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Grande image de Galois pour familles p-adiques de formes automorphes de pente positive

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Introduction

Soit N un entier positif. Soit f une forme modulaire classique, cuspidale, non-CM, de niveau N et propre pour l'action des opérateurs de Hecke $\{T_\ell\}_{\ell\nmid N}$ et $\{U_\ell\}_{\ell|N}$. Pour un nombre premier p soit $\rho_{f,p}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ la représentation de Galois p-adique associée à f. Ribet a montré un résultat de "grande image" pour la représentation $\rho_{f,p}$.

THÉORÈME 1. *Pour presque tout premier p l'image de $\rho_{f,p}$ contient le conjugué d'un sous-groupe de congruence principal non-trivial de* $\mathrm{SL}_2(\mathbb{Z}_p)$.

Dorénavant on fixe un premier $p \geq 5$ qui ne divise pas N. Dans [**Hi15**] Hida a prouvé un analogue du Théorème 1 pour une famille p-adique de formes modulaires ordinaires. On présente brièvement ce résultat. Soit $\Lambda = \mathbb{Z}_p[[T]]$ l'algèbre d'Iwasawa. Soit $\mathbb{T}^{\mathrm{ord}}$ un facteur local de la grande algèbre de Hecke ordinaire de niveau modéré N, construite en [**Hi86**] ; il s'agit d'une Λ-algèbre finie et plate. L'algèbre de Hecke abstraite $\mathcal{H}$ de niveau modéré N admet un morphisme $\mathcal{H} \to \mathbb{T}^{\mathrm{ord}}$ qui interpole les systèmes de valeurs propres associés à des formes propres classiques ordinaires. Si $\mathbb{T}^{\mathrm{ord}}$ est résiduellement non-Eisenstein les représentations de Galois p-adiques associées aux formes classiques de la composante sont interpolées par une grande représentation $\rho_{\mathbb{T}^{\mathrm{ord}}}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{T}^{\mathrm{ord}})$. Soit $\theta\colon \mathbb{T}^{\mathrm{ord}} \to \mathbb{I}$ un morphisme de Λ-algèbres définissant une composante irréductible de $\mathbb{T}^{\mathrm{ord}}$ et soit $\rho_\theta\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I})$ la représentation induite par $\rho_{\mathbb{T}^{\mathrm{ord}}}$. On dit que la famille θ est CM si toutes ses spécialisations classiques sont CM. Pour un idéal $\mathfrak{l}$ de Λ, soit $\Gamma_\Lambda(\mathfrak{l})$ le sous-groupe de congruence principal de niveau $\mathfrak{l}$ de $\mathrm{SL}_2(\Lambda)$.

THÉORÈME 2. [**Hi15**, Theorem I] *Supposons que la famille θ est non-CM. Alors il existe un élément $g \in \mathrm{GL}_2(\mathbb{I})$ et un idéal non-nul $\mathfrak{l}$ de Λ tels que*

$$\Gamma_\Lambda(\mathfrak{l}) \subset g \cdot \mathrm{Im}\,\rho_\theta \cdot g^{-1}. \tag{1}$$

Il existe un plus grand idéal $\mathfrak{l}_\theta$ de Λ entre ceux qui satisfont (1) pour un $g \in \mathrm{GL}_2(\mathbb{I})$; on l'appelle le *niveau galoisien* de θ. Hida a donné une description des facteurs premiers du niveau galoisien, comme suit. L'algèbre de Hecke $\mathbb{T}^{\mathrm{ord}}$ admet des composantes CM, pour lesquelles le Théorème 2 n'est pas valable. La famille non-CM θ peut croiser certaines de ces composantes. De tels croisements peuvent être interprétés comme des congruences entre une famille "générale" (c'est-à-dire, telle que ses spécialisations classiques ne sont pas des transferts de formes automorphes pour un groupe de rang plus pétit) et des familles "non-générales" (dans ce cas, celles qui sont induites par un caractère de Hecke d'un corps quadratique imaginaire). Dans le cas de GL_2 les seules familles non-générales sont celles qui sont CM. Hida a défini un idéal $\mathfrak{c}_\theta$ de Λ, l'*idéal de congruence CM*, qui mésure les congruences entre θ et les composantes CM.

THÉORÈME 3. [**Hi15**, Theorem II] *Les idéaux $\mathfrak{l}_\theta$ et $\mathfrak{c}_\theta$ ont le même ensemble de facteurs premiers.*

Dans sa thèse [**Lang16**] J. Lang a amélioré le Théorème 2. Son résultat est inspiré par une version plus forte du Théorème 1, due à Momose [**Mo81**] et Ribet [**Ri85**, Theorem 3.1]. Elle considère encore une famille ordinaire non-CM $\theta\colon \mathbb{T}^{\mathrm{ord}} \to \mathbb{I}$ et définit un $(2, 2, \ldots, 2)$-groupe fini Γ de *self-twists conjugués* pour θ, c'est-à-dire automorphismes de la Λ-algèbre $\mathbb{I}$ qui induisent un isomorphisme de ρ_θ avec une de ses tordues par un caractère d'ordre fini. Soit $\mathbb{I}_0$ le sous-anneau de $\mathbb{I}$ fixé par Γ. Pour un idéal $\mathfrak{l}$ de $\mathbb{I}_0$, soit $\Gamma_{\mathbb{I}_0}(\mathfrak{l})$ le sous-groupe de congruence principal de niveau $\mathfrak{l}$ de $\mathrm{SL}_2(\mathbb{I}_0)$.

THÉORÈME 4. [**Lang16**, Theorem 2.4] *Il existe un élément $g \in \mathrm{GL}_2(\mathbb{I})$ et un idéal non-nul $\mathfrak{l}$ de $\mathbb{I}_0$ tels que*

$$\Gamma_{\mathbb{I}_0}(\mathfrak{l}) \subset g \cdot \mathrm{Im}\,\rho_\theta \cdot g^{-1}. \tag{2}$$

L'existence d'un self-twist conjugué pose une restriction sur la largeur de l'image de ρ_θ, donc l'anneau $\mathbb{I}_0$ est optimal par rapport à la propriété décrite par le Théorème 4. Les preuves des Théorèmes 2 et 4 invoquent l'existence d'un élément conjugué à

$$C_T = \begin{pmatrix} u^{-1}(1+T) & * \\ 0 & 1 \end{pmatrix}$$

dans l'image par ρ_θ d'un groupe d'inertie en p ; ce fait est une conséquence de l'ordinarité de ρ_θ. La conjugaison par un élément comme ci-dessus induit une structure de Λ-module sur l'intersection de $\mathrm{Im}\,\rho_\theta$ avec les sous-groupes unipotents à un paramètre de $\mathrm{SL}_2(\mathbb{I}_0)$. Cela est combiné avec l'existence de certains éléments non-triviaux dans $\mathrm{Im}\,\rho_\theta$, construits grâce au résultat de Momose et Ribet et à la théorie de Pink des algèbres de Lie des pro-p sous-groupes de $\mathrm{SL}_2(\mathbb{I}_0)$. Une étape clé dans la preuve du Théorème 4 est un résultat de J. Lang sur le relèvements de self-twists d'un corps p-adique à l'anneau $\mathbb{I}_0$ [**Lang16**, Theorem 3.1].

Dans un travail commun avec A. Iovita et J. Tilouine (voir Chapter 2 et [**CIT15**]) on a prouvé des analogues des Théorèmes 4 et 3 pour une famille p-adique de formes propres de pente finie. Dans ce contexte on a rencontré plusieurs problèmes et phénomènes qui n'apparaissent pas dans le cas ordinaire. Les formes propres de pente finie et niveau modéré N sont interpolées par une courbe rigide analytique sur $\mathbb{Q}_p$, la courbe de Hecke construite par Coleman, Mazur [**CM98**] et Buzzard [**Bu07**], mais cet objet n'admet pas de structure entière globale analogue à l'algèbre ordinaire $\mathbb{T}^{\mathrm{ord}}$. En particulier $\mathcal{D}$ admet un morphisme vers l'espace des poids $\mathcal{W} = (\mathrm{Spf}\,\Lambda)^{\mathrm{rig}}$, le *morphisme des poids*, mais celui-ci n'est pas fini. Pour cette raison on doit fixer une paire (h, B_h), composée d'un $h \in \mathbb{Q}^{+,\times}$ et d'un disque ouvert de centre 0 et rayon r_h dépendant de h, telle que le morphisme des poids soit fini si on le restreint au sous-domaine $\mathcal{D}^h_{B_h}$ des points de $\mathcal{D}$ de pente $\leq h$ et poids dans B_h. Celle-ci est une "paire adaptée" dans la terminologie de [**Be12**, Section 2.1]. Pour tout $h \in \mathbb{Q}^{+,\times}$ il existe un rayon r_h tel que (h, B_h) soit adaptée, mais pour l'instant on ne sait pas donner une borne inférieure pour r_h en termes de h.

Soient Λ_h et $\mathbb{T}_h$ les anneaux des fonctions bornées par 1 sur les espaces rigides analytiques (sur $\mathbb{Q}_p$) B_h et $\mathcal{D}^h_{B_h}$, respectivement. L'algèbre $\mathbb{T}_h$ a une structure de Λ_h-algèbre finie et elle est notre analogue de la Λ-algèbre $\mathbb{T}^{\mathrm{ord}}$ de Hida. On appelle *famille de pente bornée par h* une composante irréductible de pente positive de $\mathbb{T}_h$, décrite par un morphisme de Λ_h-algèbres $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$. Dans le cas résiduellement irréductible on définit une représentation $\rho_\theta\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$. La notation $^\circ$ signale qu'on travaille avec une structure entière ; plus tard on aura besoin d'inverser p. Pour simplifier notre présentation on suppose que $\mathbb{I}^\circ$ est normal, mais cette hypothèse n'est pas essentielle. Une différence importante avec le cas ordinaire est donnée par le fait que les formes CM de pente finie ne forment pas des familles, mais elles définissent un sous-ensemble discret de $\mathrm{Spec}\,\mathbb{I}^\circ$ (Corollary 2.2.8). On peut donc prouver un résultat de grande image de Galois pour toute famille de pente positive.

On définit un groupe Γ de self-twists pour θ et on note $\mathbb{I}_0^\circ$ le sous-anneau de $\mathbb{I}^\circ$ fixé par Γ. Il est nécessaire ici d'utiliser les arguments de J. Lang sur le relèvement des self-twists ; ils s'appliquent aussi à des familles non-ordinaires. Vu que la représentation de Galois est non-ordinaire, on ne sait pas si son image contient un élément conjugué à la matrice C_T qui apparaît dans le travail de Hida et Lang. Cependant on construit un opérateur avec des propriétés analogues grâce à la théorie relative de Sen. Pour pouvoir le faire on a besoin d'étendre nos coefficients à l'anneau des fonctions rigides analytiques sur un disque fermé de rayon r plus pétit que r_h, puis de considérer des produits tensoriels complétés avec $\mathbb{C}_p$; on note ces opérations par l'adjonction d'indices r et $\mathbb{C}_p$ en bas. À partir de $\mathbb{I}_0$ et ρ_θ, on définit un anneau $\mathbb{B}_r$ et une sous-algèbre de

Lie $\mathfrak{G}_r$ de $\mathfrak{gl}_2(\mathbb{B}_r)$ associée à $\mathrm{Im}\,\rho_\theta$ (Section 2.4.1). Ensuite on construit un opérateur de Sen $\phi \in \mathrm{M}_2(\mathbb{B}_{r,\mathbb{C}_p})$ dont l'exponentiel normalise $\mathfrak{G}_{r,\mathbb{C}_p}$ et a des valeurs propres explicites (Section 2.4). La conjugaison par cet élément induit une structure de Λ_h-module sur les sous-algèbres nilpotentes de $\mathfrak{G}_{r,\mathbb{C}_p}$ associées aux racines de SL_2. Sous une hypothèse technique qu'on appelle $(H_0, \mathbb{Z}_p)$-régularité (voir Definition 2.3.6), les sous-algèbres nilpotentes de $\mathfrak{G}_{r,\mathbb{C}_p}$ contiennent des éléments non-triviaux, grâce au résultat résiduel de Momose et Ribet et à un argument d'approximation dû à Hida et Tilouine. Notre premier résultat est le théorème ci-dessous.

THÉORÈME 5. (Theorem 2.5.2) *Supposons que $\mathbb{I}^\circ$ est normal et que ρ_θ est $(H_0, \mathbb{Z}_p)$-régulière. Il existe un idéal non-nul $\mathfrak{l}$ de $\mathbb{I}_0$ tel que*

$$\mathfrak{l} \cdot \mathfrak{sl}_2(\mathbb{B}_r) \subset \mathfrak{G}_r. \tag{3}$$

On remarque qu'on arrive à se débarasser de l'extension de scalaires à $\mathbb{C}_p$, mais pas de l'inversion de p. On appelle *niveau galoisien* de θ l'idéal $\mathfrak{l}_\theta$ de $\mathbb{I}_0$ le plus grand entre ceux qui satisfont (3).

On décrit le lieu des points CM de la famille par un *idéal de congruence CM accidentel* $\mathfrak{c}_\theta$ de $\mathbb{I}$ (voir Definition 2.2.12), où le terme "accidentel" met en évidence le fait que ses facteurs premiers ne correspondent pas à des congruences entre une famille générale et des familles non-générales, mais à des points CM isolés. Dans notre deuxième résultat on compare le niveau galoisien et l'idéal de congruence pour la famille θ.

THÉORÈME 6. (Theorem 2.6.1) *Supposons que $\overline{\rho}_\theta$ n'est pas induite par un caractère du groupe de Galois absolue d'un corps quadratique réel. Les ensembles de facteurs premiers de $\mathfrak{c}_\theta \cap \mathbb{I}_0$ et $\mathfrak{l}_\theta$ coïncident en dehors des diviseurs de $(1 + T - u) \cdot \mathbb{I}_0$ (les facteurs de poids 1).*

Le but des chapitres 3 et 4 est d'étudier le problème de la définition et comparaison du niveau galoisien et de l'idéal de congruence pour des familles p-adiques de formes modulaires de Siegel de genre 2 et pente positive finie. Dans [**HT15**] Hida et Tilouine ont étudié le problème pour des familles qui sont résiduellement de type "Yoshida tordu et grand" : dans ce cas les seules congruences possibles sont celles avec des transferts p-adiques de familles de formes modulaires de Hilbert pour $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2/F}$, où F est un corps quadratique réel. Les transferts des formes classiques sont interpolés par des composantes irréductibles de dimension 2 de la grande algèbre de Hecke ordinaire, donc l'idéal de congruence qui en résulte est un analogue de l'idéal de congruence CM défini par Hida : il décrit des congruences entre une famille générale et les familles non-générales. Il serait possible d'étudier les phénomènes de congruence pour des familles résiduellement de type Yoshida tordu et grand et de pente positive, avec les mêmes techniques développées ici ; on trouverait encore des familles non-générales de dimension 2 et une généralisation directe de l'idéal de congruence défini dans le cas ordinaire. Pour avoir plutôt un analogue de l'idéal de congruence CM accidentel associé à des familles de pente positive pour GL_2, on considère des familles de formes modulaires de Siegel qui sont résiduellement de type "cube symétrique grand", comme expliqué ci-dessous. En plus du type différent de congruences permises, le cas de pente positive présente de nombreux nouveaux aspects par rapport au cas ordinaire.

Les formes de niveau modéré N, propres pour l'action de Hecke, sont interpolées par un espace rigide analytique $\mathcal{D}$ de dimension 2, construit par Andreatta, Iovita et Pilloni [**AIP15**]. Cet espace est muni d'un morphisme non-fini vers l'espace des poids de dimension 2, $\mathcal{W}_2 = (\mathrm{Spf}\,\Lambda_2)^{\mathrm{rig}}$. Comme dans le cas de pente positive pour GL_2, on doit fixer dans une étape préliminaire $h \in \mathbb{Q}^{+,\times}$ et un disque B_h de centre 0 et rayon suffisamment petit r_h dans l'espace des poids. Soit $\mathcal{D}^h_{B_h}$ le sous-domaine admissible de $\mathcal{D}$ des points de pente $\leq h$ et poids dans B_h. Le morphisme de poids induit un morphisme fini $\mathcal{D}^h_{B_h} \to B_h$. Soient Λ_h et $\mathbb{T}_h$ les anneaux des fonction bornées par 1 sur les espaces rigides analytiques (sur $\mathbb{Q}_p$) B_h et $\mathcal{D}^h_{B_h}$, respectivement. On appelle famille de pente bornée par h une composante irréductible de $\mathbb{T}_h$, définie par un morphisme de Λ_h-algèbres $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$. Dans le cas résiduellement irréductible l'interpolation des représentations de Galois p-adiques associées aux points classiques de θ donne une représentation

$\rho_\theta\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ)$. On suppose que la représentation résiduelle associée $\overline{\rho}_\theta$ est de type cube symétrique grand, dans le sens où

$$\mathrm{Sym}^3\mathrm{SL}_2(\mathbb{F}) \subset \mathrm{Im}\,\overline{\rho}_\theta \subset \mathrm{Sym}^3\mathrm{GL}_2(\mathbb{F})$$

pour un corps fini non-trivial $\mathbb{F}$. Cette condition crée une restriction forte sur les congruences possibles : si un point classique est non-général, il doit correspondre à l'image d'une forme propre pour GL_2 par le transfert associé au cube symétrique, construit par Kim et Shahidi [**KS02**]. Une telle forme définit un point sur une courbe de Hecke pour GL_2 d'un niveau modéré dépendant de N. On trouve que la notion d'être un transfert de type cube symétrique a un sens aussi pour des points non-classiques : si la représentation associé à un point de $\mathcal{D}$ est de la forme $\mathrm{Sym}^3\rho'$ pour quelque $\rho'\colon G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$, alors ρ' est associée à une forme surconvergente pour GL_2, donc à un point d'une courbe de Hecke pour GL_2. Cela suit de la conjecture de Fontaine-Mazur surconvergente [**Em14**, Theorem 1.2.4] après avoir adapté des résultats de Di Matteo [**DiM13**] pour montrer que ρ' est trianguline (Theorem 3.10.30).

Les points non-généraux sont contenus dans une sous-variété de dimension 1 de $\mathcal{D}$ en raison d'une restriction sur leur poids, donc il n'y a pas de famille non-générale de dimension 2. Cependant il y a des points de type cube symétrique qui forment des familles à un paramètre. On peut construire de telles familles grâce à des résultats de Bellaïche et Chenevier [**BC09**, Section 7.2.3]. Soit $\mathcal{D}_1^{\mathrm{non-CM}}$ la partie non-CM de la courbe de Hecke pour GL_2 d'un certain niveau auxiliaire dépendant de N. On interpole les transfert des points classiques en construisant un morphisme $\mathcal{D}_1^{\mathrm{non-CM}} \to \mathcal{D}$ d'espaces rigides analytiques (Section 3.9.2). L'image dans $\mathcal{D}$ d'une famille de $\mathcal{D}_1^{\mathrm{non-CM}}$ est une famille non-générale à un paramètre de formes modulaires de Siegel.

On définit le groupe Γ de self-twists pour θ (Section 4.3) et on note $\mathbb{I}_0^\circ$ le sous-anneau de $\mathbb{I}^\circ$ fixé par Γ. On montre que le résultat crucial de Lang sur le relèvement des self-twists peut être adapté à ce contexte (Proposition 4.4.1). Le théorème de Momose and Ribet admet un analogue pour les formes modulaires de Siegel (Theorem 3.11.3) ; ceci est une conséquence d'un résultat très général de Pink sur les sous-groupes compacts et Zariski-denses des groupes algébriques linéaires (Theorem 3.11.4). On le combine avec le résultat de relèvement des self-twists et l'argument d'approximation de Hida et Tilouine (Proposition 4.7.1) pour construire des éléments non-triviaux dans les sous-groupes unipotents de $\mathrm{Im}\,\rho_\theta$. On fixe un rayon $r \in p^\mathbb{Q}$ plus petit que r_h. On définit un anneau $\mathbb{B}_r$ et une sous-algèbre de Lie $\mathfrak{G}_r$ de $\mathfrak{gsp}_4(\mathbb{B}_r)$ associée à l'image de ρ_θ (Section 4.10.1). Grâce à la théorie rélative de Sen on construit un élément ϕ de $\mathrm{GSp}_4(\mathbb{B}_{r,\mathbb{C}_p})$ qui normalise $\mathfrak{G}_{r,\mathbb{C}_p}$ et a des valeurs propres explicites (Proposition 4.10.20). La conjugaison par ϕ induit une structure de Λ_h-module sur les sous-algèbres nilpotentes de $\mathfrak{G}_{r,\mathbb{C}_p}$ associées aux racines de Sp_4. Cela nous amène au résultat suivant.

Théorème 7. (Theorem 4.11.1) *Il existe un idéal non-nul* $\mathfrak{l}$ *de* $\mathbb{I}_0$ *tel que*

$$\mathfrak{l} \cdot \mathfrak{sp}_4(\mathbb{B}_r) \subset \mathfrak{G}_r. \tag{4}$$

On appelle *niveau galoisien* de θ l'idéal $\mathfrak{l}_\theta$ de $\mathbb{I}_0$ le plus grand entre ceux qui satisfont (4). On définit un *idéal de congruence* Sym^3 *accidentel* $\mathfrak{c}_\theta$ de $\mathbb{I}_0$ qui décrit le lieu des points non-généraux de θ (Definition 4.8.7). Grâce à l'interpolation p-adique du transfert associé au cube symétrique, on sait qu'il existe des familles pour lesquelles l'idéal de congruence admet des composantes de dimension 0 et 1. On compare le niveau galoisien et l'idéal de congruence en dehors d'un ensemble fini de premiers mauvais.

Théorème 8. (voir Theorem 4.12.1 pour le résultat précis) *Les ensembles de facteurs premiers de* $\mathfrak{l}_\theta$ *et* $\mathfrak{c}_\theta$ *coïncident en dehors d'un ensemble fini et explicite de premiers mauvais.*

On espère revenir sur la question de décrire les paires adaptées pour GL_2 et GSp_4, c'est-à-dire de trouver une estimation pour le rayon du disque B_h en fonction de la pente h. Des bornes pour le rayon analogue existent pour les variétés de Hecke associées aux groupes unitaires définis, grâce à des résultats de Chenevier [**Ch04**, Section 5]. Pour GL_2 ce problème est lié à des résultats de Wan [**Wa98**] dans le cadre des conjectures de type Gouvêa-Mazur, mais les estimations disponibles concernent seulement les fibres aux poids classiques et ne s'appliquent pas à l'étude de la finitude du morphisme des poids sur des voisinages p-adiques.

Il apparaît comme naturel de généraliser les Théorèmes 7 et 8 à des familles p-adiques de formes automorphes sur des groupes réductifs G pour lesquels la variétés de Hecke a été construite. Un cadre général pourrait être comme suit. Soit $\mathcal{W}_G$ l'espace des poids pour G et $\mathcal{D}_G$ la variété de Hecke paramétrisant les formes surconvergentes de pente finite pour G ; elle est munie d'un morphisme des poids $\mathcal{D}_G \to \mathcal{W}_G$. Supposons qu'on puisse fixer une paire adaptée (h, B_h), composée de $h \in \mathbb{Q}^{+,\times}$ et d'un disque B_h dans $\mathcal{W}_g$, définissant un sous-domaine $\mathcal{D}_G^h$ de $\mathcal{D}_G$ tel que le morphisme des poids $\mathcal{D}_G^h \to B_h$ soit fini. Soit $\mathbb{T}_h$ l'anneau des fonctions rigides analytiques bornées par 1 sur $\mathcal{D}_G^h$. Considérons une composante irréductible de $\mathbb{T}_h$, définie par un morphisme $\theta \colon \mathbb{T}_h \to \mathbb{I}$. Supposons qu'on puisse associer à θ une représentation de Galois $\rho_\theta \colon G_\mathbb{Q} \to {}^L G(\mathbb{I})$, où ${}^L G$ est le groupe dual de Langlands de G. L'image de ρ_θ est Zariski-dense dans les $\mathbb{I}$-points de son groupe de Mumford-Tate, qu'on écrit sous la forme ${}^L H$ pour un certain groupe réductif H. Les techniques développées pour GL_2 et GSp_4 pourraient être adaptées pour montrer qu'un H-niveau galoisien existe pour la famille θ. Pour tout groupe réductif H' de rang plus petit que celui de H et pour tout morphisme de L-groupes ${}^L H' \to {}^L H$ pour lequel le transfert de Langlands classique est connu, il semble possible de définir un transfert p-adique par interpolation. Cela nous amènerait à la définition d'un idéal de congruence qui mésure soit les congruences entre la H-famille θ et les H'-familles, soit des congruences accidentelles dues à l'existence de H'-sous-familles de θ, soit une combinaison des deux phénomènes. Dans ce contexte il y a un sens à comparer le niveau galoisien at l'idéal de congruence pour la famille θ.

Introduction

Let N be a positive integer. Let f be a non-CM, cuspidal classical modular form of level N that is an eigenform for the action of the Hecke operators $\{T_\ell\}_{\ell\nmid N}$ and $\{U_\ell\}_{\ell|N}$. For a prime p let $\rho_{f,p}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{Q}_p)$ be the p-adic Galois representation associated with f. Ribet proved a result of "big image" for the representation $\rho_{f,p}$.

THEOREM 1. *For almost every prime p the image of $\rho_{f,p}$ contains the conjugate of a non-trivial principal congruence subgroup of* $\mathrm{SL}_2(\mathbb{Z}_p)$.

From now on, fix a prime $p \geq 5$ not dividing N. In [**Hi15**] Hida proved an analogue of Theorem 1 for a p-adic family of ordinary modular forms. We briefly present his result. Let $\Lambda = \mathbb{Z}_p[[T]]$ be the Iwasawa algebra. Let $\mathbb{T}^{\mathrm{ord}}$ be a local factor of the big ordinary Hecke algebra of tame level N constructed in [**Hi86**]; it is a finite flat Λ-algebra. The abstract Hecke algebra $\mathcal{H}$ of tame level N admits a morphism $\mathcal{H} \to \mathbb{T}^{\mathrm{ord}}$ that interpolates systems of eigenvalues associated with classical ordinary eigenforms. If $\mathbb{T}_h$ is residually non-Eisenstein the representations associated with the classical forms of the component are interpolated by a big Galois representation $\rho_{\mathbb{T}^{\mathrm{ord}}}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{T}^{\mathrm{ord}})$. Let $\theta\colon \mathbb{T}^{\mathrm{ord}} \to \mathbb{I}$ be a morphism of Λ-algebras defining an irreducible component of $\mathbb{T}^{\mathrm{ord}}$ and let $\rho_\theta\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I})$ be the representation induced by $\rho_{\mathbb{T}^{\mathrm{ord}}}$. We say that the family θ is CM if all its classical specializations are CM. For an ideal $\mathfrak{l}$ of Λ, let $\Gamma_\Lambda(\mathfrak{l})$ be the principal congruence subgroup of level $\mathfrak{l}$ of $\mathrm{SL}_2(\Lambda)$.

THEOREM 2. [**Hi15**, Theorem I] *Suppose that the family θ is non-CM. Then there exists an element $g \in \mathrm{GL}_2(\mathbb{I})$ and a non-zero ideal $\mathfrak{l}$ of Λ such that*

$$\Gamma_\Lambda(\mathfrak{l}) \subset g \cdot \mathrm{Im}\,\rho_\theta \cdot g^{-1}. \tag{5}$$

There exists a largest ideal $\mathfrak{l}_\theta$ of Λ among those satisfying (5) for some $g \in \mathrm{GL}_2(\mathbb{I})$; we call it the *Galois level* of θ. Hida gave a description of the prime factors of the Galois level, as follows. The Hecke algebra $\mathbb{T}^{\mathrm{ord}}$ admits some CM components, for which Theorem 2 does not hold. The non-CM family θ may intersect some of these components. Such crossings can be interpreted as congruences between a "general" family (i.e. such that its specializations are not lifts of eigenforms for a group of smaller rank) and "non-general" ones (in this case, those induced by a Grössencharacter of an imaginary quadratic field). In the case of GL_2 the only non-general families are the CM ones. Hida defined an ideal $\mathfrak{c}_\theta$ of Λ, the *CM-congruence ideal* of θ, that measures the amount of congruences between θ and the CM components.

THEOREM 3. [**Hi15**, Theorem II] *The ideals $\mathfrak{l}_\theta$ and $\mathfrak{c}_\theta$ have the same set of prime factors.*

In her Ph.D. thesis [**Lang16**] J. Lang improved Theorem 2. Her result is inspired by a stronger version of Theorem 1, due to Momose [**Mo81**] and Ribet [**Ri85**, Theorem 3.1]. She considered again a non-CM ordinary family $\theta\colon \mathbb{T}^{\mathrm{ord}} \to \mathbb{I}$ and defined a finite $(2, 2, \ldots, 2)$-group Γ of *conjugate self-twists* for θ, i.e. automorphisms of the Λ-algebra $\mathbb{I}$ that induce an isomorphism of ρ_θ with one of its twists by a finite order character. Let $\mathbb{I}_0$ be the subring of $\mathbb{I}$ fixed by Γ. For an ideal $\mathfrak{l}$ of $\mathbb{I}_0$ let $\Gamma_{\mathbb{I}_0}(\mathfrak{l})$ be the principal congruence subgroup of level $\mathfrak{l}$ of $\mathrm{SL}_2(\mathbb{I}_0)$.

THEOREM 4. [**Lang16**, Theorem 2.4] *There exists an element $g \in \mathrm{GL}_2(\mathbb{I})$ and a non-zero ideal $\mathfrak{l}$ of $\mathbb{I}_0$ such that*

$$\Gamma_{\mathbb{I}_0}(\mathfrak{l}) \subset g \cdot \mathrm{Im}\,\rho_\theta \cdot g^{-1}. \tag{6}$$

The existence of a conjugate self-twist gives a restriction on the size of the image of ρ_θ, so the ring $\mathbb{I}_0$ is optimal with respect to the property described in Theorem 4. The proofs of Theorems 2 and 4 rely on the existence of an element conjugate to

$$C_T = \begin{pmatrix} u^{-1}(1+T) & * \\ 0 & 1 \end{pmatrix}$$

in the image of an inertia group at p via ρ_θ; this is a consequence of the ordinarity of ρ_θ. Conjugation by such an element induces a structure of Λ-module on the intersection of $\operatorname{Im}\rho_\theta$ with the one-parameter unipotent subgroups of $\mathrm{SL}_2(\mathbb{I}_0)$. This is combined with the existence of some non-trivial unipotent elements in $\operatorname{Im}\rho_\theta$, constructed via the result of Momose and Ribet and Pink's theory of Lie algebras of pro-p subgroups of $\mathrm{SL}_2(\mathbb{I})$. A key step in the proof of Theorem 4 is a result of J. Lang on lifting self-twist from a p-adic field to the ring $\mathbb{I}_0$ [**Lang16**, Theorem 3.1].

In a joint work with A. Iovita and J. Tilouine (see Chapter 2 and [**CIT15**]) we proved analogues of Theorems 4 and 3 for a p-adic family of finite slope eigenforms. In this setting we encountered various problems and phenomena that do not appear in the ordinary case. First, finite slope eigenforms of tame level N are interpolated by a rigid analytic curve over $\mathbb{Q}_p$, the eigencurve $\mathcal{D}$ constructed by Coleman, Mazur [**CM98**] and Buzzard [**Bu07**], but this object does not admit a global integral structure analogue to the ordinary algebra $\mathbb{T}^{\mathrm{ord}}$. In particular, $\mathcal{D}$ admits a morphism to the weight space $\mathcal{W} = (\operatorname{Spf}\Lambda)^{\mathrm{rig}}$ (the *weight map*), but this is not finite. For this reason we need to fix a pair (h, B_h), consisting of $h \in \mathbb{Q}^{+,\times}$ and a disc of centre 0 and radius r_h depending on h, such that the weight map is finite when restricted to the subdomain $\mathcal{D}^h_{B_h}$ of points of $\mathcal{D}$ of slope $\leq h$ and weight in B_h. This is an "adapted pair" in the terminology of [**Be12**, Section 2.1]. For every $h \in \mathbb{Q}^{+,\times}$ there exists a radius r_h such that (h, B_h) is adapted, but at the moment we cannot give a lower bound for r_h in terms of h.

Let Λ_h and $\mathbb{T}_h$ be the rings of functions bounded by 1 on the rigid analytic spaces (over $\mathbb{Q}_p$) B_h and $\mathcal{D}^h_{B_h}$, respectively. The algebra $\mathbb{T}_h$ has a structure of finite Λ_h-algebra and is our analogue of Hida's Λ-algebra $\mathbb{T}^{\mathrm{ord}}$. We call *family of slope bounded by h* a positive slope irreducible component of $\mathbb{T}_h$, described by a morphism of Λ_h-algebras $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$. In the residually irreducible case we define a representation $\rho_\theta\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$. The notation $^\circ$ signals that we work with an integral structure; later we will need to invert p. To simplify our presentation we suppose that $\mathbb{I}^\circ$ is normal, but this hypothesis is not essential. An important difference with respect to the ordinary case is that the finite slope CM eigenforms do not form families, but they define a discrete subset of $\operatorname{Spec}\mathbb{I}^\circ$ (Corollary 2.2.8). Hence we are able to prove a result of big Galois image for all positive slope families.

We define a group Γ of self-twists for θ and we denote by $\mathbb{I}_0^\circ$ the subring of $\mathbb{I}^\circ$ fixed by Γ. We have to use the arguments of J. Lang on the lifting of self-twists; they can also be applied to non-ordinary families. Since the Galois representation ρ_θ is not ordinary, we do not know whether its image contains an element conjugate to the matrix C_T appearing in Hida and Lang's work. However we can construct an operator with similar properties via relative Sen theory. In order to do this we need first to extend our coefficients to the ring of rigid analytic functions on a closed disc of radius r smaller than r_h, then to consider completed tensor products with $\mathbb{C}_p$; we denote these operations by adding subscripts r and $\mathbb{C}_p$. Starting with $\mathbb{I}_0$ and ρ_θ, we define a ring $\mathbb{B}_r$ and a Lie subalgebra $\mathfrak{G}_r$ of $\mathfrak{gl}_2(\mathbb{B}_r)$ associated with $\operatorname{Im}\rho_\theta$ (Section 2.4.1). Then we construct a Sen operator $\phi \in \mathrm{M}_2(\mathbb{B}_{r,\mathbb{C}_p})$ such that its exponential normalizes $\mathfrak{G}_{r,\mathbb{C}_p}$ and has some explicit eigenvalues (Section 2.4). Conjugation by this element induces a Λ_h-module structure on the nilpotent subalgebras of $\mathfrak{G}_{r,\mathbb{C}_p}$ associated with the roots of SL_2. Under a technical hypothesis called $(H_0, \mathbb{Z}_p)$-regularity (see Definition 2.3.6) the nilpotent subalgebras of $\mathfrak{G}_{r,\mathbb{C}_p}$ contain some non-trivial elements, thanks to the residual result of Momose and Ribet and to an approximation argument due to Hida and Tilouine. Our first result is given by the theorem below.

THEOREM 5. (Theorem 2.5.2) *Suppose that* $\mathbb{I}^\circ$ *is normal and that* ρ_θ *is* $(H_0,\mathbb{Z}_p)$*-regular. There exists a non-zero ideal* $\mathfrak{l}$ *of* $\mathbb{I}_0$ *such that*

$$\mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{B}_r)\subset\mathfrak{G}_r. \tag{7}$$

Note that we manage to get rid of the extension of scalars to $\mathbb{C}_p$, but not of the inversion of p. We call *Galois level* of θ the largest ideal $\mathfrak{l}_\theta$ of $\mathbb{I}_0$ satisfying (7).

We describe the locus of CM points of the family by a *fortuitous CM-congruence ideal* $\mathfrak{c}_\theta$ of $\mathbb{I}_0$ (see Definition 2.2.12), where the term "fortuitous" higlights the fact that its prime factors do not correspond to congruences between a general family and the non-general ones, but to isolated CM points. In our second result we compare the Galois level and the congruence ideal for the family θ.

THEOREM 6. (Theorem 2.6.1) *Suppose that* $\overline{\rho}_\theta$ *is not induced by a character of the absolute Galois group of a real quadratic field. Then the sets of prime factors of* $\mathfrak{c}_\theta\cap\mathbb{I}_0$ *and* $\mathfrak{l}_\theta$ *coincide outside of the divisors of* $(1+T-u)\cdot\mathbb{I}_0$ *(the factors of weight* 1*).*

The goal of Chapters 3 and 4 is to study the problem of the definition and comparison of the Galois level and the congruence ideal for p-adic families of Siegel eigenforms of genus 2 and finite positive slope. In [**HT15**] Hida and Tilouine studied this problem for families that are residually of "large twisted Yoshida type": in this case the only possible congruences are those with p-adic lifts of families of Hilbert modular forms for $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2/F}$, where F is a real quadratic field. The lifts of classical forms are interpolated by two-dimensional irreducible components of the big ordinary Hecke algebra, hence the resulting congruence ideal is an analogue of the CM-congruence ideal defined by Hida: it describes congruences between a general family and the non-general ones. It would be possible to study the congruence phenomena for positive slope families of twisted Yoshida type with the same techniques developed here; we would find again two dimensional non-general families and a direct generalization of the congruence ideal defined in the ordinary case. In order to obtain instead an analogue of the fortuitous CM-congruence ideal associated with positive slope families for GL_2, we focus on families of GSp_4-eigenforms that are residually of large symmetric cube type, as explained below. In addition to the different type of congruences that we allow, the positive slope case presents several new features with respect to the ordinary one.

The eigenforms of tame level N are interpolated by a rigid analytic space $\mathcal{D}$ of dimension 2 constructed by Andreatta, Iovita and Pilloni [**AIP15**]. This space is endowed with a non-finite map to the two-dimensional weight space $\mathcal{W}_2=(\mathrm{Spf}\,\Lambda_2)^{\mathrm{rig}}$. As in the positive slope case for GL_2, a preliminary step requires us to fix $h\in\mathbb{Q}^{+,\times}$ and a disc B_h of centre 0 and sufficiently small radius r_h in the weight space. Let $\mathcal{D}^h_{B_h}$ be the admissible subdomain of $\mathcal{D}$ consisting of the points of slope $\leq h$ and weight in B_h. The weight map induces a finite morphism $\mathcal{D}^h_{B_h}\to B_h$. Let Λ_h and $\mathbb{T}_h$ be the rings of functions bounded by 1 on the rigid analytic spaces (over $\mathbb{Q}_p$) B_h and $\mathcal{D}^h_{B_h}$, respectively. We call family of slope bounded by h an irreducible component of $\mathbb{T}_h$, defined by a morphism of Λ_h-algebras $\theta\colon\mathbb{T}_h\to\mathbb{I}^\circ$. In the residually irreducible case the interpolation of the representations attached to the classical points of θ gives a representation $\rho\colon G_\mathbb{Q}\to\mathrm{GSp}_4(\mathbb{I}^\circ)$. We suppose that the associated residual representation $\overline{\rho}_\theta$ is of large symmetric cube type, in the sense that it satisfies

$$\mathrm{Sym}^3\mathrm{SL}_2(\mathbb{F})\subset\mathrm{Im}\,\overline{\rho}_\theta\subset\mathrm{Sym}^3\mathrm{GL}_2(\mathbb{F})$$

for a non-trivial finite field $\mathbb{F}$. This condition creates a strong restriction on the possible congruences: if a classical point is non-general, then it must correspond to the symmetric cube lift of a GL_2-eigenform via the transfer constructed by Kim and Shahidi [**KS02**]. This GL_2-eigenform defines a point on a GL_2-eigencurve of a suitable tame level, depending on N. It turns out that the notion of being a symmetric cube lift also makes sense for non-classical points: if the representation associated with a point of $\mathcal{D}$ is of the form $\mathrm{Sym}^3\rho'$ for some $\rho'\colon G_\mathbb{Q}\to\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$, then ρ' is associated with an overconvergent GL_2-eigenform, hence with a point of an eigencurve for GL_2. This follows from the overconvergent Fontaine-Mazur conjecture [**Em14**, Theorem 1.2.4] after adapting some results of Di Matteo [**DiM13**] to show that ρ' is trianguline (Theorem 3.10.30).

The non-general points are contained in a one-dimensional subvariety of $\mathcal{D}$ due to a restriction on their weight, so there are no two-dimensional non-general families. However some symmetric cube lifts form one-parameter families. We can construct such families thanks to some results by Bellaïche and Chenevier [**BC09**, Section 7.2.3]. Let $\mathcal{D}_1^{\mathrm{non-CM}}$ be the non-CM part of the GL_2-eigencurve of a suitable level depending on N. We interpolate the symmetric cube lifts of the classical points to construct a morphism $\mathcal{D}_1^{\mathrm{non-CM}} \to \mathcal{D}$ of rigid analytic spaces (Section 3.9.2). The image in $\mathcal{D}$ of a family in $\mathcal{D}_1^{\mathrm{non-CM}}$ is a non-general one-parameter family of GSp_4-eigenforms.

We define the group Γ of self-twists for θ (Section 4.3) and we let $\mathbb{I}_0^\circ$ be the subring of $\mathbb{I}^\circ$ fixed by Γ. We show that the crucial result of Lang on lifting self-twists can be adapted to this setting (Proposition 4.4.1). The theorem of Momose and Ribet admits an analogue for Siegel modular forms (Theorem 3.11.3); this is a consequence of a very general result of Pink on Zariski-dense compact subgroups of linear algebraic groups (Theorem 3.11.4). We combine it with the lifting result for self-twists and with the approximation argument by Hida and Tilouine (Proposition 4.7.1) to construct some non-trivial elements in the unipotent subgroups of $\mathrm{Im}\,\rho_\theta$. We fix a radius $r \in p^{\mathbb{Q}}$ smaller than r_h. We define a ring $\mathbb{B}_r$ and a Lie subalgebra $\mathfrak{G}_r$ of $\mathfrak{gsp}_4(\mathbb{B}_r)$ associated with the image of ρ_θ (Section 4.10.1). Thanks to relative Sen theory we can construct an element ϕ of $\mathrm{GSp}_4(\mathbb{B}_{r,\mathbb{C}_p})$ that normalizes $\mathfrak{G}_{r,\mathbb{C}_p}$ and has some explicit eigenvalues (Proposition 4.10.20). Conjugation by ϕ induces a Λ_h-module structure on the nilpotent subalgebras of $\mathfrak{G}_{r,\mathbb{C}_p}$ associated with the roots of Sp_4. This leads to the following result.

Theorem 7. (Theorem 4.11.1) *There exists a non-zero ideal* $\mathfrak{l}$ *of* $\mathbb{I}_0$ *such that*

$$\mathfrak{l} \cdot \mathfrak{sp}_4(\mathbb{B}_r) \subset \mathfrak{G}_r. \tag{8}$$

We call *Galois level* of θ the largest ideal $\mathfrak{l}_\theta$ of $\mathbb{I}_0$ satisfying (8). We define a *fortuitous* Sym^3-*congruence ideal* $\mathfrak{c}_\theta$ of $\mathbb{I}_0$ describing the locus of non-general points of the family θ (Definition 4.8.7). Thanks to the p-adic interpolation of the symmetric cube lift we know that there exist families for which the congruence ideal admits both zero- and one-dimensional components. We can compare the Galois level and the congruence ideal outside of a finite set of bad primes.

Theorem 8. (see Theorem 4.12.1 for the precise result) *The sets of prime divisors of* $\mathfrak{l}_\theta$ *and* $\mathfrak{c}_\theta$ *coincide outside of a finite and explicit set of bad primes.*

We hope to return to the question of describing the adapted pairs for GL_2 and GSp_4, namely of finding an estimate for the radius of the disc B_h as a function of the slope h. Some bounds for the analogous radius exist for the eigenvarieties associated with the eigenvarieties for definite unitary groups, thanks to some results by Chenevier [**Ch04**, Section 5]. For GL_2 this problem is related to some results of Wan [**Wa98**] in the context of conjectures of Gouvêa-Mazur type, but the available estimates only concern the fibres at the classical weights and do not apply to the finiteness of the weight map over p-adic neighborhoods.

It seems natural to try to generalize Theorems 7 and 8 to p-adic families of automorphic forms on reductive groups G for which eigenvarieties have been constructed. A general setup could be as follows. Let $\mathcal{W}_G$ be the weight space for G and let $\mathcal{D}_G$ be the eigenvariety parametrizing finite slope overconvergent eigenforms for G; it is equipped with a weight map $\mathcal{D}_G \to \mathcal{W}_G$. Suppose that we can fix an adapted pair (h, B_h), consisting of $h \in \mathbb{Q}^{+,\times}$ and of a disc B_h in $\mathcal{W}_G$, defining a subdomain $\mathcal{D}_G^h$ of $\mathcal{D}_G$ such that the weight map $\mathcal{D}_G^h \to B_h$ is finite. Let $\mathbb{T}_h$ be the ring of rigid analytic functions bounded by 1 on $\mathcal{D}_G^h$ and consider an irreducible component of $\mathbb{T}_h$, defined by a morphism $\theta\colon \mathbb{T}^h \to \mathbb{I}$. Suppose that we can attach to θ a Galois representation $\rho_\theta\colon G_{\mathbb{Q}} \to {}^LG(\mathbb{I})$, where ${}^LG$ denotes the Langlands dual group of G. The image of ρ_θ is Zariski-dense in the $\mathbb{I}$-points of its Mumford-Tate group, that we write as ${}^LH$ for some reductive group H. The techniques developed for GL_2 and GSp_4 could be adapted to show that an H-Galois level exists for the family θ. For every reductive group H' of rank smaller than the rank of H, and for every morphism of L-groups ${}^L H' \to {}^L H$ for which the classical Langlands transfer is known, it seems possible to define a p-adic transfer via interpolation. This would lead us to the definition of a congruence ideal that can either measure congruences between the H-family θ and the H'-families or fortuitous congruences due to the existence of lower-dimensional H'-subfamilies of θ, or a combination of the two phenomena. In this setting it makes sense to compare the Galois level and the congruence ideal associated with the family θ.

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CHAPTER 1

The eigenvarieties for GL_2 and GSp_4

1.1. Preliminaries

We fix some notations and conventions. In the text p will always denote a prime number strictly larger than 3. Most argument work for every odd p; we specify when this is not sufficient. We choose algebraic closures $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}$ and $\mathbb{Q}_p$ respectively. If K is a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_p$ we denote by $G_K = \mathrm{Gal}(\overline{K}/K)$ its absolute Galois group. We equip G_K with its profinite topology. We denote by $\mathcal{O}_K$ the ring of integers of K. If K is local, we denote by $\mathfrak{m}_K$ the maximal ideal of $\mathcal{O}_K$. For every prime p we fix an embedding $\iota_p \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, identifying $G_{\mathbb{Q}_p}$ with a decomposition group of $G_{\mathbb{Q}}$. This identification will be implicit everywhere. We fix a valuation v_p on $\overline{\mathbb{Q}}_p$ normalized so that $v_p(p) = 1$. It defines a norm given by $|\cdot| = p^{-v_p(\cdot)}$. We denote by $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$ with respect to this norm.

All rigid analytic spaces will be considered in the sense of Tate (see [**BGR84**, Part C]). Let $K/\mathbb{Q}_p$ be a field extension and let X be a rigid analytic space over K. We denote by $\mathcal{O}(X)$ the K-algebra of rigid analytic functions on X, and by $\mathcal{O}(X)^\circ$ the $\mathcal{O}_K$-subalgebra of functions with norm bounded by 1 (we often say "functions bounded by 1" meaning that they are bounded in norm). When $f \colon X \to Y$ is a map of rigid analytic spaces, we denote by $f^* \colon \mathcal{O}(Y) \to \mathcal{O}(X)$ the map induced by f. There is a Grothendieck topology on X, called the Tate topology; we refer to [**BGR84**, Proposition 9.1.4/2] for the definition of its admissible open sets and admissible coverings.

We say that X is a wide open rigid analytic space if there exists an admissible covering $\{X_i\}_{i\in\mathbb{N}}$ of X by affinoid domains X_i such that, for every i, $X_i \subset X_{i+1}$ and the map $\mathcal{O}(X_{i+1}) \to \mathcal{O}(X_i)$ induced by the previous inclusion is compact.

There is a notion of irreducible components for a rigid analytic space X; see [**Con99**] for the details. We say that X is equidimensional of dimension d if all its irreducible components have dimension d.

We will denote by $\mathbb{A}^d$ the d-dimensional rigid analytic affine space over $\mathbb{Q}_p$. Given a point $x \in \mathbb{A}^d(\mathbb{C}_p)$ and $r \in p^{\mathbb{Q}}$, we denote by $B_d(x,r)$ the d-dimensional closed disc of centre x and radius r. It is an affinoid domain defined over $\mathbb{C}_p$. We denote by $B_d(x,r^-)$ the d-dimensional wide open disc of centre x and radius r, defined as the rigid analytic space over $\mathbb{C}_p$ given by the increasing union of the d-dimensional affinoid discs of centre x and radii $\{r_i\}_{i\in\mathbb{N}}$ with $r_i < r$ and $\lim_{i\mapsto+\infty} r_i = r$. With an abuse of terminology we refer to $B_d(x,r)$ as the d-dimensional "closed disc" and to $B_d(x,r^-)$ as the d-dimensional "open disc", even though both are open sets in the Tate topology.

Let X be an affinoid or a wide open rigid analytic space. We denote by $\mathcal{O}(X)\{\{T\}\}$ the ring of power series $\sum_{i\geq 0} a_i T^i$ with $a_i \in \mathcal{O}(X)$ and $\lim_i |a_i| r^i \to 0$ for every $r \in \mathbb{R}^+$. This is the ring of rigid analytic functions on $X \times \mathbb{A}^1$.

Let S be any subset of $X(\mathbb{C}_p)$. We say that S is:

(1) a discrete subset of $X(\mathbb{C}_p)$ if $S \cap A$ is a finite set for any open affinoid $A \subset X(\mathbb{C}_p)$;
(2) a Zariski-dense subset of $X(\mathbb{C}_p)$ if, for every $f \in \mathcal{O}(X)$ vanishing at every point of S, f is identically zero;

(3) an accumulation subset of $X(\mathbb{C}_p)$ if for every $x \in S$ there exists a basis $\mathcal{B}$ of affinoid neighborhoods of x in X such that for every $A \in \mathcal{B}$ the set $S \cap A(\mathbb{C}_p)$ is Zariski-dense in A.

Terminology (3) is borrowed from [**BC09**, Section 3.3.1]. The subsets of $X(\mathbb{C}_p)$ that are accumulation and Zariski-dense are called "very Zariski-dense" in [**Ch05**, Section 4.4], but we do not use this phrase.

Let $g \geq 1$ be an integer and let s be the $g \times g$ antidiagonal unit matrix $(\delta_{i,n-i}(i,j))_{1\leq i,j\leq g}$. Let J_g be the $2g\times 2g$ matrix $\begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix}$. We denote by GSp_{2g} the algebraic group of symplectic similitudes for J_g, defined over $\mathbb{Z}$; for every ring R the R-points of this group are given by

$$\mathrm{GSp}_{2g}(R) = \{A \in \mathrm{GL}_4(R) \,|\, \exists\, \nu(A) \in R^\times \text{ s.t. } {}^tAJA = \nu(A)J\}.$$

For $g = 1$ we have $\mathrm{GSp}_2 = \mathrm{GL}_2$. The map $A \to \nu(A)$ defines a character $\nu\colon \mathrm{GSp}_4(R) \to R^\times$. We refer to ν as the similitude factor and we set $\mathrm{Sp}_{2g}(R) = \{A \in \mathrm{GSp}_{2g}(R) \,|\, \nu(A) = 1\}$.

We denote by B_g the Borel subgroup of GSp_{2g} such that for every ring R the R-points of B_g are the upper triangular matrices in $\mathrm{GSp}_{2g}(R)$. We let T_g be the maximal torus such that for every ring R the R-points of T_g are the diagonal matrices in $\mathrm{GSp}_{2g}(R)$. We write U_g for the unipotent radical of B_g. We have $B_g = T_gU_g$. We will always speak of weights and roots for GSp_{2g} with respect to the previous choice of Borel subgroup and torus. For every root α we denote by U^α the corresponding one-parameter unipotent subgroup of GSp_{2g}. For every prime ℓ, we write $I_{g,\ell}$ for the Iwahori subgroup of $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$ corresponding to our choice of Borel subgroup. For every $n \geq 1$ we denote by $\mathbb{1}_n$ the $n \times n$ unit matrix.

For every integer $g \geq 1$, we identify $\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$ with a maximal compact subgroup of $\mathrm{GSp}_{2g}(\mathbb{A}_\mathbb{Q})$ via the natural inclusion $\widehat{\mathbb{Z}} \hookrightarrow \mathbb{A}_\mathbb{Q}$. For every prime ℓ and every integer $n \geq 0$ we define some smaller compact open subgroups of $\mathrm{GSp}_{2g}(\mathbb{A}_\mathbb{Q})$ by:

(1) $\Gamma_0^{(g)}(\ell^n) = \{h \in \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}) \,|\, h_\ell \pmod{\ell^n} \in B_g(\mathbb{Z}/\ell^n\mathbb{Z})\}$;
(2) $\Gamma_1^{(g)}(\ell^n) = \{h \in \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}) \,|\, h_\ell \pmod{\ell^n} \in U_g(\mathbb{Z}/\ell^n\mathbb{Z})\}$;
(3) $\Gamma^{(g)}(\ell^n) = \{h \in \mathrm{GSp}_{2g}(\widehat{\mathbb{Z}}) \,|\, h_\ell \cong \mathbb{1}_{2g} \pmod{\ell^n}\}$.

In particular for $n = 1$ the ℓ-component of $\Gamma_0(\ell)$ is the Iwahori subgroup of $\mathrm{GSp}_4(\mathbb{Q}_\ell)$. Let N be an arbitrary integer. Write $N = \prod_i \ell_i^{n_i}$ for some distinct primes ℓ_i and some $n_i \in \mathbb{N}$. We set $\Gamma_?^{(g)}(N) = \bigcap_i \Gamma_?^{(g)}(\ell_i^{n_i})$ for $? = \varnothing, 0, 1$. For $g = 1$ we will omit the upper index (1).

We denote by $\mathfrak{gsp}_{2g}$ the Lie algebra of GSp_{2g} and by $\mathfrak{sp}_{2g}$ its derived Lie algebra, which is the Lie algebra of $\mathfrak{sp}_{2g}$. We denote by $\mathrm{Ad}^0\colon \mathrm{GSp}_{2g} \to \mathrm{Aut}(\mathfrak{sp}_{2g})$ the adjoint action of GSp_{2g} on $\mathfrak{sp}_{2g}$. It is an irreducible representation of GSp_{2g}.

By "classical modular form for GSp_4" we will always mean a *vector-valued* modular form.

1.2. The eigenvarieties

1.2.1. The weight spaces. There is an isomorphism $\mathbb{Z}_p^\times \cong (\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{Z}_p$ depending on the choice of a generator u of $\mathbb{Z}_p^\times$. We choose once and for all $u = 1 + p$. Let g be a positive integer. We write Λ_g for the Iwasawa algebra $\mathbb{Z}_p[[T_1, T_2, \ldots, T_g]]$ of formal series in g variables over $\mathbb{Z}_p$. A construction by Berthelot attaches to the formal scheme $\mathrm{Spf}\,\Lambda_g$ a rigid analytic space that we denote by $\mathcal{W}_g$ (see [**dJ95**, Section 7] and Section 2.2.1 below). Denote by $(\mathbb{Z}/\widehat{(p-1)}\mathbb{Z})^g$ the group of characters $(\mathbb{Z}/(p-1)\mathbb{Z})^g \to \mathbb{C}_p^\times$. The following map gives an isomorphism from $\mathcal{W}_g$ to a disjoint union of g-dimensional open discs $B_g(0,1^-)$ indexed by $(\mathbb{Z}/\widehat{(p-1)}\mathbb{Z})^g$:

$$\begin{aligned} \eta_g\colon \mathcal{W}_g &\to (\mathbb{Z}/\widehat{(p-1)}\mathbb{Z})^g \times B_g(0,1^-) \\ \kappa &\mapsto (\kappa|_{(\mathbb{Z}/(p-1)\mathbb{Z})^g}, (\kappa(u,1,\ldots,1)-1, \kappa(1,u,1,\ldots,1)-1, \ldots, \kappa(1,\ldots,1,u)-1)) \end{aligned}$$

For $x \in (\mathbb{Z}/\widehat{(p-1)}\mathbb{Z})^g \times B_g(0,1^-)$ we denote by κ_x the only character $(\mathbb{Z}_p^\times)^g \to \mathbb{C}_p^\times$ such that $\eta_g(\kappa_x) = x$. From now on we will work on the connected component $\mathcal{W}_g^\circ$ of $\mathcal{W}_g$ containing unity. We identify $\mathcal{W}_g^\circ$ with $B_g(1,1^-)$ via η_g. The classical weights in $\mathcal{W}_g$ are the characters of the form

$$(x_1, x_2, \ldots, x_g) \mapsto (x_1^{(p-1)k_1}, x_2^{(p-1)k_2}, \ldots, x_g^{(p-1)k_g})$$

for some $k_1, k_2, \ldots, k_g \in \mathbb{N}$. If κ is of the above form for some $k_1, k_2, \ldots, k_g \in \mathbb{N}$, we write $\omega(\kappa) = ((p-1)k_1, (p-1)k_2, \ldots, (p-1)k_g)$ for the corresponding element of $\mathbb{N}^g$. We have $\eta_g(\kappa) = (u^{(p-1)k_1} - 1, u^{(p-1)k_2} - 1, \ldots, u^{(p-1)k_g} - 1) \in B_g(0,1^-)$. The set of classical weights is an accumulation and Zariski-dense subset of $\mathcal{W}_g^\circ$.

The ring of analytic functions bounded by 1 on $B(0,1^-)$ is isomorphic to Λ_g as a $\mathbb{Z}_p$-algebra. We call *arithmetic primes* the primes of $\Lambda_g \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{C}_p$ of the form $P_{\underline{k},\varepsilon} = (1 + T_1 - \varepsilon_1(u)u^{k_1}, 1 + T_2 - \varepsilon_2(u)u^{k_2}, \ldots, 1 + T_g - \varepsilon_g(u)u^{k_g})$ for some g-tuple of integers $\underline{k} = (k_1, k_2, \ldots, k_g)$ and some finite order character $\varepsilon\colon (\mathbb{Z}_p^\times)^g \to \mathbb{C}_p^\times$. We will usually take ε to be the trivial character 1; in this case we simply write $P_{\underline{k}} = P_{\underline{k},1}$.

There exists a character $\kappa_{\mathcal{W}_g}\colon (\mathbb{Z}_p^\times)^g \to \Lambda_g^\times$ with the following universal property: for every $x \in \mathcal{W}_g(\mathbb{C}_p)$ there exists a unique character $\alpha_x\colon \Lambda_g^\times \to \mathbb{C}_p^\times$ such that $\kappa_x = \alpha_x \circ \kappa_{\mathcal{W}_g}$. The character $\kappa_{\mathcal{W}_g}$ maps $(a_1, a_2, \ldots a_g) \in (\mathbb{Z}_p^\times)^g$ to the analytic function $\kappa_{\mathcal{W}_g}(a_1, a_2, \ldots, a_g)$ on $\mathcal{W}_g$ defined by $\kappa_{\mathcal{W}_g}(a_1, a_2, \ldots, a_g)(x) = \kappa_x(a_1, a_2, \ldots, a_g)$ for every $x \in \mathcal{W}_g$. We call $\kappa_{\mathcal{W}_g}$ the universal character of $\mathcal{W}_g$.

Let $A = \operatorname{Spm} R$ be an affinoid subdomain of $\mathcal{W}_g$. The inclusion $\iota_A\colon A \hookrightarrow \mathcal{W}$ induces a map $\iota_A^*\colon \Lambda_g \to R$. Define a character $\kappa_A\colon (\mathbb{Z}_p^\times) \to R^\times$ by $\kappa_A = \iota_A^* \circ \kappa_{\mathcal{W}_g}$. Then κ_A has the following universal property: for every $x \in A(\mathbb{C}_p)$ there exists a unique character $\alpha_x\colon R^\times \to \mathbb{C}_p^\times$ such that $\kappa_x = \alpha_x \circ \kappa_A$. We call κ_A the universal character of A. By [**Bu07**, Proposition 8.3] there exists $r \in p^{\mathbb{Q}}$ such that κ_A is r-analytic, in the sense that it extends to a character $(\mathbb{Z}_p^\times \cdot B_g(1,r)) \to R^\times$. The radius of analyticity of κ_A is the largest such r; we denote it by r_{κ_A}.

1.2.2. The eigenvariety machine. We recall some elements of Buzzard's "eigenvariety machine" [**Bu07**]. We call *eigenvariety datum* a 5-tuple $(\mathcal{W}, \mathcal{H}, (M(A,w))_{A,w}, (\phi_{A,w})_{A,w}, \eta)$ where:

(1) there exists an integer $g \geq 1$ such that $\mathcal{W} = \mathcal{W}_g$ is the g-dimensional weight space defined in the previous section;
(2) (A, w) varies over the couples consisting of an affinoid $A \subset \mathcal{W}$ and a sufficiently large $w \in \mathbb{Q}$;
(3) for every (A, w) with $A = \operatorname{Spm} R$, $M(A, w)$ is a projective Banach R-module;
(4) $\mathcal{H}$ is a commutative ring;
(5) $\phi_{A,w}\colon \mathcal{H} \to \operatorname{End}_{R,\mathrm{cont}}(M(A,w))$ is an action of $\mathcal{H}$ on $M(A,w)$;
(6) $\eta \in \mathcal{H}$ is an element such that $\phi_{A,w}(\eta)$ is a compact operator on $M(A,w)$ for every (A,w);
(7) when A and w vary the modules $M(A,w)$ with their $\mathcal{H}$-actions satisfy the compatibility properties assumed in [**Bu07**, Lemma 5.6].

Buzzard's construction allows for more general weight spaces, but we only need to work with those of the form $\mathcal{W}_g$ for some g.

Remark 1.2.1. *The rational w in the couple (A, w) must satisfy $p^{-w} \leq r_{\kappa_A}$ for the radius of analyticity r_{κ_A} of κ_A.*

Let K be a finite extension of $\mathbb{Q}_p$.

Definition 1.2.2. *A homomorphism $\lambda\colon \mathcal{H} \to K$ is called a* K-system of eigenvalues *for the given datum if there exists a point $\kappa \in \mathcal{W}(K)$, an affinoid $A = \operatorname{Spm} R$ containing κ, a rational w and an element $m \in M(A,w) \otimes_R K$ (where $R \to K$ is the evaluation at κ) such that $\phi_{A,w}(T)m = \lambda(T)m$ for all $T \in \mathcal{H}$.*

Theorem 1.2.3. *For every eigenvariety datum $(\mathcal{W}, \mathcal{H}, (M(A,w))_{A,w}, (\phi_{A,w})_{A,w}, \eta)$ there exists a triple $(\mathcal{D}, \psi, w)$ consisting of*

(1) a rigid analytic space $\mathcal{D}$ over $\mathbb{Q}_p$,
(2) a morphism of $\mathbb{Q}_p$-algebras $\psi\colon \mathcal{H} \to \mathcal{O}(\mathcal{D})^\circ$,
(3) a morphism of rigid analytic spaces $w\colon \mathcal{D} \to \mathcal{W}$ (called the weight morphism*),*
with the following properties:
(1) $\psi(\eta)$ is invertible in $\mathcal{O}(\mathcal{D})$;
(2) for every finite extension $K/\mathbb{Q}_p$ the map

$$\mathcal{D}(K) \to \mathrm{Hom}(\mathcal{H}, K) \tag{1.1}$$
$$x \mapsto (T \mapsto \psi(T)(x))$$

induces a bijection between the K-points of $\mathcal{D}$ and the K-systems of eigenvalues for the given datum.

We call $(\mathcal{D}, \psi, w)$ the eigenvariety *for the given datum.*

We often leave ψ and w implicit and just refer to $\mathcal{D}$ as the eigenvariety.

Since the space $\mathcal{W}_g$ is equidimensional of dimension g, [**Ch04**, Proposition 6.4.2] implies the following.

Proposition 1.2.4. *The eigenvariety $\mathcal{D}$ associated with the datum*

$$(\mathcal{W}_g, \mathcal{H}, (M(A,w))_{A,w}, (\phi_{A,w})_{A,w}, \eta)$$

is equidimensional of dimension g.

We briefly review Buzzard's construction. Let (A, w) be a pair appearing in the eigenvariety datum, with $A = \mathrm{Spm}\, R$. Let $P_{A,w}(\eta; X)$ be the characteristic series of the operator $\phi_{A,w}(\eta)$ acting on $M(A, w)$; it is a well-defined element of $R\{\{X\}\}$ because $M(A, w)$ is a projective R-module and $\phi_{A,w}(\eta)$ is a compact operator. Since the actions of $\mathcal{H}$ are compatible when varying the pair (A, w), there exists an element $P_{\mathcal{W}}(\eta; X)$ that restricts to $P_{A,w}(\eta; X)$ for every A.

Let $\mathcal{Z}_{A,w}$ be the subvariety of $A \times \mathbb{G}_m$ defined by the equation $P_{A,w}(\eta; X) = 0$. Let $\mathcal{Z}$ be the subvariety of $\mathcal{W} \times \mathbb{G}_m$ defined by the equation $P_{\mathcal{W}}(\eta; X) = 0$, which can also be obtained by gluing the varieties $\mathcal{Z}_{A,w}$ when (A, w) varies. We call $\mathcal{Z}$ the *spectral variety.* It is endowed with two natural maps $w_{\mathcal{Z}}\colon \mathcal{Z} \to \mathcal{W}$ and $\nu_{\mathcal{Z}}\colon \mathcal{Z} \to \mathbb{G}_m$. Note that in general $w_{\mathcal{Z}}$ is not finite. Consider the set $\mathcal{C}$ of affinoid subdomains Y of $\mathcal{Z}$ satisfying the following conditions:
(1) the map $w|_Y\colon Y \to w(Y)$ is finite and surjective;
(2) Y is disconnected from its complement in $w^{-1}(w(Y))$.
Buzzard showed in [**Bu07**, Theorem 4.6] that $\mathcal{C}$ is an admissible covering of $\mathcal{Z}$. Now let (A, w) be a pair appearing in the eigenvariety datum, with $A = \mathrm{Spm}\, R$. For every element $Y \in \mathcal{C}$ satisfying $w_{\mathcal{Z}}(Y) = A$, consider the ideal of functions in $R[X]$ that vanish on Y. By the discussion in [**Bu07**, Section 5] this ideal is generated by a polynomial $Q(X)$, and there is a decomposition $P_{A,w}(\eta; X) = Q(X)S(X)$ for some $S(X) \in R\{\{X\}\}$ prime to $Q(X)$. Note that the constant term of $Q(X)$ is invertible in R. Let d be the degree of Q and let $Q^*(X) = X^d Q(1/X)$. Then Riesz theory for Banach modules [**Bu07**, Theorem 3.3] gives a decomposition

$$M(A, w) = N_Y(A, w) \oplus F_Y(A, w) \tag{1.2}$$

where
- $N_Y(A, w)$ and $F_Y(A, w)$ are R-submodules stable under the action of $\mathcal{H}$;
- $N_Y(A, w)$ is projective of rank d over R;
- $Q^*(\phi_{A,w}(\eta))$ is zero on $N_Y(A, w)$ and it is invertible on $F_Y(A, w)$;
- the characteristic power series of $\phi_{A,w}(\eta)$ on $N_Y(A, w)$ is $Q(X)$.

Let $\mathbb{T}_Y(A, w)$ be the R-subalgebra of $\mathrm{End}_{R,\mathrm{cont}}(N_Y(A, w))$ generated by the image of $\mathcal{H}$. Then $\mathbb{T}_Y(A, w)$ is a $\mathbb{Q}_p$-affinoid algebra and it is finite over R. Since the constant term of $Q(X)$ is invertible in R, $\phi_{A,w}(\eta)$ is invertible in $\mathbb{T}_Y(A, w)$. The projection $A\{\{X\}\}/P_{A,w}(\eta; X) \to A\{\{X\}\}/Q(X)$ induces a finite map $\mathrm{Spm}\, \mathbb{T}_Y(A, w) \to Y$. When A, w and Y vary the affinoid varieties $\operatorname{Spm} \mathbb{T}_Y(A, w)$ and the morphisms $\operatorname{Spm} \mathbb{T}_Y(A, w) \to \mathcal{Z}_{A,w}$ glue into a rigid analytic variety $\mathcal{D}$ over $\mathbb{Q}_p$ and a finite morphism $\mathcal{D} \to \mathcal{Z}$ [**Bu07**, Construction 5.7]. The composition of the last morphism with $w_{\mathcal{Z}} \colon \mathcal{Z} \to \mathcal{W}$ gives the weight map $\mathcal{D} \to \mathcal{W}$. The natural maps $\mathcal{H} \to \mathbb{T}_Y(A, w)$ are compatible when A, w and Y vary, hence they glue to give a global map $\psi \colon \mathcal{H} \to \mathcal{O}(\mathcal{D})$. Moreover $\psi(\eta)$ is invertible in $\mathcal{O}(\mathcal{D})$ since it is invertible in $\mathbb{T}_Y(A, w)$ for every A, w, Y.

REMARK 1.2.5. *The weight map $\mathcal{D} \to \mathcal{W}$ is not finite in general, but it is locally-on-the-domain finite: by construction every point of $\mathcal{D}$ has a neighborhood of the form $\operatorname{Spm} \mathbb{T}_Y(A, w)$ for some A, w, Y as above. The weight map $\operatorname{Spm} \mathbb{T}_Y(A, w) \to A$ is finite since it is the composition of the finite maps $\operatorname{Spm} \mathbb{T}_Y(A, w) \to \mathcal{Z}_{A,w}$ and $\mathcal{Z}_{A,w} \to A$.*

Thanks to property (1) in Theorem 1.2.3, we can give the following definitions.

DEFINITION 1.2.6.
(1) Let $\nu \in \mathcal{O}(D)$ be the function defined by $\nu = \psi(\eta)^{-1}$. We see ν as a map of rigid analytic spaces $\mathcal{D} \to \mathbb{G}_m$.
(2) Let $\operatorname{sl} \colon \mathcal{D}(\mathbb{C}_p) \to \mathbb{R}^{\geq 0}$ be the function defined by $\operatorname{sl}(x) = -v_p(\nu(x)) = v_p(\psi(\eta)(x))$ for every $x \in \mathcal{D}(\mathbb{C}_p)$. We call $\operatorname{sl}(x)$ the slope *of x.*

REMARK 1.2.7.
(1) The morphism ν is the composition of the maps $\mathcal{D} \to \mathcal{Z}$ and $\nu_{\mathcal{Z}} \colon \mathcal{Z} \to \mathbb{G}_m$.
(2) The function $\operatorname{sl} \colon \mathcal{D}(\mathbb{C}_p) \to \mathbb{R}^+$ is locally constant since it is the p-adic valuation of the rigid analytic function $\psi(\eta)$. In particular sl is bounded over $A(\mathbb{C}_p)$ for every affinoid subdomain A of $\mathcal{D}$.

DEFINITION 1.2.8. *We call* ordinary eigenvariety *for the given datum the largest open subvariety $\mathcal{D}^{\mathrm{ord}}$ of $\mathcal{D}$ with the property that $\psi(\eta)|_{\mathcal{D}^{\mathrm{ord}}} \in (\mathcal{O}(\mathcal{D}^{\mathrm{ord}})^\circ)^\times$.*

We give an extra property of the weight morphism.

PROPOSITION 1.2.9. [**Ch04**, Corollary 6.4.4] *If I is an irreducible component of $\mathcal{D}$, then $w(I)$ is a Zariski-open subset of the weight space $\mathcal{W}$.*

1.2.3. Accumulation and Zariski-dense sets on eigenvarieties.

Let $(\mathcal{D}, \psi, w)$ be the eigenvariety associated with some data $(\mathcal{W}, \mathcal{H}, (M(A, w))_{A,w}, (\phi_{A,w})_{A,w}, \eta)$. For every weight κ we denote by $\mathcal{D}_\kappa$ the set-theoretic fibre of $w \colon \mathcal{D} \to \mathcal{W}$ at κ. Let S be a subset of $\mathcal{D}(\mathbb{C}_p)$. For every weight κ and every $h \in \mathbb{R}$ we write
(1) $S_\kappa = \{x \in S \mid w(x) = \kappa\}$;
(2) $S^{\leq h} = \{x \in S \mid \operatorname{sl}(x) \leq h\}$;
(3) $S_\kappa^{\leq h} = S_\kappa \cap S^{\leq h}$.

We will work with various sets S satisfying the "control" condition below, that was introduced in [**Ch05**, Section 4.4]. Note that in *loc. cit.* the condition is called (Cl) and it is defined for a classical structure on the eigenvariety, rather than for a set of points.

(Class) There exists an accumulation and Zariski-dense set $\Sigma \subset \mathcal{W}(\mathbb{C}_p)$ with the following property: for every $h \in \mathbb{R}^{\geq 0}$ the set of weights $\kappa \in \Sigma$ such that the inclusion $\mathcal{D}_\kappa^{\leq h} \subset S$ holds is the complement of a finite set in Σ.

The following result is proved in [**Ch05**, Proposition 4.5].

PROPOSITION 1.2.10. *Let S be a subset of $\mathcal{D}(\mathbb{C}_p)$ satisfying condition* (Class)*. Then S is an accumulation and Zariski-dense subset of $\mathcal{D}(\mathbb{C}_p)$.*

PROOF. Let Σ be the set given by condition (Class). Let $\mathcal{I}$ denote any irreducible component of $\mathcal{D}$. By Proposition 1.2.9 $w(\mathcal{I})(\mathbb{C}_p)$ is a Zariski-open subset of $\mathcal{W}(\mathbb{C}_p)$. Since Σ is Zariski-dense in $\mathcal{W}(\mathbb{C}_p)$, there exists a weight $\kappa \in \Sigma \cap w(\mathcal{I})(\mathbb{C}_p)$. Let $x \in \mathcal{I}(\mathbb{C}_p)$ be a point of weight κ. By

Remark 1.2.7(2) we can choose an affinoid neighborhood A of x such that the slope is constant equal to $h(x)$ on A. By Remark 1.2.5 we can suppose, up to restricting A, that $w\colon A \to w(A)$ is finite. The image $w(A)$ is an affinoid neighborhood of κ. Since Σ is accumulation and Zariski-dense in $\mathcal{W}(\mathbb{C}_p)$, $\Sigma \cap w(A)(\mathbb{C}_p)$ is Zariski-dense in $w(A)(\mathbb{C}_p)$. By condition (Class) with $h = \mathrm{sl}(x)$ we have $\mathcal{D}_\kappa^{\leq h} \subset S$ for all $\kappa \in \Sigma_h$ where Σ_h is the complement of a finite subset of Σ. In particular $A_\kappa \subset S$ for all $\kappa \in \Sigma_h$, which means that $w^{-1}(\Sigma_h) \subset S$. Since Σ_h is Zariski-dense in $w(A)$ and the morphism $A \to w(A)$ is finite and flat, the set $w^{-1}(\Sigma_h)$ is Zariski-dense in A, hence also in the irreducible component $\mathcal{I}$ containing A. Note that the same is true if we replace A by any smaller affinoid, hence it is true for all the affinoids in a basis of neighborhoods of x contained in A.

Since S is Zariski-dense in every irreducible component of $\mathcal{D}$, it is Zariski-dense in $\mathcal{D}$. By the results of the previous paragraph, for every point $x \in S$ there is a basis of affinoid neghborhoods of x such that, for every A in the basis, the set $A(\mathbb{C}_p) \cap S$ is Zariski-dense in $A(\mathbb{C}_p)$. We conclude that S is accumulation and Zariski-dense in $\mathcal{D}(\mathbb{C}_p)$. □

For later use we state a simple lemma.

Lemma 1.2.11. *Let $f\colon X \to Y$ be a finite morphism of schemes or rigid analytic spaces over $\mathbb{C}_p$. Suppose that X and Y are both equidimensional and their dimensions coincide. Let S_Y be a Zariski-dense subset of $Y(\mathbb{C}_p)$. Let S_X be a subset of $X(\mathbb{C}_p)$ such that $f(S_X) = S_Y$. Then the Zariski-closure of S_X contains an irreducible component of X.*

1.2.4. The Hecke algebras. We describe the spherical and Iwahoric Hecke algebras associated with the group GSp_{2g}. We follow the conventions of [**GT05**, Section 3]. See also the standard reference [**CF90**, Chapter VII] for the unramified algebras, but some conventions there are different.

1.2.4.1. *The abstract spherical Hecke algebra.* Let ℓ be a prime. Let G be a $\mathbb{Z}$-subgroup scheme of GSp_{2g} and let $K \subset G(\mathbb{Q}_\ell)$ be a compact open subgroup. For $\gamma \in G(\mathbb{Q}_\ell)$ we denote by $\mathbb{1}([K\gamma K])$ the characteristic function of the double coset $[K\gamma K]$. Let $\mathcal{H}(G(\mathbb{Q}_\ell), K)$ be the $\mathbb{Q}$-algebra generated by the functions $\mathbb{1}([K\gamma K])$ for $\gamma \in G(\mathbb{Q}_\ell)$, equipped with the convolution product.

Definition 1.2.12. *The* spherical Hecke algebra *of* GSp_{2g} *at* ℓ *is* $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell), \mathrm{GSp}_{2g}(\mathbb{Z}_\ell))$.

The algebra $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell), \mathrm{GSp}_{2g}(\mathbb{Z}_\ell))$ is generated by the elements

$$T_{\ell,i}^{(g)} = \mathbb{1}([\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)\mathrm{diag}\,(\mathbb{1}_i, \ell\mathbb{1}_{2g-2i}, \ell^2\mathbb{1}_i)\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)]),$$

for $i = 0, 1, \dots g$, and $((T_{\ell,0}^{(g)})^{-1})$. Note that our operator $T_{\ell,0}^{(g)}$ is often denoted by $S_\ell^{(g)}$ in the literature.

1.2.4.2. *The dilating Iwahori Hecke algebra.* The Hecke algebra $\mathcal{H}(T_g(\mathbb{Q}_\ell), T_g(\mathbb{Z}_\ell))$ carries a natural action of the Weyl group $W_g = \mathscr{S}_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g$ of GSp_{2g}, where $\mathscr{S}_g$ is the group of permutations of $\{1, 2, \dots, g\}$: if $\mathrm{diag}\,(\nu t_1, \dots, \nu t_g, t_g^{-1}, \dots, t_1^{-1})$ is an element of the torus, $\mathscr{S}_g$ acts by permuting the t_i's and the non-trivial element in each $\mathbb{Z}/2\mathbb{Z}$ sends t_i to t_i^{-1}. We denote the action of $w \in W_g$ on $t \in T(\mathbb{Q}_\ell)$ by $t \mapsto w.t$. We define a character $e^\rho\colon T(\mathbb{Q}_\ell) \to \mathbb{Q}_\ell^\times$ by

$$e^\rho(\mathrm{diag}\,(\nu t_1, \dots, \nu t_g, t_g^{-1}, \dots, t_1^{-1})) = \nu^{-g(g+1)/2} \prod_{1 \leq i \leq g} t_i^{g-i+1}.$$

We define a twisted action of the Weyl group on $\mathcal{H}(T_g(\mathbb{Q}_\ell), T_g(\mathbb{Z}_\ell))$ by $\phi^w(t) = e^\rho(w^{-1}.t)e^{-\rho}(t)\phi(t)$ for all $w \in W_g$, $\phi \in \mathcal{H}(T_g(\mathbb{Q}_\ell), T_g(\mathbb{Z}_\ell))$ and $t \in T_g(\mathbb{Q}_\ell)$.

The twisted Satake transform $S_{\mathrm{GSp}_{2g}}^{T_g}$ is a morphism of $\mathbb{Q}$-algebras

$$\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell), \mathrm{GSp}_{2g}(\mathbb{Z}_\ell)) \to \mathcal{H}(T_g(\mathbb{Q}_\ell), T_g(\mathbb{Z}_\ell))$$

defined by

$$S^{T_g}_{\mathrm{GSp}_{2g}}(\phi)(t) = e^{\rho}(t)\int_{U_g}\phi(tu)du$$

for all $\phi \in \mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell), \mathrm{GSp}_{2g}(\mathbb{Z}_\ell))$ and $t \in T_g(\mathbb{Q}_\ell)$. The morphism $S^{T_g}_{\mathrm{GSp}_{2g}}$ induces an isomorphism of $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell), \mathrm{GSp}_{2g}(\mathbb{Z}_\ell))$ onto its image, which is the subalgebra of $\mathcal{H}(T_g(\mathbb{Q}_\ell), T_g(\mathbb{Z}_\ell))$ consisting of W_g-invariant elements. In particular $\mathcal{H}(T_g(\mathbb{Q}_\ell), T_g(\mathbb{Z}_\ell))$ is a Galois extension of $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell), \mathrm{GSp}_{2g}(\mathbb{Z}_\ell))$ of Galois group W_g.

For $i = 0, 1, \ldots, g$ let $t^{(g)}_{\ell,i} = \mathbb{1}([\mathrm{diag}\,(\mathbb{1}_i, \ell\mathbb{1}_{2g-2i}, \ell^2\mathbb{1}_i)\mathbb{T}_g(\mathbb{Z}_\ell)])$. Note that the element $t^{(g)}_{\ell,0}$ is $S^{T_g}_{\mathrm{GSp}_{2g}}(T^g_{\ell,0})$. The set $(t^{(g)}_{\ell,i})_{i=1,\ldots,g}$ generates $\mathcal{H}(T_g(\mathbb{Q}_\ell), T_g(\mathbb{Z}_\ell))$ over $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell), \mathrm{GSp}_{2g}(\mathbb{Z}_\ell))$.

We call an element $\gamma \in T(\mathbb{Z}_\ell)$ dilating if $v_p(\alpha(\gamma)) \leq 0$ for every positive root α. Let $T(\mathbb{Z}_\ell)^-$ be the subset of $T(\mathbb{Z}_\ell)$ consisting of dilating elements and let $\mathcal{H}(T_g(\mathbb{Q}_\ell), T_g(\mathbb{Z}_\ell))^-$ be the $\mathbb{Q}$-subalgebra of $\mathcal{H}(T_g(\mathbb{Q}_\ell), T_g(\mathbb{Z}_\ell))$ generated by the functions $\mathbb{1}([\gamma T(\mathbb{Z}_\ell)])$ with $\gamma \in T(\mathbb{Z}_\ell)^-$. Since $\mathbb{1}([\gamma T(\mathbb{Z}_\ell)])\mathbb{1}([\gamma' T(\mathbb{Z}_\ell)]) = \mathbb{1}([\gamma\gamma' T(\mathbb{Z}_\ell)])$ for $\gamma, \gamma' \in T(\mathbb{Q}_\ell)^-$, the functions $\mathbb{1}([\gamma T(\mathbb{Z}_\ell)])$ with $\gamma \in T(\mathbb{Q}_\ell)^-$ form a basis of $\mathcal{H}(T_g(\mathbb{Q}_\ell), T_g(\mathbb{Z}_\ell))^-$ as a $\mathbb{Q}$-vector space.

Remark 1.2.13. *Every $\gamma \in T(\mathbb{Q}_\ell)$ can be written in the form $\gamma = \gamma_1\gamma_2^{-1}$ with $\gamma_1, \gamma_2 \in T(\mathbb{Z}_\ell)^-$. A character $\chi\colon \mathcal{H}(T_g(\mathbb{Q}_\ell), T_g(\mathbb{Z}_\ell))^- \to \overline{\mathbb{Q}}_p$ can be extended uniquely to a character $\chi^{\mathrm{ext}}\colon \mathcal{H}(T_g(\mathbb{Q}_\ell), T_g(\mathbb{Z}_\ell)) \to \overline{\mathbb{Q}}_p$ by setting $\chi^{\mathrm{ext}}([\gamma T(\mathbb{Z}_\ell)]) = \chi([\gamma_1 T(\mathbb{Z}_\ell)])\chi([\gamma_2 T(\mathbb{Z}_\ell)]^{-1})$ for some γ_1 and γ_2 as before. It can be easily checked that χ^{ext} is well-defined.*

Definition 1.2.14. *Let $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell), I_{g,\ell})^-$ be the subalgebra of $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell), I_{g,\ell})$ generated by the functions $\mathbb{1}([I_{g,\ell}\gamma I_{g,\ell}])$ with $\gamma \in T(\mathbb{Z}_\ell)^-$. We call $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell), I_{g,\ell})^-$ the* dilating Iwahori Hecke algebra *at ℓ.*

The algebra $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell), I_{g,\ell})^-$ is generated by the elements

$$U^{(g)}_{\ell,i} = \mathbb{1}([I_{g,\ell}\mathrm{diag}\,(\mathbb{1}_i, \ell\mathbb{1}_{2g-2i}, \ell^2\mathbb{1}_i)I_{g,\ell}]),$$

for $i = 0, 1, \ldots, g$, and $(U^{(g)}_{\ell,0})^{-1}$.

We define a map $\iota^{T_g}_{I_{g,\ell}}\colon \mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell), I_{g,\ell})^- \to \mathcal{H}(T_g(\mathbb{Q}_\ell), T_g(\mathbb{Z}_\ell))^-$ by sending $\mathbb{1}(I_{g,\ell}\gamma I_{g,\ell})$ to $\mathbb{1}(T_g(\mathbb{Z}_\ell)\gamma T_g(\mathbb{Z}_\ell))$ for every $\gamma \in T(\mathbb{Z}_\ell)^-$.

Lemma 1.2.15. *The map $\iota^{T_g}_{I_{g,\ell}}\colon \mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell), I_{g,\ell})^- \to \mathcal{H}(T_g(\mathbb{Q}_\ell), T_g(\mathbb{Z}_\ell))^-$ is an isomorphism of $\mathbb{Q}$-algebras.*

Proof. The argument is the same as in [**BC09**, Proposition 6.4.1]. By the calculation in [**Ca95**][Lemma 4.1.5] we have $\mathbb{1}([I_{g,\ell}\gamma I_{g,\ell}]) \cdot \mathbb{1}([I_{g,\ell}\gamma' I_{g,\ell}]) = \mathbb{1}([I_{g,\ell}\gamma\gamma' I_{g,\ell}])$ for all $\gamma, \gamma' \in T_g(\mathbb{Q}_\ell)^-$. This implies that $\iota^{T_g}_{I_{g,\ell}}$ is a morphism of $\mathbb{Q}$-algebras and that the functions $\mathbb{1}([I_{g,\ell}\gamma I_{g,\ell}])$ for $\gamma \in T(\mathbb{Z}_\ell)^-$ form a basis of $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell), I_{g,\ell})^-$ as a $\mathbb{Q}$-vector space. We deduce that $\iota^{T_g}_{I_{g,\ell}}$ is bijective since it sends a $\mathbb{Q}$-basis of $\mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell), I_{g,\ell})^-$ to a $\mathbb{Q}$-basis of $\mathcal{H}(T_g(\mathbb{Q}_\ell), T_g(\mathbb{Z}_\ell))^-$. $\square$

Let p be a prime and N be a positive integer such that $(N, p) = 1$.

Definition 1.2.16. *Set*

$$\mathcal{H}^{Np}_g = \bigotimes_{\mathbb{Q}, \ell\nmid Np} \mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_\ell), \mathrm{GSp}_{2g}(\mathbb{Z}_\ell))$$

and

$$\mathcal{H}^N_g = \mathcal{H}^{Np}_g \otimes_{\mathbb{Q}} \mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_p), I_{g,p})^-.$$

We call $\mathcal{H}^N_g$ the abstract Hecke algebra spherical outside N and Iwahoric dilating at p.

The algebra $\mathcal{H}_g^N$ acts on the space of classical vector-valued modular forms for $\mathrm{GSp}_{2g}(\mathbb{Q})$ of level $\Gamma_1(N)\cap\Gamma_0(p)$. With an abuse of notation we will consider the elements of one of the local algebras as elements of $\mathcal{H}_g^N$ via the natural inclusion (tensoring by 1 at all the other primes).

1.2.4.3. *The Hecke polynomials.* In the following we will specialize to the cases $g = 1, 2$. We record here some results and calculations that we will need later.

For $g = 1$, the degree two extension $\mathcal{H}(T_1(\mathbb{Q}_\ell), T_1(\mathbb{Z}_\ell))$ over $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_\ell), \mathrm{GL}_2(\mathbb{Z}_\ell))$ is generated by the element $t_{\ell,1}^{(1)} = \mathbb{1}([\mathrm{diag}\,(1,\ell)T_1(\mathbb{Z}_\ell)])$. Let w denote the only non-trivial element of the Weil group W_1. The minimal polynomial of $t_{\ell,1}^{(1)}$ is

$$P_{\min}(t_{\ell,1}^{(1)})(X) = (X - t_{\ell,1}^{(1)})(X - (t_{\ell,1}^{(1)})^w).$$

By an explicit calculation we obtain

$$P_{\min}(t_{\ell,1}^{(1)})(X) = X^2 - T_\ell^{(1)}X + \ell T_{\ell,0}^{(1)}. \tag{1.3}$$

For $g = 2$, the degree eight extension $\mathcal{H}(T_2(\mathbb{Q}_\ell), T_2(\mathbb{Z}_\ell))$ over $\mathcal{H}(\mathrm{GSp}_4(\mathbb{Q}_\ell), \mathrm{GSp}_4(\mathbb{Z}_\ell))$ is generated by the two eleemnts $t_{\ell,1}^{(2)} = \mathbb{1}([\mathrm{diag}\,(1,\ell,\ell,\ell^2)T_2(\mathbb{Z}_\ell)])$ and $t_{\ell,2}^{(2)} = \mathbb{1}([\mathrm{diag}\,(1,1,\ell,\ell)T_2(\mathbb{Z}_\ell)])$. Each of them has an orbit of order four under the action of the Weyl group, hence degree four over $\mathcal{H}(\mathrm{GSp}_4(\mathbb{Q}_\ell), \mathrm{GSp}_4(\mathbb{Z}_\ell))$. We denote by $P_{\min}(t_{\ell,i}^{(2)})$ the minimal polynomials of $t_{\ell,i}^{(2)}$ over $\mathcal{H}(\mathrm{GSp}_4(\mathbb{Q}_\ell), \mathrm{GSp}_4(\mathbb{Z}_\ell))$ by $P_{\min}(t_{\ell,i}^{(2)})$.

If $t = \mathrm{diag}\,(\nu t_1, \nu t_2, t_1^{-1}, t_2^{-1})$ is an element of the torus we denote by w_0, w_1, w_2 the generators of the Weyl group satisfying $t^{w_0} = \mathrm{diag}\,(\nu t_2, \nu t_1, t_2^{-1}, t_1^{-1})$, $t^{w_1} = \mathrm{diag}\,(\nu t_1^{-1}, \nu t_2, t_1, t_2^{-1})$, $t^{w_2} = \mathrm{diag}\,(\nu t_1, \nu t_2^{-1}, t_1^{-1}, t_2)$. Note that $t_{\ell,2}^{(2)}$ is invariant under w_0. Its minimal polynomial is

$$P_{\min}(t_{\ell,2}^{(2)})(X) = (X - t_{\ell,2}^{(2)})(X - (t_{\ell,2}^{(2)})^{w_1})(X - (t_{\ell,2}^{(2)})^{w_2})(X - (t_{\ell,2}^{(2)})^{w_1 w_2}). \tag{1.4}$$

By an explicit calculation we obtain (see [**An87**, Lemma 3.3.35]):

$$P_{\min}(t_{\ell,2}^{(2)}) = X^4 - T_{\ell,2}^{(2)}X^3 + ((T_{\ell,2}^{(2)})^2 - T_{\ell,1}^{(2)} - \ell^2 T_{\ell,0}^{(2)})X^2 - \ell^3 T_{\ell,2}^{(2)}T_{\ell,0}^{(2)}X + \ell^6 (T_{\ell,0}^{(2)})^2. \tag{1.5}$$

Note that $t_{\ell,1}^{(2)} = (t_{\ell,2}^{(2)})(t_{\ell,2}^{(2)})^{w_1}$ is invariant under w_1. Its minimal polynomial is

$$\begin{aligned} P_{\min}(t_{\ell,1}^{(2)})(X) &= (X - t_{\ell,1}^{(2)})(X - (t_{\ell,1}^{(2)})^{w_2})(X - (t_{\ell,1}^{(2)})^{w_3})(X - (t_{\ell,1}^{(2)})^{w_2 w_3}) = \\ = (X - t_{\ell,2}^{(2)}(t_{\ell,1}^{(2)})^{w_1})&(X - (t_{\ell,2}^{(2)})^{w_2}(t_{\ell,1}^{(2)})^{w_1 w_2})(X - t_{\ell,2}^{(2)}(t_{\ell,1}^{(2)})^{w_2})(X - (t_{\ell,1}^{(2)})^{w_1}(t_{\ell,1}^{(2)})^{w_1 w_2}). \end{aligned} \tag{1.6}$$

1.2.5. The cuspidal GL_2-eigencurve. Fix a prime p and an integer $N \geq 1$ such that $(N, p) = 1$. Let $\mathcal{H}_1^N$ be the abstract Hecke algebra for GL_2, spherical outside N and Iwahoric dilating at p, defined in Section 1.2.4. For every affinoid $A = \mathrm{Spm}\, R \subset \mathcal{W}_1$ and every sufficiently large rational number w, Coleman and Mazur [**CM98**, Section 2.4] defined a Banach R-module $M_1(A, w)$ of w-overconvergent cuspidal modular forms for GL_2 of weight κ_A and tame level N. For each (A, w) there is an action $\phi_{A,w}^1 \colon \mathcal{H}_1^N \to \mathrm{End}_{R,\mathrm{cont}} M_1(A, w)$. Set $U_p^{(1)} = U_{p,1}^{(1)}$. Then $(\mathcal{W}_1, \mathcal{H}_1^N, (M_1(A,w))_{A,w}, (\phi^1)_{A,w}, U_p^{(1)})$ is an eigenvariety datum. The eigenvariety machine produces from this datum a triple $(\mathcal{D}_1^N, \psi_1, w_1)$, consisting of a rigid analytic variety $\mathcal{D}_1^N$ over $\mathbb{Q}_p$ and maps $\psi_1 \colon \mathcal{H}_1^N \to \mathcal{O}(\mathcal{D}_1^N)$ and $w_1 \colon \mathcal{D}_1^N \to \mathcal{W}_1$ (the weight morphism) with the properties given by Theorem 1.2.3. Note that $\mathcal{D}_1^N$ is equidimensional of dimension 1 by Proposition 1.2.4, hence its classical name "eigencurve". The eigencurve was constructed for $p > 2$ and $N = 1$ by Coleman and Mazur in [**CM98**] and for every p and N by Buzzard in [**Bu07**, Part II].

The weight map $w_1 \colon \mathcal{D}_1^N \to \mathcal{W}_1$ is neither finite nor étale. It is locally-on-the-domain finite by Remark 1.2.5. Moreover it satisfies the valuative criterion for properness, as proved by the recent work of Diao and Liu [**DL16**].

Let f be a classical GL_2-eigenform of level $\Gamma_1(N)\cap\Gamma_0(p)$ and weight $k\geq 2$. Let $\chi_1\colon\mathcal{H}_1^N\to\overline{\mathbb{Q}}_p$ be the system of Hecke eigenvalues associated with f.

Definition 1.2.17. *Let $\chi_1^{\mathrm{norm}}\colon\mathcal{H}_1^N\to\overline{\mathbb{Q}}_p$ be the character defined by*

$$\chi_{1,\ell}^{\mathrm{norm}}=\chi_{1,\ell}^{\mathrm{norm}}\ \textit{for every}\ \ell\nmid Np,$$
$$\chi_{1,p}^{\mathrm{norm}}(U_{p,0}^{(1)})=p^{1-k}\chi_{1,p}^{\mathrm{norm}}(U_{p,0}^{(1)}),$$
$$\chi_{1,p}^{\mathrm{norm}}(U_{p,1}^{(1)})=\chi_{1,p}^{\mathrm{norm}}(U_{p,1}^{(1)}).$$

We call χ_1^{norm} the normalized *system of Hecke eigenvalues associated with f.*

Remark 1.2.18. *The eigenvariety $\mathcal{D}_1^N$ interpolates the normalized systems of Hecke eigenvalues of the classical eigenforms, rather than the usual systems. More precisely, for every f as in Definition 1.2.17 there exists a point x of $\mathcal{D}_1^N$ such that $\mathrm{ev}_x\circ\psi_1=\chi_1^{\mathrm{norm}}$, where ev_x is the evaluation of a rigid analytic function at x and χ_1^{norm} is the normalized system of Hecke eigenvalues of f.*

We call a point $x\in\mathcal{D}_1^N(\mathbb{C}_p)$ *classical* if the system of Hecke eigenvalues associated with x by the map (1.1) is that of a classical modular form f of level $\Gamma_1(N)\cap\Gamma_0(p)$ and weight $w(x)$. In this case $w_1(x)$ is clearly a classical weight. In the proposition below we recall two important results. As before $\omega(w_1(x))$ denotes the only integer k such that $w_1(x)\colon\mathbb{Z}_p^\times\to\mathbb{Z}_p^\times$ is the character $a\mapsto a^k$. By an abuse of terminology we say that a classical point x has weight k if $\omega(w_1(x))=k$.

Proposition 1.2.19. *Let x be a $\mathbb{C}_p$-point of $\mathcal{D}_1^N$ such that $w_1(x)$ is classical. Let $k=\omega(w_1(x))$. Suppose that $h(x)<k-1$. Then:*
(1) x is a classical point [**Co96**, Theorem 6.1]*;*
(2) the weight map $w_1\colon\mathcal{D}_1^N\to\mathcal{W}_1$ is étale at x [**Ki03**, Theorem 11.10].

Note that a classical point x of weight k has slope $\mathrm{sl}(x)\leq k-1$, so the only classical points not satisfying the hypotheses of Proposition 1.2.19 are those of weight k and slope $k-1$ for some $k\in\mathbb{N}$. From Proposition 1.2.19 we deduce the following well-known result.

Proposition 1.2.20. *Let S^{cl} denote the set of classical points in $\mathcal{D}_1^N(\mathbb{C}_p)$. Then S^{cl} is accumulation and Zariski-dense in $\mathcal{D}_1^N(\mathbb{C}_p)$.*

Proof. By Proposition 1.2.19, for every $h\in\mathbb{R}$, the inclusion $\mathcal{D}_k^{\leq h}\subset S_k^{\mathrm{cl}}$ is satisfied by all classical weights k such that $k-1>h$. Since the set of such weights is accumulation and Zariski-dense in $\mathcal{W}_1$, the set S^{cl} satisfies the hypothesis (Class) where we take as X the set of classical weights. By applying Proposition 1.2.10 we deduce that S^{cl} is accumulation and Zariski-dense in $\mathcal{D}_1^{N,f}(\mathbb{C}_p)$. $\square$

1.2.5.1. *The ordinary eigencurve.* Let $\mathcal{D}_1^{N,\mathrm{ord}}$ be the ordinary eigencurve obtained by applying Definition 1.2.8 to $\mathcal{D}_1^N$. For every pair (A,w) appearing in the eigenvariety datum, with $A=\mathrm{Spm}\,R$, let $\mathbb{T}_{A,w}^1$ be the R-algebra generated by $\phi_{A,w}^1(\mathcal{H}_1^N)$ in $\mathrm{End}_{R,\mathrm{cont}}(M_1(A,w))$. For every such (A,w) there exists an idempotent element $e_{A,w}^{\mathrm{ord}}\in\mathbb{T}_{A,w}^1$ such that $\phi_{A,w}^1(U_p^{(1)})$ is invertible on $e^{\mathrm{ord}}(M_1(A,w))$ and topologically nilpotent on $(1-e^{\mathrm{ord}})(M_1(A,w))$. The element $e_{A,w}^{\mathrm{ord}}$ is defined as the limit of $(\phi_{A,w}^1(U_p^{(1)}))^{n!}$ for $n\to+\infty$. When (A,w) varies the elements $e_{A,w}^{\mathrm{ord}}$ glue to give a global nilpotent element $e^{\mathrm{ord}}\in\mathcal{O}(\mathcal{D}_1^N)^\circ$ with the property that $\mathcal{D}_1^{N,\mathrm{ord}}$ is the subvariety of $\mathcal{D}_1^N$ defined by $e^{\mathrm{ord}}-1=0$. In particular $\mathcal{D}_1^{N,\mathrm{ord}}$ is a connected component of $\mathcal{D}_1^N$.

Much earlier then the work of Coleman and Mazur, Hida interpolated ordinary GL_2-eigenforms in p-adic analytic families (see [**Hi86**]). It follows from Hida theory that the weight map $w_1|_{\mathcal{D}_1^{N,\mathrm{ord}}}\colon\mathcal{D}_1^{N,\mathrm{ord}}\to\mathcal{W}_1$ is finite and that it is étale at every classical point of weight $k\geq 2$.

1.2.5.2. *The eigencurve of slope bounded by h.* In Definition 1.2.6 we introduced a slope function $\mathrm{sl}\colon \mathcal{D}_1^N(\mathbb{C}_p) \to \mathbb{R}^+$. Let $h \in \mathbb{Q}^+$. The set of $\mathbb{C}_p$-points $x \in \mathcal{D}_1^N$ satisfying $\mathrm{sl}(x) \leq h$ admits a structure of rigid analytic subvariety of $\mathcal{D}_1^N$. We denote this subvariety by $\mathcal{D}_{1,h}^N$.

1.2.5.3. *The non-CM eigencurve.* We recall the following standard definitions.

Definition 1.2.21. *We say that a classical point of $\mathcal{D}_1^N$ is CM if it corresponds to a classical CM modular form. We say that an irreducible component of $\mathcal{D}_1^N$ is CM if all its classical points are CM.*

Remark 1.2.22. *By* [**Hi15**, Proposition 5.1]*, if an irreducible component contains a classical ordinary CM eigenform then the component is CM. In particular there exist CM irreducible components of the ordinary eigencurve, and every ordinary CM classical point belongs to a CM component. On the contrary, the CM classical points of the positive slope eigencurve form a discrete set (recall that this means that they are finite in each affinoid domain). This is a consequence of Corollary 2.2.8, where it is shown that the eigencurve $\mathcal{D}^{+,\leq h}$ contains a finite number of CM classical points.*

The goal of the next chapter will be to interpolate the classical Langlands transfer associated with the symmetric cube map $\mathrm{Sym}^3\colon \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GSp}_4(\mathbb{C})$. The existence of this transfer has been proved by Kim and Shahidi in [**KS02**] (see Theorem 3.3.3). Ramakrishnan and Shahidi proved in [**RS07**] that a GL_2-eigenform can be lifted to a GSp_4-eigenform via such a transfer, but if the starting form is CM form we do not know whether its lift is cuspidal. Since non-cuspidal Siegel modular forms are not interpolated by an eigenvariety at the moment, we will need to work on the part of the eigencurve in which non-CM classical points are Zariski-dense. For this reason we give the following definition.

Definition 1.2.23. *Let $\mathcal{D}_1^{N,\mathcal{G}}$ be the Zariski-closure in $\mathcal{D}_1^N$ of the set of non-CM classical points. We call $\mathcal{D}_1^{N,\mathcal{G}}$ the* non-CM eigencurve.

The upper index $\mathcal{G}$ stands for "general", since CM components are exceptional among the irreducible components of $\mathcal{D}_1^N$.

Remark 1.2.24.

(1) By Remark 1.2.22 the set of non-CM classical points is Zariski-dense in an irreducible component of $\mathcal{D}_1^N$ if and only if the component is non-CM, and every positive slope irreducible component is non-CM. Hence the non-CM eigencurve is simply the union of all non-CM irreducible components of $\mathcal{D}_1^N$, and it contains the positive slope eigencurve.

(2) Since the set of classical points is accumulation and Zariski-dense in every irreducible component of $\mathcal{D}_1^N$ and the set of CM classical points is discrete in $\mathcal{D}_1^{N,\mathcal{G}}$, the set of non-CM classical points is an accumulation and Zariski-dense subset of $\mathcal{D}_1^{N,\mathcal{G}}$.

1.2.6. The cuspidal GSp_4-eigenvariety. Let p be an odd prime. Fix an integer $M \geq 1$ such that $(M,p) = 1$. Let $\mathcal{H}_2^M$ be the abstract Hecke algebra for GSp_4, spherical outside M and Iwahoric dilating at p, defined in Section 1.2.4. For every affinoid $A = \mathrm{Spm}\, R \subset \mathcal{W}_2$ and every sufficiently large rational number w, Andreatta, Iovita and Pilloni [**AIP15**, Section 8.2] defined a Banach R-module $M_2(A,w)$ of w-overconvergent cuspidal GSp_4-modular forms of weight κ_A and tame level $\Gamma_1(M)$. For each (A,w) there is an action $\phi^2_{A,w}\colon \mathcal{H}_2^M \to \mathrm{End}_{R,\mathrm{cont}} M_2(A,w)$. Set $U_p^{(2)} = U_{p,1}^{(2)} U_{p,2}^{(2)}$. Then $(\mathcal{W}_2, \mathcal{H}_2^M, (M_2(A,w))_{A,w}, (\phi^2)_{A,w}, U_p^{(2)})$ is an eigenvariety datum. The eigenvariety machine constructs from this datum a rigid analytic variety over $\mathbb{Q}_p$. We call it the GSp_4-eigenvariety of tame level M and we denote it by $\mathcal{D}_2^M$. It is endowed with a weight morphism $w_2\colon \mathcal{D}_2^M \to \mathcal{W}_2$ and a map $\psi_2\colon \mathcal{H}_2^M \to \mathcal{O}(\mathcal{D}_2^M)^\circ$. By Proposition 1.2.4 $\mathcal{D}_2^M$ is equidimensional of dimension 2. The weight map $w_2\colon \mathcal{D}_2^M \to \mathcal{W}_2$ is neither finite nor étale. It is locally-on-the-domain finite by Remark 1.2.5.

Let F be a classical GSp_4-eigenform of level $\Gamma_{B_2}(M)\cap\Gamma_0(p)$ and weight (k_1,k_2) with $k_1\geq k_2\geq 3$. Let $\chi_2\colon \mathcal{H}_2^M\to\overline{\mathbb{Q}}_p$ be the system of Hecke eigenvalues associated with F.

DEFINITION 1.2.25. *Let $\chi_2^{\mathrm{norm}}\colon\mathcal{H}_2^M\to\overline{\mathbb{Q}}_p$ be the character defined by*

$$\begin{aligned}\chi_{2,\ell}^{\mathrm{norm}}&=\chi_{2,\ell}^{\mathrm{norm}} \text{ for every } \ell\nmid Mp\\ \chi_{2,p}^{\mathrm{norm}}(U_{p,0}^{(2)})&=p^{3-k_1-k_2}\chi_{2,p}(U_{p,0}^{(2)}),\\ \chi_{2,p}^{\mathrm{norm}}(U_{p,1}^{(2)})&=p^{1-k_1}\chi_{2,p}(U_{p,1}^{(2)}),\\ \chi_{2,p}^{\mathrm{norm}}(U_{p,2}^{(2)})&=\chi_{2,p}(U_{p,2}^{(2)}).\end{aligned}$$

We call χ_2^{norm} the normalized *system of Hecke eigenvalues associated with F.*

REMARK 1.2.26. *The eigenvariety $\mathcal{D}_2^M$ interpolates the normalized systems of Hecke eigenvalues of the classical eigenforms, rather than the usual systems. More precisely, for every F as in Definition 1.2.25 there exists a point x of $\mathcal{D}_2^M$ such that $\mathrm{ev}_x\circ\psi_2=\chi_2^{\mathrm{norm}}$, where ev_x is the evaluation of a rigid analytic function at x and χ_2^{norm} is the normalized system of Hecke eigenvalues of F.*

An analogue of Proposition 1.2.19(1) holds for the GSp_4-eigenvariety.

PROPOSITION 1.2.27. ([**BPS16**, Theorem 5.3.1], see also Remark 1 in the introduction of *loc. cit.*) *Let x be a $\overline{\mathbb{Q}}_p$-point of $\mathcal{D}_2^N$ such that $w_2(x)=(k_1,k_2)$ is classical and* $\mathrm{sl}(x)<k_2+3$. *Then x is a classical point.*

Unfortunately we do not know of an analogue of Proposition 1.2.19(2). A partial result in this direction for the tame level 1 eigenvariety is given by [**AIP15**, Proposition 8.3.2].

1.2.6.1. *The ordinary* GSp_4*-eigenvariety.* Let $\mathcal{D}_2^{M,\mathrm{ord}}$ be the ordinary GSp_4-eigenvariety obtained by applying Definition 1.2.8 to $\mathcal{D}_2^M$. For every pair (A,w) appearing in the eigenvariety datum, with $A=\mathrm{Spm}\,R$, let $\mathbb{T}_{A,w}^2$ be the R-algebra generated by the image $\phi_{A,w}^2(\mathcal{H}_2^M)$ in $\mathrm{End}_{R,\mathrm{cont}}(M_2(A,w))$. For every such (A,w) there exists an idempotent element $e_{A,w}^{\mathrm{ord}}\in\mathbb{T}_{A,w}^2$ such that $\phi_{A,w}^2(U_p^{(2)})$ is invertible on $e^{\mathrm{ord}}(M_2(A,w))$ and topologically nilpotent on $(1-e^{\mathrm{ord}})(M_2(A,w))$. The element $e_{A,w}^{\mathrm{ord}}$ is the limit of $(\phi_{A,w}^2(U_p^{(2)}))^{n!}$ for $n\to+\infty$. When (A,w) varies the elements $e_{A,w}^{\mathrm{ord}}$ glue to give a global nilpotent element $e^{\mathrm{ord}}\in\mathcal{O}(\mathcal{D}_2^M)^\circ$ with the property that $\mathcal{D}_2^{M,\mathrm{ord}}$ is the subvariety of $\mathcal{D}_2^M$ defined by $e^{\mathrm{ord}}-1=0$. In particular $\mathcal{D}_2^{M,\mathrm{ord}}$ is a connected component of $\mathcal{D}_2^M$.

As in the case of GL_2, the ordinary eigenvariety enjoys better properties than the whole eigenvariety. This is a consequence of Hida theory for GSp_4, which is a result of the papers [**TU99**], [**Hi02**], [**Ti06**] and [**Pil12a**]. It follows from Hida theory that the weight map $w_2|_{\mathcal{D}_2^{M,\mathrm{ord}}}\colon\mathcal{D}_2^{M,\mathrm{ord}}\to\mathcal{W}_2$ is finite and that it is étale at every classical point of weight (k_1,k_2) with $k_1\geq k_2\geq 3$.

1.2.6.2. *The eigenvariety of slope bounded by h.* Let $\mathrm{sl}\colon\mathcal{D}_2^M(\mathbb{C}_p)\to\mathbb{R}^+$ be the slope function given by Definition 1.2.6. Let $h\in\mathbb{Q}^+$. The set of $\mathbb{C}_p$-points $x\in\mathcal{D}_2^M$ satisfying $\mathrm{sl}(x)\leq h$ admits a structure of rigid analytic subvariety of $\mathcal{D}_2^M$. We denote this subvariety by $\mathcal{D}_{2,h}^M$.

1.2.7. Newforms and oldforms on the eigencurve.

We recall the following classical result.

PROPOSITION 1.2.28. [**Li75**, Theorem 3] *The slope of a p-new eigenform of level $\Gamma_1(N)\cap\Gamma_0(p)$ and weight $k\geq 2$ is $(k-2)/2$.*

We say that a classical point of $\mathcal{D}_1^N$ is p-old if it corresponds to a p-old eigenform; we say that it is p-new otherwise. Let $S^{\mathrm{cl}}_{\mathrm{old}}$ denote the set of p-old classical points in $\mathcal{D}_1^N(\mathbb{C}_p)$. We apply Proposition 1.2.10 to obtain a corollary of Proposition 1.2.28.

Corollary 1.2.29. *The set $S^{\mathrm{cl}}_{\mathrm{old}}$ is accumulation and Zariski-dense in $\mathcal{D}_1^N(\mathbb{C}_p)$.*

Proof. Let $S^{\mathrm{cl}}_{\mathrm{old},k}$ be the subset of points of weight k in $S^{\mathrm{cl}}_{\mathrm{old},k}$. For any $h \in \mathbb{R}$, the inclusion $\mathcal{D}_k^{\leq h} \subset S^{\mathrm{cl}}_{\mathrm{old},k}$ is verified for all weights k satisfying $(k-2)/2 > h$ by Proposition 1.2.28. Since the set of such weights is accumulation and Zariski-dense in $\mathcal{W}_1$, condition (Class) is satisfied for $S^{\mathrm{cl}}_{\mathrm{old}}$ and we conclude by applying Proposition 1.2.10. □

Corollary 1.2.30. *The set of p-old, non-CM classical points is accumulation and Zariski-dense in $\mathcal{D}_1^{N,\mathcal{G}}$.*

Proof. Since $S^{\mathrm{cl}}_{\mathrm{old}}$ is accumulation and Zariski-dense in $\mathcal{D}_1^N$, its intersection with $\mathcal{D}_1^{N,\mathcal{G}}$ is accumulation and Zariski-dense in $\mathcal{D}_1^{N,\mathcal{G}}$. The set of CM points is discrete in $\mathcal{D}_1^{N,\mathcal{G}}$, so its complement in $S^{\mathrm{cl}}_{\mathrm{old}}$ is still an accumulation and Zariski-dense subset of $\mathcal{D}_1^{N,\mathcal{G}}(\mathbb{C}_p)$. □

CHAPTER 2

Galois level and congruence ideal for finite slope families of modular forms

This chapter contains the results of a joint work of the author with A. Iovita and J. Tilouine (see [**CIT15**]). Our goal is to define a Galois level and a CM-congruence ideal for a p-adic family of finite slope GL_2-eigenforms, and to compare them. There are a few differences in the notations with respect to [**CIT15**].

2.1. The eigencurve

All rigid analytic spaces we consider are implicitly $\mathbb{Q}_p$-analytic. Before Section 2.2 all spaces are defined over $\mathbb{Q}_p$: indeed the weight space and the eigencurve can be admissibly covered by affinoid subdomains defined over $\mathbb{Q}_p$.

In this chapter we work with the one-dimensional weight space given by the construction in Section 1.2.1 for $g = 1$.

2.1.1. Adapted pairs and the eigencurve. Let N be a positive integer prime to p. We recall the definition of the spectral curve Z^N and of the cuspidal eigencurve $\mathcal{D}^N$ of tame level $\Gamma_1(N)$. These objects were constructed in [**CM98**] for $p > 2$ and $N = 1$ and in [**Bu07**] in general. We follow the presentation of [**Bu07**, Part II], but we give a description of the admissible covering of the spectral variety as in [**Be12**, Part II]. Let $\mathrm{Spm}\, R \subset \mathcal{W}$ be an affinoid domain and let $r = p^{-s}$ for $s \in \mathbb{Q}$ be a radius smaller than the radius of analyticity of κ_R. We denote by $M_{R,r}$ the R-module of r-overconvergent modular forms of weight κ_R. It is endowed it with a continuous action of the Hecke operators T_ℓ, $\ell \nmid Np$, and U_p. The action of U_p on $M_{R,r}$ is completely continuous, so we can consider its associated Fredholm series $F_{R,r}(T) = \det(1 - U_pT|M_{R,r}) \in R\{\{T\}\}$. These series are compatible when R and r vary, in the sense that there exists $F \in \Lambda\{\{T\}\}$ that restricts to $F_{R,r}(T)$ for every R and r.

The series $F_{R,r}(T)$ converges everywhere on the R-affine line $\mathrm{Spm}\, R \times \mathbb{A}^{1,\mathrm{an}}$, so it defines a rigid curve $Z^N_{R,r} = \{F_{R,r}(T) = 0\}$ in $\mathrm{Spm}\, R \times \mathbb{A}^{1,\mathrm{an}}$. When R and r vary, these curves glue into a rigid space Z^N endowed with a quasi-finite and flat morphism $w_Z \colon Z^N \to \mathcal{W}$. The curve Z^N is called the spectral curve associated with the U_p-operator. For every $h \geq 0$, let us consider

$$Z_R^{N,\leq h} = Z_R^N \cap \left(\mathrm{Spm}\, R \times B(0, p^h)\right).$$

By [**Bu07**, Lemma 4.1] $Z_R^{N,\leq h}$ is quasi-finite and flat over $\mathrm{Spm}\, R$.

We now recall how to construct an admissible covering of Z^N.

Definition 2.1.1. *We denote by $\mathcal{C}$ the set of affinoid subdomains $Y \subset Z$ such that:*

- *there exists an affinoid domain $\mathrm{Spm}\, R \subset \mathcal{W}$ such that Y is a union of connected components of $w_Z^{-1}(\mathrm{Spm}\, R)$;*
- *the map $w_Z|_Y \colon Y \to \mathrm{Spm}\, R$ is finite.*

Proposition 2.1.2. [**Bu07**, Theorem 4.6] *The covering $\mathcal{C}$ is admissible.*

Note in particular that an element $Y \in \mathcal{C}$ must be contained in $Z_R^{N,\le h}$ for some h.

For every R and r as above and every $Y \in \mathcal{C}$ such that $w_Z(Y) = \operatorname{Spm} R$, we can associate to Y a direct factor M_Y of $M_{R,r}$ by the construction in [**Bu07**, Section I.5]. The abstract Hecke algebra $\mathcal{H} = \mathbb{Z}[T_\ell]_{\ell\nmid Np}$ acts on $M_{R,r}$ and M_Y is stable with respect to this action. Let $\mathbb{T}_Y$ be the R-algebra generated by the image of $\mathcal{H}$ in $\operatorname{End}_R(M_Y)$ and let $\mathcal{D}_Y^N = \operatorname{Spm} \mathbb{T}_Y$. Note that it is reduced as all Hecke operators are self-adjoint for a certain pairing and mutually commute.

For every Y the finite covering $\mathcal{D}_Y^N \to \operatorname{Spm} R$ factors through $Y \to \operatorname{Spm} R$. The eigencurve $\mathcal{D}^N$ is defined by gluing the affinoids $\mathcal{D}_Y^N$ into a rigid curve, endowed with a finite morphism $\mathcal{D}^N \to Z^N$. The curve $\mathcal{D}^N$ is reduced and flat over $\mathcal{W}$ since it is so locally.

We borrow the following terminology from Bellaïche.

Definition 2.1.3. [**Be12**, Def. II.1.8] *Let $\operatorname{Spm} R \subset \mathcal{W}$ be an affinoid open subset and $h > 0$ be a rational number. The couple (R, h) is called adapted if $Z_R^{N,\le h}$ is an element of $\mathcal{C}$.*

The sets of the form $Z_R^{N,\le h}$ are actually sufficient to admissibly cover the spectral curve by [**Be12**, Corollary II.1.13].

Now we fix a finite slope h. We want to work with families of slope $\le h$ which are finite over a wide open subset of the weight space. In order to do this it will be useful to know which pairs (R, h) in a connected component of $\mathcal{W}$ are adapted. If $\operatorname{Spm} R' \subset \operatorname{Spm} R$ are affinoid subdomains of $\mathcal{W}$ and (R, h) is adapted then (R', h) is also adapted by [**Be12**, Proposition II.1.10]. By [**Bu07**, Lemma 4.3], the affinoid $\operatorname{Spm} R$ is adapted to h if and only if the weight map $Z_R^{N,\le h} \to \operatorname{Spm} R$ has fibres of constant degree.

Remark 2.1.4. *Given a slope h and a classical weight k, it would be interesting to have a lower bound for the radius of a disc of centre k adapted to h. A result of Wan* [**Wa98**, Theorem 2.5] *asserts that for a certain radius r_h depending only on h, N and p, the degree of the fibres of $Z_{B(k,r_h)}^{N,\le h} \to \operatorname{Spm} B(k, r_h)$ at classical weights is constant. Unfortunately we do not know whether the degree is constant at all weights of $B(k, r_h)$, so this is not sufficient to answer our question. Estimates for the radii of adapted discs exist in the case of eigenvarieties for groups different than GL_2; see for example the results of Chenevier on definite unitary groups* [**Ch05**, Section 5].

2.1.2. Pseudo-characters and Galois representations. Let K be a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_K$. Let X be a rigid analytic variety defined over K. We denote by $\mathcal{O}(X)$ the ring of global analytic functions on X equipped with the coarsest locally convex topology making the restriction map $\mathcal{O}(X) \to \mathcal{O}(U)$ continuous for every affinoid $U \subset X$. It is a Fréchet space isomorphic to the inverse limit over all affinoid domains U of the K-Banach spaces $\mathcal{O}(U)$. We denote by $\mathcal{O}(X)^\circ$ the $\mathcal{O}_K$-algebra of functions bounded by 1 on X, equipped with the topology induced by that on $\mathcal{O}(X)$.

Lemma 2.1.5. [**BC09**, Lemma 7.2.11(ii)] *If X is reduced and wide open, then $\mathcal{O}(X)^\circ$ is a compact (hence profinite) $\mathcal{O}_K$-algebra.*

Note that "wide open" rigid analytic spaces are called "nested" in [**BC09**].

We will be able to apply Lemma 2.1.5 to the eigenvariety thanks to the following.

Proposition 2.1.6. [**BC09**, Corollary 7.2.12] *The eigenvariety $\mathcal{D}^N$ is nested for $K = \mathbb{Q}_p$.*

Given a reduced nested subvariety X of $\mathcal{D}^N$ defined over a finite extension K of $\mathbb{Q}_p$ there is a pseudo-character on X obtained by interpolating the classical ones. Let $\mathbb{Q}^{Np}$ be the largest algebraic extension of $\mathbb{Q}$ unramified outside Np and let $G_{\mathbb{Q},Np} = \operatorname{Gal}(\mathbb{Q}^{Np}/\mathbb{Q})$.

Proposition 2.1.7. [**Be12**, Theorem IV.4.1] *There exists a unique pseudo-character*

$$\tau\colon G_{\mathbb{Q},Np} \to \mathcal{O}(X)^\circ$$

of dimension 2 such that for every ℓ prime to Np, $\tau(\mathrm{Frob}_\ell) = \psi_X(T_\ell)$, where ψ_X is the composition of $\psi\colon \mathcal{H} \to \mathcal{O}(C^N)^\circ$ with the restriction map $\mathcal{O}(\mathcal{D}^N)^\circ \to \mathcal{O}(X)^\circ$.

REMARK 2.1.8. *One can take as an example of X a union of irreducible components of C^N in which case $K = \mathbb{Q}_p$. Later we will consider other examples where $K \neq \mathbb{Q}_p$.*

2.2. The fortuitous congruence ideal

In this section we will define families with slope bounded by a finite constant and coefficients in a suitable profinite ring. We will show that any such family admits at most a finite number of classical specializations which are CM modular forms. Later we will define what it means for a point (not necessarily classical) to be CM and we will associate with a family a congruence ideal describing its CM points. Contrary to the ordinary case, the non-ordinary CM points do not come in families so the points detected by the congruence ideal do not correspond to a crossing between a CM and a non-CM family. For this reason we call our ideal the "fortuitous congruence ideal".

2.2.1. The adapted slope $\leq h$ Hecke algebra. Throughout this section we fix $h \in \mathbb{R}^{>0}$. Let $\mathcal{D}^{N,\leq h}$ be the subvariety of $\mathcal{D}^N$ consisting of the points of slope $\leq h$. Unlike the ordinary case treated in [**Hi15**] the weight map $w^{\leq h}\colon \mathcal{D}^{N,\leq h} \to \mathcal{W}$ is not finite which means that a family of slope $\leq h$ is not in general defined by a finite map over the entire weight space. The best we can do in the finite slope situation is to place ourselves over the largest possible wide open subdomain U of $\mathcal{W}$ such that the restriction of the weight map $w^{\leq h}|_U\colon C^{N,\leq h} \times_{\mathcal{W}} U \to U$ is finite. This is a domain "adapted to h" in a sense analogous to that of Definition 2.1.3 where only affinoid domains were considered. The finiteness property will be necessary in order to apply going-up and going-down theorems.

Let us fix a rational number s_h such that for $r_h = p^{-s_h}$ the closed disc $B(0, r_h)$ is adapted for h. We assume that $s_h > \frac{1}{p-1}$ (this will be needed later to assure the convergence of the exponential map). Let B_h be the open disc of centre 0 and radius p^{-s_h} in the weight space. We give a model of B_h over $\mathbb{Q}_p$, adapting the construction of Berthelot [**dJ95**, Section 7] of rigid analytic spaces associated with formal schemes. For $i \geq 1$, let $s_i = s_h + 1/2^i$ and $B_i = B(0, p^{-s_i})$. The open disc B_h is the increasing union of the affinoid discs B_i. Write $s_h = \frac{b}{a}$ for some $a, b \in \mathbb{N}$. For each i a model for B_i over $\mathbb{Q}_p$ is given by $\mathrm{Spm}\, A^\circ_{r_i}[p^{-1}]$, where

$$A^\circ_{r_i} = \mathbb{Z}_p\langle t, X_i\rangle/(t^{2^i a} - p^{a+2^i b} X_i).$$

For every i we define a morphism $\mathrm{res}_i\colon A^\circ_{r_{i+1}} \to A^\circ_{r_i}$ given by

$$t \mapsto t,$$
$$X_{i+1} \mapsto p^a X_i^2.$$

The morphisms res_i induce compact morphisms $A^\circ_{r_{i+1}}[p^{-1}] \to A^\circ_{r_i}[p^{-1}]$, hence open immersions $B_i \to B_{i+1}$ defined over K_h. We define the wide open disc B_h as the inductive limit of the affinoids B_i with respect to the transition maps above. Let Λ_h be the ring of rigid analytic functions bounded by 1 on B_h. There is an isomorphism

$$\Lambda_h = \varprojlim_i A^\circ_{r_i}$$

where the transition maps are the res_i's. We define an element $t \in \Lambda_h$ as the projective limit over i of the variables t of the $A^\circ_{r_i}$'s.

REMARK 2.2.1. *Let $\eta_h \in \overline{\mathbb{Q}}_p$ be an element of p-adic valuation s_h and let $\mathcal{O}_h = \mathbb{Z}_p[\eta]$. The ring Λ_h is not a power series ring over $\mathbb{Z}_p$. However there is an isomorphism $\Lambda_h \otimes_{\mathbb{Z}_p} \mathcal{O}_h \cong \mathcal{O}_h[[t]]$. Note that in* [**CIT15**, Section 3.1] *a ring $\Lambda_h^{\mathrm{CIT}} = \mathcal{O}_h[[t]]$ is defined (we write an upper index to distinguish it from the ring Λ_h defined here) and it is stated in* [**CIT15**, Section 4.1] *that the self-twists of ρ over $\mathbb{Z}_p[[\eta]]$ fix a form of Λ_h^{CIT} over a subring $\mathcal{O}_{h,0}$ of $\mathcal{O}_h$. Thanks to the construction of this section we can identify such a form with $\mathcal{O}_{h,0} \cdot \Lambda_h$.*

Since the s_i are strictly bigger than s_h for each i, $B(0, p^{-s_i}) = \mathrm{Spm}\, A_{r_i}^\circ[p^{-1}]$ is adapted to h. Therefore for every $r > 0$ sufficiently small and for every $i \geq 1$ the image of the abstract Hecke algebra acting on $M_{A_{r_i},r}$ provides a finite affinoid $A_{r_i}^\circ$-algebra $\mathbb{T}^{\leq h}_{A_{r_i}^\circ,r}$. The morphism $w_{A_{r_i}^\circ,r}\colon \mathrm{Spm}\, \mathbb{T}^{\leq h}_{A_{r_i}^\circ,r} \to \mathrm{Spm}\, A_{r_i}^\circ$ is finite. For $i < j$ we have natural inclusions $\mathrm{Spm}\, \mathbb{T}^{\leq h}_{A_{r_j}^\circ,r} \to \mathrm{Spm}\, \mathbb{T}^{\leq h}_{A_{r_i}^\circ,r}$ and corresponding restriction maps $\mathbb{T}^{\leq h}_{A_{r_i}^\circ,r} \to \mathbb{T}^{\leq h}_{A_{r_j}^\circ,r}$. We denote by $\mathcal{D}_h$ the increasing union $\bigcup_{i\in\mathbb{N},r>0} \mathrm{Spm}\, \mathbb{T}^{\leq h}_{A_{r_i}^\circ,r}$; it is a wide open subvariety of $\mathcal{D}^N$. We denote by $\mathbb{T}_h$ the ring of rigid analytic functions bounded by 1 on $\mathcal{D}_h$. We have $\mathbb{T}_h = \mathcal{O}(\mathcal{D}_h)^\circ = \varprojlim_{i,r} \mathbb{T}^{\leq h}_{A_{r_i}^\circ,r}$. There is a natural weight map $w_h\colon \mathcal{D}_h \to B_h$ that restricts to the maps $w_{A_{r_i}^\circ,r}$. It is finite because the closed ball of radius r_h is adapted to h. Since $\mathcal{O}(B_h)^\circ = \Lambda_h$, the map w_h gives $\mathbb{T}_h$ the structure of a finite Λ_h-algebra; in particular $\mathbb{T}_h$ is profinite.

There is a natural map $\Lambda \to \Lambda_h$ given by the restriction to B_h of analytic functions bounded by 1 on the open unit disc.

DEFINITION 2.2.2. *We say that a prime of Λ_h is arithmetic if it lies over an arithmetic prime P_k of Λ. By an abuse of notation we still denote by P_k an arithmetic prime of Λ_h lying over P_k.*

REMARK 2.2.3. *An arithmetic prime P_k of Λ satisfies $P_k\Lambda_h \neq \Lambda_h$ if and only if the weight k belongs to the open disc B_h.*

2.2.2. The Galois representation associated with a family of finite slope. Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_h$. The residue field $k = \mathbb{T}_h/\mathfrak{m}$ is finite. Let $\mathbb{T}_\mathfrak{m}$ denote the localization of $\mathbb{T}_h$ at $\mathfrak{m}$. Since Λ_h is henselian, $\mathbb{T}_\mathfrak{m}$ is a direct factor of $\mathbb{T}_h$, hence it is finite over Λ_h; it is also local noetherian and profinite. It is the ring of functions bounded by 1 on a connected component of $\mathcal{D}_h$. Let $W = W(k)$ be the ring of Witt vectors of k. By the universal property of W, $\mathbb{T}_\mathfrak{m}$ is a W-algebra. The affinoid domain $\mathrm{Spm}\, \mathbb{T}_\mathfrak{m}$ contains a Zariski-dense and accumulation subset of points x corresponding to cuspidal eigenforms f_x of weight $w(x) = k_x \geq 2$ and level Np. The Galois representations ρ_{f_x} associated with f_x give rise to a residual representation $\overline{\rho}\colon G_{\mathbb{Q},Np} \to \mathrm{GL}_2(k)$ that is independent of f_x. Since $\mathcal{D}_h$ is wide open, Proposition 2.1.7 gives a pseudocharacter

$$\tau_{\mathbb{T}_\mathfrak{m}}\colon G_{\mathbb{Q},Np} \to \mathbb{T}_\mathfrak{m}$$

such that for every classical point $x\colon \mathbb{T}_\mathfrak{m} \to L$, defined over some finite extension $L/\mathbb{Q}_p$, the specialization of $\tau_{\mathbb{T}_\mathfrak{m}}$ at x is the trace of the usual representation $G_{\mathbb{Q},Np} \to \mathrm{GL}_2(L)$ attached to x.

PROPOSITION 2.2.4. *If $\overline{\rho}$ is absolutely irreducible there exists a unique continuous irreducible Galois representation*

$$\rho_{\mathbb{T}_\mathfrak{m}}\colon G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathbb{T}_\mathfrak{m}),$$

lifting $\overline{\rho}$ and whose trace is $\tau_{\mathbb{T}_\mathfrak{m}}$.

This follows from a result of Nyssen [**Ny96**] and Rouquier [**Ro96**, Corollary 5.2] because $\mathbb{T}_\mathfrak{m}$ is local henselian.

We call *family of* GL_2*-eigenforms of slope bounded by* h an irreducible component of $\mathrm{Spec}\, \mathbb{T}_h$ defined by a surjective morphism $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ of Λ_h-algebras for a finite torsion-free Λ_h-algebra.

Since such a map factors via $\mathbb{T}_\mathfrak{m} \to \mathbb{I}^\circ$ for a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_h$, we can define a residual representation $\overline{\rho}\colon G_\mathbb{Q} \to \mathrm{GL}_2(k)$ associated with θ, where k is the residue field of $\mathbb{T}_\mathfrak{m}$. Suppose that $\overline{\rho}$ is irreducible. Thanks to Proposition 2.2.4 we can define a Galois representation $\rho\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ associated with θ.

2.2.3. Finite slope CM modular forms. In this section we study non-ordinary finite slope CM modular forms. We say that a family is CM if all its classical points are CM. We prove that there are no CM families of positive slope. However, contrary to the ordinary case, a non-CM family of finite positive slope may contain classical CM points of weight $k \geq 2$. Let F be an imaginary quadratic field, $\mathfrak{f}$ an integral ideal in F, $I_\mathfrak{f}$ the group of fractional ideals of F prime to $\mathfrak{f}$. Let σ_1, σ_2 be the embeddings of F into $\mathbb{C}$ (say that $\sigma_1 = \mathrm{Id}_F$) and let $(k_1, k_2) \in \mathbb{Z}^2$. A Grössencharacter ψ of infinity type (k_1, k_2) defined modulo $\mathfrak{f}$ is a homomorphism $\psi\colon I_\mathfrak{f} \to \mathbb{C}^*$ such that $\psi((\alpha)) = \sigma_1(\alpha)^{k_1}\sigma_2(\alpha)^{k_2}$ for all $\alpha \equiv 1 \pmod{^\times \mathfrak{f}}$. Consider the q-expansion

$$\sum_{\mathfrak{a}\subset\mathcal{O}_F,(\mathfrak{a},\mathfrak{f})=1} \psi(\mathfrak{a})q^{N(\mathfrak{a})},$$

where the sum is over ideals $\mathfrak{a}$ of $\mathcal{O}_F$ and $N(\mathfrak{a})$ denotes the norm of $\mathfrak{a}$. Let $F/\mathbb{Q}$ be an imaginary quadratic field of discriminant D and let ψ be a Grössencharacter of exact conductor $\mathfrak{f}$ and infinity type $(k-1, 0)$. By [**Sh71**, Lemma 3] the expansion displayed above defines a cuspidal newform $f(F, \psi)$ of level $N(\mathfrak{f})D$.

Ribet proved in [**Ri77**, Theorem 4.5] that if a newform g of weight $k \geq 2$ and level N has CM by an imaginary quadratic field F, one has $g = f(F, \psi)$ for some Grössencharacter ψ of F of infinity type $(k-1, 0)$.

Definition 2.2.5. *We say that a classical modular eigenform g of weight k and level Np has CM by an imaginary quadratic field F if its Hecke eigenvalues for the operators T_ℓ, $\ell \nmid Np$, coincide with those of $f(F, \psi)$ for some Grössencharacter ψ of F of infinity type $(k-1, 0)$. We also say that g is CM without specifying the field.*

Remark 2.2.6. *If g, F and ψ are as in the definitions above, the Galois representations $\rho_g, \rho_{f(F,\psi)}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ associated with g and $f(F, \psi)$ are isomorphic. We deduce that the image of the representation associated with a classical eigenform is contained in the normalizer of a torus in GL_2 if and only if the form is CM.*

Proposition 2.2.7. *Let g be a CM modular eigenform of weight k and level Np^m with N prime to p and $m \geq 0$. Then its p-slope is either 0, $\frac{k-1}{2}$, $k-1$ or infinite.*

Proof. Let F be the quadratic imaginary field and ψ the Grössencharacter of F associated with the CM form g by Definition 2.2.5. Let $\mathfrak{f}$ be the conductor of ψ.

We assume first that g is p-new, so that $g = f(F, \psi)$. Let a_p be the U_p-eigenvalue of g. If p is inert in F we have $a_p = 0$, so the p-slope of g is infinite. If p splits in F as $\mathfrak{p}\overline{\mathfrak{p}}$, then $a_p = \psi(\mathfrak{p}) + \psi(\overline{\mathfrak{p}})$. We can find an integer n such that $\mathfrak{p}^n$ is a principal ideal (α) with $\alpha \equiv 1 \,(\mathrm{mod}^\times \mathfrak{f})$. Hence $\psi((\alpha)) = \alpha^{k-1}$. Since α is a generator of $\mathfrak{p}^n$ we have $\alpha \in \mathfrak{p}$ and $\alpha \notin \overline{\mathfrak{p}}$; moreover $\alpha^{k-1} = \psi((\alpha)) = \psi(\mathfrak{p})^n$, so we also have $\psi(\mathfrak{p}) \in \mathfrak{p} - \overline{\mathfrak{p}}$. In the same way we find $\psi(\overline{\mathfrak{p}}) \in \overline{\mathfrak{p}} - \mathfrak{p}$. We conclude that $\psi(\mathfrak{p}) + \psi(\overline{\mathfrak{p}})$ does not belong to $\mathfrak{p}$, so its p-adic valuation is 0.

If p ramifies as $\mathfrak{p}^2$ in F, then $a_p = \psi(\mathfrak{p})$. As before we find n such that $\mathfrak{p}^n = (\alpha)$ with $\alpha \equiv 1 \,(\mathrm{mod}^\times \mathfrak{f})$. Then $(\psi(\mathfrak{p}))^n \psi(\mathfrak{p}^n) = \psi((\alpha)) = \alpha^{k-1} = \mathfrak{p}^{n(k-1)}$. By looking at p-adic valuations we find that the slope is $\frac{k-1}{2}$.

If g is not p-new, it is the p-stabilization of a CM form $f(F, \psi)$ of level prime to p. If a_p is the T_p-eigenvalue of $f(F, \psi)$, the U_p-eigenvalue of g is a root of the Hecke polynomial $X^2 - a_pX + \zeta p^{k-1}$ for some root of unity ζ. By our discussion of the p-new case, the valuation of a_p belongs to the set $\left\{0, \frac{k-1}{2}, k-1\right\}$. Then it is easy to see that the valuations of the roots of the Hecke polynomial belong to the same set. $\square$

We state a useful corollary.

Corollary 2.2.8. *There are no CM families of strictly positive slope.*

Proof. We show that the eigencurve $\mathcal{D}_h$ contains only a finite number of points corresponding to classical CM forms. It will follow that almost all classical points of a family in $\mathcal{D}_h$ are non-CM. Let f be a classical CM form of weight k and positive slope. By Proposition 2.2.7 its slope is at least $\frac{k-1}{2}$. If f corresponds to a point of $\mathcal{D}_h$ its slope must be $\leq h$, so we obtain an inequality $\frac{k-1}{2} \leq h$. The set of weights $\mathcal{K}$ satisfying this condition is finite. Since the weight map $\mathcal{D}_h \to B_h$ is finite, the set of points of $\mathcal{D}_h$ with weight in $\mathcal{K}$ is finite. Hence the number of CM forms in $\mathcal{D}_h$ is also finite. $\square$

We conclude that, in the finite positive slope case, classical CM forms can appear only as isolated points in an irreducible component of the eigencurve $\mathcal{D}_h$. In the ordinary case, the congruence ideal of a non-CM irreducible component is defined as the intersection ideal of the CM irreducible components with the given non-CM component. In the case of a positive slope family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$, we need to define the congruence ideal in a different way.

2.2.4. Construction of the congruence ideal. Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a family. We write $\mathbb{I} = \mathbb{I}^\circ[p^{-1}]$.

Fix an imaginary quadratic field F where p is inert or ramified; let $-D$ be its discriminant. Let $\mathfrak{Q}$ be a primary ideal of $\mathbb{I}$; then $\mathfrak{q} = \mathfrak{Q} \cap \Lambda_h$ is a primary ideal of Λ_h. The projection $\Lambda_h \to \Lambda_h/\mathfrak{q}$ defines a point of B_h (possibly non-reduced) corresponding to a weight $\kappa_{\mathfrak{Q}}\colon \mathbb{Z}_p^* \to (\Lambda_h/\mathfrak{q})^*$. For $r > 0$ we denote by B_r the ball of centre 1 and radius r in $\mathbb{C}_p$. By [**Bu07**, Proposition 8.3] there exists $r > 0$ and a character $\kappa_{\mathfrak{Q},r}\colon \mathbb{Z}_p^\times \cdot B_r \to (\Lambda_h/\mathfrak{q})^\times$ extending $\kappa_{\mathfrak{Q}}$.

Let σ be an embedding $F \hookrightarrow \mathbb{C}_p$. Let r and $\kappa_{\mathfrak{Q},r}$ be as above. For m sufficiently large $\sigma(1 + p^m\mathcal{O}_F)$ is contained in $\mathbb{Z}_p^\times \cdot B_r$, the domain of definition of $\kappa_{\mathfrak{Q},r}$.

For an ideal $\mathfrak{f} \subset \mathcal{O}_F$ let $I_{\mathfrak{f}}$ be the group of fractional ideals prime to $\mathfrak{f}$. For every prime ℓ not dividing Np we denote by $a_{\ell,\mathfrak{Q}}$ the image of the Hecke operator T_ℓ in $\mathbb{I}^\circ/\mathfrak{Q}$. We define here a notion of non-classical CM point of θ (hence of the eigencurve $\mathcal{D}_h$) as follows.

Definition 2.2.9. *Let $F, \sigma, \mathfrak{Q}, r, \kappa_{\mathfrak{Q},r}$ be as above. We say that $\mathfrak{Q}$ defines a CM point of weight $\kappa_{\mathfrak{Q},r}$ if there exists an integer $m > 0$, an ideal $\mathfrak{f} \subset \mathcal{O}_F$ with norm $N(\mathfrak{f})$ such that $DN(\mathfrak{f})$ divides N, a quadratic extension $(\mathbb{I}/\mathfrak{Q})'$ of $\mathbb{I}/\mathfrak{Q}$ and a homomorphism $\psi\colon I_{\mathfrak{f}p^m} \to (\mathbb{I}/\mathfrak{Q})'^\times$ such that:*

(1) $\sigma(1 + p^m\mathcal{O}_F) \subset \mathbb{Z}_p^\times \cdot B_r$;

(2) for every $\alpha \in \mathcal{O}_F$ with $\alpha \equiv 1 \,(\mathrm{mod}^\times \mathfrak{f}p^m)$, $\psi((\alpha)) = \kappa_{\mathfrak{Q},r}(\alpha)\alpha^{-1}$;

(3) $a_{\ell,\mathfrak{Q}} = 0$ if ℓ is a prime inert in F and not dividing Np;

(4) $a_{\ell,\mathfrak{Q}} = \psi(\mathfrak{l}) + \psi(\bar{\mathfrak{l}})$ if ℓ is a prime splitting as $\mathfrak{l}\bar{\mathfrak{l}}$ in F and not dividing Np.

Note that $\kappa_{\mathfrak{Q},r}(\alpha)$ is well defined thanks to condition (1).

Remark 2.2.10. *If $\mathfrak{P}$ is a prime of $\mathbb{I}$ corresponding to a classical form f then $\mathfrak{P}$ is a CM point if and only if f is a CM form in the sense of Section 2.2.3.*

Proposition 2.2.11. *The set of CM points of* $\operatorname{Spec}\mathbb{I}$ *is finite.*

Proof. Let S be the set of CM points of $\operatorname{Spec}\mathbb{I}$. By contradiction assume that S is infinite. Since $\mathbb{I}$ has Krull dimension 1, the set S is Zariski-dense in $\operatorname{Spec}\mathbb{I}$. Hence we have an injection $\mathbb{I} \hookrightarrow \prod_{\mathfrak{P}\in S} \mathbb{I}/\mathfrak{P}$. We can assume that the imaginary quadratic field of complex multiplication is constant along $\mathbb{I}$. We can also assume that the ramification of the associated Galois characters $\lambda_{\mathfrak{P}}\colon G_F \to (\mathbb{I}/\mathfrak{P})^\times$ is bounded (in support and in exponents). On the density one set of primes of F prime to $\mathfrak{f}p$ and of degree one, the characters $\lambda_{\mathfrak{P}}$ take values in the image of $\mathbb{I}^\times$, hence they define a continuous Galois character $\lambda\colon G_F \to \mathbb{I}^\times$ such that $\rho_\theta = \operatorname{Ind}_{G_F}^{G_{\mathbb{Q}}} \lambda$. We find that this is absurd by specialing at a non-CM classical point, that exists by Corollary 2.2.8. $\square$

Definition 2.2.12. *The* fortuitous CM-congruence ideal $\mathfrak{c}_\theta$ *associated with the family θ is defined as the intersection of all the primary ideals of $\mathbb{I}$ corresponding to CM points.*

We will usually refer to $\mathfrak{c}_\theta$ simply as the "congruence ideal".

Remark 2.2.13. *(Characterizations of the CM locus)*

(1) *Assume that* $\overline{\rho}_\theta = \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \overline{\lambda}$ *for a unique imaginary quadratic field* K. *Then the closed subscheme* $V(\mathfrak{c}_\theta) = \mathrm{Spec}\,\mathbb{I}/\mathfrak{c}_\theta \subset \mathrm{Spec}\,\mathbb{I}$ *is the largest subscheme on which there is an isomorphism of Galois representations* $\rho_\theta \cong \rho_\theta \otimes \left(\frac{K/\mathbb{Q}}{\bullet}\right)$. *Indeed, for every artinian* $\mathbb{Q}_p$*-algebra* A, *a CM point* $x\colon \mathbb{I} \to A$ *is characterized by the conditions* $x(T_\ell) = x(T_\ell)\left(\frac{K/\mathbb{Q}}{\ell}\right)$ *where* ℓ *varies over the primes not dividing* Np.

(2) *Note that* N *is divisible by the discriminant* D *of* K. *Assume that* $\mathbb{I}$ *is* N*-new and that* D *is prime to* N/D. *Let* W_D *be the Atkin-Lehner involution associated with* D. *Conjugation by* W_D *defines an automorphism* ι_D *of* $\mathbb{T}_h$ *and of* $\mathbb{I}$. *Then* $V(\mathfrak{c}_\theta)$ *coincides with the (schematic) invariant locus* $(\mathrm{Spec}\,\mathbb{I})^{\iota_D=1}$.

2.3. The image of the representation associated with a finite slope family

In [**Lang16**, Theorem 2.4] J. Lang shows that, under some technical hypotheses, the image of the Galois representation $\rho\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ associated with a non-CM ordinary family $\theta\colon \mathbb{T} \to \mathbb{I}^\circ$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$, where $\mathbb{I}_0^\circ$ is the subring of $\mathbb{I}^\circ$ fixed by certain "symmetries" of the representation ρ. In order to study the Galois representation associated with a non-ordinary family we will adapt some of the results in [**Lang16**] to this situation. Since the crucial step [**Lang16**, Theorem 4.3] requires the Galois ordinarity of the representation (as in [**Hi15**, Lemma 2.9]), the results of this section will not imply the existence of a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$ contained in the image of ρ. However, we will prove in later sections the existence of a "congruence Lie subalgebra" of $\mathfrak{sl}_2(\mathbb{I}_0^\circ)$ contained in a suitably defined Lie algebra of the image of ρ, by means of relative Sen theory.

For every ring R we denote by $Q(R)$ its total ring of fractions.

2.3.1. The group of self-twists of a family.

We follow [**Lang16**, Section 2] in this subsection. Let $h \in \mathbb{Q}^{+,\times}$ and let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a non-CM family of slope $\leq h$ defined over a finite torsion free Λ_h-algebra $\mathbb{I}^\circ$.

Definition 2.3.1. *We say that* $\sigma \in \mathrm{Aut}_{Q(\Lambda_h)}(Q(\mathbb{I}^\circ))$ *is a conjugate self-twist for* θ *if there exists a finite order character* $\eta_\sigma\colon G_\mathbb{Q} \to \mathbb{I}^{\circ,\times}$ *such that*

$$\sigma(\theta(T_\ell)) = \eta_\sigma(\ell)\theta(T_\ell)$$

for all but finitely many primes ℓ.

The conjugate self-twists for θ form a subgroup of $\mathrm{Aut}_{Q(\Lambda_h)}(Q(\mathbb{I}^\circ))$. We recall the following result which holds without assuming the ordinarity of θ.

Lemma 2.3.2. [**Lang16**, Lemma 7.1] Γ *is a finite abelian* $(2,2,\ldots,2)$*-group.*

We suppose from now on that $\mathbb{I}^\circ$ is normal. The only reason for this hypothesis is that in this case $\mathbb{I}^\circ$ is stable under the action of Γ on $Q(\mathbb{I}^\circ)$, which is not true in general. This makes it possible to define the subring $\mathbb{I}_0^\circ$ of elements of $\mathbb{I}^\circ$ fixed by Γ.

Remark 2.3.3. *The hypothesis of normality of* $\mathbb{I}^\circ$ *is just a simplifying one. We could work without it by introducing the* Λ_h*-order* $\mathbb{I}^\circ_{\mathrm{Tr}} = \Lambda_h[\theta(T_\ell), \ell \nmid Np]$ *in* $\mathbb{I}^\circ$*: this is an analogue of the* Λ*-order* $\mathbb{I}'$ *defined in* [**Lang16**, Section 2] *and it is stable under the action of* Γ. *This is what we will do when we study families of* GSp_4*-eigenforms in Chapter 4, where we will give a Galois-theoretic definition of* $\mathbb{I}^\circ_{\mathrm{Tr}}$.

We define two open normal subgroups of $G_{\mathbb{Q}}$ by:

$$H_0 = \bigcap_{\sigma\in\Gamma} \ker \eta_\sigma;$$
$$H = H_0 \cap \ker(\det \overline{\rho}).$$

Note that H_0 and H are open normal subgroup of $G_{\mathbb{Q}}$. a pro-p open normal subgroup of H_0 and of $G_{\mathbb{Q}}$.

2.3.2. The level of a general ordinary family. We recall the main result of [**Lang16**]. Denote by $\mathbb{T}$ the big ordinary Hecke algebra, which is finite over $\Lambda = \mathbb{Z}_p[[T]]$. Let $\theta\colon \mathbb{T} \to \mathbb{I}^\circ$ be an ordinary family with associated Galois representation $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I}^\circ)$. Recall that we fixed an embedding $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$. The representation ρ is p-ordinary, which means that its restriction $\rho|_{G_{\mathbb{Q}_p}}$ is reducible. More precisely there exist two characters $\varepsilon, \delta\colon \mathbb{G}_{\mathbb{Q}_p} \to \mathbb{I}^{\circ,\times}$, with δ unramified, such that $\rho|_{\mathbb{G}_{\mathbb{Q}_p}}$ is an extension of ε by δ.

Denote by $\mathbb{F}$ the residue field of $\mathbb{I}^\circ$ and by $\overline{\rho}$ the representation $G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ obtained by reducing ρ modulo the maximal ideal of $\mathbb{I}^\circ$. Lang introduces the following technical condition.

Definition 2.3.4. *The p-ordinary representation $\overline{\rho}$ is called H_0-regular if*

$$\overline{\varepsilon}|_{\mathbb{G}_{\mathbb{Q}_p}\cap H_0} \neq \overline{\delta}|_{G_{\mathbb{Q}_p}\cap H_0}.$$

The following is a "big image" result for ρ.

Theorem 2.3.5. ([**Lang16**, Theorem 2.4], improving [**Hi15**, Theorem I]) *Let $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ be the representation associated with an ordinary, non-CM family $\theta\colon \mathbb{T} \to \mathbb{I}^\circ$. Assume that $p > 2$, the cardinality of $\mathbb{F}$ is not 3 and the residual representation $\overline{\rho}$ is absolutely irreducible and H_0-regular. Then there exists $\gamma \in \mathrm{GL}_2(\mathbb{I}^\circ)$ such that $\gamma \cdot \mathrm{Im}\,\rho \cdot \gamma^{-1}$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$.*

One ingredient of the proof is the analogous result proved by Momose [**Mo81**] and Ribet [**Ri75**, Theorem 3.1] for the p-adic representation associated with a classical modular form (see the Introduction).

2.3.3. An approximation lemma. In this subsection we prove an analogue of [**HT15**, Lemma 4.5]. It replaces Pink's Lie algebra theory, which is relied upon in the proof of Theorem 2.3.5. Let A be a local domain that is finite torsion free over Λ_h. It does not need to be related to a Hecke algebra for the moment.

Let N be an open normal subgroup of $G_{\mathbb{Q}}$ and let $\rho\colon N \to \mathrm{GL}_2(A)$ be an arbitrary continuous representation. We denote by $\mathfrak{m}_A$ the maximal ideal of A, by $\mathbb{F}$ the residue field $A/\mathfrak{m}_A$ and by q its cardinality. In the lemma we do not suppose that ρ comes from a family of modular forms. We only assume that it satisfies the condition given by the following definition.

Definition 2.3.6. *Keep the notations as above. We say that the representation $\rho\colon N \to \mathrm{GL}_2(\mathbb{I}_0^\circ)$ is $\mathbb{Z}_p$-*regular *if there exists $d \in \mathrm{Im}\,\rho$ with eigenvalues $d_1, d_2 \in \mathbb{Z}_p^\times$ such that $d_1^2 \not\equiv d_2^2 \pmod p$. We call d a $\mathbb{Z}_p$-*regular element*. If N' is an open normal subgroup of N then we say that ρ is $(N', \mathbb{Z}_p)$-regular if $\rho|_{N'}$ is $\mathbb{Z}_p$-regular.*

Let $B^\pm$ denote the Borel subgroups consisting of upper, respectively lower, triangular matrices in GL_2. Let $U^\pm$ be the unipotent radical of $B^\pm$.

Proposition 2.3.7. *Suppose that ρ is $\mathbb{Z}_p$-regular and that a $\mathbb{Z}_p$-regular element $d \in \mathrm{Im}\,\rho$ is diagonal. Let $\mathbf{P}$ be an ideal of A and $\rho_{\mathbf{P}}\colon N \to \mathrm{GL}_2(A/\mathbf{P})$ be the representation given by the reduction of ρ modulo $\mathbf{P}$. Let $U^\pm(\rho)$ and $U^\pm(\rho_{\mathbf{P}})$ be the upper and lower unipotent subgroups of $\mathrm{Im}\,\rho$ and $\mathrm{Im}\,\rho_{\mathbf{P}}$, respectively. Then the natural maps $U^+(\rho) \to U^+(\rho_{\mathbf{P}})$ and $U^-(\rho) \to U^-(\rho_{\mathbf{P}})$ are surjective.*

REMARK 2.3.8. *The ideal* $\mathbf{P}$ *in the proposition is not necessarily prime. At a certain point we will need to take* $A = \mathbb{I}_0^\circ$ *and* $\mathbf{P} = P \cdot \mathbb{I}_0^\circ$ *for a prime ideal* P *of* Λ_h.

As in [**HT15**, Lemma 4.5] we need two lemmas. Since the argument is the same for U^+ and U^-, we only treat here the upper triangular case $U = U^+$ and $B = B^+$.

For $* = U, B$ and every $j \geq 1$ we define the groups

$$\Gamma_*(\mathbf{P}^j) = \{x \in \mathrm{SL}_2(A) \,|\, x \pmod{\mathbf{P}^j} \in *(A/\mathbf{P}^j)\}.$$

Let $\Gamma_A(\mathbf{P}^j)$ be the kernel of the reduction morphism $\pi_j \colon \mathrm{SL}_2(A) \to \mathrm{SL}_2(A/\mathbf{P}^j)$. Note that $\Gamma_U(\mathbf{P}^j) = \Gamma_A(\mathbf{P}^j)U(A)$. Let $K = \operatorname{Im}\rho$ and

$$K_U(\mathbf{P}^j) = K \cap \Gamma_U(\mathbf{P}^j), \quad K_B(\mathbf{P}^j) = K \cap \Gamma_B(\mathbf{P}^j).$$

Since $U(\mathbb{I}_0^\circ)$ and $\Gamma_{\mathbb{I}_0^\circ}(\mathbf{P})$ are p-profinite, the groups $\Gamma_U(\mathbf{P}^j)$ and $K_U(\mathbf{P}^j)$ for all $j \geq 1$ are also p-profinite. Note that

$$\left[\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} e & f \\ g & -e \end{pmatrix}\right] = \begin{pmatrix} bg - cf & 2(af - be) \\ 2(ce - ag) & cf - bg \end{pmatrix}.$$

From this we obtain the following.

LEMMA 2.3.9. *If* $X, Y \in \mathfrak{sl}_2(\mathbb{I}_0^\circ) \cap \begin{pmatrix} \mathbf{P}^j & \mathbf{P}^k \\ \mathbf{P}^i & \mathbf{P}^j \end{pmatrix}$ *for some natural numbers* i, j, k *satisfying* $i \geq j \geq k$, *then* $[X, Y] \in \begin{pmatrix} \mathbf{P}^{i+k} & \mathbf{P}^{j+k} \\ \mathbf{P}^{i+j} & \mathbf{P}^{i+k} \end{pmatrix}$.

We denote by $\mathrm{D}\Gamma_U(\mathbf{P}^j)$ the topological commutator subgroup $(\Gamma_U(\mathbf{P}^j), \Gamma_U(\mathbf{P}^j))$. Lemma 2.3.9 tells us that

$$\mathrm{D}\Gamma_U(\mathbf{P}^j) \subset \Gamma_B(\mathbf{P}^{2j}) \cap \Gamma_U(\mathbf{P}^j). \tag{2.1}$$

By assumption, there exists a diagonal $\mathbb{Z}_p$-regular element $d \in K$. Consider the element $\delta = \lim_{n \to \infty} d^{p^n}$, which belongs to K since this is p-adically complete. In particular δ normalizes K. It is also diagonal with coefficients in $\mathbb{Z}_p^\times$, so it normalizes $K_U(\mathbf{P}^j)$ and $\Gamma_B(\mathbf{P}^j)$. Since $\delta^p = \delta$, the eigenvalues δ_1 and δ_2 of δ are roots of unity of order dividing $p-1$. They still satisfy $\delta_1^2 \neq \delta_2^2$ as $p \neq 2$.

Set $\alpha = \delta_1/\delta_2 \in \mathbb{F}_p^\times$ and let a be the order of α as a root of unity. We see α as an element of $\mathbb{Z}_p^\times$ via the Teichmüller lift. Let H be a p-profinite group normalized by δ. Since H is p-profinite, every $x \in H$ has a unique a-th root. We define a map $\Delta \colon H \to H$ given by

$$\Delta(x) = [x \cdot \mathrm{ad}\,(\delta)(x)^{\alpha^{-1}} \cdot \mathrm{ad}\,(\delta^2)(x)^{\alpha^{-2}} \cdot \cdots \cdot \mathrm{ad}\,(\delta^{a-1})(x)^{\alpha^{1-a}}]^{1/a}$$

LEMMA 2.3.10. *If* $u \in \Gamma_U(\mathbf{P}^j)$ *for some* $j \geq 1$, *then* $\Delta^2(u) \in \Gamma_U(\mathbf{P}^{2j})$ *and* $\pi_j(\Delta(u)) = \pi_j(u)$.

PROOF. If $u \in \Gamma_U(\mathbf{P}^j)$, we have $\pi_j(\Delta(u)) = \pi_j(u)$ as Δ is the identity map on $U(\mathbb{I}_0^\circ/\mathbf{P}^j)$. Let $\mathrm{D}\Gamma_U(\mathbf{P}^j)$ be the topological commutator subgroup of $\Gamma_U(\mathbf{P}^j)$. Since Δ induces the projection of the $\mathbb{Z}_p$-module $\Gamma_U(\mathbf{P}^j)/\mathrm{D}\Gamma_U(\mathbf{P}^j)$ onto its α-eigenspace for $\mathrm{ad}\,(d)$, it is a projection onto $U(\mathbb{I}_0^\circ)\mathrm{D}\Gamma_U(\mathbf{P}^j)/\mathrm{D}\Gamma_U(\mathbf{P}^j)$. The fact that this is exactly the α-eigenspace comes from the Iwahori decomposition of $\Gamma_U(\mathbf{P}^j)$, that gives a similar direct sum decomposition for the abelianization $\Gamma_U(\mathbf{P}^j)/\mathrm{D}\Gamma_U(\mathbf{P}^j)$.

By (2.1) we have $\mathrm{D}\Gamma_U(\mathbf{P}^j) \subset \Gamma_B(\mathbf{P}^{2j}) \cap \Gamma_U(\mathbf{P}^j)$. Since the α-eigenspace of $\Gamma_U(\mathbf{P}^j)/\mathrm{D}\Gamma_U(\mathbf{P}^j)$ is inside $\Gamma_B(\mathbf{P}^{2j})$, Δ projects $u\Gamma_U(\mathbf{P}^j)$ to

$$\overline{\Delta}(u) \in (\Gamma_B(\mathbf{P}^{2j}) \cap \Gamma_U(\mathbf{P}^j))/\mathrm{D}\Gamma_U(\mathbf{P}^j).$$

In particular, $\Delta(u) \in \Gamma_B(\mathbf{P}^{2j}) \cap \Gamma_U(\mathbf{P}^j)$. Again apply Δ. Since $\Gamma_B(\mathbf{P}^{2j})/\Gamma_{\mathbb{I}_0^\circ}(\mathbf{P}^{2j})$ is sent to $\Gamma_U(\mathbf{P}^{2j})/\Gamma_{\mathbb{I}_0^\circ}(\mathbf{P}^{2j})$ by Δ, we get $\Delta^2(u) \in \Gamma_U(\mathbf{P}^{2j})$ as desired. $\square$

PROOF. We can now prove Proposition 2.3.7. Let $\overline{u} \in U(\mathbb{I}_0^\circ/\mathbf{P}) \cap \operatorname{Im}(\rho_{\mathbf{P}})$. Since the reduction map $\operatorname{Im}(\rho) \to \operatorname{Im}(\rho_{\mathbf{P}})$ induced by π_1 is surjective, there exists $v \in \operatorname{Im}(\rho)$ such that $\pi_1(v) = \overline{u}$. Take $u_1 \in U(\mathbb{I}_0^\circ)$ such that $\pi_1(u_1) = \overline{u}$. This is possible because $\pi_1 \colon U(\Lambda_h) \to U(\Lambda_h/P)$ is surjective. Then $vu_1^{-1} \in \Gamma_{\mathbb{I}_0^\circ}(\mathbf{P})$, so $v \in K_U(\mathbf{P})$.

By compactness of $K_U(\mathbf{P})$ and by Lemma 2.3.10, if we start with v as above the sequence $\Delta^m(v)$ converges $\mathbf{P}$-adically to an element $\Delta^\infty(v) \in U(\mathbb{I}_0^\circ) \cap K$ when $m \mapsto \infty$. Such an element satisfies $\pi_1(\Delta^\infty(v)) = \overline{u}$. $\square$

As a first application of Proposition 2.3.7 we give a result that we will need in the next subsection.

PROPOSITION 2.3.11. *Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a family of slope $\leq h$ and $\rho_\theta\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ be the representation associated with θ. Suppose that ρ_θ is $(H_0, \mathbb{Z}_p)$-regular and let ρ be a conjugate of ρ_θ such that $\mathrm{Im}\,\rho|_{H_0}$ contains a diagonal $\mathbb{Z}_p$-regular element. Then $U^+(\rho)$ and $U^-(\rho)$ are both non-trivial.*

PROOF. By density of classical points in $\mathbb{T}_h$ we can choose a prime ideal $\mathbf{P} \subset \mathbb{I}^\circ$ corresponding to a classical modular form f. The modulo $\mathbf{P}$ representation $\rho_\mathbf{P}$ is the p-adic representation classically associated with f. By the results of [**Ri75**] and [**Mo81**] and the $\mathbb{Z}_p$-regularity condition, there exists an ideal $\mathfrak{l}_\mathbf{P}$ of $\mathbb{Z}_p$ such that $\mathrm{Im}\,\rho_\mathbf{P}$ contains the congruence subgroup $\Gamma_{\mathbb{Z}_p}(\mathfrak{l}_\mathbf{P})$. In particular $U^+(\rho_\mathbf{P})$ and $U^-(\rho_\mathbf{P})$ are both non-trivial. BY Proposition 2.3.7 applied to $A = \mathbb{I}^\circ$ and the representation ρ the maps $U^+(\rho) \to U^+(\rho_\mathbf{P})$ and $U^-(\rho) \to U^-(\rho_\mathbf{P})$ are surjective, so we can find non-trivial elements in $U^+(\rho)$ and $U^-(\rho)$. $\square$

We adapt the work in [**Lang16**, Section 7] to show the following.

PROPOSITION 2.3.12. *Suppose that the representation $\rho\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ is $(H_0, \mathbb{Z}_p)$-regular. Then there exists $g \in \mathrm{GL}_2(\mathbb{I}^\circ)$ such that the conjugate representation $g\rho g^{-1}$ satisfies the following two properties:*
(1) the image of $g\rho g^{-1}|_{H_0}$ is contained in $\mathrm{GL}_2(\mathbb{I}_0^\circ)$;
(2) the image of $g\rho g^{-1}|_{H_0}$ contains a diagonal $\mathbb{Z}_p$-regular element.

PROOF. As usual we choose a $\mathrm{GL}_2(\mathbb{I}^\circ)$-conjugate of ρ such that the $\mathbb{Z}_p$-regular element d is diagonal. We still write ρ for this conjugate representation. It will turn out to have property (1).

By the definition of self-twist, for every $\sigma \in \Gamma$ there is a character $\eta_\sigma\colon G_\mathbb{Q} \to (\mathbb{I}^\circ)^\times$ and an equivalence $\rho^\sigma \cong \rho \otimes \eta_\sigma$. Then for every $\sigma \in \Gamma$ there exists $\mathbf{t}_\sigma \in \mathrm{GL}_2(\mathbb{I}^\circ)$ such that, for all $g \in G_\mathbb{Q}$,

$$\rho^\sigma(g) = \mathbf{t}_\sigma \eta_\sigma(g) \rho(g) \mathbf{t}_\sigma^{-1}. \tag{2.2}$$

We prove that the matrices $\mathbf{t}_\sigma$ are diagonal. Choose $t \in G_\mathbb{Q}$ such that $\rho(t)$ is a non-scalar diagonal element in $\mathrm{Im}\,\rho$ (for example d). Evaluating (2.2) at $g = t$ we find that $\mathbf{t}_\sigma$ must be either a diagonal or an antidiagonal matrix. Now by Proposition 2.3.11 there exists $u^+ \in G_\mathbb{Q}$ such that $\rho(u^+)$ is a non-trivial element of $\in \mathrm{Im}\,\rho \cap U^+(\mathbb{I}^\circ)$. Evaluating (2.2) at $g = u^+$ we find that $\mathbf{t}_\sigma$ cannot be antidiagonal.

It is shown in [**Lang16**, Lemma 7.3] that there exists an extension A of $\mathbb{I}^\circ$, at most quadratic, and a function $\zeta\colon \Gamma \to A^\times$ such that $\sigma \to \mathbf{t}_\sigma \zeta(\sigma)^{-1}$ defines a cocycle with values in $\mathrm{GL}_2(A)$. The proof of this result does not require the ordinarity of ρ. Equation (2.2) remains true if we replace $\mathbf{t}_\sigma$ with $\mathbf{t}_\sigma \zeta(\sigma)^{-1}$, so we can and do suppose from now on that $\mathbf{t}_\sigma$ is a cocycle with values in $\mathrm{GL}_2(A)$. In the rest of the the proof we assume for simplicity that $A = \mathbb{I}^\circ$, but everything works in the same way if A is a quadratic extension of $\mathbb{I}^\circ$ and $\mathbb{F}$ is the residue field of A.

Let $V = (\mathbb{I}^\circ)^2$ be the space on which $G_\mathbb{Q}$ acts via ρ. As in [**Lang16**, Section 7] we use the cocycle $\mathbf{t}_\sigma$ to define a twisted action of Γ on $(\mathbb{I}^\circ)^2$. For $v = (v_1, v_2) \in V$ we denote by v^σ the vector (v_1^σ, v_2^σ) with Γ acting on each coordinate. We write $v^{[\sigma]}$ for the vector $\mathbf{t}_\sigma^{-1} v^\sigma$. Then $v \to v^{[\sigma]}$ gives an action of Γ since $\sigma \mapsto \mathbf{t}_\sigma$ is a cocycle. Note that this action is $\mathbb{I}_0^\circ$-linear.

Since $\mathbf{t}_\sigma$ is diagonal for every $\sigma \in \Gamma$, the submodules $V_1 = \mathbb{I}^\circ(1,0)$ and $V_2 = \mathbb{I}^\circ(0,1)$ are stable under the action of Γ. In the following we show that each V_i contains an element fixed by Γ. We denote by $\mathfrak{m}_{\mathbb{I}^\circ}$ the maximal ideal of $\mathbb{I}^\circ$ and by $\mathbb{F}$ the residue field $\mathbb{I}^\circ/\mathfrak{m}_\mathbb{I}^\circ$. Note that the action of Γ on V_i induces an action of Γ on the one-dimensional $\mathbb{F}$-vector space $V_i \otimes \mathbb{I}^\circ/\mathfrak{m}_{\mathbb{I}^\circ}$.

We show that for each i the space $V_i \otimes \mathbb{I}^\circ/\mathfrak{m}_{\mathbb{I}^\circ}$ contains a non-zero element $\overline{v}_i$ fixed by Γ. This is a consequence of the following argument, a form of which appeared in an early preprint of [**Lang16**]. Fix a non-zero element w of $V_i \otimes \mathbb{I}^\circ/\mathfrak{m}_{\mathbb{I}^\circ}$. for $a \in \mathbb{F}$ the sum

$$S_{aw} = \sum_{\sigma\in\Gamma}(aw)^{[\sigma]}$$

is clearly Γ-invariant. We show that we can choose a such that $S_{aw} \neq 0$. Since $V_i \otimes \mathbb{I}^\circ/\mathfrak{m}_{\mathbb{I}^\circ}$ is one-dimensional, for every $\sigma \in \Gamma$ there exists $\alpha_\sigma \in \mathbb{F}$ such that $w^{[\sigma]} = \alpha_\sigma w$. Then

$$S_{aw} = \sum_{\sigma\in\Gamma}(aw)^{[\sigma]} = \sum_{\sigma\in\Gamma} a^\sigma w^{[\sigma]} = \sum_{\sigma\in\Gamma} a^\sigma \alpha_\sigma w = \left(\sum_{\sigma\in\Gamma} a^\sigma \alpha_\sigma a^{-1}\right) aw.$$

By Artin's lemma on the independence of characters, the function $f(a) = \sum_{\sigma\in\Gamma} a^\sigma\alpha_\sigma a^{-1}$ cannot be identically zero on $\mathbb{F}$. By choosing a value of a such that $f(a) \neq 0$ we obtain a non-zero element $\overline{v}_i = S_{aw}$ fixed by Γ.

We show that $\overline{v}_i$ lifts to an element $v_i \in V_i$ fixed by Γ. Let $\sigma_0 \in \Gamma$. By Lemma 2.3.2 Γ is a finite abelian 2-group, so the minimal polynomial $P_m(X)$ of $[\sigma_0]$ acting on V_i divides $X^{2^k} - 1$ for some integer k. In particular the factor $X - 1$ appears with multiplicity at most 1. We show that its multiplicity is exactly 1. If $\overline{P_m}$ is the reduction of P_m modulo $\mathfrak{m}_{\mathbb{I}^\circ}$ then $\overline{P_m}([\sigma_0]) = 0$ on $V_i \otimes \mathbb{I}^\circ/\mathfrak{m}_{\mathbb{I}^\circ}$. By our previous argument there is an element of $V_i \otimes \mathbb{I}^\circ/\mathfrak{m}_{\mathbb{I}^\circ}$ fixed by Γ (hence by $[\sigma_0]$), so we have $(X-1) \mid \overline{P_m(X)}$. Since $p > 2$ the polynomial $X^{2^k} - 1$ has no double roots modulo $\mathfrak{m}_{\mathbb{I}^\circ}$, so neither does $\overline{P_m}$. By Hensel's lemma the factor $X - 1$ lifts to a factor $X - 1$ in P_m and $\overline{v}_i$ lifts to an element $v_i \in V_i$ fixed by $[\sigma_0]$. Note that $\mathbb{I}^\circ \cdot v_i = V_i$ since $\overline{v}_i \neq 0$.

We show that v_i is fixed by Γ. Let $W_{[\sigma_0]} = \mathbb{I}^\circ v_i$ be the one-dimensional eigenspace for $[\sigma_0]$ in V_i. Since Γ is abelian $W_{[\sigma_0]}$ is stable under Γ. Let $\sigma \in \Gamma$. Since σ has order 2^k in Γ for some $k \geq 0$, there exists a root of unity ζ_σ of order 2^k satisfying $v_i^{[\sigma]} = \zeta_\sigma v_i$. Since $\overline{v}_i^{[\sigma]} = \overline{v}_i$, the reduction of ζ_σ modulo $\mathfrak{m}_{\mathbb{I}^\circ}$ must be 1. As before we conclude that $\zeta_\sigma = 1$ since $p \neq 2$.

We found two elements $v_1 \in V_1$, $v_2 \in V_2$ fixed by Γ. We show that every element of $v \in V$ fixed by Γ must belong to the $\mathbb{I}_0^\circ$-submodule generated by v_1 and v_2. We proceed as in the end of the proof of [**Lang16**, Theorem 7.5]. Since V_1 and V_2 are Γ-stable we must have $v \in V_1$ or $v \in V_2$. Suppose without loss of generality that $v \in V_1$. Then $v = \alpha v_1$ for some $\alpha \in \mathbb{I}^\circ$. If $\alpha \in \mathbb{I}_0^\circ$ then $v \in \mathbb{I}_0^\circ v_1$, as desired. If $\alpha \notin \mathbb{I}_0^\circ$ then there exists $\sigma \in \Gamma$ such that $\alpha^\sigma \neq \alpha$. Since v_1 is $[\sigma]$-invariant we obtain $(\alpha v_1)^{[\sigma]} = \alpha^\sigma v_1^{[\sigma]} = \alpha^\sigma v_1 \neq \alpha v_1$, so αv_1 is not fixed by $[\sigma]$, a contradiction.

Since (v_1, v_2) is a basis for V over $\mathbb{I}^\circ$, the $\mathbb{I}_0^\circ$-submodule $V_0 = \mathbb{I}_0^\circ v_1 + \mathbb{I}_0^\circ v_2$ is an $\mathbb{I}_0^\circ$-lattice in V. Recall that $H_0 = \bigcap_{\sigma\in\Gamma} \ker \eta_\sigma$. We show that V_0 is stable under the action of H_0 via $\rho|_{H_0}$, i.e. that if $v \in V$ is fixed by Γ, so is $\rho(h)v$ for every $h \in H_0$. This is a consequence of the following computation, where v and h are as before and $\sigma \in \Gamma$:

$$(\rho(g)v)^{[\sigma]} = \mathbf{t}_\sigma^{-1}\eta_\sigma(g)\rho(g)^\sigma v^\sigma = \mathbf{t}_\sigma^{-1}\mathbf{t}_\sigma\rho(g)\mathbf{t}_\sigma^{-1}v^\sigma = \rho(g)v^{[\sigma]}.$$

Since V_0 is an $\mathbb{I}_0^\circ$-lattice in V stable under $\rho|_{H_0}$, we conclude that $\operatorname{Im}\rho|_{H_0} \subset \mathrm{GL}_2(\mathbb{I}_0^\circ)$. □

2.3.4. Fullness of the unipotent subgroups. Upon replacing ρ by an element in its $\mathrm{GL}_2(\mathbb{I}^\circ)$ we can suppose that $\rho|_{H_0} \in \mathrm{GL}_2(\mathbb{I}_0^\circ)$. Recall that $H = \ker(\det|_{H_0})$. As in [**Lang16**, Section 4] we define a representation $H \to \mathrm{SL}_2(\mathbb{I}_0^\circ)$ by

$$\rho_0 = \rho|_H \otimes (\det\rho|_H^{-1/2}.$$

The square root of the determinant is defined thanks to the definition of H. We will use the results of [**Lang16**] to deduce that the Λ_h-module generated by the unipotent subgroups of the image of ρ_0 is big. Later we will deduce the same for ρ.

We fix from now on a height one prime $P \subset \Lambda_h$ with the following properties:

(1) there is an arithmetic prime $P_k \subset \Lambda$ satisfying $k > h + 1$ and $P = P_k\Lambda_h$;

(2) every prime $\mathfrak{P} \subset \mathbb{I}^\circ$ lying above P corresponds to a non-CM point.

Such a prime always exists. Indeed, by Remark 2.2.3 every classical weight $k > h+1$ contained in the disc B_h defines a prime $P = P_k\Lambda_h$ satisfying (1), hence such primes are Zariski-dense in Λ_h while the set of CM primes in $\mathbb{I}^\circ$ is finite by Proposition 2.2.11.

REMARK 2.3.13. *Since $k > h+1$, every point of* $\operatorname{Spec}\mathbb{T}_h$ *above P_k is classical by* [**Co96**, Theorem 6.1]. *Moreover the weight map is étale at every such point by* [**Ki03**, Theorem 11.10]. *In particular the prime $P\mathbb{I}_0^\circ = P_k\mathbb{I}_0^\circ$ splits as a product of distinct primes of $\mathbb{I}_0^\circ$.*

Make the technical assumption that the order of the residue field $\mathbb{F}$ of $\mathbb{I}^\circ$ is not 3. For every ideal $\mathbf{P}$ of $\mathbb{I}_0^\circ$ over P we let $\pi_{\mathbf{P}}$ be the projection $\mathrm{SL}_2(\mathbb{I}_0^\circ) \to \mathrm{SL}_2(\mathbb{I}_0^\circ/\mathbf{P})$. We still denote by $\pi_{\mathbf{P}}$ the restricted maps $U^\pm(\mathbb{I}_0^\circ) \to U^\pm(\mathbb{I}_0^\circ/\mathbf{P})$.

Let $G = \operatorname{Im}\rho_0$. For every ideal $\mathbf{P}$ of $\mathbb{I}_0^\circ$ we denote by $\rho_{0,\mathbf{P}}$ the representation $\pi_{\mathbf{P}} \circ \rho_0$ and by $G_{\mathbf{P}}$ the image of $\rho_{\mathbf{P}}$, so that $G_{\mathbf{P}} = \pi_{\mathbf{P}}(G)$. We state two results from Lang's work that come over unchanged to the non-ordinary setting.

PROPOSITION 2.3.14. [**Lang16**, Corollary 6.3] *Let $\mathfrak{P}$ be a prime of $\mathbb{I}_0^\circ$ over P. Then $G_{\mathfrak{P}}$ contains a congruence subgroup $\Gamma_{\mathbb{I}_0^\circ/\mathfrak{P}}(\mathfrak{a}) \subset \mathrm{SL}_2(\mathbb{I}_0^\circ/\mathfrak{P})$. In particular $G_{\mathfrak{P}}$ is open in $\mathrm{SL}_2(\mathbb{I}_0^\circ/\mathfrak{P})$.*

PROPOSITION 2.3.15. [**Lang16**, Proposition 5.1] *Assume that for every prime $\mathfrak{P} \subset \mathbb{I}_0^\circ$ over P the subgroup $G_{\mathfrak{P}}$ is open in $\mathrm{SL}_2(\mathbb{I}_0^\circ/\mathfrak{P})$. Then the image of G in $\prod_{\mathfrak{P}|P}\mathrm{SL}_2(\mathbb{I}_0^\circ/\mathfrak{P})$ through the map $\prod_{\mathfrak{P}|P}\pi_{\mathfrak{P}}$ contains a product of congruence subgroups $\prod_{\mathfrak{P}|P}\Gamma_{\mathbb{I}_0^\circ/\mathfrak{P}}(\mathfrak{a}_{\mathfrak{P}})$.*

REMARK 2.3.16. *The proofs of Propositions 2.3.14 and 2.3.15 rely on the fact that the big ordinary Hecke algebra is étale over Λ at every arithmetic point. In order for these proofs to work in the non-ordinary setting it is essential that the prime P satisfies the properties given above Remark 2.3.13.*

We let $U^\pm(\rho_0) = G \cap U^\pm(\mathbb{I}_0^\circ)$ and $U^\pm(\rho_{\mathbf{P}}) = G_{\mathbf{P}} \cap U^\pm(\mathbb{I}_0^\circ/\mathbf{P})$. We denote by $U(\rho_{\mathbf{P}})$ either the upper or lower unipotent subgroups of $G_{\mathbf{P}}$ (the choice will be fixed throughout the proof). By projecting to the upper right element we identify $U^+(\rho_0)$ with a $\mathbb{Z}_p$-submodule of $\mathbb{I}_0^\circ$ and $U^+(\rho_{0,\mathbf{P}})$ with a $\mathbb{Z}_p$-submodule of $\mathbb{I}_0^\circ/\mathbf{P}$. We make analogous identifications for the lower unipotent subgroups. We will use Proposition 2.3.15 and Proposition 2.3.7 to show that, for both signs, $U^\pm(\rho)$ spans $\mathbb{I}_0^\circ$ over Λ_h.

First we state a version of [**Lang16**, Lemma 4.10], with the same proof. Let A and B be Noetherian rings with B integral over A. We call *A-lattice* an A-submodule of B generated by the elements of a basis of $Q(B)$ over $Q(A)$.

LEMMA 2.3.17. *Any A-lattice in B contains a non-zero ideal of B. Conversely, every non-zero ideal of B contains an A-lattice.*

We prove the following proposition by means of Proposition 2.3.7. We could also use Pink theory as in [**Lang16**, Section 4].

PROPOSITION 2.3.18. *Consider $U^\pm(\rho_0)$ as subsets of $Q(\mathbb{I}_0^\circ)$. For each choice of sign the $Q(\Lambda_h)$-span of $U^\pm(\rho_0)$ is $Q(\mathbb{I}_0^\circ)$. Equivalently the Λ_h-span of $U^\pm(\rho_0)$ contains a Λ_h-lattice in $\mathbb{I}_0^\circ$.*

PROOF. Keep the notations as above. We omit the sign when writing unipotent subgroups and we refer to either the upper or lower ones (the choice is fixed throughout the proof). Let P be the prime of Λ_h chosen above. By Remark 2.3.13 the ideal $P\mathbb{I}_0^\circ$ splits as a product of distinct primes in $\mathbb{I}_0^\circ$. When $\mathfrak{P}$ varies among these primes, the map $\bigoplus_{\mathfrak{P}|P}\pi_{\mathfrak{P}}$ gives embeddings of Λ_h/P-modules $\mathbb{I}_0^\circ/P\mathbb{I}_0^\circ \hookrightarrow \bigoplus_{\mathfrak{P}|P}\mathbb{I}_0^\circ/\mathfrak{P}$ and $U(\rho_{P\mathbb{I}_0^\circ}) \hookrightarrow \bigoplus_{\mathfrak{P}|P}U(\rho_{\mathfrak{P}})$. The following diagram commutes:

$$\begin{tikzcd} U(\rho_{P\mathbb{I}_0^\circ}) \arrow[r, hook, "\bigoplus_{\mathfrak{P}|P}\pi_{\mathfrak{P}}"] \arrow[d, hook] & \bigoplus_{\mathfrak{P}|P}U(\rho_{\mathfrak{P}}) \arrow[d, hook] \\ \mathbb{I}_0^\circ/P\mathbb{I}_0^\circ \arrow[r, hook, "\bigoplus_{\mathfrak{P}|P}\pi_{\mathfrak{P}}"] & \bigoplus_{\mathfrak{P}|P}\mathbb{I}_0^\circ/\mathfrak{P} \end{tikzcd} \tag{2.3}$$

By Proposition 2.3.15 there exist ideals $\mathfrak{a}_{\mathfrak{P}} \subset \mathbb{I}_0^\circ/\mathfrak{P}$ such that

$$\left(\bigoplus_{\mathfrak{P}|P} \pi_{\mathfrak{P}}\right)(G_{P\mathbb{I}_0^\circ}) \supset \bigoplus_{\mathfrak{P}|P} \Gamma_{\mathbb{I}_0^\circ/\mathfrak{P}}(\mathfrak{a}_{\mathfrak{P}}).$$

In particular $(\bigoplus_{\mathfrak{P}|P} \pi_{\mathfrak{P}})(U(\rho_{P\mathbb{I}_0^\circ})) \supset \bigoplus_{\mathfrak{P}|P}(\mathfrak{a}_{\mathfrak{P}})$. By Lemma 2.3.17 each ideal $\mathfrak{a}_{\mathfrak{P}}$ contains a basis of $Q(\mathbb{I}_0^\circ/\mathfrak{P})$ over $Q(\Lambda_h/P)$, so that the $Q(\Lambda_h/P)$-span of $\bigoplus_{\mathfrak{P}|P} \mathfrak{a}_{\mathfrak{P}}$ is the whole $\bigoplus_{\mathfrak{P}|P} Q(\mathbb{I}_0^\circ/\mathfrak{P})$. Then the $Q(\Lambda_h/P)$-span of $(\bigoplus_{\mathfrak{P}|P} \pi_{\mathfrak{P}})(G_{\mathfrak{P}} \cap U(\rho_{\mathfrak{P}}))$ is also $\bigoplus_{\mathfrak{P}|P} Q(\mathbb{I}_0^\circ/\mathfrak{P})$. By commutativity of diagram (2.3) we deduce that the $Q(\Lambda_h/P)$-span of $G_P \cap U(\rho_{P\mathbb{I}_0^\circ})$ is $Q(\mathbb{I}_0^\circ/P\mathbb{I}_0^\circ)$. In particular $G_{P\mathbb{I}_0^\circ} \cap U(\rho_{P\mathbb{I}_0^\circ})$ contains a Λ_h/P-lattice, hence by Lemma 2.3.17 a non-zero ideal $\mathfrak{a}_P$ of $\mathbb{I}_0^\circ/P\mathbb{I}_0^\circ$.

Note that the representation $\rho_0\colon H \to \mathrm{SL}_2(\mathbb{I}_0^\circ)$ satisfies the hypotheses of Proposition 2.3.7. Indeed we assumed that the image of $\rho|_{H_0}$ contains a diagonal $\mathbb{Z}_p$-regular element d. Since H is a normal subgroup of H_0, $\rho(H)$ is a normal subgroup of $\rho(H_0)$ and it is normalized by d. By a trivial computation we see that the image of $\rho_0 = \rho \otimes (\det \rho)^{-1/2}\colon H \to \mathrm{SL}_2(\mathbb{I}_0^\circ)$ is also normalized by d.

Let $\mathfrak{a}$ be an ideal of $\mathbb{I}_0^\circ$ projecting to $\mathfrak{a}_P \subset U(\rho_{0,P\mathbb{I}_0^\circ})$. By Proposition 2.3.7 applied to ρ_0 we obtain that the map $U(\rho_0) \to U(\rho_{0,P\mathbb{I}_0^\circ})$ is surjective, so the $\mathbb{Z}_p$-module $\mathfrak{a} \cap U(\rho_0)$ also surjects to $\mathfrak{a}_P$. Since Λ_h is local we can apply Nakayama's lemma to the Λ_h-module $\Lambda_h(\mathfrak{a} \cap U(\rho_0))$ to conclude that it coincides with $\mathfrak{a}$. Hence $\mathfrak{a} \subset \Lambda_h \cdot U(\rho_0)$, so the Λ_h-span of $U(\rho_0)$ contains a Λ_h-lattice in $\mathbb{I}_0^\circ$. $\square$

We show that Proposition 2.3.18 is true if we replace ρ_0 by $\rho|_H$. This is done in [**Lang16**, Proposition 4.2] for an ordinary representation by using the description of subnormal sugroups of $\mathrm{GL}_2(\mathbb{I}^\circ)$ presented in [**Taz83**]. We will also follow this approach, but since we cannot induce a Λ_h-module structure on the unipotent subgroups of G we need a preliminary step. For a subgroup $\mathcal{G} \subset \mathrm{GL}_2(\mathbb{I}_0^\circ)$ define $\mathcal{G}^p = \{g^p,\, g \in G\}$ and $\widetilde{\mathcal{G}} = \mathcal{G}^p \cap (1 + p\mathrm{M}_2(\mathbb{I}_0^\circ))$. Let $\widetilde{\mathcal{G}}^{\Lambda_h}$ be the subgroup of $\mathrm{GL}_2(\mathbb{I}^\circ)$ generated by the set $\{g^\lambda\colon g \in \widetilde{\mathcal{G}}, \lambda \in \Lambda_h\}$ where $g^\lambda = \exp(\lambda \log g)$. We have the following.

Lemma 2.3.19. *The group $\widetilde{\mathcal{G}}^{\Lambda_h}$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$ if and only if both the unipotent subgroups $\mathcal{G} \cap U^+(\mathbb{I}_0^\circ)$ and $\mathcal{G} \cap U^-(\mathbb{I}_0^\circ)$ contain a basis of a Λ_h-lattice in $\mathbb{I}_0^\circ$.*

Proof. It is easy to see that $\mathcal{G} \cap U^+(\mathbb{I}_0^\circ)$ contains the basis of a Λ_h-lattice in $\mathbb{I}_0^\circ$ if an only if the same is true for $\widetilde{\mathcal{G}} \cap U^+(\mathbb{I}_0^\circ)$. The same is true for U^-. By a standard argument, used in the proofs of [**Hi15**, Lemma 2.9] and [**Lang16**, Proposition 4.3], $\mathcal{G}^{\Lambda_h} \subset \mathrm{GL}_2(\mathbb{I}_0^\circ)$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$ if and only if both its upper and lower unipotent subgroup contain an ideal of $\mathbb{I}_0^\circ$. We have $U^+(\mathbb{I}_0^\circ) \cap \mathcal{G}^{\Lambda_h} = \Lambda_h(\mathcal{G} \cap U^+(\mathbb{I}_0^\circ))$, so by Lemma 2.3.17 $U^+(\mathbb{I}_0^\circ) \cap \mathcal{G}^{\Lambda_h}$ contains an ideal of $\mathbb{I}_0^\circ$ if and only if $\mathcal{G} \cap U^+(\mathbb{I}_0^\circ)$ contains a basis of a Λ_h-lattice in $\mathbb{I}_0^\circ$. We proceed in the same way for U^-. $\square$

Now let $G_0 = \mathrm{Im}\,\rho|_H$, $G = \mathrm{Im}\,\rho_0$. Note that $G_0 \cap \mathrm{SL}_2(\mathbb{I}_0^\circ)$ is a normal subgroup of G. Let $f\colon \mathrm{GL}_2(\mathbb{I}_0^\circ) \to \mathrm{SL}_2(\mathbb{I}_0^\circ)$ be the homomorphism sending g to $\det(g)^{-1/2}g$. We have $G = f(G_0)$ by definition of ρ_0. We show the following.

Proposition 2.3.20. *The subgroups $G_0 \cap U^\pm(\mathbb{I}_0^\circ)$ both contain the basis of a Λ_h-lattice in $\mathbb{I}_0^\circ$ if and only if $G \cap U^\pm(\mathbb{I}_0^\circ)$ both contain the basis of a Λ_h-lattice in $\mathbb{I}_0^\circ$.*

Proof. Since $G = f(G_0)$ we have $\widetilde{G} = f(\widetilde{G_0})$. This implies $\widetilde{G}^{\Lambda_h} = f(\widetilde{G_0}^{\Lambda_h})$. We remark that $\widetilde{G_0}^{\Lambda_h} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ)$ is a normal subgroup of $\widetilde{G}^{\Lambda_h}$. Indeed $\widetilde{G_0}^{\Lambda_h} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ)$ is normal in $\widetilde{G_0}^{\Lambda_h}$, so its image $f(G_0^{\Lambda_h} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ)) = G_0^{\Lambda_h} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ)$ is normal in $f(G_0^{\Lambda_h}) = \widetilde{G}^{\Lambda_h}$.

By [**Taz83**, Corollary 1] a subgroup of $\mathrm{GL}_2(\mathbb{I}_0^\circ)$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$ if and only if it is subnormal in $\mathrm{GL}_2(\mathbb{I}_0^\circ)$ and it is not contained in the centre. We note that $\widetilde{G_0}^{\Lambda_h} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ) = (\widetilde{G_0} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ))^{\Lambda_h}$ is not contained in the subgroup $\{\pm 1\}$. Otherwise also $\widetilde{G_0} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ)$ would be contained in $\{\pm 1\}$ and $\mathrm{Im}\,\rho \cap \mathrm{SL}_2(\mathbb{I}_0^\circ)$ would be finite, since $\widetilde{G_0}$ is of finite index in G_0^p. This would give a contradiction: indeed if $\mathfrak{P}$ is an arithmetic prime of $\mathbb{I}^\circ$ of weight greater than 1 and $\mathfrak{P}' = \mathfrak{P} \cap \mathbb{I}_0^\circ$, the image of ρ modulo $\mathfrak{P}'$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ/\mathfrak{P}')$ by the result of [**Ri75**].

Now since $\widetilde{G_0}^{\Lambda_h} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ)$ is a normal subgroup of $\widetilde{G}^{\Lambda_h}$, we deduce by Tazhetdinov's result that $\widetilde{G_0}^{\Lambda_h} \cap \mathrm{SL}_2(\mathbb{I}_0^\circ)$ (hence $\widetilde{G_0}^{\Lambda_h}$) contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$ if and only if $\widetilde{G}^{\Lambda_h}$ does. By applying Lemma 2.3.19 to $\mathcal{G} = G_0$ and $\mathcal{G} = G$ we obtain the desired equivalence. □

By combining Propositions 2.3.18 and 2.3.20 we obtain the following.

COROLLARY 2.3.21. *The Λ_h-span of each of the unipotent subgroups $U^\pm(\rho)$ contains a Λ_h-lattice in $\mathbb{I}_0^\circ$.*

Unlike in the ordinary case we cannot deduce from the corollary that $\mathrm{Im}\,\rho$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_0^\circ)$, since we cannot induce a Λ_h-module structure (not even a Λ-module structure) on $\mathrm{Im}\,\rho \cap U^\pm$. The proofs of [**Hi15**, Lemma 2.9] and [**Lang16**, Proposition 4.3] rely on the existence, in the image of the Galois group, of an element inducing by conjugation a Λ-module structure on $\mathrm{Im}\,\rho \cap U^\pm$. In their situation this is predicted by the condition of Galois ordinarity of ρ. In the non-ordinary case we will find an element with a similar property via relative Sen theory. This will force us to state a "big image" result in terms of Lie algebras rather than groups.

2.4. Relative Sen theory

We use the notations of Section 2.2.1.

We defined in Section 2.3.1 a subring $\mathbb{I}_0^\circ \subset \mathbb{I}^\circ$, finite over Λ_h. Let $\mathbb{I}_{r_i}^\circ = \mathbb{I}^\circ \widehat{\otimes}_{\Lambda_h} A_{r_i}^\circ$ and $\mathbb{I}_{0,r_i}^\circ = \mathbb{I}_0^\circ \widehat{\otimes}_{\Lambda_h} A_{0,r_i}^\circ$, both endowed with their p-adic topology. Note that $(\mathbb{I}_{r_i}^\circ)^\Gamma = \mathbb{I}_{r_i,0}^\circ$.

Consider the representation $\rho\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ associated with a family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$. We observe that ρ is continuous with respect to the profinite topology of $\mathbb{I}^\circ$ but not with respect to the p-adic topology. For this reason we cannot apply Sen theory to ρ. We fix instead an arbitrary radius r among the r_i defined above and consider the representation $\rho_r\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}_r^\circ)$ obtained by composing ρ with the inclusion $\mathrm{GL}_2(\mathbb{I}^\circ) \hookrightarrow \mathrm{GL}_2(\mathbb{I}_r^\circ)$. This inclusion is continuous, hence the representation ρ_r is continuous with respect to the p-adic topology of $\mathbb{I}_{r,0}^\circ$.

Recall from Proposition 2.3.12 that, possibly after replacing ρ by a conjugate, the restriction $\rho|_{H_0}$ takes values in $\mathrm{GL}_2(\mathbb{I}_0^\circ)$ and is $\mathbb{Z}_p$-regular. Then $\rho_r|_{H_0}\colon H_0 \to \mathrm{GL}_2(\mathbb{I}_{r,0}^\circ)$ is continuous with respect to the p-adic topology on $\mathrm{GL}_2(\mathbb{I}_{r,0}^\circ)$.

2.4.1. Big Lie algebras. Recall that we fixed an embedding $G_{\mathbb{Q}_p} \subset G_\mathbb{Q}$. Let G_r and G_r^{loc} be the images of H_0 and $G_p \cap H_0$, respectively, under the representation $\rho_r|_{H_0}\colon H_0 \to \mathrm{GL}_2(\mathbb{I}_{r,0}^\circ)$. Note that they are actually independent of r as topological Lie groups.

For every ring R and ideal $I \subset R$ we denote by $\Gamma_{\mathrm{GL}_2(R)}(I)$ the congruence subgroup of $\mathrm{GL}_2(R)$ consisting of the elements $g \in \mathrm{GL}_2(R)$ such that $g \equiv \mathbb{1}_2 \pmod{I}$. Let $G_r' = G_r \cap \Gamma_{\mathrm{GL}_2(\mathbb{I}_{r,0}^\circ)}(p)$ and $G_r^{\prime,\mathrm{loc}} = G_r^{\mathrm{loc}} \cap \Gamma_{\mathrm{GL}_2(\mathbb{I}_{r,0}^\circ)}(p)$, so that G_r' and $G_r^{\prime,\mathrm{loc}}$ are pro-p groups. Note that the congruence subgroups $\Gamma_{\mathrm{GL}_2(\mathbb{I}_{r,0})}(p^m)$ are open in $\mathrm{GL}_2(\mathbb{I}_{r,0})$ for the p-adic topology. In particular G_r' and $G_r^{\prime,\mathrm{loc}}$ can be identified with the images under ρ of the absolute Galois groups of finite extensions of $\mathbb{Q}$ and $\mathbb{Q}_p$, respectively.

REMARK 2.4.1. *We choose an arbitrary r_0 and we set $G_r' = G_r \cap \Gamma_{\mathrm{GL}_2(\mathbb{I}_{0,r_0}^\circ)}(p)$ for every r. Then G_r' is independent of r as a topological group, since G_r is, and it is a pro-p subgroup of G_r for every r. We define in the same way $G_r^{\prime,\mathrm{loc}}$. This will be important in Section 2.6.1 when we take projective limits over r of various objects.*

We set $A_r = A_r^\circ[p^{-1}]$ and $\mathbb{I}_{r,0} = \mathbb{I}_{r,0}^\circ[p^{-1}]$. We consider from now on G_r' and $G_r'^{,\mathrm{loc}}$ as subgroups of $\mathrm{GL}_2(\mathbb{I}_{r,0})$ via the inclusion $\mathrm{GL}_2(\mathbb{I}_{r,0}^\circ) \hookrightarrow \mathrm{GL}_2(A_r)$.

We define Lie algebras associated with the groups G_r' and $G_r'^{,\mathrm{loc}}$. For every non-zero ideal $\mathfrak{a}$ of A_r we denote by $G'_{r,\mathfrak{a}}$ and $G'^{,\mathrm{loc}}_{r,\mathfrak{a}}$ the images of G_r' and $G_r'^{,\mathrm{loc}}0$, respectively, under the natural projection $\mathrm{GL}_2(\mathbb{I}_{r,0}) \to \mathrm{GL}_2(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})$. The pro-$p$ groups $G'_{r,\mathfrak{a}}$ and $G'^{,\mathrm{loc}}_{r,\mathfrak{a}}$ are topologically of finite type so we can define the corresponding $\mathbb{Q}_p$-Lie algebras $\mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a}}$ using the p-adic logarithm map. We set $\mathfrak{G}_{r,\mathfrak{a}} = \mathbb{Q}_p \cdot \mathrm{Log}\, G'_{r,\mathfrak{a}}$ and $\mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a}} = \mathbb{Q}_p \cdot \mathrm{Log}\, G'^{,\mathrm{loc}}_{r,\mathfrak{a}}$. They are closed Lie subalgebras of the finite dimensional $\mathbb{Q}_p$-Lie algebra $\mathrm{M}_2(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})$.

Let $P_1 = (u^{-1}(1+T)-1)\cdot A_r$. Let $B_r = \varprojlim_{(\mathfrak{a},P_1)=1} A_r/\mathfrak{a}A_r$ where the inverse limit is taken over the non-zero ideals $\mathfrak{a} \subset A_r$ prime to P_1, with respect to the natural transition maps. The reason for excluding P_1 will be clear later. We endow B_r with the projective limit topology coming from the p-adic topology on each quotient. We have an isomorphism of $\mathbb{Q}_p$-algebras

$$B_r \cong \prod_{P \neq P_1} \widehat{(A_r)}_P,$$

where the product is over primes P of A_r and $\widehat{(A_r)}_P = \varprojlim_{m\geq 1} A_r/P^m$, that is an inverse limit of finite dimensional $\mathbb{Q}_p$-vector spaces, hence a $\mathbb{Q}_p$-Fréchet space for the natural family of seminorms. Similarly, let $\mathbb{B}_r = \varprojlim_{(\mathfrak{a},P_1)=1} \mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0}$, where as before $\mathfrak{a}$ varies over all the non-zero ideals of A_r prime to P_1. We have an isomorphism of $\mathbb{Q}_p$-algebras

$$\mathbb{B}_r \cong \prod_{P \neq P_1} \widehat{(\mathbb{I}_{r,0})}_{P\mathbb{I}_{r,0}} \cong \prod_{\mathfrak{P}\nmid P_1} \widehat{(\mathbb{I}_{r,0})}_{\mathfrak{P}} \cong \varprojlim_{(\mathfrak{Q},P_1)=1} \mathbb{I}_{r,0}/\mathfrak{Q},$$

where the second product is over primes $\mathfrak{P}$ of $\mathbb{I}_{r,0}$ and the projective limit is over primary ideals $\mathfrak{Q}$ of $\mathbb{I}_{r,0}$. Here $\widehat{(\mathbb{I}_{r,0})}_{\mathfrak{P}} = \varprojlim_{m\geq 1} \mathbb{I}_{r,0}/\mathfrak{P}^m$, that is again an inverse limit of finite dimensional $\mathbb{Q}_p$-vector spaces, hence a $\mathbb{Q}_p$-Fréchet space for the natural family of seminorms. The rightmost isomorphism follows from the fact that $\mathbb{I}_{r,0}$ is finite over A_r, so there is an isomorphism of $\mathbb{Q}_p$-FrÃ©chet spaces $\mathbb{I}_{r,0} \otimes \widehat{(A_r)}_P = \prod_{\mathfrak{P}} \widehat{(\mathbb{I}_{r,0})}_{\mathfrak{P}}$ where P is a prime of A_r and $\mathfrak{P}$ varies among the primes of $\mathbb{I}_{r,0}$ above P. We have natural continuous inclusions $A_r \hookrightarrow B_r$ and $\mathbb{I}_{r,0} \hookrightarrow \mathbb{B}_r$, both with dense image. The map $A_r \hookrightarrow \mathbb{I}_{r,0}$ induces an inclusion $B_r \hookrightarrow \mathbb{B}_r$ with closed image. We will work with $\mathbb{B}_r$ for the rest of this section, but we will need B_r later.

For every $\mathfrak{a}$ we defined Lie algebras $\mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a}}$ associated with the finite type Lie groups $G'_{r,\mathfrak{a}}$ and $G'^{,\mathrm{loc}}_{r,\mathfrak{a}}$. We take the projective limit of these algebras to obtain Lie subalgebras of $\mathrm{M}_2(\mathbb{B}_r)$.

Definition 2.4.2. *The Lie algebras associated with G_r' and $G_r'^{,\mathrm{loc}}$ are the closed $\mathbb{Q}_p$-Lie subalgebras of $\mathrm{M}_2(\mathbb{B}_r)$ given respectively by*

$$\mathfrak{G}_r = \varprojlim_{(\mathfrak{a},P_1)=1} \mathfrak{G}_{r,\mathfrak{a}}$$

and

$$\mathfrak{G}_r^{\mathrm{loc}} = \varprojlim_{(\mathfrak{a},P_1)=1} \mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a}},$$

where as usual the limits are taken over the non-zero ideals $\mathfrak{a} \subset A_r$ prime to P_1.

For every ideal $\mathfrak{a}$ of A_r prime to P_1, we have continuous surjective homomorphisms $\mathfrak{G}_r \to \mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{G}_r^{\mathrm{loc}} \to \mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a}}$.

Remark 2.4.3. *The limits in Definition 2.4.2 can be replaced by limits over primary ideals of $\mathbb{I}_{r,0}$. Explicitly, let $\mathfrak{Q}$ be a primary ideal of $\mathbb{I}_{r,0}$. Let $G'_{r,\mathfrak{Q}}$ be the image of G_r' via the natural* *projection* $\mathrm{GL}_2(\mathbb{I}_{r,0}) \to \mathrm{GL}_2(\mathbb{I}_{r,0}/\mathfrak{Q})$ *and let* $\mathfrak{G}_{r,\mathfrak{Q}}$ *be the* $\mathbb{Q}_p$*-Lie algebra associated with* $G'_{r,\mathfrak{Q}}$ *(which is a finite type Lie group). We have an isomorphism of topological* $\mathbb{Q}_p$*-Lie algebras*

$$\mathfrak{G}_r = \varprojlim_{(\mathfrak{Q},P_1)=1} \mathfrak{G}_{r,\mathfrak{Q}},$$

where the limit is taken over primary ideals $\mathfrak{Q}$ *of* $\mathbb{I}_{r,0}$ *and the topology on the right is the projective limit one.*

2.4.2. The Sen operator associated with a Galois representation. Let K and L be two p-adic fields, following [**Sen93**]. We recall the definition of the Sen operator associated with a representation $\tau\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_m(\mathscr{R})$ where $\mathscr{R}$ is an L-Banach algebra. We can suppose $L \subset K$; if this is not true we restrict τ to the open subgroup $\mathrm{Gal}(\overline{K}/KL) \subset \mathrm{Gal}(\overline{K}/K)$.

Let L_∞ be a totally ramified $\mathbb{Z}_p$-extension of L. Let γ be a topological generator of $\Gamma = \mathrm{Gal}(L_\infty/L)$, Γ_n be the subgroup of Γ generated by γ^{p^n} and $L_n = L_\infty^{\gamma^{p^n}}$, so that $L_\infty = \cup_n L_n$. Let $L'_n = L_nK$ and $G'_n = \mathrm{Gal}(\overline{L}/L'_n)$. If $\mathscr{R}^m$ is the $\mathscr{R}$-module over which $\mathrm{Gal}(\overline{K}/K)$ acts via τ, define an action of $\mathrm{Gal}(\overline{K}/K)$ on $\mathscr{R}\widehat{\otimes}_L\mathbb{C}_p$ by letting $\sigma \in \mathrm{Gal}(\overline{K}/K)$ map $x\otimes y$ to $\tau(\sigma)(x)\otimes\sigma(y)$. Then by the results of [**Sen73**] and [**Sen93**] there is a matrix $M \in \mathrm{GL}_m\left(\mathscr{R}\widehat{\otimes}_L\mathbb{C}_p\right)$, an integer $n \geq 0$ and a representation $\delta\colon \Gamma_n \to \mathrm{GL}_m(\mathscr{R} \otimes_L L'_n)$ such that for all $\sigma \in G'_n$

$$M^{-1}\tau(\sigma)\sigma(M) = \delta(\sigma).$$

Definition 2.4.4. *The Sen operator associated with* τ *is*

$$\phi = \lim_{\sigma\to 1} \frac{\log(\delta(\sigma))}{\log(\chi(\sigma))} \in \mathrm{M}_m(\mathscr{R}\widehat{\otimes}_L\mathbb{C}_p).$$

The limit exists as for σ close to 1 the map $\sigma \mapsto \dfrac{\log(\delta(\sigma))}{\log(\chi(\sigma))}$ is constant. It is proved in [**Sen93**, Section 2.4] that ϕ does not depend on the choice of δ and M.

If $L = \mathscr{R} = \mathbb{Q}_p$, we define the Lie algebra $\mathfrak{g}$ associated with $\tau(\mathrm{Gal}(\overline{K}/K))$ as the $\mathbb{Q}_p$-vector space generated by $\mathrm{Log}\,(\tau(\mathrm{Gal}(\overline{K}/K)))$ in $\mathrm{M}_m(\mathbb{Q}_p)$. In this situation the Sen operator ϕ associated with τ has the following property.

Theorem 2.4.5. [**Sen73**, Theorem 1] *For a continuous representation* $\tau\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_m(\mathbb{Q}_p)$*, the Lie algebra* $\mathfrak{g}$ *of the group* $\tau(\mathrm{Gal}(\overline{K}/K))$ *is the smallest* $\mathbb{Q}_p$*-subspace of* $\mathrm{M}_m(\mathbb{Q}_p)$ *such that* $\mathfrak{g}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$ *contains* ϕ*.*

The proof of this theorem relies heavily on the fact that the image of the Galois group is a finite dimensional Lie group. It is doubtful that its proof can be generalized to the relative case.

2.4.3. The Sen operator associated with ρ_r. Recall that we fixed a finite extension K_r of $\mathbb{Q}_p$ such that $G_r^{\prime,\mathrm{loc}}$ is the image of $\rho|_{\mathrm{Gal}(\overline{K_r}/K_r)}$ and, for an ideal $P \subset A_r$ and $m \geq 1$, $G_{r,P^m}^{\prime,\mathrm{loc}}$ is the image of $\rho_{r,P^m}|_{\mathrm{Gal}(\overline{K_r}/K_r)}$. From now on we imply write $K = K_r$, noting that for the moment r is fixed. Following [**Sen73**] and [**Sen93**] we can define a Sen operator associated with $\rho_r|_{\mathrm{Gal}(\overline{K}/K)}$ and $\rho_{r,P^m}|_{\mathrm{Gal}(\overline{K}/K)}$ for every ideal $P \subset A_r$ and every $m \geq 1$. We will see that these operators satisfy a compatibility property. We write for the rest of the section ρ_r and ρ_{r,P^m} while implicitly taking the domain to be $\mathrm{Gal}(\overline{K}/K)$.

Set $\mathbb{I}_{0,r,\mathbb{C}_p} = \mathbb{I}_{r,0}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$. It is a $\mathbb{C}_p$-Banach space. Let $\mathbb{B}_{r,\mathbb{C}_p} = \mathbb{B}_r\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$; it is the topological $\mathbb{C}_p$-algebra completion of $\mathbb{B}_r \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ for the (uncountable) set of nuclear seminorms $p_\mathfrak{a}$ induced by the p-adic norms on the quotients $\mathbb{I}_{0,r,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{0,r,\mathbb{C}_p}$ via the specialization morphisms $\pi_\mathfrak{a}\colon \mathbb{B}_r \otimes_{\mathbb{Q}_p} \mathbb{C}_p \to \mathbb{I}_{0,r,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{0,r,\mathbb{C}_p}$. Let $\mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p} = \mathfrak{G}_{r,\mathfrak{a}} \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ and $\mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p}^{\mathrm{loc}} = \mathfrak{G}_{r,\mathfrak{a},}^{\mathrm{loc}} \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. Then we define $\mathfrak{G}_{r,\mathbb{C}_p} = \mathfrak{G}_r\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$ as the topological $\mathbb{C}_p$-Lie algebra completion of $\mathfrak{G}_r \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ for the (uncountable) set of seminorms $p_\mathfrak{a}$ induce by the p-adic norms on $\mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p}$ via the specialization morphisms $\pi_{\mathfrak{a}}\colon \mathfrak{G}_{r,} \otimes_{\mathbb{Q}_p} \mathbb{C}_p \to \mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p}$. We also define $\mathfrak{G}^{\mathrm{loc}}_{r,\mathbb{C}_p} \mathfrak{G}_r \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p$ and give it the topology induced by the p-adic norms $\mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}$. Note that we have isomorphisms of $\mathbb{C}_p$-Banach spaces

$$\mathfrak{G}_{r,\mathbb{C}_p} \cong \varprojlim_{(\mathfrak{a},P_1)=1} \mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p} \quad \text{and} \quad \mathfrak{G}^{\mathrm{loc}}_{r,\mathbb{C}_p} \cong \varprojlim_{(\mathfrak{a},P_1)=1} \mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}.$$

We apply the construction of the previous subsection to $L = \mathbb{Q}_p$, $\mathscr{R} = \mathbb{I}_{r,0}$, which is a $\mathbb{Q}_p$-Banach algebra for the p-adic topology, and $\tau = \rho_r$. We obtain an operator $\phi_r \in \mathrm{M}_2(\mathbb{I}_{0,r,\mathbb{C}_p})$. Recall that we have a natural continuous inclusion $\mathbb{I}_{r,0} \hookrightarrow \mathbb{B}_r$, inducing inclusions $\mathbb{I}_{0,r,\mathbb{C}_p} \hookrightarrow \mathbb{B}_{r,\mathbb{C}_p}$ and $\mathrm{M}_2(\mathbb{I}_{0,r,\mathbb{C}_p}) \hookrightarrow \mathrm{M}_2(\mathbb{B}_{r,\mathbb{C}_p})$. We denote all these inclusions by $\iota_{\mathbb{B}_r}$ since it will be clear each time to which one we are referring. We will prove in this section that $\iota_{\mathbb{B}_r}(\phi_r)$ is an element of $\mathfrak{G}^{\mathrm{loc}}_{r,\mathbb{C}_p}$.

Let $\mathfrak{a}$ be a non-zero ideal of A_r. Let us apply Sen's construction to $L = \mathbb{Q}_p$, $\mathscr{R} = \mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0}$ and $\tau = \rho_{r,\mathfrak{a}}\colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})$; we obtain an operator $\phi_{r,\mathfrak{a}} \in \mathrm{M}_2((\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p)$.

Let

$$\pi_{\mathfrak{a}}\colon \mathrm{M}_2(\mathbb{I}_{r,0}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p) \to \mathrm{M}_2((\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p)$$

and

$$\pi^{\times}_{\mathfrak{a}}\colon \mathrm{GL}_2(\mathbb{I}_{r,0}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p) \to \mathrm{GL}_2((\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p)$$

be the natural projections.

Proposition 2.4.6. *We have $\phi_{r,\mathfrak{a}} = \pi_{\mathfrak{a}}(\phi_r)$ for all $\mathfrak{a}$.*

Proof. Recall from the construction of ϕ_r that there exists $M \in \mathrm{GL}_2\left(\mathbb{I}_{0,r,\mathbb{C}_p}\right)$, $n \geq 0$ and $\delta\colon \Gamma_n \to \mathrm{GL}_2(\mathbb{I}_{r,0}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{Q}_{p}{}'_n)$ such that for all $\sigma \in G'_n$ we have

$$M^{-1}\rho_r(\sigma)\sigma(M) = \delta(\sigma) \tag{2.4}$$

and

$$\phi_r = \lim_{\sigma\to 1} \frac{\log(\delta(\sigma))}{\log(\chi(\sigma))}. \tag{2.5}$$

Let $M_{\mathfrak{a}} = \pi^{\times}_{\mathfrak{a}}(M) \in \mathrm{GL}_2(\mathbb{I}_{0,r,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{0,r,\mathbb{C}_p})$ and $\delta_{\mathfrak{a}} = \pi^{\times}_{\mathfrak{a}} \circ \delta\colon \Gamma_n \to \mathrm{GL}_2((\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{Q}_{p}{}'_n)$ Denote by $\phi_{r,\mathfrak{a}} \in \mathrm{M}_2((\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{Q}_{p}{}'_n)$ the Sen operator associated with $\rho_{r,\mathfrak{a}}$. Now (2.4) gives

$$M_{\mathfrak{a}}^{-1}\rho_{r,\mathfrak{a}}(\sigma)\sigma(M_{\mathfrak{a}}) = \delta_{\mathfrak{a}}(\sigma) \tag{2.6}$$

so we can calculate $\phi_{r,\mathfrak{a}}$ as

$$\phi_{r,\mathfrak{a}} = \lim_{\sigma\to 1} \frac{\log(\delta_{\mathfrak{a}}(\sigma))}{\log(\chi(\sigma))}, \tag{2.7}$$

that is an element of $\mathrm{M}_2(\mathscr{R}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p)$.
By comparing this with (2.5) we see that $\phi_{r,\mathfrak{a}} = \pi_{\mathfrak{a}}(\phi_r)$. $\square$

Let $\phi_{r,\mathbb{B}_r} = \iota_{\mathbb{B}_r}(\phi_r)$. For a non-zero ideal $\mathfrak{a}$ of A_r let $\pi_{\mathbb{B}_r,\mathfrak{a}}$ be the natural projection $\mathbb{B}_r \to \mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0}$. Clearly $\pi_{\mathbb{B}_r,\mathfrak{a}}(\phi_{r,\mathbb{B}_r}) = \pi_{\mathfrak{a}}(\phi_r)$ and $\phi_{r,\mathfrak{a}} = \pi_{\mathfrak{a}}(\phi_r)$ by Proposition 2.4.6, so we have $\phi_{r,\mathbb{B}_r} = \varprojlim_{(\mathfrak{a},P_1)=1} \phi_{r,\mathfrak{a}}$.

We use Theorem 2.4.5 to show the following.

Proposition 2.4.7. *Let $\mathfrak{a}$ be a non-zero ideal of A_r prime to P_1. The operator $\phi_{r,\mathfrak{a}}$ belongs to the Lie algebra $\mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}$.*

Proof. Let n be the dimension over $\mathbb{Q}_p$ of $\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0}$. By choosing a $\mathbb{Q}_p$-basis $(\omega_1, \ldots, \omega_n)$ of this algebra, we can define an injective ring morphism $\alpha\colon \mathrm{M}_2(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0}) \hookrightarrow \mathrm{M}_{2n}(\mathbb{Q}_p)$ and an injective group morphism $\alpha^{\times}\colon \mathrm{GL}_2(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{Q}_p)$. In fact, an endomorphism f of the $(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})$-module $(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})^2 = (\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0}) \cdot e_1 \oplus (\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0}) \cdot e_2$ is $\mathbb{Q}_p$-linear, so it induces an endomorphism $\alpha(f)$ of the $\mathbb{Q}_p$-vector space $(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})^2 = \bigoplus_{i,j} \mathbb{Q}_p \cdot \omega_i e_j$; furthermore if α is an automorphism then $\alpha(f)$ is one too. In particular $\rho_{r,\mathfrak{a}}$ induces a representation

$\rho_{r,\mathfrak{a}}^{\alpha} = \alpha^{\times} \circ \rho_{r,\mathfrak{a}} \colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{2n}(\mathbb{Q}_p)$. The image of $\rho_{r,\mathfrak{a}}^{\alpha}$ is the group $G_{r,\mathfrak{a}}^{\mathrm{loc},\alpha} = \alpha^{\times}(G_{r,\mathfrak{a}}^{\mathrm{loc}})$. We consider its Lie algebra $\mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc},\alpha} = \mathbb{Q}_p \cdot \mathrm{Log}\ (G_{r,\mathfrak{a}}^{\mathrm{loc},\alpha}) \subset \mathrm{M}_{2n}(\mathbb{Q}_p)$. The p-adic logarithm commutes with α in the sense that $\alpha(\mathrm{Log}\ x) = \mathrm{Log}\ (\alpha^{\times}(x))$ for every $x \in \Gamma_{\mathbb{I}_{r_0,0}/\mathfrak{a}\mathbb{I}_{r_0,0}}(p)$, where r_0 is the radius chosen in Remark 2.4.1, so we have $\mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc},\alpha} = \alpha(\mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc}})$ (recall that $\mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc}} = \mathbb{Q}_p \cdot \mathrm{Log}\ G_{r,\mathfrak{a}}^{\mathrm{loc}}$).

Let $\phi_{r,\mathfrak{a}}^{\alpha} \in \mathrm{M}_{2n}(\mathbb{C}_p)$ be the Sen operator associated with $\rho_{r,\mathfrak{a}}^{\alpha} \colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_{2n}(\mathbb{Q}_p)$. By Theorem 2.4.5 we have $\phi_{r,\mathfrak{a}}^{\alpha} \in \mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p}^{\mathrm{loc},\alpha} = \mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc},\alpha} \widehat{\otimes} \mathbb{C}_p$. Let $\alpha_{\mathbb{C}_p} = \alpha \widehat{\otimes} 1 \colon \mathrm{M}_2(\mathbb{I}_{0,r,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{0,r,\mathbb{C}_p}) \hookrightarrow \mathrm{M}_{2n}(\mathbb{C}_p)$. We show that $\phi_{r,\mathfrak{a}}^{\alpha_{\mathbb{C}_p}} = \alpha_{\mathbb{C}_p}(\phi_{r,\mathfrak{a}})$, from which it follows that $\phi_{r,\mathfrak{a}} \in \mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p}^{\mathrm{loc}}$ since $\mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p}^{\mathrm{loc},\alpha_{\mathbb{C}_p}} = \alpha_{\mathbb{C}_p}(\mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p}^{\mathrm{loc}})$ and $\alpha_{\mathbb{C}_p}$ is injective. Now let $M_{\mathfrak{a}}$, $\delta_{\mathfrak{a}}$ be as in (2.6) and $M_{\mathfrak{a}}^{\alpha_{\mathbb{C}_p}} = \alpha_{\mathbb{C}_p}(M_{\mathfrak{a}})$, $\delta_{\mathfrak{a}}^{\alpha_{\mathbb{C}_p}} = \alpha_{\mathbb{C}_p} \circ \delta_{\mathfrak{a}}$. By applying α_C to (2.4) we obtain $(M_{\mathfrak{a}}^{\alpha_{\mathbb{C}_p}})^{-1} \rho_{r,\mathfrak{a}}^{\alpha_{\mathbb{C}_p}}(\sigma)\sigma(M_{\mathfrak{a}}^{\alpha_{\mathbb{C}_p}}) = \delta_{\mathfrak{a}}^{\alpha_{\mathbb{C}_p}}(\sigma)$ for every $\sigma \in G_n'$, so we can compute

$$\phi_{r,\mathfrak{a}}^{\alpha_{\mathbb{C}_p}} = \lim_{\sigma \to 1} \frac{\log(\delta_{\mathfrak{a}}^{\alpha_{\mathbb{C}_p}}(\sigma))}{\log(\chi(\sigma))},$$

that coincides with $\alpha_{\mathbb{C}_p}(\phi_{r,\mathfrak{a}})$. □

Proposition 2.4.8. *The element $\phi_{r,\mathbb{B}_r}$ belongs to $\mathfrak{G}_{r,\mathbb{C}_p}^{\mathrm{loc}}$, hence to $\mathfrak{G}_{r,\mathbb{C}_p}$.*

Proof. Recall that $\mathfrak{G}_{r,\mathbb{C}_p}^{\mathrm{loc}} = \varprojlim_{(\mathfrak{a},P_1)=1} \mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p}^{\mathrm{loc}}$. By Proposition 2.4.6 we have $\phi_{r,\mathbb{B}_r} = \varprojlim_{\mathfrak{a}} \phi_{r,\mathfrak{a}}$ and by Proposition 2.4.7 we have $\phi_{r,\mathfrak{a}} \in \mathfrak{G}_{r,\mathfrak{a},\mathbb{C}_p}^{\mathrm{loc}}$ for every $\mathfrak{a}$. We conclude that $\phi_{r,\mathbb{B}_r} \in \mathfrak{G}_{r,\mathbb{C}_p}^{\mathrm{loc}}$. □

Remark 2.4.9. *In order to prove that our Lie algebras are "big" it will be useful to work with primary ideals of A_r, as we did in this subsection. However, in light of Remark 2.4.3, all of the results can be rewritten in terms of primary ideals $\mathfrak{Q}$ of $\mathbb{I}_{r,0}$. This will be useful in the next subsection, when we will interpolate the Sen operators corresponding to the representations attached to the classical modular forms.*

From now on we identify $\mathbb{I}_{0,r,\mathbb{C}_p}$ with a subring of $\mathbb{B}_{r,\mathbb{C}_p}$ via $\iota_{\mathbb{B}_r}$, so we also identify $\mathrm{M}_2(\mathbb{I}_{r,0})$ with a subring of $\mathrm{M}_2(\mathbb{B}_r)$ and $\mathrm{GL}_2(\mathbb{I}_{0,r,\mathbb{C}_p})$ with a subgroup of $\mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$. In particular we identify ϕ_r with $\phi_{r,\mathbb{B}_r}$ and we consider ϕ_r as an element of $\mathfrak{G}_{r,\mathbb{C}_p} \cap \mathrm{M}_2(\mathbb{I}_{0,r,\mathbb{C}_p})$.

2.4.4. The characteristic polynomial of the Sen operator. Sen proved the following result.

Theorem 2.4.10. *Let L_1 and L_2 be two p-adic fields. Assume that L_2 contains the normal closure of L_1. Let $\tau \colon \mathrm{Gal}(\overline{L}_1/L_1) \to \mathrm{GL}_m(L_2)$ be a continuous representation. For each embedding $\sigma \colon L_1 \to L_2$, there is a Sen operator $\phi_{\tau,\sigma} \in \mathrm{M}_m(\mathbb{C}_p \otimes_{L_1,\sigma} L_2)$ associated with τ and σ. If τ is Hodge-Tate and its Hodge-Tate weights with respect to σ are $h_{1,\sigma}, \ldots, h_{m,\sigma}$ (with multiplicities, if any), then the characteristic polynomial of $\phi_{\tau,\sigma}$ is $\prod_{i=1}^{m}(X - h_{i,\sigma})$.*

Now let $k \in \mathbb{N}$ and let $P_k = (u^{-k}(1+T) - 1)$ be the corresponding arithmetic prime of A_r. Let $\mathfrak{P}_f$ a prime of $\mathbb{I}_r$ above P, associated with the system of Hecke eigenvalues of a classical modular form f. The specialization of ρ_r modulo $\mathfrak{P}$ is the representation $\rho_f \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I}_r/\mathfrak{P}_f)$ classically associated with f, defined over the field $K_f = \mathbb{I}_r/\mathfrak{P}_f\mathbb{I}_r$. By a theorem of Faltings [**Fa87**], when the weight of the form f is k, the representation ρ_f is Hodge-Tate of Hodge-Tate weights 0 and $k-1$. In such a case, by Theorem 2.4.10, the Sen operator ϕ_f associated with ρ_f has characteristic polynomial $X(X-(k-1))$. Let $\mathfrak{P}_{f,0} = \mathfrak{P}_f \cap \mathbb{I}_{r,0}$. The specialization of ρ_r modulo $\mathfrak{P}_{f,0}$ gives a representation $\rho_{r,\mathfrak{P}_{f,0}} \colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{I}_{r,0}/\mathfrak{P}_{f,0})$, that coincides with $\rho_f|_{\mathrm{Gal}(\overline{K}/K)}$. In particular the Sen operator $\phi_{r,\mathfrak{P}_{f,0}}$ associated with $\rho_{r,\mathfrak{P}_{f,0}}$ is ϕ_f.

By Proposition 2.4.6 and Remark 2.4.9, the Sen operator $\phi_r \in \mathrm{M}_2(\mathbb{I}_{0,r,\mathbb{C}_p})$ specializes modulo $\mathfrak{P}_{f,0}$ to the Sen operator $\phi_{r,\mathfrak{P}_{f,0}}$ associated with $\rho_{r,\mathfrak{P}_{f,0}}$, for every f as in the previous paragraph.

Since the primes of the form $\mathfrak{P}_{f,0}$ are dense in $\mathbb{I}_{0,r,\mathbb{C}_p}$, the eigenvalues of ϕ_r are given by the unique interpolation of those of $\rho_{r,\mathfrak{P}_{f,0}}$.

Given $f \in A_r$ we define its p-adic valuation by $v_p'(f) = \inf_{x\in B(0,r)} v_p(f(x))$, where v_p is our chosen valuation on $\mathbb{C}_p$. Then if $v'(f-1) \leq p^{-\frac{1}{p-1}}$ there are well-defined elements $\log(f)$ and $\exp(\log(f))$ in A_r, and $\exp(\log(f)) = f$.

Let $\phi_r' = \log(u)\phi_r$. Note that ϕ_r' is a well-defined element of $\mathrm{M}_2(\mathbb{B}_{r,\mathbb{C}_p})$ since $\log(u) \in \mathbb{Q}_p$. Recall that we denote by C_T the matrix $\mathrm{diag}\,(u^{-1}(1+T), 1)$. We have the following.

Proposition 2.4.11.

(1) The eigenvalues of ϕ_r' are $\log(u^{-1}(1+T))$ and 0. In particular the exponential $\Phi_r = \exp(\phi_r')$ is defined in $\mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$. Moreover Φ_r' is conjugate to C_T in $\mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$.

(2) The element Φ_r' of part (1) normalizes $\mathfrak{G}_{r,\mathbb{C}_p}$.

Proof. For every $\mathfrak{P}_{f,0}$ as in the discussion above, the element $\log(u)\phi_r$ specializes to $\log(u)\phi_{r,\mathfrak{P}_{f,0}}$ modulo $\mathfrak{P}_{f,0}$. If $\mathfrak{P}_{f,0}$ is a divisor of P_k, the eigenvalues of $\log(u)\phi_{r,\mathfrak{P}_{f,0}}$ are $\log(u)(k-1)$ and 0. Since $1+T = u^k$ modulo $\mathfrak{P}_{f,0}$ for every prime $\mathfrak{P}_{f,0}$ dividing P_k, we have $\log(u^{-1}(1+T)) = \log(u^{k-1}) = (k-1)\log(u)$ modulo $\mathfrak{P}_{f,0}$. Hence the eigenvalues of $\log(u)\phi_{r,\mathfrak{P}_{f,0}}$ are interpolated by $\log(u^{-1}(1+T))$ and 0.

Recall that in Section 2.2.1 we chose r_h smaller than $p^{-\frac{1}{p-1}}$. Since $r < r_h$ we have $v_p'(T) < p^{-\frac{1}{p-1}}$. In particular $\log(u^{-1}(1+T))$ is defined and $\exp(\log(u^{-1}(1+T))) = u^{-1}(1+T)$, so $\Phi_r = \exp(\phi_r')$ is also defined and its eigenvalues are $u^{-1}(1+T)$ and 1. The difference between the two is $u^{-1}(1+T)-1$; this element belongs to P_1, hence it is invertible in $\mathbb{B}_r$. This proves (1).

By Proposition 2.4.8, $\phi_r \in \mathfrak{G}_{r,\mathbb{C}_p}$. Since $\mathfrak{G}_{r,\mathbb{C}_p}$ is a $\mathbb{Q}_p$-Lie algebra, $\log(u)\phi_r$ is also an element of $\mathfrak{G}_{r,\mathbb{C}_p}$. Hence its exponential Φ_r' normalizes $\mathfrak{G}_{r,\mathbb{C}_p}$. $\square$

2.5. Existence of the Galois level for a family with finite positive slope

Let $r_h \in p^{\mathbb{Q}} \cap]0, p^{-\frac{1}{p-1}}[$ be the radius chosen in Section 2.2. As usual we write r for any one of the radii r_i of Section 2.2.1. Recall that $\mathfrak{G}_r \subset \mathrm{M}_2(\mathbb{B}_r)$ is the Lie algebra we attached to the image of ρ_r (see Definition 2.4.2) and that $\mathfrak{G}_{r,\mathbb{C}_p} = \mathfrak{G}_r \widehat{\otimes} \mathbb{C}_p$. Let $\mathfrak{u}^{\pm}$, respectively $\mathfrak{u}^{\pm}_{\mathbb{C}_p}$, be the upper and lower nilpotent subalgebras of $\mathfrak{G}_r$ and $\mathfrak{G}_{r,\mathbb{C}_p}$, respectively. As before we suppose that $r_0 \leq r \leq r_h$, where r_0 is the radius chosen in Remark 2.4.1.

Remark 2.5.1. *The Lie algebras $\mathfrak{G}_r$ and $\mathfrak{G}_{r,\mathbb{C}_p}$ are independent of r since the groups G_r are, by Remark 2.4.1. Hence the same is true for the commutative Lie subalgebras $\mathfrak{u}^{\pm}$.*

For $r < r'$ there is a natural inclusion $\mathbb{I}_{0,r'} \hookrightarrow \mathbb{I}_{r,0}$. Since $\mathbb{B}_r = \varprojlim_{(\mathfrak{a}P_1)=1} \mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0}$ this induces an inclusion $\mathbb{B}_{r'} \hookrightarrow \mathbb{B}_r$. We will consider from now on $\mathbb{B}_{r'}$ as a subring of $\mathbb{B}_r$ for every $r < r'$. We will also consider $\mathrm{M}_2(\mathbb{I}_{0,r',\mathbb{C}_p})$ and $\mathrm{M}_2(\mathbb{B}_{r'})$ as subsets of $\mathrm{M}_2(\mathbb{I}_{0,r,\mathbb{C}_p})$ and $\mathrm{M}_2(\mathbb{B}_r)$ respectively. These inclusions still hold after taking completed tensor products with $\mathbb{C}_p$.

Recall the elements $\phi_r' = \log(u)\phi_r \in \mathrm{M}_2(\mathbb{B}_{r,\mathbb{C}_p})$ and $\Phi_r' = \exp(\phi_r') \in \mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$ defined at the end of the previous section. The Sen operator ϕ_r is independent of r in the following sense: if $r < r' < r_h$ and $\mathbb{B}_{r',\mathbb{C}_p} \to \mathbb{B}_{r,\mathbb{C}_p}$ is the natural inclusion then the image of $\phi_{r'}$ under the induced map $\mathrm{M}_2(\mathbb{B}_{r',\mathbb{C}_p}) \to \mathrm{M}_2(\mathbb{B}_{r,\mathbb{C}_p})$ is ϕ_r. We deduce that ϕ_r' and Φ_r' are also independent of r (in the same sense).

By Proposition 2.4.11, for every $r < r_h$ there exists an element $\beta_r \in \mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$ such that $\beta_r\Phi_r'\beta_r^{-1} = C_T$. By Proposition 2.4.11(2) Φ_r' normalizes $\mathfrak{G}_{r,\mathbb{C}_p}$, so $C_T = \beta_r\Phi_r'\beta_r^{-1}$ normalizes $\beta_r\mathfrak{G}_{r,\mathbb{C}_p}\beta_r^{-1}$.

We denote by $\mathfrak{U}^{\pm}$ the upper and lower nilpotent subalgebras of $\mathfrak{sl}_2$. Note that $1+T$ is invertible in A_r since $T = p^{s_h}t$ with $r_h = p^{-s_h}$, therefore C_T is invertible. The action of C_T on $\mathfrak{G}_{r,\mathbb{C}_p}$ by conjugation is semisimple, so we can decompose $\beta_r\mathfrak{G}_{r,\mathbb{C}_p}\beta_r^{-1}$ as a sum of eigenspaces for C_T:

$$\beta_r\mathfrak{G}_{r,\mathbb{C}_p}\beta_r^{-1} = \left(\beta_r\mathfrak{G}_{r,\mathbb{C}_p}\beta_r^{-1}\right)[1] \oplus \left(\beta_r\mathfrak{G}_{r,\mathbb{C}_p}\beta_r^{-1}\right)[u^{-1}(1+T)] \oplus \left(\beta_r\mathfrak{G}_{r,\mathbb{C}_p}\beta_r^{-1}\right)[u(1+T)^{-1}]$$

with

$$\left(\beta_r\mathfrak{G}_{r,\mathbb{C}_p}\beta_r^{-1}\right)[u^{-1}(1+T)] \subset \mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p}) \qquad \text{and} \qquad \left(\beta_r\mathfrak{G}_{r,\mathbb{C}_p}\beta_r^{-1}\right)[u(1+T)^{-1}] \subset \mathfrak{U}^-(\mathbb{B}_{r,\mathbb{C}_p}).$$

Moreover, the formula

$$\begin{pmatrix} u^{-1}(1+T) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u^{-1}(1+T) & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & u^{-1}(1+T)\lambda \\ 0 & 1 \end{pmatrix}$$

shows that the action of C_T on $\mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p})$ by conjugation coincides with multiplication by $u^{-1}(1+T)$. By linearity this gives an action of the polynomial ring $\mathbb{C}_p[T]$ on $\beta_r\mathfrak{G}_{r,\mathbb{C}_p}\beta_r^{-1} \cap \mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p})$, compatible with the action of $\mathbb{C}_p[T]$ on $\mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p})$ induced by the inclusions $\mathbb{C}_p[T] \subset \Lambda_h\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p \subset B_{r,\mathbb{C}_p} \subset \mathbb{B}_{r,\mathbb{C}_p}$. The first two inclusions in the previous chain are of dense image, so $\mathbb{C}_p[T]$ is dense in $B_{r,\mathbb{C}_p}$. Since $\mathfrak{G}_{r,\mathbb{C}_p}$ is a closed Lie subalgebra of $\mathrm{M}_2(\mathbb{B}_{r,\mathbb{C}_p})$ we can define by continuity a $B_{r,\mathbb{C}_p}$-module structure on $\beta_r\mathfrak{G}_{r,\mathbb{C}_p}\beta_r^{-1} \cap \mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p})$ compatible with that on $\mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p})$. Similarly we have

$$\begin{pmatrix} u^{-1}(1+T) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}\begin{pmatrix} u^{-1}(1+T) & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ u(1+T)^{-1}\mu & 1 \end{pmatrix}.$$

By twisting by $(1+T) \mapsto (1+T)^{-1}$ we can also give $\beta_r\mathfrak{G}_{r,\mathbb{C}_p}\beta_r^{-1} \cap \mathfrak{U}^-(\mathbb{B}_{r,\mathbb{C}_p})$ a structure of $B_{r,\mathbb{C}_p}$-module compatible with that on $\mathfrak{U}^-(\mathbb{B}_{r,\mathbb{C}_p})$.

By combining the previous remarks with Corollary 2.3.21, we prove the following "fullness" result for the Lie algebra $\mathfrak{G}_r$.

THEOREM 2.5.2. *Suppose that the representation ρ is $(H_0,\mathbb{Z}_p)$-regular. Then there exists a non-zero ideal $\mathfrak{l}$ of $\mathbb{I}_0$ such that for every $r \in \{r_i\}_{i\geq 1}$ the Lie algebra $\mathfrak{G}_r$ contains $\mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{B}_r)$.*

PROOF. Since $U^{\pm}(\mathbb{B}_r) \cong \mathbb{B}_r$, we can and shall identify $\mathfrak{u}^+ = \mathbb{Q}_p\cdot\mathrm{Log}\,G_r'\cap\mathfrak{U}^+(\mathbb{B}_r)$ with a $\mathbb{Q}_p$-vector subspace of $\mathbb{B}_r$ (actually of $\mathbb{I}_0$), and $\mathfrak{u}^+_{\mathbb{C}_p}$ with a $\mathbb{C}_p$-vector subspace of $\mathbb{B}_{r,\mathbb{C}_p}$. By Corollary 2.3.21, $\mathfrak{u}^{\pm}\cap\mathbb{I}_0$ contains a basis $\{e_{i,\pm}\}_{i\in I}$ of $Q(\mathbb{I}_0)$ over $Q(\Lambda_h)$. In particular $\mathfrak{u}^+$ contains the basis of a Λ_h-lattice in $\mathbb{I}_0$. From Lemma 2.3.17 we deduce that $\Lambda_h\mathfrak{u}^+$ contains a non-zero ideal $\mathfrak{a}^+$ of $\mathbb{I}_0$. Hence we also have $B_{r,\mathbb{C}_p}\mathfrak{u}^+_{\mathbb{C}_p} \supset B_{r,\mathbb{C}_p}\mathfrak{a}^+$. Now $\mathfrak{a}^+$ is an ideal of $\mathbb{I}_0$ and $B_{r,\mathbb{C}_p}\mathbb{I}_0 = \mathbb{B}_{r,\mathbb{C}_p}$, so $B_{r,\mathbb{C}_p}\mathfrak{a}^+ = \mathbb{B}_{r,\mathbb{C}_p}\mathfrak{a}^+$ is an ideal of $\mathbb{B}_{r,\mathbb{C}_p}$. We conclude that $B_{r,\mathbb{C}_p}\mathfrak{u}^+ \supset \mathbb{B}_{r,\mathbb{C}_p}\mathfrak{a}^+$ for a non-zero ideal $\mathfrak{a}^+$ of $\mathbb{I}_0$. We proceed in the same way for the lower unipotent subalgebra, obtaining $B_{r,\mathbb{C}_p}\mathfrak{u}^- \supset \mathbb{B}_{r,\mathbb{C}_p}\mathfrak{a}^-$ for a non-zero ideal $\mathfrak{a}^-$ of $\mathbb{I}_0$.

Consider now the Lie algebra $B_{r,\mathbb{C}_p}\mathfrak{G}_{\mathbb{C}_p} \subset \mathrm{M}_2(\mathbb{B}_{r,\mathbb{C}_p})$. Its nilpotent subalgebras are $B_{r,\mathbb{C}_p}\mathfrak{u}^+$ and $B_{r,\mathbb{C}_p}\mathfrak{u}^-$ and we showed that $B_{r,\mathbb{C}_p}\mathfrak{u}^+ \supset \mathbb{B}_{r,\mathbb{C}_p}\mathfrak{a}^+$ and $B_{r,\mathbb{C}_p}\mathfrak{u}^- \supset \mathbb{B}_{r,\mathbb{C}_p}\mathfrak{a}^-$. Denote by $\mathfrak{t} \subset \mathfrak{sl}_2$ the subalgebra of diagonal matrices over $\mathbb{Z}$. By taking a Lie bracket, we see that

$$[\mathfrak{a}^+\cdot\mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p}), \mathfrak{a}^-\cdot\mathfrak{U}^-(\mathbb{B}_{r,\mathbb{C}_p})] = \mathfrak{a}^+\cdot\mathfrak{a}^-\cdot\mathfrak{t}(\mathbb{B}_{r,\mathbb{C}_p}).$$

From the decomposition $\mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p}) = \mathfrak{U}^-(\mathbb{B}_{r,\mathbb{C}_p})\oplus\mathfrak{t}(\mathbb{B}_{r,\mathbb{C}_p}\oplus\mathfrak{U}^+(\mathbb{B}_{r,\mathbb{C}_p})$ we deduce that $B_{r,\mathbb{C}_p}\mathfrak{G}_{\mathbb{C}_p} \supset \mathfrak{a}^+\cdot\mathfrak{a}^-\mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$. Let $\mathfrak{a} = \mathfrak{a}^+\cdot\mathfrak{a}^-$. Now $\mathfrak{a}\cdot\mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$ is a $\mathbb{B}_{r,\mathbb{C}_p}$-Lie subalgebra of $\mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$. Since $\beta_r \in \mathrm{GL}_2(\mathbb{B}_{r,\mathbb{C}_p})$ we have $\beta_r\left(\mathfrak{a}\cdot\mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})\right)\beta_r^{-1} = \mathfrak{a}\cdot\mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$. Thus $B_{r,\mathbb{C}_p}\left(\beta_r\mathfrak{G}_{r,\mathbb{C}_p}\beta_r^{-1}\right) \supset \mathfrak{a}\cdot\mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$. In particular, if $\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p}$ denote the nilpotent subalgebras of $\beta_r\mathfrak{G}_{r,\mathbb{C}_p}\beta_r^{-1}$, we have $B_{r,\mathbb{C}_p}\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p} \supset \mathbb{B}_{r,\mathbb{C}_p}\mathfrak{a}$ for both signs. By the discussion preceding the proposition the subalgebras $\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p}$ have a structure of $B_{r,\mathbb{C}_p}$-modules, which means that $\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p} = B_{r,\mathbb{C}_p}\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p}$. We conclude that $\mathfrak{u}^{\pm,\beta_r}_{\mathbb{C}_p} \supset \mathfrak{a}\cdot\mathfrak{U}^{\pm}(\mathbb{B}_{r,\mathbb{C}_p})$ for both signs. By taking a Lie bracket as before we obtain $\beta_r\mathfrak{G}_{r,\mathbb{C}_p}\beta_r^{-1} \supset$

$\mathfrak{a}^2\mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$. We can untwist by the invertible matrix β_r to conclude that $\mathfrak{G}_{r,\mathbb{C}_p} \supset \mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{B}_{r,\mathbb{C}_p})$ for $\mathfrak{l} = \mathfrak{a}^2$.

Let us get rid of the completed extension of scalars to $\mathbb{C}_p$. For every ideal $\mathfrak{a} \subset \mathbb{I}_{r,0}$ not dividing P_1, let $\mathfrak{G}_{r,\mathfrak{a}}$ be the image of $\mathfrak{G}_r$ in $\mathrm{M}_2(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})$. Consider the two finite dimensional $\mathbb{Q}_p$-vector spaces $\mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})$. Note that they are both subspaces of the finite dimensional $\mathbb{Q}_p$-vector space $\mathrm{M}_2(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})$. After extending scalars to $\mathbb{C}_p$, we have

$$\mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \subset \mathfrak{G}_{r,\mathfrak{a}} \otimes \mathbb{C}_p. \tag{2.8}$$

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of the $\mathbb{Q}_p$-Banach space $\mathbb{C}_p$, for an index set I such that $1 \in \{e_i\}_{i\in I}$. Let $\{v_j\}_{j=1,\ldots,n}$ be a $\mathbb{Q}_p$-basis of $\mathrm{M}_2(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})$ such that, for some $d \leq n$, $\{v_j\}_{j=1,\ldots,d}$ is a $\mathbb{Q}_p$-basis of $\mathfrak{G}_{r,\mathfrak{a}}$.

Let v be an element of $\mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})$. Then $v\otimes 1 \in \mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})\otimes\mathbb{C}_p$ and by (2.8) we have $v\otimes 1 \in \mathfrak{G}_{r,\mathfrak{a}}\otimes\mathbb{C}_p$. As $\{v_j\otimes e_i\}_{1\leq j\leq d, i\in I}$ and $\{v_j\otimes e_i\}_{1\leq j\leq n, i\in I}$ are orthonormal $\mathbb{Q}_p$-bases of $\mathfrak{G}_{r,\mathfrak{a}}\otimes\mathbb{C}_p$ and $\mathrm{M}_2(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})\otimes\mathbb{C}_p$, respectively, there exist $\lambda_{j,i}\in\mathbb{Q}_p$, $(j,i)\in\{1,2,...d\}\times I$ converging to 0 in the filter of complements of finite subsets of $\{1,2,...,d\}\times I$ such that $v\otimes 1 = \sum_{j=1,\ldots,d; i\in I}\lambda_{j,i}(v_j\otimes e_i)$.

But $v\otimes 1 \in \mathrm{M}_2(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})\otimes 1 \subset \mathrm{M}_2(\mathbb{I}_{r,0}/\mathfrak{a}\mathbb{I}_{r,0})\otimes\mathbb{C}_p$ and therefore $v\otimes 1 = \sum_{1\leq j\leq n} a_j(v_j\otimes 1)$, for some $a_j\in\mathbb{Q}_p$, $j = 1, n$. By the uniqueness of a representation of an element in a $\mathbb{Q}_p$-Banach space in terms of a given orthonormal basis we have

$$v\otimes 1 = \sum_{j=1}^{d} a_j(v_j\otimes 1) \quad \text{i.e.} \quad v = \sum_{j=1}^{d} a_j v_j \in \mathfrak{G}_{r,\mathfrak{a}}.$$

By taking the projective limit over $\mathfrak{a}$, we conclude that

$$\mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{B}_r) \subset \mathfrak{G}_r.$$

□

Definition 2.5.3. *The* Galois level *of the family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ is the largest ideal $\mathfrak{l}_\theta$ of $\mathbb{I}_0[P_1^{-1}]$ such that $\mathfrak{G}_r \supset \mathfrak{l}_\theta\cdot\mathfrak{sl}_2(\mathbb{B}_r)$ for every $r\in\{r_i\}_{i\geq 1}$.*

It follows from the previous remarks that $\mathfrak{l}_\theta$ is non-zero.

2.6. Comparison between the Galois level and the fortuitous congruence ideal

Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a family of GL_2-eigenforms of slope bounded by h. We keep all the notations from the previous sections. In particular $\rho\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ)$ is the Galois representation associated with θ. We suppose that the restriction of ρ to H_0 takes values in $\mathrm{GL}_2(\mathbb{I}_0^\circ)$. Recall that $\mathbb{I} = \mathbb{I}^\circ[p^{-1}]$ and $\mathbb{I}_0 = \mathbb{I}_0^\circ[p^{-1}]$. Also recall that P_1 is the prime of Λ_h generated by $u^{-1}(1+T)-1$. Let $\mathfrak{c}\subset\mathbb{I}$ be the fortuitous CM-congruence ideal associated with θ (see Definition 2.2.12). Set $\mathfrak{c}_0 = \mathfrak{c}\cap\mathbb{I}_0$ and $\mathfrak{c}_1 = \mathfrak{c}_0\mathbb{I}_0[P_1^{-1}]$. Let $\mathfrak{l} = \mathfrak{l}_\theta \subset \mathbb{I}_0[P_1^{-1}]$ be the Galois level of the family θ (see Definition 2.5.3). For an ideal $\mathfrak{a}$ of $\mathbb{I}_0[P_1^{-1}]$ we denote by $V(\mathfrak{a})$ the set of prime ideals of $\mathbb{I}_0[P_1^{-1}]$ containing $\mathfrak{a}$. We prove the following.

Theorem 2.6.1. *Suppose that*

(1) ρ is $(H_0,\mathbb{Z}_p)$-regular;

(2) there exists no pair (F,ψ), where F is a real quadratic field and $\psi\colon \mathrm{Gal}(\overline{F}/F)\to\mathbb{F}^\times$ is a character, such that $\overline{\rho}\colon G_\mathbb{Q}\to\mathrm{GL}_2(\mathbb{F})\cong\mathrm{Ind}_F^\mathbb{Q}\psi$.

Then we have $V(\mathfrak{l}) = V(\mathfrak{c}_1)$.

Before giving the proof we make some remarks. Let P be a prime of $\mathbb{I}_0[P_1^{-1}]$ and Q be a prime factor of $P\mathbb{I}[P_1^{-1}]$. We consider ρ as a representation $G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}[P_1^{-1}])$ by composing it with the inclusion $\mathrm{GL}_2(\mathbb{I}) \hookrightarrow \mathrm{GL}_2(\mathbb{I}[P_1^{-1}])$. We have a representation $\rho_Q\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}[P_1^{-1}]/Q)$ obtained by reducing ρ modulo Q. Its restriction $\rho_Q|_{H_0}$ takes values in $\mathrm{GL}_2(\mathbb{I}_0[P_1^{-1}]/(Q \cap \mathbb{I}_0[P_1^{-1}])) = \mathrm{GL}_2(\mathbb{I}_0[P_1^{-1}]/P)$ and coincides with the reduction ρ_P of $\rho|_{H_0}\colon H_0 \to \mathrm{GL}_2(\mathbb{I}_0[P_1^{-1}])$ modulo P. In particular $\rho_Q|_{H_0}$ is independent of the chosen prime factor Q of $P\mathbb{I}[P_1^{-1}]$.

Let K be a p-adic field and A be a finite-dimensional K-algebra. We say that a subgroup of $\mathrm{GL}_2(A)$ is *small* if it admits a finite index abelian subgroup. Let P, Q be as above, G_P be the image of $\rho_P\colon H_0 \to \mathrm{GL}_2(\mathbb{I}_0[P_1^{-1}]/P)$ and G_Q be the image of $\rho_Q\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}[P_1^{-1}]/Q)$. By our previous remark ρ_P coincides with the restriction $\rho_Q|_{H_0}$, so G_P is a finite index subgroup of G_Q for every Q. In particular G_P is small if and only if G_Q is small for all prime factors Q of $P\mathbb{I}[P_1^{-1}]$.

If Q is a CM point the representation ρ_Q is induced by a character of $\mathrm{Gal}(F/\mathbb{Q})$ for an imaginary quadratic field F. Hence G_Q admits an abelian subgroup of index 2 and G_P is also small.

Conversely, if G_P is small then $G_{Q'}$ is small for every prime Q' above P. Choose any such prime Q'; by the argument in [**Ri77**, Proposition 4.4] $G_{Q'}$ has an abelian subgroup of index 2. It follows that $\rho_{Q'}$ is induced by a character of $\mathrm{Gal}(\overline{F}_{Q'}/F_{Q'})$ for a quadratic field $F_{Q'}$. If we suppose that the residual representation $\overline{\rho}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F})$ is not induced by a character of $\mathrm{Gal}(\overline{F}/F)$ for a real quadratic field F then $F_{Q'}$ is imaginary and Q' is CM. Under assumption (2) of Theorem 2.6.1, the above argument proves that G_P is small if and only if all points $Q' \subset \mathbb{I}[P_1^{-1}]$ above P are CM.

Proof. Fix a radius $r \in \{r_i\}_{i\geq 1}$. We prove first that $V(\mathfrak{c}_1) \subset V(\mathfrak{l})$. By contradiction, suppose that a prime P of $\mathbb{I}_0[P_1^{-1}]$ contains $\mathfrak{c}_1 \cdot \mathbb{I}[P_1^{-1}]$ but not $\mathfrak{l}$. Then there exists a prime factor Q of $P\mathbb{I}[P_1^{-1}]$ such that $\mathfrak{c} \subset Q$. By definition of $\mathfrak{c}$ the point Q is CM in the sense of Section 2.2.4, hence the representation $\rho_{\mathbb{I}[P_1^{-1}],Q}$ has small image in $\mathrm{GL}_2(\mathbb{I}[P_1^{-1}]/Q)$. Then its restriction $\rho_{\mathbb{I}[P_1^{-1}],Q}|_{H_0} = \rho_P$ also has small image in $\mathrm{GL}_2(\mathbb{I}_0[P_1^{-1}]/P)$. We deduce that there is no non-zero ideal $\mathfrak{I}_P$ of $\mathbb{I}_0[P_1^{-1}]/P$ such that the Lie algebra $\mathfrak{G}_{r,P}$ contains $\mathfrak{I}_P \cdot \mathfrak{sl}_2(\mathbb{I}_{r,0}[P_1^{-1}]/P)$.

By definition of $\mathfrak{l}$ we have $\mathfrak{l} \cdot \mathfrak{sl}_2(\mathbb{B}_r) \subset \mathfrak{G}_r$. Since reduction modulo P gives a surjection $\mathfrak{G}_r \to \mathfrak{G}_{r,P}$, by looking at the previous inclusion modulo P we find $\mathfrak{l}\cdot\mathfrak{sl}_2(\mathbb{I}_{r,0}[P_1^{-1}]/P\mathbb{I}_{r,0}[P_1^{-1}]) \subset \mathfrak{G}_{r,P}$. If $\mathfrak{l} \not\subset P$ we have $\mathfrak{l}/P \neq 0$, which contradicts our earlier statement. We deduce that $\mathfrak{l} \subset P$.

We prove now that $V(\mathfrak{l}) \subset V(\mathfrak{c}_1)$. Let $P \subset \mathbb{I}_0[P_1^{-1}]$ be a prime containing $\mathfrak{l}$. Recall that $\mathbb{I}_0[P_1^{-1}]$ has Krull dimension one, so $\kappa_P = \mathbb{I}_0[P_1^{-1}]/P$ is a field. Let Q be a prime of $\mathbb{I}[P_1^{-1}]$ above P. As before ρ reduces to representations $\rho_Q\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}[P_1^{-1}]/Q)$ and $\rho_P\colon H_0 \to \mathrm{GL}_2(\mathbb{I}_0[P_1^{-1}]/P)$. Let $\mathfrak{P} \subset \mathbb{I}_0[P_1^{-1}]$ be the P-primary component of $\mathfrak{l}$ and let $\mathfrak{A}$ be an ideal of $\mathbb{I}_0[P_1^{-1}]$ containing $\mathfrak{P}$ such that the localization at P of $\mathfrak{A}/\mathfrak{P}$ is one-dimensional over κ_P. Let $\mathfrak{s} = \mathfrak{A}/\mathfrak{P} \cdot \mathfrak{sl}_2(\mathbb{I}_{r,0}[P_1^{-1}]/\mathfrak{P}) \cap \mathfrak{G}_{r,\mathfrak{P}}$, that is a Lie subalgebra of $\mathfrak{A}/\mathfrak{P} \cdot \mathfrak{sl}_2(\mathbb{I}_{r,0}[P_1^{-1}]/\mathfrak{P})$.

We show that $\mathfrak{s}$ is stable under the adjoint action $\mathrm{Ad}\,(\rho_Q)$ of $G_\mathbb{Q}$. Let $\mathfrak{Q}$ be the Q-primary component of $\mathfrak{l} \cdot \mathbb{I}[P_1^{-1}]$. Recall that $\mathfrak{G}_{r,\mathfrak{P}}$ is the Lie algebra associated with the pro-p group $\mathrm{Im}\,\rho_{r,\mathfrak{Q}}|_{H_0} \cap \Gamma_{\mathrm{GL}_2(\mathbb{I}_{r_0,0}[P_1^{-1}]/\mathfrak{P})}(p)$, where the radius r_0 was fixed in Remark 2.4.1. Since the above group is open in $\mathrm{Im}\,\rho_{r,\mathfrak{Q}} \subset \mathrm{GL}_2(\mathbb{I}_r[P_1^{-1}]/\mathfrak{Q})$, the Lie algebra associated with $\mathrm{Im}\,\rho_{r,\mathfrak{Q}}$ is again $\mathfrak{G}_{r,\mathfrak{P}}$. In particular $\mathfrak{G}_{r,\mathfrak{P}}$ is stable under $\mathrm{Ad}\,(\rho_Q)$. Since $\mathfrak{G}_{r,\mathfrak{P}} \subset \mathfrak{sl}_2(\mathbb{I}_{r,0}[P_1^{-1}]/\mathfrak{P})$ we have $\mathfrak{A}/\mathfrak{P}\cdot\mathfrak{sl}_2(\mathbb{I}_{r,0}[P_1^{-1}]/\mathfrak{P})\cap\mathfrak{G}_{r,\mathfrak{P}} = \mathfrak{A}/\mathfrak{P}\cdot\mathfrak{sl}_2(\mathbb{I}_r[P_1^{-1}]/\mathfrak{Q})\cap\mathfrak{G}_{r,\mathfrak{P}}$. Now $\mathfrak{A}/\mathfrak{P}\cdot\mathfrak{sl}_2(\mathbb{I}_r[P_1^{-1}]/\mathfrak{Q})$ is clearly stable under $\mathrm{Ad}\,(\rho_Q)$, so the same is true for $\mathfrak{A}/\mathfrak{P} \cdot \mathfrak{sl}_2(\mathbb{I}_r[P_1^{-1}]/\mathfrak{Q}) \cap \mathfrak{G}_{r,\mathfrak{P}}$, as desired.

We consider from now on $\mathfrak{s}$ as a Galois representation via $\mathrm{Ad}\,(\rho_Q)$. By the proof of Theorem 2.5.2 we can assume, possibly considering a sub-Galois representation, that $\mathfrak{G}_r$ is a $\mathbb{B}_r$-submodule of $\mathfrak{sl}_2(\mathbb{B}_r)$ containing $\mathfrak{l} \cdot \mathfrak{sl}_2(\mathbb{B}_r)$ but not $\mathfrak{a} \cdot \mathfrak{sl}_2(\mathbb{B}_r)$ for any ideal $\mathfrak{a}$ of $\mathbb{I}_0[P_1^{-1}]$ strictly bigger than

$\mathfrak{l}$. This allows us to speak of the localization $\mathfrak{s}_P$ of $\mathfrak{s}$ at P. Note that, since $\mathfrak{P}$ is the P-primary component of $\mathfrak{l}$ and $\mathfrak{A}_P/\mathfrak{P}_P \cong \kappa_P$, by P-localizing we find $\mathfrak{G}_{r,(P)} \supset \mathfrak{P}_P \cdot \mathfrak{sl}_2(\mathbb{B}_{r,(P)})$ and $\mathfrak{G}_{r,(P)} \not\supset \mathfrak{A}_P \cdot \mathfrak{sl}_2(\mathbb{B}_{r,(P)})$.

The localization at P of $\mathfrak{A}/\mathfrak{P} \cdot \mathfrak{sl}_2(\mathbb{I}_{r,0}[P_1^{-1}]/\mathfrak{P})$ is $\mathfrak{sl}_2(\kappa_P)$, so $\mathfrak{s}_P$ is contained in $\mathfrak{sl}_2(\kappa_P)$. It is a κ_P-representation of $G_\mathbb{Q}$ (via $\mathrm{Ad}\,(\rho_Q)$). We study its dimension, which is at most 3.

We cannot have $\mathfrak{s}_P = 0$. By exchanging the quotient with the localization we would obtain $(\mathfrak{A}_P \cdot \mathfrak{sl}_2(\mathbb{B}_{r,(P)}) \cap \mathfrak{G}_{r,(P)})/\mathfrak{P}_P = 0$. By Nakayama's lemma $\mathfrak{A}_P \cdot \mathfrak{sl}_2(\mathbb{B}_{r,(P)}) \cap \mathfrak{G}_{r,(P)} = 0$, which is absurd since $\mathfrak{A}_P \cdot \mathfrak{sl}_2(\mathbb{B}_{r,(P)}) \cap \mathfrak{G}_{r,(P)} \supset \mathfrak{P}_P \cdot \mathfrak{sl}_2(\mathbb{B}_{r,(P)}) \neq 0$.

We also exclude the three-dimensional case. If $\mathfrak{s}_P = \mathfrak{sl}_2(\kappa_P)$, by exchanging the quotient with the localization we obtain $(\mathfrak{A}_P \cdot \mathfrak{sl}_2(\mathbb{B}_{r,P}) \cap \mathfrak{G}_{r,P})/\mathfrak{P}_P = (\mathfrak{A}_P \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r,P}[P_1^{-1}]))/\mathfrak{P}_P\mathbb{I}_{0,r,P}[P_1^{-1}]$ because $\mathfrak{A}_P\mathbb{I}_{0,r,P}[P_1^{-1}]/\mathfrak{P}_P\mathbb{I}_{0,r,P}[P_1^{-1}] = \kappa_P$. By Nakayama's lemma we conclude that $\mathfrak{G}_{r,P} \supset \mathfrak{A} \cdot \mathfrak{sl}_2(\mathbb{B}_{r,P})$, which contradicts our choice of $\mathfrak{A}$.

We are left with the one and two-dimensional cases. If $\mathfrak{s}_P$ is two-dimensional we can always replace it by its orthogonal in $\mathfrak{sl}_2(\kappa_P)$ which is one-dimensional. Since the action of $G_\mathbb{Q}$ via $\mathrm{Ad}\,(\rho_Q)$ is isometric with respect to the scalar product $\mathrm{Tr}(XY)$ on $\mathfrak{sl}_2(\kappa_P)$.

Suppose that $\mathfrak{sl}_2(\kappa_P)$ contains a one-dimensional stable subspace. Let ϕ be a generator of this subspace over κ_P. Let $\chi\colon G_\mathbb{Q} \to \kappa_P$ be the character satisfying $\rho_Q(g)\phi\rho_Q(g)^{-1} = \chi(g)\phi$ for all $g \in G_\mathbb{Q}$. Now ϕ induces a non-trivial morphism of representations $\rho_Q \to \rho_Q \otimes \chi$. Since ρ_Q and $\rho_Q \otimes \chi$ are irreducible, ϕ must be invertible by Schur's lemma. Hence we obtain an isomorphism $\rho_Q \cong \rho_Q \otimes \chi$. By taking determinants we see that χ must be quadratic. If $F_0/\mathbb{Q}$ is the quadratic extension fixed by $\ker \chi$, then ρ_Q is induced by a character ψ of $\mathrm{Gal}(\overline{F_0}/F_0)$. By assumption the residual representation $\rho_{\mathfrak{m}_\mathbb{I}}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F})$ is not of the form $\mathrm{Ind}_F^\mathbb{Q}\psi$ for a real quadratic field F and a character $\mathrm{Gal}(\overline{F}/F) \to \mathbb{F}^\times$. We deduce that F_0 is imaginary, so Q is a CM point by Remark 2.2.13(1). By construction of the congruence ideal we obtain $\mathfrak{c} \subset Q$ and $\mathfrak{c}_1 \subset (Q \cap \mathbb{I}_0) \cdot \mathbb{I}_0[P_1^{-1}] = P$. $\square$

We prove a corollary.

Corollary 2.6.2. *If the residual representation $\overline{\rho}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F})$ is not dihedral then $\mathfrak{l} = 1$.*

Proof. Since $\overline{\rho}$ is not dihedral there cannot be any CM point in the family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$. By Theorem 2.6.1 we deduce that $\mathfrak{l}$ has no non-trivial prime factor, hence it is trivial. $\square$

Remark 2.6.3. *Theorem 2.6.1 gives another proof of Proposition 2.2.11. Indeed the CM points of a family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ correspond to the prime factors of its Galois level, which are finite in number.*

We also give a partial result about the comparison of the exponents of the prime factors of $\mathfrak{c}_1$ and $\mathfrak{l}$. This is an analogous of what is proved in [**Hi15**, Theorem 8.6] for an ordinary family; our proof also relies on the strategy there. For every prime P of $\mathbb{I}_0[P_1^{-1}]$ we denote by $\mathfrak{c}_1^P$ and $\mathfrak{l}^P$ the P-primary components of $\mathfrak{c}_1$ and $\mathfrak{l}$, respectively.

Theorem 2.6.4. *Suppose that $\overline{\rho}$ is not induced by a character of G_F for a real quadratic field $F/\mathbb{Q}$. Then $(\mathfrak{c}_1^P)^2 \subset \mathfrak{l}^P \subset \mathfrak{c}_1^P$.*

Proof. The inclusion $\mathfrak{l}^P \subset \mathfrak{c}_1^P$ is proved in the same way as the first inclusion of Theorem 2.6.1.

We show that the inclusion $(\mathfrak{c}_1^P)^2 \subset \mathfrak{l}^P$ holds. If $\mathfrak{c}_1^P$ is trivial this reduces to Theorem 2.6.1, so we can suppose that P is a factor of $\mathfrak{c}_1$. Let Q denote any prime of $\mathbb{I}[P_1^{-1}]$ above P. Let $\mathfrak{c}_1^Q$ be a Q-primary ideal of $\mathbb{I}[P_1^{-1}]$ satisfying $\mathfrak{c}_1^Q \cap \mathbb{I}_0[P_1^{-1}] = \mathfrak{c}_1^P$. Since P divides $\mathfrak{c}_1$, Q is a CM point, so we have an isomorphism $\rho_P \cong \mathrm{Ind}_F^\mathbb{Q}\psi$ for an imaginary quadratic field $F/\mathbb{Q}$ and a character $\psi\colon G_F \to \mathbb{C}_p^\times$. Choose any $r < r_h$. Consider the κ_P-vector space $\mathfrak{s}_{\mathfrak{c}_1^P} = \mathfrak{G}_r \cap \mathfrak{c}_1^P \cdot \mathfrak{sl}_2(\mathbb{I}_{r,0})/\mathfrak{G}_r \cap \mathfrak{c}_1^P P \cdot \mathfrak{sl}_2(\mathbb{I}_{r,0})$. We see it as a subspace of $\mathfrak{sl}_2(\mathfrak{c}_1^P/\mathfrak{c}_1^P P) \cong \mathfrak{sl}_2(\kappa_P)$. By the same argument as in the proof of Theorem 2.6.1, $\mathfrak{s}_{\mathfrak{c}_1^P}$ is stable under the adjoint action $\mathrm{Ad}\,(\rho_{\mathfrak{c}_1^Q Q})\colon G_\mathbb{Q} \to \mathrm{Aut}(\mathfrak{sl}_2(\kappa_P))$.

Let $\chi_{F/\mathbb{Q}}\colon G_\mathbb{Q} \to \mathbb{C}_p^\times$ be the quadratic character defined by the extension $F/\mathbb{Q}$. Let $\varepsilon \in G_\mathbb{Q}$ be an element projecting to the generator of $\mathrm{Gal}(F/\mathbb{Q})$. Let $\psi^\varepsilon\colon G_F \to \mathbb{C}_p^\times$ be given by $\psi^\varepsilon(\tau) = \psi(\varepsilon\tau\varepsilon^{-1})$. Set $\psi^- = \psi/\psi^\varepsilon$. Since $\rho_Q \cong \mathrm{Ind}_F^\mathbb{Q}\psi$, we have a decomposition $\mathrm{Ad}\,(\rho_Q) \cong \chi_{F/\mathbb{Q}} \oplus \mathrm{Ind}_F^\mathbb{Q}\psi^-$, where the two factors are irreducible. Now we have three possibilities for the Galois isomorphism class of $\mathfrak{s}_{\mathfrak{c}_1^P}$: either it is $\mathrm{Ad}\,(\rho_Q)$ or it is isomorphic to one of the two irreducible factors.

If $\mathfrak{s}_{\mathfrak{c}_1^P} \cong \mathrm{Ad}\,(\rho_Q)$ then $\mathfrak{s}_{\mathfrak{c}_1^P} = \mathfrak{sl}_2(\kappa_P)$ as κ_P-vector spaces. By Nakayama's lemma $\mathfrak{G}_r \supset \mathfrak{c}_1^P \cdot \mathfrak{sl}_2(\mathbb{B}_r)$. This implies $\mathfrak{c}_1^P \subset \mathfrak{l}^P$, hence $\mathfrak{c}_1^P = \mathfrak{l}^P$ in this case.

If $\mathfrak{s}_{\mathfrak{c}_1^P}$ is one-dimensional then we proceed as in the proof of Theorem 2.6.1 to show that $\rho_{\mathfrak{c}_1^Q Q}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}_r[P_1^{-1}]/\mathfrak{c}_1^Q Q\mathbb{I}_r[P_1^{-1}])$ is induced by a character $\psi_{\mathfrak{c}_1^Q Q}\colon G_F \to \mathbb{C}_p^\times$. In particular the image of $\rho_{\mathfrak{c}_1^P P}\colon H \to \mathrm{GL}_2(\mathbb{I}_{r,0}[P_1^{-1}]/\mathfrak{c}_1^P P\mathbb{I}_{r,0})$ is small. This is a contradiction, since $\mathfrak{c}_1^P$ is the P-primary component of $\mathfrak{c}_1$, hence it is the smallest P-primary ideal $\mathfrak{A}$ of $\mathbb{I}_{r,0}[P_1^{-1}]$ such that the image of $\rho_\mathfrak{A}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}_r[P_1^{-1}]/\mathfrak{A}\mathbb{I}_r[P_1^{-1}])$ is small.

Finally, suppose that $\mathfrak{s}_{\mathfrak{c}_1^P} \cong \mathrm{Ind}_F^\mathbb{Q}\psi^-$. Let $d = \mathrm{diag}\,(d_1, d_2) \in \rho(G_\mathbb{Q})$ be the image of a $\mathbb{Z}_p$-regular element. Since d_1 and d_2 are non-trivial modulo the maximal ideal of $\mathbb{I}_0^\circ$, the image of d modulo $\mathfrak{c}_1^Q Q$ is a non-trivial diagonal element $d_{\mathfrak{c}_1^Q Q} = \mathrm{diag}\,(d_{1,\mathfrak{c}_1^Q Q}, d_{2,\mathfrak{c}_1^Q Q}) \in \rho_{\mathfrak{c}_1^Q Q}(G_\mathbb{Q})$. We decompose $\mathfrak{s}_{\mathfrak{c}_1^P}$ in eigenspaces for the adjoint action of $d_{\mathfrak{c}_1^Q Q}$, writing $\mathfrak{s}_{\mathfrak{c}_1^P} = \mathfrak{s}_{\mathfrak{c}_1^P}[a] \oplus \mathfrak{s}_{\mathfrak{c}_1^P}[1] \oplus \mathfrak{s}_{\mathfrak{c}_1^P}[a^{-1}]$ where $a = d_{1,\mathfrak{c}_1^Q Q}/d_{2,\mathfrak{c}_1^Q Q}$. Now $\mathfrak{s}_{\mathfrak{c}_1^P}[1]$ is contained in the diagonal torus of $\mathfrak{sl}_2(\kappa_P)$, on which the adjoint action of $G_\mathbb{Q}$ is given by the character $\chi_{F/\mathbb{Q}}$. Since $\chi_{F/\mathbb{Q}}$ does not appear as a factor of $\mathfrak{s}_{\mathfrak{c}_1^P}$, we must have $\mathfrak{s}_{\mathfrak{c}_1^P}[1] = 0$. This implies that $\mathfrak{s}_{\mathfrak{c}_1^P}[a] \neq 0$ and $\mathfrak{s}_{\mathfrak{c}_1^P}[a^{-1}] \neq 0$. Since $\mathfrak{s}_{\mathfrak{c}_1^P}[a] = \mathfrak{s}_{\mathfrak{c}_1^P} \cap \mathfrak{u}^+(\kappa_P)$ and $\mathfrak{s}_{\mathfrak{c}_1^P}[a^{-1}] = \mathfrak{s}_{\mathfrak{c}_1^P} \cap \mathfrak{u}^-(\kappa_P)$, we deduce that $\mathfrak{s}_{\mathfrak{c}_1^P}$ contains non-trivial upper and lower nilpotent elements $\overline{u^+}$ and $\overline{u^-}$. Then $\overline{u^+}$ and $\overline{u^-}$ are the images of some elements u^+ and u^- of $\mathfrak{G}_r \cap \mathfrak{c}_1^P \cdot \mathfrak{sl}_2(\mathbb{I}_{r,0}[P_1^{-1}])$ non-trivial modulo $\mathfrak{c}_1^P P$. The Lie bracket $t = [u^+, u^-]$ is an element of $\mathfrak{G}_r \cap \mathfrak{t}(\mathbb{I}_{r,0}[P_1^{-1}])$ (where $\mathfrak{t}$ denotes the diagonal torus) and it is non-trivial modulo $(\mathfrak{c}_1^P)^2 P$. Hence the κ_P-vector space $\mathfrak{s}_{(\mathfrak{c}_1^P)^2} = \mathfrak{G}_r \cap (\mathfrak{c}_1^P)^2 \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r,\mathbb{C}_p}[P_1^{-1}])/\mathfrak{G}_r \cap (\mathfrak{c}_1^P)^2 P \cdot \mathfrak{sl}_2(\mathbb{I}_{0,r,\mathbb{C}_p}[P_1^{-1}])$ contains non-trivial diagonal, upper nilpotent and lower nilpotent elements, so it is three-dimensional. By Nakayama's lemma we conclude that $\mathfrak{G}_r \supset (\mathfrak{c}_1^P)^2 \cdot \mathfrak{sl}_2(\mathbb{I}_{r,0}[P_1^{-1}])$, so $(\mathfrak{c}_1^P)^2 \subset \mathfrak{l}^P$. $\square$

CHAPTER 3

A p-adic interpolation of the symmetric cube transfer

Let N be a positive integer. The goal of this chapter is to define a morphism of rigid analytic spaces $\mathcal{D}_1^N \to \mathcal{D}_2^M$, for an integer M depending on N, interpolating the classical Langlands lift of automorphic representations associated with the symmetric cube map $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{GSp}_4(\mathbb{C})$. The existence of this lift was proven by Kim and Shahidi in [**KS02**]. A technique for the p-adic interpolation of a lift defined at classical points was first developed by Chenevier in [**Ch05**], where he applied it to the Jacquet-Langlands correspondence. His arguments have been adapted to other known cases of classical Langlands functoriality by White [**Wh12**] and Ludwig ([**Lu14**], [**Lu14**]). In our context it will be more convenient to use the approach presented by Bellaïche and Chenevier in [**BC09**, Section 7.2.3], which is a reformulation of Chenevier's idea in terms of a notion of uniqueness of eigenvarieties. The advantage of this method is that it allows to work with Zariski-closed subspaces of the eigenvarieties that are not themselves eigenvarieties in the sense of Buzzard.

3.1. Galois representations attached to classical automorphic forms

We recall here the main properties of the Galois representations attached to classical eigenforms for GL_2 and GSp_4.

We recall that the cohomological weights are the integers k with $k \geq 2$ for GL_2 and the pairs of integers (k_1, k_2) with $k_1 \geq k_2 \geq 3$ for GSp_4.

Given two rings R and S and a morphism $\chi\colon R \to S$, we extend χ to a morphism of polynomial algebras $R[X] \to S[X]$ by applying it to the coefficients of each polynomial. We still denote this map by χ. In most cases R will be an abstract Hecke algebra, S a subfield of $\mathbb{C}_p$ or a ring of analytic functions on a rigid analytic space, and χ the system of Hecke eigenvalues associated with an eigenform or a family of eigenforms.

Theorem 3.1.1. *Let $g = 1$ or 2 and let f be a GSp_{2g}-eigenform of level N and cohomological weight. Let $\chi(f)\colon \mathcal{H}^N \to \overline{\mathbb{Q}}$ be the system of Hecke eigenvalues of f and, for a prime ℓ, let $\chi_\ell(f) = \iota_\ell \circ \chi(f)\colon \mathcal{H}^N \to \overline{\mathbb{Q}}_\ell$. When q varies over the rational primes, there exists a system of Galois representations*

$$\rho_{f,q}\colon G_{\mathbb{Q}} \to \mathrm{GSp}_{2g}(\overline{\mathbb{Q}}_q)$$

with the following properties:

(1) if ℓ is a prime not dividing Nq, $\rho_{f,q}$ is unramified at ℓ;

(2) if ℓ is a prime not dividing Nq and $\mathrm{Frob}_\ell \in G_{\mathbb{Q}}$ is a lift of the Frobenius automorphism at ℓ, then

$$\det(1 - X\rho_{f,q}(\mathrm{Frob}_\ell)) = \chi_\ell(f)(P_{\min}(t_{\ell,g}^{(g)}; X)). \tag{3.1}$$

This result is due to Eichler and Shimura for $g = 1$ and weight 2 [**Sh73**], Deligne for $g = 1$ and arbitrary weight [**De71**], Taylor [**Tay93**], Laumon [**Lau05**] and Weissauer [**Weiss05**] for $g = 2$.

There is an analogue of Theorem 3.1.1 for the local representation at q. See Section 3.10.1 for a summary of the basic definitions in p-adic Hodge theory, or the reference [**Fo94**]. Given $n \geq 1$ and a crystalline representation $\rho\colon G_{\mathbb{Q}_q} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_q)$, we denote by $\mathbf{D}_{\mathrm{cris}}(\rho)$ the module associated with ρ by Fontaine's theory: it is an n-dimensional $\overline{\mathbb{Q}}_q$-vector space endowed with a $\overline{\mathbb{Q}}_q$-linear Frobenius automorphism $\varphi_{\mathrm{cris}}(\rho)$. Faltings proved that if q does not divide N the representation $\rho_{f,q}|_{D_q}$ is crystalline (see Theorem 3.10.5(2)).

THEOREM 3.1.2. [**Ur05**, Theorem 1] *The Frobenius map $\varphi_{\mathrm{cris}}(\rho_{f,q})$ acting on $\mathbf{D}_{\mathrm{cris}}(\rho_{f,q})$ satisfies*

$$\det(1 - X\varphi_{\mathrm{cris}}(\rho_{f,q})) = \chi_q(f)P_{\min}(t_{q,g}^{(g)}; X). \tag{3.2}$$

REMARK 3.1.3. *Because of the analogy between Equations* (3.1) *and* (3.2) *the element $t_{q,g}^{(g)}$ is sometimes called the "Hecke-Frobenius" element.*

We recall some conditions for the representations $\rho_{f,\ell}$ to be irreducible.

THEOREM 3.1.4.

(1) [**Ri77**, Theorem 2.3] *Let f be a cuspidal GL_2-eigenform of cohomological weight. Then the ℓ-adic representation $\rho_{f,\ell}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ is irreducible for every prime ℓ.*

(2) ([**HT15**, Proposition 3.1], [**CG13**, Theorem 4.1]) *Let f be a cuspidal GSp_4-eigenform of cohomological weight. Suppose that f is neither CAP nor endoscopic and that the Langlands functoriality transfer from GSp_4 to GL_4 holds. Then the representation $\rho_{f,\ell}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ is absolutely irreducible for every prime ℓ.*

In the following we will always take q to be our fixed prime p not dividing N.

3.2. Generalities on the symmetric cube map

If R is a ring and M is a free R-module, we denote by $\mathrm{Sym}^3 M$ the symmetric cube of M. It is the quotient of the tensor product $M^{\otimes 3} = M \otimes M \otimes M$ by the R-submodule generated by the set

$$I_{\mathrm{Sym}^3} = \{m_1 \otimes m_2 \otimes m_3 - m_{\varepsilon(1)} \otimes m_{\varepsilon(2)} \otimes m_{\varepsilon(3)} \,|\, m_1, m_2, m_3 \in M,\, \varepsilon \in \mathscr{S}_3\},$$

where $\mathscr{S}_3$ is the group of permutations of $\{1, 2, 3\}$. There is a non-canonical isomorphism between $\mathrm{Sym}^3 M$ and the R-module $R[e_1, e_2, \dots, e_n]^{\deg 3}$ of homogeneous polynomials of degree 3 in n variables. If $\{e_1, e_2, \dots, e_n\}$ is an R-basis of M, one such isomorphism is given by the unique R-linear map sending $e_i \otimes e_j \otimes e_k$ to $e_ie_je_k$. We will often identify an element of $\mathrm{Sym}^3 M$ with its image in $R[e_1, e_2, \dots, e_n]^{\deg 3}$ via the isomorphism above.

If G is a group acting on the module M, we define an action of G on $M^{\otimes 3}$ by $g.(m_1 \otimes m_2 \otimes m_3) = g.m_1 \otimes g.m_2 \otimes g.m_3$. The module I_{Sym^3} is G-stable, hence there is a well-defined action of G on $\mathrm{Sym}^3 M$. We call it the symmetric cube of the G-module M. When M is two-dimensional and $\{e_1, e_2\}$ is a basis for M, the set $\{e_1^3, e_1^2e_2, e_1e_2^2, e_2^3\}$ is a basis for $\mathrm{Sym}^3 M$. These choices give identifications $\mathrm{GL}_2(R) \cong \mathrm{Aut}_R(M)$ and $\mathrm{GL}_4(R) \cong \mathrm{Aut}_R(\mathrm{Sym}^3 M)$. The action of $\mathrm{GL}_2(R)$ on M induces an action of $\mathrm{GL}_2(R)$ on $\mathrm{Sym}^3 M$, hence a group morphism $\mathrm{GL}_2(R) \to \mathrm{GL}_4(R)$. We call it the symmetric cube map and we denote it by $\mathrm{Sym}^3_R\colon \mathrm{GL}_2(R) \to \mathrm{GL}_4(R)$. Explicitly, for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)$ we have

$$\mathrm{Sym}^3_R \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ 3a^2c & a^2d + 2abc & b^2c + 2abd & 3b^2d \\ 3ac^2 & c^2b + 2acd & d^2a + 2bcd & 3bd^2 \\ c^3 & 3c^2d & 3cd^2 & d^3 \end{pmatrix}$$

A direct calculation shows that $\ker(\mathrm{Sym}^3_R) = \mu_3(R)$. Since $\mathrm{GL}_2(R)$ preserves the symplectic form on M defined by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the image of Sym^3_R preserves the symplectic form on $\mathrm{Sym}^3 M$ defined by the matrix

$$\mathrm{Sym}^3_R \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Hence Sym^3_R defines a morphism $\mathrm{Sym}^3_R \colon \mathrm{GL}_2(R) \to \mathrm{GSp}_4(R)$. This map arises from a morphism $\mathrm{Sym}^3 \colon \mathrm{GL}_2 \to \mathrm{GL}_4$ of group schemes over $\mathbb{Z}$. The following is an exact sequence of group schemes over $\mathbb{Z}$:

$$0 \to \mu_3 \to \mathrm{GL}_2 \to \mathrm{GSp}_4.$$

From now on we will drop the subscript R and simply write $\mathrm{Sym}^3 \colon \mathrm{GL}_2(R) \to \mathrm{GSp}_4(R)$ for the morphism induced by Sym^3. For every representation ρ of a group with values in $\mathrm{GL}_2(R)$ we set $\mathrm{Sym}^3\rho = \mathrm{Sym}^3 \circ \rho$.

Let $g \in \mathrm{GL}_2(R)$. Let g act on R^2 via the standard representation and let

$$P(g; X) = \det(\mathbb{1} - X \cdot g) = X^2 - TX + D$$

be the characteristic polynomial of g. If the eigenvalues of g are elements α and β of an extension of R, the eigenvalues of the element $\mathrm{Sym}^3 g$ of $\mathrm{GSp}_4(R)$ acting on R^4 via the standard representation are α^3, $\alpha^2\beta$, $\alpha\beta^2$, β^3. Then a simple calculation with the symmetric functions T and D of α and β shows that the characteristic polynomial of $\mathrm{Sym}^3 g$ is

$$\begin{aligned} &P(\mathrm{Sym}^3 g; X) = \det(1 - X \cdot \mathrm{Sym}^3 g) = \\ &= X^4 - (T^3 - 2TD)X^3 + (T^4 - 3DT^2 + 2D^2)X^2 - D^3(T^3 + 2TD)X + D^6. \end{aligned} \tag{3.3}$$

If $T, D \in R$ are arbitrary and $P(X) = X^2 - TX + D$, we define the symmetric cube of $P(X)$ as

$$\mathrm{Sym}^3 P(X) = X^4 - (T^3 - 2TD)X^3 + (T^4 - 3DT^2 + 2D^2)X^2 - D^3(T^3 + 2TD)X + D^6.$$

3.3. The classical symmetric cube transfer

Kim and Shahidi proved the existence of a Langlands functoriality transfer from GL_2 to GL_4 associated with $\mathrm{Sym}^3 \colon \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_4(\mathbb{C})$ [**KS02**, Theorem B]. Thanks to an unpublished result by Jacquet, Piatetski-Shapiro and Shalika [**KS02**, Theorem 9.1], this transfer descends to GSp_4. We briefly recall these results.

Let $\pi = \bigotimes_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, where v varies over the places of $\mathbb{Q}$. Let ρ_v be the two-dimensional representation of the Weil-Deligne group of $\mathbb{Q}_v$ attached to π_v. Consider the four-dimensional representation $\mathrm{Sym}^3\rho_v = \mathrm{Sym}^3 \circ \rho_v$ of the same group. By the local Langlands correspondence for GL_4, $\mathrm{Sym}^3\rho_v$ is attached to an automorphic representation $\mathrm{Sym}^3\pi_v$ of $\mathrm{GL}_4(\mathbb{Q}_v)$. Define a representation of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$ as $\mathrm{Sym}^3\pi = \bigotimes_v \mathrm{Sym}^3\pi_v$. Then we have the following theorems.

Theorem 3.3.1. [**KS02**, Theorem B] *The representation* $\mathrm{Sym}^3\pi$ *is an automorphic representation of* $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$. *If* π *is attached to a non-CM eigenform of weight* $k \geq 2$, *then* $\mathrm{Sym}^3\pi$ *is cuspidal.*

Theorem 3.3.2. [**KS02**, after Theorem 9.1] *If* π *is attached to a non-CM eigenform of weight* $k \geq 2$, *then there exists a globally generic cuspidal automorphic representation* Π *of* $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ *such that* $\mathrm{Sym}^3\pi$ *is the functorial lift of* Π *under the embedding* $\mathrm{GSp}_4(\mathbb{C}) \hookrightarrow \mathrm{GL}_4(\mathbb{C})$.

If K is a compact open subgroup of $\mathrm{GSp}_4(\widehat{\mathbb{Z}})$, we call level of K the smallest integer M such that K contains the principal congruence subgroup of $\mathrm{GSp}_4(\widehat{\mathbb{Z}})$ of level M. Given an automorphic representation Π of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$, we call level of Π the smallest integer M such that the finite component of Π admits an invariant vector by a compact open subgroup of $\mathrm{GSp}_4(\widehat{\mathbb{Z}})$ of level M.

Recall that we fixed for every prime ℓ an embedding $G_{\mathbb{Q}_\ell} \hookrightarrow G_\mathbb{Q}$. If $\sigma\colon G_\mathbb{Q} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is a representation and ℓ is a prime different from p, set $\sigma_\ell = \sigma|_{G_{\mathbb{Q}_\ell}}$. We denote by $N(\sigma,\ell)$ the conductor of σ_ℓ, defined in [**Ser70**]. The prime-to-p conductor of σ is defined as $N(\sigma) = \prod_{\ell\neq p} N(\sigma,\ell)$. We recall a standard formula giving $N(\sigma,\ell)$ for every ℓ prime to p (see [**Liv89**, Proposition 1.1]). Let $I \subset G_{\mathbb{Q}_\ell}$ be an inertia subgroup and for $k \geq 1$ let I_k be its higher inertia subgroups. Let V be the two-dimensional $\overline{\mathbb{Q}}_p$-vector space on which $G_\mathbb{Q}$ acts via σ. For every subgroup $H \subset G_Q$ let $d_{H,\sigma}$ be the codimension of the subspace of V fixed by $\sigma(H)$. Then $N(\sigma,\ell) = \ell^{n_{\sigma,\ell}}$, where

$$n_{\sigma,\ell} = d_{I,\sigma_\ell} + \sum_{k\geq 1} \frac{d_{I_k,\overline{\sigma_\ell}}}{[I\colon I_k]}. \tag{3.4}$$

Write Π_f for the component of Π at the finite places and Π_∞ for the component of Π at ∞. Since the representation Π given by the above theorem is globally generic, it does not correspond to a holomorphic modular form for GSp_4. However Ramakrishnan and Shahidi showed that the generic representation Π_∞ can be replaced by a holomorphic representation $\Pi_\infty^{\mathrm{hol}}$ such that $\Pi_f \otimes \Pi_\infty^{\mathrm{hol}}$ belongs to the L-packet of Π. This is the content of [**RS07**, Theorem A′], that we recall below. Note that in *loc. cit.* the theorem is stated only for π associated with a form f of level $\Gamma_0(N)$ and even weight $k \geq 2$, but Ramakrishnan pointed out that the proof also works when f has level $\Gamma_1(N)$ and arbitrary weight $k \geq 2$. The theorem also gives an information on the level of the representation produced by the lift.

Let π be the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ associated with a cuspidal, non-CM eigenform f of weight $k \geq 2$ and level $\Gamma_1(N)$ for some $N \geq 1$. Let p be a prime not dividing N and let $\rho_{f,p}$ be the p-adic Galois representation attached to f.

Theorem 3.3.3. (see [**RS07**, Theorem A′]) *There exists a cuspidal automorphic representation* $\Pi^{\mathrm{hol}} = \bigotimes_v \Pi_v^{\mathrm{hol}}$ *of* $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$, *satisfying:*

(1) $\Pi_\infty^{\mathrm{hol}}$ *is in the holomorphic discrete series;*
(2) $L(s,\Pi^{\mathrm{hol}}) = L(s,\pi,\mathrm{Sym}^3)$*;*
(3) Π^{hol} *is unramified at primes not dividing* N*;*
(4) Π^{hol} *admits an invariant vector by a compact open subgroup* K *of* $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ *of level* $N(\mathrm{Sym}^3\rho_{f,p})$.

We deduce the following corollary.

Corollary 3.3.4. *Let* f *be a cuspidal, non-CM* GL_2*-eigenform of weight* $k \geq 2$. *For every prime* ℓ *let* $\rho_{f,\ell}$ *be the* ℓ*-adic Galois representation associated with* f. *There exists a cuspidal* GSp_4*-eigenform* F *of weight* $(2k-1,k+1)$ *with associated* ℓ*-adic Galois representation* $\mathrm{Sym}^3\rho_{f,\ell}$ *for every prime* ℓ. *For every prime* p *not dividing* N, *the level of* F *is a divisor of the prime-to-*p *conductor of* $\mathrm{Sym}^3\rho_{f,p}$.

Note that the weight $(2k-1,k+1)$ is cohomological since $k \geq 2$.

Proof. Everything except for the weight of F follows immediately from Theorem 3.3.3. We obtain the weight of F by looking at the Galois representation $\rho_{F,p}$ for a prime $p \nmid N$. See Section 3.10.1 below for a summary of the definitions and results we need from p-adic Hodge theory. Let E be a finite extension of $\mathbb{Q}_p$ such that the representation $\rho_{f,p}$ is defined with coefficients in E and let V be a two-dimensional E-vector space on which $G_{\mathbb{Q}_p}$ acts via $\rho_{f,p}$. Let $G_{\mathbb{Q}_p}$ act on Sym^3V via the representation $\mathrm{Sym}^3\rho_{f,p} \cong \rho_{F,p}$. By Remark 3.10.6 V is a

Hodge-Tate representation with Hodge-Tate weights $(0, k-1)$, which means that the $\mathbb{Q}_p$-vector space

$$(\mathbb{C}_p t^i \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$$

is one-dimensional if $i = 0$ or $i = -(k-1)$ and zero-dimensional otherwise. Let $v_0, v_{k-1} \in V$ be two elements such that $t^i \otimes v_0$ and $t^i \otimes v_{k-1}$ are $G_{\mathbb{Q}_p}$-invariant. Then the elements

$$\begin{gathered}
1 \otimes v_0 \otimes v_0 \otimes v_0 \in \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{Sym}^3 V \\
t^{-(k-1)} \otimes v_0 \otimes v_0 \otimes v_{k-1} \in \mathbb{C}_p t^{-(k-1)} \otimes_{\mathbb{Q}_p} \mathrm{Sym}^3 V \\
t^{-2(k-1)} \otimes v_0 \otimes v_{k-1} \otimes v_{k-1} \in \mathbb{C}_p t^{-2(k-1)} \otimes_{\mathbb{Q}_p} \mathrm{Sym}^3 V \\
t^{-3(k-1)} \otimes v_{k-1} \otimes v_{k-1} \otimes v_{k-1} \in \mathbb{C}_p t^{-3(k-1)} \otimes_{\mathbb{Q}_p} \mathrm{Sym}^3 V
\end{gathered}$$

are $G_{\mathbb{Q}_p}$-invariant, hence the Hodge-Tate weights of $\mathrm{Sym}^3 V$ are $(0, k-1, 2(k-1), 3(k-1))$. By Remark 3.10.6 we deduce that the weight of F is $(k_1, k_2) = (2k-1, k+1)$. □

We denote by $\mathrm{Sym}^3 f$ the cuspidal Siegel eigenform given by the corollary. Let $N(f)$ and $N(\mathrm{Sym}^3 f)$ be the levels of f and $\mathrm{Sym}^3 f$, respectively. Thanks to the property (4) in Theorem 3.3.3 we can give an upper bound for $N(\mathrm{Sym}^3 f)$ in terms of $N(f)$ by comparing $N(\mathrm{Sym}^3 \rho_{f,p})$ and $N(\rho_{f,p})$ for a prime p not dividing $N(f)$.

As before let $\sigma\colon G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ be a representation and for every prime ℓ let $\sigma_\ell = \sigma|_{G_{\mathbb{Q}_\ell}}$.

Lemma 3.3.5. *For every prime ℓ different from p we have $N(\mathrm{Sym}^3 \sigma_\ell) \mid N(\sigma_\ell)^3$. In particular $N(\mathrm{Sym}^3 \sigma) \mid N(\sigma)^3$.*

Proof. We use the notations of formula (3.4). We see immediately that, for every subgroup H of $G_{\mathbb{Q}}$:

(1) if $d_{H,\sigma} = 0$ then $d_{H,\mathrm{Sym}^3\sigma} = 0$;
(2) if $d_{H,\sigma} = 1$ then $d_{H,\mathrm{Sym}^3\sigma} \leq 3$;
(3) if $d_{H,\sigma} = 2$ then trivially $d_{H,\mathrm{Sym}^3\sigma} \leq 4$.

In all cases $d_{H,\mathrm{Sym}^3\sigma} \leq 3 d_{H,\sigma}$, so formula (3.4) gives $N(\mathrm{Sym}^3\sigma, \ell) \mid N(\sigma,\ell)^3$. Since the prime-to-$p$ conductor is defined as the product of the conductors at the primes ℓ different from p, we obtain that $N(\mathrm{Sym}^3\sigma) \mid N(\sigma)^3$. □

Definition 3.3.6. *Let N be a positive integer and let $N = \prod_{i=1}^d \ell_i^{a_i}$ be its decomposition in prime factors, with $\ell_i \neq \ell_j$ if $i \neq j$. For every $i \in \{1, 2, \dots, d\}$ set:*

- $a_i' = 1$ *if* $a_i = 1$;
- $a_i' = 3a_i$ *if* $a_i > 1$.

We define an integer M, depending on N, by $M = \prod_{i=1}^d \ell_i^{a_i'}$.

Corollary 3.3.7. *Let $N = N(f)$ and let $M = M(N)$ be the integer given by Definition 3.3.6. Then $N(\mathrm{Sym}^3 f) \mid M$.*

Proof. Let $\pi_f = \bigotimes_\ell \pi_{f,\ell}$ be the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ associated with f. Let $\pi_{\mathrm{Sym}^3 f} = \bigotimes_\ell \pi_{\mathrm{Sym}^3 f,\ell}$ be the automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ associated with $\mathrm{Sym}^3 f$. For every prime ℓ the Galois representations associated with the local components $\pi_{f,\ell}$ and $\pi_{\mathrm{Sym}^3 f,\ell}$ are $\rho_{f,\ell}$ and $\mathrm{Sym}^3 \rho_{f,\ell}$, respectively. As before let $N = \prod_{i=1}^d \ell_i^{a_i}$ be the decomposition of N in prime factors. If $\ell \nmid N$ the representation $\pi_{f,\ell}$ is unramified, so $\pi_{\mathrm{Sym}^3 f,\ell}$ is also unramified.

Let $i \in \{1, 2, \dots, d\}$. If $a_i = 1$ the local component π_{f,ℓ_i} is Iwahori-spherical, hence Steinberg. Then the image of the inertia subgroup at ℓ_i via ρ_{f,ℓ_i} contains a regular unipotent element u. The image of the inertia subgroup at ℓ_i via $\mathrm{Sym}^3 \rho_{f,\ell_i}$ contains the regular unipotent element $\mathrm{Sym}^3 u$, so the automorphic representation $\pi_{\mathrm{Sym}^3 f,\ell_i}$ is Iwahori-spherical. Hence the factor ℓ appears with exponent one in $N(\mathrm{Sym}^3 f)$.

Now suppose that $a_i > 1$. By a classical result the conductor of the representation ρ_{f,ℓ_i} is a divisor of $\ell_i^{a_i}$. Let p be a prime not dividing N. By Lemma 3.3.5 the conductor $N(\mathrm{Sym}^3\rho_{f,p},\ell)$ divides $N(\rho_{f,p},\ell)^3$. Hence $N(\mathrm{Sym}^3\rho_{f,p}\,ell)$ divides $\ell_i^{3a_i}$. By Corollary 3.3.4 the power of ℓ_i dividing the level of $\mathrm{Sym}^3 f$ is at most the power of ℓ_i dividing the conductor $N(\mathrm{Sym}^3\rho_{f,p},\ell)$, which is $\ell_i^{3a_i}$. Hence the factor ℓ appears with exponent at most $3a_i$ in $N(\mathrm{Sym}^3 f)$. □

3.4. The morphisms of Hecke algebras

As usual we fix an integer $N \geq 1$ and a prime p not dividing N. We work with the abstract Hecke algebras $\mathcal{H}_1^N$, $\mathcal{H}_2^N$ defined in Section 1.2.4. Recall that they are spherical outside N and Iwahoric dilating at p.

Let f^{st} be the stabilization of a non-CM GL_2-eigenform f of level $\Gamma_1(N)$ and weight $k \geq 2$. Let $\chi_1^{Np}\colon \mathcal{H}_1^{Np} \to \overline{\mathbb{Q}}_p$ and $\chi_1^N\colon \mathcal{H}_1^N \to \overline{\mathbb{Q}}_p$ be the systems of Hecke eigenvalues of f and f^{st}, respectively. In general, and conjecturally always, there are two different forms p-stabilizations of f.

Let M be the integer given by Definition 3.3.6, depending on N. Let $\mathrm{Sym}^3 f$ be the classical, cuspidal GSp_4-eigenform of level M given by Corollary 3.3.4. Since M and N have the same prime factors, $\mathrm{Sym}^3 f$ is an eigenform for the action of $\mathcal{H}_2^{Np}$ and thus it defines a system of Hecke eigenvalues $\chi_2^{Np}\colon \mathcal{H}_2^{Np} \to \overline{\mathbb{Q}}_p$. We p-stabilize $\mathrm{Sym}^3 f$ to obtain a form of Iwahoric level. There are in general eight different p-stabilizations of $\mathrm{Sym}^3 f$. Each of them defines a system of Hecke eigenvalues $\mathcal{H}_2^N \to \overline{\mathbb{Q}}_p$. In Propositions 3.4.2 and 3.4.5 we will compute the systems of eigenvalues of all possible p-stabilizations of $\mathrm{Sym}^3 f$ in terms of that of χ_1^{Np}.

If χ is a system of Hecke eigenvalues, we write χ_ℓ for its local component at the prime ℓ.

Definition 3.4.1. *For every prime $\ell \nmid Np$, let*

$$\lambda_\ell\colon \mathcal{H}(\mathrm{GSp}_4(\mathbb{Q}_\ell),\mathrm{GSp}_4(\mathbb{Z}_\ell)) \to \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_\ell),\mathrm{GL}_2(\mathbb{Z}_\ell))$$

be the homomorphism defined by

$$\begin{gathered} T_{\ell,0}^{(2)} \mapsto (T_{\ell,0}^{(1)})^3, \\ T_{\ell,1}^{(2)} \mapsto -(T_{\ell,1}^{(1)})^6 + (4\ell-2)T_{\ell,0}^{(1)}(T_{\ell,1}^{(1)})^4 + (6\ell-4\ell^2)(T_{\ell,1}^{(1)})^2(T_{\ell,1}^{(1)})^2 - 3\ell^2(T_{\ell,0}^{(1)})^3, \\ T_{\ell,2}^{(2)} \mapsto (T_{\ell,1}^{(1)})^3 - 2\ell T_{\ell,1}^{(1)}T_{\ell,0}^{(1)}. \end{gathered}$$

Let $\lambda^{Np}\colon \mathcal{H}_2^{Np} \to \mathcal{H}_1^{Np}$ be the homomorphism defined by $\lambda^{Np} = \bigotimes_{\ell\nmid Np}\lambda_\ell$.

Proposition 3.4.2. *Let R be a ring. Let $\chi_1^{Np}\colon \mathcal{H}_1^{Np} \to R$, $\chi_2^{Np}\colon \mathcal{H}_2^{Np} \to R$ be two homomorphisms and let $\rho_1\colon G_\mathbb{Q} \to \mathrm{GL}_2(R)$, $\rho_2\colon G_\mathbb{Q} \to \mathrm{GSp}_4(R)$ be two representations satisfying:*
(1) for $g = 1,2$ ρ_g is unramified outside Np;
(2) for $g = 1,2$, every prime $\ell \nmid Np$ and a lift $\mathrm{Frob}_\ell \in G_\mathbb{Q}$ of the Frobenius at ℓ,

$$\det(1 - X\rho_i(\mathrm{Frob}_\ell)) = \chi_i^{Np}(P_{\min}(t_{\ell,g}^{(g)};X));$$

(3) there is an isomorphism $\rho_2 \cong \mathrm{Sym}^3\rho_1$.
Then λ^{Np} is the only homomorphism $\mathcal{H}_2^{Np} \to \mathcal{H}_1^{Np}$ such that $\chi_2^{Np} = \chi_1^{Np}\circ\lambda^{Np}$.

Proof. Let ℓ be a prime not dividing Np. By Equation (1.3) we have $P_{\min}(t_{\ell,1}^{(1)};X) = X^2 - T_{\ell,1}^{(1)}(f)X + \ell T_{\ell,0}^{(1)}$. Hence hypothesis (2) with $g = 1$ gives

$$\det(1 - X\rho_i(\mathrm{Frob}_\ell)) = \chi_1^{Np}(X^2 - T_{\ell,1}^{(1)}(f)X + \ell T_{\ell,0}^{(1)}). \tag{3.5}$$

Then Equation (3.3) allows us to compute
(3.6)

$$\det(1 - X\mathrm{Sym}^3\rho(\mathrm{Frob}_\ell)) = X^4 - (T_{\ell,1}^{(1)} - 2\ell T_{\ell,1}^{(1)}T_{\ell,0}^{(1)})X^3 + \\ +((T_{\ell,1}^{(1)})^4 - 3\ell T_{\ell,0}^{(1)}(T_{\ell,1}^{(1)})^2 + 2\ell^2(T_{\ell,0}^{(1)})^2)X^2 - \ell^3(T_{\ell,0}^{(1)})^3((T_{\ell,1}^{(1)})^3 + 2\ell T_{\ell,1}^{(1)}T_{\ell,0}^{(1)})X + \ell^6(T_{\ell,0}^{(1)})^6.$$

By Equation (1.5) we have

$$P_{\min}(t_{\ell,2}^{(2)}; X) = X^4 - T_{\ell,2}^{(2)}X^3 + ((T_{\ell,2}^{(2)})^2 - T_{\ell,1}^{(2)} - \ell^2 T_{\ell,0}^{(2)})X^2 + \\ -\ell^3 T_{\ell,2}^{(2)}T_{\ell,0}^{(2)}X + \ell^6(T_{\ell,0}^{(2)})^2,$$

so hypothesis (2) with $g = 2$ gives

$$\det(1 - X\mathrm{Sym}^3\rho(\mathrm{Frob}_\ell)) = \chi_2^{Np}(X^4 - T_{\ell,2}^{(2)}X^3 + \\ +((T_{\ell,2}^{(2)})^2 - T_{\ell,1}^{(2)} - \ell^2 T_{\ell,0}^{(2)})X^2 - \ell^3 T_{\ell,2}^{(2)}T_{\ell,0}^{(2)}X + \ell^6(T_{\ell,0}^{(2)})^2). \tag{3.7}$$

By comparing the coefficients of the right hand sides of Equations (3.6) and (3.7) we obtain the relations

$$\chi_2^{Np}(T_{\ell,1}^{(2)}) = \chi_1^{Np}(-(T_{\ell,1}^{(1)})^6 + (4\ell - 2)T_{\ell,0}^{(1)}(T_{\ell,1}^{(1)})^4 + (6\ell - 4\ell^2)(T_{\ell,1}^{(1)})^2(T_\ell^{(1)})^2 - 3\ell^2(T_{\ell,0}^{(1)})^3),$$

$$\chi_2^{Np}(T_{\ell,2}^{(2)}) = \chi_1^{Np}((T_\ell^{(1)})^3 - 2\ell T_{\ell,1}^{(1)}T_{\ell,0}^{(1)}),$$

$$\chi_2^{Np}(T_{\ell,0}^{(2)}) = \chi_1^{Np}((T_{\ell,0}^{(1)})^3).$$

We deduce that λ_ℓ is the only homomorphism $\mathcal{H}(\mathrm{GSp}_4(\mathbb{Q}_\ell), \mathrm{GSp}_4(\mathbb{Z}_\ell)) \to \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_\ell), \mathrm{GL}_2(\mathbb{Z}_\ell))$ satisfying $\chi_2^{Np}(\mathrm{Sym}^3 f) = \chi_1^{Np} \circ \lambda_\ell$. Since this is true for every $\ell \nmid Np$, we conclude that λ^{Np} is the only homomorphism $\mathcal{H}_2^{Np} \to \mathcal{H}_1^{Np}$ satisfying $\chi_2^{Np} = \chi_1^{Np} \circ \lambda^{Np}$. □

As a special case of Proposition 3.4.2 we obtain the following corollary.

Corollary 3.4.3. *Let f be a classical, non-CM GL_2-eigenform f of level $\Gamma_1(N)$ and system of eigenvalues $\chi_1^{Np}\colon \mathcal{H}_1^{Np} \to \overline{\mathbb{Q}}_p$ outside Np. Let $\mathrm{Sym}^3 f$ be the symmetric cube lift of f given by Corollary 3.3.4. Then the system of eigenvalues χ_2^{Np} of $\mathrm{Sym}^3 f$ outside Np is $\chi_1^{Np} \circ \lambda^{Np}\colon \mathcal{H}_2^{Np} \to \overline{\mathbb{Q}}_p$.*

Proof. The corollary follows from Proposition 3.4.2 applied to $R = \overline{\mathbb{Q}}_p$, χ_1^{Np} and χ_2^{Np} as in the statement, $\rho_1 = \rho_{f,p}$ and $\rho_2 = \rho_{\mathrm{Sym}^3 f,p}$. □

Now we study the systems of Hecke eigenvalues of the p-stabilizations of $\mathrm{Sym}^3 f$.

Definition 3.4.4. *For $i \in \{1, 2, \ldots, 8\}$ we define morphisms*

$$\lambda_{i,p}\colon \mathcal{H}(T_2(\mathbb{Q}_p), T_2(\mathbb{Z}_p))^- \to \mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p)).$$

For $i \in \{1, 2, 3, 4\}$ the morphism $\lambda_{i,p}$ is defined on a set of generators of $\mathcal{H}(T_2(\mathbb{Q}_p), T_2(\mathbb{Z}_p))^-$ as follows:

(1) $\lambda_{1,p}$ maps

$$t_{p,0}^{(2)} \mapsto (t_{p,0}^{(1)})^3,$$
$$t_{p,1}^{(2)} \mapsto t_{p,0}^{(1)}(t_{p,1}^{(1)})^4,$$
$$t_{p,2}^{(2)} \mapsto (t_{p,1}^{(1)})^3;$$

(2) $\lambda_{2,p}$ maps

$$t_{p,0}^{(2)} \mapsto (t_{p,0}^{(1)})^3,$$
$$t_{p,1}^{(2)} \mapsto (t_{p,0}^{(1)})^2(t_{p,1}^{(1)})^4,$$
$$t_{p,2}^{(2)} \mapsto (t_{p,1}^{(1)})^3;$$

(3) $\lambda_{3,p}$ maps

$$t_{p,0}^{(2)} \mapsto (t_{p,0}^{(1)})^3,$$
$$t_{p,1}^{(2)} \mapsto t_{p,0}^{(1)}(t_{p,1}^{(1)})^4,$$
$$t_{p,2}^{(2)} \mapsto t_{p,0}^{(1)}t_{p,1}^{(1)};$$

(4) $\lambda_{4,p}$ maps

$$t_{p,0}^{(2)} \mapsto (t_{p,0}^{(1)})^3,$$
$$t_{p,1}^{(2)} \mapsto (t_{p,0}^{(1)})^4(t_{p,1}^{(1)})^{-2},$$
$$t_{p,2}^{(2)} \mapsto t_{p,0}^{(1)}t_{p,1}^{(1)}.$$

For $i \in \{5,6,7,8\}$ the morphism $\lambda_{i,p}\colon \mathcal{H}(T_2(\mathbb{Q}_p), T_2(\mathbb{Z}_p)) \to \mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p))$ is given by

$$\lambda_{i,p} = \delta \circ \lambda_{i-4,p}$$

where δ is the automorphism of $\mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p))$ defined on a set of generators of the subalgebra $\mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p))^-$ by

$$\delta(t_{p,0}^{(1)}) = t_{p,0}^{(1)} \qquad (3.8)$$
$$\delta(t_{p,1}^{(1)}) = t_{p,0}^{(1)}(t_{p,1}^{(1)})^{-1}$$

and extended in the unique way.

Let f^{st} be the p-stabilization of a classical, non-CM GL_2-eigenform f of level $\Gamma_1(N)$. Let $\chi_{1,p}\colon \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{GL}_2(\mathbb{Z}_p)) \to \overline{\mathbb{Q}}_p$ and $\chi_{1,p}^{\mathrm{st}}\colon \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), I_{1,p})^- \to \overline{\mathbb{Q}}_p$ be the system of Hecke eigenvalues at p of f and f^{st}, respectively. Note that $\chi_{1,p}$ is the restriction of $\chi_{1,p}^{\mathrm{st}}$ to the abstract spherical Hecke algebra at p. Let $(\mathrm{Sym}^3 f)^{\mathrm{st}}$ be a p-stabilization of $\mathrm{Sym}^3 f$. Let $\chi_{2,p}\colon \mathcal{H}(\mathrm{GSp}_4(\mathbb{Q}_p), \mathrm{GSp}_4(\mathbb{Z}_p)) \to \overline{\mathbb{Q}}_p, \chi_{2,p}^{\mathrm{st}}\colon \mathcal{H}(\mathrm{GSp}_4(\mathbb{Q}_p), I_{2,p})^- \to \overline{\mathbb{Q}}_p$ be the systems of Hecke eigenvalues at p of $\mathrm{Sym}^3 f$ and $(\mathrm{Sym}^3 f)^{\mathrm{st}}$, respectively. Again $\chi_{2,p}$ is the restriction of $\chi_{2,p}^{\mathrm{st}}$ to the abstract spherical Hecke algebra at p.

By Lemma 1.2.15, for $g = 1,2$ there is an isomorphism

$$\iota_{I_{2,p}}^{T_2}\colon \mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_p), I_{g,p})^- \to \mathcal{H}(T_g(\mathbb{Q}_p), T_g(\mathbb{Z}_p))^-.$$

Let $\iota_{T_2}^{I_{2,p}}\colon \mathcal{H}(T_g(\mathbb{Q}_p), T_g(\mathbb{Z}_p))^- \to \mathcal{H}(\mathrm{GSp}_{2g}(\mathbb{Q}_p), I_{g,p})^-$ be its inverse. In particular $\chi_g^{\mathrm{st}} \circ \iota_{T_g}^{I_{g,p}}$ is a character $\mathcal{H}(T_g(\mathbb{Q}_p), T_g(\mathbb{Z}_p))^- \to \overline{\mathbb{Q}}_p$. By Remark 1.2.13 the character $\chi_{g,p}^{\mathrm{st}} \circ \iota_{T_g}^{I_{g,p}}$ can be extended uniquely to a character $(\chi_{g,p}^{\mathrm{st}} \circ \iota_{T_g}^{I_{g,p}})^{\mathrm{ext}}\colon \mathcal{H}(T_g(\mathbb{Q}_p), T_g(\mathbb{Z}_p)) \to \overline{\mathbb{Q}}_p$.

Proposition 3.4.5. *There exists $i \in \{1,2,\dots,8\}$ such that*

$$\chi_2^{\mathrm{st}} \circ \iota_{I_{2,p}}^{T_2} = (\chi_1^{\mathrm{st}} \circ \iota_{I_{1,p}}^{T_1})^{\mathrm{ext}} \circ \lambda_{i,p}.$$

Moreover, if $\lambda_p \mathcal{H}(T_2(\mathbb{Q}_p), T_2(\mathbb{Z}_p)) \to \mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p))$ is another morphism satisfying

$$\chi_2^{\mathrm{st}} \circ \iota_{I_{2,p}}^{T_2} = (\chi_1^{\mathrm{st}} \circ \iota_{I_{1,p}}^{T_1})^{\mathrm{ext}} \circ \lambda_p,$$

then there exists $i \in \{1,2,\dots,8\}$ such that $\lambda_p = \lambda_{i,p}$.

Proof. We will use Equation (3.2) in order to construct the local morphisms. In this proof we will leave the composition with the isomorphism $\iota_{I_{1,p}}^{T_1}$ and $\iota_{I_{2,p}}^{T_2}$ implicit and we will consider $\chi_{1,p}^{\mathrm{st}}$ and $\chi_{2,p}^{\mathrm{st}}$ as characters respectively of $\mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p))^-$ and $\mathcal{H}(T_2(\mathbb{Q}_p), T_2(\mathbb{Z}_p))^-$ for notational ease. Let $\rho_{f,p}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be the p-adic Galois representation associated with f, so that the p-adic Galois representation associated with $\mathrm{Sym}^3 f$ is $\mathrm{Sym}^3 \rho_{f,p}$. Via p-adic Hodge theory (see Section 3.10.1 below) we attach to $\rho_{f,p}$ a two-dimensional $\overline{\mathbb{Q}}_p$-vector space

$\mathbf{D}_{\mathrm{cris}}(\rho_{f,p})$ endowed with a $\overline{\mathbb{Q}}_p$-linear Frobenius endomorphism $\varphi_{\mathrm{cris}}(\rho_{f,p})$. By Equation (3.2) specialized to $g=1$ and to the form f we obtain

$$\det(1-X\varphi_{\mathrm{cris}}(\rho_{f,p}))=\chi_{1,p}(P_{\min}(t_{p,2}^{(2)};X)).$$

We will use the notations of Section 1.2.4.3 for the elements of the Weyl groups of GL_2 and GSp_4. Let α_p and β_p be the two roots of $\chi_{1,p}(P_{\min}(t_{p,2}^{(2)};X))$, ordered so that $\chi_{1,p}^{\mathrm{st}}(t_{p,1}^{(1)})=\alpha_p$. With this choice we have $\beta_p=\chi_{1,p}^{\mathrm{st}}((t_{p,1}^{(1)})^w)$.

Let $\mathbf{D}_{\mathrm{cris}}(\rho_{\mathrm{Sym}^3 f,p})$ be the 4-dimensional $\overline{\mathbb{Q}}_p$-vector space attached to $\rho_{\mathrm{Sym}^3 f,p}$ by p-adic Hodge theory. Denote by $\varphi_{\mathrm{cris}}(\rho_{\mathrm{Sym}^3 f,p})$ the Frobenius endomorphism acting on $\mathbf{D}_{\mathrm{cris}}(\rho_{\mathrm{Sym}^3 f,p})$. By Equation (3.2) specialized to $g=2$ and to the form $\mathrm{Sym}^3 f$ we obtain

$$\det(1-X\varphi_{\mathrm{cris}}(\rho_{\mathrm{Sym}^3 f,p}))=\chi_{2,p}(P_{\min}(t_{p,2}^{(2)};X)).$$

Note that the coefficients of $P_{\min}(t_{p,2}^{(2)};X)$ belong to the spherical Hecke algebra at p, so we have $\chi_{2,p}^{\mathrm{st}}(P_{\min}(t_{p,2}^{(2)};X))=\chi_{2,p}(P_{\min}(t_{p,2}^{(2)};X))$. From $\rho_{\mathrm{Sym}^3 f,p}=\mathrm{Sym}^3\rho_{f,p}$ we deduce that

$$\chi_{2,p}^{\mathrm{st}}(P_{\min}(t_{p,2}^{(2)};X))=\det(1-X\varphi_{\mathrm{cris}}(\rho_{\mathrm{Sym}^3 f,p}))=(X-\alpha_p^3)(X-\alpha_p^2\beta_p)(X-\alpha_p\beta_p^2)(X-\beta_p^3). \tag{3.9}$$

By developing the left hand side via Equation (1.4) and the right hand side via Equation (3.9) we obtain

$$\begin{aligned}(X-\chi_{2,p}^{\mathrm{st}}(t_{\ell,2}^{(2)}))(X-\chi_{2,p}^{\mathrm{st}}((t_{\ell,2}^{(2)})^{w_1}))\cdot(X-\chi_{2,p}^{\mathrm{st}}((t_{\ell,2}^{(2)})^{w_2}))(X-\chi_{2,p}^{\mathrm{st}}((t_{\ell,2}^{(2)})^{w_1w_2}))=\\=(X-\alpha_p^3)(X-\alpha_p^2\beta_p)(X-\alpha_p\beta_p^2)(X-\beta_p^3).\end{aligned}$$

In particular the sets

$$\{\chi_{2,p}^{\mathrm{st}}(t_{\ell,2}^{(2)}),\chi_{2,p}^{\mathrm{st}}((t_{\ell,2}^{(2)})^{w_1}),\chi_{2,p}^{\mathrm{st}}((t_{\ell,2}^{(2)})^{w_2}),\chi_{2,p}^{\mathrm{st}}((t_{\ell,2}^{(2)})^{w_1w_2})\}$$

and

$$\{\alpha_p^3,\alpha_p^2\beta_p,\alpha_p\beta_p^2,\beta_p^3\}$$

must coincide. Since $t_{\ell,2}^{(2)}(t_{\ell,2}^{(2)})^{w_1w_2}=(t_{\ell,2}^{(2)})^{w_1}(t_{\ell,2}^{(2)})^{w_2}$ we are reduced to eight possible choices. Four possibilities for the 4-tuple

$$\chi_{2,p}^{\mathrm{st}}(t_{\ell,2}^{(2)}),\chi_{2,p}^{\mathrm{st}}((t_{\ell,2}^{(2)})^{w_1}),\chi_{2,p}^{\mathrm{st}}((t_{\ell,2}^{(2)})^{w_2}),\chi_{2,p}^{\mathrm{st}}((t_{\ell,2}^{(2)})^{w_1w_2})$$

are

$$\begin{gathered}(\alpha_p^3,\alpha_p^2\beta_p,\alpha_p\beta_p^2,\beta_p^3),(\alpha_p^3,\alpha_p\beta_p^2,\alpha_p^2\beta_p,\beta_p^3),\\(\alpha_p^2\beta_p,\alpha_p^3,\beta_p^3,\alpha_p\beta_p^2),(\alpha_p^2\beta_p,\beta_p^3,\alpha_p^3,\alpha_p\beta_p^2).\end{gathered}$$

The other four possibilities are obtained by exchanging α_p with β_p in the ones above.

Since $t_{p,1}^{(2)}=t_{\ell,2}^{(2)}(t_{\ell,2}^{(2)})^{w_1}$ and $t_{p,0}^{(2)}=t_{\ell,2}^{(2)}(t_{\ell,2}^{(2)})^{w_1w_2}$, the displayed 4-tuples give for

$$(\chi_{2,p}^{\mathrm{st}}(t_{\ell,0}^{(2)}),\chi_{2,p}^{\mathrm{st}}(t_{\ell,1}^{(2)}),\chi_{2,p}^{\mathrm{st}}((t_{\ell,2}^{(2)})))$$

the choices

$$\begin{gathered}(\alpha_p^3\beta_p^3,\alpha_p^5\beta_p,\alpha_p^3),(\alpha_p^3\beta_p^3,\alpha_p^4\beta_p^2,\alpha_p^3),\\(\alpha_p^3\beta_p^3,\alpha_p^5\beta_p,\alpha_p^2\beta_p),(\alpha_p^3\beta_p^3,\alpha_p^2\beta_p^4,\alpha_p^2\beta_p).\end{gathered}$$

By writing $\alpha_p = \chi^{\mathrm{st}}_{1,p}(t^{(1)}_{p,1})$, $\beta_p = \chi^{\mathrm{st}}_{1,p}((t^{(1)}_{p,1})^w)$ and recalling that $t^{(1)}_{p,0} = t^{(1)}_{p,1}(t^{(1)}_{p,1})^w$, the previous choices take the form

$$\begin{aligned}&(\chi^{\mathrm{st}}_{1,p}(t^{(1)}_{p,0})^3, \chi^{\mathrm{st}}_{1,p}(t^{(1)}_{p,0}(t^{(1)}_{p,1})^4), \chi^{\mathrm{st}}_{1,p}((t^{(1)}_{p,1})^3)),\\ &(\chi^{\mathrm{st}}_{1,p}(t^{(1)}_{p,0})^3, \chi^{\mathrm{st}}_{1,p}((t^{(1)}_{p,0})^2(t^{(1)}_{p,1})^2), \chi^{\mathrm{st}}_{1,p}((t^{(1)}_{p,1})^3)),\\ &(\chi^{\mathrm{st}}_{1,p}(t^{(1)}_{p,0})^3, \chi^{\mathrm{st}}_{1,p}(t^{(1)}_{p,0}(t^{(1)}_{p,1})^4), \chi^{\mathrm{st}}_{1,p}(t^{(1)}_{p,0}t^{(1)}_{p,1})),\\ &(\chi^{\mathrm{st}}_{1,p}(t^{(1)}_{p,0})^3, \chi^{\mathrm{st}}_{1,p}((t^{(1)}_{p,0})^4(t^{(1)}_{p,1})^{-2}), \chi^{\mathrm{st}}_{1,p}(t^{(1)}_{p,0}t^{(1)}_{p,1})).\end{aligned} \tag{3.10}$$

The triples corresponding to the other four possibilities are obtained by replacing $t^{(1)}_{p,0}$ and $t^{(1)}_{p,1}$ in the triples above by their images via the automorphism δ of $\mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p))$ defined by Equation (3.8).

Let $\lambda_p\colon \mathcal{H}(T_2(\mathbb{Q}_p), T_2(\mathbb{Z}_p))^- \to \mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p))$ be a morphism that satisfies

$$\chi^{\mathrm{st}}_2 = (\chi^{\mathrm{st}}_1)^{\mathrm{ext}} \circ \lambda_p \circ \iota^-_{2,p},$$

where we leave the maps $\iota^{T_g}_{I_{g,p}}$ implicit as before. By the arguments of the previous paragraph this happens if and only if the triple $(\lambda_{i,p}(t^{(2)}_{p,0}), \lambda_{i,p}(t^{(2)}_{p,1}), \lambda_{i,p}(t^{(2)}_{p,2}))$ coincides with one of the four listed in (3.10) or the four derived by applying δ. A simple check shows that these triples correspond to the choices $\lambda_p = \lambda_{i,p}$ for $i \in \{1, 2, \ldots, 8\}$. □

REMARK 3.4.6. *Since all the Hecke actions we consider are for the algebras $\mathcal{H}^N_g$, $g = 1, 2$, that are dilating Iwahoric at p, we want to know whether the morphisms $\lambda_{i,p}$, $i \in \{1, 2, \ldots, 8\}$, can be replaced by morphisms $\lambda^-_{i,p}$ of dilating Hecke algebras that satisfy $\chi^{\mathrm{st}}_{2,p} = \chi^{\mathrm{st}}_{1,p} \circ \lambda^-_{i,p}$. Equivalently, we look for the values of i such that there exists a morphism $\lambda^-_{i,p}\colon \mathcal{H}(\mathrm{GSp}_4(\mathbb{Q}_p), I_{2,p})^- \to \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), I_{1,p})^-$ making the following diagram commute:*

$$\begin{tikzcd} \mathcal{H}(\mathrm{GSp}_4(\mathbb{Q}_p), I_{2,p})^- \arrow[r, "\iota^{I_{2,p}}_{T_2}"] \arrow[d, "\lambda^-_{i,p}"] & \mathcal{H}(T_2(\mathbb{Q}_p), T_2(\mathbb{Z}_p))^- \arrow[dr, "\lambda_{i,p}"] & \\ \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), I_{1,p})^- \arrow[r, "\iota^{I_{1,p}}_{T_1}"] & \mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p))^- \arrow[r, "\iota^-_{1,p}"] & \mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p)). \end{tikzcd}$$

Clearly $\lambda^-_{i,p}$ exists if and only if the image of $\lambda^-_{i,p}$ lies in $\mathcal{H}(T_1(\mathbb{Q}_p), T_1(\mathbb{Z}_p))^-$. A simple check shows that this is true only for $i \in \{1, 2, 3\}$.

DEFINITION 3.4.7. *Let $i \in \{1, 2, 3\}$. Let $\lambda^-_{i,p}\colon \mathcal{H}(\mathrm{GSp}_4(\mathbb{Q}_p), I_{2,p})^- \to \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p), I_{1,p})^-$ be the morphisms making diagram (3.4.6) commute. Let $\lambda_i\colon \mathcal{H}^N_2 \to \mathcal{H}^N_1$ be the morphism defined by $\lambda_i = \lambda^{Np} \otimes \lambda^-_{i,p}$.*

Keep the notations as before. Let $\chi^{\mathrm{st},i}_2\colon \mathcal{H}^N_2 \to \overline{\mathbb{Q}}_p$ be the character defined by

(1) $\chi^{\mathrm{st},i}_{2,\ell} = \chi^{\mathrm{st}}_{1,\ell} \circ \lambda_i$ for every prime $\ell \nmid Np$;
(2) $\chi^{\mathrm{st},i}_{2,p} = (\chi^{\mathrm{st}}_{1,p} \circ \iota^{T_1}_{I_{1,p}})^{\mathrm{ext}} \circ \lambda_{i,p} \circ \iota^{I_{2,p}}_{T_2}$.

We combine Propositions 3.4.2 and 3.4.5 to prove the following.

COROLLARY 3.4.8. *For every $i \in \{1, 2, \ldots, 8\}$, the form $\mathrm{Sym}^3 f$ admits a p-stabilization $(\mathrm{Sym}^3 f)^{\mathrm{st}}_i$ with associated system of Hecke eigenvalues $\chi^{\mathrm{st},i}_2$. Conversely, if $(\mathrm{Sym}^3 f)^{\mathrm{st}}$ is a p-stabilization of $\mathrm{Sym}^3 f$ with associated system of Hecke eigenvalues χ^{st}_2, then there exists $i \in \{1, 2, \ldots, 8\}$ such that $\chi^{\mathrm{st}}_2 = \chi^{\mathrm{st},i}_2$.*

PROOF. The systems of Hecke eigenvalues of the p-stabilizations of $\mathrm{Sym}^3 f$ are the characters $\chi^{\mathrm{st}}_2\colon \mathcal{H}^N_2 \to \overline{\mathbb{Q}}_p$ that satisfy the following conditions:

(1) $\chi^{\mathrm{st}}_{2,\ell} = \chi_{2,\ell}$ for every $\ell \nmid Np$;
(2) the restriction of $\chi^{\mathrm{st}}_{2,p}$ to $\mathcal{H}(\mathrm{GSp}_4(\mathbb{Q}_p), \mathrm{GSp}_4(\mathbb{Z}_p))$ is $\chi_{2,p}$.

By Propositions 3.4.2 and 3.4.5 a character χ_2^{st} satisfies (1) and (2) if and only if $\chi_2^{\mathrm{st}} = \chi_2^{\mathrm{st},i}$ for some i. $\square$

Recall form Remark 1.2.26 that the map $\psi_2\colon \mathcal{H}_2^N \to \mathcal{O}(\mathcal{D}_2)$ interpolates p-adically the systems of normalized Hecke eigenvalues associated with the classical GSp_4-eigenforms.

Keep the notations of Definition 3.4.7. For $i \in \{1,2,\ldots,8\}$ set $F_i = (\mathrm{Sym}^3 f)_i^{\mathrm{st}}$. Let $h = \mathrm{sl}(f)$ be the slope of f and $\mathrm{sl}(F_i)$ be the slope of F_i for every i. Recall that $\mathrm{sl}(f) = v_p(\chi_{1,p}^{\mathrm{st,norm}}(U_p^{(1)}))$ and $\mathrm{sl}(F_i) = v_p(\chi_{2,p}^{\mathrm{st},i,\mathrm{norm}}(U_p^{(2)}))$, with $U_p^{(1)} = U_{p,1}^{(1)}$ and $U_p^{(2)} = U_{p,1}^{(2)} U_{p,2}^{(2)}$.

Corollary 3.4.9. *The slopes of the eight p-stabilizations of* $\mathrm{Sym}^3 f$ *are:*

$$\mathrm{sl}(F_1) = 7h, \mathrm{sl}(F_2) = \mathrm{sl}(F_3) = k-1+5h, \mathrm{sl}(F_4) = 4(k-1)-h,$$
$$\mathrm{sl}(F_5) = 7(k-1-h), \mathrm{sl}(F_6) = \mathrm{sl}(F_7) = 6(k-1)-5h, \mathrm{sl}(F_8) = 3(k-1)+h.$$

Proof. For f as in the statement we have $v_p(\chi(f)(U_{p,0}^{(1)})) = k-1$ and $v_p(\chi(f)(U_{p,0}^{(1)})) = h$. By definition $U_p^{(2)} = U_{p,1}^{(2)} U_{p,2}^{(2)}$. The corollary follows from Proposition 3.4.5 via simple calculations. $\square$

We make explicit the dependence on f of the characters $\chi_2^{\mathrm{st},i}$ by adding a lower index f and writing $\chi_{2,f}^{\mathrm{st},i}$. For a $\overline{\mathbb{Q}}_p$-point x of $\mathcal{D}_2^M$ let $\chi_x\colon \mathcal{H}_2^N \to \overline{\mathbb{Q}}_p$ be the system of Hecke eigenvalues associated with x.

For $i \in \{1,2,\ldots,8\}$ let $S_i^{\mathrm{Sym}^3}$ be the set of $\overline{\mathbb{Q}}_p$-points x of $\mathcal{D}_2^M$ defined by the condition

$$x \in S_i^{\mathrm{Sym}^3} \iff \exists \text{ a cuspidal, classical, non-CM } \mathrm{GL}_2\text{-eigenform } f \text{ such that } \chi_x = \chi_{2,f}^{\mathrm{st},i}.$$

Then we have the following.

Corollary 3.4.10. *If $2 \leq i \leq 8$, the set $S_i^{\mathrm{Sym}^3}$ is discrete in $\mathcal{D}_2^M$.*

Proof. Let $i \in \{2,3,\ldots,8\}$. Let A be an affinoid subdomain of $\mathcal{D}_2^M$ and let x be a point of the set $S_i^{\mathrm{Sym}^3} \cap A(\overline{\mathbb{Q}}_p)$. Let f be a classical, cuspidal, non-CM GL_2-eigenform of weight k and level $\Gamma_1(N) \cap \Gamma_0(p)$ satisfying $\chi_x = \chi_{2,f}^{\mathrm{st},i}$. Let h be the slope of f. By Remark 1.2.7(2) the slope $v_p(\psi_A(U_p^{(2)}))$ is bounded on A by a constant c_A. Then Corollary 3.4.9 together with the inequality $0 \leq h \leq k-1$ gives a finite upper bound for k (e.g. for $i=2$ we obtain $k-1+5h \leq c_A$, so $k \leq c_A+1$). There is only a finite number of classical GL_2-eigenforms of given weight and level, so there is only a finite number of choices for f as above. We conclude that the set $S_{\mathrm{Sym}^3,i} \cap A(\overline{\mathbb{Q}}_p)$ is finite, as desired. $\square$

Remark 3.4.11. *As a consequence of Corollary 3.4.10 the only symmetric cube lifts that we can hope to interpolate p-adically are those in the set $S_1^{\mathrm{Sym}^3}$. We will prove in Section 3.12 that the Zariski closure of this set intersects each irreducible component of $\mathcal{D}_2^M$ in a subvariety of dimension* 0 *or* 1.

3.5. The Galois pseudocharacters on the eigenvarieties

In this section p is a fixed prime, M is a positive integer prime to p and g is 1 or 2. For a point $x \in \mathcal{D}_g^M(\mathbb{C}_p)$ we denote by $\mathrm{ev}_x\colon \mathcal{O}(\mathcal{D}_g^M)^\circ \to \mathbb{C}_p$ both the evaluation at x and the map $\mathrm{GSp}_{2g}(\mathcal{O}(\mathcal{D}_g^M)^\circ) \to \mathrm{GSp}_{2g}(\mathbb{C}_p)$ induced by ev_x. Recall that the GSp_{2g}-eigenvariety $\mathcal{D}_g^M$ is endowed with a homomorphism

$$\psi_g\colon \mathcal{H}_g^M \to \mathcal{O}(\mathcal{D}_g^M)^\circ$$

that interpolates the systems of Hecke eigenvalues associated with the cuspidal GSp_{2g}-eigenforms of level $\Gamma_1(N) \cap \Gamma_0(p)$. For a classical point $x \in \mathcal{D}_g^M(\overline{\mathbb{Q}}_p)$ let $\psi_x = \mathrm{ev}_x \circ \psi_g$. Let f_x be the classical GSp_{2g}-eigenform having system of Hecke eigenvalues ψ_x and let $\rho_x\colon G_{\mathbb{Q}} \to \mathrm{GSp}_{2g}(\overline{\mathbb{Q}}_p)$ be the p-adic Galois representation attached to f_x. When x varies, the traces of the representations ρ_x can be interpolated into a *pseudocharacter* with values in $\mathcal{O}(\mathcal{D}_g^M)^\circ$; this is the main result of this section. Unfortunately the pseudocharacter obtained this way cannot be lifted to a representation with coefficients in $\mathcal{O}(\mathcal{D}_g^M)^\circ$. We will be able to obtain a lift only by working over a sufficiently small admissible subdomain of cD_g^M (see Section 4.1.4).

3.5.1. Classical results on pseudocharacters. We recall the definitions and some classical results in the theory of pseudocharacters. In this subsection A is a commutative ring with unit and R is an A-algebra with unit (not necessarily commutative). Let k be any positive integer and let $\mathscr{S}_k$ be the group of permutations of the set $\{1,2,\ldots,k\}$. Given any $\nu\in\mathscr{S}_k$ we write $\varepsilon(\nu)$ for its sign and we decompose it in cycles as $\nu=\prod_{i=1}^{i_\nu}(j_{i,1}j_{i,2}\cdots j_{i,\ell_i})$. Set

$$T_\nu(x_1,x_2,\ldots,x_k)=\prod_{i=1}^{i_\nu}(x_{j_{i,1}}x_{j_{i,2}}\cdots x_{j_{i,\ell_i}})$$

for every $(x_1,x_2,\ldots,x_k)\in R^k$. We define a map $S_k\colon R^k\to A$ by letting

$$S_k(x_1,\ldots,x_k)=\sum_{\nu\in\mathscr{S}_k}\varepsilon(\nu)T_\nu(x_1,\ldots,x_k)$$

for every $(x_1,x_2,\ldots,x_k)\in R^k$.

Definition 3.5.1. *Let d be a positive integer. We say that a map $T\colon R\to A$ is a* pseudocharacter of dimension d *if it satisfies the following conditions:*

(1) T is A-linear;
(2) $T(xy)=T(yx)$ for every $x,y\in R$;
(3) the map $S_{i+1}(T)\colon R^{i+1}\to A$ is identically zero for $i=d$ and d is the smallest value of i such that this happens.

This definition is motivated by thefollowing result.

Proposition 3.5.2. *Let $\tau\colon R\to \mathrm{M}_d(A)$ be a representation. The map $\mathrm{Tr}(\tau)\colon R\to A$ is a pseudocharacter of dimension d.*

The only non-trivial property to prove is (3). The proposition above was first proved by Frobenius, who showed that $S_k(T)$ is identically zero if and only if $d\geq k+1$. We call $\mathrm{Tr}(\tau)$ the pseudocharacter associated with τ. Thanks to the following result of Carayol a representation is uniquely determined by its associated pseudocharacter.

Theorem 3.5.3. [**Ca94**] *Suppose that A is a complete noetherian local ring. Let A' be a semilocal extension of A. Let $\tau'\colon R\to\mathrm{M}_d(A')$ be a representation. Suppose that the traces of τ' belong to A. Then there exists a representation $\tau\colon R\to\mathrm{M}_d(A)$, unique up to isomorphism over A, such that τ is isomorphic to τ' over A'.*

Let G be a group. By an abuse of terminology, we will say that a map $T\colon G\to A^\times$ is a pseudocharacter of dimension d if it can be extended A-linearly to a pseudocharacter $A[G]\to A$ of dimension d.

Under some hypotheses on the ring A it is known that every pseudocharacter arises as the trace of a representation. Let d be a positive integer. The following theorem is due to Taylor when $\mathrm{char}(A)=0$ and Rouquier when $\mathrm{char}(A)>d$.

Theorem 3.5.4. ([**Ta91**],[**Ro96**]) *Suppose that A is an algebraically closed field of characteristic either 0 or greater than d. Let $T\colon R\to A$ be a d-dimensional pseudocharacter. Then there exists a representation $\tau\colon R\to\mathrm{M}_d(A)$ such that $\mathrm{Tr}(\tau)=T$.*

The following result was proved independently by Nyssen and Rouquier.

THEOREM 3.5.5. ([**Ny96**],[**Ro96**, Corollary 5.2]) *Suppose that A is a local henselian ring in which $d!$ is invertible and let $\mathbb{F}$ denote the residue field of A. Let $T\colon R \to A$ be a pseudocharacter of dimension d and $\overline{T}\colon R \to \mathbb{F}$ be its reduction modulo the maximal ideal of A. Suppose that there exists an irreducible representation $\overline{\tau}\colon R \to \mathrm{M}_d(\mathbb{F})$ such that $\mathrm{Tr}(\overline{\tau}) = \overline{T}$. Then there is an isomorphism $R/\ker T \cong \mathrm{M}_d(A)$ and the projection $R \to R/\ker T$ is a representation lifting $\overline{\tau}$.*

We mention for the sake of completeness that Chenevier studied the case where $0 < \mathrm{char}(A) \leq d$ in [**Ch14**]. He introduced the notion of *determinant*, which is a generalization of that of pseudocharacter. He showed that analogues of Theorems 3.5.4 and 3.5.5 hold if we replace pseudocharacters with determinants and A is an algebraically closed field or a local henselian ring with algebraically closed residue field, without any assumptions on the characteristic of A (see [**Ch14**, Theorems A and B]).

We introduce a notion of characteristic polynomial of a pseudocharacter. Note that if $\tau\colon G \to \mathrm{GL}_d(A)$ is a representation and $T = \mathrm{Tr}(\tau)$, we can recover the characteristic polynomial of $\tau(g)$ from T for every $g \in G$. Indeed, let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be the eigenvalues of $\tau(g)$. For every $n \in \mathbb{N}$ we have $T(g^n) = \sum_{i=1}^d \alpha_i^n$, so the functions $T(g^n)$ generate over $\mathbb{Q}$ the ring of symmetric polynomials with rational coefficients in the variables $\alpha_1, \alpha_2, \ldots, \alpha_n$. We deduce that every coefficient of $\det(1 - X\tau(g))$ can be written as a polynomial in the variables $T(g^n)$, with $1 \leq n \leq d$.

For τ as above, let $f_1, f_2, \ldots, f_d \in \mathbb{Q}[x_1, x_2, \ldots, x_d]$ be polynomials satisfying $\det(1 - X\tau(g)) = 1 + \sum_{i=1}^d f_1(T(g), T(g^2), \ldots, T(g^d))X^i$. Clearly $f_1, f_2, \ldots, f_d$ are unique and independent of τ.

DEFINITION 3.5.6. *If $T\colon G \to A$ is a d-dimensional pseudocharacter, we let $P_{\mathrm{char}}(T)\colon G \to A[X]^{\deg = d}$ be the polynomial defined by*

$$P_{\mathrm{char}}(T) = 1 + \sum_{i=1}^{d} f_1(T(g), T(g^2), \ldots, T(g^d))X^i.$$

We call $P_{\mathrm{char}}(T)$ the characteristic polynomial *of T.*

For example for $d = 2$ we have

$$P_{\mathrm{char}}(T)(g) = 1 - T(g)X + \left(\frac{T(g)^2 - T(g^2)}{2}\right)X^2. \tag{3.11}$$

For later use (especially in Section 3.12) we introduce the notion of symmetric cube of a two-dimensional pseudocharacter.

DEFINITION 3.5.7. *Let $T\colon G \to A$ be a two-dimensional pseudocharacter. The symmetric cube of T is the pseudocharacter $\mathrm{Sym}^3 T\colon G \to A$ defined by*

$$\mathrm{Sym}^3 T(g) = \frac{T(g)^2(3T(g^2) - T(g)^2)}{2}$$

This definition is justified by the lemma below.

LEMMA 3.5.8. *Let $\tau\colon G \to \mathrm{GL}_2(A)$ be a representation and let $T = \mathrm{Tr}(\tau)$. Then the trace of the representation $\mathrm{Sym}^3\tau\colon G \to \mathrm{GSp}_4(A)$ is $\mathrm{Sym}^3 T$.*

PROOF. Let $g \in G$. Thanks to formula (3.11) we can write the characteristic polynomial of $\tau(g)$ as

$$\det(1 - X \cdot \tau(g)) = 1 - T(g)X + \left(\frac{T(g)^2 - T(g^2)}{2}\right)X^2.$$

Then the trace of $\mathrm{Sym}^3\tau(g)$ can be computed from Equation (3.3). $\square$

REMARK 3.5.9. *If $T = \mathrm{Tr}(\tau)$, Lemma 3.5.8 and the definition of P_{char} give*

$$P_{\mathrm{char}}(\mathrm{Sym}^3 T)(g) = P_{\mathrm{char}}(\mathrm{Sym}^3\tau(g)) = \mathrm{Sym}^3 P_{\mathrm{char}}(\tau(g)) = \mathrm{Sym}^3 P_{\mathrm{char}}(T)(g).$$

By definition of P_{char} this implies that $P_{\mathrm{char}}(\mathrm{Sym}^3 T) = \mathrm{Sym}^3 P_{\mathrm{char}}(T)$ for every pseudocharacter $T\colon G \to A$. This can also be checked by a direct calculation.

3.5.2. Interpolation of the classical pseudocharacters. Every classical point of $\mathcal{D}_g^M$ admits an associated Galois representation given by Theorem 3.1.1. In this subsection we show how to interpolate the trace pseudocharacters attached to these representations to construct a pseudocharacter over the eigenvariety.

As before let $g \in \{1,2\}$. We remind the reader that for every ring R we implicitly extend a character of the Hecke algebra $\mathcal{H}_g^M \to R^\times$ to a morphism of polynomial algebras $\mathcal{H}_g^M[X] \to R[X]$ by applying it to the coefficients. Recall that we fixed an embedding $G_{\mathbb{Q}_\ell} \hookrightarrow G_{\mathbb{Q}}$ for every prime ℓ, hence an embedding of the inertia subgroup I_ℓ in $G_{\mathbb{Q}}$. As usual Frob_ℓ denotes a lift of the Frobenius at ℓ to $G_{\mathbb{Q}_\ell}$.

Let S^{cl} denote the set of classical points of $\mathcal{D}_g^M$. Let $x \in S^{\mathrm{cl}}$. We keep the notations ev_x, ψ_x, ρ_x as in the beginning of the section. We let $T_x\colon G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ be the pseudocharacter defined by $T_x = \mathrm{Tr}(\rho_x)$.

Theorem 3.5.10. *There exists a pseudocharacter*

$$T_{\mathcal{D}_g^M}\colon G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{D}_g^M)$$

of dimension $2g$ with the following properties:

(1) for every prime ℓ not dividing Np and every $h \in I_\ell$ we have $T_{\mathcal{D}_g^M}(h) = 2$, where $2 \in \mathcal{O}(\mathcal{D}_g^M)$ denotes the function constantly equal to 2;

(2) for every prime ℓ not dividing Np we have

$$P_{\mathrm{char}}(T_{\mathcal{D}_g^M})(\mathrm{Frob}_\ell)(X) = \psi_g(P_{\min}(t_{\ell,g}^{(g)}; X));$$

(3) for every $x \in S^{\mathrm{cl}}$ we have

$$\mathrm{ev}_x \circ T_{\mathcal{D}_g^M} = T_x.$$

The proof of the theorem relies on an interpolation argument due to Chenevier, who applied it to the eigenvarieties for definite unitary groups in [**Ch04**, Proposition 7.1.1].

Proof. The set S^{cl} is Zariski-dense in $\mathcal{D}_g^M$ by Proposition 1.2.20, so there is an injection $\mathrm{ev}\colon \mathcal{O}(\mathcal{D}_g^M) \hookrightarrow \prod_{x\in S^{\mathrm{cl}}} \mathbb{C}_p$ given by the product of the evaluations at $x \in S^{\mathrm{cl}}$.

We define a map $\mathrm{Tr}_g\colon G_{\mathbb{Q}} \to \prod_{x\in S^{\mathrm{cl}}} \mathbb{C}_p$ by

$$\mathrm{Tr}_g(\gamma) = (T_x(\gamma))_{x\in S^{\mathrm{cl}}}.$$

We show that:

(i) for every prime $\ell \nmid Np$ and every $h \in I_\ell$ we have $\mathrm{Tr}_g(h) = 2$, where 2 denotes the image of the constant function 2 via ev;

(ii) Tr_g is a pseudocharacter of dimension $2g$;

(iii) for every prime $\ell \nmid Np$ we have $P_{\mathrm{char}}(T_{\mathcal{D}_g^M})(\mathrm{Frob}_\ell)(X) = \psi_g(P_{\min}(t_{\ell,g}^{(g)}; X))$;

(iv) there exists a map $T_{\mathcal{D}_g^M}\colon G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{D}_g^M)$ such that $\mathrm{Tr}_g = \mathrm{ev} \circ T_{\mathcal{D}_g^M}$.

By Proposition 3.1.1 we have, for every $x \in S^{\mathrm{cl}}$:

(a) $\rho_x(h) = \mathrm{Id}_2$ for every prime $\ell \nmid Np$ and every $h \in I_\ell$;

(b) $\mathrm{Tr}\rho_x(\mathrm{Frob}_\ell) = \psi_x(T_{\ell,g}^{(g)})$ and $\det \rho_x = \psi_x(\ell^6 (T_{\ell,0}^{(2)})^2)$ for every prime $\ell \nmid Np$.

Now (a) gives $\mathrm{Tr}(\rho_x(h)) = 2$ for every prime $\ell \nmid Np$ and every $h \in I_\ell$, hence (i) above. To prove (ii) we observe that conditions (1-3) in the definition of a pseudocharacter of dimension $2g$ can be checked separately on each factor $\mathbb{C}_p$. This does not require any work: definition the component of Tr_g corresponding to a single factor $\mathbb{C}_p$ is the trace of a representation of dimension $2g$, so it is a pseudocharacter by Proposition 3.5.2.

By Theorem 3.1.1 and Remark 3.5.9 we have, for every $x \in S^{\mathrm{cl}}$,

$$P_{\mathrm{char}}(T_x)(\mathrm{Frob}_\ell)(X) = \det(1 - X\rho_x(\mathrm{Frob}_\ell)) = \psi_x(P_{\min}(t_{\ell,g}^{(g)}; X)).$$

Since $\psi_x = \mathrm{ev}_x \circ \psi_g$ we deduce that

$$P_{\mathrm{char}}(\mathrm{Tr}_g)(X)(\mathrm{Frob}_\ell) = (P_{\mathrm{char}}(T_x)(\mathrm{Frob}_\ell)(X))_{x\in S^{\mathrm{cl}}} =$$
$$= (\psi_x(P_{\min}(t_{\ell,g}^{(g)}; X)))_{x\in S^{\mathrm{cl}}} = \mathrm{ev} \circ \psi_g(P_{\min}(t_{\ell,g}^{(g)}; X)),$$

hence (iii).

We show that (iv) holds. Note that $\mathcal{D}_g^M$ is a BC-eigenvariety by Corollary 3.6.5, so the ring $\mathcal{O}(\mathcal{D}_g^M)$ is compact by [**BC09**, Corollary 7.2.12 and Lemma 7.2.11(ii)]. The injection $\mathrm{ev}\colon \mathcal{O}(\mathcal{D}_g^M) \hookrightarrow \prod_{x\in S^{\mathrm{cl}}} \mathbb{C}_p$ is a continuous map from a compact topological space to a separated one, so it is closed by a standard topological argument. In particular $\mathrm{ev}(\mathcal{O}(\mathcal{D}_g^M))$ is closed in $\prod_{x\in S^{\mathrm{cl}}} \mathbb{C}_p$. By part (i) we have $\mathrm{Tr}_g(\gamma\mathrm{Frob}_\ell\gamma^{-1}) = \mathrm{ev}(\psi_g(T_{\ell,g}^{(g)})) \in \mathrm{ev}(\mathcal{O}(\mathcal{D}_g^M))$ for every $\ell \nmid Np$ and $\gamma \in G_\mathbb{Q}$, so the image of the set $\{\gamma\mathrm{Frob}_\ell\gamma^{-1}\}_{\ell\nmid Np;\, \gamma\in G_\mathbb{Q}}$ via Tr_g is contained in $\mathrm{ev}(\mathcal{O}(\mathcal{D}_g^M))$. Since this set is dense in $G_\mathbb{Q}$ by Chebotarev's theorem, the image of $G_\mathbb{Q}$ via Tr_g is contained in the closure of $\mathrm{ev}(\mathcal{O}(\mathcal{D}_g^M))$, which is just $\mathrm{ev}(\mathcal{O}(\mathcal{D}_g^M))$ by the argument above. Hence there exists a map $T_{\mathcal{D}_g^M}\colon G_\mathbb{Q} \to \mathcal{O}(\mathcal{D}_g^M)$ such that $\mathrm{Tr}_g = \mathrm{ev} \circ T_{\mathcal{D}_g^M}$.

We conclude the proof of the theorem. The map $T_{\mathcal{D}_g^M}$ given by (iv) is a pseudocharacter of dimension $2g$ since Tr_g is. Then (i) and (iii) give the properties (1) and (2) stated in the theorem. Property (3) follows from the definitions of Tr_g and $T_{\mathcal{D}_g^M}$. □

For every rigid analytic subvariety $\mathcal{V}_g$ of $\mathcal{D}_g^M$ we denote by $r_{\mathcal{V}_g}\colon \mathcal{O}(\mathcal{D}_g^M) \to \mathcal{O}(\mathcal{V}_g)$ the restriction of analytic functions on $\mathcal{D}_g^M$ to $\mathcal{V}_g$ and by $\psi_{\mathcal{V}_g} = r_{\mathcal{V}_g} \circ \psi_g\colon \mathcal{H}_g^M \to \mathcal{O}(\mathcal{V}_g)$ the system of Hecke eigenvalues associated with $\mathcal{V}_g$. Then Theorem 3.5.10 allows us to define a pseudocharacter associated with $\mathcal{V}_g$.

Corollary 3.5.11. *Let $\mathcal{V}_g$ be any rigid analytic subvariety of $\mathcal{D}_g^M$. There exists a pseudocharacter*

$$T_{\mathcal{V}_g}\colon G_\mathbb{Q} \to \mathcal{O}(\mathcal{V}_g)$$

of dimension $2g$ with the following properties:

(1) for every prime ℓ not dividing Np and every $h \in I_\ell$ we have $T_{\mathcal{V}_g}(h) = 2$, where $2 \in \mathcal{O}(\mathcal{V}_g)$ denotes the function constantly equal to 2;

(2) for every prime ℓ not dividing Np we have

$$P_{\mathrm{char}}(T_{\mathcal{V}_g})(\mathrm{Frob}_\ell)(X) = \psi_{\mathcal{V}_g}(P_{\min}(t_{\ell,g}^{(g)}; X));$$

(3) for every classical point x of $\mathcal{V}_g$ we have

$$\mathrm{ev}_x \circ T_{\mathcal{V}_g} = T_x.$$

Proof. It is easily checked that the pseudocharacter $T_{\mathcal{V}_g} = r_{\mathcal{V}_g} \circ T_{\mathcal{D}_g^M}$ has the desired properties. □

As a special case of Corollary 3.5.11, by choosing $\mathcal{V}_g$ to be a point of $\mathcal{D}_g^M$ we can associate a pseudocharacter with every overconvergent GL_2- or GSp_4-eigenform. From this pseudocharacter we can construct a p-adic Galois representation, as precised in the following remark.

Remark 3.5.12. *Let $x \in \mathcal{D}_g^M(\overline{\mathbb{Q}}_p)$. Consider the $2g$-dimensional pseudocharacter $T_x\colon G_\mathbb{Q} \to \overline{\mathbb{Q}}_p$ defined by*

$$T_x = \mathrm{ev}_x \circ T_{2g}.$$

By Theorem 3.5.4 there exists a Galois representation $\rho_x\colon G_\mathbb{Q} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_p)$ satisfying

$$T_x = \mathrm{Tr}(\rho_x).$$

We will see in Section 4.1.4 that, when $\overline{\rho}_x$ is absolutely irreducible, ρ_x is isomorphic to a representation $G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$.

Remark 3.5.13. *Let $x \in \mathcal{D}_g^M(Qp)$. When x varies in a connected component of $\mathcal{D}_g^M$, the residual representation $\overline{\rho}_x\colon G_{\mathbb{Q}} \to \mathrm{GSp}_{2g}(\overline{\mathbb{Q}}_p)$ is independent of x. We call it the residual representation associated with the component.*

Let $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a representation. Let $\mathcal{D}_{1,\overline{\rho}}^N$ be the union of the connected components of $\mathcal{D}_1^N$ having $\overline{\rho}$ as associated residual representation. From now on we replace $\mathcal{D}_1^N$ by a subspace of the form $\mathcal{D}_{1,\overline{\rho}}^N$ for some $\overline{\rho}$; we do this implicitly, so we still write $\mathcal{D}_1^N$ for $\mathcal{D}_{1,\overline{\rho}}^N$. We make the following assumption on $\overline{\rho}$:

(3-twist) there exists no character $\overline{\eta}\colon G_{\mathbb{Q}} \to \overline{\mathbb{F}}_p^\times$ of order 3 satisfying $\eta \otimes \overline{\rho} \cong \overline{\rho}$.

Remark 3.5.14. *There is a map Sym_1^3 from the set of classical, cuspidal non-CM eigenforms of $\mathcal{D}_{1,\rho}^N$ to the set $S_1^{\mathrm{Sym}^3}$ of Corollary 3.4.10; it is defined by $f \mapsto (\mathrm{Sym}^3 f)_1^{\mathrm{st}}$. Thanks to condition (3-twist), if x_1 and x_2 are two points of $\mathcal{D}_{1,\rho}^N$ satisfying $\mathrm{Sym}^3\rho_{x_1} \cong \mathrm{Sym}^3\rho_{x_2}$, then $\rho_{x_1} \cong \rho_{x_2}$. In particular Sym_1^3 is injective.*

3.6. Eigenvarieties as interpolation spaces of systems of Hecke eigenvalues

In this section we recall Bellaïche and Chenevier's definition of eigenvarieties and some of their results, following [**BC09**, Section 7.2.3]. We refer to their eigenvarieties as *BC-eigenvarieties*, in order to distinguish this notion from the definition of eigenvariety we gave in Section 1.2.2 (a product of Buzzard's eigenvariety machine).

As usual fix a prime $p \geq 5$. We call "BC-datum" a 4-tuple $(g, \mathcal{H}, \eta, \mathscr{S}^{\mathrm{cl}})$ where:
- g is a positive integer;
- $\mathcal{H}$ is a commutative ring;
- η is a distinguished element of $\mathcal{H}$;
- $\mathscr{S}^{\mathrm{cl}}$ is a subset of $\mathrm{Hom}(\mathcal{H}, \overline{\mathbb{Q}}_p) \times \mathbb{Z}^g$.

The superscript "cl" stands for "classical". In our applications $\mathcal{H}$ will be a Hecke algebra and $\mathscr{S}^{\mathrm{cl}}$ will be a set of couples $(\psi, \underline{k})$ each consisting of the system of eigenvalues ψ and the weight $\underline{k}$ of a classical eigenform. In the proposition below $\mathcal{W}_g^\circ$ is the connected component of unity in the g-dimensional weight space, introduced in Section 1.2.1. Recall that we identify $\mathbb{Z}^g$ with the set of classical weights in $\mathcal{W}_G$. Also recall that for an extension L of $\overline{\mathbb{Q}}_p$ and an L-point x of a rigid analytic space X we denote by $\mathrm{ev}_x\colon \mathcal{O}(X) \to L$ the evaluation morphism at x.

Definition 3.6.1. [**BC09**, Definition 7.2.5] *A* BC-eigenvariety *for the datum $(g, \mathcal{H}, \eta, \mathscr{S}^{\mathrm{cl}})$ is a 4-tuple $(\mathcal{D}, \psi, w, S^{\mathrm{cl}})$ consisting of*
- *a reduced rigid analytic space $\mathcal{D}$ over $\mathbb{Q}_p$,*
- *a ring morphism $\psi\colon \mathcal{H} \to \mathcal{O}(\mathcal{D})$ such that $\psi(\eta)$ is invertible,*
- *a morphism $w\colon \mathcal{D} \to \mathcal{W}_g^\circ$ of rigid analytic spaces over $\mathbb{Q}_p$,*
- *an accumulation and Zariski-dense subset $S^{\mathrm{cl}} \subset \mathcal{D}(\overline{\mathbb{Q}}_p)$ such that $w(S^{\mathrm{cl}}) \subset \mathbb{Z}^g$,*

satisfying the following conditions:

(1) the map

$$\widetilde{\nu} = (w, \psi(\eta)^{-1})\colon \mathcal{D} \to \mathcal{W}_g^\circ \times \mathbb{G}_m \tag{3.12}$$

induces a finite morphism $\mathcal{D} \to \widetilde{\nu}(\mathcal{D})$;

(2) there exists an admissible affinoid covering $\mathcal{C}$ of $\widetilde{\nu}(\mathcal{D})$ such that, for every $V \in \mathcal{C}$, the map

$$\psi \otimes \widetilde{\nu}^*\colon \mathcal{H} \otimes_{\mathbb{Z}} \mathcal{O}(V) \to \mathcal{O}(\widetilde{\nu}^{-1}(V))$$

is surjective;

(3) the evaluation map

$$\widetilde{\mathrm{ev}} S^{\mathrm{cl}} \to \mathrm{Hom}(\mathcal{H}, \overline{\mathbb{Q}}_p) \times \mathbb{Z}^g, \qquad x \mapsto (\psi_x, w(x)), \tag{3.13}$$

where $\psi_x = \mathrm{ev}_x \circ \psi$, *induces a bijection* $S^{\mathrm{cl}} \to \mathscr{S}^{\mathrm{cl}}$.

We often refer to $\mathcal{D}$ as the BC-eigenvariety for the given BC-datum and leave the other elements of the BC-eigenvariety implicit.

We recall a few properties of BC-eigenvarieties. Let $(g, \mathcal{H}, \eta, \mathscr{S}^{\mathrm{cl}})$ be a BC-datum and let $(\mathcal{D}, \psi, w, S^{\mathrm{cl}})$ be a BC-eigenvariety for this datum (it may not exist).

Lemma 3.6.2. [**BC09**, Lemma 7.2.7]

(1) The rigid analytic space $\mathcal{D}$ *is an admissible union of affinoid domains of the form* $\widetilde{\nu}^{-1}(V)$ *for an affinoid subdomain* V *of* $\mathcal{W}_g^\circ \times \mathbb{Z}^g$.

(2) Two points $x, y \in \mathcal{D}(\overline{\mathbb{Q}}_p)$ *coincide if and only if* $w(x) = w(y)$ *and* $\psi_x = \psi_y$.

If a BC-eigenvariety for the given BC-datum exists then it is unique in the sense of the proposition below.

Proposition 3.6.3. [**BC09**, Proposition 7.2.8] *Let* $(\mathcal{D}_1, \psi_1, w_1, S_1^{\mathrm{cl}})$ *and* $(\mathcal{D}_2, \psi_2, w_2, S_2^{\mathrm{cl}})$ *be two BC-eigenvarieties for the same BC-datum* $(g, \mathcal{H}, \eta, \mathscr{S}^{\mathrm{cl}})$. *Then there is a unique isomorphism* $\zeta\colon \mathcal{D}_1 \to \mathcal{D}_2$ *of rigid analytic spaces over* $\mathbb{Q}_p$ *such that* $\psi_1 = \zeta^* \circ \psi_2$, $w_1 = w_2 \circ \zeta$ *and* $\zeta(S_1^{\mathrm{cl}}) = S_2^{\mathrm{cl}}$.

In the previous sections we defined various rigid analytic spaces via Buzzard's eigenvariety machine and further operations. We check that these spaces are BC-eigenvarieties for a suitable choice of BC-datum. As a first step we prove the lemma below. Consider an eigenvariety datum $(\mathcal{W}^\circ, \mathcal{H}, (M(A,w))_{A,w}, (\phi_{A,w})_{A,w}, \eta)$ and let $(\mathcal{D}, \psi, w)$ be the eigenvariety produced from this datum by Theorem 1.2.3.

Lemma 3.6.4. *The triple* $(\mathcal{D}, \psi, w)$ *satisfies conditions (1) and (2) of Definition 3.6.1.*

Proof. We refer to Buzzard's construction summarized in Section 1.2.2. Let $\mathcal{Z}$ be the spectral variety for the given datum. Let $\widetilde{\nu}$ be the map defined by Equation (3.12). By construction of $\mathcal{D}$ we have $\widetilde{\nu}(\mathcal{D}) = \mathcal{Z}$ and the map $\widetilde{\nu}\colon \mathcal{D} \to \mathcal{Z}$ is finite, so condition (1) of Definition 3.6.1 holds.

Let $\mathcal{C}$ be the admissible affinoid covering of $\mathcal{Z}$ defined by Buzzard. For $V \in \mathcal{C}$ let $A = \mathrm{Spm}\, R = w_{\mathcal{Z}}(V)$ be its image in $\mathcal{W}^\circ$. Let $w \in \mathbb{Q}$ be sufficiently large, so that the module $M(A,w)$ is defined. Let $M(A,w) = N_V(A,w) \oplus F_V(A,w)$ be the decomposition given by Equation (1.2). Then $\mathcal{O}(\widetilde{\nu}^{-1}(V))$ is the R-span of the image of $\mathcal{H}$ in $\mathrm{End}_{R,\mathrm{cont}} N_V$. Since $\mathcal{O}(V)$ is an R-module, the map $\psi\colon \mathcal{H} \otimes \mathcal{O}(V) \to \mathcal{O}(\widetilde{\nu}^{-1}V)$ is surjective, hence condition (2) is also satisfied. □

Suppose that there exists an accumulation and Zariski-dense subset S^{cl} of $\mathcal{D}$ such that the set

$$\mathscr{S}^{\mathrm{cl}} = \{(\psi_x, w(x)) \mid x \in S^{\mathrm{cl}}\}$$

is contained in $\mathrm{Hom}(\mathcal{H}, \overline{\mathbb{Q}}_p) \times \mathbb{Z}^g$. Then $(\mathcal{D}, \psi, w, S^{\mathrm{cl}})$ clearly satisfies condition (3) of Definition 3.6.1 with respect to the set $\mathscr{S}^{\mathrm{cl}}$, hence the following.

Corollary 3.6.5. *The* 4*-tuple* $(\mathcal{D}, \psi, w, S^{\mathrm{cl}})$ *is a BC-eigenvariety for the datum* $(g, \mathcal{H}, \eta, \mathscr{S}^{\mathrm{cl}})$.

3.7. Changing the BC-datum

Let $(\mathcal{D}, \psi, w, S^{\mathrm{cl}})$ be a BC-eigenvariety for the datum $(g, \mathcal{H}, \eta, \mathscr{S}^{\mathrm{cl}})$. Let S_0^{cl} be an accumulation subset of S^{cl} and let $\mathcal{D}_0$ be the Zariski closure of S_0^{cl} in $\mathcal{D}$. Let $\mathscr{S}_0^{\mathrm{cl}}$ be the image of S_0^{cl} via the bijection $S^{\mathrm{cl}} \to \mathscr{S}^{\mathrm{cl}}$. Let $\psi_0 \colon \mathcal{H} \to \mathcal{O}(\mathcal{D}_0)$ be the composition of $\psi \colon \mathcal{H} \to \mathcal{O}(\mathcal{D})$ with the restriction $\mathcal{O}(\mathcal{D}) \to \mathcal{O}(\mathcal{D}_0)$. Let $w_0 = w|_{\mathcal{D}_0}$.

LEMMA 3.7.1. *The 4-tuple $(\mathcal{D}_0, \psi_0, w_0, S_0^{\mathrm{cl}})$ is a BC-eigenvariety for the datum $(g, \mathcal{H}, \eta, \mathscr{S}_0^{\mathrm{cl}})$.*

PROOF. We check that the conditions of Definition 3.6.1 are satisfied by $(\mathcal{D}_0, \psi_0, w_0, S_0^{\mathrm{cl}})$, knowing that they are satisfied by $(\mathcal{D}, \psi, w, S^{\mathrm{cl}})$. Let $\widetilde{\nu} = (w, \psi(\eta)^{-1}) \colon \mathcal{D} \to \mathcal{W}^\circ \times \mathbb{G}_m$ and let $\mathcal{Z} = \widetilde{\nu}(\mathcal{D})$. Let $\mathcal{Z}_0 = \widetilde{\nu}(\mathcal{D}_0)$. Since $\widetilde{\nu} \colon \mathcal{D} \to \mathcal{Z}$ is finite and $\mathcal{D}_0$ is Zariski-closed in $\mathcal{D}$, the map $\widetilde{\nu}|_{\mathcal{D}_0} \colon \mathcal{D}_0 \to \mathcal{Z}_0$ is also finite, hence (1) holds.

Consider an admissible covering $\mathcal{C}$ of $\mathcal{Z}$ satisfying condition (2). Then $\{V \cap \mathcal{Z}_0\}_{V \in \mathcal{C}}$ is an admissible covering of $\mathcal{Z}_0$. Let $V \in \mathcal{C}$ and $V_0 = V \cap \mathcal{Z}_0$. Consider the diagram

$$\begin{tikzcd} \mathcal{H} \otimes \mathcal{O}(V) \arrow[r, "\psi\otimes\widetilde{\nu}^*"] \arrow[d] & \mathcal{O}(\widetilde{\nu}^{-1}(V)) \arrow[d] \\ \mathcal{H} \otimes \mathcal{O}(V_0) \arrow[r, "\psi_0\otimes\widetilde{\nu}_0^*"] & \mathcal{O}(\widetilde{\nu}_0^{-1}(V_0)) \end{tikzcd}$$

The horizontal arrows are given by the restriction of analytic functions. Since the left vertical arrow is surjective, the right one is also surjective, giving (2).

By definition of S_0^{cl} the map $\widetilde{\mathrm{ev}}$ induces a bijection $S_0^{\mathrm{cl}} \to \mathscr{S}_0^{\mathrm{cl}}$, so (3) is also true. $\square$

We prove some relations between BC-eigenvarieties associated with different BC-data.

LEMMA 3.7.2. *Let g_1 and g_2 be two positive integers with $g_1 \leq g_2$. Let $\Theta \colon \mathcal{W}_{g_1}^\circ \to \mathcal{W}_{g_2}^\circ$ be an immersion of rigid analytic spaces that maps classical points of $\mathcal{W}_{g_1}^\circ$ to classical points of $\mathcal{W}_{g_2}^\circ$. Let $(g_1, \mathcal{H}, \eta, \mathscr{S}_1^{\mathrm{cl}})$ and $(g_2, \mathcal{H}, \eta, \mathscr{S}_2^{\mathrm{cl}})$ be two BC-data satisfying*

$$\{(\psi, \Theta(\underline{k})) \in \mathrm{Hom}(\mathcal{H}, \overline{\mathbb{Q}}_p) \times \mathbb{Z}^{g_2} \mid (\psi, \underline{k}) \in \mathscr{S}_1^{\mathrm{cl}}\} \subset \mathscr{S}_2^{\mathrm{cl}}$$

Let $(\mathcal{D}_1, \psi_1, w_1, S_1^{\mathrm{cl}})$ and $(\mathcal{D}_2, \psi_2, w_2, S_2^{\mathrm{cl}})$ be the BC-eigenvarieties for the two data. Then there exists a closed immersion of rigid analytic spaces $\xi_\Theta \colon \mathcal{D}_1 \to \mathcal{D}_2$ such that $\psi_1 = \xi_\Theta^* \circ \psi_2$, $w_1 = w_2 \circ \xi_\Theta$ and $\xi_\Theta(S_1^{\mathrm{cl}}) \subset S_2^{\mathrm{cl}}$.

PROOF. Let $\mathcal{D}_1^\Theta = \mathcal{D}_2 \times_{\mathcal{W}_{g_2}^\circ} \mathcal{W}_{g_1}^\circ$, where the map $\mathcal{W}_{g_1}^\circ \to \mathcal{W}_{g_1}^\circ$ is Θ. Let $\zeta^\Theta \colon \mathcal{D}_1^\Theta \to \mathcal{D}_2$ and $w_1^\Theta \colon \mathcal{D}_1^\Theta \to \mathcal{W}_{g_1}^\circ$ be the natural maps fitting into the cartesian diagram

$$\begin{tikzcd} \mathcal{D}_1^\Theta \arrow[r, hook, "\zeta^\Theta"] \arrow[d, "w_1^\Theta"] & \mathcal{D}_2 \arrow[d, "w_2"] \\ \mathcal{W}_{g_1}^\circ \arrow[r, hook, "\Theta"] & \mathcal{W}_{g_2}^\circ \end{tikzcd}$$

Then ζ^Θ induces a ring morphism $\zeta^{\Theta,*} \colon \mathcal{O}(\mathcal{D}_2) \to \mathcal{O}(\mathcal{D}_1^\Theta)$. Let $\psi_1^\Theta = \zeta^{\Theta,*} \circ \psi_2$. Note that ζ^Θ is a closed immersion.

Let

$$\mathscr{S}_1^\Theta = \{(\psi, \underline{k}) \in \mathrm{Hom}(\mathcal{H}, \overline{\mathbb{Q}}_p) \times \mathbb{Z}^{g_1} \mid (\psi, \Theta(\underline{k})) \in \mathscr{S}_2^{\mathrm{cl}}\}.$$

Then the 4-tuple $(\mathcal{D}_1^\Theta, \zeta_\Theta^* \circ \psi_2, w_1^\Theta, \zeta_\Theta^{-1}(S_2^{\mathrm{cl}}))$ is a BC-eigenvariety for the datum $(g_1, \mathcal{H}, \eta, \mathscr{S}_1^\Theta)$. By assumption $\mathscr{S}_1^{\mathrm{cl}} \subset \mathscr{S}_1^\Theta$. Consider the Zariski-closure $\mathcal{D}_1'$ of $\widetilde{\mathrm{ev}}^{-1}(\mathscr{S}_1^{\mathrm{cl}})$ in $\mathcal{D}_1^\Theta$. Let $\iota' \colon \mathcal{D}_1' \to \mathcal{D}_1^\Theta$ be the natural closed immersion and let $w_1' = w_1^\Theta|_{\mathcal{D}_1'}$, $\psi_1' = (\iota')^* \circ \psi_1^\Theta$. By Lemma 3.7.1 the 4-tuple $(\mathcal{D}_1', \psi_1', w_1', \widetilde{\mathrm{ev}}^{-1}(\mathscr{S}_1^{\mathrm{cl}}))$ is a BC-eigenvariety for the BC-datum $(g_1, \mathcal{H}, \eta, \mathscr{S}_1^{\mathrm{cl}})$. Since

$(\mathcal{D}_1, \psi_1, w_1, \mathscr{S}_1^{\mathrm{cl}})$ is a BC-eigenvariety for the same datum, Proposition 3.6.3 gives an isomorphism of rigid analytic spaces $\zeta\colon \mathcal{D}_1 \to \mathcal{D}_1'$ compatible with all the extra structures. The composition $\xi_\Theta = \zeta_\Theta \circ \iota' \circ \zeta\colon \mathcal{D}_1 \to \mathcal{D}_2$ is a closed immersion with the desired properties. □

Let $(g, \mathcal{H}, \eta_1, \mathscr{S}^{\mathrm{cl}})$ and $(g, \mathcal{H}, \eta_2, \mathscr{S}^{\mathrm{cl}})$ be two BC-data that differ only by the choice of the distinguished elements of $\mathcal{H}$. Let $(\mathcal{D}_1, \psi_1, w_1, S_1^{\mathrm{cl}})$ and $(\mathcal{D}_2, \psi_2, w_2, S_2^{\mathrm{cl}})$ be BC-eigenvarieties for the two data. Recall that for $i = 1, 2$ the map

$$\widetilde{\nu}_i\colon \mathcal{D}_i \to \mathcal{W}_g^\circ \times \mathbb{G}_m,$$
$$x \mapsto (w_i(x), \mathrm{ev}_x \circ \psi_i(\eta_i)^{-1})$$

induces a finite morphism $\mathcal{D}_i \to \widetilde{\nu}_i(\mathcal{D}_i)$. Make the following assumption:
(Fin) the map

$$\widetilde{\nu}_{1,2}\colon \mathcal{D}_1 \to \mathcal{W}_g^\circ \times \mathbb{G}_m,$$
$$x \mapsto (w_1(x), \mathrm{ev}_x \circ \psi_1(\eta_2)^{-1})$$

induces a finite morphism $\mathcal{D}_1 \to \widetilde{\nu}_{1,2}(\mathcal{D}_1)$.

Lemma 3.7.3. *Under hypothesis* (Fin)*, there exists an isomorphism of rigid analytic spaces* $\xi_\eta\colon \mathcal{D}_1 \to \mathcal{D}_2$ *such that* $\psi_1 = \xi_\eta^* \circ \psi_2$*,* $w_1 = w_2 \circ \xi_\eta$ *and* $\xi_2(S_1^{\mathrm{cl}}) = S_2^{\mathrm{cl}}$*.*

Proof. We check that the 4-tuple $(\mathcal{D}_1, \psi_1, w_1, S_1^{\mathrm{cl}})$ is a BC-eigenvariety for the datum $(g, \mathcal{H}, \eta_2, \mathscr{S}^{\mathrm{cl}})$. All properties of Definition 3.6.1 except (1) are satisfied because $(\mathcal{D}_1, \psi_1, w_1, S_1^{\mathrm{cl}})$ is a BC-eigenvariety for the datum $(g, \mathcal{H}, \eta_1, \mathscr{S}^{\mathrm{cl}})$. Property (1) is satisfied thanks to hypothesis (Fin). Then $(\mathcal{D}_1, \psi_1, w_1, S_1^{\mathrm{cl}})$ and $(\mathcal{D}_2, \psi_2, w_2, S_2^{\mathrm{cl}})$ are BC-eigenvarieties for the same datum, and Proposition 3.6.3 gives an isomorphism of rigid analytic spaces $\mathcal{D}_1 \to \mathcal{D}_2$ with the desired properties. □

Lemma 3.7.4. *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be two commutative rings and let* $\lambda\colon \mathcal{H}_2 \to \mathcal{H}_1$ *be a ring morphism. Let* $(g, \mathcal{H}_1, \eta_1, \mathscr{S}_1^{\mathrm{cl}})$ *and* $(g, \mathcal{H}_2, \eta_2, \mathscr{S}_2^{\mathrm{cl}})$ *be two BC-data that satisfy* $\eta_1 = \lambda(\eta_2)$ *and*

$$\mathscr{S}_1^{\mathrm{cl}} = \{(\psi \circ \lambda, \underline{k}) \mid (\psi, \underline{k}) \in \mathscr{S}_2^{\mathrm{cl}}\}. \tag{3.14}$$

Let $(\mathcal{D}_1, \psi_1, w_1, S_1^{\mathrm{cl}})$ *and* $(\mathcal{D}_2, \psi_2, w_2, S_2^{\mathrm{cl}})$ *be BC-eigenvarieties for the two data. Suppose that the map* $\mathscr{S}_2^{\mathrm{cl}} \to \mathscr{S}_1^{\mathrm{cl}}$ *defined by* $(\psi, \underline{k}) \mapsto (\psi \circ \lambda, \underline{k})$ *is a bijection. Then there exists an isomorphism of rigid analytic spaces* $\xi_\lambda\colon \mathcal{D}_1 \to \mathcal{D}_2$ *such that* $\psi_1 \circ \lambda = \xi_\lambda^* \circ \psi_2$*,* $w_1 = w_2 \circ \xi_\lambda$ *and* $\xi_\lambda(S_1^{\mathrm{cl}}) = S_2^{\mathrm{cl}}$*.*

Proof. Consider the 4-tuple $(\mathcal{D}_1, \psi_1 \circ \lambda, w_1, S_1^{\mathrm{cl}})$. We show that it defines a BC-eigenvariety for the datum $(g, \mathcal{H}_2, \eta_2, \mathscr{S}_2^{\mathrm{cl}})$. Property (1) of Definition 3.6.1 is satisfied since $\psi_1 \circ \lambda(\eta_2) = \psi_1(\eta_1)$ and the map $(w, \psi_1(\eta_1)^{-1})$ is finite by property (1) relative to the datum $(g, \mathcal{H}_1, \eta_1, \mathscr{S}_1^{\mathrm{cl}})$. Property (2) is a consequence of equality (3.14) together with the fact that S_1^{cl} is Zariski-dense in $\mathcal{D}_1$. Property (3) follows immediately from equality (3.14).

Now the 4-tuples $(\mathcal{D}_1, \psi_1 \circ \lambda, w_1, S_1^{\mathrm{cl}})$ and $(\mathcal{D}_2, \psi_2, w_2, S_2^{\mathrm{cl}})$ define two BC-eigenvarieties for the datum $(g, \mathcal{H}_2, \eta_2, \mathscr{S}_2^{\mathrm{cl}})$, so Proposition 3.6.3 gives a morphism $\xi_\lambda\colon \mathcal{D}_1 \to \mathcal{D}_2$ of rigid analytic spaces such that $\psi_1 \circ \lambda = \xi_\lambda^* \circ \psi_2$, $w_1 = w_2 \circ \xi_\lambda$ and $\xi_\lambda(S_1^{\mathrm{cl}}) = S_2^{\mathrm{cl}}$, as desired. □

3.8. Auxiliary eigenvarieties

Fix a prime p and an integer $N \geq 1$ prime to p. Let M be the integer given as a function of N by Definition 3.3.6. Set $\lambda = \lambda_1$, where $\lambda_1\colon \mathcal{H}_2^N \to \mathcal{H}_1^N$ is the morphism given by Definition 3.4.7.

We will work from now on with the curves $\mathcal{D}_1^N \times_{\mathcal{W}_1} \mathcal{W}_1^\circ$ and $\mathcal{D}_2^M \times_{\mathcal{W}_2} \mathcal{W}_2^\circ$. We still denote them by $\mathcal{D}_1^N$ and $\mathcal{D}_2^M$ in order not to complicate notations. Our aim is to construct a closed immersion $\mathcal{D}_1^N \to \mathcal{D}_2^M$ interpolating the map defined by the symmetric cube transfer on the classical points. As in [**Lu14**] we define two auxiliary eigenvarieties.

3.8.1. The first auxiliary eigenvariety. Recall that for every affinoid subdomain $A = \operatorname{Spm} R$ of $\mathcal{W}_1$ and for every sufficiently large rational number w there is a Banach R-module $M_1(A,w)$ of w-overconvergent modular forms of weight κ_A and level N, carrying an action $\phi^1_{A,w}\colon \mathcal{H}_1^N \to \mathrm{End}_{R,\mathrm{cont}} M_1(A,w)$. We let $\mathcal{H}_2^N$ act on $M_1(A,w)$ through the map

$$\phi^{1,\mathrm{aux}}_{A,w} = \phi^1_{A,w} \circ \lambda\colon \mathcal{H}_2^N \to \mathrm{End}_{R,\mathrm{cont}} M_1(A,w).$$

We have

$$\phi^{1,\mathrm{aux}}_{A,w}(U_p^{(2)}) = \phi^{1,\mathrm{aux}}_{A,w}(U_{p,1}^{(2)}U_{p,2}^{(2)}) = \phi^1_{A,w}(\lambda(U_{p,1}^{(2)}U_{p,2}^{(2)})) = \phi^1_{A,w}(U_{p,0}^{(1)}(U_{p,1}^{(1)})^7).$$

This operator is compact on $M_1(A,w)$ since it is the composition of the compact operator $\phi^{1,\mathrm{aux}}_{A,w}(U_{p,1}^{(1)})$ with a continuous operator.

Definition 3.8.1. *Let $(\mathcal{D}_{1,\lambda}^N, \psi_{1,\lambda}, w_{1,\lambda})$ be the eigenvariety associated with the datum*

$$(\mathcal{W}_1^\circ, \mathcal{H}_2^N, (M_1(A,w))_{A,w}, (\phi^{1,\mathrm{aux}}_{A,w})_{A,w}, U_p^{(2)})$$

by the eigenvariety machine.

Since $\mathcal{W}_1^\circ$ is equidimensional of dimension 1, the eigenvariety $\mathcal{D}_{1,\lambda}^N$ is also equidimensional of dimension 1 by Proposition 1.2.4.

We denote by S_1^{cl} the set of classical points of $\mathcal{D}_1^N$ and by $S_1^{\mathrm{cl},\mathcal{G}}$ the set of classical non-CM points of $\mathcal{D}_1^N$. Recall that we defined a non-CM eigencurve $\mathcal{D}_1^{N,\mathcal{G}}$ as the Zariski-closure of $S_1^{\mathrm{cl},\mathcal{G}}$. The set $S_1^{\mathrm{cl},\mathcal{G}}$ is an accumulation subset of $\mathcal{D}_1^{N,\mathcal{G}}$ by Remark 1.2.24(2) and the weight map $w_1^{\mathcal{G}}\colon \mathcal{D}_1^{N,\mathcal{G}} \to \mathcal{W}_1^\circ$ is surjective by Remark 1.2.24(1).

We define two subsets of $\mathcal{D}_{1,\lambda}^N$ by

$$S_{1,\lambda}^{\mathrm{cl}} = \{x \in \mathcal{D}_{1,\lambda}^N \mid \psi_x = \chi_f \circ \lambda \\ \text{for a classical } \mathrm{GL}_2 \text{ eigenform } f\}.$$

and

$$S_{1,\mathrm{aux}}^{\mathrm{cl}} = \{x \in \mathcal{D}_{1,\lambda}^N \mid \psi_x = \chi_f \circ \lambda \\ \text{for a classical non-CM } \mathrm{GL}_2 \text{ eigenform } f\}.$$

Definition 3.8.2. *Let $\mathcal{D}_{1,\mathrm{aux}}^N$ be the Zariski-closure of the set $S_{1,\mathrm{aux}}^{\mathrm{cl}}$ in $\mathcal{D}_{1,\lambda}^N$.*

We denote by $\psi_{1,\mathrm{aux}}\colon \mathcal{H}_2^N \to \mathcal{O}(\mathcal{D}_{1,\mathrm{aux}}^N)$ and $w_{1,\mathrm{aux}}\colon \mathcal{D}_{1,\mathrm{aux}}^N \to \mathcal{W}_1^\circ$ the morphisms obtained from the corresponding morphisms for $\mathcal{D}_{1,\lambda}^N$.

3.8.2. The second auxiliary eigenvariety. We identify $\mathcal{W}_1^\circ$ with $B_1(1,1^-)$ and $\mathcal{W}_2^\circ$ with $B_2(1,1^-)$ via the fixed isomorphisms η_1 and η_2. We compose η_1 and η_2 with the translations $B_1(1,1^-) \to B_1(0,1^-)$ and $B_2(1,1^-) \to B_2(0,1^-)$. The resulting isomorphisms $\mathcal{W}_1^\circ \cong B_1(0,1^-)$ and $\mathcal{W}_2^\circ \cong B_2(0,1^-)$ give coordinates T on $\mathcal{W}_1^\circ$ and (T_1,T_2) on $\mathcal{W}_2^\circ$. We use from now on these coordinates.

Let $k \geq 2$ be an integer. Let f be a cuspidal GL_2-eigenform of weight k and level $\Gamma_1(N)$ and let f^{st} be a p-stabilization of f. Let $F = (\mathrm{Sym}^3 f)_i^{\mathrm{st}}$ be one of the p-stabilizations of $\mathrm{Sym}^3 f$ defined in Corollary 3.4.8. By Corollary 3.3.3 $(\mathrm{Sym}^3 f)_i^{\mathrm{st}}$ has weight $(2k-1,k+1)$. In particular f^{st} defines a point of the fibre of $\mathcal{D}_1^N$ at $T = u^k - 1$, and $(\mathrm{Sym}^3 f)_i^{\mathrm{st}}$ defines a point of the fibre of $\mathcal{D}_2^M$ at $(T_1,T_2) = (u^{2k-1}-1, u^{k+1}-1)$.

The map $u^k - 1 \mapsto (u^{2k-1} - 1, u^{k+1} - 1)$ is interpolated by the morphism of rigid analytic spaces

$$\iota\colon \mathcal{W}_1^\circ \hookrightarrow \mathcal{W}_2^\circ,$$
$$T \mapsto (u^{-1}(1+T)^2 - 1, u(1+T) - 1).$$

The map ι induces an isomorphism of $\mathcal{W}_1^\circ$ onto its image, which is the rigid analytic curve in $\mathcal{W}_2^\circ$ defined by the equation $u^{-3}(1+T_2)^2 - (1+T_1) = 0$. By construction ι induces a bijection between the classical weights of $\mathcal{W}_1^\circ$ and the classical weights of $\mathcal{W}_2^\circ$ belonging to $\iota(\mathcal{W}_1^\circ)$. Since the classical weights form an accumulation and Zariski-dense subset of $\mathcal{W}_1^\circ$, they also form an accumulation and Zariski-dense subset of $\iota(\mathcal{W}_1^\circ)$.

After Corollary 3.4.9 we defined for $i \in \{1, 2, \ldots, 8\}$ a set $S_i^{\mathrm{Sym}^3} \subset \mathcal{D}_2^M(\overline{\mathbb{Q}}_p)$. By construction of ι, for every i the weight of every point in $S_i^{\mathrm{Sym}^3}$ is a classical weight belonging to $\iota(\mathcal{W}_1^\circ)$. Since $\iota(\mathcal{W}_1^\circ)$ is a one-dimensional Zariski-closed subvariety of $\mathcal{W}_2^\circ$, the image of the Zariski-closure in $\mathcal{D}_2^M$ of $S_i^{\mathrm{Sym}^3}$ under the weight map is contained in $\iota(\mathcal{W}_1^\circ)$. By Remark 3.4.11 the set $S_i^{\mathrm{Sym}^3}$ is discrete if $i \geq 2$, so the only interesting Zariski-closure is that of $S_1^{\mathrm{Sym}^3}$.

Definition 3.8.3. *Let $\mathcal{D}_{2,\mathrm{aux}}^M$ be the Zariski closure of $S_1^{\mathrm{Sym}^3}$ in $\mathcal{D}_2^M$ and let $\iota_{2,\mathrm{aux}}\colon \mathcal{D}_{2,\mathrm{aux}}^M \to \mathcal{D}_2^M$ be the natural closed immersion. Define $w_{2,\mathrm{aux}}\colon \mathcal{D}_{2,\mathrm{aux}}^M \to \mathcal{W}_1^\circ$ and $\psi_{2,\mathrm{aux}}\colon \mathcal{H}_2^N \to \mathcal{O}(\mathcal{D}_{2,\mathrm{aux}}^M)$ as $w_{2,\mathrm{aux}} = \iota^{-1} \circ w_2|_{\mathcal{D}_{2,\mathrm{aux}}^M}$ and $\psi_{2,\mathrm{aux}} = \iota_{2,\mathrm{aux}}^* \circ \psi_2$.*

3.9. The morphisms between the eigenvarieties

In this section we construct morphisms of rigid analytic spaces

$$\xi_1\colon \mathcal{D}_1^{N,\mathcal{G}} \to \mathcal{D}_{1,\mathrm{aux}}^N,$$
$$\xi_2\colon \mathcal{D}_{1,\mathrm{aux}}^N \to \mathcal{D}_{2,\mathrm{aux}}^M,$$
$$\xi_3\colon \mathcal{D}_{2,\mathrm{aux}}^M \to \mathcal{D}_2^M,$$

making the following diagrams commute:

(3.15)
$$\begin{tikzcd}
\mathcal{D}_1^{N,\mathcal{G}} \arrow[r, "\xi_1"] \arrow[d] & \mathcal{D}_{1,\mathrm{aux}}^N \arrow[r, "\xi_2"] \arrow[d] & \mathcal{D}_{2,\mathrm{aux}}^M \arrow[r, "\xi_3"] \arrow[d] & \mathcal{D}_2^M \arrow[d] \\
\mathcal{W}_1^\circ \arrow[r, "="] & \mathcal{W}_1^\circ \arrow[r, "="] & \mathcal{W}_1^\circ \arrow[r, "\iota"] & \mathcal{W}_2^\circ
\end{tikzcd}$$
$$\begin{tikzcd}
\mathcal{H}_2^N \arrow[r, "="] \arrow[d] & \mathcal{H}_2^N \arrow[r, "="] \arrow[d] & \mathcal{H}_2^N \arrow[r, "\lambda"] \arrow[d] & \mathcal{H}_1^N \arrow[d] \\
\mathcal{O}(\mathcal{D}_2^M) \arrow[r, "\xi_3^*"] & \mathcal{O}(\mathcal{D}_{2,\mathrm{aux}}^M) \arrow[r, "\xi_2^*"] & \mathcal{O}(\mathcal{D}_{1,\mathrm{aux}}^N) \arrow[r, "\xi_1^*"] & \mathcal{O}(\mathcal{D}_1^{N,\mathcal{G}})
\end{tikzcd}$$

In order to construct ξ_1, ξ_2 and ξ_3 we will interpret the eigenvarieties appearing in the diagrams as BC-eigenvarieties for suitably chosen BC-data and rely on the results of Section 3.6.

3.9.1. The BC-eigenvarieties. We define two subsets $\mathscr{S}_1^{\mathrm{cl}}$ and $\mathscr{S}_1^{\mathrm{cl},\mathcal{G}}$ of $\mathrm{Hom}(\mathcal{H}_1^N,\overline{\mathbb{Q}}_p)\times\mathbb{Z}$ by

$$\mathscr{S}_1^{\mathrm{cl}}=\{(\psi,k)\in\mathrm{Hom}(\mathcal{H}_1^N,\overline{\mathbb{Q}}_p)\times\mathbb{Z}\,|\,\psi=\chi_f$$
$$\text{for a cuspidal classical }\mathrm{GL}_2\text{-eigenform }f\text{ of weight }k\},$$

$$\mathscr{S}_1^{\mathrm{cl},\mathcal{G}}=\{(\psi,k)\in\mathrm{Hom}(\mathcal{H}_1^N,\overline{\mathbb{Q}}_p)\times\mathbb{Z}\,|\,\psi=\chi_f$$
$$\text{for a cuspidal classical non-CM }\mathrm{GL}_2\text{-eigenform }f\text{ of weight }k\}.$$

We define two subsets $\mathscr{S}_{1,\lambda}$ and $\mathscr{S}_{1,\mathrm{aux}}$ of $\mathrm{Hom}(\mathcal{H}_2^N,\overline{\mathbb{Q}}_p)\times\mathbb{Z}$ by

$$\mathscr{S}_{1,\lambda}^{\mathrm{cl}}=\{(\psi,k)\in\mathrm{Hom}(\mathcal{H}_2^N,\overline{\mathbb{Q}}_p)\times\mathbb{Z}\,|\,\psi=\chi_f\circ\lambda$$
$$\text{for a cuspidal classical }\mathrm{GL}_2\text{-eigenform }f\text{ of weight }k\},$$

$$\mathscr{S}_{1,\mathrm{aux}}^{\mathrm{cl}}=\{(\psi,k)\in\mathrm{Hom}(\mathcal{H}_2^N,\overline{\mathbb{Q}}_p)\times\mathbb{Z}\,|\,\psi=\chi_f\circ\lambda$$
$$\text{for a cuspidal classical non-CM }\mathrm{GL}_2\text{-eigenform }f\text{ of weight }k\}.$$

LEMMA 3.9.1.
(1) The 4-tuple $(\mathcal{D}_1^N,\psi_1,w_1,S_1^{\mathrm{cl}})$ is a BC-eigenvariety for the datum $(1,\mathcal{H}_1^N,U_p^{(1)},\mathscr{S}_1^{\mathrm{cl}})$.
(2) The 4-tuple $(\mathcal{D}_1^N,\psi_1,w_1,S_1^{\mathrm{cl}})$ is a BC-eigenvariety for the datum $(1,\mathcal{H}_1^N,\lambda(U_p^{(2)}),\mathscr{S}_1^{\mathrm{cl}})$.
(3) The 4-tuple $(\mathcal{D}_1^{N,\mathcal{G}},\psi_1,w_1,S_1^{\mathrm{cl},\mathcal{G}})$ is a BC-eigenvariety for the datum $(1,\mathcal{H}_1^N,\lambda(U_p^{(2)}),\mathscr{S}_1^{\mathrm{cl},\mathcal{G}})$.
(4) The 4-tuple $(\mathcal{D}_{1,\lambda}^N,\psi_{1,\lambda},w_{1,\lambda},S_{1,\lambda}^{\mathrm{cl}})$ is a BC-eigenvariety for the datum $(1,\mathcal{H}_2^N,U_p^{(2)},\mathscr{S}_{1,\lambda}^{\mathrm{cl}})$.
(5) The 4-tuple $(\mathcal{D}_{1,\mathrm{aux}}^N,\psi_{1,\mathrm{aux}},w_{1,\mathrm{aux}},S_{1,\mathrm{aux}}^{\mathrm{cl}})$ is a BC-eigenvariety for the datum $(1,\mathcal{H}_2^N,U_p^{(2)},\mathscr{S}_{1,\mathrm{aux}}^{\mathrm{cl}})$.

PROOF. Part (1) follows from Lemma 3.6.5.

For part (2), observe that the couple $(w_1,\psi_1(U_p^{(1)}))$ satisfies condition (Fin) since $\lambda(U_p^{(2)})=U_{p,0}^{(1)}(U_{p,1}^{(1)})^7$. Hence Lemma 3.7.3 gives an isomorphism between the eigenvarieties for the data $(1,\mathcal{H}_1^N,\lambda(U_p^{(2)}),\mathscr{S}_1^{\mathrm{cl}})$ and $(1,\mathcal{H}_1^N,U_p^{(1)},\mathscr{S}_1^{\mathrm{cl}})$, as desired.

We prove part (3). Let $\widetilde{\mathrm{ev}}\colon S_1^{\mathrm{cl}}\to\mathscr{S}_1^{\mathrm{cl}}$ be the evaluation map given in property (3) of Definition 3.6.1. By definition the eigenvariety $\mathcal{D}_1^{N,\mathcal{G}}$ is the Zariski-closure in $\mathcal{D}_1^N$ of the set $S_1^{\mathrm{cl},\mathcal{G}}$. The image of $S_1^{\mathrm{cl},\mathcal{G}}$ in $\mathscr{S}_1^{\mathrm{cl}}$ via $\widetilde{\mathrm{ev}}$ is $\mathscr{S}_1^{\mathrm{cl},\mathcal{G}}$, so our statement follows from Lemma 3.7.1 applied to $S^{\mathrm{cl}}=S_1^{\mathrm{cl}}$ and $S_0^{\mathrm{cl}}=S_1^{\mathrm{cl},\mathcal{G}}$.

Part (4) follows from Definition 3.8.1 and Corollary 3.6.5.

The proof of part (5) is analogous to that of part (3). Let $\widetilde{\mathrm{ev}}\colon S_{1,\lambda}^{\mathrm{cl}}\to\mathscr{S}_{1,\lambda}^{\mathrm{cl}}$ be the evaluation map. By definition the eigenvariety $\mathcal{D}_{1,\mathrm{aux}}^N$ is the Zariski-closure in $\mathcal{D}_1^N$ of the set $S_{1,\mathrm{aux}}^{\mathrm{cl}}$. The image of $S_{1,\mathrm{aux}}^{\mathrm{cl}}$ in $\mathscr{S}_{1,\lambda}^{\mathrm{cl}}$ via $\widetilde{\mathrm{ev}}$ is $\mathscr{S}_{1,\mathrm{aux}}^{\mathrm{cl}}$, so the desired conclusion follows from Lemma 3.7.1 applied to $S^{\mathrm{cl}}=S_{1,\lambda}^{\mathrm{cl}}$ and $S_0^{\mathrm{cl}}=S_{1,\mathrm{aux}}^{\mathrm{cl}}$. □

Now consider the second auxiliary eigenvariety $\mathcal{D}_{2,\mathrm{aux}}^M$. Recall that $\mathcal{D}_{2,\mathrm{aux}}^M$ is defined as the Zariski-closure in $\mathcal{D}_2^M$ of the set $S_1^{\mathrm{Sym}^3}$. It is equipped with maps $\psi_{2,\mathrm{aux}}\colon\mathcal{H}_2^N\to\mathcal{O}(\mathcal{D}_{2,\mathrm{aux}}^M)$ and $w_{2,\mathrm{aux}}\colon\mathcal{D}_{2,\mathrm{aux}}^M\to\mathcal{W}_2^\circ$. Define a subset $\mathscr{S}_{2,\mathrm{aux}}^{\mathrm{cl}}$ of $\mathrm{Hom}(\mathcal{H}_2^N,\overline{\mathbb{Q}}_p)\times\mathbb{Z}$ by

$$\mathscr{S}_{2,\mathrm{aux}}^{\mathrm{cl}}=\{(\psi,k)\in\mathrm{Hom}(\mathcal{H}_2^N,\overline{\mathbb{Q}}_p)\times\mathbb{Z}\,|\,\psi=\chi_F$$
$$\text{where }F=(\mathrm{Sym}^3f)_1^{\mathrm{st}}\text{ for a cuspidal classical non-CM }\mathrm{GL}_2\text{-eigenform }f\text{ of weight }k\}.$$

LEMMA 3.9.2. *The 4-tuple $(\mathcal{D}_{2,\mathrm{aux}}^M,\psi_2,w_2,S_1^{\mathrm{Sym}^3})$ defines a BC-eigenvariety for the datum $(1,\mathcal{H}_2^N,U_p^{(2)},\mathscr{S}_{2,\mathrm{aux}}^{\mathrm{cl}})$.*

PROOF. It is clear from the definitions of S_2^{cl} and $\mathscr{S}_2^{\mathrm{cl}}$ that the evaluation of $(\psi_{2,\mathrm{aux}},w_{2,\mathrm{aux}})$ at a point $x\in S_1^{\mathrm{Sym}^3}$ induces a bijection $S_1^{\mathrm{Sym}^3}\to\mathscr{S}_2^{\mathrm{cl}}$. Then the lemma follows from Corollary 3.7.1 applied to the choices $\mathcal{D}=\mathcal{D}_2^M$, $S_0^{\mathrm{cl}}=S_1^{\mathrm{Sym}^3}$, $g_0=1$ and $\iota_0=\iota$. □

REMARK 3.9.3. *The sets* $\mathscr{S}^{\mathrm{cl}}_{1,\lambda}$ *and* $\mathscr{S}^{\mathrm{cl}}_{2,\mathrm{aux}}$ *coincide. Indeed* $(\mathrm{Sym}^3 f)^{\mathrm{st}}_1$ *is well-defined for every cuspidal non-CM* GL_2*-eigenform* f*, and a* GSp_4*-eigenform* F *satisfies* $\chi_F = \chi_f \circ \lambda$ *if and only if* $F = (\mathrm{Sym}^3 f)^{\mathrm{st}}_1$*.*

Let S^{cl}_2 be the set of classical points of $\mathcal{D}^M_2$. Define a subset $\mathscr{S}^{\mathrm{cl}}_2$ of $\mathrm{Hom}(\mathcal{H}^N_2, \overline{\mathbb{Q}}_p) \times \mathbb{Z}^2$ by

$$\mathscr{S}^{\mathrm{cl}}_2 = \{(\psi, \underline{k}) \in \mathrm{Hom}(\mathcal{H}^N_2, \overline{\mathbb{Q}}_p) \times \mathbb{Z}^2 \,|\, \psi = \chi_F \text{ for a cuspidal classical } \mathrm{GSp}_4\text{-eigenform } F \text{ of weight } \underline{k}\}.$$

LEMMA 3.9.4. *The 4-tuple* $(\mathcal{D}^M_2, \psi_2, w_2, S^{\mathrm{cl}}_2)$ *is a BC-eigenvariety associated with the datum* $(2, \mathcal{H}^N_2, U^{(2)}_p, \mathscr{S}^{\mathrm{cl}}_2)$*.*

PROOF. This is an immediate consequence of Corollary 3.6.5. □

3.9.2. Existence of the morphisms. We are ready to prove the existence of the morphisms fitting into diagram (3.15).

PROPOSITION 3.9.5. *There exists an isomorphism* $\xi_1\colon \mathcal{D}^{N,\mathcal{G}}_1 \to \mathcal{D}^N_{1,\mathrm{aux}}$ *of rigid analytic spaces over* $\mathbb{Q}_p$ *such that the following diagrams commute:*

$$\begin{tikzcd} \mathcal{D}^{N,\mathcal{G}}_1 \arrow[r, "\xi_1"] \arrow[d] & \mathcal{D}^N_{1,\mathrm{aux}} \arrow[d] & \mathcal{H}^N_2 \arrow[r, "\lambda"] \arrow[d] & \mathcal{H}^N_1 \arrow[d] \\ \mathcal{W}^\circ_1 \arrow[r, "="] & \mathcal{W}^\circ_1 & \mathcal{O}(\mathcal{D}^N_{1,\mathrm{aux}}) \arrow[r, "\xi_1^*"] & \mathcal{O}(\mathcal{D}^{N,\mathcal{G}}_1) \end{tikzcd} \tag{3.16}$$

PROOF. Note that the map $\mathscr{S}^{\mathrm{cl},\mathcal{G}}_1 \to \mathscr{S}^{\mathrm{cl}}_{1,\mathrm{aux}}$ defined by $(\psi, k) \mapsto (\psi \circ \lambda, k)$ is a bijection by Remark 3.5.14. Thanks to Lemma 3.9.1(3,5) we know that the 4-tuples $(\mathcal{D}^{N,\mathcal{G}}_1, \psi_1, w_1, S^{\mathrm{cl},\mathcal{G}}_1)$ and $(\mathcal{D}^N_{1,\mathrm{aux}}, \psi_{1,\mathrm{aux}}, w_{1,\mathrm{aux}}, S^{\mathrm{cl}}_{1,\mathrm{aux}})$ are BC-eigenvarieties for the data $(1, \mathcal{H}^N_1, \lambda(U^{(2)}_p), \mathscr{S}^{\mathrm{cl},\mathcal{G}}_1)$ and $(1, \mathcal{H}^N_2, U^{(2)}_p, \mathscr{S}^{\mathrm{cl}}_{1,\mathrm{aux}})$, respectively. Hence Lemma 3.7.4 applied to the morphism $\lambda\colon \mathcal{H}^N_2 \to \mathcal{H}^N_1$ and the two data above gives an isomorphism $\xi_1\colon \mathcal{D}^{N,\mathcal{G}}_1 \to \mathcal{D}^N_{1,\mathrm{aux}}$ that makes diagrams (3.16) commute. □

PROPOSITION 3.9.6. *There exists an isomorphism* $\xi_2\colon \mathcal{D}^N_{1,\mathrm{aux}} \to \mathcal{D}^M_{2,\mathrm{aux}}$ *of rigid analytic spaces over* $\mathbb{Q}_p$ *such that the following diagrams commute:*

$$\begin{tikzcd} \mathcal{D}^N_{1,\mathrm{aux}} \arrow[r, "\xi_2"] \arrow[d] & \mathcal{D}^M_{2,\mathrm{aux}} \arrow[d] & \mathcal{H}^N_2 \arrow[r, "="] \arrow[d] & \mathcal{H}^N_2 \arrow[d] \\ \mathcal{W}^\circ_1 \arrow[r, "="] & \mathcal{W}^\circ_1 & \mathcal{O}(\mathcal{D}^M_{2,\mathrm{aux}}) \arrow[r, "\xi_2^*"] & \mathcal{O}(\mathcal{D}^N_{1,\mathrm{aux}}) \end{tikzcd}$$

PROOF. Lemmas 3.9.1(5) and 3.9.2 together with Remark 3.9.3 imply that the 4-tuples $(\mathcal{D}^N_{1,\mathrm{aux}}, \psi_1, w_1, S^{\mathrm{cl}}_{1,\mathrm{aux}})$ and $(\mathcal{D}^M_{2,\mathrm{aux}}, \psi_2, w_2, S^{\mathrm{cl}}_{2,\mathrm{aux}})$ are both BC-eigenvarieties for the datum $g = 1$, $\mathcal{H} = \mathcal{H}^N_2$, $\eta = U^{(2)}_p$ and $\mathscr{S}^{\mathrm{cl}} = \mathscr{S}^{\mathrm{cl}}_{1,\mathrm{aux}} = \mathscr{S}^{\mathrm{cl}}_{2,\mathrm{aux}}$. Now the proposition follows from Proposition 3.6.3. □

PROPOSITION 3.9.7. *There exists a closed immersion* $\xi_3\colon \mathcal{D}^N_{2,\mathrm{aux}} \to \mathcal{D}^M_2$ *of rigid analytic spaces over* $\mathbb{Q}_p$ *such that the following diagrams commute:*

$$\begin{tikzcd} \mathcal{D}^M_{2,\mathrm{aux}} \arrow[r, "\xi_3"] \arrow[d] & \mathcal{D}^M_2 \arrow[d] & \mathcal{H}^N_2 \arrow[r, "="] \arrow[d] & \mathcal{H}^N_2 \arrow[d] \\ \mathcal{W}^\circ_1 \arrow[r, "\iota"] & \mathcal{W}^\circ_2 & \mathcal{O}(\mathcal{D}^M_2) \arrow[r, "\xi_3^*"] & \mathcal{O}(\mathcal{D}^M_{2,\mathrm{aux}}) \end{tikzcd}$$

Proof. This is a consequence of Lemma 3.7.2 applied to the BC-data $(2, \mathcal{H}_2^N, U_p^{(2)}, \mathscr{S}_2)$ and $(1, \mathcal{H}_2^N, U_p^{(2)}, \mathscr{S}_2^{\mathrm{cl}})$, with the morphism $\mathcal{W}_1 \to \mathcal{W}_2$ being ι. □

Finally, we can define the desired p-adic interpolation of the symmetric cube transfer.

Definition 3.9.8. *We define a morphism $\xi\colon \mathcal{D}_1^{N,\mathcal{G}} \to \mathcal{D}_2^M$ of rigid analytic spaces over $\mathbb{Q}_p$ by*

$$\xi = \xi_3 \circ \xi_2 \circ \xi_1.$$

The properties of ξ_1, ξ_2, ξ_3 imply that ξ is a morphism of eigenvarieties, in the sense that the following diagrams commute:

$$\begin{tikzcd} \mathcal{D}_1^{N,\mathcal{G}} \arrow[r, "\xi"] \arrow[d, "w_1"] & \mathcal{D}_2^M \arrow[d, "w_2"] \\ \mathcal{W}_1^\circ \arrow[r, "\iota"] & \mathcal{W}_2^\circ \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H}_2^N \arrow[r, "\lambda"] \arrow[d, "\psi_2"] & \mathcal{H}_1^N \arrow[d, "\psi_1"] \\ \mathcal{O}(\mathcal{D}_2^M) \arrow[r, "\xi^*"] & \mathcal{O}(\mathcal{D}_1^{N,\mathcal{G}}) \end{tikzcd} \tag{3.17}$$

Remark 3.9.9. *Since ξ_1 and ξ_2 are isomorphisms and ξ_3 is a closed immersion, the morphism ξ is a closed immersion.*

3.10. Overconvergent eigenforms and trianguline representations

In this section V is a finite-dimensional $\mathbb{Q}_p$-vector space endowed with the p-adic topology and with a continuous action of $G_{\mathbb{Q}_p}$. For every vector space or module W carrying an action (not necessarily linear) of $G_{\mathbb{Q}_p}$, we denote by $\mathrm{Sym}^3 W$ the symmetric cube of W. We always equip $\mathrm{Sym}^3 W$ with an action of $G_{\mathbb{Q}_p}$ in the standard way. We recall some definitions and results from p-adic Hodge theory and the theory of (φ, Γ)-modules. We always write invariants under a group action by an upper index.

3.10.1. Fontaine's rings and admissible representations.

Let K and E be two p-adic fields with $E \subset K$. Let $\mathbf{B}$ be a topological E-algebra equipped with a continuous action of G_K. We say that $\mathbf{B}$ is (E, G)-regular if:

(1) B is a domain;
(2) $B^{G_K} = (\mathrm{Frac} B)^{G_K}$;
(3) if $b \in B$ is non-zero and the line $\mathbf{B} \cdot b$ is G_K-stable, then b is invertible in $\mathbf{B}$.

The simpler examples of (E, G_K)-regular rings are $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$.

We suppose from now on that $\mathbf{B}$ is (E, G_K)-regular. Let V be a finite dimensional E-representation of G_K. We introduce the notion of $\mathbf{B}$-admissibility of V; our reference is [**Fo94**, Chapitre 1]. For every K-representation V the $\mathbf{B}$-module $\mathbf{B} \otimes_E V$ is free and carries a semilinear action of G_K. Set

$$\mathbf{D}(V) = (\mathbf{B} \otimes_E V)^{G_K}.$$

Then $\mathbf{D}$ is a K-vector space and there is a natural K-linear map

$$\alpha_{\mathbf{B},V}\colon \mathbf{B} \otimes_E \mathbf{D}(V) \to \mathbf{B} \otimes_K V.$$

The map $\alpha_{\mathbf{B},V}$ is always injective. We say that V is a $\mathbf{B}$-admissible representation of G_K if $\alpha_{\mathbf{B},V}$ is an isomorphism. Then

$$V \text{ is } \mathbf{B}\text{-admissible} \iff \dim_K \mathbf{D}(V) = \dim_E V \iff$$
$$\iff \mathbf{B} \otimes_E V \text{ is a trivial } \mathbf{B}\text{-representation of } G_K.$$

Consider the following condition:

$(*)$ the ring $\mathbf{B}$ is $(E, G_{K'})$-regular for every finite extension K' of K.

For $\mathbf{B}$ satisfying $(*)$, we say that V is potentially $\mathbf{B}$-admissible if there exists a finite extension K' of K such that V is $\mathbf{B}$-admissible as a representation of $G_{K'}$.

Proposition 3.10.1. *If $\mathbf{B}$ is a $\overline{\mathbb{Q}}_p$-algebra that satisfies $(*)$, then a potentially $\mathbf{B}$-admissible representation V is $\mathbf{B}$-admissibile.*

Fontaine defined some (E, G_K)-regular rings $\mathbf{B}_{\mathrm{HT}}$, $\mathbf{B}_{\mathrm{dR}}$, $\mathbf{B}_{\mathrm{st}}$, $\mathbf{B}_{\mathrm{cris}}$. The lower indices stand respectively for Hodge-Tate, de Rham, semi-stable and crystalline. We refer to [**Fo94**] for the details of the constructions. All the rings above satisfy condition $(*)$. We recall that

$$\mathbf{B}_{\mathrm{HT}} = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}_p t^i,$$

where $g.t = \chi(g)t$ for the cyclotomic character χ, and that $\mathbf{B}_{\mathrm{dR}}$ is a field. All of Fontaine's rings are independent of E, and the rings $\mathbf{B}_{\mathrm{HT}}$ and $\mathbf{B}_{\mathrm{dR}}$ are also independent of K. When $\mathbf{B}$ is one of Fontaine's rings, we replace the notation $\mathbf{D}_{\mathbf{B}}$ by $\mathbf{D}_{\mathrm{HT}}$, $\mathbf{D}_{\mathrm{dR}}$, $\mathbf{D}_{\mathrm{st}}$ or $\mathbf{D}_{\mathrm{cris}}$ depending on $\mathbf{B}$.

We say that the representation V is Hodge-Tate, de Rham, semi-stable or crystalline if it is $\mathbf{B}$-admissible respectively for $\mathbf{B}_{\mathrm{HT}}$, $\mathbf{B}_{\mathrm{dR}}$, $\mathbf{B}_{\mathrm{st}}$ or $\mathbf{B}_{\mathrm{cris}}$. We recall some basic results.

Proposition 3.10.2. *There is a chain of implications between the admissibility properties of V:*

$$\textit{crystalline} \implies \textit{semi-stable} \implies \textit{de Rham} \implies \textit{Hodge-Tate.}$$

Note that B_{dR} satisfies the assumptions of Proposition 3.10.1. By combining this with Proposition 3.10.2 we obtain that a potentially semi-stable V is de Rham. The converse is also true and is a result by Berger.

Theorem 3.10.3. [**Be02**, Théorème 0.7] *A finite dimensional E-representation of G_K is de Rham if and only if it is potentially semi-stable.*

Suppose that V is Hodge-Tate and let $d = \dim_E V$. Then the K-vector space

$$\mathbf{D}_{\mathrm{HT}} = (\mathbf{B}_{\mathrm{HT}} \otimes_E V)^{G_K} = \left(\bigoplus_{i\in\mathbb{Z}} \mathbb{C}_p t^i \otimes_E V\right)^{G_K}$$

is d-dimensional.

Definition 3.10.4. *The Hodge-Tate weights of V are the values of $i \in \mathbb{Z}$ such that the dimension*

$$d_i = \dim_K(\mathbb{C}_p t^{-i} \otimes_E V)^{G_K}$$

is positive. The multiplicity of a Hodge-Tate weight i of V is d_i.

It can be shown that

$$\mathbf{D}_{\mathrm{HT}} = \bigoplus_{i\in\mathbb{Z}} (\mathbb{C}_p t^i \otimes_E V)^{G_K},$$

hence the sum of the Hodge-Tate weights of V with multiplicities is d.

An important class of de Rham representations is given by the Galois representations associated with classical automorphic forms. We state the result only for the cases we need. Let $g = 1$ or 2 and let N be a positive integer. Let F be a classical, cuspidal GSp_{2g}-eigenform of level $\Gamma_1(N)$. Let

$$\rho_{F,p}\colon G_{\mathbb{Q}} \to \mathrm{GSp}_{2g}(\overline{\mathbb{Q}}_p)$$

be the p-adic Galois representation associated with F.

Theorem 3.10.5.

(1) for every prime p not dividing N, $\rho_{F,p}|_{G_{\mathbb{Q}_p}}$ is a crystalline representation;

(2) for every prime p, $\rho_{F,p}|_{G_{\mathbb{Q}_p}}$ is a de Rham representation.

The first statement is a consequence of Faltings's proof of Fontaine's C_{cris} conjecture [**Fa89**]. The second one follows from Tsuji's proof of the C_{st} conjecture formulated by Fontaine and Jannsen [**Ts99**]. In the case $g = 1$ this is the confirmation of one implication of the Fontaine-Mazur conjecture [**FM95**]. For the converse see Emerton's result (Theorem 3.10.18(2)).

REMARK 3.10.6. *Since $\rho_{F,p}$ is a de Rham representation it is also Hodge-Tate. Its Hodge-Tate weights can be given in terms of the weight of F:*

- *if $g = 1$ and F is a form of weight k, then the Hodge-Tate weights of $\rho_{F,p}$ are 0 and $k - 1$;*
- *if $g = 2$ and F is a form of weight (k_1, k_2), then the Hodge-Tate weights of $\rho_{F,p}$ are 0, $k_2 - 2$, $k_1 - 1$ and $k_1 + k_2 - 3$.*

3.10.2. Trianguline representations and overconvergent modular forms. We recall a few results from the theory of (φ, Γ)-modules. We refer mainly to [**Fo90**], [**Be02**] and [**Col08**]. As before E is a finite extension of $\mathbb{Q}_p$, fixed throughout the section. Let Γ be the Galois group over E of a $\mathbb{Z}_p$-extension of E and let $H_E = G_E/\Gamma$. Let $\mathscr{R}$ be the Robba ring over E. Let $\mathscr{E}^\dagger$ be the field of bounded elements of $\mathscr{R}$. The rings $\mathscr{R}$ and $\mathscr{E}^\dagger$ carry commuting actions of Γ and of a Frobenius operator φ.

A (φ, Γ)-module over $\mathscr{E}^\dagger$ or $\mathscr{R}$ is a free module D of finite type carrying commuting actions of Γ and φ and such that $\varphi(D)$ generates D as a module (over $\mathscr{E}^\dagger$ or $\mathscr{R}$). We say that a (φ, Γ)-module D over $\mathscr{R}$ is triangulable if it is obtained via successive extensions of (φ, Γ)-modules of rank one over $\mathscr{R}$.

We refer to [**Fo90**] and [**Be02**] for the definitions of the rings $\mathbf{B}^\dagger$ and $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ and of the categories of étale (φ, Γ)-modules over $\mathscr{E}^\dagger$ and of (φ, Γ)-modules of slope 0 over $\mathscr{R}$. For a finite-dimensional E-representation V of $G_{\mathbb{Q}_p}$, let

$$\mathbf{D}^\dagger(V) = (\mathbf{B}^\dagger \otimes_{\mathbb{Q}_p} V)^{H_{\mathbb{Q}_p}}.$$

Then $\mathbf{D}^\dagger(V)$ carries a natural structure of étale (φ, Γ)-module over $\mathscr{E}^\dagger$, and

$$V \mapsto \mathbf{D}^\dagger(V)$$

defines a functor $\mathbf{D}^\dagger$ from the category of finite-dimensional E-representations of $G_{\mathbb{Q}_p}$ to the category of (φ, Γ)-modules on $\mathscr{E}^\dagger$. Conversely, for every (φ, Γ)-module $\mathbf{D}$, let

$$\mathbf{V}^\dagger(D) = (D \otimes_{\mathscr{E}^\dagger_{\mathbb{Q}_p}} D)^{\varphi=1}.$$

Then $\mathbf{V}^\dagger(D)$ is a finite-dimensional E-vector space with a natural action of $G_{\mathbb{Q}_p}$, and

$$D \mapsto \mathbf{V}^\dagger(D)$$

defines a functor $\mathbf{V}^\dagger$ from the category of (φ, Γ)-modules on $\mathscr{E}^\dagger$ to the category of finite-dimensional E-representations of $G_{\mathbb{Q}_p}$.

THEOREM 3.10.7. [**Fo90**, Proposition 1.2.6] *The functors $\mathbf{D}^\dagger$ and $\mathbf{V}^\dagger$ are inverses of one another and induce an equivalence between the category of finite-dimensional E-representations of $G_{\mathbb{Q}_p}$ and that of (φ, Γ)-modules on $\mathscr{E}^\dagger$.*

Now for a finite-dimensional E-representation V of $G_{\mathbb{Q}_p}$ let

$$\mathbf{D}_{\mathrm{rig}}(V) = \mathscr{R} \otimes_{\mathscr{E}^\dagger} \mathbf{D}^\dagger(V)$$

with its natural structure of (φ, Γ)-module over $\mathscr{R}$. Then

$$V \mapsto \mathbf{D}_{\mathrm{rig}}(V)$$

defines a functor from the category of finite-dimensional E-representations of $G_{\mathbb{Q}_p}$ to the category of (φ, Γ)-modules over $\mathscr{R}$. For a (φ, Γ)-module D over $\mathscr{R}$, let

$$\mathbf{V}(D) = (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathscr{R}} D)^{\varphi=1}$$

and equip it with its natural structure of E-representation of $G_{\mathbb{Q}_p}$. Then

$$D \mapsto \mathbf{V}(D)$$

defines a functor from the category of (φ, Γ)-modules over $\mathscr{R}$ to the category of finite-dimensional E-representations of $G_{\mathbb{Q}_p}$.

THEOREM 3.10.8. [**Col08**, Proposition 1.7] *The functors $\mathbf{D}_{\mathrm{rig}}$ and $\mathbf{V}$ are inverses of one another and induce an equivalence between the category of finite-dimensional E-representations of $G_{\mathbb{Q}_p}$ and that of (φ, Γ)-modules of slope 0 on $\mathscr{R}$.*

We recall an important definition.

DEFINITION 3.10.9. [**Col08**, Section 0.4] *A finite-dimensional E-representation V of $G_{\mathbb{Q}_p}$ is trianguline if the (φ, Γ)-module $\mathbf{D}_{\mathrm{rig}}(V)$ is triangulable.*

Thanks to the results of [**Be02**], we can recover Fontaine's modules $\mathbf{D}_{\mathrm{cris}}(V)$ and $\mathbf{D}_{\mathrm{st}}(V)$ from $\mathbf{D}_{\mathrm{rig}}(V)$. Note that we formulate this result as in [**Col08**, Proposition 1.8], since we did not introduce the ring $\mathbf{B}_{\log}$.

LEMMA 3.10.10. ([**Be02**, Theorem 0.2], see [**Col08**, Proposition 1.8])

(1) The structure of (φ, Γ)-module on $\mathbf{D}_{\mathrm{rig}}(V)$ induces a structure of filtered φ-module on $(\mathscr{R}[1/t] \otimes_{\mathscr{R}} \mathbf{D}_{\mathrm{rig}}(V))^{\Gamma}$ such that there is an isomorphism

$$\mathbf{D}_{\mathrm{cris}}(V) \cong (\mathscr{R}[1/t] \otimes_{\mathscr{R}} \mathbf{D}_{\mathrm{rig}}(V))^{\Gamma}.$$

(2) The structure of (φ, Γ)-module on $\mathbf{D}_{\mathrm{rig}}(V)$ induces a structure of filtered (φ, N)-module on $(\mathscr{R}[1/t, \log T] \otimes_{\mathscr{R}} \mathbf{D}_{\mathrm{rig}}(V))^{\Gamma}$ such that there is an isomorphism

$$\mathbf{D}_{\mathrm{st}}(V) \cong (\mathscr{R}[1/t, \log T] \otimes_{\mathscr{R}} \mathbf{D}_{\mathrm{rig}}(V))^{\Gamma}.$$

The cyclotomic character χ induces an isomorphism $\Gamma \to \mathbb{Z}_p^{\times}$ that we still denote by χ. Let γ_u be the pre-image of our chosen generator u of $\mathbb{Z}_p^{\times}$. Let $\delta\colon \mathbb{Q}_p^{\times} \to E^{\times}$ be a continuous character. We define $\mathscr{R}(\delta)$ as the rank one (φ, Γ)-module having a basis element e_δ such that $\varphi.e_\delta = \delta(p)e_\delta$ and $\gamma_u.e_\delta = \delta(\chi(\gamma_u))e_\delta$.

PROPOSITION 3.10.11. [**Col08**, Théorème 0.2(i)] *For every rank one (φ, Γ)-module $\mathbf{D}$ over $\mathscr{R}$, there exists a unique continuous character $\delta\colon \mathbb{Q}_p^{\times} \to E^{\times}$ such that $\mathbf{D} \cong \mathscr{R}(\delta)$.*

If $\eta\colon G_K \to E^{\times}$ is a character, we denote by $V(\eta)$ the E-representation of G_K obtained by twisting V by η. We recall a result by Colmez.

LEMMA 3.10.12. [**Col08**, Proposition 4.3] *When V is two-dimensional the following conditions are equivalent:*

(i) V is trianguline;

(ii) there exists a continuous character $\eta\colon G_{\mathbb{Q}_p} \to \mathcal{O}_E^{\times}$ such that $\mathbf{D}_{\mathrm{cris}}(V(\eta)) \neq 0$.

As an immediate consequence we have the following.

COROLLARY 3.10.13. *If V is two-dimensional and trianguline and $\eta\colon G_{\mathbb{Q}_p} \to \mathcal{O}_E^{\times}$ is a continuous character, then $V(\eta)$ is also trianguline.*

Some potentially trianguline representations are provided by p-adic Hodge theory.

PROPOSITION 3.10.14. *If V is a de Rham representation then it is potentially trianguline.*

An important class of trianguline representations is given by the Galois representations associated with overconvergent modular forms. Let f be an overconvergent GL_2-eigenform and let $\rho_{f,p}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be the p-adic Galois representation associated with f. As Berger observed in [**Be11**, Section 4.3], the following result is a combination of [**Ki03**, Theorem 6.3] and [**Col08**, Proposition 4.3].

THEOREM 3.10.15. *The representation $\rho_{f,p}|_{G_{\mathbb{Q}_p}}$ is trianguline.*

The analogous result for an overconvergent GSp_4-eigenform can be deduced from a recent work of Kedlaya, Pottharst and Xiao [**KPX**]. Keep all notations as before. Let Σ be the set of embeddings $K \hookrightarrow E$. Every $\sigma \in \Sigma$ restricts to a character $x_\sigma\colon K^\times \to E^\times$.

THEOREM 3.10.16. [**KPX**, Theorem 6.3.13] *Let X be a rigid analytic space over L. Let $\mathbb{M}$ be a (φ,Γ)-module over $\mathscr{R}_X(\pi_E)$ of rank d. Suppose that $\mathbb{M}$ is densely pointwise strictly trianguline with respect to a Zariski-dense subset X_{alg} of X and ordered parameters $\delta_1,\ldots,\delta_d\colon K^\times \to \mathcal{O}(X)^\times$. Then for every $z \in X$ the specialization $\mathbb{M}_z$ is trianguline with parameters $\delta_{1,z},\ldots,\delta_{d,z}$, where $\delta'_{i,z} = \delta_{i,z}\prod_{\sigma\in\Sigma} x_\sigma^{n_{i,z,\sigma}}$ for some integers $n_{i,z,\sigma}$.*

We specialize the theorem to the GSp_4-eigenvariety. Since we cannot construct a big Galois representation over the whole eigenvariety, we need to rely on the construction of families that we will explain in Section 4.1.2; we use the notations defined there. Let F be an overconvergent, finite slope GSp_4-eigenform and let $\rho_{F,p}\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$ be the p-adic Galois representations associated with F.

COROLLARY 3.10.17. *The restriction $\rho_{F,p}|_{G_{\mathbb{Q}_p}}$ is trianguline.*

PROOF. Let $K = E = \mathbb{Q}_p$ and $d = 4$. Let V be a neighborhood of the weight of F such that the weight map $\mathcal{D}_{2,V}^{M,h} \to V$ is finite. Then the construction in Section 4.1.4 gives a Galois representation $\rho_{\mathcal{D}_{2,V}^{M,h}}\colon G_\mathbb{Q} \to \mathrm{GL}_4(\mathcal{O}(\mathcal{D}_{2,V}^{M,h}))$. Let $\mathbb{M}$ be the (φ,Γ)-module over $\mathscr{R}_{\mathcal{D}_{2,V}^{M,h}}(\pi_{\mathbb{Q}_p})$ associated with the local Galois representation $\rho_{\mathcal{D}_{2,V}^{M,h}}|_{G_{\mathbb{Q}_p}}$. Let X_{alg} be the set of classical points of $\mathcal{D}_{2,V}^{M,h}$ with cohomological weight and distinct Hodge-Tate weights. Let $w^*\colon \mathcal{O}(V) \to \mathcal{O}(\mathcal{D}_{2,V}^{M,h})$ be the morphism of $\mathbb{Q}_p$-algebras induced by the weight map $w\colon \mathcal{D}_{2,V}^{M,h} \to V$. Let $(\delta_1^\circ,\ldots,\delta_4^\circ)$ be the 4-tuple of characters $\mathbb{Z}_p^\times \to \mathcal{O}(\mathcal{D}_2^M)$ determined by

$$(\delta_1^\circ,\ldots,\delta_4^\circ)(u) = w^\times \circ (1, u^{-1}(1+T_1), u^{-2}(1+T_2), u^{-3}((1+T_1)(1+T_2))).$$

For $i = 1,\ldots,4$ let δ_i be an extension of δ_i° to a character $\mathbb{Q}_p^\times \to \mathcal{O}(\mathcal{D}_2^M)^\times$. We obtain the corollary by applying Theorem 3.10.16 to the above data and then specializing the resulting triangulation to the (φ,Γ)-module associated with $\rho_{F,p}|_{G_{\mathbb{Q}_p}}$. □

3.10.3. Modularity results. We recall an important theorem, the proof of which is a combination of an overconvergent modularity result by Emerton [**Em14**, Corollary 1.2.2] and a promodularity result deriving from the work of Böckle, Diamond-Flach-Guo, Khare-Wintenberger and Kisin [**Em14**, Theorem 1.2.3]. Here E is a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$ and residue field $\mathbb{F}$. We denote by $\chi\colon G_\mathbb{Q} \to \mathbb{Z}_p^\times$ the cyclotomic character and by $\overline{\chi}\colon G_\mathbb{Q} \to \mathbb{F}_p^\times$ its reduction modulo p.

THEOREM 3.10.18. [**Em14**, Theorem 1.2.4] *Let $\tau\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_E)$ be a continuous, irreducible, odd representation unramified outside a finite set of primes. Let $\overline{\tau}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F})$ be the residual representation associated with τ. Suppose that:*

(a) $p > 2$;
(b) $\overline{\tau}|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible;
(c) there exists no character $\eta\colon G_\mathbb{Q} \to \mathbb{F}^\times$ such that $\overline{\tau}$ is an extension of η by itself or of η by $\eta\overline{\chi}$.

In this setting the following conclusions hold:

(1) if $\tau|_{G_{\mathbb{Q}_p}}$ is trianguline, then τ is the twist by a character of the Galois representation attached to a finite slope, cuspidal, overconvergent GL_2-eigenform;
(2) if $\tau|_{G_{\mathbb{Q}_p}}$ is de Rham with distinct Hodge-Tate weights, then τ is the twist by a character of the Galois representation attached to a cuspidal classical GL_2-eigenform of weight $k \geq 2$.

Part (2) of the theorem is a confirmation of one implication of the Fontaine-Mazur conjecture [**FM95**, Conjecture]. A different proof of this statement was given by Kisin [**Ki03**, Theorem 6.6].

An analogue of Theorem 3.10.18 is not yet available for the representations associated with overconvergent GSp_4-eigenforms.

3.10.4. Non-abelian cohomology and semilinear group actions. We recall a few results from the theory of non-abelian cohomology. call *pointed set* a set with a distinguished element. Let S and T be two pointed sets with distinguished elements s and t, respectively. Let $f\colon S \to T$ be a map of pointed sets. We define the kernel of f by $\ker f = \{s \in S \mid f(s) = t\}$. Thanks to this notion we can speak of exact sequences of pointed sets.

Let G be a topological group. Let A be a topological group endowed with a continuous action of G, compatible with the group structure. For $i \in \{0,1\}$ let $H^i(G,A)$ be the continuous cohomology of G with values in A. Then $H^i_{\mathrm{cont}}(G,A)$ has the structure of a pointed set with distinguished element given by the class of the trivial cocycle. For $i = 0$ we have $H^0(G,A) = A^G$, the pointed set of G-invariant elements in A; its distinguished point is the identity. Since A is not necessarily abelian, we have no notion of continuous cohomology in degree greater than 1. Let B, C be two other topological groups with the same additional structures as A, and let

$$1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1 \tag{3.18}$$

be a G-equivariant short exact sequence of topological groups. Then there is an exact sequence of pointed sets

$$1 \to A^G \to B^G \to C^G \xrightarrow{\delta} H^1(G,A) \to H^1(G,B) \to H^1(G,C). \tag{3.19}$$

The connecting map δ is defined as follows. Let $c \in C^G$ and let $b \in B$ such that $\beta(b) = c$. Then $\delta(c)$ is the map given by

$$g \mapsto \alpha^{-1}(b^{-1} \cdot g.b)$$

for every $g \in G$. It is easy to check that this is a good definition and that δ is a cocycle. We call (3.19) the long exact sequence in cohomology associated with (3.18).

Now suppose that A and B are topological groups with the same structures as before, but C is just a topological pointed set with a continuous action of G that fixes the distinguished element of C. Since C is not a group we cannot define $H^1(G,C)$. However the pointed set $H^0(G,C) = C^G$ of G-invariant elements of C is well-defined; its distinguished element is the distinguished element of C.

Proposition 3.10.19. *Let A, B, C be as in the discussion above. Suppose that*

$$1 \to A \to B \to C \to 1$$

is an exact sequence of topological pointed sets. Then there is an exact sequence of pointed sets

$$1 \to A^G \to B^G \to C^G \xrightarrow{\delta} H^1(G,A) \to H^1(G,B).$$

The connecting map δ is defined as in the case of an exact sequence of groups. This definition does not rely on the group structure of C.

Proof. We check exactness at every term as in the case of an exact sequence of groups. None of these checks relies on the group structure of C. $\square$

Let R be a ring and let $\sigma\colon R \to R$ be an automorphism. Let M be an R-module. We say that a map $f\colon M \to R$ is σ-semilinear if:

- $f(x+y) = f(x) + f(y)$ for every $x, y \in M$;
- $f(rx) = \sigma(r)f(x)$ for every $r \in R$ and every $x \in M$.

Let G be a topological group. Let $\mathbf{B}$ be a topological ring equipped with a continuous action of G, compatible with the ring structure. Let M be a $\mathbf{B}$-module. A semilinear action of G on M is a map that associates with every $g \in G$ a g-semilinear map $g(\cdot)\colon M \to M$, in such a way that $gh(x) = g(h(x))$ for every $g, h \in G$ and $x \in M$. When M is free we also say that M is a semilinear $\mathbf{B}$-representation of G. We say that M is irreducible if the only G-stable $\mathbf{B}$-submodules of M are 0 and M.

Let n be a positive integer and let M be a free $\mathbf{B}$-module of rank n, endowed with the topology induced by that on $\mathbf{B}$. We say that two semilinear actions of G on M are equivalent if they can be obtained by one another via a change of basis. We choose a basis $(e_1, e_2, \ldots, e_n)$ of M, hence an isomorphism $\mathrm{GL}(M) \cong \mathrm{GL}_n(\mathbf{B})$. We let G act on $\mathrm{GL}_n(\mathbf{B})$ via its action on $\mathbf{B}$. Two semilinear actions $g(\cdot)_1$ and $g(\cdot)_2$ of G on M are equivalent if and only if there exists $A \in \mathrm{GL}(M)$ such that $g(x)_1 = M \cdot g(x)_2 \cdot (g(A))^{-1}$ for every $g \in G$ and $x \in M$. There is a bijection

(3.20)

$$\{\text{Equivalence classes of semilinear and continuous actions of } G \text{ on } M\} \leftrightarrow H^1(G, \mathrm{GL}_n(\mathbf{B})).$$

Given a semilinear action of G on M, we define $a \in H^1(G, \mathrm{GL}_n(\mathbf{B}))$ as the class of the cocycle that maps $g \in G_{\mathbb{Q}_p}$ to the matrix $(a_{ij}^g)_{i,j} \in \mathrm{GL}_2(\mathbf{B})$ satisfying

$$g(e_i) = \sum_j a_{ij}^g e_j$$

for every $i \in \{1, 2, \ldots, n\}$.

We say that G acts trivially on M if there exists a basis $(e'_1, e'_2, \ldots, e'_n)$ such that $g.e'_i = e'_i$ for every $g \in G$ and every $i \in \{1, 2, \ldots, n\}$. The action of G is trivial if and only if the corresponding class in $H^1(G, \mathrm{GL}_n(\mathbf{B}))$ is trivial. We say that the action of G is triangular if there exists a basis with respect to which the matrix $(a_{ij}^g)_{i,j}$ is upper triangular for every $g \in G$.

3.10.5. Representations with a de Rham symmetric cube. Now suppose that $\mathbf{B}$ is a $\mathbb{C}_p$-algebra equipped with a continuous action of $G_{\mathbb{Q}_p}$, compatible with the ring structure and with the natural action of $G_{\mathbb{Q}_p}$ on $\mathbb{C}_p$. Suppose that the subring of $G_{\mathbb{Q}_p}$-invariant elements in $\mathbf{B}$ is $\mathbb{Q}_p$.

Recall that there is an exact sequence of algebraic groups over $\mathbb{Z}$:

$$1 \to \mu_3 \to \mathrm{GL}_2 \to \mathrm{GL}_4, \tag{3.21}$$

where $\mu_3 \to \mathrm{GL}_2$ sends ζ to $\zeta \cdot \mathbb{1}_2$ and $\mathrm{GL}_2 \to \mathrm{GL}_4$ is the symmetric cube representation. Consider the exact sequence induced by (3.21) on the $\mathbf{B}$-points:

$$1 \to \mu_3(\mathbf{B}) \to \mathrm{GL}_2(\mathbf{B}) \to \mathrm{GL}_4(\mathbf{B}). \tag{3.22}$$

Let $G_{\mathbb{Q}_p}$ act on each term via its action on $\mathbf{B}$; this action is clearly continuous and compatible with the group structure on each term. The above sequence is $G_{\mathbb{Q}_p}$-equivariant. We split it into the short exact sequence

$$1 \to \mu_3(\mathbf{B}) \xrightarrow{\iota} \mathrm{GL}_2(\mathbf{B}) \xrightarrow{\pi} (\mathrm{GL}_2/\mu_3)(\mathbf{B}) \to 1 \tag{3.23}$$

and the injection

$$1 \to (\mathrm{GL}_2/\mu_3)(\mathbf{B}) \xrightarrow{\mathrm{Sym}^3} \mathrm{GL}_4(\mathbf{B}). \tag{3.24}$$

Both this sequences are $G_{\mathbb{Q}_p}$-equivariant. Since $\mathrm{Sym}^3\mathrm{GL}_2(\mathbf{B})$ is not normal in $\mathrm{GL}_4(\mathbf{B})$ we cannot complete (3.24) to a short exact sequence of groups. However we can complete it to an exact sequence of pointed sets. Let H be the algebraic group $\mathrm{Sym}^3\mathrm{GL}_2$. Let $[\mathrm{GL}_4, H](\mathbf{B})$ be the set of right classes $\{M \cdot H(\mathbf{B}) \,|\, M \in \mathrm{GL}_4(\mathbf{B})\}$. We equip $[\mathrm{GL}_4, H]$ with a structure of topological pointed set by giving it the quotient topology and letting the class $H(\mathbf{B})$ be the distinguished point. Let $G_{\mathbb{Q}_p}$ act on $[\mathrm{GL}_4, H](\mathbf{B})$ by $g.(M \cdot H(\mathbf{B})) = (g.M) \cdot H(\mathbf{B})$; this action is continuous and it leaves the distinguished point fixed. Then there is a $G_{\mathbb{Q}_p}$-equivariant exact sequence of topological pointed sets

$$1 \to (\mathrm{GL}_2/\mu_3)(\mathbf{B}) \to \mathrm{GL}_4(\mathbf{B}) \to [\mathrm{GL}_4, H](\mathbf{B}) \to 1, \tag{3.25}$$

where the first two non-trivial terms also have a group structure compatible with the action of $G_{\mathbb{Q}_p}$. Thanks to Proposition 3.10.19 there is an exact sequence of pointed sets

$$\begin{aligned} &1 \to ((\mathrm{GL}_2/\mu_3)(\mathbf{B}))^{G_{\mathbb{Q}_p}} \to (\mathrm{GL}_4(\mathbf{B}))^{G_{\mathbb{Q}_p}} \to ([\mathrm{GL}_4, H](\mathbf{B}))^{G_{\mathbb{Q}_p}} \to \\ &\to H^1(G_{\mathbb{Q}_p}, \mathrm{GL}_2/\mu_3(\mathbf{B})) \xrightarrow{H^1(\mathrm{Sym}^3)} H^1(G_{\mathbb{Q}_p}, \mathrm{GL}_4(\mathbf{B})). \end{aligned} \tag{3.26}$$

REMARK 3.10.20. *Let* $[\mathrm{GL}_4, H](\mathbb{Q}_p)$ *be the subset of* $[\mathrm{GL}_4, H](\mathbf{B})$ *consisting of right classes* $\{M \cdot H(\mathbf{B}) \mid M \in \mathrm{GL}_4(\mathbb{Q}_p)\}$*. Since* $G_{\mathbb{Q}_p}$ *acts on each term of* (3.25) *via its action on* $\mathbf{B}$*, we have*

$$\begin{aligned} ((\mathrm{GL}_2/\mu_3)(\mathbf{B}))^{G_{\mathbb{Q}_p}} &= (\mathrm{GL}_2/\mu_3)(\mathbb{Q}_p) \\ (\mathrm{GL}_4(\mathbf{B}))^{G_{\mathbb{Q}_p}} &= \mathrm{GL}_4(\mathbb{Q}_p) \\ ([\mathrm{GL}_4, H](\mathbf{B}))^{G_{\mathbb{Q}_p}} &= [\mathrm{GL}_4, H](\mathbb{Q}_p). \end{aligned}$$

In particular the map $(\mathrm{GL}_4(\mathbf{B}))^{G_{\mathbb{Q}_p}} \to ([\mathrm{GL}_4, H](\mathbf{B}))^{G_{\mathbb{Q}_p}}$ *that appears in the exact sequence* (3.26) *is surjective. Hence the kernel of the map* $H^1(\mathrm{Sym}^3)$ *is trivial, i.e. it contains only the distinguished point of* $H^1(G_{\mathbb{Q}_p}, \mathrm{GL}_2/\mu_3(\mathbf{B}))$*.*

Now consider the short exact sequence of topological groups (3.23):

$$1 \to \mu_3(\mathbf{B}) \xrightarrow{\iota} \mathrm{GL}_2(\mathbf{B}) \to (\mathrm{GL}_2/\mu_3)(\mathbf{B}) \to 1.$$

The associated long exact sequence of pointed sets is

$$\begin{aligned} &1 \to (\mu_3(\mathbf{B}))^{G_{\mathbb{Q}_p}} \to (\mathrm{GL}_2(\mathbf{B}))^{G_{\mathbb{Q}_p}} \to ((\mathrm{GL}_2/\mu_3)(\mathbf{B}))^{G_{\mathbb{Q}_p}} \to \\ &\to H^1(G_{\mathbb{Q}_p}, \mu_3(\mathbf{B})) \xrightarrow{H^1(\iota)} H^1(G_{\mathbb{Q}_p}, \mathrm{GL}_2(\mathbf{B}) \xrightarrow{H^1(\pi)} H^1(G_{\mathbb{Q}_p}, \mathrm{GL}_2/\mu_3)(\mathbf{B})). \end{aligned} \tag{3.27}$$

Let M be a free $\mathbf{B}$-module of rank 2, endowed with the topology induced by $\mathbf{B}$. Suppose that $G_{\mathbb{Q}_p}$ acts continuously on M. Then $\mathrm{Sym}^3 M$ is a free $\mathbf{B}$-module of rank 4 endowed with the natural semilinear action of $G_{\mathbb{Q}_p}$ induced by that on M. We use the exact sequences we constructed, together with the bijection (3.20), to prove the second part of the following proposition.

PROPOSITION 3.10.21.

(1) If the action of $G_{\mathbb{Q}_p}$ *on* M *is trivial then the action of* $G_{\mathbb{Q}_p}$ *on* $\mathrm{Sym}^3 M$ *is trivial.*
(2) If the action of $G_{\mathbb{Q}_p}$ *on* $\mathrm{Sym}^3 M$ *is trivial then there exists a subgroup* H *of* $G_{\mathbb{Q}_p}$ *of index* 3 *that acts trivially on* M*.*

PROOF. If (m_1, m_2) is a $\mathbf{B}$-basis of M on which $G_{\mathbb{Q}_p}$ acts trivially, then the image in $\mathrm{Sym}^3 M$ of the set $(m_1 \otimes m_1 \otimes m_1, m_1 \otimes m_1 \otimes m_2, m_1 \otimes m_2 \otimes m_2, m_2 \otimes m_2 \otimes m_2)$ is a $\mathbf{B}$-basis of $\mathrm{Sym}^3 M$ on which $G_{\mathbb{Q}_p}$ acts trivially. This proves the first part of the proposition.

We prove the second statement. The bijection (3.20) associates with the action of $G_{\mathbb{Q}_p}$ on M a class $\sigma \in H^1(G_{\mathbb{Q}_p}, \mathrm{GL}_2(\mathbf{B}))$. Recall the maps

$$H^1(\pi)\colon H^1(G_{\mathbb{Q}_p}, \mathrm{GL}_2(\mathbf{B})) \to H^1(G_{\mathbb{Q}_p}, \mathrm{GL}_2/\mu_3(\mathbf{B}))$$

and

$$H^1(\mathrm{Sym}^3)\colon H^1(G_{\mathbb{Q}_p}, \mathrm{GL}_2/\mu_3(\mathbf{B})) \to H^1(G_{\mathbb{Q}_p}, \mathrm{GL}_4(\mathbf{B}))$$

that appear in the sequences (3.27) and (3.26). The class in $H^1(G_{\mathbb{Q}_p}, \mathrm{GL}_4(\mathbf{B}))$ associated with the action of $G_{\mathbb{Q}_p}$ on $\mathrm{Sym}^3 M$ is $(H^1(\mathrm{Sym}^3) \circ H^1(\pi))(\sigma)$; by assumption it is trivial. By Remark 3.10.20 the kernel of $H^1(\mathrm{Sym}^3)$ is trivial, hence $(H^1(\pi))(\sigma)$ is trivial. Now by the exactness of

(3.27) the class σ belongs to the image of $H^1(\iota)\colon H^1(G_{\mathbb{Q}_p},\mu_3(\mathbf{B}))\to H^1(G_{\mathbb{Q}_p},\mathrm{GL}_2(\mathbf{B}))$. Let τ be an element of $H^1(G_{\mathbb{Q}_p},\mu_3(\mathbf{B}))$ satisfying $(H^1(\iota))(\tau)=\sigma$.

Since $\mathbb{C}_p\subset\mathbf{B}$, $\mu_3(\mathbf{B})$ is the group of cubic roots of 1, that we simply denote by μ_3. We have $H^1(G_{\mathbb{Q}_p},\mu_3)\cong\mathbb{Q}_p/\mathbb{Q}_p^3$. An isomorphism $\mathbb{Q}_p/\mathbb{Q}_p^3\to H^1(G_{\mathbb{Q}_p},\mu_3)$ is defined as follows: given $y\in\mathbb{Q}_p/\mathbb{Q}_p^3$ we choose a representative $x\in\mathbb{Q}_p$ and a cubic root $x^{1/3}\in\mathbb{C}_p$ and we send y to the cocycle $g\mapsto g.x^{1/3}/x^{1/3}$. Now let $y\in\mathbb{Q}_p/\mathbb{Q}_p^3$ be the element that corresponds to τ via the given isomorphism, and let $x\in\mathbb{Q}_p$ be a representative of y. Then the cocycle τ is trivial on the subgroup $H=\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p[x^{1/3}])$ of $G_{\mathbb{Q}_p}$. Since $\sigma=(H^1(\iota))(\tau)$, σ is also trivial on H. By definition of the bijection (3.20), the above implies that the action of H on Sym^3M is trivial. Since H is a subgroup of index 1 (if y is trivial) or 3, this concludes the proof. □

Remark 3.10.22. *There is a $G_{\mathbb{Q}_p}$-equivariant isomorphism of $\mathbf{B}_{\mathrm{dR}}$-vector spaces*

$$P\colon \mathrm{Sym}^3(\mathbf{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}V)\cong\mathbf{B}_{\mathrm{dR}}\otimes\mathrm{Sym}^3V.$$

It is defined by

$$P\left(\sum_i(b_{i,1}\otimes v_{i,1})\otimes(b_{i,2}\otimes v_{i,2})\otimes(b_{i,3}\otimes v_{i,3})\right)=\sum_i b_{i,1}b_{i,2}b_{i,3}\otimes(v_{i,1}\otimes v_{i,2}\otimes v_{i,3})$$

for every $b_{i,j}\in\mathbf{B}_{\mathrm{dR}}$ and $v_{i,j}\in V$, with $j\in\{1,2,3\}$ and i in a finite set.

Proposition 3.10.23. *The representation V of $G_{\mathbb{Q}_p}$ is de Rham if and only if Sym^3V is de Rham.*

Proof. By definition V is de Rham if and only if the semilinear action of $G_{\mathbb{Q}_p}$ on $\mathbf{B}_{\mathrm{dR}}\otimes V$ is trivial, and the analogous statement is true for Sym^3V. By Proposition 3.10.21(1), if $G_{\mathbb{Q}_p}$ acts trivially on $\mathbf{B}_{\mathrm{dR}}\otimes V$ then it also acts trivially on $\mathrm{Sym}^3(\mathbf{B}_{\mathrm{dR}}\otimes V)$. By the $G_{\mathbb{Q}_p}$-equivariant isomorphisms of Remark 3.10.22 we obtain that $G_{\mathbb{Q}_p}$ acts trivially on $\mathbf{B}_{\mathrm{dR}}\otimes\mathrm{Sym}^3V$. Conversely, if $G_{\mathbb{Q}_p}$ acts trivially on $\mathbf{B}_{\mathrm{dR}}\otimes\mathrm{Sym}^3V$, it acts trivially on $\mathrm{Sym}^3(\mathbf{B}_{\mathrm{dR}}\otimes V)$. Then Proposition 3.10.21(2) gives a subgroup $H_{\mathbb{Q}_p}$ of $G_{\mathbb{Q}_p}$ of index 3 that acts trivially on $\mathbf{B}_{\mathrm{dR}}\otimes V$. This means that the representation $G_{\mathbb{Q}_p}$ is potentially de Rham, hence it is de Rham by Proposition 3.10.1. □

3.10.6. Symmetric cube of a (φ,Γ)-module. Let E be a p-adic field and let V be an E-vector space carrying an E-linear action of $G_{\mathbb{Q}_p}$. Let D be a (φ,Γ)-module over $\mathscr{R}$. We define a (φ,Γ)-module Sym^3D over $\mathscr{R}$ as follows. The underlying $\mathscr{R}$-module of Sym^3D is the symmetric cube of the underlying $\mathscr{R}$-module of D. The action of Γ on Sym^3D is defined as follows: for $\gamma\in\Gamma$ and $d_1,d_2,d_3\in D$ we set $\gamma.v_1\otimes v_2\otimes v_3=(\gamma.d_1)\otimes(\gamma.d_2)\otimes(\gamma.d_3)$ and then extend by semilinearity with respect to the action of Γ on $\mathscr{R}$. If φ_D is the Frobenius of D, the Frobenius $\varphi_{\mathrm{Sym}^3D}$ of Sym^3D is defined by setting $\varphi_{\mathrm{Sym}^3D}(v_1\otimes v_2\otimes v_3)=\varphi_D(v_1)\otimes\varphi_D(v_2)\otimes\varphi_D(v_3)$ for $v_1,v_2,v_3\in D$ and extending by semilinearity with respect to the Frobenius of $\mathscr{R}$. The action of Γ on Sym^3D commutes with $\varphi_{\mathrm{Sym}^3D}$ since the action of Γ on D commutes with φ_D. We can check that $\varphi_{\mathrm{Sym}^3D}(\mathrm{Sym}^3D)$ generates D as an $\mathscr{R}$-module.

Remark 3.10.24. *There is an isomorphism $\mathrm{Sym}^3(\mathbf{D}_{\mathrm{rig}}(V))\cong\mathbf{D}_{\mathrm{rig}}(\mathrm{Sym}^3V)$ of (φ,Γ)-modules over $\mathscr{R}$. Indeed the isomorphism $P\colon\mathrm{Sym}^3(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}\otimes_{\mathbb{Q}_p}V)\to\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}\otimes_{\mathbb{Q}_p}\mathrm{Sym}^3V$ given by*

$$P\left(\sum_i(b_{i,1}\otimes v_{i,1})\otimes(b_{i,2}\otimes v_{i,2})\otimes(b_{i,3}\otimes v_{i,3})\right)=\sum_i b_{i,1}b_{i,2}b_{i,3}\otimes(v_{i,1}\otimes v_{i,2}\otimes v_{i,3})$$

for every $b_{i,j}\in\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ and $v_{i,j}\in V$, with $j\in\{1,2,3\}$ and i in a finite set, is seen to be $G_{\mathbb{Q}_p}$-equivariant for the natural actions on the two sides. The morphism induced by P on the $\mathscr{R}$-modules of $H_{\mathbb{Q}_p}$-invariants is compatible with the Frobenius maps, hence it is the desired isomorphism of (φ,Γ)-modules.

3.10.7. Representations with a trianguline symmetric cube. We consider now the case where $\mathrm{Sym}^3 V$ is trianguline. The goal of this subsection is to prove the following.

PROPOSITION 3.10.25. *Suppose that V is irreducible.*

(i) If the representation V is trianguline then $\mathrm{Sym}^3 V$ is trianguline.

(ii) If the representation $\mathrm{Sym}^3 V$ is trianguline then either V is trianguline or V is a twist of a de Rham representation. In particular V is a twist of a trianguline representation.

The first statement is immediate. The proof of the second one relies on a technique used by Di Matteo in [**DiM13**], together with the classification of two-dimensional potentially trianguline representations carried on by Berger and Chenevier in [**BC10**]. Di Matteo considers two representations V and W such that the tensor product representation $V \otimes W$ is trianguline, and proves that in this case V and W are potentially trianguline. We will adapt his arguments to our situation.

Let K be a p-adic field. Let $\mathbf{B}$ be a topological field equipped with a continuous action of G_K. Let $\mathcal{C}_{\mathbf{B}}^K$ be the category of semilinear $\mathbf{B}$-representations of G_K. The $\mathbf{B}$-linear dual of an object of $\mathcal{C}_{\mathbf{B}}^K$ and the tensor product over $\mathbf{B}$ of two objects of $\mathcal{C}_{\mathbf{B}}^K$ define new objects in the usual way. In this section all duals and tensor products are in $\mathcal{C}_{\mathbf{B}}^K$, except when otherwise stated.

Let $\eta\colon G_K \to \mathbf{B}^\times$ be a cocycle. Let $\mathbf{B}(\eta)$ be a one-dimensional $\mathbf{B}$-vector space with a generator e, equipped with the semilinear action of G_K defined by $g.e = \eta(g)e$ for every $g \in G_K$. We simply write $\mathbf{B}$ when η is the trivial cocycle. Clearly every one-dimensional object in $\mathcal{C}_{\mathbf{B}}^K$ is isomorphic to $\mathbf{B}(\eta)$ for some cocycle η. Note that if η takes values in $\mathbf{B}^{G_K}$ then η is a character. For every object M of $\mathcal{C}_{\mathbf{B}}^K$ we set $M(\eta) = M \otimes \mathbf{B}(\eta)$.

For every object M of $\mathcal{C}_{\mathbf{B}}^K$ and every finite extension K' of K, we consider M as an object of $\mathcal{C}_{\mathbf{B}}^{K'}$ with the action induced by the inclusion $G_{K'} \subset G_K$.

We say that an n-dimensional object M of $\mathcal{C}_{\mathbf{B}}^K$ is *triangulable* if there exists a filtration

$$M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_{n-1} \supset M_n = 0$$

where, for every $i \in \{1, 2, \ldots, n\}$, M_i is a G_K-stable subspace of M and M_{i-1}/M_i is one-dimensional. If there exists such a filtration that satisfies $M_{i-1}/M_i \cong \mathbf{B}(\eta_i)$ for some characters $\eta_1, \eta_2, \ldots, \eta_n\colon G_K \to \mathbf{B}^{G_K}$, then we say that M is *triangulable by characters*. These definitions are analoguous to those in the beginning of [**DiM13**, Section 3], but we omit the specification "split" since we use Colmez's terminology for trianguline representations rather than Berger's.

From now on M is a two-dimensional *irreducible* object in $\mathcal{C}_{\mathbf{B}}^K$.

LEMMA 3.10.26. *Let X and X' be two irreducible objects in $\mathcal{C}_{\mathbf{B}}^K$. If $X \otimes X'$ has a one-dimensional quotient in $\mathcal{C}_{\mathbf{B}}^K$, then $\dim_{\mathbf{B}} X = \dim_{\mathbf{B}} X'$.*

PROOF. The one-dimensional quotient of $X \otimes X'$ is isomorphic to $\mathbf{B}(\eta)$ for a cocycle $\eta\colon G_K \to \mathbf{B}$. Consider the tautological exact sequence in $\mathcal{C}_{\mathbf{B}}^K$:

$$0 \to \ker \phi \to X \otimes X' \xrightarrow{\phi} \mathbf{B}(\eta) \to 0.$$

There is a G_K-equivariant map $\phi'\colon X \to (X')^*(\eta)$ sending $x \in X$ to the function $\phi'(x) \in (X')^*(\eta)$ defined by $x' \mapsto \phi(x \otimes x')$ for every $x' \in X'$. Since ϕ is non-zero, ϕ' is also non-zero. The representations X and $(X')^*(\eta)$ are irreducible, hence the non-zero G_K-equivariant map ϕ' is an isomorphism. We conclude that $\dim_{\mathbf{B}} X = \dim_{\mathbf{B}} (X')^*(\eta) = \dim_{\mathbf{B}} X'$. ☐

LEMMA 3.10.27. *Suppose that $\mathrm{Sym}^3 M$ is triangulable by characters. Let $\eta_1, \eta_2, \eta_3, \eta_4\colon G_K \to \mathbf{B}^{G_K}$ be the characters appearing in the triangulation of $\mathrm{Sym}^3 M$. Then:*

(i) there exists an irreducible object M_1 of $\mathcal{C}_{\mathbf{B}}^K$ such that $\mathrm{Sym}^3 M \cong M_1 \otimes M$;

(ii) there is a decomposition $\mathrm{Sym}^3 M \cong \bigoplus_{i=1}^4 \mathbf{B}(\eta_i)$ in $\mathcal{C}_{\mathbf{B}}^K$.

The central ingredients in the proof are [**DiM13**, Lemma 3.1.3] and the proof of [**DiM13**, Corollary 3.1.4].

Proof. Let

$$\mathrm{Sym}^3 M = Y \supset Y_1 \supset Y_2 \supset Y_3 \supset Y_4 = 0$$

be a filtration of $\mathrm{Sym}^3 M$ satisfying $Y_{i-1}/Y_i \cong \mathbf{B}(\eta_i)$ for $1 \le i \le 4$. In particular $\mathbf{B}(\eta_1)$ is a quotient of $\mathrm{Sym}^3 M$ and $\mathbf{B}(\eta_4)$ is a subobject of $\mathrm{Sym}^3 M$. Let $\pi_{\eta_1}\colon \mathrm{Sym}^3 M \to \mathbf{B}(\eta_1)$ and $\pi\colon \mathrm{Sym}^2 M \otimes M \to \mathrm{Sym}^3 M$ be the natural projections.

Consider the following exact sequence in $\mathcal{C}_{\mathbf{B}}^K$:

$$0 \to \ker \pi \to \mathrm{Sym}^2 M \otimes M \xrightarrow{\pi} \mathrm{Sym}^3 M \to 0$$

The surjection $\pi_{\eta_1} \circ \pi\colon \mathrm{Sym}^2 M \otimes M \to \mathbf{B}(\eta_1)$ defines a one-dimensional quotient of $\mathrm{Sym}^2 M \otimes M$. If $\mathrm{Sym}^2 M$ is irreducible then Lemma 3.10.26 implies that $\dim_{\mathbf{B}} \mathrm{Sym}^2 M = \dim_{\mathbf{B}} M$, which is a contradiction since $\mathrm{Sym}^2 M$ is three-dimensional. Then $\mathrm{Sym}^2 M$ is reducible; this means that it admits a non-trivial filtration in $\mathcal{C}_{\mathbf{B}}^K$ (i.e. a filtration in G_K-stable subspaces). For simplicity, set $X = \mathrm{Sym}^2 M$. All the maps and the filtrations we write are in $\mathcal{C}_{\mathbf{B}}^K$. There are three possibilities:

(1) there is a filtration
$$X = X_0 \supset X_1 \supset X_2 \supset X_3 = 0$$
with $\dim_{\mathbf{B}}(X_{i-1}/X_i) = 1$ for $i = 1, 2, 3$;

(2) there is a filtration
$$X = X_0 \supset X_1 \supset X_2 = 0$$
with $\dim_{\mathbf{B}}(X/X_1) = 1$, $\dim_{\mathbf{B}} X_1 = 2$ and X_1 irreducible.

(3) there is a filtration
$$X = X_0 \supset X_1 \supset X_2 = 0$$
with $\dim_{\mathbf{B}}(X/X_1) = 2$, $\dim_{\mathbf{B}} X_1 = 1$ and X/X_1 irreducible;

Suppose that (1) holds. Since X is obtained from X/X_1, X_1/X_2 and X_2 by successive extensions, $X \otimes M$ is obtained by successive extensions of $(X/X_1) \otimes M$, $(X_1/X_2) \otimes M$ and $X_2 \otimes M$. Hence there exists $i \in \{1, 2, 3\}$ such that the surjection $X \otimes M \to \mathbf{B}(\eta_1)$ induces a surjection $X_{i-1}/X_i \otimes M \to \mathbf{B}(\eta_1)$. Since X_{i-1}/X_i and M are irreducible, Lemma 3.10.26 implies that $\dim_{\mathbf{B}}(X_{i-1}/X_i) = \dim_{\mathbf{B}} M = 2$, a contradiction since $\dim_{\mathbf{B}}(X_{i-1}/X_i) = 1$ for every i.

Suppose that we are in case (2). As before, there exists $i \in \{1, 2\}$ such that $X \otimes M \to \mathbf{B}(\eta_1)$ induces a surjection $\pi'_{\eta_1}(X_{i-1}/X_i) \otimes M \to \mathbf{B}(\eta_1)$. If $i = 1$ Lemma 3.10.26 implies that $\dim_{\mathbf{B}}(X/X_1) = \dim_{\mathbf{B}} M$, a contradiction. Hence there is an exact sequence

$$0 \to \ker \pi'_{\eta_1} \to X_1 \otimes M \xrightarrow{\pi'_{\eta_1}} \mathbf{B}(\eta_1).$$

Since X_1 and M are irreducible, this sequence splits by [**DiM13**, Lemma 3.1.3]. In particular there is a section $\mathbf{B}(\eta_1) \hookrightarrow X_1 \otimes M$. By composing this section with the inclusion $X_1 \otimes M \hookrightarrow X \otimes M$ and the projection $X \otimes M \to \mathrm{Sym}^3 M$ we obtain a section of the map π_{η_1}, hence a splitting of the exact sequence

$$0 \to \ker \pi_{\eta_1} \to \mathrm{Sym}^3 M \xrightarrow{\pi_{\eta_1}} \mathbf{B}(\eta_1) \to 0.$$

By definition of π_{η_1} we have $Y_1 = \ker \pi_{\eta_1}$, so $\mathrm{Sym}^3 M \cong Y_1 \oplus \mathbf{B}(\eta_1)$. Now Y_2 is a subobject of Y_1, hence $Y_2 \oplus \mathbf{B}(\eta_1)$ is a subobject of $\mathrm{Sym}^3 M$. There is an isomorphism $\mathrm{Sym}^3 M/(Y_2 \oplus \mathbf{B}(\eta_1)) \cong Y_1/Y_2 \cong \mathbf{B}(\eta_2)$, giving a projection $\pi_{\eta_2}\colon \mathrm{Sym}^3 M \to \mathbf{B}(\eta_2)$. By replacing π_{η_1} with π_{η_2} in the above argument, we obtain that the sequence

$$0 \to \ker \pi_{\eta_2} \to \mathrm{Sym}^3 M \xrightarrow{\pi_{\eta_2}} \mathbf{B}(\eta_2) \to 0$$

splits. Then $\mathrm{Sym}^3 M \cong \ker \pi_{\eta_2} \oplus \mathbf{B}(\eta_2)$. Since $\ker \pi_{\eta_2} \cong Y_2 \oplus \mathbf{B}(\eta_1)$ we obtain $\mathrm{Sym}^3 M \cong Y_2 \oplus \mathbf{B}(\eta_1) \oplus \mathbf{B}(\eta_2)$. We repeat the argument for the projection to $\mathbf{B}(\eta_3)$ and we obtain a decomposition $\mathrm{Sym}^3 M \cong \bigoplus_{i=1}^4 \mathbf{B}(\eta_i)$, together with maps $\pi_{\eta_i}\colon X_1 \otimes M \to \mathbf{B}(\eta_i)$.

Now consider the map $\psi\colon X_1 \otimes M \to \mathrm{Sym}^3 M$ obtained by composing the inclusion $X_1 \otimes M \hookrightarrow X \otimes M$ with $\pi\colon X \otimes M \to \mathrm{Sym}^3 M$. By the results of the previous paragraph, $\mathrm{Sym}^3 M \cong \bigoplus_{i=1}^4 \mathbf{B}(\eta_i)$ and for every $i \in \{1, 2, 3, 4\}$ there is a map $\pi_{\eta_i}\colon X_1 \otimes M \to \mathbf{B}(\eta_i)$. Hence ψ is surjective. Since $X_1 \otimes M$ and $\mathrm{Sym}^3 M$ are both 4-dimensional, ψ is an isomorphism. Moreover X_1 is irreducible, so part (1) of the lemma is true with $M_1 = X_1$.

Suppose that we are in case (3). Consider the map $\psi\colon X_1 \otimes M \to \mathrm{Sym}^3 M$ obtained by composing the inclusion $X_1 \otimes M \to \mathrm{Sym}^2 M \otimes M$ with the projection $\pi\colon \mathrm{Sym}^2 M \otimes M \to \mathrm{Sym}^3 M$. Since X_1 is one-dimensional and M is irreducible, $X_1 \otimes M$ is irreducible. Hence the kernel of ψ is either 0 or $X_1 \otimes M$. In the first case the image of ψ defines a two-dimensional irreducible subobject of $\mathrm{Sym}^3 M$, contradicting the fact that $\mathrm{Sym}^3 M$ is triangulable. In the second case π factors via a surjective map $\pi_1\colon (X/X_1) \otimes M \to \mathrm{Sym}^3 M$. Since $\dim_{\mathbf{B}}((X/X_1) \otimes M) = \dim_{\mathbf{B}} \mathrm{Sym}^3 M$, π_1 is an isomorphism. Now X/X_1 is irreducible, so part (1) of the lemma is true with $M_1 = X/X_1$.

The decomposition of $\mathrm{Sym}^3 M$ given in part (2) of the lemma follows from part (1) and [**DiM13**, Corollary 3.1.4]. $\square$

We recall another result of [**DiM13**].

Lemma 3.10.28. [**DiM13**, Lemma 3.2.1] *Let N and N' be two objects of $\mathcal{C}_{\mathbf{B}}^K$ such that $N \otimes N'$ is triangulable by characters. Let $\{\eta_i\}_{i=1}^d$ be the set of characters $G_K \to \mathbf{B}^{G_K}$ appearing in the triangulation of $N \otimes N'$. Then $\eta_1^{-1}\eta_i$ is a finite order character for every $i \in \{1, 2, \ldots, d\}$.*

The following lemma is proved in the same way as [**DiM13**, Theorem 3.2.2], with the difference that we work in the language of (φ, Γ)-modules rather than in that of B-pairs. Recall that E is a p-adic field and V is a two-dimensional E-representation of $G_{\mathbb{Q}_p}$.

Lemma 3.10.29. *Suppose that V is irreducible. If $\mathrm{Sym}^3 V$ is trianguline, then V is potentially trianguline.*

Proof. Consider the (φ, Γ)-modules $\mathbf{D}_{\mathrm{rig}}(V)$ and $\mathbf{D}_{\mathrm{rig}}(\mathrm{Sym}^3 V)$. They are free $\mathscr{R}$-modules carrying a semilinear action of $G_{\mathbb{Q}_p}$. By Remark 3.10.24 there is an isomorphism of (φ, Γ)-modules $\mathbf{D}_{\mathrm{rig}}(\mathrm{Sym}^3 V) \cong \mathrm{Sym}^3 \mathbf{D}_{\mathrm{rig}}(V)$. In particular this is an isomorphism of semilinear representations of $G_{\mathbb{Q}_p}$, where we let $G_{\mathbb{Q}_p}$ act via $\mathbb{G}_{\mathbb{Q}_p} \twoheadrightarrow \Gamma$.

Since $\mathrm{Sym}^3 V$ is trianguline, $\mathbf{D}_{\mathrm{rig}}(\mathrm{Sym}^3 V)$ is obtained by successive extensions of rank one (φ, Γ)-modules D_i, $1 \le i \le 4$. By Proposition 3.10.11, for every $i \in \{1, 2, 3, 4\}$ there exists a character $\eta_i\colon \mathbb{Q}_p^\times \to E^\times$ such that $D_i \cong \mathscr{R}(\eta_i)$. Note that $E^\times = \mathscr{R}^{G_E}$, so $\eta_i|_{G_E}$ takes values in $\mathscr{R}^{G_E}$.

Since V is irreducible, [**DiM13**, Corollary 2.2.2] implies that $\mathbf{D}_{\mathrm{rig}}(V)$ is irreducible as a semilinear $\mathscr{R}$-representation of $G_{\mathbb{Q}_p}$. In particular the choice $M = \mathbf{D}_{\mathrm{rig}}(V)$ satisfies the assumptions of Lemma 3.10.27, hence part (2) of that lemma gives a $G_{\mathbb{Q}_p}$-equivariant decomposition $\mathbf{D}_{\mathrm{rig}}(\mathrm{Sym}^3 V) \cong \bigoplus_{i=1}^4 \mathscr{R}(\eta_i)$.

Now by Lemma 3.10.28 there exists a finite extension L of E such that $\eta_1^{-1}\eta_i|_{G_L}$ is trivial for every i. Hence there is an isomorphism $\mathbf{D}_{\mathrm{rig}}(\mathrm{Sym}^3 V)(\eta_1^{-1}) \cong \bigoplus_{i=1}^4 \mathscr{R}$ of $\mathscr{R}$-representations of G_L. This means that $\mathbf{D}_{\mathrm{rig}}(\mathrm{Sym}^3 V)(\eta_1^{-1})$ is a trivial $\mathscr{R}$-representation of G_L. Let $\eta'\colon G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_p^\times$ be a character satisfying $\mathbf{D}_{\mathrm{rig}}(\mu) = \mathscr{R}(\eta_1)$. Then $\mathbf{D}_{\mathrm{rig}}((\mathrm{Sym}^3 V)(\mu^{-1})) = (\mathbf{D}_{\mathrm{rig}}(\mathrm{Sym}^3 V))(\eta_1^{-1})$. By Lemma 3.10.10(2) there is an isomorphism

$$\mathbf{D}_{\mathrm{st}}(\mathrm{Sym}^3 V(\mu^{-1})) = (\mathscr{R}[1/t, T] \otimes_{\mathscr{R}} \mathbf{D}_{\mathrm{rig}}(\mathrm{Sym}^3 V))^{\Gamma_L}$$

of filtered (φ, N)-modules. We know that G_L acts trivially on $\mathbf{D}_{\mathrm{rig}}((\mathrm{Sym}^3 V)(\eta_1^{-1}))$, so the module $\mathbf{D}_{\mathrm{st}}((\mathrm{Sym}^3 V)(\eta_1^{-1}))$ is four-dimensional. This means that $(\mathrm{Sym}^3 V)(\mu^{-1})$ is a semi-stable representation of G_L. In particular it is a de Rham representation of G_L.

Let $\mu'(x) = \mu(x)/|\mu(x)|\colon \mathbb{Q}_p^\times \to \mathcal{O}_E^\times$. Let E_1 be a finite extension of E that contains $p^{1/6}$ and let L_1 be a finite extension of L such that $\mu'|_{G_{L_1}}$ is trivial modulo the maximal ideal of $\mathcal{O}_E$. Then there exists a character $\mu^{-1/6}\colon \mathbb{Q}_p^\times \to E_1^\times$ such that $(\mu^{-1/6})^6 = \mu^{-1}$. Since $\mathrm{Sym}^3(V(\mu^{-1/6})) \cong (\mathrm{Sym}^3 V)(\mu^{-1})$ and $(\mathrm{Sym}^3 V)(\mu^{-1})$ is de Rham, $V(\mu^{-1/6})$ is also de Rham by Proposition 3.10.23. Hence $V(\mu^{-1/6})$ is potentially trianguline by Proposition 3.10.14. The twist V of $V(\mu^{-1/6})$ is still potentially trianguline by Corollary 3.10.13. $\square$

Now we can prove Proposition 3.10.25

Proof. We prove (i). Suppose that V is trianguline. By definition there is a basis $\{v_1, v_2\}$ of $\mathbf{D}_{\mathrm{rig}}(V)$ in which the actions of $G_{\mathbb{Q}_p}$ and φ are described by upper triangular matrices. By Remark 3.10.24 there is an isomorphism $\mathbf{D}_{\mathrm{rig}}(\mathrm{Sym}^3 V) \cong \mathrm{Sym}^3 \mathbf{D}_{\mathrm{rig}}(V)$. Hence the set

$$\{v_1 \otimes v_1 \otimes v_1, v_1 \otimes v_1 \otimes v_2, v_1 \otimes v_2 \otimes v_2, v_2 \otimes v_2 \otimes v_2\}$$

is a basis of $\mathbf{D}_{\mathrm{rig}}(\mathrm{Sym}^3 V)$. We see immediately that the actions of $G_{\mathbb{Q}_p}$ and φ on $\mathbf{D}_{\mathrm{rig}}(\mathrm{Sym}^3 V)$ are described by upper triangular matrices in this basis.

We prove (ii). Since $\mathrm{Sym}^3 V$ is trianguline, V is potentially trianguline by Lemma 3.10.29. Then V satisfies one of the three conditions listed in [**BC10**, Théorème A]. By assumption V is irreducible, so it cannot satisfy (2). Hence (1) or (3) must hold, as desired. □

3.10.8. Representations with symmetric cube of automorphic origin. Consider two continuous representations $\rho_1 \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ and $\rho_2 \colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$.

Theorem 3.10.30. *Suppose that:*

(1) ρ_2 is odd and it is unramified outside a finite set of primes;
(2) the residual representation $\overline{\rho}_2$ associated with ρ_2 is absolutely irreducible;
(3) $\rho_2 \cong \mathrm{Sym}^3 \rho_1$.

Then the following conclusions hold.

- *(i) If ρ_2 is associated with an overconvergent cuspidal GSp_4-eigenform, then ρ_1 is associated with an overconvergent cuspidal GL_2-eigenform.*
- *(ii) If ρ_2 is associated with a classical cuspidal GSp_4-eigenform, then ρ_1 is associated with a classical cuspidal GL_2-eigenform.*

Proof. Note that assumption (1) implies that the residual representation $\overline{\rho}_1$ is absolutely irreducible.

We prove part (i). The representation ρ_2 is associated with an overconvergent cuspidal GSp_4-eigenform F, so it is trianguline by Theorem 3.10.15. By Proposition 3.10.25 the representation ρ_1 is a twist of a trianguline representation. Then Theorem 3.10.18(2) implies that ρ_1 is the twist by a character of a representations associated with an overconvergent cuspidal GL_2-eigenform. We show that the character occurring here can be taken to be trivial.

Let V be a two-dimensional E-vector space carrying an action of $G_{\mathbb{Q}_p}$ via ρ_1 and let $\overline{V}$ be the associated residual representation. Let $\alpha \colon G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_p^\times$ be a character and N be a positive integer such that $V(\alpha)$ is associated with an overconvergent cuspidal GL_2-eigenform f of level $\Gamma_1(N) \cap \Gamma_0(p)$. Let x be the point of the eigencurve $\mathcal{D}_1^N$ corresponding to f. Let M be the positive integer associated with N by Definition 3.3.6. Let $\xi \colon \mathcal{D}_1^(N, \mathcal{G}) \to \mathcal{D}_2^M$ be the morphism of Definition 3.9.8. Let $\mathrm{Sym}^3 f$ be the overconvergent GSp_4-eigenform corresponding to the point $\xi(x)$. The Galois representation associated with $\mathrm{Sym}^3 f$ is $\mathrm{Sym}^3(V(\alpha))$.

For a continuous representation W of $G_{\mathbb{Q}_p}$, we denote by ϕ_W the generalized Sen operator associated with W (see [**Ki03**, Section 2.2] for the construction). Let (κ_1, κ_2) be the eigenvalues of ϕ_V. A calculation shows that $\phi_{\mathrm{Sym}^3 V}$ has eigenvalues $(3\kappa_1, \kappa_1 + 2\kappa_2, 2\kappa_1 + \kappa_2, 3\kappa_2)$. Since $\mathrm{Sym}^3 V$ is attached to an overconvergent GSp_4-eigenform we must have $3\kappa_1 = 0$, hence $\kappa_1 = 0$. Set $\kappa = \kappa_2$, so that the eigenvalues of ϕ_V are $(0, \kappa)$. Recall that the weight of the character α is defined by $w(\alpha) = \log(\alpha(u))/\log(u)$, where u is a generator of $\mathbb{Z}_p^\times$. The eigenvalues of $\phi_{V(\alpha)}$ are $(w(\alpha), \kappa + w(\alpha))$. Since V comes from an overconvergent GL_2-eigenform we must have $w(\alpha) = 0$. In particular the eigenvalues of $\phi_{\mathrm{Sym}^3 V}$ and $\phi_{\mathrm{Sym}^3(V(\alpha))}$ are the same. This means that $\mathrm{Sym}^3 V$ and $\mathrm{Sym}^3(V(\alpha))$ are associated with two overconvergent GSp_4-eigenforms F and $\mathrm{Sym}^3 f$ of the same weight, given in our usual coordinates by $(\kappa+1, 2\kappa-1)$. Let χ_{κ_1,κ_2} be the specialization at $(\kappa+1, 2\kappa-1)$ of the p-adic deformation of the cyclotomic character. The determinants of $\mathrm{Sym}^3 V$ and $\mathrm{Sym}^3(V(\alpha))$ are given by the product of χ_{κ_1,κ_2} with the central characters of F and $\mathrm{Sym}^3 f$, respectively. In particular the two determinants differ by a finite order character. We deduce that α^6, hence α, is a finite order character. By twisting the overconvergent GL_2-eigenform f by the finite order character α^{-1} we obtain an overconvergent GL_2-eigenform with associated Galois representation V.

We prove part (ii). Since ρ_2 is associated with a classical cuspidal GSp_4-eigenform, it is a de Rham representation by Theorem 3.10.5. Then Proposition 3.10.23 implies that ρ_1 is also a de Rham representation. The representation ρ_2 is trianguline because it is de Rham, so part (i) of the theorem implies that ρ_1 is attached to an overconvergent GL_2-eigenform f. Since ρ_1 is de Rham, the form f is classical. $\square$

3.11. Big image for Galois representations attached to classical modular forms of residual Sym^3 type

Let N be a positive integer and let p be a prime not dividing N. Let F be a GSp_4-eigenform of level $\Gamma_1(N)$. Let $\rho_{F,p}\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$ be the p-adic Galois representation associated with F. It is defined over a p-adic field K. In this section we prove that if $\rho_{F,p}$ is "$\mathbb{Z}_p$-regular" (see Definition 3.11.1) and "of Sym^3 type" (see Definition 3.11.2), the image of $\rho_{F,p}$ is "big", in the sense that it contains a congruence subgroup of $\mathrm{Sp}_4(\mathcal{O}_E)$ for the ring of integers $\mathcal{O}_E$ of a suitable p-adic field $E \subset K$. The main ingredient of our proof is a theorem of Pink [**Pink98**, Theorem 0.7].

In the following definitions, let E be a finite extensions of $\mathbb{Q}_p$. Let R be a local ring with maximal ideal $\mathfrak{m}_R$ and residue field $\mathbb{F}$. Let $\tau\colon G_E \to \mathrm{GSp}_4(R)$ be a representation. Let $\mathrm{PGSp}_4(R) = \mathrm{GSp}_4(R)/R^\times$, where $R^\times$ is identified with the subgroup of scalar matrices; note that this group is in general different from the group of R-points of the algebraic group PGSp_4. We denote by $\overline{\tau}\colon G_E \to \mathrm{GSp}_4(\mathbb{F})$ the reduction of τ modulo $\mathfrak{m}_R$ and by $\mathrm{Ad}\,\tau^0\colon G_E \to \mathrm{PGSp}_4(R)$ the adjoint representation of τ. Recall that T_2 denotes the torus consisting of diagonal matrices in GSp_4.

We give a notion of $\mathbb{Z}_p$-regularity of τ, analogous to that in [**HT15**, Lemma 4.5(2)].

Definition 3.11.1. *We say that τ is* $\mathbb{Z}_p$-regular *if there exists $d \in \mathrm{Im}\,\tau \cap T_2(R)$ with the following property: if α and α' are two distinct roots of GSp_4 then $\alpha(d) \neq \alpha'(d) \pmod{\mathfrak{m}_R}$. If d has this property we call it a* $\mathbb{Z}_p$-regular element.

From now on we focus on representations that are of residual symmetric cube type in the sense of the definition below. Note that this type of assumption already appeared in [**Pil12**, Proposition 5.9].

Definition 3.11.2. *We say that τ is of* residual Sym^3 type *if there exists a non-trivial subfield $\mathbb{F}'$ of $\mathbb{F}$ and an element $g \in \mathrm{GSp}_4(\mathbb{F})$ such that*

$$\mathrm{Sym}^3\mathrm{SL}_2(\mathbb{F}') \subset g(\mathrm{Im}\,\overline{\tau})g^{-1} \subset \mathrm{Sym}^3\mathrm{GL}_2(\mathbb{F}').$$

Recall that we write $\mathfrak{sp}_4(K)$ for the Lie algebra of $\mathrm{Sp}_4(K)$ and $\mathrm{Ad}^0\colon \mathrm{GSp}_4(K) \to \mathrm{End}(\mathfrak{sp}_4(K))$ for the adjoint representation. Let F and $\rho_{F,p}\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathcal{O}_K)$ be as in the beginning of the section. Let E be the subfield of K generated over $\mathbb{Q}_p$ by the set $\{\mathrm{Tr}(\mathrm{Ad}\,(\rho(g)))\}_{g\in G_{\mathbb{Q}}}$. Let $\mathcal{O}_E$ be the ring of integers of E. We will prove the following result.

Theorem 3.11.3. *Assume that:*

(1) $\rho_{F,p}$ is $\mathbb{Z}_p$-regular;
(2) $\rho_{F,p}$ is of residual Sym^3 type;
(3) there is no GL_2-eigenform f such that $\rho_{F,p} \cong \mathrm{Sym}^3\rho_{f,p}$, where $\rho_{f,p}$ is the p-adic Galois representation associated with f.

Then the image of $\rho_{F,p}$ contains a principal congruence subgroup of $\mathrm{Sp}_4(\mathcal{O}_E)$.

We recall a result of Pink that plays a crucial role in the proof of Theorem 3.11.3.

THEOREM 3.11.4. [**Pink98**, Theorem 0.7] *Let F be a local field and let H be an absolutely simple connected adjoint group over F. Let Γ be a compact Zariski-dense subgroup of $H(F)$. Suppose that the adjoint representation of Γ is irreducible. Then there exists a closed subfield E of F and a model H_E of H over E such that Γ is an open subgroup of $H_E(E)$.*

We prove a lemma that we will use repeatedly in the text.

LEMMA 3.11.5. *Let $\mathcal{G}$ be a profinite group and let $\mathcal{G}_1$ be a normal open subgroup of $\mathcal{G}$. Let F be a field. Let $\tau\colon \mathcal{G} \to \mathrm{GSp}_4(F)$ be a continuous representation. Suppose that:*

(1) there exists a representation $\tau_1'\colon \mathcal{G}_1 \to \mathrm{GL}_2(F)$ such that $\tau|_{\mathcal{G}_1} \cong \mathrm{Sym}^3\tau_1'$;
(2) the image of τ_1' contains a principal congruence subgroup of $\mathrm{SL}_2(F)$;
(3) there exists a character $\eta\colon \mathcal{G} \to F^\times$ such that $\det\tau \cong \eta^6$.

Then there exists a finite extension $\iota\colon F \hookrightarrow F'$ and a representation $\tau'\colon \mathcal{G} \to \mathrm{GL}_2(F')$ such that $\iota\circ\tau \cong \mathrm{Sym}^3\tau'$.

PROOF. We denote by $\mathrm{Sym}^3\mathrm{GL}_2(F)$ the copy of $\mathrm{GL}_2(F)$ embedded in $\mathrm{GSp}_4(F)$ via the symmetric cube map. In order to prove the lemma it is sufficient to find a finite extension F' of F such that $\iota\circ\tau(\mathcal{G}) \subset \mathrm{Sym}^3\mathrm{GL}_2(F')$. For $g \in \mathrm{GSp}_4(F)$ let $\mathrm{Ad}\,(g)\colon \mathrm{GSp}_4(F) \to \mathrm{GSp}_4(F)$ be conjugation by g. Since $\mathcal{G}_1$ is an open normal subgroup of $\mathcal{G}$, $\tau(\mathcal{G})$ normalizes $\tau(\mathcal{G}_1)$. Let g be an arbitrary element of $\tau(\mathcal{G})$. The map $\mathrm{Ad}\,(g)$ restricts to an automorphism $\mathrm{Ad}\,(g)|_{\tau(\mathcal{G}_1)}$ of $\tau(\mathcal{G}_1)$. Since $\tau|_{\mathcal{G}_1} \cong \mathrm{Sym}^3\tau_1'$, the symmetric cube map induces an isomorphism $\tau(\mathcal{G}_1) \cong \tau_1'(\mathcal{G}_1)$. Hence $\mathrm{Ad}\,(g)$ induces an automorphism $\mathrm{Ad}\,(g)'$ of $\tau_1'(\mathcal{G}_1)$, which is a subgroup of $\mathrm{GL}_2(F)$ containing a congruence subgroup of $\mathrm{SL}_2(F)$. By applying Corollary 4.6.5 to the map $\mathrm{Ad}\,(g)'\colon \tau_1'(\mathcal{G}_1) \to \tau_1'(\mathcal{G}_1)$ we deduce that there exists $h_g \in \mathrm{GL}_2(F)$, a field automorphism σ of F and a character $\varphi\colon \tau_1'(\mathcal{G}_1) \to F^\times$ such that

$$\mathrm{Ad}\,(g)'(x) = \varphi(x) h_g x^\sigma h_g^{-1} \tag{3.28}$$

for every $x \in \mathcal{G}_1$. Since every operation in Equation (3.28) is F-linear, the automorphism σ must be the identity. Moreover $\mathrm{Ad}\,(g)'$ is induced by $\mathrm{Ad}\,(g)$, so by taking characteristic polynomials on both sides of the equation we obtain that φ^3 is trivial. Hence by applying the symmetric cube map to both sides of Equation (3.28) we obtain

$$\mathrm{Ad}\,(g)|_{\tau(\mathcal{G}_1)} = \mathrm{Ad}\,(\mathrm{Sym}^3 h_g)|_{\tau(\mathcal{G}_1)},$$

so the element $g(\mathrm{Sym}^3 h_g)^{-1}$ centralizes $\tau(\mathcal{G}_1)$.

By Schur's lemma $g(\mathrm{Sym}^3 h_g)^{-1}$ is a scalar for every $g \in \tau(\mathcal{G})$. Let γ_g be the element of the field F satisfying $g(\mathrm{Sym}^3 h_g)^{-1} = \gamma_g \mathbb{1}_4$. Choose a set of representatives S for the finite group $\mathcal{G}/\mathcal{G}_1$. Let F' be the finite extension of F obtained by adding the cubic roots of all the elements in the set $\{\gamma_g \mid g \in \tau(S)\}$. Let $\iota\colon F \to F'$ be the inclusion. For $g \in \iota\circ\tau(S)$ we have $\iota(\gamma_g \mathbb{1}_4) \in \mathrm{Sym}^3\mathrm{GL}_2(F')$ by construction of F', so $\iota(g) = \iota(\gamma_g \mathbb{1}_4 \cdot \mathrm{Sym}^3 h_g) \in \mathrm{Sym}^3\mathrm{GL}_2(F')$. For every $g \in \tau(\mathcal{G})$ we can write $g = g_1 g_2$ with $g_1 \in \tau(\mathcal{G}_1)$ and $g_2 \in \tau(S)$. Since $\tau(\mathcal{G}_1) \subset \mathrm{Sym}^3\mathrm{GL}_2(F)$ we obtain $\iota(g) = \iota(g_1)\iota(g_2) \in \mathrm{Sym}^3\mathrm{GL}_2(F')$. We conclude that $\iota\circ\tau(\mathcal{G}) \subset \mathrm{Sym}^3\mathrm{GL}_2(F')$.

For every $g \in G_{\mathbb{Q}}$, let $\tau'(g)$ be the unique element of $\mathrm{GL}_2(F')$ that satisfies:

(1) $\mathrm{Sym}^3\tau'(g) = \iota\circ\tau(g)$;
(2) $\det\tau'(g) = \iota\circ\eta(g)$.

Such an element exists by the result of the previous paragraph. Then the map $\tau'\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(F')$ defined by $g \mapsto \tau'(g)$ is a representation satisfying $\mathrm{Sym}^3\tau' \cong \iota\circ\tau$. □

The rest of the section is devoted to the proof of Theorem 3.11.3. Let $(\mathrm{Im}\,\rho_{F,p})'$ be the derived subgroup of $\mathrm{Im}\,\rho_{F,p}$ and let $G = (\mathrm{Im}\,\rho_{F,p}) \cap \mathrm{Sp}_4(K)$. We denote by $\overline{G}$ the Zariski-closure of G in $\mathrm{Sp}_4(K)$. As in [**HT15**, Section 3], we will show first that under the hypotheses of Theorem 3.11.3 we have $\overline{G} = \mathrm{Sp}_4(K)$, and second that G is p-adically open in $\overline{G}$. We will replace the ordinarity assumption in *loc. cit.* by that of $\mathbb{Z}_p$-regularity. Let $\overline{G}^{\circ}$ denote the connected component of the identity in $\overline{G}$.

Let H be any connected, Zariski-closed subgroup of Sp_4, defined over K. As in [**HT15**, Section 3.4] we have six possibilities for the isomorphism class of H over K:
(1) $H \cong \mathrm{Sp}_4$;
(2) $H \cong \mathrm{SL}_2 \times \mathrm{SL}_2$;
(3) $H \cong \mathrm{SL}_2$ embedded in a Klingen parabolic subgroup;
(4) $H \cong \mathrm{SL}_2$ embedded in a Siegel parabolic subgroup;
(5) $H \cong \mathrm{SL}_2$ embedded via the symmetric cube representation $\mathrm{SL}_2 \to \mathrm{Sp}_4$;
(6) $H \cong \{1\}$.
When (5) holds we write $H \cong \mathrm{Sym}^3\mathrm{SL}_2$. We show that for $H = \overline{G}^{\circ}$ only two of the choices listed above are possible.

PROPOSITION 3.11.6. *We have either* $\overline{G}^{\circ} \cong \mathrm{Sp}_4$ *or* $\overline{G}^{\circ} \cong \mathrm{Sym}^3\mathrm{SL}_2$.

PROOF. Let $\mathfrak{m}_K$ be the maximal ideal of $\mathcal{O}_K$ and let $\mathbb{F}_K = \mathcal{O}_K/\mathfrak{m}_K$. The group $(\mathrm{Im}\,\rho_{F,p})'$ is contained in $H(\mathcal{O}_K)$. By reducing modulo $\mathfrak{m}_K$ we obtain that the derived subgroup $(\mathrm{Im}\,\overline{\rho}_{F,p})'$ of $\mathrm{Im}\,\overline{\rho}_{F,p}$ is contained in $H(\mathbb{F}_K)$. Since $\rho_{F,p}$ is of residual Sym^3 type H cannot satisfy any one of the conditions (2,3,4,6) of the discussion above. $\square$

We show that if $\overline{G}^{\circ} \cong \mathrm{Sym}^3\mathrm{SL}_2$ then the GSp_4-eigenform F does not satisfy assumptions (3) of Theorem 3.11.3.

PROPOSITION 3.11.7. *Suppose that* $\overline{G}^{\circ} \cong \mathrm{Sym}^3\mathrm{SL}_2$. *Then there exists a* GL_2*-eigenform* f *such that* $\rho_{F,p} \cong \mathrm{Sym}^3\rho_{f,p}$.

PROOF. Since $\overline{G}^{\circ}(K)$ is of finite index in $\overline{G}(K)$, Lemma 3.11.5 implies that $\overline{G}(K) \subset \mathrm{Sym}^3\mathrm{SL}_2(K)$, so $\mathrm{Im}\,\rho_{F,p} \subset \mathrm{Sym}^3\mathrm{GL}_2(K)$. Hence there exists a representation ρ' satisfying $\rho_{F,p} \cong \mathrm{Sym}^3\rho'$. Since $\rho_{F,p}$ is associated with a GSp_4-eigenform, Theorem 3.10.30 implies that ρ' is associated with a GL_2-eigenform f. $\square$

Theorem 3.11.3 is a consequence of the following proposition.

PROPOSITION 3.11.8. *Suppose that* $\overline{G} \cong \mathrm{Sp}_4(K)$. *Then the group* G *contains an open subgroup (for the* p*-adic topology) of* $\mathrm{Sp}_4(E)$.

PROOF. Consider the image G^{ad} of G under the projection $\mathrm{Sp}_4(K) \to \mathrm{PGSp}_4(K)$. It is a compact subgroup of $\mathrm{PGSp}_4(K)$. Since $\overline{G} \cong \mathrm{Sp}_4(K)$, the group G^{ad} is Zariski-dense in $\mathrm{PGSp}_4(K)$. By Theorem 3.11.4 there is a model H of PGSp_4 over E such that G^{ad} is an open subgroup of $H(E)$. By the assumption of $\mathbb{Z}_p$-regularity of ρ, there is a diagonal element d with pairwise distinct eigenvalues. The group $H(E)$ must contain the centralizer of d in $\mathrm{PGSp}_4(E)$, which is a split torus in $\mathrm{PGSp}_4(E)$. Since H is split and $H \times_E K \cong \mathrm{PGSp}_{4/K}$, H is a split form of PGSp_4 over E. Then H must be isomorphic to PGSp_4 over E by unicity of the quasi-split form of a reductive group. Hence G^{ad} is an open subgroup of $\mathrm{PGSp}_4(E)$. Since the map $\mathrm{Sp}_4(K) \to \mathrm{PGSp}_4(K)$ has degree 2 and $G \cap \mathrm{Sp}_4(E)$ surjects onto $G^{\mathrm{ad}} \cap \mathrm{PGSp}_4(E)$, G must contain an open subgroup of $\mathrm{Sp}_4(E)$. In particular G contains a principal congruence subgroup of $\mathrm{Sp}_4(\mathcal{O}_E)$. $\square$

3.12. The symmetric cube locus on the GSp_4-eigenvariety

In this section p is a prime number, N is a positive integer prime to p and M is the integer, depending on N, given by Definition 3.3.6. By an abuse of notation, if $\mathcal{V}_1$ and $\mathcal{V}_2$ are subvarieties of $\mathcal{D}_1^N$ and $\mathcal{D}_2^M$, respectively, we write $\psi_1\colon \mathcal{H}_1^N \to \mathcal{O}(\mathcal{V}_1)$ and $\psi_2\colon \mathcal{H}_2^N \to \mathcal{O}(\mathcal{V}_2)$

for the compositions of ψ_1 and ψ_2 with the restrictions of analytic functions to $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively.

THEOREM 3.12.1. *Let $\mathcal{V}_2$ be a rigid analytic subvariety of $\mathcal{D}_2^M$. Consider the following four conditions.*

(1a) There exists a morphism of rings $\psi_2^{(1)}\colon \mathcal{H}_1^{Np} \to \mathcal{O}(\mathcal{V}_2)$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{H}_2^{Np} \arrow[r, "\lambda^{Np}"] \arrow[d, "\psi_2"] & \mathcal{H}_1^{Np} \arrow[dl, "\psi_2^{(1)}"] \\ \mathcal{O}(\mathcal{V}_2) & \end{tikzcd} \tag{3.29}$$

(1b) There exists a pseudocharacter $T_{\mathcal{V}_2,1}\colon G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{V}_2)$ of dimension 2 such that

$$T_{\mathcal{V}_2} = \mathrm{Sym}^3 T_{\mathcal{V}_2,1}. \tag{3.30}$$

(2a) There exists a rigid analytic subvariety $\mathcal{V}_1$ of $\mathcal{D}_1^N$ and a morphism of rings $\phi\colon \mathcal{O}(\mathcal{V}_1) \to \mathcal{O}(\mathcal{V}_2)$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{H}_2^{Np} \arrow[r, "\lambda^{Np}"] \arrow[rrr, bend left, "\psi_2"] & \mathcal{H}_1^{Np} \arrow[r, "\psi_1"] & \mathcal{O}(\mathcal{V}_1) \arrow[r, "\phi"] & \mathcal{O}(\mathcal{V}_2) \end{tikzcd} \tag{3.31}$$

(2b) There exists a rigid analytic subvariety $\mathcal{V}_1$ of $\mathcal{D}_1^N$ and a morphism of rings $\phi\colon \mathcal{O}(\mathcal{V}_1) \to \mathcal{O}(\mathcal{V}_2)$ such that

$$T_{\mathcal{V}_2} = \mathrm{Sym}^3(\phi \circ T_{\mathcal{V}_1}). \tag{3.32}$$

Then:

(i) (1a) and (1b) are equivalent;
(ii) (2a) and (2b) are equivalent;
(iii) (2b) implies (1b);
(iv) when $\mathcal{V}_2$ is a point, the four conditions are equivalent.

PROOF. We prove (i), (ii), (iii) for an arbitrary rigid analytic subvariety $\mathcal{V}_2$ of $\mathcal{D}_2^M$.

(1a) $\implies$ (1b). Let $\psi_2^{(1)}\colon \mathcal{H}_1^{Np} \to \mathcal{O}(\mathcal{V}_2)$ be a morphism of rings making diagram (3.29) commute. By the argument in the proof of Proposition 3.4.2, the commutativity of diagram (3.29) gives an equality

$$\psi_2(P_{\min}(t_{\ell,2}^{(2)}; X)) = \mathrm{Sym}^3(\psi_2^{(1)}(P_{\min}(t_{\ell,1}^{(1)}; X))). \tag{3.33}$$

Choose a character ε_1 satisfying $\varepsilon_1^6 = \varepsilon$. For every ℓ not dividing Np, let P_ℓ be a polynomial in $\mathcal{H}_2^{Np}[X]^{\deg=2}$ satisfying:

$$\mathrm{Sym}^3 P_\ell(X) = \psi_2(P_{\min}(t_{\ell,2}^{(2)}; X)); \tag{3.34}$$

and

$$P_\ell(0) = \varepsilon_1 \cdot (1+T)^{\log(\chi(g))/\log(u)}. \tag{3.35}$$

Such a polynomial exists thanks to Equation (3.33) and to Remark 4.1.20, and it is clearly unique. The roots of P_ℓ differ from those of $\psi_2(P_{\min}(t_{\ell,2}^{(2)}; X))$ by a factor equal to a cubic root of 1.

By Chebotarev's theorem the set $\{\gamma \mathrm{Frob}_\ell \gamma^{-1}\}_{\ell\nmid Np;\, \gamma\in G_{\mathbb{Q}}}$ is dense in $G_{\mathbb{Q}}$. The map

$$P\colon \{\gamma \mathrm{Frob}_\ell \gamma^{-1}\}_{\ell\nmid Np;\, \gamma\in G_{\mathbb{Q}}} \to \mathcal{O}(\mathcal{V}_2)[X]^{\deg=2},$$
$$\gamma \mathrm{Frob}_\ell \gamma^{-1} \mapsto P_\ell,$$

is continuous with respect to the restriction of the profinite topology on $G_{\mathbb{Q}}$. This follows from the fact that the maps

$$\begin{gathered}\{\gamma \mathrm{Frob}_\ell \gamma^{-1}\}_{\ell\nmid Np;\,\gamma\in G_{\mathbb{Q}}} \to \mathcal{O}(\mathcal{V}_2)[X]^{\deg=4} \\ \gamma\mathrm{Frob}_\ell\gamma^{-1} \mapsto \psi_2(P_{\min}(t_{\ell,2}^{(2)};X)) = \mathrm{Sym}^3 P(\gamma\mathrm{Frob}_\ell\gamma^{-1})(X)\end{gathered}$$

and

$$\begin{gathered}\{\gamma \mathrm{Frob}_\ell \gamma^{-1}\}_{\ell\nmid Np;\,\gamma\in G_{\mathbb{Q}}} \to \mathcal{O}(\mathcal{V}_2)^\times \\ \gamma\mathrm{Frob}_\ell\gamma^{-1} \mapsto P(\gamma\mathrm{Frob}_\ell\gamma^{-1})(0) = \varepsilon_1\cdot(1+T)^{\log(\chi(g))/\log(u)}\end{gathered}$$

are continuous on $\{\gamma \mathrm{Frob}_\ell \gamma^{-1}\}_{\ell\nmid Np;\,\gamma\in G_{\mathbb{Q}}}$. Hence P can be extended to a continuous map

$$P\colon G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{V}_2)[X]^{\deg=2}.$$

Now define a map $T_{\mathcal{V}_2,1}\colon G_{\mathbb{Q}} \to \mathcal{O}(\mathcal{V}_2)$ by

$$T_{\mathcal{V}_2,1}(g) = \frac{P(g)(1)+P(g)(-1)}{2}.$$

The right hand side is simply the sum of the roots of $P(g)$.

We can check that $T_{\mathcal{V}_2,1}$ is a pseudocharacter of dimension 2. Its characteristic polynomial is P, so the fact that $T_{\mathcal{V}_2} = \mathrm{Sym}^3 T_{\mathcal{V}_2,1}$ follows from Equation (3.34).

(1b) $\Longrightarrow$ (1a). Suppose that there exists a pseudocharacter $T_{\mathcal{V}_2,1}\colon G_{\mathbb{Q}} \to \mathcal{O}_{\mathcal{V}_2}$ such that $T_{\mathcal{V}_2} = \mathrm{Sym}^3 T_{\mathcal{V}_2,1}$. Then $P_{\mathrm{char}}(T_{\mathcal{V}_2}) = \mathrm{Sym}^3 P_{\mathrm{char}}(T_{\mathcal{V}_2,1})$. By evaluating the two polynomials at Frob_ℓ we obtain

$$\begin{aligned}\psi_2(P_{\min}(t_{\ell,2}^{(2)};X)) &= P_{\mathrm{char}}(T_{\mathcal{V}_2})(\mathrm{Frob}_\ell) = \mathrm{Sym}^3 P_{\mathrm{char}}(T_{\mathcal{V}_2,1})(\mathrm{Frob}_\ell) = \\ &= \mathrm{Sym}^3\left(X^2 - T_{\mathcal{V}_2,1}(\mathrm{Frob}_\ell)X + \frac{T_{\mathcal{V}_2,1}(\mathrm{Frob}_\ell)^2 - T_{\mathcal{V}_2,1}(\mathrm{Frob}_\ell^2)}{2}\right),\end{aligned} \tag{3.36}$$

where the first equality is given by Corollary 3.5.11 and the last one comes from Equation (3.11). Let $\psi_2^{(1)}\colon \mathcal{H}_1^{Np}\colon \mathcal{O}(\mathcal{V}_2)$ be a morphism of rings satisfying

$$X^2 - T_{\mathcal{V}_2,1}(\mathrm{Frob}_\ell)X + \frac{T_{\mathcal{V}_2,1}(\mathrm{Frob}_\ell)^2 - T_{\mathcal{V}_2,1}(\mathrm{Frob}_\ell^2)}{2} = X^2 - \psi_2^{(1)}(T_{\ell,1}^{(1)})X + \ell\psi_2^{(1)}(T_{\ell,0}^{(1)}) \tag{3.37}$$

for every $\ell\nmid Np$. It is clear that such a morphism exists and is unique. Note that the right hand side of Equation (3.37) is $\psi_2^{(1)}(P_{\min}(t_{\ell,1}^{(1)};X))$. Then Equation (3.36) gives

$$\psi_2(P_{\min}(t_{\ell,2}^{(2)};X)) = \mathrm{Sym}^3(\psi_2^{(1)}(P_{\min}(t_{\ell,1}^{(1)};X))).$$

Exactly as in the proof of Proposition 3.4.2, by developing the two polynomials and comparing their coefficients we obtain that $\psi_2 = \psi_2^{(1)}\circ\lambda^{Np}$. Hence $\psi_2^{(1)}$ fits into diagram (3.29).

(2a) $\Longleftrightarrow$ (2b). Let $\mathcal{V}_1$ be a subvariety of $\mathcal{D}_1^N$ and let $\phi\colon \mathcal{O}(\mathcal{V}_1)\to\mathcal{O}(\mathcal{V}_2)$ be a morphism of rings. We show that the couple $(\mathcal{V}_1,\phi)$ satisfies (2a) if and only if it satisfies (2b). For every prime $\ell\nmid Np$ Corollary 3.5.11 gives

$$P_{\mathrm{char}}(T_{\mathcal{V}_1})(\mathrm{Frob}_\ell) = \psi_1(P_{\min}(t_{\ell,1}^{(1)};X)) \tag{3.38}$$

and

$$P_{\mathrm{char}}(T_{\mathcal{V}_2})(\mathrm{Frob}_\ell) = \psi_2(P_{\min}(t_{\ell,2}^{(2)};X)). \tag{3.39}$$

The argument in the proof of Proposition 3.4.2 gives an equality

$$\lambda^{Np}(P_{\min}(t_{\ell,2}^{(2)};X)) = \mathrm{Sym}^3(P_{\min}(t_{\ell,1}^{(1)};X)). \tag{3.40}$$

Since the set $\{\gamma \mathrm{Frob}_\ell \gamma^{-1}\}_{\ell\nmid Np;\, \gamma\in G_{\mathbb{Q}}}$ is dense in $G_{\mathbb{Q}}$, the pseudocharacters $\mathrm{Sym}^3(\phi\circ T_{\mathcal{V}_1})$ and $T_{\mathcal{V}_2}$ coincide if and only if their characteristic polynomials coincide on Frob_ℓ for every $\ell\nmid Np$. By Equations (3.38) and (3.39) the condition above is equivalent to

$$\mathrm{Sym}^3(\phi\circ\psi_1(P_{\min}(t_{\ell,1}^{(1)};X))) = \psi_2(P_{\min}(t_{\ell,2}^{(2)};X))$$

for every $\ell\nmid Np$. Thanks to Equation (3.40) the left hand side can be rewritten as

$$\mathrm{Sym}^3(\phi\circ\psi_1(P_{\min}(t_{\ell,1}^{(1)};X))) = \phi\circ\psi_1(\mathrm{Sym}^3(P_{\min}(t_{\ell,1}^{(1)};X))) = \phi\circ\psi_1\circ\lambda^{Np}(P_{\min}(t_{\ell,2}^{(2)};X)).$$

When ℓ varies over the primes not dividing Np the coefficients of the polynomials $P_{\min}(t_{\ell,2}^{(2)};X)$ generate the Hecke algebra $\mathcal{H}_2^{Np}$. Hence the equality

$$\phi\circ\psi_1\circ\lambda^{Np}(P_{\min}(t_{\ell,2}^{(2)};X)) = \psi_2(P_{\min}(t_{\ell,2}^{(2)};X))$$

holds for every $\ell\nmid Np$ if and only if

$$\phi\circ\psi_1\circ\lambda^{Np} = \psi_2.$$

This is precisely the relation describing the commutativity of diagram (3.31).

(2b) $\implies$ (1b). Suppose that condition (2b) is satisfied by some closed subvariety $\mathcal{V}_1$ of $\mathcal{D}_1^N$ and some morphism of rings $\phi\colon \mathcal{O}(\mathcal{V}_1)\to\mathcal{O}(\mathcal{V}_2)$. Consider the pseudocharacter $T_{\mathcal{V}_2,1} = \phi\circ T_{\mathcal{V}_1}\colon G_{\mathbb{Q}}\to\mathcal{O}(\mathcal{V}_2)$. Clearly $T_{\mathcal{V}_2,1}$ satisfies condition (1b).

It remains to prove that (1b) $\implies$ (2b) when $\mathcal{V}_2$ is a $\overline{\mathbb{Q}}_p$-point of $\mathcal{D}_2^M$. For this step we will need the results we recalled in Section 3.10. Write x_2 for the point $\mathcal{V}_2$; the system of eigenvalues ψ_{x_2} is that of a classical GSp_4-eigenform. By Remark 3.5.12 T_{x_2} is the pseudocharacter associated with a representation $\rho_{x_2}\colon G_{\mathbb{Q}}\to \mathrm{GL}_4(\overline{\mathbb{Q}}_p)$. Let E be a finite extension of $\mathbb{Q}_p$ over which ρ_{x_2} is defined. Suppose that x_2 satisfies condition (1b). Let $T_{x_2,1}\colon G_{\mathbb{Q}}\to\overline{\mathbb{Q}}_p$ be a pseudocharacter such that $T_{x_2}\cong \mathrm{Sym}^3 T_{x_2,1}$. By Theorem 3.5.4 there exists a representation $\rho_{x_2,1}\colon G_{\mathbb{Q}}\to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ such that $T_{x_2,1} = \mathrm{Tr}(\rho_{x_2,1})$. Then Lemma 3.5.8 implies that $\rho_{x_2}\cong \mathrm{Sym}^3\rho_{x_2,1}$. Since ρ_{x_2} is attached to an overconvergent GSp_4-eigenform, Theorem 3.10.30 implies that $\rho_{x_2,1}$ is the p-adic Galois representation attached to an overconvergent GL_2-eigenform f. Such a form defines a point x_1 of the eigencurve $\mathcal{D}_1^N$.Thus the subvariety $\mathcal{V}_1 = x_1$ satisfies condition (2b). □

The four properties stated in the theorem are stable when passing to a subvariety.

Lemma 3.12.2. *Let $\mathcal{V}_2$ and $\mathcal{V}_2'$ be two rigid analytic subvarieties of $\mathcal{D}_2^M$ satisfying $\mathcal{V}_2'\subset\mathcal{V}_2$. Let $(*)$ denote one of the conditions of Theorem 3.12.1. If $(*)$ holds for $\mathcal{V}_2$ then it holds for $\mathcal{V}_2'$.*

Proof. Thanks to the theorem it is sufficient to prove the statement for $* = 1b$ and $* = 2b$. Let $r_{\mathcal{V}_2'}\colon\mathcal{O}(\mathcal{V}_2)\to\mathcal{O}(\mathcal{V}_2')$ denote the restriction of analytic functions. It is easy to check that:

(i) if $T_{\mathcal{V}_2,1}\colon G_{\mathbb{Q}}\to \mathrm{GL}_2(\mathcal{O}_{\mathcal{V}_2})$ is a pseudocharacter satisfying condition (1b) for the variety $\mathcal{V}_2$, then the pseudocharacter $r_{\mathcal{V}_2'}\circ T_{\mathcal{V}_2,1}\colon G_{\mathbb{Q}}\to\mathcal{O}(\mathcal{V}_2')$ satisfies condition (1b) for the variety $\mathcal{V}_2'$;

(ii) if $\phi\colon\mathcal{O}(\mathcal{V}_2)\to\mathcal{O}(\mathcal{V}_1)$ is a morphism satisfies condition (2b) for the variety $\mathcal{V}_2$, then $r_{\mathcal{V}_2'}\circ\phi$ satisfies condition (2b) for the variety $\mathcal{V}_2'$.

□

In light of Theorem 3.12.1 we give the following definitions.

Definition 3.12.3. *(1) We say that a subvariety $\mathcal{V}_2$ of $\mathcal{D}_2^M$ is of* Sym^3 type *if it satisfies the equivalent conditions (2a) and (2b) of Theorem 3.12.1.*
(2) The Sym^3-locus *of $\mathcal{D}_2^M$ is the set of points of $\mathcal{D}_2^M$ of* Sym^3 *type.*

Remark 3.12.4. *A variety $\mathcal{V}_2$ of* Sym^3 *type also satisfies conditions (1a) and (1b) of Theorem 3.12.1 thanks to the implication (2b) $\implies$ (1b).*

Let $\iota\colon \mathcal{W}_1^\circ \to \mathcal{W}_2^\circ$ is the closed immersion constructed in Section 3.8.2. Let $\mathcal{D}^M_{2,\iota}$ be the one-dimensional subvariety of $\mathcal{D}_2^M$ fitting in the cartesian diagram

$$\begin{tikzcd} \mathcal{D}^M_{2,\mathrm{aux}} \arrow[r] \arrow[d] & \mathcal{D}_2^M \arrow[d, "w_2"] \\ \iota(\mathcal{W}_1^\circ) \arrow[r, "\iota"] & \mathcal{W}_2^\circ \end{tikzcd}$$

Corollary 3.12.5. *The* Sym^3*-locus of* $\mathcal{D}_2^M$ *is contained in the one-dimensional subvariety* $\mathcal{D}^M_{2,\iota}$.

Proof. By definition of the Sym^3-locus, if $x \in \mathcal{D}_2^M$ then the associated representation ρ_x is isomorphic to $\mathrm{Sym}^3\rho_{x_1}$ for a point x_1 of $\mathcal{D}_1^N$. By calculating the generalized Hodge-Tate weights of $\mathrm{Sym}^3\rho_{x_1}$ in terms of those of ρ_{x_1} we obtain that the weight of x belongs to the locus $\iota(\mathcal{W}_1^\circ)$. □

The Sym^3-locus of $\mathcal{D}_2^M$ admits a Hecke-theoretic definition thanks to condition (2b) of Theorem 3.12.1. We elaborate on this. Consider the following maps:

$$\begin{tikzcd} \mathcal{H}_2^{Np} \arrow[r, "\psi_2"] \arrow[d, "\lambda^{Np}"] & \mathcal{O}(\mathcal{D}_2^M) \\ \mathcal{H}_1^{Np} \arrow[r, "\psi_1"] & \mathcal{O}(\mathcal{D}_1^N) \end{tikzcd}$$

We define an ideal $\mathcal{I}_{\mathrm{Sym}^3}$ of $\mathcal{O}(\mathcal{D}_2^M)$ by

$$\mathcal{I}_{\mathrm{Sym}^3} = \psi_1(\ker(\psi_2 \circ \lambda^{Np})) \cdot \mathcal{O}(\mathcal{D}_2^M).$$

We denote by $\mathcal{D}^M_{2,\mathrm{Sym}^3}$ the analytic Zariski subvariety of $\mathcal{D}_2^M$ defined as the zero locus of the ideal $\mathcal{I}_{\mathrm{Sym}^3}$.

Proposition 3.12.6. *(i) The* Sym^3*-locus of* $\mathcal{D}_2^M$ *is the set of points underlying* $\mathcal{D}^M_{2,\mathrm{Sym}^3}$.

(ii) The variety $\mathcal{D}^M_{2,\mathrm{Sym}^3}$ *is of* Sym^3 *type.*

(iii) A rigid analytic subvariety $\mathcal{V}_2$ *of* $\mathcal{D}_2^M$ *is of* Sym^3 *type if and only if it is a subvariety of* $\mathcal{D}^M_{2,\mathrm{Sym}^3}$.

(iv) A rigid analytic subvariety $\mathcal{V}_2$ *of* $\mathcal{D}_2^M$ *satisfies conditions (1a) and (1b) of Theorem 3.12.1 if and only if it is a subvariety of* $\mathcal{D}^M_{2,\mathrm{Sym}^3}$.

Proof. We prove (i). Let x_2 be any $\overline{\mathbb{Q}}_p$-point of $\mathcal{D}_2^M$ and let $\mathrm{ev}_{x_2}\colon \mathcal{O}(\mathcal{D}_2^M) \to \overline{\mathbb{Q}}_p$ be the evaluation at x_2. The system of eigenvalues corresponding to x_2 is $\psi_{x_2} = \mathrm{ev}_{x_2} \circ \psi_2\colon \mathcal{H}_2^{Np} \to \overline{\mathbb{Q}}_p$. By definition x_2 is of Sym^3 type if and only if there exists a morphism of rings $\mathrm{ev}_{x_1}\colon \mathcal{O}(\mathcal{D}_1^N) \to \overline{\mathbb{Q}}_p$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{H}_2^{Np} \arrow[r, "\psi_2"] \arrow[d, "\lambda^{Np}"] & \mathcal{O}(\mathcal{D}_2^M) \arrow[r, "\mathrm{ev}_{x_2}"] & \overline{\mathbb{Q}}_p \\ \mathcal{H}_1^{Np} \arrow[r, "\psi_1"] & \mathcal{O}(\mathcal{D}_1^N) \arrow[ur, "\mathrm{ev}_{x_1}"'] & \end{tikzcd}$$

By elementary algebra the map ev_{x_1} exists if and only if $\mathrm{ev}_{x_2}(\ker(\psi_2 \circ \lambda^{Np})) = 0$. This is equivalent to the fact that the point x_2 is in the zero locus of the ideal $\mathcal{I}_{\mathrm{Sym}^3}$.

For (ii) it is sufficient to observe that there exists a morphism of rings $\Xi^*_{\mathrm{Sym}^3}\colon \mathcal{O}(\mathcal{D}_1^N) \to \mathcal{O}(\mathcal{D}^M_{2,\mathrm{Sym}^3})$ fitting into the commutative diagram

$$\begin{tikzcd} \mathcal{H}_2^{Np} \arrow[r, "\lambda^{Np}"] \arrow[rrr, bend left, "r_{\mathrm{Sym}^3} \circ \psi_2"] & \mathcal{H}_1^{Np} \arrow[r, "\psi_1"] & \mathcal{O}(\mathcal{D}_1^N) \arrow[r, "\Xi^*_{\mathrm{Sym}^3}"] & \mathcal{O}(\mathcal{D}^M_{2,\mathrm{Sym}^3}) \end{tikzcd} \tag{3.41}$$

Such a $\Xi^*_{\mathrm{Sym}^3}$ exists since by definition of $\mathcal{D}^M_{2,\mathrm{Sym}^3}$ we have $r_{\mathcal{D}^M_{2,\mathrm{Sym}^3}} \circ \psi_2(\ker(\lambda^{Np} \circ \phi_{\mathrm{Sym}^3})) = 0$.

Note that the "if" implications of (iii) and (iv) follow from Lemma 3.12.2, together with Remark 3.12.4 for (iv).

To prove the other direction of (iii) we look again at diagram (3.31) for a subvariety $\mathcal{V}_2$ of $\mathcal{D}_2^M$. In order for $\mathcal{V}_2$ to satisfy condition (2a) of Theorem 3.12.1 we must have $r_{\mathcal{V}_2}(\ker(\lambda^{Np} \circ \Xi^*_{\mathrm{Sym}^3})) = 0$, so $\mathcal{V}_2$ is contained in $\mathcal{D}^M_{2,\mathrm{Sym}^3}$.

Finally, let $\mathcal{V}_2$ be a rigid analytic subvariety of $\mathcal{D}_2^M$ satisfying conditions (1a) and (1b) of Theorem 3.12.1. Let x_2 by a point of $\mathcal{V}_2$. By Lemma 3.12.2 x_2 satisfies conditions (1a) and (1b). By Theorem 3.12.1, x_2 also satisfies conditions (2a,2b), so it is a point of $\mathcal{D}^M_{2,\mathrm{Sym}^3}$. We conclude that $\mathcal{V}_2$ is a subvariety of $\mathcal{D}^M_{2,\mathrm{Sym}^3}$. □

Remark 3.12.7. *By Proposition 3.12.6 the* Sym^3*-locus in* $\mathcal{D}_2^M$ *can be given the structure of a Zariski-closed rigid analytic subspace. From now on we will always consider the* Sym^3*-locus as equipped with this structure.*

Corollary 3.12.8. *The* Sym^3*-locus intersects each irreducible component of* $\mathcal{D}_2^M$ *in a proper analytic Zariski subvariety of dimension at most* 1.

Proof. By Proposition 3.12.6 the Sym^3-locus intersects each irreducible component of $\mathcal{D}_2^M$ in an analytic Zariski subvariety. By Corollary 3.12.5 this subvariety has dimension at most 1. □

Proposition 3.12.6 allows us to improve the result of Theorem 3.12.1.

Corollary 3.12.9. *For every rigid analytic subvariety* $\mathcal{V}_2$ *of* $\mathcal{D}_2^M$ *the conditions (1a), (1b), (2a), (2b) of Theorem 3.12.1 are equivalent.*

Proof. This follows immediately from Proposition 3.12.6(iii) and (iv). □

Consider the map $\Xi^*_{\mathrm{Sym}^3}\colon \mathcal{O}(\mathcal{D}_1^N) \to \mathcal{O}(\mathcal{D}^M_{2,\mathrm{Sym}^3})$ appearing in the commutative diagram (3.41); it induces a map of rigid analytic spaces $\Xi_{\mathrm{Sym}^3}\colon \mathcal{D}^M_{2,\mathrm{Sym}^3} \to \mathcal{D}_1^N$. Our choice of notation for this map is due to the fact that Ξ_{Sym^3} is related to the map ξ given by Definition 3.9.8.

CHAPTER 4

Galois level and congruence ideal for finite slope families of Siegel modular forms

In this chapter we prove our main theorems. In section 4.1.2 we define families of GSp_4-eigenforms. We show that the representation associated with such a family has "big image" (Theorem 4.11.1), in a Lie-algebraic sense, and that the size of the image can be described in terms of congruences of symmetric cube type (Theorem 4.12.1).

4.1. Finite slope families of eigenforms

Let p be a prime number and let N and M be two positive integers prime to p. Let $h \in \mathbb{R}^{\geq 0}$. In Sections 1.2.5.2 and 1.2.6.2 we defined the slope $\leq h$ eigenvarieties $\mathcal{D}_1^{N,h}$ and $\mathcal{D}_2^{M,h}$ as subvarieties of $\mathcal{D}_1^N$ and $\mathcal{D}_2^M$, respectively. The slope $\leq h$ eigenvarieties are in general not finite over the respective weight space if $h > 0$. In order to have finiteness we will restrict the weights of our families to a sufficiently small disc in the corresponding weight space. Recall that we always identify the g-dimensional weight space with a disjoint union of open discs of centre 0 and radius 1. As in Section 2.2.1 we centre the weights of our families at 0, but this is not necessary and the same construction can be carried out for any other centre.

4.1.1. Families of GL_2-eigenforms. In Section 2.2.1 we defined finite slope families of GL_2-eigenforms of level $\Gamma_1(N)\cap\Gamma_0(p)$. We briefly recall this construction, adapting the notations for their use in this chapter. We rely on the proofs in Section 2.2.1.

As before fix $h \in \mathbb{Q}^{+,\times}$. For a radius $r \in p^{\mathbb{Q}}$ let $\mathcal{D}^{N,h}_{1,B_1(0,r^-)} = \mathcal{D}_1^{N,h} \times_{\mathcal{W}_1^\circ} B_1(0,r^-)$. There exists $r_h \in p^{\mathbb{Q}}$ such that the weight map $w_1|_{\mathcal{D}^{N,h}_{1,B_1(0,r_h^-)}} \colon \mathcal{D}^{N,h}_{1,B_1(0,r_h^-)} \to B_{1,h}$ is finite. We simply write $B_{1,h} = B_1(0,r_h^-)$. The open disc $B_{1,h}$ admits a $\mathbb{Q}_p$-model thanks to Berthelot's construction; from now on we work with this model. Let K_h be a finite extension of $\mathbb{Q}_p$ containing an element η_h of norm r_h. Let $\mathcal{O}_h$ be the ring of integers of K_h. We call *genus* 1*, h-adapted Iwasawa algebra* the ring of analytic functions $\Lambda_{1,h} = \mathcal{O}(B_{1,h})^\circ$.

Let $\mathbb{T}_{1,h} = \mathcal{O}(\mathcal{D}^{N,h}_{1,B_1(0,r_h^-)})^\circ$. We call $\mathbb{T}_{1,h}$ the *genus* 1*, h-adapted Hecke algebra.*

Definition 4.1.1. *A* family of GL_2-eigenforms of slope bounded by h *is an irreducible component $\mathcal{I}$ of $\mathcal{D}^{N,h}_{1,B_1(0,r_h^-)}$.*

The ring of analytic functions bounded by 1 on $\mathcal{I}$ is a profinite local ring $\mathbb{I}^\circ$ with a structure of finite $\Lambda_{1,h}$-algebra induced by the weight map $w_\mathcal{I} = w_2|_\mathcal{I} \colon \mathcal{I} \to B_1(0,r_h^-)$. The component $\mathcal{I}$ is described by the surjective map $\mathbb{T}_h \to \mathbb{I}^\circ$ defined by the restriction of analytic functions. We sometimes refer to this morphism when speaking of a finite slope family.

For every ideal $\mathfrak{P}$ of $\mathbb{I}^\circ$ let $\mathrm{ev}_\mathfrak{P} \colon \mathbb{I}^\circ \to \mathbb{I}^\circ/\mathfrak{P}$ be the natural projection. Let $r_{\mathcal{D}_{1,B_{1,h}}} \colon \mathcal{O}(\mathcal{D}_1^N) \to \mathcal{O}(\mathcal{D}^{N,h}_{1,B_1(0,r_h^-)})$ be the restriction of analytic functions and set

$$\psi_\theta = \theta \circ r_{\mathcal{D}^{N,h}_{1,B_{1,h}}} \circ \psi_1 \colon \mathcal{H}_1^N \to \mathbb{I}^\circ.$$

Definition 4.1.2. *We say that a prime ideal $\mathfrak{P}$ of $\mathbb{I}^\circ$ is classical if the ring morphism* $\mathrm{ev}_{\mathfrak{P}} \circ \psi_\theta \colon \mathcal{H}_1^N \to \mathbb{I}^\circ/\mathfrak{P}$ *defines the system of Hecke eigenvalues associated with a classical* GSp_4*-eigenform.*

4.1.2. Families of GSp_4-eigenforms. We define families of finite slope GSp_4-eigenforms of level $\Gamma_1(M) \cap \Gamma_0(p)$. Fix again $h \in \mathbb{Q}^{+,\times}$. Note that in statements involving families of GL_2- and GSp_4-eigenforms at the same time we may need to take different bounds on the slope for the two groups. The restriction of the weight map $w_2 \colon \mathcal{D}_2^{M,h} \to \mathcal{W}_2^\circ$ is in general not finite if $h > 0$. To solve this problem we will restrict the weights of the family to a sufficiently small disc in the weight space.

For every affinoid subdomain V of $\mathcal{W}_2^\circ$, let $\mathcal{D}_{2,V}^{M,h} = \mathcal{D}_2^{M,h} \times_{\mathcal{W}_2^\circ} V$ and let

$$w_{2,V} = w_{2,h}|_{\mathcal{D}_{2,V}^{M,h}} \colon \mathcal{D}_{2,V}^{M,h} \to V.$$

Proposition 4.1.3. [**Be12**, Proposition II.1.12] *For every $\kappa \in \mathcal{W}_2^\circ(\overline{\mathbb{Q}}_p)$ there exists an affinoid neighborhood V of κ in $\mathcal{W}_2^\circ$ such that the map $w_{2,V}$ is finite.*

Remark 4.1.4.

(1) Every affinoid neighborhood of $\kappa \in \mathcal{W}_2^\circ$ contains a wide open disc centred in κ. Then Proposition 4.1.3 implies that there exists a radius $r_{h,\kappa} \in p^{\mathbb{Q}}$ such that

$$w_{2,B_2(\kappa,r_{h,\kappa}^-)} \colon \mathcal{D}_{2,B_2(\kappa,r_{h,\kappa}^-)}^{M,h} \to B_2(\kappa, r_{h,\kappa}^-)$$

is a finite morphism.

(2) Thanks to Hida theory for GSp_4 *we know that the ordinary eigenvariety $\mathcal{D}_2^{M,0}$ is finite over $\mathcal{W}_2^\circ$. Hence for $h = 0$ we can just take $r_{0,\kappa} = 1$ for every κ.*

(3) We would like to have an estimate for $r_{h,\kappa}$ independent of κ and with the property that $r_{h,\kappa} \to 0$ for $h \to 0$, in order to recover the ordinary case in this limit. This is not available at the moment for the group GSp_4*. An estimate of the analogue of this radius is known for the eigenvarieties associated with unitary groups compact at infinity by the work of Chenevier* [**Ch04**, Théorème 5.3.1].

From now on we set $\kappa = (0,0) \in B_2(0,1^-)$; we write in short $\kappa = 0$ and $r_{h,(0,0)} = r_h$. Let r_h be the largest radius in $p^{\mathbb{Q}}$ such that:

- $w_{2,B_2(0,r_h^-)} \colon \mathcal{D}_{2,B_2(0,r_h^-)}^{M,h} \to B_2(0, r_h^-)$ is finite;
- $r_h < p^{-\frac{1}{p-1}}$.

Such a radius is non-zero thanks to Remark 4.1.4(1). Let s_h be a rational number satisfying $r_h = p^{s_h}$. Let η_h be an element of $\overline{\mathbb{Q}}_p$ satisfying $v_p(\eta_h) = s_h$. Let $K_h = \mathbb{Q}_p(\eta_h)$ and let $\mathcal{O}_h$ be the ring of integers of K_h. Let T_1, T_2 be the coordinates of $\mathcal{W}_2^\circ$ defined in Section 3.8.2 and let $t_1 = \eta_h^{-1} T_1$, $t_2 = \eta_h^{-1} T_2$.

We write in short $B_{2,h} = B_2(0, r_h^-)$. We define a model for $B_{2,h}$ over $\mathbb{Q}_p$ by adapting Berthelot's construction (see [**dJ95**, Section 7]). Write $s_h = \frac{b}{a}$ for some $a, b \in \mathbb{N}$. For $i \geq 1$, let $s_i = s_h + 1/2^i$ and $r_i = p^{-s_i}$. Set

$$A_{r_i}^\circ = \mathbb{Z}_p\langle t_1, t_2, X_i \rangle / (t_1^{2^i a} - p^{a+2^i b} X_i, t_2^{2^i a} - p^{a+2^i b} X_i)$$

and $A_{r_i} = A_{r_i}^\circ[p^{-1}]$. Set $B_i = \mathrm{Spm}\, A_{r_i}$. Then B_i is a $\mathbb{Q}_p$-model of the disc of centre 0 and radius r_i. We define morphisms $A_{r_{i+1}}^\circ \to A_{r_i}^\circ$ by

$$X_{i+1} \mapsto p^a X_i^2,$$
$$t_1 \mapsto t_1,$$
$$t_2 \mapsto t_2.$$

They induce compact maps $A_{r_{i+1}} \to A_{r_i}$ which give open immersions $B_i \hookrightarrow B_{i+1}$. We define $B_{2,h} = \varinjlim_i B_i$ where the limit is taken with respect to the above immersions. We have $\mathcal{O}(B_{2,h})^\circ = \varprojlim_i \mathcal{O}(\mathrm{Spm}\, B_i)^\circ = \varprojlim_i A_{r_i}^\circ$.

Definition 4.1.5. *Let $\Lambda_{2,h} = \mathcal{O}(B_2(0, r_h^-))^\circ$. We call $\Lambda_{2,h}$ the* genus 2, h-adapted Iwasawa algebra. *We define $t_1, t_2 \in \Lambda_{2,h}$ as the projective limits of the variables t_1, t_2, respectively, of $A_{r_i}^\circ$.*

Since we will mainly work with the genus 2 algebra from now on, we drop the superscript and simply write $\Lambda_h = \Lambda_{2,h}$. The algebra Λ_h is not a ring of formal power series over $\mathbb{Z}_p$, but there is an isomorphism

$$\Lambda_h \otimes_{\mathbb{Z}_p} \mathcal{O}_h \cong \mathcal{O}_h[[t_1, t_2]].$$

Definition 4.1.6. *Let $\mathcal{D}^{M,h}_{2,B_{2,h}} = \mathcal{D}^{M,h}_2 \times_{\mathcal{W}_2^\circ} B_{2,h}$ and let $\mathbb{T}_{2,h} = \mathcal{O}(\mathcal{D}^{M,h}_{2,B_{2,h}})^\circ$. We call $\mathbb{T}_{2,h}$ the* genus 2, h-adapted Hecke algebra.

By definition $\mathbb{T}_{2,h}$ has a structure of Λ_h-algebra. Thanks to our choice of r_h, $\mathbb{T}_{2,h}$ is a finite Λ_h-algebra.

Definition 4.1.7. *We call* family of GSp_4-eigenforms of finite slope (bounded by h) *an irreducible component I of $\mathcal{D}^{M,h}_{2,B_{2,h}}$.*

We will usually refer to I simply as a finite slope family. For such an I let $\mathbb{I}^\circ = \mathcal{O}(I)$. Then $\mathbb{I}^\circ$ is a finite Λ_h-algebra and I is determined by the surjective morphism $\mathbb{T}_{2,h} \twoheadrightarrow \mathbb{I}^\circ$. We sometimes refer to this morphism as a finite slope family. The notation $^\circ$ denotes the fact that we are working with integral objects, i.e. that p is not invertible. When introducing relative Sen theory in Section 4.10 we will need to invert p and we will drop the $^\circ$ from all rings.

Remark 4.1.8. *Since Λ_h is profinite and local and $\mathbb{T}_{2,h}$ is finite over Λ_h, $\mathbb{T}_{2,h}$ is profinite and semilocal. The connected components of $\mathcal{D}^{M,h}_{2,B_{2,h}}$ are in bijection with the maximal ideals of $\mathbb{T}_{2,h}$. Let $\theta\colon \mathbb{T}_{2,h} \twoheadrightarrow \mathbb{I}^\circ$ be the morphism of Λ_h-algebras defining a finite slope family I. Then* $\ker\theta$ *is contained in the unique maximal ideal $\mathfrak{m}_\theta$ corresponding to the connected component of $\mathcal{D}^{M,h}_{2,B_{2,h}}$ containing I. The Λ_h-algebra $\mathbb{I}^\circ$ is profinite and local with maximal ideal $\mathfrak{m}_{\mathbb{I}^\circ} = \theta(\mathfrak{m}_\theta)$.*

From now on we replace implicitly $\mathbb{T}_{2,h}$ by one of its local components.

Definition 4.1.9. *Let $g = 1$ or 2. We say that a prime of $\Lambda_{g,h}$ is arithmetic if it lies over an arithmetic prime of Λ_g.*

By an abuse of notation we will write again $P_{\underline{k}}$ for an arithmetic prime of $\Lambda_{g,h}$ lying over the arithmetic prime $P_{\underline{k}}$ of Λ_g.

Remark 4.1.10. *Let $\underline{k} = (k_1, k_2)$ with $k_1 \geq k_2 \geq 3$. Consider the arithmetic prime $P_{\underline{k}} \subset \Lambda_2$ and the ideal $P_{\underline{k}}\Lambda_h$ defined via the natural inclusion $\Lambda_2 \hookrightarrow \Lambda_h$. Then there exists an arithmetic prime $\mathfrak{P}$ of Λ_h lying over $P_{\underline{k}}$ if and only if the classical weight $\underline{k}$ belongs to the disc $B(0, r_h^-)$; otherwise we have $P_{\underline{k}}\Lambda_h = \Lambda_h$. Since $P_{\underline{k}} = (1 + T_1 - u^{k_1}, 1 + T_2 - u^{k_2})$, the weight $\underline{k}$ belongs to $B(0, r_h^-)$ if and only if $v_p(u^{k_1} - 1) > v_p(r_h)$ and $v_p(u^{k_2} - 1) > v_p(r_h)$. Now $v_p(u^{k_1} - 1) = 1 + v_p(k_1)$ and $v_p(u^{k_2} - 1) = 1 + v_p(k_2)$, so the previous inequalities become $v_p(k_1) > -v_p(r_h) - 1$ and $v_p(k_2) > -v_p(r_h) - 1$. Note that the closed disc of centre 0 and radius $1/p$ contains all the classical weights.*

For every ideal $\mathfrak{P}$ of $\mathbb{I}^\circ$ we denote by $\mathrm{ev}_{\mathfrak{P}}\colon \mathbb{I}^\circ \to \mathbb{I}^\circ/\mathfrak{P}$ the natural projection. Set

$$\psi_\theta = \theta \circ r_{\mathcal{D}^{M,h}_{2,B_h}} \circ \psi_2\colon \mathcal{H}_2^M \to \mathbb{I}^\circ.$$

Definition 4.1.11. *We say that a prime ideal $\mathfrak{P}$ of $\mathbb{I}^\circ$ is classical if the ring morphism $\mathrm{ev}_{\mathfrak{P}} \circ \psi_\theta\colon \mathcal{H}_2^M \to \mathbb{I}^\circ/\mathfrak{P}$ defines the system of Hecke eigenvalues associated with a classical* GSp_4-*eigenform.*

Remark 4.1.12. *The set of classical primes of* $\mathbb{I}^\circ$ *is Zariski-dense by the following argument. We have* $\mathbb{I}^\circ = \mathcal{O}(I)^\circ$ *for an admissible subdomain* I *of* $\mathcal{D}_2^M$. *Then* I *contains at least one classical point by Proposition 1.2.27, so it contains a Zariski-dense susbset of classical points by their accumulation property. If* f *is an element of* $\mathbb{I}^\circ$ *such that* $f \in \mathfrak{P}$ *for every classical prime* $\mathfrak{P}$ *of* $\mathbb{I}^\circ$, *then* f *is a function on* I *that vanishes on a Zariski-dense subset, so* $f = 0$.

4.1.3. Non-critical points on families.

Let $\theta\colon \mathbb{T}_{2,h} \to \mathbb{I}^\circ$ be a family of GSp_4-eigenforms.

Definition 4.1.13. *We call an arithmetic prime* $P_{\underline{k}} \subset \Lambda_h$ non-critical for $\mathbb{I}^\circ$ *if:*

(1) every point of the fibre of $w^*_{2,B_{2,h}}\colon \Lambda_h \to \mathbb{I}^\circ$ *at* $P_{\underline{k}}$ *is classical;*

(2) $w^*_{2,B_{2,h}}\colon \Lambda_h \to \mathbb{I}^\circ$ *is étale at every point of the fibre of* $w^*_{2,B_{2,h}}$ *at* $P_{\underline{k}}$.

We call $P_{\underline{k}}$ *critical for* $\mathbb{I}^\circ$ *if it is not non-critical. We also say that a classical weight* $\underline{k}$ *is critical or non-critical for* $\mathcal{D}_2^h$ *if the corresponding arithmetic prime has the same property.*

Remark 4.1.14. *By Proposition 1.2.27, if* $\underline{k}$ *is a classical weight belonging to* $B_{2,h}$ *and* $k_2 \geq h-3$ *then every point of the fibre of* $\mathcal{D}^{M,h}_{2,B_{2,h}}$ *at* $\underline{k}$ *is classical. Since the weight* $\underline{k}$ *corresponds to the prime* $P_{\underline{k}}$ *via the identification* $B_{2,h} = (\mathrm{Spf}\,\Lambda_h)^{\mathrm{rig}}$, *the first condition of Definition 4.1.13 is satisfied by* $P_{\underline{k}}$ *when* $k_2 \geq h-3$. *However we do not know of a simple assumption on the weight that guarantees that the second condition is satisfied. We will still know that there are sufficiently many non-critical classical points thanks to Proposition 4.1.15 below.*

Later we will have to choose a non-critical arithmetic prime of Λ_h satisfying certain additional properties (see Section 4.9.1). We will need the following result in order to show that such a point exist.

Proposition 4.1.15. *The set of non-critical arithmetic primes is Zariski-dense in* Λ_h.

Proof. Suppose by contradiction that the conclusion is not true. Then the set of critical classical weights must be Zariski-dense in B_h, since the set of all classical weights is Zariski-dense in B_h. Consider the subset Σ^{crit} of critical classical weights $\underline{k} = (k_1, k_2)$ in B_h satisfying $h < k_2 + 3$. This condition excludes only a finite number of weights, so Σ^{crit} is still Zariski-dense in B_h. Let S^{crit} be the set of points $x \in \mathcal{D}^h_{2,B_h}$ such that $w(x) \in \Sigma^{\mathrm{crit}}$ and w is not étale at x. Let $\mathcal{D}^{\text{n-ét}} \subset \mathcal{D}^h_{2,B_h}$ denote the locus of non-étaleness of w. It is Zariski-closed of non-zero codimension in $\mathcal{D}^h_{2,B_h}$ and it contains S^{crit}.

By Proposition 1.2.27 condition (i) of Definition 4.1.13 is satisfied for all $\underline{k} \in \Sigma^{\mathrm{crit}}$, so condition (ii) must be false for all $\underline{k} \in \Sigma^{\mathrm{crit}}$. In particular the weight map gives a surjection of S^{crit} onto the Zariski-dense subset $S_{\mathrm{crit}} \subset B_h$. Since $w\colon \mathcal{D}^{M,h}_{2,B_h} \to B_h$ is finite we can apply Lemma 1.2.11 to find that some irreducible component of $\mathcal{D}^{M,h}_{2,B_h}$ must be contained in the Zariski closure of S^{crit}, hence in $\mathcal{D}^{\text{n-ét}}$. This is a contradiction. □

4.1.4. The Galois representation associated with a finite slope family.

Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a finite slope family of GSp_4-eigenforms. Let $\mathbb{F}_{\mathbb{T}_h}$ denote the residue field of $\mathbb{T}_h$. Let $T_{\mathcal{D}_2^M}\colon G_\mathbb{Q} \to \mathcal{O}(\mathcal{D}_2^M)^\circ$ be the pseudocharacter given by Theorem 3.5.10. Let

$$r_{\mathcal{D}^{M,h}_{2,B_h}} \colon \mathcal{O}(\mathcal{D}_2^M) \to \mathcal{O}(\mathcal{D}^{M,h}_{2,B_h})$$

be the map given by the restriction of analytic functions. Define a pseudocharacter $T_{\mathbb{T}_h}\colon G_\mathbb{Q} \to \mathbb{T}_h$ by setting

$$T_{\mathbb{T}_h} = r_{\mathcal{D}^{M,h}_{2,B_h}} \circ T_{\mathcal{D}_2^M}.$$

By reducing $T_{\mathbb{T}_h}$ modulo the maximal ideal of $\mathbb{F}_{\mathbb{T}_h}$ we obtain a pseudocharacter $\overline{T}_{\mathbb{T}_h}\colon G_{\mathbb{Q}} \to \mathbb{F}_{\mathbb{T}_h}$. By Theorem 3.5.4 the pseudocharacter $\overline{T}_{\mathbb{T}_h}$ is associated with a unique representation $\overline{\rho}_{\mathbb{T}_h}\colon G_{\mathbb{Q}} \to \mathrm{GL4}(\overline{\mathbb{F}}_p)$. We call $\overline{\rho}_{\mathbb{T}_h}$ the residual Galois representation associated with $\mathbb{T}_h$.

We assume from now on that:

the residual representation $\overline{\rho}_{\mathbb{T}_h}$ is absolutely irreducible.

By compactness of $G_{\mathbb{Q}}$ there exists a finite extension $\mathbb{F}'$ of $\mathbb{F}_{\mathbb{T}_h}$ such that $\overline{\rho}_{\mathbb{T}_h}$ is defined on $\mathbb{F}'$. Let $W(\mathbb{F}_{\mathbb{T}_h})$ and $W(\mathbb{F}')$ be the rings of Witt vectors of $\mathbb{F}_{\mathbb{T}_h}$ and $\mathbb{F}'$, respectively. Let $\mathbb{T}_h' = \mathbb{T}_h \otimes_{W(\mathbb{F}_{\mathbb{T}_h})} W(\mathbb{F}')$. We consider $T_{\mathbb{T}_h}$ as a pseudocharacter $G_{\mathbb{Q}} \to \mathbb{T}_h'$ via the natural inclusion $\mathbb{T}_h \hookrightarrow \mathbb{T}_h'$. Then $T_{\mathbb{T}_h}$ satisfies the hypotheses of Theorem 3.5.5, so there exists a representation

$$\rho_{\mathbb{T}_h'}\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\mathbb{T}_h')$$

such that $\mathrm{Tr}\rho_{T_h'} = \mathbb{T}_{\mathbb{T}_h}$. By Theorem 3.5.10, for every prime ℓ not dividing Np we have

$$\mathrm{Tr}(D_{\mathbb{T}_h})(\mathrm{Frob}_\ell) = r_{\mathcal{D}^{M,h}_{2,B_h}} \circ \psi_2(T^{(2)}_{\ell,2}). \tag{4.1}$$

In particular $\mathrm{Tr}(D_{\mathbb{T}_h})(\mathrm{Frob}_\ell)$ is an element of $\mathbb{T}_h$. Since $\mathbb{T}_h$ is complete, Chebotarev's theorem implies that $\mathbb{T}_{\mathbb{T}_h}(g)$ is an element of $\mathbb{T}_h$ for every $g \in G_{\mathbb{Q}}$. By Theorem 3.5.3 there exists a representation

$$\rho_{\mathbb{T}_h}\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\mathbb{T}_h)$$

that is isomorphic to $\rho_{\mathbb{T}_h}$ over $\mathbb{T}_h'$.

The morphism $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ induces a morphism $\mathrm{GL}_4(\mathbb{T}_h) \to \mathrm{GL}_4(\mathbb{I}^\circ)$ that we still denote by θ. Let $\rho_{\mathbb{I}^\circ}\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\mathbb{I}^\circ)$ be the representation defined by

$$\rho_{\mathbb{I}^\circ} = \theta \circ \rho_{\mathbb{T}_h}.$$

Recall that we set $\psi_\theta = \theta \circ r_{\mathcal{D}^{M,h}_{2,B_h}} \circ \psi_2\colon \mathcal{H}_2^M \to \mathbb{I}^\circ$. Let

$$\mathbb{I}^\circ_{\mathrm{Tr}} = \Lambda_h[\{\mathrm{Tr}(\rho_\theta(g))\}_{g \in G_{\mathbb{Q}}}].$$

Since $\Lambda_h \subset \mathbb{I}^\circ_{\mathrm{Tr}} \subset \mathbb{I}^\circ$, the ring $\mathbb{I}^\circ_{\mathrm{Tr}}$ is a finite Λ_h-algebra. In particular $\mathbb{I}^\circ_{\mathrm{Tr}}$ is complete. By Corollary 3.5.11 we have

$$P_{\mathrm{char}}(\mathrm{Tr}(\rho_{\mathbb{I}^\circ})(\mathrm{Frob}_\ell)) = \psi_\theta(P_{\min}(t^{(2)}_{\ell,2}; X)).$$

We deduce that

$$\mathbb{I}^\circ_{\mathrm{Tr}} = \Lambda_h[\{\mathrm{Tr}(\rho_\theta(g))\}_{g \in G_{\mathbb{Q}}}].$$

Since the traces of $\rho_{\mathbb{I}^\circ}$ belong to $\mathbb{I}^\circ_{\mathrm{Tr}}$, Theorem 3.5.3 provides us with a representation

$$\rho_\theta\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$$

that becomes isomorphic to $\rho_{\mathbb{I}^\circ}$ over $\mathbb{I}^\circ$.

We keep our usual notation for the reduction modulo an ideal $\mathfrak{P}$ of $\mathbb{I}^\circ_{\mathrm{Tr}}$.

Definition 4.1.16. *We say that a prime $\mathfrak{P}$ of $\mathbb{I}^\circ_{\mathrm{Tr}}$ is classical if it lies under a classical prime of $\mathbb{I}^\circ$.*

Remark 4.1.17.

(1) By Remark 4.1.12 the set of classical primes of $\mathbb{I}^\circ$ is Zariski-dense. Since the map $\mathbb{I}^\circ_{\mathrm{Tr}} \to \mathbb{I}^\circ$ is injective, the set of classical primes of $\mathbb{I}^\circ_{\mathrm{Tr}}$ is also Zariski-dense.

(2) Let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ be two classical primes of $\mathbb{I}^\circ$ lying over the same prime $\mathfrak{P}_{\mathrm{Tr}}$ of $\mathbb{I}^\circ_{\mathrm{Tr}}$. Let $\rho_{\mathfrak{P}} = \mathrm{ev}_{\mathfrak{P}} \circ \rho_\theta\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})$. Then $\rho_{\mathfrak{P}}$ becomes isomorphic to the reductions of $\rho_{\mathbb{I}^\circ}$ modulo $\mathfrak{P}_1$ and $\mathfrak{P}_2$ over $\mathbb{I}^\circ/\mathfrak{P}_1$ and $\mathbb{I}^\circ/\mathfrak{P}_2$, respectively. For this reason we will say that $\rho_{\mathfrak{P}}$ is the representation associated with $\mathfrak{P}'$ for every prime $\mathfrak{P}'$ of $\mathbb{I}^\circ$ lying over $\mathfrak{P}$.

Thanks to the following lemma we can attach to θ a *symplectic* representation. The argument here is similar to that in [**GT05**, Lemma 4.3.3] and [**Pil12**, Proposition 6.4].

Lemma 4.1.18. *There exists a non-degenerate symplectic bilinear form on $(\mathbb{I}^\circ_{\mathrm{Tr}})^4$ that is preserved up to a scalar by the image of ρ_θ.*

Proof. Let S^{cl} be the set of classical primes of $\mathbb{I}^\circ_{\mathrm{Tr}}$. It is Zariski-dense in $\mathbb{I}^\circ_{\mathrm{Tr}}$ by Remark 4.1.17(1). Let $\mathfrak{P} \in S^{\mathrm{cl}}$. For every $\mathfrak{P} \in S^{\mathrm{cl}}$ the representation $\rho_{\mathfrak{P}}$ is symplectic, since it is the p-adic Galois representation attached to a classical GSp_4-eigenform. In particular $\rho_{\mathfrak{P}}$ is essentially self-dual: if $\rho_{\mathfrak{P}}^\vee$ denotes the dual representation of $\rho_{\mathfrak{P}}$, there exists a character $\nu_{\mathfrak{P}}\colon \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}) \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}$ and an isomorphism

$$\rho_{\mathfrak{P}} \cong \nu_{\mathfrak{P}} \otimes \rho_\theta^\vee. \tag{4.2}$$

We can write explicitly $\nu_{\mathfrak{P}} = \varepsilon\chi^{k_1+k_2-3}$, where ε is the central character of the eigenform corresponding to $\mathfrak{P}$ and $\chi\colon G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character. Note that the central character ε is independent of the chosen classical prime $\mathfrak{P}$ of $\mathbb{I}^\circ$.

Consider the representation $\rho_\theta^\vee\colon G_{\mathbb{Q}} \to \mathrm{GL}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ dual to ρ_θ. Let $\nu_\theta\colon G_{\mathbb{Q}} \to \mathbb{I}^\circ_{\mathrm{Tr}}$ be the character defined by $\nu_\theta = \varepsilon(1+T)^{\log(k_1+k_2-3)/\log(u)}$. When $\mathfrak{P}$ varies in S^{cl} the representations $\rho_{\mathfrak{P}}^\vee$ are interpolated by $\rho_\theta^\vee$ and the characters $\nu_{\mathfrak{P}}$ are interpolated by ν_θ. By Equation (4.2), for every $g \in G_{\mathbb{Q}}$ and every $\mathfrak{P} \in S^{\mathrm{cl}}$ the reductions $\mathrm{ev}_{\mathfrak{P}} \circ \mathrm{Tr}(\rho_\theta)(g)$ and $\mathrm{ev}_{\mathfrak{P}} \circ (\mathrm{Tr}(\rho_\theta^\vee) \otimes \nu_\theta(g))$ coincide. Since S^{cl} is Zariski-dense in $\mathbb{I}^\circ_{\mathrm{Tr}}$ we deduce that

$$\mathrm{Tr}(\rho_\theta) = \mathrm{Tr}(\rho_\theta^\vee \otimes \nu_\theta),$$

so the representations ρ_θ and $\rho_\theta^\vee \otimes \nu_\theta$ are isomorphic by Theorem 3.5.3. This means that the representation ρ_θ is essentially self-dual. Since $\overline{\rho}_\theta$ is irreducible by assumption, ρ_θ is also irreducible. Hence there exists a non-degenerate bilinear form $b\colon (\mathbb{I}^\circ_{\mathrm{Tr}})^4 \times (\mathbb{I}^\circ_{\mathrm{Tr}})^4 \to \mathbb{I}^\circ_{\mathrm{Tr}}$ that is preserved by $\mathrm{Im}\,\rho_\theta$ up to a scalar. If $\mathfrak{P} \in S^{\mathrm{cl}}$ the form b specializes modulo $\mathfrak{P}$ to a bilinear form $b_{\mathfrak{P}}\colon (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})^4 \times (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})^4 \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}$ that is preserved by $\mathrm{Im}\,\rho_{\mathfrak{P}}$ up to a scalar. Since $\rho_{\mathfrak{P}}$ is irreducible the form $b_{\mathfrak{P}}$ is non-degenerate. We know that $\rho_{\mathfrak{P}}$ is symplectic since it is the p-adic Galois representation associated with a classical GSp_4-eigenform. Hence $b_{\mathfrak{P}}$ is symplectic. We deduce that b is symplectic too. $\square$

Thanks to the lemma, up to replacing it by a conjugate representation, we can suppose that ρ_θ takes values in $\mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$. We call $\rho_\theta\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ the Galois representation associated with the family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ_{\mathrm{Tr}}$. In the following we will work mainly with this representation, so we denote it simply by ρ. We write $\mathbb{F}$ for the residue field of $\mathbb{I}^\circ_{\mathrm{Tr}}$ and $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{F})$ for the residual representation associated with ρ.

Remark 4.1.19. *There is an inclusion $\mathbb{F} \hookrightarrow \mathbb{F}_{\mathbb{T}_h}$ and the representations $\overline{\rho}$ and $\overline{\rho}_{\mathbb{T}_h}$ are isomorphic over $\mathbb{F}_{\mathbb{T}_h}$. In particular the representation $\overline{\rho}$ is absolutely irreducible.*

Remark 4.1.20. *Let f be a GSp_4-eigenform appearing in the family θ. Let ε_f be the central character, (k_1,k_2) the weight and $\psi_f\colon \mathcal{H}_2^M \to \overline{\mathbb{Q}}_p$ the system of Hecke eigenvalues of f. Let $\rho_{f,p}$ be the p-adic Galois representation attached to f and let ℓ be a prime not dividing Mp. Then*

$$\det \rho_{f,p}(\mathrm{Frob}_\ell) = \ell^6\psi_f(T^{(2)}_{\ell,0}) = \varepsilon_f(\ell)\chi(\ell)^{2(k_1+k_2-3)}.$$

The determinant of $\rho(\mathrm{Frob}_\ell)$ interpolates the determinants of $\rho_{f,p}(\mathrm{Frob}_\ell)$ when f varies over the forms corresponding to the classical primes of the family. Note that ε_f is independent of the choice of the form f in the family. Since the classical primes are Zariski-dense in $\mathbb{I}^\circ_{\mathrm{Tr}}$ the interpolation is unique and coincides with

$$\det \rho(\mathrm{Frob}_\ell) = \ell^6\psi_2(T^{(2)}_{\ell,0}) = \varepsilon(\ell)(u^{-6}(1+T_1)(1+T_2))^{\log(\chi(\ell))/\log(u)} \in \Lambda_h,$$

where ε is the central character of the family. By density of the conjugates of the Frobenius elements in $G_{\mathbb{Q}}$, we deduce that

$$\det \rho(g) = \varepsilon(g)(u^{-6}(1+T_1)(1+T_2))^{2\log(\chi(g))/\log(u)} \in \Lambda_h$$

for every $g \in G_{\mathbb{Q}}$.

4.2. The congruence ideal of a finite slope family

Let $\theta\colon \mathbb{T}_h \twoheadrightarrow \mathbb{I}^\circ$ be a finite slope family and let $\rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ be the representation associated with θ in the previous section. Recall that $\overline{\rho}$ is absolutely irreducible by assumption. We make two more hypotheses on ρ, that will hold throughout the whole text:

($\mathbb{Z}_p$-regularity) ρ is $\mathbb{Z}_p$-regular as in Definition 3.11.1;

(residual Sym^3 type) $\overline{\rho}$ is of residual Sym^3 type as in Definition 3.11.2.

In this section we define a "fortuitous congruence ideal" for the family θ. It is the ideal describing the intersection of the Sym^3-locus of $\mathcal{D}_2^M$ with the family θ. Recall that the Sym^3-locus is the zero locus of the ideal $\mathcal{I}_{\mathrm{Sym}^3}$ of $\mathcal{O}(\mathcal{D}_2^M)^\circ$ defined in Section 3.12 and that there is a map $r_{\mathcal{D}^{M,h}_{2,B_h}}\colon \mathcal{O}(\mathcal{D}_2^M)^\circ \to \mathbb{T}_h$ given by the restriction of analytic functions.

Definition 4.2.1. *The* fortuitous Sym^3-congruence ideal *for the family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ is the ideal of $\mathbb{I}^\circ$ defined by*

$$\mathfrak{c}_\theta = (\theta \circ r_{\mathcal{D}^{M,h}_{2,B_h}})(\mathcal{I}_{\mathrm{Sym}^3}) \cdot \mathbb{I}^\circ.$$

The reason for this terminology will be explained after the proof of Proposition 4.2.4. In most cases we will simply refer to $\mathfrak{c}_\theta$ as the "congruence ideal".

Remark 4.2.2. *As before we denote by I the irreducible component of $\mathcal{D}^{M,h}_{2,B_h}$ defined by θ. There is a map $r_I\colon \mathcal{O}(\mathcal{D}_2^M)^\circ \to \mathbb{I}^\circ$ given by the restriction of analytic functions on $\mathcal{D}_2^M$ to I. Clearly $r_I = \theta \circ r_{\mathcal{D}^{M,h}_{2,B_h}}$, so we can also define $\mathfrak{c}_\theta$ as $r_I(\mathcal{I}_{\mathrm{Sym}^3}) \cdot \mathbb{I}^\circ$.*

The following proposition describes the main properties of the congruence ideal. Let $\mathfrak{I}$ be an ideal of $\mathbb{I}^\circ$ and let $\mathfrak{I}_{\mathrm{Tr}} = \mathfrak{I} \cap \mathbb{I}^\circ_{\mathrm{Tr}}$. Let $\rho_\mathfrak{I}\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{I}_{\mathrm{Tr}})$ be the reduction of ρ modulo $\mathfrak{I}$. If $\theta_1\colon \mathbb{T}_{h,1} \to \mathbb{J}$ is a finite slope family of GL_2-eigenforms we denote by $\rho_{\theta_1}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{J})$ the associated Galois representation. For an ideal $\mathcal{J}$ of $\mathbb{J}$ we let $\rho_{\theta_1,\mathcal{J}}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{J}/\mathcal{J})$ be the reduction of ρ_{θ_1} modulo $\mathcal{J}$.

Proposition 4.2.3. *The following are equivalent:*

(i) $\mathfrak{I} \supset \mathfrak{c}_\theta$;

(ii) there exists a finite extension $\mathbb{I}'$ of $\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{I}_{\mathrm{Tr}}$ and a representation $\rho_{\mathfrak{I},1}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}')$ such that $\rho_\mathfrak{I} \cong \mathrm{Sym}^3\rho_{\mathfrak{I},1}$ over $\mathbb{I}'$;

(iii) there exists a finite slope family of GL_2-eigenforms $\theta_1\colon \mathbb{T}_{h/7,1} \to \mathbb{J}^\circ$, an ideal $\mathfrak{J}$ of $\mathbb{J}^\circ$ and a map $\phi\colon \mathbb{J}^\circ/\mathfrak{J} \to \mathbb{I}^\circ_{\mathrm{Tr}}$ such that $\rho_\mathfrak{I} \cong \phi \circ \mathrm{Sym}^3\rho_{\theta_1,\mathfrak{J}}$ over $\mathbb{I}^\circ_{\mathrm{Tr}}$.

Note that we did not specify the image in the weight space of the admissible subdomain of $\mathcal{D}_1^N$ associated with the family θ_1. It is the preimage in $\mathcal{W}_1^\circ$ of the disc $B_{2,h}$ via the immersion $\iota\colon \mathcal{W}_1^\circ \to \mathcal{W}_2^\circ$ defined in Section 3.8.2.

Proof. Since all the coefficient rings are local and all the residual representations are absolutely irreducible, we can apply the results of Section 3.12 by replacing the pseudocharacters everywhere with the associated representations, that exist by Theorem 3.5.5 and are defined over the ring of coefficients of the pseudocharacter by Theorem 3.5.3 (see the argument in the beginning of Section 4.1.4).

Now the equivalence (i) $\iff$ (ii) follows from Proposition 3.12.6(iv) applied to the rigid analytic variety $\mathcal{V}_2 = I$. The equivalence (ii) $\iff$ (iii) follows from Proposition 3.12.6(iii) by checking that the slopes satisfy the required inequality: this is a consequence of Corollary 3.4.9 and Remark 3.4.11. $\square$

Proposition 4.2.4. *The ideal $\mathfrak{c}_\theta$ is non-zero.*

Proof. Suppose by contradiction that $\mathfrak{c}_\theta = 0$. By Remark 4.2.2 $\mathfrak{c}_\theta = r_I(\mathcal{I}_{\mathrm{Sym}^3}) \cdot \mathbb{I}^\circ$, so we must have $r_I(\mathcal{I}_{\mathrm{Sym}^3}) = 0$. This means that the 2-dimensional rigid analytic variety I is contained in the zero locus $\mathcal{D}^M_{2,\mathrm{Sym}^3}$ of $\mathcal{I}_{\mathrm{Sym}^3}$. Since $\mathcal{D}^M_{2,\mathrm{Sym}^3}$ is Zariski closed in $\mathcal{D}^M_2$ the Zariski closure of I is also contained in $\mathcal{D}^M_{2,\mathrm{Sym}^3}$. By Corollary 3.12.5 $\mathcal{D}^M_{2,\mathrm{Sym}^3}$ has no components of dimension 2, so we obtain a contradiction. □

The fortuitous Sym^3-congruence ideal is an analogue of the congruence ideal of Definition 2.2.12. There is an important difference between the situation studied here and in Chapter 2 and those treated in [**Hi15**] and [**HT15**]. In [**Hi15**] and [**HT15**] the congruence ideal describes the locus of intersection between a fixed "general" family (i.e. such that its specializations are not lifts of forms from a smaller group) and the "non-general" families. Such non-general families are obtained as the p-adic lift of families of overconvergent eigenforms for smaller groups (e.g. $\mathrm{GL}_{1/K}$ for an imaginary quadratic field K in the case of CM families of GL_2-eigenforms, as in [**Hi15**], and $\mathrm{GL}_{2/F}$ for a real quadratic field F in the case of "twisted Yoshida type" families of GSp_4-eigenforms, as in [**HT15**]). In our setting there are no non-general families: the overconvergent GSp_4-eigenforms that are lifts of overconvergent eigenforms for smaller groups must be of Sym^3 type by Theorems 3.11.6 and 3.10.30, and we know that the Sym^3-locus on the GSp_4-eigenvariety does not contain any two-dimensional irreducible component by Proposition 4.2.4. Hence the ideal $\mathfrak{c}_\theta$ measures the locus of points that are of Sym^3-type, without belonging to a two-dimensional family of Sym^3 type. For this reason we call it the "fortuitous" Sym^3-congruence ideal. This is a higher-dimensional analogue of the situation of Chapter 2, where we showed that the positive slope CM points do not form one-dimensional families but appear as isolated points on the irreducible components of the eigencurve (see Corollary 2.2.8).

Note that conditions (ii) and (iii) in Proposition 4.2.3 only depend on the ideal $\mathfrak{I}\cap\mathbb{I}^\circ_{\mathrm{Tr}}$, so we expect $\mathfrak{c}_\theta$ to be generated by elements of $\mathbb{I}^\circ_{\mathrm{Tr}}$. We prove this in the following.

Proposition 4.2.5. *Let $\mathfrak{c}_{\theta,\mathrm{Tr}}=\mathfrak{c}_\theta\cap\mathbb{I}^\circ_{\mathrm{Tr}}$. Then $\mathfrak{c}_\theta=\mathfrak{c}_{\theta,\mathrm{Tr}}\cdot\mathbb{I}^\circ$.*

Proof. By definition $\mathfrak{c}_{\theta,\mathrm{Tr}}=\theta\circ r_I(\mathcal{I}_{\mathrm{Sym}^3})\cdot\mathbb{I}^\circ$. By definition $\mathcal{I}_{\mathrm{Sym}^3}=\psi_2(\ker(\psi_1\circ\lambda^{Mp}))$, where the notations are as in diagram (3.12). Since $\ker(\psi_1\circ\lambda^{Mp})\subset\mathcal{H}_2^{Mp}$ we have

$$\theta\circ r_I(\mathcal{I}_{\mathrm{Sym}^3})=\theta\circ r_I\circ\psi_2(\ker(\psi_1\circ\lambda^{Mp}))\subset\theta\circ r_I\circ\psi_2(\mathcal{H}_2^{Mp}).$$

By the remarks of Section 4.1.4 the ring $\mathbb{I}^\circ_{\mathrm{Tr}}$ contains $\theta\circ r_I\circ\psi_2(\mathcal{H}_2^{Mp})$ in $\mathbb{I}^\circ$, so $\theta\circ r_I(\mathcal{I}_{\mathrm{Sym}^3})$ is a subset of $\mathbb{I}^\circ_{\mathrm{Tr}}$ and the ideal $\mathfrak{c}_{\theta,\mathrm{Tr}}=\theta\circ r_I(\mathcal{I}_{\mathrm{Sym}^3})\cdot\mathbb{I}^\circ_{\mathrm{Tr}}$ satisfies $\mathfrak{c}_\theta=\mathfrak{c}_{\theta,\mathrm{Tr}}\cdot\mathbb{I}^\circ$. □

Proposition 4.2.3 can be translated into a characterization of the ideal $\mathfrak{c}_{\theta,\mathrm{Tr}}$. For an ideal $\mathfrak{I}$ of $\mathbb{I}^\circ_{\mathrm{Tr}}$ let $\rho_{\mathfrak{I}}\colon G_{\mathbb{Q}}\to\mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{I})$ be the reduction of ρ modulo $\mathfrak{I}$.

Corollary 4.2.6. *Let $\mathfrak{I}$ be an ideal of $\mathbb{I}^\circ_{\mathrm{Tr}}$. The following are equivalent:*

(i) $\mathfrak{I}\supset\mathfrak{c}_{\theta,\mathrm{Tr}}$;
(ii) there exists a finite extension $\mathbb{I}'$ of $\mathbb{I}^\circ/\mathfrak{I}$ and a representation $\rho_{\mathfrak{I},1}\colon G_{\mathbb{Q}}\to\mathrm{GL}_2(\mathbb{I}')$ such that $\rho_{\mathfrak{I}}\cong\mathrm{Sym}^3\rho_{\mathfrak{I},1}$ over $\mathbb{I}'$;
(iii) there exists a finite slope family of GL_2-eigenforms $\theta_1\colon\mathbb{T}_{h/7,1}\to\mathbb{J}^\circ$, an ideal $\mathfrak{J}$ of $\mathbb{J}^\circ$ and a map $\phi\colon\mathbb{J}^\circ/\mathfrak{J}\to\mathbb{I}^\circ_{\mathrm{Tr}}$ such that $\rho_{\mathfrak{I}}\cong\phi\circ\mathrm{Sym}^3\rho_{\theta_1,\mathfrak{J}}$.

We use the results of Chapter 3 to obtain some information on the height of the prime divisors of $\mathfrak{c}_\theta$. Here $\iota\colon\mathcal{W}_1^\circ\to\mathcal{W}_2^\circ$ is the inclusion defined in Section 3.8.2. For a classical weight k in $\mathcal{W}_1^\circ$ we have $\iota(k)=(k+1,2k-1)$, with the obvious abuse of notation.

Proposition 4.2.7. *Suppose that there exists a non-CM classical point $x\in\mathcal{D}_1^N$ of weight k such that $\mathrm{sl}(x)\le h/7$ and $\iota(k)\in B_{2,h}$ and $k>h-4$. Then the ideal $\mathfrak{c}_\theta$ has a prime divisor of height 1.*

Proof. Let x be a point satisfying the assumptions of the proposition and let f be the corresponding classical GL_2-eigenform. Let Sym^3x be the point of $\mathcal{D}_2^M$ that corresponds to the form $(\mathrm{Sym}^3f)_1^{\mathrm{st}}$ defined in Corollary 3.4.8. Let $\xi\colon\mathcal{D}_1^{N,\mathcal{G}}\to\mathcal{D}_2^M$ be the map of rigid analytic spaces given by Definition 3.9.8. The image of an irreducible component J of $\mathcal{D}_1^{N,\mathcal{G}}$ containing

x is an irreducible component $\xi(J)$ of $\mathcal{D}_2^M$ that contains $\mathrm{Sym}^3 x$. By Corollary 3.4.9 we have $\mathrm{sl}(\mathrm{Sym}^3 x) \leq h$. Since $k+1 > h-3$ the weight map is étale at the point $\mathrm{Sym}^3 x$, so there exists only one finite slope family of GSp_4-eigenforms containing $\mathrm{Sym}^3 x$. This means that $\xi(J)$ intersects the admissible domain I in a one-dimensional subspace. The ideal of $\mathbb{I}^\circ = \mathcal{O}(I)^\circ$ consisting of elements that vanish on $\xi(J)$ is a height one ideal of I that divides the congruence ideal $\mathfrak{c}_\theta$. In particular $\mathfrak{c}_\theta$ admits a height one prime divisor. $\square$

4.3. The self-twists of a Galois representation

Given any ring R, we denote by $Q(R)$ its total ring of fractions and by R^{norm} its normalization. Now let R be an integral domain. For every homomorphism $\sigma\colon R \to R$ and every $\gamma \in \mathrm{GSp}_4(R)$ we define $\gamma^\sigma \in \mathrm{GSp}_4(R)$ by applying σ to each coefficient of the matrix γ. This way σ induces an automorphism $[\cdot]^\sigma\colon G(R) \to G(R)$ for every algebraic subgroup $G \subset \mathrm{GSp}_4$ defined over R. For such a G and any representation $\rho\colon G_\mathbb{Q} \to G(R)$, we define a representation $\rho^\sigma\colon G_\mathbb{Q} \to G(R)$ by setting $\rho^\sigma(g) = (\rho(g))^\sigma$ for every $g \in G_\mathbb{Q}$.

Let S be a subring of R. We say that a homomorphism $\sigma\colon R \to R$ is a homomorphism of R over S if the restriction of σ to S is the identity. The following definition is inspired by [**Ri85**, Section 3] and [**Lang16**, Definition 2.1].

Definition 4.3.1. *Let $\rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(R)$ be a representation. We call self-twist for ρ over S an automorphism σ of R over S such that there is a finite order character $\eta_\sigma\colon G_\mathbb{Q} \to R^\times$ and an isomorphism of representations over R:*

$$\rho^\sigma \cong \eta_\sigma \otimes \rho. \tag{4.3}$$

We list some basic facts about self-twists.

Proposition 4.3.2. *Let $\rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(R)$ be a representation.*

(1) The self-twists for ρ over S form a group.
(2) If R is finite over S then the group of self-twists for ρ over S is finite.
(3) Suppose that the identity of R is not a self-twist for ρ over S. Then for any self-twist σ the character η_σ satisfying the equivalence (4.3) *is uniquely determined.*
(4) Under the same hypotheses as part (3), the association $\sigma \mapsto \eta_\sigma$ defines a cocycle on the group of self-twist with values in $R^\times$.

Proof. (1) Let τ, τ' be two self-twists for ρ over S and let η_τ, $\eta_{\tau'}$ be characters satisfying Equation (4.3) for $\sigma = \tau$ and $\sigma = \tau'$, respectively. Then there are equivalences $\rho^{\tau\tau'} = (\rho^\tau)^{\tau'} \cong (\eta_\tau\rho)^{\tau'} \cong \eta_\tau^{\tau'}\rho^{\tau'} \cong \eta_\tau^{\tau'}\eta_{\tau'}\rho$. In particular $\tau\tau'$ is a self-twist with associated finite order character $\eta_\tau^{\tau'}\eta_{\tau'}$.

(2) Every self-twist can be extended to an automorphism of $Q(R)$ fixing $Q(S)$. Since R is finite over S, $Q(R)$ is finite over $Q(S)$. In particular there exists only a finite number of distinct automorphisms of $Q(R)$ over $Q(S)$, so the number of distinct self-twists is also finite.

(3) Let σ be a self-twist. Suppose that there exist two finite order characters η_σ and η'_σ satisfying $\rho^\sigma \cong \eta_\sigma \otimes \rho \cong \eta'_\sigma \otimes \rho$. From the second equivalence we deduce that $\rho \cong \eta_\sigma^{-1} \otimes \eta'_\sigma \otimes \rho$, so the identity is a self-twist with associated finite order character $\eta_\sigma^{-1}\eta'_\sigma$. This contradicts our assumption.

(4) Let τ and τ' be two self-twists and let η_τ, $\eta_{\tau'}$, $\eta_{\tau\tau'}$ be characters satisfying Equation (4.3) for $\sigma = \tau$, $\sigma = \tau'$ and $\sigma = \tau\tau'$ respectively. By part (3) these three characters are uniquely determined. By the calculation of part (1) the character $\eta_\tau^{\tau'}\eta_{\tau'}$ satisfies Equation (4.3) for $\sigma = \tau\tau'$, so we must have $\eta_{\tau\tau'} = \eta_\tau^{\tau'}\eta_{\tau'}$. $\square$

Let $\Gamma_{\rho,S}$ denote the group of self-twists for the representation ρ over S. Let $S[\mathrm{Tr}\mathrm{Ad}\,\rho]$ denote the ring generated over S by the set $\{\mathrm{Tr}(\mathrm{Ad}\,(\rho)(g))\}_{g \in G_\mathbb{Q}}$.

Proposition 4.3.3. *There is an inclusion* $S[\mathrm{TrAd}\,\rho] \subset R^{\Gamma_{\rho,S}}$.

Proof. Let $\sigma \in \Gamma_{\rho,S}$. By definition of self-twist there exists a character $\eta_\sigma\colon G_\mathbb{Q} \to R^\times$ and an isomorphism $\rho^\sigma \cong \eta_\sigma \otimes \rho$. Passing to the adjoint representations we obtain an isomorphism $\mathrm{Ad}\,\rho^\sigma \cong \mathrm{Ad}\,\rho$. The traces of the representations on the two sides must coincide, so we can write $(\mathrm{Tr}(\mathrm{Ad}\,\rho)(g))^\sigma = \mathrm{Tr}(\mathrm{Ad}\,\rho^\sigma(g)) = \mathrm{Tr}(\mathrm{Ad}\,\rho(g))$ for every $g \in G_\mathbb{Q}$. Hence σ leaves $\mathrm{Tr}(\mathrm{Ad}\,\rho(g))$ fixed for every $g \in G_\mathbb{Q}$. By definition σ leaves S fixed, so it also leaves $S[\mathrm{TrAd}\,\rho]$ fixed. Since this holds for every $\sigma \in \Gamma_{\rho,S}$ we conclude that $S[\mathrm{TrAd}\,\rho]$ is fixed by $\Gamma_{\rho,S}$. $\square$

Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a family of GSp_4-eigenforms as defined in Section 4.1.2. Let $\rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ be the Galois representation associated with θ. Recall that $\mathbb{I}^\circ_{\mathrm{Tr}}$ is generated over Λ_h by the traces of ρ. We always work under the assumption that $\overline{\rho}\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{F})$ is absolutely irreducible.

Let Γ be the group of self-twists for ρ over Λ_h. We omit the reference to Λ_h from now on and we just speak of the self-twists for ρ.

Definition 4.3.4. *Let* $\mathbb{I}^\circ_0 = (\mathbb{I}^\circ_{\mathrm{Tr}})^\Gamma$ *be the subring of* $\mathbb{I}^\circ_{\mathrm{Tr}}$ *consisting of the elements fixed by every* $\sigma \in \Gamma$.

Lemma 4.3.5. *There is a tower of finite ring extensions*

$$\Lambda_h \subset \mathbb{I}^\circ_0 \subset \mathbb{I}^\circ_{\mathrm{Tr}} \subset \mathbb{I}^\circ.$$

Proof. Since Γ is the group of self-twists for ρ over Λ_h we have $\Lambda_h \subset \mathbb{I}^\circ_0$. The other inclusions follow trivially from the definitions. Since $\mathbb{I}^\circ$ is finite over Λ_h, all of the extensions in the tower are finite. $\square$

We can study the order of Γ thanks to an argument similar to that in [**Lang16**, Proposition 7.1].

Lemma 4.3.6. *The only prime factors of* $\mathrm{card}(\Gamma)$ *are* 2 *and* 3.

Proof. Let ℓ be any prime not dividing Np. Consider the element

$$a_\ell = \frac{(\mathrm{Tr}\rho(\mathrm{Frob}_\ell))^4}{\det\rho(\mathrm{Frob}_\ell)} \tag{4.4}$$

of $\mathbb{I}^\circ_{\mathrm{Tr}}$. For every $\sigma \in \Gamma$ and every $g \in G_\mathbb{Q}$ Equation (4.3) gives $\mathrm{Tr}\rho^\sigma(g) = \eta(g)\mathrm{Tr}\rho(g)$ and $\det\rho^\sigma(g) = \eta(g)^4\det\rho(g)$. In particular $a_\ell^\sigma = a_\ell$ for every $\sigma \in \Gamma$, so $a_\ell \in \mathbb{I}^\circ_0$. By Remark 4.1.20 we have

$$\det\rho(\mathrm{Frob}_\ell) = \varepsilon(\ell)\chi(\ell)^{2(k_1+k_2-3)} \in \Lambda_h,$$

where ε is the central character of the family θ and $\chi\colon G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ denotes the cyclotomic character. In particular $\det\rho(\mathrm{Frob}_\ell) \in \mathbb{I}^\circ_0$.

Let

$$\mathbb{I}' = \mathbb{I}^\circ_0[a_\ell^{1/4}, \det\rho(\mathrm{Frob}_\ell)^{1/4}, \mu_4].$$

It is a Galois extension of $\mathbb{I}^\circ_0$. Equation (4.4) gives an inclusion $\mathbb{I}^\circ_{\mathrm{Tr}} \subset \mathbb{I}'$, hence an inclusion $\Gamma \subset \mathrm{Gal}(\mathbb{I}'/\mathbb{I}^\circ_0)$. Since $\mathbb{I}'$ is obtained from $\mathbb{I}^\circ_0$ by adding some fourth roots, the order of an element of $\mathrm{Gal}(\mathbb{I}'/\mathbb{I}^\circ_0)$ cannot have prime divisors greater than 3. This concludes the proof. $\square$

Later we will construct from ρ a representation with values in $\mathrm{GSp}_4(\mathbb{I}^\circ_0)$. One of our main goals is to prove, for the image of such a representation, a fullness result analogous to Theorem 2.5.2.

4.4. Lifting self-twists

This section is largely inspired by [**Lang16**, Section 3]. Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a family, $\rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ be the associated Galois representation and Γ be the group of self-twists for ρ over Λ_h. Let $P_{\underline{k}} \subset \Lambda_h$ be any non-critical arithmetic prime, as in Definition 4.1.13. The representation ρ reduces modulo $P_{\underline{k}}\mathbb{I}^\circ_{\mathrm{Tr}}$ to a representation $\rho_{P_{\underline{k}}}\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}}/P_{\underline{k}}\mathbb{I}^\circ_{\mathrm{Tr}})$. Let $\widetilde{\sigma} \in \Gamma$ and let $\widetilde{\eta}\colon G_\mathbb{Q} \to (\mathbb{I}^\circ_{\mathrm{Tr}})^\times$ be the character associated with $\widetilde{\sigma}$ (we will use the notations σ and η for different objects). The automorphism $\widetilde{\sigma}$ fixes Λ_h by assumption, so it induces via reduction modulo $P_{\underline{k}}\mathbb{I}^\circ_{\mathrm{Tr}}$ a ring automorphism $\widetilde{\sigma}_{P_{\underline{k}}}$ of $\mathbb{I}^\circ_{\mathrm{Tr}}/P_{\underline{k}}\mathbb{I}^\circ_{\mathrm{Tr}}$. The character $\widetilde{\eta}\colon G_\mathbb{Q} \to \mathbb{I}^\circ_{\mathrm{Tr}}$ induces modulo $P_{\underline{k}}$ a character $\widetilde{\eta}_{P_{\underline{k}}}\colon G_\mathbb{Q} \to (\mathbb{I}^\circ_{\mathrm{Tr}}/P_{\underline{k}}\mathbb{I}^\circ_{\mathrm{Tr}})^\times$, and the isomorphism $\rho^{\widetilde{\sigma}} \cong \widetilde{\eta} \otimes \rho$ over $\mathbb{I}^\circ_{\mathrm{Tr}}$ gives an isomorphism of representations over $\mathbb{I}^\circ_{\mathrm{Tr}}/P_{\underline{k}}\mathbb{I}^\circ_{\mathrm{Tr}}$:

$$\rho_{P_{\underline{k}}}^{\widetilde{\sigma}_{P_{\underline{k}}}} \cong \widetilde{\eta}_{P_{\underline{k}}} \otimes \rho_{P_{\underline{k}}}. \tag{4.5}$$

Since $P_{\underline{k}}$ is non-critical $\mathbb{I}^\circ$ is étale over Λ_h at $P_{\underline{k}}$, hence $\mathbb{I}^\circ_{\mathrm{Tr}}$ is also étale over Λ_h at $P_{\underline{k}}$. In particular $P_{\underline{k}}$ can be decomposed as a product of distinct primes in $\mathbb{I}^\circ_{\mathrm{Tr}}$; denote them by $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_d$. Since $\widetilde{\sigma}_{P_{\underline{k}}}$ is an automorphism of $\mathbb{I}^\circ_{\mathrm{Tr}}/P_{\underline{k}}\mathbb{I}^\circ_{\mathrm{Tr}} \cong \prod_{i=1}^d \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i$, there is a permutation s of the set $\{1, 2, \ldots, d\}$ and an isomorphisms $\widetilde{\sigma}_{\mathfrak{P}_i}\colon \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_{s(i)}$ for $i = 1, 2, \ldots, d$ such that $\widetilde{\sigma}|_{\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i}$ factors through $\widetilde{\sigma}_{\mathfrak{P}_i}$. The character $\widetilde{\eta}_{\widetilde{\sigma}_{P_{\underline{k}}}}$ can be written as a product $\prod_{i=1}^d \widetilde{\eta}_{\mathfrak{P}_i}$ for some characters $\widetilde{\eta}_{\mathfrak{P}_i}\colon G_\mathbb{Q} \to (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i)^\times$. From the equivalence (4.5) we deduce that

$$\rho_{\mathfrak{P}_i}^{\widetilde{\sigma}_{\mathfrak{P}_i}} \cong \widetilde{\eta}_{\mathfrak{P}_{s(i)}} \otimes \rho_{\mathfrak{P}_{s(i)}}.$$

The goal of this section is to prove that, if we are given, for a single value of i, data $s(i)$, $\widetilde{\sigma}_{\mathfrak{P}_i}$ and $\widetilde{\eta}_{\mathfrak{P}_i}$ satisfying the isomorphism above for a single value of i, there exists an element of Γ giving rise to $\widetilde{\sigma}_{\mathfrak{P}_i}$ and $\widetilde{\eta}_{\mathfrak{P}_{s_i}}$ via reduction modulo $P_{\underline{k}}$. We state this precisely in the proposition below, which is an analogue of [**Lang16**, Theorem 3.1]. The notations are those of the discussion above.

Proposition 4.4.1. *Let $i, j \in \{1, 2, \ldots, d\}$. Let $\sigma\colon \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$ be a ring isomorphism and $\eta_\sigma\colon G_\mathbb{Q} \to (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j)^\times$ be a character satisfying*

$$\rho_{\mathfrak{P}_i}^{\sigma} \cong \eta_\sigma \otimes \rho_{\mathfrak{P}_j}. \tag{4.6}$$

Then there exists $\widetilde{\sigma} \in \Gamma$ with associated character $\widetilde{\eta}\colon G_\mathbb{Q} \to (\mathbb{I}^\circ_{\mathrm{Tr}})^\times$ such that, via the construction of the previous paragraph, $s(i) = j$, $\widetilde{\sigma}_{\mathfrak{P}_i} = \sigma$ and $\widetilde{\eta}_{\mathfrak{P}_j} = \eta_\sigma$.

We will apply Proposition 4.4.1 in the proofs of two key results, Propositions 4.6.1 and 4.9.8. Note that in the statement i and j are not necessarily distinct. We proceed to prove the proposition in a way similar to Lang's, taking care of some complications that arise when adapting her work to the group GSp_4. The strategy is the following:

(1) we lift σ to an automorphism Σ of a deformation ring for $\overline{\rho}$;
(2) we show that Σ descends to a self-twist for ρ.

Before proving Proposition 4.4.1 we give a corollary. Keep the notations introduced above and let $\mathfrak{P} \in \{\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_d\}$. Let $\rho_\mathfrak{P}\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})$ be the reduction of ρ modulo $\mathfrak{P}$ and let $\Gamma_\mathfrak{P}$ be the group of self-twists for $\rho_\mathfrak{P}$ over $\mathbb{Z}_p$. Let $\Gamma(\mathfrak{P}) = \{\sigma \in \Gamma \mid \sigma(\mathfrak{P}) = \mathfrak{P}\}$; it is a subgroup of Γ. Let $\widetilde{\sigma} \in \Gamma$ and let $\widetilde{\eta}\colon G_\mathbb{Q} \to (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})^\times$ be the finite order character associated with $\widetilde{\sigma}$. Via reduction modulo $\mathfrak{P}$, $\widetilde{\sigma}$ and $\widetilde{\eta}$ induce a ring automorphism $\widetilde{\sigma}_\mathfrak{P}$ of $\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}$ and a finite order character $\widetilde{\eta}_\mathfrak{P}\colon G_\mathbb{Q} \to (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})^\times$ satisfying $\rho_{\mathfrak{P}_i}^{\sigma_\mathfrak{P}} \cong \eta_{\sigma_\mathfrak{P}} \otimes \rho_\mathfrak{P}$. Hence $\widetilde{\sigma}_\mathfrak{P}$ is an element of $\Gamma_\mathfrak{P}$. The map $\Gamma(\mathfrak{P}) \to \Gamma_\mathfrak{P}$ defined by $\widetilde{\sigma} \mapsto \widetilde{\sigma}_\mathfrak{P}$ is clearly a morphism of groups.

Corollary 4.4.2. *The morphism $\Gamma(\mathfrak{P}) \to \Gamma_\mathfrak{P}$ is surjective.*

Proof. This results from Proposition 4.4.1 by choosing $\mathfrak{P}_i = \mathfrak{P}_j = \mathfrak{P}$. $\square$

4.4.1. Lifting self-twists to the deformation ring. This subsection follows closely [**Lang16**, Section 3.1]. Keep the notations from the beginning of the section. In particular let $i, j, \mathfrak{P}_i, \mathfrak{P}_j, \sigma$ and η_σ be as in Proposition 4.4.1. Let $\mathbb{Q}^{Np}$ denote the maximal extension of $\mathbb{Q}$ unramified outside Np and set $G_{\mathbb{Q}}^{Np} = \mathrm{Gal}(\mathbb{Q}^{Np}/\mathbb{Q})$. Then ρ factors via $G_{\mathbb{Q}}^{Np}$ by Theorem 3.1.1. In this subsection we consider ρ as a representation $G_{\mathbb{Q}}^{Np} \to \mathrm{GL}_4(\mathbb{I}_{\mathrm{Tr}}^\circ)$ via the natural inclusion $\mathrm{GSp}_4(\mathbb{I}_{\mathrm{Tr}}^\circ) \hookrightarrow \mathrm{GL}_4(\mathbb{I}_{\mathrm{Tr}}^\circ)$. Coherently, we consider $G_{\mathbb{Q}}^{Np}$ as the domain of all the representations induced by ρ and we take as their range the points of GL_4 on the corresponding coefficient ring. Note that the equivalence (4.6) implies that η_σ also factors via $G_{\mathbb{Q}}^{Np}$, so we see it as a character of this group. For simplicity we will write $\eta = \eta_\sigma$.

Recall that we write $\mathfrak{m}_{\mathbb{I}_{\mathrm{Tr}}^\circ}$ for the maximal ideal of $\mathbb{I}_{\mathrm{Tr}}^\circ$ and $\mathbb{F}$ for the residue field $\mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{m}_{\mathrm{Tr}^\circ}$. The residual representation $\overline{\rho}\colon G_{\mathbb{Q}}^{Np} \to \mathrm{GL}_4(\mathbb{F})$ is absolutely irreducible by assumption.

We briefly recall the definition of deformation ring for the classical representations we work with. Our reference is [**Ma89**]. Let W denote the ring of Witt vectors of $\mathbb{F}$. Let $\mathcal{C}$ denote the category of local, p-profinite W-algebras with residue field $\mathbb{F}$. Fix a positive integer n. Let $\overline{\pi}\colon G_{\mathbb{Q}}^{Np} \to \mathrm{GL}_n(\mathbb{F})$ be a representation. Consider a couple (R, τ) consisting of an object $R \in \mathcal{C}$ with maximal ideal $\mathfrak{m}_R$ and a representation $\tau\colon G_{\mathbb{Q}}^{Np} \to \mathrm{GL}_n(R)$. Denote by $\overline{\tau}$ the representation $G_{\mathbb{Q}}^{Np} \to \mathrm{GL}_n(\mathbb{F})$ obtained by reducing τ modulo $\mathfrak{m}_R$. We call (R, τ) a *universal couple* for $\overline{\pi}$ if:

(1) there is an equivalence $\overline{\tau} \cong \overline{\pi}$;

(2) for every $A \in \mathcal{C}$ and every representation $r\colon G_{\mathbb{Q}}^{Np} \to \mathrm{GL}_n(A)$ satisfying $\overline{r} \cong \overline{\pi}$, there exists a unique W-algebra homomorphism $\alpha(r)\colon R \to A$ such that $r \cong \alpha(r) \circ \tau$.

We call a representation r as in (2) a *deformation* of $\overline{\pi}$. If (R, τ) is a universal couple for $\overline{\pi}$, we call R the *universal deformation ring* and τ the *universal deformation* of $\overline{\pi}$. We will usually write such a couple as $(R_{\overline{\pi}}, \overline{\pi}^{\mathrm{univ}})$.

We define in the natural way the isomorphisms of two couples (R, τ) and (R', τ') (not necessarily universal). The following is [**Ma89**, Proposition 1].

Theorem 4.4.3. (Mazur) *If $\overline{\pi}$ is absolutely irreducible, there exists a universal couple $(R_\pi, \overline{\pi}^{\mathrm{univ}})$ for $\overline{\pi}$. Moreover $(R_{\overline{\pi}}, \overline{\pi}^{\mathrm{univ}})$ is unique up to isomorphism.*

Let $\mathcal{O}$ be the subring of $\mathbb{I}_{\mathrm{Tr}}^\circ$ generated over W by the image of η_σ. Since $W = W(\mathbb{F}) \subset \mathcal{O} \subset \mathbb{I}_{\mathrm{Tr}}^\circ$, the residue field of $\mathcal{O}$ is $\mathbb{F}$. For any commutative W-algebra A we set ${}^{\mathcal{O}}A = \mathcal{O} \otimes_W A$.

Since $\sigma\colon \mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{P}_i \to \mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{P}_j$ is an isomorphism, it maps the maximal ideal of $\mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{P}_i$ onto that of $\mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{P}_j$. In particular σ induces an automorphism $\overline{\sigma}$ of the residue field $\mathbb{F}$. Let $\overline{\eta}_\sigma\colon G_{\mathbb{Q}}^{Np} \to \mathbb{F}^\times$ be the reduction of η_σ modulo the maximal ideal of $\mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{P}_j$. By the properties of Witt vectors $\overline{\sigma}$ lifts to an automorphism $W(\overline{\sigma})$ of W. For every commutative W-algebra A we set $A^{\overline{\sigma}} = A \otimes_{W, W(\overline{\sigma})} W$, where the tensor product is taken through the map $W(\overline{\sigma})\colon W \to W$. We denote by $\iota(\overline{\sigma}, A)\colon A \to A^{\overline{\sigma}}$ the map defined by $\iota(\overline{\sigma}, A)(a) = a \otimes 1$ for every $a \in A$. It is an isomorphism of rings with inverse given by $\iota(\overline{\sigma}^{-1}, A)$.

The representations $\overline{\rho}$, $\overline{\rho}^{\overline{\sigma}}$ and $\overline{\eta}_\sigma \otimes \overline{\rho}$ are all absolutely irreducible. By Theorem 4.4.3 the universal couples for the three representations exist. We denote them respectively by $(R_{\overline{\rho}}, \overline{\rho}^{\mathrm{univ}})$, $(R_{\overline{\rho}^{\overline{\sigma}}}, (\overline{\rho}^{\overline{\sigma}})^{\mathrm{univ}})$ and $(R_{\overline{\eta}_\sigma \otimes \overline{\rho}}, (\overline{\eta}_\sigma \otimes \overline{\rho})^{\mathrm{univ}})$.

The equivalence (4.6) induces an equivalence $\overline{\rho}^{\overline{\sigma}} \cong \overline{\eta}_\sigma \otimes \overline{\rho}$. Then Theorem 4.4.3 gives an isomorphism $(R_{\overline{\rho}^{\overline{\sigma}}}, (\overline{\rho}^{\overline{\sigma}})^{\mathrm{univ}}) \cong (R_{\overline{\eta}_\sigma \otimes \overline{\rho}}, (\overline{\eta}_\sigma \otimes \overline{\rho})^{\mathrm{univ}})$. From now on we identify the two couples via the isomorphism above. The following lemma is [**Lang16**, Lemma 3.2] with GL_2 replaced by GL_4. The proof is unchanged, since it relies only on the properties of deformation rings that we recalled above.

Lemma 4.4.4. (cf. [**Lang16**, Lemma 3.2])

(1) There is a canonical isomorphism $\phi\colon R_{\overline{\rho}}^{\overline{\sigma}} \to R_{\overline{\rho}^{\overline{\sigma}}}$ of right W-algebras such that

$$(\overline{\rho}^{\overline{\sigma}})^{\mathrm{univ}} \cong \phi \circ \iota(\overline{\sigma}, R_{\overline{\rho}}) \circ \overline{\rho}^{\mathrm{univ}}$$

as representations $G_{\mathbb{Q}}^{Np} \to \mathrm{GL}_4(R_{\overline{\rho}}^{\overline{\sigma}})$.

(2) Consider $(\overline{\eta}_\sigma \otimes \overline{\rho})^{\mathrm{univ}}$ *as a representation with values in* $\mathrm{GL}_4({}^{\mathcal{O}}R_{\overline{\eta}\otimes\overline{\rho}})$ *via the natural map* $R_{\overline{\eta}\otimes\overline{\rho}} \to {}^{\mathcal{O}}R_{\overline{\eta}\otimes\overline{\rho}}$. *Consider* η_σ *as a character* $G_{\mathbb{Q}}^{Np} \to ({}^{\mathcal{O}}R_{\overline{\rho}})^\times$ *by letting it act on the left on the* $\mathcal{O}$*-coefficients. Then there is a natural* W*-algebra morphism* $\psi\colon R_{\overline{\eta}_\sigma\otimes\overline{\rho}} \to {}^{\mathcal{O}}R_{\overline{\rho}}$ *such that*

$$\eta \otimes \overline{\rho}^{\mathrm{univ}} \cong (1 \otimes \psi) \circ (\overline{\eta} \otimes \overline{\rho})^{\mathrm{univ}}$$

as representations $G_{\mathbb{Q}}^{Np} \to \mathrm{GL}_4({}^{\mathcal{O}}R_{\overline{\rho}})$.

Proof. Exactly as the proof of [**Lang16**, Lemma 3.2]. □

We use Lemma 4.4.4 to show that the automorphism $\overline{\sigma}$ of $\mathbb{F}$ can be lifted to an automorphism Σ of the W-algebra ${}^{\mathcal{O}}R_{\overline{\rho}}$. We need an intermediate step. Define an isomorphism $m(\overline{\sigma}, \mathbb{F})\colon \mathbb{F}^{\overline{\sigma}} \to \mathbb{F}$ by $m(\overline{\sigma}, \mathbb{F})(x \otimes y) = \overline{\sigma}(x)y$. Let $\phi\colon R^{\overline{\sigma}\overline{\rho}} \to R_{\overline{\rho}}^{\overline{\sigma}}$ and $\psi\colon R_{\overline{\eta}_\sigma\otimes\overline{\rho}} \to {}^{\mathcal{O}}R_{\overline{\rho}}$ be the ring morphisms given by Lemma 4.4.4. Define a ring morphism

$$m(\overline{\sigma}, {}^{\mathcal{O}}R_{\overline{\rho}})\colon {}^{\mathcal{O}}R_{\overline{\rho}}^{\overline{\sigma}} \to {}^{\mathcal{O}}R_{\overline{\rho}}$$

by $m(\overline{\sigma}, {}^{\mathcal{O}}R_{\overline{\rho}}) = (1 \otimes \psi) \circ (1 \otimes \phi)$.

Lemma 4.4.5. (cf. [**Lang16**, Lemma 3.3]) *The morphism* $m(\overline{\sigma}, {}^{\mathcal{O}}R_{\overline{\rho}})$ *is a lift of* $m(\overline{\sigma}, \mathbb{F})$.

Proof. We follow the proof of [**Lang16**, Lemma 3.3]). Since $\mathbb{F}$ is the residue field of $\mathcal{O}$, the tensor products with $\mathcal{O}$ become trivial after reduction by the maximal ideals of the various deformation rings. In particular the morphisms of residue fields induced by $1 \otimes \psi$ and $1 \otimes \phi$ coincide with those induced by ϕ and ψ, respectively. Denote these morphisms by $\overline{\phi}\colon \mathbb{F}\otimes_{\overline{\sigma}}\mathbb{F} \to \mathbb{F}$ and $\overline{\psi}\colon \mathbb{F} \to \mathbb{F}$. Then $m(\overline{\sigma}, {}^{\mathcal{O}}R_{\overline{\rho}})$ induces $\overline{\psi} \circ \overline{\phi}$ on the residue fields. It is sufficient to show that $\overline{\phi} = m(\overline{\sigma}, \mathbb{F})$ and ψ is the identity on $\mathbb{F}$.

By definition of ϕ there is an isomorphism $(\overline{\rho}^{\overline{\sigma}})^{\mathrm{univ}} \cong \phi \circ \iota(\overline{\sigma}, R_{\overline{\rho}}) \circ \overline{\rho}^{\mathrm{univ}}$. By reducing modulo the maximal ideal of $R_{\overline{\rho}}^{\overline{\sigma}}$ we obtain $\overline{\rho}^{\overline{\sigma}} \cong \overline{\phi} \circ \overline{\iota}(\overline{\sigma}, R_{\overline{\rho}}) \circ \overline{\rho}^{\mathrm{univ}}$. By the universal property of $\overline{\rho}^{\mathrm{univ}}$ we have $\overline{\rho}^{\overline{\sigma}} = \overline{\phi} \circ \overline{\iota}(\overline{\sigma}, \mathbb{F})$. Since $\overline{\sigma} = m(\overline{\sigma}, \mathbb{F}) \circ \iota(\overline{\sigma}, \mathbb{F})$ and $\iota(\overline{\sigma}, \mathbb{F})$ is an isomorphism, we conclude that $\overline{\phi} = m(\overline{\sigma}, \mathbb{F})$.

By definition of ψ there is an isomorphism $(1 \otimes \psi) \circ (\eta \otimes \overline{\rho})^{\mathrm{univ}} \cong \eta \otimes \overline{\rho}^{\mathrm{univ}}$. By reducing modulo the maximal ideal of ${}^{\mathcal{O}}R_{\overline{\rho}}$ we obtain $\overline{\psi} \circ (\overline{\eta} \otimes \overline{\rho}) \cong \overline{\eta} \otimes \overline{\rho}$. In particular $\overline{\psi}$ acts trivially on the traces of $\overline{\eta} \otimes \overline{\rho}$. These traces generate $\mathbb{F}$ since the traces of ρ generate $\mathbb{I}_{\mathrm{Tr}}^\circ$ over Λ_h. We conclude that ψ is trivial on $\mathbb{F}$. □

Define an automorphism $\Sigma\colon R_{\overline{\rho}} \to R_{\overline{\rho}}$ by $\Sigma = m(\overline{\sigma}, {}^{\mathcal{O}}R_{\overline{\rho}}) \circ \iota(\overline{\sigma}, R_{\overline{\rho}})$.

Corollary 4.4.6. *The morphism* Σ *induces* $\overline{\sigma}$ *upon reduction by the maximal ideal of* $R_{\overline{\rho}}$.

Proof. By Lemma 4.4.5 the morphism $m(\overline{\sigma}, {}^{\mathcal{O}}R_{\overline{\rho}})$ is a lift of $m(\overline{\sigma}, \mathbb{F})$. By definition $\Sigma = m(\overline{\sigma}, {}^{\mathcal{O}}R_{\overline{\rho}}) \circ \iota(\overline{\sigma}, R_{\overline{\rho}})$. Since $\overline{\sigma} = m(\overline{\sigma}, \mathbb{F}) \circ \iota(\overline{\sigma}, \mathbb{F})$, the morphism Σ is a lift of $\overline{\sigma}$. □

We prove some additional properties of Σ that we will need in the following. Let ${}^{\mathcal{O}}\Sigma = \Sigma = m(\overline{\sigma}, {}^{\mathcal{O}}R_{\overline{\rho}}) \circ (1 \otimes \iota(\overline{\sigma}, R_{\overline{\rho}}))\colon {}^{\mathcal{O}}R_{\overline{\rho}} \to R_{\overline{\rho}}$.

Proposition 4.4.7. (cf. [**Lang16**, Proposition 3.4])

(1) For all $w \in W$ *we have* ${}^{\mathcal{O}}\Sigma(1 \otimes w) = 1 \otimes W(\overline{\sigma})(w)$.
(2) For all $x \in \mathcal{O}$ *we have* ${}^{\mathcal{O}}\Sigma(x \otimes 1) = \sigma \otimes 1$.
(3) The automorphism $\overline{\sigma}$ *of* $\mathbb{F}$ *is trivial.*
(4) There is an isomorphism $\overline{\rho} \cong \overline{\eta} \otimes \overline{\rho}$.
(5) The automorphism Σ *of* $R_{\overline{\rho}}$ *satisfies* $\Sigma \circ \overline{\rho}^{\mathrm{univ}} = \eta \circ \overline{\rho}^{\mathrm{univ}}$.
(6) The automorphism Σ *of* $R_{\overline{\rho}}$ *is a lift of* σ.

Proof. The proof is similar to that of [**Lang16**, Proposition 3.4]. Part (1) follows from a direct calculation, by recalling that ϕ is a right W-algebra morphism and ψ is a W-algebra morphism. Part (2) follows immediately from the definition of ${}^{\mathcal{O}}\Sigma$.

We use (1) and (2) to deduce (3). Indeed, for every $x \in \mathcal{O}$,

$$w \otimes 1 = {}^{\mathcal{O}}\Sigma(w \otimes 1) = {}^{\mathcal{O}}\Sigma(1 \otimes w) = 1 \otimes W(\overline{\sigma})(w) = W(\overline{\sigma})(w) \otimes 1.$$

Hence the morphism $W(\overline{\sigma}) \otimes 1 \colon W \otimes_W R_{\overline{\rho}} \to W \otimes_W R_{\overline{\rho}}$ is trivial. Since the map $W \to R_{\overline{\rho}}$ is injective, we conclude that $\overline{\sigma}$ is trivial.

Part (4) is obtained by reducing $\rho^\sigma \cong \eta \otimes \rho$ modulo the maximal ideal of $\mathbb{I}^\circ_{\mathrm{Tr}}$ and applying part (3).

By taking determinants in the equivalence of (4) we deduce that $\overline{\eta}^4$ is trivial. In particular $\mathcal{O}$ is an unramified extension of W, which means that $\mathcal{O} = W$ since both rings have residue field $\mathbb{F}$. In particular we have equalities $R_{\overline{\rho}} = {}^{\mathcal{O}}R_{\overline{\rho}}$ and ${}^{\mathcal{O}}\Sigma = \Sigma = (1 \otimes \phi) \circ (1 \otimes \psi) \circ \iota(\overline{\sigma}, R_{\overline{\rho}}) = \psi$. By definition of ψ there is an isomorphism $\psi \circ (\overline{\eta} \otimes \overline{\rho})^{\mathrm{univ}} \cong \eta \otimes \overline{\rho}^{\mathrm{univ}}$. By part (4) and the equality $\Sigma = \psi$ we deduce that $\Sigma \circ \overline{\rho}^{\mathrm{univ}} \cong \eta \otimes \overline{\rho}^{\mathrm{univ}}$, hence (5).

Let $\alpha \colon R_{\overline{\rho}} \to \mathbb{I}^\circ_{\mathrm{Tr}}$ be the unique morphism of W-algebras satisfying $\rho \cong \alpha \circ \overline{\rho}^{\mathrm{univ}}$. Let $\pi_{\mathfrak{P}_i} \colon \mathbb{I}^\circ_{\mathrm{Tr}} \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i$ and $\pi_{\mathfrak{P}_j} \colon \mathbb{I}^\circ_{\mathrm{Tr}} \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$ be the natural projections. From the isomorphism $\rho^\sigma \cong \eta \otimes \rho$ and the previous remarks we deduce that

$$\sigma \circ \pi_{\mathfrak{P}_i} \circ \alpha \circ \overline{\rho}^{\mathrm{univ}} \cong \eta \otimes (\pi_{\mathfrak{P}_j} \circ \alpha \circ \overline{\rho}^{\mathrm{univ}}) \cong \pi_{\mathfrak{P}_j} \circ \alpha \circ (\eta \otimes \overline{\rho}^{\mathrm{univ}}) \cong \pi_{\mathfrak{P}_j} \circ \alpha \circ \Sigma \circ \overline{\rho}^{\mathrm{univ}}.$$

By the universal property of $\overline{\rho}^{\mathrm{univ}}$ we conclude that $\sigma \circ \pi_{\mathfrak{P}_i} \circ \alpha \cong \pi_{\mathfrak{P}_j} \circ \alpha \circ \Sigma$, which means that Σ is a lift of σ. ☐

Let A be an object of $\mathcal{C}$ and $\tau \colon G_{\mathbb{Q}} \to \mathrm{GL}_4(A)$ be a representation satisfying $\overline{\tau} = \overline{\rho}$. Let $\alpha_\tau \colon R_{\overline{\rho}} \to A$ be the morphism of local, pro-p W-algebras associated with τ by the universal property of $(R_{\overline{\rho}, \overline{\rho}^{\mathrm{univ}}})$. We define a representation $\tau^\Sigma \colon G_{\mathbb{Q}} \to \mathrm{GL}_4(A)$ by $\tau^\Sigma = \alpha_\tau \circ \Sigma \circ \overline{\rho}^{\mathrm{univ}}$. The following is a corollary of Proposition 4.4.7(4).

Corollary 4.4.8. *There is an isomorphism $\tau^\Sigma \cong \eta \otimes \tau$.*

Proof. By applying Proposition 4.4.7(5) we obtain

$$\tau^\Sigma = \alpha_\tau \circ \Sigma \circ \overline{\rho}^{\mathrm{univ}} \cong \alpha_\tau \circ (\eta \otimes \overline{\rho}^{\mathrm{univ}}) \cong \eta \otimes (\alpha_\tau \circ \overline{\rho}^{\mathrm{univ}}) = \eta \otimes \tau,$$

as desired. ☐

Recall that ρ is the Galois representation associated with the finite slope family θ. The goal of the next section is to show that the representation ρ^Σ is associated with a family of GSp_4-eigenforms of a suitable tame level and slope bounded by h. Thanks to Corollary 4.4.8 it is sufficient to show that the representation $\eta \otimes \rho$ is associated with such a family.

4.5. Twisting Galois representations by finite order characters

We show that the twist of a representation associated with a classical Siegel eigenform by a finite order Galois character is again the Galois representation associated with a classical Siegel eigenform, of the same weight but possibly of a different level. By interpolation we will deduce the analogous result for the representations associated with families of eigenforms.

Remark 4.5.1. *We regard a Galois character $\chi \colon G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_p^\times$ of finite order m as a Dirichlet character of conductor m and vice versa via the isomorphism $(\mathbb{Z}/m\mathbb{Z})^\times \cong \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, where ζ_m is an m-th root of unity. We will switch implicitly between the two points of view as convenient.*

Let f be a cuspidal GSp_4-eigenform of weight (k_1, k_2) and level $\Gamma_1(M)$ and let $\rho_{f,p} \colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$ be the p-adic Galois representation attached to f. Let $\eta \colon G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_p^\times$ be a character of finite order m_0 prime to p. Thanks to the following proposition we can give a notion of the twist of f by η.

PROPOSITION 4.5.2. *There exists a cuspidal Siegel eigenform $f \otimes \eta$ of weight (k_1, k_2) and level $\Gamma_1(\mathrm{lcm}(M, m_0)^2)$ such that the p-adic Galois representation associated with $f \otimes \eta$ is $\eta \otimes \rho_f$.*

Recall that we regard Galois characters of finite order as Dirichlet characters and vice versa when convenient.

Our proof relies on the calculations made by Andrianov in [**An09**, Section 1]. He only considers the case $k_1 = k_2$, but as we will remark his work adapts to forms of any classical weight. For $A \in \mathrm{M}_n(R)$ we write $A \geq 0$ if A is positive semi-definite and $A > 0$ if A is positive-definite. Recall that f, seen as a function on a variable Z in the Siegel upper half-plane

$$\mathbb{H}^n = \{X + iY \mid X, Y \in \mathrm{M}_n(\mathbb{R}) \text{ and } Y > 0\}$$

admits a Fourier expansion of the form $f(Z) = \sum_{A \in \mathbb{A}^n,\, A \geq 0} a_A q^A$, where $q = e^{2\pi i \mathrm{Tr}(AZ)}$ and

$$\mathbb{A}^n = \left\{ A = (a_{jk})_{j,k} \in \mathrm{M}_n(\frac{1}{2}\mathbb{Z}) \mid {}^t A = A \text{ and } a_{jj} \in \mathbb{Z} \text{ for } 1 \leq j \leq n \right\}.$$

The weight (k_1, k_2) action of $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_4(\mathbb{C})$ on f is defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} .f = (\mathrm{Sym}^{k_1 - k_2}(\mathrm{Std}) \otimes \det{}^{k_2}(\mathrm{Std}))(CZ + D) f\left(\frac{AZ + B}{CZ + D}\right), \tag{4.7}$$

where Std denotes the standard representation of GL_2. As in [**An09**], we define the twist by η of the expansion of f by

$$f \otimes \eta = \sum_{A \in \mathbb{A}^n,\, A \geq 0} \eta(\mathrm{Tr}(A)) a_A q^A.$$

Note that Andrianov considers a family of twists by η depending on an additional 2×2 matrix L, but we only need the case $L = \mathbb{1}_2$.

The notation $\widetilde{\Gamma}(m)$ in [**An09**] stands for the congruence subgroup

$$\widetilde{\Gamma}(m) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} | A \equiv D \equiv \mathbb{1}_2 \pmod{m}, C \equiv 0 \pmod{m^2} \right\}$$

where all blocks are two-dimensional. In particular we have inclusions

$$\Gamma_1(m^2) \subset \widetilde{\Gamma}(m) \subset \Gamma_1(m). \tag{4.8}$$

We recall some results of [**An09**]. For $A \in \mathrm{GSp}_4(\mathbb{C})$ and a congruence subgroup $\Gamma \subset \mathrm{GSp}_4(\mathbb{C})$, we let the double class $[\Gamma A \Gamma]$ act as a Hecke operator on forms of level Γ by the usual formulae. Recall that $\mu(A)$ denotes the similitude factor of M.

PROPOSITION 4.5.3. *Let η be a Dirichlet character of conductor m and f be a cuspidal form of weight (k, k) and level $\widetilde{\Gamma}(m)$.*

(1) The expansion $f \otimes \eta$ defines a cuspidal form of level $\widetilde{\Gamma}(m)$ [**An09**, Proposition 1.4]. *In particular $f \otimes \eta$ defines a form of level $\Gamma_1(m^2)$ via the first inclusion of* (4.8).

(2) If $A \in \mathrm{GSp}_4(\mathbb{C})$, $[\Gamma_1(m^2) A \Gamma_1(m^2)].(f \otimes \eta) = \eta(\mu(A))[\widetilde{\Gamma}(m) A \widetilde{\Gamma}(m)].f$ [**An09**, Theorem 2.3].

We remark that the same result holds for a form f of arbitrary classical weight (k_1, k_2), with the same proof. Indeed all the steps in the proofs of [**An09**, Proposition 1.4] and [**An09**, Theorem 2.3] only involve the action on f of matrices of the form $\begin{pmatrix} 1 & U \\ 0 & 1 \end{pmatrix}$ via formula (4.7). The action of such a matrix is clearly independent of the weight of f, hence all calculations are still true upon replacing the weight (k, k) action with the weight (k_1, k_2) action.

By the second inclusion of (4.8), a form of level $\Gamma_1(m)$ can be seen as a form of level $\widetilde{\Gamma}(m)$. We can thus rewrite Proposition 4.5.3 for a general weight and in the form that we will need.

PROPOSITION 4.5.4. *Let η be a Dirichlet character of conductor m and f be a cuspidal form of weight (k_1, k_2) and level $\Gamma_1(M)$. Let $M' = \mathrm{lcm}(m_0, N)^2$.*

(1) The expansion $f \otimes \eta$ defines a cuspidal form of level $\Gamma_1(M')$.

(2) If $A \in \mathrm{GSp}_4(\mathbb{C})$, $[\Gamma_1(m^2)A\Gamma_1(m^2)].(f \otimes \eta) = \eta(\mu(A))([\Gamma_1(m)A\Gamma_1(m)].f) \otimes \eta$.

We are now ready to prove Proposition 4.5.2.

Proof. We see the form f of level $\Gamma_1(M)$ as a form of level $\Gamma_1(\mathrm{lcm}(M, m_0))$ and the character η of conductor m as a character of conductor $\mathrm{lcm}(M, m_0)$. By applying Proposition 4.5.4(1) with $m = \mathrm{lcm}(M, m_0)$ we can construct a form $f \otimes \eta$ of level $\Gamma_1(\mathrm{lcm}(M, m_0)^2)$. Let $\rho_{f\otimes\eta,p} \colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$ be the p-adic Galois representation associated with $f \otimes \eta$. We have to show that $\rho_{f\otimes\eta,p} \cong \eta \otimes \rho_{f,p}$.

For every congruence subgroup $\Gamma \subset \mathrm{GSp}_4(\mathbb{C})$ and every prime ℓ, we denote by $T_{\ell,0}$, $T_{\ell,1}$ and $T_{\ell,2}$ the Hecke operators associated respectively with the double classes $[\Gamma \mathrm{diag}\,(\ell, \ell, \ell, \ell)\Gamma]$, $[\Gamma \mathrm{diag}\,(1, \ell, \ell, \ell^2)\Gamma]$ and $[\Gamma \mathrm{diag}\,(1, 1, \ell, \ell)\Gamma]$. By abusing notations we do not specify the congruence subgroup with respect to which we work, since this does not create confusion in the following. Now Proposition 4.5.4(2) gives, for every prime $\ell \nmid Mm_0$, the relations

$$\begin{aligned}
T_{\ell,0}(f \otimes \eta) &= \eta(\ell^2)T_{\ell,0}(f) \otimes \eta,\\
T_{\ell,1}(f \otimes \eta) &= \eta(\ell^2)T_{\ell,1}(f) \otimes \eta,\\
T_{\ell,2}(f \otimes \eta) &= \eta(\ell)T_{\ell,2}(f) \otimes \eta.
\end{aligned}$$

Since f is a Hecke eigenform we can write $T_{\ell,i}(f) = \lambda_{\ell,i}f$ for $i = 1, 2, 3$ and some $\lambda_{\ell,i} \in \mathbb{C}$. Then the previous equalities become

$$\begin{aligned}
T_{\ell,0}(f \otimes \eta) &= \eta(\ell)^2\lambda_{\ell,0}f \otimes \eta,\\
T_{\ell,1}(f \otimes \eta) &= \eta(\ell)^2\lambda_{\ell,1}T_{\ell,1}f \otimes \eta,\\
T_{\ell,2}(f \otimes \eta) &= \eta(\ell)\lambda_{\ell,2}T_{\ell,2}f \otimes \eta.
\end{aligned} \tag{4.9}$$

Recall from Proposition 3.1.1 that for $\ell \nmid Mm_0p$ we have

$$\det(1 - \rho_{f,p}(\mathrm{Frob}_\ell)X) = \chi_f(X^4 - T_{\ell,2}X^3 + ((T_{\ell,2})^2 - T_{\ell,1} - \ell^2T_{\ell,0})X^2 - \ell^3T_{\ell,2}T_{\ell,0}X + \ell^6(T_{\ell,0})^2)$$

where χ_f is the character of the Hecke algebra defining the system of eigenvalues of f. It follows that

$$\begin{aligned}
&\det(1 - (\eta \otimes \rho_{f,p})(\mathrm{Frob}_\ell)X) =\\
&= \chi_f(X^4 - \eta(\ell)T_{\ell,2}X^3 + \eta(\ell)^2((T_{\ell,2})^2 - T_{\ell,1} - \ell^2T_{\ell,0})X^2 - \eta(\ell)^3\ell^3T_{\ell,2}T_{\ell,0}X + \eta(\ell)^4\ell^6(T_{\ell,0})^2).
\end{aligned} \tag{4.10}$$

Again by Proposition 3.1.1 together with formulae (4.9) we can compute

$$\begin{aligned}
&\det(1 - \rho_{f\otimes\eta,p}(\mathrm{Frob}_\ell)X) =\\
&= \chi_{f\otimes\eta}(X^4 - T_{\ell,2}X^3 + ((T_{\ell,2})^2 - T_{\ell,1} - \ell^2T_{\ell,0})X^2 - \ell^3T_{\ell,2}T_{\ell,0}X + \ell^6(T_{\ell,0})^2) =\\
&= \chi_f(X^4 - \eta(\ell)T_{\ell,2}X^3 + ((\eta(\ell)T_{\ell,2})^2 - \eta(\ell)^2T_{\ell,1} - \ell^2\eta(\ell)^2T_{\ell,0})X^2 +\\
&\quad -\ell^3(\eta(\ell)T_{\ell,2})(\eta(\ell)^2T_{\ell,0})X + \ell^6(\eta(\ell)^2T_{\ell,0})^2).
\end{aligned}$$

Since this polynomial coincides with that in Equation (4.10) for every $\ell \nmid Mm_0p$, the representations $\eta \otimes \rho_{f,p}$ and $\rho_{f\otimes\eta,p}$ are equivalent. □

Under the hypotheses of the previous proposition we prove the following.

Corollary 4.5.5. *Let $M' = \mathrm{lcm}(m_0, M)^2$. Let x be a classical p-old point of $\mathcal{D}_2^M$ with weight (k_1, k_2), slope h and associated Galois representation ρ_x. Then there exists a classical p-old point x_η of $\mathcal{D}_2^{M'}$ with weight (k_1, k_2), slope h and associated Galois representation $\rho_{x_\eta} = \eta \otimes \rho_x$.*

Proof. Since x is p-old, it corresponds to the p-stabilization of a GSp_4-eigenform f of level M and weight (k_1, k_2). Let $f \otimes \eta$ be the eigenform of weight (k_1, k_2) and level M' given by Proposition 4.5.2. We show that it admits a p-stabilization of slope h.

We are working under the assumption that the conductor of η is prime to p, so Equations 4.9 hold for $\ell = p$. In particular

$$\begin{aligned}(4.11)\quad \chi_{f\otimes\eta}(P_{\min}(t_{p,2}^{(2)})) = \chi_f(X^4 - \eta(p)T_{p,2}X^3 + ((\eta(p)T_{p,2})^2 - \eta(p)^2T_{p,1} - p^2\eta(p)^2T_{p,0})X^2 + \\ -p^3(\eta(p)T_{p,2})(\eta(p)^2T_{p,0})X + p^6(\eta(p)^2T_{p,0})^2).\end{aligned}$$

Let $\{\alpha_i\}_{i=1,\dots,4}$ be the four roots of $\chi_f(P_{\min}(t_{p,2}^{(2)}))$. Then Equation (4.11) shows that the roots of $\chi_{f\otimes\eta}(P_{\min}(t_{p,2}^{(2)}))$ are $\{\eta(p)\alpha_i\}_{i=1,\dots,4}$.

Suppose that f is p-old. Recall that we identify $U_{p,2}^{(2)}$ with $t_{p,2}^{(2)}$ via the isomorphism $\iota_{I_{g,\ell}}^{T_g}$ of Section 1.2.4.2. By the discussion in the proof of Prop. 3.4.5 there are eight p-stabilizations of $f\otimes\eta$, one for each compatible choice of $U_{p,2}^{(2)}$ and $(U_{p,2}^{(2)})^{w_1}$ among the roots of $\chi_f(P_{\min}(t_{p,2}^{(2)}))$. Let f^{st} be a p-stabilization of f with slope $h = v_p(\chi_{f^{\mathrm{st}}}(U_p^{(2)}))$. Since $U_p^{(2)} = (U_{p,2}^{(2)})^2(U_{p,2}^{(2)})^{w_1}$, there are $i,j \in \{1,2,3,4\}$ such that $\chi_{f^{\mathrm{st}}}(U_p^{(2)}) = \alpha_i^2\alpha_j$. Then by the remark of the previous paragraph there exists a p-stabilization $(f\otimes\eta)^{\mathrm{st}}$ of $f\otimes\eta$ such that

$$\chi_{(f\otimes\eta)^{\mathrm{st}}}(U_p^{(2)}) = (\eta(p)\alpha_i)^2(\eta(p)\alpha_j) = \eta(p)^3\alpha_i^2\alpha_j.$$

In particular the slope of $(f\otimes\eta)^{\mathrm{st}}$ is

$$v_p(\chi_{(f\otimes\eta)^{\mathrm{st}}}(U_p^{(2)})) = v_p(\eta(p)^3)v_p(\alpha_i^2\alpha_j) = 3v_p(\eta(p)) + h.$$

Since p is prime to the conductor of η we have that $\eta(p)$ is a unit, hence the slope of $(f\otimes\eta)^{\mathrm{st}}$ is h.

The level of the eigenform $(f\otimes\eta)^{\mathrm{st}}$ is $\Gamma_1(M')\cap\Gamma_0(p)$, so it defines a point of the eigenvariety $\mathcal{D}_2^{M'}$, as desired. $\square$

Consider the family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ fixed in the beginning of the section. For every p-old classical point x of θ, let x_η be the point of the eigenvariety $\mathcal{D}_2^{M'}$ provided by Corollary 4.5.5. Let r_h' be a radius adapted to h for the eigenvariety $\mathcal{D}_2^{M'}$. Let Λ_h' be the genus 2, h-adapted Iwasawa algebra for $\mathcal{D}_2^{M'}$ and let $\mathbb{T}_h'$ be the genus 2, h-adapted Hecke algebra of level M'. Note that $r_h' \le r_h$, so there is a natural map $\iota_h\colon \Lambda_h \to \Lambda_h'$.

Lemma 4.5.6. *There exists a finite Λ_h'-algebra $\mathbb{J}^\circ$, a family $\theta'\colon \mathbb{T}_h' \to \mathbb{J}^\circ$ and an isomorphism $\alpha\colon \mathbb{I}_{\mathrm{Tr}}^\circ\widehat{\otimes}_{\Lambda_h}\Lambda_h' \to \mathbb{J}_{\mathrm{Tr}}^\circ$ such that the representation $\rho_{\theta'}\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{J}_{\mathrm{Tr}}^\circ)$ associated with θ' satisfies*

$$(4.12)\qquad \rho_{\theta'} \cong \eta\otimes\alpha\circ\rho_\theta.$$

Proof. Let S be the set of p-old classical points of θ. Let S' be the subset of S consisting of the points with weight in the disc $B(0,r_h')$. We see S' as a subset of the set of classical points of $\mathcal{D}_2^{M'}$ via the natural inclusion $\mathcal{D}_2^M \hookrightarrow \mathcal{D}_2^{M'}$. Thanks to the conditions on the weight and the slope we can identify S' with a set of classical points of $\mathbb{T}_h'$. Note that S' is infinite.

Let

$$S_\eta' = \{x_\eta \,|\, x\in S'\},$$

which is also contained in the set of classical points of $\mathcal{D}_2^{M'}$. For every $x\in S'$ the weight and slope of x_η coincide with the weight and slope of x. In particular S_η' can be identified with an infinite set of classical points of $\mathbb{T}_h'$. Since $\mathbb{T}_h'$ is a finite Λ_h'-algebra, the Zariski-closure of S_η' in $\mathbb{T}_h'$ contains an irreducible component of $\mathbb{T}_h'$. Such a component is a family defined by a finite Λ_h'-algebra $\mathbb{J}^\circ$ and a morphism $\theta'\colon \mathbb{T}_h' \to \mathbb{J}^\circ$.

Let $\rho_{\theta'}\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{J}_{\mathrm{Tr}}^\circ)$ be the Galois representation associated with θ'. Let $S_\eta^{\theta'}$ be the subset of S_η' consisting of the points that belong to θ'; it is Zariski-dense in $\mathbb{J}^\circ$ by definition of θ'. Let $S^{\theta'} = \{x\in S' \,|\, x_\eta \in S_\eta^{\theta'}\}$. For every $x\in S^{\theta'}$ let $\rho_{\theta,x}$ be the specialization of ρ_θ at x and let ρ_{θ',x_η} be the specialization of $\rho_{\theta'}$ at x_η. By the definition of the correspondence $x \mapsto x_\eta$ we have

$$\rho_{\theta',x_\eta} \cong \eta \otimes \rho_{\theta,x}$$

over $\overline{\mathbb{Q}}_p$ for every $x \in S^{\theta'}$. Hence the representation $\eta \otimes \rho_{\theta,x}$ coincides with $\iota_h \circ \rho_{\theta'}$ on the set $S_\eta^{\theta'}$. Since this set is Zariski-dense in $\mathbb{J}$, there exists an isomorphism $\alpha\colon \mathbb{I}^\circ_{\mathrm{Tr}}\widehat{\otimes}_{\Lambda_h}\Lambda'_h \to \mathbb{J}^\circ_{\mathrm{Tr}}$ such that $\rho_{\theta'} \cong \eta \otimes \alpha \circ \rho_\theta$, as desired. ☐

Remark 4.5.7. *With the notation of the proof of Lemma 4.5.6, Equation* (4.12) *implies that all points of the set S'_η belong to the family θ', because of the unicity of a point of $\mathcal{D}_2^{M'}$ given its associated Galois representation and slope.*

By combining Lemma 4.5.6 and Corollary 4.4.8 we obtain the following.

Corollary 4.5.8. *There exists a finite Λ'_h-algebra $\mathbb{J}^\circ$, a family $\theta'\colon \mathbb{T}'_h \to \mathbb{J}^\circ$ and an isomorphism $\alpha\colon \mathbb{I}^\circ_{\mathrm{Tr}}\widehat{\otimes}_{\Lambda_h}\Lambda'_h \to \mathbb{J}^\circ_{\mathrm{Tr}}$ such that the representation $\rho_{\theta'}\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{J}^\circ_{\mathrm{Tr}})$ associated with θ' satisfies*

$$\rho_{\theta'} \cong \alpha \circ \rho^\Sigma. \tag{4.13}$$

4.5.1. Descending to a self-twist of the family.

We show that the automorphism Σ of $R_{\overline{\rho}}$ defined in the previous subsection induces a self-twist for ρ. This will prove Proposition 4.4.1. Our argument is an analogue for GSp_4 of that in the end of the proof of [**Lang16**, Theorem 3.1]; it also appears in similar forms in [**Fi02**, Proposition 3.12] and [**DG12**, Proposition A.3]. Here the non-criticality of the prime $P_{\underline{k}}$ plays an important role.

Proof. (of Proposition 4.4.1 Let $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{F})$ be the residual representation associated with ρ. Let $R_{\overline{\rho}}$ be the universal deformation ring associated with $\overline{\rho}$ and let $\overline{\rho}^{\mathrm{univ}}$ be the corresponding universal deformation. By the universal property of $R_{\overline{\rho}}$ there exists a unique morphism of W-algebras $\alpha_I\colon R_{\overline{\rho}} \to \mathbb{I}^\circ_{\mathrm{Tr}}$ satisfying $\rho \cong \alpha_I \circ \overline{\rho}^{\mathrm{univ}}$.

Consider the morphism of W-algebras $\alpha_I^\Sigma = \alpha_I \circ \Sigma\colon R_{\overline{\rho}} \to \mathbb{I}^\circ_{\mathrm{Tr}}$. We show that there exists an automorphism $\widetilde{\sigma}\colon \mathbb{I}^\circ_{\mathrm{Tr}} \to \mathbb{I}^\circ_{\mathrm{Tr}}$ fitting in the following commutative diagram:

$$\begin{array}{ccc} R_{\overline{\rho}} & \xrightarrow{\alpha_I} & \mathbb{I}^\circ_{\mathrm{Tr}} \\ \big\downarrow{\scriptstyle\Sigma} & & \big\downarrow{\scriptstyle\widetilde{\sigma}} \\ R_{\overline{\rho}} & \xrightarrow{\alpha_I^\Sigma} & \mathbb{I}^\circ_{\mathrm{Tr}} \end{array} \tag{4.14}$$

We use the notations of the discussion preceding Lemma 4.5.6. Consider the morphism $\theta \otimes 1\colon \mathbb{T}_h\widehat{\otimes}_{\Lambda_h}\Lambda'_h \to \mathbb{I}^\circ\widehat{\otimes}_{\Lambda_h}\Lambda'_h$, where the completed tensor products are taken via the map $\iota_h\colon \Lambda_h \to \Lambda'_h$. For every Λ_h-algebra A we denote again by ι_h the natural map $A \to A\widehat{\otimes}_{\Lambda_h}\Lambda'_h$. The natural inclusion $\mathcal{D}_2^M \hookrightarrow \mathcal{D}_2^{M'}$ induces a surjection $s_h\colon \mathbb{T}'_h \to \mathbb{T}_h\widehat{\otimes}_{\Lambda_h}\Lambda'_h$. We define a family of tame level $\Gamma_1(M')$ and slope bounded by h by

$$\theta^{M'} = (\theta \otimes 1) \circ s_h\colon \mathbb{T}'_h \to \mathbb{I}^\circ\widehat{\otimes}_{\Lambda_h}\Lambda'_h.$$

The Galois representation associated with $\theta^{M'}$ is $\rho_{\theta^{M'}} = \iota_h \circ \rho\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}}\widehat{\otimes}_{\Lambda_h}\Lambda'_h)$. Let $\theta'\colon \mathbb{T}'_h \to \mathbb{J}^\circ$ be the family given by Corollary 4.5.8. We identify $\mathbb{I}^\circ_{\mathrm{Tr}}\widehat{\otimes}_{\Lambda_h}\Lambda'_h$ with $\mathbb{J}^\circ_{\mathrm{Tr}}$ via the isomorphism α given by the same corollary; in particular the Galois representation associated with θ' is $\rho_{\theta'} = \rho^\Sigma\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}}\widehat{\otimes}_{\Lambda_h}\Lambda'_h)$.

Recall that we are working under the assumptions of Proposition 4.4.1. In particular we are given two primes $\mathfrak{P}_i$ and $\mathfrak{P}_j$ of $\mathbb{I}^\circ_{\mathrm{Tr}}$, an isomorphism $\sigma\colon \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$ and a character $\eta_\sigma\colon G_{\mathbb{Q}} \to (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j)^\times$ such that $\rho^\sigma_{\mathfrak{P}_i} \cong \eta_\sigma \otimes \rho_{\mathfrak{P}_j}$. Let $\mathfrak{P}'_i$ be the image of $\mathfrak{P}_i$ via the map $\iota_h\colon \mathbb{I}^\circ_{\mathrm{Tr}}\widehat{\otimes}_{\Lambda_h}\Lambda'_h$. The specialization of $\rho_{\theta^{M'}}$ at $\mathfrak{P}'_i$ is $\rho_{\mathfrak{P}_i}$. Let f' be the eigenform corresponding to $\mathfrak{P}'_i$. By Remark 4.5.7 there is a point of the family θ' corresponding to the twist of f by η; let $\mathfrak{P}'_{i,\eta}$ be the prime of $\mathbb{I}^\circ_{\mathrm{Tr}}\widehat{\otimes}_{\Lambda_h}\Lambda'_h$ defining this point. The specialization of $\rho_{\theta'}$ at $\mathfrak{P}'_{i,\eta}$ is

$\eta \otimes \rho_{\mathfrak{P}_i}$, which is isomorphic to $\rho_{\mathfrak{P}_i}^{\sigma}$ by assumption. Let f'_η be the eigenform corresponding to the prime $\mathfrak{P}'_{i,\eta}$. The forms f' and f'_η have the same slope by Corollary 4.5.5 and their associated representations are obtained from one another via Galois conjugation (given by the isomorphism σ). Hence f' and f'_η define the same point of the eigenvariety $\mathcal{D}_2^{M'}$. Such a point belongs to both the families $\theta^{M'}$ and θ'. Since $\mathfrak{P}_{\underline{k}}$ is non-critical, $\mathbb{T}'_h$ is étale at every point lying over $P_{\underline{k}}$, so the families $\theta^{M'}$ and θ' must coincide. This means that there is an isomorphism

$$\widetilde{\sigma}'\colon \mathbb{I}^\circ_{\mathrm{Tr}}\widehat{\otimes}_{\Lambda_h}\Lambda'_h \to \mathbb{I}^\circ_{\mathrm{Tr}}\widehat{\otimes}_{\Lambda_h}\Lambda'_h$$

such that $\rho_{\theta'} = \widetilde{\sigma}' \circ \rho^{M'}$. Then $\widetilde{\sigma}'$ induces by restriction an isomorphism $\Lambda'_h[\mathrm{Tr}(\rho^{M'})] \to \Lambda'_h[\mathrm{Tr}(\rho_{\theta'})]$. Note that $\Lambda'_h[\mathrm{Tr}(\rho^{M'})] = \iota_h(\mathbb{I}^\circ_{\mathrm{Tr}})$ and

$$\begin{aligned}\Lambda'_h[\mathrm{Tr}(\rho_{\theta'})] &= \Lambda'_h[\mathrm{Tr}(\widetilde{\sigma}' \circ \rho^{M'})] = \widetilde{\sigma}'(\Lambda'_h[\mathrm{Tr}(\rho^{M'})]) = \\ &= \widetilde{\sigma}'(\Lambda'_h[\mathrm{Tr}(\iota_h \circ \rho)]) = \widetilde{\sigma}'(\iota_h(\Lambda_h[\mathrm{Tr}\rho])) = \widetilde{\sigma}'(\iota_h(\mathbb{I}^\circ_{\mathrm{Tr}})).\end{aligned}$$

In particular $\widetilde{\sigma}'$ induces by restriction an isomorphism $\iota_h(\mathbb{I}^\circ_{\mathrm{Tr}}) \to \iota_h(\mathbb{I}_\circ^{\mathrm{Tr}})$. Since ι_h is injective we can identify $\widetilde{\sigma}'$ with an isomorphism $\widetilde{\sigma}\colon \mathbb{I}^\circ_{\mathrm{Tr}} \to \mathbb{I}^\circ_{\mathrm{Tr}}$. By construction $\widetilde{\sigma}$ fits in diagram (4.14). $\square$

4.6. Rings of self-twists for representations attached to classical eigenforms

Let f be a classical GSp_4-eigenform, K be a completion of the field of coefficients of f at a place above p and $\rho_{f,p}\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathcal{O}_K)$ the p-adic Galois representation associated with f. Suppose that f satisfies the hypotheses of Theorem 3.11.3, i.e. $\overline{\rho}_{f,p}$ is of Sym^3 type but f is not the symmetric cube lift of a GL_2-eigenform. Let Γ_f be the group of self-twists for ρ over $\mathbb{Z}_p$ and let $\mathcal{O}_K^{\Gamma_f}$ be the subring of elements of $\mathcal{O}_K$ fixed by Γ_f. As in in Section 3.11 we define another subring of $\mathcal{O}_K$ by $\mathcal{O}_E = \mathbb{Z}_p[\mathrm{Tr}(\mathrm{Ad}\,\rho)]$. We prove that the two subrings of $\mathcal{O}_K$ we just defined are actually the same.

Proposition 4.6.1. *There is an equality $\mathcal{O}_K^{\Gamma_f} = \mathcal{O}_E$.*

Before proving the proposition we recall a theorem of O'Meara about isomorphisms of congruence subgroups. It is a generalization to symplectic groups of arbitrary genus of a result of Merzljakov for GL_2 [**Me73**, Theorem], cited in the proof of [**Lang16**, Proposition 5.3]. The notations of [**OM78**, Theorem 5.6.4-5] are as follows: $\mathfrak{o}$ is any integral domain and F is its quotient field, n is an even positive integer, V is an n-dimensional F-vector space with an alternating bilinear form, M is an $\mathfrak{o}$-module contained in a free $\mathfrak{o}$-submodule of V, $\mathrm{Sp}_n(V)$ and $\Gamma\mathrm{Sp}_n(V)$ are respectively the groups of symplectic isometries and similitudes for V, $\mathrm{RL}_n(V)$ is the group of scalar endomorphisms of V, $\mathfrak{a}$ is any ideal of $\mathfrak{o}$, $\mathrm{Sp}_n(M,\mathfrak{a})$ is the subgroup of $\mathrm{Sp}_n(V)$ consisting of elements σ satisfying $\sigma M = M$ and $(\sigma - 1)M \subset \mathfrak{a}M$. As usual let PSp_n and PGSp_n be the projective symplectic groups. Let $(\mathfrak{o}_1, F_1, M_1, n_1, V_1, \mathfrak{a}_1)$ be another choice of the above data.

Let $\sigma\colon F \to F_1$ be an isomorphism. We say that a map S of V into V_1 is σ-semilinear if it is additive and satisfies $S(\lambda v) = \sigma(\lambda)S(v)$ for every $v \in V$ and $\lambda \in F$.

In the following we choose $V = F^{2g}$, equipped with the bilinear alternating form defined by the matrix J_g of Section 1.1, and $M = \mathfrak{o}^{2g}$, so that $\mathrm{Sp}_n(M,\mathfrak{a})$ becomes the usual congruence subgroup of level $\mathfrak{a}$ of $\mathrm{Sp}_{2g}(\mathfrak{o})$. We choose $V_1 = F_1^{2g}$, again with the form defined by the matrix J_g, and $M_1 = \mathfrak{o}_1^{2g}$. We suppose that the characteristics of F and F_1 are different from 2. In this setting [**OM78**, Theorem 5.6.4] implies the following result for isomorphisms of projective congruence subgroups.

THEOREM 4.6.2. cf. [**OM78**, Theorem 5.6.4] *Let* Δ *and* Δ_1 *be subgroups of* $\mathrm{PGSp}_{2g}(F)$ *and* $\mathrm{PGSp}_{2g}(F_1)$, *respectively, satisfying*

$$\mathrm{PSp}_{2g}(\mathfrak{o},\mathfrak{a}) \subset \Delta$$

and

$$\mathrm{PSp}_{2g}(\mathfrak{o}_1,\mathfrak{a}_1) \subset \Delta_1.$$

Let $\Theta\colon \Delta \to \Delta_1$ *be an isomorphism of groups. Then there exists an isomorphism of fields* $\sigma\colon F \to F_1$ *and a bijective, symplectic,* σ*-semilinear map* $S\colon V \to V_1$ *satisfying*

$$\Theta x = SxS^{-1}$$

for every $x \in \Delta$.

REMARK 4.6.3. *Let* $\sigma\colon F \to F_1$ *be an isomorphism. Denote by* $x \mapsto x^\sigma$ *the isomorphism* $\mathrm{GSp}_{2g}(F) \to \mathrm{GSp}_{2g}(F_1)$ *obtained by applying* σ *to the matrix coefficients. For every bijective, symplectic,* σ*-semilinear map* $S\colon V \to V_1$ *there exists* $\gamma \in \mathrm{GSp}_{2g}(F_1)$ *such that* $SxS^{-1} = \gamma x^\sigma \gamma^{-1}$ *for every* $x \in \mathrm{GSp}_4(F)$.

Thanks to Remark 4.6.3 we can rewrite the theorem as follows.

COROLLARY 4.6.4. *Let* Δ *and* Δ_1 *be subgroups of* $\mathrm{PGSp}_{2g}(F)$ *and* $\mathrm{PGSp}_{2g}(F_1)$, *respectively, satisfying*

$$\mathrm{PSp}_{2g}(\mathfrak{o},\mathfrak{a}) \subset \Delta$$

and

$$\mathrm{PSp}_{2g}(\mathfrak{o}_1,\mathfrak{a}_1) \subset \Delta_1.$$

Let $\Theta\colon \Delta \to \Delta_1$ *be an isomorphism of groups. Then there exists an automorphism* σ *of* F *and an element* $\gamma \in \mathrm{PGSp}_{2g}(F)$ *satisfying*

$$\Theta x = \gamma x^\sigma \gamma^{-1}$$

for every $x \in \Delta$.

From Theorem 4.6.2 we deduce a result on isomorphisms of congruence subgroups of $\mathrm{Sp}_{2g}(F)$.

COROLLARY 4.6.5. [**OM78**, Theorem 5.6.5] *Let* Δ *and* Δ_1 *be two subgroups of* $\mathrm{GSp}_{2g}(F)$ *satisfying*

$$\mathrm{Sp}_{2g}(\mathfrak{o},\mathfrak{a}) \subset \Delta$$

and

$$\mathrm{Sp}_{2g}(\mathfrak{o},\mathfrak{a}_1) \subset \Delta_1.$$

Let $\Theta\colon \Delta \to \Delta_1$ *be an isomorphism of groups. Then there exists an automorphism* σ *of* F, *a character* $\chi\colon \Delta \to F^\times$ *and an element* $\gamma \in \mathrm{GSp}_{2g}(F)$ *satisfying*

$$\Theta x = \chi(x)\gamma x^\sigma \gamma^{-1}$$

for every $x \in \Delta$.

Before proving Proposition 4.6.1 we fix some notations. Let $\mathrm{End}(\mathfrak{sp}_4(K))$ be the K-vector space of K-linear maps $\mathfrak{sp}_4(K) \to \mathfrak{sp}_4(K)$ and let $\mathrm{GL}(\mathfrak{sp}_4(K))$ be the subgroup consisting of the bijective ones. Let $\mathrm{Aut}(\mathfrak{gsp}_4(K))$ be the subgroup of $\mathrm{GL}(\mathfrak{sp}_4(K))$ consisting of the Lie algebra automorphisms of $\mathfrak{sp}_4(K)$. Let π_{Ad} be the natural projection $\mathrm{GSp}_4(\mathcal{O}_K) \to \mathrm{PGSp}_4(\mathcal{O}_K)$ and let $\mathrm{Ad} : \mathrm{PGSp}_4(K) \hookrightarrow \mathrm{GL}(\mathfrak{sp}_4(K))$ be the injective group morphism given by the adjoint representation. By definition the image of Ad is the group of inner automorphisms of the Lie algebra $\mathfrak{sp}_4(K)$. The group of automorphisms of a the Lie algebra associated with a classical group is the semidirect product of the group of inner automorphisms with the group of outer automorphisms (i.e. the automorphisms of the associated Dynkin diagram). Since $\mathfrak{sp}_4$ admits no outer automorphisms, Ad induces an isomorphism of $\mathrm{PGSp}_4(K)$ onto $\mathrm{Aut}(\mathfrak{sp}_4(K))$. For simplicity we write $\rho = \rho_{f,p}$ in the following proof (but recall that in the other sections ρ is the Galois representation attached to a family).

PROOF. (of Proposition 4.6.1) The inclusion $\mathcal{O}_E \subset \mathcal{O}_K^{\Gamma_f}$ follows from Proposition 4.3.3. To prove that $\mathcal{O}_K^{\Gamma_f} \subset \mathcal{O}_E$ we need the following lemma.

LEMMA 4.6.6. *Let R be an integral domain and let R_1 and R_2 be two subrings of R. Suppose that every automorphism of R over R_1 leaves R_2 fixed. Then $R_2^{\mathrm{norm}} \subset R_1^{\mathrm{norm}}$.*

Note that $\mathcal{O}_K^{\Gamma_f}$ and $\mathcal{O}_E$ are normal since they are the rings of integers of finite extensions of $\mathbb{Q}_p$. Hence by Lemma 4.6.6 it is sufficient to show that an automorphism of $\mathcal{O}_K$ over $\mathcal{O}_E$ leaves $\mathcal{O}_K^{\Gamma_f}$ fixed. Consider such an automorphism σ. Since $\mathcal{O}_E$ is fixed by σ we have $(\mathrm{Tr}(\mathrm{Ad}\,\rho)(g))^\sigma = \mathrm{Tr}(\mathrm{Ad}\,\rho(g))$ for every $g \in G_\mathbb{Q}$, hence $\mathrm{Tr}(\mathrm{Ad}\,\rho^\sigma(g)) = \mathrm{Tr}(\mathrm{Ad}\,\rho(g))$. The equality of traces induces an isomorphism between the adjoint representations $\mathrm{Ad}\,\rho, \mathrm{Ad}\,\rho^\sigma \colon G_\mathbb{Q} \to \mathrm{GL}(\mathfrak{sp}_4)$:

$$\mathrm{Ad}\,\rho^\sigma \cong \mathrm{Ad}\,\rho.$$

This means that there exists $\phi \in \mathrm{GL}(\mathfrak{sp}_4(K))$ satisfying

$$\mathrm{Ad}\,\rho^\sigma = \phi \circ \mathrm{Ad}\,\rho \circ \phi^{-1}. \tag{4.15}$$

We show that ϕ is actually an inner automorphism of $\mathfrak{sp}_4(K)$.

Clearly Ad induces an isomorphism $\pi_{\mathrm{Ad}}(\mathrm{Im}\,\rho) \cong \mathrm{Im}\,\mathrm{Ad}\,\rho$. For every $x \in \mathrm{GL}(\mathfrak{sp}_4(K))$ we denote by Θ_x the automorphism of $\mathrm{GL}(\mathfrak{sp}_4(K))$ given by conjugation by x. In particular we write Equation (4.15) as $\mathrm{Ad}\,\rho^\sigma = \Theta_\phi(\mathrm{Ad}\,\rho)$. By combining Theorem 3.11.3 and Corollary 4.6.4 we show that we can replace ϕ by an element $\phi' \in \mathrm{Aut}(\mathfrak{sp}_4(K))$ still satisfying $\mathrm{Ad}\,\rho^\sigma = \Theta_{\phi'}(\mathrm{Ad}\,\rho(\phi'))$.

We identify $\mathrm{PGSp}_4(\mathcal{O}_E)$ with a subgroup of $\mathrm{PGSp}_4(\mathcal{O}_K^{\Gamma_f})$ via the inclusion $\mathcal{O}_E \subset \mathcal{O}_K^{\Gamma_f}$ given in the beginning of the proof. Consider the group $\Delta = (\pi_{\mathrm{Ad}}\,\mathrm{Im}\,\rho) \cap \mathrm{PGSp}_4(\mathcal{O}_E) \subset \mathrm{PGSp}_4(\mathcal{O}_K)$ and its isomorphic image $\mathrm{Ad}\,(\Delta) \subset \mathrm{GL}(\mathfrak{sp}_4)$. By assumption f satisfies the hypotheses of Theorem 3.11.3, so $\mathrm{Im}\,\rho$ contains a congruence subgroup $\Gamma_{\mathcal{O}_E}(\mathfrak{a})$ of $\mathrm{GSp}_4(\mathcal{O}_E)$ of some level $\mathfrak{a} \subset \mathcal{O}_E$. Hence $\pi_{\mathrm{Ad}}\,\mathrm{Im}\,\rho$ contains the projective congruence subgroup $\mathrm{P}\Gamma_{\mathcal{O}_E}(\mathfrak{a})$ of $\mathrm{PGSp}_4(\mathcal{O}_E)$ and Δ also contains $\mathrm{P}\Gamma_{\mathcal{O}_E}(\mathfrak{a})$. In particular Δ satisfies the hypotheses of Corollary 4.6.4. Since $\mathrm{Ad}\,\rho^\sigma = \Theta_\phi(\mathrm{Ad}\,\rho)$ we have an equality $(\mathrm{Ad}\,(\Delta))^\sigma = \Theta_\phi(\mathrm{Ad}\,(\Delta))$, where we identify the two sides with subgroups of $\mathrm{PGSp}_4(\mathcal{O}_E)$. Now σ acts as the identity on $\mathrm{PGSp}_4(\mathcal{O}_E)$, so the previous equality reduces to $\mathrm{Ad}\,(\Delta) = \Theta_\phi(\mathrm{Ad}\,(\Delta))$. Let $\Theta = \mathrm{Ad}^{-1} \circ \Theta_\phi \circ \mathrm{Ad} \colon \Delta \to \Delta$. Since Ad is an isomorphism, the composition Θ is an automorphism. Moreover it satisfies

$$\Theta_\phi(\mathrm{Ad}\,(\delta)) = \mathrm{Ad}\,(\Theta(\delta)) \tag{4.16}$$

for every $\delta \in \Delta$. By Corollary 4.6.4 applied to $F = F_1 = K$, $\Delta_1 = \Delta$ and $\Theta \colon \Delta \to \Delta$, there exists an automorphism τ of K and an element $\gamma \in \mathrm{GSp}_4(K)$ such that

$$\Theta(\delta) = \gamma\delta^\tau\gamma^{-1}$$

for every $\delta \in \Delta$. We see from Equation (4.16) that τ is trivial. It follows that $\Theta_\phi(y) = \mathrm{Ad}\,(\gamma) \circ y \circ \mathrm{Ad}\,(\gamma)^{-1}$ for all $y \in \mathrm{Ad}\,(\Delta)$. By K-linearity we can extend Θ_ϕ and $\Theta_{\mathrm{Ad}\,(\gamma)}$ to identical automorphisms of the K-span of $\mathrm{Ad}\,(\Delta)$ in $\mathrm{End}(\mathfrak{sp}_4(K))$. Since Δ contains the projective congruence subgroup $\mathrm{P}\Gamma_{\mathcal{O}_E}(\mathfrak{a})$, its K-span contains $\mathrm{Ad}\,(\mathrm{GSp}_4(K))$; in particular it contains the image of $\mathrm{Ad}\,\rho$. Hence Θ_ϕ and $\Theta_{\mathrm{Ad}\,(\gamma)}$ agree on $\mathrm{Ad}\,\rho$, which means that Equation (4.15) implies

$$\mathrm{Ad}\,\rho^\sigma = \Theta_{\mathrm{Ad}\,(\gamma)}(\mathrm{Ad}\,\rho).$$

Then by definition of $\Theta_{\mathrm{Ad}\,(\gamma)}$ we have

$$\mathrm{Ad}\,\rho^\sigma = \mathrm{Ad}\,(\gamma) \circ \mathrm{Ad}\,\rho \circ (\mathrm{Ad}\,(\gamma))^{-1} = \mathrm{Ad}\,(\gamma\rho\gamma^{-1}).$$

From the displayed equation we deduce that there exists a character $\eta_\sigma \colon G_\mathbb{Q} \to \mathcal{O}_K^\times$ satisfying $\rho^\sigma(g) = \eta_\sigma(g)\gamma\rho(g)\gamma^{-1}$ for every $g \in G_\mathbb{Q}$, hence that $\rho^\sigma \cong \eta_\sigma \otimes \rho$. We conclude that σ is a self-twist for ρ. In particular σ acts as the identity on $\mathcal{O}_K^{\Gamma_f}$, as desired. $\square$

REMARK 4.6.7. *Let* $\rho\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ *be the big Galois representation associated with a family* $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$. *We can define a ring* $\Lambda_h[\mathrm{Tr}(\mathrm{Ad}\,\rho)]$ *analogue to the ring* $\mathcal{O}_E$ *defined above. We have an inclusion* $\Lambda_h[\mathrm{Tr}(\mathrm{Ad}\,\rho)] \subset \mathbb{I}^\circ_0$ *given by Proposition 4.3.3. However the proof of the inclusion* $\mathcal{O}_K^{\Gamma_f} \subset \mathcal{O}_E$ *in Proposition 4.6.1 relied on the fact that* $\mathrm{Im}\,\rho_{f,p}$ *contains a congruence subgroup of* $\mathrm{GSp}_4(\mathcal{O}_E)$. *Since we do not know if an analogue for* ρ *is true, we do not know whether an equality between the normalizations of* $\Lambda_h[\mathrm{Tr}(\mathrm{Ad}\,\rho)]$ *and* $\mathbb{I}^\circ_0$ *holds.*

Suppose that the GSp_4-eigenform f appears in a finite slope family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$. Let $\mathfrak{P}$ be the prime of $\mathbb{I}^\circ_{\mathrm{Tr}}$ associated with f and suppose that $\mathfrak{P}\cap\Lambda_h$ is a non-critical arithmetic prime $P_{\underline{k}}$. Let $\mathfrak{P}_0 = \mathfrak{P}\cap\mathbb{I}^\circ_0$. Theorem 3.11.3 gives a fullness result with respect to the ring $\mathcal{O}_E$. Thanks to Proposition 4.6.1, this implies fullness with respect to the ring $\mathcal{O}_K^{\Gamma_f}$. We use Proposition 4.4.1 to compare $\mathcal{O}_K^{\Gamma_f}$ and the residue ring of $\mathbb{I}^\circ_0$ at $\mathfrak{P}_0$, as in [**Lang16**, Proposition 6.2].

PROPOSITION 4.6.8. *There is an inclusion* $\mathbb{I}^\circ_0/\mathfrak{P}_0 \subset \mathcal{O}_K^{\Gamma_f}$.

PROOF. Let $\sigma \in \Gamma_f$ and let $\eta_\sigma\colon G_{\mathbb{Q}} \to (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})^\times$ be the character associated with σ. We use the notations of Section 4.4. By Corollary 4.4.2 there exists a self-twist $\widetilde{\sigma}\colon \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P} \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}$ with associated character $\eta_{\widetilde{\sigma}}\colon G_{\mathbb{Q}} \to (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})^\times$ such that $\mathfrak{P}$ is fixed under $\widetilde{\sigma}$, $\widetilde{\sigma}_{\mathfrak{P}} = \sigma$ and $\eta_{\widetilde{\sigma},\mathfrak{P}} = \eta_\sigma$. Since $\widetilde{\sigma} \in \Gamma$ and $\mathbb{I}^\circ_0 = (\mathbb{I}^\circ_{\mathrm{Tr}})^\Gamma$ we have $\mathbb{I}^\circ_0 \subset (\mathbb{I}^\circ_{\mathrm{Tr}})^{\langle\widetilde{\sigma}\rangle}$, where $\langle\widetilde{\sigma}\rangle$ is the cyclic group generated by $\widetilde{\sigma}$. Since $\widetilde{\sigma}$ leaves $\mathfrak{P}$ fixed, we can reduce modulo $\mathfrak{P}$ the previous inclusion to obtain $\mathbb{I}^\circ_0/\mathfrak{P}_0 \subset (\mathbb{I}^\circ_{\mathrm{Tr}})^{\langle\widetilde{\sigma}\rangle}/\mathfrak{P}$. Again since $\widetilde{\sigma}$ leaves $\mathfrak{P}$ fixed and $\widetilde{\sigma}$ induces σ modulo $\mathfrak{P}$, we have $(\mathbb{I}^\circ_{\mathrm{Tr}})^{\langle\widetilde{\sigma}\rangle}/\mathfrak{P} = (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})^{\langle\sigma\rangle}$, hence $\mathbb{I}^\circ_0/\mathfrak{P}_0 \subset (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})^{\langle\sigma\rangle}$. This holds for every σ, so $\mathbb{I}^\circ_0/\mathfrak{P}_0 \subset (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})^{\Gamma_f}$. □

The following corollary summarizes the work of this section.

COROLLARY 4.6.9. *Let* $\rho \cong G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ *be the representation associated with the family* θ. *Let* $\mathfrak{P}$ *be a prime of* $\mathbb{I}^\circ_{\mathrm{Tr}}$ *corresponding to a classical eigenform* f *which is not a symmetric cube lift of a* GL_2*-eigenform. Let* $\mathfrak{P}_0 = \mathfrak{P}\cap\mathbb{I}^\circ_0$. *Then the image of* $\rho_{\mathfrak{P}}\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})$ *contains a non-trivial congruence subgroup of* $\mathrm{GSp}_4(\mathbb{I}^\circ_0/\mathfrak{P}_0)$.

PROOF. As before let $\mathcal{O}_E = \mathbb{Z}_p[\mathrm{Tr}\mathrm{Ad}\,\rho_{\mathfrak{P}}]$. By Theorem 3.11.3 the image of $\rho_{\mathfrak{P}}$ contains a congruence subgroup of $\mathrm{GSp}_4(\mathcal{O}_E)$. By combining Propositions 4.6.1 and 4.6.8 we obtain $\mathbb{I}^\circ_0/\mathfrak{P}_0 \subset \mathcal{O}_E$, hence the corollary. □

REMARK 4.6.10. *In* [**Lang16**] *and in Chapter 2, where Galois images for families of* GL_2*-eigenforms are studied, the intermediate step given by Proposition 4.6.1 is not necessary. Indeed the fullness result for the representation attached to a* GL_2*-eigenform, due to Ribet and Momose* ([**Mo81**] *and* [**Ri85**, Theorem 3.1]) *is stated in terms of the ring fixed by the self-twists of the representation, hence an analogue of Proposition 4.6.8 is sufficient.*

4.7. An approximation argument

In this section we prove an easy generalization of the the approximation argument presented in the proof of [**HT15**, Lemma 4.5]. An analogue for GL_2 was given in Proposition 2.3.7.

Here g is an arbitrary positive integer. Recall that we fixed a maximal torus T_g and a Borel subgroup B_g of GSp_{2g}, determining a set of roots and a subset of positive roots.

PROPOSITION 4.7.1. *Let* A *be a profinite local ring of residual characteristic* p *endowed with its profinite topology. Let* G *be a compact subgroup of the level* p *principal congruence subgroup* $\Gamma_{\mathrm{GSp}_{2g}(A)}(p)$ *of* $\mathrm{GSp}_{2g}(A)$. *Suppose that:*

(1) the ring A *is complete with respect to the* p*-adic topology;*
(2) the group G *is normalized by a diagonal* $\mathbb{Z}_p$*-regular element of* $\mathrm{GSp}_{2g}(A)$.

Let α be a root of GSp_{2g}. *For every ideal Q of A, let* $\pi_Q\colon \mathrm{GSp}_{2g}(A) \to \mathrm{GSp}_{2g}(A/Q)$ *be the natural projection, inducing a map* $\pi_{Q,\alpha}\colon U^\alpha(A) \to U^\alpha(A/Q)$. *Then*

$$\pi_Q(G) \cap U^\alpha(A/Q) = \pi_Q(G \cap U^\alpha(A)).$$

Since the inclusion $\pi_Q(G \cap U^\alpha(A)) \subset \pi_Q(G) \cap U^\alpha(A/Q)$ is trivial, we can rephrase the conclusion of Proposition 4.7.1 by saying that the natural projection $\pi_Q\colon G \cap U^\alpha(A) \to \pi_Q(G) \cap U^\alpha(A/Q)$ is surjective for every α. In our applications G will be the image of a continuous representation of a Galois group in $\mathrm{GSp}_{2g}(A)$.

Proof. Let α be a root of GSp_4. As stated above, it is sufficient to show that $\pi_Q\colon G \cap U^\alpha(A) \to \pi_Q(G) \cap U^\alpha(A/Q)$ is surjective. The unipotent subgroups U^α and $U^{-\alpha}$ generate a subgroup of $\mathrm{GSp}_{2g}(A)$ isomorphic to $\mathrm{SL}_2(A)$. We denote it by $\mathrm{SL}_2^\alpha(A)$. We write $\Gamma_A(p)$ for the level p principal congruence subgroup of $\mathrm{SL}_2^\alpha(A)$. Throughout the proof we identify $U^{\pm\alpha}$ with subgroups of $\mathrm{SL}_2^\alpha(A)$. In this proof we write $T = T_g$ and $B = B_g$. Let $T^\alpha = T \cap \mathrm{SL}_2^\alpha$ and $B^\alpha = T^\alpha U^\alpha$. We also write $\mathfrak{sl}_2^\alpha, \mathfrak{u}^{\pm\alpha}, \mathfrak{t}^\alpha, \mathfrak{b}^{\pm\alpha}$ for the Lie algebras of the $\mathrm{SL}_2^\alpha, U^{\pm\alpha}, T^\alpha, B^{\pm\alpha}$, respectively. For every positive integer j, we denote by π_{Q^j} the natural projection $\mathrm{GSp}_{2g}(A) \to \mathrm{GSp}_{2g}(A/Q^j)$, as well as its restriction $\mathrm{SL}_2^\alpha(A) \to \mathrm{SL}_2^\alpha(A/Q^j)$. We define some congruence subgroups of $\mathrm{SL}_2^\alpha(A)$ of level pQ^j by setting

$$\begin{aligned}
\Gamma_A(Q^j) &= \{x \in \mathrm{SL}_2^\alpha \cap \Gamma_A(p) \mid \pi_{Q^j} x = \mathbb{1}_{2g}\},\\
\Gamma_{U^\alpha}(Q^j) &= \{x \in \mathrm{SL}_2^\alpha \cap \Gamma_A(p) \mid \pi_{Q^j} x \in U^\alpha(A/Q^j)\},\\
\Gamma_{B^\alpha}(Q^j) &= \{x \in \mathrm{SL}_2^\alpha \cap \Gamma_A(p) \mid \pi_{Q^j} x \in B^\alpha(A/Q^j)\}.
\end{aligned}$$

Note that we leave the level at p implicit. All the groups we consider in this proof are trivial modulo p. We set $G_{U^\alpha}(Q^j) = G \cap \Gamma_{U^\alpha}(Q^j)$ and $G_{B^\alpha}(Q^j) = G \cap \Gamma_{B^\alpha}(Q^j)$. Given two elements $X, Y \in \mathrm{GSp}_{2g}(A)$, we denote by $[X,Y]$ their commutator $XYX^{-1}Y^{-1}$. For every subgroup $H \subset \mathrm{GSp}_{2g}(A)$ we denote by $\mathrm{D}H$ its commutator subgroup $\{[X,Y] \mid X, Y \in H\}$. We write $[\cdot,\cdot]_{\mathrm{Lie}}$ for the Lie bracket on $\mathfrak{gsp}_{2g}(A)$.

We prove the following lemma.

Lemma 4.7.2. *For every $j \geq 1$ we have*

$$\mathrm{D}\Gamma_{U^\alpha}(Q) \subset \Gamma_{B^\alpha}(Q^{2j}) \cap \Gamma_{U^\alpha}(Q^j).$$

Proof. A matrix $X \in \Gamma_{U^\alpha}(Q^j)$ can be written in the form $X = UM$ where $U \in U^\alpha$ and $M \in \Gamma_A(Q^j)$. In particular its logarithm is defined, it satisfies $\exp(\log X) = X$ and it is of the form $\log X = u + m$ with $u \in \mathfrak{u}^\alpha(A) \subset \mathfrak{sl}_2^\alpha(A)$ and $m \in Q^j\mathfrak{sl}_2^\alpha(A)$. Now let $X, X_1 \in \Gamma_{U^\alpha}(Q^j)$ and let $\log X = u + m$ and $\log X_1 = u_1 + m_1$ be decompositions of the type described above. Modulo Q^{2j} we can calculate

$$\log[X, X_1] \cong [\log X, \log X_1]_{\mathrm{Lie}} \cong [u, u_1]_{\mathrm{Lie}} + [m, u_1]_{\mathrm{Lie}} + [u, m_1]_{\mathrm{Lie}} + [m, m_1]_{\mathrm{Lie}}.$$

Since $u, u_1 \in \mathfrak{u}^\alpha$ and $m, m_1 \in Q^j\mathfrak{sl}_2^\alpha(A)$ we have $[u, u_1]_{\mathrm{Lie}} = 0$ and $[m, m_1]_{\mathrm{Lie}} \in Q^{2j}\mathfrak{sl}_2^\alpha(A)$, so

$$\log[X, X_1] \cong [m, u_1]_{\mathrm{Lie}} + [u, m_1]_{\mathrm{Lie}} \pmod{Q^{2j}}.$$

Now write $m = u^{-\alpha} + b^\alpha$ and $m_1 = u_1^{-\alpha} + b_1^\alpha$ with $u^{-\alpha}, u_1^{-\alpha} \in Q^j\mathfrak{u}^{-\alpha}(A)$ and $b^{-\alpha}, b_1^{-\alpha} \in Q^j\mathfrak{b}^\alpha(A)$. Then $[m, u_1]_{\mathrm{Lie}} = [u^{-\alpha}, u_1]_{\mathrm{Lie}} + [b^\alpha, u_1]_{\mathrm{Lie}}$, which belongs to $Q^j\mathfrak{b}^\alpha(A)$ since $[u^{-\alpha}, u_1]_{\mathrm{Lie}} \in Q^j\mathfrak{t}^\alpha(A)$ and $[b^\alpha, u_1]_{\mathrm{Lie}} \in Q^j\mathfrak{b}^\alpha(A)$. In the same way we see that $[u, m_1]_{\mathrm{Lie}} \in \mathfrak{b}^\alpha(A)$. We conclude that $\log[X, X_1] \in Q^j\mathfrak{b}^\alpha \pmod{Q^{2j}}$, so $[X, X_1] \in \Gamma_{B^\alpha}(Q^{2j})$. Trivially $[X, X_1] \in \Gamma_{U^\alpha}(Q^j)$, so this proves the lemma. □

Let $d \in G$ be a diagonal $\mathbb{Z}_p$-regular element. Since A is p-adically complete the limit $\lim_{n\to\infty} d^{p^n}$ defines a diagonal element $\delta \in \mathrm{GSp}_{2g}(A)$. Clearly $\delta^p = \delta$, so $\delta^{p-1} = \mathbb{1}_{2g}$ and the order of δ in $\mathrm{GSp}_{2g}(A)$ is a divisor a of $p-1$. By hypothesis G is a compact subgroup of $\Gamma_{\mathrm{GSp}_{2g}(A)}(p)$, so G is a pro-p group and δ normalizes G. We denote by $\mathrm{ad}\,(\delta)$ the adjoint action of δ on $\mathrm{GSp}_{2g}(A)$.

Consider the pro-p subgroup $\Gamma_A(p)$ of SL_2^α. Every element of $\Gamma_A(p)$ has a unique a-th root in $\Gamma_A(p)$. Since δ is diagonal, it normalizes $\Gamma_A(p)$. We define a map $\Delta\colon \Gamma_A(p)\to\Gamma_A(p)$ by setting

$$\Delta(x)=\Big(x\cdot(\mathrm{ad}\,(\delta)(x))^{\alpha(\delta)^{-1}}\cdot(\mathrm{ad}\,(\delta^2)(x))^{\alpha(\delta)^{-2}}\cdots(\mathrm{ad}\,(\delta^{a-1})(x))^{\alpha(\delta)^{1-a}}\Big)^{1/a}$$

for every $x\in\Gamma_A(p)$. Note that Δ is not a homomorphism, but it induces a homomorphism of abelian groups $\Delta^{\mathrm{ab}}\colon \Gamma_A(p)/\mathrm{D}\Gamma_A(p)\to\Gamma_A(p)/\mathrm{D}\Gamma_A(p)$.

The following lemma is the analogue of [**HT15**, Lemma 4.7].

Lemma 4.7.3. *If $u\in\Gamma_{U^\alpha}(Q^j)$ for some positive integer j, then $\pi_{Q^j}(\Delta(u))=\pi_{Q^j}(u)$ and $\Delta^2(u)\in\Gamma_{U^\alpha}(Q^{2j})$.*

Proof. Let $u\in\Gamma_{U^\alpha}(Q^j)$. By the definition we see that Δ maps $Q^j\Gamma_A(p)$ to itself, so it induces a map $\Delta_{Q^j}\colon\Gamma_A(p)/Q^j\Gamma_A(p)\to\Gamma_A(p)/Q^j\Gamma_A(p)$. For $x\in U^\alpha(A/Q^j)$ we have $\pi_{Q^j}(\mathrm{ad}\,(\delta)(x))=\mathrm{ad}\,(\pi_{Q^j}(\delta))(x)=\pi_{Q^j}(\alpha(\delta))(x)$. From this we deduce that $\Delta_{Q^j}(x)=x$ for $x\in U^\alpha(A/Q^j)$. Since $\pi_{Q^j}(u)\in U^\alpha(A/Q^j)$ we obtain $\pi_{Q^j}(\Delta(u))=\Delta_{Q^j}(\pi_{Q^j}(u))=\pi_{Q^j}(u)$.

Now consider the homomorphism $\Delta^{\mathrm{ab}}\colon\Gamma_A(p)/\mathrm{D}\Gamma_A(p)\to\Gamma_A(p)/\mathrm{D}\Gamma_A(p)$. By a direct computation we see that $\mathrm{ad}\,(\delta)(\Delta^{\mathrm{ab}}(x))=\alpha(\delta)(\Delta^{\mathrm{ab}}(x))$ for every $x\in\Gamma_A(p)/\mathrm{D}\Gamma_A(p)$, so the image of Δ^{ab} lies in the $\alpha(\delta)$-eigenspace for the action of $\mathrm{ad}\,(\delta)$ on $\Gamma_A(p)/\mathrm{D}\Gamma_A(p)$. This space is just $U^\alpha(A)\mathrm{D}\Gamma_A(p)/\mathrm{D}\Gamma_A(p)$, as we can see by looking at the Iwahori decomposition of $\Gamma_A(p)$.

Note that by the first part of the proposition it follows that Δ^{ab} induces a homomorphism $\Delta^{\mathrm{ab}}_{\Gamma_{U^\alpha}}\colon\Gamma_{U^\alpha}(Q^j)/\mathrm{D}\Gamma_{U^\alpha}(Q^j)\to\Gamma_{U^\alpha}(Q^j)/\mathrm{D}\Gamma_{U^\alpha}(Q^j)$. By the remark of the previous paragraph

$$\Delta^{\mathrm{ab}}_{\Gamma_{U^\alpha}}(\Gamma_{U^\alpha}(Q^j)/\mathrm{D}\Gamma_{U^\alpha}(Q^j))\subset\Gamma_{U_\alpha}(Q^j)\mathrm{D}\Gamma_{U^\alpha}(Q^j)/\mathrm{D}\Gamma_{U^\alpha}(Q^j).$$

By Lemma 4.7.2 we have $\mathrm{D}\Gamma_{U^\alpha}(Q^j)\subset\Gamma_{B^\alpha}(Q^{2j})\cap\Gamma_{U^\alpha}(Q^j)$, so

$$\Delta^{\mathrm{ab}}_{\Gamma_{U^\alpha}}(\Gamma_{U^\alpha}(Q^j)/\mathrm{D}\Gamma_{U^\alpha}(Q^j))\subset\Gamma_{B^\alpha}(Q^{2j})\cap\Gamma_{U^\alpha}(Q^j)/\mathrm{D}\Gamma_{U^\alpha}(Q^j).$$

We deduce that $\Delta(u)\in\Gamma_{B^\alpha}(Q^{2j})\cap\Gamma_{U^\alpha}(Q^j)$.

By the same reasoning as above, Δ induces a map

$$\Delta^{\mathrm{ab}}_{\Gamma_{B^\alpha}}\colon\Gamma_{B^\alpha}(Q^{2j})/\mathrm{D}\Gamma_{B^\alpha}(Q^{2j})\to\Gamma_{B^\alpha}(Q^{2j})/\mathrm{D}\Gamma_{B^\alpha}(Q^{2j}).$$

The image of $\Delta^{\mathrm{ab}}_{\Gamma_{B^\alpha}}$ is in the $\alpha(\delta)$-eigenspace for the action of $\mathrm{ad}\,(\delta)$. This space is just $U^\alpha(Q^{2j})\mathrm{D}\Gamma_{B^\alpha}(Q^{2j})/\mathrm{D}\Gamma_{B^\alpha}(Q^{2j})$. Note that $\mathrm{D}\Gamma_{B^\alpha}(Q^{2j})\subset U^\alpha(Q^{2j})$, so

$$\Delta^{\mathrm{ab}}_{\Gamma_{B^\alpha}}(\Gamma_{B^\alpha}(Q^{2j})/\mathrm{D}\Gamma_{B^\alpha}(Q^{2j}))\subset\Gamma_{U^\alpha}(Q^{2j})/\mathrm{D}\Gamma_{B^\alpha}(Q^{2j}).$$

Since $\Delta(u)\in\Gamma_{B^\alpha}(Q^{2j})$ we conclude that $\Delta^2(u)\in\Gamma_{U^\alpha}(Q^{2j})$. $\square$

We complete the proof of the proposition. We look at $G\cap U^\alpha(A)$ and $\pi_Q(G)\cap\mathrm{SL}_2(A/Q)$ as subgroups respectively of $\mathrm{SL}_2^\alpha(A)$ and $\mathrm{SL}_2^\alpha(A/Q)$. From this point of view the statement of the proposition stays the same. Let $\overline{u}\in\pi_Q(G)\cap U^\alpha(A/U)$. Choose $u_1\in G$ and $u_2\in U^\alpha(A)$ such that $\pi_Q(u_1)=\pi_Q(u_2)=\overline{u}$. Then $u_1u_2^{-1}\in\Gamma_A(Q)$, so $u_1\in G\cap\Gamma_{U^\alpha}(Q)$. Note that $G\cap\Gamma_{U^\alpha}(Q)$ is compact since G and $\Gamma_{U^\alpha}(Q)$ are pro-p groups. By Lemma 4.7.3 we have $\pi_Q(\Delta^{2^m}(u_1))=\overline{(}u)$ and $\Delta^{2^m}(u_1)\in\Gamma_{U^\alpha}(Q^{2m})$ for any positive integer m. Hence the limit $\lim_{m\to\infty}\Delta^{2^m}(u_1)$ defines an element $u\in\mathrm{SL}_2(A)$ satisfying $\pi_Q(u)=\overline{u}$. We have $u\in G\cap\Gamma_{U^\alpha}(Q)$ since $G\cap\Gamma_{U^\alpha}(Q)$ is compact. This proves the surjectivity of the map $G\cap U^\alpha(A)\to\pi_Q(G)\cap\mathrm{SL}_2(A/Q)$. $\square$

We give a simple corollary.

Corollary 4.7.4. *Let $\rho\colon G_\mathbb{Q}\to\mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ be the Galois representation associated with a finite slope family $\theta\colon\mathbb{T}_h\to\mathbb{I}^\circ$. For every root α of GSp_4 the group $\mathrm{Im}\,\rho\cap U^\alpha(\mathbb{I}^\circ_{\mathrm{Tr}})$ is non-trivial.*

Proof. Let $\mathfrak{P}$ be a prime of $\mathbb{I}^\circ$ corresponding to a classical eigenform f which is not the symmetric cube lift of a GL_2-eigenform. The reduction $\rho_\mathfrak{P}\colon G_\mathbb{Q}\to\mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})$ of ρ modulo $\mathfrak{P}$ is the p-adic Galois representation associated with f. Let $\mathcal{O}_E=\mathbb{Z}_p[\mathrm{Tr}(\mathrm{Ad}\,\rho_\mathfrak{P})]$. By Theorem 3.11.3 $\mathrm{Im}\,\rho_\mathfrak{P}$ contains a non-trivial congruence subgroup of $\mathrm{Sp}_4(\mathcal{O}_E)$. In particular $\mathrm{Im}\,\rho_\mathfrak{P}\cap U^\alpha(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})$

is non-trivial for every root α. Now we apply Proposition 4.7.1 to $g = 2$, $A = \mathbb{I}^\circ_{\mathrm{Tr}}$, $G = \mathrm{Im}\,\rho$ and $Q = \mathfrak{P}$. We deduce that the projection $\mathrm{Im}\,\rho \cap U^\alpha(\mathbb{I}^\circ_{\mathrm{Tr}}) \to \mathrm{Im}\,\rho_{\mathfrak{P}} \cap U^\alpha(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})$ is surjective for every α. In particular $\mathrm{Im}\,\rho \cap U^\alpha(\mathbb{I}^\circ_{\mathrm{Tr}})$ must be non-trivial for every α. □

4.8. A representation with image fixed by the self-twists

Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a finite slope family. As before let $\rho\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ be the representation associated with ρ, that we supposed to be residually irreducible and $\mathbb{Z}_p$-regular (see Definition 3.11.1). Consider the group Γ of self-twists for ρ and the subring $\mathbb{I}_0^\circ$ of $\mathbb{I}^\circ_{\mathrm{Tr}}$ consisting of the elements fixed by Γ. By restricting the domain of ρ and replacing it with a suitable conjugate representation, we will obtain a $\mathbb{Z}_p$-regular representation for which the image is fixed by the action of Γ. This is the main result of this section. In an imporant intermediate step we will need to apply Corollary 4.7.4.

We write η_σ for the finite order Galois character associated with a self-twist $\sigma \in \Gamma$. Let $H_0 = \bigcap_{\sigma\in\Gamma} \ker \eta_\sigma$. Since Γ is finite the subgroup H_0 is open and normal in $G_{\mathbb{Q}}$. The following is an immediate consequence of the definition of H_0.

Lemma 4.8.1. *For every $g \in H_0$ we have $\mathrm{Tr}(\rho(g)) \in \mathrm{GSp}_4(\mathbb{I}_0^\circ)$.*

Proof. Let $g \in H_0$ and $\sigma \in \Gamma$. By definition of self-twist we have an equivalence of representations $\rho^\sigma \cong \eta_\sigma \otimes \rho$. In particular the traces of the two representations must coincide, so $(\mathrm{Tr}(\rho(g))^\sigma = \mathrm{Tr}(\rho^\sigma(g)) = \eta_\sigma(g)\mathrm{Tr}(\rho(g))$. Since $H_0 \subset \ker \eta_\sigma$ we deduce that $(\mathrm{Tr}(\rho(g)))^\sigma = \mathrm{Tr}(\rho(g))$. Then $\mathrm{Tr}(\rho(g))$ is fixed by all self-twists, so it is an element of $\mathbb{I}_0^\circ$. □

Consider the restrictions $\rho|_{H_0}\colon H_0 \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ and $\overline{\rho}|_{H_0}\colon H_0 \to \mathrm{GSp}_4(\mathbb{F})$. If $\overline{\rho}|_{H_0}$ is irreducible, then by Theorem 3.5.3 there exists $g \in \mathrm{GL}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ such that the representation $\rho^g = g\rho g^{-1}$ satisfies $\mathrm{Im}\,\rho^g|_{H_0} \subset \mathrm{GL}_4(\mathbb{I}_0^\circ)$. Since we prefer not to assume that $\overline{\rho}|_{H_0}$ is irreducible we follow the approach of Proposition 2.3.12, that comes in part from the proof of [**Lang16**, Theorem 7.5]. Our result is the following.

Proposition 4.8.2. *There exists an element $g \in \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ such that:*
(1) $g\rho g^{-1}(H_0) \subset \mathrm{GSp}_4(\mathbb{I}_0)$;
(2) $g\rho g^{-1}(H_0)$ contains a diagonal $\mathbb{Z}_p$-regular element.

In the proof of the proposition we will need the following lemma.

Lemma 4.8.3. *Let F be a field and α be a root of GSp_4. Suppose that there exist $u_0 \in U^\alpha(F)$ and $g \in \mathrm{GSp}_4(F)$ such that $gu_0g^{-1} \in U^\alpha(F)$. Then g normalizes $U^\alpha(F)$.*

Proof. Consider the subgroup of $\mathrm{M}_4(F)$ defined by $N^\alpha(F) = \{u - \mathbb{1}_4 \mid u \in U^\alpha(F)\}$. For $n_0 = u_0 - \mathbb{1}_4$, we have $N^\alpha(F) = \{fn_0 \mid f \in F\}$ and $U^\alpha(F) = \{\mathbb{1}_4 + n \mid n \in N^\alpha(F)\}$. Conjugation by g on $\mathrm{M}_4(F)$ is F-linear, so for every $f \in F$ we have

$$g(\mathbb{1}_4+fn_0)g^{-1} = g\mathbb{1}_4g^{-1}+gfn_0g^{-1} = \mathbb{1}_4+fgn_0g^{-1} = \mathbb{1}_4+fg(u_0-\mathbb{1}_4)g^{-1} = \mathbb{1}_4+f(gu_0g^{-1}-\mathbb{1}_4).$$

By hypothesis $gu_0g^{-1} \in U^\alpha(F)$, so $gu_0g^{-1} - \mathbb{1}_4 \in N^\alpha(F)$. Hence $f(gu_0g^{-1} - \mathbb{1}_4) \in N^\alpha(F)$ and $\mathbb{1}_4 + f(gu_0g^{-1} - \mathbb{1}_4) \in U^\alpha(F)$. This concludes the proof. □

Proof. (of Proposition 4.8.2) Let V be a free, rank four $\mathbb{I}^\circ_{\mathrm{Tr}}$-module. The choice of a basis of V determines an isomorphism $\mathrm{GL}_4(\mathbb{I}^\circ_{\mathrm{Tr}}) \cong \mathrm{Aut}(V)$, hence an action of ρ on V. Let d be a $\mathbb{Z}_p$-regular element contained in $\mathrm{Im}\,\rho$. We denote by $\{e_i\}_{i=1,\dots,4}$ a symplectic basis of V such that d is diagonal. Until further notice we work in this basis.

By definition of self-twist, for each $\sigma \in \Gamma$ there exists a character $\eta_\sigma\colon G_{\mathbb{Q}} \to (\mathbb{I}^\circ_{\mathrm{Tr}})^\times$ satisfying $\rho^\sigma \cong \eta_\sigma \otimes \rho$. This equivalence of representations implies that there exists a matrix $C_\sigma \in \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ such that

$$\rho^\sigma(g) = \eta_\sigma C_\sigma \rho(g) C_\sigma^{-1}. \tag{4.17}$$

Recall that we write $\mathfrak{m}_{\mathbb{I}^\circ_{\mathrm{Tr}}}$ for the maximal ideal of $\mathbb{I}^\circ_{\mathrm{Tr}}$ and $\mathbb{F}$ for the residue field of $\mathbb{I}^\circ_{\mathrm{Tr}}$. Let $\overline{C}_\sigma$ be the image of C_σ under the natural projection $\mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}}) \to \mathrm{GSp}_4(\mathbb{F})$. We prove the following lemma.

LEMMA 4.8.4. *For every $\sigma \in \Gamma$ the matrix C_σ is diagonal and the matrix $\overline{C_\sigma}$ is scalar.*

PROOF. Let α be any root of GSp_4 and u^α be a non-trivial element of $\mathrm{Im}\,\rho \cap U^\alpha(\mathbb{I}^\circ_{\mathrm{Tr}})$. Such a u^α exists thanks to Corollary 4.7.4. Let g^α be an element of $G_{\mathbb{Q}}$ such that $\rho(g^\alpha) = u^\alpha$. By evaluating Equation (4.17) at g^α we obtain $C_\sigma u^\alpha C_\sigma^{-1} = (u^\alpha)^\sigma$, which is again an element of $U^\alpha(\mathbb{I}^\circ_{\mathrm{Tr}})$. From Lemma 4.8.3 applied to $F = Q(\mathbb{I}^\circ_{\mathrm{Tr}})$, $u_0 = u^\alpha$ and $g = C_\sigma$ we deduce that C_σ normalizes $U^\alpha(Q(\mathbb{I}^\circ_{\mathrm{Tr}}))$. This holds for every root α, so C_σ normalizes the Borel subgroups of upper and lower triangular matrices in $\mathrm{GSp}_4(Q(\mathbb{I}^\circ_{\mathrm{Tr}}))$. Since a Borel subgroup is its own normalizer, we conclude that C_σ is diagonal.

By Proposition 4.4.7(3) the action of Γ on $\mathbb{I}^\circ_{\mathrm{Tr}}$ induces the trivial action of Γ on $\mathbb{F}$. By evaluating Equation (4.17) at g^α and modulo $\mathfrak{m}_{\mathbb{I}^\circ_{\mathrm{Tr}}}$ we obtain, with the obvious notations, $\overline{C}_\sigma \overline{u}^\alpha (\overline{C}_\sigma)^{-1} = (\overline{u}^\alpha)^\sigma = \overline{u}^\alpha$. Since C_σ is diagonal and $\overline{u}^\alpha \in U^\alpha(\mathbb{F})$, the left hand side is equal to $\alpha(\overline{C}_\sigma)\overline{u}^\alpha$. We deduce that $\alpha(\overline{C}_\sigma) = 1$. Since this holds for every root α, we conclude that $\overline{C}_\sigma$ is scalar. □

We write C for the map $\Gamma \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ defined by $C(\sigma) = C_\sigma$. We show that C can be modified into a 1-cocycle C' such that $C'(\sigma)$ still satisfies Equation (4.17). Define a map $c\colon \Gamma^2 \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ by $c(\sigma,\tau) = C_{\sigma\tau}^{-1} C_\sigma^\tau C_\tau$ for every $\sigma, \tau \in \Gamma$. By using multiple times Equation (4.17) we find that for every $g \in G_{\mathbb{Q}}$

$$\eta_{\sigma\tau} C_{\sigma\tau} \rho(g) C_{\sigma\tau}^{-1} = \rho^{\sigma\tau}(g) = \eta_\sigma^\tau \eta_\tau C_\sigma^\tau C_\tau \rho(g) C_\tau^{-1} (C_\sigma^\tau)^{-1}.$$

By rearranging the terms we obtain

$$\rho(g) = \eta_{\sigma\tau}^{-1} \eta_\sigma^\tau \eta_\tau c(\sigma,\tau) \rho(g) c(\sigma,\tau)^{-1}$$

Recall that $\eta_\sigma^\tau \eta_\tau = \eta_{\sigma\tau}$ by Proposition 4.3.2(4), so the matrix $c(\sigma,\tau)$ commutes with the image of ρ. Since ρ is irreducible, $c(\sigma,\tau)$ must be a scalar.

For every $\sigma \in \Gamma$ and every $i \in \{1,2,3,4\}$ let $(C_\sigma)_i$ denote the i-th diagonal entry of C_σ. Define a map $C_i'\colon \Gamma \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ by $C_i'(\sigma) = (C_\sigma)_i^{-1} C_\sigma$. Let $c_i'(\sigma,\tau) = C_i'(\sigma\tau)^{-1} C_i'(\sigma)^\tau C_i'(\tau)$ for every $\sigma, \tau \in \Gamma$ and $i \in \{1,2,3,4\}$. Then

$$c_i'(\sigma,\tau) = ((C_{\sigma\tau})_i^{-1} (C_\sigma)_i (C_\tau)_i)^{-1} c(\sigma,\tau). \tag{4.18}$$

Since $(C_{\sigma\tau})_i^{-1} (C_\sigma)_i^\tau (C_\tau)_i$ is the i-th diagonal entry of $c(\sigma,\tau)$ and $c(\sigma,\tau)$ is scalar, the quantity $(C_{\sigma\tau})_i^{-1} (C_\sigma)_i (C_\tau)_i$ is independent of i and $((C_{\sigma\tau})_i^{-1} (C_\sigma)_i (C_\tau)_i)^{-1} c(\sigma,\tau) = \mathbb{1}_4$ for every i. From Equation (4.18) we deduce that C_i' is a 1-cocycle.

Set $C_\sigma' = C_i'(\sigma)$. We have

$$\rho^\sigma(g) = \eta_\sigma C_\sigma \rho(g) C_\sigma^{-1} = \eta_\sigma C_\sigma' \rho(g) (C_\sigma')^{-1}. \tag{4.19}$$

By Lemma 4.8.4 $\overline{C}_\sigma$ is scalar, so we $\overline{C_\sigma'} = (\overline{C}_\sigma)_i^{-1} \overline{C}_\sigma = \mathbb{1}_4$ with the obvious notations.

Recall that $\{e_i\}_{i=1,\dots,4}$ is our chosen basis of the free $\mathbb{I}^\circ_{\mathrm{Tr}}$-module V, on which $G_{\mathbb{Q}}$ acts via ρ. For every $v \in V$ we write as $v = \sum_{i=1}^4 \lambda_i(v) e_i$ its unique decomposition in the basis $(e_i)_{i=1,\dots,4}$, with $\lambda_i(v) \in \mathbb{I}^\circ_{\mathrm{Tr}}$ for $1 \le i \le 4$. For every $v \in V$ and every $\sigma \in \Gamma$ we set $v^{[\sigma]} = (C_\sigma')^{-1} \sum_{i=1}^4 \lambda_i(v)^\sigma e_i$. This defines an action of Γ on V since C_σ' is a 1-cocycle. Let $V^{[\Gamma]}$ denote the set of elements of V fixed by Γ. The action of Γ is clearly $\mathbb{I}_0^\circ$-linear, so $V^{[\Gamma]}$ has a structure of $\mathbb{I}_0^\circ$-submodule of V.

Let $v \in V^{[\Gamma]}$ and $h \in H_0$. Then $\rho(h)v$ is also in $V^{[\Gamma]}$, as we see by computing for every $\sigma \in \Gamma$

$$(\rho(h)v)^{[\sigma]} = (C_\sigma')^{-1} (\rho(h)v)^\sigma = (C_\sigma')^{-1} \rho^\sigma(h) v^\sigma = (C_\sigma')^{-1} (C_\sigma' \rho(h) (C_\sigma')^{-1}) v^\sigma = \rho(h) v^{[\sigma]},$$

where we used Equation (4.19) as an intermediate step. We deduce that the action of $G_{\mathbb{Q}}$ on V via ρ induces an action of H_0 on $V^{[\Gamma]}$. We will conclude the proof of the proposition after having studied the structure of $V^{[\Gamma]}$.

LEMMA 4.8.5. *The $\mathbb{I}_0^\circ$-submodule $V^{[\Gamma]}$ of V is free of rank four and its $\mathbb{I}_{\mathrm{Tr}}^\circ$-span is V.*

PROOF. Choose $i \in \{1, \ldots, 4\}$. We construct a non-zero, Γ-invariant element $w_i \in \mathbb{I}_{\mathrm{Tr}}^\circ e_i$. The submodule $\mathbb{I}_{\mathrm{Tr}}^\circ e_i$ is stable under Γ because C'_σ is diagonal. The action of Γ on $\mathbb{I}_{\mathrm{Tr}}^\circ e_i$ induces an action of Γ on the one-dimensional $\mathbb{F}$-vector space $\mathbb{I}_{\mathrm{Tr}}^\circ e_i \otimes_{\mathbb{I}_{\mathrm{Tr}}^\circ} \mathbb{F}$. Recall that the self-twists induce the identity on $\mathbb{F}$ by Proposition 4.4.7(3) and that the matrix $\overline{C'_\sigma}$ is trivial for every $\sigma \in \Gamma$, so Γ acts trivially on $\mathbb{I}_{\mathrm{Tr}}^\circ e_i \otimes_{\mathbb{I}_{\mathrm{Tr}}^\circ} \mathbb{F}$.

Now choose any $v_i \in \mathbb{I}_{\mathrm{Tr}}^\circ e_i$. Let $w_i = \sum_{\sigma\in\Gamma} v_i^{[\sigma]}$. Clearly w_i is invariant under the action of Γ. We show that $w_i \neq 0$. Let $\overline{v}_i$, $\overline{w}_i$ denote respectively the images of v_i and w_i via the natural projection $\mathbb{I}_{\mathrm{Tr}}^\circ e_i \to \mathbb{I}_{\mathrm{Tr}}^\circ e_i \otimes_{\mathbb{I}_{\mathrm{Tr}}^\circ} \mathbb{F}$. Then $\overline{w}_i = \sum_{\sigma\in\Gamma} \overline{v}_i^{[\sigma]} = \sum_{\sigma\in\Gamma} \overline{v}_i = \mathrm{card}(\Gamma) \cdot \overline{v}_i$ because Γ acts trivially on $\mathbb{I}_{\mathrm{Tr}}^\circ e_i \otimes_{\mathbb{I}_{\mathrm{Tr}}^\circ} \mathbb{F}$. By Lemma 4.3.6 the only prime factors of $\mathrm{card}(\Gamma)$ are 2 and 3. Since we supposed that $p \geq 5$ we have $\mathrm{card}(\Gamma) \neq 0$ in $\mathbb{F}$. We deduce that $\overline{w}_i = \mathrm{card}(\Gamma)\overline{v}_i \neq 0$ in $\mathbb{F}$, so $w_i \neq 0$.

Note that $\{w_i\}_{i=1,\ldots,4}$ is an $\mathbb{I}_{\mathrm{Tr}}^\circ$-basis of V since $\overline{w_i} \neq 0$ for every i. In particular the $\mathbb{I}_0^\circ$-span of the set $\{w_i\}_{i=1,\ldots,4}$ is a free, rank four $\mathbb{I}_0^\circ$-submodule of V. Since $V^{[\Gamma]}$ has a structure of $\mathbb{I}_0^\circ$-module and $w_i \in V^{[\Gamma]}$ for every i, there is an inclusion $\sum_{i=1}^4 \mathbb{I}_0^\circ w_i \subset V^{[\Gamma]}$. We show that this is an equality. Let $v \in V^{[\Gamma]}$. Write $v = \sum_{i=1}^4 \lambda_i w_i$ for some $\lambda_i \in \mathbb{I}_{\mathrm{Tr}}^\circ$. Then for every $\sigma \in \Gamma$ we have $v = v^{[\sigma]} = \sum_{i=1}^4 \lambda_i^\sigma w_i^{[\sigma]} = \sum_{i=1}^4 \lambda_i^\sigma w_i$. Since $\{w_i\}_{i=1,\ldots,4}$ is an $\mathbb{I}_{\mathrm{Tr}}^\circ$-basis of V, we must have $\lambda_i = \lambda_i^\sigma$ for every i. This holds for every σ, we obtain $\lambda_i \in \mathbb{I}_0^\circ$ for every i. Hence $v = \sum_{i=1}^4 \lambda_i w_i \in \sum_{i=1}^4 \mathbb{I}_0^\circ w_i$.

The second assertion of the lemma follows immediately from the fact that the set $\{w_i\}_{i=1,\ldots,4}$ is contained in $V^{[\Gamma]}$ and is an $\mathbb{I}_{\mathrm{Tr}}^\circ$-basis of V. $\square$

Now let $h \in H_0$. Let $\{w_i\}_{i=1,\ldots,4}$ be an $\mathbb{I}_0^\circ$-basis of $V^{[\Gamma]}$ satisfying $w_i \in \mathbb{I}_{\mathrm{Tr}}^\circ e_i$, such as that provided by the lemma. Since $\mathbb{I}_{\mathrm{Tr}}^\circ \cdot V^{[\Gamma]} = V$, $\{w_i\}_{i=1,\ldots,4}$ is also an $\mathbb{I}_{\mathrm{Tr}}^\circ$-basis of V. Moreover $\{w_i\}_{i=1,\ldots,4}$ is a symplectic basis of V, since $w_i \in \mathbb{I}_{\mathrm{Tr}}^\circ e_i$ for every i and $\{e_i\}$ is a symplectic basis. We show that the basis $\{w_i\}_{i=1,\ldots,4}$ has the two properties we want.

(1) By previous remarks $V^{[\Gamma]}$ is stable under ρ, so $\rho(h)w_i = \sum_{i=1}^4 a_{ij}w_j$ for some $a_{ij} \in \mathbb{I}_0^\circ$. This implies that the matrix coefficients of $\rho(h)$ in the basis $\{w_i\}_{i=1,\ldots,4}$ belong to $\mathbb{I}_0^\circ$. Since $\{w_i\}_{i=1,\ldots,4}$ is a symplectic basis, we have $\rho(h) \in \mathrm{GSp}_4(\mathbb{I}_0^\circ)$.
(2) By our choice of $\{e_i\}_{i=1,\ldots,4}$, the $\mathbb{Z}_p$-regular element d is diagonal in this basis. Since $w_i \in \mathbb{I}_{\mathrm{Tr}}^\circ e_i$, d is still diagonal in the basis $\{w_i\}_{i=1,\ldots,4}$.

$\square$

4.8.1. The $\mathbb{I}_0^\circ$-congruence ideal. Starting from Corollary 4.2.6 we can descend further and prove that $\mathfrak{c}_\theta$ is generated by elements invariant under the action of the group of self-twist.

PROPOSITION 4.8.6. *Let $\mathfrak{c}_{\theta,0} = \mathfrak{c}_{\theta,\mathrm{Tr}} \cap \mathbb{I}_0^\circ$. Then $\mathfrak{c}_{\theta,\mathrm{Tr}} = \mathfrak{c}_{\theta,0} \cdot \mathbb{I}_{\mathrm{Tr}}^\circ$.*

PROOF. Let σ be a self-twist and let $\eta_\sigma \colon G_\mathbb{Q} \to (\mathbb{I}_{\mathrm{Tr}}^\circ)^\times$ be the associated finite order character. Let $\mathfrak{c}_{\theta,\mathrm{Tr}}^\sigma = \sigma(\mathfrak{c}_{\mathrm{Tr}}^\theta)$. Since σ is an automorphism of $\mathbb{I}_{\mathrm{Tr}}^\circ$, it induces an isomorphism $\mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{c}_{\theta,\mathrm{Tr}} \cong \mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{c}_{\theta,\mathrm{Tr}}^\sigma$. In particular we can consider the two representations $\rho_{\mathfrak{c}_{\theta,\mathrm{Tr}},1} \colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{c}_{\theta,\mathrm{Tr}}^\sigma)$ and $\rho_{\mathfrak{c}_{\theta,\mathrm{Tr}},1}^\sigma = \sigma \circ \rho_{\mathfrak{c}_{\theta,\mathrm{Tr}},1} \colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{c}_{\theta,\mathrm{Tr}}^\sigma)$. By Corollary 4.2.6 applied to the ideal $\mathfrak{I} = \mathfrak{c}_{\theta,\mathrm{Tr}}$ there exists a representation $\rho_{\mathfrak{c}_{\theta,\mathrm{Tr}},1} \colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ/\mathfrak{c}_{\theta,\mathrm{Tr}})$ such that $\rho_{\mathfrak{c}_{\theta,\mathrm{Tr}}} \cong \mathrm{Sym}^3\rho_{\mathfrak{c}_{\theta,\mathrm{Tr}},1}$. We apply σ to both sides of this equivalence and we obtain

$$\rho_{\mathfrak{c}_{\theta,\mathrm{Tr}}}^\sigma \cong \mathrm{Sym}^3\rho_{\mathfrak{c}_{\theta,\mathrm{Tr}},1}^\sigma.$$

By definition of self-twist we have $\rho^\sigma \cong \eta_\sigma \otimes \rho$. By reducing modulo $\mathfrak{c}_{\theta,\mathrm{Tr}}^\sigma$ we obtain, with the obvious notations,

$$(\rho^\sigma)_{\mathfrak{c}_{\theta,\mathrm{Tr}}^\sigma} \cong \eta_{\sigma,\mathfrak{c}_{\theta,\mathrm{Tr}}^\sigma} \otimes \rho_{\mathfrak{c}_{\theta,\mathrm{Tr}}^\sigma}.$$

Now $(\rho^\sigma)_{\mathfrak{c}^\sigma_{\theta,\mathrm{Tr}}} = (\rho_{\mathfrak{c}_{\theta,\mathrm{Tr}}})^\sigma$, so by combining the two displayed equations we deduce

$$(\rho^\sigma)_{\mathfrak{c}^\sigma_{\theta,\mathrm{Tr}}} \cong \eta_{\sigma,\mathfrak{c}^\sigma_{\theta,\mathrm{Tr}}} \otimes \mathrm{Sym}^3 \rho^\sigma_{\mathfrak{c}_{\theta,\mathrm{Tr}},1}.$$

Since $\eta_{\sigma,\mathfrak{c}^\sigma_{\theta,\mathrm{Tr}}}$ is a finite order character, there exists an extension $\mathbb{I}_1$ of $\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{c}^\sigma_{\theta,\mathrm{Tr}}$ of degree at most 3 and a character $\eta_{\sigma,\mathfrak{c}^\sigma_{\theta,\mathrm{Tr}},1}$ satisfying $(\eta_{\sigma,\mathfrak{c}^\sigma_{\theta,\mathrm{Tr}},1})^3 = \eta_{\sigma,\mathfrak{c}^\sigma_{\theta,\mathrm{Tr}}}$. Then

$$(\rho^\sigma)_{\mathfrak{c}^\sigma_{\theta,\mathrm{Tr}}} \cong \mathrm{Sym}^3(\eta_{\sigma,\mathfrak{c}^\sigma_{\theta,\mathrm{Tr}},1} \otimes \rho^\sigma_{\mathfrak{c}_{\theta,\mathrm{Tr}},1}),$$

so the implication (ii) $\implies$ (i) of Corollary 4.2.6 gives $\mathfrak{c}^\sigma_{\theta,\mathrm{Tr}} \supset \mathfrak{c}_{\theta,\mathrm{Tr}}$. This holds for every $\sigma \in \Gamma$, hence $\bigcap_{\sigma\in\Gamma} \mathfrak{c}^\sigma_{\theta,\mathrm{Tr}} \supset \mathfrak{c}_{\theta,\mathrm{Tr}}$. This is an equality because the inclusion in the other direction is trivial. We conclude that $\mathfrak{c}_{\theta,\mathrm{Tr}}$ is Γ-stable, so the ideal $\mathfrak{c}_{\theta,\mathrm{Tr}} \cap \mathbb{I}^\circ_0$ of $\mathbb{I}^\circ_0$ satisfies $(\mathfrak{c}_{\theta,\mathrm{Tr}} \cap \mathbb{I}^\circ_0)\cdot \mathbb{I}^\circ_{\mathrm{Tr}} = \mathfrak{c}_{\theta,\mathrm{Tr}}$. $\square$

Definition 4.8.7. *We call $\mathfrak{c}_{\theta,0}$ the* fortuitous $(\mathrm{Sym}^3, \mathbb{I}^\circ_0)$-congruence ideal *for the family $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$.*

As for $\mathfrak{c}_\theta$, we will usually refer to $\mathfrak{c}_{\theta,0}$ simply as the "congruence ideal" (we will always specify which one we are considering).

Recall that we always work with the conjugate of ρ satisfying $\rho(H_0) \subset \mathrm{GSp}_4(\mathbb{I}^\circ_0)$. For an ideal $\mathfrak{I}$ of $\mathbb{I}^\circ_0$ we denote by $\rho_\mathfrak{I}\colon H_0 \to \mathrm{GSp}_4(\mathbb{I}^\circ_0/\mathfrak{I})$ the reduction of $\rho|_{H_0}$ modulo $\mathfrak{I}$. The ideal $\mathfrak{c}_{\theta,0}$ admits a characterization similar to that of $\mathfrak{c}_\theta$ and $\mathfrak{c}_{\theta,\mathrm{Tr}}$.

Proposition 4.8.8. *Let P_0 be a prime ideal of $\mathbb{I}^\circ_0$. The following are equivalent.*

(i) $P_0 \supset \mathfrak{c}_{\theta,0}$;
(ii) there exists a finite extension $\mathbb{I}'$ of $\mathbb{I}^\circ_{\mathrm{Tr}}/P_0\mathbb{I}^\circ_{\mathrm{Tr}}$ and a representation $\rho_{P_0\mathbb{I}^\circ_{\mathrm{Tr}},1}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}')$ such that $\rho_{P_0\mathbb{I}^\circ_{\mathrm{Tr}}} \cong \mathrm{Sym}^3\rho_{\mathbb{I}'}$ over $\mathbb{I}'$;
(iii) for one prime P of $\mathbb{I}^\circ_{\mathrm{Tr}}$ lying above P_0 there exists a finite extension I' of $\mathbb{I}^\circ_{\mathrm{Tr}}/P$ and a representation $\rho_{P,1}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}')$ such that $\rho_P \cong \mathrm{Sym}^3\rho_{P,1}$ over $\mathbb{I}'$;
(iv) there exists a representation $\rho_{P_0,1}\colon H_0 \to \mathrm{GL}_2(\mathbb{I}^\circ_0/\mathfrak{I})$ such that $\rho_{P_0} \cong \mathrm{Sym}^3\rho_{P_0,1}$ over $\mathbb{I}^\circ_0/\mathfrak{I}$.

Proof. We prove the chain of implications (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv). If $P_0 \supset \mathfrak{c}_{\theta,0}$ then $P_0 \cdot \mathbb{I}^\circ_{\mathrm{Tr}} \supset \mathfrak{c}_{\theta,0} \cdot \mathbb{I}^\circ_{\mathrm{Tr}} = \mathfrak{c}_{\theta,\mathrm{Tr}}$. Now (ii) follows from Corollary 4.2.6.

If (ii) holds for some $\mathbb{I}'$ and $\rho_{P_0\mathbb{I}^\circ_{\mathrm{Tr}},1}$ and if P is a prime of $\mathbb{I}^\circ_{\mathrm{Tr}}$ lying above P_0 then $P \supset P_0\mathbb{I}^\circ_{\mathrm{Tr}}$, so it makes sense to reduce $\rho_{P_0\mathbb{I}^\circ_{\mathrm{Tr}},1}$ modulo $P\mathbb{I}'$. The resulting representation $\rho_{P,1}\colon G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}^\circ_{\mathrm{Tr}}/P)$ satisfies (iii).

If (iii) is satisfied by some $\rho_{P_0,1}$ then $\rho_{P_0,1} = \rho_{P,1}|_{H_0}$ satisfies (iv).

We complete the proof by showing that (iv) $\implies$ (ii) and (iii) $\implies$ (i). If (iv) holds then the image of ρ_{P_0} is contained in $\mathrm{Sym}^3\mathrm{GL}_2(\mathbb{I}^\circ_0/\mathfrak{I})$. Since $\rho_{P_0} = \rho_{P_0\mathbb{I}^\circ_{\mathrm{Tr}}}|_{H_0}$ Lemma 3.11.5 implies that, after extending the coefficients to a finite extension $\mathbb{I}'_0$ of $\mathbb{I}^\circ_{\mathrm{Tr}}/P_0\mathbb{I}^\circ_{\mathrm{Tr}}$ the image of $\rho_{P_0\mathbb{I}^\circ_{\mathrm{Tr}}}$ is contained in $\mathrm{Sym}^3\mathrm{GL}_2(\mathbb{I}'_0)$. This proves (ii).

Suppose that (iii) holds. By Corollary 4.2.6 $P \supset \mathfrak{c}_{\theta,\mathrm{Tr}}$. Hence $P_0 = P \cap \mathbb{I}^\circ_0 \supset \mathfrak{c}_{\theta,0}$, which is condition (i). $\square$

The following is a corollary of Proposition 4.2.4.

Corollary 4.8.9. *The ideal $\mathfrak{c}_{\theta,0}$ is non-zero.*

Proof. If $\mathfrak{c}_{\theta,0} = 0$, then $\mathfrak{c}_\theta = \mathfrak{c}_{\theta,0} \cdot \mathbb{I}^\circ = 0$. This contradicts Proposition 4.2.4. $\square$

4.9. Unipotent subgroups of the image of a big Galois representation

We give a definition and a lemma that will be important in the following. Let $B \hookrightarrow A$ an integral extension of Noetherian integral domains.

DEFINITION 4.9.1. *An* A-lattice in B *is an* A*-submodule of* B *generated by the elements of a basis of* $Q(B)$ *over* $Q(A)$.

The following is essentially [**Lang16**, Lemma 4.10]. The proof is the same as that given in *loc. cit.*.

LEMMA 4.9.2. *Every* A*-lattice in* B *contains a non-zero ideal of* B*. Conversely, every non-zero ideal of* B *contains an* A*-lattice in* B*.*

Let $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ be a finite slope family of GSp_4-eigenforms and let $\rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ be the representation associated with θ. For every root α, we identify the unipotent group $U^\alpha(\mathbb{I}^\circ_0)$ with $\mathbb{I}^\circ_0$ and $\mathrm{Im}\,\rho \cap U^\alpha(\mathbb{I}^\circ_0)$ with a $\mathbb{Z}_p$-submodule of $\mathbb{I}^\circ_0$. The goal of this section is to show that, for every α, $\mathrm{Im}\,\rho \cap U^\alpha$ contains a basis of a Λ_h-lattice in $\mathbb{I}^\circ_0$. Our strategy is similar to that of Section 2.3.4, which in turn is inspired by [**HT15**] and [**Lang16**]. We proceed in two main steps, by showing that:

(1) there exists a non-critical arithmetic prime $P_{\underline{k}} \subset \Lambda_h$ such that $\mathrm{Im}\,\rho_{P_{\underline{k}}\mathbb{I}^\circ_0} \cap U^\alpha(\mathbb{I}^\circ_0/P_{\underline{k}}\mathbb{I}^\circ_0)$ contains a basis of a $\Lambda_h/P_{\underline{k}}$-lattice in $\mathbb{I}^\circ_0/P_{\underline{k}}\mathbb{I}^\circ_0$;
(2) the natural morphism $\mathrm{Im}\,\rho \cap U^\alpha(\mathbb{I}^\circ_0) \to \mathrm{Im}\,\rho_{P_{\underline{k}}\mathbb{I}^\circ_0} \cap U^\alpha(\mathbb{I}^\circ_0/P_{\underline{k}}\mathbb{I}^\circ_0)$ is surjective, so we can lift a basis as in point (1) to a basis of a Λ_h-lattice in $\mathbb{I}^\circ_0$.

Part (1) is proved by adapting the work of [**Lang16**, Sections 3 and 5] to our situation and combining it with Theorem 3.11.3. Part (2) will result from an application of Proposition 4.7.1.

4.9.1. Big image at a non-critical arithmetic prime.

We choose an arithmetic prime $P_{\underline{k}} \subset \Lambda_h$ satisfying the following conditions:

(1) $P_{\underline{k}}$ is non-critical in the sense of Definition 4.1.13;
(2) for every prime $\mathfrak{P} \subset \mathbb{I}^\circ$ lying above $P_{\underline{k}}$, the classical eigenform corresponding to $\mathfrak{P}$ satisfies the assumptions of Theorem 3.11.3 (i.e. it is not the symmetric cube lift of a GL_2-eigenform).

Note that the form corresponding to the prime $\mathfrak{P}$ in (2) is necessarily classical because $P_{\underline{k}}$ is non-critical. We have to show that a prime with the desired properties exists.

LEMMA 4.9.3. *There exists an arithmetic prime* $P_{\underline{k}} \subset \Lambda_h$ *satisfying conditions (1) and (2) above.*

PROOF. Let Σ^{ncr} be the set of non-critical arithmetic primes of Λ_h. By Proposition 4.1.15 Σ^{ncr} is Zariski-dense in Λ_h. Consider the set S^{Sym^3} of prime ideals $\mathfrak{P}$ of $\mathbb{I}^\circ$ satisfying the following conditions:

(1) $\mathfrak{P} \cap \Lambda_h \in \Sigma^{\mathrm{ncr}}$;
(2) the classical eigenform associated with $\mathfrak{P}$ is the symmetric cube lift of a GL_2-eigenform.

The inclusion $\Lambda_h \hookrightarrow \mathbb{I}^\circ$ is finite and defines a map $w\colon S^{\mathrm{Sym}^3} \to \Sigma^{\mathrm{ncr}}$ (the usual weight map). It is sufficient to show that w is not surjective. By contradiction suppose that it is. Then Lemma 1.2.11 implies that the Zariski-closure of S^{Sym^3} contains an irreducible component of $\mathbb{I}^\circ$. Since $\mathbb{I}^\circ$ is irreducible, this means that S^{Sym^3} is Zariski-dense in $\mathbb{I}^\circ$. By definition the congruence ideal $\mathfrak{c}_\theta$ is contained in the intersection of the primes in the set S^{Sym^3}, so it must be 0. This contradicts Proposition 4.2.4. $\square$

Let $\mathfrak{m}_0$ denote the maximal ideal of $\mathbb{I}^\circ_0$. We define a subgroup H of H_0 by

$$H = \{g \in H_0 \mid \rho(g) \equiv 1 \pmod{\mathfrak{m}_0}\}.$$

This is a normal open subgroup of H_0, hence of $G_{\mathbb{Q}}$. Thanks to Proposition 4.8.2 we can suppose that $\rho(H_0) \subset \mathrm{GSp}_4(\mathbb{I}_0^\circ)$. We define a representation $\rho_0\colon H \to \mathrm{Sp}_4(\mathbb{I}_0^\circ)$ by setting

$$\rho_0 = \rho|_H \otimes \det(\rho|_H)^{-1/2}.$$

Here the square root is defined via the usual power series, that converges on $\rho(H)$.

Even though our results are all stated for the representation ρ, in an intermediate step we will need to work with ρ_0 and its reduction modulo a prime ideal of $\mathbb{I}_0^\circ$. In order to transfer our results to ρ_0 we need to relate the images of the two representations to each other.

4.9.2. Subnormal subgroups of symplectic groups. Let R be a local ring in which 2 is a unit. In the proof of [**Lang16**, Proposition 5.3], the author compares the images of ρ and ρ_0 via the classification of the subnormal subgroups of $\mathrm{GL}_2(R)$ by Tazhetdinov [**Taz83**]. Our technique relies on the analogous classification of the subnormal subgroups of $\mathrm{Sp}_4(R)$, which is also due to Tazhetdinov [**Taz85**]. We recall the main result of his paper. If N and K are two groups, we write $N \lhd K$ if N is a normal subgroup of K. Let m be a positive integer. We write $N \lhd^m K$ if there exist subgroups K_i of K, for $i = 0, 1, 2, \ldots, m$, that fit into a chain

$$N = K_0 \lhd K_1 \lhd K_2 \lhd \ldots \lhd K_m = K.$$

We say that a subgroup N of K is subnormal if $N \lhd^m K$ for some m.

For an ideal J of R, let $\Gamma_R(J)$ be the principal congruence subgroup of $\mathrm{Sp}_4(R)$ of level J. For $M \in \mathrm{Sp}_4(R)$, let $J(M)$ be the smallest ideal of R such that $M \in \{\pm 1\} \cdot \Gamma_R(J)$. If N is a subgroup of $\mathrm{Sp}_4(R)$ let $J(N) = \sum_{M\in N} J(M)$, so that $N \subset \{\pm 1\} \cdot \Gamma_R(J(N))$. For a positive integer m, let $f(m) = \frac{1}{10}(11^m - 1)$.

Theorem 4.9.4. [**Taz85**, Theorem] *If N is a subgroup of $\mathrm{Sp}_4(R)$ such that $N \lhd^m \mathrm{Sp}_4(R)$, then*

$$\Gamma_R(J(N)^{f(m)}) \subset N \subset \{\pm 1\} \cdot \Gamma_R(J(N)).$$

We will need the following corollary.

Corollary 4.9.5. *If N is a subnormal subgroup of $\mathrm{Sp}_4(R)$ and N is not contained in $\{\pm 1\}$, then N contains a non-trivial congruence subgroup of $\mathrm{Sp}_4(R)$.*

Proof. If N is not contained in $\{\pm 1\}$, the ideal $J(N)$ is non-zero. The conclusion follows from Theorem 4.9.4. □

Let $P_{\underline{k}}$ be the arithmetic prime we chose in the beginning of the section. By the étaleness condition in Definition 4.1.13, $P_{\underline{k}}\mathbb{I}^\circ$ is an intersection of distinct primes of $\mathbb{I}^\circ$, so $P_{\underline{k}}\mathbb{I}_0^\circ$ is an intersection of distinct primes of $\mathbb{I}_0^\circ$. Let $\mathfrak{Q}_1, \mathfrak{Q}_2, \ldots, \mathfrak{Q}_d$ be the prime divisors of $P_{\underline{k}}\mathbb{I}_0^\circ$.

Let $\mathfrak{I}$ be either $P_{\underline{k}}\mathbb{I}_0^\circ$ or $\mathfrak{Q}_i$ for some $i \in \{1, 2, \ldots, d\}$. Let $\rho_{\mathfrak{I}}\colon H_0 \to \mathrm{GSp}_4(\mathbb{I}_0^\circ/\mathfrak{I})$ and $\rho_{0,\mathfrak{I}}\colon H \to \mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{I})$ be the reductions modulo $\mathfrak{I}$ of ρ and ρ_0, respectively. Let $\mathcal{G} = \rho_{\mathfrak{I}}(H)$ and $\mathcal{G}_0 = \rho_{0,\mathfrak{I}}(H)$. Let $f\colon \mathrm{GSp}_4(\mathbb{I}_0^\circ) \to \mathrm{Sp}_4(\mathbb{I}_0^\circ)$ be the homomorphism sending g to $\det(g)^{-1/2}g$. We have $\mathcal{G} = f(\mathcal{G}_0)$ by definition of ρ_0. We show an analogue of Lemma 2.3.20.

Lemma 4.9.6. *The group $\mathcal{G}$ contains a non-trivial congruence subgroup of $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{I})$ if and only if the group $\mathcal{G}_0$ contains a non-trivial congruence subgroup of $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{I})$.*

Proof. Clearly the group $\mathcal{G} \cap \mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{I})$ is a normal subgroup of $\mathcal{G}$. Then the group $f(\mathcal{G} \cap \mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{I}))$ is a normal subgroup of $f(\mathcal{G})$. Now $f(\mathcal{G}) = \mathcal{G}_0$ and $f(\mathcal{G} \cap \mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{I})) = \mathcal{G} \cap \mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{I})$ since the restriction of f to $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{I})$ is the identity. Hence $\mathcal{G} \cap \mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{I})$ is a subnormal subgroup of $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{I})$ if and only if $\mathcal{G}_0$ is a subnormal subgroup of $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{I})$. By Corollary 4.9.5 a subnormal subgroup of $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{I})$ contains a non-trivial congruence subgroup of $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{I})$ if and only if it is not contained in $\{\pm 1\}$. Neither $\mathcal{G} \cap \mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{I})$ nor $\mathcal{G}_0$ is contained in $\{\pm 1\}$, since the image of $\rho_{\mathfrak{P}_i}$ contains a non-trivial congruence subgroup of $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{P}_i)$ by Theorem 3.11.3. Hence Corollary 4.9.5 gives the desired equivalence. □

The following is a consequence of Proposition 4.6.9 and Lemma 4.9.6, together with our choice of $P_{\underline{k}}$.

LEMMA 4.9.7. *Let $\mathfrak{Q}$ be a prime of $\mathbb{I}_0^\circ$ lying over $P_{\underline{k}}$. Then the image of $\rho_{0,\mathfrak{Q}}$ contains a non-trivial congruence subgroup of* $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{Q})$.

PROOF. Let $\mathfrak{P}$ be a prime of $\mathbb{I}_{\mathrm{Tr}}^\circ$ lying above $\mathfrak{Q}$. Let $\rho_{\mathfrak{P}}$ be the reduction of $\rho\colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{I}_{\mathrm{Tr}}^\circ)$ modulo $\mathfrak{P}$. The representation $\rho_{\mathfrak{Q}}$ is the restriction of $\rho_{\mathfrak{P}}$ to H_0. By Proposition 4.6.9 the group $\rho_{\mathfrak{P}}(G_{\mathbb{Q}})$ contains a non-trivial congruence subgroup of $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{Q})$. Since H is a finite index subgroup of $G_{\mathbb{Q}}$, the group $\rho_{\mathfrak{Q}}(H)$ is a finite index subgroup of $\rho_{\mathfrak{P}}(G_{\mathbb{Q}})$, so it also contains a non-trivial congruence subgroup of $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{Q})$. Now the conclusion follows from Lemma 4.9.6 applied to $\mathfrak{I} = \mathfrak{Q}$. $\square$

4.9.3. Big image in a product. Lifting the congruence subgroup of Proposition 4.9.7 to $\mathbb{I}^\circ$ does not provide the information we need on the image of ρ_0. We need the following fullness result for $\rho_{P_{\underline{k}}}$.

PROPOSITION 4.9.8. *The image of the representation $\rho_{P_{\underline{k}}}$ contains a non-trivial congruence subgroup of* $\mathrm{Sp}_4(\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ)$.

The strategy of the proof is similar to that of [**Lang16**, Proposition 5.1]. There is an injective morphism $\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ \hookrightarrow \prod_{i=1}^d \mathbb{I}_0^\circ/\mathfrak{Q}_i$. Let G be the image of $\mathrm{Im}\,\rho_{0,P_{\underline{k}}}$ in $\prod_{i=1}^d \mathbb{I}_0^\circ/\mathfrak{Q}_i$ via the previous injection. Proposition 4.9.8 will follow from Lemma 4.9.6, once we prove that G is an open subgroup of $\prod_{i=1}^d \mathbb{I}_0^\circ/\mathfrak{Q}_i$. This is a consequence of a lemma of Ribet (Lemma 4.9.18) and the following.

LEMMA 4.9.9. *Let $1 \leq i < j \leq d$. Then the image of G in $\mathbb{I}_0^\circ/\mathfrak{Q}_i \times \mathbb{I}_0^\circ/\mathfrak{Q}_j$ is open.*

We will show that if the conclusion of the lemma is not true, then there is a self-twist σ of ρ such that $\sigma(\mathfrak{Q}_i) = \mathfrak{Q}_j$, which is a contradiction since $\mathbb{I}_0^\circ$ is fixed by all self-twists. The first part of the proof follows the strategy of [**Lang16**, Proposition 5.3], which is inspired by [**Ri75**, Theorem 3.5].

We will need Goursat's Lemma, that we recall here. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two groups and let $\mathcal{G}$ be a subgroup of $\mathcal{K}_1 \times \mathcal{K}_2$ such that the two projections $\pi_1\colon \mathcal{G} \to \mathcal{K}_1$ and $\pi_2\colon \mathcal{G} \to \mathcal{K}_2$ are surjective. Let $\mathcal{N}_1 = \ker \pi_2$ and $\mathcal{N}_2 = \ker \pi_1$. We identify $\mathcal{N}_1$ and $\mathcal{N}_2$ with $\pi_1(\mathcal{N}_1)$ with $\pi_2(\mathcal{N}_2)$, hence with subgroups of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Clearly $\mathcal{N}_1 \times \mathcal{N}_2 \supset \mathcal{G}$. The natural projections induce a map $\mathcal{G} \to \mathcal{K}_1/\mathcal{N}_1 \times \mathcal{K}_2/\mathcal{N}_2$.

LEMMA 4.9.10. (Goursat's Lemma, [**Go1889**, Sections 11 and 12], [**La02**, Exercise 5, p. 75]) *The image of $\mathcal{G}$ in $\mathcal{K}_1/\mathcal{N}_1 \times \mathcal{K}_2/\mathcal{N}_2$ is the graph of an isomorphism $\mathcal{K}_1/\mathcal{N}_1 \cong \mathcal{K}_2/\mathcal{N}_2$.*

Another ingredient is the the isomorphism theory of open subgroups of GSp_4 over local rings, due to O'Meara [**OM78**]. This replaces the analogous theory for SL_2, that is due to Merzljakov [**Me73**] and appears in the proof of [**Lang16**, Proposition 5.3].

PROOF. (of Lemma 4.9.9) By Lemma 4.9.7 there exist two non-zero ideals $\mathfrak{l}_1$ and $\mathfrak{l}_2$ of $\mathbb{I}_0^\circ/\mathfrak{Q}_i$ and $\mathbb{I}_0^\circ/\mathfrak{Q}_j$, respectively, such that $\Gamma_{\mathbb{I}_0^\circ/\mathfrak{Q}_i}(\mathfrak{l}_1) \subset \mathrm{Im}\,\rho_{0,\mathfrak{Q}_i}$ and $\Gamma_{\mathbb{I}_0^\circ/\mathfrak{Q}_j}(\mathfrak{l}_2) \subset \mathrm{Im}\,\rho_{0,\mathfrak{Q}_j}$. Recall that the domain of the representation ρ_0 is the open normal subgroup H of $G_{\mathbb{Q}}$ defined in the beginning of this subsection. Consider the group

$$H_1 = \{h \in H \mid h \pmod{\mathfrak{Q}_i} \in \Gamma_{\mathbb{I}_0^\circ/\mathfrak{Q}_i}(\mathfrak{l}_1) \text{ and } h \pmod{\mathfrak{Q}_j} \in \Gamma_{\mathbb{I}_0^\circ/\mathfrak{Q}_j}(\mathfrak{l}_2)\}.$$

Since the subgroups $\Gamma_{\mathbb{I}_0^\circ/\mathfrak{Q}_i}(\mathfrak{l}_1)$ and $\Gamma_{\mathbb{I}_0^\circ/\mathfrak{Q}_j}(\mathfrak{l}_2)$ are normal and of finite index in $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{Q}_i)$ and $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{Q}_j)$, respectively, the subgroup H_1 is normal and of finite index in H. It is clearly closed, hence it is also open.

Let $1 \leq i < j \leq d$. The couple (i,j) will be fixed throughout the proof. Let $\mathcal{K}_1 = \rho_{0,\mathfrak{Q}_i}(H_1)$, $\mathcal{K}_2 = \rho_{0,\mathfrak{Q}_j}(H_1)$ and let $\mathcal{G}_0$ be the image of $\rho_0(H_1)$ in $\mathcal{K}_1 \times \mathcal{K}_2$. Note that $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{G}_0$ are profinite and closed since they are continuous images of a Galois group. By definition of $\mathfrak{l}_1$, $\mathfrak{l}_2$ and H_1 we have $\mathcal{K}_1 = \Gamma_{\mathbb{I}_0^\circ/\mathfrak{Q}_i}(\mathfrak{l}_1)$ and $\mathcal{K}_2 = \Gamma_{\mathbb{I}_0^\circ/\mathfrak{Q}_j}(\mathfrak{l}_2)$. In particular $\mathcal{K}_1$ and $\mathcal{K}_2$ are normal and finite index subgroups of $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{Q}_i)$ and $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{Q}_j)$ respectively. Define $\mathcal{N}_1$ and $\mathcal{N}_2$ as in the discussion preceding Lemma 4.9.10. They are normal closed subgroups of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, since they are defined as kernels of continuous maps. In particular $\mathcal{N}_1$ and $\mathcal{N}_2$ are subnormal subgroups of $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{Q}_i)$ and $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{Q}_j)$, respectively.

Suppose that $\mathcal{N}_1$ is open in $\mathcal{K}_1$ and $\mathcal{N}_2$ is open in $\mathcal{K}_2$. Then the product $\mathcal{N}_1 \times \mathcal{N}_2$ is open in $\mathcal{K}_1 \times \mathcal{K}_2$. Since $\mathcal{G}_0$ contains $\mathcal{N}_1 \times \mathcal{N}_2$, it is also open in $\mathcal{K}_1 \times \mathcal{K}_2$. The subgroup $\mathcal{K}_1 \times \mathcal{K}_2 = \Gamma_{\mathbb{I}_0^\circ/\mathfrak{Q}_i}(\mathfrak{l}_1) \times \Gamma_{\mathbb{I}_0^\circ/\mathfrak{Q}_j}(\mathfrak{l}_2)$ is an open subgroup of $\mathbb{I}_0^\circ/\mathfrak{Q}_i \times \mathbb{I}_0^\circ/\mathfrak{Q}_j$, so $\mathcal{G}_0$ is open in $\mathbb{I}_0^\circ/\mathfrak{Q}_i \times \mathbb{I}_0^\circ/\mathfrak{Q}_j$. Then the conclusion of Lemma 4.9.9 is true in this case.

Now suppose that one among $\mathcal{N}_1$ and $\mathcal{N}_2$ is not open. Without loss of generality, let it be $\mathcal{N}_1$. Since $\mathcal{N}_1$ is closed in the profinite group $\mathcal{K}_1$, it is not of finite index in $\mathcal{K}_1$. By Lemma 4.9.10 there is an isomorphism $\mathcal{K}_1/\mathcal{N}_1 \cong \mathcal{K}_2/\mathcal{N}_2$, hence $\mathcal{N}_2$ is not of finite index in $\mathcal{K}_2$. In particular $\mathcal{N}_1$ and $\mathcal{N}_2$ are not of finite index in $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{Q}_i)$ and $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{Q}_j)$, respectively. Since $\mathcal{N}_1$ is subnormal and not of finite index in $\mathrm{Sp}_4(\mathbb{I}_0^\circ/\mathfrak{Q}_i)$, it is contained in $\{\pm 1\}$ by Corollary 4.9.5. The same reasoning gives that $\mathcal{N}_2$ is contained in $\{\pm 1\}$. By definition of H the image of ρ_0 lies in $\Gamma_{\mathbb{I}_0^\circ}(\mathfrak{m}_{\mathbb{I}_0^\circ})$, so the centres of $\mathcal{K}_1$ and $\mathcal{K}_2$ are trivial since $p > 2$. We conclude that $\mathcal{N}_1 = \{1\}$ and $\mathcal{N}_2 = \{1\}$.

By the result of the previous paragraph we have $\mathcal{K}_1/\mathcal{N}_1 \cong \mathcal{K}_1$ and $\mathcal{K}_2/\mathcal{N}_2 \cong \mathcal{K}_2$. Hence Lemma 4.9.10 gives an isomorphism $\alpha\colon \mathcal{K}_1 \cong \mathcal{K}_2$ such that, for every $(x,y) \in \mathcal{K}_1 \times \mathcal{K}_2$, $(x,y) \in \mathcal{G}_0$ if and only if $y = \alpha(x)$. By Corollary 4.6.5, applied to $F = Q(\mathbb{I}_0^\circ/\mathfrak{Q}_i)$, $F_1 = Q(\mathbb{I}_0^\circ/\mathfrak{Q}_j)$, $\Delta = \mathcal{K}_1$, $\Delta_1 = \mathcal{K}_2$, there exists an isomorphism $\alpha\colon Q(\mathbb{I}_0^\circ/\mathfrak{Q}_i) \to Q(\mathbb{I}_0^\circ/\mathfrak{Q}_j)$, a character $\chi\colon \mathcal{K}_1 \to Q(\mathbb{I}_0^\circ/\mathfrak{Q}_j)^\times$ and an element $\gamma \in \mathrm{GSp}_4(Q(\mathbb{I}_0^\circ/\mathfrak{Q}_i))$ such that for every $z \in \mathcal{K}_1$ we have

$$\alpha(z) = \chi(z)\gamma\alpha(z)\gamma^{-1}, \tag{4.20}$$

where as usual we define $\alpha\colon \mathrm{Sp}_4(Q(\mathbb{I}_0^\circ/\mathfrak{Q}_i)) \to \mathrm{Sp}_4(Q(\mathbb{I}_0^\circ/\mathfrak{Q}_j))$ by applying α to the matrix coefficients. Since the centre of $\mathcal{K}_2$ is trivial, the character χ is also trivial. By recalling the definitions of $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{G}_0$ we can rewrite Equation (4.20) as

$$\rho_{0,\mathfrak{Q}_j}(h) = \gamma_0^{-1}\alpha(\rho_{0,\mathfrak{Q}_i}(h))\gamma_0^{-1}$$

for every $h \in H_1$. The last equation gives an isomorphism

$$\rho_{0,\mathfrak{Q}_i}|_{H_1}^{\alpha} \cong \rho_{0,\mathfrak{Q}_j}|_{H_1} \tag{4.21}$$

of representations of H_1 over $Q(\mathbb{I}_0^\circ/\mathfrak{Q}_j)$. Denote by π_j the projection $\mathbb{I}_0^\circ \to \mathbb{I}_0^\circ/\mathfrak{Q}_j$. By definition of ρ_0 we have $\rho_0|_{H_1} = \rho|_{H_1} \otimes (\det \rho|_{H_1})^{-1/2}$. Define a character $\varphi\colon H_1 \to Q(\mathbb{I}_0^\circ/\mathfrak{Q}_j)^\times$ by setting

$$\varphi(h) = \pi_j\left(\frac{\det \rho(h)}{\alpha(\det \rho(h))}\right)$$

for every $h \in H_1$. Then Equation 4.21 implies that

$$\rho_{\mathfrak{Q}_i}|_{H_1}^{\alpha} \cong \varphi \otimes \rho_{\mathfrak{Q}_j}|_{H_1} \tag{4.22}$$

We will use the isomorphism (4.22), together with Proposition 4.4.1, to construct a self-twist for ρ. Let $\mathfrak{P}_i$ and $\mathfrak{P}_j$ be primes of $\mathbb{I}_{\mathrm{Tr}}^\circ$ that lie above $\mathfrak{Q}_i$ and $\mathfrak{Q}_j$, respectively.

Lemma 4.9.11. *The isomorphism $\alpha\colon Q(\mathbb{I}_0^\circ/\mathfrak{Q}_i) \to Q(\mathbb{I}_0^\circ/\mathfrak{Q}_j)$ and the character $\varphi\colon H_1 \to Q(\mathbb{I}_0^\circ/\mathfrak{Q}_j)$ can be extended to an isomorphism $\widetilde{\alpha}\colon Q(\mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{P}_i) \to Q(\mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{P}_j)^\times$ and a character $\widetilde{\varphi}\colon G_\mathbb{Q} \to Q(\mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{P}_j)^\times$, respectively, such that*

$$\rho_{\mathfrak{P}_i}^{\widetilde{\alpha}} \cong \widetilde{\varphi} \otimes \rho_{\mathfrak{P}_j}. \tag{4.23}$$

We prove Lemma 4.9.11 by using obstruction theory, following the strategy presented in [**Lang16**, Section 5]. The proofs in *loc. cit.* only need a few changes. Let n be a positive integer. Let N be a normal subgroup of $G_\mathbb{Q}$. Let K be a finite extension of $\mathbb{Q}_p$ and let $r\colon N \to \mathrm{GL}_n(K)$ be a continuous, absolutely irreducible representation. For every $g \in G_\mathbb{Q}$ let

$r^g\colon N \to \mathrm{GL}_n(K)$ be the representation defined by $r^g = r(ghg^{-1})$ for every $h \in N$. Assume that the following condition holds:

(obstr) for every $g \in G_\mathbb{Q}$ there is an isomorphism $r^g \cong r$ over K.

PROPOSITION 4.9.12. *There exists a map $c\colon G_\mathbb{Q} \to \mathrm{GL}_n(K)$ with the following properties:*
(1) $c(1) = 1$;
(2) $c(hg) = c(h)c(g)$ for every $h \in N$, $g \in G_\mathbb{Q}$;
(3) $r = c(g)^{-1}r^g c(g)$ for every $g \in G_\mathbb{Q}$.

Let $\Delta = G_\mathbb{Q}/N$. The map $b\colon (G_\mathbb{Q})^2 \to \mathrm{GL}_n(K)$ defined by $b(g_1, g_2) = c(g_1)c(g_2)c(g_1g_2)^{-1}$ is trivial on N^2, hence we can and do consider it as a map $b\colon \Delta^2 \to \mathrm{GL}_n(K)$. Since r is absolutely irreducible, b is a 2-cocycle with values in $K^\times$. We denote by $\mathrm{Ob}\,(r)$ the class of b in the cohomology group $H^2(\Delta, K^\times)$. We denote by 1 the class of the trivial cocycle in $H^2(\Delta, K^\times)$.

An *extension* of r to $G_\mathbb{Q}$ is a representation $\widetilde{r}\colon G_\mathbb{Q} \to \mathrm{GL}_n(K)$ satisfying $\widetilde{r}|_N = r$.

PROPOSITION 4.9.13.
(1) There exists an extension $\widetilde{r}$ of r to $G_\mathbb{Q}$ if and only if $\mathrm{Ob}\,(r) = 1$.
(2) If $\widetilde{r}$ is an extension of r to $G_\mathbb{Q}$ then every other extension $\widetilde{r}'$ of r to $G_\mathbb{Q}$ satisfies

$$\widetilde{r}' \cong \widetilde{r} \otimes \psi$$

for a character $\psi\colon G_\mathbb{Q} \to K^\times$ that is trivial on N.

Let $E_1 = Q(\mathbb{I}_0^\circ/\mathfrak{Q}_i)$, $E_2 = Q(\mathbb{I}_0^\circ/\mathfrak{Q}_j)$, $K_1 = Q(\mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{P}_i)$, $K_2 = Q(\mathbb{I}_{\mathrm{Tr}}^\circ/\mathfrak{P}_j)$. They are all p-adic fields and there are natural inclusions $E_1 \subset K_1$ and $E_2 \subset K_2$. Recall that there is an isomorphism $\alpha\colon E_1 \to E_2$ and a character $\varphi\colon H_1 \to E_2^\times$ that fit in Equation (4.22). Let L_1 and L_2 be arbitrary finite extensions of K_1 and K_2 respectively. Consider $\rho_1|_{H_1}$ as a representation $H_1 \to \mathrm{GL}_4(L_1)$, then $\rho_2|_{H_1}$ and $\rho_1|_{H_1}^\alpha$ as representations $H_1 \to \mathrm{GL}_4(L_2)$ and φ as a character $H_1 \to L_2^\times$. We check that each of these representations satisfies condition (obstr) with $N = H_1$ and K equal to the corresponding coefficient field.

LEMMA 4.9.14. (cf. [**Lang16**, Lemma 5.5]) *The representations $\rho_1|_{H_1}$, $\rho_2|_{H_1}$, $\rho_1|_{H_1}^\alpha$ and φ satisfy condition* (obstr). *Moreover* $\mathrm{Ob}\,(\rho_1|_{H_1})$ *and* $\mathrm{Ob}\,(\rho_1|_{H_1})$ *are trivial, and* $\mathrm{Ob}\,(\varphi \otimes \rho_2|_{H_1}) = \mathrm{Ob}\,(\varphi)\mathrm{Ob}\,(\rho_2|_{H_1})$.

PROOF. For every $g \in G_\mathbb{Q}$ there is an isomorphism $\rho_1^g \cong \rho_1$ over L_1. By restriction we obtain an isomorphism $\rho_1^g|_{H_1} \cong \rho_1|_{H_1}$ over L_1, so ρ_1 satisfies (obstr). The same reasoning shows that ρ_2 satisfies (obstr). Moreover the classes $\mathrm{Ob}\,(\rho_1|_{H_1})$ and $\mathrm{Ob}\,(\rho_1|_{H_1})$ are trivial by Proposition 4.9.13, since $\rho_1|_{H_1}$ and $\rho_2|_{H_1}$ both admit extensions to $G_\mathbb{Q}$.

Let $\tau\colon K_1 \to \overline{\mathbb{Q}}_p$ be an extension of α. Then the representation ρ_1^τ is an extension to $G_\mathbb{Q}$ of $\rho_1|_{H_1}^\alpha$. In particular $(\rho_1^\tau)^g \cong \rho_1^\tau$ over $\overline{\mathbb{Q}}_p$ for every $g \in G_\mathbb{Q}$, so

$$\rho_1|_{H_1}^\alpha \cong (\rho_1^\tau)^g|_{H_1} \cong (\rho_1^\tau)|_{H_1} = \rho_1|_{H_1}^\alpha$$

over $\overline{\mathbb{Q}}_p$. The previous isomorphism also holds over K_2.

Since $\rho_1|_{H_1}^\alpha$ and $\rho_2|_{H_1}$ satisfy (obstr), for every $g \in G_\mathbb{Q}$ we have

$$\varphi \otimes \rho_2|_{H_1} \cong \rho_1|_{H_1}^\alpha \cong (\rho_1|_{H_1}^\alpha)^g \cong \varphi^g \otimes (\rho_2|_{H_1})^g \cong \varphi^g \otimes \rho_2|_{H_1},$$

so $\rho_2|_{H_1} \cong \varphi^{-1} \otimes \varphi^g \otimes \rho_2|_{H_1}$. Recall that the representation ρ_2 is the p-adic Galois representation associated with a classical GSp_4-eigenform. Hence by Theorem 3.11.3 the image of ρ_2 is open in $\mathrm{GSp}_4(K_2)$. This implies that ρ_2 cannot be isomorphic to a twist of itself by a non-trivial character, so the previous equality gives $\varphi^g \cong \varphi$. We conclude that φ satisfies (obstr).

Let c_2 and c_φ be maps $G_\mathbb{Q} \to L_2^\times$ satisfying the conditions of Proposition 4.9.12 for $r = \rho_2|_{H_1}$ and $r = \varphi$ respectively. Then an easy check shows that $c_\varphi \cdot c_2$ satisfies the conditions of Proposition 4.9.12 for $r = \phi \otimes \rho_2|_{H_1}$, so that $\mathrm{Ob}\,(\varphi \otimes \rho_2|H_1) = \mathrm{Ob}\,(\varphi)\mathrm{Ob}\,(\rho_2|_{H_1})$. $\square$

We show that for a certain choice of L_1 and L_2 there exists an isomorphism $\widetilde{\alpha}\colon L_1 \to L_2$ extending $\alpha\colon E_1 \to E_2$ and a character $\widetilde{\varphi}'\colon G_{\mathbb{Q}} \to L_2^\times$ extending $\varphi\colon G_{\mathbb{Q}} \to L_2^\times$. Let $\tau\colon K_1 \to \overline{\mathbb{Q}}_p$ be an arbitrary extension of α to K_1. Let $L_2 = K_2 \cdot \tau(K_1)$. Let $\tau'\colon L_2 \to \overline{\mathbb{Q}}_p$ be an extension of $\tau^{-1}\colon \tau(K_1) \to K_1$ and let $L_1 = \tau'(L_2)$. Set $\widetilde{\alpha} = (\tau')^{-1}\colon L_1 \to L_2$. Then $\widetilde{\alpha}$ is an extension of α. In particular $\rho_1^{\widetilde{\alpha}}\colon G_{\mathbb{Q}} \to L_2^\times$ is an extension of $\rho_1|_{H_1}^{\alpha}$, so $\mathrm{Ob}\,(\rho_1|_{H_1}^{\alpha}) = 1$. Thanks to Lemma 4.9.14 we have

$$1 = \mathrm{Ob}\,(\rho_1|_{H_1}^{\alpha}) = \mathrm{Ob}\,(\varphi)\mathrm{Ob}\,(\rho_2|_{H_1}) = \mathrm{Ob}\,(\varphi),$$

so φ can be extended to a character $\widetilde{\varphi}'\colon G_{\mathbb{Q}} \to L_2^\times$.

Thanks to the following proposition we can modify $\widetilde{\varphi}'$ in order to satisfy Equation (4.23).

Lemma 4.9.15. (cf. [**Lang16**, Lemma 5.6]) *There exists an extension $\widetilde{\varphi}\colon G_{\mathbb{Q}} \to L_2^\times$ of $\varphi\colon H_1 \to L_2^\times$ such that there is an isomorphism*

$$\rho_1^{\widetilde{\alpha}} \cong \widetilde{\varphi} \otimes \rho_2 \tag{4.24}$$

over $L_2^\times$.

Proof. Since $\widetilde{\varphi}'$ is an extension of φ, the representation $\widetilde{\varphi}' \otimes \rho_2$ is an extension of $\rho_1|_{H_1}$. Since $\rho_1^{\widetilde{\alpha}}$ is also an extension of $\rho_1|_{H_1}$, Proposition 4.9.13 implies that there is a character $\psi\colon G_{\mathbb{Q}} \to L_2^\times$, trivial on H_1, such that $\rho_1^{\widetilde{\alpha}} \cong \psi \otimes \widetilde{\varphi}' \otimes \rho_2$. Then the character $\widetilde{\varphi}$ defined as $\psi\widetilde{\varphi}'$ satisfies Equation (4.24). □

In order to show Lemma 4.9.11 it is sufficient to show that $\widetilde{\alpha}$ restricts to an isomorphism $\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$ and that $\widetilde{\varphi}$ takes values in $\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$. We write $(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j)[\widetilde{\varphi}]$ for the subring of L_2 generated over $\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$ by the values of $\widetilde{\varphi}$.

Remark 4.9.16. *Since $\widetilde{\varphi} \otimes \rho_2$ takes values in $\mathrm{GL}_4((\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j)[\widetilde{\varphi}])$, the representation $\rho_1^{\widetilde{\alpha}}$ also takes values in $\mathrm{GL}_4((\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j)[\widetilde{\varphi}])$. In particular $\widetilde{\alpha}(\mathrm{Tr}(\rho_1(h))) \in (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j)[\widetilde{\varphi}]$ for every $h \in H_1$. Since the traces of the representation ρ_1 generate the ring $\mathbb{I}_{\mathrm{Tr}}/\mathfrak{P}_i$ over $\mathbb{Z}_p$, we conclude that $\widetilde{\alpha}$ restricts to an isomorphism $(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i)[\widetilde{\varphi}] \to (\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j)[\widetilde{\varphi}]$.*

Lemma 4.9.17. (cf. [**Lang16**, Lemma 5.7]) *There are equalities $(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i)[\widetilde{\varphi}] = \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i$ and $(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j)[\widetilde{\varphi}] = \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$.*

Proof. As before let χ be the p-adic cyclotomic character. Recall that $\mathfrak{P}_i$ and $\mathfrak{P}_j$ lie over the prime $P_{\underline{k}}$ of Λ, with $\underline{k} = (k_1, k_2)$. By taking determinants in Equation (4.24) and using Remark 4.1.20 we obtain

$$\widetilde{\varphi}^4 = \frac{\det(\rho_1^{\widetilde{\alpha}})}{\det(\rho_2)} = \frac{\widetilde{\alpha}(\chi^{2(k_1+k_2-3)})}{\chi^{2(k_1+k_2-3)}}. \tag{4.25}$$

Since the quantity on the right defines an element of $\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$, the degree $[(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j)[\widetilde{\varphi}]\colon \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j]$ is at most 4. In particular the extension $(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j)[\widetilde{\varphi}]$ is obtained from $\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$ by adding a 2-power root of unit, hence it is an unramified extension. The same is true for the extension $(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i)[\widetilde{\varphi}]$ over $\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i$ thanks to the isomorphism $\widetilde{\alpha}$.

Note that the residue fields of $(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i)[\widetilde{\varphi}]$ and $(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j)[\widetilde{\varphi}]$ are identified by $\widetilde{\alpha}$ and those of $\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i$ and $\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$ coincide by the non-criticality of $P_{\underline{k}}$ (see the étaleness condition in Definition 4.1.13). Let $\mathbb{E}$ and $\mathbb{F}$ be the residue fields of $(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i)[\widetilde{\varphi}]$ and $\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i$ respectively. To conclude the proof it is sufficient to show that $\mathbb{E} = \mathbb{F}$. The isomorphism $\widetilde{\alpha}$ induces an automorphism $\overline{\alpha}$ of the residue field $\mathbb{E}$ and the character $\widetilde{\varphi}$ induces a character $\overline{\varphi}\colon G_{\mathbb{Q}} \to \mathbb{E}^\times$. Then $\mathbb{E}$ is the field $\mathbb{F}[\overline{\varphi}]$ generated over $\mathbb{F}$ by the values of $\overline{\varphi}$. Let s be an integer such that $\overline{\alpha}$ is the s-th power of the Frobenius automorphisms. By reducing Equation 4.25 modulo the maximal ideal of $(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j)[\widetilde{\varphi}]$ we obtain

$$\overline{\varphi}^4 = \frac{\overline{\alpha}(\chi^{2(k_1+k_2-3)})}{\chi^{2(k_1+k_2-3)}} = \chi^{2(p^s-1)(k_1+k_2-3)}.$$

Since p is odd, $2(p^s-1)$ is a multiple of 4. In particular $\mathbb{F}[\overline{\varphi}^4] \subset \mathbb{F}[\chi^4]$, that implies $\mathbb{F}[\overline{\varphi}] \subset \mathbb{F}$. We conclude that $\mathbb{E} = \mathbb{F}$, as desired. □

Thanks to Remark 4.9.16 and Lemma 4.9.17, $\widetilde{\alpha}\colon L_1 \to L_2$ restricts to an isomorphism $\widetilde{\alpha}\colon \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$ and $\widetilde{\varphi}$ takes values in $\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$. Hence $\widetilde{\alpha}$ and $\widetilde{\varphi}$ satisfy the hypotheses of Lemma 4.9.11.

We conclude the proof of Lemma 4.9.9. Set $\sigma = \widetilde{\alpha}\colon \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_i \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$ and $\eta = \widetilde{\varphi}\colon G_{\mathbb{Q}} \to \mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P}_j$. Thanks to Lemma 4.9.11, σ and η satisfy the hypotheses of Proposition 4.4.1. Hence there exists a self-twist $\widetilde{\sigma}\colon \mathbb{I}^\circ_{\mathrm{Tr}} \to \mathbb{I}^\circ_{\mathrm{Tr}}$ for ρ over Λ_h that induces σ. In particular $\widetilde{\sigma}(\mathfrak{P}_i) = \mathfrak{P}_j$. Since $\mathfrak{P}_i$ and $\mathfrak{P}_j$ lie over different primes of $\mathbb{I}^\circ_0$, the self-twist $\widetilde{\sigma}$ does not fix $\mathbb{I}^\circ_0$, a contradiction. Recall that the assumption of this argument is that $\mathcal{N}_1$ is not open in $\mathcal{K}_1$ or $\mathcal{N}_2$ is not open in $\mathcal{K}_2$. When this is not the case we already observed that the conclusion of Lemma 4.9.9 holds, so the proof of the lemma is complete. $\square$

We recall a lemma of Ribet. Let k be an integer greater than 2 and let $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k$ be profinite groups. Suppose that for every $i \in \{1, 2, \ldots, k\}$ the following condition holds:

(comm) if $\mathcal{K}$ is an open subgroup of $\mathcal{G}_i$, then the closure of the commutator subgroup of $\mathcal{K}$ is also open in $\mathcal{G}_i$.

Let $\mathcal{G}_0$ be a closed subgroup of $\mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \times \mathcal{G}_k$.

Lemma 4.9.18. [**Ri75**, Lemma 3.4] *Suppose that for every i, j with $1 \le i < j \le k$ the image of $\mathcal{G}_0$ in $\mathcal{G}_i \times \mathcal{G}_j$ is an open subgroup of $\mathcal{G}_i \times \mathcal{G}_j$. Then $\mathcal{G}_0$ is an open subgroup of $\mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \times \mathcal{G}_k$.*

We are ready to complete the proof of Proposition 4.9.7.

Proof. For $1 \le i \le k$ let $\mathcal{G}_i$ be the image of $\rho_{0,\mathfrak{P}_i}\colon H \to \mathrm{Sp}_4(\mathbb{I}^\circ_0/\mathfrak{Q}_i)$. As before let $\mathcal{G}_0$ be the image of $\mathrm{Im}\,\rho_{0,P_{\underline{k}}}$ via the inclusion $\mathrm{Sp}_4(\mathbb{I}^\circ_0/P_{\underline{k}}\mathbb{I}^\circ_0) \hookrightarrow \prod_i \mathrm{Sp}_4(\mathbb{I}^\circ_0/\mathfrak{Q}_i\mathbb{I}^\circ_0)$. The groups $\mathcal{G}_i$ are profinite and they satisfy condition (comm). The group $\mathcal{G}_0$ is closed since it is the continuous image of H. By Lemma 4.9.9 it is open in $\mathcal{G}_i \times \mathcal{G}_j$ for every i, j with $1 \le i < j \le d$. Hence Lemma 4.9.18 implies that $\mathcal{G}_0$ is open in $\prod_i \mathcal{G}_i = \prod_i \mathcal{G}_i$.

By Proposition 4.9.7 the group $\mathcal{G}_i$ is open in $\mathrm{Sp}_4(\mathbb{I}^\circ_0/P_{\underline{k}}\mathbb{I}^\circ_0)$ for every i, hence $\prod_i \mathcal{G}_i$ is open in $\prod_i \mathrm{Sp}_4(\mathbb{I}^\circ_0/\mathfrak{Q}_i\mathbb{I}^\circ_0)$. We deduce that $\mathcal{G}_0$ is open in $\prod_i \mathrm{Sp}_4(\mathbb{I}^\circ_0/P_{\underline{k}}\mathbb{I}^\circ_0)$, so $\mathrm{Im}\,\rho_{0,P_{\underline{k}}}$ is open in $\mathrm{Sp}_4(\mathbb{I}^\circ_0/P_{\underline{k}}\mathbb{I}^\circ_0)$. In particular $\mathrm{Im}\,\rho_{0,P_{\underline{k}}}$ contains a non-trivial congruence subgroup of $\mathrm{Sp}_4(\mathbb{I}^\circ_0/P_{\underline{k}}\mathbb{I}^\circ_0)$. Now Lemma 4.9.6 applied to $\mathfrak{I} = P_{\underline{k}}$ implies that $\mathrm{Im}\,\rho_{P_{\underline{k}}}$ contains a non-trivial congruence subgroup of $\mathrm{Sp}_4(\mathbb{I}^\circ_0/P_{\underline{k}}\mathbb{I}^\circ_0)$. $\square$

4.9.4. Unipotent subgroups and fullness. Recall that for a root α of GSp_4 we denote by U^α the corresponding one-parameter unipotent subgroup of GSp_4.

We relate the fullness of the image of a representation to the fullness of its unipotent subgroups. This way we can gather useful informations by lifting unipotent elements in the image of a residual representation to unipotent elements in the image of the "big" representation. This is the same strategy that was used in [**Hi15**], [**Lang16**] and Chapter 2 for GL_2, and in [**HT15**] for GSp_4. It is based on the simple result below. We call "congruence subalgebra" of $\mathfrak{sp}_4(R)$ a Lie algebra of the form $\mathfrak{a} \cdot \mathfrak{sp}_4(R)$ for some ideal $\mathfrak{a}$ of R.

Lemma 4.9.19. *Let R be an integral domain and let $\mathfrak{G}$ be a Lie subalgebra of $\mathfrak{sp}_4(R)$. The following are equivalent:*

(1) the Lie algebra $\mathfrak{G}$ contains a congruence Lie subalgebra $\mathfrak{a} \cdot \mathfrak{sp}_4(R)$ of level a non-zero ideal $\mathfrak{a}$ of R;

(2) for every root α of Sp_4, the nilpotent Lie algebra $\mathfrak{G} \cap \mathfrak{u}^\alpha(R)$ contains a non-zero ideal $\mathfrak{a}_\alpha$ of R via the identification $\mathfrak{u}^\alpha(R) \cong R$.

Moreover:

(i) if condition (1) is satisfied for an ideal $\mathfrak{a}$ then condition (2) is satisfied if we choose $\mathfrak{a}_\alpha = \mathfrak{a}$ for every α;

(ii) if condition (2) is satisfied for a set of ideals $\{\mathfrak{a}_\alpha\}_\alpha$ then condition (1) is satisfied for the ideal $\mathfrak{a} = \prod_\alpha \mathfrak{a}^\alpha$, where the product is over all roots α of Sp_4.

Proof. It is clear that if $\mathfrak{a}\cdot\mathfrak{gsp}_4(R)\subset\mathfrak{G}$ for a non-zero ideal $\mathfrak{a}$ of R then $\mathfrak{a}\subset\mathfrak{G}\cap\mathfrak{u}^\alpha(R)$ for every root α. For the converse, suppose that $\mathfrak{a}^\alpha\subset\mathfrak{G}\cap\mathfrak{u}^\alpha(R)$ for every α and a set of non-zero ideals $\{\mathfrak{a}_\alpha\}_\alpha$. Let $\{\alpha_1,\alpha_2,\ldots,\alpha_g\}$ be a set of simple roots for Sp_{2g}. For $i=1,2,\ldots,g$ let $u_{\pm\alpha_i}$ be generators of the unipotent subalgebras $\mathfrak{u}_{\pm\alpha_i}(R)$ as R-modules and let $t_{\alpha_i}=[u_{\alpha_i},u_{-\alpha_i}]$. The elements $\{t_{\alpha_i}\}_{i=1,2,\ldots,g}$ generate the toral subalgebra $\mathfrak{t}$ of $\mathfrak{sp}_4(R)$. Since $\mathfrak{a}_{\pm\alpha_i}\subset\mathfrak{G}\cap\mathfrak{u}_{\pm\alpha_i}(R)$ for every i, we can write a chain of inclusions

$$\mathfrak{G}\supset\{[X_+,X_-]\,|\,X_+\in\mathfrak{G}\cap\mathfrak{u}_{\alpha_i}(R),X_-\in\mathfrak{G}\cap\mathfrak{u}_{-\alpha_i}(R)\}\supset$$
$$\{[X_+,X_-]\,|\,X_+\in\mathfrak{a}_{\alpha_i}\cdot\mathfrak{u}_{\alpha_i}(R),X_-\in\mathfrak{a}_{-\alpha_i}\cdot\mathfrak{u}_{-\alpha_i}(R)\}=\mathfrak{a}_{\alpha_i}\mathfrak{a}_{-\alpha_i}\cdot t_{\alpha_i}.$$

Set $\mathfrak{a}=\prod_\alpha\mathfrak{a}^\alpha$, where the product is over all roots of Sp_4. Since R is an integral domain $\mathfrak{a}$ is a non-zero ideal. The hypotheses of the lemma and the displayed inclusions give $\mathfrak{G}\supset\mathfrak{a}\cdot t_{\alpha_i}$ for every i and $\mathfrak{G}\supset\mathfrak{a}\cdot\mathfrak{u}_\alpha$ for every α. Since the set $\bigcup_{i=1,2,\ldots,g}(\mathfrak{a}\cdot t_{\alpha_i})\cup\bigcup_\alpha(\mathfrak{a}\cdot\mathfrak{u}_\alpha)$ generates $\mathfrak{a}\cdot\mathfrak{sp}_4(R)$ as an additive group, we conclude that $\mathfrak{G}\supset\mathfrak{a}\cdot\mathfrak{sp}_4(R)$. □

Lemma 4.9.19 admits an analogue dealing with unipotent and congruence subgroups rather than Lie algebras.

Lemma 4.9.20. *Let R be an integral domain and let G be a subgroup of $\mathrm{GSp}_4(R)$. The following are equivalent:*

(1) the group G contains a principal congruence subgroup $\Gamma_R(\mathfrak{a})$ of level a non-zero ideal $\mathfrak{a}$ of R;
(2) for every root α of Sp_4, the unipotent group $G\cap U^\alpha(R)$ contains a non-zero ideal $\mathfrak{a}_\alpha$ of R via the identification $U^\alpha(R)\cong R$.

Moreover:

(i) if condition (1) is satisfied for an ideal $\mathfrak{a}$ then condition (2) is satisfied if we choose $\mathfrak{a}_\alpha=\mathfrak{a}$ for every α;
(ii) if condition (2) is satisfied for a set of ideals $\{\mathfrak{a}_\alpha\}_\alpha$ then condition (1) is satisfied for the ideal $\mathfrak{a}=\prod_\alpha\mathfrak{a}_\alpha$, where the product is taken over all roots of Sp_4.

Proof. This follows from an argument analogous to that of Lemma 4.9.19, by replacing the Lie bracket with the commutator. □

Remark 4.9.21. *In both Lemma 4.9.19 and Lemma 4.9.20, if there is an ideal $\mathfrak{a}'$ of R such that the choice $\mathfrak{a}_\alpha=\mathfrak{a}'$ for every α satisfies condition (2), then the choice $\mathfrak{a}=(\mathfrak{a}')^2$ satisfies condition (1). This follows immediately from the arguments of the proofs.*

By applying Proposition 4.9.8 and Lemma 4.9.20 to $R=\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ$ and $G=\mathrm{Im}\,\rho_{0,P_{\underline{k}}}$ we obtain the following corollary.

Corollary 4.9.22. *For every root α of GSp_4, the group $\mathrm{Im}\,\rho_{P_{\underline{k}}}\cap U^\alpha(\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ)$ contains the image of an ideal of $\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ$.*

4.9.5. Lifting the congruence subgroup. If α is a root of Sp_4, we write U^α for the one-parameter unipotent subgroup of Sp_4 associated with α. If G is a group, R is a ring and $\tau\colon G\to\mathrm{GSp}_4(R)$ is a representation, let $U^\alpha(\tau)=\tau(G)\cap U^\alpha(R)$. We always identify $U^\alpha(R)$ with R, hence $U^\alpha(\tau)$ with an additive subgroup of R.

Recall that ρ is the representation associated with a finite slope family $\theta\colon\mathbb{T}_h\to\mathbb{I}^\circ$ and that $\rho_{P_{\underline{k}}}$ is the reduction of $\rho\colon H_0\to\mathrm{GSp}_4(\mathbb{I}_0^\circ)$ modulo $P_{\underline{k}}\mathbb{I}_0^\circ$. We use Corollary 4.9.22 together with Proposition 4.7.1 to obtain a result on the unipotent subgroups of the image of ρ.

Proposition 4.9.23. *For every root α of GSp_4, the group $U^\alpha(\rho)$ contains a basis of a Λ_h-lattice in $\mathbb{I}_0^\circ$.*

Proof. Let $\pi_{\underline{k}}\colon \mathbb{I}_0^\circ \to \mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ$ be the natural projection. We denote also by $\pi_{\underline{k}}$ the induced map $\mathrm{GSp}_4(\mathbb{I}_0^\circ) \to \mathrm{GSp}_4(\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ)$. For a root α of GSp_4, let $\pi_{\underline{k}}^\alpha\colon U^\alpha(\mathbb{I}_0^\circ) \to U^\alpha(\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ)$ be the projection induced by $\pi_{\underline{k}}$.

Let $G = \mathrm{Im}\,\rho \cap \Gamma_{\mathrm{GSp}_4(\mathbb{I}_0^\circ)}(p)$ and $G_{P_{\underline{k}}} = \pi_{\underline{k}}(G)$. We check that the choices $A = \mathbb{I}_0^\circ$, $g = 2$, $G = \mathrm{Im}\,\rho \cap \Gamma_{\mathrm{GSp}_4(\mathbb{I}_0^\circ)}(p)$ and $Q = P_{\underline{k}}$ satisfy the hypotheses of Proposition 4.7.1:

- the group G is compact since $\mathrm{Im}\,\rho$ is the continuous image of a Galois group and $\Gamma_{\mathrm{GSp}_4(\mathbb{I}_0^\circ)}(p)$ is a pro-p group;
- by assumption $\mathrm{Im}\,\rho$ contains a diagonal $\mathbb{Z}_p$-regular element d, and since $\Gamma_{\mathrm{GSp}_4(\mathbb{I}_0^\circ)}(p)$ is a normal subgroup of $\mathrm{GSp}_4(A)$ the element d normalizes $\mathrm{Im}\,\rho \cap \Gamma_{\mathrm{GSp}_4(\mathbb{I}_0^\circ)}(p)$.

Hence by Proposition 4.7.1 $\pi_{\underline{k}}^\alpha$ induces a surjection $G \cap U^\alpha(\mathbb{I}_0^\circ) \to G_{\underline{k}} \cap U^\alpha(\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ)$. Let $G^\alpha = G \cap U^\alpha(\mathbb{I}_0^\circ)$ and $G_{\underline{k}}^\alpha = G_{\underline{k}} \cap U^\alpha(\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ)$. As usual we identify them with $\mathbb{Z}_p$-submodules of $\mathbb{I}_0^\circ$ and $\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ$, respectively.

By Corollary 4.9.22 there exists a non-zero ideal $\mathfrak{a}_{\underline{k}}$ of $\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ$ such that $\mathfrak{a}_{\underline{k}} \subset \mathrm{Im}\,\rho_{P_{\underline{k}}} \cap U^\alpha(\mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ)$. Set $\mathfrak{b}_{\underline{k}} = p\mathfrak{a}_{\underline{k}}$. Then $\mathfrak{b}_{\underline{k}} \subset G_{\underline{k}}^\alpha$. By the result of the previous paragraph the map $G^\alpha \to G_{\underline{k}}^\alpha$ induced by $\pi_{\underline{k}}^\alpha$ is surjective, so we can choose a subset A of G^α that surjects onto $\mathfrak{b}_{\underline{k}}$. Let M be the Λ_h-span of A in $\mathbb{I}_0^\circ$. Let $\mathfrak{b}$ be the pre-image of $\mathfrak{b}_{\underline{k}}$ via $\pi_{\underline{k}}^\alpha\colon \mathbb{I}_0^\circ \to \mathbb{I}_0^\circ/P_{\underline{k}}\mathbb{I}_0^\circ$. Clearly $A \subset \mathfrak{b}$, so M is a Λ_h-submodule of $\mathfrak{b}$. Moreover $M/P_{\underline{k}}M = \mathfrak{b}_{\underline{k}}$ by the definition of A. Since Λ is local Nakayama's lemma implies that the inclusion $M \hookrightarrow \mathfrak{b}$ is an equality. In particular the Λ_h-span of G^α contains an ideal of $\mathbb{I}_0^\circ$. By Lemma 4.9.2 this implies that G^α contains a basis of a Λ_h-lattice in $\mathbb{I}_0^\circ$. $\square$

4.10. Relative Sen theory

We recall some of the notations of Section 4.1.2. We write $r_h = p^{-s_h}$ for the radius of a disc adapted to h and η_h for an element of $\overline{\mathbb{Q}}_p$ of valuation s_h. For $i \geq 1$, let $s_i = s_h + 1/i$ and $r_i = p^{-s_i}$. We constructed $\mathbb{Q}_p$-models B_h and B_i, $i \geq 1$ of the open discs $B(0, r_h^-)$ and the affinoid discs $B(1, p^{-s_i})$, $i \geq 1$. We write $\Lambda_h = \mathcal{O}^\circ(B_h)$ and $A_{r_i}^\circ = \mathcal{O}^\circ(B_i)$. We have $\Lambda_h = \varprojlim_i A_{r_i}^\circ$, where the transition maps in the projective limit correspond to the restrictions of analytic functions from larger to smaller discs. If $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ is a family, the rings $\mathbb{I}_{\mathrm{Tr}}^\circ$ and $\mathbb{I}_0^\circ$ defined in Sections 4.1.2 and 4.3 are finite Λ_h-algebras. All the Λ_h-algebras we listed are endowed with their profinite topology.

We give some new definitions. For every i there is a natural map $\iota_{r_i}\colon \Lambda_h \to A_{r_i}$. Set $\mathbb{I}_{r_i,0}^\circ = \mathbb{I}_0^\circ \widehat{\otimes}_{\Lambda_h} A_{r_i}^\circ$. We endow $\mathbb{I}_{r_i,0}^\circ$ with its p-adic topology.

Remark 4.10.1.

(1) *The ring $\mathbb{I}_0^\circ$ admits two inequivalent topologies: the profinite one and the p-adic one. The representation ρ is continuous with respect to the profinite topology on $\mathbb{I}_0^\circ$ but it is not necessarily continuous with respect to the p-adic one.*

(2) *Since $\mathbb{I}_0^\circ$ is a finite Λ_h-algebra, $\mathbb{I}_{r_i,0}^\circ$ is a finite $A_{r_i}^\circ$-algebra. There is an injective ring morphism $\iota'_{r_i}\colon \mathbb{I}_0^\circ \hookrightarrow \mathbb{I}_{r_i,0}^\circ$ sending f to $f \otimes 1$. This map is continuous with respect to the profinite topology on $\mathbb{I}_0^\circ$ and the p-adic topology on $\mathbb{I}_{r_i,0}^\circ$.*

We will still write ι'_{r_i} for the map $\mathrm{GSp}_4(\mathbb{I}_0^\circ) \hookrightarrow \mathrm{GSp}_4(\mathbb{I}_{r_i,0}^\circ)$ induced by ι'_{r_i}.

We associated with θ a representation $\rho|_{H_0}\colon H_0 \to \mathrm{GSp}_4(\mathbb{I}_0^\circ)$ that is continuous with respect to the profinite topologies on both its domain and target. By Remark 4.10.1(1) $\rho|_{H_0}$ needs not be continuous with respect to the p-adic topology on $\mathrm{GSp}_4(\mathbb{I}_0^\circ)$. This poses a problem when trying to apply Sen theory. For this reason we introduce for every i the representation

$$\rho_{r_i}\colon H_0 \to \mathrm{GSp}_4(\mathbb{I}_{r_i,0}^\circ)$$

defined by $\rho_{r_i} = \iota'_{r_i} \circ \rho|_{H_0}$. We deduce from the continuity of ι'_{r_i} that ρ_{r_i} is continuous with respect to the profinite topology on H_0 and the p-adic one on $\mathbb{I}^\circ_{r_i,0}$. It is clear from the definition that the image of ρ_{r_i} is independent of i as a topological group.

There is a good notion of Lie algebra for a pro-p group that is topologically of finite type. For this reason we further restrict H_0 so that the image of ρ_{r_i} is a pro-p group. Let $H_{r_1} = \{g \in H_0 \,|\, \rho_{r_1}(g) \cong \mathbb{1}_4 \pmod p\}$ and set $H_{r_i} = H_{r_1}$ for every $i \geq 1$. The subgroup $\{M \in \mathrm{GSp}_4(\mathbb{I}^\circ_{r_1,0}) \,|\, M \cong \mathbb{1}_4 \pmod p\}$ is of finite index in $\mathrm{GSp}_4(\mathbb{I}^\circ_{r_1,0})$. Note that this depends on the fact that we extended the coefficients to $\mathbb{I}_{r_1,0}$, since $\{M \in \mathrm{GSp}_4(\mathbb{I}^\circ_0) \,|\, M \cong \mathbb{1}_4 \pmod p\}$ is *not* of finite index in $\mathrm{GSp}_4(\mathbb{I}_0)$. We deduce that H_{r_1} is a normal open subgroup of $G_\mathbb{Q}$. Let $K_{H_{r_i}}$ be the subfield of $\overline{\mathbb{Q}}$ fixed by H_{r_i}. It is a finite Galois extension of $\mathbb{Q}$.

Recall that we fixed an embedding $G_{\mathbb{Q}_p} \hookrightarrow G_\mathbb{Q}$, identifying $G_{\mathbb{Q}_p}$ with a decomposition subgroup of $G_\mathbb{Q}$ at p. Let $H_{r_i,p} = H_{r_i} \cap G_{\mathbb{Q}_p}$. Let $K_{H_{r_i},p}$ be the subfield of $\overline{\mathbb{Q}}_p$ fixed by $H_{r_i,p}$. The field $K_{H_{r_i},p}$ will play a role when we apply Sen theory.

For every i, let $G_{r_i} = \rho_{r_i}(H_{r_i})$ and $G^{\mathrm{loc}}_{r_i} = \rho_{r_i}(H_{r_i,p})$.

Remark 4.10.2. *The topological Lie groups $\mathfrak{G}_r$ and $\mathfrak{G}^{\mathrm{loc}}_r$ are independent of r, in the following sense. For positive integers i,j with $i \leq j$ let $\iota^{r_i}_{r_j} \colon \mathbb{I}_{r_j,0} \to \mathbb{I}_{r_i,0}$ be the natural morphism induced by the restriction of analytic functions $A_{r_j} \to A_{r_i}$. Since $H_{r_i} = H_{r_j} = H_{r_1}$ by definition, $\iota^{r_i}_{r_j}$ induces isomorphisms*

$$\iota^{r_i}_{r_j} \colon G_{r_j} \xrightarrow{\sim} G_{r_i}$$

and

$$\iota^{r_i}_{r_j} \colon G^{\mathrm{loc}}_{r_j} \xrightarrow{\sim} G^{\mathrm{loc}}_{r_i}.$$

4.10.1. Big Lie algebras. As before let r be a radius among the r_i, $i \in \mathbb{N}^{>0}$. We will associate with $\rho_r(H_{r_i})$ some Lie algebras that will give the context in which to apply Sen's results. Our methods require that we work with $\mathbb{Q}_p$-Lie algebras, so we define the rings $A_r = A^\circ_r[p^{-1}]$ and $\mathbb{I}_{r,0} = \mathbb{I}^\circ_{r,0}[p^{-1}]$.

Let $\mathfrak{a}$ be a height two ideal of $\mathbb{I}_{r,0}$. The quotient $\mathbb{I}_{r,0}/\mathfrak{a}$ is a finite-dimensional $\mathbb{Q}_p$-algebra. Let $\pi_\mathfrak{a} \colon \mathbb{I}_{r,0} \to \mathbb{I}_{r,0}/\mathfrak{a}$ be the natural projection. We still denote by $\pi_\mathfrak{a}$ the induced map $\mathrm{GSp}_4(\mathbb{I}_{r,0}) \to \mathrm{GSp}_4(\mathbb{I}_{r,0}/\mathfrak{a})$. Consider the subgroups $G_{r,\mathfrak{a}} = \pi_\mathfrak{a}(G_r)$ and $G^{\mathrm{loc}}_{r,\mathfrak{a}} = \pi_\mathfrak{a}(G^{\mathrm{loc}}_r)$ of $\mathrm{GSp}_4(\mathbb{I}_{r,0}/\mathfrak{a})$. They are both pro-$p$ groups and they are topologically of finite type since $\mathrm{GSp}_4(\mathbb{I}_{r,0}/\mathfrak{a})$ is. Note that it makes sense to consider the logarithm of an element of $G_{r,\mathfrak{a}}$ (or $G^{\mathrm{loc}}_{r,\mathfrak{a}}$) since this group is contained in $\{M \in \mathrm{GSp}_4(\mathbb{I}^\circ_{r,0}) \,|\, M \cong \mathbb{1}_4 \pmod p\}$.

We attach to $G_{r,\mathfrak{a}}$ and $G^{\mathrm{loc}}_{r,\mathfrak{a}}$ the following $\mathbb{Q}_p$-vector subspaces $\mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a}}$ of $\mathfrak{gsp}_4(\mathbb{I}_{r,0}/\mathfrak{a})$:

$$\mathfrak{G}_{r,\mathfrak{a}} = \mathbb{Q}_p \cdot \log G_{r,\mathfrak{a}},$$
$$\mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a}} = \mathbb{Q}_p \cdot \log G^{\mathrm{loc}}_{r,\mathfrak{a}}.$$

The $\mathbb{Q}_p$-Lie algebra structure of $\mathfrak{gsp}_4(\mathbb{I}_{r,0}/\mathfrak{a})$ restricts to a $\mathbb{Q}_p$-Lie algebra structure on $\mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a}}$. These two Lie algebras are finite-dimensional over $\mathbb{Q}_p$ since $\mathfrak{gsp}_4(\mathbb{I}_{r,0}/\mathfrak{a})$ is.

Remark 4.10.3. *The Lie algebras $\mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a}}$ are independent of r, in the following sense. For positive integers i,j with $i \leq j$ let $\iota^{r_i}_{r_j} \colon \mathbb{I}_{r_j,0} \to \mathbb{I}_{r_i,0}$ be the natural morphism. By Remark 4.10.2 $\iota^{r_i}_{r_j}$ induces isomorphisms*

$$\iota^{r_i}_{r_j} \colon \mathfrak{G}_{r_j,\mathfrak{a}} \xrightarrow{\sim} \mathfrak{G}_{r_i,\mathfrak{a}}$$

and

$$\iota^{r_i}_{r_j} \colon \mathfrak{G}^{\mathrm{loc}}_{r_j,\mathfrak{a}} \xrightarrow{\sim} \mathfrak{G}^{\mathrm{loc}}_{r_i,\mathfrak{a}}.$$

Remark 4.10.4. *The definitions of $\mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a}}$ do not make sense if $\mathfrak{a}$ is not a height two ideal. In this case $\mathbb{I}_{r,0}/\mathfrak{a}$ is not a finite extension of $\mathbb{Q}_p$ and G_r and G^{loc}_r need not be topologically of finite type. We can define subsets $\mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a}}$ of $\mathfrak{gsp}_4(\mathbb{I}_{r,0}/\mathfrak{a})$ as above but they do not have* *in general a Lie algebra structure. In particular the choice $\mathfrak{a} = 0$ does not give Lie algebras for G_r and G_r^{loc}. We will construct these Lie algebras via a different approach, which consists in taking a suitable limit of the finite-dimensional $\mathbb{Q}_p$-Lie algebras $\mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc}}$ when $\mathfrak{a}$ varies over certain height two ideals of $\mathbb{I}_{r,0}$. Another reason for defining our algebras this way is that some results of Sen theory are available only for finite-dimensional Lie algebras over a p-adic field (see Remark 4.10.12).*

Recall that there is a natural injection $\Lambda_2 \hookrightarrow \Lambda_h$, hence an injection $\Lambda_2[p^{-1}] \hookrightarrow \Lambda_h[p^{-1}]$. For every $\underline{k} = (k_1, k_2)$ the ideal $P_{\underline{k}}\Lambda_h[p^{-1}]$ is either prime in $\Lambda_h[p^{-1}]$ or equal to $\Lambda_h[p^{-1}]$. We define the set of "bad" ideals S_Λ^{bad} of $\Lambda_2[p^{-1}]$ as

$$S_\Lambda^{\mathrm{bad}} = \{(1+T_1-u), (1+T_2-u^2), (1+T_2-u(1+T_1)), ((1+T_1)(1+T_2)-u^3)\}.$$

Then we define the set of bad prime ideals of $\Lambda_h[p^{-1}]$ as

$$S^{\mathrm{bad}} = \{P \text{ prime of } \Lambda_h[p^{-1}] \,|\, P \cap \Lambda_2[p^{-1}] \in S_\Lambda^{\mathrm{bad}}\}.$$

We will take care to define rings where the images of the ideals in S^{bad} consist of invertible elements. The reason for this will be clear in Section 4.10.4. Let S_2 be the set of ideals $\mathfrak{a}$ of $\mathbb{I}_{r,0}$ of height two such that $\mathfrak{a}$ is prime to P for every $P \in S^{\mathrm{bad}}$. Let S_2' be the subset of prime ideals in S_2. We define the ring

$$\mathbb{B}_r = \varprojlim_{\mathfrak{a} \in S_2} \mathbb{I}_{r,0}/\mathfrak{a},$$

where the limit of finite-dimensional $\mathbb{Q}_p$-Banach spaces is taken with respect to the natural transition maps $\mathbb{I}_{r,0}/\mathfrak{a}_1 \to \mathbb{I}_{r,0}/\mathfrak{a}_2$ defined for every inclusion of ideals $\mathfrak{a}_1 \subset \mathfrak{a}_2$. We equip $\mathbb{I}_{r,0}/\mathfrak{a}$ with the p-adic topology for every $\mathfrak{a}$ and $\mathbb{B}_r$ with the projective limit topology. There is a natural injection $\iota_{\mathbb{B}_r} \colon \mathbb{I}_{r,0} \hookrightarrow \mathbb{B}_r$ with dense image.

Remark 4.10.5. *There is an isomorphism of* rings

$$\mathbb{B}_r \cong \prod_{P \in S_2'} \widehat{(\mathbb{I}_{r,0})}_P, \tag{4.26}$$

where $\widehat{(\mathbb{I}_{r,0})}_P = \varprojlim_i \mathbb{I}_{r,0}/P^i$ with respect to the natural transition maps, but (4.26) *is* not *an isomorphism of topological rings if we equip $\widehat{(\mathbb{I}_{r,0})}_P$ with the P-adic topology for every P. In this case the resulting product topology is not the topology on $\mathbb{B}_r$, which is the p-adic one.*

For later use we define an analogue of the ring $\mathbb{B}_r$ constructed from A_r rather than $\mathbb{I}_{r,0}$. We begin by defining the sets

$$\begin{aligned} S_A^{\mathrm{bad}} &= \{P \cap A_r \,|\, P \in S^{\mathrm{bad}}\}, \\ S_{2,A} &= \{\mathfrak{a} \cap A_r \,|\, \mathfrak{a} \in S_2\}, \\ S_{2,A}' &= \{\mathfrak{a} \cap A_{r,} \,|\, \mathfrak{a} \in S_2'\}. \end{aligned}$$

We define a ring

$$B_r = \varprojlim_{\mathfrak{a} \in S_{2,A}} A_r/\mathfrak{a},$$

where the limit of finite-dimensional $\mathbb{Q}_p$-Banach spaces is taken with respect to the natural transition maps $A_r/\mathfrak{a}_1 \to A_r/\mathfrak{a}_2$ defined for every inclusion of ideals $\mathfrak{a}_1 \subset \mathfrak{a}_2$. We equip $A_r/\mathfrak{a}$ with the p-adic topology for every $\mathfrak{a}$ and B_r with the projective limit topology. There is a natural injection $\iota_{B_r} \colon A_r \hookrightarrow B_r$ with dense image.

Remark 4.10.6. *There is an isomorphism of* rings

$$B_r \cong \prod_{P \in S_{2,A}'} \widehat{(A_r)}_P \tag{4.27}$$

where $\widehat{(A_r)}_P = \varprojlim_i A_r/P^i$ *with respect to the natural transition maps, but* (4.27) *is* not *an isomorphism of topological rings if we equip* $\widehat{(A_r)}_P$ *with the* P*-adic topology for every* P*. In this case the resulting product topology is not the topology on* B_r*, which is the* p*-adic one.*

REMARK 4.10.7. *For every* $P \in S^{\mathrm{bad}}$ *we have* $P \cdot \mathbb{B}_r = \mathbb{B}_r$*, since the limit defining* $\mathbb{B}_r$ *is taken over ideals prime to* P*. In the same way we have* $P \cdot B_r = B_r$ *for every* $P \in S_A^{\mathrm{bad}}$*.*

Recall that $\mathbb{I}_{r,0}$ is a finite A_r-algebra. Then $\mathbb{I}_{r,0}/\mathfrak{a}$ is a finite $A_r/(\mathfrak{a} \cap A_r)$-algebra for every $\mathfrak{a} \in S_2$, so the ring $\mathbb{B}_r$ has a natural structure of topological B_r-algebra. For every $\mathfrak{a} \in S_2$ the degree of the extension $\mathbb{I}_{r,0}/\mathfrak{a}$ over $A_r/(\mathfrak{a} \cap A_r)$ is bounded by that of $\mathbb{I}_{r,0}$ over A_r. We deduce that $\mathbb{B}_r$ is a finite B_r-algebra.

We work with the ring $\mathbb{B}_r$ for the moment, but B_r will play an important role in Section 4.11.

We proceed to define the Lie algebras of G_r and G_r^{loc} as subalgebras of $\mathfrak{gsp}_4(\mathbb{B}_r)$. Let

$$\mathfrak{G}_r = \varprojlim_{\mathfrak{a} \in S_2} \mathfrak{G}_{r,\mathfrak{a}}$$

and

$$\mathfrak{G}_r^{\mathrm{loc}} = \varprojlim_{\mathfrak{a} \in S_2} \mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc}},$$

where $\mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc}}$ are the Lie algebras we attached to $G_{r,\mathfrak{a}}$ and $G_{r,\mathfrak{a}}^{\mathrm{loc}}$. The $\mathbb{Q}_p$-Lie algebra structures on $\mathfrak{G}_{r,\mathfrak{a}}$ and $\mathfrak{G}_{r,\mathfrak{a}}^{\mathrm{loc}}$ induce $\mathbb{Q}_p$-Lie algebra structures on $\mathfrak{G}_r$ and $\mathfrak{G}_r^{\mathrm{loc}}$. We endow $\mathfrak{G}_r$ and $\mathfrak{G}_r^{\mathrm{loc}}$ with the p-adic topology induced by that on $\mathfrak{gsp}_4(\mathbb{B}_r)$.

When we introduce the Sen operators we will have to extend the scalars of the various rings and Lie algebras to $\mathbb{C}_p$. We denote this operation by adding a lower index $\mathbb{C}_p$. Explicitly, we set $\mathbb{I}_{r,0,\mathbb{C}_p} = \mathbb{I}_{r,0}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$, $A_{r,\mathbb{C}_p} = A_r\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$, $\mathbb{B}_{r,\mathbb{C}_p} = \mathbb{B}_r \otimes_{\mathbb{Q}_p} \mathbb{C}_p$, $B_{r,\mathbb{C}_p} = B_r\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$, $\mathfrak{G}_{r,\mathbb{C}_p} = \mathfrak{G}_r\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$ and $\mathfrak{G}_{r,\mathbb{C}_p}^{\mathrm{loc}} = \mathfrak{G}_r^{\mathrm{loc}}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$. We still endow all these rings with their p-adic topology. Clearly $\mathbb{I}_{r,0,\mathbb{C}_p}$ has a structure of finite $A_{r,\mathbb{C}_p}$-algebra and $\mathbb{B}_{r,\mathbb{C}_p}$ has a structure of finite $B_{r,\mathbb{C}_p}$-algebra. The injection $\iota_{\mathbb{B}_r}$ and ι_{B_r} induce injections with dense image $\iota_{\mathbb{B}_r,\mathbb{C}_p} \colon \mathbb{I}_{r,0,\mathbb{C}_p} \hookrightarrow \mathbb{B}_{r,\mathbb{C}_p}$ and $\iota_{B_r,\mathbb{C}_p} \colon A_{r,\mathbb{C}_p} \hookrightarrow \mathbb{B}_{r,\mathbb{C}_p}$. The Lie algebras $\mathfrak{G}_{r,\mathbb{C}_p}$ and $\mathfrak{G}_{r,\mathbb{C}_p}^{\mathrm{loc}}$ are $\mathbb{C}_p$-Lie subalgebras of $\mathfrak{gsp}_4(\mathbb{B}_{r,\mathbb{C}_p})$.

REMARK 4.10.8. *The* $\mathbb{Q}_p$*-Lie algebras* $\mathfrak{G}_r$ *and* $\mathfrak{G}_r^{\mathrm{loc}}$ *do not have a priori any* $\mathbb{B}_r$ *or* B_r*-module structure. As a crucial step in our arguments we will use Sen theory to induce a* $B_{r,\mathbb{C}_p}$*-vector space (hence a* $B_{r,\mathbb{C}_p}$*-Lie algebra) structure on suitable subalgebras of* $\mathfrak{G}_{r,\mathbb{C}_p}$*.*

4.10.2. The Sen operator associated with a p-adic Galois representation. Let L be a p-adic field and let $\mathscr{R}$ be a Banach L-algebra. Let K be another p-adic field, m be a positive integer and $\tau \colon \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_m(\mathscr{R})$ be a continuous representation. We recall the construction of the Sen operator associated with τ, following [**Sen93**].

We fix embeddings of K and L in $\overline{\mathbb{Q}}_p$. The constructions that follow will depend on these choices. We suppose that the Galois closure L^{Gal} of L over $\mathbb{Q}_p$ is contained in K. If this is not the case we simply restrict τ to the open subgroup $\mathrm{Gal}(\overline{K}/KL^{\mathrm{Gal}}) \subset \mathrm{Gal}(\overline{K}/K)$.

We denote by $\chi \colon \mathrm{Gal}(\overline{L}/L) \to \mathbb{Z}_p^\times$ the p-adic cyclotomic character. Let L_∞ be a totally ramified $\mathbb{Z}_p$-extension of L. Let γ be a topological generator of $\Gamma = \mathrm{Gal}(L_\infty/L)$. For a positive integer n, let $\Gamma_n \subset \Gamma$ be the subgroup generated by γ^{p^n} and $L_n = L_\infty^{\langle \gamma^{p^n} \rangle}$ be the subfield of L_∞ fixed by Γ_n. We have $L_\infty = \cup_n L_n$. Let $L_n' = L_n K$ and $G_n' = \mathrm{Gal}(\overline{L}/L_n')$.

Write $\mathscr{R}^m$ for the $\mathscr{R}$-module over which $\mathrm{Gal}(\overline{K}/K)$ acts via τ. We define an action of $\mathrm{Gal}(\overline{K}/K)$ on $\mathscr{R}^m\widehat{\otimes}_L\mathbb{C}_p$ by letting $\sigma \in \mathrm{Gal}(\overline{K}/K)$ send $x \otimes y$ to $\tau(\sigma)(x) \otimes \sigma(y)$. Then by [**Sen93**] there exists a matrix $M \in \mathrm{GL}_m\left(\mathscr{R}\widehat{\otimes}_L\mathbb{C}_p\right)$, an integer $n \geq 0$ and a representation $\delta \colon \Gamma_n \to \mathrm{GL}_m(\mathscr{R} \otimes_L L_n')$ such that for all $\sigma \in G_n'$ we have

$$M^{-1}\tau(\sigma)\sigma(M) = \delta(\sigma). \tag{4.28}$$

Definition 4.10.9. *The Sen operator associated with τ is the element*
$$\phi=\lim_{\sigma\to 1}\frac{\log(\delta(\sigma))}{\log(\chi(\sigma))}$$
of $\mathrm{M}_m(\mathscr{R}\widehat{\otimes}_L\mathbb{C}_p)$.

The limit exists as for σ close to 1 the map $\sigma\mapsto\dfrac{\log(\delta(\sigma))}{\log(\chi(\sigma))}$ is constant. It is proved in [**Sen93**, Section 2.4] that ϕ does not depend on the choice of δ and M.

Recall that we fixed some embeddings $L\hookrightarrow L^{\mathrm{Gal}}\hookrightarrow K\hookrightarrow\overline{\mathbb{Q}}_p$. Suppose that $\mathscr{R}=L$ and that τ is a Hodge-Tate representation with Hodge-Tate weights $h_1,h_2,\ldots,h_m$. Let ϕ be the Sen operator associated with τ; it is an element of $\mathrm{M}_m(\mathbb{C}_p)$. The following theorem is a consequence of the results of [**Sen80**]; see in particular the discussion in the beginning of Section 2.2 and the Corollary to Theorem 6 in *loc. cit.*.

Theorem 4.10.10. *The characteristic polynomial of ϕ is $\prod_{i=1}^m(X-h_i)$.*

We restrict now to the case $L=\mathscr{R}=\mathbb{Q}_p$, so that τ is a continuous representation $\mathrm{Gal}(\overline{K}/K)\to\mathrm{GL}_m(\mathbb{Q}_p)$. Define a $\mathbb{Q}_p$-Lie algebra $\mathfrak{g}\subset\mathrm{M}_m(\mathbb{Q}_p)$ by $\mathfrak{g}=\mathbb{Q}_p\cdot\log(\tau(\mathrm{Gal}(\overline{K}/K)))$. We say that $\mathfrak{g}$ is the Lie algebra of $\tau(\mathrm{Gal}(\overline{K}/K))$. For any $\mathbb{Q}_p$-vector space V contained in $\mathrm{M}_m(\mathbb{Q}_p)$ we consider $V\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$ as a $\mathbb{C}_p$-subspace of $\mathrm{M}_m(\mathbb{C}_p)$. The Sen operator ϕ associated with τ has the property given by the following result.

Theorem 4.10.11. [**Sen73**, Theorem 1] *The $\mathbb{Q}_p$-vector space underlying the Lie algebra $\mathfrak{g}$ is the smallest among the $\mathbb{Q}_p$-subspaces V of $\mathrm{M}_m(\mathbb{Q}_p)$ with the property that $V\widehat{\otimes}\mathbb{C}_p$ contains ϕ.*

Remark 4.10.12. *The proof of Theorem 4.10.11 relies on the fact that $\tau(\mathrm{Gal}(\overline{K}/K))$ is a finite dimensional Lie group. It is doubtful that this proof can be generalized to the relative case.*

4.10.3. The relative Sen operator associated with ρ_r. Fix a radius r in the set $\{r_i\}_{i\in\mathbb{N}^{>0}}$. Consider as usual the representation $\rho_r\colon H_0\to\mathrm{GSp}_4(\mathbb{I}_{r,0})$. We defined earlier a p-adic field $K_{H_r,p}$. Write $G_{K_{H_r,p}}$ for its absolute Galois group. We look at the restriction $\rho_r|_{G_{K_{H_r,p}}}\colon G_{K_{H_r,p}}\to\mathrm{GSp}_4(\mathbb{I}_{r,0})$ as a representation with values in $\mathrm{GL}_4(\mathbb{I}_{r,0})$. Recall that $\mathfrak{G}_r^{\mathrm{loc}}$ is the Lie algebra associated with the image of $\rho_r|_{G_{K_{H_r,p}}}$. The goal of this section is to prove an analogue of Theorem 4.10.11 for this representation, i.e. to attach to $\rho_r|_{G_{K_{H_r,p}}}$ a "$\mathbb{B}_r$-Sen operator" belonging to $\mathfrak{G}_{r,\mathbb{C}_p}^{\mathrm{loc}}$.

We begin by constructing various Sen operators via Definition 4.10.9.

(1) The $\mathbb{Q}_p$-algebra $\mathbb{I}_{r,0}$ is complete for the p-adic topology. We associate with $\rho_r|_{G_{K_{H_r,p}}}$ a Sen operator $\phi_r\in\mathrm{M}_4(\mathbb{I}_{r,0,\mathbb{C}_p})$.
(2) Let $\mathfrak{a}\in S_2$. Then $\mathbb{I}_{r,0}/\mathfrak{a}$ is a finite-dimensional $\mathbb{Q}_p$-algebra. As usual write $\pi_\mathfrak{a}\colon\mathbb{I}_{r,0}\to\mathbb{I}_{r,0}/\mathfrak{a}$ for the natural projection. Denote by $\rho_{r,\mathfrak{a}}$ the representation $\pi_\mathfrak{a}\circ\rho_r|_{G_{K_{H_r,p}}}\colon G_{K_{H_r,p}}\to\mathrm{GL}_4(\mathbb{I}_{r,0}/\mathfrak{a})$. We associate with $\rho_{r,\mathfrak{a}}$ a Sen operator $\phi_{r,\mathfrak{a}}\in\mathrm{M}_4((\mathbb{I}_{r,0}/\mathfrak{a})\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p)$.
(3) Let $\mathfrak{a}\in S_2$. Let d be the $\mathbb{Q}_p$-dimension of $\mathbb{I}_{r,0}/\mathfrak{a}$. Let k be a positive integer. An $\mathbb{I}_{r,0}/\mathfrak{a}$-linear endomorphism of $(\mathbb{I}_{r,0}/\mathfrak{a})^k$ is also $\mathbb{Q}_p$-linear, so it defines a $\mathbb{Q}_p$-linear endomorphism of the underlying $\mathbb{Q}_p$-vector space $\mathbb{Q}_p^{kd}$. This gives natural maps $\alpha_{\mathbb{Q}_p}\colon\mathrm{M}_k(\mathbb{I}_{r,0}/\mathfrak{a})\to\mathrm{M}_{kd}(\mathbb{Q}_p)$ and $\alpha_{\mathbb{Q}_p}^\times\colon\mathrm{GL}_k(\mathbb{I}_{r,0}/\mathfrak{a})\to\mathrm{GL}_{kd}(\mathbb{Q}_p)$ (we omit the index k in the symbol of the morphism since this does not generate confusion). Choose $k=4$ and consider the representation $\rho_{r,\mathfrak{a}}^{\mathbb{Q}_p}=\alpha_{\mathbb{Q}_p}^\times\circ\rho_{r,\mathfrak{a}}\colon G_\mathbb{Q}\to\mathrm{GL}_{4d}(\mathbb{Q}_p)$. We associate with $\rho_{r,\mathfrak{a}}^{\mathbb{Q}_p}$ a Sen operator $\phi_{r,\mathfrak{a}}^{\mathbb{Q}_p}\in\mathrm{M}_{4d}(\mathbb{C}_p)$.

Note that Theorem 4.10.11 applies only to representations with coefficients in $\mathbb{Q}_p$, hence to construction (3) above. We will prove that the operators constructed in (1), (2) and (3) are related, so it is possible to transfer information from one to the others. We write $\pi_{\mathfrak{a},\mathbb{C}_p}=\pi_\mathfrak{a}\otimes 1\colon\mathbb{I}_{r,0,\mathbb{C}_p}\to\mathbb{I}_{r,0,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{r,0,\mathbb{C}_p}$. We still write $\pi_{\mathfrak{a},\mathbb{C}_p}$ for the maps $\mathrm{M}_4(\mathbb{I}_{r,0,\mathbb{C}_p})\to\mathrm{M}_4(\mathbb{I}_{r,0,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{r,0,\mathbb{C}_p})$

and $\mathrm{GL}_4(\mathbb{I}_{r,0,\mathbb{C}_p}) \to \mathrm{GL}_4(\mathbb{I}_{r,0,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{r,0,\mathbb{C}_p})$ obtained by applying $\pi_{\mathfrak{a},\mathbb{C}_p}$ to the matrix coefficients. As before we let d be the $\mathbb{Q}_p$-dimension of $\mathbb{I}_{r,0}/\mathfrak{a}$. For every positive integer k, we set $\alpha_{\mathbb{C}_p} = \alpha_{\mathbb{Q}_p} \otimes 1 \colon \mathrm{M}_k(\mathbb{I}_{r,0,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{r,0,\mathbb{C}_p}) \to \mathrm{M}_{kd}(\mathbb{C}_p)$ and $\alpha^\times_{\mathbb{C}_p} = \alpha^\times_{\mathbb{Q}_p} \otimes 1 \colon \mathrm{GL}_k(\mathbb{I}_{r,0,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{r,0,\mathbb{C}_p}) \to \mathrm{GL}_{kd}(\mathbb{C}_p)$.

Remark 4.10.13. *For every positive integer k, the map $\alpha_{\mathbb{Q}_p}$ commutes with the logarithm map $\log\colon \mathrm{M}_{kd}(\mathbb{I}_{r,0}/\mathfrak{a}) \to \mathrm{GL}_{kd}(\mathbb{I}_{r,0}/\mathfrak{a})$ in the sense that*

$$\log \circ \alpha^\times_{\mathbb{Q}_p} = \alpha_{\mathbb{Q}_p} \circ \log .$$

The same is true for the maps $\alpha_{\mathbb{C}_p}$ and $\alpha^\times_{\mathbb{C}_p}$:

$$\log \circ \alpha^\times_{\mathbb{C}_p} = \alpha_{\mathbb{C}_p} \circ \log .$$

Our result is the following.

Proposition 4.10.14. *For every $\mathfrak{a} \in S_2$ the following relations hold:*

(i) $\phi_{r,\mathfrak{a}} = \pi_{\mathfrak{a},\mathbb{C}_p}(\phi_r)$;

(ii) $\phi^{\mathbb{Q}_p}_{r,\mathfrak{a}} = \alpha_{\mathbb{C}_p}(\phi_{r,\mathfrak{a}})$.

Proof. Recall the general construction of the Sen operator presented in Section 4.10.2. We specialize it to the representation $\rho_r|_{G_{K_{H_r,p}}} \colon G_{K_{H_r,p}} \to \mathrm{GL}_4(\mathbb{I}_{r,0})$; in particular we choose $m = 4$, $K = K_{H_r,p}$ and $L = \mathbb{Q}_p$. By the discussion preceding Definition 4.10.9, there exists a matrix $M_0 \in \mathrm{GL}_4(\mathbb{I}_{r,0,\mathbb{C}_p})$, an integer $n_0 \geq 0$ and a representation $\delta_0 \colon \Gamma_{n_0} \to \mathrm{GL}_4(\mathbb{I}_{r,0}\widehat{\otimes}_{\mathbb{Q}_p}(\mathbb{Q}_p)'_{n_0})$ such that for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/(\mathbb{Q}_p)'_{n_0})$ we have

$$M_0^{-1}\rho_r(\sigma)\sigma(M_0) = \delta_0(\sigma). \tag{4.29}$$

Let $M_{0,\mathfrak{a}} = \pi_{\mathfrak{a},\mathbb{C}_p}(M_0) \in \mathrm{M}_4(\mathbb{I}_{r,0,\mathbb{C}_p}/\mathfrak{a}\mathbb{I}_{r,0,\mathbb{C}_p})$ and $\delta_{0,\mathfrak{a}} = \pi_{\mathfrak{a},\mathbb{C}_p} \circ \delta_0 \colon \Gamma_{n_0} \to \mathrm{GL}_4((\mathbb{I}_{r,0}/\mathfrak{a})\widehat{\otimes}_{\mathbb{Q}_p}(\mathbb{Q}_p)'_{n_0})$. By applying $\pi_{\mathfrak{a},\mathbb{C}_p}$ to both sides of Equation (4.29) we obtain

$$M_{0,\mathfrak{a}}^{-1}\rho_{r,\mathfrak{a}}(\sigma)\sigma(M_{0,\mathfrak{a}}) = \delta_{0,\mathfrak{a}}(\sigma) \tag{4.30}$$

for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/(\mathbb{Q}_p)'_{n_0})$. Hence the choices $M = M_{0,\mathfrak{a}}$, $n = n_0$ and $\delta = \delta_{0,\mathfrak{a}}$ satisfy Equation (4.28) specialized to the representation $\rho_{r,\mathfrak{a}}$. Then, by definition, the Sen operator associated with $\rho_{r,\mathfrak{a}}$ is

$$\phi_{r,\mathfrak{a}} = \lim_{\sigma\to 1} \frac{\log(\delta_{0,\mathfrak{a}}(\sigma))}{\log(\chi(\sigma))},$$

that coincides with

$$\lim_{\sigma\to 1} \frac{\log(\pi_{\mathfrak{a},\mathbb{C}_p} \circ \delta(\sigma))}{\log(\chi(\sigma))} = \pi_{\mathfrak{a},\mathbb{C}_p}\left(\lim_{\sigma\to 1} \frac{\log(\delta(\sigma))}{\log(\chi(\sigma))}\right) = \pi_{\mathfrak{a},\mathbb{C}_p}(\phi_r).$$

This proves (i).

For (ii), keep notations as in the previous paragraph. Let $M^{\mathbb{Q}_p}_{0,\mathfrak{a}} = \alpha^\times_{\mathbb{C}_p}(M_{0,\mathfrak{a}})$ and $\delta^{\mathbb{Q}_p}_{0,\mathfrak{a}} = \alpha^\times_{\mathbb{C}_p} \circ \delta_{0,\mathfrak{a}}$. By applying $\alpha^\times_{\mathbb{C}_p}$ to both sides of Equation (4.30) we obtain

$$(M^{\mathbb{Q}_p}_{0,\mathfrak{a}})^{-1}\rho^{\mathbb{Q}_p}_{r,0,\mathfrak{a}}(\sigma)\sigma(M^{\mathbb{Q}_p}_{0,\mathfrak{a}}) = \delta^{\mathbb{Q}_p}_{0,\mathfrak{a}}(\sigma) \tag{4.31}$$

for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/(\mathbb{Q}_p)'_{n_0})$. Then the choices $M = M^{\mathbb{Q}_p}_{0,\mathfrak{a}}$, $n = n_0$ and $\delta = \delta^{\mathbb{Q}_p}_{0,\mathfrak{a}}$ satisfy Equation (4.28) specialized to the representation $\rho^{\mathbb{Q}_p}_{r,0,\mathfrak{a}}$, so by definition the Sen operator associated with $\rho^{\mathbb{Q}_p}_{r,0,\mathfrak{a}}$ is

$$\phi^{\mathbb{Q}_p}_{r,0,\mathfrak{a}} = \lim_{\sigma\to 1} \frac{\log(\delta^{\mathbb{Q}_p}_{0,\mathfrak{a}}(\sigma))}{\log(\chi(\sigma))}.$$

Thanks to Remark 4.10.13 the right hand side of the equation above can be rewritten as

$$\lim_{\sigma\to 1}\frac{\log(\alpha_{\mathbb{C}_p}^{\times}\circ\delta_{0,\mathfrak{a}}^{\mathbb{Q}_p}(\sigma))}{\log(\chi(\sigma))}=\lim_{\sigma\to 1}\frac{\alpha_{\mathbb{C}_p}\circ\log(\delta_{0,\mathfrak{a}}(\sigma))}{\log(\chi(\sigma))}=$$
$$=\alpha_{\mathbb{C}_p}\left(\lim_{\sigma\to 1}\frac{\log(\delta_{0,\mathfrak{a}}(\sigma))}{\log(\chi(\sigma))}\right)=\alpha_{\mathbb{C}_p}(\phi_{r,\mathfrak{a}}).$$

This concludes the proof. □

Recall that there is a natural inclusion $\iota'_{\mathbb{B}_r,\mathbb{C}_p}\colon \mathbb{I}_{r,0,\mathbb{C}_p}\hookrightarrow \mathbb{B}_{r,\mathbb{C}_p}$. It induces an injection $\mathrm{M}_4(\mathbb{I}_{r,0,\mathbb{C}_p})\hookrightarrow \mathrm{M}_4(\mathbb{B}_{r,\mathbb{C}_p})$ that we still denote by $\iota_{\mathbb{B}_r,\mathbb{C}_p}$. We define the $\mathbb{B}_r$-Sen operator attached to $\rho_r|_{G_{K_{H_r,p}}}$ as

$$\phi_{\mathbb{B}_r}=\iota'_{\mathbb{B}_r,\mathbb{C}_p}(\phi_r).$$

By definition $\phi_{\mathbb{B}_r}$ is an element of $\mathrm{M}_4(\mathbb{B}_{r,\mathbb{C}_p})$. Since $\mathbb{B}_{r,\mathbb{C}_p}=\varprojlim_{\mathfrak{a}\in S_2}\mathbb{I}_{r,0}/\mathfrak{a}$, it is clear that $\phi_{\mathbb{B}_r}=\varprojlim_{\mathfrak{a}\in S_2}\pi_{\mathfrak{a},\mathbb{C}_p}(\phi_r)$. Then Proposition 4.10.14(i) implies that

$$\phi_{\mathbb{B}_r}=\varprojlim_{\mathfrak{a}\in S_2}\phi_{r,\mathfrak{a}}. \tag{4.32}$$

We use Proposition 4.10.14(ii) to show the following.

Proposition 4.10.15. *The operator $\phi_{\mathbb{B}_r}$ belongs to the Lie algebra $\mathfrak{G}^{\mathrm{loc}}_{r,\mathbb{C}_p}$.*

Proof. For every $\mathfrak{a}\in S_2$, let $d_\mathfrak{a}$ be the degree of the extension $\mathbb{I}_{r,0}/\mathfrak{a}$ over $\mathbb{Q}_p$. Let $\mathfrak{G}^{\mathrm{loc},\mathbb{Q}_p}_{r,\mathfrak{a}}$ be the Lie subalgebra of $\mathrm{M}_{4d_\mathfrak{a}}$ associated with the image of $\rho^{\mathbb{Q}_p}_{r,\mathfrak{a}}$, defined by

$$\mathfrak{G}^{\mathrm{loc},\mathbb{Q}_p}_{r,\mathfrak{a}}=\mathbb{Q}_p\cdot\log(\mathrm{Im}\,\rho^{\mathbb{Q}_p}_{r,\mathfrak{a}}).$$

Let $\mathfrak{G}^{\mathrm{loc},\mathbb{Q}_p}_{r,\mathfrak{a},\mathbb{C}_p}=\mathfrak{G}^{\mathrm{loc},\mathbb{Q}_p}_{r,\mathfrak{a}}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$. Since $\mathrm{Im}\,\rho^{\mathbb{Q}_p}_{r,\mathfrak{a}}=\alpha^{\times}_{\mathbb{Q}_p}(\mathrm{Im}\,\rho_{r,\mathfrak{a}}$, Remark 4.10.13 implies that

$$\mathfrak{G}^{\mathrm{loc},\mathbb{Q}_p}_{r,\mathfrak{a},\mathbb{C}_p}=\alpha_{\mathbb{C}_p}(\mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}). \tag{4.33}$$

The representation $\rho^{\mathbb{Q}_p}_{r,\mathfrak{a}}$ satisfies the assumptions of Theorem 4.10.11, so the Sen operator $\phi^{\mathbb{Q}_p}_{r,0,\mathfrak{a}}$ belongs to $\mathfrak{G}^{\mathrm{loc},\mathbb{Q}_p}_{r,\mathfrak{a},\mathbb{C}_p}$. By Proposition 4.10.14(ii) $\phi^{\mathbb{Q}_p}_{r,\mathfrak{a}}=\alpha_{\mathbb{C}_p}(\phi_{r,\mathfrak{a}})$. Then Equation (4.33) and the injectivity of $\alpha_{\mathbb{C}_p}$ give

$$\phi_{r,\mathfrak{a}}\in\mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}. \tag{4.34}$$

Since $\mathfrak{G}^{\mathrm{loc}}_{r,\mathbb{C}_p}=\varprojlim_{\mathfrak{a}\in S_2}\mathfrak{G}^{\mathrm{loc}}_{r,\mathfrak{a},\mathbb{C}_p}$, Equations (4.32) and (4.34) imply that $\phi_{\mathbb{B}_r}\in\mathfrak{G}^{\mathrm{loc}}_{r,\mathbb{C}_p}$. □

The following corollary follows trivially from the inclusion $\mathfrak{G}^{\mathrm{loc}}_r\subset\mathfrak{G}_r$.

Corollary 4.10.16. *The operator $\phi_{\mathbb{B}_r}$ belongs to the Lie algebra $\mathfrak{G}_{r,\mathbb{C}_p}$.*

4.10.4. The exponential of the Sen operator. We use the work of the previous section to construct an element of $\mathrm{GL}_4(\mathbb{B}_r)$ that has some specific eigenvalues and normalizes the Lie algebra $\mathfrak{G}^{\mathrm{loc}}_{r,\mathbb{C}_p}$. Such an element will be used in Section 4.11 to induce a $B_{r,\mathbb{C}_p}$-module structure on some subalgebra of $\mathfrak{G}_{r,\mathbb{C}_p}$, thus replacing the matrix "$\rho(\sigma)$" of [**HT15**], that is not available in the non-ordinary setting. The candidate for our special matrix will be the exponential of the $\mathbb{B}_r$-Sen operator, that we will define shortly.

Recall that $A_{r,\mathbb{C}_p}$ is a subring of the ring of $\mathbb{C}_p$-analytic functions on the affinoid disc $B(0,r)$. Denote by $\widetilde{v}_p$ the p-adic valuation on $A_{r,\mathbb{C}_p}$ defined by $\widetilde{v}_p(f)=\inf_{x\in B(0,r)}v_p(f(x))$. We still denote by $\widetilde{v}_p$ an extension of v_p to $\mathbb{I}_{r,0,\mathbb{C}_p}$. Consider the two subrings

$$\mathbb{I}^{>\frac{1}{p-1}}_{r,0,\mathbb{C}_p}=\left\{f\in\mathbb{I}_{r,0,\mathbb{C}_p}\,|\,\widetilde{v}_p(f)>\frac{1}{p-1}\right\}$$

and

$$1+\mathbb{I}_{r,0,\mathbb{C}_p}^{>\frac{1}{p-1}}=\left\{f\in\mathbb{I}_{r,0,\mathbb{C}_p}\,|\,\widetilde{v}_p(f-1)>\frac{1}{p-1}\right\}.$$

The exponential series is convergent on $\mathbb{I}_{r,0,\mathbb{C}_p}^{>\frac{1}{p-1}}$ and defines a map

$$\exp\colon \mathbb{I}_{r,0,\mathbb{C}_p}^{>\frac{1}{p-1}}\to 1+\mathbb{I}_{r,0,\mathbb{C}_p}^{>\frac{1}{p-1}}.$$

The logarithmic series is convergent on $1+\mathbb{I}_{r,0,\mathbb{C}_p}^{>\frac{1}{p-1}}$ and defines a map

$$\log\colon 1+\mathbb{I}_{r,0,\mathbb{C}_p}^{>\frac{1}{p-1}}\to \mathbb{I}_{r,0,\mathbb{C}_p}^{>\frac{1}{p-1}}.$$

For $f\in\mathbb{I}_{r,0,\mathbb{C}_p}^{>\frac{1}{p-1}}$ we have $\log(\exp(f))=f$ and $\exp(\log(1+f))=1+f$.

Let $\mathrm{M}_4(\mathbb{I}_{r,0,\mathbb{C}_p})^{>\frac{1}{p-1}}$ be the subring of $\mathrm{M}_4(\mathbb{I}_{r,0,\mathbb{C}_p})$ consisting of matrices having all their eigenvalues in $\mathbb{I}_{r,0,\mathbb{C}_p}^{>\frac{1}{p-1}}$. For a matrix $M\in\mathrm{M}_4(\mathbb{I}_{r,0,\mathbb{C}_p})^{>\frac{1}{p-1}}$, the exponential series defines an element $\exp(M)\in\mathrm{GL}_4(\mathbb{I}_{r,0,\mathbb{C}_p})$.

Let $\phi_r\in\mathrm{M}_4(\mathbb{I}_{r,0,\mathbb{C}_p})$ be the Sen operator defined in the previous section. We rescale it to define an element $\phi_r'=\log(u)\phi_r$, where $u=1+p$.

Proposition 4.10.17. *The eigenvalues of $\phi_{r,0}'$ are 0, $\log(u^{-2}(1+T_2))$, $\log(u^{-1}(1+T_1))$ and $\log(u^{-3}(1+T_1)(1+T_2))$.*

Remark 4.10.18. *The logarithms in Proposition 4.10.17 are well-defined. The reason is that in Section 4.1.2 we chose a radius r_h satisfying $r_h<p^{-\frac{1}{p-1}}$. Using this inequality we can compute $\widetilde{v}_p((1+T_1)-1)=\inf_{x\in B_{0,r}}v_p(T_1(x))<p^{-\frac{1}{p-1}}$, hence $(1+T_1)\in 1+\mathbb{I}_{r,0}^{>\frac{1}{p-1}}$ and $\log(1+T_1)$ is defined. Clearly the same is true for $\log(u^{-1}(1+T_1))$. An analogous calculation shows that $\log(u^{-2}(1+T_2))$ and $\log(u^{-3}(1+T_1)(1+T_2))$ are also defined.*

Proof. (of Proposition 4.10.17) Let $P_{\underline{k}}$ be an arithmetic prime of Λ_h, with $\underline{k}=(k_1,k_2)$. Let $\mathcal{P}$ be a prime of $\mathbb{I}^\circ$ that lies above $P_{\underline{k}}$ and corresponds to a classical GSp_4-eigenform $f_{\mathcal{P}}$ of weight $\underline{k}$. Let $\mathfrak{P}=\mathcal{P}\cap\mathbb{I}^\circ_{\mathrm{Tr}}$. as usual $\rho_{\mathfrak{P}}\colon G_{\mathbb{Q}}\to\mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})$ denotes the reduction of $\rho\colon G_{\mathbb{Q}}\to\mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ modulo $\mathfrak{P}$.

Let $\phi_{\mathfrak{P}}$ be the Sen operator associated with $\rho_{f_{\mathfrak{P}}}$. It is an element of $\mathrm{M}_4((\mathbb{I}^\circ_{\mathrm{Tr}}/\mathfrak{P})\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p)$. By Remark 3.10.6 $\rho_{f_{\mathfrak{P}}}$ is a Hodge-Tate representation with Hodge-Tate weights $(0,k_2-2,k_1-1,k_1+k_2-3)$. By Theorem 4.10.10 these weights are the eigenvalues of the operator $\phi_{\mathfrak{P}}$.

Now let $\mathfrak{P}_{r,0}=(\mathfrak{P}\cap\mathbb{I}^\circ_0)\cdot\mathbb{I}^\circ_{r,0}$. Recall that $\iota'_{r,0,\mathfrak{P}}\colon\mathbb{I}^\circ_0/\mathfrak{P}\to\mathbb{I}^\circ_{r,0}/\mathfrak{P}\mathbb{I}^\circ_{r,0}$ is the natural inclusion. There is an isomorphism of Galois representation $\rho_{r,\mathfrak{P}_{r,0}}\cong\iota'_{r,0,\mathfrak{P}}\circ\rho_{\mathfrak{P}}|_{H_{r,p}}$. Then the eigenvalues of the Sen operator $\phi_{r,\mathfrak{P}_{r,0}}$ attached to $\rho_{r,\mathfrak{P}_{r,0}}$ are the images of those of $\phi_{\mathfrak{P}}$ via $\iota'_{r,0,\mathfrak{P}}$.

Let S^{class} be the set of primes of $\mathbb{I}^\circ$ that correspond to classical GSp_4-eigenforms. Consider the set

$$S^{\mathrm{class}}_{r,0}=\{(\mathcal{P}\cap\mathbb{I}^\circ_0)\cdot\mathbb{I}^\circ_{r,0}\,|\,\mathcal{P}\in S^{\mathrm{class}}\}.$$

Since S^{class} is Zariski-dense in $\mathbb{I}^\circ$, $S^{\mathrm{class}}_{r,0}$ is Zariski-dense in $\mathbb{I}^\circ_{r,0}$. In particular the eigenvalues of the Sen operator ϕ_r are given by the unique interpolation of the eigenvalues of $\phi_{r,\mathfrak{P}_{r,0}}$ when $\mathfrak{P}_{r,0}$ varies in $S^{\mathrm{class}}_{r,0}$. If $\mathfrak{P}_{r,0}\cap A_r=P_{\underline{k}}\cdot A_r$, the eigenvalues of $\phi_{r,\mathfrak{P}_{r,0}}$ are $(0,k_2-2,k_1-1,k_1+k_2-3)$ by the discussion above. Let (T_1,T_2) be the images in A_r of the usual coordinate functions on the weight space. For every arithmetic prime $P_{\underline{k}}$ of Λ_h, the element

$$(0,k_2-2,k_1-1,k_1+k_2-3)$$

is the evaluation at $P_{\underline{k}}\cdot A_r$ of the function $A_r\to\mathbb{C}_p^4$ defined by

$$(T_1,T_2)\mapsto(0,\log(u^{-2}(1+T_2)),\log(u^{-1}(1+T_1)),\log(u^{-3}(1+T_1)(1+T_2))). \tag{4.35}$$

Hence this function gives the desired interpolation. $\square$

COROLLARY 4.10.19.

(1) The operator $\phi'_{r,0}$ *belongs to* $\mathrm{M}_4(\mathbb{I}_{r,0,\mathbb{C}_p})^{\geq\frac{1}{p-1}}$. *In particular the exponential series defines an element* $\exp(\phi'_{r,0}) \in \mathrm{GL}_4(\mathbb{I}_{r,0,\mathbb{C}_p})$.

(2) The eigenvalues of $\exp(\phi'_{r,0})$ *are* 1, $u^{-2}(1+T_2)$, $u^{-1}(1+T_1)$ *and* $u^{-3}(1+T_1)(1+T_2)$.

PROOF. By Proposition 4.10.17 the eigenvalues of $\phi'_{r,0}$ are in the image of the logarithm map, so (i) holds. By exponentiating them we obtain (ii). $\square$

Let $\Phi_{r,0} = \iota_{\mathbb{B}_{r,\mathbb{C}_p}}(\exp(\phi'_{r,0}))$. By definition $\Phi_{r,0}$ is an element of $\mathrm{GL}_4(\mathbb{B}_{r,\mathbb{C}_p})$. We show that it has the two properties we need. We define a matrix $C_{T_1,T_2} \in \mathrm{GSp}_4(\mathbb{B}_{r,\mathbb{C}_p})$ by

$$C_{T_1,T_2} = \begin{pmatrix} u^{-3}(1+T_1)(1+T_2) & 0 & 0 & 0 \\ 0 & u^{-1}(1+T_1) & 0 & 0 \\ 0 & 0 & u^{-2}(1+T_2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

PROPOSITION 4.10.20. *(1) There exists* $\gamma \in \mathrm{GSp}_4(\mathbb{B}_{r,\mathbb{C}_p})$ *satisfying*

$$\Phi_{\mathbb{B}_r} = \gamma C_{T_1,T_2}\gamma^{-1}. \tag{4.36}$$

(2) The element $\Phi_{\mathbb{B}_r}$ *normalizes the Lie algebra* $\mathfrak{G}_{r,\mathbb{C}_p}$.

PROOF. The matrices $\Phi_{\mathbb{B}_r}$ and C_{T_1,T_2} have the same eigenvalues by Corollary 4.10.19(2). Hence there exists $\gamma \in \mathrm{GL}_4(\mathbb{B}_{r,\mathbb{C}_p})$ satisfying (4.36) if and only if the difference between any two of the eigenvalues of $\Phi_{\mathbb{B}_r}$ is invertible in $\mathbb{B}_r$. We check by a direct calculation that each one of these differences belongs to an ideal of the form $P\cdot\mathbb{B}_r$ with $P \in S^{\mathrm{bad}}$, hence it is invertible in $\mathbb{B}_r$ by Remark 4.10.7. Since both $\Phi_{\mathbb{B}_r}$ and C_{T_1,T_2} are elements of $\mathrm{GSp}_4(\mathbb{B}_{r,\mathbb{C}_p})$, we can take $\gamma \in \mathrm{GSp}_4(\mathbb{B}_{r,\mathbb{C}_p})$.

Part (2) of the proposition follows from the fact that $\phi'_{\mathbb{B}_r} \in \mathfrak{G}_{r,\mathbb{C}_p}$, given by Corollary 4.10.16. $\square$

4.11. Existence of the Galois level

We have all the ingredients we need to state and prove our first main theorem.

THEOREM 4.11.1. *Let* $h \in \mathbb{Q}^{+,\times}$. *Let* $\theta\colon \mathbb{T}_h \to \mathbb{I}^\circ$ *be a family of cuspidal Siegel modular eigenforms of level* $\Gamma_1(N)\cap\Gamma_0(p)$ *and slope bounded by* h. *Suppose that the residual Galois representation associated with* θ *is absolutely irreducible. Let* $\rho\colon G_\mathbb{Q} \to \mathrm{GSp}_4(\mathbb{I}^\circ_{\mathrm{Tr}})$ *be the Galois representation associated with* θ. *Suppose that:*

(1) ρ *is residually of* Sym^3 *type in the sense of Definition 3.11.2;*

(2) ρ *is* $\mathbb{Z}_p$*-regular in the sense of Definition 3.11.1.*

For every radius r *in the set* $\{r_i\}_{i\in\mathbb{N}^{>0}}$ *defined in Section 4.1.2, let* $\mathfrak{G}_r$ *be the Lie algebra that we attached to* $\mathrm{Im}\,\rho$ *in Section 4.10.1. Then there exists a non-zero ideal* $\mathfrak{l}$ *of* $\mathbb{I}_0$ *such that*

$$\mathfrak{l}\cdot\mathfrak{sp}_4(\mathbb{B}_r) \subset \mathfrak{G}_r \tag{4.37}$$

for every r *as above.*

Let Δ be the set of roots of GSp_4 with respect to our choice of maximal torus. Recall that for $\alpha \in \Delta$ we denote by $\mathfrak{u}^\alpha$ the nilpotent subalgebra of $\mathfrak{gsp}_4$ corresponding to α. Let r be a radius in the set $\{r_i\}_{i\geq1}$. We set $\mathfrak{U}^\alpha_r = \mathfrak{G}_r \cap \mathfrak{u}^\alpha(\mathbb{B}_r)$ and $\mathfrak{U}^\alpha_{r,\mathbb{C}_p} = \mathfrak{G}_{r,\mathbb{C}_p} \cap \mathfrak{u}^\alpha(\mathbb{B}_{r,\mathbb{C}_p})$, which coincides with $\mathfrak{U}^\alpha_r\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$. Via the isomorphisms $\mathfrak{u}^\alpha(\mathbb{B}_r)\cong\mathbb{B}_r$ and $\mathfrak{u}^\alpha(\mathbb{B}_{r,\mathbb{C}_p})\cong\mathbb{B}_{r,\mathbb{C}_p}$ we see $\mathfrak{U}^\alpha_r$ as a $\mathbb{Q}_p$-vector subspace of $\mathbb{B}_r$ and $\mathfrak{U}^\alpha_{r,\mathbb{C}_p}$ as a $\mathbb{C}_p$-vector subspace of $\mathbb{B}_{r,\mathbb{C}_p}$.

Recall that U^α denotes the one-parameter unipotent subgroup of GSp_4 associated with the root α. Let H_r be the normal open subgroup of $G_\mathbb{Q}$ defined in the beginning of Section 4.10.

Note that Proposition 4.9.23 holds with $\rho|_{H_0}$ replaced by $\rho|_{H_r}$ since H_r is open in $G_{\mathbb{Q}}$. Let $U^\alpha(\rho|_{H_r}) = U^\alpha(\mathbb{I}_0^\circ) \cap \rho(H_r)$ and $U^\alpha(\rho_r) = U^\alpha \cap \rho_r(H_r)$. Via the isomorphisms $U^\alpha(\mathbb{I}_0) \cong \mathbb{I}_0$ and $U^\alpha(\mathbb{I}_{r,0}) \cong \mathbb{I}_{r,0}$ we identify $U^\alpha(\rho|_{H_0})$ and $U^\alpha(\rho_r)$ with $\mathbb{Z}_p$-submodules of $\mathbb{I}_0$ and $\mathbb{I}_{r,0}$, respectively. Note that the injection $\mathbb{I}_0^\circ \hookrightarrow \mathbb{I}_{r,0}^\circ$ induces an isomorphism of $\mathbb{Z}_p$-modules $U^\alpha(\rho|_{H_0}) \cong U^\alpha(\rho_r)$.

We define a nilpotent subalgebra of $\mathfrak{gsp}_4(\mathbb{I}_{r,0})$ by

$$\mathfrak{U}^\alpha_{\mathbb{I}_{r,0}} = \mathbb{Q}_p \cdot \log(U^\alpha(\rho_r)).$$

As usual we identify $\mathfrak{U}^\alpha_{\mathbb{I}_{r,0}}$ with a $\mathbb{Q}_p$-vector subspace of $\mathbb{I}_{r,0}$. Note that the natural injection $\iota_{\mathbb{B}_r} \colon \mathbb{I}_{r,0} \hookrightarrow \mathbb{B}_r$ induces an injection $\mathfrak{U}^\alpha_{\mathbb{I}_{r,0}} \hookrightarrow \mathfrak{U}^\alpha_r$ for every α.

Lemma 4.11.2. *For every $\alpha \in \Delta$ and every r there exists a non-zero ideal $\mathfrak{l}^\alpha$ of $\mathbb{I}_0$, independent of r, such that the B_r-span of $\mathfrak{U}^\alpha_r$ contains $\mathfrak{l}^\alpha \mathbb{B}_r$.*

Proof. Let d be the dimension of $Q(\mathbb{I}_0^\circ)$ over $Q(\Lambda_h)$. Let $\alpha \in \Delta$. By Proposition 4.9.23 the unipotent subgroup $U^\alpha(\rho|_{H_r})$ contains a basis $E = \{e_i\}_{i=1,\ldots,d}$ of a Λ_h-lattice in $\mathbb{I}_0^\circ$. Lemma 4.9.2 implies that the $\Lambda_h[p^{-1}]$-span of E contains a non-zero ideal $\mathfrak{l}^\alpha$ of $\mathbb{I}_0$. Consider the map $\iota^\alpha \colon U^\alpha(\mathbb{I}_0) \to \mathfrak{u}^\alpha(\mathbb{B}_r)$ given by the composition

$$U^\alpha(\mathbb{I}_0) \hookrightarrow U^\alpha(\mathbb{I}_{r,0}) \xrightarrow{\log} \mathfrak{u}^\alpha(\mathbb{I}_{r,0}) \hookrightarrow \mathfrak{u}^\alpha(\mathbb{B}_r),$$

where all the maps have been introduced above. Note that $\iota^\alpha(U^\alpha(\rho|_{H_0})) \subset \mathfrak{U}^\alpha_r$. Let $E_{\mathbb{B}_r} = \iota^\alpha(E)$. Since ι^α is a morphism of $\mathbb{I}_0$-modules we have

$$B_r \cdot \mathfrak{U}^\alpha_r \supset B_r \cdot E_{\mathbb{B}_r} = B_r \cdot (\Lambda_h[p^{-1}] \cdot E_{\mathbb{B}_r}) = B_r \cdot \iota^\alpha(\Lambda_h[p^{-1}] \cdot E) \supset \mathbb{B}_r \cdot \iota^\alpha(\mathfrak{l}^\alpha) = \mathfrak{l}^\alpha \mathbb{B}_r.$$

By construction and by Remark 4.10.2 the ideal $\mathfrak{l}^\alpha$ can be chosen independently of r. □

By Proposition 4.10.20(1) there exists $\gamma \in \mathrm{GSp}_4(\mathbb{B}_{r,\mathbb{C}_p})$ such that

$$\Phi_{\mathbb{B}_r} = \gamma C_{T_1,T_2} \gamma^{-1}.$$

Let $\mathfrak{G}^\gamma_{r,\mathbb{C}_p} = \gamma^{-1} \mathfrak{G}_{r,\mathbb{C}_p} \gamma$. For each $\alpha \in \Delta$ let $\mathfrak{U}^{\gamma,\alpha}_{r,\mathbb{C}_p} = \mathfrak{u}^\alpha(\mathbb{B}_{r,\mathbb{C}_p}) \cap \mathfrak{G}^\gamma_{r,\mathbb{C}_p}$.

We prove the following lemma by an argument similar to that of [**HT15**, Theorem 4.8].

Lemma 4.11.3. *For every $\alpha \in \Delta$ the Lie algebra $\mathfrak{U}^{\gamma,\alpha}_{r,\mathbb{C}_p}$ is a $B_{r,\mathbb{C}_p}$-submodule of $\mathbb{B}_{r,\mathbb{C}_p}$.*

Proof. By Proposition 4.10.20(2) the operator $\Phi_{\mathbb{B}_r}$ normalizes $\mathfrak{G}_{r,\mathbb{C}_p}$, hence C_{T_1,T_2} normalizes $\mathfrak{G}^\gamma_{r,\mathbb{C}_p}$. Since C_{T_1,T_2} is diagonal it also normalizes $\mathfrak{U}^{\gamma,\alpha}_{r,\mathbb{C}_p}$. Moreover

$$\mathrm{Ad}\,(C_{T_1,T_2}) u^\alpha = \alpha(C_{T_1,T_2}) u^\alpha$$

for every $u^\alpha \in \mathfrak{U}^{\gamma,\alpha}_{r,\mathbb{C}_p}$. Let α_1 and α_2 be the roots sending $\mathrm{diag}\,(t_1, t_2, \nu t_2^{-1}, \nu t_1^{-1}) \in T_2$ to t_1/t_2 and $\nu^{-1} t_2^2$, respectively. With respect to our choice of Borel subgroup, the set of positive roots of GSp_4 is $\{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}$. The Lie bracket gives an identification

$$[\mathfrak{U}^{\gamma,\alpha_1}_{r,\mathbb{C}_p}, \mathfrak{U}^{\gamma,\alpha_2}_{r,\mathbb{C}_p}] = \mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}.$$

Conjugation by C_{T_1,T_2} on the $\mathbb{C}_p$-vector space $\mathfrak{u}^{\alpha_1}(\mathbb{B}_{r,\mathbb{C}_p})$ induces multiplication by $\alpha_1(C_{T_1,T_2}) = u^{-2}(1+T_2)$. Since $u^{-2} \in \mathbb{Z}_p^\times$ and $\mathfrak{U}^{\gamma,\alpha_1}_{r,\mathbb{C}_p}$ is stable under $\mathrm{Ad}\,(C_{T_1,T_2})$, multiplication by $1+T_2$ on $\mathfrak{u}^{\alpha_1}(\mathbb{B}_{r,\mathbb{C}_p})$ leaves $\mathfrak{U}^{\gamma,\alpha_1}_{r,\mathbb{C}_p}$ stable. Now we compute

$$\begin{aligned}(1+T_2) \cdot \mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p} &= (1+T_2) \cdot [\mathfrak{U}^{\gamma,\alpha_1}_{r,\mathbb{C}_p}, \mathfrak{U}^{\gamma,\alpha_2}_{r,\mathbb{C}_p}] = \\ &= [(1+T_2) \cdot \mathfrak{U}^{\gamma,\alpha_1}_{r,\mathbb{C}_p}, \mathfrak{U}^{\gamma,\alpha_2}_{r,\mathbb{C}_p}] \subset [\mathfrak{U}^{\gamma,\alpha_1}_{r,\mathbb{C}_p}, \mathfrak{U}^{\gamma,\alpha_2}_{r,\mathbb{C}_p}] = \mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p},\end{aligned}$$

where the inclusion $(1+T_2) \cdot \mathfrak{U}^{\gamma,\alpha_1}_{r,\mathbb{C}_p} \subset \mathfrak{U}^{\gamma,\alpha_1}_{r,\mathbb{C}_p}$ is the result of the previous sentence. We deduce that multiplication by $1+T_2$ on $\mathfrak{u}^{\alpha_1+\alpha_2}(\mathbb{B}_{r,\mathbb{C}_p})$ leaves $\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}$ stable.

Similarly, conjugation by C_{T_1,T_2} on the $\mathbb{C}_p$-vector space $\mathfrak{u}^{\alpha_2}(\mathbb{B}_{r,\mathbb{C}_p})$ induces multiplication by $\alpha_2(C_{T_1,T_2}) = u \cdot \frac{1+T_1}{1+T_2}$. Since $u \in \mathbb{Z}_p^\times$ and $\mathfrak{U}^{\gamma,\alpha_2}_{r,\mathbb{C}_p}$ is stable under $\mathrm{Ad}\,(C_{T_1,T_2})$, multiplication by

$1+T_2$ on $\mathfrak{u}^{\alpha_2}(\mathbb{B}_{r,\mathbb{C}_p})$ leaves $\mathfrak{U}^{\gamma,\alpha_2}_{r,\mathbb{C}_p}$ stable. The same calculation as above shows that multiplication by $\frac{1+T_1}{1+T_2}$ on $\mathfrak{u}^{\alpha_1+\alpha_2}(\mathbb{B}_{r,\mathbb{C}_p})$ leaves $\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}$ stable.

Having proved that multiplication by both $1+T_2$ and $\frac{1+T_1}{1+T_2}$ leaves $\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}$ stable, we deduce that multiplication by $(1+T_2)\cdot\frac{1+T_1}{1+T_2}=1+T_1$ also leaves $\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}$ stable. Since $\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}$ is a $\mathbb{C}_p$-vector space, we obtain that the $\mathbb{C}_p[T_1,T_2]$-module structure on $\mathfrak{u}^{\alpha_1+\alpha_2}(\mathbb{B}_{r,\mathbb{C}_p})$ induces a $\mathbb{C}_p[T_1,T_2]$-module structure on $\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}$. With respect to the p-adic topology $\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}$ is complete and $\mathbb{C}_p[T_1,T_2]$ is dense in $B_{r,\mathbb{C}_p}$, so the $B_{r,\mathbb{C}_p}$-module structure on $\mathfrak{u}^{\alpha_1+\alpha_2}(\mathbb{B}_{r,\mathbb{C}_p})$ induces a $B_{r,\mathbb{C}_p}$-module structure on $\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}$.

If β is another root, we can write

$$\begin{aligned}B_{r,\mathbb{C}_p}\cdot\mathfrak{U}^{\gamma,\beta}_{r,\mathbb{C}_p}&=B_{r,\mathbb{C}_p}\cdot[\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p},\mathfrak{U}^{\gamma,\beta-\alpha_1-\alpha_2}_{r,\mathbb{C}_p}]\subset\\ \subset[B_{r,\mathbb{C}_p}\cdot\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p},\mathfrak{U}^{\gamma,\beta-\alpha_1-\alpha_2}_{r,\mathbb{C}_p}]&\subset[\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p},\mathfrak{U}^{\gamma,\beta-\alpha_1-\alpha_2}_{r,\mathbb{C}_p}]=\mathfrak{U}^{\gamma,\beta}_{r,\mathbb{C}_p},\end{aligned}$$

where the inclusion $B_{r,\mathbb{C}_p}\cdot\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}\subset\mathfrak{U}^{\gamma,\alpha_1+\alpha_2}_{r,\mathbb{C}_p}$ is the result of the previous paragraph. □

Proof. (of Theorem 4.11.1) Let $E_{\mathbb{B}_r}\subset\mathfrak{U}^\alpha_r$ be the set defined in the proof of Lemma 4.11.2. Let $E_{\mathbb{B}_r,\mathbb{C}_p}=\{e\otimes 1\,|\,e\in E_{\mathbb{B}_r}\}\subset\mathfrak{U}^\alpha_{r,\mathbb{C}_p}$. Consider the Lie subalgebra $B_{r,\mathbb{C}_p}\cdot\mathfrak{G}_{r,\mathbb{C}_p}$ of $\mathfrak{gsp}_4(\mathbb{B}_{r,\mathbb{C}_p})$. For every $\alpha\in\Delta$ we have $B_{r,\mathbb{C}_p}\cdot\mathfrak{G}_{r,\mathbb{C}_p}\cap\mathfrak{u}^\alpha(B_{r,\mathbb{C}_p})=B_{r,\mathbb{C}_p}\cdot\mathfrak{U}^\alpha_r$. By Lemma 4.11.2 there exists an ideal $\mathfrak{l}^\alpha$ of $\mathbb{I}_0$, independent of r, such that $\mathfrak{l}^\alpha\cdot\mathbb{B}_{r,\mathbb{C}_p}\subset B_{r,\mathbb{C}_p}\cdot\mathfrak{U}^\alpha_r$. Let $\mathfrak{l}_0=\prod_{\alpha\in\Delta}\mathfrak{l}^\alpha$. Then Lemma 4.9.19 gives an inclusion

$$\mathfrak{l}_0\cdot\mathfrak{sp}_4(\mathbb{B}_{r,\mathbb{C}_p})\subset B_{r,\mathbb{C}_p}\cdot\mathfrak{G}_{r,\mathbb{C}_p}. \tag{4.38}$$

Let $\gamma\in\mathrm{GSp}_4(\mathbb{B}_{r,\mathbb{C}_p})$ be the element satisfying $\Phi_{\mathbb{B}_r}=\gamma C_{T_1,T_2}\gamma^{-1}$. The Lie algebra $\mathfrak{l}_0\cdot\mathfrak{sp}_4(\mathbb{B}_{r,\mathbb{C}_p})$ is stable under $\mathrm{Ad}\,(\gamma^{-1})$, so Equation 4.38 implies that

$$\begin{aligned}\mathfrak{l}_0\cdot\mathfrak{sp}_4(\mathbb{B}_{r,\mathbb{C}_p})&=\gamma^{-1}(\mathfrak{l}_0\cdot\mathfrak{sp}_4(\mathbb{B}_{r,\mathbb{C}_p}))\gamma\subset\gamma^{-1}(B_{r,\mathbb{C}_p}\cdot\mathfrak{G}_r)\gamma=\\ &=B_{r,\mathbb{C}_p}\cdot\gamma^{-1}\mathfrak{G}_r\gamma=B_{r,\mathbb{C}_p}\cdot\mathfrak{G}^\gamma_r.\end{aligned}$$

We deduce that, for every $\alpha\in\Delta$,

$$\begin{aligned}\mathfrak{l}_0\cdot\mathfrak{u}^\alpha(\mathbb{B}_{r,\mathbb{C}_p})&=\mathfrak{u}^\alpha(\mathbb{B}_{r,\mathbb{C}_p})\cap\mathfrak{l}_0\cdot\mathfrak{sp}_4(\mathbb{B}_{r,\mathbb{C}_p})\subset\mathfrak{u}^\alpha(\mathbb{B}_{r,\mathbb{C}_p})\cap B_{r,\mathbb{C}_p}\cdot\mathfrak{G}^\gamma_{r,\mathbb{C}_p}=\\ &=B_{r,\mathbb{C}_p}\cdot(\mathfrak{u}^\alpha(\mathbb{B}_{r,\mathbb{C}_p})\cap\mathfrak{G}^\gamma_{r,\mathbb{C}_p})=B_{r,\mathbb{C}_p}\cdot\mathfrak{U}^{\gamma,\alpha}_{r,\mathbb{C}_p}.\end{aligned} \tag{4.39}$$

By Lemma 4.11.3 $\mathfrak{U}^{\alpha,\gamma}_{r,\mathbb{C}_p}$ is a $B_{r,\mathbb{C}_p}$-submodule of $\mathfrak{u}_r(\mathbb{B}_{r,\mathbb{C}_p})$, so $B_{r,\mathbb{C}_p}\cdot\mathfrak{U}^{\gamma,\alpha}_{r,\mathbb{C}_p}=\mathfrak{U}^{\gamma,\alpha}_{r,\mathbb{C}_p}$. Hence Equation (4.39) gives

$$\mathfrak{l}_0\cdot\mathfrak{u}^\alpha(\mathbb{B}_{r,\mathbb{C}_p})\subset\mathfrak{U}^{\gamma,\alpha}_{r,\mathbb{C}_p} \tag{4.40}$$

for every α. Set $\mathfrak{l}_1=\mathfrak{l}_0^2$. By Lemma 4.9.19 and Remark 4.9.21, applied to the Lie algebra $\mathfrak{G}_{r,\mathbb{C}_p}$ and the set of ideals $\{\mathfrak{l}_1\mathbb{B}_r\}_{\alpha\in\Delta}$, Equation (4.40) implies that

$$\mathfrak{l}_1\cdot\mathfrak{sp}_4(\mathbb{B}_{r,\mathbb{C}_p})\subset\mathfrak{G}^\gamma_{r,\mathbb{C}_p}.$$

Observe that the left hand side of the last equation is stable under $\mathrm{Ad}\,(\gamma)$, so we can write

$$\mathfrak{l}_1\cdot\mathfrak{sp}_4(\mathbb{B}_{r,\mathbb{C}_p})=\gamma(\mathfrak{l}_1\cdot\mathfrak{sp}_4(\mathbb{B}_{r,\mathbb{C}_p}))\gamma^{-1}\subset\gamma\mathfrak{G}^\gamma_{r,\mathbb{C}_p}\gamma^{-1}=\mathfrak{G}_{r,\mathbb{C}_p}. \tag{4.41}$$

To complete the proof we show that the extension of scalars to $\mathbb{C}_p$ in Equation 4.41 is unnecessary, up to restricting the ideal $\mathfrak{l}_1$. By Equation 4.41 we have, for every α,

$$\mathfrak{l}_1\cdot\mathbb{B}_{r,\mathbb{C}_p}\subset\mathfrak{U}^\alpha_{r,\mathbb{C}_p}. \tag{4.42}$$

We prove that the above inclusion of $\mathbb{C}_p$-vector spaces descends to an inclusion $\mathfrak{l}_1\cdot\mathbb{B}_r\subset\mathfrak{U}^\alpha_r$ of $\mathbb{Q}_p$-vector spaces. Let I be some index set and let $\{f_i\}_{i\in I}$ be an orthonormal basis of $\mathbb{C}_p$ as a $\mathbb{Q}_p$-Banach space, satisfying $1\in\{f_i\}_{i\in I}$. Let $\mathfrak{a}$ be any ideal of $\mathbb{I}_{r,0}$ belonging to the set S_2. Recall that the $\mathbb{Q}_p$-vector space $\mathbb{B}_r/\mathfrak{a}\mathbb{B}_r\cong\mathbb{I}_{r,0}/\mathfrak{a}$ is finite-dimensional. We write $\pi_\mathfrak{a}$ for the projection $\mathbb{B}_r\to\mathbb{I}_{r,0}/\mathfrak{a}$ and also for its restriction $\mathbb{I}_{r,0}\to\mathbb{I}_{r,0}/\mathfrak{a}$. Let n and d be the $\mathbb{Q}_p$-dimensions of

$\mathbb{I}_{r,0}/\mathfrak{a}$ and $\pi_\mathfrak{a}(\mathfrak{U}_r^\alpha)$, respectively. Choose a $\mathbb{Q}_p$-basis $\{v_j\}_{j=1,\ldots,n}$ of $\mathbb{I}_{r,0}/\mathfrak{a}$ such that $\{v_j\}_{j=1,\ldots,d}$ is a $\mathbb{Q}_p$-basis of $\mathfrak{U}_r^\alpha$.

Let v be any element of $\pi_\mathfrak{a}(\mathfrak{l}_1)$. Then $v\otimes 1\in\pi_\mathfrak{a}(\mathfrak{l}_1)\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$ and by Equation (4.42) we have $v\otimes 1\in\pi_\mathfrak{a}(\mathfrak{U}_r^\alpha)\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$. Now $\{v_j\otimes f_i\}_{1\leq j\leq n;\, i\in I}$ and $\{v_j\otimes f_i\}_{1\leq j\leq d;\, i\in I}$ are orthonormal $\mathbb{Q}_p$-basis of $\mathbb{B}_r/\mathfrak{a}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$ and $\pi_\mathfrak{a}(\mathfrak{U}_r^\alpha)\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$, respectively. Hence there exists a set $\{\lambda_{j,i}\}_{1\leq j\leq d;\, i\in I}\subset\mathbb{Q}_p$ converging to 0 in the filter of complements of finite subsets of $\{1,2,\ldots,d\}\times I$ such that $v\otimes 1=\sum_{j=1,\ldots,d;\, i\in I}\lambda_{j,i}(v_j\otimes f_i)$. By setting $\lambda_{j,i}=0$ for $d<j\leq n$ we obtain a representation $v\otimes 1=\sum_{j=1,\ldots,n;\, i\in I}\lambda_{j,i}(v_j\otimes f_i)$ with respect to the basis $\{v_j\otimes f_i\}_{1\leq j\leq n;\, i\in I}$ of $(\mathbb{B}_r/\mathfrak{a})\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$.

On the other hand there exist $a_j\in\mathbb{Q}_p$, $j=1,2,\ldots,n$, such that $v=\sum_{j=1}^n a_jv_j$, so $v\otimes 1=\sum_{j=1}^n a_j(v_j\otimes 1)$ is another representation of $v\otimes 1$ with respect to the basis $\{v_j\otimes f_i\}_{1\leq j\leq n;\, i\in I}$. By the uniqueness of the representation of an element in a $\mathbb{Q}_p$-Banach space in terms of a given orthonormal basis we must have $a_j=\lambda_{j,i}$ if $f_i=1$. In particular $a_j=0$ for $d<j\leq n$, so $v=\sum_{j=1}^d a_jv_j$ is an element of $\pi_\mathfrak{a}(\mathfrak{U}_r^\alpha)$.

The discussion above proves that $\pi_\mathfrak{a}(\mathfrak{l}_1)\subset\pi_\mathfrak{a}(\mathfrak{U}_r^\alpha)$ for every $\mathfrak{a}\in S_2$. By taking a projective limit over $\mathfrak{a}$ with respect to the natural maps we obtain $\mathfrak{l}_1\cdot\mathbb{B}_r\subset\mathfrak{U}_r^\alpha$. Let $\mathfrak{l}=\mathfrak{l}_1^2$. From Lemma 4.9.19 and Corollary 4.9.21, applied to the Lie algebra $\mathfrak{G}_{r,\mathbb{C}_p}$ and the set of ideals $\{\mathfrak{l}_1\mathbb{B}_r\}_{\alpha\in\Delta}$, we deduce that

$$\mathfrak{l}\cdot\mathfrak{sp}_4(\mathbb{B}_r)\subset\mathfrak{G}_{r,\mathbb{C}_p}.$$

By definition we have

$$\mathfrak{l}=\mathfrak{l}_1^2=\mathfrak{l}_0^4=\left(\prod_{\alpha\in\Delta}\mathfrak{l}_\alpha\right)^4.$$

For every α the ideal $\mathfrak{l}^\alpha$ provided by Lemma 4.11.2 is independent of r, so $\mathfrak{l}$ is also independent of r. This concludes the proof of Theorem 4.11.1. □

Definition 4.11.4. *We call* Galois level *of θ and denote by $\mathfrak{l}_\theta$ the largest ideal of $\mathbb{I}_0$ satisfying the inclusion* (4.37).

4.12. Galois level and congruence ideal in the residual symmetric cube case

We work in the setting of Theorem 4.11.1. In particular h is a positive rational number, $\theta\colon\mathbb{T}_h\to\mathbb{I}^\circ$ is a family of GSp_4-eigenforms of slope bounded by h and $\rho\colon G_\mathbb{Q}\to\mathrm{GSp}_4(\mathbb{I}_{\mathrm{Tr}}^\circ)$ is the Galois representation associated with θ. We make the same assumptions on θ and ρ as in Theorem 4.11.1.

With the family θ we associate two ideals of $\mathbb{I}_0$:

- the ideal $\mathfrak{c}_{\theta,0}\cdot\mathbb{I}_0$, where $\mathfrak{c}_{\theta,0}$ is the fortuitous $(\mathrm{Sym}^3,\mathbb{I}_0^\circ)$-congruence ideal (see Definition 4.8.7);
- the Galois level $\mathfrak{l}_\theta$ (see Definition 4.11.4).

In the theorem below we prove that the prime divisors of these two ideals are the same outside of a finite explicit set of bad primes. This is an analogue of Theorem 2.5.2.For every ring R and every ideal $\mathfrak{I}$ of R we denote by $V_R(\mathfrak{I})$ the set of primes of R containing $\mathfrak{I}$. The set of bad primes of $\mathbb{I}_0$ already appeared in Section 4.10.1: it is

$$S^{\mathrm{bad}}=\{P\text{ prime of }\Lambda_h[p^{-1}]\,|\,P\cap\Lambda_2[p^{-1}]\in S_\Lambda^{\mathrm{bad}}\},$$

where S_Λ^{bad} is the set of prime ideals of $\Lambda_2[p^{-1}]$ defined by

$$S_\Lambda^{\mathrm{bad}}=\{(1+T_1-u),(1+T_2-u^2),(1+T_2-u(1+T_1)),((1+T_1)(1+T_2)-u^3)\}.$$

To simplify notations we write $\mathfrak{c}_{\theta,0}$ for $\mathfrak{c}_{\theta,0}\cdot\mathbb{I}_0$.

Theorem 4.12.1. *The following equality holds:*

$$V_{\mathbb{I}_0}(\mathfrak{c}_{\theta,0})-S^{\mathrm{bad}}=V_{\mathbb{I}_0}(\mathfrak{l}_\theta)-S^{\mathrm{bad}}.$$

Recall that there is a natural inclusion $\iota_r\colon \mathbb{I}_0 \hookrightarrow \mathbb{I}_{r,0}$.

PROOF. First we prove that $V_{\mathbb{I}_0}(\mathfrak{c}_{\theta,0}) - S^{\mathrm{bad}} \subset V_{\mathbb{I}_0}(\mathfrak{l}_\theta) - S^{\mathrm{bad}}$. Choose a radius r in the set $\{r_i\}_{i\in\mathbb{N}^{>0}}$ defined in Section 4.1.2. Let $P \in V_{\mathbb{I}_0}(\mathfrak{c}_{\theta,0}) - S^{\mathrm{bad}}$ and let ρ_P be the reduction of $\rho|_{H_0}\colon H_0 \to \mathrm{GSp}_4(\mathbb{I}_0)$ modulo P. By Proposition 4.8.8 there exists a representation $\rho_{P,1}\colon H_0 \to \mathrm{GL}_2(\mathbb{I}_0/P)$ such that $\rho_P \cong \mathrm{Sym}^3\rho_{P,1}$. Let $\rho_{r,P} = \iota_r \circ \rho_P$ and $\rho_{r,P,1} = \iota_r \circ \rho_{P,1}$. The isomorphism above gives $\rho_{r,P} \cong \mathrm{Sym}^3\rho_{r,P,1}$.

Suppose by contradiction that $\mathfrak{l}_\theta \not\subset P$. By definition of $\mathfrak{l}_\theta$ we have $\mathfrak{G}_r \supset \mathfrak{l}_\theta \cdot \mathfrak{sp}_4(\mathbb{B}_r)$. Recall that $\mathbb{B}_r/P = \mathbb{I}_{r,0}/P$ by the construction of $\mathbb{B}_r$. By looking at the previous inclusion modulo P we obtain

$$\mathfrak{G}_{r,P} \supset (\mathfrak{l}_\theta/(P\cap\mathfrak{l}_\theta)) \cdot \mathfrak{sp}_4(\mathbb{I}_{r,0}/P). \tag{4.43}$$

Since $\mathfrak{l}_\theta \not\subset P$ we have $\mathfrak{l}_\theta/(P\cap\mathfrak{l}_\theta) \neq 0$. By definition $\mathfrak{G}_{r,P} = \mathbb{Q}_p \cdot \log \mathrm{Im}\,\rho_{r,P}$. By our previous argument $\mathrm{Im}\,\rho_{r,P} \subset \mathrm{Sym}^3\mathrm{GL}_2(\mathbb{I}_{r,0}/P\mathbb{I}_{r,0})$, so $\log\mathrm{Im}\,\rho_{r,P}$ cannot contain a subalgebra of the form $\mathfrak{I}\cdot\mathfrak{sp}_4(\mathbb{I}_{r,0}/P\mathbb{I}_{r,0})$ for a non-zero ideal $\mathfrak{I}$ of $\mathbb{I}_{r,0}/P\mathbb{I}_{r,0}$. This contradicts Equation (4.43).

We prove the inclusion $V_{\mathbb{I}_0}(\mathfrak{l}_\theta) - S^{\mathrm{bad}} \subset V_{\mathbb{I}_0}(\mathfrak{c}_{\theta,0}) - S^{\mathrm{bad}}$. Let P be a prime of $\mathbb{I}_0$. We have to show that if $P \notin S^{\mathrm{bad}}$ and $\mathfrak{l}_\theta \subset P$ then $\mathfrak{c}_{\theta,0} \subset P$. Every prime of $\mathbb{I}_0$ is the intersection of the maximal ideals that contain it, so it is sufficient to show the previous implication when P is a maximal ideal.

Let P be a maximal ideal of $\mathbb{I}_0$ such that $P \notin S^{\mathrm{bad}}$ and $\mathfrak{l}_\theta \subset P$. Let κ_P be the residue field $\mathbb{I}_0/P$. We define two ideals of $\mathbb{I}_{r,0}$ by $\mathfrak{l}_{\theta,r} = \iota_r(\mathfrak{l}_\theta)\mathbb{I}_{r,0}$ and $P_r = \iota_r(P)\mathbb{I}_{r,0}$. Note that ι_r induces an isomorphism $\mathbb{I}_0/P \cong \mathbb{I}_{r,0}/P_r$. In particular P_r is maximal in $\mathbb{I}_{r,0}$ and $\mathbb{I}_{r,0}/P_r \cong \kappa_P$, which is a local field.

As before let $\rho_{r,P} = \iota_r \circ \rho_P$. The residual representation $\overline{\rho}_{r,P}\colon H_0 \to \mathrm{GSp}_4(\mathbb{I}^\circ_{r,0}/\mathfrak{m}_{\mathbb{I}^\circ_{r,0}})$ associated with $\rho_{r,P}$ coincides with $\overline{\rho}|_{H_0}$. In particular $\rho_{r,P}$ is of residual Sym^3 type in the sense of Definition 3.11.2. Let $G_{r,P} = \mathrm{Im}\,\rho_{r,P}$ and $G^\circ_{r,P}$ be the connected component of the identity in $G_{r,P}$. Let $\overline{G^\circ_{r,P}}^{\mathrm{Zar}}$ be the Zariski closure of $G^\circ_{r,P}$ in $\mathrm{GSp}_4(\mathbb{I}_{r,0}/P_r)$. Since $\rho_{r,P}$ satisfies the hypotheses of Proposition 3.11.6, one of the following must hold:

(i) the algebraic group $\overline{G^\circ_{r,P}}^{\mathrm{Zar}}$ is isomorphic to $\mathrm{Sym}^3\mathrm{SL}_2$ over $\mathbb{I}_{r,0}/P_r$;

(ii) the algebraic group $\overline{G^\circ_{r,P}}^{\mathrm{Zar}}$ is isomorphic to Sp_4 over $\mathbb{I}_{r,0}/P_r$.

In the two cases let H^0 denote the normal open subgroup of H_0 satisfying $\mathrm{Im}\,\rho_{r,P}|_{H^0} = G^\circ_{r,P}$. Since H_0 is open and normal in $G_\mathbb{Q}$, H^0 is also open and normal in $G_\mathbb{Q}$. In case (i) there exists a representation $\rho^0_{r,P}\colon H^0 \to \mathrm{GL}_2(\mathbb{I}_{r,0}/P_r)$ such that $\rho_{r,P}|_{H^0} \cong \mathrm{Sym}^3\rho^0_{r,P}$. Since the image of $\rho_{r,P}|_{H^0}$ is Zariski-dense in the copy of $\mathrm{SL}_2(\mathbb{I}_{r,0}/P_r)$ embedded via the symmetric cube map, the image of $\rho^0_{r,P}$ is Zariski-dense in $\mathrm{SL}_2(\mathbb{I}_{r,0}/P_r)$. From Lemma 3.11.5 we deduce that $\mathrm{Im}\,\rho^0_{r,P}$ contains a congruence subgroup of $\mathrm{SL}_2(\mathbb{I}_{r,0}/P_r)$. Now the hypotheses of Lemma 3.11.5 are satisfied by the representation $\rho^0_{r,P}$ and the group H^0, so we conclude that there exists a representation $\rho'_{H_0,r,P}\colon H_0 \to \mathrm{GL}_2(\mathbb{I}_{r,0}/P_r)$ such that $\rho_{H_0,r,P} \cong \mathrm{Sym}^3\rho'_{H_0,r,P}$. By Proposition 4.8.8 the prime P must contain $\mathfrak{c}_{\theta,0}$, as desired.

We show that case (ii) never occurs. Suppose by contradiction that $\overline{G^\circ_{H_0,r,P}}^{\mathrm{Zar}} \cong \mathrm{Sp}_4$ over $\mathbb{I}_{r,0}/P_r$. By Propositions 4.6.1 and 4.6.8 we know that the field $\mathbb{I}_0/P$ is generated over $\mathbb{Q}_p$ by the traces of $\mathrm{Ad}\,(\rho_P|_{H_0})$. Hence the field $\mathbb{I}_{r,0}/P_r$ is generated over $\mathbb{Q}_p$ by the traces of $\mathrm{Ad}\,\rho_{r,P}$. By Theorem 3.11.4 applied to $\mathrm{Im}\,\rho_{r,P}$ there exists a non-zero ideal $\mathfrak{l}_{r,P}$ of $\mathbb{I}_{r,0}/P_r$ such that $G_{r,P}$ contains the principal congruence subgroup $\Gamma_{\mathbb{I}_{r,0}/P_r}(\mathfrak{l}_{r,P})$ of $\mathrm{Sp}_4(\mathbb{I}_{r,0}/P_r)$. By definition $\mathfrak{G}_{r,P} = \mathbb{Q}_p \cdot \log(\mathrm{Im}\,\rho_{r,P}|_{H_r})$ where H_r is an open $G_\mathbb{Q}$, so up to replacing $\mathfrak{l}_{r,P}$ by a smaller non-zero ideal we have

$$\mathfrak{l}_{r,P}\cdot\mathfrak{sp}_4(\mathbb{I}_{r,0}/P_r) \subset \log(\Gamma_{\mathbb{I}_{r,0}/P_r}(\mathfrak{l}_{r,P})) \subset \log(\iota_{r,0}(G_P)) \subset \mathfrak{G}_{r,P}. \tag{4.44}$$

The algebras $\mathfrak{G}_{r,P}$ are independent of r in the sense of Remark 4.10.3, so there exists an ideal $\mathfrak{l}_P$ of $\mathbb{I}_0/P$ such that, for every r in the set $\{r_i\}_{i\geq 1}$, the ideal $\mathfrak{l}_{r,P} = \iota_r(\mathfrak{l}_P)$ satisfies Equation (4.44). We choose the ideals $\mathfrak{l}_{r,P}$ of this form.

As before Δ is the set of roots of GSp_4 with respect to the chosen maximal torus. Let $\alpha \in \Delta$. Let $\mathfrak{U}_r^\alpha$ and $\mathfrak{U}_{r,P_r}^\alpha$ be the nilpotent Lie subalgebras respectively of $\mathfrak{G}_r$ and $\mathfrak{G}_{r,P_r}$ corresponding to α. We denote by π_{P_r} the projection $\mathfrak{gsp}_4(\mathbb{B}_r) \to \mathfrak{gsp}_4(\mathbb{B}_r/P_r\mathbb{B}_r)$. Clearly $\mathfrak{G}_{r,P_r} = \pi_{P_r}(\mathfrak{G}_r)$, so $\mathfrak{U}_{r,P_r}^\alpha = \pi_{P_r}(\mathfrak{U}_r^\alpha)$. Equation (4.44) gives $\mathfrak{l}_{r,P}\mathfrak{u}^\alpha(\mathbb{I}_{r,0}/P_r) \subset \mathfrak{U}_{r,P_r}^\alpha$. Choose a subset A_P^α of $\mathfrak{u}^\alpha(\mathbb{I}_0)$ such that, for every r, $\iota_r(A_P^\alpha) \subset \mathfrak{U}_r^\alpha$ and $\pi_{P_r}(\iota_r(A_P^\alpha)) = \mathfrak{l}_{r,P}\mathfrak{u}^\alpha(\mathbb{I}_{r,0}/P_r)$. Such a set exists because the algebras $\mathfrak{U}_r^\alpha$ are independent of r by Remark 4.10.3 and the ideals $\mathfrak{l}_{r,P}$ have been chosen of the form $\iota_r(\mathfrak{l}_P)$. Set $\mathfrak{A}_P = \left(\prod_{\alpha\in\Delta} \mathfrak{A}_P^\alpha\right)^4$. By the same argument as in the proof of Theorem 4.11.1, the ideal $\mathfrak{A}_P^\alpha$ satisfies

$$\iota_r(\mathfrak{A}_P^\alpha) \cdot \mathfrak{sp}_4(\mathbb{B}_r) \subset \mathfrak{G}_r.$$

Since $\mathfrak{l}_\theta \cdot \mathfrak{sp}_4(\mathbb{B}_r) \subset \mathfrak{G}_r$ for every r, we also have $(\mathfrak{l}_\theta + \mathfrak{A}_{H_0,P}^\alpha)\mathfrak{sp}_4(\mathbb{B}_r) \subset \mathfrak{G}_r$ for every r.

By assumption $\mathfrak{l}_\theta \subset P$, so $\pi_P(\mathfrak{l}_\theta) = 0$. By definition of $\mathfrak{A}_{H_0,P}^\alpha$ we have $\pi_P(\mathfrak{A}_P^\alpha) \supset \pi_P(A_P) = \mathfrak{l}_P$, so $\pi_P(\mathfrak{l}_\theta + \mathfrak{A}_P) = \mathfrak{l}_P$. We deduce that $\mathfrak{l}_\theta + \mathfrak{A}_P^\alpha$ is strictly larger than $\mathfrak{l}_\theta$. This contradicts the fact that $\mathfrak{l}_\theta$ is the largest among the ideals $\mathfrak{l}$ of $\mathbb{I}_0$ satisfying $\mathfrak{l} \cdot \mathfrak{sp}_4(\mathbb{B}_r) \subset \mathfrak{G}_r$. $\square$

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Résumé

Soit $g = 1$ ou 2 et $p > 3$ un nombre premier. Pour le groupe symplectique GSp_{2g}, les sytèmes de valeurs propres de Hecke apparaissant dans les espaces de formes automorphes classiques, d'un niveau modéré fixé et de poids variable, sont interpolés p-adiquement par un espace rigide analytique, la variété de Hecke pour GSp_{2g}. Un sous-domaine suffisamment petit de cette variété peut être décrit comme l'espace rigide analytique associé à une algèbre profinie $\mathbb{T}$. Une composante irréductible de $\mathbb{T}$ est définie par un anneau profini $\mathbb{I}$ et un morphisme $\theta\colon \mathbb{T} \to \mathbb{I}$. Dans le cas résiduellement irréductible on peut associer à θ une représentation $\rho_\theta\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}_{2g}(\mathbb{I})$. On étudie l'image de ρ_θ quand θ décrit une composante de pente positive de $\mathbb{T}$. Pour $g = 1$ il s'agit d'un travail en commun avec A. Iovita et J. Tilouine. On suppose que $g = 1$ où que $g = 2$ et θ est résiduellement de type cube symétrique. On montre que $\mathrm{Im}\,\rho_\theta$ est "grande" et que sa taille est liée aux "congruences fortuites" de θ avec les transferts de familles pour groupes de rang plus petit. Plus précisement, on agrandit un sous-anneau $\mathbb{I}_0$ de $\mathbb{I}[1/p]$ en un anneau $\mathbb{B}$ et on définit une sous-algèbre de Lie $\mathfrak{G}$ de $\mathfrak{gsp}_{2g}(\mathbb{B})$ associée à $\mathrm{Im}\,\rho_\theta$. On prouve qu'il existe un idéal non-nul $\mathfrak{l}$ de $\mathbb{I}_0$ tel que $\mathfrak{l} \cdot \mathfrak{sp}_{2g}(\mathbb{B}) \subset \mathfrak{G}$. Pour $g = 1$ les facteurs premiers de $\mathfrak{l}$ correspondent aux points CM de la famille θ. Pour $g = 2$ les facteurs premiers de $\mathfrak{l}$ correspondent à des congruences fortuites de θ avec des sous-familles de dimension 0 ou 1, obtenues par des transferts de type cube symétrique de points ou familles de la courbe de Hecke pour GL_2.

Abstract

Let $g = 1$ or 2 and $p > 3$ be a prime. For the symplectic group GSp_{2g} the Hecke eigensystems appearing in the spaces of classical automorphic forms, of a fixed tame level and varying weight, are p-adically interpolated by a rigid analytic space, the GSp_{2g}-eigenvariety. A sufficiently small subdomain of the eigenvariety can be described as the rigid analytic space associated with a profinite algebra $\mathbb{T}$. An irreducible component of $\mathbb{T}$ is defined by a profinite ring $\mathbb{I}$ and a morphism $\theta\colon \mathbb{T} \to \mathbb{I}$. In the residually irreducible case we can attach to θ a representation $\rho_\theta\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}_{2g}(\mathbb{I})$. We study the image of ρ_θ when θ describes a positive slope component of $\mathbb{T}$. In the case $g = 1$ this is a joint work with A. Iovita and J. Tilouine. Suppose either that $g = 1$ or that $g = 2$ and θ is residually of symmetric cube type. We prove that $\mathrm{Im}\,\rho_\theta$ is "big" and that its size is related to the "accidental congruences" of θ with the subfamilies that are obtained as lifts of families for groups of smaller rank. More precisely, we enlarge a subring $\mathbb{I}_0$ of $\mathbb{I}[1/p]$ to a ring $\mathbb{B}$ and we define a Lie subalgebra $\mathfrak{G}$ of $\mathfrak{gsp}_{2g}(\mathbb{B})$ associated with $\mathrm{Im}\,\rho_\theta$. We prove that there exists a non-zero ideal $\mathfrak{l}$ of $\mathbb{I}_0$ such that $\mathfrak{l} \cdot \mathfrak{sp}_{2g}(\mathbb{B}) \subset \mathfrak{G}$. For $g = 1$ the prime factors of $\mathfrak{l}$ correspond to the CM points of the family θ. Such points do not define congruences between θ and a CM family, so we call them accidental congruence points. For $g = 2$ the prime factors of $\mathfrak{l}$ correspond to accidental congruences of θ with subfamilies of dimension 0 or 1 that are symmetric cube lifts of points or families of the GL_2-eigencurve.