UNIVERSITÉ PARIS-PARIS NORD

THÈSE
pour obtenir le grade de
Docteur de l'Université Paris 13
Displine : **Mathématiques**
préparée au laboratoire **LAGA**
dans la cadre de lÉcole Doctorale **Galilée**
présentée et soutenue publiquement
par
DENG Taiwang
le 24 Juin 2016

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Induction Parabolique et Géométrie des Variétés Orbitales pour GL_n
Directeur de thèse : Pascal BOYER
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Résumé

Ariki et Ginzburg, en se basant sur les travaux de Zelevinsky sur les variétés orbitales, ont démontré que les multiplicités dans une representation induite *totale* sont données par les valeurs en $q = 1$ des polynômes de Kazhdan-Lusztig associés aux groupes symétriques. Dans ma thèse, j'ai introduit la notion de dérivée partielle qui raffine celle de Zelevinksy et s'identifie en $q = 1$, à l'exponentielle formelle de la q-dérivée de Kashiwara sur l'algèbre quantique. A l'aide de cette notion et en explorant la géométrie des variétés orbitales, je construis une procédure de *symétrisation* des multisegments me permettant, en particulier, de prouver une conjecture de Zelevinsky portant sur une propiété d'indépendance de l'induite parabolique *totale*. Je développe par ailleurs une stratégie afin de calculer les multiplicités dans une induite parabolique générale en utilisant le produit de faisceaux pervers de Lusztig.

Abstract

Ariki and Ginzburg, after the previous work of Zelevinsky on orbital varieties, proved that multiplicities in a *total* parabolically induced representations are given by the value at $q = 1$ of Kazhdan-Lusztig Polynomials associated to the symmetric groups. In my thesis I introduce the notion of partial derivative which refines the Zelevinsky derivative and show that it can be identified with the formal exponential of the q-derivative of Kashiwara with q=1. With the help of this notion, I exploit the geometry of the nilpotent orbital varieties to construct a symmetrization process for the multi-segments, which allows me to proove a conjecture of Zelevinsky on the property of the independence of the *total* parabolic induction. On the other hand, I develop a strategy to calculate the multiplicity in a general parabolic induction by using the Lusztig product of perverse sheaves.

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Introduction

This thesis deals with the computation of the Jordan-Hölder decomposition of a parabolic induced representation of GL_n over a p-adic field F. Starting with irreducible cuspidal representations, Zelevinsky classified the irreducible representations in terms of multisegments

$$\mathbf{a} \mapsto L_{\mathbf{a}},$$

where $L_{\mathbf{a}}$ is the irreducible representation of $GL_n(F)$ associated to the multiset $\mathbf{a}$, which is a set with multiplicities, of segments

$$\Delta_{\rho,r} = \{\rho, \rho\nu, \cdots, \rho\nu^{r-1}\},$$

where ρ is an irreducible cuspidal representation of $GL_g(F)$, $n = rg$ and $\nu : GL_g(F) \to \mathbb{C}$ is the character given

$$x \mapsto |\det(x)|.$$

For example, $L_{\Delta_{\nu^{(1-r)/2},r}}$ is the trivial representation of $GL_r(F)$. Given a multisegment $\mathbf{a} = \{\Delta_1, \cdots, \Delta_s\}$ the total parabolic associated induced representation is

$$\pi(\mathbf{a}) = L_{\Delta_1} \times L_{\Delta_2} \times \cdots \times L_{\Delta_s}$$

and one wants to compute the multiplicity $m(\mathbf{b}, \mathbf{a})$ of $L_{\mathbf{b}}$ in $\pi(\mathbf{a})$.
Zelevinsky introduced the geometry of nilpotent orbits and conjectured that the coefficients $m(\mathbf{b}, \mathbf{a})$ is the value at $q = 1$ of the Poincaré series $P_{\sigma(\mathbf{a}),\sigma(\mathbf{b})}(q)$ where $\sigma(\mathbf{a})$ and $\sigma(\mathbf{b})$ are the associated orbits. Moreover, he proved that these orbital varieties admit an open immersion into some Schubert varieties of type A. This conjecture was proved by Chriss-Ginzburg and Ariki, see [12], [1].

In the first part of this thesis, we are interested in another conjecture of Zelevinsky stated in the last sentence of §8 of [34].

Conjecture. *The* $m(\mathbf{b}, \mathbf{a})$ *depend only on natural relationships between segments of* $\mathbf{a}$ *and* $\mathbf{b}$*.*

Note :

— first that using types theory, the $m(\mathbf{b}, \mathbf{a})$ are independent of the Zelevinsky lines considered , cf. [30] for example, so that one is reduced to the case where the cuspidal support of all the segment considered are contained in the Zelevinsky line of the trivial representation.

— Using this reduction, this conjecture can now be viewed as a special case of a conjecture of Lusztig about *combinatorial invariance of Kazhdan-Lusztig polynomials* which can be stated in these terms :

let $x \leq y$ two elements of the symmetric group S_n, the Kazhdan-Lusztig polynomial $P_{x,y}(q)$ depends only on the poset structure of $[x, y] := \{z \in S_n : x \leq z \leq y\}$.

The main application of the results of this part of this thesis is then *the proof of the above conjecture of Zelevinsky*, cf. theorem 4.4.5 : the results is already interesting in the symmetric case, cf. the corollary 4.4.7.

Our approach rests on the use of some truncation functors

$$\mathbf{a} \mapsto \mathbf{a}^{(k)},$$

and the notion of partial derivation

$$\mathscr{D}^k \text{ indexed by integers } k \in \mathbb{Z},$$

which allows us, starting from general multisegments $\mathbf{a}$ and $\mathbf{b}$, to reduce to a symmetric situation where $\mathbf{a}$ and $\mathbf{b}$ are parametrized by $\sigma, \tau \in S_n$ for some n usually less than the degree of $\mathbf{a}$. In this symmetric case we obtain, using the result of Chriss-Ginzburg and Ariki, the equality

$$m(\mathbf{a}_\tau, \mathbf{a}_\sigma) = P_{\tau,\sigma}(1),$$

where $P_{\tau,\sigma}$ is the Kazhdan Lusztig polynomial associated to the permutations $\tau, \sigma \in S_n$.

Let us recall that these $m(\mathbf{a}_\tau, \mathbf{a}_\sigma)$ are given, using Chriss-Ginzburg and Ariki, by Kazhdan-Lusztig polynomials for the symmetric group S_m where m is the degree of a. So our formula can be also viewed as equalities between Kazhdan-Lusztig polynomials for different symmetric groups : these equalities were also obtained by Henderson [16], but instead of using the Billey-Warrington cancellation for the symmetric group, we investigate the geometry of nilpotent symmetric orbits.

Remark: using our truncation method, it should be possible to find a new algorithm for computing the general $m(\mathbf{b}, \mathbf{a})$.

In the second part we give some applications of our method, the main aim is to give *a formula for the computation of an induced representation*

$$L_{\mathbf{a}} \times L_{\mathbf{b}} = \sum m(\mathbf{c}, \mathbf{b}, \mathbf{a}) L_{\mathbf{c}}.$$

in terms of the coefficients of the "highest degree term" of some explicit Kazhdan-Lusztig polynomials. For the moment we treat the case where $\mathbf{b}$ is a segment and leave the general case for future work. To give an impression, the most simple formula in the case where $\mathbf{b} = [k+1]$ from proposition 8.1.5, looks like

$$L_{\mathbf{a}} \times L_{\mathbf{b}} = L_{\mathbf{a}+\mathbf{b}} + \sum_{\mathbf{c} \in \Gamma^{\ell_k - 1}(\mathbf{a},k)} (\theta_k(\mathbf{c}, \mathbf{a}) - \theta_k(\mathbf{c}^{[k+1]_1}, \mathbf{a}+\mathbf{b})) L_{\mathbf{c}^{[k+1]_1 [k]_{\ell_k - 1}}}.$$

where the $\theta_k(\mathbf{c}, \mathbf{a})$ are defined thanks to partial derivative, cf. notation 7.8.17. It would be interesting to compare our results with the known criteria of the irreducibility for parabolic induced representations, cf. [29], [22] and [17]. Moreover,

- in chapter 5, we obtain a geometric interpretation of the 5 relations defining Kazhdan-Lusztig polynomials.
- *In view of the conjecture of Lusztig*, which can be viewed as a generalization of Zelevinsky's conjecture, in chapter 6, we give a classification of the posets $S(\mathbf{a}) = \{\mathbf{b} : \mathbf{b} \leq \mathbf{a}\}$, in the sense of notation 1.3.2. We prove that they can be identified with either an interval in the symmetric group S_n or an interval in a double quotient of S_n, which corresponds to parabolic orbits in a generalized flag variety.
- Concerning partial derivation, in Chapter 7, using the Lusztig product of perverse sheaves (cf. [27]), we give a geometric meaning of the multiplicities appearing in the partial derivatives. In the general case we then obtain an explicit formula for the derivative $\mathscr{D}^k(L_{\mathbf{a}})$, cf. corollary 7.8.16. The main application is to calculate the coefficient $m(\mathbf{c}, \mathbf{b}, \mathbf{a})$ in chapter 8.

Let us now give more details. For a p-adic field F and $g > 1$, an irreducible admissible representation ρ of $GL_g(F)$ is called cuspidal if for all proper parabolic subgroup P, the corresponding Jacquet functor J_P^G sends ρ to 0. We write

$$\nu : GL_g(F) \to \mathbb{C}, \qquad \nu(x) = |\det(x)|$$

and for $k \geq 1$ and ρ a cuspidal irreducible representation of $GL_g(F)$, we call the set

$$\Delta_{\rho,k} = \{\rho, \rho\nu, \cdots, \rho\nu^{k-1}\}$$

a segment. For such a segment, the normalized induction functor

$$\mathrm{ind}_{P_{g,\cdots,g}}^{GL_{kg}(F)}(\rho \otimes \cdots \otimes \rho\nu^{k-1})$$

contains an unique irreducible sub-representation denoted by $L_{[\rho,\nu^{k-1}\rho]}$, where $P_{g,\cdots,g}$ is the standard parabolic subgroup with Levi subgroup isomorphic to

k blocks of GL_g. Then a multisegment is a multiset of segments that is a set with multiplicities. For $i = 1, \cdots, r$, let ρ_i be an irreducible cuspidal representation of $GL_{n_i}(F)$ and for $k_i \in \mathbb{N}$, by definition, the multisegment

$$\mathbf{a} = \{\Delta_{\rho_i,k_i} : i = 1, \cdots, r\},$$

is of degree $\deg(\mathbf{a}) = \sum n_i k_i$. In [34], the author gave a parametrization $\mathbf{a} \mapsto L_{\mathbf{a}}$ of irreducible admissible representations of $GL_n(F)$ in terms of multisegments of degree n, where for a well ordered multisegment $\mathbf{a}$(cf. definition 1.1.10), the representation $L_{\mathbf{a}}$ is the unique irreducible submodule of the parabolic induced representation

$$\pi(\mathbf{a}) = \mathrm{ind}_P^{GL_n(F)}(L_{\Delta_{\rho_1,k_1}} \otimes \cdots \otimes L_{\Delta_{\rho_r,k_r}}).$$

Now given two multisegments $\mathbf{a}$ and $\mathbf{b}$, one wants to determine the multiplicity $m(b, \mathbf{a})$ of $L_{\mathbf{b}}$ in $\pi(\mathbf{a})$.
Thanks to the Bernstein central decomposition, one is reduced to the case where the cuspidal representation ρ_i of $\mathbf{a}$ and $\mathbf{b}$ belongs to the same Zelevinsky line $\{\rho_0\nu^k : k \in \mathbb{Z}\}$. Zelevinsky also conjectured that $m(b, \mathbf{a})$ is independent of ρ_0 and depends only on the relative position of $\mathbf{a}$ and $\mathbf{b}$: this conjecture now follows from the theory of types, cf. [30]. So one is reduced to the simplest case where ρ_0 is the trivial representation.

Let us now explain what is known about these coefficients $m(\mathbf{b}, \mathbf{a})$ where the cuspidal support of $\mathbf{a}, \mathbf{b}$ belongs the Zelevinsky line of the trivial representation. First of all, it is proved in [34] that there exists a poset structure on the set of multisegments such that $m_{\mathbf{b},\mathbf{a}} > 0$ if and only if $\mathbf{b} \leq \mathbf{a}$. And we let

$$S(\mathbf{a}) = \{\mathbf{b} : \mathbf{b} \leq \mathbf{a}\}.$$

In [35], Zelevinsky introduced the nilpotent orbit associated to a multisegment $\mathbf{a}$. More precisely, to a multisegment $\mathbf{a}$, one can associate $\varphi_{\mathbf{a}} : \mathbb{Z} \to \mathbb{N}$ with $\varphi_{\mathbf{a}}(k)$ the multiplicities of ν^k appearing in $\mathbf{a}$. For each φ, V_φ is a $\mathbb{C}$-vector space of dimension $\deg \varphi := \sum_{k \in \mathbb{Z}} \varphi(k)$ with graded k-part of dimension $\varphi(k)$. Then E_φ is the set of endomorphisms T of degree $+1$, which admits a natural action of the group $G_\varphi = \prod_k GL(V_{\varphi,k})$. Then the orbits of E_φ under G_φ are parametrized by multisegments $\mathbf{a} = \sum_{i \leq j} a_{ij} \Delta_{\nu^i, j-i+1}$ such that $\varphi = \varphi_{\mathbf{a}}$ consisting of T with a_{ij} Jordan cells starting from $V_{\varphi,i}$ and ending in $V_{\varphi,j}$. We denote by $O_{\mathbf{a}}$ this orbit and we have the nice following property

$$\overline{O}_{\mathbf{a}} = \bigsqcup_{\mathbf{b} \geq \mathbf{a}} O_{\mathbf{b}}.$$

Now given a local system $\mathcal{L}_{\mathbf{a}}$ on $O_{\mathbf{a}}$, we can consider its intermediate extension $IC(\mathcal{L}_{\mathbf{a}})$ on $\overline{O}_{\mathbf{a}}$ and its fiber at a geometric point $z_{\mathbf{b}}$ of $O_{\mathbf{b}}$ and form the Kazhdan-Lusztig polynomial

$$P_{\mathbf{a},\mathbf{b}}(q) = \sum_i q^{i/2} \dim_{\mathbb{C}} \mathcal{H}^i(IC(\mathcal{L}_{\mathbf{a}}))_{z_{\mathbf{b}}}.$$

Zelevinsky then conjectured that $m_{\mathbf{b},\mathbf{a}} = P_{\mathbf{a},\mathbf{b}}(1)$ and call it the p-adic analogue of Kazhdan Lusztig Conjecture. This conjecture is a special case of a more general multiplicities formula proved by Chriss and Ginzburg in [12], chapter 8.

In this work, we first introduce the notion of a symmetric multisegment (cf. definition 2.1.5), which is, roughly speaking, a multisegment such that the beginnings and the ends of its segments are distinct and its segments admit non-empty intersections. We show that for a well chosen [1] symmetric multisegment $\mathbf{a}_{\mathrm{Id}}$, there is a natural bijection between the symmetric group S_n to the set of symmetric multisegments $S(\mathbf{a}_{\mathrm{Id}})$, cf. proposition 2.1.8, where n is the number of segments contained in $\mathbf{a}_{\mathrm{Id}}$.
When we restrict to the geometry of the nilpotent orbits to the symmetric locus, we recover the geometric situation of the Schubert varieties associated to S_n and obtain that for two symmetric multisegment $\mathbf{a}_\sigma, \mathbf{a}_\tau$ associated to $\sigma, \tau \in S_n$, the coefficient $m_{\mathbf{a}_\sigma,\mathbf{a}_\tau} = P_{\sigma,\tau}(1)$.

The next step *in chapter 3* is to try to reach non symmetric cases, starting with a symmetric one. For example for $\mathbf{a} \geq \mathbf{b}$ two multisegments and ν^k in the supercuspidal support of $\mathbf{a}$, one can eliminate every ν^k which appears at the end of some segments in $\mathbf{a}$ and $\mathbf{b}$ to obtain respectively a new pair of multisegments $\mathbf{a}^{(k)}$, $\mathbf{b}^{(k)}$ and try to prove that that $m(\mathbf{b},\mathbf{a}) = m(\mathbf{b}^{(k)},\mathbf{a}^{(k)})$. This result is almost true if we demand that $\mathbf{b}$ belongs to some subset $S(\mathbf{a})_k$ of $S(\mathbf{a})$, cf. Prop.3.4.1. The proof relies on the study of the geometry of nilpotent orbits and their links with the Grassmannian, cf. the introduction of chapter 3.

I*n chapter 4*, we iterate the process in chapter 3. In fact, for a multisegment $\mathbf{a}$ and $k_1, \cdots, k_r$ integers such that ν^{k_i} appears in the supercuspidal support of $\mathbf{a}$, let

$$\mathbf{a}^{(k_1,\cdots,k_r)} = (((\mathbf{a}^{(k_1)})\cdots)^{(k_r)}),$$

and

$$S(\mathbf{a})_{k_1,\cdots,k_r} = \{\mathbf{c} \in S(\mathbf{a}) : \mathbf{c}^{(k_1,\cdots,k_i)} \in S(\mathbf{a}^{(k_1,\cdots,k_i)})_{k_{i+1}}, \text{ for } i = 1, \cdots, r\}.$$

1. Thanks to corollary 4.4.7 which is a particular case of the Zelevinsky's conjecture, the results are independent of the choice of $\mathbf{a}_{\mathrm{Id}}$.

Then we show that for $\mathbf{b} \in S(\mathbf{a})_{k_1,\cdots,k_r}$, we always have

$$m(\mathbf{b}, \mathbf{a}) = m(\mathbf{a}^{(k_1,\cdots,k_r)}, \mathbf{b}^{(k_1,\cdots,k_r)}),$$

Reciprocally, we show, cf. proposition 4.2.4, that for any pair of multisegments $\mathbf{a} > \mathbf{b}$, we can find $\mathbf{a}^{\text{sym}}$ and $\mathbf{b}^{\text{sym}} < \mathbf{a}^{\text{sym}}$ such that

$$m(\mathbf{b}, \mathbf{a}) = m(\mathbf{b}^{\text{sym}}, \mathbf{a}^{\text{sym}}).$$

In the end of chapter 4, following an example, we present an algorithm to find $(\mathbf{a}^{\text{sym}}, \mathbf{b}^{\text{sym}})$. Finally the main application of the first part of this thesis, is, cf. theorem 4.4.5, the proof of the Zelevinsky's conjecture stated before.

In the second part, we consider the application of our result from the first four chapters. *In chapter 5*, as a first application, using the relation between symmetric groups and symmetric multisegments we try to give a new proof of the fact that the Poincaré polynomial $P_{\mathbf{a}_\tau,\mathbf{a}_\sigma}(q)$ of the intersection cohomology groups $\mathcal{H}^i(IC(\overline{O}_{\mathbf{a}_\tau}))_{\mathbf{a}_\sigma}$ for

— $\mathbf{a}_\sigma > \mathbf{a}_\tau$ a pair of symmetric multisegments with $\sigma, \tau \in S_n$,

— where the index $\mathbf{a}_\sigma$ indicates that we localize at a point in $O_{\mathbf{a}_\sigma}$,

satisfies the axioms defining the Kazhdan Lusztig polynomials for a Hecke algebra. We succeed in proving that $P_{\mathbf{a}_\tau,\mathbf{a}_\sigma}(q)$ satisfies the first four relations satisfying by $P_{\tau,\sigma}(q)$ and leave the last one (see the introduction of chapter 5). As for the last relation, we give an interpretation in terms of the decomposition theorem in our contexte.

In Chapter 6, we classify the poset $S(\mathbf{a})$. First of all, we single out the case where the multisegment $\mathbf{a}$ contains segments with different beginnings and endings and call it ordinary multisegment, cf. definition 2.1.1. In this case we prove that, as a poset,

$$S(\mathbf{a}) \simeq S(\mathbf{a}^{\text{sym}}, \mathbf{a}^{\text{sym}}_{\text{min}}) := \{\mathbf{d} \in S(\mathbf{a}^{\text{sym}}) : \mathbf{d} \geq \mathbf{a}^{\text{sym}}_{\text{min}}\},$$

where $\mathbf{a}_{\text{min}}$ is the minimal element in $S(\mathbf{a})$ and $\mathbf{a}^{\text{sym}}$(resp. $\mathbf{a}^{\text{sym}}_{\text{min}}$) is the symmetric multisegment associated to $\mathbf{a}$ (resp. $\mathbf{a}_{\text{min}}$) constructed in Chapter 4. Recall that in Chapter 2, we showed that $S(\mathbf{a}^{\text{sym}}) \subseteq S(\mathbf{a}_{\text{Id}})$, for some $\mathbf{a}_{\text{Id}}$, and $S(\mathbf{a}_{\text{Id}})$ as a poset is isomorphic to S_n with n equal to the number of segments contained in $\mathbf{a}_{\text{Id}}$. In this way, we identify the poset $S(\mathbf{a})$ with some Bruhat interval in S_n, where n is the number of segments contained in $\mathbf{a}$.
In the general case, as the ordinary case, we can reduce to parabolic multisegments where a multisegment $\mathbf{a}$ is called parabolic if all of its segments contain

a common point, cf. definition 6.2.5 and 6.2.22. Then all our construction for symmetric multisegments can be carried out with parabolic multisegments. Finally, we show that the poset $S(\mathbf{a})$ is isomorphic to a Bruhat interval in $S_{J_2}\backslash S_n/S_{J_1}$, where $J_i(i=1,2)$ is a subset of generators and S_{J_i} is the subgroup generated by J_i, see proposition 6.3.6 for details.

In chapter 7, if one is interested in calculating the multiplicities in $L_{\mathbf{a}} \times L_{\mathbf{b}}$, it might be interesting to first compute $\mathscr{D}^k(L_{\mathbf{a}})$. Using the formula of $\pi(\mathbf{a}) = \sum_{\mathbf{b}} m(\mathbf{b},\mathbf{a})L_{\mathbf{b}}$, one is reduced to compute

$$\mathscr{D}^k(\pi(\mathbf{a})) = \sum_{\mathbf{b}} n(\mathbf{b},\mathbf{a})L_{\mathbf{b}}$$

for some coefficients $n(\mathbf{b},\mathbf{a}) \geq 0$. As expected we can introduce a poset structure $\preceq_k$ on the set of multisegments so that $n(\mathbf{b},\mathbf{a}) \geq 0 \Leftrightarrow \mathbf{b} \preceq_k \mathbf{a}$, cf. proposition 7.1.4. Then using the notion of Lusztig's product of two perverse sheaves we prove, cf. proposition 7.3.8, that $n(\mathbf{b},\mathbf{a})$ is the value at $q=1$ of the Poincaré series of Lusztig product of two explicit perverse sheaves. In the parabolic case, we give an explicit description of this Lusztig product. As a consequence, for case $\deg(\mathbf{b}) < \deg(\mathbf{a})$, we show that the coefficient $n(\mathbf{b},\mathbf{a})$ is related to some $\mu(x,y)$, which is the coefficient of degree $\frac{1}{2}(\ell(y)-\ell(x)-1)$ in $P_{x,y}(q)$ defined to be zero if $\ell(y)-\ell(x)$ is even), where x,y are elements in certain symmetric group and are related to $\mathbf{a},\mathbf{b}$.

In the chapter 8 we use the computation of the partial derivatives in chapter 7 to give a recursive formula for the coefficients in the induced representation

$$L_{\mathbf{a}} \times L_{\mathbf{b}} = \sum m(\mathbf{c},\mathbf{b},\mathbf{a})L_{\mathbf{c}}.$$

It should be possible to treat the general case, but here we only consider the case where $\mathbf{b}$ is a segment. The idea is to pass to lower degree by applying the partial derivatives. The formulas are complicated, cf. proposition 8.1.12, even in the simplest case where $\mathbf{b}$ is a point. It should be interesting to implement the algorithm on a computer.
In the last chapter, using previous results, I give a proof of a conjecture of Lapid and Mínguez as well as its generalizations. Also, we give a counter example to a conjecture of Badulescu in section 9.2 as well as prove a particular case of this conjecture. In section 9.3, we give an example of an imaginary multisegment due to Leclerc and relate it the the Langlands-Jacquet correspondence. We end the chapter with the following conjecture

Conjecture : $\mathbf{a}$ is real if and only if $LJ(L_{\mathbf{a}})$ is irreducible for all D.

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Première partie

Multiplicities in Induced Representations

Chapitre 1

Induced Representations of GL_n

The aim of this section is to present our main object of study which are some integral coeffcients introduced by Zelevinsky, and defined by the formula 1.2.4, relating to some multisegments $\mathbf{a}, \mathbf{b}$ with cupsidal support contained in the Zelevinsky line associated to a cuspidal representation ρ.
Recall that the set of irreducible representations of GL_n breaks into pieces according to the super-cuspidal support (Bernstein Center), and, thanks to the theory of types, we are reduced to study the unipotent block, cf. [30], that is induced representations with super-cuspidal support contained in the Zelevinsky line attach to the representation $\rho = 1$.
Every unipotent irreducible representation is parametrized by a multisegment $\mathbf{a}$, that can be viewed as a function from the set of segments $\mathcal{C}$ to $\mathbb{N}$. For a multisegment $\mathbf{a}$, we denote by $L_{\mathbf{a}}$ the corresponding irreducible representation and $\pi(\mathbf{a})$ the induced representation, cf. notations 1.1.16. The question is then to calculate the image of such an induced representation in the associated Grothendieck group, that is to compute the multiplicity $m(\mathbf{b}, \mathbf{a})$ of $L_{\mathbf{b}}$ in $\pi(\mathbf{a})$.

To begin, let us fix some notations. Let p be a prime number, $F/\mathbb{Q}_p$ be a finite extension. We fix an absolute value $|.|$ on F such that $|\varpi_F| = 1/q$, where ϖ_F is a uniformizer of F, and q is the order of its residue field. For an integer $n \geq 1$, we denote by ν the character of $GL_n(F)$ defined by $\nu(g) = |det(g)|$.

1.1 Zelevinsky Classification

Notation 1.1.1. *We denote a partition of n by $\underline{n} = \{r_1, \cdots, r_\alpha\}$ with $\sum_{i=1}^{\alpha} r_i = n$. For a divisor m of n, the partition $(m, \cdots, m)$ will be denoted $\underline{n_m}$. We will also use the notation $\underline{n+m} = (n, m)$.*

Definition 1.1.2. *For a partition $\underline{n}$, let*

$$P_{\underline{n}} = P_{\underline{n}}(F) = M_{\underline{n}} U_{\underline{n}}$$

be the corresponding parabolic subgroup of $GL_n(F)$ with its decomposition into the product of its Levi subgroup $M_{\underline{n}} = GL_{r_1}(F) \times \cdots GL_{r_\alpha}(F)$ and its unipotent radical $U_{\underline{n}}$. Let $\delta_{P_{\underline{n}}}$ be the modular character of $P_{\underline{n}}$, given by

$$\delta_{P_{\underline{n}}}(-) = |det(ad(-)|_{\mathrm{Lie}\, U_{\underline{n}}})|^{-1}$$

For a topological group G, we recall that a representation (π, V) of G is

— *smooth* if for any vectors v, the stabilizer of v in G is an open subgroup,

— *admissible* if for any open compact subgroup K of G, $V^K = \{v : k.v = v, \forall k \in K\}$ is of finite dimension.

According to [6] theorem 4.1, a smooth representation of $GL_n(F)$ is of finite length if and only if it is admissible and finitely generated.

Definition 1.1.3. *For $\underline{n} = \{r_1, \cdots, r_\alpha\}$ and $\rho = \rho_1 \otimes \cdots \otimes \rho_{r_\alpha}$ a smooth representation of $M_{\underline{n}}$, where the ρ_i are representations of $GL_{r_i}(F)$, trivially extended to $P_{\underline{n}}$, we define the normalized induction functor which associates to ρ the representation $\pi = \mathrm{ind}_{P_{\underline{n}}}^{GL_n(F)}(\rho)$ of G such that*

$$\pi = \left\{ f : G \to V | \quad \begin{array}{c} f(pg) = \delta_{P_{\underline{n}}}(p)^{-1/2}\rho(p)f(g), \forall p \in P_{\underline{n}}, f(gk) = f(g) \\ \textit{for all } k \in K, \textit{ with } K \textit{ a certain open subgroup.} \end{array} \right\},$$

here G acts on f by $\pi(g)f(x) = f(xg)$.

Definition 1.1.4. *Let (π, V) be a representation of $GL_n(F)$ and $P_{\underline{n}}$ a parabolic subgroup. Let $J_{P_{\underline{n}}}^{GL_n(F)}(\pi)$ be the Jacquet functor of π defined by*

$$J_{P_{\underline{n}}}^{GL_n(F)}(\pi) = V/V(U_{\underline{n}}),$$

where $V(U_{\underline{n}}) = \{u.v - u | u \in U_{\underline{n}}, v \in V\}$.

Remark: Both parabolic induction and Jacquet functor are additive exact functors between the category of smooth representations of $M_{\underline{n}}$ and $GL_n(F)$. Moreover, they preserve admissible representations and finitely generated representations.

Proposition 1.1.5. *(cf. [31] theorem 2.7, 4.1 and 5.3.) For π a smooth representation of $GL_n(F)$, and σ a smooth representation of $M_{\underline{n}}$, we have the following Frobenius reciprocity,*

$$\mathrm{Hom}_G(\pi, \mathrm{ind}_{P_{\underline{n}}}^{GL_n(F)}(\sigma)) = \mathrm{Hom}_{M_{\underline{n}}}(J_{P_{\underline{n}}}^{GL_n(F)}(\pi), \sigma\delta_{P_{\underline{n}}}^{-1/2}).$$

Definition 1.1.6. *A smooth representation of $GL_n(F)$ is called cuspidal if for all nontrivial parabolic subgroup $P_{\underline{n}}$ of $GL_n(F)$,*

$$J_{P_{\underline{n}}}^{GL_n(F)}(\pi) = 0.$$

We denote by $\mathscr{C}_n$ the set of irreducible cuspidal representations of $GL_n(F)$, and

$$\mathscr{C} = \coprod_{n\geq 1} \mathscr{C}_n.$$

Proposition 1.1.7. *(cf. [7] 4.1) Let π be an irreducible representation of $GL_n(F)$, then there exists a partition $\underline{n} = \{r_1, \cdots, r_\alpha\}$ and a cuspidal representation $\rho = \rho_1 \otimes \cdots \otimes \rho_\alpha$, of $M_{\underline{n}}$, such that π can be embedded into $\mathrm{ind}_{P_{\underline{n}}}^{GL_n(F)}(\rho)$. The set $\{\rho_1, \cdots, \rho_r\}$ is determined by π up to permutation, we call it the cuspidal support of π.*

According to Harish Chandra, the study of irreducible representations of GL_n is thus divided into two parts, the cuspidal representations and the parabolically induced representations. We will not discuss here the classification of cuspidal representations of $GL_n(F)$, which rests on the theory of types for which the reader can refer to for example [10].

Definition 1.1.8. *By a multiset, we mean a pair (S, r) where S is a set and $r : S \to \mathbb{N}$ is a map. We say $(S_1, r_1) \subseteq (S_2, r_2)$ if $S_1 \subseteq S_2$ and $r_1(s) \leq r_2(s)$ for all $s \in S_1$. We define a bijection of multisets from (S_1, r_1) to (S_2, r_2) to be a bijection $\xi : S_1 \to S_2$ satisfying*

$$r_2(\xi(x)) = r_1(x).$$

Definition 1.1.9. *Let (S, r) be a multi-set, then we define*

$$\sharp(S, r) = \sum_{s\in S} r(s).$$

Convention : Naturally, we write a multiset as a set with repetition. For example, for $S = \{a, b\}$ and $r(a) = 2, r(b) = 1$, then we write the multiset (S, r) by $\{a, a, b\}$.

Definitions 1.1.10.

- *By a segment, we mean a subset Δ of $\mathscr{C}$ of the form $\Delta = \{\rho, \nu\rho, \cdots, \nu^k\rho = \rho'\}$. We denote it by $\Delta = [\rho, \rho']$ where $b(\Delta) := \rho$ is called its beginning and $e(\Delta) := \rho'$ its end. Let Σ^{univ} be the set of segments.*
- *We say that two segments Δ_1 and Δ_2 are linked if none of them is contained in the other and the union is again a segment.*

— *For* $\Delta_1 = [\rho_1, \rho_1']$ *and* $\Delta_2 = [\rho_2, \rho_2']$, *we say* Δ_1 *proceeds* Δ_2 *if they are linked and* $\rho_2 = \nu^k \rho_1$ *with* $k > 0$.

— *By a multisegment, we mean a finite multiset* $\mathbf{a} = \{\Delta_1, \cdots, \Delta_r\}$. *Let* $\mathcal{O}^{univ}$ *be the set of multisegments.*

— *We say a multisegment* $\mathbf{a} = \{\Delta_1, \cdots, \Delta_r\}$ *is well ordered if for each pair of indices* i, j *such that* $i < j$, Δ_i *does not proceeds* Δ_j.

Remark: for a given multisegment, we may have several ways to arrange it to be a well ordered multisegment.

Notation 1.1.11. *Let* $\mathbf{a} = \{\Delta_1, \cdots, \Delta_r\}$. *We call*

$$e(\mathbf{a}) = \big\{e(\Delta_1), \cdots, e(\Delta_r)\big\} \quad \text{and} \quad b(\mathbf{a}) = \big\{b(\Delta_1), \cdots, b(\Delta_r)\big\}$$

respectively the end and the beginning of $\mathbf{a}$ *as a multiset.*

Definition-Proposition 1.1.12. *([34]3.1) Let* ρ *be a cuspidal representation of* $GL_m(F)$ *then for* $n = rm$

$$\mathrm{ind}_{P_{\underline{nm}}}^{GL_{rm}(F)}(\rho \otimes \nu\rho \otimes \cdots \otimes \nu^{r-1}\rho)$$

contains a unique irreducible sub-representation, denoted by $L_{[\rho, \nu^{m-1}\rho]}$.

Notation 1.1.13. *Let* $\underline{n} = (r_1, \cdots, r_\alpha)$ *be a partition. Let* π_i *be a representation of* $GL_{r_i}(F)$ *for* $i = 1, \cdots, \alpha$. *Then we denote*

$$\pi_1 \times \cdots \times \pi_\alpha = \mathrm{ind}_{P_{\underline{n}}}^{GL_n(F)}(\pi_1 \otimes \cdots \otimes \pi_\alpha).$$

Proposition 1.1.14. *([34] Theorem 4.2) Let* $\Delta_1, \cdots, \Delta_r$ *be segments, then the following two conditions are equivalent :*

(1) *The representation* $L_{\Delta_1} \times \cdots \times L_{\Delta_r}$ *is irreducible.*

(2) *For each* $1 \leq i, j \leq r$, Δ_i *and* Δ_j *are not linked.*

The following theorem gives a complete classification of the induced irreducible representations of $GL_n(F)$ in terms of multisegments.

Theorem 1.1.15. *([34] Theorem 6.1) Let* $\mathbf{a} = \{\Delta_1, \cdots, \Delta_r\}$ *be a well ordered multisegment.*

(1) *Then the representation*

$$L_{\Delta_1} \times \cdots \times L_{\Delta_r}$$

contains a unique sub-representation, which we denote by $L_{\mathbf{a}}$.

(2) *The representations* $L_{\mathbf{a}}$ *and* $L_{\mathbf{a}'}$ *are isomorphic if and only if* $\mathbf{a} = \mathbf{a}'$ *as well ordered multisegments, which means that there is a way to well order* $\mathbf{a}'$ *to obtain* $\mathbf{a}$.

(3) *Any irreducible representation of* $GL_n(F)$ *is isomorphic to some representation of the form* $L_{\mathbf{a}}$.

Remark: according to (2), the irreducible representation $L_{\mathbf{a}}$ does not depend on the well ordered form of $\mathbf{a}$.

Notation 1.1.16. *From now on, for* $\mathbf{a} = \{\Delta_1, \cdots, \Delta_r\}$ *being well ordered, we denote*

$$\pi(\mathbf{a}) = L_{\Delta_1} \times \cdots \times L_{\Delta_r}.$$

1.2 Coefficients $m(\mathbf{b}, \mathbf{a})$

Notation 1.2.1. *We denote by* $\mathcal{R}_n$ *the Grothendieck group of the category of finite length representations of* $GL_n(F)$ *and*

$$\mathcal{R}^{univ} = \oplus_{n \geq 1} \mathcal{R}_n.$$

Proposition 1.2.2. *The set* $\mathcal{R}^{univ}$ *is a bi-algebra with the multiplication* μ *and co-multiplication* c *given by*

$$\mu(\pi_1 \otimes \pi_2) = \pi_1 \times \pi_2, \qquad c(\pi) = \sum_{r=0}^{n} J_{P_{r,n-r}}^{GL_n(F)}(\pi).$$

A consequence of theorem 1.1.15 is :

Corollary 1.2.3. *The algebra* $\mathcal{R}^{univ}$ *is a polynomial ring with indeterminates* $\{L_\Delta : \Delta \in \Sigma^{univ}\}$. *Moreover, as a* $\mathbb{Z}$*-module, the set* $\{L_{\mathbf{a}} : \mathbf{a} \in \mathcal{O}^{univ}\}$ *form a basis for* $\mathcal{R}^{univ}$.

Remark: Note that this implies the Bernstein Center theorem, i.e, we have a decomposition

$$\mathcal{R}^{univ} = \prod_{\rho} \mathcal{R}(\rho),$$

where ρ runs through the equivalent classes of irreducible supercuspidal representations. Here we say two irreducible super-cuspidal representations are equivalent if they lie in the same Zelevinsky line, and $\mathcal{R}(\rho)$ is the sub-algebra with support contained in the Zelevinsky line Π_ρ generated by ρ. We denote by $\mathcal{O}(\rho)$ the set of multisegments supported on Π_ρ.

Using theorem 1.1.15, let $\mathbf{a} = \{\Delta_1, \cdots, \Delta_r\}$ be a multisegment with support contained in some Zelevinsky line Π_ρ, then we can write

$$\pi(\mathbf{a}) = \sum_{\mathbf{b} \in \mathcal{O}(\rho)} m(\mathbf{b}, \mathbf{a}) L_{\mathbf{b}} \tag{1.2.4}$$

where $\pi(\mathbf{a}) = \Delta_1 \times \cdots \times \Delta_r,\ m(\mathbf{b}, \mathbf{a}) \in \mathbb{N}$. One of the aims of this thesis is to give some new insights on these $m(\mathbf{b}, \mathbf{a})$.

Remark: For our purpose, note that we can also rewrite the equation 1.2.4 in the following form

$$L_{\mathbf{a}} = \sum_{\mathbf{b} \in \mathcal{O}(\rho)} \widetilde{m}(\mathbf{b}, \mathbf{a}) \pi(\mathbf{b}). \tag{1.2.5}$$

The simplest example is given by

Proposition 1.2.6. *(cf. [35] section 4.6) Let Δ_1 and Δ_2 be two linked segments, then*

$$\Delta_1 \times \Delta_2 = L_{\mathbf{a}_1} + L_{\mathbf{a}_2}$$

with $\mathbf{a}_1 = \{\Delta_1,\ \Delta_2\}$, $\mathbf{a}_2 = \{\Delta_1 \cup \Delta_2,\ \Delta_1 \cap \Delta_2\}$.

Remark: it is conjectured in [34] 8.7 that the coefficient $m(\mathbf{b}, \mathbf{a})$ depends only on the combinatorial relations of $\mathbf{b}$ and $\mathbf{a}$, and not on the specific cuspidal representation ρ. The independence of specific cuspidal representation can be showed by type theory, see for example [30]. **In other words, as far as we are concerned with the coefficient $m(\mathbf{b}, \mathbf{a})$, we can restrict ourselves to the special case $\rho = 1$, the trivial representation of $GL_1(F)$.**

Definitions 1.2.7. *Let*

$$\Pi = \{\nu^k : k \in \mathbb{Z}\}$$

denote the Zelevinsky line of $\rho = 1$. We note

- *Σ the set of segments associated to Π,*
- *$\mathcal{O}$ the set of multisegments associated to Σ,*
- *$\mathcal{R}$ the subalgebra of $\mathcal{R}^{univ}$ generate by the elements in $L_{\mathbf{a}}$ with $\mathbf{a} \in \mathcal{O}$,*
- *$\mathcal{C} = \{f : \Sigma \to \mathbb{N}$ with finite support$\}$,*
- *$\mathcal{S} = \{\varphi : \mathbb{Z} \to \mathbb{N}\}$.*

Notation 1.2.8. *For $i \leq j$, we will identify $L_{[\nu^i, \nu^j]} \in \mathcal{R}$ with $[i, j]$(for simplicity we let $[i] = [i, i]$). More generally we denote a multisegment $\mathbf{a}$ by $\sum_{i \leq j} a_{ij}[i, j]$.*

Proposition 1.2.9. *By associating to $f \in \mathcal{C}$ the multisegment*

$$\sum_{\Delta \in \Sigma} f(\Delta)\Delta,$$

we can identify $\mathcal{C}$ with $\mathcal{O}$. For every element $\mathbf{b} \in \mathcal{O}$, we set $f_{\mathbf{b}}$ for the associated function in $\mathcal{C}$.

Definition 1.2.10. *For a multisegment*

$$\mathbf{a} = \sum_{i \leq j} a_{ij}[i, j]$$

with $f_{\mathbf{a}}$ associated function in $\mathcal{C}$, let

$$\varphi_{\mathbf{a}} = \sum_{\Delta \in \mathbf{a}} f_{\mathbf{a}}(\Delta)\chi_{\Delta} \in \mathcal{S}.$$

We call $\varphi_{\mathbf{a}}$ the weight of $\mathbf{a}$, and we call $\deg(\mathbf{a}) = \sum_{k \in \mathbb{N}} \varphi_{\mathbf{a}}(k)$ the degree of $\mathbf{a}$(or, the degree of $L_{\mathbf{a}}$).

Definition 1.2.11. *For $\varphi \in \mathcal{S}$, let $S(\varphi)$ be the set of multisegments with weight φ.*

1.3 A partial order on $\mathcal{O}$

Definition 1.3.1. *For $\mathbf{a}$ a multisegment, by an elementary operation, we mean replacing two linked segments $\{\Delta_1, \Delta_2\}$ by $\{\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2\}$ in $\mathbf{a}$.*

Notation 1.3.2. *Let $\mathbf{b}$ be a multisegment such that it can be obtained from $\mathbf{a}$ by a series of elementary operations, then we say $\mathbf{b} \leq \mathbf{a}$. We denote*

$$S(\mathbf{a}) = \{\mathbf{b} : \mathbf{b} \leq \mathbf{a}\}.$$

Definition 1.3.3. *We define for $\mathbf{b} \leq \mathbf{a}$,*

$$\ell(\mathbf{b}, \mathbf{a}) = \max_n \{n : \mathbf{a} = \mathbf{b}_0 \geq \mathbf{b}_1 \cdots \geq \mathbf{b}_n = \mathbf{b}\},$$

and $\ell(a) = \ell(\mathbf{a}_{\min}, \mathbf{a})$.

Definition 1.3.4. *We define the following total order relations on Σ :*

$$\begin{cases} [j,k] \prec [m,n], \text{ if } k < n, \\ [j,k] \prec [m,n], \text{ if } j > m, n = k. \end{cases}$$

Lemma 1.3.5. *Let* $\mathbf{b} \in S(\mathbf{a})$*, then* $\pi(\mathbf{a}) - \pi(\mathbf{b}) \geq 0$ *in* $\mathcal{R}$*.*

Démonstration. By choosing a maximal chain of multisegments between $\mathbf{a}$ and $\mathbf{b}$, we can assume that

$$\mathbf{a} = \{\Delta_1, \cdots, \Delta_r\},$$

$$\mathbf{b} = (\mathbf{a} \backslash \{\Delta_j, \Delta_k\}) \cup \{\Delta_j \cap \Delta_k, \Delta_j \cup \Delta_k\}.$$

Then by proposition 1.2.6,

$$\pi(\mathbf{a}) = \pi(\mathbf{b}) + L_{\Delta_1} \times \cdots \times \widehat{L}_{\Delta_j} \times \cdots \times \widehat{L}_{\Delta_k} \times \cdots \times L_{\Delta_r} \times L_{\{\Delta_j, \Delta_k\}}$$

□

Proposition 1.3.6. *The set* $S(\mathbf{a})$ *is a partially ordered finite set with unique minimal element* $\mathbf{a}_{\min}$*. Furthermore,* $\mathbf{a}_{\min}$ *is the unique multisegment in* $S(\mathbf{a})$ *in which no segment is linked to the others.*

Remark: in particular by proposition 1.1.14 a multisegment $\mathbf{a}$ is minimal if and only if $\pi(\mathbf{a})$ is irreducible.

Démonstration. For a proof of the fact that $\leq$ is a partial order, we refer to [34] 7.1. Let $X_{\mathbf{a}} := \cup_{\Delta \in \mathbf{a}} \Delta$ be a subset of the Zelevinsky line Π. Let $\varphi_{\mathbf{a}}$ be the weight function of $\mathbf{a}$. Let $\Sigma(\mathbf{a})$ be the set of segments with support in $X_{\mathbf{a}}$: this is a finite set. For every $\Delta \in \Sigma(\mathbf{a})$, we note χ_Δ the characteristic function of the set Δ. Now we consider the set

$$\Gamma(\mathbf{a}) = \{f \in \mathcal{C} : \ \varphi_{\mathbf{a}} = \sum_{\Delta \in \Sigma} f(\Delta) \chi_\Delta\}.$$

Then $\Gamma(\mathbf{a})$ is a finite set. Clearly, for any $\mathbf{b} \in S(\mathbf{a})$, we have $f_{\mathbf{b}} \in \mathcal{C}$ since the elementary operation does not change the weight function, note that $\mathbf{b}$ is uniquely determined by $f_{\mathbf{b}}$, so $S(\mathbf{a})$ is finite since $\Gamma(\mathbf{a})$ is finite.
We define $\mathbf{a}_{\min} = \{\Delta_1, \Delta_2, \cdots, \Delta_r\}$ with $\Delta_1 \succeq \cdots \succeq \Delta_r$, where for $\Delta_0 = \emptyset$, we set Δ_i be the maximal segment with respect to the total order $\prec$, such that χ_{Δ_i} is supported in $\mathrm{Supp}(\varphi_{\mathbf{a}} - \chi_{\Delta_0} - \cdots - \chi_{\Delta_{i-1}})$.
We only need to show that for all $\mathbf{b} \in S(\mathbf{a})$, we have $\mathbf{a}_{\min} \leq \mathbf{b}$. To see this, we look at a maximal segment Δ' in $\mathbf{b}$, if it is linked to some segments Δ'', then we apply the elementary operation to them and get $\mathbf{b}_1$. Now repeat the same procedure, in finite steps we get a multisegment $\mathbf{b}' \leq \mathbf{b}$ in which no segments are linked to the others. It remains to show that $\mathbf{b}' = \mathbf{a}_{\min}$.
In fact, we have

$$\varphi_{\mathbf{a}} = \sum_{\Delta \in \Sigma(\mathbf{a})} f_{\mathbf{a}_{\min}}(\Delta) \chi_\Delta = \sum_{\Delta \in \Sigma(\mathbf{a})} f_{\mathbf{b}'}(\Delta) \chi_\Delta. \tag{1.3.7}$$

Let $\mathbf{b}' = \{\Delta'_1, \cdots, \Delta'_t\}$ with $\Delta'_1 \succeq \cdots \succeq \Delta'_t$. Put $\Delta'_0 = \emptyset$ and suppose by induction that there is an s with $1 \leq s \leq \min\{r,\ t\}$ such that for all $0 \leq i < s$, $\Delta'_i = \Delta_i$. By construction, we have $\Delta'_s \preceq \Delta_s$ and we assume that $\Delta'_s \prec \Delta_s$. By the equality (1.3.7), $e(\Delta_s) = e(\Delta'_s)$, then $\chi_{\Delta'_s} - \chi_{\Delta_s}$ is negative. Let $\Delta = \Delta_s \setminus \Delta'_s$. Now by the equality (1.3.7), there exists a minimal $i > s$ such that the segment Δ'_i satisfies the property that $b(\Delta'_i) \leq b(\Delta) \leq e(\Delta) \leq e(\Delta'_i)$. But this implies that Δ'_s is linked to Δ'_i, contradiction. Therefore $\Delta'_s = \Delta_s$. We conclude by the same argument that

$$r = s,\ \Delta'_i = \Delta_i, 1 \leq i \leq r.$$

□

Concerning the coefficient $m(\mathbf{b}, \mathbf{a})$, we have

Proposition 1.3.8. *(cf.[34] 7.1) The coefficient $m(\mathbf{b}, \mathbf{a})$ is*
— *nonzero if and only if $\mathbf{b} \leq \mathbf{a}$, and*
— *equal to 1 if $\mathbf{b} = \mathbf{a}$.*

1.4 Partial Derivatives

In this section we show how to define some analogue of the Zelevinsky derivation. This section will not be used until Chapter 7 but some of the properties of partial derivation will appear all along the text.

Definition 1.4.1. *We define a left partial derivation with respect to index i to be a morphism of algebras*

$$\begin{aligned} &{}^i\mathscr{D} : \mathcal{R} \to \mathcal{R}, \\ &{}^i\mathscr{D}(L_{[j,k]}) = L_{[j,k]} + \delta_{i,j} L_{[j+1,k]} \ \textit{if } (k > j), \\ &{}^i\mathscr{D}(L_{[j]}) = L_{[j]} + \delta_{[i],[j]}. \end{aligned}$$

Also we define a right partial derivation with respect to index i to be a morphism of algebras

$$\begin{aligned} &\mathscr{D}^i : \mathcal{R} \to \mathcal{R} \\ &\mathscr{D}^i(L_{[j,k]}) = L_{[j,k]} + \delta_{i,k} L_{[j,k-1]} \ \textit{if } (j < k) \\ &\mathscr{D}^i(L_{[j]}) = L_{[j]} + \delta_{[i],[j]}. \end{aligned}$$

Definition 1.4.2. *We define*

$$\mathscr{D}^{[i,j]} = \mathscr{D}^j \circ \cdots \circ \mathscr{D}^i$$

$$^{[i,j]}\mathscr{D} = (^{i}\mathscr{D}) \circ \cdots \circ (^{j}\mathscr{D})$$

And for $\mathbf{c} = \{\Delta_1, \cdots, \Delta_s\}$ *with*

$$\Delta_1 \preceq \cdots \preceq \Delta_s,$$

we define

$$\mathscr{D}^{\mathbf{c}} = \mathscr{D}^{\Delta_1} \circ \cdots \circ \mathscr{D}^{\Delta_s}$$

and

$$^{\mathbf{c}}\mathscr{D} = (^{\Delta_s}\mathscr{D}) \circ \cdots \circ (^{\Delta_1}\mathscr{D}).$$

Remark: we recall that in [7] 4.5, Zelevinsky defines a derivative $\mathscr{D}$ to be an algebraic morphism

$$\mathscr{D} : \mathcal{R} \to \mathcal{R},$$

which plays a crucial role in Zelevinsky's classification theorem.
The relation between Jacquet functor and derivative is given by

Proposition 1.4.3. *(cf. [34]3.8) Let* δ *be the algebraic morphism such that* $\delta(\rho) = 1$ *for all* $\rho \in \mathscr{C}$ *and* $\delta(L_\Delta) = 0$ *for all non cuspidal representations* L_Δ*. Then*

$$\mathscr{D} = (1 \otimes \delta) \circ c,$$

where c *is the co-multiplication.*

The main advantage to work with partial derivatives instead of the derivative defined by Zelevinsky is that they are much more simpler but share the following positivity properties :

Theorem 1.4.4. *Let* $\mathbf{a}$ *be any multisegment, then we have*

$$\mathscr{D}^i(L_{\mathbf{a}}) = \sum_{\mathbf{b} \in \mathcal{O}} n(\mathbf{b}, \mathbf{a}) L_{\mathbf{b}},$$

such that $n(\mathbf{b}, \mathbf{a}) \geq 0$*, for all* $\mathbf{b}$*.*

Remark: the same property of positivity holds for $^{i}\mathscr{D}$.
The theorem follows from the following two lemmas

Definition 1.4.5. *For* $i \in \mathbb{Z}$*, let* ϕ_i *be the morphism of algebras defined by*

$$\phi_i : \mathcal{R} \to \mathbb{Z}$$
$$\phi_i([j,k]) = \delta_{[i],[j,k]}.$$

Lemma 1.4.6. *For all multisegment* $\mathbf{a}$*, we have* $\phi_i(L_{\mathbf{a}}) = 1$ *if and only if* $\mathbf{a}$ *contains no other segments than* $[i]$*, otherwise it is zero.*

Démonstration. We prove this result by induction on the cardinality of $S(\mathbf{a})$, denoted by $|S(\mathbf{a})|$. If $|S(\mathbf{a})| = 1$, then $\mathbf{a} = \mathbf{a}_{min}$, hence $\phi_i(L_{\mathbf{a}}) = \phi_i(\pi(\mathbf{a}))$, which is nonzero if and only if $\mathbf{a}$ contains no other segments than $[i]$, and in latter case it is 1. Let $\mathbf{a}$ be a general multi-segment,

$$\pi(\mathbf{a}) = L_{\mathbf{a}} + \sum_{\mathbf{b}<\mathbf{a}} m(\mathbf{b}, \mathbf{a}) L_{\mathbf{b}}.$$

Now $|S(\mathbf{a})| > 1$, we know that $\mathbf{a}$ is not minimal in $S(\mathbf{a})$, hence $\mathbf{a}$ contains segments other than $[i]$, which implies $\phi_i(\pi(\mathbf{a})) = 0$.
Since $|S(\mathbf{b})| < |S(\mathbf{a})|$ for any $\mathbf{b} < \mathbf{a}$, by induction, we know that $\phi_i(L_{\mathbf{b}}) = 0$ because $\mathbf{b}$ must contain segments other than $[i]$. So we are done.

□

Lemma 1.4.7. *We have* $\mathscr{D}^i = (1 \otimes \phi_i) \circ c$.

Démonstration. Since both are algebraic morphisms, we only need to check that they coincide on generators. We recall the equation from [34], proposition 3.4

$$c(L_{[j,k]}) = 1 \otimes L_{[j,k]} + \sum_{r=j}^{k-1} L_{[j,r]} \otimes L_{[r+1,k]} + L_{[j,k]} \otimes 1.$$

Now applying ϕ_i,

$$\begin{aligned} (1 \otimes \phi_i) c(L_{[j,k]}) =& L_{[j,k]} + \delta_{i,k} L_{[j,k-1]} \text{ if } (k > j) \\ (1 \otimes \phi_i) c(L_{[j]}) =& L_{[j,k]} + \delta_{i,j}, \end{aligned}$$

where $\delta_{i,j}$ is the Kronecker symbol. Comparing this with the definition of $\mathscr{D}^i$ yields the result. □

Remark: We have the following relation between partial derivative and derivative of Zelevinsky. Let $e(\mathbf{a}) = \{[i_1], \cdots, [i_\alpha] : i_1 \leq \cdots \leq i_\alpha\}$ be the end of $\mathbf{a}$, then

$$\mathscr{D}(\mathbf{a}) = \mathscr{D}^{[i_1, i_\alpha]}(\mathbf{a}).$$

Chapitre 2

Schubert varieties and KL polynomials

In this chapter we recall some of the geometric constructions of Zelevinsky : the nilpotent orbital varieties and their relation with Schubert varieties.

Concretely, for $\mathbf{a}, \mathbf{b}$ multisegments of degree n such that the nilpotent orbit $O_{\mathbf{a}}$ is included in the closure of $\overline{O}_{\mathbf{b}}$, the germs of intersection complexe $IC(\overline{O_{\mathbf{b}}})$ at a generic point of $O_{\mathbf{a}}$ gives the Poincaré polynomial $P_{\mathbf{a},\mathbf{b}}(q)$ and Zelevinsky conjectured that

$$m_{\mathbf{b},\mathbf{a}} = P_{\mathbf{a},\mathbf{b}}(1) = P_{\sigma(\mathbf{a}),\sigma(\mathbf{b})}(1)$$

viewed in the Schubert variety associated to the symmetric group S_n, where $\sigma(\mathbf{a})$ and $\sigma(\mathbf{b})$ are certain permutations attached to $\mathbf{a}$ and $\mathbf{b}$. This conjecture was proved by Chriss-Ginzburg [12], and Ariki [1].

In the following, we study the case of symmetric multisegments in the sense of definition 2.1.5. The set of symmetric multisegment of some specific weight φ is indexed by S_m, where $m = \max_{k\in\mathbb{Z}} \varphi(k)$, which is in general strictly smaller than its degree$=\sum_k \varphi(k)$. In this symmetric situation, we construct a fibration from the symmetric locus in the orbital varieties E_φ to some smooth variety, where the stratification of E_φ gives rise to a stratification of the fibers. And we show that the fiber is isomorphic to some Schubert variety of type A_{m-1}, which identifies the stratification of fiber with the stratification by Schubert cells.

2.1 Symmetric multisegments

Before we introduce the symmetric multisegments, we present a type of multisegments which is more general and will be used in Chapter 6.

Definition 2.1.1. *We say a multisegment* $\mathbf{a}$ *is ordinary if there exists no two segments in* $\mathbf{a}$ *that possesses the same beginning or end.*

Example 2.1.2. *Some typical examples of ordinary multisegments : let* $\mathbf{a} = \{\Delta_1, \Delta_2, \Delta_3\}$*, and* $\mathbf{b} = \{\Delta_4, \Delta_5, \Delta_6\}$

$$\Delta_1 = [1,4],\ \Delta_2 = [2,5],\ \Delta_3 = [3,6],$$

$$\Delta_4 = [1,2],\ \Delta_5 = [2,4],\ \Delta_6. = [4,5]$$

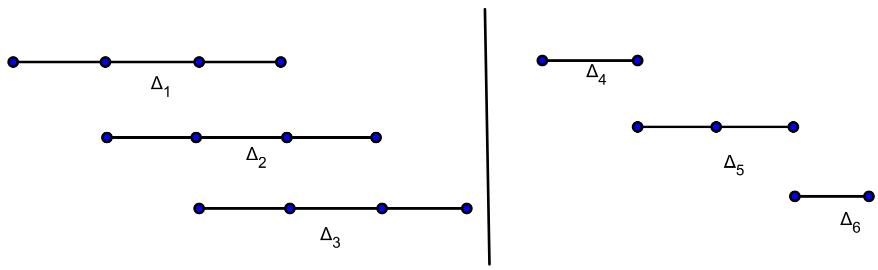

FIGURE 2.1 – Ordinary multi-segments

Proposition 2.1.3. *If* $\mathbf{a}$ *is ordinary then every* $\mathbf{b} \leq \mathbf{a}$ *is ordinary.*

Démonstration. From the definition, $\mathbf{b}$ is ordinary if and only if each element in $e(\mathbf{b})$ and $b(\mathbf{b})$ appears with multiplicity one. We deduce from the following lemma that $\mathbf{b} \leq \mathbf{a}$ is also ordinary. □

Lemma 2.1.4. *Note that for* $\mathbf{b} \leq \mathbf{a}$*, we have* $e(\mathbf{b}) \subseteq e(\mathbf{a})$ *and* $b(\mathbf{b}) \subseteq b(\mathbf{a})$ *(cf. notation 1.1.11).*

Démonstration. In fact, by transitivity, we only need to check this for case where $\mathbf{b}$ can be obtained from $\mathbf{a}$ by applying the elementary operation to the pair $\{\Delta_1 \prec \Delta_2\}$. Hence

$$\mathbf{b} = \mathbf{a} \setminus \{\Delta_1, \Delta_2\} \cup \{\Delta_1 \cup \Delta_2,\ \Delta_1 \cap \Delta_2\}.$$

Note that $e(\Delta_1 \cup \Delta_2) = e(\Delta_2),\ b(\Delta_1 \cup \Delta_2) = b(\Delta_1)$, and if $\Delta_1 \cap \Delta_2 \neq \emptyset$, $e(\Delta_1 \cap \Delta_2) = e(\Delta_1),\ b(\Delta_1 \cap \Delta_2) = b(\Delta_2)$. Hence $b(\mathbf{b}) \subseteq b(\mathbf{a}),\ e(\mathbf{b}) \subseteq e(\mathbf{a})$. □

Definition 2.1.5. *Let* $\mathbf{a} = \{\Delta_1, \cdots, \Delta_n\}$ *be ordinary. We say that* $\mathbf{a}$ *is symmetric if*

$$\max\{b(\Delta_i) : i = 1, \cdots, n\} \leq \min\{e(\Delta_i) : i = 1, \cdots, n\}.$$

To explain the link with the symmetric group, we recall some basic facts about the symmetric group S_n(cf. [8]). Let (i, j) be the transposition exchanging i and j, then

$$S = \{\sigma_i := (i, i+1) : i = 1, \cdots n-1\}$$

form a system of generators of S_n.

Definition 2.1.6. *For* $w \in S_n$*, its length* $\ell(w)$ *is the smallest integer* k *such that*

$$w = s_1 s_2 \cdots s_k, \text{ with } s_i \in S, \text{ for } i = 1, \cdots, k.$$

On S_n, we have the famous Bruhat order which is defined as follow :

Definition 2.1.7. *Let* $T = \{wsw^{-1} : w \in S_n, s \in S\}$*. For* $u, w \in S_n$*,*

- **(i)** *We write* $u \xrightarrow{t} w$ *, if* $u^{-1}w = t \in T$ *and* $\ell(u) < \ell(w)$*.*
- **(ii)** *We write* $u \longrightarrow w$ *, if* $u \xrightarrow{t} w$ *for some* $t \in T$*.*
- **(iii)** *We write* $u \leq w$ *if there exists a sequence of* $w_i \in S_n$*, such that*

$$u \to w_1 \to w_2 \to \cdots \to w_k = w.$$

This defines a partial order on S_n*, which is called the Bruhat order.*

Proposition 2.1.8. *Let* $\mathbf{a}_{\mathrm{Id}} = \{\Delta_1, \cdots, \Delta_n\}$ *be symmetric, such that*

$$b(\Delta_1) < \cdots < b(\Delta_n),$$

$$e(\Delta_1) < \cdots < e(\Delta_n).$$

Then for $w \in S_n$*, the formula*

$$\Phi(w) = \sum_{i=1}^{n} [b(\Delta_i), e(\Delta_{w(i)})]$$

defines a bijection between S_n *and* $S(\mathbf{a}_{\mathrm{Id}})$*. Moreover, the order relation on* $S(\mathbf{a}_{\mathrm{Id}})$ *induces the inverse Bruhat order, i.e.,*

$$w \leq v \Leftrightarrow \Phi(w) \geq \Phi(v).$$

Example 2.1.9. *Let $n = 3$ and $\mathbf{a}_{\mathrm{Id}} = \{\Delta_1, \Delta_2, \Delta_3\}$ with*

$$\Delta_1 = [1, 4], \ \ \Delta_2 = [2, 5], \ \ \Delta_3 = [3, 6].$$

Then $\Phi(\sigma_1) = \{\Delta_4, \Delta_5, \Delta_6\}$ with

$$\Delta_4 = [1, 5], \ \ \Delta_5 = [2, 4], \ \ \Delta_6 = [3, 6].$$

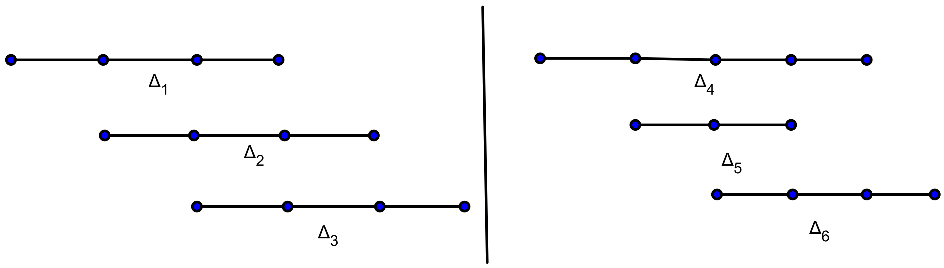

FIGURE 2.2 – Symmetric multi-segments

Démonstration. The injectivity is clear. We observe that $\Phi(\mathrm{Id}) = \mathbf{a}_{\mathrm{Id}}$. We show now that $\Phi(w) \in S(\mathbf{a}_{\mathrm{Id}})$ for general w and the partial order on $S(\mathbf{a}_{\mathrm{Id}})$ induces the inverse Bruhat order.

(1) For $v \leq w$, by the chain property of Bruhat order(cf. [8] Theorem 2.2.6), we have
$$v = w_0 < w_1 < \cdots < w_\alpha = w,$$
such that $w_\gamma = \sigma_{i_{\gamma-1}, j_{\gamma-1}} w_{\gamma-1}$ for some $i_{\gamma-1} < j_{\gamma-1}$ and $\ell(w_\gamma) = \ell(w_{\gamma-1}) + 1$. Now by lemma 2.1.4 of [8], we know that
$$w_{\gamma-1}^{-1}(i_{\gamma-1}) < w_{\gamma-1}^{-1}(j_{\gamma-1}).$$
Hence the segments
$$[b(\Delta_{w_{\gamma-1}^{-1}(i_{\gamma-1})}), e(\Delta_{i_{\gamma-1}})]$$
$$[b(\Delta_{w_{\gamma-1}^{-1}(j_{\gamma-1})}), e(\Delta_{j_{\gamma-1}})]$$
are linked in $\Phi(w_{\gamma-1})$. Moreover, by performing the elementary operation on the two segments, we obtain $\Phi(w_\gamma)$, hence
$$\Phi(w_{\gamma-1}) > Phi(w_\gamma).$$

Again by transitivity of partial orders, we are done. Note that we proved that all $\Phi(w)$ are in $S(\mathbf{a}_{\mathrm{Id}})$. Moreover, for any $\mathbf{b} \in S(\mathbf{a}_{\mathrm{Id}})$, the fact that $\mathbf{a}_{\mathrm{Id}}$ is symmetric implies $b(\mathbf{a}_{\mathrm{Id}}) = b(\mathbf{b})$ since no segment is juxtaposed to the others. The same reason shows that $e(\mathbf{a}_{\mathrm{Id}}) = e(\mathbf{b})$. Hence there is a unique $w \in S_n$ such that

$$\mathbf{b} = \sum_{i=1}^{n} [b(\Delta_i), e(\Delta_{w(i)})].$$

This proves the surjectivity.

(2) Let $\Phi(w) \geq \Phi(v)$, we choose

$$\Phi(w) = \Phi(w_0) > \Phi(w_1) > \cdots > \Phi(w_\alpha) = \Phi(v)$$

to be a maximal chain of multisegments, where $\Phi(w_\gamma)$ is obtained from $\Phi(w_{\gamma-1})$ by performing the elementary operation to segments

$$\{[b(\Delta_{i_{\gamma-1}}), e(\Delta_{w_{\gamma-1}(i_{\gamma-1})})], \quad [b(\Delta_{j_{\gamma-1}}), e(\Delta_{w_{\gamma-1}(j_{\gamma-1})})]\}$$

in $\Phi(w_{\gamma-1})$ with $i_{\gamma-1} < j_{\gamma-1}$. Then

$$w_{\gamma-1}(i_{\gamma-1}) < w_{\gamma-1}(j_{\gamma-1}).$$

Hence

$$w_\gamma = \sigma_{w_{\gamma-1}(i_{\gamma-1}), w_{\gamma-1}(j_{\gamma-1})} w_{\gamma-1}.$$

Note that in this case, we have either

$$w_\gamma < w_{\gamma-1}$$

or

$$w_\gamma > w_{\gamma-1},$$

by (1), the former implies $\Phi(w_{\gamma-1}) < \Phi(w_\gamma)$, contradiction to our assumption.

Hence we must have

$$w_\gamma > w_{\gamma-1}.$$

We conclude by transitivity of partial order that $w < v$.

□

2.2 Nilpotent Orbits

In this section we shall introduce the nilpotent orbits constructed in [35] and discuss their geometry and relations with multisegments.

Definition 2.2.1. **(1)** *Let* $\varphi \in \mathcal{S}$ *(cf. Def. 1.2.7) such that* $\operatorname{supp}\varphi = \{1, \cdots, h\}$*. Let* $V_\varphi = \oplus_{k\in\mathbb{Z}} V_{\varphi,k}$ *be a* $\mathbb{Z}$*-graded* $\mathbb{C}$ *vector space such that* $\dim V_{\varphi,k} = \varphi(k)$*.*

(2) *Let* E_φ *be the set of endomorphism* T *of* V_φ *of degree* 1*, i.e. such that* $TV_{\varphi,k} \subseteq V_{\varphi,k+1}$*.*

Remark: (cf.[35], 1.8) $G_\varphi(\mathbb{C}) = \prod_{k\in\mathbb{Z}} GL(V_{\varphi,k})$ acts on E_φ by conjugation. For each element T in E_φ, there exists a basis of V_φ that consists of homogeneous elements, under which T is of the Jordan form .

Notation 2.2.2. *From now on, for simplicity, in all circumstances, we will write* G_φ *for* $G_\varphi(\mathbb{C})$*,* GL_n *for* $GL_n(\mathbb{C})$ *and* $M_{i,j}$ *for* $M_{i,j}(\mathbb{C})$*.*

Lemma 2.2.3. *By fixing a basis for each* V_k*, we have*

$$E_\varphi \simeq M_{\varphi(2),\varphi(1)} \times \cdots \times M_{\varphi(h),\varphi(h-1)}$$

Here we suppose that $\operatorname{supp}\varphi \subseteq [1, h]$ *and* $M_{k,\ell}$ *denotes the vector space of matrices over* $\mathbb{C}$ *with* k *rows and* ℓ *columns.*

Remark: In this case, the group

$$G_\varphi = GL_{\varphi(1)} \times \cdots \times GL_{\varphi(h)}$$

acts by

$$(g_1, \cdots, g_h).(f_1, \cdots, f_{h-1}) = (g_2 f_1 g_1^{-1}, g_3 f_2 g_2^{-1}, \cdots, g_h f_{h-1} g_{h-1}^{-1}).$$

Démonstration. It follows directly from the definition of E_φ. □

Example 2.2.4. *Consider the function* $\varphi = \chi_1 + 2\chi_2 + \chi_3 \in \mathcal{S}$*(cf. Def 1.3.4), where* χ_k *denotes the characteristic function of* k*. To* φ *we can attach the space* $V_\varphi = V_1 \oplus V_2 \oplus V_3$ *such that*

$$V_1 = \mathbb{C}v_1,\ V_2 = \mathbb{C}v_2 \oplus \mathbb{C}v_3,\ V_3 = \mathbb{C}v_4.$$

Consider the operator $T \in E_\varphi$*, such that*

$$T(v_1) = v_2 - v_3,\ T(v_2) = T(v_3) = v_4.$$

Then by choosing a new basis

$$v_1' = v_1,\ v_2' = v_1 - v_2,\ v_3' = v_1 + v_3,\ v_4' = 2v_4,$$

we get

$$T(v_1') = v_2',\ T(v_2') = 0, T(v_3') = v_4',$$

which gives the Jordan form J_T of T

$$J_T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Proposition 2.2.5. *(cf.[35], 2.3) The orbits of E_φ under G_φ are naturally parametrized by multisegments of weight φ.*

Démonstration. Let $\mathbf{a} = \sum_{i\leq j} a_{ij}[i,j]$ such that $\varphi_{\mathbf{a}} = \varphi$, then the orbit associated consists of the operators having exactly a_{ij} Jordan cells starting from $V_{\varphi,j}$ and ending in $V_{\varphi,i}$. □

Notation 2.2.6. *We denote by $O_{\mathbf{a}}$ the orbit associated to the multisegment* $\mathbf{a}$.

Example 2.2.7. *We take the same function $\varphi = \chi_1 + 2\chi_2 + \chi_3$ as in example 2.2.4. Then the multisegments of weight φ are listed below(cf. [34] section 11.4)*

$$\mathbf{a}_{\max} = \{[1],[2],[2],[3]\},\ \mathbf{a}_\ell = \{[1,2],[2],[3]\},$$

$$\mathbf{a}_r = \{[1],[2],[2,3]\},\ \mathbf{a}_0 = \{[1,2],[2,3]\},\ \mathbf{a}_{\min} = \{[1,3],[2]\}.$$

And the corresponding Jordan forms are given by

$$J_{\mathbf{a}_{\max}} = 0,\ J_{\mathbf{a}_\ell} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$J_{\mathbf{a}_r} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ J_{\mathbf{a}_0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ J_{\mathbf{a}_{\min}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Proposition 2.2.8. *(cf. [35], 2.2) In E_φ, we have $\overline{O}_{\mathbf{b}} = \coprod_{\mathbf{a}\geq\mathbf{b}} O_{\mathbf{a}}$.*

Definition 2.2.9. *For any $T \in E_\varphi$, and $i \le j$, denote by $T^{[i,j]}$ the composition map :*

$$V_i \xrightarrow{T} V_{i+1} \cdots \xrightarrow{T} V_j,$$

we define

$$r_{ij}(T) = \mathrm{rank}(T^{[i,j]}).$$

Remark: For $\mathbf{a}$ a multisegment, $r_{ij}(T)$ remains constant for any $T \in O_{\mathbf{a}}$, we denote it by $r_{ij}(\mathbf{a})$.
We recall the following combinatorial results

Proposition 2.2.10. *(cf. [35]section 2.5) Let $\mathbf{a}, \mathbf{b}$ be two multisegments such that*

$$\varphi_{\mathbf{a}} = \varphi_{\mathbf{b}}.$$

Then the following two conditions are equivalent :

(1) $\mathbf{b} \le \mathbf{a}$ *;*

(2) $r_{ij}(\mathbf{a}) \le r_{ij}(\mathbf{b})$ *for all $i \le j$.*

In symmetric case, we have the following interesting description of r_{ij}.

Lemma 2.2.11. *Let $w \in S_n$. Then we have $r_{i,j+n-1}(w) := r_{i,j+n-1}(\Phi(w)) = \{k \le i : w(k) \ge j\}$.*

Démonstration. In fact, take

$$\mathbf{a}_{\mathrm{Id}} = \sum_{k=1}^{n} [k, k+n-1],$$

and consider the bijection

$$\Phi : S_n \to S(\mathbf{a}_{\mathrm{Id}})$$

with

$$\Phi(w) = \sum_{k=1}^{n} [k, w(k)+n-1].$$

By definition, $r_{i,j+n-1}(w)$ is the number of segments in $L_{\Phi(w)}$ which contains $[i, j+n-1]$, hence is of the form $[k, w(k)+n-1]$ with

$$k \le i, \quad w(k) \ge j.$$

□

Now combining with the proposition 2.2.10, gives the following known results,

Proposition 2.2.12. *([28] Proposition 2.1.12) In S_n, $v \le w \Leftrightarrow r_{ij}(v) \le r_{ij}(w)$, for all $i \le j$.*

2.3 Schubert Varieties and KL Polynomials

Let Y be an algebraic variety over $\mathbb{C}$.

Definition 2.3.1. *By a stratification* $\mathfrak{H}$ *on* Y*, we mean a decomposition of* Y *into locally closed smooth sub-varieties* Y_i*. An element of* $\mathfrak{H}$ *is called a stratum.*

Remark: We require a variety to be irreducible.

Definition 2.3.2. *Let* $D^b(Y) = D_c^b(Y)$ *be the bounded derived category of sheaves with values in complex vector spaces over* Y*. And let* $D(Y)$ *be the subcategory consisting of those complexes whose cohomology sheaves are constructible.*

Given a stratification $\mathfrak{H}$, we let U_ℓ denote the set of strata whose dimension is $\geq \ell$.

Definition 2.3.3. *(cf.[13] Remark 3.8.1) Given a local system on the open stratum* U_d *with* $d = \dim(Y)$*, we define inductively a complex* $IC(Y, L)$ *in* $D(Y)$ *as follows.*
We start by letting $IC(U_d, L) := L[\dim Y]$*. Assuming that we already defined* $IC(U_{\ell+1}, L)$*, let* $j : U_{\ell+1} \to U_\ell$ *be the open immersion, then we define*

$$IC(U_\ell, L) := \tau_{\leq -\ell-1} Rj_* IC(U_{\ell+1}, L),$$

here $\tau_{\leq k}$ *is the truncation from the right in degree* k*. In finite step, we get* $IC(Y, L)$*.*

Notation 2.3.4. *When we take* $L = \mathbb{C}$*, which is always the case for us, we denote* $IC(Y, \mathbb{C})$ *by* $IC(Y)$*. In this case we denote*

$$\mathcal{H}^i(Y) := \mathcal{H}^i(IC(Y)).$$

Remark: The cohomology sheaves $\mathcal{H}^i(Y)$ are locally constant over each stratum in $\mathfrak{H}$.

Definition 2.3.5. *Let* $n \geq 1$*. By a Schubert variety of type* A_{n-1}*, we mean a closed sub-variety of the projective variety* GL_n/B_n *which is stable under the multiplication by* B_n *from the left, where* B_n *is the Borel subgroup consisting of upper triangular matrices.*

Remark: Let V be a $\mathbb{C}$ vector space. Note that GL_n acts transitively on the set of complete flags $\mathcal{F}(V) := \{(U^i : i = 0, \cdots, n) : 0 = U^0 \subset U^1 \subset \cdots \subset U^n = V, \dim(U^i) = i\}$ and the stabilizer of a complete flag is given by a Borel subgroup. Hence by fixing a complete flag $(V^i : i = 0, \cdots, n)$ and denoting its stabilizer by B, we identify the variety GL_n/B with $\mathcal{F}(V)$, in this way, we can consider the Schubert variety as a subset of $\mathcal{F}(V)$.

Proposition 2.3.6. *(cf. [11] page 148.) We identify S_n with the set of the permutation matrices in GL_n. Then we have the Bruhat decomposition $GL_n = \coprod_{w \in S_n} B_n w B_n$. Moreover, we have*

$$\overline{B_n w B_n} = \coprod_{v \leq w} B_n v B_n.$$

Definition 2.3.7. *We denote $C_w := B_n w B_n / B_n$ in GL_n/B_n and the Schubert variety $X_w = \overline{C_w}$.*

Then for the Schubert variety X_w, we have a stratification given by $\mathfrak{H} = \{C_v : v \leq w\}$.

Definition 2.3.8. *Let $v \leq w$, we define the Kazhdan Lusztig polynomial for the pair v, w :*

$$P_{v,w}(q) = \sum_i q^{(i+d_w)/2} \dim \mathcal{H}^i(X_w)_{x_v},$$

where x_v is an element in C_v and $d_w = \dim(X_w) = \ell(w)$.

Concerning the intersection cohomology of Schubert varieties, we have the following purity theorem due to Kazhdan and Lusztig.

Theorem 2.3.9. *([19]) If $i + \ell(w)$ is odd, then the cohomology group*

$$\mathcal{H}^i(X_w) = 0.$$

Remark: This implies that $P_{v,w}(q)$ is a polynomial in q.

2.4 Orbital Varieties and Schubert Varieties

Note that on the orbital variety $\overline{O}_{\mathbf{b}}$, we have a stratification given by $\mathfrak{H}_{\mathbf{b}} = \{O_{\mathbf{a}} : \varphi_{\mathbf{a}} = \varphi_{\mathbf{b}}, \mathbf{b} \leq \mathbf{a}\}$.

Definition 2.4.1. *Let $\mathbf{a}$, $\mathbf{b}$ be two multisegments such that $\mathbf{b} \in S(\mathbf{a})$. Then we define the polynomial*

$$P_{\mathbf{a},\mathbf{b}}(q) = \sum_i q^{(i+d_{\mathbf{b}})/2} \dim \mathcal{H}^i(\overline{O}_{\mathbf{b}})_{x_{\mathbf{a}}},$$

where $x_{\mathbf{a}} \in O_{\mathbf{a}}$ is an arbitrary point and $d_{\mathbf{b}} = \dim(O_{\mathbf{b}})$. We call it the Kazhdan Lusztig polynomial associated to $\{\mathbf{a}, \mathbf{b}\}$.

Remark: In [37] Theorem 1, Zelevinsky showed that the varieties $\overline{O}_{\mathbf{b}}$ are locally isomorphic to some Schubert varieties of type A_m, where $m = \deg(\mathbf{b})$. Hence again by theorem 2.3.9, we know that $P_{\mathbf{a},\mathbf{b}}$ is a polynomial in q.

Here, we briefly recall Zelevinsky's results in [37]. Let φ be a function in $\mathcal{S}$(cf. Def. 1.2.7) such that $\mathrm{supp}(\varphi) \subseteq [1, r]$. We consider the flag variety

$$\mathcal{F}(\varphi) = \{0 = U^0 \subset U^1 \subset \cdots U^r = V_\varphi : \dim(U^i/U^{i-1}) = \varphi(i), 1 \leq i \leq r\}$$

We fix the standard flag

$$F_\varphi = \{0 = V_\varphi^0 \subseteq V_\varphi^1 \cdots \subseteq V_\varphi^r : V_\varphi^i = V_{\varphi,1} \oplus \cdots \oplus V_{\varphi,i}\} \in \mathcal{F}(\varphi).$$

Definition 2.4.2. *Let $\mathcal{G}(\varphi)$ be the subset of $\mathcal{F}(\varphi)$ containing the elements $(U^i : 0 \leq i \leq r) \in \mathcal{F}(\varphi)$ such that $U^i \supseteq V_\varphi^{i-1}$ for $i = 1, \cdots, r$.*

Zelevinsky defined a map $\tau : E_\varphi \to \mathcal{G}(\varphi)$, by associating to $T \in E_\varphi$ the element $\tau(T) = (U^i : 0 \leq i \leq r)$ such that

$$U^i = \{(v_1, \cdots, v_r) \in V_{\varphi,1} \oplus \cdots \oplus V_{\varphi,r} : v_{j+1} = T(v_j), j \geq i\}.$$

Theorem 2.4.3. *(cf.[37]Theorem 1) The morphism τ is an open immersion into the Schubert variety $\mathcal{G}(\varphi)$.*

In fact, for $\mathbf{b}$ a multisegment of weight φ, we can describe explicitly the image of $O_{\mathbf{b}}$ in terms of Schubert cells in $\mathcal{G}(\varphi)$. Let $\mathbf{b} = \sum_{1 \leq i \leq j \leq r} b_{ij}[i, j], X^{\mathbf{b}} = (x_{ij})$ with

$$\begin{aligned} x_{ij} =& b_{ij}, \quad \text{for } i \leq j \\ x_{ij} =& 0, \quad \text{for } i > j+1 \\ x_{i,i-1} =& \sum_{n \leq i-1, i \leq m} b_{nm}. \end{aligned}$$

Example 2.4.4. *Let $\varphi = \chi_{[1]} + 2\chi_{[2]} + \chi_{[3]}, \mathbf{a} = [1,2] + [2,3],\ \mathbf{b} = [1,3] + [2]$. And $X_{\mathbf{a}} = (x_{ij}^{\mathbf{a}}),\ X_{\mathbf{b}} = (x_{ij}^{\mathbf{b}})$ be the matrix such that*

$$X^{\mathbf{a}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ X^{\mathbf{b}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Definition 2.4.5. *Let $\mathbf{b}$ be a multisegment of weight φ and*

$$X^{\mathbf{b}} = (x_{i,j})_{1 \leq i,j \leq r}.$$

We define $Y_{\mathbf{b}}$ *to be the set of flags*

$$(U^i : i = 0, 1, \cdots, r) \in \mathcal{G}(\varphi)$$

such that

$$\dim((U^i \cap V_\varphi^j)/(U^i \cap V_\varphi^{j-1} + U^{i-1} \cap V_\varphi^j)) = x_{ij}, \text{ for all } 1 \leq i, j \leq r.$$

Example 2.4.6. *Let* $\mathbf{a}$ *be the multisegment in example 2.4.4. We have* $Y_{\mathbf{a}}$ *be the set of flags* $(U^i : i = 0, 1, 2, 3)$ *such that*

$$\begin{aligned} U^0 &= 0; \\ \dim(U^1 \cap V_\varphi^1) &= x_{11}^{\mathbf{a}} = 0 \Rightarrow U^1 \cap V_\varphi^1 = 0; \\ \dim(U^1 \cap V_\varphi^2) &= x_{12}^{\mathbf{a}} = 1 \Rightarrow U^1 \subseteq V_\varphi^2; \\ \dim(U^2 \cap V_\varphi^1) &= x_{21}^{\mathbf{a}} = 1 \Rightarrow U^2 \supseteq V_\varphi^1. \end{aligned}$$

And

$$\dim(U^2 \cap V_\varphi^2/(U^2 \cap V_\varphi^1 + U^1 \cap V_\varphi^2)) = x_{22}^{\mathbf{a}} = 0,$$

which implies

$$U^2 \cap V_\varphi^2 = U^2 \cap V_\varphi^1 + U^1 \cap V_\varphi^2;$$

hence $U^2 \cap V_\varphi^2 = V_\varphi^1 + U^1$*, which is of dimension 2. Hence* $Y_{\mathbf{a}}$ *is the set of flags* $(U^i : i = 0, 1, 2, 3)$ *satisfying*

$$U^0 = 0, \ U^1 \cap V_\varphi^1 = 0, \ U^2 \cap V_\varphi^2 = V_\varphi^1 + U^1, \ U^3 = V_\varphi^3.$$

Proposition 2.4.7. *(cf. [37] Theorem 1.) We have* $O_{\mathbf{b}} = Y_{\mathbf{b}} \cap E_\varphi$.

Example 2.4.8. *Again, let* $\mathbf{a} = [1, 2] + [2, 3]$. *Let* $T \in E_\varphi \cap Y_{\mathbf{a}}$*, then we have* $\tau(T) = (U^i : i = 0, 1, 2, 3)$*, satisfying*

$$U^0 = 0, \ U^1 \cap V_\varphi^1 = 0, \ U^2 \cap V_\varphi^2 = V_\varphi^1 + U^1, \ U^3 = V_\varphi^3.$$

By definition, if we write $T = (T_1, T_2)$ *such that*

$$T_i : V_{\varphi,i} \to V_{\varphi,i+1}, \quad i = 1, 2,$$

then

$$U^1 = \{(v, T_1 v, T_2 T_1 v) \in V_\varphi : v \in V_{\varphi,1}\},$$

and $U^1 \cap V_\varphi^1 = 0$ *is equivalent to* $T_1 v \neq 0$. *Also, we have*

$$U^2 = \{(v_1, v_2, T_2 v_2) \in V_\varphi : v_1 \in V_{\varphi,1}, v_2 \in V_{\varphi,2}\}.$$

Note that $U^2 \cap V_\varphi^2 = V_\varphi^1 + U^1$ is equivalent to the following conditions

$$U^1 \subseteq V_\varphi^2, \quad U^2 \not\supseteq V_{\varphi,2}.$$

We know that $U^1 \subseteq V_\varphi^2$ is equivalent to the fact that for any $v \in V_{\varphi,1}$, $(v, T_1 v, T_2 T_1 v) \in V_{\varphi,2}$, hence $T_2 T_1 v = 0$. Furthermore, we know that $U^2 \not\supseteq V_{\varphi,2}$ is equivalent to the fact that there exits $v \in V_{\varphi,2}$ such that $(0, v, T_2 v) \notin V_{\varphi,2}$, hence $T_2 v \neq 0$. Hence we obtain that $T \in E_\varphi \cap Y_{\mathbf{a}}$ is equivalent to the following facts

$$T_1 \neq 0, \quad T_2 T_1 = 0, \quad T_2 \neq 0.$$

The latter is the same as to say that $T \in O_{\mathbf{a}}$.

Definition 2.4.9. *Let $B_i(\varphi) = \{j : \sum_{m \leq i-1} \varphi(m) < j \leq \sum_{m \leq i} \varphi(m)\}$.*

$$S^{\mathbf{b}} = \{w \in S_{\deg(\mathbf{b})} : Card(w(B_i(\varphi)) \cap B_j(\varphi)) = x_{ij}, 1 \leq i, j \leq r\}.$$

We denote by $w(\mathbf{b})$ the unique element in $S^{\mathbf{b}}$ of maximal length.

Example 2.4.10. *In the example 2.4.4, we have*

$$B_1(\varphi) = \{1\}, B_2(\varphi) = \{2, 3\}, B_3(\varphi) = \{4\}.$$

Let $\mathbf{a} = [1, 2] + [2, 3]$. Then by definition

$$S^{\mathbf{a}} = \{w \in S_4 : Card(w(B_i(\varphi)) \cap B_j(\varphi)) = x_{ij}^{\mathbf{a}}, 1 \leq i, j \leq 3\}$$

Therefore, for $w \in S^{\mathbf{a}}$, we have

$$w(1), w(4) \in \{2, 3\}, \ \{w(2), w(3)\} \cap \{2, 3\} = \emptyset,$$

therefore,

$$\{w(1), w(4)\} = \{2, 3\}, \ \{w(2), w(3)\} = \{1, 4\},$$

hence

$$w = (13)(24), \ \text{or } w = (12)(34),$$

compare the length, we have

$$w(\mathbf{a}) = (13)(24).$$

The same method shows that

$$w(\mathbf{b}) = (1423)$$

Note in the picture we denote a permutation by its image.

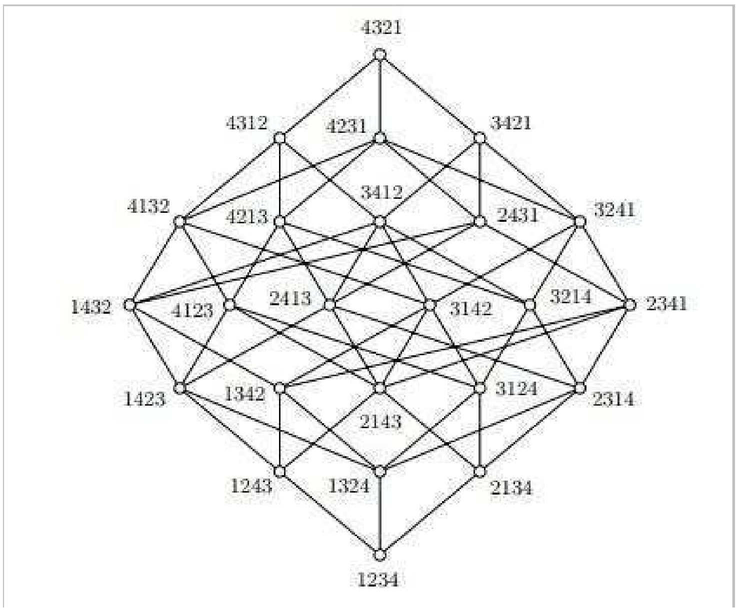

FIGURE 2.3 – Bruhat Order for S_4

Theorem 2.4.11. *(cf. [37]) Let* $\mathbf{b}' \geq \mathbf{b}$ *such that* $Y_{\mathbf{b}'} \subseteq \overline{Y}_{\mathbf{b}}$*, we have*

$$P_{\mathbf{b}',\mathbf{b}}(q) = P_{w(\mathbf{b}'),w(\mathbf{b})}(q).$$

Theorem 2.4.12. *([35], [12]) Let* $\mathcal{H}^i(\overline{O}_{\mathbf{b}})_{\mathbf{a}}$ *denote the stalk at a point* $x \in O_{\mathbf{a}}$ *of the* i*-th intersection cohomology sheaf of the variety* $\overline{O}_{\mathbf{b}}$*. Then*

$$m(\mathbf{b},\mathbf{a}) = P_{\mathbf{b},\mathbf{a}}(1).$$

Remark: The intersection cohomology is nonzero only if $i + \dim(O_{\mathbf{b}})$ is even. Hence $m(\mathbf{b},\mathbf{a})$ is the value at $v = 1$ of a certain Kazhdan Lusztig polynomial for the symmetric group S_m with $m = \deg(\mathbf{b})$.
Remark: Combining with theorem 2.4.11, this theorem gives a complete calculation of the coefficients $m(\mathbf{b},\mathbf{a})$. But as we have seen, this often involves elements in a huge symmetric group, which is too clumsy. Moreover, another difficulty arise from the description of the element $w(\mathbf{b})$, which is not explicit.
Remark: In this chapter, for symmetric multisegments $\mathbf{a}$ and $\mathbf{b}$, we will give more concrete description about the coefficient $m_{\mathbf{b},\mathbf{a}}$ in terms of elements in S_n with n equals to the number of segments contained in $\mathbf{a}$, cf. corollary 2.5.9. For general case, we will give use the reduction method from chapter 4 to give a more concrete description.

2.5 Geometry of Symmetric Nilpotent Orbits

For the moment, we consider a special case of symmetric multisegments, we assume that

$$\mathbf{a}_{\mathrm{Id}} = \sum_{i=1}^{n} [i, n+i-1], \quad \varphi = \sum_{\Delta \in \mathbf{a}} f_{\mathbf{a}}(\Delta) \chi_{\Delta}.$$

We remind that we already constructed a bijection

$$\Phi : S_n \to S(\mathbf{a}_{\mathrm{Id}})$$

such that $\Phi(\mathrm{Id}) = \mathbf{a}_{\mathrm{Id}}$.

Definition 2.5.1. *Let*

$$O_w = O_{\Phi(w)}, \quad \text{and} \quad O_\varphi^{\mathrm{sym}} = \coprod_{w \in S_n} O_w \subseteq E_\varphi.$$

Also, let

$$\overline{O}_w^{sym} = \overline{O}_w \cap O_\varphi^{\mathrm{sym}}.$$

Definition 2.5.2. *Let*

$$\begin{array}{l} E_\varphi = M_{2,1} \times \cdots M_{n-1,n-2} \times M_{n,n-1} \times M_{n-1,n} \times \cdots \times M_{1,2} \\ \Big\downarrow p_\varphi \\ Z_\varphi := M_{2,1} \times \cdots M_{n-1,n-2} \times M_{n-1,n} \times \cdots \times M_{1,2}. \end{array}$$

be the natural projection with fiber $M_{n,n-1}$.
Now we want to describe the fiber of the restriction $p_\varphi|_{O_\varphi^{\mathrm{sym}}}$.

Definition 2.5.3. *We define $GL_{n,n-1}$ to be the subset of $M_{n,n-1}$ consisting of the matrices of rank $n-1$.*

We denote by $p_n : M_{n,n} \twoheadrightarrow M_{n,n-1}$ the morphism of forgetting the last column of elements in $M_{n,n}$.
Remark: Now by restriction to GL_n, we have the morphism

$$p_n : GL_n \twoheadrightarrow GL_{n,n-1},$$

which satisfies the property that $p_n(g_1 g_2) = g_1 p_n(g_2)$ for $g_1, g_2 \in GL_n$.

Proposition 2.5.4. *The morphism*

$$p_n : GL_n \twoheadrightarrow GL_{n,n-1},$$

is a fibration. Furthermore, it induces a bijection

$$p_n : B_n \backslash GL_n / B_n \to B_n \backslash GL_{n,n-1} / B_{n-1}.$$

Démonstration. To see that it is locally trivial, note that p_n is GL_n equivariant with GL_n acting by multiplication from the left. Since GL_n acts transitively on itself, it acts also transitively on $GL_{n,n-1}$. Now p_n is equivariant implies that all the fibers of p_n are isomorphic. Let H be the stabilizer of $p_n(\mathrm{Id})$, then $GL_{n,n-1} \simeq GL_n/H$, it is a étale locally trivial fibration according to Serre [32] proposition 3. By Bruhat decomposition, every $g \in GL_n$ admits a decomposition

$$g = b_1 w b_2, \quad b_i \in B_n, i = 1, 2, \quad w \in S_n,$$

here we identify S_n with the set of permutation matrices in GL_n. We can decompose $b_2 = b_3 v$, where $b_3 \in GL_{n-1}$, which is identified with the direct summand in the Levi subgroup $GL_{n-1} \times \mathbb{C}^\times$, and $v - \mathrm{Id}$ only contains non zero elements in the last column, by definition,

$$p_n(g) = b_1 p_n(w) b_3.$$

We obtain that p_n induces

$$p_n : B_n \backslash GL_n / B_n \to B_n \backslash GL_{n,n-1} / B_{n-1}.$$

It is bijective because given $p_n(w)$, there is a unique way to reconstruct an element which belongs to S_n. □

Theorem 2.5.5. *The morphism*

$$p_\varphi|_{O_\varphi^{\mathrm{sym}}}$$

is smooth with fiber $GL_{n,n-1}$. Moreover, the morphism $p_\varphi|_{O_w} : O_w \to p_\varphi(O_\varphi^{\mathrm{sym}})$ is surjective with fiber $B_n p_n(w) B_{n-1}$.

Démonstration. Note that smoothness follows from that $p_\varphi : E_\varphi \to Z_\varphi$ is smooth and that O_φ^{sym} is open in E_φ. To see the rest of the properties, we fix an element e_w in each orbit O_w as follow. Let $(v_{ij})(i = 1, \cdots, 2n-1, j = 1, \cdots, \varphi(i))$ be a basis of $V_{\varphi,i}$, and an element e_w satisfying

$$\begin{cases} e_w(v_{ij}) = v_{i+1,j}, & \text{for } i < n-1 \\ e_w(v_{n-1,j}) = v_{n,w(j)}, & \\ e_w(v_{ij}) = v_{i+1,j-1}, & \text{for } i \geq n. \end{cases},$$

here we put $v_{i,0} = 0$.

Example 2.5.6. *Let $w = (1, 2)$, then by the strategy in the proof, e_w is given by the following picture :*

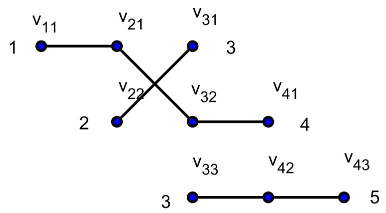

FIGURE 2.4 –
Construction of e_w in case $n=3$

We claim that $e_w \in O_w$. In fact, it suffices to observe that

$$e_w : v_{ii} \to \cdots \to v_{n-1,i} \to v_{n,w(i)} \to v_{n+1,w(i)-1} \to \cdots v_{n+w(i)-1,1},$$

which by proposition 2.2.5, implies that the multisegment indexing e_w contains $[i, w(i)+n-1]$ for all $i = 1, \cdots, n$, hence it must be $\Phi(w)$. Note that, by definition, we have

$$p_\varphi(e_{\mathrm{Id}}) = p_\varphi(e_w), \text{ for all } w \in S_n.$$

Since p_φ is compatible with the action of G_φ, we get

$$p_\varphi(O_\varphi^{\mathrm{sym}}) = p_\varphi(O_w), \text{ for all } w \in S_n,$$

which implies that $p|_{O_w}$ is surjective. Now it remains to characterize its fiber. Let $T' \in p_\varphi(O_\varphi^{\mathrm{sym}})$, then $p_\varphi^{-1}(T') \simeq M_{n,n-1}$ in E_φ. Moreover, for $T = (T_1, \cdots, T_{2n-2}) \in p_\varphi^{-1}(T')$, then $T \in O_\varphi^{\mathrm{sym}}$ if and only if

$$T_{n-1} \in GL_{n,n-1}.$$

Therefore, the map $T \mapsto T_{n-1}$ induces

$$p_\varphi^{-1}(T') \cap O_\varphi^{\mathrm{sym}} \simeq GL_{n,n-1}.$$

Consider the variety $p_\varphi^{-1}(T') \cap O_w$. Note that since G_φ acts transitively on $p_\varphi(O_\varphi^{\mathrm{sym}})$, we may assume that $T' = p_\varphi(e_{\mathrm{Id}})$.

Lemma 2.5.7. *The set of $f_w \in O_w$ satisfying*

$$\begin{cases} f_w(v_{ij}) &= v_{i+1,j}, \quad \text{for } i < n-1 \\ f_w(v_{ij}) &= v_{i+1,j-1}, \quad \text{for } i \geq n. \end{cases}$$

is in bijection with $B_n p_n(w) B_{n-1}$ via $p_\varphi^{-1}(p_\varphi(e_{\mathrm{Id}})) \cap O_\varphi^{\mathrm{sym}} \simeq GL_{n,n-1}$.

Démonstration. Now the element $f_w \in O_w$ is completely determined by the component

$$f_{w,n-1} : V_{\varphi,n-1} \to V_{\varphi,n}.$$

We know by proposition 2.2.5 that $f_{w,n-1}$ is injective hence of rank $n-1$. Hence we have $f_{w,n-1} \in GL_{n,n-1}$.
Now by proposition 2.5.4 we get $B_n \backslash GL_{n,n-1}/B_{n-1}$ is indexed by S_n, it remains to see that $f_{w,n-1}$ is in the class indexed by $p_n(w)$.
Finally, we note that p_φ is a morphism equivariant under the action of

$$G_\varphi = GL_1 \times GL_2 \times \cdots \times GL_{n-1} \times GL_n \times \cdots \times GL_2 \times GL_1.$$

Since G_φ acts transitively on O_w, the image of O_w is $G_\varphi.(p_\varphi(e_w))$, hence is $p_\varphi(O_{\mathrm{Id}})$. Now we prove that the stabilizer of $p_\varphi(e_w)$ is $B_n \times B_{n-1}$. Let $e_{\mathrm{Id}} = (e_1, \cdots, e_{n-1}, e_n, \cdots, e_{2n-2})$ with $e_i \in M_{i,i+1}$ if $i < n$ and $e_i \in M_{i,i-1}$ if $i \geq n$. We have

$$p_\varphi(e_{\mathrm{Id}}) = (e_1, \cdots, e_{n-2}, e_n, \cdots, e_{2n-2}).$$

Let $g = (g_1, \cdots, g_n, g_{n+1}, \cdots, g_{2n-1})$ such that $g.p_\varphi(e_{\mathrm{Id}}) = p_\varphi(e_{\mathrm{Id}})$. Then by definition for $i < n-1$ we know that $g_{i+1}e_ig_i^{-1} = e_i$. We prove by induction on i that $g_i \in B_i \in GL_i$ for $i \leq n-1$. For $i = 1$, we have nothing to prove. Now assume that $i \leq n-2$, and $g_i \in B_i$, we show that $g_{i+1} \in B_{i+1}$. Consider

$$g_{i+1}e_ig_i^{-1}(g_i(v_{ij})) = g_{i+1}e_i(v_{ij}) = g_{i+1}(v_{i+1,j}).$$

On the other hand, by induction, we know that

$$g_{i+1}e_ig_i^{-1}(g_i(v_{ij})) = e_i(g_i(v_{ij})) \in \oplus_{k \leq j}\mathbb{C}v_{i+1,k}.$$

Therefore we have $g_{i+1} \in B_{i+1}$. Actually, since e_i is injective, the equality $e_i(g_i(v_{ij})) = g_{i+1}(v_{i+1,j})$, implies that g_i is completely determined by g_{i+1}. This shows that $g_{n-1} \in B_{n-1}$ it determines all g_i for $i \geq n-1$. The same method proves that $g_n \in B_n$ and it determines all g_i for $i \geq n$. We conclude that the fiber of the morphism $p_\varphi|_{O_w}$ is isomorphic to $B_np_n(w)B_{n-1}$. □

□

Corollary 2.5.8. *We have for $v \leq w$ in S_n, and X_w the closure of B_nwB_n in GL_n,*

$$\dim \mathcal{H}^i(\overline{O}_w^{sym})_v = \dim \mathcal{H}^i(X_w)_v,$$

for all $i \in \mathbb{Z}$, here the index v on the left hand side means that we localize at a generic point in O_v and on the right hand side means that we localize at a generic point in C_v.

Démonstration. Since $p_\varphi|_{O_\varphi^{\mathrm{sym}}}$ is a fibration with fiber $GL_{n,n-1}$ over Z_φ, we apply the smooth base change theorem to the following Cartesian diagram

$$\begin{array}{ccc} GL_{n,n-1} & \longrightarrow & O_\varphi^{\mathrm{sym}} \\ \downarrow & & \downarrow \\ p_\varphi(\Phi(\mathrm{Id})) & \longrightarrow & Z_\varphi. \end{array}$$

We get

$$\dim \mathcal{H}^i(\overline{O}_w^{sym})_v = \dim \mathcal{H}^i(\overline{B_n p_n(w) B_{n-1}})_{B_n p_n(v) B_{n-1}}.$$

Now apply proposition 2.5.4, we have

$$\dim \mathcal{H}^i(\overline{B_n p_n(w) B_{n-1}})_{B_n p_n(v) B_{n-1}} = dim \mathcal{H}^i(X_w)_v.$$

□

Corollary 2.5.9. *We have for $v \leq w$ in S_n,*

$$m_{\Phi(v),\Phi(w)} = P_{v,w}(1).$$

Démonstration. This follows from the fact that

$$\dim \sum_i \mathcal{H}^i(X(w))_v = P_{v,w}(1)$$

(cf. [20]). □

Chapitre 3

Descent of Degrees for Multisegment

To attack the question of calculating the coefficient $m(\mathbf{b}, \mathbf{a})$, this first naive idea, which can trace back to Zelevinsky [34], is to use the (partial) derivation. If we believe that for $\mathbf{b} \in S(\mathbf{a})$, the coefficient $m(\mathbf{b}, \mathbf{a})$ only depends on the relative position between the segments in $\mathbf{a}$, but not on the exact multisegment $\mathbf{a}$, we should be allowed to do some sort of truncation on the multisegments simultaneously without changing the coefficient $m(\mathbf{b}, \mathbf{a})$. It is reasonable to think that the partial derivative should play the role of truncation.

However, it is not true that we can always truncate. For example if we take $\mathbf{a} = \{[1, 2], [2, 3]\}$ and we replace the segment $[2, 3]$ by $[2]$(truncate at the place 3), then we get $\mathbf{a}' = \{[1, 2], [2]\}$, this should not allowed because we changed the linkedness relation between the two segments. And simple calculation shows that

$$\mathscr{D}^3(L_{\mathbf{a}}) = L_{\mathbf{a}},$$

we quickly notice that $\mathscr{D}^3(L_{\mathbf{a}})$ does not achieve its minimal degree term $L_{\mathbf{a}'}$, which are supposed to appear.

Such examples lead us to think that we can do truncation only when our partial derivative achieve its minimal degree terms. More explicitly, we should avoid applying truncation to the multisegments as $\mathbf{a}$ above. This gives us the hypothesis $H_k(\mathbf{a})$(definition 3.1.3). And satisfying the hypothesis $H_k(\mathbf{a})$ means that we can apply the truncation without changing the coefficients.

3.1 Morphism for Descent of Degree of multisegment

For a multisegment $\mathbf{a}$ and $k \in \mathbb{Z}$, we will introduce a hypothesis called $H_k(\mathbf{a})$ and let $S(\mathbf{a})_k$ be the set of elements in $S(\mathbf{a})$ satisfying the hypothesis $H_k(\mathbf{a})$. We construct a multisegment $\mathbf{a}^{(k)}$ and a morphism $\psi_k : S(\mathbf{a})_k \to S(\mathbf{a}^{(k)})$. We show that the morphism ψ_k is surjective.

Notation 3.1.1. *For $\Delta = [i, j]$ a segment, we put*

$$\begin{aligned} \Delta^- &= [i, j-1], \quad {}^-\Delta = [i+1, j], \\ \Delta^+ &= [i, j+1], \quad {}^+\Delta = [i-1, j]. \end{aligned}$$

Definition 3.1.2. *Let $k \in \mathbb{Z}$ and Δ be a segment, we define*

$$\Delta^{(k)} = \begin{cases} \Delta^-, & \text{if } e(\Delta) = k; \\ \Delta, & \text{otherwise} . \end{cases}$$

For a multisegment $\mathbf{a} = \{\Delta_1, \cdots, \Delta_r\}$, we define

$$\mathbf{a}^{(k)} = \{\Delta_1^{(k)}, \cdots, \Delta_r^{(k)}\}.$$

Definition 3.1.3. *We say that the multisegment $\mathbf{b} \in S(\mathbf{a})$ satisfies **the hypothesis** $H_k(\mathbf{a})$ if the following two conditions are verified*

(1) $\deg(\mathbf{b}^{(k)}) = \deg(\mathbf{a}^{(k)})$ *;*

(2) *there exists not a pair of linked segments $\{\Delta, \Delta'\}$ such that $e(\Delta) = k-1,\ e(\Delta') = k$.*

Definition 3.1.4. *Let*

$$\widetilde{S}(\mathbf{a})_k = \{\mathbf{c} \in S(\mathbf{a}) : \deg(\mathbf{c}^{(k)}) = \deg(\mathbf{a}^{(k)})\}.$$

Lemma 3.1.5. *Let $\mathbf{c} \in \widetilde{S}(\mathbf{a})_k$. Then*

$$\sharp\{\Delta \in \mathbf{a} : e(\Delta) = k\} = \sharp\{\Delta \in \mathbf{c} : e(\Delta) = k\}.$$

Démonstration. Note that

$$\deg(\mathbf{a}) = \deg(\mathbf{a}^{(k)}) + \sharp\{\Delta \in \mathbf{a} : e(\Delta) = k\}.$$

Now that for $\mathbf{c} \in \widetilde{S}(\mathbf{a})_k$

$$\deg(\mathbf{c}) = \deg(\mathbf{a}), \quad \deg(\mathbf{c}^{(k)}) = \deg(\mathbf{a}^{(k)}),$$

we have

$$\sharp\{\Delta \in \mathbf{a} : e(\Delta) = k\} = \sharp\{\Delta \in \mathbf{c} : e(\Delta) = k\}.$$

□

Lemma 3.1.6. *Let* $k \in \mathbb{Z}$

(a) *For any* $\mathbf{b} \in S(\mathbf{a})$*, we have* $\deg(\mathbf{b}^{(k)}) \geq \deg(\mathbf{a}^{(k)})$*.*

(b) *Let* $\mathbf{c} \in \widetilde{S}(\mathbf{a})_k$*, then for* $\mathbf{b} \in S(\mathbf{a})$ *such that* $\mathbf{b} > \mathbf{c}$*, we have* $\mathbf{b} \in \widetilde{S}(\mathbf{a})_k$*.*

(c) *Let* $\mathbf{b} \in \widetilde{S}(\mathbf{a})_k$*, then* $\mathbf{b}^{(k)} \in S(\mathbf{a}^{(k)})$*. Moreover, if we suppose that* $\mathbf{a}$ *satisfies the hypothesis* $H_k(\mathbf{a})$ *and* $\mathbf{b} \neq \mathbf{a}$*, then*

$$\mathbf{b}^{(k)} \in S(\mathbf{a}^{(k)}) - \{\mathbf{a}^{(k)}\}$$

.

(d) *Suppose that* $\mathbf{a}$ *does not verify the hypothesis* $H_k(\mathbf{a})$*, then there exists a* $\mathbf{b} \in S(\mathbf{a})$ *satisfying the hypothesis* $H_k(\mathbf{a})$*, such that* $\mathbf{b}^{(k)} = \mathbf{a}^{(k)}$*.*

Démonstration. For (a), by lemma 2.1.4 for any $\mathbf{b} \in S(\mathbf{a})$, $e(\mathbf{b})$ is a sub-multisegment of $e(\mathbf{a})$. And from $\mathbf{b}$ to $\mathbf{b}^{(k)}$, we replace those segments Δ such that $e(\Delta) = k$ by Δ^-. Now (a) follows by counting the segments ending in k.

For (b), by (a), we have

$$\deg(\mathbf{a}^{(k)}) \leq \deg(\mathbf{b}^{(k)}) \leq \deg(\mathbf{c}^{(k)}).$$

The fact that $\mathbf{c} \in \widetilde{S}(\mathbf{a})_k$ implies that $\deg(\mathbf{a}^{(k)}) = \deg(\mathbf{c}^{(k)})$, hence $\deg(\mathbf{a}^{(k)}) = \deg(\mathbf{b}^{(k)})$ and $\mathbf{b} \in \widetilde{S}(\mathbf{a})_k$.

As for (c), suppose that $\deg(\mathbf{b}^{(k)}) = \deg(\mathbf{a}^{(k)})$, we prove $\mathbf{b}^{(k)} < \mathbf{a}^{(k)}$. Let

$$\mathbf{a} = \mathbf{a}_0 > \cdots > \mathbf{a}_r = \mathbf{b}$$

be a maximal chain of multisegments, then by (b), we know $\deg(\mathbf{a}_j^{(k)}) = \deg(\mathbf{a}^{(k)})$, for all $j = 1, \cdots, r$. Our proof breaks into two parts.

(1)We show that

$$\deg(\mathbf{a}_j^{(k)}) = \deg(\mathbf{a}_{j+1}^{(k)}) \Rightarrow \mathbf{a}_j^{(k)} \geq \mathbf{a}_{j+1}^{(k)}.$$

Let $\mathbf{a}_{j+1}$ be obtained from $\mathbf{a}_j$ by applying the elementary operation to two linked segments Δ, Δ'.

- If none of them ends in k, then $\mathbf{a}_j^{(k)}$ contains both of them. And we obtain $\mathbf{a}_{j+1}^{(k)}$ by applying the elementary operation to them.
  If one of them ends in k, we assume $e(\Delta') = k$.
- If Δ precedes Δ', we know that if $e(\Delta) < k-1$, Δ is still linked to Δ'^-, and one obtains $\mathbf{a}_{j+1}^{(k)}$ by applying elementary operation to $\{\Delta,\ \Delta'^-\}$, otherwise $e(\Delta) = k-1$, which implies $\mathbf{a}_{j+1}^{(k)} = \mathbf{a}_j^{(k)}$.

— If Δ is preceded by Δ', then the fact that

$$\deg(\mathbf{a}_{j+1}^{(k)}) = \deg(\mathbf{a}_j^{(k)})$$

implies $b(\Delta) \leq k$, hence Δ'^{-} is linked to Δ, and we obtain $\mathbf{a}_{j+1}^{(k)}$ from $\mathbf{a}_j^{(k)}$ by applying elementary operation to them.

Here we conclude that $\mathbf{b}^{(k)} \in S(\mathbf{a}^{(k)})$.

(2)Assuming that $\mathbf{a}$ satisfies the hypothesis $H_k(\mathbf{a})$, we show that

$$\mathbf{a}_1^{(k)} < \mathbf{a}^{(k)}.$$

Let $\mathbf{a}_1$ be obtained from $\mathbf{a}$ by performing the elementary operation to Δ, Δ'. We do it as in (1) but put $j = 0$. Note that in (1), the only case where we can have $\mathbf{a}_1^{(k)} = \mathbf{a}^{(k)}$ is when Δ precedes Δ' and $e(\Delta') = k,\ e(\Delta) = k - 1$. But such a case can not exist since $\mathbf{a}$ verifies the hypothesis $H_k(\mathbf{a})$. Hence we are done.

Finally, for (d), we construct $\mathbf{b}$ in the following way. Suppose that $\mathbf{a}$ does not satisfy the hypothesis $H_k(\mathbf{a})$, then there exists a pair of linked segments $\{\Delta, \Delta'\}$ such that

$$e(\Delta) = k - 1, \quad e(\Delta') = k,$$

let $\mathbf{a}_1$ be the multisegment obtained by applying the elementary operation to Δ and Δ'. We have

$$\mathbf{a}_1^{(k)} = \mathbf{a}^{(k)}.$$

If again $\mathbf{a}_1$ fails the hypothesis $H_k(\mathbf{a})$, we repeat the same construction to get $\mathbf{a}_2, \cdots$, since

$$\mathbf{a} > \mathbf{a}_1 > \cdots.$$

In finite step, we get $\mathbf{b}$ satisfying the conditions in the theorem and

$$\mathbf{b}^{(k)} = \mathbf{a}^{(k)}.$$

□

Remark: Actually, the multisegment constructed in (d) is unique, as we shall see later(proposition 3.4.1).

Definition 3.1.7. *We define a morphism*

$$\psi_k : \widetilde{S}(\mathbf{a})_k \to S(\mathbf{a}^{(k)})$$

by sending $\mathbf{c}$ *to* $\mathbf{c}^{(k)}$.

Proposition 3.1.8. *The morphism ψ_k is surjective.*

Démonstration. Let $\mathbf{d} \in S(\mathbf{a}^{(k)})$, such that we have a maximal chain of multisegments,

$$\mathbf{a}^{(k)} = \mathbf{d}_0 > \cdots > \mathbf{d}_r = \mathbf{d}.$$

By induction, we can assume that there exists $\mathbf{c}_i \in \widetilde{S}(\mathbf{a})_k$ such that $\mathbf{c}_i^{(k)} = \mathbf{d}_i$, for all $i < r$. Assume we obtain $\mathbf{d}$ from $\mathbf{d}_{r-1}$ by performing the elementary operation on the pair of linked segments $\{\Delta \prec \Delta'\}$.

- If $e(\Delta) \neq k-1$ and $e(\Delta') \neq k-1$, then we observe that the pair of segments are actually contained in $\mathbf{c}_{r-1}$. Let $\mathbf{c}_r$ be the multisegment obtained by performing the elementary operation to them . We conclude that $\mathbf{c}_r^{(k)} = \mathbf{d}_r$, and $\mathbf{c} \in \widetilde{S}(\mathbf{a})_k$.
- If $e(\Delta) = k-1$, then $\Delta \in \mathbf{c}_{r-1}$ or $\Delta^+ \in \mathbf{c}_{r-1}$ and $\Delta' \in \mathbf{c}_{r-1}$. The fact that $\mathbf{d}_{r-1} = \mathbf{c}_{r-1}^{(k)}$ implies that $k \notin e(\mathbf{d}_{r-1})$, hence $e(\Delta') > k$. Hence both Δ and Δ^+ are linked to Δ'. In either case we perform the elementary operation to get $\mathbf{c}_r$ such that $\mathbf{c}_r^{(k)} = \mathbf{d}$.
- If $e(\Delta') = k-1$, then $\Delta' \in \mathbf{c}_{r-1}$ or $\Delta'^+ \in \mathbf{c}_{r-1}$ and $\Delta \in \mathbf{c}_{r-1}$. The same argument as in the second case shows that there exists $\mathbf{c}_r$ such that $\mathbf{c}_r^{(k)} = \mathbf{d}$.

□

Actually, the proof in proposition 3.1.8 yields the following refinement.

Corollary 3.1.9. *Let $\mathbf{c} \in \widetilde{S}(\mathbf{a})_k, \mathbf{d} \in S(\mathbf{a}^{(k)})$ such that*

$$\mathbf{c}^{(k)} > \mathbf{d},$$

then there exists a multisegment $\mathbf{e} \in \widetilde{S}(\mathbf{a})_k$ such that

$$\mathbf{c} > \mathbf{e},\ \mathbf{e}^{(k)} = \mathbf{d}.$$

Démonstration. Note that $\mathbf{c} \in \widetilde{S}(\mathbf{a})_k$ implies $\widetilde{S}(\mathbf{a})_k \supseteq \widetilde{S}(\mathbf{c})_k$. Combine with the surjectivity of

$$\psi_k : \widetilde{S}(\mathbf{c})_k \to S(\mathbf{c}^{(k)}),$$

we get the result. □

Definition 3.1.10. *For $\mathbf{a}$ a multisegment, and $k \in \mathbb{Z}$ we define*

$$S(\mathbf{a})_k = \{\mathbf{c} \in \widetilde{S}(\mathbf{a})_k : \mathbf{c} \textit{ satisfies the hypothesis } H_k(\mathbf{a})\}.$$

Proposition 3.1.11. *The restriction*

$$\psi_k : S(\mathbf{a})_k \to S(\mathbf{a}^{(k)})$$
$$\mathbf{c} \mapsto \mathbf{c}^{(k)}$$

is also surjective.

Démonstration. For $\mathbf{d} \in S(\mathbf{a}^{(k)})$, by proposition 3.1.8, we know that there exists $\mathbf{c} \in \widetilde{S}(\mathbf{a})_k$ such that $\mathbf{c}^{(k)} = \mathbf{d}$. But by (d) in lemma 3.1.6, we know that there exists $\mathbf{c}' \in S(\mathbf{c})_k$ such that $\mathbf{c}'^{(k)} = \mathbf{c}^{(k)} = \mathbf{d}$. We conclude by the observation that if $\mathbf{c} \in \widetilde{S}(\mathbf{a})_k$, then

$$S(\mathbf{c})_k \subseteq S(\mathbf{a})_k.$$

□

Also, concerning the corollary 3.1.9, we have the following

Corollary 3.1.12. *Let $\mathbf{c} \in \widetilde{S}(\mathbf{a})_k$ and $\mathbf{d} \in S(\mathbf{a}^{(k)})$ such that $\mathbf{c}^{(k)} > \mathbf{d}$. Then there exists a multisegment $\mathbf{e} \in S(\mathbf{c})_k$ such that $\mathbf{e}^{(k)} = \mathbf{d}$.*

Démonstration. By corollary 3.1.9, we know that there exists an $\mathbf{e}' \in \widetilde{S}(\mathbf{c})_k$ such that $\mathbf{e}'^{(k)} = \mathbf{d}$. By (d) in lemma 3.1.6, we know that there exists $\mathbf{e} \in S(\mathbf{e}')_k$ such that $\mathbf{e}^{(k)} = \mathbf{e}'^{(k)} = \mathbf{d}$. Hence we conclude by the fact that if $\mathbf{e}' \in \widetilde{S}(\mathbf{a})_k$, then

$$S(\mathbf{e}')_k \subseteq S(\mathbf{a})_k.$$

□

Definition 3.1.13. *Let $k \in \mathbb{Z}$ and Δ be a segment.*

$$^{(k)}\Delta = \begin{cases} {}^{-}\Delta, & \text{if } b(\Delta) = k; \\ \Delta, & \text{otherwise .} \end{cases}$$

Let

$$\mathbf{a} = \{\Delta_1, \cdots, \Delta_r\},$$

be a multisegment, we define

$$^{(k)}\mathbf{a} = \{^{(k)}\Delta_1, \cdots, {}^{(k)}\Delta_r, \}.$$

Definition 3.1.14. *We say that the multisegment $\mathbf{b} \in S(\mathbf{a})$ satisfies **the hypothesis** $_kH(\mathbf{a})$ if the following two conditions are verified*

(1) $\deg(^{(k)}\mathbf{b}) = \deg(^{(k)}\mathbf{a})$ *;*

(2) *there exists no pair of linked segments $\{\Delta, \Delta'\}$ such that*

$$b(\Delta) = k, \; b(\Delta') = k + 1.$$

Remark: There exists a version of lemma 3.1.6 for $^{(k)}a$. In the following sections, we will work exclusively with $\mathbf{a}^{(k)}$ and the hypothesis $H_k(\mathbf{a})$. But all our results will remain valid if we replace $\mathbf{a}^{(k)}$ by $^{(k)}\mathbf{a}$ and $H_k(\mathbf{a})$ by $_kH(\mathbf{a})$.

3.2 Injectivity of ψ_k : First Step

By previous section, we know there exists $\mathbf{c} \in S(\mathbf{a})_k$ such that $\mathbf{c}^{(k)} = (\mathbf{a}^{\min})_{\min}$, the minimal element in $S(\mathbf{a}^{(k)})$. In this section, we give an explicit construction of such a $\mathbf{c}$ and show that it is the unique multisegment in $S(\mathbf{a})_k$ which is set to $(\mathbf{a}^{(k)})_{\min}$ by ψ_k.

- In proposition 3.2.3, we construct a multisegment $\mathbf{c} \in S(\mathbf{a}_1)_k$ such that $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$, where $\mathbf{a}_1$ is a multisegment such that $\mathbf{a} \in S(\mathbf{a}_1)$.
- We prove that there exists a unique element in $S(\mathbf{a})_k$ which is sent to $(\mathbf{a}^{(k)})_{\min}$ by ψ_k.
- Then we apply the uniqueness result to $S(\mathbf{a}_1)_k$ to prove that the constructed $\mathbf{c}$ before is in $S(\mathbf{a})_k$.[1]

Notation 3.2.1. *Let* $\ell_k = f_{e(\mathbf{a})}(k)$ *(cf. Def.1.2.7).*

Definition 3.2.2. *Let*

$$\mathbf{a}_0 = \{\Delta \in (\mathbf{a}^{(k)})_{\min} : e(\Delta) = k-1\}.$$

Proposition 3.2.3. *Let* $\mathbf{a}_0 = \{\Delta_1 \succeq \cdots \succeq \Delta_r\}$. *Let* $\mathbf{c}$ *be a multisegment such that*

(1) *If* $\varphi_{\mathbf{a}}(k-1) \geq \varphi_{\mathbf{a}}(k)$, *then* $r = \varphi_{\mathbf{a}}(k-1) - \varphi_{\mathbf{a}}(k) + \ell_k$. *Let*

$$\mathbf{c} = ((\mathbf{a}^{(k)})_{\min} \setminus \mathbf{a}_0) \cup \{\Delta_1^+ \succeq \cdots \succeq \Delta_{\ell_k}^+ \succeq \Delta_{m+1} \succeq \cdots \succeq \Delta_r\}.$$

(2) *If* $\varphi_{\mathbf{a}}(k) - \ell_k < \varphi_{\mathbf{a}}(k-1) < \varphi_{\mathbf{a}}(k)$, *then* $r = \varphi_{\mathbf{a}}(k-1) - \varphi_{\mathbf{a}}(k) + \ell_k$. *Let*

$$\mathbf{c} = ((\mathbf{a}^{(k)})_{\min} \setminus \mathbf{a}_0) \cup \{\Delta_1^+ \succeq \cdots \succeq \Delta_r^+ \succ \underbrace{[k] = \cdots = [k]}_{\ell_k - r}\}$$

(3) *If* $\varphi_{\mathbf{a}}(k-1) \leq \varphi_{\mathbf{a}}(k) - \ell_k$, *then* $\mathbf{a}_0 = \emptyset$ *and*

$$\mathbf{c} = \mathbf{a}^{(k)} + \ell_k[k].$$

Then $\mathbf{c}$ *satisfies the hypothesis* $H_k(\mathbf{c})$ *and* $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$.

Démonstration. We show only the case $\varphi_{\mathbf{a}}(k-1) > \varphi_{\mathbf{a}}(k)$, the proof for other cases is similar. Note that we have the following equality

$$\varphi_{\mathbf{a}}(k-1) = \varphi_{(\mathbf{a}^{(k)})_{\min}}(k-1) = r + \sharp\{\Delta \in (\mathbf{a}^{(k)})_{\min} : \Delta \supseteq [k-1,k]\}.$$

1. Here we use partial derivative to prove our result, but it can also be done in a purely combinatorial way, which is less elegant and more lengthy though.

Moreover, $\varphi_{\mathbf{a}}(k-1) > \varphi_{\mathbf{a}}(k)$ implies that no segment in $(\mathbf{a}^{(k)})_{\min}$ starts at k by minimality, hence we also have

$$\varphi_{\mathbf{a}}(k) = \varphi_{(\mathbf{a}^{(k)})_{\min}}(k) + \ell_k = \sharp\{\Delta \in (\mathbf{a}^{(k)})_{\min} : \Delta \supseteq [k-1,k]\} + \ell_k.$$

Now comparing the two formulas gives the equality $r = \varphi_{\mathbf{a}}(k-1) - \varphi_{\mathbf{a}}(k) + \ell_k$. By definition we have $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$. To check that $\mathbf{c}$ satisfies the hypothesis $H_k(\mathbf{c})$, it suffices to note that $(\mathbf{a}^{(k)})_{\min} \setminus \mathbf{a}_0$ does not contain segment which ends in $k-1$. □

Lemma 3.2.4. *Let $\mathbf{c} \in S(\mathbf{c})_k$ be a multisegment such that $\mathbf{c}^{(k)}$ is minimal. Then if $\mathbf{d} \in S(\mathbf{c})$ such that $\mathbf{d}^{(k)} = \mathbf{c}^{(k)}$, then $\mathbf{c} = \mathbf{d}$*

Démonstration. Suppose that $\mathbf{d} < \mathbf{c}$ is a multisegment such that $\mathbf{d}^{(k)} = \mathbf{c}^{(k)}$. Consider the maximal chain of multisegments

$$\mathbf{c} = \mathbf{c}_0 > \cdots > \mathbf{c}_t = \mathbf{d}.$$

Our assumption implies that $\mathbf{c}_i^{(k)} = \mathbf{c}^{(k)}$ for all $i = 1, \cdots, t$ by lemma 3.1.6. Hence we can assume $t = 1$ and consider $\mathbf{d} \in S(\mathbf{c})$ to be a multisegment obtained by applying the elementary operation to the pair of linked segments $\{\Delta \prec \Delta'\}$.

— If $e(\Delta) \neq k, e(\Delta') \neq k$, then the pair $\{\Delta, \Delta'\}$ also appears in $\mathbf{c}^{(k)}$, contradict the fact that $\mathbf{c}^{(k)}$ is minimal.

— If $e(\Delta') = k$, then by the fact that $\mathbf{c} \in S(\mathbf{c})_k$, we know that $e(\Delta) < k-1$, which implies that the pair $\{\Delta, \Delta^-\}$ is linked and belongs to $\mathbf{c}^{(k)}$,contradiction.

— If $e(\Delta') = k$ and $b(\Delta') < k+1$, then the pair $\{\Delta^-, \Delta'\}$ is still linked and belongs to $\mathbf{c}^{(k)}$, contradiction.

Hence we must have $e(\Delta') = k$ and $b(\Delta') = k+1$, this implies that $\deg(\mathbf{d}^{(k)}) > \deg(\mathbf{c}^{(k)})$ and $\mathbf{d} \notin \widetilde{S}(\mathbf{c})_k$. Finally, (b) of lemma 3.1.6 implies that for all $\mathbf{d} < \mathbf{c}$, we have $\mathbf{d} \notin \widetilde{S}(\mathbf{c})_k$. □

Proposition 3.2.5. *Let $\mathbf{c} \in S(\mathbf{c})_k$ be a multisegment such that $\mathbf{c}^{(k)}$ is minimal. Then the partial derivative $\mathscr{D}^k(L_{\mathbf{c}})$ contains in $\mathcal{R}$ a unique term of minimal degree $L_{\mathbf{c}^{(k)}}$, which appears with multiplicity one.*

Démonstration. Let $\mathbf{c} = \{\Delta_1, \cdots, \Delta_r\}$ such that $e(\Delta_t) = k$ if and only if $t = i, \cdots, j$ with $i \leq j$. Then

$$\mathscr{D}^k(\pi(\mathbf{c})) = \Delta_1 \times \cdots \times \Delta_{i-1} \times (\Delta_i + \Delta_i^-) \times \cdots \times (\Delta_j + \Delta_j^-) \times \Delta_{j+1} \times \cdots \times \Delta_r$$

with minimal degree term given by

$$\pi(\mathbf{c}^{(k)}) = \Delta_1 \times \cdots \times \Delta_{i-1} \times \Delta_i^- \times \cdots \times \Delta_j^- \times \Delta_{j+1} \times \cdots \times \Delta_r.$$

The same calculation shows that for any $\mathbf{d} \in S(\mathbf{c})$, the minimal degree terms in $\mathscr{D}^k(\pi(\mathbf{d}))$ is given by $\pi(\mathbf{d}^{(k)})$, whose degree is strictly greater than that of $\mathbf{c}^{(k)}$ since by previous lemma we know that $\mathbf{d} \notin \widetilde{S}(\mathbf{c})_k$. Note that $\mathscr{D}^k(L_{\mathbf{d}})$ is a non-negative sum of irreducible representations (Theorem 1.4.4), which cannot contain any representation of degree equal to that of $\mathbf{c}^{(k)}$, by comparing the minimal degree terms in $\mathscr{D}^k(\pi(\mathbf{d}))$ and $\sum_{\mathbf{e} \in S(\mathbf{d})} m(\mathbf{e}, \mathbf{d})\mathscr{D}^k(L_{\mathbf{e}})$. Finally, comparing the minimal degree terms in $\mathscr{D}^k(\pi(\mathbf{c}))$ and $\sum_{\mathbf{e} \in S(\mathbf{c})} m(\mathbf{e}, \mathbf{c})\mathscr{D}^k(L_{\mathbf{e}})$ gives the proposition.

□

Proposition 3.2.6. *Let* $\mathbf{a}$ *be a multisegment. Then* $S(\mathbf{a})_k$ *contains a unique multisegment* $\mathbf{c}$ *such that* $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$.

Démonstration. Let $\mathbf{a} = \{\Delta'_1, \cdots, \Delta'_s\}$ such that $e(\Delta'_t) = k$ if and only if $n = i, \cdots, j$ with $i \leq j$. Then

$$\mathscr{D}^k(\pi(\mathbf{a})) = \Delta'_1 \times \cdots \times \Delta'_{i-1} \times (\Delta'_i + \Delta'^{-}_i) \times \cdots \times (\Delta'_j + \Delta'^{-}_j) \times \Delta'_{j+1} \times \cdots \times \Delta'_s$$

with minimal degree term given by

$$\pi(\mathbf{a}^{(k)}) = \Delta'_1 \times \cdots \times \Delta'_{i-1} \times \Delta'^{-}_i \times \cdots \times \Delta'^{-}_j \times \Delta'_{j+1} \times \cdots \times \Delta'_r.$$

Note that in $\pi(\mathbf{a}^{(k)})$, $m((\mathbf{a}^{(k)})_{\min}, \mathbf{a}^{(k)}) = 1$(cf. [35]). Now compare with the terms of minimal degree in $\sum_{\mathbf{d} \in S(\mathbf{a})} m(\mathbf{d}, \mathbf{a})\mathscr{D}^k(L_{\mathbf{d}})$ and apply the proposition 3.2.6 yields the uniqueness of $\mathbf{c}$ such that $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$. □

Proposition 3.2.7. *Let* $\mathbf{c}$ *be the multisegment constructed in proposition 3.2.3. Then* $\mathbf{c} \in S(\mathbf{a})$.

Démonstration. Let

$$\mathbf{a}_1 = \mathbf{a}^{(k)} + m[k],$$

then we observe that $\mathbf{a} \in S(\mathbf{a}_1)$. Because of $\mathbf{c} \in S((\mathbf{a}^{(k)})_{\min} + m[k])$, we have $\mathbf{c} \in S(\mathbf{a}_1)$. Note that since $\deg((\mathbf{a}_1)^{(k)}) = \deg(\mathbf{c}^{(k)})$, the fact that $\mathbf{c} \in S(\mathbf{c})_k$ implies that $\mathbf{c} \in S(\mathbf{a}_1)_k$. Now let $\mathbf{d} \in S(\mathbf{a})_k$, then we have $\mathbf{d} \in S(\mathbf{a}_1)_k$ since $\deg(\mathbf{d}^{(k)}) = \deg(\mathbf{a}_1^{(k)}) = \deg(\mathbf{a}^{(k)})$. Assume furthermore that $\mathbf{d}^{(k)}$ is minimal, then by proposition 3.2.6, we know that such a multisegment in $S(\mathbf{a}_1)_k$ is unique, which implies $\mathbf{d} = \mathbf{c}$. □

Corollary 3.2.8. *Let* $\mathbf{c} \in S(\mathbf{a})_k$ *such that* $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$, *then* $\mathbf{c}$ *is minimal in* $\widetilde{S}(\mathbf{a})_k$.

Démonstration. By corollary 3.1.12, we know that for any $\mathbf{d} \in \widetilde{S}(\mathbf{a})_k$, there exists a multisegment $\mathbf{c}' \in S(\mathbf{a})_k$ with $\mathbf{c}'^{(k)} = (\mathbf{a}^{(k)})_{\min}$, such that $\mathbf{d} > \mathbf{c}'$. By uniqueness, we must have $\mathbf{c} = \mathbf{c}'$. □

3.3 Geometry of Nilpotent Orbits : General Cases

In this section, we show geometrically that the morphism

$$\psi_k : S(\mathbf{a})_k \to S(\mathbf{a}^{(k)})$$
$$\mathbf{c} \mapsto \mathbf{c}^{(k)}$$

is bijective, satisfying the properties

(1) For $\mathbf{c} \in S(\mathbf{a})_k$, we have $m(\mathbf{c}, \mathbf{a}) = m(\mathbf{c}^{(k)}, \mathbf{a}^{(k)})$.

(2) The morphism ψ_k preserves the order, i.e, for $\mathbf{c}, \mathbf{d} \in S(\mathbf{a})_k$, $\mathbf{c} > \mathbf{d}$ if and only if $\mathbf{c}^{(k)} > \mathbf{d}^{(k)}$.

To achieve this, firstly we consider the sub-variety $X_{\mathbf{a}}^k = \coprod_{\mathbf{c} \in \widetilde{S}(\mathbf{a})_k} O_{\mathbf{c}}$, and construct a fibration α from $X_{\mathbf{a}}^k$ to $Gr(\ell_k, V_{\varphi_{\mathbf{a}},k})$, the latter is the space of the ℓ_k-dimensional subspace of $V_{\varphi_{\mathbf{a}},k}$. Secondly, we construct an open immersion

$$\tau_W : (X_{\mathbf{a}}^k)_W \to Y_{\mathbf{a}^{(k)}} \times \mathrm{Hom}(V_{\varphi_{\mathbf{a}},k-1}, W),$$

where $(X_{\mathbf{a}}^k)_W$ is the fiber over W with respect to α and $Y_{\mathbf{a}^{(k)}} = \coprod_{\mathbf{c} \in S(\mathbf{a}^{(k)})} O_{\mathbf{c}}$.
Our main difficulty here lies in proving that τ_W is actually an open immersion. The idea is to apply Zariski Main theorem, to do this, we have to prove the normality and irreducibility of both varieties. Irreducibility of $(X_{\mathbf{a}}^k)_W$ follows from our results in previous section, and normality follows from the fibration α and the fact that orbital varieties are locally isomorphic to some Schubert varieties, by Zelevinsky, cf. [37].
Once we prove that τ_W is an open immersion. All the desired properties of ψ_k follow.

Here we fix a multisegment $\mathbf{a}$ and let $\varphi = \varphi_{\mathbf{a}}$.

Definition 3.3.1. [2]

2. In this section we only work with $X_{\mathbf{a}}^{(k)}$ instead of $\widetilde{X}_{\mathbf{a}}^{(k)} = \coprod_{\mathbf{b} \in S(\mathbf{a})_k} O_{\mathbf{b}}$. The reason

— *Let*

$$X_{\mathbf{a}}^k = \coprod_{\mathbf{c}\in\widetilde{S}(\mathbf{a})_k} O_{\mathbf{c}},$$

— *Let* $Y_{\mathbf{a}^{(k)}} = \coprod_{c\in S(\mathbf{a}^{(k)})} O_{\mathbf{c}}$.

— *For* $\mathbf{b} > \mathbf{c}$ *in* $\widetilde{S}(\mathbf{a})_k$, *we define*

$$X_{\mathbf{b},\mathbf{c}}^k = \coprod_{\mathbf{b}\geq\mathbf{d}\geq\mathbf{c}} O_{\mathbf{d}}.$$

Let $\mathbf{c} \in \widetilde{S}(\mathbf{a})_k, T \in O_{\mathbf{c}}$, then

Lemma 3.3.2. *Let* $\varphi = \varphi_{\mathbf{a}}$. *We have* $\dim(\ker(T|_{V_{\varphi,k}})) = \sharp\{\Delta \in \mathbf{a} : e(\Delta) = k\} = \ell_k$ *(Notation 3.2.1), which does not depend on the choice of* T.

Démonstration. The fact $T \in O_{\mathbf{c}}$ implies

$$\dim(\ker(T|_{V_{\varphi,k}})) = \sharp\{\Delta \in \mathbf{c} : e(\Delta) = k\}.$$

Then our lemma follows from lemma 3.1.5.

□

Definition 3.3.3. *Let*

$$Gr(\ell_k, V_\varphi) = \{W \subseteq V_{\varphi,k} : \dim(W) = \ell_k\},$$

and for $W \in Gr(\ell_k, V_\varphi)$, *let*

$$V_\varphi/W = V_{\varphi,1} \oplus \cdots V_{\varphi,k-1} \oplus V_{\varphi,k}/W \oplus \cdots .$$

Also, we denote by

$$p_W : V_\varphi \to V_\varphi/W$$

the canonical projection.

---

can be seen from the simple example of the affine plane $\mathbb{A}^2$ endowed with the stratification

$$X_1 = \mathbb{A}^2 - \mathbb{A}^1, \quad X_2 = \mathbb{A}^1 - pt, \quad X_3 = pt.$$

If we are interested in $X_1 \coprod X_3$, it is better to study $\mathbb{A}^2$, because there is no nontrivial directed extension of X_1 by X_3. Instead, if we are interested in $X_1 \coprod X_2$, we can study $\mathbb{A}^2 - pt$, which is already a nontrivial extension.

Definition 3.3.4. *We define*

$$\widetilde{Z}^k = \{(T,W) : W \in Gr(\ell_k, V_\varphi), T \in End(V/W) \text{ of degree } +1\},$$

and the canonical projection

$$\begin{aligned} \pi : \widetilde{Z}^k &\to Gr(\ell_k, V_\varphi) \\ (T,W) &\mapsto W. \end{aligned}$$

Proposition 3.3.5. *The morphism π is a fibration with fiber*

$$E_{\varphi_{\mathbf{a}^{(k)}}}(\text{ Def.2.2.1}).$$

Démonstration. This follows from the definition. □

Definition 3.3.6. *Assume $\mathbf{b}, \mathbf{c} \in S(\mathbf{a}^{(k)})$.*

— *Let*
$$Z^{k,\mathbf{a}} = \{(T,W) \in \widetilde{Z}^k : T \in Y_{\mathbf{a}^{(k)}}\}.$$

— *Let*

$$Z^{k,\mathbf{a}}_{\mathbf{b},\mathbf{c}} = \{(T,W) \in \widetilde{Z}^k : T \in \coprod_{\mathbf{b}\geq\mathbf{d}\geq\mathbf{c}} O_{\mathbf{d}}\},\ Z^{k,\mathbf{a}}_{\mathbf{b}} = \{(T,W) \in \widetilde{Z}^k : T \in \coprod_{\mathbf{d}\geq\mathbf{b}} O_{\mathbf{d}}\}.$$

— *Let*
$$Z^{k,\mathbf{a}}(\mathbf{c}) = \{(T,W) \in Z^{k,\mathbf{a}}, T \in O_{\mathbf{c}}\}.$$

Remark: The restriction of π to $Z^{k,\mathbf{a}}$ is a fibration with fiber $Y_{\mathbf{a}^{(k)}}$.

Definition 3.3.7. *Now we define $T^{(k)} \in End(V/\ker(T|_{V_{\varphi,k}}))$ such that*

$$T^{(k)}|_{V_{\varphi,i}} = \begin{cases} T|_{V_{\varphi,i}}, \text{ for } i \neq k, k-1, \\ p_{T,k} \circ T|_{V_{\varphi,i}}, \text{ for } i = k-1 \\ T|_{V_{\varphi,i}} \circ p_{T,k}, \text{ for } i = k. \end{cases}$$

where $p_{T,k} : V_\varphi \to V_\varphi/\ker(T|_{V_{\varphi,k}})$ is the canonical projection.

This gives naturally an element $(T^{(k)}, \ker(T|_{V_{\varphi,k}}))$ in $Z^{k,\mathbf{a}}$. We construct a morphism
$$\gamma_k : X^k_{\mathbf{a}} \to Z^{k,\mathbf{a}}.$$
by
$$\gamma_k(T) = (T^{(k)}, \ker(T|_{V_{\varphi,k}})).$$

Definition 3.3.8. *We define*

$$\alpha : X_{\mathbf{a}}^k \to Gr(\ell_k, V_\varphi),$$

with $\alpha(T) = \ker(T|_{V_{\varphi,k}})$.

Remark: We have a commutative diagram

$$\begin{array}{ccc} X_{\mathbf{a}}^k & \xrightarrow{\gamma_k} & Z^{k,\mathbf{a}} \\ \downarrow{\scriptstyle \alpha} & \swarrow{\scriptstyle \pi} & \\ Gr(\ell_k, V_\varphi). & & \end{array}$$

where γ_k maps fibers to fibers.

Proposition 3.3.9. *The morphism α is a fiber bundle such that $\alpha|_{O_{\mathbf{c}}}$ is surjective for any $\mathbf{c} \in \widetilde{S}(\mathbf{a})_k$.*

Démonstration. We have to show that α is locally trivial. We fix $W \in Gr(\ell_k, V_\varphi)$ Note that $GL_{\varphi(k)}$ acts transitively on $Gr(\ell_k, V_\varphi)$. Let P_W be the stabilizer of W. Then by Serre [32] proposition 3, we know that the principle bundle

$$GL_{\varphi(k)} \to GL_{\varphi(k)}/P_W$$

is étale-locally trivial. Here the base $GL_{\varphi(k)}/P_W$ is isomorphic to $Gr(\ell_k, V_\varphi)$. It is even Zariski-locally trivial because P_W is parabolic, which is special in the sense of Serre [32], § 4. Now we can write

$$\begin{array}{ccc} X_{\mathbf{a}}^k & \xleftarrow{\delta} & GL_{\varphi(k)} \times_{P_W} \alpha^{-1}(W) \\ \downarrow & \swarrow & \\ Gr(\ell_k, V_\varphi) & & \end{array}$$

where

$$\delta([g, T]) = g.T.$$

We claim that δ is an isomorphism. In fact, for any $T \in X_{\mathbf{a}}^k$, we choose $g \in GL_{\varphi(k)}$ such that

$$g(\ker(T|_{V_{\varphi,k}})) = W.$$

This implies $g.T \in \alpha^{-1}(W)$, thus

$$\delta([g^{-1}, g.T]) = T.$$

This shows the surjectivity. For injectivity, it is enough to show that

$$\delta([g,T]) = g.T \in \alpha^{-1}(W)$$

implies $g \in P_W$. But this is by definition of P_W.
The fact that α is locally trivial then can be deduced from that of

$$GL_{\varphi(k)} \times_{P_W} \alpha^{-1}(W),$$

while the latter is a consequence of the fact that $GL_{\varphi(k)}$ is locally trivial over $Gr(\ell_k, V_\varphi)$.
Finally, we want to show the surjectivity of the orbit $\alpha|_{O_{\mathbf{c}}}$. This is a consequence the fact that $GL_{\varphi(k)}$ acts transitively on $Gr(\ell_k, V_\varphi)$. □

Proposition 3.3.10. *Let $\mathbf{c} \in \widetilde{S}(\mathbf{a})_k$. The restriction map*

$$\gamma_k : O_{\mathbf{c}} \to Z^{k,\mathbf{a}}(\mathbf{c}^{(k)})$$

is surjective.

Démonstration. Let $(T_0, W) \in Z^{k,\mathbf{a}}(\mathbf{c}^{(k)})$. Consider

$$m = \sharp\{\Delta \in \mathbf{c} : e(\Delta) = k, \deg(\Delta) \geq 2\} \leq \min\{\ell_k, \dim(\ker(T_0|_{V_{\varphi,k-1}}))\}.$$

We choose a splitting $V_{\varphi,k} = W \oplus V_{\varphi,k}/W$ and let $T' : V_{\varphi,k-1} \to W$ be a linear morphism of rank m. Finally, we define $T \in \gamma_k^{-1}((T_0, W))$ by letting

$$T|_{V_{\varphi,k-1}} = T' \oplus T_0|_{V_{\varphi,k-1}},$$

$$T|_{V_{\varphi,k}} = T_0|_{V_{\varphi,k}/W} \circ p_W,$$

$$T|_{V_{\varphi,i}} = T|_{V_{\varphi,i}}, \text{ for } i \neq k-1, k.$$

Let

$$\{\Delta \in \mathbf{c} : e(\Delta) = k, \deg(\Delta) \geq 2\} = \{\Delta_1, \cdots, \Delta_m\}, \quad b(\Delta_1) \leq \cdots \leq b(\Delta_m).$$

We denote $W_i = T_0^{[b(\Delta_1),k-1]}(V_{\varphi,b(\Delta_1)}) \cap \ker(T_0|_{V_{\varphi,k-1}})$, then

$$W_1 \subseteq \cdots \subseteq W_r \subseteq \ker(T_0|_{V_{\varphi,k-1}}).$$

Then we have $T \in O_{\mathbf{c}}$ if and only if

$$\dim(T'(W_i)) - \dim(T(W_{i-1})) = \dim(W_i/W_{i-1}), \quad i = 1, \cdots, m.$$

Since such T' always exists, we are done. □

Notation 3.3.11. *We fix $W \in Gr(\ell_k, V_\varphi)$, and denote*

$$(X_{\mathbf{a}}^k)_W, \quad (Z^{k,\mathbf{a}})_W$$

the fibers over W.

Proposition 3.3.12. *The fiber $(X_{\mathbf{a}}^k)_W$ is normal and irreducible as an algebraic variety over $\mathbb{C}$.*

Démonstration. Note that since $\widetilde{S}(\mathbf{a})_k$ contains a unique minimal element $\mathbf{c}$, the variety $X_{\mathbf{a}}^k$ is contained and is open in the irreducible variety $\overline{O}_{\mathbf{c}}$. Now by [37] theorem 1, we know that $X_{\mathbf{a}}^k$ is actually normal.
By proposition 3.3.9, we know that α is a fibration between two varieties $X_{\mathbf{a}}^k$ and $Gr(\ell_k, V_\varphi)$. The fact that both are normal and irreducible implies that the fiber $(X_{\mathbf{a}}^k)_W$ is normal and irreducible. □

Remark: Note that by definition, we are allowed to identify $(Z^{k,\mathbf{a}})_W$ with $Y_{\mathbf{a}^{(k)}}$. This is what we do from now on.

Definition 3.3.13. *We choose a splitting $V_{\varphi,k} = W \oplus V_{\varphi,k}/W$ and denote by $q_W : V_{\varphi,k} \to W$ the projection. We define a morphism τ_W*

$$\tau_W(T) = ((\gamma_k)_W(T), q_W \circ T|_{V_{\varphi,k-1}}).$$

Remark: Then we have the following commutative diagram

$$\begin{array}{ccc} (X_{\mathbf{a}}^k)_W & \xrightarrow{\tau_W} & (Z^{k,\mathbf{a}})_W \times \mathrm{Hom}(V_{\varphi,k-1}, W) \\ \downarrow{\scriptstyle (\gamma_k)_W} & \swarrow{\scriptstyle s} & \\ (Z^{k,\mathbf{a}})_W & & \end{array}$$

where s is the canonical projection.

Lemma 3.3.14. *The morphism τ_W is injective.*

Démonstration. Note that any $T \in (X_{\mathbf{a}}^k)_W$ is determined by $(\gamma_k)_W(T)$ and $T|_{V_{\varphi,k-1}}$. Furthermore, $T|_{V_{\varphi,k-1}}$ is determined by $p_W \circ T|_{V_{\varphi,k-1}}$ and $q_W \circ T|_{V_{\varphi,k-1}}$. Since $p_W \circ T|_{V_{\varphi,k-1}}$ is a component of $(\gamma_k)_W(T)$, it is determined by $(\gamma_k)_W(T)$ and $q_W \circ T|_{V_{\varphi,k-1}}$. This gives us the injectivity. □

Lemma 3.3.15. *Let $\mathbf{c} \in S(\mathbf{a})_k$ such that $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$. Then The image of $O_{\mathbf{c}} \cap (X_{\mathbf{a}}^k)_W$ is open in $O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W)$.*

Démonstration. Let $\mathbf{c} \in S(\mathbf{a})_k$ such that $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$. We shall use the description in proposition 3.2.3. We show that the image of

$$O_{\mathbf{c}} \cap (X_{\mathbf{a}}^k)_W$$

is open in $O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W)$.
Let $T \in (O_{\mathbf{c}})_W$. We check case by case :

(1) If $\varphi(k-1) \leq \varphi(k) - \ell_k$, the fact $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$ implies that $T^{(k)}|_{V_{\varphi,k-1}}$ is injective. As a consequence we have $\mathrm{Im}(T|_{V_{\varphi,k-1}}) \cap W = 0$. Hence for any element $T_0 \in \mathrm{Hom}(V_{\varphi,k-1}, W)$, we define $T_0 \in O_{\mathbf{c}}$, such that

$$T_0|_{V_{\varphi,k-1}} = T_0 \oplus T^{(k)}|_{V_{\varphi,k-1}},$$

which lies in the fiber over $(\gamma_k)_W^{-1}((T^{(k)}, W))$. Since by proposition 3.3.10, every element in $O_{\mathbf{c}^{(k)}}$ comes from some element in $O_{\mathbf{c}}$, hence

$$\tau_W(O_{\mathbf{c}} \cap (X_{\mathbf{a}}^k)_W) = O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W),$$

which is open.

(2) If $\varphi(k) - \ell_k < \varphi(k-1) < \varphi(k)$, the fact $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$ implies that the morphism

$$T^{(k)}|_{V_{\varphi,k-1}}$$

contains a kernel of dimension

$$\varphi(k-1) - \varphi(k) + \ell_k.$$

Our description of $\mathbf{c}$ in proposition 3.2.3 shows that in this case

$$\dim(\mathrm{Im}(T|_{V_{\varphi,k-1}}) \cap W) = \varphi(k-1) - \varphi(k) + \ell_k.$$

In this situation, given an element $T_0 \in \mathrm{Hom}(V_{\varphi,k-1}, W)$ we define $T' \in E_\varphi$, such that

$$T'|_{V_{\varphi,k-1}} = T_0 \oplus T^{(k)}|_{V_{\varphi,k-1}},$$

$$T'|_{V_{\varphi,k}} = T^{(k)}|_{V_{\varphi,k}/W} \circ p_W,$$

$$T'|_{V_{\varphi,i}} = T^{(k)}, \text{ for } i \neq k-1, k.$$

By construction and proposition 2.2.5, we know that $T' \in O_{\mathbf{c}}$ if and only if $T'|_{V_{\varphi,k-1}}$ is injective, since no segment in $\mathbf{c}$ ends in $k-1$, as described in proposition 3.2.3. And this is equivalent to say

$$T_0|_{\ker(T^{(k)}|_{V_{\varphi,k-1}})}$$

is injective. This is an open condition, hence $O_{\mathbf{c}} \cap (X_{\mathbf{a}}^k)_W$ is open in $O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W)$.

(3) If $\varphi(k-1) \geq \varphi(k)$, then by proposition 3.2.3

$$\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$$

implies

$$\operatorname{Im}(T|_{V_{\varphi,k-1}}) \supseteq W.$$

Recall the notation from proposition 3.2.3

$$\mathbf{a}_0 = \{\Delta_1 \succeq \cdots \succeq \Delta_r\}.$$

with $r = \varphi(k-1) - \varphi(k) + \ell_k$. Then

$$\mathbf{c} = ((\mathbf{a}^{(k)})_{\min} \setminus \mathbf{a}_0) \cup \{\Delta_1^+ \succeq \cdots \succeq \Delta_{\ell_k}^+ \succeq \Delta_{\ell_k+1} \succeq \cdots \succeq \Delta_r\}.$$

Let $T_0 \in \operatorname{Hom}(V_{\varphi,k-1}, W)$, we define $T' \in E_\varphi$

$$T'|_{V_{\varphi,k-1}} = T_0 \oplus T^{(k)}|_{V_{\varphi,k-1}},$$

$$T'|_{V_{\varphi,k}} = T^{(k)}|_{V_{\varphi,k}/W} \circ p_W,$$

$$T'|_{V_{\varphi,i}} = T^{(k)}, \text{ for } i \neq k-1, k.$$

Consider the following flag over $V_{\varphi,k-1}$,

$$\ker(T^{(k)}|_{\varphi,k-1}) = V_r \supseteq \cdots \supseteq V_1 \supseteq V_0 = 0,$$

where $V_i = \operatorname{Im}((T^{(k)})^{\Delta_i}) \cap \ker(T^{(k)}|_{\varphi,k-1})$, with $i = 1, \cdots, r$, for the notation $(T^{(k)})^{\Delta}$, we refer to definition 2.2.9.
Now by proposition 2.2.5, we know that $T' \in O_{\mathbf{c}}$ if and only if

$$\dim(T_0(V_i)) - \dim(T_0(V_{i-1})) = \dim(V_i/V_{i-1}),$$

for $i = 1, \cdots, \ell_k$. In fact, if $V_i \neq V_{i-1}$, then

$$\dim(V_i/V_{i-1}) = \sharp\{j : \Delta_j = \Delta_i\}.$$

And by construction, if $i \leq \ell_k$, by proposition 2.2.5, the fact that $\mathbf{c}$ contains Δ_i^+ implies that if $T' \in O_{\mathbf{c}}$,

$$\dim(T_0(V_i)) - \dim(T_0(V_{i-1})) = \dim(V_i/V_{i-1}).$$

The converse holds by the same reason.
Again, this is an open condition, which proves that $O_{\mathbf{c}} \cap (X_{\mathbf{a}}^k)_W$ is open in $O_{\mathbf{c}^{(k)}} \times \operatorname{Hom}(V_{\varphi,k-1}, W)$.

□

Proposition 3.3.16. *The morphism τ_W is an open immersion.*

Démonstration. To see that it is open immersion, we shall use Zariski's main theorem. Since all Schubert varieties are normal, we observe that

$$(Z^{k,\mathbf{a}})_W \times \mathrm{Hom}(V_{\varphi,k-1}, W)$$

are normal by theorem 1 of [37]. Also, by proposition 3.3.12, we know that $(X^k_{\mathbf{a}})_W$ is irreducible and normal, hence τ_W is an open immersion. □

Proposition 3.3.17. *Let $\mathbf{c} \in \widetilde{S}(\mathbf{a})_k$. Then $\mathbf{c} \in S(\mathbf{a})_k$ if and only if*

$$O_{\mathbf{c}} \cap (X^k_{\mathbf{a}})_W$$

is open in

$$(O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W)).$$

Démonstration. We already showed that

$$O_{\mathbf{c}} \cap (X^k_{\mathbf{a}})_W$$

is a sub-variety of

$$O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W).$$

Moreover, we know that

$$(O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W)) \cap (X^k_{\mathbf{a}})_W$$

is open in

$$O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W)$$

since τ_W is open. Finally, by proposition 3.3.10,

$$\begin{aligned}(O_{\mathbf{c}^{(k)}} &\times \mathrm{Hom}(V_{\varphi,k-1}, W)) \cap (X^k_{\mathbf{a}})_W \\ &= \coprod_{\mathbf{d}\in\widetilde{S}(\mathbf{a})_k, \mathbf{d}^{(k)}=\mathbf{c}^{(k)}} O_{\mathbf{d}} \cap (X^k_{\mathbf{a}})_W.\end{aligned}$$

The variety $(O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W)) \cap (X^k_{\mathbf{a}})_W$ is irreducible because $(O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W))$ is irreducible, hence the stratification $\coprod_{\mathbf{d}\in\widetilde{S}(\mathbf{a})_k, \mathbf{d}^{(k)}=\mathbf{c}^{(k)}} O_{\mathbf{d}} \cap (X^k_{\mathbf{a}})_W$ by locally closed sub-varieties can only contain one term which is open, from the point of view of Zariski topology. Since for any element

$$\mathbf{d}' \in \{\mathbf{d} \in \widetilde{S}(\mathbf{a})_k, \mathbf{d}^{(k)} = \mathbf{c}^{(k)}\},$$

by (d) of lemma 3.1.6, we know that there exists $\mathbf{c}' \in S(\mathbf{a})_k$ such that $\mathbf{d}' > \mathbf{c}'$. Hence we conclude that

$$\{\mathbf{d} \in \widetilde{S}(\mathbf{a})_k, \mathbf{d}^{(k)} = \mathbf{c}^{(k)}\},$$

contains a unique minimal element, which lies in $S(\mathbf{a})_k$. Now our proposition follows. □

Corollary 3.3.18. *Let* $\mathbf{a}$ *be a multisegment and*

$$\mathbf{c} \in S(\mathbf{a})_k,$$

then

$$P_{\mathbf{a},\mathbf{c}}(q) = P_{\mathbf{a}^{(k)},\mathbf{c}^{(k)}}(q).$$

Démonstration. First of all, by proposition 3.3.9 and Kunneth formula, we know that

$$\mathcal{H}^j(\overline{O}_{\mathbf{c}})_{\mathbf{a}} = \mathcal{H}^j(\overline{O}_{\mathbf{c}} \cap (X_{\mathbf{a}}^{(k)})_W)_{\mathbf{a}},$$

the localization being taken at a point in $O_{\mathbf{a}} \cap (X_{\mathbf{a}}^{(k)})_W$. Now by proposition 3.3.16 and proposition 3.3.17 , we may regard $\overline{O}_{\mathbf{c}} \cap (X_{\mathbf{a}}^{(k)})_W$ as an open subset of $\overline{O}_{\mathbf{c}^{(k)}} \times Hom(V_{\varphi,k-1}, W)$, hence

$$\mathcal{H}^j(\overline{O}_{\mathbf{c}} \cap (X_{\mathbf{a}}^{(k)})_W)_{\mathbf{a}} = \mathcal{H}^j(\overline{O}_{\mathbf{c}^{(k)}} \times Hom(V_{\varphi,k-1}, W))_{\mathbf{a}^{(k)}}$$

and Kunneth formula implies that the latter is equal to

$$\mathcal{H}^j(\overline{O}_{\mathbf{c}^{(k)}})_{\mathbf{a}^{(k)}}.$$

□

Corollary 3.3.19. *Let* $\mathbf{d} \in S(\mathbf{a})$ *such that*

$$\mathbf{d}^{(k)} = \mathbf{a}^{(k)},$$

and

$$\mathbf{c} \in S(\mathbf{a})_k,$$

then $\mathbf{c} < \mathbf{d}$, *and*

$$P_{\mathbf{a},\mathbf{c}}(q) = P_{\mathbf{d},\mathbf{c}}(q).$$

Démonstration. By corollary 3.1.12, we know that there exists $\mathbf{c}' \in S(\mathbf{a})_k$ such that

$$\mathbf{d} > \mathbf{c}', \ \mathbf{c}'^{(k)} = \mathbf{c}^{(k)}.$$

And proposition 3.3.17 implies $\mathbf{c}' = \mathbf{c}$. Finally, applying the corollary 3.3.18 to the pairs $\{\mathbf{a}, \mathbf{c}\}$ and $\{\mathbf{d}, \mathbf{c}\}$ yields the result. □

3.4 Conclusion

In this section, we draw some conclusions from what we have done before, espectially the properties related to ψ_k.

Proposition 3.4.1. *The map*

$$\psi_k : S(\mathbf{a})_k \to S(\mathbf{a}^{(k)})$$
$$\mathbf{c} \mapsto \mathbf{c}^{(k)}$$

is bijective. Moreover,

— *for* $\mathbf{c} \in S(\mathbf{a})_k$
$$m(\mathbf{c}, \mathbf{a}) = m(\mathbf{c}^{(k)}, \mathbf{a}^{(k)}).$$

— *for* $\mathbf{b}, \mathbf{c} \in S(\mathbf{a})_k$*, we have* $\mathbf{b} > \mathbf{c}$ *if and only if* $\mathbf{b}^{(k)} > c^{(k)}$*.*

Démonstration. By proposition 3.3.17, we know that ψ_k is injective. Surjectivity is given by proposition 3.1.11.
For $\mathbf{c} \in S(\mathbf{a})_k$,
$$m(\mathbf{c}, \mathbf{a}) = m(\mathbf{c}^{(k)}, \mathbf{a}^{(k)})$$
is by corollary 3.3.18 by putting $q = 1$, and applying theorem 2.4.12.
Finally, for $\mathbf{b}, \mathbf{c} \in S(\mathbf{a})_k$, if $\mathbf{b} > \mathbf{c}$, then $\mathbf{c} \in S(\mathbf{b}^{(k)}, \mathbf{b})$, and by lemma 3.1.6, we know that $\mathbf{b}^{(k)} > \mathbf{c}^{(k)}$. Reciprocally, if $\mathbf{b}^{(k)} > \mathbf{c}^{(k)}$, by proposition 3.3.17, we know that $\overline{O}_{\mathbf{b}} \subseteq \overline{O}_{\mathbf{c}}$, hence $\mathbf{b} > \mathbf{c}$.

□

Corollary 3.4.2. *We have*

—
$$\pi(\mathbf{a}^{(k)}) = \sum_{\mathbf{c} \in S(\mathbf{a})_k} m(\mathbf{c}, \mathbf{a}) L_{\mathbf{c}^{(k)}}, \tag{3.4.3}$$

— *let* $\mathbf{b} \in S(\mathbf{a})$ *such that* $\mathbf{b}$ *satisfies the hypothesis* $H_k(\mathbf{a})$ *and* $\mathbf{b}^{(k)} = \mathbf{a}^{(k)}$*, then*
$$m(\mathbf{b}, \mathbf{a}) = 1, \ S(\mathbf{a})_k = S(\mathbf{b})_k.$$

Démonstration. The first part follows from the fact that ψ_k is bijective and $m(\mathbf{c}, \mathbf{a}) = m(\mathbf{c}^{(k)}, \mathbf{a}^{(k)})$. For the second part of the lemma, we note that $L_{\mathbf{b}^{(k)}} = L_{\mathbf{a}^{(k)}}$ appears with multiplicity one in $\pi(\mathbf{a}^{(k)})$, then equation (3.4.3) implies $m(\mathbf{b}, \mathbf{a}) = m(\mathbf{b}^{(k)}, \mathbf{a}^{(k)}) = 1$. To see that $S(\mathbf{a})_k = S(\mathbf{b})_k \subseteq S(\mathbf{b})$. Note that we have $S(\mathbf{b})_k \subseteq S(\mathbf{a})_k$ and two bijection
$$\psi_k : S(\mathbf{a})_k \to S(\mathbf{a}^{(k)}),$$
$$\psi_k : S(\mathbf{b})_k \to S(\mathbf{b}^{(k)}) = S(\mathbf{a}^{(k)}),$$
Hence comparing the cardinality gives $S(\mathbf{a})_k = S(\mathbf{b})_k$. □

3.5 Minimal Degree Terms in Partial Derivatives

Proposition 3.5.1. **(i)** *Suppose that* $\mathbf{a}$ *satisfies the hypothesis* $H_k(\mathbf{a})$. *Then* $\mathscr{D}^k(L_{\mathbf{a}})$ *contains in* $\mathcal{R}$ *a unique irreducible representation of minimal degree, which is* $L_{\mathbf{a}^{(k)}}$*, and it appears with multiplicity one.*

(ii) *If* $\mathbf{a}$ *fails to satisfy the hypothesis* $H_k(\mathbf{a})$*, then* $L_{\mathbf{a}^{(k)}}$ *will not appear in* $\mathscr{D}^k(L_{\mathbf{a}})$*, and the irreducible representations appearing are all of degree* $> \deg(\mathbf{a}^{(k)})$.

Démonstration. Let $\mathbf{a} = \{\Delta_1 \preceq \cdots \preceq \Delta_r\}$, such that

$$e(\Delta_1) \leq \cdots < e(\Delta_i) = \cdots = e(\Delta_j) < \cdots \leq e(\Delta_r),$$

with $k = e(\Delta_i)$.
We prove the proposition by induction on $\ell(\mathbf{a})$(cf. definition 1.3.3). For, $\ell(\mathbf{a}) = 0$, which means that $\mathbf{a} = \mathbf{a}_{\min}$, in this case $\mathbf{a}$ satisfies the $H_k(\mathbf{a})$, and

$$\mathscr{D}^k(L_{\mathbf{a}}) = \mathscr{D}^k(\pi(\mathbf{a})) = \Delta_1 \times \cdots \times (\Delta_i + \Delta_i^-) \times \cdots \times (\Delta_j + \Delta_j^-) \times \cdots$$

which contains

$$L_{\mathbf{a}^{(k)}} = \pi(\mathbf{a}^{(k)}) = \Delta_1 \times \cdots \times \Delta_i^- \times \cdots \Delta_j^- \times \cdots .$$

Hence we are done.
For general $\mathbf{a}$, we have refer to the lemma 3.1.6.
We write

$$\pi(\mathbf{a}) = L_{\mathbf{a}} + \sum_{\mathbf{b}<\mathbf{a}} m(\mathbf{b}, \mathbf{a}) L_{\mathbf{b}}. \tag{3.5.2}$$

Now applying $\mathscr{D}^k$ to both sides and consider only the lowest degree terms, on the left hand side, we get

$$\pi(\mathbf{a}^{(k)}) = \Delta_1 \times \cdots \times \Delta_{i-1} \times \Delta_i^- \times \cdots \times \Delta_j^- \times \cdots \Delta_r. \tag{3.5.3}$$

By theorem 1.4.4, both sides are positive sum of irreducible representations, then

— If $\mathbf{a}$ satisfies the hypothesis $H_k(\mathbf{a})$, on the right hand side, from our lemma 3.1.6 and induction, we know that for all $\mathbf{b} < \mathbf{a}$, $\mathscr{D}^k(L_{\mathbf{b}})$ does not contain $L_{\mathbf{a}^{(k)}}$ as subquotient, hence $\mathscr{D}^k(L_{\mathbf{a}})$ must contain $L_{\mathbf{a}^{(k)}}$ with multiplicity one. We have to show that it does not contain other subquotients of $\pi(\mathbf{a}^{(k)})$. Note that by induction, we have the following formula

$$\pi(\mathbf{a}^{(k)}) = X + \sum_{\mathbf{c}\in S(\mathbf{a})_k\setminus\mathbf{a}} m(\mathbf{c},\mathbf{a})L_{\mathbf{c}^{(k)}},$$

where X denotes the minimal degree terms in $\mathscr{D}^k(L_{\mathbf{a}})$. Now apply corollary 3.4.2, we conclude that $X = L_{\mathbf{a}^{(k)}}$.

— Now if $\mathbf{a}$ fails to satisfy the hypothesis $H_k(\mathbf{a})$, $\mathbf{a} \notin S(\mathbf{a})_k$, by proposition 3.4.1 and induction, we know that there exists $\mathbf{b} \in S(\mathbf{a})_k$, such that $\mathbf{a}^{(k)} = \mathbf{b}^{(k)}$ and $\mathscr{D}^k(L_{\mathbf{b}})$ contains $L_{\mathbf{a}^{(k)}}$ as a subquotient with multiplicity one.
Now by the lemma 1.3.5, $\pi(\mathbf{a})-\pi(\mathbf{b})$ is a positive sum of irreducible representations which contain $L_{\mathbf{a}}$: by the positivity of partial derivative, we know that we obtain a positive sum of irreducible representations after applying $\mathscr{D}^k$. Now

$$\mathscr{D}^k(\pi(\mathbf{a}) - \pi(\mathbf{b})) = \pi(\mathbf{a}^{(k)}) - \pi(\mathbf{b}^{(k)}) + \text{ higher degree terms}$$

contains only terms of degree $> \deg(\mathbf{a}^{(k)})$, so does $\mathscr{D}^k(L_{\mathbf{a}})$.

This finishes our induction.

□

Corollary 3.5.4. *Let* $\mathbf{a}$ *be a multisegment such that* $\varphi_{e(\mathbf{a})}(k) = 1$. *Then*

— *If* $\mathbf{a} \in S(\mathbf{a})_k$, *then* $\mathscr{D}^k(L_{\mathbf{a}}) = L_{\mathbf{a}} + L_{\mathbf{a}^{(k)}}$.

— *If* $\mathbf{a} \notin S(\mathbf{a})_k$, *then* $\mathscr{D}^k(L_{\mathbf{a}}) = L_{\mathbf{a}}$.

Démonstration. First of all, we observe that the highest degree term in $\mathscr{D}^k(L_{\mathbf{a}})$ is given by $L_{\mathbf{a}}$. In fact, we have

$$\mathscr{D}^k(\pi(\mathbf{a})) = \mathscr{D}^k(L_{\mathbf{a}}) + \sum_{\mathbf{b}<\mathbf{a}} m(\mathbf{b},\mathbf{a})\mathscr{D}^k(L_{\mathbf{b}}),$$

meanwhile we have

$$\mathscr{D}^k(\pi(\mathbf{a})) = \pi(\mathbf{a}) + \text{ lower terms.}$$

By induction on $\ell(\mathbf{a})$ we conclude that the highest degree terms in $\mathscr{D}^k(L_{\mathbf{a}})$ is $L_{\mathbf{a}}$.
If $\mathbf{a} \in S(\mathbf{a})_k$, then proposition 3.5.1 implies that the minimal degree term of $\mathscr{D}^k(L_{\mathbf{a}})$, but since $\deg(\mathbf{a}^{(k)}) = \deg(\mathbf{a}) - 1$, therefore we must have

$$\mathscr{D}^k(L_{\mathbf{a}}) = L_{\mathbf{a}} + L_{\mathbf{a}^{(k)}}.$$

On the contrary, if $\mathbf{a} \notin S(\mathbf{a})_k$, then by (ii) of the proposition 3.5.1, we know that all irreducible representations appearing in $\mathscr{D}^k(L_{\mathbf{a}})$ are of degree $> \deg(\mathbf{a}^{(k)}) = \deg(\mathbf{a}) - 1$, which implies

$$\mathscr{D}^k(L_{\mathbf{a}}) = L_{\mathbf{a}}.$$

$\square$

Chapitre 4

Reduction to symmetric cases

- In the first paragraph of this chapter, we generalize the construction of chapter 3 by iterating the truncation functor to obtain for $\mathbf{c}_1, \mathbf{c}_2$ two multisegments, the truncation ${}^{(\mathbf{c}_1)}\mathbf{b}^{(\mathbf{c}_2)}$ of a multisegment $\mathbf{b}$.
- Then we give an algorithm to, starting from two multisegments $\mathbf{a}$ and $\mathbf{b} \in S(\mathbf{a})$, construct two symmetric multisegments $\mathbf{a}^{\mathrm{sym}}$ and $\mathbf{b}^{\mathrm{sym}} \in S(\mathbf{a}^{\mathrm{sym}})$ such that we have the following equality

$$m(\mathbf{b}, \mathbf{a}) = m(\mathbf{b}^{\mathrm{sym}}, \mathbf{a}^{\mathrm{sym}}).$$

- Then we study some examples and we show how our algorithm works for finding the coefficient $m(\mathbf{b}, \mathbf{a})$.
- Finally, in the last paragraph, we give a proof of the Zelevinsky's conjecture stated in the introduction.

4.1 Minimal Degree Terms

The goal of this section is to define the set $S(\mathbf{a})_{\mathbf{d}} \subseteq S(\mathbf{a})$ and describe some of its properties.

Definition 4.1.1. *Let $(k_1, \cdots, k_r)$ be a sequence of integers. We define*

$$\mathbf{a}^{(k_1, \cdots, k_r)} = (((\mathbf{a}^{(k_1)}) \cdots)^{(k_r)}).$$

Notation 4.1.2. *Let $\Delta = [k, \ell]$, we denote*

$$\mathbf{a}^{(\Delta)} = \mathbf{a}^{(k, \cdots, \ell)}.$$

More generally, for $\mathbf{d} = \{\Delta_1 \preceq \cdots \preceq \Delta_r\}$, let

$$\mathbf{a}^{(\mathbf{d})} = (\cdots((\mathbf{a}^{(\Delta_r)})^{(\Delta_{r-1})}) \cdots)^{(\Delta_1)}.$$

Definition 4.1.3. *Let* $(k_1, \cdots, k_r)$ *be a sequence of integers , then we define*

$$S(\mathbf{a})_{k_1,\cdots,k_r} = \{\mathbf{c} \in S(\mathbf{a}) : \mathbf{c}^{(k_1,\cdots,k_i)} \in S(\mathbf{a}^{(k_1,\cdots,k_i)})_{k_{i+1}}, \text{ for } i = 1, \cdots, r\}.$$

and

$$\psi_{k_1,\cdots,k_r} : S(\mathbf{a})_{k_1,\cdots,k_r} \to S(\mathbf{a}^{(k_1,\cdots,k_r)}),$$

sending $\mathbf{c}$ *to* $\mathbf{c}^{(k_1,\cdots,k_r)}$.

Notation 4.1.4. *Let* $\mathbf{d} = \{\Delta_1 \preceq \cdots \preceq \Delta_r\}$ *such that* $\Delta_i = [k_i, \ell_i]$. *We denote*

$$S(\mathbf{a})_{\mathbf{d}} := S(\mathbf{a})_{k_r,\cdots,\ell_r,k_{r-1},\cdots,k_1,\cdots,\ell_1}$$

and

$$\psi_{\mathbf{d}} := \psi_{k_r,\cdots,\ell_r,k_{r-1},\cdots,k_1,\cdots,\ell_1}.$$

Proposition 4.1.5. *Let* $(k_1, \cdots, k_r)$ *be a sequence of integers . Then the set* $S(\mathbf{a})_{k_1,\cdots,k_r}$ *is non-empty. In fact, we have a bijective morphism*

$$\psi_{k_1,\cdots,k_r} : S(\mathbf{a})_{k_1,\cdots,k_r} \to S(\mathbf{a}^{(k_1,\cdots,k_r)}).$$

Moreover,

(1) *For* $\mathbf{c} \in S(\mathbf{a})_{k_1,\cdots,k_r}$, *we have*

$$m(\mathbf{c}, \mathbf{a}) = m(\mathbf{c}^{(k_1,\cdots,k_r)}, \mathbf{a}^{(k_1,\cdots,k_r)}).$$

(2) *For* $\mathbf{b}, \mathbf{c} \in S(\mathbf{a})_{k_1,\cdots,k_r}$, *then* $\mathbf{b} > \mathbf{c}$ *if and only if* $\mathbf{b}^{(k_1,\cdots,k_r)} > \mathbf{c}^{(k_1,\cdots,k_r)}$.

(3) *We have*

$$\pi(\mathbf{a}^{(k_1,\cdots,k_r)}) = \sum_{\mathbf{c} \in S(\mathbf{a})_{k_1,\cdots,k_r}} m(\mathbf{c}, \mathbf{a}) L_{\mathbf{c}^{(k_1,\cdots,k_r)}}.$$

(4) *Let* $\mathbf{b} \in S(\mathbf{a})_{k_1,\cdots,k_r}$ *and* $\mathbf{b}^{(k_1,\cdots,k_r)} = \mathbf{a}^{(k_1,\cdots,k_r)}$, *then*

$$S(\mathbf{a})_{k_1,\cdots,k_r} = S(\mathbf{b})_{k_1,\cdots,k_r}.$$

Démonstration. Injectivity follows from the fact

$$\psi_{k_1,\cdots,k_r} = \psi_{k_r} \circ \psi_{k_{r-1}} \circ \cdots \circ \psi_{k_1}.$$

For surjectivity, let $\mathbf{d} \in S(\mathbf{a}^{(k_1,\cdots,k_r)})$, we construct $\mathbf{b}$ inductively such that $\psi_{k_1,\cdots,k_r}(\mathbf{b}) = \mathbf{d}$. Let $\mathbf{a}_r = \mathbf{d}$, assume that we already construct $\mathbf{a}_i \in S(\mathbf{a}^{(k_1,\cdots,k_i)})_{k_{i+1}}$ satisfying that

$$\mathbf{a}_i^{(k_{i+1},\cdots,k_j)} \in S(\mathbf{a}^{(k_1,\cdots,k_j)}))_{k_{j+1}}$$

for all $i < j \leq r$ and $\mathbf{a}_i^{(k_{i+1},\cdots,k_r)} = \mathbf{d}$.
Note that by the bijectivity of the morphism

$$\psi_{k_i} : S(\mathbf{a}^{(k_1,\cdots,k_{i-1})})_{k_i} \to S(\mathbf{a}^{(k_1,\cdots,k_i)}),$$

there exists a unique $\mathbf{a}_{i-1} \in S(\mathbf{a}^{(k_1,\cdots,k_{i-1})})_{k_i}$, such that

$$\mathbf{a}_{i-1}^{(k_i)} = \mathbf{a}_i.$$

Finally, take $\mathbf{b} = \mathbf{a}_0 \in S(\mathbf{a})_{k_1,\cdots,k_r}$. We show (1) by induction on r. The case for $r = 1$ is by proposition 3.4.1. For general r, by induction

$$m(\mathbf{c},\mathbf{a}) = m(\mathbf{c}^{(k_1,\cdots,k_{r-1})},\mathbf{a}^{(k_1,\cdots,k_{r-1})}),$$

and now apply the case $r = 1$ to the pair $\mathbf{c}^{(k_1,\cdots,k_{r-1})}, \mathbf{a}^{(k_1,\cdots,k_{r-1})}$ gives

$$m(\mathbf{c}^{(k_1,\cdots,k_{r-1})},\mathbf{a}^{(k_1,\cdots,k_{r-1})}) = m(\mathbf{c}^{(k_1,\cdots,k_r)},\mathbf{a}^{(k_1,\cdots,k_r)}).$$

Hence

$$m(\mathbf{c},\mathbf{a}) = m(\mathbf{c}^{(k_1,\cdots,k_r)},\mathbf{a}^{(k_1,\cdots,k_r)}).$$

Also, to show (2), it suffices to apply successively the proposition 3.4.1. And (3) follows from the bijectivity of $\psi_{k_1,\cdots,k_r}$ and (1). As for (4), we know by definition,

$$S(\mathbf{a})_{k_1,\cdots,k_r} \supseteq S(\mathbf{b})_{k_1,\cdots,k_r}.$$

We know that any for $\mathbf{c} \in S(\mathbf{a})_{k_1,\cdots,k_r}$, we have $\mathbf{c}^{(k_1,\cdots,k_r)} \leq \mathbf{b}^{(k_1,\cdots,k_r)}$, by (2), this implies that $\mathbf{c} \leq \mathbf{b}$. Hence we are done. □

Similarly, we have

Definition 4.1.6. *Let $(k_1, \cdots, k_r)$ be a sequence of integers, then we define*

$${}_{k_r,\cdots,k_1}S(\mathbf{a}) = \{\mathbf{c} \in S(\mathbf{a}) : {}^{(k_i,\cdots,k_1)}\mathbf{c} \in {}_{k_{i+1}}S({}^{(k_i,\cdots,k_1)}\mathbf{a}),\ \textit{for } i = 1,\cdots,r\}.$$

and

$${}_{k_r,\cdots,k_1}\psi : {}_{k_r,\cdots,k_1}S(\mathbf{a}) \to S({}^{(k_r,\cdots,k_1)}\mathbf{a}),$$

sending $\mathbf{c}$ *to* ${}^{(k_r,\cdots,k_1)}\mathbf{c}$.

Notation 4.1.7. *Let $\mathbf{d} = \{\Delta_1, \cdots, \Delta_r\}$ such that $\Delta_i = [k_i, \ell_i]$ with $k_1 \leq \cdots \leq k_r$ We denote*

$${}_{\mathbf{d}}S(\mathbf{a}) :=_{k_r,\cdots,\ell_r,k_{r-1},\cdots,k_1,\cdots,\ell_1} S(\mathbf{a}),$$

and

$${}_{\mathbf{d}}\psi :=_{k_r,\cdots,\ell_r,k_{r-1},\cdots,k_1,\cdots,\ell_1} \psi.$$

Remark: Let k_1, k_2 be two integers. In general, we do not have

$$_{k_2}(S(\mathbf{a})_{k_1}) = (_{k_2}S(\mathbf{a}))_{k_1}.$$

For example, let $k_1 = k_2 = 1$, $\mathbf{a} = \{[1], [2]\}$, then

$$_{k_2}(S(\mathbf{a})_{k_1}) = \{\mathbf{a}\},\ (_{k_2}S(\mathbf{a}))_{k_1} = \{[1, 2]\}.$$

Notation 4.1.8. *We write for multisegments $\mathbf{d}_1, \mathbf{d}_2, \mathbf{a}$,*

$$_{\mathbf{d}_2}S(\mathbf{a})_{\mathbf{d}_1} := (_{\mathbf{d}_2}S(\mathbf{a}))_{\mathbf{d}_1},\ S(\mathbf{a})_{\mathbf{d}_1,\mathbf{d}_2} := (S(\mathbf{a})_{\mathbf{d}_1})_{\mathbf{d}_2}.$$

and

$$_{\mathbf{d}_2}\psi_{\mathbf{d}_1} := (_{\mathbf{d}_2}\psi)_{\mathbf{d}_1},\ \psi_{\mathbf{d}_1,\mathbf{d}_2} := (\psi_{\mathbf{d}_1})_{\mathbf{d}_2}$$

And for $\mathbf{b} \in S(\mathbf{a})$,

$$^{(\mathbf{d}_2)}\mathbf{b}^{(\mathbf{d}_1)} := (^{\mathbf{d}_2}\mathbf{b})^{(\mathbf{d}_1)},\ \mathbf{b}^{(\mathbf{d}_1,\mathbf{d}_2)} := (\mathbf{b}^{(\mathbf{d}_1)})^{(\mathbf{d}_2)}.$$

4.2 Reduction to symmetric case

Now we return to the main question, i.e., the calculation of the coefficient $m(\mathbf{c}, \mathbf{a})$ for $\mathbf{c} \in S(\mathbf{a})$. Before we go into the details, we describe our strategies :

(i) Find a symmetric multisegment, denoted by $\mathbf{a}^{\mathrm{sym}}$, such that $L_{\mathbf{a}}$ is the minimal degree term in some partial derivative of $L_{\mathbf{a}^{\mathrm{sym}}}$.

(ii) For $\mathbf{c} \in S(\mathbf{a})$, find $\mathbf{c}^{\mathrm{sym}} \in S(\mathbf{a}^{\mathrm{sym}})$, such that we have $m(\mathbf{c}, \mathbf{a}) = m(\mathbf{c}^{\mathrm{sym}}, \mathbf{a}^{\mathrm{sym}})$.

Proposition 4.2.1. *Let $\mathbf{a}$ be any multisegment, then there exists an ordinary multisegment $\mathbf{b}$, and two multisegments $\mathbf{c}_i, i = 1, 2$ such that*

$$\mathbf{b} \in {}_{\mathbf{c}_2}S(\mathbf{b})_{\mathbf{c}_1},\ \mathbf{a} = {}^{(\mathbf{c}_2)}\mathbf{b}^{(\mathbf{c}_1)}$$

Démonstration. Let $\mathbf{a} = \{\Delta_1, \cdots, \Delta_r\}$ be such that

$$\Delta_1 \preceq \cdots \preceq \Delta_r,$$

and

$$e(\Delta_1) \leq \cdots < e(\Delta_j) = \cdots = e(\Delta_i) < e(\Delta_{i+1}) \leq \cdots,$$

such that Δ_j is the smallest multisegment in $\mathbf{a}$ such that $e(\Delta_j)$ appears in $e(\mathbf{a})$ with multiplicity greater than 1. Let $\Delta^1 = [e(\Delta_i) + 1, \ell]$ be a segment, where ℓ is the maximal integer such that for any m such that $e(\Delta_i) \leq m \leq \ell - 1$, there is a segment in $\mathbf{a}$ which ends in m. Let $\mathbf{a}_1$ be the multisegment obtained by replacing Δ_i by Δ_i^+, and all $\Delta \in \mathbf{a}$ such that $e(\Delta) \in (e(\Delta_i), \ell]$ by Δ^+. Now we continue the previous construction with $\mathbf{a}_1$ to get $\mathbf{a}_2 \cdots$, until we get a multisegment $\mathbf{a}_{r_1}$ such that $e(\mathbf{a}_{r_1})$ contains no segment with multiplicity greater than 1. Let

$$c_1 = \{\Delta^1, \Delta^2, \cdots, \Delta^{r_1}\}.$$

Note that by construction, we have

$$\Delta^1 \prec \Delta^2 \prec \cdots \prec \Delta^{r_1}.$$

And we show that $\mathbf{a}_{r_1} \in S(\mathbf{a}_{r_1})_{\mathbf{c}_1}$. Note that

$$\mathbf{a}_i = \mathbf{a}_{r_1}^{(\Delta^{r_1}, \cdots, \Delta^{i+1})},$$

by induction on r_1, we can assume that $\mathbf{a}_1 \in S(\mathbf{a}_{r_1})_{\Delta^{r_1}, \cdots, \Delta^2}$ and show that $\mathbf{a} \in S(\mathbf{a}_1)_{\Delta^1}$. We observe that in $\mathbf{a}_1$, by construction, with the notations above, $\Delta_j, \cdots, \Delta_{i-1}$ are the only segments in $\mathbf{a}_1$ that ends in $e(\Delta_i)$, and Δ_i^+ is the only segment in $\mathbf{a}_1$ that ends in $e(\Delta_i) + 1$. Hence we conclude that $\mathbf{a}_1 \in S(\mathbf{a}_1)_{e(\Delta_i)+1}$. And for $e(\Delta_i) + 1 < m \leq \ell$, we know that $\mathbf{a}_1^{(e(\Delta_1)+1, \cdots, m-1)}$ does not contain a segment which ends in $m - 1$, hence $\mathbf{a}_1^{(e(\Delta_1)+1, \cdots, m-1)} \in S(\mathbf{a}_1^{(e(\Delta_1)+1, \cdots, m-1)})_m$. We are done by putting $m = \ell$.
Now same construction can be applied to show that there exists a multisegment $\mathbf{a}_{r_2}$ such that $b(\mathbf{a}_{r_2})$ contains no segment with multiplicity greater than 1, and

$$c_2 = \{{}^1\Delta, \cdots, {}^{r_2}\Delta\},$$

such that

$$\mathbf{a}_{r_2} \in {}^{\mathbf{c}_2}S(\mathbf{a}_2),\ \mathbf{a}_{r_1} = {}^{(\mathbf{c}_2)}\mathbf{a}_{r_2}$$

as minimal degree component.
Note that in this way we construct an ordinary multisegment $\mathbf{b} = \mathbf{a}_{r_2}$,

$$\mathbf{b} \in {}_{\mathbf{c}_2}S(\mathbf{b})_{\mathbf{c}_1},\ \mathbf{a} = {}^{(\mathbf{c}_2)}\mathbf{b}^{(\mathbf{c}_1)}$$

□

To finish our strategy (i), we are reduced to consider the case of ordinary multisegments.

Proposition 4.2.2. *Let* $\mathbf{b}$ *be an ordinary multisegment, then there exists a symmetric multisegment* $\mathbf{b}^{\mathrm{sym}}$*, and a multisegment* $\mathbf{c}$ *such that such that*

$$\mathbf{b}^{\mathrm{sym}} \in S(\mathbf{b}^{\mathrm{sym}})_{\mathbf{c}},\ \mathbf{b} = \mathbf{b}^{\mathrm{sym},\ (\mathbf{c})}.$$

Démonstration. In general $\mathbf{b}$ is not symmetric, i.e, we do not have $\min\{e(\Delta) : \Delta \in \mathbf{b}\} \geq \max\{b(\Delta) : \Delta \in \mathbf{b}\}$. Let

$$\mathbf{b} = \{\Delta_1, \cdots, \Delta_r\}, \quad b(\Delta_1) > \cdots > b(\Delta_r).$$

so that

$$b(\Delta_1) = \max\{b(\Delta_i) : i = 1, \cdots, r\}.$$

If $\mathbf{b}$ is not symmetric, let $\Delta^1 = [\ell, b(\Delta_1) - 1]$ with ℓ maximal satisfying that for any m such that $\ell - 1 \leq m \leq b(\Delta_1)$, there is a segment in $\mathbf{b}$ starting in m. We construct $\mathbf{b}_1$ by replacing every segment Δ in $\mathbf{b}$ ending in Δ^1 by $^+\Delta$. Repeat this construction with $\mathbf{b}_1$ to get $\mathbf{b}_2 \cdots$, until we get $\mathbf{b}^{\text{sym}} = \mathbf{b}_s$, which is symmetric. Let $\mathbf{c} = \{\Delta^1, \cdots, \Delta^s\}$, then as before, we have

$$\mathbf{b}^{\text{sym}} \in {}_{\mathbf{c}}S(\mathbf{b}^{\text{sym}}), \ \mathbf{b} = {}^{(\mathbf{c})}(\mathbf{b}^{\text{sym}}).$$

□

As a corollary, we know that

Corollary 4.2.3. *For any multisegment* $\mathbf{a}$*, we can find a symmetric multisegment* $\mathbf{a}^{\text{sym}}$ *and three multisegments* $\mathbf{c}_i, i = 1, 2, 3$*, such that*

$$\mathbf{a}^{\text{sym}} \in {}_{\mathbf{c}_2,\mathbf{c}_3}S(\mathbf{a}^{\text{sym}})_{\mathbf{c}_1}, \ \mathbf{a} = {}^{(\mathbf{c}_2,\mathbf{c}_3)}\mathbf{a}^{\text{sym},(\mathbf{c}_1)}.$$

Now applying proposition 4.1.5

Proposition 4.2.4. *The morphism*

$${}_{\mathbf{c}_2,\mathbf{c}_3}\psi_{\mathbf{c}_1} : {}_{\mathbf{c}_2,\mathbf{c}_3}S(\mathbf{a}^{\text{sym}})_{\mathbf{c}_1} \to S(\mathbf{a})$$

is bijective, and for $\mathbf{b} \in S(\mathbf{a})$*, there exists a unique* $\mathbf{b}^{\text{sym}} \in S(\mathbf{a}^{\text{sym}})$ *such that*

$$m(\mathbf{b}, \mathbf{a}) = m(\mathbf{b}^{\text{sym}}, \mathbf{a}^{\text{sym}}).$$

4.3 Examples

In this section we shall give some examples to illustrate the idea of reduction to symmetric case.
We first take $\mathbf{a} = \{[1], [2], [2], [3]\}$ to show how to reduce a general multisegment to an ordinary multisegment. The procedure is showed in the following picture.
Here we have $\mathbf{a}_2 = \{[0, 1], [1, 3], [2], [3, 4]\}$, such that

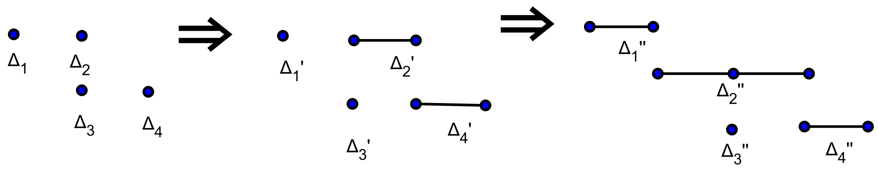

FIGURE 4.1 –

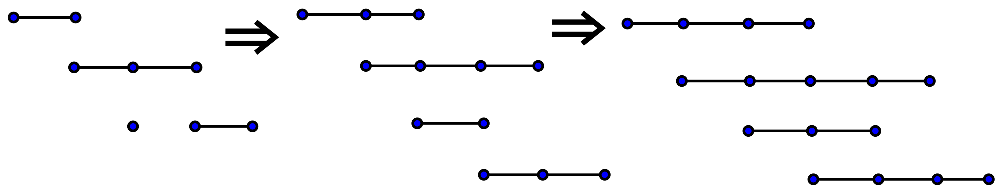

FIGURE 4.2 –

$$\mathbf{a}_2 \in {}_{[0,1]}S(\mathbf{a}_2)_{[3,4]}, \ \ \mathbf{a} = {}^{([0,1])}\mathbf{a}_2^{([3,4])}$$

Next, we reduce the ordinary multisegment $\mathbf{a}_2$ to a multisegment $\mathbf{a}^{\mathrm{sym}}$, as is showed in the following picture.
Here,we have

$$\mathbf{a}^{\mathrm{sym}} = \{[0,3],[1,5],[2,4],[3,6]\} = \Phi(w)$$

where $w = \sigma_2 \in S_4$.
Now we take $\mathbf{b} = \{[1,2],[2,3]\}$, we want to find $\mathbf{b}^{\mathrm{sym}} \in S(\mathbf{a}^{\mathrm{sym}})$ such that $m(\mathbf{b},\mathbf{a}) = m(\mathbf{b}^{\mathrm{sym}},\mathbf{a}^{\mathrm{sym}})$. Actually, following the procedure in Figure 2 above, we have

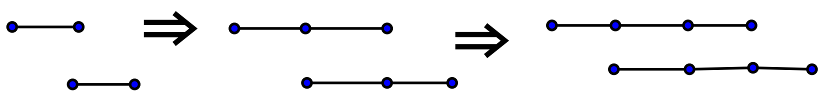

FIGURE 4.3 –

Here we get $\mathbf{b}_2 = \{[0,3],[1],[2,4]\}$. Again, follow the procedure in Figure 3 above gives
Hence we get

$$\mathbf{b}^{\mathrm{sym}} = \{[0,5],[1,3],[2,6],[3,4]\} = \Phi(v)$$

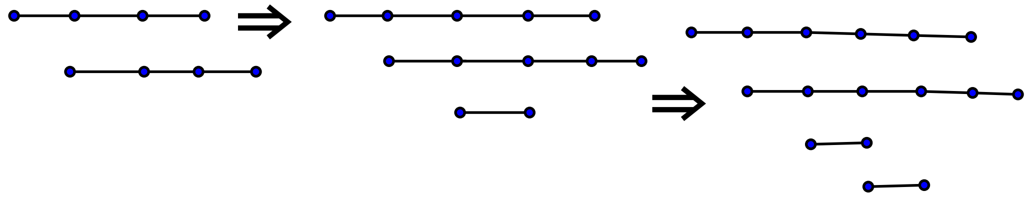

FIGURE 4.4 –

with $v = (13)(24) \in S_4$. From [34] section 11.3, we know that $m(\mathbf{b}, \mathbf{a}) = 2$, hence we get $m(\mathbf{b}^{\text{sym}}, \mathbf{a}^{\text{sym}}) = 2$.
Remark: We showed in section 2 that

$$m(\mathbf{b}^{\text{sym}}, \mathbf{a}^{\text{sym}}) = P_{v,w}(1),$$

where $P_{v,w}(q)$ is the Kazhdan Lusztig polynomial associated to v, w. One knows that $P_{v,w}(q) = 1 + q$, hence $P_{v,w}(1) = 2$.
As we have seen, to each multisegment, we have (at least) two different ways to attach a Kazhdan Lusztig polynomial :
(1)To use the Zelevinsky construction as described in section 4.2.
(2)To first construct an associated symmetric multisegment, and then attach the corresponding Kazhdan Lusztig polynomial.

Remark: In general, for $\mathbf{a} > \mathbf{b}$, (1) gives a polynomial $P^Z_{\mathbf{a},\mathbf{b}}$ which is a Kazhdan Lusztig polynomial for the symmetric group $S_{\deg(\mathbf{a})}$. And (2) gives a polynomial $P^S_{\mathbf{a},\mathbf{b}}$, which is a KL polynomial for a symmetric group S_n with $n \leq \deg(\mathbf{a})$. It may happen that $n = \deg(\mathbf{a})$. By corollary 3.3.18, we always have $P^Z_{\mathbf{a},\mathbf{b}} = P^S_{\mathbf{a},\mathbf{b}}$.

Example 4.3.1. *Consider* $\mathbf{a} = \{1, 2, 2, 3\}, \mathbf{b} = \{[1, 2], [3, 4]\}$*, then by [35] section 3.4, we know that* $P^Z_{\mathbf{a},\mathbf{b}} = 1 + q$*. And the symmetrization of* $\mathbf{a}$ *and* $\mathbf{b}$ *are given by*

$$\mathbf{a}^{\text{sym}} = \Psi((2, 3)), \quad \mathbf{b}^{\text{sym}} = \Psi((1, 3)(2, 4)).$$

Hence $P^S_{\mathbf{a},\mathbf{b}} = P_{(2,3),(1,3)(2,4)} = 1 + q$*, which is the Kazhdan Lusztig polynomial for the pair* $((2, 3), (1, 3)(2, 4))$ *in* S_4

4.4 Proof of the Zelevinsky's conjecture

Definition 4.4.1. *The relation type between 2 segments* $\{\Delta, \Delta'\}$ *is one of the following*

— *Δ cover Δ' if $\Delta \supseteq \Delta'$;*
— *linked but not juxtaposed if Δ does not cover Δ' and $\Delta \cup \Delta'$ is a segment but $\Delta \cap \Delta' \neq \emptyset$;*
— *juxtaposed if $\Delta \cup \Delta'$ is a segment but $\Delta \cap \Delta' = \emptyset$;*
— *unrelated if $\Delta \cap \Delta' = \emptyset$ and Δ, Δ' are not linked.*

Definition 4.4.2. *Two multisegments*

$$\mathbf{a} = \{\Delta_1, \cdots, \Delta_r\} \qquad \textit{and} \qquad \mathbf{a}' = \{\Delta'_1, \cdots, \Delta'_{r'}\}$$

have the same relation type if
— *$r = r'$;*
— *there exists a bijection*

$$\xi : \mathbf{a} \to \mathbf{a}'$$

of multisets which preserves the partial order $\preceq$ and relation type of segments and induces bijection of multisets

$$e(\xi) : e(\mathbf{a}) \to e(\mathbf{a}'), \quad b(\xi) : b(\mathbf{a}) \to b(\mathbf{a}').$$

satisfying

$$e(\xi)(e(\Delta)) = e(\xi(\Delta)), \quad b(\xi)(b(\Delta)) = b(\xi(\Delta)).$$

Lemma 4.4.3. *Let $\mathbf{a}$ and $\mathbf{a}'$ be of the same relation type induced by ξ. Let $\{\Delta_1 \preceq \Delta_2\}$ be linked in $\mathbf{a}$. Denote by $\mathbf{a}_1$($\mathbf{a}'_1$, resp.) the multisegment obtained by applying the elementary operation to $\{\Delta_1, \Delta_2\}$($\{\xi(\Delta_1), \xi(\Delta_2)\}$, resp.). Then $\mathbf{a}_1$ and $\mathbf{a}'_1$ also have the same relation type.*

Démonstration. We define a bijection

$$\xi_1 : \mathbf{a}_1 \to \mathbf{a}'_1$$

by

$$\xi_1(\Delta_1 \cup \Delta_2) = \xi(\Delta_1) \cup \xi(\Delta_2), \quad \xi_1(\Delta_1 \cap \Delta_2) = \xi(\Delta_1) \cap \xi(\Delta_2)$$

and

$$\xi_1(\Delta) = \xi(\Delta), \qquad \text{for all } \Delta \in \mathbf{a} \setminus \{\Delta_1, \Delta_2\}.$$

It induces a bijection between the end multisets $e(\mathbf{a}_1)$ and $e(\mathbf{a}'_1)$ as well as the beginning multisets $b(\mathbf{a}_1)$ and $b(\mathbf{a}'_1)$. Also the morphism ξ preserves the partial order follows from the fact that for $x, y \in e(\mathbf{a})$ such that $x \leq y$, then $e(\xi_1)(x) = e(\xi)(x) \leq e(\xi_1)(y) = e(\xi)(y)$(The same fact holds for $b(\xi_1)$). Finally, it remains to show that ξ_1 respects the relation type. Let $\Delta \preceq \Delta'$ be two segments in $\mathbf{a}_1$, if non of them is contained in $\{\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2\}$, then $\xi_1(\Delta) = \xi(\Delta)$ and $\xi_1(\Delta') = \xi(\Delta')$ and they are in the same relation type as $\{\Delta, \Delta'\}$ by assumption. For simplicity, we only discuss the case where $\Delta = \Delta_1 \cup \Delta_2$ but Δ' is not contained in $\{\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2\}$, other cases are similar.

— If Δ' cover Δ, then Δ cover Δ_1 and Δ_2, hence $\xi_1(\Delta) = \xi(\Delta)$ cover $\xi(\Delta_1)$ and $\xi(\Delta_2)$, which implies $\xi_1(\Delta')$ covers $\xi_1(\Delta)$.
— If Δ' is linked to Δ but not juxtaposed, then either Δ' covers Δ_2 and linked to Δ_1, or Δ' is linked to Δ_2 but not juxtaposed. In both cases we have $\xi(\Delta')$ is linked to $\xi(\Delta_1) \cup \xi(\Delta_2)$ and not juxtaposed.
— If Δ' is juxtaposed to Δ, then Δ' is juxtaposed to Δ_2 since $\Delta_2 \succeq \Delta_1$. Therefore $\xi(\Delta')$ is juxtaposed to $\xi(\Delta_2)$ which implies $\xi_1(\Delta')$ is juxtaposed to the segment $\xi_1(\Delta)$.
— If Δ' is unrelated to $\Delta_1 \cup \Delta_2$, then it is unrelated to both Δ_1 and Δ_2 with $\Delta_2 \preceq \Delta'$, this implies that $\xi(\Delta')$ is unrelated to $\xi(\Delta_1) \cup \xi(\Delta_2)$.

□

Remark: As every element $\mathbf{b} \in S(\mathbf{a})$ is obtained from $\mathbf{a}$ by a sequence of elementary operations, we can define an application of poset

$$\Xi : S(\mathbf{a}) \longrightarrow S(\mathbf{a}').$$

Lemma 4.4.4. *The application Ξ is well defined and bijective.*

Démonstration. We give a new definition of Ξ in the following way. For $\mathbf{b} \in S(\mathbf{a})$, we define

$$\Xi(\mathbf{b}) = \{[b(\xi)(b(\Delta)), e(\xi)(e(\Delta))] : \Delta \in \mathbf{b}\}$$

such a definition is independent of the choice of elementary operations. It remains to see that it coincides with the one using elementary operation. In fact, let $\mathbf{a}_1$ be a multisegment obtained by applying the elementary operation to the pair of segments $\{\Delta_1 \preceq \Delta_2\}$, then by our original definition of Ξ, it sends $\mathbf{a}_1$ to $\mathbf{a}_1'$ in the previous lemma. Now by the new definition, we have $\Xi(\mathbf{a}_1)$ given by

$$\{\xi(\Delta) : \Delta \in \mathbf{a}\backslash\{\Delta_1, \Delta_2\}\}\cup\{[b(\xi)(b(\Delta_1)), b(\xi)(b(\Delta_2))], [b(\xi)(b(\Delta_2)), b(\xi)(b(\Delta_1))]\}.$$

By our definition of ξ, we get

$$[b(\xi)(b(\Delta_1)), b(\xi)(b(\Delta_2))] = \xi(\Delta_1) \cup \xi(\Delta_2),$$

and

$$[b(\xi)(b(\Delta_2)), b(\xi)(b(\Delta_1))] = \xi(\Delta_1) \cap \xi(\Delta_2).$$

Hence we conclude that Ξ is well defined. Note that by our definition, ξ is invertible, which gives ξ^{-1}, and in the same way we can construct Ξ^{-1}. Now we have

$$\Xi\Xi^{-1} = \mathrm{Id}, \quad \Xi^{-1}\Xi = \mathrm{Id}$$

by our definition above using $b(\xi)$ and $e(\xi)$. This shows that Ξ is bijective.

□

Theorem 4.4.5. *For* $\mathbf{a}$ *and* $\mathbf{a}'$ *having the same relation type, then for* $\mathbf{b} \in S(\mathbf{a})$ *with* $\mathbf{b}' = \Xi(\mathbf{b})$*, we have*

$$m(\mathbf{b}, \mathbf{a}) = m(\mathbf{b}', \mathbf{a}').$$

Démonstration. First of all, we consider the case where $\mathbf{a}$ and $\mathbf{a}'$ are symmetric multisegments. Let $\mathbf{a} = \Phi(w)$ by fixing a map

$$\Phi : S_n \to S(\mathbf{a}_{\mathrm{Id}}).$$

Now since $\mathbf{a}$ and $\mathbf{a}'$ have the same relation type, we know that $\mathbf{a}' = \Phi'(w)$ for some fixe map

$$\Phi' : S_n \to S(\mathbf{a}'_{\mathrm{Id}}).$$

Finally, let $\mathbf{a} = \{\Delta_1, \cdots, \Delta_n\}$ and $\mathbf{a}' = \{\Delta'_1, \cdots, \Delta'_n\}$ such that

$$b(\Delta_1) < \cdots < b(\Delta_n), \quad \Delta'_i = \xi(\Delta_i).$$

Without loss of generality, we assume that $b(\Delta_1) = b(\Delta'_1)$. We can assume that $b(\Delta_i) = b(\Delta_{i-1}) + 1$. In fact, if $b(\Delta_i) > b(\Delta_{i-1}) + 1$, then by replacing Δ_i by ${}^+\Delta_i$, we get a new symmetric multisegment $\mathbf{a}_1$ which has the same relation type as $\mathbf{a}$. Moreover, let $\mathbf{b} \in S(\mathbf{a})$ and $\mathbf{b}_1$ be the corresponding multisegment in $S(\mathbf{a}_1)$, then

$$m(\mathbf{b}, \mathbf{a}) = m(\mathbf{b}_1, \mathbf{a}_1)$$

by proposition 3.4.1. We note that the equality

$$m(\mathbf{b}_1, \mathbf{a}_1) = m(\mathbf{b}', \mathbf{a}')$$

implies that

$$m(\mathbf{b}', \mathbf{a}') = m(\mathbf{b}, \mathbf{a}).$$

Therefore it suffices to prove the theorem for $\mathbf{a}_1$ and $\mathbf{a}'$. From now on, let $b(\Delta_i) = b(\Delta_{i-1}) + 1$ and $b(\Delta_i) = b(\Delta'_i)$. The same argument shows that we can furthermore assume that

$$e(\Delta_{w^{-1}(i)}) = e(\Delta_{w^{-1}(i-1)}) + 1, \quad e(\Delta'_{w^{-1}(i)}) = e(\Delta'_{w^{-1}(i-1)}) + 1.$$

Now if $e(\Delta_{w^{-1}(1)}) < e(\Delta'_{w^{-1}(1)})$, then consider the truncation functor $\mathbf{a}' \mapsto \mathbf{a}'^{(e(\Delta_{w^{-1}(1)})+1, \cdots, e(\Delta_{w^{-1}(1)}))}$, the latter is a symmetric multisegment having the same relation type as $\mathbf{a}'$, and

$$m(\mathbf{b}', \mathbf{a}') = m(\mathbf{b}'^{(e(\Delta_{w^{-1}(1)})+1, \cdots, e(\Delta_{w^{-1}(1)}))}, \mathbf{a}'^{(e(\Delta_{w^{-1}(1)})+1, \cdots, e(\Delta_{w^{-1}(1)}))})$$

by proposition 4.1.5. Repeat the same procedure, in finite step, we find $\mathbf{c}$, such that

$$\mathbf{a} = \mathbf{a}'^{(\mathbf{c})}$$

and

$$m(\mathbf{b}, \mathbf{a}) = m(\mathbf{b}', \mathbf{a}').$$

by proposition 4.1.5.
Remark: an interesting application of this computation is given in the corollary 4.4.7.
For general case, note that in section 4.4, we construct a symmetric multisegment $\mathbf{a}^{\mathrm{sym}}$ and three multisegments $\mathbf{c}_i, i = 1, 2, 3$ such that

$$\mathbf{a}^{\mathrm{sym}} \in {}_{\mathbf{c}_2,\mathbf{c}_3}S(\mathbf{a}^{\mathrm{sym}})_{\mathbf{c}_1}, \ \mathbf{a} = {}^{(\mathbf{c}_2,\mathbf{c}_3)}\mathbf{a}^{\mathrm{sym},(\mathbf{c}_1)}.$$

(cf. Corollary 4.2.3). The same for $\mathbf{a}'$, we have

$$\mathbf{a}'^{\mathrm{sym}} \in {}_{\mathbf{c}'_2,\mathbf{c}'_3}S(\mathbf{a}'^{\mathrm{sym}})_{\mathbf{c}'_1}, \ \mathbf{a}' = {}^{(\mathbf{c}'_2,\mathbf{c}'_3)}\mathbf{a}'^{\mathrm{sym},(\mathbf{c}'_1)}.$$

□

Lemma 4.4.6. *The two multisegment $\mathbf{a}^{\mathrm{sym}}$ and $\mathbf{a}'^{\mathrm{sym}}$ have the same relation type. And let $\Xi^{\mathrm{sym}} : S(\mathbf{a}^{\mathrm{sym}}) \to \mathbf{a}'^{\mathrm{sym}}$ be the bijection constructed above, then we have the following commutative diagram*

$$\begin{array}{ccc}
{}_{\mathbf{c}_2,\mathbf{c}_3}S(\mathbf{a}^{\mathrm{sym}})_{\mathbf{c}_1} & \xrightarrow{\Xi^{\mathrm{sym}}} & {}_{\mathbf{c}'_2,\mathbf{c}'_3}S(\mathbf{a}'^{\mathrm{sym}})_{\mathbf{c}'_1} \\
\big\downarrow {}_{\mathbf{c}_2,\mathbf{c}_3}\psi_{\mathbf{c}_1} & & \big\downarrow {}_{\mathbf{c}'_2,\mathbf{c}'_3}\psi_{\mathbf{c}'_1} \\
S(\mathbf{a}) & \xrightarrow{\quad\Xi\quad} & S(\mathbf{a}').
\end{array}$$

Admitting the lemma, we have

$$m(\mathbf{b}, \mathbf{a}) = m(\mathbf{b}^{\mathrm{sym}}, \mathbf{a}^{\mathrm{sym}}), \quad m(\mathbf{b}', \mathbf{a}') = m(\mathbf{b}'^{\mathrm{sym}}, \mathbf{a}'^{\mathrm{sym}})$$

by proposition 4.2.4. Now by what we have proved before and the above lemma, we have

$$m(\mathbf{b}^{\mathrm{sym}}, \mathbf{a}^{\mathrm{sym}}) = m(\mathbf{b}'^{\mathrm{sym}}, \mathbf{a}'^{\mathrm{sym}}),$$

which implies $m(\mathbf{b}, \mathbf{a}) = m(\mathbf{b}', \mathbf{a}')$.

Démonstration. Note that by construction we know that the number of segments in $\mathbf{a}^{\mathrm{sym}}$ is the same as that of $\mathbf{a}$. Let $\mathbf{a}^{\mathrm{sym}} = \{\Delta_1 \preceq \cdots \preceq \Delta_r\}$, then $\mathbf{a} = \{{}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_1^{(\mathbf{c}_1)} \preceq \cdots \preceq {}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_r^{(\mathbf{c}_1)}\}$. Also let $\mathbf{a}'^{\mathrm{sym}} = \{\Delta'_1 \preceq \cdots \preceq \Delta'_r\}$. We define

$$\begin{aligned} \xi^{\mathrm{sym}} : &\mathbf{a}^{\mathrm{sym}} \to \mathbf{a}'^{\mathrm{sym}} \\ &\Delta_i \mapsto \Delta'_i. \end{aligned}$$

This automatically induces bijections

$$e(\xi^{\mathrm{sym}}) : e(\mathbf{a}^{\mathrm{sym}}) \to e(\mathbf{a}'^{\mathrm{sym}}), \quad b(\xi^{\mathrm{sym}}) : b(\mathbf{a}^{\mathrm{sym}}) \to b(\mathbf{a}'^{\mathrm{sym}}),$$

since all of them are sets. Note that we definitely have

$$\xi({}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_i^{(\mathbf{c}_1)}) = {}^{(\mathbf{c}_2',\mathbf{c}_3')}\Delta_i'^{(\mathbf{c}_1')}$$

It remains to show that ξ^{sym} preserve the relation type. Let $i \leq j$. Then Δ_i and Δ_j are linked if and only if one of the following happens

— ${}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_i^{(\mathbf{c}_1)}$ and ${}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_j^{(\mathbf{c}_1)}$ are linked, juxtaposed or not ;

— ${}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_i^{(\mathbf{c}_1)}$ and ${}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_j^{(\mathbf{c}_1)}$ are unrelated.

And Δ_j covers Δ_i if and only if ${}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_j^{(\mathbf{c}_1)}$ covers ${}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_i^{(\mathbf{c}_1)}$. Since ξ preserves relation types, this shows that ξ^{sym} also preserves relation types. Hence we conclude that $\mathbf{a}^{\mathrm{sym}}$ and $\mathbf{a}'^{\mathrm{sym}}$ have same relation type. To see that the map Ξ^{sym} sends ${}_{\mathbf{c}_2,\mathbf{c}_3}S(\mathbf{a}^{\mathrm{sym}})_{\mathbf{c}_1}$ to ${}_{\mathbf{c}_2',\mathbf{c}_3'}S(\mathbf{a}'^{\mathrm{sym}})_{\mathbf{c}_1'}$, consider $\mathbf{b} \in S(\mathbf{a})$ and its related element $\mathbf{b}^{\mathrm{sym}} \in {}_{\mathbf{c}_2,\mathbf{c}_3}S(\mathbf{a}^{\mathrm{sym}})_{\mathbf{c}_1}$.

- First of all, we assume that $l(\mathbf{b}) = 1$, i.e. $\mathbf{b}$ can be obtained from $\mathbf{a}$ by applying the elementary operation to the pair $\{{}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_i^{(\mathbf{c}_1)}, {}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_j^{(\mathbf{c}_1)}\}(i < j)$. Let $\widetilde{\mathbf{b}}$ be the element in $S(\mathbf{a}^{\mathrm{sym}})$ obtained by applying the elementary operation to the pair of segments $\{\Delta_i, \Delta_j\}$ in $\mathbf{a}^{\mathrm{sym}}$. Then we have

$$\mathbf{b} = {}^{(\mathbf{c}_2,\mathbf{c}_3)}\widetilde{\mathbf{b}}^{(\mathbf{c}_1)}.$$

Let $\widetilde{\mathbf{b}}' = \Xi^{\mathrm{sym}}(\widetilde{\mathbf{b}})$. By construction, we have

$$\mathbf{b}' = \Xi(\mathbf{b}) = {}^{(\mathbf{c}_2',\mathbf{c}_3')}\widetilde{\mathbf{b}}'^{(\mathbf{c}_1')}.$$

Now consider

$$\widetilde{\mathbf{b}}_0 = \widetilde{\mathbf{b}} > \cdots > \widetilde{\mathbf{b}}_n = \mathbf{b}^{\mathrm{sym}}$$

be a maximal chain of multisegments and let $\widetilde{\mathbf{b}}_i' = \Xi^{\mathrm{sym}}(\widetilde{\mathbf{b}}_i')$, then

$$\widetilde{\mathbf{b}}_0' > \cdots > \widetilde{\mathbf{b}}_n'.$$

Let

$$\widetilde{\mathbf{b}}_i = \{\Delta_{i,1} \preceq \cdots \preceq \Delta_{i,r_i}\}, \quad \widetilde{\mathbf{b}}_i' = \{\Delta_{i,1}' \preceq \cdots \preceq \Delta_{i,r_i}'\}.$$

We prove by induction that

$$\mathbf{b}' = {}^{(\mathbf{c}_2',\mathbf{c}_3')}\widetilde{\mathbf{b}}_i'^{(\mathbf{c}_1')}.$$

We already showed the case where $i=0$. Assume that we have

$$\mathbf{b}' = {}^{(\mathbf{c}_2',\mathbf{c}_3')}\widetilde{\mathbf{b}}_j'^{(\mathbf{c}_1')}$$

for $j<i$. Suppose that $\widetilde{\mathbf{b}}_i$ is obtained from $\widetilde{\mathbf{b}}_{i-1}$ by applying the elementary operation to the pair of segments $\{\Delta_{i-1,\alpha_{i-1}} \preceq \Delta_{i-1,\beta_{i-1}}\}$. We deduce from the fact $\widetilde{\mathbf{b}}_i \geq \mathbf{b}^{\mathrm{sym}}$ that we are in one of the following situatios

— ${}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{(\mathbf{c}_1)}_{i-1,\alpha_{i-1}} = \emptyset$ or ${}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{(\mathbf{c}_1)}_{i-1,\beta_{i-1}} = \emptyset$;

— $b({}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{(\mathbf{c}_1)}_{i-1,\beta_{i-1}}) = b({}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{(\mathbf{c}_1)}_{i-1,\alpha_{i-1}})$;

— $e({}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{(\mathbf{c}_1)}_{i-1,\beta_{i-1}}) = e({}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{(\mathbf{c}_1)}_{i-1,\alpha_{i-1}})$.

According the our assumption that $\widetilde{\mathbf{b}}_i' = \Xi^{\mathrm{sym}}(\widetilde{\mathbf{b}}_i')$, we have

$$\xi({}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{(\mathbf{c}_1)}_{i-1,j}) = {}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta'^{(\mathbf{c}_1)}_{i-1,j},$$

therefore the pair $\{{}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta'^{(\mathbf{c}_1)}_{i-1,\alpha_{i-1}}, {}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta'^{(\mathbf{c}_1)}_{i-1,\beta_{i-1}}\}$ also satisfies one of the listed properties above. And this shows that $\widetilde{\mathbf{b}}_i'$ is sent to $\mathbf{b}'$ by ${}_{\mathbf{c}_2',\mathbf{c}_3'}\psi_{\mathbf{c}_1'}$. Therefore by proposition 3.3.17, we know that

$$\mathbf{b}_n' \geq \mathbf{b}'^{\mathrm{sym}}.$$

Conversely, we have

$$\Xi^{\mathrm{sym}\,-1}(\mathbf{b}'^{\mathrm{sym}}) \geq \mathbf{b}^{\mathrm{sym}}.$$

Combine the two inequalities to get

$$\Xi^{\mathrm{sym}}(\mathbf{b}^{\mathrm{sym}}) = \mathbf{b}'^{\mathrm{sym}}.$$

- The general case where $\ell(\mathbf{b})>1$, we can choose a maximal chain of multisegments

$$\mathbf{a} = \mathbf{a}_0 > \cdots > \mathbf{a}_{\ell(\mathbf{b})} = \mathbf{b}.$$

Let $\mathbf{a}_i' = \Xi(\mathbf{a}_i)$, by assumption, we can assume that for $i<\ell(\mathbf{b})$, we have

$$\Xi^{\mathrm{sym}}(\mathbf{a}_i^{\mathrm{sym}}) = \mathbf{a}_i'^{\mathrm{sym}}.$$

By considering the set $S(\mathbf{a}_{\ell(\mathbf{b})-1})$, we are reduce to the case where $\ell(\mathbf{b})=1$. Hence we are done. □

Corollary 4.4.7. *Let $\mathbf{a}_{\mathrm{Id}}$ be a symmetric multisegment associated to the identity in S_n and*

$$\Phi : S_n \to S(\mathbf{a}_{\mathrm{Id}}).$$

Then

$$m(\Phi(v),\Phi(w)) = P_{w,v}(1).$$

Démonstration. The special case where

$$\mathbf{a}_{\mathrm{Id}} = \sum_{i=1}^{n} [i, i+n-1]$$

is already treated in corollary 2.5.9. The general case can be deduced from the theorem above. □

Deuxième partie

Applications

Chapitre 5

Geometric Proof of KL Relations

For $n \geq 1$, recall that the permutation group S_n of $\{1, \cdots, n\}$ and that $S = \{\sigma_i = (i, i{+}1) : i = 1, \cdots, n{-}1\}$ is a set of generators. It is followed from [19] that the following properties characterize a unique family of polynomials $P_{x,y}(q)$ of $\mathbb{Z}[q]$ for $x, y \in S_n$

(1) $P_{x,x} = 1$ for all $x \in S_n$;

(2) if $x < y$ and $s \in S$, are such that $sy < y$, $sx > x$, then $P_{x,y} = P_{sx,y}$;

(3) if $x < y$ and $s \in S$, are such that $ys < y$, $xs > x$, then $P_{x,y} = P_{xs,y}$;

(4) if $x < y$ and $s \in S$, are such that $sy < y$, $sx < x$, and x is not comparable to sy, then $P_{x,y} = P_{sx,sy}$;

(5) if $x < y$ and $s \in S$, are such that $sy < y$, $sx < x$, and $x < sy$, then

$$P_{x,y} = P_{sx,sy} + qP_{x,sy} - \sum_{x \leq z < sy, sz < z} q^{1/2(\ell(y) - \ell(z))} \mu(z, sy) P_{x,z},$$

here $\mu(z, sy)$ is the coefficient of degree $1/2(\ell(sy) - \ell(z) - 1)$ in $P_{z,sy}$ defined to be zero if $\ell(sy) - \ell(z)$ is even).

In this chapter , we shall prove by using our results in section 3.3 that the polynomial

$$P_{x,y}(q) := q^{\frac{1}{2}(\dim(O_{\Phi(y)}) - \dim(O_{\Phi(x)}))} \sum_i q^{\frac{1}{2}i} \mathcal{H}^i(\overline{O}_{\Phi(y)})_{\Phi(x)}$$

satisfies the first 4 conditions and we give an interpretation geometric for the fifth condition which will be used in the Chapter 7.

Remark: The condition $P_{x,x} = 1$ is trivial.

The set up for through this chapter is the following. Assume that $k, k_1 \in \mathbb{N}$ such that $1 < k_1 \leq n, k = n + k_1 - 1$, and $\mathbf{a}_{\mathrm{Id}}$ be a multisegment such that we have an isomorphism

$$\Phi : S_n \to S(\mathbf{a}_{\mathrm{Id}}).$$

Note that we have $n < k \leq 2n - 1$.

5.1 Relation (2) and (3)

Since the relation (2) and (3) are symmetric to each other, we only prove (2). By [8] (1.26), the conditions

$$\sigma_{k_1-1}w > w, \ \sigma_{k_1-1}v < v.$$

are equivalent to

$$w^{-1}(k_1-1) < w^{-1}(k_1), \quad v^{-1}(k_1-1) > v^{-1}(k_1).$$

Proposition 5.1.1. *Let* $\mathbf{a} = \Phi(w), \mathbf{c} = \Phi(v) \in S(\mathbf{a})$*, such that*

$$w^{-1}(k_1-1) < w^{-1}(k_1), \quad v^{-1}(k_1-1) > v^{-1}(k_1),$$

then

$$P_{w,v}(q) = P_{\sigma_{k_1-1}w,v}(q).$$

Démonstration. Suppose that

$$\Phi(\mathrm{Id}) = \{\Delta_1 \preceq \cdots \preceq \Delta_n\}.$$

Let $\mathbf{b} = \Phi(\sigma_{k_1-1}w)$, then

$$\begin{aligned}
\mathbf{b} &= \sum_j [b(\Delta_j), e(\Delta_{\sigma_{k_1-1}w(j)})] \\
&= \sum_j [b(\Delta_{w^{-1}\sigma_{k_1-1}(j)}), e(\Delta_j)] \\
&= \sum_{j \neq k_1-1, k_1} [b(\Delta_{w^{-1}(j)}), e(\Delta_j)] + [b(\Delta_{w^{-1}(k_1-1)}), e(\Delta_{k_1})] + [b(\Delta_{w^{-1}(k_1)}), e(\Delta_{k_1-1})].
\end{aligned}$$

Note that

$$e(\Delta_{k_1-1}) = n + k_1 - 2 = k - 1, \ e(\Delta_{k_1}) = n + k_1 - 1 = k,$$

then $\mathbf{b}^{(k)} = \mathbf{a}^{(k)}$. Now applying the corollary 3.3.19 gives the result. □

5.2 Relation (4)

Let $\mathbf{a} = \Phi(\mathrm{Id})$, $\varphi = \varphi_{\mathbf{a}}$, As in section 3.3, we know that for fixed W, by proposition 3.3.16, we have an open immersion

$$\tau_W : (X_{\mathbf{a}}^k)_W \to (Z^{k,\mathbf{a}})_W \times \mathrm{Hom}(V_{\varphi,k-1}, W).$$

Definition 5.2.1. *By composing with the canonical projection*

$$(Z^{k,\mathbf{a}})_W \times \mathrm{Hom}(V_{\varphi,k-1}, W) \to (Z^{k,\mathbf{a}})_W,$$

we have a morphism

$$\phi_W : (X^k_{\mathbf{a}})_W \to (Z^{k,\mathbf{a}})_W.$$

Proposition 5.2.2. *For any* $\mathbf{b} = \Phi(w) \in S(\mathbf{a})_k$*, we have*

$$\psi_k^{-1}(\mathbf{b}^{(k)}) = \{\mathbf{b}, \mathbf{b}' = \Phi(\sigma_{k_1-1}w)\}.$$

Moreover, ϕ_W *is a fibration such that*

(1) *We have an isomorphism* $\phi_W^{-1}(O_{\mathbf{b}^{(k)}}) \simeq (\mathbb{C}^2 - \{0\}) \times \mathbb{C}^{2n-k-1}$*.*

(2) *We have* $\phi_W^{-1}(O_{\mathbf{b}^{(k)}}) \cap O_{\mathbf{b}'} \simeq \mathbb{C}^\times \times \mathbb{C}^{2n-k-1}$*.*

Démonstration. Note that we have

$$\psi_k^{-1}(\mathbf{b}^{(k)}) \subseteq S(\mathbf{b}^{(k)} + [k]),$$

we observe that

$$S(\mathbf{b}^{(k)} + [k]) \cap S(\mathbf{a}) = S(\mathbf{b}'),$$

Since $\mathbf{b}$ is minimal in $\psi_k^{-1}(\mathbf{b}^{(k)})$ (See Prop. 3.3.17), we have

$$\psi_k^{-1}(\mathbf{b}^{(k)}) = \{\mathbf{b}, \mathbf{b}'\}.$$

Then consider the restricted morphism

$$\phi_W : (O_{\mathbf{b}} \cup O_{\mathbf{b}'})_W \to O_{\mathbf{b}^{(k)}}.$$

Let $T \in O_{\mathbf{b}} \cup O_{\mathbf{b}'},\ T_0 \in \mathrm{Hom}(V_{\varphi,k-1}, W)$. Define $T' \in E_\varphi$ by

$$T'|_{V_{\varphi,k-1}} = T_0 \oplus T^{(k)}|_{V_{\varphi,k-1}},$$

$$T'|_{V_{\varphi,k}} = T^{(k)}|_{V_{\varphi,k}/W} \circ p_W,$$

$$T'|_{V_{\varphi,i}} = T^{(k)}, \text{ for } i \neq k-1, k.$$

We know that $\dim(W) = \ell_k = 1$, and for $\dim(\ker(T^{(k)}|_{V_{\varphi,k-1}})) = 2$. Now let

$$\Delta_1 < \Delta_2$$

be the two segments in $\mathbf{b}^{(k)}$ which ends in $k-1$. And we consider the following flag

$$V_0 = \ker(T^{(k)}|_{V_{\varphi,k-1}}) \supseteq V_1 = \mathrm{Im}(T^{(k)})^{\Delta_2} \cap \ker(T^{(k)}|_{V_{\varphi,k-1}}).$$

And we have $\dim(V_1) = 1$. Then for $T' \in O_{\mathbf{b}} \cup O_{\mathbf{b}'}$, it is necessary and sufficient that

$$T_0(V_0) \neq 0.$$

This amounts to give a nonzero element in $\mathrm{Hom}(V_0, W) \simeq \mathbb{C}^2$, which proves that the fiber $\phi_W^{-1}(T^{(k)}) \simeq (\mathbb{C}^2 - 0) \times \mathbb{C}^{2n-k-1}$, where the factor $\mathbb{C}^{2n-k-1}$ comes from the fact that $\dim(V_{\varphi,k-1}) = 2n - (k-1) = 2n - k + 1$. As for $T' \in O_{\mathbf{b}'}$, it is necessary and sufficient that

$$T_0(V_1) = 0, \; T_0(V_0) \neq 0,$$

which amounts to give a zero element in $\mathrm{Hom}(V_0/V_1, W) \simeq \mathbb{C}$. Hence $\phi_W^{-1}(T^{(k)}) \cap O_{\mathbf{b}'} \simeq \mathbb{C}^\times \times \mathbb{C}^{2n-k-1}$. To see that ϕ_W is a fibration, fix $V \subseteq V_{\varphi,k-1}$ such that $\dim(V) = 2$. Consider the sub-scheme of Z_W^k given by

$$U_V = \{T \in Z_W^k : \ker(T|_{V_{\varphi,k-1}}) = V\}.$$

Note that since $\dim(V_{\varphi,k-1}) = \dim(V_{\varphi,k}/W)+2$, the fact that $\dim(\ker(T|_{V_{\varphi,k-1}})) = 2$ implies that U_V is actually open in Z_W^k. In this case

$$\phi_W^{-1}(U_V) = U_V \times (\mathrm{Hom}(V, W) - \{0\}) \times \mathrm{Hom}(V_{\varphi,k-1}/V, W).$$

□

Proposition 5.2.3. *Let* $\mathbf{b} = \Phi(w), \mathbf{c} = \Phi(v) \in S(\mathbf{a})$, *such that*

$$w^{-1}(k_1 - 1) > w^{-1}(k_1), \quad v^{-1}(k_1 - 1) > v^{-1}(k_1), \quad w < v,$$

and w *is not comparable with* $\sigma_{k_1-1}v$, *then*

$$P_{w,v}(q) = P_{\sigma_{k_1-1}w, \sigma_{k_1-1}v}(q).$$

Remark: As before, our conditions are equivalent to

$$\sigma_{k_1-1}w > w, \; \sigma_{k_1-1}v > v.$$

Démonstration. Note that our assumption implies that both $\mathbf{b}$ and $\mathbf{c}$ are in $S(\mathbf{a})_k$. Let $b' = \Phi(\sigma_{k_1-1}w), \mathbf{c}' = \Phi(\sigma_{k_1-1}v)$. Then $\mathbf{b}' > \mathbf{c}'$.
For $\mathbf{b} > \mathbf{d} > \mathbf{c}$, we must have $\mathbf{d} = \Phi(\alpha)$ with $\sigma_{k_1-1}\alpha < \alpha$. In fact, $\sigma_{k_1-1}\alpha > \alpha$ would imply $\mathbf{d} > \mathbf{c}'$ by lifting property of Bruhat order (cf. [8] proposition 2.2.7). Now that we have $\mathbf{b} > \mathbf{d} > \mathbf{c}'$, contradicting to our assumption that $\mathbf{b}$ is not comparable to $\mathbf{c}'$. Let $\mathbf{d}' = \Phi(\sigma_{k_1-1}\alpha)$. Note that we create actually by this way construct a morphism between the sets

$$\rho : \{\mathbf{d} : \mathbf{b} \geq \mathbf{d} \geq \mathbf{c}\} \to \{\mathbf{d}' : \mathbf{b}' \geq \mathbf{d}' \geq \mathbf{c}'\}$$

sending $\mathbf{d}$ to $\mathbf{d}'$.

Lemma 5.2.4. *The morphism ρ is a bijection.*

Démonstration. Let $\mathbf{e}' = \Phi(\beta) \in S(\mathbf{b}')$ with $\mathbf{e}' > \mathbf{c}'$. We show that $\sigma_{k_1-1}\beta > \beta$. In fact, assume that $\sigma_{k_1-1}\beta < \beta$. Then the lifting property of Bruhat order implies $\mathbf{b} > \mathbf{e}' > \mathbf{c}'$, which is a contradiction to the fact that $\mathbf{b}$ is not comparable to $\mathbf{c}'$. Hence we have $\mathbf{e} = \Phi(\sigma_{k_1-1}\beta) < \mathbf{e}'$. Moreover, since $\sigma_{k_1-1}w < \beta < \sigma_{k_1-1}v$, and $w > \sigma_{k_1-1}w,\ v > \sigma_{k_1-1}v$, we have

$$w < \sigma_{k_1-1}\beta < v,$$

hence $\mathbf{b} > \mathbf{e} > \mathbf{c}$. This proves the surjectivity. The injectivity is clear from the definition.

□

As a corollary, we have

Lemma 5.2.5. *The restricted morphism*

$$\phi_W : X^k_{\mathbf{b}',\mathbf{c}'} \to Z^k_{b^{(k)},\mathbf{c}^{(k)}} (cf.\ Def.\ 3.3.6)$$

is a fibration with fibers isomorphic to $\mathbb{C}^\times \times \mathbb{C}^{n-k}$.

Démonstration. Since ϕ_W is a composition of τ_W, which is an open immersion, and a canonical projection, to show that it is a fibration, it suffices to show that all of its fibers are isomorphic to $\mathbb{C}^\times \times \mathbb{C}^{n-k}$. This follows from proposition 5.2.2 and the fact that for any $\mathbf{d}' \in S(\mathbf{b}')$ we have $\mathbf{d}' \notin S(\mathbf{a})_k$. □

Hence we get

$$P_{\mathbf{b}',\mathbf{c}'}(q) = P_{\mathbf{b}^{(k)},\mathbf{c}^{(k)}}(q).$$

Now we are done by applying corollary 3.3.19, i.e,

$$P_{\mathbf{b}^{(k)},\mathbf{c}^{(k)}}(q) = P_{\mathbf{b},\mathbf{c}}(q).$$

Hence we are done. □

5.3 Relation (5)

Finally, we arrive at the relation (5). We will give an interpretation of this relation in terms of the decomposition theorem (See [4]).

Definition 5.3.1. *Let*

$$\mathfrak{Z}_W = \{(T, z) \in Z^{k,\mathbf{a}}_W \times \mathrm{Hom}(V^*_{\varphi,k-1}, W^*) :\ \textit{and } z \textit{ factors through the canonical projection } V^*_{\varphi,k-1} \to \ker(T|_{V_{\varphi,k-1}})^*\}.$$

Proposition 5.3.2. *The canonical projection* $\mathfrak{Z}_W \to Z_W^{k,\mathbf{a}}$ *turns* $\mathfrak{Z}_W$ *into a vector bundle of rank 2 over* Z_W^k.

Démonstration. Note that we have $\dim(\ker(T|_{V_{\varphi,k-1}})) = 2$ and $\dim(W) = 1$. Note that by taking dual, as a scheme, $\mathcal{Z}_W$ is isomorphic to the scheme parametrize the data $(T, z) \in Z_W^k \times V_{\varphi,k-1}$ such that $z \in \ker(T|_{V_{\varphi,k-1}})$. Fix $V \subseteq V_{\varphi,k-1}$ such that $\dim(V) = 2$. Consider the sub-scheme of Z_W^k given by

$$U_V = \{T \in Z_W^k : \ker(T|_{V_{\varphi,k-1}}) = V\}.$$

As is showed in proposition 5.2.2, U_V is actually open in Z_W^k. Using the previous interpretation of $\mathfrak{Z}_W^k$, we observe that the open set U_V trivializes the projection $\mathfrak{Z}_W \to Z_W^{k,\mathbf{a}}$.

□

Definition 5.3.3. *Let* $\mathcal{Z}_W^k = Proj_{Z_W^{k,\mathbf{a}}}(\mathfrak{Z}_W)$ *be the projectivization of the vector bundle* $\mathfrak{Z}_W \to Z_W^k$. *And we shall denote the structure morphism by* $\kappa_W^k : \mathcal{Z}_W^k \to Z_W^k$.

Definition 5.3.4. *From now on, we fix a pair of non-degenerate bi-linear forms*

$$\zeta_{k-1} : V_{\varphi,k-1} \times V_{\varphi,k-1} \to \mathbb{C},\ \zeta_k : V_{\varphi,k} \times V_{\varphi,k} \to \mathbb{C}.$$

which allows us to have an identification $\eta_i : V_{\varphi,i} \simeq V_{\varphi,i}^*$, *for* $i = k-1, k$.

Remark: Here our definition $X_{\mathbf{a}}^k$ depends on the choice of V_φ. If we choose V'_φ such that

$$V'_{\varphi,i} = V_{\varphi,i},\ \text{for } i \neq k-1, k,\ V'_{\varphi,k-1} = V^*_{\varphi,k-1},\ V'_{\varphi,k} = V^*_{\varphi,k},$$

we can get $X_{\mathbf{a}}^k(V'_\varphi)$, which is isomorphic to $X_{\mathbf{a}}^k$ after we choose an isomorphism $V^*_{\varphi,k-1} \simeq V_{\varphi,k-1}$ and $V^*_{\varphi,k} \simeq V_{\varphi,k}$. This is what we do here. Note that once we fix $V^*_{\varphi,k-1} \simeq V_{\varphi,k-1}$ and $V^*_{\varphi,k} \simeq V_{\varphi,k}$. Our morphism η_i will become an inner automorphism, but in general we have $\eta_k(W) = W^* \neq W$.

Definition 5.3.5. *Let* $T \in (X_{\mathbf{a}}^k)_W$, *then we define*

$$\lambda : (X_{\mathbf{a}}^k)_W \to (X_{\mathbf{a}}^k)_{\eta_k(W)},$$

by letting

$$\lambda(T)|_{V_{\varphi,k-2}} = \eta_{k-1} \circ T|_{\varphi,k-2}$$

$$\lambda(T)|_{V_{\varphi,k-1}} = \eta_k \circ T|_{\varphi,k-1} \circ \eta_{k-1}^{-1},$$
$$\lambda(T)_{V_{\varphi,k}} = T|_{\varphi,k} \circ \eta_k^{-1},$$

and

$$\lambda(T)_{V_{\varphi,i}} = T|_{\varphi,i},\ \textit{for } i \neq k-2, k-1, k.$$

Lemma 5.3.6. *We have* $\ker(\lambda(T)|_{V_{\varphi,k}}) = \eta_k(W)$, *and*

$$\ker(\lambda(T)^{(k)}|_{V_{\varphi,k-1}}) = \eta_{k-1}(\ker(T|_{V_{\varphi,k-1}})).$$

Démonstration. The fact $\ker(\lambda(T)|_{V_{\varphi,k}}) = \eta_k(W)$ follows from definition. Note that

$$\ker(T^{(k)}|_{V_{\varphi,k-1}}) = \{v \in V_{\varphi,k-1} : T(v) \in W\} = T|^{-1}_{V_{\varphi,k-1}}(W).$$

Since

$$(\lambda(T)|_{V_{\varphi,k-1}})^{-1}(\eta_k(W)) = \eta_{k-1}(T|_{V_{\varphi,k-1}})^{-1}(W) = \eta_{k-1}(\ker(T|_{V_{\varphi,k-1}})),$$

hence

$$\ker(\lambda(T)^{(k)}|_{V_{\varphi,k-1}}) = \eta_{k-1}(\ker(T|_{V_{\varphi,k-1}})).$$

□

Definition 5.3.7. *We define*

$$\xi_W : (X^k_{\mathbf{a}})_W \to \mathcal{Z}^k_W,$$

for $T \in (X^k_{\mathbf{a}})_W$, *then*

$$\xi_W(T) = (T^{(k)}, \lambda(T)|_{\ker(\lambda(T)^{(k)}|_{V_{\varphi,k-1}})}).$$

This is well defined since

$$\lambda(T)|_{\ker(\lambda(T)^{(k)}|_{V_{\varphi,k-1}})} \in \mathrm{Hom}(\ker(\lambda(T)^{(k)}|_{V_{\varphi,k-1}}), \eta_k(W)),$$

and

$$\mathrm{Hom}(\ker(\lambda(T)^{(k)}|_{V_{\varphi,k-1}}), \eta_k(W)) \simeq \mathrm{Hom}(\ker(T^{(k)}|_{V_{\varphi,k-1}})^*, W^*),$$

and $\lambda(T)|_{\ker(\lambda(T)^{(k)}|_{V_{\varphi,k-1}})} \neq 0$.

Proposition 5.3.8. *The morphism* ξ_W *is a fibration with fibers isomorphic to* $\mathbb{C}^\times \times \mathbb{C}^{n-k}$.

Démonstration. Let $V \subseteq V_{\varphi,k-1}$ be a subspace such that $\dim(V) = 2$. Consider the open sub-scheme of $\mathfrak{Z}_W$

$$U_{1,V} = \{(T, z) \in \mathfrak{Z}_W : z \neq 0,\ \ker(T|_{V_{\varphi,k-1}}) = \eta^{-1}_{k-1}(V)\}.$$

$$U_V = \{T \in Z^k_W,\ \ker(T|_{V_{\varphi,k-1}}) = \eta^{-1}_{k-1}(V)\}.$$

Let $\widetilde{U}_{1,V}$ be the image of $U_{1,V}$ in $\mathcal{Z}^k_W$ by the canonical projection. As indicated in the proof of proposition 5.3.2, the set U_V trivialize the morphism $\mathfrak{Z}_W \to Z^k_W$, hence

$$\widetilde{U}_{1,V} \simeq U_V \times (\mathrm{Hom}(V, \eta_k(W)) - \{0\}/\mathbb{C}^\times)$$

Note that we have

$$\mathrm{Hom}(V, \eta_k(W)) \simeq \mathrm{Hom}(\eta^{-1}_{k-1}(V), W).$$

And by proposition 3.3.16 and proposition 5.2.2, we have the following isomorphism

$$\xi^{-1}_W(\widetilde{U}_{1,V}) \simeq U_V \times (\mathrm{Hom}(\eta^{-1}_{k-1}(V), W) - 0) \times \mathrm{Hom}(V_{\varphi,k-1}/\eta^{-1}_{k-1}(V), W).$$

Hence for any $(T, z) \in \mathcal{Z}^k_W$ such that $\ker(T|_{V_{\varphi,k-1}}) = \eta^{-1}_{k-1}(V)$, let $U_{2,V}$ be an open subset of $(\mathrm{Hom}(\eta^{-1}_{k-1}(V), W) - 0)/\mathbb{C}$ which trivializes the bundle

$$(\mathrm{Hom}(\eta^{-1}_{k-1}(V), W) - 0) \to (\mathrm{Hom}(\eta^{-1}_{k-1}(V), W) - 0)/\mathbb{C},$$

then the open sub-scheme $U_V \times U_{2,V}$ of $\widetilde{U}_{V,1}$ trivialize the morphism ϕ_W as a neighborhood of (T, z).

□

Definition 5.3.9. *Let* $\mathbf{b} > \mathbf{c}$ *be two elements in* $\widetilde{S}(\mathbf{a})_k$*, then we define*

$$\mathcal{Z}^k_{\mathbf{b},\mathbf{c}} = \xi_W((X^k_{\mathbf{b},\mathbf{c}})_W).$$

And

$$\mathcal{Z}^k(\mathbf{b}) = \xi_W((O_{\mathbf{b}})_W).$$

Definition 5.3.10. *Let* $w < v$ *be two elements in* S_n *such that* $\sigma_{k_1-1}v < v$*. We define*

$$R(w, v)_{k_1} = \{z : w \leq z < \sigma_{k_1-1}v, \sigma_{k_1-1}z < z\}.$$

And we denote $R(\mathrm{Id}, v)_{k_1}$ *by* $R(v)_{k_1}$*.*

Now let $\mathbf{b} = \Phi(w)$, $\mathbf{c} = \Phi(v)$ such that

$$w(k_1 - 1) > w(k_1),\ v(k_1 - 1) > v(k_1).$$

And let $\mathbf{b}' = \Phi(\sigma_{k_1-1}w), \mathbf{c}' = \Phi(\sigma_{k_1-1}v)$. We assume that

$$\mathbf{b} > \mathbf{c},\ \mathbf{b} > \mathbf{c}',$$

which coincide with the assumption in relation (5) at the beginning of this chapter.

Now we apply the decomposition theorem to the projective morphism

$$\kappa_W : \mathcal{Z}^k_{\mathbf{b}',\mathbf{c}'} \to Z^{k,\mathbf{a}}_{\mathbf{b}^{(k)},\mathbf{c}^{(k)},W}.$$

which asserts that there exists a finite collection of triples $(\mathbf{d}_i, L_i, h_i : i = 1, \cdots, r)$, with $\mathbf{d} \in S(\mathbf{a})_k$, $\mathbf{b}^{(k)} \leq \mathbf{d}_i^{(k)} < \mathbf{c}^{(k)}$, where L_i is a vector spaces over $\mathbb{C}$, such that

$$R(\kappa_W)_* IC(\mathcal{Z}^k_{\mathbf{b}',\mathbf{c}'}) = IC(Z^{k,\mathbf{a}}_{\mathbf{b}^{(k)},\mathbf{c}^{(k)},W}) \oplus_{i=1}^r IC(Z^{k,\mathbf{a}}_{\mathbf{b}^{(k)},\mathbf{d}_i^{(k)},W}, L_i)[h_i]. \quad (5.3.11)$$

Now localize at a point $x_{\mathbf{b}^{(k)}} \in O_{\mathbf{b}^{(k)}}$, we have know that the Poincaré series of $(IC(Z^{k,\mathbf{a}}_{\mathbf{b}^{(k)},\mathbf{c}^{(k)},W}))_{x_{\mathbf{b}^{(k)}}}$ is given by $P_{\mathbf{b},\mathbf{c}}(q) = P_{w,v}(q)$. And

Lemma 5.3.12. *The Poincaré series of $R\Gamma(\kappa_W^{-1}(x_{\mathbf{b}^{(k)}}), IC(\mathcal{Z}^k_{\mathbf{b}',\mathbf{c}'}))$ is given by $P_{\sigma_{k_1-1}w,\sigma_{k_1-1}v}(q) + qP_{w,\sigma_{k_1-1}v}(q)$, where Γ is the functor of taking global sections.*

Démonstration. Note that by assumption, we have

$$\kappa_W^{-1}(x_{\mathbf{b}^{(k)}}) \simeq \mathbb{P}^1$$

such that $\kappa_W^{-1}(x_{\mathbf{b}^{(k)}}) \cap \mathcal{Z}^k(\mathbf{b}') = \{pt\}$ and $\kappa_W^{-1}(x_{\mathbf{b}^{(k)}}) \cap \mathcal{Z}^k(\mathbf{b}) \simeq \mathbb{P}^1 - \{pt\}$. And we have the following exact sequence

$$0 \to IC(\mathcal{Z}^k_{\mathbf{b}',\mathbf{c}'})|_{pt} \to IC(\mathcal{Z}^k_{\mathbf{b}',\mathbf{c}'}) \to IC(\mathcal{Z}^k_{\mathbf{b}',\mathbf{c}'})|_{\kappa_W^{-1}(x_{\mathbf{b}^{(k)}})-\{pt\}} \to 0.$$

Taking the Poincaré series gives the result.

□

Now it is clear that our equation (5.3.11) will give rise to an equation of the form as that in (5) in the introduction of this chapter. Comparing the two equations, we get

Proposition 5.3.13. *The collection of triples $(\mathbf{d}_i, L_i, h_i : i = 1, \cdots, r)$ are given by*

(1) *We have $\{\mathbf{d}_i : i = 1, \cdots, r\} = \{z \in R(w, v)_k : \mu(z, \sigma_{k-1}v) \neq 0\}$.*
(2) *If $\mathbf{d}_i = \Phi(z)$, then $L_i \simeq \mathbb{C}^{\mu(z,\sigma_{k-1}v)}$.*
(3) *If $\mathbf{d}_i = \Phi(z)$, then $h_i = \ell(v) - \ell(z)$.*

Démonstration. Note that the Poincaré series of the intersection complex $IC(Z^{k,\mathbf{a}}_{\mathbf{b}^{(k)},\mathbf{d}_i^{(k)},W}, L_i)[h_i]$ is

$$\dim(L_i) q^{1/2h_i} P_{w,z_i}(q),$$

where $\mathbf{d}_i = \Phi(z_i)$. Now compare the polynomials given by 5.3.11 and the relation (5) in the beginning of this chapter, we get our results. □

Remark: Note that one should be able to deduce the above results from a general statement about the decomposition theorem. We leave this for future work.

Remark: It seems that we have done here may be generalized to give the normality of for general $\overline{O}_{\mathbf{b}}$ instead of using the results of Zelevinsky.

Chapitre 6

Classification of Poset $S(\mathbf{a})$

Let $\mathbf{a}$ be a multisegment and $S(\mathbf{a}) = \{\mathbf{b} \leq \mathbf{a}\}$ the associated poset defined in 1.3.2. The aim of this chapter is to identify the poset structure of $S(\mathbf{a})$.
In the first section we consider the case where $\mathbf{a}$ is ordinary and prove that $S(\mathbf{a})$ is an interval in $S_m \simeq B\backslash GL_m/B$, where m is the number of segments in $\mathbf{a}$. and B is the Borel subgroup.
In the general case we identify $S(\mathbf{a})$ with an interval in a parabolic quotient $S_{J_1}\backslash S_m/S_{J_2}$ of S_m given in section 2 related to the double quotient $P_{J_1}\backslash GL_m/P_{J_2}$, where P_{J_1} and P_{J_2} are parabolic subgroups.

6.1 Ordinary Case

Our goal in this section is to prove that for general ordinary multisegment $\mathbf{a}$, the set $S(\mathbf{a})$ is isomorphic to some Bruhat interval $[x, y]$ for $x, y \in S_n$, where n depends on $\mathbf{a}$.

Lemma 6.1.1. *Assume that $\mathbf{b} \in S(\mathbf{b})_k$ such that $\mathbf{b}$ and $\mathbf{b}^{(k)}$ are both ordinary. Let $\mathbf{c} \in S(\mathbf{b})_k$. Then for $\mathbf{d} \in S(\mathbf{b})$ and $\mathbf{d} > \mathbf{c}$, we have $\mathbf{d} \in S(\mathbf{b})_k$.*

Démonstration. It suffices to show that $\mathbf{d}$ satisfies the hypothesis $H_k(\mathbf{b})$. Note that $e(\mathbf{d}) = \{e(\Delta) : \Delta \in \mathbf{d}\}$ is a set because $\mathbf{d}$ is ordinary and by lemma 2.1.4 we have $e(\mathbf{d}) \subseteq e(\mathbf{b})$.
Note that $k - 1 \notin e(\mathbf{b})$ since $\mathbf{b} \in S(\mathbf{b})_k$ and $\mathbf{b}^{(k)}$ is ordinary , therefore it is not in $e(\mathbf{d})$ either. Hence to show that $\mathbf{d} \in S(\mathbf{b})_k$ hence it is equivalent to show that $k \in e(\mathbf{d})$. Since $\mathbf{c} \in S(\mathbf{d})$, we know that $e(\mathbf{c}) \subseteq e(\mathbf{d})$. Now that $k \in e(\mathbf{c})$, we conclude that $k \in e(\mathbf{d})$. We are done. □

Now let $\mathbf{b}' \in S(\mathbf{b})_k$ such that $\psi_k(\mathbf{b}') = (\mathbf{b}^{(k)})_{\min}$, then

Lemma 6.1.2. *We have*

$$S(\mathbf{b})_k = \{\mathbf{c} \in S(\mathbf{b}) : \mathbf{c} \geq \mathbf{b}'\}.$$

Démonstration. By the lemma above, we know that $S(\mathbf{b})_k \supseteq \{\mathbf{c} \in S(\mathbf{b}) : \mathbf{c} \geq \mathbf{b}'\}$. We conclude that we have equality since ψ preserve the order. □

Proposition 6.1.3. *Assume that* $\mathbf{a}$ *is ordinary. Then*

(1) *There exists a symmetric multisegment* $\mathbf{a}^{\text{sym}}$ *such that*

$$S(\mathbf{a}) \simeq S(\mathbf{a}^{\text{sym}})_{k_r,\cdots,k_1}.$$

(2) *There exists an element* $\mathbf{a}' \in S(\mathbf{a}^{\text{sym}})$ *such that*

$$S(\mathbf{a}^{\text{sym}})_{k_r,\cdots,k_1} = \{\mathbf{c} \in S(\mathbf{a}^{\text{sym}}) : \mathbf{c} \geq \mathbf{a}'\}.$$

Démonstration. Note that (1) follows directly from proposition 4.2.2 and (2) follows from applying successively lemma 6.1.2 to the sequence obtained in the lemma below. □

Lemma 6.1.4. *There exists a sequence of multisegments* $\mathbf{a}_0 = \mathbf{a}, \cdots, \mathbf{a}_r = \mathbf{a}^{\text{sym}}$ *such that* $\mathbf{a}^{\text{sym}}$ *is symmetric, with* $\mathbf{a}_i \in S(\mathbf{a}_i)_{k_i}$ *and* $\mathbf{a}_{i-1} = \mathbf{a}_i^{(k_i)}$ *for some* k_i*. Moreover,* $\mathbf{a}_i$ *is ordinary for all* $i = 1, \cdots, r$

Démonstration. Recall that in proposition 4.2.2 that every ordinary multisegment $\mathbf{a}$ can be obtained as

$$\mathbf{a} = \mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_r,$$

where $\mathbf{a}_r$ is symmetric, with $\mathbf{a}_i \in S(\mathbf{a}_i)_{k_i}$ and $\mathbf{a}_{i-1} = \mathbf{a}_i^{(k_i)}$ for some k_i.
The statement (1) follows directly from proposition 4.2.2. Note that the ordinarity of $\mathbf{a}_i$'s follows from construction.

□

6.2 The parabolic KL polynomials

For fixed $n \in \mathbb{N}$ and a pair of elements in S_n, we can associate a Kazhdan Lusztig Polynomial $P_{x,y}(q)$. We know also that the coefficients of such a polynomial are given by the dimensions of the intersection cohomology of corresponding Schubert varieties in GL_n/B.
Similar construction can give rise to a polynomial related to the Poincaré series of the intersection cohomology of the Schubert varieties in GL_n/P, where P is a standard parabolic subgroup. This has been done in Deodhar

[14] for general Coxeter System (W, S). However, as indicated in the same article, in our case where $G = GL_n$, this is not so interesting because we have a good fibration $G/P \to G/B$, so basically everything boils down to the Borel case.
In this section, for certain multisegment $\mathbf{a}$, we shall relate the set $S(\mathbf{a})$ to the orbits in GL_n/P, where the multiplicities appear to be the corresponding Parabolic Kazhdan Lusztig Polynomials.

Notation 6.2.1. *Let $S = \{\sigma_i : i = 1, \cdots, n-1\}$ be a set of generators for S_n. For $J \subseteq S$, let $S_J =< J >$ be the subgroup generated by J and $S_n^J = \{w \in S_n : ws > w$ for all $s \in J\}$.*

Proposition 6.2.2. *(cf. [8] Prop. 2.4.4) We have*

(1) $S_n = \coprod_{w \in S_n^J} wS_J$;

(2) *for $w \in S_n^J$, and $x \in S_J$, $\ell(wx) = \ell(w) + \ell(x)$.*

Remark: Now we can identify S_n^J with S_n/S_J, hence it is in bijection with the Borel orbits in GL_n/P, where P is the parabolic subgroup determined by J.

Notation 6.2.3. *Let $\mathbf{a}_{\mathrm{Id}}^J = \{\Delta_1, \cdots, \Delta_n\}$ such that*

$$e(\Delta_1) < \cdots < e(\Delta_n),$$

and

$$b(\Delta_1) \leq \cdots \leq b(\Delta_n),$$

such that

$$b(\Delta_i) = b(\Delta_{i+1}) \text{ if and only if } \sigma_i \in J$$

and $b(\Delta_n) \leq e(\Delta_1)$.

Example 6.2.4. *Let $n = 4$, and $J = \{\sigma_1, \sigma_3\}$, then we can choose*

$$\mathbf{a}_{\mathrm{Id}}^J = [1, 3] + [1, 4] + [2, 5] + [2, 6].$$

Definition 6.2.5. *We call a multisegment $\mathbf{a} \in S(\mathbf{a}_{\mathrm{Id}}^J)$ a multisegment of parabolic type J.*

Proposition 6.2.6. *For $w \in S_n^J$, let $\mathbf{a}_w^J = \sum [b(\Delta_i), e(\Delta_{w(i)})]$, then $\mathbf{a}_w^J \in S(\mathbf{a}_{\mathrm{Id}}^J)$.*

Example 6.2.7. *Let $\mathbf{a}_{\mathrm{Id}}^J$ as in example 6.2.4. For $w = \sigma_1\sigma_2$, then*

$$\mathbf{a}_w^J = [1, 4] + [1, 5] + [2, 3] + [2, 6].$$

Démonstration. We proceed by induction on $|J|$. If $|J| = 0$, we are in the symmetric case, so we are done by Proposition 2.1.8. And in general, let $J = J_1 \cup \{\sigma_{i_0}\}$ with $i_0 = \min\{i : \sigma_i \in J\}$ and $i_1 = \max\{i : b(\Delta_i) = b(\Delta_{i_0})\}$. Let $\mathbf{a}_1 = \{\Delta_1^1, \cdots, \Delta_n^1\}$, such that

$$\Delta_i^1 =^+ (\Delta_i), \text{ for } i \leq i_0,$$
$$\Delta_i^1 = \Delta_i, \text{ otherwise. (cf. Nota. 3.1.1).}$$

Example 6.2.8. *Let $\mathbf{a}_{\mathrm{Id}}^J$ be a multisegment as in example 6.2.4. Then*

$$\mathbf{a}_1 = [0,3] + [1,4] + [2,5] + [2,6].$$

Let $\mathbf{a}_{\mathrm{Id}}^{J_1} = \mathbf{a}_1$ with

$$b(\Delta_i^1) = \begin{cases} b(\Delta_i) - 1, & \text{for } i \leq i_0, \\ b(\Delta_i), & \text{for } i > i_0. \end{cases}$$

Then we have

$$\mathbf{a}_{\mathrm{Id}}^J = {}^{(b(\Delta_1^1), \cdots, b(\Delta_{i_0}^1))}\mathbf{a}_1.$$

Let $w_1 = (i_1, \cdots, i_0+1, i_0)$, then $w_1 \in S_n^{J_1}$. Note that we have also $ww_1 \in S_n^{J_1}$, since

$$ww_1(i) = w(i-1) < ww_1(i+1) = w(i), \text{ for } i = i_0 + 1, \cdots, i_1 - 1.$$

Then by induction, we know that

$$\mathbf{a}_{ww_1}^{J_1} = \sum_i [b(\Delta_i^1), e(\Delta_{ww_1(i)}^1)] \in S(\mathbf{a}_1).$$

Example 6.2.9. *Let $\mathbf{a}_{\mathrm{Id}}^J$ as in the previous example. Then $i_1 = 2$, and $J_1 = \{\sigma_3\}$. In this case, we have $w_1 = \sigma_1$ and $ww_1 = \sigma_1\sigma_2\sigma_1$, with*

$$\mathbf{a}_{ww_1}^{J_1} = [0,5] + [1,4] + [2,3] + [3,6].$$

Moreover,

$$\mathbf{a}_w^J = {}^{(b(\Delta_1^1), \cdots, b(\Delta_{i_0}^1))}\mathbf{a}_{ww_1}^{J_1}.$$

The result, that is the fact $\mathbf{a}_w^J \in S(\mathbf{a}_{\mathrm{Id}}^J)$ follows from the next lemma.

□

Lemma 6.2.10. *We have*

$$\mathbf{a}_{ww_1}^{J_1} \in {}_{b(\Delta_1^1), \cdots, b(\Delta_{i_0}^1)}S(\mathbf{a}_1).$$

Démonstration. In fact, let $\mathbf{a}_{1,0} = \mathbf{a}^J_{\mathrm{Id}}$ and for $j \leq i_0$, $\mathbf{a}_{1,j} = \{\Delta_{1,j}, \cdots, \Delta_{n,j}\}$, such that

$$\begin{aligned} \Delta_{i,j} &=^+ (\Delta_i), \text{ for } i \leq j, \\ \Delta_{i,j} &= \Delta_i, \text{ otherwise.} \end{aligned}$$

Then we have $\mathbf{a}_{1,j} = {}^{(b(\Delta^1_{j+1}),\cdots,b(\Delta^1_{i_0}))}\mathbf{a}_1$, for $j = 0, 1, \cdots, i_0$. For $j < i_0 - 1$, let

$$\mathbf{b}_j = \sum_{j<i\leq i_0} [b(\Delta^1_i) + 1, e(\Delta^1_{w(i)})] + \sum_{i>i_0, \text{ or } i\leq j} [b(\Delta^1_i), e(\Delta^1_{w(i)})],$$

and $\mathbf{b}_{i_0} = \mathbf{a}^{J_1}_{ww_1}$ so that $\mathbf{b}_j = {}^{(b(\Delta^1_{j+1}),\cdots,b(\Delta^1_{i_0}))}\mathbf{a}_2$. We show that $\mathbf{b}_j \in {}_{b(\Delta^1_j)}S(\mathbf{a}_{1,j})$ by induction on j.

(1) For $j = i_0$, we have

$$b(\Delta^1_{i_0}) = b(\Delta^1_{i_0+1}) - 1 = \cdots = b(\Delta^1_{i_1-1}) - 1 = b(\Delta^1_{i_1}) - 1.$$

And $ww_1(i_0) > ww_1(i_1) > ww_1(i_1 - 1) > \cdots > ww_1(i_0 + 1)$, hence

$$e(\Delta^1_{ww_1(i_0)}) > e(\Delta^1_{ww_1(i_1)}) > e(\Delta^1_{ww_1(i_1-1)}) > \cdots > e(\Delta^1_{ww_1(i_0+1)}),$$

because $w \in S^J$. This implies that $\mathbf{b}_{i_0}$ satisfies the hypothesis $({}_{b(\Delta^1_{i_0})}H(\mathbf{a}_{1,i_0}))$.

(2) For general $j \leq i_0 - 1$, By induction, we may assume that $\mathbf{b}_{j+1} \in {}_{b(\Delta^1_{j+1})}S(\mathbf{a}_{1,j+1})$. Now to show $\mathbf{b}_j \in {}_{b(\Delta^1_j)}S(\mathbf{a}_{1,j})$, we know that $b(\Delta^1_j) + 1 < b(\Delta^1_{j+1}) + 1$ in $\mathbf{b}_{j+1}$ (we have inequality by assumption on i_0), which proves that $\mathbf{b}_j \in {}_{b(\Delta^1_j)}S(\mathbf{a}_{1,j})$. Hence we are done.

□

Lemma 6.2.11. *Let $J = \{\sigma_{i_0}\} \cup J_1$ such that $i_0 = \min\{i : \sigma_i \in J\}$. Let $i_1 \in \mathbb{Z}$ be the maximal integer satisfying for $i_0 \leq i < i_1$ we have $\sigma_i \in J$. Then*

$$S^{J_1}_J = \{w_i : i = 1, \cdots, i_1 - i_0 + 1\}$$

with

$$w_i = (i_1 - i + 1, \cdots, i_0 + 1, i_0) \in S_J.$$

As a consequence, we have

$$S^{J_1}_n = \coprod_i S^J_n w_i.$$

Démonstration. By proposition 6.2.2, we only need to show that $S_J = \coprod_j w_j S_{J_1}$ and $w_j \in S^{J_1}$. The fact that $w_j \in S^{J_1}$ follows from

$$w_j(i) = i-1, \text{ for } i = i_0+1, \cdots, i_1-j+1, \; w_j(i_0) = i_1-j+1,$$

and $w_j(i) = i$ for $i \notin \{i_0, \cdots, i_1-j+1\}$. Finally, to see that $S_J = \coprod_j w_j S_{J_1}$, we compare the cadinalities. Let $J_0 = \{\sigma_i : i = i_0 \cdots, i_1 - 1\}$, then

$$S_J \simeq S_{J_0} \times S_{J \setminus J_0},$$

$$S_{J_1} \simeq S_{J_0 \setminus \{\sigma_{i_0, i_0+1}\}} \times S_{J \setminus J_0}.$$

Hence $\sharp S_J / \sharp S_{J_1} = \dfrac{\sharp S_{J_0}}{\sharp S_{J_0 \setminus \{\sigma_{i_0,i_0+1}\}}} = (i_1 - i_0 + 1)!/(i_1 - i_0)! = i_1 - i_0 + 1$. Finally, by proposition 6.2.2, we know that

$$S_n = \coprod_{v \in S_n^J} v S_J = \coprod_{j=i_0}^{i_1-i_0+1} \coprod_{v \in S_n^J} v w_j S_{J_1} = \coprod_j S_n^J w_j S_{J_1}.$$

□

Keeping the notations of proposition 6.2.6, we have

Lemma 6.2.12. *For $i = 1, \cdots, i_1 - i_0 + 1$, we have*

$$\mathbf{a}_w^J = {}^{(b(\Delta_1^1), \cdots, b(\Delta_{i_0}^1))} \mathbf{a}_{ww_i}^{J_1}.$$

Démonstration. Note that by definition We have

$$\mathbf{a}_{ww_j}^{J_1} = \sum_i b(\Delta_i^1), e(\Delta_{ww_j(i)}^1)].$$

As noted before, we have

$$b(\Delta_i^1) = b(\Delta_i) - 1, \text{ for } i \leq i_0, \; b(\Delta_i^1) = b(\Delta_i), \text{ for } i > i_0.$$

Also,we observe that $e(\Delta_i^1) = e(\Delta_i)$. Hence

$${}^{(b(\Delta_1^1), \cdots, b(\Delta_{i_0}^1))} \mathbf{a}_{ww_i}^{J_1} = \sum_i b(\Delta_i), e(\Delta_{ww_j(i)})].$$

It remains to see that we have

$$\sum_{i=i_0}^{i_1-j+1} b(\Delta_i), e(\Delta_{ww_j(i)})] = \sum_{i_0}^{i_1-j+1} b(\Delta_i), e(\Delta_{w(i)})]$$

since $b(\Delta_{i_0}) = \cdots = b(\Delta_{i_1-j+1})$. Hence we have

$$\mathbf{a}_w^J = {}^{(b(\Delta_1^1),\cdots,b(\Delta_{i_0}^1))}\mathbf{a}_{ww_i}^{J_1}.$$

□

Definition 6.2.13. *As in the symmetric cases, we have the following map*

$$\Phi_J : S_n^J \to S(\mathbf{a}_{\mathrm{Id}}^J)$$
$$w \mapsto \mathbf{a}_w^J.$$

Proposition 6.2.14. *The morphism Φ_J is bijective and translate the inverse Bruhat order on S_n^J to the order on $S(\mathbf{a}_{\mathrm{Id}}^J)$.*

Démonstration. Again, we do this by induction on $|J|$. If $|J| = 0$, we are in the symmetric case, so everything is done in section 2.3. In general, we keep the notation in the proposition 6.2.6. We have $J = J_1 \cup \{\sigma_{i_0}\}$. And as we proved above,

$$\mathbf{a}_{ww_1}^{J_1} \in {}_{b(\Delta_1^1),\cdots,b(\Delta_{i_0}^1)}S(\mathbf{a}_1).$$

Also, we note that the morphism ${}_{b(\Delta_1^1),\cdots,b(\Delta_{i_0}^1)}\psi$ sends $\Phi_{J_1}(ww_1)$ to $\Phi_J(w)$ for $w \in J$, as is proved in the proposition above. Therefore

$$\Phi_J = {}_{b(\Delta_1^1),\cdots,b(\Delta_{i_0}^1)}\psi \circ \Phi_{J_1},$$

and the injectivity of follows from that of ${}_{b(\Delta_1^1),\cdots,b(\Delta_{i_0}^1)}\psi$ and induction on J_1. For surjectivity, let $\mathbf{b} \in S(\mathbf{a}_{\mathrm{Id}}^J)$, by surjectivity of the map

$${}_{b(\Delta_1^1),\cdots,b(\Delta_{i_0}^1)}\psi : {}_{b(\Delta_1^1),\cdots,b(\Delta_{i_0}^1)}S(\mathbf{a}_1) \to S(\mathbf{a}_{\mathrm{Id}}^J),$$

we know that there exists a $w' \in S^{J_1}$, such that $\Phi_{J_1}(w') \in {}_{b(\Delta_1^1),\cdots,b(\Delta_{i_0}^1)}S(\mathbf{a}_1)$, and is sent to $\mathbf{b}$ by ${}_{b(\Delta_1^1),\cdots,b(\Delta_{i_0}^1)}\psi$. By lemma 6.2.11, every $w' \in S^{J_1}$ can be write as ww_j for some $w \in S^J$ and $w_j \in S^{J_1}$. Now by lemma 6.2.12,

$$\mathbf{b} = \mathbf{a}_w^J.$$

Note that for $w > w'$ in S^J, then $ww_1 > w'w_1$ in S^{J_1}, hence by induction

$$\Phi_{J_1}(ww_1) < \Phi_{J_1}(ww_1),$$

we get

$$\Phi_{J_1}(w) < \Phi_{J_1}(w),$$

since the morphism ${}_{b(\Delta_1^1),\cdots,b(\Delta_{i_0}^1)}\psi$ preserves the order.

□

Proposition 6.2.15. *Let $v_1, v_2 \in S^J$, then we have*

$$P_{\Phi_J(v_1),\Phi_J(v_2)}(q) = P^J_{v_1,v_2}(q)$$

where on the right hand side is the parabolic KL polynomial indexed by v_1, v_2.

Démonstration. As is proved in [14], we have $P^J_{v_1,v_2}(q) = P_{v_1v_J,w_2v_J}(q)$, where v_J is the maximal element in S_J. So it suffices to show that we have the equality $P_{\Phi_J(v_1),\Phi_J(v_2)}(q) = P_{v_1v_J,v_2v_J}(q)$. Also, from lemma 6.2.10, we know that

$$\Phi_J(v_1) = {}_{b(\Delta^1_1),\cdots,b(\Delta^1_{i_0})}\psi(\Phi_{J_1}(v_1w_1)),$$

where w_1 is described in lemma 6.2.11. Hence we have

$$P_{\Phi_{J_1}(v_1w_1),\Phi_{J_1}(v_2w_1)}(q) = P_{\Phi_J(v_1),\Phi_J(v_2)}(q)$$

by corollary 3.3.18.
By induction, we have

$$P_{\Phi_{J_1}(v_1w_1),\Phi_{J_1}(v_2w_1)}(q) = P_{v_1w_1v_{J_1},v_1w_1v_{J_1}}(q).$$

Now to finish, we have to show $v_J = w_1v_{J_1}$. But we know that

$$S_J = \coprod_j w_jS_{J_1}$$

with $w_1 = \max\{w_j : j = 1, \cdots, i_1 - i_0 + 1\}$, we surely have

$$v_J = w_1v_{J_1}.$$

□

More generally, for $J_i \subseteq S$, $i = 1, 2$, we can consider the P_{J_1} orbit in GL_n/P_{J_2}. We state the related result without proving.

Definition 6.2.16. *Let $S_n^{J_1,J_2} = \{w \in S_n : s_1vs_2 > v$ for all $s_i \in J_i, i = 1, 2\}$.*

Definition 6.2.17. *Let $v \in S_n^{J_1,J_2}$. We define*

$$S^{J_2,v}_{J_1} = \{w \in S_{J_1} : ws > w, \text{ for all } s \in S_{J_1} \cap vS_{J_2}v^{-1}\}.$$

Remark: If we let M_J be the Levi subgroup of P_J, then the set $S^{J_2,v}_{J_1}$ corresponds to the Borel orbits in $M_{J_1}/(M_{J_1} \cap vM_{J_2}v^{-1})$.

Proposition 6.2.18. *We have*

(1) $S_n = \coprod_{v \in S_n^{J_1,J_2}} S_{J_1} v S_{J_2}$;

(2) $\ell(xvy) = \ell(v) + \ell(x) + \ell(y)$ *for* $v \in S^{J_1,J_2}$, $x \in S_{J_1}^{J_2,v}, y \in S_{J_2}$.

(3) *The* P_{J_1} *orbits in* GL_n/P_{J_2} *are indexed by* $S_n^{J_1,J_2}$.

Definition 6.2.19. *For* $v_1, v_2 \in S^{J_1,J_2}$ *such that* $v_1 \leq v_2$, *we let* $P_{v_1,v_2}^{J_1,J_2}(q)$ *be the Poincaré series of the localized intersection cohomology*

$$\mathcal{H}^\bullet(\overline{P_{J_1} v_2 P_{J_2}})_{v_1 P_{J_2}}.$$

Lemma 6.2.20. *For* $v_1, v_2 \in S^{J_1,J_2}$ *such that* $v_1 \leq v_2$, *we have*

$$P_{v_1,v_2}^{J_1,J_2}(q) = P_{w_1,w_2}(q),$$

where w_i *is the element of maximal length in* $S_{J_1} v_i S_{J_2}$.

Notation 6.2.21. *Let* $\mathbf{a}_{\mathrm{Id}}^{J_1,J_2} = \{\Delta_1, \cdots, \Delta_n\}$ *such that*

$$e(\Delta_1) \leq \cdots \leq e(\Delta_n),$$

such that

$$e(\Delta_i) = e(\Delta_{i+1}) \text{ if and only if } \sigma_i \in J_1$$

and

$$b(\Delta_1) \leq \cdots \leq b(\Delta_n),$$

such that

$$b(\Delta_i) = b(\Delta_{i+1}) \text{ if and only if } \sigma_i \in J_2$$

and $b(\Delta_n) \leq e(\Delta_1)$.

Definition 6.2.22. *We call a multisegment* $\mathbf{a} \in S(\mathbf{a}_{\mathrm{Id}}^{J_1,J_2})$ *a multisegment of parabolic type* (J_1, J_2).

Lemma 6.2.23. *For* $w \in S^{J_1,J_2}$, *let* $\mathbf{a}_w^{J_1,J_2} = \sum [b(\Delta_i), e(\Delta_{w(i)})]$, *then* $\mathbf{a}_w^{J_1,J_2} \in S(\mathbf{a}_{\mathrm{Id}}^{J_1,J_2})$. *Therefore we have an application*

$$\begin{aligned} \Phi_{J_1,J_2} : S^{J_1,J_2} &\to S(\mathbf{a}_{\mathrm{Id}}^{J_1,J_2}) \\ w &\mapsto \mathbf{a}_w^{J_1,J_2}. \end{aligned}$$

Proposition 6.2.24. *The morphism* Φ_{J_1,J_2} *is bijective and translate the inverse Bruhat order on* S^{J_1,J_2} *to the order on* $S(\mathbf{a}_{\mathrm{Id}}^{J_1,J_2})$.

Proposition 6.2.25. *Let $w_1, w_2 \in S_n^{J_1,J_2}$, then we have*

$$P_{\Phi_{J_1,J_2}(w_1),\Phi_{J_1,J_2}(w_2)}(q) = P^{J_1,J_2}_{w_1,w_2}(q)$$

where on the right hand side is the parabolic KL polynomial indexed by w_1, w_2.

Example 6.2.26. *We are now ready to interpret the following results (due to Zelevinsky, see [35] Section 3.3) : let $\mathbf{a} = k[0,1] + (n-k)[1,2]$ then $\mathbf{a}$ corresponding to the identity in $S_n^{J,J}$ with*

$$J = \{\sigma_i : i \neq k\}.$$

Note that in this case, we have GL_n/P_J is the Grassmanian $G_k(\mathbb{C}^n)$, where as the P_J orbits correspond to the stratification, for $r \leq r_0 = \min\{k, n-k\}$ and fixed $\mathbb{C}^k \in G_k(\mathbb{C}^n)$,

$$X_r = \{U \in G_k(\mathbb{C}^n) : \dim(U \cap \mathbb{C}^k) = k - r\}$$

with $\overline{X_r} = \coprod_{r' \leq r} X_{r'}$.

Remark: There is another way to obtain the result of this section , i.e., by direct geometric construct, as in section 4.3, where we prove the same result for symmetric case. In this situation, instead having the flag variety G/B in the fibers, we will find G/P_J in the fibers. There is one advantage in this geometric construction, i.e, by employing the same proof as in section 4.4, one can get a resolution for G/P_J by pulling back that of the corresponding orbit variety. This shows for example, that the resolution can not be small when the associated quiver is of type A_n, $n \geq 3$, by the example constructed by Zelevinsky for flag variety, which does not admit any small resolution. We remark that the resolution is always small for type A_2, as is proved by Zelevinsky.

Remark: Note that in [35], Zelevinsky constructed a small resolution for the $\overline{O}_{\mathbf{a}}$ with $\mathbf{a} = \{[1,2],[2,3]\}$, which corresponds to a Schubert varieties of 2-step . Now with our interpretation, we should be able to construct a small resolution for all 2-step Schubert varieties. We return to this question later.

Remark: With the help of partial derivative which we will develop in next section, we will be able to give inverse parabolic KL polynomials combining results of this section, which is described in [14] . See next section for more details.

6.3 Non Ordinary Case

In this section, for a general multisegment $\mathbf{a}$, we will relate the poset $S(\mathbf{a})$ to a Bruhat interval $[x, y]$ with $x < y$ in some $S_r^{J_1,J_2}$.
Now let $\mathbf{a}$ be a multisegment. First of all, we decide the set J_1, J_2.

Definition 6.3.1. *We define two sets $J_1(\mathbf{a}), J_2(\mathbf{a})$.*
- *Let $b(\mathbf{a}) = \{k_1 \leq \cdots \leq k_r\}$. Then let $J_2(\mathbf{a}) \subseteq S_r$ be the set such that $\sigma_i \in J_2(\mathbf{a})$ if and only if $k_i = k_{i+1}$.*
- *Let $e(\mathbf{a}) = \{\ell_1 \leq \cdots \leq \ell_r\}$. Then let $J_1(\mathbf{a}) \subseteq S_r$ be the set such that $\sigma_i \in J_1(\mathbf{a})$ if and only if $\ell_i = \ell_{i+1}$.*

Keeping the notations in definition 6.3.1,

Proposition 6.3.2. *There exists a unique $w \in S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}$, such that*

$$\mathbf{a} = \sum_j [k_j, \ell_{w(j)}].$$

Démonstration. We observe that there exists an element $w' \in S_r$, such that

$$\mathbf{a} = \sum_j [k_j, \ell_{w'}(j)].$$

Now by proposition 6.2.18, we know that there exists $w' = w_{J_1(\mathbf{a})} w w_{J_2(\mathbf{a})}$ with $w_{J_i(\mathbf{a})} \in S_{J_i(\mathbf{a})}$ for $i = 1, 2$ and $w \in S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}$. Now we only need to prove that

$$\mathbf{a} = \sum_j [k_j, \ell_w(j)].$$

In fact, by definition of $J_i(\mathbf{a}), i = 1, 2$, we know that

$$k_j = k_{v(j)}, \text{ for all } v \in S_{J_2(\mathbf{a})},$$
$$\ell_j = \ell_{v(j)}, \text{ for all } v \in S_{J_1(\mathbf{a})}.$$

Hence

$$\begin{aligned} \mathbf{a} &= \sum_j [k_j, \ell_{w_{J_1(\mathbf{a})} w w_{J_2(\mathbf{a})}(j)}] \\ &= \sum_j [k_j, \ell_{w(\mathbf{a}) w_{J_2(\mathbf{a})}(j)}] \\ &= \sum_j [k_{w_{J_2(\mathbf{a})}^{-1} w^{-1}(j)}, \ell_j] \\ &= \sum_j [k_j, \ell_{w(j)}]. \end{aligned}$$

□

Next we show how to reduce a general multisegment $\mathbf{a}$ to a multisegment $\mathbf{a}_w^{J_1(\mathbf{a}),J_2(\mathbf{a})}$ of parabolic type $(J_1(\mathbf{a}), J_2(\mathbf{a}))$ without changing the poset structure $S(\mathbf{a})$.

Proposition 6.3.3. *Let $\mathbf{a}$ be a multisegment, then there exists a multisegment $\mathbf{c}$, and a multisegment $\mathbf{a}_w^{J_1(\mathbf{a}),J_2(\mathbf{a})}$ of parabolic type $(J_1(\mathbf{a}), J_2(\mathbf{a}))$, such that*

$$\mathbf{a}_w^{J_1(\mathbf{a}),J_2(\mathbf{a})} \in S(\mathbf{a}_w^{J_1(\mathbf{a}),J_2(\mathbf{a})})_{\mathbf{c}},\ \mathbf{a} = (\mathbf{a}_w^{J_1(\mathbf{a}),J_2(\mathbf{a})})^{(\mathbf{c})}$$

Démonstration. In general $\mathbf{a}$ is not of parabolic type, i.e, we do not have $\min\{e(\Delta) : \Delta \in \mathbf{a}\} \geq \max\{b(\Delta) : \Delta \in \mathbf{a}\}$. Now we show how to construct $\mathbf{a}_w^{J_1(\mathbf{a}),J_2(\mathbf{a})}$.
In fact, let

$$\mathbf{a} = \{\Delta_1, \cdots, \Delta_n\}, \Delta_1 \prec \cdots \prec \Delta_n.$$

Then

$$e(\Delta_1) = \min\{k : i = 1, \cdots, n\}.$$

If $\mathbf{a}$ is not of parabolic type, let $\Delta^1 = [e(\Delta_1)+1, \ell]$ with ℓ maximal satisfying that for any m such that $e(\Delta_1) \leq m \leq \ell - 1$, there is a segment in $\mathbf{a}$ ending in m. We construct $\mathbf{a}^1$ by replacing every segment Δ in $\mathbf{a}$ ending in Δ^1 by Δ^+. Repeat this construction with $\mathbf{b}_1$ to get $\mathbf{a}^2 \cdots$, until we get $\mathbf{a}^s$, which is of parabolic type. Let $\mathbf{c} = \{\Delta^1, \cdots, \Delta^s\}$, then we do as in proposition 4.2.1 to get

$$\mathbf{a}^s \in S(\mathbf{a}^s)_{\mathbf{c}},\ \mathbf{a} = (\mathbf{a}^s)^{(\mathbf{c})}.$$

Note that by our construction we have

$$J_1(\mathbf{a}^i) = J_1(\mathbf{a}),\ J_2(\mathbf{a}^i) = J_2(\mathbf{a}),$$

for $i = 1, \cdots, s$. □

Lemma 6.3.4. *Assume that $\mathbf{a} \in S(\mathbf{a})_k$ such that*

$$J_1(\mathbf{a}) = J_1(\mathbf{a}^{(k)}),\ J_2(\mathbf{a}) = J_2(\mathbf{a}^{(k)}).$$

Let $\mathbf{c} \in S(\mathbf{b})_k$. Then for $\mathbf{d} \in S(\mathbf{b})$ and $\mathbf{d} > \mathbf{c}$, we have $\mathbf{d} \in S(\mathbf{b})_k$.

Démonstration. It suffices to show that $\mathbf{d}$ satisfies the hypothesis $H_k(\mathbf{a})$. Note that $e(\mathbf{d}) \subseteq e(\mathbf{a})$ by lemma 2.1.4. Assume that $k \in e(\mathbf{a})$ to avoid triviality. Now that $k - 1 \notin e(\mathbf{a})$ since $\mathbf{a} \in S(\mathbf{a})_k$ and

$$J_1(\mathbf{a}) = J_1(\mathbf{a}^{(k)}),\ J_2(\mathbf{a}) = J_2(\mathbf{a}^{(k)}),$$

so it is also not in $e(\mathbf{d})$. Hence to show that $\mathbf{d} \in S(\mathbf{b})_k$ hence it is equivalent to show that $\varphi_{e(\mathbf{d})}(k) = e_{e(\mathbf{a})}(k)$. Since $\mathbf{c} \in S(\mathbf{d})$, we know that $e(\mathbf{c}) \subseteq e(\mathbf{d})$ hence $\varphi_{e(\mathbf{d})} \leq \varphi_{e(\mathbf{d})}(k)$. Now that $\mathbf{c} \in S(\mathbf{a})_k$ implies $\varphi_{e(\mathbf{c})} = \varphi_{e(\mathbf{a})}$, we conclude that $\varphi_{e(\mathbf{d})}(k) = e_{e(\mathbf{a})}(k)$. We are done. □

Now let $\mathbf{a}' \in S(\mathbf{a})_k$ such that $\psi_k(\mathbf{a}') = (\mathbf{a}^{(k)})_{\min}$, then

Lemma 6.3.5. *We have*

$$S(\mathbf{a})_k = \{\mathbf{c} \in S(\mathbf{a}) : \mathbf{c} \geq \mathbf{a}'\}.$$

Démonstration. By the lemma above, we know that $S(\mathbf{a})_k \supseteq \{\mathbf{c} \in S(\mathbf{a}) : \mathbf{c} \geq \mathbf{a}'\}$. We conclude that we have equality since ψ preserve the order. $\square$

Proposition 6.3.6. *Assume that* $\mathbf{a}$ *is a multisegment. Then*

(1) *There exists a multisegment* $\mathbf{a}_w^{J_1(\mathbf{a}),J_2(\mathbf{a})}$ *of parabolic type* $(J_1(\mathbf{a}), J_2(\mathbf{a}))$ *and a sequence of integers* $k_1, \cdots, k_r$ *such that*

$$S(\mathbf{a}) \simeq S(\mathbf{a}_w^{J_1(\mathbf{a}),J_2(\mathbf{a})})_{k_r,\cdots,k_1}.$$

(2) *There exists an element* $\mathbf{a}' \in S(\mathbf{a}_w^{J_1(\mathbf{a}),J_2(\mathbf{a})})$ *such that*

$$S(\mathbf{a}_w^{J_1(\mathbf{a}),J_2(\mathbf{a})})_{k_r,\cdots,k_1} = \{\mathbf{c} \in S(\mathbf{a}_w^{J_1(\mathbf{a}),J_2(\mathbf{a})}) : \mathbf{c} \geq \mathbf{a}'\}.$$

Démonstration. Note that (1) follows from proposition 6.3.3 and proposition 4.1.5. And (2) follows from applying the lemma 6.3.5 successively to the lemma below. $\square$

Lemma 6.3.7. *There exists a sequence of multisegments* $\mathbf{a}_0 = \mathbf{a}, \cdots, \mathbf{a}_r = \mathbf{a}_w^{J_1(\mathbf{a}),J_2(\mathbf{a})}$ *such that* $\mathbf{a}_w^{J_1(\mathbf{a}),J_2(\mathbf{a})}$ *is of parabolic type* $(J_1(\mathbf{a}), J_2(\mathbf{a}))$*,* $\mathbf{a}_i \in S(\mathbf{a}_i)_{k_i}$ *and* $\mathbf{a}_{i-1} = \mathbf{a}_i^{(k_i)}$ *for some* k_i*. Moreover,*

$$J_1(\mathbf{a}_i) = J_1(\mathbf{a}), \ J_2(\mathbf{a}) = J_2(\mathbf{a})$$

for all $i = 1, \cdots, r$*.*

Démonstration. This follows from our construction in the proof of proposition 6.3.3. $\square$

Chapitre 7

Computation of Partial Derivatives

In this chapter, we study the problem of computing the partial derivatives $\mathscr{D}^k(L_{\mathbf{a}})$ of the irreducible representation $L_{\mathbf{a}}$ attached to a multisegment $\mathbf{a}$. The idea is to use these computations to calculate the multiplicities in the induced representation $L_{\mathbf{a}} \times L_{\mathbf{b}}$, cf. the next chapter. Recall that we have already given a way of computing $L_{\mathbf{a}}$ as a sum, cf. (1.3.7)

$$L_{\mathbf{a}} = \sum_{\mathbf{b}} \widetilde{m}_{\mathbf{b},\mathbf{a}} \pi(\mathbf{a}).$$

So one is reduced to the calculate

$$\mathscr{D}^k(\pi(\mathbf{a})) = \sum_{\mathbf{b}} n_{\mathbf{b},\mathbf{a}} L_{\mathbf{b}}, \qquad n_{\mathbf{b},\mathbf{a}} \geq 0.$$

As for the coefficient $m_{\mathbf{b},\mathbf{a}}$, we first introduce a new poset structure $\preceq_k$ on the set of multisegments so that we have the equivalence between $n_{\mathbf{b},\mathbf{a}} > 0$ and $\mathbf{b} \preceq_k \mathbf{a}$, cf. proposition 7.1.4.
The principal result of this chapter is the interpretation of the coefficient $n_{\mathbf{b},\mathbf{a}}$ as the value at $q = 1$ of some Poincaré series of the Lusztig product of two explicit perverse sheaves on orbital varieties, cf. proposition 7.3.8.
In 7.4, we compute these Lusztig products as the push forward by a projection β'', cf. corollary 7.4.19, of some concrete perverse sheaf on an orbital variety.
In §7.6 we first study the geometry of the case where the multisegments are of Grassmanian type. In this case the projection β'' is simply cf. proposition 7.6.8, the natural projection

$$GL_n/P \to GL_n/P'$$

with $P \subseteq P'$ two parabolic subgroups. The geometry of the parabolic case is treated in §7.7 : the constructions and proofs are the same as the Grassmanian type.
Finally in the last section 7.8, we obtain a complete formula for $\mathscr{D}^k(L_{\mathbf{a}})$ in the general case, cf. corollary 7.8.16

7.1 New Poset Structure on Multisegments

In this section we define a new poset structure $\preceq_k$ depending on an integer k on the set of multisegments and show that the term $L_{\mathbf{b}}$ appears in $\mathscr{D}^k(\pi(\mathbf{a}))$ if and only if $\mathbf{b} \preceq_k \mathbf{a}$.

Definition 7.1.1. *For a well ordered multisegment* $\mathbf{a} = \{\Delta_1, \cdots, \Delta_s\}$ *with* $\Delta_1 \preceq \cdots \preceq \Delta_s$*, let*

$$\mathbf{a}(k) := \{\Delta \in \mathbf{a} : e(\Delta) = k\} = \{\Delta_{i_0}, \Delta_{i_0+1}, \cdots, \Delta_{i_1}\}.$$

Now let $\Gamma \subseteq \mathbf{a}(k)$*, let*

$$\mathbf{a}(k)_\Gamma := (\mathbf{a}(k) \setminus \Gamma) \cup \{\Delta^{(k)} : \Delta \in \Gamma\},$$

and

$$\mathbf{a}_\Gamma := (\mathbf{a} \setminus \mathbf{a}(k)) \cup \mathbf{a}(k)_\Gamma.$$

We say $\mathbf{b} \preceq_k \mathbf{a}$ *if there exist a multisegment* $\mathbf{c} \in S(\mathbf{a})$ *such that*

$$\mathbf{b} \leq \mathbf{a}_\Gamma$$

for some Γ*.*

Lemma 7.1.2. *We have*

$$\mathscr{D}^k(\pi(\mathbf{a})) = \pi(\mathbf{a}) + \sum_{\Gamma \subseteq \mathbf{a}(k), \Gamma \neq \emptyset} \pi(\mathbf{a}_\Gamma). \tag{7.1.3}$$

Démonstration. Let

$$\mathbf{a} = \{\Delta_1, \cdots, \Delta_r, \Delta_{r+1}, \cdots, \}.$$

Then

$$\pi(\mathbf{a}) = \prod_{i=1}^{r} L_{\Delta_i} \times \prod_{i>r} L_{\Delta_i}$$

and

$$\begin{aligned}\mathscr{D}^k(\pi(\mathbf{a})) &= \prod_{i=1}^{r}(L_{\Delta_i} + L_{\Delta_i^{(k)}}) \times \prod_{i>r} L_{\Delta_i} \\ &= \pi(\mathbf{a}) + \sum_{\Gamma\subseteq \mathbf{a}(k),\Gamma\neq\emptyset} \pi(\mathbf{a}_\Gamma).\end{aligned}$$

□

Proposition 7.1.4. *Let*

$$\mathscr{D}^k(\pi(\mathbf{a})) = \sum_{\mathbf{b}} n_{\mathbf{b},\mathbf{a}} L_{\mathbf{b}}. \tag{7.1.5}$$

Then $n_{\mathbf{b},\mathbf{a}} > 0$ *if and only if* $\mathbf{b} \preceq_k \mathbf{a}$.

Démonstration. Let $\mathbf{b} \preceq \mathbf{a}$, then by definition we have $\mathbf{b} \leq \mathbf{a}_\Gamma$ for some Γ. Therefore $m_{\mathbf{b},\mathbf{a}_\Gamma} > 0$, now we have $n_{\mathbf{b},\mathbf{a}} > 0$ by equation 7.1.3. Conversely, if $n_{\mathbf{b},\mathbf{a}} > 0$, then by equation 7.1.3, we know that $\mathbf{b} \leq \mathbf{a}_\Gamma$ for some Γ. □

Corollary 7.1.6. *We have* $\mathbf{b} \preceq_k \mathbf{a}$ *if and only if* $\mathscr{D}^k(\pi(\mathbf{a})) - \pi(\mathbf{b}) \geq 0$ *in* $\mathcal{R}$.

Démonstration. We keep the notations in the proof of proposition 7.1.4. We know that $\mathbf{b} \preceq_k \mathbf{a}$ implies $\mathbf{b} \leq \mathbf{a}_\Gamma$ for some $\Gamma \subseteq \mathbf{a}(k)$. By lemma 1.3.5, we know that $\mathbf{b} \leq \mathbf{a}_\Gamma$ implies that $\pi(\mathbf{a}_\Gamma) - \pi(\mathbf{b}) \geq 0$ in $\mathcal{R}$. Since $\mathscr{D}^k(\pi(\mathbf{a})) - \pi(\mathbf{a}_\Gamma) \geq 0$ by equation (7.1.3), we have $\mathscr{D}^k(\pi(\mathbf{a})) - \pi(\mathbf{b}) \geq 0$. Conversely, if $\mathscr{D}^k(\pi(\mathbf{a})) - \pi(\mathbf{b}) \geq 0$, we have $n(\mathbf{b}, \mathbf{a}) > 0$, hence $\mathbf{b} \preceq_k \mathbf{a}$ by proposition 7.1.4. □

Proposition 7.1.7. *For any* $\mathbf{b} \preceq_k \mathbf{a}$*, there exists* $\mathbf{c} \in S(\mathbf{a})$*, and some subset* $\Gamma \subseteq \mathbf{c}(k)$*, such that*

$$\mathbf{b} = \mathbf{c}_\Gamma.$$

Conversely, if $\mathbf{b} = \mathbf{c}_\Gamma$ *for some* $\mathbf{c} \in S(\mathbf{a})$*, then* $\mathbf{b} \preceq_k \mathbf{a}$.

Démonstration. For the converse part, suppose $\mathbf{c} \neq \mathbf{a}$, by equation 7.1.3, we have $\mathscr{D}^k(\pi(\mathbf{c})) - \pi(\mathbf{b}) \geq 0$ in $\mathcal{R}$. By lemme 1.3.5, we know that $\pi(\mathbf{a}) - \pi(\mathbf{c}) \geq 0$ in $\mathcal{R}$, hence $\mathscr{D}^k(\pi(\mathbf{a})) - \mathscr{D}^k(\pi(\mathbf{c})) \geq 0$ by theorem 1.4.4. Therefore $n_{\mathbf{b},\mathbf{a}} > 0$. Hence we have $\mathbf{b} \preceq_k \mathbf{a}$.
For the direct part, suppose that $\mathbf{b} \preceq_k \mathbf{a}$, hence $\mathbf{b} < \mathbf{a}_{\Gamma_1}$ for some Γ_1. We prove by induction on $\ell(\mathbf{b}, \mathbf{a}_T)$. If $\ell(\mathbf{b}, \mathbf{a}_{\Gamma_1}) = 0$, then $\mathbf{b} = \mathbf{a}_{\Gamma_1}$, we are done. Now let $\mathbf{b} < \mathbf{d} \leq \mathbf{a}_{\Gamma_1}$ such that $\ell(\mathbf{b}, \mathbf{d}) = 1$, by induction, we know that

$$\mathbf{d} = \mathbf{c}'_{\Gamma_0},$$

for some $\mathbf{c}' \in S(\mathbf{a})$. Note that by replacing $\mathbf{c}'$ by $\mathbf{a}$, we can assume that $\mathbf{d} = \mathbf{a}_{\Gamma_1}$ and $\ell(\mathbf{b}, \mathbf{a}_{\Gamma_1}) = 1$.
By definition, we know that $\mathbf{b}$ is obtained by applying the elementary operation to a pair of segments $\{\Delta \preceq \Delta'\}$ in $\mathbf{a}_T$. Now we set out to construct $\mathbf{c}$.

— If $\{\Delta, \Delta'\} \subseteq \mathbf{a} \setminus \{\Delta^{(k)} : \Delta \in \Gamma_1\} \subseteq \mathbf{a}$, let $\mathbf{c}$ be the multisegment obtained by applying the elementary operations to $\{\Delta, \Delta'\}$. And we have
$$\mathbf{b} = \mathbf{c}_{\Gamma_1}.$$

— If $\{\Delta, \Delta'\} \cap \{\Delta^{(k)} : \Delta \in \Gamma_1\} = \{\Delta'\}$, then $\{\Delta, \Delta'^{+}\} \in \mathbf{a}$ let $\mathbf{c}$ be the multisegment obtained by applying the elementary operations to $\{\Delta, \Delta'^{+}\}$. Then let
$$\Gamma = (\Gamma_1 \setminus \{\Delta'^{+}\}) \cup \{\Delta \cup \Delta'^{+}\}$$
and we have
$$\mathbf{b} = \mathbf{c}_{\Gamma}.$$

— If $\{\Delta, \Delta'\} \cap \{\Delta^{(k)} : \Delta \in \Gamma_1\} = \{\Delta\}$, then $\{\Delta^{+}, \Delta'\} \in \mathbf{a}$ let $\mathbf{c}$ be the multisegment obtained by applying the elementary operations to $\{\Delta^{+}, \Delta'\}$. Then let
$$\Gamma = (\Gamma_1 \setminus \{\Delta^{+}\}) \cup \{\Delta \cap \Delta'\}$$
and we have
$$\mathbf{b} = \mathbf{c}_{\Gamma}.$$

Hence we are done.

□

Proposition 7.1.8. *The relation $\preceq_k$ defines a poset structure on $\mathcal{O}$.*

Démonstration. By definition we have $\mathbf{a} \preceq_k \mathbf{a}$ for any $\mathbf{a} \in \mathcal{O}$. Suppose $\mathbf{a}_1 \preceq_k \mathbf{a}_2, \mathbf{a}_2 \preceq_k \mathbf{a}_3$, we want to show that $\mathbf{a}_1 \preceq_k \mathbf{a}_3$. By proposition 7.1.7, there exists $\mathbf{c} \in S(\mathbf{a}_2)$ and $\Gamma_1 \subseteq \mathbf{c}(k)$, such that
$$\mathbf{a}_1 = \mathbf{c}_{\Gamma_1}.$$
Note that by corollary 7.1.6, the fact $\mathbf{a}_2 \preceq_k \mathbf{a}_3$ implies $\mathscr{D}^k(\pi(\mathbf{a}_3)) - \pi(\mathbf{a}_2) \geq 0$. Hence we have $n(\mathbf{a}_3, \mathbf{c}) > 0$, therefore $\mathbf{c} \preceq_k \mathbf{a}_3$ by proposition 7.1.4. In turn, we know that there exists a multisegment $\mathbf{c}' \in S(\mathbf{a}_3)$ and $\Gamma_2 \subseteq \mathbf{c}'(k)$, such that
$$\mathbf{c} = \mathbf{c}'_{\Gamma_2}.$$

Since we have $\mathbf{c}(k) \subseteq \mathbf{c}'(k)$, we take

$$\Gamma_3 := \Gamma_1 \cup \Gamma_2 \subseteq \mathbf{c}'(k).$$

Now we get

$$\mathbf{a}_1 = \mathbf{c}'_{\Gamma_3},$$

which implies $\mathbf{a}_1 \preceq_k \mathbf{a}_3$ by proposition 7.1.7. Finally, if $\mathbf{a} \preceq_k \mathbf{b}$ and $\mathbf{b} \preceq_k \mathbf{a}$, then by definition we have $\mathbf{a} = \mathbf{b}$.

□

Definition 7.1.9. *We let*

$$\Gamma(\mathbf{a}, k) = \{\mathbf{b} : \mathbf{b} \preceq_k \mathbf{a}\}.$$

7.2 Canonical Basis and Quantum Algebras

In this section, following [25], we recall the results of Lusztig on canonical basis, the relation of quantum algebras and the algebra $\mathcal{R}$. We are especially interested in the construction of a product of perverse sheaves over orbital varieties defined by Lusztig [27], which is closely related to the product defined by induction in $\mathcal{R}$.

Definition 7.2.1. *Let $\mathbb{N}^{(\mathbb{Z})}$ be the semi-group of sequences $(d_j)_{j\in\mathbb{Z}}$ of non negative integers which are zero for all but finitely many j. Let α_i be the element whose i-th term is 1 and other terms are zero.*

Definition 7.2.2. *We define a symmetric bilinear form on $\mathbb{N}^{(\mathbb{Z})}$ given by*

$$(\alpha_i, \alpha_j) = \begin{cases} 2, & \text{for } i = j; \\ -1, & \text{for } |i-j| = 1; \\ 0, & \text{otherwise }. \end{cases}$$

Definition 7.2.3. *Let q be an indeterminate and $\mathbb{Q}(q^{1/2})$ be the fractional field of $\mathbb{Z}[q^{1/2}]$. Let $U_q^{\geq 0}$ be the $Q(q^{1/2})$-algebra generated by the elements E_i and $K_i^{\pm 1}$ for $i \in \mathbb{Z}$ with the following relations :*

$$K_iK_j = K_jK_i, \; K_iK_i^{-1} = 1;$$
$$K_iE_i = q^{1/2(\alpha_i,\alpha_j)}E_iK_i;$$
$$E_iE_j = E_jE_i, \text{ if } |i-j| > 1;$$
$$E_i^2E_j - (q^{1/2} + q^{-1/2})E_iE_jE_i + E_jE_i^2 = 0, \text{ if } |i-j| = 1.$$

and let U^+ be the subalgebra generated by the E_i's.

Remark: This is the + part of the quantized enveloping algebra U associated by Drinfeld and Jimbo to the root system A_∞ of SL_∞. And for $q = 1$, this specializes to the classical enveloping algebra of the nilpotent radical of a Borel subalgebra.

Definition 7.2.4. *We define a new order on the set of segments* Σ

$$\begin{cases} [j,k] \lhd [m,n], & \text{if } k < n, \\ [j,k] \rhd [m,n], & \text{if } j < m, n = k. \end{cases}$$

We also denote $[j,k] \lhd [m,n]$ *or* $[j,k] = [m,n]$ *by* $\unlhd [m,n]$.

Lemma 7.2.5. *The algebra* U_q^+ *is* $\mathbb{N}^{(\mathbb{Z})}$*-graded via the weight function* $\mathrm{wt}(E_i) = \alpha_i$. *Moreover, for a given weight* α, *the homogeneous component of* U_q^+ *with weight* α *is of finite dimension, and its basis are naturally parametrized by the multisegments of the same weight.*

Démonstration. Let $\mathbf{a} = \sum_{s=1}^{r} m_{i_s,j_s}[i_s, j_s]$ be a multisegment of weight α, note that here we identify the weight φ_i with α_i, and that

$$[i_1, j_1] \unlhd \cdots \unlhd [i_r, j_r] (\text{ cf. Def. 7.2.4})$$

Then we associate to $\mathbf{a}$ the element

$$(E_{j_1} \cdots E_{i_1}) \cdots (E_{j_r} \cdots E_{i_r}).$$

□

Notation 7.2.6. *For* $x \in U^+$ *be an element of degree* α, *we will denote* $\mathrm{wt}(x) = \alpha$.

Example 7.2.7. *For* $i \leq j$, *let* $\alpha_{ij} = \alpha_i + \cdots + \alpha_j$. *Consider the homogeneous components of* U^+ *with weight* $\alpha = 2\alpha_{12}$, *whose basis is given by*

$$E_1E_2E_1E_2, \ E_1E_1E_2E_2.$$

The element $E_1E_2E_1E_2$ *is parametrized by the multisegment* $[1] + [1,2] + [2]$, *while* $E_1E_1E_2E_2$ *is parametrized by the multisegment* $2[1] + 2[2]$.

In [27], Lusztig has defined certain bases for U_q^+ associated to the orientations of a Dynkin diagram, called PBW(Poincaré-Birkhoff-Witt) basis, which specializes to the classical PBW type bases. Following [25], we describe the PBW-basis

Definition 7.2.8. *We define*

$$E([i]) = E_i,\ E([i,j]) = [E_j[\cdots[E_{i+1}, E_i]_{q^{1/2}}\cdots]_{q^{1/2}}]_{q^{1/2}},$$

where $[x,y]_{q^{1/2}} = xy - q^{-1/2(\mathrm{wt}(x),\mathrm{wt}(y))}yx$. *More generally, let* $\mathbf{a} = \sum_s a_{i_s,j_s}[i_s,j_s]$ *be a multisegment, such that*

$$[i_1,j_1] \trianglelefteq \cdots \trianglelefteq [i_r,j_r] (\text{ cf. Def. 7.2.4}),$$

we define

$$E(\mathbf{a}) = \frac{1}{\prod_s [a_{i_s,j_s}]_{q^{1/2}}!} E([i_1,j_1])^{a_{i_1,j_1}} \cdots E([i_r,j_r])^{a_{i_r,j_r}},$$

here $[m]_{q^{1/2}} = \dfrac{q^{1/2m} - q^{-1/2m}}{q^{1/2} - q^{-1/2}}$ *for* $m \in \mathbb{Z}$ *and* $[m]_{q^{1/2}}! = [m]_{q^{1/2}}[m-1]_{q^{1/2}}\cdots[2]_{q^{1/2}}$.

Definition 7.2.9. *Let* $x \mapsto \overline{x}$ *be the involution defined as the unique ring automorphism of* U_q^+ *defined by*

$$\overline{q^{1/2}} = q^{-1/2},\ \overline{E_i} = E_i.$$

Proposition 7.2.10. *(cf. [27]) Let* $\mathcal{L} := \bigoplus_{\mathbf{a}\in\mathcal{O}} \mathbb{Z}[q^{1/2}]E(\mathbf{a}) \subseteq U_q^+$. *Then there exists a unique* $\mathbb{Q}(q^{1/2})$*-basis* $\{G(\mathbf{a}) : \mathbf{a} \in \mathcal{O}\}$ *of* U_q^+ *such that*

$$\overline{G(\mathbf{a})} = G(\mathbf{a}),\ G(\mathbf{a}) = E(\mathbf{a}) \text{ modulo } q^{1/2}\mathcal{L}.$$

This is called Lusztig's canonical basis.

Lusztig also gave a geometric description of his canonical basis in terms of the orbital varieties $\overline{O}_{\mathbf{a}}$.

Definition 7.2.11. *Let* $\mathbf{A}$ *be the group ring of* $\overline{\mathbb{Q}}_\ell^*$ *over* $\mathbb{Z}$. *Let* $\mathbf{K}_\varphi$ *be the Grothendieck group over* $\mathbf{A}$ *of the category of constructible,* G_φ*-equivariant* $\mathbb{Q}_\ell$ *sheaves over* E_φ*, considered as a variety over a finite field* $\mathbb{F}_q$.

Lemma 7.2.12. *(cf. [27]) The* $\mathbf{A}$*-module* K_φ *admits a basis* $\{\gamma_{\mathbf{a}} : \mathbf{a} \in S(\varphi)\}$ *indexed by the* G_φ *orbits of* E_φ*, where* $\gamma_{\mathbf{a}}$ *corresponds to the constant sheaf* $\overline{\mathbb{Q}}_\ell$ *on the orbit* $O_{\mathbf{a}}$*, extending by* 0 *to the complement.*

Definition 7.2.13. *Let $\varphi = \varphi_1 + \varphi_2 \in \mathcal{S}$. We define a diagram of varieties*

$$E_{\varphi_1} \times E_{\varphi_2} \xleftarrow{\beta} E' \xrightarrow{\beta'} E'' \xrightarrow{\beta''} E_{\varphi}, \tag{7.2.14}$$

where

$$\begin{aligned} E'' :=& \{(T, W) : W = \bigoplus W_i,\ W_i \subseteq V_{\varphi,i},\ T(W_i) \subseteq W_{i+1},\ \dim(W_i) = \varphi_2(i)\}, \\ E' :=& \{(T, W, \mu, \mu') : (T, W) \in E'',\ \mu : W \simeq V_{\varphi_2},\ \mu' : V_{\varphi}/W \simeq V_{\varphi_1}\}, \end{aligned}$$

and

$$\beta''((T, W)) = W,\ \beta'((T, W, \mu, \mu')) = (T, W),\ \beta((T, W, \mu, \mu')) = (T_1, T_2),$$

such that

$$T_1 = \mu' \circ T \circ \mu'^{-1},\ T_2 = \mu \circ T \circ \mu^{-1}.$$

Proposition 7.2.15. *(cf. [27]) The group $G_{\varphi} \times G_{\varphi_1} \times G_{\varphi_2}$ acts naturally on the varieties in the diagram (7.2.14) with G_{φ} acting trivially on $E_{\varphi_1} \times E_{\varphi_2}$ and $G_{\varphi_1} \times G_{\varphi_2}$ acting trivially on E_{φ}. And all the maps there are compatible with such actions. Moreover, we have*

(1) *The morphism β' is a principle $G_{\varphi_1} \times G_{\varphi_2}$-fibration.*

(2) *The morphism β is a locally trivial trivial fibration with smooth connected fibers.*

(3) *The morphism β'' is proper.*

Example 7.2.16. *Let $\varphi_1 = \chi_1,\ \varphi_2 = \chi_2$. Then $\varphi = \chi_1 + \chi_2$ and*

$$E_{\varphi_1} = E_{\varphi_2} = 0,\ E_{\varphi} = \overline{\mathbb{F}}_q.$$

Moreover, we have

$$E'' = \{(T, W) : W = V_{\varphi_2}, T \in \overline{\mathbb{F}}_q\} \simeq \overline{\mathbb{F}}_q,$$

and

$$E' = \{(T, W, \mu, \mu') : (T, W) \in E'', \mu, \mu' \in \overline{\mathbb{F}}_q^{\times}\} \simeq \overline{\mathbb{F}}_q \times (\overline{\mathbb{F}}_q^{\times})^2.$$

Corollary 7.2.17. *(cf. [27]) Let $\mathbf{a} \in \mathcal{O}(\varphi_1), \mathbf{a}' \in \mathcal{O}(\varphi_2)$. There exists a simple perverse sheaf(up to shift) $\mathcal{P}$ such that*

$$\beta^*(IC(\overline{O}_{\mathbf{a}}) \otimes IC(\overline{O}_{\mathbf{a}'})) = \beta'^*(\mathcal{P}).$$

Example 7.2.18. *As in example 7.2.16, let* $\mathbf{a} = \{[1]\}, \mathbf{a}' = \{[2]\}$*, then*

$$IC(\overline{O}_{\mathbf{a}}) = \overline{\mathbb{Q}}_\ell,\ IC(\overline{O}_{\mathbf{a}'}) = \overline{\mathbb{Q}}_\ell.$$

Hence if we let

$$\mathcal{P} = \overline{\mathbb{Q}}_\ell,$$

then

$$\beta^*(IC(\overline{O}_{\mathbf{a}}) \otimes IC(\overline{O}_{\mathbf{a}'})) = \beta''^*(\mathcal{P}).$$

Definition 7.2.19. *We define a multiplication*

$$IC(\overline{O}_{\mathbf{a}}) \star IC(\overline{O}_{\mathbf{a}'}) = \beta''_*(\mathcal{P}).$$

Example 7.2.20. *As in the example 7.2.16, we have*

$$IC(\overline{O}_{\mathbf{a}}) \star IC(\overline{O}_{\mathbf{a}'}) = \beta''_*(\mathcal{P}) = IC(E_\varphi),$$

note that here β'' *is an isomorphism.*

Proposition 7.2.21. *(cf. [27]) Let* $\mathbf{a} \in \mathcal{O}(\varphi_1), \mathbf{a}' \in \mathcal{O}(\varphi_2)$*. We associate to the intersection cohomology complex* $IC(\overline{O}_{\mathbf{a}})$

$$\widetilde{\gamma}_{\mathbf{a}} = \sum_{\mathbf{b} \geq \mathbf{a}} p_{\mathbf{b},\mathbf{a}}(q)\gamma_{\mathbf{b}},$$

where $p_{\mathbf{b},\mathbf{a}}(q)$ *is the formal alternative sum of eigenvalues of the Frobenius map on the stalks of the cohomology sheaves of* $IC(\overline{O}_{\mathbf{a}})$ *at any* $\mathbb{F}_q$ *rational point of* $O_{\mathbf{b}}$*. Moreover, the multiplication* $\star$ *gives a* $\mathbf{A}$*-bilinear map*

$$\mathbf{K}_{\varphi_1} \times \mathbf{K}_{\varphi_2} \to \mathbf{K}_\varphi,$$

which defines an associative algebra structure over $\mathbf{K} = \bigoplus_\varphi \mathbf{K}_\varphi$*.*

To relate the algebra K and $U^{\geq 0}$

Proposition 7.2.22. *([27] Prop. 9.8, Thm. 9.13)*

- *The elements* $\gamma_i := \gamma_{[i]}$ *for all* $i \in \mathbb{Z}$ *generate the algebra* $\mathbf{K}$ *over* $\mathbf{A}$*.*
- *Let* $U_{\mathbf{A}}^{\geq 0} = U_q^{\geq 0} \otimes_{\mathbb{Z}} \mathbf{A}$*. Then we have a unique* $\mathbf{A}$*-algebra morphism* $\Gamma : \mathbf{K} \to U_{\mathbf{A}}^{\geq 0}$ *such that*

$$\Gamma(\gamma_j) = K_j^{-j} E_j;$$

for all $j \in \mathbb{Z}$*. Moreover, for* $\varphi \in \mathcal{S}$*, let*

$$S(\varphi) = \sum_{i\in\mathbb{Z}} (\varphi(i) - 1)\varphi(i)/2 - \sum_{i\in\mathbb{Z}} \varphi(i)\varphi(i+1).$$

Then there is an $\mathbf{A}$*-linear map* $\Theta : K_\varphi \to U_{\mathbf{A}}^+$*, such that*

$$\Gamma(\xi) = q^{1/2S(\varphi)} K(\varphi)\Theta(\xi),$$

where $K(\varphi) = \prod_{i\in\mathbb{Z}} K_i^{-i\varphi(i)}$*.*

— *We have*

$$\Gamma(\gamma_{\mathbf{c}}) = q^{1/2(r-\delta_{\mathbf{c}})} K(\varphi_{\mathbf{c}}) E(\mathbf{c}),$$

where

$$r = \sum_i \varphi_{\mathbf{c}}(i)(\varphi_{\mathbf{c}}(i) - 1)(2i-1)/2 - \sum_i i\varphi_{\mathbf{c}}(i-1)\varphi_{\mathbf{c}}(i),$$

and $\delta_{\mathbf{c}}$ *is the co-dimension of the orbit* $O_{\mathbf{c}}$ *in* $E_{\varphi_{\mathbf{c}}}$*.*

— *We have*

$$\Theta(\gamma_{\mathbf{a}}) = q^{1/2\dim(O_{\mathbf{a}})} E(\mathbf{a}),\ \Theta(\widetilde{\gamma}_{\mathbf{a}}) = q^{1/2\dim(O_{\mathbf{a}})} G(\mathbf{a}).$$

Hence

$$G(\mathbf{a}) = \sum_{\mathbf{b}\geq\mathbf{a}} P_{\mathbf{b},\mathbf{a}}(q) E(\mathbf{b}).$$

Proposition 7.2.23. *The canonical basis of* U_q^+ *are almost orthogonal with respect to a scalar product introduced by Kashiwara [18], which are given by*

$$(E(\mathbf{a}), E(\mathbf{b})) = \frac{(1-q)^{\deg(\mathbf{a})}}{\prod_{i\leq j} h_{a_{ij}}(q)} \delta_{\mathbf{a},\mathbf{b}},$$

where $\mathbf{a} = \sum_{i\leq j} a_{ij}[i,j]$*,* $h_k(z) = (1-z)\cdots(1-z^k)$ *and* δ *is the Kronecker symbol([26]). And we have*

$$(G(\mathbf{a}), G(\mathbf{b})) = \delta_{\mathbf{a},\mathbf{b}} \ mod\ q^{1/2}\mathbf{A}.$$

Notation 7.2.24. *We denote by* $\{E^*(\mathbf{a})\}$ *and* $\{G^*(\mathbf{a})\}$ *the dual basis of* $\{E(\mathbf{a})\}$ *and* $\{G(\mathbf{a})\}$ *with respect to the Kashiwara scalar product.*

Proposition 7.2.25. *(cf. [25]) Let* $\mathbf{a} = \sum_{i\leq j} a_{ij}[i,j]$*. Then*

— *We have*

$$E^*(\mathbf{a}) = \frac{\prod_{i\le j} h_{a_{ij}}(q)}{(1-q)^{\deg(\mathbf{a})}} E(\mathbf{a}) = \overrightarrow{\prod_{ij}} q^{1/2\binom{a_{ij}}{2}} E^*([i,j])^{a_{ij}},$$

here the product is taken with respect to the order $\leq$.

— *And*

$$E^*(\mathbf{a}) = \sum_{\mathbf{b}\le\mathbf{a}} P_{\mathbf{a},\mathbf{b}}(q) G^*(\mathbf{b}).$$

Example 7.2.26. *Let* $\mathbf{a} = [1] + [2]$. *Then*

$$E^*(\mathbf{a}) = E([1])E([2]) = E(\mathbf{a}),$$

and

$$G^*([1,2]) = E^*([1,2]),$$
$$G^*(\mathbf{a}) = E^*(\mathbf{a}) - q^{1/2}E^*([1,2]).$$

Finally, we establish the relation of between the algebras $\mathcal{R}$ and U^+.

Definition 7.2.27. *Let* B *be the polynomial algebra generated by the set of coordinate functions* $\{t_{ij} : i < j\}$. *Following [25], we write* $t_{ii} = 1$, $t_{ij} = 0$ *if* $i > j$, *and indexed the non-trivial* $t_{i,j}$*'s by segments, namely,* $t_{[ij]} = t_{i,j-1}$ *for* $i < j$.

Now by corollary 1.2.3, we have the following

Proposition 7.2.28. *We have an algebra isomorphism* $\phi : B \simeq \mathcal{R}$ *by identifying* $t_{[ij]}$ *with* $L_{[ij]}$ *for all* $i < j$.

Definition 7.2.29. *Let* B_q *be the quantum analogue of* B *generate by* $\{T_{ij} : i < j\}$, *where* T_{ij} *is considered as the q-analogue of* t_{ij}. *Also, we write* $T_{ii} = 1$ *and* $T_{ij} = 0$ *if* $i > j$. *And we will indexed the non-trivial* T_{ij} *by* $T_{[i,j-1]}$. *The generators* T_s*'s satisfies the following relations (cf. [5]). Let* $s > s'$ *be two segments. Then*

$$T_{s'}T_s = \begin{cases} q^{-1/2(\mathrm{wt}(s'),\mathrm{wt}(s))}T_sT_{s'} + (q^{-1/2} - q^{1/2})T_{s\cap s'}T_{s\cup s'}, & \text{if } s \text{ and } s' \text{ are linked,} \\ q^{-1/2(\mathrm{wt}(s'),\mathrm{wt}(s))}T_sT_{s'}, \text{ otherwise}. \end{cases}$$

Proposition 7.2.30. *(cf. [25] Section 3.5) There exist an algebra isomorphic morphism*

$$\iota : U_q^+ \to B_q,$$

given by $\iota(E^*([i,j])) = T_{[i,j]}$. *Moreover, for* $\mathbf{a} = \sum_{i\leq j} a_{ij}[i,j]$, *we have*

$$\iota(E^*(\mathbf{a})) = \overrightarrow{\prod_{i\leq j}} q^{1/2\binom{a_{ij}}{2}} T_{[i,j]}^{a_{ij}},$$

here the multiplication is taken with respect to the order $<$.

Example 7.2.31. *Let* $\mathbf{a} = [1] + [2]$, *then*

$$\iota(E^*(\mathbf{a})) = T_{[1]}T_{[2]}.$$

Proposition 7.2.32. *By specializing at* $q = 1$, *the dual canonical basis* $\{G^*(\mathbf{a}) : \mathbf{a} \in \mathcal{O}\}$ *gives rise to a well defined basis for* B, *denoted by* $\{g^*(\mathbf{a}) : \mathbf{a} \in \mathcal{O}\}$. *Moreover, the morphism* ϕ *sends* $g^*(\mathbf{a})$ *to* $L_{\mathbf{a}}$ *for all* $\mathbf{a} \in \mathcal{O}$. [1]

7.3 Partial Derivatives and Poincaré's series

In this section we will deduce a geometric description for the partial derivatives, using results of last section.

Definition 7.3.1. *Kashiwara [18] introduced some* q*-derivations* E_i' *in* $\mathrm{End}(U_q^+)$ *for all* $i \in \mathbb{Z}$ *satisfying*

$$E_i'(E_j) = \delta_{ij},\ E_i'(uv) = E_i'(u)v + q^{-1/2(\alpha_i,\mathrm{wt}(u))}uE_i'(v).$$

Example 7.3.2. *Simple calculation shows that*

$$E_i'(E([j,k])) = \delta_{i,k}(1-q)E([j,k-1]),$$

by taking dual, we get

$$E_i'(E^*([j,k])) = \delta_{i,k}E^*([j,k-1]),$$

Proposition 7.3.3. *We have*

$$(E_i'(u), v) = (u, E_i v),$$

where $(,)$ *is the scalar product introduced in proposition 7.2.28.*

1. It is surprising that an isomorphism in the commutative world is governed by a non-commutative one, such phenomenon also happens in the theory of periods, where a period be a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}_n$ given by polynomial inequalities with rational coefficients, then there is a conjecture saying that two rational functions give the same period if and only if they can be transformed to each other according to three simple rules, see [21] chapter 1.

Note that by identifying the algebra U_q^+ and B_q via ι, we get a version of q-derivations in $\mathrm{End}(B_q)$.

Definition 7.3.4. *By specializing at $q = 1$, the q derivation E_i' gives a derivation e_i' of the algebra B by*

$$e_i'(t_{[jk]}) = \delta_{ik} t_{[j,k-1]}, \; e_i'(uv) = e_i'(u)v + ue_i'(v).$$

Proposition 7.3.5. *Let*

$$D^i := \sum_{n=0}^{\infty} \frac{1}{n!} e_i'^n.$$

Then the morphism $D^i : B \to B$ is an algebraic morphism. Moreover, if we identify the algebras $\mathcal{R}$ and B via ϕ, then the morphism D^i coincides with the partial derivative $\mathscr{D}^i$.

Démonstration. For $n \in \mathbb{N}$, we have

$$e'^n(uv) = \sum_{r+s=n} \binom{n}{r} e_i'^r(u) e_i'^s(v),$$

therefore

$$D^i(uv) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r+s=n} \binom{n}{r} e_i'^r(u) e_i'^s(v) = D^i(u) D^i(v).$$

Finally, to show that D^i and $\mathscr{D}^i$ coincides, it suffices to prove that

$$\phi \circ D^i(t_{[j,k]}) = \mathscr{D}^i \circ \phi(t_{[j,k]}),$$

but we have

$$D^i(t_{[j,k]} = t_{[j,k]} + \delta_{i,k} t_{[j,k-1]},$$

and

$$\mathscr{D}^i(L_{[j,k]}) = L_{j,k} + \delta_{i,k} L_{[j,k-1]}.$$

Therefore, we have

$$\phi \circ D^i(t_{[j,k]}) = \mathscr{D}^i \circ \phi(t_{[j,k]}).$$

□

Remark: Without specializing at $q = 1$, the operator D^i is not an algebraic morphism. To get an algebraic morphism at the level of U_q^+, one should consider not only the summation of the iteration of e_i''s but all the derivations, which gives rise to an embedding into the quantum shuffle algebras, cf. [24]. Next we show how to determine $\mathscr{D}^i(L_{\mathbf{a}})$ by the algebra $\mathbf{K}$ of Lusztig.

Lemma 7.3.6. *Let $n \in \mathbb{N}$, and $\mathbf{d} \in \mathcal{O}$. Then we have*

$$E_i^n G(\mathbf{b}) = \sum_{\mathbf{d}} (E_i^n G(\mathbf{b}), E^*(\mathbf{d})) E(\mathbf{d}) = \sum_{\mathbf{d}} (G(\mathbf{b}), E_i'^n E^*(\mathbf{d})) E(\mathbf{d}),$$

where $(,)$ is the Kashiwara scalar product. Moreover, for each $\mathbf{b}$ such that $(G(\mathbf{b}), E_i'^n E^*(\mathbf{d})) \neq 0$, we have

$$\mathrm{wt}(\mathbf{d}) = \mathrm{wt}(\mathbf{b}) + n\alpha_i.$$

Démonstration. This is by definition. □

Corollary 7.3.7. *Let $\mathbf{b} \preceq_k \mathbf{d}$ such that $wt(\mathbf{d}) = \mathrm{wt}(\mathbf{b}) + n\alpha_i$. Then $L_{\mathbf{b}}$ appears as a factor of $\frac{1}{r!} e_i'^r(\pi(\mathbf{d}))$ if and only if $r = n$.*

Démonstration. We know that for each $\mathbf{b} \preceq_k \mathbf{d}$, the representation $L_{\mathbf{b}}$ is a factor of $\mathscr{D}^i(\pi(\mathbf{d}))$. Now by proposition 7.3.5, $\mathscr{D}^i = \sum_r \frac{1}{r!} e_i'^r$, moreover, by lemma 7.3.6, factors of $\frac{1}{r!} e_i'^r(\pi(\mathbf{d}))$ always have weight $\mathrm{wt}(\mathbf{d}) - r\alpha_i$. Therefore we are done. □

Proposition 7.3.8. *Let $\mathbf{b} \preceq_k \mathbf{a}$, then there exists $\mathbf{c} \in S(\mathbf{a})$ such that $\mathbf{c} = \mathbf{b} + \ell[k]$. Then*

$$n_{\mathbf{b},\mathbf{a}} = \sum_i \dim \mathcal{H}^{2i}(IC(\overline{O}_{\ell[k]}) \star IC(\overline{O}_{\mathbf{b}}))_{\mathbf{a}}.$$

Démonstration. Note that by proposition 7.2.22, we have

$$\begin{aligned}
\Gamma(\widetilde{\gamma}_{\ell[k]} \star \widetilde{\gamma}_{\mathbf{b}}) &= \Gamma(\widetilde{\gamma}_{\ell[k]})\Gamma(\widetilde{\gamma}_{\mathbf{b}}) \\
&= q^{1/2(S(\varphi_{\ell[k]})+S(\varphi_{\mathbf{b}}))} K(\varphi_{\ell[k]}) K(\varphi_{\mathbf{b}}) \Theta(\widetilde{\gamma}_{\ell[k]}) \Theta(\widetilde{\gamma}_{\mathbf{b}}) \\
&= q^{1/2(S(\varphi_{\ell[k]})+S(\varphi_{\mathbf{b}}))} K(\varphi_{\ell[k]} + \varphi_{\mathbf{b}}) q^{1/2(\dim(O_{\ell[k]})+\dim(O_{\mathbf{b}}))} G(\ell[k]) G(\mathbf{b}).
\end{aligned}$$

Since we have

$$S(\varphi_{\ell[k]}) = \dim(O_{\ell[k]}) = 0,\ G(\ell[k]) = E(\ell[k]) = \frac{1}{[\ell]_{q^{1/2}}!} E_k^{\ell},\ \varphi_{\ell[k]} + \varphi_{\mathbf{b}} = \varphi_{\mathbf{a}},$$

so

$$\Gamma(\widetilde{\gamma}_{\ell[k]} \star \widetilde{\gamma}_{\mathbf{b}}) = \frac{1}{[\ell]_{q^{1/2}}!} q^{1/2(S(\varphi_{\mathbf{b}})+\dim(O_{\mathbf{b}}))} K(\varphi_{\mathbf{a}}) E_k^{\ell} G(\mathbf{b}).$$

And

$$\begin{aligned}
\Gamma(\gamma_{\mathbf{d}}) &= q^{1/2S(\varphi_{\mathbf{d}})} K(\varphi_{\mathbf{d}}) \Theta(\gamma_{\mathbf{d}}) \\
&= q^{1/2(S(\varphi_{\mathbf{d}})+\dim(O_{\mathbf{d}}))} K(\varphi_{\mathbf{d}}) E(\mathbf{d}).
\end{aligned}$$

Now write

$$\widetilde{\gamma}_{\ell[k]} \star \widetilde{\gamma}_{\mathbf{b}} = \sum_{\mathbf{b} \preceq_k \mathbf{d}, \varphi_{\mathbf{d}} = \varphi_{\mathbf{a}}} p_{\mathbf{d},\mathbf{b}}(q) \gamma_{\mathbf{d}},$$

with

$$p_{\mathbf{d},\mathbf{b}}(q) = \sum_i q^i \mathcal{H}^{2i}(IC(\overline{O}_{\ell[k]}) \star IC(\overline{O}_{\mathbf{b}}))_{\mathbf{d}}.$$

Applying Γ gives

$$\frac{1}{[\ell]_{q^{1/2}}!} q^{1/2(S(\varphi_{\mathbf{b}})+\dim(O_{\mathbf{b}}))} K(\varphi_{\mathbf{a}}) E_k^{\ell} G(\mathbf{b}) = \sum_{\mathbf{b} \preceq_k \mathbf{d}, \varphi_{\mathbf{d}} = \varphi_{\mathbf{a}}} p_{\mathbf{d},\mathbf{b}}(q) q^{1/2(S(\varphi_{\mathbf{d}})+\dim(O_{\mathbf{d}}))} K(\varphi_{\mathbf{d}}) E(\mathbf{d}).$$

Hence

$$E_k^{\ell} G(\mathbf{b}) = [\ell]_{q^{1/2}}! \sum_{\mathbf{b} \preceq_k \mathbf{d}, \varphi_{\mathbf{d}} = \varphi_{\mathbf{a}}} p_{\mathbf{d},\mathbf{b}}(q) q^{1/2(S(\varphi_{\mathbf{d}})+\dim(O_{\mathbf{d}})-S(\varphi_{\mathbf{b}})-\dim(O_{\mathbf{b}}))} E(\mathbf{d}),$$

now compare with lemma 7.3.6, we get

$$(G(\mathbf{b}), E_i'^n E^*(\mathbf{d})) = [\ell]_{q^{1/2}}! p_{\mathbf{d},\mathbf{b}}(q) q^{1/2(S(\varphi_{\mathbf{d}})+\dim(O_{\mathbf{d}})-S(\varphi_{\mathbf{b}})-\dim(O_{\mathbf{b}}))}.$$

Finally, we write

$$\frac{1}{[\ell]_{q^{1/2}}!} E_i'^n E^*(\mathbf{d}) = \sum_{\mathbf{b}} n_{\mathbf{b},\mathbf{d}}(q) G^*(\mathbf{b}),$$

by applying the scalar product, we get

$$n_{\mathbf{b},\mathbf{d}}(q) = (G(\mathbf{b}), \frac{1}{[\ell]_{q^{1/2}}!} E_i'^n E^*(\mathbf{d})) = p_{\mathbf{d},\mathbf{b}}(q) q^{1/2(S(\varphi_{\mathbf{d}})+\dim(O_{\mathbf{d}})-S(\varphi_{\mathbf{b}})-\dim(O_{\mathbf{b}}))}.$$

Hence, by specializing at $q = 1$, we have

$$n_{\mathbf{b},\mathbf{d}} = p_{\mathbf{d},\mathbf{b}}(1).$$

Now take $\mathbf{d} = \mathbf{a}$, we get the formula in our proposition. □

7.4 A formula for Lusztig's product

In this section we will find a geometric way to calculate Lusztig's product in special case, which allows us to determine the partial derivatives in the following sections.

Definition 7.4.1. *Let $k \in \mathbb{Z}$. We say that* $\mathbf{a}$ *satisfies the assumption* $(\mathbf{A}_k)$ *if it satisfies the following conditions*[2]

(1) *We have*

$$\max\{b(\Delta) : \Delta \in \mathbf{a}\} + 1 < \min\{e(\Delta) : \Delta \in \mathbf{a}\}.$$

(2) *Moreover, we have* $\varphi_{e(\mathbf{a})}(k) \neq 0$ *and* $\varphi_{e(\mathbf{a})}(k+1) = 0$.

Lemma 7.4.2. *Let* $\mathbf{a}$ *be a multisegment satisfying the assumption* $(\mathbf{A}_k)$. *Then* $\mathbf{a}$ *is of parabolic type. Moreover, The set* $S(\varphi_{\mathbf{a}})$ *contains a unique maximal element satisfying the assumption* $(\mathbf{A}_k)$, *denoted by* $\mathbf{a}_{\mathrm{Id}}$.

Démonstration. Let $b(\mathbf{a}) = \{k_1 \leq \cdots \leq k_r\}$, $e(\mathbf{a}) = \{\ell_1 \leq \cdots \leq \ell_r\}$.
Then by proposition 6.3.2, we know that there exists an element $w \in S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}$, such that

$$\mathbf{a} = \sum_j [k_j, \ell_{w(j)}].$$

Let

$$\mathbf{a}_{\mathrm{Id}} = \sum_j [k_j, \ell_j],$$

now by proposition 6.2.24, we know that $\mathbf{a} \leq \mathbf{a}_{\mathrm{Id}}$. Finally, $\mathbf{a}_{\mathrm{Id}}$ depends only on $b(\mathbf{a})$ and $e(\mathbf{a})$, not on $\mathbf{a}$, which shows that $\mathbf{a}_{\mathrm{Id}}$ is the maximal element in $S(\varphi_{\mathbf{a}})$ satisfying the assumption $(\mathbf{A}_k)$. □

Lemma 7.4.3. *Suppose that* $\mathbf{a}$ *is a multisegment satisfying the hypothesis* $(\mathbf{A}_k)$, *then*

(1) $\widetilde{S}(\mathbf{a})_k = S(\mathbf{a})$ *;*

(2) *we have*

$$X_{\mathbf{a}}^k = Y_{\mathbf{a}} = \coprod_{\mathbf{c} \in S(\mathbf{a})} O_{\mathbf{c}}.$$

Démonstration. Note that by assumption

$$\max\{b(\Delta) : \Delta \in \mathbf{a}\} < \min\{e(\Delta) : \Delta \in \mathbf{a}\}.$$

This ensures that for any $\mathbf{c} \in S(\mathbf{a})$, we have $\varphi_{e(\mathbf{c})}(k) = \varphi_{e(\mathbf{a})}(k)$, hence by definition $\mathbf{c} \in \widetilde{S}(\mathbf{a})_k$. This proves (1), and (2) follows from (1). □

2. Since here we only work with the partial derivative $\mathscr{D}^k$ with $k \in \mathbb{Z}$, for every multisegment, we can always use the reduction method to increase the length of segments from the left, so that at some point we arrive at the situation of our assumption $(\mathbf{A}_k)$, therefore we do not lose the generality.

Lemma 7.4.4. *Let* $\mathbf{a}$ *be a multisegment satisfying the assumption* $(\mathbf{A}_k)$ *and* $\mathbf{a} = \mathbf{a}_{\mathrm{Id}}$. *Let* $\ell \in \mathbb{N}$ *such that* $\ell \leq \varphi_{e(\mathbf{a})}(k)$ *and* $\varphi \in \mathcal{S}$ *such that*

$$\varphi + \ell\chi_{[k]} = \varphi_{\mathbf{a}}.$$

Then for $\mathbf{b} \in S(\varphi)$, *we have* $\mathbf{b} \preceq_k \mathbf{a}$ *if and only if* $\mathbf{b}^{(k)} \leq \mathbf{a}^{(k)}$ *and* $\varphi_{e(\mathbf{b})}(k-1) = \ell + \varphi_{e(\mathbf{a})}(k-1)$.

Démonstration. Let $\mathbf{b} \in S(\varphi)$ such that $\mathbf{b} \preceq_k \mathbf{a}$, then by proposition 7.1.7, we know that $\mathbf{b} = \mathbf{c}_\Gamma$ for some $\mathbf{c} \in S(\mathbf{a})$ and $\Gamma \subseteq \mathbf{c}(k)$. Therefore

$$\mathbf{b}^{(k)} = \mathbf{c}^{(k)} \leq \mathbf{a}^{(k)}$$

by the lemma above. And by definition of $\mathbf{c}_\Gamma$, we know that

$$\varphi_{e(\mathbf{b})}(k-1) = \ell + \varphi_{e(\mathbf{c})}(k-1).$$

Now applying the fact that $\mathbf{a}$ satisfies the assumption $(\mathbf{A}_k)$, we deduce that

$$\varphi_{e(\mathbf{c})}(k-1) = \varphi_{e(\mathbf{a})}(k-1).$$

Conversely, let $\mathbf{b} \in S(\varphi)$ be a multisegment such that $\mathbf{b}^{(k)} \leq \mathbf{a}^{(k)}$ and $\varphi_{e(\mathbf{b})}(k-1) = \ell + \varphi_{e(\mathbf{a})}(k-1)$.
We deduce from $\mathbf{b}^{(k)} \leq \mathbf{a}^{(k)}$ that

$$\mathbf{b} \leq \mathbf{a}^{(k)} + \varphi_{e(\mathbf{b})}(k)[k],$$

from which we obtain

$$\varphi_{\mathbf{b}} = \varphi_{\mathbf{a}^{(k)}} + \varphi_{e(\mathbf{b})}(k)\chi_{[k]}.$$

By assumption, we know that

$$\varphi_{\mathbf{b}} + \ell\chi_{[k]} = \varphi_{\mathbf{a}}.$$

Combining with the formula

$$\varphi_{\mathbf{a}} = \varphi_{\mathbf{a}^{(k)}} + \varphi_{e(\mathbf{a})}(k)\chi_{[k]},$$

we have

$$\varphi_{e(\mathbf{a})}(k) = \varphi_{e(b)}(k) + \ell.$$

Now that for any $\Delta \in \mathbf{a}$, if $e(\Delta) = k$, then $b(\mathbf{a}) \leq k-1$. Therefore we have

$$\varphi_{e(\mathbf{a}^{(k)})}(k-1) = \varphi_{e(\mathbf{a})}(k-1) + \varphi_{e(\mathbf{a})}(k).$$

Applying the formula $\varphi_{e(\mathbf{b})}(k-1) = \ell + \varphi_{e(\mathbf{a})}(k-1)$, $\varphi_{e(\mathbf{b}^{(k)})}(k-1) = \varphi_{e(\mathbf{a}^{(k)})}(k-1)$, we get

$$\varphi_{e(\mathbf{b}^{(k)})}(k-1) = \varphi_{e(\mathbf{b})}(k-1) + \varphi_{e(\mathbf{b})}(k).$$

Such a formula implies that for $\Delta \in \mathbf{b}$, if $e(\Delta) = k$, then $b(\Delta) \leq k-1$.
Let $b(\mathbf{a}) = \{k_1 \leq \cdots \leq k_r\}$, $e(\mathbf{a}) = \{\ell_1 \leq \cdots \leq \ell_r\}$. The assumption that $\mathbf{a} = \mathbf{a}_{\mathrm{Id}}$ implies that

$$\mathbf{a} = \sum_i [k_i, \ell_i]$$

Suppose that

$$\mathbf{a}(k) = \{[k_i, \ell_i] : i_0 \leq i \leq i_1\}.$$

Take $\Gamma = \{[k_i, \ell_i] : i_0 + \ell \leq i \leq i_1\}$ and

$$\mathbf{a}' = \mathbf{a}_\Gamma.$$

Then $\mathbf{a}' \preceq_k \mathbf{a}$. Note that $\mathbf{a}'$ is a multisegment of parabolic type which corresponds to the identify in some symmetric group, cf. notation 6.2.21. Finally, proposition 6.2.24 implies that $\mathbf{b} \in S(\mathbf{a}')$.

□

Lemma 7.4.5. *Assume that* $\mathbf{a}$ *is a multisegment satisfying* $(\mathbf{A}_k)$. *Let* $r \leq \varphi_{e(\mathbf{a})}(k)$ *and* $\mathbf{d} = \mathbf{a} + r[k+1]$. *Then we have* $X_{\mathbf{d}}^{k+1} = Y_{\mathbf{d}}$ *and for a fixed subspace* W *of* $V_{\varphi_{\mathbf{d}},k+1}$ *of dimension* r, *the open immersion*

$$\tau_W : (X_{\mathbf{d}}^{k+1})_W \to (Z^{k+1,\mathbf{d}})_W \times \mathrm{Hom}(V_{\varphi_{\mathbf{d}},k}, W)$$

is an isomorphism.

Démonstration. Note that our assumption on $\mathbf{a}$ ensures that $X_{\mathbf{d}}^{k+1} = Y_{\mathbf{d}}$ since we have $\mathbf{d}_{\min} \in \widetilde{S}(\mathbf{d})_{k+1}$. It suffices to show that τ_W is surjective. Let $(T^{(k)}, T_0) \in (Z^{k+1,\mathbf{d}})_W \times \mathrm{Hom}(V_{\varphi_{\mathbf{d}},k}, W)$, by fixing a splitting $V_{\varphi_{\mathbf{d}},k+1} = W \oplus V_{\varphi_{\mathbf{d}},k+1}/W$, we define

$$T'|_{V_{\varphi_{\mathbf{d}},k}} = T_0 \oplus T^{(k+1)}|_{V_{\varphi_{\mathbf{d}},k}},$$

$$T'|_{V_{\varphi_{\mathbf{d}},k+1}} = T^{(k+1)}|_{V_{\varphi_{\mathbf{d}},k+1}/W} \circ p_W,$$

$$T'|_{V_{\varphi_{\mathbf{d}},i}} = T^{(k+1)}, \text{ for } i \neq k, k+1,$$

where $p_w : V_{\varphi_{\mathbf{d}}} \to V_{\varphi_{\mathbf{d}},k}/W$ is the canonical projection. Then we have $T' \in Y_{\mathbf{d}}$ hence $T' \in (X_{\mathbf{d}}^{k+1})_W$. Now since by construction we have $\tau_W(T') = (T^{(k)}, T_0)$, we are done. □

Definition 7.4.6. *Assume that* $\mathbf{a}$ *is a multisegment satisfying* $(\mathbf{A}_k)$ *and* $\mathbf{d} = \mathbf{a} + r[k+1]$ *for some* $r \leq \varphi_{e(\mathbf{a})}(k)$. *Let* $\mathfrak{X}_{\mathbf{d}}$ *be the open sub-variety of* $X_{\mathbf{d}}^{k+1}$ *consisting of those orbits* $O_{\mathbf{c}}$ *with* $\mathbf{c} \in S(\mathbf{d})$, *such that* $\varphi_{e(\mathbf{c})}(k) + r = \varphi_{e(\mathbf{a})}(k)$.

Definition 7.4.7. *Let* V *be a vector space and* $\ell_1 < \ell_2 < \dim(V)$ *be two integers. We define*

$$Gr(\ell_1, \ell_2, V) = \{(U_1, U_2) : U_1 \subseteq U_2 \subseteq V, \dim(U_1) = \ell_1, \dim(U_2) = \ell_2\}.$$

Definition 7.4.8. *Let* ℓ *be an integer and* $\mathbf{a}$ *be a multisegment. We let*

$$E''_{\mathbf{a}} = \{(T', W') : T' \in Y_{\mathbf{a}}, W' \in Gr(\ell, \ker(T'|_{V_{\varphi_{\mathbf{a}},k}}))\}.$$

Note that we have a canonical morphism

$$\alpha' : E''_{\mathbf{a}} \to Gr(\ell, \varphi_{e(\mathbf{a})}(k), V_{\varphi_{\mathbf{a}},k})$$

sending (T', W') *to* $(W', \ker(T'|_{V_{\varphi_{\mathbf{a}},k}}))$.

Proposition 7.4.9. *The morphism* α' *is a fibration.*

Démonstration. The morphism α' is equivariant under the action of $GL(V_{\varphi_{\mathbf{a}},k})$. The same proof as in proposition 3.3.9 shows that the morphism α' is actually a $P_{(U_1,U_2)}$ bundle, where $P_{(U_1,U_2)}$ is a subgroup of $GL(V_{\varphi_{\mathbf{a}},k})$ which fixes the given element (U_1, U_2). Now we take a Zariski neighborhood $\mathfrak{U}$ of (U_1, U_2) over which we have the trivialization

$$\gamma : \alpha'^{-1}(\mathfrak{U}) \simeq \alpha'^{-1}((U_1, U_2)) \times \mathfrak{U},$$

such an isomorphism comes from a section

$$s : \mathfrak{U} \to GL(V_{\varphi_{\mathbf{a}},k}), \ s((U_1, U_2)) = Id,$$

by $\gamma((T, W')) = [(g^{-1}T, g^{-1}W'), \alpha'((T, W'))]$, where $g = s(\alpha'((T, W')))$. We remark that the existence of the section s is guaranteed by local triviality of $GL(V_{\varphi_{\mathbf{a}},k}) \to GL(V_{\varphi_{\mathbf{a}},k})/P_{(U_1,U_2)}$, cf. [32], § 4. □

Proposition 7.4.10. *Assume that* $\mathbf{a}$ *is a multisegment satisfying* $(\mathbf{A}_k)$ *and* $\mathbf{d} = \mathbf{a} + r[k+1]$ *for some* $r \leq \varphi_{e(\mathbf{a})}(k)$. *Let* $\ell \in \mathbb{N}$ *such that* $r + \ell = \varphi_{e(\mathbf{a})}(k)$ *and* W *a subspace of* $V_{\varphi_{\mathbf{d}},k+1}$ *such that* $\dim(W) = r$. *We have a canonical projection*

$$p : (\mathfrak{X}_{\mathbf{d}})_W \to E''_{\mathbf{a}}$$

where for $T \in (\mathfrak{X}_{\mathbf{d}})_W$ *with* $\tau_W(T) = (T_1, T_0) \in (Z^{k+1,\mathbf{d}})_W \times \mathrm{Hom}(V_{\varphi_{\mathbf{d}},k}, W)$, *we define* $p(T) = (T_1, \ker(T_0|_{W_1}))$, *where* $W_1 = \ker(T_1|_{V_{\varphi_{\mathbf{d}},k}})$ *(Note that here we identify* $(Z^{k+1,\mathbf{d}})_W$ *with* $Y_{\mathbf{a}}$, *see the remark after proposition 3.3.12). Moreover, let* $U_1 \subseteq U_2 \subseteq V_{\varphi_{\mathbf{d}},k}$ *be subspaces such that* $\dim(U_1) = \ell$, $\dim(U_2) = \varphi_{e(\mathbf{a})}(k)$, *then* p *is a fibration with fiber*

$$\{T \in \mathrm{Hom}(V_{\varphi_{\mathbf{d}},k}, W) : \ker(T|_{U_2}) = U_1\}.$$

Démonstration. We show that p is well defined. Since by definition of $(\mathfrak{X}_{\mathbf{d}})_W$ we know that

$$\dim(W) + \dim(\ker(T_0|_{\ker(T_1|_{V_{\varphi_{\mathbf{d}},k}})})) = \dim(\ker(T_1|_{V_{\varphi_{\mathbf{d}},k}})),$$

hence to see that

$$\ell = \dim(\ker(T_0|_{\ker(T_1|_{V_{\varphi_{\mathbf{d}},k}})})),$$

it suffices to show that

$$\varphi_{e(\mathbf{a})}(k) = \dim(\ker(T_1|_{V_{\varphi_{\mathbf{d}},k}})),$$

this follows from the fact that $\mathbf{a} = \mathbf{d}^{(k+1)}$. Finally, we show that p is a fibration. Note that by definition, the fiber of p is isomorphic to

$$\{T \in \mathrm{Hom}(V_{\varphi_{\mathbf{d}},k}, W) : \ker(T|_{U_2}) = U_1\}.$$

So it suffices to show that it is locally trivial. To show this, we consider the open subset $\mathfrak{U}$ in $E''_{\mathbf{a}}$ as constructed in the proof of proposition 7.4.9.
Now we construct a trivialization for p

$$\varrho : p^{-1}(\alpha'^{-1}(\mathfrak{U})) \to \alpha'^{-1}(\mathfrak{U}) \times \{T \in \mathrm{Hom}(V_{\varphi_{\mathbf{d}},k}, W) : \ker(T|_{U_2}) = U_1\}$$

with $\varrho(T) = [(T_1, W'), g^{-1}(T_0)]$, where $g = s((W', W_1))$, $W_1 = \ker(T_1|_{V_{\varphi_{\mathbf{d}},k}})$.
Note that given

$$[(T_1, W'), T_0] \in \alpha'^{-1}(\mathfrak{U}) \times \{T \in \mathrm{Hom}(V_{\varphi_{\mathbf{d}},k}, W) : \ker(T|_{U_2}) = U_1\},$$

take $W_1 = \ker(T_1|_{V_{\varphi_{\mathbf{d}},k}})$ then $(W', W_1) \in \mathfrak{U}$, hence $g = s((W', W_1))$ exists. Let $T'_0 = gT_0$. Then $T = \tau_W^{-1}((T_1, T'_0)) \in p^{-1}(\alpha'^{-1}(\mathfrak{U}))$.

□

Definition 7.4.11. *Let $\ell + \dim(W) = \varphi_{e(\mathbf{a})}(k)$. We define $\mathfrak{Y}_{\mathbf{a}}$ to be the set of pairs (T, U) satisfying*

(1) *$U \in Gr(\ell, V_{\varphi_{\mathbf{a}},k})$, $T \in \mathrm{End}(V_{\varphi_{\mathbf{a}}}/U)$ of degree 1 ;*

(2) *$T \in O_{\mathbf{b}}$ for some $\mathbf{b} \preceq_k \mathbf{a}$.*

And we have a canonical projection

$$\sigma : E''_{\mathbf{a}} \to \mathfrak{Y}_{\mathbf{a}}$$

for $(T, U) \in E''_{\mathbf{a}}$, we associate

$$\sigma((T, U)) = (T', U)$$

where $T' \in \mathrm{End}(V_{\varphi_{\mathbf{a}}}/U)$ *is the quotient of* T. *Also, we have a morphism*

$$\sigma' : \mathfrak{Y}_{\mathbf{a}} \to Gr(\ell, \varphi_{e(\mathbf{a})}(k), V_{\varphi_{\mathbf{a}},k}),$$

by

$$\sigma'((T,U)) = (U, \pi^{-1}(\ker(T|_{V_{\varphi_{\mathbf{a}},k}/U}))).$$

where $\pi : V_{\varphi_{\mathbf{a}},k} \to V_{\varphi_{\mathbf{a}},k}/U$ *be the canonical projection.*

Lemma 7.4.12. *We have for* $T \in \mathfrak{Y}_{\mathbf{a}}$,

(1) $\gamma_k(T) \in Z^{k,\mathbf{a}}$ *;*

(2) $T|_{V_{\varphi_{\mathbf{a}},k-1}}$ *is surjective ;*

(3) $\dim(\ker(T|_{V_{\varphi_{\mathbf{a}},k}/U})) = \dim(W)$ *;*

for γ_k, *see definition 3.3.7.*

Démonstration. (1)To show $\gamma_k(T) \in Z^{k,\mathbf{a}}$, it suffices to show that for $\mathbf{b} \preceq_k \mathbf{a}$, we have $\mathbf{b}^{(k)} \leq \mathbf{a}^{(k)}$. Note that by proposition 7.1.7, there exists $\mathbf{c} \in S(\mathbf{a})$ and $\Gamma \subseteq \mathbf{c}(k)$, such that

$$\mathbf{b} = \mathbf{c}_{\Gamma}.$$

Now by lemma 7.4.3, we have $\mathbf{c} \in \widetilde{S}(\mathbf{a})_k$, which implies that

$$\mathbf{b}^{(k)} = \mathbf{c}^{(k)} \leq \mathbf{a}^{(k)}.$$

(2)By definition for any $T \in \mathfrak{Y}_{\mathbf{a}}$, we have $Y \in O_{\mathbf{b}}$ for some $\mathbf{b} \preceq_k \mathbf{a}$. By the fact that $\mathbf{a}$ satisfies the assumption $(\mathbf{A}_k)$, we know that any $\mathbf{c} \in S(\mathbf{a})$ satisfies $(\mathbf{A}_k)$, hence

$$\mathbf{b} = \mathbf{c}_{\Gamma}$$

cannot contain a segment which starts at k, therefore $T|_{V_{\varphi_{\mathbf{a}},k-1}}$ is surjective. (3) Note that from the definition of $\mathfrak{Y}$, we know that for $T \in \mathfrak{Y}_{\mathbf{a}}$, we have $T \in O_{\mathbf{b}}$ for some $\mathbf{b} \preceq_k \mathbf{a}$. Now it follows

$$\ker(T|_{V_{\varphi_{\mathbf{a}},k}/U}) = \varphi_{e(\mathbf{a})}(k) - \ell = \dim(W).$$

□

Proposition 7.4.13. *Let* $\mathbf{a}$ *be a multisegment satisfying the assumption* $(\mathbf{A}_k)$. *Then the morphism* σ' *is a fibration. Moreover, if we assume that* $\mathbf{a} = \mathbf{a}_{\mathrm{Id}}$, *cf. lemma 7.4.2, then the morphism* σ *is also a fibration.*

Démonstration. We first show that σ' is locally trivial. We observe that the group $GL(V_{\varphi_{\mathbf{a}},k})$ acts both on the source and target of σ' in such a way that σ' is $GL(V_{\varphi_{\mathbf{a}},k})$-equivariant. As in the proof of proposition 7.4.9, let $\mathfrak{U} \subseteq Gr(\ell, \varphi_{e(\mathbf{a})}(k), V_{\varphi_{\mathbf{a}},k})$ be a neighborhood of a given element (U_1, U_2) such that we have a section

$$s : \mathfrak{U} \to GL(V_{\varphi_{\mathbf{a}},k}),\ \ s((U_1, U_2)) = Id.$$

Note that in this case we have a natural trivialization of σ' by

$$\sigma' : \beta'^{-1}(\mathfrak{U}) \simeq \mathfrak{U} \times \beta'^{-1}((U_1, U_2))$$

by $\sigma'((T, U)) = [(U, \pi^{-1}(\ker(T|_{V_{\varphi_{\mathbf{a}},k}}))), g^{-1}((T, U))]$ with $g = s((U, \pi^{-1}(\ker(T|_{V_{\varphi_{\mathbf{a}},k}}))))$.
Finally, we show that σ is surjective and locally trivial.
We observe that $\alpha' = \sigma'\sigma$ and σ preserves fibers. Now we fix a neighborhood $\mathfrak{U}$ as above and get a commutative diagram

$$\begin{array}{ccc} \alpha'^{-1}(\mathfrak{U}) & \xrightarrow{\gamma} & \mathfrak{U} \times \alpha'^{-1}((U_1, U_2)) \\ \downarrow & & \downarrow \delta \\ \sigma'^{-1}(\mathfrak{U}) & \xrightarrow{\gamma'} & \mathfrak{U} \times \sigma'^{-1}((U_1, U_2)) \end{array}$$

where $\delta([x, T]) = [x, \sigma(T)]$ for any $x \in \mathfrak{U}$ and $T \in \alpha'^{-1}((U_1, U_2))$. Therefore to show that σ is locally trivial , it suffices to show that it is locally trivial when restricted to the fiber $\alpha'^{-1}((U_1, U_2))$. Note that we have

$$\alpha'^{-1}((U_1, U_2)) \simeq \{T \in Y_{\mathbf{a}} : \ker(T|_{V_{\varphi_{\mathbf{a}},k}}) = U_2\} \simeq (X_{\mathbf{a}}^k)_{U_2} \hookrightarrow Y_{\mathbf{a}^{(k)}} \times \mathrm{Hom}(V_{\varphi_{\mathbf{a}},k-1}, U_2)$$

and

$$\begin{aligned}\sigma'^{-1}((U_1, U_2)) \simeq \{T : T \in \mathrm{End}(V_{\varphi_{\mathbf{a}}}/U_1) \text{ of degree } 1, \ker(T|_{V_{\varphi_{\mathbf{a}},k}/U_1}) = U_2/U_1, \\ T \in O_{\mathbf{b}},\ \text{for some } \mathbf{b} \preceq_k \mathbf{a}\} \hookrightarrow Y_{\mathbf{a}^{(k)}} \times \mathrm{Hom}(V_{\varphi_{\mathbf{a}},k-1}, U_2/U_1).\end{aligned}$$

Note that the canonical morphism

$$\mathrm{Hom}(V_{\varphi_{\mathbf{a}},k-1}, U_2) \to \mathrm{Hom}(V_{\varphi_{\mathbf{a}},k-1}, U_2/U_1)$$

is a fibration. Hence to show that

$$\alpha'^{-1}((U_1, U_2)) \to \sigma'^{-1}((U_1, U_2))$$

is a fibration, it suffices to show that $\sigma|_{\alpha'^{-1}((U_1,U_2))}$ is surjective with isomorphic fibers everywhere . Let $(T, U_1) \in \sigma'^{-1}((U_1, U_2))$ with

$$\tau_{U_2/U_1}(T) = (T_0, q_0) \in Y_{\mathbf{a}^{(k)}} \times \mathrm{Hom}(V_{\varphi_{\mathbf{a}},k-1}, U_2/U_1),$$

where τ_{U_2/U_1} is the morphism in definition 3.3.13. We fix a splitting $U_2 \simeq U_2/U_1 \oplus U_1$. Now to give $(T', U_1) \in \sigma^{-1}((T, U_1))$ amounts to give $q_1 \in \mathrm{Hom}(V_{\varphi_{\mathbf{a}},k-1}, U_1)$ such that

$$\tau_{U_2}(T') = (T_0, q_0 \oplus q_1).$$

Note that by lemma 7.4.4, the condition $\mathbf{a} = \mathbf{a}_{\mathrm{Id}}$ implies that T' lies in $(X^k_{\mathbf{a}})_{U_2}$ if and only if q_1 satisfies

$$\dim(\ker(q_0 \oplus q_1|_{\ker(T_0|_{V_{\varphi_{\mathbf{a}},k-1}})})) = \varphi_{e(\mathbf{a})}(k-1),$$

which is an open condition. Therefore σ is surjective. By definition of $\mathfrak{Y}_{\mathbf{a}}$, we know that

$$\dim(\ker(q_0|_{\ker(T_0|_{V_{\varphi_{\mathbf{a}},k-1}})})) = \varphi_{e(\mathbf{a})}(k-1) + \ell,$$

therefore if we denote $W_1 = \ker(q_0|_{\ker(T_0|_{V_{\varphi_{\mathbf{a}},k-1}})})$, then q_1 satisfies that

$$\dim(\ker(q_1|_{\ker(T_0|_{V_{\varphi_{\mathbf{a}},k-1}})}) \cap W_1) = \varphi_{e(\mathbf{a})}(k-1).$$

Such a condition is independent of the pair (T_0, q_0) since we always have $\dim(\ker(T_0|_{V_{\varphi_{\mathbf{a}},k-1}})) = \varphi_{e(\mathbf{a})}(k-1) + \varphi_{e(\mathbf{a})}(k)$ and $\dim(W_1) = \varphi_{e(\mathbf{a})}(k-1) + \ell$. □

We return to the morphism p and σ.

Lemma 7.4.14. *Note that an element of G_{φ_d} stabilizes $(\mathfrak{X}_{\mathbf{d}})_W$ if and only if it stabilizes W. Let $G_{\varphi_{\mathbf{d}},W}$ be the stabilizer of W, then for $\mathbf{c} \leq \mathbf{d}$, and $T \in O_{\mathbf{c}} \cap (\mathfrak{X}_{\mathbf{d}})_W$, we have*

$$O_{\mathbf{c}} \cap (\mathfrak{X}_{\mathbf{d}})_W = G_{\varphi_{\mathbf{d}},W} T.$$

Démonstration. Recall that from proposition 5.3.2, we have

$$\begin{array}{ccc} X^{k+1}_{\mathbf{d}} & \xleftarrow{\ \delta\ } & GL_{\varphi_{\mathbf{d}}(k+1)} \times_{P_W} \alpha^{-1}(W) \\ \downarrow & \swarrow & \\ Gr(\ell_{k+1}, V_{\varphi_{\mathbf{d}}}) & & \end{array}$$

where $\ell_{k+1} = \varphi_{e(\mathbf{d})}(k+1)$. Note that we have

$$G_{\varphi_{\mathbf{d}},W} = \cdots \times G_{\varphi_{\mathbf{d}},k} \times P_W \times G_{\varphi_{\mathbf{d}},k+2} \times \cdots,$$

where $G_{\varphi_{\mathbf{d}},i} = GL(V_{\varphi_{\mathbf{d}},i})$. From this diagram we observe that the orbits there is a one to one correspondance between the $G_{\varphi_{\mathbf{d}}}$ orbits on $X^{k+1}_{\mathbf{d}}$ and $G_{\varphi_{\mathbf{d}},W}$ orbits on $\alpha^{-1}(W)$. Finally, since $\mathfrak{X}_{\mathbf{d}}$ is an open subvariety consisting of $G_{\varphi_{\mathbf{d}}}$ orbits, we are done. □

Definition 7.4.15. *The canonical projection*

$$\pi : V_{\varphi_{\mathbf{d}}} \to V_{\varphi_{\mathbf{d}}}/W$$

induces a projection

$$\pi_* : G_{\varphi_{\mathbf{d}},W} \to G_{\varphi_{\mathbf{a}}},$$

where we identify $V_{\varphi_{\mathbf{d}}}/W$ *with* $V_{\varphi_{\mathbf{a}}}$.

Proposition 7.4.16. *The morphism* p *is equivariant under the action of* $G_{\varphi_{\mathbf{d}},W}$ *and* $G_{\varphi_{\mathbf{a}}}$ *via* π_*, *i.e,*

$$p(gx) = \pi_*(g)p(x).$$

Moreover, it induces a one to one correspondance between orbits.

Démonstration. Note that for $T \in (\mathfrak{X}_{\mathbf{d}})_W$, such that $\tau_W(T) = (T_1, T_0) \in (Z^{k+1,\mathbf{d}})_W \times \mathrm{Hom}(V_{\varphi_{\mathbf{d}},k}, W)$, let

$$U_1 = \ker(T_1|_{V_{\varphi_{\mathbf{d}},k}}),\ U_0 = \ker(T_0|_{U_1})$$

we have

$$p(T) = (T_1, U_0).$$

Now it follows from the definition that we have

$$p(gT) = \pi_*(g)p(T).$$

Hence p sends orbits to orbits. It remains to show that the pre-image of an orbit is an orbit instead of unions of orbits.
We proved in proposition 7.4.10 that

$$p^{-1}p(T) = \{(T_1, q) : q \in \mathrm{Hom}(V_{\varphi_{\mathbf{d}},k}, W), \ker(q|_{U_1}) = U_0\},$$

note that here we identify elements of $(\mathfrak{X}_{\mathbf{d}})_W$ with its image under τ_W. Let $(T_1, q) \in p^{-1}p(T)$. Then we want to find $g \in G_{\varphi_{\mathbf{d}},W}$ such that $g(T_1, T_0) = (T_1, q)$. Note that by fixing a splitting $V_{\varphi_{\mathbf{d}},k+1} = W \oplus V_{\varphi_{\mathbf{d}},k+1}/W$, we can choose $g \in G_{\varphi_{\mathbf{d}}}$ such that $g_i = Id \in GL(V_{\varphi_{\mathbf{d}},i})$ for all $i \neq k+1$, and

$$g_{k+1} = \begin{pmatrix} g_1 & g_{12} \\ 0 & Id_{V_{\varphi_{\mathbf{d}},k+1}/W} \end{pmatrix} \in P_W,$$

where $g_1 \in GL(W)$, and $g_{12} \in \mathrm{Hom}(V_{\varphi_{\mathbf{d}},k+1}/W, W)$. By hypothesis, we know that the restrictions of q and T_0 to U_1 are surjective with kernel U_0, so we can choose $g_1 \in GL(W)$, such that

$$g_1 T_0(v) = q(v), \text{ for all } v \in U_1.$$

Finally, for $v_1 \in V_{\varphi_{\mathbf{d}},k+1}/W$, by our assumption at the beginning of this section on $\mathbf{a}$, we know that $T_1|V_{\varphi_{\mathbf{d}},k}$ is surjective, hence there exists $v \in V_{\varphi_{\mathbf{d}},k}$ such that $T_1(v) = v_1$. Then we define

$$g_{12}(v_1) = q(v) - g_1 T_0(v).$$

We check that this is well defined, i.e, for another $v' \in V_{\varphi_{\mathbf{d}},k}$ such that $T_1(v') = v_1$, we have

$$q(v) - g_1 T_0(v) = q(v') - g_1 T_0(v'),$$

this is the same as to say that

$$q(v - v') = g_1 T_0(v - v').$$

We observe that $T_1(v-v') = 0$, hence $v - v' \in U_1$, now $q(v-v') = g_1 T_0(v-v')$ follows from our definition of g_1. Under such a choice, we have

$$g((T_1, T_0)) = (T_0, q).$$

Hence we are done. □

Proposition 7.4.17. *The morphism σ is equivariant under the action of $G_{\varphi_{\mathbf{a}}}$. Assume that $\mathbf{a}$ is a multisegment which satisfies the assumption $(\mathbf{A}_k)$. Let $\varphi \in \mathcal{S}$ such that*

$$\varphi + \ell\chi_{[k]} = \varphi_{\mathbf{a}},$$

where χ is the characteristic function. Then there exists a one to one correspondance between the orbits of $\mathfrak{Y}_{\mathbf{a}}$ and the set

$$S := \{\mathbf{b} \in S(\varphi) : \mathbf{b} \preceq_k \mathbf{a}\}.$$

Moreover, for each orbit $\mathfrak{Y}(\mathbf{b})$ indexed by $\mathbf{b}$, $\sigma^{-1}(\mathfrak{Y}(\mathbf{b}))$ is irreducible hence contains a unique orbit in $E''_{\mathbf{a}}$ as (Zariski) open subset.

Démonstration. The fact that σ is equivariant under the action of $G_{\varphi_{\mathbf{a}}}$ follows directly from the definition. To show that the orbits of $\mathfrak{Y}$ under $G_{\varphi_{\mathbf{a}}}$ is indexed by S, consider the morphism

$$p' : \mathfrak{Y}_{\mathbf{a}} \to Gr(\ell, V_{\varphi_{\mathbf{a}},k}),\ (T, U) \mapsto U$$

As in the proposition 5.3.2, we have the following diagram

which shows that p' is a $GL_{\varphi_{\mathbf{a}},k}$ bundle. Moreover, the same proof as in lemma 7.4.14 shows that the orbits of $\mathfrak{Y}$ are in in one to one correspondance with that of the fibers

$$p'^{-1}(U) \simeq \{T \in \mathrm{End}(V_{\varphi_{\mathbf{a}}}/U) : T \text{ is of degree } 1, T \in O_{\mathbf{b}} \text{ for some } \mathbf{b} \preceq_k \mathbf{a}\},$$

under the action of stabilizer $G_{\varphi_{\mathbf{a}},U}$ of U. Let $\varphi \in \mathcal{S}$ be the such that $\varphi + \ell\chi_{[k]} = \varphi_{\mathbf{a}}$. Then by identifying V_φ with $V_{\varphi_{\mathbf{a}}}/U$, we can view $p'^{-1}(U)$ as an open subvariety of E_φ. Note that we are identifying orbits with orbits by the canonical projection

$$G_{\varphi_{\mathbf{a}},U} \to G_\varphi.$$

Now it follows that the fibers are parametrized by the set S. Finally, let $\mathbf{b} \in S$. We have to show that $\sigma^{-1}(\mathfrak{Y}(\mathbf{b}))$ is irreducible, which is a consequence of the following lemma.

□

Lemma 7.4.18. *Let $\mathbf{a}, \mathbf{b}$ be the multisegments as above. Then there exists a bijection between the set*

$$Q(\mathbf{a}, \mathbf{b}) = \{\mathbf{c} \in S(\mathbf{a}) : \mathbf{b} = \mathbf{c}_\Gamma \text{ for some } \Gamma \subseteq \mathbf{c}(k)\},$$

and the orbits in $\sigma^{-1}(\mathfrak{Y}(\mathbf{b}))$ which respects the poset structure, given by

$$\mathbf{c} \mapsto E''_{\mathbf{a}}(\mathbf{c}^\sharp),$$

where for $\mathbf{b} = \mathbf{c}_\Gamma$,

$$\mathbf{c}^\sharp = (\mathbf{c} \setminus \mathbf{c}(k)) \cup \Gamma \cup \{\Delta^+ : \Delta \in \mathbf{c}(k) \setminus \Gamma\},$$

and $E''_{\mathbf{a}}(\mathbf{c}^\sharp)$ is the orbit indexed by $\mathbf{c}^\sharp$. Moreover, the set $Q(\mathbf{a}, \mathbf{b})$ contains a unique minimal element.

Remark:
We remark that $S(\varphi)$ contains a unique maximal element.

Démonstration. Recall that we constructed in proposition 7.4.10 a morphism p, consider the composition

$$(\mathfrak{X}_{\mathbf{d}})_W \xrightarrow{p} E''_{\mathbf{a}} \xrightarrow{\sigma} \mathfrak{Y}_{\mathbf{a}},$$

which sends $(O_{\mathbf{c}})_W$ to $\mathfrak{Y}(\mathbf{b})$, where $\mathbf{b} = \mathbf{c}^{(k,k+1)}$ for $\mathbf{c} \in S(\mathbf{d})$. Hence we have

$$\mathbf{b} = (\mathbf{c}^{(k+1)})_\Gamma$$

for $\Gamma = \{\Delta \in \mathbf{c} : e(\Delta) = k\}$. Note that $\mathbf{c} \in S(\mathbf{d})$ implies that $\mathbf{c}^{(k+1)} \leq \mathbf{a} = \mathbf{d}^{(k+1)}$. Conversely, for $\mathbf{c} \in Q(\mathbf{a}, \mathbf{b})$, such that

$$\mathbf{b} = \mathbf{c}_\Gamma,$$

there is a unique element

$$\mathbf{c}' = \mathbf{c}^\sharp$$

in $S(\mathbf{d})$ such that $O_{\mathbf{c}'} \subseteq \mathfrak{X}_\mathbf{d}$ and $\mathbf{c} = \mathbf{c}'^{(k+1)}$. Therefore we conclude that there is a bijection between the $G_{\varphi_\mathbf{a}}$-orbits in $\sigma^{-1}(\mathfrak{Y}(\mathbf{b}))$ and $Q(\mathbf{a}, \mathbf{b})$.
Finally, for

$$\varphi_\mathbf{a} = \varphi_\mathbf{b} + \ell\chi_{[k]},$$

we show by induction on ℓ that the set $Q(\mathbf{a}, \mathbf{b})$ contains a unique minimal element.
For case $\ell = 1$, let

$$\mathbf{b}(k) := \{\Delta \in \mathbf{b} : e(\Delta) = k\} = \{\Delta_1 \preceq \cdots \preceq \Delta_h\}.$$

and $\mathbf{c}_i = (\mathbf{b} \setminus \Delta_i) \cup \Delta_i^+$. Then

$$Q(\mathbf{a}, \mathbf{b}) \subseteq \{\mathbf{c}_i : i = 1, \cdots, h\},$$

and $\mathbf{c}_h$ is minimal in the latter, which implies that $\mathbf{c}_h \in Q(\mathbf{a}, \mathbf{b})$ and is minimal. In general, let

$$\varphi = \varphi_\mathbf{b} + \chi_{[k]}.$$

Note that there exists $\mathbf{c}' \in S(\varphi)$ satisfying the assumption $(\mathbf{A}_k)$ and $\Gamma' \subseteq \mathbf{c}'(k)$ such that

$$\mathbf{b} = \mathbf{c}'_{\Gamma'}.$$

In fact, by assumption, we know that

$$\mathbf{b} = \mathbf{c}_\Gamma$$

for some $\mathbf{c} \in S(\mathbf{a})$ and $\Gamma \subseteq \mathbf{c}(k)$. Let

$$\Gamma \supseteq \Gamma_1,$$

such that $\ell = \sharp\Gamma = \sharp\Gamma' + 1$ and

$$\mathbf{c}' = \mathbf{c}_{\Gamma_1},$$

then we have

$$\mathbf{b} = \mathbf{c}'_{\Gamma \setminus \Gamma_1}.$$

Now we apply our induction to the case

$$Q_1 := \{\mathbf{c} \in S(\varphi) : \mathbf{c} \text{ satisfies the assumption } (\mathbf{A}_k), \mathbf{b} = \mathbf{c}_\Gamma \text{ for some } \Gamma \subseteq \mathbf{c}(k)\},$$

from which we know that there exists a unique minimal element $\mathbf{c}_1$ in Q_1. Now by assumption, we know that

$$\mathbf{b}_1 \leq \mathbf{c}' \preceq_k \mathbf{a},$$

and by induction, we know that the set $Q(\mathbf{a}, \mathbf{b}_1)$ contains a unique element $\mathbf{b}_2$. We claim that $\mathbf{b}_2$ is minimal in $Q(\mathbf{a}, \mathbf{b})$. In fact, let $\mathbf{e} \in Q(\mathbf{a}, \mathbf{b})$, then

$$\mathbf{b} = \mathbf{e}_{\Gamma'}$$

for some $\Gamma' \subseteq \mathbf{e}(k)$. Again let

$$\Gamma_1' \subseteq \Gamma', \ \mathbf{e}' = \mathbf{e}_{\Gamma_1'}$$

such that $\ell = \sharp\Gamma' = \sharp\Gamma_1' + 1$. Now we obtain

$$\mathbf{e}' \in Q_1, \ \mathbf{b} = \mathbf{e}'_{\Gamma' \setminus \Gamma_1'}.$$

By minimality of $\mathbf{c}_1$, we know that

$$\mathbf{c}_1 \leq \mathbf{e}'.$$

Note that this implies $\mathbf{c}_1 \preceq \mathbf{e}'$, and by transitivity of poset relation, we get $\mathbf{c}_1 \preceq_k \mathbf{e}$. Now we apply proposition 7.1.7 to get

$$\mathbf{c}_1 = \mathbf{f}_{\Gamma''},$$

for some $\mathbf{f} \in S(\mathbf{e})$ and $\Gamma'' \subseteq \mathbf{f}(k)$. Again we deduce from induction that

$$\mathbf{f} \geq \mathbf{c}_2.$$

Hence $\mathbf{c}_2 \leq \mathbf{e}$. □

Now we return to the calculation of product of perverse sheaves, cf. corollary 7.2.17.

Corollary 7.4.19. *Let* $\mathbf{a}$ *be a multisegment satisfying the assumption* $(\mathbf{A}_k)$ *and* $\mathbf{b} \preceq_k \mathbf{a}$ *such that*

$$\varphi_{\mathbf{a}} = \varphi_{\mathbf{b}} + \ell\chi_{[k]}.$$

Let $\mathbf{c}$ *the minimal element in* $Q(\mathbf{a}, \mathbf{b})$ *and* $E''_{\mathbf{a}}(\mathbf{c})$ *be the* $G_{\varphi_{\mathbf{a}}}$ *orbit indexed by* $\mathbf{c}$ *in* $E''_{\mathbf{a}}$*. Then we have*

$$IC(\overline{O}_{\mathbf{b}}) \star IC(\overline{O}_{\ell[k]}) = \beta''_*(IC(\overline{E''_{\mathbf{a}}(\mathbf{c}^\sharp)})).$$

Démonstration. First of all, by definition

$$E'' = \{(T, U) : T \in E_{\varphi_{\mathbf{a}}},\ T(U) = 0,\ \dim(U) = \ell\},$$

therefore we have

$$E''_{\mathbf{a}} \subseteq E''.$$

Furthermore, the variety $E''_{\mathbf{a}}$ is open in E''. In fact, consider the canonical morphism

$$\beta'' : E'' \to E_{\varphi_{\mathbf{a}}},$$

then $E''_{\mathbf{a}} = \beta''^{-1}(Y_{\mathbf{a}})$. Since $Y_{\mathbf{a}}$ is open in $E_{\varphi_{\mathbf{a}}}$, we know that $E''_{\mathbf{a}}$ is open in E''. Now we have two morphisms

$$\begin{aligned} \sigma\beta' : & \beta'^{-1}(E''_{\mathbf{a}}) \to \mathfrak{Y}_{\mathbf{a}}, \\ \beta : & E' \to E_{\varphi_{\mathbf{b}}} \times E_{\varphi_{\ell[k]}} \simeq E_{\varphi_{\mathbf{b}}}. \end{aligned}$$

We claim that $\beta^{-1}(O_{\mathbf{b}}) \cap \beta'^{-1}(E''_{\mathbf{a}}) = \beta'^{-1}\sigma^{-1}(\mathfrak{Y}(\mathbf{b}))$, where $\mathfrak{Y}(\mathbf{b})$ is the orbit in $\mathfrak{Y}(\mathbf{b})$ under the action of $G_{\varphi_{\mathbf{a}}}$.
By definition of β, we know that

$$\begin{aligned} \beta^{-1}(O_{\mathbf{b}}) \cap \beta'^{-1}(E''_{\mathbf{a}}) = \{(T, W, \mu, \mu') : & \mu : W \simeq V_{\varphi_{\ell[k]}},\ \mu' : V_{\varphi_{\mathbf{a}}}/W \simeq V_{\varphi_{\mathbf{b}}}, \\ & T \in O_{\mathbf{f}} \text{ for some } \mathbf{f} \in S(\mathbf{a}), \mathbf{b} \preceq_k \mathbf{f}\}. \end{aligned}$$

Now by definition of σ and β', we know that $\beta^{-1}(O_{\mathbf{b}}) \cap \beta'^{-1}(E''_{\mathbf{a}}) = \beta'^{-1}\sigma^{-1}(\mathfrak{Y}(\mathbf{b}))$.
Now by proposition 7.4.17, $\sigma^{-1}(\mathfrak{Y}_{\mathbf{b}})$ contains $E''_{\mathbf{a}}(\mathbf{c}^{\sharp})$ as the unique open suborbit, where $\mathbf{c}$ is the minimal element in $Q(\mathbf{a}, \mathbf{b})$. Therefore we conclude that

$$\beta'^{*}(IC(\overline{E''_{\mathbf{a}}(\mathbf{c}^{\sharp})})) = \beta^{*}(IC(\overline{O_{\mathbf{b}}}) \otimes IC(E_{\varphi_{\ell[k]}})).$$

Now by definition

$$IC(\overline{O}_{\mathbf{b}}) \star IC(\overline{O}_{\ell[k]}) = \beta''_{*}(IC(\overline{E''_{\mathbf{a}}(\mathbf{c}^{\sharp})})).$$

□

7.5 Multisegments of Grassmanian Type

In order to precisely describe the previous corollary concerning Lusztig's product in the Grassmanian case in the next section, we generalize the construction in section 3.3 to get more general results concerning the the set $S(\mathbf{a})$ for general multisegment $\mathbf{a}$.
Let V a $\mathbb{C}$ vector space of dimension $r + \ell$ and $Gr_r(V)$ be the variety of r-dimensional subspaces of V.

Definition 7.5.1. *By a partition of ℓ, we mean a sequence $\lambda = (\ell_1, \cdots, \ell_r)$ for some r, where $\ell_i \in \mathbb{N}$, $0 \leq \ell_1 \leq \cdots \ell_r \leq \ell$. And for $\mu = (\mu_1, \cdots, \mu_s)$ be another partition, we say $\mu \leq \lambda$ if and only if $\mu_i \leq \lambda_i$ for all $i = 1, \cdots ,$. Let $\mathcal{P}(\ell, r)$ be the set of partitions of ℓ into r parts.*

Definition 7.5.2. *We fix a complete flag*

$$0 = V^0 \subset V^1 \subset \cdots \subset V^{r+\ell} = V.$$

This flag provides us a stratification of the variety $Gr_r(V)$ by Schubert varieties, labeling by partitions , denoted by $\overline{X}_\lambda$,

$$\overline{X}_\lambda = \{U \in Gr_r(V) : \dim(U \cap V^{\ell_i + i}) \geq i, \text{ for all } i = 1, \cdots, r\}.$$

Lemma 7.5.3. *(cf. [36]) We have*

$$\mu \leq \lambda \Longleftrightarrow \overline{X}_\mu \subseteq \overline{X}_\lambda.$$

And the Schubert cell

$$X_\lambda = \overline{X}_\lambda - \sum_{\mu < \lambda} \overline{X}_\mu$$

is open in $\overline{X}_\lambda$.

Definition 7.5.4. *Let $\Omega^{r,\ell}$ be the set*

$$\Omega^{r,\ell} = \{(a_1, \cdots, a_m; b_0, \cdots, b_{m-1}) : \sum_i a_i = r, \sum_j, b_j = \ell, \\ \text{for } 0 < i < m, a_i > 0, b_i > 0\}.$$

Lemma 7.5.5. *(cf. [36]) There exists a bijection*

$$\Omega^{r,\ell} \to \mathcal{P}(\ell, r),$$

which sends $(a_1, \cdots, a_m; b_0, \cdots, b_{m-1})$ to a partition of ℓ given by b_0, $b_0 + b_1, \cdots,\ , b_0 + \cdots + b_{m-1}$, and that the elements $b_0 + \cdots + b_{i-1}$ figures in λ with multiplicity a_i.

Notation 7.5.6. *From now on, we will also write*

$$\lambda = (a_1, \cdots, a_m; b_0, \cdots, b_{m-1}),$$

with notations as in the previous lemma.

We introduce the formula in [36] to calculate the Kazhdan Lusztig polynomials for Grassmannians.

Definition 7.5.7. *Let $\lambda = (a_1, \cdots, a_m; b_0, \cdots, b_{m-1})$ be a partition. Following [36], we represent a partition as a broken line in the plane (x, y), i.e, the graph of the piecewise-linear function $y = \lambda(x)$ which equals $|x|$ for large $|x|$, has everywhere slope ± 1, and whose ascending and decreasing segments are precisely $b_0, \cdots, b_{m-1}$ and $a_1, \cdots, a_m$, respectively. Moreover, we call the local maximum and minimum of the graph $y = \lambda(x)$ the peaks and depressions of λ.*

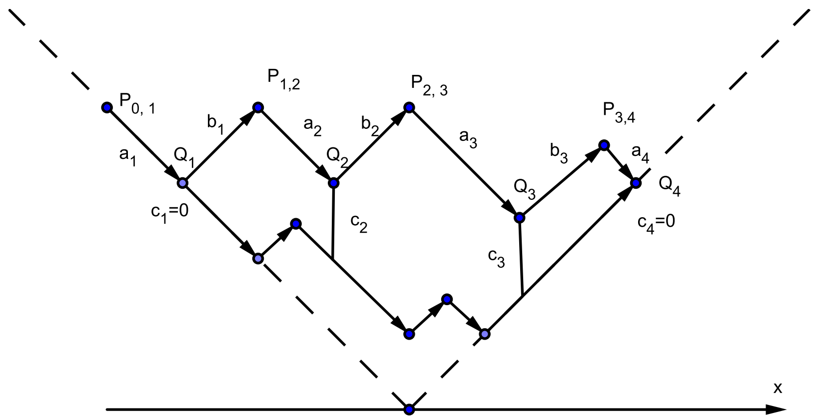

FIGURE 7.1 –

Lemma 7.5.8. *(cf. [36]) For $\lambda, \mu \in \Omega^{r,\ell}$, then*

$$\lambda \geq \mu \Longleftrightarrow \lambda(x) \geq \mu(x), \text{ for all } x.$$

From now on until the end of this section, we let

$$J = \{\sigma_i : i = 1, \cdots, r-1\} \cup \{\sigma_i : i = r+1 \cdots, r+\ell-1\},$$

and

$$\mathbf{a} := \mathbf{a}_{\mathrm{Id}}^{J,\emptyset} = \{\Delta_1, \cdots, \Delta_r, \cdots, \Delta_{r+\ell}\}$$

be a multisegment of parabolic type $(J, \emptyset)$, where

$$e(\Delta_i) = k-1, \text{ for } i = 1, \cdots, r,$$

and

$$e(\Delta_i) = k, \text{ for } i = r+1, \cdots, r+\ell.$$

Definition 7.5.9. *Then to each partition $\lambda \in \Omega^{r_1,\ell_1}$ such that $r_1 \geq r$ and $r_1 + \ell_1 = r + \ell$, we associate*

$$\begin{aligned}\mathbf{a}_\lambda = &\sum_{i=1}^{b_0}[b(\Delta_i), k] + \sum_{i=b_0+1}^{b_0+a_1}[b(\Delta_i), k-1] + \cdots \\ &+ \sum_{i=b_0+a_1\cdots+b_{j-1}+1}^{b_0+a_1\cdots+b_{j-1}+a_j}[b(\Delta_i), k-1] + \sum_{i=b_0+\cdots+b_{j-1}+a_j+1}^{b_0+a_1\cdots+b_j}[b(\Delta_i), k] + \cdots .\end{aligned}$$

Definition 7.5.10. *Let $r, n \in \mathbb{N}$ such that $r \leq n$. Let*

$$R_r(n) = \{(x_1, \cdots, x_r) : 1 \leq x_1 < \cdots < x_r \leq n\}.$$

(1) *Let $x = (x_1, \cdots, x_{r_1}) \in R_{r_1}(n)$ and $x' = (x'_1, \cdots, x'_{r_2}) \in R_{r_2}(n)$ such that $r_1 \geq r_2$. We say $x \supseteq x'$ if $\{x_1, \cdots, x_{r_1}\} \supseteq \{x'_1, \cdots, x'_{r_2}\}$.*

(2) *Let $x = (x_1, \cdots, x_r) \in R_r(n)$ and $x' = (x'_1, \cdots, x'_r) \in R_r(n)$. We say $x \geq x'$ if $x_i \geq x'_i$ for all $i = 1, \cdots, r$.*

(3) *We define $x \succeq y$, if $x \geq y' \supseteq y$ for some y'.*

Remark: The set $R_r(n)$ is a poset with respect to the relation $\geq$. And the set $\cup_{r\leq n} R_r(n)$ is a poset with respect to the relation $\supseteq$.

Proposition 7.5.11. *For $J = \{\sigma_i : i = 1, \cdots, r-1\} \cup \{\sigma_i : i = r+1 \cdots, r+\ell-1\}$, we have an isomorphism of posets*

$$\varsigma_1 : S^{J,\emptyset}_{r+\ell} \to R_r(r+\ell),$$

by associating the element w with $x_w := (w^{-1}(1), \cdots, w^{-1}(r))$.

Démonstration. Note that by definition

$$S^{J,\emptyset}_{r+\ell} = \{w \in S_{r+\ell} : w^{-1}(1) < \cdots < w^{-1}(r) \text{ and } w^{-1}(r+1) < \cdots < w^{-1}(r+\ell)\}.$$

Therefore, ς is a bijection. This preserves the partial order, for a proof, see [8] proposition 2.4.8. □

Definition 7.5.12. *For $\lambda \in \Omega^{r,\ell}$ and $\lambda' \in \Omega^{r_1,\ell_1}$ such that $r + \ell = r_1 + \ell_1$. We define $\lambda \supseteq \lambda'$ if and only if $x_\lambda \supseteq x_{\lambda'}$, and $\lambda \succeq \lambda'$ if and only if $x_\lambda \succeq x_{\lambda'}$.*

Definition 7.5.13. *Let $\lambda = (a_1, \cdots, a_m; b_0, \cdots, b_{m-1})$, consider the set*

$$\{b(\Delta) : \Delta \in \mathbf{a}_\lambda, e(\Delta) = k-1\} = \{x_1 < \cdots < x_r\},$$

here we have r segments ending in $k-1$ since $\sum_i a_i = r$, we associate λ with the element

$$x_\lambda := (x_1, \cdots, x_r).$$

This allows us to get a morphism $\varsigma_2 : \Omega^{r,\ell} \to R_r(r+\ell)$ sending λ to x_λ.

Lemma 7.5.14. *The map ς_2 is an isomorphism of posets.*

Démonstration. To see that ς is a bijection, we only need to construct an inverse. Given $x = (x_1, \cdots, x_r) \in R_r(r+\ell)$, we have $y = (y_1, \cdots, y_\ell) \in R_\ell(r+\ell)$ such that $\{1, \cdots, r+\ell\} = \{x_1, \cdots, x_r, y_1, \cdots, y_\ell\}$. We can associate a multisegment to x

$$\mathbf{a}_x = \sum_{j=1}^{r}[b(\Delta_{x_j}), k-1] + \sum_{j=1}^{\ell}[b(\Delta_{y_j}), k].$$

Note that this allows us to construct a partition $\lambda(x) \in \Omega^{r,\ell}$ by counting the segments ending in k and $k+1$ alternatively.
A simple calculation shows that if we write $\lambda = (\ell_1, \cdots, \ell_r)$ with $0 \leq \ell_1 \leq \cdots \leq \ell_r$, then

$$\varsigma_2(\lambda) = (\ell_1 + 1, \cdots, \ell_r + r),$$

as described in [9]. This shows that

$$\mu \geq \lambda \Leftrightarrow \varsigma_2(\mu) \geq \varsigma_2(\lambda).$$

□

Proposition 7.5.15. *For $\lambda \in \Omega^{r,\ell}$, we have $\mathbf{a}_\lambda \in S(\mathbf{a})$, moreover, all the elements in $S(\mathbf{a})$ are of this form. Moreover, we have $S(\mathbf{a}_\lambda) = \{a_\mu : \mu \geq \lambda\}$.*

Démonstration. Let $w \in S^{J,\emptyset}$, by definition, we have

$$w^{-1}(1) < \cdots < w^{-1}(r),\ w^{-1}(r+1) < \cdots < w^{-1}(r+\ell).$$

By definition, we have

$$\begin{aligned}\Phi_{J,\emptyset}(w) &= \sum_j [b(\Delta_j), e(\Delta_{w(j)})] \\ &= \sum_j [b(\Delta_{w^{-1}(j)}), e(\Delta_j)] \\ &= \sum_{j=1}^{r}[b(\Delta_{w^{-1}(j)}), k-1] + \sum_{j=r+1}^{r+\ell}[b(\Delta_{w^{-1}(j)}), k] \\ &= \mathbf{a}_{\varsigma_2^{-1}(x_w)}\end{aligned}$$

Now that $\varsigma_2^{-1} \circ \varsigma_1$ preserves the partial order, we have

$$S(\mathbf{a}_\lambda) = \{\mathbf{a}_\mu : \mu \geq \lambda\}$$

by proposition 6.2.14.

□

Example 7.5.16. *For example, for $r = 1, \ell = 3$, with $J = \{\sigma_2, \sigma_3\}$ and*

$$\mathbf{a} = \mathbf{a}_{\mathrm{Id}}^{J,\emptyset} = [1,4] + [2,5] + [3,5] + [4,5].$$

Let $\lambda = (a_1, a_2; b_0, b_1) = (1, 0; 2, 1)$, then $\mathbf{a}_\lambda = [1,5] + [2,5] + [3,4] + [4,5]$. This corresponds to the element $\varsigma_1^{-1} \circ \varsigma_2(\lambda) = \sigma_1\sigma_2$ in $S_4^{J,\emptyset}$.

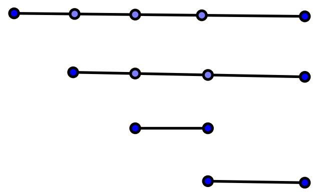

FIGURE 7.2 –

Proposition 7.5.17. *Let $\lambda, \mu \in \Omega^{r,\ell}$ such that $\lambda < \mu$. We have*

$$P_{\mathbf{a}_\lambda, \mathbf{a}_\mu}(q) = P_{\lambda,\mu}(q).$$

Démonstration. We can also prove this proposition in the following way. Let $w, v \in S_{r+\ell}^{J,\emptyset}$, such that

$$\lambda = \varsigma_2^{-1}\varsigma_1(w), \ \ \mu = \varsigma_2^{-1}\varsigma_1(v).$$

Let P_J be the parabolic subgroup of GL_n, then by fixing an element in $V_0 \in Gr_r(\mathbb{C}^{r+\ell})$, we can identify $P_J \backslash GL_n$ with $Gr_r(\mathbb{C}^{r+\ell})$. Moreover, the B-orbits $P_J \backslash wB$ corresponds to the varieties X_λ, see [9] for a precise description. Hence we have

$$P_{\lambda,\mu}(q) = P_{w,v}^{J,\emptyset}(q) = P_{a_\lambda, \mathbf{a}_\mu}(q).$$

□

Remark: One can surely prove this result using the open immersion we constructed in section 3.3.

Definition 7.5.18. *Let $\lambda \in \Omega^{r,\ell}$.*

(1) *We define*

$$\Gamma(\lambda) = \{\mu \in \Omega^{r_1,\ell_1} : r_1 + \ell_1 = r + \ell, r_1 \geq r, \ \mu \succeq \lambda\}.$$

and

$$\Gamma^{\mu}(\lambda) = \{\mu' : \mu \geq \mu', \mu' \succeq \lambda\},$$
$$\Gamma_1^{\mu}(\lambda) = \{\mu' : \mu \geq \mu', \mu' \supseteq \lambda\}.$$

(2) *For $\mu \in \Gamma(\lambda)$, we define*

$$S^{\mu}(\lambda) = \{\lambda' \in \Omega^{r,\ell} : \lambda' \geq \lambda, \mu \succeq \lambda'\},$$

and let

$$S_1^{\mu}(\lambda) = \{\lambda' \in \Omega^{r,\ell} : \lambda' \geq \lambda, \mu \supseteq \lambda'\}.$$

Proposition 7.5.19. *Let $\lambda \in \Omega^{r,\ell}$ and $\mu \in \Omega^{r_1,\ell_1}$ with $r_1 \geq r$ and $r_1 + \ell_1 = r+\ell$. Then $\pi(\mathbf{a}_\mu)$ appears as a summand of $\mathscr{D}^k(\pi(\mathbf{a}_\lambda))$ if and only if $\mu \in \Gamma(\lambda)$.*

Démonstration. Let $x_\lambda = (x_1^\lambda, \cdots, x_r^\lambda) = \varsigma_2(\lambda)$ and $y_\lambda = (y_1^\lambda, \cdots, y_\ell^\lambda) \in R_r(r+\ell)$ such that

$$\{1, \cdots, r+\ell\} = \{x_1^\lambda, \cdots, x_r^\lambda, y_1^\lambda, \cdots, y_\ell^\lambda\}.$$

As described in proposition 7.5.15, we have

$$\mathbf{a}_\lambda = \sum_{j=1}^{r} [b(\Delta_{x_j^\lambda}), k-1] + \sum_{j=1}^{\ell} [b(\Delta_{y_j^\lambda}), k].$$

therefore

$$\mathscr{D}^k(\pi(\mathbf{a}_\lambda)) = \pi(\mathbf{a}_\lambda) + \sum_{y \supseteq x_\lambda} \pi(\mathbf{a}_{\varsigma_2^{-1}(y)}).$$

Now by lemma 1.3.5, we know that $\pi(\mu)$ is a summand of $\mathscr{D}^k(\pi(\mathbf{a}_\lambda))$ if and only if $\mu \geq \varsigma_2^{-1}(y)$ for some $y \supseteq x_\lambda$, i.e, $\mu \succeq \lambda$. □

Corollary 7.5.20. *We have $\mu \succeq \lambda$ if and only if $\mathbf{a}_\mu \preceq_k \mathbf{a}_\lambda$.*

Démonstration. By corollary 7.1.6, we know that $\mathbf{a}_\mu \preceq_k \mathbf{a}_\lambda$ if and only if $\mathscr{D}^k(\pi(\mathbf{a}_\lambda)) - \pi(\mathbf{a}_\mu) \geq 0$ in $\mathcal{R}$, which is equivalent to say that $\mu \preceq \lambda$ by the previous proposition. □

Proposition 7.5.21. *Let $\lambda \in \Omega^{r,\ell}$ and $\mu \in \Omega^{r_1,\ell_1}$. Then we have $\mathbf{a}_\mu = (\mathbf{a}_\lambda)_\Gamma$ for some $\Gamma \subseteq \mathbf{a}_\lambda(k)$. if and only if we have $\mu \supseteq \lambda$.*

Démonstration. Let $x_\lambda = (x_1^\lambda, \cdots, x_r^\lambda) = \varsigma_2(\lambda)$ and $y_\lambda = (y_1^\lambda, \cdots, y_\ell^\lambda) \in R_r(r+\ell)$ such that

$$\{1, \cdots, r+\ell\} = \{x_1^\lambda, \cdots, x_r^\lambda, y_1^\lambda, \cdots, y_\ell^\lambda\}.$$

As described in proposition 7.5.15, we have

$$\mathbf{a}_\lambda = \sum_{j=1}^{r} [b(\Delta_{x_j^\lambda}), k-1] + \sum_{j=1}^{\ell} [b(\Delta_{y_j^\lambda}), k].$$

And we have

$$\mathbf{a}_\lambda(k) = \sum_{j=1}^{\ell} [b(\Delta_{y_j^\lambda}), k].$$

Let $\Gamma = \sum_{m=1}^{t} [b(\Delta_{y_{j_m}^\lambda}), k]$. If $\mathbf{a}_\mu = (\mathbf{a}_\lambda)_\Gamma$, then

$$\mathbf{a}_\mu = \sum_{j=1}^{r} [b(\Delta_{x_j^\lambda}), k-1] + \sum_{m=1}^{t} [b(\Delta_{y_{j_m}^\lambda}), k-1] + \sum_{j \notin \{j_1, \cdots, j_t\}} [b(\Delta_{y_j^\lambda}), k].$$

Therefore

$$x_\mu \supseteq x_\lambda$$

as a set. The converse is also true. □

7.6 Grassmanian case

As before, let

$$J = \{\sigma_i : i = 1, \cdots, r-1\} \cup \{\sigma_i : i = r+1 \cdots, r+\ell-1\},$$

and

$$\mathbf{a} := \mathbf{a}_{\mathrm{Id}}^{J,\emptyset} = \{\Delta_1, \cdots, \Delta_r, \cdots, \Delta_{r+\ell}\}$$

be a multisegment of parabolic type $(J, \emptyset)$, where

$$e(\Delta_i) = k-1, \text{ for } i = 1, \cdots, r,$$

and

$$e(\Delta_i) = k, \text{ for } i = r+1, \cdots, r+\ell.$$

Moreover, for $\lambda \in \mathcal{P}(\ell, r)$, let $x_\lambda = (x_1^\lambda, \cdots, x_r^\lambda) = \varsigma_2(\lambda) \in R_r(r+\ell)$ and $y_\lambda = (y_1^\lambda, \cdots, y_\ell^\lambda) \in R_\ell(r+\ell)$ such that

$$\{1, \cdots, r+\ell\} = \{x_1^\lambda, \cdots, x_r^\lambda, y_1^\lambda, \cdots, y_\ell^\lambda\}.$$

As described in proposition 7.5.15, we have

$$\mathbf{a}_\lambda = \sum_{j=1}^{r}[b(\Delta_{x_j^\lambda}), k-1] + \sum_{j=1}^{\ell}[b(\Delta_{y_j^\lambda}), k].$$

Let $0 < r_0 \leq \ell$ and $r_1 = r + r_0,\ \ell_1 = \ell - r_0$.

Proposition 7.6.1. *Let $\mu \in \mathcal{P}(\ell_1, r_1)$. Then there exists $\mu^\flat \in \mathcal{P}(\ell, r)$, such that*

$$\{\mathbf{b} \in S(\mathbf{a}) : \mathbf{a}_\mu \preceq_k \mathbf{b}\} = \{\mathbf{a}_\lambda : \lambda \in \mathcal{P}(\ell, r),\ \lambda \leq \mu^\flat\}.$$

More explicitly, if $x_\mu = (x_1^\mu, \cdots, x_{r_1}^\mu) = \varsigma_2(\mu)$, then

$$x_{\mu^\flat} = \varsigma_2(\mu^\flat) = (x_{r_0+1}^\mu, \cdots, x_{r_1}^\mu).$$

Démonstration. By lemma 7.4.18, we know that the set

$$\{\mathbf{b} \in S(\mathbf{a}) : \mathbf{a}_\mu \preceq_k \mathbf{b}\}$$

contains a unique minimal element $\mathbf{a}_{\mu^\flat} \in S(\mathbf{a})$ for some $\mu^\flat \in \mathcal{P}(\ell, r)$. Therefore we have

$$\{\mathbf{b} \in S(\mathbf{a}) : \mathbf{a}_\mu \preceq_k \mathbf{b}\} = \{\mathbf{a}_\lambda : \lambda \in \mathcal{P}(\ell, r),\ \lambda \leq \mu^\flat\}.$$

Note that if we write

$$\mathbf{a}_\mu = \sum_{j=1}^{r_1}[b(\Delta_{x_j^\mu}), k-1] + \sum_{j=1}^{\ell_1}[b(\Delta_{y_j^\mu}), k],$$

then

$$\mathbf{a}_{\mu^\flat} = \sum_{j=1}^{r_0}[b(\Delta_{x_j^\mu}), k] + \sum_{j=r_0+1}^{r_1}[b(\Delta_{x_j^\mu}), k-1] + \sum_{j=1}^{\ell_1}[b(\Delta_{y_j^\mu}), k]$$

is the minimal element in $S(\mathbf{a})$ satisfying

$$\mathbf{a}_\mu = (\mathbf{a}_{\mu^\flat})_\Gamma$$

for some $\Gamma \subseteq \mathbf{a}_{\mu^\flat}(k)$. □

Definition 7.6.2. *Let*

$$J_1 = \{\sigma_i : i = 1, \cdots, r-1\} \cup \{\sigma_i : i = r+1, \cdots, r_1-1\} \cup \{\sigma_i : r_1+1, \cdots, r+\ell-1\},$$

and

$$\mathbf{a}_1 =: \mathbf{a}_{\mathrm{Id}}^{J_1,\emptyset} = \{\Delta_1, \cdots, \Delta_{r_1}, \Delta_{r_1+1}^+, \cdots, \Delta_{r+\ell}^+\},$$

where $\mathbf{a} = \{\Delta_1, \cdots, \Delta_{r+\ell}\}$ *with* $\Delta_1 \trianglelefteq \Delta_2 \trianglelefteq \cdots \trianglelefteq \Delta_r$ *(cf. Def. 7.2.4).*

Lemma 7.6.3. *Let* $\mathbf{d} = \mathbf{a} + \ell_1[k+1]$*, then*

— *we have* $\mathbf{a} = \mathbf{a}_1^{(k+1)}$*;*

— *and* $\mathfrak{X}_{\mathbf{d}} = \coprod_{w \in S_{r+\ell}^{J_1,\emptyset}} O_{\mathbf{a}_w}$*, where* $\mathbf{a}_w = \mathbf{a}_w^{J_1,\emptyset} \in S(\mathbf{a}_1)$ *is the element associated to* w *by lemma 6.2.23.*

Démonstration. Note that by definition we have

$$\mathbf{a} = \mathbf{a}_1^{(k+1)}.$$

And by definition of $\mathfrak{X}_{\mathbf{d}}$, we know that $\mathfrak{X}_{\mathbf{d}}$ consists of the orbit $O_{\mathbf{c}}$ with $\mathbf{c} \in S(\mathbf{d})$ such that $\varphi_{e(\mathbf{c})}(k) + \ell_1 = \varphi_{e(\mathbf{a})}(k)$, and the latter condition implies that there exists $w \in S_{r+\ell}^{J_1,\emptyset}$ such that $\mathbf{c} = \mathbf{a}_w^{J_1,\emptyset}$. □

Proposition 7.6.4. *Let* $\mathbf{d} = \mathbf{a} + \ell_1[k+1]$ *and* $W \subseteq V_{\varphi_{\mathbf{d}},k+1}$ *such that* $\dim(W) = \ell_1$ *(which implies that* $W = V_{\varphi_{\mathbf{d}},k+1}$*). Then the composition of morphisms*

$$\mathfrak{X}_{\mathbf{d}} = (\mathfrak{X}_{\mathbf{d}})_W \xrightarrow{p} E''_{\mathbf{a}} \xrightarrow{\beta''} E_{\varphi_{\mathbf{a}}},$$

sends $O_{\mathbf{a}_w} \cap (\mathfrak{X}_{\mathbf{d}})_W$ *to* $O_{\mathbf{a}_w^{(k+1)}}$*.*

Démonstration. This is by definition. □

Proposition 7.6.5. *Let* $\mu \in \mathcal{P}(\ell_1, r_1)$ *and* $x_\mu = \varsigma_2(\mu) = (x_1^\mu, \cdots, x_{r_1}^\mu)$*,* $y_\mu = (y_1^\mu, \cdots, y_{\ell_1}^\mu)$ *such that*

$$\{1, \cdots, r_1 + \ell_1\} = \{x_1^\mu, \cdots, x_{r_1}^\mu, y_1^\mu, \cdots, y_{\ell_1}^\mu\}.$$

Then

$$(\mathbf{a}_{\mu^\flat})^\sharp = \sum_{j=1}^{r_0}[b(\Delta_{x_j^\mu}), k] + \sum_{j=r_0+1}^{r_1}[b(\Delta_{x_j^\mu}), k-1] + \sum_{j=1}^{\ell_1}[b(\Delta_{y_j^\mu}), k+1],$$

for definition of $(\mathbf{a}_{\mu^\flat})^\sharp$*, cf. lemma 7.4.18.*

Démonstration. Note that by proposition 7.6.1, we know that

$$\mathbf{a}_{\mu^\flat} = \sum_{j=1}^{r_0}[b(\Delta_{x_j^\mu}), k] + \sum_{j=r_0+1}^{r_1}[b(\Delta_{x_j^\mu}), k-1] + \sum_{j=1}^{\ell_1}[b(\Delta_{y_j^\mu}), k]$$

and

$$\mathbf{a}_\mu = (\mathbf{a}_{\mu^\flat})_\Gamma$$

for $\Gamma = \sum_{j=1}^{r_0}[b(\Delta_{x_j^\mu}), k]$. Now by construction in lemma 7.4.18, we know that

$$(\mathbf{a}_{\mu^\flat})^\sharp = \sum_{j=1}^{r_0}[b(\Delta_{x_j^\mu}), k] + \sum_{j=r_0+1}^{r_1}[b(\Delta_{x_j^\mu}), k-1] + \sum_{j=1}^{\ell_1}[b(\Delta_{y_j^\mu}), k+1].$$

□

Proposition 7.6.6. *We have*

$$n(\mathbf{a}_\mu, \mathbf{a}_{\mu^\flat}) = \sharp\{\mathbf{c} \in S(\mathbf{a}_1) : \mathbf{c}^{(k+1)} = \mathbf{a}_{\mu^\flat}, \mathbf{c} \geq (\mathbf{a}_{\mu^\flat})^\sharp\}.$$

Démonstration. Consider the composed morphism

$$h : \mathfrak{X}_{\mathbf{d}} = (\mathfrak{X}_{\mathbf{d}})_W \xrightarrow{p} E''_{\mathbf{a}} \xrightarrow{\beta''} E_{\varphi_{\mathbf{a}}},$$

then the orbits contained in $h^{-1}(O_{\mathbf{a}_{\mu^\flat}})$ is indexed by the set

$$\{\mathbf{c} \in S(\mathbf{a}_1) : \mathbf{c}^{(k+1)} = \mathbf{a}_{\mu^\flat}, \mathbf{c} \geq (\mathbf{a}_{\mu^\flat})^\sharp\}$$

Note that by corollary 7.4.19 and proposition 7.3.8, the number

$$n(\mathbf{a}_\mu, \mathbf{a}_{\mu^\flat}) = \sum_i \dim \mathcal{H}^{2i}(\beta''_*(IC(\overline{E''_{\mathbf{a}}((\mathbf{a}_{\mu^\flat})^\sharp)})))_x$$

for some $x \in O_{\mathbf{a}_{\mu^\flat}}$. Finally, note that the morphism β'' is smooth when restricted to the variety $\beta''^{-1}(O_{\mathbf{a}_{\mu^\flat}})$. Moreover, the fibers are open in some Schubert variety, therefore, we are reduced to the counting of orbits. □

More generally, we have

Definition 7.6.7. *Let $w_\mu \in S_{r+\ell}^{J_1,\emptyset}$ be the element such that*

$$\mathbf{a}_{w_\mu} = (\mathbf{a}_{\mu^\flat})^\sharp.$$

Proposition 7.6.8. *Let P_J and P_{J_1} be the parabolic subgroups corresponding to J, J_1 respectively. Consider the natural morphism*

$$\pi : P_{J_1}\backslash GL_{r+\ell} \to P_J\backslash GL_{r+\ell}.$$

Then

$$n(\mathbf{a}_\mu, \mathbf{a}_\lambda) = \sum_i \dim \mathcal{H}^{2i}(\pi_*(IC(\overline{P_{J_1}w_\mu B})))_x$$

for some $x \in P_J t_\lambda B$, here t_λ is the element in $S_{r+\ell}^{J,\emptyset}$ associated to the partition λ.

Démonstration. Consider the composed morphism

$$h : \mathfrak{X}_{\mathbf{d}} = (\mathfrak{X}_{\mathbf{d}})_W \xrightarrow{p} E''_{\mathbf{a}} \xrightarrow{\beta''} E_{\varphi_{\mathbf{a}}}.$$

This proposition can be deduced from a construction of fibration similar to the one we did in Chapter 2 for symmetric multisegments, cf.§2.5.

□

7.7 Parabolic Case

In this section, as in the Grassmannian case, we deduce a formula for calculating the coefficient $n(\mathbf{b}, \mathbf{a})$.
Let

$$J \subseteq S$$

be a subset of generators and

$$\mathbf{a} = \mathbf{a}_{\mathrm{Id}}^{J,\emptyset}$$

be some multisegment of parabolic type $(J, \emptyset)$ associated to the identity, satisfying $f_{e(\mathbf{a})}(k) \neq 0, f_{e(\mathbf{a})}(k+1) = 0$.

Notation 7.7.1. *For $k \in \mathbb{Z}$, we let $\ell_k = f_{e(\mathbf{a})}(k)$.*

Definition 7.7.2. *Let $\mathbf{a}(k) = \{\Delta_1, \cdots, \Delta_{\ell_k}\}$ with $\Delta_1 \trianglelefteq \cdots \trianglelefteq \Delta_{\ell_k}$ and $r_0 \in \mathbb{N}$ with $1 \leq r_0 \leq \ell_k$. Then let*

$$\mathbf{a}_1 = (\mathbf{a} \setminus \mathbf{a}(k)) \cup \{\Delta \in \mathbf{a}(k) : \Delta \trianglelefteq \Delta_{\ell_k - r_0}\} \cup \{\Delta^+ \in \mathbf{a}(k) : \Delta \trianglerighteq \Delta_{\ell_k - r_0 + 1}\},$$
$$\mathbf{a}_2 = (\mathbf{a} \setminus \mathbf{a}(k)) \cup \{\Delta^- \in \mathbf{a}(k) : \Delta \trianglelefteq \Delta_{r_0}\} \cup \{\Delta \in \mathbf{a}(k) : \Delta \trianglerighteq \Delta_{r_0+1}\}$$

and $J_i(r_0, k)(i = 1, 2)$ be a subset of S such that $\mathbf{a}_i$ is a multisegment of parabolic type $(J_i(r_0, k), \emptyset)$. Moreover, let

$$\mathbf{a}_{\mathrm{Id}}^{J_i(r_0,k),\emptyset} = \mathbf{a}_i, \ \textit{for } i = 1, 2.$$

Lemma 7.7.3. *Let* $\ell_1 = \ell_k - r_0$ *and* $\mathbf{d} = \mathbf{a} + \ell_1[k+1]$, *then*

— *we have* $\mathbf{a} = \mathbf{a}_1^{(k+1)}$ *;*

— *and* $\mathfrak{X}_{\mathbf{d}} = \coprod_{w \in S_n^{J_1(r_0,k),\emptyset}} O_{\mathbf{a}_w}$, *where* $\mathbf{a}_w = \mathbf{a}_w^{J_1(r_0,k),\emptyset} \in S(\mathbf{a}_1)$ *is the element associated to* w *by lemma 6.2.23.*

Proposition 7.7.4. *Let* $w \in S_n^{J_2(r_0,k),\emptyset}$. *Then there exists* $w^\flat \in S_n^{J,\emptyset}$, *such that*

$$\{\mathbf{b} \in S(\mathbf{a}) : \mathbf{a}_w \preceq_k \mathbf{b}\} = \{\mathbf{a}_v : v \in S_n^{J,\emptyset},\ v \leq w^\flat\}.$$

More explicitly, if $\mathbf{a}_w(k-1) = \{\Delta_1, \cdots, \Delta_{\ell_{k-1}}\}$ *with* $\Delta_1 \trianglelefteq \cdots \trianglelefteq \Delta_{\ell_{k-1}}$, *then*

$$\mathbf{a}_{w^\flat} = (\mathbf{a}_w \backslash \mathbf{a}_w(k-1)) \cup \{\Delta^+ \in \mathbf{a}_w(k-1) : \Delta \trianglelefteq \Delta_{r_0}\} \cup \{\Delta \in \mathbf{a}_w(k-1) : \Delta \trianglerighteq \Delta_{r_0+1}\}.$$

Proposition 7.7.5. *Let* $w \in S_n^{J_2(r_0,k),\emptyset}$. *Then*

$$(\mathbf{a}_{\mu^\flat})^\sharp = (\mathbf{a}_{w^\flat} \setminus \mathbf{a}_w(k)) \cup \{\Delta^+ : \Delta \in \mathbf{a}_w(k)\}$$

for definition of $(\mathbf{a}_{\mu^\flat})^\sharp$, *cf. lemma 7.4.18.*

Definition 7.7.6. *Let* $t_w \in S_n^{J_1(\ell_k - r_0,k),\emptyset}$ *be the element such that*

$$\mathbf{a}_{t_w} = (\mathbf{a}_{w^\flat})^\sharp.$$

Proposition 7.7.7. *Let* P_J *and* $P_{J_1(\ell_k - r_0,k)}$ *be the parabolic subgroups corresponding to* J, $J_1(\ell_k - r_0, k)$ *respectively. Consider the natural morphism*

$$\pi : P_{J_1(\ell_k - r_0,k)} \backslash GL_n \to P_J \backslash GL_n.$$

Then

$$n(\mathbf{a}_w, \mathbf{a}_v) = \sum_i \dim \mathcal{H}^{2i}(\pi_*(IC(\overline{P_{J_1(\ell_k - r_0,k)} t_w B})))_x$$

for some $x \in P_J v B$.

Démonstration. Consider the composed morphism

$$h : \mathfrak{X}_{\mathbf{d}} = (\mathfrak{X}_{\mathbf{d}})_W \xrightarrow{p} E''_{\mathbf{a}} \xrightarrow{\beta''} E_{\varphi_{\mathbf{a}}}.$$

This proposition can be deduced from a construction of fibration similar to the one we did in Chapter 2 for symmetric multisegments, cf.§2.5.

□

7.8 Calculation of Partial Derivatives

Again, as previous section, we restrict ourselves to the case of multisegment of parabolic type.

Definition 7.8.1. *Let $J_1 \subseteq J_2 \subseteq S$ be two subsets of generators of S_n. Let $v \in S_n^{J_1,\emptyset}, w \in S_n^{J_2,\emptyset}$, we define $\theta_{J_2}^{J_1}(w,v)$ to be the multiplicities of $IC(\overline{P_{J_2}wB})$ in $\pi_*(IC(\overline{P_{J_1}vB}))$, where*

$$\pi : P_{J1}\backslash GL_n \to P_{J_2}\backslash GL_n$$

be the canonical projection.

Remark: By proposition 5.3.13, we know that in case where $J_1 = \emptyset$, $J_2 = \{s_i\}$ we have $\theta_{J_2}^{J_1}(w,v) = \mu(s_iw,v)$ if $\ell(v) \leq \ell(s_iv)$, where $\mu(x,y)$ is the coefficient of degree $(\ell(y)-\ell(x)-1)/2$ in $P_{x,y}(q)$.

Proposition 7.8.2. *Let $J \subseteq S$ be a subset of generators in S_n. Let $k \in \mathbb{Z}$ and* **a** *be a multisegment satisfies all the assumptions in the beginning of section 7.7. Then for any $w \in S_n^{J,\emptyset}$, we have*

$$\mathscr{D}^k(L_{\Phi(w)}) = \sum_{r_0=0}^{\ell_k} \sum_{v\in S_n^{J_2(r_0,k),\emptyset}} \theta_J^{J_1(\ell_k-r_0,k)}(w,t_v)L_{\Phi(v)}.$$

Démonstration. Note that by proposition 7.1.4

$$\mathscr{D}^k(\pi(\Phi(w))) = \sum_{\mathbf{b}\preceq_k\Phi(w)} n(\mathbf{b},\mathbf{a})L_{\mathbf{b}}.$$

Note that by proposition proposition 7.1.7, we know that $\mathbf{b} \preceq_k \Phi(w)$ implies that

$$\mathbf{b} = \Phi(v),$$

for some $v \in J_2(\ell_k - r_0, k)$. Moreover, according to the proposition 7.7.7

$$n(\Phi(v),\Phi(w)) = \sum_i \dim \mathcal{H}^{2i}(\pi_*(IC(\overline{P_{J_1(\ell_k-r_0,k)}t_vB})))_x$$

for some $x \in P_JwB$. In fact, by the decomposition theorem, we have

$$\pi_*(IC(\overline{P_{J_1(\ell_k-r_0,k)}t_vB}) = \bigoplus_{u\in S_n^J} \oplus_i IC(\overline{P_JuB})^{h_i(u,t_v)}[d_u^i] \tag{7.8.3}$$

therefore

$$\theta_J^{J_1(\ell_k-r_0,k)}(u,t_v) = \sum_i h_i(u,t_v).$$

Furthermore, if we denote by

$$\theta_J^{J_1(\ell_k-r_0,k)}(u,t_v)(q)=\sum_i h_i(u,t_v)q^{-d_u^i/2}$$

by localizing at a point of P_JwB and applying proper base change, we get

$$\sum_{\rho\in S_{J/J_1(\ell_k-r_0,k)}} q^{\ell(\rho)}P_{\rho w,t_v}^{J_1(\ell_k-r_0,k),\emptyset}(q)=\sum_u \theta_J^{J_1(\ell_k-r_0,k)}(u,t_v)(q)P_{w,u}^{J,\emptyset}(q). \quad (7.8.4)$$

Now we return to the formula

$$\pi(\Phi(w))=\sum_u P_{w,u}^{J,\emptyset}(1)L_{\Phi(u)}. \quad (7.8.5)$$

By induction, we can assume that for $u>w$, we have that $L_{\Phi(v)}$ appears in $\mathscr{D}^k(L_{\Phi(u)})$ with multiplicity $\theta_J^{J_1(\ell_k-r_0,k)}(u,t_v)$. Then by applying the derivation $\mathscr{D}^k$ to equation (7.8.5), on the right hand side we get the multiplicity of $L_{\Phi(v)}$ given by

$$x+\sum_{u>w}\theta_J^{J_1(\ell_k-r_0,k)}(u,t_v)P_{w,u}^{J,\emptyset}(1),$$

where x denotes the multiplicity of $L_{\Phi(v)}$ in the derivative $\mathscr{D}^k(L_{\Phi(w)})$. And on the right hand side, applying corollary 3.3.19, we get

$$\sum_{\rho\in S_{J/J_1(\ell_k-r_0,k)}} P_{\rho w,t_v}^{J_1(\ell_k-r_0,k),\emptyset}(1).$$

Now compare with the equation (7.8.4) to get $x=\theta_J^{J_1(\ell_k-r_0,k)}(w,t_v)$ □

From now on we consider the derivative $\mathscr{D}^k(L_{\mathbf{c}})$ for a general multisegment $\mathbf{c}$ such that $f_{e(\mathbf{c})}(k)>0$.

Proposition 7.8.6. *There exists a multisegment $\mathbf{c}'$ which is of parabolic type $(J_1(\mathbf{c}),\emptyset)$(cf. definition 6.3.1) and a sequence of integers $k_1,\ldots,k_r,k_{r+1},\ldots,k_{r+\ell}$ such that $L_{\mathbf{c}}$ is the minimal degree term with multiplicity one in*

$$^{k_1}\mathscr{D}\ldots{}^{k_r}\mathscr{D}\mathscr{D}^{k_{r+1}}\ldots\mathscr{D}^{k_{r+\ell}}(L_{\mathbf{c}'}),$$

and

$$f_{e(\mathbf{c}')}(i)=f_{e(\mathbf{c})}(i), \quad \text{if } i\leq k,$$
$$f_{e(\mathbf{c}')}(k+1)=0,$$
$$k_i>k+1, \quad \text{if } i>r.$$

Démonstration. Let $i_0 = \min\{i : f_{b(\mathbf{c})}(i) > 1\}$ and $\Delta_0 = \max_{\prec}\{\Delta \in \mathbf{c} : b(\Delta) = i_0\}$. Then replace all segments $\Delta \in \mathbf{c}$ with $b(\mathbf{c}) < i_0$ by ${}^+\Delta$ and Δ_0 by ${}^+\Delta$ to get a new multisegment $\mathbf{c}_1$. Then if we let $\{i \in b(\mathbf{c}) : i < i_0\} = \{j_1 < \cdots < j_r\}$, we have $L_{\mathbf{c}}$ is the minimal degree terms in

$$^{j_1-1}\mathscr{D} \ldots {}^{j_r-1}\mathscr{D}(L_{\mathbf{c}_1}),$$

Repeat this procedure to get $\mathbf{c}_0$ and a sequence of integers $k_1, \cdots, k_r$ such that $L_{\mathbf{c}}$ is the minimal degree term with multiplicity one in

$$^{k_1}\mathscr{D} \ldots {}^{k_r}\mathscr{D}(L_{\mathbf{c}_0}).$$

Suppose that $f_{e(\mathbf{c}_0)}(k+1) > 0$. Then replace all segments Δ in $\mathbf{c}_0$ with $e(\Delta) > k$ by Δ^+ to obtain $\mathbf{c}'$, we are done. □

Definition 7.8.7. *We define*

$$\Gamma^i(\mathbf{a}, k) = \{\mathbf{b} \in \Gamma(\mathbf{a}, k) : \deg(\mathbf{b}) + i = \deg(\mathbf{a})\},$$

where $\ell_k = f_{e(\mathbf{a})}(k)$*.*

Definition 7.8.8. *Let* $\mathbf{a}$ *be a multisegment and* $k, k_1 \in \mathbb{Z}$*. Then we define*

$$\begin{aligned}\Gamma^i(\mathbf{a}, k)_{k_1} &= \{\mathbf{b} \in \Gamma^i(\mathbf{a}, k) : \mathbf{b} \in S(\mathbf{b})_{k_1}, \mathbf{b}^{(k_1)} \in \Gamma^i(\mathbf{a}^{(k_1)}, k)\},\\ \Gamma(\mathbf{a}, k)_{k_1} &= \cup_i \Gamma^i(\mathbf{a}, k)_{k_1}.\end{aligned}$$

More generally for a sequence of integers $k_1, \cdots, k_r$*, we define*

$$\Gamma(\mathbf{a}, k)_{k_1, \cdots, k_r} = \{\mathbf{b} \preceq_k \mathbf{a} : \mathbf{b}^{(k_1, \cdots, k_{i-1})} \in \Gamma(\mathbf{a}^{(k_1, \cdots, k_{i-1})}, k)_{k_i} \text{ for } 1 \leq i \leq r\}.$$

Similarly, we can define ${}_{k_1}\Gamma(\mathbf{a}, k)$ *and* ${}_{k_1, \cdots, k_r}\Gamma(\mathbf{a}, k)$*.*

Remark: We can also talk about the set ${}_{k_{r+1}, \cdots, k_{r+\ell}}(\Gamma(\mathbf{a}, k)_{k_1, \cdots, k_r})$.

Lemma 7.8.9. *Let* $k_1 \neq k-1$*, then the map*

$$\begin{aligned}\psi_{k_1} : &\Gamma(\mathbf{a}, k)_{k_1} \to \Gamma(\mathbf{a}^{(k_1)}, k)\\ &\mathbf{b} \mapsto \mathbf{b}^{(k_1)}\end{aligned}$$

is bijective.

Démonstration. In fact we have $\Gamma^i(\mathbf{a}, k) = S(\mathbf{a}_i)$ where $\mathbf{a}_i$ is constructed in the following way : let $\mathbf{a}(k) = \{\Delta_1 \succeq \cdots \succeq \Delta_r\}$, then

$$\mathbf{a}_i = (\mathbf{a} \setminus \mathbf{a}(k)) \cup \{\Delta_j^- : j \leq i\} \cup \{\Delta_j : j > i\}.$$

Note that $\Gamma^i(\mathbf{a}, k) = S(\mathbf{a}_i)$, which implies that we have

$$\Gamma^i(\mathbf{a}, k)_{k_1} = S(\mathbf{a}_i)_{k_1}.$$

Finally, note that by proposition 3.4.1 we have a bijection

$$\psi_{k_1} : S(\mathbf{a}_i)_{k_1} \to S(\mathbf{a}_i^{(k_1)}).$$

Note that $k_1 \neq k-1, k$ implies that $\mathbf{a}_i^{(k_1)} \in \Gamma^i(\mathbf{a}^{(k_1)}, k)$ and

$$\Gamma(\mathbf{a}^{(k_1)}, k) = \bigcup_i S(\mathbf{a}_i^{(k_1)}).$$

And if $k_1 = k$, then

$$\Gamma(\mathbf{a}, k)_k = S(\mathbf{a})_k, \quad \Gamma(\mathbf{a}^{(k_1)}, k) = S(\mathbf{a}^{(k)}).$$

Hence we are done. □

Lemma 7.8.10. *Let $k_1, k \in \mathbb{Z}$ then the map*

$$\begin{aligned} {}_{k_1}\psi :{}_{k_1}\Gamma(\mathbf{a}, k) &\to \Gamma({}^{(k_1)}\mathbf{a}, k) \\ \mathbf{b} &\mapsto {}^{(k_1)}\mathbf{b} \end{aligned}$$

is bijective.

Démonstration. If $k_1 \neq k$, the proof is the same as that of the previous lemma. Consider the case where $k_1 = k$. Let $\mathbf{a}(k) = \{\Delta_1 \succeq \cdots \succeq \Delta_{r_0} \succ \underbrace{[k] = \cdots = [k]}_{r_1}\}$. Then for $i \leq \ell_k$, we have

$$\mathbf{a}_i = (\mathbf{a} \setminus \mathbf{a}(k)) \cup \{\Delta_j^- : j \leq i\} \cup \{\Delta_j : j > i\},$$

where $\Delta_j = [k]$ if $j > r_0$. And we have $\Gamma^i(\mathbf{a}, k) = S(\mathbf{a}_i)$. By definition, we have $\mathbf{b} \in {}_k\Gamma^i(\mathbf{a}, k)$ if and only if

$$\mathbf{b} \in {}_kS(\mathbf{b}), \quad {}^{(k)}\mathbf{b} \in \Gamma^i({}^{(k)}\mathbf{a}, k).$$

Since ${}^{(k)}\mathbf{a}(k) = \{\Delta_1, \cdots, \Delta_{r_0}\}$, we know that for $\mathbf{b} \in {}_k\Gamma^i(\mathbf{a}, k)$, we must have $i \leq r_0$. Also, let

$$({}^{(k)}\mathbf{a})_i = ({}^{(k)}\mathbf{a} \setminus {}^{(k)}\mathbf{a}(k)) \cup \{\Delta_j^- : j \leq i\} \cup \{\Delta_j : r_0 \geq j > i\}.$$

And we have $\Gamma^i({}^{(k)}\mathbf{a}, k) = S(({}^{(k)}\mathbf{a})_i)$. Then we have

$${}^{(k)}\mathbf{a}_i = ({}^{(k)}\mathbf{a})_i.$$

Finally, we conclude that $\mathbf{b} \in {}_k\Gamma^i(\mathbf{a}, k)$ if and only if $\mathbf{b} \in {}_kS(\mathbf{a}_i)$. Since the map

$${}_kS(\mathbf{a}_i) \to S({}^{(k)}\mathbf{a}_i)$$

is bijective, we are done. □

Proposition 7.8.11. *Let* $\mathbf{b}, \mathbf{c}$ *be two multisegments and* $k_1 \in \mathbb{Z}$ *such that*

$$\mathbf{b} =^{(k_1)} \mathbf{c}, \quad \mathbf{c} \in {}_{k_1}S(\mathbf{c}).$$

If we write

$$\mathscr{D}^k(L_{\mathbf{c}}) = L_{\mathbf{c}} + \sum_{\mathbf{d} \in \Gamma(\mathbf{c},k) \setminus \{\mathbf{c}\}} \widetilde{n}(\mathbf{d}, \mathbf{c}) L_{\mathbf{d}}, \tag{7.8.12}$$

then

$$\mathscr{D}^k(L_{\mathbf{b}}) = L_{\mathbf{b}} + \sum_{\mathbf{d} \in {}_{k_1}\Gamma(\mathbf{c},k) \setminus \{\mathbf{c}\}} \widetilde{n}(\mathbf{d}, \mathbf{c}) L_{{}^{(k_1)}\mathbf{d}}.$$

Démonstration. -Suppose that $\deg(\mathbf{c}) = \deg(\mathbf{b}) + 1$. In fact, by corollary 3.5.4, we have

$${}^{k_1}\mathscr{D}(L_{\mathbf{c}}) = L_{\mathbf{c}} + L_{\mathbf{b}}$$

By applying the derivation $\mathscr{D}^k$ and using the fact $\mathscr{D}^k({}^{k_1}\mathscr{D}) = {}^{k_1}\mathscr{D}\mathscr{D}^k$, we have

$$\mathscr{D}^k(L_{\mathbf{c}}) + \mathscr{D}^k(L_{\mathbf{b}}) = L_{\mathbf{c}} + L_{\mathbf{b}} + \sum_{\mathbf{d} \in \Gamma(\mathbf{c},k) \setminus \{\mathbf{c}\}} \widetilde{n}(\mathbf{d}, \mathbf{c})^{k_1}\mathscr{D}(L_{\mathbf{d}})$$

By assumption that $\deg(\mathbf{b}) + 1 = \deg(\mathbf{c})$, we have

$${}^{k_1}\mathscr{D}(L_{\mathbf{d}}) = L_{\mathbf{d}} + L_{{}^{(k_1)}\mathbf{d}} \text{ or } L_{\mathbf{d}},$$

where ${}^{k_1}\mathscr{D}(L_{\mathbf{d}}) = L_{\mathbf{d}} + L_{{}^{(k_1)}\mathbf{d}}$ if and only if $\mathbf{d} \in {}_{k_1}S(\mathbf{d})$ and $\deg({}^{(k_1)}\mathbf{d}) = \deg(\mathbf{d}) - 1$. This is equivalent to say that $\mathbf{d} \in {}_{k_1}\Gamma(\mathbf{a}, k)$.

-For general case, consider

$$\{\Delta \in \mathbf{c} : b(\Delta) = k_1\} = \{\Delta_1 \succeq \cdots \succeq \Delta_r\}.$$

Now by proposition 3.5.1 and proposition 7.1.4, we know that

$${}^{k_1}\mathscr{D}(L_{\mathbf{c}}) = L_{\mathbf{b}} + \sum_{f_{\mathbf{d}}(k_1) > f_{\mathbf{b}}(k_1)} \widetilde{n}(\mathbf{d}, \mathbf{c}) L_{\mathbf{d}},$$

for some $\widetilde{n}(\mathbf{d}, \mathbf{c}) \in \mathbb{N}$.
If $k_1 \neq k$, then We observe that for any $\mathbf{d}$ such that $f_{\mathbf{d}}(k_1) > f_{\mathbf{b}}(k_1)$ and $\mathbf{d}' \preceq_k \mathbf{d}$, we have

$$f_{\mathbf{d}'}(k_1) > f_{\mathbf{b}}(k_1),$$

which implies that $L_{\mathbf{d}'}$ can not be a summand of $\mathscr{D}^k(L_{\mathbf{b}})$. Therefore we know that

$$\mathscr{D}^k(L_{\mathbf{b}})$$

is the sum of all irreducible representations $L_{\mathbf{d}''}$ contained in $\mathscr{D}^k({}^{k_1}\mathscr{D})(L_{\mathbf{c}})$ satisfying

$$f_{\mathbf{d}''}(k_1) = f_{\mathbf{b}}(k_1).$$

Applying the derivation ${}^{k_1}\mathscr{D}$ to the equation (7.8.12), we get

$${}^{k_1}\mathscr{D}\mathscr{D}^k(L_{\mathbf{c}}) = {}^{k_1}\mathscr{D}(L_{\mathbf{c}}) + \sum_{\mathbf{d}\preceq_k \mathbf{c}} \widetilde{n}(\mathbf{d},\mathbf{c})({}^{k_1}\mathscr{D})(L_{\mathbf{d}}).$$

Note that in this case the sub-quotient of ${}^{k_1}\mathscr{D}\mathscr{D}^k(L_{\mathbf{c}})$ consisting of irreducible representations $L_{\mathbf{d}''}$ satisfying

$$f_{\mathbf{d}''}(k_1) = f_{\mathbf{b}}(k_1)$$

is given by

$$L_{\mathbf{b}} + \sum_{\mathbf{d}\in_{k_1}\Gamma(\mathbf{c},k)\setminus\{\mathbf{c}\}} \widetilde{n}(\mathbf{d},\mathbf{c})L_{{}^{(k_1)}\mathbf{d}}.$$

Compare the equation ${}^{k_1}\mathscr{D}\mathscr{D}^k(L_{\mathbf{c}}) = \mathscr{D}^k({}^{k_1}\mathscr{D})(L_{\mathbf{c}})$ gives the results.
If $k_1 = k$, consider

$$\{\Delta \in \mathbf{c} : b(\Delta) = k_1\} = \{\Delta_1 \succeq \cdots \succeq \Delta_r\}.$$

Let $\mathbf{c}'$ be the multisegment obtained by replacing all segments Δ in $\mathbf{c}$ such that $b(\Delta) < k_1$ by ${}^+\Delta$, and Δ_1 by ${}^+\Delta_1$. Then there exists

$$k_2 = k_1 - 1 > k_3 > \cdots > k_r$$

such that

$$\mathbf{c} = {}^{(k_r,\cdots,k_2)}\mathbf{c}',$$

and

$$\mathbf{b} = {}^{(k_r,\cdots,k_3,k_1,k_2,k_1)}\mathbf{c}'.$$

Let $\mathbf{b}' ={}^{(k_1)} \mathbf{c}'$, then by induction on $f_{b(\mathbf{c})}(k)$, we can assume that

$$\mathscr{D}^k(L_{\mathbf{b}'}) = L_{\mathbf{b}'} + \sum_{\mathbf{d}\in_k\Gamma(\mathbf{c}',k)\setminus\mathbf{c}'} \widetilde{n}(\mathbf{d},\mathbf{c}')L_{{}^{(k)}\mathbf{d}}.$$

Applying what we have proved before, we get

$$\mathscr{D}^k(L_{\mathbf{b}}) = L_{\mathbf{b}} + \sum_{\mathbf{d}\in_{k_r,\cdots,k_3,k,k_2,k}\Gamma(\mathbf{c}',k)\setminus\{\mathbf{c}'\}} \widetilde{n}(\mathbf{d},\mathbf{c}')L_{{}^{(k_r,\cdots,k_3,k,k_2,k)}\mathbf{d}}.$$

Also, we have

$$\mathscr{D}^k(L_{\mathbf{c}}) = L_{\mathbf{c}} + \sum_{\mathbf{d}\in_{k_r,\cdots,k_3,k_2}\Gamma(\mathbf{c}',k)\setminus\{\mathbf{c}'\}} \widetilde{n}(\mathbf{d},\mathbf{c}')L_{{}^{(k_r,\cdots,k_3,k_2)}\mathbf{d}}.$$

Since for any multisegment $\mathbf{d}$, we have

$$^{(k,k_r,\cdots,k_3,k_2)}\mathbf{d} =^{(k_r,\cdots,k_3,k,k_2,k)} \mathbf{d},$$

it remains to show that

$$_{k_r,\cdots,k_3,k,k_2,k}\Gamma(\mathbf{c}',k) =_{k,k_r,\cdots,k_3,k_2} \Gamma(\mathbf{c}',k).$$

By definition and the following lemma, we can assume that $r = 2$. In this case we argue by contradiction. Suppose that $\mathbf{d} \in {}_{k,k-1,k}\Gamma^i(\mathbf{c}',k)$ and $\mathbf{d} \notin {}_{k,k-1}\Gamma(\mathbf{c}',k)$, which is equivalent to say that $\mathbf{d} \notin {}_{k,k-1}S(\mathbf{d})$. Note that $\mathbf{d} \notin {}_{k,k-1}S(\mathbf{d})$ implies that there exists two linked segments $\{\Delta,\Delta'\}$, such that

$$b(\Delta) = k, \quad b(\Delta') = k-1.$$

Then $^{(k-1,k)}\mathbf{d}$ contains the pair of segments $\{^-\Delta, {}^-\Delta'\}$. The fact that $^{(k-1,k)}\mathbf{d} \in {}_kS(^{(k-1,k)}\mathbf{d})$ implies that $^-\Delta' = \emptyset$, i.e. $\Delta' = [k-1]$. However, this implies that $^{(k,k-1,k)}\mathbf{d} \notin \Gamma^i(^{(k,k-1,k)}\mathbf{c}',k)$ since $\deg(^{(k,k-1,k)}\mathbf{d}) + i = \deg(^{(k,k-1,k)}\mathbf{a}) + 1$, which is a contradiction.
Conversely, assume that $\mathbf{d} \in {}_{k,k-1}\Gamma(\mathbf{c}',k)$ and $\mathbf{d} \notin {}_{k,k-1,k}\Gamma^i(\mathbf{c}',k)$, which by definition is equivalent to $\mathbf{d} \notin {}_{k,k-1,k}S(\mathbf{d})$. Note that $\mathbf{d} \notin {}_{k,k-1,k}S(\mathbf{d})$ implies that $\mathbf{d} \notin {}_kS(\mathbf{d})$, which contradicts to $\mathbf{d} \in {}_{k,k-1}S(\mathbf{d})$. □

Lemma 7.8.13. *Let $k > k-1 > k'$ be two integers. Then for any multisegment $\mathbf{c}$, we have*

$$_{k,k'}\Gamma(\mathbf{c},k) = {}_{k',k}\Gamma(\mathbf{c},k).$$

Démonstration. Note that since for any multisegment $\mathbf{d}$

$$^{(k',k)}\mathbf{d} = {}^{k,k'}\mathbf{d},$$

the fact

$$_{k,k'}\Gamma(\mathbf{c},k) = {}_{k',k}\Gamma(\mathbf{c},k)$$

is equivalent to

$$\mathbf{d} \in {}_{k,k'}S(\mathbf{d}) \Leftrightarrow \mathbf{d} \in {}_{k',k}S(\mathbf{d})$$

for all $\mathbf{d} \in {}_{k,k'}\Gamma(\mathbf{c},k)$. But for any multisegment $\mathbf{d}$ and $k > k-1 > k'$, we have

$$\mathbf{d} \in {}_{k,k'}S(\mathbf{d}) \Leftrightarrow \mathbf{d} \in {}_{k',k}S(\mathbf{d}).$$

Hence we are done. □

Proposition 7.8.14. *Let $k_1 \neq k-1, k, k+1$. Let $\mathbf{b}, \mathbf{c}$ be two multisegments such that*

$$\mathbf{b} = \mathbf{c}^{(k_1)}, \quad \mathbf{c} \in S(\mathbf{c})_{k_1}.$$

If we write

$$\mathscr{D}^k(L_{\mathbf{c}}) = L_{\mathbf{c}} + \sum_{\mathbf{d} \in \Gamma(\mathbf{c},k) \setminus \{\mathbf{c}\}} \widetilde{n}(\mathbf{d}, \mathbf{c}) L_{\mathbf{d}}, \tag{7.8.15}$$

then

$$\mathscr{D}^k(L_{\mathbf{b}}) = L_{\mathbf{b}} + \sum_{\mathbf{d} \in \Gamma(\mathbf{c},k)_{k_1} \setminus \{\mathbf{c}\}} \widetilde{n}(\mathbf{d}, \mathbf{c}) L_{d^{(k_1)}}.$$

Démonstration. The proof is the same as the proposition above. □

Now let $\mathbf{c}' = \Phi(w)$ for some $w \in S_n^{J,\emptyset}$.

Corollary 7.8.16. *We have*

$$\mathscr{D}^k(L_{\mathbf{a}}) = \sum_{r_0=0}^{\ell_k} \sum_{v \in S_n^{J_2(r_0,k),\emptyset}, \Phi(v) \in_{k_1,\cdots,k_r} (\Gamma(\Phi(w),k)_{k_{r+1},\cdots,k_{r+\ell}})} \theta_J^{J_1(\ell_k - r_0, k)}(w, t_v) L_{(k_1,\cdots,k_r)\Phi(v)^{(k_{r+1},\cdots,k_{r+\ell})}}.$$

Notation 7.8.17. *For $\mathbf{b} \preceq_k \mathbf{a}$, we denote*

$$\theta_k(\mathbf{b}, \mathbf{a}) = \theta_J^{J_1(\ell_k - r_0, k)}(w, t_v)$$

if $\mathbf{b} =^{(k_1,\cdots,k_r)} \Phi(v)^{(k_{r+1},\cdots,k_{r+\ell})}$. Otherwise, put $\theta_k(\mathbf{b}, \mathbf{a}) = 0$.

Remark: The same way we define ${}_k\theta(\mathbf{b}, \mathbf{a})$ by the formula

$$({}^k\mathscr{D})(L_{\mathbf{a}}) = \sum_{\mathbf{b}} {}_k\theta(\mathbf{b}, \mathbf{a}) L_{\mathbf{b}}.$$

And let

$$\Gamma(k, \mathbf{a}) = \{\mathbf{b} : {}_k\theta(\mathbf{b}, \mathbf{c}) \neq 0 \text{ for some } \mathbf{c} \in S(\mathbf{a})\},$$

it shares similar properties with $\Gamma(\mathbf{a}, k)$.

Chapitre 8

Multiplicities in induced representations : case of a segment

In this chapter we will consider the multiplicities $m(\mathbf{c}, \mathbf{b}, \mathbf{a})$ of irreducible components in the induced representation $L_{\mathbf{a}} \times L_{\mathbf{b}}$,

$$L_{\mathbf{a}} \times L_{\mathbf{b}} = \sum m(\mathbf{c}, \mathbf{b}, \mathbf{a}) L_{\mathbf{c}}.$$

Our goal in this chapter is then to determine a formula for the coefficient $m(\mathbf{c}, \mathbf{b}, \mathbf{a})$ in case where $\mathbf{b} = [k - i_0 + 1, k + 1] (i_0 \geq 0)$ is a segment. Roughly speaking, there are two major cases to discuss

(1) $\max b(\mathbf{a}) \leq k - i_0 + 1$,

(2) $\max b(\mathbf{a}) > k - i_0 + 1$.

In §8.1 we treat the first case, which is simpler to deal with. We have an explicit formula for the case where $\mathbf{b} = [k+1]$ (cf. lemma 8.1.7 and proposition 8.1.5), and then we deduce by induction the general case (cf. proposition 8.1.12). For example the formula of proposition 8.1.5 looks like

$$L_{\mathbf{a}} \times L_{\mathbf{b}} = L_{\mathbf{a}+\mathbf{b}} + \sum_{\mathbf{c} \in \Gamma^{\ell_k - 1}(\mathbf{a}, k)} \Big(\theta_k(\mathbf{c}, \mathbf{a}) - \theta_k(\mathbf{c}^{[k+1]_1}, \mathbf{a} + \mathbf{b}) \Big) L_{\mathbf{c}^{[k+1]_1 [k]_{\ell_k - 1}}}.$$

where the $\theta_k(\mathbf{c}, \mathbf{a})$ are defined thanks to partial derivative, cf. notation 7.8.17. Here our main tool is the derivatives for which we have complete formulas, cf. proposition 7.8.16. Note that even in the case where $\mathbf{b} = [k+1]$ is a point, we come across the difficulty that we have $\mathscr{D}^k(L_{\mathbf{c}}) = L_{\mathbf{c}}$ for certain multisegments, cf. example 8.1.10, which prevents us from applying the partial

derivations. Our idea here is to first treat the case where $f_{e(\mathbf{a})}(k-1)=0$, cf. proposition 8.1.5, and then reduce everything to such case.
In §8.2, we describe a procedure to compute $m(\mathbf{c},\mathbf{b},\mathbf{a})$ for the second case, combining the first case and partial derivatives.

Finally, we remark that our method could be used to deduce the general multiplicities for case where $\mathbf{b}$ is not a segment. We intend to study this general case in some future work.

8.1 When $\max b(\mathbf{a}) \leq k-i_0+1$

In this section we consider the case $L_{\mathbf{a}} \times L_{\mathbf{b}}$ where $\mathbf{b}=[k-i_0+1,k+1]$, with $i_0 \geq 0$, is a segment and $\mathbf{a}$ is a multisegment satisfying

$$\max b(\mathbf{a}) \leq k-i_0+1.$$

Definition 8.1.1. *Let* $\mathbf{b}$ *be a multisegment such that* $f_{e(\mathbf{b})}(k+1)=0$. *Then we denote by* $\mathbf{b}^{[k+1]_i}$ *the unique element in* $S(\mathbf{b}+i[k+1])_k$ *such that*

$$\mathbf{c}=(\mathbf{c}^{[k+1]_i})^{(k+1)}.$$

Proposition 8.1.2. *Let* $\mathbf{a}$ *be a multisegment satisfying the condition*

$$f_{e(\mathbf{a})}(k-i_0-1)\neq 0.$$

If we assume that

$$\{t\in e(\mathbf{a}): t\leq k-i_0-1\}=\sum_{i=1}^{r}\ell_{k_i}[k_i]$$

with $k_1<\cdots<k_r=k-1$, *then*

$$m(\mathbf{c}^{[k_r]_{\ell_{k_r}}[k_{r-1}]_{\ell_{k_{r-1}}}\cdots[k_1]_{\ell_1}},\mathbf{b},\mathbf{a})=m(\mathbf{c},\mathbf{b},\mathbf{a}^{(k_1,\cdots,k_r)}).$$

Démonstration. We prove by induction on i that

$$m(\mathbf{c}^{[k_i]_{\ell_{k_i}}[k_{i-1}]_{\ell_{k_{i-1}}}\cdots[k_1]_{\ell_{k_1}}},\mathbf{b},\mathbf{a})=m(\mathbf{c},\mathbf{b},\mathbf{a}^{(k_1,\cdots,k_i)})$$

For $i=1$, since $\mathbf{a}$ satisfies the hypothesis $H_{k_1}(\mathbf{a})$, by proposition 3.5.1, $\mathscr{D}^{k_1}(L_{\mathbf{a}})$ contains a unique minimal degree term with multiplicity one, which is $L_{\mathbf{a}^{(k_1)}}$, now apply $\mathscr{D}^{k_1}$ to

$$L_{\mathbf{a}}\times L_{\mathbf{b}}=\sum_{\mathbf{c}}m(\mathbf{c},\mathbf{b},\mathbf{a})L_{\mathbf{c}}$$

and consider the minimal degree terms on both sides, we obtain

$$L_{\mathbf{a}^{(k_1)}} \times L_{\mathbf{b}} = \sum_{\mathbf{c}\in S(\mathbf{a}+\mathbf{b})_{k_1}} m(\mathbf{c},\mathbf{b},\mathbf{a}) L_{\mathbf{c}^{(k_1)}}$$

which gives the formula. Now for general $i < r$, assume that we have

$$m(\mathbf{c}^{[k_i]_{\ell_{k_i}}[k_{i-1}]_{\ell_{k_{i-1}}}\cdots[k_1]_{\ell_{k_1}}}, \mathbf{b}, \mathbf{a}) = m(\mathbf{c}, \mathbf{b}, \mathbf{a}^{(k_1,\cdots,k_i)}),$$

that is to say

$$L_{\mathbf{a}^{(k_1,\cdots,k_i)}} \times L_{\mathbf{b}} = \sum_{\mathbf{c}\in S(\mathbf{a}+\mathbf{b})_{k_1,\cdots,k_i}} m(\mathbf{c},\mathbf{b},\mathbf{a}) L_{\mathbf{c}^{(k_1,\cdots,k_i)}}.$$

Now apply $\mathscr{D}^{k_{i+1}}$ and the same argument as in the case where $i = 1$ gives

$$L_{\mathbf{a}^{(k_1,\cdots,k_{i+1})}} \times L_{\mathbf{b}} = \sum_{\mathbf{c}\in S(\mathbf{a}+\mathbf{b})_{k_1,\cdots,k_{i+1}}} m(\mathbf{c},\mathbf{b},\mathbf{a}) L_{\mathbf{c}^{(k_1,\cdots,k_{i+1})}}.$$

□

Remark: If we assume that **a** is of parabolic type, i.e

$$\bigcap_{\Delta\in\mathbf{a}} \Delta \neq \emptyset$$

then

$$S(\mathbf{a})_{k_1,\cdots,k_r} = S(\mathbf{a}).$$

Then by replacing **a** by $\mathbf{a}^{(k_1,\cdots,k_r)}$, we are reduced to the case where

$$f_{e(\mathbf{a})}(k - i_0 - 1) = 0.$$

Proposition 8.1.3. *Let* **a** *be a multisegment such that*

$$f_{e(\mathbf{a})}(k+1) \neq 0.$$

And let

$$\{t \in e(\mathbf{a}) : t \geq k+1\} = \sum_{i=1}^{s} \ell_{k_i}[k_i]$$

with $k_1 < k_2 < \cdots < k_s$. *Then*

$$m(\mathbf{c},\mathbf{b},\mathbf{a}) = m(\mathbf{c}^{[k_r]_{\ell_{k_r}}[k_{r-1}]_{\ell_{k_{r-1}}}\cdots[k_1]_{\ell_{k_1}}}, \mathbf{b}, \mathbf{a}^{[k_r]_{\ell_{k_r}}[k_{r-1}]_{\ell_{k_{r-1}}}\cdots[k_1]_{\ell_{k_1}}})$$

Remark: This proposition allows us to reduce to the case where

$$f_{e(\mathbf{a})}(k+1) = 0.$$

Démonstration. The proof is the same as that of the proposition above. □

As usual, we reduce to the parabolic case by the following proposition.

Proposition 8.1.4. *Let* $\mathbf{a}$ *be a multisegment satisfying* $\max b(\mathbf{a}) \leq k - i_0 + 1$, *then there exists a sequence of integers* $k_1, k_2, \cdots, k_r$ *and a parabolic multisegments* $\mathbf{c}$ *of type* $(J_1(\mathbf{a}), \emptyset)$*such that*

$$\mathbf{a} = {}^{(k_1,\cdots,k_r)}\mathbf{c}, \quad \mathbf{c} \in {}_{k_1,\cdots,k_r}S(\mathbf{c})$$

and if

$$L_{\mathbf{c}} \times L_{\mathbf{b}} = \sum_{\mathbf{d}} m(\mathbf{d}, \mathbf{c}, \mathbf{b}) L_{\mathbf{d}}$$

then

$$L_{\mathbf{a}} \times L_{\mathbf{b}} = \sum_{\mathbf{d} \in {}_{k_1,\cdots,k_r}S(\mathbf{c}+\mathbf{b})} m(\mathbf{d}, \mathbf{c}, \mathbf{b}) L_{{}^{(k_1,\cdots,k_r)}\mathbf{d}}.$$

Démonstration. The existence of $\mathbf{c}$ follows from proposition 6.3.3. To deduce our result, it suffices to apply the derivation

$$({}^{k_1}\mathscr{D})({}^{k_2}\mathscr{D}) \cdots ({}^{k_r}\mathscr{D})$$

to $L_{\mathbf{c}} \times L_{\mathbf{b}} = \sum_{\mathbf{d}} m(\mathbf{d}, \mathbf{c}, \mathbf{b}) L_{\mathbf{d}}$ and then apply proposition 3.5.1. □

Proposition 8.1.5. *Assume that* $\mathbf{a}$ *is a* ***parabolic*** *multisegment such that*

$$f_{e(\mathbf{a})}(k-i+1) = 0$$

for some $1 \leq i \leq i_0$. *Then*

$$m(\mathbf{c}, \mathbf{b}, \mathbf{a}) = m(\mathbf{c}^{(k-i+2,\cdots,k-1,k)}, \mathbf{b}^{(k+1)}, \mathbf{a}^{(k-i+2,\cdots,k-1,k)}).$$

Démonstration. The proof is the same as that of proposition 8.1.2 □

Remark: Combining the proposition 8.1.2, 8.1.4, 8.1.5, and 8.1.3, the calculation of the coefficients $m(\mathbf{c}, \mathbf{b}, \mathbf{a})$ for case (1) can be reduced to the case where $\mathbf{a}$ is a parabolic multisegment such that

$$f_{e(\mathbf{a})}(k-i_0-1) = f_{e(\mathbf{a})}(k+1) = 0, \quad f_{e(\mathbf{a})}(k-i+1) \neq 0, \text{ for all } 1 \leq i \leq i_0+1.$$

From now on until the end of the section, assume that

$$\mathbf{a}_{\mathrm{Id}}^{J,\emptyset}$$

be a multisegment of type $(J,\emptyset)$ associated to the identity in S_n, which satisfies

$$f_{e(\mathbf{a}_{\mathrm{Id}}^{J,\emptyset})}(k-i_0-1)=f_{e(\mathbf{a}_{\mathrm{Id}}^{J,\emptyset})}(k+1)=0,\quad f_{e(\mathbf{a}_{\mathrm{Id}}^{J,\emptyset})}(i)>0 \text{ for } k-i_0\leq i\leq k,$$

and fix a bijection

$$\Phi^{i_0}:S_n^{J,\emptyset}\to S(\mathbf{a}_{\mathrm{Id}}^{J,\emptyset})$$

and $\mathbf{a}_{i_0}=\Phi^{i_0}(w)$.

Lemma 8.1.6. *Under the above assumption, we have*

$$J_1(\ell_{k-i_0}-r_0,k-i_0)=J_2(\ell_{k-i_0}-r_0,k-i_0).$$

Démonstration. This follows directly from the definition. □

Lemma 8.1.7. *Let* $\mathbf{b}=[k+1]$ *and* $\ell_k=f_{e(\mathbf{a}_0)}(k)$. *Then*

$$L_{\mathbf{a}_0}\times L_{\mathbf{b}}=L_{\mathbf{a}_0+\mathbf{b}}+\sum_{\mathbf{c}\in\Gamma^{\ell_k-1}(\mathbf{a}_0,k)}\Big(\theta_k(\mathbf{c},\mathbf{a}_0)-\theta_k(\mathbf{c}^{[k+1]_1},\mathbf{a}_0+\mathbf{b})\Big)L_{\mathbf{c}^{[k+1]_1[k]_{\ell_k-1}}}$$

Démonstration. Note that

$$({}^{k+1}\mathscr{D})(L_{\mathbf{a}_0}\times L_{\mathbf{b}})=L_{\mathbf{a}_0}\times L_{\mathbf{b}}+L_{\mathbf{a}_0}.$$

And for each $\mathbf{c}\in S(\mathbf{a}_0+\mathbf{b})$ if $[k+1]\in\mathbf{c}$, then

$$({}^{k+1}\mathscr{D})L_{\mathbf{c}}=L_{\mathbf{c}}+L_{(k+1)\mathbf{c}}.$$

This implies that if $\mathbf{c}\neq\mathbf{a}_0+\mathbf{b}$ and $[k+1]\in\mathbf{c}$, then $L_{\mathbf{c}}$ can not be a direct summand of $L_{\mathbf{a}_0}\times L_{\mathbf{b}}$. Furthermore, by assumption on $\mathbf{a}_0$, we know that for any $\mathbf{c}\in S(\mathbf{a}_0+\mathbf{b})$ and $[k+1]\notin\mathbf{c}$, we have $\mathbf{c}\in S(\mathbf{a}_0+\mathbf{b})_k$ and hence $\mathbf{c}\in S(\mathbf{a}_0+\mathbf{b})_{k,k+1}$. Moreover, we know that $\mathbf{c}^{(k,k+1)}\in\Gamma^{\ell_k-1}(\mathbf{a}_0,k)$. Therefore we have

$$L_{\mathbf{a}_0}\times L_{\mathbf{b}}=L_{\mathbf{a}_0+\mathbf{b}}+\sum_{\mathbf{c}\in\Gamma^{\ell_k-1}(\mathbf{a},k)}m(\mathbf{c},\mathbf{b},\mathbf{a}_0)L_{\mathbf{c}^{[k+1]_1[k]_{\ell_k-1}}}.$$

Now apply the derivation $\mathscr{D}^k$ to both sides of the equation to get

$$\begin{aligned}\mathscr{D}^k(L_{\mathbf{a}_0}\times L_{\mathbf{b}})&=(\sum_{\mathbf{c}\preceq_k\mathbf{a}_0}\theta_k(\mathbf{c},\mathbf{a}_0)L_{\mathbf{c}})\times L_{\mathbf{b}}\\&=\sum_{\mathbf{c}\in\Gamma^{\ell_k-1}(\mathbf{a}_0,k)}\theta_k(\mathbf{c},\mathbf{a}_0)L_{\mathbf{c}}\times L_{\mathbf{b}}+\text{ other degree terms}\end{aligned}$$

and the right hand side we get

$$\sum_{\mathbf{c}\in\Gamma^{\ell_k-1}(\mathbf{c},\mathbf{a}_0+\mathbf{b})}\theta_k(\mathbf{c},\mathbf{a}_0+\mathbf{b})L_{\mathbf{c}}+\sum_{\mathbf{c}\in\Gamma^{\ell_k-1}(\mathbf{a}_0,k)}m(\mathbf{c},\mathbf{b},\mathbf{a}_0)L_{\mathbf{c}^{[k+1]_1}}+\text{ other degree terms }.$$

Now by the following lemma we know that for $\mathbf{c}\in\Gamma^{\ell_k-1}(\mathbf{a}_0,k)$

$$L_{\mathbf{c}}\times L_{\mathbf{b}}=L_{\mathbf{c}+\mathbf{b}}+L_{\mathbf{c}^{[k+1]_1}},$$

therefore by comparing the two sides, we obtain that for $\mathbf{c}\in\Gamma^{\ell_k-1}(\mathbf{a}_0,k)$

$$m(\mathbf{c},\mathbf{b},\mathbf{a}_0)+\theta_k(\mathbf{c}^{[k+1]},\mathbf{a}_0+\mathbf{b})=\theta_k(\mathbf{c},\mathbf{a}_0).$$

Hence we are done.

□

Lemma 8.1.8. *Let* $\mathbf{a}$ *be a multisegment such that*

$$\max b(\mathbf{a})\leq k+1,\quad f_{e(\mathbf{a})}(k)=1,\quad f_{e(\mathbf{a})}(k+1)=0.$$

Then we have

$$L_{\mathbf{a}}\times L_{[k+1]}=L_{\mathbf{a}+[k+1]}+L_{\mathbf{a}^{[k+1]_1}}.$$

Démonstration. First of all, it is known by Zelevinsky that $L_{\mathbf{a}+[k+1]}$ appears in $L_{\mathbf{a}}\times L_{[k+1]}$ with multiplicity one. Also, since

$$\mathscr{D}^{k+1}(L_{\mathbf{a}}\times L_{[k+1]})=L_{\mathbf{a}}\times L_{[k+1]}+L_{\mathbf{a}},$$

we know that $L_{\mathbf{a}^{[k+1]_1}}$ is the only element in $S(\mathbf{a}+[k+1])$ which appears as a subquotient in $L_{\mathbf{a}}\times L_{[k+1]}$ and does not contain $[k+1]$ as a beginning. Finally, since

$$^{k+1}\mathscr{D}(L_{\mathbf{a}}\times L_{[k+1]})=L_{\mathbf{a}}\times L_{[k+1]}+L_{\mathbf{a}},$$

we conclude that $\mathbf{a}+[k+1]$ is the only multisegment in $S(\mathbf{a}+[k+1])$ which is a subquotient of $L_{\mathbf{a}}\times L_{[k+1]}$ and contains $[k+1]$ as a beginning. □

In particular,gathering all the calculation in case where $\mathbf{b}=[k+1]$, we obtain the following formula.

Corollary 8.1.9. *Let* $\mathbf{a}$ *be a parabolic multisegment satisfying the condition*

$$f_{e(\mathbf{a})}(k)\neq 0,\quad f_{e(\mathbf{a})}(k-1)=f_{e(\mathbf{a})}(k+1)=0,$$

and $\mathbf{b}=[k+1]$*. Then*

$$L_{\mathbf{a}}\times L_{\mathbf{b}}=L_{\mathbf{a}+\mathbf{b}}+\sum_{\mathbf{c}\in\Gamma^{\ell_k-1}(\mathbf{a},k)}\Big(\theta_k(\mathbf{c},\mathbf{a})-\theta_k(\mathbf{c}^{[k+1]_1},\mathbf{a}+\mathbf{b})\Big)L_{\mathbf{c}^{[k+1]_1[k]_{\ell_k-1}}}.$$

Remark: The proposition is no longer true if we remove the condition

$$f_{e(\mathbf{a})}(k-1)=0.$$

Example 8.1.10. *Let* $\mathbf{a}=[0,2]+[1,3]+[2,3]$ *and* $\mathbf{b}=[4]$, *and*

$$\mathbf{c}_1=[0,3]+[1,4]+[2],\quad \mathbf{c}_2=[0,2]+[1,4]+[2,3],\quad \mathbf{d}=[0,2]+[2]+[1,3],$$

then

$$L_{\mathbf{a}}\times L_{\mathbf{b}}=L_{\mathbf{a}+\mathbf{b}}+L_{\mathbf{c}_1}+L_{\mathbf{c}_2}$$

and

$$\mathscr{D}^3(L_{\mathbf{a}})=L_{\mathbf{a}}+L_{\mathbf{d}},\quad \mathscr{D}^3(L_{\mathbf{c}_2})=L_{\mathbf{c}_2}.$$

In this case we cannot compute the multiplicity of $L_{\mathbf{c}_2}$ *using directly the partial derivatives.*

Remark: The proposition is also false if we remove the condition

$$f_{e(\mathbf{a})}(k+1)=0$$

Example 8.1.11. *Let* $\mathbf{a}=[1,2]+[2,3]$ *and* $\mathbf{b}=[3]$, *then*

$$L_{\mathbf{a}}\times L_{\mathbf{b}}=L_{\mathbf{a}+\mathbf{b}}$$

which contradicts our formula.

Proposition 8.1.12. *Let* $\mathbf{a}_{i_0}=\Phi^{i_0}(w)$ *and* $\mathbf{b}=[k-i_0,k+1]$. *Then*

$$L_{\mathbf{a}_{i_0}}\times L_{\mathbf{b}}=\sum_{\mathbf{e}} m(\mathbf{e},{}^{(k-i_0+1)}\mathbf{b},\mathbf{a})L_{[k-i_0+1]_1\mathbf{e}}+$$
$$\sum_{\mathbf{c}} m(\mathbf{c}^{[k-i_0+1]_{\ell_{k-i_0+1}}[k-i_0]_{\ell_{k-i_0}-1}},\mathbf{b},\mathbf{a}_{i_0})L_{\mathbf{c}^{[k-i_0+1]_{\ell_{k-i_0+1}}[k-i_0]_{\ell_{k-i_0}-1}}},$$

with

$$m(\mathbf{c}^{[k-i_0+1]_{\ell_{k-i_0+1}}[k-i_0]_{\ell_{k-i_0}-1}},\mathbf{b},\mathbf{a}_{i_0})=\sum_{\mathbf{d}\in\Gamma^{\ell_{k-i_0}-1}(\mathbf{a},k-i_0)_{k-i_0+1}}\theta_{k-i_0}(\mathbf{d},\mathbf{a})m(\mathbf{c},\mathbf{b},\mathbf{d}^{(k-i_0+1)})-$$
$$\sum_{\mathbf{e}}\theta_{k-i_0}(\mathbf{c}^{[k-i_0+1]_{\ell_{k-i_0+1}}},{}^{[k-i_0+1]_1}\mathbf{e})m(\mathbf{e},{}^{(k-i_0+1)}\mathbf{b},\mathbf{a}_{i_0})$$

where $\mathbf{c}$ *runs through all the terms such that* $m(\mathbf{c},\mathbf{b},\mathbf{d}^{(k-i_0+1)})\neq 0$ *for some* $\mathbf{d}$ *and* $f_{b(\mathbf{c})}(k-i_0+1)=0$, $\mathbf{e}$ *runs through all the terms such that* $m(\mathbf{e},{}^{(k-i_0+1)}\mathbf{b},\mathbf{a}_{i_0})\neq 0$.

Démonstration. Consider the formula

$$L_{\mathbf{a}_{i_0}} \times L_{\mathbf{b}} = \sum_{\mathbf{c}} m(\mathbf{c}, \mathbf{b}, \mathbf{a}_{i_0}) L_{\mathbf{c}}. \tag{8.1.13}$$

In case $k - i_0 + 1 \in b(\mathbf{c})$, we know that $\mathbf{c} \in {}_{k-i_0+1}S(\mathbf{a}+\mathbf{b})$, and moreover

$$m(\mathbf{c}, \mathbf{b}, \mathbf{a}) = m({}^{(k-i_0+1)}\mathbf{c}, {}^{(k-i_0+1)}\mathbf{b}, \mathbf{a}),$$

this gives the first part of the formula in our proposition. Now if $k - i_0 + 1 \notin b(\mathbf{c})$, then we have

$$f_{e(\mathbf{c})}(i) = f_{e(\mathbf{a})}(i), \text{ for all } k - i_0 + 1 \leq i \leq k, \quad f_{e(\mathbf{c})}(k - i_0) = f_{e(\mathbf{a})}(k - i_0) - 1.$$

In this case, we apply the derivative $\mathscr{D}^{k-i_0+1}\mathscr{D}^{k-i_0}$ to the equation (8.1.13) and consider terms of degree equal to $\deg(\mathbf{c}^{(k-i_0,k-i_0+1)})$. On the left hand side we find

$$\sum_{\mathbf{c}} \sum_{\mathbf{d} \in \Gamma^{\ell_{k-i_0}-1}(\mathbf{a},k-i_0)_{k-i_0+1}} \theta_{k-i_0}(\mathbf{d}, \mathbf{a}) m(\mathbf{c}, \mathbf{b}, \mathbf{d}^{(k-i_0+1)}) L_{\mathbf{c}}.$$

While for fix $\mathbf{c}$, on the right hand side we find

$$\Big(\sum_{\mathbf{e}} \theta_{k-i_0}(\mathbf{c}^{[k-i_0+1]_{\ell_{k-i_0+1}}}, {}^{[k-i_0+1]_1}\mathbf{e}) m(\mathbf{e}, {}^{(k-i_0+1)}\mathbf{b}, \mathbf{a}) \\ + m(\mathbf{c}^{[k-i_0+1]_{\ell_{k-i_0+1}}[k-i_0]_{\ell_{k-i_0}-1}}, \mathbf{b}, \mathbf{a}_{i_0})\Big) L_{\mathbf{c}}$$

here $\mathbf{e}$ runs through all the terms such that $m(\mathbf{e}, {}^{(k-i_0+1)}\mathbf{b}, \mathbf{a}) \neq 0$. The first part in the coefficient comes from the part

$$\sum_{\mathbf{e}} m(\mathbf{e}, {}^{(k-i_0+1)}\mathbf{b}, \mathbf{a}) L_{{}^{[k-i_0+1]_1}\mathbf{e}}$$

in the induction $L_{\mathbf{a}_{i_0}} \times L_{\mathbf{b}}$ so that by taking the difference, we get our results. □

Remark: In general the multisegment $\mathbf{d}^{(k-i_0+1)}$ in the the formula does not satisfies the condition

$$f_{e(\mathbf{d}^{(k-i_0+1)})}(i) = 0, \text{ for all } k - i_0 \leq i \leq k.$$

In order to proceed our calculation, we have to apply proposition 8.1.5.
Remark: Combining all the propositions above, we finish the computation of $m(\mathbf{c}, \mathbf{b}, \mathbf{a})$ in case where

$$\mathbf{b} = [k - i_0 + 1, k + 1], \quad \max b(\mathbf{a}) \leq k - i_0 + 1.$$

8.2 General case

Now we consider the case (2) in the introduction of this chapter.

Proposition 8.2.1. *Let $k \in \mathbb{Z}$ and $\mathbf{a}$ be a multisegment. Then there exists a multisegment $\mathbf{a}'$ and a sequence of integers $k_1, \cdots, k_r$ such that*

$$\mathbf{a} = {}^{(k_1,\cdots,k_r)}\mathbf{a}', \quad \mathbf{a}' \in {}_{k_1,\cdots,k_r}S(\mathbf{a}'),$$

and for any $1 \leq i \leq r$,

$$\deg({}^{(k_i,\cdots,k_r)}\mathbf{a}) = \deg({}^{(k_{i+1},\cdots,k_r)}\mathbf{a}) - 1, \quad \max b(\mathbf{a}') \leq k.$$

Démonstration. This is proved by applying successively the truncation functor, which is the same as the proof of proposition 6.3.3. □

Proposition 8.2.2. *Let $\mathbf{a}$ be a multisegment such that*

$$\mathbf{a} \in {}_{k-i_0+1}S(\mathbf{a}), \quad f_{e(\mathbf{a})}(k - i_0 + 1) = 1.$$

If we assume that $\mathbf{b} = [k - i_0 + 1, k + 1](i_0 \geq 0)$ and

$$L_{\mathbf{a}} \times L_{\mathbf{b}} = \sum_{\mathbf{c}} m(\mathbf{c}, \mathbf{b}, \mathbf{a}) L_{\mathbf{c}},$$

then

$$m(\mathbf{d}, \mathbf{b}, {}^{(k-i_0+1)}\mathbf{a}) = \sum_{\mathbf{c}} m(\mathbf{c}, \mathbf{b}, \mathbf{a})({}_{k-i_0+1}\theta(\mathbf{d}, \mathbf{c})) - m(\mathbf{d}, {}^{(k-i_0+1)}\mathbf{b}, \mathbf{a}).$$

Démonstration. Note that by assumption we have

$${}^{k-i_0+1}\mathscr{D} L_{\mathbf{a}} = L_{\mathbf{a}} + L_{{}^{(k-i_0+1)}\mathbf{a}}.$$

If we apply ${}^{k-i_0+1}\mathscr{D}$ to

$$L_{\mathbf{a}} \times L_{\mathbf{b}} = \sum_{\mathbf{c}} m(\mathbf{c}, \mathbf{b}, \mathbf{a}) L_{\mathbf{c}},$$

on the left hand side get

$$L_{\mathbf{a}} \times L_{\mathbf{b}} + L_{{}^{(k-i_0+1)}\mathbf{a}} \times L_{\mathbf{b}},$$

while on the right hand side we get

$$\sum_{\mathbf{d}} \sum_{\mathbf{c}} m(\mathbf{c}, \mathbf{b}, \mathbf{a})({}_{k-i_0+1}\theta(\mathbf{d}, \mathbf{c})) L_{\mathbf{d}}$$

by comparing the two hand side, we get our result. □

Proposition 8.2.3. *Let* $k_1 \neq k - i_0 + 1$ *and* $\mathbf{a}$ *be a multisegment such that*

$$\mathbf{a} \in {}_kS(\mathbf{a}), \quad f_{e(\mathbf{a})}(k) = 1.$$

If we assume that $\mathbf{b} = [k - i_0 + 1, k + 1] (i_0 \geq 0)$ *and*

$$L_{\mathbf{a}} \times L_{\mathbf{b}} = \sum_{\mathbf{c}} m(\mathbf{c}, \mathbf{b}, \mathbf{a}) L_{\mathbf{c}},$$

then

$$m(\mathbf{d}, \mathbf{b}, {}^{(k_1)}\mathbf{a}) = m({}^{[k_1]_1}\mathbf{d}, \mathbf{b}, \mathbf{a}).$$

Démonstration. The proof is the same as that of proposition 8.1.4. □

Remark: Combining the three proposition we get the computation of $m(\mathbf{c}, \mathbf{b}, \mathbf{a})$ for any $\mathbf{a}$ and $\mathbf{b}$ a segment.

Chapitre 9

On Several Conjectures

9.1 Conjecture of Mínguez et Lapid

In this section we recall a conjecture in [15] (Conjecture 1)and give a proof for it.
To summarize what we have proved in chapter 5,

Proposition 9.1.1. *We fix a symmetric multisegment* $\mathbf{a}_{\mathrm{Id}}$ *and a bijection*

$$\Phi : S_n \to S(\mathbf{a}_{\mathrm{Id}})$$

if we denote by $\mathbf{a}_\tau = \Phi(\tau)$*, then*

$$\pi(\mathbf{a}_\tau) = \sum_{\sigma \geq \tau} P_{\tau,\sigma}(1) L_{\mathbf{a}_\sigma},$$

and reciprocally,

$$L_{\mathbf{a}_\tau} = \sum_{\sigma \geq \tau} (-1)^{\ell(\sigma)-\ell(\tau)} P_{w_0\sigma, w_0\tau}(1) \pi(\mathbf{a}_\sigma),$$

where w_0 *is the maximal element in the symmetric group* S_n*.*

Démonstration. The first formula is by corollary 4.4.7, and the second is known to be equivalent to the first (cf. [19]).

□

Remark: By letting $\tau = Id$, we get the determinantal formula by Mínguez and Lapid

$$L_{\mathbf{a}_{\mathrm{Id}}} = \sum_{\sigma} (-1)^{\ell(\sigma)} \pi(\mathbf{a}_\sigma)$$

More generally, for an multisegment $\mathbf{a}$, following proposition 6.3.2, we can associate an element τ in $S_n^{J_1(\mathbf{a}),J_2(\mathbf{a})}$, such that

$$\mathbf{a} = \sum_{i=1}^{n} [k_i, \ell_{\tau(i)}]$$

with $k_1 \leq k_2 \leq \cdots k_n$, $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_n$. Again, by proposition [**?**], we know that every multisegment in $S(\mathbf{a})$ is of the following form

$$\mathbf{a}_\sigma = \sum_{i=1}^{n} [k_i, \ell_{\sigma(i)}]$$

with $\sigma \geq \tau$, here we let $[i,j] = \emptyset$ if $i > j$ and $\mathbf{a}_\sigma = \emptyset$ if $\ell_{\sigma(i)} < k_i$ for all i. Now we get the following proposition

Proposition 9.1.2. *We fix the notation as above, then*

$$\pi(\mathbf{a}_\tau) = \sum_{\sigma \geq \tau} P_{\tau,\sigma}^{J_1(\mathbf{a}),J_2(\mathbf{a})}(1) L_{\mathbf{a}_\sigma}.$$

Equivalently, we have

$$L_{a_\tau} = \sum_{\sigma \geq \tau} P_{\tau,\sigma}^{J_1(\mathbf{a}),J_2(\mathbf{a}),-}(1)\pi(\mathbf{a}_\sigma),$$

here we let $L_{\mathbf{a}} = \pi(\mathbf{a}) = 0$ *if* $\mathbf{a} = \emptyset$ *and*

$$P_{\tau,\sigma}^{J_1(\mathbf{a}),J_2(\mathbf{a}),-}(q) = \sum_{\gamma \in S_{J_1(\mathbf{a})}\sigma S_{J_2(\mathbf{a})}} (-1)^{\ell(\gamma)-\ell(\tau)} P_{w_0\gamma, w_0\tau}(q).$$

Démonstration. The first formula is deduced from proposition 6.2.25. The second can be obtained by applying the partial derivation to the formula for symmetric case(which reverses our procedure to produce a symmetric multisegment from a given one). □

9.2 Two conjectures of Badulescu

In this section, let D be an central division algebra of dimension d^2 over our local field F. We consider $G' =: GL_n(D)$, which is an inner form of $G = GL_{nd}(F)$.

Notation 9.2.1. *In this section, we denote by* $\mathcal{R}^{univ}(G)$ *and* $\mathcal{R}^{univ}(G')$ *the Grothendieck group of admissible representations of* G_n *and* G'_n*, respectively. Let*

$$\mathcal{R}^{univ}(F) = \sum_n \mathcal{R}^{univ}(GL_n(F)), \quad \mathcal{R}^{univ}(D) = \sum_n \mathcal{R}^{univ}(GL_n(D)).$$

Also, we denote by $\Pi(G)$ *and* $\Pi(G')$ *the set of essentially square integrable representations of* G *and* G'*, respectively.*

Remark: We can define a multiplication and a co-multiplication on $\mathcal{R}^{univ}(D)$ exactly like the case of $\mathcal{R}^{univ}(F)$ and verify that we obtain a Hopf algebra(cf. [33]).

Definition 9.2.2. *We say that two elements semi simple regular* $g \in G$ *and* $g' \in G'$ *correspond to each other if their characteristic polynomials are the same.*

Definition 9.2.3. *Let* $\pi \in \mathcal{R}^{univ}(G)$ *and* $\pi' \in \mathcal{R}^{univ}(G')$*. We say that* π' *is the Jacquet-Langlands transfer of* π *if we have*

$$\chi_\pi(g) = \chi_{\pi'}(g')$$

for all correspondent semi simple regular elements $g \in G, g' \in G'$*, where* χ_π *and* $\chi_{\pi'}$ *are the character function of* π *and* π'*, respectively.*

Theorem 9.2.4. *(cf. [2]) There exists a unique morphism of Hopf algebras*

$$LJ : \mathcal{R}(F) \to \mathcal{R}(D)$$

such that $LJ(\pi)$ *is the transfert of* π *for all* $\pi \in \mathcal{R}(F)$*. Moreover, this morphism is surjective. Moreover, if* π *is essentially square integrable of* G*, then* $LJ(\pi) = (-1)^{n(d-1)}\pi'$ *for some essentially square integrable representation* π' *of* G'*. We shall denote by* $C(\pi)$ *the representation* π'*.*

Remark: According to [2] section 3.5, we can define a Zelevinsky type involution t on $\mathcal{R}(D)$ such that we have

$$LJ(\pi^t) = (-1)^{nd-n} LJ(\pi)^t$$

for all representation $\pi \in \mathcal{R}^{univ}(G)$
Now following [3] section 4.4, we calculate the image of all irreducible representations $L_{\mathbf{a}} \in \mathcal{R}^{univ}(G)$.

Definition 9.2.5. *Let* ρ *be a cuspidal representation of* $GL_p(F)$*. We denote by* $s(\rho)$ *the minimal natural number such that* d *divides* $s(\rho)p$*.*

Notation 9.2.6. *Let Δ is a multisegment of length $s(\rho)$ in $\mathbb{Z}_\rho$. Then by theorem above, $\rho' = C(L_{\Delta^t})$ is a cuspidal representation of $GL_{\frac{s(\rho)p}{d}}(D)$. Let $\nu_{\rho'} = \nu^{s(\rho)}$ and $\mathbb{Z}_{\rho'}$ a Tadíc line generate by ρ' and $\nu_{\rho'}$.*

Proposition 9.2.7. *(cf. [3]) Let $\Delta = [a, b]_\rho$ and $\Delta' = [a', b']_{\rho'}$ such that*

$$a' = \frac{a + \frac{s(\rho)-1}{2}}{s(\rho)}, \quad b' = \frac{b + \frac{s(\rho)-1}{2}}{s(\rho)}.$$

Then $C(L_{\Delta^t}) = L_{\Delta'^t}$.

Remark: According to theorem 9.2.4 and the remark following on Zelevinsky involution, if $\Delta = [a, b]_\rho$ is a segment such that $s(\rho)$ divides $b - a + 1$, then

$$LJ(L_\Delta) = L_{[a',b']_{\rho'}},$$

with the same notations as in the previous proposition.

Notation 9.2.8. *We denote by*

$$C([a, b]_\rho)$$

the segment $[a', b']_{\rho'}$.

Remark: The map C induces a bijection between

$$\{\Delta \subseteq \mathbb{Z}_\rho : s_\rho | \deg(\Delta)\} \quad \text{and} \quad \{\Delta \subseteq \mathbb{Z}_{\rho'}\}$$

Proposition 9.2.9. *(cf. [2] 2.3) The essentially square integrable representations of G' are of the form L_{Δ^t}, where Δ is a segment on the Tadíc line generated by the cuspidal representation ρ and $t : \mathcal{R}(D) \to \mathcal{R}(D)$ is the Zelevinsky involution.*

Proposition 9.2.10. *(cf. [2] Prop. 3.5) The ring $\mathcal{R}^{univ}(D)$ is isomorphic to the commutative polynomial ring with infinitely many variables indexed by $\bigcup_{r \in \mathbb{N}} \Pi(GL_r(D))$.*

Definition 9.2.11. *Now let $\mathbf{a} = \{\Delta_1, \cdots, \Delta_r\}$ such that $\Delta_i = [a_i, b_i]_\rho$ for $i = 1, \cdots, r$, where $a_1 \leq a_1 \leq \cdots \leq a_r, b_1 \leq b_2 \leq \cdots \leq b_r$. We define $A_j(\mathbf{a}) = \{a_1^j, \cdots, a_{k_j}^j\}$ to be the multi-set in increasing order of a_i which is congrue to $j + 1$ modulo $s(\rho)$ and $B_j(\mathbf{a}) = \{b_1^j, b_2^j, \cdots, b_{\ell_j}^j\}$ be the multi-set in increasing order of b_i which are congruent to j modulo $s(\rho)$.*

Notation 9.2.12. *For each j, let $S(A_j(\mathbf{a}))$ be the set of permutations of $A_j(\mathbf{a})$.*

Lemma 9.2.13. *(cf. [3])*

(1) There exists a permutation σ of $\{1,\cdots,r\}$ such that for all i, s divide $b_{\sigma(i)}-a_i+1$ if and only if we have $\sharp A_j(\mathbf{a})=\sharp B_j(\mathbf{a})$ for all j.

(2) If $\sharp A_j(\mathbf{a})=\sharp B_j(\mathbf{a})$ for all j, let σ_0 be the permutation of minimal length which induces a bijection between $\{a_1,\cdots,a_r\}$ and $\{b_1,\cdots,b_r\}$ and sends $A_j(\mathbf{a})$ to $B_j(\mathbf{a})$ increasingly. Then the permutation $w\in S_r$ such that $w(A_j(\mathbf{a}))\subseteq B_j(\mathbf{a})$ is in bijection with the set $S(A_j(\mathbf{a})))$ via left multiplication by σ_0^{-1}.

(3) Let $\mathbf{a}_\tau=\sum_i[a_i,b_{\tau(i)}]$ for $\tau\in S_r$. Then the set $\{\mathbf{a}_\sigma:\sigma_0^{-1}\sigma\in(\times_jS(A_j(\mathbf{a})))\}$ is in bijection with the set $S_{J_1(\mathbf{a})^{\sigma_0}}\backslash(\times_jS(A_j(\mathbf{a})))/S_{J_2(\mathbf{a})}$, where

$$J_1(\mathbf{a})^{\sigma_0}=\{\sigma_0^{-1}\sigma\sigma_0:\sigma\in J_1(\mathbf{a})\}.$$

Lemma 9.2.14. *We have*

$$J_1(\mathbf{a})^{\sigma_0}=\coprod_j(J_1(\mathbf{a})^{\sigma_0}\cap S(A_j)).$$

Démonstration. This follows from the definition of $B_j(\mathbf{a})$ and $J_1(\mathbf{a})$. □

Definition 9.2.15. *We define*

$$S(A_j)^{J_1(\mathbf{a})^{\sigma_0}\cap S(\mathbf{A}_j),J_2(\mathbf{a})}=\{\sigma\in S(A_j):s_1\sigma s_2>\sigma,\forall s_1\in J_1(\mathbf{a})^{\sigma_0}\cap S(A_j),s_2\in J_2(\mathbf{a})\}$$

and

$$(\times_jS(A_j))^{J_1(\mathbf{a})^{\sigma_0},J_2(\mathbf{a})}=\{\sigma\in(\times_jS(A_j)):s_1\sigma s_2>\sigma,\ \textit{for all } s_1\in J_1(\mathbf{a})^{\sigma_0},s_2\in J_2(\mathbf{a})\}.$$

Lemma 9.2.16. *We have*

$$\sigma_0((\times_jS(A_j))^{J_1(\mathbf{a})^{\sigma_0},J_2(\mathbf{a})})=\sigma_0(\times_jS(A_j(\mathbf{a})))\cap S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}.$$

Démonstration. Note that each element $\tau\in S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}$, we have a unique element $\mathbf{a}_\tau=\sum[a_i,b_{\tau(i)}]$ associated. We know that such an element satisfies the condition $\tau(A_j(\mathbf{a}))\subseteq B_j(\mathbf{a})$ if and only if $s(\rho)$ divides $\deg(\Delta)$ for all $\Delta\in\mathbf{a}_\tau$. Then, we have $\sigma_0^{-1}\tau\in\times_jS(A_j)$. In this case, we can write

$$\mathbf{a}_\tau=\sum_j\sum_i[\mathbf{a}_i^j,\mathbf{b}^j_{\tau^{(j)}(i)}]$$

where $\tau^{(j)}$ is the induced element in $S(A_j)$ by $\sigma_0^{-1}\tau$. Note that

$$b^j_{\tau^{(j)}(i)} = \tau(a^j_i),$$

wherefore if $a^j_i = a^j_{i+1}$, then by the fact $\tau \in S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}$, we have

$$\tau(a^j_i) \leq \tau(a^j_{i+1})$$

hence $\mathbf{b}^j_{\tau^{(j)}(i)} \leq \mathbf{b}^j_{\tau^{(j)}(i+1)}$, i.e, $\tau^{(j)}(i) < \tau^{(j)}(i+1)$. This shows that $\sigma_0^{-1}\tau \in (\times_j S(A_j))^{J_1(\mathbf{a})^{\sigma_0},J_2(\mathbf{a})}$. The converse can be showed similarly.

□

Definition 9.2.17. *Let*

$$\mathcal{T}(\mathbf{a}) = \{\mathbf{a}_\sigma : \sigma \in S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}, \sigma_0^{-1}\sigma \in (\times_j S(A_j(\mathbf{a})))\}.$$

Example 9.2.18. *Let* $\mathbf{a} = [1,3] + [2,5] + [2,6]$ *and* $d = 2$ *then* $s(1) = 2$. *We have*

$$A_1(\mathbf{a}) = \{1\}, \quad A_2(\mathbf{a}) = \{2,2\}, \quad B_1(\mathbf{a}) = \{6\}, \quad B_2(\mathbf{a}) = \{3,5\}.$$

Then by the previous lemma we have

$$\sigma_0 = (132)$$

and

$$S(A_2(\mathbf{a})) =< s_2 >, \quad S_{J_2(\mathbf{a})} =< s_2 >,$$

where $< \sigma >$ *denote the subgroup generate by* σ. *Hence*

$$\mathcal{T}(\mathbf{a}) = \{\mathbf{a}_{\sigma_0}\}.$$

Remark: In [3], the authors only treat the case where

$$a_1 < a_2 < \cdots < a_r, \quad b_1 < b_2 < \cdots < b_r,$$

while we allow the a_i's and b_i's to be equal.

Definition 9.2.19. *For* $\tau \in \sigma_0(\times_j S(A_j(\mathbf{a}))) \cap S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}$, *we let* $\tau^{(j)}$ *be the induced element of* $\sigma_0^{-1}\tau$ *in* $S(A_j(\mathbf{a}))$. *We define*

$$C(\mathbf{a}_\tau, j) = \sum_i C([a^j_i, b^j_{\tau^{(j)}(i)}]_\rho), \quad C(\mathbf{a}_\tau) = \sum_j C(\mathbf{a}_\tau, j)$$

here we view $\sigma_0^{-1}\tau$ *as an element in* $(\times_j S(A_j(\mathbf{a})))$.

Notation 9.2.20. *For $\gamma \in S(A_j(\mathbf{a}))$, we denote by*

$$C(\mathbf{a}, j)_\gamma$$

the element $\sum_i C([a_i^j, b_{\gamma(i)}^j]_\rho)$. By definition, we have

$$C(\mathbf{a}_\tau, j) = C(\mathbf{a}, j)_{\tau(j)}.$$

Lemma 9.2.21. *We have*

$$\pi(C(\mathbf{a}, j)_\gamma) = \sum_{\tau \geq \gamma, \tau \in S(A_j)^{J_1(\mathbf{a})^{\sigma_0} \cap S(\mathbf{A}_j), J_2(\mathbf{a})}} P_{\gamma,\tau}(1) L_{C(\mathbf{a},j)_\tau}$$

Démonstration. Let $(\mathbf{a}, j) = \sum_i [a_i^j, b_i^j]$ and

$$(\mathbf{a}, j)_\gamma = \sum_i [a_i^j, b_{\gamma(i)}^j].$$

Note that by definition, we know that

$$C(\mathbf{a}, j)_\gamma = C((\mathbf{a}, j)_\gamma).$$

Hence C induces a bijection between $S((\mathbf{a}, j))$ and $S(C(\mathbf{a}, j))$. Now induction on $\ell(\gamma)$ gives the result.

□

Definition 9.2.22. *We say that the multisegment is simple with respect to D if each of the set $A_j(\mathbf{a})$ is a consecutive subset of $\{a_1, \cdots, a_r\}$ for all j.*

Theorem 9.2.23. *Assume that $\mathbf{a}$ is simple with respect to D. We have*

(1) *If there exists $j \in \{1, \cdots, s\}$ such that $\sharp A_j(\mathbf{a}) \neq \sharp B_j(\mathbf{a})$, then $LJ(L_\mathbf{a}) = 0$.*

(2) *Suppose for all $j \in \{1, \cdots, s\}$ we have $A_j(\mathbf{a}) = B_j(\mathbf{a})$, then*

  (i) *For $\sigma \in S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}$ such that $\mathbf{a}_\sigma \in \mathcal{T}(\mathbf{a})$, we have $LJ(L_{\mathbf{a}_\sigma}) = L_{C(\mathbf{a}_\sigma)}$;*

  (ii) *For $\sigma \in S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}$ such that $\mathbf{a}_\sigma \notin \mathcal{T}(\mathbf{a})$ and $S(\mathbf{a}_\sigma) \cap \mathcal{T}(\mathbf{a}) \neq \emptyset$, then*

$$LJ(L_{\mathbf{a}_\sigma}) = \sum_{\gamma \geq \sigma, \gamma \in \sigma_0(\times_j S(A_j(\mathbf{a}))) \cap S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}} m(L_{C(\mathbf{a}_\gamma)}, LJ(L_{\mathbf{a}_\sigma})) L_{C(\mathbf{a}_\gamma)},$$

  with

$$m(L_{C(\mathbf{a}_\gamma)}, LJ(L_{\mathbf{a}_\sigma})) = \sum_{\tau \leq \gamma, \mathbf{a}_\tau \in S(\mathbf{a}_\sigma) \cap \mathcal{T}(\mathbf{a})} P_{\sigma,\tau}^{J_1(\mathbf{a}),J_2(\mathbf{a}),-}(1) P_{\tau,\gamma}^{J_1(\mathbf{a}),J_2(\mathbf{a})}(1)$$

  see Proposition 9.1.2 for notation.

(iii) For $\sigma \in S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}$ such that $S(\mathbf{a}_\sigma)\cap\mathcal{T}(\mathbf{a})=\emptyset$, then $LJ(\mathbf{a}_\sigma)=0$.

Démonstration. (1)Note that we have

$$L_{\mathbf{a}_\sigma}=\sum_{\tau\in S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})},\tau\geq\sigma} m'(\mathbf{a}_\tau,\mathbf{a}_\sigma)\pi(\mathbf{a}_\tau)$$

the fact that there exists a j such that $B_j(\mathbf{a})\neq A_j(\mathbf{a})$ implies there is a segment $[a_i,b_{\tau(i)}]$ of whom $s(\rho)$ does not divide the length, hence $LJ(\pi(\mathbf{a}_\tau))=0$. Hence we must have $LJ(L_{\mathbf{a}_\tau})=0$.

(2)We consider the case (2).

(a) First of all we consider the case where $S(\mathbf{a}_\sigma)\cap\mathcal{T}(\mathbf{a})=\emptyset$, which means that for any $\mathbf{a}_\tau\in S(\mathbf{a}_\sigma)$, there exists a segment $[a_i,b_{\tau(i)}]$ of whom $s(\rho)$ does not divide the length. The same argument as in case (1) shows that $LJ(\mathbf{a}_\sigma)=0$.

(b) Secondly, if $\mathbf{a}_\sigma\in\mathcal{T}(\mathbf{a})$, we prove by induction on $\ell(\sigma)$ that $LJ(L_{\mathbf{a}_\sigma})=L_{C(\mathbf{a}_\sigma)}$. For case where $\ell(\sigma)$ is maximal among elements of $\sigma_0(\times_j S(A_j(\mathbf{a})))\cap S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}$, then we have $S(\mathbf{a}_\sigma)\cap\mathcal{T}(\mathbf{a})=\{\mathbf{a}_\sigma\}$, and consider the formula

$$\pi(\mathbf{a}_\sigma)=L_{\mathbf{a}_\sigma}+\sum_{\mathbf{b}<\mathbf{a}} m(\mathbf{b},\mathbf{a})L_{\mathbf{b}}$$

by applying LJ and case (a), we obtain

$$LJ(L_{\mathbf{a}_\sigma})=LJ(\pi(\mathbf{a}_\sigma))=\pi(C(\mathbf{a}_\sigma)).$$

Note that the fact $\mathbf{a}_\sigma$ is maximal in $\sigma_0(\times_j S(A_j(\mathbf{a})))\cap S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}$ implies that $S(C(\mathbf{a}_\sigma))$ contains only one element. Hence

$$LJ(L_{\mathbf{a}_\sigma})=L_{C(\mathbf{a}_\sigma)}.$$

Assume that t for all $\tau\in\sigma_0(\times_j S(A_j(\mathbf{a})))\cap S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}$ and $\tau>\sigma$, we have $LJ(L_\tau)=L_{C(\mathbf{a}_\tau)}$. Consider the formula

$$L_{\mathbf{a}_\sigma}=\pi(\mathbf{a}_\sigma)-\sum_{\tau\in S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})},\tau>\sigma} m(\mathbf{a}_\tau,\mathbf{a}_\sigma)L_{\mathbf{a}_\tau},$$

applying the morphism LJ, we get

$$LJ(L_{\mathbf{a}_\sigma})=\pi(C(\mathbf{a}_\sigma))-\sum_{\tau\in\sigma_0(\times_j S(A_j(\mathbf{a})))\cap S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})},\tau>\sigma} m(\mathbf{a}_\tau,\mathbf{a}_\sigma)L_{C(\mathbf{a}_\tau)}.$$

But we have

$$\pi(C(\mathbf{a}_\sigma)) = \prod_j \pi(C(\mathbf{a}_\sigma, j))$$
$$= \prod_j (\sum_{\gamma\geq\sigma^{(j)}, \gamma\in S(A_j)^{J_1(\mathbf{a})^{\sigma_0}\cap S(\mathbf{A}_j), J_2(\mathbf{a})}} P_{\sigma^{(j)},\gamma}^{J_1(\mathbf{a})^{\sigma_0}\cap S(\mathbf{A}_j), J_2(\mathbf{a})\cap S(A_j)}(1) L_{C(\mathbf{a},j)_\gamma})$$
$$= \sum_{\tau\in\sigma_0(\times_j S(A_j(\mathbf{a})))\cap S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}, \tau>\sigma} \prod_j P_{\sigma^{(j)},\tau^{(j)}}^{J_1(\mathbf{a})^{\sigma_0}\cap S(\mathbf{A}_j), J_2(\mathbf{a})\cap S(A_j)}(1) L_{C(\mathbf{a}_\tau)},$$

the first equality follows from the fact that the multisegment $C(\mathbf{a}_\sigma)_j$ lies in different Tadić line for different j, the second follows from lemma 9.2.21.
Therefore it suffices to show the following formula

$$\prod_j P_{\sigma^{(j)},\tau^{(j)}}(1) = P_{\sigma,\tau}(1),$$

which is proved in lemma 9.2.24. Hence we are done.
(c) Finally, we prove the case (ii). We start with the following formula

$$L_{\mathbf{a}_\sigma} = \pi(\sigma_\sigma) + \sum_{\tau>\sigma, \tau\in S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}} P_{\sigma,\tau}^{J_1(\mathbf{a}),J_2(\mathbf{a}),-}(1)\pi(\mathbf{a}_\tau).$$

Now apply the morphism LJ, we get

$$LJ(L_{\mathbf{a}_\sigma}) = \sum_{\mathbf{a}_\tau\in S(\mathbf{a}_\sigma)\cap\mathcal{T}(\mathbf{a})} P_{\sigma,\tau}^{J_1(\mathbf{a}),J_2(\mathbf{a}),-}(1) LJ(\pi(\mathbf{a}_\tau)).$$

And

$$LJ(\pi(\mathbf{a}_\tau)) = \pi(C(\mathbf{a}_\tau)) = \sum_{\gamma\geq\tau, \gamma\in\sigma_0(\times_j S(A_j(\mathbf{a})))\cap S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}} P_{\tau,\gamma}^{J_1(\mathbf{a}),J_2(\mathbf{a})}(1) L_{C(\mathbf{a}_\gamma)}$$

note that the second equality is proved in case (b). Therefore we obtain the following

$$LJ(L_{\mathbf{a}_\sigma}) = \sum_{\mathbf{a}_\tau\in S(\mathbf{a}_\sigma)\cap\mathcal{T}(\mathbf{a})} \sum_{\gamma\geq\tau, \gamma\in\sigma_0(\times_j S(A_j(\mathbf{a})))\cap S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}} P_{\sigma,\tau}^{J_1(\mathbf{a}),J_2(\mathbf{a}),-}(1) P_{\tau,\gamma}^{J_1(\mathbf{a}),J_2(\mathbf{a})}(1) L_{C(\mathbf{a}_\gamma)}.$$

Now for fixed $\gamma\in\sigma_0(\times_j S(A_j(\mathbf{a})))\cap S_r^{J_1(\mathbf{a}),J_2(\mathbf{a})}$, we get the multiplicity of $L_{(\mathbf{a}_\gamma)}$ in $LJ(L_{\mathbf{a}_\sigma})$ given by

$$m(L_{C(\mathbf{a}_\gamma)}, LJ(L_{\mathbf{a}_\tau})) = \sum_{\mathbf{a}_\tau\in S(\mathbf{a}_\sigma)\cap\mathcal{T}(\mathbf{a}), \tau\leq\gamma} P_{\sigma,\tau}^{J_1(\mathbf{a}),J_2(\mathbf{a}),-}(1) P_{\tau,\gamma}^{J_1(\mathbf{a}),J_2(\mathbf{a})}(1).$$

□

Lemma 9.2.24. *Assume that* $\mathbf{a}$ *is simple with respect to* D*. We have*

$$\prod_j P_{\sigma^{(j)},\tau^{(j)}}^{J_1(\mathbf{a})^{\sigma_0}\cap S(\mathbf{A}_j),J_2(\mathbf{a})\cap S(A_j)}(q) = P_{\sigma,\tau}(q).$$

Démonstration. Let $r_j = \sharp A_j(\mathbf{a})$, then $\sum_j r_j = r$. Let $M = \prod_j GL_{r_j}(\mathbb{C})$, which is considered as a closed subgroup in $GL_r(\mathbb{C})$ via the diagonal imbedding. To prove the lemma, we consider the following imbedding

$$\iota : M/M\cap P_{J_1(\mathbf{a})} \to GL_r(\mathbb{C})/P_{J_1(\mathbf{a})}, \quad \iota(g) = \sigma_0 g$$

here $P_{J_1(\mathbf{a})}$ is the parabolic subgroup associated to $J_1(\mathbf{a})$. We construct a retraction from the variety $X = \coprod_{\sigma\in\times_j S(A_j(\mathbf{a}))} P_{J_2(\mathbf{a})}\sigma_0\sigma P_{J_1(\mathbf{a})}$ to $M/M\cap P_{J_1(\mathbf{a})}$. In fact, by the fact that $\sigma_0^{-1}S_{J_2(\mathbf{a})}\sigma_0 \subseteq \times_j S(A_j(\mathbf{a}))$, we know that $\sigma_0^{-1}P_{J_2(\mathbf{a})}\sigma_0\cap M$ is a parabolic subgroup in M and the quotient

$$N = \sigma_0^{-1}P_{J_2(\mathbf{a})}\sigma_0/(\sigma_0^{-1}P_{J_2(\mathbf{a})}\sigma_0\cap M)$$

is unipotent. We observe that the unipotent group N admits a section into $\sigma_0^{-1}P_{J_2(\mathbf{a})}\sigma_0$, whose image is a normal subgroup in the latter. Now let $x \in \sigma_0^{-1}P_{J_2(\mathbf{a})}\sigma_0$, it admits a unique decomposition

$$x = x_M x_N, \quad x_M \in M, x_N \in N$$

and we define $p : \sigma_0^{-1}P_{J_2(\mathbf{a})}\sigma_0 \to M$ by letting $p(x) = x_M$. The morphism p is a group homomorphism. Finally, we define

$$\widetilde{p} : X \to M/M\cap P_{J_1(\mathbf{a})}, \quad \widetilde{p}(x\sigma_0\sigma P_{J_1(\mathbf{a})}) = p(\sigma_0^{-1}x\sigma_0)\sigma(P_{J_1(\mathbf{a})}\cap M).$$

Now $\widetilde{p}$ is a fibration over its target and maps $P_{J_2(\mathbf{a})}$-orbits to $P_{J_2(\mathbf{a})}\cap M$-orbits. Note that the fact that $\mathbf{a}$ is simple with respect to D implies that X is locally closed in $GL_n(\mathbb{C})/P_{J_1(\mathbf{a})}$. Now apply the construction of intersection complex gives the desired results.

□

Remark:
The formula in (2) (ii) is still valid for the non simple case if we modify

$$m(L_{C(\mathbf{a}_\gamma)}, LJ(L_{\mathbf{a}_\sigma})) = \sum_{\tau\leq\gamma,\mathbf{a}_\tau\in S(\mathbf{a}_\sigma)\cap\mathcal{T}(\mathbf{a})} P_{\sigma,\tau}^{J_1(\mathbf{a}),J_2(\mathbf{a}),-}(1)\prod_j P_{\tau^{(j)},\gamma^{(j)}}^{J_1(\mathbf{a})^{\sigma_0}\cap S(\mathbf{A}_j),J_2(\mathbf{a})\cap S(A_j)}(q).$$

Example 9.2.25. *With the remark above, we give a counter example to the conjecture 3.10 in [2]. Let*

$$\mathbf{a} = [1,2] + [2,3] + [3,4] + [4,5] + [5,6],$$

and $d = 2$. Then

$$A_0(\mathbf{a}) = \{2,4\}, \quad A_1(\mathbf{a}) = \{1,3,5\}, \quad B_0(\mathbf{a}) = \{2,4,6\}, \quad B_0(\mathbf{a}) = \{3,5\}.$$

Clearly $\mathbf{a}$ is not simple with respect to D. We have

$$\mathcal{T}(\mathbf{a}) = \{\mathbf{a}, \mathbf{a}_{(13)}, \mathbf{a}_{(24)}, \mathbf{a}_{(35)}, \mathbf{a}_{(153)}, \mathbf{a}_{(13)(24)}, \mathbf{a}_{(24)(35)}, \mathbf{a}_{(153)(24)}\}.$$

We have

$$P_{w_0(13)(24),w_0(24)}(1) = P_{(135),(15)}(1) = 1, \quad P_{w_0(35)(24),w_0(24)}(1) = P_{(153),(15)}(1) = 1,$$

and

$$P_{w_0(153)(24),w_0(24)}(1) = P_{(35),(15)}(1) = 2.$$

First of all, we have

$$L_{\mathbf{a}_{(24)}} = \pi(\mathbf{a}_{(24)}) + \text{ other terms } - \pi(\mathbf{a}_{(13)(24)}) - \pi(\mathbf{a}_{(24)(35)}) + 2\pi(\mathbf{a}_{(153)(24)}).$$

Therefore

$$LJ(L_{\mathbf{a}_{(24)}}) = \pi(C(\mathbf{a}_{(24)})) - \pi(C(\mathbf{a}_{(13)(24)})) - \pi(C(\mathbf{a}_{(24)(35)})) + 2\pi(C(\mathbf{a}_{(153)(24)})).$$

Explicit computation shows that

$$LJ(L_{\mathbf{a}_{(24)}}) = L_{C(\mathbf{a}_{(24)})} + L_{C(\mathbf{a}_{(153)(24)})}.$$

Remark: The example above also shows that lemma 9.2.24 is not true in general for $\mathbf{a}$ not simple.

9.3 Imaginary Multisegment

According to [23],

Definition 9.3.1. *A multisegment $\mathbf{a}$ is said to be imaginary if $L_{\mathbf{a}} \times L_{\mathbf{a}}$ is not simple, otherwise it is said to be real.*

In this section we give an example of an imaginary multisegment, following [23].

Example 9.3.2. *Let $d = 4$. And we consider the case*

$$\mathbf{a} = [1, 9] + [2, 11] + [4, 12] + [6, 13],$$

and $\sigma = (23)$ such that

$$\mathbf{a}_\sigma = [1, 9] + [2, 12] + [4, 11] + [6, 13].$$

Also, we have

$$A_0(\mathbf{a}) = \{4\}, \quad A_1(\mathbf{a}) = \{1\}, \quad A_2(\mathbf{a}) = \{2, 6\}$$

and

$$B_0(\mathbf{a}) = \{12\}, \quad B_1(\mathbf{a}) = \{9, 13\}, \quad B_2(\mathbf{a}) = \emptyset, \quad B_3(\mathbf{a}) = \{11\}$$

In this case $\mathbf{a}$ is not simple with respect to D with

$$\mathcal{T}(\mathbf{a}) = \{\mathbf{a}_{(132)}, \mathbf{a}_{(1324)}\}$$

Let $w_0 = (14)(23)$. One checks to see that

$$P_{w_0(132),w_0\sigma}(1) = 1, \quad P_{w_0(1324),w_0\sigma} = 2.$$

Hence we get

$$LJ(L_{\mathbf{a}_\sigma}) = -\pi(C(\mathbf{a}_{(132)})) + 2\pi(C(\mathbf{a}_{(1324)}))$$

and

$$LJ(L_{\mathbf{a}_\sigma}) = -L_{C(\mathbf{a}_{(132)})} + L_{C(\mathbf{a}_{(1324)})}.$$

We keep the notation in example 9.3.2. Now we are ready to show that $L_{\mathbf{a}_\sigma} \times L_{\mathbf{a}_\sigma}$ is not irreducible.
Assume the contrary, i.e, $L_{\mathbf{a}_\sigma} \times L_{\mathbf{a}_\sigma} = L_{\mathbf{a}_\sigma + \mathbf{a}_\sigma}$.
Consider

$$({}^6\mathscr{D})(L_{\mathbf{a}_\sigma} \times L_{\mathbf{a}_\sigma}) = L_{\mathbf{a}_\sigma} \times L_{\mathbf{a}_\sigma} + 2L_{\mathbf{a}_\sigma} \times L_{{}^{(6)}\mathbf{a}_\sigma} + L_{{}^{(6)}\mathbf{a}_\sigma} \times L_{{}^{(6)}\mathbf{a}_\sigma}.$$

And we are mainly interested in the terms of degree $\deg(\mathbf{a}_\sigma + {}^{(6)}\mathbf{a}_\sigma)$.
Let

$$\mathbf{b} = {}^{(6)}\mathbf{a}_{(13)} + \mathbf{a}_{(24)}, \quad \mathbf{b}_2 = 2({}^{(6)}\mathbf{a}_\sigma).$$

Finally, by our results in chapter 7, we know that

$$({}^6\mathscr{D})(L_{2\mathbf{a}_\sigma}) = L_{2\mathbf{a}_\sigma} + 2L_{{}^{(6)}\mathbf{a}_\sigma + \mathbf{a}_\sigma} + L_{\mathbf{b}} + \text{ other terms }.$$

If $L_{2\mathbf{a}_\sigma} = L_\sigma \times L_\sigma$, by considering the terms of degree $\deg(\mathbf{a}_\sigma + {}^{(6)}\mathbf{a}_\sigma)$, we get

$$2L_{\mathbf{a}_\sigma} \times L_{{}^{(6)}\mathbf{a}_\sigma} = 2L_{{}^{(6)}\mathbf{a}_\sigma + \mathbf{a}_\sigma} + L_{\mathbf{b}} + \text{ other terms },$$

which is clearly a contradiction.
We end this section by the following conjecture :
Conjecture : $\mathbf{a}$ is real if and only if $LJ(L_{\mathbf{a}})$ is irreducible for all D.

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Résumé

Ariki et Ginzburg, en se basant sur les travaux de Zelevinsky sur les variétés orbitales, ont démontré que les multiplicités dans une representation induite *totale* sont données par les valeurs en $q = 1$ des polynômes de Kazhdan-Lusztig associés aux groupes symétriques. Dans ma thèse, j'ai introduit la notion de dérivée partielle qui raffine celle de Zelevinksy et s'identifie en $q = 1$, à l'exponentielle formelle de la q-dérivée de Kashiwara sur l'algèbre quantique. A l'aide de cette notion et en explorant la géométrie des variétés orbitales, je construis une procédure de *symétrisation* des multisegments me permettant, en particulier, de prouver une conjecture de Zelevinsky portant sur une propiété d'indépendance de l'induite parabolique *totale*. Je développe par ailleurs une stratégie afin de calculer les multiplicités dans une induite parabolique générale en utilisant le produit de faisceaux pervers de Lusztig.

Title : Parabolic Induction and Geometry of Orbital Vaieties

Abstract

Ariki and Ginzburg, after the previous work of Zelevinsky on orbital varieties, proved that multiplicities in a *total* parabolically induced representations are given by the value at $q = 1$ of Kazhdan-Lusztig Polynomials associated to the symmetric groups. In my thesis I introduce the notion of partial derivative which refines the Zelevinsky derivative and show that it can be identified with the formal exponential of the q-derivative of Kashiwara with q=1. With the help of this notion, I exploit the geometry of the nilpotent orbital varieties to construct a symmetrization process for the multi-segments, which allows me to proove a conjecture of Zelevinsky on the property of the independence of the *total* parabolic induction. On the other hand, I develop a strategy to calculate the multiplicity in a general parabolic induction by using the Lusztig product of perverse sheaves.

Discipline : Mathématiques
Mots-clefs : théorie de représentation d'un groupe p-adique, variété de Schubert, polynôme de Kazhdan-Lusztig.

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