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Analyse multifractale de mesures faiblement Gibbs aléatoires et de leurs inverses

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Résumé

Nous montrons la validité du formalisme multifractal pour les mesures aléatoires faiblement Gibbs portées par l' attracteur associé à une dynamique aléatoire C^1 codée par un sous-shift de type fini aléatoire, et expansive en moyenne. Nous établissons également des loi de type 0-∞ pour les mesures de Hausdorff et de packing généralisées des ensembles de niveau de la dimension locale, et calculons les dimensions de Hausdorff et de packing des ensembles de points en lesquels la dimension inférieure locale et la dimension supérieure locale sont prescrites. Lorsque l'attracteur est un ensemble de Cantor de mesure de Lebesgue nulle, nous montrons la validité du formalisme multifractal pour les mesures discrètes obtenues comme inverses de ces mesures faiblement Gibbs.

Mots-clefs : Mesures et dimensions de Hausdorff et de packing, formalisme multifractal, formalisme thermodynamique, mesure faiblement Gibbs aléatoires, systèmes dynamiques aléatoires, théorie métrique de l'approximation, mesures inverses.

MULTIFRACTAL ANALYSIS OF RANDOM WEAK GIBBS MEASURES AND THEIR INVERSE

Abstract

We establish the validity of the multifractal formalism for random weak Gibbs measures supported on the attractor associated with a C^1 random dynamics coded by a random subshift of finite type, and expanding in the mean. We also prove a 0-∞ law for the generalized Hausdorff and packing measures of the level sets of the local dimension, and we compute the Hausdorff and packing dimensions of the sets of points at which the lower and upper local dimensions are prescribed. In the case that the attractor is a Cantor set of zero Lebesgue measure, we prove the validity of the multifractal formalism for the discrete measures obtained as inverse of these weak Gibbs measures.

Keywords : Hausdorff and packing measures and dimensions, multifractal formalism, thermodynamic formalism, random weak Gibbs measure, random dynamical systems, metric approximation theory, inverse measures.

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Chapter 1

Introduction

Weak Gibbs measures are conformal probability measures obtained as eigenvectors of Ruelle-Perron-Frobenius operators associated with continuous potentials on topological dynamical systems. When the system (X, f) has nice enough geometric properties, for instance in the case of a conformal repeller, these measures provide natural, and now standard examples of measures obeying the multifractal formalism: their Hausdorff spectrum and L^q-spectrum form a Legendre pair.

Specifically, for such a measure μ on (X, f), the (lower) L^q-spectrum $\tau_\mu : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$\tau_\mu(q) = \liminf_{r \to 0} \frac{\log \sup\{\sum_i (\mu(B_i))^q\}}{\log(r)}, \tag{1.1}$$

where the supremum is taken over all families of disjoint closed balls B_i of radius r with centers in $\text{supp}(\mu)$; the lower Hausdorff spectrum of μ is defined by

$$d \in \mathbb{R} \mapsto \dim_H \underline{E}(\mu, d),$$

where $\dim_H$ stands for the Hausdorff spectrum, $\underline{E}(\mu, d)$ is the level set of level d of the lower local dimension $\underline{\dim}_{\text{loc}}(\mu, x) = \liminf_{r \to 0^+} \frac{\log(\mu(B(x, r)))}{\log(r)}$, i.e.

$$\underline{E}(\mu, d) = \{x \in \text{supp}(\mu) : \underline{\dim}_{\text{loc}}(\mu, x) = d\},$$

and we have the duality relation

$$\dim_H \underline{E}(\mu, d) = \tau_\mu^*(d) := \inf_{q \in \mathbb{R}} dq - \tau_\mu(q), \quad \forall d \in \mathbb{R},$$

a negative dimension meaning that the set is empty. In fact, due to the super and submultiplicativity properties associated with μ, the same equality holds if we replace the lim inf by a lim sup or a limit in the definition of the local dimension.

The rigorous study of these measures started with the Gibbs measures case, which corresponds to Hölder continuous potentials, or continuous potentials possessing the so-called bounded distorsions property, and in particular on the so-called "cookie-cutter" Cantor sets associated with a $C^{1+\alpha}$ expanding map f on the line [19, 71] (see [68] for an extended discussion of dimension theory and multifractal analysis for hyperbolic conformal dynamical systems). This followed seminal works by physicists of turbulence and statistical mechanics pointing the accuracy of multifractals to statistically and geometrically describe the local behavior of functions and measures [33, 36]. In the case of Gibbs measures, the L^q-spectrum of the Gibbs measure is differentiable, and analytic if the potential ϕ is Hölder continuous; it is the unique solution t of the equation $P(q\phi + t\log \|Df\|) = 0$, where $P(\cdot)$ stands for the topological pressure. The general case of continuous potentials was solved later in [26, 29, 42, 63], with the same formula for the L^q-spectrum. These progress then led to the multifractal analysis of Bernoulli convolutions associated with Pisot numbers [28, 30]. Thermodynamic formalism and large deviations are central tool in these studies. It is worth mentioning that simultaneously another family of multifractal measures has been studied intensively, namely the random measures possessing scale invariance in multiplicative chaos theory (see [55, 56, 57, 41, 5, 72, 3]).

It turns out that Gibbs measures on cookie-cutter sets naturally generate a class of discrete measures obtained as their inverse (see Definition 1.12), for which the validity of the multifractal formalism was established in [11], after a partial study in [58, 73, 74]. Given such a Gibbs measure μ, the L^q-spectrum of its right continuous inverse measure ν is given by $\tau_\nu(q) = \min(0, -\tau_\mu^{-1}(q))$; in [11], an essential new ingredient is needed, namely conditioned ubiquity [8], which combines ergodic theory and metric approximation theory.

In the context of random dynamical systems, the multifractal analysis of random Gibbs measures (to be defined below) associated with random Hölder continuous potentials on attractors of random $C^{1+\alpha}$ expanding (or expanding in the mean) random conformal dynamics encoded by random subshifs of finite type has been studied in [45], [31] and [61]. These works, as well as the dimension theory of attractors of random dynamics [15, 45, 46, 61], are based on the thermodynamic formalism for random transforms [14, 15, 20, 21, 22, 35, 43, 44, 49, 61]. The multifractal analysis of random weak Gibbs measures is also implicitly considered in [31] (which deals with the multifractal analysis of Birkhoff averages), but the fibers are deterministic, and the techniques developed there seems difficult to adapt in a simple way in the case of random subshifts.

In this thesis we consider, on a base probability space $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$, random weak Gibbs measures on some class of attractors included in $[0, 1]$ and associated with C^1 random dynamics semi-conjugate (up to countably many points), or conjugate, to a random subshift of finite type, and show that these measures and their inverse obey the multifractal formalism. Compared to the above mentioned works, apart the source of new difficulties coming from the relaxation of the regularity properties of the potentials, our assumptions provide a slightly more general process of construction of the random Cantor set in terms of the distribution of the random family of intervals used to refine the construction at a given step: it can contain contiguous intervals (i.e. without gap in between, and even no gap) with positive probability; thus, it covers the natural families of Cantor sets one can obtain by picking at random a fiber in a Bedford-McMullen carpet. As a consequence, the expression and study of the inverse measure are more involved than in the standard and deterministic situation considered in [8]. Moreover, we succeed in developing ubiquity theory in this random context without assuming any mixing properties on the base space $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$. This substantially improves the approach developed in [7] to get ubiquity results associated with the special class of random Gibbs measures obtained as random Riesz products, for which the product structure of $(\Omega, \mathbb{P})$ plays an essential role.

Before stating our main results, we introduce some background about random dynamical systems and thermodynamic formalism.

1.1 Weak Gibbs measures on random subshifts

Now let us introduce the concepts of random subshifts and associated topological pressure of random continuous potentials. They have been studied by many authors [14, 15, 34, 44, 43, 49, 61].

Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and σ is a $\mathbb{P}$-preserving invertible ergodic map. In fact, assuming that $\mathbb{P}$ is σ-invariant and it has an ergodic decomposition is enough (this holds if, for example, $\mathcal{F}$ is a countably generated (separable) σ-algebra). Also, we do not really need to assume the map σ to be invertible; assuming that $\sigma\Omega = \Omega$ or that $\sigma^n\Omega$ is measurable for all $n \geq 0$ makes it possible to construct a Rokhlin natural extension which preserves the ergodicity and mixing (see [79, theorem 1.5] or [70, section 2.7]).

Let $\widetilde{\mathbb{Z}}^+ = \{1, 2, \cdots\} \cup \{\infty\}$ be the one-point compactification of $\mathbb{Z}^+ = \{1, 2, \cdots\}$. Let $\Gamma := \widetilde{\mathbb{Z}}^+ \times \widetilde{\mathbb{Z}}^+ \times \cdots$ with the metric on Γ given by

$$d(\underline{v}, \underline{v}') = \sum_{i \in \mathbb{N}} \exp(-i) \left| \frac{1}{v_i} - \frac{1}{\underline{v}'_i} \right|.$$

where $\underline{v} = v_0 v_1 \cdots v_i \cdots$, $\underline{v}' = v'_0 v'_1 \cdots v'_i \cdots$ and we set $\frac{1}{\infty} = 0$.

Let l be a $\mathbb{Z}^+$ valued random variable such that

$$\hat{l} := \int \log(l) \, \mathrm{d}\mathbb{P} < \infty \quad \text{and} \quad \mathbb{P}(\{\omega \in \Omega, l(\omega) \geq 2\}) > 0.$$

Here, $l(\omega)$ will define the number of types for a fixed ω.

Let $A = \{A(\omega) = (A_{r,s}(\omega)) : \omega \in \Omega\}$ be a random transition matrix such that $A(\omega)$ is a $l(\omega) \times l(\sigma\omega)$-matrix with entries 0 or 1. We suppose that the map

$\omega \mapsto A_{r,s}(\omega)$ is measurable for all $(r,s) \in \widetilde{\mathbb{Z}}^+ \times \widetilde{\mathbb{Z}}^+$ and each $A(\omega)$ has at least one non-zero entry in each row and each column. Let

$$\Sigma_\omega = \{\underline{v} = v_0 v_1 \cdots ; 1 \le v_k \le l(\sigma^k(\omega)) \text{ and } A_{v_k, v_{k+1}}(\sigma^k(\omega)) = 1 \; for \; k \in \mathbb{N}\},$$

and $F_\omega : \Sigma_\omega \to \Sigma_{\sigma\omega}$ be the left shift $(F_\omega \underline{v})_i = v_{i+1}$ for any $\underline{v} = v_0 v_1 \cdots \in \Sigma_\omega$. Define $\Sigma_\Omega = \{(\omega, \underline{v}) : \omega \in \Omega, \underline{v} \in \Sigma_\omega\}$ and the map $\Pi : \Sigma_\Omega \to \Omega$ as $\Pi(\omega, \underline{v}) = \omega$. Define the map $F : \Sigma_\Omega \to \Sigma_\Omega$ as $F((\omega, \underline{v})) = (\sigma\omega, F_\omega \underline{v})$. The corresponding family $\tilde{F} = \{F_\omega : \omega \in \Omega\}$ is called a random subshift.

We assume that the random subshift defined above is topologically mixing, i.e. there exists a $\mathbb{N}$-valued random variable $M = M(\omega) < +\infty$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for $\mathbb{P}$-almost every ω,

$$A(\omega)A(\sigma\omega) \cdots A(\sigma^{M-1}\omega) \text{ is positive.}$$

It is not hard to see that this implies that one can choose $M = M(\omega)$ such that for $\mathbb{P}$-almost every ω,

$$\begin{cases} A(\sigma^{-M}\omega)A(\sigma^{-M+1}\omega) \ldots A(\sigma^{-1}\omega) \\ \text{and} \\ A(\omega)A(\sigma\omega) \cdots A(\sigma^{M-1}\omega) \end{cases} \text{are positive.} \tag{1.2}$$

Define

$$\Sigma_{\omega,n} = \left\{ v = v_0 v_1 \cdots v_{n-1} : \begin{array}{c} 1 \le v_k \le l(\sigma^k(\omega)) \text{ for } 0 \le k \le n-1, \\ \text{and } A_{v_k, v_{k+1}} = 1 \text{ for } 0 \le k \le n-2. \end{array} \right\}.$$

By convention we write $\Sigma_{\omega,0} = \emptyset$. Define $\Sigma_{\omega,*} = \bigcup_{n \ge 0} \Sigma_{\omega,n}$. For $v = v_0 v_1 \cdots v_{n-1} \in \Sigma_{\omega,n}$, we denote $|v| = n$. For such v, we define the cylinder $[v]_\omega$ as

$$[v]_\omega := \{\underline{w} \in \Sigma_\omega : w_i = v_i \text{ for } i = 0, \ldots, n-1\}.$$

Now let us introduce basic notations. For any word $v = v_0 \cdots v_{r-1} v_r \cdots v_{m-1} \in \Sigma_{\omega,m}$, define v_0 to be the first character of v and v_{m-1} to be the last character of v. For $r \le m$, define $v|_r = v_0 \cdots v_{r-1}$.

For any $1 \le s \le l(\omega)$, for any $n \ge M(\omega)$, for any $1 \le t \le l(\sigma^n \omega)$, there exists at least one word $v = v(\omega, n, s, t) \in \Sigma_{\sigma^\omega, n-1}$ such that $svt \in \Sigma_{\omega, n+1}$. The choice of v may be not unique. For each such v, we denote the word svt by $s * t$.

For any $v = v_0 v_1 \cdots v_{n-1} \in \Sigma_{\omega,n}$ and $w = w_0 w_1 \cdots w_{m-1} \in \Sigma_{\sigma^{n+k}\omega, m}$, if $k \ge M(\sigma^n \omega)$, then $v_0 v_1 \cdots v_{n-2} v_{n-1} * w_0 w_1 \cdots w_{m-1} \in \Sigma_{\omega, n+k+m-1}$.

For any $\underline{v} = v_0 v_1 \cdots v_{n-1} \cdots \in \Sigma_\omega$, define $\underline{v}|_n = v_0 v_1 \cdots v_{n-1}$.

For any $v = v_0 v_1 \cdots v_{r-1} v_r \cdots v_n$, $w = w_0 w_1 \cdots w_{r-1} w_r \cdots w_n$, if for any i, $0 \le i \le r-1$ one has $v_i = w_i$ and $v_r \ne w_r$, then define we $v \wedge w =: v_0 v_1 \cdots v_{r-1}$.

	subshift of finite type	Random subshift
	———	$(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$
l (the number of types)	constant	random variable
$A = (a_{i,j})$ (Transitive matrix)	constant matrix	random matrix
M(To define the mixing property)	constant	random variable
$\underline{v} = v_0 v_1 \dots v_n \dots$ (Point)	$1 \le v_i \le l$ $A_{v_i, v_{i+1}} = 1$ $\underline{v} \in \Sigma$	$1 \le v_i \le l(\sigma^i(\omega))$ $A_{v_i, v_{i+1}}(\sigma^i(\omega)) = 1$ $\underline{v} \in \Sigma_\omega$
shift operator	$\Sigma \to \Sigma$	$\Sigma_\omega \to \Sigma_{\sigma\omega}$

Table 1.1 – The differences between subshift of finite type and random subshift

The differences between subshift of finite type and random subshift of finite type are indicated in Table 1.1.

Using the same notations as in [44, 47, 49], let

$$\mathcal{P}_{\mathbb{P}}(\Sigma_\Omega) = \{\rho, \text{ probability measure on } \Sigma_\Omega : \Pi_* \rho = \rho \circ \Pi^{-1} = \mathbb{P}\},$$

and

$$\mathcal{I}_{\mathbb{P}}(\Sigma_\Omega) = \{\rho \in \mathcal{P}_{\mathbb{P}}(\Sigma_\Omega) : \rho \text{ is } F\text{-invariant}\}.$$

Any $\rho \in \mathcal{P}_{\mathbb{P}}(\Sigma_\Omega)$ on Σ_Ω disintegrates in $d\rho(\omega, \underline{v}) = d\rho_\omega(\underline{v}) d\mathbb{P}(\omega)$ where the measures ρ_ω, $\omega \in \Omega$, are regular conditional probabilities with respect to the σ-algebra $\pi_\Omega^{-1}(\mathcal{F})$, where π_Ω is the canonical projection from Σ_Ω to Ω. This implies that for $\mathbb{P}$-almost every ω, for any measurable set $R \subset \Sigma_\Omega$, $\rho_\omega(R(\omega)) = \rho(R|\pi_\Omega^{-1}(\mathcal{F}))$, where $R(\omega) = \{x : (\omega, \underline{v}) \in R\}$.

Let $\mathcal{R} = \{R_i\}$ be a finite or countable partition of Σ_Ω into measurable sets. Then for all $\omega \in \Omega$, $\mathcal{R}(\omega) = \{R_i(\omega) : \ R_i(\omega) = \{x \in \Sigma_\omega : \ (\omega, x) \in R_i\}\}$ is a partition of Σ_ω.

Given $\rho \in \mathcal{P}_{\mathbb{P}}(\Sigma_\Omega)$, the conditional entropy of $\mathcal{R}$ given $\pi_\Omega^{-1}(\mathcal{F})$ is defined by

$$\begin{aligned} H_\rho(\mathcal{R}|\pi_\Omega^{-1}(\mathcal{F})) &= -\int \sum_i \rho(R_i|\pi_\Omega^{-1}(\mathcal{F})) \log(\rho(R_i|\pi_\Omega^{-1}(\mathcal{F}))) d\mathbb{P} \\ &= \int H_{\rho_\omega}(\mathcal{R}(\omega)) d\mathbb{P}(\omega) \end{aligned}$$

where $H_{\rho_\omega}(\mathcal{A})$ denotes the usual entropy of a partition $\mathcal{A}$.

Now, given a finite or countable partition $\mathcal{Q}$ of Σ_Ω, define the fiber entropy of F or the relative entropy of F with respect to $\mathcal{Q}$ as

$$h_\rho(F, \mathcal{Q}) = \lim_{n\to\infty} \frac{1}{n} H_\rho(\vee_{i=0}^{n-1} F^{-i} \mathcal{Q} | \pi_\Omega^{-1}(\mathcal{F}))$$

(here $\vee$ denotes the join of partitions).

Then define

$$h_\rho(F) = \sup_{\mathcal{Q}} h_\rho(F, \mathcal{Q}),$$

where the supremum is taken over all finite or countable measurable partitions $\mathcal{Q} = \{Q_i\}$ of Σ_Ω with finite conditional entropy, that is $h_\rho(F, \mathcal{Q}) < +\infty$. In our setting, we have $h_\rho(F) \leq \int \log(l)\, \mathrm{d}\mathbb{P}$. The number $h_\rho(F)$, also denoted $h(\rho|\mathbb{P})$ in the literature, is the relativized entropy of F given ρ. It is also called the fiber entropy of the bundle random dynamics F.

We say that a measurable function Φ on Σ_Ω is in $\mathbb{L}^1_{\Sigma_\Omega}(\Omega, C(\Sigma))$ if

1. $$C_\Phi =: \int_\Omega \|\Phi(\omega)\|_\infty \, \mathrm{d}\mathbb{P}(\omega) < \infty \tag{1.3}$$

   where

   $$\|\Phi(\omega)\|_\infty =: \sup_{\underline{v} \in \Sigma_\omega} |\Phi(\omega, \underline{v})|; \tag{1.4}$$

2. $$\mathrm{var}_n \Phi(\omega) \to 0 \text{ as } n \to \infty, \quad \mathbb{P}\text{-almost surely} \tag{1.5}$$

   where $\mathrm{var}_n \Phi(\omega) = \sup\{|\Phi(\omega, \underline{v}) - \Phi(\omega, \underline{w})| : v_i = w_i, \forall i < n\}$.

The topological pressure $P(\Phi)$ of Φ is defined by

$$P(\Phi) = \sup_{\rho \in \mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)} \left\{ h_\rho(F) + \int \Phi \mathrm{d}\rho \right\}.$$

Now, with $\Phi \in \mathbb{L}^1_{\Sigma_\Omega}(\Omega, C(\Sigma))$ is associated the transfer operator $\mathcal{L}^\omega_\Phi : C^0(\Sigma_\omega) \to C^0(\Sigma_{\sigma\omega})$ defined as

$$\mathcal{L}^\omega_\Phi h(\underline{v}) = \sum_{F_\omega \underline{w} = \underline{v}} \exp(\Phi(\omega, \underline{w})) h(\underline{w}), \quad \forall\ \underline{v} \in \Sigma_{\sigma\omega}.$$

By replacing if necessary Ω by a subset of full probability over which the mappings $\Phi(\sigma^k \omega, \cdot)$, $k \geq 0$ are all continuous, we have the following result:

Proposition 1.1 *[44, 61] For all $\omega \in \Omega$, there exists $\lambda(\omega) > 0$ and a probability measure $\widetilde{\mu}_\omega$ on Σ_ω such that $(\mathcal{L}^\omega_\Phi)^* \widetilde{\mu}_{\sigma\omega} = \lambda(\omega) \widetilde{\mu}_\omega$.*

We call the family $\{\widetilde{\mu}_\omega : \omega \in \Omega\}$ a random weak Gibbs measure on $\{\Sigma_\omega : \omega \in \Omega\}$ associated with Φ.

1.2 A model of random dynamical attractor

Given a random subshift of finite type as above, we can construct a random dynamical attractor. Our assumptions on the distribution of the number of intervals used in the construction, and the distribution of the lengths and positions of these intervals are more general than those used to get dynamical random Cantor sets in [45, 68, 76, 61].

For any $\omega \in \Omega$, let $U_\omega^1 = [a_{\omega,1}, b_{\omega,1}], U_\omega^2 = [a_{\omega,2}, b_{\omega,2}], \cdots U_\omega^s = [a_{\omega,s}, b_{\omega,s}] \cdots$ be closed non trivial intervals with disjoint interiors and suppose that U_ω^s is on the left side of U_ω^{s+1} for each $s \in \mathbb{N}$, i.e. $b_{\omega,s} \leq a_{\omega,s+1}$.

We assume that for each $s \geq 1$, $\omega \mapsto (a_{\omega,s}, b_{\omega,s})$ is measurable, $a_{\omega,1} \geq 0$, $b_{\omega,l(\omega)} \leq 1$ and setting $f_\omega^s(x) = \frac{x-a_{\omega,s}}{b_{\omega,s}-a_{\omega,s}}$, the mapping $\omega \mapsto \mathbf{T}_\omega^s$ is measurable from $(\Omega, \mathcal{F})$ to the space of diffeomorphisms of $[0,1]$ endowed with its Borel σ-field. Then $T_\omega^s : U_\omega^s \to [0,1]$ which is defined by $T_\omega^s := \mathbf{T}_\omega^s \circ f_\omega^s(x)$ is a C^1 diffeomorphism and we denote its inverse by $g_\omega^s := (T_\omega^s)^{-1}$.

From now on for all $\omega \in \Omega$ and $s \geq 1$, we define

$$\widetilde{\psi}(\omega, s, x) = -\log |(T_\omega^s)'(x)|, \ \forall x \in U_\omega^s.$$

Here, if x is a endpoint of U_ω^s, the derivative means the left derivative or right derivative of T_ω^s in the interval U_ω^s:

$$\widetilde{\psi}(\omega, s, x)(x) = \begin{cases} -\log |(T_\omega^s)'_+(x)|, & x \text{ is the left end point of the interval } U_\omega^s; \\ -\log |(T_\omega^s)'_-(x)|, & x \text{ is the right end point of the interval } U_\omega^s. \end{cases}$$

We say that a measurable function ψ on $\widetilde{U}_\Omega = \{(\omega, s, x) : \omega \in \Omega, 1 \leq s \leq l(\omega), x \in U_\omega^s\}$ is in $\mathbb{L}^1_{X_\omega}(\Omega, \widetilde{C}([0,1]))$ if

1. $$C_\psi := \int_\Omega \|\psi(\omega)\|_\infty \mathrm{d}\mathbb{P}(\omega) < \infty,$$
   where
   $$\|\psi(\omega)\|_\infty := \sup_{1 \leq s \leq l(\omega)} \sup_{x \in U_\omega^s} |\psi(\omega, s, x)|; \tag{1.6}$$

2. for $\mathbb{P}$-almost every $\omega \in \Omega$, $\mathrm{var}(\psi, \omega, \varepsilon) \to 0$ as $\varepsilon \to 0$, where
   $$\mathrm{var}(\psi, \omega, \varepsilon) = \sup_{1 \leq s \leq l(\omega)} \sup_{x,y \in U_\omega^s \text{ and } |x-y| \leq \varepsilon} |\psi(\omega, s, x) - \psi(\omega, s, y)|. \tag{1.7}$$

We now make our first assumption on the construction:

Assumption 1 $\psi := \widetilde{\psi}_{|\widetilde{U}_\Omega} \in \mathbb{L}^1_{X_\omega}(\Omega, \widetilde{C}([0,1]))$ *and* ψ *satisfies the contraction property in the mean*

$$c_\psi := -\int_\Omega \sup_{1 \leq s \leq l(\omega)} \sup_{x \in U_\omega^s} \psi(\omega, s, x) \mathrm{d}\mathbb{P}(\omega) > 0. \tag{1.8}$$

Define

$$\begin{aligned} U_\omega^v &= g_\omega^{v_0} \circ g_{\sigma\omega}^{v_1} \circ \cdots \circ g_{\sigma^{n-1}\omega}^{v_{n-1}}([0,1]), \ \forall v = v_0 v_1 \cdots v_{n-1} \in \Sigma_{\omega,n} \\ X_\omega &= \bigcap_{n\geq 1} \bigcup_{v\in\Sigma_{\omega,n}} U_\omega^v \\ X_\Omega &= \{(\omega, x) : \omega \in \Omega, x \in X_\omega\}. \end{aligned}$$

Now we will draw a picture to show the construction of random dynamical attractor:

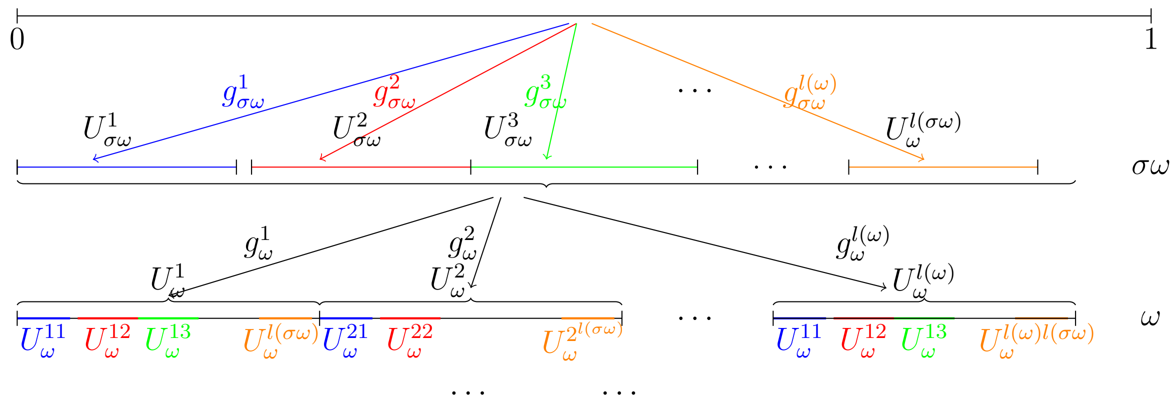


Figure 1.1 – First and second steps of the construction of the random attractor X_ω.

There is a natural projection $\pi_\omega : \Sigma_\omega \to X_\omega$ defined as

$$\pi_\omega(\underline{v}) = \lim_{n\to\infty} g_\omega^{v_0} \circ g_{\sigma\omega}^{v_1} \circ \cdots \circ g_{\sigma^{n-1}\omega}^{v_{n-1}}(0).$$

This map may not be injective, but any $x \in X_\omega$ has at most two preimages in Σ_ω.

The family $X = \{X_\omega : \omega \in \Omega\}$ is called a random dynamical attractor. Specifically, we will see that either X_ω is a Cantor set with probability 1, or $X_\omega = [0,1]$ with probability 1 (see chapter 4). In both cases, $\{X_\omega : \omega \in \Omega\}$ is the attractor of the random directed graph IFS $\{g_\omega^s : 1 \leq s \leq l(\omega), \omega \in \Omega\}$ where the edges are given by the $A(\sigma^k\omega)$, $k \geq 0$. In the first case, the mappings π_ω conjugates $\{\Sigma_\omega : \omega \in \Omega\}$ with the family of random Cantor sets $\{X_\omega : \omega \in \Omega\}$, which is endowed with the random dynamics $F_\omega \circ \pi_\omega^{-1}$; in the deterministic fullshift setting, when the intervals U^s are separated, the Cantor set X is called cookie-cutter set.

The above property 2. satisfied by ψ is weaker than the Hölder continuity assumed in [48, 61] (see chapter 3). It is also slightly weaker than the situation where the attractor would be the repeller of a family of random C^1 conformal mappings, since for two neighboring intervals U_ω^s and U_ω^{s+1} we do not require any continuity at their intersection point whenever it exists.

Define the map $\pi : \Sigma_\Omega \to X_\Omega$ as $\pi((\omega, \underline{v})) = (\omega, \pi_\omega(\underline{v}))$. For all $n \in \mathbb{N}$, for any word $v = v_0 v_1 \dots v_{n-1} \in \Sigma_{\omega,n}$, we define

$$X_\omega^v =: \pi_\omega([v]_\omega).$$

We also define

$$\Psi(\omega, \underline{v}) = \psi(\omega, v_0, \pi(\underline{v})) \text{ for } \underline{v} = v_0 v_1 \cdots \in \Sigma_\omega. \tag{1.9}$$

By Assumption 1, since $c_\psi > 0$ we have $\sup_{v \in \Sigma_{\omega,n}} |U_\omega^v| \le \exp(\frac{-nc_\psi}{2})$ for n larger than some $N(\omega)$, hence the function Ψ is in $\mathbb{L}^1_{\Sigma_\Omega}(\Omega, C(\Sigma))$. Furthermore,

$$c_\Psi := -\int_\Omega \sup_{\underline{v} \in \Sigma_\omega} \Psi(\omega, \underline{v}) \mathrm{d}\mathbb{P}(\omega) = c_\psi.$$

Theorem 1.2 *Under the Assumption 1, for $\mathbb{P}$-almost every $\omega \in \Omega$, the Bowen-Ruelle formula holds, i.e.* $\dim_H X_\omega = t_0$ *where t_0 is the unique root of the equation $P(t\Psi) = 0$. Furthermore,* $t_0 = \max_{\rho \in \mathcal{I}_\mathbb{P}(\Sigma_\Omega)} \left\{ \frac{h_\rho(F)}{-\int \Psi \mathrm{d}\rho} \right\}$.

Such a formula appeared for the first time in [17], where Bowen considered the Hausdorff dimension of quasi-circles. His method easily applies to the study of deterministic cookie-cutter sets (see [13] for instance). The Hausdorff dimensions of random dynamical Cantor sets (as well as some invariant sets of random dynamics on the torus) have been obtained through the same formula as in theorem 1.2 in [45, 46, 68, 16, 49, 61, 76] under the assumptions that the dynamics are piecewise $C^{1+\alpha}$. Thus, theorem 1.2 is expected and not difficult to get from Bowen's ideas, but for the sake of completeness, we will give a proof using an appropriate random weak Gibbs measure.

1.3 Multifractal formalism

Let us recall some general concepts of geometric measure theory. We start with Hausdorff and packing measures and dimensions in general metric spaces (we follow [59]).

Let (X, d) be a metric space, $\mathcal{F}$ a family of subsets of X and ζ a non-negative function on $\mathcal{F}$. We make the following two assumption

- For every $\delta > 0$ there are $E_1, E_2, \dots \in \mathcal{F}$ such that $X = \bigcup_{i=1}^\infty E_i$ and $d(E_i) \le \delta$, where $d(E) = \sup_{x,y \in E} d(x, y)$ for any $E \in \mathcal{F}$. The diameter $d(E)$ will be also denoted by $|E|$ is the sequel.
- For every $\delta > 0$ there is $E \in \mathcal{F}$ such that $\zeta(E) \le \delta$ and $d(E) \le \delta$.

For $0 \leq \delta \leq \infty$ and $A \subset X$, define

$$\mathcal{H}_\delta^\zeta(A) = \inf\{\sum_{i=1}^{\infty} \zeta(E_i) : A \subset \cup_{i=1}^{\infty} E_i, d(E_i) \leq \delta, E_i \in \mathcal{F}\}.$$

Then define

$$\mathcal{H}^\zeta(A) = \lim_{\delta \downarrow 0} \mathcal{H}_\delta^\zeta(A) = \sup_{\delta > 0} \mathcal{H}_\delta^\zeta(A)$$

for $A \subset X$. Then $\mathcal{H}^\zeta$ is a Borel measure further more if the member of $\mathcal{F}$ are Borel sets, then H^ζ is Borel regular.

Definition 1.3 *[59] Let X be separable, $0 \leq s < \infty$, choose*

$$\mathcal{F} = \{E : E \subset X\},$$

and

$$\zeta(E) = (d(E))^s,$$

with the convention $0^0 = 1$ and $(d(\emptyset))^s = 0$. The corresponding $\mathcal{H}_\delta^\zeta$ is called the s-dimensional pre-Hausdorff measure and is denoted by $\mathcal{H}_\delta^s$, and the resulting measure $\mathcal{H}^s$ is called the s-dimensional Hausdorff measure and is denoted by $\mathcal{H}^s$.

The Hausdorff dimension of a set $A \subset X$ is

$$\begin{aligned} \dim_H A &= \sup\{s : \mathcal{H}^s(A) > 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\} \\ &= \inf\{t : \mathcal{H}^t(A) < \infty\} = \inf\{t : \mathcal{H}^t(A) = 0\} \end{aligned}$$

Let $g : [0, \infty) \to [0, \infty)$ be a non decreasing function with $g(0) = 0$ and we will call g a gauge function. We take again

$$\mathcal{F} = \{E : E \subset X\} \quad \text{and} \quad \zeta(E) = g(d(E)),$$

then the corresponding measure $\mathcal{H}^g$ is called the Hausdorff g measure. Of course $\mathcal{H}^g = \mathcal{H}^s$ when $g(t) = t^s$.

The packing measure and packing dimension can be defined in a similar way.

Let $g : [0, \infty) \to [0, \infty)$ be a non decreasing function with $g(0) = 0$. For $0 \leq \delta \leq \infty$ and $A \subset X$, define the packing g pre-measure

$$\mathcal{P}_{0,\delta}^g(A) = \sup \left\{ \sum_{i \in I} g(d(B_i)) \;\middle|\; \begin{array}{l} \{B_i\}_{i \in I} \text{ is a countable collection of} \\ \text{pairwise disjoint closed balls with} \\ \text{diameter not larger than } \delta \text{ and centers in } A \end{array} \right\}.$$

Then define

$$\mathcal{P}_0^g(A) = \lim_{\delta \downarrow 0} \mathcal{P}_{0,\delta}^g(A) = \sup_{\delta > 0} \mathcal{P}_{0,\delta}^g(A),$$

for $A \subset X$. Then

$$\mathcal{P}^g(A) = \inf\{\sum_{j \in J} \mathcal{P}_0^g(A_j) : A \subset \cup_{j \in J} A_j, \; J \text{ countable } \}.$$

$\mathcal{P}^g$ is called the packing g measure.

Definition 1.4 *If $g(E) = (d(E))^s$, then $\mathcal{P}^g$ is also called the s-dimensional packing measure and it is denoted by $\mathcal{P}^s$.*

The packing dimension of a set $A \subset X$ is

$$\dim_p A = \sup\{s : \mathcal{P}^s(A) = \infty\} = \inf\{t : \mathcal{P}^t(A) = 0\}.$$

Now, let us set up the multifractal formalism on $\mathbb{R}$, which will be the context of this thesis. Possible references are [24, 18, 64, 50, 4].

Let μ be a compactly supported positive and finite measure on $\mathbb{R}$.

Definition 1.5 *The (lower) L^q-spectrum $\tau_\mu : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ and the upper-L^q spectrum $\overline{\tau}_\mu : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ are given by*

$$\tau_\mu(q) = \liminf_{r\to 0} \frac{\log \sup\{\sum_i (\mu(B_i))^q\}}{\log(r)} \tag{1.10}$$

$$\overline{\tau}_\mu(q) = \limsup_{r\to 0} \frac{\log \sup\{\sum_i (\mu(B_i))^q\}}{\log(r)} \tag{1.11}$$

where the supremum is taken over all families of disjoint closed balls B_i of radius r with centers in supp(μ).

By construction, the function τ_μ is non decreasing and concave over its domain, which equals $\mathbb{R}$ of $\mathbb{R}_+$ (see [50, 4]).

Definition 1.6 *The lower and upper large deviations spectra $\underline{LD}$ and $\overline{LD}$ are given by*

$$\underline{LD}_\mu(d) = \lim_{\varepsilon\to 0} \liminf_{r\to 0} \frac{\log \#\{i : r^{d+\varepsilon} \leq \mu(B(x_i, r) \leq r^{d-\varepsilon})\}}{-\log(r)} \tag{1.12}$$

$$\overline{LD}_\mu(d) = \lim_{\varepsilon\to 0} \limsup_{r\to 0} \frac{\log \#\{i : r^{d+\varepsilon} \leq \mu(B(x_i, r) \leq r^{d-\varepsilon})\}}{-\log(r)} \tag{1.13}$$

where the supremum is taken over all families of disjoint closed balls $B_i = B(x_i, r)$ of radius r with centers x_i in supp(μ).

Definition 1.7 *For all $x \in \mathrm{supp}(\mu)$, define*

$$\underline{\dim}_{\mathrm{loc}}(\mu, x) = \liminf_{r\to 0^+} \frac{\log \mu(B(x, r))}{\log r}, \quad \overline{\dim}_{\mathrm{loc}}(\mu, x) = \limsup_{r\to 0^+} \frac{\log \mu(B(x, r))}{\log r},$$

and

$$\dim_{\mathrm{loc}}(\mu, x) = \lim_{r\to 0^+} \frac{\log \mu(B(x, r))}{\log r}$$

whenever the limit exists. Then, for $d \leq d' \in \mathbb{R}$*, define*

$$\begin{aligned}
\underline{E}(\mu, d) &= \{x \in \operatorname{supp}(\mu) : \underline{\dim}_{\mathrm{loc}}(\mu, x) = d\}, \\
\overline{E}(\mu, d) &= \{x \in \operatorname{supp}(\mu) : \overline{\dim}_{\mathrm{loc}}(\mu, x) = d\}, \\
E(\mu, d) &= \underline{E}(\mu, d) \cap \overline{E}(\mu, d), \\
E(\mu, d, d') &= \{x \in \operatorname{supp}(\mu) : \underline{\dim}_{\mathrm{loc}}(\mu, x) = d, \overline{\dim}_{\mathrm{loc}}(\mu, x) = d'\}.
\end{aligned}$$

It is clear that since μ *is bounded,* $E(\mu, d, d') = \emptyset$ *if* $d' < 0$*.*

Finally, define

$$\dim_H(\mu) = \sup\{s : \text{ for } \mu\text{-almost every } x \in \operatorname{supp}(\mu), \underline{\dim}_{\mathrm{loc}}(\mu, x) \geq s\}.$$

An equivalent definition is (see [24]):

$$\dim_H(\mu) = \inf\{\dim_H E : \mu(E) > 0\}.$$

Definition 1.8 *(Legendre Transform) For any function* $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ *with non-empty domain, its Legendre transform* f^* *is defined on* $\mathbb{R}$ *by*

$$f^*(d) = \inf_{q \in \mathbb{R}} \{dq - f(q)\} \in \mathbb{R} \cup \{-\infty\}.$$

Definition 1.9 *(Multifractal formalism) We say that* μ *obeys the strong multifractal formalism at* $d \in \mathbb{R} \cup \{\infty\}$ *if*

$$\dim_H E(\mu, d) = \tau_\mu^*(d)$$

and that the strong multifractal formalism holds (globally) for μ *if it holds at any* $d \in \mathbb{R} \cup \{\infty\}$*.*

We say that μ *obeys the multifractal formalism at* $d \in \mathbb{R} \cup \{\infty\}$ *if*

$$\dim_H \underline{E}(\mu, d) = \tau_\mu^*(d)$$

and that the multifractal formalism holds (globally) for μ *if it holds at any* $d \in \mathbb{R} \cup \{\infty\}$*.*

The reader should have in mind that if the domain of τ_μ is the whole interval $\mathbb{R}$, then $\tau_\mu^*(d) \geq 0$ if and only if $\tau_\mu^*(d) > -\infty$, i.e. $d \in [\tau_\mu'(+\infty), \tau_\mu'(-\infty)]$.

It turns out that if the strong multifractal formalism holds for μ at d, then one has ([64, 50])

$$\begin{aligned}
\dim_H E(\mu, d) &= \dim_H \underline{E}(\mu, d) = \dim_H \overline{E}(\mu, d) = \dim_P E(\mu, d) = \underline{LD}_\mu(d) = \overline{LD}_\mu(d) \\
&= \tau_\mu^*(d)
\end{aligned}$$

since one always has

$$\dim_H E(\mu, d) \le \underline{LD}_\mu(d)$$

and

$$\max(\underline{LD}_\mu(d), \dim_H \underline{E}(\mu, d), \dim_H \overline{E}(\mu, d), \dim_P E(\mu, d)) \le \overline{LD}_\mu(d) \le \tau_\mu^*(d).$$

Then, it is straightforward that in this case τ_μ is a limit, i.e. $\tau_\mu = \overline{\tau}_\mu$.

A full illustration of this multifractal formalism is given in [4], where, for any concave function τ naturally candidate to be the L^q-spectrum of a compactly supported positive measure on $\mathbb{R}$ (and more generally $\mathbb{R}^d$), such a measure μ is constructed; moreover, this measure obeys the strong multifractal formalism and it is exactly dimensional.

Definition 1.10 *Given $\alpha \ge 0$, we say that the zero-infinity law holds for a set E at α in the sense of Hausdorff if for any gauge function g we have*

$$\mathcal{H}^g(E) = \begin{cases} 0 & \textit{if } \limsup_{r\to 0} \frac{\log g(r)}{\log r} > \alpha, \\ \infty & \textit{if } \limsup_{r\to 0} \frac{\log g(r)}{\log r} \le \alpha. \end{cases}$$

We say that the zero-infinity law holds for E at α in the sense of packing if for any gauge function g we have

$$\mathcal{P}^g(E) = \begin{cases} 0 & \textit{if } \liminf_{r\to 0} \frac{\log g(r)}{\log r} > \alpha, \\ \infty & \textit{if } \liminf_{r\to 0} \frac{\log g(r)}{\log r} \le \alpha. \end{cases}$$

1.4 Multifractal analysis of the random weak Gibbs measures

Let $\phi \in \mathbb{L}^1_{\Sigma_\Omega}(\Omega, \widetilde{C}([0,1]))$ and consider the function: $\Phi(\omega, \underline{v}) = \phi(\omega, v_0, \pi(\underline{v}))$, for $\underline{v} = v_0 v_1 \cdots \in \Sigma_\omega$. Then Φ is an element of $\mathbb{L}^1_{\Sigma_\Omega}(\Omega, C(\Sigma))$.

Let μ be the random weak Gibbs measure on $\{X_\omega : \omega \in \Omega\}$ obtained as $\mu_\omega = \pi_{\omega *}\widetilde{\mu}_\omega := \widetilde{\mu}_\omega \circ \pi_\omega^{-1}$, where $\widetilde{\mu}$ is obtained from proposition 1.1. Without changing the random measures $\widetilde{\mu}_\omega$ and μ_ω, we can assume $P(\Phi) = 0$ after replacing ϕ by $\phi - P(\Phi)$ and $\lambda(\omega)$ by $\lambda(\omega)e^{-P(\Phi)}$ if necessary.

Since $c_\Psi > 0$, one deduces from the definition of the topological pressure that for any $q \in \mathbb{R}$, there exists a unique $T(q) \in \mathbb{R}$ such that

$$P(q\Phi - T(q)\Psi) = 0,$$

and that the mapping T is concave and non decreasing. It follows from the variational principle that $T(q)/q$ is bounded near $-\infty$ and $+\infty$, so $(T'(+\infty), T'(-\infty))$ is a bounded subset of $\mathbb{R}_+$.

Theorem 1.11 *Under the Assumption 1, for $\mathbb{P}$-almost every $\omega \in \Omega$, one has*

1. *for any $q \in \mathbb{R}$,*

$$\tau_{\mu_\omega}(q) = \overline{\tau}_{\mu_\omega}(q) = T(q) = \min_{\rho \in \mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)} \left\{ \frac{h_\rho(F) + q \int \Phi \mathrm{d}\rho}{\int \Psi \mathrm{d}\rho} \right\}.$$

2. *The strong multifractal formalism holds globally for μ, i.e. for all $d \in [T'(+\infty), T'(-\infty)]$ we have*

$$\begin{aligned} \dim_H E(\mu_\omega, d) &= \dim_H \underline{E}(\mu_\omega, d) = \dim_H \overline{E}(\mu_\omega, d) = \dim_P E(\mu_\omega, d) \\ &= \underline{LD}_{\mu_\omega}(d) = \overline{LD}_{\mu_\omega}(d) = \tau^*_{\mu_\omega}(d) = T^*(d); \end{aligned}$$

furthermore

$$T^*(d) = \max_{\rho \in \mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)} \left\{ -\frac{h_\rho(F)}{\int \Psi \mathrm{d}\rho} : \frac{\int \Phi \mathrm{d}\rho}{\int \Psi \mathrm{d}\rho} = d \right\}.$$

3. *For any given $d, d' \in [T'(+\infty), T'(-\infty)]$,*

$$\dim_H E(\mu_\omega, d, d') = \inf\{T^*(d), T^*(d')\},$$

$$\dim_P E(\mu_\omega, d, d') = \sup\{T^*(\beta) : \beta \in [d, d']\}.$$

4. *For any given $d \in [T'(+\infty), T'(-\infty)]$,*

$$\dim_H \underline{E}(\mu_\omega, d) = T^*(d), \ \dim_P \underline{E}(\mu_\omega, d) = \sup\{T^*(d') : d' \geq d\},$$

and

$$\dim_H \overline{E}(\mu_\omega, d) = T^*(d), \ \dim_P \overline{E}(\mu_\omega, d) = \sup\{T^*(d') : d' \leq d\}.$$

5. *For any $d \in [T'(+\infty), T'(-\infty)]$ such that*

$$T^*(d) < \sup\{T^*(d') : d' \in [T'(+\infty), T'(-\infty)]\} = t_0 = \dim_H X_\omega,$$

the zero-infinity law holds for the set $E(\mu_\omega, d)$ at $T^*(d)$ in the sense of both Hausdorff and packing.

Let us put our result in perspective with respect to the existing literature.

Multifractal analysis and large deviations for Gibbs measures on cookie-cutter sets were achieved in [71], after the study of some class of multifractal invariant measures in [19] (see also[13] and [24, Ch. 4 and 5]). They used a multifractal formalism based on a L^q-spectrum defined thanks to partition functions associated with the symbolic coding of the support of the measure [18], while the more intrinsic point of view we adopt here, which consists in considering balls centered on the support of the measure, comes from the fundamental contributions by Olsen [64] and Lau-Ngai [50]. More generally, the multifractal analysis of Gibbs measures and quasi-Bernoulli measures on attractors of hyperbolic dynamics has been studied intensively (see for instance [18, 25, 67, 38, 12]; on the other hand, a lot of works have been dedicated to the closely related class of self-similar measures, see [5] for a survey). Thermodynamic formalism and multifractal analysis was deal with for the conformal infinite iterated function systems in [37] and for meromorphic functions of finite order in [62]. Graph directed Markov system was considered in [60] and then multifractal analysis of conformal measure for such system (over a subset of the limit set which is often large) was considered in [77]. The first results for weak Gibbs measures were obtained in [42], and completed in [26, 29, 30].

The study achieved in [68, 69] leads to the multifractal nature of Gibbs measures projected on some random Cantor sets whose construction assumes a strong separation condition for the pieces of the construction. About the same time, the multifractal analysis of random Gibbs measures and Birkhoff averages on random Cantor sets and the whole torus where obtained in [45, 46]; when the support of the measure is a Cantor set, a strong separation condition is assumed as well. More recently, in [31], the multifractal analysis for disintegrations of Gibbs measures on $\{1,\dots,m\}^{\mathbb{N}}\times\{1,\dots,m\}^{\mathbb{N}}$ was achieved as a consequence of the multifractal analysis of conditional Birkhoff averages of random continuous potentials (not C^{α}). The approach developed there could, with some effort, be adapted to derive our results on weak Gibbs measures if we worked with random fullshift only. However, as we already said it in the beginning of the introduction, the method can not be extended easily to the random subshift, and our view point will be different. In [31], the authors start by establishing large deviations results and use them to construct by concatenation Moran sets of arbitrary large dimension in the level sets $E(\mu_\omega, d)$; we will concatenate information provided by random Gibbs measures associated with Hölder potentials which approximate the continuous potentials associated with the random weak Gibbs measure and the random maps generating the attractor X_ω. This will provide us with a very flexible tool from which, for instance, we will deduce the result about the sets $E(\mu_\omega, d, d')$. In this sense, our results also complete a part of those obtained in [61] which, in particular, achieves the multifractal analysis of random Gibbs measures on random Cantor sets obtained as the repeller of random conformal maps.

The multifractal analysis of Birkhoff averages on random conformal repellers of C^1 expanding maps is studied in [81], where the random dynamics is in fact coded by a non random subshift of finite type, and the random potentials that are considered satisfy an equicontinuity property stronger than the one we require.

The sets $E(\mu, d, d')$ were studied for Gibbs measures on conformal repellers and for self-similar measures in [32, 65, 2, 66].

Finally, in [53], zero-infinity laws are established for Besicovitch subsets of self-similar sets of the line. This inspired theorem 1.11(5), of which the results in [53] turn out to be a special case. Also, in [52], a zero-infinity law is established for the Hausdorff and packing measure of sets of generic points of invariant measures on a conformal repeller.

1.5 Multifractal analysis of the inverse of random weak Gibbs measures

Definition 1.12 *For any positive Borel measure* μ *on* $[0,1]$*, let* F_μ *be the distribution function of the measure* μ *which is* $F_\mu(t) = \mu([0,t])$*. The inverse measure* ν *of* μ *is the unique Borel probability measure on* $[0,1]$ *such that*

$$\text{for all } x \in [0,1], F_\nu(x) = \sup\{t \in [0,1]; F_\mu(t) \le x\}.$$

After reducing the situation to the case $P(\Phi) = 0$, our second main assumptionis:

Assumption 2

$$c_\phi := -\int_\Omega \sup_{1\le s\le l(\omega)} \sup_{x\in U_\omega^s} (\phi(\omega,s,x))\mathrm{d}\mathbb{P}(\omega) > 0,$$

and

$$\mathbb{P}(\{\omega \in \Omega : \text{ The Lebesgue measure of } X_\omega \text{ is equal to } 0\}) = 1.$$

The general assumption that for $\mathbb{P}$-almost every $\omega \in \Omega$, the Lebesgue measure of X_ω vanishes ensures that, $\mathbb{P}$-almost every $\omega \in \Omega$, the inverse measure of any probability measure on X_ω is a discrete probability measure. This assumption is weaker than the essential randomness sometimes assumed in some related works (see [61]), which implies that for $\mathbb{P}$-almost every $\omega \in \Omega$ one has $\mathcal{H}^{t_0}(X_\omega) = 0$ (recall that t_0 is the almost sure Hausdorff dimension of X_ω): since $t_0 \le 1$, essential randomness implies vanishing of the Lebesgue measure. Our assumption, as well as essential randomness, seems hard to illustrate with examples for which $t_0 = 1$. It would be good to prove, or disprove, the existence of such an example.

For any $q \in \mathbb{R}$, there exists a unique $\mathcal{T}(q) \in \mathbb{R}$ such that

$$P(q\Psi - \mathcal{T}(q)\Phi) = 0.$$

This is due to the fact that $c_\Phi = c_\phi > 0$. There is an obvious relationship between T and $\mathcal{T}$ through the equation $P(q\Phi - T(q)\Psi) = 0$.

Our result about the multifractal analysis of the inverse measure for the random weak Gibbs measure is following:

Theorem 1.13 *Under the assumption 1 and 2, for $\mathbb{P}$-almost every $\omega \in \Omega$, the inverse measure ν_ω of μ_ω is a discrete measure and it satisfies the following properties:*

1. *For any $q \in \mathbb{R}$, one has that $\tau_{\nu_\omega}(q) = \min\{\mathcal{T}(q), 0\}$.*

2. *For any $d \in [0, \tau'_{\nu_\omega}(-\infty)]$, one has*

$$\dim_H(\underline{E}(\nu_\omega, d)) = \tau^*_{\nu_\omega}(d).$$

3.
   - *For any $d \in [\mathcal{T}'(+\infty), \mathcal{T}'(-\infty)]$, one has*

$$\dim_H(\overline{E}(\nu_\omega, d)) = \dim_H(E(\nu_\omega, d)) = \mathcal{T}^*(d) = dT^*(1/d).$$

   - *For any $d \in (0, \mathcal{T}'(+\infty))$, the sets $\overline{E}(\nu_\omega, d)$ and $E(\nu_\omega, d)$ are empty.*
   - *For $d = 0$,*

$$\overline{E}(\nu_\omega, 0) \;\; = \;\; E(\nu_\omega, 0) = \{\text{atoms of } \nu_\omega\}$$

so that

$$\dim_H(\overline{E}(\nu_\omega, 0)) = \dim_H(E(\nu_\omega, 0)) = 0.$$

Multifractal analysis of inverse measures started in [58], and then was developed in [73, 74]. In these papers, the local dimension is defined in a stronger sense in order to get general relations between the multifractal behavior of measure and its inverse: $\dim_{\text{loc}}(\mu, x) = \lim_{I\to\{x\}} \frac{\log(\mu(I))}{\log(|I|)}$, where I is a non trivial interval containing I. With this definition, it was shown in [74] that for the discrete inverse of a Gibbs measure on a cookie-cutter the strong multifractal formalism fails on a non trivial interval. Later, in [11] obtained the validity of the multifractal formalism using the lower local dimension. This used the so-called conditioned ubiquity theory, which combines ergodic theory and metric approximation theory, and was developed in [8]. This tool makes it possible to study a broad class of multifractal discrete measures [6, 10], to which the measures ν_ω do not belong to.

The flavor of theorem 1.13(1) and (2) is similar to that of [11] regarding the inverse of Gibbs measures on cookie-cutter sets: for the level sets of the lower local dimension, the Hausdorff spectrum is composed of two parts: a linear part with slope $\dim_H X_\omega$, which is established thanks to conditioned ubiquity theory, and a concave part which mainly reflects the multifractal structure of weak Gibbs measures or, equivalently, ratios of Birkhoff averages. Theorem 1.13(3) completes [11] results in this deterministic situation. Also, in [11] the level set $\underline{E}(\nu_\omega, \mathcal{T}'(-\infty))$ was not treated when $\mathcal{T}^*(\mathcal{T}'(-\infty)) = 0$. The study of the sets $\dim_H E(\nu_\omega, d, d')$ is in progress.

As we explained it at the beginning of this chapter, though following the main lines of the deterministic case considered in [11], the study of ν_ω requires our results on weak Gibbs measures since the potentials are not Hölder. Also, even for Hölder potentials, the study is made structurally more complexe because the inverse structure comes from a random subshift. Moreover, we need a version of the conditioned ubiquity theorem of [8] adapted to our context.

It is worth mentioning that the mutifractal analysis of discrete measures started with homogeneous sums of Dirac masses [1, 39, 40, 23], in particular the derivative of Lévy subordinators [40], and that originally heterogeneous ubiquity was elaborate with the multifractal analysis of Lévy processes in multifractal time as initial target [9].

Here are some pictures describing the Legendre pairs $(\tau_{\mu_\omega}, \tau^*_{\mu_\omega})$ and $(\tau_{\nu_\omega}, \tau^*_{\nu_\omega})$.

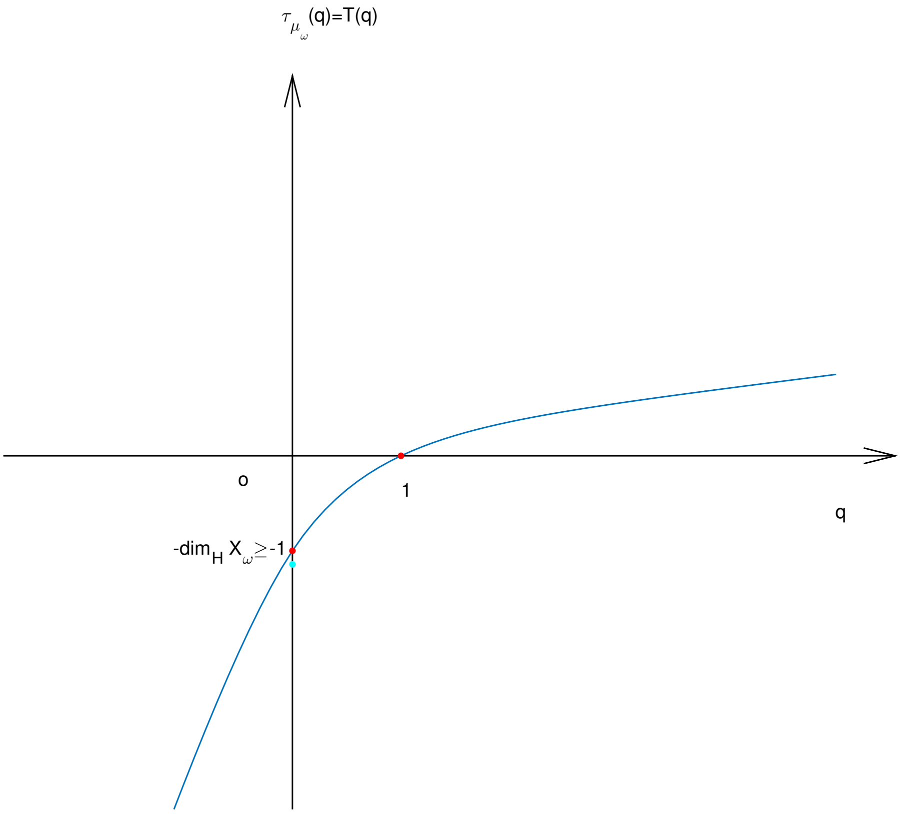


Figure 1.2 – The function of $\tau_{\mu_\omega} = T$

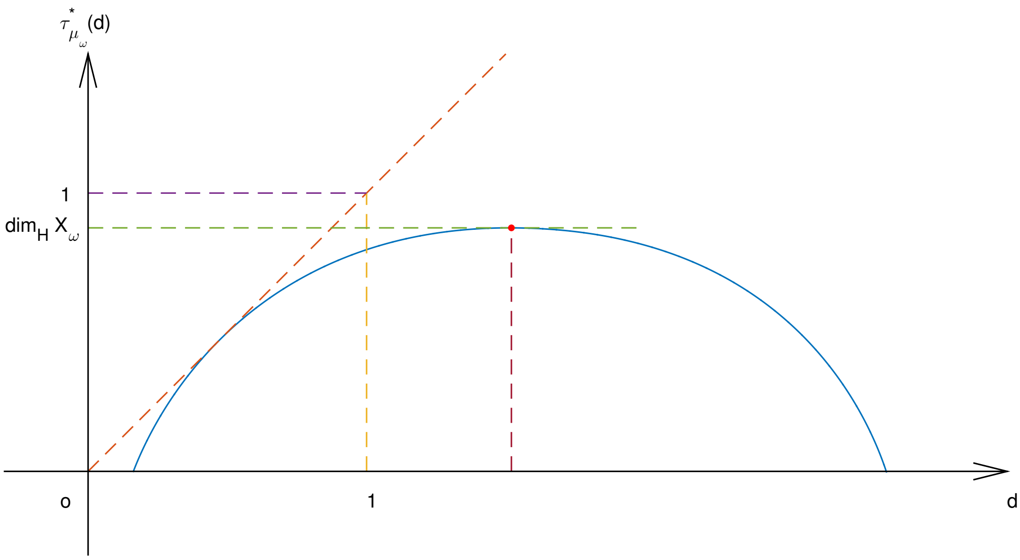


Figure 1.3 – The function of $\tau^*_{\mu_\omega}(d)$.

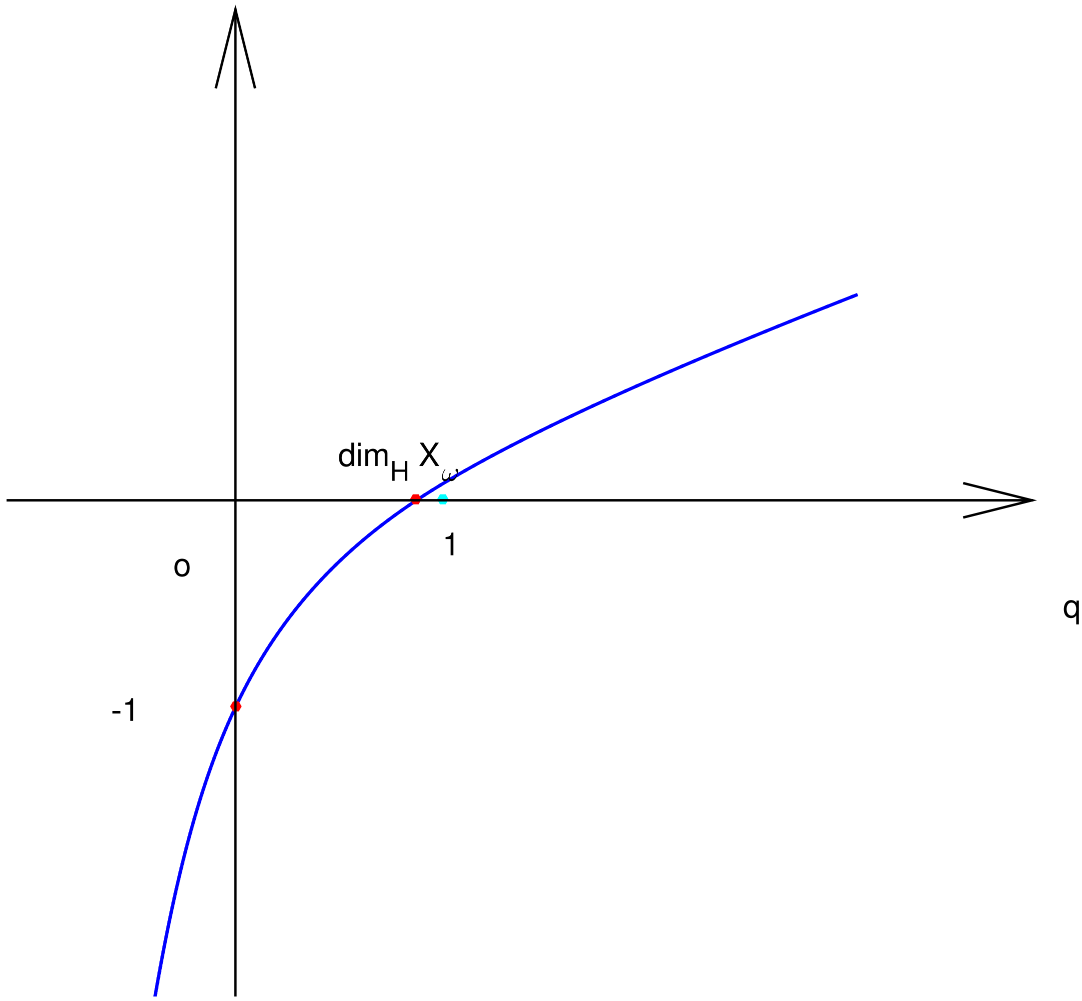


Figure 1.4 – The function $\mathcal{T}$

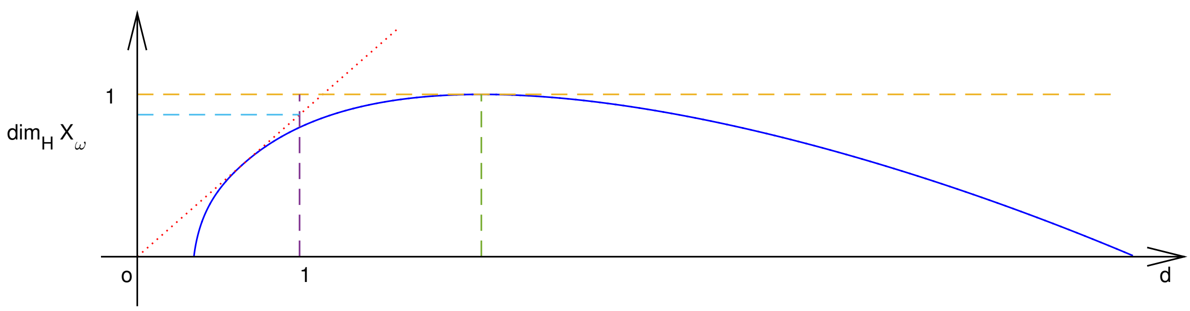


Figure 1.5 – The Hausdorff dimension of the level sets $E(\nu_\omega, d)$ and $\overline{E}(\nu_\omega, d)$: $\mathcal{T}^*(d)$

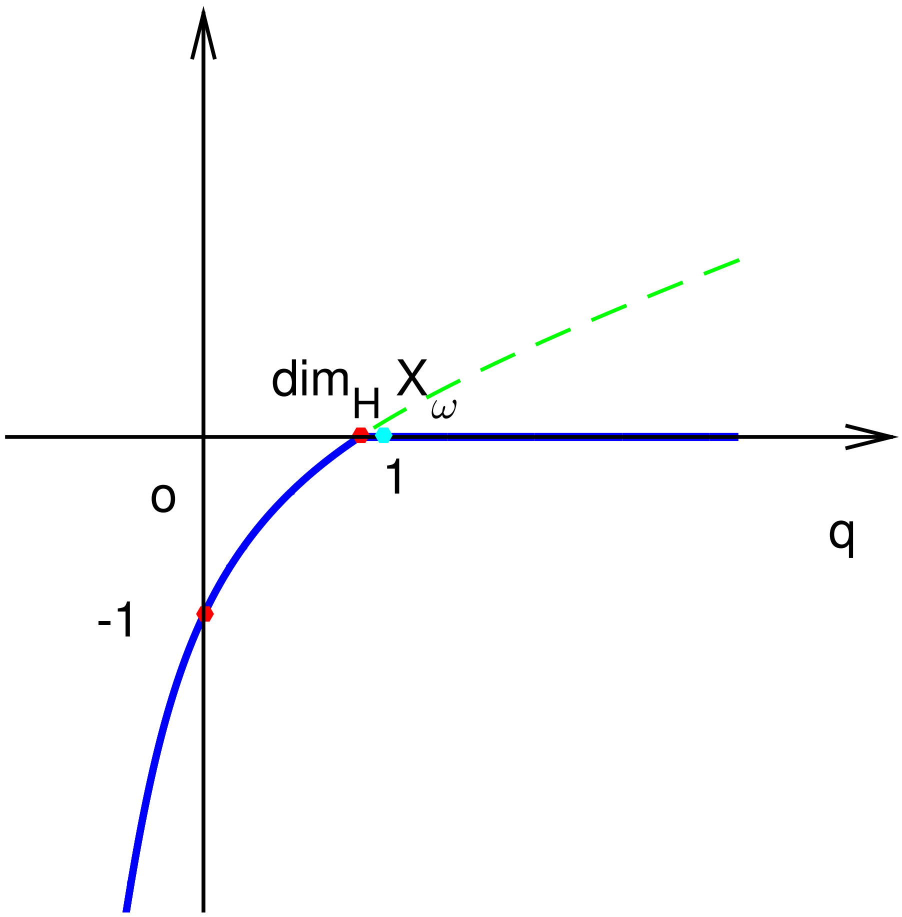


Figure 1.6 – The L^q-spectrum for the inverse measure — τ_{ν_ω}

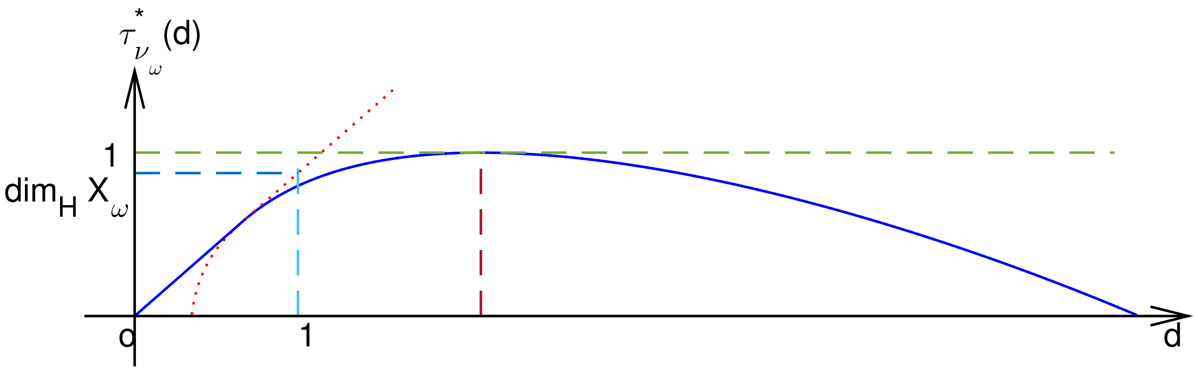


Figure 1.7 – The Hausdorff dimension of the level sets $\underline{E}(\nu_\omega, d)$: $\tau^*_{\nu_\omega}(d)$

1.6 Concrete examples of random attractors

The random attractors considered in this paper extend for instance the examples that are obtained if one considers the fibers of McMullen-Bedford self-affine carpets, and more generally the Gatzouras-Lalley self-affine carpets [82]. In particular, such fibers naturally illustrate the idea that at a given step of the construction two consecutive intervals U_ω^s and U_ω^{s+1} may touch each other. In [51], Luzia considers a class of expanding maps of the 2-torus of the form $f(x,y) = (a(x,y), b(y))$ that are C^2-perturbations of Gatzouras-Lalley carpets, whose fibers naturally illustrate our purpose with nonlinear maps, but not C^1. Also these examples are associated with random fullshift.

Let us give more explicit examples. Here we use the notations of Section 1.2.

A first example is the following. Let $(\Omega, \mathcal{F}, \mathcal{P}, \sigma)$ is the following:

$$\Omega = \Gamma := \widetilde{\mathbb{Z}}^+ \times \widetilde{\mathbb{Z}}^+ \times \cdots ,$$

$\mathcal{F}$ is the σ-algebra generated by the cylinders $[n_1 n_2 \cdots n_k]$ for any $k \in \mathbb{N}$ and any $n_i \in \tilde{\mathbb{Z}}^+$ for any $i \in \mathbb{N}$ with $1 \le i \le k$,

$$\mathbb{P}([n_1 n_2 \cdots n_k]) = \frac{1}{n_1(n_1+1)} \cdot \frac{1}{n_2(n_2+1)} \cdot \cdots \cdot \frac{1}{n_k(n_k+1)},$$

the map σ is the shift map. Such a system is ergodic. It satisfies the conditions we need.

Let $\underline{n} = n_1 n_2 \cdots n_k \cdots \in \Gamma$, define $l(\underline{n}) = n_1$ and $A(\underline{n}) = A_{n_1 \times n_2}$, where $A_{n_1 \times n_2}$ is $n_1 \times n_2$-matrix with all entries equaling to 1 if $n_2 \neq n_2 - 1$ or $n_1 = 2$, otherwise it is a matrix satisfying that the entries of the first $n_1 - 1$ rows are 1 and the entries of the n_1-th row are 0 except that $a(n_1, n_1 - 1) = 1$. It is clear that such a system can give us a random subshift system.

In fact it is easy to check that M and l are measurable. For any $k \in \mathbb{N}$,

$$\{\underline{n} \in \Omega : M(\underline{n}) = k\} = [(k+1)k(k-1)\cdots 2],$$

and it is measurable. So that M is measurable. For l, for any $k \in \mathbb{N}$,

$$\{\underline{n} \in \Omega : l(\underline{n}) = k\} = [k],$$

and it is also measurable.

Then l and M are unbounded but $\int \log l \, d\mathbb{P} < +\infty$, and we get a random subshift, which is not a fullshift . Now, we set $T_{\underline{n}}^i(x) = n_1 x \mod 1$ for $x \in [\frac{i-1}{n_1}, \frac{i}{n_1}]$ and for $i = 1, 2, \cdots, n_1$.

For any point $\omega = \underline{n}$, If $n_k = 3$, then at the k-th step the whole length of the cylinders will become $5/6$ of the whole length of $k-1$-th step. As $\mathbb{P}([3]) = 1/12$, from

Poincaré's recurrence theorem [83, theorem 1.4] or ergodic theorem [83, theorem 1.14], for $\mathbb{P}$-almost every $\omega \in \Omega$, 3 will appear infinity many times, so at last the whole length (Lebesgue measure) will be 0 for the fiber at ω.

In fact, in the previous model the measure $\mathbb{P}$ is a special example of a Gibbs measure on the (Ω, σ) (see [78, 80]). So we can enrich the previous construction by considering any such measure $\mathbb{P}$ for which $\int \log l \, \mathrm{d}\mathbb{P} < +\infty$. For the mappings maps T^s_ω, here is a way to provide a non trivial example, which seems to be not covered by the existing literature.

Start with a family $\{\varphi_{s,\omega}\}_{s\in\mathbb{N}}$ of random C^1 differeomorphisms of $[0,1]$ such that at least one $\varphi'_{s,\omega}$ is nowhere C^ϵ with positive probability. Assume that there exists a random variable a_0 taking values in $(0,1]$ and such that

$$\inf_{1\le s\le l(\omega),\, x\in[0,1]} |\varphi'_{s,\omega}(x)| \ge a_0(\omega).$$

Let $T^s_\omega = \varphi_{s,\omega} \circ f^s_\omega$, where f^s_ω is the linear map from U^s_ω onto $[0,1]$. Then, the constant c_ψ of Assumption 1 satisfies

$$c_\psi \ge \int_\Omega \left[\log(a_0(\omega)) - \sup_{1\le s\le l(\omega)} \log(|U^s_\omega|)\right] \mathrm{d}\mathbb{P}(\omega).$$

Thus, we require that

$$\int_\Omega \left[\log(a_0(\omega)) - \sup_{1\le s\le l(\omega)} \log(|U^s_\omega|)\right] \mathrm{d}\mathbb{P}(\omega) > 0.$$

This allows some T^s_ω be not uniformly expanding, but ensures expansiveness in the mean. It is easily seen that the Lebesgue measure of X_ω is almost surely bounded by $\prod_{i=0}^{n-1}\left(\sum_{1\le s\le l(\omega)} |U^s_{\sigma^i\omega}|/a_0(\sigma^i\omega)\right)$ for all $n \ge 1$. Thus, if we strengthen our requirement by assuming that

$$\int_\Omega \left[\log(a_0(\omega)) - \log\left(\sum_{1\le s\le l(\omega)} |U^s_\omega|\right)\right] \mathrm{d}\mathbb{P}(\omega) > 0,$$

then the Lebesgue measure of X_ω is 0 almost surely.

Now let us provide a completely explicit illustration of the last idea (we will work with a random fullshift for simplicity of the exposition).

We take $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$ as the fullshift $(\{0,1,2\}^{\mathbb{N}}, \mathcal{F}, \mathbb{P}, \sigma)$. For any n-th cylinder $[\omega_0\omega_1\cdots\omega_{n-1}] \subset \Omega$ we set $\mathbb{P}([\omega_0\omega_1\cdots\omega_{n-1}]) = \frac{1}{3^n}$. It is the unique ergodic measure of maximal entropy for the shift map.

Let l be a random variable depending on ω_0 only, which is given by

$$l(\omega) = \begin{cases} 4 & \omega_0 = 0 \\ 1 & \omega_0 = 1 \\ 3 & \omega_0 = 2 \end{cases}$$

The entries of the random transition matrix are always 1 (we consider the random fullshift). We assume that the map $T(\omega, x)$ just depends on ω_0 and x.

If $\omega_0 = 0$, let $\varphi_{s,\omega}(x) = x$ for $s = 1, 2, 3, 4$ and $U^1_\omega = [0, 1/4]$, $U^2_\omega = [1/4, 1/2]$, $U^3_\omega = [1/2, 3/4]$ and $U^4_\omega = [3/4, 1]$. In this case, we know that $a_0(\omega) = 1$. By the way the intervals U^s_ω, $1 \le s \le 4$ cover the interval $[0, 1]$.

If $\omega_0 = 1$, let $h(x) = 6 + \sum_{j=1}^{+\infty} j^{-2} \sin(2^j \pi x)$. Define

$$\varphi_{1,\omega}(x) = \frac{\int_0^x h(t)dt}{\int_0^1 h(t)dt},$$

and $U^1_\omega = [0, 1]$. In this case we can choose $a_0(\omega) = 1/2$. It is easy to check that T^1_ω is not expanding on some interval; furthermore it is just of class C^1 since h is nowhere ϵ-Hölder for any $\epsilon \in (0, 1)$.

If $\omega_0 = 2$, let $\varphi_{s,\omega}(x) = x$ for $s = 1, 3$ and $\varphi_{2,\omega}(x) = \frac{7x}{8} + \frac{x^2}{8}$, and $U^1_\omega = [0, 1/9]$, $U^2_\omega = [1/9, 2/9]$, $U^3_\omega = [2/3, 7/9]$. It is easy to check that the left derivative of T^1_ω and the right derivative of T^2_ω is not coincide with each other, so it can not be express as a conformal map here. In this case we can choose $a_0(\omega) = 7/8$.

Also,

$$\int_\Omega \left[\log(a_0(\omega)) - \log \left(\sum_{1 \le s \le l(\omega)} |U^s_\omega| \right) \right] \mathrm{d}\mathbb{P}(\omega) = \frac{\log 21 - \log 16}{3} > 0,$$

so that all the conditions hold.

Chapter 2

Basic properties of random weak Gibbs measures

We will use the notations of the previous chapter.

Fix a potential $\Phi \in \mathbb{L}^1_{\Sigma_\Omega}(\Omega, C(\Sigma))$ (here $P(\Phi)$ may not be 0). Since $\mathrm{var}_n\Phi(\omega) \to 0$ as $n \to 0$ and $(\mathrm{var}_n\Phi)_{n\geq 1}$ is bounded in L^1 norm, using Maker's ergodic theorem [54], we can get

$$V_n\Phi(\omega) := \sum_{i=0}^{n-1} \mathrm{var}_{n-i}\Phi(\sigma^i\omega) = o(n), \quad \mathbb{P}\text{-almost surely.} \tag{2.1}$$

Due to (1.3) and the ergodic theorem, setting $S_n\|\Phi(\omega)\|_\infty = \sum_{i=0}^{n-1} \|\Phi(\sigma^i\omega)\|_\infty$, for any positive sequence $(a_n)_{n\geq 0}$ such that $a_n = o(n)$ we have

$$\Big|S_n\|\Phi(\omega)\|_\infty - S_{n-a_n}\|\Phi(\omega)\|_\infty\Big| = nC_\Phi - (n-a_n)C_\Phi + o(n) = o(n), \quad \mathbb{P}\text{-almost surely.} \tag{2.2}$$

Definition 2.1 *A family $u = \{u_{n,\omega} : \Sigma_{\omega,n} \to \Sigma_\omega\}$ of measurable maps satisfying $(u_{n,\omega}(v))|_n = v$ for all $v \in \Sigma_{\omega,n}$ and $(n,\omega) \in \mathbb{N} \times \Omega$ is called an extension. We say that it is measurable, if the map $(\omega, x) \mapsto u_{n,\omega}(x)$ is measurable for all $n \in \mathbb{N}$.*

Let $u = \{u_{n,\omega}\}$ be an extension and $\Phi \in \mathbb{L}^1_{\Sigma_\Omega}(\Omega, C(\Sigma))$. Then for $(n,\omega) \in \mathbb{N} \times \Omega$

$$Z_{u,n}(\Phi,\omega) := \sum_{v\in\Sigma_{\omega,n}} \exp\big(S_n\Phi(\omega, u_{n,\omega}(v))\big) = \sum_{v\in\Sigma_{\omega,n}} \exp\left(\sum_{i=0}^{n-1} \Phi(F^i(\omega, u_{n,\omega}(v)))\right)$$

is called n-th partition function of Φ in ω with respect to u.

Let

$$\pi_{n,u}(\Phi,\omega) = \frac{1}{n}\log Z_{u,n}(\Phi,\omega).$$

Due to the assumption $\log(l) \in \mathbb{L}^1(\Omega, \mathbb{P})$, using the same method as in [35, 49], it is easy to prove the following lemma.

Lemma 2.2 *Let u be any extension and $\Phi \in \mathbb{L}^1_{\Sigma_\Omega}(\Omega, C(\Sigma))$.*

Then $\lim_{n\to\infty} \pi_{n,u}(\Phi, \omega) = P(\Phi)$ for $\mathbb{P}$-almost every $\omega \in \Omega$. This limit is independent of u.

Now let

$$\lambda(\omega, n) = \lambda(\omega) \cdot \lambda(\sigma\omega) \cdot \cdots \cdot \lambda(\sigma^{n-1}\omega),$$

where $\lambda(\omega)$ is defined as in proposition 1.1. The following lemma is direct when the potential Φ possesses bounded distorsions so that the Ruelle-Perron-Frobenious theorem holds for the operator $\mathcal{L}^\omega_\Phi$. For general potentials in $\mathbb{L}^1_{\Sigma_\Omega}(\Omega, C(\Sigma))$ we need a proof.

Lemma 2.3 *One has $\lim_{n\to\infty} \dfrac{\log \lambda(\omega, n)}{n} = P(\Phi)$ for $\mathbb{P}$-almost every $\omega \in \Omega$.*

Remark 2.4 *In the following proof, as well as in the rest of this text, we will use the letter M to denote the levels of the function $M(\cdot)$. Keeping this in mind should prevent from some confusion.*

Proof First, for $M > 0$, let $F_M = \{\omega \in \Omega : M(\omega) \leq M\}$. Fix M large enough so that $\mathbb{P}(F_M) > 0$. For each $\omega \in \Omega$, let $b_k(\omega)$ be the k-th return time of ω to the set A_M. From ergodic theorem we get $\lim_{k\to\infty} \frac{b_k}{k} = \frac{1}{\mathbb{P}(F_M)}$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Then $\lim_{k\to\infty} \frac{b_{k+1}-b_k}{b_k} = 0$ for $\mathbb{P}$-almost every $\omega \in \Omega$, which implies that $M(\sigma^n\omega) = o(n)$.

Second, for any $\underline{v} \in \Sigma_{\sigma^n\omega}$ we have

$$Z_{u,n-M(\sigma^n\omega))}(\Phi, \omega) \exp(-o(n)) \leq \mathcal{L}^{\omega,n}_\Phi 1(\underline{v}) \leq Z_{u,n}(\Phi, \omega) \exp(o(n)).$$

The right inequality uses the fact that we work with a subshift as well as (2.1). We just prove the left inequality: for n large enough so that $M(\sigma^n\omega) \leq n$,

$$\begin{aligned}
\mathcal{L}^{\omega,n}_\Phi 1(\underline{v}) &= \sum_{w\in\Sigma_{\omega,n}, w\underline{v}\in\Sigma_\omega} \exp(S_n\Phi(\omega, w\underline{v})) \\
&\geq \sum_{w'\in\Sigma_{\omega,n-M(\sigma^n\omega)}} \exp\left(S_{n-M(\sigma^n\omega)}(\omega, u_{\omega,n-M(\sigma^n\omega)}(w')) - o(n)\right) \\
&= Z_{u,n-M(\sigma^n\omega))}(\Phi, \omega) \exp(-o(n)).
\end{aligned}$$

The inequality follows by using (1.2), then by preserving for each $w' \in \Sigma_{\omega,n-M(\sigma^n\omega)}$ only one path of length $M(\sigma^n\omega)$ from w' to $\underline{v}$, and by using (2.1), $M(\sigma^n\omega) = o(n)$ and (2.2).

Now, since $\lambda(\omega, n) = \int \mathcal{L}^{\omega,n}_\Phi 1(\underline{v}) d\widetilde{\mu}_{\sigma^n\omega}(\underline{v})$, we can easily get the result from lemma 2.2 and the fact that $M(\sigma^n\omega) = o(n)$.

Proposition 2.5 *Let $u = \{u_{n,\omega}\}$ be an extension and $\Phi, \Psi \in \mathbb{L}^1_{\Sigma_\Omega}(\Omega, C(\Sigma))$. There exists $\Omega' \subset \Omega$ such that:*

1. *$\mathbb{P}(\Omega') = 1$.*

2. *Setting $\Phi_{q,t} = q\Phi - t\Psi$ for $(q,t) \in \mathbb{R}^2$, for any $\omega \in \Omega'$, $\pi_{n,u}(\Phi_{q,t}, \omega)$ converges uniformly to $P(\Phi_{q,t})$ over the compact subsets of $\mathbb{R}^2$ as $n \to \infty$.*

Proof We first check that that $\pi_{n,u}(\Phi_{q,t}, \omega)$ is a convex function of (q,t):

For any $(q_1,t_1), (q_2,t_2) \in \mathbb{R}^2$ and $\alpha \in [0,1]$

$$\begin{aligned}
& \pi_{n,u}(\Phi_{\alpha q_1+(1-\alpha)q_2, \alpha t_1+(1-\alpha)t_2}, \omega) \\
= \; & \frac{1}{n} \log \sum_{v \in \Sigma_{\omega,n}} \exp(S_n(\Phi_{\alpha q_1+(1-\alpha)q_2, \alpha t_1+(1-\alpha)t_2})(\omega, u_{n,\omega}(v)) \\
= \; & \frac{1}{n} \log \sum_{v \in \Sigma_{\omega,n}} \exp(S_n(\alpha\Phi_{q_1,t_1} + (1-\alpha)\Phi_{q_2,t_2})(\omega, u_{n,\omega}(v))) \\
= \; & \frac{1}{n} \log \sum_{v \in \Sigma_{\omega,n}} \exp(S_n(\alpha\Phi_{q_1,t_1})(\omega, u_{n,\omega}(v)) \cdot \exp(S_n((1-\alpha)\Phi_{q_2,t_2})(\omega, u_{n,\omega}(v))) \\
\leq \; & \frac{\alpha}{n} \log \sum_{v \in \Sigma_{\omega,n}} \exp(S_n(\Phi_{q_1,t_1})(\omega, u_{n,\omega}(v))) + \frac{1-\alpha}{n} \log \sum_{v \in \Sigma_{\omega,n}} \exp(S_n(\Phi_{q_2,t_2})(\omega, u_{n,\omega}(v))) \\
= \; & \alpha \pi_{n,u}(\Phi_{q_1,t_1}, \omega) + (1-\alpha)\pi_{n,u}(\Phi_{q_2,t_2}, \omega).
\end{aligned}$$

Fix a dense countable subset D of $\mathbb{R}^2$. For any $(q,t) \in D$, from theorem 2.2 we can find $\Omega_{q,t} \subset \Omega$ such that $\mathbb{P}(\Omega_{q,t}) = 1$ and for any $\omega \in \Omega_{q,t}$ one has $\lim_{n\to\infty} \pi_{n,u}(\Phi_{q,t}, \omega) = P(\Phi_{q,t})$.

Let $\Omega' = \cap_{(q,t)\in D} \Omega_{q,t}$. One has $\mathbb{P}(\Omega') = 1$ and for any $\omega \in \Omega_C$, $\lim_{n\to\infty} \pi_{n,u}(\Phi_{q,t}, \omega) = P_\sigma(\Phi_{q,t})$ for all $(q,t) \in D$. The uniform convergence over the compact subsets of $\mathbb{R}^2$ is now a standard result in convex analysis (theorem 10.8 of [75]).

For each $\omega \in \Omega$, let

$$D(\omega) =: \frac{1}{\lambda(\omega, M(\omega))} \exp(-S_{M(\omega)} \|\Phi(\omega)\|_\infty).$$

Then $D(\omega) > 0$ for $\mathbb{P}$-almost every $\omega \in \Omega$. From $M(\sigma^n\omega) = o(n)$, (2.2) and lemma 2.3 we can get that $\log D(\sigma^n\omega) = o(n)$ $\mathbb{P}$-almost surely.

Recall that by proposition 1.1, for $\mathbb{P}$-a.e $\omega \in \Omega$, the measures $\widetilde{\mu}_\omega$ satisfy $(\mathcal{L}^\omega_\Phi)^* \widetilde{\mu}_{\sigma\omega} = \lambda(\omega)\widetilde{\mu}_\omega$.

Proposition 2.6 *For $\mathbb{P}$-a.e $\omega \in \Omega$, for any $n \in \mathbb{N}$, for all $v = v_0 v_1 \dots v_{n-1} \in \Sigma_{\omega,n}$, one has*

$$\frac{D(\sigma^n\omega)}{\lambda(\omega,n)} \exp\big(\inf_{\underline{v}\in[v]_\omega} S_n\Phi(\omega,v)\big) \le \widetilde{\mu}_\omega([v]_\omega) \le \frac{1}{\lambda(\omega,n)} \exp\big(\sup_{\underline{v}\in[v]_\omega} S_n\Phi(\omega,\underline{v})\big),$$

so that

$$\exp(-\epsilon_n n) \le \frac{\widetilde{\mu}_\omega([v]_\omega)}{\exp(S_n\Phi(\omega,\underline{v}) - \log(\lambda(\omega,n)))} \le \exp(\epsilon_n n)$$

for any $\underline{v} \in [v]_\omega$, where ϵ_n does not depend on v and tends to 0 as $n \to \infty$.

Proof Let us deal first with the case $n = 1$.

Fix $1 \le i \le l(\omega)$. For any $1 \le j \le l(\sigma^{M(\omega)}\omega)$, there exists $w \in \Sigma_{\sigma\omega, M(\omega)-1}$ such that $iwj \in \Sigma_{\omega,1+M(\omega)}$. Due to proposition 1.1, we have

$$\widetilde{\mu}_\omega([iwj]_\omega) = \frac{1}{\lambda(\omega,M(\omega))} \int \mathcal{L}_\Phi^{\omega,M(\omega)} 1_{[iwj]_\omega} \,\mathrm{d}\widetilde{\mu}_{\sigma^{M(\omega)}\omega},$$

where $\mathcal{L}_\Phi^{\omega,n} = \mathcal{L}_\Phi^{\sigma^{n-1}\omega} \circ \dots \circ \mathcal{L}_\Phi^{\sigma\omega} \circ \mathcal{L}_\Phi^{\omega}$. This implies

$$\widetilde{\mu}_\omega([iwj]) \ge \frac{\inf_{\underline{w}\in[iwj]_\omega} \exp(S_{M(\omega)}\Phi(\omega,\underline{w}))}{\lambda(\omega,M(\omega))} \widetilde{\mu}_{\sigma^{M(\omega)}\omega}([j]_{\sigma^{M(\omega)}\omega}).$$

Then $\widetilde{\mu}_\omega([i]) \ge D(\omega)$ follows after summing over $1 \le j \le l(\sigma^{M(\omega)}\omega)$. The upper bound $\widetilde{\mu}_\omega([i]) \le 1$ is obvious.

The general case is achieved similarly: If $v \in \Sigma_{\omega,n}$, for each $1 \le j \le l(\sigma^{n+M(\sigma^n\omega)-1}\omega)$, there exists $w \in \Sigma_{\sigma^n\omega, M(\sigma^n\omega)-1}$ such that $vwj \in \Sigma_{\omega,n+M(\sigma^n\omega)}$. One has

$$\widetilde{\mu}_\omega([vwj]_\omega) = \frac{1}{\lambda(\omega,n)\lambda(\sigma^n\omega, M(\sigma^n\omega))} \int \mathcal{L}_\Phi^{\omega,n+M(\sigma^n\omega)} 1_{[vwj]_\omega} \,\mathrm{d}\widetilde{\mu}_{\sigma^{n+M(\sigma^n\omega)}\omega},$$

from which we get

$$\widetilde{\mu}_\omega([vwj]_\omega) \ge \frac{1}{\lambda(\omega,n)} \cdot D(\sigma^n\omega) \exp\big(\inf_{\underline{v}\in[v]_\omega} S_n\Phi(\omega,v)\big) \widetilde{\mu}_{\sigma^{n+M(\sigma^n\omega)}\omega}([j]_{\sigma^{n+M(\sigma^n\omega)}\omega}).$$

Then, taking the sum over $1 \le j \le l(\sigma^{n+M(\sigma^n\omega)-1}\omega)$ we get

$$\widetilde{\mu}_\omega([v]_\omega) \ge \frac{D(\sigma^n\omega)}{\lambda(\omega,n)} \exp\big(\inf_{\underline{v}\in[v]_\omega} S_n\Phi(\omega,v)\big).$$

The inequality $\widetilde{\mu}_\omega([v]_\omega) \le \frac{1}{\lambda(\omega,n)} \exp\big(\sup_{\underline{v}\in[v]_\omega} S_n\Phi(\omega,\underline{v})\big)$ is direct from the equality

$$\widetilde{\mu}_\omega([v]_\omega) = \frac{1}{\lambda(\omega,n)} \int \mathcal{L}_\Phi^{\omega,n} 1_{[v]_\omega} \,\mathrm{d}\widetilde{\mu}_{\sigma^n\omega}.$$

Finally we conclude with (2.1) and $\log D(\sigma^n\omega) = o(n)$.

For any $\gamma \in \mathbb{L}^1_{X_\Omega}(\Omega, \widetilde{C}([0,1]))$ and any $z \in U_\omega^v$, let

$$S_n\gamma(\omega, z) = \sum_{i=0}^{n-1} \gamma(\sigma^i\omega, v_i, T^{v_{i-1}}_{\sigma^{i-1}\omega} \cdots T^{v_0}_\omega z).$$

Proposition 2.7 *For $\mathbb{P}$-almost every $\omega \in \Omega$, there are positive sequences $(\epsilon(\psi, n))_{n\geq 0}$ and $(\epsilon(\phi, n))_{n\geq 0}$, that we also denote as $(\epsilon(\Psi, n))_{n\geq 0}$ and $(\epsilon(\Phi, n))_{n\geq 0}$, converging to 0 as $n \to +\infty$, such that for all $n \in \mathbb{N}$, for all $v = v_0v_1 \dots v_n \in \Sigma_{\omega,n}$, we have :*

1. *For all $z \in \mathring{U}_\omega^v$,*

$$\exp(S_n\psi(\omega, z) - n\epsilon(\psi, n)) \leq |U_\omega^v| \leq \exp(S_n\psi(\omega, z) + n\epsilon(\psi, n)),$$

hence for all $\underline{v} \in [v]_\omega$,

$$\exp(S_n\Psi(\omega, \underline{v}) - n\epsilon(\Psi, n)) \leq |U_\omega^v| \leq \exp(S_n\Psi(\omega, \underline{v}) + n\epsilon(\Psi, n)).$$

Consequently, for all $\underline{v} \in X_\omega^v$:

$$|X_\omega^v| \leq |U_\omega^v| \leq \exp(S_n\Psi(\omega, \underline{v}) + n\epsilon(\Psi, n)).$$

2. *For all $\underline{v} \in [v]_\omega$,*

$$\exp(S_n\Phi(\omega, \underline{v}) - n\epsilon(\Phi, n)) \leq \widetilde{\mu}_\omega([v]_\omega) \leq \exp(S_n\Phi(\omega, \underline{v}) + n\epsilon(\Phi, n)),$$

hence for all $z \in U_\omega^v$,

$$\exp(S_n\phi(\omega, z) - n\epsilon(\phi, n)) \leq \mu_\omega(X_\omega^v) = \mu_\omega(U_\omega^v),$$

as well as $\mu_\omega(U_\omega^v) \leq \exp(S_n\phi(\omega, z) + n\epsilon(\phi, n))$ if $\widetilde{\mu}_\omega$ is atomless.

Proof 1. For all $n \in \mathbb{N}$, for all $v = v_0v_1 \cdots v_{n-1} \in \Sigma_{\omega,n}$, define T_ω^v as

$$T^{v_{n-1}}_{\sigma^{n-1}\omega} \circ \cdots \circ T^{v_1}_{\sigma\omega} \circ T^{v_0}_\omega.$$

For any $x, y \in U_\omega^v$, from the Lagrange's finite-increment theorem we have that

$$|T_\omega^v x - T_\omega^v y| = |(T_\omega^v)'(z)||x - y|,$$

for some z between x and y. Since $T_\omega^v(U_\omega^v) =: T^{v_{n-1}}_{\sigma^{n-1}\omega} \circ \cdots \circ T^{v_0}_\omega(U_\omega^v) = [0, 1]$,

$$|U_\omega^v| = \frac{1}{|(T_\omega^v)'(y)|} = \exp(S_n\psi(\omega, z)),$$

for some z in the interior of U_ω^v. Then by definition (1.8) of c_ψ, due to the ergodicity of the system $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$, we have

$$\sup_{v\in\Sigma_{\omega,n}} |U_\omega^v| \leq \exp(\|S_n\Psi(\omega)\|_\infty) \leq \exp(-nc_\Psi/2) \tag{2.3}$$

for n larger than some $N(\omega)$.

Let $\alpha_n(\omega) = \mathrm{var}(\psi, \omega, \sup_{v\in\Sigma_{\omega,n}} |U_\omega^v|)$. Then, by definition of ψ, $\alpha_n(\omega) \to 0$ as $n \to \infty$. Consequently,

$$V(\psi, \omega, n) := \sum_{0\leq i\leq n} \alpha_i(\sigma^{n-i}\omega) = o(n)$$

by Maker's ergodic theorem, and the same holds for $V_n\Psi(\omega)$ which by definition equals $V(\psi, \omega, n)$ (see (1.7) and (2.1)).

Since $|S_n\psi(\omega, z) - S_n\psi(\omega, y)| \leq V(\psi, \omega, n)$ for any $y, z \in U_\omega^v$, we get

$$\exp(S_n\psi(\omega, z) - o(n)) \leq |U_\omega^v| \leq \exp(S_n\psi(\omega, z) + o(n))$$

for all $z \in \mathring{U}_\omega^v$. The inequality associated with $S_n\Psi$ follows immediately.

2. Noting that we reduced the situation to $P(\Phi) = 0$, the first part of this item comes from theorem 2.6, lemma 2.3, proposition 2.6, the fact that $D(\sigma^n\omega) = o(n)$, and the control (2.1) of the distorsion $V_n\Phi$ coming from the assumption $\phi \in \mathbb{L}^1_{X_\Omega}(\Omega, \widetilde{C}([0,1]))$.

The second part comes from the relation $\mu_\omega = \pi_{\omega *}\widetilde{\mu}_\omega$.

Chapter 3

Basic properties of random Gibbs measures

Random Gibbs measures are associated with random Hölder continuous potentials. We say that a function Φ is a random Hölder potential if Φ is measurable from Σ_Ω to $\mathbb{R}$,

$$\int \sup_{\underline{v}\in\Sigma_\omega} |\Phi(\omega,\underline{v})| d\mathbb{P} < \infty,$$

and there exists $\kappa \in (0,1]$ such that

$$\mathrm{var}_n \Phi(\omega) \leq K_\Phi(\omega) e^{-\kappa n}, \tag{3.1}$$

where the random variable $K_\Phi = K_\Phi(\omega) > 0$ is such that $\int \log K_\Phi(\omega) \mathrm{d}\mathbb{P}(\omega) < \infty$.

A random Hölder continuous potential is obviously in $\mathbb{L}^1_{\Sigma_\Omega}(\Omega, C(\Sigma))$.

Theorem 3.1 ([48, 49]) *Assume that $\mathcal{F}$ is a countably generated σ-algebra, F is a topological mixing subshift of finite type and Φ a random Hölder potential.*

For $\mathbb{P}$-almost every $\omega \in \Omega$, there exists some random variables $C = C^\Phi(\omega) > 0$, $\lambda = \lambda^\Phi(\omega) > 0$, a function $h = h(\omega) = h(\omega,\underline{v}) > 0$ and a measure $\widetilde{\mu} \in \mathcal{M}^1_{\mathbb{P}}(\Sigma_\Omega)$ with disintegrations $\widetilde{\mu}_\omega$ satisfying

$$\int |\log C^\Phi| \,\mathrm{d}\mathbb{P} < +\infty, \quad \int |\log \lambda^\Phi| \,\mathrm{d}\mathbb{P} < +\infty \text{ and } \log h \text{ is a Hölder potential,}$$

and such that

$$\mathcal{L}^\omega_\Phi h(\omega) = \lambda(\omega) h(\sigma\omega), \ (\mathcal{L}^\omega_\Phi)^* \widetilde{\mu}_{\sigma\omega} = \lambda(\omega)\widetilde{\mu}_\omega, \int h(\omega)\,\mathrm{d}\widetilde{\mu}_\omega = 1. \tag{3.2}$$

Let $m_\omega = m^\Phi_\omega$ be given by $dm_\omega = h(\omega)d\widetilde{\mu}_\omega$ and set $dm(\omega,\underline{v}) = d\widetilde{\mu}_\omega(\underline{v})d\mathbb{P}(\omega)$. Then $m \in \mathcal{I}^1_{\mathbb{P}}(\Sigma_\Omega)$, for $\mathbb{P}$-almost every $\omega \in \Omega$, for all $v = v_0v_1 \ldots v_{n-1} \in \Sigma_{\omega,n}$, and *for all* $\underline{v} \in [v]_\omega$

$$\frac{1}{C^\Phi} \leq \frac{m_\omega([v]_\omega)}{\exp(\sum_{i=0}^{n-1} \Phi(F^i(\omega, \underline{v})) - \log \lambda^\Phi(\sigma^{n-1}\omega) \cdot \dots \cdot \lambda^\Phi(\omega))} \leq C^\Phi. \tag{3.3}$$

The family of measures $(m_\omega)_{\omega \in \Omega}$ *is called a random (or relative) Gibbs measure (or state) for the potential* Φ*. Moreover,* m *is the unique maximizing* F*-invariant probability measure in the variational principle, i.e. such that*

$$P(\Phi) = h_F(m|\mathbb{P}) + \int \Phi \, \mathrm{d}m, \text{ and one has } P(\Phi) = \int \log \lambda(\omega) \, \mathrm{d}\mathbb{P}. \tag{3.4}$$

Each time we need to refer to the function Φ*, we denote the measures* m *and* m_ω *as* m^Φ *and* m_ω^Φ*, and denote* λ *as* λ^Φ*.*

We can also define the random Gibbs measure on the random attractor X_ω by setting $\mu_\omega = m_\omega \circ \pi_\omega^{-1}$.

In this thesis if the potential Φ (which is related to ϕ) is a random Hölder potential, then when we say the relative measures m_ω^Φ and μ_ω^ϕ, they are referred to be random Gibbs measures.

Given a random Hölder potential Φ, from (3.2) we can define the normalized potential $\Phi'(\omega, \underline{v}) = \Phi(\omega, \underline{v}) + \log h(\omega, \underline{v}) - \log h(F(\omega, \underline{v})) - \log \lambda(\omega)$, which satisfies $\mathcal{L}_{\Phi'}^\omega 1 = 1$ for $\mathbb{P}$-almost every $\omega \in \Omega$. This implies that $\Phi' \leq 0$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Also, we have the following fact:

Proposition 3.2 *Suppose that* Φ *is a random Hölder potential. If* $P(\Phi) = 0$*, there exist some* $\varpi > 0$ *such that for* $\mathbb{P}$*-almost every* $\omega \in \Omega$*, there exists* $N(\omega)$ *such that for any* $n \geq N(\omega)$ *and any* $v \in \Sigma_{\omega,n}$*, one has*

$$\sup_{\underline{v} \in [v]_\omega} S_n \Phi(\omega, \underline{v}) \leq -n\varpi.$$

As a consequence, μ_ω *is atomeless.*

If we need to refer explicitly to Φ, we will use the notations $N_\Phi(\omega)$ and ϖ_Φ instead of $N(\omega)$ and ϖ.

The main idea of the proof is from [31].

Proof Since $P(\Phi) = 0$, we have $\sup\{\int \Phi \, \mathrm{d}\rho : \rho \in \mathcal{I}_\mathbb{P}(\Sigma_\Omega)\} \leq 0$.

We claim that $\sup\{\int \Phi \, \mathrm{d}\rho : \rho \in \mathcal{I}_\mathbb{P}(\Sigma_\Omega)\} < 0$. Let M large enough such that $\mathbb{P}(\{\omega : M(\omega) < M, l(\omega) \geq 2\}) > 0$. For any $\omega \in \Omega$ such that $M(\omega) < M$ and $l(\omega) \geq 2$ we have $\mathcal{L}_{\Phi'}^{\omega,M} 1 = 1$, hence $S_M \Phi'(\omega, \underline{v}) < 0$ for any $\underline{v} \in \Sigma_\omega$ and $\int S_M \Phi'(\omega, \underline{v}) d\rho_\omega < 0$ for any probability measure ρ_ω on Σ_ω. Since, moreover, we have $S_M \Phi' \leq 0$, we conclude that $\sup\{\int \Phi' \, \mathrm{d}\rho, \rho \in \mathcal{I}_\mathbb{P}(\Sigma_\Omega)\} < 0$.

Let $-2\varpi := \sup\{\int \Phi \, \mathrm{d}\rho : \rho \in \mathcal{M}^1_{\mathbb{P}}(\Sigma_\Omega, F)\}$. If the proposition does not hold, there exists a subsequence $(n_k)_{k\geq 1}$ such that

$$\#\{v : |v| = n_k, \frac{\sup_{\underline{v}\in[v]_\omega} S_{n_k}\Phi(\omega, \underline{v})}{n_k} > -\varpi\} \geq 1.$$

For any $q \in \mathbb{R}^+$,

$$\begin{aligned} P(q\Phi) &= \lim_{k\to\infty} \frac{\log \sum_{v\in\Sigma_{\omega,n_k}} \exp\left(qS_{n_k}\Phi(\omega, u_{n_k,\omega}(v))\right)}{n_k} \\ &\geq \lim_{k\to\infty} \frac{-q\varpi n_k}{n_k} = -q\varpi. \end{aligned}$$

However,

$$\begin{aligned} P(q\Phi) &= \sup_{\rho\in\mathcal{M}^1_{\mathbb{P}}(\Sigma_\Omega,F)} \left\{h_\rho(F) + \int q\Phi \, \mathrm{d}\rho\right\} \\ &\leq \sup_{\rho\in\mathcal{M}^1_{\mathbb{P}}(\Sigma_\Omega,F)} \left\{\int q\Phi \, \mathrm{d}\rho\right\} + \sup_{\rho\in\mathcal{M}^1_{\mathbb{P}}(\Sigma_\Omega,F)} \{h_\rho(F)\} \\ &\leq -2q\varpi + \sup_{\rho\in\mathcal{M}^1_{\mathbb{P}}(\Sigma_\Omega,F)} \{h_\rho(F)\} \end{aligned}$$

Since $\sup_{\rho\in\mathcal{M}^1_{\mathbb{P}}(\Sigma_\Omega,F)}\{h_\rho(F)\} = \int \log l \, \mathrm{d}\mathbb{P} < \infty$, letting q tend to ∞ we get a contradiction.

Chapter 4

Proof of Bowen's formula

Here we first explain that for $\mathbb{P}$-almost every $\omega \in \Omega$, the set X_ω is either equal to $[0,1]$ or totally disconnected.

From the construction of X_ω we deduce that setting $\Omega' = \{\omega \in \Omega : X_\omega \neq [0,1]\}$, then we have

$$\Omega' = \{\omega \in \Omega : \text{ there exists a non trivial open interval } I_\omega \subset [0,1] \text{ such that } I_\omega \cap X_\omega = \emptyset\}.$$

For each $\omega \in \Omega'$, fix a nontrivial open interval I_ω such that $I_\omega \cap X_\omega = \emptyset$.

If $\mathbb{P}(\Omega') > 0$, for $\mathbb{P}$-almost every $\omega \in \Omega$

$$\lim_{n\to\infty} \frac{\sharp\{i < n : \sigma^i\omega \in \Omega'\}}{n} = \mathbb{P}(\Omega').$$

Then, for any $n \in \mathbb{N}$, for any $v = v_0 v_1 \cdots v_{n-1} \in \Sigma_{\omega,n}$, the interval U_ω^v contains an interval I such that $I \cap X_\omega = \emptyset$. Indeed, since we can find that $k > n + M(\sigma^n\omega)$ such that $\sigma^k\omega \in \Omega'$, i.e. $I_{\sigma^k\omega} \cap X_{\sigma^k\omega} = \emptyset$, for each $v = v_0 v_1 \cdots v_{n-1} \in \Sigma_{\omega,n}$ we can find a word v' such that $w = vv' \in \Sigma_{\omega,k}$ and the nontrivial interval

$$I = g_\omega^{w_0} \circ g_{\sigma\omega}^{v_1} \circ \cdots \circ g_{\sigma^{k-1}\omega}^{w_{k-1}}(I_{\sigma^k\omega}) \subset U_\omega^w (\subset U_\omega^v)$$

does not intersect $g_\omega^{w_0} \circ g_{\sigma\omega}^{v_1} \circ \cdots \circ g_{\sigma^{k-1}\omega}^{w_{k-1}}(X_{\sigma^k\omega}) = X_\omega \cap U_\omega^w$, hence it does not intersect $X_\omega \cap U_\omega^v$. Consequently, $\mathbb{P}$-almost surely, the set $[0,1] \setminus X_\omega$ is open and everywhere dense, i.e. the closed set X_ω is totally disconnected. Moreover, $\mathbb{P}(\Omega') = 1$.

The case $P(\Omega') = 0$ occurs only if for $\mathbb{P}$-almost every ω the intervals U_ω^s, $1 \leq s \leq l(\omega)$ form a covering of the interval $[0,1]$, and simultaneously the matrix $A(\omega)$ is positive.

Now let us come to the proof of theorem 1.2. At first we notice that the uniqueness of t_0 comes from the fact that $t \mapsto P(t\Psi)$ is decreasing because $c_\psi > 0$.

Upper bound: For any $s > 0$ and $\delta > 0$ denote by $\mathcal{H}_\delta^s$ the s-dimensional Hausdorff pre-measure computed using coverings by set of diameter less than δ. Let

$\delta_n := \sup_{v\in\Sigma_{\omega,n}} |X_\omega^v|$. Then

$$\mathcal{H}^s_{\delta_n}(X_\omega) \leq \sum_{v\in\Sigma_{\omega,n}} |X_\omega^v|^s.$$

As is shown in proposition 2.7, for any $n \in \mathbb{N}$, for any $v \in \Sigma_{\omega,n}$ we have $|X_\omega^v| \leq |U_\omega^v| \leq \exp(\Psi(\omega, \underline{v}) + o(n))$, so

$$\mathcal{H}^s_{\delta_n}(X_\omega) \leq \sum_{v\in\Sigma_{\omega,n}} \exp(sS_{n_k}\Psi(\omega, \underline{v}) + o(n)).$$

But $P(s\Psi) < 0$ when $s > t_0$. Then, from lemma 2.2, for $\mathbb{P}$-almost every $\omega \in \Omega$, for n large enough we get $\sum_{v\in\Sigma_{\omega,n}} \exp(sS_n\Psi(\omega, \underline{v}) + o(n)) \leq \exp(n\frac{P(s\Psi)}{2})$. This implies $\mathcal{H}^s(X_\omega) = 0$. So $\dim_H X_\omega \leq s$. Since $s > t_0$ is arbitrary, we get that $\dim_H X_\omega \leq t_0$.

Lower bound: If $t_0 = 0$, since for $\mathbb{P}$-almost every $\omega \in \Omega$ the set X_ω is not empty, there is nothing to prove.

Suppose that $t_0 > 0$: Since $\psi \in \mathbb{L}^1_{X_\Omega}(\Omega, \widetilde{C}([0,1]))$, $t_0\psi \in \mathbb{L}^1_{X_\omega}(\Omega, \widetilde{C}([0,1]))$ as well. For the potential $t_0\Psi$, we can consider the projection $\{\mu_\omega^{t_0\psi}\}$ of the associated weak Gibbs measure $\{\widetilde{\mu}_\omega^{t_0\Psi}\}$. We want to prove that $\dim_H(\mu_\omega^{t_0\psi}) \geq t_0$. If so, since $\mu_\omega^{t_0\psi}(X_\omega) = 1$, we get $\dim_H X_\omega \geq t_0$.

First of all, since $P(t_0\Psi) = 0$, by proposition 2.7(1) we can get that for any $n \geq 1$, $v \in \Sigma_{\omega,n}$ and $\underline{v} \in [v]_\omega$, we have

$$\exp(S_n(t_0\Psi)(\omega, \underline{v}) - o(n)) \leq \widetilde{\mu}_\omega^{t_0\psi}([v]_\omega) \leq \exp(S_n(t_0\Psi)(\omega, \underline{v}) + o(n)).$$

Since $\sup_{\underline{v}\in\Sigma_\omega} S_n(t_0\Psi)(\omega, v)$ tends to $-\infty$ as $n \to \infty$, we conclude that $\widetilde{\mu}^{t_0\psi}$ is atomless. Consequently,

$$\exp(S_n(t_0\Psi)(\omega, \underline{v}) - o(n)) \leq \mu_\omega^{t_0\Psi}(U_\omega^v) \leq \exp(S_n(t_0\Psi)(\omega, \underline{v}) + o(n)).$$

Define $V(r) = \{v \in \Sigma_{\omega,*} : |U_\omega^v| \geq r, \exists\, vs \in \Sigma_{\omega,|v|+1}, |U_\omega^{vs}| < r\}$. We have $\sup_{v\in V(r)} |v| = O(-\log r)$, and for every $x \in X_\omega$, for r small enough, there exist two words $v, v' \in V(r)$ such that

$$(B(x, r/2) \cap X_\omega) \subset (U_\omega^v \cup U_\omega^{v'}).$$

Then proposition 2.7(2) yields

$$\begin{aligned} \mu_\omega^{t_0\psi}(B(x, r/2)) &\leq \mu_{t_0\psi,\omega}(U_\omega^v) + \mu_{t_0\psi,\omega}(U_\omega^{v'}) \\ &\leq \exp(S_{|v|}(t_0\Psi)(\omega, \underline{v}) + o(|v|)) + \exp(S_{|v'|}(t_0\Psi)(\omega, \underline{v'}) + o(|v'|)) \end{aligned}$$

for any $\underline{v} \in [v]_\omega$ and $\underline{v'} \in [v']_\omega$, where $o(|v|)$ and $o(|v'|)$ depend on ω and $t_0\Psi$ only. Thus

$$\mu_\omega^{t_0\psi}(B(x, r/2)) \leq \exp(S_{|v|}(t_0\Psi)(\omega, \underline{v}) + o(|v|)) + \exp(S_{|v'|}(t_0\Psi)(\omega, \underline{v'}) + o(|v'|))$$

$$
\begin{aligned}
&\leq \exp(S_{|v|+1}(t_0\Psi)(\omega,\underline{v}) + o(|v|)) + \exp(S_{|v'|+1}(t_0\Psi)(\omega,\underline{v'}) + o(|v'|)) \\
&\leq (|U_\omega^{vs}|^{t_0}\exp(o(|v|))) + (|U_\omega^{v's'}|^{t_0}\exp(o|v'|)) \\
&\leq r^{t_0}\exp(o(-\log r)).
\end{aligned}
$$

It follows that $\liminf_{r\to 0}\frac{\log\mu_\omega^{t_0\psi}(B(x,r))}{\log r} \geq t_0$, hence $\dim_H(\mu_\omega^{t_0\psi}) \geq t_0$.

For the equation $t_0 = \sup_{\rho\in\mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)}\left\{\frac{h_\rho(F)}{-\int\Psi\mathrm{d}\rho}\right\}$, it will follow from proposition 5.3.

Chapter 5

Approximation of (Φ, Ψ) by random Hölder potentials and related properties

We mainly introduce objects and related properties which will be used in the next chapters. Also, we explain the variational formulas appearing in Bowen's formula and in the statement of theorem 1.11.

5.1 Approximation of (Φ, Ψ) by random Hölder potentials

Now we approximate the potentials Φ and Ψ associated with $\{\mu_\omega\}_{\omega\in\Omega}$ and $\{X_\omega\}_{\omega\in\Omega}$ by more regular potentials: for any $i \geq 1$, for any $\omega \in \Omega$ for any $\underline{v} = v_0 v_1 \cdots v_i \cdots \in [v]_\omega \subset \Sigma_\omega$ with $v = v_0 v_1 \cdots v_{i-1} \in \Sigma_{\omega,i-1}$ define

$$\Phi_i(\omega, \underline{v}) = \frac{\max\{\Phi(\omega, \underline{w}), \underline{w} \in [v]_\omega\} + \min\{\Phi(\omega, \underline{w}), \underline{w} \in [v]_\omega\}}{2},$$

$$\Psi_i(\omega, \underline{v}) = \frac{\max\{\Psi(\omega, \underline{w}), \underline{w} \in [v]_\omega\} + \min\{\Psi(\omega, \underline{w}), \underline{w} \in [v]_\omega\}}{2}.$$

These functions Φ_i and Ψ_i are piecewise constant with respect to the second variable. They are random Hölder continuous potentials. If we take

$$K_{\Phi_i}(\omega) = (2 \sup_{\underline{v}\in\Sigma_\omega} |\Phi(\omega, \underline{v})| + 1)e^i \quad \text{and} \quad \kappa = 1,$$

then

$$\text{var}_n \Phi_i(\omega) \leq 2 \sup_{\underline{v}\in\Sigma_\omega} |\Phi(\omega, \underline{v})| \leq K_{\Phi_i}(\omega) \exp(-n) \quad \text{if } n \leq i$$

$$\text{var}_n \Phi_i(\omega) = 0 \quad \text{if } n > i$$

Furthermore

$$\log((2\sup_{\underline{v}\in\Sigma_\omega}|\Phi(\omega,\underline{v})|+1)e^i)\le i+2\sup_{\underline{v}\in\Sigma_\omega}|\Phi(\omega,\underline{v})|,$$

and the right hand side is integrable since $\Phi\in\mathbb{L}^1_{\Sigma_\Omega}(\Omega,C(\Sigma))$.

Also, since for $\mathbb{P}$-almost every ω we have $\mathrm{var}_n\Phi(\omega)\to 0$ as $n\to+\infty$, and $\|\Phi(\omega)-\Phi_i(\omega)\|_\infty\le\mathrm{var}_i\Phi(\omega)$, we have $\Phi_i\to\Phi$ uniformly as $i\to\infty$ for $\mathbb{P}$-almost every ω. The same property holds for Ψ_i and Ψ. Consequently, without loss of generality we can also assume that $P(\Phi_i)=0$ since $P(\Phi_i)$ converges to $P(\Phi)$ as i tends to $+\infty$.

5.2 Approximation of (T,T^*) by (T_i,T_i^*)

Due to our assumptions on (Φ,Ψ) and the definition of $(\Phi_i,\Psi_i)_{i\in\mathbb{N}}$, we have $c_{\Psi_i}<0$, hence for the same reason as for (Φ,Ψ), for any $q\in\mathbb{R}$, for any $i\in\mathbb{N}$, there exists a unique $T_i(q)$ such that $P(q\Phi_i-T_i(q)\Psi_i)=0$ and the function T_i is concave and non-decreasing. Also, the function T_i is differentiable since for Hölder potentials the associated random Gibbs measure is the unique invariant measure that maximizes the variation principle (see [35, 61].)

Lemma 5.1 *For any $q\in\mathbb{R}$, one has that $T_i(q)\to T(q)$ as $i\to\infty$.*

Proof At first, we recall that for any $\Phi\in\mathbb{L}^1_{\Sigma_\Omega}(\Omega,C(\Sigma))$ one has

$$P(\Phi)=\sup_{\rho\in\mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)}\{h_F(\rho|\mathbb{P})+\int\Phi\,\mathrm{d}\rho\}.$$

Also, for any $q\in\mathbb{R}$, we have $P(q\Phi-T(q)\Psi)=P(q\Phi_i-T_i(q)\Psi_i)=0$. Thus

$$\inf_{\rho\in\mathcal{I}_{\mathbb{P}}}\Big(\int\big[q(\Phi-\Phi_i)-T(q)(\Psi-\Psi_i)-(T(q)-T_i(q))\Psi_i\big]\,\mathrm{d}\rho\Big)\le 0,\qquad(5.1)$$

and

$$\sup_{\rho\in\mathcal{I}_{\mathbb{P}}}\Big(\int\big[q(\Phi-\Phi_i)-T(q)(\Psi-\Psi_i)-(T(q)-T_i(q))\Psi_i\big]\,\mathrm{d}\rho\Big)\ge 0.\qquad(5.2)$$

The inequality (5.1) implies that for any $\varepsilon>0$, there exists a measure $\rho\in\mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)$ such that

$$\int\big[q(\Phi-\Phi_i)-T(q)(\Psi-\Psi_i)-(T(q)-T_i(q))\Psi_i\big]\,\mathrm{d}\rho\le\varepsilon.$$

Then

$$\int(T(q)-T_i(q))\Psi_i\,\mathrm{d}\rho\ge\int q(\Phi-\Phi_i)-T(q)(\Psi-\Psi_i)\,\mathrm{d}\rho-\varepsilon,$$

and

$$\begin{aligned}(T(q) - T_i(q)) &\leq \frac{\int q(\Phi - \Phi_i) - T(q)(\Psi - \Psi_i)d\rho - \varepsilon}{\int(\Psi_i)\,\mathrm{d}\rho} \\ &\leq \frac{\int -|q|(\mathrm{var}_i\Phi) - |T(q)|(\mathrm{var}_i\Psi)\,\mathrm{d}\mathbb{P} - \varepsilon}{\int(\Psi_i)\,\mathrm{d}\rho} \\ &\leq \frac{\int -|q|(\mathrm{var}_i\Phi) - |T(q)|(\mathrm{var}_i\Psi)\,\mathrm{d}\mathbb{P} - \varepsilon}{-c_\Psi}\end{aligned}$$

since $\int(\Psi_i)\,\mathrm{d}\rho \leq -c_\Psi < 0$. Letting $i \to \infty$, from the arbitrariness of ε we get

$$\liminf_{i\to\infty} T_i(q) \geq T(q).$$

Using (5.2) similarly we can get $\limsup_{i\to\infty} T_i(q) \leq T(q)$. Finally $\lim_{i\to\infty} T_i(q) = T(q)$.

Proposition 5.2 *Let $T : \mathbb{R} \to \mathbb{R}$ be a concave function. Suppose that $(T_i)_{i\geq 1}$ is a sequence of differentiable concave functions from $\mathbb{R}$ to $\mathbb{R}$ which converges pointwise to T. Then $(T_i^*)_{i\geq 1}$ converges pointwise to T^* over the interior of the domain of T^*.*

Proof Let α be an interior point of $\mathrm{dom}(T^*)$. Let $q_\alpha \in \mathbb{R}$ be the unique point such that $\alpha \in [T'(q_\alpha+), T'(q_\alpha-)]$, and $T^*(\alpha) = \alpha q_\alpha - T(q_\alpha)$.

By [27, proposition 2.5(i)], there exists a sequence $(q_i)_{i\geq 1}$ such that for i large enough one has $T_i'(q_i) = \alpha$. Without loss of generality we can assume that this sequence converges to $q_0' \in \mathbb{R}$ or diverges to $-\infty$ or ∞.

Suppose first that it converges to $q_0' \in \mathbb{R}$. If $q_0' = q_\alpha$ then we are done since $(T_i)_{i\geq 1}$ converges uniformly on compact sets. Suppose that $q_0' \neq q_\alpha$ and $q_0' > q_\alpha$. Using the uniform convergence of $(T_i)_{i\geq 1}$ in a compact neighborhood of $[q_\alpha, q_0']$ and the inequality $T_i(q) \leq T_i(q_i) + T_i'(q_i)(q - q_i)$ (T_i is concave), we can get $T(q_\alpha) \leq T(q_0') + \alpha(q_\alpha - q_0')$. On the other hand, T being concave we have $T(q_\alpha) + T'(q_\alpha+)(q_0' - q_\alpha) \geq T(q_0')$ and $T'(q_\alpha+) \leq \alpha$. This implies that $\alpha = T'(q_\alpha+)$ hence $T^*(\alpha) = \alpha q_\alpha - T(q_\alpha) = \alpha q_0' - T(q_0') = \lim_{i\to\infty}(\alpha q_i - T_i(q_i) = T_i^*(\alpha))$.

The case $q_0' \neq q_\alpha$ and $q_0' < q_\alpha$ is similar. Now suppose that $(q_i)_{i\geq 1}$ diverges to ∞ (the case where it diverges to $-\infty$ is similar). If T is affine over $[q_\alpha, \infty)$ with slope α, α is not an interior point of $\mathrm{dom}(T^*)$. Consequently, there exists q_0' and $\epsilon > 0$ such that $T'(q_0'+) < \alpha - \epsilon$, and $T(q) \leq T(q_0') + (\alpha - \epsilon)(q - q_0')$ for all $q \geq q_0'$. On the other hand, since T_i' is non increasing for all i, for i large enough we have $T_i(q) \geq T_i(q_0') + \alpha(q - q_0')$ for all $q \in [q_0', q_i]$. Since $(q_i)_{i\geq 1}$ diverges to ∞, this contradicts the convergence of $(T_i)_{i\geq 1}$ to T.

5.3 Explanation of some variational formulas

Proposition 5.3 *If Assumption 1 holds, then*

$$t_0 = \max_{\rho\in\mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)} \left\{ \frac{h_\rho(F)}{-\int \Psi \mathrm{d}\rho} \right\} \tag{5.3}$$

and

$$T(q) = \min_{\rho\in\mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)} \left\{ \frac{h_\rho(F) + q\int \Phi \mathrm{d}\rho}{\int \Psi \mathrm{d}\rho} \right\}. \tag{5.4}$$

Furthermore, for any $d \in (T'(+\infty), T'(-\infty))$,

$$T^*(d) = \max_{\rho\in\mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)} \left\{ -\frac{h_\rho(F)}{\int \Psi \mathrm{d}\rho} : \frac{\int \Phi \mathrm{d}\rho}{\int \Psi \mathrm{d}\rho} = d \right\}. \tag{5.5}$$

Proof For equation (5.3), since $P(t_0\Psi) = 0$ and $\int \Psi \mathrm{d}\rho < 0$ for any $\rho \in \mathcal{I}_{\mathbb{P}}$, then for any $\rho \in \mathcal{I}_{\mathbb{P}}$ we have

$$h_\rho(F) + t_0 \int \Psi \mathrm{d}\rho \leq 0,$$

hence

$$t_0 \geq \frac{h_\rho(F)}{-\int \Psi \mathrm{d}\rho}.$$

On the other hand, for any $\epsilon > 0$, there exists $\rho \in \mathcal{I}_{\mathbb{P}}$ such that

$$h_\rho(F) + t_0 \int \Psi \mathrm{d}\rho \geq -\epsilon,$$

so

$$t_0 \leq \frac{h_\rho(F) + \epsilon}{-\int \Psi \mathrm{d}\rho}.$$

Letting ϵ tend to 0 yields $t_0 = \sup_{\rho\in\mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)} \left\{ \frac{h_\rho(F)}{-\int \Psi \mathrm{d}\rho} \right\}$. Finally we can get equation (5.3) from the fact that $\mathcal{I}_{\mathbb{P}}$ is compact under the weak* topology and the entropy map $\rho \mapsto h_\rho(F)$ is upper semi-continuous [49, subsection 4.1].

For equation (5.4), it is almost the same as for equation (5.3).

Regarding the equation (5.5), on the one hand, for any $d \in \mathbb{R}$,

$$\begin{aligned} T^*(d) &= \inf_{q\in\mathbb{R}} \{qd - T(q)\} = \inf_{q\in\mathbb{R}} \left\{ qd - \inf_{\rho\in\mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)} \left\{ \frac{h_\rho(F) + q\int \Phi \mathrm{d}\rho}{\int \Psi \mathrm{d}\rho} \right\} \right\} \\ &= \inf_{q\in\mathbb{R}} \left\{ \sup_{\rho\in\mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)} \left\{ \frac{-h_\rho(F) - q\int \Phi \mathrm{d}\rho}{\int \Psi \mathrm{d}\rho} + qd \right\} \right\} \\ &= \inf_{q\in\mathbb{R}} \left\{ \sup_{\rho\in\mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)} \left\{ \frac{-h_\rho(F)}{\int \Psi \mathrm{d}\rho} + q\left(d - \frac{\int \Phi \mathrm{d}\rho}{\int \Psi \mathrm{d}\rho}\right) \right\} \right\} \end{aligned}$$

$$\begin{aligned} &\geq \sup_{\rho \in \mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)} \left\{ \inf_{q \in \mathbb{R}} \left\{ \frac{-h_\rho(F)}{\int \Psi \mathrm{d}\rho} + q \left(d - \frac{\int \Phi \mathrm{d}\rho}{\int \Psi \mathrm{d}\rho} \right) \right\} \right\} \\ &= \sup_{\rho \in \mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)} \left\{ \frac{-h_\rho(F)}{\int \Psi \mathrm{d}\rho} : \frac{\int \Phi \mathrm{d}\rho}{\int \Psi \mathrm{d}\rho} = d \right\}. \end{aligned}$$

On the other hand, for any $d \in (T'(+\infty), T'(-\infty))$, by the proof of proposition 5.2, there exists i large enough and $q_i \in \mathbb{R}$ such that $T_i'(q_i) = d$ and $P(q_i\Phi_i - T_i(q_i)\Psi) = 0$. Then there exists $\rho_i \in \mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)$ such that

$$h_{\rho_i}(F) + \int (q_i\Phi_i - T_i(q_i)\Psi_i)\mathrm{d}\rho = 0$$

and

$$T_i'(q_i) = d = \frac{\int \Phi_i \mathrm{d}\rho_i}{\int \Psi_i \mathrm{d}\rho_i}.$$

This implies that there exists $\rho_i \in \mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)$ such that

$$\frac{h_{\rho_i}(F)}{\int \Psi_i \mathrm{d}\rho_i} = T_i^*(d)$$

and

$$d = \frac{\int \Phi_i \mathrm{d}\rho_i}{\int \Psi_i \mathrm{d}\rho_i}.$$

Proposition 5.2 tells us that $T_i^*(d) \to T^*(d)$ as $i \to \infty$ and $\mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)$ is compact for the wear-star topology. Thus, there exists a limit point ρ' of (ρ_i) in $\mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)$ such that

$$\frac{h_{\rho'}(F)}{\int \Psi \mathrm{d}\rho} \geq T^*(d) \text{ and } d = \frac{\int \Phi \mathrm{d}\rho'}{\int \Psi \mathrm{d}\rho'},$$

since the entropy map is upper semi-continuous and (Φ_i, Ψ_i) converges uniformly to (Φ, Ψ). Finally, we get

$$T^*(d) = \max_{\rho \in \mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)} \left\{ -\frac{h_\rho(F)}{\int \Psi \mathrm{d}\rho} : \frac{\int \Phi \mathrm{d}\rho}{\int \Psi \mathrm{d}\rho} = d \right\}.$$

The case $d \in \{T'(+\infty), T'(-\infty)\}$ now follows by approximating d by a sequence $(d_k)_{k\geq 0}$ of elements of $(T'(+\infty), T'(-\infty))$ and for each k picking ρ_k which realizes $\max_{\rho \in \mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)} \left\{ -\frac{h_\rho(F)}{\int \Psi \mathrm{d}\rho} : \frac{\int \Phi \mathrm{d}\rho}{\int \Psi \mathrm{d}\rho} = d_k \right\}$. Then, since T^* is continuous at d (it is lower semi-continuous as a concave function and upper semi-continuous as a Legendre transform), any limit point of $(\rho_k)_{k\geq 0}$ is such that $-\frac{h_\rho(F)}{\int \Psi \mathrm{d}\rho} = T^*(d)$ and $\frac{\int \Phi \mathrm{d}\rho}{\int \Psi \mathrm{d}\rho} = d$. It exists since $\mathcal{I}_{\mathbb{P}}(\Sigma_\Omega)$ is compact in the weak* topology (see [47, 49]).

5.4 Simultaneous control for random Gibbs measures associated with (Φ_i, Ψ_i)

In this quite technical section, we prepare the "concatenation of random Gibbs measures" approach that will be used in the next chapters to built auxiliary measures with nice properties. We also show an almost everywhere almost doubling property for projections of random Gibbs measures on the random attractor X_ω.

Let D be a dense and countable subset of $(T'(+\infty), T'(-\infty))$. Let $\{D_i\}_{i\in\mathbb{N}}$ be a sequence of sets such that

- D_i is a finite set for each $i \in \mathbb{N}$,
- $D_i \subset D_{i+1}$, for each $i \in \mathbb{N}$,
- $\cup_{i\in\mathbb{N}} D_i = D$.

Fix a sequence $\{\varepsilon_i\}_{i\in\mathbb{N}}$ decreasing to 0 as $i \to \infty$. We saw in the proof of proposition 5.2 that for any $i \in \mathbb{N}$, there exists j_i large enough such that for any $d \in D_i$, there exists $q_i \in \mathbb{R}$ such that

1. $T'_{j_i}(q_i) = d$,
2. $|T^*_{j_i}(d) - T^*(d)| \leq \varepsilon_i$.
3. $\int_\Omega \mathrm{var}_{j_i}\Phi \; d\mathbb{P} \leq \varepsilon_i^3$ and $\int_\Omega \mathrm{var}_{j_i}\Psi \; d\mathbb{P} \leq \varepsilon_i^3$.

We can also assume that $j_{i+1} > j_i$ for each $i \in \mathbb{N}$. We set

$$Q_i = \{q_i, d_i \in D_i\}.$$

For any $q \in Q_i$, we define

$$\Lambda_{i,q} = q\Phi_{j_i} - T_{j_i}(q)\Psi_{j_i}.$$

Recall (2.1) and proposition 2.7. For any $\epsilon > 0$, there exist positive integers M, L, N, C (large enough) such that there is a set Ω_0 and a sequence $\{c_n\}_{n\geq 1}$ decreasing to 0 as $n \to \infty$ such that: $\mathbb{P}(\Omega_0) > 1 - \epsilon/4$, and for any $\omega \in \Omega_0$, one has:

- $M(\omega) < M, \; l(\omega) \leq L$,
- for any $n \geq 1$,

  $$\max(V_n\Phi(\omega), V_n\Psi(\omega)) \leq nc_n$$

  and

  $$\max\{\epsilon(\psi, n) = \epsilon(\Psi, n), \epsilon(\phi, n) = \epsilon(\Phi, n)\} \leq c_n,$$

  where we have used Egorov's theorem;

- for all $n \geq N$,

$$\begin{aligned}
\left|S_n \mathrm{var}_{j_i}\Phi(\omega) - n\int_\Omega \mathrm{var}_{j_i}\Phi(\omega)\ d\mathbb{P}\right| &\leq nc_n, \\
\left|S_n \mathrm{var}_{j_i}\Psi(\omega) - n\int_\Omega \mathrm{var}_{j_i}\Psi(\omega)\ d\mathbb{P}\right| &\leq nc_n, \\
\left|\frac{1}{n}S_n(\log l)(\omega)\right| &\leq C, \\
\max\left(\frac{1}{n}S_n\|\Phi(\omega)\|_\infty,\ \frac{1}{n}S_n\|\Phi(\sigma^{-n+1}\omega)\|_\infty\right) &\leq C, \\
\max\left(\frac{1}{n}S_n\|\Psi(\omega)\|_\infty,\ \frac{1}{n}S_n\|\Psi(\sigma^{-n+1}\omega)\|_\infty\right) &\leq C.
\end{aligned}$$

and

$$\sup_{\underline{v}\in[v]_\omega} S_n\Psi(\omega,\underline{v}) \leq (-n\varpi_\Psi),\ \forall v\in\Sigma_{\omega,n},$$

where we have applied ergodic theorem to $\mathrm{var}_{j_i}\Phi$, $\log l$, $\|\Phi(\cdot)\|_\infty$ and $\|\Psi(\cdot)\|_\infty$, (2.3) and Egorov's theorem again.

Given a finite set Q, we know that for $\mathbb{P}$-almost every ω, for s large enough one has $\sharp\Sigma_{\omega,s} \geq \sharp Q$. Denote the smallest such s by $\mathcal{S}(\omega, \sharp Q)$.

For any $i \in \mathbb{N}$, choose $\mathcal{S}(i) \in \mathbb{N}$ large enough such that there exists a set $\Omega'(i) \subset \Omega_0$ such that

- $\mathbb{P}(\Omega'(i)) \geq 1 - 2\epsilon/4$ and for any $\omega \in \Omega'(i)$, one has $\mathcal{S}(\sigma^M\omega, \sharp Q_i) \leq \mathcal{S}(i)$, where M has been fixed above.

Also, for all $i \in \mathbb{N}$, there exist $\varpi_i > 0$ and integers $N(i) > N$ and $M(i) \geq M$ large enough, as well as a positive sequence $\{c_{i,n}\}_{n\geq 1}$ converging to 0 as $n \to \infty$, and a set $\Omega(i) \subset \Omega'(i)$ such that $\mathbb{P}(\Omega(i)) \geq 1 - 3\epsilon/4$, and for any $\omega \in \Omega(i)$, one has:

- $M(\sigma^{M+\mathcal{S}(i)}\omega) \leq M(i)$;
- for any $q \in Q_i$, the random Gibbs measure $\{\tilde{\mu}^{\Lambda_{i,q}}_{\sigma^{M+\mathcal{S}(i)+M(i)}\omega}\}_{\omega\in\Omega}$ is well defined, and for any $n \geq N(i)$

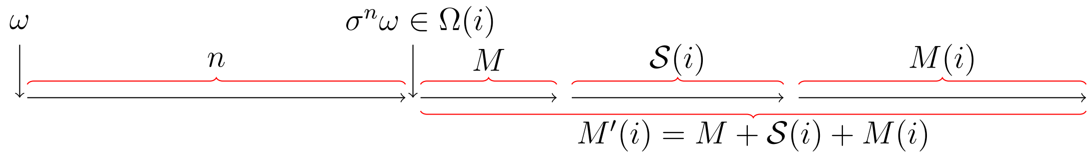
$$V_n\Lambda_{i,q}(\sigma^{M+\mathcal{S}(i)+M(i)}\omega) \leq nc_{i,n} \text{ and } \epsilon(\Lambda_{i,q}, n) \leq c_{i,n};$$

- for all $n \geq N(i)$, for all $v \in \Sigma_{\sigma^{M+\mathcal{S}(i)+M(i)}\omega,n}$, for any $q \in Q_i$,

$$\sup_{\underline{v}\in[v]_{\sigma^{M+\mathcal{S}(i)+M(i)}\omega}} S_n\Lambda_{i,q}(\sigma^{M+\mathcal{S}(i)+M(i)}\omega,\underline{v}) \leq (-n\varpi_i),$$

where again we used the finiteness of Q_i and again (2.1), proposition 2.7, proposition 3.2 and Egorov's theorem.

Here we draw a picture to illustrate some of the parameters coming into play.



Let $\theta'(i, \omega, s)$ be the s-th return time of the point ω to the set $\Omega(i)$ under the map σ, that is

$$\theta'(i, \omega, 1) = \inf\{n \in \mathbb{N} \cup \{0\} : \sigma^n\omega \in \Omega(i)\},$$

and for any $s \in \mathbb{N}$ and $s \geq 2$,

$$\theta'(i, \omega, s) = \inf\{n \in \mathbb{N} : n > \theta'(i, \omega, s-1),\ \sigma^n\omega \in \Omega(i)\}.$$

Then for any $i \in \mathbb{N}$

$$\lim_{s\to\infty} \frac{\theta'(i, \omega, s)}{s} = \frac{1}{\mathbb{P}(\Omega(i))}$$

for $\mathbb{P}$-almost every ω. Consequently,

$$\lim_{k\to\infty} \frac{\theta'(i, \omega, s) - \theta'(i, \omega, s-1)}{\theta'(i, \omega, s)} = 0. \tag{5.6}$$

Since $\mathbb{N}$ is countable, there exists $\widetilde{\Omega}' \subset \Omega$ of full probability such that for all $\omega \in \widetilde{\Omega}'$, for any $i \in \mathbb{N}$, we have

$$\lim_{s\to\infty} \frac{\theta'(i, \omega, s)}{s} = \frac{1}{\mathbb{P}(\Omega(i))},$$

hence

$$\lim_{s\to\infty} \frac{\theta'(i, \omega, s) - \theta'(i, \omega, s-1)}{\theta'(i, \omega, s-1)} = 0.$$

Let $M'(i) = M + \mathcal{S}(i) + M(i)$. Given $\omega \in \Omega(i)$, let

$$n_1^i(\omega) = \inf\{\theta'(i, \omega, s) : \theta'(i, \omega, s) \geq M'(i)\} - M'(i).$$

For $k \geq 2$, define $n_k^i(\omega) = \theta'(i, \omega, s_k) - M'(i)$, where s_k is the smallest s such that the following hold:

$$\theta'(i, \omega, s) - n_{k-1}^i(\omega) \geq \max\left(M'(i), n_{k-1}^i(\omega)(c_{i,n_{k-1}^i})^{\frac{1}{3}} + \sqrt{\theta'(i, \omega, s)}\right).$$

It is easy to show that

$$\lim_{k\to\infty} \frac{n_k^i(\omega) - n_{k-1}^i(\omega)}{n_{k-1}^i(\omega)} = 0.$$

Now we prove an almost everywhere almost doubling property for the Gibbs measures $\mu_\omega^{\Lambda_{i,q}}$.

For $v \in \Sigma_{\sigma^{M'(i)}\omega,n}$, we denote by $U^{v+}_{\sigma^{M'(i)}\omega}$ and $U^{v-}_{\sigma^{M'(i)}\omega}$ the two intervals of the n-th generation of the construction of X_ω which are neighboring $U^v_{\sigma^{M'(i)}\omega}$, whenever $U^v_{\sigma^{M'(i)}\omega}$ is neither the leftmost nor the rightmost of the whole collection, and with the convention that $U^{v-}_{\sigma^{M'(i)}\omega}$ is on the left of $U^v_{\sigma^{M'(i)}\omega}$.

We say that $B(i, \sigma^{M'(i)}\omega, k, v)$ holds if $v \in \Sigma_{\sigma^{M'(i)}\omega, n_k^i}$, and $|v \wedge v+| \le n_{k-1}^i$ or $|v \wedge v-| \le n_{k-1}^i$.

Let

$$\mathcal{U}(i, \sigma^{M'(i)}\omega, k) = \bigcup_{v\in\Sigma_{\sigma^{M'(i)}\omega,n_k^i} :\, B(i,\sigma^{M'(i)}\omega,k,v) \text{ holds}} U^v_{\sigma^{M'(i)}\omega}.$$

Lemma 5.4 *For all $i \in \mathbb{N}$, for all $\omega \in \Omega(i)$, for all $q \in Q_i$, we have*

$$\mu^{\Lambda_{i,q}}_{\sigma^{M'(i)}\omega}\Big(\bigcap_{N=1}\bigcup_{k\ge N} \mathcal{U}(i, \sigma^{M'(i)}\omega, k)\Big) = 0.$$

Proof For any v and v' such that $|v| = n_{k-1}^i, |v'| = n_k^i - n_{k-1}^i$ and $vv' \in \Sigma_{\sigma^{M'(i)}\omega, n_k^i}$, by construction of $\mu^{\Lambda_{i,q}}_{\sigma^{M'(i)}\omega}$ one has

$$\frac{\mu^{\Lambda_{i,q}}_{\sigma^{M'(i)}\omega}(U^{vv'}_{\sigma^{M'(i)}\omega})}{\mu^{\Lambda_{i,q}}_{\sigma^{M'(i)}\omega}(U^{v}_{\sigma^{M'(i)}\omega})} \le \exp(-(n_k^i - n_{k-1}^i)\varpi_i + 4n_k^i c_{i,n_k^i}).$$

We will use this fact to calculate the measure of $\mathcal{U}(i, \sigma^{M'(i)}\omega, k)$. Notice that for any $v \in \Sigma_{\sigma^{M'(i)}\omega, n_{k-1}^i}$, there are at most two v' such that $vv' \in \Sigma_{\sigma^{M'(i)}\omega, n_k^i}$ and $B(i, \sigma^{M'(i)}\omega, k, vv')$ holds. Consequently,

$$\mu^{\Lambda_{i,q}}_{\sigma^{M'(i)}\omega}(\mathcal{U}(i, \sigma^{M'(i)}\omega, k)) \le 2\exp(-(n_k^i - n_{k-1}^i)\varpi_i + 4n_k^i c_{i,n_k^i}).$$

Since $\varpi_i > 0$ and

$$n_k^i - n_{k-1}^i > n_{k-1}^i (c_{n_k^i})^{1/3} + \sqrt{n_k^i},$$

we get

$$\sum_{k=1}^{\infty} \mu^{\Lambda_{i,q}}_{\sigma^{M'(i)}\omega}(\mathcal{U}(i, \sigma^{M'(i)}\omega, k)) < +\infty.$$

By Borel-Cantelli's lemma we get $\mu^{\Lambda_{i,q}}_{\sigma^{M'(i)}\omega}(\bigcap_{N=1}\bigcup_{k\ge N} \mathcal{U}(i, \sigma^{M'(i)}\omega, k)) = 0.$

For any $\varepsilon > 0$, $\beta \geq 0$, and $k, p \geq 1$ we now define the following sets:

$$F_{i,\beta,k}(\sigma^{M'(i)}\omega, \varepsilon) = \left\{ \begin{array}{l} x \in X_{\sigma^{M'(i)}\omega} : \forall \gamma \in \{-1,1\}, \forall v \in \Sigma_{\sigma^{M'(i)}\omega, n_k^i} \text{ satisfying} \\ |v \wedge x|_{n_k^i}| \geq n_{k-1}^i, \text{ for any } \underline{v} \in [v]_{\sigma^{M'(i)}\omega}, \\ \exp(-\gamma((\beta - \gamma\varepsilon)S_{n_k^i}\Psi_{j_i}(\sigma^{M'(i)}\omega, \underline{v}) + \gamma S_{n_k^i}\Phi_{j_i}(\sigma^{M'(i)}\omega, \underline{v}))) \leq 1 \end{array} \right\},$$

$$E_{i,\beta,p}(\sigma^{M'(i)}\omega, \varepsilon) = \bigcap_{k \geq p} F_{i,\beta,k}(\sigma^{M'(i)}\omega, \varepsilon)$$

and then

$$E_{i,\beta}(\sigma^{M'(i)}\omega, \varepsilon) = \bigcup_{p \geq 1} E_{i,\beta,p}(\sigma^{M'(i)}\omega, \varepsilon).$$

Lemma 5.5 *For all $i \in \mathbb{N}$, for all $\omega \in \Omega(i)$, for all $q \in Q_i$, for any $\varepsilon > 0$, the singularity set $E_{i,T'_{j_i}(q)}(\sigma^{M'(i)}\omega, \varepsilon)$ has full $\mu^{\Lambda_{i,q}}_{\sigma^{M'(i)}\omega}$-measure.*

Proof Fix $\varepsilon > 0$. Let

$$S_{i,q,k} = \mu^{\Lambda_{i,q}}_{\sigma^{M'(i)}\omega}(X_{\sigma^{M'(i)}\omega} \setminus F_{i,T'_{j_i}(q),k}(\sigma^{M'(i)}\omega, \varepsilon)).$$

We have

$$\begin{aligned} S_{i,q,k} \;&\leq\; \sum_{\gamma \in \{-1,1\}} \sum_{v \in \Sigma_{\sigma^{M'(i)}\omega, n_k^i}} \sum_{v' \in \Sigma_{\sigma^{M'(i)}\omega, n_k^i}, |v \wedge v'| \geq n_{k-1}^i} \mu^{\Lambda_{i,q}}_{\sigma^{M'(i)}\omega}(U^v_{\sigma^{M'(i)}\omega}) \\ &\quad \cdot \exp\big(-\gamma\eta((T'_{j_i}(q) - \gamma\varepsilon)S_{n_k^i}\Psi_{j_i}(\sigma^{M'(i)}\omega, \underline{v}') - S_{n_k^i}\Phi_{j_i}(\sigma^{M'(i)}\omega, \underline{v}'))\big) \\ &= \sum_{\gamma \in \{-1,1\}} \sum_{v, v' \in \Sigma_{\sigma^{M'(i)}\omega, n_k^i}, |v \wedge v'| \geq n_{k-1}^i} \exp((q + \gamma\eta)S_{n_k^i}\Phi_{j_i}(\sigma^{M'(i)}\omega, \underline{v})) \\ &\quad \exp((-(T_{j_i}(q) + \gamma\eta T'_{j_i}(q) - \varepsilon\eta)S_{n_k^i}\Psi_{j_i})(\sigma^{M'(i)}\omega, \underline{v})) \\ &\quad \cdot \exp(-\gamma\eta((T'_{j_i}(q) - \gamma\varepsilon)(S_{n_k^i}\Psi_{j_i}(\sigma^{M'(i)}\omega, \underline{v}') - S_{n_k^i}\Psi_{j_i}(\sigma^{M'(i)}\omega, \underline{v}))) \\ &\quad \cdot \exp(\gamma\eta(S_{n_k^i}\Phi_{j_i}(\sigma^{M'(i)}\omega, \underline{v}') - S_{n_k^i}\Phi_{j_i}(\sigma^{M'(i)}\omega, \underline{v}))) + o(n_k^i)) \end{aligned}$$

Since T_{j_i} is in fact not only differentiable, but analytic [35, 61], we have

$$T_{j_i}(q + \gamma\eta) = T_{j_i}(q) + T'_{j_i}(q)\gamma\eta + o(\eta^2).$$

uniformly in $q \in Q_i$. Then there exists $b > 0$ such that for η small enough, for all $q \in Q_i$, we have

$$|T_{j_i}(q + \gamma\eta) - T_{j_i}(q) - T'_{j_i}(q)\gamma\eta| \leq b\eta^2.$$

Consider such an η in $(0, \frac{\varepsilon}{2b}]$. We have

$$S_{i,q,k} \;\leq\; \sum_{\gamma \in \{-1,1\}} \sum_{v \in \Sigma_{\sigma^{M'(i)}\omega, n_k^i}} (l(\sigma^{M'(i)+n_{k-1}^i}\omega) \cdots l(\sigma^{M'(i)+n_k^i-1}\omega))$$

$$
\begin{aligned}
& \cdot \exp(S_{n_k^i}((q+\gamma\eta)\Phi_{j_i} - T_{j_i}(q+\gamma\eta)\Psi_{j_i})(\sigma^{M'(i)}\omega, \underline{v})) \\
& \cdot \exp((\varepsilon\eta - b\eta^2)S_{n_k^i}\Psi_{j_i}(\sigma^{M'(i)}\omega, \underline{v}) + o(n_k^i)) \\
\leq & \sum_{\gamma\in\{-1,1\}} \exp((n_k^i - n_{k-1}^i)C - (\varepsilon\eta - b\eta^2)n_k^i \varpi_{\Psi_{j_i}} + o(n_k^i)) \\
\leq & \sum_{\gamma\in\{-1,1\}} \exp(n_k^i c_{i,n_k^i} - (\varepsilon\eta - b\eta^2)n_k^i \varpi_{\Psi_{j_i}} + o(n_k^i)) \quad \text{for } k \text{ large enough} \\
\leq & \; 2\exp\left(-(\frac{\varepsilon^2}{4b})n_k^i \varpi_\Psi + o(n_k^i)\right).
\end{aligned}
$$

Consequently, $\sum_{k=1}^{+\infty} S_{i,q,k} < \infty$, which yields the desired conclusion since ε is arbitrary.

Now we can collect the following facts.

Facts 5.6 Lemma 5.4 and 5.5 imply that for all $i \in \mathbb{N}$, for all $\omega \in \Omega(i)$, for any $\epsilon_i > 0$, there exists an integer $\mathcal{N}_i = \mathcal{N}_i(\sigma^{M'(i)}\omega)$ such that for any $q \in Q_i$, there exists $E_{i,q} = E_{i,q}(\sigma^{M'(i)}\omega) \subset X_{\sigma^{M'(i)}\omega}$ such that

1. $\mu^{\Lambda_{i,q}}_{\sigma^{M'(i)}\omega}(E_{i,q}) > 1 - \epsilon_i$,
2. $M'(i) \leq n^i_{\mathcal{N}_i}\varepsilon_i^3$,
3. $c_{n^i_{\mathcal{N}_i}} \leq \varepsilon_i^3$ and $c_{i,n^i_{\mathcal{N}_i}} \leq \varepsilon_i^3$,
4. $n_k^i - n_{k-1}^i \leq n_{k-1}^i\varepsilon_i^3$ for any $k \geq \mathcal{N}_i$,
5. for any $\underline{v} \in E_{i,q}$, for any $v \in \Sigma_{\sigma^{M'(i)}\omega, n_k^i}$ with $k \geq \mathcal{N}_i$ such that $\underline{v} \in [v]_{\sigma^{M'(i)}\omega}$, one has $|v \wedge v+| \geq n_{k-1}^i$ and $|v \wedge v-| \geq n_{k-1}^i$. Furthermore, for any $w \in \{v, v+, v-\}$, there exists (in fact for all) $\underline{w} \in [w]_{\sigma^{M'(i)}\omega}$ such that

$$\left|\frac{S_{n_k^i}\Phi_{j_i}(\sigma^{M'(i)}\omega, \underline{w})}{S_{n_k^i}\Psi_{j_i}(\sigma^{M'(i)}\omega, \underline{w})} - T'_{j_i}(q)\right| \leq \varepsilon_i, \tag{5.7}$$

$$\left|\frac{\log \mu^{\Lambda_{i,q}}_{\sigma^{M'(i)}\omega}(U^w_{\sigma^{M'(i)}\omega})}{S_{|v|}\Psi_{j_i}(\sigma^{M'(i)}\omega, \underline{w})} - T^*_{j_i}(T'_{j_i}(q))\right| \leq \varepsilon_i. \tag{5.8}$$

and

$$\left|\frac{S_{|v|}\Lambda_{i,q}(\sigma^{M'(i)}\omega, \underline{w}))}{S_{|v|}\Psi_{j_i}(\sigma^{M'(i)}\omega, \underline{w})} - T^*_{j_i}(T'_{j_i}(q))\right| \leq \varepsilon_i. \tag{5.9}$$

In fact with a suitable change of ε_i (take it as $2\varepsilon_i$), we can deduce from the above item 5. the following property:

For any $d_i \in D_i$, there exists $q_i \in Q_i$ such that for any $\underline{v} \in E_{i,q_i}$, for any $v \in \Sigma_{\sigma^{M'(i)}\omega, n_k^i}$ with $k \geq \mathcal{N}_i$ such that $\underline{v} \in [v]_{\sigma^{M'(i)}\omega}$, one has $|v \wedge v+| \geq n_{k-1}^i$ and $|v \wedge v-| \geq n_{k-1}^i$. Furthermore, for any $w \in \{v, v+, v-\}$, there exists (or for all) $\underline{w} \in [w]_{\sigma^{M'(i)}\omega}$ such that

$$\left| \frac{S_{n_k^i} \Phi_{j_i}(\sigma^{M'(i)}\omega, \underline{w})}{S_{n_k^i} \Psi_{j_i}(\sigma^{M'(i)}\omega, \underline{w})} - d_i \right| \leq \varepsilon_i, \tag{5.10}$$

$$\left| \frac{\log \mu_{\sigma^{M'(i)}\omega}^{\Lambda_{i,q}}(U_{\sigma^{M'(i)}\omega}^{w})}{S_{|v|} \Psi_{j_i}(\sigma^{M'(i)}\omega, \underline{w})} - T^*(d_i) \right| \leq \varepsilon_i. \tag{5.11}$$

and

$$\left| \frac{S_{|v|} \Lambda_{i,q}(\sigma^{M'(i)}\omega, \underline{w}))}{S_{|v|} \Psi_{j_i}(\sigma^{M'(i)}\omega, \underline{w})} - T^*(d_i) \right| \leq \varepsilon_i. \tag{5.12}$$

Facts 5.7 We can change $\Omega(i)$ to $\Omega_i \subset \Omega(i)$ a bit smaller such that $\mathbb{P}(\Omega_i) \geq 1 - \epsilon$ and there exist $\mathcal{N}_i$ and $W(i)$ such that for any $\omega \in \Omega_i$, $\mathcal{N}_i(\sigma^{M'(i)}\omega) \leq \mathcal{N}_i$ and $n_{\mathcal{N}_i}^i(\omega) \leq W(i)$ and the properties listed in Facts 5.6 hold.

We define $\theta(i, \omega, s)$ as being the s-th return time to the set Ω_i for the point ω.

Since $\mathbb{N}$ is countable, there exists $\widetilde{\Omega} \subset \widetilde{\Omega}'$ of full probability such that for all $\omega \in \widetilde{\Omega}$, for any $i \in \mathbb{N}$, we have

$$\lim_{s \to \infty} \frac{\theta(i, \omega, s)}{s} = \frac{1}{\mathbb{P}(\Omega_i)},$$

hence

$$\lim_{s \to \infty} \frac{\theta(i, \omega, s) - \theta(i, \omega, s-1)}{\theta(i, \omega, s-1)} = 0.$$

From now on we just deal with the point in the set $\widetilde{\Omega}$ which is a set with $\mathbb{P}$-full measure.

Chapter 6

Multifractal analysis of random weak Gibbs measures: Proof of Theorem 1.11

This chapter consists of three sections. In the first one we obtain the sharp upper bound for the L^q-spectrum of μ_ω. Next, in the second section, we prove the validity of the strong mutifractal formalism. There, our approach to construct suitable auxiliary measures already prepares the material used to establish in the third section the refinements gathered in theorem 1.11(3)(4)(5).

6.1 Lower bound for τ_{μ_ω} and upper bound for $\tau^*_{\mu_\omega}$

Fix a countable and dense subset D of $\mathbb{R}$. Let $\widehat{\Omega}$ be a set of full $\mathbb{P}$-probability, such that:

1. for all $q \in D$ the weak Gibbs measure $\{\widetilde{\mu}_\omega^{(q\Phi - T(q)\Psi)}\}_{\omega\in\widehat{\Omega}}$ are defined;

2. for all $\omega \in \widehat{\Omega}$ the conclusions of proposition 2.7 hold all the potentials $q\Phi - T(q)\Psi$, $q \in D$;

3. for all n large enough, for all $v \in \Sigma_{\omega,n}$, for all $\underline{v} \in [v]_\omega$,

$$\exp(-nC_\Psi - o(n)) \leq \exp(S_n\Psi(\omega,\underline{v})) \leq \exp(-nc_\Psi - o(n)),$$

which follows from ergodic theorem applied to the potentials $\|\Psi(\omega)\|_\infty = \sup_{\underline{v}\in\Sigma_\omega} |\Psi(\omega,\underline{v})|$ and $\sup_{\underline{v}\in\Sigma_\omega} \Psi(\omega,\underline{v})$, where $\Psi(\omega,\underline{v})$ has been defined in (1.9), and c_Ψ and C_ψ are finite due to (1.6).

We will establish the lower bound $\tau_{\mu_\omega}(q) \geq T(q)$ for all $\omega \in \widehat{\Omega}$ and $q \in D$. Since D is dense and both τ_{μ_ω} and T are continuous, this will yield $\tau_{\mu_\omega} \geq T$ for all $\omega \in \widehat{\Omega}$. By using the multifractal formalism, this immediately yields the desired upper bound T^* for $\tau^*_{\mu_\omega}$ and the various spectra we consider for μ_ω.

Let $\omega \in \widehat{\Omega}$. Let $r > 0$ and consider $\mathcal{B} = \{B_i\}$, a packing of X_ω by disjoint balls B_i with the center x_i and radius r. For each ball B_i, choose $n = n_i$ and $v(x_i) \in \Sigma_{\omega,n}$ such that $x_i \in U_\omega^{v(x_i)}$ and $|U_\omega^{v(x_i)}| \le r$, but $|U_\omega^{v(x_i)|_{n-1}}| > r$. By removing a set of probability 0 from $\widehat{\Omega}$ if necessary, for any $\underline{v} \in [v(x_i)]_\omega$, we have

$$r \ge |U_\omega^{v(x_i)}| \ge \exp(S_n\Psi(\omega, \underline{v}) - o(n)) \ge \exp(-nC_\Psi - o(n)),$$

where we have used ergodic theorem. Thus $n \ge \frac{-\log r}{2C_\Psi}$ for r small enough. On the other hand, for r small enough, for any $\underline{v} \in [v(x_i)]_\omega$ we have

$$r \le |U_\omega^{v(x_i)|_{n-1}}| \le \exp(S_{n-1}\Psi(\omega, \underline{v}) + o(n)) \le \exp(-(n-1)c_\Psi + o(n)),$$

so $n \le \frac{-2\log r}{c_\Psi}$. To resume, for r small enough, independently on $\mathcal{B}$, if $v(x_i) \in \Sigma_{\omega,n}$ and $\underline{v} \in [v(x_i)]_\omega$ we have

$$\frac{-\log r}{2C_\Psi} \le n \le \frac{-2\log r}{c_\Psi}. \tag{6.1}$$

Case $q \in D \cap (-\infty, 0)$:

For each $B_i \in \mathcal{B}$, one has $X_\omega^{v(x_i)} \subset B_i$, so for any $\underline{v} \in [v(x_i)]_\omega$

$$\begin{aligned}(\mu_\omega(B_i))^q &\le (\mu_\omega(X_\omega^{v(x_i)}))^q \\ &\le \exp(qS_n\Phi(\omega, \underline{v}) + o(n)) \\ &= \exp(S_n(q\Phi - T(q)\Psi)(\omega, \underline{v})) \cdot \exp(T(q)S_n\Psi(\omega, \underline{v}) + o(n)) \\ &\le \mu_\omega^{(q\Phi - T(q)\Psi)}(X_\omega^{v(x_i)}) r^{T(q)} \exp(o(-\log r)),\end{aligned}$$

where we have applied proposition 2.7(2) to the potential $q\Phi - T(q)\Psi$ as well as proposition 2.7(1), the fact that $|U_\omega^{v(x_i)|_{n-1}}| > r \ge |U_\omega^{v(x_i)}|$ and (6.1). It follows that $\sum_i (\mu_\omega(B_i))^q \le r^{T(q)} \exp(o(-\log r))$, and this bound does not depend on the choice of the packing $\{B_i\}$. Letting $r \to 0$, this yields $\tau_{\mu_\omega}(q) \ge T(q)$.

Case $q \in D \cap [0, +\infty)$: Define

$$V(\omega, n, r) = \{v \in \Sigma_{\omega,n} : |U_\omega^v| \ge 2r, \exists s \text{ such that } vs \in \Sigma_{\omega,n+1},\ |U_\omega^{vs}| < 2r\},$$

$$V'(\omega, n, r) = \{v \in V(\omega, n, r), \text{there is no } k < n \text{ such that } v|_k \in V(\omega, k, r)\},$$

$$V(\omega, r) = \cup_{n \ge 1} V'(\omega, n, r).$$

Then $\{U_\omega^v : v \in V(\omega, r)\}$ is a partition of $[0, 1]$. Define $n(\omega, r) = \max\{|v| : v \in V(\omega, r)\}$ and $n'(\omega, r) = \min\{|v| : v \in V(\omega, r)\}$. Then, from (6.1) we now that for some positive constants B_1 and B_2, for r small enough, we have $-B_1 \log(r) \le n'(\omega, r) \le n(\omega, r) \le -B_2 \log(r)$.

For any $v \in V(\omega, r)$, U_ω^v meets at most $\exp(o(-\log r))$ many balls of B_i and for any B_i, B_i meets at most two intervals of U_ω^v, $U_\omega^{v'}$ with $v, v' \in V(\omega, r)$. Consequently, since $(\mu_\omega(B_i))^q \leq 2^q((\mu_\omega(U_\omega^v))^q + (\mu_\omega(U_\omega^{v'}))^q)$, we have

$$\sum_{B_i \in \mathcal{B}} (\mu_\omega(B_i))^q \leq \exp(o(-\log r))2^q \sum_{n'(\omega,r) \leq n \leq n(\omega,r)} \sum_{v \in \Sigma_{\omega,n} \cap V(\omega,r)} (\mu_\omega(U_\omega^v))^q)$$

Using the same method argument as for $q < 0$, we can know get that

$$(\mu_\omega(U_\omega^v))^q) \leq \mu_\omega^{(q\Phi - T(q)\Psi)}(U_\omega^v) r^{T(q)} \exp(o(-\log r)),$$

so that

$$\begin{aligned} \sum_{B_i \in \mathcal{B}} (\mu_\omega(B_i))^q &\leq r^{T(q)} \exp(o(-\log r)) \sum_{n'(\omega,r) \leq n \leq n(\omega,r)} \sum_{v \in \Sigma_{\omega,n} \cap V(\omega,r)} \mu_\omega^{(q\Phi - T(q)\Psi)}(U_\omega^v) \\ &= r^{T(q)} \exp(o(-\log r)), \end{aligned}$$

independently on $\{B_i\}$, where we use the fact that $\{U_\omega^v : v \in V(\omega, r)\}$ is a partition of $[0, 1]$. Letting $r \to 0$, this yields $\tau_{\mu_\omega}(q) \geq T(q)$.

6.2 Lower bound for the Hausdorff spectrum

Recall facts 5.6 and facts 5.7. For any $\omega \in \widetilde{\Omega}$, for any $d \in [T'(+\infty), T'(-\infty)]$, for any sequence $\{d_i\}_{i \in \mathbb{N}}$ with $d_i \in D_i$, such that $\lim_{i \to \infty} d_i = d$, and consequently $\lim_{i \to \infty} T^*(d_i) = T^*(d)$ by continuity of T^*, we will build a measure η_ω on a set $K(\omega, \{d_i\}_{i \geq 1})$ such that

- $\eta_\omega(K(\omega, \{d_i\}_{i \geq 1})) = 1$,
- $K(\omega, \{d_i\}_{i \geq 1}) \subset E(\mu_\omega, d)$,
- For any $x \in K(\omega, \{d_i\}_{i \geq 1})$, $\lim_{r \to 0} \frac{\log(\eta_\omega(B(x,r)))}{\log r} \geq T^*(d)$.

This will imply that $\dim_H \eta_\omega \geq T^*(d)$, and then

$$\dim_H(E(\mu_\omega, d)) \geq \dim_H(K(\omega, \{d_i\}_{i \geq 1})) \geq T^*(d).$$

The construction will consist of four steps. Fix a sequence $\{\epsilon_i\}_{i \in \mathbb{N}}$ small enough such that $\Pi_{i \geq 1}(1 - \epsilon_i) \geq \frac{1}{2}$. For each $i \in \mathbb{N}$, Facts 5.6 will be applied with this ϵ_i. Notice that $\{\epsilon_i\}_{i \in \mathbb{N}}$differs from the other sequence $\{\epsilon_i\}_{i \in \mathbb{N}}$ also invoked in Facts 5.6.

In the two first steps, we build a family of Moran structures indexed by the elements of $\prod_{i \geq 1} D_i$.

First step: For any $\omega \in \widetilde{\Omega}$, recall that $\theta(1,\omega,1)$ is the smallest $n \in \mathbb{N}$ such that $\sigma^n\omega \in \Omega_1 \subset \Omega(1)$. Define $m_1 := \theta(1,\omega,1) + M'(1)$. Facts 5.6 and facts 5.7 tell us that there exists an integer $\mathcal{N}_1 = \mathcal{N}_1(\sigma^{m_1}\omega) = \mathcal{N}_1(\sigma^{M_1'}(\sigma^{\theta(1,\omega,1)}\omega))$, such that for any $d_1 \in D_1$, there exists $q = q_1 \in Q_1$ and a set

$$E_{1,q_1}(\sigma^{m_1}\omega) \subset X_{\sigma^{m_1}\omega}$$

such that

1. $\mu^{\Lambda_{1,q_1}}_{\sigma^{m_1}\omega}(E_{1,q_1}) > 1 - \epsilon_1$, recall that $\Lambda_{1,q_1} = q_1\Phi_{j_1} - T_{j_1}(q_1)\Psi_{j_1}$.
2. $M'(1) \le \varepsilon_1^3 n^1_{\mathcal{N}_1}$,
3. $c_{n^1_{\mathcal{N}_1}} \le \varepsilon_1^3$ and $c_{1,n^1_{\mathcal{N}_1}} \le \varepsilon_1^3$,
4. $n^1_k - n^1_{k-1} \le \varepsilon_1^3 n^1_{k-1}$ for any $k \ge \mathcal{N}_1$,
5. for any $\underline{v} \in E_{1,q}$, for any $v \in \Sigma_{\sigma^{m_1}\omega, n^1_k}$ with $k \ge \mathcal{N}_1$ such that $\underline{v} \in [v]_{\sigma^{m_1}\omega}$, one has $|v\wedge v+| \ge n^1_{k-1}$ and $|v\wedge v-| \ge n^1_{k-1}$. Furthermore, for any $w \in \{v, v+, v-\}$, there exists (or for all) $\underline{w} \in [w]_{\sigma^{n+M+\mathcal{S}(1)}\omega}$ such that

$$\begin{aligned}
\left|\frac{S_{n^1_k}\Phi_{j_1}(\sigma^{m_1}\omega,\underline{w})}{S_{n^1_k}\Psi_{j_1}(\sigma^{m_1}\omega,\underline{w})} - d\right| &\le \varepsilon_1.\\
\left|\frac{\log\mu^{\Lambda_{1,q}}_{\sigma^{m_1}\omega}(U^w_\omega)}{S_{|v|}\Psi_{j_1}(\sigma^{m_1}\omega,\underline{w})} - T^*(d)\right| &\le \varepsilon_1,\\
\left|\frac{\Lambda_{1,q}(\sigma^{m_1}\omega,\underline{w})}{S_{|v|}\Psi_{j_1}(\sigma^{m_1}\omega,\underline{w})} - T^*(d)\right| &\le \varepsilon_1,
\end{aligned}$$

Choose $\mathcal{N}_1' > \mathcal{N}_1$ large enough such that

- $m_1 \le \varepsilon_2^3 n^1_{\mathcal{N}_1'}$.
- $M'(2) \le \varepsilon_2^3 n^1_{\mathcal{N}_1'}$,
- $W(2) \le \varepsilon_2^3 n^1_{\mathcal{N}_1'}$,
- for any s such that the return time $\theta(2,\omega,s)$ satisfies $\theta(2,\omega,s) \ge m_1 + n^1_{\mathcal{N}_1'}$, one also has
$$\frac{\theta(2,\omega,s) - \theta(2,\omega,s-1)}{\theta(2,\omega,s-1)} \le \varepsilon_2^3.$$

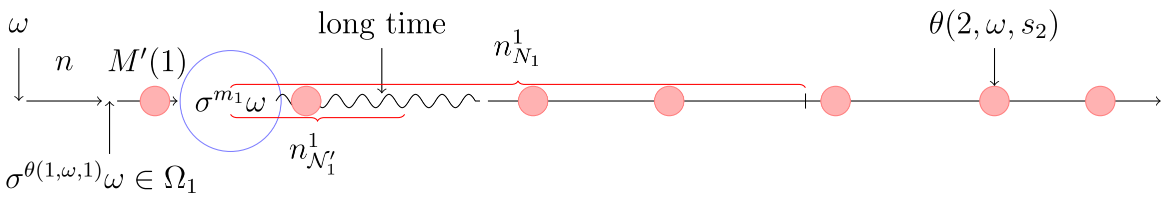
Let s_2 be the smallest s such that $\theta(2,\omega,s) \ge m_1 + n^1_{\mathcal{N}_1'}$.

Now, let N_1 be the largest k such that $m_1 + n^1_k \leq \theta(2, \omega, s_2)$ (by construction we have $N_1 \geq \mathcal{N}'_1$). Then

$$\theta(2, \omega, s_2) - m_1 - n^1_{N_1} \leq n^1_{N_1+1} - n^1_{N_1} \leq \varepsilon_1^3 n^1_{N_1}$$

by item 4. above.

Here is a picture which illustrates the beginning of the construction.



For each $q \in Q_1$ and $k \geq 1$, let

$$\mathcal{V}(\sigma^{m_1}\omega, 1, q, k) = \left\{ v \in \Sigma_{\sigma^{m_1}\omega, n^1_k} : E_{1,q}(\sigma^{m_1}\omega) \cap X^v_{\sigma^{m_1}\omega} \neq \emptyset \right\}.$$

Also, set

$$\mathcal{V}(\sigma^{m_1}\omega, 1, q) = \mathcal{V}(\sigma^{m_1}\omega, 1, q, N_1).$$

Since there exists at least $\sharp Q_1$ words in $\Sigma_{\sigma^{\theta(1,\omega,1)+M}\omega, \mathcal{S}(1)}$, for each $q \in Q_1$ we can choose $v(q) \in \Sigma_{\sigma^{\theta(1,\omega,1)+M}\omega, \mathcal{S}(1)}$ with these $v(q)$ pairwise distinct.

For any $w \in \Sigma_{\omega, \theta(1,\omega,1)}$, for any $q \in Q_1$ and $v' \in \mathcal{V}(\sigma^{\theta(1,\omega,1)+M+M(1)+\mathcal{S}(1)}\omega, 1, q)$ one can find at least one $v''(w, q) \in \Sigma_{\sigma^{\theta(1,\omega,1)}\omega, M}$ and $v'''(q, v'') \in \Sigma_{\sigma^{\theta(1,\omega,1)+M+\mathcal{S}(1)}\omega, M(1)}$ such that

$$wv''(q)v(q)v'''(q, v')v' \in \Sigma_{\omega, \theta(1,\omega,1)+M+M(1)+\mathcal{S}(1)+n^1_{N_1}},$$

For each $(w, q, v') \in \Sigma_{\omega,\theta(1,\omega,1)} \times Q_1 \times \mathcal{V}(\sigma^{\theta(1,\omega,1)+M+M(1)+\mathcal{S}(1)}\omega, 1, q)$ we choose such a couple $(v''(q), v'''(q, v'))$ by requiring that for two distinct (w, q) and $(w', \tilde{q})$ in $\Sigma_{\omega,\theta(1,\omega,1)} \times Q_1$, $v''(q) = v''(\tilde{q})$ if w and w' have the same last letter, and v' and $\tilde{v}'$ have the same first letter.

In the sequel, we denote $wv''(q)v(q)v'''(q)v'_q$ by $w * v(q) * v'_q$ or $w * v'_q$ for short.

Fix $w_0 \in \Sigma_{\omega,\theta(1,\omega,1)}$. For any $d_1 \in D_1$, there exists $q_1 \in Q_1$ such that $T'_{j_1}(q_1) = d_1$. We define:

$$R_1(d_1) = \{w_0 * v(q_1) * v'_{\cdot} v' \in \mathcal{V}(\sigma^{m_1}\omega, 1, q_1)\},$$

and

$$R_1 = \cup_{d_1 \in D_1} R_1(d_1).$$

Second step: suppose that $\theta(i+1, \omega, s_{i+1}), N_i, R_i$ have been chosen. Define

$$m_{i+1} := \theta(i+1, \omega, s_{i+1}) + M + \mathcal{S}(i+1) + M(i+1) = \theta(i+1, \omega, s_{i+1}) + M'(i+1)$$

and

$$n_k^{i+1} = n_k^{i+1}(\sigma^{\theta(i+1,\omega,s_{i+1})}\omega).$$

Facts 5.6 and facts 5.7 tell us that there exists an integer $\mathcal{N}_{i+1} = \mathcal{N}_{i+1}(\sigma^{m_{i+1}}\omega)$ such that for any $d_{i+1} \in D_{i+1}$, there exists $q_{i+1} \in Q_{i+1}$ and a set $E_{i+1,q_{i+1}}(\sigma^{m_{i+1}}\omega) \subset X_{\sigma^{m_{i+1}}\omega}$ such that

1. $\mu_{\sigma^{m_{i+1}}\omega}^{\Lambda_{i+1,q_{i+1}}}(E_{i+1,q}) > 1 - \epsilon_{i+1}$,
2. $M'(i+1) \leq \varepsilon_{i+1}^3 n_{\mathcal{N}_{i+1}}^{i+1}$
3. $c_{n_{\mathcal{N}_{i+1}}^{i+1}} \leq \varepsilon_{i+1}^3$ and $c_{i+1,n_{\mathcal{N}_{i+1}}^{i+1}} \leq \varepsilon_{i+1}^3$,
4. $n_k^{i+1} - n_{k-1}^{i+1} \leq \varepsilon_{i+1}^3 n_{k-1}^{i+1}$ for any $k \geq \mathcal{N}_{i+1}$,
5. for any $x \in E_{i+1,q_{i+1}}$, for any $v \in \Sigma_{\sigma^{m_{i+1}}\omega, n_k^{i+1}}$ with $k \geq \mathcal{N}_{i+1}$ such that $x \in X_{\sigma^{m_{i+1}}\omega}^v$, one has $|v \wedge v+| \geq n_{k-1}^{i+1}$ and $|v \wedge v-| \geq n_{k-1}^{i+1}$. Furthermore, for any $w \in \{v, v+, v-\}$, there exists (or for all) $\underline{w} \in [w]_{\sigma^{m_{i+1}}\omega}$ such that

$$\begin{aligned}
\left| \frac{S_{n_k^{i+1}}\Phi_{j_{i+1}}(\sigma^{m_{i+1}}\omega, \underline{w})}{S_{n_k^{i+1}}\Psi_{j_{i+1}}(\sigma^{m_{i+1}}\omega, \underline{w})} - d \right| &\leq \varepsilon_{i+1}, \\
\left| \frac{\log \mu_\omega^{\Lambda_{i+1,q}}(U_{\sigma^{m_{i+1}}\omega}^w)}{S_{|v|}\Psi_{j_{i+1}}(\sigma^{m_{i+1}}\omega, \underline{w})} - T^*(d) \right| &\leq \varepsilon_{i+1}. \\
\left| \frac{\Lambda_{i+1,q}(\sigma^{m_{i+1}}\omega, \underline{w})}{S_{|v|}\Psi_{j_{i+1}}(\sigma^{m_{i+1}}\omega, \underline{w})} - T^*(d) \right| &\leq \varepsilon_{i+1}.
\end{aligned}$$

Choose $\mathcal{N}'_{i+1}$ large enough such that

- $m_{i+1} \leq \varepsilon_{i+2}^3 n_{\mathcal{N}'_{i+1}}^{i+1}$,
- $M'(i+2) \leq \varepsilon_{i+2}^3 n_{\mathcal{N}'_{i+1}}^{i+1}$,
- $W(i+2) \leq \varepsilon_{i+2}^2 n_{\mathcal{N}'_{i+1}}^{i+1}$.

  The above two items ensure that we do not need to wait a long relative time to go into an other step.

- for any s with $\theta(i+2,\omega,s) \geq m_{i+1} + n_{\mathcal{N}'_{i+1}}^{i+1}$ one has

  $$\frac{\theta(i+2,\omega,s) - \theta(i+2,\omega,s-1)}{\theta(i+2,\omega,s-1)} \leq \varepsilon_{i+2}^3.$$

  Let s_{i+2} be the smallest s such that

  $$\theta(i+2,\omega,s) \geq m_{i+1} + n_{\mathcal{N}'_{i+1}}^{i+1}. \tag{6.2}$$

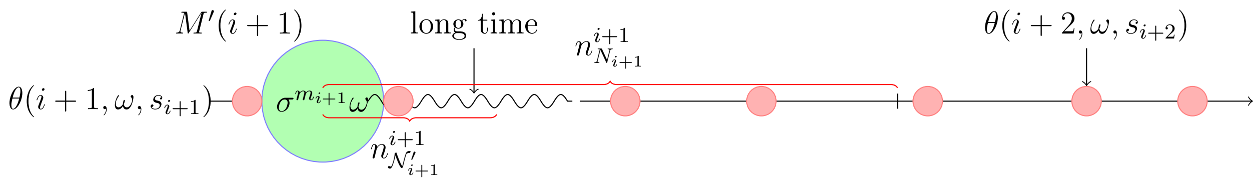
Let N_{i+1} be the largest $k \geq \mathcal{N}'_{i+1}$ such that $n_k^{i+1} \leq \theta(i+2,\omega,s_{i+2})$. Then we have

$$\theta(i+2,\omega,s_{i+2}) - m_{i+1} - n^{i+1}_{N_{i+1}} \leq n^{i+1}_{N_{i+1}} \varepsilon^3_{i+1}$$

due to item 4.

Remark 6.1 *By construction, we have* $m_{i+2} - m_{i+1} \geq n^{i+1}_{\mathcal{N}'_{i+1}}$ *and* $m_{i+1} = o(n^{i+1}_{\mathcal{N}'_{i+1}})$. *Consequently the speed we fix for the growth of* $(n^i_{\mathcal{N}'_i})_{i\in\mathbb{N}}$ *is directly related to the growth speed of* $(m_i)_{i\in\mathbb{N}}$.

Here again we draw a picture to illustrate this construction.



For $q_{i+1} \in Q_{i+1}$ and $k \geq 1$, define

$$\mathcal{V}(\sigma^{m_{i+1}}\omega, i+1, q_{i+1}, k) = \left\{ v \in \Sigma_{\sigma^{m_{i+1}}\omega, n_k^{i+1}} : E_{i+1,q}(\sigma^{m_{i+1}}\omega) \cap X^v_{\sigma^{m_{i+1}}\omega} \neq \emptyset \right\},$$

and

$$\mathcal{V}(\sigma^{m_{i+1}}\omega, i+1, q_{i+1}) = \mathcal{V}(\sigma^{m_{i+1}}\omega, i+1, q_{i+1}, N_{i+1}).$$

As in the case $i = 1$, for any $w \in R_i$, for any $d_{i+1} \in D_{i+1}$, there exists $q_{i+1} \in Q_{i+1}$ such that $T'_{j_{i+1}}(q_{i+1}) = d_{i+1}$. For any $v(q_{i+1}) \in \Sigma_{\sigma^{\theta(i+1,\omega,s_{i+1})+M}\omega,\mathcal{S}(i+1)}$ and any $v' \in \mathcal{V}(\sigma^{m_{i+1}}\omega, i+1, q_{i+1})$, we can build the word $w * v(q_{i+1}) * v'$ by using the same tule as in step 1, and denote it by $w * v'$ if there is no possible confusion.

Define

$$R_{i+1}(d_1, d_2, \cdots, d_i, d_{i+1}) = \left\{ w * v(q_{i+1}) * v' \;\middle|\; \begin{array}{l} w \in R_i(d_1, d_2 \cdots d_i), v(q_{i+1}) \in \Sigma_{\sigma^{\theta(i+1,\omega,s_{i+1})+M}\omega,\mathcal{S}(i+1)} \\ v' \in \mathcal{V}(\sigma^{m_{i+1}}\omega, i+1, q_{i+1}) \end{array} \right\}$$

and

$$R_{i+1} = \left\{ w * v(q_{d_{i+1}}) * v' \;\middle|\; \begin{array}{l} w \in R_i, v(q_{i+1}) \in \Sigma_{\sigma^{\theta(i+1,\omega,s_{i+1})+M}\omega,\mathcal{S}(i+1)} \text{ with } d_{i+1} \in D_{i+1} \\ v' \in \mathcal{V}(\sigma^{m_{i+1}}\omega, i+1, q_{i+1}) \end{array} \right\}.$$

Third step: For any $d \in [T'(+\infty), T'(-\infty)]$, there exists $\{d_i\}_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} D_i$, such that $\lim_{i\to\infty} d_i = d$ and $\lim_{i\to\infty} T^*(d_i) = T^*(d)$. Moreover, if $d \in (T'(+\infty), T'(-\infty))$,

$T^*_{j_i}(d_i)$ converges to $T^*(d)$ directly from proposition 5.2. If $d \in \{T'(+\infty), T'(-\infty)\}$, again due to proposition 5.2, we can choose $(d_i)_{i\geq 1}$ to be piecewise constant to make sure that $T^*_{j_i}(d_i) - T^*(d_i)$ tends to 0 as $i \to \infty$, so that $T^*_{j_i}(d_i)$ converges to $T^*(d)$ as well. We fix such a sequence.

Define

$$K(\omega, \{d_i\}_{i\geq 1}) = \cap_{i\geq 1} \cup_{v\in R_i(d_1,d_2,\cdots,d_i)} U^v_\omega.$$

We will prove that for any $x \in K(\{d_i\}_{i\geq 1})$, one has $\lim_{r\to 0+} \frac{\log \mu_\omega(B(x,r))}{\log r} = d$. Then $K(\omega, \{d_i\}_{i\geq 1}) \subset E(\mu_\omega, d)$.

Let us start with the following general estimate.

- For any $w \in R_i$, $v \in \mathcal{V}(\sigma^{m_{i+1}}\omega, i+1, q_{i+1}, k)$, for any $k \geq \mathcal{N}_{i+1}$, for any $\underline{v} \in [w * v]_\omega$ we can write for $\Upsilon \in \{\Phi, \Psi\}$

$$\begin{aligned}
&\Big|S_{m_{i+1}+n_k^{i+1}}\Upsilon(\omega,\underline{v}) - \sum_{p=1}^{i} S_{n^p_{N_p}}\Upsilon_{j_p}(F^{m_p}(\omega,\underline{v})) - S_{n_k^{i+1}}\Upsilon_{j_{i+1}}(F^{m_{i+1}}(\omega,\underline{v}))\Big| \\
\leq&\Big|\sum_{p=1}^{i}\sum_{t=0}^{n^p_{N_p}-1}(\Upsilon-\Upsilon_{j_p})(\sigma^{m_p+t}\omega)\Big| + \Big|\sum_{t=0}^{n_k^{i+1}-1}(\Upsilon-\Upsilon_{j_{i+1}})(\sigma^{m_{i+1}+t}\omega)\Big| \\
&+\Big|\sum_{t=0}^{m_1-1}\Upsilon(F^t(\omega,\underline{v}))\Big| + \Big|\sum_{p=1}^{i}\sum_{t=m_p+n^p_{N_p}}^{m_{p+1}-1}\Upsilon(F^t(\omega,\underline{v}))\Big| \\
\Upsilon\quad \leq&\sum_{p=1}^{i}\sum_{t=0}^{n^p_{N_p}-1}(\mathrm{var}_{j_p}\Upsilon)(\sigma^{m_p+t}\omega) + \sum_{t=0}^{n_k^{i+1}-1}(\mathrm{var}_{i+1}\Upsilon)(\sigma^{m_{i+1}+t}\omega) \quad . \\
&+(m_1)C + \Big|\sum_{p=1}^{i}(m_{p+1}-m_p-n^p_{N_p})C\Big| \\
\leq&\sum_{p=1}^{i} n^p_{N_p}\varepsilon_p^3 + n_k^{i+1}\varepsilon_{i+1}^3 + m_1 C + \sum_{p=1}^{i} n^p_{N_p}(\varepsilon_p^3+\varepsilon_{p+1}^3), \\
\leq&\sum_{p=1}^{i} n^p_{N_p}(3\varepsilon_p^3) + n_k^{i+1}\varepsilon_{i+1}^3 + m_1 C \\
\leq&(m_{i+1}+n_k^{i+1})(\varepsilon_i^2) \quad \text{for } i,k \text{ large enough,}
\end{aligned} \tag{6.3}$$

where to get the term $\sum_{p=1}^{i} n^p_{N_p}\varepsilon_p^3 + n_k^{i+1}\varepsilon_{i+1}^3$ in the upper bound we used successively the property that

$$|\Upsilon(\omega,\underline{v}) - \Upsilon_{j_p}(\omega,\underline{v})| \leq \mathrm{var}_{j_p}\Upsilon(\omega)$$

and

$$\sum_{t=0}^{n^p_{N_p}-1}(\mathrm{var}_{j_p}\Upsilon)(\sigma^{m_p+t}\omega)$$

$$
\begin{aligned}
&= S_{n^p_{N_p}} \mathrm{var}_{j_p} \Upsilon(\sigma^{m_p}\omega) \\
&\le \left| S_{n^p_{N_p}} \mathrm{var}_{j_p} \Upsilon(\sigma^{m_p}\omega) - n^p_{N_p} \int_\Omega \mathrm{var}_{j_p} \Upsilon(\omega)\ d\mathbb{P} \right| + n^p_{N_p} \int_\Omega \mathrm{var}_{j_p} \Upsilon(\omega)\ d\mathbb{P} \\
&\le 2 n^p_{N_p} \varepsilon_p^3.
\end{aligned}
$$

We also used

$$\left| S_{n^p_{N_p}} \mathrm{var}_{j_p} \Upsilon(\sigma^{m_p}\omega) - n^p_{N_p} \int_\Omega \mathrm{var}_{j_p} \Upsilon(\omega)\ d\mathbb{P} \right| \le n^p_{N_p} c_{n^p_{N_p}} \le n^p_{N_p} \varepsilon_p^3,$$

which holds since $\sigma^{m_p}\omega \in \Omega(p)$ and $c_{n^p_{N_p}} \le \varepsilon_p^3$, and

$$\int_\Omega \mathrm{var}_{j_p} \Upsilon(\omega)\ d\mathbb{P} \le \varepsilon_p^3,$$

which holds by construction of $(j_i)_{i\in\mathbb{N}}$.
The estimate of the term invoking $\mathrm{var}_{j_{i+1}} \Upsilon$ works similarly.
To obtain the term $m_1 C + \sum_{p=1}^{i} n^p_{N_p}(\varepsilon_p^3 + \varepsilon_{p+1}^3)$ in the upper bound we wrote

$$m_{p+1} - m_p - n^p_{N_p} = M'(p+1) + \theta(s+1, \omega, s_{p+1}) - m_p - n^p_{N_p}$$

and by construction, $M'(p+1) \le \varepsilon_{p+1}^3 n^p_{N^p}$ and $\theta(p+1, \omega, s_{p+1}) - m_p - n^p_{N_p} \le \varepsilon_p^3 n^p_{N^p}$.
At the end we have used the fact that $m_p \le n^p_{\mathcal{N}'_p} \varepsilon_{p+1}^3 \le m_{p+1}\varepsilon_{p+1}^3$.

We also have:

- For $i \in \mathbb{N}$ large enough, for any k with $\mathcal{N}_{i+1} < k \le N_{i+1}$ for any $v, v' \in \Sigma_{\omega, m_{i+1}+n^{i+1}_k}$ satisfying $|v \wedge v'| \ge m_{i+1} + n^{i+1}_{k-1}$, we have

$$\frac{|U^v_\omega|}{|U^{v'}_\omega|} \le \exp((m_{i+1} + n^{i+1}_k)\varepsilon_i^2). \tag{6.4}$$

Indeed, at first,

$$
\begin{aligned}
& \left| \log |U^v_\omega| - \log |U^{v'}_\omega| \right| \\
\le\ & 2V_{m_{i+1}+M'(i+1)+n^i_k} \Psi(\omega) + 2(n^{i+1}_k - n^{i+1}_{k-1})C \\
\le\ & 2V_{m_{i+1}+M'(i+1)+n^i_k} \Psi(\omega) + 2C n^{i+1}_k \varepsilon_{k+1}^3 \\
\le\ & 2\sum_{i=0}^{m_1-1} \|\Psi_{\sigma^i\omega}\|_\infty + 2\sum_{j=1}^{i} (m_{j+1} - m_j)\varepsilon_j^3 + 2C n^{i+1}_k \varepsilon_{i+1}^3 \\
\le\ & 2\sum_{i=0}^{m_1-1} \|\Psi_{\sigma^i\omega}\|_\infty + 2m_i + 2m_{i+1}\varepsilon_i^3 + 2C n^{i+1}_k \varepsilon_{i+1}^3
\end{aligned}
$$

$$\leq \ 2\sum_{i=0}^{m_1-1}\|\Psi_{\sigma^i\omega}\|_\infty + 4m_{i+1}\varepsilon_i^3 + n_k^{i+1}\varepsilon_{i+1}^2$$

for i large enough. Second, for i large enough, one can get $2\sum_{i=0}^{m_1-1}\|\Psi_{\sigma^i\omega}\|_\infty \leq m_{i+1}\varepsilon_i^3$.

At last we get $\left|\log|U_\omega^v| - \log|U_\omega^{v'}|\right| \leq (m_{i+1}+n_k^{i+1})\varepsilon_i^2$ for i large enough. Then (6.4) follows.

- Now, fix $x \in K(\omega,\{d_i\}_{i\geq 1})$. If r is small enough, we can choose the largest i, then the largest $k = k_{i+1}$, with $\mathcal{N}_{i+1} < k \leq N_{i+1}$ such that: there exists $w \in R_i(d_1,d_2,\cdots,d_i)$, $v \in \mathcal{V}(\sigma^{m_{i+1}}\omega, i+1, q_{i+1}, k)$ satisfying $x \in U_\omega^{w*v}$ and
$$|U_\omega^{w*v}| \geq 2r\exp((m_{i+1}+n_k^{i+1})\varepsilon_i^2).$$
From the construction, if U_ω^{w*v+} and U_ω^{w*v-} are the neighboring intervals of U_ω^{w*v}, then $|v\wedge v+|$, $|v \wedge v-|$ are larger than n_{k-1}^{i+1}. Then by (6.4) we have $|U_\omega^{w*v+}| \geq 2r$ and $|U_\omega^{w*v-}| \geq 2r$. So there exists $v' = v-$ or $v' = v+$ such that $B(x,r) \subset U_\omega^{w*v} \cup U_\omega^{w*v'}$.

  Now, using estimates similar to those leading to (6.4) with Ψ replaced by Φ we get that for any $\underline{w*v} \in [w*v]_\omega$,
$$\begin{aligned}\mu_\omega(B(x,r)) \leq &\mu_\omega(U_\omega^{w*v}) + \mu_\omega(U_\omega^{w*v'})\\ \leq &2\exp(S_{m_{i+1}+n_k^{i+1}}\Phi(\omega,\underline{w*v}) + (m_{i+1}+n_k^{i+1})\varepsilon_{i+1}^2).\end{aligned}$$
Consequently, using (6.3),
$$\begin{aligned}&\log\mu_\omega(B(x,r))\\ \leq\ &\log 2 + \sum_{p=1}^{i} S_{n_{N_p}^p}\Phi_{j_p}(F^{m_p}(\omega,\underline{v})) + S_{n_k^{i+1}}\Phi_{j_{i+1}}(F^{m_{i+1}}(\omega,\underline{v}))\\ &+2(m_{i+1}+n_k^{i+1})\varepsilon_i^2\\ \leq\ &\sum_{p=1}^{i} S_{n_{N_p}^p}\Phi_{j_p}(F^{m_p}(\omega,\underline{v})) + S_{n_k^{i+1}}\Phi_{j_{i+1}}(F^{m_{i+1}}(\omega,\underline{v})) + 3(m_{i+1}+n_k^{i+1})\varepsilon_i^2\end{aligned}$$
Let $\mathcal{I}_p^\Phi = S_{n_{N_p}^p}\Phi_{j_p}(F^{m_p}(\omega,\underline{v}))$ and $\mathcal{I}_{i+1,k}^\Phi = S_{n_k^{i+1}}\Phi_{j_{i+1}}(F^{m_{i+1}}(\omega,\underline{v}))$. Then
$$\log\mu_\omega(B(x,r)) \leq \left(\sum_{p=1}^{i}\mathcal{I}_p^\Phi\right) + \mathcal{I}_{i+1,k}^\Phi + 3(m_{i+1}+n_k^{i+1})\varepsilon_i^2. \tag{6.5}$$
Now let us estimate $\log r$ from below:

  - **If** $k < N_{i+1}$, there exists $\widetilde{v}$ such that $|w*\widetilde{v}| = m_{i+1}+n_{k+1}^{i+1}$, $x \in U_\omega^{w*\widetilde{v}}$ and
$$|U_\omega^{w*\widetilde{v}}| \leq 2r\exp((m_{i+1}+n_{k+1}^{i+1})\varepsilon_i^2). \tag{6.6}$$

If we notice that $n_{k+1}^{i+1} - n_k^{i+1} \leq n_k^{i+1}\varepsilon_{i+1}^3$ and $\mathcal{I}_{i+1,k+1}^{\Psi} - \mathcal{I}_{i+1,k}^{\Psi} \leq -Cn_k^{i+1}\varepsilon_{i+1}^3$ (where I_p^{Ψ} is defined similarly as I_p^{Φ}), from (6.6) and (6.3) we can get

$$\begin{aligned}\log r &\geq \left(\sum_{p=1}^{i} \mathcal{I}_p^{\Psi}\right) + \mathcal{I}_{i+1,k+1}^{\Psi} - 2(m_{i+1} + n_{k+1}^{i+1})\varepsilon_i^2 - \log 2 \\ &\geq \left(\sum_{p=1}^{i} \mathcal{I}_p^{\Psi}\right) + \mathcal{I}_{i+1,k}^{\Psi} - Cn_k^{i+1}\varepsilon_{i+1}^3 - 2(m_{i+1} + n_k^{i+1})\varepsilon_i^2 - \log 2 \\ &\geq \left(\sum_{p=1}^{i} \mathcal{I}_p^{\Psi}\right) + \mathcal{I}_{i+1,k}^{\Psi} - 3(m_{i+1} + n_k^{i+1})\varepsilon_i^2.\end{aligned}$$

So

$$\log r \geq \left(\sum_{p=1}^{i} \mathcal{I}_p^{\Psi}\right) + \mathcal{I}_{i+1,k}^{\Psi} - 3(m_{i+1} + n_k^{i+1})\varepsilon_i^2. \tag{6.7}$$

– **If** $k = N_{i+1}$, there exists $\widetilde{v}$ such that $|w * \widetilde{v}| = m_{i+2} + n_{\mathcal{N}_{i+2}+1}^{i+2}$, $x \in U_{\omega}^{w*\widetilde{v}}$ and

$$|U_{\omega}^{w*\widetilde{v}}| \leq 2r \exp((m_{i+2} + n_{\mathcal{N}_{i+2}+1}^{i+2})\varepsilon_{i+1}^2). \tag{6.8}$$

We have

$$\begin{aligned}\log r &\geq \left(\sum_{p=1}^{i+1} \mathcal{I}_p^{\Psi}\right) + \mathcal{I}_{i+2,\mathcal{N}_{i+2}+1}^{\Psi} - 2(m_{i+2} + n_{\mathcal{N}_{i+2}+1}^{i+2})\varepsilon_i^2 - \log 2 \\ &\geq \left(\sum_{p=1}^{i} \mathcal{I}_p^{\Psi}\right) + \mathcal{I}_{i+1,N_{i+1}}^{\Psi} - 3(m_{i+1} + n_{N_{i+1}}^{i+1})\varepsilon_i^2,\end{aligned}$$

where we have used (6.2). This implies that (6.7) holds as well.

Finally, for any $\underline{v} \in (w * \widetilde{v})_{\omega}$, (6.5) and (6.7) imply

$$\frac{\log \mu_{\omega}(B(x,r))}{\log r} \geq \frac{(\sum_{p=1}^{i} \mathcal{I}_p^{\Phi}) + \mathcal{I}_{i+1,k}^{\Phi} + 3(m_{i+1} + n_k^{i+1})\varepsilon_i^2}{(\sum_{p=1}^{i} \mathcal{I}_p^{\Psi}) + \mathcal{I}_{i+1,k+1}^{\Psi} - 3(m_{i+1} + n_k^{i+1})\varepsilon_i^2} \tag{6.9}$$

Due to item 5 in the second step we have $|\frac{\mathcal{I}_p^{\Phi}}{\mathcal{I}_p^{\Psi}} - d_i| \leq \varepsilon_i$ and $|\frac{\mathcal{I}_{i,k}^{\Psi}}{\mathcal{I}_{i,k}^{\Psi}} - d| \leq \varepsilon_{i+1}$ for $k \geq \mathcal{N}_{i+1}$.

It follows from Stolz-Cesàro theorem that

$$\liminf_{r \to 0} \frac{\log(\mu_{\omega}(B(x,r)))}{\log r} \geq d. \tag{6.10}$$

- Now it remains to prove that $\limsup_{r\to 0} \frac{\log \mu_\omega(B(x,r))}{\log r} \le d$. This is easier since we just need to choose the smallest i and then the smallest $k = k_{i+1}$ with $\mathcal{N}_{i+1} \le k \le N_{i+1}$, such that there exists $w \in R_i(d_1, d_2, \cdots, d_i)$ and $v \in \mathcal{V}(\sigma^{m_{i+1}}\omega, i+1, q_{i+1}, k_{i+1})$ for which $x \in U_\omega^{w*v}$ and $|U_\omega^{w*v}| \le r$. Then $\mu_\omega(B(x,r)) \ge \mu_\omega(U_\omega^{w*v})$.

  If $k_{i+1} > \mathcal{N}_{i+1}$, then $\widetilde{v}$, the father of v, belongs to $\mathcal{V}(\sigma^{m_{i+1}}\omega, i+1, d_{i+1}, k_{i+1}-1)$ and $x \in U_\omega^{w*\widetilde{v}}$. We have $|U_\omega^{w*\widetilde{v}}| \ge r$, and

$$\lim_{r\to 0} \frac{\log |U_\omega^{w*v}|}{\log |U_\omega^{w*\widetilde{v}}|} = 1. \tag{6.11}$$

  If $k_{i+1} = \mathcal{N}_{i+1}$, then there exists $w' \in R_{i-1}(d_1, d_2, \cdots, d_{i-1})$, and $v' \in \mathcal{V}(\sigma^{m_i}\omega, i, d_i, N_i)$ with $x \in U_\omega^{w'*v'}$, $|U_\omega^{w'*v'}| \ge r$, and

$$\lim_{r\to 0} \frac{\log |U_\omega^{w*v}|}{\log |U_\omega^{w'*v'}|} = 1. \tag{6.12}$$

  In any case, we get $\limsup_{r\to 0} \frac{\log(\mu_\omega(B(x,r)))}{\log r} \le d$.

Fourth step: For any $v = w_0 * v(q_1) * v' \in R_1$, where $v' \in \mathcal{V}(\sigma^{n+M+M(1)+\mathcal{S}(1)}\omega, 1, d_1) =: \mathcal{V}_1$, define:

$$\eta_\omega(U_\omega^{w_0*v(q_1)*v'}) := \frac{\mu_{\sigma^{m_1}\omega}^{\Lambda_{1,q}}(U_{\sigma^{m_1}\omega}^{v'})}{\sum_{v''\in V(\sigma^{m_1}\omega,1,q_1)} \mu_{\sigma^{m_1}\omega}^{\Lambda_{1,q}}(U_{\sigma^{m_1}\omega}^{v''})}. \tag{6.13}$$

Then inductively, for any $w \in R_i(d_1, d_2, \cdots, d_i)$, $v \in \mathcal{V}_{i+1}$, define:

$$\eta_\omega(U_\omega^{w*v(q_{i+1})*v}) := \eta_\omega(U_\omega^{w}) \frac{\mu_{\sigma^{m_{i+1}}\omega}^{\Lambda_{i+1,q_{i+1}}}(U_{\sigma^{m_{i+1}}\omega}^{v})}{\sum_{v'\in\mathcal{V}_{i+1}} \mu_{\sigma^{m_{i+1}}\omega}^{\Lambda_{i+1,q_{i+1}}}(U_{\sigma^{m_{i+1}}\omega}^{v'})}. \tag{6.14}$$

We can extend η_ω in a unique way to a probability measure on the σ-algebra generated by $\cup_{i\ge 1}\{U_\omega^v : v \in R_i(d_1, d_2, \cdots, d_i)\}$. This measure is supported on $K(\omega, \{d_i\}_{i\ge 1})$.

Since for each $i \ge 1$ we have $\sum_{v'\in\mathcal{V}_i} \mu_{\sigma^{m_i}\omega}^{\Lambda_{i,q_i}}(U_{\sigma^{m_i}\omega}^{v'}) \ge \mu_{\sigma^{m_i}\omega}^{\Lambda_{i,q_i}}(E_{i,q_i}) \ge 1 - \epsilon_i$ and $\prod_{i=1}^\infty (1-\epsilon_i) \ge \frac{1}{2}$, using the same method as in step three, we can prove that: for any $x \in K(\omega, \{d_i\}_{i\ge 1})$,

$$\liminf_{r\to 0} \frac{\log(\eta_\omega(B(x,r)))}{\log r} \ge \liminf_{i\to\infty} T^*(d_i) = T^*(d) \tag{6.15}$$

Then we get $\dim_H(E(\mu_\omega, d)) \ge T^*(d)$.

In fact, our estimates yield a positive sequence $(\varepsilon'_i)_{i\in\mathbb{N}}$ decreasing to 0 and a constant $C' > 0$ such that, independently on $\{d_i\}_{i\in\mathbb{N}}$, if in the construction the sequence $(m_i)_{i\in\mathbb{N}}$ is replaced by another one growing faster (with the effect to modify $(K(\omega,\{d_i\}_{i\in\mathbb{N}}),\eta_\omega))$, for all $x \in K(\omega,\{d_i\}_{i\in\mathbb{N}})$, for i large enough, if $\exp(-m_{i+1}c_\Psi/2) < r$, then

$$\eta_\omega(B(x,r)) \leq C' r^{\min\{T^*(d_j)-\varepsilon'_j:1\leq j\leq i\}}. \tag{6.16}$$

This property will be used in the next section.

Remark 6.2 *From the proof of theorem 1.11 we can directly get that*

$$\eta_\omega(\{x : \lim_{n\to\infty} \frac{S_n\varphi(\omega,x)}{S_n\psi(\omega,x)} = d\}) = 1,$$

and then

$$\dim_H(\{x \in \Sigma_\omega : \lim_{n\to\infty} \frac{S_n\varphi(\omega,x)}{S_n\psi(\omega,x)} = d\}) \geq T^*(d).$$

6.3 Proofs of theorems 1.11(3), (4) and (5)

Lemma 6.3 *For $\mathbb{P}$-almost every $\omega \in \Omega$, for any given $d \leq d' \in [T'(+\infty), T'(-\infty)]$,*

$$\dim_H E(\mu_\omega, d, d') = \inf\{T^*(d), T^*(d')\},$$

$$\dim_P E(\mu_\omega, d, d') = \sup\{T^*(\beta) : \beta \in [d, d']\}.$$

Proof 1. We first deal with the lower bounds for the dimensions.

At first, for the Hausdorff dimension let us take two sequences $(d_i)_{i\geq 1}$ and $(d'_i)_{i\geq 1}$ in $\prod_{i\geq 1} D_i$ such that $\lim_{i\to\infty} d_i = d$ and $\lim_{i\to\infty} d'_i = d'$, with the properties:

$$\lim_{i\to\infty} T^*(d_i) = T^*(d),\ \lim_{i\to\infty} T^*(d'_i) = T^*(d').$$

Set $\tilde{d}_{2i} = d_i$ and $\tilde{d}_{2i+1} = d'_i$.

We can use the same construction as in the previous section and get a set $K(\omega,\{\tilde{d}_i\}_{i\geq 1})$, as well as a probability measure η_ω supported on $K(\omega,\{\tilde{d}_i\}_{i\geq 1})$.

If we choose the sequence $(\mathcal{N}'_i)_{i\geq 1}$ used in the construction so that $m_i^3 \leq n^i_{\mathcal{N}'_i}\varepsilon_i^3$, and consequently $m_i^3 \leq m_{i+1}\varepsilon_i^3$), then this growth speed yields that for any $x \in K(\omega,\{\tilde{d}_i\}_{i\geq 1})$, $\frac{\log\mu(B(x,r))}{\log(r)}$ will have d and d' as accumulating points and will fluctuate asymptotically between these points, hence $K(\omega,\{\tilde{d}_i\}_{i\geq 1}) \subset E(\mu_\omega, d, d')$. Also, for all $x \in K(\omega,\{\tilde{d}_i\}_{i\geq 1})$, one has $\liminf_{r\to 0} \frac{\log(\eta_\omega(B(x,r)))}{\log r} \geq \inf\{T^*(d), T^*(d')\}$.

From proposition 10.1 in [24], this will gives us that

$$\dim_H E(\mu_\omega, d, d') \geq \inf\{T^*(d), T^*(d')\}.$$

Second, for the packing dimension, we just need to notice that we can choose three sequences $(d_i)_{i\geq 1}$ $(d'_i)_{i\geq 1}$ and $(d''_i)_{i\geq 1}$in $\prod_{i\geq 1} D_i$ such that $\lim_{i\to\infty} d_i = d$, $\lim_{i\to\infty} d'_i = d'$ and $\lim_{i\to\infty} d''_i = d''$ with the properties:

$$\lim_{i\to\infty} T^*(d_i) = T^*(d),\ \lim_{i\to\infty} T^*(d'_i) = T^*(d'),$$

and

$$\lim_{i\to\infty} T^*(d''_i) = T^*(d'') = \sup\{T^*(\beta) : \beta \in [d, d']\}.$$

Take $\tilde{d}_{3i} = d_i$, $\tilde{d}_{3i+1} = d''_i$ and $\tilde{d}_{3i+2} = d'_i$.

Here again, we get $K(\omega, \{\tilde{d}_i\}_{i\geq 1})$ and η_ω, and if m_i grows fast enough, then for any $x \in K(\omega, \{\tilde{d}_i\}_{i\geq 1})$, $\frac{\log \mu(B(x,r))}{\log(r)}$ alternatively accumulates near d, d' and d'' and fluctuates between d and d' as $r \to 0$, so that $K(\omega, \{\tilde{d}_i\}_{i\geq 1}) \subset E(\mu_\omega, d, d')$; simultaneously,

$$\limsup_{r\to 0} \frac{\log(\eta_\omega(B(x,r)))}{\log r} = \sup\{T^*(\beta) : \beta \in [d, d']\}.$$

From proposition 10.1 in [24], this gives us

$$\dim_P E(\mu_\omega, d, d') \geq \sup\{T^*(\beta) : \beta \in [d, d']\}.$$

2. For the upper bound of the dimensions they directly come from (1) of proposition 1.3 and (1.2),(1.3) in [4].

Using lemma 6.3 and (2) of proposition 1.3 in [4] we can directly get the following corollary.

Corollary 6.4 *$\mathbb{P}$-almost every $\omega \in \Omega$, for any $d \in [T'(+\infty), T'(-\infty)]$,*

$$\dim_H \underline{E}(\mu_\omega, d) = T^*(d),\ \dim_P \underline{E}(\mu_\omega, d) = \sup\{T^*(d') : d' \geq d\},$$

and

$$\dim_H \overline{E}(\mu_\omega, d) = T^*(d),\ \dim_P \overline{E}(\mu_\omega, d) = \sup\{T^*(d') : d' \leq d\}.$$

Now, to finish the proof of theorem 1.11 we just need the following proposition.

Proposition 6.5 *For $\mathbb{P}$-almost every $\omega \in \Omega$,*

1. *for any gauge function* $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *such that* $\limsup_{r\to 0} \frac{\log g(r)}{\log r} > T^*(d)$, *one has that*
$$\mathcal{H}^g(E(\mu_\omega, d)) = 0.$$

2. *for any gauge function* $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *such that* $\liminf_{r\to 0} \frac{\log g(r)}{\log r} > T^*(d)$, *one has that*
$$\mathcal{P}^g(E(\mu_\omega, d)) = 0.$$

3. *if* $\dim_H E(\mu_\omega, d) = T^*(d) < \sup\{T^*(d') : d' \in [T'(+\infty), T'(-\infty)]\} = t_0 = \dim_H X_\omega$, *one has that for any gauge function* $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *such that* $\limsup_{r\to 0} \frac{\log g(r)}{\log r} \leq T^*(d)$, *one has that*
$$\mathcal{H}^g(E(\mu_\omega, d)) = +\infty.$$

4. *if* $\dim_P E(\mu_\omega, d) = T^*(d) < \sup\{T^*(d') : d' \in [T'(+\infty), T'(-\infty)]\} = t_0 = \dim_H X_\omega$, *one has that for any gauge function* $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *such that* $\liminf_{r\to 0} \frac{\log g(r)}{\log r} \leq T^*(d)$, *one has that*
$$\mathcal{P}^g(E(\mu_\omega, d)) = +\infty.$$

Proof 1. Since $\limsup_{r\to 0} \frac{\log g(r)}{\log r} > T^*(d)$, then there exist $\varepsilon > 0$ small enough and a sequence $\{r_j\}_{j\geq 1}$ such that $\lim_{j\to\infty} r_j = 0$ and $g(2r_j) \leq (2r_j)^{T^*(d)+2\varepsilon}$.

Also, since $\overline{LD}_\mu(d) = T^*(d)$, for $\epsilon > 0$ small enough, there exists $n \in \mathbb{N}$ such that for any $r \leq 2^{-n}$ one has that $\#\{i : r^{d+\epsilon} \leq \mu(B(x_i, r) \leq r^{d-\epsilon})\} \leq r^{-T^*(d)-\varepsilon}$.

From the definition of $E(\mu_\omega, d)$ we can get $E(\mu, d) \subset \cup_{N\geq 1} E_N$, where
$$E_N = \cap_{0<r\leq 2^{-N}}\{x \in \text{supp}(\mu) : r^{d+\epsilon} \leq \mu_\omega(B(x, r)) \leq r^{d-\epsilon}\}.$$

Fix $N \geq 1$. It follows from the previous lines that for any $n \geq N$, there exists $j \geq 1$ such that $r_j \leq 2^{-n}$ and we have
$$E_N \subset \{x \in \text{supp}(\mu) : r_j^{d+\epsilon} \leq \mu_\omega(B(x, r_j)) \leq r_j^{d-\epsilon}\}$$

It follows from Besicovitch's covering theorem (see [59]) that there exists an integer Q (which is a constant just depends on m in the space $\mathbb{R}^m$) such that, defining $E_j(\varepsilon) = \{x \in \text{supp}(\mu) : r_j^{d+\epsilon} \leq \mu_\omega(B(x, r_j)) \leq r_j^{d-\epsilon}\}$, we can extract from $\{B(x, r) : x \in E_j(\varepsilon)\}$, Q families $\mathcal{E}_k (1 \leq k \leq Q)$ of disjoint balls such that $E_j(\varepsilon) \subset \cup_{k=1}^{Q} \cup_{B\in\mathcal{E}_k} B$.

Then
$$\begin{aligned}\mathcal{H}^g_{2^{-n+1}}(E_N) &\leq \sum_{k=1}^{Q} \sum_{B\in\mathcal{E}_k} g(|B|) \leq Q(\sharp\mathcal{E}_k)(2r_j)^{T^*(d)+2\varepsilon} \\ &\leq Qr^{-T^*(d)-\varepsilon}(2r_j)^{T^*(d)+2\varepsilon} \leq (2)^{T^*(d)+1} Qr^{\varepsilon}\end{aligned}$$

letting n tends to ∞ yields $\mathcal{H}^g(E_N) = 0$ for any $N \geq 1$. Finally we get $\mathcal{H}^g(E(\mu_{\omega,d})) = 0$.

2. $\liminf_{r\to 0} \frac{\log g(r)}{\log r} > T^*(d)$ implies that for $\varepsilon > 0$ small enough, there exists r_0 such that for any $0 < r \le r_0$, one has $g(r) \le r^{T^*(d)+\varepsilon}$.

$$E_N \subset \cap_{0<2^{-p}\le 2^{-N}} \{x \in \text{supp}(\mu) : 2^{-p(d+\epsilon)} \le \mu_\omega(B(x, 2^{-p})) \le 2^{-p(d-\epsilon)}\},$$

   For any $A \subset [0,1]$, for any $n \in \mathbb{N}$, let $\{B(x_i, r_i) : i \in \mathbb{N}\}$ be an 2^{-n} packing of the set $A \cap E_N$. For each $p \ge n+1$, we can define $P_p = \{i \in \mathbb{N} : 2^{-p} < r_i \le 2^{-p+1}\}$. The balls in $\{B(x_i, 2^{-p}) : i \in P_p, 2^{-p(d+\epsilon)} \le \mu_\omega(B(x, 2^{-p})) \le 2^{-p(d-\epsilon)}\}$ form a 2^{-p}-packing of $\text{supp}(\mu_\omega)$ of cardinality less than $2^{p(T^*(d)+\eta)}$.

   Consequently,

$$\begin{aligned} \sum_i g(2r_i) &\le \sum_i (2r_i)^{T^*(d)+2\eta} \le \sum_{p\ge n} \sum_{i\in P_p} (2\cdot 2^{-p+1})^{T^*(d)+2\eta} \\ &\le 4^{T^*(d)+2\eta} \sum_{p\ge n} (\sharp P_p) 2^{-p(T^*(d)+2\eta)} \\ &\le 4^{T^*(d)+2\eta} \sum_{p\ge n} 2^{-p\eta}. \end{aligned}$$

   The upper bound does not depend on the choice of the 2^{-n}-packing $\{B(x_i, r_i) : i \in \mathbb{N}\}$ and goes to 0 as $n \to \infty$. It follows that the packing g pre-measure of $E_N \cap A$ with respect to the gauge function g is 0 for any $A \subset [0,1]$.

   At last we get $\mathcal{P}^g(E(\mu_\omega, d)) = 0$.

3. Let $\{\varepsilon'_i\}_{i\in\mathbb{N}}$ and $C' > 0$ be so that (6.16) holds. Since $T^*(d) < t_0$, we can find $\{d_i\}_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} D_i$ such that $d_i \to d$ as $i \to \infty$, $T^*(d_i) - \varepsilon'_i \ge T^*(d) + \varepsilon'_i$ for i large enough, $T^*(d_i) \to T^*(d)$ as $i \to \infty$, and $T^*(d_i) - \varepsilon'_i$ is ultimately non increasing.

   For any gauge function g such that $\limsup_{r\to 0} \frac{\log g(r)}{\log r} \le T^*(d)$, there exists a positive sequence $\{\upsilon_r\}_{r>0}$ such that both υ_r and r^{υ_r} decrease to 0 as r decreases to 0 and

$$g(r) \ge r^{T^*(d)+\upsilon_r} \quad (r \le 1).$$

   Due to (6.16) for i large enough, for any r such that $\exp(-m_{i+1}c_\Psi/2) \le r \le \exp(-m_i c_\Psi/2)$, for any $x \in K(\omega, \{d_i\}_{i\ge 1})$,

$$\eta_\omega(B(x,r)) \le C' r^{\min\{T^*(d_j)-\varepsilon'_j : 1\le j\le i\}} \le C' r^{T^*(d_i)-\varepsilon'_i} \le C' r^{T^*(d)+\varepsilon'_i}.$$

   Notice that $g(r) r^{\upsilon_r} \ge r^{T^*(d)+2\upsilon_r}$. So, if we can impose $2\upsilon_r \le \varepsilon'_i$, we will have

$$\eta_\omega(B(x,r)) \le C' g(r) r^{\upsilon_r}$$

   hence

$$g(r) \ge C'^{-1} \eta_\omega(B(x,r)) r^{-\upsilon_r}.$$

Then, for any positive real number $\delta > 0$, this will yield $\mathcal{H}^g_\delta(K(\omega, \{d_i\}_{i\geq 1})) \geq \delta^{-\upsilon_\delta}$, and letting $\delta \to 0$, $\mathcal{H}^g(K(\omega, \{d_i\}_{i\geq 1})) = +\infty$, and since $K(\omega, \{d_i\}_{i\geq 1}) \subset E(\mu_\omega, d)$, item 3. will be proven.

Now, if we choose m_i large enough so that

$$\upsilon_{\exp(-m_i c_\Psi/2)} \leq \varepsilon'_i/2,$$

then for $\exp(-m_{i+1}c_\Psi/2) \leq r \leq \exp(-m_i c_\Psi/2)$, we have $2\upsilon_r \leq \varepsilon'_i$ since $\upsilon_r \leq \upsilon_{\exp(-m_i c_\Psi/2)}$.

4. Since $\liminf_{r\to 0} \frac{\log g(r)}{\log r} \leq T^*(d)$, there exist $\{r_j\}_{i\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$, and $\{\upsilon_{r_j}\}_{j\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ such that $\upsilon_{r_j} \in (0,1]$ and $r_j^{\upsilon_j}$ decrease to 0 as j tends to ∞, and

$$g(2r_j) \geq r_j^{T^*(d)+\upsilon_{r_j}}.$$

Using the same approach as for 3., we can choose $(d_i)_{\geq 1} \in \prod_{i\geq 1} D_i$ such that $\lim_{i\to\infty} d_i = d$, $T^*(d_i)$ converges slowly to $T^*(d)$ from above, and in the construction of $(K(\omega, \{d_i\}_{i\geq 1}), \eta_\omega)$ as in the previous section, m_i tends fast enough to ∞ so that, for some $j_0 \in \mathbb{N}$, for all $j \geq j_0$, for any $x \in K(\omega, \{d_i\}_{i\geq 1}$,

$$\eta_\omega(B(x, r_j)) \leq C'(2r_j)^{T^*(d)+2\upsilon_{r_j}}.$$

Now, let $A \subset K(\omega, \{d_i\}_{i\geq 1})$ be of positive η_ω-measure. For any given $\delta > 0$, take $j'_0 \geq j_0$ such that $r_{j'_0} \leq \delta$ consider the following family of closed balls

$$\mathcal{B}_k = \{B(x, r_j) : x \in A, j \geq j'_0\},$$

which is a covering of A. Due to Besicovitch covering theorem, we can extract an at most countable subfamily of pairwise disjoint balls $\{B(x_i, \rho_i)\}_{i\in I}$ such that $\eta_\omega(\bigcup_{i\in I} B_i) > 0$. This family is a δ-packing of A, and

$$\begin{aligned}
\mathcal{P}^g_{0,\delta}(A) &\geq \sum_i g(B(x_i, \rho_i)) \geq \sum_i \rho_i^{T^*(d)+\upsilon_{\rho_i}} \\
&\geq \sum_s \rho_i^{-\upsilon_{\rho_i}} \eta_\omega(B(x_i, \rho_i)) \\
&\geq \rho_{j'_0}^{-\upsilon_{\rho_{j'_0}}} \eta_\omega(A).
\end{aligned}$$

Since when $\delta \to 0$, we have $j'_0 \to \infty$ and then $\rho_{j'_0}^{-\upsilon_{\rho_{j'_0}}} \to \infty$, we can conclude that $P^g_0(A) = +\infty$. Since any at most countable covering of $K(\omega, \{d_i\}_{i\geq 1})$ must contain a set A of positive η_ω-measure, we finally get $\mathcal{P}^g(K(\omega, \{d_i\}_{i\geq 1})) = +\infty$. Finally, $\mathcal{P}^g(E(\mu_\omega, d)) = +\infty$ since $E(\mu_\omega, d) \supset K(\omega, \{d_i\}_{i\geq 1})$.

Chapter 7

Multifractal analysis of the inverse measures: Proof of Theorem 1.13

After introducing new notations in section 7.1, we give an explicit writing of the measure ν_ω and some useful estimate of the mass of its atoms in section 7.2. Then, in section 7.3. we start the multifractal analysis of ν_ω by examining the possible scenarii which lead to a given lower local dimension. This yields a first, not everywhere sharp, but very useful for the sequel, upper bound for the lower Hausdorff spectrum. Indeed, it is already related to conditioned ubiquity properties associated with the sets of atoms, and thus it provides a beginning of concrete explanation of the origin of the linear part in the lower Hausdorff spectrum. Then, in section 7.4, we derive the sharp upper bound for the L^q-spectrum of ν_ω, in which ubiquity properties remain hidden. Section 7.5 derives the sharp lower bound for the lower Hausdorff spectrum in its non linear part. This is based on the study of weak Gibbs measures achieved in chapter 6. Then section 7.6 prepares section 7.7, which provides the conditioned ubiquity theorem used in section 7.8 to get the sharp lower bound for the lower Hausdorff spectrum in the linear part. Finally, section 7.9 deals with the Hausdorff dimension of the level sets $E(\nu_\omega, d)$ and $\overline{E}(\nu_\omega, d)$.

7.1 Some notations

Since Assumption 2 implies that μ_ω is atomless $\mathbb{P}$-almost surely (due to proposition 2.7), without loss of generality, we assume that this is the case for all $\omega \in \Omega$.

For $\omega \in \Omega$, $n \geq 1$, $v \in \Sigma_{\omega,n}$ and $k \geq 1$ we define

$$S(\omega, v, k) = \{w \in \Sigma_{\sigma^n\omega,k} : vw \in \Sigma_{n+k}(\omega)\},$$

the set of words in $\Sigma_{\sigma^n\omega,k}$ which can be a suffix of v.

Next we consider the set of words w in $S(\omega, v, k)$ such that U^{vw} has a right neighbor $U^{v\tilde{w}}$, with $\tilde{w} \in S(\omega, v, k)$:

$$S'(\omega, v, k) = \left\{ w \in S(\omega, v, k) \;\middle|\; \begin{array}{l} \text{there exists } \widetilde{w} \in S(\omega, v, k) \text{ such that} \\ U_\omega^{v\widetilde{w}} \text{ is the nearest right neighbor of } U_\omega^{vw} \end{array} \right\}.$$

We need to point out that such a set can be empty. For any $w \in S'(\omega, v, k)$, we denote by $\widetilde{w}$ the element of $S(\omega, v, k)$ such that $U_\omega^{v\widetilde{w}}$ is the closest right neighbor of U_ω^{vw}.

For every $v \in \Sigma_{\omega,*}$, $k \geq 1$ and $w \in S(\omega, v, k)$, define

$$m_\omega^{vw} = \min X_\omega^{vw} \quad \text{and} \quad M_\omega^{vw} = \max X_\omega^{vw},$$

as well as

$$\operatorname{gap}(\omega, k) = \inf_{v \in \Sigma_{\omega,1}} \sup_{1 \leq m \leq k} \sup_{w \in S'(\omega, v, m)} \{m_\omega^{v\widetilde{w}} - M_\omega^{vw}\}.$$

For any $v \in \Sigma_{\omega,*}$, we define

$$I_\omega^v := [F_{\mu_\omega}(m_\omega^v), F_{\mu_\omega}(M_\omega^v)) = F_{\mu_\omega}(X_\omega^v) \setminus \{F_{\mu_\omega}(M_\omega^v)\}.$$

Since the support of μ_ω restricted to the interval $[m_\omega^v, M_\omega^v]$ (or U_ω^v) is X_ω^v, and μ_ω is atomless, from the construction, we get that I_ω^v is a non-empty interval of length $|I_\omega^v| = \mu_\omega(X_\omega^v) = \widetilde{\mu}_\omega([v]_\omega)$.

Since $\operatorname{supp}(\mu_\omega) = X_\omega$ and $\cup_{v \in \Sigma_{\omega,n}} X_\omega^v = X_\omega$, we can get that the family of intervals $\mathcal{F}_\omega^n = \{I_\omega^v\}_{v \in \Sigma_{\omega,n}}$, $n \geq 1$, form a nested interval of $[0, 1)$. For any $n \in \mathbb{N}$, for any $v \in \Sigma_{\omega,n}$, we call I_ω^v the n-th basic grid.

7.2 An explicit writing of the inverse measure ν_ω, and preliminary estimates for the mass of atoms

For any $v \in \Sigma_{\omega,*}$ and $s \in S'(\omega, v, 1)$, denote $x_\omega^{vs} = F_{\mu_\omega}(M_\omega^{vs})$. Denote $m_\omega^{\min} = \min X_\omega$ and $M_\omega^{\max} = X_\omega$.

Using the same method as [11] for the inverse measures of deterministic Gibbs measures on cookie-cutter sets, we can get the following explicit form for the inverse of the random weak Gibbs measures $\{\mu_\omega : \omega \in \Omega\}$.

Proposition 7.1 (The inverse measure ν_ω of μ_ω) *Suppose that the Assumptions (1) and (2) hold. Then the inverse measure ν_ω of the random weak Gibbs measure μ_ω is the discrete probability measure on $[0, 1]$ given by the following weighted sum of Dirac measures:*

$$\nu_\omega = m_\omega^{\min} \cdot \delta_0 + \sum_{v \in \Sigma_{\omega,*}} \sum_{s \in S'(\omega, v, 1)} (m_\omega^{v\widetilde{s}} - M_\omega^{vs}) \cdot \delta_{x_\omega^{vs}} + (1 - M_\omega^{\max})\delta_1. \tag{7.1}$$

This proposition can be easily proved if we notice the following two facts. On the one hand, from the definition we can get for each point x_ω^{vs}, the measure is at least $m_\omega^{\widetilde{vs}} - M_\omega^{vs}$. On the other hand, we the whole measure is 1 and $m_\omega^{\min} + \sum_{v\in\Sigma_{\omega,*}} \sum_{s\in S'(\omega,v,1)}(m_\omega^{\widetilde{vs}} - M_\omega^{vs}) + (1 - M_\omega^{\max}) = 1$ from the assumption (2).

Lemma 7.2 *If Assumption (2) holds, then*

$$\mathbb{P}(\{\omega \in \Omega : \sup_{k\geq 1} \operatorname{gap}(\omega, k) > 0\}) > 0.$$

Consequently, there exist some $k_\psi > 0$ and $c > 0$ such that $\mathbb{P}(\operatorname{Gap}(k_\psi, c)) > 0$ where $\operatorname{Gap}(k_\psi, c) =: \{\omega \in \Omega : \operatorname{gap}(\omega, k_\psi) > c\}$.

Remark 7.3 *We point out that the assumption (2) is not necessary to get lemma 7.2. We only need that X_ω is not equal to $[0,1]$ for $\mathbb{P}$-almost every $\omega \in \Omega$. This can deduced from the beginning of chapter 4.*

Proof By contradiction:

If the result does not hold, then for $\mathbb{P}$-almost every ω, there exists $v \in \Sigma_{\omega,1}$ such that

$$\sup_{m\in\mathbb{N}} \sup_{w\in S'(\omega,v,m)} m_\omega^{\widetilde{vw}} - M_\omega^{vw} = 0.$$

This implies that X_ω^v can not have any gaps. Then X_ω^v is either a point or an interval. From the assumption that X_ω has a Lebesgue measure 0, we get that it is a point.

Now, defining

$$B = \{\omega \in \Omega : \ M(\omega) \leq M, \ l(\omega) \geq 2\},$$

we have $\mathbb{P}(B) > 0$ for M large enough. For any $\omega \in \Omega$, define $b_k(\omega)$ the k-th return time of ω to the set B by the map σ. From ergodic theorem we have that $\lim_{k\to\infty} \frac{b_k(\omega)}{k} = \frac{1}{\mathbb{P}(B)}$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Define $\Omega' = \{\omega \in \Omega : \lim_{k\to\infty} \frac{b_k(\omega)}{k} = \frac{1}{\mathbb{P}(B)}\}$.

For any $\omega \in \Omega'$, we know that there is at least four words in $\Sigma_{\omega,b_{M+2}}$ with the prefix $v \in \Sigma_{\omega,1}$, and we denote them by w^1, w^2, w^3 and w^4. We can assume that these intervals appear from the left to the right as $U_\omega^{w^1}, U_\omega^{w^2}, U_\omega^{w^3}, U_\omega^{w^4}$. The sets $X_\omega^{w^i} \subset U_\omega^{w^i}, i = 1,2,3,4$, are not empty since by definition the random transition matrix A has at least one non-zero entry in each row and each column. Choose $x_i \in X_\omega^{w^i} \subset U_\omega^{w^i}, i = 1,2,3,4$. Since $U_\omega^{w^i}, i = 1,2,3,4$ are intervals, we have that $x_4 - x_1 > 0$, which contradicts the fact that X_ω^v is a point.

With the same method as in the proof of propositions 2.6 and 2.7 we can get the following proposition:

Proposition 7.4 *Under assumption 2, for $\mathbb{P}$-almost every $\omega \in \Omega$, for all $n \in \mathbb{N}$, for all $v \in \Sigma_{\omega,n}$, there exists k_v and $w \in S'(\omega, v, k_v)$ such that*

$$m_\omega^{v\widetilde{w}} - M_\omega^{vw} \geq \exp(S_n\Psi(\omega, \underline{v}) - o(n))$$

for any $\underline{v} \in [v]_\omega$. Here $o(n)$ is independent of v, and $k_v = o(n)$ independently of v as well.

Proof For any $N \in \mathbb{N}_+$, let

$$\Omega'_N = \left\{\omega : \; M(\omega) \leq N, \; \frac{1}{n}\sum_{k=0}^{n-1} \sup_{1\leq s\leq l(\sigma^k\omega)} \sup_{x\in U_{\sigma^k\omega}} |\psi(\omega, s, x)| \leq 2C_\psi, \forall n \geq N\right\}.$$

Choose N large enough such that $\mathbb{P}(\Omega'_N \cap \mathrm{Gap}(k_\psi, c)) > 0$, where $\mathrm{Gap}(k_\psi, c)$ was defined in lemma 7.2.

For $\mathbb{P}$-almost every $\omega \in \Omega$, for n large enough, denote by $H(n)$ the smallest integer such that $\sigma^{n+H(n)}\omega \in \Omega'_N \cap \mathrm{Gap}(k_\psi, c)$ and $H(n) \geq N$. Since $\mathbb{P}(\Omega'_N \cap \mathrm{Gap}(k_\psi, c)) > 0$, from ergodic theorem we can get that $\lim_{n\to\infty} \frac{H(n)}{n} = 0$. Moreover, since $\sigma^{n+H(n)}\omega \in \Omega'_N \cap \mathrm{Gap}(k_\psi, c)$, there exists some $1 \leq s \leq l(\sigma^{n+H(n)}\omega)$ and $v' \in S'(\sigma^{n+H(n)}\omega, s, k)$ with $k \leq k_\psi$ such that $m^{s\widetilde{v'}}_{\sigma^{n+H(n)}\omega} - M^{sv'}_{\sigma^{n+H(n)}\omega} > c$. For any $v \in \Sigma_{\omega,n}$ there exists v'' of length $H(n) - 1$ such that $vv''s \in \Sigma_{\omega,n+H(n)}$ (by definition of $M(\sigma^{n+H(n)})$ and since $H(n) \geq N \geq M(\sigma^{n+H(n)})$). Set $w = v''sv'$ and $\widetilde{w} = v''s\widetilde{v'}$. We have $w \in S'(\sigma^n\omega, v, H(n) + k)$. Moreover, $T_\omega^{vv''}([M_\omega^{vw}, m_\omega^{v\widetilde{w}}]) = [M^{sv'}_{\sigma^{n+H(n)-1}\omega}, m^{s\widetilde{v'}}_{\sigma^{n+H(n)-1}\omega}]$. Now using Lagrange's finite-increment theorem and the same approach as in lemma 2.7 we can prove that

$$m_\omega^{v\widetilde{w}} - M_\omega^{vw} \geq c\exp(S_n\Psi(\omega, \underline{v}) - o(n)).$$

for any $\underline{v} \in [v]_\omega$, since $H(n)$ is a $o(n)$. Morover, $k_v = |w| = H(n) + k = o(n)$.

Definition 7.5 *Proposition 7.4 implies that for $\mathbb{P}$-almost every $\omega \in \Omega$, for all $n \in \mathbb{N}$, for any $v \in \Sigma_{\omega,n}$, there exist some point $x = x_\omega^{vw}$ such that*

$$\nu_\omega(\{x\}) \geq \exp(S_n\Psi(\omega, \underline{v}) - o(n))$$

for any $\underline{v} \in [v]_\omega$. For each $v \in \Sigma_{\omega,*}$, we fix one such point and denote it by z_ω^v.

Arguments similar to those giving proposition 7.4 lead to the following remark.

Remark 7.6 *For $\mathbb{P}$-almost every $\omega \in \Omega$, for all $n \in \mathbb{N}$ and $v \in \Sigma_{\omega,n}$, for any $\underline{v} \in [v]_\omega$,*

$$|X_\omega^v| \geq \exp(S_n\Psi(\omega, \underline{v}) - o(n)),$$

where the $o(n)$ does not depend on the choice of v. Then, using point (1) of proposition 2.7 we get

$$\exp(S_n\Psi(\omega, \underline{v}) - o(n)) \leq |X_\omega^v| \leq \exp(S_n\Psi(\omega, \underline{v}) + o(n)).$$

7.3 Pointwise behavior of ν_ω and an upper bound for the lower Hausdorff spectrum without using of multifractal formalism

Definition 7.7 *For $v \in \Sigma_{\omega,*}$, we set*

$$\ell_\omega^v = 2|I_\omega^v| = 2\widetilde{\mu}_\omega([v]_\omega),$$
$$\xi_\omega^v = \log |I_\omega^v|,$$
$$\alpha_\omega^v = \frac{\widetilde{\Psi}(\omega, v)}{\xi_\omega^v},$$

where

$$\widetilde{\Psi}(\omega, v) = \sup_{\underline{v}\in[v]_\omega} \{S_{|v|}\Psi(\omega, \underline{v})\}.$$

For $x \in [0, 1)$ and $n \geq 1$, let $v(\omega, n, x)$ stand for the unique element v in $\Sigma_{\omega,n}$ such that $x \in I_\omega^{v(\omega,n,x)}$. If $x = 1$, $v(\omega, n, 1)$ is the unique $v \in \Sigma_{\omega,n}$ such that $1 \in \overline{I_\omega^v}$. If there is no confusion we will denote $v(\omega, n, x)$ by $v(n, x)$ or $x|_n$ for short. Let

$$\alpha_\omega^n(x) = \alpha_\omega^{x|_n},$$
$$\alpha_\omega(x) = \liminf_{n\to\infty} \alpha_\omega^n(x).$$

For $x \in [0, 1] \setminus \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}$, the approximation degrees ξ_ω^x and $\widetilde{\xi}_\omega^x$ by the system $\{(x_\omega^{vs}, \ell_\omega^v)\}_{v\in\Sigma_{\omega,*},s\in S'(\omega,v,1)}$ are defined as

$$\xi_\omega^x = \limsup_{n\to\infty} \left(\sup_{v\in\Sigma_n(\omega)} \sup_{s\in S'(\omega,v,1)} \frac{\log |x - x_\omega^{vs}|}{\log \ell_\omega^v} \right)$$

$$\widetilde{\xi}_\omega^x = \limsup_{n\to\infty} \sup_{s\in S'(\omega,x|_n,1)} \frac{\log |x - x_\omega^{x|_n s}|}{\log \ell_\omega^{x|_n}}$$

Since we use only one specific word in the definition of $\widetilde{\xi}_\omega^x$, we have $\xi_\omega^x \geq \widetilde{\xi}_\omega^x \geq 1$, for every $x \in [0, 1] \setminus \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}$.

Proposition 7.8
1. *If $x \in \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1),$ and $m_\omega^{\widetilde{vs}} - M_\omega^{vs} > 0\}$, then $\nu_\omega(\{x\}) > 0$, thus $\dim_{\mathrm{loc}}(\nu_\omega, x) = 0$.*
2. *For any $x \in [0, 1]$, if $x \notin \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}$, then*

$$\frac{\alpha_\omega(x)}{\xi_\omega^x} \leq \frac{\alpha_\omega(x)}{\widetilde{\xi}_\omega^x} \leq \underline{\dim}_{\mathrm{loc}}(\nu_\omega, x) \leq \alpha_\omega(x).$$

Here, if $\xi_\omega^x = +\infty$ then $\frac{\alpha_\omega(x)}{\xi_\omega^x} := 0$. In the same way, if $\widetilde{\xi}_\omega^x = +\infty$ then $\frac{\alpha_\omega(x)}{\widetilde{\xi}_\omega^x} := 0$.

Proof (1) is obvious, we just give the proof of (2).

Let $x \in [0,1] \setminus \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}$ and $r > 0$.

- The first inequality is obvious since one always has $\xi_\omega^x \geq \widetilde{\xi}_\omega^x \geq 1$.
- For the second inequality, let

$$n_\omega^{x,r} = \max\{n : \exists v \in \Sigma_{\omega,n}, \text{ such that } B(x,r) \subset I_\omega^v\}. \tag{7.2}$$

If $x \notin \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}$, then $n_\omega^{x,r} \leq \frac{-2\log r}{c_\psi}$ for r small enough and $n_\omega^{x,r} \to \infty$ as $r \to 0$. Since n is the largest one then there exist a unique v denoted by $v(x,r) := x|_{n_\omega^{x,r}}$ and a $s \in S'(\omega, v(x,r), 1)$ such that

$$x_\omega^{v(x,r)s} \in B(x,r) \subset I_\omega^{v(x,r)}.$$

Hence, we get

$$\begin{aligned} & \nu_\omega(B(x,r)) \leq \nu_\omega(I_\omega^{v(x,r)} \setminus \{\max I_\omega^{v(x,r)}, \min I_\omega^{v(x,r)}\}) \leq |X_\omega^{v(x,r)}| \leq |U_\omega^{v(x,r)}| \\ \leq \ & \exp(S_{|v(x,r)|}\Psi(\omega, \underline{v}) + o(|v(x,r)|)) \text{ where } \underline{v} \in [v(x,r)]_\omega, \end{aligned}$$

by proposition 2.7(1).

Now for any $\varepsilon > 0$, by definition of $\widetilde{\xi}_\omega^x$, for r small enough we have

$$r \geq |x - x_\omega^{v(x,r)s}| \geq (2|I_\omega^{v(x,r)}|)^{\widetilde{\xi}_\omega^x + \varepsilon}.$$

Moreover, again for r small enough, we have

$$\exp(\widetilde{\Psi}(\omega, v(x,r))) \leq |I_\omega^{v(x,r)}|^{\alpha_\omega(x) - \varepsilon}$$

by definition of $\alpha_\omega(x)$.

These estimates yield

$$\nu_\omega(B(x,r)) \leq \exp(\widetilde{\Psi}(\omega, v(x,r)) + o(|v(x,r)|)) \leq r^{\frac{\alpha_\omega(x)-\varepsilon}{\widetilde{\xi}_\omega^x + \varepsilon}} \exp(o(n_\omega^{x,r})),$$

and by letting r tend to zero, since $n_\omega^{x,r} \leq \frac{-2\log r}{c_\psi}$, it follows that $\underline{\dim}_{\text{loc}}(\nu_\omega, x) \geq \frac{\alpha_\omega(x)-\varepsilon}{\widetilde{\xi}_\omega^x + \varepsilon}$. From the arbitrariness of ε we get that $\underline{\dim}_{\text{loc}}(\nu_\omega, x) \geq \frac{\alpha_\omega(x)}{\widetilde{\xi}_\omega^x}$.

- Finally, for the third inequality, let $\{p_i\}_{i\geq 1}$ be an increasing sequence of integers such that $\exp(\widetilde{\Psi}(\omega, v_{p_i})) \geq |I_\omega^{v_{p_i}}|^{\alpha_\omega(x)+\epsilon}$ where $v_{p_i} \in \Sigma_{\omega,p_i}$ for all $i \geq 1$. Since $z_\omega^{v_{p_i}} \in B(x, 2|I_\omega^{v_{p_i}}|)$,

$$\begin{aligned} \nu_\omega(B(x, 2|I_\omega^{v_{p_i}}|)) &\geq \nu_\omega(\{z_\omega^{v_{p_i}}\}) \geq \exp(S_{p_i}\Psi(\omega, \underline{v}))\exp(-o(p_i)) \\ &\geq |I_\omega^{v_{p_i}}|^{\alpha_\omega(x)+\epsilon}\exp(-o(p_i)). \end{aligned}$$

Also, $|I_\omega^v| \leq \exp(-\frac{\varpi p_i}{2})$ for p_i large enough and ϵ can approximate 0 arbitrarily, so $\underline{\dim}_{\text{loc}}(\nu_\omega, x) \leq \alpha_\omega(x)$.

Definition 7.9 *Let $\alpha > 0, \xi \geq 1$ and $\varepsilon > 0$. A real number $x \in [0,1]$ is said to satisfy the property $\mathcal{P}(\alpha, \xi, \varepsilon)$ if there exists an increasing sequence of positive integers $(n_k)_{k\geq 1}$ such that for every $k \geq 1$, there exists $v \in \Sigma_{\omega,n_k}$ and $s \in S'(\omega, v, 1)$, such that $x \in B(x_\omega^{vs}, (\ell_\omega^{vs})^{\xi-\varepsilon})$ and $\alpha_\omega^v \in [\alpha - \varepsilon, \alpha + \varepsilon]$.*

Remark 7.10 *Arguments similar to the previous ones show that $\overline{\dim}_{\mathrm{loc}}(\nu_\omega, x) \leq \limsup_{n\to\infty} \alpha_\omega^n(x)$.*

We need additional definitions.

Definition 7.11 *For $d \geq 0$, let*

$$F(d) = \left\{ x \in (0,1) \,\middle|\, \begin{array}{l} \forall \varepsilon > 0, \exists \alpha \in \mathbb{Q}^+, \exists \xi \in \mathbb{Q}, \xi \geq 1 \text{ such that} \\ \alpha/\xi \leq d + 2\varepsilon \text{ and } x \text{ satisfies the property } \mathcal{P}(\alpha, \xi, \varepsilon) \end{array} \right\}.$$

Definition 7.12 *For every $\alpha, \varepsilon > 0$ and $\xi \geq 1$, let*

$$G(\alpha, \varepsilon, \xi) = \cap_{N\geq 1} \cup_{n\geq N} \cup_{v\in\Sigma_{\omega,n}:\alpha_\omega^v\in[\alpha-\varepsilon,\alpha+\varepsilon]} B(x_\omega^v, (\ell_\omega^v)^\xi).$$

It is easily seen that

$$F(d) \subset \cup_{\alpha\in\mathbb{Q}^+} \cup_{\xi\in\mathbb{Q}\cap[1,+\infty),\alpha/\xi\leq d+2\varepsilon} G(\alpha, \varepsilon, \xi).$$

Proposition 7.13 *For $\mathbb{P}$-almost every ω, for any $h \geq 0$, we have $(\underline{E}(\nu_\omega, h) \setminus \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}) \subset F(h)$.*

Proof Fix $d \geq 0$, $x \in \underline{E}(\nu_\omega, d)$ and $\varepsilon > 0$. By definition of $\underline{\dim}_{\mathrm{loc}}(\nu_\omega, x)$, there exists a sequence $(r_k)_{k\geq 1}$ of positive numbers decreasing to zero such that for all $k \geq 1$ we have $\nu_\omega(B(x, r_k)) \geq (r_k)^{d+\varepsilon}$. Let us recall the definition of $n_\omega^{x,r}$ as in (7.2):

$$n_\omega^{x,r} = \max\{n : \exists v \in \Sigma_{\omega,n} \text{ such that } B(x,r) \subset I_\omega^v\}.$$

If $x \notin \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}$, then $n_\omega^{x,r} \to \infty$ as $r \to 0$. Since $n_\omega^{x,r}$ is maximal, there exist $v = v(x,r)$ and $s \in S'(\omega, v(x,r), 1)$ such that

$$x_\omega^{v(x,r)s} \in B(x,r) \subset I_\omega^{v(x,r)}.$$

Then $\nu_\omega(B(x,r)) \leq \nu_\omega(I_\omega^{v(x,r)} \setminus \{\max I_\omega^{v(x,r)}, \min I_\omega^{v(x,r)}\}) \leq |U_\omega^{v(x,r)}|$, and

$$(r_k)^{d+\varepsilon} \leq \nu_\omega(B(x,r)) \leq \exp(\widetilde{\Psi}(\omega, v(x,r_k)) + o(|v(x,r_k)|)).$$

Consequently,

$$|I_\omega^{v(x,r_k)}|^{\alpha_\omega^{v(x,r_k)}+o(1)} \geq (r_k)^{d+\varepsilon}.$$

Since $x_\omega^{v(x,r)s} \in B(x,r)$, $|x - x_\omega^{v(x,r_k)s}| \le r_k$. Writing $r_k \ge |x - x_\omega^{v(x,r_k)s}| = (2(|I_\omega^{v(x,r_k)}|))^{\xi_k} = (\ell_{\omega,v(x,r_k)})^{\xi_k}$, we have

$$|I_\omega^{v(x,r_k)}|^{\alpha_\omega^{v(x,r_k)}+o(1)} \ge (2(|I_\omega^{v(x,r_k)}|))^{\xi_k(d+\varepsilon)},$$

and $\xi_k \ge 1$.

If $\limsup_{k\to\infty} \xi_k < \infty$, there exists $(\alpha,\xi) \in \mathbb{Q}^+ \times (\mathbb{Q} \cap [1,+\infty))$ and an increasing sequence of integer number $(k_s)_{s\ge1}$ such that

$$|\alpha_\omega^{v(x,r_{k_s})} - \alpha| \le \varepsilon,$$

$$|\xi_{k_s} - \xi| \le \varepsilon,$$

$$\alpha/\xi \le d + 2\varepsilon.$$

This means $x \in G(\alpha,\varepsilon,\xi)$, $(\alpha,\xi) \in \mathbb{Q}^+ \times (\mathbb{Q} \cap [1,+\infty))$ and $\alpha/\xi \le h + 2\varepsilon$.

If $\limsup_{k\to\infty} \xi_k = \infty$, there exists $\alpha \in \mathbb{Q}^+$ and an increasing sequence of integer number $(k_s)_{s\ge1}$ such that

$$|\widetilde{\alpha}_\omega^{v(x,r_{k_s})} - \alpha| \le \varepsilon,$$

$$\xi_{k_s} \to \infty,$$

Since $\alpha_\omega^{v(x,r_{k_s})}$ is bounded (for $\mathbb{P}$-almost every ω), there exists some $\xi \in \mathbb{Q} \cap [1,+\infty)$ with $\alpha/\xi \le d + 2\varepsilon$ such that x satisfies $\mathcal{P}(\alpha,\varepsilon,\xi)$ (because if $\xi_1 \le \xi_2$ then $\mathcal{P}(\alpha,\varepsilon,\xi_2)$ implies $\mathcal{P}(\alpha,\varepsilon,\xi_1)$).

Finally,

$$(\underline{E}(\nu_\omega,d) \setminus \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega,v,1)\}) \subset F(d). \tag{7.3}$$

Lemma 7.14 *There exists $C > 0$ such that for $\mathbb{P}$-almost every ω, for $\varepsilon > 0$ small enough, for all rationals $\alpha > 0$ and $\xi \ge 1$,*

$$\dim_H G(\alpha,\varepsilon,\xi) \le C\varepsilon + \frac{\max(\mathcal{T}^*(\alpha-\varepsilon), \mathcal{T}^*(\alpha), \mathcal{T}^*(\alpha+\varepsilon))}{\xi}.$$

When the right hand side of the inequality is negative, it means the set $G(\alpha,\varepsilon,\xi)$ is empty.

Proof We only need to deal with fixed $\epsilon > 0$ and rationals $\alpha > 0$ and $\xi \ge 1$. For any $N \ge 1$, let $\delta_N = \sup_{v\in\Sigma_{\omega,N}} \ell_\omega^v$. By construction, if $G(\alpha,\varepsilon,\xi) \ne \emptyset$, given $s \in \mathbb{R}$ we have

$$\mathcal{H}^s_{\delta_N}(G(\alpha,\varepsilon,\xi)) \le \sum_{n\ge N} \sum_{v\in\Sigma_{\omega,n}:\alpha-\varepsilon\le\widetilde{\alpha}_\omega^v\le\alpha+\varepsilon} l(\sigma^n\omega) 2^s (\ell_\omega^v)^{s\xi}$$

where we naturally extend the definition of $\mathcal{H}^s_\delta$ to negative s.

Here, to avoid confusions, we recall that $l(\omega)$ is the number of types in the subshift, and ℓ_ω^v is the length of the interval I_ω^v.

Case 1 $\alpha \leq \mathcal{T}'(0-) - \varepsilon$**:** In other words, $\alpha + \varepsilon \leq \mathcal{T}'(0-)$.

Since $\alpha_\omega^v = \frac{\widetilde{\Psi}(\omega,v)}{\log|I_\omega^v|}$, then for any $q \geq 0$ one has:

$$\begin{aligned}
\mathcal{H}^s_{\delta_N}(G(\alpha,\varepsilon,\xi)) &\leq 2^s \sum_{n\geq N} \sum_{v\in\Sigma_{\omega,n}: q\widetilde{\Psi}(\omega,v)\geq q(\alpha+\varepsilon)\log|I_\omega^v|} l(\sigma^n\omega)(\ell_\omega^v)^{s\xi} \\
&\leq 4^s \sum_{n\geq N} \sum_{v\in\Sigma_{\omega,n}} l(\sigma^n\omega)\exp(q\widetilde{\Psi}(\omega,v) - q(\alpha+\varepsilon)\log|I_\omega^v|) \\
&\quad \cdot \exp(s\xi\log|I_\omega^v| + o(n))
\end{aligned}$$

almost surely. Now take $s = (\eta + (\alpha+\varepsilon)q - \mathcal{T}(q))/\xi$ with $\eta > 0$. We get,

$$\mathcal{H}^s_{\delta_N}(G(\alpha,\varepsilon,\xi)) \leq \sum_{n\geq N}\sum_{v\in\Sigma_{\omega,n}} \exp(q\Psi(\omega,\underline{v}) - (\mathcal{T}(q)+\eta)\Phi(\omega,\underline{v}))\cdot\exp(o(n)),$$

where $\underline{v}$ is any element of $[v]_\omega$. Then

$$\begin{aligned}
\mathcal{H}^s_{\delta_N}(G(\alpha,\varepsilon,\xi)) &\leq \sum_{n\geq N}\sum_{v\in\Sigma_{\omega,n}} \widetilde{\mu}_\omega^{q\Psi-\mathcal{T}(q)\Phi}([v]_\omega)\exp(-\eta c_\Phi n + o(n)) \\
&\leq \sum_{n\geq N}\exp(-\frac{\eta c_\Phi}{2}n)
\end{aligned}$$

for n large enough (recall the assumption (2), from which $c_\Phi = c_\phi > 0$). Consequently, $\lim_{N\to\infty}\mathcal{H}^s_{\delta_N}G(\alpha,\varepsilon,\xi) = 0$. However, if $\mathcal{T}^*(\alpha+\varepsilon) < 0$, we can choose η and q such that $s < 0$, in which case it is necessary that $\lim_{N\to\infty}\mathcal{H}^s_{\delta_N} = +\infty$ if $G(\alpha,\varepsilon,\xi)$ is not empty. Consequently, if $\mathcal{T}^*(\alpha+\varepsilon) < 0$, then $G(\alpha,\varepsilon,\xi) = \emptyset$. Otherwise, $\dim_H G(\alpha,\varepsilon,\xi) \leq (\eta + (\alpha+\varepsilon)q - \mathcal{T}(q))/\xi$. This holds for all $\eta > 0$ and $q \geq 0$, so $\dim_H G(\alpha,\varepsilon,\xi) \leq \mathcal{T}^*(\alpha+\varepsilon)$.

Case 2 $\alpha \geq \mathcal{T}'(0+) + \varepsilon$**:** In other words, $\alpha - \varepsilon \geq \mathcal{T}'(0+)$. It is almost the same as before except that one needs to use $q \leq 0$ and $q\widetilde{\Psi}(\omega,v) \geq q(\alpha-\varepsilon)\log|I_\omega^v|$.

Case 3 $\alpha > \mathcal{T}'(0-) - \varepsilon$ **and** $\alpha < \mathcal{T}'(0+) + \varepsilon$ Two situations must be considered.

- if $\mathcal{T}'(0-) - \mathcal{T}'(0+) > 0$, we can assume $\varepsilon < \frac{\mathcal{T}'(0-)-\mathcal{T}'(0+)}{2}$. Then $\mathcal{T}'(0-) - \varepsilon > \mathcal{T}'(0+) + \varepsilon$, so that Case 3 is empty.
- if $\mathcal{T}'(0-) = \mathcal{T}'(0+)$, then $\mathcal{T}$ is differentiable at 0. Take $s = \frac{\eta+\mathcal{T}(0)}{\xi}$ with $\eta > 0$. Then

  $$\mathcal{H}^s_{\delta_N}G(\alpha,\varepsilon,\xi) \leq \sum_{n\geq N}\exp(-\frac{\eta n c_\Phi}{2} - nP(\mathcal{T}(0)\Phi)) = \sum_{n\geq N}\exp(-\frac{\eta n c_\phi}{2}) < \infty.$$

  Here we used the fact that by definition we have $P(\mathcal{T}(0)\Phi) = 0$.

  This yields $\dim_H G(\alpha,\varepsilon,\xi) \leq \frac{\mathcal{T}(0)}{\xi}$ since we can choose η arbitrarily close to 0. Since $\mathcal{T}^*$ is concave, for ε small enough there exists some $C > 0$ such that $\frac{\mathcal{T}(0)}{\xi} = \frac{\mathcal{T}^*(\mathcal{T}'(0))}{\xi} \leq C\varepsilon + \frac{\mathcal{T}^*(\alpha)}{\xi}$.

Corollary 7.15 *For $\mathbb{P}$-almost every ω, for all $d \geq 0$,*

$$\dim_H \underline{E}(\mu, d) \leq \dim_H F(d) \leq d \cdot \sup_{\alpha>0} \frac{\mathcal{T}^*(\alpha)}{\alpha} \leq d \cdot t_0.$$

This provides us with a first upper bound for $\dim_H \underline{E}(\mu, d)$ which turns out to be sharp on $[0, \mathcal{T}'(t_0-)]$.

Proof For any $\varepsilon > 0$, we saw that

$$F(d) \subset \cup_{\alpha\in\mathbb{Q}^+} \cup_{\xi\in\mathbb{Q}\cap[1,+\infty),\alpha/\xi\leq d+2\varepsilon} G(\alpha, \varepsilon, \xi).$$

Thus, from lemma 7.14 one has

$$\dim_H F(d) \leq \sup_{\alpha\in\mathbb{Q}^+,\xi\in\mathbb{Q}\cap[1,+\infty):\alpha/\xi\leq d+2\varepsilon} C\varepsilon + \frac{\max(\mathcal{T}^*(\alpha-\varepsilon), \mathcal{T}^*(\alpha), \mathcal{T}^*(\alpha+\varepsilon))}{\xi}.$$

Letting ε tends to 0 yields

$$\dim_H F(d) \leq \sup_{\alpha\in\mathbb{Q}^+,\xi:\alpha/\xi\leq d} \frac{\mathcal{T}^*(\alpha)}{\xi} \leq d \cdot \sup_{\alpha>0} \frac{\mathcal{T}^*(\alpha)}{\alpha} = d \cdot t_0.$$

For the last equality, at first, since $\mathcal{T}(t_0) = 0$, we have $\sup_{\alpha>0} \frac{\mathcal{T}^*(\alpha)}{\alpha} \geq \frac{\alpha t_0 - \mathcal{T}(t_0)}{\alpha} = t_0$. Next, we know that for any $\alpha > 0$, $\inf_q\{q\alpha - t_0\alpha - \mathcal{T}(q)\} \leq 0$, so $\frac{\inf_q\{q\alpha-\mathcal{T}(q)\}}{\alpha} \leq t_0$, and $\frac{\mathcal{T}^*(\alpha)}{\alpha} \leq t_0$. Finally $\sup_{\alpha>0} \frac{\mathcal{T}^*(\alpha)}{\alpha} \leq t_0$. Now, recall (7.3). Since the set of atoms of ν_ω is countable, we get the desired conclusion for $\dim_H \underline{E}(\nu_\omega, d)$.

7.4 Upper bound for the lower Hausdorff spectrum

Proposition 7.16 *For $\mathbb{P}$-almost every $\omega \in \Omega$, for every $q \in \mathbb{R}$, we have $\tau_{\nu_\omega}(q) \geq \min(\mathcal{T}(q), 0) := \widetilde{\mathcal{T}}(q)$.*

Proposition 7.16 will give the upper bound for the lower Hausdorff spectrum.

Since the function $\tau_{\nu_\omega}(q)$ and the function $\widetilde{\mathcal{T}}(q)$ are both continuous, we just need to prove that this holds on a dense and countable subset of $\mathbb{R}$, which amounts to prove it for any fixed $q \in \mathbb{R}$, almost surely.

Define

$$\widetilde{\lambda}_q(\omega, n) := \lambda^{q\Psi-\mathcal{T}(q)\Phi}(\sigma^{n-1}\omega) \cdots \lambda^{q\Psi-\mathcal{T}(q)\Phi}(\sigma\omega)\lambda^{q\Psi-\mathcal{T}(q)\Phi}(\omega).$$

Proof Let $r > 0$ and consider $\mathcal{B} = \{B_i\}$, a packing of $[0, 1]$ by disjoint intervals B_i with radii equal to r.

- First, fix $q < 0$. Denote $B_i =: B(x_i, r)$.

  There exists a unique $v^i = v(x_i, r) \in \Sigma_{\omega,*}$ such that $x_i \in I_\omega^{v^i} \subset B_i$ and $I_\omega^{(v^i)^*} \not\subset B_i$. Here the notation $*$ means that we delete the last character of the word. Then

$$\begin{aligned} 2r &\geq |I_\omega^{v^i}| = \widetilde{\mu}_\omega([v^i]_\omega) \\ &\geq \exp(S_n\Phi(\omega, \underline{v}) - \log\lambda(\sigma^{n-1}\omega) \cdot \cdots \cdot \lambda(\omega) - o(n)) \\ &\geq \exp(-2nC_\Phi), \end{aligned}$$

  where $n = |v^i|$. On the other hand, since $I_\omega^{(v^i)^*} \not\subset B_i$ we have $r \leq \exp(-\frac{(n-1)c_\Phi}{2})$ (recall that $c_\phi = c_\Phi$ was defined in Assumption 2). Consequently, $n = O(-\log(r) = O(n)$ independently of x_i.

  Also, $\nu_\omega(B_i) \geq \nu_\omega(I_\omega^{v^i}) \geq |X_\omega^{v^i}|$. Since $q < 0$, we get

$$\begin{aligned} \nu_\omega(B_i)^q \leq |X_\omega^{v^i}|^q &= \exp(q(S_n\psi(\omega, x) + o(n))) \quad (\forall x \in U_\omega^{v^i}) \\ &= \exp(q(S_n\Psi(\omega, \underline{v}) + o(n))) \quad (\forall vl \in [v^i]_\omega) \\ &= \exp((S_n(q\Psi - \mathcal{T}(q)\Phi)(\omega, \underline{v}) - \log\widetilde{\lambda}_q(\omega, n)) \\ &\quad \cdot \exp(\mathcal{T}(q)S_{n-1}\Phi(\omega, \underline{v}) - \mathcal{T}(q)\log\lambda_0(\omega, n-1)) \\ &\quad \cdot \exp(o(n)) \\ &\leq \widetilde{\mu}_\omega^{q\Psi - \mathcal{T}(q)\Phi}([v^i]_\omega)|I_\omega^{(v^i)^*}|^{\mathcal{T}(q)}\exp(o(n)) \\ &\leq \widetilde{\mu}_\omega^{q\Psi - \mathcal{T}(q)\Phi}([v^i]_\omega)r^{\mathcal{T}(q)}\exp(o(-\log r)). \end{aligned}$$

  Thus

$$\sum_{B_i \in \mathcal{B}} \nu_\omega(B_i)^q \leq r^{\mathcal{T}(q)}\exp(o(-\log r)).$$

  Letting $r \to 0$, we have $\tau_{\nu_\omega}(q) \geq \mathcal{T}(q)$.

- Second, fix $q \in (0, t_0) \subset (0, 1)$, and recall that $t_0 = \dim_H X_\omega$ is the unique real number such that $P(t\Psi) = 0$. Let

$$V(\omega, n, r) = \{v \in \Sigma_{\omega,n} : |I_\omega^v| \geq 2r, \exists s \text{ such that } vs \in \Sigma_{\omega,n+1},\ |I_\omega^{vs}| < 2r\},$$

$$V'(\omega, n, r) = \{v \in V(\omega, n, r) : \text{there is no } k \text{ such that } v|_k \in V(\omega, k, r) \text{ for some } k < n\},$$

$$V(\omega, r) = \cup_{n \geq 1} V'(\omega, n, r),$$

$$n_r = \max\{|v| : V(\omega, r)\}, \text{ and } n'_r = \min\{|v| : v \in V(\omega, r)\}.$$

  We have $n_r = O(-\log r) = O(n'_r)$.

  For any $v \in V(\omega, r)$ we have

$$|I_\omega^v| \leq 2r\exp(o(-\log r)).$$

  For $v \in V(\omega, r)$, I_ω^v meets at most $\exp(o(-\log r))$ intervals B_i of the packing $\mathcal{B}$, and every B_i is included in the union of at most two intervals belonged to

$V(\omega, r)$, denote as v and v'. Using the sub-additivity of the function $s \geq 0 \mapsto s^q$, we get

$$\nu_\omega(B_i)^q \leq \nu_\omega(1)^q + \nu_\omega(I_\omega^v)^q + \nu_\omega(I_\omega^{v'})^q, \text{ if } 1 \in B_i,$$

otherwise

$$\nu_\omega(B_i)^q \leq \nu_\omega(I_\omega^v)^q + \nu_\omega(I_\omega^{v'})^q, \text{ if } 1 \notin B_i,$$

and

$$\nu_\omega(I_\omega^v)^q \leq \nu_\omega(\mathring{I}_\omega^v)^q + \nu_\omega(\{\min(I_\omega^v)\})^q.$$

Recalling the definition of the inverse measure and proposition 7.1 we know that $\nu_\omega(\mathring{I}_\omega^v) \leq |X_\omega^v|$. Since $q > 0$, we get:

$$\begin{aligned}
\nu_\omega(\mathring{I}_\omega^v)^q &\leq |X_\omega^v|^q \\
&\leq \exp(qS_n\psi(\omega, x) + o(n)) \quad (\forall x \in U_\omega^v) \\
&\leq \exp(qS_n\Psi(\omega, \underline{v}) + o(n)) \quad (\forall \underline{v} \in [v]_\omega) \\
&\leq \exp((S_n(q\Psi - \mathcal{T}(q)\Phi)(\omega, \underline{v}) - \log \widetilde{\lambda}_q(\omega, n)) \\
&\quad \cdot \exp(\mathcal{T}(q)S_n\Phi(\omega, j) - \mathcal{T}(q)\log\lambda(\omega, n)) \\
&\quad \cdot \exp(\log \widetilde{\lambda}_q(\omega) + \mathcal{T}(q)\log\lambda(\omega, n) + o(n)) \\
&\leq \widetilde{\mu}_\omega^{q\Psi - \mathcal{T}(q)\Phi}([v]_\omega)|I_\omega^v|^{\mathcal{T}(q)} \exp(o(n)) \\
&\leq \widetilde{\mu}_\omega^{q\Psi - \mathcal{T}(q)\Phi}([v]_\omega) r^{\mathcal{T}(q)} \exp(o(-\log r)).
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{B_i \in \mathcal{B}} \nu_\omega(B_i)^q &\leq \nu_\omega(1)^q + \exp(o(-\log r))\Big(\sum_{n=n_r'}^{n_r} \sum_{v \in V(\omega,r)\cap\Sigma_{\omega,n}} \nu_\omega(\mathring{I}_\omega^v)^q \\
&\quad + \sum_{n=0}^{n_r} \sum_{v\in\Sigma_{\omega,n}} \sum_{s \in S'(\omega,v,1)} \nu_\omega(\{x_\omega^{vs}\})^q\Big).
\end{aligned}$$

On the one hand,

$$\sum_{n=n_r'}^{n_r} \sum_{v \in V(\omega,r)\cap\Sigma_{\omega,n}} \nu_\omega(\mathring{I}_\omega^v)^q \leq r^{\mathcal{T}(q)} \exp(o(-\log r)).$$

On the other hand, for any $n \leq n_r$, we have that

$$\begin{aligned}
&\sum_{v\in\Sigma_{\omega,n}} \exp(qS_n\Psi(\omega, \underline{v}) + o(n))(\forall \underline{v} \in [v]_\omega) \\
\leq &\sum_{v\in\Sigma_{\omega,n}} \widetilde{\mu}_\omega^{q\Psi - \mathcal{T}(q)\Phi}([v]_\omega) r^{\mathcal{T}(q)} \exp(o(-\log r)) \\
\leq &\; r^{\mathcal{T}(q)} \exp(o(-\log r)).
\end{aligned}$$

Consequently,

$$\sum_{n=0}^{n_r} \sum_{v\in\Sigma_{\omega,n}} \sum_{s \in S'(\omega,v,1)} \nu_\omega(\{x_\omega^{vs}\})^q)$$

$$\begin{aligned}
&\le \sum_{n=0}^{n_r}\sum_{v\in\Sigma_{\omega,n}}\sum_{s\in S'(\omega,v,1)}(m_\omega^{vs'}-M_\omega^{vs})^q\\
&\le \sum_{n=0}^{n_r}\sum_{v\in\Sigma_{\omega,n}} l(\sigma^n\omega)\exp(qS_n\Psi(\omega,\underline{v})+o(n))(\forall \underline{v}\in[v]_\omega)\\
&\le r^{\mathcal{T}(q)}\exp(o(-\log r)),
\end{aligned}$$

Using the fact that $\log n+\log l(\sigma^n\omega)=o(-\log r)$, we obtain

$$\sum_{B_i\in\mathcal{B}}\nu_\omega(B_i)^q\le r^{\mathcal{T}(q)}\exp(o(\log r)),$$

and letting $r\to 0$, we get $\tau_{\nu_\omega}(q)\ge\mathcal{T}(q)$.

- At last, for $q\ge t_0=\dim_H X_\omega$, since ν_ω is discrete, we can easily get $\tau_{\nu_\omega}(q)=0$ for every $q\ge 1$. For $q=t_0$, one has $\tau_{\nu_\omega}(q)\ge\mathcal{T}(q)=0$. Since the function τ_{ν_ω} is concave, we get $\tau_{\nu_\omega}(q)=0$ for every $q\ge t_0$.

7.5 First lower bound for the lower Hausdorff spectrum

Proposition 7.17 *For any $d\in[\mathcal{T}'(+\infty),\mathcal{T}'(-\infty)]$,*

$$\dim_H(\underline{E}(\nu_\omega,d))\ge\mathcal{T}^*(d),$$

Proof For any $d\in[\mathcal{T}'(+\infty),\mathcal{T}'(-\infty)]$ such that $\mathcal{T}^*(d)>0$, the proof is the following. Proposition 7.8 shows that

$$\underline{E}(\nu_\omega,d)\supset\widetilde{E}(d)=\{x|\lim_{n\to\infty}\alpha_\omega^n(x)=d\text{ and }\widetilde{\xi}_\omega^x=1\}.$$

The set $\widetilde{E}(d)$ can be expressed as:

$$\widetilde{E}(d)=\{x|\lim_{n\to\infty}\alpha_\omega^n(x)=d\}\setminus(\cup_{m\ge1}\{x|\lim_{n\to\infty}\alpha_\omega^n(x)=d\text{ and }\widetilde{\xi}_\omega^x\ge 1+1/m\})$$

Using the same method as in section 6.2, but reversing the roles of Ψ and Φ, we can construct a probability measure $\widetilde{\eta}_\omega$ on $[0,1]$ with the following properties.

1. $\widetilde{\eta}_\omega(\{x|\lim_{n\to\infty}\alpha_\omega^n(x)=d\})=1$,
2. $\dim_H\widetilde{\eta}_\omega\ge\mathcal{T}^*(d)$.

Since for any $\varepsilon > 0$, $\{x| \lim_{n\to\infty} \alpha_\omega^n(x) = d \text{ and } \widetilde{\xi}_\omega^x \geq 1 + 1/m\} \subset G(\alpha, \varepsilon, \xi)$, from proposition 7.14 one gets

$$\dim_H\{x| \lim_{n\to\infty} \alpha_\omega^n(x) = d \text{ and } \widetilde{\xi}_\omega^x \geq 1 + 1/m\} < \mathcal{T}^*(d).$$

Then

$$\widetilde{\eta}_\omega(\{x| \lim_{n\to\infty} \alpha_\omega^n(x) = h \text{ and } \widetilde{\xi}_\omega^x \geq 1 + 1/m\}) = 0.$$

which implies $\cup_{m\geq 1}\{x| \lim_{n\to\infty} \alpha_\omega^n(x) = h \text{ and } \widetilde{\xi}_\omega^x \geq 1 + 1/m\}$ is $\widetilde{\eta}_\omega$-negligible.

Hence, $\widetilde{\eta}_\omega(\widetilde{E}(d)) = 1$.

Finally,

$$\dim \underline{E}(\nu_\omega, d) \geq \dim_H(\widetilde{\eta}_\omega) \geq \mathcal{T}^*(d).$$

If $d \in [T'(+\infty), T'(-\infty)]$ with $\mathcal{T}^*(d) = 0$, what we need is to prove that $\dim_H E(\nu_\omega, d) \geq 0$, in the other word $E(\nu_\omega, d) \neq \emptyset$.

Here, we will use the process in the proof of the lower bound for the Hausdorff spectrum in theorem 1.11(see section 6.2).

We can fixed $\{\varepsilon_i\}_{i\in\mathbb{N}}$ such that $\frac{1}{i} \leq \varepsilon_i$, $\lim_{i\to\infty} \frac{\varepsilon_i}{\varepsilon_{i+1}} = 1$ and ε_i decreases to 0 as $i \to \infty$. We can choose suitably d_i such that $\mathcal{T}^*(d_i) = \sqrt{\varepsilon_i}$ with $\lim_{i\to\infty} d_i = d$ and build a measure η_ω and a set $K(\omega, \{d_i\}_{i\geq 1})$ such that $\eta_\omega(K(\omega, \{d_i\}_{i\geq 1})) = 1$ and

- for each n with $m_i + n_{\mathcal{N}_{i+1}}^{i+1} \leq n < m_{i+1} + n_{\mathcal{N}_{i+2}}^{i+2}$, one has

$$(|I_\omega^v|)^{\sqrt{\varepsilon_i}+\varepsilon_i} \leq \eta_\omega(I_\omega^v) \leq (|I_\omega^v|)^{\sqrt{\varepsilon_i}-\varepsilon_i},$$

  if $v \in \Sigma_{\omega,n}$ and $I_\omega^v \cap K(\omega, \{d_i\}_{i\in\mathbb{N}}) \neq \emptyset$.

- we can choose $n_{\mathcal{N}_i'}^i > \frac{m_i^3}{\varepsilon_i^3}$ large enough for each $i \in \mathbb{N}$ such that for any $\xi > 1$, and then for any $x \in [0, 1]$, for i large enough, one has

$$\eta_\omega(B(x, |I_\omega^v|^\xi)) \leq |I_\omega^v|^{\xi(\sqrt{\varepsilon_{i+1}}-\varepsilon_i)},$$

- in addition $\sup_{k\geq m_i} \frac{\log l(\sigma^k\omega)}{k} \leq \varepsilon_i$.

For any given $\xi > 1$, for i large enough, $m_i + n_{\mathcal{N}_{i+1}}^{i+1} \leq n < m_{i+1} + n_{\mathcal{N}_{i+2}}^{i+2}$ and $v \in \Sigma_{\omega,n}$, we get

$$\frac{\eta_\omega(B(x, |I_\omega^v|^\xi))}{\eta_\omega(I_\omega^v)} \leq |I_\omega^v|^{\xi(\sqrt{\varepsilon_{i+1}}-\varepsilon_i)-(\sqrt{\varepsilon_i}+\varepsilon_i)} \leq \exp(-\frac{nc_\Phi}{2} \cdot (c\sqrt{\varepsilon_i})),$$

where $c \in (0, \xi - 1)$, since $\xi > 1$ and $\lim_{i\to 0} \frac{\sqrt{\varepsilon_{i+1}}-\varepsilon_i}{\sqrt{\varepsilon_i}+\varepsilon_i} = 1$.

Since $\eta_\omega(\cup_{v\in\Sigma_{\omega,n}} I_\omega^v) = 1$, we can get

$$\eta_\omega(\cup_{v\in\Sigma_{\omega,n}} \cup_{s\in S(\omega,v,1)} B(x_\omega^{vs}, |I_\omega^v|^\xi))$$

$$
\begin{aligned}
&\leq \sum_{v\in\Sigma_{\omega,n}} \sum_{s\in S(\omega,v,1)} \eta_\omega(B(x_\omega^{vs}, |I_\omega^v|^\xi))) \\
&\leq \sum_{v\in\Sigma_{\omega,n}} l(\sigma^n\omega)\exp(-\frac{nc_\Phi}{2}\cdot(c\sqrt{\varepsilon_i}))\eta_\omega(I_\omega^v) \\
&\leq \sum_{v\in\Sigma_{\omega,n}} \exp(-\frac{nc_\Phi}{2}\cdot(c\sqrt{\varepsilon_i}-\frac{2}{c_\Phi}\varepsilon_i))\eta_\omega(I_\omega^v) \\
&\leq \exp(-\frac{nc_\phi}{2}\cdot(\frac{c\sqrt{\varepsilon_i}}{2})) \\
&\leq \exp(-n^{\frac{1}{2}})
\end{aligned}
$$

for n large enough (recall that $\varepsilon_i \geq 1/i$). Then

$$\sum_{\widetilde{N}\in\mathbb{N}} \sum_{n\geq\widetilde{N}} \eta_\omega(\cup_{v\in\Sigma_{\omega,n}} \cup_{s\in S(\omega,v,1)} B(x_\omega^{vs}, |I_\omega^v|^\xi)) < +\infty,$$

hence

$$\eta_\omega(\cap_{\widetilde{N}\geq1} \cup_{n\geq\widetilde{N}} \cup_{v\in\Sigma_{\omega,n}} \cup_{s\in S(\omega,v,1)} B(x_\omega^{vs}, |I_\omega^v|^\xi)) = 0.$$

The set

$$\cap_{\widetilde{N}\geq1} \cup_{n\geq\widetilde{N}} \cup_{v\in\Sigma_{\omega,n}} \cup_{s\in S(\omega,v,1)} B(x_\omega^{vs}, |I_\omega^v|^\xi)$$

increases as ξ decreases to 1, so

$$\{x : \widetilde{\xi}_\omega^x > 1\} = \cup_{m\geq1} \cap_{\widetilde{N}\geq1} \cup_{n\geq\widetilde{N}} \cup_{v\in\Sigma_{\omega,n}} \cup_{s\in S(\omega,v,1)} B(x_\omega^{vs}, |I_\omega^v|^{1+\frac{1}{m}}).$$

For any $m \geq 1$,

$$\eta_\omega(\cap_{\widetilde{N}\geq1} \cup_{n\geq\widetilde{N}} \cup_{v\in\Sigma_{\omega,n}} \cup_{s\in S(\omega,v,1)} B(x_\omega^{vs}, |I_\omega^v|^{1+\frac{1}{m}})) = 0,$$

so

$$\eta_\omega(\cup_{m\geq1} \cap_{\widetilde{N}\geq1} \cup_{n\geq\widetilde{N}} \cup_{v\in\Sigma_{\omega,n}} \cup_{s\in S(\omega,v,1)} B(x_\omega^{vs}, |I_\omega^v|^{1+\frac{1}{m}})) = 0,$$

and

$$\eta_\omega(K(\omega, \{d_i\}_{i\geq1} \setminus \cup_{m\geq1} \cap_{\widetilde{N}\geq1} \cup_{n\geq\widetilde{N}} \cup_{v\in\Sigma_{\omega,n}} \cup_{s\in S(\omega,v,1)} B(x_\omega^{vs}, |I_\omega^v|^{1+\frac{1}{m}})) = 1.$$

Finally, we can say that $E(\nu_\omega, d) \neq \emptyset$, since

$$
\begin{aligned}
E(\nu_\omega, d) &\supset (K(\omega, \{d_i\}_{i\geq1}) \cap \{x : \widetilde{\xi}_\omega^x = 1\} \\
&= K(\omega, \{d_i\}_{i\geq1}) \setminus \cup_{m\geq1} \cap_{\widetilde{N}\geq1} \cup_{n\geq\widetilde{N}} \cup_{v\in\Sigma_{\omega,n}} \cup_{s\in S(\omega,v,1)} B(x_\omega^{vs}, |I_\omega^v|^{1+\frac{1}{m}})
\end{aligned}
$$

has full η_ω-measure.

7.6 Some preparation to the conditioned ubiquity theorem

This section is very similar to chapter 5. Now we fix the two sequences of functions $\{\Psi_i\}_{i\geq 1}, \{\Phi_i\}_{i\geq 1}$ as in section 5.1. Since $c_\psi > 0$ and then $c_\Psi > 0$, for each $i \in \mathbb{N}$ there exists a function $\mathcal{T}_i$ such that for any $q \in \mathbb{R}$ one has $P(q\Psi_i - \mathcal{T}_i(q)\Phi_i) = 0$.

Lemma 7.18 *1. $\mathcal{T}_i$ converges poitwise to $\mathcal{T}$ as $i \to \infty$.*

2. $\mathcal{T}_i^*$ converges pointwise to $\mathcal{T}^*$ over the interior of the domain of $\mathcal{T}^*$ as $i \to \infty$.

Let $\mathcal{D}$ be a dense and countable subset of $(\mathcal{T}'(+\infty), \mathcal{T}'(-\infty))$, so that for any $d \in [\mathcal{T}'(+\infty), \mathcal{T}'(-\infty)]$, there exists $\{d_k\}_{k\in\mathbb{N}} \subset \mathcal{D}^{\mathbb{N}}$ such that $\lim_{k\to\infty} d_k = d$ and $\lim_{k\to\infty} \mathcal{T}^*(d_k) = \mathcal{T}^*(d)$.

Let $\{\mathcal{D}_i\}_{i\in\mathbb{N}}$ be a sequence of sets such that

- $\mathcal{D}_i$ is a finite set for each $i \in \mathbb{N}$,
- $\mathcal{D}_i \subset \mathcal{D}_{i+1}$, for each $i \in \mathbb{N}$,
- $\cup_{i\in\mathbb{N}} \mathcal{D}_i = \mathcal{D}$.

Let us fix a positive sequence $\{\varepsilon_i\}_{i\in\mathbb{N}}$ decreasing to 0. For each i, there exists j_i large enough such that for any $d_i \in \mathcal{D}_i$, there exists $q_i \in \mathbb{R}$ such that

1. $\mathcal{T}_{j_i}'(q_i) = d_i$,
2. $|\mathcal{T}_{j_i}^*(d_i) - \mathcal{T}^*(d_i)| \leq \varepsilon_i$.
3. $\int_\Omega \mathrm{var}_{j_i} \Psi \; d\mathbb{P} \leq \varepsilon_i^3$ and $\int_\Omega \mathrm{var}_{j_i} \Phi \; d\mathbb{P} \leq \varepsilon_i^3$

Define $\mathcal{Q}_i = \{q_i, d_i \in \mathcal{D}_i\}$ and $\widetilde{\Lambda}_{i,q_i} = q_i \Psi_{j_i} - \mathcal{T}_{j_i}(q_i)\Phi_{j_i}$ for $q_i \in \mathcal{Q}_i$.

For any $\epsilon > 0$, there exist positive integers M, L, N, C large enough, and a set $\widetilde{\Omega}_0$ such that there exists a sequence $\{c_n\}_{n\geq 1}$ where $c_n \to 0$ as $n \to \infty$ such that $\mathbb{P}(\widetilde{\Omega}_0) > 1 - \epsilon/4$, and for any $\omega \in \widetilde{\Omega}_0$, one has:

- $M(\omega) < M, \; l(\omega) \leq L$,
- the $o(n)$ with respect to the potential Φ, Ψ in proposition 2.7 is smaller than nc_n and $V_n\Phi(\omega) \leq nc_n$, $V_n\Psi(\omega) \leq nc_n$.

- for any $n \geq N$,

$$\begin{aligned}
\left|S_n \mathrm{var}_{j_i}\Phi(\omega) - n\int_\Omega \mathrm{var}_{j_i}\Phi(\omega)\ d\mathbb{P}\right| &\leq nc_n, \\
\left|S_n \mathrm{var}_{j_i}\Psi(\omega) - n\int_\Omega \mathrm{var}_{j_i}\Psi(\omega)\ d\mathbb{P}\right| &\leq nc_n, \\
\left|\frac{1}{n}S_n(\log l)(\omega)\right| &\leq C, \\
\max\left(\frac{1}{n}S_n\|\Phi(\omega)\|_\infty,\ \frac{1}{n}S_n\|\Phi(\sigma^{-n+1}\omega)\|_\infty\right) &\leq C, \\
\max\left(\frac{1}{n}S_n\|\Psi(\omega)\|_\infty,\ \frac{1}{n}S_n\|\Psi(\sigma^{-n+1}\omega)\|_\infty\right) &\leq C.
\end{aligned}$$

- $$\sup_{\underline{v}\in[v]_\omega} S_n(\Phi)(\omega,\underline{v}) \leq (-n\varpi_\Phi)$$

  and

  $$\sup_{\underline{v}\in[v]_\omega} S_n(\Psi)(\omega,\underline{v}) \leq (-n\varpi_\Psi)$$

  for any $v \in \Sigma_{\omega,n}$ with $n \geq N$.

For any given finite set $\mathcal{Q}$ we know that for s large enough one has $\sharp\Sigma_{\omega,s}$ will larger than $\sharp\mathcal{Q}$. Denote the smallest such s by $\mathcal{S}(\omega,\sharp\mathcal{Q})$.

Also, for all $i \in \mathbb{N}$, choose $\mathcal{S}(i) \in \mathbb{N}$ large enough so that there exists a set $\widetilde{\Omega}'(i) \subset \widetilde{\Omega}_0$ such that

- $\mathbb{P}(\widetilde{\Omega}'(i)) \geq 1 - 2\epsilon/4$,
- for any $\omega \in \widetilde{\Omega}'(i)$, one has $\mathcal{S}(\sigma^M\omega, \sharp\mathcal{Q}_i) \leq \mathcal{S}(i)$.

For all $i \in \mathbb{N}$, there exists $N(i) > N$ and $M(i) \geq M$ large enough and $\{c_{i,n}\}_{n\geq 1}$ decreasing 0 as $n \to \infty$, and a set $\widetilde{\Omega}(i) \subset \widetilde{\Omega}'(i)$ such that $\mathbb{P}(\widetilde{\Omega}(i)) \geq 1 - 3\epsilon/4$ and for any $\omega \in \widetilde{\Omega}(i)$,

- $M(\sigma^{M+\mathcal{S}(i)}\omega) \leq M(i)$,
- for any $q \in \mathcal{Q}_i$, the random Gibbs measures $\tilde{\mu}^{\widetilde{\Lambda}_{i,q}}_{\sigma^{M+\mathcal{S}(i)+M(i)}\omega}$ are well defined, $V_n\widetilde{\Lambda}_{i,q}(\sigma^{M+\mathcal{S}(i)+M(i)}\omega) \leq nc_{i,n}$ for any $n \geq N(i)$, and the $o(n)$ with respect to the potential $\widetilde{\Lambda}_{i,q}$ in proposition 2.7 is smaller than $nc_{i,n}$, that is

  $$\epsilon(\widetilde{\Lambda}_{i,q}, n) \leq c_{i,n}.$$

- for any $q \in Q_i$, there exist $\varpi_i > 0$ such that

$$\sup_{\underline{v} \in [v]_{\sigma^{M+M(i)+\mathcal{S}(e_i)}\omega\omega}} S_n \widetilde{\Lambda}_{i,q}(\sigma^{M+\mathcal{S}(i)+M(i)}\omega, \underline{v}) \leq (-n\varpi_i),$$

for all $n \geq N(i)$, for all $v \in \Sigma_{\sigma^{M+M(i)+\mathcal{S}(i)}\omega,n}$.

Let $\widetilde{\theta}'(i,\omega,s)$ be the s-th return time of the point ω to the set $\widetilde{\Omega}(i)$ under the map σ. Then for any $i \in \mathbb{N}$

$$\lim_{s\to\infty} \frac{\widetilde{\theta}'(i,\omega,s)}{s} = \frac{1}{\mathbb{P}(\widetilde{\Omega}(i))}$$

for $\mathbb{P}$-almost every ω. Consequently,

$$\lim_{s\to\infty} \frac{\widetilde{\theta}'(i,\omega,s) - \widetilde{\theta}'(i,\omega,s-1)}{\widetilde{\theta}'(i,\omega,s-1)} = 0.$$

Since $\mathbb{N}$ is countable, then for $\mathbb{P}$-almost every $\omega \in \Omega$, for any $i \in \mathbb{N}$,

$$\lim_{s\to\infty} \frac{\widetilde{\theta}'(i,\omega,s)}{s} = \frac{1}{\mathbb{P}(\widetilde{\Omega}(i))}$$

then

$$\lim_{s\to\infty} \frac{\widetilde{\theta}'(i,\omega,s) - \widetilde{\theta}'(i,\omega,s-1)}{\widetilde{\theta}'(i,\omega,s)} = 0.$$

In the following we always deal with the $\omega \in \Omega$ such that the above hold we denote all these ω by $\widetilde{\Omega}$ which is full of $\mathbb{P}$-measure.

Given $\omega \in \widetilde{\Omega}'(i)$, let $M'(i) = M + \mathcal{S}(i) + M(i)$ and

$$n_1^i(\omega) = \inf\{\widetilde{\theta}'(i,\omega,s) : \widetilde{\theta}(i,\omega,s) \geq M'(i)\} - M'(i).$$

For $k \geq 2$, define $n_k^i(\omega) = \widetilde{\theta}'(i,\omega,s_k) - M'(i)$, where s_k is the smallest s such that the following hold:

$$\widetilde{\theta}'(i,\omega,s) - n_{k-1}^i(\omega) \geq \max\left(M'(i), n_{k-1}^i(\omega)(c_{i,n_{k-1}^i})^{\frac{1}{3}} + \sqrt{\widetilde{\theta}'(i,\omega,s)}\right)$$

It is easy to show that

$$\lim_{k\to\infty} \frac{n_k^i(\omega) - n_{k-1}^i(\omega)}{n_{k-1}^i(\omega)} = 0.$$

Using the same method as in section 5.4, we can get the following series of properties.

Facts 7.19 For any $i \in \mathbb{N}$, for any $\omega \in \widetilde{\Omega}(i)$, there exist a sequence $\{n_k^i = n_k^i(\omega)\}_{k\geq 1}$ and a positive integer $\mathcal{N}_i = \mathcal{N}_i(\sigma^{M'(i)}\omega)$ such that for any $d = d_i \in \mathcal{D}_i$, there exists $q = q_i \in \mathcal{Q}_i$, a measure $\rho^{\widetilde{\Lambda}_{i,q}}_{\sigma^{M'(i)}\omega}$ on $[0,1]$ and a set $\widetilde{E}_{i,q}(\sigma^{M'(i)}\omega) \subset [0,1]$ satisfying

1. For $\rho^{\widetilde{\Lambda}_{i,q}}_{\sigma^{M'(i)}\omega}$-almost every $x \in [0,1]$, there exists $N(q,x)$ such that for any $k \geq N(q,x)$ for any $v \in \Sigma_{\sigma^{M'(i)}\omega, n_k^i}$ satisfying $x \in I_\omega^v$ one has: $|v \wedge v+| \geq n_{k-1}^i$ and $|v \wedge v-| \geq n_{k-1}^i$. Furthermore, for any $\underline{v} \in [v]_{\sigma^{M'(i)}\omega} \cup [v+]_{\sigma^{M'(i)}\omega} \cup [v-]_{\sigma^{M'(i)}\omega}$, one has
$$\left| \frac{S_{n_k^i}\Psi_{j_i}(\sigma^{M'(i)}\omega, \underline{v})}{S_{n_k^i}\Phi_{j_i}(\sigma^{M'(i)}\omega, \underline{v})} - \mathcal{T}'_{j_i}(q) \right| \leq \varepsilon_i,$$
and for $v' = v$, $v' = v-$ or $v' = v+$, and $\underline{v}' \in [v']_{\sigma^{M'(i)}}$
$$\left| \frac{\log \rho^{\widetilde{\Lambda}_{i,q}}_{\sigma^{M'(i)}\omega}(I^{v'}_{\sigma^{M'(i)}\omega})}{S_{|v'|}\Phi_{j_i}(\sigma^{M'(i)}\omega, \underline{v'})} - \mathcal{T}^*_{j_i}(\mathcal{T}'_{j_i}(q)) \right| \leq \varepsilon_i.$$
and
$$\left| \frac{S_{|v'|}\widetilde{\Lambda}_{i,q}(\sigma^{M'(i)}\omega, \underline{v'})}{S_{|v'|}\Phi_{j_i}(\sigma^{M'(i)}\omega, \underline{v'})} - \mathcal{T}^*_{j_i}(\mathcal{T}'_{j_i}(q)) \right| \leq \varepsilon_i.$$

2. for any $k \geq \mathcal{N}_i$ one has $c_{n^i_{\mathcal{N}_i}} \leq (\varepsilon_i)^4/8$, $M'(i) \leq n_k^i(\varepsilon_i)^4/8$, $c_{i,n_k^i} \leq (\varepsilon_i)^4/8$ and $\frac{n_k^i - n_{k-1}^i}{n_{k-1}^i} \leq (\varepsilon_i)^4/8$.

3. $\rho^{\widetilde{\Lambda}_{i,q}}_{\sigma^{M'(i)}\omega}(\widetilde{E}_{i,q}(\sigma^{M'(i)}\omega)) > \frac{1}{2}$ and:

   (a) For any $x \in \widetilde{E}_{i,q}(\sigma^{M'(i)}\omega)$, for any $k \geq \mathcal{N}_i$, for any $v \in \Sigma_{\sigma^{M'(i)}\omega, n_k^i}$ such that $x \in I_\omega^v$, one has $|v \wedge v+| \geq n_{k-1}^i$ and $|v \wedge v-| \geq n_{k-1}^i$.

   (b) For any $\underline{v} \in [v]_{\sigma^{M'(i)}\omega} \cup [v+]_{\sigma^{M'(i)}\omega} \cup [v-]_{\sigma^{M'(i)}\omega}$, one has
$$\left| \frac{S_{n_k^i}\Psi_{j_i}(\sigma^{M'(i)}\omega, \underline{v})}{S_{n_k^i}\Phi_{j_i}(\sigma^{M'(i)}\omega, \underline{v})} - \mathcal{T}'_{j_i}(q) \right| \leq \varepsilon_i,$$
   and for $v' = v$, $v' = v-$ or $v' = v+$, and $\underline{v}' \in [v']_{\sigma^{M'(i)}}$
$$\left| \frac{\log \rho^{\widetilde{\Lambda}_{i,q}}_{\sigma^{M'(i)}\omega}(I^{v'}_{\sigma^{M'(i)}\omega})}{S_{|v'|}\Phi_{j_i}(\sigma^{M'(i)}\omega, \underline{v'})} - \mathcal{T}^*_{j_i}(\mathcal{T}'_{j_i}(q)) \right| \leq \varepsilon_i.$$
   and
$$\left| \frac{S_{|v'|}\widetilde{\Lambda}_{i,q}(\sigma^{M'(i)}\omega, \underline{v'})}{S_{|v'|}\Phi_{j_i}(\sigma^{M'(i)}\omega, \underline{v'})} - \mathcal{T}^*_{j_i}(\mathcal{T}'_{j_i}(q)) \right| \leq \varepsilon_i.$$

In fact we will use the following version of the previous facts:

Facts 7.20 We can change $\widetilde{\Omega}(i)$ to $\widetilde{\Omega}_i \subset \widetilde{\Omega}(i)$ a bit smaller such that $\mathbb{P}(\Omega_i) \geq 1-\epsilon$ and there exist $\mathcal{N}_i$ and $W(i)$ such that for any $\omega \in \widetilde{\Omega}_i$, $\mathcal{N}_i(\sigma^{M'(i)}\omega) \leq \mathcal{N}_i$ and $n^i_{\mathcal{N}_i}(\omega) \leq W(i)$, and the items 2 and 3 of facts 7.19 also hold. Furthermore if we change the ε_i (for example take it to be $2\varepsilon_i$), in the inequalities of (b) in item 3 of facts 7.19 $\mathcal{T}^*_{j_i}(\mathcal{T}'_{j_i}(q))$ can be changed to $\mathcal{T}^*(\mathcal{T}'_{j_i}(q))$.

From now on we will define $\widetilde{\theta}(i,\omega,s)$ be the s-th return time to the set $\widetilde{\Omega}_i$ for the point ω

Since $\mathbb{N}$ is countable, there exists $\widetilde{\Omega} \subset \widetilde{\Omega}'$ of full probability such that for all $\omega \in \widetilde{\Omega}$, for any $i \in \mathbb{N}$, we have

- $$\lim_{n\to\infty} V_n\Psi(\omega)/n = 0,$$
- $$\lim_{n\to\infty} V_n\Phi(\omega)/n = 0,$$
- $$\lim_{s\to\infty} \frac{\widetilde{\theta}(i,\omega,s)}{s} = \frac{1}{\mathbb{P}(\Omega_i)},$$

hence
$$\lim_{s\to\infty} \frac{\widetilde{\theta}(i,\omega,s)-\widetilde{\theta}(i,\omega,s-1)}{\widetilde{\theta}(i,\omega,s-1)} = 0.$$

From now on we just deal with the points of $\widetilde{\Omega}$, which is a set with $\mathbb{P}$-full measure.

For any $w \in \Sigma_{\omega,n}$ such that $\sigma^n\omega \in \widetilde{\Omega}_i$, for any $d_i \in \mathcal{D}_i$, there exists $q_i \in \mathcal{Q}_i$ such that $\mathcal{T}'_{j_i}(q_i) = d_i$. Since $\sharp\Sigma_{\sigma^{n+M},\mathcal{S}(i)} \geq \sharp\mathcal{Q}_i$, for each q_i we can choose $v(q_i) \in \Sigma_{\sigma^{n+M},\mathcal{S}(i)}$ so that these words are pairwise distinct. Now, using the same rule as in section 6.2, for any q_i and $v \in \Sigma_{\sigma^{n+M'(i)}\omega,*}$ we can connect w with $v(q_i)$ and $v(q_i)$ with $v \in \Sigma_{\sigma^{n+M'(i)}\omega,*}$, and get a word $w * v(q_i) * v$, denoted we call $w * v$ if there is no confusion. Also, we define a new measure ζ_{ω,w,q_i} supported on $\overline{I^{w*v(q_i)}_\omega}$ by:

$$\zeta_{\omega,w,q_i}(I^{w*v(q_i)*v}_\omega) = \rho^{\widetilde{\Lambda}_{i,q_i}}_{\sigma^{n+M'(i)}\omega}(I^v_{\sigma^{n+M'(i)}\omega}). \tag{7.4}$$

Also, we define

$$E(\omega,i,w,q_i) = \{w * v(q_i) * \underline{v} : \underline{v} \in \widetilde{E}_{i,q_i}(\sigma^{|w|+M'(i)}\omega)\}.$$

We notice that the measures ζ_{ω,w,q_i}, $q_i \in \mathcal{Q}_i$, have supports which pairwise have at most one point in common. Also, $\zeta_{\omega,w,q_i}(E(\omega,i,w,q_i)) > 1/2$.

7.7 Conditioned ubiquity

We will adapt the method of [7] to get the following result, which provides us with the necessary material to get the sharp lower bound for the lower Hausdorff spectrum on $[0, \mathcal{T}'(t_0-)]$.

Definition 7.21 *If $d \geq 0$, $\xi \geq 1$ and $\widetilde{\epsilon} = \{\epsilon_i\}_{i\in\mathbb{N}}$ is a positive sequence decreasing to 0 as $i \to \infty$, we set*

$$S(\omega, d, \xi, \widetilde{\epsilon}) := \cap_{N\geq 1} \cup_{n\geq N} \cup_{v\in\Sigma_{\omega,n}, \exists \underline{v}\in[v]_\omega \text{ such that } |\frac{S_n\Psi(\omega,\underline{v})}{S_n\Phi(\omega,\underline{v})} - d|\leq \epsilon_i} B(z_\omega^v, (\ell_\omega^v)^\xi).$$

The following result concerns the ubiquity of the family of points $\{z_\omega^v\}_{v\in\Sigma_{\omega,*}}$ relatively to the radii $\{\ell_\omega\}_{v\in\Sigma_{\omega,*}}$, and conditionally on the behavior of $\frac{S_n\Psi(\omega,\underline{v})}{S_n\Phi(\omega,\underline{v})}$ in $[v]_\omega$.

Theorem 7.22 *For $\mathbb{P}$-almost every $\omega \in \Omega$, for any $\xi \geq 1$ and any $d \in [\mathcal{T}'(+\infty), \mathcal{T}'(-\infty)]$, there exists a sequence $\widetilde{\epsilon}(\omega) = \{\epsilon_i(\omega)\}_{i\in\mathbb{N}}$ decreasing to 0 as $i \to \infty$, as well as a set $K^d(\xi) \subset S(\omega, d, \xi, \widetilde{\epsilon})$ and a Borel probability measure m_ξ^d supported on $K^d(\xi)$ such that $\dim_H(m_\xi^d) \geq \frac{\mathcal{T}^*(d)}{\xi}$.*

Remark 7.23 *In fact we can choose $\widetilde{\epsilon}$ independent of ω.*

Remark 7.24 *For any $x \in S(\omega, d, \xi, \widetilde{\epsilon})$ there are infinitely many n_i and $v(i) \in \Sigma_{\omega,n_i}$ such that $x \in B(z_\omega^{v(i)}, (\ell_\omega^{v(i)})^\xi)$ and $\alpha_\omega^{n_i}(x) \leq d + \epsilon_i$, so*

$$\underline{\dim}_{\mathrm{loc}}(\nu_\omega, x) \leq \liminf_{i\to\infty} \frac{\log \nu_\omega(B(z_\omega^{v(i)}, (\ell_\omega^{v(i)})^\xi))}{\xi \log \ell_\omega^{v(i)}} \leq \liminf_{i\to\infty} \frac{\alpha_\omega^{n_i}(x)}{\xi} \leq \frac{d}{\xi}, \qquad (7.5)$$

since $\frac{\log \nu_\omega(\{z_\omega^{v_i}\})}{\log \ell_\omega^{v(i)}}$ is asymptotically not bigger than the Birkhoff average $\frac{S_{|v_i|}\Psi(\omega,\underline{v})}{S_{|v_i|}\Phi(\omega,\underline{v})}$. Consequently, $S(\omega, d, \xi, \widetilde{\epsilon}) \subset \cup_{h\leq \frac{d}{\xi}} \underline{E}(\nu_\omega, h)$.

Now, we will prove theorem 7.22.

Proof Recall that $\widetilde{\theta}(i, \omega, k)$ is the k-th return time of ω to the set $\widetilde{\Omega}_i$ under the mapping σ.

We start by constructing a generalized Cantor set $K(\xi, \widetilde{d})$ for each $\xi > 1$ and each sequence $\widetilde{d} \in \prod_{i=1}^\infty \mathcal{D}_i$.

Step 1: Let $\omega \in \widetilde{\Omega}$. Let $n = \widetilde{\theta}(1, \omega, 1)$. Fix $w^{(1)} \in \Sigma_{\omega,n}$. Recall that for each $d_1 \in \mathcal{D}_1$ there is $q_1 \in \mathcal{Q}_1$ with $\mathcal{T}'_{j_1}(q_1) = d_1$ and $v(q_1) \in \Sigma_{\sigma^{\widetilde{\theta}(1,\omega,1)+M_{\omega,S(1)}}}$ (see the end of the previous section).

From the facts 7.19 and facts 7.20, there exists $\mathcal{N}_1$ large enough, such that for each $d_1 \in \mathcal{D}_1$, there is a set $E(\omega, 1, w^{(1)}, q_1)$ which is a subset of the closure of $I_\omega^{w^{(1)}*v(q_1)}$ such that $\zeta_{\omega,w^{(1)},q_1}(E(\omega,1,w^{(1)},q_1)) > 1/2$ (recall that $\zeta_{\omega,w^{(1)},q_1}$ and $E(\omega,1,w^{(1)},q_1))$ have been defined at the end of section 7.6) and the following properties hold:

1. $M'(1) \le (\varepsilon_1)^4 n^1_{\mathcal{N}_1}$,
2. $c_{n^1_{\mathcal{N}_1}} \le (\varepsilon_1)^4$ and $c_{1,n^1_{\mathcal{N}_1}} \le (\varepsilon_1)^4$,
3. $n^1_k - n^1_{k-1} \le (\varepsilon_1)^4 n^1_{k-1}$ for any $k \ge \mathcal{N}_1$,
4. - For any $x \in E(\omega,1,w^{(1)},q_1)$, for any $k \ge \mathcal{N}_1$, for any $v \in \Sigma_{\sigma^{\widetilde{\theta}(1,\omega,1)+M'(1)}\omega,n^1_k}$ such that $x \in I_\omega^{w^{(1)}*v(q_1)*v}$, one has $|v \wedge v+| \ge n^1_{k-1}$ and $|v \wedge v-| \ge n^1_{k-1}$.
   - For any $\underline{v} \in [v]_{\sigma^{\widetilde{\theta}(1,\omega,1)+M'(1)}\omega} \cup [v+]_{\sigma^{\widetilde{\theta}(1,\omega,1)+M'(1)}\omega} \cup [v-]_{\sigma^{\widetilde{\theta}(1,\omega,1)+M'(1)}\omega}$, one has

$$\left| \frac{S_{n^i_k}\Psi_{j_1}(\sigma^{\widetilde{\theta}(1,\omega,1)+M'(1)}\omega, \underline{v})}{S_{n^i_k}\Phi_{j_1}(\sigma^{\widetilde{\theta}(1,\omega,1)+M'(1)}\omega, \underline{v})} - d_1 \right| \le \varepsilon_1,$$

   and for $v' = v$, $v' = v-$ or $v' = v+$, and $\underline{v}' \in [v']_{\sigma^{M'(i)}}$

$$\left| \frac{\log \rho^{\widetilde{\Lambda}_{1,q_1}}_{\sigma^{\widetilde{\theta}(1,\omega,1)+M'(1)}\omega}(I^{v'}_{\sigma^{\widetilde{\theta}(1,\omega,1)+M'(1)}\omega})}{S_{|v'|}\Phi_{j_1}(\sigma^{\widetilde{\theta}(1,\omega,1)+M'(1)}\omega, \underline{v'})} - \mathcal{T}^*_{j_1}(d_1) \right| \le \varepsilon_1,$$

   and

$$\left| \frac{S_{|v'|}\widetilde{\Lambda}_{1,q_1}(\sigma^{\widetilde{\theta}(1,\omega,1)+M'(1)}\omega, \underline{v'})}{S_{|v'|}\Phi_{j_1}(\sigma^{\widetilde{\theta}(1,\omega,1)+M'(1)}\omega, \underline{v'})} - \mathcal{T}^*_{j_1}(d_1) \right| \le \varepsilon_1.$$

Choose $\mathcal{N}_1' > \mathcal{N}_1$ large enough such that

- $\widetilde{\theta}(1,\omega,1) \le (\varepsilon_1)^4 n^1_{\mathcal{N}_1'}$,
- $V_p(\omega) \le (\varepsilon_1)^4 p$ for any $p \ge n^1_{\mathcal{N}_1'}$ and

$$\sum_{j=0}^{\widetilde{\theta}(1,\omega,1)+M'(1)} \|\Phi(\sigma^j\omega)\|_\infty \le (\varepsilon_1)^4 n^1_{\mathcal{N}_1'},$$

- $M'(2) \le (\varepsilon_2)^4 n^1_{\mathcal{N}_1'}$,
- $W(2) \le (\varepsilon_2)^4 n^1_{\mathcal{N}_1'}$,
- for any s such that the return time $\widetilde{\theta}(2,\omega,s)$ satisfies $\widetilde{\theta}(2,\omega,s) \ge \widetilde{\theta}(1,\omega,1) + M'(1) + n^1_{\mathcal{N}_1'}$, one also has

$$\frac{\widetilde{\theta}(2,\omega,s) - \widetilde{\theta}(2,\omega,s-1)}{\widetilde{\theta}(2,\omega,s-1)} \le (\varepsilon_2)^4.$$

Let s_2 be the smallest s such that $\widetilde{\theta}(2,\omega,s) \geq \widetilde{\theta}(1,\omega,1) + M'(1) + n^1_{\mathcal{N}'_1}$.

Now, let N_1 be the largest k such that $\widetilde{\theta}(1,\omega,1) + M'(1) + n^1_k \leq \widetilde{\theta}(2,\omega,s_2)$ (by construction we have $N_1 \geq \mathcal{N}'_1$). Then

$$\widetilde{\theta}(2,\omega,s_2) - \widetilde{\theta}(1,\omega,1) - M'(1) - n^1_{N_1} \leq n^1_{N_1+1} - n^1_{N_1} \leq (\varepsilon_1)^4 n^1_{N_1}$$

by item 3. above.

For $x \in E(\omega,1,w^{(1)},q_1)$, we denote $v(\omega,1,q_1,n^1_k,x)$ the unique word such that $x \in I_\omega^{w^{(1)}*v(q_1)*v(\omega,1,q_1,n^1_k,x)}$ and $v(\omega,1,q_1,n^1_k,x) \in \Sigma_{\sigma^{\widetilde{\theta}(1,\omega,1)+M'(1)}\omega}$.

For any $k \geq N_1$, let

$$\mathcal{F}_1(q_1,k) = \{B(y, 2\ell_\omega^{w^{(1)}*v(q_1)*v(\omega,1,q_1,n^1_k,y)}) : y \in E(\omega,1,w^{(1)},q_1)\}.$$

Then $\mathcal{F}_1(q_1,k)$ is a covering of $E(\omega,1,w^{(1)},q_1)$. From Besicovitch covering theorem [59, theorem 2.7], there are Γ_1 families of intervals $\mathcal{F}_1^1(q_1,k),\cdots,\mathcal{F}_1^{\Gamma_1}(q_1,k) \subset \mathcal{F}_1(q_1,k)$, such that $E(\omega,1,w^{(1)},q_1) \subset \cup_{s=1}^{\Gamma_1}\cup_{B\in\mathcal{F}_1^s(q_1,k)} B$ and for any $B,B' \in \mathcal{F}_1^s$, if $B \neq B'$ one has $B \cap B' = \emptyset$ (here Γ_1 just depends on the dimension 1 of the Euclidean space $\mathbb{R}$).

Since $\zeta_{\omega,w^{(1)},q_1}(E(\omega,1,w^{(1)},q_1)) > 1/2$, there exists s such that

$$\zeta_{\omega,w^{(1)},q_1}(\cup_{B\in\mathcal{F}_1^s(q_1,k)}B) \geq \frac{1}{2\Gamma_1}.$$

Among the intervals of $\mathcal{F}_1^s(k)$ we can choose finite subset

$$D^{w^{(1)}}(1,q_1,k) = \{B_1,\cdots,B_{s'}\}$$

such that

$$\zeta_{\omega,w^{(1)},q_1}(\cup_{B_l\in D^{w^{(1)}}(1,q_1,k)}B_l) \geq \frac{1}{4\Gamma_1}.$$

For any $B_l \in D^{w^{(1)}}(1,q_1,k)$, there exists $y_l \in E(\omega,1,w^{(1)},q_1)$ such that $B_l = B(y_l, 2\ell_\omega^{w^{(1)}*v(q_1)*v(k,l)})$, where $v(k,l) := v(\omega,1,q_1,n^1_k,y_l)$. Since

$$\begin{aligned} B(z_\omega^{w^{(1)}*v(q_1)*v(k,l)}, (\ell_\omega^{w^{(1)}*v(q_1)*v(k,l)})^\xi) &\subset B(z_\omega^{w^{(1)}*v(q_1)*v(k,l)}, \ell_\omega^{w^{(1)}*v(q_1)*v(k,l)}) \\ &\subset B(y_l, 2\ell_\omega^{w^{(1)}*v(q_1)*v(k,l)}) = B_l, \end{aligned}$$

using the same argument as in step 3 in section 6.2 we can obtain that

$$\zeta_{\omega,w^{(1)},q_1}(B(y_l, 2\ell_\omega^{w^{(1)}*v(q_1)*v(k,l)})) \leq (4\ell_\omega^{w^{(1)}*v(q_1)*v(k,l)})^{\mathcal{T}^*_{j_1}(d_1)-2\varepsilon_1}, \quad (7.6)$$

and

$$\zeta_{\omega,w^{(1)},q_1}(B(z_\omega^{w^{(1)}*v(q_1)*v(k,l)}, \ell_\omega^{w^{(1)}*v(q_1)*v(k,l)})) \leq (4\ell_\omega^{w^{(1)}*v(q_1)*v(k,l)})^{\mathcal{T}^*_{j_1}(d_1)-2\varepsilon_1}. \quad (7.7)$$

Choose the smallest j such that $\sigma^j\omega \in \widetilde{\Omega}_2$ and there exists $v \in \Sigma_{\omega,j}$ satisfying

- $z_\omega^{w^{(1)}*v(q_1)*v(k,l)} \in \overline{I_\omega^v}$,
- $I_\omega^v \subset B(z_\omega^{w^{(1)}*v(q_1)*v(k,l)}, (\ell_\omega^{w^{(1)}*v(q_1)*v(k,l)})^\xi)$.

Define $J_l = \overline{I_\omega^v}$, the closure of I_ω^v. From the construction we get that

$$|J_l| \leq 2(\ell_\omega^{w^{(1)}*v(q_1)*v(k,l)})^\xi.$$

Since we have chosen the smallest $j := \widetilde{\theta}(2,\omega,s)$, then for $v' = v|_{\widetilde{\theta}(2,\omega,s-1)}$ we have

$$(\ell_\omega^{w^{(1)}*v(q_1)*v(k,l)})^\xi \leq |I_\omega^{v'}| \leq |I_\omega^v| \exp(V_{\widetilde{\theta}(2,\omega,s)}(\omega)) \exp((\widetilde{\theta}(2,\omega,s) - \widetilde{\theta}(2,\omega,s-1))C).$$

Since $|I_\omega^v| \geq \exp(-2\cdot\widetilde{\theta}(2,\omega,s)C_\Phi)$, $V_{\widetilde{\theta}(2,\omega,s)}(\omega) \leq \widetilde{\theta}(2,\omega,s)\cdot(\varepsilon_1)^4$ and $\widetilde{\theta}(2,\omega,s) - \widetilde{\theta}(2,\omega,s-1) \leq 2\cdot\widetilde{\theta}(2,\omega,s-1)\cdot(\epsilon_2)^4 \leq 2\cdot\widetilde{\theta}(2,\omega,s)\cdot(\varepsilon_1)^4$ we obtain

$$(4\ell_\omega^{w^{(1)}*v(q_1)*v(k,l)})^\xi \leq |I_\omega^v|^{1-(\varepsilon_1)^3}. \tag{7.8}$$

Denote $\underline{B_l} := J_l$. Conversely, if an interval J can be written as $\underline{B}$, then we denote $B =: \overline{J}$. By construction we have

$$|J| \leq |\overline{J}|^\xi \leq |J|^{1-(\varepsilon_1)^3}. \tag{7.9}$$

Define

$$G^{w^{(1)}}(1,d_1,k) = \{\underline{B_l} : B_l \in D^{w^{(1)}}(1,q_1,k), \text{ where } \mathcal{T}'_{j_1}(q_1) = d_1\}. \tag{7.10}$$

We notice the following useful property:

By construction, if J_1 and J_2 are two distinct elements of $G^w(1,d_1,k)$, their distance is larger than $\max_{i\in\{1,2\}}(|\overline{J_i}|/2 - (|\overline{J_i}|/2)^\xi)$. Since $\xi > 1$, when k large enough, one gets that $\max_{i\in\{1,2\}}(|\overline{J_i}|/2 - (|\overline{J_i}|/2)^\xi) \geq \max_{i\in\{1,2\}} |\overline{J_i}|/3$. This implies that their distance is larger than $\max_{i\in\{1,2\}} |\overline{J_i}|/3$.

On the algebra generated by $G^w(1,d_1,k)$, we can define a probability measure $m_\xi^{d_1}$ by:

$$m_\xi^{d_1}(J) = \frac{\zeta_{\omega,w^{(1)},q_1}(\overline{J})}{\sum_{J_l\in G^{w^{(1)}}(1,q_1,k)} \zeta_{\omega,w^{(1)},q_1}(\overline{J_l})}$$

For any $J \in G^w(1,d_1,k)$, from inequities (7.6) and (7.8) we can get

$$\zeta_{\omega,w^{(1)},q_1}(\overline{J}) \leq |\overline{J}|^{\mathcal{T}^*_{j_1}(d_1)-2\varepsilon_1} \leq |J|^{\frac{(\mathcal{T}^*_{j_1}(d_1)-2\varepsilon_1)(1-(\varepsilon_1)^3)}{\xi}}$$

and

$$\sum_{J_l\in G^{w^{(1)}}(1,d_1,k)} \zeta_{\omega,w^{(1)},q_1}(\overline{J_l}) = \sum_{B_l\in D^{w^{(1)}}(1,q_1,k)} \zeta_{\omega,w^{(1)},q_1}(B_l) \geq \frac{1}{4\Gamma_1}.$$

Then for any $J \in G^w(1, d_1, k)$,

$$m_\xi^{d_1}(J) \leq 4\Gamma_1 |J|^{\frac{(\mathcal{T}_{j_1}^*(d_1)-2\varepsilon_1)(1-(\varepsilon_1)^3)}{\xi}}.$$

We can choose k large enough such that $4\Gamma_1 \leq |J|^{-\varepsilon_1}$ for any $d_1 \in D_1$ and $J \in G^{w^{(1)}}(1, d_1, k)$. We denote such a k by k_1.

The first step of the construction of $K(\xi, \widetilde{d})$ is $G(d_1) := G^{w^{(1)}}(1, d_1, k_1)$. Define:

$$G_1 = \cup_{d_1 \in \mathcal{D}_1} G(d_1).$$

From the construction we know that: for any $d_1 \in \mathcal{D}_1$, there exists $q_1 \in \mathcal{Q}_1$ such that

1. $$\forall J \in G(d_1),\ m_\xi^{d_1}(J) \leq |J|^{\frac{(\mathcal{T}_{j_1}^*(d_1)-2\varepsilon_1)(1-(\varepsilon_1)^3)}{\xi} - \varepsilon_1}. \tag{7.11}$$
2. For any $J \in G(d_1)$ there exists $y_l \in E(\omega, 1, w^{(1)}, q_1)$ such that
$$B(z_\omega^{w^{(1)}*v(q_1)*v(k_1,l)}, (\ell_\omega^{w^{(1)}*v(q_1)*v(k_1,l)})^\xi) \subset \overline{J} = B(y_l, 2\ell_\omega^{w^{(1)}*v(q_1)*v(k_1,l)}).$$

Then, for any $y \in \cup_{J \in G(d_1)} J$, there exists a word $v = v(k_1, l)$ satisfying

$$|z_\omega^{w^{(1)}*v(q_1)*v} - y| \leq (\ell_\omega^{w^{(1)}*v(q_1)*v})^\xi \leq \frac{\ell_\omega^{w^{(1)}*v(q_1)*v}}{2},$$
$$\zeta_{\omega, w^{(1)}, q_1}(B(z_\omega^{w^{(1)}*v(q_1)*v}, \ell_\omega^{w^{(1)}*v(q_1)*v})) \leq (4\ell_\omega^{w^{(1)}*v(q_1)*v})^{\mathcal{T}_{j_1}^*(d_1)-2\varepsilon_1}.$$

Step 2: Suppose that G_i is well defined and for any $\{d_j\}_{1\leq j\leq i} \in \prod_{1\leq j\leq i} \mathcal{D}_j$, the set function $m_\xi^{\{d_j\}_{1\leq j\leq i}}$ is well defined on the set $G(d_1, \ldots, d_i)$.

For any w such that $J = \overline{I_\omega^w} \in G(d_1 \cdots d_i) \subset G_i$, we set $n = |w|$.

In this step n_k^{i+1} stands for $n_k^{i+1}(\sigma^n \omega)$.

By construction:

1. $\sigma^n \omega \in \widetilde{\Omega}_{i+1}$,
2. $M'(i) + n_{\mathcal{N}_{i+1}}^{i+1} \leq n(\varepsilon_{i+1})^4/8$;

For any $d_{i+1} \in \mathcal{D}_{i+1}$, take $q_{i+1} \in \mathcal{Q}_{i+1}$ such that $\mathcal{T}'_{j_{i+1}}(q_{i+1}) = d_{i+1}$, and $v(q_{i+1}) \in \Sigma_{\sigma^{n+M}\omega, S(i+1)}$ the associated word.

From facts 7.19 and 7.20, for each $d_{i+1} \in \mathcal{D}_{i+1}$, there is a set $E(\omega, i+1, w, q_{i+1}) \subset \overline{I_\omega^{w*v(q_{i+1})}}$ such that $\zeta_{\omega,w,q_{i+1}}(E(\omega, i+1, w, q_{i+1})) > 1/2$ (Recall that the definition of $\zeta_{\omega,w,q_{i+1}}$ and $E(\omega, i+1, w, q_{i+1})$ are defined at the end of section 7.6) and:

1. $c_{n^{i+1}_{\mathcal{N}_{i+1}}} \leq (\varepsilon_{i+1})^4$ and $c_{i+1,n^{i+1}_{\mathcal{N}_{i+1}}} \leq (\varepsilon_1)^4$,
2. $n_k^{i+1} - n_{k-1}^{i+1} \leq (\varepsilon_{i+1})^4 n_{k-1}^{i+1}$ for any $k \geq \mathcal{N}_{i+1}$,
3.
   - For any $x \in E(\omega, i+1, w, q_{i+1})$, for any $k \geq \mathcal{N}_{i+1}$, for any $v \in \Sigma_{\sigma^{n+M'(i+1)}\omega, n_k^{i+1}}$ such that $x \in I_\omega^{w*v(q_{i+1})*v}$, one has $|v \wedge v+| \geq n_{k-1}^{i+1}$ and $|v \wedge v-| \geq n_{k-1}^{i+1}$.
   - For any $\underline{v} \in [v]_{\sigma^{n+M'(i+1)}\omega} \cup [v+]_{\sigma^{n+M'(i+1)}\omega} \cup [v-]_{\sigma^{n+M'(i+1)}\omega}$, one has

$$\left| \frac{S_{n_k^i} \Psi_{j_{i+1}}(\sigma^{n+M'(i+1)}\omega, \underline{v})}{S_{n_k^i} \Phi_{j_{i+1}}(\sigma^{n+M'(i+1)}\omega, \underline{v})} - d_{i+1} \right| \leq \varepsilon_{i+1},$$

and for $v' = v$, $v' = v-$ or $v' = v+$, and $\underline{v}' \in [v']_{\sigma^{n+M'(i+1)}\omega}$

$$\left| \frac{\log \rho_{\sigma^{n+M'(i+1)}\omega}^{\widetilde{\Lambda}_{i+1,q_{i+1}}}(I^{v'}_{\sigma^{n+M'(i+1)}\omega})}{S_{|v'|}\Phi_{j_{i+1}}(\sigma^{n+M'(i+1)}\omega, \underline{v'})} - \mathcal{T}^*_{j_{i+1}}(d_{i+1}) \right| \leq \varepsilon_{i+1}.$$

and

$$\left| \frac{S_{|v'|}\widetilde{\Lambda}_{i+1,q_{i+1}}(\sigma^{n+M'(i+1)}\omega, \underline{v'})}{S_{|v'|}\Phi_{j_{i+1}}(\sigma^{n+M'(i+1)}\omega, \underline{v'})} - \mathcal{T}^*_{j_{i+1}}(d_{i+1}) \right| \leq \varepsilon_{i+1}.$$

Choose $\mathcal{N}'_{i+1} > \mathcal{N}_{i+1}$ large enough such that

- $n \leq (\varepsilon_{i+1})^4 n^{i+1}_{\mathcal{N}'_{i+1}}$,
- $V_p(\omega) \leq (\varepsilon_{i+1})^4 p$ for any $p \geq n^{i+1}_{\mathcal{N}'_{i+1}}$ and

$$\sum_{j=0}^{n+M'(i+1)} \|\Phi(\sigma^j\omega)\|_\infty \leq (\varepsilon_{i+1})^4 n^{i+1}_{\mathcal{N}'_{i+1}},$$

- $M'(i+2) \leq (\varepsilon_{i+2})^4 n^{i+1}_{\mathcal{N}'_{i+1}}$,
- $W(i+2) \leq (\varepsilon_{i+2})^4 n^{i+1}_{\mathcal{N}'_{i+1}}$,
- for any s such that the return time $\widetilde{\theta}(i+2,\omega,s)$ satisfies $\widetilde{\theta}(i+2,\omega,s) \geq n + M'(i+1) + n^{i+1}_{\mathcal{N}'_{i+1}}$, one also has

$$\frac{\widetilde{\theta}(i+2,\omega,s) - \widetilde{\theta}(i+2,\omega,s-1)}{\widetilde{\theta}(i+2,\omega,s-1)} \leq (\varepsilon_{i+2})^4.$$

Let s_{i+2} be the smallest s such that $\widetilde{\theta}(i+2,\omega,s) \geq n + M'(i+1) + n^{i+1}_{\mathcal{N}'_{i+1}}$.

Now, let N_{i+1} be the largest k such that $n+M'(i+1)+n_k^{i+1} \leq \widetilde{\theta}(i+2,\omega,s_{i+2})$ (by construction we have $N_{i+1} \geq \mathcal{N}'_{i+1}$). Then

$$\widetilde{\theta}(i+2,\omega,s_{i+2}) - n - M'(i+1) - n_{N_{i+1}}^{i+1} \leq n_{N_{i+1}}^{i+1} - n_{N_{i+1}}^{i+1} \leq (\varepsilon_{i+1})^4 n_{N_{i+1}}^{i+1}$$

by item 3. above.

For any $k \geq N_{i+1}$, let

$$\mathcal{F}_{i+1}(q_{i+1}, n+M'(i+1)+n_k^{i+1}) = \{B(y, 2\ell_\omega^{w*v(q_{i+1})*v(\omega,i+1,q_{i+1},n_k^{i+1},y)}) : y \in E(\omega, i+1, w, q_{i+1})\},$$

here $v(\omega, i+1, q_{i+1}, n_k^{i+1}, y)$ is the unique word such that $y \in I_\omega^{w*v(q_{i+1})*v(\omega,i+1,q_{i+1},n_k^{i+1},y)}$ and $v(\omega, i+1, q_{i+1}, n_k^{i+1}, y) \in \Sigma_{\sigma^{n+M'(i+1)}\omega, n_k^{i+1}}$. Then $\mathcal{F}_{i+1}(q_{i+1}, n+M'(i+1)+n_k^{i+1})$ is a covering of $E(\omega, i+1, w, q_{i+1})$.

From the Besicovitch covering theorem, Γ_1 families of disjoint intervals, namely

$$\mathcal{F}_{i+1}^1(q_{i+1}, n+M'(i+1)+n_k^{i+1}), \cdots, \mathcal{F}_{i+1}^{\Gamma_1}(n+M'(i+1)+n_k^{i+1})$$

can be extracted from $\mathcal{F}_{i+1}(q_{i+1}, n+M'(i)+n_k^{i+1})$ so that $E(\omega, i+1, w, q_{i+1}) \subset \cup_{s=1}^{\Gamma_1} \cup_{B\in\mathcal{F}_{i+1}^s(q_{i+1},n+M'(i+1)+n_k^{i+1})} B$.

Since $\zeta_{\omega,w,q_{i+1}}(E(\omega, i+1, w, q_{i+1})) \geq 1/2$, there exists s such that

$$\zeta_{\omega,w,q_{i+1}}(\cup_{B\in\mathcal{F}_{i+1}^s(q_{i+1},n+M'(i)+n_k^{i+1})} B) \geq \frac{1}{2\Gamma_1}.$$

Again, we extract from $\mathcal{F}_{i+1}^s(q_{i+1}, n+M'(i)+n_k^{i+1})$ a finite family of pairwise disjoint intervals $D^w(i+1, q_{i+1}, k) = \{B_1, \cdots, B_{s'}\}$ such that

$$\zeta_{\omega,w,q_{i+1}}(\cup_{B_l\in D^w(i+1,q_{i+1},k)} B_l) \geq \frac{1}{4\Gamma_1}.$$

For each $B_l \in D^w(i+1, q_{i+1}, k)$, there exists $y_l \in E(\omega, i+1, w, q_{i+1})$ such that $B_l = B(y_l, 2\ell_\omega^{w*v(q_{i+1})*v(\omega,i+1,q_{i+1},n_k^{i+1},y_l)})$. Set $v(k,l) = v(\omega, i+1, q_{i+1}, n_k^{i+1}, y_l)$. Moreover,

$$\begin{aligned} B(z_\omega^{w*v(q_{i+1})*v(k,l)}, (\ell_\omega^{w*v(q_{i+1})*v(k,l)})^\xi) &\subset B(z_\omega^{w*v(q_{i+1})*v(k,l)}, \ell_\omega^{w*v(q_{i+1})*v(k,l)}) \\ &\subset B(y_l, 2\ell_\omega^{w*v(q_{i+1})*v(k,l)}) = B_l. \end{aligned}$$

Also, using the same argument as in the step 3 of section 6.2 and noticing the fact that $n \leq (\varepsilon_{i+1})^4 n_{N_{i+1}}^{i+1}$, we can obtain

$$\zeta_{\omega,w,q_{i+1}}(B(y_l, 2\ell_\omega^{w*v(q_{i+1})*v(k,l)})) \leq \left(\frac{4\ell_\omega^{w*v(q_{i+1})*v(k,l)}}{|I_\omega^w|}\right)^{\mathcal{T}_{j_{i+1}}^*(d_{i+1})-2\varepsilon_{i+1}}, \quad (7.12)$$

and

$$\zeta_{\omega,w,q_{i+1}}(B(z_\omega^{w*v(q_{i+1})*v(k,l)}, \ell_\omega^{w*v(q_{i+1})*v(k,l)})) \leq \left(\frac{4\ell_\omega^{w*v(q_{i+1})*v(k,l)}}{|I_\omega^w|}\right)^{\mathcal{T}_{j_{i+1}}^*(d_{i+1})-2\varepsilon_{i+1}}. \quad (7.13)$$

Choose the smallest j such that $\sigma^j\omega \in \widetilde{\Omega}_{i+2}$ and there exists $v \in \Sigma_{\omega,j}$ satisfying

- $z_\omega^{w*v(q_{i+1})*v(k,l)} \in \overline{I_\omega^v}$,
- $I_\omega^v \subset B(z_\omega^{w*v(q_{i+1})*v(k,l)}, (\ell_\omega^{w*v(q_{i+1})*v(k,l)})^\xi)$

Define $J_l =: \overline{I_\omega^v}$, the closure of the interval I_ω^v.

Denotes $\underline{B_l} = J_l$. Conversely, if an interval J can be written $\underline{B}$ for some larger interval B, we write $B = \overline{J}$.

Using the same method as in the step 1, we can get:

$$|J| \leq |\overline{J}|^\xi \leq |J|^{1-(\varepsilon_{i+1})^3}. \tag{7.14}$$

From the construction we get

$$G^w(i+1, d_{i+1}, k) = \{\underline{B_l}, B_l \in D^w(i+1, d_{i+1}, k)\}. \tag{7.15}$$

If J_1 and J_2 are two distinct elements of $G^w(i+1, d_{i+1}, k)$ then their distance is at least $\max_{i\in\{1,2\}}(|\overline{J_i}|/2 - (|\overline{J_i}|/2)^\xi)$, which is larger than $\max_{i\in\{1,2\}} |\overline{F_i}|/3$ for k large enough (since $\xi > 1$).

We can define $m_\xi^{\{d_j\}_{1\leq j\leq i+1}}$ with $d_{i+1} \in \mathcal{D}_{i+1}$ as follows,

$$m_\xi^{\{d_j\}_{1\leq j\leq i+1}}(J) = \frac{\zeta_{\omega,w,q_{i+1}}(\overline{J})}{\sum_{J_l\in G^w(i+1,d_{i+1},k)} \zeta_{\omega,w,q_{i+1}}(\overline{J_l})} \left(m_\xi^{\{d_j\}_{1\leq j\leq i}}(\overline{I_\omega^w})\right).$$

For any $J \in G^w(i+1, d_{i+1}, k)$, from the inequality (7.12) we get that

$$\begin{aligned} \zeta_{\omega,w,q_{i+1}}(\overline{J}) &\leq \left(\frac{|\overline{J}|}{|I_\omega^w|}\right)^{\mathcal{T}^*_{j_{i+1}}(d_{i+1})-2\varepsilon_{i+1}} \\ &\leq |J|^{\frac{(\mathcal{T}^*_{j_{i+1}}(d_{i+1})-2\varepsilon_{i+1})(1-(\varepsilon_{i+1})^3)}{\xi}} |I_\omega^w|^{-\mathcal{T}^*_{j_{i+1}}(d_{i+1})}. \end{aligned}$$

Then, since

$$\sum_{J_l\in G^w(n,i+1,k)} \zeta_{\omega,w,q_{i+1}}(\overline{J_l}) \geq \frac{1}{4\Gamma_1},$$

we get, $\forall J \in G^w(i+1, d_{i+1}, k)$:

$$m_\xi^{\{d_j\}_{1\leq j\leq i+1}}(J) \leq 4\Gamma_1 |J|^{\frac{(\mathcal{T}^*_{j_{i+1}}(d_{i+1})-2\varepsilon_{i+1})(1-(\varepsilon_{i+1})^3)}{\xi}} |I_\omega^w|^{-\mathcal{T}^*_{j_{i+1}}(d_{i+1})}.$$

We can also choose $k = k_{i+1}$ large enough such that for any $d_{i+1} \in \mathcal{D}_{i+1}$ and for any $J \in G^w(i+1, d_{i+1}, k_{i+1})$, one has

$$4\Gamma_1 |I_\omega^w|^{-\mathcal{T}^*_{j_{i+1}}(d_{i+1})} \leq |J|^{-\varepsilon_{i+1}}.$$

Then by construction we have

$$\forall J \in G^w(i+1, d_{i+1}, k_{i+1}),\ m_\xi^{\{d_j\}_{1\leq j\leq i+1}}(J) \leq |J|^{\frac{(\mathcal{T}^*_{j_{i+1}}(d_{i+1})-2\varepsilon_{i+1})(1-(\varepsilon_{i+1})^3)}{\xi}-\varepsilon_{i+1}}. \tag{7.16}$$

Then, for $(d_j)_{1\leq j\leq i+1} \in \prod_{j=1}^{i+1}\mathcal{D}_j$ define:

$$G(d_1, d_2, \cdots, d_{i+1}) = \cup_{w\in G(d_1,q_2,\cdots,d_i)} G^w(i+1, d_{i+1}, k_{i+1}),$$

and

$$G_{i+1} = \cup_{w\in G(i)} \cup_{q_{i+1}\in\mathcal{Q}_{i+1}} G^w(i+1, d_{i+1}, k_{i+1}).$$

The definition of $m_\xi^{\{d_j\}_{1\leq j\leq i+1}}$ can be extended to the algebra generated by $\cup_{s\leq i+1} G(d_1, d_2, \cdots, d_s)$, and for any $J = I^v_\omega \in G(d_1, d_2, \cdots, d_s)$,

$$m_\xi^{\{d_j\}_{1\leq j\leq i+1}}(J) \leq |J|^{\frac{(\mathcal{T}^*_{j_{i+1}}(d_{i+1})-2\varepsilon_{i+1})(1-(\varepsilon_{i+1})^3)}{\xi}-\varepsilon_{i+1}}.$$

Step 3: For any $\widetilde{d} = \{d_i\}_{i\in\mathbb{N}} \in \prod_{i=1}^\infty \mathcal{D}_i$, for any $J \in G(d_1, \cdots, d_i)$, define $m_\xi^{\widetilde{d}}(J) = m_\xi^{\{d_j\}_{1\leq j\leq i}}(J)$. This yields a probability measure $m_\xi^{\widetilde{d}}$ on the algebra generated by $\cup_{i\in\mathbb{N}} G(d_1, \cdots, d_i)$.

For any $i \in \mathbb{N}$, the elements in $G(d_1, \cdots, d_i)$ are closed and disjoint intervals. Also, for any $J \in G(d_1, \cdots, d_i)$, we take $\overline{J}$ to be the associated interval of J. We have the following properties:

1. - $J \subset \overline{J}$, for any $J \in G(d_1, \cdots, d_i)$,
   - for any $J \in G(d_1, \cdots, d_i)$
   $$|J| \leq |\overline{J}|^\xi \leq |J|^{1-(\varepsilon_i)^3}, \tag{7.17}$$
   - if $J_1 \neq J_2$ belong to $G(d_1, \cdots, d_i)$, their distance is at least
   $$\max_{l\in\{1,2\}} \frac{|\overline{J_l}|}{3},$$
   - The intervals $\overline{J_l}$, $J_l \in G(d_1, \cdots, d_i)$, are disjoint.
2. Any J in $G(d_1, d_2, \cdots, d_i)$ is contained in some element $L = \overline{I^w_\omega} \in G(d_1, d_2, \cdots, d_{i-1})$. Moreover , $\overline{J} \cap E(\omega, i, w, q_i) \neq \emptyset$, where $q_i \in \mathcal{Q}_i$ is such that $\mathcal{T}'_{j_i}(q_i) = d_i$ and $E(\omega, i, w, q_i)$ is the set which can be seen in step 2.
3. For any $J \in G(d_1, d_2, \cdots, d_i)$,
$$m_\xi^{\widetilde{d}}(J) \leq |J|^{\frac{(\mathcal{T}^*_{j_i}(d_i)-2\varepsilon_i)(1-(\varepsilon_i)^3)}{\xi}-\varepsilon_i}. \tag{7.18}$$

4. Any J in $G(d_1, d_2, \cdots, d_i)$ is contained by some element $L = \overline{I_\omega^w} \in G(d_1, d_2, \cdots, d_{i-1})$ such that

$$m_\xi^{\widetilde{d}}(J) \le 4\Gamma_1 m_\xi^{\widetilde{d}}(L)\zeta_{\omega,w,q_i}(\overline{J}),$$

where $q_i \in \mathcal{Q}_i$ is such that $\mathcal{T}'_{j_i}(q_i) = d_i$.

Because of the separation property 1, we get a probability measure $m_\xi^{\widetilde{d}}$ on $\sigma(J : J \in \cup_{i\ge1} G(d_1, d_2, \cdots, d_i))$ such that properties 1. to 4. hold for every $i \ge 1$. We now define

$$K(\xi, \widetilde{d}) = \cap_{i\ge1} \cup_{J\in G(d_1,\cdots,d_i)} J.$$

Then, $m_\xi^{\widetilde{d}}(K(\xi, \widetilde{d})) = 1$. The measure $m_\xi^{\widetilde{d}}$ can be extended to $[0, 1]$ by setting, for any $B \in \mathcal{B}([0, 1])$, $m_\xi^{\widetilde{d}}(B) := m_\xi^{\widetilde{d}}(B \cap K(\xi, \widetilde{d}))$.

step 4: Fix a sequence $\widetilde{d} = \{d_i\}_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} \mathcal{D}_i$ such that

$$\lim_{i\to\infty} d_i = d, \ \lim_{i\to\infty} \mathcal{T}^*(d_i) = \mathcal{T}^*(d).$$

Define $K^d(\xi) = K(\xi, \widetilde{d})$, and $m_\xi^d = m_\xi^{\widetilde{d}}$. Below we first show that $K^d(\xi) \subset S(\omega, d, \xi, \widetilde{\epsilon})$ and estimate the lower Hausdorff dimension of m_ξ^d.

Now, let $\epsilon_{i+1} = |d - d_{i+1}| + 2\varepsilon_{i+1}$, we will use the same notation as in step 2 and prove that for any $w * v(q_{i+1}) * \underline{v} \in [w * v(q_{i+1}) * v(K_{i+1}, l)]_\omega$ we have:

$$\left| \frac{S_{n+M'(i+1)+n^{i+1}_{k_{i+1}}} \Psi(\omega, w * v(q_{i+1}) * \underline{v})}{S_{n+M'(i+1)+n^{i+1}_{k_{i+1}}} \Phi(\omega, w * v(q_{i+1}) * \underline{v})} - d \right| \le \epsilon_{i+1}.$$

Indeed,

$$\begin{aligned} &\frac{S_{n+M'(i+1)+n^{i+1}_{k_{i+1}}} \Psi(\omega, w * v(q_{i+1}) * \underline{v})}{S_{n+M'(i+1)+n^{i+1}_{k_{i+1}}} \Phi(\omega, w * v(q_{i+1}) * \underline{v})} \\ = &\frac{S_{n+M'(i+1)}\Psi(\omega, w * v(q_{i+1}) * \underline{v}) + S_{n^{i+1}_{k_{i+1}}} \Psi(F^{n+M'(i+1)-1}(\omega, w * v(q_{i+1}) * \underline{v}))}{S_{n+M'(i+1)}\Phi(\omega, w * v(q_{i+1}) * \underline{v}) + S_{n^{i+1}_{k_{i+1}}} \Phi(F^{n+M'(i+1)-1}(\omega, w * v(q_{i+1}) * \underline{v}))}. \end{aligned}$$

Thus,

$$\begin{aligned} &\left| \frac{S_{n+M'(i+1)+n^{i+1}_{k_{i+1}}} \Psi(\omega, w * v(q_{i+1}) * \underline{v})}{S_{n+M'(i+1)+n^{i+1}_{k_{i+1}}} \Phi(\omega, w * v(q_{i+1}) * \underline{v})} - d_{i+1} \right| \\ \le &\frac{|S_{n+M'(i+1)}(\Psi - d_{i+1}\Phi)(\omega, w * v(q_{i+1}) * \underline{v})|}{|S_{n+M'(i+1)}\Phi(\omega, w * v(q_{i+1}) * \underline{v}) + S_{n^{i+1}_{k_{i+1}}} \Phi(F^{n+M'(i+1)-1}(\omega, w * v(q_{i+1}) * \underline{v}))|} \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{|S_{n^{i+1}_{k_{i+1}}}(\Psi - d_{i+1}\Phi)(F^{n+M'(i+1)-1}(\omega, w * v(q_{i+1}) * \underline{v}))|}{|S_{n+M'(i+1)}\Phi(\omega, w * v(q_{i+1}) * \underline{v}) + S_{n^{i+1}_{k_{i+1}}}\Phi(F^{n+M'(i+1)-1}(\omega, w * v(q_{i+1}) * \underline{v}))|}.\\
&\leq \quad \frac{(n + M'(i+1))(1 + d_{i+1})C}{|S_{n+M'(i+1)}\Phi(\omega, w * v(q_{i+1}) * \underline{v}) + S_{n^{i+1}_{k_{i+1}}}\Phi(F^{n+M'(i+1)-1}(\omega, w * v(q_{i+1}) * \underline{v}))|}\\
&+ \quad \frac{(\varepsilon_{i+1})|S_{n^{i+1}_{k_{i+1}}}\Phi(F^{n+M'(i+1)-1}(\omega, w * v(q_{i+1}) * \underline{v}))|}{|S_{n+M'(i+1)}\Phi(\omega, w * v(q_{i+1}) * \underline{v}) + S_{n^{i+1}_{k_{i+1}}}\Phi(F^{n+M'(i+1)-1}(\omega, w * v(q_{i+1}) * \underline{v}))|}\\
&\leq \quad \frac{(n^{i+1}_{k_{i+1}}(\varepsilon_{i+1})^4(1 + d_{i+1})C}{|S_{n+M'(i+1)}\Phi(\omega, w * v(q_{i+1}) * \underline{v}) + S_{n^{i+1}_{k_{i+1}}}\Phi(F^{n+M'(i+1)-1}(\omega, w * v(q_{i+1}) * \underline{v})|)}\\
&+ \quad (\varepsilon_{i+1})\\
&\leq \quad (2\varepsilon_{i+1}),
\end{aligned}
$$

where we used the fact that the Birkhoff sums arising above are negative. Finally, we get

$$
\left| \frac{S_{n+M'(i+1)+n^{i+1}_{k_{i+1}}}\Psi(\omega, w * v(q_{i+1}) * \underline{v})}{S_{n+M'(i+1)+n^{i+1}_{k_{i+1}}}\Phi(\omega, w * v(q_{i+1}) * \underline{v})} - d \right| \leq |d - d_{i+1}| + 2\varepsilon_{i+1} = \epsilon_{i+1}.
$$

By construction, we conclude that

$$
K^d(\xi) \subset S(\omega, d, \xi, \widetilde{\epsilon}).
$$

Now let us estimate the lower Hausdorff dimension of m^d_ξ. If $\mathcal{T}^*(d) = 0$, there is nothing to prove. So we assume that $\mathcal{T}^*(d) > 0$. We recall that $|\mathcal{T}^*_{j_i}(d_i) - \mathcal{T}^*(d_i)| \leq \varepsilon_i$.

For any $J \in G(d_1, d_2, \cdots, d_i)$, define $g(J) = i$. Let us fix B an interval of $[0, 1]$ of length less than that of the elements of $G(d_1)$, and assume that $B \cap \widetilde{K^d_\xi} \neq \emptyset$. Let $L = \overline{I^w_\omega}$ be the element of largest diameter in $\cup_{i\geq 1} G(d_1 \cdots d_i)$ such that B intersects at least two elements of $G(d_1 \cdots d_{g(L)+1})$ included in $L \in G_{d_1 \cdots d_{g(L)}}$. We remark that this implies that B does not intersect any other element of $G(d_1, d_2, \cdots, d_s)$, where $s = g(L)$, and as a consequence $m^d_\xi(B) \leq m^d_\xi(L)$.

Let us distinguish three cases:

- If $|B| \geq |L|$: then

$$
m^d_\xi(B) \leq m^d_\xi(L) \leq |L|^{\frac{(\mathcal{T}^*_{j_s}(d_s) - 2\varepsilon_s)(1-\varepsilon_s)}{\xi} - \varepsilon_s} \leq |B|^{\frac{(\mathcal{T}^*_{j_s}(d_s) - 2\varepsilon_s)(1-\varepsilon_s)}{\xi} - \varepsilon_s}. \tag{7.19}
$$

- If $|B| \leq \frac{1}{8}|L| \exp(-(|w|)(\varepsilon_s)^2/4)$. Assume $L_1, \ldots, L_p$ are the elements of G_{s+1} which have non-empty intersection with B.

From property (4), we can choose $q_{s+1} \in \mathcal{Q}_{s+1}$ so that $\mathcal{T}'_{j_s}(q_{s+1}) = d_{s+1}$, and get

$$m_\xi^d(B) = \sum_{l=1}^{p} m_\xi^d(B \cap L_l) \leq 4\Gamma_1 m_\xi^d(L) \sum_{i=1}^{p} \zeta_{\omega,w,q_{s+1}}(\overline{L_l}). \tag{7.20}$$

From property (1) we can also deduce that $\max\{|\overline{L_l}| : 1 \leq l \leq p\} \leq 3|B|$.

From property (2) we can get $E(\omega, s+1, w, q_{s+1}) \cap \overline{L_i} \neq \emptyset$. There exists l such that $E(\omega, s+1, w, q_{s+1}) \cap \overline{L_l} \neq \emptyset$. If y is taken in the intersection, we have $B(y, 4|B|) \supset (\cup_{l=1}^{p} \overline{L_l})$.

Now we notice that L is the closure of I_ω^w for some $w \in \Sigma_{\omega,n}$ with $n \in \mathbb{N}$. We have $\sigma^n \omega \in \widetilde{\Omega}_{s+1}$.

Now for any two intervals $I_\omega^{w*v(q_{s+1})*v}$ and $I_\omega^{w*v(q_{s+1})*v'}$ with $|v \wedge v'| \geq n_k^{s+1}$ and $|v| = |v'| = n_{k+1}^{s+1}$ with $k \geq \mathcal{N}_{s+1}$, we want to calculate $|\log|I_\omega^{w*v(q_{s+1})*v}| - \log|I_\omega^{w*v(q_{s+1})*v'}||$. We have

$$\begin{aligned} & |\log|I_\omega^{w*v(q_{s+1})*v}| - \log|I_\omega^{w*v(q_{s+1})*v'}|| \\ \leq\ & 2V_{n+M'(s+1)+n_{k+1}^{s+1}}(\omega) + 2C(n_{k+1}^{s+1} - n_k^{s+1}) \\ \leq\ & 2(n + M'(s+1) + n_k^{s+1})(\varepsilon_s)^4 \\ \leq\ & 2(n + M'(s+1) + n_k^{s+1})(\varepsilon_s)^3. \end{aligned} \tag{7.21}$$

The second inequality uses the fact that

$$V_{n+M'(s+1)+n_{k+1}^{s+1}}(\omega) \leq (\varepsilon_s)^4(n + M'(s+1) + n_{k+1}^{s+1}),$$

since $n + M'(s+1) + n_{k+1}^{s+1} \geq n_{\mathcal{N}'_s}^s$ and for $k \geq \mathcal{N}_{s+1}$ we have $|n_{k+1}^{s+1} - n_k^{s+1}| \leq (\varepsilon_s)^4 n_k^{s+1}$.

Since $|B| \leq \frac{1}{8}|L| \exp(-(|w|)(\varepsilon_s)^2/4)$, we obtain:

$$\begin{aligned} 8|B| &\leq |L| \exp(-(|w|)(\varepsilon_s)^2/4) \\ &\leq |I_\omega^{w*v(q_{s+1})*v(\omega,s+1,q_{s+1},n_k^{s+1},y)}| \exp(-2(n + M'(s+1) + n_k^{s+1})(\varepsilon_i)^3) \end{aligned} \tag{7.22}$$

for some $k \geq \mathcal{N}_{s+1}$. We denote by k_B the largest of those k such that the previous inequality holds.

From (7.21), we obtain $B(y, 4|B|) \subset I_\omega^{w*v(q_{s+1})*v(\omega,s+1,q_{s+1},n_{k_B}^{s+1},y)} \cup I_\omega^{w*v(q)*v'}$, where v' is a neighbor of $v(\omega, s+1, q_{s+1}, n_{k_B}^{s+1}, y)$ such that $|v(\omega, s+1, q_{s+1}, n_{k_B}^{s+1}, y) \wedge v'| \geq n_{k_B-1}^{s+1}$.

Now we can now give the following upper bound for $\zeta_{\omega,w,q_{s+1}}(B(y, 4|B|))$ (this is the same proof as in section 6.2):

for any $\underline{v} \in [w * v(q_{s+1}) * v(\omega, s+1, q_{s+1}, n_{k_B}^{s+1}, y)]_\omega$, one has

$$\begin{aligned} & \zeta_{\omega,w,q_{s+1}}(B(y, 4|B|)) \\ & \leq \exp(S_{n_{k_B}^{s+1}} \widetilde{\Lambda}_{s+1,q_{s+1}}(F^{n+M'(s+1)}(\omega, \underline{v})) + (\varepsilon_{s+1})^3 n_{k_B}^{s+1} + (\varepsilon_s)^3 n). \end{aligned} \tag{7.23}$$

Next we prove that (7.23) implies

$$\zeta_{\omega,w,q_{s+1}}(B(y,4|B|)) \leq \left(\frac{8|B|}{|I_\omega^w|}\right)^{\mathcal{T}^*_{j_{s+1}}(d_{s+1})-\varepsilon_s} (|B|)^{-(\varepsilon_s)^2}. \qquad (7.24)$$

By definition of k_B (see (7.22)), we have

$$\begin{aligned}
8|B| \geq\ & |I_\omega^{v(\omega,n+M'(s+1)+n^{s+1}_{k_B+1},y)}|\exp(-2(n+M'(s+1)+n^{s+1}_{k_B+1})(\varepsilon_s)^3) \\
\geq\ & |I_\omega^w|\exp(S_{n^{s+1}_{k_B+1}}\Phi(F^{n+M'(s+1)}(\omega,\underline{v}))-V_n(\omega)-V_{n+M'(s+1)+n^{s+1}_{k_B+1}}(\omega)) \\
& \cdot\exp(-2(n+M'(s+1)+n^{s+1}_{k_B+1})(\varepsilon_s)^3) \\
\geq\ & |I_\omega^w|\exp(S_{n^{s+1}_{k_B}}\Phi(F^{n+M'(s+1)}(\omega,\underline{v}))-|n^{s+1}_{k_B+1}-n^{s+1}_{k_B}|C) \\
& \cdot\exp(-4(n+M'(s+1)+n^{s+1}_{k_B+1})(\varepsilon_s)^3) \\
\geq\ & |I_\omega^w|\exp(S_{n^{s+1}_{k_B}}\Phi(F^{n+M'(s+1)}(\omega,\underline{v}))-5(n+M'(s+1)+n^{s+1}_{k_B+1})(\varepsilon_s)^3) \\
\geq\ & |I_\omega^w|\exp(S_{n^{s+1}_{k_B}}\Phi(F^{n+M'(s+1)}(\omega,\underline{v}))-5(n+M'(s+1)+n^{s+1}_{k_B+1})(\varepsilon_s)^3).
\end{aligned}$$

Thus

$$\frac{8|B|}{|I_\omega^w|} \geq \exp(S_{n^{s+1}_{k_B}}\Phi(F^{n+M'(s+1)}(\omega,\underline{v}))-5(n+M'(s+1)+n^{s+1}_{k_B+1})(\varepsilon_s)^3). \qquad (7.25)$$

Then, using (7.23) we obtain

$$\begin{aligned}
& \zeta_{\omega,w,q_{s+1}}(B(y,4|B|)) \\
\leq\ & \exp(S_{n^{s+1}_{k_B}}\widetilde{\Lambda}_{s+1,q_{s+1}}(F^{n+M'(s+1)}(\omega,\underline{v})))+(\varepsilon_{s+1})^3 n^{s+1}_{k_B}+(\varepsilon_s)^3 n) \\
\leq\ & \exp((\mathcal{T}^*_{j_{s+1}}(d_{s+1})-\varepsilon_{s+1})S_{n^{s+1}_{k_B}}\Phi(F^{n+M'(s+1)}(\omega,\underline{v})))+(\varepsilon_{s+1})^3 n^{s+1}_{k_B}+(\varepsilon_s)^3 n) \\
\leq\ & \left(\frac{8|B|}{|I_\omega^w|}\right)^{\mathcal{T}^*_{j_{s+1}}(d_{s+1})-\varepsilon_{s+1}} \exp((\varepsilon_{s+1})^3 n^{s+1}_{k_B}+(\varepsilon_s)^3 n) \\
& \cdot\exp((\mathcal{T}^*_{j_{s+1}}(d_{s+1})-\varepsilon_{s+1})\cdot(5(n+M'(s+1)+n^{s+1}_{k_B+1})(\varepsilon_s)^3) \\
\leq\ & \left(\frac{8|B|}{|I_\omega^w|}\right)^{\mathcal{T}^*_{j_{s+1}}(d_{s+1})-\varepsilon_{s+1}} |B|^{-(\varepsilon_s)^2}.
\end{aligned}$$

The last inequality is just from $\frac{c_\Phi}{2} \leq \frac{\log|B|}{n+M'(s+1)+n^{s+1}_{k_B+1}} \leq 2C_\Phi$.

Now, since $L=\overline{I_\omega^w}$ is the closure of I_ω^w, we can get:

$$\begin{aligned}
m_\xi^d(B) \leq & 4\Gamma_1 m_\xi^d(L)\sum_{l=1}^{p}\zeta_{\omega,v,q_{s+1}}(\overline{L_l})\\
\leq & 4\Gamma_1 m_\xi^d(L)\zeta_{\omega,w,q_{s+1}}(B(y,4|B|))\\
\leq & 4\Gamma_1 |L|^{\frac{(\mathcal{T}^*_{j_s}(d_s)-3\varepsilon_s)(1-\varepsilon_s)}{\xi}-\varepsilon_s}\left(\frac{8|B|}{|L|}\right)^{\mathcal{T}^*_{j_{s+1}}(d_{s+1})-\varepsilon_{s+1}}|B|^{-(\varepsilon_s)^2}\\
\leq & 4\Gamma_1 (8|B|)^{\frac{(\mathcal{T}^*_{j_s}(d_s)-3\varepsilon_s)(1-\varepsilon_s)}{\xi}}\left(\frac{8|B|}{|L|}\right)^{\alpha_s}(|B|)^{-(\varepsilon_s)^2},
\end{aligned}$$

where $\alpha_s = \mathcal{T}^*_{j_{s+1}}(d_{s+1}) - \varepsilon_{s+1} - \frac{(\mathcal{T}^*_{j_s}(d_s)-3\varepsilon_s)(1-\varepsilon_s)}{\xi}$ is positive for s large enough since $\mathcal{T}^*(d) > 0$. Moreover, $8|B|/|L| \leq 1$, so

$$m_\xi^d(B) \leq 4\Gamma_1(8|B|)^{\frac{(\mathcal{T}^*_{j_s}(d_s)-3\varepsilon_s)(1-\varepsilon_s)}{\xi}}(|B|)^{-(\varepsilon_s)^2}.$$

• If $\frac{1}{8}|L|\exp(-(|w|)(\varepsilon_s)^2/4) \leq |B| \leq |L|$:
We need at most $M(B) = \lfloor 9\exp((|w|)(\varepsilon_s)^2/4)\rfloor$ contiguous intervals $(B(k))_{1\leq k\leq M(B)}$ with diameter $\frac{1}{8}|L|\exp(-(|w|)(\varepsilon_s)^2/4)$ to cover B. For these intervals we have the estimate above. Consequently,

$$\begin{aligned}
m_\xi^d(B) \leq & \sum_{k=1}^{M(B)} 4\Gamma_1(8|B|)^{\frac{(\mathcal{T}^*_{j_s}(d_s)-3\varepsilon_s)(1-\varepsilon_s)}{\xi}}(|B|)^{-(\varepsilon_s)^2}\\
\leq & 4M(B)\Gamma_1(8|B|)^{\frac{(\mathcal{T}^*_{j_s}(d_s)-3\varepsilon_s)(1-\varepsilon_s)}{\xi}}(|B|)^{-(\varepsilon_s)^2}\\
\leq & |B|^{\mathcal{T}^*(d)/\xi-\varepsilon'_s},
\end{aligned}$$

where ε'_s tends to 0 as s tends to ∞, or as $|B|$ goes to 0 (here it is important to notice that $M(B) \leq |L|^{-\varepsilon_s} \leq |B|^{-\varepsilon_s}$ if s is large enough). It follows from the previous estimates that

$$\dim_H(m_\xi^d) \geq \frac{\mathcal{T}^*(d)}{\xi}. \tag{7.26}$$

We have finished the proof of theorem 7.22.

7.8 Conclusion on the lower bound for the lower Hausdorff spectrum

Proposition 7.25 *For $\mathbb{P}$-almost every ω, for any $d \in [0, \mathcal{T}'(t_0-)]$, one has*

$$\dim_H(\underline{E}(\nu_\omega, d)) \geq \sup_{a\in[\mathcal{T}'(+\infty),\mathcal{T}'(-\infty)]}\frac{d\mathcal{T}^*(a)}{a} = dt_0 = d\dim_H X_\omega.$$

Proof In the proof of corollary 7.15 we proved that

$$\sup_{\alpha>0} \frac{\mathcal{T}^*(\alpha)}{\alpha} = t_0$$

and the supremum is attained at $\alpha = \mathcal{T}'(t_0-)$.

If $d \in (0, \mathcal{T}'(t_0-)]$, we write $d = \mathcal{T}'(t_0-)/\xi$ with $\xi \geq 1$. We can find a suitable sequence $\widetilde{\varepsilon}$ such that theorem 7.22 and remark 7.24 can be used. This provides us a positive Borel measure $m_\xi^{\mathcal{T}'(t_0-)}$ on $K^{\mathcal{T}'(t_0-)}(\xi)$, with the following properties.

- $m_\xi^{\mathcal{T}'(t_0-)}(K^{\mathcal{T}'(t_0-)}(\xi)) = 1$ and $\dim_H(m_\xi^{\mathcal{T}'(t_0-)}) \geq \frac{\mathcal{T}^*(\mathcal{T}'(t_0-))}{\xi} = d\,t_0$.
- $m_\xi^{\mathcal{T}'(t_0-)}(E) = 0$ as soon as $\dim_H E < d\,t_0$.
- For any $x \in K^{\mathcal{T}'(t_0-)}(\xi)$, we have that $\underline{\dim}_{\mathrm{loc}}(\nu_\omega, x) \leq d$.

It follows from lemma 7.13 that

$$(K(\xi) \setminus (\cup_{0\leq h<d} F(h))) \subset (\underline{E}(\nu_\omega, d) \cup \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}).$$

Also, $\dim_H F(h) \leq ht_0 < dt_0$, for all $0 \leq h < d$, so $m_\xi^{\mathcal{T}'(t_0-)}(F(h)) = 0$, for all $0 \leq h < d$.

Moreover, the family of sets $(F(h))_{0<h<d}$ is nondecreasing. Thus, we have

$$m_\xi^{\mathcal{T}'(t_0-)}(\underline{E}(\nu_\omega, d) \cup \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}) > 0,$$

thus

$$\dim_H(\underline{E}(\nu_\omega, d) \cup \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}) \geq dt_0.$$

Finally, $\dim_H \underline{E}(\nu_\omega, d) \geq dt_0$ since $\{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}$ is a countable set.

If $d = 0$ or $t_0 = 0$, we have

$$\emptyset \neq \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1), \text{ and } m_\omega^{\widetilde{vs}} - M_\omega^{vs} > 0\} \subset \underline{E}(\nu_\omega, 0),$$

thus $\dim_H \underline{E}(\nu_\omega, d) \geq dt_0$ for $d = 0$.

Next proposition collects all the information required to conclude regarding the lower bound for the lower Hausdorff spectrum. Its claim 3. is the desired sharp lower bound.

Proposition 7.26 *For $\mathbb{P}$-almost every ω:*

1. *If $d \in [0, \mathcal{T}'(t_0-)]$, then $\dim_H(\underline{E}(\nu_\omega, d)) \geq dt_0$,*

2. *if* $d \in [\mathcal{T}'(+\infty), \mathcal{T}'(-\infty)]$, *then* $\dim_H(\underline{E}(\nu_\omega, d)) \geq \mathcal{T}^*(d)$,

3. *For any* $d \in [0, \mathcal{T}'(-\infty)]$, $\dim_H(\underline{E}(\nu_\omega, d)) \geq \widetilde{\mathcal{T}}^*(d)$.

Proof (1) and (2) come from proposition 7.25 and proposition 7.17.

To prove (3), since $\widetilde{\mathcal{T}}(q) = \min\{\mathcal{T}(q), 0\}$,$\mathcal{T}(t_0) = 0$ and $\mathcal{T}$ is increasing,

$$\widetilde{\mathcal{T}}^*(d) = \inf_q\{td - \widetilde{\mathcal{T}}(q)\} = \begin{cases} dt_0, & d \in [0, \mathcal{T}'(t_0-)], \\ \mathcal{T}^*(d), & d \in [\mathcal{T}'(t_0-), \mathcal{T}'(-\infty)]. \end{cases}$$

7.9 Hausdorff dimensions of the level sets $E(\nu_\omega, d)$ and $\overline{E}(\nu_\omega, d)$

Now we need to define another approximation rate.

For any $x \in [0,1] \setminus \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}$, define

$$\widehat{\xi}(\omega, n, x) = \frac{\log(\inf\{|x - x_\omega^{vs}| : |v| \leq n, s \in S'(\omega, v, 1)\})}{\log |I_\omega^{v(\omega,n,x)}|}$$

and then

$$\widehat{\xi}(\omega, x) = \liminf_{n\to\infty} \widehat{\xi}(\omega, n, x).$$

First of all we point out the following lemma:

Lemma 7.27 *For* $\mathbb{P}$*-almost every* $\omega \in \Omega$*, we have*

$$\{x \in [0,1] \setminus \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\} : \widehat{\xi}(\omega, x) > 1\} = \emptyset.$$

In other words, for any $x \in [0,1]$*, if* $x \notin \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}$,

$$\widehat{\xi}(\omega, x) = 1.$$

Proof We just need to prove that for any $k \in \mathbb{Z}^+$,

$$\{x \in [0,1] \setminus \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\} : \widehat{\xi}(\omega, x) > 1 + 1/k\} = \emptyset.$$

For any $x \in [0,1] \setminus \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}$ such that $\widehat{\xi}(\omega, x) > 1 + 1/k$, there exists $N(x) \in \mathbb{Z}^+$ such that for any $n \geq N(x)$ one has

$$\inf\{|x - x_\omega^{vs}| : |v| \leq n, s \in S'(\omega, v, 1)\} \leq |I_\omega^{v(\omega,n,x)}|^{1+1/k}.$$

Furthermore, the infimum must be attained at a point x_ω^{vs} which is in the closure of $I_\omega^{v(\omega,n,x)}$. We denote vs by $w(\omega, n, x)$. We just need to prove that x is the point $x_\omega^{w(\omega,n,x)}$ for n large enough. This will give a contradiction.

The choice of $w(\omega, n+1, x)$ must be made in

$$\{v(\omega,n,x)s: \ s \in S'(\omega, v, 1)\} \cup \{w(\omega,n,x)\}.$$

Otherwise it is easily seen that it is in contradiction with the choice of $w(\omega, n, x)$ and $x \in I_\omega^{w(\omega,n,x)}$.

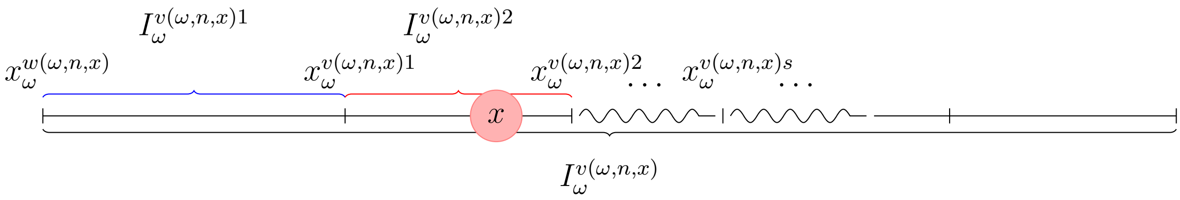


Figure 7.1 – The choice for $w(\omega, n, x)$

We have

$$\begin{aligned} & \inf\{|x - x_\omega^{vs}| : |v| \le n+1, s \in S'(\omega, v, 1)\} \\ = \ & |x_\omega^{w(\omega,n+1,x)} - x| \le |I_\omega^{v(\omega,n+1,x)}|^{1+1/k} \le |I_\omega^{v(\omega,n,x)}|^{1+1/k}. \end{aligned}$$

Now suppose that $w(\omega, n+1, x) \neq w(\omega, n, x)$. Then, on the one hand, we have

$$|I_\omega^{v(\omega,n,x)s}| \le |x_\omega^{w(\omega,n+1,x)} - x_\omega^{w(\omega,n,x)}| \le 2|I_\omega^{v(\omega,n,x)}|^{1+1/k}.$$

On the other hand, from assumption 2 and proposition 2.7, it is easy to prove that for n large enough we have

$$|I_\omega^{v(\omega,n,x)s}| = |I_\omega^{v(\omega,n,x)}| \cdot e^{o(\log(|I_\omega^{v(\omega,n,x)}|))} \ge |I_\omega^{v(\omega,n,x)}|^{1+\frac{1}{2k}} > 4|I_\omega^{v(\omega,n,x)}|^{1+1/k}.$$

We get a contradiction. So t$w(\omega, n+1, x) = w(\omega, n, x)$ for n large enough.

Lemma 7.28 *For $\mathbb{P}$-almost every $\omega \in \Omega$, for any $x \in [0,1] \backslash \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}$, we have that*

$$\overline{\dim}_{\text{loc}}(\nu_\omega, x) \ge \liminf_{n\to\infty} \frac{\log \nu(\mathring{I}_\omega^{v(\omega,n,x)})}{\log |I_\omega^{v(\omega,n,x)}|}.$$

Proof For $\mathbb{P}$-almost every $\omega \in \Omega$, since $x \in [0,1] \backslash \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}$, lemma 7.27 tells us

$$\widehat{\xi}(\omega, x) = 1.$$

Then there exists a subsequence $\{n_k\}_{k\in\mathbb{Z}^+}$ such that $\widehat{\xi}(\omega, n_k, x) \to 1$ as $k \to \infty$. Now,

$$\limsup_{r\to 0} \frac{\log \nu_\omega(B(x,r))}{\log r}$$

$$\begin{aligned}
&\geq \limsup_{k\to\infty} \frac{\log \nu(B(x, |I_\omega^{v(\omega,n_k,x)}|^{\widehat{\xi}(\omega,n_k,x)+1/n_k}))}{\log |I_\omega^{v(\omega,n_k,x)}|^{\widehat{\xi}(\omega,n_k,x)+1/n_k}} \\
&\geq \limsup_{k\to\infty} \frac{\log \nu(\mathring{I}_\omega^{v(\omega,n_k,x)})}{\log |I_\omega^{v(\omega,n_k,x)}|} \\
&\geq \liminf_{n\to\infty} \frac{\log \nu(\mathring{I}_\omega^{v(\omega,n,x)})}{\log |I_\omega^{v(\omega,n,x)}|}.
\end{aligned}$$

The second inequality follows from the fact that $\widehat{\xi}(\omega, n_k, x) \to 1$ as $k \to \infty$ and

$$B(x, |I_\omega^{v(\omega,n_k,x)}|^{\widehat{\xi}(\omega,n_k,x)+1/n_k}) \subset \mathring{I}_\omega^{v(\omega,n_k,x)},$$

by definition of $\widehat{\xi}(\omega, n_k, x)$.

Proposition 7.29 *For $\mathbb{P}$-almost every $\omega \in \Omega$,*

1. *for any $d \in [\mathcal{T}'(+\infty), \mathcal{T}'(t_0-)]$,*

$$\dim_H(\{x \in [0,1] : \overline{\dim}_{\mathrm{loc}}(\nu_\omega, x) \leq d\}) \leq \mathcal{T}^*(d).$$

2. *for any $0 < d < \mathcal{T}'(+\infty)$ one has*

$$\{x \in [0,1] : \overline{\dim}_{\mathrm{loc}}(\nu_\omega, x) = d\} = \emptyset.$$

3. *If $x \in \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1),$ and $m_\omega^{\widetilde{vs}} - M_\omega^{vs} = 0\}$, then*

$$\overline{\dim}_{\mathrm{loc}}(\nu_\omega, x) \geq \mathcal{T}'(+\infty).$$

4.

$$\begin{aligned}
\overline{E}(\nu_\omega, 0) &= E(\nu_\omega, 0) \\
&= \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1),\ \textit{and}\ m_\omega^{\widetilde{vs}} - M_\omega^{vs} > 0\}
\end{aligned}$$

so that

$$\dim_H(\overline{E}(\nu_\omega, 0)) = \dim_H(E(\nu_\omega, 0)) = 0.$$

Proof 1. First, for any $d \in [\mathcal{T}'(+\infty), \mathcal{T}'(t_0-)]$, for any $\varepsilon > 0$, there exists $q \geq 0$ such that $\mathcal{T}^*(d) \geq qd - \mathcal{T}(q) - \varepsilon/2$.

Second, choose $\epsilon > 0$ such that $q\epsilon \leq \varepsilon/4$, for any $N \in \mathbb{N}$

$$\{x \in [0,1] : \overline{\dim}_{\mathrm{loc}}(\nu_\omega, x) \leq d\} \subset \left\{x \in [0,1] : \liminf_{n\to\infty} \frac{\log \nu(\mathring{I}_\omega^{v(\omega,n,x)})}{\log |I_\omega^{v(\omega,n,x)}|} \leq d\right\}$$

$$\subset \bigcup_{n\geq N} \bigcup_{v\in\Sigma_{\omega,n}, \nu(\mathring{I}_\omega^v)\geq |I_\omega^v|^{d+\epsilon}} I_\omega^v.$$

Third, for any $\delta > 0$, for N large enough, and $v \in \Sigma_{\omega,n}$, one has $|I_\omega^v| < \delta$. Choosing $s = \mathcal{T}^*(d) + \varepsilon$ we get for N large enough,

$$\begin{aligned}
\mathcal{H}_\delta^s(\{x \in [0,1] : \overline{\dim}_{\mathrm{loc}}(\nu_\omega, x) \leq d\}) &\leq \sum_{n\geq N'} \sum_{v\in\Sigma_{\omega,n}, \nu(I_\omega^v)\geq |I_\omega^v|^{d+\epsilon}} |I_\omega^v|^{\mathcal{T}^*(d)+\varepsilon} \\
&\leq \sum_{n\geq N} \sum_{v\in\Sigma_{\omega,n}, \nu_\omega(I_\omega^v)\geq |I_\omega^v|^{d+\epsilon}} |I_\omega^v|^{qd-\mathcal{T}(q)+\varepsilon/2} \\
&\leq \sum_{n\geq N} \sum_{v\in\Sigma_{\omega,n}} |I_\omega^v|^{-\mathcal{T}(q)+\varepsilon/2-q\epsilon} (\nu_\omega(\mathring{I}_\omega^v))^q \\
&\leq \sum_{n\geq N} \sum_{v\in\Sigma_{\omega,n}} |I_\omega^v|^{-\mathcal{T}(q)+\varepsilon/4} (\nu(\mathring{I}_\omega^v))^q \\
&\leq \sum_{n\geq N} \exp(nP(q\Psi - \mathcal{T}(q)\Phi) - \frac{nc_\Phi\varepsilon}{8}) \\
&\leq \sum_{n\geq N} \exp(-\frac{nc_\Phi\varepsilon}{8}).
\end{aligned}$$

Here we used the fact that $\nu_\omega(\mathring{I}_\omega^v) \leq |X_\omega^v| \leq |U_\omega^v| \leq \exp(S_n\Psi(\omega, \underline{v}) + o(n))$ for any $\underline{v} \in [v]_\omega$ and $v \in \Sigma_{\omega,*}$.

Letting N go to ∞ we get $\mathcal{H}_\delta^s(\{x \in [0,1] : \overline{\dim}_{\mathrm{loc}}(\nu_\omega, x) \leq d\}) = 0$ for any $\delta > 0$, so $\mathcal{H}^s(\{x \in [0,1] : \underline{\dim}_{\mathrm{loc}}(\nu_\omega, x) \leq d\}) = 0$. This holds for any $s > -\mathcal{T}^*(d)$, so $\dim_H\{x \in [0,1] : \overline{\dim}_{\mathrm{loc}}(\nu_\omega, x) \leq d\} \leq \mathcal{T}^*(d)$.

2. If $0 < d < \mathcal{T}'(+\infty)$, we have $\mathcal{T}^*(d) = -\infty$. This implies that for any $s > -\infty$, for any $\varepsilon > 0$, there exists $q > 0$ such that $s > qd - \mathcal{T}(q) + \varepsilon$. Thus, we can deduce from the above calculation that we have $\{x \notin \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\} : \liminf_{n\to\infty} \frac{\log \nu(\mathring{I}_\omega^{v(\omega,n,x)})}{\log |I_\omega^{v(\omega,n,x)}|} \leq d\} = \emptyset$.

3. From the proof of item (2) if $x \in \{x_\omega^{vs} : v \in \Sigma_{\omega,*}, s \in S'(\omega, v, 1)\}$ and x do not have a positive measure, the upper local dimension will larger than $\mathcal{T}'(+\infty)$.

4. Since item (3) it is obvious.

Theorem 7.30 *For $\mathbb{P}$-almost every $\omega \in \Omega$, for any $d \in [\mathcal{T}'(+\infty), \mathcal{T}'(-\infty)]$, we have*

$$\dim_H E(\nu_\omega, d) = \dim_H \overline{E}(\nu_\omega, d) = \mathcal{T}^*(d).$$

Proof By construction, if $x \in \widetilde{E}(d)$, the set constructed in the proof of proposition 7.17, then $\lim_{n\to\infty} \alpha_\omega^n(x) = d$, so due to remark 7.10 we must have

$\overline{\dim}_{\text{loc}}(\nu_\omega, x) \leq d$. Since, moreover, $\underline{\dim}_{\text{loc}}(\nu_\omega, x) = d$, we get $\widetilde{E}(d) \subset E(\nu_\omega, d) \subset \overline{E}(\nu_\omega, d)$, and the the lower bound $\dim_H \widetilde{E}(d) \geq \mathcal{T}^*(d)$, yields the expected lower bound, while the upper bound was obtained in the previous proposition for $d \in [\mathcal{T}'(+\infty), \mathcal{T}'(t_0-)]$, and it follows from the multifractal formalism for $d \in [\mathcal{T}'(t_0-), \mathcal{T}'(-\infty)]$, since $\tau^*_{\nu_\omega} \leq \mathcal{T}^*$.

Remark 7.31 Let us explain how, to study the sets $E(\nu_\omega, d)$, we could have used some result from [58, 73, 74], which gives an inversion formula for multifractals. If the definition of local dimension is changed to the more uniform one:

$$\dim^u_{\text{loc}}(\mu, x) = \lim_{I \to \{x\}} \frac{\log \mu(I)}{\log |I|},$$

where $I \to \{x\}$ means that I is an interval containing x, and that the length of I tends to zero, we have

$$\{x \in \text{supp}(\mu) : \dim^u_{\text{loc}}(\mu, x) = \alpha\} \subset E(\mu, \alpha).$$

Theorem 7.32 (Corollary 2.2 in [74]) *Let μ be a probability measure on $[0,1]$ and ν be its inverse measure. Assume that $0 < \alpha < \infty$. Then*

$$\dim_H\{x \in \text{supp}(\nu) : \dim^u_{\text{loc}}(\nu, x) = \alpha\} = \alpha \dim_H\{x \in \text{supp}(\mu) : \dim^u_{\text{loc}}(\mu, x) = 1/\alpha\},$$

and

$$\dim_P\{x \in \text{supp}(\nu) : \dim^u_{\text{loc}}(\nu, x) = \alpha\} = \alpha \dim_P\{x \in \text{supp}(\mu) : \dim^u_{\text{loc}}(\mu, x) = 1/\alpha\}.$$

Lemma 7.33 *For $\mathbb{P}$-almost every $\omega \in \Omega$, for any $d \in [\mathcal{T}'(+\infty), \mathcal{T}'(-\infty)]$, we have*

$$\mathcal{T}^*(d) = dT^*(1/d),$$

and

$$\dim_H(\{x \in \text{supp}(\nu_\omega) : \dim^u_{\text{loc}}(\nu_\omega, x) = d\}) = \mathcal{T}^*(d).$$

Proof We have

$$\begin{aligned} \mathcal{T}^*(d) &= \inf\{qd - \mathcal{T}(q) : q \in \mathbb{R}\} \\ &= d \cdot \inf\{q - \mathcal{T}(q)/d : q \in \mathbb{R}\} \end{aligned}$$

Moreover, $T = -\mathcal{T}^{-1} \circ (-Id_{\mathbb{R}})$, so

$$\begin{aligned} \mathcal{T}^*(d) &= d \cdot \inf\{q - \mathcal{T}(q)/d : q \in \mathbb{R}\} \\ &= d \cdot \inf\{q - \mathcal{T}(q)/d : \mathcal{T}(q) \in \mathbb{R}\} \\ &= d \cdot \inf\{-T(t) + t/d : t \in \mathbb{R}\} \end{aligned}$$

$$= \quad dT^*(1/d).$$

The result about the Hausdorff dimension then follows from theorem 7.32 and the fact that in section 6.2, the item

$$K(\omega, \{d_i\}_{i\geq 1}) \subset E(\mu_\omega, d)$$

can be change to

$$K(\omega, \{d_i\}_{i\geq 1}) \subset \{x \in \mathrm{supp}(\mu) : \dim^u_{\mathrm{loc}}(\mu, x) = d\}.$$

This comes from the fact that the measure is almost doubling on $K(\omega, \{d_i\}_{i\geq 1})$.

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