$$V_{i,n+1} = \left(1 - \frac{2\tau}{h^2}\right) V_{i,n} + \frac{\tau}{h^2}\left(V_{i+1,n} + V_{i-1,n}\right) + \tau V_{i,n}^p + \frac{\tau\beta}{(2h)^q}\left|V_{i+1,n} - V_{i-1,n}\right|^q.$$

If $\beta \geq 0$ and $\tau \leq \frac{h^2}{2}$, we directly infer the desired result. If $\beta < 0$, we have for $i = 1, \ldots, I-1$,

$$\begin{aligned}
V_{i,n+1} &\geq \left(1 - \frac{2\tau}{h^2}\right) V_{i,n} + \frac{\tau}{h^2}\left(V_{i+1,n} + V_{i-1,n}\right) + \tau V_{i,n}^p - \frac{\tau|\beta|}{(2h)^q}\left(V_{i+1,n}^q + V_{i-1,n}^q\right) \\
&= \left(1 - \frac{2\tau}{h^2}\right) V_{i,n} + \tau V_{i,n}^p + \frac{\tau}{h^q}\left(\frac{1}{h^{2-q}} - \frac{|\beta|}{2^q} V_{i+1,n}^{q-1}\right) V_{i+1,n} \\
&\qquad + \frac{\tau}{h^q}\left(\frac{1}{h^{2-q}} - \frac{|\beta|}{2^q} V_{i-1,n}^{q-1}\right) V_{i-1,n}.
\end{aligned}$$

Here we used the induction assumption that $V_{i,n} \geq 0$ for $i = 0, 1, \ldots, I$. To obtain $V_{i,n+1} > 0$, then it requires the following restrictions

$$\frac{\tau}{h^2} \leq \frac{1}{2} \quad \text{and} \quad \frac{1}{h^{2-q}} - \frac{|\beta|}{2^q} M_0^{q-1} \geq 0.$$

Recall that $q \in [1, 2)$, then the last condition yields $h \leq \left(\frac{2^q}{|\beta| M_0^{q-1}}\right)^{\frac{1}{2-q}}$. This ends the proof of Lemma 4.9. ☐

The following lemma is a discrete version of the maximum principle.

Lemma 4.10. *Let $\mathbf{b}_n = (b_{0,n}, b_{1,n}, \dots, b_{I,n})^T, \mathbf{c}_n = (c_{0,n}, c_{1,n}, \dots, c_{I,n})^T$ be two vectors such that $\mathbf{b}_n \geq 0$ and $\mathbf{c}_n$ is bounded. Let $\mathbf{V}_n = (V_{0,n}, V_{1,n}, \dots, V_{I,n})^T$ satisfy*

$$\begin{aligned}
\delta_t V_{i,n} - \delta_x^2 V_{i,n} - b_{i,n} V_{i,n} - c_{i,n}\delta_x V_{i,n} &\geq 0, \quad 0 \leq i \leq I-1, \\
V_{i,0} &\geq 0, \quad 0 \leq i \leq I, \\
V_{I,n} &\geq 0, \quad n \geq 0.
\end{aligned}$$

If $\tau \leq \frac{h^2}{2}$ and $h \leq \frac{2}{\|\mathbf{c}_n\|_\infty}$, then $\mathbf{V}_n \geq 0$ for all $n \geq 0$.

Remark 4.11. *Note that as before, we handle symmetric data in this lemma. That is the reason why we focus only on $i \geq 0$. Note also that* (4.3) *is useful for this lemma.*

Proof of Lemma 4.10. We proof this lemma by induction. Assume that $\mathbf{V}_k \geq 0$ for $k = 0, 1, \dots, n$. Let us show that $\mathbf{V}_{n+1} \geq 0$. A straightforward calculation yields

$$V_{0,n+1} \geq \frac{2\tau}{h^2} V_{1,n} + \left(1 - \frac{2\tau}{h^2}\right) V_{0,n} + \tau b_{0,n} V_{0,n},$$

for i between 1 and $I-1$, we have

$$\begin{aligned}
V_{i,n+1} \geq & \left(\frac{\tau}{h^2} - \frac{\tau}{2h} c_{i,n}\right) V_{i-1,n} + \left(1 - \frac{2\tau}{h^2}\right) V_{i,n} \\
& + \tau b_{i,n} V_{i,n} + \left(\frac{\tau}{h^2} + \frac{\tau}{2h} c_{i,n}\right) V_{i+1,n}.
\end{aligned}$$

Since $\tau \leq \frac{h^2}{2}$, $h \leq \frac{2}{\|\mathbf{c}_n\|_\infty}$ and $\mathbf{b}_n, \mathbf{V}_n$ are non-negative, we deduce that $\mathbf{V}_{n+1} \geq 0$. This ends the proof. □

Let us now give the proof of Proposition 4.5.

Proof of Proposition 4.5. Under the hypothesis stated in Proposition 4.5, we see that if h is small enough, we may consider $K \leq N$ be the greatest value such that for all $n < K$,

$$\max_{0 \leq i \leq I} |V_{i,n} - v(x_i, t_n)| < 1 \text{ and } \max_{0 \leq i \leq I-1} \big| |\delta_x V_{i,n}| - |\delta_x v(x_i, t_n)| \big| < 1. \tag{4.4}$$

From the the fact that $v_0 \geq 0$ and Lemma 4.9, we see that the solution of (4.2) is non-negative. Furthermore, since $v(x,t) \in C^{4,2}([-L, L] \times [0, T-\eta])$, there exist positive constants C_1 and C_2 such that for all $(x,t) \in [-L, L] \times [0, T-\eta]$,

$$|v_x^{(i)}(x,t)| \leq C_1, \quad 0 \leq i \leq 4 \quad \text{and} \quad |v_t^{(j)}(x,t)| \leq C_2, \quad 0 \leq j \leq 2.$$

Thus, we obtain from the triangle inequality that

$$\max_{0 \leq i \leq I} |V_{i,n}| \leq 1 + C_1 \quad \text{and} \max_{0 \leq i \leq I-1} |\delta_x V_{i,n}| \leq 1 + C_2, \quad \text{for} \quad n < K.$$

Using Taylor's expansion and (4.1), we derive for all $1 \le i \le I-1, 0 < n < K$,

$$\delta_t v(x_i, t_n) \le \delta_x^2 v(x_i, t_n) + v^p(x_i, t_n) + \beta|\delta_x v(x_i, t_n)|^q + C_3 h^2 + C_4 \tau,$$

where C_3, C_4 are positive constants.

Let $e_{i,n} = V_{i,n} - v(x_i, t_n)$ be the discretization error. We have,

$$\delta_t e_{i,n} \le \delta_x^2 e_{i,n} + V_{i,n}^p - v^p(x_i, t_n) + \beta(|\delta_x V_{i,n}|^q - |\delta_x v(x_i, t_n)|^q) + C_3 h^2 + C_4 \tau.$$

Applying the mean value theorem, we get

$$\delta_t e_{i,n} \le \delta_x^2 e_{i,n} + p\xi_{i,n}^{p-1} e_{i,n} + \beta q |\theta_{i,n}^{q-2}| \theta_{i,n} \delta_x e_{i,n} + C_3 h^2 + C_4 \tau,$$

where $\xi_{i,n}$ is an intermediate value between $V_{i,n}$ and $v(x_i, t_n)$, $\theta_{i,n}$ is between $\delta_x V_{i,n}$ and $\delta_x v(x_i, t_n)$.
Since $\tau \le \frac{h^2}{2}$, we then obtain for all $i \le I-1$ and $n < K$,

$$\delta_t e_{i,n} \le \delta_x^2 e_{i,n} + p\xi_{i,n}^{p-1} e_{i,n} + \beta q |\theta_{i,n}^{q-2}| \theta_{i,n} \delta_x e_{i,n} + C_5 h^2, \tag{4.5}$$

where $C_5 = C_3 + C_4/2$.
We now consider the function

$$z(x,t) = e^{At+x^2}\left(\epsilon_1 + \epsilon_2 + Qh^2\right),$$

where A, Q are positive constants which will be chosen later.
We observe that, for $0 \le i \le I$,

$$z(x_i, 0) = e^{x_i^2}\left(\epsilon_1 + \epsilon_2 + Qh^2\right) \ge e_{i,0},$$
$$z(x_I, t_n) = e^{At_n + x_I^2}\left(\epsilon_1 + \epsilon_2 + Qh^2\right) \ge e_{I,n},$$

and

$$\begin{aligned} z_t(x,t) - z_{xx}(x,t) - p\xi_{i,n}^{p-1} z(x,t) - \beta q |\theta_{i,n}^{q-2}| \theta_{i,n} z_x(x,t) \\ = (A - 2 - p\xi_{i,n}^{p-1} - 4x^2 - 2\beta q |\theta_{i,n}^{q-2}| \theta_{i,n} x) z(x,t). \end{aligned}$$

Using Taylor's expansion, we get

$$\begin{aligned} &\delta_t z(x_i, t_n) - \delta_x^2 z(x_i, t_n) - p\xi_{i,n}^{p-1} z(x_i, t_n) - \beta q |\theta_{i,n}^{q-2}| \theta_{i,n} \delta_x z(x_i, t_n) \\ &\quad = (A - 2 - p\xi_{i,n}^{p-1} - 4x_i^2 - 2\beta q |\theta_{i,n}^{q-2}| \theta_{i,n} x_i) z(x_i, t_n) \\ &\qquad + \frac{h^2}{12} z_{xxxx}(\tilde{x}_i, t_n) + \beta q |\theta_{i,n}^{q-2}| \theta_{i,n} \frac{h^2}{6} z_{xxx}(\bar{x}_i, t_n) - \frac{\tau}{2} z_{tt}(x_i, \tilde{t}_n), \end{aligned}$$

where $\tilde{t}_n \in [t_n, t_{n+1}]$ and $\tilde{x}_i, \bar{x}_i \in [x_{i-1}, x_{i+1}]$.
By taking A, Q large enough, then h small enough such that the right-hand side of the above equation is lager than $C_5 h^2$, we obtain

$$\delta_t z(x_i,t_n) - \delta_x^2 z(x_i,t_n) - p\xi_{i,n}^{p-1} z(x_i,t_n) - \beta q|\theta_{i,n}^{q-2}|\theta_{i,n}\delta_x z(x_i,t_n) \geq C_5 h^2. \tag{4.6}$$

From (4.5) and (4.6), applying Lemma 4.10 to $z(x_i,t_n) - e_{i,n}$ with $b_{i,n} = p\xi_{i,n}^{p-1} \geq 0$ and $c_{i,n} = \beta q|\theta_{i,n}^{q-2}|\theta_{i,n}$ bounded, we get $e_{i,n} \leq z(x_i,t_n)$ for $0 \leq i \leq I$ and $0 \leq n < K$. By the same way, we also show that $-e_{i,n} \leq z(x_i,t_n)$ for $0 \leq i \leq I$ and $0 \leq n \leq K$. In conclusion, we derive

$$\max_{0\leq i\leq I} |V_{i,n} - v(x_i,t_n)| \leq z(x_i,t_n) \leq e^{AT+L^2}(\epsilon_1 + \epsilon_2 + Qh^2), \quad \text{for} \quad n < K.$$

Let us show that $K = N$. Assuming by contradiction that $K < N$, we have

$$1 \leq \max_{0\leq i\leq I} |V_{i,K} - v(x_i,t_K)| \leq z(x_i,t_K) \leq e^{AT+L^2}(\epsilon_1 + \epsilon_2 + Qh^2).$$

But this contradicts with the fact that the last term in the above inequality tends to zero as h tends to zero. This concludes the proof of Proposition 4.5. Since Theorem 4.1 is a consequence of Proposition 4.5, as we pointed in Remark 4.2, this is also the conclusion of the proof of Theorem 4.1. □

5 Numerical results

The numerical experiments presented in this section are performed with the initial data

$$u_0(x) = A\left(1 + \cos(\pi x)\right), \quad x \in (-1,1), \tag{5.1}$$

where $A = 1.2$. For the non-linearity power, we take $p = 5$ and $p = 7$. Let us recall from Remark 3.2 that the threshold M is given by $M = \lambda^{-\frac{2}{p-1}}\|u_0\|_\infty$. Therefore, the parameters of the algorithm are $\lambda = \frac{1}{2}$, $\alpha = 0.4$, $M = 2A \times 2^{1/2}$ if $p = 5$ and $M = 2A \times 2^{1/3}$ if $p = 7$. For the time step τ, we take $\tau = \frac{h^2}{4}$ where $h = \frac{2}{I}$. We perform the experiments with $I = 50, 100, 160, 250, 320, 400$.

5.1 The semilinear heat equation ($\beta = 0$ in equation (1.1)).

Note that the original paper of Berger and Kohn [9] was totally devoted to this case. We now recall the assertion that the value τ_k^* is independent of k and tends to a constant as k tends to infinity. In order to establish this assertion, we recall from Merle and Zaag [41] that

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(t)\|_\infty = \kappa, \quad \text{with} \quad \kappa = (p-1)^{-\frac{1}{p-1}}. \tag{5.2}$$

Then, using (3.9) we see that

$$u^{(k)}(\xi_k, \tau_k^*) = \lambda^{\frac{2}{p-1}} u^{(k-1)}(\lambda\xi_k, \tau_{k-1}^* + \lambda^2\tau_k^*) = \cdots = \lambda^{\frac{2k}{p-1}} u(\lambda^k\xi_k, t_k), \tag{5.3}$$

where $t_k = \tau_0^* + \lambda^2\tau_1^* + \cdots + \lambda^{2k}\tau_k^*$.
Hence, it holds that

$$(T - t_k)^{\frac{1}{p-1}} \|u(t_k)\|_\infty = (T - t_k)^{\frac{1}{p-1}} \lambda^{\frac{-2k}{p-1}} \|u^{(k)}(\tau_k^*)\|_\infty.$$

Since $\|u^{(k)}(\tau_k^*)\|_\infty = M$, we obtain

$$T - t_k = \lambda^{2k} M^{1-p}(p-1)^{-1} + o(1) \quad \text{as} \quad k \to \infty \tag{5.4}$$

on the one hand.

On the other hand, we get

$$\begin{aligned} \tau_k^* &= \lambda^{-2k}(t_k - t_{k-1}) = \lambda^{-2k}\left((T - t_{k-1}) - (T - t_k)\right) \\ &= M^{1-p}(p-1)^{-1}(\lambda^{-2} - 1) + o(1). \end{aligned}$$

Consequently, we obtain

$$\lim_{k\to+\infty} \tau_k^* = M^{1-p}(p-1)^{-1}(\lambda^{-2} - 1). \tag{5.5}$$

Figure 5.1 presents the computed values of τ_k^* when $p = 5$, for different values of I. The values of τ_k^* are tabulated in Tables 5.1 and 5.2 for some selected values of k. These experimental results are in agreement with the fact that τ_k^* tends to the constant indicated in the right-hand side of (5.5) as k tends to infinity. In Figure 5.2, we show the

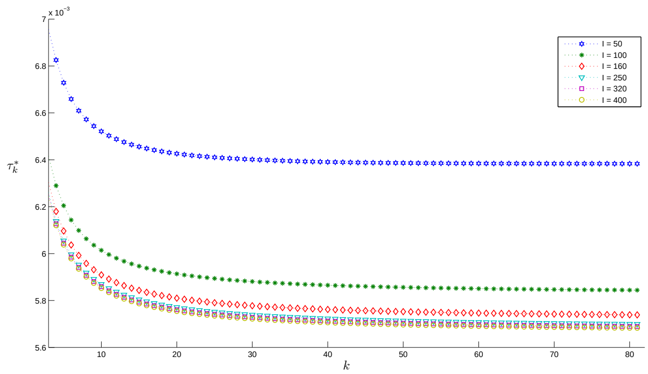

Figure 5.1. *The computed values of τ_k^* are plotted against k when $p = 5$.*

plot of $\|\mathbf{U}_{h,\tau}(t)\|_\infty$ versus $(T_{h,\tau} - t)$ in log-scale where $T_{h,\tau}$ is given by $T_{h,\tau} = \sum_{k=0}^{K} \lambda^{2k}\tau_k^*$.

k	I = 50	I = 100	I = 160	I = 250	I = 320	I = 400
20	0.6426	0.5913	0.5810	0.5771	0.5760	0.5755
30	0.6401	0.5881	0.5778	0.5739	0.5728	0.5722
40	0.6391	0.5865	0.5762	0.5723	0.5712	0.5706
50	0.6386	0.5856	0.5752	0.5713	0.5703	0.5697
60	0.6384	0.5851	0.5746	0.5707	0.5697	0.5691
70	0.6384	0.5847	0.5742	0.5703	0.5693	0.5687
80	0.6383	0.5844	0.5739	0.5700	0.5689	0.5683

Table 5.1. *The computed values of* $\tau_k^*(\times 10^{-2})$ *when* $p = 5$.

k	I = 50	I = 100	I = 160	I = 250	I = 320	I = 400
20	0.1279	0.0826	0.0726	0.0688	0.0677	0.0671
30	0.1279	0.0825	0.0724	0.0686	0.0675	0.0670
40	0.1279	0.0825	0.0724	0.0685	0.0674	0.0669
50	0.1279	0.0825	0.0723	0.0684	0.0674	0.0668
60	0.1279	0.0825	0.0723	0.0684	0.0673	0.0667
70	0.1279	0.0825	0.0723	0.0683	0.0673	0.0667
80	0.1279	0.0825	0.0723	0.0683	0.0673	0.0667

Table 5.2. *The computed values of* $\tau_k^*(\times 10^{-2})$ *when* $p = 7$.

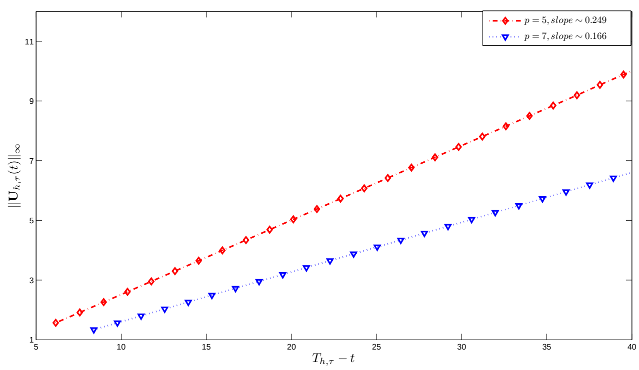

Figure 5.2. *Blow-up rate (in log-scale) when* $p = 5$ *and* $p = 7$*, for* $I = 400$.

The slope of the obtained curves measures the blow-up rate. As expected from (5.2), these slopes for $p = 5$ and $p = 7$ are $\frac{1}{4}$ and $\frac{1}{6}$ respectively.

In order to examine the theoretical profile defined in (2.5), we recall the method of Berger and Kohn [9] to consider the rescaled profile,

$$z \to u^{(k)}\left(z\lambda^{-1}\xi^{+}_{k-1}, \tau^{*}_{k}\right), \quad |z| < 1, \tag{5.6}$$

where $\xi^{+}_{q} = \xi_{q,i^{+}_{q}}$. Using the *semilarity variables* defined in (2.1) and (5.3), we get

$$u^{(k)}(\xi_k, \tau^{*}_{k}) = \lambda^{\frac{2k}{p-1}}(T - t_k)^{-\frac{1}{p-1}} w\left(\lambda^k \frac{\xi_k}{\sqrt{T - t_k}}, s_k\right), \tag{5.7}$$

where $s_k = -\log(T - t_k)$.
We recall from (5.4) that

$$T - t_k \sim \left(\lambda^{2k} M^{1-p}\right)(p-1)^{-1}. \tag{5.8}$$

Substituting (5.8) into (5.7) yields

$$u^{(k)}(\xi_k, \tau^{*}_{k}) \sim (p-1)^{\frac{1}{p-1}} M w\left(\sqrt{p-1} M^{\frac{p-1}{2}} \xi_k, s_k\right).$$

From (2.4), we replace ξ_k by $z\lambda^{-1}\xi^{+}_{k-1}$ to obtain

$$u^{(k)}(z\lambda^{-1}\xi^{+}_{k-1}, \tau^{*}_{k}) \sim (p-1)^{\frac{1}{p-1}} M f\left(\sqrt{p-1} M^{\frac{p-1}{2}} z\lambda^{-1} \frac{\xi^{+}_{k-1}}{\sqrt{s_k}}\right). \tag{5.9}$$

Assume that $\frac{\xi^{+}_{k-1}}{\sqrt{s_k}}$ tends to ζ. Using the fact that $\alpha M = u^{(k-1)}(\xi^{+}_{k-1}, \tau^{*}_{k})$ yields

$$\alpha M = M(p-1)^{\frac{1}{p-1}} f\left(\sqrt{p-1} M^{\frac{p-1}{2}} \zeta\right),$$

or

$$\alpha = (p-1)^{\frac{1}{p-1}} f(A\zeta), \quad A = \sqrt{p-1} M^{\frac{p-1}{2}}.$$

Using the definition of f in (2.5), it holds that

$$\alpha = (p-1)^{\frac{1}{p-1}} \left(p - 1 + \frac{(p-1)^2}{4p} |A\zeta|^2\right)^{\frac{-1}{p-1}}.$$

A straightforward computation gives

$$|A\zeta|^2 = \frac{4p}{p-1}\left(\alpha^{1-p} - 1\right). \tag{5.10}$$

Using (5.10) and (5.9), we arrive at

$$\begin{aligned} u^{(k)}(z\lambda^{-1}y^{+}_{k-1}, \tau^{*}_{k}) &\sim M(p-1)^{\frac{1}{p-1}} f\left(\lambda^{-1} z (A\zeta)\right)^{-\frac{1}{p-1}} \\ &\sim M(p-1)^{\frac{1}{p-1}} \left(p - 1 + \frac{(p-1)^2}{4p} \lambda^{-2} z^2 |A\zeta|^2\right)^{-\frac{1}{p-1}} \\ &\sim M\left(1 + \left(\alpha^{1-p} - 1\right)\lambda^{-2} z^2\right)^{-\frac{1}{p-1}}. \end{aligned} \tag{5.11}$$

Since $\mathbf{U}_{h,\tau}^{(k)}$ converges to $u^{(k)}$ as h goes to zero, it holds that

$$\mathbf{U}_{h,\tau}^{(k)}(z\lambda^{-1}\xi_{k-1}^{+}, \tau_k^{*}) \sim M\left(1 + \left(\alpha^{1-p} - 1\right)\lambda^{-2}z^2\right)^{-\frac{1}{p-1}}, \quad |z| < 1. \tag{5.12}$$

We expect that the left-hand side of (5.12) tends to the predicted profile as k tends to infinity. Figures 5.3 and 5.4 display this relationship after 80 iterations with $I = 400$. Figures 5.5 and 5.6 illustrate the output of our algorithm using $I = 400$ at some selected values of k. As k increases these computed profiles converge to the profile shown in Figures 5.3 and 5.4 respectively. We give in Tables 5.3 and 5.4 the error in L^{∞}-norm between the computed profiles and the predicted profile using various values of I in both cases $p = 5$ and $p = 7$. The expression of the error is given by

$$e_{h,\tau}^{(k)} = \sup_{-1 \le z \le 1} \left| \mathbf{U}_{h,\tau}^{(k)}(z\lambda^{-1}\xi_{k-1}^{+}, \tau_k^{*}) - M[1 + (\alpha^{1-p} - 1)\lambda^{-2}z^2]^{-\frac{1}{p-1}} \right|.$$

The graphs of $e_{h,\tau}^{(k)}$ versus h in log-scale are visualized in Figures 5.7 and 5.8. We observe in those figures that the error tends to zeros as $h \to 0$. We note that the error $e_{h,\tau}^{(k)}$ includes two sources: the discretization error in using the scheme (3.2) and the asymptotic error which refers to the behavior of $w(y, s)$ as s tends to infinity.

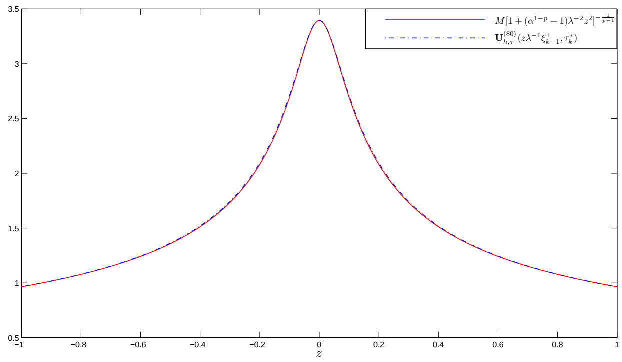

Figure 5.3. *The computed profile* (5.6) *for* $k = 80$ *with* $I = 400$ *and the predicted profile* (5.12) *with* $p = 5$.

5.2 The nonlinear heat equation in case $\beta \neq 0$

5.2.1 A formal calculation

This part gives a formal calculation to obtain the prediction given in (2.9). This kind of arguments can be found in [9], [40] and [39]. Using similarity variables defined in (2.1) with $a = 0$, we see that $w = w_{0,T}$ satisfies the following equation for all $s \geq -\log T$ and

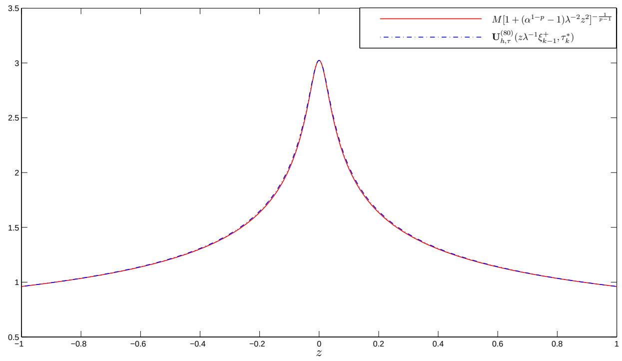

Figure 5.4. *The computed profile* (5.6) *for* $k = 80$ *with* $I = 400$ *and the predicted profile* (5.12) *with* $p = 7$.

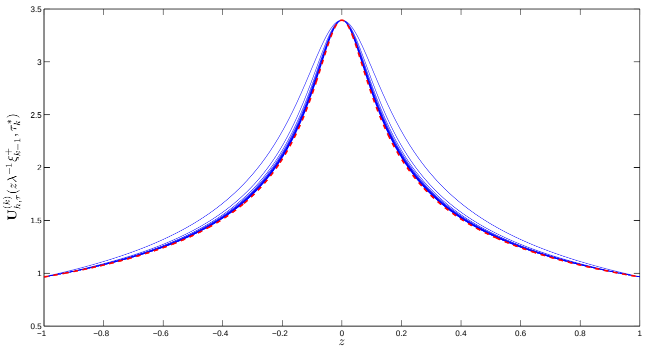

Figure 5.5. *The computed profiles as in* (5.6) *for selected values of* k *with* $I = 400$ *and* $p = 5$.

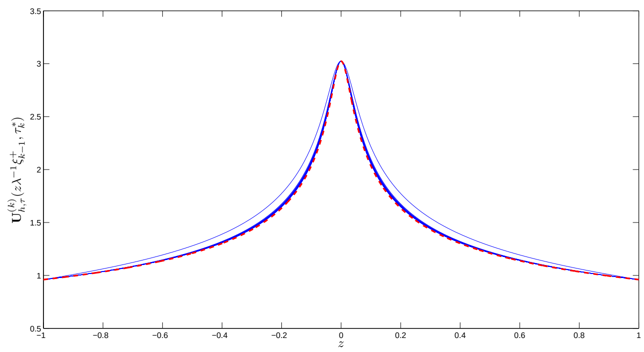

Figure 5.6. *The computed profiles as in* (5.6) *for selected values of* k *with* $I = 400$ *and* $p = 7$.

k	I = 50	I = 100	I = 160	I = 250	I = 320	I = 400
10	0.2023	0.1581	0.1501	0.1466	0.1446	0.1432
20	0.1336	0.0858	0.0787	0.0735	0.0722	0.0715
30	0.1213	0.0636	0.0517	0.0480	0.0483	0.0474
40	0.1141	0.0504	0.0404	0.0376	0.0351	0.0354
50	0.1091	0.0444	0.0341	0.0297	0.0289	0.0276
60	0.1076	0.0409	0.0300	0.0249	0.0241	0.0231
70	0.1068	0.0372	0.0255	0.0214	0.0209	0.0202
80	0.1066	0.0354	0.0232	0.0188	0.0182	0.0174

Table 5.3. *Error in L^∞-norm between the computed profile and the predicted profile for selected values of k using various values of I with $p = 5$.*

k	I = 50	I = 100	I = 160	I = 250	I = 320	I = 400
10	0.2900	0.1930	0.1205	0.0887	0.0792	0.0730
20	0.2748	0.1757	0.0970	0.0651	0.0556	0.0516
30	0.2711	0.1715	0.0880	0.0545	0.0451	0.0398
40	0.2725	0.1699	0.0843	0.0484	0.0391	0.0337
50	0.2723	0.1696	0.0822	0.0441	0.0351	0.0295
60	0.2706	0.1695	0.0810	0.0421	0.0322	0.0265
70	0.2726	0.1694	0.0803	0.0403	0.0298	0.0240
80	0.2720	0.1694	0.0802	0.0393	0.0285	0.0224

Table 5.4. *Error in L^∞-norm between the computed profile and the predicted profile for selected values of k using various values of I with $p = 7$.*

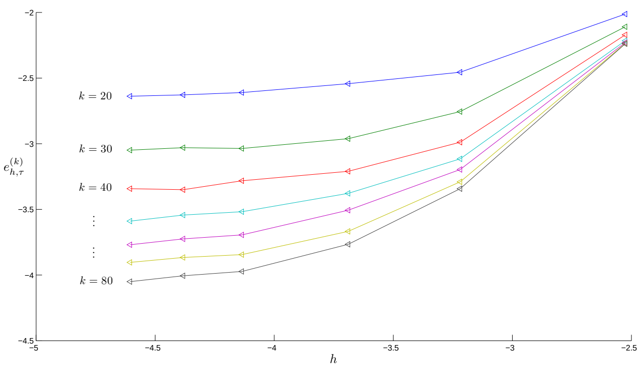

Figure 5.7. *Error between the computed profiles and the predicted profile in log-scale when $p = 5$.*

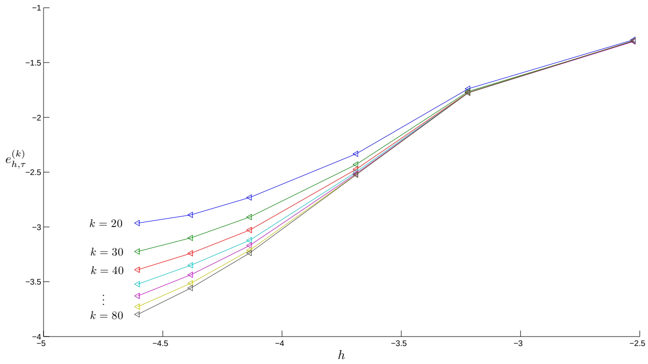

Figure 5.8. *Error between the computed profiles and the predicted profile in log-scale when $p = 7$.*

$y \in \mathbb{R}^N$:

$$w_s = \Delta w - \frac{1}{2} y.\nabla w - \frac{w}{p-1} + |w|^{p-1} w + \beta |\nabla w|^{\frac{2p}{p+1}}. \tag{5.13}$$

We try to find a solution of (5.13) in the form $v\left(\frac{y}{\sqrt{s}}\right)$, with

$$v(0) = \kappa, \qquad \lim_{|z| \to +\infty} |v(z)| = 0.$$

A computation shows that v must satisfy the following equation, for each $s \geq -\log T$ and each $z \in \mathbb{R}^N$:

$$-\frac{z.\nabla v(z)}{2s} = \frac{1}{s}\Delta v(z) - \frac{z.\nabla v(z)}{2} - \frac{v(z)}{p-1} + |v(z)|^{p-1} v(z) + \frac{\beta}{s^{\frac{p}{p+1}}} |\nabla v(z)|^{\frac{2p}{p+1}}. \tag{5.14}$$

We formally seek regular solutions of (5.13) in the form

$$V(z) = v_0(z) + \frac{1}{s^\alpha} R(z, s),$$

where $z = \frac{y}{\sqrt{s}}$, $\alpha > 0$ and $\|R\|_{L^\infty} \leq C$.
Pugging this ansatz in (5.14) and making $s \to +\infty$, we obtain the following equation satisfied by v_0,

$$-\frac{1}{2} z v_0'(z) - \frac{1}{p-1} v_0(z) + v_0(z)^p = 0. \tag{5.15}$$

Solving (5.15) yields

$$v_0(z) = \left(p - 1 + bz^2\right)^{-\frac{1}{p-1}}, \tag{5.16}$$

for some constant $b = b(p, \beta) \in \mathbb{R}$. We impose $b(p, \beta) > 0$ in order to have a bounded constant solution.

Remark 5.1. *In the case $\beta = 0$, imposing an analyticity condition, Berger and Kohn [9] have formally found $b(p,0) = \frac{(p-1)^2}{4p}$, which is the coefficient of f given in* (2.5)*. The value of $b(p,0)$ was confirmed in several contributions (Filippas and Kohn [22], Herrero and Velázquez [34], Bricmont and Kupiainen [10]). Unfortunately, we were not able to adapt the formal approach of [9] in the case $\beta \neq 0$, so we only have a numerical expression of $b(p,\beta)$ in Figure 5.12 and Figure 5.13 below.*

5.2.2 Numerical simulations

An important aim in this work is to give a numerical confirmation for the conjectured profile given in (2.7). Note that we have just given a formal argument in the previous subsection, for the existence of that profile, without, specifying the value of $b(p,\beta)$. Up to our knowledge, there is neither a rigorous proof nor a numerical confirmation for (2.7), and our paper is the first to exhibit such a solution numerically. More importantly, thanks to our computations, we are able to find a numerical approximation of $b(p,\beta)$ in the formula of $\bar{f}_\beta$ in (2.8) from our computations.

If we make the same analysis to check that the numerical profile fits with the conjecture theoretical profile (2.7) as the above analysis when $\beta = 0$, then the same result holds in this case, namely

$$u^{(k)}(z\lambda^{-1}\xi_{k-1}^{+}, \tau_k^*) \sim M\left(1 + \left(\alpha^{1-p} - 1\right)\lambda^{-2}z^2\right)^{-\frac{1}{p-1}}, \quad -1 < z < 1. \tag{5.17}$$

Figures 5.9 and 5.10 show the graphs of the computed profile $\mathbf{U}_{\mathbf{h},\tau}^{(\mathbf{80})}(z\lambda^{-1}\xi_{k-1}^{+}, \tau_k^*)$ and the predicted profile given in the right hand side of (5.17), for computations using $I = 320$, $\beta = 1$, $p = 5$ and $p = 7$.

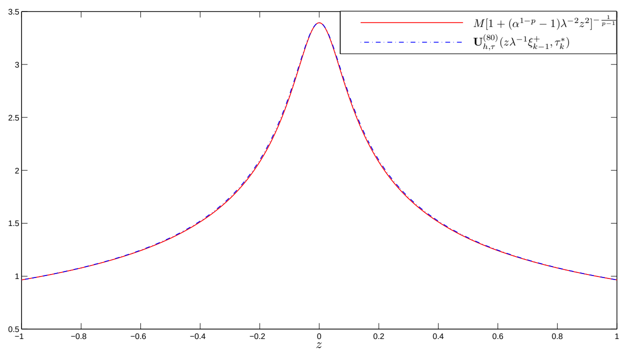

Figure 5.9. *The computed and the predicted profiles in* (5.17)*, for computations using* $I = 320$*,* $\beta = 1$ *and* $p = 5$.

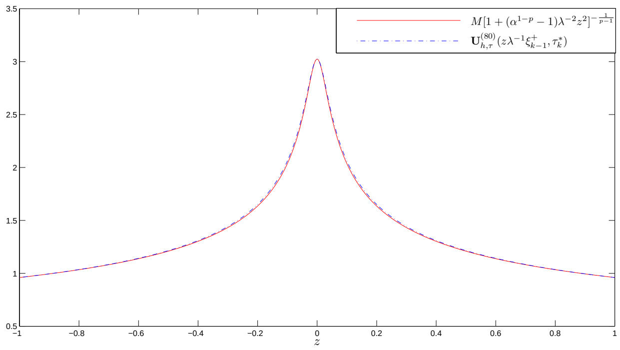

Figure 5.10. *The computed and the predicted profiles in* (5.17)*, for computations using* $I = 320$*,* $\beta = 1$ *and* $p = 7$*.*

In order to compute the value of $b(p,\beta)$ from the simulations, we use the relation (5.7) with $\xi_k = z\lambda^{-1}\xi_{k-1}^+$, we get

$$u^{(k)}(z\lambda^{-1}\xi_{k-1}^+,\tau_k^*) = \lambda^{\frac{2k}{p-1}}(T-t_k)^{-\frac{1}{p-1}} w\left(\lambda^k \frac{z\lambda^{-1}\xi_{k-1}^+}{\sqrt{T-t_k}}, s_k\right). \tag{5.18}$$

We recall from (2.8) that the predicted profile $\bar{f}_\beta$ is given by

$$\bar{f}_\beta(z) = \kappa\left(1+\frac{b(p,\beta)}{p-1}z^2\right)^{-\frac{1}{p-1}}, \quad z = \frac{x}{\sqrt{(T-t)|\log(T-t)|}}, \quad \kappa = (p-1)^{-\frac{1}{p-1}}, \tag{5.19}$$

and that

$$\sup_{|z|<K}\left|w(y,s)-\bar{f}_\beta(z)\right| \to 0 \quad \text{as } s\to\infty \quad \text{with } z = \frac{y}{\sqrt{s}}. \tag{5.20}$$

From (5.20), (5.19) and (5.18), ignoring the error of asymptotic behavior as s goes to infinity, we obtain

$$\begin{aligned} u^{(k)}(z\lambda^{-1}\xi_{k-1}^+,\tau_k^*) &= \lambda^{\frac{2k}{p-1}}(T-t_k)^{-\frac{1}{p-1}} f\left(\lambda^k\frac{z\lambda^{-1}\xi_{k-1}^+}{\sqrt{T-t_k}}\times\frac{1}{\sqrt{s_k}}\right) \\ &= \lambda^{\frac{2k}{p-1}}(T-t_k)^{\frac{-1}{p-1}}\kappa\left(1+\frac{b(p,\beta)}{p-1}\frac{\lambda^{2k}z^2\lambda^{-2}(\xi_{k-1}^+)^2}{T-t_k}\frac{1}{s_k}\right)^{-\frac{1}{p-1}}. \end{aligned}$$

After some straightforward calculations, we arrive at

$$b(p,\beta) = \frac{s_k}{(\xi_{k-1}^+)^2}\left[\frac{\kappa\lambda^{2k}(p-1)[u^{(k)}(z\lambda^{-1}\xi_{k-1}^+,\tau_k^*)]^{1-p}-(p-1)(T-t_k)}{\lambda^{2k-2}z^2}\right].$$

Setting $z=\lambda$ and taking the limit of the above equation as k goes to infinity, we get

$$b(p,\beta)=\lim_{k\to+\infty}\frac{s_k}{(\xi_{k-1}^+)^2}\zeta_k,$$

where

$$\zeta_k=(p-1)\left(\kappa[u^{(k)}(\xi_{k-1}^+,\tau_k^*)]^{1-p}-\lambda^{-2k}(T-t_k)\right).$$

Using (5.4) and (5.17), we see that ζ_k approaches a limit given by

$$\lim_{k\to+\infty}\zeta_k=M^{1-p}\left[(p-1)\kappa\alpha^{1-p}-1\right].$$

This implies that the ratio $\frac{s_k}{(\xi_{k-1}^+)^2}$ should approach a constant as k tends to infinity. This is presented in Figure 5.11. We remark that the computations of s_k and ζ_k do not depend

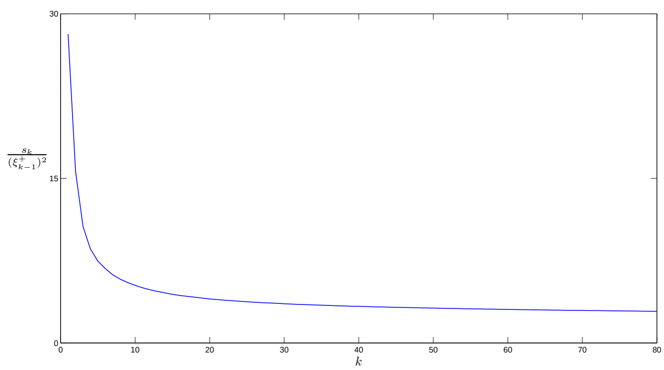

Figure 5.11. *The graph of* $\frac{s_k}{(\xi_{k-1}^+)^2}$ *versus* k*, for computations using* $I=320$*,* $p=5$ *and* $\beta=0$*.*

on β. Moreover, we know that the value of $b(p,0)$ is $\frac{(p-1)^2}{4p}$. In particular, we compute the value of $b(p,\beta)$ by

$$b(p,\beta)=\frac{C_K}{\left[\xi_{K-1}^+(p,\beta)\right]^2},$$

where $C_K=b(p,0)\left[\xi_{K-1}^+(p,0)\right]^2$ for K large.
Consequently, we have just given a numerical evidence for the following conjecture:

Conjecture 5.2. *Equation* (1.1) *has a solution* $u(x,t)$ *which blows up in finite time* T *at one blow-up point* $x=0$*, moreover,*

$$\left\|(T-t)^{1/(p-1)}u(t)-\bar f_\beta\left(\frac{\cdot}{\sqrt{(T-t)|\log(T-t)|}}\right)\right\|_{L^\infty}\to 0,\quad as\quad t\to T, \tag{5.21}$$

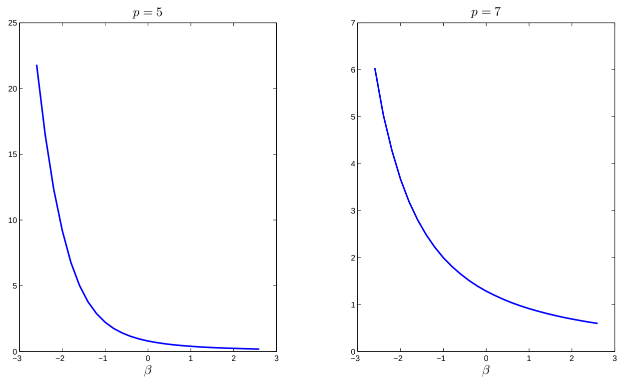

Figure 5.12. *The computed values of* $b(5, \beta)$ *(left) and* $b(7, \beta)$ *(right) for various values of* β.

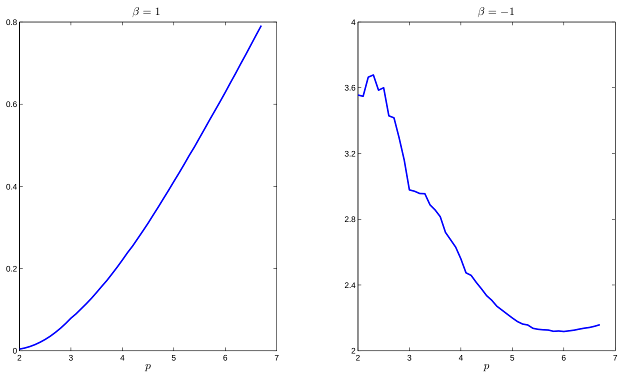

Figure 5.13. *The computed values of* $b(p, 1)$ *(left) and* $b(p, -1)$ *(right) for various values of* p.

where

$$\bar{f}_\beta(z) = \left(p - 1 + b(p, \beta)|z|^2\right)^{-\frac{1}{p-1}}, \quad \textit{with} \quad b(p, 0) = \frac{(p-1)^2}{4p},$$

and $b(p, \beta)$ *is represented in Figure 5.12 and Figure 5.13.*

While remarking numerical simulation for equation (1.1) with $\beta \neq 0$, we could never obtain the self-similar behavior (2.6) rigorously proved in [52]. On the contrary, we could exhibit the behavior (5.21), at the heart of our conjecture. In our opinion, this is probably due to the fact that the behavior (2.6) is unstable, unlike the behavior (5.21), which we suspect to be stable with respect to perturbations in initial data.

5.3 The complex Ginzburg-Landau equation

We recall that $e^{\imath\theta}\tilde{f}_{\delta,\gamma}$ is an asymptotic profile of the solution of (1.2) where $\theta \in \mathbb{R}$ and $\tilde{f}_{\delta,\gamma}$ is given in (2.12), namely

$$\tilde{f}_{\delta,\gamma} = \left(p - 1 + b(\delta,\gamma)|z|^2\right)^{-\frac{1+\imath\delta}{p-1}}, \quad b(\delta,\gamma) = \frac{(p-1)^2}{4(p-\delta^2-\gamma\delta-\gamma\delta p)} > 0. \tag{5.22}$$

Using the same analysis as Section 5.1 resulting (5.11), we have for $|z| < 1$,

$$u^{(k)}(z\lambda^{-1}y^+_{k-1}, \tau^*_k) \sim M^{1+\imath\delta}\lambda^{-\frac{2\imath k\delta}{p-1}}(p-1)^{\frac{\imath\delta}{p-1}}e^{\imath\theta}\left(1 + (\alpha^{1-p}-1)\lambda^{-2}z^2\right)^{-\frac{1+\imath\delta}{p-1}}. \tag{5.23}$$

Remark 5.3. *We remark that the rescaled profile* (5.23) *is obtained under the assumption* $p - \delta^2 - \gamma\delta(p+1) > 0$. *If this condition is not satisfied, the question is open.*

Remark 5.4. *If we take the modulus and the phase of both sides in* (5.23)*, then we get*

$$\left|u^{(k)}\right|(z\lambda^{-1}y^+_{k-1}, \tau^*_k) \sim M(1 + (\alpha^{1-p}-1)\lambda^{-2}z^2)^{-\frac{1}{p-1}}, \quad |z| < 1, \tag{5.24}$$

$$\begin{aligned} phase\left[u^{(k)}\right](z\lambda^{-1}y^+_{k-1}, \tau^*_k) \sim \theta &+ \frac{\delta}{p-1}(\ln M + \ln(p-1) - 2k\ln\alpha) \\ &- \frac{\delta}{p-1}\ln\left(1 + (\alpha^{1-p}-1)\lambda^{-2}z^2\right), \quad |z| < 1. \end{aligned} \tag{5.25}$$

The right hand side of (5.24) *is the same as in* (5.11).

5.3.1 Experiments with $p - \delta^2 - \gamma\delta(p+1) > 0$.

We first make an experiment with $\gamma = 0$, $\delta = 0.2$, $p = 5$ and the initial grid with $I = 320$. The numerical result displayed in Figure 5.14 is in agreement with the expectation obtained in (5.24) and (5.25). Both the numerical modulus and phase coincide with the predicted profile given in (5.24) and (5.25) within plotting resolution.

An experiment with $\gamma = 0$, $p = 5$ and various values of δ are performed on three grids with $I = 100, 200, 320$. The purpose is to confirm the theoretical profile $\tilde{f}_{\delta,\gamma}$ given in (5.22). More precisely, we would like to calculate values of $b(p,\delta,0)$ from our numerical simulation. We recall that the theoretical value of $b(p,\delta,0)$ is equal to $\frac{(p-1)^2}{p-\delta^2}$. In Figure 5.15, we have the computed values of $b(p,\delta,0)$ on various initial grids I. Note that these computed values tend to the predicted ones as I increases. However, as δ approaches $\sqrt{p}$ ($\sqrt{5}$ in Figure 5.15), b becomes singular, and that is the reason why the coincidence between the numerical and theoretical values becomes less clear.

A further experiment with $\gamma = 1$, $\delta = 1$ is shown in Figure 5.16. These calculations show the relationship we obtained in (5.24) and (5.25). Both the numerical phase and modulus coincide with the predicted ones given in (5.24) and (5.25) within plotting resolution.

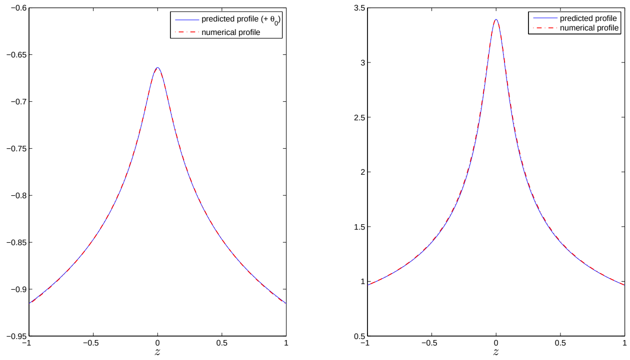

Figure 5.14. *Comparing the numerical profile with the predicted profile given in* (5.24) *and* (5.25) *after 80 iterative steps* ($\gamma = 0, \delta = 0.2, p = 5, I = 320$)*. (Left) phase* $\left[u^{(k)}\right](z\lambda^{-1}y_{k-1}^{+}, \tau_k^*)$*. (Right)* $\left|u^{(k)}\right|(z\lambda^{-1}y_{k-1}^{+}, \tau_k^*)$*.*

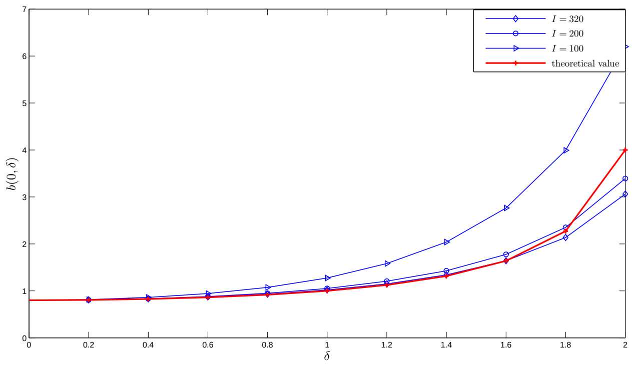

Figure 5.15. *The computed values of* $b(p, \delta, 0)$ *for various initial grids when* $p = 5$*.*

5.3.2 Experiments with $p - \delta^2 - \gamma\delta(p + 1) < 0$.

In this section, we make some experiments with $\gamma = 0$ and $\delta > \sqrt{p} = \sqrt{5}$. For δ large enough, there is no blow-up phenomenon (for example with $\delta = 3$). With δ near $\sqrt{p}$, we made two simulations with $\delta = \sqrt{p} + 0.1$ and $\delta = \sqrt{p} + 0.5$, then the blow-up phenomenon still occurs. Figure 5.17 displays the modulus of $u^{(k)}(z\lambda^{-1}y_{k-1}^{+}, \tau_k^*)$ at some selected values of k, for computations using the initial grid $I = 320$. It shows the rescaled profile $z \mapsto \left|u^{(k)}\right|(z\lambda^{-1}y_{k-1}^{+}, \tau_k^*)$. We can see that these rescaled profiles converge as k increases.

Consequently, if $p - \delta^2 - \gamma\delta - \gamma\delta p < 0$, the blow-up phenomenon may occur and there

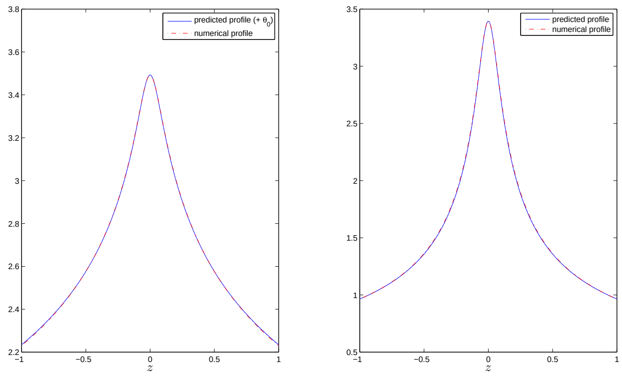

Figure 5.16. *Comparing the numerical profile with the predicted profile given in* (5.24) *and* (5.25) *after 80 iterative steps* ($\gamma = 1, \delta = 1, p = 5, I = 320$). *(Left) phase* $\left[u^{(k)}\right](z\lambda^{-1}y^{+}_{k-1}, \tau^{*}_{k})$. *(Right)* $\left|u^{(k)}\right|(z\lambda^{-1}y^{+}_{k-1}, \tau^{*}_{k})$.

may exist a blow-up profile. So far, we have no answer for this case. We wonder whether its solution behaves as the solution in the case $p - \delta^2 - \gamma\delta - \gamma\delta p > 0$ with a different function of $b(\delta, \gamma)$ in formula of $\tilde{f}_{\delta,\gamma}$ given in (5.22).

A A regularity result for equation (1.1)

We claim the following:

Proposition A.1 (Parabolic regularity). *Consider u solution of*

$$\begin{cases} u_t & = u_{xx} + |u|^{p-1}u + \beta|u_x|^q, & \text{in} \quad \Omega \times (0, T), \\ u(x,t) & = 0 & \text{on} \quad \partial\Omega \times [0, T), \\ u(x,0) & = u_0(x), & \text{on} \quad \bar{\Omega}. \end{cases} \tag{A.1}$$

where $u(t) : x \in \Omega \mapsto \mathbb{R}$ with Ω is an interval in $\mathbb{R}$, $p, q > 1$ and $\beta \in \mathbb{R}$.
Assume that $\|u_0\|_{\mathcal{C}^2(\bar{\Omega})} \leq C_0$ and $\|u\|_{\mathcal{C}(\bar{\Omega}\times[0,T_0])} + \|u_x\|_{\mathcal{C}(\bar{\Omega}\times[0,T_0])} \leq C_u$ with $T_0 < T$ (T is the existence time of the maximal solution). Then for all $t \in [0, T_0]$,
$i)\|u_{xx}(t)\|_{L^\infty(\bar{\Omega})} + \|u_t(t)\|_{L^\infty(\bar{\Omega})} \leq C$ for some $C = C(C_0, C_u, T_0, p, q, \beta)$.
Assume in addition, $\|u_0\|_{\mathcal{C}^4(\bar{\Omega})} \leq C_0$ and $p, q \geq 2$. Then for all $t \in [0, T_0]$,
$ii)$ $\|u_{xxx}(t)\|_{L^\infty(\bar{\Omega})} + \|u_{xxxx}(t)\|_{L^\infty(\bar{\Omega})} + \|u_{tt}(t)\|_{L^\infty(\bar{\Omega})} \leq C$ for some $C = C(C_0, C_u, T_0, p, q, \beta)$.

Remark A.2. *Chipot and Weissler showed in [14] (see Proposition 2.2) that for $s \in \mathbb{R}$ sufficient large and $u_0 \in W^{1,s}_0(\Omega)$, then u, a solution of* (A.1), *satisfies that*

$$\|u(t)\|_{L^\infty} \quad \text{and} \quad \|u_x(t)\|_{L^\infty} \quad \text{are bounded for any inteval } [0, T_0] \text{ with } T_0 < T.$$

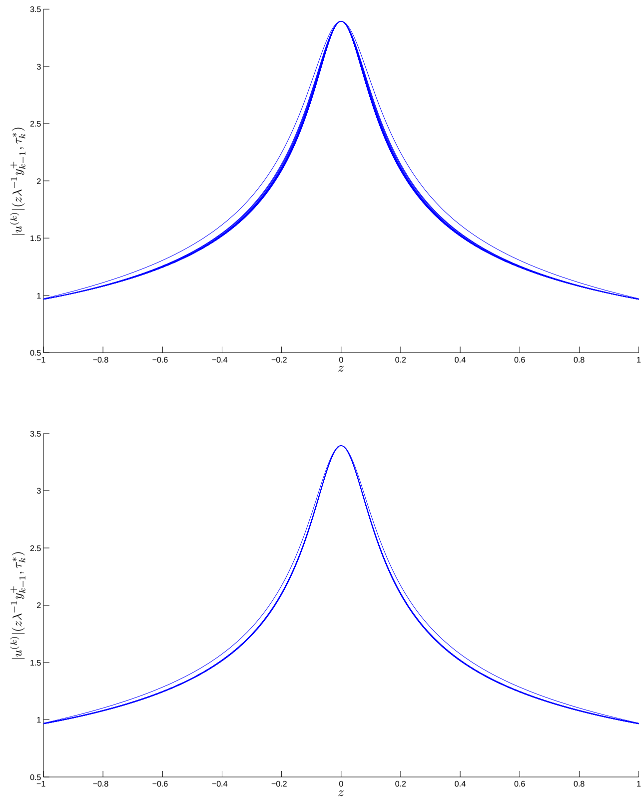

Figure 5.17. *The numerical values of* $u^{(k)}(z\lambda^{-1}y^+_{k-1}, \tau^*_k)$ *at some selected values of* k*, for computation using the initial grid* $I = 320$ *with* $p = 5$*. (Above)* $\gamma = 0, \delta = \sqrt{p} + 0.1$*. (Below)* $\gamma = 0, \delta = \sqrt{p} + 0.5$.

Proof of Proposition A.1. In what follows, we write $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\bar{\Omega})}$ for simplicity and denote by $C_1, C_2, \ldots$ constants depending only on C_0, C_u, T_0, p, q and β.

$i)$ We see from (A.1) that $\|u_t(t)\|_\infty$ is bounded on $[0, T_0]$ if $\|u_{xx}(t)\|_\infty$ is bounded on $[0, T_0]$. Let us consider $h = u_{xx}$, then h satisfies

$$h_t = h_{xx} + \partial_x \left(p|u|^{p-1}u_x + q\beta|u_x|^{q-2}u_x h \right). \tag{A.2}$$

An integral form of the solution of equation (A.2) is

$$h(t) = e^{t\Delta}h(0) + \int_0^t e^{(t-s)\Delta}\partial_x \left(p|u(s)|^{p-1}u_x(s) + q\beta|u_x(s)|^{q-2}u_x(s)h(s) \right) ds, \tag{A.3}$$

where $e^{t\Delta}$ denotes the heat semigroup on Ω with Dirichlet boundary condition.

Recall that for all $\varphi \in L^\infty$,

$$\|e^{t\Delta}\varphi\|_\infty \le \|\varphi\|_\infty \quad \text{and} \quad \|e^{t\Delta}\nabla\varphi\|_\infty \le \frac{C'}{\sqrt{t}}\|\varphi\|_\infty. \tag{A.4}$$

Since $u_0 \in \mathcal{C}^2$ and $\|u(t)\|_\infty, \|u_x(t)\|_\infty$ are bounded for all $t \in [0, T_0]$, then we have by (A.4) and (A.3) that

$$\|h(t)\|_\infty \le C_1 + C_1\int_0^t \frac{\|h(s)\|_\infty}{\sqrt{t-s}}ds, \quad \forall t \in [0, T_0].$$

Using a Growall's argument, we have

$$\|h(t)\|_\infty \le 2C_1 e^{C_1\sqrt{T_0}}, \quad \forall t \in [0, T_0].$$

Therefore, $\|u_{xx}(t)\|_\infty$ is bounded for all $t \in [0, T_0]$ which concludes the proof of i).

ii) We assume additionally in what follows that $p, q \ge 2$, $\|u_0\|_{\mathcal{C}^4(\bar\Omega)} \le C_0$. Consider $v = u_{xxx}$, let us show that $\|v(t)\|_\infty$ is bounded for all $t \in [0, T_0]$. From (A.1) we see that v satisfies the following equation

$$v_t = v_{xx} + p|u|^{p-1}v + \beta q\partial_x(|u_x|^{q-2}u_x v) + \phi + \partial_x\psi, \tag{A.5}$$

where

$$\phi = p(p-1)|u|^{p-3}uu_xu_{xx}, \psi \quad = p(p-1)|u|^{p-3}u(u_x)^2 + \beta q(q-1)|u_x|^{q-2}(u_{xx})^2.$$

We now use an integral formulation of (A.5) to write

$$\begin{aligned} v(t) = e^{t\Delta}v(0) &+ p\int_0^t e^{(t-s)\Delta}|u(s)|^{p-1}v(s)ds \\ &+ \beta q\int_0^t e^{(t-s)\Delta}\partial_x(|u_x(s)|^{q-2}u_x(s)v(s))ds \\ &+ \int_0^t e^{(t-s)\Delta}\phi(s)ds + \int_0^t e^{(t-s)\Delta}\partial_x\psi(s)ds. \end{aligned}$$

From (i) and the hypothesis on $u_0 \in \mathcal{C}^4$, we see that for all $t \in [0, T_0]$,

$$\|v(0)\|_\infty + \|u(t)\|_\infty^{p-1} + \|u_x(t)\|_\infty^{q-1} + \|\phi(t)\|_\infty + \|\psi(t)\|_\infty \le C_2.$$

Hence, the use of (A.4) yields

$$\|v(t)\|_\infty \le C_2 + C_2\int_0^t \left(1 + \frac{1}{\sqrt{t-s}}\right)\|v(s)\|_\infty ds \le 2C_2 e^{C_2(T_0+2\sqrt{T_0})}, \quad \forall t \in [0, T_0],$$

which follows that $\|u_{xxx}(t)\|_\infty$ is bounded on $[0, T_0]$.

We now bound $\|u_{tt}(t)\|_\infty$ on $[0,T_0]$. Consider $\theta = u_{tt}$, by (A.1), we see that θ satisfies

$$\theta_t = \theta_{xx} + \eta\theta + \beta q \partial_x \left(|u_x|^{q-2}u_x\theta\right) + \gamma, \tag{A.6}$$

where

$$\eta = p|u|^{p-1} - \beta q(q-1)|u_x|^{q-2}u_{xx},$$
$$\gamma = p(p-1)|u|^{p-3}u(u_t)^2 + \beta q(q-1)|u_x|^{q-2}\left(u_{xxx} + p|u|^{p-1}u_x + \beta q|u_x|^{q-2}u_x u_{xx}\right)^2.$$

An integral form of the solution of equation (A.6) is

$$\theta(t) = e^{t\Delta}\theta(0) + \int_0^t e^{(t-s)\Delta}\eta(s)\theta(s)ds$$
$$+ \beta q \int_0^t e^{(t-s)\Delta}\partial_x\left(|u_x(s)|^{q-2}u_x(s)\theta(s)\right)ds + \int_0^t e^{(t-s)\Delta}\gamma(s)ds.$$

Since $u_0 \in \mathcal{C}^4$, then $\|\theta(0)\|_\infty = \|u_{tt}(0)\|_\infty$ is bounded. Using the fact that $\|u_{xxx}(t)\|_\infty$ is bounded on $[0,T_0]$ and (i), we have by (A.4) that

$$\|\theta(t)\|_\infty \le C_3 + C_3\int_0^t \left(1 + \frac{1}{\sqrt{t-s}}\right)\|\theta(s)\|_\infty ds \le 2C_3 e^{C_3(T_0 + 2\sqrt{T_0})}, \quad \forall t \in [0,T_0].$$

Since $\|u_{tt}(t)\|_{L^\infty}$ is bounded on $[0,T_0]$, we have from (A.1) that $\|u_{xxxx}(t)\|_{L^\infty}$ is also bounded on $[0,T_0]$. This completes the proof of Proposition A.1. $\square$

Bibliography

[1] L. M. Abia, J. C. López-Marcos, and J. Martínez. Blow-up for semidiscretizations of reaction-diffusion equations. *Appl. Numer. Math.*, 20(1-2):145–156, 1996. Workshop on the method of lines for time-dependent problems (Lexington, KY, 1995).

[2] L. M. Abia, J. C. López-Marcos, and J. Martínez. On the blow-up time convergence of semidiscretizations of reaction-diffusion equations. *Appl. Numer. Math.*, 26(4):399–414, 1998.

[3] L. M. Abia, J. C. López-Marcos, and J. Martínez. The Euler method in the numerical integration of reaction-diffusion problems with blow-up. *Appl. Numer. Math.*, 38(3):287–313, 2001.

[4] G. Acosta, R. G. Durán, and J. D. Rossi. An adaptive time step procedure for a parabolic problem with blow-up. *Computing*, 68(4):343–373, 2002.

[5] G. Acosta, J. Fernández Bonder, P. Groisman, and J. D. Rossi. Numerical approximation of a parabolic problem with a nonlinear boundary condition in several space dimensions. *Discrete Contin. Dyn. Syst. Ser. B*, 2(2):279–294, 2002.

[6] L. A. Assalé, Boni T. K., and Nabongo Diabate. Numerical blow-up time for a semilinear parabolic equation with nonlinear boundary conditions. *Journal of Applied Mathematics*, 2008:Article ID 753518, 29 p.–Article ID 753518, 29 p., 2008.

[7] J. M. Ball. Remarks on blow-up and nonexistence theorems for nonlinear evolution equations. *Quart. J. Math. Oxford Ser. (2)*, 28(112):473–486, 1977.

[8] G. Baruch, G. Fibich, and N. Gavish. Singular standing-ring solutions of nonlinear partial differential equations. *Phys. D*, 239(20-22):1968–1983, 2010.

[9] M. Berger and R. V. Kohn. A rescaling algorithm for the numerical calculation of blowing-up solutions. *Comm. Pure Appl. Math.*, 41(6):841–863, 1988.

[10] J. Bricmont and A. Kupiainen. Universality in blow-up for nonlinear heat equations. *Nonlinearity*, 7(2):539–575, 1994.

[11] T. Cazenave, F. Dickstein, and F. Weissler. Finite-time blowup for a complex Ginzburg-Landau equation. *SIAM Journal on Mathematical Analysis*, 45(1):244–266, 2013.

[12] Y. G. Chen. Asymptotic behaviours of blowing-up solutions for finite difference analogue of $u_t = u_{xx} + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 33(3):541–574, 1986.

[13] Y. G. Chen. Blow-up solutions to a finite difference analogue of $u_t = \Delta u + u^{1+\alpha}$ in N-dimensional balls. *Hokkaido Math. J.*, 21(3):447–474, 1992.

[14] M. Chipot and F. B. Weissler. Some blowup results for a nonlinear parabolic equation with a gradient term. *SIAM J. Math. Anal.*, 20(4):886–907, 1989.

[15] R. G. Duran, J. I. Etcheverry, and J. D. Rossi. Numerical approximation of a parabolic problem with a nonlinear boundary condition. *Discrete Contin. Dynam. Systems*, 4(3):497–506, 1998.

[16] M. A. Ebde and H. Zaag. Construction and stability of a blow up solution for a nonlinear heat equation with a gradient term. *S$\vec{e}$MA J.*, (55):5–21, 2011.

[17] C. Fermanian Kammerer, F. Merle, and H. Zaag. Stability of the blow-up profile of non-linear heat equations from the dynamical system point of view. *Math. Ann.*, 317(2):347–387, 2000.

[18] C. Fermanian Kammerer and H. Zaag. Boundedness up to blow-up of the difference between two solutions to a semilinear heat equation. *Nonlinearity*, 13(4):1189–1216, 2000.

[19] J. Fernández Bonder, P. Groisman, and J. D. Rossi. On numerical blow-up sets. *Proc. Amer. Math. Soc.*, 130(7):2049–2055, 2002.

[20] R. Ferreira, P. Groisman, and J. D. Rossi. Numerical blow-up for a nonlinear problem with a nonlinear boundary condition. *Math. Models Methods Appl. Sci.*, 12(4):461–483, 2002.

[21] R. Ferreira, P. Groisman, and J. D. Rossi. Numerical blow-up for the porous medium equation with a source. *Numer. Methods Partial Differential Equations*, 20(4):552–575, 2004.

[22] S. Filippas and R. V. Kohn. Refined asymptotics for the blowup of $u_t - \Delta u = u^p$. *Comm. Pure Appl. Math.*, 45(7):821–869, 1992.

[23] S. Filippas and W. X. Liu. On the blowup of multidimensional semilinear heat equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(3):313–344, 1993.

[24] A. Friedman. Remarks on nonlinear parabolic equations. In *Proc. Sympos. Appl. Math., Vol. XVII*, pages 3–23. Amer. Math. Soc., Providence, R.I., 1965.

[25] H. Fujita. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I*, 13:109–124 (1966), 1966.

[26] V. A. Galaktionov and S. A. Posashkov. The equation $u_t = u_{xx} + u^\beta$. Localization, asymptotic behavior of unbounded solutions. *Akad. Nauk SSSR Inst. Prikl. Mat. Preprint*, (97):30, 1985.

[27] V. A. Galaktionov and S. A. Posashkov. Asymptotics of the process of nonlinear heat conduction with absorption in the case of a critical value of the parameter. *Akad. Nauk SSSR Inst. Prikl. Mat. Preprint*, (71):25, 1986.

[28] Y. Giga and R. V. Kohn. Asymptotically self-similar blow-up of semilinear heat equations. *Comm. Pure Appl. Math.*, 38(3):297–319, 1985.

[29] Y. Giga and R. V. Kohn. Characterizing blowup using similarity variables. *Indiana Univ. Math. J.*, 36(1):1–40, 1987.

[30] Y. Giga and R. V. Kohn. Nondegeneracy of blowup for semilinear heat equations. *Comm. Pure Appl. Math.*, 42(6):845–884, 1989.

[31] P. Groisman. Totally discrete explicit and semi-implicit Euler methods for a blow-up problem in several space dimensions. *Computing*, 76(3-4):325–352, 2006.

[32] P. Groisman and J. D. Rossi. Asymptotic behaviour for a numerical approximation of a parabolic problem with blowing up solutions. *J. Comput. Appl. Math.*, 135(1):135–155, 2001.

[33] M. A. Herrero and J. J. L. Velázquez. Generic behaviour of one-dimensional blow up patterns. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 19(3):381–450, 1992.

[34] M. A. Herrero and J. J. L. Velázquez. Blow-up behaviour of one-dimensional semilinear parabolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(2):131–189, 1993.

[35] M. A. Herrero and Juan J. L. Velázquez. Comportement générique au voisinage d'un point d'explosion pour des solutions d'équations paraboliques unidimensionnelles. *C. R. Acad. Sci. Paris Sér. I Math.*, 314(3):201–203, 1992.

[36] C. Hirota and K. Ozawa. Numerical method of estimating the blow-up time and rate of the solution of ordinary differential equations—an application to the blow-up problems of partial differential equations. *J. Comput. Appl. Math.*, 193(2):614–637, 2006.

[37] L. M. Hocking, K. Stewartson, J. T. Stuart, and S. N. Brown. A nonlinear instability burst in plane parallel flow. *Journal of Fluid Mechanics*, 51:705–735, 1 1972.

[38] H. A. Levine. Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$. *Arch. Rational Mech. Anal.*, 51:371–386, 1973.

[39] N. Masmoudi and H. Zaag. Blow-up profile for the complex Ginzburg-Landau equation. *J. Funct. Anal.*, 255(7):1613–1666, 2008.

[40] F. Merle and H. Zaag. Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$. *Duke Math. J.*, 86(1):143–195, 1997.

[41] F. Merle and H. Zaag. Optimal estimates for blowup rate and behavior for nonlinear heat equations. *Comm. Pure Appl. Math.*, 51(2):139–196, 1998.

[42] F. Merle and H. Zaag. Refined uniform estimates at blow-up and applications for nonlinear heat equations. *Geom. Funct. Anal.*, 8(6):1043–1085, 1998.

[43] T. Nakagawa. Blowing up of a finite difference solution to $u_t = u_{xx} + u_2$.. *Appl. Math. Optim.*, 2(4):337–350, 1975/76.

[44] T. Nakagawa and T. Ushijima. Finite element analysis of the semi-linear heat equation of blow-up type. *in Topics in Numerical Analysis, J. J. H. Miller, ed., Academic Press*, 1977.

[45] F. K. N'gohisse and T. K. Boni. Numerical blow-up for a nonlinear heat equation. *Acta Math. Sin. (Engl. Ser.)*, 27(5):845–862, 2011.

[46] S. Popp, O. Stiller, E. Kuznetsov, and L. Kramer. The cubic complex Ginzburg-Landau equation for a backward bifurcation. *Physica D: Nonlinear Phenomena*, 114(1-2):81 – 107, 1998.

[47] P. Quittner and P. Souplet. *Superlinear parabolic problems*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2007. Blow-up, global existence and steady states.

[48] S. Snoussi and S. Tayachi. Nonglobal existence of solutions for a generalized Ginzburg-Landau equation coupled with a Poisson equation. *J. Math. Anal. Appl.*, 254(2):558–570, 2001.

[49] S. Snoussi and S. Tayachi. Large time behavior of solutions for parabolic equations with nonlinear gradient terms. *Hokkaido Math. J.*, 36(2):311–344, 2007.

[50] S. Snoussi, S. Tayachi, and F. B. Weissler. Asymptotically self-similar global solutions of a semilinear parabolic equation with a nonlinear gradient term. *Proc. Roy. Soc. Edinburgh Sect. A*, 129(6):1291–1307, 1999.

[51] P. Souplet. Finite time blow-up for a non-linear parabolic equation with a gradient term and applications. *Mathematical Methods in the Applied Sciences*, 19(16):1317–1333, 1996.

[52] P. Souplet, S. Tayachi, and F. B. Weissler. Exact self-similar blow-up of solutions of a semilinear parabolic equation with a nonlinear gradient term. *Indiana Univ. Math. J.*, 45(3):655–682, 1996.

[53] T. K. Ushijima. On the approximation of blow-up time for solutions of nonlinear parabolic equations. *Publ. Res. Inst. Math. Sci.*, 36(5):613–640, 2000.

[54] J. J. L. Velázquez. Higher-dimensional blow up for semilinear parabolic equations. *Comm. Partial Differential Equations*, 17(9-10):1567–1596, 1992.

[55] J. J. L. Velázquez. Classification of singularities for blowing up solutions in higher dimensions. *Trans. Amer. Math. Soc.*, 338(1):441–464, 1993.

[56] H. Zaag. Blow-up results for vector-valued nonlinear heat equations with no gradient structure. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 15(5):581–622, 1998.

Address: Université Paris 13, Institut Galilée, LAGA,
99 Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France.

E-mail: vtnguyen@math.univ-paris13.fr

Chapter III

On the blow-up results for a class of strongly perturbed semilinear heat equations[1]

V. T. Nguyen

Université Paris 13, Sorbonne Paris Cité,
LAGA, CNRS (UMR 7539), F-93430, Villetaneuse, France.

Abstract
We consider in this work some class of strongly perturbed for the semilinear heat equation with Sobolev sub-critical power nonlinearity. We first derive a Lyapunov functional in similarity variables and then use it to derive the blow-up rate. We also classify all possible asymptotic behaviors of the solution when it approaches to singularity. Finally, we describe precisely the blow-up profiles corresponding to these behaviors.

Keyword: Finite-time blow-up, asymptotic behavior of solutions, nonlinear parabolic equations.

1 Introduction

We are interested in the following nonlinear parabolic equation:

$$\begin{cases} u_t &= \Delta u + |u|^{p-1}u + h(u), \\ u(0) &= u_0 \in L^\infty(\mathbb{R}^n), \end{cases} \tag{1.1}$$

where u is defined for $(x,t) \in \mathbb{R}^n \times [0,T)$, p is a sub-critical nonlinearity,

$$1 < p, \quad (n-2)p < n+2. \tag{1.2}$$

[1]submitted, arXiv:1404.4018

The function h is in $\mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}^*)$ satisfying

$$j = 0, 1,\ |h^{(j)}(z)| \leq M\left(\frac{|z|^{p-j}}{\log^a(2+z^2)} + 1\right), \quad |h''(z)| \leq M\frac{|z|^{p-2}}{\log^a(2+z^2)}, \tag{1.3}$$

where $a > 1$, $M > 0$. Typically, $h(z) = \frac{\mu|z|^{p-1}z}{\log^a(2+z^2)}$ with $\mu \in \mathbb{R}$.

By standard results, the problem (1.1) has a unique classical solution $u(x,t)$ in $L^\infty(\mathbb{R}^n)$, which exists at least for small times. The solution $u(x,t)$ may develop singularities in some finite time. We say that a function $u : \mathbb{R}^n \times [0,T) \mapsto \mathbb{R}$ is a solution of (1.1) if u solves (1.1) and satisfies

$$u, u_t, \nabla u, \nabla^2 u \text{ are bounded and continuous on } \mathbb{R}^n \times [0,\tau],\ \forall \tau < T. \tag{1.4}$$

It is said that $u(x,t)$ blows up in a finite time $T < +\infty$ if $u(x,t)$ satisfies (1.1), (1.4) and

$$\lim_{t\to T} \|u(t)\|_{L^\infty(\mathbb{R}^n)} = +\infty.$$

Here we call T the blow-up time of $u(x,t)$. In such a blow-up case, a point $x_0 \in \mathbb{R}^n$ is called a blow-up point of $u(x,t)$ if and only if there exist $(x_n, t_n) \to (x_0, T)$ such that $|u(x_n, t_n)| \to +\infty$ as $n \to +\infty$.

Consider v a positive blow-up solution of the associated ODE of (1.1). It is clear that v is given by

$$v' = v^p + h(v), \quad v(T) = +\infty, \quad \text{for some } T > 0. \tag{1.5}$$

Since the blow-up solution of (1.5) satisfies (see Lemma A.1)

$$v(t) \sim \kappa(T-t)^{-\frac{1}{p-1}} \quad \text{as } t \to T, \text{ where } \kappa = (p-1)^{-\frac{1}{p-1}}, \tag{1.6}$$

it is natural to ask whether the blow-up solution $u(t)$ of (1.1) has the same blow-up rate as $v(t)$ does. More precisely, are there constants $c, C > 0$ such that

$$c(T-t)^{-\frac{1}{p-1}} \leq \|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(T-t)^{-\frac{1}{p-1}}, \quad \forall t \in (0,T)? \tag{1.7}$$

By a simple argument based on Duhamel's formula, we can show that the lower bound in (1.7) is always satisfied (see [22]). For the upper blow-up rate estimate, it is much less simple and requires more work. Practically, we define for all $x_0 \in \mathbb{R}^n$ (x_0 may be a blow-up point of u or not) the following *similarity variables* introduced in Giga and Kohn [5], [6], [7]:

$$y = \frac{x - x_0}{\sqrt{T-t}}, \quad s = -\log(T-t), \quad w_{x_0,T}(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t). \tag{1.8}$$

Hence $w_{x_0,T}$ satisfies for all $s \geq -\log T$ and for all $y \in \mathbb{R}^n$:

$$\partial_s w_{x_0,T} = \frac{1}{\rho}\text{div}(\rho \nabla w_{x_0,T}) - \frac{w_{x_0,T}}{p-1} + |w_{x_0,T}|^{p-1} w_{x_0,T} + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}} w_{x_0,T}\right), \tag{1.9}$$

where

$$\rho(y) = \left(\frac{1}{4\pi}\right)^{n/2} e^{-\frac{|y|^2}{4}}, \quad y \in \mathbb{R}^n. \tag{1.10}$$

Here, we say that $w : \mathbb{R}^n \times [-\log T, +\infty) \mapsto \mathbb{R}$ is a solution of (1.9) if w solves (1.9) and satisfies

$$w, w_s, \nabla w, \nabla^2 w \text{ are bounded and continuous on } \mathbb{R}^n \times [-\log T, S], \ \forall S < +\infty. \tag{1.11}$$

We can see that the study of u in the neighborhood of (x_0, T) is equivalent to the study of the long-time behavior of $w_{x_0,T}$ and each result for u has an equivalent formulation in term of $w_{x_0,T}$. In particular, the proof of the upper bound in (1.7) is now equivalent to showing that there exists a time $\hat{s} \geq -\log T$ large enough such that

$$\|w_{x_0,T}(s)\|_{L^\infty(\mathbb{R}^n)} \leq C, \quad \forall s \geq \hat{s}. \tag{1.12}$$

We remark that the perturbation term added to equation (1.9) satisfies the following inequality,

$$j = 0, 1, \quad e^{-\frac{(p-j)s}{p-1}} \left| h^{(j)} \left(e^{\frac{s}{p-1}} z \right) \right| \leq \frac{C_0}{s^a} \left(|z|^{p-j} + 1 \right), \quad \forall s \geq s_0, \tag{1.13}$$

for some $C_0 > 0$ and $s_0 > 0$ (see Lemma A.3 for a proof of this fact).

When $h \equiv 0$, Giga and Kohn proved (1.12) in [6] for $1 < p < \frac{3n+8}{3n-4}$ or for non-negative initial data (so that the solution is positive everywhere) with sub-critical p (note that Weissler [21] first obtained (1.7) in the positive, radially symmetric case under the assumption that, for each $0 < t < T$, $u_t(x,t)$ achieves maximum at $x = 0$). Estimate (1.12) is extended for all p satisfying (1.2) without assuming non-negativity for initial data u_0 by Giga, Matsui and Sasayama in [8]. The proof written in [8] is strongly based on the existence of the following Lyapunov functional:

$$\mathcal{E}_0[w](s) = \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla w|^2 + \frac{1}{2(p-1)}|w|^2 - \frac{1}{p+1}|w|^{p+1} \right) \rho dy. \tag{1.14}$$

Based on this functional, some energy estimates related to this structure and a bootstrap argument given in [16], the authors in [8] have established the following key integral estimate

$$\sup_{s \geq s'} \int_s^{s+1} \|w_{x_0,T}(s)\|^{(p+1)q}_{L^{p+1}(\mathbf{B}_R)} ds \leq C_{q,s'}, \quad \forall q \geq 2, \quad s' > -\log T. \tag{1.15}$$

Since this estimate holds for all $q \geq 2$, we obtain an upper bound for $w_{x_0,T}$ which yields (1.12).
Mueller and Weissler [14], Friedman and McLeod [4] obtained related results for a general semilinear heat equation of the form

$$u_t = \Delta u + f(u), \quad u_0 \geq 0$$

under some mild assumptions on f. In particular, they showed that

$$\|u(t)\|_{L^\infty} \leq G^{-1}(C(T-t)) \quad \text{with} \quad G(s) = \int_s^{+\infty} \frac{d\tau}{f(\tau)}.$$

In this paper, we wonder whether a perturbation of the method of [8] would work for our problem (note that we have no assumption on sign of u_0 and h). A key step is to find a Lyapunov functional for equation (1.9). Following the method introduced by Hamza and Zaag in [10], [9] for perturbations of the semilinear wave equation, we introduce

$$\mathcal{J}[w](s) = \mathcal{E}[w](s)e^{\frac{\gamma}{a-1}s^{1-a}} + \theta s^{1-a}, \tag{1.16}$$

where $\gamma = 8C_0\left(\frac{p+1}{p-1}\right)^2$ and $\theta > 0$ is sufficiently large constant which will be determined later,

$$\mathcal{E}[w] = \mathcal{E}_0[w] + \mathcal{I}[w], \quad \mathcal{I}[w](s) = -e^{-\frac{p+1}{p-1}s}\int_{\mathbb{R}^n} H\left(e^{\frac{s}{p-1}}w\right)\rho dy, \tag{1.17}$$

with $H(z) = \int_0^z h(\xi)d\xi$.
With this introduction, we derive that the functional $\mathcal{J}[w]$ is a decreasing function of time for equation (1.9), provided that s is large enough. More precisely, we have the following:

Theorem 1.1 (Existence of a Lyapunov functional for equation (1.9)**).** *Let a, p, n, M be fixed, consider w a solution of equation* (1.9) *satisfying* (1.11)*. Then there exist $\hat{s}_0 = \hat{s}_0(a,p,n,M) \geq s_0$ and $\hat{\theta}_0 = \hat{\theta}_0(a,p,n,M)$ such that if $\theta \geq \hat{\theta}_0$, then $\mathcal{J}$ satisfies the following inequality, for all $s_2 > s_1 \geq \max\{\hat{s}_0, -\log T\}$,*

$$\mathcal{J}[w](s_2) - \mathcal{J}[w](s_1) \leq -\frac{1}{2}\int_{s_1}^{s_2}\int_{\mathbb{R}^n}(\partial_s w)^2 \rho dy ds. \tag{1.18}$$

As mentioned above, the existence of this Lyapunov functional $\mathcal{J}$ is a crucial step in the derivation of the blow-up rate for equation (1.1). Indeed, with the functional $\mathcal{J}$ and some more work, we are able to adapt the analysis in [8] for equation (1.1) in the case $h \equiv 0$ and get the following result:

Theorem 1.2 (Blow-up rate for equation (1.1)**).** *Let a, p, n, M be fixed, p satisfy* (1.2)*. There exists $\hat{s}_1 = \hat{s}_1(a,p,n,M) \geq \hat{s}_0$ such that if u is a blow-up solution of equation* (1.1) *with a blow-up time T, then*
(i) for all $s \geq s' = \max\{\hat{s}_1, -\log T\}$,

$$\|w_{x_0,T}(s)\|_{L^\infty(\mathbb{R}^n)} \leq C, \tag{1.19}$$

where $w_{x_0,T}$ is defined in (1.8) *and C is a positive constant depending only on n, p, M and a bound of $\|w_{x_0,T}(\hat{s}_0)\|_{L^\infty}$.*
(ii) For all $t \in [t_1, T)$ where $t_1 = T - e^{-s'}$,

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(T-t)^{-\frac{1}{p-1}}. \tag{1.20}$$

Remark 1.3. *The proof of Theorem 1.2 is far from being a straightforward adaptation of [8]. Indeed, three major difficulties arise in our case and make the heart of our contribution:*
- the existence of a Lyapunov functional in similarity variables (see Theorem 1.1 above),
- the control of the L^2-norm in terms of the energy (see (ii) of Proposition 2.3, where we rely on a new blow-up criterion greatly simplifying the approach in [6]),
- the proof of a nonlinear parabolic result (see Proposition 2.7 below).

The estimate obtained in Theorem 1.2 is a fundamental step in studying the asymptotic behavior of blow-up solutions. When $h \equiv 0$, Giga and Kohn in [6], [7] (see also [5]) obtained the following result: *For a given blow-up point x_0, it holds that*

$$\lim_{s\to+\infty} w_{x_0,T}(y,s) = \lim_{t\to T}(T-t)^{\frac{1}{p-1}} u(x_0 + y\sqrt{T-t}, t) = \pm\kappa,$$

where $\kappa = (p-1)^{\frac{1}{p-1}}$, uniformly on compact subsets of $\mathbb{R}^n$. The result is pointwise in x_0. Besides, for a.e. y, $\lim_{s\to+\infty} \nabla w_{x_0,T}(y,s) = 0$.

For our problem, when $h \not\equiv 0$ and h is given in (1.3), we also derive an analogous result on the behavior of $w_{x_0,T}$ as $s \to +\infty$. We claim the following:

Theorem 1.4 (Behavior of $w_{x_0,T}$ as $s \to +\infty$). *Let a, p, n, M be fixed, p satisfy* (1.2)*. Consider $u(t)$ a solution of equation* (1.1) *which blows up at time T and x_0 a blow-up point. Then*

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}} u(x_0 + y\sqrt{T-t}, t) = \lim_{s\to+\infty} w_{x_0,T}(y,s) = \pm\kappa,$$

holds in L^2_ρ (L^2_ρ is the weighted L^2 space associated with the weight ρ (1.10)*), and also uniformly on each compact subset of $\mathbb{R}^n$.*

Up to changing u_0 in $-u_0$ and h in $-h$, we may assume that $w \to \kappa$ in L^2_ρ as $s \to +\infty$. Let us consider ϕ a positive solution of the associated ordinary differential equation of equation (1.9)

$$\phi_s = -\frac{\phi}{p-1} + \phi^p + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}}\phi\right) \tag{1.21}$$

such that

$$\phi(s) = \kappa + \mathcal{O}\left(\frac{1}{s^a}\right) \quad \text{as} \quad s \to +\infty, \tag{1.22}$$

(see Lemma A.4 for a proof of the existence of ϕ).
Let us introduce $v_{x_0,T} = w_{x_0,T} - \phi(s)$, then $\|v_{x_0,T}(s)\|_{L^2_\rho} \to 0$ as $s \to +\infty$ and $v_{x_0,T}$ (or v for simplicity) satisfies the following equation:

$$\partial_s v = (\mathcal{L} + \omega(s))v + F(v) + H(v,s), \quad \forall y \in \mathbb{R}^n,\ \forall s \in [-\log T, +\infty),$$

where $\mathcal{L} = \Delta - \frac{y}{2}\cdot\nabla + 1$ and ω, F, H satisfy

$$|\omega(s)| = \mathcal{O}(s^{-a}) \quad \text{and} \quad |F(v)| + |H(v,s)| = \mathcal{O}(|v|^2) \quad \text{as } s \to +\infty,$$

(see the beginning of Section 3 for the proper definitions of ω, F and G).
Since the linear part will play an important role in our analysis, let us point out its properties. The operator $\mathcal{L}$ is self-adjoint on $L^2_\rho(\mathbb{R}^n)$. Its spectrum is given by

$$spec(\mathcal{L}) = \{1 - \frac{m}{2},\ m \in \mathbb{N}\},$$

and it consists of eigenvalues. The eigenfunctions of $\mathcal{L}$ are derived from Hermite polynomials:
- For $n = 1$, the eigenfunction corresponding to $1 - \frac{m}{2}$ is

$$h_m(y) = \sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{k!(m-2k)!}(-1)^k y^{m-2k}, \tag{1.23}$$

- For $n \geq 2$: we write the spectrum of $\mathcal{L}$ as

$$spec(\mathcal{L}) = \{1 - \frac{|m|}{2},\ |m| = m_1 + \cdots + m_n,\ (m_1, \ldots, m_n) \in \mathbb{N}^n\}.$$

For $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$, the eigenfunction corresponding to $1 - \frac{|m|}{2}$ is

$$H_m(y) = h_{m_1}(y_1) \ldots h_{m_n}(y_n), \tag{1.24}$$

where h_m is defined in (1.23).

By studying the behavior of v as $s \to +\infty$, we obtain the following result:

Theorem 1.5 (Classification of the behavior of w as $s \to +\infty$). *Consider $u(t)$ a solution of equation* (1.1) *which blows-up at time T and x_0 a blow-up point. Let $w(y,s)$ be a solution of equation* (1.9)*. Then one of the following possibilities occurs:*
$i)$ $w(y,s) \equiv \phi(s)$,
$ii)$ There exists $l \in \{1, \ldots, n\}$ such that up to an orthogonal transformation of coordinates, we have

$$w(y,s) = \kappa - \frac{\kappa}{4ps}\left(\sum_{j=1}^{l} y_j^2 - 2l\right) + \mathcal{O}\left(\frac{1}{s^a}\right) + \mathcal{O}\left(\frac{\log s}{s^2}\right) \quad \textit{as} \quad s \to +\infty.$$

$iii)$ There exist an integer number $m \geq 3$ and constants c_α not all zero such that

$$w(y,s) = \phi(s) - e^{-(\frac{m}{2}-1)s} \sum_{|\alpha|=m} c_\alpha H_\alpha(y) + o\left(e^{-(\frac{m}{2}-1)s}\right) \quad \textit{as} \quad s \to +\infty.$$

The convergence takes place in L^2_ρ as well as in $\mathcal{C}^{k,\gamma}_{loc}$ for any $k \geq 1$ and some $\gamma \in (0,1)$.

Remark 1.6. *Applying our result to a space-independent solution of* (1.9)*, we get the uniqueness of the solution of the ODE* (1.21) *that converges to κ as $s \to +\infty$.*

Remark 1.7. *Since both the perturbed ($h \not\equiv 0$) and the unperturbed ($h \equiv 0$) cases in equation* (1.1) *share the same convergence stated in Theorem 1.5, we wonder whether the perturbation h may have an influence on further terms of the expansion of w. From our result, if case (ii) occurs, we see no difference in the following term of the expansion. On the contrary, if case (i) or (iii) occurs, with $h(x) = \mu \frac{|x|^{p-1}x}{\log^a(2+x^2)}$, we see from Lemma A.4 that*

$$w(y,s) - \kappa \sim \frac{C_0(a,p,\mu)}{s^a} \quad \text{as } s \to +\infty,$$

which is clearly different from the unperturbed case when in case (i), we have $w \equiv \kappa$ and case (iii), we have $w - \kappa = \mathcal{O}(e^{-s})$, (see [11], [20]).

Remark 1.8. *If we linearize w around κ, which is an explicit profile, we then fall in logarithmic scales $\gamma = \frac{1}{|\log \epsilon|}$ with $\epsilon = T - t$. Further refinements in this direction should give an expansion of $w - \kappa$ in terms of powers of γ, i.e in logarithmic scales of ϵ. Therefore, we can not reach significantly small error terms in the expansion of the solution w as (iii) of Theorem 1.5 describes. In order to escape this situation, a relevant approximation is required in order to go beyond all logarithmic scales, i.e approximations up to lower order terms such as ϵ^α for some $\alpha > 0$. Our idea to capture such relevant terms is to abandon the explicit profile obtained as a first order approximation, namely κ, and take an implicit profile function as a first order description of the singular behavior, namely $\phi(s)$ introduced in* (1.21) *and* (1.22)*. A similar idea was used by Zaag [23] where the solution was linearized around a less explicit profile function in order to go beyond all logarithmic scales. For our problem, we particularly take $\phi(s)$ as the implicit profile function, which is a solution of the associated ODE of equation* (1.9) *in w such that $\phi(s) \to \kappa$ as $s \to +\infty$. By linearizing the solution w around ϕ, we can get to error terms of polynomial order $\epsilon^{\left(\frac{m}{2}-1\right)}$, as stated in (iii) of Theorem 1.5.*

Remark 1.9. *When $h(x) = |x|^q$ with $q \in (1,p)$, we see that*

$$\phi(s) - \kappa \sim C_0'(p,q)e^{-\lambda s} \quad \text{as} \quad s \to +\infty.$$

If case (ii) in Theorem 1.5 holds, we then recover the same expansion as in the unperturbed case ($h \equiv 0$). On the contrary, if case (i) or (iii) occurs, then

$$w(y,s) - \kappa \sim C_0'(p,q)e^{-\lambda s} \quad \text{as} \quad s \to +\infty.$$

Moreover, if case (iii) in Theorem 1.5 holds, we have new terms in the expansion of w which was not available in the unperturbed case, namely

$$w(y,s) = \kappa - \sum_{k=1}^{K} C_k e^{-k\lambda s} - e^{-\left(\frac{m}{2}-1\right)s} \sum_{|\alpha|=m} c_\alpha H_\alpha(y) + o\left(e^{-\left(\frac{m}{2}-1\right)s}\right) \text{ as } s \to +\infty,$$

where $C_k, k = 1,2,\dots,K$ are some constants depending on p and q, and $K \in \mathbb{N}$ is the integer part of $\frac{1}{\lambda}\left(\frac{m}{2}-1\right)$.

In the last section, we will extend the asymptotic behavior of w obtained in Theorem 1.5 to larger regions. Particularly, we claim the following:

Theorem 1.10 (Convergence extension of w_a to larger regions). *For all $K_0 > 0$,*
$i)$ if $ii)$ of Theorem 1.5 occurs, then

$$\sup_{|\xi|\leq K_0} \left|w(\xi\sqrt{s}, s) - f_l(\xi)\right| = \mathcal{O}\left(\frac{1}{s^{a-1}}\right) + \mathcal{O}\left(\frac{\log s}{s}\right) \quad \text{as } s \to +\infty, \tag{1.25}$$

where

$$f_l(\xi) = \kappa\left(1 + \frac{p-1}{4p}\sum_{j=1}^{l}\xi_j^2\right)^{-\frac{1}{p-1}}, \quad \forall \xi \in \mathbb{R}^n,$$

with l the same as in $ii)$ of Theorem 1.5.
$ii)$ if $iii)$ of Theorem 1.5 occurs, then $m \geq 4$ is even, and

$$\sup_{|\xi|\leq K_0} \left|w\left(\xi e^{\left(\frac{1}{2}-\frac{1}{m}\right)s}, s\right) - \psi_m(\xi)\right| \to 0 \quad \text{as } s \to +\infty, \tag{1.26}$$

where

$$\psi_m(\xi) = \kappa\left(1 + \kappa^{-p}\sum_{|\alpha|=m} c_\alpha \xi^\alpha\right)^{-\frac{1}{p-1}}, \quad \forall \xi \in \mathbb{R}^n,$$

with c_α the same as in Theorem 1.5.

Let us mention briefly the structure of the paper. In Section 2, we prove the existence of Lyapunov functional for equation (1.9) (Theorem 1.1), we then get Theorem 1.2 and Theorem 1.4. In Section 3, we follow the method of [3] and [20] to prove Theorem 1.5. Finally, the section 4 is devoted to the proof of Theorem 1.10.

Acknowledgement: The author is grateful to H. Zaag for his dedicated advice, suggestions and remarks during the preparation of this paper.

2 A Lyapunov functional

This section is divided in four subsections: we first prove the existence of a Lyapunov functional for equation (1.9) (Theorem 1.1); after that, we derive a blow-up criterion for equation (1.9) and some energy estimates based on this Lyapunov functional. Following the method of [8], we prove the boundedness of solution in similarity variables which determines the blow-up rate for solutions of (1.1) (Theorem 1.2). Finally, we derive the limit of w as $s \to +\infty$, which concludes Theorem 1.4.

In what follows, we denote by $C_i, i = 0, 1, \ldots$ positive constants depending only on a, n, p, M, and by $L^q_\rho(\Omega)$ the weighted $L^q(\Omega)$ space endowed with the norm

$$\|\varphi\|_{L^q_\rho(\Omega)} = \left(\int_\Omega |\varphi(y)|^q \rho(y) dy\right)^{\frac{1}{q}},$$

and by $H^1_\rho(\Omega)$ the space of function $\varphi \in L^2_\rho(\Omega)$ satisfying $\nabla\varphi \in L^2_\rho(\Omega)$, endowed with the norm

$$\|\varphi\|_{H^1_\rho(\Omega)} = \left(\|\nabla\varphi\|^2_{L^2_\rho(\Omega)} + \frac{1}{p-1}\|\varphi\|^2_{L^2_\rho(\Omega)}\right)^{\frac{1}{2}}.$$

We denote by $\mathbf{B}_R(x)$ the open ball in $\mathbb{R}^n$ with center x and radius R, and set $\mathbf{B}_R := \mathbf{B}_R(0)$.

2.1 Existence of a Lyapunov function

In this part, we aim at proving that the functional $\mathcal{J}$ defined in (1.16) is a Lyapunov functional for equation (1.9). Note that that functional is far from being trivial and it is our main contribution. We first claim the following lemma:

Lemma 2.1. *Let a, p, n, M be fixed and w be solution of equation* (1.9) *satisfying* (1.11). *There exists $\tilde{s}_0 = \tilde{s}_0(a, p, n, M) \geq s_0$ such that the functional of $\mathcal{E}$ defined in* (1.17) *satisfies the following inequality, for all $s \geq \max\{\tilde{s}_0, -\log T\}$,*

$$\frac{d}{ds}\mathcal{E}[w](s) \leq -\frac{1}{2}\int_{\mathbb{R}^n} w_s^2 \rho dy + \gamma s^{-a}\mathcal{E}[w](s) + Cs^{-a}, \tag{2.1}$$

where $\gamma = 8C_0\left(\frac{p+1}{p-1}\right)^2$, C_0 is introduced in (1.13) *and C is a positive constant depending only on a, p, n, M.*

Let us first derive Theorem 1.1 from Lemma 2.1 which will be proved later.

Proof of Theorem 1.1 admitting Lemma 2.1. Differentiating the functional $\mathcal{J}$ defined in (1.16), we obtain

$$\begin{aligned}
\frac{d}{ds}\mathcal{J}[w](s) &= \frac{d}{ds}\left\{\mathcal{E}[w](s)e^{\frac{\gamma}{a-1}s^{1-a}} + \theta s^{1-a}\right\}\\
&= \frac{d}{ds}\mathcal{E}[w](s)e^{\frac{\gamma}{a-1}s^{1-a}} - \gamma s^{-a}\mathcal{E}[w](s)e^{\frac{\gamma}{a-1}s^{1-a}} - (a-1)\theta s^{-a}\\
&\leq -\frac{1}{2}e^{\frac{\gamma}{a-1}s^{1-a}}\int_{\mathbb{R}^n} w_s^2\rho dy + \left[Ce^{\frac{\gamma}{a-1}s^{1-a}} - (a-1)\theta\right]s^{-a} \quad \text{(use (2.1))}.
\end{aligned}$$

Choosing θ large enough such that $Ce^{\frac{\gamma}{a-1}\tilde{s}_0^{1-a}} - (a-1)\theta \leq 0$ and noticing that $e^{\frac{\gamma}{a-1}s^{1-a}} \geq 1$ for all $s > 0$, we derive

$$\frac{d}{ds}\mathcal{J}[w](s) \leq -\frac{1}{2}\int_{\mathbb{R}^n} w_s^2\rho dy, \quad \forall s \geq \tilde{s}_0.$$

This implies inequality (1.18) and concludes the proof of Theorem 1.1, assuming that Lemma 2.1 holds. □

It remains to prove Lemma 2.1 in order to conclude the proof of Theorem 1.1.

Proof of Lemma 2.1 . Multiplying equation (1.9) with $w_s\rho$ and integrating by parts:

$$\int_{\mathbb{R}^n}|w_s|^2\rho = -\frac{d}{ds}\left\{\int_{\mathbb{R}^n}\left(\frac{1}{2}|\nabla w|^2+\frac{1}{2(p-1)}|w|^2-\frac{1}{p+1}|w|^{p+1}\right)\rho dy\right\} + e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)w_s\rho dy.$$

For the last term of the above expression, denoting $H(z)=\int_0^z h(\xi)d\xi$, we write in the following:

$$\begin{aligned} e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)w_s\rho dy &= e^{-\frac{(p+1)s}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)\left(e^{\frac{s}{p-1}}w_s+\frac{e^{\frac{s}{p-1}}}{p-1}w\right)\rho dy \\ &\quad -\frac{1}{p-1}e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)w\rho dy \\ &= e^{-\frac{p+1}{p-1}s}\frac{d}{ds}\int_{\mathbb{R}^n}H\left(e^{\frac{s}{p-1}}w\right)\rho dy-\frac{1}{p-1}e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)w\rho dy. \end{aligned}$$

This yields

$$\begin{aligned} \int_{\mathbb{R}^n}|w_s|^2\rho dy = &-\frac{d}{ds}\left\{\int_{\mathbb{R}^n}\left(\frac{1}{2}|\nabla w|^2+\frac{1}{2(p-1)}|w|^2-\frac{1}{p+1}|w|^{p+1}\right)\rho dy\right\} \\ &+\frac{d}{ds}\left\{e^{-\frac{p+1}{p-1}s}\int_{\mathbb{R}^n}H\left(e^{\frac{s}{p-1}}w\right)\rho dy\right\} \\ &+\frac{p+1}{p-1}e^{-\frac{p+1}{p-1}s}\int_{\mathbb{R}^n}H\left(e^{\frac{s}{p-1}}w\right)\rho dy \\ &-\frac{1}{p-1}e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)w\rho dy. \end{aligned}$$

From the definition of the functional $\mathcal{E}$ given in (1.17), we derive a first identity in the following:

$$\begin{aligned} \frac{d}{ds}\mathcal{E}[w](s) = -\int_{\mathbb{R}^n}|w_s|^2\rho dy+\frac{p+1}{p-1}e^{-\frac{p+1}{p-1}s}\int_{\mathbb{R}^n}H\left(e^{\frac{s}{p-1}}w\right)\rho dy \\ -\frac{1}{p-1}e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)w\rho dy. \end{aligned} \tag{2.2}$$

A second identity is obtained by multiplying equation (1.9) with $w\rho$ and integrating by parts:

$$\begin{aligned}\frac{d}{ds}\int_{\mathbb{R}^n}|w|^2\rho dy = -4\Big\{\int_{\mathbb{R}^n}\left(\frac{1}{2}|\nabla w|^2+\frac{1}{2(p-1)}|w|^2-\frac{1}{p+1}|w|^{p+1}\right)\rho dy\\ -e^{-\frac{(p+1)s}{p-1}}\int_{\mathbb{R}^n}H\left(e^{\frac{s}{p-1}}w\right)\rho dy\Big\}\\ +\left(2-\frac{4}{p+1}\right)\int_{\mathbb{R}^n}|w|^{p+1}\rho dy-4e^{-\frac{p+1}{p-1}s}\int_{\mathbb{R}^n}H\left(e^{\frac{s}{p-1}}w\right)\rho dy\\ +2e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)w\rho dy.\end{aligned}$$

Using again the definition of $\mathcal{E}$ given in (1.17), we derive the second identity in the following:

$$\begin{aligned}\frac{d}{ds}\int_{\mathbb{R}^n}|w|^2\rho dy = -4\mathcal{E}[w](s)+2\frac{p-1}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\rho dy\\ -4e^{-\frac{p+1}{p-1}s}\int_{\mathbb{R}^n}H\left(e^{\frac{s}{p-1}}w\right)\rho dy+2e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)w\rho dy.\end{aligned}\tag{2.3}$$

From (2.2), we estimate

$$\begin{aligned}\frac{d}{ds}\mathcal{E}[w](s)\le -\int_{\mathbb{R}^n}|w_s|^2\rho dy\\ +\frac{p+1}{p-1}\int_{\mathbb{R}^n}\left\{\left|e^{-\frac{p+1}{p-1}s}H\left(e^{\frac{s}{p-1}}w\right)\right|+\left|e^{-\frac{ps}{p-1}}h\left(e^{\frac{s}{p-1}}w\right)w\right|\right\}\rho dy.\end{aligned}$$

From (1.13) and using the fact that $|w|\le|w|^{p+1}+1$, we obtain for all $s\ge s_0$,

$$\left|e^{-\frac{p+1}{p-1}s}H\left(e^{\frac{s}{p-1}}w\right)\right|+\left|e^{-\frac{p}{p-1}s}h\left(e^{\frac{s}{p-1}}w\right)w\right|\le 2C_0s^{-a}\left(|w|^{p+1}+1\right).\tag{2.4}$$

Using (2.4) yields

$$\frac{d}{ds}\mathcal{E}[w](s)\le -\int_{\mathbb{R}^n}|w_s|^2\rho dy+C_1s^{-a}\int_{\mathbb{R}^n}|w|^{p+1}\rho dy+C_1s^{-a},\tag{2.5}$$

where $C_1=2C_0\frac{p+1}{p-1}$.
From (2.3), we have

$$\begin{aligned}\int_{\mathbb{R}^n}|w|^{p+1}\rho dy\le\frac{2(p+1)}{p-1}\mathcal{E}[w](s)+\frac{p+1}{p-1}\int_{\mathbb{R}^n}|w_sw|\rho dy\\ +\frac{2(p+1)}{p-1}\int_{\mathbb{R}^n}\left(\left|e^{-\frac{p+1}{p-1}s}H\left(e^{\frac{s}{p-1}}w\right)\right|+\left|e^{-\frac{ps}{p-1}}h\left(e^{\frac{s}{p-1}}w\right)w\right|\rho dy\right).\end{aligned}$$

Using the fact that $|w_sw|\le\epsilon(|w_s|^2+|w|^{p+1})+C_2(\epsilon)$ for all $\epsilon>0$ and (2.4), we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}|w|^{p+1}\rho dy\le\frac{2(p+1)}{p-1}\mathcal{E}[w](s)+\epsilon'\int_{\mathbb{R}^n}|w_s|^2\rho dy\\ +\left(\epsilon'+2C_1s^{-a}\right)\int_{\mathbb{R}^n}|w|^{p+1}\rho dy+2C_1s^{-a}+C_3,\end{aligned}$$

where $\epsilon' = \epsilon\frac{p+1}{p-1}$, $C_3 = 2C_1 + C_2\frac{p+1}{p-1}$.
Taking $\epsilon = \frac{p-1}{4(p+1)}$ and s_1 large enough such that $2C_1 s^{-a} \leq \frac{1}{4}$ for all $s \geq s_1$, we see that

$$\int_{\mathbb{R}^n} |w|^{p+1}\rho dy \leq \frac{4(p+1)}{p-1}\mathcal{E}[w](s) + \frac{1}{2}\int_{\mathbb{R}^n} |w_s|^2\rho dy + C_4, \quad \forall s > s_1, \tag{2.6}$$

with $C_4 = \frac{C_3}{2} + \frac{1}{8}$.
Substituting (2.6) into (2.5) yields (2.1) with $\tilde{s}_0 = \max\{s_0, s_1\}$. This concludes the proof of Lemma 2.1. Since we have already showed that Theorem 1.1 is a direct consequence of Lemma 2.1, this is also the conclusion of Theorem 1.1. ☐

2.2 A blow-up criterion for the equation in similarity variables

In this part, we give a new blow-up criterion for equation (1.9). Then, we will use it to control the L^2-norm in terms of the energy (see (ii) of Proposition 2.3). We claim the following:

Lemma 2.2. *Let a, p, n, M be fixed and w be solution of equation* (1.9) *satisfying* (1.11). *If there exists $\tilde{s}_1 = \tilde{s}_1(a, p, n, M) \geq \max\{\hat{s}_0, -\log T\}$ such that*

$$-4\mathcal{J}[w](\bar{s}) + \frac{p-1}{p+1}\left(\int_{\mathbb{R}^n} |w(y,\bar{s})|^2\rho dy\right)^{\frac{p+1}{2}} > 0 \quad \textit{for some } \bar{s} \geq \tilde{s}_1, \tag{2.7}$$

then w is not defined for all $(y, s) \in \mathbb{R}^n \times [\bar{s}, +\infty)$.

Proof. We proceed by contradiction and suppose that w is defined for all $s \in [\bar{s}, +\infty)$. From definition of $\mathcal{J}$ in (1.16) and from (2.3), (2.4), we have for all $s \geq s_0$,

$$\begin{aligned}\frac{d}{ds}\int_{\mathbb{R}^n} |w|^2\rho dy \geq & -4e^{-\frac{\gamma}{a-1}s^{1-a}}\left(\mathcal{J}[w](s) - \theta s^{1-a}\right) \\ & + 2\left(\frac{p-1}{p+1} - 4C_0 s^{-a}\right)\int_{\mathbb{R}^n} |w|^{p+1}\rho dy - 8C_0 s^{-a}.\end{aligned} \tag{2.8}$$

We take s_1 large enough such that

$$4C_0 s^{-a} \leq \frac{p-1}{2(p+1)} \quad \text{and} \quad e^{-\frac{\gamma}{a-1}s^{1-a}} - \frac{2C_0}{s} > 0 \quad \text{for all} \quad s \geq s_1.$$

Then, using Jensen's inequality and noting that $e^{-\frac{\gamma}{a-1}s^{1-a}} \leq 1$ for all $s > 0$, we get from (2.8) the following: for all $s \geq \max\{0, s_0, s_1\}$,

$$\frac{d}{ds}\int_{\mathbb{R}^n} |w|^2\rho dy \geq -4\mathcal{J}[w](s) + \frac{p-1}{p+1}\left(\int_{\mathbb{R}^n} |w|^2\rho dy\right)^{\frac{p+1}{2}}. \tag{2.9}$$

Setting $f(s) = \int_{\mathbb{R}^n} |w(y,s)|^2 \rho dy$, $A = -4\mathcal{J}[w](\bar{s})$ and $B = \frac{p-1}{p+1}$, then using the fact that $\mathcal{J}$ is decreasing in time to get that

$$f'(s) \geq A + Bf(s)^{\frac{p+1}{2}}, \quad \forall s \geq \bar{s}.$$

The hypothesis reads $A + Bf(\bar{s})^{\frac{p+1}{2}} > 0$ which implies that

$$f'(s) > 0 \quad \text{and} \quad A + Bf(s)^{\frac{p+1}{2}} > 0, \quad \forall s \geq \bar{s}.$$

By a direct integration, we obtain

$$\forall s \geq \bar{s}, \quad s - \bar{s} \leq \int_{f(\bar{s})}^{f(s)} \frac{dz}{A + Bz^{\frac{p+1}{2}}} \leq \int_{f(\bar{s})}^{+\infty} \frac{dz}{A + Bz^{\frac{p+1}{2}}} < +\infty,$$

which is a contradiction and Lemma 2.2 is proved.

□

As a consequence of Theorem 1.1 and Lemma 2.2, we obtain the following estimates which will be useful for getting Theorem 1.2:

Proposition 2.3. *Let w be solution of equation* (1.9) *satisfying* (1.11)*, it holds that*

$$-Q_0 \leq \mathcal{E}[w](s) \leq 2J_0, \quad \forall s \geq \tilde{s}_2 = \max\{\hat{s}_0, -\log T\},$$

where $J_0 = \mathcal{J}[w](\tilde{s}_2)$ and $Q_0 = \theta \tilde{s}_2^{1-a}$. Moreover, there exists a time $\tilde{s}_3 \geq \max\{\hat{s}_0, -\log T\}$ such that for all $s \geq \tilde{s}_3$

$$\begin{aligned}
&(i) \quad \int_s^{s+1} \|w_\tau(\tau)\|^2_{L^2_\rho(\mathbb{R}^n)} d\tau \leq 2J_0,\\
&(ii) \quad \|w(s)\|^2_{L^2_\rho(\mathbb{R}^n)} \leq J_1,\\
&(iii) \quad \|w(s)\|^{p+1}_{L^{p+1}_\rho(\mathbb{R}^n)} \leq J_2\left(1 + \|w(s)\|^2_{H^1_\rho(\mathbb{R}^n)}\right),\\
&(iv) \quad \|w(s)\|^2_{H^1_\rho(\mathbb{R}^n)} \leq J_3\left(1 + \|w_s(s)\|_{L^2_\rho(\mathbb{R}^n)}\right),\\
&(v) \quad \int_s^{s+1} \|w(\tau)\|^{2(p+1)}_{L^{p+1}_\rho(\mathbb{R}^n)} d\tau \leq J_4,\\
&(vi) \quad \int_s^{s+1} \|w(\tau)\|^2_{H^1_\rho(\mathbb{R}^n)} d\tau \leq J_5,
\end{aligned}$$

where J_i, $i = 1, \dots, 5$ depend only on J_0, Q_0, a, p, n, M.

Proof. The upper and lower bounds of $\mathcal{E}$, (i) and (ii) obviously follow from Theorem 1.1 and Lemma 2.2 (in fact, since w is defined for all $s \geq \tilde{s}_1$, condition (2.7) is never satisfied).

(iii) By definition of $\mathcal{E}$ given in (1.17) and (2.4), we get for all $s \geq \max\{s_0, -\log T\}$,

$$2\mathcal{E}[w](s) \leq \int_{\mathbb{R}^n} \left(|\nabla w|^2 + \frac{1}{p-1}|w|^2\right)\rho dy \\ - 2\left(\frac{1}{p-1} - C_0 s^{-a}\right)\int_{\mathbb{R}^n} |w|^{p+1}\rho dy + 2C_0 s^{-a}.$$

Let s_1 large enough such that for all $s \geq s_1$, $C_0 s^{-a} \leq \frac{1}{2(p-1)}$, then for all $s \geq \max\{s_0, s_1, -\log T\}$,

$$2\mathcal{E}[w](s) \leq \int_{\mathbb{R}^n} \left(|\nabla w|^2 + \frac{1}{p-1}|w|^2\right)\rho dy - \frac{1}{p-1}\int_{\mathbb{R}^n} |w|^{p+1}\rho dy + \frac{2}{p-1}.$$

This follows that for all $s \geq \max\{s_0, s_1, -\log T\}$,

$$\|w(s)\|^{p+1}_{L^{p+1}_\rho(\mathbb{R}^n)} \leq -2(p-1)\mathcal{E}[w](s) + (p-1)\|w(s)\|^2_{H^1_\rho(\mathbb{R}^n)} + 1.$$

Since $\mathcal{E}$ is bounded from below, then (iii) follows.
(iv) From the definition $\mathcal{E}$ in (1.17), (2.3) and (2.4), we have $\forall s \geq \max\{s_0, -\log T\}$,

$$\|w(s)\|^2_{H^1_\rho(\mathbb{R}^n)} \leq \frac{1}{p-1}\frac{d}{ds}\int_{\mathbb{R}^n} |w|^2\rho dy + \frac{2(p+1)}{p-1}\mathcal{E}[w](s) \\ + \frac{4C_0(p+1)}{p-1}s^{-a}\|w(s)\|^{p+1}_{L^{p+1}_\rho(\mathbb{R}^n)} + \frac{4C_0(p+1)}{p-1}s^{-a}.$$

Using (iii), we have for all $s \geq \tilde{s}_3$,

$$\|w(s)\|^2_{H^1_\rho(\mathbb{R}^n)} \leq \frac{1}{p-1}\frac{d}{ds}\int_{\mathbb{R}^n} |w|^2\rho dy + \frac{2(p+1)}{p-1}\mathcal{E}[w](s) \\ + \frac{4C_0J_2(p+1)}{p-1}s^{-a}\left(1 + \|w(s)\|^2_{H^1_\rho(\mathbb{R}^n)}\right) + \frac{4C_0(p+1)}{p-1}s^{-a}.$$

Let s_2 large enough such that $\frac{4C_0J_2(p+1)}{p-1}s^{-a} \leq \frac{1}{2}$ for all $s \geq s_2$ and noting that $\mathcal{E}(s)$ is bounded from above, we obtain for all $s \geq \max\{s_2, \tilde{s}_2\}$,

$$\|w(s)\|^2_{H^1_\rho(\mathbb{R}^n)} \leq \frac{4}{p-1}\int_{\mathbb{R}^n} |ww_s|\rho dy + C_1,$$

where $C_1 = \frac{4J_0(p+1)}{p-1} + \frac{1}{J_2}$.
Using Schwarz's inequality and (ii) yields

$$\|w(s)\|^2_{H^1_\rho(\mathbb{R}^n)} \leq \frac{4}{p-1}\|w(s)\|_{L^2_\rho(\mathbb{R}^n)}\|w_s(s)\|_{L^2_\rho(\mathbb{R}^n)} + C_1 \leq \frac{4\sqrt{J_1}}{p-1}\|w_s(s)\|_{L^2_\rho(\mathbb{R}^n)} + C_1,$$

which follows (iv).
Since (v) and (vi) follows directly from (i) and (iii), (iv), we end the proof of Proposition 2.3. □

2.3 Boundedness of the solution in similarity variables

This section is devoted to the proof of Theorem 1.2, which is a direct consequence of the following theorem:

Theorem 2.4. *Let a, p, n, M be fixed, p satisfy* (1.2). *There exists $\hat{s}_1 = \hat{s}_1(a, p, n, M) \geq \hat{s}_0$ such that if u is a blow-up solution of equation* (1.1) *with a blow-up time T, then for all $s \geq s' = \max\{\hat{s}_1, -\log T\}$,*

$$\|w_{x_0,T}(s)\|_{L^\infty(\mathbf{B}_R)} \leq C, \tag{2.10}$$

where C is a positive constant depending only on n, p, M, R and a bound of $\|w_{x_0,T}(\hat{s}_0)\|_{L^\infty}$.

Let us show that Theorem 1.2 follows from Theorem 2.4.

Proof of Theorem 1.2 admitting Theorem 2.4. We have from (2.10) that

$$|w_{x_0,T}(0, s)| \leq C, \quad \forall s \geq s',$$

with C independent on $x_0 \in \mathbb{R}^n$. Therefore, we get from (1.8) that

$$|u(x_0, t)| \leq C(T-t)^{-\frac{1}{p-1}}, \quad \forall x_0 \in \mathbb{R}^n, \forall t \in [T - e^{-s'}, T),$$

which is the conclusion of Theorem 1.2, assuming that Theorem 2.4 holds. □

Following the method in [8], the proof of Theorem 2.4 requests the following key integral estimate:

Lemma 2.5 (**Key integral estimate**). *Let a, p, n, M be fixed and w be solution of equation* (1.9) *satisfying* (1.11). *For all $q \geq 2$ and $R > 0$, there exists $\hat{s}_2 \geq \tilde{s}_3$ and a positive constant K_q such that,*

$$\int_s^{s+1} \|w(\tau)\|^{q(p+1)}_{L^{p+1}(\mathbf{B}_R)} d\tau \leq K_q, \quad \forall s \geq \hat{s}_2, \tag{2.11}$$

where K_q depends only on $J_0, Q_0, a, n, p, q, R, \hat{s}_2$.

Let us first show that how Theorem 2.4 follows from Lemma 2.5, then we will prove it later. In order to derive uniform bound in Theorem 2.4 for all p satisfying (1.2), we need two following techniques. The first one is an interpolation result from Cazenave and Lions [1]:

Lemma 2.6 (**Interpolation technique, Cazenave and Lions [1]**). *Assume that*

$$v \in L^\alpha\left((0,\infty); L^\beta(\mathbf{B}_R)\right), \ v_t \in L^\gamma\left((0,\infty); L^\delta(\mathbf{B}_R)\right)$$

for some $1 < \alpha, \beta, \gamma, \delta < \infty$. Then

$$v \in \mathcal{C}\left([0,\infty); L^\lambda(\mathbf{B}_R)\right)$$

for all $\lambda < \lambda_0 = \frac{(\alpha+\gamma')\beta\delta}{\gamma'\beta+\alpha\delta}$ *with* $\gamma' = \frac{\gamma}{\gamma-1}$*, and satisfies*

$$\sup_{t\geq 0} \|v(t)\|_{L^\lambda(\mathbf{B}_R)} \leq C \int_0^\infty \left(\|v(\tau)\|^\alpha_{L^\beta(\mathbf{B}_R)} + \|v_\tau(\tau)\|^\gamma_{L^\delta(\mathbf{B}_R)} \right) d\tau$$

for $\lambda < \lambda_0$*. The positive constant* C *depends only on* $\alpha, \beta, \gamma, \delta, n$ *and* R*.*

The second one is an interior regularity result for a nonlinear parabolic equation:

Proposition 2.7 (Interior regularity). *Let*

$$v(x,t) \in L^\infty\big((0,+\infty), L^2(\mathbf{B}_R)\big) \cap L^2\big((0,+\infty), H^1(\mathbf{B}_R)\big)$$

which satisfies

$$v_t - \Delta v + b.\nabla v = F, \quad (x,t) \in Q_R = \mathbf{B}_R \times (0,+\infty), \tag{2.12}$$

where $R > 0$*,* $|b(x,t)| \leq \mu_1$ *in* Q_R *and* $|F(x,t,v)| \leq g(x,t)(|v|+1)$ *with*

$$\int_t^{t+1} \|g(\tau)\|^{\beta'}_{L^{\alpha'}(\mathbf{B}_R)} d\tau \leq \mu_2, \quad \forall t \in (0,+\infty), \tag{2.13}$$

and $\frac{1}{\beta'} + \frac{n}{2\alpha'} < 1$ *and* $\alpha' \geq 1$*. If*

$$\int_t^{t+1} \|v(\tau)\|^2_{L^2(\mathbf{B}_R)} d\tau \leq \mu_3, \quad \forall t \in (0,+\infty), \tag{2.14}$$

and μ_1*,* μ_2 *and* μ_3 *are uniformly bounded in* t*, then there exists a positive constant* C *depending only on* μ_1*,* μ_2*,* μ_3*,* α'*,* β'*,* n*,* R *and* $\tau \in (0,1)$ *such that*

$$|v(x,t)| \leq C, \quad \forall (x,t) \in \mathbf{B}_{R/4} \times (\tau, +\infty).$$

Proof. Since the argument of the proof is analogous as in the corresponding part in [13], we then leave the proof to Appendix B.1. □

Let us now use Lemma 2.5 to derive the conclusion of Theorem 2.4, then we will prove it later.

Proof of Theorem 2.4 admitting Lemma 2.5. Let us recall the equation in w:

$$w_s - \Delta w + \frac{1}{2} y.\nabla w = -\frac{w}{p-1} + |w|^{p-1}w + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}} w\right),$$

where h is given in (1.3).
We now apply Proposition 2.7 to w with $b = \frac{y}{2}$ and

$$F = -\frac{w}{p-1} + |w|^{p-1}w + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}} w\right).$$

From (1.13), we see that

$$|F| \leq C'(C_0, p)(|w|^{p-1} + 1)(|w| + 1), \quad \forall s \geq s_0.$$

Thus, the first identity in (2.13) holds with $g = C'(|w|^{p-1} + 1)$ and the second condition in (2.13) turns into

$$\int_s^{s+1} \left(\int_{\mathbf{B}_R} |w(y,\tau)|^{\alpha'(p-1)} dy \right)^{\frac{\beta'}{\alpha'}} d\tau \leq C_1 \quad \text{for some } C_1 > 0,$$

for some α' and β' satisfying $\frac{1}{\beta'} + \frac{n}{2\alpha'} < 1$.
For this bound, we first use (i) of Proposition 2.3, (2.11) and apply Lemma 2.6 with $\alpha = q(p+1)$, $\beta = p+1$, $\gamma = \delta = \gamma' = 2$ to get that

$$\sup_{s \geq \hat{s}_2} \|w(s)\|_{L^\lambda(\mathbf{B}_R)} \leq C_2(R, K_q), \quad \forall \lambda < \lambda_1 = p + 1 - \frac{p-1}{q+1}. \tag{2.15}$$

Next, applying Proposition 2.7 with $\alpha'(p-1) = \lambda$, β' and q large (note that the condition $\frac{1}{\beta'} + \frac{n}{2\alpha'} < 1$ turns into $p < \frac{n+2}{n-2}$), we obtain

$$\int_s^{s+1} \left(\int_{\mathbf{B}_R} |w(y,\tau)|^{\alpha'(p-1)} dy \right)^{\frac{\beta'}{\alpha'}} d\tau \leq C_2^{\beta'(p-1)}.$$

Hence, condition (2.13) holds. Therefore, $|w(y,s)|$ is bounded for all $(y,s) \in \mathbf{B}_{R/4} \times (\tau + \hat{s}_2, +\infty)$ for some $\tau \in (0,1)$, which concludes the proof of Theorem 2.4, assuming that Lemma 2.5 holds. □

Remark 2.8. *If we use (v) of Proposition 2.3, we already have for all $s \geq \tilde{s}_3$,*

$$\int_s^{s+1} \left(\int_{\mathbf{B}_R} |w(y,\tau)|^{p+1} dy \right)^2 d\tau \leq C(R) K_1.$$

Applying Proposition 2.7 with $\alpha' = \frac{p+1}{p-1}$ and $\frac{\beta'}{\alpha'} = 2$ (noting that the condition $\frac{1}{\beta'} + \frac{n}{2\alpha'} < 1$ turns into $p < \frac{n+3}{n-1}$), we obtain w is uniformly bounded with $p \in \left(1, \frac{n+3}{n-1}\right)$.
If we use (i) and (v) in Proposition 2.3, Lemma 2.6 with $\alpha = 2(p+1)$, $\beta = p+1$, $\gamma = \delta = \gamma' = 2$, then we obtain

$$\sup_{s \geq \tilde{s}_3} \|w(s)\|_{L^\lambda(\mathbf{B}_R)} \leq C(R), \quad \forall \lambda < \lambda_1 = \frac{2(p+2)}{3}$$

Next, Proposition 2.7 applies with $\alpha'(p-1) = \lambda$ with λ approaches to $\frac{2(p+1)}{3}$ and β' very large, then the condition $\frac{1}{\beta'} + \frac{n}{2\alpha'} < 1$ now becomes

$$\exists \lambda < \frac{2(p+1)}{3}, \quad \textit{such that} \quad \frac{n}{2\alpha'} < 1.$$

This turns into $p < \frac{3n+8}{3n-4}$. This result was proved by Giga and Kohn in [6].
Relying on a bootstrap argument, [8] improved the input estimate of Proposition 2.7 covering this way the whole subcritical range $p < \frac{n+2}{n-2}$. Here, we extend their approach to a larger class of equation.

Let us now give the proof of Lemma 2.5 in order to complete the proof of Theorem 2.4 and Theorem 1.2 also. To this end, let $\psi \in \mathcal{C}^2(\mathbb{R}^n)$ be a bounded function, we introduce the following local functional, which is a perturbed version of the function of [8],

$$\begin{aligned} \mathcal{E}_\psi[w](s) = &\frac{1}{2}\int_{\mathbb{R}^n} \psi^2 \left(|\nabla w|^2 + \frac{1}{p-1}|w|^2\right)\rho dy \\ &- \frac{1}{p+1}\int_{\mathbb{R}^n} \psi^2 |w|^{p+1}\rho dy - e^{-\frac{p+1}{p-1}s}\int_{\mathbb{R}^n} \psi^2 H\left(e^{\frac{s}{p-1}}w\right)\rho dy. \end{aligned} \tag{2.16}$$

We get the following bound on the local functional $\mathcal{E}_\psi$:

Proposition 2.9. *Let a, p, n, M be fixed and w be solution of equation* (1.9) *satisfying* (1.11)*. For $\psi \in \mathcal{C}^2(\mathbb{R}^n)$ bounded, there exist positive constants Q', K' such that*

$$-Q' \le \mathcal{E}_\psi[w](s) \le K', \quad \forall s \ge \tilde{s}_3, \tag{2.17}$$

where $\tilde{s}_3$ is given in Proposition 2.3 and Q', K' depend on a, p, n, M, $\|\psi\|^2_{L^\infty}$, $\|\nabla\psi\|^2_{L^\infty}$ and J_0.

Proof. The proof is essentially the same as the corresponding part in [8], except for the control of the last term in (2.16). Since that control is a bit long and technical, we leave the proof to B.2. □

Let $R > 0$, we fix $\psi(y)$ so that it satisfies

$$\psi(y) \in \mathcal{C}_0^\infty(\mathbb{R}^n), \quad 0 \le \psi(y) \le 1, \quad \psi(y) = \begin{cases} 1 & \text{on} \quad \mathbf{B}_R \\ 0 & \text{on} \quad \mathbb{R}^n \setminus \mathbf{B}_{2R} \end{cases}. \tag{2.18}$$

We claim the following:

Lemma 2.10. *Let a, p, n, M be fixed and w be solution of equation* (1.9) *satisfying* (1.11)*. Then there exists $\tilde{s}_5 \ge \tilde{s}_3$ such that*

$$\|w(s)\|^{p+1}_{L^{p+1}_\rho(\mathbf{B}_R)} \le K_1\left(1 + \|w(s)\|^2_{H^1_\rho(\mathbf{B}_{2R})}\right), \quad \forall s \ge \tilde{s}_5, \tag{2.19}$$

where $K_1 = K_1(a, p, n, M, Q')$.

Proof. From (2.4) and the definition of $\mathcal{E}_\psi$ in (2.16), we have $\forall s \geq \max\{s_0, s_1\}$,

$$\int_{\mathbb{R}^n} \psi^2 |w|^{p+1} \rho dy \leq -2(p+1)\mathcal{E}_\psi[w](s) + (p+1)\int_{\mathbb{R}^n} \psi^2 \left(|\nabla w|^2 + \frac{1}{p-1}|w|^2 \right) \rho dy + 1, \quad (2.20)$$

where s_1 is large enough such that $2C_0 s^{-a} \leq \frac{1}{2(p+1)}$ for all $s \geq s_1$.
Thus, (2.19) follows from the lower bound of $\mathcal{E}_\psi$ and the property of ψ. This ends the proof of Lemma 2.10. ☐

Remark 2.11. *By* (2.19)*, the proof of estimate* (2.11) *is equivalent to showing that*

$$\int_s^{s+1} \|w(\tau)\|^{2q}_{H^1_\rho(\mathbf{B}_R)} d\tau \leq K_q, \quad \forall s \geq \hat{s}_2. \quad (2.21)$$

Note from (i) *and* (iv) *in Proposition 2.3 that* (2.21) *already holds in the case* $q = 2$*.*

In order to derive (2.21) for all $q \geq 2$, we need the following result:

Lemma 2.12. *Let* a, p, n, M *be fixed and* w *be solution of equation* (1.9) *satisfying* (1.11)*. Then there exists* $\tilde{s}_6 \geq \tilde{s}_3$ *such that*

$$\|w(s)\|^2_{H^1_\rho(\mathbf{B}_R)} \leq K_2 \left(1 + \|\psi^2 w(s) w_s(s)\|^2_{L^1_\rho(\mathbf{B}_{2R})} \right), \quad \forall s \geq \tilde{s}_6, \quad (2.22)$$

where $K_2 = K_2(a, p, n, M, Q', K')$*.*

Proof. Multiplying equation (1.9) with $\psi^2 w \rho$, integrating over $\mathbb{R}^n$, using the definition of $\mathcal{E}_\psi$ and estimate (2.4), we have

$$\begin{aligned}\int_{\mathbb{R}^n} \psi^2 \left(|\nabla w|^2 + \frac{1}{p-1}|w|^2 \right) \rho dy \leq \frac{2}{p-1} \int_{\mathbb{R}^n} \psi^2 w w_s \rho dy + \frac{2(p+1)}{p-1} \mathcal{E}_\psi[w](s) \\ + \frac{4}{p-1} \int_{\mathbb{R}^n} \psi w \nabla\psi . \nabla w \rho dy \\ + \frac{4(p+1)C_0}{(p-1)s^a} \int_{\mathbb{R}^n} \psi^2 (|w|^{p+1} + 1) \rho dy, \quad \forall s \geq s_0.\end{aligned}$$

Using (2.20), then taking s_2 large such that $\frac{4(p+1)^2 C_0}{(p-1)s^a} \leq \frac{1}{2}$ and noting that $\mathcal{E}$ is bounded, we have for all $s \geq \max\{s_0, s_1, s_2\}$,

$$\int_{\mathbb{R}^n} \psi^2 \left(|\nabla w|^2 + \frac{1}{p-1}|w|^2 \right) \rho dy \leq C \left(\int_{\mathbb{R}^n} \psi^2 w w_s \rho dy + \int_{\mathbb{R}^n} \psi w \nabla\psi . \nabla w \rho dy + 1 \right).$$

Let $J_\psi[w](s) = \int_{\mathbb{R}^n} \psi w \nabla\psi . \nabla w \rho dy$, then one can show that $J_\psi[w](s) \leq C_1$ (see (B.8) for a proof of this fact). Hence, we have for all $s \geq \max\{s_0, s_1, s_2\}$,

$$\int_{\mathbb{R}^n} \psi^2 \left(|\nabla w|^2 + \frac{1}{p-1}|w|^2 \right) \rho dy \leq C_2 \left(\int_{\mathbb{R}^n} \psi^2 w w_s \rho dy + 1 \right).$$

Thus, (2.22) follows from the property of ψ, and Lemma 2.12 is proved. ☐

Since the estimate (2.21) already holds in the case $q = 2$, we now use a bootstrap argument in order to get (2.21) for all $q \geq 2$.

Proof of (2.21) ***for all*** $q \geq 2$ ***by a bootstrap argument***. This part is the same as in [8]. We give it here for the sake of completeness. Suppose that (2.21) holds for some $q \geq 2$, let us show that (2.21) holds for all $\tilde{q} \in [q, q+\epsilon]$ for some $\epsilon > 0$ independent from q. We start with Hölder's inequality,

$$\|\psi^2 w w_s\|_{L^1_\rho(\mathbf{B}_{2R})} \leq \|\psi w\|_{L^\lambda_\rho(\mathbf{B}_{2R})} \times \|\psi w_s\|_{L^{\lambda'}_\rho(\mathbf{B}_{2R})}, \quad \frac{1}{\lambda} + \frac{1}{\lambda'} = 1.$$

Using (2.11) and applying Lemma 2.6, we obtain

$$\|w\|_{L^\lambda(\mathbf{B}_{2R})} \leq C'_q, \quad \forall \lambda < \lambda_1(q) = p + 1 - \frac{p-1}{q+1}.$$

Let us now bound $\|\psi w_s\|_{L^{\lambda'}_\rho(\mathbf{B}_{2R})}$. We remark that for q large then λ approaches to $p+1$ and λ' approaches to $p_1 = \frac{p+1}{p}$. Let $f = \psi w_s$ and make use Hölder's inequality,

$$\|f\|_{L^{\lambda'}} \leq \|f\|^{1-\theta}_{L^2} \times \|f\|^\theta_{L^{p_1}}, \quad \frac{1}{\lambda'} = \frac{1-\theta}{2} + \frac{\theta}{p_1}, \quad \theta \in [0,1].$$

From now on, we take $\lambda \geq 2$ and fix $\theta = \frac{(\lambda-2)(p+1)}{\lambda(p-1)}$ (note that with this choice, $\theta \in [0,1]$). From Lemma 2.12, we have

$$\|w(s)\|^2_{H^1_\rho(\mathbf{B}_R)} \leq K'_2 \left(1 + \|\psi w_s\|^{1-\theta}_{L^2_\rho(\mathbf{B}_{2R})} \times \|\psi w_s\|^\theta_{L^{p_1}_\rho(\mathbf{B}_{2R})}\right).$$

This follows that

$$\int_s^{s+1} \|w(s)\|^{2\tilde{q}}_{H^1_\rho(\mathbf{B}_R)} d\tau \leq C_{\tilde{q}} \left[1 + \underbrace{\int_s^{s+1} \|\psi w_s\|^{\tilde{q}(1-\theta)}_{L^2_\rho(\mathbf{B}_{2R})} \times \|\psi w_s\|^{\tilde{q}\theta}_{L^{p_1}_\rho(\mathbf{B}_{2R})} d\tau}_{\mathbf{G}}\right], \tag{2.23}$$

for some $\tilde{q} > q$.
Let $\alpha = \frac{2}{(1-\theta)\tilde{q}}$ and use Hölder's inequality in time to $\mathbf{G}$, we obtain

$$\begin{aligned}\mathbf{G} &\leq \left(\int_s^{s+1} \|\psi w_s\|^2_{L^2_\rho(\mathbf{B}_{2R})} d\tau\right)^{\frac{1}{\alpha}} \left(\int_s^{s+1} \|\psi w_s\|^{\tilde{q}\theta\alpha'}_{L^{p_1}_\rho(\mathbf{B}_{2R})} d\tau\right)^{\frac{1}{\alpha'}} \\ &\leq (2J_0)^{\frac{1}{\alpha}} \left(\int_s^{s+1} \|\psi w_s\|^{\tilde{q}\theta\alpha'}_{L^{p_1}_\rho(\mathbf{B}_{2R})} d\tau\right)^{\frac{1}{\alpha'}} \equiv \mathbf{G}_1,\end{aligned}$$

where we used (i) in Proposition 2.3.
Let us bound $\mathbf{G}_1$. To this end, we use the $L^p - L^q$ estimate for the heat equation (see

Lemmas 6.3 and 6.4 in [8]) to get

$$\begin{aligned}\int_s^{s+1} \|\psi w_s\|^{\tilde{q}\theta\alpha'}_{L^{p_1}_\rho(\mathbf{B}_{2R})} d\tau &\leq C'_{\tilde{q}} \left(1 + \int_s^{s+1} \||w|^p\|^{\tilde{q}\theta\alpha'}_{L^{p_1}_\rho(\mathbf{B}_{2R})} d\tau\right)\\ &= C'_{\tilde{q}} \left(1 + \int_s^{s+1} \|w\|^{p\tilde{q}\theta\alpha'}_{L^{p+1}_\rho(\mathbf{B}_{2R})} d\tau\right)\\ &\leq C''_{\tilde{q}} \left(1 + \int_s^{s+1} \|w\|^{\frac{2p\tilde{q}\theta\alpha'}{p+1}}_{H^1_\rho(\mathbf{B}_{4R})} d\tau\right) \quad \text{(using Lemma 2.20)}.\end{aligned}$$

By Proposition 6.2 in [8], we have $\frac{2p\tilde{q}\theta\alpha'}{p+1} < 2q$ for all $\tilde{q} \in [q, q + \frac{2}{p+1}]$. Then, applying Hölder's inequality again yields

$$\int_s^{s+1} \|w(s)\|^{2\tilde{q}}_{H^1_\rho(\mathbf{B}_R)} d\tau \leq C'''_{\tilde{q}} \left[1 + \left(\int_s^{s+1} \|w(s)\|^{2q}_{H^1_\rho(\mathbf{B}_{4R})} d\tau\right)^{\frac{1}{2q\alpha'}}\right] \leq \bar{C}_{\tilde{q}}.$$

Thus, inequality (2.21) is valid for all $\tilde{q} \in [q, q + \frac{2}{p+1}]$. Repeating this argument, we would obtain that (2.21) holds for all $q \geq 2$. This concludes the proof of Lemma 2.5, Theorem 2.4 and Theorem 1.2 too. □

2.4 Limit of w as $s \to +\infty$

This section is devoted to the proof of Theorem 1.4. Note in the unperturbed case ($h \equiv 0$) that Theorem 1.4 was proved in [7] (see also [5], [6]). The proof is divided into two steps. The first step is to show that the limit of solution in similarity variables exists and belongs to the set of solutions of the following equation,

$$0 = \Delta w - \frac{1}{2} y.\nabla w - \frac{1}{p-1} w + |w|^{p-1} w, \tag{2.24}$$

Then, by using a nondegeneracy result (Lemma 2.16), the blow-up criterion (Lemma 2.2) and suitable energy arguments, we shall show that the possibility of $w_a \to 0$ as $s \to +\infty$ is excluded if a is a blow-up point. Let us restate Theorem 1.4 in below:

Proposition 2.13 (Limit of w as $s \to +\infty$). *Let a, p, n, M be fixed, p be a sub-critical non-linearity given in* (1.2)*. Consider $u(t)$ a solution of equation* (1.1) *which blows up at time T and a a blow-up point. Then*

$$\lim_{s \to +\infty} w_a(y, s) = \pm\kappa, \quad \textit{uniformly on each compact subset of } \mathbb{R}^n.$$

Before going into the proof of Proposition 2.13, let us first derive some elementary results. The first one concerns the stationary solutions in $\mathbb{R}^n$ of equation (2.24). Particularly, we have the following:

Lemma 2.14 (Stationary solutions, Giga and Kohn [5]). *Let p satisfy* (1.2)*, then all bounded solutions of* (2.24) *are constants: $w \equiv 0$ or $w \equiv \pm\kappa$.*

Proof. The proof is given in Proposition 2 of [5]. For the reader's interest, we mention that the proof relies on a clever use of multiplying factors, together with a Pohozaev technique, resulting in the following identity:

$$\left(\frac{n}{p+1}-\frac{2-n}{2}\right)\int_{\mathbb{R}^n}|\nabla w|^2\rho dy+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n}|y|^2|\nabla w|^2\rho dy=0. \tag{2.25}$$

From (2.25) and the fact that p is Sopolev subcritical, it follows that $\frac{n}{p+1}-\frac{2-n}{2}>0$ and $\frac{1}{2}-\frac{1}{p+1}>0$, hence $\nabla w\equiv 0$. This implies that w is actually a constant. This concludes the proof of Lemma 2.14. □

The second one is due to parabolic estimates:

Lemma 2.15 (Parabolic estimates). *Let u be a solution to equation* (1.1). *Assume that $T=T_{\max}(u_0)<+\infty$ and that u satisfies* (1.20). *Then, there is a positive constant C such that for all $t\in[T/2,T)$,*

$$\|\nabla u(t)\|_{L^\infty(\mathbb{R}^n)}\le C(T-t)^{-\frac{1}{p-1}-\frac{1}{2}}\quad\textit{and}\quad\|\nabla^2 u(t)\|_{L^\infty(\mathbb{R}^n)}\le C(T-t)^{-\frac{1}{p-1}-1}. \tag{2.26}$$

In similarity variables, we have for all $s\in[-\log(T/2),+\infty)$ and $x_0\in\mathbb{R}^n$,

$$\|\nabla w_{x_0,T}(s)\|_{L^\infty(\mathbb{R}^n)}\le C\quad\textit{and}\quad\|\nabla^2 w_{x_0,T}(s)\|_{L^\infty(\mathbb{R}^n)}\le C. \tag{2.27}$$

Proof. Since $|h(z)|\le C(|z|^p+1)$ and $|h'(z)|\le C(|z|^{p-1}+1)$ from (1.3), the proof given in Proposition 23.15, page 189 of Souplet and Quittner [17] in the case $h\equiv 0$ extends with no difficulty in this case. □

The last one is the nondegeneracy result from Giga and Kohn [7]:

Lemma 2.16 (Nondegeneracy, Giga and Kohn [7]). *Let $p>1$, $T>0$, $r>0$, $\sigma\in(0,1)$, $a\in\mathbb{R}^n$ and denote $Q_{r,\sigma}(a)=\mathbf{B}_r(a)\times(T-\sigma,T)$. There exists $\epsilon=\epsilon(n,p)>0$ such that if u is a classical solution of*

$$u_t-\Delta u=F(u),\quad (x,t)\in Q_{r,\sigma}(a), \tag{2.28}$$

where $|F(u)|\le M(|u|^p+1)$ for some $M>0$. Assume that u satisfies

$$|u(x,t)|\le\epsilon(T-t)^{-\frac{1}{p-1}},\quad (x,t)\in Q_{r,\sigma}(a), \tag{2.29}$$

then u is uniformly bounded in a neighborhood of (a,T).

Proof. See Theorem 2.1, page 850 in Giga and Kohn [7]. □

Let us now give the proof of Proposition 2.13.

Proof of Proposition 2.13. Consider a a blow-up point and write w instead of w_a for simplicity. By Lemma 2.15 and equation (1.9), we see that $|w_s(y,s)| \le C(|y|+1)$ for some $C > 0$. Therefore, w, ∇w, $\nabla^2 w$ and w_s are bounded for all $|y| \le R$ and $s \ge s'$ for some $R > 0$ and $s' \in \mathbb{R}$. Let $\{s_j\}$ be a sequence tending to $+\infty$ and $w_j(y,s) = w(y, s+s_j)$. By the Arzela-Ascoli theorem, there is a subsequence of s_j (still denoted s_j) such that w_j converges uniformly on compact sets to some w^∞, $\nabla w_j \to \nabla w^\infty$, $\Delta w_j \to \Delta w^\infty$ and $w_{js} \to w^\infty_s$. On the other hand, by (i) and (vi) of Proposition 2.3, we see that as $j \to +\infty$,

$$\int_{\tilde{s}_3}^{+\infty}\int_{\mathbf{B}_R} |w_{js}|^2 dyds = \int_{\tilde{s}_3+s_j}^{+\infty}\int_{\mathbf{B}_R} |w_s|^2 dyds \to 0.$$

This implies that $w^\infty_s = 0$ and w^∞ satisfies (2.24). Hence, by Lemma 2.14, $w^\infty \equiv 0$ or $w^\infty \equiv \pm\kappa$.

It remains to show that $w(\cdot, s_j) \nrightarrow 0$ as $j \to +\infty$. We proceed by contradiction. Let us assume that $w(\cdot, s_j) \to 0$ as $j \to +\infty$. We observer that if $w(\cdot, s_j) \to 0$, then by the definition of $\mathcal{J}$ given in (1.16), the bound of w and ∇w and dominated convergence, then $\mathcal{J}[w](s_j) \to 0$. Since $\mathcal{J}$ is a Lyapunov functional, it follows that the whole sequence

$$\mathcal{J}[w](s) \to 0 \quad \text{as} \quad s \to +\infty. \tag{2.30}$$

Let $b \in \mathbb{R}^n$, then by (2.27), we have $w_b(y,s)$ and $\nabla w_b(y,s)$ are bounded for all $y \in \mathbb{R}^n$ and $s \ge s'$. We now use the interpolation inequality which reads

$$|w_b(0,s)| \le C\left(\|w_b(s)\|^\theta_{L^2(\mathbf{B}_R)}\|\nabla w_b(s)\|^{1-\theta}_{L^\infty(\mathbf{B}_R)} + \|w_b(s)\|_{L^2(\mathbf{B}_R)}\right),$$

where $\theta \in (0, \frac{2}{n+2})$ if $n \ge 2$ and $\theta = 1/2$ if $n = 1$.

By Lemma 2.2, we see that $\|w_b(s)\|_{L^2(\mathbf{B}_R)} \le C(p)\big(\mathcal{J}[w_b](s)\big)^{\frac{1}{p+1}}$ for all $s \ge \tilde{s}_1$. Hence,

$$|w_b(0,s)| \le C'\left(\big(\mathcal{J}[w_b](\tilde{s}_1)\big)^{\frac{\theta}{p+1}} + \big(\mathcal{J}[w_b](\tilde{s}_1)\big)^{\frac{1}{p+1}}\right), \quad \forall s \ge \tilde{s}_1.$$

Consider some $\epsilon > 0$ small. From (2.30), there is $s'(\epsilon)$ such that $\mathcal{J}[w](s) \le \epsilon$ for all $s \ge s'(\epsilon)$. Therefore, by continuity depending of $\mathcal{J}[w_b](s)$ on b and the monotonicity of $\mathcal{J}[w_b](s)$ in time s, we infer that $\mathcal{J}[w_b](s) \le 2\epsilon$ for all $s \ge s'$ and $|b-a|$ small. This implies that $|w_b(0,s)| \le \epsilon''$ for all $s \ge s'$, or $|u(b,t)| \le \epsilon''(T-t)^{-\frac{1}{p-1}}$ for (b,t) close to (a,T), where $\epsilon'' = \epsilon''(\epsilon) \to 0$ as $\epsilon \to 0$. Thus, a is not a blow-up point by Lemma 2.16, and this is a contradiction. Therefore, this concludes the proof of Proposition 2.13 and the proof of Theorem 1.4 also. ☐

3 Classification of the behavior of w as $s \to +\infty$ in L^2_ρ

This section is devoted to the proof of Theorem 1.5. Consider a a blow-up point and write w instead of w_a for simplicity. From Theorem 1.4 and up to changing the signs of

w and h, we may assume that $\|w(y,s)-\kappa\|_{L^2_\rho} \to 0$ as $s \to +\infty$, uniformly on compact subsets of $\mathbb{R}^n$. As mentioned in the introduction, by setting $v(y,s) = w(y,s) - \phi(s)$ (ϕ is a positive solution of (1.21) such that $\phi(s) \to \kappa$ as $s \to +\infty$), we see that $\|v(s)\|_{L^2_\rho} \to 0$ as $s \to +\infty$ and v solves the following equation:

$$\partial_s v = (\mathcal{L} + \omega(s))v + F(v) + H(v,s), \quad \forall y \in \mathbb{R}^n,\ \forall s \in [-\log T, +\infty), \tag{3.1}$$

where $\mathcal{L} = \Delta - \frac{y}{2}\cdot\nabla + 1$ and ω, F, H are given by

$$\begin{aligned}
\omega(s) &= p\left(\phi^{p-1} - \kappa^{p-1}\right) + e^{-s}h'\left(e^{\frac{s}{p-1}}\phi\right),\\
F(v) &= |v+\phi|^{p-1}(v+\phi) - \phi^p - p\phi^{p-1}v,\\
H(v,s) &= e^{-\frac{ps}{p-1}}\left[h\left(e^{\frac{s}{p-1}}(v+\phi)\right) - h\left(e^{\frac{s}{p-1}}\phi\right) - e^{\frac{s}{p-1}}h'\left(e^{\frac{s}{p-1}}\phi\right)v\right].
\end{aligned}$$

We remark from (1.22) and (1.13) that

$$|\omega(s)| = \mathcal{O}\left(\frac{1}{s^a}\right) \quad \text{as} \quad s \to +\infty. \tag{3.2}$$

Let us introduce for all $y \in \mathbb{R}^n$, for all $s \in [-\log T, +\infty)$,

$$\beta(s) = e^{-\int_s^{+\infty}\omega(\tau)d\tau} \quad \text{and} \quad V(y,s) = \beta(s)v(y,s), \tag{3.3}$$

(note that $\beta(s) \to 1$ as $s \to +\infty$).
By multiplying equation (3.1) to $\beta(s)$, we find the following equation satisfied by V:

$$\partial_s V = \mathcal{L}V + \bar{F}(V,s), \quad \forall y \in \mathbb{R}^n,\ \forall s \in [-\log T, +\infty), \tag{3.4}$$

where $\bar{F}(V,s) = \beta(s)(F(v) + H(v,s))$ satisfying

$$|\bar{F}(V,s)| \le CV^2. \tag{3.5}$$

Since $\|w(s)\|_{L^\infty} \le C$ from Theorem 1.2, we may use a Taylor expansion, (1.13), (1.22) and the fact that $\beta(s) = 1 + \mathcal{O}\left(\frac{1}{s^{a-1}}\right)$ as $s \to +\infty$ to write

$$\left|\bar{F}(V,s) - \frac{p}{2\kappa}V^2\right| = \mathcal{O}(|V|^3) + \mathcal{O}\left(\frac{V^2}{s^{a-1}}\right) \quad \text{as } s \to +\infty, \tag{3.6}$$

(see Lemma C.1 for the proof of (3.6), and note that (3.5) follows from (3.6)).
Since the eigenfunctions of $\mathcal{L}$ constitute a total orthonormal family of L^2_ρ, we can expand V as follows:

$$V(y,s) = \sum_{k=1}^{\infty}\pi_k(V)(y,s) = V_+(y,s) + V_{null}(y,s) + V_-(y,s), \tag{3.7}$$

where $\pi_k(V)$ is the orthogonal projector of v on the eigenspace associated to $\lambda_k = 1 - \frac{k}{2}$,

$$\begin{aligned}
V_+(y,s) &= \pi_+(V)(y,s) = \sum_{k=0}^{1} \pi_k(V)(y,s),\\
V_-(y,s) &= \pi_-(V)(y,s) = \sum_{k=3}^{\infty} \pi_k(V)(y,s),\\
V_{null}(y,s) &= \pi_2(V)(y,s) = V_2(s) \cdot H_2(y),
\end{aligned} \tag{3.8}$$

where $H_2(y) = (H_{2,ij}, i \leq j)$, with $H_{2,ii} = h_2(y_i)$ and $H_{2,ij} = h_1(y_i)h_1(y_j)$ if $i \neq j$, h_m is introduced in (1.24); $V_2(s) = (V_{2,ij}, i \leq j)$, with $V_{2,ij}$ being the projection of V on $H_{2,ij}$.

We claim that Theorem 1.5 is a direct consequence of the following:

Proposition 3.1 (Classification of the behavior of V as $s \to +\infty$). *One of the following possibilities occurs:*
$i)$ $V(y,s) \equiv 0$,
$ii)$ There exists $l \in \{1,\dots,n\}$ such that up to an orthogonal transformation of coordinates, we have

$$V(y,s) = -\frac{\kappa}{4ps}\left(\sum_{j=1}^{l} y_j^2 - 2l\right) + \mathcal{O}\left(\frac{1}{s^a}\right) + \mathcal{O}\left(\frac{\log s}{s^2}\right) \quad as \quad s \to +\infty.$$

$iii)$ There exist an integer number $m \geq 3$ and constants c_α not all zero such that

$$V(y,s) = -e^{\left(1-\frac{m}{2}\right)s} \sum_{|\alpha|=m} c_\alpha H_\alpha(y) + o\left(e^{\left(1-\frac{m}{2}\right)s}\right) \quad as \quad s \to +\infty.$$

The convergence takes place in L^2_ρ as well as in $C^{k,\gamma}_{loc}$ for any $k \geq 1$ and $\gamma \in (0,1)$.

Remark 3.2. *Let us insist on the fact that the linearizing of w around κ would generate some terms of the size $\frac{1}{s^a}$, and prevent us from reaching exponentially small terms.*

Let us first derive Theorem 1.5 assuming Proposition 3.1 and then we will prove it later.

***Proof of Theorem 1.5 assuming that Proposition 3.1 holds*.** By the definition (3.3) of V, we see that $i)$ of Proposition 3.1 directly follows that $v(y,s) \equiv \phi(s)$ which is $i)$ of Theorem 1.5. Using $ii)$ of Proposition 3.1 and the fact that $\beta(s) = 1 + \mathcal{O}(\frac{1}{s^{a-1}})$ as

$s \to +\infty$, we see that as $s \to +\infty$,

$$\begin{aligned} w(y,s) &= \phi(s) + V(y,s)\left(1 + \mathcal{O}(\frac{1}{s^{a-1}})\right) \\ &= \phi(s) - \frac{\kappa}{4ps}\left(\sum_{j=1}^{l} y_j^2 - 2l\right) + \mathcal{O}\left(\frac{1}{s^a}\right) + \mathcal{O}\left(\frac{\log s}{s^2}\right) \\ &= \kappa - \frac{\kappa}{4ps}\left(\sum_{j=1}^{l} y_j^2 - 2l\right) + \mathcal{O}\left(\frac{1}{s^a}\right) + \mathcal{O}\left(\frac{\log s}{s^2}\right), \end{aligned}$$

which yields $ii)$ of Theorem 1.5.
Using $iii)$ of Proposition 3.1 and again the fact that $\beta(s) = 1 + \mathcal{O}(\frac{1}{s^{a-1}})$ as $s \to +\infty$, we have

$$w(y,s) = \phi(s) - e^{\left(1-\frac{m}{2}\right)s} \sum_{|\alpha|=m} c_\alpha H_\alpha(y) + o\left(e^{\left(1-\frac{m}{2}\right)s}\right) \quad \text{as} \quad s \to +\infty.$$

This concludes the proof of Theorem 1.5 assuming that Proposition 3.1 holds. □

The proof of Proposition 3.1 will be very close to that in [3] and [20], thanks to (3.5) and (3.6). It happens that the proofs written in Filippas, Kohn, Liu, Herrero and Velázquez [2],[3], [11], [20] in the unperturbed case ($h \equiv 0$) hold for equation (3.4) under the general assumptions (3.5) and (3.6). For that reason, we only give the sketch of the proof below and refer to these papers for details of the proofs.
Following [3] and [20], we divide the proof into 3 steps which are given in separated subsections:
- Step 1: deriving the fact that either $\|V_+(s)\|_{L^2_\rho} + \|V_-(s)\|_{L^2_\rho} = o\left(\|V_{null}(s)\|_{L^2_\rho}\right)$, or $\|V(s)\|_{L^2_\rho} = \mathcal{O}(e^{-\mu s})$ for some $\mu > 0$.
- Step 2: assuming that $\|V(s)\|_{L^2_\rho} \sim \|V_{null}(s)\|_{L^2_\rho}$, we find an equation satisfied by $V_{null}(s)$ as $s \to +\infty$. Solving this equation, we find that $\|V(s)\|_{L^2_\rho}$ behaves like $\frac{1}{s}$ as $s \to +\infty$. Using this information, we can get a more accurate equation for $V_{null}(s)$ as $s \to +\infty$ and then $ii)$ of Proposition 3.1 follows.
- Step 3: assuming $\|V(s)\|_{L^2_\rho} = \mathcal{O}(e^{-\mu s})$ for some $\mu > 0$ as $s \to +\infty$, we derive $i)$ or $iii)$ of Proposition 3.1.

3.1 Finite dimension reduction of the problem.

We claim the following proposition:

Proposition 3.3 (**Competition between** V_+, V_- **and** V_{null}). *As* $s \to +\infty$,

$$\text{either} \quad i)\ \|V(s)\|_{L^2_\rho} = \mathcal{O}\left(e^{-\mu s}\right), \quad \text{for some } \mu > 0, \tag{3.9}$$

$$\text{or} \quad ii)\ \|V_+(s)\|_{L^2_\rho} + \|V_-(s)\|_{L^2_\rho} = o\left(\|V_{null}(s)\|_{L^2_\rho}\right). \tag{3.10}$$

Proof. Let us denote

$$Z(s) = \|V_+(s)\|_{L^2_\rho}, \quad X(s) = \|V_{null}(s)\|_{L^2_\rho}, \quad Y(s) = \|V_-(s)\|_{L^2_\rho}, \tag{3.11}$$

then the following lemma is claimed:

Lemma 3.4. *Let $\epsilon > 0$, there exists $s^* = s^*(\epsilon) \in \mathbb{R}$ such that for all $s \geq s^*$,*

$$\begin{aligned} Z' &\geq \left(\frac{1}{2} - \epsilon\right) Z - \epsilon(X + \bar{Y}) \\ |X'| &\leq \epsilon(X + \bar{Y} + Z) \\ \bar{Y}' &\leq -\left(\frac{1}{2} - \epsilon\right) \bar{Y} + \epsilon\,(X + Z) \end{aligned}$$

where $\bar{Y}(s) = Y(s) + r(s)$ with $r(s) = \left\| |y|^{\frac{k}{2}} V^2(s) \right\|_{L^2_\rho}$ for a fixed integer k.

Proof. From the fact that $|\bar{F}(V, s)| \leq CV^2$ for s large, the proof is the same as the proof of Theorem A, pages 842-847 in Filippas and Kohn [2]. □

The following lemma allows us to conclude Proposition 3.3:

Lemma 3.5. *Let $\xi(t), \nu(t), \zeta(t)$ be absolutely continuous, real-valued functions that are nonnegative and satisfy:*
$i)$ $(\xi(t), \nu(t), \zeta(t)) \to 0$ as $t \to +\infty$,
$ii)$ For all $\epsilon > 0$, there exists $t_0 \in \mathbb{R}$ such that for all $t \geq t_0$,

$$\begin{aligned} \zeta' &\geq c_0\zeta - \epsilon(\xi + \nu) \\ |\xi'| &\leq \epsilon(\xi + \nu + \zeta) \\ \nu' &\leq -c_0\nu + \epsilon(\xi + \zeta), \end{aligned}$$

for some $c_0 > 0$.
Then either $\xi + \zeta = o(\nu)$ or $\nu + \zeta = o(\xi)$ as $t \to +\infty$.

Remark 3.6. *In the first case, we clearly see that $\nu' \leq -\frac{c_0}{2}\nu$ for t large, hence ξ, υ, ζ tend to zero exponentially fast.*

Proof. The original proof is due to Filippas and Kohn [2]. For this particular statement, see Lemma A.1, page 3425 [15] for the proof. □

Since $\|V(s)\|_{L^\infty_{loc}} \to 0$ as $s \to +\infty$, we have $X(s), \bar{Y}(s), Z(s) \to 0$ as $s \to +\infty$. Thus, Lemma 3.5 applies to $X(s), \bar{Y}(s)$, and $Z(s)$ and yields the desired result (use the remark after the statement). This ends the proof of Proposition 3.3. □

3.2 Deriving conclusion $ii)$ of Proposition 3.1

In this part, we recall from Filippas and Liu the proof of $ii)$ of Proposition 3.1. We focus on the case $ii)$ of Proposition 3.3, namely that

$$\|V_+(s)\|_{L^2_\rho} + \|V_-(s)\|_{L^2_\rho} = o\left(\|V_{null}(s)\|_{L^2_\rho}\right) \quad \text{as} \quad s \to +\infty, \tag{3.12}$$

and show that it leads to case $ii)$ of Proposition 3.1.
We first claim the following proposition:

Proposition 3.7 (An ODE satisfied by $V_{null}(s)$ as $s \to +\infty$). *If $\|V_+(s)\|_{L^2_\rho}+\|V_-(s)\|_{L^2_\rho} = o\left(\|V_{null}(s)\|_{L^2_\rho}\right)$, then*
$i)$ for all $i, j \in \{1, ..., n\}$ and as $s \to +\infty$,

$$V'_{2,ij}(s) = \frac{p}{2\kappa}\int_{\mathbb{R}} V^2_{null}(y,s)\frac{H_{2,ij}(y)}{\|H_{2,ij}\|^2_{L^2_\rho}}\rho(y)dy + o\left(\|V_{null}(s)\|^2_{L^2_\rho}\right). \tag{3.13}$$

$ii)$ There exist a symmetric $n \times n$ matrix $A(s)$ such that for all $s \in \mathbb{R}$,

$$\begin{aligned} V_{null}(y,s) &= y^T A(s) y - 2tr(A(s)) \\ \text{and} \ \ c_1\|A(s)\| &\le \|V_{null}(s)\|_{L^2_\rho} \le c_2\|A(s)\| \end{aligned} \tag{3.14}$$

where c_1, c_2 are some positive constant and $\|A\|$ stands for any norm on the space of $n \times n$ symmetric matrices. Moreover,

$$A'(s) = \frac{4p}{\kappa}A^2(s) + o\left(\|A(s)\|^2\right) \quad \text{as} \quad s \to +\infty. \tag{3.15}$$

Proof. Let us remark that $ii)$ follows directly from $i)$. Here, one has to use (3.6) which is more accurate than (3.5), in order to isolate the $\mathcal{O}(V^2)$ term in the nonlinear term. Using properties of Hermites polynomials, we may project that term and obtain (3.13). ☐

In the next step, we show that although we can not derive directly from (3.13) the asymptotic behavior of $V_{null}(s)$, we can use it to show that $\|V(s)\|_{L^2_\rho}$ decays like $\frac{1}{s}$ as $s \to +\infty$. More precisely, we have the following proposition:

Proposition 3.8. *If $\|V_+(s)\|_{L^2_\rho} + \|V_-(s)\|_{L^2_\rho} = o\left(\|V_{null}(s)\|_{L^2_\rho}\right)$, then for s large, we have*

$$\frac{c_1}{s} \le \|V(s)\|_{L^2_\rho} \le \frac{c_2}{s}, \tag{3.16}$$

for some positive constants c_1 and c_2.

Proof. Since $\|V(s)\|_{L^2_\rho} \sim \|V_{null}(s)\|_{L^2_\rho}$ and because of (3.14), it is enough to show that

$$\frac{c_1}{s} \le \|A(s)\| \le \frac{c_2}{s}, \quad \text{for } s \text{ large}. \tag{3.17}$$

Since the proof of (3.17) is totally given in Section 3 of Filippas and Liu [3], we just give its steps of the proof below. The following Lemma asserts that $A(s)$ has continuously differential eigenvalues:

Lemma 3.9 ([18], [12]). *Suppose that $A(s)$ is a $n \times n$ symmetric and continuously differentiable matrix-function in some interval I, then there exists continuously differentiable functions $\lambda_1(s), \dots, \lambda_n(s)$ in I such that for all $i \in \{1, \dots, n\}$,*

$$A(s)\Phi^{(i)}(s) = \lambda_i(s)\Phi^{(i)}(s),$$

for some orthonormal system of vector-functions $\Phi^{(1)}(s), \dots, \Phi^{(n)}(s)$.

Let $\lambda_1(s), \dots, \lambda_n(s)$ be the eigenvalues of $A(s)$. We can derive from (3.15) an equation satisfied by $\lambda_i(s)$, $i \in \{1, \dots, n\}$:

Lemma 3.10 (Filippas and Liu [3]). *The eigenvalues of $A(s)$ satisfy for all $i \in \{1, \dots, n\}$,*

$$\lambda_i'(s) = \frac{4p}{\kappa}\lambda_i^2(s) + o\left(\sum_{i=1}^{n} \lambda_i^2(s)\right). \tag{3.18}$$

Using (3.18), one can show that (see the end of Section 3 in [3])

$$\frac{c_1}{s} \le \sum_{i=1}^{n} |\lambda_i(s)| \le \frac{c_2}{s}, \quad \text{for } s \text{ large.} \tag{3.19}$$

Since $\|A(s)\| = \sum_{i=1}^{n} |\lambda_i(s)|$, this concludes the proof of (3.17) and Proposition 3.8 also. ☐

Using the fact that $\|V(s)\|_{L^2_\rho}$ decays like $\frac{1}{s}$, we will show that $\|V_-(s))\|_{L^2_\rho} + \|V_+(s))\|_{L^2_\rho}$ is in fact $\mathcal{O}(\|V_{null}(s))\|^2_{L^2_\rho}$ and not only $o(\|V_{null}(s))\|_{L^2_\rho})$. This new estimate will be used then to derive a more accurate equation satisfied by V_{null}.

Proposition 3.11. *If $\|V_+(s)\|_{L^2_\rho} + \|V_-(s)\|_{L^2_\rho} = o\left(\|V_{null}(s)\|_{L^2_\rho}\right)$, then we have*

$$V_{2,ij}'(s) = \frac{p}{2\kappa}\int_{\mathbb{R}^n} V_{null}^2(y,s)\frac{H_{2,ij}(y)}{\|H_{2,ij}\|^2_{L^2_\rho}}\rho(y)dy + \mathcal{O}\left(\|V_{null}(s)\|^3_{L^2_\rho}\right) + \mathcal{O}\left(\frac{\|V_{null}(s)\|^2_{L^2_\rho}}{s^{a-1}}\right), \tag{3.20}$$

and

$$A'(s) = \frac{4p}{\kappa}A^2(s) + \mathcal{O}\left(\frac{1}{s^3}\right) + \mathcal{O}\left(\frac{1}{s^{a+1}}\right), \tag{3.21}$$

where $A(s)$ is given in (3.14).

Proof. The proof corresponds to Section 4 in [3]. Let us mention that the proof relies on the following priori estimate of solutions of (3.4) shown by Herrero and Velázquez in [11]. Although they proved their result in the case $N = 1$, their proof holds in higher dimensions under the general assumption (3.5).

Lemma 3.12 (Herrero and Valázquez [11]). *Assume that* V *solves* (3.4) *and* $|V| \leq M < +\infty$*. Then for any* $r > 1$, $q > 1$ *and* $L > 0$*, there exist* $s_0^* = s_0^*(q, r)$ *and* $C = C(q, r, L) > 0$ *such that*

$$\left(\int_{\mathbb{R}^n} |V(y, s+\tau)|^r \rho(y) dy \right)^{\frac{1}{r}} \leq C \left(\int_{\mathbb{R}^n} |V(y, s)|^q \rho(y) dy \right)^{\frac{1}{q}},$$

for any $s \geq 0$ *and any* $\tau \in [s_0^*, s_0^* + L]$.

From Proposition 3.8, we have $\|V(s)\|_{L^2_\rho}$ decays like $\frac{1}{s}$. Then Lemma 3.12 implies that

$$\left(\int_{\mathbb{R}^n} |V(y, s)|^r \rho(y) dy \right)^{\frac{1}{r}} \leq C \left(\int_{\mathbb{R}^n} |V(y, s)|^q \rho(y) dy \right)^{\frac{1}{q}}, \tag{3.22}$$

for any $r > 1$, $q > 1$ and for s large.
Using estimate (3.22), we derive the fact that

$$\|V_+(s)\|_{L^2_\rho} + \|V_-(s)\|_{L^2_\rho} = \mathcal{O}\left(\|V_{null}(s)\|^2_{L^2_\rho} \right). \tag{3.23}$$

Then, projecting (3.4) onto the null space of $\mathcal{L}$ and using (3.23), (3.22), we would obtain (3.20). Since $\|V(s)\|_{L^2_\rho} \sim \|V_{null}(s)\|_{L^2_\rho} \sim \frac{1}{s}$, we then obtain (3.21) from (3.20). This ends the proof of Proposition 3.11. □

Let us now use 3.11 to derive conclusion $ii)$ of Proposition 3.1. Using Lemma 3.9, we get from (3.21) that the eigenvalues of $A(s)$ satisfy

$$\forall i \in \{1, \dots, n\}, \quad \lambda_i'(s) = \frac{4p}{\kappa} \lambda_i^2(s) + \mathcal{O}\left(\frac{1}{s^{a+1}} \right) + \mathcal{O}\left(\frac{1}{s^3} \right), \quad \text{as } s \to +\infty,$$

then Lemma C.2 yields

$$\text{either } \lambda_i(s) = -\frac{\kappa}{4ps} + \mathcal{O}\left(\frac{1}{s^a} \right) \text{ or } \lambda_i(s) = \mathcal{O}\left(\frac{1}{s^a} \right), \text{ if } a \in (1, 2), \tag{3.24}$$

$$\text{either } \lambda_i(s) = -\frac{\kappa}{4ps} + \mathcal{O}\left(\frac{\log s}{s^2} \right) \text{ or } \lambda_i(s) = \mathcal{O}\left(\frac{1}{s^2} \right), \text{ if } a \geq 2. \tag{3.25}$$

Therefore, Proposition 5.1 in [3] yields the existence of $l \in \{1, \dots, n\}$ and a $n \times n$ orthonormal matrix Q such that

$$A(s) = -\frac{\kappa}{4ps} A_l + \mathcal{O}\left(\frac{1}{s^a} \right), \text{ if } a \in (1, 2),$$

$$A(s) = -\frac{\kappa}{4ps} A_l + \mathcal{O}\left(\frac{\log s}{s^2} \right), \text{ if } a \geq 2,$$

where

$$A_l = Q \begin{pmatrix} \mathbf{I}_l & O \\ O & O \end{pmatrix} Q^{-1}.$$

Combining this with (3.14), it yields the behavior of $V_{null}(y, s)$ and $V(y, s)$ announced in $ii)$ of Proposition 3.1. The convergence in $\mathcal{C}^{k,\gamma}_{loc}$ follows from standard parabolic regularity (see section 5 in [3] for a brief demonstration). This completes the proof of $ii)$ of Proposition 3.1.

3.3 Deriving conclusions $i)$ and $iii)$ of Proposition 3.1

In this part, we recall the proof given by Velázquez [20]. We focus on the case $i)$ of Proposition 3.3, namely $\|V(s)\|_{L^2_\rho} = \mathcal{O}(e^{-\mu s})$ for some $\mu > 0$, and we will show that it leads to either $i)$ or $iii)$ of Proposition 3.1. Let us start the first step. From equation (3.4), we write $V(y,s)$ in the integration form

$$V(y,s) = S_{\mathcal{L}}(s)V(s_0) + \int_{s_0}^{s} S_{\mathcal{L}}(s-\tau)\bar{F}(V(\tau),\tau)d\tau, \quad \text{with} \quad s_0 = -\log T,$$

where $S_{\mathcal{L}}(s)$ is the linear semigroup corresponding to the heat-type equation $\partial V = \mathcal{L}V$ given by

$$S_{\mathcal{L}}(s)V(y,\tau) = \sum_{|\alpha|=0}^{\infty} a_\alpha(\tau)e^{\left(1-\frac{|\alpha|}{2}\right)(s-\tau)}H_\alpha(y),$$

with

$$a_\alpha(\tau) = \langle V(\tau), H_\alpha\rangle := \int_{\mathbb{R}^n} V(y,\tau)H_\alpha(y)\rho(y)dy.$$

Let us fix a integer $k_0 > 2$ such that $\frac{k_0}{2} - 1 < 2\mu < \frac{k_0+1}{2} - 1$ and write $V(y,s)$ as follow:

$$\begin{aligned}
V(y,s) = &\sum_{|\alpha|\le k_0} a_\alpha(s_0)e^{\left(1-\frac{|\alpha|}{2}\right)(s-s_0)}H_\alpha(y) + \sum_{|\alpha|\ge k_0+1} a_\alpha(s_0)e^{\left(1-\frac{|\alpha|}{2}\right)(s-s_0)}H_\alpha(y)\\
&+ \sum_{|\alpha|\le k_0} H_\alpha(y)\int_{s_0}^{s} e^{\left(1-\frac{|\alpha|}{2}\right)(s-\tau)}\langle\bar{F}(V(y,\tau),\tau), H_\alpha(y)\rangle d\tau\\
&+ \sum_{|\alpha|\ge k_0+1} H_\alpha(y)\int_{s_0}^{s} e^{\left(1-\frac{|\alpha|}{2}\right)(s-\tau)}\langle\bar{F}(V(y,\tau),\tau), H_\alpha(y)\rangle d\tau\\
&:= I + II + III + IV.
\end{aligned}$$

Since $|\bar{F}(V,s)| \le C|V|^2$ and $\|V(s)\|_{L^2_\rho} \le Ce^{-\mu s}$, we derive from Lemma 3.12 that

$$\|\bar{F}(V(\tau))\|_{L^2_\rho} \le Ce^{-2\mu\tau}. \tag{3.26}$$

By a direct computation, we find that

$$\|II\|_{L^2_\rho} + \|IV\|_{L^2_\rho} \le Ce^{-2\mu s}, \quad \text{for some } C > 0.$$

For III, we write

$$\begin{aligned}
&\int_{s_0}^{s} e^{-\left(1-\frac{|\alpha|}{2}\right)\tau}\langle\bar{F}(V(y,\tau),\tau), H_\alpha(y)\rangle d\tau\\
&\qquad = \beta_\alpha - \int_{s}^{+\infty} e^{-\left(1-\frac{|\alpha|}{2}\right)\tau}\langle\bar{F}(V(y,\tau),\tau), H_\alpha(y)\rangle d\tau.
\end{aligned}$$

Using (3.26), we can bound the last term of the above expression by $Ce^{-2\mu s}$. Hence,

$$V(y,s) = \sum_{|\alpha|\leq k_0} (a_\alpha + \beta_\alpha) e^{\left(1-\frac{|\alpha|}{2}\right)s} H_\alpha(y) + Q(y,s),$$

where $\|Q(s)\|_{L^2_\rho} = \mathcal{O}(e^{-2\mu s})$.
Since $\|V(s)\|_{L^2_\rho} = \mathcal{O}(e^{-\mu s})$, it requires $a_\alpha + \beta_\alpha = 0$ for $|\alpha| \leq 2$. Thus, we have two possibilities: if there exists an integer $m \in [3, k_0]$ such that $a_\alpha + \beta_\alpha \neq 0$ for $|\alpha| = m$ and $a_\alpha + \beta_\alpha = 0$ for all $|\alpha| < m$, then we obtain $iii)$ of Proposition 3.1 for some $m \in [3, k_0]$. If this is not the case, we get $\|V(s)\|_{L^2_\rho} = \mathcal{O}(e^{-2\mu s})$. Using this new estimate and repeating the process in a finite number of steps, we may obtain either $iii)$ of Proposition 3.1 for some $m \geq 3$ or $\|V(s)\|_{L^2_\rho} = \mathcal{O}(e^{-Rs})$ for any $R > 0$. For the second case, we use the following nondegeneracy result from Herrero and Velázquez [11] in order to conclude that $V(y,s) \equiv 0$, which is $i)$ of Proposition 3.1,

Lemma 3.13 (Herrero and Velázquez [11]). *Let V be a solution to equation* (3.4). *Assume that $|V(y,s)|$ is bounded, and that for any $R > 0$ there exists $C = C(R)$ such that*

$$\|V(s)\|_{L^2_\rho} \leq Ce^{-Rs} \quad \text{if} \quad s \geq 0,$$

then $V(y,s) \equiv 0$.

Proof. Since the proof written in [11] holds under general assumption (3.5), we then refer the reader to Lemma 3.5, page 144 of [11] for detail of the proof. ☐

Since the convergence in $\mathcal{C}^{k,\gamma}_{loc}$ for any $k \geq 1$ and $\gamma \in (0,1)$ follows from a standard parabolic regularity, we end the proof of Proposition 3.1 here. This also concludes the proof of Theorem 1.5.

4 Bow-up profile for equation (1.1) in extended spaces regions

We give the proof of Theorem 1.10 in this section. Note that the derivation of Theorem 1.10 from Theorem 1.5 in the unperturbed case ($h \equiv 0$) was done by Velázquez in [19]. The idea to extend the convergence up to sets of the type $\{|y| \leq K_0\sqrt{s}\}$ or $\{|y| \leq K_0 e^{\left(\frac{1}{2}-\frac{1}{m}\right)s}\}$ is to estimate the effect of the convective term $-\frac{y}{2}\cdot\nabla w$ in the equation (1.9) in L^q_ρ spaces with $q > 1$. Since the proof of Theorem 1.10 is actually in spirit by the method given in [19], all that we need to do is to control the strong perturbation term in equation (1.9). We therefore give the main steps of the proof and focus only on the new arguments. The proof will be separated into two parts: the first part concerns case $ii)$ in Theorem 1.5 and gives the asymptotic behavior of w in the $\frac{y}{\sqrt{s}}$ variable, and the second part concerns case $iii)$ in Theorem 1.5 and gives the asymptotic behavior of w in the $ye^{-\left(\frac{1}{2}-\frac{1}{m}\right)s}$ variable. In Part 1, we stick to the method of Velázquez [19], whereas, in Part 2, where we work in the scale $e^{-\mu s}$ for $\mu > 0$, we need new ideas to get rid of the term in the scale $\frac{1}{s}$ coming from the strong perturbation.

Part 1: Case $ii)$ in Theorem 1.5 and asymptotic behavior in the $\frac{y}{\sqrt{s}}$ variable.
Let us restate $i)$ of Theorem 1.10 in the following proposition:

Proposition 4.1 (Asymptotic behavior in the $\frac{y}{\sqrt{s}}$ variable). *Assume that w is a solution of equation* (1.9) *which satisfies $ii)$ of Theorem 1.5. Then, for all $K > 0$,*

$$\sup_{|\xi|\le K} \left|w(\xi\sqrt{s}, s) - f_l(\xi)\right| = \mathcal{O}\left(\frac{1}{s^{a-1}}\right) + \mathcal{O}\left(\frac{\log s}{s}\right), \quad \textit{as} \quad s \to +\infty,$$

where $f_l(\xi) = \kappa\left(1 + \frac{p-1}{4p}\sum_{j=1}^{l}\xi_j^2\right)^{-\frac{1}{p-1}}$.

Proof. Following the method in [19], we define $q = w - \varphi$, where

$$\varphi(y, s) = \kappa\left(1 + \frac{p-1}{4ps}\sum_{j=1}^{l} y_j^2\right)^{-\frac{1}{p-1}} + \frac{\kappa l}{2ps}. \tag{4.1}$$

Using Taylor's formula in (4.1) and $ii)$ of Theorem 1.5, we find that

$$\|q(s)\|_{L^2_\rho} = \mathcal{O}\left(\frac{1}{s^a}\right) + \mathcal{O}\left(\frac{\log s}{s^2}\right), \quad \text{as} \quad s \to +\infty. \tag{4.2}$$

Straightforward calculations based on equation (1.9) yield

$$\partial_s q = (\mathcal{L} + \omega)q + F(q) + G(q, s) + R(y, s), \quad \forall (y, s) \in \mathbb{R}^n \times [-\log T, +\infty), \tag{4.3}$$

where

$$\begin{aligned}
\omega(y, s) &= p(\varphi^{p-1} - \kappa^{p-1}) + e^{-s}h'\left(e^{\frac{s}{p-1}}\varphi\right),\\
F(q) &= |q + \varphi|^{p-1}(q + \varphi) - \varphi^p - p\varphi^{p-1}q,\\
G(q, s) &= e^{-\frac{ps}{p-1}}\left[h\left(e^{\frac{s}{p-1}}(q + \varphi)\right) - h\left(e^{\frac{s}{p-1}}\varphi\right) - e^{\frac{s}{p-1}}h'\left(e^{\frac{s}{p-1}}\varphi\right)q\right],\\
R(y, s) &= -\partial_s\varphi + \Delta\varphi - \frac{y}{2}\cdot\nabla\varphi - \frac{\varphi}{p-1} + \varphi^p + e^{-\frac{ps}{p-1}}h\left(e^{\frac{s}{p-1}}\varphi\right).
\end{aligned}$$

Let $K_0 > 0$ be fixed, we consider first the case $|y| \ge 2K_0\sqrt{s}$ and then $|y| \le 2K_0\sqrt{s}$ and make a Taylor expansion for $\xi = \frac{y}{\sqrt{s}}$ bounded. Simultaneously, noticing from (1.13), we then obtain for all $s \ge s_0$,

$$\omega(y, s) \le \frac{C_1}{s},$$

$$|F(q)| + |G(q, s)| \le C_1(q^2 + \mathbf{1}_{\{|y|\ge 2K_0\sqrt{s}\}}),$$

$$|R(y,s)| \le C_1\left(\frac{|y|^2}{s^2}+\frac{1}{s^2}+\frac{1}{s^a}+\mathbf{1}_{\{|y|\ge 2K_0\sqrt{s}\}}\right),$$

where $C_1 = C_1(M_0, K_0) > 0$, M_0 is the bound of w in L^∞-norm.

Let $Q = |q|$, we then use the above estimates and Kato's inequality, i.e $\Delta f \cdot \text{sign}(f) \le \Delta(|f|)$, to derive from equation (4.3) the following: for all $K_0 > 0$ fixed, there are $C_* = C_*(K_0, M_0) > 0$ and a time $s' > 0$ large enough such that for all $s \ge s_* = \max\{s', -\log T\}$,

$$\partial_s Q \le \left(\mathcal{L}+\frac{C_*}{s}\right)Q + C_*\left(Q^2+\frac{|y|^2}{s^2}+\frac{1}{s^2}+\frac{1}{s^a}+\mathbf{1}_{\{|y|\ge 2K_0\sqrt{s}\}}\right), \quad \forall y \in \mathbb{R}^n. \tag{4.4}$$

Since

$$\left|w(y,s) - f_l\left(\frac{y}{\sqrt{s}}\right)\right| \le Q + \frac{\kappa l}{2ps},$$

the conclusion of Proposition 4.1 follows if we show that

$$\forall K_0 > 0, \quad \sup_{|y|\le K_0\sqrt{s}} Q(y,s) \to 0 \quad \text{as} \quad s \to +\infty. \tag{4.5}$$

Let us now focus on the proof of (4.5) in order to conclude Proposition 4.1. For this purpose, we introduce the following norm: for $r \ge 0$, $q > 1$ and $f \in L^q_{loc}(\mathbb{R}^n)$,

$$L^{q,r}_\rho(f) \equiv \sup_{|\xi|\le r}\left(\int_{\mathbb{R}^n}|f(y)|^q\rho(y-\xi)dy\right)^{\frac{1}{q}}.$$

Following the idea in [19], we shall make estimates on solution of (4.4) in the $L^{2,r(\tau)}_\rho$ norm where $r(\tau) = K_0 e^{\frac{\tau-\bar{s}}{2}} \le K_0\sqrt{\tau}$. Particularly, we have the following:

Lemma 4.2. *Let s be large enough and $\bar{s}$ is defined by $e^{s-\bar{s}} = s$. Then for all $\tau \in [\bar{s}, s]$ and for all $K_0 > 0$, it holds that*

$$g(\tau) \le C_0\left(e^{\tau-\bar{s}}\epsilon(\bar{s}) + \int_{\bar{s}}^{(\tau-2K_0)_+}\frac{e^{(\tau-t-2K_0)}g^2(t)}{\left(1-e^{-(\tau-t-2K_0)}\right)^{1/20}}dt\right)$$

where $g(\tau) = L^{2,r(K_0,\tau,\bar{s})}_\rho(Q(\tau))$, $r(K_0,\tau,\bar{s}) = K_0 e^{\frac{\tau-\bar{s}}{2}}$, $\epsilon(s) = \mathcal{O}\left(\frac{1}{s^a}\right) + \mathcal{O}\left(\frac{\log s}{s^2}\right)$, $C_0 = C_0(C_*, M_0, K_0)$ and $z_+ = \max\{z, 0\}$.

Proof. Multiplying (4.4) by $\alpha(\tau) = e^{\int_{\bar{s}}^{\tau}\frac{C_*}{t}dt}$, then we write $Q(y,\tau)$ for all $(y,\tau) \in \mathbb{R}^n \times [\bar{s}, s]$ in the integration form:

$$\begin{aligned}Q(y,\tau) &= \alpha(\tau)S_{\mathcal{L}}(\tau-\bar{s})Q(y,\bar{s})\\ &\quad + C_*\int_{\bar{s}}^{\tau}\alpha(\tau)S_{\mathcal{L}}(\tau-t)\left(Q^2+\frac{|y|^2}{t^2}+\frac{1}{t^2}+\frac{1}{t^a}+\mathbf{1}_{\{|y|\ge 2K_0\sqrt{t}\}}\right)dt,\end{aligned}$$

where $S_{\mathcal{L}}$ is the linear semigroup corresponding to the operator $\mathcal{L}$.
Next, we take the $L_\rho^{2,r(K_0,\tau,\bar{s})}$-norms both sides in order to get the following:

$$\begin{aligned}
g(\tau) &\leq C_0 L_\rho^{2,r}\big[S_{\mathcal{L}}(\tau-\bar{s})Q(\bar{s})\big] + C_0 \int_{\bar{s}}^{\tau} L_\rho^{2,r}\big[S_{\mathcal{L}}(\tau-t)Q^2(t)\big]dt \\
&+ C_0 \int_{\bar{s}}^{\tau} L_\rho^{2,r}\left[S_{\mathcal{L}}(\tau-t)\left(\frac{|y|^2}{t^2}+\frac{1}{t^2}+\frac{1}{t^a}\right)\right]dt \\
&+ C_0 \int_{\bar{s}}^{\tau} L_\rho^{2,r}\big[S_{\mathcal{L}}(\tau-t)\mathbf{1}_{\{|y|\geq 2K_0\sqrt{t}\}}\big]dt \\
&\equiv J_1+J_2+J_3+J_4.
\end{aligned}$$

Proposition 2.3 in [19] (with a slight modification for the estimate of J_3) yields

$$|J_1| \leq C_0 e^{\tau-\bar{s}}\|Q(\bar{s})\|_{L_\rho^2} = e^{\tau-\bar{s}}\mathcal{O}(\epsilon(\bar{s})) \quad \text{as} \quad \bar{s}\to+\infty,$$

$$|J_2| \leq \frac{C_0}{\bar{s}^2}e^{\tau-\bar{s}} + C_0\int_{\bar{s}}^{(\tau-2K_0)_+} \frac{e^{(\tau-t-2K_0)}}{\big(1-e^{-(\tau-t-2K_0)}\big)^{1/20}}\Big[L_\rho^{2,r(K_0,t,\bar{s})}Q(t)\Big]^2 dt,$$

$$|J_3| \leq C_0 e^{\tau-\bar{s}}\left(\frac{1}{\bar{s}^2}+\frac{1}{\bar{s}^a}\right)(1+(\tau-\bar{s})),$$

$$|J_4| \leq C_0 e^{-\delta\bar{s}}, \quad \text{where} \quad \delta=\delta(K_0)>0.$$

Putting together the estimates on $J_i, i=1,2,3,4,$ we conclude the proof of Lemma 4.2. □

We now use the following Gronwall lemma from Velázquez [19]:

Lemma 4.3 (Velázquez [19]). *Let ϵ, C, R and δ be positive constants, $\delta\in(0,1)$. Assume that $H(\tau)$ is a family of continuous functions satisfying*

$$\mathcal{H}(\tau) \leq \epsilon e^{\tau} + C\int_0^{(\tau-R)_+} \frac{e^{\tau-s}\mathcal{H}^2(s)}{\big(1-e^{-(\tau-s-R)}\big)^{\delta}}ds, \quad \textit{for } \tau>0.$$

Then there exist $\theta=\theta(\delta,C,R)$ and $\epsilon_0=\epsilon_0(\delta,C,R)$ such that for all $\epsilon\in(0,\epsilon_0)$ and any τ for which $\epsilon e^{\tau}\leq\theta$, we have

$$\mathcal{H}(\tau)\leq 2\epsilon e^{\tau}.$$

Applying Lemma 4.3 with $\mathcal{H}\equiv g$, we see from Lemma 4.2 that for s large enough,

$$g(\tau)\leq 2C_0 e^{\tau-\bar{s}}\epsilon(\bar{s}), \quad \forall\tau\in[\bar{s},s].$$

If $\tau=s$, then $e^{s-\bar{s}}=s$, $r=K_0\sqrt{s}$ and

$$g(s)\equiv L_\rho^{2,K_0\sqrt{s}}\big(Q(s)\big)=\mathcal{O}\left(\frac{1}{s^{a-1}}\right)+\mathcal{O}\left(\frac{\log s}{s}\right), \text{ as } \quad s\to+\infty.$$

By using the regularizing effects of the semigroup $S_{\mathcal{L}}$ (see Proposition 2.3 in [19]), we then obtain

$$\sup_{|y|\leq \frac{K_0\sqrt{s}}{2}} Q(y,s) \leq C'(C_*, K_0, M_0) L_\rho^{2,K_0\sqrt{s}}(Q(s)) = \mathcal{O}\left(\frac{1}{s^{a-1}}\right) + \mathcal{O}\left(\frac{\log s}{s}\right),$$

as $s \to +\infty$, which concludes the proof of Proposition 4.1. □

Part 2: Case $iii)$ in Theorem 1.5 and the asymptotic behavior in the $ye^{-(\frac{1}{2}-\frac{1}{m})s}$ variable.

We give the proof of $ii)$ of Theorem 1.10 in this part. Since we work in the scale $e^{-\mu s}$ for $\mu > 0$ in the case where $iii)$ in Theorem 1.5 occurs, we need new ideas to get rid of the term in the scale $\frac{1}{s}$ coming from the strong perturbation.

Let us restate $ii)$ of Theorem 1.10 in the following proposition:

Proposition 4.4 (Asymptotic behavior in the $ye^{-(\frac{1}{2}-\frac{1}{m})s}$ variable). *Assume that w is a solution of equation (1.9) and satisfies $iii)$ of Theorem 1.5. Then, for all $K > 0$,*

$$\sup_{|\xi|\leq K} \left| w(\xi e^{(\frac{1}{2}-\frac{1}{m})s}, s) - \psi_m(\xi)\right| \to 0, \quad \text{as} \quad s \to +\infty, \tag{4.6}$$

where $\psi_m(\xi) = \kappa\left(1 + \kappa^{-p}\sum_{|\alpha|=m} c_\alpha \xi^\alpha\right)^{-\frac{1}{p-1}}$, and $m \geq 4$ is an even integer.

Proof. Note that the proof will proceed in the same way as for the proof of Proposition 4.1. Let us introduce $q = w - \varphi$, where

$$\varphi(y,s) = \frac{\phi(s)}{\kappa} J(y,s), \tag{4.7}$$

with

$$J(y,s) = \frac{\phi(s)}{\kappa}\left[G\left(ye^{-(\frac{1}{2}-\frac{1}{m})s}\right) + e^{-(\frac{m}{2}-1)s}\left(\sum_{|\alpha|=m} c_\alpha y^\alpha - \sum_{|\alpha|=m} c_\alpha H_\alpha(y)\right)\right],$$

and $G(\xi) = \kappa\left(1 + \kappa^{-p}\sum_{|\alpha|=m} c_\alpha \xi^\alpha\right)^{-\frac{1}{p-1}}$ satisfying

$$-\frac{\xi}{m}\cdot\nabla G(\xi) + G^p(\xi) = \frac{G(\xi)}{p-1}. \tag{4.8}$$

Note that Velázquez [19] takes $\varphi = J$, and if we do the same, we will obtain some terms in the scale of $\frac{1}{s}$, much stronger than the $e^{-\mu s}$ scale that we intended to work in.

Using Taylor's formula in (4.7) and recalling from Lemma A.3 that fact that $\frac{\phi(s)}{\kappa} = 1 + \mathcal{O}(s^{-a})$ as $s \to +\infty$, we have by $iii)$ of Theorem 1.5,

$$\|q(s)\|_{L^2_\rho} = o\left(e^{-\left(\frac{m}{2}-1\right)s}\right), \quad \text{as } s \to +\infty. \tag{4.9}$$

Straightforward calculations based on equation (1.9) yield

$$\partial_s q = (\mathcal{L} + \omega)q + F(q) + G(q,s) + R(y,s), \quad \forall (y,s) \in \mathbb{R}^n \times [-\log T, +\infty), \tag{4.10}$$

where

$$\begin{aligned}
\omega(y,s) &= p(\varphi^{p-1} - \kappa^{p-1}) + e^{-s} h'\left(e^{\frac{s}{p-1}}\varphi\right),\\
F(q) &= |q+\varphi|^{p-1}(q+\varphi) - \varphi^p - p\varphi^{p-1}q,\\
G(q,s) &= e^{-\frac{ps}{p-1}}\left[h\left(e^{\frac{s}{p-1}}(q+\varphi)\right) - h\left(e^{\frac{s}{p-1}}\varphi\right) - e^{\frac{s}{p-1}} h'\left(e^{\frac{s}{p-1}}\varphi\right) q\right],\\
R(y,s) &= -\partial_s\varphi + \Delta\varphi - \frac{y}{2}\cdot\nabla\varphi - \frac{\varphi}{p-1} + \varphi^p + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}}\varphi\right).
\end{aligned}$$

Fix now $K_0 > 0$ and define $\chi(y,s) = 1$ if $|y| \geq 2K_0 e^{\left(\frac{1}{2}-\frac{1}{m}\right)s}$ and $\chi(y,s) = 0$ otherwise. Then, using Taylor's formula for $\xi = ye^{-\left(\frac{1}{2}-\frac{1}{m}\right)s}$ bounded, and noticing from (1.13), we then obtain for all $s \geq s_0$,

$$\omega(y,s) \leq \frac{C_1}{s},$$

$$|F(q)| + |G(q,s)| \leq C_1\left(q^2 + \chi(y,s)\right),$$

where $C_1 = C_1(M_0, K_0) > 0$.
To estimate $R(y,s)$, we write $R(y,s)$ as follow:

$$\begin{aligned}
R(y,s) &= \frac{\phi(s)}{\kappa}\left(-\partial_s J + \Delta J - \frac{y}{2}\cdot\nabla J - \frac{J}{p-1} + J^p\right)\\
&\quad + \left(-\frac{\phi'(s)}{\kappa} J - \frac{\phi(s)}{\kappa} J^p + \varphi^p + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}}\varphi\right)\right) \equiv \frac{\phi(s)}{\kappa} I + II.
\end{aligned}$$

By using Taylor's formula, (4.8) and Hermite's equation, i.e.

$$\mathcal{L}H_\alpha(y) = \left(1 - \frac{|\alpha|}{2}\right) H_\alpha(y),$$

it was proved in [19] (see Proposition 2.4) that

$$I \leq Ce^{-(m-2)s}(|y|^{2m-2} + 1)(1 - \chi(y,s)) + C\chi(y,s), \quad \text{for some } C > 0.$$

It remains to estimate II. To do so, we write $J(y,s)$ for $|y|e^{\left(\frac{1}{2}-\frac{1}{m}\right)s}$ bounded in the form:

$$J(y,s) = \kappa - e^{-\left(\frac{m}{2}-1\right)s} \sum_{|\alpha|=m} c_\alpha H_\alpha(y) + \mathcal{O}\left(e^{-(m-2)s}|y|^{2m}\right).$$

We then use Taylor's formula in II, (1.13), and the fact that $\phi(s)$ satisfies (1.21) to find that

$$II \leq \frac{C}{s^a} e^{-(\frac{m}{2}-1)s}(|y|^m+1)(1-\chi(y,s)) + C\chi(y,s), \quad \text{for some } C>0.$$

Note that $e^{-(m-2)s}|y|^{2m-2}(1-\chi(y,s)) \leq \frac{1}{s^a} e^{-(\frac{m}{2}-1)s}|y|^m(1-\chi(y,s))$ for s large, we then obtain

$$|R(y,s)| \leq C\left(\frac{1}{s^a} e^{-(\frac{m}{2}-1)s}(|y|^m+1)(1-\chi(y,s)) + \chi(y,s)\right), \quad \text{for some } C>0.$$

Let $Q=|q|$ and use Kato's inequality, we obtain from (4.10) and from the above estimates that: for all $K_0>0$ fixed, there are $C_* = C_*(K_0, M_0) > 0$ and a time $s'>0$ large enough such that for all $s \geq s_* = \max\{s', -\log T\}$,

$$\partial_s Q \leq \left(\mathcal{L} + \frac{C_*}{s}\right) Q + C_*\left(Q^2 + \frac{1}{s^a} e^{-(\frac{m}{2}-1)s}(|y|^m+1) + \chi(y,s)\right), \quad \forall y \in \mathbb{R}^n. \quad (4.11)$$

We claim the following:

Lemma 4.5. *Let s be large enough and $\bar{s} = \frac{2s}{m}$. Then for all $\tau \in [\bar{s}, s]$, $\tau - \bar{s} \geq 2$ and for all $K_0 > 0$, it holds that*

$$g(\tau) \leq e^{\tau-\bar{s}}\left(o\left(e^{-(\frac{m}{2}-1)\bar{s}}\right) + C'\int_{\bar{s}}^{(\tau-2K_0)_+} \frac{e^{(\tau-t-2K_0)}g^2(t)}{\left(1-e^{-(\tau-t-2K_0)}\right)^{1/20}}dt\right)$$

where $g(\tau) = L_\rho^{2,r(K_0,\tau,\bar{s})}(Q(\tau))$, $r(K_0,\tau,\bar{s}) = K_0 e^{\frac{\tau-\bar{s}}{2}}$, $C' = C'(C_*, M_0, K_0)$ and $z_+ = \max\{z, 0\}$.

Proof. Proceeding as in the proof of Lemma (4.2), we write

$$\begin{aligned} L_\rho^{2,r}(Q) &\leq C_0 L_\rho^{2,r}\left[S_{\mathcal{L}}(\tau-\bar{s})Q(\bar{s})\right] + C_0\int_{\bar{s}}^{\tau} L_\rho^{2,r}\left[S_{\mathcal{L}}(\tau-t)Q^2(t)\right]dt \\ &+ C_0\int_{\bar{s}}^{\tau} L_\rho^{2,r}\left[S_{\mathcal{L}}(\tau-t)\left(\frac{1}{t^a}e^{-(\frac{m}{2}-1)t}(|y|^m+1)\right)\right]dt \\ &+ C_0\int_{\bar{s}}^{\tau} L_\rho^{2,r}\left[S_{\mathcal{L}}(\tau-t)\chi(y,t)\right]dt \\ &\equiv J_1 + J_2 + J_3 + J_4. \end{aligned}$$

One can show that for $\bar{s}$ large enough (see Proposition 2.4 in [19]),

$$|J_1| = e^{\tau-\bar{s}} o\left(e^{(1-m/2)\bar{s}}\right),$$

$$|J_2| \leq Ce^{2\tau-2(m-1)\bar{s}} + C\int_{\bar{s}}^{(\tau-2K_0)_+} \frac{e^{(\tau-t-2K_0)}g^2(t)}{\left(1-e^{-(\tau-t-2K_0)}\right)^{1/20}}dt,$$

$$|J_3| \le Ce^{\tau-\bar{s}}\frac{e^{(1-m/2)\bar{s}}}{\bar{s}^a}(1+\tau-\bar{s}) = e^{\tau-\bar{s}}o\left(e^{(1-m/2)\bar{s}}\right),$$
$$|J_4| \le Ce^{-\theta e^{(1-2/m)\bar{s}}} \quad \text{for some } \theta > 0.$$

Putting together the above estimates yields the desired result. This ends the proof of Lemma 4.5. □

Applying now Lemma 4.3 and Lemma 4.5, we obtain for s large enough,

$$g(\tau) \le e^{\tau-\bar{s}}o\left(e^{-(m/2-1)\bar{s}}\right), \quad \forall \tau \in [\bar{s}, s].$$

Since $\bar{s} = \frac{2s}{m}$, if we set $\tau = s$, then $r = K_0 e^{\left(\frac{1}{2}-\frac{1}{m}\right)s}$ and

$$g(s) \equiv L_\rho^{2,r(s)}(Q(s)) = o(1) \quad \text{as} \quad s \to +\infty.$$

By the regularizing effects of the semigroup $S_{\mathcal{L}}$, we then obtain

$$\sup_{|y| \le \frac{K_0}{2}e^{\left(\frac{1}{2}-\frac{1}{m}\right)s}} Q(y,s) \le C'(C_*, K_0, M_0) L_\rho^{2,r(s)}(Q(s)) \to 0, \text{ as } s \to +\infty,$$

From (4.7), we see that for all $|y| \le \frac{K_0}{2}e^{\left(\frac{1}{2}-\frac{1}{m}\right)s}$,

$$\left| w(y,s) - \frac{\phi(s)}{\kappa}G\left(ye^{-\left(\frac{1}{2}-\frac{1}{m}\right)s}\right)\right| \le Q(y,s) + Ce^{-\left(1-\frac{2}{m}\right)s},$$

Noticing from Lemma A.4 that $\frac{\phi(s)}{\kappa} = 1 + \mathcal{O}(s^{-a})$ as $s \to +\infty$, we obtain

$$\sup_{|y| \le \frac{K_0}{2}e^{\left(\frac{1}{2}-\frac{1}{m}\right)s}} \left| w(y,s) - G\left(ye^{-\left(\frac{1}{2}-\frac{1}{m}\right)s}\right)\right| = o(1), \quad \text{as} \quad s \to +\infty.$$

It remains to show that m is even. Indeed, from (4.6), we can see that if m is not even, there would exist $\xi_0 \in \mathbb{R}^n$ such that $w\left(\xi_0 e^{\left(\frac{1}{2}-\frac{1}{m}\right)s}, s\right) \to \psi_m(\xi_0) \to +\infty$ as $s \to +\infty$, which contradicts the fact that w is bounded as stated in (2.10). Therefore, m must be even. This concludes the proof of Proposition 4.4 and Theorem 1.10 too. □

A Appendix A

The following lemma shows the asymptotic behavior of the solution of the associated ODE of equation 1.1:

Lemma A.1. *Let v be a positive blow-up solution of the following ordinary differential equation:*

$$v'(t) = v^p(t) + h(v), \quad v(T) = +\infty \quad \textit{for some } T > 0, \tag{A.1}$$

where h is defined in (1.3). *Then v satisfies*

$$v(t) \sim \kappa(T-t)^{-\frac{1}{p-1}} \quad \textit{as } t \to T, \textit{ where } \kappa = (p-1)^{-\frac{1}{p-1}}.$$

Remark A.2. *Note that for each $T > 0$, there exists an unique solution to* (A.1) *which blows up at time T. This is a consequence of the classification result (Theorem 1.5) applying to a space-independent solution.*

Proof. Divide (A.1) by v^p and note that $\frac{h(v)}{v^p} \to 0$ as $v \to +\infty$, we see that for all $\varepsilon > 0$, there exists a number $\delta = \delta(\epsilon) > 0$ such that

$$\left| \frac{v'}{v^p} - 1 \right| \leq \varepsilon, \quad \forall t \in [T - \delta, T). \tag{A.2}$$

Solving (A.2) with noting that $v(T) = +\infty$ yields

$$(1+\varepsilon)^{-\frac{1}{p-1}} \kappa (T-t)^{-\frac{1}{p-1}} \leq v(t) \leq (1-\varepsilon)^{-\frac{1}{p-1}} \kappa (T-t)^{-\frac{1}{p-1}}, \quad \forall t \in [T-\delta, T).$$

This concludes the proof of Lemma A.1. □

The following lemma gives us an estimation of the perturbation term in equation (1.9):

Lemma A.3. *Let h be the function defined in* (1.3)*, then it holds that*

$$j = 0, 1, \quad e^{-\frac{(p-j)s}{p-1}} \left| h^{(j)} \left(e^{\frac{s}{p-1}} w \right) \right| \leq C s^{-a} \left(|w|^{p-j} + 1 \right), \quad \forall s \geq \hat{s},$$

where $C = C(a, p, \mu, M) > 0$ and $\hat{s} = \hat{s}(a, p) > 0$ such that $\frac{\log s}{s} \leq \frac{p}{a(p-1)}$ for all $s \geq \hat{s}$.

Proof. We have from (1.3) that for $j = 0, 1$,

$$e^{-\frac{(p-j)s}{p-1}} \left| h^{(j)} \left(e^{\frac{s}{p-1}} w \right) \right| \leq C'(M, \mu) \left(\frac{|w|^{p-j}}{\log^a \left(2 + e^{\frac{2s}{p-1}} w^2 \right)} + e^{-\frac{(p-j)s}{p-1}} \right).$$

Considering the first case $w^2 e^{\frac{s}{p-1}} \geq 4$, we have

$$\frac{|w|^{p-j}}{\log^a \left(2 + e^{\frac{2s}{p-1}} w^2 \right)} \leq \frac{|w|^{p-j}}{\log^a \left(4 e^{\frac{s}{p-1}} \right)} \leq \frac{(p-1)^a}{s^a} |w|^{p-j}.$$

Now, considering the second case $w^2 e^{\frac{s}{p-1}} \leq 4$, we have $|w|^{p-j} \leq 2^{p-j} e^{-\frac{(p-j)s}{2(p-1)}}$ which yields

$$\frac{|w|^{p-j}}{\log^a \left(2 + e^{\frac{2s}{p-1}} w^2 \right)} \leq \frac{|w|^{p-j}}{\log^a (2)} \leq \frac{2^{p-j}}{\log^a 2} e^{-\frac{(p-j)s}{2(p-1)}}.$$

Taking $C = \max \left\{ C', \frac{2^p}{\log^a 2}, (p-1)^a \right\}$ and $\hat{s} > 0$ such that $e^{-\frac{(p-j)s}{p-1}} \leq s^{-a}$ for all $s \geq \hat{s}$, we have the conclusion. This ends the proof of Lemma A.3. □

The following lemma shows us the existence of solutions of the associated ODE of equation (1.9):

Lemma A.4. *Let ϕ be a positive solution of the following ordinary differential equation:*

$$\phi_s = -\frac{\phi}{p-1} + \phi^p + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}}\phi\right). \tag{A.3}$$

Then $\phi(s) \to \kappa$ as $s \to +\infty$ and $\phi(s)$ is given by

$$\phi(s) = \kappa(1+\eta_a(s))^{-\frac{1}{p-1}}, \quad \textit{where} \quad \eta_a(s) = \mathcal{O}\left(\frac{1}{s^a}\right). \tag{A.4}$$

If $h(x) = \mu\frac{|x|^{p-1}x}{\log^a(2+x^2)}$, we have for all $k \in \mathbb{N}$,

$$\eta_a(s) \sim C_0 \int_s^{+\infty} \frac{e^{s-\tau}}{\tau^a} d\tau = \frac{C_0}{s^a}\left(1 + \sum_{j=1}^{k} \frac{b_j}{s^j}\right) + \mathcal{O}\left(\frac{1}{s^{a+k+1}}\right),$$

where $C_0 = \mu\left(\frac{p-1}{2}\right)^a$ and $b_j = (-1)^j \prod_{i=0}^{j-1}(a+i)$.

Proof. By the following transformation

$$v(t) = (T-t)^{-\frac{1}{p-1}}\phi(s), \quad s = -\log(T-t),$$

equation (A.1) is transformed into (A.3). From Lemma A.1, we see that $\phi(s) \to \kappa$ as $s \to +\infty$.
By dividing equation (A.3) by ϕ^p, we find that

$$\left(\frac{1}{\phi^{p-1}}\right)' = \frac{1}{\phi^{p-1}} - (p-1)(1+g(s)), \quad g(s) = \frac{1}{\phi^p} e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}}\phi\right). \tag{A.5}$$

Since $\phi(s) \to \kappa$ as $s \to +\infty$, we have from Lemma A.3 that $g(s) = \mathcal{O}\left(\frac{1}{s^a}\right)$ as $s \to +\infty$. Solving equation (A.5) yields

$$\phi(s) = \kappa\left(1+\eta_a(s)\right)^{-\frac{1}{p-1}}, \quad \text{where} \quad \eta_a(s) = \int_s^{+\infty} e^{s-\tau} g(\tau) d\tau.$$

By integration by part, we find that for all $k \in \mathbb{N}$,

$$\int_s^{+\infty} e^{s-\tau}\tau^{-a} d\tau = \frac{1}{s^a}\left(1 + \sum_{j=1}^{k}\frac{b_j}{s^j}\right) + \mathcal{O}\left(\frac{1}{s^{a+k+1}}\right), \quad b_j = (-1)^j \prod_{i=0}^{j-1}(a+i), \tag{A.6}$$

which follows $\eta_a(s) = \mathcal{O}(s^{-a})$ as $s \to +\infty$. If $h(x) = \mu\frac{|x|^{p-1}x}{\log^a(2+x^2)}$, we see that $g(s) = \mu\log^{-a}\left(2 + e^{\frac{2s}{p-1}}\phi^2(s)\right) \sim \mu\left(\frac{p-1}{2s}\right)^a = \frac{C_0}{s^a}$ as $s \to +\infty$. From (A.6), we conclude the proof of Lemma A.4. □

B Appendix B

B.1 Proof of Proposition 2.7

We give the proof of Proposition 2.7 here.

Proof. The idea of the proof is given in Ladyženskaja and al. [13]. Note that we still get interior regularity even if we know nothing about the initial or boundary data. Indeed, let $\tau \in (0,1)$ and fix t_0 such that $t_0 - \tau > 0$, we denote $Q_\tau(t_0) = \mathbf{B}_{R/2} \times (t_0 - \tau, t_0) \subset Q_R$, and let $\varphi(x,t)$ be a smooth function defined in Q_R such that $0 \le \varphi(x,t) \le 1$ and $\varphi(x,t) = 0$ for all $(x,t) \in Q_R \setminus Q_\tau(t_0)$. Let $k \ge 1$, define

$$v_k(x,t) = \max\{v(x,t) - k, 0\} \quad \text{and} \quad A_k(t) = \{x \in \mathbf{B}_R : v(x,t) > k\}.$$

Then, multiplying equation (2.12) by $v_k\varphi^2$ and integrating over $Q_\tau(t_0)$, we find that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbf{B}_R} v_k^2\varphi^2 dx\Big|_{t_0-\tau}^{t_0} + \int_{t_0-\tau}^{t_0}\int_{\mathbf{B}_R} |\nabla v_k|^2 \varphi^2 dxdt \\
&= -\int_{t_0-\tau}^{t_0}\int_{\mathbf{B}_R} v_k^2\varphi\varphi_t dxdt + 2\int_{t_0-\tau}^{t_0}\int_{\mathbf{B}_R} (\nabla v_k \cdot \nabla\varphi)\, v_k\varphi dxdt \\
&\quad - \int_{t_0-\tau}^{t_0}\int_{\mathbf{B}_R} (b\cdot\nabla v_k)\, v_k\varphi^2 dxdt + \int_{t_0-\tau}^{t_0}\int_{A_k(t)} F v_k\varphi^2 dxdt.
\end{aligned}$$

Using the assumption $|F| \le g(|v|+1)$ and some elementary inequalities with noticing that $\varphi(\cdot, t_0 - \tau) = 0$, we then obtain

$$\begin{aligned}
\max_{t_0-\tau\le t\le t_0} \|v_k(t)\varphi(t)\|^2_{L^2(\mathbf{B}_R)} &+ \int_{t_0-\tau}^{t_0}\int_{\mathbf{B}_R} |\nabla v_k|^2 \varphi^2 dxdt \\
&\le 2\int_{t_0-\tau}^{t_0}\int_{\mathbf{B}_R} (4|\nabla\varphi| + \varphi|\varphi_t|)\, v_k^2 dxdt \\
&\quad + 2\int_{t_0-\tau}^{t_0}\int_{A_k(t)} (\mu_1^2 + 2g)\left(v_k^2 + k^2\right)\varphi^2 dxdt. \qquad \text{(B.1)}
\end{aligned}$$

For the last term in the right-hand side (denote by I), we use Hölder's inequality and (2.13), which reads

$$\begin{aligned}
|I| &\le \left(\int_{t_0-\tau}^{t_0} \|2\mu_1^2 + 4g\|^{\beta'}_{L^{\alpha'}(A_k(t))} dt\right)^{\frac{1}{\beta'}} \left(\int_{t_0-\tau}^{t_0} \left\|\left(v_k^2 + k^2\right)\varphi^2\right\|^{\beta_1}_{L^{\alpha_1}(A_k(t))} dt\right)^{\frac{1}{\beta_1}}, \\
&\le \gamma \left(\int_{t_0-\tau}^{t_0} \left\|\left(v_k^2 + k^2\right)\varphi^2\right\|^{\beta_1}_{L^{\alpha_1}(A_k(t))} dt\right)^{\frac{1}{\beta_1}} = \gamma II,
\end{aligned}$$

where $\gamma = \gamma(\mu_1, \mu_2, R, \alpha', \beta') > 0$, $\alpha_1 = \frac{\alpha'}{\alpha'-1}$ and $\beta_1 = \frac{\beta'}{\beta'-1}$.
From pages 184 and 185 in [13], we have the following interpolation identity:

$$II \leq \beta\theta_k^{\frac{2\epsilon}{r}} \left(\max_{t_0-\tau\leq t\leq t_0} \|v_k(t)\varphi(t)\|^2_{L^2(A_k(t))} + \int_{t_0-\tau}^{t_0}\int_{A_k(t)} |\nabla v_k|^2 \varphi^2 dxdt \right) + k^2\sigma_k^{\frac{2(1+\epsilon)}{r}},$$

where $\epsilon \in (0,1)$, $r \geq 2$, $\beta > 0$ are constants,

$$\theta_k = \int_{t_0-\tau}^{t_0} |A_k(t)|^{\beta_1/\alpha_1}\, dt, \quad \sigma_k = \int_{t_0-\tau}^{t_0} \|\varphi(t)\|^{\beta_1}_{L^{\alpha_1}(A_k(t))} dt.$$

Since $\theta_k \leq \tau R^{\beta_1/\alpha_1}$, we can take τ small enough such that

$$\gamma\beta(\tau R^{\beta_1/\alpha_1})^{\frac{2\epsilon}{r}} \leq \frac{1}{2}.$$

Then from (B.1), we have

$$\begin{aligned} \max_{t_0-\tau\leq t\leq t_0} \|v_k(t)\varphi(t)\|^2_{L^2(\mathbf{B}_R)} &+ \int_{t_0-\tau}^{t_0}\int_{\mathbf{B}_R} |\nabla v_k|^2 \varphi^2 dxdt \\ &\leq \gamma' \left[\int_{t_0-\tau}^{t_0}\int_{\mathbf{B}_R} (|\nabla\varphi| + \varphi|\varphi_t|)\, v_k^2 dxdt + k^2\sigma_k^{\frac{2(1+\epsilon)}{r}} \right]. \end{aligned} \tag{B.2}$$

By Remark 6.4, page 109 and Theorem 6.2, page 103 in [13], we know that if v satisfies (B.2) for any $k \geq 1$, then for all $(x,t) \in \mathbf{B}_{R/4} \times (t_0 - \tau/2, t_0)$,

$$\begin{aligned} |v(x,t)| \leq \gamma'' \Bigg[\left(\frac{R}{2}\right)^{-\frac{n+2}{2}} \left(1 + \frac{R}{2\sqrt{\tau}}\right) &\left(\int_{t_0-\tau}^{t_0} \|v(t)\|^2_{L^2(\mathbf{B}_R)} dt \right)^{1/2} \\ &+ \left(1 + \frac{4\tau}{R}\right)^{\frac{1+\epsilon}{r}} \Bigg] < +\infty. \end{aligned} \tag{B.3}$$

Analogous arguments with the function $-v$ would yield the same estimate. Since μ_1, μ_2 and μ_3 are uniformly bounded in t_0, this implies that estimate (B.3) holds for all $(x,t) \in \mathbf{B}_{R/4} \times (\tau/2, +\infty)$. This concludes the proof of Proposition 2.7. □

B.2 Proof of Proposition 2.9

We prove Proposition 2.9 here. Let us first derive the upper bound for $\mathcal{E}_\psi$.

Proof of the upper bound for $\mathcal{E}_\psi$. Multiplying equation (1.9) with $\psi^2 w_s$ and integrating on $\mathbb{R}^n$ yield

$$\begin{aligned}\int_{\mathbb{R}^n} \psi^2 w_s^2 \rho dy = &-\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^n} \psi^2 |\nabla w|^2 \rho dy - 2\int_{\mathbb{R}^n} \psi w_s \nabla\psi.\nabla w \rho dy\\ &- \frac{1}{2(p-1)}\frac{d}{ds}\int_{\mathbb{R}^n} \psi^2 |w|^2 \rho dy + \frac{1}{p+1}\frac{d}{ds}\int_{\mathbb{R}^n} \psi^2 |w|^{p+1} \rho dy\\ &+ e^{-\frac{p+1}{p-1}}\frac{d}{ds}\int_{\mathbb{R}^n} \psi^2 H\left(e^{\frac{s}{p-1}} w\right)\rho dy\\ &- \frac{1}{p-1} e^{-\frac{p}{p-1}s}\int_{\mathbb{R}^n} \psi^2 h\left(e^{\frac{s}{p-1}} w\right) w\rho dy.\end{aligned}$$

We derive the following identity from the definition (2.16) of the local functional $\mathcal{E}_\psi$,

$$\begin{aligned}\frac{d}{ds}\mathcal{E}_\psi[w](s) = &-\int_{\mathbb{R}^n} \psi^2 |w_s|^2 \rho dy - 2\int_{\mathbb{R}^n} \psi w_s \nabla\psi.\nabla w \rho dy\\ &+ \frac{p+1}{p-1} e^{-\frac{p+1}{p-1}s}\int_{\mathbb{R}^n} \psi^2 H\left(e^{\frac{1}{p-1}s} w\right)\rho dy\\ &- \frac{1}{p-1} e^{-\frac{p}{p-1}s}\int_{\mathbb{R}^n} \psi^2 h\left(e^{\frac{1}{p-1}s} w\right) w\rho dy. \qquad \text{(B.4)}\end{aligned}$$

Using the fact that $2ab \leq \frac{a^2}{2} + 2b^2$, we obtain

$$2\psi w_s \nabla\psi.\nabla w \leq \frac{1}{2}\psi^2 w_s^2 + 2|\nabla\psi|^2|\nabla w|^2.$$

Combining with (2.4), we get an estimate for (B.4) as follows:

$$\begin{aligned}\frac{d}{ds}\mathcal{E}_\psi[w](s) \leq &-\frac{1}{2}\int_{\mathbb{R}^n} \psi^2 |w_s|^2 \rho dy + 2\|\nabla\psi\|_{L^\infty}^2 \int_{\mathbb{R}^n} |\nabla w|^2 \rho dy\\ &+ Cs^{-a}\int_{\mathbb{R}^n} |w|^{p+1}\rho dy + Cs^{-a},\end{aligned}$$

where $C = C(a, p, n, M, \|\psi\|_{L^\infty}^2)$.
Using (iii) and (iv) of Proposition 2.3, we see that

$$\frac{d}{ds}\mathcal{E}_\psi[w](s) \leq C_1\left(1 + \|w_s\|_{L^2_\rho(\mathbb{R}^n)}\right), \quad \forall s \geq \tilde{s}_3, \qquad \text{(B.5)}$$

where $C_1 = C_1\left(a, p, n, N, J_3, J_4, \|\psi\|_{L^\infty}^2, \|\nabla\psi\|_{L^\infty}^2\right)$ and J_i is introduced in Proposition 2.3.
From the definition of $\mathcal{E}_\psi$ given in (2.16), we have

$$\begin{aligned}\mathcal{E}_\psi[w](s) &\leq \|\psi\|_{L^\infty}^2\left\{\frac{1}{2}\int_{\mathbb{R}^n}\left(|\nabla w|^2 + \frac{1}{p-1}|w|^2\right)\rho dy - e^{-\frac{p+1}{p-1}}\int_{\mathbb{R}^n} H\left(e^{\frac{s}{p-1}} w\right)\rho dy.\right\}\\ &= \|\psi\|_{L^\infty}^2\left\{\mathcal{E}[w](s) + \frac{1}{p+1}\int_{\mathbb{R}^n} |w|^{p+1}\rho dy\right\}\\ &\leq \|\psi\|_{L^\infty}^2\left\{J_0 + \frac{1}{p+1}\int_{\mathbb{R}^n} |w|^{p+1}\rho dy\right\}. \quad \forall s \geq \tilde{s}_3.\end{aligned}$$

Integrating on $[s, s+1]$, we obtain

$$\begin{aligned}\int_s^{s+1} \mathcal{E}_\psi[w](\tau)d\tau &\leq \|\psi\|_{L^\infty}^2 \left\{ J_0 + \frac{1}{p+1}\int_s^{s+1}\int_{\mathbb{R}^n} |w|^{p+1}\rho dy d\tau \right\} \\ &\leq \|\psi\|_{L^\infty}^2 \left\{ J_0 + \frac{1}{p+1}\left[\int_s^{s+1}\left(\int_{\mathbb{R}^n}|w|^{p+1}\rho dy\right)^2 d\tau\right]^{\frac{1}{2}} \right\} \\ &\leq C_2\left(\|\psi\|_{L^\infty}^2, J_0, J_5\right) \quad \text{(use } (v) \text{ of Proposition 2.3)}.\end{aligned}$$

Hence,

$$\int_s^{s+1} \mathcal{E}_\psi[w](\tau)d\tau \leq C_2, \quad \forall s \geq \tilde{s}_3. \tag{B.6}$$

Thus, there exists $\tau(s) \in [s, s+1]$ such that

$$\mathcal{E}_\psi[w](\tau(s)) = \int_s^{s+1} \mathcal{E}_\psi[w](\tau')d\tau' \leq C_2.$$

We then have

$$\begin{aligned}\mathcal{E}_\psi[w](s) &= \mathcal{E}_\psi[w](\tau(s)) + \int_{\tau(s)}^s \frac{d}{ds}\mathcal{E}_\psi[w](\tau')d\tau' \\ &\leq C_2 + \int_s^{s+1} C_1\left(1 + \|w_s\|_{L^2_\rho(\mathbb{R}^n)}\right)d\tau' \leq C_2'. \quad \text{(use } (i) \text{ of Proposition 2.3)}\end{aligned}$$

This concludes the proof of the upper bound for $\mathcal{E}_\psi$.

It remains to prove the lower bound in order to conclude the proof of Proposition (2.3).
□

Proof of the lower bound for $\mathcal{E}_\psi$. Multiplying equation (1.9) with $\psi^2 w$ and integrating on $\mathbb{R}^n$ yield

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^n}(\psi w)^2\rho dy = &-2\mathcal{E}_\psi[w](s) + \frac{p+1}{p-1}\int_{\mathbb{R}^n}\psi^2|w|^{p+1}\rho dy \\ &- 2\int_{\mathbb{R}^n}\psi w\nabla\psi.\nabla w\rho dy \\ &- 2e^{-\frac{p+1}{p-1}s}\int_{\mathbb{R}^n}\psi^2 H\left(e^{\frac{s}{p-1}}w\right)\rho dy \\ &+ e^{-\frac{p}{p-1}s}\int_{\mathbb{R}^n}\psi^2 h\left(e^{\frac{s}{p-1}}w\right)w\rho dy.\end{aligned} \tag{B.7}$$

We now control the new term $J_\psi[w](s) = 2\int_{\mathbb{R}^n} \psi w \nabla\psi.\nabla w \rho dy$ as follows:

$$\begin{aligned}
J_\psi[w](s) &= -2\int_{\mathbb{R}^n} w\nabla.\left(\psi w \nabla\psi\rho\right) dy\\
&= -2\int_{\mathbb{R}^n} |w|^2|\nabla\psi|^2\rho dy - 2\int_{\mathbb{R}^n} \psi w\nabla\psi.\nabla w\rho dy\\
&\quad - 2\int_{\mathbb{R}^n} \psi|w|^2\Delta\psi\rho dy + \int_{\mathbb{R}^n}\psi|w|^2 y.\nabla\psi\rho dy.\\
&= -\int_{\mathbb{R}^n} |w|^2|\nabla\psi|^2\rho dy - \int_{\mathbb{R}^n}\psi|w|^2\Delta\psi\rho dy + \frac{1}{2}\int_{\mathbb{R}^n}\psi|w|^2 y.\nabla\psi\rho dy.\\
&\le \left[\|\psi\|_{L^\infty}\left(\|\Delta\psi\|_{L^\infty} + \frac{1}{2}\|y.\nabla\psi\|_{L^\infty}\right)\right]\int_{\mathbb{R}^n}|w|^2\rho dy\\
&\le J_2 C_1(\psi), \quad \forall s \ge \tilde{s}_3 \quad \text{(use } (ii) \text{ of Proposition 2.3).}
\end{aligned} \tag{B.8}$$

Using (2.4) and (B.7), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^n}(\psi w)^2\rho dy \ge -2\mathcal{E}_\psi - J_2C_1(\psi) - C_2 s^{-a}&\\
+\left(\frac{p+1}{p-1} - C_2 s^{-a}\right)\int_{\mathbb{R}^n}\psi^2|w|^{p+1}\rho dy.&
\end{aligned}$$

Taking S large enough such that $C_2 s^{-a} \le \frac{p+1}{2(p-1)}$ for all $s \ge S$, we obtain for all $s \ge \max\{S, \tilde{s}_3\}$,

$$\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^n}(\psi w)^2\rho dy \ge -(2\mathcal{E}_\psi + C_3) + \frac{p+1}{2(p-1)}\int_{\mathbb{R}^n}\psi^2|w|^{p+1}\rho dy,$$

where $C_3 = J_2C_1 + \frac{p+1}{2(p-1)}$.

Let $g(s) = 2\mathcal{E}_\psi + C_3$ and $f(s) = \frac{1}{2}\int_{\mathbb{R}^n}\psi^2|w|^2\rho dy$. Using Jensen's inequality, we have

$$\begin{aligned}
f(s)^{\frac{p+1}{2}} &= 2^{-\frac{p+1}{2}}\left(\int_{\mathbb{R}^n}\psi^2|w|^2\rho dy\right)^{\frac{p+1}{2}}\\
&\le 2^{-\frac{p+1}{2}}\int_{\mathbb{R}^n}(\psi|w|)^{p+1}\rho dy \le 2^{-\frac{p+1}{2}}\|\psi\|_{L^\infty}^{p-1}\int_{\mathbb{R}^n}\psi^2|w|^{p+1}\rho dy.
\end{aligned}$$

We therefore obtain for all $s \ge S_1 = \max\{S, \tilde{s}_3\}$,

$$f'(s) \ge -g(s) + C_4 f(s)^{\frac{p+1}{2}}. \tag{B.9}$$

From (B.5), we also have

$$g'(s) \le C_5 + h(s), \quad \forall s \ge \tilde{s}_3, \tag{B.10}$$

where $h(s) = C_5\|w_s\|_{L^2_\rho(\mathbb{R}^n)}$ and $m = \int_{\tilde{s}_3}^{+\infty} h(s)ds \le C_6$ by using (i) of Proposition 2.3, where C_5, C_6 are some positive constants.
We claim that the function of g is bounded from below by some constant M. Arguing by contradiction, we suppose that there exists a time $s^* \ge S_1$ such that $g(s^*) \le -M$. Then for all $s \ge s^*$, we write

$$\begin{aligned} g(s) = g(s^*) + \int_{s^*}^{s} g'(\tau)d\tau &\le -M + \int_{s^*}^{s}(C_5 + h(\tau))d\tau \\ &\le -M + m + C_5(s - s^*). \end{aligned}$$

Thus, we have by (B.9),

$$f' \ge M - m - C_5(s - s^*) + C_4 f^{\frac{p+1}{2}}, \quad f(s^*) \ge 0.$$

On the other hand, we know that the solution of the following equation

$$f' \ge 1 + C_5 f^{\frac{p+1}{2}}, \quad f(s^*) \ge 0$$

blows up in finite time before

$$s = s^* + \int_0^{+\infty} \frac{d\xi}{1 + C_4 f^{\frac{p+1}{2}}} = s^* + T^*.$$

On the interval $[s^*, s^* + T^*]$, we have

$$M - m - C_5(s - s^*) \ge M - m - C_5 T^*.$$

Thus, we fix $M = m + C_5T^* + 1$ to get $M - m - C_5(s - s^*) \ge 1$ for all $s \in [s^*, s^* + T^*]$. Therefore, f blows up in some finite time before $s^* + T^*$. But this contradicts with the existence global of w. This follows (2.17) and we complete the proof of Proposition 2.9. ☐

C Appendix C

We claim the following:

Lemma C.1 (**Estimate on** $\bar{F}$)**.** *For s large enough, we have*

$$\left|\bar{F}(V, s) - \frac{p}{2\kappa}V^2\right| \le C|V|^3 + \frac{C|V|^2}{s^{a-1}},$$

where $C = C(a, p, M, \mu) > 0$.

Proof. Consider the Taylor expansion of the nonlinear terms F and H, we have

$$F(v) = \frac{1}{2}p(p-1)\phi^{p-2}v^2 + \gamma_1 v^3, \quad H(v,s) = \gamma_2 v^2,$$

where

$$\gamma_1 = \frac{1}{6}p(p-1)(p-2)|\phi + \theta_1 v|^{p-3}, \quad \gamma_2 = \frac{1}{2}e^{-\frac{(p-2)s}{p-1}} h''\left(e^{\frac{s}{p-1}}(\phi + \theta_2 v)\right),$$

with $\theta_i \in [0,1]$, $i = 1,2$.
We claim the following: for s large,

$$|\gamma_1| \le C \quad \text{and} \quad |\gamma_2| \le \frac{C}{s^a}. \tag{C.1}$$

Let us leave the proof of (C.1) later and continue the proof of Lemma C.1. Recalling from Lemma A.4 that $\phi(s) = \kappa + \mathcal{O}(s^{-a})$ as $s \to +\infty$, we derive

$$\left|F(v) + H(v,s) - \frac{p}{2\kappa}v^2\right| = \mathcal{O}\left(\frac{|v|^2}{s^a}\right) + \mathcal{O}(|v|^3), \quad \text{as } s \to +\infty.$$

From the definition of $\bar{F}$ and the fact that $\beta(s) = 1 + \mathcal{O}(\frac{1}{s^{a-1}})$ as $s \to +\infty$, we have for s large enough,

$$\begin{aligned}
\left|\bar{F}(V,s) - \frac{p}{2\kappa}V^2\right| &= \left|\beta(s)\left(F(v) + H(v,s)\right) - \frac{p}{2\kappa}v^2\beta^2\right| \\
&\le \left|F(v) + H(v,s) - \frac{p}{2\kappa}v^2\right| + \frac{C|v|^2}{s^{a-1}} \\
&\le \frac{C|v|^2}{s^a} + C|v|^3 + \frac{C|v|^2}{s^{a-1}} \le C|V|^3 + \frac{C|V|^2}{s^{a-1}},
\end{aligned}$$

which concludes the proof of Lemma C.1, assuming that (C.1) holds.

Let us now give the proof of (C.1). Since $\phi(s) \to \kappa$ as $s \to +\infty$, we can take $s_* > 0$ such that

$$\frac{3\kappa}{4} \le \phi(s) \le \frac{5\kappa}{4}, \quad \forall s \ge s_*.$$

Let us bound $|\gamma_1|$. If $p \ge 3$, by the boundedness of $|\phi|$ and $|v|$, then $|\gamma_1|$ is already bounded. If $p \in (1,3)$, we consider the case $|\theta_1 v| \le \frac{\kappa}{2}$, then the case $|\theta_1 v| > \frac{\kappa}{2}$. In the first case, we have $|\phi + \theta_1 v| \ge \frac{\kappa}{4}$ for all $s \ge s_*$, then $|\gamma_1| \le C|\phi + \theta_1 v|^{p-3} \le C\left(\frac{\kappa}{4}\right)^{p-3}$ for all $s \ge s_*$. Now, considering the second case where $|\theta_1 v| > \frac{\kappa}{2}$, note that in this case, we have $\theta_1 \ne 0$ and $\phi < \frac{5}{2}|\theta_1 v|$ for all $s \ge s_*$. From the definition of $F(v)$, we have

$$\begin{aligned}
|\gamma_1 v^3| &= \left||\phi + v|^{p-1}(\phi + v) - \phi^p - p\phi^{p-1}v - \frac{1}{2}p(p-1)\phi^{p-2}v^2\right| \\
&\le C(|v|^p + v^2), \quad \forall s \ge s_*.
\end{aligned}$$

This yields $|\gamma_1| \le C(|v|^{p-3} + |v|^{-1}) \le C\left((\kappa/2\theta_1)^{p-3} + (\kappa/2\theta_1)^{-1}\right)$ for all $s \ge s_*$. This concludes the proof of the first estimate of (C.1).

Let us now prove that $|\gamma_2| \le Cs^{-a}$ for s large enough. From (1.3), we have

$$|\gamma_2| \le M \frac{|\phi + \theta_2 v|^{p-2}}{\log^a(2 + e^{\frac{s}{p-1}}(\phi + \theta_2 v)^2)}. \tag{C.2}$$

If $p > 2$, by the same technique given in the proof of Lemma A.3, we can show that (C.2) implies

$$|\gamma_2| \le \frac{C}{s^a}(|\phi + \theta_2 v|^{p-2} + 1) \le \frac{2C}{s^a}, \quad \forall s \ge s'(a,p).$$

If $p \in (1,2]$, we consider the first case $|\theta_2 v| \le \frac{\kappa}{2}$, which implies $|\phi(s) + \theta_2 v| \ge \frac{\kappa}{4}$ for all $s \ge s_*$. From (C.2), we derive

$$|\gamma_2| \le \frac{C(\kappa/4)^{p-2}}{\log^a(2 + e^{\frac{s}{p-1}}(\kappa/4)^2)} \le \frac{2C}{s^a}, \quad \text{for } s \text{ large}.$$

In the case where $|\theta_2 v| > \frac{\kappa}{2}$, we note that $\theta_2 \ne 0$ and $\phi(s) \le \frac{5}{2}|\theta_2 v|$ for all $s \ge s_*$. Using the definition of $H(v,s)$ and (1.3), we find that

$$\begin{aligned}|\gamma_2 v^2| \le C \Bigg(&\frac{|\phi + v|^p}{\log^a\left(2 + e^{\frac{2s}{p-1}}(\phi + v)^2\right)} + \frac{\phi^p}{\log^a\left(2 + e^{\frac{2s}{p-1}}\phi^2\right)} \\ &+ \frac{\phi^{p-1}v}{\log^a\left(2 + e^{\frac{2s}{p-1}}\phi^2\right)} + e^{-\frac{ps}{p-1}} + e^{-s}\Bigg) \\ \le &\frac{C}{s^a}\left(|\phi + v|^p + \phi^p + \phi^{p-1}|v| + 1\right) \le \frac{2C}{s^a}(|v|^p + 1),\end{aligned}$$

for s large. This yields $|\gamma_2| \le \frac{2C}{s^a}(|v|^{p-2} + |v|^{-2}) \le \frac{2C}{s^a}\left((\frac{\kappa}{2\theta_2})^{p-2} + (\frac{\kappa}{2\theta_2})^{-2}\right) \le \frac{3C}{s^a}$. This concludes the proof of (C.1) and the proof of Lemma C.1 also. □

Lemma C.2. *Let $\alpha(s)$ be a solution of*

$$\alpha'(s) = \alpha^2(s) + \mathcal{O}\left(\frac{1}{s^q}\right), \quad q \in (2,3], \tag{C.3}$$

which exists for all time. Then

$$\textit{either} \quad \alpha(s) = -\frac{1}{s} + \mathcal{O}\left(\frac{1}{s^q}\right) \quad \textit{or} \quad \alpha(s) = \mathcal{O}\left(\frac{1}{s^q}\right), \quad \textit{if} \quad q \in (2,3), \tag{C.4}$$

$$\textit{either} \quad \alpha(s) = -\frac{1}{s} + \mathcal{O}\left(\frac{\log s}{s^2}\right) \quad \textit{or} \quad \alpha(s) = \mathcal{O}\left(\frac{1}{s^2}\right), \quad \textit{if} \quad q = 3. \tag{C.5}$$

Proof. Let us first show that

$$\text{either } \alpha(s) = \mathcal{O}\left(\frac{1}{s^{1+\sigma}}\right) \text{ or } \alpha(s) = -\frac{1}{s} + \mathcal{O}\left(\frac{1}{s^{1+\sigma'}}\right) \text{ as } s \to +\infty, \tag{C.6}$$

for some $\sigma \in \left(0, \frac{q-2}{2}\right)$ and $\sigma' = q - 2 - 2\sigma$.

Fix s_0 large enough et let $\sigma \in \left(0, \frac{q-2}{2}\right)$. If $|\alpha(s)| \leq \frac{1}{s^{1+\sigma}}$, for all $s \geq s_0$, then we are done. If not, namely there exists a time $s_1 > s_0$ such that $|\alpha(s_1)| > \frac{1}{s^{1+\sigma}}$, we have two possibilities:

$$|\alpha(s)| > \frac{1}{s^{1+\sigma}}, \quad \forall s \geq s_1, \tag{C.7}$$

or there exists a time $s_2 > s_1$ such that

$$|\alpha(s_2)| = \frac{1}{s_2^{1+\sigma}} \quad \text{and} \quad |\alpha(s_2)| \leq \frac{1}{s^{1+\sigma}}, \quad \forall s \in (s_2, s_2 + \delta),\ \delta > 0. \tag{C.8}$$

If (C.7) is the case, then we have by equation (C.3),

$$\left(\frac{1}{\alpha}\right)' = 1 + \mathcal{O}\left(\frac{1}{s^{q-2-2\sigma}}\right), \quad \forall s \geq s_1,$$

which yields (C.6) by integration.
If (C.8) is the case, we assume that $\alpha(s_2) > 0$, then $\alpha'(s_2) \leq -\frac{1+\sigma}{s_2^{2+\sigma}} < 0$. By equation (C.3) and note that $2 + 2\sigma < q$, we have $\alpha'(s_2) > 0$ and a contradiction follows. If $\alpha(s_2) < 0$, then $\alpha'(s_2) \geq \frac{1+\delta}{s_2^{2+\delta}}$, by equation (C.3), we get

$$\frac{1+\delta}{s_2^{2+\delta}} \leq \alpha'(s_2) \leq \frac{1}{s_2^{2+2\sigma}} + \frac{1}{s_2^{q}}.$$

Since $2 + \delta < 2 + 2\delta < q$, we have a contradiction and (C.6) follows.

We now use (C.6) in order to conclude Lemma C.2. Let us give the proof in the case $q = 3$. Assume $\alpha(s) = \mathcal{O}\left(\frac{1}{s^{1+\sigma}}\right)$ for some $\sigma \in (0, \frac{1}{2})$, then (C.3) yields

$$\alpha'(s) = \mathcal{O}\left(\frac{1}{s^{2+2\sigma}}\right) + \mathcal{O}\left(\frac{1}{s^3}\right) = \mathcal{O}\left(\frac{1}{s^{2+2\sigma}}\right).$$

By integration, we get $\alpha(s) = \mathcal{O}\left(\frac{1}{s^{1+2\sigma}}\right)$. Using this estimate, we obtain $\alpha'(s) = \mathcal{O}\left(\frac{1}{s^3}\right)$ and the conclusion follows.
Let us consider

$$\alpha(s) = -\frac{1}{s} + \beta(s), \quad \text{with} \quad \beta(s) = \mathcal{O}\left(\frac{1}{s^{1+\sigma'}}\right), \quad \sigma' = 1 - 2\sigma.$$

Substituting this into (C.3) yields

$$\beta'(s) = \frac{2\beta(s)}{s} + \beta^2(s) + \mathcal{O}\left(\frac{1}{s^3}\right).$$

Multiplying this equation by s^2, we find

$$\left[s^2\beta(s)\right]' = s^2\beta^2 + \mathcal{O}\left(\frac{1}{s}\right) = \mathcal{O}\left(\frac{1}{s^{2\sigma'}}\right) + \mathcal{O}\left(\frac{1}{s}\right).$$

If $\sigma' \geq \frac{1}{2}$, then $\left[s^2\beta(s)\right]' = \mathcal{O}\left(\frac{1}{s}\right)$ which follows $\beta(s) = \mathcal{O}\left(\frac{\log s}{s^2}\right)$. If $\sigma' < \frac{1}{2}$, then $\beta(s) = \mathcal{O}\left(\frac{1}{s^{1+2\sigma'}}\right)$. Using this estimate and repeating the process again, we would obtain $\beta(s) = \mathcal{O}\left(\frac{\log s}{s^2}\right)$ and (C.5) then follows. Since the argument is similar in the case $q \in (2,3)$, we escape here and concludes the proof of Lemma C.2. $\square$

Bibliography

[1] T. Cazenave and P. L. Lions. Solutions globales d'équations de la chaleur semi linéaires. *Comm. Partial Differential Equations*, 9(10):955–978, 1984.

[2] S. Filippas and R. V. Kohn. Refined asymptotics for the blowup of $u_t - \Delta u = u^p$. *Comm. Pure Appl. Math.*, 45(7):821–869, 1992.

[3] S. Filippas and W. X. Liu. On the blowup of multidimensional semilinear heat equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(3):313–344, 1993.

[4] A. Friedman and B. McLeod. Blow-up of positive solutions of semilinear heat equations. *Indiana Univ. Math. J.*, 34(2):425–447, 1985.

[5] Y. Giga and R. V. Kohn. Asymptotically self-similar blow-up of semilinear heat equations. *Comm. Pure Appl. Math.*, 38(3):297–319, 1985.

[6] Y. Giga and R. V. Kohn. Characterizing blowup using similarity variables. *Indiana Univ. Math. J.*, 36(1):1–40, 1987.

[7] Y. Giga and R. V. Kohn. Nondegeneracy of blowup for semilinear heat equations. *Comm. Pure Appl. Math.*, 42(6):845–884, 1989.

[8] Y. Giga, S. Matsui, and S. Sasayama. Blow up rate for semilinear heat equations with subcritical nonlinearity. *Indiana Univ. Math. J.*, 53(2):483–514, 2004.

[9] M. A. Hamza and H. Zaag. Lyapunov functional and blow-up results for a class of perturbations of semilinear wave equations in the critical case. *J. Hyperbolic Differ. Equ.*, 9(2):195–221, 2012.

[10] M. A. Hamza and H. Zaag. A Lyapunov functional and blow-up results for a class of perturbed semilinear wave equations. *Nonlinearity*, 25(9):2759–2773, 2012.

[11] M. A. Herrero and J. J. L. Velázquez. Blow-up behaviour of one-dimensional semilinear parabolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(2):131–189, 1993.

[12] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[13] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[14] C. E. Mueller and F. B. Weissler. Single point blow-up for a general semilinear heat equation. *Indiana Univ. Math. J.*, 34(4):881–913, 1985.

[15] N. Nouaili and H. Zaag. A Liouville theorem for vector valued semilinear heat equations with no gradient structure and applications to blow-up. *Trans. Amer. Math. Soc.*, 362(7):3391–3434, 2010.

[16] P. Quittner. A priori bounds for global solutions of a semilinear parabolic problem. *Acta Math. Univ. Comenian. (N.S.)*, 68(2):195–203, 1999.

[17] P. Quittner and P. Souplet. *Superlinear parabolic problems*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2007. Blow-up, global existence and steady states.

[18] F. Rellich. *Perturbation theory of eigenvalue problems*. Assisted by J. Berkowitz. With a preface by Jacob T. Schwartz. Gordon and Breach Science Publishers, New York, 1969.

[19] J. J. L. Velázquez. Higher-dimensional blow up for semilinear parabolic equations. *Comm. Partial Differential Equations*, 17(9-10):1567–1596, 1992.

[20] J. J. L. Velázquez. Classification of singularities for blowing up solutions in higher dimensions. *Trans. Amer. Math. Soc.*, 338(1):441–464, 1993.

[21] F. B. Weissler. An L^∞ blow-up estimate for a nonlinear heat equation. *Comm. Pure Appl. Math.*, 38(3):291–295, 1985.

[22] F. B. Weissler. Existence and nonexistence of global solutions for a semilinear heat equation. *Israel J. Math.*, 38(1-2):29–40, 1981.

[23] H. Zaag. One-dimensional behavior of singular N-dimensional solutions of semilinear heat equations. *Comm. Math. Phys.*, 225(3):523–549, 2002.

Address: Université Paris 13, Institut Galilée, LAGA,
99 Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France.

E-mail: vtnguyen@math.univ-paris13.fr

Chapter IV

Construction of a stable blow-up solution for a class of strongly perturbed semilinear heat equations[1]

V. T. Nguyen and H. Zaag[2]

Université Paris 13, Sorbonne Paris Cité,
LAGA, CNRS (UMR 7539), F-93430, Villetaneuse, France.

Abstract
We construct a solution for a class of strongly perturbed semilinear heat equations which blows up in finite time with a prescribed blow-up profile. The construction relies on the reduction of the problem to a finite dimensional one and the use of index theory to conclude.

Keywords: Blow-up profile, finite-time blow-up, stability, semilinear heat equations.

1 Introduction

We are interested in the following nonlinear parabolic equation:

$$\begin{cases} u_t &= \Delta u + |u|^{p-1}u + h(u), \\ u(0) &= u_0 \in L^\infty(\mathbb{R}^n), \end{cases} \tag{1.1}$$

[1]submitted, arXiv:1406.5233

[2]This author is supported by the ERC Advanced Grant no. 291214, BLOWDISOL and by the ANR project ANAÉ ref. ANR-13-BS01-0010-03.

where u is defined for $(x,t) \in \mathbb{R}^n \times [0,T)$, $1 < p$ and $p < \frac{n+2}{n-2}$ if $n \geq 3$, the function h is in $\mathcal{C}^1(\mathbb{R},\mathbb{R})$ satisfying

$$j = 0,1, \quad |h^{(j)}(z)| \leq M \left(\frac{|z|^{p-j}}{\log^a(2+z^2)} + 1 \right) \quad \text{with} \quad a > 1, \; M > 0, \tag{1.2}$$

or

$$h(z) = \mu \frac{|z|^{p-1}z}{\log^a(2+z^2)} \quad \text{with} \quad a > 0, \; \mu \in \mathbb{R}. \tag{1.3}$$

By standard results, the Cauchy problem for equation (1.1) can be solved in $L^\infty(\mathbb{R}^n)$. The solution $u(t)$ of (1.1) would exist either on $[0,+\infty)$ (global existence) or only on $[0,T)$, with $0 < T < +\infty$. In this case, we say that $u(t)$ blows up in finite time T, namely

$$\lim_{t \to T} \|u(t)\|_{L^\infty(\mathbb{R}^n)} = +\infty.$$

Here T is called the blow-up time, and a point $x_0 \in \mathbb{R}^n$ is called a blow-up point if and only if there exist $(x_n,t_n) \to (x_0,T)$ such that $|u(x_n,t_n)| \to +\infty$ as $n \to +\infty$. In this paper, we are interested in the finite time blow-up for equation (1.1).

When $h \equiv 0$, the blow-up result for equation (1.1) is largely well-understood. The existence of blow-up solutions has been proved by several authors (see Fujita [7], Ball [1], Levine [13]). We have a lots of results concerning the asymptotic blow-up behavior, locally near a given blow-up point (see Giga and Kohn [10], Weissler [25], Filippas, Kohn and Liu [5], [6], Herrero and Velázquez [11], [12], [23], [24], Merle and Zaag [17], [18], [19]). The notion of asymptotic profile appears also in various papers (see Bricmont and Kupiainen [3], Merle and Zaag [16], Berger and Kohn [2], Nguyen [20] for numerical studies).

Given b a blow-up point of u, we study the behavior of u near the singularity (b,T) through the following *similarity variables* introduced by Giga and Kohn [8], [9], [10]:

$$y = \frac{x-b}{\sqrt{T-t}}, \quad s = -\log(T-t), \quad w_b(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \tag{1.4}$$

and w_b satisfies for all $(y,s) \in \mathbb{R}^n \times [-\log T, +\infty)$,

$$\partial_s w_b = (\Delta - \frac{y}{2} \cdot \nabla + 1) w_b - \frac{p}{p-1} w_b + |w_b|^{p-1} w_b + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}} w_b\right). \tag{1.5}$$

In [21], the author showed that if w_b does not approach ϕ exponentially fast, where ϕ is the positive solution of the associated ordinary differential equation of equation (1.5),

$$\phi_s = -\frac{\phi}{p-1} + \phi^p + e^{-\frac{ps}{p-1}} h(e^{\frac{s}{p-1}}\phi) \quad \text{such that} \quad \phi(s) \to \kappa \quad \text{as} \quad s \to +\infty, \tag{1.6}$$

then the solution u of (1.1) would approach an explicit universal profile as follows:

$$(T-t)^{\frac{1}{p-1}} u(b + z\sqrt{(T-t)|\log(T-t)|}, t) \to f(z) \quad \text{as} \quad t \to T, \tag{1.7}$$

in L^∞_{loc} and in the case $a > 1$, where

$$f(z) = \kappa \left(1 + c_p|z|^2\right)^{-\frac{1}{p-1}}, \quad \text{with } c_p = \frac{p-1}{4p}. \tag{1.8}$$

The goal of this work is to show that the behavior (1.7) does occur. More precisely, we construct a blow-up solution of equation (1.1) satisfying the behavior described in (1.7). This is our main result:

Theorem 1.1 (**Existence of a blow-up solution for equation** (1.1) **with the description of its profile**)**.** *There exists $T > 0$ such that equation* (1.1) *has a solution $u(x,t)$ in $\mathbb{R}^n \times [0,T)$ satisfying:*
i) the solution u blows up in finite-time T at the point $b = 0$,
ii)

$$\left\| (T-t)^{\frac{1}{p-1}} u(\cdot\sqrt{T-t}, t) - f\left(\frac{\cdot}{\sqrt{|\log(T-t)|}}\right) \right\|_{W^{1,\infty}(\mathbb{R}^n)} \le \frac{C}{|\log(T-t)|^{\varrho}}, \tag{1.9}$$

for all $\varrho \in (0,\nu)$ with $\nu = \min\{a-1, \frac{1}{2}\}$ in the case (1.2) *and $\nu = \min\{a, \frac{1}{2}\}$ in the case* (1.3)*, C is some positive constant and f is defined in* (1.8)*.*
iii) There exists $u_* \in \mathcal{C}(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ such that $u(x,t) \to u_*(x)$ as $t \to T$ uniformly on compact subsets of $\mathbb{R}^n \setminus \{0\}$, where

$$u_*(x) \sim \left(\frac{8p|\log|x||}{(p-1)^2|x|^2}\right)^{\frac{1}{p-1}} \quad \text{as} \quad x \to 0.$$

Remark 1.2. *Note that i) directly follows from ii). Indeed, ii) implies that $u(0,t) \sim \kappa(T-t)^{-\frac{1}{p-1}} \to +\infty$ as $t \to T$, which means that u blows up in finite-time T at the point 0. From iii), we see that u blows up only at the point $b = 0$.*

Remark 1.3. *Note that the profile f is the same as in the nonlinear heat equation without the perturbation ($h \equiv 0$), see Bricmont and Kupiainen [3], Merle and Zaag [16].*
The estimate (1.9) *holds in $W^{1,\infty}$ and uniformly in $z \in \mathbb{R}^n$. In the previous work, Ebde and Zaag [4] gives such a uniform convergence in the case h involving a nonlinear gradient term. In fact, the convergence in $W^{1,\infty}$ comes from a parabolic regularity estimate for equation* (1.5) *(see Proposition 3.3 below). Dealing with the case $h \equiv 0$, Bricmont and Kupiainen [3], Merle and Zaag [16] also give such a uniform convergence but only in $L^\infty(\mathbb{R}^n)$. In most papers, the same kind convergence is proved, but only uniformly on a smaller subsets, $|z| \le K\sqrt{|\log(T-t)|}$ (see Velázquez [23]).*

The proof of Theorem 1.1 bases on techniques developed by Bricmont and Kupiainen in [3] and Merle and Zaag in [16] for the semilinear heat equation

$$u_t = \Delta u + |u|^{p-1}u. \tag{1.10}$$

Note that the perturbation term h certainly impacts on the construction of solutions of (1.1) satisfying (1.9). This causes some crucial modifications in [16] in order to totally control the term h. Although these modifications do not affect the general framework developed in [16], they lay in 3 crucial places:
- We modify the profile around which we study equation (1.5), so that we go beyond the order $\frac{1}{s^a}$ generated by the perturbation term. Indeed, for small $a > 0$ and with the same profile as in [16], the order $\frac{1}{s^a}$ will become too strong and will not allow us to close our estimates. See Section 2 below, particularly definition (2.1), which enables us to reach the order $\frac{1}{s^{a+1}}$.
- In order to handle the order $\frac{1}{s^{a+1}}$, we need to modify the definition of the shrinking set near the profile. See Section 3 and particularly Definition 3.1 below.
- A sharp understanding of the dynamics of the linearized operator of (1.5) around the profile (2.1), and which allows to handle the new definition of the shrinking set (see Lemma 3.5 below).

For that reason, we will stress only the main parts of the proof of Theorem 1.1 and put forward the novelties of our argument. In particular, the proof relies on the understanding of the dynamics of the self-similar version of equation (1.5) around the profile (1.8). Following the work by Merle and Zaag [16], the proof will be divided into 2 steps:
- In the first step, we reduce the problem to a finite-dimensional problem: we will show that it is enough to control a finite-dimensional variable in order to control the solution near the profile.
- In the second step, we proceed by contradiction to solve the finite-dimensional problem and conclude using index theory.

We would like to mention that Masmoudi and Zaag [14] adapted the method of [16] for the following Ginzburg-Landau equation:

$$u_t = (1 + \imath\beta)\Delta u + (1 + \imath\delta)|u|^{p-1}u, \tag{1.11}$$

where $p - \delta^2 - \beta\delta(p + 1) > 0$ and $u : \mathbb{R}^n \times [0, T) \to \mathbb{C}$. Note that the case $\beta = 0$ and $\delta \in \mathbb{R}$ small has been studied earlier by Zaag [26].
In [4], Ebde and Zaag used the same idea to show the persistence of the profile (1.8) under weak perturbations of equation (1.10) by lower order terms involving u and ∇u. More precisely, they considered the problem (1.1) in the case where $h = h(u, |\nabla u|)$ satisfies

$$|h(u, v)| \le M(1 + |u|^q + |v|^r), \quad M > 0,\ 0 \le q < p,\ 0 \le r < \frac{2p}{p+1}.$$

In some sense, the term $h(u, \nabla u)$ has a subcritical size when $q < p$ and $r < \frac{2p}{p+1}$. In the *selfsimilar* setting (1.4), we see that

$$\left| e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}} w, e^{\frac{(p+1)s}{2(p-1)}} \nabla w \right) \right| \le Ce^{-\delta s}(|w|^q + |\nabla w|^r + 1),$$

with $\delta = \min\left\{\frac{p-q}{p-1}, \frac{2p-r(p+1)}{2(p-1)}\right\} > 0$. This is the reason why we say that this perturbation is "weak", and justify why our perturbation given in (1.2) or (1.3) is called "strong".
Nouaili and Zaag [22] successfully used the method of [16] for the following complex valued semilinear heat equation:

$$u_t = \Delta u + u^2,$$

where $u(t) : x \in \mathbb{R}^n \to \mathbb{C}$.

As in [16], [26] and [14], it is possible to make the interpretation of the finite-dimensional variable in terms of the blow-up time and the blow-up point. This allows us to derive the stability of the profile f in Theorem 1.1 with respect to perturbations in the initial data. More precisely, we have the following:

Theorem 1.4 (Stability of the solution constructed in Theorem 1.1). *Let us denote by $\hat{u}(x,t)$ the solution constructed in Theorem 1.1 and by $\hat{T}$ its blow-up time. Then, there exists a neighborhood $\mathcal{V}_0$ of $\hat{u}(x,0)$ in $W^{1,\infty}$ such that for any $u_0 \in \mathcal{V}_0$, equation* (1.1) *has a unique solution $u(x,t)$ with initial data u_0, and $u(x,t)$ blows up in finite time $T(u_0)$ at one single blow-up point $b(u_0)$. Moreover, estimate* (1.7) *is satisfied by $u(x-b,t)$ and*

$$T(u_0) \to \hat{T}, \quad b(u_0) \to 0 \quad \textit{as } u_0 \to \hat{u}_0 \textit{ in } W^{1,\infty}(\mathbb{R}^n).$$

Remark 1.5. *We will not give the proof of Theorem 1.4 because the stability result follows from the reduction to a finite dimensional case as in [16] with the same proof. Hence, we only prove the reduction and refer to [16] for the stability. Note that from the parabolic regularity, our stability result holds in the larger space $L^\infty(\mathbb{R}^n)$.*

2 Formulation of the problem

As in [16], [3], we give the proof in one dimension ($n = 1$). The proof remains the same for higher dimensions ($n \geq 2$). We would like to find u_0 initial data such that the solution u of equation (1.1) blows up in finite time T and satisfies the estimate (1.9). Using similarity variables (1.4), this is equivalent to finding $s_0 > 0$ and $w_0(y) \equiv w(y, s_0)$ such that the solution w of equation (1.5) with initial data w_0 satisfies

$$\lim_{s\to+\infty} \|w(s) - f(\frac{\cdot}{\sqrt{s}})\|_{W^{1,\infty}} = 0,$$

where f is given in (1.8).
In order to prove this, we will not linearize equation (1.5) around $f + \frac{\kappa}{2ps}$ as in [16]. We will instead introduce

$$q = w - \varphi, \quad \text{where} \quad \varphi(y,s) = \frac{\phi(s)}{\kappa}\left(f(\frac{y}{\sqrt{s}}) + \frac{\kappa}{2ps}\right), \tag{2.1}$$

with ϕ and f are introduced in (1.6) and (1.8). Then, the problem is reduced to constructing a function q such that

$$\lim_{s\to+\infty} \|q(s)\|_{W^{1,\infty}} = 0$$

and q is a solution of the following equation for all $(y,s) \in \mathbb{R} \times [s_0, +\infty)$,

$$q_s = (\mathcal{L} + V)q + B(q) + R(y,s) + N(y,s), \tag{2.2}$$

where $\mathcal{L} = \Delta - \frac{y}{2}\cdot\nabla + 1$ and

$$V(y,s) = p\left(\varphi(y,s)^{p-1} - \frac{1}{p-1}\right) + \imath e^{-s} h'(e^{\frac{s}{p-1}}\varphi), \tag{2.3}$$

$$B(q) = |\varphi + q|^{p-1}(\varphi + q) - \varphi^p - p\varphi^{p-1}q, \tag{2.4}$$

$$R(y,s) = -\varphi_s + \Delta\varphi - \frac{y}{2}\cdot\nabla\varphi - \frac{\varphi}{p-1} + \varphi^p + e^{\frac{-ps}{p-1}} h\left(e^{\frac{s}{p-1}}\varphi\right), \tag{2.5}$$

$$N(q,s) = e^{\frac{-ps}{p-1}}\left[h\left(e^{\frac{s}{p-1}}(\varphi+q)\right) - h\left(e^{\frac{s}{p-1}}\varphi\right) - \imath e^{\frac{s}{p-1}} h'\left(e^{\frac{s}{p-1}}\varphi\right) q\right], \tag{2.6}$$

with $\imath = 0$ in the case (1.2) and $\imath = 1$ in the case (1.3).

One can remark that we don't linearize (1.5) around $\tilde{\varphi}(y,s) = f(\frac{y}{\sqrt{s}}) + \frac{\kappa}{2ps}$ as in the case of equation (1.10) treated in [16]. In fact, if we do the same, we may obtain some terms like $\frac{1}{s^a}$ coming from the strong perturbation h in equation (1.5), and we may not be able to control these terms in the case $a < 3$. To extend the range of a, we multiply the factor $\frac{\phi(s)}{\kappa}$ to $\tilde{\varphi}$ in order to go beyond the order $\frac{1}{s^a}$ and reach at the order $\frac{1}{s^{a+1}}$. Linearizing around φ given in (2.1) is a major novelty in our approach.

In following analysis, we will use the following integral form of equation (2.2): for each $s \geq \sigma \geq s_0$:

$$q(s) = \mathcal{K}(s,\sigma)q(\sigma) + \int_\sigma^s \mathcal{K}(s,\tau)\left[B(q(\tau)) + R(\tau) + N(q(\tau),\tau)\right] d\tau, \tag{2.7}$$

where $\mathcal{K}$ is the fundamental solution of the linear operator $\mathcal{L} + V$ defined for each $\sigma > 0$ and for each $s \geq \sigma$,

$$\partial_s \mathcal{K}(s,\sigma) = (\mathcal{L} + V)\mathcal{K}(s,\sigma), \quad \mathcal{K}(\sigma,\sigma) = Identity. \tag{2.8}$$

Since the dynamics of equation (2.2) are influenced by the linear part, we first need to recall some properties of the operator $\mathcal{L}$ from Bricmont and Kupiainen [3]. The operator $\mathcal{L}$ is self-adjoint in $L^2_\rho(\mathbb{R}^n)$, where L^2_ρ is the weighted L^2 space associated with the weight ρ defined by

$$\rho(y) = \left(\frac{1}{4\pi}\right)^{n/2} e^{-\frac{|y|^2}{4}}.$$

Its spectrum is given by

$$spec(\mathcal{L}) = \{1 - \frac{m}{2},\ m \in \mathbb{N}\},$$

and its eigenfunctions are derived from Hermite polynomials.
If $n = 1$, the eigenfunction corresponding to $1 - \frac{m}{2}$ is

$$h_m(y) = \sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{k!(m-2k)!}(-1)^k y^{m-2k}. \tag{2.9}$$

We also denote $k_m(y) = \frac{h_m(y)}{\|h_m\|^2_{L^2_\rho}}$.

If $n \geq 2$, we write the spectrum of $\mathcal{L}$ as $spec(\mathcal{L}) = \{1 - \frac{|m|}{2},\ |m| = m_1 + \cdots + m_n,\ (m_1, \ldots, m_n) \in \mathbb{N}^n\}$. Given $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$, the eigenfunction corresponding to $1 - \frac{|m|}{2}$ is

$$H_m(y) = h_{m_1}(y_1) \ldots h_{m_n}(y_n), \quad \text{where } h_m \text{ is defined in (2.9).} \tag{2.10}$$

The potential $V(y, s)$ has two fundamental properties:
$i)$ $V(\cdot, s) \to 0$ in L^2_ρ as $s \to +\infty$. In particular, the effect of V on the bounded sets or in the "blow-up" region ($|y| \leq K\sqrt{s}$) is regarded as a perturbation of the effect of $\mathcal{L}$.
$ii)$ outside of the "blow-up" region, we have the following property: for all $\epsilon > 0$, there exist $C_\epsilon > 0$ and s_ϵ such that

$$\sup_{s \geq s_\epsilon, |y| \geq C_\epsilon \sqrt{s}} |V(y, s) - (-\frac{p}{p-1})| \leq \epsilon. \tag{2.11}$$

This means that $\mathcal{L} + V$ behaves like $\mathcal{L} - \frac{p}{p-1}$ in the region $|y| \geq K\sqrt{s}$. Because 1 is the biggest eigenvalue of $\mathcal{L}$, the operator $\mathcal{L} - \frac{p}{p-1}$ has purely negative spectrum. Therefore, the control of $q(y, s)$ in L^∞ outside of the "blow-up" region will be done without difficulties.
Since the behavior of V inside and outside of the "blow-up" region are different, let us decompose q as following: Let $\chi_0 \in \mathcal{C}_0^\infty([0, +\infty))$ with $supp(\chi_0) \subset [0, 2]$ and $\chi_0 \equiv 1$ on $[0, 1]$. We define

$$\chi(y, s) = \chi_0(\frac{|y|}{K\sqrt{s}}), \tag{2.12}$$

where $K > 0$ to be fixed large enough, and write

$$q(y, s) = q_b(y, s) + q_e(y, s), \tag{2.13}$$

where $q_b(y, s) = \chi(y, s)q(y, s)$ and $q_e(y, s) = (1 - \chi(y, s))q(y, s)$. Note that $supp(q_b(s)) \subset \mathbf{B}(0, 2K\sqrt{s})$ and $supp(q_e(s)) \subset \mathbb{R} \setminus \mathbf{B}(0, K\sqrt{s})$.
In order to control q_b, we expand it with respect to the spectrum of $\mathcal{L}$ in L^2_ρ. More precisely, we write q into 5 components as follows:

$$q(y, s) = \sum_{m=0}^{2} q_m(s)h_m(y) + q_-(y, s) + q_e(y, s), \tag{2.14}$$

where q_m, q_- are coordinates of q_b (not of q), namely that q_m is the projection of q_b in h_m and $q_- = P_-(q_b)$ with P_- being the projector on the negative subspace of $\mathcal{L}$.

3 Proof of the existence of a blow-up solution with the given blow-up profile

In this section, we use the framework developed in [16] in order to prove Theorem 1.1. We proceed in 4 steps:
- In the first step, we define a shrinking set $V_A(s)$ and translate our goal of making $q(s)$ go to 0 in $L^\infty(\mathbb{R})$ in terms of belonging to $V_A(s)$. We also exhibit a two parameter initial data family for equation (2.2) whose coordinates are very small (with respect to the requirements of $V_A(s)$), except the two first q_0 and q_1. Note that the set $V_A(s)$ is different from the corresponding one in [16], and this makes the second major novelty of our work, in addition to the modification of the profile in (2.1).
- In the second step, using the spectral properties of equation (2.2), we reduce our goal from the control of $q(s)$ (an infinite dimensional variable) in $V_A(s)$ to the control of its two first components $(q_0(s), q_1(s))$ (a two-dimensional variable) in $\left[-\frac{A}{s^{1+\nu}}, \frac{A}{s^{1+\nu}}\right]^2$ with $\nu > 0$.
- In the third step, we solve the local in time Cauchy problem for equation (2.2).
- In the last step, we solve the finite dimensional problem using index theory and conclude the proof of Theorem 1.1.

In what follows, the constant C denotes a universal one independent of variables, only depending upon constants of the problems such as a, p, M, μ and K in (2.12).

3.1 Definition of a shrinking set $V_A(s)$ and preparation of initial data

Let first introduce the following definition:

Definition 3.1 (A shrinking set to zero). *Let $\nu = \min\{a-1, \frac{1}{2}\}$ in the case (1.2) and $\nu = \min\{a, \frac{1}{2}\}$ in the case (1.3), we fix $\varrho \in (0, \nu)$. For each $A > 0$, for each $s > 0$, we define $\hat{V}_A(s) = \left[-\frac{A}{s^{1+\nu}}, \frac{A}{s^{1+\nu}}\right]^2 \subset \mathbb{R}^2$, and $V_A(s)$ as being the set of all functions g in L^∞ such that:*

$$m = 0, 1, \quad |g_m(s)| \le \frac{A}{s^{1+\nu}}, \quad |g_2(s)| \le \frac{A^2}{s^{1+\nu}},$$

$$\forall y \in \mathbb{R}, \quad |g_-(y,s)| \le \frac{A}{s^{3/2+\varrho}}(1+|y|^3), \quad \|g_e(s)\|_{L^\infty} \le \frac{A^2}{s^\varrho},$$

where g_m, g_- and g_e are defined in (2.14).

As a master of fact, if $s \ge e$ and $g \in V_A(s)$, one easily derives from the definition of $V_A(s)$ and the fact that $\left|\frac{1-\chi(y,s)}{1+|y|^3}\right| \le \frac{C}{s^{3/2}}$ the following:

$$\forall y \in \mathbb{R}, \quad |g(y,s)| \le \frac{CA^2}{s^{3/2+\varrho}}(1+|y|^3) + \frac{CA^2}{s^{1+\nu}}(1+|y|^2) \quad \text{and} \quad \|g(s)\|_{L^\infty} \le \frac{CA^2}{s^\varrho}. \quad (3.1)$$

Initial data (at time $s_0 = -\log T$) for the equation (2.2) will depend on two real parameters d_0 and d_1 as given in the following proposition:

Lemma 3.2 (Decomposition of initial data on the different components)). *For each $A > 1$, there exists $\delta_1(A) > 0$ such that for all $s_0 \geq \delta_1(A)$: If we consider the following function as initial data for equation* (2.2)*:*

$$q_{d_0,d_1}(y, s_0) = \frac{\phi(s_0)}{\kappa} \left(f^p(z)(d_0 + d_1 z) - \frac{\kappa}{2ps_0} \right), \tag{3.2}$$

where $z = \frac{y}{\sqrt{s_0}}$, f and ϕ are defined in (1.8) *and* (1.6)*, then*
$i)$ There exists a constant $C = C(p) > 0$ such that the components of $q_{d_0,d_1}(s_0)$ (or $q(s_0)$ for short) satisfy:

$$q_0(s_0) = d_0 a_0(s_0) + b_0(s_0), \quad \textit{with} \quad a_0(s_0) \sim C, \quad |b_0(s_0)| \leq \frac{C}{s_0}, \tag{3.3}$$

$$q_1(s_0) = d_1 a_1(s_0) + b_1(s_0), \quad \textit{with} \quad a_1(s_0) \sim \frac{C}{\sqrt{s_0}}, \quad |b_1(s_0)| \leq \frac{C}{s_0^2}, \tag{3.4}$$

and

$$|q_2(s_0)| \leq \frac{C|d_0|}{s_0} + Ce^{-s_0}, \quad |q_-(y, s_0)| \leq \left(\frac{C|d_0|}{s_0} + \frac{C|d_1|}{s_0\sqrt{s_0}} \right)(1 + |y|^3),$$
$$\|q_e(s_0)\|_{L^\infty} \leq C|d_0| + \frac{C|d_1|}{\sqrt{s_0}}, \quad \|\nabla q(s_0)\|_{L^\infty} \leq \frac{C(|d_0| + |d_1|)}{\sqrt{s_0}}.$$

$ii)$ For each $A > 0$, if (d_0, d_1) is chosen so that $(q_0, q_1)(s_0) \in \hat{V}_A(s_0)$, then

$$|d_0| + |d_1| \leq \frac{C}{s_0},$$
$$|q_2(s_0)| \leq \frac{C}{s_0^2}, \quad \left\| \frac{q_-(y, s_0)}{1 + |y|^3} \right\|_{L^\infty} \leq \frac{C}{s_0^2}, \quad \|q_e(s_0)\|_{L^\infty} \leq \frac{C}{s_0},$$
$$q(s_0) \in V_A(s_0), \quad \|\nabla q(s_0)\|_{L^\infty} \leq \frac{C}{s_0\sqrt{s_0}},$$

where the statement $q(s_0) \in V_A(s_0)$ holds with "strict inequalities", except for $(q_0, q_1)(s_0)$, in the sense that

$$m = 0, 1, \quad |q_m(s)| \leq \frac{A}{s^{1+\nu}}, \quad |q_2(s)| < \frac{A^2}{s^{1+\nu}},$$
$$\forall y \in \mathbb{R}, \quad |q_-(y, s)| < \frac{A}{s^{3/2+\varrho}}(1 + |y|^3), \quad \|q_e(s)\|_{L^\infty} < \frac{A^2}{s^\varrho}.$$

$iii)$ There exists a rectangle $\mathcal{D}_{s_0} \subset \left[-\frac{C}{s_0}, \frac{C}{s_0} \right]^2$ such that the mapping $(d_0, d_1) \mapsto (q_0, q_1)(s_0)$ is linear and one to one from $\mathcal{D}_{s_0}$ onto $\left[-\frac{A}{s_0^{1+\nu}}, \frac{A}{s_0^{1+\nu}} \right]^2$ and maps $\partial\mathcal{D}_{s_0}$ into $\partial\left[-\frac{A}{s_0^{1+\nu}}, \frac{A}{s_0^{1+\nu}} \right]^2$. Moreover, it is of degree one on the boundary and the following equivalence holds:

$$q(s_0) \in V_A(s_0) \ \textit{if and only if} \ (d_0, d_1) \in \mathcal{D}_{s_0}.$$

Proof. $i)$ Since we have the similar expression of initial data (3.2) as in [16], we refer the reader to Lemma 3.5 of [16], except for the bound on $\|\nabla q(s_0)\|_{L^\infty}$. Note that although $i)$ is not stated explicitly in Lemma 3.5 of [16], they are clearly written in its proof. For $\|\nabla q(s_0)\|_{L^\infty}$, we use (3.2) and the fact that $f'(z) = -\frac{p-1}{2p} z f^p(z)$, $f^p(z), z f^{p-1}(z)$ and $z^2 f^{p-1}(z)$ are in $L^\infty(\mathbb{R})$ to derive

$$\begin{aligned}|\nabla q(y, s_0)| &\leq \left|\frac{\phi(s_0)}{\kappa}\right| \left|\frac{f^p(z)}{\sqrt{s_0}}\left(p d_0 z f^{p-1}(z) + d_1 + p d_1 z^2 f^{p-1}(z)\right)\right| \\ &\leq \frac{C}{\sqrt{s_0}}(|d_0| + |d_1|).\end{aligned}$$

$ii)$ We see from (3.3) and (3.4) that if (d_0, d_1) is chosen so that $(q_0, q_1)(s_0) \in \left[-\frac{A}{s_0^{1+\nu}}, \frac{A}{s_0^{1+\nu}}\right]^2$, then $|d_0|$ and $|d_1|$ are bounded by $\frac{C}{s_0}$. Substituting these bounds into the estimates stated in $i)$, we immediately derive $ii)$.
$iii)$ It follows from (3.3) and (3.4), part $ii)$ and the definition of V_A. This ends the proof of Lemma 3.2. □

As stated in Theorem 1.1, the convergence holds in $W^{1,\infty}(\mathbb{R})$, we need the following parabolic regularity estimate for equation (2.2), with $q(s_0)$ given by (2.7) and $q(s) \in V_A(s)$. More precisely, we have the following:

Proposition 3.3. *For each $A \geq 1$, there exists $\delta_2(A) > 0$ such that for all $s_0 \geq \delta_2(A)$: if $q(s)$ is a solution of equation* (2.2) *on $[s_0, s_1]$ with initial data at $s = s_0$, $q_{d_0,d_1}(s_0)$ given in* (2.7) *where $(d_0, d_1) \in \mathcal{D}_{s_0}$, assume in addition that $q(s) \in V_A(s)$ for $s \in [s_0, s_1]$, then*

$$\|\nabla q(s)\|_{L^\infty} \leq \frac{CA^2}{s^\varrho}, \quad \forall s \in [s_0, s_1],$$

for some positive constant C.

Proof. The proof is the same as Proposition 3.3 of [4]. We would like to mention that the proof bases on a Gronwall's argument and the following properties of the kernel $e^{\theta\mathcal{L}}$ defined in (B.1):

$$\forall g \in L^\infty, \quad \|\nabla(e^{\theta\mathcal{L}} g)\|_{L^\infty} \leq \frac{Ce^{\theta/2}\|g\|_{L^\infty}}{\sqrt{1 - e^{-\theta}}},$$

and

$$\forall f \in W^{1,\infty}, \quad \|\nabla(e^{\theta\mathcal{L}} f)\|_{L^\infty} \leq Ce^{\theta/2}\|\nabla f\|_{L^\infty}.$$

Although the definition of V_A is slightly different from the one defined in [4], the reader will have absolutely no difficulty to adapt their proof to the new situation. For that reason, we refer the reader to [4] for details of the proof. □

3.2 Reduction to a finite dimensional problem

We are going to the crucial step of the proof of Theorem 1.1. In this step, we will show that through a priori estimates, the control of $q(s)$ in V_A reduces to the control of $(q_0, q_1)(s)$ in $\hat{V}_A(s)$. As presented in [16] (see also[26],[14]), we would like to emphasize that this step make the heart of the contribution. Even more, here lays another major contribution of ours, in the sense that we understand better the dynamics of the fundamental solution $\mathcal{K}(s,\sigma)$ defined in (2.8). Our sharper estimates are given in Lemma 3.5 below. In fact all that we do is to rewrite the corresponding estimates of Bricmont and Kupiainen [3] without taking into account the particular form of the shrinking set they used. Furthermore, because of the difference in the definition (2.1) of φ and the difference in the definition of V_A, the proof is far from being an adaptation of the proof written in [16]. We therefore need some involved arguments to control the components of q and conclude the reduction to a finite dimensional problem.

We mainly claim the following:

Proposition 3.4 (Control of $q(s)$ by $(q_0, q_1)(s)$ in $V_A(s)$). *There exist $A_3 > 0$ such that for each $A \geq A_3$, there exists $\delta_3(A) > 0$ such that for each $s_0 \geq \delta_3(A)$, we have the following properties:*
- if (d_0, d_1) is chosen so that $(q_0, q_1)(s_0) \in \hat{V}_A(s_0)$, and
- if for all $s \in [s_0, s_1]$, $q(s) \in V_A(s)$ and $q(s_1) \in \partial V_A(s_1)$ for some $s_1 \geq s_0$, then:
$i)$ ***(Reduction to a finite dimensional problem)*** $(q_0, q_1)(s_1) \in \partial \hat{V}_A(s_1)$,
$ii)$ ***(Transversality)*** *there exists $\eta_0 > 0$ such that for all $\eta \in (0, \eta_0)$, $(q_0, q_1)(s_1 + \eta) \notin \partial \hat{V}_A(s_1 + \eta)$ (hence, $q(s_1 + \eta) \notin V_A(s_1 + \eta)$).*

The proof follows the general ideas of [16] and we proceed in three steps:
- Step 1: we give a priori estimates on $q(s)$ in $V_A(s)$: assume that for given $A > 0$ lager, $\lambda > 0$ and an initial time $s_0 \geq \sigma_2(A, \lambda) \geq 1$, we have $q(s) \in V_A(s)$ for each $s \in [\tau, \tau + \lambda]$ where $\tau \geq s_0$, then using the integral form (2.7) of $q(s)$, we derive new bounds on $q_2(s), q_-(s)$ and $q_e(s)$ for $s \in [\tau, \tau + \lambda]$.
- Step 2: we show that these new bounds are better than those defining $V_A(s)$. It then remains to control $q_0(s)$ and $q_1(s)$. This means that the problem is reduced to the control of a two dimensional variable $(q_0, q_1)(s)$ and we then conclude $i)$ of Proposition 3.4.
- Step 3: we use dynamics of $(q_0, q_1)(s)$ to show its transversality on $\partial V_A(s)$, which corresponds to part $ii)$ of Proposition 3.4.

Step 1: A priori estimates on $q(s)$ in $V_A(s)$

As indicated above, the derivation of the new bounds on the components of $q(s)$ bases on the integral formula (2.7). It is clear to see the strong influence of the kernel $\mathcal{K}$ in this formula. Therefore, it is convenient to give the following result from Bricmont and Kupiainen in [3] which gives the dynamics of the linear operator $\mathcal{L} + V$:

Lemma 3.5 (Refined understanding of the linearized operator in the decomposition (2.14)). *For all $\lambda > 0$, there exists $\sigma_0 = \sigma_0(\lambda)$ such that if $\sigma \geq \sigma_0 \geq 1$ and $\psi(\sigma)$ satisfies*

$$\sum_{m=0}^{2} |\psi_m(\sigma)| + \left\| \frac{\psi_-(y,\sigma)}{1+|y|^3} \right\|_{L^\infty} + \|\psi_e(\sigma)\|_{L^\infty} < +\infty, \tag{3.5}$$

then, $\theta(s) = \mathcal{K}(s,\sigma)\psi(\sigma)$ satisfies for all $s \in [\sigma, \sigma+\lambda]$,

$$\begin{aligned} |\theta_2(s)| \leq \left(\frac{\sigma}{s}\right)^2 |\psi_2(\sigma)| + \frac{C(s-\sigma)}{s}\left(\sum_{l=0}^{2}|\psi_l(\sigma)| + \left\|\frac{\psi_-(y,\sigma)}{1+|y|^3}\right\|_{L^\infty}\right) \\ + C(s-\sigma)e^{-s/2}\|\psi_e(\sigma)\|_{L^\infty}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \left\|\frac{\theta_-(y,s)}{1+|y|^3}\right\|_{L^\infty} \leq \frac{Ce^{s-\sigma}\left((s-\sigma)^2+1\right)}{s}\left(|\psi_0(\sigma)| + |\psi_1(\sigma)| + \sqrt{s}|\psi_2(\sigma)|\right) \\ + Ce^{-\frac{(s-\sigma)}{2}}\left\|\frac{\psi_-(y,\sigma)}{1+|y|^3}\right\|_{L^\infty} + \frac{Ce^{-(s-\sigma)^2}}{s^{3/2}}\|\psi_e(\sigma)\|_{L^\infty}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \|\theta_e(s)\|_{L^\infty} \leq Ce^{s-\sigma}\left(\sum_{l=0}^{2} s^{l/2}|\psi_l(\sigma)| + s^{3/2}\left\|\frac{\psi_-(y,\sigma)}{1+|y|^3}\right\|_{L^\infty}\right) \\ + Ce^{-\frac{(s-\sigma)}{p}}\|\psi_e(\sigma)\|_{L^\infty}. \end{aligned} \tag{3.8}$$

where $C = C(\lambda, K) > 0$ (K is given in (2.12)*), ψ_m, ψ_-, ψ_e and $\theta_m, \theta_-, \theta_e$ are defined by* (2.13) *and* (2.14).

Remark 3.6. *In view of the formula* (2.7)*, we see that Lemma 3.5 will play an important role in deriving the new bounds on the components of $q(s)$ and making our proof simpler. This means that, given bounds on the components of $q(\sigma), B(q(\tau)), R(\tau), N(q(\tau),\tau)$, we directly apply Lemma 3.5 with $\mathcal{K}(s,\sigma)$ replaced by $\mathcal{K}(s,\tau)$ and then integrate over τ to obtain estimates on the components of q.*

Proof. Let us mention that Lemma 3.5 relays mainly on the understanding of the behavior of the kernel $\mathcal{K}(s,\sigma)$. The proof is essentially the same as in [3], but those estimates did not present explicitly the dependence on all the components of $\psi(\sigma)$ which is less convenient for our analysis below. Because the proof is long and technical, we leave it to Appendix B. ☐

We now assume that for each $\lambda > 0$, for each $s \in [\sigma, \sigma+\lambda]$, we have $q(s) \in V_A(s)$ with $\sigma \geq s_0$. Applying Lemma 3.5, we get new bounds on all terms in the right hand side of (2.7), and then on q. More precisely, we claim the following:

Lemma 3.7. *There exists $A_2 > 0$ such that for each $A \geq A_2$, $\lambda^* > 0$, there exists $\sigma_2(A,\lambda^*) > 0$ with the following property: for all $s_0 \geq \sigma_2(A,\lambda^*)$, for all $\lambda \leq \lambda^*$, assume*

that for all $s \in [\sigma, \sigma+\lambda]$, $q(s) \in V_A(s)$ *with* $\sigma \geq s_0$, *then we have for all* $s \in [\sigma, \sigma+\lambda]$,
$i)$ ***(linear term)***

$$|\alpha_2(s)| \leq \left(\frac{\sigma}{s}\right)^{1-\nu} \frac{A^2}{s^{1+\nu}} + \frac{CA^2(s-\sigma)}{s^{2+\nu}},$$
$$\left\|\frac{\alpha_-(y,s)}{1+|y|^3}\right\|_{L^\infty} \leq \frac{C}{s^{3/2+\varrho}} + \frac{C}{s^{3/2+\varrho}}\left(Ae^{-\frac{s-\sigma}{2}} + A^2 e^{-(s-\sigma)^2}\right),$$
$$\|\alpha_e(s)\|_{L^\infty} \leq \frac{C}{s^\varrho} + \frac{C}{s^\varrho}\left(Ae^{s-\sigma} + A^2 e^{-\frac{s-\sigma}{p}}\right),$$

where

$$\mathcal{K}(s,\sigma)q(\sigma) = \alpha(y,s) = \sum_{m=0}^{2} \alpha_m(s)h_m(y) + \alpha_-(y,s) + \alpha_e(y,s).$$

If $\sigma = s_0$, *we assume in addition that* (d_0, d_1) *is chosen so that* $(q_0, q_1)(s_0) \in \hat{V}_A(s_0)$. *Then for all* $s \in [s_0, s_0+\lambda]$, *we have*

$$|\alpha_2(s)| \leq \frac{C}{s^2}, \quad \left\|\frac{\alpha_-(y,s)}{1+|y|^3}\right\|_{L^\infty} \leq \frac{C}{s^2}, \quad \|\alpha_e(s)\|_{L^\infty} \leq \frac{Ce^{s-s_0}}{\sqrt{s}}.$$

$ii)$ ***(remaining terms)***

$$|\beta_2(s)| \leq \frac{C(s-\sigma)}{s^{2+\nu}}, \quad \left\|\frac{\beta_-(y,s)}{1+|y|^3}\right\|_{L^\infty} \leq \frac{C}{s^{3/2+\varrho}}, \quad \|\beta_e(s)\|_{L^\infty} \leq \frac{C}{s^\varrho},$$

where

$$\int_\sigma^s \mathcal{K}(s,\tau)\left[B(q(\tau)) + R(\tau) + N(q(\tau),\tau)\right] d\tau$$
$$= \beta(y,s) = \sum_{m=0}^{2} \beta_m(s)h_m(y) + \beta_-(y,s) + \beta_e(y,s).$$

Proof. $i)$ It immediately follows from the definition of $V_A(\sigma)$ and Lemma 3.5. Note that in the case $\sigma = s_0$, we use in addition part $ii)$ of Lemma 3.2 to have the conclusion. For part $ii)$, all what we need to do is to find the estimates on the components of different terms appearing in equation (2.2), then we use Lemma 3.5 and the linearity to have the conclusion. We claim the following:

Lemma 3.8. *We have the following properties:*
$i)$ ***(Estimates on** $B(q)$**)*** *For all* $A > 0$, *there exists* $\sigma_3(A)$ *such that for all* $\tau \geq \sigma_3(A)$, $q(\tau) \in V_A(\tau)$ *implies*

$$m = 0,1,2, \ |B_m(\tau)| \leq \frac{CA^4}{\tau^{2+2\nu}}, \ \left|\frac{B_-(y,\tau)}{1+|y|^3}\right| \leq \frac{CA^4}{\tau^{3/2+2\varrho}}, \ \|B_e(\tau)\|_{L^\infty} \leq \frac{CA^{2p'}}{\tau^{\varrho p'}}, \tag{3.9}$$

where $p' = \min\{p, 2\}$.
$ii)$ ***(Estimates on** R**)** There exists* $\sigma_4 > 0$ *such that for all* $\tau \geq \sigma_4$,

$$\begin{aligned} m = 0, 1, \; |R_m(\tau)| \leq \frac{C}{\tau^2}, \quad |R_2(\tau)| \leq \frac{C}{\tau^{2+\nu}}, \\ \textit{and} \quad \left\| \frac{R_-(y,\tau)}{1+|y|^3} \right\|_{L^\infty} \leq \frac{C}{\tau^2}, \quad \|R_e(\tau)\|_{L^\infty} \leq \frac{C}{\tau^\nu}. \end{aligned} \tag{3.10}$$

$iii)$ ***(Estimates on** $N(q,\tau)$**)** For all* $A > 0$*, there exists* $\sigma_5(A)$ *such that for all* $\tau \geq \sigma_5(A)$*,* $q(\tau) \in V_A(\tau)$ *implies*

$$m = 0, 1, 2, \; |N_m(\tau)| \leq \frac{CA^4}{\tau^{2+2\nu}}, \; \left\| \frac{N_-(y,\tau)}{1+|y|^3} \right\|_{L^\infty} \leq \frac{CA^4}{\tau^{2+2\varrho}}, \; \|N_e(\tau)\|_{L^\infty} \leq \frac{CA^4}{\tau^{2\varrho}}. \tag{3.11}$$

Proof. Since the proof is technical, we leave it to Appendix C. □

Substituting the estimates stated in Lemma 3.8 into Lemma 3.5, then integrating over $[\sigma, s]$ with respect to τ, and taking $\sigma_2(A, \lambda^*) \geq \max\{\sigma_3, \sigma_4, \sigma_5\}$ such that

$$\forall s \geq \sigma_2(A, \lambda^*), \;\; (A^4+1)e^{\lambda^*}((\lambda^*)^3+1)\left(s^{-\varrho(p'-1)} + s^{-(\nu-\varrho)}\right) \leq 1,$$

with $p' = \min\{p, 2\}$, we have the conclusion. This ends the proof of Lemma 3.7. □

Thanks to Lemma 3.8, we obtain the following equations satisfied by the expanding modes:

Lemma 3.9 (ODE satisfied by the expanding modes). *For all* $A > 0$*, there exists* $\sigma_6(A)$ *such that for all* $s \geq \sigma_6(A)$*,* $q(s) \in V_A(s)$ *implies that for all* $s \geq \sigma_6(A)$*,*

$$m = 0, 1, \quad \left| q_m'(s) - (1 - \frac{m}{2})q_m(s) \right| \leq \frac{C}{s^{3/2+\nu}}, \tag{3.12}$$

and

$$\left| q_2'(s) + \frac{2}{s}\, q_2(s) \right| \leq \frac{C}{s^{2+\nu}}. \tag{3.13}$$

Proof. The proof is very close to that in [16]. We therefore give the sketch of the proof. By the definition (2.14), we write

$$m = 0, 1, 2, \quad \frac{dq_m(s)}{ds} = \int \frac{\partial \chi(y,s)}{\partial s} q(s) k_m \rho dy + \int \chi(y,s) \frac{\partial q(s)}{\partial s} k_m \rho dy := I + II.$$

Since the support of $\frac{\partial \chi(y,s)}{\partial s}$ is the set $K\sqrt{s} \leq |y| \leq 2K\sqrt{s}$ (see (2.12)), using the fact that $\|q(s)\|_{L^\infty} \leq \frac{CA^2}{s^\varrho}$ (see (3.1)), we obtain

$$|I| \leq \int \left| \frac{\partial \chi(y,s)}{\partial s} \right| |q(s)||k_m|\rho dy \leq CA^2 e^{-s} s^{-\varrho},$$

for K large enough.
For II, we have by equation (2.2),

$$II = \int \chi(y,s)\mathcal{L}q(s)k_m\rho dy + \int \chi(y,s)V(s)q(s)k_m\rho dy$$
$$+ \int \chi(y,s)\left[B(q(s)) + R(s) + N(q(s),s)\right]k_m\rho dy := IIa + IIb + IIc.$$

Since $\mathcal{L}$ is self-adjoint on L^2_ρ and $\mathcal{L}(\chi(y,s)k_m) = (1-\frac{m}{2})\chi(y,s)k_m + \frac{\partial^2\chi(y,s)}{\partial s^2}k_m + \frac{\partial\chi(y,s)}{\partial s}(2\frac{\partial k_m}{dy} - \frac{y}{2}k_m)$, we obtain

$$IIa = \int \mathcal{L}(\chi(y,s)k_m)q(s)\rho dy = (1-\frac{m}{2})q_m(s) + \mathcal{O}(CA^2e^{-s}),$$

where $\mathcal{O}(r)$ stands for quantity whose absolute value is bounded precisely by r and not Cr.
Recalling from part $c)$ of Lemma B.1 that $|V(y,s)| \le \frac{C}{s}(1+|y|^2)$ and from (3.1) that $|q(y,s)| \le \frac{CA^2}{s^{1+\nu}}(1+|y|^3)$, we derive

$$m=0,1,\quad |IIb| \le \frac{CA^2}{s^{2+\nu}}\int(1+|y|^5)|k_m|\rho dy \le \frac{CA^2}{s^{2+\nu}}.$$

For $m=2$, using the second estimate in part $c)$ of Lemma B.1, namely that $V(y,s) = -\frac{h_2(y)}{4s} + \mathcal{O}\left(\frac{C(1+|y|^4)}{s^{1+\bar{a}}}\right)$ with $\bar{a} = \min\{a-1,a\}$ in the case (1.2) and $\bar{a} = \min\{a,1\}$ in the case (1.3), simultaneously noting that $\int h_2^2\rho dy = 8$, $\int h_2^3\rho dy = 64$ and $2+\bar{a}+\nu \ge 2+2\nu$, we obtain

$$m=2,\quad IIb = -\frac{2}{s}\,q_2(s) + \mathcal{O}\left(\frac{CA^2}{s^{2+2\nu}}\right).$$

The bound for IIc already obtained from (3.9), (3.10) and (3.11). Adding all these bounds and taking $\sigma_6(A)$ large enough such that for all $s \ge \sigma_6(A)$, $A^4s^{-\nu} + A^2s^{2+\nu}e^{-s} \le 1$, we then have the conclusion. This ends the proof of Lemma 3.9. □

Step 2: Deriving conclusion $i)$ of Proposition 3.4

Here we use Lemma 3.7 in order to derive conclusion $i)$ of Proposition 3.4. Indeed, from equation (2.7) and Lemma 3.7, we derive new bounds on $|q_2(s)|$, $\left\|\frac{q_-(y,s)}{1+|y|^3}\right\|_{L^\infty}$ and $\|q_e(s)\|_{L^\infty}$, assuming that for all $s \in [\sigma,\sigma+\lambda]$, $q(s) \in V_A(s)$, for $\lambda \le \lambda^*$ and $\sigma \ge s_0 \ge \sigma_1(A,\lambda^*)$ (σ_1 is given in Lemma 3.7). The key estimate is to show that for $s = \sigma+\lambda$ (or $s \in [\sigma,\sigma+\lambda]$ if $\sigma = s_0$), these bounds are better than those defining $V_A(s)$, provided that $\lambda \le \lambda^*(A)$. More precisely, we claim that following proposition which directly follows $i)$ of Proposition 3.4:

Proposition 3.10 (Control of $q(s)$ by $(q_0,q_1)(s)$ in $V_A(s)$). *There exist $A_4 > 1$ such that for each $A \ge A_4$, there exists $\delta_4(A) > 0$ such that for each $s_0 \ge \delta_4(A)$, we have the*

following properties:
- if (d_0, d_1) is chosen so that $(q_0, q_1)(s_0) \in \hat{V}_A(s_0)$, and
- if for all $s \in [s_0, s_1]$, $q(s) \in V_A(s)$ for some $s_1 \geq s_0$, then: for all $s \in [s_0, s_1]$,

$$|q_2(s)| < \frac{A^2}{s^{1+\nu}}, \quad \left\| \frac{q_-(y,s)}{1+|y|^3} \right\|_{L^\infty} \leq \frac{A}{2s^{3/2+\varrho}}, \quad \|q_e(s)\|_{L^\infty} \leq \frac{A^2}{2s^{\varrho}}. \tag{3.14}$$

Let us now derive the conclusion $i)$ of Proposition 3.4 from Proposition 3.10, and we then prove it later.

***Proof of** $i)$ **of Proposition 3.4**.* Indeed, if $q(s_1) \in \partial V_A(s_1)$, we see from (3.14) and the definition of $V_A(s)$ that the first two components of $q(s_1)$ must be in $\partial \hat{V}_A(s_1)$, which is the conclusion of part $i)$ of Proposition 3.4, assuming Proposition 3.10 holds. ☐

We now give the proof of Proposition 3.10 in order to conclude the proof of part $i)$ of Proposition 3.4.

***Proof of Proposition 3.10**.* Note that the conclusion of this proposition is very similar to Proposition 3.7, pages 157 in [16]. However, its proof is far from being an adaptation of the proof given in the case of the semilinear heat equation treated in [16] because of the difference of the definition of $V_A(s)$ and the presence of the strong perturbation term. In fact, the argument given in [16] does not work here to control $|q_2(s)|$ in this new situation, we use instead equation (3.13) to handle this term.

Let $\lambda_1 \geq \lambda_2$ be two positive numbers which will be fixed in term of A later. It is enough to show that (3.14) holds in two cases: $s - s_0 \leq \lambda_1$ and $s - s_0 \geq \lambda_2$. In both cases, we use Lemma 3.7 and equation and suppose $A \geq A_2 > 0$, $s_0 \geq \max\{\sigma_2(A, \lambda_1), \sigma_2(A, \lambda_2), \sigma_6(A), 1\}$.

Case $s - s_0 \leq \lambda_1$: Since we have for all $\tau \in [s_0, s]$, $q(\tau) \in V_A(\tau)$, we apply Lemma 3.7 with A and $\lambda^* = \lambda_1$, and $\lambda = s - s_0$. From (2.7) and Lemma 3.7, we have

$$|q_2(s)| \leq \frac{C}{s^2} + \frac{C\lambda_1}{s^{2+\nu}}, \qquad \left\| \frac{q_-(y,s)}{1+|y|^3} \right\|_{L^\infty} \leq \frac{C}{s^{3/2+\varrho}}, \quad \|q_e(s)\|_{L^\infty} \leq \frac{Ce^{\lambda_1}}{\sqrt{s}} + \frac{C}{s^{\varrho}}.$$

If we fix $\lambda_1 = \frac{3}{2}\log A$ and A large enough, then (3.14) satisfies.

Case $s - s_0 \geq \lambda_2$: Since we have for all $\tau \in [\sigma, s]$, $q(\tau) \in V_A(\tau)$, we apply Lemma 3.7 with A, $\lambda = \lambda^* = \lambda_2$, $\sigma = s - \lambda_2$. From (2.7) and Lemma 3.7, we have

$$\left\| \frac{q_-(y,s)}{1+|y|^3} \right\|_{L^\infty} \leq \frac{C}{s^{3/2+\varrho}} \left(1 + Ae^{-\frac{\lambda_2}{2}} + A^2 e^{-\lambda_2^2}\right),$$
$$\|q_e(s)\|_{L^\infty} \leq \frac{C}{s^{\varrho}} \left(1 + Ae^{\lambda_2} + A^2 e^{-\frac{\lambda_2}{p}}\right).$$

To obtain (3.14), except for $|q_2(s)|$ which will be treated later, it is enough to have $A \geq 4C$ and

$$C\left(Ae^{-\frac{\lambda_2}{2}} + A^2 e^{-\lambda_2^2}\right) \leq \frac{A}{4},$$
$$C\left(Ae^{\lambda_2} + A^2 e^{-\frac{\lambda_2}{p}}\right) \leq \frac{A^2}{4}.$$

If we fix $\lambda_2 = \log(A/8C)$ and take A large enough, we see that these requests are satisfied. This follows the last two estimates in (3.14).

It remains to show that if $q(s) \in V_A(s)$ for all $s \in [s_0, s_1]$ then $|q_2(s)| < \frac{A^2}{s^{1+\nu}}$ for all $s \in [s_0, s_1]$. We proceed by contradiction, assume that for all $s \in [s_0, s_*)$, $|q_2(s)| < \frac{A^2}{s^{1+\nu}}$ and $|q_2(s_*)| = \frac{A^2}{s_*^{1+\nu}}$. Considering the case $q_2(s_*) = -\frac{A^2}{s_*^{1+\nu}}$, we have

$$q_2'(s_*) \leq \frac{d}{ds}\left(\frac{-A^2}{s_*^{1+\nu}}\right) \leq \frac{(1+\nu)A^2}{s_*^{2+\nu}}. \tag{3.15}$$

On the other hand, we have from (3.13),

$$q_2'(s_*) \geq -\frac{2}{s}\, q_2(s_*) - \frac{C}{s_*^{2+\nu}} = \frac{2A^2 - C}{s_*^{2+\nu}},$$

which contradicts (3.15) if we take A large enough.
Using the same argument in the case where $q_2(s_*) = \frac{A^2}{s_*^{1+\nu}}$, we also have a contradiction. This completes the proof of Proposition 3.10 and part $i)$ of Proposition 3.4 too. □

Step 3: Deriving conclusion $ii)$ of Proposition 3.4.

We prove part $ii)$ of Proposition 3.4 here. In order to prove this, we follow the ideas of [16] to show that for each $m \in \{0, 1\}$ and each $\iota \in \{-1, 1\}$, if $q_m(s_1) = \frac{\iota A}{s_1^{3/2+\nu}}$, then $\frac{dq_m}{ds}(s_1)$ has the opposite sign of $\frac{d}{ds}\left(\frac{\iota A}{s^{3/2+\nu}}\right)(s_1)$ so that $(q_0, q_1)(s_1)$ actually leaves $\hat{V}_A$ at s_1 for $s_1 \geq s_0$ where s_0 will be large enough. Indeed, from equation (3.12), we take $A = 2C + 1$. If $\iota = 1$, then $\frac{dq_m}{ds}(s_1) > 0$ and if $\iota = -1$, then $\frac{dq_m}{ds}(s_1) < 0$. This implies that $(q_0, q_1)(s_1 + \eta) \notin \partial\hat{V}_A(s_1 + \eta)$ which yields conclusion $ii)$ of Proposition 3.4.

3.3 Local in time solution of equation (2.2)

In the following, we find a local in time solution for equation (2.2).

Proposition 3.11 (Local in time solution of equation (2.2)). *For all $A > 1$, there exists $\delta_5(A)$ such that for all $s_0 \geq \delta_5(A)$, the following holds: For all $(d_0, d_1) \in \mathcal{D}_{s_0}$, there exists $s_{max}(d_0, d_1) > s_0$ such that equation (2.2) with initial data $q(s_0)$ given in (3.2) has a unique solution satisfying $q(s) \in V_{A+1}(s)$ for all $s \in [s_0, s_{max})$.*

Proof. Using the definition (2.1) of q and the equivalent formulation (1.4), we see that the Cauchy problem of (2.2) is equivalent to the Cauchy problem of equation (1.1). Note that the initial data for (1.1) is derived the initial data for (2.2) at $s = s_0$ given in (3.2), namely

$$u_{d_0,d_1}(x) = \frac{T^{-\frac{1}{p-1}}\phi(-\log T)}{\kappa}\left\{f(z)\left(1 + \frac{d_0 + d_1 z}{p - 1 + \frac{(p-1)^2}{4p}z^2}\right)\right\},$$

where f is defined in (1.8), $T = e^{-s_0}$ and $z = \frac{x}{\sqrt{T|\log T|}}$.
This initial data is belong to $L^\infty(\mathbb{R})$ which insures the local existence of u in $L^\infty(\mathbb{R})$ (see the introduction). From part $iii)$ of Lemma 3.2, we have $q_{d_0,d_1}(s_0) \in V_A(s_0) \subseteq V_{A+1}(s_0)$. Then there exists s_{max} such that for all $s \in [s_0, s_{max})$, we have $q(s) \in V_{A+1}(s)$. This concludes the proof of Proposition 3.11. □

3.4 Deriving conclusion $ii)$ of Theorem 1.1

In this subsection, we derive conclusion $ii)$ of Theorem 1.1 using the previous subsections. Although the derivation of the conclusion is the same as in [16], we would like to give details of its proof for the reader's convenience and for explaining the two-point strategy: reduction to a finite dimensional problem and the conclusion $ii)$ of Theorem 1.1 using index theory.

***Proof of** $ii)$ **of Theorem 1.1**.* We first solve the finite-dimensional problem and show the existence of $A > 1$, $s_0 > 0$ and $(d_0, d_1) \in \mathcal{D}_{s_0}$ such that problem (2.2) with initial data at $s = s_0$, $q_{d_0,d_1}(s_0)$ is given in (3.2) has a solution $q(s)$ defined for all $s \in [s_0, +\infty)$ such that

$$q(s) \in V_A(s), \quad \forall s \in [s_0, +\infty). \tag{3.16}$$

For this purpose, let us take $A \geq A_1$ and $s_0 \geq \delta_3$, where A_1 and δ_3 are given in Proposition 3.4, we will find the parameter (d_0, d_1) in the set $\mathcal{D}_{s_0}$ defined in Lemma 3.2 such that (3.16) holds. We proceed by contradiction and assume from $iii)$ of Lemma 3.2 that for all $(d_0, d_1) \in \mathcal{D}_{s_0}$, there exists $s_*(d_0, d_1) \geq s_0$ such that $q_{d_0,d_1}(s) \in V_A(s)$ for all $s \in [s_0, s_*]$ and $q_{d_0,d_1}(s_*) \in \partial V_A(s_*)$. Applying Proposition 3.4, we see that $q_{d_0,d_1}(s_*)$ can leave $V_A(s_*)$ only by its first two components, hence,

$$(q_0, q_1)(s_*) \in \partial \hat{V}_A(s_*).$$

Therefore, we can define the following function:

$$\begin{aligned} \Phi \ &: \mathcal{D}_{s_0} \mapsto \partial([-1,1]^2) \\ (d_0, d_1) &\to \frac{s_*^{1+\nu}}{A}(q_0, q_1)(s_*). \end{aligned}$$

Since $q(y,s_0)$ is continuous in (d_0,d_1) (see Lemma 3.2), we have that $(q_0,q_1)(s)$ is continuous with respect to (d_0,d_1,s). Then using the transversality property of (q_0,q_1) on $\partial\hat{V}_A$ (part ii) of Proposition 3.4), we claim that $s_*(d_0,d_1)$ is continuous. Therefore, Φ is continuous.

If we manage to prove that Φ is of degree one on the boundary, then we have a contradiction from the degree theory. Let us prove that. From Lemma 3.2, we see that if (d_0,d_1) is on the boundary of $\mathcal{D}_{s_0}$, then

$$(q_0,q_1)(s_0)\in\partial\hat{V}_A(s_0),\quad \text{and}\quad q(s_0)\in V_A(s_0),$$

where the statement $q(s)\in V_A(s)$ holds with strict inequalities for q_2,q_- and q_e. Using again ii) of Proposition 3.4, we see that $q(s)$ can leave $V_A(s)$ at $s=s_0$, hence $s_*=s_0$. Using iii) of Lemma 3.2, we have that the restriction of Φ to the boundary is of degree 1. This gives us a contradiction (by the index theory). Thus, there exists $(d_0,d_1)\in\mathcal{D}_{s_0}$ such that for all $s\geq s_0$, $q_{d_0,d_1}(s)\in V_A(s)$, which is the conclusion (3.16).

Since $q_{d_0,d_1}(s)$ satisfies (3.16), we use the parabolic estimate in Proposition 3.3, the transformations (2.1) and (1.4) and the fact that $\frac{\phi(s)}{\kappa}=1+\mathcal{O}(s^{-a})$ with $a>0$ to derive estimate (1.9). This concludes the proof of ii) of Theorem 1.1. □

3.5 Deriving conclusion iii) of Theorem 1.1

We give the proof of part iii) of Theorem 1.1 in this subsection. We consider $u(t)$ solution of equation (1.1) which blows-up in finite-time $T>0$ at only one blow-up point $x=0$ and satisfies (1.9). Adapting the techniques used by Merle in [15] to equation (1.1) without the perturbation ($h\equiv 0$), we show the existence of a profile $u_*\in\mathcal{C}(\mathbb{R}\setminus\{0\},\mathbb{R})$ such that $u(x,t)\to u_*(x)$ as $t\to T$ uniformly on compact subsets of $\mathbb{R}\setminus\{0\}$, where u_* is given in iii) of Theorem 1.1. Note that Zaag [26], Masmoudi and Zaag [14] successfully used these techniques to equation (1.11). Since the proof is very similar to that written in [26] and [14], and no new idea is needed, we just give the key argument and kindly refer the reader to see Section 4 in [26] for details.

For each $x_0\in\mathbb{R}\setminus\{0\}$ small enough, we define for all $(\xi,\tau)\in\mathbb{R}\times\left[-\frac{t(x_0)}{T-t(x_0)},1\right)$ the following function:

$$v(x_0,\xi,\tau)=(T-t(x_0))^{\frac{1}{p-1}}u(x_0+\xi\sqrt{T-t(x_0)},t(x_0)+(T-t(x_0))\tau),\tag{3.17}$$

where $t(x_0)$ is uniquely defined by

$$|x_0|=K_0\sqrt{(T-t(x_0))|\log(T-t(x_0))|},\tag{3.18}$$

with $K_0>0$ to be fixed large enough later.

Note that v blows up at time $\tau=1$ at only one blow-up point $x_0=0$. From (1.1) and (3.17), we see that $v(x_0,\xi,\tau)$ satisfies the following equation: for all $\tau\in\left[-\frac{t(x_0)}{T-t(x_0)},1\right)$,

$$\frac{\partial v}{\partial\tau}=\Delta_\xi v+|v|^{p-1}v+(T-t(x_0))^{\frac{p}{p-1}}h\left((T-t(x_0))^{-\frac{1}{p-1}}v\right).\tag{3.19}$$

From estimate (1.9), the definition (3.17) of v and (3.18), we have the following:

$$\sup_{|\xi|<|\log(T-t(x_0))|^{\frac{\varrho}{2}}} |v(x_0,\xi,0) - f(K_0)| \leq \frac{C}{|\log(T-t(x_0))|^{\frac{\varrho}{2}}} \to 0 \quad \text{as } x_0 \to 0,$$

where f is given in (1.8).
Using the continuity with respect to initial data for equation (1.1) (also for equation (3.19)) associated to a space-localization in the ball $B(0, |\xi| < |\log(T-t(x_0))|^{\frac{\varrho}{2}})$, it is showed in Section 4 of [26] that

$$\sup_{|\xi|<|\log(T-t(x_0))|^{\frac{\varrho}{2}}, 0\leq\tau<1} |v(x_0,\xi,\tau) - \hat{f}_{K_0}(\tau)| \leq \epsilon(x_0) \to 0 \quad \text{as } x_0 \to 0,$$

where $\hat{f}_{K_0}(\tau) = \kappa(1-\tau+\frac{p-1}{4p}K_0^2)^{-\frac{1}{p-1}}$.
Then letting $\tau \to 1$ and using the definition (3.17) of v, we have

$$\begin{aligned} u_*(x_0) = \lim_{t'\to T} u(x,t') &= (T-t(x_0))^{-\frac{1}{p-1}} \lim_{\tau\to 1} v(x_0,0,\tau) \\ &\sim (T-t(x_0))^{-\frac{1}{p-1}} \hat{f}_{K_0}(1), \quad \text{as } x_0 \to 0. \end{aligned}$$

From (3.18), we have

$$(T-t(x_0))^{-\frac{1}{p-1}} \sim \left(\frac{|x_0|^2}{2K_0^2|\log x_0|}\right)^{-\frac{1}{p-1}}, \quad \text{as } x_0 \to 0.$$

Hence,

$$u_*(x_0) \sim \left(\frac{8p|\log x_0|}{(p-1)^2|x_0|^2}\right)^{\frac{1}{p-1}}, \quad \text{as } x_0 \to 0,$$

which concludes the proof of part $iii)$ of Theorem 1.1.

A Appendix A

We claim the following:

Lemma A.1. *Let $\varepsilon \in (0,p]$, there exist $C = C(a,p,\mu,M) > 0$ and $s_0 = s_0(a,\varepsilon) > 0$ such that for all $s \geq s_0$,*
$i)$ if h is given by (1.2)*,*

$$j = 0,1, \quad e^{-\frac{(p-j)s}{p-1}} \left| h^{(j)}\left(e^{\frac{s}{p-1}}w\right)\right| \leq Cs^{-a}\left(|w|^{p-j}+1\right),$$

$ii)$ if h is given by (1.3)*,*

$$\sum_{j=0}^{3} e^{-\frac{(p-j)s}{p-1}} |w|^j \left| h^{(j)}\left(e^{\frac{s}{p-1}}w\right)\right| \leq Cs^{-a}(|w|^p + |w|^{p-\varepsilon}).$$

Proof. We see that the proof directly follows from the following key estimate:

$$\frac{|w|^\varepsilon}{\log^a\left(2+e^{\frac{2s}{p-1}}w^2\right)} \le \frac{C}{s^a}(|w|^\varepsilon+1), \quad \forall s \ge s_0(a,\varepsilon). \tag{A.1}$$

Indeed, considering the case $w^2 e^{\frac{s}{p-1}} \ge 4$, we have

$$\frac{|w|^\varepsilon}{\log^a\left(2+e^{\frac{2s}{p-1}}w^2\right)} \le \frac{|w|^\varepsilon}{\log^a\left(4e^{\frac{s}{p-1}}\right)} \le \frac{C|w|^\varepsilon}{s^a},$$

then the case $w^2 e^{\frac{s}{p-1}} \le 4$ which follows that

$$\frac{|w|^\varepsilon}{\log^a\left(2+e^{\frac{2s}{p-1}}w^2\right)} \le \frac{|w|^\varepsilon}{\log^a(2)} \le Ce^{-\frac{\varepsilon s}{p-1}} \le Cs^{-a}.$$

This concludes the proof of (A.1) and the proof of Lemma A.1 also. □

The following lemma shows the existence of solutions of the associated ODE of equation (1.5):

Lemma A.2. *Let ϕ be a positive solution of the following ordinary differential equation:*

$$\phi_s = -\frac{\phi}{p-1} + \phi^p + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}}\phi\right). \tag{A.2}$$

Then $\phi(s) \to \kappa$ as $s \to +\infty$ and $\phi(s)$ is given by

$$\phi(s) = \kappa(1+\eta_a(s))^{-\frac{1}{p-1}}, \quad \text{where} \quad \eta_a(s) = \mathcal{O}\left(\frac{1}{s^a}\right). \tag{A.3}$$

If $h(x) = \mu \frac{|x|^{p-1}x}{\log^a(2+x^2)}$, then

$$\eta_a(s) \sim C_0 \int_s^{+\infty} \frac{e^{s-\tau}}{\tau^a} d\tau = \frac{C_0}{s^a}\left(1+\sum_{j\ge 1}\frac{b_j}{s^j}\right),$$

where $C_0 = \mu\left(\frac{p-1}{2}\right)^a$ and $b_j = (-1)^j \prod_{i=0}^{j-1}(a+i)$.

Proof. See Lemma A.3 in [21]. □

B Proof of Lemma 3.5

In this appendix, we give the proof of Lemma 3.5. The proof follows from the techniques of Bricmont and Kupiainen [3] with some additional care, since we have a different profile function φ defined in (2.1), and since we give the explicit dependence of the bounds in terms of all the components of initial data. As mentioned early, the proof relies mainly on the understanding of the behavior of the kernel $\mathcal{K}(s,\sigma,y,x)$ (see (2.8)). This behavior follows from a perturbation method around $e^{(s-\sigma)\mathcal{L}}(y,s)$, where the kernel of $e^{t\mathcal{L}}$ is given by Mehler's formula:

$$e^{t\mathcal{L}}(y,x) = \frac{e^t}{\sqrt{4\pi(1-e^{-t})}} exp\left[-\frac{(ye^{-t/2}-x)^2}{4(1-e^{-t})}\right]. \tag{B.1}$$

By definition (2.8) of $\mathcal{K}$, we use a Feynman-Kac representation for $\mathcal{K}$:

$$\mathcal{K}(s,\sigma,y,x) = e^{(s-\sigma)\mathcal{L}}(y,x)\int d\mu_{yx}^{s-\sigma}(\omega)e^{\int_0^{s-\sigma}V(\omega(\tau),\sigma+\tau)d\tau}, \tag{B.2}$$

where $d\mu_{yx}^{s-\sigma}$ is the oscillator measure on the continuous paths $\omega : [0, s-\sigma] \to \mathbb{R}$ with $\omega(0) = x$, $\omega(s-\sigma) = y$, i.e. the Gaussian probability measure with covariance kernel

$$\begin{aligned}\Gamma(\tau,\tau') &= \omega_0(\tau)\omega_0(\tau')\\ &+ 2\left(e^{-\frac{1}{2}|\tau-\tau'|} - e^{-\frac{1}{2}|\tau+\tau'|} + e^{-\frac{1}{2}|2(s-\sigma)+\tau-\tau'|} - e^{-\frac{1}{2}|2(s-\sigma)-\tau-\tau'|}\right),\end{aligned}$$

which yields $\int d\mu_{yx}^{s-\sigma}\omega(\tau) = \omega_0(\tau)$, with

$$\omega_0(\tau) = (\sinh((s-\sigma)/2))^{-1}\left(y\sinh(\frac{\tau}{2}) + x\sinh(\frac{s-\sigma-\tau}{2})\right).$$

In view of (B.2), we can consider the expression for $\mathcal{K}$ as a perturbation of $e^{(s-\sigma)\mathcal{L}}$. Since our profile φ defined in (2.1) is different from the one defined in [3], we have here a potential V defined in (2.3) which is different as well. Thus, we first estimate the potential V, then we restate some basic properties of the kernel $\mathcal{K}$.

Lemma B.1 (**Estimates on the potential** V)**.** *For s large enough, we have*
a) $V(y,s) \le \frac{C}{s^{a'}}$ *with* $a' = \min\{a,1\}$.
b) $\left|\frac{d^m V(y,s)}{dy^m}\right| \le \frac{C}{s^{m/2}}$ *for* $m = 0,1,2$.
c) $|V(y,s)| \le \frac{C}{s}(1+|y|^2)$, $\quad V(y,s) = -\frac{h_2(y)}{4s} + \tilde{V}(y,s)$,
where $|\tilde{V}(y,s)| = \mathcal{O}\left(\frac{1+|y|^4}{s^2}\right) + \mathcal{O}\left(\frac{1}{s^a}\right)$ *in the case* (1.2) *and* $|\tilde{V}(y,s)| = \mathcal{O}\left(\frac{1+|y|^4}{s^2}\right) + \mathcal{O}\left(\frac{1+|y|^2}{s^{a+1}}\right)$ *in the case* (1.3). *In particular, in both cases* $|\tilde{V}(y,s)| \le \frac{C(1+|y|^4)}{s^{1+\bar{a}}}$, *where* $\bar{a} = \min\{a-1,1\}$ *in the case* (1.2) *and* $\bar{a} = \min\{a,1\}$ *in the case* (1.3).

Proof. $a)$ From the definition (2.3) of V, we see that

$$V(y,s) \le p(\varphi(0,s)^{p-1} - \kappa^{p-1}) + \imath \left| e^{-s} h'\left(e^{\frac{s}{p-1}} \varphi \right) \right|,$$

where $\imath$ is defined in (2.3). From Lemma A.2, we have

$$\varphi(0,s)^{p-1} - \kappa^{p-1} = \kappa^{p-1} \left[(1+\eta_a(s))^{-1} \left(1 + \frac{1}{2ps} \right)^{p-1} - 1 \right] \le \frac{C}{s^{a'}}.$$

Since $|\varphi|$ is bounded, Lemma A.1 yields $\imath \left| e^{-s} h'\left(e^{\frac{s}{p-1}} \varphi \right) \right| \le \frac{\imath C}{s^a}$. This concludes part $a)$.
$b)$ We introduce $W(z,s) = V(y,s)$ with $z = \frac{y}{\sqrt{s}}$. In order to derive part $b)$, it is enough to show that $|\frac{d^m W}{dz^m}| \le C$ for $m = 0,1,2$, which follows easily from Lemma A.1 and the following key estimates

$$\frac{\partial f(z)}{\partial z} = -\frac{z f(z)}{2p(1+c_p z^2)},$$

where f and c_p are defined in (1.8).
$c)$ Since $|V(y,s)| \le C$ for all $y \in \mathbb{R}$ and $s \ge 1$, considering the cases $|y| \le \sqrt{s}$, then $|y| \ge \sqrt{s}$, we directly see that the first estimate follows from the second. Hence, we only prove the second. To do so, we introduce $\tilde{W}(Z,s) = V(y,s)$ with $Z = \frac{|y|^2}{s}$. By the definition (2.1) and by a direct calculation, we find that

$$\begin{aligned}\frac{d^2\tilde{W}(Z,s)}{dZ^2} &= p(p-1)(p-2)\varphi^{p-3}(Z,s)\left(\frac{d\varphi(Z,s)}{dZ}\right)^2 \\ &+ \imath e^{-\frac{(p-3)s}{p-1}} h'''\left(e^{\frac{s}{p-1}}\varphi(Z,s)\right)\left(\frac{d\varphi(Z,s)}{dZ}\right)^2 \\ &+ \left[p(p-1)\varphi^{p-2}(Z,s) + \imath e^{-\frac{(p-2)s}{p-1}} h''\left(e^{\frac{s}{p-1}}\varphi(Z,s)\right)\right]\frac{d^2\varphi(Z,s)}{dZ^2}.\end{aligned}$$

Applying Lemma A.1 with $\varepsilon = \frac{p-1}{2}$, we see that

$$\begin{aligned}\left|\frac{d^2\tilde{W}(Z,s)}{dZ^2}\right| &\le C\left(\varphi^{p-3}(Z,s) + \imath\varphi^{p-3-\frac{p-1}{2}}(Z,s)\right)\left(\frac{d\varphi(Z,s)}{dZ}\right)^2 \\ &+ C\left(\varphi^{p-2}(Z,s) + \imath\varphi^{p-2-\frac{p-1}{2}}(Z,s)\right)\left|\frac{d^2\varphi(Z,s)}{dZ^2}\right|, \quad \forall s \ge s_0.\end{aligned}$$

From the definition (2.1) of φ, we note that $\frac{d\varphi}{dZ} = -\frac{c_p\phi}{\kappa}F^p(Z)$, where $c_p = \frac{p-1}{4p}$ and $F(Z) = \kappa(1+c_pZ)^{-\frac{1}{p-1}}$, we derive

$$\varphi(Z,s)^{p-3-\frac{p-1}{2}}\left(\frac{d\varphi(Z,s)}{dZ}\right)^2 \le C\left(F + \frac{\kappa}{2ps}\right)^{p-3-\frac{p-1}{2}} F^{2p} \le 2C,$$

and

$$\varphi(Z,s)^{p-2-\frac{p-1}{2}}\left|\frac{d^2\varphi(Z,s)}{dZ^2}\right| \le C\left(F+\frac{\kappa}{2ps}\right)^{p-2-\frac{p-1}{2}} F^{2p} \le 2C.$$

Hence, $\left|\frac{d^2\tilde{W}(Z,s)}{dZ^2}\right|$ is bounded for all $Z \in [0,+\infty)$ and for all $s \ge s_0$. Then by a Taylor expansion, we have

$$\left|\tilde{W}(Z,s) - \tilde{W}(0,s) - Z\frac{\partial \tilde{W}(0,s)}{\partial Z}\right| \le CZ^2, \quad \forall Z \in [0,+\infty), \ \forall s \ge s_0.$$

From the definition (2.1) of φ and from Lemma A.2, we have

$$\begin{aligned} W(0,s) &= \frac{p}{p-1}\left[\frac{1}{(1+\eta_a)}\left(1+\frac{1}{2ps}\right)^{p-1} - 1\right] + \imath e^{-s}h'\left(e^{\frac{s}{p-1}}(\phi+\frac{\phi}{2ps})\right) \\ &= \frac{1}{2s} - \frac{p}{p-1}\left(\frac{\eta_a(s)}{1+\eta_a(s)}\right) + \imath e^{-s}h'\left(e^{\frac{s}{p-1}}\phi\right) + \mathcal{O}\left(\frac{1}{s^{a+1}}\right) + \mathcal{O}\left(\frac{1}{s^2}\right). \end{aligned}$$

Recalling from Lemma A.2 that $\eta_a(s) = \mathcal{O}\left(\frac{1}{s^a}\right)$, this immediately yields $W(0,s) = \frac{1}{2s} + \mathcal{O}\left(\frac{1}{s^a}\right) + \mathcal{O}\left(\frac{1}{s^2}\right)$ in the case (1.2). In the case (1.3), we obtain by a direct calculation,

$$\begin{aligned} &\left|-\frac{p}{p-1}\left(\frac{\eta_a(s)}{1+\eta_a(s)}\right) + \imath e^{-s}h'\left(e^{\frac{s}{p-1}}\phi\right)\right| \\ &\qquad = \left|-\frac{p}{(p-1)(1+\eta_a(s))}\left(\eta_a(s) - \frac{C_0}{s^a}\right)\right| + \mathcal{O}\left(\frac{1}{s^{a+1}}\right) = \mathcal{O}\left(\frac{1}{s^{a+1}}\right). \end{aligned}$$

In the last estimate, we used that fact that $\eta_a(s) = \frac{C_0}{s^a} + \mathcal{O}\left(\frac{1}{s^{a+1}}\right)$ in the case (1.3) (see Lemma A.2). Hence, $W(0,s) = \frac{1}{2s} + \mathcal{O}\left(\frac{1}{s^{a+1}}\right) + \mathcal{O}\left(\frac{1}{s^2}\right)$ in the case (1.3).
For $\frac{\partial W(0,s)}{\partial Z}$, we use Lemmas A.1 and A.2 to derive

$$\begin{aligned} \frac{\partial W(0,s)}{\partial Z} &= -\frac{1}{4(1+\eta_a(s))}\left(1+\frac{1}{2ps}\right)^{p-2} - \imath e^{-\frac{(p-2)s}{p-1}}\frac{\phi}{4p}h''\left(e^{\frac{s}{p-1}}(\phi+\frac{\phi}{2ps})\right) \\ &= -\frac{1}{4} + \mathcal{O}\left(\frac{1}{s^a}\right) + \mathcal{O}\left(\frac{1}{s}\right). \end{aligned}$$

Returning to V, we conclude part c). This ends the proof of Lemma B.1. ☐

In what follows, we denote $\int f(y)g(y)\rho(y)dy$ by $\langle f,g\rangle$ and write $\chi(y,s) = \chi(s)$ (χ is defined in (2.12)). Let us now recall some basic properties of the kernel $\mathcal{K}$ stated in [3]:

Lemma B.2 (Bricmont and Kupiainen [3]). *For all $s \ge \sigma \ge \max\{s_0, 1\}$ with $s \le 2\sigma$ and s_0 given in Lemma A.1, for all $(y,x) \in \mathbb{R}^2$,*
a) $|\mathcal{K}(s,\sigma,y,x)| \le Ce^{(s-\sigma)\mathcal{L}}(y,x)$.
b) $\mathcal{K}(s,\sigma,y,x) = e^{(s-\sigma)\mathcal{L}}(y,x)\,(1+P_2(y,x)+P_4(y,x))$, where

$$|P_2(y,x)| \le \frac{C(s-\sigma)}{s}(1+|y|+|x|)^2,$$

$$\textit{and} \ \ |P_4(y,x)| \leq \frac{C(s-\sigma)(1+s-\sigma)}{s^2}(1+|y|+|x|)^4.$$

Moreover, $\left|\left\langle k_2, \left(\mathcal{K}(s,\sigma) - \left(\frac{\sigma}{s}\right)^2\right) h_2\right\rangle\right| \leq \frac{C(s-\sigma)(1+s-\sigma)}{s^{1+\bar{a}}}$*, with* $\bar{a} = \min\{a-1, 1\}$ *in the case* (1.2) *and* $\bar{a} = \min\{a, 1\}$ *in the case* (1.3).
$c)$ $\|\mathcal{K}(s,\sigma)(1-\chi)\|_{L^\infty} \leq Ce^{-\frac{(s-\sigma)}{p}}$.

Proof. $a)$ From part $a)$ of Lemma B.1 and the definition (B.2) of $\mathcal{K}$, we have

$$\begin{aligned} |\mathcal{K}(s,\sigma,y,x)| &\leq e^{(s-\sigma)\mathcal{L}}(y,x) \int d\mu_{yx}^{s-\sigma}(\omega) e^{\int_0^{s-\sigma} C(\sigma+\tau)^{-a'} d\tau} \\ &\leq Ce^{(s-\sigma)\mathcal{L}}(y,x) \int d\mu_{yx}^{s-\sigma}(\omega) \leq Ce^{(s-\sigma)\mathcal{L}}(y,x), \end{aligned}$$

since $s \leq 2\sigma$ and $d\mu_{yx}^{s-\sigma}$ is a probability.
$b)$ The proof is exactly the same as the corresponding one written in [3]. Although there is the difference of $\tilde{V}(y,s)$ given in part $c)$ of Lemma B.1, this change does not affect the argument given in [3]. For that reason, we refer the reader to Lemma 5, page 555 in [3] for details of the proof.
$c)$ Our potential V given in (2.3) has the same behavior as the potential in [3] for $\frac{|y|^2}{s}$ and s large (see (2.11)). For that reason, we refer to Lemma, page 559 in [3] for its proof. □

Before going to the proof of Lemma 3.5, we would like to state some basic estimates which will be used frequently in the proof.

Lemma B.3. *For K large enough, we have the following estimates:*
$a)$ *For any polynomial P,*

$$\int P(y)\mathbf{1}_{\{|y|\geq K\sqrt{s}\}}\rho(y)dy \leq C(P)e^{-s}. \tag{B.3}$$

$b)$ *Let $p \geq 0$ and $|f(x)| \leq (1+|x|)^p$, then*

$$|(e^{t\mathcal{L}}f)(y)| \leq Ce^t(1+e^{-t/2}|y|)^p, \tag{B.4}$$

Proof. $i)$ follows from a direct calculation. $ii)$ follows from the explicit expression (B.1) by a simple change of variable. □

Let us now give the proof of Lemma 3.5.

Proof of Lemma 3.5. We consider $\lambda > 0$, let $\sigma_0 \geq \lambda$, $\sigma \geq \sigma_0$ and $\psi(\sigma)$ satisfying (3.5). We want to estimate some components of $\theta(y,s) = \mathcal{K}(s,\sigma)\psi(\sigma)$ for each $s \in [\sigma, \sigma+\lambda]$. Since $\sigma \geq \sigma_0 \geq \lambda$, we have

$$\sigma \leq s \leq 2\sigma. \tag{B.5}$$

Therefore, up to a multiplying constant, any power of any $\tau \in [\sigma, s]$ will be bounded systematically by the same power of s.
$a)$ **Estimate of** θ_2: We first write

$$\begin{aligned}\theta_2(s) &= \langle k_2, \chi(s)\mathcal{K}(s,\sigma)\psi(\sigma)\rangle \\ &= \sigma^2 s^{-2}\psi_2(\sigma) + \left\langle k_2, (\chi(s)-\chi(\sigma))\sigma^2 s^{-2}\psi(\sigma)\right\rangle \\ &\quad + \left\langle k_2, \chi(s)(\mathcal{K}(s,\sigma) - \sigma^2 s^{-2})\psi(\sigma)\right\rangle := \sigma^2 s^{-2}\psi_2(\sigma) + Ib + IIb.\end{aligned}$$

To bound Ib, we write $\psi(x,\sigma) = \sum_{l=0}^{2}\psi_l(\sigma)h_l(x) + \frac{\psi_-(x,\sigma)}{1+|x|^3}(1+|x|^3) + \psi_e(x,\sigma)$ and use (B.3) to derive

$$|Ib| \le C(s-\sigma)e^{-s}\sigma^2 s^{-2}\left(\sum_{l=0}^{2}|\psi_l(\sigma)| + \left\|\frac{\psi_-(x,\sigma)}{1+|x|^3}\right\|_{L^\infty} + \|\psi_e(\sigma)\|_{L^\infty}\right).$$

For IIb, we write

$$\begin{aligned}IIb &= \sum_{l=0}^{2}\left\langle k_2, \chi(s)(\mathcal{K}(s,\sigma) - \sigma^2 s^{-2})h_l\right\rangle \psi_l(\sigma) \\ &\quad + \left\langle k_2, \chi(s)(\mathcal{K}(s,\sigma) - \sigma^2 s^{-2})\psi_-(\sigma)\right\rangle \\ &\quad + \left\langle k_2, \chi(s)(\mathcal{K}(s,\sigma) - \sigma^2 s^{-2})\psi_e(\sigma)\right\rangle := IIb.1 + IIb.2 + IIb.3.\end{aligned}$$

Let us bound $IIb.1$. For $l=2$, we already get from part $b)$ of Lemma B.2 and (B.3) that

$$\left|\left\langle k_2, \chi(s)(\mathcal{K}(s,\sigma) - \sigma^2 s^{-2})h_2\right\rangle \psi_2(\sigma)\right| \le \frac{C(s-\sigma)(1+s-\sigma)}{s^{1+\bar{a}}}|\psi_2(\sigma)|,$$

with $\bar{a} > 0$.
For $l = 0$ or 1, we use $b)$ of Lemma B.2, (B.4), (B.3) and the fact that $\langle k_2, h_l\rangle = 0$ and $e^{(s-\sigma)\mathcal{L}}h_l = e^{(1-l/2)(s-\sigma)}h_l$ to find that

$$\begin{aligned}\left|\left\langle k_2, \chi(s)(\mathcal{K}(s,\sigma) - \sigma^2 s^{-2})h_l\right\rangle \psi_l(\sigma)\right| &\le \left|\left\langle k_2, \chi(s)\left(\mathcal{K}(s,\sigma) - e^{(s-\sigma)\mathcal{L}}\right)h_l\right\rangle\right| |\psi_l(\sigma)| \\ &\quad + \left|\left\langle k_2, \chi(s)\left(e^{(s-\sigma)\mathcal{L}} - \sigma^2 s^{-2}\right)h_l\right\rangle\right| |\psi_l(\sigma)| \\ &\le C(s-\sigma)\left(s^{-1} + e^{-s}\right)|\psi_l(\sigma)| \\ &\le \frac{C(s-\sigma)}{s}|\psi_l(\sigma)|.\end{aligned}$$

This yields

$$|IIb.1| \le \frac{C(s-\sigma)}{s}\sum_{l=0}^{2}|\psi_l(\sigma)|.$$

If we write $\psi_-(x,\sigma) = \frac{\psi_-(x,\sigma)}{1+|x|^3}(1+|x|^3)$ and use the same arguments as for $l=0$, we obtain

$$|IIb.2| \le \frac{C(s-\sigma)}{s}\left\|\frac{\psi_-(x,\sigma)}{1+|x|^3}\right\|_{L^\infty}.$$

For $IIb.3$, we write

$$IIb.3 = \left\langle k_2, \chi(s)\left(\mathcal{K}(s,\sigma) - e^{(s-\sigma)\mathcal{L}}\right)\psi_e(\sigma)\right\rangle$$
$$+ \left\langle k_2, \chi(s)\left(e^{(s-\sigma)\mathcal{L}} - 1\right)\psi_e(\sigma)\right\rangle + \left\langle k_2, \chi(s)(1-\sigma^2 s^{-2})\psi_e(\sigma)\right\rangle.$$

Using (B.3), we bound the last term by $C(s-\sigma)e^{-\sigma}\|\psi_e(\sigma)\|_{L^\infty} \le C(s-\sigma)e^{-s/2}\|\psi_e(\sigma)\|_{L^\infty}$ from (B.5). For the second term, we write $e^{(s-\sigma)\mathcal{L}} - 1 = \int_0^{s-\sigma} d\tau \mathcal{L} e^{\tau\mathcal{L}}$ and use the fact that

$$\sup_{|y|\le 2K\sqrt{s}, |x|\ge K\sqrt{\sigma}} e^{-\frac{|y|^2}{4} - \frac{(ye^{-\tau/2}-x)^2}{4(1-e^{-\tau})}} \le e^{-2s}, \tag{B.6}$$

for K large enough, then it is also bounded by $C(s-\sigma)e^{-s}\|\psi_e(\sigma)\|_{L^\infty}$. For the first term, we use $b)$ of Lemma B.2, (B.4) and again (B.6) to bound it by $C(s-\sigma)s^{-1}e^{-s}\|\psi_e(\sigma)\|_{L^\infty}$. This yields

$$|IIb.3| \le C(s-\sigma)e^{-s/2}\|\psi_e(\sigma)\|_{L^\infty}.$$

Collecting all these bounds yields the bound for $\theta_2(s)$ as stated in (3.6).

$b)$ **Estimate of θ_-:** By definition,

$$\theta_-(y,s) = P_-\left[\chi(s)\mathcal{K}(s,\sigma)\psi(\sigma)\right] = \sum_{l=0}^{2}\psi_l(\sigma)P_-\left[\chi(s)\mathcal{K}(s,\sigma)h_l\right]$$
$$+ P_-\left[\chi(s)\mathcal{K}(s,\sigma)\psi_-(\sigma)\right] + P_-\left[\chi(s)\mathcal{K}(s,\sigma)\psi_e(\sigma)\right] := Ic + IIc + IIIc.$$

In order to bound Ic, we write $\mathcal{K}(s,\sigma) = \mathcal{K}(s,\sigma) - e^{(s-\sigma)\mathcal{L}} + e^{(s-\sigma)\mathcal{L}}$, then we use the fact that $e^{(s-\sigma)\mathcal{L}}h_l = e^{(1-l/2)(s-\sigma)}h_l$, part $b)$ of Lemma B.2 and (B.4) to derive for $l = 0,1,2$,

$$\left|\left(\mathcal{K}(s,\sigma) - e^{(s-\sigma)(1-l/2)}\right)h_l\right| = \left|e^{(s-\sigma)\mathcal{L}}\left(P_2 + P_4\right)h_l\right|$$
$$\le \frac{Ce^{s-\sigma}(s-\sigma)}{s}\left(1 + e^{-(s-\sigma)/2}|y|\right)^{2+l}$$
$$+ \frac{Ce^{s-\sigma}(s-\sigma)(1+s-\sigma)}{s^2}\left(1 + e^{-(s-\sigma)/2}|y|\right)^{4+l}.$$

On the support of $\chi(s)$, namely when $|y| \le 2K\sqrt{s}$, we can bound $s^{-k/2}|y|^k$ by C for $k \in \mathbb{N}$. Then, from the easy-to-check fact that

$$\text{if } |f(y)| \le m(1+|y|^3), \text{ then } P_-\left[f(y)\right] \le Cm(1+|y|^3), \tag{B.7}$$

we obtain

$$l = 0,1, \quad P_-\left[\psi_l(\sigma)\chi(s)\mathcal{K}(s,\sigma)h_l - \psi_l(\sigma)e^{(s-\sigma)(1-l/2)}(\chi(s)h_l)\right]$$
$$\le \frac{Ce^{s-\sigma}(s-\sigma)(1+s-\sigma)}{s}(1+|y|^3)|\psi_l(\sigma)|,$$

and

$$P_-\left[\psi_2(\sigma)\chi(s)\mathcal{K}(s,\sigma)h_2 - \psi_2(\sigma)e^{(s-\sigma)(1-l/2)}(\chi(s)h_2)\right] \\ \leq \frac{Ce^{s-\sigma}(s-\sigma)(1+s-\sigma)}{\sqrt{s}}(1+|y|^3)|\psi_2(\sigma)|.$$

Since $P_-(h_l)=0$ and $|(1-\chi(y,s))h_l(y)| \leq Cs^{-3/2+l/2}(1+|y|^3)$, we have

$$l=0,1,2,\quad \left|\psi_l(\sigma)e^{(s-\sigma)(1-l/2)}P_-\left[\chi(s)h_l(y)\right]\right| \leq \frac{Ce^{s-\sigma}}{s^{3/2-l/2}}|\psi_l(\sigma)|(1+|y|^3).$$

Hence,

$$|Ic| \leq \frac{Ce^{s-\sigma}\left((s-\sigma)^2+1\right)}{s}\left(|\psi_0(\sigma)|+|\psi_1(\sigma)|+\sqrt{s}|\psi_2(\sigma)|\right)(1+|y|^3).$$

To bound $IIIc$, we use $a)$ of Lemma B.2 and the definition (B.1) of $e^{(s-\sigma)\mathcal{L}}$ to write

$$\begin{aligned}\left\|\frac{\chi(y,s)\mathcal{K}(s,\sigma)\psi_e(x,\sigma)}{1+|y|^3}\right\|_{L^\infty} &\leq Ce^{s-\sigma}\|\psi_e(\sigma)\|_{L^\infty} \\ &\qquad \sup_{|y|\leq 2K\sqrt{s},|x|\geq K\sqrt{\sigma}} e^{-\frac{1}{2}\frac{(ye^{-(s-\sigma)/2}-x)^2}{4(1-e^{-(s-\sigma)})}}(1+|y|^3)^{-1} \\ &\leq \begin{cases} Cs^{-3/2}\|\psi_e(\sigma)\|_{L^\infty} & \text{if } s-\sigma\leq s_* \\ Ce^{-s}\|\psi_e(\sigma)\|_{L^\infty} & \text{if } s-\sigma\geq s_*\end{cases}\end{aligned}$$

for a suitable constant s_*.
Exploiting again (B.7), we obtain the bound on this term which can be written as

$$|IIIc| \leq Cs^{-3/2}e^{-(s-\sigma)^2}\|\psi_e(\sigma)\|_{L^\infty}(1+|y|^3) \text{ for } \sigma \text{ large enough}.$$

We still have to consider IIc. In order to bound this term, we proceed as in [3]. We write

$$\mathcal{K}(s,\sigma)\psi_-(\sigma) = \int dx e^{x^2/4}\mathcal{K}(s,\sigma)(\cdot,x)f(x) = \int dx N(\cdot,x)E(\cdot,x)f(x), \tag{B.8}$$

where $f(x)=e^{-x^2/4}\psi_-(x,\sigma)$ and

$$N(y,x) = \frac{e^{s-\sigma}e^{x^2/4}}{\sqrt{4\pi(1-e^{-(s-\sigma)})}}e^{-\frac{(ye^{-(s-\sigma)/2}-x)^2}{4(1-e^{-(s-\sigma)})}},$$

$$E(y,x) = \int d\mu_{yx}^{s-\sigma}(\omega)e^{\int_0^{s-\sigma}V(\omega(\tau),\sigma+\tau)d\tau}.$$

Let $f^0=f$ and for $m\geq 1$, $f^{(-m-1)}(y)=\int_{-\infty}^y dx f^{(-m)}(x)$, then we have the following:

Lemma B.4. $|f^{(-m)}(y)| \leq C \left\| \frac{\psi_-(x,\sigma)}{1+|x|^3} \right\|_{L^\infty} (1+|y|)^{(3-m)} e^{-y^2/4}$ *for* $m \leq 3$.

Proof. See Lemma 6, page 557 in [3]. □

We now rewrite (B.8) by integrating by parts as follows:

$$\mathcal{K}(s,\sigma)\psi_-(\sigma) = \sum_{l=0}^{2} (-1)^{l+1} \int dx \partial_x^l N(y,x) \partial_x E(y,x) f^{(-l-1)}(x) + \int dx \partial_x^3 N(y,x) E(y,x) f^{-3}(x). \quad \text{(B.9)}$$

From the definition of $N(y,x)$, we have for $l = 0, 1, 2, 3$,

$$|\partial_x^l N(y,x)| \leq C e^{-l(s-\sigma)/2} (1+|y|+|x|)^l e^{x^2/4} e^{(s-\sigma)\mathcal{L}}(y,x).$$

Now using the integration by parts formula for Gaussian measures to write

$$\partial_x E(y,x) = \frac{1}{2} \int_0^{s-\sigma} \int_0^{s-\sigma} d\tau d\tau' \partial_x \Gamma(\tau,\tau') \int d\mu_{yx}^{s-\sigma}(\omega) V'(\omega(\tau), \sigma+\tau) V'(\omega(\tau'), \sigma+\tau') e^{\int_0^{s-\sigma} d\tau'' V(\omega(\tau''),\sigma+\tau'')} + \frac{1}{2} \int_0^{s-\sigma} d\tau \partial_x \Gamma(\tau,\tau) \int d\mu_{yx}^{s-\sigma}(\omega) V''(\omega(\tau), \sigma+\tau) e^{\int_0^{s-\sigma} d\tau'' V(\omega(\tau''),\sigma+\tau'')}.$$

Recalling from Lemma B.1 that $V(y,s) \leq \frac{C}{s^{a'}}$ with $a' > 0$ and $\left|\frac{d^m V(y,s)}{dy^m}\right| \leq \frac{C}{s^{m/2}}$ for $m = 0, 1, 2$. Since $s \leq 2\sigma$, this yields $\int_0^{s-\sigma} V(\omega(\tau), \sigma+\tau) d\tau \leq C$. Because $d\mu_{yx}^{s-\sigma}$ is a probability, we then obtain

$$|E(y,x)| \leq C \quad \text{and} \quad |\partial_x E(y,x)| \leq \frac{C}{s}(s-\sigma)(1+s-\sigma)(|y|+|x|).$$

Substituting all these bounds into (B.9), then using (B.4), Lemma B.4, the fact that $s^{-1}(s-\sigma)(1+s-\sigma) \leq e^{-3/2(s-\sigma)}$ for s large and then (B.7), we derive

$$|IIc| \leq C e^{-(s-\sigma)/2} \left\| \frac{\psi_-(x,\sigma)}{1+|x|^3} \right\|_{L^\infty} (1+|y|^3).$$

Collecting all the bounds for Ic, IIc and $IIIc$, we obtain the bound (3.7).
$c)$ **Estimate for** θ_e: By definition, we write

$$\theta_e(y,s) = (1-\chi(y,s))\mathcal{K}(s,\sigma)\psi(\sigma) = (1-\chi(y,s))\mathcal{K}(s,\sigma)\left(\psi_b(\sigma) + \psi_e(\sigma)\right).$$

Using $c)$ of Lemma B.2, we have

$$\|(1-\chi(y,s))\mathcal{K}(s,\sigma)\psi_e(\sigma)\|_{L^\infty} \leq C e^{-(s-\sigma)/p} \|\psi_e(\sigma)\|_{L^\infty}.$$

It remains to bound $(1-\chi(y,s))\mathcal{K}(s,\sigma)\psi_b(\sigma)$. To this end, we write

$$\psi_b(x,\sigma) = \sum_{l=0}^{2} \psi_l(\sigma)h_l(x) + \frac{\psi_-(x,\sigma)}{1+|x|^3}(1+|x|^3),$$

then we use $\chi(x,\sigma)|x|^k \leq C\sigma^{k/2} \leq Cs^{k/2}$ for $k \in \mathbb{N}$, and $a)$ of Lemma B.2 to derive

$$\|(1-\chi(y,s))\mathcal{K}(s,\sigma)\psi_b(x,\sigma)\|_{L^\infty} \leq Ce^{s-\sigma}\sum_{l=0}^{2} s^{l/2}|\psi_l(\sigma)| + Ce^{s-\sigma}s^{3/2}\left\|\frac{\psi_-(x,\sigma)}{1+|x|^3}\right\|_{L^\infty}.$$

This yields the bound (3.8) and concludes the proof of Lemma 3.5. □

C Proof of Lemma 3.8

We give the proof of Lemma 3.8 here.

Proof of Lemma 3.8. $i)$ From the definition (2.4) of B, we use a Taylor expansion and the boundedness of $|\varphi|$ and $|q|$ to find that

$$|\chi(\tau)B(q(\tau))| \leq C|q(\tau)|^2 \quad \text{and} \quad |B(q(\tau))| \leq C|q(\tau)|^{p'}, \tag{C.1}$$

where $p' = \min\{2,p\}$.
(Since we have the same definition of B as in [16], we do not give the proof of (C.1) and kindly refer the reader to Lemma 3.15, page 168 of [16] for its proof.)
Using (C.1) and (3.1), we have

$$|\chi(\tau)B(q(\tau))| \leq \frac{CA^4}{\tau^{3+2\varrho}}(1+|y|^6) + \frac{CA^4}{\tau^{2+2\nu}}(1+|y|^4). \tag{C.2}$$

From (C.2), we then derive for $m = 0,1,2$,

$$|B_m(\tau)| = \left|\int \chi(\tau)B(q(\tau))k_m\rho dy\right| \leq \frac{CA^4}{\tau^{2+2\nu}}. \tag{C.3}$$

Since $B_-(y,\tau) = \chi(\tau)B(q(\tau)) - \sum_{m=0}^{2} B_m(\tau)h_m(y)$, we have from (C.2) and (C.3),

$$\left|\frac{B_-(y,\tau)}{1+|y|^3}\right| \leq \left|\frac{\chi(\tau)B(q(\tau))}{1+|y|^3}\right| + \left|\frac{\sum_{m=0}^{2} B_m(\tau)h_m(y)}{1+|y|^3}\right| \leq \chi(\tau)\left[\frac{CA^4}{\tau^{3+2\varrho}}(1+|y|^3) + \frac{CA^4}{\tau^{2+2\nu}}(1+|y|)\right] + \frac{CA^4}{\tau^{2+2\nu}}\left(\frac{|\sum_{m=0}^{2} h_m(y)|}{1+|y|^3}\right)$$

If we use $|y|^l\chi(y,\tau) \le C\tau^{l/2}$ for $l \in \mathbb{N}$, and $|\sum_{m=0}^{2} h_m(y)| \le C(1+|y|^2)$, then we obtain

$$\left\|\frac{B_-(y,\tau)}{1+|y|^3}\right\|_{L^\infty} \le \frac{CA^4}{\tau^{3/2+2\varrho}}.$$

Using the second estimates in (C.1) and (3.1), we obviously obtain $\|B(\tau)\|_{L^\infty} \le \frac{CA^{2p'}}{\tau^{\varrho p'}}$ which yields $\|B_e(\tau)\|_{L^\infty} \le \frac{CA^{2p'}}{\tau^{\varrho p'}}$. This ends the proof of part $i)$.

$ii)$ From the definition (2.5) of R, we write $\varphi(y,\tau) = \frac{\phi(\tau)}{\kappa}\vartheta(y,\tau)$ and $R(y,\tau) = \frac{\phi(\tau)}{\kappa}Q + G$, where $\vartheta(y,\tau) = f(\frac{y}{\sqrt{\tau}}) + \frac{\kappa}{2p\tau}$ and

$$Q(y,\tau) = -\vartheta_\tau + \Delta\vartheta - \frac{y}{2}\nabla\vartheta - \frac{\vartheta}{p-1} + \vartheta^p, \tag{C.4}$$

$$G(y,\tau) = -\frac{\phi'}{\kappa}\vartheta - \frac{\phi}{\kappa}\vartheta^p + \phi^p\left(\frac{\vartheta}{\kappa}\right)^p + e^{\frac{-p\tau}{p-1}}h\left(e^{\frac{\tau}{p-1}}\frac{\phi}{\kappa}\vartheta\right). \tag{C.5}$$

The conclusion of part $ii)$ is a direct consequence of the following:

Lemma C.1. *There exists $\sigma_7 > 0$ such that for all $\tau \ge \sigma_7$, we have*
$i)$ ***(Estimates on** Q**)***

$$m = 0,1,\ |Q_m(\tau)| \le \frac{C}{\tau^2}, \quad |Q_2(\tau)| \le \frac{C}{\tau^3},$$
$$\left\|\frac{Q_-(y,\tau)}{1+|y|^3}\right\|_{L^\infty} \le \frac{C}{\tau^2}, \quad \|Q_e(\tau)\|_{L^\infty} \le \frac{C}{\sqrt{\tau}}. \tag{C.6}$$

$ii)$ ***(Estimates on** G**)***

$$m = 0,1,2,\ |G_m(\tau)| \le \frac{C}{\tau^{1+a'}}, \quad \left\|\frac{G_-(y,\tau)}{1+|y|^3}\right\|_{L^\infty} \le \frac{C}{\tau^{1+a'}}, \ \|G_e(\tau)\|_{L^\infty} \le \frac{C}{\tau^a}, \tag{C.7}$$

where $a' = a > 1$ in the case (1.2) *and $a' = a+1 > 1$ in the case* (1.3).

Proof. $i)$ See page 563 in [3]. For part $ii)$, one can see that it is a direct consequence of the following:

$$|G(y,\tau)| \le \frac{C}{\tau^a} \quad \text{and} \quad |\chi(\tau)G(y,\tau)| \le \frac{C}{\tau^{1+a'}}(1+|y|^2). \tag{C.8}$$

By the definition of G_m, G_- and G_e, part $ii)$ simply follows from (C.8). By the linearity, this also concludes the proof of part $ii)$ of Lemma 3.8.

Let us now give the proof of (C.8). For the first estimate, we use the definition (C.5) of G, Lemmas A.1 and A.2,

$$|G(y,\tau)| \le \left|\frac{\phi'\vartheta}{\kappa}\right| + \left|\frac{\phi\vartheta}{\kappa}\right|\left|1 - \frac{\phi^{p-1}}{\kappa^{p-1}}\right| + \left|e^{-\frac{ps}{p-1}}h\left(e^{\frac{s}{p-1}}\frac{\phi\vartheta}{\kappa}\right)\right| \le \frac{C}{s^a}.$$

For the second estimate in (C.8), we use the fact that ϕ satisfies (1.6) and write

$$\begin{aligned} G(y,\tau) &= \frac{\vartheta\phi}{\kappa^p}(\kappa^{p-1}-\phi^{p-1})(\kappa^{p-1}-\vartheta^{p-1}) \\ &+ e^{-\frac{p\tau}{p-1}}\left[h\left(e^{\frac{\tau}{p-1}}\frac{\phi\vartheta}{\kappa}\right) - h\left(e^{\frac{\tau}{p-1}}\phi\right)\right] \\ &+ \left(1-\frac{\vartheta}{\kappa}\right)e^{-\frac{p\tau}{p-1}}h\left(e^{\frac{\tau}{p-1}}\phi\right) := \bar{G}+\tilde{G}+\hat{G}. \end{aligned}$$

Noting that $\vartheta(y,\tau) = \kappa\left(1-\frac{h_2(y)}{4p\tau}+\mathcal{O}\left(\frac{|y|^4}{\tau^2}\right)\right)$ uniformly for $y\in\mathbb{R}$ and $\tau\geq 1$, and recalling from Lemma A.2 that $\phi(\tau) = \kappa(1+\eta_a(\tau))^{-\frac{1}{p-1}}$ where $\eta_a(\tau) = \mathcal{O}(\tau^{-a})$, then using a Taylor expansion, we derive

$$\begin{aligned} \bar{G}(y,\tau) &= \frac{\phi\eta_a(\tau)}{1+\eta_a(\tau)}\left(\frac{h_2(y)}{4p\tau}+\mathcal{O}\left(\frac{|y|^4}{\tau^2}\right)\right), \\ \tilde{G}(y,\tau) &= -\phi e^{-\tau}h'\left(e^{\frac{\tau}{p-1}}\phi\right)\left(\frac{h_2(y)}{4p\tau}+\mathcal{O}\left(\frac{|y|^4}{\tau^2}\right)\right), \\ \hat{G}(y,\tau) &= e^{-\frac{p\tau}{p-1}}h\left(e^{\frac{\tau}{p-1}}\phi\right)\left(\frac{h_2(y)}{4p\tau}+\mathcal{O}\left(\frac{|y|^4}{\tau^2}\right)\right). \end{aligned}$$

This yields the second estimate in (C.8) in the case (1.2). If h is given by (1.3), we have furthermore

$$\begin{aligned} &\left|\frac{\phi\eta_a(\tau)}{1+\eta_a(\tau)} - e^{-s}h'(e^{\frac{s}{p-1}}\phi)\phi + e^{-\frac{ps}{p-1}}h(e^{\frac{s}{p-1}}\phi)\right| \\ &\leq \left|\frac{\phi}{1+\eta_a(\tau)}\right|\left|\eta_a(\tau) - \frac{\mu}{\log^a\left(2+e^{\frac{2\tau}{p-1}}\phi^2(\tau)\right)}\right| + \frac{C}{\tau^{a+1}} \leq \frac{2C}{\tau^{1+a}}, \end{aligned}$$

which yields the second estimate in (C.8) in the case (1.3). This concludes the proof of (C.8) and the proof of part $ii)$ of Lemma 3.8 also. □

$iii)$ From the definition (2.6) of N, we use a Taylor expansion for N to find that in the case (1.2),

$$N(q(\tau),\tau) = e^{-\tau}h'\left(e^{\frac{\tau}{p-1}}(\phi(\tau)+\theta_1 q(\tau))\right)q(\tau) \quad \text{with} \quad \theta_1\in[0,1],$$

and in the case (1.3),

$$N(q(\tau),\tau) = e^{-\frac{(p-2)\tau}{p-1}}h''\left(e^{\frac{\tau}{p-1}}(\phi(\tau)+\theta_2 q(\tau))\right)q^2(\tau) \quad \text{with} \quad \theta_2\in[0,1].$$

Since $\varphi(\tau)\to\kappa$ and $\|q(\tau)\|_{L^\infty(\mathbb{R})}\to 0$ as $\tau\to+\infty$, this implies that there exists τ_0 large enough such that $\frac{\kappa}{2}\leq|\phi(\tau)+\theta_i q(\tau)|\leq\frac{3\kappa}{2}$ for all $\tau\geq\tau_0$ and $y\in\mathbb{R}$. Then by Lemma A.1, we have $|N(q(\tau),\tau)|\leq\frac{C|q|^\beta}{\tau^a}$ where $\beta=1$ in the case (1.2) and $\beta=2$ in the case (1.3), which implies part $iii)$ of Lemma 3.8. This concludes the proof of Lemma 3.8. □

Bibliography

[1] J. M. Ball. Remarks on blow-up and nonexistence theorems for nonlinear evolution equations. *Quart. J. Math. Oxford Ser. (2)*, 28(112):473–486, 1977.

[2] M. Berger and R. V. Kohn. A rescaling algorithm for the numerical calculation of blowing-up solutions. *Comm. Pure Appl. Math.*, 41(6):841–863, 1988.

[3] J. Bricmont and A. Kupiainen. Universality in blow-up for nonlinear heat equations. *Nonlinearity*, 7(2):539–575, 1994.

[4] M. A. Ebde and H. Zaag. Construction and stability of a blow up solution for a nonlinear heat equation with a gradient term. *S$\vec{e}$MA J.*, (55):5–21, 2011.

[5] S. Filippas and R. V. Kohn. Refined asymptotics for the blowup of $u_t - \Delta u = u^p$. *Comm. Pure Appl. Math.*, 45(7):821–869, 1992.

[6] S. Filippas and W. X. Liu. On the blowup of multidimensional semilinear heat equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(3):313–344, 1993.

[7] H. Fujita. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I*, 13:109–124 (1966), 1966.

[8] Y. Giga and R. V. Kohn. Asymptotically self-similar blow-up of semilinear heat equations. *Comm. Pure Appl. Math.*, 38(3):297–319, 1985.

[9] Y. Giga and R. V. Kohn. Characterizing blowup using similarity variables. *Indiana Univ. Math. J.*, 36(1):1–40, 1987.

[10] Y. Giga and R. V. Kohn. Nondegeneracy of blowup for semilinear heat equations. *Comm. Pure Appl. Math.*, 42(6):845–884, 1989.

[11] M. A. Herrero and J. J. L. Velázquez. Blow-up profiles in one-dimensional, semilinear parabolic problems. *Comm. Partial Differential Equations*, 17(1-2):205–219, 1992.

[12] M. A. Herrero and J. J. L. Velázquez. Generic behaviour of one-dimensional blow up patterns. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 19(3):381–450, 1992.

[13] H. A. Levine. Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$. *Arch. Rational Mech. Anal.*, 51:371–386, 1973.

[14] N. Masmoudi and H. Zaag. Blow-up profile for the complex Ginzburg-Landau equation. *J. Funct. Anal.*, 255(7):1613–1666, 2008.

[15] F. Merle. Solution of a nonlinear heat equation with arbitrarily given blow-up points. *Comm. Pure Appl. Math.*, 45(3):263–300, 1992.

[16] F. Merle and H. Zaag. Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$. *Duke Math. J.*, 86(1):143–195, 1997.

[17] F. Merle and H. Zaag. Optimal estimates for blowup rate and behavior for nonlinear heat equations. *Comm. Pure Appl. Math.*, 51(2):139–196, 1998.

[18] F. Merle and H. Zaag. Refined uniform estimates at blow-up and applications for nonlinear heat equations. *Geom. Funct. Anal.*, 8(6):1043–1085, 1998.

[19] F. Merle and H. Zaag. A Liouville theorem for vector-valued nonlinear heat equations and applications. *Math. Ann.*, 316(1):103–137, 2000.

[20] V. T. Nguyen. Numerical analysis of the rescaling method for parabolic problems with blow-up in finite time. *submitted, arXiv:1403.7547*.

[21] V. T. Nguyen. On the blow-up results for a class of strongly perturbed semilinear heat equation. *submitted, arXiv:1404.4018*.

[22] N. Nouaili and H. Zaag. Profile for a simultaneously blowing up solution for a complex valued semilinear heat equation. *submitted*, 2014.

[23] J. J. L. Velázquez. Higher-dimensional blow up for semilinear parabolic equations. *Comm. Partial Differential Equations*, 17(9-10):1567–1596, 1992.

[24] J. J. L. Velázquez. Classification of singularities for blowing up solutions in higher dimensions. *Trans. Amer. Math. Soc.*, 338(1):441–464, 1993.

[25] F. B. Weissler. Single point blow-up for a semilinear initial value problem. *J. Differential Equations*, 55(2):204–224, 1984.

[26] H. Zaag. Blow-up results for vector-valued nonlinear heat equations with no gradient structure. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 15(5):581–622, 1998.

Address: Université Paris 13, Institut Galilée, LAGA,
99 Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France.

E-mail: vtnguyen@math.univ-paris13.fr
Hatem.Zaag@univ-paris13.fr

Chapter V

Blow-up results for a strongly perturbed semilinear heat equation: Theoretical analysis and numerical method[1]

V. T. Nguyen and H. Zaag[2]

Université Paris 13, Sorbonne Paris Cité,
LAGA, CNRS (UMR 7539), F-93430, Villetaneuse, France.

Abstract
We consider a blow-up solution for a strongly perturbed semilinear heat equation with Sobolev subcritical power nonlinearity. Working in the framework of similarity variables, we find a Lyapunov functional for the problem. Using this Lyapunov functional, we derive the blow-up rate and the blow-up limit of the solution. We also classify all asymptotic behaviors of the solution at the singularity and give precisely blow-up profiles corresponding to these behaviors. Finally, we attain the blow-up profile numerically, thanks to a new mesh-refinement algorithm inspired by the rescaling method of Berger and Kohn [3]. Note that our method is applicable to more general equations, in particular those with no scaling invariance.

Keywords: Blow-up, Lyapunov functional, asymptotic behavior, blow-up profile, semilinear heat equation, lower order term.

---

[1]submitted, arXiv:1410.4079

[2]This author is supported by the ERC Advanced Grant no. 291214, BLOWDISOL and by the ANR project ANAÉ ref. ANR-13-BS01-0010-03.

1 Introduction

We are concerned in this paper with blow-up phenomena arising in the following nonlinear heat problem:

$$\begin{cases} u_t &= \Delta u + |u|^{p-1}u + h(u), \\ u(.,0) &= u_0 \in L^\infty(\mathbb{R}^n), \end{cases} \tag{1.1}$$

where $u(t) : x \in \mathbb{R}^n \to u(x,t) \in \mathbb{R}$ and Δ stands for the Laplacian in $\mathbb{R}^n$. The exponent $p > 1$ is subcritical (that means that $p < \frac{n+2}{n-2}$ if $n \geq 3$) and h is given by

$$h(z) = \mu \frac{|z|^{p-1}z}{\log^a(2+z^2)}, \quad \text{with } a > 0,\ \mu \in \mathbb{R}. \tag{1.2}$$

By standard results, the problem (1.1) has a unique classical solution $u(x,t)$ in $L^\infty(\mathbb{R}^n)$, which exists at least for small times. The solution $u(x,t)$ may develop singularities in some finite time. We say that $u(x,t)$ blows up in a finite time T if $u(x,t)$ satisfies (1.1) in $\mathbb{R}^n \times [0,T)$ and

$$\lim_{t\to T} \|u(t)\|_{L^\infty(\mathbb{R}^n)} = +\infty.$$

T is called the blow-up time of $u(x,t)$. In such a blow-up case, a point $b \in \mathbb{R}^n$ is called a blow-up point of $u(x,t)$ if and only if there exist $(x_n, t_n) \to (b, T)$ such that $|u(x_n,t_n)| \to +\infty$ as $n \to +\infty$.

In the case $\mu = 0$, the equation (1.1) is the semilinear heat equation,

$$u_t = \Delta u + |u|^{p-1}u. \tag{1.3}$$

Problem (1.3) has been addressed in different ways in the literature. The existence of blow-up solutions has been proved by several authors (see Fujita [10], Levine [21], Ball [1]). Consider $u(x,t)$ a solution of (1.3) which blows up at a time T. The very first question to be answered is the blow-up rate, i.e. there are positive constants C_1, C_2 such that

$$C_1(T-t)^{-\frac{1}{p-1}} \leq \|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq C_2(T-t)^{-\frac{1}{p-1}}, \quad \forall t \in (0,T). \tag{1.4}$$

The lower bound in (1.4) follows by a simple argument based on Duhamel's formula (see Weissler [31]). For the upper bound, Giga and Kohn proved (1.4) in [11] for $1 < p < \frac{3n+8}{3n-4}$ or for non-negative initial data with subcritical p (note that Weissler [30] first obtained (1.4) in the positive, radially symmetric case under the assumption that, for each $0 < t < T$, $u_t(x,t)$ achieves maximum at $x = 0$, see also Mueller and Weissler [24], Friedman and McLeod [9]). Then, this result was extended to all subcritiacal p without assuming non-negativity for initial data u_0 by Giga, Matsui and Sasayama in [13]. The estimate (1.4) is a fundamental step to obtain more information about the asymptotic blow-up behavior, locally near a given blow-up point $\hat{b}$. Giga and Kohn showed in [12] that for a given blow-up point $\hat{b} \in \mathbb{R}^n$,

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}} u(\hat{b} + y\sqrt{T-t}, t) = \pm\kappa,$$

where $\kappa = (p-1)^{-\frac{1}{p-1}}$, uniformly on compact sets of $\mathbb{R}^n$.
This result was specified by Filippas ans Liu [8] (see also Filippas and Kohn [7]) and Velázquez [28], [29] (see also Herrero and Velázquez [19], [20], [17]). Using the renormalization theory, Bricmont and Kupiainen showed in [4] the existence of a solution of (1.3) such that

$$\left\|(T-t)^{\frac{1}{p-1}}u(\hat{b}+z\sqrt{(T-t)|\log(T-t)|},t)-f_0(z)\right\|_{L^\infty(\mathbb{R}^n)}\to 0 \quad \text{as} \quad t\to T, \tag{1.5}$$

where

$$f_0(z)=\kappa\left(1+\frac{p-1}{4p}|z|^2\right)^{-\frac{1}{p-1}}. \tag{1.6}$$

Merle and Zaag in [23] obtained the same result through a reduction to a finite dimensional problem. Moreover, they showed that the profile (1.6) is stable under perturbations of initial data (see also [5], [6] and [22]).

In the case where the function h satisfies

$$j=0,1,\ \ |h^{(j)}(z)|\le M\left(\frac{|z|^{p-j}}{\log^a(2+z^2)}+1\right),\quad |h''(z)|\le M\frac{|z|^{p-2}}{\log^a(2+z^2)}, \tag{1.7}$$

with $a>1$ and $M>0$, the first author proved in [26] the existence of a Lyapunov functional in *similarity variables* for the problem (1.1) which is a crucial step in deriving the estimate (1.4). We also gave a classification of possible blow-up behaviors of the solution when it approaches to singularity. In [27], we constructed a blow-up solution of the problem (1.1) satisfying the behavior described in (1.5) in the case where h satisfies the first estimate in (1.7) or h is given by (1.2).

In this paper, we aim at extending the results of [26] to the case $a\in(0,1]$. As we mentioned above, the first step is to derive the blow-up rate of the blow-up solution. As in [13] and [26], the key step is to find a Lyapunov functional in *similarity variables* for equation (1.1). More precisely, we introduce for all $b\in\mathbb{R}^n$ (b may be a blow-up point of u or not) the following *similarity variables*:

$$y=\frac{x-b}{\sqrt{T-t}},\quad s=-\log(T-t),\quad w_{b,T}=(T-t)^{\frac{1}{p-1}}u(x,t). \tag{1.8}$$

Hence $w_{b,T}$ satisfies for all $s\ge-\log T$ and for all $y\in\mathbb{R}^n$:

$$\partial_s w_{b,T}=\frac{1}{\rho}\text{div}(\rho\nabla w_{b,T})-\frac{w_{b,T}}{p-1}+|w_{b,T}|^{p-1}w_{b,T}+e^{-\frac{ps}{p-1}}h\left(e^{\frac{s}{p-1}}w_{b,T}\right), \tag{1.9}$$

where

$$\rho(y)=\left(\frac{1}{4\pi}\right)^{n/2}e^{-\frac{|y|^2}{4}}. \tag{1.10}$$

Following the method introduced by Hamza and Zaag in [14], [15] for perturbations of the semilinear wave equation, we introduce

$$\mathcal{J}_a[w](s) = \mathcal{E}[w](s)e^{\frac{\gamma}{a}s^{-a}} + \theta s^{-a}, \tag{1.11}$$

where γ, θ are positive constants depending only on p, a, μ and n which will be determined later, and

$$\mathcal{E}[w] = \mathcal{E}_0[w] + \mathcal{I}[w], \tag{1.12}$$

where

$$\mathcal{E}_0[w](s) = \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla w|^2 + \frac{1}{2(p-1)}|w|^2 - \frac{1}{p+1}|w|^{p+1} \right) \rho dy, \tag{1.13}$$

and

$$\mathcal{I}[w](s) = -e^{-\frac{p+1}{p-1}s} \int_{\mathbb{R}^n} H\left(e^{\frac{s}{p-1}}w\right) \rho dy, \quad H(z) = \int_0^z h(\xi)d\xi. \tag{1.14}$$

The main novelty of this paper is to allow values of a in $(0,1]$, and this is possible at the expense of taking the particular form (1.2) for the perturbation h. We aim at the following:

Theorem 1.1 **(Existence of a Lyapunov functional for equation** (1.9)**).** *Let a, p, n, μ be fixed, consider w a solution of equation* (1.9)*. Then, there exist $\hat{s}_0 = \hat{s}_0(a,p,n,\mu) \geq s_0$, $\hat{\theta}_0 = \hat{\theta}_0(a,p,n,\mu)$ and $\gamma = \gamma(a,p,n,\mu)$ such that if $\theta \geq \hat{\theta}_0$, then $\mathcal{J}_a$ satisfies the following inequality, for all $s_2 > s_1 \geq \max\{\hat{s}_0, -\log T\}$,*

$$\mathcal{J}_a[w](s_2) - \mathcal{J}_a[w](s_1) \leq -\frac{1}{2}\int_{s_1}^{s_2}\int_{\mathbb{R}^n} (\partial_s w)^2 \rho dy ds. \tag{1.15}$$

As in [13] and [26], the existence of the Lyapunov functional is a crucial step for deriving the blow-up rate (1.4) and then the blow-up limit. In particular, we have the following:

Theorem 1.2. *Let a, p, n, μ be fixed and u be a blow-up solution of equation* (1.1) *with a blow-up time T.*
(i) ***(Blow-up rate)*** *There exists $\hat{s}_1 = \hat{s}_1(a,p,n,\mu) \geq \hat{s}_0$ such that for all $s \geq s' = \max\{\hat{s}_1, -\log T\}$,*

$$\|w_{b,T}(s)\|_{L^\infty(\mathbb{R}^n)} \leq C, \tag{1.16}$$

where $w_{b,T}$ is defined in (1.8) *and C is a positive constant depending only on n, p, μ and a bound of $\|w_{b,T}(\hat{s}_0)\|_{L^\infty}$.*
(ii) ***(Blow-up limit)*** *If $\hat{a}$ is a blow-up point, then*

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}}u(\hat{a}+y\sqrt{T-t},t) = \lim_{s\to+\infty} w_{\hat{a},T}(y,s) = \pm\kappa, \tag{1.17}$$

holds in L^2_ρ (L^2_ρ is the weighted L^2 space associated with the weight ρ (1.10)*), and also uniformly on each compact subset of $\mathbb{R}^n$.*

Remark 1.3. *We will not give the proof of Theorem 1.2 because its proof follows from Theorem 1.1 as in [26]. Hence, we only give the proof of Theorem 1.1 and refer the reader to Section 2 in [26] for the proofs of* (1.16) *and* (1.17) *respectively.*

The next step consists in obtaining an additional term in the asymptotic expansion given in (ii) of Theorem 1.2. Given b a blow-up point of $u(x,t)$, and up to changing u_0 by $-u_0$ and h by $-h$, we may assume that $w_{b,T} \to \kappa$ in L^2_ρ as $s \to +\infty$. As in [26], we linearize $w_{b,T}$ around ϕ, where ϕ is the positive solution of the ordinary differential equation associated to (1.9),

$$\phi_s = -\frac{\phi}{p-1} + \phi^p + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}}\phi\right) \tag{1.18}$$

such that

$$\phi(s) \to \kappa \quad \text{as} \quad s \to +\infty, \tag{1.19}$$

(see Lemma A.3 in [26] for the existence of ϕ, and note that ϕ is unique. For the reader's convenience, we give in Lemma A.1 the expansion of ϕ as $s \to +\infty$).

Let us introduce $v_{b,T} = w_{b,T} - \phi(s)$, then $\|v_{b,T}(s)\|_{L^2_\rho} \to 0$ as $s \to +\infty$ and $v_{b,T}$ (or v for simplicity) satisfies the following equation:

$$\partial_s v = (\mathcal{L} + \omega(s))v + F(v) + H(v,s), \quad \forall y \in \mathbb{R}^n,\ \forall s \in [-\log T, +\infty),$$

where $\mathcal{L} = \Delta - \frac{y}{2}\cdot\nabla + 1$ and ω, F, H satisfy

$$\omega(s) = \mathcal{O}(\frac{1}{s^{a+1}}) \quad \text{and} \quad |F(v)| + |H(v,s)| = \mathcal{O}(|v|^2) \quad \text{as } s \to +\infty,$$

(see the beginning of Section 3 for the proper definitions of ω, F and G).
It is well known that the operator $\mathcal{L}$ is self-adjoint in $L^2_\rho(\mathbb{R}^n)$. Its spectrum is given by

$$spec(\mathcal{L}) = \{1 - \frac{m}{2},\ m \in \mathbb{N}\},$$

and it consists of eigenvalues. The eigenfunctions of $\mathcal{L}$ are derived from Hermite polynomials:
- For $n = 1$, the eigenfunction corresponding to $1 - \frac{m}{2}$ is

$$h_m(y) = \sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{k!(m-2k)!}(-1)^k y^{m-2k}, \tag{1.20}$$

- For $n \geq 2$: we write the spectrum of $\mathcal{L}$ as

$$spec(\mathcal{L}) = \{1 - \frac{|m|}{2},\ |m| = m_1 + \cdots + m_n,\ (m_1, \ldots, m_n) \in \mathbb{N}^n\}.$$

For $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$, the eigenfunction corresponding to $1 - \frac{|m|}{2}$ is

$$H_m(y) = h_{m_1}(y_1) \ldots h_{m_n}(y_n), \tag{1.21}$$

where h_m is defined in (1.20).
We also denote $c_m = c_{m_1} c_{m_2} \dots c_{m_n}$ and $y^m = y_1^{m_1} y_2^{m_2} \dots y_n^{m_n}$ for any $m = (m_1, \dots, m_n) \in \mathbb{N}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$.
By this way, we derive the following asymptotic behaviors of $w_{b,T}(y,s)$ as $s \to +\infty$:

Theorem 1.4 (Classification of the behavior of $w_{b,T}$ as $s \to +\infty$). *Consider $u(t)$ a solution of equation* (1.1) *which blows-up at time T and b a blow-up point. Let $w_{b,T}(y,s)$ be a solution of equation* (1.9)*. Then one of the following possibilities occurs:*
$i)$ $w_{b,T}(y,s) \equiv \phi(s)$,
$ii)$ There exists $l \in \{1,\dots,n\}$ such that up to an orthogonal transformation of coordinates, we have

$$w_{b,T}(y,s) = \phi(s) - \frac{\kappa}{4ps}\left(\sum_{j=1}^{l} y_j^2 - 2l\right) + \mathcal{O}\left(\frac{1}{s^{a+1}}\right) + \mathcal{O}\left(\frac{\log s}{s^2}\right) \quad as \quad s \to +\infty,$$

$iii)$ There exist an integer number $m \geq 3$ and constants c_α not all zero such that

$$w_{b,T}(y,s) = \phi(s) - e^{-(\frac{m}{2}-1)s} \sum_{|\alpha|=m} c_\alpha H_\alpha(y) + o\left(e^{-(\frac{m}{2}-1)s}\right) \quad as \quad s \to +\infty.$$

The convergence takes place in L^2_ρ as well as in $\mathcal{C}^{k,\gamma}_{loc}$ for any $k \geq 1$ and some $\gamma \in (0,1)$.

Remark 1.5. *In our previous paper [26], we were unable to get this result in the case where h satisfies* (1.7) *with $a \in (0,1]$. Here, by taking the particular form of the perturbation (see* (1.2)*), we are able to overcome technical difficulties in order to derive the result.*

Remark 1.6. *From $ii)$ of Theorem 1.2, we would naturally try to find an equivalent for $w - \kappa$ as $s \to +\infty$. A posteriori from our results in Theorem 1.4, we see that in all cases $\|w(s) - \kappa\|_{L^2_\rho} \sim \frac{C}{s^{a'}}$ with $a' = \min\{a, 1\}$. This is indeed a new phenomenon observed in our equation* (1.1)*, and which is different from the case of the unperturbed semilinear heat equation where either $w - \kappa \equiv 0$, or $\|w(s) - \kappa\|_{L^2_\rho} \sim \frac{C}{s}$ or $\|w - \kappa\|_{L^2_\rho} \sim Ce^{(1-m/2)s}$ for some even $m \geq 4$. This shows the originality of our paper. In our case, linearizing around κ would keep us trapped in the $\frac{1}{s}$ scale. In order to escape that scale, we forget the explicit function κ which is not a solution of equation* (1.9)*, and linearizing instead around the non-explicit function ϕ, which happens to be an exact solution of* (1.9)*. This way, we escape the $\frac{1}{s}$ scale and reach exponentially decreasing order.*

Using the information obtained in Theorem 1.4, we can extend the asymptotic behavior of $w_{b,T}$ to larger regions. Particularly, we have the following:

Theorem 1.7 (Convergence extension of $w_{b,T}$ to larger regions). *For all $K_0 > 0$,*
$i)$ if $ii)$ of Theorem 1.4 occurs, then

$$\sup_{|\xi| \leq K_0} \left| w_{b,T}(\xi\sqrt{s}, s) - f_l(\xi) \right| = \mathcal{O}\left(\frac{1}{s^a}\right) + \mathcal{O}\left(\frac{\log s}{s}\right), \quad as \quad s \to +\infty, \tag{1.22}$$

where

$$\forall \xi \in \mathbb{R}^n, \quad f_l(\xi) = \kappa \left(1 + \frac{p-1}{4p} \sum_{j=1}^{l} \xi_j^2 \right)^{-\frac{1}{p-1}}, \tag{1.23}$$

with l given in $ii)$ of Theorem 1.4.
$ii)$ if $iii)$ of Theorem 1.4 occurs, then $m \geq 4$ is even, and

$$\sup_{|\xi| \leq K_0} \left| w_{b,T}\left(\xi e^{\left(\frac{1}{2}-\frac{1}{m}\right)s}\right) - \psi_m(\xi) \right| \to 0 \quad \text{as} \quad s \to +\infty, \tag{1.24}$$

where

$$\forall \xi \in \mathbb{R}^n, \quad \psi_m(\xi) = \kappa \left(1 + \kappa^{-p} \sum_{|\alpha|=m} c_\alpha \xi^\alpha \right)^{-\frac{1}{p-1}}, \tag{1.25}$$

with c_α the same as in Theorem 1.4, and the multilinear for $\sum_{|\alpha|=m} c_\alpha \xi^\alpha$ is nonnegative.

Remark 1.8. *As in the unperturbed case ($h \equiv 0$), we expect that* (1.22) *is stable (see the previous remarks, particularly the paragraph after* (1.5)*), and* (1.24) *should correspond to unstable behaviors (the unstability of* (1.24) *was proved only in one space dimension by Herrero and Velázquez in [16] and [18] for $h \equiv 0$). While remarking numerical simulations for equation* (1.1) *in one space dimension (see Section 4.2 below), we see that the numerical solutions exhibit only the behavior* (1.22)*, we could never obtain the behavior* (1.24)*. This is probably due to the fact that the behavior* (1.24) *is unstable.*

At the end of this work, we give numerical confirmations for the asymptotic profile described in Theorem 1.7. For this purpose, we propose a new mesh-refinement method inspired by the rescaling algorithm of Berger and Kohn [3]. Note that, their method was successful to solve blowing-up problems which are invariant under the following transformation,

$$\forall \gamma > 0, \quad \gamma \mapsto u_\gamma(\xi, \tau) = \gamma^{\frac{2}{p-1}} u(\gamma \xi, \gamma^2 \tau). \tag{1.26}$$

However, there are a lot of equations whose solutions blow up in finite time but the equation does not satisfy the property (1.26), one of them is the equation (1.1) because of the presence of the perturbation term h. Although our method is very similar to Berger and Kohn's algorithm in spirit, it is better in the sense that it can be applied to a larger class of blowing-up problems which do not satisfy the rescaling property (1.26). Up to our knowledge, there are not many papers on the numerical blow-up profile, apart from the paper of Berger and Kohn [3], who already obtained numerical results for equation (1.1) without the perturbation term. Recently, the author in [25] successfully used the rescaling method and gave numerical confirmations on blow-up profiles of various problems satisfying (1.26). There is also the work of Baruch et al. [2] studying standing-ring solutions.

This paper is organized as follows: Section 2 is devoted to the proof of Theorem 1.1. Theorem 1.2 follows from Theorem 1.1. Since all the arguments presented [26] remain valid for the case (1.7), except the existence of the Lyapunov functional for equation (1.9) (Theorem 1.1), we kindly refer the reader to Section 2.3 and 2.4 in [26] for details of the proof. Section 3 deals with results on asymptotic behaviors (Theorem 1.4 and Theorem 1.7). In Section 4, we describe the new mesh-refinement method and give some numerical justifications for the theoretical results.

Acknowledgement: The authors are grateful to M. A. Hamza for several helpful conversations pertaining to this work, and especially for giving the idea for the proof of Theorem 1.1 in this paper.

2 Existence of a Lyapunov functional for equation (1.9)

In this section, we mainly aim at proving that the functional $\mathcal{J}_a$ defined in (1.11) is a Lyapunov functional for equation (1.9) (Theorem 1.1). Note that this functional is far from being trivial and makes our main contribution.

In what follows, we denote by C a generic constant depending only on a, p, n and μ. We first give the following estimates on the perturbation term appearing in equation (1.9):

Lemma 2.1. *Let h be the function defined in* (1.2). *For all $\epsilon \in (0,p]$, there exists $C_0 = C_0(a,\mu,p,\epsilon) > 0$ and $\bar{s}_0 = \bar{s}_0(a,p,\epsilon) > 0$ large enough such that for all $s \geq \bar{s}_0$,*
$i)$

$$\left|e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}} z\right)\right| \leq \frac{C_0}{s^a}(|z|^p + |z|^{p-\epsilon}),$$
$$\text{and} \quad \left|e^{-\frac{(p+1)s}{p-1}} H\left(e^{\frac{s}{p-1}} z\right)\right| \leq \frac{C_0}{s^a}(|z|^{p+1} + 1),$$

where H is defined in (1.14).
$ii)$

$$\left|(p+1)e^{-\frac{(p+1)s}{p-1}} H\left(e^{\frac{s}{p-1}} z\right) - e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}} z\right) z\right| \leq \frac{C_0}{s^{a+1}}(|z|^{p+1} + 1).$$

Proof. Note that $i)$ obviously follows from the following estimate,

$$\forall q > 0, b > 0, \quad \frac{|z|^q}{\log^b(2 + e^{\frac{2s}{p-1}} z^2)} \leq \frac{C}{s^b}(|z|^q + 1), \quad \forall s \geq \bar{s}_0, \tag{2.1}$$

where $C = C(b,q) > 0$ and $\bar{s}_0 = \bar{s}_0(b,q) > 0$.
In order to derive estimate (2.1), considering the first case $z^2 e^{\frac{s}{p-1}} \geq 4$, then the case $z^2 e^{\frac{s}{p-1}} \leq 4$, we would obtain (2.1).

$ii)$ directly follows from an integration by part and estimate (2.1). Indeed, we have

$$\begin{aligned}H(\xi) &= \int_0^\xi h(x)dx = \mu \int_0^\xi \frac{|x|^{p-1}x}{\log^a(2+x^2)}dx \\ &= \frac{\mu|\xi|^{p+1}}{(p+1)\log^a(2+\xi^2)} + \frac{2a\mu}{p+1}\int_0^\xi \frac{|x|^{p+1}x}{(2+x^2)\log^{a+1}(2+x^2)}dx.\end{aligned}$$

Replacing ξ by $e^{\frac{s}{p-1}}z$ and using (2.1), we then derive $ii)$. This ends the proof of Lemma 2.1. ☐

We assert that Theorem 1.1 is a direct consequence of the following lemma:

Lemma 2.2. *Let a, p, n, μ be fixed and w be solution of equation* (1.9). *There exists $\tilde{s}_0 = \tilde{s}_0(a, p, n, \mu) \geq s_0$ such that the functional of $\mathcal{E}$ defined in* (1.12) *satisfies the following inequality, for all $s \geq \max\{\tilde{s}_0, -\log T\}$,*

$$\frac{d}{ds}\mathcal{E}[w](s) \leq -\frac{1}{2}\int_{\mathbb{R}^n} w_s^2 \rho dy + \gamma s^{-(a+1)}\mathcal{E}[w](s) + Cs^{-(a+1)}, \qquad (2.2)$$

where $\gamma = \frac{4C_0(p+1)}{(p-1)^2}$, C_0 is given in Lemma 2.1.

Let us first derive Theorem 1.1 from Lemma 2.2 and we will prove it later.

Proof of Theorem 1.1 admitting Lemma 2.2. Differentiating the functional $\mathcal{J}$ defined in (1.11), we obtain

$$\begin{aligned}\frac{d}{ds}\mathcal{J}_a[w](s) &= \frac{d}{ds}\left\{\mathcal{E}[w](s)e^{\frac{\gamma}{a}s^{-a}} + \theta s^{-a}\right\} \\ &= \frac{d}{ds}\mathcal{E}[w](s)e^{\frac{\gamma}{a}s^{-a}} - \gamma s^{-(a+1)}\mathcal{E}[w](s)e^{\frac{\gamma}{a}s^{-a}} - a\theta s^{-(a+1)} \\ &\leq -\frac{1}{2}e^{\frac{\gamma}{a}s^{-a}}\int_{\mathbb{R}^n} w_s^2\rho dy + \left[Ce^{\frac{\gamma}{a}s^{-a}} - a\theta\right]s^{-(a+1)} \quad \text{(use (2.2))}.\end{aligned}$$

Choosing θ large enough such that $Ce^{\frac{\gamma}{a}\tilde{s}_0^{-a}} - a\theta \leq 0$ and noticing that $e^{\frac{\gamma}{a}s^{-a}} \geq 1$ for all $s > 0$, we derive

$$\frac{d}{ds}\mathcal{J}_a[w](s) \leq -\frac{1}{2}\int_{\mathbb{R}^n} w_s^2\rho dy, \quad \forall s \geq \tilde{s}_0.$$

This implies inequality (1.15) and concludes the proof of Theorem 1.1, assuming that Lemma 2.2 holds. ☐

It remains to prove Lemma 2.2 in order to conclude the proof of Theorem 1.1.

Proof of Lemma 2.2 . Multiplying equation (1.9) with $w_s\rho$ and integrating by parts:

$$\int_{\mathbb{R}^n}|w_s|^2\rho = -\frac{d}{ds}\left\{\int_{\mathbb{R}^n}\left(\frac{1}{2}|\nabla w|^2+\frac{1}{2(p-1)}|w|^2-\frac{1}{p+1}|w|^{p+1}\right)\rho dy\right\}$$
$$+e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)w_s\rho dy.$$

For the last term of the above expression, we write in the following:

$$e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)w_s\rho dy = e^{-\frac{(p+1)s}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)\left(e^{\frac{s}{p-1}}w_s+\frac{e^{\frac{s}{p-1}}}{p-1}w\right)\rho dy$$
$$-\frac{1}{p-1}e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)w\rho dy$$
$$= e^{-\frac{p+1}{p-1}s}\frac{d}{ds}\int_{\mathbb{R}^n}H\left(e^{\frac{s}{p-1}}w\right)\rho dy-\frac{1}{p-1}e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)w\rho dy.$$

This yields

$$\int_{\mathbb{R}^n}|w_s|^2\rho dy = -\frac{d}{ds}\left\{\int_{\mathbb{R}^n}\left(\frac{1}{2}|\nabla w|^2+\frac{1}{2(p-1)}|w|^2-\frac{1}{p+1}|w|^{p+1}\right)\rho dy\right\}$$
$$+\frac{d}{ds}\left\{e^{-\frac{p+1}{p-1}s}\int_{\mathbb{R}^n}H\left(e^{\frac{s}{p-1}}w\right)\rho dy\right\}$$
$$+\frac{p+1}{p-1}e^{-\frac{p+1}{p-1}s}\int_{\mathbb{R}^n}H\left(e^{\frac{s}{p-1}}w\right)\rho dy$$
$$-\frac{1}{p-1}e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)w\rho dy.$$

From the definition of the functional $\mathcal{E}$ given in (1.12), we derive a first identity in the following:

$$\frac{d}{ds}\mathcal{E}[w](s) = -\int_{\mathbb{R}^n}|w_s|^2\rho dy+\frac{p+1}{p-1}e^{-\frac{p+1}{p-1}s}\int_{\mathbb{R}^n}H\left(e^{\frac{s}{p-1}}w\right)\rho dy$$
$$-\frac{1}{p-1}e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)w\rho dy. \tag{2.3}$$

A second identity is obtained by multiplying equation (1.9) with $w\rho$ and integrating by parts:

$$\frac{d}{ds}\int_{\mathbb{R}^n}|w|^2\rho dy = -4\left\{\int_{\mathbb{R}^n}\left(\frac{1}{2}|\nabla w|^2+\frac{1}{2(p-1)}|w|^2-\frac{1}{p+1}|w|^{p+1}\right)\rho dy\right.$$
$$\left.-e^{-\frac{(p+1)s}{p-1}}\int_{\mathbb{R}^n}H\left(e^{\frac{s}{p-1}}w\right)\rho dy\right\}$$
$$+\left(2-\frac{4}{p+1}\right)\int_{\mathbb{R}^n}|w|^{p+1}\rho dy-4e^{-\frac{p+1}{p-1}s}\int_{\mathbb{R}^n}H\left(e^{\frac{s}{p-1}}w\right)\rho dy$$
$$+2e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n}h\left(e^{\frac{s}{p-1}}w\right)w\rho dy.$$

Using again the definition of $\mathcal{E}$ given in (1.12), we rewrite the second identity in the following:

$$\frac{d}{ds}\int_{\mathbb{R}^n}|w|^2\rho dy = -4\mathcal{E}[w](s) + 2\frac{p-1}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\rho dy \\ - 4e^{-\frac{p+1}{p-1}s}\int_{\mathbb{R}^n} H\left(e^{\frac{s}{p-1}}w\right)\rho dy + 2e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n} h\left(e^{\frac{s}{p-1}}w\right)w\rho dy. \quad (2.4)$$

From (2.3), we estimate

$$\frac{d}{ds}\mathcal{E}[w](s) \le -\int_{\mathbb{R}^n}|w_s|^2\rho dy \\ + \frac{1}{p-1}\int_{\mathbb{R}^n}\left\{\left|(p+1)e^{-\frac{(p+1)s}{p-1}}H\left(e^{\frac{s}{p-1}}w\right) - e^{-\frac{ps}{p-1}}h\left(e^{\frac{s}{p-1}}w\right)w\right|\right\}\rho dy.$$

Using $ii)$ of Lemma 2.1, we have for all $s \ge \bar{s}_0$,

$$\frac{d}{ds}\mathcal{E}[w](s) \le -\int_{\mathbb{R}^n}|w_s|^2\rho dy + \frac{C_0 s^{-(a+1)}}{p-1}\int_{\mathbb{R}^n}|w|^{p+1}\rho dy + Cs^{-(a+1)}. \quad (2.5)$$

On the other hand, we have by (2.4),

$$\int_{\mathbb{R}^n}|w|^{p+1}\rho dy \le \frac{2(p+1)}{p-1}\mathcal{E}[w](s) + \frac{p+1}{p-1}\int_{\mathbb{R}^n}|w_s w|\rho dy \\ + \frac{2(p+1)}{p-1}\int_{\mathbb{R}^n}\left(\left|e^{-\frac{p+1}{p-1}s}H\left(e^{\frac{s}{p-1}}w\right)\right| + \left|e^{-\frac{ps}{p-1}}h\left(e^{\frac{s}{p-1}}w\right)w\right|\rho dy\right).$$

Using the fact that $|w_s w| \le \epsilon(|w_s|^2 + |w|^{p+1}) + C(\epsilon)$ for all $\epsilon > 0$ and $i)$ of Lemma 2.1, we obtain

$$\int_{\mathbb{R}^n}|w|^{p+1}\rho dy \le \frac{2(p+1)}{p-1}\mathcal{E}[w](s) + \epsilon\int_{\mathbb{R}^n}|w_s|^2\rho dy \\ + \left(\epsilon + Cs^{-a}\right)\int_{\mathbb{R}^n}|w|^{p+1}\rho dy + C.$$

Taking $\epsilon = \frac{1}{4}$ and s_1 large enough such that $Cs^{-a} \le \frac{1}{4}$ for all $s \ge s_1$, we have

$$\int_{\mathbb{R}^n}|w|^{p+1}\rho dy \le \frac{4(p+1)}{p-1}\mathcal{E}[w](s) + \frac{1}{2}\int_{\mathbb{R}^n}|w_s|^2\rho dy + C, \quad \forall s > s_1. \quad (2.6)$$

Substituting (2.6) into (2.5) yields (2.2) with $\tilde{s}_0 = \max\{\bar{s}_0, s_1\}$. This concludes the proof of Lemma 2.2 and Theorem 1.1 also. □

3 Blow-up behavior

This section is devoted to the proof of Theorem 1.4 and Theorem 1.7. Consider b a blow-up point and write w instead of $w_{b,T}$ for simplicity. From (ii) of Theorem 1.2

and up to changing the signs of w and h, we may assume that $\|w(y,s)-\kappa\|_{L^2_\rho} \to 0$ as $s \to +\infty$, uniformly on compact subsets of $\mathbb{R}^n$. As mentioned in the introduction, by setting $v(y,s) = w(y,s) - \phi(s)$ (ϕ is the positive solution of (1.18) such that $\phi(s) \to \kappa$ as $s \to +\infty$), we see that $\|v(s)\|_{L^2_\rho} \to 0$ as $s \to +\infty$ and v solves the following equation:

$$\partial_s v = (\mathcal{L} + \omega(s))v + F(v) + G(v,s), \quad \forall y \in \mathbb{R}^n,\ \forall s \in [-\log T, +\infty), \tag{3.1}$$

where $\mathcal{L} = \Delta - \frac{y}{2}\cdot\nabla + 1$ and ω, F, G are given by

$$\begin{aligned}
\omega(s) &= p\left(\phi^{p-1} - \kappa^{p-1}\right) + e^{-s}h'\left(e^{\frac{s}{p-1}}\phi\right),\\
F(v) &= |v+\phi|^{p-1}(v+\phi) - \phi^p - p\phi^{p-1}v,\\
G(v,s) &= e^{-\frac{ps}{p-1}}\left[h\left(e^{\frac{s}{p-1}}(v+\phi)\right) - h\left(e^{\frac{s}{p-1}}\phi\right) - e^{\frac{s}{p-1}}h'\left(e^{\frac{s}{p-1}}\phi\right)v\right].
\end{aligned}$$

By a direct calculation, we can show that

$$|\omega(s)| = \mathcal{O}(\frac{1}{s^{a+1}}), \quad \text{as } s \to +\infty, \tag{3.2}$$

(see Lemma B.1 for the proof of this fact, note also that in the case where h is given by (1.7) and treated in [26], we just obtain $|\omega(s)| = \mathcal{O}(s^{-a})$ as $s \to +\infty$, and that was a major reason preventing us from deriving the result in the case $a \in (0,1]$) in [26].
Now introducing

$$V(y,s) = \beta(s)v(y,s), \quad \text{where } \beta(s) = exp\left(-\int_s^{+\infty}\omega(\tau)d\tau\right), \tag{3.3}$$

then V satisfies

$$\partial_s V = \mathcal{L}V + \bar{F}(V,s), \tag{3.4}$$

where $\bar{F}(V,s) = \beta(s)(F(V) + G(V,s))$ satisfying

$$\left|\bar{F}(V,s) - \frac{p}{2\kappa}V^2\right| = \mathcal{O}\left(\frac{V^2}{s^a}\right) + \mathcal{O}(|V|^3), \quad \text{as } s \to +\infty. \tag{3.5}$$

(see Lemma C.1 in [26] for the proof of this fact, note that in the case where h is given by (1.7), the first term in the right-hand side of (3.5) is $\mathcal{O}\left(\frac{V^2}{s^{a-1}}\right)$).

Since $\beta(s) \to 1$ as $s \to +\infty$, each equivalent for V is also an equivalent for v. Therefore, it suffices to study the asymptotic behavior of V as $s \to +\infty$. More precisely, we claim the following:

Proposition 3.1 (Classification of the behavior of V as $s \to +\infty$). *One of the following possibilities occurs:*
i) $V(y,s) \equiv 0$,

$ii)$ There exists $l \in \{1, \ldots, n\}$ such that up to an orthogonal transformation of coordinates, we have

$$V(y,s) = -\frac{\kappa}{4ps}\left(\sum_{j=1}^{l} y_j^2 - 2l\right) + \mathcal{O}\left(\frac{1}{s^{a+1}}\right) + \mathcal{O}\left(\frac{\log s}{s^2}\right) \quad \textit{as} \quad s \to +\infty.$$

$iii)$ There exist an integer number $m \geq 3$ and constants c_α not all zero such that

$$V(y,s) = -e^{\left(1-\frac{m}{2}\right)s} \sum_{|\alpha|=m} c_\alpha H_\alpha(y) + o\left(e^{\left(1-\frac{m}{2}\right)s}\right) \quad \textit{as} \quad s \to +\infty.$$

The convergence takes place in L^2_ρ as well as in $\mathcal{C}^{k,\gamma}_{loc}$ for any $k \geq 1$ and $\gamma \in (0,1)$.

Proof. Because we have the same equation (3.4) and a similar estimate (3.5) to the case treated in [26], we do not give the proof and kindly refer the reader to Section 3 in [26]. □

Let us derive Theorem 1.4 from Proposition 3.1.

Proof of Theorem 1.4. By the definition (3.3) of V, we see that $i)$ of Proposition 3.1 directly follows that $v(y,s) \equiv \phi(s)$ which is $i)$ of Theorem 1.4. Using $ii)$ of Proposition 3.1 and the fact that $\beta(s) = 1 + \mathcal{O}(\frac{1}{s^a})$ as $s \to +\infty$, we see that as $s \to +\infty$,

$$\begin{aligned} w(y,s) &= \phi(s) + V(y,s)\left(1 + \mathcal{O}(\frac{1}{s^a})\right) \\ &= \phi(s) - \frac{\kappa}{4ps}\left(\sum_{j=1}^{l} y_j^2 - 2l\right) + \mathcal{O}\left(\frac{1}{s^{a+1}}\right) + \mathcal{O}\left(\frac{\log s}{s^2}\right), \end{aligned}$$

which yields $ii)$ of Theorem 1.4.
Using $iii)$ of Proposition 3.1 and again the fact that $\beta(s) = 1 + \mathcal{O}(\frac{1}{s^a})$ as $s \to +\infty$, we have

$$w(y,s) = \phi(s) - e^{\left(1-\frac{m}{2}\right)s} \sum_{|\alpha|=m} c_\alpha H_\alpha(y) + o\left(e^{\left(1-\frac{m}{2}\right)s}\right) \quad \text{as} \quad s \to +\infty.$$

This concludes the proof of Theorem 1.4. □

We now give the proof of Theorem 1.7 from Theorem 1.4. Note that the derivation of Theorem 1.7 from Theorem 1.4 in the unperturbed case ($h \equiv 0$) was done by Velázquez in [28]. The idea to extend the convergence up to sets of the type $\{|y| \leq K_0\sqrt{s}\}$ or $\{|y| \leq K_0 e^{\left(\frac{1}{2}-\frac{1}{m}\right)s}\}$ is to estimate the effect of the convective term $-\frac{y}{2}\cdot\nabla w$ in the equation (1.9) in L^q_ρ spaces with $q > 1$. Since the proof of Theorem 1.7 is actually in spirit by the method given in [28], all that we need to do is to control the strong perturbation term in equation (1.9). We therefore give the main steps of the proof and focus only on the new arguments. Note also that we only give the proof of $ii)$ of Theorem 1.4 because the proof of $iii)$ is exactly the same as written in Proposition 34 in [26].

Let us restate $i)$ of Theorem 1.7 in the following proposition:

Proposition 3.2 (Asymptotic behavior in the $\frac{y}{\sqrt{s}}$ variable). *Assume that w is a solution of equation (1.9) which satisfies $ii)$ of Theorem 1.4. Then, for all $K > 0$,*

$$\sup_{|\xi|\le K} \left| w(\xi\sqrt{s}, s) - f_l(\xi)\right| = \mathcal{O}\left(\frac{1}{s^a}\right) + \mathcal{O}\left(\frac{\log s}{s}\right), \quad \textit{as} \quad s \to +\infty,$$

where $f_l(\xi) = \kappa\left(1 + \frac{p-1}{4p}\sum_{j=1}^{l}\xi_j^2\right)^{-\frac{1}{p-1}}$.

Proof. Define $q = w - \varphi$, where

$$\varphi(y,s) = \frac{\phi(s)}{\kappa}\left[\kappa\left(1 + \frac{p-1}{4ps}\sum_{j=1}^{l} y_j^2\right)^{-\frac{1}{p-1}} + \frac{\kappa l}{2ps}\right], \tag{3.6}$$

and ϕ is the unique positive solution of (1.18) satisfying (1.19).
Note that in [28] and [26], the authors took $\varphi(y,s) = \kappa\left(1 + \frac{p-1}{4ps}\sum_{j=1}^{l} y_j^2\right)^{-\frac{1}{p-1}} + \frac{\kappa l}{2ps}$. But this choice just works in the case where $a > 1$. In the particular case (1.2), we use in additional the factor $\frac{\phi(s)}{\kappa}$ which allows us to go beyond the order $\frac{1}{s^a}$ coming from the strong perturbation term in order to reach $\frac{1}{s^{a+1}}$ in many estimates in the proof.

Using Taylor's formula in (3.6) and $ii)$ of Theorem 1.4, we find that

$$\|q(s)\|_{L^2_\rho} = \mathcal{O}\left(\frac{1}{s^{a+1}}\right) + \mathcal{O}\left(\frac{\log s}{s^2}\right), \quad \text{as} \quad s \to +\infty. \tag{3.7}$$

Straightforward calculations based on equation (1.9) yield

$$\partial_s q = (\mathcal{L} + \alpha)q + F(q) + G(q,s) + R(y,s), \quad \forall (y,s) \in \mathbb{R}^n \times [-\log T, +\infty), \tag{3.8}$$

where

$$\begin{aligned}
\alpha(y,s) &= p(\varphi^{p-1} - \kappa^{p-1}) + e^{-s} h'\left(e^{\frac{s}{p-1}}\varphi\right),\\
F(q) &= |q+\varphi|^{p-1}(q+\varphi) - \varphi^p - p\varphi^{p-1}q,\\
G(q,s) &= e^{-\frac{ps}{p-1}}\left[h\left(e^{\frac{s}{p-1}}(q+\varphi)\right) - h\left(e^{\frac{s}{p-1}}\varphi\right) - e^{\frac{s}{p-1}} h'\left(e^{\frac{s}{p-1}}\varphi\right) q\right],\\
R(y,s) &= -\partial_s\varphi + \Delta\varphi - \frac{y}{2}\cdot\nabla\varphi - \frac{\varphi}{p-1} + \varphi^p + e^{-\frac{ps}{p-1}} h\left(e^{\frac{s}{p-1}}\varphi\right).
\end{aligned}$$

Let $K_0 > 0$ be fixed, we consider first the case $|y| \geq 2K_0\sqrt{s}$ and then $|y| \leq 2K_0\sqrt{s}$ and make a Taylor expansion for $\xi = \frac{y}{\sqrt{s}}$ bounded. Simultaneously, we obtain for all $s \geq s_0$,

$$\alpha(y,s) \leq \frac{C_1}{s^{a'}},$$

$$|F(q)| + |G(q,s)| \leq C_1(q^2 + \mathbf{1}_{\{|y|\geq 2K_0\sqrt{s}\}}),$$

$$|R(y,s)| \leq C_1\left(\frac{|y|^2+1}{s^{1+a'}} + \mathbf{1}_{\{|y|\geq 2K_0\sqrt{s}\}}\right),$$

where $a' = \min\{1, a\}$, $C_1 = C_1(M_0, K_0) > 0$, M_0 is the bound of w in L^∞-norm. Note that we need to use in addition the fact that ϕ satisfies equation (1.18) to derive the bound for R (see Lemma B.2).

Let $Q = |q|$, we then use the above estimates and Kato's inequality, i.e $\Delta f \cdot \text{sign}(f) \leq \Delta(|f|)$, to derive from equation (3.8) the following: for all $K_0 > 0$ fixed, there are $C_* = C_*(K_0, M_0) > 0$ and a time $s' > 0$ large enough such that for all $s \geq s_* = \max\{s', -\log T\}$,

$$\partial_s Q \leq \left(\mathcal{L} + \frac{C_*}{s^{a'}}\right)Q + C_*\left(Q^2 + \frac{(|y|^2+1)}{s^{1+a'}} + \mathbf{1}_{\{|y|\geq 2K_0\sqrt{s}\}}\right), \quad \forall y \in \mathbb{R}^n. \tag{3.9}$$

Since

$$\left|w(y,s) - f_l\left(\frac{y}{\sqrt{s}}\right)\right| \leq Q + \frac{C}{s^{a'}},$$

the conclusion of Proposition 3.2 follows if we show that

$$\forall K_0 > 0, \quad \sup_{|y|\leq K_0\sqrt{s}} Q(y,s) \to 0 \quad \text{as} \quad s \to +\infty. \tag{3.10}$$

Let us now focus on the proof of (3.10) in order to conclude Proposition 3.2. For this purpose, we introduce the following norm: for $r \geq 0$, $q > 1$ and $f \in L^q_{loc}(\mathbb{R}^n)$,

$$L^{q,r}_\rho(f) \equiv \sup_{|\xi|\leq r}\left(\int_{\mathbb{R}^n} |f(y)|^q \rho(y-\xi)dy\right)^{\frac{1}{q}}.$$

Following the idea in [28], we shall make estimates on solution of (3.9) in the $L^{2,r(\tau)}_\rho$ norm where $r(\tau) = K_0 e^{\frac{\tau-\bar{s}}{2}} \leq K_0\sqrt{\tau}$. Particularly, we have the following:

Lemma 3.3. *Let s be large enough and $\bar{s}$ is defined by $e^{s-\bar{s}} = s$. Then for all $\tau \in [\bar{s}, s]$ and for all $K_0 > 0$, it holds that*

$$g(\tau) \leq C_0\left(e^{\tau-\bar{s}}\epsilon(\bar{s}) + \int_{\bar{s}}^{(\tau-2K_0)_+} \frac{e^{(\tau-t-2K_0)}g^2(t)}{\left(1-e^{-(\tau-t-2K_0)}\right)^{1/20}}dt\right)$$

where $g(\tau) = L^{2,r(K_0,\tau,\bar{s})}_\rho(Q(\tau))$, $r(K_0,\tau,\bar{s}) = K_0 e^{\frac{\tau-\bar{s}}{2}}$, $\epsilon(s) = \mathcal{O}\left(\frac{1}{s^{a+1}}\right) + \mathcal{O}\left(\frac{\log s}{s^2}\right)$, $C_0 = C_0(C_*, M_0, K_0)$ and $z_+ = \max\{z, 0\}$.

Proof. Multiplying (3.9) by $\beta(\tau) = e^{\int_{\bar{s}}^{\tau} \frac{C_*}{t^{a'}}dt}$, then we write $Q(y,\tau)$ for all $(y,\tau) \in \mathbb{R}^n \times [\bar{s}, s]$ in the integration form:

$$\begin{aligned} Q(y,\tau) &= \beta(\tau) S_{\mathcal{L}}(\tau - \bar{s}) Q(y,\bar{s}) \\ &\quad + C_* \int_{\bar{s}}^{\tau} \beta(\tau) S_{\mathcal{L}}(\tau - t) \left(Q^2 + \frac{|y|^2}{t^{1+a'}} + \frac{1}{t^{1+a'}} + \mathbf{1}_{\{|y| \ge 2K_0\sqrt{t}\}} \right) dt, \end{aligned}$$

where $S_{\mathcal{L}}$ is the linear semigroup corresponding to the operator $\mathcal{L}$.
Next, we take the $L_\rho^{2,r(K_0,\tau,\bar{s})}$-norms both sides in order to get the following:

$$\begin{aligned} g(\tau) &\le C_0 L_\rho^{2,r}\left[S_{\mathcal{L}}(\tau - \bar{s}) Q(\bar{s})\right] + C_0 \int_{\bar{s}}^{\tau} L_\rho^{2,r}\left[S_{\mathcal{L}}(\tau - t) Q^2(t)\right] dt \\ &\quad + C_0 \int_{\bar{s}}^{\tau} L_\rho^{2,r}\left[S_{\mathcal{L}}(\tau - t) \left(\frac{|y|^2}{t^{1+a'}} + \frac{1}{t^{1+a'}} \right) \right] dt \\ &\quad + C_0 \int_{\bar{s}}^{\tau} L_\rho^{2,r}\left[S_{\mathcal{L}}(\tau - t) \mathbf{1}_{\{|y| \ge 2K_0\sqrt{t}\}}\right] dt \equiv J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Proposition 2.3 in [28] yields

$$|J_1| \le C_0 e^{\tau - \bar{s}} \|Q(\bar{s})\|_{L_\rho^2} = e^{\tau - \bar{s}} \mathcal{O}(\epsilon(\bar{s})) \quad \text{as} \quad \bar{s} \to +\infty,$$

$$|J_2| \le \frac{C_0}{\bar{s}^{1+a'}} e^{\tau - \bar{s}} + C_0 \int_{\bar{s}}^{(\tau - 2K_0)_+} \frac{e^{(\tau - t - 2K_0)}}{\left(1 - e^{-(\tau - t - 2K_0)}\right)^{1/20}} \left[L_\rho^{2,r(K_0,t,\bar{s})} Q(t)\right]^2 dt,$$

$$|J_3| \le \frac{C_0 e^{\tau - \bar{s}}}{\bar{s}^{1+a'}} (1 + (\tau - \bar{s})),$$

$$|J_4| \le C_0 e^{-\delta \bar{s}}, \quad \text{where} \quad \delta = \delta(K_0) > 0.$$

Putting together the estimates on $J_i, i = 1,2,3,4$, we conclude the proof of Lemma 3.3. □

We now use the following Gronwall lemma from Velázquez [28]:

Lemma 3.4 (Velázquez [28]). *Let ϵ, C, R and δ be positive constants, $\delta \in (0,1)$. Assume that $H(\tau)$ is a family of continuous functions satisfying*

$$\mathcal{H}(\tau) \le \epsilon e^{\tau} + C \int_0^{(\tau - R)_+} \frac{e^{\tau - s} \mathcal{H}^2(s)}{\left(1 - e^{-(\tau - s - R)}\right)^{\delta}} ds, \quad \textit{for } \tau > 0.$$

Then there exist $\theta = \theta(\delta, C, R)$ and $\epsilon_0 = \epsilon_0(\delta, C, R)$ such that for all $\epsilon \in (0, \epsilon_0)$ and any τ for which $\epsilon e^{\tau} \le \theta$, we have

$$\mathcal{H}(\tau) \le 2\epsilon e^{\tau}.$$

Applying Lemma 3.4 with $\mathcal{H} \equiv g$, we see from Lemma 3.3 that for s large enough,

$$g(\tau) \leq 2C_0 e^{\tau - \bar{s}} \epsilon(\bar{s}), \quad \forall \tau \in [\bar{s}, s].$$

If $\tau = s$, then $e^{s-\bar{s}} = s$, $r = K_0\sqrt{s}$ and

$$g(s) \equiv L_\rho^{2,K_0\sqrt{s}}(Q(s)) = \mathcal{O}\left(\frac{1}{s^a}\right) + \mathcal{O}\left(\frac{\log s}{s}\right), \text{ as } \quad s \to +\infty.$$

By using the regularizing effects of the semigroup $S_{\mathcal{L}}$ (see Proposition 2.3 in [28]), we then obtain

$$\sup_{|y| \leq \frac{K_0\sqrt{s}}{2}} Q(y,s) \leq C'(C_*, K_0, M_0) L_\rho^{2,K_0\sqrt{s}}(Q(s)) = \mathcal{O}\left(\frac{1}{s^a}\right) + \mathcal{O}\left(\frac{\log s}{s}\right),$$

as $s \to +\infty$, which concludes the proof of Proposition 3.2. □

4 Numerical method

We give in this section a numerical study of the blow-up profile of equation (1.1) in one dimension. Though our method is very similar to Berger and Kohn's algorithm [3] in spirit, it is better in the sense that is can be applied to equations which are not invariant under the transformation (1.26). Our method differs from Berger and Kohn's in the following way: we step the solution forward until its maximum value multiplied by a power of its mesh size reaches a preset threshold, where the mesh size and the preset threshold are linked; for the rescaling algorithm, the solution is stepped forward until its maximum value reaches a preset threshold, and the mesh size and the preset threshold do not need to be linked. For more clarity, we present in the next subsection the mesh-refinement technique applied to equation (1.1), then give various numerical experiments to illustrate the effectiveness of our method for the problem of the numerical blow-up profile. Note that our method is more general than Berger and Kohn's [3], in the sense that it applies to non scale invariant equations. However, when applied to the unperturbed case $F(u) = |u|^{p-1}u$, our method gives exactly the same approximation as that of [3].

4.1 Mesh-refinement algorithm

In this section, we describe our refinement algorithm to solve numerically the problem (1.1) with initial data $\varphi(x) > 0$, $\varphi(x) = \varphi(-x)$, $x\frac{d\varphi(x)}{dx} < 0$ for $x \neq 0$, which gives a positive symmetric and radially decreasing solution. Let us rewrite the problem (1.1) (with $\mu = 1$) in the following:

$$\begin{cases} u_t &= u_{xx} + F(u), \quad (x,t) \in (-1,1) \times (0,T), \\ u(1,t) &= u(-1,t) = 0, \quad t \in (0,T), \\ u(x,0) &= \varphi(x), \quad x \in (-1,1), \end{cases} \tag{4.1}$$

where $p > 1$ and

$$F(u) = u^p + \frac{u^p}{\log^a(2+u^2)} \quad \text{with } \; a > 0. \tag{4.2}$$

Let $\hbar$ and τ be the initial space and time steps, we define $C_\Delta = \frac{\tau}{\hbar^2}$, $x^i = i\hbar$, $t^n = n\tau$, $I = \frac{1}{\hbar}$ and $u^{i,n}$ as the approximation of $u(x^i, t^n)$, where $u^{i,n}$ is defined for all $n \geq 0$, for all $i \in \{-I, \dots, I\}$ by

$$\begin{aligned} u^{i,n+1} &= u^{i,n} + C_\Delta \left[u^{i-1,n} - 2u^{i,n} + u^{i+1,n}\right] + \tau F\left(u^{i,n}\right), \\ u^{I,n} &= u^{-I,n} = 0, \quad u^{i,0} = \varphi_i. \end{aligned} \tag{4.3}$$

Note that this scheme is first order accurate in time and second order in space, and it requests the stability condition $C_\Delta = \frac{\tau}{\hbar^2} \leq \frac{1}{2}$.
Our algorithm needs to fix the following parameters:

- $\lambda < 1$: the refining factor with λ^{-1} being a small integer.
- M: the threshold to control the amplitude of the solution,
- α: the parameter controlling the width of interval to be refined.

The parameters λ and M must satisfy the following relation:

$$M = \lambda^{-\frac{2}{p-1}} M_0, \quad \text{where} \quad M_0 = \hbar^{\frac{2}{p-1}} \|\varphi\|_\infty. \tag{4.4}$$

Note that the relation (4.4) is important to make our method works. In [3], the typical choice is $M_0 = \|\varphi\|_\infty$, hence $M = \lambda^{-\frac{2}{p-1}} \|\varphi\|_\infty$.

In the initial step of the algorithm, we simply apply the scheme (4.3) until $\hbar^{\frac{2}{p-1}} \|u(\cdot, t^n)\|_\infty$ reaches M (note that in [3] the solution is stepped forward until $\|u(\cdot, t^n)\|_\infty$ reaches M; in this first step, the thresholds of the two methods are the same, however, they will split after the second step; roughly speaking, for the threshold we shall use the quantity $\hbar^{\frac{2}{p-1}} \|u(\cdot, t^n)\|_\infty$ in our method instead of $\|u(\cdot, t^n)\|_\infty$ in [3]). Then, we use a linear interpolation in time to find τ_0^* such that

$$t^n - \tau \leq \tau_0^* \leq t^n \quad \text{and} \quad \hbar^{\frac{2}{p-1}} \|u(\cdot, \tau_0^*)\| = M.$$

Afterward, we determine two grid points y_0^- and y_0^+ such that

$$\begin{cases} \hbar^{\frac{2}{p-1}} u(y_0^- - \hbar, \tau_0^*) < \alpha M \leq \hbar^{\frac{2}{p-1}} u(y_0^-, \tau_0^*) \\ \hbar^{\frac{2}{p-1}} u(y_0^+ + \hbar, \tau_0^*) < \alpha M \leq \hbar^{\frac{2}{p-1}} u(y_0^+, \tau_0^*). \end{cases} \tag{4.5}$$

Note that $y_0^- = -y_0^+$ because of the symmetry of the solution. This closes the initial step. Let us begin the first refining step. Define

$$u_1(y_1, t_1) = u(y_1, \tau_0^* + t_1), \quad y_1 \in (y_0^-, y_0^+),\ t_1 \geq 0, \tag{4.6}$$

and setting $h_1 = \lambda\hbar$, $\tau_1 = \lambda^2\tau$ as the space and time step for the approximation of u_1 (note that $\frac{\tau_1}{h_1^2} = \frac{\tau}{\hbar^2} = C_\Delta$ which is a constant), $y_1^i = ih_1$, $t_1^n = n\tau_1$, $I_1 = \frac{y_0^+}{h_1}$ and $u_1^{i,n}$ as the approximation of $u_1(y_1^i, t_1^n)$ (note that in the unperturbed case, Berger and Kohn used the transformation (1.26) to define $u_1(y_1, t_1) = \lambda^{\frac{2}{p-1}} u(\lambda y_1, \tau_0^* + \lambda^2 t_1)$, and then applied the same scheme for u to u_1. However, we can not do the same because the equation (4.1) is not in fact invariant under the transformation (1.26)). Then applying the scheme (4.3) to u_1 which reads

$$u_1^{i,n+1} = u_1^{i,n} + C_\Delta \left[u_1^{i-1,n} - 2u_1^{i,n} + u_1^{i+1,n} \right] + \tau_1 F\left(u_1^{i,n} \right), \tag{4.7}$$

for all $n \geq 0$ and for all $i \in \{-I_1 + 1, \dots, I_1 - 1\}$.
Note that the computation of u_1 requires the initial data $u_1(y_1, 0)$ and the boundary condition $u_1(y_0^\pm, t_1)$. For the initial condition, it is determined from $u(x, \tau_0^*)$ by using interpolation in space to get values at the new grid points. For the boundary condition, since $\tau_1 = \lambda^2\tau$, we then have from (4.6),

$$u_1(y_0^\pm, n\tau_1) = u(y_0^\pm, \tau_0^* + n\lambda^2\tau). \tag{4.8}$$

Since u and u_1 will be stepped forward, each on its own grid (u_1 on (y_0^-, y_0^+) with the space and time step h_1 and τ_1, and u on $(-1, 1)$ with the space and time step $\hbar$ and τ), the relation (4.8) will provide us with the boundary values for u_1. In order to better understand how it works, let us consider an example with $\lambda = \frac{1}{2}$. After closing the initial phase, the two solutions u_1 and u are stepped forward independently, each on its own grid, in other words, u_1 on (y_0^-, y_0^+) with the space and time step h_1 and τ_1, and u on $(-1, 1)$ with the space and time step $\hbar$ and τ. Then using the linear interpolation in time for u, we get the boundary values for u_1 by (4.8). Since $\tau_1 = \lambda^2\tau = \frac{1}{4}\tau$. This means that u is stepped forward once every 4 time steps of u_1. After 4 steps forward of u_1, the values of u on the interval (y_0^-, y_0^+) must be updated to agree with the calculations of u_1. In other words, the approximation of u is used to assist in computing the boundary values for u_1. At each successive time step for u, the values of u on the interval (y_0^-, y_0^+) must be updated to make them agree with the more accurate fine grid solution u_1. When $h_1^{\frac{2}{p-1}} \|u_1(\cdot, n\tau_1)\|_\infty$ first exceeds M, we use a linear interpolation in time to find $\tau_1^* \in [\tau_1^{n-1}, \tau_1^n]$ such that $h_1^{\frac{2}{p-1}} \|u_1(\cdot, \tau_1^*)\|_\infty = M$. On the interval where $h_1^{\frac{2}{p-1}} \|u_1(\cdot, \tau_1^*)\|_\infty > \alpha M$, the grid is refined further and the entire procedure as for u_1 is repeated to yield u_2 and so forth.

Before going to a general step, we would like to comment on the relation (4.4). Indeed, when $\hbar^{\frac{2}{p-1}} \|u(t)\|_\infty$ reaches the given threshold M in the initial phase, namely when $\hbar^{\frac{2}{p-1}} \|u(\cdot, \tau_0^*)\|_\infty = M$, we want to refine the grid such that the maximum values of $h_1^{\frac{2}{p-1}} u_1(y_1, 0)$ equals to M_0. By (4.6), this request turns into $h_1^{\frac{2}{p-1}} \|u(\cdot, \tau_0^*)\|_\infty = M_0$. Since $h_1 = \lambda\hbar$, it follows that $M = \lambda^{-\frac{2}{p-1}} M_0$, which yields (4.4).

Let $k \geq 0$, we set $h_{k+1} = \lambda^{-1} h_k$ and $\tau_{k+1} = \lambda^2 \tau_k$ (note that $\frac{\tau_{k+1}}{h_{k+1}^2} = \frac{\tau_k}{h_k^2} = C_\Delta$ which is a constant), y_{k+1} and t_{k+1} as the variables of u_{k+1}, $y_k^i = i h_k$, $t_k^n = n\tau_k$. The index $k = 0$ means that $u_0(y_0, t_0) \equiv u(x, t)$, $h_0 \equiv \hbar$ and $\tau_0 \equiv \tau$. The solution u_{k+1} is related to u_k by

$$u_{k+1}(y_{k+1}, t_{k+1}) = u_k(y_{k+1}, \tau_k^* + t_{k+1}), \quad y_{k+1} \in (y_k^-, y_k^+),\ t_{k+1} \geq 0. \tag{4.9}$$

Here, the time $\tau_k^* \in [t_k^{n-1}, t_k^n]$ satisfies $h_k^{\frac{2}{p-1}} \|u_k(\cdot, \tau_k^*)\|_\infty = M$, and y_k^-, y_k^+ are two grid points determined by

$$\begin{cases} h_k^{\frac{2}{p-1}} u_k(y_k^- - h_k, \tau_k^*) < \alpha M \leq h_k^{\frac{2}{p-1}} u_k(y_k^-, \tau_k^*), \\ h_k^{\frac{2}{p-1}} u_k(y_k^+ + h_k, \tau_k^*) < \alpha M \leq h_k^{\frac{2}{p-1}} u_k(y_k^+, \tau_k^*). \end{cases} \tag{4.10}$$

The approximation of $u_{k+1}(y_{k+1}^i, t_{k+1}^n)$ (denoted by $u_{k+1}^{i,n}$) uses the scheme (4.3) with the space step h_{k+1} and the time step τ_{k+1}, which reads

$$u_{k+1}^{i,n+1} = u_{k+1}^{i,n} + C_\Delta \left[u_{k+1}^{i-1,n} - 2u_{k+1}^{i,n} + u_{k+1}^{i+1,n}\right] + \tau_{k+1} F\left(u_{k+1}^{i,n}\right), \tag{4.11}$$

for all $n \geq 1$ and $i \in \{-I_k + 1, \cdots, I_k - 1\}$ with $I_k = \frac{y_k^+}{h_{k+1}}$ (note from introduction that I_k is an integer since $\lambda^{-1} \in \mathbb{N}$).
As for the approximation of u_k, the computation of $u_{k+1}^{i,n}$ needs the initial data and the boundary condition. From (4.9) and the fact that $\tau_{k+1} = \lambda^2 \tau_k$, we see that

$$u_{k+1}(y_{k+1}, 0) = u_k(y_{k+1}, \tau_k^*) \quad \text{and} \quad u_{k+1}(y_k^\pm, n\tau_{k+1}) = u_k(y_k^\pm, \tau_k^* + n\lambda^2\tau_k). \tag{4.12}$$

Hence, from the first identity in (4.12), the initial data is simply calculated from $u_k(\cdot, \tau_k^*)$ by using a linear interpolation in space in order to assign values at new grid points. The essential step in this new mesh-refinement method is to determine the boundary condition through the second identity in (4.12). This means by a linear interpolation in time of u_k. Therefore, the previous solutions u_k, u_{k-1}, $\cdots$ are stepped forward independently, each on its own grid. More precisely, since $\tau_{k+1} = \lambda^2 \tau_k = \lambda^4 \tau_{k-1} = \cdots$, then u_k is stepped forward once every λ^{-2} time steps of u_{k+1}; u_{k-1} once every λ^{-4} time steps of u_{k+1}, ... On the other hand, the values of u_k, u_{k-1}, ... must be updated to agree with the calculation of u_{k+1}. When $h_{k+1}^{\frac{2}{p-1}} \|u_{k+1}(\cdot, \tau_{k+1})\|_\infty > M$, then it is time for the next refining phase.

We would like to comment on the output of the refinement algorithm:

i) Let τ_k^* be the time at which the refining takes place, then the ratio $\frac{\tau_k^*}{\tau_k}$, which indicates the number of time steps until $h_k^{\frac{2}{p-1}} \|u_k\|_\infty$, reaches the given threshold M, is independent of k and tends to a constant as $k \to \infty$.

ii) Let $u_k(\cdot, \tau_k^*)$ be the *refining solution*. If we plot $h_k^{\frac{2}{p-1}} u_k(\cdot, \tau_k^*)$ on $(-1, 1)$, then their graphs are eventually independent of k and converge as $k \to \infty$.

iii) Let (y_k^-, y_k^+) be the interval to be refined, then the quality $(h_k^{-1}y_k^+)^2$ behaves as a linear function of k.

These assertions can be well understood by the following theorem:

Theorem 4.1 (Formal analysis)**.** *Let u be a blowing-up solution to equation (4.1), then the output of the refinement algorithm satisfies:*
$i)$ The ratio $\frac{\tau_k^*}{\tau_k}$ is independent of k and tends to a constant as $k \to \infty$, namely

$$\frac{\tau_k^*}{\tau_k} \to \frac{(\lambda^{-2}-1)M^{1-p}}{C_\Delta(p-1)}, \quad \textit{as} \;\; k \to +\infty. \tag{4.13}$$

Assume in addition that $i)$ of Theorem 1.7 holds,
$ii)$ Defining $v_k(z) = h_k^{\frac{2}{p-1}} u_k(zy_{k-1}^+, \tau_k^*)$ for all $k \geq 1$, we have

$$\forall |z| < 1, \;\; v_k(z) \sim M\left(1 + (\alpha^{1-p} - 1)\lambda^{-2}z^2\right)^{-\frac{1}{p-1}} \quad \textit{as} \;\; k \to +\infty. \tag{4.14}$$

$iii)$ The quality $(h_k^{-1}y_k^+)^2$ behaves as a linear function, namely

$$(h_k^{-1}y_k^+)^2 \sim \gamma k + B \quad \textit{as} \;\; k \to +\infty. \tag{4.15}$$

where $\gamma = \frac{2M^{1-p}(\alpha^{1-p}-1)|\log\lambda|}{c_p(p-1)\lambda^2}$, $B = -\frac{M^{1-p}(\alpha^{1-p}-1)}{c_p(p-1)\lambda^2}\log\left(\frac{M^{1-p}\hbar^2}{p-1}\right)$ and $c_p = \frac{p-1}{4p}$.

Remark 4.2. *Note that there is no assumption on the value of a in the hypothesis in Theorem 4.1. It is understood in the sense that u blows up in finite time and its profile is described in Theorem 1.7.*

Proof. As we will see in the proof that the statement $i)$ concerns the blow-up limit of the solution and the second one is due to the blow-up profile stated in Theorem 1.7.
$i)$ Let σ_k is the real time when the refinement from u_k to u_{k+1} takes place, we have by (4.9),

$$\sigma_k = \tau_0^* + \tau_1^* + \cdots + \tau_k^*,$$

where τ_j^* is such that $h_j^{\frac{2}{p-1}}\|u_k(\cdot, \tau_j^*)\|_\infty = M$. This means that

$$u_k(\cdot, \tau_k^*) = u(\cdot, \sigma_k). \tag{4.16}$$

On the other hand, from $i)$ of Theorem 1.7 and the definition (1.23) of f, we see that

$$\lim_{t \to T}(T-t)^{\frac{1}{p-1}}\|u(t)\|_{L^\infty} = \kappa. \tag{4.17}$$

Combining (4.17) and (4.16) yields

$$(T-\sigma_k)^{\frac{1}{p-1}}\|u_k(\cdot, \tau_k^*)\|_\infty = \kappa + o(1), \tag{4.18}$$

where $o(1)$ represents a term that tends to 0 as $k \to +\infty$.
Since $\|u_k(\cdot, \tau_k^*)\|_\infty = M h_k^{\frac{-2}{p-1}}$, we then derive

$$T - \sigma_k = \left(M^{-1}\kappa\right)^{p-1} h_k^2 + o(1). \tag{4.19}$$

By the definition of σ_k and (4.16), we infer that $\tau_k^* = \sigma_k - \sigma_{k-1}$ (we can think τ_k^* as the *live time* of u_k in the k-th refining phase). Hence,

$$\begin{aligned}\frac{\tau_k^*}{\tau_k} = \frac{\sigma_k - \sigma_{k-1}}{\tau_k} &= \frac{1}{\tau_k}\left[(T - \sigma_{k-1}) - (T - \sigma_k)\right]\\ &= \frac{1}{\tau_k}\left(M^{-1}\kappa\right)^{p-1}(h_{k-1}^2 - h_k^2) + o(1)\\ &= \frac{h_k^2}{\tau_k}\left(M^{-1}\kappa\right)^{p-1}(\lambda^{-2} - 1) + o(1).\end{aligned}$$

Since the ratio $\frac{\tau_k}{h_k^2}$ is always fixed by the constant C_Δ, we finally obtain

$$\lim_{k\to+\infty} \frac{\tau_k^*}{\tau_k} = \frac{(\lambda^{-2} - 1)M^{1-p}}{C_\Delta(p-1)},$$

which concludes the proof of part $i)$ of Theorem 4.1.

$ii)$ Since the symmetry of the solution, we have $y_{k-1}^- = y_{k-1}^+$. We then consider the following mapping: for all $k \geq 1$,

$$\forall |z| \leq 1, \quad z \mapsto v_k(z), \quad \text{where} \quad v_k(z) = h_k^{\frac{2}{p-1}} u_k(z y_{k-1}^+, \tau_k^*).$$

We will show that $v_k(z)$ is independently of k and converges as $k \to +\infty$. For this purpose, we first write $u_k(y_k, \tau_k *)$ in term of $w(\xi, s)$ thanks to (4.16) and (1.8),

$$u_k(y_k, \tau_k^*) = u(y_k, \sigma_k) = (T - \sigma_k)^{-\frac{1}{p-1}} w(\xi_k, s_k), \tag{4.20}$$

where $\xi_k = \frac{y_k}{\sqrt{T - \sigma_k}}$ and $s_k = -\log(T - \sigma_k)$.
If we write $i)$ of Theorem 1.7 in the variable $\frac{y}{\sqrt{s}}$ through (1.8), we have the following equivalence:

$$\left\| w(y,s) - f\left(\frac{y}{\sqrt{s}}\right)\right\|_{L^\infty} \to 0 \quad \text{as } s \to +\infty, \tag{4.21}$$

where f is given in (1.23).
From (4.21), (4.19) and (4.20), we derive

$$u_k(y_k, \tau_k^*) = M\kappa^{-1} h_k^{-\frac{2}{p-1}} f\left(\frac{y_k}{(M^{-1}\kappa)^{\frac{p-1}{2}} h_k \sqrt{s_k}}\right) + o(1).$$

Then multiplying both of sides by $h_k^{\frac{2}{p-1}}$ and replacing y_k by zy_{k-1}^+, we obtain

$$h_k^{\frac{2}{p-1}} u_k(zy_{k-1}^+, \tau_k^*) = M\kappa^{-1} f\left(\frac{zy_{k-1}^+}{(M^{-1}\kappa)^{\frac{p-1}{2}} h_k \sqrt{s_k}}\right) + o(1). \tag{4.22}$$

From the definition (4.10) of y_{k-1}^+, we may assume that

$$h_{k-1}^{\frac{2}{p-1}} u_{k-1}(y_{k-1}^+, \tau_{k-1}^*) = \alpha M.$$

Combining this with (4.22), we have

$$\alpha = \kappa^{-1} f\left(\frac{y_{k-1}^+}{(M^{-1}\kappa)^{\frac{p-1}{2}} h_{k-1}\sqrt{s_{k-1}}}\right) + o(1).$$

Since $s_k = -\log(T - \sigma_k)$ and the fact that $h_k = \lambda^k \hbar$, we have from (4.19) that

$$s_k = 2k|\log\lambda| - \log\left(\frac{M^{1-p}\hbar^2}{p-1}\right) + o(1), \tag{4.23}$$

which follows $\lim_{k\to+\infty} \frac{s_{k-1}}{s_k} = 1$. Thus, it is reasonable to assume that $\frac{y_{k-1}^+}{\sqrt{s_{k-1}}}$ and $\frac{y_{k-1}^+}{\sqrt{s_k}}$ tend to the positive root ζ as $k \to +\infty$. Hence,

$$\alpha = \kappa^{-1} f\left(\frac{\zeta}{(M^{-1}\kappa)^{\frac{p-1}{2}} h_k \lambda^{-1}}\right) + o(1).$$

Using the definition (1.23) of f, we have

$$\alpha = \left(1 + c_p \left|\frac{\zeta}{(M^{-1}\kappa)^{\frac{p-1}{2}} h_k}\right|^2 \lambda^2\right)^{-\frac{1}{p-1}} + o(1),$$

which follows

$$\left|\frac{\zeta}{(M^{-1}\kappa)^{\frac{p-1}{2}} h_k}\right|^2 = \frac{1}{c_p}\left[(\alpha^{1-p} - 1)\lambda^{-2}\right] + o(1), \tag{4.24}$$

where c_p is the constant given in the definition (1.23) of f.
Substituting this into (4.22) and using again the definition (1.23) of f, we arrive at

$$\begin{aligned} v_k(z) &= M\left(1 + c_p \left|\frac{\zeta}{(M^{-1}\kappa)^{\frac{p-1}{2}} h_k}\right|^2 z^2\right)^{-\frac{1}{p-1}} + o(1) \\ &= M\left(1 + (\alpha^{1-p} - 1)\lambda^{-2} z^2\right)^{-\frac{1}{p-1}} + o(1). \end{aligned}$$

Let $k \to +\infty$, we then obtain the conclusion ii).

iii) From (4.24) and the fact that $\frac{y_k^+}{\sqrt{s_k}} \to \zeta$ as $k \to +\infty$, we have

$$(h_k^{-1} y_k^+)^2 = \frac{(\alpha^{1-p} - 1)M^{1-p}}{c_p \lambda^2 (p-1)} \log s_k + o(1).$$

Using (4.23), we then derive

$$(h_k^{-1} y_k^+)^2 = \frac{2k|\log\lambda|(\alpha^{1-p} - 1)M^{1-p}}{c_p \lambda^2 (p-1)} - \frac{(\alpha^{1-p} - 1)M^{1-p}}{c_p \lambda^2 (p-1)} \log\left(\frac{M^{1-p}\hbar^2}{p-1}\right) + o(1),$$

which yields the conclusion iii) and completes the proof of Theorem 4.1. □

4.2 The numerical results

This subsection gives various numerical confirmation for the assertions stated in the previous subsection (Theorem 4.1). All the experiments reported here used $\varphi(x) = 2(1 + cos(\pi x))$ as the initial data, $\alpha = 0.6$ as the parameter for controlling the interval to be refined, $\lambda = \frac{1}{2}$ as the refining factor, $C_\Delta = \frac{1}{4}$ as the stability condition for the scheme (4.3), $p = 3$ and $a = 0.1, 1, 10$ in the nonlinearity F given in (4.2). The threshold M is chosen to be satisfied the condition (4.4). In Table 4.1, we give some values of M corresponding the initial data and the initial space step $\hbar$. We generally stop the

$\hbar$	0.040	0.020	0.010	0.005
M	0.320	0.160	0.080	0.040

Table 4.1. *The value of M corresponds to the initial data and the initial space step.*

computation after 40 refining phases. Indeed, since $h_k^{\frac{2}{p-1}} \|u_k(\cdot, \tau_k^*)\|_\infty = M$ and the fact that $h_k = \lambda h_{k-1}$, we have by induction,

$$\|u_k(\cdot, \tau_k^*)\|_\infty = h_k^{-\frac{2}{p-1}} M = (\lambda h_{k-1})^{-\frac{2}{p-1}} M = \cdots = (\lambda^k \hbar)^{-\frac{2}{p-1}} M.$$

With these parameters, we see that the corresponding amplitude of u approaches 10^{12} after 40 iterations.

i) **The value $\frac{\tau_k^*}{\tau_k}$ is independent of k and tends to the constant as $k \to +\infty$.**

It is convenient to denote the computed value of $\frac{\tau_k^*}{\tau_k}$ by N_k and the predicted value given in the statement i) of Theorem 4.1 by N_{pre}. Note that the values of N_{pre} does not depend on a but depend on $\hbar$ because of the relation (4.4). More precisely,

$$N_{pre}(\hbar) = \frac{(1-\lambda^2)\|\varphi\|_\infty^{1-p}}{C_\Delta (p-1)\hbar^2}.$$

Then considering the quality $\frac{N_k}{N_{pre}}$, theoretically, it is expected to converge to 1 as k tends to infinity. Table 4.2 provides computed values of $\frac{N_k}{N_{pre}}$ at some selected indexes of k, for computing with $\hbar = 0.005$ and three different values of a. According to the numerical results given in Table 4.2, the computed values in the case $a = 10$ and $a = 1.0$ approach to 1 as expected which gives us a numerical answer for the statement (4.17). However the numerical results in the case $a = 0.1$ is not good due to the fact that the speech of convergence to the blow-up limit (4.17) is $\frac{1}{|\log(T-t)|^{a'}}$ with $a' = \min\{a, 1\}$ (see Theorem 1.4).

k	$a = 10$	$a = 1.0$	$a = 0.1$
10	1.0325	0.9699	0.5853
15	1.0203	0.9771	0.5885
20	1.0149	0.9816	0.5923
25	1.0117	0.9845	0.5957
30	1.0096	0.9867	0.5989
35	1.0080	0.9885	0.6016
40	1.0072	0.9899	0.6043

Table 4.2. *The values of* $\frac{N_k}{N_{pre}}$ *at some selected indexes of k, for computing with* $\hbar = 0.005$ *and three different values of* a.

$ii)$ **The function** $v_k(z)$ **introduced in part** $ii)$ **of Theorem 4.1 converges to a predicted profile as** $k \to +\infty$**.**
As stated in part $ii)$ of Theorem 4.1, if we plot $v_k(z)$ over the fixed interval $(-1, 1)$, then the graph of v_k would converge to the predicted one. Figure 4.1 gives us a numerical confirmation for this fact, for computing with $\hbar = 0.005$ and $a = 10$. Looking at Figure 4.1, we see that the graph of v_k evidently converges to the predicted one given in the right-hand side of (4.14) as k increases. The last curve v_{40} seemly coincides to the prediction. Figure 4.2 shows the graph of v_{40} and the predicted profile for an other experiment with $\hbar = 0.005$ and $a = 0.1$. They coincide to within plotting resolution.

In Table 4.3, we give the error in L^∞ between $v_k(z)$ at index $k = 40$ and the predicted profile given in the right hand-side of (4.14), namely

$$e_{\hbar,a} = \sup_{z\in(-1,1)} \left| v_{40}(z) - M\left(1 + (\alpha^{1-p} - 1)\lambda^{-2}z^2\right)^{-\frac{1}{p-1}} \right|. \tag{4.25}$$

These numerical computations give us a confirmation that the computed profiles v_k converges to the predicted one. Since the error $e_{\hbar,a}$ tends to 0 as $\hbar$ goes to zero, the numerical computations also answer to the stability of the blow-up profile stated in $i)$ of Theorem 1.7. In fact, the stability makes the solution visible in numerical simulations.

$iii)$ **The quality** $(h_k^{-1}y_k^+)^2$ **behaves like a linear function.**

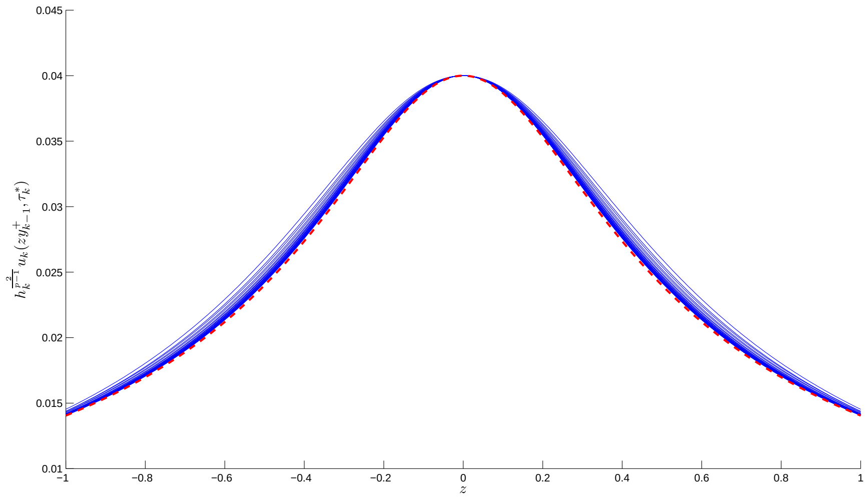

Figure 4.1. *The graph of $v_k(z)$ at some selected indexes of k, for computing with $\hbar = 0.005$ and $a = 10$. They converge to the predicted profile (the dash line) as stated in* (4.14) *as k increases.*

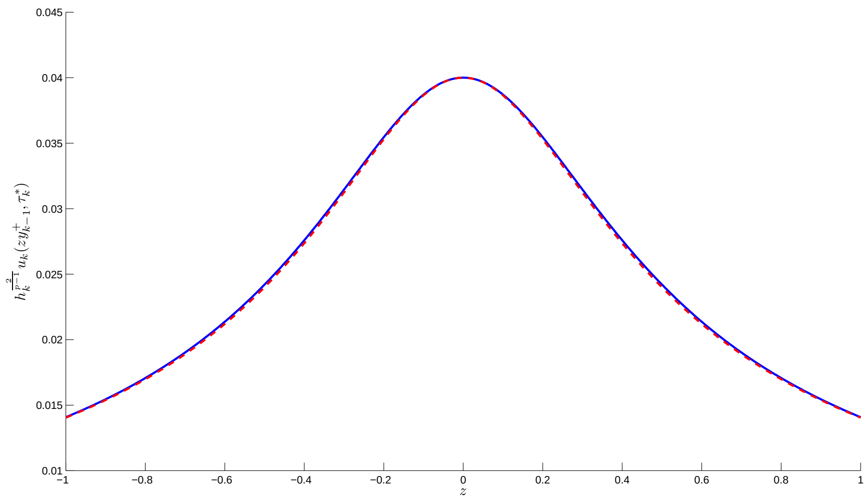

Figure 4.2. *The graph of $v_k(z)$ at $k = 40$ and the predicted profile given in* (4.14)*, for computing with $\hbar = 0.005$ and $a = 0.1$. They coincide within plotting resolution.*

$\hbar$	$a = 10$	$a = 1.0$	$a = 0.1$
0.04	0.002906	0.001769	0.002562
0.02	0.000789	0.000671	0.000687
0.01	0.000470	0.000359	0.000380
0.005	0.000238	0.000213	0.000235

Table 4.3. *Error in L^∞ between the computed and predicted profiles, $e_{\hbar,a}$ defined in* (4.25).

For making a quantitative comparison between our numerical results and the predicted behavior as stated in $iii)$ of Theorem 4.1, we plot the graph of $(h_k^{-1}y_k^+)^2$ against k and denote by $\gamma_{\hbar,a}$ the slope of this curve. Then considering the ratio $\frac{\gamma_{\hbar,a}}{\gamma}$, where γ is given in part $iii)$ of Theorem 4.1. As expected, this ratio $\frac{\gamma_{\hbar,a}}{\gamma}$ would approach one. Figure 4.3 shows $(h_k^{-1}y_k^+)^2$ as a function of k for computing with the initial space step $\hbar = 0.005$ for different values of a. Looking at Figure 4.3, we see that the two middle curves corresponding the case $a = 10$ and $a = 1$ behave like the predicted linear function (the top line), while this is not true in the case $a = 0.1$ (the bottom curve). In order to make this clearer, let us see Table 4.4 which lists the values of $\frac{\gamma_{\hbar,a}}{\gamma}$ for computing with various values of the initial space step $\hbar$ for three different values of a. Here, the value of $\gamma_{\hbar,a}$ is calculated for $20 \leq k \leq 40$. As Table 4.4 shows that the numerical values in the case $a = 10$ and $a = 1$ agree with the prediction stated in $ii)$ of Theorem 4.1, while the numerical values in the case $a = 0.1$ is far from the predicted one.

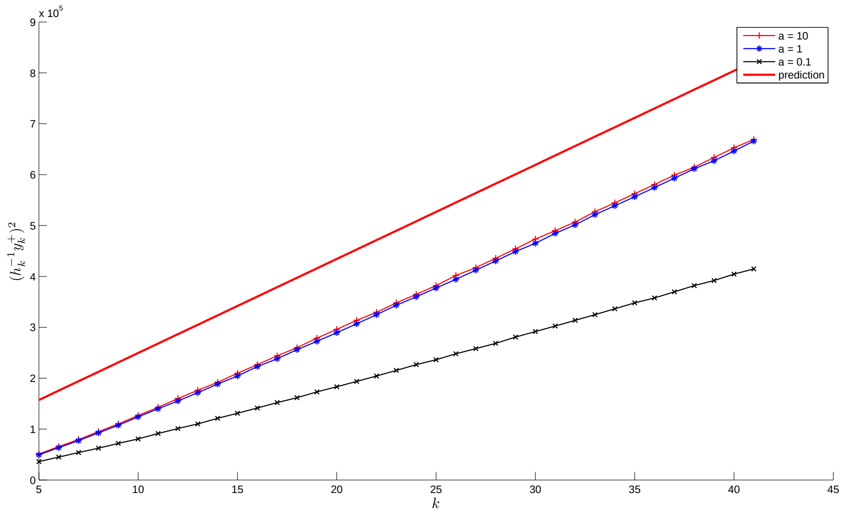

Figure 4.3. *The graph of $(h_k^{-1}y_k^+)^2$ against k, for computing with $\hbar = 0.005$ for three different values of a.*

$\hbar$	$a = 10$	$a = 1.0$	$a = 0.1$
0.04	1.9514	1.9863	1.9538
0.02	1.1541	1.1436	0.8108
0.01	0.9991	1.0052	0.6417
0.005	0.9669	0.9682	0.5986

Table 4.4. *The values of* $\frac{\gamma_{\hbar,a}}{\gamma}$ *for computing with various values of the initial space step* $\hbar$ *for three different values of* a.

A Appendix A

The following lemma from [26] gives the expansion of $\phi(s)$, the unique solution of equation (1.18) satisfying (1.19):

Lemma A.1. *Let* ϕ *be a positive solution of the following ordinary differential equation:*

$$\phi_s = -\frac{\phi}{p-1} + \phi^p + \frac{\mu\phi^p}{\log^a(2 + e^{\frac{2s}{p-1}}\phi^2)}.$$

Assuming in addition $\phi(s) \to \kappa$ *as* $s \to +\infty$*, then* $\phi(s)$ *takes the following form:*

$$\phi(s) = \kappa(1 + \eta_a(s))^{-\frac{1}{p-1}} \quad \textit{as} \quad s \to +\infty,$$

where

$$\eta_a(s) \sim C_* \int_s^{+\infty} \frac{e^{s-\tau}}{\tau^a} d\tau = \frac{C_*}{s^a}\left(1 + \sum_{j=1}^{k} \frac{b_j}{s^j}\right) + \mathcal{O}\left(\frac{1}{s^{a+k+1}}\right), \quad \forall k \in \mathbb{N},$$

with $C_* = \mu\left(\frac{p-1}{2}\right)^a$ *and* $b_j = (-1)^j \prod_{i=0}^{j-1}(a+i)$.

Proof. See Lemma A.3 in [26]. □

B Appendix B

We aim at proving the following:

Lemma B.1 (Estimate of $\omega(s)$). *We have*

$$|\omega(s)| = \mathcal{O}\left(\frac{1}{s^{a+1}}\right), \quad \textit{as} \quad s \to +\infty.$$

Proof. From Lemma A.1, we write

$$p(\phi(s)^{p-1} - \kappa^{p-1}) = -\frac{p\eta_a(s)}{p-1}(1+\eta_a(s))^{-1} = -\frac{pC_*}{(p-1)s^a}(1+\eta_a(s))^{-1} + \mathcal{O}\left(\frac{1}{s^{a+1}}\right).$$

A direct calculation yields,

$$\begin{aligned} e^{-s}h'\left(e^{\frac{p}{p-1}}\phi(s)\right) &= \frac{\mu p\phi^{p-1}(s)}{\log^a(2+e^{\frac{2s}{p-1}}\phi^2(s))} - \frac{2a\mu e^{\frac{2s}{p-1}}\phi^{p+1}(s)}{(2+e^{\frac{2s}{p-1}}\phi^2(s))\log^{a+1}(2+e^{\frac{2s}{p-1}}\phi^2(s))} \\ &= \frac{pC_*}{(p-1)s^a}(1+\eta_a(s))^{-1} + \mathcal{O}\left(\frac{1}{s^{a+1}}\right). \end{aligned}$$

Adding the two above estimates, we obtain the desired result. This ends the proof of Lemma B.1. □

Lemma B.2 (Estimate of $R(y,s)$). *We have*

$$|R(y,s)| = \mathcal{O}\left(\frac{|y|^2+1}{s^{a'+1}}\right), \quad \textit{as} \quad s \to +\infty,$$

with $a' = \min\{1, a\}$.

Proof. Let us write $\varphi(y,s) = \frac{\phi(s)}{\kappa}\nu(y,s)$ where

$$\nu(y,s) = \kappa\left(1 + \frac{p-1}{4ps}\sum_{j=1}^{l} y_j^2\right)^{-\frac{1}{p-1}} + \frac{\kappa l}{2ps}.$$

Then, we write $R(y,s) = \frac{\phi(s)}{\kappa}R_1(y,s) + R_2(y,s)$ where

$$\begin{aligned} R_1(y,s) &= \nu_s - \Delta\nu - \frac{y}{2}\cdot\nabla\nu - \frac{\nu}{p-1} + \nu^p, \\ R_2(y,s) &= -\frac{\phi'}{\kappa}\nu - \frac{\phi}{\kappa}\nu^p + \phi^p\left(\frac{\nu}{\kappa}\right)^p + e^{-\frac{ps}{p-1}}h'\left(e^{\frac{s}{p-1}}\frac{\phi\nu}{\kappa}\right). \end{aligned}$$

The term $R_1(y,s)$ is already treated in [28], and it is bounded by

$$|R_1(y,s)| \le \frac{C(|y|^2+1)}{s^2} + C\mathbf{1}_{\{|y|\ge 2K_0\sqrt{s}\}}.$$

To bound R_2, we use the fact that ϕ satisfies (1.19) to write

$$\begin{aligned} R_2(y,s) = &\frac{\nu\phi}{\kappa^p}(\kappa^{p-1} - \phi^{p-1})(\kappa^{p-1} - \nu^{p-1}) \\ &+ e^{-\frac{ps}{p-1}}\left[h\left(e^{\frac{s}{p-1}}\frac{\phi\nu}{\kappa}\right) - h\left(e^{\frac{s}{p-1}}\phi\right)\right] \\ &+ \left(1 - \frac{\nu}{\kappa}\right)e^{-\frac{ps}{p-1}}h\left(e^{\frac{s}{p-1}}\phi\right). \end{aligned}$$

Noting that $\nu(y,s) = \kappa + \bar{\nu}(y,s)$ with $|\bar{\nu}(y,s)| \leq \frac{C}{s}(|y|^2+1)$, uniformly for $y \in \mathbb{R}$ and $s \geq 1$, and recalling from Lemma A.1 that $\phi(s) = \kappa(1+\eta_a(s))^{-\frac{1}{p-1}}$ where $\eta_a(s) = \mathcal{O}(s^{-a})$, then using a Taylor expansion, we derive

$$|R_2(y,s)| \leq C\left(\frac{|y|^2+1}{s^{a+1}} + \mathbf{1}_{\{|y|\geq 2K_0\sqrt{s}\}}\right).$$

This concludes the proof of Lemma B.2. □

Bibliography

[1] J. M. Ball. Remarks on blow-up and nonexistence theorems for nonlinear evolution equations. *Quart. J. Math. Oxford Ser. (2)*, 28(112):473–486, 1977.

[2] G. Baruch, G. Fibich, and N. Gavish. Singular standing-ring solutions of nonlinear partial differential equations. *Phys. D*, 239(20-22):1968–1983, 2010.

[3] M. Berger and R. V. Kohn. A rescaling algorithm for the numerical calculation of blowing-up solutions. *Comm. Pure Appl. Math.*, 41(6):841–863, 1988.

[4] J. Bricmont and A. Kupiainen. Universality in blow-up for nonlinear heat equations. *Nonlinearity*, 7(2):539–575, 1994.

[5] C. Fermanian Kammerer, F. Merle, and H. Zaag. Stability of the blow-up profile of non-linear heat equations from the dynamical system point of view. *Math. Ann.*, 317(2):347–387, 2000.

[6] C. Fermanian Kammerer and H. Zaag. Boundedness up to blow-up of the difference between two solutions to a semilinear heat equation. *Nonlinearity*, 13(4):1189–1216, 2000.

[7] S. Filippas and R. V. Kohn. Refined asymptotics for the blowup of $u_t - \Delta u = u^p$. *Comm. Pure Appl. Math.*, 45(7):821–869, 1992.

[8] S. Filippas and W. X. Liu. On the blowup of multidimensional semilinear heat equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(3):313–344, 1993.

[9] A. Friedman and B. McLeod. Blow-up of positive solutions of semilinear heat equations. *Indiana Univ. Math. J.*, 34(2):425–447, 1985.

[10] H. Fujita. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I*, 13:109–124 (1966), 1966.

[11] Y. Giga and R. V. Kohn. Characterizing blowup using similarity variables. *Indiana Univ. Math. J.*, 36(1):1–40, 1987.

[12] Y. Giga and R. V. Kohn. Nondegeneracy of blowup for semilinear heat equations. *Comm. Pure Appl. Math.*, 42(6):845–884, 1989.

[13] Y. Giga, S. Matsui, and S. Sasayama. Blow up rate for semilinear heat equations with subcritical nonlinearity. *Indiana Univ. Math. J.*, 53(2):483–514, 2004.

[14] M. A. Hamza and H. Zaag. Lyapunov functional and blow-up results for a class of perturbations of semilinear wave equations in the critical case. *J. Hyperbolic Differ. Equ.*, 9(2):195–221, 2012.

[15] M. A. Hamza and H. Zaag. A Lyapunov functional and blow-up results for a class of perturbed semilinear wave equations. *Nonlinearity*, 25(9):2759–2773, 2012.

[16] M. A. Herrero and J. J. L. Velázquez. Generic behaviour of one-dimensional blow up patterns. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 19(3):381–450, 1992.

[17] M. A. Herrero and J. J. L. Velázquez. Blow-up profiles in one-dimensional, semilinear parabolic problems. *Comm. Partial Differential Equations*, 17(1-2):205–219, 1992.

[18] M. A. Herrero and Juan J. L. Velázquez. Comportement générique au voisinage d'un point d'explosion pour des solutions d'équations paraboliques unidimensionnelles. *C. R. Acad. Sci. Paris Sér. I Math.*, 314(3):201–203, 1992.

[19] M. A. Herrero and J. J. L. Velázquez. Flat blow-up in one-dimensional semilinear heat equations. *Differential Integral Equations*, 5(5):973–997, 1992.

[20] M. A. Herrero and J. J. L. Velázquez. Blow-up behaviour of one-dimensional semilinear parabolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(2):131–189, 1993.

[21] H. A. Levine. Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$. *Arch. Rational Mech. Anal.*, 51:371–386, 1973.

[22] N. Masmoudi and H. Zaag. Blow-up profile for the complex Ginzburg-Landau equation. *J. Funct. Anal.*, 255(7):1613–1666, 2008.

[23] F. Merle and H. Zaag. Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$. *Duke Math. J.*, 86(1):143–195, 1997.

[24] C. E. Mueller and F. B. Weissler. Single point blow-up for a general semilinear heat equation. *Indiana Univ. Math. J.*, 34(4):881–913, 1985.

[25] V. T. Nguyen. Numerical analysis of the rescaling method for parabolic problems with blow-up in finite time. *submitted, arXiv:1403.7547*.

[26] V. T. Nguyen. On the blow-up results for a class of strongly perturbed semilinear heat equation. *submitted, arXiv:1404.4018*.

[27] V. T. Nguyen and H. Zaag. Contruction of a stable blow-up solution for a class of strongly perturbed semilinear heat equation. *submitted, arXiv:1406.5233*.

[28] J. J. L. Velázquez. Higher-dimensional blow up for semilinear parabolic equations. *Comm. Partial Differential Equations*, 17(9-10):1567–1596, 1992.

[29] J. J. L. Velázquez. Classification of singularities for blowing up solutions in higher dimensions. *Trans. Amer. Math. Soc.*, 338(1):441–464, 1993.

[30] F. B. Weissler. An L^∞ blow-up estimate for a nonlinear heat equation. *Comm. Pure Appl. Math.*, 38(3):291–295, 1985.

[31] F. B. Weissler. Existence and nonexistence of global solutions for a semilinear heat equation. *Israel J. Math.*, 38(1-2):29–40, 1981.

Address: Université Paris 13, Institut Galilée, LAGA,
99 Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France.

E-mail: vtnguyen@math.univ-paris13.fr
Hatem.Zaag@univ-paris13.fr

Étude numérique et théorique du profil à l'explosion dans les équations paraboliques non linéaires.

On s'intéresse au phénomène d'explosion en temps fini dans les équations aux dérivées partielles paraboliques non linéaires, particulièrement au profil à l'explosion, des points de vue numérique et théorique.

Dans la partie théorique, on s'intéresse au phénomène d'explosion en temps fini pour une classe d'équations semi-linéaires de la chaleur perturbées fortement avec l'exposant sous-critique de Sobolev. Travaillant dans le cadre des *variables auto-similaires*, on obtient d'abord l'existence d'une fonctionnelle de Lyapunov, ce qui constitue une étape cruciale pour établir le taux d'explosion de la solution. Dans une seconde étape, on s'intéresse à la structure de la solution au voisinage du temps et du point d'explosion. On classifie tous les comportements asymptotiques possibles pour la solution quand elle s'approche de la singularité. Ensuite, on décrit les profils à l'explosion correspondant à ces comportements asymptotiques. Dans une troisième étape, on construit pour cette équation une solution qui explose en temps fini en un seul point avec un profil d'explosion prescrit. Cette construction s'appuie sur la réduction en dimension finie du problème et sur l'utilisation du théorème de l'indice pour conclure.

Dans la partie numérique, on se propose de développer des méthodes afin de donner des réponses numériques à la question du profil à l'explosion pour certaines équations paraboliques, y compris le modèle de Ginzburg-Landau. Nous proposons deux méthodes. La première est *l'algorithme de remise à l'échelle* (rescaling) proposé par Berger et Kohn en 1988, appliqué à des équations paraboliques satisfaisant une propriété d'*invariance d'échelle*. Cette propriété nous permet de faire un zoom de la solution quand elle est proche de la singularité, tout en gardant la même équation. Le principal avantage de cette méthode est sa capacité à donner une très bonne approximation numérique qui nous permet d'atteindre numériquement le profil à l'explosion. Le profil à l'explosion que l'on obtient numériquement est en bon accord avec le profil théorique. De plus, en considérant une équation de la chaleur non linéaire critique avec un terme de gradient non linéaire, avec peu de résultats théoriques, nous énonçons une conjecture sur le profil à l'explosion, grâce à nos simulations numériques. La deuxième méthode numérique s'appuie aussi sur un raffinement de maillage, dans l'esprit de *l'algorithme de remise à l'échelle* de Berger et Kohn. Cette méthode est applicable à une plus grande classe d'équations dont les solutions explosent en temps fini sans la propriété d'*invariance d'échelle*.
Mots clés : Équation semi-linéaire de la chaleur, perturbation d'ordre inférieur, singularité, explosion numérique, explosion en temps fini, profil, stabilité, comportement asymptotique, équation complexe de Ginzburg-Landau.

Numerical and theoretical study of the blow-up profile in nonlinear parabolic equations.

We are interested in finite-time blow-up phenomena arising in the study of Nonlinear Parabolic Partial Differential Equations, in particular in the blow-up profile, under the theoretical and numerical aspects.

In the theoretical direction, we are interested in particular in finite-time blow-up phenomena for some class of strongly perturbed semilinear heat equations with Sobolev subcritical power nonlinearity. Working in the framework of *similarity variables*, we first derive a Lyapunov functional in similarity variables which is a crucial step to derive the blow-up rate of the solution. In a second step, we are interested in the structure of the solution near blow-up time and point. We classify all possible asymptotic behaviors of the solution when it approaches to the singularity. Then we describe blow-up profiles corresponding to these asymptotic behaviors. In a third step, we construct for this equation a solution which blows up in finite time at only one blow-up point with a prescribed blow-up profile. The construction relies on the reduction of the problem to a finite dimensional one and the use of index theory to conclude.

In the numerical direction, we intend to develop methods in order to give numerical answers to the question of the blow-up profile for some parabolic equations including the Ginzburg-Landau model. We propose two methods. The first one is the *rescaling algorithm* proposed by Berger and Kohn in 1988 applied to parabolic equations which are invariant under a scaling transformation. This scaling property allows us to make a zoom of the solution when it is close to the singularity, still keeping the same equation. The main advantage of this method is its ability to give a very good numerical approximation allowing to attain the numerical blow-up profile. The blow-up profile we obtain numerically is in good accordance with the theoretical one. Moreover, by applying the method to a critical nonlinear heat equation with a nonlinear gradient term, where almost nothing is known, we give a conjecture for its blow-up profile thanks to our numerical simulations. The second one is a new mesh-refinement method inspired by the *rescaling algorithm* of Berger and Kohn, which is applicable to more general equations, in particular those with no scaling invariance.
Keyword: Semilinear heat equations, lower order perturbation, singularity, numerical blow-up, finite-time blow-up, profile, stability, asymptotic behavior, complex Ginzburg-Landau equation.

Université Paris 13 - Institut Galilée,
Laboratoire Analyse, Géométrie et Applications, UMR 7539.