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<u>Titre</u>

Généralisation du théorème de Greenberg-Stevens au cas du carré symétrique d'une forme modulaire et application au groupe de Selmer

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Chapter 1

Introduction

1.1 Introduction en français

Soit E une courbe elliptique sur $\mathbb{Q}$ et $p \geq 3$ un nombre premier fixé ; si on suppose que E est ordinaire en p, alors il existe une fonction L p-adique $L_p(s,E)$ interpolant les valeurs spécials $\frac{L(1,E,\varepsilon)}{\Omega_\varepsilon(E)}$, lorsque ε parcourt l'ensemble des caractères de conducteur une puissance de p et $\Omega_\varepsilon(E)$ dans $\mathbb{C}^\times$ est une période explicite. En particulier, on a en $s=0$

$$L_p(0,E) = (1-a_p^{-1})\frac{L(1,E)}{\Omega(E)},$$

où a_p désigne la racine unitaire du polynome de Hecke en p associé à E. Si E a mauvaise réduction multiplicative déployée en p, alors $a_p = 1$ et $L_p(0,E) = 0$ même si $L(1,E)$ n'est pas nul. Soit q_E dans $\mathbb{Q}_p$ telle que

$$E(\overline{\mathbb{Q}}_p) = \overline{\mathbb{Q}}_p^\times / q_E^{\mathbb{Z}};$$

dans [MTT86] les auteurs formulent une conjecture sur le comportement de $L_p(s,E)$ en $s=0$ dont une forme faible est la suivante

$$\left.\frac{\mathrm{d}L_p(s,E)}{\mathrm{d}s}\right|_{s=0} = \mathcal{L}(E)\frac{L(1,E)}{\Omega(E)},$$

pour $\mathcal{L}(E) = \log_p(q_E)/\mathrm{ord}_p(q_E)$. Cette forme faible de la conjecture a été démontrée dans [GS93] et a ouvert les portes à une série de conjectures sur le comportement des fonctions L p-adiques en présence des *zéros triviaux*, généralisant la conjecture de [MTT86]. On va maintenant énoncer cette conjecture formulée en première instance par Greenberg [Gre94b] dans le cas ordinaire et par Benois [Ben10] dans le cas semistable ou potentiellement cristallin.
Soit K un corps p-adique, $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ et

$$V : G_{\mathbb{Q}} \to \mathrm{GL}_n(K)$$

une représentation continue, irreductible et semistable en p que l'on suppose *motivique* ; c'est-à-dire qu'il existe un motif dont la réalisation p-adique coïncide avec V. On peut donc associer à V une fonction L analytique $L(s,V)$ de la variable complexe s. On suppose de plus que $L(s,V)$ satisfait l'équation fonctionelle predite par Deligne [Del79], que $s=0$ est critique et qu'il existe une période de Deligne $\Omega(V)$ telle que $L(s,V)/\Omega(V) \in \overline{\mathbb{Q}}$.

Sous ces hypothèses, Coates et Perrin Riou [CPR89, PR95] ont conjecturé l'existence d'une fonction L p-adique $L_p(s,V,D)$ (pour D un sous (φ,N)-module semistable de $\mathbf{D}_{\mathrm{st}}(V)$) interpolant les valeurs spéciales de V tordue par des caractères de Dirichlet de conducteur une puissance de p.
En particulier, on conjecture que

$$L_p(0,V,D) = E(V,D)\frac{L(0,V)}{\Omega(V)},$$

pour $E(V,D)$ un produit fini de facteurs eulériens en p, explicitement décrit dans [Ben13]. On suppose maintenant que $L(0,V) \neq 0$.
Il pourrait bien se passer qu'un facteur de $E(V,D)$ s'annule ; dans ce cas, on dit que des zéros triviaux apparaissent. Comme ce qui nous intéresse est la valeur spéciale $\frac{L(0,V)}{\Omega(V)}$, on voudrait pouvoir récupérer des informations sur cette valeur à partir de $L_p(s,V,D)$. En étudiant ce problème, Greenberg et Benois ont défini un nombre $\mathcal{L}(V)$ qui ne dépend que de la cohomologie galoisienne de V (dans le cas ordinaire) resp. que de la cohomologie du (φ,Γ)-module $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ (dans le cas semi-stable). Ce nombre, qu'on appelle l'invariant $\mathcal{L}$, est conjecturé comme étant non nul et il apparaît dans la conjecture suivante.

Conjecture 1.1.1. *Soit g le nombre de facteurs de $E(V,D)$ qui s'annulent, alors on a*

$$\left.\frac{\mathrm{d}^g L_p(s,E)}{g!\mathrm{d}s^g}\right|_{s=0} = \mathcal{L}(V)E^*(V,D)\frac{L(0,V)}{\Omega(V)},$$

où $E^*(V,D)$ est le produit des facteurs de $E(V,D)$ qui sont non nuls.

On ne connaît pas beaucoup de représentations V pour lesquelles la conjecture a été démontrée : à part le cas mentionné ci-dessus, on connaît la conjecture pour un caractère de Dirichlet [FG79], une forme modulaire dont la représentation automorphe est Steinberg en p [Ste10] ou cristalline [Ben12], le carré symétrique d'une courbe elliptique à mauvaise réduction multiplicative en p (Greenberg et Tilouine, non publié), les puissances symétriques de formes modulaires CM [Har12, HL13] et pour les généralisations aux corps totalement réels de certains cas précédents [DDP11, Mok09, Spi13a, Spi13b].
Le but de cette thèse est la généralisation du résultat de Greenberg et Tilouine dans deux cas : le premier, pour une courbe elliptique E ordinaire sur un corps totalement réel, sous l'hypothèse qu'il y a un seul ideal premier au dessus de p et, le deuxième, une forme modulaire de poids pair $k \geq 2$ dont la représentation automorphe est Steinberg en p (noter aque si $k > 2$ alors f n'est pas ordinaire).
Plus précisément, on a les théorèmes suivants

Théorème 1.1.2 (Théorème A)**.** *Soit F un corps totalement réel tel qu'il y a un seul idéal premier $\mathfrak{p}$ au-dessus de p ; soit f une forme modulaire de Hilbert de poids parallèle* 2*, de Nebentypus trivial et de conducteur $\mathfrak{N}\mathfrak{p}$. On suppose $\mathfrak{N}$ et $\mathfrak{p}$ premiers entre eux et $\mathfrak{N}$ sans facteur carré et divisible par tous les ideaux premiers au-dessus de* 2*. Alors la Conjecture 1.1.1 est vraie pour* $\mathrm{Ind}_F^{\mathbb{Q}}\mathrm{Sym}^2(f)(1)$.

Théorème 1.1.3 (Théorème B)**.** *Soit f une forme modulaire elliptique de poids k, de Nebentypus trivial et de conducteur Np, avec N et p premiers entre eux et avec N sans facteur carré. Alors la Conjecture 1.1.1 est vraie pour* $\mathrm{Sym}^2(f)(k-1)$.

Nous donnons maintenant une esquisse des démonstrations dans les deux cas, qui suivent la méthode de Greenberg et Stevens. On pose $V = \mathrm{Ind}_F^{\mathbb{Q}}\mathrm{Sym}^2(f)(1)$ ou $V = \mathrm{Sym}^2(f)(k-1)$ selon que l'on considère le premier cas ou le deuxième.
Tout d'abord, on remarque que la forme peut être déformée dans une famille p-adique, soit de Hida, soit de Coleman. On note cette famille par $\mathbf{F}(k)$ (k est la variable du poids) et on suppose $\mathbf{F}(0) = f$.
On construit donc une fonction L p-adique, que l'on voit comme une fonction analytique en deux variables $L_p(s,k)$ telle que

- $L_p(s,0) = L_p(s,V)$,
- $L_p(k,k) \equiv 0$.

La deuxième égalité nous donne

$$\left.\frac{\mathrm{d}L_p(s,k)}{\mathrm{d}s}\right|_{k=s=0} = -\left.\frac{\mathrm{d}L_p(s,k)}{\mathrm{d}k}\right|_{k=s=0},$$

ce qui permet de réduire le problème du calcul de la dérivée par rapport à s au calcul de la dérivée par rapport à k. Pour calculer cette dérivée on remarque que le facteur qui donne le zéro trivial est la spécialisation en $k=0$ de la fonction analytique $(1 - a_p^{-2}(k))$, où $a_p(k)$ est la valeur propre de la famille pour l'opérateur U_p. On arrive donc à démontrer une factorisation

$$L_p(0,k) = (1 - a_p^{-2}(k))L_p^*(k)$$

localement au voisinage de $k = 0$ pour une fonction $L_p^*(k)$ et analytique au voisinage de 0 et telle que $L_p^*(0) = \frac{L(0,V)}{\Omega(V)}$. On peut donc conclure grâce au théorème dû à Hida (dans le cas ordinaire) resp. Mok et Benois (dans le cas semistable) qui donne l'égalité

$$\mathcal{L}(V) = -2\mathrm{d}\log_p(a_p(k)).$$

Remarquons que dans le cas d'une forme de Hilbert il est très important de travailler avec la représentation induite, c'est-à-dire avec une représentation de $G_{\mathbb{Q}}$, pour pouvoir calculer l'invariant $\mathcal{L}$.
Insistons sur la robustesse de cette méthode qui peut être utilisée dans plusieurs situations. En fait, il suffit de pouvoir déformer V p-adiquement et que la fonction L p-adique à plusieurs variables corréspondante satisfaisse les trois propriétées precédents :

- spécialisation à $L_p(s,V)$,
- annulation le long une droite,
- factorisation.

Malheureusement, cette méthode semble difficile à généraliser au cas des zéros trivaux d'ordre supérieur ; c'est pour cela que l'on doit supposer pour le moment qu'il n'y a que un seul premier au dessus de p dans F.
Une option pour étudier les dérivées d'ordre supérieur, par exemple dans le cas d'une forme modulaire de Hilbert, pourrait être d'utiliser le fait que f admet des déformations en famille à $[F:\mathbb{Q}]$ variables. L' obstacle principale est alors que en général la valeur propre $a_p(\underline{k})$ n'est plus une fonction analytique de $\underline{k}$; il est donc très difficile de trouver un analogue de la factorisation comme on l'a fait dans le cas précédent.
Avant d'expliquer les hypothèses sur le conducteur, rappelons comment la fonction L p-adique est construite dans le cas du carré symétrique. La clef est la formule intégrale

$$\mathcal{L}(s, V\otimes\varepsilon) = \int_{X_0(\mathfrak{N})} \overline{f}(z)\theta(z,\varepsilon)E_k(z,s,\varepsilon)\mathrm{d}\mu \tag{1.1.4}$$

où $\mathcal{L}(s,V)$ désigne la fonction L de V à laquelle on a enlevé des facteurs eulériens en les premiers divisant le niveau et en 2 (et qu'on appellera dans la suite *imprimitive*), et $\theta(z)$ et $E_k(z,s)$ sont respectivement une série thêta et une série d'Eisenstein de poids demi-entier. [Comme les formes de poids demi-entier ne sont définiés que par les niveaux divisible par 4, il faut considérer f comme une forme de niveau divisible par 4. Cela nous fait perdre un facteur d'Euler en 2 qu'il est difficile de rétablir en général. On verra dans le Chapitre 5 comme le rétablir.]
La méthode la plus naturelle pour construire $L_p(s,k)$ est donc d'utiliser un produit de Petersson p-adique analogue à 1.1.4 comme dans [Hid91, Pan03], en prenant une convolution de deux mesures à valeur dans

certains espaces de formes modulaires p-adiques. La construction de $L_p^*(k)$ est faite de la même façon que dans [HT01] ; on remplace la convolution des deux mesures par le produit d'une forme fixée et d'une mesure d'Eisenstein.
Malheureusement, on n'arrive à démontrer une factorisation le long de la variable du poids que pour la fonction imprimitive et les facteurs qui manquent sont souvent nuls. Les hypothèses sur le conducteur de f garantisssent qu'il n'y a pas de facteurs eulériens qui manquent.
La structure de la thèse est la suivante : dans le premier chapitre on traite le cas d'une forme modulaire de Hilbert ordinaire et de poids parallèle 2 et dans le deuxième le cas d'une forme elliptique de poids $k \geq 2$ et Steinberg en p. Dans le troisième, on donne la définition de l'invariant $\mathcal{L}$ à la Greenberg et à la Benois et on calcule l'invariant $\mathcal{L}$ dans le cas du carré symétrique.

Donnons plus de détails sur le contenu des chapitres : le Chapitre 2 est un compte rendu publié dans *Comptes Rendus Mathématique*, Vol. 351, nÂ° 7-8, p. 251-254 et fait d'introduction au Chapitre 3. Il commence avec des rappels sur les formes modulaires de Hilbert de poids entier ou demi-entier et sur les familles de formes quasi-ordinaires. Ensuite on définit l'espace (naïf) des formes de Hilbert p-adiques de poids demi-entier et on construit des mesures à valeurs dans cet espace. Soit δ le default de la conjecture de Leopoldt pour F et p. On construit donc une fonction L p-adique en $[F : \mathbb{Q}] + 1 + \delta$ qui interpole les valeurs spéciales de la fonction L primitive du carré symétrique, en généralisant des travaux de Hida [Hid90] et Wu [Wu01]. Par la méthode de Dabrowski et Delbourgo [DD97] on démontre, que, si la famille de formes qu'on considère n'a pas de multiplication complexe par un caractère quadratique imaginaire, cette fonction L p-adique est holomorphe en dehors d'un nombre fini de points correspondant aux intersections de la famille avec d'autres familles. Soulignons que dans le cas $F = \mathbb{Q}$ on a un resultat nouveau, puisque on traite le cas $p = 3$ précédement exclu dans [Hid90]. On peut donc démontrer le Théorème 1.1.2 dans la façon esquissée avant.
Ensuite, on démontre que la Conjecture 1.1.1 est vraie pour tous les changements de base abéliens h d'une forme de Hilbert f comme dans le Théorème 1.1.2. Pour cela, on utilise le fait que la fonction L p-adique de h admet, comme la fonction L complexe, une factorisation comme produit de fonctions L p-adiques associées à f. L'intérêt de ce résultat est qu'on a une preuve de la Conjecture 1.1.1 dans des cas de zéros triviaux d'ordre supérieur, c'est-à-dire pour $g > 1$.
On conclut avec un paragraphe qui explique comment le Théorème 1.1.2 permet de compléter la preuve de la conjecture principale de Greenberg-Iwasawa par Urban [Urb06] dans le cas $F = \mathbb{Q}$.
Dans le Chapitre 4, on commence par rappeler la toute récente théorie des formes modulaires quasi-surconvergentes de Urban [Urb], en soulignant comment cette théorie peut être utilisée pour construire des fonctions L p-adiques.
Plus précisément, dans le cas ordinaire on dispose d'un projecteur ordinaire défini sur tout l'espace des formes modulaires p-adiques qui permet de se ramener à un espace de dimension finie où l'on peut définir par des méthodes d'algèbre lineaire un produit de Petersson p-adique. Dans le cas de pente finie, Panchishkin [Pan03] arrive à construire un produit de Petersson p-adique défini seulement sur le sous-espace des formes p-adiques qui sont surconvergentes. Cela provient du fait que U_p agit sur ce sous-espace comme un opérateur complétement continu. De l'analyse fonctionelle p-adique élémentaire permet de projeter cet espace sur le sous-espace de dimension finie correspondant aux formes de pente finie.
Malheureusement, les formes $\theta(z)E_k(z, s)$ ne sont pas surconvergentes ; dans [Urb], Urban construit l'espace des formes quasi-surconvergentes qui contient les formes $\theta(z)E_k(z, s)$ et où U_p agit de façon complétement continue. Cela suffit pour généraliser la construction de Panchishkin et pour constuire un produit de Petersson p-adique quasi-surconvergent et par conséquent la fonction L p-adique pour le carré symétrique d'une famille de formes de pente finie. On démontre par la même méthode que [DD97] que la fonction qui interpole les valeurs spéciales de la fonction L primitive est holomorphe en dehors d'un nombre fini de points correspondants aux intersections avec d'autres familles de formes quasi-surconvergentes.
Une telle fonction L p-adique a été déjà construite dans [Kim06] à l'aide d'un projecteur distinct de celui d'Urban, mais on préfère donner une nouvelle preuve par la théorie de formes quasi-surconvergentes pour pouvoir faire la factorisation le long du diviseur $s = 0$.

Par rapport à [DD97], on arrive à construire une fonction L p-adique d'une variable aussi pour les formes dont la pente α est grande par rapport au poids (c'est-à-dire $2\alpha \geq k-2$). La distribution définie dans [DD97] est, dans la terminologie de Amice-Vélu, $(2\alpha+1)$-admissible et pour qu'elle définisse de façon unique une fonction analytique on a besoin de connaître au moins $2\alpha+2$ moments de la distribution. Les entiers critiques pour le carré symétrique sont $k-1$ et donc quand $2\alpha \geq k-2$ on peut pas déterminer la fonction L p-adique seulement en term d'interpolation de valeurs spéciales avec leur construction.
Dans le Chapitre 5 on utilise la méthode de Böecherer-Schidmt [BS00] (qui exploit une formule de *pullback*) pour construire la fonction $L_p(k)$. Il nous permet d'enlever l'hypothèse de conducteur paire (et de traiter le cas $p=2$).
Le Chapitre 6 a deux parties : dans la première on rappelle la définition de Greenberg de l'invariant $\mathcal{L}$ et les calculs de Hida de cet invariant dans le cas du carré symétrique d'une forme de Hilbert quasi-ordinaire.
Dans la deuxième partie on rappelle la définition de l'invariant $\mathcal{L}$ de Benois (qui généralise celle de Greenberg au cas non-ordinaires) pour l'invariant $\mathcal{L}$ et son calcul dans le cas où le (φ,Γ)-module de la représentation de V est une extension de $\mathcal{R}(|x|x^m)$ par $\mathcal{R}(x^{-n})$, oú $\mathcal{R}$ est l'anneau de Robba et $m \geq 1$, $n \geq 0$ deux entiers. Ce type de représentations détermine l'invariant $\mathcal{L}$ dans le cas du carré symétrique d'une forme qui est Steinberg en p. Une application de ce résultat est le calcul de l'invariant $\mathcal{L}$ de la représentation standard associée à une forme de Siegel qui est de type Steinberg en p.

1.2 English Introduction

Since the seminal work of Euler on the value of Riemann zeta function $\zeta(s)$ at the positive even integers, mathematicians have been deeply fascinated by the behaviour of $\zeta(s)$ at the integers. Thanks to the functional equation, we know that its values at negative odd integers are well understood rational numbers. More explicitly, we have $\zeta(-n) = -\frac{B_{n+1}}{n+1}$.

Let p be an odd prime number and let n, m be two positive integers. Suppose that $n \equiv m \bmod p^t(p-1)$ and $n \not\equiv m \bmod p-1$. In 1895, Kummer discovered the following remarkable congruences:

$$(1-p^n)\frac{B_{n+1}}{n+1} \equiv (1-p^m)\frac{B_{m+1}}{m+1} \bmod p^{t+1}.$$

In 1964, Kubota and Leopoldt understood that these congruences allow one to define a p-adic analytic function $\zeta_p(s)$ which can be thought of as the p-adic avatar of the classical Riemann zeta function.
In the twentieth century, mathematicians associated to interesting objects such as Dirichlet characters, Hecke characters and modular forms a complex function, usually called the L-function. These functions are defined similarly to $\zeta(s)$ as an infite product over all primes of certain polynomial factors which encode arithmetic properties of the object we are interested in. Many important theorems about their properties and behaviour at the integers have been proved.

These results lead Deligne to formulate in 1979 an impressive set of conjectures on the L-function of motives. A motive M can be thought of as an interesting arithmetic object, such as a variety defined over $\mathbb{Q}$, and it is possible to associate with it an L-function $L(M, s)$. Deligne conjectured that for certain integers associated to the motive, the so-called *critical integers*, there exists an almost canonical non-zero complex number $\Omega(M, s)$ such that $\frac{L(M,s)}{\Omega(M,s)}$ is an algebraic number.
If Deligne's conjecture is true, it makes sense to ask about the p-adic valuation of the algebraic numbers $\frac{L(M,s)}{\Omega(M,s)}$. In 1989, Coates and Perrin-Riou conjectured that for any motive there should exist a p-adic analogue of the complex L-function $L_p(M, s)$ which interpolates the algebraic part of the values of $L(M, s)$.
As in the case of the Kubota-Leopoldt p-adic L-function, conjecturally, to interpolate the special values of $L(M, s)$ we need to remove certain factors at p. It may happen that some of these factors vanish and we say that we are in the presence of *trivial zeros*. Motivated by conjectures of Birch and Swynnerton-Dyer and Bloch and Kato which use the L-functions and their special values to express certain important arithmetic properties of the motive, Greenberg and Benois formulated a precise conjecture to explain these trivial zeros. Before giving the exact form, we offer an example.

Let E be an elliptic curve over $\mathbb{Q}$ and $p \geq 3$ a fixed pime number. If we suppose that E is ordinary at p then there exists a p-adic L-function $L_p(s, E)$ interpolating the special values $\frac{L(1,E,\varepsilon)}{\Omega_\varepsilon(E)}$, where ε ranges between Dirichlet characters of conductor a power of p and $\Omega_\varepsilon(E)$ in $\mathbb{C}^\times$ is an explicit period. In particular, at $s = 0$

$$L_p(0, E) = (1 - a_p^{-1})\frac{L(1, E)}{\Omega(E)},$$

where a_p is the unit root of the Hecke polynomial at p associated to E.
If E has split, multiplicative reduction at p then $a_p = 1$ and $L_p(0, E) = 0$ even if $L(1, E)$ is not zero. Let q_E in $\mathbb{Q}_p$ such that

$$E(\overline{\mathbb{Q}}_p) = \overline{\mathbb{Q}}_p^\times / q_E^{\mathbb{Z}}.$$

In [MTT86] the authors make a conjecture concerning the behaviour of $L_p(s,E)$ at $s=0$ and a weak form of it is the following:

$$\left.\frac{\mathrm{d}L_p(s,E)}{\mathrm{d}s}\right|_{s=0} = \mathcal{L}(E)\frac{L(1,E)}{\Omega(E)},$$

for $\mathcal{L}(E) = \log_p(q_E)/\mathrm{ord}_p(q_E)$.

This weak form of the conjecture has been proved in [GS93] and opened the door for the study of p-adic L-functions which present trivial zeros, generalizing the conjecture of [MTT86]. We now state this generalization as proposed by Greenberg [Gre94b] (resp. Benois [Ben10]) in the ordinary (resp. semistable or potentially crystalline) case.
Let K be a p-adic field, $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and

$$V : G_{\mathbb{Q}} \to \mathrm{GL}_n(K)$$

a continuous, semistable p-adic representation which we suppose *motivic*. That is, there exists a motive whose p-adic realization coincides with V. Consequently, we can associate to V a complex analytic L-function $L(s,V)$. We suppose moreover that $L(s,V)$ satisfies the functional equation conjectured by Deligne [Del79], that $s=0$ is critical and that there exists a Deligne's period $\Omega(V)$ as before such that $L(s,V)/\Omega(V) \in \overline{\mathbb{Q}}$.

Under these hypotheses, Coates et Perrin Riou [CPR89, PR95] conjectured the existence of a p-adic L-function $L_p(s,V,D)$ (for D a regular (φ,N)-submodule of $\mathbf{D}_{\mathrm{st}}(V)$) interpolating the special values of V twisted by characters whose conductor is a p-power.
In particular, it is conjectured that

$$L_p(0,V,D) = E(V,D)\frac{L(0,V)}{\Omega(V)},$$

for $E(V,D)$ a finite product of Euler factors at p, explicitly described in [Ben13]. We suppose now that $L(0,V) \neq 0$.
It could happen that one of the factors of $E(V,D)$ vanishes; in this case, we say that trivial zeros appear. As what we are interested in is the special value $\frac{L(0,V)}{\Omega(V)}$, we would like to be able to recover information about it from $L_p(s,V,D)$. Studying this problem, Greenberg and Benois define $\mathcal{L}(V)$, depending only on the Galois cohomology of V. This number, called the $\mathcal{L}$-invariant, is conjecture to be non-zero and it appears in the following conjecture.

Conjecture 1.2.1. *Let g be the number of vanishing factors of $E(V,D)$. We have then that*

$$\left.\frac{\mathrm{d}^g L_p(s,E)}{g!\mathrm{d}s^g}\right|_{s=0} = \mathcal{L}(V)E^*(V,D)\frac{L(0,V)}{\Omega(V)},$$

for $E^*(V,D)$ the product of non-vanishing factors of $E(V,D)$.

We do not know many V's for which the conjecture has been proven. Aside from the above mentioned case, we know the conjecture for Dirichlet characters [FG79], modular forms which are Steinberg [Ste10] or potentially crystalline [Ben12] at p, the symmetric square of an elliptic curve with bad multiplicative reduction at p (Greenberg et Tilouine, unpublished), symmetric powers of CM forms [Har12, HL13] and certain generalizations to totally real fields of some of these cases [DDP11, Mok09, Spi13a, Spi13b].

The goal of this thesis is the generalization of the work of Greenberg and Tilouine in two cases: the first one, for an elliptic ordinary curve E over a totally real field, under the hypothesis that there is only one

prime above p and, the second, a modular form of even weight $k \geq 2$ whose automorphic representation is Steinberg at p (note that if $k > 2$ then f is not ordinary).
More precisely, we have the following theorems:

Theorem 1.2.2 (Theorem A)**.** *Let F be a totally real field where there is only one prime ideal $\mathfrak{p}$ above p; let f be a Hilbert modular form of parallel weight 2, trivial Nebentypus and conductor $\mathfrak{N}\mathfrak{p}$, for $\mathfrak{N}$ et $\mathfrak{p}$ coprime. We suppose $\mathfrak{N}$ squarefree and divisible by all the prime ideal above 2. Then Conjecture 1.2.1 is true for* $\mathrm{Ind}_F^{\mathbb{Q}}\mathrm{Sym}^2(f)(1)$.

Theorem 1.2.3 (Theorem B)**.** *Let f be an elliptic modular form of weight k, trivial Nebentypus and conductor Np, with $(N,p)=1$ and N squarefree. Then Conjecture 1.2.1 is true for* $\mathrm{Sym}^2(f)(k-1)$.

We now sketch the proof, which follows the method of Greenberg et Stevens. We set $V = \mathrm{Ind}_F^{\mathbb{Q}}\mathrm{Sym}^2(f)(1)$ or $V = \mathrm{Sym}^2(f)(k-1)$ according to whether we are in the first or the second situation.
First of all, we note that in both cases f can be deformed into a p-adic family. We shall denote this family by $\mathbf{F}(k)$ (k is the weight variable) and we suppose $\mathbf{F}(0) = f$.
We construct a two variable p-adic L-function which we see as an analytic function $L_p(s,k)$ such that

- $L_p(s,0) = L_p(s,V)$,
- $L_p(k,k) \equiv 0$.

The second property gives us

$$\left.\frac{\mathrm{d}L_p(s,k)}{\mathrm{d}s}\right|_{k=s=0} = -\left.\frac{\mathrm{d}L_p(s,k)}{\mathrm{d}k}\right|_{k=s=0}.$$

This allows us to calculate the derivative with respect to s in terms of the one with respect to k. To calculate the latter we remark that the Euler factor is the specialization at $k=0$ of the analytic function $(1 - a_p^{-2}(k))$, where $a_p(k)$ is the eigenvalue of $F(k)$ for U_p. We can prove a factorization

$$L_p(0,k) = (1 - a_p^{-2}(k))L_p^*(k),$$

locally around $k=0$, for an analytic function $L_p^*(k)$ satisfying $L_p^*(0) = \frac{L(0,V)}{\Omega(V)}$. We can conclude with the theorem, due to Hida (in the ordinary case) and Mok and Benois (in the semistable case), which says

$$\mathcal{L}(V) = -2\mathrm{d}\log_p(a_p(k)).$$

We remark that it is important, in the Hilbert case, to work with the induced representation, that is, with a representation of $G_{\mathbb{Q}}$, to calculate the $\mathcal{L}$-invariant.
This method is quite robust and easily adaptable in many situations. Unluckily, it seems a hard problem to generalize to higher order zeros. It is for this reason that we have to suppose that there is only one prime above p in F.
We refer to Chapter 5 for a deeper discussion on the subject. Before explaining the hypothesis on the conductor, we recall an integral formulation of the symmetric square L-function. We have

$$\mathcal{L}(s, V \otimes \varepsilon) = \int_{X_0(\mathfrak{N})} \overline{f}(z)\theta(z,\varepsilon)E_k(z,s,\varepsilon)\mathrm{d}\mu \tag{1.2.4}$$

where $\mathcal{L}(s,V)$ is the L-function of V from which we remove some Euler factors at 2 and the bad primes of f (that we shall call *imprimitive*), and $\theta(z)$ and $E_k(z,s)$ are respectively a theta series and an Eisenstein series of half-integral weight. [As half-integral forms are defined only for level divisible by 4, we have to consider f as a form of level divisible by 4. This makes us lose an Euler factor at 2 which is hard to restore in general. We

shall restore it in Chapter 5.]
The most natural method to construct $L_p(s,k)$ is to use a p-adic Petersson product as in [Hid91, Pan03], taking the convolution of two measures with values in the space of half-integral weight p-adic modular forms. The construction of $L_p^*(k)$ is performed in the same spirit of [HT01]; we replace the convolution by the product of a fixed form and an Eisenstein measure.
We can only prove a factorization along the weight variable for the imprimitive function, the missing Euler factors often vanish and the hypotheses on the conductor of f are sufficent to ensure that no Euler factor is missing.

The structure of the thesis is the following: Chapter 2 is a *compte rendu* published in *Comptes Rendus Mathématique*, Vol. 351, n 7-8, p. 251-254, and is an introduction to the following chapter in which we deal with the Hilbert case. In Chapter 4 we treat the case of a form of finite slope. In Chapter 5 we always deal with the case of finite slope, but we construct the p-adic L-function using a pullback formula, inspired by [BS00]. This allows us to remove the hypothesis on 2 (and also to consider the case $p = 2$). In Chapter 6 we give the definition of the $\mathcal{L}$-invariant *à la* Greenberg and *à la* Benois and we compute the $\mathcal{L}$-invariant for the symmetric square following [Ben10, Mok12, Hid06]. We also compute the $\mathcal{L}$-invariant for a genus two Siegel modular form. This last result was not known.
We refer to the introduction of each chapter for more details.

1.3 Nederlandse Inleiding

Zij E een elliptische kromme over $\mathbb{Q}$ en $p \geq 3$ een vast priemgetal; als we veronderstellen dat E ordinair is bij p, dan bestaat er een p-adische L-functie $L_p(s, E)$ die de speciale waarden $\frac{L(1,E,\varepsilon)}{\Omega_\varepsilon(E)}$ interpoleert, wanneer ε de verzameling karakters doorloopt met conductor een macht van p, en $\Omega_\varepsilon(E)$ in $\mathbb{C}^\times$ een expliciete periode is. In het bijzonder geldt in $s = 0$ dat

$$L_p(0, E) = (1 - a_p^{-1}) \frac{L(1, E)}{\Omega(E)},$$

waar a_p de unitaire wortel is van de Heckeveelterm bij p geassocieerd met E. Als E gespleten multiplicative reductie heeft bij p, dan $a_p = 1$ en $L_p(0, E) = 0$ zelfs als $L(1, E)$ verschillend is van nul. Zij q_E in $\mathbb{Q}_p$ zó dat

$$E(\overline{\mathbb{Q}}_p) = \overline{\mathbb{Q}}_p^\times / q_E^{\mathbb{Z}}.$$

In [MTT86] formuleren de auteurs een vermoeden over het gedrag van $L_p(s, E)$ bij $s = 0$, waarvan de volgende uitspraak een zwakke vorm is:

$$\left.\frac{\mathrm{d}L_p(s, E)}{\mathrm{d}s}\right|_{s=0} = \mathcal{L}(E) \frac{L(1, E)}{\Omega(E)},$$

voor $\mathcal{L}(E) = \log_p(q_E)/\mathrm{ord}_p(q_E)$. Deze zwakke vorm van het vermoeden is bewezen in [GS93] en heeft de deur geopend naar een reeks vermoedens over het gedrag van de p-adische L-functies wanneer er *triviale nulpunten* zijn. Deze veralgemenen het vermoeden in [MTT86]. We formuleren nu een veralgemening uit Greenberg [Gre94b] (het ordinaire geval) en Benois [Ben10] (het semi-stabiele of potentieel kristallijne geval). Zij K een p-adisch veld, stel $G_\mathbb{Q} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ en zij

$$V : G_\mathbb{Q} \to \mathrm{GL}_n(K)$$

een continue representatie, irreducibel en semi-stabiel bij p. We nemen aan dat de representatie *motivisch* is. Dit betekent dat er een motief bestaat waarvan de p-adische realisatie gelijk is aan V. We kunnen dus aan V een p-adische L-functie $L(s, V)$ toekennen, met complexe veranderlijke s. We veronderstellen bovendien dat $L(s, V)$ voldoet aan Delignes functionaalvergelijking [Del79], dat $s = 0$ kritisch is en dat er een Deligne-periode $\Omega(V)$ bestaat zodat $L(s, V)/\Omega(V) \in \overline{\mathbb{Q}}$.
Onder deze voorwaarden voorspellen Coates en Perrin Riou [CPR89, PR95] het bestaan van een p-adische L-functie $L_p(s, V, D)$ (voor D een semistabiele (φ, N)-deelmodule van $\mathbf{D}_{\mathrm{st}}(V)$) die de speciale waarden interpoleert van V, getwist met Dirichletkarakters met conductor een macht van p.
In het bijzonder verwachten we dat

$$L_p(0, V, D) = E(V, D) \frac{L(0, V)}{\Omega(V)},$$

met $E(V, D)$ een eindig product van Eulerfactoren bij p, expliciet beschreven in [Ben13]. We nemen nu aan dat $L(0, V) \neq 0$.
Het is goed mogelijk dat één van de factoren van $E(V, D)$ nul is; in dat geval zeggen we dat er triviale nulpunten voorkomen. Omdat we geïnterreseerd zijn in de speciale waarde $\frac{L(0,V)}{\Omega(V)}$, zouden we graag informatie over deze speciale waarde verkrijgen uit $L_p(s, V, D)$. Om dit probleem te bestuderen, hebben Greenberg en Benois een waarde $\mathcal{L}(V)$ gedefinieerd die alleen afhangt van de Galoiscohomologie van V (in het ordinaire geval), resp. de cohomologie van de (φ, Γ)-module $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ (in het semistabiele geval). Van deze waarde, die we de $\mathcal{L}$-invariant noemen, vermoeden we dat hij verschillend is van 0. Bovendien komt de $\mathcal{L}$-invariant voor in het volgende vermoeden.

Vermoeden 1.3.1. *Noteer met g het aantal factoren van $E(V,D)$ die gelijk zijn aan 0. Dan hebben we*

$$\left.\frac{\mathrm{d}^g L_p(s,E)}{g!\mathrm{d}s^g}\right|_{s=0} = \mathcal{L}(V)E^*(V,D)\frac{L(0,V)}{\Omega(V)},$$

waar $E^*(V,D)$ het product is van de factoren van $E(V,D)$ verschillend van 0.

Er zijn niet veel representaties V waarvoor het vermoeden is bewezen: behalve de gevallen die hierboven zijn vermeld, is het vermoeden waar voor een Dirichletkarakter [FG79], een modulaire vorm waarvan de automorfe representatie Steinberg is bij p [Ste10] of potentieel kristallijn [Ben12], het symmetrisch kwadraat van een elliptische kromme met multiplicatieve reductie in p (Greenberg et Tilouine, ongepubliceerd), symmetrische machten van CM vormen [Har12, HL13] en voor veralgemeningen naar totaal reële velden in een aantal van de voorgaande gevallen [DDP11, Mok09, Spi13a, Spi13b].
Het doel van dit proefschift is het resultaat van Greenberg et Tilouine te veralgemenen in twee gevallen: ten eerste, een elliptische kromme E over een totaal reëel veld, met de hypothese dat er slechts één priem ligt boven p; en ten tweede, een modulaire vorm van even gewicht $k \geq 2$ waarvan de automorfe representatie Steinberg is bij p (merk op dat f niet ordinair is als $k > 2$).
Meer precies bewijzen we de volgende stellingen:

Stelling 1.3.2 (Stelling A)**.** *Zij F een totaal reëel veld zodat er slechts één priemideaal $\mathfrak{p}$ boven p ligt; zij f een Hilbert modulaire vorm van parallel gewicht 2, met triviaal Nebentypus en conductor $\mathfrak{N}\mathfrak{p}$. We veronderstellen dat $\mathfrak{N}$ en $\mathfrak{p}$ copriem zijn, en $\mathfrak{N}$ kwadraatvrij en deelbaar door alle priemen boven 2. Dan is Vermoeden 1.3.1 waar voor* $\mathrm{Ind}_F^{\mathbb{Q}}\mathrm{Sym}^2(f)(1)$.

Stelling 1.3.3 (Stelling B)**.** *Zij f een elliptische modulaire vorm van gewicht k, met triviaal Nebentypus en conductor Np, met N en p copriem, en N kwadraatvrij. Dan is Vermoeden 1.3.1 waar voor* $\mathrm{Sym}^2(f)(k-1)$.

We geven nu een schets van de bewijzen in de twee gevallen. Onze bewijzen volgen de methode van Greenberg et Stevens. We definiëren $V = \mathrm{Ind}_F^{\mathbb{Q}}\mathrm{Sym}^2(f)(1)$ of $V = \mathrm{Sym}^2(f)(k-1)$, naargelang we het eerste of tweede geval beschouwen.
Om te beginnen merken we op dat in beude gevallen f kan worden vervormd in een p-adische familie; hetzij een Hida-familie, hetzij een Coleman-familie. We schrijven $\mathbf{F}(k)$ voor deze familie (k is de variabele van het gewicht) en we nemen aan dat $\mathbf{F}(0) = f$.
We construëren een p-adische L-functie, die we zien als analytische functie $L_p(s,k)$ in twee veranderlijken zodat

- $L_p(s,0) = L_p(s,V)$,
- $L_p(k,k) \equiv 0$.

De tweede eigenschap geeft ons

$$\left.\frac{\mathrm{d}L_p(s,k)}{\mathrm{d}s}\right|_{k=s=0} = -\left.\frac{\mathrm{d}L_p(s,k)}{\mathrm{d}k}\right|_{k=s=0},$$

en dit herleidt de berekening van de afgeleide ten opzichte van s naar de berekening van de afgeleide ten opzichte van k. Om deze laatste afgeleide te berekenen, merken we op dat de Eulerfactor die de triviale nulpunt geeft de specialisatie is bij $k = 0$ van de analytische functie $(1 - a_p^{-2}(k))$, waar $a_p(k)$ de eigenwaarde van de familie $F(k)$ is voor de U_p operator. We bekomen een ontbinding van de vorm

$$L_p(0,k) = (1 - a_p^{-2}(k))L_p^*(k)$$

in een omgeving van $k = 0$ met $L_p^*(k)$ een analytische functie in een omgeving van 0 zodat $L_p^*(0) = \frac{L(0,V)}{\Omega(V)}$.
De stelling van Hida (in het ordinaire geval), resp. Mok en Benois (in het semi-stabiele geval), zegt nu dat

$$\mathcal{L}(V) = -2\mathrm{d}\log_p(a_p(k)).$$

We merken op dat het in het geval van een Hilbert modulaire vorm om de invariant $\mathcal{L}$ te berekenen erg belangrijk is om te werken met de geïnduceerde representatie, dat wil zeggen met een representatie van $G_{\mathbb{Q}}$. We benadrukken dat onze methode robuust is, en in verschillende situaties nuttig kan zijn. Inderdaad, het volstaat om de representatie V p-adisch te vervormen zodat de corresponderende p-adische L-functie met meerdere veranderlijken de volgende drie eigenschappen heeft:

- specialisatie bij $L_p(s, V)$,
- nul worden langs een rechte,
- factorisatie.

Jammer genoeg lijkt deze methode moeilijk te veralgemenen naar het geval van nulpunten van hogere orde; daarom moeten we veronderstellen dat er slechts één priem in F boven p ligt. Een mogelijkheid om hogere orde afgeleiden te bestuderen, bijvoorbeeld in het geval van een Hilbert modulaire vorm, zou kunnen zijn dat f deformeert in een familie met $[F : \mathbb{Q}]$-variabelen. Het belangrijkste obstakel is dan dat in het algemeen de eigenwaarde $a_p(\underline{k})$ geen analytische functie van $\underline{k}$ is; het is dus erg moeilijk een ontbinding te vinden zoals we die vonden in het geval $[F : \mathbb{Q}] = 1$.
Voordat we de voorwaarden op de conductor verklaren, leggen we uit hoe de p-adische L-functie geconstrueerd is in het geval van de symmetrische kwadratische macht. De sleutel is de integraalformule

$$\mathcal{L}(s, V \otimes \varepsilon) = \int_{X_0(\mathfrak{N})} \overline{f}(z)\theta(z, \varepsilon)E_k(z, s, \varepsilon)\mathrm{d}\mu \tag{1.3.4}$$

waar $\mathcal{L}(s, V)$ de L-functie van V is waarbij we Eulerfactoren bij 2 en de slechte priemen van f verwijderen, en $\theta(z)$ en $E_k(Z, s)$ zijn respectievelijk een thetareeks en een Eisensteinreeks van half-geheel gewicht. [Omdat de vormen van half-geheel gewicht alleen gedefinieerd zijn voor niveau deelbaar door 4, moeten we f zien als een vorm van gewicht deelbaar door 4. Hierdoor verliezen we een Eulerfactor bij 2, die in het algemeen moeilijk te herstellen is. We zullen in Hoofdstuk 5 zien hoe je deze factor terugvindt.]
De methode die het meest natuurlijk is om $L_p(s, k)$ te construëren is om een p-adisch Peterssonproduct te gebruiken zoals in [Hid91, Pan03], waarbij we een convolutieproduct nemen van twee maten met waarden in bepaalde ruimten van p-adische modulaire vormen. De constructie van $L_p^*(k)$ gebeurt op dezelfde manier als in [HT01]; we vervangen de convolutie van twee maten met het product van een vaste vorm en een Eistensteinmaat.
Helaas slagen we er enkel in om de factorisatie langs de gewichtsvariabele te bewijzen voor de imprimitieve functie; de factoren die ontbreken zijn vaak nul en de hypothesen op de conductor van f garanderen dan dat er geen Eulerfactoren zijn die ontbreken.
De structuur van dit proefschrift is als volgt: in het eerste hoofdstuk behandelen we het geval van een gewone Hilbert modulaire vorm en parallel gewicht 2 en in het tweede hoofdstuk het geval van een elliptische vorm van gewicht $k \geq 2$ en Steinberg bij p. In het derde hoofdstuk geven we de definitie van de $\mathcal{L}$-invariant à la Greenberg et à la Benois en we berekenen de $\mathcal{L}$-invariant in het geval van de kwadratisch symmetrische macht.

De thesis is opgebouwd als volgt. Hoofdstuk 2 is gepubliceerd in *Comptes Rendus Mathématique*, Vol. 351, nÂ° 7-8, p. 251-254 en vormt een inleiding tot Hoofdstuk 3. Het begint met een overzicht van de theorie van Hilbert modulaire vormen van geheel of half-geheel gewicht en families van quasi-ordinaire vormen. Vervolgens definiëren we (naïef) de Hilbertvormen van half-geheel gewicht en contruëren we maten met waarden in deze ruimte. Zij δ de correctieterm in het Leopoldt-vermoeden voor F en p. We construëren een p-adische L-functie in $[F : \mathbb{Q}] + 1 + \delta$ variabelen die de speciale waarden van de primitieve L-functie van de kwadratisch symmetrische macht interpoleert. Dit veralgemeent het werk van Hida [Hid90] en Wu [Wu01]. Via de methode van Dabrowski et Delbourgo [DD97] bewijzen we dat, als de familie van vormen die we beschouwen geen complexe vermenigvuldiging met een imaginair kwadratisch karakter heeft, deze p-adische L-functie holomorf is buiten een eindig aantal punten die corresponderen met intersecties van de familie met andere families. We benadrukken dat we voor $F = \mathbb{Q}$ een nieuw resultaat bekomen, omdat we ook het geval $p = 3$ behandelen dat is uitgesloten in [Hid90]. We kunnen dus Stelling 1.3.2 bewijzen op de manier die we hierboven hebben geschetst.

Vervolgens bewijzen we dat Vermoeden 1.3.1 waar is voor alle abelse basisveranderingen h van een Hilbert-modulaire vorm f, zoals in Stelling 1.3.2. Hiervoor gebruiken we dat de p-adische L-functie van h, net als de complexe L-functie, een ontbinding heeft in een product van p-adische L-functies geassocieerd met f. Dit resultaat is interessant omdat we een bewijs hebben voor Vermoeden 1.3.1 ingeval er nulpunten van hogere orde zijn, dat wil zeggen voor $g > 1$.
We leggen nu uit hoe je Stelling 1.3.2 kan gebruiken om het bewijs van het Hoofdvermoeden van Greenberg-Iwasawa door Urban [Urb06] ingeval $F = \mathbb{Q}$ te vervolledigen.
In Hoofdstuk 4 introduceren we de recente theorie van quasi-overconvergente modulaire vormen van Urban [Urb], in het bijzonder leggen we uit hoe deze theorie gebruikt kan worden om p-adische L-functies te contruëren.
In het ordinaire geval maken we een ordinaire projector gedefinieerd op de gehele ruimte van p-adische modulaire vormen. Dankzij deze projector kunnen we reduceren naar een ruimte van eindige dimensie, waar we lineaire algebra kunnen gebruiken om een p-adisch Peterssonproduct te definiëren. In het geval van eindige helling (*slope*), lukte het Panchishkin [Pan03] enkel om een p-adisch Peterssonproduct te definieren op de deelruimte van overconvergente p-adische vormen. Dit komt omdat U_p op deze ruimte werkt als complete, continue operator. Dankzij elementaire p-adische functionaalanalyse kan je deze ruimte projecteren op de eindigdimensionale ruimte van vormen met eindige helling.
Helaas zijn de vormen $\theta(z)E_k(z,s)$ niet overconvergent; in [Urb] construeert Urban de ruimte van quasi-overconvergente vormen die de vormen $\theta(z)E_k(z,s)$ bevat, en waar U_p compleet continu werkt. Dit volstaat om de constructie van Panchiskhkin te veralgemenen en om een quasi-overconvergent p-adisch Peterssonproduct te definiëren, en dus de p-adische L-functie voor het symmetrisch kwadraat van een familie van vormen van eindige helling. We bewijzen met dezelfde methode [DD97] dat de functie die de speciale waarden van de primitieve L-functie interpoleert holomorf is buiten een eindige verzameling van punten corresponderend met intersecties van andere families van quasi-overconvergente vormen.
Een dergelijke p-adische L-functie was al geconstrueerd in [Kim06] met hehulp van een projector, verschillend van die van Urban, maar we geven liever een nieuw bewijs in termen van de theorie van quasi-overconvergente vormen, om de ontbinding langs de divisor $s = 0$ te verkrijgen.

Ten opzichte van [DD97] lukt het ons om een p-adische L-functie te construëren met één variable, ook voor de vormen waarvoor de helling α groot is ten opzichte van het gewicht (dat wil zeggen: $2\alpha \geq k - 2$). De distributie gedefinieerd in [DD97] is, in de terminologie van Amice-Vélu, $(2\alpha + 1)$-admissibel et opdat ze op een unieke manier een analytische functie zou definiëeren, moeten we ten minste $2\alpha + 2$ momenten van de distributie kennen. Het kritieke gehele getal voor het symmetrisch kwadraat is $k - 1$; dus als $2\alpha \geq k - 2$ kunnen we de p-adische L-functie niet uitdrukken als een interpolatie van speciale waarden.

In Hoofdstuk 5 gebruiken we de methode van Böecherer-Schidmt [BS00] (die een *pullback* formule gebruikt) om de functie $L_p(k)$ te construëren. Hierdoor kunnen we de hypothèse dat de conductor even is wegwerken (en het geval $p = 2$ behandelen).
Het Hoofdstuk 6 heeft twee delen: in het eerste deel herhalen we de definitie van Greenberg voor de invariant $\mathcal{L}$ en de berekeningen van Hida van deze invariant, in het geval van een symmetrisch kwadraat van een quasi-ordinaire Hilbert modulaire vorm.
In het tweede deel herhalen we de definitie van de invariant $\mathcal{L}$ van Benois (die de invariant van Greenberg veralgemeent naar het niet-ordinaire geval) voor de $\mathcal{L}$-invariant in het geval van het symmetrisch kwadraat van een vorm die Steinberg is bij p. Een toepassing van dit resultaat is de berekening van de $\mathcal{L}$-invariant van de standaardrepresentatie geassocieerd met een Siegelvorm die Steinberg is bij p.

Chapter 2

Dérivée en $s = 1$ de la fonction L p-adique du carré symétrique d'une courbe elliptique sur un corps totalement réel

2.1 Introduction

Dans la théorie de l'interpolation p-adique de valeurs spéciales de fonctions L, on a des conjectures très précises sur les propriétés que les fonctions L p-adiques doivent satisfaire ; en particulier, il y a une conjecture de Greenberg qui prédit le comportement de la fonction L p-adique lorsque des zéros triviaux apparaissent. Le but de cette note est de donner l'esquisse de la preuve de cette conjecture dans un cas particulier.
Avant d'expliciter la conjecture, on fait quelques rappels sur les fonctions L p-adiques ; soit M un motif, et soit $L(s, M)$ la fonction L complexe associée à M selon la recette de Deligne [Del79]. Pour tout caractère d'ordre fini ε, soit $L(s, M, \varepsilon)$ la fonction L du motif M tordu par ε. Si $s = 0$ est critique pour $L(s, M, \varepsilon)$, au sens de [Del79], on suppose l'existence d'une période $\Omega(0, M, \varepsilon)$ tel que

$$\frac{L(0, M, \varepsilon)}{\Omega(0, M, \varepsilon)} \in \overline{\mathbb{Q}}.$$

Fixons un nombre premier p et un corps p-adique K contenant les coefficients de M, et soit $\mathcal{O}$ l'anneau des entiers de K. Soit u un générateur topologique de $1 + p\mathbb{Z}_p$ et identifions $\mathcal{O}[[1 + p\mathbb{Z}_p]]$ avec l'algèbre d'Iwasawa $\mathcal{O}[[T]]$ via $u \mapsto 1 + T$. Si on suppose que M est *ordinaire* en p (voir [Gre94b]), on conjecture l'existence d'une série formelle $G(T, M)$ telle que pour tout caractère ε de $1 + p\mathbb{Z}_p$ d'ordre fini on a

$$G(\varepsilon(u) - 1, M) = C(0, M, \varepsilon)E(0, M, \varepsilon)\frac{L(0, M, \varepsilon)}{\Omega(0, M, \varepsilon)},$$

où $C(0, M, \varepsilon)$ est un nombre algebrique non-nul et $E(0, M, \varepsilon)$ est un produit explicite de facteurs de type d'Euler en p [Gre94b]. Il est possible, quand $\varepsilon = \mathbf{1}$ est le caractère trivial, que certains facteurs de $E(0, M, \mathbf{1})$ s'annulent et dans ce cas la formule d'interpolation ne donne aucune information sur la valeur qui nous intéresse. Quand on a un tel phénomène, on dit que la fonction L p-adique a un zéro trivial. Soit $L_p(s, M) = G(u^s - 1, M)$; dans [Gre94b] l'auteur formule la conjecture suivante sur le comportement de $L_p(s, M)$ en $s = 0$.

Conjecture 2.1.1. *Soit g le nombre des facteurs de $E(0,M,\mathbf{1})$ qui s'annulent. On a*

$$L_p(s,M) \equiv \mathcal{L}(M)E^*(0,M)\frac{L(0,M)}{\Omega(0,M,\mathbf{1})}s^g \bmod s^{g+1}$$

où $\mathcal{L}(M)$ est un terme d'erreur non nul défini en termes de la cohomologie galoisienne et $E^*(0,M)$ est le produit des facteurs de $E(0,M,\mathbf{1})$ qui ne s'annulent pas.

Il n'y a pas beaucoup de cas connus de cette conjecture : quand M est le motif d'une courbe elliptique sur un corps totalement réel ayant mauvaise réduction multiplicative scindée exactement à une place au-dessus de p [GS93, Gre94b] et quand M est le motif du carré symétrique d'une courbe elliptique sur $\mathbb{Q}$ ayant conducteur pair et sans facteur carré et mauvaise réduction multiplicative en p (Greenberg et Tilouine, non publié). Récemment, Dasgupta a annoncé une preuve de la conjecture dans les cas où M est le motif du carré symétrique d'une forme modulaire elliptique ordinaire, sans restriction sur le conducteur.

2.2 Le cas du carré symétrique d'une forme modulaire de Hilbert

Soit F un corps totalement réel de degré d et $\mathbf{f}$ une forme modulaire de Hilbert de poids $k=(k_1,\ldots,k_d)$ et de niveau $\mathfrak{N}$. Soit k^0 le maximum des k_i et k_0 le minimun des k_i ; on suppose que $k_i \equiv k_0 \bmod 2$ pour tout i et $k_0 \geq 2$. On pose $2v := (k^0-k_1,\ldots,k^0-k_d)$. Suppons que $\mathbf{f}$ soit propre pour l'action de l'algèbre de Hecke engendrée par tous les opérateurs $T(\mathfrak{q})$, et soit K_0 un corps de nombres qui contient les coefficients de Fourier de $\mathbf{f}$. On fixe un nombre premier p, $p>2$ et des plongements $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ et $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$; soit K la complétion de K_0 dans $\overline{\mathbb{Q}}$ par rapport au plongement fixé et $\mathcal{O}$ l'anneau des entiers de K. On peut associer à $\mathbf{f}$ une représentation galoisienne $\rho_{\mathbf{f}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(K)$, dont le determinant est $\psi\chi^{k^0-1}$, où ψ est un caractère de Hecke d'ordre fini qui correspond au Nebentypus de $\mathbf{f}$ et χ est le caractère cyclotomique p-adique. Pour tout $\mathfrak{q}$ idéal premier de F, notons $\mathbf{a}_{\mathfrak{q}}(\mathbf{f})$ la valeur propre de $T(\mathfrak{q})$ sur $\mathbf{f}$ et $\alpha_{\mathfrak{q}}$ et $\beta_{\mathfrak{q}}$ les racines du polynôme caractéristique de $T(\mathfrak{q})$.
Soit $\mathrm{Sym}^2(M_{\mathbf{f}})$ le carré symétrique du motif $M_{\mathbf{f}}$ associé à $\mathbf{f}$, et soit ε un caractère de Hecke d'ordre fini ; on peut associer à ces données, selon la recette de Deligne, une fonction L notée par $L(s,\mathrm{Sym}^2(\mathbf{f}),\varepsilon)$ et définie par un produit eulérien sur F. Le facteur en $\mathfrak{q}$, pour presque tous $\mathfrak{q}$, est

$$(1-\varepsilon(\mathfrak{q})\alpha_{\mathfrak{q}}^2\mathcal{N}(\mathfrak{q})^{-s})(1-\varepsilon(\mathfrak{q})\alpha_{\mathfrak{q}}\beta_{\mathfrak{q}}\mathcal{N}(\mathfrak{q})^{-s})(1-\varepsilon(\mathfrak{q})\beta_{\mathfrak{q}}^2\mathcal{N}(\mathfrak{q})^{-s}).$$

Soit n un entier dans l'intervalle $\left[k^0-k_0+1, k^0-1\right]$ tel que $\varepsilon_v(-1)=(-1)^{n-1}$ pour tout $v \mid \infty$. D'après [Im91], on connaît l'existence de périodes $\Omega(n,\mathrm{Sym}^2(\mathbf{f}),\varepsilon)$ telles que

$$\frac{L(n,\mathrm{Sym}^2(\mathbf{f}),\varepsilon)}{\Omega(n,\mathrm{Sym}^2(\mathbf{f}),\varepsilon)} \in \overline{\mathbb{Q}}.$$

Supposons que $\mathbf{f}$ soit quasi-ordinaire, c'est-à-dire que p divise le niveau de $\mathbf{f}$ et la valeur propre de l'opérateur de Hecke normalisé $T_0(p)=p^{-v}T(p)$ sur $\mathbf{f}$ est une unité p-adique. Soit ω le caractère de Teichmüller de $\mathbb{Z}_p^*$. On a le résultat suivant [Ros13a] :

Théorème 2.2.1. *Il existe une série formelle $G(T,\mathrm{Sym}^2(\mathbf{f}))$ dans $\mathcal{O}((T))[p^{-1}]$ telle que pour presque tout caractère d'ordre fini ε de $1+p\mathbb{Z}_p$ et n entier tel que $k^0-k_0+1 \leq n \leq k^0-1$ on a*

$$G(\varepsilon(u)u^n-1,\mathrm{Sym}^2(\mathbf{f})) = C(n,\varepsilon\omega^{1-n})E(n,\varepsilon\omega^{1-n})\frac{L(n,\mathrm{Sym}^2(\mathbf{f}),\varepsilon\omega^{1-n})}{\Omega(n,\mathrm{Sym}^2(\mathbf{f}),\varepsilon\omega^{1-n})}$$

où $C(n,\varepsilon\omega^{1-n})$ est un nombre algébrique non-nul et $E(n,\varepsilon\omega^{1-n})$ est le produit des facteurs d'Euler en p prédit par [Gre94b].

En particulier, ce théorème nous donne des fonctions L p-adiques (au sens de l'introduction) pour les $\mathbb{Q}$-motifs tordus à la Tate $(\mathrm{Ind}_F^{\mathbb{Q}}(\mathrm{Sym}^2(M_{\mathbf{f}})))(-n)$. Dans la suite on pose $L_p(s, \mathrm{Sym}^2(\mathbf{f})) := G(u^s - 1, \mathrm{Sym}^2(\mathbf{f}))$, avec $s \in \mathbb{Z}_p$ (sauf un nombre fini de valeurs).
Le preuve du théorème suit la méthode classique utilisée par Hida dans [Hid91] et se déroule de la façon suivante ; on définit la fonction L imprimitive $\mathcal{L}(s, \mathrm{Sym}^2(\mathbf{f}), \varepsilon)$ comme $E_{\mathfrak{N}}(s, \varepsilon) L(s, \mathrm{Sym}^2(\mathbf{f}), \varepsilon)$, où $E_{\mathfrak{N}}(s, \varepsilon)$ est un produit fini de certains facteurs d'Euler pour des idéaux premiers divisant $\mathfrak{N}$ et 2. La fonction $\mathcal{L}(s, \mathrm{Sym}^2(\mathbf{f}), \varepsilon)$ peut être écrite comme le produit de Petersson de $\mathbf{f}$ avec le produit de deux formes modulaires de Hilbert quasi-holomorphes de poids demi-entier : une série thêta $\theta(\varepsilon)$ et une série d'Eisenstein $E(k, s, \varepsilon)$. On a que les coefficients de Fourier de ces deux formes définissent des fonctions p-adiques continues quand s et ε varient p-adiquement. On peut ainsi construire deux mesures - une mesure thêta et une mesure d'Eisenstein, à valeurs dans les formes modulaires Hilbert p-adiques de poids demi-entier ; on conclut en appliquant le produit de Petersson p-adique (voir [Hid91]). On utilise la méthode de Schmdit [Sch88] pour démontrer que, quand $\mathbf{f}$ n'a pas multiplication complexe, la série formelle $G(T, \mathrm{Sym}^2(\mathbf{f}))$ appartient à $\mathcal{O}[[T]][p^{-1}]$.
Sous certaines hypothèses sur F, cette méthode a déjà été exploitée dans [Wu01] pour la construction de la fonction L p-adique associée à $\mathrm{Sym}^2(M_{\mathbf{f}})$.
Dans le cas du carré symétrique d'une forme de Hilbert, les zéros triviaux apparaissent quand la forme $\mathbf{f}$ a Nebentypus trivial en p et $n = k^0 - 1$.
Soit E une courbe elliptique *modulaire* sur F ; c'est-à-dire qu'il existe une forme modulaire de Hilbert $\mathbf{f}$ de poids $(2, \ldots, 2)$ telle que $\rho_{\mathbf{f}}$ soit la représentation de Galois associée au module de Tate de E. Supposons $p \geq 5$ et inerte dans F ; on a alors le théorème suivant :

Théorème 2.2.2. *Suppons que le conducteur de E soit sans facteurs carrés et divisible par* 2 *et p. Alors*

$$\frac{\mathrm{d}L_p(s, \mathrm{Sym}^2(\mathbf{f}))}{\mathrm{d}s}|_{s=1} = \mathcal{L}(\mathrm{Sym}^2(\mathbf{f})) E^*(1, \mathbf{1}) \frac{L(1, \mathrm{Sym}^2(\mathbf{f}))}{\Omega(1, \mathrm{Sym}^2(\mathbf{f}))}.$$

Avant d'esquisser la preuve du théorème, on remarque que l'hypothèse sur le conducteur de E en p nous donne $\mathbf{a}_p(\mathbf{f}) = \pm 1$, et donc $\mathbf{f}$ est ordinaire. On sait qu'il existe une extension finie et plate $\mathbf{I}$ de $\mathcal{O}[[X]]$, une famille $\mathbf{I}$-adique de formes de Hilbert ordinaires $\mathbf{F}$ et un idéal premier P_2 de $\mathbf{I}$ au dessus de $X + 1 - u^2$ telle que $\mathbf{F} \bmod P_2 = \mathbf{f}$. On construit alors une fonction $L_p(T) \in \mathbf{I}[[T]]$; comme l'extension $\mathbf{I}/\mathcal{O}[[T]]$ est étale en P_2, on peut voir $L_p(T)$ comme une fonction analytique $L_p(s, k)$, avec s dans $\mathbb{Z}_p$ et k voisin de 2 p-adiquement. On a les propriétés suivantes :

i) $L_p(s, 2) = L_p(s, \mathrm{Sym}^2(\mathbf{f}))$,

ii) $L_p(s, k)$ est identiquement nulle sur la droite $s = k - 1$,

iii) $L_p(1, k) = (1 - \mathbf{a}_p(\mathbf{F})^{-2}(k)) L_p^*(k)$ dans un voisinage de 2,

iv) $L_p^*(2) = \frac{L(1, \mathrm{Sym}^2(\mathbf{f}))}{\Omega(1, \mathrm{Sym}^2(\mathbf{f}))}$.

Une telle fonction étant construite, la démonstration du théorème suit la méthode de [GS93] ; *i)* et *ii)* nous permettent de relier la dérivée de $L_p(s, \mathrm{Sym}^2(\mathbf{f}))$ par rapport à s à la dérivée de $L_p(1, k)$ par rapport à k. Le calcul de cette dérivée est fait avec *iii)* et *iv)*, et on conclut avec un résultats de [Mok09] qui relie la dérivée par rapport à k de $\mathbf{a}_p(\mathbf{F})(k)$ à l'invariant $\mathcal{L}$, que l'on sait être non nul grâce à un théorème de théorie des nombres transcendants [BSDGP96].
La construction de la fonction $L_p(T)$ est faite comme dans le théorème 2.2.1 via le produit de Petersson p-adique de $\mathbf{F}$ avec la convolution d'une mesure thêta et d'une mesure d'Eisenstein. La construction de la fonction $L_p^*(k)$ est faite en généralisant la construction de Greenberg-Tilouine ; dans ce cas, on prend le produit de Petersson p-adique de $\mathbf{F}$ avec un produit d'une série thêta fixée et d'une mesure d'Eisenstein.
On remarque que la fonction $L_p^*(k)$ *a priori* n'interpole que la valeur spéciale de la fonction L imprimitive $\mathcal{L}(s, \mathrm{Sym}^2(\mathbf{f}))$ et que les hypothèses sur le conducteur de E nous assurent que $\mathcal{L}(s, \mathrm{Sym}^2(\mathbf{f}))$ coïncide avec

$L(s, \mathrm{Sym}^2(\mathbf{f}))$.
L'hypothèse que p soit inerte est due au fait que cette méthode ne semble pas se généraliser au cas des zéros triviaux d'ordre supérieur.

Chapter 3

Derivative at $s = 1$ of the p-adic L-function of the symmetric square of a Hilbert modular form

3.1 Introduction

The aim of this paper is to prove a conjecture of Greenberg on trivial zeros of p-adic L-functions [Gre94b], generalizing the classical conjecture of Mazur-Tate-Teitelbaum [MTT86] to more general, ordinary motives. Let $V = \{V_l\}$ be a compatible system of continous, finite-dimensional l-adic representations of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let us denote by V^* the dual motive whose l-adic realizations are $V_l^* = \mathrm{Hom}(V_l, \mathbb{Q}_l(1))$.
We can associate with V in a standard way a complex L-function $L(s,V)$, converging for $Re(s) \gg 0$; we suppose that $L(s,V)$ extends to an holomorphic function on the whole complex plane and that it satisfies the functional equation

$$L(s,V)\Gamma(s,V) = \varepsilon(s)L(1-s,V^*)\Gamma(s,V^*)$$

where $\Gamma(V,s)$ is a product of complex and real Gamma functions and $\varepsilon(s)$ a function of the form ζN^s, for ζ a root of unity and N a positive integer. We suppose moreover that both $\Gamma(0,V)$ and $\Gamma(1,V^*)$ are defined; in this case $s = 1$ is critical à la Deligne [Del79] and we suppose in addition the existence of a Deligne period Ω such that

$$\frac{L(V,0)}{\Omega} \in \overline{\mathbb{Q}}.$$

Let us fix a prime number p; we will fix once and for all an isomorphism $\mathbb{C} \cong \mathbb{C}_p$. This in particular defines an embedding of $\overline{\mathbb{Q}}$ in $\mathbb{C}$ and $\mathbb{C}_p$. We say that V is ordinary at p if V_p as $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representation admits a decreasing and exhaustive filtration $\mathrm{Fil}^i V_p$ such that I_p acts on $\mathrm{Gr}^i V_p$ as χ_{cycl}^i, where χ_{cycl} denotes the p-adic cyclotomic character.
Let us denote by K the coefficient field of V_p and by $\mathcal{O}$ its valuation ring. Under these hypotheses, Coates and Perrin-Riou [CPR89] have formulated the following conjecture:

Conjecture 3.1.1. *There exists a formal series $G(T,V)$ in $\mathcal{O}[[T]]$ such that for all finite-order, non-trivial characters ε of $1 + p\mathbb{Z}_p$ we have*

$$G(\varepsilon(1+p)-1,T) = C_{\varepsilon}\frac{L(V\otimes\varepsilon,0)}{\Omega};$$

here C_ε is an explicit determined algebraic number. Moreover

$$G(0) = \mathcal{E}(V)\frac{L(V,0)}{\Omega}$$

where $\mathcal{E}(V)$ is a finite product of Euler type factors at p.

A description of $\mathcal{E}(V)$ when V_p is semistable can be found in [Ben13, §0.1].
It is clear from the above interpolation formula that if one of the factors of $\mathcal{E}(V)$ vanishes, then the p-adic L-function does not provide any infomation about the special value $\frac{L(V,0)}{\Omega}$. In this case, we say that we have an "exceptional" or "trivial" zero. In [Gre94b], Greenberg proposes a conjecture which generalizes the well-known conjecture of Mazur-Tate-Teitelbaum.

Conjecture 3.1.2. *Let us denote by g the number of factors of $\mathcal{E}(V)$ which vanish. We have then*

$$G(T,V) = \mathcal{L}(V)\mathcal{E}^*(V)\frac{L(V,0)}{\Omega}\frac{T^g}{g!} + (T^{g+1})$$

where $\mathcal{E}^*(V)$ denotes the non-zero factor of $\mathcal{E}(V)$ and $\mathcal{L}(V)$ is a non-zero error factor.

The factor $\mathcal{L}(V)$, usually called the $\mathcal{L}$-invariant, has a conjectural interpretation in terms of the Galois cohomology of V_p. At present, not many cases of this conjecture are known; up to the 1990's, the conjecture was only known where V is a Dirichlet character [FG79], or an elliptic curve over $\mathbb{Q}$ with split multiplicative reduction at p [GS93], or the symmetric square of an elliptic curve with multiplicative reduction at p (Greenberg and Tilouine, unpublished).
Recently, several people [DDP11, Mok09, Spi13a, Spi13b] have obtained positive results on this conjecture in the case where V is an induction from a totally real field of the Galois representation associated with a Hecke character or a modular elliptic curve; these cases can be seen both as an attempt to test the conjecture for higher rank V and as a (highly non-trivial) generalization of the known cases. The natural next step is the case of the induction of the symmetric square representation associated with an elliptic curve over a totally real field; this is the main subject of this paper.
Let F be a totally real number field and $\mathfrak{r}$ its ring of integers, let I be the set of its real embeddings. Let $\mathbf{f}$ be a Hilbert modular form of weight $k = (k_\sigma)_{\sigma \in I}$, $k_\sigma \geq 2$ for all σ. Let $K \subset \mathbb{C}_p$ be a p-adic field containing the Hecke eigenvalues of $\mathbf{f}$ and let $\mathcal{O}$ be its valuation ring. It is well known [BR89, Tay89] that there exists a 2-dimensional p-adic Galois representation

$$\rho_{\mathbf{f}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathcal{O})$$

associated with $\mathbf{f}$. Let $\mathrm{Sym}^2(\rho_{\mathbf{f}})$ be the symmetric square of $\rho_{\mathbf{f}}$, it is a 3-dimensional Galois representation. For almost all primes $\mathfrak{q}$ of $\mathfrak{r}$, the Euler factor of $L(s, \mathrm{Sym}^2(\rho_{\mathbf{f}}))$ is

$$(1-\alpha(\mathfrak{q})^2\mathcal{N}(\mathfrak{q})^{-s})(1-\alpha\beta(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})(1-\beta(\mathfrak{q})^2\mathcal{N}(\mathfrak{q})^{-s})$$

where $\alpha(\mathfrak{q})$ and $\beta(\mathfrak{q})$ are the two roots of the Hecke polynomial of $\mathbf{f}$ at $\mathfrak{q}$.
We define similarly the twisted L-function $L(s, \mathrm{Sym}^2(\rho_{\mathbf{f}}), \chi)$ for χ a grössencharacter of $\mathbb{A}_F^\times$. We shall write $L(s, \mathrm{Sym}^2(\mathbf{f}), \chi) = L(s, \mathrm{Sym}^2(\rho_{\mathbf{f}}), \chi)$. Let us denote by $k^0 = \max(k_\sigma)$ and $k_0 = \min(k_\sigma)$; in [Im91], Im proves Deligne's conjecture for this L-function, i.e. he shows for any integer n in the critical strip $[k^0 - k_0 + 1, k^0 - 1]$ such that $\chi_v(-1) = (-1)^{n+1}$ for all $v \mid \infty$ the existence of non-zero complex number $\Omega(\mathbf{f}, n)$ such that the ratio

$$\frac{L(n, \mathrm{Sym}^2(\mathbf{f}), \chi)}{\Omega(\mathbf{f}, n)}$$

is an algebraic number. This algebraic number will be called the special value of $L(s, \mathrm{Sym}^2(\mathbf{f}), \chi)$ at n.
We can make a "good choice" of the periods $\Omega(\mathbf{f}, n)$ such that the special values can be p-adically interpolated. In fact, there exists a power series $G(T, \mathrm{Sym}^2(\mathbf{f}))$ in $\mathcal{O}[[T]]$ constructed in [Wu01] and in Section 3.7 of the present work which satisfies the following interpolation formula

$$G(\varepsilon(1+p)(1+p)^n - 1, \mathrm{Sym}^2(\mathbf{f})) = C_{\varepsilon,n} E(n, \varepsilon, \mathbf{f}) \frac{L(n, \mathrm{Sym}^2(\mathbf{f}), \varepsilon)}{\Omega(\mathbf{f}, n)}.$$

for n critical. Here ε is a finite-order character of $1 + p\mathbb{Z}_p$, $C_{\varepsilon,n}$ is a non-zero explicit number, and $E(n, \varepsilon, \mathbf{f})$ is a product of some Euler-type factors at primes above p as predicted by the above conjecture.
Let us pose $L_p(s, \mathrm{Sym}^2(\mathbf{f})) := G(u^s - 1, \mathrm{Sym}^2(\mathbf{f}))$. The trivial zeros occur when the Nebentypus of $\mathbf{f}$ is trivial at p and $s = k^0 - 1$. In this case the number of vanishing factors is equal to the number of primes above p in F. The problem of trivial zeros for the symmetric square has already been studied in great detail by Hida [Hid00a, Hid06, Hid09]. Hida calculated the $\mathcal{L}$-invariant, defined according to the cohomological definition of Greenberg [Gre94b], in [Hid06]. It is difficult to show that it is non-zero, but we know that it is "frequently" non-zero when $\mathbf{f}$ varies in a p-adic family which is Steinberg for all primes above p except possibly one.
The main result of the paper is the following

Theorem 3.1.3. *Let $p \geq 3$ be a prime such that there is only one prime ideal $\mathfrak{p}$ of F above p and let $\mathbf{f}$ be a Hilbert cuspidal eigenform of parallel weight 2 and conductor $\mathfrak{N}\mathfrak{p}$ such that the Nebentypus of $\mathbf{f}$ is trivial. Suppose that $\mathfrak{N}$ is squarefree and divisible by all the primes of F above 2; suppose moreover that $\pi(\mathbf{f})_{\mathfrak{p}}$ is a Steinberg representation. Then the formula for the derivative in Conjecture 3.1.2 is true.*

In the case where $\mathbf{f}$ corresponds to an elliptic curve, such a theorem has been claimed in [Ros13c]. We were able to substitute the hypothesis *p inert* by *there is only one prime ideal of F above p* thanks to a abstute observation by Éric Urban to whom we are therefore indebted.
In this particular case of multiplicative reduction, the $\mathcal{L}$-invariant depends only on the restriction of V_p to $G_{\mathbb{Q}_p}$ and was already calculated in [GS93]. By the "*Théorème de Saint-Étienne*" [BSDGP96], we know that $\mathcal{L}(\mathrm{Sym}^2(\mathbf{f}))$ is non-zero when $\mathbf{f}$ is associated with an elliptic curve.
In reality, we could weaken the hypothesis on the conductor slightly; see Section 3.8 for an example.

We point out that this theorem is a generalization of Greenberg and Tilouine's unpublished theorem where $F = \mathbb{Q}$. In fact, our proof is the natural generalization of theirs to the totally real case.
Recently, again in the case $F = \mathbb{Q}$, Dasgupta has shown Conjecture 3.1.2 in the case the symmetric square of modular forms of any weights. He uses the strategy outlined in [Cit08] which suggests to factor the three variable p-adic L-function $L_p(k, l, s)$ associated with the Rankin product of a Hida family with itself [Hid88c] when $k = l$. To do this, he employs the recent results of Bertolini, Darmon and Rotger relating p-adic regulators of Beilinson-Flach elements (and circular units) to the value of $L_p(k, l, s)$ evaluated at forms of weight 2 and non trivial Nebentypus. Evidently at this moment such a method is almost impossible to generalize to an arbitrary totally real field.

The proof of Theorem 3.1.3 follows the same line as in the work of Greenberg and Stevens for the standard L-function.
We view the p-adic L-function $L_p(s, \mathrm{Sym}^2(\mathbf{f}))$ as the specialization of a two-variable p-adic L-function $L_p(s, k)$. Such a function vanishes identically on the line $s = k - 1$ so that we can relate the derivative with respect to s to the one with respect to k. In order to evaluate the derivative with respect to k, we find a factorization of $L_p(1, k)$ into two p-analytic functions, one which gives rise to the trivial zero (it corresponds to the product of some Euler factors at primes above p) and one which interpolates the complex L-value (the *improved* p-adic L-function in the terminology of Greenberg-Stevens). The construction of the improved L-functions is done as in [HT01] by substituting the convolution of two measures used in the construction

of $L_p(s, m)$ by a product of a function and a measure.
The main limitation of this approach arise from the fact that we are using only two variables and so, such a method can not go beyond the first derivative without a new idea. In fact, one can define a p-adic L-function associated with the nearly ordinary family deforming $\mathbf{f}$ which has $d + 2 + \delta$ variables; however, we can only find a factorization of this multi-variable p-adic L-function only after restricting it to the ordinary variable. This is because the Euler factors which we remove from our p-adic L-functions are not p-adic analytic functions of the *nearly ordinary* variables. Moreover, this method works only when $\mathbf{f}$ is Steinberg at p and such representation at p in the nearly ordinary setting can appear only when the weight is parallel and equal to $2t$.
Still, we want to point out that in the case when $\mathbf{f}$ is Steinberg at all primes above p we can show a formula similar to one in Conjecture 3.1.2 but for the derivative with respect to the weight. Relating this derivative to the one with respect to s appears to be a quite a challenging problem.
As this method relies on Shimura's integral formula which relates the symmetric square L-function to a suitable Petersson product, it can be generalized to all the cases of finite slope families, assuming similar hypothesis and the existence of the many-variable p-adic L-function. In particular, thanks to the recent work of Urban [Urb] on families of nearly-overconvergent modular forms, this method can now be successfully applied in the case where $F = \mathbb{Q}$. This is the main subject of [Ros13d].
We point out that understanding the order of this zero and an exact formula for the derivative of the p-adic L-function has recently become more important after the work of Urban on the main conjecture for the symmetric square representation [Urb06] of an elliptic modular form. We shall explain how this proof can be completed with Theorem 3.1.3.
We shall also extend the result of Theorem 3.1.3 to the base change of a Hilbert modular form as in Theorem 3.1.3, providing examples of the conjecture for trivial zeros of higher order.

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The author would like to thank also Arno Kret for his great patience in answering all the author's questions and Prof. Ralph Greenberg, Prof. Haruzo Hida, Chung Pang Mok, Prof. Johannes Nicaise, Prof. Éric Urban and John Welliaveetil for useful discussions and suggestions. Part of this work has been written during a stay at the Hausdorff Institute during the *Arithmetic and Geometry* program; the author would like to thank the organizers and the institute for the support and for providing such optimal working conditions.

The structure of the paper is as follows: in Section 3.2 we recall the theory of complex and p-adic modular Hilbert modular forms of integral weight and in Section 3.3 we do the same for the half-integral weight, presenting our definition of p-adic modular form of half-integral weight. In Section 3.4 we review the definition of the L-function for the symmetric square of $\mathbf{f}$ and relate it to the Rankin product of $\mathbf{f}$ with a theta series. In Section 3.5 we present some operators which will be used in the construction of p-adic L-function, and in Section 3.6 we summarise briefly the theory of p-adic measures. In Section 3.7 we use Hida and Wu's method to construct a many-variable p-adic L-function for the symmetric square (relaxing also some of their hypotheses) and the *improved* one, while in Section 3.8 we prove Theorem 3.1.3. In Section 3.9 we deal with the case of base change. In Section 3.10 we explain how our result allows one to complete the proof of the main conjecture given by Urban. Finally, in Appendix 3.11 and 3.12 we prove a theorem on the holomorphicity of the p-adic L-functions constructed in Section 3.7.

3.2 Classical Hilbert modular forms

In this section we review the theory of complex and p-adic Hilbert modular forms.

3.2.1 Complex Hilbert modular forms

As our interest lies more in Hecke algebras than in modular forms *per se*, we look at adelic Hilbert modular forms. Let F be a totally real field of degree d and $\mathfrak{r}$ its ring of integers. Let I be the set of its real embeddings. An element $w \in \mathbb{Z}[I]$ is called a weight. For two weights $k = \sum_{\sigma \in I} k_\sigma \sigma$ and $w = \sum_{\sigma \in I} w_\sigma$, we say that $k \geq w$ if $k_\sigma \geq w_\sigma$ for all σ in I. In particular $k \geq 0$ if $k_\sigma \geq 0$ for all σ in I. Define $t = \sum_{\sigma \in I} \sigma$.
Let $m \geq 0$ be an integer and v in $\mathbb{Z}[I]$. Consider a Hilbert cuspidal eigenform $\mathbf{f}$ of weight $k = (m+2)t - 2v$ and define k_0 as the minimum of k_σ for σ in I. Let $\mathfrak{d}$ be the different of F and $D_F = \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{d})$ the discriminant of F. By abuse of notation, we shall denote by $\mathfrak{d}$ also a fixed idèle which is 1 at the primes which do not divide the different, and is a generator of the local different at the primes dividing it.
We denote by $\mathbb{A}_F$ the adèles of F, and we factor $\mathbb{A}_F = F_f \times F_\infty$ into its finite component and infinite component. For y in $\mathbb{A}_F$, we denote by y_f resp. y_∞ the projection of y to F_f resp. F_∞.
Let GL_2 be the algebraic group, defined over F, of 2×2 invertible matrices. For y in $\mathrm{GL}_2(\mathbb{A}_F)$, we denote by y_f, resp. y_∞, the projection of y to $\mathrm{GL}_2(F_f)$ resp. $\mathrm{GL}_2(F_\infty)$.
Let S be a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})$ and let $\mathrm{GL}_2(F_\infty)^+$ be the connected component of $\mathrm{GL}_2(\mathbb{A}_{F,\infty})$ containing the identity. Let $\mathcal{H}$ be the upper half plane then $\mathrm{GL}_2(F_\infty)^+$ acts on $\mathcal{H}^I$ by linear fractional transformation. Let $C_{\infty+}$ be the stabilizer of the point $z_0 = (i, \ldots, i)$ in $\mathcal{H}^I$ under this action. Fix two elements k and w of $\mathbb{Z}[I]$ such that $k \geq 0$. Let $t = \sum_{\sigma \in I} \sigma$. We say that an element of $\mathbb{Z}[I]$ is parallel if it is of the form mt with m an integer.
We use the multi-index notation; for x in F and $k = \sum_{\sigma \in I} k_\sigma \sigma$, z^k will denote $\prod_{\sigma \in I} \sigma(z)^{k_\sigma}$.
We define $\mathbf{M}_{k,w}(S, \mathbb{C})$ as the space of complex-valued functions on $\mathrm{GL}_2(\mathbb{A}_F)$, holomorphic on $\mathrm{GL}_2(\mathbb{A}_{F,\infty})$ such that

$$\mathbf{f}(\alpha x u) = \mathbf{f}(x) j_{k,w}(u_\infty, z_0)^{-1} \text{ for } \alpha \in \mathrm{GL}_2(F) \text{ and } u \in SC_{\infty+},$$

where $j_{k,w}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = (ad - bc)^{-w}(cz + d)^k$, under the multi-index convention.

We say that $\mathbf{f}$ is cuspidal if

$$\int_{F \backslash \mathbb{A}_F} f(ga) \mathrm{d}a = 0 \text{ for all } g \in \mathrm{GL}_2(\mathbb{A}_F);$$

we shall denote by $\mathbf{S}_{k,w}(S, \mathbb{C})$ the subset of cuspidal modular forms.
Fix a pair (n, v) such that $k = n + 2t$ and $w = t - v$, and suppose that $n + 2v = mt$ (otherwise there are no non-zero Hilbert modular forms). The choice of $v \in \mathbb{Z}[I]$ guarantees that the Fourier coefficients of $\mathbf{f}$, defined below, belong to a number field.
For an element $k \in \mathbb{Q}[I]$, we will denote by $[k]$ the integer such that $k = [k]t$ if k is parallel or 0 otherwise.
We define some congruence subgroups of $\mathrm{GL}_2(\mathbb{A}_F)$. Let $\hat{\mathfrak{r}}$ be the profinite completion of $\mathfrak{r}$, then let

$$\begin{aligned}
\tilde{U}_0(\mathfrak{N}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathfrak{r}}) | c \in \mathfrak{N}\hat{\mathfrak{r}} \right\}, \\
\tilde{U}_0(\mathfrak{N}, \mathfrak{M}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(\mathfrak{N}) | b \in \mathfrak{M}\hat{\mathfrak{r}} \right\}, \\
\tilde{V}_1(\mathfrak{N}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(\mathfrak{N}) | d \equiv 1 \bmod \mathfrak{N}\hat{\mathfrak{r}} \right\}, \\
\tilde{U}(\mathfrak{N}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V_1(\mathfrak{N}) | a \equiv 1 \bmod \mathfrak{N}\hat{\mathfrak{r}} \right\}, \\
\tilde{U}(\mathfrak{N}, \mathfrak{M}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(\mathfrak{N}) | b \in \mathfrak{M}\hat{\mathfrak{r}} \text{ and } a \equiv d \equiv 1 \bmod \mathfrak{N}\mathfrak{M}\hat{\mathfrak{r}} \right\}
\end{aligned}$$

for two integral ideals $\mathfrak{N}, \mathfrak{M}$ of $\mathfrak{r}$. The subgroup $\tilde{U}(\mathfrak{N}, \mathfrak{M})$ is isomorphic to $\tilde{U}(\mathfrak{N}\mathfrak{M})$ and we will use it in the following to simplify some calculations in the construction of the p-adic L-function.

We say that $\tilde{S}$ is congruence subgroup if $\tilde{S}$ contains $\tilde{U}(\mathfrak{N})$ for a certain ideal $\mathfrak{N}$. For such a $\tilde{S}$, we pose $\det\tilde{S} = \left\{\det(s)|s \in \tilde{S}\right\} \subset \mathbb{A}^{\times}_{F,\mathbf{f}}$, and decompose the idèles into

$$\mathbb{A}^{\times}_{F} = \bigcup_{i=1}^{h(S)} F^{\times} a_i \det\tilde{S}_F F^{\times}_{\infty+}. \tag{3.2.1}$$

Here $h(S)$ is a positive integer and $\{a_i\}$ is a set of idèles such that the $\{\mathfrak{a}_i = a_i\mathfrak{r}\}$ form a set of representatives. We can and we will choose each a_i such that $a_{i,p} = 1$ and $a_{i,\infty} = 1$. When $\tilde{S} = \tilde{U}(\mathfrak{N})$, we will use the notation $h(\mathfrak{N})$ for $h(S)$, as in this case $h(S)$ is the strict class number of F of level $\mathfrak{N}$. Define now

$$E = \left\{\varepsilon \in \mathfrak{r}^{\times} | \varepsilon \gg 0\right\}, \ \mathfrak{r}^{\times}(\mathfrak{N}) = \left\{\varepsilon \in \mathfrak{r}^{\times} | \varepsilon \equiv 1 \bmod \mathfrak{N}\right\},$$
$$E(\mathfrak{N}) = E \cap \mathfrak{r}^{\times}(\mathfrak{N}) \text{ and}$$
$$\tilde{\Gamma}[\mathfrak{N}, \mathfrak{a}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathfrak{r} & \mathfrak{a} \\ \mathfrak{N} & \mathfrak{r} \end{pmatrix} | ad - bc \in E, \right\}.$$

In this way we can associate to each adelic form $\mathbf{f}$ for $\tilde{U}(\mathfrak{N})$ exactly $h(\mathfrak{N})$ complex Hilbert modular forms,

$$\mathbf{f}_i(z) = j_{k,w}(u_\infty, z)\mathbf{f}(t_i u_\infty) = y_\infty^{-w}\mathbf{f}\left(t_i \begin{pmatrix} y_\infty & x_\infty \\ 0 & 1 \end{pmatrix}\right)$$

where u_∞ is such that $u_\infty(z_0) = z$. Each $\mathbf{f}_i$ is an element of $\mathbf{M}_{k,w}(\tilde{\Gamma}^{\equiv}[\mathfrak{N}\mathfrak{a}_i, \mathfrak{a}_i^{-1}])$, where

$$\tilde{\Gamma}^{\equiv}[\mathfrak{N}\mathfrak{a}_i, \mathfrak{a}_i^{-1}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}[\mathfrak{N}, \mathfrak{a}] | a \equiv d \bmod \mathfrak{N}, \right\}.$$

This allows us to define the following isomorphism

$$\begin{array}{cccc} \mathcal{I}_{\mathfrak{N}}^{-1}: & \mathbf{M}_{k,v}(U(\mathfrak{N}), \mathbb{C}) & \rightarrow & \bigoplus_i \mathbf{M}_{k,v}(\tilde{\Gamma}^{\equiv}[\mathfrak{N}\mathfrak{a}_i, \mathfrak{a}_i^{-1}], \mathbb{C}) \\ & \mathbf{f} & \mapsto & (\mathbf{f}_1, \dots, \mathbf{f}_{h(\mathfrak{N})}) \end{array}$$

We point out that each $\mathbf{f}_i(z)$ is a Hilbert modular form on a different connected component of the Shimura variety associated with $\tilde{U}(\mathfrak{N})$.
Each $\mathbf{f}_i(z)$ has a Fourier expansion

$$\mathbf{f}_i(z) = a(0, \mathbf{f}_i) + \sum_{0 \ll \xi \in \mathfrak{a}\mathfrak{d}^{-1}} a(\xi, \mathbf{f}_i)\mathbf{e}_F(\xi z).$$

As in the classical case, $\mathbf{f}$ is cuspidal if and only if $a(0, \mathbf{f}_i|\gamma) = 0$ for $i = 1, \dots, h(\mathfrak{N})$ and all the matrices $\alpha \in \mathrm{GL}_2(F_\infty)^+$.
Let K_0 be a finite extension of the Galois closure of F, and suppose that in its ring of integers all the integral ideals of F are principal. We choose a compatible system of generators $\{y^\sigma\}$ for all idèles y as in [Hid88b, §3]. Let K be the p-adic completion of K_0 with respect to the embedding of $\overline{\mathbb{Q}}$ in $\mathbb{C}_p$ fixed in the introduction. When we will be working with coefficients in K, we will suppose $\{y\} = 1$ if y is prime to p.
We define now functions on $F^{\times}_{\mathbb{A}+}$; let y be an integral idèle, by the decomposition given in (3.2.1) write $y = \xi a_i \mathfrak{d} u$, for u in $U_F(N)F^{\times}_{\infty+}$, and define

$$\mathbf{a}(y, \mathbf{f}) = a(\xi, \mathbf{f}_i)\left\{y^{-v}\right\}\xi^v |a_i|_{\mathbb{A}}^{-1},$$
$$\mathbf{a}_p(y, \mathbf{f}) = a(\xi, \mathbf{f}_i) y_p^{-v} \xi^v \mathcal{N}_p(a_i)^{-1}$$

and 0 if y is not integral. Here $\mathcal{N}_p$ is the cyclotomic character such that $\mathcal{N}_p(y) = y_p^{-t}|y_f|_{\mathbb{A}}^{-1}$. If the infinity-part and the p-part of a_i are 1 as we chose before, then

$$\mathbf{a}_p(y,\mathbf{f}) = \mathbf{a}(y,\mathbf{f})\left\{y^v\right\}y_p^{-v}.$$

Multiplication by ξ^v is necessary to ensure that these functions are independent of the choice of the decomposition of y. We define also the constant term

$$\mathbf{a}_0(y,\mathbf{f}) = a(0,\mathbf{f}_i)|a_i|_{\mathbb{A}}^{1-[v]}, \mathbf{a}_{0,p}(y,\mathbf{f}) = \mathbf{a}_0(y,\mathbf{f})\mathcal{N}_p(y\mathfrak{d}^{-1})^{[v]},$$

and we can now state a proposition on the Fourier expansion of $\mathbf{f}$.

Proposition 3.2.2. *[Hid91, Theorem 1.1] Let $\mathbf{f}$ be an Hilbert modular form of level $\tilde{V}_1(\mathfrak{N})$, then $\mathbf{f}$ has a Fourier expansion of the form*

$$\begin{aligned}\mathbf{f}\left(\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\right) =& |y|_{\mathbb{A}}\left\{\mathbf{a}_0(y\mathfrak{d},\mathbf{f})|y|_{\mathbb{A}}^{-[v]}+\right. \\ &\left.+\sum_{0\ll\xi\in F^\times}\mathbf{a}(\xi\mathfrak{d}y,\mathbf{f})\{\xi\mathfrak{d}y\}^v(\xi y_\infty)^{-v}\mathbf{e}_\infty(i\xi y_\infty)\mathbf{e}_F(\xi x)\right\}.\end{aligned} \tag{3.2.3}$$

and by formal substitution we obtain a p-adic q-expansion

$$\mathbf{f} = \mathcal{N}_p(y)^{-1}\left\{\mathbf{a}_{0,p}(y\mathfrak{d},\mathbf{f})|y|_{\mathbb{A}}^{-[v]} + \sum_{0\ll\xi\in F^\times}\mathbf{a}_p(\xi\mathfrak{d}y,\mathbf{f})q^\xi\right\}.$$

Here $\mathbf{e}_F$ is the standard additive character of $\mathbb{A}_F/F$ such that $\mathbf{e}_\infty(x_\infty) = \exp(2\pi i x_\infty)$.

We define

$$\mathbf{f}^u(x) = |D_F|^{-(m/2)-1}\overline{\mathbf{f}^\rho}(x)j(x_\infty,z_0)^{k\rho}|det(x)|_{\mathbb{A}}^{m/2};$$

$\mathbf{f}^u$ is the unitarization of (the complex conjugate of) $\mathbf{f}$. If $\mathbf{f}$ is of weight (k,w), $\mathbf{f}^u$ is of weight $(k,k/2)$. For arithmetic applications, we define the normalized Fourier coefficient as

$$C(\mathfrak{a},\mathbf{f}) = \xi^{-k/2}\mathbf{a}(\xi,\mathbf{f}^u),$$

where we have $\mathfrak{a} = \xi a_i^{-1}$. From [Hid91, (1.3 a)(4.3 b)] we have the relation

$$C(\mathfrak{a},\mathbf{f}) = a(\xi,\mathbf{f}_i)\mathcal{N}(\mathfrak{a})^{-m/2-1}\xi^v|a_i|_{\mathbb{A}}^{-1}. \tag{3.2.4}$$

From now on, following Shimura, we conjugate all the previous subgroups by $\begin{pmatrix}\mathfrak{d} & 0 \\ 0 & 1\end{pmatrix}$. The conjugated groups will be denoted by the same symbol without the tilde. The advantage of this choice is that the Fourier expansion of a Hilbert modular form $\mathbf{f}$ for $V_1(\mathfrak{N})$ is now indexed by totally positive elements in $\mathfrak{r}$ instead of totally positive elements belonging to the fractional ideal $\mathfrak{d}^{-1}$.

3.2.2 The p-adic theory

In [Hid88b, Hid89b, Hid02] Hida develops the theory of nearly ordinary modular forms. It is constructed via the duality between modular forms and their associated Hecke algebra. Let $v \geq 0$ and $k > 0$ be a couple of weights and $m \geq 0$ an integer such that $mt = k - 2t - 2v$; let $w = t - v$. Let S be a compact-open subgroup

of $\mathrm{GL}_2(\hat{\mathfrak{r}})$, $U_0(\mathfrak{N}) \supset S \supset U(\mathfrak{N})$. We suppose $\mathfrak{N}$ prime to p.
Let K be a p-adic field containing the Galois closure of F and $\mathcal{O}$ its ring of integers. For all integral domain A which are $\mathcal{O}$-algebra we define :

$$\begin{aligned}
\mathbf{M}_{k,w}(S,A) &= \{\mathbf{f} \in \mathbf{M}_{k,w}(S,\mathbb{C}) \mid \mathbf{a}_0(y,\mathbf{f}), \mathbf{a}(y,\mathbf{f}) \in A\}\,,\\
\mathbf{S}_{k,w}(S,A) &= \{\mathbf{f} \in \mathbf{S}_{k,w}(S,\mathbb{C}) \mid \mathbf{a}(y,\mathbf{f}) \in A\}\,,\\
\mathbf{m}_{k,w}(S,A) &= \{\mathbf{f} \in \mathbf{M}_{k,w}(S,\mathbb{C}) \mid \mathbf{a}(y,\mathbf{f}) \in A\}\,.
\end{aligned}$$

For all $y \in \mathbb{A}_{F,\mathbf{f}}^{\times}$, we define the Hecke operator $T(y)$ as in [Hid88b, §3] and we pose $T_0(y) = \{y^{-v}\}T(y)$. $T(y)$ is an operator on $\mathbf{M}_{k,w}(S,A)$ and $\mathbf{m}_{k,w}(S,A)$ and, if $\{y^{-v}\}$ belongs to A for all y, then $T_0(y)$ acts on both spaces. Furthermore, we define $T(a,b)$ as in [Hid91, §2].
We define the Hecke algebra $\mathbf{H}_{k,w}(S,A)$ (resp. $\mathbf{h}_{k,w}(S,A)$) as the sub-algebra of $\mathrm{End}_A(\mathbf{M}_{k,w}(S,A))$ (resp. $\mathrm{End}_A(\mathbf{S}_{k,w}(S,A))$) generated by $T(y)$ and $T(a,b)$.
For a positive integer α, we pose $S(p^\alpha) = S \cap U(p^\alpha)$ and define:

$$\begin{aligned}
\mathbf{M}_{k,w}(S(p^\infty),A) &= \varinjlim_{\alpha} \mathbf{M}_{k,w}(S(p^\alpha),A),\\
\mathbf{S}_{k,w}(S(p^\infty),A) &= \varinjlim_{\alpha} \mathbf{S}_{k,w}(S(p^\alpha),A),\\
\mathbf{m}_{k,w}(S(p^\infty),A) &= \varinjlim_{\alpha} \mathbf{m}_{k,w}(S(p^\alpha),A),\\
\mathbf{h}_{k,w}(S(p^\infty),A) &= \varprojlim_{\alpha} \mathbf{h}_{k,w}(S(p^\alpha),A),\\
\mathbf{H}_{k,w}(S(p^\infty),A) &= \varprojlim_{\alpha} \mathbf{H}_{k,w}(S(p^\alpha),A).
\end{aligned}$$

Hida shows in [Hid89b, Theorem 2.3] that when $k \geq 2t$ we have an isomorphism $\mathbf{h}_{k,w}(S(p^\infty),A) \cong \mathbf{h}_{2t,t}(S(p^\infty),A)$ which sends $T_0(y)$ into $T_0(y)$. Let $\mathbf{h}_{k,w}^{\mathrm{n.ord}}(S(p^\infty),A)$ be the direct summand of $\mathbf{h}_{k,w}(S(p^\infty),A)$ where $T_0(p)$ acts as a unit. We will drop the weight from the notation in what follows.
We let $e = \lim_n T_0(p)^{n!}$ be the idempotent of $\mathbf{h}(S(p^\infty),A)$ which defines $\mathbf{h}^{\mathrm{n.ord}}(S(p^\infty),A)$.
Let $\mathbf{S}^{\mathrm{n.ord}}(S(p^\infty),A)$ be defined as $e\overline{\mathbf{S}}_{k,w}(S(p^\infty),A)$, where the line denotes the completion with respect to the p-adic topology on the function $\mathbf{a}_p(y,\mathbf{f})$. This completion is independent of (k,w) as shown in [Hid91, Theorem 3.1].
Let now $S_0(p^\alpha) = S \cap U_0(p^\alpha)$. We define

$$\begin{aligned}
\mathbf{G}^\alpha &= S(p^\alpha)\mathfrak{r}^\times / S_0(p^\alpha)\mathfrak{r}^\times,\\
\mathbf{G} &= \varprojlim_{\alpha} \mathbf{G}^\alpha.
\end{aligned}$$

Let S_F (resp. $S(p^\alpha)_F$) be $S \cap \mathbb{A}_{F,\mathbf{f}}^{\times}$ (resp. $S(p^\alpha) \cap \mathbb{A}_{F,\mathbf{f}}^{\times}$) and

$$\mathrm{Cl}_S(p^\infty) = \varprojlim_{\alpha} S_F\mathfrak{r}^\times / S(p^\alpha)_F\mathfrak{r}^\times.$$

From [Hid89b, Lemma 2.1] we have an isomorphism $\mathbf{G} \cong \mathfrak{r}_p^\times \times \mathrm{Cl}_S(p^\infty)$ via the map

$$S \ni s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a_p^{-1}d_p, a). \tag{3.2.5}$$

If $S = U(\mathfrak{N})$, then $\mathrm{Cl}_S(p^\infty) = \mathrm{Cl}_{\mathfrak{N}}(p^\infty)$; in general, $\mathrm{Cl}_S(p^\infty)$ is always a subgroup of $\mathrm{Cl}_{\mathfrak{N}}(p^\infty)$ of finite index.
We have an action of $\mathbf{G}^\alpha$ on $\mathbf{M}_{k,w}(S(p^\alpha),A)$; the couple (a,z) acts via $T(z,z)T(a^{-1},1)$. In the following, we will denote $T(z,z)$ by $\langle z \rangle$. For two characters ψ, ψ' of finite order modulo $\mathfrak{N}p^\alpha$, we define

$\mathbf{S}^{\text{n.ord}}_{k,w}(S(p^\alpha),\psi,\psi';\mathcal{O})$ as the forms on which (a,z) acts via $\psi(z)\mathcal{N}(z)^m\psi'(a)a^v$ and $\mathbf{h}^{\text{n.ord}}_{k,w}(S(p^\alpha),\psi,\psi';\mathcal{O})$ in the obvious way.
We let $\mathbf{W}$ be the torsion-free part of $\mathbf{G}$. Let Λ be the Iwasawa algebras of $\mathbf{W}$; we have that $\mathbf{h}^{\text{n.ord}}(S(p^\infty),A)$ is a module over Λ and $\mathcal{O}[[\mathbf{G}]]$. We say that a $\mathcal{O}$-linear morphism $P:\mathcal{O}[[\mathbf{G}]]\to\mathcal{O}$ is arithmetic of type (m,v,ψ,ψ') if $P(z,a)=\psi(z)\mathcal{N}(z)^m\psi'(a)a^v$ for $m\geq 0$ and ψ, ψ' finite-order characters of $\mathbf{G}^\alpha$.
We have the following theorem, which subsumes several results of Hida [Hid89b, Theorem 2.3, 2.4], [Hid91, Corollary 3.3], [Hid02, §4]

Theorem 3.2.6. *Let S as above, then $\mathbf{h}^{\text{n.ord}}(S(p^\infty),A)$ is torsion-free over Λ, and for all points P of type (m,v,ψ,ψ') we have*

$$\mathbf{h}^{\text{n.ord}}(S(p^\infty),\mathcal{O})_P/P\mathbf{h}^{\text{n.ord}}(S(p^\infty),\mathcal{O})_P\cong\mathbf{h}^{\text{n.ord}}_{k,w}(S(p^\alpha),\psi,\psi';\mathcal{O}).$$

Moreover, there is a duality as $\mathcal{O}$-module between $\overline{\mathbf{S}}_{k,w}(S(p^\infty),\mathcal{O})$ (resp. $\overline{\mathbf{m}}_{k,w}(S(p^\infty),\mathcal{O})$) and $\mathbf{h}(S(p^\infty),\mathcal{O})$ (resp. $\mathbf{H}(S(p^\infty),\mathcal{O})$). When $S=U(\mathfrak{N})$, then for all points P of type (m,v,ψ,ψ') we have also

$$\mathbf{S}^{\text{n.ord}}(S(p^\infty),\mathcal{O})[P]\cong\mathbf{S}^{\text{n.ord}}_{k,w}(S(p^\alpha),\psi,\psi';\mathcal{O}).$$

If moreover $p\nmid 6D_F$, then $\mathbf{h}^{\text{n.ord}}(U(\mathfrak{N})(p^\infty),\mathcal{O})$ is free over $\mathcal{O}[[\mathbf{W}]]$.

In the following, we will concentrate on the case $S=V_1(\mathfrak{N})$, in what follows we will use the notation $\mathbf{h}^{\text{n.ord}}(\mathfrak{N},\mathcal{O})$ for $\mathbf{h}^{\text{n.ord}}(V_1(\mathfrak{N})(p^\infty),\mathcal{O})$. We will denote by $\mathbf{h}^{\text{ord}}(\mathfrak{N},\mathcal{O})$ the ordinary part of the Hecke algebra. Via the above mentioned duality, it corresponds to the Hilbert modular forms of parallel weight; it is a module over $\mathcal{O}[[\text{Cl}_{\mathfrak{N}}(p^\infty)]]$. We have $\mathbf{h}^{\text{ord}}(\mathfrak{N},\mathcal{O})=\mathbf{h}^{\text{n.ord}}(\mathfrak{N},\mathcal{O})/P^{\text{ord}}$, where P^{ord} is the ideal of $\mathcal{O}[[\mathbf{G}]]$ corresponding to the projection on the second component $\mathbf{G}\to\text{Cl}_{\mathfrak{N}}(p^\infty)$.

Let $\mathbf{K}$ be a field extension of $\text{Frac}(\Lambda)$ and $\mathbf{I}$ the integral closure of Λ in $\mathbf{K}$. By duality, to each morphism $\lambda:\mathbf{h}^{\text{n.ord}}(\mathfrak{N},\mathcal{O})\to\mathbf{I}$ corresponds an $\mathbf{I}$-adic modular form $\mathbf{F}$ which is characterized by its $\mathbf{I}$-adic Fourier coefficient $\mathbf{a}_p(y,\mathbf{F})=\lambda(T_0(y))$. To such a λ corresponds an irreducible component of $\text{Spec}(\mathbf{h}^{\text{n.ord}}(\mathfrak{N},\mathcal{O}))$, determined by the prime ideal $\text{Ker}(\lambda)$. Let $X(\mathbf{I})$ be the set of arithmetic points of $\text{Spec}(\mathbf{I})$, i.e. the $\overline{\mathbb{Q}}_p$-points that restrained to Λ correspond to an arithmetic point of type (m,v,ψ,ψ'). By the previous theorem, to each arithmetic point P corresponds a Hilbert modular eigenform $\mathbf{f}_P=P(\mathbf{F})$ such that $\mathbf{a}_p(y,\mathbf{f}_P)=P(\lambda(T_0(y)))$, $\mathbf{f}_P|T(a^{-1},1)=a^v\psi'(a)\mathbf{f}_P$ and $\mathbf{f}_P|\left\langle z\right\rangle=\mathcal{N}^m(z)\psi(z)\mathbf{f}_P$. In particular, the center of $\text{GL}_2(\hat{\mathfrak{r}})$ acts via $\mathcal{N}^m(-)\psi(-)$.

Following [Hid89a, Theorem I], we can associate to each family of nearly ordinary Hilbert modular forms with coefficients in a complete noetherian local integral domain of characteristic different from 2 (i.e. to each $\lambda:\mathbf{h}^{n.ord}(\mathfrak{N},\mathcal{O})\to A$), a 2-dimensional semisimple and continuous Galois representation ρ_λ, unramified outside $\mathfrak{N}p$ such that for all $\mathfrak{q}\nmid\mathfrak{N}p$, we have

$$det(1-\rho_\lambda(Fr_{\mathfrak{q}}))=1-\lambda(T(\mathfrak{q}))X+\lambda(\langle\mathfrak{q}\rangle)\mathcal{N}(\mathfrak{q})X^2.$$

Moreover this representation is nearly-ordinary at all $\mathfrak{p}|p$. For the exact definition of nearly ordinary, see [Hid89a, Theorem I (iv)]. Essentially, it means that its restriction to $D_{\mathfrak{p}}$, the decomposition group at $\mathfrak{p}$, is upper triangular and unramified after a twist by a finite-order character of $I_{\mathfrak{p}}$.

As in Section 3.8 we will discuss the relation between the analytic and the arithmetic $\mathcal{L}$-invariant, we give now briefly the definition of the cyclotomic Hecke algebra $\mathbf{h}^{\text{cycl}}(\mathfrak{N},\mathcal{O})$; it has been introducted in [Hid06, 3.2.9]. Roughly speaking, $\mathbf{h}^{\text{cycl}}(\mathfrak{N},\mathcal{O})$ corresponds to the subspace of $\mathbf{h}^{\text{n.ord}}(\mathfrak{N},\mathcal{O})$ defined by the equation $v_{\sigma_1}=v_{\sigma_2}$, for σ_1 and σ_2 which induce the same p-adic place $\mathfrak{p}$ of F.
Consequently, we say that an arithmetic point P is locally cyclotomic if P, as a character of $T(\mathbb{Z}_p)\equiv\left\{\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}\in\text{GL}_2(\mathfrak{r}_p)\right\}$, factors through the local norms, i.e.,

$$P|_{T(\mathbb{Z}_p)}=\prod_{\mathfrak{p}|p}(\mathcal{N}(a_p)^{v_{\mathfrak{p}}},\mathcal{N}(d_p)^{m_{\mathfrak{p}}}),$$

up to a finite-order character.
From the deformation-theoretic point of view, requiring that a Hilbert modular form $\mathbf{f}$ belongs to $\mathbf{h}^{\mathrm{cycl}}(\mathfrak{N},\mathcal{O})$ is equivalent to demand that, for all $\mathfrak{p} \mid p$ the local Galois representation at $\mathfrak{p}$, $\rho_{\mathbf{f},\mathfrak{p}}$ is of type $\begin{pmatrix} \varepsilon_{\mathfrak{p}} & * \\ 0 & \delta_{\mathfrak{p}} \end{pmatrix}$, with the two characters $\varepsilon_{\mathfrak{p}}$ and δ_p of fixed type outside $\mathrm{Gal}(F_{\mathfrak{p}}(\mu_{p^\infty})/F_{\mathfrak{p}})$ ([Hid06, (Q4')]).

3.3 Hilbert modular forms of half-integral weights

In this section we first recall the theory of Hilbert modular forms of half integral weight, with particular interest in theta series and Eisenstein series. We will use these series to give, in the next section, an integral expression for the L-function of the symmetric square representation. In the second part of the section, we develop a p-adic theory for half integral weight Hilbert modular forms which will be used in Section 3.6.

3.3.1 Complex forms of half-integral weights

We will follow in the exposition [Shi87]. Half-integral weight modular forms are defined using the metaplectic groups as defined by Weil. One can define $M_{\mathbb{A}_F}$ as a certain non trivial extension of $SL_2(\mathbb{A}_F)$ by the complex torus $\mathbb{S}^1$ with a faithful unitary representation (called the Weil representation) on the space $L^2(\mathbb{A}_F)$.
Similarly, for any place v, M_v is defined as a certain extension of $SL_2(F_v)$ by the complex torus $\mathbb{S}^1$ together with a faithful unitary representation on $L^2(F_v)$. We denote by the same symbol pr the projection of $M_{\mathbb{A}}$ onto $SL_2(\mathbb{A}_F)$ and of M_v onto $SL_2(F_v)$. In the following, we will sometimes use G to denote the algebraic group SL_2.

Denote by $P_{\mathbb{A}_F}$ the subgroup of upper triangular matrices in $SL_2(\mathbb{A}_F)$ and by $\Omega_{\mathbb{A}_F}$ the subset of $SL_2(\mathbb{A}_F)$ consisting of matrices which have a bottom left entry invertible.
The extension $M_{\mathbb{A}_F}$ is not split, but we have three liftings r, r_P and r_Ω of $SL_2(F)$, $P_{\mathbb{A}_F}$ and $\Omega_{\mathbb{A}_F}$ such that $pr \circ r_?$ is the identity, for $? = \emptyset, P, \Omega$. Their explicit description is given in [Shi87, (1.9),(1.10)] and they are compatible in the sense that $r = r_P$ on $G \cap P_{\mathbb{A}_F}$ and $r = r_\Omega$ on $G \cap \Omega_{\mathbb{A}_F}$.
For x in $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}) \times \mathrm{GL}_2(F_\infty)^+$ and z in $\mathcal{H}^I$ we define $j(x,z) = det(x_\infty)^{-1/2}(cz+d)$, where $x_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the projection of x to $\mathrm{GL}_2(F_\infty)^+$. For τ in $M_{\mathbb{A}_F}$ we pose $j(\tau,z) = j(pr(\tau),z)$.

We define C' as the matrices γ in $G(\mathbb{A}_F)$ such that γ_v belongs to the group C'_v defined as follows

$$C'_v = \begin{cases} \{\gamma \in SL_2(F_v) | \gamma i = i\} & \text{if } v|\infty \\ \left\{\gamma \in SL_2(F_v) | \gamma \in \begin{pmatrix} * & 2\mathfrak{d}_v \\ 2\mathfrak{d}_v^{-1} & * \end{pmatrix}\right\} & \text{if } v \text{ finite} \end{cases}.$$

Let η_0 be defined by

$$(\eta_0)_v = \begin{cases} 1 & \text{if } v|\infty \\ \begin{pmatrix} 0 & -\mathfrak{d}_v^{-1} \\ \mathfrak{d}_v & 0 \end{pmatrix} & \text{if } v \text{ finite} \end{cases}.$$

C'' is define as the union of C' and $C'\eta_0$. By [Shi87, Proposition 2.3], we can define for τ in $pr^{-1}(P_{\mathbb{A}_F}C'')$ an holomorphic function $h(\tau,z)$ such that

$$h(\tau,z)^2 = \zeta_\tau j(\tau,z),$$

where ζ_τ is a certain fourth root of 1.
Let Γ be a congruence subgroup contained in $C''G(F_\infty)$, we define a Hilbert modular form of half-integral

weight $k+\frac{1}{2}t$, k in $\mathbb{Z}[I]$ and $k \geq 0$, as a holomorphic function on $\mathcal{H}^I$ such that

$$f|_{k+\frac{1}{2}t}\gamma(z) = f(\gamma z)j(\gamma, z)^{-k}h(\gamma, z)^{-1} = f(z)$$

for all γ in Γ. Contrary to the case of GL_2, such complex modular forms are the same as the functions $\mathbf{f} : M_{\mathbb{A}} \to \mathbb{C}$ such that

$$\mathbf{f}(\alpha x w) = \mathbf{f}(x)j(w, z_0)^{-k}h(w, z_0)^{-1} \text{ for all } \alpha \in \mathrm{SL}_2(F), w \in pr^{-1}(B),$$

where B is a compact open subgroups of C''.
In fact, we can pass from one formulation to the other in the following way: to such a $\mathbf{f}$ we associate a Hilbert modular form f on $G \cap B\Gamma_\infty$ by $f(z) = \mathbf{f}(u)j(u, z_0)^{-k}h(u, z_0)^{-1}$ for any u such that $u(z_0) = z$. Conversely, to such f we can associate $\mathbf{f}$ such that

$$\mathbf{f}(\alpha x) = f|_{(k+\frac{1}{2}t)}x(z_0) \text{ for all } \alpha \in \mathrm{SL}_2(F), x \in pr^{-1}(BG(\mathbb{A}_{F,\infty})).$$

Let $\mathfrak{N}$ and $\mathfrak{M}$ be two fractional ideals such that $\mathfrak{N}\mathfrak{M}$ is integral and divisible by 4. Congruence subgroups of interest are $D(\mathfrak{N}, \mathfrak{M})$ which are defined as the intersection of $G(\mathbb{A})$ and $U_0(\mathfrak{N}, \mathfrak{M}) \times SO(2)^{\mathbf{a}}$. The intersection of $D(\mathfrak{N}, \mathfrak{M})$ with G shall be denote by $\Gamma^1[\mathfrak{N}, \mathfrak{M}]$. In particular, assuming $4 \mid \mathfrak{N}$, $h(-, z)$ is a factor of automorphy and the map $\chi_{-1} : \tau \mapsto \zeta_\tau^2$, for the root of unit ζ_τ defined above, is a quadratic character of $D(\mathfrak{N}, 4)$ depending only on the image of d_γ modulo 4.
Let $k' = k + \frac{1}{2}t$, with $k \in \mathbb{Z}[I]$ and $k \geq 0$ and choose a Hecke character ψ of $\mathbb{A}_F^\times$ (i.e. ψ in trivial on $F^\times$) of conductor dividing $\mathfrak{M}$ and such that $\psi_\infty(-1) = (-1)^{\sum_{\sigma \in I} k_\sigma}$. Let $\psi_{\mathfrak{M}} = \prod_{v|\mathfrak{M}} \psi_v$. We define $\mathbf{M}_k(\Gamma^1[\mathfrak{N}, \mathfrak{M}], \psi, \mathbb{C})$ (resp. $\mathbf{S}_k(\Gamma^1[\mathfrak{N}, \mathfrak{M}], \psi, \mathbb{C})$) as the set of all holomorphic functions f (resp. cuspidal) such that

$$f|_k\gamma = \psi_{\mathfrak{M}\mathfrak{N}}(a_\gamma)f \text{ for all } \gamma \in \Gamma^1[\mathfrak{N}, \mathfrak{M}].$$

This definition implies that the corresponding adelic form $\mathbf{f}$ satisfies $\mathbf{f}(x\gamma) = \psi(d_\gamma)\mathbf{f}(x)$ for $\gamma \in D(\mathfrak{N}, \mathfrak{M})$.
To have a non-zero Hilbert modular form on $D(\mathfrak{N}, \mathfrak{r})$, we need to suppose that $4 \mid \mathfrak{N}$. So we will define $\mathbf{M}_k(\mathfrak{N}, \psi, \mathbb{C}) = \mathbf{M}_k(\Gamma^1[2^{-1}\mathfrak{N}, 2], \psi, \mathbb{C})$ and similarly for $\mathbf{S}_k(\mathfrak{N}, \psi, \mathbb{C})$
We can define a Fourier expansion also for half-integral weight.

Proposition 3.3.1. *Let $k \in \mathbb{Z}[I]$ and let $\mathbf{f}$ be a Hilbert modular form in $\mathbf{M}_{k'}(\mathfrak{N}, \psi, \mathbb{C})$, for $k' = \frac{t}{2} + k$, or in $\mathbf{M}_{k,-k/2}(U_0(2^{-1}\mathfrak{N}, 2), \psi, \mathbb{C})$, then we have (ignore r_P if the weight is integral)*

$$\mathbf{f}\left(r_P\begin{pmatrix} y & x \\ 0 & y^{-1}\end{pmatrix}\right) = \psi_\infty^{-1}(y)y_\infty^k|y|_{\mathbb{A}}^{[k'-k]}\sum_{\xi \in F}\lambda(\xi, y\mathfrak{r}; \mathbf{f}, \psi)\mathbf{e}_\infty(i\xi y_\infty^2/2)\mathbf{e}_F(yx\xi/2).$$

The following properties hold

- $\lambda(\xi, \mathfrak{m}; \mathbf{f}, \psi) \neq 0$ *only if* $\xi \in \mathfrak{m}^{-2}$ *and* $\xi = 0$ *or* $\xi \gg 0$,
- $\lambda(\xi b^2, \mathfrak{m}; \mathbf{f}, \psi) = b^k\psi_\infty(b)\lambda(\xi, b\mathfrak{m}; \mathbf{f}, \psi)$ *for all* $b \in F^\times$.

In particular

$$f(z) = \sum_{\xi \in F}\lambda(\xi, \mathfrak{r}; \mathbf{f}, \psi)\mathbf{e}_\infty(\xi z/2)$$

We can compare with the previous Fourier expansion when $\mathbf{f}$ has integral weight and we have

$$\lambda(\xi, y\mathfrak{r}; \mathbf{f}, \psi) = \psi_\infty^{-1}(y)|y|_{\mathbb{A}}^{2+m}\mathbf{a}(y^2\xi, \mathbf{f})\left\{y^2\xi\right\}^v(y_\infty^2\xi)^{-v}. \quad (3.3.2)$$

Proof. The Fourier expansion and the first properties are contained in [Im91, Proposition 2.1]. For the last formula, we point out that $\mathbf{f}$ can be written as $\mathbf{f}'|\begin{pmatrix} 2^{-1} & 0 \\ 0 & 1 \end{pmatrix}_f$ with $\mathbf{f}'$ a Hilbert modular form for $V_1(\mathfrak{N})$. Then we note that

$$\begin{pmatrix} y & x \\ 0 & y^{-1} \end{pmatrix} = \begin{pmatrix} y^2 & xy \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix}$$

and, if the central character of $\mathbf{f}$ is $\psi|\ |_{\mathbb{A}}^{-m}$, we obtain the formula by comparison with 3.2.3. □

The interest of this adelic Fourier expansion is that it can give rise to the Fourier expansions at all cusps. Let $\mathfrak{N}$ be an integral ideal such that $4|\mathfrak{N}$ and $\mathfrak{n}$ a finite idèle representing $\mathfrak{N}$: $\mathfrak{n}\mathfrak{r} = \mathfrak{N}$ and $\mathfrak{n}$ is 1 at the place outside $\mathfrak{N}$. We shall define now two operators $[\mathfrak{n}^2]$ and $\tau(\mathfrak{n}^2)$ which will be useful to simplify the calculations of Eisenstein series and p-adic L functions we shall perform later. The latter operator will be used to define a Hecke equivariant pairing on integral weight modular forms. Let $k' = \frac{t}{2} + k$. We set

$$\mathbf{f}|[\mathfrak{n}^2](x) = \mathcal{N}(\mathfrak{n})^{-\frac{1}{2}}\mathbf{f}\left(xr_P\begin{pmatrix} \mathfrak{n}^{-1} & 0 \\ 0 & \mathfrak{n} \end{pmatrix}_f\right).$$

This operator sends a form of level group $D(\mathfrak{C},\mathfrak{M})$ into one of level group $D(\mathfrak{C}\mathfrak{N}^2,\mathfrak{N}^{-2}\mathfrak{M})$ and its the analogue of [Hid90, §2 h3]. It is easy to see how the Fourier expansion changes;

$$\lambda(\xi, y\mathfrak{r}; \mathbf{f}|[\mathfrak{n}^2], \psi) = \lambda(\xi, \mathfrak{n}^{-1}y\mathfrak{r}; \mathbf{f}, \psi). \tag{3.3.3}$$

In particular, we have from [Im91, Proposition 2.1 (i)] that a necessary condition for $\lambda(\xi, \mathfrak{r}; \mathbf{f}|[\mathfrak{n}^2], \psi) \neq 0$ is that $\xi \in \mathfrak{n}^2$ and $\xi \gg 0$. We write f resp. f' for the complex version of $\mathbf{f}$ resp. $\mathbf{f}|[\mathfrak{n}^2]$; we easily see that the Fourier expansion at infinity is

$$f'(z) = \sum_{0 \ll \xi \in F^\times, \xi \in \mathfrak{n}^2} \lambda(\xi, \mathfrak{n}^{-1}\mathfrak{r}; \mathbf{f}, \psi)\mathbf{e}_\infty(\xi z/2).$$

If moreover $\mathfrak{N} = (b)$, for a totally positive element b, then $f'(z) = b^k f(b^2 z)$.
Important remark: in the next section we shall often use a similar operator $\left[\frac{\mathfrak{n}^2}{4}\right]$. This operator is defined exactly as above with the difference that 4 does not represent the ideal $4\mathfrak{r}$, but 4 is *diagonally embedded* in F_f.

We define the half-integral weight Atkin-Lehner involution

$$\begin{aligned} \mathbf{f}|\tau(\mathfrak{n}^2) =& \mathbf{f}|\eta_2|r_P\begin{pmatrix} \mathfrak{n}^{-1} & 0 \\ 0 & \mathfrak{n} \end{pmatrix}_f \\ =& \mathbf{f}|r_\Omega\begin{pmatrix} 0 & -2\mathfrak{d}^{-1}\mathfrak{n}^{-1} \\ 2^{-1}\mathfrak{d}\mathfrak{n} & 0 \end{pmatrix}_f. \end{aligned}$$

The matrix η_2 is η_0 conjugated with the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}_f$ (embedded diagonally).
This second operator sends $\mathbf{M}_{k'}(\mathfrak{N}^2, \psi, \mathbb{C})$ to $\mathbf{M}_{k'}(\mathfrak{N}^2, \psi^{-1}, \mathbb{C})$, since the corresponding matrix normalizes $D(2^{-1}\mathfrak{N}^2, 2)$. Note that $\tau(\mathfrak{n}^2)$ is almost an involution:

$$\mathbf{f}|\tau(\mathfrak{n}^2)\tau(\mathfrak{n}^2) = \mathbf{f}|\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (-1)^k\mathbf{f}.$$

3.3.2 Theta Series

We give now a first example of half-integral weight modular forms which will be used for the integral formulation of the symmetric square L-function. We follow [Shi87, §4]. Recall that we have a unitary action of $M_{\mathbb{A}_F}$ on the space $L^2(\mathbb{A}_F)$. For a Schwartz function on the finite adèles η and $n \in \mathbb{Z}[I]$, we define a function for $\tau \in M_{\mathbb{A}_F}$

$$\theta_n(\tau, \eta) = \sum_{\xi \in F} \tau\eta_n(\xi, i)$$

where we define for ξ in $\mathbb{A}_F$ and w in $\mathcal{H}^I$

$$\eta_n(\xi, w) = \eta(\xi_f) \prod_{\sigma \in I} \phi_{n_\sigma}(\xi_\sigma, w_\sigma),$$

$\phi_{n_\sigma}(\xi_\sigma, w_\sigma) = y_{n_\sigma}^{-n_\sigma/2} H_{n_\sigma}(\sqrt{4\pi y_{n_\sigma}}\xi_\sigma)\mathbf{e}(\xi_\sigma^2 w_\sigma/2)$, where H_n is the n-th Hermite polynomial. It is the adelic counterpart of the complex form

$$\theta_n(z, \eta) = (4\pi y)^{-n/2} \sum_{\xi \in F} \eta(\xi, i) H_n(\sqrt{4\pi y}\xi)\mathbf{e}_\infty(\xi^2 z/2)$$

Choose a Hecke character χ of F of conductor $\mathfrak{N}$ such that $\chi_\infty(-1) = (-1)^{\sum_I n_\sigma}$, we denote also by χ the corresponding character on the group of fractional ideals. Recall that we defined $t = \sum_{\sigma \in I} \sigma$; for $n = 0, t$ we have

$$\theta_0(\chi) = \sum_{\xi \in F} \chi_\infty(\xi)\chi(\xi\mathfrak{r})\mathbf{e}_\infty(\xi^2 z/2)$$
$$\theta_t(\chi) = \sum_{\xi \in F} \chi_\infty(\xi)\chi(\xi\mathfrak{r})\xi^t\mathbf{e}_\infty(\xi^2 z/2).$$

From now on $\theta_n(\chi) := \theta_{nt}(\chi)$; $\theta_n(\chi)$ is holomorphic. Moreover, from [Shi87, Lemma 4.3], we have that for all γ in $\Gamma^1[2\mathfrak{N}^2, 2]$ we have

$$\theta_n(\chi)|_{(n+1/2)t}\gamma = \chi_{4\mathfrak{N}^2}(a_\gamma)\theta_n(\chi).$$

This tells us that $\theta_n(\chi)$ is a modular form in $\mathbf{M}_{n+\frac{t}{2}}(4\mathfrak{N}^2, \chi_{4\mathfrak{N}^2}^{-1}, \mathbb{C})$. The explicit coefficients, given by [Shi87, (4.21)], are as follows:

$$\lambda(\xi, \mathfrak{m}, \theta_n(\chi), \chi) = \begin{cases} 2\chi_\infty(\eta)\chi(\eta\mathfrak{m})\xi^n & \text{if } 0 \neq \xi = \eta^2 \in \mathfrak{m}^{-2}, \\ \chi(\mathfrak{m}) & \text{if } 0 = \xi, \mathfrak{f} = \mathfrak{r} \text{ and } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For a Hecke character χ, we define, following [Shi93, (4.8)], the Gauß sum of χ

$$G(\chi) = \sum_{x \in \mathfrak{f}^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}} \chi_\infty(x)\chi_f(x\mathfrak{d}\mathfrak{f})\mathbf{e}_F(x).$$

We conclude with the following proposition which generalizes a well-known result for $F = \mathbb{Q}$.

Proposition 3.3.4. *Let χ be a Hecke character of conductor $\mathfrak{N}$. Then*

$$\theta(\tau, \chi)|\tau(4\mathfrak{n}^2) = C(\chi)\theta(\tau, \chi^{-1})$$

for $C(\chi) = G(\chi)\mathcal{N}(\mathfrak{N})^{-1/2}\chi(\mathfrak{d}\mathfrak{N})$.

Proof. We decompose
$$\begin{pmatrix} 0 & -\mathfrak{d}^{-1}\mathfrak{n}^{-1} \\ \mathfrak{n}\mathfrak{d} & 0 \end{pmatrix} = \begin{pmatrix} \mathfrak{d}^{-1}\mathfrak{n}^{-1} & 0 \\ 0 & \mathfrak{n}\mathfrak{d} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
As in [Shi87, Lemma 4.2] we have
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\chi_f(x) = \mathcal{N}(\mathfrak{d})^{-1/2}\mathcal{N}(\mathfrak{n})^{-1}G(\chi)\chi(\mathfrak{d}\mathfrak{n})\chi_f^{-1}(\mathfrak{d}\mathfrak{n}x).$$
Now, we are left to study the action of the diagonal matrix on χ. As this matrix is in $P_{\mathbb{A}_F}$, we can use [Shi87, (1.12)] to reduce ourself to local calculations
$$r_P\begin{pmatrix} \mathfrak{d}^{-1}\mathfrak{n}^{-1} & 0 \\ 0 & \mathfrak{n}\mathfrak{d} \end{pmatrix}\chi_f = \prod_v r_P\begin{pmatrix} \mathfrak{d}^{-1}\mathfrak{n}^{-1} & 0 \\ 0 & \mathfrak{n}\mathfrak{d} \end{pmatrix}_v \chi_v.$$
With [Shi87, (1.9)] we see that the action of this diagonal matrix sends $\chi_v(x)$ to $|\mathfrak{d}\mathfrak{n}|_v{}^{-1/2}\chi_v(\mathfrak{d}_v^{-1}\mathfrak{n}_v^{-1}x)$, then
$$\tau(4\mathfrak{n}^2)\chi_f(x) = G(\chi)\mathcal{N}(\mathfrak{n})^{-1/2}\chi(\mathfrak{d}\mathfrak{N})\chi_f^{-1}(x).$$
□

3.3.3 Eisenstein series

We give a second example of half-integral weight forms which we shall use in the next sections. The aim of this section is to find a normalization of these series which shall allow us to p-adically interpolated its Fourier coefficients 3.3.10.
Let us fix an integral weight $k \in \mathbb{Z}[I]$, a congruence subgroup of level $\mathfrak{c}$, a Hecke character ψ of finite conductor dividing $\mathfrak{c}$ and of infinite part of type $(x/|x|)^k$. Decompose
$$G \cap P_{\mathbb{A}_F}D(2^{-1}\mathfrak{N}, 2) = \coprod_{\mathfrak{a}_i \in \mathrm{Cl}(F)} P\beta_i\Gamma^1[2^{-1}\mathfrak{N}, 2],$$
where P is the Borel of G and $\beta_i = \begin{pmatrix} t_i & 0 \\ 0 & 1 \end{pmatrix}$, for t_i an idèle representing $\mathfrak{a}_i$.
We pose
$$E(z,s;k,\psi,\mathfrak{c}) = \sum_{\mathfrak{a}_i \in \mathrm{Cl}(F)} N(\mathfrak{a}_i)^{2s+\frac{1}{2}} \sum_{\gamma \in P_i \backslash \beta_i\Gamma^1[2^{-1}\mathfrak{N},2]} \psi(d_\gamma \mathfrak{a}_i^{-1})\psi_\infty(d_\gamma)\frac{y^{st-k/2}h(\gamma,z)^{-1}j(\gamma,z)^{-k}}{|j(\gamma,z)|^{2st-k}},$$
where $P_i = P \cap \beta_i\Gamma^1[2^{-1}\mathfrak{N},2]\beta_1^{-1}$. We follow mainly the notation of [Im91, §1], but we point out that our Eisenstein series $E(z,s;k,\psi,\mathfrak{c})$ coincides with Im's series E evaluated at $s+1/4$ and $\mathfrak{a} = 2^{-1}\mathfrak{r}$, and we refer to *loc. cit.* for the statements without proof which will follow.

Take an element η_1 in $\mathrm{SL}_2(F)$ such that $\eta_1^{-1} \in Z(\mathbb{A}_F)\eta_0$ and an element $\tilde{\eta}$ in $M_{\mathbb{A}_F}$ such that $pr(\tilde{\eta}) = \eta_0$; we can define
$$E'(z,s;k,\psi,\mathfrak{c}) = E(z,s;k,\psi,\mathfrak{c})|_{k+\frac{1}{2}}\eta_1.$$
We can see $E(z,s;k,\psi,\mathfrak{c})$ as an adelic form, as in Section 3.3.1: for all non-negative integers ρ we define
$$J(\tau,z) = h(\tau,z)^{2\rho+1}(j(\tau,z)/|j(\tau,z)|)^k$$

and a function f on $M_{\mathbb{A}}$ as
$$f(\tau) = \psi_f(d_p)\psi_{\mathfrak{c}}(d_w^{-1})|J(\tau,z_0)|J(\tau,z_0)^{-1}$$
if $\tau = pw$ is in $P_{\mathbb{A}_F}D(2^{-1}\mathfrak{c}\mathfrak{d}, 2\mathfrak{d}^{-1})$ and 0 otherwise. We define moreover, for τ in $P_{\mathbb{A}_F}C$, $\tau = pw'$, $\delta_\tau = |d_p|_{\mathbb{A}}$ and for $\tau \in pr^{-1}P_{\mathbb{A}_F}C$, $\delta_\tau = \delta_{pr(\tau)}$.
The adelic Eisenstein series is defined by
$$E_{\mathbb{A}}(\xi, s; (2\rho+1)/2, k, \psi, \mathfrak{c}) = \sum_{g \in P \backslash SL_2(F)} f(g\xi)\delta_{g\xi}{}^{-2s-\rho-1/2}.$$
We have therefore $E(z,s;k,\psi,\mathfrak{c}) = E_{\mathbb{A}}(u,s;1/2,k,\psi,\mathfrak{c})J(u,z_0)$, where u is such that $z = u(z_0)$. Moreover,
$$E'(z,s;k,\psi,\mathfrak{c}) = E_{\mathbb{A}}(u\tilde{\eta}, s; \frac{1}{2}t, k, \psi, \mathfrak{c})J(u,z_0).$$
For $k = \kappa t$, $\kappa > 0$ and $s = \kappa/2$ we have:
$$\begin{aligned} E(z,\kappa/2;\kappa t,\psi,\mathfrak{c}) \in & \mathbf{M}_{k+\frac{1}{2}}(\mathfrak{c};\psi_{\mathfrak{c}}^{-1}), \\ E'(z,\kappa/2;\kappa t,\psi,\mathfrak{c}) \in & \mathbf{M}_{k+\frac{1}{2}}(2,2^{-1}\mathfrak{c};\psi_{\mathfrak{c}}^{-1}). \end{aligned}$$
We define a normalization of the previous Eisenstein series
$$\mathcal{E}'(z,s;k,\psi,\mathfrak{c}) = L_{\mathfrak{c}}(4s,\psi^2)E'(z,s;k,\psi,\mathfrak{c}),$$
where $L_{\mathfrak{c}}(s,\psi)$ stands for $L(s,\psi)\prod_{\mathfrak{q}|\mathfrak{c}}(1-\psi(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})$. We define $\mathcal{E}(z,s;k,\psi,\mathfrak{c})$ analogously. Let $A = \psi(\mathfrak{d})D_F{}^{\kappa-1}\mathcal{N}(2\mathfrak{c}^{-1})i^{\kappa d}\pi^d 2^{d\left(\kappa-\frac{1}{2}\right)}$. We consider now the case when s is negative; we have the following result on the algebraicity of these series

Theorem 3.3.5 ([Im91], Proposition 1.5)**.** *The series $A^{-1}\mathcal{E}'(z,\frac{1-\kappa}{2};\kappa t,\psi,\mathfrak{c})$ has the following Fourier development*
$$L_{\mathfrak{c}}(1-2\kappa) + \sum_{0 \ll \xi \in 2\mathfrak{c}^{-1}} L_{\mathfrak{c}}(1-\kappa,\psi\omega_\xi)\beta\left(\xi,\frac{1-\kappa}{2}\right)\mathbf{e}_\infty(\xi x),$$
where ω_ξ is the Hecke character associated with $F(\sqrt{\xi})$ and
$$\beta(\xi,s) = \sum_{\substack{\mathfrak{a}^2\mathfrak{b}^2|\mathfrak{c}\xi \\ \mathfrak{a},\mathfrak{b} \text{ prime to } \mathfrak{c}}} \mu(\mathfrak{a})\psi\omega_\xi(\mathfrak{a})\psi(\mathfrak{b})^2N(\mathfrak{b})^{-2s}N(\mathfrak{a})^{1-4s}.$$

The fact that the Fourier expansion is shifted by $\mathfrak{c}$ is due to the fact that we are working with forms for the congruence subgroup $D(2,2^{-1}\mathfrak{c})$. Similarly, if $\mathfrak{c}$ is the square of a principal ideal in $\mathfrak{r}$ and $\mathfrak{c}_0$ an idèle representing this ideal, we can deduce the Fourier expansion of $\mathcal{N}(\mathfrak{c}_0 2^{-1})^{-\kappa}A^{-1}\mathcal{E}'(z,\frac{1-\kappa}{2};\kappa t,\psi,\mathfrak{c})[\mathfrak{c}_0^2 4^{-1}]$ simply by taking the same sum but over all $0 \ll \xi \in 2^{-1}\mathfrak{r}$.

We introduce now the notion of nearly-holomorphic modular form. It has been extensively studied by Shimura. Let k be in $\mathbb{Z}\left[\frac{1}{2}\right][I]$, and v in $\mathbb{Z}[I]$ when k is integral. Let Γ be a congruence subgroup of $\mathrm{GL}_2(F_\infty)$ or $\mathrm{SL}_2(F_\infty)$, according to the fact that the weight is integral or half-integral, and ψ a finite-order character of this congruence subgroup. Let $\beta \in \mathbb{Z}[I]$, $\beta \geq 0$, we define the space of nearly-holomorphic forms of degree at most β, $\mathbf{N}^\beta_{k,v}(\Gamma,\psi;\mathbb{C})$ as the set of all functions
$$f : \mathcal{H}^I \to \mathbb{C}$$

such that $f|_{k,v}\gamma(z) = \psi(\gamma)f(z)$ for all γ in Γ and f admits a generalized Fourier expansion

$$f = \sum_{i=0}^{\beta}\sum_{n}^{\infty} a_n^{(i)} q^n \frac{1}{(4\pi y)^i}$$

at all cups for Γ.
In case of integral weight, it is possible to define adelic nearly-holomorphic forms for compact open subgroups of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})$ as in [Hid91, §1]; to each such form we can associate form an h-tuple of complex forms as we defined.
We warn the reader that the function $g_i(z) = \sum_n^{\infty} a_n^{(i)} q^n$ are not modular forms for Γ, except when $i = \beta$.
Let now be l in $\mathbb{Z}[I]$, λ in $\mathbb{R}$ and σ in I, we define a (non-holomorphic) differential operator on $\mathcal{H}^I$ by

$$\begin{aligned}\partial_\lambda^\sigma =& \frac{1}{2\pi i}\left(\frac{\lambda}{2iy_\sigma} + \frac{\mathrm{d}}{\mathrm{d}z_\sigma}\right),\\ \partial_k^l =& \prod_{\sigma\in I}\left(\partial_{k_\sigma+2l_\sigma-2}^\sigma \cdots \partial_{k_\sigma}^\sigma\right),\end{aligned}$$

and here $z_\sigma = x_\sigma + iy_\sigma$ is the variable on the copy of $\mathcal{H}$ indexed by σ.
Let f be in $\mathbf{N}_{k,v}^s(\Gamma,\psi;\mathbb{C})$, if $k > 2s$ (when $F \neq \mathbb{Q}$ this condition is automatically satisfied) then we have the following decomposition

$$f = \sum_{i=0}^{s} \partial_{k-2i}^i f_i$$

with each f_i holomorphic and modular. A proof of this, when $F = \mathbb{Q}$, is given in [Shi76, Lemma 7] and for general F the proof is the same.
For such a form f we define the *holomorphic projection* $H(f) := f_0$. For any f in $\mathbf{N}_{k,v}^s(\Gamma,\psi;\mathbb{C})$ we define the *constant term projection* $c(f) = g_0$. If we see f as a polynomial in $\frac{1}{(4\pi y)^i}$, $c(f)$ is then the constant term. We will see when the weight is integral that $c(f)$ is a p-adic modular form. This two operators will be used to calculate the values of the p-adic L-function in Section 3.7.
We have the relation

$$\partial_k^l(f|_{k,v}\gamma) =(\partial_k^l f)|_{k+2l,v-l}\gamma \tag{3.3.6}$$

which tells us that ∂_k^l sends nearly-holomorphic modular forms for Γ of weight (k,v) to nearly-holomorphic modular forms for Γ of weight $(k+2r, v-r)$.
This operator allows us to find some relation between Eisenstein series of different weights. Let k, l be in $\mathbb{Z}[I]$, $k,l > 0$. Then we have by the proof of [Im91, Proposition 1.1]

$$\delta_{k+\frac{1}{2}t}^l E(z,s;k,\psi,\mathfrak{c}) =(4\pi)^{-l} b_l\left(-\frac{k}{2} - \left(s+\frac{1}{2}\right)t\right) E(z,s;k+2l,\psi,\mathfrak{c}) \tag{3.3.7}$$

where

$$b_l(x) = \prod_{\sigma\in I}\prod_{j=0}^{l_\sigma-1}(x_\sigma - j) \text{ for } x = (x_\sigma) \in \mathbb{C}^I.$$

If $k = \kappa t$ and $2s = 1-\kappa$ then we obtain $b_l(-t) = (-1)^{\sum l_v}\prod_{v\in\mathbf{a}}(l_v)!$.
In particular, in the case we will need later, this coefficient is not zero. Let $k = m + 2t - 2v \geq 0$ be an integral weight, let k_0 be the minimum of the k_σ's and s an integer such that $m - k_0 - 1 < s$, we have that $b_v\left(\frac{2v-st-2t}{2}\right) \neq 0$.

For x in $\mathbb{R}^I$, we pose $\Gamma_\infty(x) = \prod_{\sigma\in I}\Gamma(x_\sigma)$.
Let $k = m+2t-2v \geq 0$ be an integral weight, let k_0 be the minimum of the k_σ's and s an integer such that $m+1 \leq s \leq m+k_0-1$, and $n \in \{0,1\}$, $n \not\equiv s(2)$ and $\psi(-1) = (-1)^{k-d-nd}$.
In the notation of [Shi85, Lemma 4.2] we pose, for σ in I, $\alpha_\sigma = \frac{s-m-n+k_\sigma}{2}$, $\beta_\sigma = \frac{s-m-k_\sigma+n+1}{2}$.

Proposition 3.3.8. *For k, s, m, n and ψ as above we have*

$$\mathcal{E}'\left(z, \frac{s-m}{2}; k-(n+1)t, \psi, \mathfrak{c}\right) = A'_0 \sum_{0\leq j\leq -\beta} e_j(k,s,\psi)\frac{\Gamma_\infty(1-\alpha-j)/\Gamma_\infty(1-\alpha)}{(4\pi y)^j}$$

where

$$\begin{aligned} A'_0 =& i^{(n+1/2)d-k}\pi^\alpha\Gamma_\infty(\alpha)^{-1}2^{k-\left(n+\frac{1}{2}\right)d}(-1)^{-d(n+1)+\sum_\sigma(k_\sigma)}\times \\ &\times \psi(\mathfrak{d})\mathcal{N}(\mathfrak{d})^{m-s}\mathcal{N}(2\mathfrak{c}^{-1})\mathbf{e}_\infty([F:\mathbb{Q}]/8), \\ e_j(k,s,\psi) =& \sum_{\xi\in 2\mathfrak{c}^{-1},\xi\gg 0} \xi^{-\beta-j}g_f(\xi,(s-m)/2)L_{\mathfrak{c}}(s-m,\psi\omega_{2\xi})\mathbf{e}_\infty(\xi z). \end{aligned}$$

Here $g_f(\xi,s)$ is a product over the primes $\mathfrak{q}$ dividing $\xi\mathfrak{c}$ but prime with $\mathfrak{c}$ of polynomials in $\psi(\mathfrak{q})\mathcal{N}(\mathfrak{q})^s$.

Proof. We just have to explicitate the Fourier coefficients denoted by $y^\beta\xi(y,\xi;\alpha,\beta)$ in [Shi85, Theorem 6.1]. They are essentially hypergeometric functions.
Notice that our definition is different from his; in fact we have

$$E'(z,s;k-(n+1)t,\psi,\mathfrak{c}) = y^{k/2}E'_{\mathrm{Sh}}(z,s;1/2,k,\psi).$$

Note that in our situation the α_σ's are positive half-integers and the β_σ's are non-positive integers, for all σ. Using [Hid91, (6.9 b)], the evaluation of $y_\sigma^{\beta_\sigma}\xi(y_\sigma,\xi_\sigma;\alpha_\sigma,\beta_\sigma)$ reduces to the following cases: if $\xi_\sigma > 0$, we find

$$i^{\beta_\sigma-\alpha_\sigma}(2\pi)^{\alpha_\sigma}\Gamma(\alpha_\sigma)^{-1}2^{-\beta_\sigma}\xi_\sigma^{\alpha_\sigma-1}e^{-4\pi y\xi_\sigma/2}W(4\pi y\xi_\sigma,\alpha_\sigma,\beta_\sigma)$$

for W the Whittaker function given in [Hid91, page 359]

$$W(y,\alpha_\sigma,\beta_\sigma) = \Gamma(\beta_\sigma)^{-1}y^{-\beta_\sigma}\int_{\mathbb{R}^+} e^{-yx}(x+1)^{\alpha_\sigma-1}x^{\beta_\sigma-1}.$$

If $\xi_\sigma = 0$ we find:

$$i^{\beta_\sigma-\alpha_\sigma}\pi^{\alpha_\sigma}2^{\alpha_\sigma+\beta_\sigma}\frac{\Gamma(\beta_\sigma+\alpha_\sigma-1)}{\Gamma(\alpha_\sigma)\Gamma(\beta_\sigma)}(4\pi y)^{1-\alpha_\sigma}$$

If $\xi_\sigma < 0$ we find :

$$i^{\beta_\sigma-\alpha_\sigma}(2\pi)^{\beta_\sigma}\Gamma(\beta_\sigma)^{-1}2^{-\alpha_\sigma}y^{\beta_\sigma-\alpha_\sigma}|\xi_\sigma|^{\alpha_\sigma-1}e^{-4\pi y\xi_\sigma/2}W(-4\pi y\xi_\sigma,\beta_\sigma,\alpha_\sigma).$$

Notice that $W(y,1,\beta_\sigma) = 1$.
If ξ is not zero and not a totally positive element, then for at least one σ the corresponding hypergeometric function is 0 (because of the zero of $\Gamma(\beta_\sigma)^{-1}$).
So we are reduced to evaluate these W's only for $\xi_\sigma \geq 0$. By [Shi82, Theorem 3.1] we have $W(y,\alpha_v,\beta_v) = W(y,1-\beta_v,1-\alpha_v)$. As $1-\beta_\sigma \geq 1$, we can use [Hid91, (6.5)] to obtain

$$y^\beta\xi(y,\xi;\alpha,\beta) = i^{\beta-\alpha}\pi^\alpha\Gamma(\alpha)^{-1}2^{\alpha-\beta}e^{\frac{-4\pi y\xi}{2}}\sum_{j=0}^{-\beta}\binom{-\beta}{j}\frac{\Gamma_\infty(1-\alpha+j)}{\Gamma_\infty(1-\alpha)}\frac{\xi^{\alpha-1-j}}{(4\pi y)^j}.$$

□

We point out that $\mathcal{E}'\left(z,\frac{k_0}{2};k_0 t,\psi,\mathfrak{c}\right)$ is an holomorphic Hilbert modular form for all $k_0 \geq 1$. Suppose now $\mathfrak{c}=\mathfrak{n}^2$, we are interested now in the Fourier expansion of $\mathcal{E}'(z,\frac{s-m}{2};k-(n+1)t,\psi,\mathfrak{N}^2)|[\mathfrak{n}^2]$. Let us define, for all u in $\mathbb{A}_F$, $t(u)=r_P\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}$ and recall the function on $M_{\mathbb{A}_F}$

$$f(\tau)=\psi_f(d_p)\psi_{\mathfrak{c}}(d_w^{-1})|J(\tau,z_0)|J(\tau,z_0)^{-1} \qquad (\tau=pw)$$

defined above. Let $\iota:=r_\Omega\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}_f$. Let $\tilde{z}=r_P\begin{pmatrix}y^{1/2} & x\\ 0 & y^{-1/2}\end{pmatrix}$ for $y=(y_\sigma)\in\mathbb{R}_+^I$ and $x=(x_\sigma)\in\mathbb{R}^I$. We give now a useful lemma

Lemma 3.3.9. *For u in F_f, we have $f(\iota t(u)\tilde{z}\tau(\mathfrak{n}^2/4))\neq 0$ if and only if $u\in 2\mathfrak{d}^{-1}\mathfrak{r}$, and then*

$$f(\gamma\tau(\mathfrak{n}^2/4))|\delta_{\gamma\tau(\mathfrak{n}^2/4)}|_{\mathbb{A}}^{-2s-\frac{1}{2}}=\psi_f(2^{-1}\mathfrak{n})\mathcal{N}(\mathfrak{n}2^{-1})^{-2s-\frac{1}{2}}f(\gamma'\eta_0)|\delta_{\gamma'\eta_0}|_{\mathbb{A}}^{-2s-\frac{1}{2}}$$

for $u'_v=(u\mathfrak{n}^2 4^{-1})_v$, $\gamma=\iota t(u)\tilde{z}$ and $\gamma'=\iota t(u')\tilde{z}$.

Proof. For all finite place v, we have

$$(\iota t(u)\tau(\mathfrak{n}^2/4))_v=\begin{pmatrix}-(2^{-1}\mathfrak{d}\mathfrak{n})_v & 0\\ (u2^{-1}\mathfrak{d}\mathfrak{n})_v & -(2\mathfrak{d}^{-1}\mathfrak{n}^{-1})_v\end{pmatrix}.$$

We have by definition that $f(\xi)\neq 0$ if and only if $\xi\in P_{\mathbb{A}_F}D(2^{-1}\mathfrak{N}^2\mathfrak{d},2)$. Let us write then

$$\iota t(u)z\tau(\mathfrak{n}^2/4)=r_P\begin{pmatrix}* & *\\ 0 & e\end{pmatrix}r_\Omega\begin{pmatrix}* & *\\ c & d\end{pmatrix}.$$

We know that d can be chosen as a v-adic unit for all $v\mid\mathfrak{d}\mathfrak{N}$; combining this with the explicit expression of $\iota t(u)z\tau(\mathfrak{n}^2/4)$, we obtain $u\in 2\mathfrak{d}^{-1}\mathfrak{r}$ and this proves the first part of the lemma.
Concerning the second part of the lemma, the right hand side has been calculated in [Shi85, Lemma 4.2]. We have (compare this formula with [Shi85, (4.19)])

$$\delta_{(\iota t(u)\tau(\mathfrak{n}^2/4))_v}=|u\mathfrak{d}\mathfrak{n}2^{-1}|_v=\mathcal{N}(\mathfrak{n}/2)|u'\mathfrak{d}|_v$$

and $h(\tau(\mathfrak{n}^2/4),z)=j(\tau(\mathfrak{n}^2/4),z)=1$. If we write

$$\iota t(u')\eta_0=r_P\begin{pmatrix}* & *\\ 0 & e'\end{pmatrix}r_\Omega\begin{pmatrix}* & *\\ c' & d'\end{pmatrix},$$

we see we can choose $e=2\mathfrak{n}^{-1}e'$ and $d=d'$, and this allows us to conclude. □

We can now state the following proposition:

Proposition 3.3.10. *Let k, s, m, n and ψ as in Proposition 3.3.8, we have*

$$\mathcal{E}'\left(z,\frac{s-m}{2};k-(n+1)t,\psi,\mathfrak{n}^2\right)\mid\left[\frac{\mathfrak{n}^2}{4}\right]=A_0\sum_{0\leq j\leq-\beta}e'_j(k,s,\psi)\frac{\Gamma_\infty(1-\alpha-j)/\Gamma_\infty(1-\alpha)}{(4\pi y)^j}$$

where

$$\begin{aligned}A_0=&i^{\left(n+\frac{1}{2}\right)d-k}\pi^\alpha\Gamma_\infty(\alpha)^{-1}2^{k-\left(n+\frac{3}{2}\right)d}(-1)^{-d(n+1)+\sum_\sigma(k_\sigma)}\times\\ &\times\psi(\mathfrak{d}\mathfrak{n}2^{-1})D_F\mathcal{N}(\mathfrak{d}\mathfrak{n}2^{-1})^{m-s-1}\mathbf{e}_\infty([F:\mathbb{Q}]/8),\\ e'_j(k,s,\psi)=&L_{\mathfrak{c}}(2(s-m)-1,\psi\omega_{2\xi})+\sum_{\xi\in 2^{-1}\mathfrak{r},\xi\gg 0}\xi^{-\beta-j}g_f(\xi,(s-m)/2)L_{\mathfrak{c}}(s-m,\psi\omega_{2\xi})\mathbf{e}_\infty(\xi z).\end{aligned}$$

Here $g_f(\xi,s)$ is a product over the primes $\mathfrak{q}$ dividing 2ξ but prime with 2 of polynomials in $\psi(\mathfrak{q})\mathcal{N}(\mathfrak{q})^s$. Moreover, let κ be an integer, $\kappa \geq 2$, then

$$A_0^{-1}\mathcal{E}'\left(z,\frac{1-\kappa}{2};\kappa t,\psi,\mathfrak{c}\right) \mid \left[\frac{\mathfrak{n}^2}{4}\right] = L_{\mathfrak{c}}(1-2\kappa) + \sum_{0\ll\xi\in 2^{-1}} L_{\mathfrak{c}}(1-\kappa,\psi\omega_{\xi})\beta\left(\xi,\frac{1-\kappa}{2}\right)\mathbf{e}_{\infty}(\xi x),$$

for $A_0 = i^{\kappa d}\pi^d 2^{d\left(\kappa-\frac{1}{2}\right)}\psi_f(\mathfrak{d}\mathfrak{n}2^{-1})D_F\mathcal{N}(\mathfrak{d}\mathfrak{n}2^{-1})^{\kappa-2}$.

Proof. We give a proof only of the first formula as the proofs of the two formulae are identical.
The fact that the Fourier expansion is indexed by $\xi \gg 0$ and ξ in $2^{-1}\mathfrak{r}$ can be proved exactly as in [Shi83, page 430]. Using the adelic expression, [Shi85, pag. 300] and Lemma 3.3.9, we see that up to a factor $\psi_f(2\mathfrak{n}^{-1})\mathcal{N}(\mathfrak{n}2^{-1})^{m-s-1}$, the sum which gives $\mathcal{E}'(z,\frac{s-m}{2};k-(n+1)t,\psi,\mathfrak{n}^2)|[\mathfrak{n}^2/4]$ is the same as the one for $\mathcal{E}'(z,\frac{s-m}{2};k-(n+1)t,\psi,\mathfrak{c})$ which can be found in [Shi85, pag. 300]. We can calculate than the Fourier coefficients as in [Shi85, §5]; we choose a measure on F_v which gives volume 1 to the valuation ring of F_v.
Then we have, in the notation [Shi85, §5], for $\sigma \in 2^{-1}\mathfrak{d}^{-1}\mathfrak{r}$,

$$c_v(\sigma,s) = |2|_{F_v}, \quad v \mid 2\mathfrak{n}.$$

This implies that we have to substitute the factor $\mathcal{N}(2\mathfrak{n}^{-2})$ which appears in the factor A_0' defined in Proposition 3.3.8 with 2^{-d}. The integrals at the other places are unchanged. □

3.3.4 The p-adic theory again

We want now define the notion of a p-adic modular form of half-integral weight. This is done in order to construct in Section 3.6 some measures interpolating the forms introduced in the previous section and which will be used to construct the p-adic L-function in Section 3.7. In particular, we shall construct a p-adic trace 3.3.14 to pass from "SL_2"-type p-adic modular forms (such as the product of two half-integral weight forms) to "GL_2"-type p-adic modular forms. This is done interpolating p-adically the traces defined in [Im91, Proposition 4.1].
We will construct half-integral weight p-adic modular forms as in [Wu01]; we will define them them via q-expansion. Although this approach is hardly generalizable to more general context where no q-expansion principle ia available, it will be enough for our purpose.
For a geometric approach using the Igusa tower, we refer the reader to [DT04].
We fix as above an ideal $\mathfrak{N}$ of $\mathfrak{r}$ such that $4 \mid \mathfrak{N}$ and let k be an half integral weight. For all f in $\mathbf{M}_k(\mathfrak{N},\psi,\mathbb{C})$, consider the Fourier expansion from Proposition 3.3.1

$$f(z) = \sum_{\xi\in F}\lambda(\xi,\mathfrak{r};\mathbf{f},\psi)\mathbf{e}_{\infty}(\xi z/2).$$

This gives us an embedding (because of the uniqueness of the Fourier expansion and the transformation properties of f) of $\mathbf{M}_k(\mathfrak{N},\psi,\mathbb{C})$ into $\mathbb{C}[[q]]$.
For a subalgebra A of $\mathbb{C}$ we define $\mathbf{M}_k(\mathfrak{N},\psi,A)$ as $\mathbf{M}_k(\mathfrak{N},\psi,\mathbb{C})\cap A[[q]]$. By the q-expansion principle [DT04, Proposition 8.7], if p is unramified, we are indeed considering geometric modular forms of half-integral weight defined over A.
Let K a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}$ its ring of integer, choose a number field K_0, of the same degree as K, dense in K for the p-adic topology induced by the fixed embeddings chosen at the beginning and let $\mathcal{O}_0$ its ring of integers. We define

$$\begin{aligned}\mathbf{M}_k(\mathfrak{N},\psi,\mathcal{O}) =& \mathbf{M}_k(\mathfrak{N},\psi,\mathcal{O}_0)\otimes\mathcal{O},\\ \mathbf{M}_k(\mathfrak{N},\psi,K) =& \mathbf{M}_k(\mathfrak{N},\psi,\mathcal{O})\otimes K.\end{aligned}$$

It is possible, multiplying by a suitable θ series, to show the independence of the choice of K_0 and that these spaces coincide with the completion w.r.t. the p-adic topology of $\mathbf{M}_k(\mathfrak{N},\psi,\mathcal{O}_0)$ and $\mathbf{M}_k(\mathfrak{N},\psi,K_0)$. The p-adic topology is defined as the maximum of the p-adic norm of the coefficients in the q-expansion. We define

$$\overline{\mathbf{M}}_{\mathrm{half}}(\mathfrak{N},\mathcal{O}) = \overline{\sum_{k\in\mathbb{Z}[I]+\frac{1}{2}t}\bigcup_{r\in\mathbb{N}_{>0},\psi}\mathbf{M}_k(\mathfrak{N}p^r,\psi,\mathcal{O})},$$
$$\overline{\mathbf{M}}_{\mathrm{half}}(\mathfrak{N},K) = \overline{\sum_{k\in\mathbb{Z}[I]+\frac{1}{2}t}\bigcup_{r\in\mathbb{N}_{>0},\psi}\mathbf{M}_k(\mathfrak{N}p^r,\psi,K)}.$$

If p is unramified in F, this is again compatible with the geometric construction [DT04, Application page 608]. From the considerations in [DT04, §9], it could be possible in most cases to prove that the reunion on both p-level and weight is superfluous.
The same construction allows us to define $\overline{\mathbf{S}}_{\mathrm{half}}(\mathfrak{N},\mathcal{O})$ and $\overline{\mathbf{S}}_{\mathrm{half}}(\mathfrak{N},K)$.
Let χ be an Hecke character of conductor $\mathfrak{c}$ of finite-order. We consider it both as an idèle character $\chi=\chi_f\chi_\infty$ for two character of $\mathbb{A}_{F,\mathbf{f}}^\times$ and $F_\infty^\times$ such that χ is trivial on $F^\times$ and as a function on $\mathfrak{r}/\mathfrak{c}$ which is the character $\chi_{\mathfrak{c}}=\prod_{\mathfrak{q}|\mathfrak{c}}\chi_{\mathfrak{q}}$ on $(\mathfrak{r}/\mathfrak{c})^\times$ and 0 otherwise.

Lemma 3.3.11. *Let χ be an Hecke character of* $\mathrm{Cl}(\mathfrak{c})$ *and*

$$f=\sum_\xi \lambda(\xi,\mathfrak{r};f,\psi)q^{\xi/2}$$

be an element of $\mathbf{M}_k(\mathfrak{N},\psi,\mathbb{C})$, k an half-integral weight. Then

$$\sum_\xi \chi(\xi\mathfrak{r})\lambda(\xi,\mathfrak{r};f,\psi)q^{\xi/2}$$

belongs to $\mathbf{M}_k(\mathfrak{N}\mathfrak{c}^2,\psi\chi^2,\mathbb{C})$ and we will denote it by $f|\chi$.

Proof. We proceed as in [Hid91, §7.F]. We define

$$f|\chi = G(\chi)^{-1}\sum_u \chi(u)f|\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix},$$

for $u\in\mathfrak{r}_{\mathfrak{c}}$ which runs over a set of representative of $\mathfrak{c}^{-1}2\mathfrak{d}^{-1}/2\mathfrak{d}^{-1}$. We have

$$\lambda(\xi,y\mathfrak{r};f|\chi,\psi) = G(\chi)^{-1}\lambda(\xi,y\mathfrak{r};f,\psi)\sum_u\chi(u)\mathbf{e}_F(y^2\xi u/2)$$

Recall that $\mathbf{e}_{\mathfrak{q}}(x_{\mathfrak{q}})=1$ if $x_{\mathfrak{q}}$ is in $\mathfrak{d}_{\mathfrak{q}}^{-1}$, for $\mathfrak{q}$ a non-archimedean place, and $\sum_u\chi(u)=0$ if χ is not trivial.
We have in particular $\lambda(\xi,\mathfrak{r};f|\chi,\psi)=\chi(\xi)\lambda(\xi,\mathfrak{r};f,\psi)$. □

Let $\mathcal{C}(\mathfrak{r}_{\mathfrak{p}}/\mathfrak{p}^e,\mathcal{O})$ be the set of functions from $\mathfrak{r}_{\mathfrak{p}}/\mathfrak{p}^e$ to $\mathcal{O}$, as a $\mathcal{O}$-module it is spanned by the characters of $\mathfrak{r}_{\mathfrak{p}}^\times/\mathfrak{p}^r$ and the constant function. So we can define the operator $|\phi$ for all function ϕ in $\mathcal{C}(\mathfrak{r}_{\mathfrak{p}}/\mathfrak{p}^e,\mathcal{O})$. Let $\mathcal{C}(\mathfrak{r}_{\mathfrak{p}},\mathcal{O})$ be the set of continuous functions from $\mathfrak{r}_p$ to $\mathcal{O}$, the locally constant function are dense in this set. For all ϕ in $\mathcal{C}(\mathfrak{r}_{\mathfrak{p}},\mathcal{O})$ we define the operator $|\phi$ on $\overline{\mathbf{M}}_{\mathrm{half}}(\mathfrak{N},\mathcal{O})$ in the following way: let $\{\phi_n\}$ be some locally constant functions such that $\phi=\lim_n\phi_n$, then we pose $\mathbf{f}|\phi=\lim_n\mathbf{f}|\phi_n$.
We can also define on $\overline{\mathbf{M}}_{\mathrm{half}}(\mathfrak{N},\mathcal{O})$ the differential operator d^σ, for $\sigma\in I$; take y in $\mathfrak{r}_p^\times$, we have the continuous

map $(\ )^\sigma : y \mapsto y^\sigma$. We define $\mathrm{d}^\sigma \mathbf{f} = \mathbf{f}|(\ )^\sigma$. We have then $\lambda(\xi, \mathfrak{r}; \mathrm{d}^\sigma \mathbf{f}, \psi) = \xi^\sigma \lambda(\xi, \mathfrak{r}; \mathbf{f}, \psi)$. As $y \mapsto y^{-1}$ is a continuous map, we can define also $\mathrm{d}^{-\sigma}$ such that

$$\lambda(\xi, \mathfrak{r}; \mathrm{d}^{-\sigma}\mathbf{f}, \psi) = \iota_p(\xi)\xi^{-\sigma}\lambda(\xi, \mathfrak{r}; \mathbf{f}, \psi),$$

where $\iota_p(\xi) = 0$ if $(p, \xi) \neq \mathfrak{r}$ and 1 otherwise. We define for all $r \in \mathbb{Z}[I]$ the operator $\mathrm{d}^r = \prod_{\sigma \in I} \mathrm{d}^{\sigma r_\sigma}$.
We have an action of $\mathfrak{r}_p^\times$ on $\overline{\mathbf{M}}_{\mathrm{half}}(\mathfrak{N}, \mathcal{O})$; it acts via the Hecke operator $T(b, b^{-1})$ defined in 3.2.2. $T(b, b^{-1})$ acts by right multiplication by the matrix $r_P \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}_f$, as this element normalizes the congruence subgroup $D(2^{-1}\mathfrak{N}, 2)$. This actions commutes with the action of $(\mathfrak{r}/\mathfrak{N}p^r\mathfrak{r})^\times$ via the diamond operators. We can describe this action in the complex setting as follows.
Let (b', b) in $(\mathfrak{r}/\mathfrak{N}\mathfrak{r})^\times \times \mathfrak{r}_p^\times$ and take b_0 in $\mathbb{A}_{F,\mathbf{f}}^\times$ which is projected to (b', b). For f in $\mathbf{M}_k(\mathfrak{N}p^r, \psi, \mathcal{O})$ we pose $f|(b', b) = b^k f|\gamma_b$ with $\gamma_b \in \mathrm{SL}_2(F)$ such that

$$\gamma_b^{-1} r_P \begin{pmatrix} b_0 & 0 \\ 0 & b_0^{-1} \end{pmatrix}_f \in D(2^{-1}\mathfrak{N}, 2)\mathrm{SL}_2(F_\infty)$$

and then extend it by continuity on the whole $\overline{\mathbf{M}}_{\mathrm{half}}(\mathfrak{N}, \mathcal{O})$. Here we are implicitly assuming that the coefficients ring $\mathcal{O}$ contains all the square roots of elements of $\mathfrak{r}_p^\times$.
Such an action is compatible with the action of $\mathrm{SL}_2(\hat{\mathfrak{r}})$ on the forms of integral weight as defined in [Hid91, §7.C].

Take now f in $\mathbf{M}_{k_1+\frac{t}{2}}(\mathfrak{N}, \psi_1, A)$ and g in $\mathbf{M}_{k_2+\frac{t}{2}t}(\mathfrak{N}, \psi_2, A)$; the product fg is a modular form in $\mathbf{M}_{k_1+k_2+t}(\Gamma^1[2^{-1}\mathfrak{N}, 2], \psi_1\psi_2\chi_{-1}, A)$ where χ_{-1} is the quadratic character modulo 4 defined before via the automorphy factor $h(\gamma, z)$.
It is clear that this product induces a map

$$\overline{\mathbf{M}}_{k_1+\frac{t}{2}}(\mathfrak{N}, \mathcal{O}) \times \overline{\mathbf{M}}_{k_2+\frac{t}{2}}(\mathfrak{N}, \mathcal{O}) \to \overline{\mathbf{M}}_{k_1+k_2+t}(\Gamma^1[2^{-1}\mathfrak{N}, 2], \mathcal{O})$$

which on the level of q-expansion is just multiplication of formal series.

We point out that when $F = \mathbb{Q}$, the congruence subgroups which are considered for the integral weights and the ones for the half-integral weights are the same, because in $\mathbb{Z}$ the only totally positive unit is 1, while when $F \neq \mathbb{Q}$ the totally positive units are of positive rank.
This poses a problem because in the sequel we shall need to consider the product of two half-integral weight modular forms as a "GL_2-type modular form" whereas it is only a form of "SL_2-type".
We follow the ideas of [Im91, Proposition 4.1] to work out this problem.
For simplicity, we pose $\Gamma[\mathfrak{N}] := \Gamma[2^{-1}\mathfrak{N}, 2]$ and $\Gamma^1[\mathfrak{N}] := \Gamma^1[2^{-1}\mathfrak{N}, 2]$. Let k, v be *integral weights* and A a p-adic $\mathcal{O}$-algebra of characteristic 0, we define

$$\begin{aligned}
\overline{\mathbf{M}}_k(\Gamma^1[\mathfrak{N}], A) &= \overline{\bigcup_{r \in \mathbb{N}_{>0}, \psi_0} \mathbf{M}_k(\Gamma^1[\mathfrak{N}p^r], \psi_0, A)}, \\
\overline{\mathbf{M}}_{k,v}(\Gamma[\mathfrak{N}], A) &= \overline{\bigcup_{r \in \mathbb{N}_{>0}, \psi, \psi'} \mathbf{M}_k(\Gamma[\mathfrak{N}p^r], \psi, \psi', A)}, \\
\overline{\mathbf{M}}(\Gamma^1[\mathfrak{N}], A) &= \overline{\sum_{k \in \mathbb{Z}[I]} \overline{\mathbf{M}}_k(\Gamma^1[\mathfrak{N}], A)}, \qquad (3.3.12) \\
\overline{\mathbf{M}}(\Gamma[\mathfrak{N}], A) &= \overline{\sum_{k, v \in \mathbb{Z}[I]} \mathbf{M}_{k,v}(\Gamma[\mathfrak{N}], A)},
\end{aligned}$$

where completion is taken w.r.t. the p-adic topology defined as the maximum of the p-adic norm of the coefficients in the q-expansion.
We consider now the three following tori of $\mathrm{GL}_2(\mathfrak{r}/\mathfrak{N}p^r\mathfrak{r})$:

$$\mathbb{T}_{\mathrm{ss}} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} | a \in (\mathfrak{r}/\mathfrak{N}p^r\mathfrak{r})^\times \right\},$$
$$\mathbb{T}_Z = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} | a \in (\mathfrak{r}/\mathfrak{N}p^r\mathfrak{r})^\times \right\},$$
$$\mathbb{T}_\alpha = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} | a \in (\mathfrak{r}/\mathfrak{N}p^r\mathfrak{r})^\times \right\}.$$

Let ψ_0 resp. ψ, ψ' be a character of the torus $\mathbb{T}_{\mathrm{ss}}$, resp. $\mathbb{T}_Z$, $\mathbb{T}_\alpha$. We say that they form a *compatible triplet* if

$$\psi_0\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = \psi\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix}\right)\psi'\left(\begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Proposition 3.3.13. *Let ψ_0, ψ and ψ' as above forming a compatible triplet. Suppose that ψ factors through $(\mathfrak{r}/\mathfrak{N}p^r\mathfrak{r})^\times/E$. Let $w \in \mathbb{Z}^d$ be such that $k - 2w = mt$, $m \geq 0$ and pose, as in Section 3.2.1, $v = t - w$. We have a map*

$$\mathrm{Tr}_{\mathrm{SL}_2}^{\mathrm{GL}_2}(v, \psi') : \quad \overline{\mathbf{M}}_k(\Gamma^1[\mathfrak{N}], \mathcal{O}) \quad \to \quad \overline{\mathbf{M}}_{k,v}(U[\mathfrak{N}], \mathcal{O})$$

which is equivariant for the action of $\Gamma[\mathfrak{N}]$ and $\mathfrak{r}_p^\times$ and such that if f is a modular form in $\mathbf{M}_k(\Gamma^1[\mathfrak{N}p^r], \mathcal{O}; \psi_0)$, then $\mathrm{Tr}_{\mathrm{SL}_2}^{\mathrm{GL}_2}(v, \psi')(f)$ is a modular form in $\mathbf{M}_{k,v}(U[\mathfrak{N}p^r], \mathcal{O}; \psi, \psi')$.

Proof. We point out that we are adding just one variable, namely m.

We extend f to $E\Gamma^1[\mathfrak{N}]$ simply requiring that it is invariant for the diagonal action of the units.
Let R be a set of representatives of $\Gamma[\mathfrak{N}p^r]/E\Gamma_1[\mathfrak{N}p^r] \cong E/E^2$, we define

$$\mathrm{Tr}_E(f) = \frac{1}{2^{d-1}} \sum_{\varepsilon_i \in R} \psi'(\varepsilon_i^{-1}) f|_{k,w}\varepsilon_i.$$

For all γ in $\Gamma_1[\mathfrak{N}p^r]$, we have $\varepsilon_i\gamma = \gamma_1\varepsilon_i$, with $\psi_0(\gamma) = \psi_0(\gamma_1)$. Moreover, the matrix $\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$ acts via $\psi'(\varepsilon)$.
We call the space of such forms $\mathbf{M}_{k,v}(\Gamma[\mathfrak{N}p^r], \mathcal{O}; \psi_0, \psi')$.

It is easy to see that Tr_E commutes with the following inclusions:

$$\mathbf{M}_k(\Gamma^1[\mathfrak{N}p^r], \mathcal{O}; \psi_0) \hookrightarrow \mathbf{M}_k(\Gamma^1[\mathfrak{N}p^{r+1}], \mathcal{O}; \psi_0),$$
$$\mathbf{M}_{k,v}(\Gamma[\mathfrak{N}p^r], \mathcal{O}; \psi_0, \psi') \hookrightarrow \mathbf{M}_{k,v}(\Gamma[\mathfrak{N}p^{r+1}], \mathcal{O}; \psi_0, \psi').$$

At the beginning of Section 3.2.1 we decomposed the space $\mathbf{M}_{k,w}(U(\mathfrak{N}p^r))$ of adelic modular forms for $U(\mathfrak{N}p^r)$ into complex ones. Let $\mathfrak{a}_i$ be a set of representatives for the strict class group of F modulo $\mathfrak{N}p^r$. As ψ factors through $(\mathfrak{r}/\mathfrak{N}p^r\mathfrak{r})^\times/E$, we can see it as a character of $\mathrm{Cl}(\mathfrak{N}p^r)$. We define

$$\begin{array}{rccc} \mathcal{I}: & \mathbf{M}_{k,w}(\Gamma[\mathfrak{N}], \psi_0, \psi') & \to & \mathbf{M}_{k,w}(U(\mathfrak{N}), \psi, \psi') \\ & f & \mapsto & \mathcal{I}_{\mathfrak{N}p^r}(f, \ldots, \psi^{-1}(\mathfrak{a}_i)f, \ldots, \psi^{-1}(\mathfrak{a}_{h(\mathfrak{N}p^r)})f), \end{array}$$

where $\mathcal{I}_{\mathfrak{N}p^r}$ is the isomorphism defined in Section 3.2.1. We can choose $\mathfrak{a}_i$ in a compatible way when r grows such that it is compatible with the obvious inclusion

$$\mathbf{M}_{k,w}(U(\mathfrak{N}p^r)) \hookrightarrow \mathbf{M}_{k,w}(U(\mathfrak{N}p^{r+1}))$$

and compatible also with the Nebentypus decomposition.
We define $\mathrm{Tr}_{\mathrm{SL}_2}^{\mathrm{GL}_2}(v,\psi') := \mathcal{I}\circ\mathrm{Tr}_E$. We can extend this map to the space of p-adic modular forms Nebentypus by Nebentypus and then to the completion, as it is clearly continuous. It is equivariant by construction.

$\square$

Let us show that the trace morphisms defined above vary p-adically continuously. We decompose $\mathfrak{r}_p^\times$ as the product of a torsion part μ and its free part $\mathbf{W}'$ which we identify with $(1+p\mathbb{Z}_p)^d$. Fix elements a_j for $j=1,\ldots,d$ which generates $\mathbf{W}'$.
Choose as before a set R of representative (independent of r) for $\Gamma[\mathfrak{N}p^r]/E\Gamma_1[\mathfrak{N}p^r]$. Let $q=p^f$ and s big enough such that all the torsion of $\mathfrak{r}_p^\times$ is killed by $(q-1)p^s$. Fix ψ_0, ψ and ψ' as above, then we have if $v\equiv v' \bmod (q-1)p^s$

$$\begin{aligned}\mathrm{Tr}_E(v,\psi')f =& \frac{1}{2^{d-1}}\sum_R \psi'(\varepsilon_i^{-1})\det(\varepsilon_i)^w f|_{k,0}\varepsilon_i\\ \equiv& \frac{1}{2^{d-1}}\sum_R \psi'(\varepsilon_i^{-1})\det(\varepsilon_i)^{w'} f|_{k,0}\varepsilon_i \bmod p^s\\ =&\mathrm{Tr}_E(v',\psi')f \bmod p^s.\end{aligned}$$

This shows that the trace morphism $\mathrm{Tr}_E(v,\psi')$ can be p-adically continuously interpolated over $\mathfrak{r}_p^\times$.
For each y in $\mathfrak{r}_p^\times$, we write the projection of y to $\mathbf{W}'$ as $(\langle y\rangle_1,\ldots,\langle y\rangle_d)$, we have by definition

$$\langle y\rangle_j = a_j^{\log_p(\langle y\rangle_j)/\log_p(a_j)}.$$

Let us identify $\mathcal{O}[[\mathbf{W}']]$ with $\mathcal{O}[[X_1,\ldots,X_d]]$, in such a way that a_j corresponds to $1+X_j$ and define $g_i=\det(\varepsilon_i)$. We pose

$$\begin{aligned}\mathrm{Tr}_{\mathrm{SL}_2}^{\mathrm{GL}_2} : \overline{\mathbf{M}}_k(\Gamma^1[\mathfrak{N}],\mathcal{O}) &\to \overline{\mathbf{M}}_k(U(\mathfrak{N}),\mathcal{O}[[\mathfrak{r}_p^\times]])\\ f &\mapsto \mathcal{I}\left(\frac{1}{2^{d-1}}\sum_R \psi'(\varepsilon_i^{-1})A_{g_i}(X)f|_{k,0}\varepsilon_i\right)\end{aligned}$$

where $A_y(X)=\prod_j(1+X_j)^{\log_p(\langle y\rangle_j)/\log_p(a_j)}$.
For all points P of $\mathrm{Spec}(\mathcal{O}[[\mathfrak{r}_p^\times]])$ of type $a\mapsto\psi'(a)a^v$, we obtain a commutative diagram

$$\begin{tikzcd}\overline{\mathbf{M}}_k(\Gamma^1[\mathfrak{N}],\mathcal{O}) \arrow[r, "\mathrm{Tr}_{\mathrm{SL}_2}^{\mathrm{GL}_2}"] & \overline{\mathbf{M}}_k(U(\mathfrak{N}),\mathcal{O}[[\mathfrak{r}_p^\times]]) \arrow[d, "P"]\\ \mathbf{M}_k(\Gamma^1[\mathfrak{N}p^r],\psi_0,\mathcal{O}) \arrow[u] \arrow[r, "\mathrm{Tr}_{\mathrm{SL}_2}^{\mathrm{GL}_2}(v,\psi'_P)"] & \mathbf{M}_k(U(\mathfrak{N}),\psi,\psi'_P,\mathcal{O})\end{tikzcd}$$

for $\psi'_P(\zeta)=\psi'(\zeta)\zeta^{-v}$ ([Hid91, page 337]).

Consider the action of $\mathfrak{r}_p^\times$ defined above, we see exactly as in [Hid91, page 334] that $b\in\mathfrak{r}_p^\times$ acts via b^k. This shows that the sum over k in (3.3.12), before taking completion, is a direct sum. Substituting in the above definition $A_{g_i}(X)f|_{k,0}\varepsilon_i$ by $f|T(g_i,1)$, for $T(g_i,1)$ the Hecke operator defined in Section 3.2.2, we can extend $\mathrm{Tr}_{\mathrm{SL}_2}^{\mathrm{GL}_2}$ to the map below

$$\mathrm{Tr}_{\mathrm{SL}_2}^{\mathrm{GL}_2} : \overline{\mathbf{M}}(\Gamma^1[\mathfrak{N}],\mathcal{O}) \to \overline{\mathbf{M}}(U(\mathfrak{N}),\mathcal{O}[[\mathfrak{r}_p^\times]]). \tag{3.3.14}$$

We show now some compatibility with the Hecke action; Shimura in [Shi87, §5] has defined Hecke operators $T'(p)$ in the half-integral weight case.
There is a correspondence between the integral weight Hecke operator $T_0(p^2)$ and the half-integral weight Hecke operator $T'(p)$. Notice that we can define an operator $T(p)$ on $\mathbf{M}_{k,v}(\Gamma[\mathfrak{N}], \mathcal{O})$ because $T(p)$, as defined in Section 3.2.2, does not permute the connected components of the Shimura variety associated with $V_1(\mathfrak{N})$. Using [Hid91, Proposition 7.4] (note that there is a misprint in the expression of $\mathbf{a}(y, \mathbf{f}|T_0^n(p))$ and that the correct expression can be found in [Hid91, (2.2b)]), we see that $T_0(p^n)$ is given by

$$f = \sum_{0 \ll \xi \in F^\times} a(\xi, f) q^{\xi/2} \mapsto f|T_0(p^n) = p^{nv}\{p\}^{-nv} \sum_{0 \ll \xi \in F^\times} a(p^n\xi, f) q^{\xi/2}$$

Using [Shi87, Proposition 5.4] we see that the operator $T'(p^n)$ defined by Shimura is given by

$$f = \sum_{0 \ll \xi \in F^\times} a(\xi, f) q^{\xi/2} \mapsto f|T'(p^n) = p^{-nk} \sum_{0 \ll \xi \in F^\times} a(p^{2n}\xi, f) q^{\xi/2}.$$

So we have then $T(p^{2n}) = \mathcal{N}(p^n)^{(m+2)} T'(p^n)$. We can think of $T'(p^n)$ as the operator on "unitarized" Hilbert modular forms (for which $v = -k/2$).
This compatibility allows us to notice the following; let χ be a Hecke character of conductor p^n and f and g two forms in $\mathbf{M}_{k_1+\frac{t}{2}}(\mathfrak{N}, \psi_1, \mathcal{O})$ and $\mathbf{M}_{k_2+\frac{t}{2}}(\mathfrak{N}, \psi_2, \mathcal{O})$, then as in [Hid91, Proposition 7.4] we obtain

$$T_0(p^n)(f\ g|\chi) = \chi_\infty(-1) T_0(p^n)(f|\chi\ g).$$

A similar statement applies to linear combinations of characters. In particular for the idempotent e of Section 3.2.2 and all r in $\mathbb{Z}[I]$ we have

$$e(f\ g|d^r) = (-1)^r e(f|d^r\ g).$$

3.4 The L-function for the symmetric square

Nothing new is presented in this section; we shall first recall the definition of the symmetric square L-function and present in details the Euler factors at place of bad reduction for $\mathbf{f}$. We explain in formula 3.4.5 the origin of one of the Euler type factor at p which appears in the interpolation formula of the p-adic L-function 3.7.2.

3.4.1 The imprimitive L-function

Let $\mathbf{f}$ be a Hilbert modular form of level $\mathfrak{N}$, weight (k, w). We put $v = t - w$, and let m be the non-negative integer such that $(m+2)t = k + 2v$. Suppose that $\mathbf{f}$ is an eigenvector for the whole Hecke algebra and let ψ be the finite-order character such that $\mathbf{f}|T(z,z) = \psi(z)\mathbf{f}$. Let λ be the morphism of the Hecke algebra such that $\mathbf{f}|h = \lambda(h)$ and pose

$$\sum_{\mathfrak{m} \subset \mathfrak{r}} \frac{\lambda(T_0(\mathfrak{m}))}{\mathcal{N}(\mathfrak{m})^s} = \prod_{\mathfrak{q} \in Spec^\circ(\mathfrak{r})} \left(1 - \lambda(T(\mathfrak{q}))\mathcal{N}(\mathfrak{q})^{-s} + \psi(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{m+1-2s}\right)^{-1}.$$

We decompose each Euler factor as

$$(1 - \lambda(T(\mathfrak{q}))\mathcal{N}(\mathfrak{q})^{-s} + \psi(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{m+1-2s}) = (1 - \alpha(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})(1 - \beta(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s}).$$

Take a Hecke character χ of F such that $\chi_\infty(-1) = (-1)^{nt}$ with $0 \leq n \leq 1$ and $nt \equiv k \bmod 2$. The L-function of the symmetric square of the Galois representation associated with $\mathbf{f}$ coincides, up to some Euler factor at the bad primes, with

$$\mathcal{L}(s, \mathbf{f}, \chi) = \prod_{\mathfrak{q} \in Spec^\circ(\mathfrak{r})} D_{\mathfrak{q}}(\chi(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})^{-1}$$

where

$$D_{\mathfrak{q}}(X) = (1-\alpha(\mathfrak{q})^2 X)(1-\alpha(\mathfrak{q})\beta(\mathfrak{q})X)(1-\beta(\mathfrak{q})^2 X).$$

The following elementary identity holds

$$\mathcal{L}(s,\mathbf{f},\chi) = L_{\mathfrak{N}}(2s-2,\mathcal{N}^{2m}\psi^2\chi^2)\sum_{\mathfrak{m}\subset\mathfrak{r}}\lambda(T(\mathfrak{m}^2))\chi(\mathfrak{m})\mathcal{N}(\mathfrak{m})^{-s}.$$

We want to express such a L-function in terms of the Petersson product of $\mathbf{f}$ with a product of two half-integral weight modular forms. We define now the Rankin product of two modular forms for a congruence subgroup of $\mathrm{SL}_2(\mathbb{A}_F)$ following [Im91, §3]. Let $\mathbf{f}$ be a modular form of integral weight k and $\mathbf{g}$ of half-integral weight $l' = l + \frac{t}{2}$, using the Fourier coefficients defined in Proposition 3.3.1, we define the Rankin product $D(s,\mathbf{f},\mathbf{g})$ as

$$D(s,\mathbf{f},\mathbf{g}) = \sum_{W}\lambda(\xi,\mathfrak{m};\mathbf{f},\psi)\lambda(\xi,\mathfrak{m};\mathbf{g},\phi)\xi^{-(l+k)/2}\mathcal{N}(\xi\mathfrak{m}^2)^{-s}.$$

where W is the set of classes $(\xi,\mathfrak{m})$, modulo the equivalence relation $(\xi,\mathfrak{m}) = (\xi\xi_0^2,\xi_0^{-1}\mathfrak{m})$. Notice that we have changed the definition of Im, as we do not conjugate the coefficients of $\mathbf{f}$.
Suppose now that we have $\mathbf{f}$ of weight k and $\mathbf{g} = \theta_{nt}(\chi)$ with $n \equiv k \bmod 2$, we obtain

$$\begin{aligned} D(s,\mathbf{f},\theta_n(\chi)) &= \sum_{W}\lambda(\xi^2,\mathfrak{m};\mathbf{f})2\chi_\infty(\xi)\chi(\xi\mathfrak{m})\xi^{-k}\mathcal{N}(\xi^2\mathfrak{m}^2)^{-s} \\ &= \sum_{W}\lambda(1,\xi\mathfrak{m};\mathbf{f})2\chi(\xi\mathfrak{m})\mathcal{N}(\xi^2\mathfrak{m}^2)^{-s} \\ &= \sum_{\mathfrak{m}\subset\mathfrak{r}}\lambda(1,\mathfrak{m};\mathbf{f})2\chi(\mathfrak{m})\mathcal{N}(\mathfrak{m}^2)^{-s}. \end{aligned}$$

If $\mathbf{f} = (\mathbf{f}_1,\cdots,\mathbf{f}_{h(\mathfrak{N})})$ is a modular form for $U(\mathfrak{N})$, then define $\mathbf{f}^{(2)} = \mathbf{f}|\begin{pmatrix} 2^{-1} & 0 \\ 0 & 1\end{pmatrix}_f$.
We have then $\mathbf{f}_1^{(2)}(z) = \mathbf{f}_1(2z)$. Similarly to [Im91, Proposition 2.2], using (3.3.2) for $\mathbf{f}^{(2)}$ we obtain that

$$\lambda(1,y;\mathbf{f}_1(2z),\psi) = 2^{-v}\psi_\infty^{-1}(y)\mathcal{N}(y)^{-2-m}y_p^{2v}\mathbf{a}_p(y^2,\mathbf{f}). \tag{3.4.1}$$

Using the explicit form of $\lambda(T(\mathfrak{q}))$ in terms of the Fourier coefficient given in 3.2.2, we conclude

$$\begin{aligned} D(s,\mathbf{f}_1(2z),\theta_n(\chi)) &= 2^{-v+1}\sum_{\mathfrak{m}\subset\mathfrak{r}}\lambda(T(\mathfrak{m}^2))\chi(\mathfrak{m})\mathcal{N}(\mathfrak{m})^{-2s-m-2}, \\ D\left(\frac{s-m-1}{2},\mathbf{f}_1(2z),\theta_n(\chi)\right) &= 2^{-v+1}\frac{\mathcal{L}(s+1,\mathbf{f};\chi)}{L_{\mathfrak{N}}(2s-2m,\psi^2\chi^2)}. \end{aligned} \tag{3.4.2}$$

We quote the following proposition [Im91, (3.13)] which gives an integral expression of the Rankin product defined above

Proposition 3.4.3. *Let f be a Hilbert modular form in $\mathbf{M}_k(\Gamma^1[2^{-1}\mathfrak{N},2],\psi_1,\mathbb{C})$ for an integral weight k, and g a Hilbert modular form of half integral weight l' in $\mathbf{M}_k(\mathfrak{N},\psi_2,\mathbb{C})$. Define*

$$R = \left\{\gamma\in\Gamma^1[2^{-1}\mathfrak{N},2] | a_\gamma \equiv 1 \bmod \mathfrak{N}\right\}.$$

and let Φ be a fundamental domain for $R\setminus\mathcal{H}^I$. Then

$$\int_\Phi fg\mathcal{E}y^k\mathrm{d}\mu(z) = BD_F^{-1/2}(2\pi)^{-d(s-3/4)-\frac{k+l'}{2}}\Gamma_\infty\left(\left(s-\frac{3}{4}\right)t+\frac{k+l'}{2}\right)D\left(s-\frac{1}{2},f,g\right)$$

where

$$\begin{aligned}B =& 2\left[\Gamma^1[2^{-1}\mathfrak{N},2]:\{\pm 1\}R\right]N(2\mathfrak{d}^{-1}),\\ \mathcal{E} =& \mathcal{E}\left(z,s;k-l'-\frac{1}{2}t,\chi_{-1}\psi_2\psi_1^{-1},\mathfrak{N}\right).\end{aligned}$$

Let χ be a Hecke character of level $\mathfrak{c}$ and χ_0 be the associated primitive character of conductor $\mathfrak{c}_0$. Let us investigate now the relation between $D(s,\mathbf{f}_1(2z),\theta_n(\chi))$ and $D(s,\mathbf{f}_1(2z),\theta_n(\chi_0))$ in order to make explicit some of the Euler factors at p (more precisely, the one denoted by E_1) which will appear in the interpolation formula of the p-adic L-function of Section 3.7.
Suppose that p divides the level of $\mathbf{f}$ and suppose that $\mathfrak{c}/\mathfrak{c}_0$ is divisible only by primes above p. Let $\left[\mathfrak{e}^2\right]$ be the operator defined in 3.3.4 and suppose $\mathfrak{e}\mathfrak{r}|p$, we have

$$\begin{aligned}D(s,\mathbf{f}_1(2z),\theta_{nt}(\chi_0)|\left[\mathfrak{e}^2\right]) =& 2^{-v+1}\sum_{\mathfrak{m}\subset\mathfrak{e}\mathfrak{r}}\chi(\mathfrak{m}d^{-1})\lambda(T(\mathfrak{m}^2))\mathcal{N}(\mathfrak{m})^{-2s-m-2}\\ =& \lambda(T(\mathfrak{e}^2))\mathcal{N}(\mathfrak{e})^{-2s-m-2}2^{v+1}\frac{\mathcal{L}(2s+m+2,\mathbf{f},\chi)}{L_{p\mathfrak{N}}(4s+2,\psi^2\chi^2)}.\end{aligned} \tag{3.4.4}$$

and using the formula

$$\theta_{nt}(\chi)=\sum_{\mathfrak{e}|p}\mu(\mathfrak{e})\chi_0(\mathfrak{e})\theta_{nt}(\chi_0)|[\mathfrak{e}^2]$$

we can conclude that

$$D(s,\mathbf{f}_1(2z),\theta_{nt}(\chi_0))=\prod_{\mathfrak{p}_i|\mathfrak{c}/\mathfrak{c}_0}(1-\lambda(T(\mathfrak{p}_i^2))\mathcal{N}(\mathfrak{p}_i)^{-2s-m-2})D(s,\mathbf{f}_1(2z),\theta_{nt}(\chi)) \tag{3.4.5}$$

In [Im91, Theorem 5.3] it is shown, for a Hilbert modular form $\mathbf{f}$ of weight $(k,-k/2)$, the algebraicity and Galois equivariance of the values $\mathcal{L}(s_0,\mathbf{f},\chi)$ when divided by suitable periods, where s_0 ranges over the critical integers (in the sense of Deligne [Del79]) for the symmetric square.
Let k_0 be the minimum of k_σ for σ in I and $n\in\{0,1\}$ such that $\chi(-1)=(-1)^{nt}$. Using Deligne's formalism, one finds (see [Im91]) that the critical integers are

$$\{0\leq s<k_0-n-3/2, s\equiv k_0-n(2)\text{ or }-k_0+n+1/2<s\leq -1, s\not\equiv k_0-n(2)\}.$$

As $\mathcal{L}(s,\mathbf{f},\chi)=\mathcal{L}(s-m-1,\mathbf{f}^u,\chi)$, supposing that $m\geq s\geq 0$, we have that $s+1$ is a critical integer when $s\not\equiv n \bmod 2$ and $m+1\leq s\leq m+k_0-1$. The other half of the critical values corresponds to $s\equiv n \bmod 2$ and $m-k_0+n+2\leq s\leq m$.

For any ρ two dimensional representation, $\mathrm{Ad}(\rho)$ denotes the adjoint representation of ρ on $\mathfrak{sl}_2$, the Lie algebra of SL_2. It is known that $\mathrm{Ad}(\rho)$ is the twist of $\mathrm{Sym}^2(\rho)$ by the inverse of the $\det(\rho)$. We can define a naïve L-function $\mathcal{L}(s,\mathrm{Ad}(\mathbf{f}),\chi)$ for $\mathrm{Ad}(\rho_\mathbf{f})$ which is then

$$\begin{aligned}\mathcal{L}(s,\mathrm{Ad}(\mathbf{f}),\chi) =& \mathcal{L}(s,\mathbf{f},\chi\mathcal{N}^{-m-1}\psi^{-1})\\ =& L_{\mathfrak{N}}(2s,\chi^2)\sum_{\mathfrak{m}\subset\mathfrak{r}}\frac{\lambda(T(\mathfrak{m}^2))\chi\psi^{-1}(\mathfrak{m})}{\mathcal{N}(\mathfrak{m})^{s+m+1}}.\end{aligned}$$

We have introduced the L-function of $\mathrm{Ad}(\rho)$ because in the next subsection we will describe explicitly the Euler factors at the bad primes for it and in the literature such a classification is given in term of the adjoint representation, and we prefer to follow the classical references.

3.4.2 The completed L-function

Let $\mathbf{f}$ be a Hilbert modular form of weight (k, v), with $(m+2)t = k+2v$, and of character ψ. Let λ be the corresponding morphism of the Hecke algebra and let $\pi(\mathbf{f})$ be the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ spanned by $\mathbf{f}$. In [GJ78], the authors construct an automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$ denoted $\hat{\pi}(\mathbf{f})$ and usually called the base change to GL_3 of $\pi(\mathbf{f})$. It is standard to associate to $\hat{\pi}(\mathbf{f})$ a complex L-function $\Lambda(s, \hat{\pi}(\mathbf{f}))$ which satisfies a nice functional equation and coincides, up to some Euler factors at the primes for which $\pi(\mathbf{f})$ is ramified and factors at infinity, with $\mathcal{L}(s, Ad(\mathbf{f}))$. We explicit now the functional equation as we will need it in the Appendix 3.11 to show that the p-adic L-function that we will construct is holomorphic (unless $\mathbf{f}$ has CM).
For a Hecke character of F, the automorphic representation $\hat{\pi}(\mathbf{f}) \otimes \chi$ is defined via its L-factor at all primes. For any place v of F, we pose

$$L_v(s, \hat{\pi}(\mathbf{f}), \chi) = \frac{L_v(s, \pi(\mathbf{f})_v \otimes \chi_v \times \tilde{\pi}(\mathbf{f})_v)}{L_v(s, \chi_v)},$$

where $\tilde{}$ denotes the contragredient and $\pi(\mathbf{f})_v \times \tilde{\pi}(\mathbf{f})_v$ is a representation of $\mathrm{GL}_2(F_v) \times \mathrm{GL}_2(F_v)$.
The completed L-function

$$\Lambda(s, \hat{\pi}(\mathbf{f}), \chi) = \prod_v L_v(s, \hat{\pi}(\mathbf{f}), \chi)$$

is holomorphic over $\mathbb{C}$ except in a few cases which correspond to CM-forms with complex multiplication by χ [GJ78, Theorem 9.3].
In addition to the original article by Gelbart and Jacquet, two very good references for the classification of the L-factors at bad prime are [Sch88, §1] for $F = \mathbb{Q}$ and [HT93, §7], where the authors work in the context of nearly-holomorphic forms.
Let $\pi = \pi(\mathbf{f})$ and let $\mathfrak{q}$ be a place where π ramifies and let $\pi_{\mathfrak{q}}$ be the component at $\mathfrak{q}$. By twisting by a character of $F_{\mathfrak{q}}^\times$, we may assume that $\pi_{\mathfrak{q}}$ has minimal conductor among its twist. In fact, this does not change the factor at $\mathfrak{q}$, as one sees from the explicit calculation given in [GJ78, Proposition 1.4]. We distinguish the following four cases

(i) $\pi_{\mathfrak{q}}$ is a principal series $\pi(\eta, \nu)$, with both η and ν unramified,

(ii) $\pi_{\mathfrak{q}}$ is a principal series $\pi(\eta, \nu)$ with η unramified,

(iii) $\pi_{\mathfrak{q}}$ is a special representation $\sigma(\eta, \nu)$ with η, ν unramified and $\eta\nu^{-1} = | \ |_{\mathfrak{q}}$,

(iv) $\pi_{\mathfrak{q}}$ is supercuspidal.

We will partition the set of primes dividing the conductor of $\mathbf{f}$ as $\Sigma_1, \cdots, \Sigma_4$ according to this decomposition.
Let $\varpi_{\mathfrak{q}}$ be a uniformizer of $F_{\mathfrak{q}}$. Just for the next three lines, to lighten notation, we assume, by abuse of notation, that when a character is ramified its value on $\varpi_{\mathfrak{q}}$ will be 0. The Euler factor $L_{\mathfrak{q}}(\hat{\pi}_{\mathfrak{q}} \otimes \chi_{\mathfrak{q}}, s)^{-1}$ is then

(i) $(1 - \chi_{\mathfrak{q}}\nu^{-1}\eta(\varpi_{\mathfrak{q}})\mathcal{N}(\mathfrak{q})^{-s})(1 - \chi_{\mathfrak{q}}(\varpi_{\mathfrak{q}})\mathcal{N}(\mathfrak{q})^{-s})(1 - \chi_{\mathfrak{q}}\nu\eta^{-1}(\varpi_{\mathfrak{q}})\mathcal{N}(\mathfrak{q})^{-s})$,

(ii) $(1 - \chi_{\mathfrak{q}}\nu^{-1}\eta(\varpi_{\mathfrak{q}})\mathcal{N}(\mathfrak{q})^{-s})(1 - \chi_{\mathfrak{q}}(\varpi_{\mathfrak{q}})\mathcal{N}(\mathfrak{q})^{-s})(1 - \chi_{\mathfrak{q}}\nu\eta^{-1}(\varpi_{\mathfrak{q}})\mathcal{N}(\mathfrak{q})^{-s})$,

(iii) $(1 - \chi_{\mathfrak{q}}(\varpi)\mathcal{N}(\mathfrak{q})^{-s-1})$,

The supercuspidal factors are slightly more complicated and depend on the ramification of $\chi_{\mathfrak{q}}$. They are classified by [Sch88, Lemma 1.6]; we recall them briefly. Let $\mathfrak{q}$ be a prime such that $\pi_{\mathfrak{q}}$ is supercuspidal, and let $\xi_{\mathfrak{q}}$ be the unramified quadratic character of $F_{\mathfrak{q}}$. If $\chi_{\mathfrak{q}}^2$ is unramified, let λ_1 and λ_2 the two ramified characters such that $\chi_{\mathfrak{q}}\lambda_i$ is unramified (for completeness, we can suppose $\lambda_1 = \chi_{\mathfrak{q}}$ and $\lambda_2 = \chi_{\mathfrak{q}}\xi_{\mathfrak{q}}$). We consider the following disjoint subsets of Σ_4, the set of cuspidal primes:

$$\Sigma_4^0 = \{\mathfrak{q} \in \Sigma_4 : \chi_{\mathfrak{q}} \text{ is unramified and } \pi_{\mathfrak{q}} \cong \pi_{\mathfrak{q}} \otimes \xi_{\mathfrak{q}}\},$$
$$\Sigma_4^1 = \left\{\mathfrak{q} \in \Sigma_4 : \chi_{\mathfrak{q}}^2 \text{ is unramified and } \pi_{\mathfrak{q}} \cong \pi_{\mathfrak{q}} \otimes \lambda_i \text{ for } i = 1,2\right\},$$
$$\Sigma_4^2 = \left\{\mathfrak{q} \in \Sigma_4 : \chi_{\mathfrak{q}}^2 \text{ is unramified and } \pi_{\mathfrak{q}} \ncong \pi_{\mathfrak{q}} \otimes \lambda_1 \text{ and } \pi_{\mathfrak{q}} \cong \pi_{\mathfrak{q}} \otimes \lambda_2\right\},$$
$$\Sigma_4^3 = \left\{\mathfrak{q} \in \Sigma_4 : \chi_{\mathfrak{q}}^2 \text{ is unramified and } \pi_{\mathfrak{q}} \ncong \pi_{\mathfrak{q}} \otimes \lambda_2 \text{ and } \pi_{\mathfrak{q}} \cong \pi_{\mathfrak{q}} \otimes \lambda_1\right\}.$$

If $\mathfrak{q}$ is in Σ_4 but not in Σ_4^i, for $i = 0, \cdots, 3$, then $L_{\mathfrak{q}}(s, \hat{\pi}_{\mathfrak{q}}, \chi_{\mathfrak{q}}) = 1$. If $\mathfrak{q}$ is in Σ_4^0, then

$$L_{\mathfrak{q}}(s, \hat{\pi}_{\mathfrak{q}}, \chi_{\mathfrak{q}})^{-1} = 1 + \chi(\varpi_{\mathfrak{q}})\mathcal{N}(\mathfrak{q})^{-s}$$

and if $\mathfrak{q}$ is in Σ_4^i, for $i = 1, 2, 3$ then

$$L_{\mathfrak{q}}(s, \hat{\pi}_{\mathfrak{q}}, \chi_{\mathfrak{q}})^{-1} = \prod_{j \text{ s.t.} \pi_{\mathfrak{q}} \cong \pi_{\mathfrak{q}} \otimes \lambda_j} (1 - \chi_{\mathfrak{q}}\lambda_j(\varpi_{\mathfrak{q}})\mathcal{N}(\mathfrak{q})^{-s}).$$

If σ is an infinite place, the L-factor at σ depends only on the parity of the character by which we twist. As we are interested in the symmetric square, we consider the twist by $\psi_\sigma\chi_\sigma$, χ as before. We suppose that the parity of $\psi_\sigma^{-1}\chi_\sigma$ is independent of σ. Let $\kappa = 0, 1$ according to the parity of m, from [Sch88, Lemma 1.1] we have $L(s - m - 1, \hat{\pi}_\sigma, \chi_\sigma\psi_\sigma) = \Gamma_{\mathbb{R}}(s - m - \kappa)\Gamma_{\mathbb{C}}(s - m - 2 + k_\sigma)$ for the complex and real Γ-functions

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2),$$
$$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

We define

$$\mathcal{E}_{\mathfrak{N}}(s, \mathbf{f}, \chi) = \prod_{\mathfrak{q}|\mathfrak{N}} (1 - \chi(\mathfrak{q})\lambda(T(\mathfrak{q}))^2\mathcal{N}(\mathfrak{q})^{-s})L_{\mathfrak{q}}(s - m - 1, \hat{\pi}_{\mathfrak{q}}, \psi_{\mathfrak{q}}^{-1}\chi_{\mathfrak{q}}).$$

Note that $\lambda(T(\mathfrak{q})) = 0$ if π is not minimal at $\mathfrak{q}$ or if $\pi_{\mathfrak{q}}$ is a supercuspidal representation. We multiply then $\mathcal{L}(s, \mathbf{f}, \chi)$, the imprimitive L-function, by $\mathcal{E}_{\mathfrak{N}}(s, \mathbf{f}, \chi)$ to get

$$\begin{aligned} L(s, \mathrm{Sym}^2(\mathbf{f}), \chi) :=& L(s - m - 1, \hat{\pi}(\mathbf{f}) \otimes \chi\psi) \\ =& \mathcal{L}(s, \mathbf{f}, \chi)\mathcal{E}_{\mathfrak{N}}(s, \mathbf{f}, \chi). \end{aligned}$$

We can now state the functional equation

$$\Lambda(s, \hat{\pi}(\mathbf{f}), \chi) = \varepsilon(s, \hat{\pi}(\mathbf{f}), \chi)\Lambda(1 - s, \hat{\pi}(\mathbf{f}), \chi^{-1}),$$
$$\Lambda(s, \mathrm{Sym}^2(\mathbf{f}), \chi) = \varepsilon(s - m - 1, \hat{\pi}(\mathbf{f}), \chi\psi)\Lambda(2m + 3 - s, \mathrm{Sym}^2(\mathbf{f}^c), \chi^{-1}).$$

For a finite place $\mathfrak{q}$ of F, let us recall the proper normalization $\varepsilon_{\mathfrak{q}}$ of the ε-factor which makes it algebraic and Galois equivariant. In [GJ78], the authors use the Langlands normalization, see [Tat79, §3.6], which we will denote by $\varepsilon_{L,\mathfrak{q}}$. For a fixed place $\mathfrak{q}$, it gives to $\mathcal{O}_{F_{\mathfrak{q}}}$ the measure $|\mathfrak{d}_{F_{\mathfrak{q}}/\mathbb{Q}_q}|^{-1/2}$. This is not the ε-factor we want to use, so we multiplying $\varepsilon_{L,\mathfrak{q}}$ by the factor $\mathcal{N}(\mathfrak{d}_{F_{\mathfrak{q}}/\mathbb{Q}_q})^{-s+1/2}$ and we denote this new factor by $\varepsilon_{\mathfrak{q}}$. This is the ε-factor used in [Sch88].
If $\mathfrak{q}$ is such that $\pi_{\mathfrak{q}}$ is a principal series $\pi(\eta, \nu)$ or a special representation $\sigma(\eta, \nu)$, we have explicitly [GJ78, Proposition 1.4, 3.1.2]

$$\varepsilon_{\mathfrak{q}} = \varepsilon(s, \hat{\pi}_{\mathfrak{q}}, \chi_{\mathfrak{q}}) = \frac{\varepsilon(s, \pi_{\mathfrak{q}} \otimes \chi_{\mathfrak{q}}\eta^{-1})\varepsilon(s, \pi_{\mathfrak{q}} \otimes \chi_{\mathfrak{q}}\nu^{-1})}{\varepsilon(s, \chi_{\mathfrak{q}})}.$$

In fact, if $\pi_{\mathfrak{q}}$ and $\chi_{\mathfrak{q}}$ are both unramified, then $\varepsilon(\hat{\pi}_{\mathfrak{q}}, \chi_{\mathfrak{q}}) = 1$.
We know from [BH06, §6.26.1, Theorem] that if $\pi_{\mathfrak{q}}$ is a principal series

$$\varepsilon(s, \pi_{\mathfrak{q}} \otimes \chi_{\mathfrak{q}}\eta^{-1}) = \varepsilon(s, \chi_{\mathfrak{q}})\varepsilon(s, \nu\chi_{\mathfrak{q}}\eta^{-1})$$

and that if $\pi_{\mathfrak{q}} = St$ is a special representation and $\chi_{\mathfrak{q}}$ is ramified then

$$\varepsilon(s, \pi_{\mathfrak{q}} \otimes \chi_{\mathfrak{q}}) = \varepsilon(s, \chi_{\mathfrak{q}})^2.$$

For a supercuspidal $\pi_{\mathfrak{q}}$ we do not need an explicit formula for the ε-factor.
For all prime ideal $\mathfrak{q}$, we know from [GJ78, page 475] that $\varepsilon_{\mathfrak{q},L}(s, \pi_{1,\mathfrak{q}} \times \pi_{2,\mathfrak{q}})$ is $\mathcal{N}(\mathfrak{q}^c)^{-s}\varepsilon_{\mathfrak{q},L}(0, \pi_{1,\mathfrak{q}} \times \pi_{2,\mathfrak{q}})$, where c is a positive integer such that $\mathfrak{q}^c$ is the exact conductor of $\pi_{1,\mathfrak{q}} \times \pi_{2,\mathfrak{q}}$.
The ε-factor at infinity is given by [Sch88, 1.12] $\varepsilon(s, \hat{\pi}_\infty, \chi_\infty) = (-1)^{md}\chi(-1)^{-1/2}$.
Our interest is to see how the ε-factor changes under twist by grössencharacters χ' of p^r-level. This behavior is studied in [Sch88, Lemma 1.4 b)], but there the form $\mathbf{f}$ is supposed to be of level prime to p. We have then to make explicit the ε-factor $\varepsilon(s, \chi_{\mathfrak{q}})$; let $\alpha_{\mathfrak{q}}$ be the conductor of $\chi_{\mathfrak{q}}$ and $e_{\mathfrak{q}}$ the ramification index of $F_{\mathfrak{q}}$ over $\mathbb{Q}_q$. If we define

$$G(\chi_{\mathfrak{q}}) =\chi_{\mathfrak{q}}^{-1}(\varpi_{\mathfrak{q}}^{e_{\mathfrak{q}}+\alpha_{\mathfrak{q}}}) \sum_{x \bmod \varpi_{\mathfrak{q}}^{\alpha_{\mathfrak{q}}}} \chi_{\mathfrak{q}}(x)\mathbf{e}_{F_{\mathfrak{q}}}\left(\frac{x}{\varpi^{e_{\mathfrak{q}}+\alpha_{\mathfrak{q}}}}\right)$$

we have then

$$\varepsilon(s, \chi_{\mathfrak{q}}) = \mathcal{N}(\varpi_{\mathfrak{q}}^{e_{\mathfrak{q}}+\alpha_{\mathfrak{q}}})^{-s}\mathcal{N}(\varpi_{\mathfrak{q}}^{e_{\mathfrak{q}}})^{1/2}G(\chi_{\mathfrak{q}}).$$

We can summarize this discussion in the following lemma

Lemma 3.4.6. *Let $\hat{\pi}$ be the automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$ associated with $\mathbf{f}$. Suppose that $\pi_p := \otimes_{\mathfrak{p}|p}\pi_{\mathfrak{p}}$ is a product of principal series $\pi_{\mathfrak{p}} \cong \pi(\eta, \nu)$ or special representations $\pi_{\mathfrak{p}} \cong \sigma(\eta, \nu)$. Let χ be any grössencharacter and χ' a finite-order character of the class-group modulo p^∞, we have*

$$\begin{aligned}\varepsilon(s, \hat{\pi}, \chi\chi') =&\varepsilon(s, \hat{\pi}, \chi) \prod_{\mathfrak{p}} \frac{C(\chi_{\mathfrak{p}}\chi'_{\mathfrak{p}})^{-s}C(\nu\chi'_{\mathfrak{p}}\chi_{\mathfrak{q}})^{-s}C(\nu^{-1}\chi'_{\mathfrak{p}}\chi_{\mathfrak{p}})^{-s}}{C(\chi_{\mathfrak{p}})^{-s}C(\nu\chi_{\mathfrak{q}})^{-s}C(\nu^{-1}\chi_{\mathfrak{p}})^{-s}} \times \\ &\times \frac{G(\chi_{\mathfrak{p}}\chi'_{\mathfrak{p}})G(\nu\chi'_{\mathfrak{p}}\chi_{\mathfrak{q}})G(\nu^{-1}\chi'_{\mathfrak{p}}\chi_{\mathfrak{p}})}{G(\chi_{\mathfrak{p}})G(\nu\chi_{\mathfrak{q}})G(\nu^{-1}\chi_{\mathfrak{p}})}.\end{aligned}$$

3.5 Some useful operators

We define in this section certain operators which will be useful for the construction of p-adic L-functions in Section 3.7.

3.5.1 The Atkin-Lehner involution on GL_2

Let $\mathfrak{N}$ be an integral ideal and $\mathfrak{n}$ a finite idèle which represents $\mathfrak{N}$: $\mathfrak{n}\mathfrak{r} = \mathfrak{N}$.
Let $\mathbf{f}$ be an element of $\mathbf{M}_{k,w}(U_0(\mathfrak{N}), \psi, \psi')$, where k and w are two integral weights. We define

$$\mathbf{f}|\tau'(\mathfrak{n}) =\psi^{-1}(det(x))\mathbf{f}\left(x\begin{pmatrix} 0 & -1 \\ \mathfrak{d}^2\mathfrak{n} & 0\end{pmatrix}\right).$$

This operator does not change the level but the Nebentypus of $\mathbf{f}|\tau'(\mathfrak{n})$ is (ψ^{-1}, ψ'^{-1}).

Take an in integral ideal $\mathfrak{L}$ prime to p, and a finite idèle $\mathfrak{l}$ such that $\mathfrak{L} = \mathfrak{l}\mathfrak{r}$ and $\mathfrak{l}_p = 1$. We define a *level raising* operator

$$[\mathfrak{l}] : \mathbf{f} \mapsto \mathcal{N}(\mathfrak{L})^{-1}\mathbf{f}| \begin{pmatrix} \mathfrak{l}^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

If the level of $\mathbf{f}$ is big enough, at least $V_1(\mathfrak{N})$, we have independence of the choice of $\mathfrak{l}$.

Which is the relation between $\tau'(\mathfrak{n})$, $[\mathfrak{l}]$ and $\tau'(\mathfrak{n}\mathfrak{l})$? We have

$$\begin{aligned}\mathbf{f}|\tau'(\mathfrak{n})|[\mathfrak{l}](x) =& \mathcal{N}(\mathfrak{l})^{-1}\chi^{-1}(det(x)\mathfrak{l}^{-1})\mathbf{f}\left(x \begin{pmatrix} \mathfrak{l}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \mathfrak{d}^2\mathfrak{n} & 0 \end{pmatrix}\right) \\ =& \mathcal{N}(\mathfrak{l})^{-m-1}\mathbf{f}|\tau'(\mathfrak{n}\mathfrak{l}).\end{aligned}$$

If $\mathfrak{m}\mathfrak{r}|\mathfrak{n}\mathfrak{r}$ we have also

$$\mathbf{f}|[\mathfrak{m}]|\tau'(\mathfrak{n})(x) = \mathcal{N}(\mathfrak{m})^{-1}\mathbf{f}|\tau'(\mathfrak{n}\mathfrak{m}^{-1}).$$

The operator $\tau'(\mathfrak{n}^2)$ differs from the operator of half-integral weight $\tau(\mathfrak{n}^2)$ defined at the end of Section 3.3.1 by a constant which corresponds to the central character of $\mathbf{f}$: $\mathbf{f}|\tau'(\mathfrak{n}^2) = \psi(\mathfrak{n}\mathfrak{d})\mathcal{N}(\mathfrak{n}\mathfrak{d})^m\mathbf{f}|\tau(\mathfrak{n}^2)$.

Suppose now that l' is an half-integral weight, $l' = l + \frac{t}{2}$, and let $\mathfrak{n}, \mathfrak{m}$ be two idèles which represent $\mathfrak{N}$ and $\mathfrak{M}$, with $\mathfrak{M}|\mathfrak{N}$. For the operators $\tau(\mathfrak{n}^2)$ and $[\mathfrak{m}^2]$ defined at the end of Section 3.3.1, we have

$$\begin{aligned}f|[\mathfrak{m}^2]|\tau(\mathfrak{n}^2) =& f|\tau(\mathfrak{n}^2\mathfrak{m}^{-2})\mathcal{N}(\mathfrak{M})^{-1/2}, \\ f|\tau(\mathfrak{n}^2)|[\mathfrak{m}^2] =& f|\tau(\mathfrak{n}^2\mathfrak{m}^2)\mathcal{N}(\mathfrak{M})^{-1/2}.\end{aligned}$$

3.5.2 Some trace operators

We recall some trace operators defined in [Hid91, 7.D,E]. Let $\mathfrak{N}$ and $\mathfrak{L}$ two integral ideals of level prime to p such that $\mathfrak{L}\mathfrak{N} = \mathfrak{M}$, we define a trace operator

$$\begin{array}{rccc} Tr_{\mathfrak{M}/\mathfrak{N}}: & \overline{\mathbf{M}}_{k,w}(U(\mathfrak{N},\mathfrak{L})) & \rightarrow & \overline{\mathbf{M}}_{k,w}(U(\mathfrak{N})) \\ & \mathbf{f} & \mapsto & \sum_{x \in U(\mathfrak{N},\mathfrak{L})/U(\mathfrak{N})} \mathbf{f}|x \end{array}.$$

It naturally extends to p-adic modular form if p is coprime with $\mathfrak{M}$.
We define then a *twisted* trace operator

$$T_{\mathfrak{M}/\mathfrak{N}} = Tr_{\mathfrak{M}/\mathfrak{N}} \circ \begin{pmatrix} \mathfrak{l} & 0 \\ 0 & 1 \end{pmatrix} : \overline{\mathbf{M}}_{k,w}(U(\mathfrak{M})) \rightarrow \overline{\mathbf{M}}_{k,w}(U(\mathfrak{N})),$$

where $\mathfrak{l}$ and $\mathfrak{n}$ are two idèles representing the ideals $\mathfrak{L}$ and $\mathfrak{N}$.

3.5.3 The Petersson product

We recall briefly the definition of the Petersson inner product given in [Hid91, §4]. For $\mathbf{f}$ and $\mathbf{g}$ in $\mathbf{M}_{k,v}(U_0(\mathfrak{N}), \psi, \psi')$, we define

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathfrak{N}} = \int_{X_0(\mathfrak{N})} \overline{\mathbf{f}(x)}\mathbf{g}(x)|\det(x)|_{\mathbb{A}}^m \mathrm{d}\mu_{\mathfrak{N}}(x),$$

where $X_0(\mathfrak{N})$ is the Shimura variety associated with $U_0(\mathfrak{N})C_{\infty+}$ and $\mu_{\mathfrak{N}}$ is a measure on $X_0(\mathfrak{N})$ which is induced from the standard measure on the Borel of $\mathrm{GL}_2(\mathbb{A}_F)$. Let us point out that we do not divide by the

volume of the corresponding Shimura variety.
Let h be the strict class number of F and $\mathfrak{a}_i$ a set of representatives, using the decomposition

$$X_0(\mathfrak{N}) = \bigcup_{i=1}^{h} \mathcal{H}^I/\Gamma[\mathfrak{N}\mathfrak{a}_i, \mathfrak{a}_i^{-1}],$$

we have

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathfrak{N}} = \sum_{i=1}^{h} \langle \mathbf{f}_i, \mathbf{g}_i \rangle_{\mathfrak{N}\mathfrak{a}_i}$$

where

$$\langle \mathbf{f}_i, \mathbf{g}_i \rangle_{\mathfrak{N}\mathfrak{a}_i} = \mathcal{N}(\mathfrak{a}_i)^m \int_{\mathcal{H}^I/\Gamma[\mathfrak{N}\mathfrak{a}_i, \mathfrak{a}_i^{-1}]} \overline{\mathbf{f}_i(z)} \mathbf{g}_i(z) y^k \mathrm{d}\mu(z)$$

$y = \mathrm{Im}(z)$ and $\mu(z)$ is the standard measure on $\mathcal{H}^I$ invariant under linear fractional transformations.
Denote by $\mathbf{f}^c$ the Hilbert modular form whose Fourier coefficients are the complex conjugate of $\mathbf{f}$. If we define $(\mathbf{f}, \mathbf{g}) = \langle \mathbf{f}^c | \tau'(\mathfrak{n}), \mathbf{g} \rangle$ (we dropped from the notation the dependence on the level), we have then that the Hecke algebra is self-adjoint this Petersson product $(-,-)$. We have, [Hid91, 7.2], the following adjunction formula

$$\left\langle \mathbf{f}, \mathbf{g} | T_{\mathfrak{L}/\mathfrak{N}} \right\rangle_{\mathfrak{N}} = \{\mathfrak{L}/\mathfrak{N}\}^{-v} \mathcal{N}(\mathfrak{L}/\mathfrak{N})^{1-m} \langle \mathbf{f} | [\mathfrak{m}], \mathbf{g} \rangle_{\mathfrak{L}}$$

and consequently, if $\mathfrak{L}$ and $\mathfrak{N}$ are prime to p,

$$\left\langle \mathbf{f} | \tau'(\mathfrak{l}), \mathbf{g} | T_{\mathfrak{N}/\mathfrak{L}} \right\rangle_{\mathfrak{N}} = \langle \mathbf{f} | \tau'(\mathfrak{n}), \mathbf{g} \rangle_{\mathfrak{L}}.$$

Let $\mathbf{f}$ be a Hilbert modular form in $\mathbf{M}_{k,v}(U_0(2^{-1}\mathfrak{N}p^r, 2), \psi, \psi')$ and g be a Hilbert modular form of half integral weight l' in $\mathbf{M}_{l'}(\mathfrak{N}p^r, \psi_2, \mathbb{C})$, we can now restate Proposition 3.4.3 in the following way

$$\left\langle \mathbf{f}^c, \mathrm{Tr}_{\mathrm{SL}_2}^{\mathrm{GL}_2}(v, \psi') \left(g\mathcal{E}\left(z, s; k - l' - \frac{1}{2}t, \chi_{-1}\psi_2\psi\psi'^{-2}, \mathfrak{N}\right)\right)\right\rangle_{\mathfrak{N}} =$$

$$\begin{aligned} =& 2^d D_F^{-3/2} (2\pi)^{-d(s-3/4) - \frac{k+l'}{2}} \times \\ & \times \Gamma_\infty \left(\left(s - \frac{3}{4} \right) t + \frac{k+l'}{2} \right) \mathcal{D}\left(s - \frac{1}{2}, \mathbf{f}_1, g \right). \end{aligned}$$

3.6 Arithmetic measures

In this section we recall the notion of an arithmetic measure and construct some of them, in the spirit of [Hid90, Wu01]. We will first construct a many-variable measure $\mathcal{E}_c^{\chi\chi_{-1}} * \Theta_\chi | [\mathfrak{l}^2]$ which will be used for the construction of the p-adic L-function in $[F : \mathbb{Q}] + 2 + \delta$ variable; then, we will construct a one variable measure $E_c^{\chi,+}$ will be used for the construction of the "improved" p-adic L-function (see Section 3.7).
An *arithmetic measure of half integral weight* is a measure from a p-adic space V on which $\mathfrak{r}_p^\times$ acts to the space of p-adic modular forms of half-integral weight which satisfies certain conditions (cfr. [Hid90, §4]).
More precisely, an arithmetic measure of half integral weight is a $\mathcal{O}$-linear map $\mu : \mathcal{C}(V, \mathcal{O}) \to \overline{\mathbf{M}}_{\mathrm{half}}(\mathfrak{N}, \mathcal{O})$ such that

A.1 There exist a non-negative integer κ such that for all $\phi \in \mathcal{LC}(V, \mathcal{O})$, there exists an integer r such that

$$\mu(\phi) \in \mathbf{M}_{(\kappa+\frac{1}{2})t}(\mathfrak{N}p^r, \overline{\mathcal{O}})$$

A.2 There is a finite-order character $\psi : \mathfrak{r}_p^\times \to \mathcal{O}^\times$ such that for the action $|$ of $\mathfrak{r}_p^\times$ defined in Section 3.3.4,

$$\mu(\phi)|b = b^{\kappa t+\frac{1}{2}t}\psi(b)\mu(\phi|b),$$

where $\phi|b(v) = \phi(b^{-1}v)$.

A.3 There is a continuous function $\nu : V \to \mathcal{O}$ such that

$$(\nu|b)(v) = b^{2t}\nu(v) \text{ and } \mathrm{d}^t(\mu(\phi)) = \mu(\nu\phi).$$

We say that such a measure is *supersingular* if $\iota_p\mu = \mu$ and *cuspidal* if μ has values in $\overline{\mathbf{S}}_{\mathrm{half}}(\mathfrak{N}, \mathcal{O})$. Under some hypotheses, such as Leopodt's conjecture (but even under weaker hypotheses, cfr. [Hid91, (8.2)]), it is possible to show that supersingular implies cuspidal as in [Hid90, Lemma 4.1].
We can define an arithmetic measure of half-integral weight for $\mathrm{Cl}_{\mathfrak{N}}(p^\infty)$ after obvious changes in the properties *A.1 - A.3*. Note that the action of $\mathfrak{r}_p^\times$ is trivial on the closure of global units.
Before giving some examples, we recall that we have the following theorem on the existence of p-adic L-function for Hecke character of a totally real field [DR80]

Theorem 3.6.1. *Let χ be a primitive character of finite-order of conductor $\mathfrak{c}$. For all $c \in \mathrm{Cl}_{\mathfrak{N}}(p^\infty)$, we have a measure $\zeta_{\chi,c}$ on $\mathrm{Cl}_{\mathfrak{N}}(p^\infty)$ such that*

$$\int_{\mathrm{Cl}_{\mathfrak{N}}(p^\infty)} \psi(z)\langle\mathcal{N}_p(z)\rangle^n \mathrm{d}\zeta_{\chi,c}(z) = (1-\chi\psi(c)\mathcal{N}(c)^{n+1})\prod_{\mathfrak{p}|p}(1-(\psi\chi)_0(\mathfrak{p})\mathcal{N}(\mathfrak{p})^n)L(-n,(\psi\chi)_0).$$

for all $n \geq 0$ and for all finite-order character ψ, where $(\chi\psi)_0$ denotes the primitive character associated with $\chi\psi$.

Here $\mathcal{N}_p$ is the p-adic cyclotomic character. As in the case of Kubota-Leopoldt p-adic L-functions, if χ is odd, then the above measure is 0 for all n. In the sequel, fix $c \in \mathrm{Cl}_{\mathfrak{N}}(p^\infty)$ such that $\langle\mathcal{N}_p(c)\rangle$ generates the free part of the p^∞-cyclotomic extension of F.
It is possible to interpolate the values of the imprimitive L-function. What we have to do is to remove the factor at prime ideals $\mathfrak{q}$, for $\mathfrak{q}$ prime to p and ranging in a fixed finite set of prime ideals. We multiply the measure $\zeta_{\chi,c}$ by the factor

$$(1-(\psi\chi)_0(\mathfrak{q})A_{\mathcal{N}_p(\mathfrak{q})}(X))$$

for $\mathfrak{q}$ in this fixed set. Here $A_z(X)$ denotes the formal power series $(1+X)^{\log_p(z)/\log_p(u)}$.
Let us explain how this theorem implies that the Eisenstein series of Theorem 3.3.5 can be p-adically interpolated, by interpolating their Fourier coefficients as p-adic analytic functions.
Fix an integral ideal $\mathfrak{M}$ of F prime to p and divisible by 4. Up to enlarging $\mathfrak{M}$, we can suppose that it is the square of a principal ideal. Call $\mathfrak{a}_p$ and $\mathfrak{b}_p$ the image of $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathcal{N}_p(\mathrm{Cl}_{\mathfrak{M}}(p^\infty))$.
The first *Eisenstein measure* (of level $\mathfrak{M}$) which we define is

$$\int_{\mathrm{Cl}_{\mathfrak{M}}(p^\infty)} \psi(z)\mathrm{d}E^{\chi}_{ss,c}(z) = \sum_{\substack{0 \ll \xi \in \mathfrak{r}, \\ (\xi,p)=\mathcal{O}}} q^{\xi/2} \sum_{\substack{\mathfrak{a}^2\mathfrak{b}^2|\xi \\ \mathfrak{a},\,\mathfrak{b} \text{ prime to } p\mathfrak{M}}} \mu(\mathfrak{a})\omega_\xi(\mathfrak{a})\mathcal{N}(\mathfrak{b})\int_{\mathrm{Cl}_{\mathfrak{M}}(p^\infty)} \psi|(\mathfrak{a}_p\mathfrak{b}_p^2)\mathrm{d}\zeta_{\chi\omega_\xi,c}$$

Here the action of z in $\mathrm{Cl}_{\mathfrak{M}}(p^\infty)$ on $\mathcal{C}(\mathrm{Cl}_{\mathfrak{M}}(p^\infty), \mathcal{O})$ is given by $\phi|z(v) = \phi(z^{-1}v)$.
We see that for a character ψ of $\mathrm{Cl}_{\mathfrak{M}}(p^\infty)$ of finite order of p-conductor p^α, $\alpha = (\alpha_{\mathfrak{p}})_{\mathfrak{p}|p}$, with $\alpha_{\mathfrak{p}} \geq 0$ for all prime ideal $\mathfrak{p}$, we have

$$\begin{aligned} E^{\chi}_{ss,c}(\psi(z)\mathcal{N}(z)^{m-1}) =& (1-\omega^{-n}\chi\psi(c)\mathcal{N}(c)^m)A_0^{-1} \times \\ & \times \mathcal{E}'\left(z, \frac{1-m}{2}; mt, \psi\chi, \mathfrak{M}^2 p^{2s}\right)|[\mathfrak{m}^2\varpi^{2\alpha}4^{-1}]|\iota_p \\ & \in \mathbf{M}_{mt+\frac{1}{2}t}(\mathfrak{M}^2p^{2\alpha}, \chi^{-1}\psi^{-1}), \end{aligned}$$

for $A_0 = i^{md}\pi^d 2^{(m-\frac{1}{2})d}\psi(\mathfrak{d}\mathfrak{n}2^{-1})D_F\mathcal{N}(\mathfrak{d}\mathfrak{n}2^{-1})^{m-2}$ of Proposition 3.3.10.
Here ϖ is a product of fixed uniformizers at $\mathfrak{p}_i$ for $\mathfrak{p}_i|p$. In general, we shall use the notation $p^\alpha = \prod_{i=1}^{e}\mathfrak{p}_i^{\alpha_i}$, with $\alpha = (\alpha_i)$ and $\mathfrak{p}_1, \dots, \mathfrak{p}_e$ the divisors of p in $\mathfrak{r}$.
We define then the measure $\mathcal{E}^{\chi}_c$ on $\mathbf{G} = \mathrm{Cl}_{\mathfrak{M}}(p^\infty) \times \mathfrak{r}_p^\times$; let ψ be a finite-order character of $\mathrm{Cl}_{\mathfrak{M}}(p^\infty)$ and ψ' a finite-order character of $\mathfrak{r}_p^\times$ which we supposed induced by a finite-order Hecke character of F which we denote by the same symbol. We define for the function $a^v\psi'(a)\psi(z)\mathcal{N}_p(z)^m$

$$\int_{\mathbf{G}} \psi(z)\mathcal{N}_p(z)^m a^v\psi'(a)\mathrm{d}\mathcal{E}^{\chi}_c(z,a) = \mathrm{d}^{-v}\left(\int_{\mathrm{Cl}_{\mathfrak{N}}(p^\infty)} \psi(z)\psi'^{-2}(z)\mathcal{N}_p(z)^m \mathrm{d}E^{\chi}_{ss,c}\right).$$

Note that such a functions are dense between the continuous function on $\mathbf{G}$. We have that $\mathcal{E}^{\chi}_c$ is an arithmetic measure of half-integral weight. In fact, it verifies *A.1* with $\kappa = 1$.
For *A.2*, we define an action of $b \in \mathfrak{r}_p^\times$ on $\mathcal{C}(\mathbf{G}, \mathcal{O})$ as $\phi_1(a)\phi_2(z)|b = \phi_1(ab^{-2})\phi_2(zb)$. To show that

$$\mathcal{E}^{\chi}_c(\phi_1, \phi_2)|b = \mathcal{E}^{\chi}_c((\phi_1, \phi_2)|b)$$

it is enough to check the formula on functions of type $a^v\psi'(a)\psi(z)$ with ψ and ψ' characters of finite-order. Notice that $(\mathrm{d}^\sigma f)|b = b^{2\sigma}\mathrm{d}^\sigma(f|b)$; we have then

$$\begin{aligned} \mathcal{E}^{\chi}_c(a^v\psi'(a)\psi(z))|b =& (\mathrm{d}^{-v}E^{\chi}_{ss,c}(\psi'^{-2}\psi(z)))|b \\ =& A'b^{-2v}\mathrm{d}^{-v}(\mathcal{E}'\left(z, 0; t, \psi\psi'^{-2}, \mathfrak{M}^2p^{2\alpha}\right)|[\varpi^{2\alpha}\mathfrak{m}^2 4^{-1}]|\iota_p|b) \\ =& A'b^{-2v}\psi^{-1}\psi'^2(b)\mathrm{d}^{-v}\mathcal{E}'\left(z, 0; t, \psi\psi'^{-2}, \mathfrak{M}^2p^{2\alpha}\right)|[\varpi^{2\alpha}\mathfrak{m}^2 4^{-1}]|\iota_p \\ =& \mathcal{E}^{\chi}_c((a^v\psi'(a)\psi(z))|b), \end{aligned}$$

for $A' = (1-\omega^{-n}\chi\psi(c)\mathcal{N}(c)^m)A_0^{-1}$.
Then *A.3* is verified by $(a,z) \mapsto a^{-t}$.

Furthermore, we define a third Eisenstein measure $E^{\chi,+}_c$ on $\mathrm{Cl}_{\mathfrak{N}}(p^\infty)$. Let χ be a Hecke character of finite order, we define $E^{\chi,+}_c$ as the restriction on the divisor $D := (Y + 1 - \langle\mathcal{N}_p(c)\rangle(X+1)^2)$ of the following measure μ

$$\int_{\mathrm{Cl}_{\mathfrak{N}}(p^\infty)^2} \phi_1(z_1)\phi_2(z_2)\mathrm{d}\mu = (1-\chi(c)\mathcal{N}_p(c)(1+X)) \int_{\mathrm{Cl}_{\mathfrak{N}}(p^\infty)} \phi_2 \mathrm{d}\zeta_{\chi^2,c} + \\ + (1-\chi^2(c)\mathcal{N}_p^2(c)(1+Y)) \sum_{\substack{\xi \gg 0,\\ \xi \in \mathfrak{r}}} q^{\xi/2} \times \\ \left(\sum_{\substack{\mathfrak{a}^2\mathfrak{b}^2 | \sigma \\ (\mathfrak{a},\mathfrak{b},p\mathfrak{c})=1}} \mu(\mathfrak{a})\omega_\xi(\mathfrak{a})\mathcal{N}(\mathfrak{b}) \int_{\mathrm{Cl}_{\mathfrak{N}}(p^\infty)} \phi_1 | \mathfrak{a}_p \mathfrak{b}_p^2 \mathrm{d}\zeta_{\chi\omega_\xi,c} \right).$$

We define now a *theta measure*; fix a Hecke character χ of level $\mathfrak{c}$, we pose

$$\begin{array}{rccc} \Theta_\chi : & \mathcal{C}(\mathrm{Cl}_{\mathfrak{c}}(p^\infty),\mathcal{O}) & \rightarrow & \overline{\mathbf{M}}_{\mathrm{half}}(4\mathfrak{c}^2,\mathcal{O}) \\ & \varepsilon & \mapsto & \sum_{\xi \gg 0} \chi\varepsilon(\xi)\mathcal{N}(\xi)^\alpha q^{\frac{\xi^2}{2}}, \end{array}$$

where $\alpha \in \{0,1\}$ and ε a character of finite order of conductor p^s, such that $\chi\varepsilon(-1) = (-1)^\alpha$. We have seen that $\Theta_\chi(\varepsilon) \in \mathbf{S}_{\frac{t}{2}+\alpha t}(4\mathfrak{c}^2p^{2s},\chi\varepsilon)$.
We define now the convolution of two measures of half-integral weight; let μ_1 and μ_2 be two measures, defined respectively on $\mathbf{G}$ and $\mathrm{Cl}_{\mathfrak{N}}(p^\infty)$, the *convolution* $\mu_1 * \mu_2$ is defined by

$$\int_{\mathrm{Cl}_{\mathfrak{N}}(p^\infty)} \left(\int_{\mathbf{G}} \Phi(z^{-1}z_1)(z,a)\mathrm{d}\mu_1 \right) \mathrm{d}\mu_2,$$

for $z_1 \in \mathrm{Cl}_{\mathfrak{N}}(p^\infty)$, $(z,a) \in \mathbf{G} = \mathrm{Cl}_{\mathfrak{N}}(p^\infty) \times \mathfrak{r}_p^\times$ and $\Phi : \mathrm{Cl}_{\mathfrak{N}}(p^\infty)^* \rightarrow \mathcal{C}(\mathbf{G},\mathcal{O})$ a continuous morphism. Here * denotes the $\mathcal{O}$-dual. If we let $\mathbf{G}$ acts on $\mathrm{Cl}_{\mathfrak{N}}(p^\infty) \times \mathbf{G}$ via

$$(z_0,a_0)(z_1,z,a) = (z_1z_0^{-1}, zz_0, aa_0^{-1}),$$

we see that this action is compatible with the action of $\mathfrak{r}_p^\times$ on $\mathcal{C}(\mathbf{G},\mathcal{O})$ defined above if we send

$$\mathfrak{r}_p^\times \ni b \mapsto (b^{-1},b^2) \in \mathrm{Cl}_{\mathfrak{N}}(p^\infty) \times \mathfrak{r}_p^\times.$$

Let χ be a character of conductor $\mathfrak{c}$ and $\mathfrak{N}$ a fixed ideal (which in what follows will be the level of our Hilbert modular form), and pose $\mathfrak{M} = lcm(4,\mathfrak{c}^2,\mathfrak{N}^2)$ Let us write $\mathfrak{L}^2 = \frac{\mathfrak{M}}{4\mathfrak{c}^2}$ and let $\mathfrak{l}$ be an idèle representing $\mathfrak{L}$. We take $\mu_1 = \mathcal{E}_c^{\chi\chi_{-1}}$ (of level $\mathfrak{M}$) and $\mu_2 = \Theta_\chi|[\mathfrak{l}^2]$. Recall that χ_{-1} is the character corresponding to the extension $F(i)$ defined in Section 3.3.1. We have then that $\mathcal{E}_c^{\chi\chi_{-1}} * \Theta_\chi|[\mathfrak{l}^2]$ is a measure which takes values in $\overline{\mathbf{M}}_k(\Gamma^1[\mathfrak{M}],\mathcal{O})$.

We compose now with $\mathrm{Tr}_{\mathrm{SL}_2}^{\mathrm{GL}_2}$ and we see that such an action is compatible with the action of $\mathbf{G}$ on $\overline{\mathbf{m}}(\mathfrak{M}(p^\infty),\mathcal{O})$, so what we have just constructed is a morphism of $\mathcal{O}[[\mathbf{G}]]$-modules.

3.7 Some p-adic L-functions

In this section we construct two p-adic L-functions for the symmetric square (Theorem 3.7.2); to do this, we first recall the definition of the *p-adic Petersson product* l_λ which is a key tool in the construction of p-adic L-function *à la Hida*.
Let us denote by δ Leopoldt's defect for F and p; the first p-adic L-function $\mathcal{L}_p(Q,P)$ has $[F:\mathbb{Q}]+2+2\delta$ and to construct it we use the method of Hida [Hid90] as generalized by Wu [Wu01]. We improve the result of Wu (which requires the strict class number of F to be equal to 1) and correct some minor errors.
We construct also a one variable p-adic L-function $\mathcal{L}_p^+(Q,P)$ which we call the "improved" p-adic L-function.

It is constructed similarly to [HT01]; instead of considering the convolution of measures, we multiply the measure $E_c^{\chi,+}$ of the previous section by a fixed theta series $\theta(\chi)$. By doing so, we lose the cyclotomic variable but in return we have that when χ is not of conductor divisible by p we do not have to consider theta series of level divisble by p. As a consequence, in the notation of Theorem 3.7.2, the factor $E_1(Q,P)$ (whose origin has been explained in 3.4.5) does not appear. These two p-adic L-functions are related by Corollary 3.7.3 which is the key for the proof of Theorem 3.1.3; indeed, $E_1(Q,P)$ is exactly the Euler factor which brings the trivial zero for $\mathbf{f}$ as in Theorem 3.1.3.
We recall that we defined $\Lambda = \mathcal{O}[[\mathbf{W}]]$, for $\mathbf{W}$ the free part of $\mathbf{G}$. Let $\mathbf{I}$ be a finite, integrally closed extension of Λ and let $\mathbf{F}$ be a family of nearly-ordinary forms which corresponds by duality to a morphism $\lambda : \mathbf{h}^{\mathrm{n.ord}}(\mathfrak{N},\mathcal{O}) \to \mathbf{I}$ as in Section 3.2.2. We suppose that λ is associated with a family of $\mathfrak{N}$-new forms. In the follow, we will denote by $\mathbf{h}^{\mathrm{ord}}$, resp. $\mathbf{I}^{\mathrm{ord}}$, $\mathbf{F}^{\mathrm{ord}}$ the ordinary part of $\mathbf{h}^{\mathrm{n.ord}}$, resp. $\mathbf{I}$, $\mathbf{F}$, i.e. the specialization at $v=0$.
Such a morphism λ induces two finite-order characters ψ, resp. ψ' of the torsion part of $\mathrm{Cl}_{\mathfrak{N}}(p^{\infty})$, resp. $\mathfrak{r}_p^{\times}$. We define, following [Hid91, §9], the congruence ideal of λ. By abuse of notation, we denote again by λ the following morphism

$$\lambda : \mathbf{h}^{\mathrm{n.ord}}(\mathfrak{N},\mathcal{O}) \otimes_{\Lambda} \mathbf{I} \to \mathbf{I}$$

which is the composition of multiplication $\mathbf{I}\otimes\mathbf{I}\to\mathbf{I}$ and $\lambda\otimes \mathrm{id}_{\mathbf{I}}$.
Let $\mathbf{K}$ be the field of fraction of $\mathbf{I}$; such a morphism λ induce a decomposition of Λ-algebra

$$\mathbf{h}^{\mathrm{n.ord}}(\mathfrak{N},\mathcal{O}) \otimes_{\Lambda} \mathbf{K} \cong \mathbf{K} \oplus \mathbf{B},$$

where the projection on $\mathbf{K}$ is induced by λ and $\mathbf{B}$ is a complement. Let us call 1_{λ} the idempotent corresponding to the projection onto $\mathbf{K}$ and proj the projection on $\mathbf{B}$. We define the congruence ideal

$$C(\lambda) :=(\mathbf{I} \oplus \mathrm{proj}(\mathbf{h}^{\mathrm{n.ord}}(\mathfrak{N},\mathcal{O}) \otimes_{\Lambda} \mathbf{I}))/(\mathbf{h}^{\mathrm{n.ord}}(\mathfrak{N},\mathcal{O}) \otimes_{\Lambda} \mathbf{I}).$$

We know that $C(\lambda)$ is a $\mathbf{I}$-module of torsion and we fix an element H of $\mathbf{I}$ such that $HC(\lambda)=0$.
Let us define $\overline{\mathbf{S}}(\mathfrak{N}p^{\infty},\psi,\psi',\mathcal{O})$ as the part of $\overline{\mathbf{S}}(\mathfrak{N}p^{\infty},\mathcal{O})$ on which the torsion of $\mathbf{G}$ acts via the finite-order characters ψ and ψ' defined as above. Write $\hat{\mathbf{I}} = \mathrm{Hom}_{\Lambda}(\mathbf{I},\Lambda)$; we have a $\mathcal{O}$-linear form

$$l_{\lambda} : \overline{\mathbf{S}}(\mathfrak{N}p^{\infty},\psi,\psi',\mathcal{O}) \otimes_{\Lambda} \hat{\mathbf{I}} \to \mathcal{O}.$$

Let us denote by $X(\mathbf{I})$ the subset of arithmetic points of $\mathrm{Spec}(\mathbf{I})$; let P be in $X(\mathbf{I})$ such that P resticted to Λ is of type $(m_P, v_P, \varepsilon_P, \varepsilon'_P)$. Let p^{α} be the smallest ideal divisible by the conductors of ε_P and ε'_P. The point P corresponds by duality to a form $\mathbf{f}_P$ of weight $(k_P = (m_P+2)t - 2v_P = \sum_{\sigma\in I} k_{P,\sigma}\sigma, w_P = t - v_P)$, level $\mathfrak{N}p^{\alpha}$ and character $\psi_P = \psi\omega^{-m}\varepsilon_P$, $\psi'_P(\zeta,w) = \varepsilon'_P(w)\psi'(\zeta)\zeta^{-v}$, where we have decomposed $\mathfrak{r}_p^{\times} = \mu \times \mathbf{W}'$ as in the end of section 3.3.4, and $a \in \mathfrak{r}_p^{\times}$ corresponds to (ζ, w). When there is no possibility of confusion, in the follow we will drop the subscript $_P$.
We have an explicit formula for $l_{\lambda_P} := P \circ l_{\lambda}$ given by [Hid91, Lemma 9.3]

$$l_{\lambda_P}(\mathbf{g}) = H(P)\frac{\langle \mathbf{f}_P^c | \tau'(\mathfrak{n}\varpi^{\alpha}), \mathbf{g}\rangle_{\mathfrak{N}p^{\alpha}}}{\langle \mathbf{f}_P^c | \tau'(\mathfrak{n}\varpi^{\alpha}), \mathbf{f}_P\rangle_{\mathfrak{N}p^{\alpha}}}$$

for all $\mathbf{g} \in \mathbf{S}_{k_P,w_P}(\mathfrak{N}p^{\alpha},\psi_P,\psi'_P,\overline{\mathbb{Q}})$. Here ϖ denotes the product over $\mathfrak{p} \mid p$ of some fixed uniformizers $\varpi_{\mathfrak{p}}$.

It is clear that we can extend this morphism in a unique way to $\overline{\mathbf{m}}_{k,w}(\mathfrak{N}p^{\infty},\mathcal{O})$, using the duality with $\mathbf{H}_{k,w}(\mathfrak{N}p^{\infty},\mathcal{O})$ [Hid91, Theorem 3.1] and then projecting onto $\mathbf{h}_{k,w}(\mathfrak{N}p^{\infty},\mathcal{O})$. We have

Lemma 3.7.1. *Let $\mathbf{f}_P$ and λ as above, let $\mathbf{g} \in \mathbf{M}_{k_P,w_P}(\mathfrak{N}p^{\beta},\psi_P,\psi'_P,\overline{\mathbb{Q}})$ with $\beta \geq \alpha$. Then*

$$l_{\lambda_P}(\mathbf{g}) = H(P)\lambda(T(\varpi^{\beta-\alpha}))^{-1}\frac{\langle \mathbf{f}_P^c | \tau(\mathfrak{n}\varpi^{\beta}), \mathbf{g}\rangle_{\beta}}{\langle \mathbf{f}_P^c | \tau(\mathfrak{n}\varpi^{\alpha}), \mathbf{f}_P\rangle_{\alpha}}$$

Proof. This is proven in [Mok09] on pages 29-30 in the case $v = 0$. If $\alpha = \beta$ it is clear. Otherwise, we write $\mathbf{g}' = \mathbf{g}|T_0(p^{\beta-\alpha})$ and proceed as in [Hid85, Proposition 4.5]

$$\begin{aligned}H(P)^{-1}\lambda(T_0(\varpi^{\beta-\alpha}))l_{\lambda_P}(\mathbf{g}) &= l_{\lambda_P}(\mathbf{g}') \\ &= \frac{\langle \mathbf{f}_P^c|\tau'(\mathfrak{n}\varpi^{\alpha}), \mathbf{g}'\rangle_{\alpha}}{\langle \mathbf{f}_P^c|\tau'(\mathfrak{n}\varpi^{\alpha}), \mathbf{f}_P\rangle_{\alpha}} \\ &= \frac{\left\langle \mathbf{f}_P^c|\tau'(\mathfrak{n}\varpi^{\alpha})|T_0^*(p^{\beta-\alpha}), \mathbf{g}\right\rangle_{\beta}}{\langle \mathbf{f}_P^c|\tau'(\mathfrak{n}\varpi^{\alpha}), \mathbf{f}_P\rangle_{\alpha}} \\ &= \left\{\varpi^{-v(\beta-\alpha)}\right\}\frac{\left\langle \mathbf{f}_P^c|\tau'(\mathfrak{n}\varpi^{\beta}), \mathbf{g}\right\rangle_{\beta}}{\langle \mathbf{f}_P^c|\tau'(\mathfrak{n}\varpi^{\alpha}), \mathbf{f}_P\rangle_{\alpha}}.\end{aligned}$$

□

We can now state the main theorem of the section

Theorem 3.7.2. *Fix an adelic character χ of level $\mathfrak{c}$, such that $\chi_\sigma(-1) = 1$ for all $\sigma|\infty$. We have two p-adic L-functions $\mathcal{L}_p(Q, P)$ in the total ring of fractions of $\mathcal{O}[[X]]\hat{\otimes}\mathbf{I}$ and $\mathcal{L}_p^+(P)$ in the fraction field of $\mathbf{I}^{ord}$ such that the following interpolation properties hold*

i) for (almost) all arithmetic points (Q, P) of type $(s_Q, \varepsilon_Q; m_P, \varepsilon_P, v_P, \varepsilon'_P)$, with $m_P - k_{P,0} + 2 \leq s_Q \leq m_P$ (for $k_{P,0}$ equal to the minimum of $k_{P,\sigma}$'s) and such that the p-part of the conductor of $\omega^s\varepsilon_Q^{-1}\chi^{-1}\psi_P{\psi'_P}^{-2}$ is p^α, α positive integer, the following interpolation formula holds

$$\mathcal{L}_p(Q, P) = C_1 E_1(Q, P)E_2(Q, P)\frac{2^d\mathcal{L}(s_Q + 1, \mathbf{f}_P, \varepsilon_Q^{-1}\omega^{s_Q}\chi^{-1})}{(2\pi)^{ds}\Omega(\mathbf{f}_P)},$$

where the Euler factor $E_1(Q, P)$ and $E_2(Q, P)$ are defined below.
Suppose now that χ is of conductor not divisible by all $\mathfrak{p}|p$. Suppose moreover that $\lambda|_{\mathrm{Cl}_{\mathfrak{N}}(p^\infty)} = \psi^{(p)}\omega^{m_0}$, with $\psi^{(p)}$ of conductor coprime to p, then we have a generalized p-adic L-function $\mathcal{L}_p(P)$ in the fraction field of $\mathbf{I}^{ord}$ satisfying the following interpolation property

ii) for (almost) all arithmetic points P of type (m_P, ε_P), with $m_P \geq 0$ and $m_P \equiv m_0 \bmod p - 1$, the following formula holds

$$\mathcal{L}_p(P) = C_1 E'_1(P)E_2(P)\frac{\mathcal{L}(1, \mathbf{f}_P, \chi_0)}{\Omega(\mathbf{f}_P)},$$

where the Euler factor $E'_1(P)$ and $E_2(P)$ are defined below.

The term $C_1 = C_1(Q, P)$ can be found in the proof of the theorem and it is a non-zero algebraic number; $\Omega(\mathbf{f}_P)$ is the Petersson norm of $\mathbf{f}_P^\circ$, the primitive form associated with $\mathbf{f}_p$, times $(2\pi)^{d-2v}$.
We want to point out that number of independent variables of $\mathcal{L}_p(Q, P)$ is $[F : \mathbb{Q}] + 1 + \delta$, as along the lines $s_Q - m_P = c$, for c a fixed integer, this function is constant.
In the statement of the theorem we have to exclude a finite number of points which are the zeros of an Iwasawa function interpolating an Euler factor at 2 for which the interpolation formula, a priori, does not hold. If 2 divides the conductor of $\mathbf{f}_P$ or χ, then this function is identically 1. If 2 does not divide neither the conductor of $\mathbf{f}_P$ or of χ, then a zero of this Iwasawa function appears for P such that $\mathbf{f}_P$ is not primitive at p and $Q = (m_P, \mathbf{1})$. In the Appendix we will weaken this assumption.
The power $(2\pi)^{ds}$ corresponds essentially to the periods of the Tate's motive; this corresponds, from the point of view of L-functions, to consider the values at $s + 1$ instead that in 1.

According to Deligne's conjecture [Del79, §7], we would expect as a period for the symmetric square, instead of $\Omega(\mathbf{f}_P)$, the product of the plus and minus period associated with $\mathbf{f}$ via the Eichler-Harder-Shimura isomorphism (see for example [Dim13a, Definition 2.9]); but it is well-known that the Petersson norm of $\mathbf{f}_P$ differs from Deligne's period by π^{d-2v} and a non-zero algebraic number (see [Yos95, Main Theorem]). The power π^{2v} gives (part of) the factor at infinity of the automorphic L-function of $\mathrm{Sym}^2(\mathbf{f}_P)$ (see Section 3.4.2).
The term $E_1(Q,P)$ and $E_2(Q,P)$ are the Euler factor at p which has to be removed to allow the p-adic interpolation of the special values as predicted by Coates and Perrin-Riou [CPR89].
The factor $E_2(Q,P)$ contains the Euler factor of the primitive L-function which are missing when $\mathbf{f}_P$ is not primitive; we define, if $\mathbf{f}_P$ is not primitive at $\mathfrak{p}_i$

$$\begin{aligned} E_{\mathfrak{p}_i}(Q,P) =& (1-(\chi^{-1}\varepsilon_Q^{-1}\omega^s\psi_P)_0(\mathfrak{p}_i)\mathcal{N}(\mathfrak{p}_i)^{m-s})\times \\ & \times (1-(\chi^{-1}\varepsilon_Q^{-1}\omega^s\psi_P^2)_0(\mathfrak{p}_i)\lambda(T(\varpi_{\mathfrak{p}_i}))^{-2}\mathcal{N}(\varpi_i)^{2m+1-s}) \end{aligned}$$

and if $\mathbf{f}_P$ is primitive at $\mathfrak{p}_i$ then $E_{\mathfrak{p}_i}(Q,P)=1$.
The factor $E_2(Q,P)$ is consequently

$$E_2(Q,P) = \prod_{\mathfrak{p}_i | p} E_{\mathfrak{p}_i}(Q,P).$$

The factor $E_1(Q,P)$ comes from the fact that we are using theta series of level divisible by p but whose conductor is not necessarily divisible by p. For two points Q and P as above, let us denote by p^β the minimum power of p such that the conductors of all the characters appearing in Q and P and the ideal $\varpi\mathfrak{r}$ divide p^β, and let us denote by p^{α_0} the p-part of the conductor of $\chi\varepsilon_Q\omega^{-s}$. We have

$$E_1(Q,P) = \lambda(T(\varpi^{2\beta-2\alpha_0}))\mathcal{N}(\varpi^{\beta-\alpha_0})^{-(s+1)}\prod_{\mathfrak{p}|p}(1-(\chi\varepsilon_Q\omega^{-s})_0(\mathfrak{p})\mathcal{N}(\mathfrak{p})^s\lambda(T(\varpi_{\mathfrak{p}}))^{-2}).$$

The factor E_1 is the Euler factor which gives the trivial zero in the case of Theorem 3.1.3 and that we remove in the second part of the above theorem. The factor $E_1'(P)$ which appears in that second part is $\lambda(T(\varpi^{2\alpha-2\alpha_0}))\mathcal{N}(\varpi^{\alpha-\alpha_0})^{-2}$.
We point out that the factor E_1 is in reality the second term of the factor E_2 when evaluated at $2m+3-s$ (so it an Euler factor of the dual motive of $\mathrm{Sym}^2(\mathbf{f})$).

This theorem deals with the imprimitive L-functions. We will show the analog of Theorem 3.7.2, i) in the Appendix 3.12 for the p-adic L-function interpolating the primitive one.

As an immediate consequences we have the following

Corollary 3.7.3 (Important Corollary). *Let Q_0 be the point $(0,\mathbf{1})$ of $\mathrm{Spec}(\mathcal{O}[[\mathrm{Cl}_{\mathfrak{N}}(p^\infty)]])$, then we have the following factorization in the fraction field of $\mathbf{I}^{\mathrm{ord}}$:*

$$\mathcal{L}_p(Q_0,P) = \prod_{\mathfrak{p}_i|p}(1-\lambda(T(\varpi_i))^{-2})\mathcal{L}_p^+(P). \tag{3.7.4}$$

Note that a similar formula could be proved for all fixed v.

Theorem 3.7.5. *Let $\mathbf{f}=\mathbf{f}_P$ be a nearly ordinary form of Nebentypus (ψ,ψ') and weight $k \geq 2t$ which we decompose as $k=m+2t-2v$. Let χ be a Hecke character of level $\mathfrak{c}$, such that $\chi_\sigma(-1)=1$ for all $\sigma|\infty$. Then we have a a formal series $G(X,\mathbf{f},\chi)$ in $\mathcal{O}((X))\left[\frac{1}{p}\right]$ such that for all finite-order character ε of $1+p\mathbb{Z}_p \cong u^{\mathbb{Z}_p}$,* *of conductor p^{α_0}, and $s \in [m-k_0+2, m]$ with $n \equiv s$ which are not a pole of $G(X)$ we have*

$$\begin{aligned}G(\varepsilon(u)u^s-1,\mathbf{f},\chi) =& i^{(s+1)d-k}2^{-d\frac{s+n}{2}}s!D_F^{s-\frac{3}{2}}\times \\ & G(\chi\varepsilon\omega_0^{-s})\eta^{-1}(\mathfrak{m}\varpi^{\alpha}\mathfrak{d}2^{-1})\chi\varepsilon\omega_0^{-s}(\mathfrak{d}\mathfrak{c}\varpi^{\alpha_0})\mathcal{N}(\mathfrak{m})^s\times \\ & \frac{\psi(\mathfrak{d}\mathfrak{m}\varpi^{\beta})\mathcal{N}(\mathfrak{l})^{-2[s/2]}\mathcal{N}(\varpi)^{(s+1)\beta-\alpha_0+\frac{(\alpha-\alpha')m}{2}}}{\lambda(T(\varpi^{2\beta-\alpha'}))W'(\mathbf{f})S(P)\prod_{\mathfrak{p}}\frac{\eta\nu(\mathfrak{d}_{\mathfrak{p}})}{|\eta\nu(\mathfrak{d}_{\mathfrak{p}})|}\prod_J G(\nu\psi')}\times \\ & E_1(s+1)E_2(s+1)\frac{2^d\mathcal{L}(s+1,\mathbf{f},\chi^{-1}\varepsilon^{-1}\omega^s)}{(2\pi)^{ds}\Omega(\mathbf{f})}.\end{aligned}$$

for $\eta = \omega^s\varepsilon^{-1}\chi_{-1}^{-1}\chi^{-1}\psi\psi'^{-2}$.

Here $\beta = \alpha_0$ if ε is not trivial, and $\beta = 1$ otherwise. The factors $E_1(s+1)$ and $E_2(s+1)$ are the factors $E_1(Q,P)$ and $E_2(Q,P)$ above, for P such that $\mathbf{f}_P = \mathbf{f}$ and Q of type (s,ε). The factor $S(P)$ is defined below. We explain the p-part of the fudge factor. Fix a prime $\mathfrak{p}$ dividing p and let $\pi_{\mathfrak{p}}$ be the local component of $\pi(\mathbf{f})$, as in Section 3.4.2. Then $\pi_{\mathfrak{p}} = \pi(\eta,\nu)$ or $\pi_{\mathfrak{p}} = \sigma(\eta,\nu)$, according to the fact that $\pi_{\mathfrak{p}}$ is a principal series or a special representation. In particular we have $\mathbf{f}|T(\varpi_{\mathfrak{p}}) = \eta(\varpi_{\mathfrak{p}})\mathbf{f}$.
Let $\psi'_{\mathfrak{p}}$ be the local component at $\mathfrak{p}$ of ψ'. We know that $\eta\psi'_{\mathfrak{p}}$ is unramified. Let us denote by $\varpi_{\mathfrak{p}}^{\alpha'_{\mathfrak{p}}}$ the conductor of $\nu\psi'_{\mathfrak{p}}$. Then $p^{\alpha'}$ is the p-part of the conductor of the representation $\pi(\mathbf{f})\otimes\psi'$.

The factor $S(P)$ is defined in [Hid91, page 355] as a product over $\mathfrak{p}|p$ and each factor depends only on the local representation at $\mathfrak{p}$; if $\pi_{\mathfrak{p}}$ is special the factor is -1, if $\pi_{\mathfrak{p}}$ is a ramified principal series, the factor is $\eta^{-1}\nu(\varpi_{\mathfrak{p}}^{\alpha'_{\mathfrak{p}}})|\varpi_{\mathfrak{p}}^{\alpha'_{\mathfrak{p}}}|_{\mathfrak{p}}^{-1}$ and finally if $\pi_{\mathfrak{p}}$ is unramified the factor is

$$(1-\eta^{-1}\nu(\varpi_{\mathfrak{p}})|\varpi_{\mathfrak{p}}|_{\mathfrak{p}}^{-1})(1-\eta^{-1}\nu(\varpi_{\mathfrak{p}})).$$

Moreover, let us denote by $\mathbf{f}^u$ the unitarization of $\mathbf{f}$, as defined in Section 3.2.1, and if we suppose $\mathbf{f}$ primitive of conductor $\mathfrak{M}p^{\alpha}$ then we know that $\mathbf{f}^u|\tau'(\mathfrak{m}\varpi^{\alpha}) = W(\mathbf{f})\mathbf{f}^{u,c}$, for an algebraic number $W(\mathbf{f})$ of complex absolute value equal to 1 (here c stands for complex conjugation). Moreover $W(\mathbf{f})$ can be written as a product of local $W_{\mathfrak{q}}(\mathbf{f})$, and we write it as $W(\mathbf{f}) = W'(\mathbf{f})\prod_{\mathfrak{p}|p}W_{\mathfrak{p}}(\mathbf{f})$. For the explicit expression of $W_{\mathfrak{p}}(\mathbf{f})$, we refer to [Hid91, §4]. From the formulae in *loc. cit.* we see that the factors $\frac{\eta\nu(\mathfrak{d}_{\mathfrak{p}})}{|\eta\nu(\mathfrak{d}_{\mathfrak{p}})|}$ and $G(\nu\psi')$ come from $W_{\mathfrak{p}}$. The remaining part of $W_{\mathfrak{p}}$ is incorporated in the Hecke eigenvalue $\lambda(T(\varpi^{\alpha'}))$.
We warn the reader that the proof of Theorem 3.7.2 is computational, and the reader is advised to skip it on a first reading. We recall in the following lemma some well-known results which we will use when evaluating the p-adic L-function; recall from Section 3.3.3 that we have defined H, the *holomorphic projector*, ∂_p^q, the Mass-Shimura differential operator, for $\sigma \in I$, d^{σ}, the holomorphic differential operator on $\mathcal{H}^I$, and c, the *constant term projection* of a nearly-holomorphic modular form.

Lemma 3.7.6. *Let $f \in \mathbf{S}_k(\Gamma^1[\mathfrak{N}])$, $g \in \mathbf{N}^{s_1}_{l+\frac{t}{2}}(\Gamma^1[\mathfrak{N}])$ and $h \in \mathbf{N}^{s_2}_{m+\frac{t}{2}}(\Gamma^1[\mathfrak{N},\mathfrak{a}])$ with $k,l,m \in \mathbb{Z}[I]$ and $l > 2s_1$, $m > 2s_2$ (recall that these two condition are automatic if $F \neq \mathbb{Q}$). Let $r \in \mathbb{Z}[I]$.
Then we have*

$$\begin{aligned}\langle f, H(g)\rangle =& \langle f, g\rangle, \\ H(g\partial^r_{m+\frac{t}{2}}h) =& (-1)^r H(\partial^r_{l+\frac{t}{2}}gh), \\ eH(g\partial^r_{m+\frac{t}{2}}h) =& e(g\mathrm{d}^r h).\end{aligned}$$

Proof. The first formula is [Shi78, Lemma 4.11].
The second formula can be proven exactly as [Hid91, Proposition 7.2].
The last one can be shown as in [Hid91, Proposition 7.3], writing $\partial_{m+\frac{t}{2}}$ in term of d^j for $0 \leq j \leq r$ (in $\mathbb{Z}[I]$) and noticing that $e(\mathrm{d}^{\sigma}f) = 0$ for all σ in I. □

Proof of Theorem 3.7.2 i). We pose $\mathfrak{M} = lcm(4, \mathfrak{c}^2, \mathfrak{N}^2) = 4\mathfrak{c}^2\mathfrak{L}^2$ as in the end of Section 3.6. Fix two idèles $\mathfrak{m}$ and $\mathfrak{l}$ such that $\mathfrak{m}^2$ represents $\mathfrak{M}$ and $\mathfrak{l}$ represents $\mathfrak{L}$. We suppose moreover that $\mathfrak{m}\mathfrak{l}^{-1}$ represents $\mathfrak{c}$. To lighten the following formulae, let us pose

$$\mathrm{Tr}_1 = T_{\mathfrak{M}/\mathfrak{N}} \ , \mathrm{Tr}_2 = Tr^{\mathrm{GL}_2}_{\mathrm{SL}_2}.$$

For each $\mathfrak{q}$ dividing 2 in $\mathfrak{r}$, we pose

$$\begin{aligned}\mathcal{E}_{\mathfrak{q}}(X,Y) =&(1-\psi^2\chi^{-2}(\mathfrak{q})A^{-2}_{\mathcal{N}(\mathfrak{q})}(X)A^{2}_{\mathcal{N}(\mathfrak{q})}(Y)),\\ \mathcal{E}_2(X,Y) =&\prod_{\mathfrak{q}|2}\mathcal{E}_{\mathfrak{q}}(X,Y),\end{aligned}$$

where X is a variable on the free part of $\mathbb{Z}_p^\times$ in $\mathrm{Cl}_{\mathfrak{N}}(p^\infty)$ (it corresponds to the variable Q) and Y on the free part of $\mathbb{Z}_p^\times$ embedded in the fist component of $\mathbf{G} = \mathrm{Cl}_{\mathfrak{M}}(p^\infty) \times \mathfrak{r}_p^\times$ (it corresponds to the variable m of P).
As in the end of Section 3.3.4, we have $A_z(X) = (1+X)^{\log_p(z)/\log_p(u)}$, for u a fixed topological generator of $1+p\mathbb{Z}_p$.
We pose moreover $\Delta(X,Y) = \left(1-\psi'^{-2}\chi_{-1}\chi^{-1}\psi(c)\mathcal{N}(c)\frac{1+Y}{1+X}\right)$ where, as in Section 3.6, c is chosen such that $\langle\mathcal{N}(c)\rangle$ correspond to the generator u fixed above.
We define a first p-adic L-function

$$\mathcal{L}_p = (\Delta(X,Y)\mathcal{E}_2(X,Y)H)^{-1}l_\lambda e\mathrm{Tr}_1(((\mathrm{Tr}_2(\mathcal{E}_c^{\chi\chi_{-1}} * \Theta_\chi|[\mathfrak{l}^2])|))\Xi_2)$$

where $\Xi_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}_f$.
The operator Ξ_2 is necessary as the family $\mathbf{F}$ is of level $U(\mathfrak{N})$ while our convolution of measures is of level $U(2^{-1}\mathfrak{M}, 2)$.

This p-adic L-function is an element of the total fraction field of $\mathcal{O}[[\mathrm{Cl}_{\mathfrak{N}}(p^\infty)]]\hat{\otimes}\mathbf{I}$. In fact, $\mathbf{I}$ is a reflexive Λ-module as it is integrally closed and finite over Λ. We want to evaluate $\mathcal{L}_p$ on an arithmetic point $(Q,P) \in X(\mathrm{Cl}_{\mathfrak{N}}(p^\infty)) \times X(\mathbf{I})$ of type $(s, \varepsilon_Q; m, \varepsilon_P, v, \varepsilon'_P)$ with $s \leq m$. We let ψ_P and ψ'_P as in the beginning of this section. Let $n \in \{0,1\}$, $n \equiv s \bmod 2$, we evaluate

$$\begin{aligned}g =&(\Delta(X,Y)^{-1}\mathcal{E}^{\chi\chi_{-1}} * \Theta_\chi|[\mathfrak{l}^2])(Q,P)\\ =&\Delta(Q,P)^{-1}\left(\left(\left(\int_{\mathrm{Cl}_{\mathfrak{N}}(p^\infty)}\varepsilon_Q\omega^{-s}(z_1)\mathcal{N}_p(z_1)^s\mathrm{d}\Theta_\chi(z_1)\right)|\,[\mathfrak{l}^2]\times\right.\right.\\ &\left.\times\int_{\mathbf{G}}\varepsilon_Q^{-1}\omega^s\psi_P(z)\langle\mathcal{N}_p(z)\rangle^{m-s}\psi'_P(a)a^v\mathrm{d}\mathcal{E}_c^{\chi\chi_{-1}}(z,a)\right)\\ =&A_0^{-1}\mathcal{N}(\mathfrak{l})^{-2[s/2]}\left(\mathrm{d}^{[s/2]t}(\theta_n(\chi\varepsilon_Q\omega^{-s})|\,[\mathfrak{l}^2])\times\right.\\ &\left.\times\mathrm{d}^{-v}\left(\mathcal{E}'\left(z,\frac{s-m}{2};s_0t,\eta\right)|[\mathfrak{m}^2\varpi^{2\beta}4^{-1}]\right)\right)\end{aligned}$$

for $s_0 = m-s+1$, $\eta = \omega^s\varepsilon_Q^{-1}\chi_{-1}^{-1}\chi^{-1}\psi_P{\psi'_P}^{-2}$, β such that the conductors of ε_Q, ε_P and ε'_P and the level of $\mathbf{f}_P$ divide p^β and, we recall,

$$A_0 = \eta(\mathfrak{m}\varpi^\alpha\mathfrak{d}2^{-1})D_F\mathcal{N}(\mathfrak{m}\varpi^\beta\mathfrak{d}2^{-1})^{m-s-1}i^{s_0 d}\pi^d 2^{d\left(s_0-\frac{1}{2}\right)}.$$

For an integral idèle y we use, by abuse of notation, the expression $\mathcal{N}(y)$ to denote the norm of the corresponding ideal $y\mathfrak{r}$.

Here we have used the relation $\mathrm{d}^t[\mathfrak{l}^2] = \mathcal{N}(\mathfrak{l})^2[\mathfrak{l}^2]\mathrm{d}^t$. We use Lemma 3.7.6, formula (3.3.6), the property that e commutes with the operator $\mathcal{I}$ and Tr_2, as seen at the end of Section 3.3.4, and the formula (3.3.7) applied two times to obtain

$$\begin{aligned}\mathcal{L}_p(Q,P) =& \frac{\left\langle \mathbf{f}_P^c|\tau'(\mathfrak{n}\varpi^{2\alpha}), e\mathrm{Tr}_1(\mathrm{Tr}_2(g)|\Xi_2)\right\rangle_{\mathfrak{N}p^{2\alpha}}}{\langle \mathbf{f}_P^c|\tau'(\mathfrak{n}\varpi^{\alpha}), \mathbf{f}_P\rangle_{\mathfrak{N}p^{\alpha}}},\\ =& C\frac{\left\langle \mathbf{f}_P^c|\tau(\mathfrak{m}^2\varpi^{2\beta}), \mathrm{Tr}_2(g)|\Xi_2\right\rangle_{\mathfrak{M}p^{2\beta}}}{\langle \mathbf{f}_P^c|\tau'(\mathfrak{n}\varpi^{\alpha}), \mathbf{f}_P\rangle_{\mathfrak{N}p^{\alpha}}},\\ =& C\frac{\left\langle \mathbf{f}_{P,1}^c|\Xi_2^{-1}, h|\tau(\mathfrak{m}^2\varpi^{2\beta})\right\rangle_{\mathfrak{M}p^{2\beta}}}{\langle \mathbf{f}_P^c|\tau'(\mathfrak{n}\varpi^{\alpha}), \mathbf{f}_P\rangle_{\mathfrak{N}p^{\alpha}}},\end{aligned}$$

for

$$\begin{aligned} h =& \theta_n(\chi\varepsilon\omega^{-s})|\left[\mathfrak{l}^2\right] \times \mathcal{E}\left(z, \frac{s-m}{2}; k-(n+1)t, \eta\right)|\tau(\mathfrak{m}\varpi^{2\beta}),\\ C =& \psi_P(\mathfrak{m}\varpi^{\beta}\mathfrak{d})\Gamma_\infty(([s/2]+1)t-v)2^{2v-sd+nd-ds_0+\frac{1}{2}d}\pi^{v-[s/2]d-d}i^{(s+1)d-k}\times\\ & \eta^{-1}(\mathfrak{m}\varpi^{\alpha}\mathfrak{d}2^{-1})\mathcal{N}(\mathfrak{l})^{-2[s/2]}\mathcal{N}(\mathfrak{m}\varpi^{\beta}2^{-1})^{1/2}\mathcal{N}(\mathfrak{m}\varpi^{\beta}\mathfrak{d}2^{-1})^{s}\lambda(T(\varpi^{\alpha-2\beta}))\end{aligned}$$

where we used the formula for the change of variable for the Petersson product and the relation

$$\Xi_2\tau'(\mathfrak{m}^2)\Xi_2^{-1} = \begin{pmatrix} 0 & -2 \\ 2^{-1}\mathfrak{d}^2\mathfrak{m}^2 & 0 \end{pmatrix}_f.$$

We apply the duplication formula of the Γ function to the factors $\Gamma_\infty\left(\left(\left[\frac{s}{2}\right]+n+\frac{1}{2}\right)t-v\right)$ coming from Propostion 3.4.3 and to the Γ factor appearing in the constant C above, to obtain, for $z=\left[\frac{s}{2}\right]-v_\sigma+\frac{n+1}{2}$, that

$$\Gamma(z)\Gamma(z+1/2) = 2^{2v_\sigma - s}(s-2v_\sigma)!\sqrt{\pi}.$$

We are left to evaluate $\theta_n(\chi\varepsilon\omega^{-s})|\left[\mathfrak{l}^2\right]|\tau(\mathfrak{m}^2\varpi^{2\beta})$. Let us denote by χ' the primitive character associated with $\chi\varepsilon\omega^{-s}$ and let us denote by p^{α_0} the p-part of the conductor of χ'. Recall the relations given at the end on Section 3.5.1, Proposition 3.3.4 and the formula

$$\theta_n(\chi\varepsilon\omega^{-s}) = \sum_{\mathfrak{e}\mathfrak{r}|p}\mu(\mathfrak{e})\chi'(\mathfrak{e})\theta_n(\chi')|[\mathfrak{e}^2].$$

We have then

$$\begin{aligned}\theta_n(\chi')|\left[\mathfrak{l}^2\right]|\tau(\mathfrak{m}^2\varpi^{2\beta}) =& G(\chi')\mathcal{N}(2^{-1}\mathfrak{m}\varpi^{\alpha_0})^{-1/2}\chi'(\mathfrak{d}\mathfrak{c}\varpi^{\alpha_0})\mathcal{N}(\varpi^{\beta-\alpha_0})^{1/2}\times\\ & \times\sum\mu(\mathfrak{e})\chi'(\mathfrak{e})\mathcal{N}(\mathfrak{e})^{-1}\theta_n({\chi'}^{-1})|\left[\frac{\varpi^{2\beta}}{\varpi^{2\alpha_0}\mathfrak{e}^2}\right].\end{aligned}$$

We know from [Hid91, Lemma 5.3 (vi)] an explicit expression for $\langle \mathbf{f}_P^c|\tau'(\mathfrak{n}\varpi^{\alpha}), \mathbf{f}_P\rangle_{\mathfrak{N}\varpi^{\alpha}}$ in terms of the Petersson norm of the primitive form $\mathbf{f}_P^\circ$ associated with $\mathbf{f}_P$ (and ψ'_P).
We refer to the discussion after Theorem 3.7.5 for the notation, we have then

$$\begin{aligned}\frac{\langle \mathbf{f}_P^c | \tau'(\mathfrak{n}\varpi^\alpha), \mathbf{f}_P\rangle_{\mathfrak{N}\varpi^\alpha}}{\langle \mathbf{f}_P^\circ, \mathbf{f}_P^\circ\rangle_{\mathfrak{N}\varpi^{\alpha'}}} =& \mathcal{N}(\varpi)^{-(\alpha-\alpha')m/2}\lambda(T(\varpi^{\alpha-\alpha'}))\psi_\infty(-1)\times \\ & \times W'(\mathbf{f}_P)S(P)\prod_{\mathfrak{p}} \frac{\eta\nu(\mathfrak{d}_\mathfrak{p})}{|\eta\nu(\mathfrak{d}_\mathfrak{p})|}\prod_J G(\nu\psi'),\end{aligned}$$

where J is the set of $\mathfrak{p}|p$ such that $\pi_\mathfrak{p}$ is a ramified principal series.
We can conclude that

$$\begin{aligned}\mathcal{L}_p(Q,P) =& i^{(s+1)d-k}2^{1+2v-(s+m)d+nd-d\frac{s+n}{2}}\pi^{2v-d(s+1)}(st-2v)!D_F^{s-3/2}\times \\ & G(\chi\varepsilon\omega_0^{-s})\eta^{-1}(\mathfrak{m}\varpi^\alpha\mathfrak{d}2^{-1})\chi\varepsilon\omega_0^{-s}(\mathfrak{d}\mathfrak{c}\varpi^{\alpha_0})\psi_P(\mathfrak{d}\mathfrak{m}\varpi^\beta)\times \\ & \frac{\mathcal{N}(\mathfrak{l})^{-2[s/2]}\mathcal{N}(\mathfrak{m})^s\mathcal{N}(\varpi)^{(s+1)\beta-\alpha_0+\frac{(\alpha-\alpha')m}{2}}}{\lambda(T(\varpi^{2\beta-\alpha'}))W'(\mathbf{f}_P)S(P)\prod_\mathfrak{p}\frac{\eta\nu(\mathfrak{d}_\mathfrak{p})}{|\eta\nu(\mathfrak{d}_\mathfrak{p})|}\prod_J G(\nu\psi')\psi_\infty(-1)}\times \\ & E_1(Q,P)E_2(Q,P)\frac{\mathcal{L}(s+1,\mathbf{f},\chi^{-1}\varepsilon_Q^{-1}\omega^s)}{2^{d-2v}\langle \mathbf{f}_P^\circ, \mathbf{f}_P^\circ\rangle_{\mathfrak{N}\varpi^{\alpha'}}}.\end{aligned}$$

The factors $E_1(Q,P)$ and $E_2(Q,P)$ are the ones defined after 3.7.2. To obtain the explicit expression of $E_1(Q,P)$ we have used the formula (3.4.5). □

Proof of ii). We define the improved p-adic L-function on the ordinary Hecke algebra; let $\mathbf{I}^{\mathrm{ord}}$ be the ordinary part of $\mathbf{I}$, i.e. the fiber above $v=0$. It is a finite flat extension of $\mathcal{O}[[\mathbf{W}_0]]$, where $\mathbf{W}_0$ is the free part of $\mathrm{Cl}_\mathfrak{N}(p^\infty)$ corresponding, via class field theory, to the cyclotomic extension.
We pose $\mathfrak{M} = lcm(4,\mathfrak{c}^2,\mathfrak{N}^2) = 4\mathfrak{c}^2\mathfrak{L}^2$ as above. We fix the same two idèles $\mathfrak{m}$ such that $\mathfrak{m}^2$ represents $\mathfrak{M}$ and $\mathfrak{l}$ which represents $\mathfrak{L}$ as before. We pose again

$$\mathrm{Tr}_1 = T_{\mathfrak{M}/\mathfrak{N}}\ , \mathrm{Tr}_2 = Tr_{\mathrm{SL}_2}^{\mathrm{GL}_2}$$

For each $\mathfrak{q}$ dividing 2 in $\mathfrak{r}$, we pose

$$\begin{aligned}\mathcal{E}_\mathfrak{q}(Y) =& (1-\psi^2\chi_0^{-2}(\mathfrak{q})A^2_{\mathcal{N}(\mathfrak{q})}(Y)) \\ \mathcal{E}_2(Y) =& \prod_{\mathfrak{q}|2} E_\mathfrak{q}(Y),\end{aligned}$$

where Y is a coordinate of $\mathcal{O}[[\mathbf{W}_0]]$ (it corresponds to the variable m of the weight).
As before we set

$$\begin{aligned}\Delta(Y) =& 1-\chi_{-1}\chi^{-1}\psi(c)\mathcal{N}(c)(1+Y), \\ \Delta'(Y) =& 1-\chi(c)^{-2}\psi^2\mathcal{N}(c)^2\langle\mathcal{N}(c)\rangle(1+Y)^2.\end{aligned}$$

We define the improved p-adic L-function

$$\mathcal{L}_p^+ = (\Delta(Y)\Delta'(Y)\mathcal{E}_2(Y)H)^{-1}l_\lambda e\mathrm{Tr}_1\left(\mathrm{Tr}_2\left(\theta(\chi_0)|\left[\mathfrak{l}^2\right]E^+\right)\right)|\Xi_2 \in \mathbf{I}^{\mathrm{ord}}.$$

It can be seen that it is an element of $\mathbf{I}^{\mathrm{ord}}$ as in [Mok09, §6.2] from the duality of [Hid91, Theorem 3.1].
Let P be an arithmetic point of type (ε_P, m), such that ε_P factors through the cyclotomic character. Let $k=m+2$, we have

$$l_{\lambda_P}e\mathrm{Tr}_1(\mathcal{I}\circ\mathrm{Tr}_2\theta(\chi_0)|\left[\mathfrak{l}^2\right]E^+)|\Xi_2 = \frac{\langle \mathbf{f}_P^c|\tau'(\mathfrak{n}\varpi^\alpha), e\mathrm{Tr}_1(\mathrm{Tr}_2 g|\Xi_2)\rangle}{\langle \mathbf{f}_P^c|\tau(\mathfrak{N}\omega^\alpha), \mathbf{f}_P\rangle},$$

for

$$g = \theta(\chi_0)| \left[\mathfrak{l}^2\right] \mathcal{N}(\mathfrak{m}\varpi^\beta 2^{-1})^{-1/2} A_0^{-1} \mathcal{E}\left(z, \frac{2-k}{2}, (k-1)t, \psi_P \chi_1^{-1} \chi_0^{-1}\right) |\tau(\mathfrak{m}\varpi^{2\alpha}).$$

With calculations analogous to the previous one we obtain

$$\mathcal{L}_p^+ = \frac{\chi_{-1}\chi_0^{-1}(\mathfrak{d}\varpi^\alpha \mathfrak{m} 2^{-1})\chi_0(\mathfrak{d}\mathfrak{c}\varpi^{\alpha_0}) D_F^{-\frac{3}{2}} 2^{1-k+2d} \mathcal{N}(\varpi)^{\frac{\alpha-\alpha'}{2} m}}{\lambda(T(\varpi^{2\alpha_0 - \alpha'}))W'(\mathbf{f}_P) S(P) \prod_{\mathfrak{p}} \frac{\eta\nu(\mathfrak{d}_\mathfrak{p})}{|\eta\nu(\mathfrak{d}_\mathfrak{p})|} \prod_J G(\nu\psi')} \times$$
$$\times\, i^{d-k} G(\chi_0)\psi_P^{-1}(2) \frac{\mathcal{L}(1, \mathbf{f}, \chi_0^{-1})}{(2\pi)^d \langle \mathbf{f}_P^\circ, \mathbf{f}_P^\circ \rangle_{\mathfrak{N}\varpi^{\alpha'}}}.$$

□

Before the end of the section, we give a proposition about the behavior of $\mathcal{L}_p(Q,P)$ along the element

$$\Delta(X,Y) = \left(1 - \chi_{-1}^{-1}\chi^{-1}\psi'^{-2}\psi(c)\mathcal{N}(c)\frac{1+Y}{1+X}\right).$$

Proposition 3.7.7. *Suppose that $\omega\chi^{-1}\psi\psi'^{-2}$ is quadratic imaginary, that its conductor is prime to p and that the family associated with λ has CM by this character. Then $\mathcal{L}_p(Q,P)$ has a pole along $\Delta(X,Y)=0$ of the same order as the p-adic zeta function. Otherwise $\mathcal{L}_p(Q,P)$ is holomorphic along $\Delta(X,Y)=0$.*

Proof. We proceed as in [Hid90, Proposition 5.2]. First of all, notice that $\Delta(X,Y)$ has a simple zero along $s = m+1$, $\varepsilon_Q = \varepsilon_P = \varepsilon'_P = \mathbf{1}$ if and only if the p-part of $\omega\chi^{-1}\psi\psi'^{-2}$ is trivial.
The ξ-th Fourier coefficient of the Eisenstein series is a multiple of $\int_{\mathrm{Cl}_{\mathfrak{N}}(p^\infty)} \langle \mathcal{N}_p(z)\rangle^{-1} \mathrm{d}\zeta_\eta$, where $\eta = \omega\chi^{-1}\psi\psi'^{-2}\omega_{-\xi}$ and $\omega_{-\xi}$ is the primitive quadratic character associated with $F(\sqrt{-\xi})$. So if $\psi\psi'^{-2}\chi^{-1}$ is not imaginary quadratic, then

$$\int_{\mathrm{Cl}_{\mathfrak{N}}(p^\infty)} \langle \mathcal{N}_p(z)\rangle^{-1} \mathrm{d}\zeta_\eta = 0$$

for all ξ. In fact, we can take $s_0 \equiv 0 \bmod p^n$, s_0 odd, and we have then

$$\int_{\mathrm{Cl}_{\mathfrak{N}}(p^\infty)} \langle \mathcal{N}_p(z)\rangle^{s_0-1} \mathrm{d}\zeta_\eta = *L(1-s_0, \eta).$$

which is 0 as η is even. We know then that the Eisenstein series we obtain is 0.
If $\psi\psi'^{-2}\chi^{-1} = \omega_{-\xi_0}$, then the value of this integral is a non zero multiple (because the missing Euler factors at primes dividing $\mathfrak{N}$ but not the conductor of $\omega_{-\xi_0}$ do not vanish) of the p-adic regulator of the global units of F [Col88].
Let

$$g = (\mathrm{d}^{[s/2]t}\theta(\chi\omega^{-s}))|[\mathfrak{l}^2] \times \mathrm{d}^{-v}\left(\mathcal{E}'(z, \frac{1}{2}; s_0 t, \eta))|[\mathfrak{m}^2\varpi^{2\beta}4^{-1}]\right);$$

we have to show that if $\mathbf{f}_P$ has not CM by a $\omega_{-\xi_0}$ then g is killed by the Petersson product with $\mathbf{f}_P$.
A necessary condition for $\lambda(\xi, \mathfrak{r}; g, \psi_P\psi_P^{-2})$ to be different from 0 is $\xi = \xi_2^2 + \xi_1^2\xi_0$, with ξ_1 and ξ_2 in $\mathfrak{r}$. Then ξ is a norm from $F(\sqrt{-\xi_0})$ to F. Take ξ such that $\xi\mathfrak{r}$ is a prime ideal of F which remains prime in $F(\sqrt{-\xi_0})$ (i.e. it is not a norm) and such that $\lambda_P(T(\xi\mathfrak{r})) \neq 0$. So $T(\xi\mathfrak{r})$ acts as a non-zero scalar on $\mathbf{f}_P$ and as 0 on g, and g must be orthogonal to $\mathbf{f}_P$. □

If λ has CM by $\omega_{-\xi_0}$, then Leopoldt's conjecture for p and F is equivalent to the fact that $\mathcal{L}_p(Q,P)$ has a simple pole along $\Delta(X,Y)=0$.

3.8 A formula for the derivative

We can apply now the classic method of Greenberg and Stevens [GS93] to the formula (3.7.4) in order to prove Theorem 3.1.3 which has been stated in the introduction and which we recall now.

Theorem 3.8.1. *Let $p \geq 3$ be a prime such that there is only one prime ideal of F above p and let $\mathbf{f}$ be a Hilbert cuspidal eigenform of parallel weight 2, trivial Nebentypus and conductor $\mathfrak{N}p$. Suppose that $\mathfrak{N}$ is squarefree and divisible by all the primes of F above 2; suppose moreover that $\pi(\mathbf{f})_p$ is the Steinberg representation. Then the formula for the derivative in Conjecture 3.1.2 is true (i.e. when $g = 1$).*

We fix a Hilbert modular form $\mathbf{f}$ of parallel weight 2 and of Nebentypus ψ trivial at p.
Let $\mathbf{I}^{\mathrm{ord}}$ be the integral closure of an irreducible component of $\mathbf{h}^{\mathrm{ord}}$ and let $P_{\mathbf{f}}$ be an arithmetic point of $X(\mathbf{I}^{\mathrm{ord}})$ corresponding to $\mathbf{f}$. Let us denote by λ the corresponding structural morphism from $\mathbf{h}^{\mathrm{ord}}$ to $\mathbf{I}^{\mathrm{ord}}$. Let $L_p^{\mathrm{ord}}(Q, P)$ be the p-adic L functions obtained by specializing the p-adic L-function of Theorem 3.7.2 to $\mathcal{O}[[\mathrm{Cl}_{\mathfrak{N}}(p^{\infty})]]\hat{\otimes}\mathbf{I}^{\mathrm{ord}}$. In particular, we have that, up to some non-zero factor, $L_p^{\mathrm{ord}}(s, P_{\mathbf{f}})$ coincides with $L_p(s, \mathrm{Sym}^2(\mathbf{f}))$, the p-adic L function associated with $\mathbf{f}$.
We remark that if $\mathbf{f}$ is primitive, then the first statement of Theorem 3.2.6 tell us that $\mathbf{I}$ is étale at $P_{\mathbf{f}}$ over Λ.This allows us to define a $\mathbf{I}^{\mathrm{ord}}$-algebra structure φ (which depends on the point $P_{\mathbf{f}}$) on the field of meromorphic functions around 2 of fixed, positive radius of convergence $\mathrm{Mer}(D_2)$, which is a subfield of $\overline{\mathbb{Q}}_p((k-2))$.
Namely we consider the continuous group homomorphism

$$\begin{array}{rccc} \varphi_0 : & \mathrm{Cl}_{\mathfrak{N}}(p^{\infty}) & \rightarrow & \mathrm{Mer}(D(2))^{\times} \\ & \mathfrak{a} & \mapsto & (k \mapsto \psi(\mathcal{N}(\mathfrak{a}))\langle \mathcal{N}(\mathfrak{a})\rangle^{k-2}). \end{array}$$

This morphism extends to a continuous algebra homomorphism

$$\varphi_0 : \quad \mathcal{O}[[\mathrm{Cl}_{\mathfrak{N}}(p^{\infty})]] \quad \rightarrow \quad \mathrm{Mer}(D(2)).$$

Let us denote by $\mathbf{I}^{\mathrm{ord}}_{P_{\mathbf{f}}}$ the localization-completion of $\mathbf{I}^{\mathrm{ord}}$ at $P_{\mathbf{f}}$.
As $\mathbf{I}^{\mathrm{ord}}$ is étale around $P_{\mathbf{f}}$ over $\mathcal{O}[[X]]$ and because the ring of meromorphic functions of fixed, positive radius of convergence is henselian [Nag62, Theorem 45.5] we have that it exist a morphism φ such that

$$\begin{tikzcd} \mathbf{I}^{\mathrm{ord}}_{P_{\mathbf{f}}} \arrow[r, "\varphi"] & \mathrm{Mer}(D(2)) \\ \mathcal{O}[[\mathrm{Cl}_{\mathfrak{N}}(p^{\infty})]]. \arrow[u] \arrow[ur, "\varphi_0"'] & \end{tikzcd}$$

As $\mathbf{I}^{\mathrm{ord}}$ is generated by a finite number of elements $\lambda(T(\mathfrak{q}))$, we deduce that there exists a disc of positive radius around 2 such that $\varphi(i)$ is convergent for all i in $\mathbf{I}^{\mathrm{ord}}$.
For each k in this disc, we can define a new point P_k of $\mathbf{I}^{\mathrm{ord}}$; $P_k(i) = \varphi(i)(k)$. It is obvious that $P_2 = P_{\mathbf{f}}$. So, it makes sense now to derive elements of $\mathbf{I}^{\mathrm{ord}}$ with respect to k.
For all arithmetic points P of $\mathbf{I}^{\mathrm{ord}}$ and i in $\mathbf{I}^{\mathrm{ord}}$, we have that $P(i) = \varphi(i)(2)$.
We can do the same also for the cyclotomic variable, that is we define

$$\begin{array}{rccc} \varphi' : & \mathrm{Cl}_{\mathfrak{N}}(p^{\infty}) & \rightarrow & \mathrm{Mer}(D(0))^{\times} \\ & \mathfrak{a} & \mapsto & (s \mapsto \langle \mathcal{N}(\mathfrak{a})\rangle^{s}), \end{array}$$

where $\mathrm{Mer}(D(0))$ denotes the ring of meromorphic functions of fixed, positive radius of convergence around 0, and then we extend φ' to a continuous algebra homomorphism

$$\varphi' : \quad \mathcal{O}[[\mathrm{Cl}_{\mathfrak{N}}(p^{\infty})]] \quad \rightarrow \quad \mathrm{Mer}(D(0)).$$

We define then

$$\varphi\otimes\varphi' : \quad \mathcal{O}[[\mathrm{Cl}_{\mathfrak{N}}(p^{\infty})]]\hat{\otimes}\mathbf{I}^{\mathrm{ord}} \quad \rightarrow \quad \mathrm{Mer}(D(0,2)).$$

where $\mathrm{Mer}(D(0,2))$ denotes the ring of meromorphic functions of fixed, positive radius of convergence around $(0,2)$. We consider $\mathrm{Mer}(D(0,2))$ as a subring of $\overline{\mathbb{Q}}_p((s,k-2))$. Let us denote by $L_p(s,k)$ the image of $L_p^{\mathrm{ord}}(Q,P)$ through $\varphi\otimes\varphi'$ (notice that we have a change of variable $s\mapsto s-1$ from the p-adic L function of the introduction).
Before proving the theorem, we recall some facts about the arithmetic $\mathcal{L}$-invariant of the Galois representation of $\mathrm{Ind}_F^{\mathbb{Q}}(\mathrm{Ad}(\rho_{\mathbf{f}}))$; it has been calculated, under certain hypotheses, in [Hid06, Theorem 3.73] following the definition given in [Gre94b].
Untill further notice, suppose p unramified. The main hypothesis required in the proof of [Hid06, Theorem 3.73] is the so-called $R=T$ theorem ([Hid06, (vsl)]). Namely, let $\mathbf{h}^{\mathrm{cycl}}(\mathfrak{N},\mathcal{O})$ be the cyclotomic Hecke algebra (defined in Section 3.2.2 of this paper). Let

$$Def_{\overline{\rho}_{\mathbf{f}}} : \quad CLN(\mathcal{O}) \quad \rightarrow \quad Set$$

be the deformation functor from the category of local, profinite $\mathcal{O}$-algebra with same residue fields as $\mathcal{O}$ into sets which associate to A in $CNL(\mathcal{O})$ the set of A-deformations which satisfy certain ramification conditions outside p and an ordinarity condition at $\mathfrak{p}$ dividing p. Let R^{ver} be the minimal versal hull for $Def_{\overline{\rho}_{\mathbf{f}}}$ and let $\mathfrak{m}$ be the ideal of R^{ver} corresponding to $\rho_{\mathbf{f}}$. Let $T_{\mathbf{f}}$ be the localization-completion of $\mathbf{h}^{\mathrm{cycl}}(\mathfrak{N},\mathcal{O})$ at $P_{\mathbf{f}}$. We have then a morphism $u: R^{\mathrm{ver}}\rightarrow T_{\mathbf{f}}$. Let R be the localization-completion of R^{ver} at $\mathfrak{m}$ and let T be the localization-completion of $T_{\mathbf{f}}$ at $\mathfrak{m}$. Then the above mentioned condition (vsl) states that $u:R\rightarrow T$ is an isomorphism.
Such a condition holds in many cases thanks to the work of Fujiwara [Fuj06]. For the Hilbert modular form of Theorem 3.1.3, a proof can be found in [Hid06, Theorem 3.50].
Let $\mathbf{I}^{\mathrm{cycl}}$ be the irreducible component of $\mathbf{h}^{\mathrm{cycl}}(\mathfrak{N},\mathcal{O})$ to which $\mathbf{f}$ belong and let λ^{cycl} be the corresponding structural morphism; $\mathbf{I}^{\mathrm{cycl}}$ is finite flat over $\mathcal{O}[[x_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. The $\mathcal{L}$-invariant is expressed in term of partial derivative of $\lambda^{\mathrm{cycl}}(T_0(\mathfrak{p}))$ with respect to $x_{\mathfrak{p}}$. This is not surprising, as from its own definition, the $\mathcal{L}$-invariant is defined in term of the Galois cohomology of the restriction of the representation to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{p^{\infty}}))$ and, as we said at the end of Section 3.2.2, the cyclotomic Hecke-algebra takes into account only modular forms whose local Galois representation at $\mathfrak{p}$ is of fixed type outside $\mathrm{Gal}(F_{\mathfrak{p}}(\mu_{p^{\infty}})/F_{\mathfrak{p}})$.
In the particular case of an elliptic curve E with multiplicative reduction at all $\mathfrak{p}|p$, the calculation of the $\mathcal{L}$-invariant does not require to consider all the all the cyclotomic deformations, but only the ordinary, as explained below.
If E has split multiplicative reduction at $\mathfrak{p}$, let us denote by $Q_{\mathfrak{p}}$ the Tate period of E over $F_{\mathfrak{p}}$ and pose $q_{\mathfrak{p}}=\mathcal{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(Q_{\mathfrak{p}})$. If the reduction at $\mathfrak{p}$ is not split, then $\lambda^{\mathrm{cycl}}(T(\varpi_{\mathfrak{p}}))(2)=-1$; we consider a quadratic twist E_d such that E_d has split multiplicative reduction and we define $Q_{\mathfrak{p}}$ and $q_{\mathfrak{p}}$ as above. We have then a Galois theoretic formula for the $\mathcal{L}$-invariant [Hid06, Corollary 3.74]

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}(\mathrm{Ad}(\rho_{\mathbf{f}})))=\prod_{\mathfrak{p}|p}\frac{\log_p(q_{\mathfrak{p}})}{ord_p(q_{\mathfrak{p}})}.$$

The right hand side as been calculated by Mok in term of logarithmic derivative of the character

$$\lambda:\mathbf{h}^{\mathrm{ord}}\rightarrow\mathbf{I}^{\mathrm{ord}}.$$

We obtain from [Mok09, Proposition 8.7]

$$\frac{\partial\lambda(T(\varpi_{\mathfrak{p}}))}{\partial k}(2)\lambda(T(\varpi_{\mathfrak{p}}))(2)^{-1}=-\frac{1}{2}f_{\mathfrak{p}}\frac{\log_p(q_{\mathfrak{p}})}{ord_p(q_{\mathfrak{p}})}$$

where $f_{\mathfrak{p}} = [\mathbb{F}_{\mathfrak{p}} : \mathbb{F}_p]$.
We remark that the proof of [Mok09, Proposition 8.7] works without any change when p is 2, 3 or ramified in F.
The presence of $f_{\mathfrak{p}}$ has been explained in the introduction of [Hid09] and is related to the Euler factors which we removed at p in the formula presented in Conjecture 3.1.2. More precisely, in this formula appears the product $E^*(m+1, \mathbf{f})$ of all the non-zero Euler factors which have to be removed to obtain p-adic interpolation. Among the factor which we removed, there is $(1-p^{(1-s)f_{\mathfrak{p}}})$ for the primes $\mathfrak{p}$ such that $\lambda(T(\varpi_{\mathfrak{p}}))^{-2} = 1$. We can rewrite this as $\prod_{\zeta \in \mu_{f_{\mathfrak{p}}}} (1-\zeta p^{1-s})$. In the case of the elliptic curve E, we have the trivial zeros at $s=1$; hence the factor $E^*(1, \mathbf{f})$ should then contain $\prod_{1 \neq \zeta \in \mu_{f_{\mathfrak{p}}}} (1-\zeta) = f_{\mathfrak{p}}$.
We remark that the fact that the $\mathcal{L}$-invariants of $\rho_{\mathbf{f}}$ and $\mathrm{Ad}(\rho_{\mathbf{f}})$ are the same is not a coincidence. Let W be the part of Hodge-Tate weights 1 and 0 of this two representations restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (which are isomorphic); we have that W is a non-split extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$ and this ensures that the $\mathcal{L}$-invariant is determined only by the restriction to W (as said at the end of page 169 of [Gre94b]).

Proof of Theorem 3.1.3. Using the interpolation formula of 3.7.2 we see that $L_p(k-2,k) = 0$. In particular, we apply the operator $\frac{\mathrm{d}}{\mathrm{d}k}$ and we get

$$\frac{\mathrm{d}L_p(s,k)}{\mathrm{d}s}|_{s=k-2} = -\frac{\mathrm{d}L_p(s,k)}{\mathrm{d}k}|_{s=k-2}.$$

If we suppose that 2 divides the level of $\mathbf{f}$, than we have that $E_2(0,k)$, defined in the proof of Theorem 3.7.2*i)*, is identically 1. Let Σ_p be set of $\mathfrak{p} \mid p$ such that $\mathbf{f}$ is a Steinberg representation at $\mathfrak{p}$, and Π_p the set $\mathfrak{p} \mid p$ such that $\mathbf{f}$ is a principal series representation at $\mathfrak{p}$. Let us denote by g_1 the cardinality of Σ_p, from the factorization in 3.7.4, we also have

$$\begin{aligned} L_p(0,k) =& 2^{g_1} \prod_{\mathfrak{p} \in \Sigma_p} \frac{\lambda'(T(\varpi_{\mathfrak{p}}))(2)}{\lambda(T(\varpi_{\mathfrak{p}}))(2)^3} \prod_{\mathfrak{p} \in \Pi_p} (1-\lambda(T(\varpi_{\mathfrak{p}}))(2)^{-2}) E_{\mathfrak{p}}(Q_0, P_{\mathbf{f}}) \frac{\mathcal{L}(1,\mathbf{f})}{\Omega(\mathbf{f})} (k-2)^{g_1} + \\ &+ ((k-2)^{g_1+1}). \end{aligned}$$

If we suppose that the Nebentypus of $\mathbf{f}$ is trivial and that the conductor of $\mathbf{f}$ is squarefree, then we have that $\mathbf{f}$ is Steinberg at all primes dividing its conductor. Using the explicit description of the Euler factors at the ramification primes for $\mathbf{f}$ given in Section 3.4.2, case *(iii)*, we see that there aren't any missing Euler factors and consequently the primitive L-function and the imprimitive one coincide.
Using the explicit expression of the $\mathcal{L}$-invariant in term derivative of Hecke eigenvalue given above, we obtain for $\mathbf{f}$ as in the theorem

$$\frac{\mathrm{d}L_p(s,2)}{\mathrm{d}s}|_{s=0} = \mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}(\mathrm{Ad}(\rho_{\mathbf{f}}))) f_p \frac{L(1, \mathrm{Sym}^2(\mathbf{f}))}{\Omega(\mathbf{f})}.$$

□

We remark that we have something more; if there is at least a prime $\mathfrak{p}$ at which $\mathbf{f}$ is Steinberg and $g \geq 2$, then the order at $s=0$ of $L_p(s, \mathrm{Sym}^2(\mathbf{f}))$ is at least 2 (keeping the assumptions on the conductor outside p). Let us point out that the hypothesis that 2 divides the conductor of $\mathbf{f}$ is necessary, as the missing Euler factor at 2 which we remove when interpolating the imprimitive L-function is zero. But a more carefull study of the missing Euler factor at $\mathfrak{N}$ could allow us to weaken the hypothesis on the conductor of Theorem 3.1.3, as not all the missing Euler factors have to vanish.
For example, we could prove Conjecture 3.1.2 for $L(s, \mathrm{Sym}^2(\mathbf{f}), \chi)$ when the character χ is very ramified modulo $2\mathfrak{N}$.
Let us give now the example of an elliptic curve E over $\mathbb{Q}$. Let q be a prime of potentially good reduction of E, we have to exclude the cases when one of the missing Euler factor at q is $(1-q^{1-s})$. We have that in this

case the missing Euler factors have been explicitly calculated in [CS87]; let l be an auxiliary prime different from 2 and q, $\mathbb{Q}_q(E[l])$ the extension of $\mathbb{Q}_q$ generated by the coordinates of the torsion points of order l of E and I_q the inertia of $\mathrm{Gal}(\mathbb{Q}_q(E[l])/\mathbb{Q}_q)$. By [CS87, Lemma 1.3, 1.4] we have that there is no Euler factor of type $(1-q^{1-s})$ at q if and only if one of the following is satisfied:

a) I_q is not cyclic,

b) I_q is cyclic and $\mathbb{Q}_q(E[l])/\mathbb{Q}_q$ is not abelian.

We have the following corollary:

Corollary 3.8.2. *Let E be an elliptic curve over $\mathbb{Q}$ and let $\mathbf{f}$ be the associated modular form. Suppose that E has multiplicative reduction at p, even conductor and that for all prime of additive reduction one between a) or b) is satisfied, then 3.1.2 for $\mathbf{f}$ is true.*

3.9 A formula for higher order derivative

In this section we want to provide an example of 3.1.2 in which $g \geq 2$. We begin considering a form $\mathbf{f}$ has in 3.1.3 and we show that 3.1.2 is true (under mild hypothesis) for $\mathrm{Sym}^2(\mathbf{g})$, where $\mathbf{g}$ denotes an abelian base change (is a sense we specify below) of $\mathbf{f}$. We follow closely the strategy of [Mok09, §9]. Let F be a totally real number field as before, and E a totally real abelian extension of F of degree n. For this whole section, we will suppose for simplicity that p is unramified in E, even though we belive this is not necessary, to simplify the calculations. We fix, for this section, $H = \mathrm{Gal(E/F)}$. Let $\mathbf{f}$ be a Hilbert modular form for F, following [AC89, Chapter 3, §6] we can define its base-change $\mathbf{g}$ to E. Let $\mathfrak{q}'$ be a prime of E above a prime $\mathfrak{q}$ of F, suppose that $\mathbf{f}$ at $\mathfrak{q}$ is minimal but not supercuspidal, then $\mathbf{a}(\mathfrak{q}',\mathbf{g}) = \mathbf{a}(\mathcal{N}_{E/F}(\mathfrak{q}'),\mathbf{f})$. Let us denote by $\hat{H}$ the group of characters of H; via class field field theory, we can identify each element of $\hat{H}$ with a Hecke character of F factoring through $F^\times\mathcal{N}_{E/F}(\mathbb{A}_E^\times)$. We have then

$$L(s,\mathrm{Sym}^2(\mathbf{g})) = \prod_{\phi\in\hat{H}} L(s,\mathrm{Sym}^2(\mathbf{f}),\phi).$$

Note that if $\mathbf{f}$ is nearly-ordinary (resp. ordinary), then $\mathbf{g}$ is nearly-ordinary (resp. ordinary) too.
Assuming that Conjecture 3.1.2 holds for $\mathbf{f}$ we shall show that the same is true for $\mathbf{g}$ by factoring the p-adic L-function $L_p(s,\mathrm{Sym}^2(\mathbf{g}))$ in terms of $L_p(s,\mathrm{Sym}^2(\mathbf{f}),\phi)$.

Lemma 3.9.1. *Let χ a grössencharacter of F of conductor $\mathfrak{c}$ and $\chi_E = \chi\circ\mathcal{N}_{E/F}$. Assuming $D_{E/F}$ and $\mathfrak{c}$ coprime, for the Gauß sum defined before Proposition 3.3.4, we have*

$$G(\chi_E) = \chi(D_{E/F})G(\chi)^n.$$

Proof. As in [Mok07, Lemma 9.4], we use the functional equation for an Hecke character which we recall in Appendix 3.11 and the factorization $L(\chi_E,s)$ in term of $L(\chi\phi,s)$, for ϕ in $\hat{H}$.
We let $\mathfrak{c}_\phi$ be the conductor of $\chi\phi$ and $\mathfrak{f}_\phi$ the conductor of ϕ. Using the formula for the discriminant in tower of extensions

$$D_E = \mathcal{N}_{F/\mathbb{Q}}(D_{E/F})D_F^n,$$

we obtain the equality

$$\prod_{\phi\in\hat{H}} G(\chi\phi)\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{c}_\phi)^{-s} = G(\chi_E)\mathcal{N}_{E/\mathbb{Q}}(\mathfrak{c})^{-s}\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{d}_{E/F})^{-s+1/2}.$$

As $\mathfrak{c}$ in unramified in E, we can split the Gauß sums and the conductor of $\chi\phi$ for all ϕ into a $\mathfrak{c}$-part and a prime-to-$\mathfrak{c}$-part. We conclude with the following formulae from class field theory [Neu99, §7 (11.9),(6.4)]

$$\prod_{\phi\in\hat{H}} \mathfrak{f}_\phi = D_{E/F}, \quad \prod_{\phi\in\hat{H}} G(\phi) = \mathcal{N}_{F/\mathbb{Q}}(D_{E/F}^{1/2}).$$

□

Let us denote by $S(\mathbf{f})$ (resp. $S(\mathbf{g})$) the factor $S(P)$ defined after Theorem 3.7.5 for P the point corresponding to $\mathbf{f}$ (resp. $\mathbf{g}$). We show now the following

Proposition 3.9.2. *For E, F, $\mathbf{g}$, $\mathbf{f}$ as above. Let χ be a finite-order Hecke character of F, and suppose that the conductor of $\mathbf{f}$ and χ are both unramified in the extension E/F. We have then, for all point $u^s\varepsilon(u)-1$ in the interpolation range of Theorem 3.7.5,*

$$G(u^s\varepsilon(u)-1,\mathbf{g},\chi_E) = C\prod_{\phi\in\hat{H}} G(u^s\varepsilon(u)-1,\mathbf{f},\chi\phi),$$

where $G(X,\mathbf{f},\chi\phi)$ (resp. $G(X,\mathbf{g},\chi_E)$) is the Iwasawa function of Theorem 3.7.5 giving the p-adic L-function $L_p(s,\mathrm{Sym}^2(\mathbf{f}),\chi\phi)$ (resp. $L_p(s,\mathrm{Sym}^2(\mathbf{g}),\chi_E)$) and

$$C = \mathcal{N}_{F/\mathbb{Q}}(D_{E/F})^{-2}\frac{(S(\mathbf{f})W(\mathbf{f})'\Omega(\mathbf{f}))^n}{S(\mathbf{g})W(\mathbf{g})'\Omega(\mathbf{g})}.$$

Proof. We use Theorem 3.7.5 to evaluate both sides and show that they coincide.
We note that we are implicitly identifying the $\mathbb{Z}_p$-cyclotomic extension of E with the one of F via $\mathcal{N}_{E/F}$; in particular if p^{α_0} is the conductor of the character ε, seen as a character of $\mathrm{Cl}_{\mathfrak{N}}(p^\infty)$ of F, then the conductor of ε_E is $p^{n\alpha_0}$. This is important to write correctly the evaluation formula of 3.7.5.
We see that the contributions of the factors with 2 and i match as $d_E = nd_F$. The complex formula for the L-function of $\mathrm{Sym}^2(\mathbf{g})$ gives the equality of the L-values. We use Lemma 3.9.1 and the relation between D_F and D_E to control the contribution of the Gauß sums and discriminants. □

In particular this tells us that $\Omega(\mathbf{f})^n$ and $\Omega(\mathbf{g})$ differ by a non-zero algebraic number. We suppose that the conductor of $\mathbf{f}$ and χ are both unramified in the extension E/F to simplify the calculation, although it may not be necessary.
It is well known that for E/F an abelian extension of Galois group H and for $\mathfrak{p}$ a prime ideal of F which is unramified in E, we have

$$\left|\left\{\phi\in\hat{H} \text{ s.t. } \phi(\mathfrak{p})=1\right\}\right| = g,$$

where g is the number of primes of E above $\mathfrak{p}$.
From now on, suppose that p is inert in F. Let $\mathfrak{p}$ be a prime ideal of E above p and let us denote by $f_\mathfrak{p}$ the residual degree of $E_\mathfrak{p}$, we have the following

Theorem 3.9.3. *Let E, F, $\mathbf{g}$, $\mathbf{f}$ be as in Proposition 3.9.2. Suppose that $\mathbf{f}$ satisfies the hypothesis of Theorem 3.1.3. Then we have*

$$\frac{\mathrm{d}^g L_p(s,\mathrm{Sym}^2(\mathbf{g}))}{\mathrm{d}s^g}|_{s=0} = C\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}(\mathrm{Ad}(\rho_\mathbf{g})))f_\mathfrak{p}^g\frac{L_p(1,\mathrm{Sym}^2(\mathbf{g}))}{\Omega(\mathbf{g})},$$

where g denotes the number of primes above p in E and C is a non-zero algebraic number.

Proof. First of all from [Mok09, Proposition 8.7] we obtain

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}(\mathrm{Ad}(\rho_{\mathbf{g}}))) = \mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}(\mathrm{Ad}(\rho_{\mathbf{f}})))^g.$$

Let us denote by f_p the residual degree of F_p; whenever $\phi(p) = 1$, we can show, in the same way as we proved Theorem 3.1.3, that

$$\frac{\mathrm{d}L_p(s, \mathrm{Sym}^2(\mathbf{f}), \phi)}{\mathrm{d}s}|_{s=0} = C_\phi \mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}(\mathrm{Ad}(\rho_{\mathbf{f}}))) f_p \frac{L(1, \mathrm{Sym}^2(\mathbf{f}), \phi)}{\Omega(\mathbf{f})}.$$

If $\phi(p) \neq 1$, we have instead

$$L_p(0, \mathrm{Sym}^2(\mathbf{f}), \phi) = C_\phi (1 - \phi(p)) \frac{L(1, \mathrm{Sym}^2(\mathbf{f}), \phi)}{\Omega(\mathbf{f})}$$

If we write $f = n/g$, we have that

$$\prod_{\phi \in \hat{H}, \phi(p) \neq 1} (1 - \phi(p)) = f^g$$

then we use Proposition 3.9.2 and the fact that $f_{\mathfrak{p}} = f_p f$ to conclude. □

We remark that this method does not work for an abelian base change $\mathbf{h}$ of $\mathbf{g}$. In fact, if we want to apply the same strategy as in the proof of Theorem 3.9.3, we would need a formula for the derivative of $L_p(s, \mathrm{Sym}^2(\mathbf{g}), \psi)$. But for the complex L-function $L(s, \mathrm{Sym}^2(\mathbf{g}), \psi)$ there is no factorization in term of L-functions of $\mathrm{Sym}^2(\mathbf{f})$ (unless ψ is a base change from F too).

3.10 Application to the Main Conjecture

In the introduction we said that Theorem 3.1.3 has application to the main conjecture for the symmetric square and the aim of this section is to explain how.
We suppose now $F = \mathbb{Q}$; let $G(T, \mathrm{Sym}^2(\mathbf{f}))$ be the Iwasawa function for $\mathbf{f}$ of Theorem 3.7.5. We shall write

$$L_p^{\mathrm{an}}(T, \mathrm{Sym}^2(\mathbf{f})) = \frac{\langle \mathbf{f}, \mathbf{f} \rangle \pi^2}{i\Omega^+\Omega^-} G(T, \mathrm{Sym}^2(\mathbf{f})),$$

for $\Omega^{\pm}$ the $\pm$ period associated with $\mathbf{f}$ via the Eichler-Shimura isomorphism. This change of periods is necessary for what follows and is due to the fact that our p-adic L-function has a denominator $H_{\mathbf{f}}$. Moreover, the evaluations of $L_p^{\mathrm{an}}(T, \mathrm{Sym}^2(\mathbf{f}))$ are compatible with Deligne's conjecture on special values and periods.
Let $L_p^{\mathrm{al}}(T, \mathrm{Sym}^2(\mathbf{f}))$ be the algebraic p-adic L-function defined as $\Psi(s)$ in [Hid00a, Theorem 6.3]; it is the characteristic ideal of a certain Selmer group. The Greenberg-Iwasawa main conjecture for $\mathrm{Ind}_F^{\mathbb{Q}}(\mathrm{Sym}^2(\mathbf{f}))$ says that the ideal generated by $L_p^{\mathrm{al}}(T, \mathrm{Sym}^2(\mathbf{f}))$ in $\mathcal{O}[[T]]$ is the same as the one generated by $L_p^{\mathrm{an}}(T, \mathrm{Sym}^2(\mathbf{f}))$.
In what follows we will denote by $L_p^{\mathrm{al}}(0, \mathrm{Sym}^2(\mathbf{f}))^*$ resp. $L_p^{\mathrm{an}}(T, \mathrm{Sym}^2(\mathbf{f}))^*$ the first non-zero coefficient of $L_p^{\mathrm{al}}(T, \mathrm{Sym}^2(\mathbf{f}))$ resp. $L_p^{\mathrm{an}}(T, \mathrm{Sym}^2(\mathbf{f}))$.
Let η be the characteristic ideal of the Pontryagin dual of $\mathrm{Sel}(\mathrm{Ad}(\rho_{\mathbf{f}}))$ (defined as in [Hid00a]); in [Hid00a, Theorem 6.3 (3)] it is shown that $\mathrm{ord}_{T=0} L_p^{\mathrm{al}}(T, \mathrm{Sym}^2(\mathbf{f})) = 1$ and

$$L_p^{\mathrm{al}}(T, \mathrm{Sym}^2(\mathbf{f}))^* \mid \mathcal{L}(\mathrm{Ad}(\rho_{\mathbf{f}}))\eta.$$

From the work of Urban [Urb06] we know that, in most of the cases for $F = \mathbb{Q}$ (for example when $\mathbf{f}$ is as in Theorem 3.1.3), $L_p^{\mathrm{an}}(T, \mathrm{Sym}^2(\mathbf{f}))$ divides $L_p^{\mathrm{al}}(T, \mathrm{Sym}^2(\mathbf{f}))$ in $\mathcal{O}[[T]][T^{-1}]$.

We have, from works of Hida and Wiles, a formula which equals the p-adic valuation of the special value $L_p^{\mathrm{an}}(0, \mathrm{Sym}^2(\mathbf{f}))$ and of η [Urb06, 1.2.3]. For totally real fields, we mention that in [Dim09] the special value is related to the cardinality of the Fitting ideal.
If in addition we also know Conjecture 3.1.2, then we deduce that the two series have the same order at T and

$$L_p^{\mathrm{al}}(0, \mathrm{Sym}^2(\mathbf{f}))^* | L_p^{\mathrm{an}}(0, \mathrm{Sym}^2(\mathbf{f}))^* \quad \text{in } \mathcal{O}.$$

We can write $L_p^{\mathrm{al}}(T, \mathrm{Sym}^2(\mathbf{f})) = A(T) L_p^{\mathrm{an}}(0, \mathrm{Sym}^2(\mathbf{f}))$, with $A(T)$ in $\mathcal{O}[[T]]$; the above divisibility tells us that $A(0)^{-1}$ belongs to $\mathcal{O}$, i.e. $A(T)$ is a unit (because its constant term is a unit) and the main conjecture is proven.

3.11 Holomorphicity in one variable

In this sectio, we want to show that the p-adic L-function of Theorem 3.7.2 *i)* can be modified (dividing by suitable Iwasawa functions interpolating the missing Euler factors) into an Iwasawa function interpolating special values of the primitive complex L-function.
To do this, we follow an idea of Schmidt [Sch88] and Dabrowski and Delbourgo [DD97]; first we construct another p-adic L-function which interpolates the other half of critical values and then we relate the two p-adic L-functions via the functional equation of Section 3.4.2. We shal use this results in the second part of the appendix to show that the many-variable p-adic L-function is holomorphic too, following [Hid90].
Before doing this, we recall the functional equation satisfied by $L(s, \chi)$, for a primitive unitary grössencaracter χ. Let $\mathfrak{f}$ be its conductor, we define the complete L-function

$$\Lambda(s, \chi) = \prod_{v|\infty} \left(\pi^{-(s+p_v)/2} \Gamma\left(\frac{s+p_v}{2} \right) \right) L(s, \chi),$$

for $p_v = 0$ or 1 such that $\chi_v(-1) = (-1)^{p_v}$. It is well known since Hecke (see also Tate's thesis [Tat67]) the existence of a functional equation

$$\Lambda(s, \chi) = \varepsilon(s, \chi) \Lambda(1-s, \chi^{-1})$$

for $\varepsilon(s, \chi) = i^{\sum_v p_v} \chi_\infty(-1) G(\chi) \mathcal{N}(\mathfrak{f})^{-s} |\mathcal{N}(\mathfrak{d})|^{-s+1/2}$ where as before

$$\begin{aligned} G(\chi) =& \sum_{x \in \mathfrak{f}^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}} \chi_\infty(x) \chi_f(x\mathfrak{d}\mathfrak{f}) \mathbf{e}_F(x) \\ =& \prod G(\chi_{\mathfrak{q}}), \\ G(\chi_{\mathfrak{q}}) =& \chi_{\mathfrak{q}}^{-1}(\varpi_{\mathfrak{q}}^{\alpha_{\mathfrak{q}}+e_{\mathfrak{q}}}) \sum_{x \bmod \varpi_{\mathfrak{q}}^{\alpha_{\mathfrak{q}}}} \chi_{\mathfrak{q}}(x) \mathbf{e}_{F_{\mathfrak{q}}} \left(\frac{x}{\varpi^{e_{\mathfrak{q}}+\alpha_{\mathfrak{q}}}} \right), \end{aligned}$$

for $e_{\mathfrak{q}}$ the ramification index of $F_{\mathfrak{q}}$ over $\mathbb{Q}_q$ and $\alpha_{\mathfrak{q}}$ such that $\mathfrak{q}^{\alpha_{\mathfrak{q}}}$ divides $\mathfrak{f}$ exactly. We can now modify Theorem 3.6.1 to see

Proposition 3.11.1. *Let χ be a character of finite order of conductor $\mathfrak{f}$. Suppose $\chi_\sigma(-1) = 1$ for all $\sigma \in I$. For all $c \in \mathrm{Cl}_{\mathfrak{N}}(p^\infty)$, we have a measure $\zeta'_{\chi,c}$ on $\mathrm{Cl}_{\mathfrak{N}}(p^\infty)$ such that, for all finite-order characters ε and for all $s \geq 0$, we have*

$$\begin{aligned} \int_{\mathrm{Cl}_{\mathfrak{N}}(p^\infty)} \varepsilon(z) \mathcal{N}_p(z)^s \mathrm{d}\zeta'_{\chi,c}(z) =& (1 - (\chi\varepsilon)_0(c) \mathcal{N}_p(c)^{s+1}) \prod_{\mathfrak{p}|p} (1 - (\chi\varepsilon)_0(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{-s}) \times \\ & \times \Omega(\varepsilon\chi, s) \frac{\prod_{\mathfrak{p}|p} G((\chi\varepsilon_0)_{\mathfrak{p}})}{\mathcal{N}(\varpi^{\alpha_0})^{-s}} L(1+s, (\chi^{-1}\varepsilon^{-1})_0). \end{aligned}$$

where $(\chi\varepsilon)_0$ *is the primitive character associated with the character* $\chi\varepsilon$ *of* $\mathfrak{r}/\mathfrak{c}p^{\alpha}$*,* p^{α_0} *is the p-part of its conductor, and*

$$\Omega(\varepsilon\chi, s) = i^{\kappa d} {D_{F,p}}^{s+\frac{1}{2}} \varepsilon\chi_{\infty}(-1) \frac{\left(\pi^{-(s+1+\kappa)/2}\Gamma((s+1+\kappa)/2)\right)^d}{\left(\pi^{(s-\kappa)/2}\Gamma(-(s-\kappa)/2)\right)^d}.$$

for $\kappa = 0, 1$*, congruent to* s *modulo* 2

Proof. Let $\mathfrak{f}^{(p)}$ be the conductor of χ outside p and let $\mathfrak{d}^{(p)}$ (resp. $\mathfrak{d}_p$) be the prime-to-p part (the p-part) of $\mathfrak{d}$. Let us denote $\mathcal{N}(\mathfrak{d}_p)$ by $D_{F,p}$ and $\mathcal{N}(\mathfrak{d}^{(p)})$ by D'_F. From Theorem 3.6.1 and the functional equation above, we simply divide by the functions $A_{\mathcal{N}(\mathfrak{d}^{(p)}\mathfrak{f}^{(p)})}(X)$, $N(\mathfrak{d})^{-1/2}$ and the Gauß sums for $\mathfrak{q}|\mathfrak{f}'$. □

We recall that for an element $\mathfrak{c}$ of $\mathrm{Cl}_{\mathfrak{N}}(p^{\infty})$, we have defined $\langle \mathcal{N}_p(\mathfrak{c})\rangle = \omega^{-1}(\mathcal{N}_p(\mathfrak{c}))\mathcal{N}_p(\mathfrak{c})$ and that $\langle \mathcal{N}_p(\mathfrak{c})\rangle$ is an element of $1 + p\mathbb{Z}_p$, so it makes sense to define

$$A_{\mathcal{N}_p(\mathfrak{c})}(X) = (1+X)^{\log_p(\langle \mathcal{N}_p(\mathfrak{c})\rangle)/\log_p(u)},$$

where u is a generator of $1 + p\mathbb{Z}_p$. In case we have to consider the L-values of imprimitive characters, we can obtain the p-adic interpolation of the values $L_{\mathfrak{N}}(1+n, (\psi\varepsilon)^{-1})$ simply multiplying by $(1 - (\psi\varepsilon)_0(\mathfrak{q})^{-1}\langle \mathcal{N}_p(\mathfrak{q})\rangle^{-1} A_{\mathcal{N}_p(\mathfrak{q})}(X)^{-1})$.

Fix now $\mathbf{f}$, a nearly ordinary form of Nebentypus (ψ, ψ') and weight $k \geq 2t$ which we decompose as $k = (m+2)t - 2v$ as in Theorem 3.7.5. Suppose that ψ' comes from a character of $\mathrm{Cl}_{\mathfrak{N}}(p^{\infty})$. We give now the analogue of Theorem 3.7.5 for the other critical values, namely the one in the strip $[m + 2m + k_0]$.

Theorem 3.11.2. *Fix a primitive adelic character* χ *of level* $\mathfrak{c}$*, such that* $\chi_{\sigma}(-1) = 1$ *for all* $\sigma|\infty$*. We have a p-adic L-functions* $\mathcal{L}_p^-(Q)$ *in the fraction field of* $\mathcal{O}[[\mathrm{Cl}_{\mathfrak{N}}(p^{\infty})]]$ *such that the following interpolation property hold; for all arithmetic points* Q *of type* (s, ε_Q)*, with* $m - k_0 + 2 \leq s \leq m$ *the following interpolation formula holds*

$$\mathcal{L}_p^-(Q) = C_2 E_1^-(Q)E_2^-(Q)E_3(Q)\frac{\mathcal{L}(2m+2-s, \mathbf{f}, \varepsilon_Q\omega^{-s}\psi^{-2}\chi)}{(2\pi)^{ds}\Omega(\mathbf{f})}.$$

The Euler factors $E_1^-(Q)$ and $E_2^-(Q)$ at p could be thought as the factors $E_1(\tilde{Q})$ and $E_2(\tilde{Q})$, for $\tilde{Q} = \varepsilon_Q^{-1}(u)u^{2m-2-s} - 1$, the *symmetric* of Q; more precisely, they are

$$\begin{aligned}
E_1(Q) =& \lambda(T(\varpi^{2\beta-2\alpha_0}))\mathcal{N}(\varpi^{\beta-\alpha_0})^{s-2m-2}\times \\
&\times \prod_{\mathfrak{p}|p}(1 - (\chi^{-1}\psi^2\varepsilon_Q^{-1}\omega^s)_0(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{2m+1-s}\lambda(T(\mathfrak{p}))^{-2}),\\
E_2(Q) =& \prod_{\mathfrak{p}|p}(1 - (\chi\varepsilon_Q\omega^{-s}\psi^{-1})_0(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{s-m-1})\times \\
&\times (1 - (\chi\varepsilon_Q\omega^{-s})_0(\mathfrak{p})\lambda(T(\varpi_{\mathfrak{p}}))^{-2}\mathcal{N}(\varpi)^s).
\end{aligned}$$

The Euler factor $E_3(Q)$ is defined as

$$\prod_{\mathfrak{p}|p}\frac{G({\eta^{-1}}_{\mathfrak{p}})(1 - (\eta^{-1})_0(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{m-s})}{(1 - (\eta)_0(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{s-m-1})},$$

for $\eta = \varepsilon_Q\omega^{-s}\chi_{-1}\chi\psi^{-1}\psi'^{-2}$. The constant C_2 can be found the following proof of Theorem 3.11.2.
Let us point out that this symmetric square p-adic L-function interpolates the imprimitive special values, but that none of the Euler factors which we have removed outside p is zero. This means that if we could find a formula for the first derivative of it as we did in Theorem 3.1.3, we could then prove Conjecture 3.1.2 in the case in which p is inert *without any assumption on the conductor of* $\mathbf{f}$.

Proof. We use the same notation of Sections 3.5 and 3.7. In particular we have $\mathfrak{M} = 4\mathfrak{c}^2\mathfrak{L}^2$.
We let α such that $\mathbf{f}$ is of level p^α. Let Q be a point of type (s, ε_Q), with $m - k_0 + 2 \leq s \leq m$ and ε_Q a finite-order character as in the statement of the theorem; in what follows, we shall write ε for ε_Q. We can define an Eisenstein measure $E_c^{\chi,-}$ on $\mathrm{Cl}_{\mathfrak{N}}(p^\infty)$ similarly to Section 3.6; let us define its value on Q as

$$\int_{\mathrm{Cl}_{\mathfrak{N}}(p^\infty)} \varepsilon(z)\langle\mathcal{N}_p(z)\rangle^s \mathrm{d}E_c^{\chi,-}(z) = (1 - \eta^{-1}(c)\mathcal{N}_p(c)^{s_0 - m})\Omega(\eta^{-1}, s_0 - m - 1)\times$$
$$\prod_{\mathfrak{p}|p} G(\eta_{\mathfrak{p}}^{-1}) \times \mathcal{N}(\varpi)^{(s_0-m-1)\beta_0} e_0'(k, s_0, \eta)|\nu_Q,$$

where $\eta = \varepsilon\omega^{-s}\chi_{-1}\chi\psi^{-1}\psi'^{-2}$, β is such that η is a character modulo $\mathfrak{r}/\mathfrak{M}p^\beta$ (we do not assume η primitive), β_0 is the exact p-power of the conductor of η_0 and $s_0 = 2m + 1 - s$ and ν_Q is the continuous function on $\mathfrak{r}_p^\times$ define on $\xi \in F$, $\xi \gg 0$ by

$$\nu_Q(\xi) = \frac{\prod_{\mathfrak{p}|p}(1 - (\eta^{-1}\omega_\xi)_0(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{s_0-m-1})}{\prod_{\mathfrak{p}|p}(1 - (\omega_\xi\eta)_0(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{m-s_0})}.$$

We recall from Proposition 3.3.10

$$e_0'(k, s_0, \eta) = \sum_{\xi\in 2^{-1}, \xi\gg 0, (p,\xi)=\mathfrak{r}} \xi^{-2v} g_f(\xi, s_0 - m, \eta)\, L_{p\mathfrak{M}}(s_0 - m, \eta\omega_{2\xi})q^\xi.$$

Let us point out that it is a p-adic measure as its Fourier coefficients are the p-adic L-functions of Proposition 3.11.1 (twisted by $\varepsilon(z) \mapsto \varepsilon^{-1}(z)\left\langle \mathcal{N}_p(z)^{m+2}\right\rangle$) and some exponential functions as in Section 3.6. The presence of the twist by the function ν_Q is due to the fact that the factor appearing in the interpolation formula of Proposition 3.11.1 for $\chi = \eta\omega_{2\xi}$ is not independent of ξ.
The value of this measure corresponds, up to some normalization factors, with the *constant term projection* (cf. Section 3.3.3) of $\mathcal{E}'\left(\frac{s_0-m}{2}, k-(n+1)t, \eta\right)|[\mathfrak{m}^2\varpi^{2\beta}/4]$, with $n = 0, 1$ and $n \equiv s \bmod 2$.

Let $\mathfrak{C}_0$ be the conductor (outside p) of $\chi^{-1}\psi^2$ and let $\mathfrak{c}_0$ be an idèle representing it. Let $\Theta_{\chi\psi^2}$ be the theta measure of Section 3.6 *which we consider now of level* $\mathfrak{C}_0$. We define $\Theta'_{\chi^{-1}\psi^2}(\varepsilon) = \Theta_{\chi^{-1}\psi^2}(\varepsilon^{-1})$ and then we define the product measure $\mu := \Theta'_{\chi^{-1}\psi^2}|\left[\frac{\mathfrak{m}^2}{\mathfrak{c}_0^2}\right] \times E_c^{\chi,-}$ as $p-1$ measures μ_i on $1 + p\mathbb{Z}_p$; for s in $\mathbb{Z}_{\geq 0}$, $s \equiv i \bmod p-1$, we pose

$$\int_{1+p\mathbb{Z}_p} \varepsilon_Q(z)\langle\mathcal{N}_p(z)\rangle^s \mathrm{d}\mu_i(x) =$$

$$= \left(\int_{\mathrm{Cl}_{\mathfrak{N}}(p^\infty)} \omega^{-s}\varepsilon_Q(z)\mathrm{d}\Theta'_{\chi^{-1}\psi^2}(z)\right)|\left[\frac{\mathfrak{m}^2}{4\mathfrak{c}_0^2}\right] \times \int_{\mathrm{Cl}_{\mathfrak{N}}(p^\infty)} \varepsilon_Q\mathcal{N}_p(z)^s \mathrm{d}E_c^{\chi,-}.$$

Evaluating at Q as above we obtain

$$A_0{}^{-1}(1 - \eta^{-1}(c)\mathcal{N}_p(c)^{s_0-m})\frac{\Omega(\eta^{-1}, s_0 - m - 1)}{\mathcal{N}(\varpi)^{-(s_0-m-1)\beta_0}}\times$$
$$\times\, \theta(\chi^{-1}\omega^s\varepsilon_Q^{-1}\psi^2)|\left[\frac{\mathfrak{m}^2}{4\mathfrak{c}_0^2}\right] c\left(\mathcal{E}'\left(\frac{s_0 - m}{2}, k - (n+1)t, \eta|[\mathfrak{m}^2\varpi^{2\beta}2^{-1}]\right)\right)|\nu_Q.$$

where A_0 is given in Proposition 3.3.10 and is equal to

$$\begin{aligned}A_0 =& i^{k-(n+1)d}\pi^{\alpha'}\Gamma_\infty(\alpha')^{-1}2^{k-\left(n+\frac{3}{2}\right)d}\times\\ &\times\eta(\mathfrak{m}\varpi^\beta\mathfrak{d}2^{-1})\mathcal{N}(\mathfrak{d})^{m-s_0}\mathcal{N}(\mathfrak{m}\varpi^\beta 2^{-1})^{m-s_0-1},\\ \alpha' =& \frac{(m-s+1-n)t+k}{2}.\end{aligned}$$

Let us denote by χ' the primitive character associated with $\omega^s\varepsilon^{-1}\chi^{-1}\psi^2$. We use the relations given in Section 3.4.1 to obtain

$$\begin{aligned}\theta(\chi^{-1}\omega^s\varepsilon^{-1}\psi^2)\left[\frac{\mathfrak{m}^2}{4\mathfrak{c}_0^2}\right]|\tau(\mathfrak{m}^2\varpi^{2\beta}) = C(\chi')\mathcal{N}(2^{-1}\mathfrak{m}\mathfrak{c}_0^{-1})^{-1/2}\mathcal{N}(\varpi^{\beta-\alpha_0})^{1/2}\times\\ \times\sum_{\mathfrak{e}|p}\mathcal{N}(\mathfrak{e})\mu(\mathfrak{e})\chi'(\mathfrak{e})\theta(\chi'^{-1})|\left[\frac{\varpi^{2\beta}}{\varpi^{2\alpha_0}\mathfrak{e}^2}\right].\end{aligned}$$

For each $\mathfrak{q}$ dividing 2 in $\mathfrak{r}$, we pose

$$\begin{aligned}\mathcal{E}_{\mathfrak{q}}^-(X) =& (1-\psi^{-2}\chi^2(\mathfrak{q})A^2_{\mathcal{N}(\mathfrak{q})}(X)2^{-2m-2})\\ \mathcal{E}_2^-(X) =& \prod_{\mathfrak{q}|2}\mathcal{E}_{\mathfrak{q}}^-(X),\end{aligned}$$

where X is a variable on the free part of the cyclotomic extension in $\mathrm{Cl}_{\mathfrak{N}}(p^\infty)$ and corresponds to the variable Q. We put moreover $\Delta^-(Y) = (1-\chi_{-1}\chi^{-1}\psi^{-1}\psi'^{-2}(c)\mathcal{N}_p(c)^{m+2}(1+Y)^{-1})$ where c is chosen such that $\langle\mathcal{N}(c)\rangle$ correspond to the fixed generator u.
We define a p-adic L-function

$$\mathcal{L}_p^-(X) = (\Delta^-(X)\mathcal{E}_2^-(X)H_{\mathbf{f}})^{-1}l_{\lambda_{\mathbf{f}}}e\mathrm{Tr}_1(\mathrm{Tr}_2\Theta'_\chi\times E_c^{\chi^{-1},-}|\Xi_2).$$

where

$$\mathrm{Tr}_1 = T_{\mathfrak{M}/\mathfrak{N}}\,,\mathrm{Tr}_2 = Tr^{\mathrm{GL}_2}_{\mathrm{SL}_2}(v,\psi'),$$

and $H_{\mathbf{f}}$ is the congruence number of $\mathbf{f}$ (the specialization of H, defined at the beginning of Section 3.7, at the point corresponding to $\mathbf{f}$).
Let us point out that we can move the twist by ν_Q from the Eisenstein series to the theta series as we did to prove the formulas at the end of Section 3.3.4, and that on $\theta(\chi^{-1}\omega^s\varepsilon_Q^{-1}\psi^2)|\left[\frac{\mathfrak{m}^2}{4\mathfrak{c}_0^2}\right]$ this twist is simply multiplication by $\psi^{(p)}(\chi^{(p)})^{-1}\chi_{-1}(p^{\alpha_0})E_3(Q)$.
We have then for $Q = X+1-\varepsilon(u)u^s$

$$\begin{aligned}\mathcal{L}_p^-(\varepsilon(u)u^s-1) =& \mathcal{E}_2^-(\varepsilon(u)u^s-1)^{-1}\frac{\mathcal{N}(\mathfrak{m}\varpi^\beta 2^{-1})^{-\frac{1}{2}}E_3(Q)\Omega(\eta^{-1},s_0-m-1)}{A_0\lambda(T(\varpi^{2\beta-\alpha}))\mathcal{N}(\varpi)^{(m-s_0+1)\beta_0}}\times\\ &\times\frac{\left\langle\mathbf{f}^c,\mathrm{Tr}_1\circ\mathrm{Tr}_2\left(\theta_n(\chi')|\left[\frac{\mathfrak{m}^2}{4\mathfrak{c}_0^2}\right]\mathcal{E}\left(\frac{s_0-m}{2},k-(n+1)t,\eta\right)\tau(\mathfrak{m}^2\varpi^{2\beta})\right)\right\rangle_{\mathfrak{M}p^{2\beta}}}{\langle\mathbf{f}^c|\tau'(\mathfrak{n}\varpi^\alpha),\mathbf{f}\rangle_{\mathfrak{N}p^\alpha}},\\ =& \frac{C^{(p)}}{C_p}E_1^-(Q)E_2^-(Q)E_3(Q)\frac{\mathcal{L}(s_0+1,\mathbf{f},\varepsilon_Q\omega^{-s}\psi^{-2}\chi)}{\langle\mathbf{f}^\circ,\mathbf{f}^\circ\rangle_{\mathfrak{N}p^\alpha}}.\end{aligned}$$

Here we have

$$
\begin{aligned}
C^{(p)} =&2^{1-v} i^{(n+1)d-k} 2^{\left(n-\frac{3}{2}\right)d-k} \pi^{\frac{(s+n-m-1)t-k}{2}} \Gamma_\infty \left(\frac{(m-s-n+1)t+k}{2} \right) \times \\
&\times (2\pi)^{\frac{(s-m-n)d-k}{2}} \Gamma_\infty \left(\frac{(m-s+n)t+k}{2} \right) \times \\
&\times \eta^{-1}(\mathfrak{m}\varpi^\beta \mathfrak{d} 2^{-1}) \mathcal{N}(\mathfrak{d})^{m-s-\frac{1}{2}} \mathcal{N}(\mathfrak{m}\varpi^\beta 2^{-1})^{m+\frac{3}{2}-s} \times \\
&\times \frac{\Omega(\eta^{-1}, m-s)}{\mathcal{N}(\varpi)^{(s-m)\beta_0}} G(\chi') \mathcal{N}(\mathfrak{m}2^{-1})^{-1/2} \mathcal{N}(\varpi^{\frac{1}{2}\beta-\alpha_0}) \\
&\times \mathcal{N}(2^{-1}\mathfrak{d}\mathfrak{m}\varpi^\beta)^m \psi(\mathfrak{m}\varpi^\beta \mathfrak{d}) \\
=& i^{(n+1)d-k} 2^{1+d(s+n-4m+\frac{s-n-11}{2})-2v} \pi^{(s-m)d-k} 2^{(s-m+1)d-k} ((m-s-1)t+k)! \times \\
&\times \eta^{-1}(\mathfrak{m}\varpi^\beta \mathfrak{d} 2^{-1}) \psi(\mathfrak{m}\varpi^\alpha \mathfrak{d}) \mathcal{N}(\mathfrak{d})^{2m-s-\frac{1}{2}} \mathcal{N}(\varpi)^{(m-s)\beta_0+(2m-s+2)\beta-\alpha_0} \\
&\times G(\omega^s \varepsilon^{-1}\chi^{-1}\psi^2)(\omega^s \varepsilon^{-1}\chi^{-1}\psi^2)(\mathfrak{d}\mathfrak{c}_0\varpi^{\alpha_0}) \\
&\times \Omega(\eta^{-1}, m-s) \mathcal{N}(\mathfrak{m})^{2m+1-s}, \\
C_p =& \mathcal{N}(\varpi)^{-(\alpha-\alpha')m/2} \lambda(T(\varpi^{2\beta-\alpha'})) \psi_\infty(-1) \times \\
&\times W'(\mathbf{f}_P) S(P) \times \prod_{\mathfrak{p}} \frac{\eta\nu(\mathfrak{d}_\mathfrak{p})}{|\eta\nu(\mathfrak{d}_\mathfrak{p})|} \prod_J G(\nu\psi'),
\end{aligned}
$$

where we applied the duplication formula

$$
\begin{aligned}
&\Gamma_\infty \left(\frac{(m-s-n+1)t+k}{2} \right) \Gamma_\infty \left(\frac{(m-s+n)t+k}{2} \right) = \\
=&\Gamma_\infty((m-s)t+k) 2^{d(s-m+1)-k} \pi^{1/2}.
\end{aligned}
$$

□

We can now construct the primitive symmetric square p-adic L-function as a one variable Iwasawa function.
Recall the factor $\mathcal{E}_\mathfrak{N}(s, \mathbf{f}, \chi)$ defined in Section 3.4.2 and the partition of the set of primes dividing $\mathfrak{N}$ in the four subsets *i)*, *ii)*, *iii)*, *iv)*.
For each prime $\mathfrak{q}$ dividing $\mathfrak{N}$, let $\mathbf{f}_\mathfrak{q}$ be a twist of $\mathbf{f}$ which is minimal among all the twists at $\mathfrak{q}$ (in the sense of Section 3.4.2), and let $\lambda_\mathfrak{q}(T(\mathfrak{q}))$ be the Hecke eigenvalues of $\mathbf{f}_\mathfrak{q}$ at $T(\mathfrak{q})$. In this case, let us denote by $\alpha_\mathfrak{q}$ and $\beta_\mathfrak{q}$ the two roots of the Hecke polynomial at $\mathfrak{q}$. The existence of a global character η such that $\mathbf{f} \otimes \eta$ is of the desired type at $\mathfrak{q}$ is guaranteed by [Che51].
We define the following factors; if $\mathfrak{q}$ belongs to cases *i)* or *ii)* we have two possibilities, according to $\lambda(T(\mathfrak{q})) = 0$ or not. Only for these two cases, we mean $\chi(\mathfrak{q}) = 0$ if χ is ramified. In the first case, we put

$$
\begin{aligned}
\mathcal{E}^+_\mathfrak{q}(X) =& \left(1 - \chi^{-1}(\mathfrak{q})\alpha_\mathfrak{q}^2 \mathcal{N}(\mathfrak{q})^{-1} A^{-1}_{\mathcal{N}(\mathfrak{q})}(X)\right) \left(1 - \psi\chi^{-1}(\mathfrak{q})\mathcal{N}(\mathfrak{q})^m A^{-1}_{\mathcal{N}(\mathfrak{q})}(X)\right) \times \\
&\left(1 - \psi^2\chi^{-1}(\mathfrak{q})\alpha_\mathfrak{q}(T(\mathfrak{q}))^{-2}\mathcal{N}(\mathfrak{q})^{2m+1} A^{-1}_{\mathcal{N}(\mathfrak{q})}(X)\right), \\
\mathcal{E}^-_\mathfrak{q}(X) =& \left(1 - \psi^{-2}\chi(\mathfrak{q})\alpha_\mathfrak{q}^2 \mathcal{N}(\mathfrak{q})^{-2m-2} A_{\mathcal{N}(\mathfrak{q})}(X)\right) \left(1 - \psi^{-1}\chi(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-1-m} A_{\mathcal{N}(\mathfrak{q})}(X)\right) \times \\
&\left(1 - \chi(\mathfrak{q})\alpha_\mathfrak{q}^{-2} A_{\mathcal{N}(\mathfrak{q})}(X)\right),
\end{aligned}
$$

while if $\lambda(T(\mathfrak{q})) \neq 0$

$$\mathcal{E}_{\mathfrak{q}}^{+}(X) = \left(1 - \psi\chi^{-1}(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{m} A_{\mathcal{N}(\mathfrak{q})}^{-1}(X)\right)\left(1 - \psi^{2}\chi^{-1}(\mathfrak{q})\lambda_{\mathfrak{q}}(T(\mathfrak{q}))^{-2}\mathcal{N}(\mathfrak{q})^{2m+1} A_{\mathcal{N}(\mathfrak{q})}^{-1}(X)\right),$$
$$\mathcal{E}_{\mathfrak{q}}^{-}(X) = \left(1 - \psi^{-1}\chi(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-1-m} A_{\mathcal{N}(\mathfrak{q})}(X)\right)\left(1 - \chi(\mathfrak{q})\lambda_{\mathfrak{q}}(T(\mathfrak{q}))^{-2} A_{\mathcal{N}(\mathfrak{q})}(X)\right).$$

If $\mathfrak{q}$ is in case *iv)*, recall that there are several subcases; if $(\psi\chi)^2$ is ramified, we do not need to define any extra factors.
If $\psi\chi$ is unramified, we define

$$\mathcal{E}_{\mathfrak{q}}^{+}(X) = 1 + \psi\chi^{-1}(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{m} A_{\mathcal{N}(\mathfrak{q})}^{-1}(X),$$
$$\mathcal{E}_{\mathfrak{q}}^{-}(X) = 1 + \chi\psi^{-2}(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-m-1} A_{\mathcal{N}(\mathfrak{q})}(X).$$

For the remaining case, $(\psi\chi)^2$ unramified and $\psi\chi$ ramified, recall that in Section 3.4.2 we have defined two characters λ_i, $i = 1, 2$, such that $\psi\chi\lambda_i$ is unramified. The extra factor we need to define are

$$\mathcal{E}_{\mathfrak{q},i}^{+}(X) = 1 - \psi\chi^{-1}\lambda_i(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{m} A_{\mathcal{N}(\mathfrak{q})}(X)^{-1},$$
$$\mathcal{E}_{\mathfrak{q},i}^{-}(X) = 1 - \chi\psi^{-1}\lambda_i(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-m-1} A_{\mathcal{N}(\mathfrak{q})}(X),$$

and then

$$\mathcal{E}_{\mathfrak{q}}^{+}(X) = \prod_{i \text{ s.t.} \pi_{\mathfrak{q}} \cong \pi_{\mathfrak{q}} \otimes \lambda_i} \mathcal{E}_{\mathfrak{q},i}^{+}(X),$$
$$\mathcal{E}_{\mathfrak{q}}^{-}(X) = \prod_{i \text{ s.t.} \pi_{\mathfrak{q}} \cong \pi_{\mathfrak{q}} \otimes \lambda_i} \mathcal{E}_{\mathfrak{q},i}^{-}(X).$$

Let $G(X)$ the formal Laurent series of Theorem 3.7.5. Then the p-adic L-function interpolating the primitive values is

$$F(X) := G(X) \prod_{\mathfrak{q} \mid \mathfrak{N}} \mathcal{E}_{\mathfrak{q}}^{+}(X)^{-1}.$$

We give the following proposition;

Proposition 3.11.3. *Let* $\mathbf{f}$ *be a neearly-ordinary Hilbert eigenform. For all fixed* s *in the interpolation range and* ε *a finite-order character of* $1 + p\mathbb{Z}_p$*, we have the following identity*

$$F(\varepsilon(u)u^s - 1) = C_s \mathcal{L}_p^{-}(\varepsilon(u)u^s - 1) \prod_{\mathfrak{q} \mid \mathfrak{N}} \mathcal{E}_{\mathfrak{q}}^{-}(\varepsilon(u)u^s - 1)^{-1},$$

with $\mathcal{L}_p^{-}(X)$ *of Theorem 3.11.2 and* C_s *in* K*.*

Proof. We fix s in the interpolation range and we compare the two evaluations formula given in Theorem 3.7.5 and 3.11.2 at points of type (ε, s) such that ε is not trivial and its conductor is bigger than the p-part of the conductor of $\mathbf{f}$, ψ and χ.
We will then use the complex functional equation given in 3.4.2.
Recall that the factors at infinity that we need are

$$\begin{aligned}
\Gamma_{\mathbb{R},\infty}(s + 1 - m - \kappa_S) =& \pi^{-\frac{s+1-m+\kappa_S}{2}} \Gamma_{\infty}\left(\frac{s + 1 - m - \kappa_S}{2}\right), \\
\Gamma_{\mathbb{C},\infty}(s - m - 1 + k) =& 2^{2v - sd}(st - 2v)!\pi^{2v - d(s+1)}, \\
\Gamma_{\mathbb{R},\infty}(m + 3 - \kappa_S) =& \pi^{d\frac{s-m+\kappa_S-2}{2}} \Gamma_{\infty}\left(\frac{m + 2 - s - \kappa_S}{2}\right), \\
\Gamma_{\mathbb{C},\infty}(m - s + k) =& 2^{(s-m+1)d-k}\pi^{(s-m)d-k}(k + (m - s - 1)t)!,
\end{aligned}$$

where $\kappa_{\mathrm{S}} = 0, 1$ is congruent to $s+m$ modulo 2 (notice that we are evaluating the functional equation at $s+1$).

In the notation of Theorem 3.7.5 and 3.11.2 we have then $\beta = \alpha = \beta_0 = \alpha_0$ and

$$\begin{aligned}\frac{F(X)\prod_{\mathfrak{q}|\mathfrak{N}}\mathcal{E}_{\mathfrak{q}}^{-}(X)}{\mathcal{L}_p^{-}(X)} =& i^{(s-n)d}2^{3md-2sd+\frac{9}{2}d}D_{F,p}^{3s-3m+\frac{1}{2}}{D'_F}^{2s-2m-1}\times\\ &\times\frac{\mathcal{N}(\mathfrak{l})^{n-s}\mathcal{N}(\mathfrak{m})^{2s-2m-1}\mathcal{N}(\varpi^{\alpha})^{3s-3m-1}G(\chi\varepsilon\omega^{-s})}{G(\chi^{-1}\varepsilon^{-1}\omega^{s}\psi^{2})\chi\psi^{-1}\omega^{-s}\varepsilon(\varpi^{\alpha+e})\prod_{\mathfrak{p}}G(\chi_{\mathfrak{p}}^{-1}\psi_{\mathfrak{p}}\varepsilon^{-1}\omega^{s})}\\ &\times\frac{(\chi\varepsilon\omega^{-s})_0(\mathfrak{d}\mathfrak{c}\varpi^{\alpha})\chi\varepsilon\omega^{-s}\psi^{-1}(\mathfrak{d}\mathfrak{m}\varpi^{\alpha}2^{-1})}{(\chi^{-1}\varepsilon^{-1}\omega^{s}\psi^{2})_0(\mathfrak{d}\mathfrak{c}_0\varpi^{\alpha})\chi^{-1}\varepsilon^{-1}\omega^{s}\psi(\mathfrak{d}\mathfrak{m}\varpi^{\alpha}2^{-1})}\\ &\times\frac{\Gamma_{\mathbb{R},\infty}(s+1-m-\kappa_S)\Gamma_{\mathbb{C},\infty}(s-m-1+k)}{\Gamma_{\mathbb{R},\infty}(m+3-\kappa_S)\Gamma_{\mathbb{C},\infty}(m-s+k)}\\ &\times\frac{L(s+1,\mathrm{Sym}^2(\mathbf{f}),\chi^{-1}\varepsilon^{-1}\omega^{s})}{L(2m-s+2,\mathrm{Sym}^2(\mathbf{f}^c),\chi\varepsilon\omega^{-s})}.\end{aligned}$$

The occurence of the quotient of real gamma factors comes from the expression of the period $\Omega(\eta^{-1}, m-s)$ in the evaluation formula of Proposition 3.11.1 (noticing that if $\kappa = 0, 1$ is such that $\eta(-1) = (-1)^{\kappa}$, we have $\kappa = 1 - \kappa_{\mathrm{S}}$). As we have

$$\frac{\Lambda(s+1,\mathrm{Sym}^2(\mathbf{f}),\chi^{-1}\varepsilon^{-1}\omega^{s})}{\Lambda(2m-s+2,\mathrm{Sym}^2(\mathbf{f}^c),\chi\varepsilon\omega^{-s})} = \varepsilon(s-m,\hat{\pi}(\mathbf{f}),\chi^{-1}\varepsilon^{-1}\omega^{s}\psi)$$

we can use Lemma 3.4.6 and after noticing that (in the notation of Lemma 3.4.6) $\nu = \psi_{\mathfrak{q}}$ we obtain that the only part on the right hand side which depends on ε or p is

$$\frac{(\chi\psi^{-1}\varepsilon\omega^{-s}\psi)_p^4(\varpi^{\alpha+e})\mathcal{N}(\varpi^{\alpha_1+\alpha_2+\alpha_3})^{s-m}}{\chi_p(\varpi^{\alpha_1+e})\psi_p^2\chi_p^{-1}(\varpi^{\alpha_2+e})\psi_p\chi_p^{-1}(\varpi^{\alpha_3+e})G(\chi_p^{-1})G(\psi_p^2\chi_p^{-1})G(\psi_p\chi_p^{-1})}$$

where α_1 (resp. α_2, α_3) is the conductor of χ_p (resp. $\chi_p\psi_p^{-2}$, $\chi_p\psi_p^{-1}$). If we choose ϖ_p such that $\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}}) = \psi_{\mathfrak{p}}(\varpi_{\mathfrak{p}}) = 1$ and $\mathcal{N}(\varpi_p^e) = p^d$, we obtain that this quantity is independent of ε and we are done. $\square$

The main theorem of this section is the following;

Theorem 3.11.4. *Let $\mathbf{f} = \mathbf{f}_P$ be a nearly ordinary form of Nebentypus ψ and weight $k \geq 2t$ which we decompose as $k = m + 2t - 2v$. Let χ be a Hecke character such that $\chi_{\sigma}(-1) = 1$ for all $\sigma|\infty$. Suppose that $\mathbf{f}$ has not CM by $\chi\psi^{-1}$ (resp. has CM), then we have a a formal series $F(X, \mathbf{f}, \chi)$ in $\mathcal{O}[[X]]\left[\frac{1}{p}\right]$ (resp. $\mathcal{O}[[X]]\left[\frac{1}{p}, \frac{1}{1+X-u^{m+1}}\right]$) such that for all finite-order character ε of $1 + p\mathbb{Z}_p \cong u^{\mathbb{Z}_p}$, of conductor p^{α_0}, and $s \in [m - k_0 + 2, m]$ with $n \equiv s$ we have*

$$\begin{aligned}F(\varepsilon(u)u^s-1,\mathbf{f},\chi) =& i^{(s+1)d+k}2^{2v-(s+m)d+nd-\frac{s+n}{2}d}(st-2v)!D_F^{s-\frac{3}{2}}\times\\ & G(\chi\varepsilon\omega_0^{-s})\eta^{-1}(\mathfrak{m}\varpi^{\alpha_0}\mathfrak{d}2^{-1})\chi\varepsilon\omega_0^{-s}(\mathfrak{d}\mathfrak{c}\varpi^{\alpha_0})\times\\ & \frac{\psi(\mathfrak{d}\mathfrak{m}\varpi^{\beta})\mathcal{N}(\mathfrak{l})^{-2[s/2]}\mathcal{N}(\varpi)^{(s+1)\beta-\alpha_0}}{\lambda(T(\varpi^{2\beta-\alpha}))W'(\mathbf{f})S(P)\prod_{\mathfrak{p}}\frac{\eta\nu(\mathfrak{d}_{\mathfrak{p}})}{|\eta\nu(\mathfrak{d}_{\mathfrak{p}})|}\prod_J G(\nu)}\times\\ & E_1(s+1)E_2(s+1)\frac{L(s+1,\mathrm{Sym}^2(\mathbf{f}),\chi^{-1}\varepsilon^{-1}\omega^{s})}{(2\pi)^{ds}\Omega(\mathbf{f})}.\end{aligned}$$

for $\eta = \omega^s\varepsilon^{-1}\chi_1^{-1}\chi^{-1}\psi\psi^{-2}$ and, we recall, α such that ϖ^{α} is the conductor of $\mathbf{f}$ and β such that ϖ^{β} is divisible by the l.c.m. of ϖ^{α} and ϖ^{α_0}.

Proof. Recall the functional equation of Proposition 3.11.3

$$F((X+1)u^s-1) = C_s\mathcal{L}_p^-((X+1)u^s-1)\prod_{\mathfrak{q}|\mathfrak{N}}\mathcal{E}_{\mathfrak{q}}^-((X+1)u^s-1)^{-1}.$$

We shall show that the possible poles of the left hand side and right hand side are disjoint (are reduce to $X = u^{m+1}-1$ if $\mathbf{f}$ has CM by $\psi^{-1}\chi$).
Take for example $u = p+1$. The possible poles of $F(X)$ are the zeros of $\prod_{\mathfrak{q}|\mathfrak{N}}\mathcal{E}_{\mathfrak{q}}^+(X)$, $\mathcal{E}_2(X)$ and $\Delta(X)$. We recall that

$$\begin{aligned}\mathcal{E}_2(X) =&\prod_{\mathfrak{q}|2}(1-\psi^2\chi^{-2}(\mathfrak{q})A_{\mathcal{N}(\mathfrak{q})}^{-2}(X)\mathcal{N}(\mathfrak{q})^{2m}),\\ \Delta(X) =&(1-\chi(c)\mathcal{N}(c)^{m+1}(1+X)^{-1}).\end{aligned}$$

Therefore the possible poles must be either

- $s = m$ (occurring in one of the factors $\mathcal{E}_{\mathfrak{q}}^+$ where neither $\lambda(T(\mathfrak{q}))$ nor $\alpha_{\mathfrak{q}}$ do not appear),
- or $s = m+1$ (occurring in the factor $\Delta(X)$),
- or s such that there exists a p^∞-root of unit ζ such that one the following occurs:

$$\begin{aligned}\psi^2\chi^{-1}(\mathfrak{q}){K_{\mathfrak{q}}}^{-2}\mathcal{N}(\mathfrak{q})^{2m+1}A_{\mathcal{N}(\mathfrak{q})}^{-1}(\zeta u^s-1) =&1,\\ \chi^{-1}(\mathfrak{q}){K_{\mathfrak{q}}}^{2}\mathcal{N}(\mathfrak{q})^{-1}A_{\mathcal{N}(\mathfrak{q})}^{-1}(\zeta u^s-1) =&1,\end{aligned}$$

  for $K_{\mathfrak{q}} = \lambda_{\mathfrak{q}}(T(\mathfrak{q}))$ or $\alpha_{\mathfrak{q}}$.

On the other hand, the possible poles of the right hand side are instead

- $s = m+1$ (occurring in one of the factors $\mathcal{E}_{\mathfrak{q}}^-$ where $\lambda(T(\mathfrak{q}))$ does not appear),
- or when there is a p^∞-root of unit such that as

$$\begin{aligned}\chi(\mathfrak{q}){K_{\mathfrak{q}}}^{-2}A_{\mathcal{N}(\mathfrak{q})}(\zeta u^s-1) =&1,\\ \psi^{-2}\chi(\mathfrak{q}){K_{\mathfrak{q}}}^{2}\mathcal{N}(\mathfrak{q})^{-2-2m}A_{\mathcal{N}(\mathfrak{q})}(X)(\zeta u^s-1) =&1,\end{aligned}$$

  for $K_{\mathfrak{q}} = \lambda_{\mathfrak{q}}(T(\mathfrak{q}))$ or $\alpha_{\mathfrak{q}}$.

From Weierstrass preparation theorem, we know that there is only a finite number of possibilities of the above cases.
We proceed now exactly as in [DD97, §3.1]. Let $\mathfrak{q}$ be a prime ideal such that

$$\chi(\mathfrak{q})\lambda(T(\mathfrak{q}))^{-2}A_{\mathcal{N}(\mathfrak{q})}(\zeta u^m-1) = 1$$

and $\mathfrak{q}'$ a prime ideal such that $\mathcal{E}_{\mathfrak{q}'}$ has a zero at $s = m$, $\varepsilon(u) = \zeta$. Write $\langle\mathcal{N}(\mathfrak{q})\rangle = u^{z_{\mathfrak{q}}}$, so

$$A_{\mathcal{N}(\mathfrak{q})}(X) = (X+1)^{z_{\mathfrak{q}}}.$$

Recall that $|\lambda_{\mathfrak{q}}(T(\mathfrak{q}))|_{\mathbb{C}}^2 = \mathcal{N}(\mathfrak{q})^{m+1}$. In particular, we obtain the following relations: $|u^m|_{\mathbb{C}} = \mathcal{N}(\mathfrak{q}')^m$ and $|u^m|_{\mathbb{C}} = \mathcal{N}(\mathfrak{q})^{m+1}$. But this is a contradiction as $|u|_{\mathbb{C}} \neq 1$. The other cases are similar.
The case $s = m+1$ has already been treated in Proposition 3.7.7, and we can then conclude. ☐

3.12 Holomorphicity in many variables

In this section we will show that the many-variable p-adic L-function constructed in Theorem 3.7.2 is holomorphic (when the family has not complex multiplication). We adapt to the totally real case the strategy of proof of Hida [Hid90, §6].

Fix a family of $\mathbf{I}$-adic eigenforms $\mathbf{F}$ and let λ be the structural morphism as in Section 3.7; let us denote by ψ the restriction of λ to the torsion of $\mathrm{Cl}_{\mathfrak{N}}(p^{\infty})$. Before defining the functions interpolating the missing Euler factors, we have to show that we can interpolate p-adically the correct Satake parameter at the primes $\mathfrak{q}$ at which $\mathbf{F}$ is not minimal.

Let $\mathbf{F}_{\mathfrak{q}}$ be a twist of $\mathbf{F}$ by a character $\eta_{\mathfrak{q}}$ such that $\mathbf{F}_{\mathfrak{q}}$ is minimal at $\mathfrak{q}$ and primitive outside p. This family corresponds to an $\mathbf{I}_{\mathfrak{q}}$-family of eigenforms, where $\mathbf{I}_{\mathfrak{q}}$ is finite flat over the Iwasawa algebra in $d+1+\delta$ variables (here δ is the default of Leopoldt's conjecture for p and F).

Since we have $\lambda_{\mathfrak{q}}(T(\mathfrak{q}')) = \eta_{\mathfrak{q}}(\mathfrak{q}')\lambda(T(\mathfrak{q}'))$ for almost all prime ideals $\mathfrak{q}'$, we see that, after possibly enlarging the coefficients ring $\mathcal{O}$, we have $\mathbf{I}_{\mathfrak{q}} = \mathbf{I}$. For all arithmetic points P of $\mathrm{Spec}(\mathbf{I})$, we have then an element $\lambda_{\mathfrak{q}}(T(\mathfrak{q})) \in \mathbf{I}$ such that the values $\lambda_{\mathfrak{q}}(T(\mathfrak{q}))$ mod P are the Hecke eigenvalues, still denoted by $\lambda_{\mathfrak{q}}(T(\mathfrak{q}))$, of a form $\mathbf{f}_P$ which is minimal at $\mathfrak{q}$. For the prime ideal $\mathfrak{q}$ for which $\mathbf{F}$ is not minimal, let us denote by $\alpha_{\mathfrak{q}}$ and $\beta_{\mathfrak{q}}$ the two roots of the Hecke polynomial at $\mathfrak{q}$. After possibly enlarging $\mathbf{I}$ by a quadratic extension, we may assume that $\alpha_{\mathfrak{q}}$ belongs to $\mathbf{I}$.

We can now define the functions interpolating the missing Euler factors; as in the proof of Theorem 3.7.2 we use X (resp. Y) to denote a variable on the free part of $\mathbb{Z}_p^{\times}$ inside $\mathrm{Cl}_{\mathfrak{N}}(p^{\infty})$ (resp. $\mathrm{Cl}_{\mathfrak{N}}(p^{\infty})$ embedded in $\mathbf{G} = \mathrm{Cl}_{\mathfrak{N}}(p^{\infty}) \times \mathfrak{r}_p^{\times}$).

We recall that in Section 3.4.2 we partitioned the set of primes dividing $\mathfrak{N}$ in the four subsets *i)*, *ii)*, *iii)*, *iv)*. If $\mathfrak{q}$ belongs to cases *i)* or *ii)* we have two possibilities, according to $\lambda(T(\mathfrak{q})) = 0$ or not. Only for these two cases, we mean $\chi(\mathfrak{q}) = 0$ if χ is ramified. In the first case, we put

$$\begin{aligned}
\mathcal{E}^{+}_{\mathfrak{q}}(X,Y) = & \left(1 - \chi^{-1}(\mathfrak{q})\alpha_{\mathfrak{q}}^{2}\mathcal{N}(\mathfrak{q})^{-1}A^{-1}_{\mathcal{N}(\mathfrak{q})}(X)\right)\left(1 - \psi\chi^{-1}(\mathfrak{q})A_{\mathcal{N}(\mathfrak{q})}(Y)A^{-1}_{\mathcal{N}(\mathfrak{q})}(X)\right) \times \\
& \left(1 - \psi^{2}\chi^{-1}(\mathfrak{q})\alpha_{\mathfrak{q}}^{-2}\mathcal{N}(\mathfrak{q})A^{2}_{\mathcal{N}(\mathfrak{q})}(Y)A^{-1}_{\mathcal{N}(\mathfrak{q})}(X)\right), \\
\mathcal{E}^{-}_{\mathfrak{q}}(X,Y) = & \left(1 - \psi^{-2}\chi(\mathfrak{q})\alpha_{\mathfrak{q}}^{2}\mathcal{N}(\mathfrak{q})^{-2}A^{-2}_{\mathcal{N}(\mathfrak{q})}(Y)A_{\mathcal{N}(\mathfrak{q})}(X)\right)\left(1 - \chi(\mathfrak{q})\alpha_{\mathfrak{q}}^{-2}A_{\mathcal{N}(\mathfrak{q})}(X)\right) \times \\
& \left(1 - \psi^{-1}\chi(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-1}A^{-1}_{\mathcal{N}(\mathfrak{q})}(Y)A_{\mathcal{N}(\mathfrak{q})}(X)\right),
\end{aligned}$$

while if $\lambda(T(\mathfrak{q})) \neq 0$

$$\begin{aligned}
\mathcal{E}^{+}_{\mathfrak{q}}(X,Y) = & \left(1 - \psi\chi^{-1}(\mathfrak{q})A_{\mathcal{N}(\mathfrak{q})}(Y)A^{-1}_{\mathcal{N}(\mathfrak{q})}(X)\right)\left(1 - \psi^{2}\chi^{-1}(\mathfrak{q})\lambda_{\mathfrak{q}}(T(\mathfrak{q}))^{-2}\mathcal{N}(\mathfrak{q})A^{2}_{\mathcal{N}(\mathfrak{q})}(Y)A^{-1}_{\mathcal{N}(\mathfrak{q})}(X)\right), \\
\mathcal{E}^{-}_{\mathfrak{q}}(X,Y) = & \left(1 - \psi^{-1}\chi(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-1}A^{-1}_{\mathcal{N}(\mathfrak{q})}(Y)A_{\mathcal{N}(\mathfrak{q})}(X)\right)\left(1 - \chi(\mathfrak{q})\lambda_{\mathfrak{q}}(T(\mathfrak{q}))^{-2}A_{\mathcal{N}(\mathfrak{q})}(X)\right).
\end{aligned}$$

If $\mathfrak{q}$ is in case *iv)*, recall that there are several subcases; if $(\psi\chi)^2$ is ramified, we do not need to define any extra factors.

If $\psi\chi$ is unramified, we define

$$\begin{aligned}
\mathcal{E}^{+}_{\mathfrak{q}}(X,Y) &= 1 + \psi\chi^{-1}(\mathfrak{q})A_{\mathcal{N}(\mathfrak{q})}(Y)A^{-1}_{\mathcal{N}(\mathfrak{q})}(X), \\
\mathcal{E}^{-}_{\mathfrak{q}}(X,Y) &= 1 + \chi\psi^{-2}(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-1}A^{-1}_{\mathcal{N}(\mathfrak{q})}(Y)A_{\mathcal{N}(\mathfrak{q})}(X).
\end{aligned}$$

For the remaining case, $(\psi\chi)^2$ unramified and $\psi\chi$ ramified, recall that in Section 3.4.2 we have defined two characters λ_i, $i = 1, 2$, such that $\psi\chi\lambda_i$ is unramified. The extra factor we need to define are

$$\begin{aligned}
\mathcal{E}^{+}_{\mathfrak{q},i}(X,Y) &= 1 - \psi\chi^{-1}\lambda_i(\mathfrak{q})A_{\mathcal{N}(\mathfrak{q})}(Y)A_{\mathcal{N}(\mathfrak{q})}(X)^{-1}, \\
\mathcal{E}^{-}_{\mathfrak{q},i}(X,Y) &= 1 - \chi\psi^{-1}\lambda_i(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-1}A^{-1}_{\mathcal{N}(\mathfrak{q})}(Y)A_{\mathcal{N}(\mathfrak{q})}(X),
\end{aligned}$$

and then

$$\mathcal{E}_{\mathfrak{q}}^{+}(X,Y) = \prod_{i \text{ s.t.} \pi_{\mathfrak{q}} \cong \pi_{\mathfrak{q}} \otimes \lambda_i} \mathcal{E}_{\mathfrak{q},i}^{+}(X,Y),$$
$$\mathcal{E}_{\mathfrak{q}}^{-}(X,Y) = \prod_{i \text{ s.t.} \pi_{\mathfrak{q}} \cong \pi_{\mathfrak{q}} \otimes \lambda_i} \mathcal{E}_{\mathfrak{q},i}^{-}(X,Y).$$

We define

$$\mathcal{E}^{\pm}(X,Y) = \prod_{\mathfrak{q}|\mathfrak{N}} \mathcal{E}_{\mathfrak{q}}^{\pm}(X,Y).$$

We refer to Section 3.7 for all the notation used in the following theorem.

Theorem 3.12.1. *Fix an adelic character χ, such that $\chi_\sigma(-1) = 1$ for all $\sigma|\infty$ and fix $\mathbf{F}$, a $\mathbf{I}$-adic family of Hilbert eigenforms. If $\chi\psi^{-1}$ is imaginary quadratic, suppose that $\mathbf{F}$ has not (resp. has) complex multiplication by $\chi\psi^{-1}$; then we have a p-adic L-functions $H(P)L_p(Q,P)$ in $\mathcal{O}[[X]]\hat{\otimes}\mathbf{I}$ (resp. $\mathcal{O}[[X]]\hat{\otimes}\mathbf{I}[(1+X)-u(1+Y)^{-1}]$) such that, for all arithmetic points (Q,P) of type $(s_Q, \varepsilon_Q; m_P, \varepsilon_P, v_P, \varepsilon'_P)$, with $m_P - k_{P,0} + 2 \leq s_Q \leq m_P$ (for $k_{P,0}$ equal to the minimum of $k_{P,\sigma}$'s) such that ε'_P factors through $\mathrm{Cl}_{\mathfrak{N}}(p^\infty)$ the following interpolation formula holds*

$$L_p(Q,P) = C_1 E_1(Q,P) E_2(Q,P) \frac{2^d L(s_Q+1, \mathrm{Sym}^2(\mathbf{f}_P), \varepsilon_Q^{-1}\omega^{s_Q}\chi^{-1})}{(2\pi)^{ds}\Omega(\mathbf{f}_P)}.$$

It is clear that this is generalization of [Hid90, Theorem].
Remarks:

i) If Leopoldt's conjecture for F and p does not hold, then $L_p(Q,P)$ is holomorphic also in the case where $\mathbf{F}$ has CM by $\chi\psi^{-1}$.

ii) If $\mathcal{E}^{+}(X,Y)\mathcal{E}_2(X,Y) \equiv 0 \bmod (Q,P)$ and $s_Q = m_P$, then we are in the case of a trivial zero and the formula above for this point is not of great interest.

iii) If we could show an holomorphic analogue of Theorem 3.7.2 *ii)*, we could prove Conjecture 3.1.2 for simple zeros without any assumption on the conductor.

Proof. We follow Hida's strategy [Hid90, §6]. We define

$$L_p(Q,P) = \frac{\mathcal{L}_p(Q,P)}{\mathcal{E}^{+}(X,Y)},$$

We shall denote $\mathcal{E}^{+}(X,Y)\mathcal{E}_2(X,Y)$ (resp. $\mathcal{E}^{+}(X,Y)$) by $A(Q,P)$ (resp. $B(Q,P)$). Let us define

$$\varepsilon_0(\hat{\pi}, \chi^{-1}\psi) = \prod_{\mathfrak{q}\nmid p} \varepsilon(0, \hat{\pi}_{\mathfrak{q}}, \psi_{\mathfrak{q}}\chi_{\mathfrak{q}}^{-1}),$$
$$\varepsilon(X,Y) = \frac{A_{\mathcal{N}(C_{\pi,\chi})}(X) A^3_{D'_F}(X)}{A_{\mathcal{N}(C_{\pi,\chi})}(Y) A^3_{D'_F}(Y)},$$
$$C(X,Y) = \frac{\mathcal{N}(\mathfrak{m}) A^2_{\mathcal{N}(\mathfrak{m})}(Y) A_{\mathcal{N}(\mathfrak{l})}(X) D'_F A^2_{D'_F}(Y) A^2_{\mathcal{N}(2)}(X)}{A^2_{\mathcal{N}(\mathfrak{m})}(Y) 2^{\frac{9}{2}d} A^2_{D'_F}(X) \mathcal{N}(\mathfrak{l})^n A^3_{\mathcal{N}(2)}(X)},$$

where $C_{\pi,\chi}$ is the conductor outside p of $\hat{\pi}\otimes\psi\chi^{-1}$, and, as above, D'_F is the prime-to-p part of the discriminant and $\mathfrak{m} = 2\mathfrak{l}\mathfrak{c}$ is an idèle representing *l.c.m.* $(\mathfrak{N}, 4, \mathfrak{c})$ where $\mathfrak{c}$ is the conductor of χ .
It is plain that $\varepsilon(X,Y)$ and $C(X,Y)$ are units in $\mathcal{O}[[X]]\hat{\otimes}\mathbf{I}$. We define

$$L'_p(Q,P) = i^{nd}C_sL_p(Q,P)\varepsilon_0\varepsilon(X,Y)C(X,Y).$$

Suppose that $P \in X(\mathbf{I})$ is such that $\mathbf{f}_P$ is not primitive at p, then m_P is in a fixed class modulo $p-1$ and ε_P is trivial. We see, using the formula in Proposition 3.11.3, that $C_sB(Q,P)L'_p(Q,P) \equiv \mathcal{L}_p^-(Q) \bmod P$, where $\mathcal{L}_p^-(Q)$ is the p-adic L-function of Theorem 3.11.2 for $\mathbf{f} = \mathbf{f}_P$.
Let us denote by R the localization of $\mathcal{O}[[X]]\hat{\otimes}\mathbf{I}$ at $((1+X) - u(1+Y))$. As $B(Q,P)L'_p(Q,P) \bmod P$ belongs to $\mathcal{O}[[X]][p^{-1}]$ (or to $\mathcal{O}[[X]][p^{-1}, (1+X-u^{m+1})^{-1}]$ if $\mathbf{F}$ has CM by $\chi\psi^{-1}$), arguing as in [Hid90, pag. 137], we see that there exists H' in $\mathbf{I}$ such that $H'(P)B(Q,P)L'_p(Q,P)$ belongs to R, (i.e. the possible denominators do not involve the variable X); from the construction in Theorem 3.7.2 we see that the choice $H' = H$, the congruence ideal of the family $\mathbf{F}$, works.
We know then that

$$R[B^{-1}(Q,P)H^{-1}(P)] \ni L'_p(Q,P) \stackrel{\times}{=} L_p(Q,P) \in R[A^{-1}(Q,P)H^{-1}(P)],$$

where $\stackrel{\times}{=}$ means *equality up to a unit.*
It remains to show that $A(Q,P)$ and $B(Q,P)$ are coprime. As in [Hid90, Lemma 6.2], we know that a prime factor of $A(Q,P)$ (resp. $B(Q,P)$) is of the form $(1+X) - \zeta(1+Y)$ (resp. $(1+X) - \zeta u^{-1}(1+Y)$), for ζ a p-power-order root of unit, or it is of the form $(1+X) - \alpha$ with α in $\mathbf{I}$ such that $\alpha = \zeta {K_{\mathfrak{q}}}^2\mathcal{N}(\mathfrak{q})^{-1} \bmod P$ or $\alpha = \zeta\lambda_{\mathfrak{q}}(T(\mathfrak{q}))^{-2}\mathcal{N}(\mathfrak{q})^{2m+1} \bmod P$ (resp. $\alpha = \zeta\lambda_{\mathfrak{q}}(T(\mathfrak{q}))^{-2} \bmod P$ or $\alpha = \zeta K_{\mathfrak{q}}^2\mathcal{N}(\mathfrak{q})^{-2m-2} \bmod P$), where ζ is a root of unit and $K_{\mathfrak{q}} = \lambda_{\mathfrak{q}}(T(\mathfrak{q}))$ or $\alpha_{\mathfrak{q}}$.
Let us denote by ϖ_K a uniformizer of $\mathcal{O}$, then $\mathcal{E}_{\mathfrak{q}}^{\pm}(X,Y) \not\equiv 0 \bmod \varpi_K$. By contradiction, suppose that it is not true, then we would obtain $1+X \equiv \alpha \bmod \varpi_K$. But this is not possible as X does not belong to $\mathbf{I}/\varpi_K$. If $(1+X)-\alpha$ is a factor of both $A(Q,P)$ and $B(Q,P)$, we must have $\alpha = \zeta u^{-1}(1+Y)$ (or $\alpha = \zeta(1+Y)$). Reasoning with complex absolute values as in the proof of Theorem 3.11.4, we see that this is a contradiction. The behavior at $((1+X) - u(1+Y))$ has been studied in Proposition 3.7.7 and we can conclude the first part of the theorem.
The interpolation formula at points (Q,P) for which $\mathcal{E}^+(X,Y)\mathcal{E}_2(X,Y) \equiv 0 \bmod (Q,P)$ is a consequence of Theorem 3.11.4. $\square$

Chapter 4

A formula for the derivative of the p-adic L-function of the symmetric square of a finite slope modular form

4.1 Introduction

The aim of this paper is to prove a conjecture of Benois on trivial zeros in the particular case of the symmetric square representation of a modular form whose associated automorphic representation at p is Steinberg.
We begin by recalling the statement of Benois' conjecture. Let $G_{\mathbb{Q}}$ be the absolute Galois group of $\mathbb{Q}$. We fix an odd prime number p and two embeddings

$$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p, \ \ \overline{\mathbb{Q}} \hookrightarrow \mathbb{C},$$

and we let $G_{\mathbb{Q}_p}$ be the absolute Galois group of $\mathbb{Q}_p$. Let

$$V : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$$

be a continuous, irreducible, p-adic Galois representation of $G_{\mathbb{Q}}$. We suppose that V is the p-adic realization of a pure motive $M_{/\mathbb{Q}}$ of weight 0. We can associate a complex L-function $L(M,s)$. Let $M^* = \mathrm{Hom}(M,-)(1)$ be the dual motive of M. Conjecturally, if M is not trivial, $L(M,s)$ is a holomorphic function on the whole complex plane satisfying a holomorphic functional equation

$$L(M,s)\Gamma(M,s) = \varepsilon(M,s)L(M^*,1-s)\Gamma(M^*,1-s),$$

where $\Gamma(M,s)$ denotes a product of Gamma functions and $\varepsilon(M,s) = \zeta N^s$, for N a positive integer and ζ a root of unit. We say that M is *critical* at $s = 0$ if neither $\Gamma(M,s)$ nor $\Gamma(M^*,1-s)$ have a pole at $s = 0$. In this case the complex value $L(M,s)$ is not forced to be 0 by the functional equation; we shall suppose, moreover, that $L(M,s)$ is not zero. Similarly, we can say that M is critical at an integer n if $s = 0$ is critical for $M(n)$.
Deligne [Del79] has defined a non-zero complex number $\Omega(M)$ (defined only modulo multiplication by a non zero algebraic number) depending only on the Betti and de Rham realizations of M, such that conjecturally

$$\frac{L(M,0)}{\Omega(M)} \in \overline{\mathbb{Q}}.$$

We now suppose that all the above conjectures are true for M and all its twists $M\otimes\varepsilon$, where ε ranges among the finite-order characters of $1+p\mathbb{Z}_p$. We suppose moreover that V is a semi-stable representation of $G_{\mathbb{Q}_p}$. Let $\mathbf{D}_{\mathrm{st}}(V)$ be the semistable module associated to V; it is a filtered (ϕ,N)-module, i.e. it is endowed with a filtration and with the action of two operators: a Frobenius ϕ and a monodromy operator N. We say that a filtered (ϕ,N)-sub-module D of $\mathbf{D}_{\mathrm{st}}(V)$ is regular if

$$\mathbf{D}_{\mathrm{st}}(V)=\mathrm{Fil}^0(\mathbf{D}_{\mathrm{st}}(V))\bigoplus D.$$

To these data Perrin-Riou [PR95] associates a p-adic L-function $L_p(s,V,D)$ which is supposed to *interpolate* the special values $\frac{L(M\otimes\varepsilon,0)}{\Omega(M)}$, for ε as above. In particular, it should satisfy

$$L_p(0,V,D)=E_p(V,D)\frac{L(M,0)}{\Omega(M)},$$

where $E_p(V,D)$ denotes a finite product of Euler-type factors, corresponding to a subset of the eigenvalues of ϕ on D and on the dual regular submodule D^* of $\mathbf{D}_{\mathrm{st}}(V^*)$ (see [Ben13, §0.1]).
It may happen that some of these Euler factors vanish. In this case, we say that we are in the presence of a *trivial zero*. When trivial zeros appear, we would like to be able to retrieve information about the special value $\frac{L(M,0)}{\Omega(M)}$ from the p-adic derivative of $L_p(s,V,D)$.
Under certain suitable hypotheses (denoted by $\mathbf{C1}-\mathbf{C4}$ in [Ben10, §0.2]) on the representation V, Benois states the following conjecture:

Conjecture 4.1.1. *[Trivial zeros conjecture] Let e be the number of Euler-type factors of $E_p(V,D)$ which vanish. Then*

$$\lim_{s\to 0}\frac{L_p(V,D,s)}{s^e e!}=\mathcal{L}(V^*,D^*)E^*(V,D)\frac{L(V,0)}{\Omega(V)}.$$

Here $\mathcal{L}(V^*,D^*)$ is a non-zero number defined in term of the cohomology of the (ϕ,Γ)-module associated with V.

We remark that the conjectures of Bloch and Kato tell us that the aforementioned hypotheses $\mathbf{C1}-\mathbf{C4}$ in [Ben10] are a consequence of all the assumptions we have made about M. In the case V is ordinary this conjecture has already been stated by Greenberg in [Gre94b]. In this situation, the $\mathcal{L}$-invariant can be calculated in terms of the Galois cohomology of V. Conjecturally, the $\mathcal{L}$-invariant is non-zero, but even in the cases when $\mathcal{L}(V,D)$ has been calculated it is hard to say whether it vanishes or not.
We now describe the Galois representation for which we will prove the above-mentioned conjecture.
Let f be a modular eigenform of weight $k\geq 2$, level N and Nebertypus ψ. Let K_0 be the number field generated by the Fourier coefficients of f. For each prime λ of K_0 the existence of a continuous Galois representatation associated to f is well-known

$$\rho_{f,\lambda}:G_{\mathbb{Q}}\to\mathrm{GL}_2(K_{0,\lambda}).$$

Let $\mathfrak{p}$ be the prime above p in K_0 induced by the above embedding $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}_p$, we shall write $\rho_f:=\rho_{f,\mathfrak{p}}$.
The adjoint action of GL_2 on the Lie algebra of SL_2 induces a three-dimensional representation of GL_2 which we shall denote by Ad. We shall denote by $\mathrm{Ad}(\rho_f)$ the 3-dimensional Galois representation obtained by composing Ad and ρ_f.
The L-function $L(s,\mathrm{Ad}(\rho_f))$ has been studied in [GJ78]; unless f has complex multiplication, $L(s,\mathrm{Ad}(\rho_f))$ satisfies the conjectured functional equation and the Euler factors at primes dividing the conductor of f are well-known. For each $s=2-k,\dots,0$, s even, we have that $L(s,\mathrm{Ad}(\rho_f))$ is critical à la Deligne and the algebraicity of the special values has been shown in [Stu80].

If $p \nmid N$, we choose a p-stabilization $\tilde{f}$ of f; i.e. a form of level Np such that f and $\tilde{f}$ have the same Hecke eigenvalues outside p and $U_p\tilde{f} = \lambda_p \tilde{f}$, where λ_p is one of the roots of the Hecke polynomial at p for f. From now on, we shall suppose that f is of level Np and primitive at N. We point out that the choice of a p-stabilization of f induces a regular sub-module D of $\mathbf{D}_{\mathrm{st}}(\rho_f)$. So, from now on, we shall drop the dependence on D in the notation for the p-adic L-function.
Following the work of many people [Sch88, Hid90, DD97], the existence of a p-adic L-function associated to $\mathrm{Ad}(\rho_f)$ when f is ordinary (i.e. $v_p(\lambda_p) = 0$) or when $2v_p(\lambda_p) < k-2$ is known.
In what follows, we shall not work directly with $\mathrm{Ad}(\rho_f)$ but with $\mathrm{Sym}^2(\rho_f) = \mathrm{Ad}(\rho_f) \otimes \det(\rho_f)$. For each prime l, let us denote by α_l and β_l the roots of the Hecke polynomial at l associated to f. We define

$$D_l(X) := (1-\alpha_l^2 X)(1-\alpha_l\beta_l X)(1-\beta_l^2 X).$$

For each Dirichlet character ξ we define

$$\mathcal{L}(s, \mathrm{Sym}^2(f), \xi) := (1-\psi^2\xi^2(2)2^{2k-2-2s}) \prod_l D_l(\xi(l)l^{-s})^{-1}.$$

This L-function differs from $L(s, \mathrm{Sym}^2(\rho_f) \otimes \xi)$ by a finite number of Euler factors at prime dividing N and for the Euler factor at 2. The advantage of dealing with this *imprimitive* L-function is that it admits an integral expression (see Section 4.3.3) as the Petersson product of f with a certain product of two half-integral weight forms. The presence of the Euler factor at 2 in the above definition is due to the fact that forms of half-integral weight are defined only for levels divisible by 4. This forces us to consider f as a form of level divisible by 4, thus losing one Euler factor at 2 if $\psi\xi(2) \neq 0$.
Let us suppose that $\lambda_p \neq 0$; then we know that f can be *interpolated* in a "Coleman family". Indeed, let us denote by $\mathcal{W}$ the weight space. It is a rigid analytic variety over $\mathbb{Q}_p$ such that $\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{cont}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$. In [CM98], Coleman and Mazur constructed a rigid-analytic curve $\mathcal{C}$ which is locally finite above $\mathcal{W}$ and whose points are in bijection with overconvergent eigenforms.
If f is a classical form of non-critical weight (i.e. if $v_p(\lambda_p) < k-1$), then there exists a unique irreducible component of $\mathcal{C}$ such that f belongs to it. We fix a neighbourhood $\mathcal{C}_F$ of f in this irreducible component, it gives rise to an analytic function $F(\kappa)$ which we shall call a family of eigenforms. Let us denote by $\lambda_p(\kappa)$ the U_p-eigenvalue of $F(\kappa)$. We know that $v_p(\lambda_p(\kappa))$ is constant on $\mathcal{C}_F$. For any k in $\mathbb{Z}_p$, let us denote by $[k]$ the weight corresponding to $z \mapsto z^k$. Then for all κ' above $[k']$ such that $v_p(\lambda_p(\kappa)) < k-1$ we know that $F(\kappa')$ is classical.
Let us fix an even Dirichlet character ξ. We fix a generator u of $1+p\mathbb{Z}_p$ and we shall denote by $\langle z \rangle$ the projection of z in $\mathbb{Z}_p^\times$ to $1+p\mathbb{Z}_p$. We prove the following theorem in Section 4.4.3:

Theorem 4.1.2. *We have a function $L_p(\kappa, \kappa')$ on $\mathcal{C}_F \times \mathcal{W}$, meromorphic in the first variable and of logarithmic growth $h = [2v_p(\lambda_p)] + 2$ in the second variable (i.e. $L_p(\kappa, [s])/\prod_{i=0}^h \log_p(u^{s-i}-1)$ is holomorphic on the open unit ball). For any point $(\kappa, \varepsilon(\langle z \rangle)z^s)$ such that κ is above $[k]$, ε is a finite-order character of $1+p\mathbb{Z}_p$ and s is an integer such that $1 \leq s \leq k-1$, we have the following interpolation formula*

$$L_p(\kappa, \varepsilon(\langle z \rangle)z^s) = C_{\kappa,\kappa'} E_1(\kappa,\kappa')E_2(\kappa,\kappa') \frac{\mathcal{L}(s, \mathrm{Sym}^2(F(\kappa)), \xi^{-1}\varepsilon^{-1}\omega^{1-s})}{\pi^s S(F(\kappa))W'(F(\kappa)) \langle F^\circ(\kappa), F^\circ(\kappa)\rangle}.$$

Here $E_1(\kappa,\kappa')$ and $E_2(\kappa,\kappa')$ are two Euler-type factors at p. We refer to Section 4.4.3 for the notations. Here we want to point out that this theorem fits perfectly within the framework of p-adic L-functions for motives and their p-adic deformations [CPR89, Gre94a, PR95].
Our first remark is that such a two variable p-adic L-function has been constructed in [Hid90] in the ordinary case and in [Kim06] in the non-ordinary case. Its construction is quite classical: first, one constructs a measure interpolating p-adically the half-integral weight forms appearing in the integral expression of the L-function, and then one applies a p-adic version of the Petersson product.

Unless $s = 1$, the half-integral weight forms involved are not holomorphic but, in Shimura terminology, *nearly holomorphic*. It is well known that nearly holomorphic forms can be seen as p-adic modular forms (see [Kat76, §5.7]).
In the ordinary case, we have Hida's ordinary projector which is defined on the whole space of p-adic modular forms and which allows us to project p-adic forms on a finite dimensional vector space where a p-adic Petersson product can be defined.
If f is not ordinary, the situation is more complicated; f is annihilated by the ordinary projector, and there exists no other projector which could possibly play the role of Hida's projector. The solution is to consider instead of the whole space of p-adic forms, the smaller subspace of *overconvergent* ones.
On this space U_p acts as a completely continuous operator, and elementary p-adic functional analysis allows us to define, for any given $\alpha \in \mathbb{Q}_{\geq 0}$, a projector to the finite dimensional subspace of forms whose slopes with respect to U_p are smaller than α. Then it is easy to construct a p-adic analogue of the Petersson product as in [Pan03].
The problem in our situation is that nearly holomorphic forms are not overconvergent. Kim's idea is to construct a space of *nearly holomorphic and overconvergent forms* which projects, thanks to a p-adic analogue of the holomorphic projector for nearly holomorphic forms, to the space of overconvergent forms. Unfortunately, some of his constructions and proofs are sketched-out, and we prefer to give a new proof of this result using the recent work of Urban.
In [Urb], an algebraic theory for nearly holomorphic forms has been developed; it allows this author to construct a space of *nearly overconvergent* forms in which all classical nearly holomorphic forms appear and where one can define an *overconvergent* projector to the subspace of overconvergent forms. This is enough to construct, as sketched above, the p-adic L-function.
We expect that the theory of nearly overconvergent forms will be very usefully applies for the conscturction of p-adic L-functions; as an example, we can give the generalization of the work of Niklas [Nik10] on values of p-adic L-function at non-critical integers to finite slope families, or the upcoming work of Eischen, Harris, Li and Skinner on p-adic L-functions for unitary groups.
A second remark is that for all weights such that $k > h$ we obtain, by specializing the weight variable, the p-adic L-functions constructed in [DD97]. They construct several distribution μ_i, for $i = 1, \ldots, k-1$, satisfying Kummer congruences and the p-adic L-function is defined via the Mellin transform. The μ_i define an h-admissible measure μ in the sense of Amice-Vélu; in this case the Mellin transform is uniquely determined once one knows μ_i for $i = 1, ..., h$.
If $k \leq h$, then the number of special values is not enough to determine uniquely an analytic one-variable function. Nevertheless, as in Pollack-Stevens [PS11], we can construct a well-defined one variable p-adic L-function for eigenforms such that $k \leq h$ (see Section 4.4.3).
Let κ_0 be a point of $\mathcal{C}_F$ above $[k_0]$, and $f := F(\kappa_0)$. We shall write

$$L_p(s, \mathrm{Sym}^2(f), \xi) := L_p(\kappa_0, [s]).$$

We now deal with the trivial zeros of this p-adic L-function. Let κ be above $[k]$ and suppose that $F(\kappa)$ has trivial Nebentypus at p, then either $E_1(\kappa, \kappa')$ or $E_2(\kappa, \kappa')$ vanishes when $\kappa'(u) = u^{k-1}$. The main theorem of the paper is:

Theorem 4.1.3. *Let f be a modular form of trivial Nebentypus, weight k_0 and conductor Np, N square-free, even and prime to p. Then Conjecture 4.1.1 (up to the non-vanishing of the $\mathcal{L}$-invariant) is true for $L_p(s, \mathrm{Sym}^2(f), \omega^{2-k_0})$.*

In this case, the form f is Steinberg at p and the trivial zero is brought by E_1. The proof of this theorem is the natural generalization of the (unpublished) proof of Greenberg and Tilouine in the ordinary case (which has already been generalized to the Hilbert setting in [Ros13a]).
When we fix $\kappa_0'(u) = u^{k_0}$ we see that $E_1(\kappa, \kappa_0')$ is an analytic function of κ. We can then find a factorization $L_p(\kappa, \kappa_0') = E_1(\kappa, \kappa_0') L_p^*(\kappa)$, where $L_p^*(\kappa)$ is an *improved* p-adic L-function in the sense of [GS93] (see Section

4.4.3 for the exact meaning). The construction of the improved p-adic L-function is similar to [HT01]; we substitute the convolution of two measures with the product of a modular form with an Eisenstein measure. We note that the two-variable p-adic L-function vanishes on the line $\kappa = [k]$ and $\kappa' = [k-1]$ (the *line of trivial zeros*) and we are left to follow the method of [GS93].
The hypotheses on the conductor are to ensure that $\mathcal{L}(s, \mathrm{Sym}^2(f))$ coincides with $L(s-k+1, \mathrm{Ad}(\rho_f))$. The same proof gives a proof of Conjecture 4.1.1 for $\mathrm{Sym}^2(f) \otimes \xi$ for many ξ, and f not necessarily of even weight. We refer to Section 4.5 for a list of such a ξ.
Recently, Dasgupta has shown Conjecture 4.1.1 for all weights in the ordinary case. He uses the strategy outlined in [Cit08].

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4.2 Nearly holomorphic modular forms

The aim of this section is to recall the theory of nearly holomorphic modular forms from the analytic and geometric point of view, and construct their p-adic analogous, the *nearly overconvergent* modular forms. We shall use them in Section 4.4 to construct a two variable p-adic L-function for the symmetric square, generalizing the construction of [Pan03]. We want to remark that, contrary to the situation of [Pan03], the theory of nearly overconvergent forms is *necessary* for the construction of the two variable p-adic L-function. We will also construct an eigenvariety parameterizing finite slope nearly overconvergent eigenforms. The main reference is [Urb]; we are very grateful to Urban for sharing this paper, from its earliest versions, with the author. We point out that there is nothing really new in this section; however, we shall give a proof of all the statements which we shall need in the rest of the paper in the attempt to make it self-contained. We will also emphasize certain aspects of the theory we find particularly interesting. For all the unproven propositions we refer to the aforementioned paper.

4.2.1 The analytic definition

Nearly-holomorphic forms for GL_2 have been introduced and extensively studied by Shimura. His definition is of analytic nature, but he succeeded in proving several algebraicity results. Later, Harris [Har85, Har86] studied them in terms of coherent sheaves on Shimura varieties.
Let Γ be a congruence subgroups of $\mathrm{GL}_2(\mathbb{Z})$ and k a positive integer. Let $\mathcal{H}$ be the complex upper-half plane, we let $\mathrm{GL}_2(\mathbb{Q})^+$ act on the space of $\mathcal{C}^\infty$ functions $f : \mathcal{H} \to \mathbb{C}$ in the usual way

$$f|_k\gamma(z) = \det(\gamma)^{k/2}(cz+d)^{-k}f(\gamma(z))$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma(z) = \frac{az+b}{cz+d}$. We now give the precise definition of nearly holomorphic form.

Definition 4.2.1. *Let $r \geq 0$ be an integer. Let $f : \mathcal{H} \to \mathbb{C}$ be a $\mathcal{C}^\infty$-function, we say that f is a nearly holomorphic form for Γ of weight k and degree r if*

i) for all γ in Γ, we have $f|_k\gamma = f$, *ii) there are holomorphic $f_i(z)$ such that*

$$f(z) = \sum_{i=0}^{r} \frac{1}{y^i} f_i(z),$$

for $y = \mathrm{Im} z$,

iii) f is finite at the cusps of Γ.

Let us denote by $\mathcal{N}_k^r(\Gamma, \mathbb{C})$ the space of nearly holomorphic forms of weight k and degree at most r for Γ. When $r = 0$, we will write $\mathcal{M}_k(\Gamma, \mathbb{C})$.
A simple calculation, as in the case of holomorphic modular forms, tells us that $k \geq 2r$.
Finally, let us notice that we can substitute condition $ii)$ by

$$\varepsilon^{r+1}(f) = 0$$

for ε the differential operator $-4y^2 \frac{\partial f}{\partial \overline{z}}$. If f belongs to $\mathcal{N}_k^r(\Gamma, \mathbb{C})$, then $\varepsilon(f)$ belongs to $\mathcal{N}_{k-1}^{r-1}(\Gamma, \mathbb{C})$.
We warn the reader that except for $i = r$, the f_i's are not modular forms.
Let us denote by δ_k the Maaß-Shimura differential operator

$$\begin{array}{rccc} \delta_k : & \mathcal{N}_k^r(\Gamma, \mathbb{C}) & \rightarrow & \mathcal{N}_{k+2}^{r+1}(\Gamma, \mathbb{C}) \\ & f & \mapsto & \frac{1}{2\pi i}\left(\frac{\partial}{\partial z} + \frac{k}{2yi}\right) f. \end{array}$$

For any integer s, we define

$$\begin{array}{rccc} \delta_k^s : & \mathcal{N}_k^r(\Gamma, \mathbb{C}) & \rightarrow & \mathcal{N}_{k+2s}^{r+s}(\Gamma, \mathbb{C}) \\ & f & \mapsto & \delta_{k+2s-2} \circ \cdots \circ \delta_k f. \end{array}$$

Let us denote by $E_2(z)$ the nearly holomorphic form

$$E_2(z) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n + \frac{1}{8\pi y}, \quad \left(\text{where } \forall n \geq 1 \sigma_1(n) = \sum_{d|n, d>0} d \right).$$

It belongs to $\mathcal{N}_2^1(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C})$. It is immediate to see that for any Γ and for any form $f \neq 0$ in $\mathcal{N}_2^1(\Gamma, \mathbb{C})$, it does not exist a nearly holomorphic form g such that $\partial_0 g(z) = f(z)$. This is an exception, as the following proposition, due to Shimura [Shi87, Lemma 8.2], tells us.

Proposition 4.2.2. *Let f in $\mathcal{N}_k^r(\Gamma, \mathbb{C})$ and suppose that $(k, r) \neq (2, 1)$. If $k \neq 2r$, then there exists a sequence $(g_i(z))$, $i = 0, \ldots, r$, where g_i is in $\mathcal{M}_{k-2i}(\Gamma, \mathbb{C})$ such that*

$$f(z) = \sum_{i=0}^{r} \delta_{k-2i}^i g_i(z),$$

while if $r = 2k$ there exists a sequence $(g_i(z))$, $i = 0, \ldots, r-1$, where g_i is in $\mathcal{M}_{k-2i}(\Gamma, \mathbb{C})$ and c in $\mathbb{C}^\times$ such that

$$f(z) = \sum_{i=0}^{r} \delta_{k-2i}^i g_i(z) + c\delta_2^{r-1} E_2(z).$$

Moreover, such a decomposition is unique.

The importance of such a decomposition is given by the fact that the various $\delta_{k-2i}^{i} g_i(z)$ are nearly holomorphic forms. This decomposition will be very useful for the study of the Hecke action on the space $\mathcal{N}_k^r(\Gamma, \mathbb{C})$.
We can define, as in the case of holomorphic modular forms, the Hecke operators as double coset operators. For all l positive integer, we decompose

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix} \Gamma = \cup_i \Gamma \alpha_i.$$

We define $f(z)|_k T_l = l^{\frac{k}{2}-1} \sum_i f(z)|_k \alpha_i$. We have the following relations

$$\begin{aligned} l\delta_k(f(z)|_k T_l) =& (\delta_k f)|_{k+2} T_l, \\ \varepsilon(f|_k T_l) =& l(\varepsilon f)|_{k-2} T_l. \end{aligned}$$

Lemma 4.2.3. *Let $f(z) = \sum_{i=0}^{r} \delta_{k-2i} g_i(z)$ in $\mathcal{N}_k^r(\Gamma)$ be an eigenform for T_l of eigenvalue $\lambda_f(l)$, then g_i is an eigenform for T_l of eigenvalue $l^{-i}\lambda_f(l)$.*

Proof. It is an immediate consequence of the uniqueness of the decomposition in the previous proposition and of the relation between δ_k and T_l. □

Following Urban, we give an alternative construction of nearly holomorphic forms as section of certain coherent sheaves. Such a description will allow us to define a notion of nearly holomorphic forms over any ring R.
Let $Y = Y(\Gamma)$ be the open modular curve of level Γ defined over $\mathrm{Spec}\,(\mathbb{Z})$, and let $\mathbf{E}$ be the universal elliptic curve over Y. Let us consider a minimal compactification $X = X(\Gamma)$ of Y and the Kuga-Sato compactification $\overline{\mathbf{E}}$ of $\mathbf{E}$. Let us denote by $\mathbf{p}$ the projection of $\overline{\mathbf{E}}$ to X and by ω the sheaf of invariant differential over X, i.e. $\omega = \mathbf{p}_* \Omega^1_{\overline{\mathbf{E}}/X}(\log(\overline{\mathbf{E}}/\mathbf{E}))$.
We define

$$\mathcal{H}^1_{\mathrm{dR}} = R^1 \mathbf{p}_* \Omega^{\bullet}_{\overline{\mathbf{E}}/X}(\log(\overline{\mathbf{E}}/\mathbf{E}));$$

it is the algebraic De Rham cohomology. Let us denote by $\pi : \mathcal{H} \to \mathcal{H}/\Gamma$ the quotient map, we have over the $\mathcal{C}^\infty$-topos of $\mathcal{H}$ the splitting

$$\pi^* \mathcal{H}^1_{\mathrm{dR}} \cong \omega \oplus \overline{\omega} \cong \omega \oplus \omega^\vee$$

Let us denote by $\pi^* \mathbf{E}$ the fiber product of $\mathcal{H}$ and $\mathbf{E}$ above Y. The fiber above $z \in \mathcal{H}$ is the elliptic curve $\mathbb{C}/(\mathbb{Z} + z\mathbb{Z})$. If we denote by τ a coordinate on $\mathbb{C}$, the first isomorphism is given in the basis $\mathrm{d}\tau$, $\mathrm{d}\overline{\tau}$, while the second isomorphism is induced by the Poincaré duality. Let us define

$$\mathcal{H}_k^r = \omega^{k-r} \otimes \mathrm{Sym}^r(\mathcal{H}^1_{\mathrm{dR}}).$$

The above splitting induces

$$\mathcal{H}_k^r \cong \omega^k \oplus \omega^{k-2} \oplus \cdots \oplus \omega^{k-2r}.$$

We have the Gauß-Manin connexion

$$\nabla : \mathrm{Sym}^k(\mathcal{H}^1_{\mathrm{dR}}) \to \mathrm{Sym}^k(\mathcal{H}^1_{\mathrm{dR}}) \otimes \Omega^1_{X/\mathbb{Z}[N^{-1}]}(\log(\mathrm{Cusp})).$$

We recall briefly how it is defined. For sake of notation, we shall write $\Omega^{\bullet}_{\tilde{\mathbf{E}}/X}$ for $\Omega^{\bullet}_{\tilde{\mathbf{E}}/X}(\log(\overline{\mathbf{E}}/\mathbf{E}))$ and $\Omega^{\bullet}_{X/\mathbb{Z}[N^{-1}]}$ for $\Omega^{\bullet}_{X/\mathbb{Z}[N^{-1}]}(\log(\mathrm{Cusp}))$. We shall denote by $\Omega^{\bullet,-i}_{\tilde{\mathbf{E}}/X}$ the complex $\Omega^{\bullet}_{\tilde{\mathbf{E}}/X}$ shifted by i degrees to the right. We have an injection

$$\mathbf{p}^* \Omega^1_{X/\mathbb{Z}[N^{-1}]} \hookrightarrow \Omega^1_{\tilde{\mathbf{E}}/X}$$

and consequently for avery i a map

$$\Omega^{\bullet,-i}_{\tilde{\mathbf{E}}/X} \otimes_{\mathcal{O}(\tilde{\mathbf{E}})} \mathbf{p}^*\Omega^i_{X/\mathbb{Z}[N^{-1}]} \to \Omega^{\bullet}_{\tilde{\mathbf{E}}/X}.$$

We shall denote by $\mathrm{Im}\left(\Omega^{\bullet,-i}_{\tilde{\mathbf{E}}/X} \otimes_{\mathcal{O}(\tilde{\mathbf{E}})} \mathbf{p}^*\Omega^i_{X/\mathbb{Z}[N^{-1}]}\right)$ the image of the above map. We have a short exact sequence

$$0 \to \Omega^{\bullet,-1}_{\tilde{\mathbf{E}}/X} \otimes_{\mathcal{O}(\tilde{\mathbf{E}})} \mathbf{p}^*\Omega^1_{X/\mathbb{Z}[N^{-1}]} \to \frac{\Omega^{\bullet}_{\tilde{\mathbf{E}}/X}}{\mathrm{Im}\left(\Omega^{\bullet,-2}_{\tilde{\mathbf{E}}/X} \otimes_{\mathcal{O}(\tilde{\mathbf{E}})} \mathbf{p}^*\Omega^2_{X/\mathbb{Z}[N^{-1}]}\right)} \to \Omega^{\bullet}_{\tilde{\mathbf{E}}/X} \to 0.$$

We apply the functor $R\mathbf{p}_*$ to obtain a long exact sequence of complexes of sheaves on X. We let

$$\nabla : R^1\mathbf{p}_*\Omega^{\bullet}_{\tilde{\mathbf{E}}/X} \to R^2\mathbf{p}_*\left(\Omega^{\bullet,-1}_{\tilde{\mathbf{E}}/X} \otimes_{\mathcal{O}(\tilde{\mathbf{E}})} \mathbf{p}^*\Omega^1_{X/\mathbb{Z}[N^{-1}]}\right) = R^1\mathbf{p}_*\Omega^{\bullet}_{\tilde{\mathbf{E}}/X} \otimes_{\mathcal{O}(X)} \Omega^1_{X/\mathbb{Z}[N^{-1}]}$$

be the connection homorphism from degree one to degree two (for the equality on the right hand side have used the projection formula). For any f in $\mathcal{O}_X$ and e in $R^1\mathbf{p}_*\Omega^{\bullet}_{\tilde{\mathbf{E}}/X}$ we have from the definition

$$\nabla(fe) = \mathrm{d}f \otimes e + f\nabla(e).$$

In particular, ∇ defines a connection. We have also the following exact sequence

$$0 \to \Omega^{\bullet,-2}_{\tilde{\mathbf{E}}/X} \otimes_{\mathcal{O}(\tilde{\mathbf{E}})} \mathbf{p}^*\Omega^2_{X/\mathbb{Z}[N^{-1}]} \to \frac{\Omega^{\bullet,-1}_{\tilde{\mathbf{E}}/X} \otimes_{\mathcal{O}(\tilde{\mathbf{E}})} \mathbf{p}^*\Omega^1_{X/\mathbb{Z}[N^{-1}]}}{\mathrm{Im}\left(\Omega^{\bullet,-3}_{\tilde{\mathbf{E}}/X} \otimes_{\mathcal{O}(\tilde{\mathbf{E}})} \mathbf{p}^*\Omega^3_{X/\mathbb{Z}[N^{-1}]}\right)} \to \Omega^{\bullet,-1}_{\tilde{\mathbf{E}}/X} \otimes_{\mathcal{O}(\tilde{\mathbf{E}})} \mathbf{p}^*\Omega^1_{X/\mathbb{Z}[N^{-1}]} \to 0.$$

We define

$$\nabla_1 : R^1\mathbf{p}_*\Omega^{\bullet}_{\tilde{\mathbf{E}}/X} \otimes_{\mathcal{O}(X)} \Omega^1_{X/\mathbb{Z}[N^{-1}]} \to R^1\mathbf{p}_*\Omega^{\bullet}_{\tilde{\mathbf{E}}/X} \otimes_{\mathcal{O}(X)} \Omega^2_{X/\mathbb{Z}[N^{-1}]}$$

to be the connecting morphism from degree two to degree three. From the construction we have $\nabla_1 \circ \nabla = 0$ and consequently ∇ is integrable. This connection is usually called the Gauß-Manin connexion. We have the following, more concrete, description of the Gauß-Manin connection. Let us denote by z a variable on $\mathcal{H}$ and let us write $\pi^*\mathbf{E}$ as $\mathbb{C}/(\mathbb{Z}+z\mathbb{Z})$. Let us fix the basis γ_1, γ_2 of $H_1(\pi^*\mathbf{E},\mathbb{Z})$ given by the paths from 0 to 1 and from 0 to z. The connection ∇ corresponds to the following action of $\frac{\mathrm{d}}{\mathrm{d}z}$ on $H^1_{dR}(\pi^*\mathbf{E})$;

$$\frac{\mathrm{d}}{\mathrm{d}z}\int_{\gamma_i}\omega = \int_{\gamma_i}\nabla\omega, \quad \forall\omega \in H^1_{\mathrm{dR}}(\pi^*\mathbf{E}), i = 1,2.$$

We shall denote by the same symbol and call it by the same name the connection induced on $\mathrm{Sym}^k(\mathcal{H}^1_{\mathrm{dR}})$. Recall the descending Hodge filtration on $\mathrm{Sym}^k\left(\mathcal{H}^1_{\mathrm{dR}}\right)$ given by

$$\mathrm{Fil}^{k-r}\left(\mathrm{Sym}^k\left(\mathcal{H}^1_{\mathrm{dR}}\right)\right) = \mathcal{H}^r_k.$$

In particular, we have

$$0 \to \omega^k \to \mathcal{H}^r_k \stackrel{\tilde{\varepsilon}}{\to} \mathcal{H}^{r-1}_{k-2} \to 0.$$

By definition, ∇ satisfies Griffiths transversality;

$$\nabla\mathrm{Fil}^{k-r}\left(\mathrm{Sym}^k\left(\mathcal{H}^1_{\mathrm{dR}}\right)\right) \subset \mathrm{Fil}^{k-r-1}\left(\mathrm{Sym}^k\left(\mathcal{H}^1_{\mathrm{dR}}\right)\right) \otimes \Omega^1_{X/\mathbb{Z}[N^{-1}]}(\log(\mathrm{Cusp})).$$

Recall the Kodaira-Spencer isomorphism $\Omega^1_{X/\mathbb{Z}[N^{-1}]}(\log(\mathrm{Cusp})) \cong \omega^{\otimes 2}$; then the map ∇ induces a differential operator

$$\tilde{\delta}_k : \mathcal{H}^r_k \to \mathcal{H}^{r+1}_{k+2},$$

We have the following proposition [Urb, Proposition 2.2.3]

Proposition 4.2.4. *We have a natural isomorphism $H^0\left(X, \mathcal{H}_k^r\right) \cong \mathcal{N}_k^r(\Gamma, \mathbb{C})$. Once Hecke correspondances are defined on $(X, \mathcal{H}_k^r)$, the above isomorphism is Hecke-equivariant.*

Proof. Let us denote $R_1\mathbf{p}_*\mathbb{Z} = \mathcal{H}om(R^1\mathbf{p}_*\mathbb{Z}, \mathbb{Z})$. Via Poincaré duality we can identify

$$\pi^*\mathcal{H}^1_{\mathrm{dR}} = \mathcal{H}om(R_1\mathbf{p}_*\mathbb{Z}, \mathcal{O}_{\mathcal{H}}),$$

where $\mathcal{O}_{\mathcal{H}}$ denote the sheaf of holomorphic functions on $\mathcal{H}$. For all $z \in \mathcal{H}$, we have

$$\pi^*\left(R_1\mathbf{p}_*\mathbb{Z}\right)_z = H_1(\mathbb{C}/(\mathbb{Z} + z\mathbb{Z}), \mathbb{Z}) = \mathbb{Z} + z\mathbb{Z}.$$

Let us denote by α resp. β the linear form in $\mathrm{Hom}(R_1\mathbf{p}_*\mathbb{Z}, \mathcal{O}_{\mathcal{H}})$ which at the stalk at z sends $a + bz$ to a resp. b. It is the dual basis of γ_1, γ_2. Let $\eta \in H^0(X, \mathcal{H}_k^r)$, we can write

$$\pi^*\eta = \sum_{i=0}^{r} f_i(z) \mathrm{d}\tau^{\otimes^{k-i}} \beta^{\otimes^i}.$$

We remark that we have $\beta = \frac{\mathrm{d}\tau - \mathrm{d}\overline{\tau}}{2iy}$. Using the Hodge decomposition of $\mathcal{H}^r_{k/\mathcal{C}^\infty}$ given above, we project $\pi^*\eta$ onto $\omega^{\otimes^k}$ and we obtain

$$f(z) = \sum_{i=0}^{r} \frac{f_i(z)}{(2iy)^i}.$$

It is easy to check that $f(z)$ belongs to $\mathcal{N}_k^r(\Gamma, \mathbb{C})$ and that such a map is bijective. $\square$

This proposition allows us to identify $\tilde{\varepsilon}$ with the differential operator ε and $\tilde{\delta}_k$ with the Maaß-Shimura operator δ_k.

We now give another description of the sheaf $\mathcal{H}_k^r$; for any ring R we shall denote by $R[X]_r$ the group of polynomial with coefficients in R of degree at most r. Let us denote by B the standard Borel of upper triangular matrices of SL_2. We have a left action of $B(R)$ over $\mathbb{A}^1(R) \subset \mathbb{P}^1(R)$ via the usual fractional linear transformations

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} .X = \frac{aX + b}{a^{-1}}.$$

We define then a right action of weight $k \geq 0$ of $B(R)$ on $R[X]_r$ as

$$P(X)|_k \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = a^k P\left(a^{-2}X + ba^{-1}\right).$$

If we see $P(X)$ as a function on $\mathbb{A}^1(R)$, then

$$P(X)|_k\gamma = a^k P\left(\begin{pmatrix} a^{-1} & b \\ 0 & a \end{pmatrix} .X\right).$$

We will denote by $R[X]_r(k)$ the group $R[X]_r$ endowed with this action of the Borel. We now use this representation of B to give another description of $\mathcal{H}_k^r$.
We can define a B-torsor $\mathcal{T}$ over Y_{Zar} which consists of isomorphism $\psi_U : \mathcal{H}^1_{DR/U} \cong \mathcal{O}(U) \oplus \mathcal{O}(U)$ such that on the first component it induces $\mathcal{O}(U) \cong \omega_{/U}$ and on the quotient it induces $\mathcal{O}(U) \cong \omega^\vee_{/U}$, for U a Zariski open of Y. That is, $\mathcal{T}$ is the set of trivialization of $\mathcal{H}^1_{\mathrm{dR}}$ which preserves the line spanned by a fixed invariant differential ω and the Poincaré pairing. We have a right action of B on such a trivialization given by

$$(\omega, \omega') \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = (a\omega, a^{-1}\omega' + b\omega).$$

We can define similarly an action of the Borel of $\mathrm{GL}_2(R)$ but this would not respect the Poincaré pairing. Then we define the product $\mathcal{T}\times^B R[X]_r(k)$, consisting of couples $(t, P(X))$ modulo the relation $(t\gamma, P(X)) \sim (t, P(X)|_k\gamma^{-1})$, for γ in B. It is isomorphic to $\mathcal{H}_k^r$ as R-sheaf over Y. In fact, a nearly holomorphic modular form can be seen as a function

$$f : \mathcal{T} \to R[X]_r(k)$$

which is B-equivariant. That is, f associates to an element $(E, \mu, \omega, \omega')$ in $\mathcal{T}$ (μ denotes a level structure) an element $f(E, \mu, \omega, \omega')(X)$ in $R[X]_r$ such that

$$f(E, \mu, a\omega, a^{-1}\omega' + b\omega) = a^{-k} f(E, \mu, \omega, \omega')(a^2X - ba).$$

We are now ready to introduce a *polynomial* q-expansion principle for nearly holomorphic forms. Let us pose $A = \mathbb{Z}\left[\frac{1}{N}\right]$; let $\mathrm{Tate}(q)$ be the Tate curve over $A[[q]]$, ω_{can} the canonical differential and α_{can} the canonical N-level structure.
We can construct as before (take $\mathrm{Tate}(q)$ and $A[[q]]$ in place of $\tilde{\mathbf{E}}$ and X) the Gauß-Manin connection ∇, followed by the contraction associated to the vector field $q\frac{\mathrm{d}}{\mathrm{d}q}$

$$\nabla\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right) : \mathcal{H}^1_{\mathrm{dR}}(\mathrm{Tate}(q)_{/A((q))}) \to \mathcal{H}^1_{\mathrm{dR}}(\mathrm{Tate}(q)_{/A((q))}).$$

We pose $u_{\mathrm{can}} := \nabla\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)(\omega_{\mathrm{can}})$. We remark that $(\omega_{\mathrm{can}}, u_{\mathrm{can}})$ is a basis of $\mathcal{H}^1_{\mathrm{dR}}(\mathrm{Tate}(q)_{/A((q))})$ and that u_{can} is horizontal for the Gauß-Manin connection (moreover u_{can} is a basis for the *unit root subspace*, defined by Dwork, which we will describe later). For any A-algebra R and f in $\mathcal{N}_k^r(\Gamma, R)$, we say that

$$f(q, X) := f(\mathrm{Tate}(q), \mu_{\mathrm{can}}, \omega_{\mathrm{can}}, u_{\mathrm{can}})(X) \in R[[q]][X]$$

is the *polynomial* q-expansion of f. If we take a form f in $\mathcal{N}_k^r(\Gamma, \mathbb{C})$ written in the form $\sum_i^r f_i(z)\frac{1}{(-4\pi i)^i}$ we obtain

$$f(q, X) = \sum_i^r f_i(q)X^i \quad (i.e.\ X \text{``}=\text{''} -\frac{1}{4\pi y}).$$

For example, we have

$$E_2(q, X) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)q^n - \frac{X}{2},$$

We have the following proposition [Urb, Proposition 2.3]

Proposition 4.2.5. *Let f be in $\mathcal{N}_k^r(\Gamma, R)$ and let $\varepsilon(f)$ in $\mathcal{N}_{r-1}^{k-2}(\Gamma, R)$. Then for all $(E, \mu, \omega, \omega')$ in $\mathcal{T}$ we have*

$$\varepsilon(f)(E, \mu, \omega, \omega') = \frac{\mathrm{d}}{\mathrm{d}X} f(E, \mu, \omega, \omega')(X).$$

Note that if $r!$ is not invertible in R, then ε is NOT surjective. We have that E_2 is defined over $\mathbb{Z}_p$ for $p \geq 5$. As $\varepsilon(2E_2(q, X)) = -1$ we have that $-2E_2(q, X)$ gives a section for the map $\varepsilon : \mathcal{H}^1_{\mathrm{dR}} \to \omega$.

Remark 4.2.6. *There is also a representation-theoretic interpretation of $y\delta_k$ in term of Lie operator and representation of the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$. More details can be found in [Bum97, §2.1, 2.2].*

4.2.2 Nearly overconvergent forms

In this section we give the notion of nearly overconvergent modular forms à la Urban. Let N be a positive integer and p a prime number coprime with N. Let X be $X(\Gamma)$ for $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$ and let X_{rig} the generic fiber of the associated formal scheme over $\mathbb{Z}_p$. Let A be a lifting of the Hasse invariant in characteristic 0. If $p \geq 5$, we can take $A = E_{p-1}$, the Eisenstein series of weight $p-1$. For all v in $\mathbb{Q}$ such that $v \in [0, \frac{p}{p+1}]$ we define $X_N(v)$ as the set of x in $X(\Gamma_1(N))_{rig}$ such that $|A(x)| \geq p^{-v}$. The assumption that $v \leq \frac{p}{p+1}$ is necessary to ensure the existence of the canonical subgroup of level p. Consequently, we have that $X_N(v)$ can be seen as an affinoid of X_{rig} via the map

$$u : (E, \mu_N) \mapsto (E, \mu_N, C),$$

where C is the canonical subgroup. Let us define $X(v) := u(X_N(v))$. We define X_{ord} as the ordinary multiplicative locus of X_{rig}, i.e. $X_{\mathrm{ord}} = X(0)$. For all v as above, $X(v)$ is a rigid variety and a strict neighborhood of X_{ord}.

We remark that the set of x in X_{rig} such that $|A(x)| \geq p^{-v}$ consists of two disjoint connected component, isomorphic via the Fricke involution, and that $X(v)$ is the connected component containing ∞. We define, following Katz [Kat73], the space of p-adic modular forms of weight k as

$$\mathcal{M}_k^{p-\mathrm{adic}}(N) = H^0(X_{\mathrm{ord}}, \omega^{\otimes^k}).$$

We say that a p-adic modular form f is *overconvergent* if f can be extended to a strict neighborhood of X_{ord}. That is, there exists $v > 0$ such that f belongs to $H^0(X(v), \omega^{\otimes^k})$. Let us define the space of overconvergent modular forms

$$\mathcal{M}_k^{\dagger}(N) = \varinjlim_{v>0} H^0(X(v), \omega^{\otimes^k}). \tag{4.2.7}$$

In the same way, we define the set of *nearly overconvergent* modular forms as

$$\mathcal{N}_k^{r,\dagger}(N) = \varinjlim_{v>0} H^0(X(v), \mathcal{H}_k^r).$$

The sheaf $\mathcal{H}_k^r$ is locally free and as the fact that $E_2(q, X)$ give a splitting of $\mathcal{H}^1_{\mathrm{dR}}$ we can find an isomorphism $H^0(X(v), \mathcal{H}_k^r) \cong \mathcal{O}(X(v))^M$. For $v' < v$, these isomorphisms are compatible with the restiction maps $X(v) \to X(v')$. The supremum norm on $X(v)$ induces a norm on each $H^0(X(v), \mathcal{H}_k^r)$ which makes this space a Banach module over $\mathbb{Q}_p$. This allows moreover to define an integral structure on $H^0(X(v), \mathcal{H}_k^r)$. For all $\mathbb{Z}_p$-algebra R, we shall denote by $\mathcal{M}_k^{\dagger}(N, R)$, $\mathcal{N}_k^{r,\dagger}(N, R)$ the global section of the previous sheaves when they are seen as sheaves over $X(v)_{/R}$.

We have a correspondence

$$\begin{tikzcd} & C_p \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X(v) & & X(v). \end{tikzcd}$$

On the non-compactified modular curve, over $\mathbb{Q}_p$, C_p is the rigid curve classifying quadruplets (E, μ_N, C, H) with $|A(E)| \geq p^{-v}$, μ_N a $\Gamma_1(N)$-structure, C the canonical subgroup and H a subgroup of $E[p]$ which intersects C trivially. The projections are explicitly given by

$$\begin{aligned} p_1(E, \mu_N, C, H) =& (E, \mu_N, C), \\ p_2(E, \mu_N, C, H) =& (E/H, Im(\mu_N), E[p]/H). \end{aligned}$$

We remark that the theory of canonical subgroups ensure us that if $v \leq \frac{1}{p+1}$ then $E[p]/H$ is the canonical subgroup of E/H (and the image of C modulo H, of course). The map p_2 induces an isomorphism $C_p \cong X\left(\frac{v}{p}\right)$.
We define the operator U_p on $H^0(X(v), \mathcal{H}_k^r)$ as the following map

$$H^0(X(v), \mathcal{H}_k^r) \to H^0(X(v), p_2^*\mathcal{H}_k^r) \stackrel{p^{-1}Trace(p_1)}{\to} H^0(X(v), \mathcal{H}_k^r).$$

The fact that p_2 is an isomorphism implies the well known property that U_p *improves overconvergence*.
We can construct correspondences as in [Pil13, §4] to define operators T_l for $l \nmid Np$ and U_l for $l \mid N$.
Let A be a Banach ring, and let $U : M_1 \to M_2$ be a continuous morphism of A-Banach modules. We pose

$$|U| = \sup_{m \neq 0} \frac{|U(m)|}{|m|}.$$

This norm induces a topology on the module of continuous morphisms of A-Banach modules. We say that an operator U is *of finite rank* if is a continuous morphism of A-Banach modules such that its image is of finite rank over A. We say that U is *completely continuous* if it is a limit of finite rank operators. Completely continous operators admit a Fredholm determinant [Ser62, Proposition 7].
We give to $H^0(X(v), \omega^{\otimes^k})$ the structure of Banach space for norm induced by the supremum norm on $X(v)$; the transition maps in 4.2.7 are completely continuos and we complete $M_k^\dagger(N)$ for this norm. It is known that U_p acts as a completely continuous operator on this completion; its Fredholm determinant is independent of v, for v big enough [Col97, Theorem B]. Similarly, we have that U_p is completely continuous on $\mathcal{N}_k^{r,\dagger}(N)$. Indeed, U_p is the composition of the restriction to $X\left(\frac{v}{p}\right)$ and a trace map.
On q-expansion, U_p amounts to

$$\sum_{i=0}^{r}\sum_{n} a_n^{(i)} q^n X^i \mapsto \sum_{i=0}^{r}\sum_{n} a_{pn}^{(i)} q^n p^i X^i.$$

We now recall that we have on $\mathcal{H}^1_{\mathrm{dR}/X_{\mathrm{ord}}}$ a splitting $\omega \oplus U$. Here U is a Frobenius stable line where the Frobenius is invertible. Some authors call this splitting the *unit root splitting*. It induces $\mathcal{H}^r_{k/X_{\mathrm{ord}}} = \omega^{\otimes^k} \oplus \cdots \oplus U^{\otimes^k} \otimes \omega^{\otimes^{k-2r}}$. We have then [Urb, Proposition 3.2.4]

Proposition 4.2.8. *The morphism*

$$\begin{array}{ccccc} H^0(X(v), \mathcal{H}_k^r) & \to & H^0(X_{\mathrm{ord}}, \mathcal{H}_k^r) & \to & H^0(X_{\mathrm{ord}}, \omega^{\otimes^k}) \\ f(X) & \mapsto & f(X)_{|X_{\mathrm{ord}}} & \mapsto & f(0) \end{array}$$

is injective and commutes with q-expansion.

Note that the injectivity of the composition is a remarkable result. A consequence of this is that every nearly overconvergent form has a unique degree r [Urb, Corollary 3.2.5].
We remark that we have two differential maps

$$\begin{array}{rccc} \varepsilon : & \mathcal{N}_k^{r,\dagger}(\Gamma) & \to & \mathcal{N}_{k-2}^{r-1,\dagger}(\Gamma), \\ \delta_k : & \mathcal{N}_k^{r,\dagger}(\Gamma) & \to & \mathcal{N}_{k+2}^{r+1,\dagger}(\Gamma). \end{array}$$

Both of them are induced by functoriality from the maps defined in Section 4.2.1 at the level of sheaves. We want to mention that Cameron in his PhD thesis [Cam11, Definition 4.3.6] gives an analogue of the Maaß-Shimura differential operator for rigid analytic modular forms on the Cerednik-Drinfeld p-adic upper

half plane. It would be interesting to compare his definition with this one.
The above mentioned splitting allows us to define a map

$$\Theta : \mathcal{M}^{\dagger}_{k}(N) \stackrel{\delta_k}{\to} \mathcal{N}^{1,\dagger}_{k+2}(N) \to \mathcal{M}^{p-\mathrm{adic}}_{k+2}(N)$$

which at level of q-expansion is $q\frac{\mathrm{d}}{\mathrm{d}q}$. We have the following application of Proposition 4.2.8

Corollary 4.2.9. *Let f be an overconvergent form, then Θf is not overconvergent.*

We have the following proposition [Urb, Lemma 3.3.4];

Proposition 4.2.10. *Let (k,r) different form $(2,1)$ and f in $\mathcal{N}^{r,\dagger}_{k}(N,R)$. If $k \neq 2r$, then there exist g_i, $i = 0,\dots,r$, in $M^{\dagger}_{k-2i}(N,R)$ such that*

$$f = \sum_{i=0}^{r} \delta_{k-2i} g_i,$$

while if $r = 2k$ there exists a sequence (g_i), $i = 0,\dots,r-1$, with each g_i in $M^{\dagger}_{k-2i}(N,R)$ and c in R such that

$$f = \sum_{i=0}^{r-1} \delta_{k-2i} g_i + c\delta_2^{r-1} E_2.$$

Moreover, such a decomposition is unique.

We conclude with a sufficient condition for a nearly overconvergent modular form to be classical;

Proposition 4.2.11. *Let k be a classical weight, f in $\mathcal{N}^{r,\dagger}_{k}(N)$ an eigenform for U_p of slope α. Then $r \leq \alpha$. If $\alpha < k-1+r$, then f is classical.*

Proof. The first part is a trivial consequence of the above formula for U_p acting on q-expansion. For the second part, the hypotheses of Proposition 4.2.10 are satisfied. We apply ε^r to f to see that g_r is of degree 0, slope $\alpha - r$ and weight $k-2r$. It is then known that g_r is classical. We conclude by induction on the degree. □

Let $\alpha \in \mathbb{Q}_{\geq 0}$ and $r \geq 0$ be a positive integer such that $r \leq \alpha$. We say that a positive integer k is a *non critical weight* with respect to α and r if $\alpha < k-1+r$.
In particular if $\alpha = 0$, we have $r = 0$. This should convince the reader of the fact that the ordinary projector is a p-adic analogue of the holomorphic projector.

4.2.3 Families

In this subsection we construct families of nearly overconvergent forms. We start recalling the construction of families of overconvergent modular forms as done in Andreatta-Iovita-Stevens [AIS12] and Pilloni [Pil13]. The authors of the first paper use p-adic Hodge theory to construct their families, while Pilloni's approach is more in the spirit of Hida's theory. We will follow in our exposition the article [Pil13].
Let us denote by $\mathcal{W}$ the weight space. It is a rigid analytic variety over $\mathbb{Q}_p$ such that $\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{cont}(\mathbb{Z}_p^{\times}, \mathbb{C}_p^{\times})$.
For all integer k, we denote the continuos homomorphism $z \mapsto z^k$ by $[k]$.
Let $\Delta = \mu_{p-1}$ if $p > 2$ (resp. $\Delta = \mu_2$ if $p = 2$) let $B(1,1^-)$ be the open unit ball centered in 1. It is known that $\mathcal{W}$ is an analytic space isomorphic to $\Delta \times B(1,1^-)$; let us denote by $\mathcal{A}(\mathcal{W})$ the ring of analytic function on $\mathcal{W}$. We define for t in $(0,\infty)$,

$$\mathcal{W}(t) := \left\{(\zeta,z) \in \mathcal{W}(\mathbb{C}_p) | |z-1| \leq p^{-t}\right\}.$$

Let Δ be the cyclic group of q-roots of unity. We define κ, the universal weight, as

$$\begin{array}{rccc} \kappa: & \mathbb{Z}_p^\times & \to & (\mathbb{Z}_p[\Delta][[S]])^\times \\ & (\zeta, z) & \mapsto & \tilde{\zeta}(1+S)^{\frac{\log_p(z)}{\log_p(u)}}, \end{array}$$

where $\tilde{\zeta}$ is the image of ζ via the tautological character $\Delta \to (\mathbb{Z}_p[\Delta])^\times$. We can see κ as a local coordinate on the open unit ball $\{1\} \times B(1, 1^-)$.
For any weight κ_0, both of the aforementioned papers construct an invertible sheaf ω^{κ_0} over $X(v)$ whose sections over $X(v)$ correspond to overconvergent forms of weight κ_0. This construction can be globalized over $\mathcal{W}(t)$ into a coherent sheaf ω^κ over $X(v) \times \mathcal{W}(t)$ (for suitable v and t) such that the corresponding sections will give rise to families of holomorphic modular forms.
We describe more in detail Pilloni's construction. Let n, v be such that $0 \leq v < \frac{1}{p^{n-2}(p+1)}$; there exists then a *canonical subgroup* H_n of level n over $X(v)$. It is possible to define a rigid variety $F_n^\times(v)$ above $X(v)$ whose $\mathbb{C}_p$-points are triplets $(x, y; \omega)$ where x is an element of $X(v)$ corresponding to an elliptic curve E_x, y a generator of H_n^D (the Cartier dual of H_n) and ω is an element of $e^*\Omega_{E_x/\mathbb{C}_p}$ (for e the unit section $X(v) \to E_x$) whose restriction to $e^*\Omega_{H_n/\mathbb{C}_p}$ is the image of y via the Hodge-Tate map [Pil13, §3.3]. Locally, $F_n^\times(v)$ is a trivial fibration of $X(v)$ in $p^{n-1}(p-1)$ balls.
On $F_n(v)^\times$ we have an action of $(\mathbb{Z}/p^n)^\times$. This induce an action of $\mathbb{Z}_p^\times$. For each t, there exist v and n satisfying the above condition such that any κ_0 in $\mathcal{W}(t)$ acts on $F_n^\times(v)$. Let us denote by $\pi_n(v)$ the projection from $F_n^\times(v)$ to $X(v)$; ω^{κ_0} is by definition the κ_0 eigenspace of $(\pi_n(v)_* \mathcal{O}F_n^\times(v))$ (which we shall denote by $\left(\pi_n(v)_* \mathcal{O}_{F_n^\times(v)}\right)\langle\kappa_0\rangle$) for the action of $\mathbb{Z}_p^\times$. If k is a positive integer, then $\omega^{[k]} = \omega^{\otimes k}$, for ω the sheaf defined in Section 4.2.1.
A family of overconvergent modular forms is then an element of

$$\mathcal{M}(N, \mathcal{A}(\mathcal{W}(t))) := \varinjlim_v H^0\left(X(v) \times \mathcal{W}(t), \omega^\kappa\right),$$

$$\omega^\kappa = \left(\pi_n(v)_* (\mathcal{O}_{F_n^\times(v)} \hat{\otimes} \mathcal{O}_W)\right)\langle\kappa\rangle.$$

The construction commutes to base change in the sense that for weights $\kappa_0 \in \mathcal{W}(t)(K)$ we have

$$\omega^\kappa \otimes_{\kappa_0} K = \omega^{\kappa_0}_{/K}.$$

The operator U_p defined in the previous section is completely continuous on $\mathcal{M}(N, \mathcal{A}(\mathcal{W}(t)))$. Let $Q_0(\kappa, T)$ be its Fredholm determinant; it is independent of v and belongs to $\mathbb{Z}_p[[\kappa]][[T]]$ [CM98, Theorem 4.3.1].
This definition includes the family of overconvergent modular forms à la Coleman. Let $\zeta^*(\kappa)$ be the p-adic ζ-function, we pose

$$\tilde{E}(\kappa) = \frac{\zeta^*(\kappa)}{2} + \sum_n \sigma_n^*(\kappa) q^n, \tag{4.2.12}$$

where $\sigma_n^*(\kappa) = \sum_{1 \leq d | n, (d,p)=1} \kappa(d) d^{-1}$, and

$$E(\kappa) = \frac{2}{\zeta^*(\kappa)} \tilde{E}(\kappa).$$

It is known that the zeros of $E(\kappa)$ are *far enough* from the ordinary locus [Col97, B1], in particular it exist v such $E(\kappa)$ is invertible on $X(v) \times \mathcal{W}(t)$. In [Col97, B4] a family of modular forms $F(\kappa)$ is defined as an element of $\mathcal{A}(\mathcal{W}(t))[[q]]$ such that for all $\kappa \in \mathcal{W}(t)$, we have $\frac{F(\kappa)}{E(\kappa)}$ in $H^0(X(v) \times \mathcal{W}(t), \mathcal{O}_{X(v) \times \mathcal{W}})$. The fact that $E(\kappa)$ is invertible induces an isomorphism

$$H^0(X(v) \times \mathcal{W}(t), \mathcal{O}_{X(v) \times \mathcal{W}}) \stackrel{\times E(\kappa)}{\longrightarrow} H^0\left(X(v) \times \mathcal{W}(t), \omega^\kappa\right).$$

Let us define the following coherent sheaf

$$\mathcal{H}^r_\kappa = \omega^{\kappa[-r]} \otimes \mathrm{Sym}^r(\mathcal{H}^1_{\mathrm{dR}});$$

we define then for all affinoid $\mathcal{U} \subset \mathcal{W}(t)$ the family of nearly overconvergent forms of degree r

$$\mathcal{N}^r(N, \mathcal{A}(\mathcal{U})) = \varinjlim_v H^0(X(v) \times \mathcal{U}, \mathcal{H}^r_\kappa \hat{\otimes} \mathcal{O}_\mathcal{U}).$$

We remark that we can choose v small enough such that $H^0(X(v) \times \mathcal{U}, \omega^{[-r]} \otimes \mathrm{Sym}^r(\mathcal{H}^1_{\mathrm{dR}})\hat{\otimes}\mathcal{O}(\mathcal{U}))$ is isomorphic via multiplication by $E(\kappa)$ to $H^0(X(v) \times \mathcal{U}, \mathcal{H}^r_\kappa \otimes \mathcal{O}(\mathcal{U}))$. We shall call the elements of the former space families of nearly overconvergent forms *à la Coleman*.
We can define $\mathcal{N}^\infty(N, \mathcal{A}(\mathcal{U}))$ as the completion of $\cup_r \mathcal{N}^r(N, \mathcal{A}(\mathcal{U}))$ with respect to the Frechet topology. For the interested reader, let us mention that there exist forms in $\mathcal{N}^\infty(N, \mathcal{A}(\mathcal{U}))$ whose polynomial q-expansion is no longer a polynomial in X but an effective formal series. Indeed, we can trivialize $\mathcal{H}^r_{\kappa/X(v)\times\mathcal{W}(t)}$ as $\oplus_{i=0}^r \omega^{\kappa[-2i]}$ and take a sequence of $f_r = (f_{r,0}, \dots, f_{r,r})$, f_r in $\mathcal{N}^r(N, \mathcal{A}(\mathcal{U}))$, such that $f_{r,i} = f_{r+1,i}$ and $f_{r,r}$ smaller and smaller for the norm induced by $X(v)$.
There is a sheaf-theoretic interpretation of $\mathcal{N}^\infty(N, \mathcal{A}(\mathcal{U}))$. Let $\mathcal{A}n(\mathbb{Z}_p)$ be the ring of analytic function on $\mathbb{Z}_p$ with values in $\mathcal{A}(\mathcal{U})$; we can define the vector bundle in Frechet space

$$\mathcal{H}^\infty_\kappa = \mathcal{T} \times^B \mathcal{A}n(\mathbb{Z}_p).$$

As in the rest of the paper we will work with nearly overconvergent forms of bounded slope, there is no particular interest in taking $r = \infty$, as we have already mentioned the degree gives a lower bound on the slopes which can appear. However, we think that the case $r = \infty$ could have some interesting applications, both geometric or representation-theoretic.
We can see that U_p acts completely continuously on $\mathcal{N}^r_\kappa(N, \mathcal{A}(\mathcal{W}(t)))$ using [Col97, Proposition A5.2], as it is defined via the correspondence C_p. We have on $\mathcal{N}^r(N, \mathcal{A}(\mathcal{W}(t)))$ an action of the Hecke algebra $\mathbb{T}^r(N, \mathcal{A}(\mathcal{W}(t)))$ generated by the Hecke operators T_l, for l coprime with Np, and U_l for l dividing Np. We will denote by $Q_r(\kappa, T)$ the Fredholm determinant of U_p on $\mathcal{N}^r_\kappa(\Gamma, \mathcal{A}(\mathcal{W}(t)))$. To lighten the notation, we will write sometimes $Q_r(T)$ for $Q_r(\kappa, T)$ if there is no possibility of confusion.

Lemma 4.2.13. *For any $t \in (0, \infty)$ and suitable v (see [Pil13, §5.1]) small enough and t big enough, $H^0(X(v) \times \mathcal{W}(t), \mathcal{H}^r_\kappa \otimes \mathcal{O}_\mathcal{W})$ is a direct factor of a potentially orthonormalizable $\mathcal{A}(\mathcal{W}(t))$-module (see [Buz07, page 7] for the definition of potentially orthonormalizable).*

Proof. The proof is exactly the same as [Pil13, Corollary 5.2], so we only sketch it. Let

$$M := H^0(X(v) \times \mathcal{W}(t), \mathcal{H}^r_\kappa \otimes \mathcal{O}_\mathcal{W}), \quad A := \mathcal{A}(\mathcal{W}(t)).$$

Let us denote by B the function ring of $X(v)$ and by B' the function ring of $(H^D_n)^\times$ above $X(v)$. We know that B' is an étale B-algebra of Galois group $(\mathbb{Z}/p^n\mathbb{Z})^\times$. As M is a direct summand of $M' = M \otimes_B B'$, it will be enough to show that the latter is potentially orthonormalizable. Let $(\mathcal{U}_i)_{i=1,\dots,I} \to (H^D_n)^\times$ be a finite cover by open sets such that for all i's $F^\times_n(v) \times_{X(v)} \mathcal{U}_i$ is a disjoint union of $p^{n-1}(p-1)$ copies of $\mathcal{U}_i$. The augmented Čech complex associated to this cover is then

$$0 \to M' \to M_1 \to \cdots M_I \to 0$$

and it is exact. Let $k \geq 1$ be an integer and $\underline{i}$ be a subset of $\{1, 2, \dots, I\}$ of cardinality k. By construction M_k is a sum of modules of the type $M' \hat{\otimes}_{B'} B_{\underline{i}}$ for $B_{\underline{i}} = \hat{\otimes}_{j \in \underline{i}} \mathcal{O}(\mathcal{U}_j)$ where the tensor product is taken over B'. By the choice of $\mathcal{U}_i$, each one of these modules is free of rank $r+1$ over $A \hat{\otimes} B_{\underline{i}}$. As $B_{\underline{i}}$ is potentially orthonormalizable over $\mathbb{Q}_p$ we know that $A \hat{\otimes} B_{\underline{i}}$ is potentially orthonomalizable over A. We can conclude by [Pil13, Lemma 5.1].

□

We can thus apply Buzzard's eigenvariety machinery [Buz07, Construction 5.7]. This means that to the data

$$(\mathcal{A}(\mathcal{W}(t)), \mathcal{N}^r(N, \mathcal{A}(\mathcal{W}(t))), \mathbb{T}^r(N, \mathcal{A}(\mathcal{W}(t))), U_p)$$

we can associate a rigid-analytic one-dimensional variety $\mathcal{C}^r(t)$. Let us denote by Z the zero-locus of $Q_r(T)$ on $\mathcal{W}(t) \times \mathbb{A}^1_{\mathrm{An}}$ (see [Urb, §3.4]). The rigid-analytic variety $\mathcal{C}^r(t)$ is characterized by the following properties.

- We have a finite map $\mathcal{C}^r(t) \to Z$.
- There is a cover of Z by affinoid Y_i such $X_i = Y_i \times_Z \mathcal{W}(t)$ is an open affinoid of $\mathcal{W}(t)$ and $Y_i \to X_i$ is finite.
- Above X_i we can write $Q_r(T) = R_r(T)S_r(T)$ with $R_r(T)$ a polynomial in T whose constant term is 1 and $S_r(T)$ power series in T coprime to $R_r(T)$.
- Let $R_r^*(T) = T^{\deg(R_r(T))} R_r(T^{-1})$. Above X_i we have a U_p-invariant decomposition

  $$\mathcal{N}^r(N, \mathcal{A}(X_i)) = \mathcal{N}^r(N, \mathcal{A}(X_i))^* \bigoplus \mathcal{N}^r(N, \mathcal{A}(X_i))',$$

  such that $R_r^*(U_p)$ acts on $\mathcal{N}^r(N, \mathcal{A}(X_i))'$ invertibly and on $\mathcal{N}^r(N, \mathcal{A}(X_i))^*$ is 0. Moreover the rank of $\mathcal{N}^r(N, \mathcal{A}(X_i))^*$ on $\mathcal{A}(X_i)$ is $\deg(R_r(T))$.
- There exists a coherent sheaf $\mathcal{N}^r(\widetilde{N, \mathcal{A}(\mathcal{W}}(t)))$ above $\mathcal{C}^r(t)$.
- To each K-point x of $\mathcal{C}^r(t) \times_Z Y_i$ above $\kappa(x) \in \mathcal{W}(t)$ corresponds a system of Hecke eigenvalues for $\mathbb{T}^r_\kappa(N, K)$ on $\mathcal{N}^r_{\kappa(x)}(Np, K)$ such that the U_p-eigenvalue is a zero of $R_r^*(T)$ (in particular it is not zero).
- To each K-point x as above, the fiber $\mathcal{N}^r(\widetilde{N, \mathcal{A}(\mathcal{W}}(t)))_x$ is the generalized eigenspace in $\mathcal{N}^r_{\kappa(x)}(Np, K)$ for the system of eigenvalues associated to x.

Taking the limit for t which goes to 0, we obtain the eigencurve $\mathcal{C}^r \to \mathcal{W}$. When $r = 0$ this is the Coleman-Mazur eigencurve which we shall denote by $\mathcal{C}$.
For a Banach module M, a completely continuous operator U and $\alpha \in \mathbb{Q}_{\geq 0}$, we define $M^{\leq \alpha}$ resp. $M^{> \alpha}$ as the subspace which contains all the generalized eigenspaces of eigenvalues of U of valuation less or equal than α resp. strictly bigger than α. Then the above discussion gives us the following proposition which is essentially all we need in what follows;

Proposition 4.2.14. *For all $\alpha \in \mathbb{Q}_{>0}$ we have the a direct sum decomposition*

$$\mathcal{N}^r(N, \mathcal{A}(\mathcal{U})) = \mathcal{N}^r(N, \mathcal{A}(\mathcal{U}))^{\leq \alpha} \bigoplus \mathcal{N}^r(N, \mathcal{A}(\mathcal{U}))^{> \alpha},$$

where $\mathcal{N}^r(N, \mathcal{A}(\mathcal{U}))^{\leq \alpha}$ is a finite dimensional, free Banach module over $\mathcal{A}(\mathcal{U})$. Moreover the projector on $\mathcal{N}^r(N, \mathcal{A}(\mathcal{U}))^{\leq \alpha}$ is given by a formal series in U_p which we shall denote by $\mathrm{Pr}^{\leq \alpha}$.

Remark 4.2.15. *As $\mathcal{N}^r(N, \mathcal{A}(\mathcal{U}))^{\leq \alpha}$ is of finite rank and $\mathcal{A}(\mathcal{U})$ is noetherian there exists v such that $\mathcal{N}^r(N, \mathcal{A}(\mathcal{U}))^{\leq \alpha} = H^0(X(v) \times \mathcal{U}, \mathcal{H}^r_\kappa \hat{\otimes} \mathcal{O}_{\mathcal{U}})^{\leq \alpha}$.*

If we want to consider forms with Nebentypus ψ whose p-part is non-trivial, we need to apply the above construction to an affinoid $\mathcal{U}$ of $\mathcal{W}$ where ψ is constant. This is because finite-order characters do not define Tate functions on $\mathcal{W}$.
It is well known that on a finite dimensional vector space over a complete field, all the norms are equivalent. In particular the overconvergent norm on $\mathcal{N}^r(N, \mathcal{A}(\mathcal{U}))^{\leq \alpha}$ is equivalent to sup-norm on the coefficients of the q-expansion. We call it the q-expansion norm; a unit ball for this norm defines a natural integral structure $\mathcal{N}^r(N, \mathcal{A}(\mathcal{U}))^{\leq \alpha}$, which coincides with the one defined in the previous subsection.
We now give a useful lemma.

Lemma 4.2.16. *Let f be a nearly overconvergent form in $\mathcal{N}_k^{r,\dagger}(N)$ and let $f^{\leq \alpha}$ be its projection to $\mathcal{N}_k^{r,\dagger}(N)^{\leq \alpha}$. If $f(q,X) \in p^n \mathbb{Z}_p[[q]][X]$, then $f^{\leq \alpha}(q,X) \in p^n \mathbb{Z}_p[[q]][X]$.*

Proof. Let f be as in the statement of the lemma, then we have $U_p f(q,X) \in p^n \mathbb{Z}_p[[q]][X]$. As $f^{\leq \alpha} = \mathrm{Pr}^{\leq \alpha} f$, we conclude. $\square$

Let $\mathcal{A}^0(\mathcal{U})$ be the unit ball in $\mathcal{A}(\mathcal{U})$. We have the following proposition which, roughly speaking, guarantees us that the limit for the q-expansion norm of nearly overconvergent forms of bounded slope is nearly overcovergent.

Proposition 4.2.17. *Let $F(\kappa) = \sum_{i=0}^r F_i(\kappa) X^i$, with $F_i(\kappa)$ in $\mathcal{A}^0(\mathcal{U})[[q]]$. Suppose that for a set $\{\kappa_i\}$ of $\overline{\mathbb{Q}}_p$-points of $\mathcal{U}$ which are dense we have $F(\kappa_i) \in \mathcal{N}_{\kappa_i}^r(N, \overline{\mathbb{Q}}_p)^{\leq \alpha}$. Then*

$$F(\kappa) \in \mathcal{N}^r(N, \mathcal{A}(\mathcal{U}))^{\leq \alpha}.$$

Proof. It is enough to show that for every κ_0 in $\mathcal{U}$, $F(\kappa_0)$ is nearly overconvergent (and the radius of overconvergence can be chosen independently of κ_0 by Remark 4.2.15).
We follow the proof of [Til06, Corollary 4.8]. We have for all κ in $\mathcal{U}$ the Eisenstein series $E(\kappa)$. It is know that $E(\kappa)$ has no zeros on $X(v)$ for $v > 0$ small enough.
We will write $f \equiv 0 \bmod p^n$ for $f(q,X) \in p^n \mathbb{Z}_p[[q]][X]$.
Let κ_0 be a L-point of $\mathcal{U}$, and fix a sequence of points κ_i, $i > 0$, of $\mathcal{U}$ such that κ_i converges to κ_0. In particular, $F(\kappa_i)$ converges to $F(\kappa_0)$ for the q-expansion topology. Let us consider the nearly overconvergent modular forms $G(\kappa_i) := \frac{F(\kappa_i)E(\kappa_0)}{E(\kappa_i)}$ of weight κ_0, we want to show that $G(\kappa_i)^{\leq \alpha}$ converge to $F(\kappa_0)$ in the q-expansion topology. This will prove that $F(\kappa_0)$ is nearly overconvergent because, as already said, in the space of nearly overceonvergent forms of slope bounded by α all the norms are equivalent.
If $|\kappa_i - \kappa_0| < p^{-n}$, we have $E(\kappa_i) \equiv E(\kappa_0) \bmod p^n$, hence $E(\kappa_i)^{-1} \equiv E(\kappa_0)^{-1} \bmod p^n$; consequently, it is clear that $G(\kappa_i) \equiv F(\kappa) \bmod p^n$. We apply Lemma 4.2.16 to the forms $G(\kappa_i) - F(\kappa_i)$ to see that $G(\kappa_i)^{\leq \alpha}$ is a sequence of overconvergent forms of weight κ_0 and bounded slope which converges to $F(\kappa_0)$ for the q-expansion topology.

$\square$

We remark that in [Pan03, §1] the author defines *rigid analytic nearly holomorphic modular forms* as elements of $\mathcal{A}(\mathcal{U})[[q]][X]$, which on classical points give classical nearly holomorphic forms. It would be interesting to compare his definition with the one here, especially understanding necessary and sufficient conditions to detect when a specialization at a non classical weight of a rigid analytic nearly holomorphic modular form is nearly overconvergent or not.

One application of the above proposition is that it allows us to define a Maaß-Shimura operator of weight κ as follows. Let us define

$$\log(\kappa) = \frac{\log_p(\kappa(u^r))}{\log_p(u^r)}$$

for u any topological generator of $1 + p\mathbb{Z}_p$ and r any integer big enough.
For any open affinoid $\mathcal{U}$ of $\mathcal{W}$ and κ_0 in $\mathcal{W}(\mathbb{C}_p)$, we define the κ_0-translate $\mathcal{U}\kappa_0$ of $\mathcal{U}$ as the composition $\mathcal{U} \to \mathcal{W}$ with $\mathcal{W} \stackrel{\times \kappa_0}{\to} \mathcal{W}$.

Proposition 4.2.18. *We have an operator*

$$\begin{array}{rccc} \delta_\kappa : & \mathcal{N}^r(N, \mathcal{A}(\mathcal{U}))^{\leq \alpha} & \to & \mathcal{N}^{r+1}(N, \mathcal{A}(\mathcal{U}[2]))^{\leq \alpha+1} \\ & \sum_{i=0}^r F_i(\kappa) X^i & \mapsto & \sum_{i=0}^r \Theta F_i(\kappa) X^i + (\log(\kappa) - i) F_i(\kappa) X^{i+1} \end{array}$$

which is $\mathcal{A}(\mathcal{U})$-linear.

Note that δ_κ is not $\mathcal{A}(X(v))$-linear.

Proof. It is an application of the fact that for classical $\overline{\mathbb{Q}}_p$-points of $\mathcal{U}$ above $[k]$ we have $[k+2](\delta_\kappa) = \delta_k[k]$ and Proposition 4.2.17. □

We point out that there are other possible constructions of the Maaß-Shimura operator on nearly overconvergent forms which are defined on the whole space $\mathcal{N}^r(N, \mathcal{A}(\mathcal{U}))$ and not only on the part of finite slope. In [HX13], the authors construct an overconvergent Gauß-Manin connection

$$\mathcal{H}^r_\kappa \to \mathcal{H}^{r+1}_{\kappa[2]}$$

using the existence of the canonical splitting of $\mathcal{H}^1_{\mathrm{dR}/X(v)}$ given by E_2 (which exists because $X(v)$ is affinoid, [Kat73, Appendix 1]).
Let $r \geq 0$ be an integer, we define

$$\log^{[r]}(\kappa) = \prod_{j=0}^{r-1} \log(\kappa[-2r+j]).$$

Let us denote by $\mathcal{K}(\mathcal{U})$ the total fraction field of $\mathcal{A}(\mathcal{U})$; we define

$$\mathcal{N}^r(N, \mathcal{K}(\mathcal{U}))^{\leq \alpha} = \mathcal{N}^r(N, \mathcal{A}(\mathcal{U}))^{\leq \alpha} \otimes_{\mathcal{A}(\mathcal{U})} \mathcal{K}(\mathcal{U}).$$

Proposition 4.2.19. *Let $F(\kappa)$ in $\mathcal{N}^r_\kappa(N, \mathcal{K}(\mathcal{U}))^{\leq \alpha}$, then*

$$F(\kappa) = \sum_{i=0}^{r} \frac{\delta^i_{\kappa[-2i]} G_i(\kappa)}{\log^{[i]}(\kappa)}$$

for a unique sequence $(G_i(\kappa))$, $i = 0, \ldots, r$, with $G_i(\kappa)$ in $\mathcal{M}(N, \mathcal{K}(\mathcal{U}[-2i]))$.

Proof. The proposition is clear if $r = 0$. For $r \geq 1$, we proceed by induction; write

$$F(\kappa) = \sum_{i=0}^{r} F_i(\kappa) X^i,$$

we have then $\varepsilon^r F(\kappa) = r! F_r(\kappa)$, so $F_r(\kappa)$ is a family of overconvergent forms.
We pose $G_r(\kappa) := F_r(\kappa)$ and we see easily that

$$F(\kappa) - \frac{\delta^r_{\kappa[-2r]} G_r(\kappa)}{\log^{[r]}(\kappa)}$$

has degree $r-1$ and by induction there exist $G_i(\kappa)$ as in the statement.
For uniqueness, suppose

$$\sum_{i=0}^{r} \frac{\delta^i_{\kappa[-2i]} G_i(\kappa)}{\log^{[i]}(\kappa)} = 0,$$

by applying ε^r we obtain $G_r(\kappa) = 0$ and uniqueness follows by induction. □

We have the following corollaries.

Corollary 4.2.20. *We have an isomorphism of Hecke-modules*

$$\bigoplus_{i=0}^{r} \delta^{i}_{\kappa[-2i]}\mathcal{M}(N, \mathcal{K}(\mathcal{U}[-2i]))^{\leq \alpha - i} \cong \mathcal{N}^r(N, \mathcal{K}(\mathcal{U}))^{\leq \alpha}.$$

and consequently the characteristic series of U_p is given by $Q_r(\kappa, T) = \prod_{i=0}^{r} Q_0(\kappa[-2i], p^i T)$.

Corollary 4.2.21. *We define a projector*

$$H : \mathcal{N}^r\left(N, \mathcal{A}(\mathcal{U})\right)^{\leq \alpha} \to \mathcal{M}\left(N, \mathcal{A}(\mathcal{U})\left[\frac{1}{\prod_{j=0}^{2r}\log(\kappa[-j])}\right]\right)^{\leq \alpha}$$

by sending $F(\kappa)$ to $G_0(\kappa)$. It is called the overconvergent projection.

Proof. We use the same notation of the proof of Proposition 4.2.19. If $F(\kappa)(X)$ has q-expansion in $\mathcal{A}(\mathcal{U})[[q]][X]$, we can see by induction on the degree that then the only possible poles of $G_0(\kappa)$ are the zero of $\prod_{j=0}^{2r}\log(\kappa[-j])$. □

It is clear that this projector is a p-adic version of the classical holomorphic projector.
We remark that it is not possible to improve Proposition 4.2.19 allowing holomorphic coefficients, as shown by the following example; let us write $E_2^{\mathrm{cr}}(z)$ for the critical p-stabilization $E_2(z) - E_2(pz)$. We have that the polynomial q-expansion of $E_2^{\mathrm{cr}}(z)$ is

$$E_2^{\mathrm{cr}}(q, X) = \frac{p-1}{2p}X + \sum_{np^m,(n,p)=1} p^m \sigma_1(n) q^{np^m}.$$

Recall the Eisenstein family $\tilde{E}(\kappa)$ defined in (4.2.12). We have

$$E_2^{\mathrm{cr}}(q, X) = \delta_\kappa \tilde{E}(\kappa)|_{\kappa=\mathbf{1}},$$

as the residue at $\kappa = \mathbf{1}$ of $\zeta^*(\kappa)$ is $\frac{p-1}{p}$. The fact that the overconvergent projector has denominators in the weight variable was already known to Hida [Hid85, Lemma 5.1].
We now give the following proposition

Proposition 4.2.22. *Let $F(\kappa)$ be an element of $\mathcal{N}^r(N, \mathcal{A}(\mathcal{U}))$ and suppose that $F(\kappa)$ is an eigenform for the whole Hecke algebra and of finite slope for U_p. Then $F(\kappa) = \delta_\kappa^r G(\kappa)$, for $G(\kappa) \in \mathcal{M}(N, (\mathcal{K}(\mathcal{U}[-2r])))$ a family of overconvergent eigenforms.*

Proof. Let $\lambda_F(n)$ be the Hecke eigenvalue of T_n; we have from Proposition 4.2.19 that

$$F(\kappa)(X) = \sum_{i=0}^{r} \delta^{i}_{\kappa[-2i]} G_i(\kappa)$$

with $G_i(\kappa)$ overconvergent. Moreover, we know from Proposition 4.2.3 that $G_i(\kappa) = a_0(G_i)\sum_{i=1}^{\infty} n^{-i}\lambda_F(n)q^n$. We have then

$$G_i(\kappa) = \frac{a_0(G_i)}{a_0(G_r)}\Theta^{r-i}G_r(\kappa).$$

By restriction to the ordinary locus and projecting to ω^κ by $X \mapsto 0$ we find:

$$F(\kappa)(0) = \left(\sum_{i=0}^{r}\frac{a_0(G_i)}{a_0(G_r)}\right)\Theta^r G_0(\kappa).$$

This is the same q-expansion of $\left(\sum_{i=0}^{r}\frac{a_0(G_i)}{a_0(G_r)}\right)\delta^r_{\kappa[-2r]}G_r(\kappa)$; hence we can conclude by Proposition 4.2.8. □

For any $\alpha < \infty$ and for $i = 0, \ldots, r$ we define a map $s_i : \mathcal{C}^{\leq \alpha} \to \mathcal{C}^{i^{\leq \alpha + i}}$ induced by

$$\delta_\kappa^i : \mathcal{M}(N, \mathcal{A}(\mathcal{U}))^{\leq \alpha} \to \mathcal{N}^i(N, \mathcal{A}(\mathcal{U}[2i]))^{\leq \alpha + i}.$$

The interest of the above proposition lies in the fact that it tells us that $\mathcal{C}^r$ minus a finite set of points (such as E_2^{cr}) can be covered by the images of s_i. The images of these maps are not disjoint; it may indeed happen that two families of different degrees meet.
Here is an example; let $k \geq 2$ be an integer, we have that $\delta_{1-k}^k = \Theta^k$ (see formula 4.4.6). It is well known that Θ^k preserve overconvergence [Col96, Proposition 4.3]. Let $F(\kappa)$ be a family of overconvergent forms of finite slope, then the specialization at $\kappa = [1-k]$ of the nearly overconvergent family $\delta_\kappa^k F(\kappa)$ is overconvergent and consequently belongs to an overconvergent family.
From the polynomial q-expansion principle for the degree of near holomorphicity [Urb, Corollary 3.2.5], we see that intersections between families of different degrees may happen only when the coefficients of the higher terms in X of δ_κ^i vanish. It is clear from Formula 4.4.6 that this can happen only for points $[1-k]$, for $i \geq k \geq 2$. Note that these points lie above the poles of the overconvergent projection H. This is not a coincidence; in fact $H(\delta_\kappa^k F(\kappa)) = 0$ for all points in $\mathcal{W}$ for which H is defined. If we could extend H over the whole $\mathcal{W}$, we should have then $H\left(\delta_{[1-k]}^k F([1-k])\right) = 0$ but we have just seen that $\delta_{[1-k]}^k F([1-k])$ is already overconvergent.

4.3 Half-integral weight modular forms and symmetric square L-function

In this section we first recall the definition and some examples of half-integral weight modular forms. Then we use them to give an integral expression of $\mathcal{L}(s, \mathrm{Sym}^2(f), \xi)$. We conclude the section studying the Euler factor by which $\mathcal{L}(s, \mathrm{Sym}^2(f), \xi)$ and $L(s, \mathrm{Sym}^2(f), \xi)$ differ.

4.3.1 Half-integral weight modular forms

We recall the definition of half-integral weight modular forms. We define an holomorphic function on $\mathcal{H}$

$$\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2}, \quad q = e^{2\pi i z}.$$

Note that this theta series has no relations with the operator Θ of the previous section. We hope that this will cause no confusion.
We define a factor of automorphy

$$h(\gamma, z) = \frac{\theta(\gamma(z))}{\theta(z)} \ \ \gamma \in \Gamma_0(4), z \in \mathcal{H}.$$

It satisfies

$$h(\gamma, z)^2 = \sigma_{-1}(d)(c + zd).$$

Let $k \geq 0$ be an integer and Γ a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. We define the space of half-integral weight nearly holomorphic modular forms $\mathcal{N}_{k+\frac{1}{2}}^r(\Gamma, \mathbb{C})$ as the set of $\mathcal{C}^\infty$-functions

$$f : \mathcal{H} \to \mathbb{C}$$

such that

- $f|_{k+\frac{1}{2}}\gamma(z) := f(\gamma(z))h(\gamma,z)^{-1}(c+zd)^{-k} = f(z)$ for all γ in Γ,
- f has a finite limit at all cusp of Γ,
- there exist holomorphic $f_i(z)$ such that

$$f(z) = \sum_{i=0}^{r} f_i(z)\frac{1}{(4\pi y)^i}, \quad y = \mathrm{Im}(z).$$

When $r=0$, one simply writes $\mathcal{M}_{k+\frac{1}{2}}(\Gamma,\mathbb{C})$ for the space of holomorphic forms of weight $k+\frac{1}{2}$.
As Γ is a congruence subgroup, then there exists N such that each $f_i(z)$ as above admits a Fourier expansion of the form

$$f_i(z) = \sum_{n=0}^{\infty} a_n(f_i)q^{\frac{n}{N}}.$$

This allows us to embed $\mathcal{N}^r_{k+\frac{1}{2}}(\Gamma,\mathbb{C})$ into $\mathbb{C}[[q]][q^{\frac{1}{N}},X]$. For all $\mathbb{C}$-algebra A containing the N-th roots of unity, we define

$$\mathcal{N}^r_{k+\frac{1}{2}}(\Gamma,A) = \mathcal{N}^r_{k+\frac{1}{2}}(\Gamma,\mathbb{C}) \cap A[[q]][q^{\frac{1}{N}},X].$$

For a geometric definition, see [DT04, Proposition 8.7]. In the following, we will drop the variable z from f.
Let us consider a non trivial Dirichlet character χ of level N and let β be 0 resp. 1 if ξ is even, resp. odd. We define

$$\theta(\xi) = \sum_{n=1}^{\infty} n^{\beta}\xi(n)q^{n^2} \in \mathcal{M}_{\beta+\frac{1}{2}}(\Gamma_1(4N^2),\xi,\mathbb{Z}[\zeta_N]).$$

Another example of half-integral weight forms is given by Eisenstein series; we recall their definition. Let $k>0$ be an integer and χ be a Dirichlet character modulo Lp^r (L a positive integer prime to p) such that $\chi(-1)=(-1)^{k-1}$, we set

$$\begin{aligned}
\frac{E^*_{k-1/2}(z,s;\chi)}{L(2s+2k-2,\chi^2)} &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Lp^r)} \chi\sigma_{Lp^r}{\sigma_{-1}}^{k-1}(\gamma)h(\gamma,z)^{-2k+1}|h(\gamma,z)|^{-2s},\\
E_{k-1/2}(z,m;\chi) &= C_{m,k}\left\{(2y)^{\frac{-m}{2}}E^*_{k-1/2}(z,-m;\chi)\right\}|_{k-1/2}\tau_{Lp^r},\\
C_{m,k} &= (2\pi)^{\frac{m-2k+1}{2}}(Lp^r)^{\frac{2k-1-2m}{4}}\Gamma\left(\frac{2k-1-m}{2}\right),
\end{aligned}$$

where τ_{Lp^r} is the Atkin-Lehner involution for half-integral weight modular forms normalized as in [Hid90, §2 h4] and σ_n is the quadratic character corresponding via class field theory to the quadratic extension $\mathbb{Q}(\sqrt{n}/\mathbb{Q})$.
If we set $E_{k-1/2}(\chi)=E_{k-1/2}(z,3-2k;\chi)$, then $E_{k-1/2}(\chi)$ is a holomorphic modular form of half-integral weight $k-1/2$, level Lp^r and nebentypus χ. Let us denote by μ the Möbius function. The Fourier expansion of $E_{k-1/2}(\chi)$ is given by

$$L_{Lp}(3-2k,\chi^2) + \sum_{n=1}^{\infty} q^n L_{Lp}\left(2-k,\chi\chi_n\right) \sum_{\substack{t_1^2t_2^2|n,\\ (t_1t_2,Lp)=1,\\ t_1>0,\, t_2>0}} \mu(t_1)\chi(t_1t_2^2)\chi_n(t_1)t_2(t_1t_2^2)^{k-2},$$

where $L_{Lp}(s,\chi)=\prod_{q|Lp}(1-\chi_0(q)q^{-s})L(s,\chi_0)$, for χ_0 the primitive character associated to χ .
Let s be an odd integer, $1 \leq s \leq k-1$. We have the following key formula, for the compatibility with the Maaß-Shimura operators as defined in Section 4.2.1;

$$\delta_{k-s+\frac{1}{2}}^{\frac{s+1}{2}-1} E_{k-s+\frac{1}{2}}(\chi) = E_{k-\frac{1}{2}}(z, 2k-s-2;\chi).$$

In particular $E_{k-\frac{1}{2}}(z,2k-s-2;\chi) \in \mathcal{N}_{k-\frac{1}{2}}^{\frac{s+1}{2}-1}(\Gamma_1(Np^r),\chi,\overline{\mathbb{Q}})$.
If g_1 resp. g_2 denotes a form in $\mathcal{N}_{k_1+\frac{1}{2}}^{r_1}(\Gamma_1(N),\psi_1,A)$ resp. $\mathcal{N}_{k_2-\frac{1}{2}}^{r_2}(\Gamma_1(N),\psi_2,A)$, then g_1g_2 belongs to $\mathcal{N}_{k_1+k_2}^{r_1+r_2}(\Gamma_1(N),\psi_1\psi_2\sigma_{-1},A)$.

4.3.2 An integral formula

In this subsection we use the half-integral weight forms we have defined before to express $\mathcal{L}(s,\mathrm{Sym}^2(f),\xi)$ as the Petersson product of f with the product of two half-integral weight forms. Let f be a cusp form of integral weight k and Nebentypus ψ_1, g a modular form of half-integral weight $l/2$ and Nebentypus ψ_2. Let N be the least common multiple of the levels of f and g and suppose $k > l/2$. We define the Rankin product of f and g

$$D(s,f,g) = L_N(2s-2k-l+3,(\psi\xi)^2)\sum_n \frac{a(n,f)a(n,g)}{n^{s/2}}.$$

The Eisenstein series introduced above allow us to give an integral formulation for this Rankin product.

Lemma 4.3.1. *Let f, g and $D(s,f,g)$ as above. We have the equality*

$$\begin{aligned}(4\pi)^{-s/2}\Gamma(s/2)D(s,f,g) &= \left\langle f^c, gE^*_{k-l/2}(z,s+2-2k;\psi_1\psi_2\sigma_{-N})y^{(s/2)+1-k}\right\rangle_N,\\ &= (-i)^k\left\langle f^c|_k\tau_N, g|_{l/2}\tau_N\left(E^*_{k-l/2}(z,s+2-2k;\psi_1\psi_2\sigma_{-N})y^{(s/2)+1-k}\right)|_{k-l/2}\tau_N\right\rangle_N.\end{aligned}$$

Here $\langle f,g\rangle$ denotes the complex Petersson product

$$\langle f,g\rangle = \int_{X(\Gamma)}\overline{f(z)}g(z)y^{k-2}\mathrm{d}x\mathrm{d}y$$

and it is defined for any couple (f,g) in $\mathcal{N}_k^r(N,\mathbb{C})^2$ such that at least one between f and g is cuspidal.
If we take for g a theta series $\theta(\xi)$ as defined above we have then

$$D(\beta+s,f,\theta(\xi)) = \mathcal{L}(s,\mathrm{Sym}^2(f),\xi),$$

for $\mathcal{L}(s,\mathrm{Sym}^2(f),\xi)$ the imprimitive L-function defined in the introduction. The interest of writing $\mathcal{L}(s,\mathrm{Sym}^2(f),\xi)$ as a Petersson product lies in the fact that such a product, properly normalized, is algebraic. This allowed Sturm to show Deligne's conjecture for the symmetric square [Stu80] and it is at the base of our construction of p-adic L-function for the symmetric square. We conclude with the following relation which can be easily deduced from [Hid90, (5.1)] and which is fundamental for the proof of Theorem 4.1.3.

Lemma 4.3.2. *Let f be an Hecke eigenform of level divisible by p and let ξ be a character defined modulo Cp of conductor C. Let us denote by ξ' the primitive character associated to ξ, then*

$$D(s,f,\theta(\xi)) = (1-\lambda_p^2p^{1-s})D(s,f,\theta(\xi')).$$

4.3.3 The L-function for the symmetric square

Let f be a modular form of weight k and of Nebentypus ψ and $\pi(f)$ the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ spanned by f. Let us denoted by λ_q the associated set of Hecke eigenvalues. In [GJ78], the authors construct an automorphic representation of $\mathrm{GL}_3(\mathbb{A})$ denoted $\hat{\pi}(f)$ and usually called the base change to GL_3 of $\pi(f)$. It is standard to associate to $\hat{\pi}(f)$ a complex L-function $\Lambda(s, \hat{\pi}(f))$ which satisfies a nice functional equation and coincides with $\mathcal{L}(s, \mathrm{Sym}^2(f), \psi^{-1})$ up to some Euler factors. The problem is that some of these Euler factors could vanish at critical integers. We recall very briefly the L-factors at primes of bad reduction of $\Lambda(s, \hat{\pi}(f))$ in order to determine in Section 4.5 whether the L-value we interpolate vanishes or not. We shall also use them in section 4.7 to generalize the results of [DD97, Hid90]. For a more detailed exposition, we refer to [Ros13a, §4.2]. Fix an adelic Hecke character $\tilde{\xi}$ of $\mathbb{A}_{\mathbb{Q}}$. For any place v of $\mathbb{Q}$, we pose

$$L_v(s, \hat{\pi}(f), \xi) = \frac{L_v(s, \pi(f)_v \otimes \tilde{\xi}_v \times \check{\pi}(f)_v)}{L_v(s, \tilde{\xi}_v)},$$

where $\check{}$ denotes the contragredient and $\pi(f)_v \times \check{\pi}(f)_v$ is a representation of $\mathrm{GL}_2(\mathbb{Q}_v) \times \mathrm{GL}_2(\mathbb{Q}_v)$.
The completed L-function

$$\Lambda(s, \hat{\pi}(f), \tilde{\xi}) = \prod_v L_v(s, \hat{\pi}(f), \xi)$$

is holomorphic over $\mathbb{C}$ except in a few cases which correspond to CM-forms with complex multiplication by ξ [GJ78, Theorem 9.3].
Let $\pi = \pi(f)$ and let q be a place where π ramifies and let π_q be the component at q. By twisting by a character of $\mathbb{Q}_q^\times$, we may assume that π_q has minimal conductor among its twist; this does not change the L-factor $\hat{\pi}_q$. Let ψ' be the Nebentypus of the minimal form associated with f.
We distinguish the following four cases

(i) π_q is a principal series $\pi(\eta, \nu)$, with both η and ν unramified,

(ii) π_q is a principal series $\pi(\eta, \nu)$ with η unramified,

(iii) π_q is a special representation $\sigma(\eta, \nu)$ with η, ν unramified and $\eta\nu^{-1} = |\ |_q$,

(iv) π_q is supercuspidal.

We will partition the set of primes dividing the conductor of f as $\Sigma_1, \cdots, \Sigma_4$ according to these cases. Let us denote by ξ the primitive Dirichlet character corresponding to $\tilde{\xi}$. When π_q is a ramified principal series we have $\eta(q) = \lambda_q q^{\frac{1-k}{2}}$ and $\nu = \eta^{-1}\tilde{\psi}'_l$, where $\tilde{\psi}'$ is the adelic character corresponding to ψ'. In case *i)*, if $\tilde{\xi}_q$ is unramified, the Euler factor $L_q(\hat{\pi}_q \otimes \tilde{\xi}_q, s)^{-1}$ is

$$(1 - \xi_q\nu^{-1}\eta(q)q^{-s})(1 - \xi_q(q)q^{-s})(1 - \xi_q\nu\eta^{-1}(q)q^{-s})$$

and 1 othewise. In case *ii)* we have that $L_q(\hat{\pi}_q \otimes \tilde{\xi}_q, s)^{-1}$ equals

$$(1 - (\xi\psi'^{-1})_0(q)\lambda_q^2 q^{1-k-s})(1 - \xi(q)q^{-s})(1 - (\xi\psi')_0(q)\lambda_q^{-2}q^{k-1-s})$$

While in the third case if $\tilde{\xi}_q$ is unramified we have $(1 - \xi_q(q)q^{-s-1})$ and 1 otherwise. The supercuspidal factors are slightly more complicated and depend on the ramification of ξ_q. They are classified by [Sch88, Lemma 1.6]; we recall them briefly. Let q be a prime such that π_q is supercuspidal. If ξ_q^2 is unramified, let λ_1

and λ_2 the two ramified characters such that $\xi_q \lambda_i$ is unramified. We consider the following disjoint subsets of Σ_4:

$$\begin{aligned}
\Sigma_4^0 &= \{q \in \Sigma_4 : \xi_q \text{ is unramified and } \pi_q \cong \pi_q \otimes \xi_q\}, \\
\Sigma_4^1 &= \left\{q \in \Sigma_4 : \xi_q^2 \text{ is unramified and } \pi_q \cong \pi_q \otimes \lambda_i \text{ for } i = 1, 2\right\}, \\
\Sigma_4^2 &= \left\{q \in \Sigma_4 : \xi_q^2 \text{ is unramified and } \pi_q \not\cong \pi_q \otimes \lambda_1 \text{ and } \pi_q \cong \pi_q \otimes \lambda_2\right\}, \\
\Sigma_4^3 &= \left\{q \in \Sigma_4 : \xi_q^2 \text{ is unramified and } \pi_q \not\cong \pi_q \otimes \lambda_2 \text{ and } \pi_q \cong \pi_q \otimes \lambda_1\right\}.
\end{aligned}$$

If q is in Σ_4 but not in Σ_4^i, for $i = 0, \cdots, 3$, then $L_q(s, \hat{\pi}_q, \xi_q) = 1$. If q is in Σ_4^0, then

$$L_q(s, \hat{\pi}_q, \xi_q)^{-1} = 1 + \xi_q(q) q^{-s}$$

and if q is in Σ_4^i, for $i = 1, 2, 3$ then

$$L_q(s, \hat{\pi}_q, \xi_q)^{-1} = \prod_{j \text{ s.t.} \pi_q \cong \pi_q \otimes \lambda_j} (1 - \xi_q \lambda_j(q) q^{-s}).$$

If $v = \infty$, the L-factor depends only on the parity of the character by which we twist. Let $\kappa = 0, 1$ according to the parity of $\xi_\infty \psi_\infty$, from [Sch88, Lemma 1.1] we have $L(s - k + 1, \hat{\pi}_\infty, \xi_\infty \psi_\infty) = \Gamma_{\mathbb{R}}(s - k + 2 - \kappa) \Gamma_{\mathbb{C}}(s)$ for the complex and real Γ-functions

$$\begin{aligned}
\Gamma_{\mathbb{R}}(s) =& \pi^{-s/2} \Gamma(s/2), \\
\Gamma_{\mathbb{C}}(s) =& 2(2\pi)^{-s} \Gamma(s).
\end{aligned}$$

We define

$$\mathcal{E}_N(s, f, \xi) = \frac{\prod_{q|N} (1 - \xi(q) \lambda(T(q))^2 q^{-s}) L_q(s - m - 1, \hat{\pi}_q, \psi_q^{-1} \xi_q)}{(1 - \psi^2 \xi^2(2) 2^{2k-2-2s})}.$$

Note that $\lambda(T(q)) = 0$ if π is not minimal at q or if π_q is a supercuspidal representation. We multiply then $\mathcal{L}(s, f, \xi)$, the imprimitive L-function, by $\mathcal{E}_N(s, f, \xi)$ to get

$$\begin{aligned}
L(s, \mathrm{Sym}^2(f), \xi) :=& L(s - k + 1, \hat{\pi}(f) \otimes \xi \psi) \\
=& \mathcal{L}(s, f, \xi) \mathcal{E}_N(s, f, \xi).
\end{aligned}$$

We can now state the functional equation

$$\begin{aligned}
\Lambda(s, \hat{\pi}(f), \xi) &= \varepsilon(s, \hat{\pi}(f), \xi) \Lambda(1 - s, \hat{\pi}(f), \xi^{-1}), \\
\Lambda(s, \mathrm{Sym}^2(f), \xi) &= \varepsilon(s - k + 1, \hat{\pi}(f), \xi \psi) \Lambda(2k + 1 - s, \mathrm{Sym}^2(f^c), \xi^{-1}),
\end{aligned}$$

for $\varepsilon(s, \hat{\pi}(f), \xi)$ of [DD97, Theorem 1.3.2].

4.4 p-adic measures and p-adic L-functions

The aim of this section is to construct the p-adic L-functions which we have described in the introduction. We first review the notion of an h-admissible distribution, and we generalize this notition to measures with values in nearly overconvergent forms; we then produce two such distributions. We shall use these distributions in the sequel in order to construct the p-adic L-functions for the symmetric square.

4.4.1 Admissibility condition

We now give the definition of the admissibility condition for measures with value in the space of nearly overconvergent modular forms. We will follow the approach of [Pan03, §3]. Let us denote by A a $\mathbb{Q}_p$-Banach algebra, by M a Banach module over A and by Z_L the p-adic space $(\mathbb{Z}/Lp\mathbb{Z})^\times \times (1+p\mathbb{Z}_p)$. Let h be an integer, we define $\mathcal{C}^h(Z_L,A)$ as the space of locally polynomial function on Z_L of degree strictly less than h in the variable $z_p \in 1+p\mathbb{Z}_p$. Let us define $\mathcal{C}^h_n(Z_L,A)$ as the space of functions from Z_L to A which are polynomial of degree stricty less than h when restricted to ball of radius p^n. It is a compact Banach space and we have

$$\mathcal{C}^h(Z_L,A) = \varinjlim_n \mathcal{C}^h_n(Z_L,A).$$

If $h \leq h'$, we have an isometric immersion of $\mathcal{C}^h_n(Z_L,A)$ into $\mathcal{C}^{h'}_n(Z_L,A)$

Definition 4.4.1. *Let μ be an M-valued distribution on Z_L, i.e. a A-linear continuous map*

$$\mu : \mathcal{C}^1(Z_L,A) \to M.$$

We say that μ is an h-admissible measure if μ can be extended to a continuous morphism (which we shall denote by the same letter) $\mu : \mathcal{C}^h(Z_L,A) \to M$ such that for all n positive integer, any $a \in (\mathbb{Z}/Lp^n\mathbb{Z})^\times$ and $h' = 0,\ldots,h-1$ we have

$$\left| \int_{a+(Lp^n)} (z_p - a)^{h'} \mathrm{d}\mu \right| = o(p^{-n(h'-h)}).$$

If we denote by $\mathbf{1}_U$ the characteristic function of a open set U of Z_L, we shall sometimes write $\int_U \mathrm{d}\mu$ for $\mu(\mathbf{1}_U)$.
The definition of h-admissible measure for $A = \mathcal{O}_{\mathbb{C}_p}$ is due to Amice and Vélu.
There are many different (equivalent) definitions of a h-admissible measure; we refer to [Col10a, §II] for a detail exposition of them.
The following proposition will be very usefull in the following ([Col10a, Proposition II.3.3]);

Proposition 4.4.2. *Let μ be an h-admissible measure, let $\tilde{h} \geq h$ be a positive integer, then μ satisfies*

$$\left| \int_{a+Lp^n} (z_p - a_p)^{h'} \mathrm{d}\mu \right| = o(p^{-n(h'-\tilde{h})})$$

for any $n \in \mathbb{N}$, $h' \in \mathbb{N}$ and $a \in (\mathbb{Z}/Lp^n\mathbb{Z})^\times$.

It is known that any h-admissible measure is uniquely determined by the values $\int_{Z_L} \chi(z)\varepsilon(z_p){z_p}^{h'}\mathrm{d}\mu$, for all integers h' in $[0,\ldots,h-1]$, all χ in $\widehat{(\mathbb{Z}/Lp\mathbb{Z})}^\times$ and all finite-order characters ε of $1+p\mathbb{Z}_p$.
Let us fix now $\mathcal{U}$, an affinoid subset of $\mathcal{W}$. We have the following proposition about the behavior of U_p on $\mathcal{N}^r(Np^n,\mathcal{A}(\mathcal{U}))$

Proposition 4.4.3. *Let $n \geq 1$ be an integer, we have that U_p^n sends $\mathcal{N}^r(Np^{n+1},\mathcal{A}(\mathcal{U}))$ into $\mathcal{N}^r(Np,\mathcal{A}(\mathcal{U}))$. In particular, the map*

$$\begin{array}{cccc} \mathrm{Pr}^{\leq\alpha,p^\infty} : & \bigcup_{n=0}^\infty \mathcal{N}^r(Np^{n+1},\mathcal{A}(\mathcal{U})) & \to & \bigcup_{n=0}^\infty \mathcal{N}^r(Np^{n+1},\mathcal{A}(\mathcal{U})) \\ & G(\kappa) & \mapsto & U_p^{-n}\mathrm{Pr}^{\leq\alpha}U_p^n G(\kappa). \end{array}$$

is well-defined and induces an equality $\mathcal{N}^r(Np^{n+1},\mathcal{A}(\mathcal{U}))^{\leq\alpha} = \mathcal{N}^r(Np,\mathcal{A}(\mathcal{U}))^{\leq\alpha}$

Proof. The same proof as [Pan03, Proposition 1.6] applies, so we shall only sketch it. The first part is due to the fact that U_p induces a map

$$\mathcal{N}^r(Np^{n+1}, \mathcal{A}(\mathcal{U})) \to \mathcal{N}^r(Np^n, \mathcal{A}(\mathcal{U})). \tag{4.4.4}$$

This is checked on classical points.
Let $G(\kappa) \in \bigcup_{n=0}^{\infty} \mathcal{N}^r(Np^{n+1}, \mathcal{A}(\mathcal{U}))$ and $n \in \mathbb{N}$ such that $U_p^n G(\kappa)$ belongs to $\mathcal{N}^r(Np, \mathcal{A}(\mathcal{U}))$. For all $i \geq 0$ we have

$$U_p^{-i-n}\mathrm{Pr}^{\leq \alpha} U_p^{n+i} G(\kappa) = U_p^{-i-n} U_p^i \mathrm{Pr}^{\leq \alpha} U_p^n G(\kappa)$$

because $\mathrm{Pr}^{\leq \alpha}$ commutes with U_p and U_p is invertible on the part of slope less or equal than α. The final statement is a straightforward consequence of the fact that U_p^{-n} is the inverse of the map 4.4.4. ☐

We remark that the trick to use U_p to lower the level was already known to Shimura and is a fundamental tool in the study of family of p-adic modular forms. We conclude the section with the following theorem, which is exactly [Pan03, Theorem 3.4] in the nearly overconvergent context. Let $\mathcal{U}$ be an open affinoid of $\mathcal{W}$; we let $A = \mathcal{A}(\mathcal{U})$ and $M = \mathcal{N}^r(\Gamma, \mathcal{A}(\mathcal{U}))$.

Theorem 4.4.5. *Let α be a positive rational number and let μ_s, $s = 0, 1, \ldots$, be a set of distributions on $\mathcal{C}^1(Z_L, M)$. Suppose there exists a positive integer h_1 such that the following two conditions are satisfied:*

$$\mu_s(a + (Lp^n)) \in \mathcal{N}^r\left(Lp^{h_1 n}, \mathcal{A}(\mathcal{U})\right),$$
$$\left| U^{h_1 n} \sum_{i=0}^{s} \binom{s}{i} (-a_p)^{s-i} \mu_i(a + (Lp^n)) \right|_p < Cp^{-ns}.$$

Let h be such that $h > h_1\alpha + 1$; then there exists an h-admissible measure μ such that

$$\int_{a+(Lp^n)} (z_p - a_p)^s \mathrm{d}\mu = U_p^{-h_1 n} \mathrm{Pr}^{\leq \alpha}(U_p^{h_1 n} \mu_s(a + (Lp^n))).$$

4.4.2 Nearly overconvergent measures

In this subsection we will define two measures with values in the space of nearly overconvergent forms. We begin by studying the behavior of the Maaß-Shimura operator modulo p^n. We have from [Hid88c, (6.6)] the following expression of the Maaß-Shimura on polynomial q-expansion

$$\delta_k^s = \sum_{j=0}^{s} \binom{s}{j} \frac{\Gamma(k+s)}{\Gamma(k+s-j)} \Theta^{s-j} X^j, \tag{4.4.6}$$

for $\Theta = q\frac{\mathrm{d}}{\mathrm{d}q}$ as in Section 4.2.2. We now give an elementary lemma

Lemma 4.4.7. *We have for all integers s*

$$\mathrm{v}_p(s) \leq \mathrm{v}_p\left(\binom{s}{j}(k+s-1)\cdots(k+s-j)\right)$$

for all $1 \leq j \leq s$.

Proof. Simply notice that the valuation of $(k+s-1)\cdots(k+s-j)$ is bigger than that of $j!$ and $s|(s!/(s-j)!)$. ☐

The following two propositions are almost straightforward;

Proposition 4.4.8. *Let k, k' be two integers, $k \equiv k' \bmod p^n(p-1)$ and f_k and $f_{k'}$ two nearly holomorphic modular forms, algebraic such that $f_k \equiv f_{k'} \bmod p^m$. Then $\delta_k f_k \equiv \delta_{k'} f_{k'} \bmod p^{\min(n,m)}$.*

Proof. Direct computation from the formula in Proposition 4.2.18. □

Proposition 4.4.9. *Let k, k' be two integers, $k \equiv k' \bmod p^n(p-1)$ and f_k and $f_{k'}$ two nearly holomorphic modular forms, algebraic of same degree such that $f_k \equiv f_{k'} \bmod p^n$. Let s, s' be two positive integers, $s' = s + s_0 p^n(p-1)$. Then $(\delta_k^s f_k)|\iota_p \equiv \delta_{k'}^{s'} f_{k'} \bmod p^n$.*

Proof. Iterating the above proposition we get $\delta_k^s f_k \equiv \delta_{k'}^s f_{k'} \bmod p^n$. But $s' - s \equiv 0 \bmod p^n$, so by the above lemma and (4.4.6) we have $\delta_{k+2s}^{s'-s}\delta_k^s f_k \equiv \Theta^{s'-s}\delta_{k'}^s f_{k'}$. We conclude as $\Theta^{s'-s} \equiv \iota_p \bmod p^n$. □

Before constructing the aforementioned measures, we recall the existence of the Kubota-Leopoldt p-adic L-function.

Proposition 4.4.10. *Let χ be a primitive character modulo Cp^r, with C and p coprime and $r \geq 0$. Then for any $b \geq 2$ coprime with p, there exists a measure $\zeta_{\chi,b}$ such that for every finite-order character ε of Z_L and any integer $m \geq 1$ we have*

$$\int_{Z_L} \varepsilon(z) z_p^{m-1} \mathrm{d}\zeta_{\chi,b}(z) = (1 - \varepsilon'\chi'(b)b^m) L_{Lp}(1-m, \chi\varepsilon),$$

where χ' denote the prime-to-p part of χ.

To such a measure and to each character ε modulo Np^r, we can associate by p-adic Mellin transform a formal series

$$G(S, \varepsilon, \chi, b) = \int_{Z_L} \varepsilon(z)(1+S)^{z_p} \mathrm{d}\zeta_{\chi,b}(z)$$

in $\mathcal{O}_K[[S]]$, where K is a finite extension of $\mathbb{Q}_p$. We have a natural map from $\mathcal{O}_K[[S]]$ to $\mathcal{A}(\mathcal{W})$ induced by $S \mapsto (\kappa \mapsto \kappa(u) - 1)$. We shall denote by $L_p(\kappa, \varepsilon, \chi, b)$ the image of $G(S, \varepsilon, \chi, b)$ by this map.
We define an element of $\mathcal{A}(\mathcal{W})[[q]]$

$$\mathcal{E}_\kappa(\varepsilon) = \sum_{n=1,(n,p)=1}^{\infty} L_p(\kappa[-2], \varepsilon, \sigma_n, b) q^n \sum_{\substack{t_1^2 t_2^2 | n, \\ (t_1 t_2, Lp) = 1, \\ t_1 > 0, t_2 > 0}} t_1^{-2} t_2^{-3} \mu(t_1) \varepsilon(t_1 t_2^2) \sigma_n(t_1) \kappa(t_1 t_2^2).$$

If $\kappa = [k]$, we have then $[k](\mathcal{E}_\kappa(\varepsilon)) = (1 - \varepsilon'(b)b^{k-1}) E_{k-\frac{1}{2}}(\varepsilon\omega^{-k})|\iota_p$, where ι_p is the trivial character modulo p.
We fix two even Dirichlet characters: ξ is primitive modulo $\mathbb{Z}/Cp^\delta\mathbb{Z}$ ($\delta = 0, 1$) and ψ is defined modulo $\mathbb{Z}/pN\mathbb{Z}$. Fix also a positive slope α and an integer L which is a square and divisible by 4, C^2 and N.
Let h be an integer, $h > 2\alpha + 1$. For $s = 0, 1, \dots$ we now define distributions μ_s on $\mathbb{Z}_p^\times$ with value in $\mathcal{N}^r(L, \mathcal{A}(\mathcal{W}))^{\leq \alpha}$. For any finite-order character ε of conductor p^n we pose

$$\mu_s(\varepsilon) = \mathrm{Pr}^{\leq \alpha} U_p^{2n-1} \left(\theta(\varepsilon\xi\omega^s)| \left[\frac{L}{4C^2} \right] \delta_{\kappa\left[-s-\frac{1}{2}\right]}^{\frac{s-\beta}{2}} \mathcal{E}_{\kappa[-s]}(\psi\xi\varepsilon\sigma_{-1}) \right)$$

with $\beta = 0$, 1 such that $s \equiv \beta \bmod 2$. The projector $\mathrm{Pr}^{\leq \alpha}$ is *a priori* defined only on $\mathcal{N}^r(L, \mathcal{A}(\mathcal{U}))$, but it makes perfect sense to apply it to a formal polynomial q-expansion, as it is a formal power series in U_p and we know how U_p acts on a polynomial q-expansion.
Define $t_0 \in \mathbb{Q}$ to be the smaller rational such that $z_p^{\log(\kappa)}$ converges for all z_p in $1 + p\mathbb{Z}_p$ and κ in $\mathcal{W}(t_0)$.

Proposition 4.4.11. *The above defined-distributions μ_s define an h-admissible measure μ with values in $\mathcal{N}^r(L, \mathcal{A}(\mathcal{W}(t_0) \times \mathcal{W}))^{\leq \alpha}$.*

Proof. We have to check that the two conditions of Theorem 4.4.5 are verified. The calculations are similar to the one of [DD97, Theorem 2.7.6, 2.7.7] or, more precisely, to the one made by [Gor06, §3.5.6] and [CP04, §4.6.8] which study in detail the growth condition. We have the discrete Fourier expansion

$$\mathbf{1}_{a_p+p^n\mathbb{Z}_p}(x) = \frac{1}{p^{n-1}} \sum_{\varepsilon} \varepsilon(a_p^{-1}x)$$

By integration, together with the fact that each $\mu_s(\varepsilon)$ belongs to $\mathcal{N}^r(Lp^{2r}, \mathcal{A}(\mathcal{U}))$, we obtain *i)*.
For the estimate *ii)*, we have to show that for all $n \geq 0$, $0 \leq s \leq h-1$

$$\left| U_p^{2n} \sum_{i=0}^{s} \binom{s}{i} (-a_p)^{s-i} \mu_i(a + (Lp^n)) \right|_p < Cp^{-ns}$$

where the norm $| \ |_p$ is the q-expansion norm defined in Section 4.2.3. The coefficients of $\sum_{i=0}^{s} \binom{s}{i} (-a_p)^{s-i} \mu_i(a + (Lp^n))$ are a combination of $L_p(\kappa, \varepsilon, \psi\xi\sigma_{-1}, b)$'s and $\log(\kappa[-2i])$. Weierstraß Preparation Theorem applied to the Tate algebra $\mathcal{A}(\mathcal{W}(t))$ tells us that $L_p(\kappa, \xi, \chi, b) = P(\kappa)S(\kappa - 1)$ where P is a polynomial and S a unit power series. This is enough to show that $|L_p(\kappa, \xi, \chi, b)|_{\mathcal{W}} < C_t$. On the contrary, the p-adic logarithm $\log(\kappa)$ is not bounded on $\mathcal{W}$; the maximum modulus principle [BGR84, §3.8.1, Proposition 7] tells us that $|\log(\kappa)|_{\mathcal{W}(t)} < C_t$ for all $0 < t < \infty$.
We now have to study more in detail the coefficient of $X^j q^n$, for $j \geq 1$. We shall call this coefficient b_n^j. Explicitly, from (4.4.6), we have

$$b_n^j = \sum_{i=0}^{s} \binom{s}{i} (-a_p)^{s-i} a(i,n) \binom{\frac{i-\beta}{2}}{j} \log(\kappa[-\frac{i+1+\beta}{2} - 1]) \cdots \log(\kappa[-\frac{i+1+\beta}{2} - j]),$$

where $a(i,n) = \int z_p^i \mathrm{d}\nu_n$, for ν_n a measure, namely a linear combination of Kubota-Leopoldt p-adic L-functions. For $\beta = 0$, 1 and for $j \geq 1$ we shall write:

$$D_{\kappa,\beta}^j = \left(z_p^{\log(\kappa^2)-2-\beta} \frac{\partial}{\partial z_p} \cdots z_p^{-1} \frac{\partial}{\partial z_p} \cdot z_p^{-1} \frac{\partial}{\partial z_p} z_p^{1+\beta+2j-2\log(\kappa)} \right),$$

where we have applied $\frac{\partial}{\partial z_p}$ j-times and multiplied $j-1$ times by z_p^{-1}. We note that we have for any positive integer i:

$$D_{\kappa,\beta}^j(z_p^i) = \log(\kappa^{-2}[i+1+\beta+2]) \cdots \log(\kappa^{-2}[i+1+\beta+2j]) z_p^i.$$

Similarly,

$$\mathfrak{D}_{\kappa,\beta}^j = \left(z_p^{2j+\beta-1} \frac{\partial}{\partial z_p} \cdots z_p^{-1} \frac{\partial}{\partial z_p} \cdot z_p^{-1} \frac{\partial}{\partial z_p} z_p^{-\beta} \right),$$
$$\mathfrak{D}_{\kappa,\beta}^j(z_p^i) = (i-\beta)(i-\beta-2) \cdots (i-\beta-2j+2) z_p^i.$$

Summing up

$$\sum_{i=0}^{s} \binom{s}{i} (-a_p)^{s-i} \binom{\frac{i-\beta}{2}}{j} \log(\kappa[-\frac{i+1+\beta}{2} - 1]) \cdots \log(\kappa[-\frac{i+1+\beta}{2} - j]) z_p^i =$$
$$= \frac{2^{2j}}{j!} (-1)^{-j} \mathfrak{D}_{\kappa,\beta}^j D_{\kappa,\beta}^j \left((z_p - a_p)^s \right).$$

We have that $|\frac{\partial}{\partial z_p}(z_p - a_p)^s \mathbf{1}_{a+(Lp^n)}|_p = p^{-n(s-1)}$. As μ_n is a measure, we have

$$\left| \int_{a+(Lp^n)} \mathfrak{D}^j_{\kappa,\beta} D^j_{\kappa,\beta} \left((z_p - a_p)^s \right) \mathrm{d}\mu_n \right|_p = o(p^{-n(s-2j)}).$$

As we have

$$\begin{aligned} \sum_{i=0, i \equiv 0 \bmod 2}^{s} \binom{s}{i} (-a_p)^{s-i} z_p^i =& \frac{1}{2}((z_p - a_p)^s + (-z_p - a_p)^s), \\ \sum_{i=0, i \equiv 1 \bmod 2}^{s} \binom{s}{i} (-a_p)^{s-i} z_p^i =& \frac{1}{2}((z_p - a_p)^s - (-z_p - a_p)^s), \end{aligned}$$

we deduce the same estimate on $|b_n^j|_p$ because

$$\begin{aligned} \frac{j!}{2^{2j}}(-1)^{-j} b_n^j =& \mathfrak{D}^j_{\kappa,0} D^j_{\kappa,0} \left(\frac{1}{2}((z_p - a_p)^s + (-z_p - a_p)^s) \right) + \\ & + \mathfrak{D}^j_{\kappa,1} D^j_{\kappa,1} \left(\frac{1}{2}((z_p - a_p)^s - (-z_p - a_p)^s) \right). \end{aligned}$$

Recall that $U_p^{2n} X^j = p^{2nj} X^j$. Then for all $s \geq 0$, $n \geq 0$ we have the growth condition of Proposition 4.4.5. This assures us that these distributions define a unique h-admissible measure μ with values in $\mathcal{A}(\mathcal{W}(t_0) \times \mathcal{W}(t))[[q]]$. We can see using Proposition 4.4.9 that $\mu_s(\varepsilon)$ satisfies the hypothesis of Proposition 4.2.17 and hence $\mu_s(\varepsilon)$ belongs to $\mathcal{N}^r(L, \mathcal{A}(\mathcal{W}))^{\leq \alpha}$. We take then the limit for t which goes to 0. The Mellin transform

$$\kappa' \mapsto \int_{\mathbb{Z}_p^\times} \kappa'(u)^z \mathrm{d}\mu(z)$$

gives us the desired two variables family. □

Note that if $\alpha = 0$, we do not need to introduce the differential operators $\mathfrak{D}^j_{\kappa,\beta}$ and $D^j_{\kappa,\beta}$ and the above families are defined over the whole $\mathcal{W} \times \mathcal{W}$ (see also the construction in [Urb, §4.3]).
We define then an improved one variable family $\theta.E(\kappa) = \theta.E(b, \xi', \psi')$. We call this measure improved because it will allow us to construct a one-variable p-adic L-function which does not present a trivial zero. Fix a weight k_0 and, to define $\theta.E(\kappa)$, suppose that $\xi = \xi' \omega^{2-k_0}$, with ξ' a character of conductor C such that $\xi'(-1) = (-1)^{k_0}$. We define

$$\mathrm{Pr}^{\leq \alpha} \left(\theta(\xi') | \left[\frac{L}{4C^2} \right] \delta_{\kappa[-k_0 - \frac{3}{2}]}^{\frac{k_0 - \beta}{2} - 1} \tilde{\mathcal{E}}_{[\kappa]}(\sigma_{-1} \psi' \xi') \right),$$

where

$$\begin{aligned} \tilde{\mathcal{E}}_{[\kappa]}(\chi') = & (1 - \chi'(b)\kappa(b) b^{-k_0+1}) L_p(\kappa^2[-4 - 2k_0], {\chi'}^2, \mathbf{1}, b) + \\ & (1 - (\chi')^2(b)\kappa(b^2) b^{-2k_0-4}) \sum_{n=1}^{\infty} L_p(\kappa[-k_0], \chi, \sigma_n, b) q^n \\ & \times \sum_{\substack{t_1^2 t_2^2 | n, \\ (t_1 t_2, Lp) = 1, \\ t_1 > 0, t_2 > 0}} t_1^{-2} t_2^{-3} \mu(t_1) \chi(t_1 t_2^2) \sigma_n(t_1) \kappa(t_1 t_2^2). \end{aligned}$$

Let $F(\kappa)$ be a family of overconvergent eigenforms with coefficients in $\mathcal{U}$. We define a linear form l_F on $\mathcal{M}(N,\mathcal{K}(\mathcal{U}))^{\leq\alpha}$ as in [Pan03, Proposition 6.7]. Note that the evaluation formula holds also for weights which are, in Panchishkin's notation, *critical*, i.e. when $\alpha = (k_0-2)/2$, because at such point $\mathcal{C}$ is étale above $\mathcal{W}$. In [Pan03] this case is excluded because a trivial zero appears in his interpolation formula. Such a trivial zero is studied in [Ste10], where Conjecture 4.1.1 for $\rho_f(k_0/2)$ is proven.
We can define linear forms for nearly overconvergent families, in a way similar to [Urb, §4.2] but without the restriction $N=1$. For this, let $\mathbb{T}^r(N,\mathcal{K}(\mathcal{U}))^{(Np)}$ be the sub-algebra of $\mathrm{End}_{\mathcal{K}(\mathcal{U})}(\mathcal{N}^r(N,\mathcal{K}(\mathcal{U}))^{\leq\alpha})$ generated by the Hecke operators outside Np. It is a commutative and semisimple algebra; hence we can diagonalize $\mathcal{N}^r(N,\mathcal{K}(\mathcal{U}))^{\leq\alpha}$ for the action of this Hecke algebra. Let F be an eigenform for $\mathbb{T}^r(N,\mathcal{K}(\mathcal{U}))^{(Np)}$, we have a linear form l_F^r corresponding to the projection of an element of $\mathcal{N}^r(N,\mathcal{K}(\mathcal{U}))^{\leq\alpha}$ to the $\mathcal{K}(\mathcal{U})$-line spanned by F.
We say that a family $F(\kappa)$ is *primitive* if it is a family of eigenforms and all its specializations at non critical weights are the Maaß-Shimura derivative of a primitive form. This implies that the system of eigenvalues for $\mathbb{T}^r(N,\mathcal{K}(\mathcal{U}))^{(Np)}$ corresponding to $F(\kappa)$ appears with multiplicity one in $\mathcal{N}^r(N,\mathcal{K}(\mathcal{U}))^{\leq\alpha}$. We can see l_F^r as a p-adic analogue of the normalized Petersson product; more precisely we have the following proposition.
We recall that τ_N is the Atkin-Lehner involution of level N normalized as in [Hid90, h4]. When the level will be clear from the context, we shall simply write τ.

Proposition 4.4.12. *Let $F(\kappa)$ be an overconvergent family of primitive eigenform of finite slope α, degree r and conductor N. Let k be a classical non critical weight, then for all $G(\kappa)$ in $\mathcal{N}^r(N,\mathcal{K}(\mathcal{U}))^{\leq\alpha}$ we have*

$$l_F^r(G(k)) = \frac{\langle F(k)^c|\tau, G(k)\rangle}{\langle F(k)^c|\tau, F(k)\rangle}.$$

Proof. Let f be an element of $\mathcal{N}_k^r(N,\mathbb{C})$, the linear form

$$g \mapsto \frac{\langle f^c|\tau, g\rangle}{\langle f^c|\tau, f\rangle}$$

is Hecke equivariant and takes the value 1 on f. It is the unique one with these two properties.
For any pairs of forms g_1 and g_2 of weights $k-2r_1$ and $k-2r_2$, $r_1\neq r_2$, using Proposition 4.2.2 and Lemma 4.2.3, we see that $\delta_{k-2r_1}^{r_1} g_1$ and $\delta_{k-2r_2}^{r_2} g_2$ are automatically orthogonal for the Petersson product normalized as above.
Then, as $k>2r$, we have for any f in $\mathcal{N}_k^r(N,\mathbb{C})$ a linear form

$$g \mapsto \frac{\langle f^c|\tau, g\rangle}{\langle f^c|\tau, f\rangle}$$

which is Hecke equivariant and takes the value 1 on f. Moreover if f is defined over $\overline{\mathbb{Q}}$ then both linear forms are defined over $\overline{\mathbb{Q}}$.
Let $l_{F(k)}^r$ be the specialization of l_F^r at weight k. As we have $l_{F(k)}^r(F(k))=1$, we deduce that $l_{F(k)}^r$ must coincide, after extending scalars if necessary, with the previous one and we are done. □

In particular, we deduce from the above proof the p-adic analogue of the theorem which say that holomorphic forms are orthogonal to Maaß-Shimura derivatives.
We have the following lemma

Lemma 4.4.13. *Let H be the overconvergent projector of Corollary 4.2.21 and $F(\kappa)$ a family of overconvergent primitive eigenforms, then*

$$l_F \circ H = l_F^r.$$

Proof. Let us write $G(\kappa) = \sum_{i=0}^{r} \delta^{i}_{\kappa[-2i]} G_i(\kappa)$. The above proposition tells us $l^r_F(G(\kappa)) = l^r_F(G_0(\kappa))$. By definition, $l^r_F = l_F$ when restricted to $\mathcal{M}(N, \mathcal{K}(\mathcal{U}))^{\leq \alpha}$ and we are done. $\square$

We remark that l_F is defined over $\mathcal{K}(\mathcal{U})$ but not over $\mathcal{A}(\mathcal{U})$.
The linear forms l_F defines a splitting of $\mathcal{K}(\mathcal{U})$-algebras

$$\mathbb{T}^r(N, \mathcal{A}(\mathcal{W}(t)))^{\leq \alpha} \otimes \mathcal{K}(\mathcal{U}) = \mathcal{K}(\mathcal{U}) \times C$$

and consequently an idempotent $1_F \in \mathbb{T}^r(N, \mathcal{A}(\mathcal{W}(t)))^{\leq \alpha} \otimes \mathcal{K}(\mathcal{U})$. It is possible to find an element $H_F(\kappa) \in \mathcal{A}^{\circ}(\mathcal{U})$ such that $H_F(\kappa) 1_F$ belongs to $\mathbb{T}^r(N, \mathcal{A}(\mathcal{W}(t)))^{\leq \alpha}$. Then we can say that l_F is not holomorphic in the sense that it is not defined for certain κ in $\mathcal{U}$. We hope that the above lemma helps the reader to understand why the overconvergent projectors cannot be defined for all weights.
We will discuss later some possible relations between the poles of l_F and another p-adic L-function for the symmetric square.

4.4.3 The two p-adic L-functions

We shall now construct the two variable p-adic L-function $L_p(\kappa, \kappa')$ of Theorem 4.1.2 and, in the case where $\xi = \xi' \omega^{2-k_0}$, with ξ' a character of conductor C such that $\xi'(-1) = (-1)^{k_0}$, an *improved* p-adic L-function $L^*_p(\kappa)$. We call this p-adic L-function, in the terminology of Greenberg-Stevens, improved because it has no trivial zero and at κ_0 is a non zero multiple of the value $\mathcal{L}(k_0 - 1, \mathrm{Sym}^2(F(\kappa)), {\xi'}^{-1})$.
These two p-adic L-functions are related by the key Corollary 5.4.3. Allowing a cyclotomic variable forces us to use theta series of level divisible by p even when the conductor of the character is not divisible by p; Lemma 4.3.2 tells us that the trivial zero for f as in Theorem 4.1.3 comes precisely from this fact. The construction of the one-variable p-adic L-function is done in the spirit of [HT01], using the measure $\theta.E(\kappa)$ which is not a convolution of two measures but a product of a measure by a constant theta series whose level is not divisible by p. We warn the reader that the proof of Theorem 4.4.14 below is very technical and is not necessary for the following.
Before constructing the p-adic L-functions, we introduce the generalization to nearly overconvergent forms of the *twisted* trace operator defined in [Hid88c, §1 VI]. It will allow us to simplify certain calculations we will perform later.
Fix two prime-to-p integers L and N, with $N|L$. We define for classical k, r

$$\begin{array}{cccc} T_{L/N,k}: & \mathcal{N}^r_k(Lp, A) & \rightarrow & \mathcal{N}^r_k(Np, A) \\ & f & \mapsto & (L/N)^{k/2} \sum_{[\gamma] \in \Gamma(N)/\Gamma(N, L/N)} f|_k \begin{pmatrix} 1 & 0 \\ 0 & L/N \end{pmatrix} |_k \gamma \end{array}.$$

As L is prime to p, it is clear that $T_{L/N,k}$ commutes with U_p. It extends uniquely to a linear map

$$T_{L/N}: \quad \mathcal{N}^{\infty}(Lp, \mathcal{A}(\mathcal{U})) \quad \rightarrow \quad \mathcal{N}^{\infty}(Np, \mathcal{A}(\mathcal{U}))$$

which in weight k specializes to $T_{L/N,k}$. In particular, it preserves the slope decomposition.
Let us fix a p-stabilized eigenform f of weight k as in the introduction such that $k - 1 > v_p(\lambda_p)$. Let $\mathcal{C}_F$ be a neighbourhood of f in $\mathcal{C}$ contained in a unique irreducible component of $\mathcal{C}$. It corresponds by duality to a family of overconvergent modular forms $F(\kappa)$. We have that the slope of U_p on $\mathcal{C}_F$ is constant; let us denote by α this slope. We shall denote by $\lambda_p(\kappa)$ the eigenvalue of U_p on $\mathcal{C}_F$.
Let u be a generator of $1 + p\mathbb{Z}_p$ such that $u = b\omega^{-1}(b)$, where b is the positive integer we have chosen in Proposition 4.4.10. Let us define

$$\begin{aligned} \Delta(\kappa, \kappa') = & \left(1 - \psi'\xi'(b) \frac{\kappa(u)}{b\kappa'(u)}\right), \\ \Delta_0(\kappa) = & (1 - \xi'\psi'(b) b^{-k_0+1} \kappa(u))(1 - \xi'\psi'(b) b^{-2k_0-4} \kappa(u)^2). \end{aligned}$$

The two p-adic L-functions that we define are

$$\begin{aligned}L_p(\kappa,\kappa') =& L^{-1}\Delta(\kappa,\kappa')^{-1} l_F(T_{L/N}\theta * E(\kappa,\kappa')) \in \mathcal{K}(\mathcal{C}_F \times \mathcal{W}),\\ L_p^*(\kappa) =& L^{-1}\Delta_0(\kappa)^{-1} l_F(T_{L/N}\theta . E(\kappa)) \in \mathcal{K}(\mathcal{C}_F).\end{aligned}$$

We say that a point (κ,κ') of $\mathcal{A}(\mathcal{C}_F \times \mathcal{W})$ is classical if κ is a non-critical weight and $\kappa'(z) = \varepsilon(\langle z\rangle)z^s$, for ε a finite-order character of $1+p\mathbb{Z}_p$ and s an integer such that $1 \leq s+1 \leq k-1$. This ensures that $s+1$ is a critical integer à la Deligne for $\mathrm{Sym}^2(f)\otimes\omega^{-s}$. We define certain numbers which will appear in the following interpolation formulae. Suppose that (κ,κ') is classical in the above sense and let n be such that ε factors through $1+p^n\mathbb{Z}_p$. Let $n_0 = n$ resp. $n = 0$ if ε is not trivial resp. is trivial. For a Dirichlet character η, we denote by η_0 the associated primitive character. Let us pose

$$E_1(\kappa,\kappa') = \lambda_p(\kappa)^{-2n_0}(1-(\xi\varepsilon\omega^s)_0(p)\lambda_p(\kappa)^{-2}p^s);$$

if $F(\kappa)$ is primitive at p we define $E_2(\kappa,\kappa') = 1$, otherwise

$$\begin{aligned}E_2(\kappa,\kappa') =& (1-(\xi^{-1}\varepsilon^{-1}\omega^{-s}\psi)_0(p)p^{k-2-s})\times\\ & (1-(\xi^{-1}\varepsilon^{-1}\omega^{-s}\psi^2)_0(p)\lambda_p(\kappa)^{-2}p^{2k-3-s}).\end{aligned}$$

We shall denote by $F^\circ(\kappa)$ the primitive form associated to $F(\kappa)$. We shall write $W'(F(\kappa))$ for the prime-to-p part of the root number of $F^\circ(\kappa)$. If $F(\kappa)$ is not primitive at p we pose

$$S(F(\kappa)) = (-1)^k\left(1-\frac{\psi_0(p)p^{k-1}}{\lambda_p(\kappa)^2}\right)\left(1-\frac{\psi_0(p)p^{k-2}}{\lambda_p(\kappa)^2}\right),$$

and $S(F(\kappa)) = (-1)^k$ otherwise. Let L be a positive integer divisible by $4C^2$ and N, we shall write $L = 4C^2L'$. Let $\beta = 0$, 1 such that $s \equiv \beta \bmod 2$, we pose

$$\begin{aligned}C_{\kappa,\kappa'} &= s! i^k G(\xi\varepsilon\omega^s)C(\xi\varepsilon\omega^s)^s N^{-k/2}L'^{\frac{s-\beta}{2}}2^{-2s-k-\frac{1}{2}},\\ C_\kappa &= C_{\kappa,[k_0-2]}.\end{aligned}$$

Theorem 4.4.14. *i) The function $L_p(\kappa,\kappa')$ is defined on $\mathcal{C}_F\times\mathcal{W}$, it is meromorphic in the first variable and of logarithmic growth $h = [2\alpha]+2$ in the second variable (i.e., as function of s, $L_p(\kappa,[s])/\prod_{i=0}^h \log_p(u^{s-i}-1)$ is holomorphic on the open unit ball). For all classical points (κ,κ'), we have the following interpolation formula*

$$L_p(\kappa,\kappa') = C_{\kappa,\kappa'}E_1(\kappa,\kappa')E_2(\kappa,\kappa')\frac{\mathcal{L}(s+1,\mathrm{Sym}^2(F(\kappa)),\xi^{-1}\varepsilon^{-1}\omega^{-s})}{\pi^{s+1}S(F(\kappa))W'(F(\kappa))\langle F^\circ(\kappa),F^\circ(\kappa)\rangle}.$$

ii) The function $L_p^*(\kappa)$ is meromorphic on $\mathcal{C}_F$. For $k \geq k_0-1$, we have the following interpolation formula

$$L_p^*(\kappa) = C_\kappa E_2(\kappa,[k_0-2])\frac{\mathcal{L}(k_0-1,\mathrm{Sym}^2(F(\kappa)),\xi'^{-1})}{\pi^{k_0-1}S(F(\kappa))W'(F(\kappa))\langle F^\circ(\kappa),F^\circ(\kappa)\rangle}.$$

If $\alpha = 0$, using the direct estimate in [DD97, Theorem 2.7.6], we see that we can take $h = 1$ and the first part of this theorem is [Hid90, Theorem].
The poles on $\mathcal{C}_F$ of these two functions come from the poles of the overconvergent projection and of l_F; if (κ,κ') corresponds to a couple of points which are classical, then locally around this point no poles appear. Let us fix a point κ of $\mathcal{C}_F$ above $[k]$ and let f be the corresponding form. If $k > 2\alpha+2$ then, by specializing at κ the first variable, we recover the one variable p-adic L-function of the symmetric square of f constructed in [DD97] (up to some Euler factors). If instead $k \leq 2\alpha+2$, the method of [DD97] cannot give a well-defined one

variable p-adic L-function because, as we said in the introduction, the Mellin transform of an h-admissible measure μ is well-defined only if the first $h+1$ moments are specified. But in this situation the number of critical integers is $k-1$ and consequently we do not have enough moments. What we have to do is to choose the extra moments $\int \varepsilon(u)u^s \mathrm{d}\mu$ for all ε finite-order character of $1+p\mathbb{Z}_p$ and $s = k-1, \ldots, h$. We proceed as in [PS11]; the two-variable p-adic L-function $L_p(\kappa, \kappa')$ is well defined for all (κ, κ'), so we decide that

$$L_p(s, \mathrm{Sym}^2(f), \xi) := L_p(\kappa, [s]).$$

This amounts to say that the extra moments for μ are

$$\int \varepsilon(u)u^s \mathrm{d}\mu = L_p(\kappa, \varepsilon(\langle z \rangle)z^s).$$

To justify such a choice, we remark that if a classical point κ'' of $\mathcal{C}_F$ is sufficiently close to κ, then κ'' is above $[k'']$ with $k'' > h+1$. In this case $L_p(\kappa'', \varepsilon(\langle z \rangle)z^s)$ interpolates the special values $L(s, \mathrm{Sym}^2(f'') \otimes \xi)$ which are critical à la Deligne. We are then choosing the extra moments by p-adic intepolation along the weight variable. Fix f as in Theorem 4.1.3 and let κ_0 in $\mathcal{U}$ such that $F(\kappa_0) = f$. We have the following important corollary

Corollary 4.4.15. *We have the following factorization of locally analytic functions around κ_0 in $\mathcal{C}_F$:*

$$L_p(\kappa, [k_0 - 1]) = (1 - \xi'(p)\lambda_p(\kappa)^{-2}p^{k_0-2})L_p^*(\kappa).$$

We recall that this corollary is the key for the proof of Theorem 4.1.3.
The rest of the section will be devoted to the proof of Theorem 4.4.14

Proof of Theorem 4.4.14. Let (κ, κ') be a classical point as in the statement of the theorem. In particular $\kappa(u) = u^k$ and $\kappa'(u) = \varepsilon(u)u^s$, with $0 \leq s \leq k-2$.
We point out that all the calculations we need have already been performed in [Pan03, Hid90].
If ε is not trivial at p, we shall write p^n for the conductor of ε. If ε is trivial, then we let $n = 1$. Let $\beta = 0, 1$, $\beta \equiv s \bmod 2$.
We have

$$\begin{aligned} L_p(\kappa, \kappa') =& L^{-1} \frac{\left\langle F(\kappa)^c | \tau_{Np}, T_{L/N,k} U_p^{-2n+1} \mathrm{Pr}^{\leq \alpha} U_p^{2n-1} g \right\rangle}{\langle F(\kappa)^c | \tau, F(\kappa) \rangle}, \\ g =& \theta(\varepsilon\xi\omega^s) | [L/4C^2] E_{k - \frac{2\beta+1}{2}}(2k - s - \beta - 3, \xi\psi\sigma_{-1}\omega^{-s}\varepsilon) | \iota_p. \end{aligned}$$

We have as in [Pan03, (7.11)]

$$\left\langle F(k)^c | \tau_{Np}, U_p^{-2n+1} \mathrm{Pr}^{\leq \alpha} U_p^{2n-1} g \right\rangle = \lambda_p(\kappa)^{1-2n} p^{(2n-1)(k-1)} \left\langle F(\kappa)^c | \tau_{Np} | [p^{2n-1}], g \right\rangle,$$

where $f|[p^{2n-1}](z) = f(p^{2n-1}z)$. We recall the well-known formulae [Hid88c, page 79]

$$\begin{aligned} \left\langle f | [p^{2n}], T_{L/N,k} g \right\rangle =& (L/N)^k \left\langle f | [(p^{2n}L)/N], g \right\rangle, \\ \tau_{Np} | [(p^{2n-1}L)/N] =& \left(\frac{p^{2n-1}L}{N} \right)^{-k/2} \tau_{Lp^{2n}}, \\ \frac{\langle F(\kappa)^c | \tau_{Np}, F(\kappa) \rangle}{\langle F(\kappa)^\circ, F(\kappa)^\circ \rangle} =& (-1)^k W'(F(\kappa)) p^{(2-k)/2} \lambda_p(\kappa) \times \\ & \times \left(1 - \frac{\psi(p)p^{k-1}}{\lambda_p(\kappa)^2} \right) \left(1 - \frac{\psi(p)p^{k-2}}{\lambda_p(\kappa)^2} \right). \end{aligned}$$

Combining these with Lemma 4.3.1, we have

$$\begin{aligned}L_p(\kappa,\kappa') =& L^{-1} i^k 2^{\frac{s+1+\beta}{2}+1-k}(4\pi)^{-\frac{s+1+\beta}{2}}(2\pi)^{-\frac{s+2-\beta}{2}}\Gamma\left(\frac{s+1+\beta}{2}\right)\Gamma\left(\frac{s+2-\beta}{2}\right)\times\\ &\times \lambda_p(\kappa)^{1-2n}p^{(2n-1)\left(\frac{k}{2}-1\right)}(L/N)^{k/2}(Lp^{2n})^{\frac{2s-2k+5}{4}}\times\\ &\times \frac{D(\beta+s+1,f,\theta(\xi\omega^s\varepsilon)|[L/(4C^2)]|\tau(Lp^{2n}))}{\langle F(\kappa)^c|\tau,F(\kappa)\rangle}.\end{aligned}$$

Let $\eta=\xi\omega^s\varepsilon$. We recall from the transformation formula for theta series (see [Hid90, (5.1 c)])

$$\theta(\eta)|\tau_{4C^2p^{2n}} = \begin{cases} (-i)^\beta(Cp^n)^{-1/2}G(\eta)\theta(\eta^{-1}) & \text{if } \eta \text{ primitive } \bmod p\\ -(-i)^\beta(Cp)^{-1/2}G(\eta)\eta_0(p)\times & \\ \times(\theta(\eta_0^{-1})-p^{\beta+1}\eta_0^{-1}(p)\theta(\eta_0^{-1})|[p^2]) & \text{if not.}\end{cases}$$

We have the following relations for weight $\frac{2\beta+1}{2}$:

$$\tau_{Lp^n}=\tau_L[p^n]p^{n\frac{2\beta+1}{4}},\quad [L']\,|_{\frac{2\beta+1}{2}}\tau_L=\tau_{4C^2}(L')^{-\frac{2\beta+1}{4}};$$

when η is trivial modulo p we obtain

$$\begin{aligned}D(\beta+s,f,\theta(\eta)|\tau_{4C^2p^{2n}}) =& (-i)^\beta(Cp)^{-1/2}(1-\lambda_p(\kappa)^2p^{1-s}\eta_0^{-1}(p))\times\\ & G(\eta)\eta_0(p)D(\beta+s,f,\theta(\eta_0^{-1})).\end{aligned}$$

We recall the well-known duplication formula

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=2^{1-2z}\pi^{1/2}\Gamma(2z).$$

Summing up, we let $\delta=0$ resp. $\delta=1$ if η_0 has conductor divisible resp. not divisible by p. We obtain

$$\begin{aligned}L_p(\kappa,\kappa') =& L^{\frac{2s-2\beta}{4}} i^k N^{-k/2}C^\beta 2^{-s-\frac{1}{2}-k}2^{1-s-1}s!\times\\ &\times(-1)^\delta p^{sn}\lambda_p(\kappa)^{-2n}(1-\lambda_p(\kappa)^2p^{-s}\eta_0^{-1}(p))\eta_0(p)\times\\ &\times E_2(\kappa,\kappa')\frac{\mathcal{L}(s+1,\mathrm{Sym}^2(F(\kappa)),\xi^{-1}\varepsilon^{-1}\omega^{-s})}{\pi^{s+1}S(F(\kappa))W'(F(\kappa))\left\langle F^\circ(\kappa),F^\circ(\kappa)\right\rangle}.\end{aligned}$$

We now evaluate the second p-adic L-function; we have

$$\begin{aligned}L_p^*(\kappa) =& \frac{\left\langle F(\kappa)^c|\tau_{Np},T_{L/N,k}\mathrm{Pr}^{\leq\alpha}g\right\rangle}{\langle F(\kappa)^c|\tau,F(\kappa)\rangle},\\ g =& \theta(\xi')|\left[\frac{L}{4C^2}\right]\delta_{k-k_0-\frac{3}{2}}^{\frac{k_0-\beta}{2}-1}E_{k-k_0-1}(\sigma_{-1}\psi'\xi').\end{aligned}$$

The relation

$$\theta(\xi')|\left[\frac{L}{4C^2}\right]|\tau_{Lp}=\left(\frac{4C^2}{L}\right)^{\frac{2\beta+1}{4}}\theta(\xi')|\tau_{4C^2}[p]p^{\frac{2\beta+1}{4}}.$$

gives us

$$\begin{aligned}D\left(\beta+k_0-1,f,\theta(\xi')|\left[\frac{L}{4C^2}\right]|\tau_{Lp}\right) =& p^{\frac{2\beta+1}{4}}\lambda_p(\kappa)p^{-\frac{k_0-1}{2}}\left(\frac{4C^2}{L}\right)^{\frac{2\beta+1}{4}}(-i)^\beta\\ &\times C^{-1/2}G(\eta)D(\beta+k_0-1,f,\theta({\xi'}^{-1})).\end{aligned}$$

$\square$

We now give a proposition on the behavior of $L_p(\kappa,\kappa')$ along $\Delta(\kappa,\kappa')$. We say that $F(\kappa)$ has complex multiplication by a quadratic imaginary field K if $F(\kappa)|T_l = 0$ for almost all l inert in K. In particular, if $F(\kappa)$ has complex multiplication it is ordinary; indeed, all the non critical specializations are classical CM forms and a finite slope CM form of weight k can only have slope 0, $(k-1)/2$ or $k-1$. As we supposed that the slope of $F(\kappa)$ is fixed, then it must be zero.

Proposition 4.4.16. *Unless $\psi\xi\omega^{-1}$ is quadratic imaginary and $F(\kappa)$ has complex multiplication by the field corresponding to $\psi\xi\omega^{-1}$, $H(\kappa)L_p(\kappa,\kappa')$ is holomorphic along $\Delta(\kappa,\kappa')$, except possibly for a finite number of points.*

Proof. The proof is essentially the same as [Hid90, Proposition 5.2].
The poles of Δ lies on the line $\kappa = \kappa'[-1]$ inside $\mathcal{W}\times\mathcal{W}$. So we are left to evaluate $g := \theta * E(\kappa,\kappa[1])$. The well-known formula

$$L_p([-1],\chi,\sigma_n,b) = \begin{cases} 0 & \text{if } \chi \neq \sigma_n, \\ -\frac{\phi(C(\chi)Lp)}{C(\chi)Lp}\log_p(\omega^{-1}(b)b) & \text{otherwise.} \end{cases}$$

tells us that $g = 0$ unless we have $\psi\xi\omega^{-1} = \sigma_{-m}$, for a certain m.
If $\psi\xi\omega^{-1} = \sigma_{-m}$, the only non zero Fourier coefficients of g could possibly be the one of the form $n = c^2\frac{L}{4C^2} + md^2$; in particular n is norm in $\mathbb{Q}(\sqrt{-m})$. If $F(\kappa)$ has not complex multiplication by $\psi\xi\omega^{-1}$, we can find a prime l such that $a_l(F(\kappa)) \neq 0$ and l inert in $\mathbb{Q}(\sqrt{m})$ (i.e. it is not a norm). The first condition implies that $F(\kappa)|T_l \neq 0$ while the second implies $g|T_l = 0$.
If $F(\kappa)$ has complex multiplication by σ_{-m} this is [Hid90, Proposition 5.2] $\square$

We want to point out that $L_p(\kappa,\kappa')$ could have a pole on a point of type $([k],[s+1])$. Indeed, it could happen that the family $F(\kappa)$ specializes to a CM form. If the form is moreover critical, then it is known to be in the image of the operator Θ^{k-1} and therefore should correspond to a zero of $H_F(\kappa)$. Moreover such a point on $\mathcal{C}_F$ is ramified above the weight space [Bel12b, Proposition 1].

4.5 The proof of Benois' conjecture

In this section we shall prove a more general version of Theorem 4.1.3. Once one know Corollary 5.4.3, what we are left to do is to reproduce *mutatis mutandis* the method of Greenberg-Stevens. We remark that we have a shift $s \mapsto s-1$ between the p-adic L-function of the previous section and the one of the introduction. From the interpolation formula given in Theorem 4.4.14, we see that we have a trivial zero when $s = k_0 - 2$ and $(\omega^{2-k_0}\xi)_0(p) = 1$. Let us denote by $\mathcal{L}(f)$ the $\mathcal{L}$-invariant of $\mathrm{Sym}^2(f)\otimes\xi(k_0-1)$ as defined in [Ben11]. We have the following theorem

Theorem 4.5.1. *Fix f in $\mathcal{M}_{k_0}(Np,\psi)$ and suppose that f is Steinberg at p. Let $\xi = \xi'\omega^{k_0-2}$ be a character such that $\xi(-1) = (-1)^{k_0}$ and $\xi'(p) = 1$. Then*

$$\lim_{s\to k_0-1}\frac{L_p(s,\mathrm{Sym}^2(f),\xi)}{s-k_0+1} = \mathcal{L}(f)\frac{\mathcal{L}(s,\mathrm{Sym}^2(f),\xi^{-1})}{\pi^{k_0-1}W'(f)S(f)\Omega(f)}.$$

We shall leave the proof of this theorem for the end of the section. We now give the proof of the main theorem of the paper.

Proof of Theorem 4.1.3. We let $\xi' = \mathbf{1}$. Assuming Theorem 4.5.1, what we are left to show is that $\mathcal{L}(s,\mathrm{Sym}^2(f))$ coincides with the completed L-function $L(s,\mathrm{Sym}^2(f))$.
As we said in Section 4.3.3, the two L-functions differ only for some Euler factors at primes dividing $2N$.

As $2 \mid N$, we have $(1-\psi^2(2)) = 1$ as $\psi(2)=0$.
We have seen in Section 4.3.3 that when $\pi(f)_q$ is a Steinberg representation, the Euler factors at q of $\mathcal{L}(s, \mathrm{Sym}^2(f))$ and $L(s, \mathrm{Sym}^2(f))$ are the same.
As the form f has trivial Nebentypus and squarefree conductor, we have that $\pi(f)_q$ is Steinberg for all $q \mid N$ and we are done. $\square$

More precisely, we have that Theorem 4.5.1 implies Conjecture 4.1.1 any time that the factor $\mathcal{E}_N(k-1, f, \chi)$ is not zero, for the same reasoning as above. This is true if, for example, the character ξ is very ramified modulo $2N$.
If we choose $\xi = \psi^{-1}$, we are then considering the L-function for the representation $\mathrm{Ad}(\rho_f)$. In this case the conditions for $\mathcal{E}_N(k-1, f, \chi)$ to be non-zero are quite restrictive. For example 2 must divide the level of f. If moreover we have that the weight is odd, then there exist at least a prime q for which, in the notation of Section 4.3.3, π_q is a ramified principal series. From the explicit description of the Euler factors at q given in Section 4.3.3, we see that $L_q(0, \hat{\pi}(f))^{-1}$ is always zero.

The $\mathcal{L}$-invariant for the adjoint representation has been calculated in the ordinary case in [Hid04]. This approach has been generalized to calculate Benois' $\mathcal{L}$-invariant in [Ben10, Mok12]. These results can be subsumed as follows

Theorem 4.5.2. *Let $F(\kappa)$ be a family of overconvergent eigenforms such that $F(\kappa_0) = f$ and let $\lambda_p(\kappa)$ be its U_p eigenvalue. We have*

$$\mathcal{L}(f) = -2 \frac{\mathrm{d} \log \lambda_p(\kappa)}{\mathrm{d}\kappa}|_{\kappa=\kappa_0}.$$

We remark that it is very hard to determine whether $\mathcal{L}(f)$ is not zero, even though the above theorem tells us that this is always true except for a finite number of points.
If we suppose $k_0 = 2$, we are considering an ordinary form and in this case $\rho_f|_{\mathbb{Q}_p}$ is an extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$. The $\mathcal{L}$-invariant can be described via Kummer theory. Let us denote by q_f the universal norm associated to the extension $\rho_f|_{\mathbb{Q}_p}$; we have then $\mathcal{L}(f) = \frac{\log_p(q_f)}{\mathrm{ord}_p(q_f)}$. Let A_f be the abelian variety associated to f, in [GS93, §3] the two authors give a description of q_f in term of the p-adic uniformization of A_f. When A_f is an elliptic curve, then q_f is Tate's uniformizer and a theorem of transcendental number theory [BSDGP96] tells us that $\log_p(q_f) \neq 0$.

Proof of Theorem 4.5.1. Let κ_0 be the point on $\mathcal{C}$ corresponding to f. As the weight of f is not critical, we have that $w : \mathcal{C} \to \mathcal{W}$ is étale at κ_0. We have $w(\kappa_0) = [k_0]$. Let us write $t_0 = (z \mapsto \omega^{-k_0}(z) z^{k_0})$; t_0 is a local uniformizer in $\mathcal{O}_{\mathcal{W},[k_0-1]}$. As the map w is étale at κ_0, t_0 is a local uniformizer for $\mathcal{O}_{\mathcal{C},\kappa_0}$. Let us write $A = \mathcal{O}_{\mathcal{C},\kappa_0}/(T_0^2)$, for $T_0 = \kappa - t_0$. We have an isomorphism between the tangent spaces; this induces an isomorphism on derivations

$$\mathrm{Der}_K(\mathcal{O}_{\mathcal{W},[k_0-1]}, \mathbb{C}_p) \cong \mathrm{Der}_K(\mathcal{O}_{\mathcal{C},\kappa_0}, \mathbb{C}_p).$$

The isomorphism is made explict by fixing a common basis $\frac{\partial}{\partial T_0}$.
We take the local parameter at $[k_0-2]$ in $\mathcal{W}$ to be $t_1 = (z \mapsto \omega^{-k_0+2}(z) z^{k_0-2})$.
Let ξ be as in the hypothesis of the theorem and let $L_p(\kappa, \kappa')$ be the p-adic L-function constructed in Theorem 4.4.14. We can see, locally at $(\kappa_0, [k_0-2])$, the two variable p-adic L-function $L_p(\kappa, \kappa')$ as a function $L_p(t_0, t_1)$ of the two local parameters (t_0, t_1). Let us define $t_0(k) = (z \mapsto \omega^{-k_0}(z) z^k)$ resp. $t_1(s) = (z \mapsto \omega^{-k_0+2}(z) z^s)$ for k resp. s p-adically close to k_0 resp. $k_0 - 2$. Consequently, we pose $L_p(k, s) = L_p(t_0(k), t_1(s))$; this is a locally analytic function around (k_0, k_0-2).

We have $\frac{\partial}{\partial k} = \log_p(u)\frac{\partial}{\partial \log T_0}$. The interpolation formula of Theorem 4.4.14 *i)* tells us that locally $L(k,k-2) \equiv 0$. We derive this identity with respect to k to obtain

$$\left.\frac{\partial L(k,s)}{\partial s}\right|_{s=k_0-1,k=k_0} = -\left.\frac{\partial L(k,s)}{\partial k}\right|_{s=k_0-1,k=k_0}.$$

Using Corollary 5.4.3 and Theorem 4.5.2 we see that

$$L_p(k,k_0-1) = \mathcal{L}(f)L_p^*(k_0) + O((k-k_0)^2)$$

and we can conclude thanks to the second interpolation formula of Theorem 4.4.14. □

4.6 Relation with other symmetric square p-adic L-functions

As we have already said, the p-adic L-function of the previous section $L_p(\kappa,\kappa')$ has some poles on $\mathcal{C}_F$ coming from the *"denominator"* $H_F(\kappa)$ of l_F^r. In this section we will see how we can modify it to obtain an holomorphic function using a one-variable p-adic L-function for the symmetric square constructed by Kim and, more recently, Bellaïche. The modification we will perform to $L_p(\kappa,\kappa')$ will also change the interpolation formula of Theorem 4.4.14, changing the automorphic period (the Petersson norm of f) with a motivic one. We shall explain in the end of the section why, in the ordinary case, this change is important for the Greenberg-Iwasawa-Coates-Schimdt Main Conjecture [Urb06, Conjecture 1.3.4].
We want to point out that all we will say in this section is conditional on Kim's thesis [Kim06] which has not been published yet.
In the ordinary settings, overconvergent projector and ordinary projector coincides and it is known in many cases, thanks to Hida [Hid88a, Theorem 0.1], that $H_F(\kappa)$ interpolates, up to a p-adic unit, the special value

$$(k-1)!W'(F(\kappa))E_3^*(\kappa)\frac{L(k,\mathrm{Sym}^2(F(\kappa)),\psi^{-1})}{\pi^{k+1}\Omega^+\Omega^-},$$

where $W'(F(\kappa))$ is the root number of $F^\circ(\kappa)$ and $E_3^*(\kappa)=1$ if $F(\kappa)$ is primitive at p and

$$\left(1-\frac{\psi(p)p^{k-1}}{\lambda_p(\kappa)^2}\right)\left(1-\frac{\psi(p)p^{k-2}}{\lambda_p(\kappa)^2}\right)$$

otherwise (note that, up to a sign, it coincides with $S(F(\kappa))$). Here $\Omega^+ = \Omega^+(F(\kappa))$ and $\Omega^- = \Omega^-(F(\kappa))$ are two complex periods defined via the Eichler-Shimura isomorphism.
Kim in his thesis [Kim06] and recently Bellaïche generalize Hida's construction to obtain a one variable p-adic L-function for the symmetric square. The aim of this section is to confront the p-adic L-function of section 4.4.3 with theirs.
Kim's idea is very beautiful and at the same time quite simple; we will sketch it now. Its construction relies on two key ingredients; the first one is the formula, due to Shimura,

$$\langle f^\circ, f^\circ\rangle = (k-1)!\frac{L_N(k,\mathrm{Sym}^2(f),\psi^{-1})}{N^2 2^{2k_0}\pi^{k_0+1}}. \tag{4.6.1}$$

The second one is the sheaf over the eigencurve $\mathcal{C}$ of distribution-valued modular symbol, sometimes called the *overconvergent modular symbol*, constructed by Stevens [PS11, Bel12a]. It is a sheaf interpolating the sheaves $\mathrm{Sym}^{k-2}(\mathbb{Z}_p^2)$ appearing in the classical Eichler-Shimura isomorphism. When modular forms are seen as sections on this sheaf, the Petersson product is induced by the natural pairing on $\mathrm{Sym}^{k-2}(\mathbb{Z}_p^2)$. Kim's idea is to interpolate these pairings when k varies in the weight space to construct a pairing on the space of locally analytic functions on $\mathbb{Z}_p$. This will induce a (non-perfect) pairing $\langle\ ,\ \rangle_\kappa$ on the sheaf of overconvergent modular symbol. For a family $F(\kappa)$ we can define two modular symbols $\Phi^{\pm}(F(\kappa))$. Kim defines

$$L_p^{KB}(\kappa) = \left\langle \Phi^+(F(\kappa)), \Phi^-(F(\kappa)) \right\rangle_\kappa .$$

This p-adic L-function satisfies the property that its zero locus contains the ramification points of the map $\mathcal{C} \to \mathcal{W}$ [Kim06, Theorem 1.3.3] and for all classical non critical point κ of weight k we have the following interpolation formula [Kim06, Theorem 3.3.9]

$$L_p^{KB}(\kappa) = E_3^*(\kappa)W'(F(\kappa))\frac{(k-1)!L_N(k, \mathrm{Sym}^2(F(\kappa)), \psi^{-1})}{N^2 2^{2k}\pi^{k+1}\Omega^+\Omega^-}.$$

The period $\Omega^+\Omega^-$ is the one predicted by Deligne's conjecture for the symmetric square motive and it is probably a better choice than the Petersson norm of f for at least two reasons. The first one is that, as we have seen in (4.6.1), the Petersson norm of f essentially coincides with $L(k, \mathrm{Sym}^2, \psi^{-1})$ and such a choice as a period is not particulary enlightening when one is interested in Bloch-Kato style conjectures.
The second reason is related to the the Main Conjecture. Under certain hypotheses, such a conjecture is proven in [Urb06] for the p-adic L-function with motivic period. In fact, in [Urb06, §1.3.2] the author is force to make a change of periods from the p-adic L-function of [CS87, Hid90, DD97] to obtain equality of μ-invariant.
It seems reasonable to the author that in many cases, away from the zero of the overconvergent projection, we could choose $H_F(\kappa) = L_p^{KB}(\kappa)$; in any case, we can define a function

$$\tilde{L}_p(\kappa, \kappa') := L_p^{KB}(\kappa)L_p(\kappa, \kappa')$$

which is locally holomorphic in κ and at classical points interpolates, up to some explicit algebraic number which we do write down explicitly, the special values $\frac{\mathcal{L}(s, \mathrm{Sym}^2(F(\kappa)), \varepsilon^{-1}\omega^{s-1})}{\pi^s\Omega^+\Omega^-}$.

4.7 Functional equation and holomorphy

The aim of this section is to show that we can divide the two-variable p-adic L-function constructed in Section 4.4.3 by suitable two-variable functions to obtain an holomorphic p-adic L-function interpolating the special values of the primitive L-function, as defined in Section 4.3.3.
The method of proof follows closely the one used in [DD97] and [Hid90]. We shall first construct another two variable p-adic L-function, interpolating the other set of critical values. The construction of this two variables p-adic L-function has its own interest. The missing Euler factors do not vanish, and if one could prove a formula for the derivative of this function would obtain a proof of 4.1.1 without hypotheses on the conductor.
We will show that, after dividing by suitable functions, this p-adic L-function and the one of Section 4.4.3 satisfy a functional equation. We shall conclude by showing that the poles of these two functions are distinct.
We start recalling the Fourier expansion of some Eisenstein series from [Ros13a, Proposition 3.10];

$$\begin{aligned} E_{k-\frac{1}{2}}(z, 0; \chi) =& L_{Lp}(2k-3, \chi^2) + \sum_{n=1}^{\infty} q^n L_{Lp}\left(k-1, \chi\sigma_n\right) \times \\ & \times \left(\sum_{\substack{t_1^2 t_2^2 \mid n, \\ (t_1 t_2, Lp) = 1, \\ t_1 > 0, t_2 > 0}} \mu(t_1)\chi(t_1 t_2^2)\sigma_n(t_1)t_2(t_1 t_2^2)^{1-k} \right). \end{aligned}$$

We have for $0 \le s \le k/2$

$$\delta^s_{k+\frac{1}{2}} E_{k+\frac{1}{2}}(z,0,\chi) = E_{k+2s+\frac{1}{2}}(z,2s;\chi).$$

We give this well known lemma;

Lemma 4.7.1. *Let χ be a even primitive character modulo Cp^r, with C and p coprime and $r \geq 0$. Then for any $b \geq 2$ coprime with p, there exists a measure $\zeta^+_{\chi,b}$ such that for every finite-order character ε of Z_L and any integer $m \geq 1$ we have*

$$\int_{Z_L} \varepsilon(z) z_p^{m-1} \mathrm{d}\zeta^+_{\chi,b}(z) = (1-\varepsilon'\chi'(b)b^m)(1-(\varepsilon\chi)_0(p)p^{m-1})\times$$
$$\times \frac{G((\varepsilon\chi)_p)}{p^{(1-m)c_p}} \frac{L_L(m,\chi^{-1}\varepsilon^{-1})}{\Omega(m)},$$

where χ' denote the prime-to-p part of χ, χ_p the p-part of χ and c_p the p-part of the conductor of $\chi\varepsilon$. If we let $a = 0$, 1 such that $\varepsilon\chi(-1) = (-1)^a$, we have

$$\Omega(m) = \Omega(m,\varepsilon\chi) = i^a \pi^{1/2-m} \frac{\Gamma(\frac{m+a}{2})}{\Gamma(\frac{1-m+a}{2})}.$$

As before, we can associate to this measure a formal series

$$G^+(S,\xi,\chi,b) = \int_{Z_L} \xi(z)(1+S)^{z_p} \mathrm{d}\zeta^+_{\chi,b}(z).$$

We shall denote by $L^+_p(\kappa,\xi,\chi,b)$ the image of $G^+(S,\xi,\chi,b)$ by the map $S \mapsto (\kappa \mapsto \kappa(u)-1)$.
We define an element of $\mathcal{A}(\mathcal{W})[[q]]$

$$\mathcal{E}^+_\kappa(\chi) = \sum_{n=1,(n,p)=1}^{\infty} L^+_p(\kappa[-2],\chi,\sigma_n,b)q^n \sum_{\substack{t_1^2 t_2^2 | n, \\ (t_1 t_2, Lp)=1, \\ t_1>0, t_2>0}} t_1^2 t_2^3 \mu(t_1)\chi(t_1 t_2^2)\sigma_n(t_1)\kappa^{-1}(t_1 t_2^2).$$

If $\kappa = [k]$, we have then

$$[k](\mathcal{E}^+_\kappa(\chi)) = \frac{G((\chi)_p)}{p^{(2-k)c_p}\Omega(k-1)}(1-\chi'(b)b^{k-1})E_{k-\frac{1}{2}}(z,0,\omega^k\chi^{-1})|\nu_k$$
$$\nu_k(n) = \frac{(1-\omega^{\frac{p-1}{2}}(n)(\chi\omega^k)_0(p)p^{k-3})}{(1-\omega^{\frac{p-1}{2}}(n)(\chi^{-1}\omega^{-k})_0(p)p^{2-k})},$$

where the twist by ν_k is defined as in [Hid90, h5]. We fix two even Dirichlet characters as in Section 4.4.3: ξ is primitive modulo $\mathbb{Z}/Cp^\delta\mathbb{Z}$ ($\delta = 0,1$) and ψ is defined modulo $\mathbb{Z}/pN\mathbb{Z}$. Fix also a positive slope α and a positive integer L which is a square and divisible by C^2, 4 and N. Let us denote by C_0 the conductor of $\xi\psi^{-2}$ and let us write $L = 4C_0^2 L'_0$.
For $s = 0, 1, \ldots$ we now define distributions μ^+_s on $\mathbb{Z}_p^\times$ with values in $\mathcal{N}^r(L,\mathcal{A}(\mathcal{W}))^{\leq\alpha}$. For any ε of conductor p^n we pose

$$\mu^+_s(\varepsilon) = \mathrm{Pr}^{\leq\alpha} U_p^{2n-1}\left(\theta(\psi^2\xi^{-1}\varepsilon^{-1}\omega^{-s})|\left[\frac{L}{4C_0^2}\right]\delta^{\frac{s-\beta}{2}}_{\kappa[-s-\frac{1}{2}]}\mathcal{E}^+_{\kappa[-s]}(\psi\xi^{-1}\varepsilon^{-1}\sigma_{-1})\right)$$

with $\beta = 0$, 1 such that $s \equiv \beta \bmod 2$.

Proposition 4.7.2. *For any $0 < t < \infty$, the distributions μ_s^+ define an h-admissible measure μ^+ with values in $\mathcal{N}^r(L, \mathcal{A}(\mathcal{W}(t_0) \times \mathcal{W}(t)))^{\leq \alpha}$.*

We take the Mellin transform

$$\kappa' \mapsto \int_{\mathbb{Z}_p^\times} \kappa'(u)^z \mathrm{d}\mu^+(z)$$

to obtain an element $\theta * E^+(\kappa, \kappa')$ of $\mathcal{N}^r(L, \mathcal{A}(\mathcal{W}(t_0) \times \mathcal{W}(t)))^{\leq \alpha}$.
Let F be a family of finite slope eigenforms. We refer to Section 4.4.3 for all the unexplained notation and terminology. We pose

$$\Delta(\kappa, \kappa') = \left(1 - \psi'\xi'^{-1}(b)\frac{\kappa(u)}{b\kappa'(u)}\right).$$

We define a new p-adic L-function

$$L_p^+(\kappa, \kappa') = L^{-1}\Delta(\kappa, \kappa')^{-1} l_F(T_{L/N}\theta * E^+(\kappa, \kappa')) \in \mathcal{K}(\mathcal{C}_F \times \mathcal{W}).$$

Let us define

$$E_1^+(\kappa, \kappa') = \lambda_p(\kappa)^{-2n}(1 - (\xi^{-1}\varepsilon^{-1}\omega^{-s}\psi^2)_0(p)\lambda_p(\kappa)^{-2}p^{2k-3-s});$$

when $F(\kappa)$ is primitive at p we define $E_2^+(\kappa, \kappa') = 1$, otherwise

$$E_2^+(\kappa, \kappa') = (1 - \xi^{-1}\varepsilon^{-1}\omega^{-s}\psi(p)p^{k-2-s})(1 - (\xi\varepsilon\omega^s)_0(p)\lambda_p(\kappa)^{-2}p^s).$$

Let $\beta = 0, 1$ such that $s \equiv \beta \bmod 2$, we pose

$$\begin{aligned} C_{\kappa,\kappa'}^+ = &(2k-3-s)! i^k p^{n(3k-2s-5)} G(\psi\xi^{-1}\varepsilon^{-1}\omega^{-s}) G(\psi^2\xi^{-1}\varepsilon^{-1}\omega^{-s}) \times \\ &\times C_0(\xi'\psi^{-2'})^{2k-s-1} N^{-k/2} {L'_0}^{k-1-\frac{s+\beta+1}{2}} 2^{2s+5-5k+\frac{1}{2}}. \end{aligned}$$

Theorem 4.7.3. *$L_p^+(\kappa, \kappa')$ is a function on $\mathcal{C}_F \times \mathcal{W}$, meromorphic in the first variable and of logarithmic growth $h = [2\alpha] + 2$ in the second variable. For all classical points (κ, κ'), we have the following interpolation formula*

$$L_p^+(\kappa, \kappa') = C_{\kappa,\kappa'}^+ \frac{E_1^+(\kappa, \kappa')E_2^+(\kappa, \kappa')\mathcal{L}(2k-2-s, \mathrm{Sym}^2(F(\kappa)), \xi\varepsilon\omega^s)}{\Omega(k-s-1)\pi^{2k-2-s} S(F(\kappa)) W'(F(\kappa)) \langle F^\circ(\kappa), F^\circ(\kappa)\rangle}.$$

Proof. The calculation are essentially the same as Theorem 4.4.14; the only real difference is the presence of the twist by ν_k. We can deal with it as we did in [Ros13a, Appendix A] so we shall only sketch the calculations. We first remark the following; let χ be any character modulo p^r, then it is immediate to see the following identity of q-expansions

$$U_{p^r}\left(\sum_n \chi(n)a_n q^n \sum_m a_m q^m\right) = \chi(-1)U_{p^r}\left(\sum_n a_n q^n \sum_m \chi(m) a_m q^m\right).$$

We can write

$$\frac{1}{(1 - \omega^{\frac{p-1}{2}}(n)(\chi)_0(p)p^{k-2})} = 1 + \omega^{\frac{p-1}{2}}(n)(\chi^{-1}\omega^k)_0(p)p^{k-2} + \dots.$$

We apply this to

$$\theta(\psi^2\xi^{-1}\varepsilon^{-1}\omega^{-s})|\left[\frac{L}{4C_0^2}\right][k]\delta_{\kappa[-s-\frac{1}{2}]}^{\frac{s-\beta}{2}}\mathcal{E}^{+}_{\kappa[-s]}(\psi\xi^{-1}\varepsilon^{-1}\sigma_{-1})$$

and we see that we can move the twist ν_k to $\theta(\psi^2\xi^{-1}\varepsilon^{-1}\omega^{-s})|\left[\frac{L}{4C^2}\right]$, and we conclude noticing that

$$\nu_k\left(\frac{L}{4C^2}n^2\right)=\frac{(1-(\psi\xi^{-1}\varepsilon^{-1}\sigma_{-1})_0(p)p^{k-s-2})}{(1-(\psi^{-1}\xi\varepsilon\sigma_{-1})_0(p)p^{s+1-k})}$$

is independent of n. We have

$$\begin{aligned}
L_p^+(\kappa,\kappa') =& i^k C_{s-\beta,k-\frac{2\beta+1}{2}}G(\eta_0^{-1})L^{-1}(L/N)^{k/2}{L_0'}^{-\frac{2\beta+1}{4}}(-i)^{\beta}(C_0p^n)^{-1/2}\frac{G(\psi\xi^{-1}\varepsilon^{-1}\omega^{-s})}{p^{n(1-k+s)}\Omega(k-s-1)}\times\\
&\times p^{-(2n-1)\frac{k}{2}}(1-\lambda_p(\kappa)^2p^{s-2k-2}\eta_0^{-1}(p))\lambda_p(\kappa)^{1-2n}p^{(2n-1)(k-1)}\frac{\langle F(\kappa)^c,g\rangle}{\langle F(\kappa)^c|\tau,F(\kappa)\rangle},\\
g =& 2^{\frac{\beta-s}{2}}\theta(\psi^{-2}\xi\varepsilon\omega^s)E^*_{k-\frac{2\beta+1}{2}}(\beta-s,\psi^{-1}\xi\varepsilon\omega^s\sigma_{-1})y^{\frac{\beta-s}{2}},\\
C_{s-\beta,k-\frac{2\beta+1}{2}} =& (2\pi)^{\frac{s-2k+1+\beta}{2}}(Lp^{2n})^{\frac{2k-2s-1}{4}}\Gamma\left(\frac{2k-s-1-\beta}{2}\right).
\end{aligned}$$

We recall the well-known duplication formula

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=2^{1-2z}\pi^{1/2}\Gamma(2z)$$

which we apply for $z=\frac{2k-s-2}{2}$. Summing up, we obtain

$$\begin{aligned}
L_p^+(\kappa,\kappa') =& (2k-3-s)!i^k2^{2s+5-5k+\frac{1}{2}}N^{-k/2}C_0^{2k-s-1}{L_0'}^{k-1-\frac{s+\beta+1}{2}}\times\\
&\times G(\psi\xi^{-1}\varepsilon^{-1}\omega^{-s})G(\psi^2\xi^{-1}\varepsilon^{-1}\omega^{-s})p^{n(3k-2s-5)}\lambda_p(\kappa)^{-2n}\times\\
&\times\frac{(1-\eta_0^{-1}(p)\lambda_p(\kappa)^2p^{s-2k-2})E_2^+(\kappa,\kappa')\mathcal{L}(2k-2-s,\mathrm{Sym}^2(F(\kappa)),\psi^2\xi\varepsilon\omega^s)}{\pi^{2k-2-s}S(F(\kappa))W'(F(\kappa))\left\langle F^\circ(\kappa),F^\circ(\kappa)\right\rangle\Omega(k-s-1)}.
\end{aligned}$$

□

The interest in interpolating directly the critical values in the strip $[k,2k-2]$ lies in the fact that on the line of trivial zeros $s=k-2$ none of the missing Euler factors of $\mathcal{L}(k,\mathrm{Sym}^2(F(\kappa)),\psi^2\xi\varepsilon\omega^s)$ (which we recalled in Section 4.3.3) vanish. To prove Conjecture 4.1.1 it would then be enough to calculate the derivative of $L_p^+(\kappa,\kappa')$ with respect to κ at $(\kappa_0,[k_0-2])$.
To interpolate the primitive L-function we have to divide $L_p(\kappa,\kappa')$ by some functions which interpolate the extra factors given in Section 4.3.3. Let $F(\kappa)$ be as above and let us denote by $\{\lambda_n(\kappa)\}$ the corresponding system of Hecke eigenvalues. For any Dirichlet character of prime-to-p conductor χ, let us denote by $F_\chi(\kappa)$ the primitive family of eigenforms associated to the system of Hecke eigenvalues $\{\lambda_n(\kappa)\chi(n)\}$. Let q be a prime number and f a classical modular form, we say that f is minimal at q if the local representation $\pi(f)_q$ has minimal conductor among its twist. Let χ be a Dirichlet character such that $F_\chi(\kappa)$ is minimal everywhere for every non critical point κ. As the Hecke algebra $\mathbb{T}^r(N,\mathcal{K}(\mathcal{U}))$ is generated by a finite number of Hecke operators, if K is big enough to contain the values of χ, then the Hecke eigenvalues of $F(\kappa)$ and $F_\chi(\kappa)$ all belong to $\mathcal{A}(\mathcal{U})$. We shall denote by $\lambda_q^\circ(\kappa)$ the Hecke eigenvalue corresponding to the family which is minimal at q and by $\alpha_q^\circ(\kappa)$ and $\beta_q^\circ(\kappa)$ the two roots of the corresponding Hecke polynomial; enlarging $\mathcal{A}(\mathcal{U})$ if necessary, we can suppose that both of them belong to $\mathcal{A}(\mathcal{U})$.

For each prime q, let us write $l_q = \frac{\log_p(q)}{\log_p(u)}$. Recall the partition of the primes dividing the level of F *i), ..., iv)* given in Section 4.3.3, we define

$$\begin{aligned} E_q(\kappa,\kappa') =& \left(1-\xi_0^{-1}(q)q^{-1}\alpha_q^\circ(\kappa)^2\kappa'(u^{-l_q})\right)^{-1}\times \\ & \times\left(1-(\psi\xi^{-1})_0(q)q^{-2}\frac{\kappa(u^{l_q})}{\kappa'(u^{l_q})}\right)^{-1}\left(1-\xi_0^{-1}(q)q^{-1}\beta_q^\circ(\kappa)^2\kappa'(u^{-l_q})\right)^{-1} \text{ (if } q \text{ in case } i), \\ E_q(\kappa,\kappa') =& \left(1-(\psi\xi^{-1})_0(q)q^{-2}\frac{\kappa(u^{l_q})}{\kappa'(u^{l_q})}\right)^{-1}\left(1-(\psi^2\xi^{-1})_0(q)q^{-1}\lambda_q^\circ(\kappa)^{-2}\kappa'(u^{l_q})\right)^{-1} \text{ (if } q \text{ in case } ii), \\ E_q(\kappa,\kappa') =& \prod_{j \text{ s.t. } \pi_q\cong\pi_q\otimes\lambda_j}\left(1-(\psi\lambda_j\xi^{-1})_0(q)q^{-2}\frac{\kappa(u^{l_q})}{\kappa'(u^{l_q})}\right)^{-1} \text{ (if } q \text{ in case } iv) \end{aligned}$$

and we pose

$$A(\kappa,\kappa') = \left(1-\psi^{-2}\xi^2(2)2^2\frac{\kappa'(u^{2l_2})}{\kappa(u^{2l_2})}\right)\prod_q E_q(\kappa,\kappa')^{-1}.$$

We define also

$$\begin{aligned} E_q^+(\kappa,\kappa') =& \left(1-(\psi^{-2}\xi)_0(q)q^2\alpha_q^\circ(\kappa)^2\frac{\kappa'(u^{l_q})}{\kappa(u^{2l_q})}\right)^{-1}\times \\ & \times\left(1-(\psi^{-1}\xi)_0(q)q\frac{\kappa'(u^{l_q})}{\kappa(u^{l_q})}\right)^{-1}\left(1-(\psi^{-2}\xi)_0(q)q^2\beta_q^\circ(\kappa)^2\frac{\kappa'(u^{l_q})}{\kappa(u^{2l_q})}\right)^{-1} \text{ (if } q \text{ in case } i), \\ E_q^+(\kappa,\kappa') =& \left(1-(\psi^{-1}\xi)_0(q)q\frac{\kappa'(u^{l_q})}{\kappa(u^{l_q})}\right)^{-1}\left(1-(\psi^{-1}\xi)_0(q)q^{-1}\lambda_q^\circ(\kappa)^{-2}\kappa(u^{l_q})\kappa'(u^{l_q})\right)^{-1} \text{ (if } q \text{ in case } ii), \\ E_q^+(\kappa,\kappa') =& \prod_{j \text{ s.t. } \pi_q\cong\pi_q\otimes\lambda_j}\left(1-(\psi^{-1}\lambda_j\xi)_0(q)q\frac{\kappa'(u^{l_q})}{\kappa(u^{l_q})}\right)^{-1} \text{ (if } q \text{ in case } iv) \end{aligned}$$

and we pose

$$B(\kappa,\kappa') = \left(1-\psi^2\xi^{-2}(2)2^{-4}\frac{\kappa(u^{2l_2})}{\kappa'(u^{2l_2})}\right)\prod_q E_q^+(\kappa,\kappa')^{-1}.$$

Proposition 4.7.4. *We have the following equality of meromorphic functions on $\mathcal{C}_F\times\mathcal{W}$*

$$L_p(\kappa,\kappa')A(\kappa,\kappa')^{-1} = \varepsilon(\kappa,\kappa')L_p^+(\kappa,\kappa')B(\kappa,\kappa')^{-1}$$

where $\varepsilon(\kappa,\kappa')$ is the only Iwasawa function such that

$$\varepsilon(u^k,u^s) = \frac{G({\chi'}^{-1})G({\xi'}^{-1}\psi')^2 G({\xi'}^{-1}{\psi'}^2)^2}{G(\xi')}{L_0'}^{\frac{s+1+\beta}{2}+1-k}{L'}^{\frac{s-\beta}{2}}C'(\pi\otimes\xi)^{s-k+1}2^{4k-4s-6},$$

for $C'(\pi\otimes\xi)$ the conductor outside p of $\hat{\pi}\otimes\psi\xi^{-1}$.

Proof. The explicit epsilon factor of the functional equation stated in Section 4.3.3 can be found in [DD97, Theorem 1.3.2].
Recall from *loc. cit.* that

$$\frac{L_\infty(s+1)}{L_\infty(2k-2-s)} = \frac{s!(2\pi)^{-s-1}\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s-k+2+a}{2}\right)}{(2k-3-s)!(2\pi)^{-2k+2+s}\pi^{-\frac{2k-s-2}{2}}\Gamma\left(\frac{k-1-s+a}{2}\right)}$$

We have, on all classical points

$$\begin{aligned}\frac{L_p(\kappa,\kappa')B(\kappa,\kappa')}{L_p^+(\kappa,\kappa')A(\kappa,\kappa')} &= \frac{\pi^{2k-s-2}s!G(\xi\varepsilon\omega^s)C(\xi\varepsilon\omega^{-s})^s {L'}^{\frac{s-\beta}{2}} 2^{-2s-k-\frac{1}{2}}2^{-(2s+5-5k+\frac{1}{2})}p^{ns}\Omega(k-s-1)}{\pi^{s+1}(2k-3-s)!G(\psi\xi^{-1}\varepsilon^{-1}\omega^{-s})G(\psi^2\xi^{-1}\varepsilon^{-1}\omega^{-s})C_0^{2k-s-1}{L_0'}^{k-\beta-\frac{2s-3}{4}}p^{n(2k-s-3)}}\times \\ &\times \frac{L(s+1,\mathrm{Sym}^2(f),\xi^{-1}\varepsilon^{-1}\omega^{-s})}{L(2k-2-s,\mathrm{Sym}^2(f),\psi^2\xi\varepsilon\omega^s)} \\ &= \frac{p^{n(3(s+1)+3k+2)}G(\xi\varepsilon\omega^s)2^{4k-4s-6}C(\xi\varepsilon\omega^{-s})^s {L'}^{\frac{s-\beta}{2}}}{G(\psi\xi^{-1}\varepsilon^{-1}\omega^{-s})G(\psi^2\xi^{-1}\varepsilon^{-1}\omega^{-s})C_0^{2k-s-1}{L_0'}^{k-\beta-\frac{2s-3}{4}}}\varepsilon(s-k+2,\hat{\pi}(f),\xi^{-1}\varepsilon^{-1}\omega^{-s}\psi)\end{aligned}$$

To conclude we use [Sch88, Lemma 1.4] and the relations

$$p^n = G(\tilde{\psi})G(\tilde{\psi}^{-1}), \quad G(\psi_1\psi_2) = \psi_1(C_2)\psi_2(C_1)G(\psi_1)G(\psi_2),$$

for $\tilde{\psi}$ a character of conductor p^n and ψ_i a character of conductor C_i, with $(C_1, C_2) = 1$. □

Proposition 4.7.5. *The elements $A(\kappa,\kappa')$ and $B(\kappa,\kappa')$ are mutually coprime in $\mathcal{A}(\mathcal{U}\times\mathcal{W})$.*

Proof. We follow closely the proof of [DD97, §3.1].
During the proof of this proposition we shall identify $\mathcal{A}(\mathcal{U}\times\mathcal{W})$ with $\mathcal{A}(\mathcal{U})[[T]]$ and we will see $\mathcal{A}(\mathcal{U})$ as a $\mathcal{O}[[S]]$-algebra via $S \mapsto (\kappa \mapsto \kappa(u)-1)$.
Consider one of the factors of $A(\kappa,\kappa')$ in which neither $\lambda_q(\kappa)$, nor $\alpha_q^\circ(\kappa)$, nor $\beta_q^\circ(\kappa)$ appear. Then such a factor belongs to $\mathcal{O}[[S,T]]$ and a prime factor of it is of the form $(1+T)-z(1+S)$, with $z \in \mu_{p^\infty}$.
A prime divisor of the excluded factors of $A(\kappa,\kappa')$ is $(1+T)-j(\kappa)$, with $j(\kappa)$ in $\mathcal{A}(\mathcal{U})$.
Similarly, a prime factor of $B(\kappa,\kappa')$ is $(1+T)-z'(1+S)$, with $z \in u^{-1}\mu_{p^\infty}$ or $(1+T)-j'(\kappa)$.
If a prime elements divides both elements, we must have $z(1+S) = j'(\kappa)$ or $z'(1+S) = j(\kappa)$. We deal with the fist case. Suppose that this prime elements divides $\left(1-(\psi\xi^{-1})_0(q)q^{-2}\frac{\kappa(u^{l_q})}{\kappa'(u^{l_q})}\right)$ and $\left(1-(\psi^{-2}\xi)_0(q'){q'}^2\alpha_{q'}^\circ(\kappa)^2\frac{\kappa'(u^{l_{q'}})}{\kappa(u^{2l_{q'}})}\right)$.
Specializing at any classical point (κ,κ') we obtain $q^{k-s-2} = \zeta {q'}^{s-2k+2}\alpha_{q'}^\circ(\kappa)^2$, for ζ a root of unity. Noticing that $|\alpha_{q'}^\circ(\kappa)^2|_{\mathbb{C}} = {q'}^{k-1}$ we obtain $|q^{k-s-2}|_{\mathbb{C}} = |{q'}^{s-k+1}|_{\mathbb{C}}$, contradiction. All the other cases are analogous. □

We can than state the main theorem of the section. We exclude the case where $\psi\xi\omega^{-1}$ is quadratic imaginary and $F(\kappa)$ has complex multiplication by the corresponding quadratic field because this case has already been treated in [Hid90]. Recall the "denominator" $H_F(\kappa)$ of l_F^r defined at the end of section 4.4.2.

Theorem 4.7.6. *We have a two-variable p-adic L-function $H_F(\kappa)\Lambda_p(\kappa,\kappa')$ on $\mathcal{C}_F\times\mathcal{W}$, holomorphic in the first variable and of logarithmic growth $h = [2\alpha]+2$ in the second variable such that for all classical points (κ,κ') we have the following interpolation formula*

$$\Lambda_p(\kappa,\kappa') = C_{\kappa,\kappa'}E_1(\kappa,\kappa')E_2(\kappa,\kappa')\frac{L(s+1,\mathrm{Sym}^2(F(\kappa)),\xi^{-1}\varepsilon^{-1}\omega^{-s})}{\pi^{s+1}S(F(\kappa))W'(F(\kappa))\left\langle F^\circ(\kappa),F^\circ(\kappa)\right\rangle}.$$

Proof. We pose

$$\Lambda_p(\kappa,\kappa') := L_p(\kappa,\kappa')A(\kappa,\kappa')^{-1}.$$

We begin by showing that $\Lambda_p(\kappa,\kappa')$ is holomorphic. We know from the definition of $L_p(\kappa,\kappa')$ and Proposition 4.4.16 that all the poles of $L_p(\kappa,\kappa')$ are controlled by $H_F(\kappa)$, that is $H_F(\kappa)L_p(\kappa,\kappa')$ is holomorphic in κ. We have moreover that $A(\kappa,\kappa')^{-1}$ brings no extra poles; indeed, because of the functional equation of Proposition 4.7.4, a zero of $A(\kappa,\kappa')$ induces a pole of $H_F(\kappa)L_p^+(\kappa,\kappa')B(\kappa,\kappa')^{-1}$. But the only poles of the latter could be the zeros of $B(\kappa, \kappa')$. Proposition 4.7.5 tells us that the zeros of $A(\kappa, \kappa')$ and $B(\kappa, \kappa')$ are disjoint and we are done.

To conclude, we have to show the interpolation formula at zeros of $A(\kappa, \kappa')$; for this, it is enough to combine Proposition 4.7.4 and Theorem 4.7.3. $\square$

Chapter 5

Derivative of symmetric square p-adic L-functions via pull-back formula

5.1 Introduction

Let M be a motive over $\mathbb{Q}$ and suppose that it is pure of weight zero and irreducible. We suppose also that $s = 0$ is a critical integer à la Deligne.
Fix a prime number p and let V be the p-adic representation associated to M. We fix once and for all an isomorphism $\mathbb{C} \cong \mathbb{C}_p$. If V is semistable, it is conjectured that for each regular submodule D [Ben11, §0.2] there exists a p-adic L-function $L_p(V, D, s)$. It is supposed to interpolate the special values of the L-function of M twisted by finite-order characters of $1 + p\mathbb{Z}_p$ [PR95], multiplied by a corrective factor (to be thought of as a part of the local epsilon factor at p) which depends on D. In particular, we expect the following interpolation formula at $s = 0$;

$$L_p(V, D, s) = E(V, D)\frac{L(V, 0)}{\Omega(V)},$$

for $\Omega(V)$ a complex period and $E(V, D)$ some Euler type factors which conjecturally have to be removed in order to permit p-adic interpolation (see [Ben11, §2.3.2] for the case when V is crystalline). It may happen that certain of these Euler factors vanish. In this case the connection with what we are interested in, the special values of the L-function, is lost. Motivated by the seminal work of Mazur-Tate-Teitelbaum [MTT86], Greenberg, in the ordinary case [Gre94b], and Benois [Ben11] have conjectured the following;

Conjecture 5.1.1. *[Trivial zeros conjecture] Let e be the number of Euler-type factors of $E_p(V, D)$ which vanish. Then the order of zeros at $s = 0$ of $L_p(V, D, s)$ is e and*

$$\lim_{s\to 0} \frac{L_p(V, D, s)}{s^e} = \mathcal{L}(V, D)E^*(V, D)\frac{L(V, 0)}{\Omega(V)} \tag{5.1.2}$$

for $E^*(V, D)$ the non-vanishing factors of $E(V, D)$ and $\mathcal{L}(V, D)$ a non-zero number called the $\mathcal{L}$-invariant.

There are many different ways in which the $\mathcal{L}$-invariant can be defined. A first attempt at such a definition could that of an *analytic* $\mathcal{L}$-invariant

$$\mathcal{L}^{\mathrm{an}}(V, D) = \frac{\lim_{s\to 0} \frac{L_p(V,D,s)}{s^e}}{E^*(V, D)\frac{L(V,0)}{\Omega(V)}}.$$

Clearly, with this definition, the above conjecture reduces to the statement on the order of $L_p(V, D, s)$ at $s = 0$ and the non-vanishing of the $\mathcal{L}$-invariant.

In [MTT86] the authors give a more arithmetic definition of the $\mathcal{L}$-invariant for an elliptic curve, in terms of an *extended* regulator on the *extended* Mordell-Weil group. The search for an intrinsic, Galois theoretic interpretation of this error factor led to the definition of the *arithmetic* $\mathcal{L}$-invariant $\mathcal{L}^{\mathrm{ar}}(V, D)$ given by Greenberg [Gre94b] (resp. Benois [Ben11]) in the ordinary case (resp. semistable case) using Galois cohomology (resp. cohomology of (φ, Γ)-module).
For 2-dimensional Galois representations many more definitions have been proposed and we refer to [Col05] for a detailed exposition.

When the p-adic L-function can be constructed using an Iwasawa cohomology class and the big exponential [PR95], one can use the machinery developed in [Ben12, §2.2] to prove formula (5.1.2) with $\mathcal{L} = \mathcal{L}^{\mathrm{al}}$. Unluckily, it is a very hard problem to construct classes in cohomology which are related to special values. Kato's Euler system has been used in this way in [Ben12] to prove many instances of Conjecture 5.1.1 for modular forms. It might be possible that the construction of Lei-Loeffler-Zerbes [LLZ12] of an Euler system for the Rankin product of two modular forms could produce such Iwasawa classes for other Galois representations; in particular, for $V = \mathrm{Sym}^2(V_f)(1)$, where V_f is the Galois representation associated to a weight two modular form (see also [Ber98] and the upcoming work of Dasgupta on Greenberg's conjecture for the symmetric square p-adic L-functions of ordinary forms). We also refer the reader to [BDR14].

We present in this paper a different method which has already been used extensively in many cases and which we think to be more easy to apply at the current state: the method of Greenberg and Stevens [Gre94b]. Under certain hypotheses which we shall state in the next section, it allows us to calculate the derivative of $L_p(V, D, s)$.

The main ingredient of their method is the fact that V can be p-adically deformed in a one dimensional family. For example, modular forms can be deformed in a Hida-Coleman family. We have decided to present this method because of the recent developments on families of automorphic forms [AIP12, Bra13] have opened the door to the construction of families of p-adic L-function in many different settings. For example, we refer to the ongoing PhD thesis of Z. Liu on p-adic L-functions for Siegel modular forms.
Consequently, we expect that one could prove many new instances of Conjecture 5.1.1.

In Section 5.4 we shall apply this method to the case of the symmetric square of a modular form which is Steinberg at p. The theorem which will be proved is the following;

Theorem 5.1.3. *Let f be a modular form of trivial Nebentypus, weight k_0 and conductor Np, N squarefree and prime to p. If $p = 2$, suppose then $k = 2$. Then Conjecture 5.1.1 (up to the non-vanishing of the $\mathcal{L}$-invariant) is true for $L_p(s, \mathrm{Sym}^2(f))$.*

In this case there is only one choice for the regular submodule D and the trivial zero appears at $s = k_0 - 1$. This theorem generalizes [Ros13d, Theorem 1.3] in two ways: we allow $p = 2$ in the ordinary case and when $p \neq 2$ we do not require N to be even. In particular, we cover the case of the symmetric square 11-adic L-function for the elliptic curve $X_0(11)$ and of the 2-adic L-function for the symmetric square of $X_0(14)$.

In the ordinary case (*i.e.* for $k = 2$), the proof is completly independent of [Ros13d] as we can construct directly a two-variable p-adic L-function. In the finite slope case we can not construct a two-variable function with the method described below and consequently we need the two-variable p-adic L-function of [Ros13d, Theorem 4.14], which has been constructed only for $p \neq 2$.

In the ordinary setting, the same theorem (but with the hypothesis at 2) has been proved by Greenberg and Tilouine (unpublished). The importance of this formula for the proof of the Greenberg-Iwasawa Main Conjecture [Urb06] has been put in evidence in [HTU97].

We improve the result of [Ros13d] a different construction of the p-adic L-function, namely that of [BS00]. We express the complex L-function using Eisenstein series for GSp_4, a pullback formula to the Igusa divisor and a double Petersson product. We are grateful to É. Urban for having suggested this approach to us. In Section 5.3 we briefly recall the theory of Siegel modular forms and develop a theory of p-adic modular forms for $\mathrm{GL}_2 \times \mathrm{GL}_2$ necessary for the construction of p-adic families of Eisenstein series.

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5.2 The method of Greenberg and Stevens

The aim of this section is to recall the method of Greenberg and Stevens [GS93] to calculate analytic $\mathcal{L}$-invariant. This method has been used successfully many other times [Mok09, Ros13a, Ros13d]. It is very robust and easily adaptable to many situations in which the expected order of the trivial zero is one. We also describe certain obstacles which occur while trying to apply this method to higher order zeros.

We let K be a p-adic local field, $\mathcal{O}$ its valuation ring and Λ the Iwasawa algebra $\mathcal{O}[[T]]$. Let V be a p-adic Galois representation as before.
We denote by $\mathcal{W}$ the rigid analytic space whose $\mathbb{C}_p$-points are $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$. We have a map $\mathbb{Z} \to \mathcal{W}$ defined by $k \mapsto [k] : z \to z^k$. Let us fix once and for all, if $p \neq 2$ (resp. $p = 2$) a generator u of $1 + p\mathbb{Z}_p$ (resp. $1 + 4\mathbb{Z}_2$) and a decomposition $\mathbb{Z}_p^* = \mu \times 1 + p\mathbb{Z}_p$ (resp $\mathbb{Z}_2^* = \mu \times 1 + 4\mathbb{Z}_2$). Here $\mu = \mathbb{G}_m^{\mathrm{tors}}(\mathbb{Z}_p)$. We have the following isomorphism of rigid spaces:

$$\begin{aligned} \mathcal{W} \cong & \mathbb{G}_m^{\mathrm{tors}}(\mathbb{Z}_p)^{\wedge} \times B(1, 1^-) \\ & \kappa \mapsto (\kappa_{|_\mu}, \kappa(u)), \end{aligned}$$

where the first set has the discrete topology and the second is the rigid open unit ball around 1. Let $0 < r < \infty$, $r \in \mathbb{R}$, we define

$$\mathcal{W}(r) = \left\{ (\zeta, z) | \zeta \in \mathbb{G}_m^{\mathrm{tors}}(\mathbb{Z}_p), |z - 1| \leq p^{-r} \right\}.$$

We fix an integer h and we denote by $\mathcal{H}_h$ the algebra of h-admissible distributions over $1 + p\mathbb{Z}_p$ (or $1 + 4\mathbb{Z}_2$ if $p = 2$) with values in K. Here we take the definition of admissibility as in [Pan03, §3], so measures are one-admissible, i.e. $\Lambda \otimes \mathbb{Q}_p \cong \mathcal{H}_1$. The Mellin transform gives us a map

$$\mathcal{H}_h \to \mathrm{An}(0),$$

where $\mathrm{An}(0)$ stands for the algebra of $\mathbb{Q}_p$-analytic, locally convergent functions around 0. If we see $\mathcal{H}_h$ as a subalgebra of the ring of formal series, this amounts to $T \mapsto u^s - 1$.

We suppose that we can construct a p-adic L-function for $L_p(s, V, D)$ and that it presents a single trivial zero.

We suppose also that V can be deformed in a p-adic family $V(\kappa)$. Precisely, we suppose that we are given an affinoid $\mathcal{U}$, finite over $\mathcal{W}(r)$. Let us write $\pi : \mathcal{U} \to \mathcal{W}(r)$. Let us denote by $\mathcal{I}$ the Tate algebra corresponding to $\mathcal{U}$. We suppose that $\mathcal{I}$ is integrally closed and that there exists a *big* Galois representation $V(\kappa)$ with values in $\mathcal{I}$ and a point $\kappa_0 \in \mathcal{U}$ such that $V = V(\kappa_0)$.
We define $\mathcal{U}^{\mathrm{cl}}$ to be the set of $\kappa \in \mathcal{U}$ satisfying the followings conditions:

- $\pi(\kappa) = [k]$, with $k \in \mathbb{Z}$,
- $V(\kappa)$ is motivic,
- $V(\kappa)$ is semistable as $G_{\mathbb{Q}_p}$-representation,
- $s = 0$ is a critical integer for $V(\kappa)$.

We make the following assumption on $\mathcal{U}^{\mathrm{cl}}$.

(**CI**) For every $n > 0$, there are infinitely many κ in $\mathcal{U}^{\mathrm{cl}}$ and $s \in \mathbb{Z}$ such that:

- $|\kappa - \kappa_0|_{\mathcal{U}} < p^{-n}$
- s critical for $V(\kappa)$,
- $s \equiv 0 \bmod p^n$.

This amounts to asking that the couples $(\kappa, [s])$ in $\mathcal{U} \times \mathcal{W}$ with s critical for $V(\kappa)$ accumulate at $(\kappa_0, 0)$.

We suppose that there is a global triangulation $D(\kappa)$ of the (φ, Γ)-module associated to $V(\kappa)$ [Liu13] and that this induces the regular submodule used to construct $L_p(s, V, D)$.
Under these hypotheses, it is natural to conjecture the existence of a two-variable p-adic L-function (depending on $D(\kappa)$) $L_p(\kappa, s) \in \mathcal{I}\hat{\otimes}\mathcal{H}_h$ interpolating the p-adic L-functions of $V(\kappa)$, for all κ in $\mathcal{U}^{\mathrm{cl}}$. Conjecturally [Pan94, Pot13], h should be defined solely in terms of the p-adic Hodge theory of $V(\kappa)$ and $D(\kappa)$.

We make two hypotheses on this p-adic L-function.

i) There exists a subspace of dimension one $(\kappa, s(\kappa))$ containing $(\kappa_0, 0)$ over which $L_p(\kappa, s(\kappa))$ vanishes identically.

ii) There exists an *improved* p-adic L-function $L_p^*(\kappa)$ in $\mathcal{I}$ such that $L_p(\kappa, 0) = E(\kappa) L_p^*(\kappa)$, for $E(\kappa)$ a non-zero element which vanishes at κ_0.

The idea is that *i)* allows us to express the derivative we are interested in in terms of the "derivative with respect to κ." The latter can be calculated using *ii)*. In general, we expect that $s(\kappa)$ is a simple function of $\pi(\kappa)$.

Let $\log_p(z) = -\sum_{n=1}^{\infty} \frac{(1-z)^n}{n}$, for $|z-1|_p < 1$ and

$$\mathrm{Log}_p(\kappa) = \frac{\log_p(\kappa(u^r))}{\log_p(u^r)},$$

for r any integer big enough.
For example, in [Gre94b, Mok09] we have $s(\kappa) = \frac{1}{2}\mathrm{Log}_p(\pi(\kappa))$ and in [Ros13a, Ros13d] we have $s(\kappa) = \mathrm{Log}_p(\pi(\kappa)) - 1$. In the first case the line corresponds to the vanishing on the central critical line which is a consequence of the fact that the ε-factor is constant in the family. In the second case, the vanishing is due to a line of trivial zeros, as all the motivic specializations present a trivial zero.

The idea behind *ii)* is that the Euler factor which brings the trivial zero for V varies analytically along $V(\kappa)$ once one fixes the cyclotomic variable. This is often the case with one dimensional deformations. If we allow deformations of V in more than one variable, it is unlikely that the removed Euler factors define p-analytic functions, due to the fact that *eigenvalues of the crystalline Frobenius do not vary p-adically* or equivalently, that the Hodge-Tate weights are not constant.

We now give the example of families of Hilbert modular forms. For simplicity of notation, we consider a totally real field F of degree d where p is split. Let $\mathbf{f}$ be a Hilbert modular form of weight $(k_1, \ldots, k_d)$ such that the parity of k_i does not depend on i. We define $m = \max(k_i - 1)$ and $v_i = \frac{m+1-k_i}{2}$. We suppose that $\mathbf{f}$ is nearly-ordinary [Hid89b] and let $\mathbf{F}$ be the only Hida family to which $\mathbf{f}$ belongs. For each p-adic place $\mathfrak{p}_i$ of F, the corresponding Hodge-Tate weights are $(v_i, m - v_i)$. This implies that the Fourier coefficient $a_{\mathfrak{p}_i}(\mathbf{F})$ is a p-adic analytic function only if it is divided by p^{v_i}. Unluckily, $a_{\mathfrak{p}_i}(\mathbf{F})$ is the number which appears in the Euler type factor of the evaluation formula for the p-adic L-function of $\mathbf{F}$ or $\mathrm{Sym}^2(\mathbf{F})$. This is why in [Mok09, Ros13a] the authors deal only with forms of parallel weight. It seems very hard to generalize the method of Greenberg and Stevens to higher order derivatives without new ideas.

It may happen that the Euler factor which brings the trivial zero for V is (locally) identically zero on the whole family; this is the case for the symmetric square of a modular form of prime-to-p conductor and more generally for the standard L-function of parallel weight Siegel modular forms of prime-to-p level. That's why in [Ros13a, Ros13d] and in this article we can deal only with forms which are Steinberg at p.

We have seen in the examples above that $s(\kappa)$ is a linear function of the weight. Consequently, one needs to evaluate the p-adic L-function $L_p(V(\kappa), s)$ at s which are big for the archimedean norm. When s is not a critical integer it is quite a hard problem to evaluate the p-adic L-function. This is why we have supposed (**CI**). It is not a hypothesis to be taken for granted. One example is the spinor L-function for genus two Siegel modular forms of any weight, which has only one critical integer.
The improved p-adic L-function is said so because $L_p^*(\kappa_0)$ is supposed to be exactly the special value we are interested in.

The rest of the section is devoted to make precise the expression "derive with respect to κ."
We recall some facts about differentials. We fix a Λ-algebra $\mathcal{I}_1$. We suppose that $\mathcal{I}_1$ is a DVR and a K-algebra. Let $\mathcal{I}_2$ be an integral domain and a local ring which is finite, flat and integrally closed over $\mathcal{I}_1$. We have the first fundamental sequence of Kähler differentials

$$\Omega_{\mathcal{I}_1/K} \otimes_{\mathcal{I}_1} \mathcal{I}_2 \to \Omega_{\mathcal{I}_2/K} \to \Omega_{\mathcal{I}_2/\mathcal{I}_1} \to 0.$$

Under the hypotheses above, we can write $I_2 = \frac{I_1[X]}{P(X)}$. Then, every K-linear derivation of $\mathcal{I}_1$ can be extended to a derivation of $\mathcal{I}_2$ and [Mat80, Theorem 57 ii)] ensures us that the first arrow is injective.

Let P_0 be the prime ideal of $\mathcal{I}$ corresponding to the point κ_0 and P the corresponding ideal in $\mathcal{O}(\mathcal{W}(r))$. We take $\mathcal{I}_1 = \mathcal{O}(\mathcal{W}(r))_P$ and $\mathcal{I}_2 = \mathcal{I}_{P_0}$. The assumption that $\mathcal{I}_2$ is integrally closed is equivalent to ask that $\mathcal{U}$ is smooth at κ_0. In many cases, we expect that $\mathcal{I}_1 \to \mathcal{I}_2$ is étale; under this hypothesis, we can appeal to the fact that locally convergent series are Henselian [Nag62, Theorem 45.5] to define a morphism $\mathcal{I}_2 \to \mathrm{An}(k_0)$ to the ring of meromorphic functions around k_0 which extend the natural inclusion of $\mathcal{I}_1$. Once this morphism is defined, we can derive elements of $\mathcal{I}_2$ as if they were locally analytic functions.

There are some cases in which this morphism is known not to be étale; for example, for certain weight one forms [Dim13b, DG12] and some critical CM forms [Bel12b, Proposition 1]. (Note that in these cases no regular submodule D exists.)

We would like to explain what we can do in the case when $\mathcal{I}_1 \to \mathcal{I}_2$ is not étale. We also hope that what we say will clarify the situation in the case where the morphism is étale.
We have that $\Omega_{\Lambda/K}$ is a free rank 1 Λ-module. Using the universal property of differentials

$$\mathrm{Hom}_\Lambda(\Omega_{\Lambda/K}, \Lambda) = \mathrm{Der}_K(\Lambda, \Lambda),$$

we shall say, by slight abuse of notation, that $\Omega_{\Lambda/K}$ is generated as Λ-module by the derivation $\frac{\mathrm{d}}{\mathrm{d}T}$. Similarly, we identify $\Omega_{\mathcal{I}_1/K}$ with the free $\mathcal{I}_1$-module generated by $\frac{\mathrm{d}}{\mathrm{d}T}$. As the first arrow in the first fundamental sequence is injective, there exists an element $d \in \Omega_{\mathcal{I}_2/K}$ (which we see as K-linear derivation from $\mathcal{I}_2$ to $\mathcal{I}_2$) which extends $\frac{\mathrm{d}}{\mathrm{d}T}$. If $\mathcal{I}_1 \to \mathcal{I}_2$ is étale, then $\Omega_{\mathcal{I}_2/\mathcal{I}_1} = 0$ and the choice of d is unique.

Under the above hypotheses, we can then define a new analytic $\mathcal{L}$-invariant (which a priori depends on the deformation $V(\kappa)$ and d) by

$$\mathcal{L}_d^{\mathrm{an}}(V) := -\log_p(u)\ d(s(\kappa))^{-1} d(E(\kappa))\big|_{\kappa=\kappa_0} = -\ d(s(\kappa))^{-1} \frac{d(L_p(0,\kappa))}{L_p^*(\kappa)}\bigg|_{\kappa=\kappa_0}.$$

We remark that with the notation above we have $\log_p(u)\frac{\mathrm{d}}{\mathrm{d}T} = \frac{\mathrm{d}}{\mathrm{d}s}$. We apply d to $L_p(\kappa, s(\kappa)) = 0$ to obtain

$$0 = d(L_p(\kappa, s(\kappa))) = \log_p(u)^{-1}\ \frac{\mathrm{d}}{\mathrm{d}s} L_p(\kappa, s)\bigg|_{s=s(\kappa)} d(s(\kappa)) + d(L_p(\kappa, s))\big|_{s=s(\kappa)}.$$

Evaluating at $\kappa = \kappa_0$ we deduce

$$\frac{\mathrm{d}}{\mathrm{d}s} L_p(V, s)\bigg|_{s=0} = \mathcal{L}_d^{\mathrm{an}}(V) L_p^*(\kappa_0)$$

and consequently

$$\mathcal{L}^{\mathrm{an}}(V) = \mathcal{L}_d^{\mathrm{an}}(V).$$

In the cases in which $\mathcal{L}^{\mathrm{al}}$ has been calculated, namely symmetric powers of Hilbert modular forms, it is expressed in terms of the logarithmic derivative of Hecke eigenvalues at p of certain finite slope families [Ben10, HJ13, Hid06, Mok12]. Consequently, the above formula should allow us to prove the equality $\mathcal{L}^{\mathrm{an}} = \mathcal{L}^{\mathrm{al}}$.
Moreover, the fact that the $\mathcal{L}$-invariant is a derivative of a non constant function shows that $\mathcal{L}^{\mathrm{an}} \neq 0$ outside a codimension 1 subspace of the weight space. In this direction, positive results for a given V have been obtained only in the cases of a Hecke character of a quadratic imaginary field and of an elliptic curve with p-adic uniformization, using a deep theorem in transcendent number theory [BSDGP96].

5.3 Eisenstein measures

In this section we first fix the notation concerning genus two Siegel forms. We then recall a normalization of certain Eisenstein series for $\mathrm{GL}_2 \times \mathrm{GL}_2$ and develop a theory of p-adic families of modular forms (of parallel weight) on $\mathrm{GL}_2 \times \mathrm{GL}_2$. Finally we construct two Eisenstein measures which will be used in the next section to construct two p-adic L-functions.

5.3.1 Siegel modular forms

We now recall the basic theory of Siegel modular forms. We follow closely the notation of [BS00] and we refer to the first section of *loc. cit.* for more details. Let us denote by $\mathbb{H}_1$ the complex upper half-plane and by $\mathbb{H}_2$ the Siegel space for GSp_4. We have explicitly

$$\mathbb{H}_2 = \left\{ Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \middle| Z^t = Z \text{ and } \mathrm{Im}(Z) > 0 \right\}.$$

It has a natural action of $\mathrm{GSp}_4(\mathbb{R})$ via fractional linear transformation; for any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $\mathrm{GSp}_4^+(\mathbb{R})$ and Z in $\mathbb{H}_2$ we define

$$M(Z) = (AZ + B)(CZ + D)^{-1}.$$

Let $\Gamma = \Gamma_0^{(2)}(N)$ be the congruence subgroup of $\mathrm{Sp}_4(\mathbb{Z})$ of matrices whose lower block C is congruent to 0 modulo N. We consider the space $M_k^{(2)}(N, \phi)$ of scalar Siegel forms of weight k and Nebentypus ϕ:

$$\left\{ F : \mathbb{H}_2 \to \mathbb{C} \middle| F(M(Z))(CZ + D)^{-k} = \phi(M)F(Z) \ \forall M \in \Gamma, f \text{ holomorphic} \right\}.$$

Each F in $M_k(\Gamma, \phi)$ admits a Fourier expansion

$$F(Z) = \sum_T a(T) e^{2\pi i \mathrm{tr}(TZ)},$$

where $T = \begin{pmatrix} T_1 & T_2 \\ T_2 & T_4 \end{pmatrix}$ ranges over all matrices T positive and semi-defined, with T_1, T_4 integer and T_2 half-integer.

We have two embeddings (of algebraic groups) of SL_2 in Sp_4:

$$\mathrm{SL}_2^{\uparrow}(R) = \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(R) \right\},$$
$$\mathrm{SL}_2^{\downarrow}(R) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(R) \right\}.$$

We can embed $\mathbb{H}_1 \times \mathbb{H}_1$ in $\mathbb{H}_2$ in the following way

$$(z_1, z_4) \mapsto \begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix}.$$

If γ belongs to $\mathrm{SL}_2(\mathbb{R})$, we have

$$\gamma^{\uparrow} \begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix} = \begin{pmatrix} \gamma(z_1) & 0 \\ 0 & z_4 \end{pmatrix},$$
$$\gamma^{\downarrow} \begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix} = \begin{pmatrix} z_1 & 0 \\ 0 & \gamma(z_4) \end{pmatrix}.$$

Consequently, evaluation at $z_2 = 0$ gives us a map

$$M_k^{(2)}(N, \phi) \hookrightarrow M_k(N, \phi) \otimes_{\mathbb{C}} M_k(N, \phi),$$

where $M_k(N,\phi)$ denotes the space of elliptic modular forms of weight k, level N and Nebentypus ϕ. This also induces a closed embedding of two copies of the modular curve in the Siegel threefold. We shall call its image the Igusa divisor. On points, it corresponds to abelian surfaces which decompose as the product of two elliptic curves.

We consider the following differential operators on $\mathbb{H}_2$:

$$\partial_1 = \frac{\partial}{\partial z_1}, \quad \partial_2 = \frac{1}{2}\frac{\partial}{\partial z_2}, \quad \partial_4 = \frac{\partial}{\partial z_4}.$$

We define

$$\begin{aligned}\mathfrak{D}_l =& z_2\left(\partial_1\partial_4 - \partial_2^2\right) - \left(l - \frac{1}{2}\right)\partial_2,\\ \mathfrak{D}_l^s =& \mathfrak{D}_{l+s-1} \circ \ldots \circ \mathfrak{D}_l,\\ \mathring{\mathfrak{D}}_l^s =& \mathfrak{D}_l^s|_{z_2=0}.\end{aligned}$$

The importance of $\mathring{\mathfrak{D}}_l^s$ is that it preserves holomorphicity.
Let $I = \begin{pmatrix} T_1 & T_2 \\ T_2 & T_4 \end{pmatrix}$. We define $\mathfrak{b}_l^s(I)$ to be the only homogeneous polynomial in the indeterminates T_1, T_2, T_4 of degree s such that

$$\mathring{\mathfrak{D}}_l^s e^{T_1 z_1 + 2T_2 z_2 + T_4 z_4} = \mathfrak{b}_l^s(I) e^{T_1 z_1 + T_4 z_4}.$$

We need to know a little bit more about the polynomial $\mathfrak{b}_l^s(I)$. Let us write

$$\mathfrak{D}_l^s e^{T_1 z_1 + 2T_2 z_2 + T_4 z_4} = P_l^s(z_2, I) e^{T_1 z_1 + 2T_2 z_2 + T_4 z_4},$$

we have

$$\mathfrak{D}_l^{s+1} e^{\mathrm{tr}(ZI)} = \left[P_{l+s}^1(z_2,I)P_l^s(z_2,I) - \left(z_2\frac{1}{4}\frac{\partial^2}{\partial z_2^2} + \left(l + s - \frac{1}{2}\right)\frac{1}{2}\frac{\partial}{\partial z_2}\right)P_l^s(z_2,I)\right]e^{tr(ZI)}. \tag{5.3.1}$$

We obtain easily

$$\begin{aligned}P_l^1(z_2,I) =& \left(z_2\det(I) - \left(l - \frac{1}{2}\right)T_2\right),\\ \mathfrak{b}_l^s(I) =& P_l^s(0,I).\end{aligned} \tag{5.3.2}$$

We have

$$\begin{aligned}\mathfrak{b}_l^{s+1}(I) =& \mathfrak{b}_l^s(I)\mathfrak{b}_{l+s}^1(I) + (l+s-1/2)\partial_2 P_l^s(z_2,I)\\ =& (l+s-1/2)(-T_2\mathfrak{b}_l^s(I) + \partial_2 P_l^s(z_2,I)|_{z_2=0}).\end{aligned}$$

Let $J = \{j_1, j_2, j_4\}$. We shall write ∂^J resp. z^J for $\partial_1^{j_1}\partial_2^{j_2}\partial_4^{j_4}$ resp. $z_1^{j_1}z_2^{j_2}z_4^{j_4}$. We can write [BS00, page 1381]

$$\mathring{\mathfrak{D}}_l^s = \sum_{j_1+j_2+j_4=s} c_l^J \partial^J, \tag{5.3.3}$$

where $c_l^J = \frac{\mathring{\mathfrak{D}}_l^s(z^J)}{\partial^J(z^J)}$. We have easily:

$$\begin{aligned}\partial^J(z^J) =& j_1!j_2!j_4!2^{-j_2},\\ \partial^J(e^{\mathrm{tr}(ZI)}) =& T_1^{j_1}T_2^{j_2}T_4^{j_4}(e^{\mathrm{tr}(ZI)}),\end{aligned}$$

We pose

$$
\begin{aligned}
c_l^s :=& c_l^{0,s,0},\\
(-1)^s c_l^s =& \prod_{i=1}^{s}\left(l-1+s-\frac{i}{2}\right) = 2^{-s}\frac{(2l-2+2s-1)!}{(2l-2+s-1)!}.
\end{aligned}
$$

Consequently, for $L \mid T_1$, T_4 and for any positive integer d, we obtain

$$4^s \mathfrak{b}_{t+1}^s(I) \equiv (-1)^s 4^s \sum_{j_1+j_4<d} c_{t+1}^J T_1^{j_1} T_2^{s-j_1-j_4} T_4^{j_4} \bmod L^d.$$

5.3.2 Eisenstein series

The aim of this section is to recall certain Eisenstein series which can be used to construct the p-adic L-functions, as in [BS00]. In *loc. cit.* the authors consider certain Eisenstein series for GSp_{4g} whose pullback to the Igusa divisor is a **holomorphic** Siegel modular form.
We now fix a (parallel weight) Siegel modular form f for GSp_{2g}. We write the standard L-function of f as a double Petersson product between f and these Eisenstein series (see Proposition 5.3.5). When $g=1$, the standard L-function of f coincides, up to a twist, with the symmetric square L-function of f we are interested in.

In general, for an algebraic group bigger than GL_2 it is quite hard to find the normalization of the Eisenstein series which maximizes, in a suitable sense, the p-adic behavior of its Fourier coefficients. In [BS00, §2] the authors develop a *twisting method* which allow them to define Eisenstein series whose Fourier coefficients satisfy Kummer's congruences when the character associated with the Eisenstein series varies p-adically. This is the key for their construction of the one variable (cyclotomic) p-adic L-function and of our two-variable p-adic L-function.
When the character is trivial modulo p there exists a simple relation between the twisted and the not-twisted Eisenstein series [BS00, §6 Appendix]. To construct the improved p-adic L-function, we shall simply interpolate the not-twisted Eisenstein series.

Let us now recall these Fourier developments.
We fix a weight k, an integer N prime to p and a Nebentypus ϕ. Let f be an eigenform in $M_k(Np,\phi)$, of finite slope for the Hecke operator U_p. We write $N = N_{\mathrm{ss}}N_{\mathrm{nss}}$, where N_{ss} (resp. N_{nss}) is divisible by all primes $q \mid N$ such that $U_q f = 0$ (resp. $U_q f \neq 0$). Let R be an integer coprime with N and p and N_1 a positive integer such that $N_{\mathrm{ss}} \mid N_1 \mid N$. We fix a Dirichlet character χ modulo $N_1 R p$ which we write as $\chi_1\chi'\varepsilon_1$, with χ_1 defined modulo N_1, χ' primitive modulo R and ε_1 defined modulo p. We shall explain after Proposition 5.3.5 why we introduce χ_1.
Let $t \geq 1$ be an integer and $\mathbb{F}^{t+1}\left(w,z,R^2N^2p^{2n},\phi,u\right)^{(\chi)}$ be the twisted Eisenstein series of [BS00, (5.3)].
We define

$$\mathcal{H}'_{L,\chi}(z,w) := L(t+1+2s,\phi\chi)\mathring{\mathfrak{D}}_{t+1}^s\left(\mathbb{F}^{t+1}\left(w,z,R^2N^2p^{2n},\phi,u\right)^{(\chi)}\right)|^z U_{L^2}\,|^w U_{L^2}$$

for s a non-negative integer and $p^n \mid L$, with L a p-power. It is a form for $\Gamma_0(N^2R^2p)\times\Gamma_0(N^2R^2p)$ of weight $t+1+s$.
We shall sometimes choose $L=1$ and in this case the level is $N^2R^2p^{2n}$.
For any prime number q and matrices I as in the previous section, let $B_q(X,I)$ be the polynomial of degree at most 1 of [BS00, Proposition 5.1]. We pose

$$B(t) = (-1)^{t+1}\frac{2^{1+2t}}{\Gamma(3/2)}\pi^{\frac{5}{2}}.$$

We deduce easily from [BS00, Theorem 7.1] the following theorem

Theorem 5.3.4. *The Eisenstein series defined above has the following Fourier development;*

$$\begin{aligned}\mathcal{H}'_{L,\chi}(z,w)|_{u=\frac{1}{2}-t} =& B(t)(2\pi i)^s G(\chi) \sum_{T_1\geq 0}\sum_{T_4 \geq 0} \\ & \left(\sum_I \mathfrak{b}^s_{t+1}(I)(\chi)^{-1}(2T_2) \sum_{G\in \mathrm{GL}_2(\mathbb{Z})\backslash \mathbf{D}(I)} (\phi\chi)^2(\det(G))|\det(G)|^{2t-1} \right. \\ & \left. L(1-t, \sigma_{-\det(2I)}\phi\chi) \prod_{q|\det(2G^{-t}IG^{-1})} B_q\left(\chi\phi(q)q^{t-2}, G^{-t}IG^{-1}\right)\right) e^{2\pi i (T_1 z + T_2 w)},\end{aligned}$$

where the sum over I runs along the matrices $\begin{pmatrix} L^2T_1 & T_2 \\ T_2 & L^2T_4 \end{pmatrix}$ positive definite and with $2T_2 \in \mathbb{Z}$, and

$$\mathbf{D}(I) = \left\{ G \in M_2(\mathbb{Z}) | G^{-t}IG^{-1} \text{ is a half-integral symmetric matrix} \right\}.$$

Proof. The only difference from *loc. cit.* is that we do not apply $\left|\begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}\right.$. □

In fact, contrary to [BS00], we prefer to work with $\Gamma_0(N^2S)$ and not with the opposite congruence subgroup.
In particular each sum over I is finite because I must have positive determinant. Moreover, we can rewrite it as a sum over T_2, with $(2T_2, p) = 1$ and $T_1T_4 - T_2^2 > 0$.

It is proved in [BS00, Theorem 8.5] that (small modifications of) these functions $\mathcal{H}'_{L,\chi}(z,w)$ satisfy Kummer's congruences. The key fact is what they call the twisting method [BS00, (2.18)]; the Eisenstein series $\mathbb{F}^{t+1}\left(w,z,R^2N^2p^{2n},\phi,u\right)^{(\chi)}$ are obtained weighting $\mathbb{F}^{t+1}\left(w,z,R^2N^2p^{2n},\phi,u\right)$ with respect to χ over integral matrices modulo NRp^n. To ensure these Kummer's congruences, even when p does not divide the conductor of χ, the authors are forced to consider χ of level divisible by p. Using nothing more than Tamagawa's rationality theorem for GL_2, they find the relation [BS00, (7.13')]:

$$\mathbb{F}^{t+1}\left(w,z,R^2N^2p,\phi,u\right)^{(\chi)} = \mathbb{F}^{t+1}\left(w,z,R^2N^2p,\phi,u\right)^{(\chi_1\chi')} \left| \left(\mathrm{id} - p\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}\right)\right. .$$

So the Eisenstein series we want to interpolate to construct the improved p-adic L-function is

$$\mathcal{H}'^{*}_{L,\chi'}(z,w) := L(t+1+2s,\phi\chi)\mathring{\mathfrak{D}}^s_{t+1}\left(\mathbb{F}^{t+1}\left(w,z,R^2N^2p,\phi,u\right)^{(\chi_1\chi')}\right).$$

In what follows, we shall specialize $t = k - k_0 + 1$ (for k_0 the weight of the form in the theorem of the introduction) to construct the *improved* one variable p-adic L-function.
For each prime q, let us denote by α_q and β_q the roots of the Hecke polynomial at q associated to f. We define

$$D_q(X) := (1-\alpha_q^2X)(1-\alpha_q\beta_qX)(1-\beta_q^2X).$$

For each Dirichlet character χ we define

$$\mathcal{L}(s, \mathrm{Sym}^2(f), \chi) := \prod_q D_q(\chi(q)q^{-s})^{-1}.$$

This L-function differs from the motivic L-function $L(s, \mathrm{Sym}^2(\rho_f)\otimes\chi)$ by a finite number of Euler factors at prime dividing N. We conclude with the integral formulation of $\mathcal{L}(s, \mathrm{Sym}^2(f), \chi)$ [BS00, Theorem 3.1, Proposition 7.1 (7.13)].

Proposition 5.3.5. *Let f be a form of weight k, Nebentypus ϕ. We put $t+s=k-1$, $s_1=\frac{1}{2}-t$ and $\mathcal{H}'=\mathcal{H}'_{1,\chi}(z,w)|_{u=s_1}$; we have*

$$\begin{aligned}\left\langle f(w)\middle|\begin{pmatrix}0 & -1\\ N^2p^{2n}R^2 & 0\end{pmatrix},\mathcal{H}'\right\rangle_{N^2p^{2n}R^2} =&\frac{\Omega_{k,s}(s_1)p_{s_1}(t+1)}{\chi(-1)d_{s_1}(t+1)}(RN_1p^n)^{s+3-k}\left(\frac{N}{N_1}\right)^{2-k}\times\\ &\times\mathcal{L}(s+1,\mathrm{Sym}^2(f),\chi^{-1})f(z)|U_{N^2/N_1^2}\end{aligned}$$

for

$$\begin{aligned}\frac{p_{s_1}(t+1)}{d_{s_1}(t+1)}=&\frac{c^s_{t+1}}{c^s_{\frac{3}{2}}}=\frac{\prod_{i=1}^s\left(s+t-\frac{i}{2}\right)}{\prod_{i=1}^s\left(s-\frac{i-1}{2}\right)},\\ \Omega_{k,s}\left(\frac{1}{2}-t\right)=&2^{2t}(-1)^{\frac{k}{2}}\pi\frac{\Gamma\left(k-t\right)\Gamma\left(k-t-\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}.\end{aligned}$$

Proof. With the notation of [BS00, Theorem 3.1] we have $M=R^2N^2p^{2n}$ and $N=N_1Rp^n$. We have that $\frac{d_{s_1}(t+1)}{p_{s_1}(t+1)}\mathcal{H}'$ is the holomorphic projection of the Eisenstein series of [BS00, Theorem 3.1] (see [BS00, (1.30),(2.1),(2.25)]). The final remark to make is the following relation between the standard (or adjoint) L-function of f and the symmetric square one:

$$\mathcal{L}(1-t,\mathrm{Ad}(f)\otimes\phi,\chi)=\mathcal{L}(k-t,\mathrm{Sym}^2(f),\chi).$$

□

The authors of [BS00] prefer to work with an auxiliary character modulo N to remove all the Euler factors at bad primes of f, but we do not want to do this. Still, we have to make some assumptions on the level χ. Suppose that there is a prime q dividing N such that $q\nmid N_1$ and $U_qf=0$, then the above formula would give us zero. That is why we introduce the character χ_1 defined modulo a multiple of N_{ss}.
At the level of L-function this does not change anything as for $q\mid N_{\mathrm{ss}}$ we have $D_q(X)=1$.
For f as in Theorem 5.1.3 we can take $N_1=1$.

5.3.3 Families for $\mathrm{GL}_2\times\mathrm{GL}_2$

The aim of this section is the construction of families of modular forms on two copies of the modular curves. Let us fix a tame level N, and let us denote by X the compactified modular curve of level $\Gamma_1(N)\cap\Gamma_0(p)$. For $n\geq 2$, we shall denote by $X(p^n)$ the modular curve of level $\Gamma_1(N)\cap\Gamma_0(p^n)$.
We denote by $X(v)$ the tube of ray p^{-v} of the ordinary locus. We fix a p-adic field K. We recall from [AIS12, Pil13] that, for v and r suitable, there exists an invertible sheaf ω^κ on $X(v)\times_K\mathcal{W}(r)$. This allows us to define families of overconvergent forms as

$$M_\kappa(N):=\varinjlim_v H^0(X(v)\times_K\mathcal{W}(r),\omega^\kappa).$$

We denote by $\omega^{\kappa,(2)}$ the sheaf on $X(v)\times_K X(v)\times_K\mathcal{W}(r)$ obtained by base change over $\mathcal{W}(r)$.
We define

$$\begin{aligned}M^{(2)}_{\kappa,v}(N):=&H^0(X(v)\times_K X(v)\times_K\mathcal{W}(r),\omega^{\kappa,(2)})\\ =&H^0(X(v)\times\mathcal{W}(r),\omega^\kappa)\hat{\otimes}_{\mathcal{O}(\mathcal{W}(r))}H^0(X(v)\times\mathcal{W}(r),\omega^\kappa);\\ M^{(2)}_\kappa(N):=&\varinjlim_v M^{(2)}_{\kappa,v}(N).\end{aligned}$$

We believe that this space should correspond to families of Siegel modular forms [AIP12] of parallel weight restricted on the Igusa divisor, but for shortness of exposition we do not examine this now.

We have a correspondence C_p above $X(v)$ defined as in [Pil13, §4.2]. We define by fiber product the correspondence C_p^2 on $X(v) \times_K X(v)$ which we extend to $X(v) \times_K X(v) \times_K \mathcal{W}(r)$. This correspondence induces a Hecke operator $U_p^{\otimes^2}$ on $H^0(X(v) \times_K X(v) \times_K \mathcal{W}(r), \omega^{\kappa,(2)})$ which corresponds to $U_p \otimes U_p$. These are potentially orthonormalizable $\mathcal{O}(\mathcal{W}(r))$-modules [Pil13, §5.2] and $U_p^{\otimes^2}$ acts on these spaces as a completely continuous operator (or compact, in the terminology of [Buz07, §1]) and this allows us to write

$$M_\kappa^{(2)}(N)^{\leq \alpha} = \bigoplus_{\alpha_1+\alpha_2 \leq \alpha} M_\kappa(N)^{\leq \alpha_1} \hat{\otimes}_{\mathcal{O}(\mathcal{W}(r))} M_\kappa(N)^{\leq \alpha_2}. \qquad (5.3.6)$$

Here and in what follows, for A a Banach ring, M a Banach A-module and U a completely continuous operator on M, we write $M^{\leq \alpha}$ for the finite dimensional submodule of generalized eigenspaces associated to the eigenvalues of U of valuation smaller or equal than α. We write $\mathrm{Pr}^{\leq \alpha}$ for the corresponding projection. We remark [Urb11, Lemma 2.3.13] that there exists $v > 0$ such that

$$M_\kappa^{(2)}(N)^{\leq \alpha} = M_{\kappa,v'}^{(2)}(N)^{\leq \alpha}$$

for all $0 < v' < v$. We define similarly $M_\kappa^{(2)}(Np^n)$.

We now use the above Eisenstein series to give examples of families. More precisely, we shall construct a two-variable measure (which will be used for the two variables p-adic L-function in the ordinary case) and a one variable measure (which will be used to construct the improved one variable L-function) without the ordinary assumption.
Let us fix $\chi = \chi_1 \chi' \varepsilon_1$ as before. We suppose χ even. We recall the Kubota-Leopoldt p-adic L-function;

Theorem 5.3.7. *Let η be a even Dirichlet character. There exists a p-adic L-function $L_p(\kappa, \eta)$ satisfying for any integer $t \geq 1$ and finite-order character ε of $1 + p\mathbb{Z}_p$*

$$L_p(\varepsilon(u)[t], \eta) = (1 - (\varepsilon\omega^{-t}\eta)_0(p))L(1-t, \varepsilon\omega^{-t}\eta),$$

where η_0 stands for the primitive character associated to η. If η is not trivial then $L_p([t], \eta)$ is holomorphic. Otherwise, it has a simple pole at $[0]$.

We can consequently define a p-adic analytic function interpolating the Fourier coefficients of the Eisenstein series defined in the previous section; for any z in $\mathbb{Z}_p^*$, we define $l_z = \frac{\log_p(z)}{\log_p(u)}$. We define also

$$\begin{aligned} a_{T_1,T_4,L}(\kappa, \kappa') = \Bigg(& \sum_I \kappa(u_{2T_2}) \chi^{-1}(2T_2) \times \\ & \times \sum_{G \in \mathrm{GL}_2(\mathbb{Z}) \backslash \mathbf{D}(I)} (\phi\chi)^2(\det(G)) |\det(G)|^{-1} {\kappa'}^2(u^{l_{|\det(G)|}}) \\ & L_p(\kappa', \sigma_{-\det(2I)}\phi\chi) \prod_{q | \det(2G^{-t}IG^{-1})} B_q\left(\phi(q)\kappa'(u^{l_q})q^{-2}, G^{-t}IG^{-1}\right) \Bigg). \end{aligned}$$

If $p = 2$, then $\chi^{-1}(2T_2)$ vanishes when T_2 is an integer, so the above above sum is only on half-integral T_2 and $\kappa(u_{2T_2})$ is a well-define 2-analytic function. Moreover, it is 2-integral.
We recall that if $p^j \mid T_1, T_4$ we have [BS00, (1.21, 1.34)]

$$4^s \mathfrak{b}_{t+1}^s(I) \equiv (-1)^s 4^s c_{t+1}^s T_2^s \bmod p^d,$$

for $s = k - t - 1$. Consequently, if we define

$$\mathcal{H}_L(\kappa, \kappa') = \sum_{T_1 \geq 0} \sum_{T_4 \geq 0} a_{T_1,T_4,L}(\kappa[-1]{\kappa'}^{-1}, \kappa') q_1^{T_1} q_2^{T_4}$$

we have,

$$(-1)^s 2^s c_{t+1}^s A \mathcal{H}_L([k], \varepsilon[t]) \equiv 2^s \mathcal{H}'_{L,\chi\varepsilon\omega^{-s}}(z, w) \bmod L^r,$$

with

$$A = A(t, k, \varepsilon) = B(t)(2\pi i)^s G\left(\chi\varepsilon\omega^{-s}\right).$$

We have exactly as in [Pan03, Definition 1.7] the following lemma;

Lemma 5.3.8. *There exists a projector*

$$\mathrm{Pr}_{\infty}^{\leq\alpha} : \bigcup_n M_{\kappa}^{(2)}(Np^n) \to M_{\kappa}^{(2)}(N)^{\leq\alpha}$$

which on $M_{\kappa}^{(2)}(Np^n)$ is $(U_p^{\otimes^2})^{-i} \mathrm{Pr}^{\leq\alpha} (U_p^{\otimes^2})^i$, independent of $i \geq n$.

When $\alpha = 0$, we shall write $\mathrm{Pr}_{\infty}^{\mathrm{ord}}$.

We shall now construct the improved Eisenstein family. Fix $k_0 \geq 2$ and $s_0 = k_0 - 2$. It is easy to see from (5.3.2) and (5.3.1) that when k varies p-adically the value $\mathfrak{b}_k^{s_0}(I)$ varies p-adically analytic too. We define $\mathfrak{b}_{\kappa}^{s_0}(I)$ to be the only polynomial in $\mathcal{O}(\mathcal{W}(r))[T_i]$, homogeneous of degree s_0 such that $\mathfrak{b}_{[t]}^{s_0}(I) = \mathfrak{b}_{t+1}^{s_0}(I)$. Its coefficients are products of $\mathrm{Log}_p(\kappa[i])$. We let $\chi = \chi_1 \chi' \omega^{k_0-2}$ and we define

$$a^*_{T_1,T_4}(\kappa) = \left(\sum_I \mathfrak{b}_{\kappa}^{s_0}(I) (\chi'\chi_1)^{-1}(2T_2) \sum_{G \in \mathrm{GL}_2(\mathbb{Z}) \backslash \mathbf{D}(I)} (\phi\chi'\chi_1)^2(\det(G)) |\det(G)|^{-1} \kappa(u^{l_{2|\det(G)|}}) \times \right.$$
$$\left. \times L_p(\kappa, \sigma_{-\det(2I)}\phi\chi'\chi_1) \prod_{q | \det(2G^{-t}IG^{-1})} B_q\left(\phi\chi'\chi_1(q)\kappa(u^{l_q})q^{-2}, G^{-t}IG^{-1}\right) \right).$$

We now construct another p-adic family of Eisenstein series;

$$\mathcal{H}^*(\kappa) = \mathrm{Pr}^{\leq\alpha} (-1)^{k_0} \sum_{T_1 \geq 0} \sum_{T_4 \geq 0} a^*_{T_1,T_4}(\kappa[1 - k_0]) q_1^{T_1} q_2^{T_2}.$$

Proposition 5.3.9. *Suppose $\alpha = 0$. We have a p-adic family $\mathcal{H}(\kappa, \kappa') \in M_{\kappa}^{(2)}(N^2R^2)^{\mathrm{ord}}$ such that*

$$\mathcal{H}([k], \varepsilon[t]) = \frac{1}{(-1)^s c_{t+1}^s A} (U_p^{\otimes^2})^{-2i} \mathrm{Pr}^{\mathrm{ord}} \mathcal{H}'_{p^i,\chi\varepsilon\omega^{-s}}(z, w).$$

For any α, we have $\mathcal{H}^*(\kappa) \in M_{\kappa}^{(2)}(N^2R^2)^{\leq\alpha}$ such that

$$\mathcal{H}^*([k]) = {A^*}^{-1} (-1)^{k_0} \mathrm{Pr}^{\leq\alpha} {\mathcal{H}'}^*_{1,\chi_1\chi'}(z, w)|_{u = \frac{1}{2} - k + k_0 - 1},$$

for

$$A^* = B(k - k_0 + 1)(2\pi i)^{k_0 - 2} G(\chi'\chi_1).$$

Proof. From its own definition we have $(U_p^{\otimes^2})^{2j}\mathcal{H}'_{1,\chi}(z,w) = \mathcal{H}'_{p^j,\chi}(z,w)$. We define

$$\mathcal{H}(\kappa,\kappa') = \varinjlim_j (U_p^{\otimes^2})^{-2j}\mathrm{Pr}^{\mathrm{ord}}\mathcal{H}_{p^j,\chi}(\kappa,\kappa').$$

With Lemma 5.3.8 and the previous remark we obtain

$$\begin{aligned}4^s(U_p^{\otimes^2})^{-2i}\mathrm{Pr}^{\mathrm{ord}}\mathcal{H}'_{p^i,\chi\varepsilon\omega^{-s}}(z,w) =& 4^s(U_p^{\otimes^2})^{-2j}\mathrm{Pr}^{\mathrm{ord}}\mathcal{H}'_{p^j,\chi\varepsilon\omega^{-s}}(z,w)\\ \equiv& A(-1)^s 4^s c_t^s(U_p^{\otimes^2})^{-2j}\mathrm{Pr}^{\mathrm{ord}}\mathcal{H}_{p^j}([k],\varepsilon[t]) \bmod p^{2j},\end{aligned}$$

as U_p^{-1} acts on the ordinary part with norm 1.
The punctual limit is then a well-defined classical, finite slope form. By the same method of proof of [Urb, Corollary 3.4.7] or [Ros13d, Proposition 2.17] (in particular, recall that when the slope is bounded, the ray of overconvergence v can be fixed), we see that this q-expansion defines a family of finite slope forms.
For the second family, we remark that $\mathrm{Log}_p(\kappa)$ is bounded on $\mathcal{W}(r)$, for any r and we reason as above. □

We want to explain briefly why the construction above works in the ordinary setting and not in the finite slope one.
It is slightly complicated to explicitly calculate the polynomial $\mathfrak{b}_{t+1}^s(I)$ and in particulat to show that they vary p-adically when varying s. But we know that $\mathring{\mathfrak{D}}_l^s$ is an homogeneous polynomial in ∂_i of degree s. Suppose for now $\alpha = 0$, *i.e.* we are in the ordinary case. We have a single monomial of $\mathring{\mathfrak{D}}_l^s$ which does not involve ∂_1 and ∂_4, namely $c_{t+1}^s\partial_2^s$. Consequently, in $\mathfrak{b}_{t+1}^s(I)$ there is a single monomial without T_1 and T_4. When the entries on the diagonal of I are divisible by p^i, $4^s\mathfrak{b}_{t+1}^s(I)$ reduces to $(-1)^s 4^s c_{t+1}^s$ modulo p^i. Applying $U_p^{\otimes^2}$ many times ensures us that T_1 and T_4 are very divisible by p. Speaking p-adically, we approximate $\mathring{\mathfrak{D}}_l^s$ by ∂_2 (multiplied by a constant). The more times we apply $U_p^{\otimes^2}$, the better we can approximate p-adically $\mathring{\mathfrak{D}}_l^s$ by ∂_2^s. At the limit, we obtain equality.

For $\alpha > 0$, when we apply $U_p^{\otimes^2 -1}$ we introduce denominators of the order of p^α. In order to construct a two-variable family, we should approximate $\mathring{\mathfrak{D}}_l^s$ with higher precision. For example, it would be enough to consider the monomials ∂^J for $j_1 + j_4 \leq \alpha$. In fact, ∂_1 and ∂_4 increase the slope of U_p; we have the relation

$$\left(U_p^{\otimes^2}\partial_1\partial_4\right)_{|z_2=0} = p^2\left(\partial_1\partial_4 U_p^{\otimes^2}\right)_{|z_2=0}.$$

Unluckily, it seems quite hard to determine explicitly the coefficients c_{t+1}^J of (5.3.3) or even show that they satisfies some p-adic congruences as done in [CP04, Gor06, Ros13d]. We guess that it could be easier to interpolate p-adically the projection to the ordinary locus of the Eisenstein series of [BS00, Theorem 3.1] rather than the holomorphic projection as we are doing here (see also [Urb, §3.4.5] for the case of nearly holomorphic forms for GL_2).
This should remind the reader of the fact that on p-adic forms the Maaß-Shimura operator and Dwork Θ-operator coincide [Urb, §3.2].

5.4 p-adic L-functions

We now construct two p-adic L-functions using the above Eisenstein measure: the two-variable one in the ordinary case and the improved one for any finite slope. Necessary for the construction is a p-adic Petersson product [Pan03, §6] which we now recall. We fix a family $F = F(\kappa)$ of finite slope modular forms which we suppose primitive, i.e. all its classical specialization are primitive forms, of prime-to-p conductor N. We consider characters χ, χ' and χ_1 as in Section 5.3.2. We keep the same decomposition for N (because the local behavior at q is constant along the family). We shall write N_0 for the conductor of χ_1.

(**notCM**) We suppose that F has not complex multiplication by χ.

Let $\mathcal{C}_F$ be the corresponding irreducible component of the Coleman-Mazur eigencurve. It is finite flat over $\mathcal{W}(r)$, for a certain r. Let $\mathcal{I}$ be the coefficients ring of F and $\mathcal{K}$ its field of fraction. For a classical form f, let us denote by f^c the complex conjugated form. We denote by τ_N the Atkin-Lehner involution of level N normalized as in [Hid90, h4]. When the level is clear from the context, we shall simply write τ.
Standard linear algebra allows us to define a $\mathcal{K}$-linear form l_F on $M_\kappa(N)^{\leq \alpha} \otimes_{\mathcal{I}} \mathcal{K}$ with the following property [Pan03, Proposition 6.7];

Proposition 5.4.1. *For all $G(\kappa)$ in $M_\kappa(N)^{\leq \alpha} \otimes_{\mathcal{I}} \mathcal{K}$ and any κ_0 classical point, we have*

$$l_F(G(\kappa))_{|\kappa=\kappa_0} = \frac{\langle F(\kappa_0)^c|\tau_{Np}, G(\kappa_0)\rangle_{Np}}{\langle F(\kappa_0)^c|\tau_{Np}, F(\kappa_0)\rangle_{Np}}.$$

We can find $H_F(\kappa)$ in $\mathcal{I}$ such that $H_F(\kappa)l_F$ is defined over $\mathcal{I}$. We shall refer sometimes to $H_F(\kappa)$ as the *denominator* of l_F.
We define consequently a $\mathcal{K}$-linear form on $M_\kappa^{(2)}(N)^{\leq \alpha} \otimes_{\mathcal{I}} \mathcal{K}$ by

$$l_{F\times F} := l_F \otimes l_F$$

under the decomposition in (5.3.6).

Before defining the p-adic L-functions, we need an operator to lower the level of the Eisenstein series constructed before. We follow [Hid88c, §1 VI]. Fix a prime-to-p integer L, with $N|L$. We define for classical weights k:

$$\begin{array}{rlcl} T_{L/N,k}: & M_k(Lp,A) & \rightarrow & M_k(Np,A) \\ & f & \mapsto & (L/N)^{k/2}\sum_{[\gamma]\in\Gamma(N)/\Gamma(N,L/N)} f|_k \left(\begin{array}{cc} 1 & 0 \\ 0 & L/N \end{array}\right)|_k\gamma \end{array}.$$

As L is prime to p, it is clear that $T_{L/N,k}$ commutes with U_p. It extends uniquely to a linear map

$$T_{L/N}: \quad M_\kappa(L) \quad \rightarrow \quad M_\kappa(N)$$

which in weight k specializes to $T_{L/N,k}$.

We have a map $M_\kappa(N)^{\leq\alpha} \hookrightarrow M_\kappa(N^2R^2)^{\leq\alpha}$. We define $1_{N^2R^2/N}$ to be one left inverse. We define

$$\begin{aligned} L_p(\kappa,\kappa') &= \frac{N_0}{N^2R^2N_1} l_F \otimes l_F \left((U_{N^2/N_1^2}^{-1} \circ 1_{N^2R^2/N}) \otimes T_{N^2R^2/N}(\mathcal{H}(\kappa,\kappa')) \right) \text{(for } \alpha = 0), \\ L_p^*(\kappa) &= \frac{N_0}{N^2R^2N_1} l_F \otimes l_F \left((U_{N^2/N_1^2}^{-1} \circ 1_{N^2R^2/N}) \otimes T_{N^2R^2/N}(\mathcal{H}^*(\kappa) \right) \text{(for } \alpha < +\infty). \end{aligned}$$

We will see in the proof of the following theorem that it is independent of the left inverse $1_{N^2R^2/N}$ which we have chosen.

We fix some notations. For a Dirichlet character η, we denote by η_0 the associated primitive character. Let $\lambda_p(\kappa) \in \mathcal{I}$ be the U_p-eigenvalue of F. We say that $(\kappa,\kappa') \in \mathcal{C}_F \times \mathcal{W}$ is of type $(k;t,\varepsilon)$ if :

- $\kappa_{|_{\mathcal{W}(r)}} = [k]$ with $k \geq 2$,
- $\kappa' = \varepsilon[t]$ with $1 \leq t \leq k-1$ and ε finite order character defined modulo p^n, $n \geq 1$.

Let as before $s = k - t - 1$. We define

$$E_1(\kappa, \kappa') = \lambda_p(\kappa)^{-2n_0}(1 - (\chi\varepsilon\omega^{-s})_0(p)\lambda_p(\kappa)^{-2}p^s))$$

where $n_0 = 0$ (resp. $n_0 = n$) if $\chi\varepsilon\omega^{-s}$ is (resp. is not) trivial at p.
If $F(\kappa)$ is primitive at p we define $E_2(\kappa, \kappa') = 1$, otherwise

$$\begin{aligned} E_2(\kappa, \kappa') = &(1 - (\chi^{-1}\varepsilon^{-1}\omega^s\phi)_0(p)p^{k-2-s})\times \\ &(1 - (\chi^{-1}\varepsilon^{-1}\omega^s\phi^2)_0(p)\lambda_p(\kappa)^{-2}p^{2k-3-s}). \end{aligned}$$

We denote by $F^\circ(\kappa)$ the primitive form associated to $F(\kappa)$. We shall write $W'(F(\kappa))$ for the prime-to-p part of the root number of $F^\circ(\kappa)$. If $F(\kappa)$ is not p-primitive we pose

$$S(F(\kappa)) = (-1)^k \left(1 - \frac{\phi_0(p)p^{k-1}}{\lambda_p(\kappa)^2}\right)\left(1 - \frac{\phi_0(p)p^{k-2}}{\lambda_p(\kappa)^2}\right),$$

and $S(F(\kappa)) = (-1)^k$ otherwise. We pose

$$\begin{aligned} C_{\kappa,\kappa'} &= i^{1-k}s!G\left(\left(\chi\varepsilon\omega^{-s}\right)^{-1}\right)(\chi\varepsilon\omega^{-s})_0(p^{n-n_0})(N_1Rp^{n_0})^sN^{-k/2}2^{-s}, \\ C_\kappa &= C_{\kappa,[k-k_0+1]}, \\ \Omega(F(\kappa), s) &= W'(F(\kappa))(2\pi i)^{s+1}\left\langle F^\circ(\kappa), F^\circ(\kappa)\right\rangle. \end{aligned}$$

We have the following theorem, which will be proven at the end of the section.

Theorem 5.4.2. *i) The function $L_p(\kappa, \kappa')$ is defined on $\mathcal{C}_F \times \mathcal{W}$, it is meromorphic in the first variable and bounded in the second variable. For all classical points (κ, κ') of type $(k; t, \varepsilon)$ with $k \geq 2$, $1 \leq t \leq k-1$, we have the following interpolation formula*

$$L_p(\kappa, \kappa') = C_{\kappa,\kappa'}E_1(\kappa, \kappa')E_2(\kappa, \kappa')\frac{\mathcal{L}(s+1, \mathrm{Sym}^2(F(\kappa)), \chi^{-1}\varepsilon^{-1}\omega^s)}{S(F(\kappa))\Omega(F(\kappa), s)}.$$

ii) The function $L_p^*(\kappa)$ is meromorphic on $\mathcal{C}_F$. For κ of type k with $k \geq k_0$, we have the following interpolation formula

$$L_p^*(\kappa) = C_\kappa E_2(\kappa, [k - k_0 + 1])\frac{\mathcal{L}(k_0 - 1, \mathrm{Sym}^2(F(\kappa)), {\chi'}^{-1}\chi_1^{-1})}{S(F(\kappa))\Omega(F(\kappa), k_0 - 2)}.$$

Let us denote by $\tilde{L}_p(\kappa, \kappa')$ the two-variable p-adic L-function of [Ros13d, Theorem 4.14], which is constructed for any slope but NOT for $p = 2$. We can deduce the fundamental corollary which allows us to apply the method of Greenberg and Stevens;

Corollary 5.4.3. *For $\alpha = 0$ (resp. $\alpha > 0$ and $p \neq 2$) we have the following factorization of locally analytic functions around κ_0 in $\mathcal{C}_F$:*

$$L_p(\kappa, [k - k_0 + 1]) = (1 - \chi'\chi_1(p)\lambda_p(\kappa)^{-2}p^{k_0-2})L_p^*(\kappa)$$

$$(\textit{resp. } C\frac{\kappa^{-1}(2)\tilde{L}_p(\kappa, [k - k_0 + 1])}{1 - \phi^{-2}\chi^2\kappa(4)2^{-2k_0}} = (1 - \chi'\chi_1(p)\lambda_p(\kappa)^{-2}p^{k_0-2})L_p^*(\kappa)),$$

where C is a constant independent of κ, explicitly determined by the comparison of C_κ here and in [Ros13d, Theorem 4.14].

We can now prove the main theorem of the paper;

Proof of Theorem 5.1.3. Let f be as in the statement of the theorem; we take $\chi' = \chi_1 = \mathbf{1}$. For $\alpha = 0$ resp. $\alpha > 0$ we define

$$L_p(\mathrm{Sym}^2(f), s) = C^{-1}_{\kappa_0,[1]} L_p(\kappa_0, [k_0 - s]),$$

$$L_p(\mathrm{Sym}^2(f), s) = C \frac{\kappa(2)\tilde{L}_p(\kappa, [k - k_0 + 1])}{C_{\kappa_0,[1]}(1 - \phi^{-2}\chi^2\kappa(4)2^{-2k_0})}$$

The two variables p-adic L-function vanishes on $\kappa' = [1]$. As f is Steinberg at p, we have $\lambda_p(\kappa_0)^2 = p^{k_0-2}$. Consequently, the following formula is a straightforward consequence of Section 5.2 and Corollary 5.4.3;

$$\lim_{s\to 0} \frac{L_p(\mathrm{Sym}^2(f), s)}{s} = -2 \frac{\mathrm{d}\log \lambda_p(\kappa)}{\mathrm{d}\kappa}|_{\kappa=\kappa_0} \frac{\mathcal{L}(\mathrm{Sym}^2(f), k_0 - 1)}{S(F(\kappa))\Omega(f, k_0 - 2)}.$$

From [Ben10, Mok12] we obtain

$$\mathcal{L}^{\mathrm{al}}(\mathrm{Sym}^2(f)) = -\, 2\frac{\mathrm{d}\log \lambda_p(\kappa)}{\mathrm{d}\kappa}|_{\kappa=\kappa_0}.$$

Under the hypotheses of the theorem, f is Steinberg at all primes of bad reduction and we see from [Ros13d, §3.3] that

$$\mathcal{L}(\mathrm{Sym}^2(f), k_0 - 1) = L(\mathrm{Sym}^2(f), k_0 - 1)$$

and we are done. □

Proof of Theorem 5.4.2. We point out that most of the calculations we need in this proof and have not already been quoted can be found in [Hid90, Pan03].
If ε is not trivial at p, we shall write p^n for the conductor of ε. If ε is trivial, then we let $n = 1$.
We recall that $s = k - t - 1$; we have

$$L_p(\kappa, \kappa') = \frac{N_0 \left\langle F(\kappa)^c|\tau_{Np}, U^{-1}_{N^2/N_1^2} \left\langle F(\kappa)^c|\tau_{Np}, T_{N^2R^2/N,k}\mathcal{H}([k], \varepsilon[t])\right\rangle\right\rangle}{N^2R^2N_1 \langle F(\kappa)^c|\tau, F(\kappa)\rangle^2}.$$

We have as in [Pan03, (7.11)]

$$\left\langle F(k)^c|\tau_{Np}, U_p^{-2n+1}\mathrm{Pr}^{\mathrm{ord}}U_p^{2n-1}g\right\rangle = \lambda_p(\kappa)^{1-2n}p^{(2n-1)(k-1)} \left\langle F(\kappa)^c|\tau_{Np}|[p^{2n-1}], g\right\rangle, \tag{5.4.4}$$

where $f|[p^{2n-1}](z) = f(p^{2n-1}z)$. We recall the well-known formulae [Hid88c, page 79]:

$$\begin{aligned}
\left\langle f|[p^{2n-1}], T_{N^2R^2/N,k}g\right\rangle =& (NR^2)^k \left\langle f|[p^{2n}NR^2], g\right\rangle,\\
\tau_{Np}|[p^{2n-1}N^2R^2] =& \left(NR^2p^{2n-1}\right)^{-k/2}\tau_{N^2R^2p^{2n}},\\
\frac{\langle F(\kappa)^c|\tau_{Np}, F(\kappa)\rangle}{\langle F(\kappa)^\circ, F(\kappa)^\circ\rangle} =& (-1)^k W'(F(\kappa))p^{(2-k)/2}\lambda_p(\kappa)\times\\
& \left(1 - \frac{\phi_0(p)p^{k-1}}{\lambda_p(\kappa)^2}\right)\left(1 - \frac{\phi_0(p)p^{k-2}}{\lambda_p(\kappa)^2}\right).
\end{aligned}$$

So we are left to calculate

$$\frac{N_0(NR^2)^{k/2}\lambda_p(\kappa)^{1-2n}p^{(2n-1)\left(\frac{k}{2}-1\right)}}{A(-1)^s c^s_{t+1}N_1N^2R^2} \left\langle F(\kappa)^c|\tau_{Np}, U^{-1}_{N^2/N_1^2} \left\langle F(\kappa)^c|\tau_{N^2R^2p^n}, \mathcal{H}'_{1,\chi\varepsilon\omega^{-t}}(z,w)|_{s=\frac{1}{2}-t}\right\rangle\right\rangle.$$

We use Proposition 5.3.5 and [BS00, (3.29)] (with the notation of *loc. cit.* $\beta_1 = \frac{\lambda_p^2(\kappa)}{p^{k-1}}$) to obtain that the interior Petersson product is a scalar multiple of

$$\frac{\mathcal{L}(\mathrm{Sym}^2(f), s+1)\tilde{E}(\kappa,\kappa')E_2(\kappa,\kappa')F(\kappa)|U_{N^2/N_1^2}}{\langle F(\kappa)^\circ, F(\kappa)^\circ\rangle\, W'(F(\kappa))p^{(2-k)/2}\lambda_p(\kappa)S(F(\kappa))}. \tag{5.4.5}$$

Here

$$\tilde{E}(\kappa,\kappa') = -p^{-1}(1-\chi^{-1}\varepsilon^{-1}\omega^s(p)\lambda_p(\kappa)^2p^{t+1-k})$$

if $\chi\varepsilon\omega^{-s}$ is trivial modulo p, and 1 otherwise. The factor $E_2(\kappa,\kappa')$ appears because $F(\kappa)$ could not be primitive. Clearly it is independent of $1_{N^2R^2/N}$ and we have $l_F(F(\kappa)) = 1$. We explicit the constant which multiplies (5.4.5);

$$\begin{aligned}
&(-1)^s\frac{N_0(NR^2)^{k/2}\lambda_p(\kappa)^{1-2n}p^{(2n-1)\left(\frac{k}{2}-1\right)}N_1^{s+1}(Rp^n)^{s+3-k}N^{2-k}}{B(t)(2\pi i)^s c_{t+1}^s G\left((\chi\varepsilon\omega^{-s})\right)N^2R^2N_1}\frac{\omega^{k-t-1}(-1)\Omega_{k,s}(s_1)p_{s_1}(t+1)}{d_{s_1}(t+1)}\\
=&(-1)^{-\frac{k}{2}}\frac{N^{-\frac{k}{2}}N_0N_1^sR^{s+1}\lambda_p(\kappa)^{1-2n}p^{(2-k)/2}p^{n(s+1)}(2s)!2^{-2s}2^{2t}\pi^{3/2}}{2^{1+2t}\pi^{\frac{5}{2}}(2\pi i)^s G\left((\chi\varepsilon\omega^{-s})\right)2^{-s}\frac{(2s)!}{s!}}.
\end{aligned}$$

If $\varepsilon_1\varepsilon\omega^{-s}$ is not trivial we obtain

$$= i^{1-k}\frac{N^{-\frac{k}{2}}N_1^sR^s\lambda_p(\kappa)^{1-2n}p^{(2-k)/2}p^{ns}s!G\left((\chi\varepsilon\omega^{-s})^{-1}\right)}{(2\pi i)^{s+1}2^s},$$

otherwise

$$= i^{1-k}\frac{N^{-\frac{k}{2}}N_1^sR^s\lambda_p(\kappa)^{-1}p^{(2-k)/2}p^{s+1}s!G\left((\chi_1\chi')^{-1}\right)}{(2\pi i)^{s+1}2^s\chi_1\chi'(p)}.$$

The calculations for $L_p^*(\kappa)$ are similar. We have to calculate

$$\frac{N_0(NR^2)^{k/2}}{A^*N^2R^2N_1}\left\langle F(\kappa)^c|\tau_{Np}, U_{N^2/N_1^2}^{-1}\left\langle F(\kappa)^c|\tau_{N^2R^2p}, (-1)^{k_0}{\mathcal{H}'}^{*}_{1,\chi_1\chi'}(z,w)|_{u=s_1}\right\rangle\right\rangle,$$

where $s_1 = \frac{1}{2} - k + k_0 - 1$. The interior Petersson product equals (see [BS00, Theorem 3.1] with M, N of *loc. cit.* as follows: $M = R^2N^2p$, $N = N_1R$)

$$\frac{R^{k_0+1-k}(N^2p)^{\frac{2-k}{2}}(-1)^{k_0}\Omega_{k,k_0-2}(s_1)p_{s_1}(k-k_0+2)}{d_{s_1}(k-k_0+2)\langle F(\kappa)^\circ, F(\kappa)^\circ\rangle\, W'(F(\kappa))p^{(2-k)/2}\lambda_p(\kappa)S(F(\kappa))}F(\kappa)|U_p|U_{N^2/N_1^2},$$

so we have

$$L_p^*(\kappa) = i^{1-k}\frac{N_1^{k_0-2}R^{k_0-2}N^{k/2}(k_0-2)!}{(2\pi i)^{k_0-1}G(\chi_1\chi')2^{k_0-2}}\frac{E_2(\kappa,[k-k_0+1])\mathcal{L}(k_0-1,\mathrm{Sym}^2(F(\kappa)),(\chi_1\chi')^{-1})}{S(F(\kappa))W'(F(\kappa))\langle F(\kappa)^\circ, F(\kappa)^\circ\rangle}.$$

□

We give here some concluding remarks. As $F(\kappa)$ has not complex multiplication by χ, we can see exactly as in [Hid90, Proposition 5.2] that $H_F(\kappa)L_p(\kappa,\kappa')$ is holomorphic along $\kappa' = [0]$ (which is the pole of the Kubota Leopoldt p-adic L-function).
We point out that the analytic $\mathcal{L}$-invariant for CM forms has already been studied in literature [DD97, Har12, HL13].
Note also that our choice of periods is not optimal [Ros13d, §6].

Chapter 6

The arithmetic $\mathcal{L}$-invariant

In the previous chapters we have been concentrating on the study of p-adic L-functions and their derivatives; the aim of this chapter is the deal with the arithmetic side of L-functions: the Selmer group. Given a a Galois representation V, we shall recall the definition of the arithmetic $\mathcal{L}$-invariant of V, following Greenberg [Gre94b] and Benois [Ben11]. This invariant, defined independently of the p-adic L-function, is conjectured to be the *error factor* (sometimes called analytic $\mathcal{L}$-invariant) in the interpolation formula of a p-adic L-function for V which presents trivial zeros. We will reproduce here the computation of the $\mathcal{L}$-invariant, as made by Hida, Mok and Benois, in the case of the symmetric square of a (Hilbert) modular form. The only new part of this chapter is the final section where we combine a formula of Benois and the method of [HJ13] to calculate in Theorem 6.3.3 the $\mathcal{L}$-invariant of the two representations associated with a genus 2 Siegel modular form whose automophic representation at p is Steinberg.

6.1 $\mathcal{L}$-invariant in the ordinary case

Let K be a p-adic field and V and n-dimensional K vector space. We fix a p-adic representation over K:

$$G_{\mathbb{Q}} \to GL_K(V).$$

By abuse of notation, we shall sometimes refer to this representation simply by V.
We suppose that it is motivic and sastifies all the nice properties which are conjectured (functional equation, existence of a critical integer which we assume to be $s = 0$, existence of periods, ...).

6.1.1 Local condition at p

We suppose that V is ordinary at p, that is when V is restricted to a decomposition group at p it satisfies the following conditions:

- there exists a decreasing and exhaustive filtration Fil^i which is stable for the action of $G_{\mathbb{Q}_p}$,
- the inertia acts on $\mathrm{Gr}^i := \mathrm{Fil}^i/\mathrm{Fil}^{i+1}$ by the i-th power χ_{cycl}^i the p-adic cyclotomic character.

It could be useful for the reader to think of ordinary representations as representations which are "upper triangular" at p.
We make the following assumption

S) The action of $G_{\mathbb{Q}_p}$ on Gr^0 and $\mathrm{Gr}^1(-1)$ is Frobenius-semisimple.

For a profinite group G and a G-representation M, we shall write $H^i(G,M)$ for the continuos cohomology. Let us denote by $H^1_{\mathrm{unit}}(G_{\mathbb{Q}_p},\mathbb{Q}_p(1))$ the image of $\varprojlim_n \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^{p^n}$ via the Kummer map and let U_p be the unique extension (up to isomorphism)

$$0 \to \mathbb{Q}_p(1) \to U_p \to \mathbb{Q}_p \to 0$$

such that, in the long exact sequence in cohomology, we have

$$\mathrm{Ker}\left(H^1(G_{\mathbb{Q}_p},\mathbb{Q}_p(1)) \to H^1(G_{\mathbb{Q}_p},U_p)\right) = H^1_{\mathrm{unit}}(G_{\mathbb{Q}_p},\mathbb{Q}_p(1)).$$

We suppose moreover

U) V as $G_{\mathbb{Q}_p}$-representation does not admit any subquotient isomorphic to U_p.

Let us denote Fil^1 by F^+. From assumption **S** we know that we can find two subspace

$$\mathrm{Fil}^2 \subset F^{11} \subset F^+ \subset F^{00} \subset \mathrm{Fil}^0$$

such that F^{00} resp. F^{11} is maximal resp. minimal for the property that the quotient $W := F^{00}/F^{11}$ is isomorphic to

$$\mathbb{Q}_p(1)^{t_1} \oplus M \oplus {\mathbb{Q}_p}^{t_0},$$

with M a non-split extension of $\mathbb{Q}_p(1)^t$ by ${\mathbb{Q}_p}^t$. In other words, W is the biggest subquotient of V where the Frobenius has eigenvalues 1 or p. We make the additional assumption

T) either t_1 or t_0 is zero.

It is under all these assumptions that Greenberg defines an $\mathcal{L}$-invariant.
We shall write g for $t+t_1+t_0$.

6.1.2 Selmer groups

Let us choose a stable lattice T inside V and let us denote V/T by A; we fix some local conditions for the Galois cohomology of A. Let us denote by $\mathbb{Q}_\infty$ the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Let $G_\infty = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_\infty)$ and $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. If v is a finite place in $\mathbb{Q}_\infty$ not dividing p, let us fix a decomposition group at v in G_∞ which we shall denote by D_v. We pose

$$L_v := \mathrm{Ker}\left(H^1(D_v,A) \to H^1(I_v,A)\right),$$

where I_v denotes the inertia subgroup of D_v. Let us denote by F^+A the image of F^+ inside A, and we shall write $\overline{A} := A/F^+A$. At the only prime above p in $\mathbb{Q}_\infty$ (which we shall denote bt $\mathfrak{p}$) we define

$$\begin{aligned} L_{\mathfrak{p}} :=& \mathrm{Ker}\left(H^1(G_{\mathfrak{p}},A) \to H^1(I_{\mathfrak{p}},\overline{A})\right), \\ L_{\mathfrak{p}}^{\mathrm{st}} :=& \mathrm{Ker}\left(H^1(G_{\mathfrak{p}},A) \to H^1(G_{\mathfrak{p}},\overline{A})\right). \end{aligned}$$

Here st stands for *strict*. We define the following Selmer groups:

$$S_A^?(\mathbb{Q}_\infty) := \mathrm{Ker}\left(H^1(G_\infty,A) \to \prod_v \frac{H^1(D_v,\overline{A})}{L_v^?}\right),$$

for $? = \emptyset$ or st. Let $\Lambda := \mathcal{O}[[\Gamma]] \cong \mathcal{O}[[T]]$. By normal action of Γ, these Selmer groups continuos discrete Λ-modules. Let $S_A^?(\mathbb{Q}_\infty)^*$ be its Pontryagin dual which is a continuos compact Λ-module. In particular, it is of finite type over Λ [Gre89, Proposition 6].
Let us denote by $f_A(T)$ the characteristic ideal of $S_A(\mathbb{Q}_\infty)$; if it is not a Λ-torsion module, then $f_A(T)=0$. The following proposition is meant to link all what we have defined up to now [Gre94b, Proposition 1]

Proposition 6.1.1. *Assume* $\boldsymbol{S}$, $\mathbf{T}$, $\mathbf{U}$, then $T^g \mid f_A(T)$.

We are supposed to interpret this proposition in the light of the Greenberg-Iwasawa Main Conjecture for V. One conjectures the existence of a p-adic L-function $L_p(T, V) \in \mathcal{O}[[T]]$ interpolating the special values at $s = 0$ of the complex L-function of V twisted by characters whose conductor is a power of p. The Greenberg-Iwasawa Main Conjecture can be stated as

$$(L_p(T, V)) = (f_A(T)) \subset \mathcal{O}[[T]].$$

6.1.3 Trivial zeros and $\mathcal{L}$-invariants

Conjecturally, the p-adic L-function of V should have a trivial zero of order g and this proposition confirms it.
If we suppose moreover that $L(0, V) \neq 0$, we would expect

$$\frac{L_p(T, V)}{T^e}_{|T=0} = \mathcal{L}^{\mathrm{an}}(V) \times E^*(V) \times \frac{L(0, V)}{\Omega},$$

where $\mathcal{L}^{\mathrm{an}}$ denotes a possible error factor. The aim of the rest of the chapter is to give a conjectural description of the error term *independently* of the p-adic L-function.
In what follows, we shall write

$$F^? H^i(D_p, B) := \mathrm{Im}\left(H^i(D_p, F^? B) \to H^i(D_p, B)\right),$$

for $i = 0, 1, 2$, $? = 11, +, 00$ and $B = W, V, A$.
To lighten the notation, we shall write $H^i(B)$ for $H^i(D_p, B)$.
Let us fix a basis $(\log_p \chi_{\mathrm{cycl}}, \mathrm{ord}_p)$ of $H^1(\mathbb{Q}_p) = \mathrm{Hom}_{\mathrm{cont}}(G_{\mathbb{Q}_p}, \mathbb{Q}_p)$ corresponding (via class field theory) to the cyclotomic extension and the unramified one. We shall write

$$H^1(\mathbb{Q}_p) = H^1_{\mathrm{cyc}}(\mathbb{Q}_p) \oplus H^1_{\mathrm{unr}}(\mathbb{Q}_p).$$

Let us define

$$\tilde{L}_p(W) := F^+ H^1(W) \times H^1_{\mathrm{unit}}(\mathbb{Q}_p(1)^{t_1}) \times H^1_{\mathrm{unr}}(\mathbb{Q}_p^{t_0}) \subset H^1(W).$$

The dimension of $\tilde{L}_p(W)$ is exactly g, while the dimension of $H^1(W)$ is $2g$. We define $\tilde{L}_p$ in $H^1(V)$ such that its image in $H^1(W)$ is $\tilde{L}_p(W)$. We have

$$F^{11} H^1(V) \subset \tilde{L}_p \subset F^{00} H^1(V).$$

We define also the Selmer K-vector space over $\mathbb{Q}$

$$S^?_V(\mathbb{Q}) := \mathrm{Ker}\left(H^1(G_{\mathbb{Q}}, V) \to \prod_{l \neq p} \frac{H^1(D_l, V)}{L_l} \times \frac{H^1(D_p, V)}{L^?_p}\right), \tag{6.1.2}$$

for $? = \emptyset, \mathrm{st}$, where the local conditions are defined as before. We suppose moreover the following

- **Sel**) $S^{\mathrm{st}}_V(\mathbb{Q}) = S^{\mathrm{st}}_{V^*(1)}(\mathbb{Q}) = 0$,
- **NInv**) $H^0(V) = H^0(V^*(1)) = 0$.

The second assumption is not particularly restricting but the first one implies, according to the conjectures of Bloch-Kato, the non vanishing of $L(0, V)$. This is not verified for example if $V = V_p(E)$, where E is an elliptic curve over $\mathbb{Q}$ of positive rank. This problem can be overcome by working separately for the two cases $t_1 = t_0 = 0$ and $t_0 \neq 0$. We will came back to the case $L(0, V) = 0$ at the end of Section 6.2.5.
We shall suppose for now $t_1 = 0$. The case $t_0 = 0$ can be dealt working with $V^*(1)$.
It is easy to see that $L_p^{\mathrm{st}}(V) = L_p^{\mathrm{st}}(V^*(1))^{\perp}$, where $\perp$ denotes local Tate duality. We deduce from the Poitou-Tate exact sequence [Hid00b, Theorem 3.37] that

$$H^1(G_{\mathbb{Q}}, V) \cong \prod_{l \neq p} \frac{H^1(D_v, V)}{L_l} \times \frac{H^1(D_p, V)}{L_p^{\mathrm{st}}}.$$

Consider the K-vector spaces $\overline{V} = V/F^+V$ and $\overline{W} = W/F^+W$. Let us define $\mathbf{T}(V) \subset H^1(G_{\mathbb{Q}}, V)$ which projects isomorphically via the above isomorphism to $\frac{F^{00}H^1(\overline{V})}{L_p^{\mathrm{st}}}$. The localization map send $\mathbf{T}(V)$ into $H^1(\overline{W})$. By definition, we have

$$H^1(\overline{W}) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^{\times}, \mathbb{Q}_p^g).$$

We shall denote by $(\log_p \chi_{\mathrm{cycl}})^g$, $(\mathrm{ord}_p)^g$ the basis $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^{\times}, \mathbb{Q}_p^g)$ induced from the basis of $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^{\times}, \mathbb{Q}_p)$ fixed before. A simple calculation tells us that $F^{00}H^1(\overline{W})$ has dimension g. Let $c \in \mathbf{T}(V)$, if $c_p \in F^{00}H^1(\overline{W})$ belongs to the space generated by $(\mathrm{ord}_p)^g$ then c_p is unramified and therefore $c \in S_V^{\mathrm{st}}(\mathbb{Q}) = 0$, so $c = 0$. This implies that the projection of $\mathbf{T}(V)$ to $(\log_p \chi_{\mathrm{cycl}})^g$ is surjective. Let us denote by ι_{cyc} resp. ι_{unr} the projection to $(\log_p \chi_{\mathrm{cycl}})^g$ resp. $(\mathrm{ord}_p)^g$. We define

$$\mathcal{L} = \mathcal{L}(V) := \det(\iota_{\mathrm{unr}} \circ \iota_{\mathrm{cyc}}^{-1})$$

where $\iota_{\mathrm{unr}} \circ \iota_{\mathrm{cyc}}^{-1}$ is seen as a linear transformation of $\mathbb{Q}_p^g$ with respect to the basis $(\log_p \chi_{\mathrm{cycl}})^g$ of the source and $(\mathrm{ord}_p)^g$ of the target. We have a lemma

Lemma 6.1.3. *If $t_0 = 0$ too, then $\mathcal{L}(V)$ depends only on $V_{|G_{\mathbb{Q}_p}}$.*

Proof. By definition $F^{00}H^1(\overline{V})$ is contained in $H^1(W)$. The dimension of the latter is $2t_0 + t$. If $t_0 = 0$, then $\mathbf{T}(V)$ and $H^1(W)$ have the same dimension, so they have to coincide. The second space depends only on W and in particular only on $V_{|G_{\mathbb{Q}_p}}$ and we are done. □

We conclude with the following proposition [Gre94b, Proposition 3]

Proposition 6.1.4. *We have $\mathrm{ord}_{T=0}(f_A(T)) = e$ if and only if $\mathcal{L}(V) \neq 0$.*

Let us denote by $S_A(\mathbb{Q})$ the p-divisible Selmer group of V over $\mathbb{Q}$ defined similarly to 6.1.2.
Assuming the non-vanishing of $\mathcal{L}$ and some other assumptions (see [Gre94b, Proposition 4]) Greenberg has shown that the p-adic valuation of $\frac{f_A(T)}{T^e}|_{T=0}$ and the one of $(\mathcal{L}(V)/p^e)|S_A(\mathbb{Q})|$ coincide.

6.1.4 The $\mathcal{L}$-invariant of the adjoint representation of a Hilbert modular form

In this section we recall the calculation by Hida [Hid06] of the $\mathcal{L}$-invariant for the adjoint representation of a Hilbert modular form. Let us fix a totally real field F of degree d, and let us denote by $\mathfrak{p}_1, \ldots, \mathfrak{p}_g$ the prime ideals above p. Let $\mathbf{f}$ be a Hilbert modular form of weight $k = (k_\sigma)$. We shall suppose that if $\sigma_1 : F \to \overline{\mathbb{Q}}$ and $\sigma_2 : F \to \overline{\mathbb{Q}}$ induce same p-adic place $\mathfrak{p}$, then $k_{\sigma_1} = k_{\sigma_2}$. We shall write then $k_{\mathfrak{p}}$ for k_σ. We suppose $k_{\mathfrak{p}} \geq 2$ for all $\mathfrak{p}$ and we shall denote for simplicity of notation $m := \max(k_{\mathfrak{p}} - 1)$.
We have a two-dimensional p-adic Galois representation $\rho_{\mathbf{f}}$ associated with $\mathbf{f}$ [Tay89, BR89]; let us consider the Galois representation $V_F := \mathrm{Ad}(\rho_{\mathbf{f}})$ which is a 3-dimensional Galois representation by our convention.

We will calculate the $\mathcal{L}$-invariant of $V := \mathrm{Ind}_F^{\mathbb{Q}} V_F$. If $\mathbf{f}$ is nearly ordinary in the sense of Chapter 3 and if p is unramified in F, then V is ordinary in the above sense. The convenience of working with the induced representation is that over $\mathbb{Q}_p$ we have the above mentioned decomposition in two one-dimensional subspaces

$$H^1(\mathbb{Q}_p) = H^1_{\mathrm{cyc}}(\mathbb{Q}_p) \oplus H^1_{\mathrm{unr}}(\mathbb{Q}_p)$$

while for a p-adic field $F_{\mathfrak{p}} \neq \mathbb{Q}_p$ we have

$$H^1(G_{F_{\mathfrak{p}}}, \mathbb{Q}_p) \cong \mathbb{Q}_p^{[F_{\mathfrak{p}}:\mathbb{Q}_p]} \oplus H^1_{\mathrm{unr}}(\mathbb{Q}_p).$$

where one summand is higher dimensional. This creates difficulty; hence to study the $\mathcal{L}$-invariant of V_F we need to work with V. On the level of L-functions, this changes nothing.
We will see soon that it is possible to define a $G_{\mathbb{Q}_p}$-invariant filtration on V.
Notice that if p is ramified in F then the inertia acts on the graded piece as Gr^i via χ^i_{cycl} times a finite quotient. Consequently V is not ordinary in the above sense. We believe that it should be possible to to generalize Greenberg's if one suppose that, for $i = 0$,1, the part of Gr^i where I_p acts as a power of the cyclotomic character is a direct summand of Gr^i. As this is not necessary for the cases we have been studying in the rest of this thesis, we shall not bother to generalize Greenberg definition to these cases.
We begin by studying the representation $V_{F|G_{F_{\mathfrak{p}}}}$. As $\mathbf{f}$ is nearly ordinary, we have a filtration of $G_{F_{\mathfrak{p}}}$-modules:

$$0 \to \mathcal{F}^+_{\mathfrak{p}} \to \rho_{\mathbf{f}} \to \rho_{\mathbf{f}}/\mathcal{F}^+_{\mathfrak{p}} \to 0. \tag{6.1.5}$$

We fix a basis of $\rho_{\mathbf{f}}$ adapted to this filtration. Then the representation space of V_F identifies to the space of trace zero endomorphisms of the space of $\rho_{\mathbf{f}}$ with respect to the above basis and 6.1.5 induces a filtration on V_F

$$V_F = \mathrm{Fil}^{1-k_{\mathfrak{p}}}_{\mathfrak{p}} \supset \mathrm{Fil}^0_{\mathfrak{p}} \supset \mathrm{Fil}^{k_{\mathfrak{p}}-1}_{\mathfrak{p}} \supset 0,$$

where $\mathrm{Fil}^0_{\mathfrak{p}}$ denotes the upper triangular matrices and $\mathrm{Fil}^{k_{\mathfrak{p}}-1}_{\mathfrak{p}}$ the upper nilpotent matrices. We shall define $F^?_{\mathfrak{p}}$, for $? = 00, 11$ as in the previous section.
We introduce now some notation. For any p-adic representation V_F of G_F we fix a G_F-stabel lattice T_F. A decomposition group D for p in $G_{\mathbb{Q}}$ induces a decomposition group D' at a certain prime $\mathfrak{p}$ above p in F. We identify the set $D \setminus \mathrm{Hom}(F, \overline{\mathbb{Q}})$ with the primes above p in F; for each σ in $D \setminus \mathrm{Hom}(F, \overline{\mathbb{Q}})$ let us write $D_\sigma = \sigma D \sigma^{-1}$ and $D'_\sigma = D_\sigma \cap G_F$. The latter is the decomposition group for $\mathfrak{p}_\sigma$. We choose a direct summand X_σ of T_F, stable under D'_σ. We shall write $\mathcal{E} = \mathrm{Hom}(F, \overline{\mathbb{Q}})$. Let us pose $X = \{X_\sigma\}_{\sigma \in D \setminus \mathcal{E}}$, we define

$$\mathrm{Ind}_F^{\mathbb{Q}} X = \bigoplus_{\sigma \in D\setminus\mathcal{E}} \sigma^{-1} \mathrm{Ind}_{D'_\sigma}^{D_\sigma} X_\sigma.$$

Shapiro's lemma tells us that

$$H^i(D, \sigma^{-1}\mathrm{Ind}_{D'_\sigma}^{D_\sigma} X_\sigma) = H^i(D'_\sigma, X_\sigma).$$

From now on $V_F = \mathrm{Ad}(\rho_{\mathbf{f}})$ and we suppose $\mathbf{f}$ not CM.
We shall apply this formalism to $X_\sigma = \mathrm{Fil}^i_{\mathfrak{p}}$ for the calculation of the $\mathcal{L}$-invariant of V as defined in Section 6.1.3. The hypotheses **S, T, U** and $H^0(V) = H^0(V^*(1)) = 0$ are easily verified. However, notice that the condition **Sel** which says $S^{\mathrm{st}}_V(\mathbb{Q}) = S^{\mathrm{st}}_{V^*(1)}(\mathbb{Q}) = 0$, even though conjecturally true since $L(0, V_F)$ does not vanish, is not yet been proven in full generality, although there are many cases in which this has been proven to be true [Hid06, Theorem 3.50], [Dim09, Theorem B]. We will therefore assume that **Sel** holds.
We give the following lemmas [Hid06, Lemma 3.84]

Lemma 6.1.6. *Let L be a finite Galois extension of $\mathbb{Q}_p$, and let K be a p-adic fields viewed as a trivial $G_{\mathbb{Q}_p}$-module, then*

$$\mathrm{Hom}_{\mathrm{cont}}(L^{\times}, K)^{\mathrm{Gal}(L/\mathbb{Q}_p)} = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^{\times}, K).$$

Lemma 6.1.7. *Let K be a characteristic 0 field. Let G be a group and H a normal subgroup of G; let us denote by $\overline{G}$ the quotient. Then we have*

$$\mathrm{Res} : H^1(G, K[\overline{G}]) \cong H^0(\overline{G}, H^1(H, K[\overline{G}])) = H^1(H, K)$$

where K is a H-module with the trivial action. Moreover the image of $\mathrm{Hom}(G, K)$ *in $H^1(H, K)$ is $H^0(\overline{G}, H^1(H, K))$.*

If we let $F^{00}(V_F) = \left\{F^{00}_{\mathfrak{p}}\right\}_{\sigma \in I/D}$, we remark that by definition we have

$$F^{00}V \subset \mathrm{Ind}_F^{\mathbb{Q}} F^{00}(V_F).$$

The above lemmas are useful to prove the following proposition which allows us to study the $\mathcal{L}$-invariant, defined for the representation V, in terms of V_F. The key point is to find the image of $\mathbf{T}(V)$ in the cohomology of V_F.

Proposition 6.1.8. *We have a commutative diagram*

$$\begin{tikzcd}
H^1(G_{\mathbb{Q}}, V) \arrow[d] & \mathbf{T}(V) \arrow[l] \arrow[r, "\mathrm{Res}"] \arrow[d] & H^1\left(G_{\mathbb{Q}_p}, \frac{F^{00}V}{\mathrm{Ind}_F^{\mathbb{Q}} F^+ V_F}\right) \arrow[d, "\iota_p"] \\
H^1(G_F, V_F) & \mathbf{T}(V_F) \arrow[l] \arrow[r, "\mathrm{Res}"] & \prod_{\sigma \in D \setminus \mathcal{E}} \sigma^{-1}\left(H^1\left(G_{F_{p_\sigma}}, \frac{F^0_{p_\sigma}0}{F^+_{p_\sigma}}\right)^{D_\sigma}\right)
\end{tikzcd}$$

whose vertical arrows are isomorphism.

Proof. The isomorphism on the left is simply Shapiro's Lemma.
To define ι_p we note that

$$\begin{aligned}
H^1\left(G_{\mathbb{Q}_p}, \frac{F^{00}V}{\mathrm{Ind}_F^{\mathbb{Q}} F^+ V_F}\right) =& H^1\left(G_{\mathbb{Q}_p}, \bigoplus_{\sigma \in D \setminus \mathcal{E}} \sigma^{-1} \mathrm{Ind}_{D'_{\mathfrak{p}}}^{D_{\mathfrak{p}}} \frac{F^{00}_{\mathfrak{p}} V}{F^+_{\mathfrak{p}} V}\right) \\
\cong & \bigoplus_{\sigma \in D \setminus \mathcal{E}} H^1\left(G_{F_{\mathfrak{p}}}, \frac{F^{00}_{\mathfrak{p}} V}{F^+_{\mathfrak{p}} V}\right).
\end{aligned}$$

We have the short exact sequence

$$0 \to \frac{F^{00}_{\mathfrak{p}}}{F^+_{\mathfrak{p}}} \to \frac{V}{F^+_{\mathfrak{p}}} \to \frac{V}{F^{00}_{\mathfrak{p}}} \to 0.$$

We have then

$$H^1\left(G_{F_{\mathfrak{p}}}, \frac{F^{00}_{\mathfrak{p}}}{F^+_{\mathfrak{p}}}\right) = \mathrm{Hom}\left(G_{F_{\mathfrak{p}}}, \frac{F^{00}_{\mathfrak{p}}}{F^+_{\mathfrak{p}}}\right) \cong \left(\frac{F^{00}_{\mathfrak{p}}}{F^+_{\mathfrak{p}}}\right)^{1+[F_{\mathfrak{p}}:\mathbb{Q}_p]}.$$

It remains to characterize the image $\mathbf{T}(V_F)$ of $\mathbf{T}(V)$ as a g-dimensional subspace of

$$\bigoplus_{\sigma \in D \setminus \mathcal{E}} \left(\frac{F_{\mathfrak{p}}^{00} V}{F_{\mathfrak{p}}^{+} V} \right)^{1+[F_{\mathfrak{p}}:\mathbb{Q}_p]}.$$

Suppose first that $F_{\mathfrak{p}}$ is Galois over $\mathbb{Q}_p$; we want to show that the image of $\mathbf{T}(V)$ via $\iota_p \circ \mathrm{Res}$ lies in the subspace of $\mathrm{Gal}(F_{\mathfrak{p}}/\mathbb{Q}_p)$-invariants (which by Lemma 6.1.6 is 2-dimensional). We apply Lemma 6.1.7 for $G = G_{\mathbb{Q}_p}$ and $H = G_{F_{\mathfrak{p}}}$; then $\overline{G} = \mathrm{Gal}(F_{\mathfrak{p}}/\mathbb{Q}_p)$ and we are done.
If $F_{\mathfrak{p}}$ is not Galois over $\mathbb{Q}_p$, we can make the same consideration as before with $H = G_{F_{\mathfrak{p}}^{gal}}$ where $F_{\mathfrak{p}}^{gal}$ denotes the Galois closure of $F_{\mathfrak{p}}$. Let us denote by Res' the restriction from $H^1(G_{F_{\mathfrak{p}}}, \;)$ to $H^1(G_{F_{\mathfrak{p}}^{gal}}, \;)$. We have then that the image of $\mathbf{T}(V)$ via $\mathrm{Res}' \circ \iota_p \circ \mathrm{Res}$ is contained in the $\overline{G}$-invariant part, which from Lemma 6.1.7 coincides with $\mathrm{Hom}\left(G_{\mathbb{Q}_p}, \frac{F_{\mathfrak{p}}^{00}}{F_{\mathfrak{p}}^{+}}\right)$. Then the image of $\mathbf{T}(V)$ via $\iota_p \circ \mathrm{Res}$ must be contained in $\mathrm{Hom}\left(G_{\mathbb{Q}_p}, \frac{F_{\mathfrak{p}}^{00}}{F_{\mathfrak{p}}^{+}}\right)$. □

The importance of the proposition is that we can calculate the space $\mathbf{T}(V)$ defined à la Greenberg in the cohomology of V_F and not of V, as the latter space will be easier to describe explicitly.
Before doing this, we impose some extra conditions on $\mathbf{f}$; this is done to ensure that we dispose of a modularity theorem in what follows. We shall suppose indeed that $\mathbf{f}$ is non-cuspidal at each prime of ramification, and we will write

$$\rho_{|D_{\mathfrak{q}}} \cong \begin{pmatrix} \eta_{2,\mathfrak{q}} & * \\ 0 & \eta_{1,\mathfrak{q}} \end{pmatrix}$$

for the local representation at $\mathfrak{q}$. We shall denote by $\varepsilon_{i,\mathfrak{q}}$ the restriction of $\eta_{i,\mathfrak{q}}$ to $I_{\mathfrak{q}}$. The $\varepsilon_{i,\mathfrak{q}}$'s are two finite order characters.
Let us denote by $CLN(\mathcal{O})$ the category of local, profinite $\mathcal{O}$-algebra A whose residue field is the same as $\mathcal{O}$ and let us fix a deformation problem Φ for $\overline{\rho}_{\mathbf{f}}$; Φ is a functor from $CLN(\mathcal{O})$ to the category of sets such that

$$\Phi(A) = \{\rho : G_F \to GL_2(A) | \rho \text{ satisfying } Q(1-5)\}$$

where $Q(1-5)$ are the following conditions:

Q1) if $\mathfrak{m}_A$ denotes the maximal ideal of A, $\rho \equiv \overline{\rho}_{\mathbf{f}} \bmod \mathfrak{m}_A$,

Q2) ρ is unramified outside $p\mathfrak{N}$,

Q3) the determinant of ρ is $\chi_{\mathrm{cycl}}^{m'}\varepsilon$, for $m' \in \mathbb{Z}$,

Q4) $\rho_{|D_{\mathfrak{p}}} \cong \begin{pmatrix} \varepsilon_{\mathfrak{p}} & * \\ 0 & \delta_{\mathfrak{p}} \end{pmatrix}$ such that $\varepsilon_{\mathfrak{p}}|_{I_{\mathfrak{p}}}\varepsilon_{2,\mathfrak{p}}^{-1}$ and $\delta_{\mathfrak{p}}|_{I_{\mathfrak{p}}}\varepsilon_{1,\mathfrak{p}}^{-1}$ factor through the p^{∞}-cyclotomic extension of $F_{\mathfrak{p}}$,

Q5) $\rho_{|D_{\mathfrak{q}}} \cong \begin{pmatrix} \varepsilon_{\mathfrak{q}} & * \\ 0 & \delta_{\mathfrak{q}} \end{pmatrix}$ such that if $\mathfrak{q} \mid c(\varepsilon)\mathfrak{N}$ then $\delta_{\mathfrak{q}}|_{I_{\mathfrak{q}}} = \varepsilon_{1,\mathfrak{q}}$, and $\rho|_{I_{\mathfrak{q}}} \otimes \varepsilon_1^{-1}$ is unramified if $\mathfrak{q} \mid c(\varepsilon)$.

Let us denote by $h_{\mathcal{N}}^{\mathrm{n.ord}}(\mathfrak{N}, \psi)$ the *big cyclotomic Hecke algebra* of [Hid06, §3.2.8]; it is a module over $\mathcal{O}[[\Gamma_F]]$, where

$$\Gamma_F = \prod_{\mathfrak{p}} \Gamma_{\mathfrak{p}};$$
$$\Gamma_{\mathfrak{p}} = p - \text{Sylow of } \mathrm{Gal}(F_{\mathfrak{p}}^{\mathrm{ur}}(\mu_{p^{\infty}})/F_{\mathfrak{p}}^{\mathrm{ur}}).$$

We identify $\mathcal{O}[[\Gamma_F]]$ with $\mathcal{O}[[x_{\mathfrak{p}}]]_{\mathfrak{p}|p}$ via $\gamma_{\mathfrak{p}} \mapsto x_{\mathfrak{p}}+1$, for $\gamma_{\mathfrak{p}}$ a topological generator of $\Gamma_{\mathfrak{p}}$ seen, via class field theory, as a subgroup of $1+p\mathbb{Z}_p$.
We shall denote by $\mathbb{T}$ the irreducible component of $h^{\mathrm{n.ord}}_{\mathcal{N}}(\mathfrak{N},\psi)$ to which $\mathbf{f}$ belongs and by $\rho_{\mathbb{T}}$ the associated Galois representation.
We suppose moreover that $\overline{\rho}_{\mathbf{f}}$ satisfies the following properties:

R1) $\overline{\rho}_{\mathbf{f}|F(\mu_p)}$ is absolutely irreducible,

R2) the character of $(\hat{\mathbb{Z}}^{\times})^2$ defined by $\varepsilon_1(a)\varepsilon_2(d)$ is of order prime to p,

R3) $\overline{\rho}_{\mathbf{f}}$ is $\mathfrak{p}$-distinguished for all $\mathfrak{p} \mid p$ (i.e. $\overline{\delta}_{\mathfrak{p}} \neq \overline{\varepsilon}_{\mathfrak{p}}$),

R4) if $\mathbf{f}$ is Steinberg at $\mathfrak{q}$, then $\overline{\rho}_{\mathbf{f}|I_{\mathfrak{q}}} \cong \begin{pmatrix} \varepsilon_{\mathfrak{q}} & * \\ 0 & \delta_{\mathfrak{q}} \end{pmatrix}$ with non-trivial $*$.

We have the following theorem [Hid06, Theorem 3.50]

Theorem 6.1.9. *The deformation problem Φ is representable by $(\mathbb{T},\rho_{\mathbb{T}})$; moreover $\mathbb{T}$ is a reduced complete intersection ring.*

This theorem is not directly related to the definition of the $\mathcal{L}$-invariant but it will be necessary to produce explicit elements of $\mathbf{T}(V_F)$.
Let us denote by ϕ the structural morphism $\mathbb{T} \to \mathcal{O}$ corresponding to $\mathbf{f}$, we have then [Hid06, Proposition 3.87]

Proposition 6.1.10. *We have an isomorphism*

$$\Omega_{\mathbb{T}/\mathcal{O}[[\Gamma_F]]} \otimes_{\phi} \mathcal{O} \cong Sel^*_F(A_F),$$

where $A_F := V_F/T_F$ and * denotes Pontryagin dual.

We will not give the proof, but we want to point out that the key ingredient is the following isomorphism already noticed in [MT90]. Let us denote by $R := \mathbb{T}[s]/(s^2)$ the infinitesimal deformation space of $\mathbb{T}$. We consider the set

$$\{\rho \in \Phi(R) | \rho \bmod s = \rho_{\mathbb{T}}\};$$

explicitly, a representation ρ in this set can be written $\rho_{\mathbb{T}} \oplus u'$. If we define $u(\sigma) = u'(\sigma)\rho_{\mathbb{T}}(\sigma)$, we see that

$$u(\sigma\tau) = \rho_{\mathbb{T}}(\sigma)u(\tau)\rho_{\mathbb{T}}^{-1}(\sigma) + u(\sigma).$$

This maps clearly satisfies the cocycle condition for V_F. Moreover, we can think u as a cocycle with value in $T_F \otimes_{\mathbb{T}} s$ (recall that T_F is a stable lattice in V_F) via the exponential map. One can check that these cocycles lie in the Selmer group thanks to the deformation conditions imposed by Φ.
We conclude noticing that

$$\{\rho \in \Phi(R) | \rho \bmod s = \rho_{\mathbb{T}}\} \cong Der_{\mathcal{O}}(\mathbb{T},\mathcal{O}).$$

Let us choose some local coordinates $t_{\mathfrak{p}}$ on $\mathbb{T}$ such that $t_{\mathfrak{p}} = 1 + x_{\mathfrak{p}} - \gamma_{\mathfrak{p}}^{k_{\mathfrak{p}}}$. Let us denote by $a_{\mathfrak{p}}(x)$ the Hecke eigenvalue of $T_0(\mathfrak{p})$, as defined in Chapter 3. Let us denote by x_0 the point corresponding to $\mathbf{f}$ (i.e. $t = 0$).

Theorem 6.1.11. *We have*

$$\mathcal{L}(V) = \det(\partial a_{\mathfrak{p}'}(x)/\partial x_{\mathfrak{p}''})_{|x=x_0} \prod_{\mathfrak{p}} \log_p(\gamma_{\mathfrak{p}})\delta_{\mathfrak{p}}([\gamma_{\mathfrak{p}} : F_{\mathfrak{p}}])a_{\mathfrak{p}}(x_0)^{-1},$$

where $[- : F_{\mathfrak{p}}]$ denotes the local Artin symbol for $F_{\mathfrak{p}}$.

Proof. Let us write $T_{\mathbf{f}}$ for the tangent space at $\mathbf{f}$ of $\mathbb{T}$. We want to identify it with $\mathbf{T}(V_F)$. As the two spaces have the same dimension, it is enough to show $\mathbf{T}(V_F) \subset T_{\mathbf{f}}$. Each cocycle c in $\mathbf{T}(V_F)$ can be written in the form

$$c_{|D_{\mathfrak{p}}} = B_{\mathfrak{p}} \begin{pmatrix} a_{\mathfrak{p}} & b_{\mathfrak{p}} \\ 0 & -a_{\mathfrak{p}} \end{pmatrix} B_{\mathfrak{p}}$$

for a suitable matrix $B_{\mathfrak{p}} \in GL_2(K)$ and an additive character $a_{\mathfrak{p}}$ (recall that $F_{\mathfrak{p}}^{00}$ consists of trace zero matrices). From the definition of Φ, we have that the elements of $\Phi(R)$ are upper triangular at all $\mathfrak{p}$. The representation

$$\rho_{\mathbf{f}} + c\rho_{\mathbf{f}} s$$

belongs to $\Phi(\mathcal{O}[s]/s^2) = Der_{\mathcal{O}}(\mathbb{T}, \mathcal{O})$ which can be identified with the tangent space $T_{\mathbf{f}}$. It is esay to see that this gives us an injective map $\mathbf{T}(V_F) \to T_{\mathbf{f}}$.
As we know that $\left\{ \frac{\partial}{\partial x_{\mathfrak{p}'}} \right\}$ is a basis of $T_{\mathbf{f}}$, we can write

$$c(\sigma)\rho_{\mathbf{f}}(\sigma) = \sum_{\mathfrak{p}} c_{\mathfrak{p}'} \frac{\partial \rho_{\mathbb{T}}}{\partial x_{\mathfrak{p}'}}, \quad c_{\mathfrak{p}'} \in K.$$

Explicitly, we have

$$\begin{aligned} a_{\mathfrak{p}}([p : F_{\mathfrak{p}}]) =& \delta_{\mathfrak{p}}([p : F_{\mathfrak{p}}])^{-1} \sum_{\mathfrak{p}'} c_{\mathfrak{p}'} \frac{\partial \delta_{\mathbb{T},\mathfrak{p}}([p : F_{\mathfrak{p}}])}{\partial x_{\mathfrak{p}'}} \\ a_{\mathfrak{p}}([\gamma_{\mathfrak{p}} : F_{\mathfrak{p}}]) =& \delta_{\mathfrak{p}}([\gamma_{\mathfrak{p}} : F_{\mathfrak{p}}])^{-1} \sum_{\mathfrak{p}'} c_{\mathfrak{p}'} \frac{\partial \delta_{\mathbb{T},\mathfrak{p}}([\gamma_{\mathfrak{p}} : F_{\mathfrak{p}}])}{\partial x_{\mathfrak{p}'}}. \end{aligned}$$

So

$$\begin{aligned} \iota_{\mathrm{unr}} &= \left(\delta_{\mathfrak{p}}([p : F_{\mathfrak{p}}])^{-1} \frac{\partial \delta_{\mathbb{T},\mathfrak{p}}([p : F_{\mathfrak{p}}])}{\partial x_{\mathfrak{p}'}} \right)_{\mathfrak{p},\mathfrak{p}'}, \\ \iota_{\mathrm{cylc}} &= \left(\log_p(\gamma_{\mathfrak{p}})^{-1} \delta_{\mathfrak{p}}([\gamma_{\mathfrak{p}} : F_{\mathfrak{p}}])^{-1} \frac{\partial \delta_{\mathbb{T},\mathfrak{p}}([\gamma_{\mathfrak{p}} : F_{\mathfrak{p}}])}{\partial x_{\mathfrak{p}'}} \right)_{\mathfrak{p},\mathfrak{p}'}. \end{aligned}$$

To conclude, we recall that $\delta_{\mathfrak{p}}([p : F_{\mathfrak{p}}]) = x_0 \circ \delta_{\mathbb{T},\mathfrak{p}}([p : F_{\mathfrak{p}}]) = a_{\mathfrak{p}}(\mathbf{f})$, and that the $\mathcal{O}[[\Gamma_F]]$-structure on $\mathbb{T}$ induces $\delta_{\mathfrak{p}}([\gamma_{\mathfrak{p}}^s : F_{\mathfrak{p}}]) = (1 + x_{\mathfrak{p}})^s$. □

From [Hid06, Corollary 3.57] we know that if a form in the family is Steinberg at $\mathfrak{p}$, then $\delta_{\mathfrak{p}}([p : F_{\mathfrak{p}}])$ depends only on $x_{\mathfrak{p}}$. In particular, if the family is Steinberg at all primes above p except possibly one, the $\mathcal{L}$-invariant is zero only for finitely many forms in the family [Hid00a, Proposition 7.1].

6.2 $\mathcal{L}$-invariant for semistable representations

In this section we want to deal with the case where V is not ordinary. Let us begin with an example: let f be an elliptic modular forms of even weight $k > 2$ and Steinberg at p. Then $a_p(f) = \pm p^{(k-2)/2}$ and it is known that V_f as a $G_{\mathbb{Q}_p}$-representation is irreducible. In particular it is not ordinary, but the p-adic L-function which has been constructed for f has a trivial zero at $s = k/2$ if $a_p(f) = p^{(k-2)/2}$. Fontaine and Mazur defined an $\mathcal{L}$-invariant for this particular case in term of the filtered (φ, N)-module $\mathbf{D}^{\dagger}(V_f)$. In [Ste10] Stevens proved that the derivative of this p-adic L-function is exactly the $\mathcal{L}$-invariant times the special value.

Recently, Benois [Ben11] succeeded in generalizing the definitions of the $\mathcal{L}$-invariant given by Greenberg and by Fontaine-Mazur in a uniform way.
His generalization is based on the observation that p-adic representations of $G_{\mathbb{Q}_p}$ which are irreducible may not have an irreducible (φ, Γ)-module (remark originally due to Colmez). In particular, if V is semistable, then it is trianguline, i.e. the associated (φ, Γ)-module is a successive extensions of rank one étale (φ, Γ)-modules (see [BC09, Remark 2.3.5, Proposition 2.4.1]). The analogy between an ordinary representation and the (φ, Γ)-module of a semistable representation is now evident to the reader; the definition of the $\mathcal{L}$-invariant is then done similarly as before using cohomology of (φ, Γ)-modules in place of Galois cohomology.

6.2.1 Reminder of (φ, Γ)-modules

We begin by recalling some period rings; we fix a completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}}_p$ and a valuation v_p extending the valuation on $\mathbb{Q}_p$; we shall denote by $\mathcal{O}_{\mathbb{C}_p}$ its valuation ring. We define

$$\tilde{\mathbf{E}}^+ := \varprojlim_{x \mapsto x^p} \frac{\mathcal{O}_{\mathbb{C}_p}}{p\mathcal{O}_{\mathbb{C}_p}}.$$

Each element can be written in the form $(x_0, x_1, x_2, \ldots)$ with $x_i \in \mathcal{O}_{\mathbb{C}_p}$ and $x_i = x_{i+1}^p$. It is a domain, integrally closed in its fraction field $\tilde{\mathbf{E}}$ and has a valuation $v_{\tilde{\mathbf{E}}}(x) = v_p(x_0)$. We let $\tilde{\mathbf{A}} = W(\tilde{\mathbf{E}})$ be the ring of Witt vector of $\tilde{\mathbf{E}}$ and we shall denote by $[\ ]$ the Teichmüller character. Let (ζ_{p^i}) be a coherent sequence of primitive p-th roots of unity; we define $[\epsilon] = (1, \zeta_p, \zeta_p^2, \ldots)$ and $\pi = [\epsilon] - 1$. We shall denote by φ the Frobenius on $\mathbb{C}_p$ and by abuse of notation the Frobenius on $\tilde{\mathbf{A}}$, and by Γ the Galois group (isomorphic to $\mathbb{Z}_p^*$) of $\mathbb{Q}_{p^\infty} = \bigcup_i \mathbb{Q}_p(\zeta_{p^i})$.
We warn the readear that Γ here is different from Γ in the previous section.
We have

$$\begin{aligned}\varphi(\pi) =& (1+\pi)^p - 1,\\ \gamma(\pi) =& (1+\pi)^{\chi_{\mathrm{cycl}}(\gamma)} - 1.\end{aligned}$$

We define $\tilde{\mathbf{B}} = \tilde{\mathbf{A}}[p^{-1}]$; we introduce now several rings:

$$\begin{aligned}\mathbf{B}^{\dagger,r} =& \left\{ f = \sum_{n \in \mathbb{Z}} a_n \pi^n | a_n \in \mathbb{Q}_p, \text{ such that } f(X) = \sum_{n \in \mathbb{Z}} a_n X^n \right. \\ & \left. \text{is holomorphic and bounded on } p^{-1/r} \leq |X|_p < 1 \right\}, \\ \mathbf{B}^{\dagger} =& \bigcup_r \mathbf{B}^{\dagger,r}.\end{aligned}$$

Note that by definition $\mathbf{B}^\dagger \subset \tilde{\mathbf{B}}$. We also define

$$\begin{aligned}\mathbf{B}^{\dagger,r}_{\mathrm{rig}} =& \left\{ f = \sum_{n \in \mathbb{Z}} a_n \pi^n | a_n \in \mathbb{Q}_p, \text{ such that } f(X) = \sum_{n \in \mathbb{Z}} a_n X^n \right. \\ & \left. \text{is holomorphic on } p^{-1/r} \leq |X|_p < 1 \right\}, \\ \mathcal{R} =& \bigcup_r \mathbf{B}^{\dagger,r}_{\mathrm{rig}}.\end{aligned}$$

Note that $\mathbf{B}^\dagger \subset \mathcal{R} \not\subset \tilde{\mathbf{B}}$. We define a φ-module over $\mathbf{B}^\dagger$ to be a finitely generated, free $\mathbf{B}^\dagger$-module D with a semilinear action of a Frobenius φ extending the Frobenius on $\mathbf{B}^\dagger$.

Let us fix a rational $r = a/b$, with $a \in \mathbb{Z}$, $b > 0$, a and b coprime. We define $D^{[r]}$ as the free $\mathbf{B}^{\dagger}$-module of rank b, with basis $e_1, \ldots, e_b$ such that

$$\varphi(e_1) = e_2, \varphi(e_2) = e_3, \ldots, \varphi(e_b) = p^a e_1.$$

We shall call such a φ-module pure of slope r.
Dieudonné-Manin theory gives us the following decomposition

$$D \otimes_{\mathbf{B}^{\dagger}} \tilde{\mathbf{B}} = \sum_{i \in I} D^{[r_i]}.$$

The theory classifying φ-modules over the Robba ring $\mathcal{R}$ is way more complicated and it has been recently developed by Kedlaya [Ked04]. We have the following structure theorem

Theorem 6.2.1. *Let D be a free, finitely generated module over $\mathcal{R}$ with a semilinear Frobenius, then there exists a canonical filtration*

$$0 = D_0 \subset D_1 \subset \ldots \subset D_h = D$$

such that each graded piece D_i/D_{i+1} contains a unique $\mathbf{B}^{\dagger}$-submodule Δ_i satisfying

i) Δ_i is a φ-module over $\mathbf{B}^{\dagger}$, pure of slope r_i and $\Delta \otimes_{\mathbf{B}^{\dagger}} \mathcal{R} = D_i/D_{i+1}$,

ii) $r_1 < r_2 \ldots < r_h$.

We shall say that D is pure of slope r if $h = 1$, $r_1 = r$. A (φ, Γ)-module over $\mathcal{R}$ is a φ-module over $\mathcal{R}$ with a semilinear action of Γ, commuting with φ. We shall say that a (φ, Γ)-module is étale if it is pure of slope 0. This is equivalent to ask that the determinant of φ is of valuation 0 in $\mathbf{B}^{\dagger}$.
Let us denote by $H_{\mathbb{Q}_p}$ the absolute Galois group of $\mathbb{Q}_{p^{\infty}}$.

Theorem 6.2.2. *The functor*

$$D^{\dagger} : V \mapsto \mathbf{D}^{\dagger}(V) := \left(\mathbf{B}^{\dagger} \otimes V\right)^{H_{\mathbb{Q}_p}}$$

defines an equivalence of Tannakian categories between the category of continuous, finite dimensional, Galois representations of $G_{\mathbb{Q}_p}$ and étale (φ, Γ)-module over $\mathbf{B}^{\dagger}$.
The functor

$$D^{\dagger}_{\mathrm{rig}} : V \mapsto \mathbf{D}^{\dagger}_{\mathrm{rig}}(V) := \mathbf{D}^{\dagger}(V) \otimes_{\mathbf{B}^{\dagger}} \mathcal{R}$$

defines an equivalence of Tannakian category between the category of continuous, finite dimensional, Galois representations of $G_{\mathbb{Q}_p}$ and (φ, Γ)-module over $\mathcal{R}$ pure of slope 0.

6.2.2 Cohomology of (φ, Γ)-modules

Let D be a (φ, Γ)-module over a ring A; we recall the definition of cohomology of (φ, Γ)-module [Her98, Her01]. We associate to D a complex

$$C_{\varphi,\gamma}(D) : 0 \to D \xrightarrow{f} D \oplus D \xrightarrow{g} D \to 0,$$

where $f(x) = ((1-\varphi)x, (1-\gamma)x)$ and $g(y, z) = (1-\varphi)z - (1-\gamma)y$, for γ in Γ.
We shall write $H^i(D)$ for the i-th cohomology of $C_{\varphi,\gamma}(D)$. For any short exact sequence

$$0 \to D' \to D \to D'' \to 0,$$

we have a corresponding long exact sequence of cohomology groups. Each $H^i(D)$ is a finite dimensional $\mathbb{Q}_p$-vector space and there exists a cup product [Her01] which generalizes Tate duality for local fields. We have in particular a duality

$$\begin{array}{ccccc} H^1(D) & \times & H^1(D^*(1)) & \to & \mathbb{Q}_p \\ (d_1, d_2) & \times & (e_1, e_2) & \mapsto & (d_2 \otimes \gamma(e_1) - d_1 \otimes \varphi(e_2)) \end{array}$$

after identifying $H^2(\mathcal{R}(x|x|))$ with $\mathbb{Q}_p$.
Moreover if D is a (φ, Γ)-module over $\mathbf{B}^\dagger$, we have

$$H^i(D) = H^i(D \otimes_{\mathbf{B}^\dagger} \mathcal{R})$$

and for all p-adic representation V

$$H^i(G_{\mathbb{Q}_p}, V) = H^i(\mathbf{D}^\dagger(V)).$$

Let us write $\log \pi$ for a transcendent element over the fraction field of $\mathcal{R}$ such that

$$\begin{aligned} \varphi(\log \pi) =& p \log \pi + \log\left(\frac{\varphi(\pi)}{\pi^p}\right), \\ \gamma(\log \pi) =& \log \pi + \log\left(\frac{\gamma(\pi)}{\pi}\right). \end{aligned}$$

Note that $\log\left(\frac{\varphi(\pi)}{\pi^p}\right)$ and $\log\left(\frac{\gamma(\pi)}{\pi}\right)$ converge in $\mathcal{R}$. We shall write $\mathcal{R}_{\log}$ for $\mathcal{R}[\log \pi]$. We can define a monodromy operator N on $\mathcal{R}_{\log}$: $N = -\left(1 - \frac{1}{p}\right)^{-1} \frac{\mathrm{d}}{\mathrm{d} \log \pi}$.
Let us write $\mathbb{Q}_{p^\infty}((t))$ for the field of Laurent series equipped with the filtration $\mathrm{Fil}^i(\mathbb{Q}_{p^\infty}((t))) := t^i \mathbb{Q}_{p^\infty}[[t]]$.
We have a natural action of Γ on t given by $\gamma t = \chi_{\mathrm{cycl}}(\gamma) t$.
Let $n > 0$ and $r_n = p^{n-1}(p-1)$, we have an injection

$$\iota_n : \mathbf{B}_{\mathrm{rig}}^{\dagger, r_n}[\log \pi] \to \mathbb{Q}_{p^\infty}[[t]]$$

characterized by $\iota_n(\pi) = \zeta_{p^n} \exp(t/p^n) - 1$ and $\iota_n(\log \pi) = \log(\iota_n(\pi))$. It is known that for each (φ, Γ)-module there exists $n \gg 0$ such that D has a structure of $\mathbf{B}_{\mathrm{rig}}^{\dagger, r_n}$-module [CC98]. We shall denote this module over $\mathbf{B}_{\mathrm{rig}}^{\dagger, r_n}$ by $D^{(n)}$. We can define the following filtered $\mathbb{Q}_p$- vector space

$$\mathcal{D}_{\mathrm{dR}}(D) = \left(D^{(n)} \otimes_{\mathbf{B}_{\mathrm{rig}}^{\dagger, r_n}, \iota_n} \mathbb{Q}_{p^\infty}((t))\right)^\Gamma.$$

It is independent of n and its dimension over $\mathbb{Q}_p$ is smaller or equal to the rank of D. We pose moreover

$$\mathcal{D}_{\mathrm{st}}(D) = (D \otimes_{\mathcal{R}} \mathcal{R}_{\log}[1/t])^\Gamma.$$

It is a finite dimensional $\mathbb{Q}_p$-vector space contained in $\mathcal{D}_{\mathrm{dR}}(D)$. We say that D is semistable if the dimension of $\mathcal{D}_{\mathrm{st}}(D)$ is equal to the rank of D.
We have a filtration on $\mathcal{D}_{\mathrm{st}}(D)$, induced by the one on $\mathbb{Q}_{p^\infty}((t))$, which is independent of n.
For further use, we define $\mathcal{D}_{\mathrm{cris}}(D) = \mathcal{D}_{\mathrm{st}}(D)^{N=0}$, and we say that D is crystalline if the dimension of $\mathcal{D}_{\mathrm{cris}}(D)$ is equal to the rank of D.

Remark 6.2.3. *When $D = \mathbf{D}^\dagger(V)$, D is semistable (resp. crystalline) if and only if V is semistable (resp. crystalline) à la Fontaine.*

We introduce now the theory of filtered (φ, N)-modules. We say that a finite dimensional vector space M over $\mathbb{Q}_p$ is a filtered (φ, N)-module if the following are satisfied:

- there exists an exhaustive, decreasing filtration $\mathrm{Fil}^i M$,
- there exists a $\mathbb{Q}_p$-linear Frobenius $\varphi : M \to M$,
- there is a $\mathbb{Q}_p$-linear monodromy operator $N : M \to M$ such that $N\varphi = p\varphi N$.

A morphism of (φ, N)-modules is morphism commuting to φ and N. We can define a tensor product in this category. We define on $M \otimes M'$ the following

- $\mathrm{Fil}^i(M \otimes M') = \sum_{j+j'=i} \mathrm{Fil}^j M \otimes \mathrm{Fil}^{j'} M'$,
- $\varphi(m \otimes m') = \varphi(m) \otimes \varphi(m')$,
- $N(m \otimes m') = m \otimes N(m') + N(m) \otimes m'$.

Proposition 6.2.4. *The functor $\mathcal{D}_{\mathrm{st}}(D)$ induces an equivalence of additive and tensor categories between the category of semistable (φ, Γ)-modules over $\mathcal{R}$ and filtered (φ, N)-modules over $\mathbb{Q}_p$.*

We are ready now to study more in detail the cohomology of certain semistable (φ, Γ)-modules D. Let $\alpha = (a, b)$ be an homogeneous cocycle in $Z^1(C_{\varphi,\gamma})$, we can associate to it the extension

$$0 \to D \to D_\alpha \to \mathcal{R} \to 0,$$

defined by

$$D_\alpha = D \oplus \mathcal{R}e, \;\; (\varphi - 1)e = a, \;\; (\gamma - 1)e = b.$$

This defines an isomorphism

$$H^1(D) = \mathrm{Ext}^1_{\mathcal{R}}(\mathcal{R}, D).$$

We say that class α in $H^1(D)$ is crystalline resp. semistable if

$$\dim_{\mathbb{Q}_p} \mathcal{D}_{\mathrm{cris}}(D_\alpha) = \dim_{\mathbb{Q}_p} \mathcal{D}_{\mathrm{cris}}(D) + 1$$

resp. $\dim_{\mathbb{Q}_p} \mathcal{D}_{\mathrm{st}}(D_\alpha) = \dim_{\mathbb{Q}_p} \mathcal{D}_{\mathrm{st}}(D)+1$. We define consequently $H^1_f(D)$ resp. $H^1_{\mathrm{st}}(D)$ as the set of crystalline resp. semistable classes in $H^1(D)$. When $D = \mathbf{D}^\dagger_{\mathrm{rig}}(V)$, we have that

$$\begin{aligned} H^1_{\mathrm{f}}(D) = H^1_{\mathrm{f}}(V) :=& \mathrm{Ker}\left(H^1(G_{\mathbb{Q}_p}, V) \to H^1(G_{\mathbb{Q}_p}, V \otimes \mathbf{B}_{\mathrm{cris}})\right), \\ H^1_{\mathrm{st}}(D) = H^1_{\mathrm{st}}(V) :=& \mathrm{Ker}\left(H^1(G_{\mathbb{Q}_p}, V) \to H^1(G_{\mathbb{Q}_p}, V \otimes \mathbf{B}_{\mathrm{st}})\right), \end{aligned}$$

where $\mathbf{B}_{\mathrm{st}}$ and $\mathbf{B}_{\mathrm{cris}}$ denote Fontaine's rings.
Let us denote by t_D the quotient $\mathcal{D}_{\mathrm{dR}}(D)/\mathrm{Fil}^0\mathcal{D}_{\mathrm{dR}}(D)$. We define the following complexes

$$\begin{aligned} C^\bullet_{\mathrm{cris}} :& \mathcal{D}_{\mathrm{cris}}(D_\alpha) \xrightarrow{f} t_D \oplus \mathcal{D}_{\mathrm{cris}}(D_\alpha), \\ C^\bullet_{\mathrm{st}} :& \mathcal{D}_{\mathrm{st}}(D_\alpha) \xrightarrow{g} t_D \oplus \mathcal{D}_{\mathrm{st}}(D_\alpha) \oplus \mathcal{D}_{\mathrm{st}}(D_\alpha) \xrightarrow{h} \mathcal{D}_{\mathrm{st}}(D_\alpha) \end{aligned}$$

where

$$\begin{aligned} f(x) =& (x \bmod \mathrm{Fil}^0(\mathcal{D}_{\mathrm{dR}}(D)), (1 - \varphi)x), \\ g(x) =& (x \bmod \mathrm{Fil}^0(\mathcal{D}_{\mathrm{dR}}(D)), (\varphi - 1)x, N(x)), \\ h(x, y, z) =& N(y) - p(\varphi - 1)(z). \end{aligned}$$

Proposition 6.2.5. *We have* $H^0(D) = H^0(C^\bullet_{\mathrm{cris}}) = H^0(C^\bullet_{\mathrm{st}})$, $H^1_{\mathrm{f}}(D) = H^1(C^\bullet_{\mathrm{cris}})$, *and* $H^1_{\mathrm{st}}(D) = H^1(C^\bullet_{\mathrm{st}})$.

The above proposition is useful to calculate the dimension of the space of crystalline and semistable extensions;

Corollary 6.2.6. *Let D be a de Rham (φ,Γ)-module, then we have*

$$\begin{aligned}\dim_{\mathbb{Q}_p} H^1_{\mathrm{f}}(D) - \dim_{\mathbb{Q}_p} H^0(D) =& \dim_{\mathbb{Q}_p} t_D,\\ \dim_{\mathbb{Q}_p} H^1_{\mathrm{st}}(D) - \dim_{\mathbb{Q}_p} H^1_{\mathrm{f}}(D) =& \dim_{\mathbb{Q}_p} \mathcal{D}_{\mathrm{cris}}(D^*(1))^{\varphi=1}.\end{aligned}$$

Moreover $H^1_{\mathrm{f}}(D)$ is orthogonal to $H^1_{\mathrm{f}}(D^*(1))$ for the cup product defined above.

6.2.3 Rank one (φ,Γ)-modules

We want to study now the cohomology of rank one semistable (φ,Γ)-module. Let δ be any continuous character of $\mathbb{Q}_p^\times$, we denote by $\mathcal{R}(\delta)$ the rank one (φ,Γ)-module $\mathcal{R}e_\delta$ characterized by $\varphi e_\delta = \delta(p)e_\delta$ and $\gamma e_\delta = \delta(\chi_{\mathrm{cycl}}(\gamma))e_\delta$.
We denote by x the identity map and by $|x|$ the character $x \mapsto |x|$. By the above equivalence of categories, $\mathcal{R}(x|x|)$ corresponds to the p-adic cyclotomic character.
We have that $\mathcal{R}(\delta)$ is semistable if and only if there exist $m \in \mathbb{Z}$ such that for all $u \in \mathbb{Z}_p^\times$ we have $\delta(u) = u^m$.
We have the following proposition

Proposition 6.2.7. *We have $H^0(\mathcal{R}(\delta)) \neq 0$ unless $\delta = x^{-m}$, for $m \geq 0$, in which case $H^0(\mathcal{R}(x^{-m})) = \mathbb{Q}_p(t^m e)$.*
If $m \geq 1$, then, unless $\delta = x^m|x|$, we have that $H^1(\mathcal{R}(\delta))$ has dimension one and $H^1(\mathcal{R}(\delta)) = H^1_{\mathrm{f}}(\mathcal{R}(\delta))$.
If $m \leq 0$, then, unless $\delta = x^m$, we have that $H^1(\mathcal{R}(\delta))$ has dimension one and $H^1_{\mathrm{f}}(\mathcal{R}(\delta)) = 0$.

We are left to study the cases $\delta = x^m|x|$, $m \geq 1$ and $\delta = x^m$, $m \leq 0$. This amounts to study the extension between $\mathcal{R}(\delta)$ and $\mathcal{R}$ and is the main subject of [Col08].
We introduce the differential operator $\partial = (1+\pi)\frac{\mathrm{d}}{\mathrm{d}\pi}$. We notice now that

$$\partial\varphi = p\varphi\partial, \quad \partial\gamma = \chi(\gamma)\gamma\partial.$$

This implies that ∂ induces an isomorphism of (φ,Γ)-modules between $\mathcal{R}(x^{-1}\delta)$ and $\mathcal{R}(\delta)$ and, by fonctoriality, a morphism between cocycles and in cohomology. Let us denote by $\mathrm{Res}(f)$ the residue at $\pi = 0$ of $(1+\pi)^{-1}f(\pi)\mathrm{d}\pi$. We have the following proposition

Proposition 6.2.8. *If $\delta \neq x|x|$ and $\delta \neq x$, then*

$$\partial : H^1(\mathcal{R}(x^{-1}\delta)) \to H^1(\mathcal{R}(\delta))$$

is an isomorphism, otherwise it is identically zero.

Proof. We begin by studying the kernel of this morphism; if $\delta = x$, $x^{-1}\delta = 1$ and $H^1(1)$ is generated by the constants $(1,0)$ and $(0,1)$ which are killed by ∂.
If $\delta = x|x|$, we have that $H^1(|x|)$ is one-dimensional, generated by the class of $\left(\log\frac{\gamma(\pi)}{\pi}, \frac{1}{p}\log\frac{\varphi(\pi)}{\pi^p}\right)$. We have consequently

$$\begin{aligned}\partial\frac{1}{p}\log\frac{\varphi(\pi)}{\pi^p} &= \frac{(1+\pi)-(1+\pi)^p}{\pi\varphi(p)}\\ &= \varphi\left(\frac{\pi+1}{\pi}\right) - \frac{\pi+1}{\pi},\\ \partial\log\frac{\gamma(\pi)}{\pi} &= \gamma\left(\frac{1+\pi}{\pi}\right) - \frac{1+\pi}{\pi}\end{aligned}$$

which is a cobord.
From now on, we suppose that $\delta \neq x|x|$ and $\delta \neq x$. If $\partial(a,b)$ is a coboundary, we have $c \in \mathcal{R}$ such that $\partial a = (\delta(p)\varphi - 1)c$ and $\partial b = (\delta(\chi_{\mathrm{cycl}}(\gamma))\gamma - 1)c$. As $\delta \neq x|x|$ and $\delta \neq x$, at least one between $(\delta(p)-1)$ and $(\delta(\chi_{\mathrm{cycl}}(\gamma))\chi_{\mathrm{cycl}}^{-1}(\gamma) - 1)$ is not zero. Using the fact that the image of ∂ coincide with the series f which satisfies $\mathrm{Res}(f) = 0$ [Col08, Proposition A.6], we find that $c = \partial c'$. So, up to a constant, (a,b) is a coboundary and we are done.
We show now that it is surjective. Let (a,b) be a cocycle in $Z^1(C_{\varphi,\gamma}(\mathcal{R}(x^{-1}\delta)))$, we have the relation $(\delta(\gamma)\gamma - 1)a = (\delta(p)\varphi - 1)b$. As $\delta \neq x|x|$, we can change (a,b) by a coboundary such that $\mathrm{Res}(b) = \mathrm{Res}(a) = 0$. Then we can find two primitive a' and b' of a and b. The couple (a',b') does not belong necessarily to $Z^1(C_{\varphi,\gamma}(\mathcal{R}(x^{-1}\delta)))$ but satisfy the relation

$$\partial((\delta(\gamma)\chi_{\mathrm{cycl}}(\gamma)\gamma - 1)a' - (\delta(p)\varphi p^{-1} - 1)b') = 0,$$

or equivalently

$$(\delta(\gamma)\chi_{\mathrm{cycl}}(\gamma)\gamma - 1)a' - (\delta(p)\varphi p^{-1} - 1)b' \in \mathbb{Q}_p.$$

Changing (a',b') by a constant, we have then $\partial(a',b') = (a,b)$. □

Proposition 6.2.9. *Suppose $\delta = x^m|x|$ for $m \geq 1$, then $H^1(\mathcal{R}(\delta))$ is two dimensional, generated by $\mathrm{cl}(\alpha_m)$ and $\mathrm{cl}(\beta_m)$, for α_m and β_m in $Z^1(C_{\varphi,\gamma}(\mathcal{R}(\delta)))$ explicitly given by*

$$\alpha_m = \frac{(-1)^{m-1}}{(m-1)!}\partial^{m-1}\left(\frac{1}{\pi} + \frac{1}{2}, a\right)e_\delta,\ (1-\varphi)a = (1 - \chi_{\mathrm{cycl}}(\gamma)\gamma)\left(\frac{1}{\pi} + \frac{1}{2}\right),$$
$$\beta_m = \frac{(-1)^{m-1}}{(m-1)!}\partial^{m-1}\left(b, \frac{1}{\pi}\right)e_\delta,\ (1-\varphi)\frac{1}{\pi} = (1 - \chi_{\mathrm{cycl}}(\gamma)\gamma)b.$$

Suppose $\delta = x^{-m}$ for $m \geq 0$, then $H^1(\mathcal{R}(\delta))$ is two dimensional, generated by $\mathrm{cl}(t^m, 0)e$ and $\mathrm{cl}(0, t^m)e$.

Proof. We begin with the case $\delta = x|x|$. Once we find a basis for $H^1(\mathcal{R}(x|x|))$, the iteration of ∂ will give us a basis for $H^1(\mathcal{R}(x^m|x|))$.
For the same reasoning as in the proof of the above proposition, we have that ∂ induces a surjection from $H^1(|x|)$ into $Z^1(C_{\varphi,\gamma}(\mathcal{R}(x|x|)))_{\mathrm{Res}(b)=\mathrm{Res}(a)=0}$. But we have seen before that in this case ∂ is identically zero, so the map

$$(a,b) \mapsto (\mathrm{Res}(a), \mathrm{Res}(b))$$

is an injection of $H^1(\mathcal{R}(x|x|))$ into K^2. We are left to exhibit two distinct elements. We have from [Col08, Lemme 2.3 (i)] (choosing the measure λ such that $\int_{\mathbb{Z}_p^\times}(1+\pi)^x \mathrm{d}\lambda(x) = (\varphi - 1)\frac{1}{\pi}$) that the equation $(\chi_{\mathrm{cycl}}(\gamma)\gamma - 1)a = (\varphi - 1)\frac{1}{\pi}$ has a unique solution, and clearly $\mathrm{Res}\left(\frac{1}{\pi} + \frac{1}{2}\right) = 1$ and moreover $\mathrm{Res}(a) = 0$.
We have from [Col08, Lemme 2.4] that the equation $(\varphi - 1)b = (\chi_{\mathrm{cycl}}(\gamma)\gamma - 1)\left(\frac{1}{\pi} + \frac{1}{2}\right)$ has a unique solution in $\pi\mathcal{R}^+$. Clearly $\mathrm{Res}\left(\frac{1}{\pi}\right) = 1$ and $\mathrm{Res}(b) = 0$ and we are done.
The case $\delta = x^{-m}$ is [Col08, Proposition 2.6] and can be done by hand.

□

We want to point out that among the cases considered above, $\mathcal{R}(\delta)$ admits an extension by $\mathcal{R}$ which corresponds, via the above mentioned equivalence of category, to an ordinary 2-dimensional local Galois representations only if $\delta = x|x|$ or $\delta = 1$.
We are left to study $H^2(\mathcal{R}(\delta))$;

Proposition 6.2.10. *If $\delta = |x|x^m$, $m \geq 1$, then $H^2(\mathcal{R}(x^m|x|))$ is one-dimensional, and the isomorphism with $\mathbb{Q}_p$ is induced by*

$$\begin{array}{rccc} \mathrm{inv}: & \mathcal{R}(\delta) & \to & \mathbb{Q}_p \\ & f & \mapsto & -\left(1-\frac{1}{p}\right)^{-1} \log \chi_{\mathrm{cycl}} \gamma^{-1} \mathrm{res}(f t^{m-1} \mathrm{d}t) \end{array}$$

and a generator is

$$w_m = (-1)^m \left(1 - \frac{1}{p}\right) \frac{\log \chi_{\mathrm{cycl}} \gamma}{(m-1)!} \partial^{m-1} \mathrm{cl}\left(\frac{1}{\pi}\right).$$

Otherwise $H^2(\mathcal{R}(\delta)) = 0$.

Proof. The statement about the dimension is immediate from the duality with $H^0(\delta^{-1}x|x|)$.
We start checking that Ker(inv) contains all the 2-coboundaries. We let e be the basis of $\mathcal{R}(\delta)$. If α is 2-coboundary, then α must be of the form

$$\alpha = xe = (1-\gamma)ye$$

for some x and y in $\mathcal{R}$. Then x and y must satisfy $x = (1 - \chi_{\mathrm{cycl}}^m(\gamma)\gamma)y$. We multiply by t^{m-1}, and obtain $xt^{m-1} = (1 - \chi_{\mathrm{cycl}}(\gamma)\gamma)(yt^{m-1})$. From [Col08, Appendix A] we have $\mathrm{Res}(\gamma x) = \chi_{\mathrm{cycl}}^{-1}(\gamma)\mathrm{Res}(x)$ and we conclude $\mathrm{res}(\alpha t^{m-1}\mathrm{d}t) = 0$. The same is true if $xe = (1-\varphi)ye$.
Clearly w_1 is a basis of $H^2(\mathcal{R}(x|x|))$. The explicit calculation of $\partial^{m-1}\frac{1}{\pi}$ allows us to conclude. $\square$

We conclude with the study of the space $H^1_{\mathrm{f}}(\delta)$; we modify slightly the above basis, let us pose

$$x_m^* = -\mathrm{cl}(t^m, 0)w_m,\ y_m^* = \log \chi_{\mathrm{cycl}}(\gamma)\mathrm{cl}(0, t^m)w_m$$
$$\alpha_m^* = -\left(1 - \frac{1}{p}\right)\mathrm{cl}\alpha_m,\ \beta_m^* = \left(1 - \frac{1}{p}\right)\log \chi_{\mathrm{cycl}}(\gamma)\mathrm{cl}\beta_m.$$

Proposition 6.2.11. *We have that α_m^* generates $H^1_{\mathrm{f}}(\mathcal{R}(x^m|x|))$ and x_m^* generates $H^1_{\mathrm{f}}(\mathcal{R}(x^{-m}))$.*

Proof. We have that both space are 1-dimensional from Corollary 6.2.6. It is immediate to see that x_m^* defines a crystalline extension. The cup product

$$H^1(\mathcal{R}(x^m|x|)) \times H^1(\mathcal{R}(x^{1-m})) \to H^2(\mathcal{R}(x|x|)),$$

sending $(a,b),(f,g)$ into $b\gamma(f) - a\varphi(g)$ defines a perfect duality. Using the definition of the isomorphism of $H^2(\mathcal{R}(x|x|))$ with $\mathbb{Q}_p$ we see that $\alpha_m^* \cup x_{m-1}^* = 0$. From Corollary 6.2.6 we conclude. $\square$

We are almost ready to define the analogue of the decomposition

$$H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p) = H^1_{\mathrm{cyc}}(\mathbb{Q}_p) \oplus H^1_{\mathrm{unr}}(\mathbb{Q}_p).$$

We consider a semistable (φ,Γ)-module D of rank g such that $\mathcal{D}_{\mathrm{st}}(D)^{\varphi=1} = \mathcal{D}_{\mathrm{st}}(D)$ and whose Hodge-Tate weight are non-negatives. We have the following theorem;

Theorem 6.2.12. *i) D is crystalline and*

$$D \cong \bigoplus_{i=1}^{g} \mathcal{R}(x^{-k_i}),$$

where k_i, are the Hodge-Tate weights ordered such that $k_i \geq k_{i-1}$.

ii) $H^0(D) \cong \mathcal{D}_{\mathrm{cris}}(D)$ *is of dimension* g, $H^2(D) = 0$, $H^1(D)$ *is of dimension* $2g$ *and*

$$\iota_D : \mathcal{D}_{\mathrm{cris}}(D) \oplus \mathcal{D}_{\mathrm{cris}}(D) \to H^1(D)$$

induced by $\iota_D(x,y) = \mathrm{cl}(-x, \log(\chi_{\mathrm{cycl}}(\gamma)y))$ *is an isomorphism.*

iii) Let us denote by $\iota_{D,\mathrm{f}}$ *resp.* $\iota_{D,\mathrm{c}}$ *the restriction to the first resp. second copy of* $\mathcal{D}_{\mathrm{cris}}(D)$*. Then* $\mathrm{Im}(\iota_{D,\mathrm{f}}) \cong H^1_{\mathrm{f}}(D)$ *and* ι_D *induces a canonical splitting*

$$H^1(D) = H^1_{\mathrm{f}}(D) \bigoplus H^1_{\mathrm{c}}(D),$$

where $H^1_{\mathrm{c}}(D) = \mathrm{Im}(\iota_{D,\mathrm{c}})$.

Proof. We prove now *i)* . The relation $N\varphi = p\varphi N$ combined with the fact that $\mathcal{D}_{\mathrm{st}}(D)^{\varphi=1} = \mathcal{D}_{\mathrm{st}}(D)$ tells us that $N = 0$ and D is crystalline. We proceed by induction and we suppose now that $g = 2$. By twisting, we suppose also that the Hodge-Tate weights are 0 and k. Then D corresponds, up to isomorphism, to the class x_k defined above. Let e_2 be a lift in D of $1 \in \mathcal{R}$. Then Γ acts trivially on e_2 because the extension corresponds to x_k. By hypothesis φ acts trivially too, so the extension is split.
We suppose now that *i)* holds for all D' of rank $d-1$. We choose a vector v in D corresponding to a Hodge-Tate weight k_d. This defines an extension

$$0 \to \mathcal{R}(x^{-k_d}) \to D \to D' \to 0.$$

By inductive hypothesis, $D' \cong \oplus_{i=1}^{d-1}\mathcal{R}(x^{-k_i})$. By fonctoriality

$$\mathrm{Ext}^1(D', \mathcal{R}(x^{-k_d})) \cong \bigoplus_{i=1}^{d-1} \mathrm{Ext}^1(\mathcal{R}(x^{-k_i}), \mathcal{R}(x^{-k_d})).$$

For the same reasoning as before, we see that the extensions on the right hand side are split, and we are done.
For *ii), iii)* we can reduce to the case $d = 1$, which has already been studied in detail. $\square$

We consider now the dual case, i.e. $D' = D^*(1)$. We have $\mathcal{D}_{\mathrm{st}}(D')^{\varphi=p^{-1}} = \mathcal{D}_{\mathrm{st}}(D')$ and all the Hodge-Tate weights are non-positive. We have a pairing

$$[\ ,\ ]_D : \mathcal{D}_{\mathrm{cris}}(D') \times \mathcal{D}_{\mathrm{cris}}(D) \to \mathbb{Q}_p$$

induced from the pairing between D' and D.
We define

$$\iota_{D'} : \mathcal{D}_{\mathrm{cris}}(D') \oplus \mathcal{D}_{\mathrm{cris}}(D') \to H^1(D')$$

as the only linear map such that $\iota_{D'}(\alpha,\beta) \cup \iota(x,y) = [\beta, x]_D - [\alpha, y]_D$. Comparing dimensions, we deduce that $\iota_{D'}$ is non-degenerated and so an isomorphism. Because $H^1_{\mathrm{f}}(D)$ and $H^1_{\mathrm{f}}(D')$ are orthogonal, we have a decomposition as before

$$H^1(D') = H^1_{\mathrm{f}}(D') \bigoplus H^1_{\mathrm{c}}(D'),$$

where the two direct summands are the images of the restrictions of $\iota_{D'}$ to the first and second components.
Moreover, as $H^0(D) = \mathcal{D}_{\mathrm{cris}}(D)$, using the cup product we identify $H^2(D')$ with $\mathcal{D}_{\mathrm{cris}}(D')$ via the unique map $\mathrm{inv}_{D'}$ such that

$$x \cup y = [\mathrm{inv}_{D'}(x), y]$$

for all x in $H^2(D')$ and y in $H^0(D)$. For the rank one case, we have that inv_D coincides with the the morphism defined before and denoted by inv_m.

6.2.4 Definition of the $\mathcal{L}$-invariant

Let us fix a p-adic representation

$$V : G_{\mathbb{Q}} \to GL_n(K)$$

where K is a p-adic field. We will suppose that V is continuous, unramified outside a finite number of places S, semistable as a $G_{\mathbb{Q}_p}$-representation.
All the above theory extends without any change to (φ, Γ)-modules over $\mathcal{R} \otimes_{\mathbb{Q}_p} K$. For simplicity of notation, we shall write $\mathcal{R}$ instead of $\mathcal{R} \otimes_{\mathbb{Q}_p} K$.
We can now define the $\mathcal{L}$-invariant associated with V. As before, we must assume certain conditions on V to be able to define it. Let $m \geq 0$, $k \geq 1$ be two integers. We have a unique (up to isomorphism) crystalline extension

$$0 \to \mathcal{R}(|x|x^k) \to U_{k,m} \to \mathcal{R}(x^{-m}) \to 0.$$

For each $l \neq p$ we fix as before the same local conditions L_l as in Section 6.1.2. At p we define now

$$L_p = \mathrm{Ker}(H^1(D_p, V) \to H^1(D_p, V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{cris}})).$$

If $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ denotes the (φ, Γ)-module associated with V, then $L_p = H^1_{\mathrm{f}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$. We define then the Bloch-Kato Selmer group

$$H^1_{\mathrm{f}}(V) := \mathrm{Ker}\left(H^1(G_S, V) \to \prod_{l \in S} \frac{H^1(G_l, V)}{L_l} \right).$$

We shall suppose the following properties for V

C1) $H^1_{\mathrm{f}}(V) = H^1_{\mathrm{f}}(V^*(1)) = 0$,

C2) $H^0(G_S, V) = H^0(G_S, V^*(1)) = 0$,

C3) φ on $\mathbf{D}_{\mathrm{st}}(V)$ is semisimple at 1 and p^{-1},

C4) $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ has no saturated subquotient of type $U_{m,k}$.

It should be clear to the reader the analogy with the hypotheses in the ordinary setting. Assuming Bloch-Kato conjectures, we have that **C1** implies that the L-function of V is holomorphic on $\mathbb{C}$ and **C2** implies the non-vanishing of $L(0, V)$.
The hypothesis **C3** should be a consequence of Tate's conjecture, while **C4** is the generalization of **U** in the non-ordinary case (in fact, the case $k = 1$ and $m = 0$ reduces to Kummer theory). In particular if V comes from the étale cohomology of a variety X with good reduction, then V is crystalline (thanks to a deep theorem of Faltings). Because of Weil conjectures for $X_{/\mathbb{F}_p}$, the eigenvalues of φ are Weil numbers of the same weight and consequently $U_{m,k}$ can not appear as subquotient of V (because the eigenvalues of φ on $U_{m,k}$ are p^{k-1} and p^{-m}).
We say that a (φ, N)-submodule D of $\mathbf{D}_{\mathrm{st}}(V)$ is regular if it is isomorphic to $t_{\mathbf{D}_{\mathrm{st}}(V)}$, defined as before Proposition 6.2.5, via the canonical map. We give examples of regular D for $V = \rho_f(j)$, where ρ_f is the Galois representation associated with a modular form f of weight k and $1 \leq j \leq k-1$. If f has conductor prime to p, to each eigenvalue of φ corresponds a unique regular submodule of $\mathbf{D}_{\mathrm{st}}(V)$. In particular, we have a bijection between p-stabilization of f and regular submodule. If f is a newform at p with trivial nebentypus, then the only regular submodule of $\mathbf{D}_{\mathrm{st}}(V)$ corresponds to the unique crystalline period [Col10b, §3.1].
When $V = \mathrm{Ad}(\rho_f)$, we have the regular submodules induced by the ones of ρ_f. If f has conductor prime to p, we have also a regular submodule D corresponding to the eigevalue 1 of φ. To our knowledge, no p-adic

L-function has been constructed for this choice of D; conjecturally, this p-adic L-function should not present a trivial zero. We wonder if such a p-adic L-function could be constructed from the would be Euler system for $f_\alpha \otimes f_\beta$ (the tensor product of the two p-stabilizations of f) to be constructed as in [LLZ12].
For each regular submodule D we can define a filtration (D_i) of $\mathbf{D}_{\mathrm{st}}(V)$.

$$D_i = \begin{cases} 0 & i=-2, \\ (1-p^{-1}\varphi^{-1})D + N(D^{\varphi=1}) & i=-1, \\ D & i=0, \\ D+\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1} \cap N^{-1}(D^{\varphi=p^{-1}}) & i=1, \\ \mathbf{D}_{\mathrm{st}}(V) & i=2. \end{cases}$$

We have that D_1/D_{-1} coincides with the part of $\mathbf{D}_{\mathrm{st}}(V)$ where φ acts as p^{-1} or 1. The reader should think of D_{-1} and D_1 as non ordinary F^{11} and F^{00} (note that the filtration is now increasing).
This filtration induces a filtration on $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$. Namely, we pose

$$F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V) = \mathbf{D}^\dagger_{\mathrm{rig}}(V) \cap (D_i \otimes \mathcal{R}_{\log}[t^{-1}]).$$

As in the ordinary situation, we pose

$$W := F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)/F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V).$$

We have as before

$$W = W_0 \bigoplus W_1 \bigoplus M$$

where $t_0 = \dim_{\mathbb{Q}_p} H^0(W) = \mathrm{rank}_{\mathcal{R}} W_0$, $t_1 = \dim_{\mathbb{Q}_p} H^0(W^*(1)) = \mathrm{rank}_{\mathcal{R}} W_1$ and M sits in a non split sequence

$$0 \to M_0 \xrightarrow{f} M \xrightarrow{g} M_1 \to 0$$

such that $\mathrm{gr}^0(\mathbf{D}^\dagger_{\mathrm{rig}}(V)) = W_0 \oplus M_0$ and $\mathrm{gr}^1(\mathbf{D}^\dagger_{\mathrm{rig}}(V)) = W_1 \oplus M_1$. Moreover $2\,\mathrm{rank}_{\mathcal{R}} M_0 = 2\,\mathrm{rank}_{\mathcal{R}} M_1 = \mathrm{rank}_{\mathcal{R}} M = 2t$.
We give now more properties of M

Lemma 6.2.13. *We have* $\dim_{\mathbb{Q}_p} H^1(M) = 2\dim_{\mathbb{Q}_p} H^1_{\mathrm{f}}(M) = 2t$.
Let us denote by δ_i the connecting homomorphisms of the long exact sequence associated with M, and by f_1 and g_1 the induced morphism on the H^1. We have then

$$\begin{aligned} H^1(M_0) \cong& \mathrm{Im}(\delta_0) \oplus H^1_{\mathrm{f}}(M_0), \\ \mathrm{Im}(f_1) =& H^1_{\mathrm{f}}(M), \\ H^1(M_1) \cong& \mathrm{Im}(g_1) \oplus H^1_{\mathrm{f}}(M_1) \end{aligned}$$

We want to point out that the hypothesis **C4** is necessary to ensure that $\mathrm{Im}(\mathrm{f}_1) = H^1_{\mathrm{f}}(M)$.
We are left to define the analogue of the space $\mathbf{T}(V)$. We consider the short exact sequence

$$0 \to F_1(\mathbf{D}^\dagger_{\mathrm{rig}}(V)) \to \mathbf{D}^\dagger_{\mathrm{rig}}(V) \to \mathrm{gr}^2(\mathbf{D}^\dagger_{\mathrm{rig}}(V)) \to 0.$$

Using Proposition 6.2.5 and the fact that D corresponds to a section of t_V, we deduce that $\mathrm{gr}^2(\mathbf{D}^\dagger_{\mathrm{rig}}(V))$ has only non-negative Hodge-Tate weights, so $H^0(\mathrm{gr}^2(\mathbf{D}^\dagger_{\mathrm{rig}}(V))) = H^1_{\mathrm{f}}(\mathrm{gr}^2(\mathbf{D}^\dagger_{\mathrm{rig}}(V))) = 0$. The long exact sequence in cohomology tells us

$$H^1_{\mathrm{f}}(\mathbf{D}^\dagger_{\mathrm{rig}}(V)) = H^1_{\mathrm{f}}(F_1(\mathbf{D}^\dagger_{\mathrm{rig}}(V))).$$

We consider now the short exact sequence

$$0 \to F_{-1}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \to F_1(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \to W \to 0.$$

All the Hodge-Tate weights of $F_{-1}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$ are negative, its dual has no H^0 and by duality $H^2(F_{-1}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))) = 0$. This tells us

$$H^1(W) = \mathrm{Coker}\left(H^1(F_{-1}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))) \to H^1(F_1(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)))\right).$$

Moreover

$$H^1(F_{-1}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))) = H^1_{\mathrm{f}}(F_{-1}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))).$$

and this allows us to apply [Ben11, Corollary 1.4.6] to obtain

$$H^1_{\mathrm{f}}(W) = \mathrm{Coker}\left(H^1(F_{-1}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))) \to H^1_{\mathrm{f}}(F_1(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)))\right).$$

Summing up we obtain

$$\frac{H^1(W)}{H^1_{\mathrm{f}}(W)} \cong \frac{H^1(F_1(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)))}{L_p}$$

because $L_p = H^1_{\mathrm{f}}(V) = H^1_{\mathrm{f}}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$. Note that the dimension of this space is $g := t_0 + t_1 + t$.
Hypotheses **C1** and **C2** combined with the Poitou-Tate exact sequence tell us

$$H^1(G_{\mathbb{Q}_S}, V) \cong \bigoplus_{l \in S} \frac{H^1(G_l, V)}{L_l},$$

where L_l as in Section 6.1.2. There exists then a unique subspace $H^1(D, V)$ in $H^1(G_{\mathbb{Q}_S}, V)$ projecting isomorphically to $\frac{H^1(F_1(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)))}{L_p}$. This is the non-ordinary analogue of $\mathbf{T}(V)$; we chose a different notation in order to emphasize the dependence from D.
Let us denote by κ_p the localization map $H^1(G_{\mathbb{Q}_S}, V) \to H^1(G_p, V)$. This induces an injection of $H^1(D, V)$ into $H^1(W)$ which we shall denote by κ_D. A simple dimension argument as in Lemma 6.1.3 tells us that $H^1(D, V)$ depends only on $V_{|\mathbb{Q}_p}$ when $g = t$.
We suppose now

C5) $W_0 = 0$.

If $W_0 \neq 0$, but $W_1 = 0$, we can deal then with $V^*(1)$ in place of V. As in the ordinary case, we are not able to deal with V such that W_0 and W_1 are both non zero.
Let us denote by $\tilde{\kappa}_D$ the composition of κ_D with the projection to $\overline{W} := \mathrm{gr}^1(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$. If $W_0 = 0$, from the short exact sequence

$$0 \to M \to W \to W_1 \to 0$$

and the explicit description of $H^1(M)$ and $H^1(M_1)$ we obtain

$$H^1(\overline{W}) = \frac{H^1(W)}{H^1_{\mathrm{f}}(M)}.$$

As $H^1_{\mathrm{f}}(M) \subset H^1_{\mathrm{f}}(W)$, then $\tilde{\kappa}_D$ is injective. We a diagram

$$\begin{tikzcd}
\mathcal{D}_{\mathrm{cris}}(\overline{W}) \arrow[r, "\iota_{D,\mathrm{f}}"] & H^1_{\mathrm{f}}\left(\overline{W}\right) \\
H^1(D,V) \arrow[u, "\rho_f"] \arrow[r, "\tilde{\kappa}_D"] \arrow[d, "\rho_c"'] & H^1(\overline{W}) \arrow[u, "p_f"'] \arrow[d, "p_c"] \\
\mathcal{D}_{\mathrm{cris}}(\overline{W}) \arrow[r, "\iota_{D,\mathrm{f}}"] & H^1_{\mathrm{c}}\left(\overline{W}\right)
\end{tikzcd}$$

where ρ_f and ρ_c are the only arrows making the diagram commute. From the definition of $H^1(D,V)$ we have that $p_f \circ \tilde{\kappa}_D$ is surjective. We can then invert ρ_c and we define

$$\mathcal{L}(V,D) = \det_{\mathbb{Q}_p}(\rho_f \circ {\rho_c}^{-1}),$$

where the determinant is calculated with respect to the basis defined in Theorem 6.2.12.
If V is ordinary and we choose the regular submodule D to be the one induced by $\mathrm{Fil}^1(V)$ (recall that the filtration on $\mathcal{D}_{\mathrm{st}}(V)$ is increasing, so all the indexes have to be inverted), then Benois $\mathcal{L}$-invariant coincides with that of Greenberg.

6.2.5 A particular case

In this section we want to deal with representation V such that $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) = W = M$ and $t = 1$. We will calculate explicitly the $\mathcal{L}$-invariant in term of *infinitesimal deformation* of V. This covers the case of the representation associated to a modular form Steinberg at p and the case of its adjoint representation.
We suppose that M is of the form

$$0 \to \mathcal{R}(\delta) \to M \to \mathcal{R}(\psi) \to 0,$$

where $\delta(x) = x^m|x|$, $m \geq 1$, and $\psi(x) = x^{-n}$, $n \geq 0$. We have no choice for D but $\mathcal{R}(\delta)$.
We suppose that we dispose of an infinitesimal deformation V_A of V over $A = K[T]/(T^2)$ equipped with a filtration on $F^i\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_A)$ such that

$$F^i\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_A) \otimes_{A,T=0} K = F^i\mathbf{D}^{\dagger}_{\mathrm{rig}}(V).$$

We have then

$$0 \to \mathcal{R}(\delta_A) \to \mathbf{D}^{\dagger}_{\mathrm{rig}}(V_A) \to \mathcal{R}(\psi_A) \to 0$$

for δ_A resp. ψ_A an infinitesimal deformation of δ resp. ψ. We remark that we can write tautologically $\delta_A(x) = \delta(x) + T\frac{\mathrm{d}\delta_A(x)}{\mathrm{d}T}_{|T=0}$.

Theorem 6.2.14. *Suppose* $\left.\frac{\mathrm{d}\delta_A\psi_A^{-1}(u)}{\mathrm{d}T}\right|_{T=0} \neq 0$ *for any* $u \equiv 1 \bmod p^2$, *then*

$$\mathcal{L}(V) = -\log_p(u)\frac{\mathrm{d}\log\delta_A\psi_A^{-1}(p)}{\mathrm{d}\log\delta_A\psi_A^{-1}(u)}.$$

Such a theorem has been proved also in [Mok12, Hid04], and in a different way in [Col10b]. The proof is not really different from the original one of Greenberg and Stevens [GS93].

Proof. We write as before M_0 resp. M_1 for $\mathcal{R}(\delta)$ resp. $\mathcal{R}(\psi)$. We have

$$\begin{tikzcd} H^1(M_0) \arrow[r, "f_1"] & H^1(M) \arrow[r, "g_1"] & H^1(M_1) \arrow[r, "\delta_1"] & H^2(M_0) \arrow[r, "f_2"] & 0 \\ & H^1(V,D) \arrow[u, "\kappa_D"] \arrow[ur, "\tilde{\kappa}_D"'] & & & \end{tikzcd}$$

Because $\mathrm{Im}(f_1) = H^1_{\mathrm{f}}(M)$, we have $\mathrm{Im}(\tilde{\kappa}_D) = \mathrm{Im}(g_1)$. What we have to do is to calculate this image. We shall consider the dual module $M^*(1)$; it sits in the exact sequence

$$0 \to \mathcal{R}(|x|x^{n+1}) \to M^*(1) \to \mathcal{R}(x^{1-m}) \to 0.$$

Let $t^{m-1}e_m$ be the basis of $H^0(\mathcal{R}(x^{1-m}))$. We can write

$$\delta_0^*(t^{m-1}e_m) = a\alpha^*_{n+1} + b\beta^*_{n+1}$$

where δ_0^* is the first connecting morphism and α^*_{n+1} and β^*_{n+1} are the canonical basis of $H^1(\mathcal{R}(|x|x^{n+1}))$. We have that $\mathrm{Im}(\delta_0^*)$ is orthogonal to $\mathrm{Ker}(\delta_1)$ for the pairing

$$H^1(\mathcal{R}(x^{n+1}|x|)) \times H^1(\mathcal{R}(x^{-m})) \to H^2(\mathcal{R}(x|x|)),$$

so $\mathrm{Im}(g_1) = \mathrm{Ker}(\delta_1) = ax^*_{n+1} + by^*_{n+1}$. By definition

$$\mathcal{L}(V,D) = b^{-1}a.$$

Let us write W_A for $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V_A)$. We have

$$0 \to \mathcal{R}(\delta) \to \mathcal{R}(\delta_A) \to \mathcal{R}(\delta) \to 0.$$

Let us denote by B^i_δ the connecting map $H^i(\mathcal{R}(\delta)) \to H^{i+1}(\mathcal{R}(\delta))$ coming from the above exact sequence. We have the relation $B^1_{|x|x^{n+1}}\delta_0^* = -\delta_1^* B^0_{x^{1-m}}$. We want now to apply inv_{n+1} to this relation evaluated on $t^{m-1}e_m$. We have for $\delta(x) = x^m|x|$ and δ_A as above

$$\begin{aligned} \mathrm{inv}_m(B^1_\delta(\alpha^*_m)) &= \log(\chi_{\mathrm{cycl}}(\gamma))^{-1}\mathrm{d}\log(\delta_A(\chi_{\mathrm{cycl}}(\gamma)))_{|T=0}, \\ \mathrm{inv}_m(B^1_\delta(\beta^*_m)) &= \mathrm{d}\log(\delta_A(p))_{|T=0}. \end{aligned}$$

So

$$\mathrm{inv}_{n+1}(B^1_{|x|x^{n+1}}\delta_0^*)(t^{m-1}e) = a\log(\chi_{\mathrm{cycl}}(\gamma))^{-1}\mathrm{d}\log(\psi_A^{-1}(\chi_{\mathrm{cycl}}(\gamma)))_{|T=0} + b\mathrm{d}\log(\psi_A^{-1}(p))_{|T=0}. \tag{6.2.15}$$

Let $e_{A,m}$ be a lift of e_m to $\mathcal{R}(\delta^*_A(1))$. So we have

$$\begin{aligned} B^0_{x^{1-m}}(t^{m-1}e_{A,m}) &= \frac{1}{T}\mathrm{cl}((\varphi-1)t^{m-1}e_{A,m}, (\gamma-1)t^{m-1}e_{A,m}) \\ &= \frac{1}{T}\mathrm{cl}((\delta_A^{-1}(p)p^{m-1}-1)t^{m-1}e_{A,m}, (\chi^{m-1}_{\mathrm{cycl}}(\gamma)\delta_A^{-1}(\chi^{m-1}_{\mathrm{cycl}}(\gamma))-1)t^{m-1}e_{A,m}) \\ &= \mathrm{cl}(\mathrm{d}\delta_A^{-1}(p)_{|T=0}t^{m-1}e_m, \mathrm{d}\delta_A^{-1}{}_{|T=0}(\chi_{\mathrm{cycl}}(\gamma))t^{m-1}e_m). \end{aligned}$$

We already know that

$$\delta_1(\mathrm{cl}(x,y)) = \mathrm{cl}((\gamma-1)\hat{x} - (\varphi-1)\hat{y})$$

where $\hat{x}$ resp. $\hat{y}$ is a lift of x resp. y.
We already know that $\mathrm{res}(xt^{m-1}\mathrm{d}t)$ and $\mathrm{res}(yt^{m-1}\mathrm{d}t)$ give the coordinates of $\mathrm{cl}(x,y)$ with respect to the basis x_m^* and y_m^*. If we write e_M for a lift of e_m to $M^*(1)$ we have

$$\mathrm{res}((\gamma-1)e_M t^{m-1}\mathrm{d}t) = \log(\chi_{\mathrm{cycl}}(\gamma))\left(1+\frac{1}{p}\right)b,$$
$$\mathrm{res}((\varphi-1)e_M t^{m-1}\mathrm{d}t) = \left(1+\frac{1}{p}\right)a.$$

So we have

$$\mathrm{inv}_{n+1}(\delta_1^* B^0_{x^{1-m}})(t^{m-1}e) = -a\log(\chi_{\mathrm{cycl}}(\gamma))\mathrm{d}\log(\delta_A^{-1}(\chi_{\mathrm{cycl}}(\gamma)))_{|T=0} + b\mathrm{d}\log(\delta_A^{-1}(p))_{|T=0}. \tag{6.2.16}$$

Summing up 6.2.15 and 6.2.16 we obtain

$$a\log(\chi_{\mathrm{cycl}}(\gamma))^{-1}\mathrm{d}\log(\psi_A^{-1}\delta_A(\chi_{\mathrm{cycl}}(\gamma)))_{|T=0} = -b\mathrm{d}\log(\psi_A^{-1}\delta_A(p))_{|T=0}$$

□

We want to point out that to define the $\mathcal{L}$-invariant in this case we do not need to assume **C4**. This covers the case of $V = V_p(E)(1)$, for E an elliptic curve with positive rank, which was not covered in the first section.

6.3 A Siegel modular forms of genus two

The calculation of the $\mathcal{L}$-invariant requires to produce explicit cocycles in $H^1(D,V)$; when V appears in $\mathrm{Ad}(V')$ for a certain representation V' we can use the same reasoning as we did before Theorem 6.1.11 to produce these cocycles. This is done for symmetric powers of the Galois representation associated with Hilbert modular forms in [HJ13]. The main limit of this approach is that most of the representations V do not appear as quotient of adjoint representations.
In the case $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) = W = M$ the situation is way simpler; if $t=1$ we have seen above that to produce the cocycle in $H^1(V,D)$ it is enough to find deformations of $V|_{\mathbb{Q}_p}$.
We want now to apply the above calculations in the case of the Galois representations associated with a Siegel modular form of genus 2. We thank A. Jorza for pointing out that the calculation of the $\mathcal{L}$-invariant in this case is currently within reach. Thanks to the work of Scholze [Sch13], we should dispose soon of Galois representation for (each L-packet of) Siegel-Hilbert modular forms of any genus (see for example [HJ13, Theorem 18]), but we limit ourselves to Siegel forms of genus 2 for simplicity of exposition and to avoid some technicalities. We are confident that the same argument should generalized, with minimal changes, to Siegel-Hilbert modular forms of any genus, assuming p split and the form Steinberg at all prime above p.
We fix a Siegel modular form f of parallel weight k as in [BS00, Preliminaries]. It is known thanks to works of Taylor [Tay93], Laumon [Lau05] and Weissauer [Wei05] that to such a form we can associate a spin Galois representation V_{spin} whose image is contained in $\mathrm{GSp}_4(K)$. We can define a standard Galois representation V_{sta} whose image is contained in $\mathrm{GL}_5(K)$ using the isomorphism between GSp_4 and GSpin_5 followed by the orthogonal representation of degree 5.
Let π be the automorphic representation of GSp_4 spanned by f. We suppose that π at p is not spherical, and we suppose that the Satake parameters of f at p, normalized as in [BS00, Corollary 3.2] (note in particulat that π is unitary) are (α_1,α_2). We let $\alpha_0^2 = p^{2k-3}\alpha_1^{-1}\alpha_2^{-1}$. We define $\alpha = \alpha_0$ and $\beta = \alpha_1\alpha_0$. We suppose that $\alpha_1 = p$; this implies that π_p is the Steinberg representation. As π is generic, using [Whi05], we can lift π to an automorphic representation $\pi^{(4)}$ to GL_4, induced from the embedding of GSp_4 into GL_4. We suppose also that we can lift π to an automorphic representation $\pi^{(5)}$ of GL_5, using Arthur's transfer.
Let $V = V_{\mathrm{spin}}$ or V_{sta}. We make the following assumption

LGp) V is semistable at p and strong local-global compatibility at $l = p$ holds.

These hypotheses are conjectured to be always true for f as above.
Roughly speaking, we require that

$$\mathrm{WD}(V)^{ss} \cong \iota_n^{-1}\pi_p^{(n)},$$

where $\mathrm{WD}(V)$ is the Weil-Deligne representation associated with $V_{|\mathbb{Q}_p}$ à la Berger, $\pi_p^{(n)}$ is the component at p of $\pi^{(n)}$, and ι_n is the local Langlands correspondence associated for $\mathrm{GL}_n(\mathbb{Q}_p)$ geometrically normalized ($n = 5$ when V is the standard representation and $n = 4$ when V is the spinorial representation). Consequently, the Frobenius eigenvalues on $\mathrm{WD}(V_{\mathrm{spin}})^{ss}$ are $\alpha, \beta, p^{2k-3}\beta^{-1}, p^{2k-3}\alpha^{-1}$, while the one on $\mathrm{WD}(V_{\mathrm{sta}})^{ss}$ are

$$\left(\frac{\alpha\beta}{p^{2k-3}}, \frac{\alpha}{\beta}, 1, \frac{\beta}{\alpha}, \frac{p^{2k-3}}{\alpha\beta}\right).$$

Moreover, the monodromy operator should have maximal rank (i.e. one-dimensional kernel). (This is also a consequence of the weight-monodromy conjecture for V).
If $\pi^{(n)}$ is square-integrable at another finite place other than p, then $\pi^{(n)}$ is the image via the Jacquet-Langlands correspondence of an automorphic representation on a unitary group and we can then use Caraiani's theorem [Car13] to obtain semistability and local-global compatibility.
More generally, Arthur's transfer from Sp_{2g} to Gl_{2g+1} should allow us to apply Caraiani's theorem in full generality.
The Hodge-Tate weights of V_{spin} are then $(0, k-2, k-1, 2k-3)$ while the one of V_{sta} are $(1-k, 2-k, 0, k-2, k-1)$.
Thanks to work of Tilouine-Urban [TU99], Urban [Urb11] and Andreatta-Iovita-Pilloni [AIP12] we have families of Siegel modular forms;

Theorem 6.3.1. *Let $\mathcal{W} = \mathrm{Hom}_{cont}\left((\mathbb{Z}_p^\times)^2, \mathbb{C}_p\right)$ be the weight space. There exist an affinoid neighborhood U of $\kappa_0 = \left((z_1, z_2) \mapsto z_1^k z_2^k\right)$ in $\mathcal{W}$, an equidimensional rigid variety $\mathcal{X}_f$ of dimension 2, a finite surjective map $w : \mathcal{X}_f \to U$, a character $\Theta : \mathcal{H}^{Np} \to \mathcal{O}(\mathcal{X}_f)$, and a point x_0 in $\mathcal{X}_f$ above κ_0 such that $x_0 \circ \Theta$ corresponds to the Hecke eigensystem of f.*
Moreover, there exists a dense set of points x of $\mathcal{X}_f$ coming from classical cuspidal Siegel modular forms of weight (k_1, k_2) which are regular (i.e. $(k_2 > k_1 \geq 3)$) and unramified at p.

This allows us to define two pseudo-representations $R_? : G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{X}_f)$, for ? = spin, sta, interpolating the trace of the representations associated with classical Siegel forms [BC09, Proposition 7.5.4]. Since V_{spin} is irreducible, we have, shrinking $\mathcal{U}$ around κ_0 if necessary, a *big* Galois representation ρ_{spin} with value in $\mathrm{GL}_4(\mathcal{O}(\mathcal{X}_f))$ such that $\mathrm{Tr}(\rho_{\mathrm{spin}}) = R_{\mathrm{spin}}$ [BC09, page 214]. As V_{sta} is irreducible too, there exists a representation ρ_{sta} with value in $\mathrm{GL}_5(\mathcal{O}(\mathcal{X}_f))$ such that $\mathrm{Tr}(\rho_{\mathrm{sta}}) = R_{\mathrm{sta}}$.
We proceed now as in [HJ13]. We recall the following theorem [Liu13, Theorem 5.3.2]

Theorem 6.3.2. *Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathcal{O}(\mathcal{X}))$ be a continuos representation. Suppose that there exist $\kappa_1(x), \ldots, \kappa_n(x)$ in $\mathcal{O}(\mathcal{X})$, $F_1(x), \ldots, F_d(x)$ in $\mathcal{O}(\mathcal{X})$, and a Zariski dense set of points $Z \subset \mathcal{X}$ such that*

- *for any x in $\mathcal{X}$, the Hodge-Tate weights of ρ_x are $\kappa_1(x), \ldots, \kappa_n(x)$,*
- *for any z in Z, ρ_z is crystalline,*
- *for any z in Z, $\kappa_1(z) < \ldots < \kappa_n(z)$,*
- *for any z in Z, the eigenvalues of φ on $\mathcal{D}_{\mathrm{cris}}(V_z)$ are $p^{\kappa_1(x)}F_1(x), \ldots, p^{\kappa_n(x)}F_n(x)$,*
- *for any C in $\mathbb{R}$, defines $Z_C \subset Z$ as set of points z such that for all $I, J \subset \{1, \ldots, n\}$ such that $|\sum_{i\in I}\kappa_i(z) - \sum_{j\in J}\kappa_j(z)| > C$. We require that for all $z \in Z$ and $C \in \mathbb{R}$, Z_C accumulates at z.*

- *for $1 \leq i \leq n$ there exist $\chi_i : \mathbb{Z}_p^\times \to \mathcal{O}(\mathcal{X})^\times$ such that $\chi_i(u) = u^{\kappa_i(x)}$.*

Then, for all x in $\mathcal{X}$ non-critical and regular ($\kappa_1(x) < \ldots < \kappa_n(x)$ and the eigenvalues of φ on $\mathcal{D}_{\mathrm{cris}}(V_x)$ are distinct) there exists a neighborhood U of x such that ρ_U is trianguline and its graded pieces are $\mathcal{R}_U(\chi_i)$.

We can apply this theorem to show that the (φ,Γ)-module associated with $\rho_{?|G_{\mathbb{Q}_p}}$ is trianguline. We explicit now the triangulation; let U_1 and U_2 be the two Hecke operators at p normalized as in [HJ13, Lemma 16] and let us write $F_1 = F_1(x) = \Theta(U_1)$ and $F_2 = F_2(x) = \Theta(U_2)\Theta(U_1)^{-1}$. We have that the graded pieces of $\mathbf{D}^\dagger_{\mathrm{rig}}(\rho_{\mathrm{spin}})$ are given by $\mathcal{R}_U(\delta_i)$, where

$$\begin{aligned}
&\delta_1(p) = F_1, \quad \delta_1(\gamma) = 1,\\
&\delta_2(p) = F_2, \quad \delta_2(\gamma) = \gamma^{k_1-2},\\
&\delta_3(p) = F_2^{-1}, \quad \delta_3(\gamma) = \gamma^{k_2-1},\\
&\delta_4(p) = F_1^{-1}, \quad \delta_4(\gamma) = \gamma^{k_1+k_2-3}.
\end{aligned}$$

While the graded pieces of $\mathbf{D}^\dagger_{\mathrm{rig}}(\rho_{\mathrm{sta}})$ are given by $\mathcal{R}_U(\psi_i)$, for

$$\begin{aligned}
&\psi_1(p) = F_1F_2, \quad \psi_1(\gamma) = \gamma^{1-k_2},\\
&\psi_2(p) = F_1/F_2, \quad \psi_2(\gamma) = \gamma^{2-k_1},\\
&\psi_3(p) = 1, \quad \psi_3(\gamma) = 1,\\
&\psi_4(p) = F_2/F_1, \quad \psi_4(\gamma) = \gamma^{k_1-2},\\
&\psi_5(p) = F_2^{-1}F_1^{-1}, \quad \psi_5(\gamma) = \gamma^{k_2-1}.
\end{aligned}$$

We suppose $\alpha = p^{k-3}$. As we supposed that the monodromy operator has maximal rank, there is only one choice as regular (φ, N)-submodule D of $\mathbf{D}_{\mathrm{st}}(V)$, where V is one of the two representations associated with f described above.
We begin by considering the Galois representation $V_{\mathrm{spin}}(k-1)$ ($s = k-1$ is the only critical integer); the only choice for the regular submodule D is the one spanned by the Frobenius eigenvalue corresponding to $p^{1-k}F_1$ (which is the also the only crystalline period [Nak11, Corollary 3.1 (2-ii)]) and $p^{1-k}F_2$. According to the conjectural description of the modifying Euler factors at p given in [Ben13], the p-adic L-function for this representation should presents a trivial zero. We have just seen that $\mathbf{D}^\dagger_{\mathrm{rig}}(V_{\mathrm{spin}}(k-1))$ contains a sub-module of type M which is an extension of δ_2 by δ_1 and the above triangulation gives (in the notation of the previous section) the desired filtration $F^i(\mathbf{D}^\dagger_{\mathrm{rig}}(\rho_{\mathrm{spin}}))$.
Concerning V_{sta}, the only regular submodule D is the one spanned by the eigenvectors of eigenvalue F_1F_2 and F_1/F_2 and the corresponding p-adic L-function has a trivial zero (we will come back on this p-adic L-function later). Similarly to the previous case, $\mathbf{D}^\dagger_{\mathrm{rig}}(V_{\mathrm{st}})$ contains a sub-module of type M which is an extension of ψ_3 by ψ_2. The two submodules are isomorphic (twist the first one by δ_2^{-1}). Summing up, we can apply Theorem 6.2.14 to the two-dimensional subspace M to obtain the following theorem;

Theorem 6.3.3. *For f as above, assuming **LGp*** for V_{spin}, we have

$$\mathcal{L}(V_{\mathrm{st}}) = \mathcal{L}(V_{\mathrm{spin}}) = -\mathrm{d}\log((F_1/F_2)(x)).$$

6.3.1 Speculations on analytic $\mathcal{L}$-invariant

As the title says, we now want to speculate on the analytic $\mathcal{L}$-invariant. We concentrate on the case of the standard representation but we want to point out that a p-adic L-function for the spin representation could be constructed similarly to [AG94] (in particular, see [AG94, (3.1)] for the Euler factors at p which have to

be removed). We remark that the method of Greenberg-Stevens can hardly be adapted to study the analytic $\mathcal{L}$-invariant for V_{spin} without new ideas, because for each Siegel form we have only one critical value.
But the method of Greenberg-Stevens should work quite well for V_{sta}. For a Dirichlet charater ε, we denote by ε_0 the associated primitive character. We have the following theorem concerning p-adic L-functions for the standard representation [BS00, Theorem 9.5 b)];

Theorem 6.3.4. *Let f an ordinary Siegel eigenform of level Np, Nebentypus ψ and parallel weight k which is of finite slope for U_1. We have a distribution $\mu = \mu_f$ on $\mathbb{Z}_p^\times$ such that for all finite order character χ and $s = 1, \dots, k-2$ we have*

$$\int \varepsilon(z) z^{s-1} \mathrm{d}\mu(z) = C_\varepsilon E_1(s,\varepsilon) E_2(s,\varepsilon) \frac{L_N(1-s, V_{\mathrm{sta}} \otimes \varepsilon^{-1})}{\Omega(f,s)}$$

where C_ε is an algebraic number, $\Omega(f,s)$ a trascendent period,

$$E_1(s,\varepsilon) = (1-\alpha_1\varepsilon_0(p)p^{-s})(1-\alpha_2\varepsilon_0(p)p^{-s}),$$
$$E_2(s,\varepsilon) = \begin{cases} 1 & f \text{ primitive at } p, \\ (1-\alpha_1\varepsilon_0^{-1}(p)p^{s-1})(1-\alpha_2\varepsilon_0^{-1}(p)p^{s-1})(1-\varepsilon_0^{-1}(p)p^{s-1}) & \text{otherwise,} \end{cases}$$

and $L_N(1-s, V_{\mathrm{sta}} \otimes \varepsilon^{-1})$ denotes an "incomplete" L-function for $V_{\mathrm{sta}} \otimes \varepsilon^{-1}$ (see [BS00, (3.23)]).

A priori this p-adic L-function is constructed only for f ordinary and unramified at p. In our situation f is ordinary only when $k = 3$, i.e. $\alpha_0 = 1$; the same construction holds a perfectly well defined p-adic L-function. If $k > 3$, then the distributions μ_f are not bounded but it should be possibile to generalize their construction, assuming either a deep knowledge of the explicit form of certain differential operators or a complete theory of nearly overconvergent Siegel forms. We assume that such a p-adic L-function, with the same interpolation formulae, can be constructed.
The interpolation formula for $s = 1$ gives a factor of type $(1-\alpha_1 p^{-1})$. But $\alpha_1 = p$ and so we are in the presence of a trivial zero. If we consider the p-adic L-function associated with a form f' of higher weight k' and evaluate $\mu_{f'}$ at z^s for $s = k'-k+1$ this factor becomes $(1-\alpha_1 p^{k-k'-1})$. But

$$\alpha_1 p^{k-k'-1} = \beta\alpha^{-1} p^{k-k'-1} = (F_1^{-1}F_2)(x_{k'})p^{k-3}$$

is a p-adic analytic function of k'. Indeed, if we can construct a two-variable p-adic L-function then the factor $E_1(k'-k+1, \mathbf{1})$ can be removed as in [Ros13b]. Finally, the classical method of Greenberg and Stevens applies because for k' big enough the factor $E_2(1,\mathbf{1})$ vanishes.
Thus we see (ignoring the fact that the p-adic L-function of [BS00] is not the *completed one*, as some Euler factors are missing) a confirmation of Benois' conjecture.

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Résumé : Dans cette thèse, on démontre une conjecture de Greenberg et Benois sur les zéros triviaux des fonctions L p-adiques dans certains cas. Pour cela, on utilise la méthode de Greenberg et Stevens. Plus précisément, on démontre d'abord cette conjecture pour une forme de Hilbert de poids parallèle 2 sur un corps totalement réel où p est inerte, quand la forme est Steinberg en p et sous d'autres hypothèses sur le conducteur. Ce résultat est une généralisation de travaux non publiés de Greenberg et Tilouine. On démontre ensuite cette conjecture pour une forme modulaire elliptique de pente finie et Steinberg en p et sous des hypothèses similaires. Pour construire la fonction L p-adique en deux variables (construction nécessaire à l'utilisation de la méthode de Greenberg-Stevens), on utilise la récente théorie des formes quasi surconvergentes d'Urban. On améliore le précédent résultat en enlevant l'hypothèse de conducteur pair et en utilisant la construction de la fonction L p-adique de Böcherer et Schmidt. Dans le chapitre final, on rappelle la définition et les calculs de l'invariant $\mathcal{L}$ de Greenberg-Benois et on explique comment certains résultats précédement énoncés peuvent être généralisés aux formes modulaires de Siegel.

Abstract: This thesis is devoted to the study of certain cases of a conjecture of Greenberg and Benois on derivative of p-adic L-functions using the method of Greenberg and Stevens. We first prove this conjecture in the case of the symmetric square of a parallel weight 2 Hilbert modular form over a totally real field where p is inert and whose associated automorphic representation is Steinberg in p, assuming certain hypotheses on the conductor. This is a direct generalization of (unpublished) results of Greenberg and Tilouine. Subsequently, we deal with the symmetric square of a finite slope, elliptic, modular form which is Steinberg at p. To construct the two-variable p-adic L-function, necessary to apply the method of Greenberg and Stevens, we have to appeal to the recently developed theory of nearly overconvergent forms of Urban. We further strengthen the above result, removing the assumption that the conductor of the form is even, using the construction of the p-adic L-function by Böcherer and Schmidt. In the final chapter we recall the definition and the calculation of the algebraic $\mathcal{L}$-invariant *à la* Greenberg-Benois, and explain how some of the above-mentioned results could be generalized to higher genus Siegel modular forms.

Mots clefs : Formes modulaires - Familles p-adiques de formes automorphes - Fonctions L p-adiques - Représentations Galoisiennes - Théorie d'Iwasawa