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RELATIONAL GRAPH MODELS AND MORRIS'S OBSERVABILITY

RESOURCE-SENSITIVE SEMANTIC INVESTIGATIONS ON THE UNTYPED λ-CALCULUS

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CONTENTS

INTRODUCTION

Alonzo Church introduced the *untyped λ-calculus* in the early 30's, within an attempt to give mathematics a logical foundation [Chu32]. In fact, he was unintentionally setting a milestone for the study of the logical foundation of a new discipline: computer science. The system turned out to provide a model of computation through λ-definability (Church's thesis), a notion perfectly equivalent to other mathematical definitions of computability, as proved by Stephen Kleene in [Kle36] and Alan Turing in [Tur37]. Moreover, the late 60's saw the rise of an abstract theory of programming having the λ-calculus as central core. Over the last five decades λ-calculi have played a prominent role in the conception, implementation and analysis of functional programming languages, but also in a number of impressive theoretical insights into the concepts of computation, program and proof. And nothing suggests that this will not be the case in the decades to come.

This thesis is a contribution to the purely mathematical study of the untyped λ-calculus, as a term rewriting system having the *β-reduction* (the formal counterpart of the idea of *execution of programs*) as main rule. The λ-calculus is a rich field of research, which uses tools from algebra, computability, rewriting theory, type theory and shares deep connections with proof theory and category theory thanks to the Curry-Howard isomorphism [How80].

Our main focus is on *denotational semantics* [Sco70, SS71], namely the investigation of mathematical models of the λ-calculus giving the same denotation to β-convertible λ-terms. Dana Scott discovered the first denotational model in 1969 [Sco69]. Since then, a large number of such models, lying in many different categories, have been studied. In most of them λ-terms are interpreted as functions between some order-theoretic, algebraic or topological structures. This is not the case for the denotational semantics that we study in this thesis, which is called *relational semantics*. It has its roots in Jean-Yves Girard's *linear logic* [Gir87, Gir88], a major source of inspiration for recent developments in denotational semantics. Relational semantics interprets λ-terms as *relations*, where their inputs are grouped together in *multisets*. As a result of this usage of multisets, relational models are *resource-sensitive*, in that they represent explicitly the consumption of input resources during the execution of programs.

We study a proper subclass of relational models, which we call *relational graph models* (rgm's). On the one hand, rgm's rephrase in the relational setting the graph models *à la* Plotkin-Scott-Engeler [Eng81, Lon83, Plo93]. On the other hand, they can be studied using some *(non-idempotent) intersection type systems*, so they can be seen as a resource-sensitive reformulation of filter models [BDS13, Part III]. Graph models and filter models were introduced in the late 70's and 80's to study Scott continuous semantics respectively in a more set-theoretical way and in a more type-theoretical way. In particular, the choice of handling intersection types is perhaps not necessary here, but we find it convenient, so we fully embrace it. As a matter of fact, this thesis can also be seen as a work on intersection type theory. The idea of using non-idempotent intersection types to study some relational models is not exclusively ours. Most notably, it has been advocated by Luca Paolini, Mauro Piccolo and Simona Ronchi in [PPRDR15].

The study of the untyped λ-calculus is not restricted to the sole β-rule. One is more often interested in λ-*theories*, which are congruences on λ-terms that include β-conversion. All λ-theories form a complete lattice of cardinality $2^{\aleph_0}$, mostly still unexplored. From the point of view of computer science, *observational equivalences* have a certain relevance among λ-theories. Indeed, they provide an answer to a nontrivial question: when two programs are equivalent? The answer is *behavioural*: they are equivalent if they look to behave in the same way in every possible case of execution. Formally, two λ-terms M and N are observationally (or contextually) equivalent with respect to some fixed set $\mathcal{O}$ of *observable terms* when, for every possible context of evaluation $C[-]$, the λ-term $C[M]$ β-reduces to an observable in $\mathcal{O}$ if and only if $C[N]$ β-reduces to an observable in $\mathcal{O}$. The choice of $\mathcal{O}$ is not unique. The most studied instance (not only for the untyped λ-calculus) is the one where the observables are λ-terms in *head normal form*. This λ-theory is denoted by $\mathcal{H}^*$. An alternative choice is to take as $\mathcal{O}$ the set of λ-terms in β-*normal form*. We call this last *Morris's observational equivalence*. Basically John H. Morris introduced it (together with the general notion of observational equivalence) in his PhD thesis [Mor68] in 1968. This λ-theory, which we call $\mathcal{H}^+$, has been a bit neglected in comparison to $\mathcal{H}^*$. This is why we decided to investigate it in this thesis.

Every denotational model *induces* a λ-theory, defined by equating λ-terms that have the same interpretation. If in particular the λ-theory is an observational equivalence, then the model is declared *fully abstract* for that equivalence. The main aim of this work is to find rgm's fully abstract for $\mathcal{H}^+$. We address the problem in two different ways.

In Chapter 4 there are rgm's in which β-normalizability can be characterized. As we handle the interpretation *via* intersection types, this reduces to characterize β-normalizable λ-terms through some specific kind of typings. The characterization that we find involves the occurrence of the *empty intersection* in some typing judgments. The (infinitely many) rgm's in which this characterization holds are called *uniformly bottomless* and they are all fully abstract for $\mathcal{H}^+$. This approach to the full abstraction problem is similar to the one used by Coppo, Dezani and Zacchi to find their filter model [CDZ87], so far the only fully abstract denotational model for $\mathcal{H}^+$ that has appeared in the literature.

In Chapter 5 we take a more radical approach, and as a payoff we get a much more general, basically *exhaustive*, result: we find *necessary and sufficient* conditions on rgm's to be fully abstract for $\mathcal{H}^+$. Precisely, our main theorem states that an rgm is fully abstract for $\mathcal{H}^+$ if and only if it is extensional (i.e. a model of η-conversion) and λ-*König*. Intuitively an rgm is λ-König when every infinite computable tree has an infinite branch *witnessed* by some type of the model. This *witnessing* can be seen as a property of non-well-foundedness on the type. The theorem actually characterizes the full abstraction for $\mathcal{H}^+$ in the *whole* class of relational models. (Since extensional rgm's coincide with all extensional relational models.) The idea of *characterizing* a certain full abstraction property within a fixed semantics (in this case relational semantics), rather than just finding some instances, is quite a novelty. It has been first advocated by one of the co-authors of our result, Flavien Breuvart, in his paper [Bre14], where a similar theorem is shown about $\mathcal{H}^*$.

Some further results on $\mathcal{H}^+$ are proved in this thesis.

- We prove that $\mathcal{H}^+$ satisfies the ω-rule, a strong form of extensionality. This solves a long-standing open question [Bar84, §17.4].
- We define yet another model of $\mathcal{H}^+$, which we call *extensional Taylor expansion*.

We also provide a couple of results concerning other λ-theories.

- We show that the first rgm that appeared in the literature [HNPR06, dC09] , here called *rgm à la Engeler*, induces the λ-theory equating λ-terms with the same Böhm tree, and that this λ-theory is minimal among the λ-theories represented by rgm's.
- We define a fully abstract rgm for $\mathcal{H}^*$. To prove its full abstraction we rely on another such relational model, introduced by Bucciarelli, Ehrhard and Manzonetto in [BEM07].

Here is a more detailed plan of the manuscript, with credits.

Chapter 1

In this preliminary chapter we recall basic notions and results on the λ-calculus.

A certain emphasis is given to the many ways of defining *extensional* versions of Böhm trees. In 1968 Corrado Böhm proved that, as far as we consider β-normalizable λ-terms M and N, we have $M =_{\mathcal{H}^+} N$ exactly when their β-normal forms, i.e. their Böhm trees, only differs by η-conversion (Böhm's theorem). This fact was generalized in the 70's mainly by Hyland and Nakajima to characterize $\mathcal{H}^+$, and even $\mathcal{H}^*$, on all λ-terms, even those with infinite Böhm trees. The various notions of η-reduction on generic Böhm-like trees so obtained play a prominent role here.

We also recall the *linear resource calculus* and the related notion of *Taylor expansion* of λ-terms. This calculus is an alternative syntax for the linear fragment of the *differential* λ-*calculus*, introduced by Ehrhard and Regnier in [ER03]. Despite not being the main focus of this work, the linear resource calculus can be interpreted in rgm's. This circumstance helps us when proving some important facts about the semantics.

Chapter 2

We define the class of rgm's and the corresponding class of intersection type systems (see the informal introduction above). We show how the typing derivations can be used to interpret λ-terms. We also prove that they are suitable to model the linear resource calculus. Through the notion of Taylor expansion, this last fact helps us to prove the *approximation theorem* for all rgm's, without using the technique of reducibility candidates.

The content of this chapter has been developed together with Giulio Manzonetto, with the obvious exception of the basic generalities about relational semantics, which are mainly due to Girard and Ehrhard. It was presented in [MR14], but with most of the proofs omitted. We present all the technical details for the first time here.

Chapter 3

In the first part of this chapter we study an rgm that we call *à la Engeler*, since it can be seen as a relational version of Engeler's graph model [Eng81]. We find out that its λ-theory is the one equating λ-terms with the same Böhm tree. Also, this is the *minimal* λ-theory (with reference to inclusion) that can be induced by any rgm. Actually, we focus on the *preorder theory* induced by the model, namely the preorder defined by $M \sqsubseteq N$ if and only if the interpretation of M is included in the interpretation of N. The preorder theory of the rgm *à la* Engeler is not just given by the usual order between Böhm trees, but turns out to involve η-expansions on Böhm trees.

In the second part of the chapter we remark that the *maximal* λ-theory represented by rgm's is $\mathcal{H}^*$. As a matter of facts, there is already in the literature an rgm fully abstract for $\mathcal{H}^*$. It was introduced by Bucciarelli, Ehrhard and Manzonetto in [BEM07] and proved to induce $\mathcal{H}^*$ in [Man09]. We define yet another rgm doing that.

Everything in this chapter is unpublished material by the author.

Chapter 4

In this chapter we introduce the notion of *uniformly bottomless* (extensional) rgm and show, through a characterization of β-normalizable λ-terms, that such a model is fully abstract for $\mathcal{H}^+$. We see some examples, with a particular attention to the simplest of them, an rgm built up from one single atomic type $\star$ satisfing $\star \to \star \simeq \star$.

Looking for reflexive objects in some cartesian closed category is not the only possible approach to reformulate $\mathcal{H}^+$. At the end of this chapter we present a characterization of $\mathcal{H}^+$ that relies on an *extensional* version of the Taylor expansion of λ-terms.

This chapter is a revisited version of results published with Giulio Manzonetto in [MR14].

Chapter 5

This chapter contains the main result of the thesis, as already described above: an rgm is fully abstract for $\mathcal{H}^+$ if and only if it is extensional and λ-*König*. In order to prove this, we study what distinguishes two λ-terms that are equated in $\mathcal{H}^*$ but not in $\mathcal{H}^+$ (something that we formalize with the notion of *Morris's separator*) and then we extract such a difference thanks to an *ad hoc* refined version of the Böhm-out technique.

As a byproduct of our version of the Böhm-out, we get another purely syntactic result, already mentioned above: $\mathcal{H}^+$ satisfies the ω-rule.

The results in this chapter are a joint work with Flavien Breuvart, Giulio Manzonetto and Andrew Polonsky published in [BMPR16] (with a minor error, fixed here). In particular, applying the Böhm-out to prove the validity of the ω-rule in $\mathcal{H}^+$ is an idea of Polonsky.

Paris, October 2016

EDIT

In this revisited version of the manuscript we corrected a few minor errors and implemented some useful suggestions received from the jury members.

Meanwhile - as if to prove how quickly a PhD thesis can become obsolete! - most of the results in this manuscript were represented in a more suitable form in our article [BMR17]. Most relevantly, there we provided a brand new direct proof of the *approximation theorem*, which avoids not only reducibility candidates, but even any reference to the resource calculus and the Taylor expansion.

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1 PRELIMINARIES

In this chapter we recall the notions and results on the λ-calculus used throughout the thesis. Of course, this is in no way intended to be an actual presentation of the field. We just want to fix the notations and make this work as much self-contained as possible. As a matter of fact, all definitions and theorems that we use in this thesis are properly stated within it, with the exception of some basic notions from set theory (sets, functions, relations, orders), category theory (categories, functors, natural transformations, products, monads) and theoretical computer science (computable functions, rewritings, grammars). This basic notions can be found in any standard manual, like [ML97, Bor94a, Bor94b, BW90] for categories or [Odi89] for computability theory.

Nevertheless, here and there one will find some informal references to facts from proof theory, type theory and semantics of the λ-calculus that we will not even try to present. Even if formally those facts are not used in any of our proofs, having an idea of them helps to understand the whole picture.

It is needless to say that the reader should have some familiarity with the kind of proofs typically encountered in the field of the logical foundations of proofs and programs. In particular, this work contains a great number of proofs by induction on different kinds of structures, and even a few ones by coinduction. We write IH as an abbreviation for *inductive hypothesis*, whereas coIH stands for *coinductive hypothesis*.

The symbol $:=$ is used with the meaning of *equality by definition*.

1.1 GENERALITIES

Sets

Given a set X we write $\mathcal{P}(X)$ for the set of all subsets of X and $\mathcal{P}_f(X)$ for the set of all finite subsets of X. Given two sets X and Y their intersection is denoted by $X \cap Y$, their union by $X \cup Y$, their cartesian product by $X \times Y$, their disjoint union by $X \uplus Y$ and the relative complement of Y w.r.t. X by $X - Y$. The empty set is represented by the symbol $\emptyset$.

Given a function $f : X \to Y$, we denote its domain by $\mathrm{dom}(f)$ and its range (in the sense of image of the domain) by $\mathrm{rng}(f)$.

Partial functions are written as $f : X \rightharpoonup Y$, meaning that $\mathrm{dom}(f) \subseteq X$.

We recall that given a structure (X, R) composed of a set X and a binary relation R on X, an *ideal* of (X, R) is any non-empty subset Y of X that is *downward closed* (if $x \mathrel{R} y \in Y$ then $x \in Y$) and *directed* (if $x, y \in Y$ then there is $z \in Y$ such that $x, y \mathrel{R} z$).

We write $\mathbb{N}$ for the set of natural numbers.

Sequences of natural numbers

We call $\mathbb{N}^*$ the set of finite sequences of natural numbers. The symbol ε denotes the empty sequence in $\mathbb{N}^*$. Let $\varphi = \langle n_1, \dots, n_k \rangle, \varphi' = \langle m_1, \dots, m_h \rangle \in \mathbb{N}^*$ and let $n \in \mathbb{N}$. We write

- $|\varphi|$ for the length k of φ,
- $\varphi.n$ for the sequence $\langle n_1, \dots, n_k, n\rangle$,
- $\varphi\,\varphi'$ for the concatenation of φ and φ', namely $\langle n_1, \dots, n_k, m_1, \dots, m_h\rangle$,
- $\varphi' \preccurlyeq \varphi$ whenever φ' is a *prefix* of φ, i.e. when $\varphi = \varphi'\psi$ for some $\psi \in \mathbb{N}^*$. In particular $\varphi' \prec \varphi$ if such a ψ is not ε.

Trees
A *(naked) tree* is a partial function $T : \mathbb{N}^* \rightharpoonup \mathbb{N}$ such that $\mathrm{dom}(T)$ is downward closed under prefixes and for all $\sigma \in \mathrm{dom}(T)$ and $n \in \mathbb{N}$ we have $\sigma.n \in \mathrm{dom}(T)$ if and only if $n < T(\sigma)$.

The elements of $\mathrm{dom}(T)$ are called *positions* on T. For all $\sigma \in \mathrm{dom}(T)$, $T(\sigma)$ gives the number of children of the node in position σ. In particular σ is a *leaf* of T when $T(\sigma) = 0$.

A naked tree is *computable* (or *recursive*) if it is computable as a partial function.

We denote by $\mathbb{T}$ the set of all naked trees, by $\mathbb{T}_{\mathsf{rec}}$ the set of all the computable naked trees, by $\mathbb{T}^{\infty}_{\mathsf{rec}}$ the set of all infinite computable naked trees.

Multisets
Let X be a set. A *multiset* over X is any map $m : X \to \mathbb{N}$. For all $x \in X$ the natural number $m(x)$ is the *multiplicity* of x in m. The *support* of m is the set $\{\, x \in A \mid m(x) \neq 0 \,\}$.

A multiset m is called *finite* if its support is finite. We represent a finite multiset m by the unordered list of its elements, possibly with repetitions, between square brackets, like this: $m = [x_1, \dots, x_n]$. Accordingly the *empty multiset*, i.e. the function $m : X \to \mathbb{N}$ mapping all elements of X to 0, is denoted by $[\,]$.

We write $\mathcal{M}_f(X)$ for the set of all finite multisets over X. Given $m, m' \in \mathcal{M}_f(X)$, their *multiset union* is the pointwise sum $m + m' : x \in X \mapsto m(x) + m'(x) \in \mathbb{N}$.

Rewriting
Consider a reduction rule $\to_{\mathsf{R}}$ in a rewriting system.

The R-*reduction* $\twoheadrightarrow_{\mathsf{R}}$ is the transitive-reflexive closure of $\{\to_{\mathsf{R}}\}$. In other words, given two terms t, t' of the system we have $t \twoheadrightarrow_{\mathsf{R}} t'$ if and only if there are $n \in \mathbb{N}$ and terms $t_0, \dots, t_{n-1}$ such that $t_0 = t$, $t_{n-1} = t'$ and $t_i \to_{\mathsf{R}} t_{i+1}$ for all $i \in \{0, \dots, n-1\}$. The term t is in R-*normal form* (R-*nf*) if there is no $t' \neq t$ such that $t \to_{\mathsf{R}} t'$. The term t is R-*normalizable* if $\mathrm{nf}_{\mathsf{R}}(t) := \{t' \mid t \twoheadrightarrow_{\mathsf{R}} t' \text{ and } t' \text{ R-nf}\}$ is not empty. For a set X of terms $\mathrm{nf}_{\mathsf{R}}(X) := \bigcup_{t \in X} \mathrm{nf}_{\mathsf{R}}(t)$.

The R-*conversion* $=_{\mathsf{R}}$ is the transitive-reflexive closure of $\{\to_{\mathsf{R}}\} \cup \{{}_{\mathsf{R}}\!\leftarrow\}$. So, given two terms t, t' we have $t =_{\mathsf{R}} t'$ if and only if there are $n \in \mathbb{N}$ and terms $t_0, \dots, t_{n-1}$ such that $t_0 = t$, $t_{n-1} = t'$ and $\left(t_i \to_{\mathsf{R}} t_{i+1} \text{ or } t_i \;{}_{\mathsf{R}}\!\leftarrow t_{i+1}\right)$ for all $i \in \{0, \dots, n-1\}$.

Categories
Given a (small) category $\mathcal{C}$, the hom-set of two objects A and B of $\mathcal{C}$, i.e. the set of morphisms (also called arrows) in $\mathcal{C}$ from A to B, is denoted by $\mathcal{C}(A, B)$.

Given $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$ their composition is written as $g \circ f \in \mathcal{C}(A, C)$. Whenever f is an isomorphism we denote its inversion by $f^{-1} \in \mathcal{C}(B, A)$.

Consider two functors $F, G : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\alpha : F \to G$. For every object A of $\mathcal{C}$ the corresponding component of α is denoted by $\alpha_A \in \mathcal{D}(FA, GA)$.

The *untyped* (or *type-free*) *λ-calculus* [Chu41] studies the purely computational behavior of functions. It is a formal system whose terms can be all considered to have the same status of 'functions freely applicable to one another'.

The λ-calculus first appeared in [Chu32], as a fragment of a wider system conceived by Church as a logical foundation for mathematics. Church's idea was to take functions, rather than sets, as primitive objects, but his system turned out to be logically inconsistent [SCK35]. Nevertheless, the fragment dealing only with functions became relevant in the study of the logical foundations of computer science. The untyped λ-calculus has a prominent role in computability theory, since it can represent all computable functions (*Church's thesis*). But also in the theory of programming, as a suitable framework to study from an abstract perspective many aspects of functional programming languages and their implementation. Actually, the untyped λ-calculus can be seen as a programming language on its own.

The terms of the system are built up from variables by means of two freely applicable constructors: the *application* MN, intuitively corresponding to the operation of applying a function M to (another function used as) an argument N; the *λ-abstraction* $\lambda x.M$, which allows the effective substitutions of arguments for the variable x in an expression M.

The syntax

We generally use the notation of Barendregt's book [Bar84] for the untyped λ-calculus.

We fix an infinite set Var. Its elements are called *variables* and denoted by x, y, z or occasionally other lowercase Latin letters, possibly with apexes and pedexes.

The set Λ of *λ-terms* is defined by the following grammar:

$$\Lambda : \quad M, N ::= x \mid \lambda x.M \mid MN \qquad \text{for all } x \in \text{Var}. \tag{1}$$

A λ-term of the form MN is called an *application* . One of the form $\lambda x.M$ is a *λ-abstraction*.

We assume that the application associates to the left, namely we write $M_1 M_2 M_3 \cdots M_n$ for the λ-term $(\cdots((M_1 M_2)M_3)\cdots)M_n$. Obviously the λ-abstraction associates to the right, in that $\lambda x_1.\lambda x_2 \ldots \lambda x_n.M$ stands for the λ-term $\lambda x_1.(\lambda x_2.(\ldots(\lambda x_n.M)\ldots))$. In fact, such a λ-terms will be always denoted by the compact notation $\lambda x_1 x_2 \ldots x_n.M$.

The set $\text{fv}(M)$ of *free variables* of $M \in \Lambda$ is defined by induction on the structure of M as follows: $\text{fv}(x) := \{x\}$, $\text{fv}(\lambda x.M') := \text{fv}(M') - \{x\}$ and $\text{fv}(M_1 M_2) := \text{fv}(M_1) \cup \text{fv}(M_2)$. When a variable x occurs in M but $x \notin \text{fv}(M)$ we say that x is *bound* in M.

A λ-term M is *closed* whenever $\text{fv}(M) = \emptyset$. Closed λ-terms are also called *combinators*. We denote the set of all closed λ-terms by Λ^0.

Given $M, N \in \Lambda$ we denote by $M\{N/x\}$ the capture-free simultaneous substitution of N for all free occurrences of x in M. Formally $M\{N/x\}$ is defined by induction on M as follows: $x\{N/x\} := N$, $y\{N/x\} := y$ if $y \neq x$, $(M_1 M_2)\{N/x\} := M_1\{N/x\}M_2\{N/x\}$, $(\lambda y.M')\{N/x\} := \lambda y.M'\{N/x\}$ for $y \neq x$.

Let $\lambda x.M \in \Lambda$ and y not occurring in M, neither free nor bound. One may define in this hypothesis the rewriting rule $\lambda x.M \to_\alpha \lambda y.M\{y/x\}$, called *α-rule*. By convention we consider α-convertible λ-terms to be the *same* λ-term, namely if $M =_\alpha N$ then actually $M = N$. For instance $\lambda xz.xxz = \lambda yz.yyz = \lambda yx.yyx$. In other words, we assume the α-convertion to be implicit in the syntax of the λ-calculus.

We also adopt the so-called *variable convention* [Bar84, Conv. 2.1.13]: when in the same context (in the same definition, statement, proof, and so on) we are dealing with certain λ-terms $M_1, \dots, M_n$ the set of all variables that are bound in any of them is disjoint from $\mathrm{fv}(M_1 \cdots M_n)$, which is the set of all variables that are free in any of $M_1, \dots, M_n$. For instance, if we are talking of $M_1 = \lambda x.x\,y$ and $M_2 = xx$ then the variable x occurring in M_1 is not the same as the one in M_2.

The dynamics

In the λ-calculus the idea of computation is represented by the β-*rule*, a rewriting rule that defines the dynamics of substitution. It is the contextual closure of $(\lambda x.M)\,N \to_\beta M\{N/x\}$. More explicitly, this means that $\to_\beta$ is defined by the clauses: $(\lambda x.M)\,N \to_\beta M\{N/x\}$; whenever $M \to_\beta M'$ then $MN \to_\beta M'N$, $NM \to_\beta NM'$ and $\lambda x.M \to_\beta \lambda x.M'$ for all $N \in \Lambda$ and $x \in \mathrm{Var}$. See § 1.1 for the meaning of $\twoheadrightarrow_\beta$, $=_\beta$ and β-nf.

A λ-term M is *solvable* if and only if it has a *head normal form* (*hnf*), which means that $M \twoheadrightarrow_\beta \lambda x_1 \dots x_n.x\,M_1 \cdots M_m$ for some $n, m \in \mathbb{N}$. Otherwise M is called *unsolvable*.

Also relevant is the η-*rule*, which is the contextual closure of $\lambda x.Mx \to_\eta M$ for $x \notin \mathrm{fv}(M)$. The η-conversion provides a way to axiomatize *extensionality*, the idea that every λ-term actually behaves like a function.

Here some notable closed λ-terms:

$$
\begin{array}{rclrclrcl}
\mathbf{I} & := & \lambda x.x & \overline{n} & := & \lambda fx.\overbrace{f(f(\cdots f}^{n \text{ times}}(x)\cdots)) & \mathsf{succ} & := & \lambda nfx.f(nfx) \\
\mathbf{K} & := & \lambda xy.x & \mathbf{1}_n & := & \lambda xx_1 \dots x_n.xx_1 \dots x_n & \mathsf{sum} & := & \lambda nmfx.nf(mfx) \\
\mathbf{S} & := & \lambda xyz.xz(yz) & \mathbf{Y} & := & \lambda f.\big(\lambda x.f(xx)\big)\big(\lambda x.f(xx)\big) & \mathbf{\Omega} & := & (\lambda x.xx)(\lambda x.xx) \\
\mathbf{F} & := & \lambda xy.y & \mathbf{\Theta} & := & \big(\lambda xf.(f(xxf))\big)\big(\lambda xf.f(f(xxf))\big) & \mathbf{J} & := & \mathbf{\Theta}\big(\lambda zxy.x(zy)\big)
\end{array}
$$

The combinator $\mathbf{I}$ is called the *identity* and gives $\mathbf{I}\,M \to_\beta M$ for all $M \in \Lambda$. Schönfinkel's combinators $\mathbf{K}$ and $\mathbf{S}$ play a key role in combinatory logic. Notice that $\mathbf{SKK} \twoheadrightarrow_\beta \mathbf{I}$. In particular $\mathbf{K}$ is sometimes denoted by $\mathbf{T}$, a reference to the truth value *true*. The other boolean *false* is $\mathbf{F}$. For every $n \in \mathbb{N}$ the combinator $\overline{n}$ is called the n-*th Church's numeral*. We have $\mathsf{succ}\,\overline{n} \twoheadrightarrow_\beta \overline{n+1}$ and $\mathsf{sum}\,\overline{n}\,\overline{m} \twoheadrightarrow_\beta \overline{n+m}$. For every $n \in \mathbb{N}$ clearly $\mathbf{1}_n \twoheadrightarrow_\eta \mathbf{I}$. In particular $\mathbf{1}_1 = \overline{1} \to_\eta \mathbf{I}$. A prominent example of unsolvable λ-term is $\mathbf{\Omega}$, since $\mathbf{\Omega} \to_\beta \mathbf{\Omega}$. For every $M \in \Lambda$ Church's fixpoint combinator $\mathbf{Y}$ satisfies $\mathbf{Y}M =_\beta M(\mathbf{Y}M)$, whereas the more refined Turing's fixpoint combinator $\mathbf{\Theta}$ gives $\mathbf{\Theta}M \twoheadrightarrow_\beta M(\mathbf{\Theta}M)$. Fixpoint combinators help us define solvable but not β-normalizable λ-terms, such as Wadsworth's combinator $\mathbf{J}$. It has the property $\mathbf{J} \twoheadrightarrow_\beta \lambda xy.x(\mathbf{J}y) \twoheadrightarrow_\beta \lambda xy.x\big(\lambda y_1.y(\mathbf{J}y_1)\big) \twoheadrightarrow_\beta \lambda xy.x(\lambda y_1.y\big(\lambda y_2.y_1(\mathbf{J}y_2))\big) \twoheadrightarrow_\beta \dots$ and so on to infinity. The λ-term $\mathbf{J}$ will play a prominent role in this thesis.

As already mentioned, the untyped λ-calculus is a model of computation, like Turing machines and recursive functions. This is established through the following notion. For any $k \in \mathbb{N}$ we say that $f : \mathbb{N}^k \to \mathbb{N}$ is λ-*definable* if and only if there exists $F \in \Lambda$ such that $F\,\overline{n_1} \cdots \overline{n_k} =_\beta \overline{f(n_1, \dots, n_k)}$ for all $n_1, \dots, n_k \in \mathbb{N}$. Kleene proved that f is λ-definable if and only if it is recursive [Bar84, Theorem 6.3.13], hence when it is Turing-recognizable. *Church's thesis* states that a function is (intuitively) computable if and only if it is λ-definable.

Böhm trees

A tree representation can be very handy when reasoning on β-convertibility. This successful approach was pioneered by Barendregt [Bar84] relying on a seminal work of Böhm [Böh68].

Let π_1 and π_2 be respectively the left and right projection operator for the cartesian product of sets. Let $\Sigma := \{ \lambda x_1 \ldots x_n.x \mid x_1, \ldots, x_n, x \in \mathrm{Var} \text{ and } n \in \mathbb{N} \} \cup \{\bot\}$. A *Böhm-like tree* is a partial function $A : \mathbb{N}^* \rightharpoonup \Sigma \times \mathbb{N}$ such that $\pi_2 \circ A$ is a naked tree and $(\pi_2 \circ A)(\sigma) = 0$ whenever $(\pi_1 \circ A)(\sigma) = \bot$. We call $\lceil A \rceil := (\pi_2 \circ A)$ the *(naked) tree underlying* A. For convenience, with an abuse of language we will usually write $A(\sigma)$ to denote $(\pi_1 \circ A)(\sigma)$.

Intuitively, a Böhm-like tree is just a labelled tree where every node is given a label of the form $\lambda x_1 \ldots x_n.x$ or the label $\bot$, with the latter case possible only if the node is a leaf.

Just like λ-terms, Böhm-like trees are considered up to α-conversion and with the variable convention. We denote by $\Lambda^{\mathcal{B}}$ the set of all Böhm-like trees.

Given a position $\varphi \in \mathrm{dom}(A)$ on $A \in \Lambda^{\mathcal{B}}$, the *subtree of* A *at position* φ is the Böhm-like tree A_φ defined by $A_\varphi(\psi) := A(\varphi\,\psi)$ for all $\psi \in \mathbb{N}^*$.

As an alternative, Böhm-like trees can be defined coinductively as follows: $\bot \in \Lambda^{\mathcal{B}}$; for all $m \in \mathbb{N}$ if $A_1, \ldots, A_m \in \Lambda^{\mathcal{B}}$ then $\lambda x_1 \ldots x_n.x\, A_1, \ldots, A_m \in \Lambda^{\mathcal{B}}$ for all $n \in \mathbb{N}$ and for all $x_1, \ldots, x_n, x \in \mathrm{Var}$. This definition is formally different from the one above, since these coinductive terms have nothing to do with functions of the form $A : \mathbb{N}^* \rightharpoonup \Sigma \times \mathbb{N}$. Nevertheless, we will freely use one or the other depending on the need.

We denote by $\mathcal{N}$ the set of *finite* Böhm-like trees, also called *(finite) approximants*. Formally $A \in \mathcal{N}$ if and only if $A \in \Lambda^{\mathcal{B}}$ and $\mathrm{dom}(A)$ is finite.

Finite approximants have an alternative definition through a rewriting system. They can be seen as the $\beta\bot$-nf's of the *$\lambda\bot$-calculus*, which is the extension of the λ-calculus obtained by adding the constant $\bot$ to the grammar (1) and the new rules $\lambda x.\bot \to_\bot \bot$ and $\bot M \to_\bot \bot$ for all $\lambda\bot$-term M.

We use upper case Latin letters $A, B, \ldots$ for generic elements of $\Lambda^{\mathcal{B}}$ and lower case Latin letters $a, b, \ldots$ for elements of $\mathcal{N}$, possibly with pedexes and apexes.

The set $\Lambda^{\mathcal{B}}$ can be given a structure of order. We set $A \leqslant_\bot B$ if and only if A results from B by replacing some subtrees with $\bot$. The intuitive reading of $A \leqslant_\bot B$ is that the Böhm-like tree A is a less refined approximation of B.

We call $A^* := \{a \in \mathcal{N} \mid a \leqslant_\bot A\}$ the set of *finite approximants of* $A \in \Lambda^{\mathcal{B}}$. Notice that $\mathcal{S} \subseteq \mathcal{N}$ is an ideal of $(\mathcal{N}, \leqslant_\bot)$ if and only if there exist $A \in \Lambda^{\mathcal{B}}$ such that $\mathcal{S} = A^*$.

Some elements of $\Lambda^{\mathcal{B}}$ can be used to represent the result of the (possibly infinite) complete β-reduction of λ-terms. Let $M \in \Lambda$. The *Böhm tree of* M is the tree $\mathrm{BT}(M) \in \Lambda^{\mathcal{B}}$ defined coinductively as follows:

if M is unsolvable then $\mathrm{BT}(M) := \bot$; if M is solvable and $M \twoheadrightarrow_\beta \lambda x_1 \ldots x_n.x\, N_1 \cdots N_m$ then $\mathrm{BT}(M) := \lambda x_1 \ldots x_n.x\, \mathrm{BT}(N_1) \cdots \mathrm{BT}(N_m)$. Such a definition is independent of the hnf $\lambda x_1 \ldots x_n.x\, N_1 \cdots N_m$ chosen in the second clause.

Notice that not every $A \in \Lambda^{\mathcal{B}}$ is the Böhm tree of a λ-term, see [Bar84, Theorem 10.1.23]. In Figure 1 we provide some examples of Böhm trees.

One should think of $\mathrm{BT}(M)$ as the possibly infinite β-normal form of M, with the unsolvable parts of M represented by the symbol $\bot$. In particular, whenever M has a β-nf N then actually $\mathrm{BT}(M)$ is N.

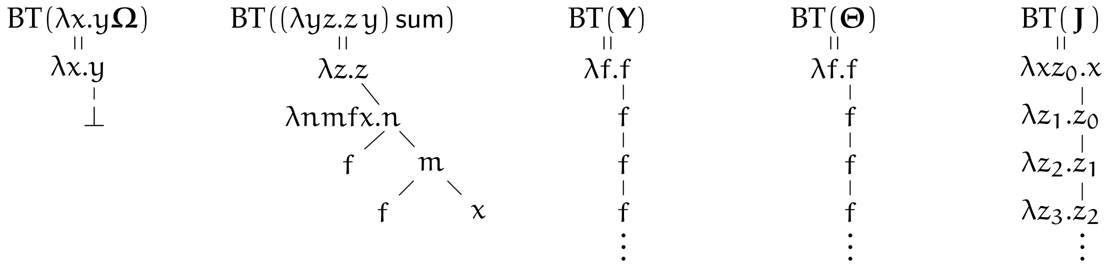


Figure 1: Some examples of Böhm trees.

For every position σ in $\mathrm{BT}(M)$, we want some λ-term M_σ such that $\mathrm{BT}(M_\sigma) = \mathrm{BT}(M)_\sigma$. Of course, such an M_σ is not unique. So we define a canonical one. Firstly, the *principal head normal form* of a λ-term M, denoted $\mathrm{phnf}(M)$, is the head normal form obtained from M by the *head reduction* strategy [Bar84, Def. 8.3.10]. Then for any σ on $\mathrm{BT}(M)$ we define the *subterm* M_σ *of* M *at* σ by: $M_\varepsilon = M$; $M_{i.\sigma} = (M_{i+1})_\sigma$ whenever $\mathrm{phnf}(M) = \lambda\vec{x}.yM_1 \cdots M_n$.

Preorder and λ-theories

The study of the untyped λ-calculus is not restricted to the sole β-rule. For a variety of reasons, one may want to identify more λ-terms than β-convertibility does. So typically the focus is on some equational extension of β-convertibility. Such extensions are known as λ-*theories*, formally defined below. A further refinement consists in investigating inequational extensions of β-convertibility, which we call *preorder theories*.

Remember that a binary relation is a preorder if it is reflexive and transitive, whereas an equivalence is a symmetric preorder.

A binary relation $\mathcal{R}$ on Λ is *compatible (with abstraction and application)* if it satisfies $M \,\mathcal{R}\, N \Rightarrow \lambda x.M \,\mathcal{R}\, \lambda x.N$ and $(M \,\mathcal{R}\, N) \wedge (M' \,\mathcal{R}\, N') \Rightarrow (MM' \,\mathcal{R}\, NN')$ for all $M, N, M', N' \in \Lambda$.

A *preorder theory* is any compatible preorder on Λ including $=_\beta$.

A λ-*theory* is any compatible equivalence on Λ including $=_\beta$.

Given a preorder theory $\sqsubseteq_\mathcal{T}$, its induced equivalence $=_\mathcal{T}$ is defined by $M =_\mathcal{T} N$ if and only if $M \sqsubseteq_\mathcal{T} N$ and $N \sqsubseteq_\mathcal{T} M$, and it is always a λ-theory. We often use the symbol $\mathcal{T}$ itself when referring to the λ-theory $=_\mathcal{T}$.

A λ-theory is: *consistent* if it does not equate all λ-terms (hence $\Lambda \times \Lambda$ is the only *inconsistent* λ-theory); *extensional* if it includes $=_\eta$; *sensible* if it equates all unsolvables. A preorder theory is *consistent*, *extensional* or *sensible* if such is its induced λ-theory.

We call $\boldsymbol{\lambda}$ the least λ-theory. It is in fact the λ-theory just equating β-convertible λ-terms, namely $\boldsymbol{\lambda} = \{(M, N) \in \Lambda \times \Lambda \mid M =_\beta N\}$.

We call $\boldsymbol{\lambda\eta}$ the least extensional λ-theory. Such $\boldsymbol{\lambda\eta}$ is nothing but the transitive closure of $\boldsymbol{\lambda} \cup \{(M, N) \in \Lambda \times \Lambda \mid M =_\eta N\}$. Actually, it is not complicated to prove that $\boldsymbol{\lambda\eta} = \{(M, N) \in \Lambda \times \Lambda \mid M =_{\beta\eta} N\}$, where $=_{\beta\eta}$ is the notion of convertibility of the rewriting system defined by both the β-rule and the η-rule.

More generally, given a λ-theory $\mathcal{T}$ we denote by $\mathcal{T}\eta$ the least λ-theory including $\mathcal{T} \cup \boldsymbol{\lambda\eta}$.

The preorder theory $\sqsubseteq_\mathcal{B}$ is defined as $M \sqsubseteq_\mathcal{B} N$ if and only if $\mathrm{BT}(M) \leqslant_\perp \mathrm{BT}(N)$. By antisymmetry of $\leqslant_\perp$ the induced λ-theory is then: $M =_\mathcal{B} N$ if and only if $\mathrm{BT}(M) = \mathrm{BT}(N)$. Notice that $\mathcal{B}$ is sensible, because when M and N are unsolvable $\mathrm{BT}(M) = \perp = \mathrm{BT}(N)$.

We call $\mathcal{H}$ the least sensible λ-theory. In other words, $\mathcal{H}$ is the least λ-theory that includes $\{(M,N) \mid M,N \text{ unsolvables}\}$. We have $\mathcal{H} \subset \mathcal{B}$ (since $\mathcal{B}$ is sensible, $\mathcal{H} \subseteq \mathcal{B}$; moreover this inclusion is proper, as $\mathrm{BT}(\mathbf{\Theta}) = \mathrm{BT}(\mathbf{Y})$ but $\mathbf{\Theta} \neq_{\mathcal{H}} \mathbf{Y}$).

The *ω-rule* is a strong form of extensionality defined as follows:

$$\text{for all } M,N \in \Lambda \left(\left(\text{for all } P \in \Lambda^0 \; MP = NP\right) \text{ implies } M = N \right).$$

The *ω^0-rule* is the restriction of the ω-rule to combinators $M,N \in \Lambda^0$. Given a λ-theory $\mathcal{T}$ we denote its closure under the ω-rule by $\mathcal{T}\omega$, and we say that $\mathcal{T}$ *satisfies the ω-rule* if $\mathcal{T} = \mathcal{T}\omega$. An analogous notation is used for the ω^0-rule. By collecting some results in [Bar84, §4.1], for all λ-theories $\mathcal{T}$ we have: $\mathcal{T}\eta \subseteq \mathcal{T}\omega$; $\mathcal{T}$ satisfies the ω-rule if and only if $\mathcal{T}$ satisfies the ω^0-rule; $\mathcal{T} \subseteq \mathcal{T}'$ entails $\mathcal{T}\omega \subseteq \mathcal{T}'\omega$.

Categorical semantics

Most of this thesis concerns denotational semantics of the untyped λ-calculus. Among the many notions of model for such system (weakly extensional λ-algebras, λ-models, syntactical λ-models, categorical models, see [Bar84, Ch. 5]) we will only use the categorical one.

We recall that a *cartesian closed category* is a category $\mathcal{C}$ with finite products $\times$ and a bifunctor $- \Rightarrow - : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ right adjoint to the product. For any product $\prod_{i \in F} A_i$ we denote by $\pi_i^{\prod_{i\in F} A_i} : \prod_{i\in F} A_i \to A_i$ its i-th projection. The adjunction $- \times - \dashv - \Rightarrow -$ assures the existence of a bijection $\Lambda_{A,B,C} : \mathcal{C}(A \times B, C) \to \mathcal{C}(A, B \Rightarrow C)$ natural in A, B and C. We denote by $\mathrm{Ev}_{A,B} : (A \Rightarrow B) \times A \to B$ the counit of the adjunction. Cartesian closed categories provide the categorical semantics of the simply typed λ-calculus.

Definition 1.2.1. Let $\mathcal{C}$ be a cartesian closed category. A *reflexive object* in $\mathcal{C}$ is a triple $(D, \mathtt{Abs}, \mathtt{App})$ composed of an object D of $\mathcal{C}$ and morphisms $\mathtt{Abs} \in \mathcal{C}(D \Rightarrow D, D)$ and $\mathtt{App} \in \mathcal{C}(D, D \Rightarrow D)$ such that $\mathtt{App} \circ \mathtt{Abs} = \mathrm{id}_{D \Rightarrow D}$. The reflexive object is *extensional* if $\mathtt{Abs} \circ \mathtt{App} = \mathrm{id}_D$.

Definition 1.2.2. Let $\mathcal{D} = (D, \mathtt{Abs}, \mathtt{App})$ be a reflexive object in a cartesian closed category. For all $M \in \Lambda$ and for all finite sequence of variables $x_1, \ldots, x_n$ such that $\mathrm{fv}(M) \subseteq \vec{x}$, the *interpretation* $[\![M]\!]_{\mathcal{D}}^{\vec{x}} : D^n \to D$ is given by induction on M as: $[\![x_i]\!]_{\mathcal{D}}^{\vec{x}} := \pi_i^{D^n}$, $[\![\lambda x.M]\!]_{\mathcal{D}}^{\vec{x}} := \mathtt{Abs} \circ \Lambda_{D,D^n,D}([\![M]\!]_{\mathcal{D}}^{x,\vec{x}})$ and $[\![MN]\!]_{\mathcal{D}}^{\vec{x}} := \mathrm{Ev}_{D,D} \circ \langle \mathtt{App} \circ [\![M]\!]_{\mathcal{D}}^{\vec{x}}, [\![N]\!]_{\mathcal{D}}^{\vec{x}} \rangle$.

We could not find any reference for the following definition in the literature. Nevertheless, we are convinced that it must have already appeared elsewhere, and that it is considered by many as mathematical folklore.

Definition 1.2.3. Let $\mathcal{D} = (D, \mathtt{Abs}, \mathtt{App})$ and $\mathcal{D}' = (D', \mathtt{Abs}', \mathtt{App}')$ be reflexive objects in a given cartesian closed category. An *isomorphism of reflexive objects* $f : \mathcal{D} \to \mathcal{D}'$ is an isomorphism $f : D \to D'$ making the following two diagrams commute:

$$\begin{array}{ccccc} D \Rightarrow D & \xrightarrow{\mathtt{Abs}} & D & \xrightarrow{\mathtt{App}} & D \Rightarrow D \\ \downarrow {\scriptstyle f^{-1} \Rightarrow f} & & \downarrow {\scriptstyle f} & & \downarrow {\scriptstyle f^{-1} \Rightarrow f} \\ D' \Rightarrow D' & \xrightarrow{\mathtt{Abs}'} & D' & \xrightarrow{\mathtt{App}'} & D' \Rightarrow D' \end{array} \tag{2}$$

Theorem 1.2.4. *Let $\mathcal{D}$ and $\mathcal{D}'$ be isomorphic reflexive objects in a cartesian closed category. Then for all $M, N \in \Lambda$ and for all finite sequence of variables $x_1, \ldots, x_n$ such that $\mathrm{fv}(MN) \subseteq \vec{x}$ we have $[\![M]\!]^{\vec{x}}_{\mathcal{D}} = [\![N]\!]^{\vec{x}}_{\mathcal{D}}$ if and only if $[\![M]\!]^{\vec{x}}_{\mathcal{D}'} = [\![N]\!]^{\vec{x}}_{\mathcal{D}'}$.*

Proof. See Appendix A. □

When it comes to isomorphisms between *extensional* reflexive objects we do not need to check the commutativity of the right diagram of (2) in Definition 1.2.3.

Lemma 1.2.5. *Let $\mathcal{D}$ and $\mathcal{D}'$ be reflexive objects in a cartesian closed category. In particular let $\mathcal{D}$ be extensional. If the isomorphism $f : D \to D'$ makes the left diagram of (2) commute, then f is an isomorphism of reflexive objects.*

Proof. See Appendix A. □

The research on denotational semantics of λ-calculi has derived much benefit from the discovery of *Linear Logic* (LL) by Girard [Gir87]. The natural deduction of propositional intuitionistic logic can be embedded into LL. This is true in particular for its implicative fragment, which corresponds to the simply typed λ-calculus *via* the Curry-Howard (or proofs-as-programs) isomorphism [How80, SU06]. From the semantic perspective, this means that from a categorical model of LL one must always be able to construct a categorical model of the simply typed λ-calculus, namely a cartesian closed category.

There are different categorical axiomatizations of LL, all of them sharing Bierman's *linear-non-linear principle* [Bie95], as explained in [Mel09, Ch. 7]. We briefly recall here the categorical model of LL known by the name of *Seely category* (skipping most technical details). The notion is in fact a reformulation by Bierman [Bie95] of a definition of Seely [See89].

A *symmetric monoidal category* (*smc*) $(\mathcal{S}, \otimes, 1, \alpha, \lambda, \gamma, \sigma)$ is composed of a category $\mathcal{S}$, a bifunctor $\otimes : \mathcal{S} \otimes \mathcal{S} \to \mathcal{S}$ (called tensor), an object 1 (the unity of the tensor), and natural isomorphisms $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $\lambda_A : 1 \otimes A \to A$, $\rho_A : A \otimes 1 \to A$, $\sigma_{A,B} : A \otimes B \to B \otimes A$ that make commute some well-known diagrams axiomatizing associativity, left and right neutrality of 1 and commutativity (see for instance [Mel09] or [ML97]). An smc as above is *closed* (*smcc*) if there exists a bifunctor $- \multimap - : \mathcal{S}^{\mathrm{op}} \times \mathcal{S} \to \mathcal{S}$ right adjoint to the tensor $-\otimes-$. This provides a bijection $\lambda_{A,B,C} : \mathcal{S}(A \otimes B, C) \to \mathcal{S}(A, B \multimap C)$ natural in A, B and C. We denote by $\mathsf{ev}_{A,B} : (A \multimap B) \otimes A \to B$ the counit of this adjunction. In particular one can define $\mathsf{dual}^B_A := \lambda_{A,A\multimap B,B}(\mathsf{ev}_{A,B} \circ \sigma_{A,A\multimap B}) : A \to (A \multimap B) \multimap B$. The smcc is a $*$-*autonomous category* if there is an object $\perp$ (called the dualizing object) such that for every A the arrow $\mathsf{dual}^{\perp}_A$ is an isomorphism. In such a case one also writes $A^{\perp} := A \multimap \perp$. Finally, a *Seely category* is a $*$-autonomous category $\mathcal{S}$ with finite products (usually denoted by $\&$ rather than $\times$ in this context, with $\top$ being the final object) and a symmetric monoidal comonad $(!, \mathsf{der}, \mathsf{dig}, m) : (\mathcal{S}, \otimes, 1) \to (\mathcal{S}, \&, \top)$ that satisfies a further technical condition relating dig and m, see diagram (72) in [Mel09, § 7.3]. We refer to [Mel09] for the unfolding of what *symmetric monoidal comonad* means. We just recall that the counit $\mathsf{der}_A : !A \to A$ of the comonad is called *dereliction*, the comultiplication $\mathsf{dig}_A : !A \to !!A$ is called *digging*, whereas the natural isomorphisms $m^2_{A,B} : !A \otimes !B \to !(A \,\&\, B)$ and $m^0 : 1 \to !\top$ making the functor $!$ monoidal go by the name of *Seely isomorphisms*. Also, using these ingredients one can define $\mathsf{prom}_{A,B} : !A \multimap B \to !A \multimap !B$, which we call *promotion*. Let

$(\mathcal{S}, \otimes, 1, \alpha, \lambda, \gamma, \sigma, \perp, \&, \top, !, \mathsf{der}, \mathsf{dig}, m)$ be a Seely category. Then the *co-Klesli category of* ! *on* $\mathcal{S}$, here denoted by $\mathsf{Kl}_!(\mathcal{S})$, is a cartesian closed category. Remember that its objects are the same as $\mathcal{S}$, its morphisms are given by $\mathsf{Kl}_!(\mathcal{S})(A, B) := \mathcal{S}(!A, B)$ with $\mathsf{der}_A \in \mathsf{Kl}_!(\mathcal{S})(A, A)$ as identity, and the composition of $f \in \mathsf{Kl}_!(\mathcal{S})(A, B)$ and $g \in \mathsf{Kl}_!(\mathcal{S})(B, C)$ is provided by the arrow $g \circ !f \circ \mathsf{dig}_A$ existing in $\mathcal{S}$. The product & of $\mathcal{S}$ lifts to a product in $\mathsf{Kl}_!(\mathcal{S})$, when taking as i-th projection of $\&_{i\in F} A_i$ just the morphism $\pi_i^{\&_{i\in F} A_i} \circ \mathsf{der}_{\&_{i\in F} A_i} : !\&_{i\in F} A_i \to A_i$ in $\mathcal{S}$. As concerns the closure $- \Rightarrow -$, it is defined on objects as $A \Rightarrow B := !A \multimap B$ (also known as *Girard's first translation* [Gir87]), whereas given $g \in \mathsf{Kl}_!(\mathcal{S})^{\mathsf{op}}(A, A') = \mathsf{Kl}_!(\mathcal{S})(A', A) = \mathcal{S}(!A', A)$ and $h \in \mathsf{Kl}_!(\mathcal{S})(B, B') = \mathcal{S}(!B, B')$ the morphism

$$g \Rightarrow h \in \mathsf{Kl}_!(\mathcal{S})(A \Rightarrow B, A' \Rightarrow B') = \mathcal{S}(!(A \Rightarrow B), A' \Rightarrow B') = \mathcal{S}(!(!A \multimap B), !A' \multimap B')$$

is the following composition of arrows in $\mathcal{S}$:

$$!(!A \multimap B) \xrightarrow{\mathsf{der}_{!A \multimap B}} !A \multimap B \xrightarrow{\mathsf{prom}_{A,B}} !A \multimap !B \xrightarrow{!g \multimap h} !!A' \multimap B' \xrightarrow{\mathsf{der}_{!A'} \multimap B'} !A' \multimap B'.$$

1.3 OBSERVATIONAL EQUIVALENCES

The problem of determining when two programs are equivalent is crucial in computer science. For λ-calculi and related systems, it has become standard to consider two terms M and N as equivalent programs when they are *observationally* (or *contextually*) *equivalent* with respect to some fixed set $\mathcal{O}$ of *observable terms*. This means that one can plug either M or N into any context $C[-]$ without noticing any difference in behaviour through the glasses of $\mathcal{O}$: the program $C[M]$ reduces to an observable in $\mathcal{O}$ exactly when $C[N]$ does. The underlying intuition is that terms in $\mathcal{O}$ have a form with a certain stable amount of information, so they can be used as observable outputs of the computation. The choice of the set $\mathcal{O}$ is not unique.

Several interesting preorder theories and λ-theories are obtained when one applies this approach to the untyped λ-calculus, as we do now.

A *context* $C[-]$ is a λ-term with exactly one *hole* $[-]$ occurring as a subterm in it. A way to formally define this is by first adding the constant $[-]$ to the grammar defining λ-terms (1), and then taking from the terms so obtained only those where $[-]$ occurs once. Given a context $C[-]$ and $M \in \Lambda$, we denote by $C[M]$ the λ-term obtained by replacing the hole with M. It is important to stress that we do not apply the α-conversion and the variable conventions to contexts. For instance, if we have the context $\lambda x.[-]$ and the λ-term xx, the occurrences of x in both of them refer to the same variable. The reason is that we do want to be able to capture bound variables when filling a hole.

A context $E[-]$ is a *head context* if it has the form $\lambda x_1 \dots x_m.[-] M_1 \cdots M_n$ for $m, n \in \mathbb{N}$. In particular such a head context is *applicative* whenever $m = 0$, namely if it has the form $[-] M_1 \cdots M_n$.

Consider a set $\mathcal{O} \subseteq \Lambda$. We write $M \in_R \mathcal{O}$ when there exists $M' \in \mathcal{O}$ such that $M \twoheadrightarrow_R M'$. The $\mathcal{O}$-*observational preorder* $\sqsubseteq^{\mathcal{O}}$ is defined as

$$M \sqsubseteq^{\mathcal{O}} N \quad \text{if and only if} \quad \text{for every context } C[-] \left(C[M] \in_\beta \mathcal{O} \Rightarrow C[N] \in_\beta \mathcal{O}\right).$$

Notice that we are not asking $C[M]$ and $C[N]$ to reduce to the *same* element of $\mathcal{O}$.

The $\mathcal{O}$-*observational equivalence* $=^{\mathcal{O}}$ is the equivalence induced by $\sqsubseteq^{\mathcal{O}}$, namely $M =^{\mathcal{O}} N$ if and only if $M \sqsubseteq^{\mathcal{O}} N$ and $N \sqsubseteq^{\mathcal{O}} M$.

It is easy to check that $\sqsubseteq^{\mathcal{O}}$ is a preorder theory, hence $=_{\mathcal{O}}$ is a λ-theory.

We have not asked any condition on $\mathcal{O}$. Nevertheless, it seems reasonable to take some set $\mathcal{O}$ closed by β-convertibility. We are interested in two specific cases.

Hyland's Observability

Let $\mathcal{O}$ be the set of λ-terms in hnf. The corresponding $\mathcal{O}$-observational preorder is denoted by $\sqsubseteq_{\mathcal{H}^*}$ and the corresponding $\mathcal{O}$-observational equivalence by $=_{\mathcal{H}^*}$. More explicitly

$$M \sqsubseteq_{\mathcal{H}^*} N \quad \text{if and only if} \quad \text{for every } C[-] \; \Big(C[M] \text{ has a hnf } \;\Rightarrow\; C[N] \text{ has a hnf}\Big).$$

It is easy to realize that $\sqsubseteq_{\mathcal{H}^*}$ and $\mathcal{H}^*$ are consistent, extensional and sensible.

It is a well-known fact that one can focus attention to head contexts.

Lemma 1.3.1 (Context Lemma). *Let $M, N \in \Lambda$. Then $M \sqsubseteq_{\mathcal{H}^*} N$ if and only if for all head context $E[-]$ whenever $E[M]$ has a hnf then $E[N]$ has a hnf.*

Corollary 1.3.2. *Let $M, N \in \Lambda^0$. Then $M \sqsubseteq_{\mathcal{H}^*} N$ if and only if for all applicative and closed context $\vec{X}$ whenever $M\vec{X}$ has a hnf then $N\vec{X}$ has a hnf.*

The λ-theory $\mathcal{H}^*$ was extensively studied in the 70's, primarily by Hyland [Hyl75, Hyl76] and Wadsworth [Wad76]. In particular Hyland proved that $\mathcal{H}^*$ is the *maximal* consistent sensible λ-theory. The proof of this fact, as can be found in [Bar84, Theorem 16.2.6], can also be straightforwardly adapted to the preorder case. So $\sqsubseteq_{\mathcal{H}^*}$ is the maximal consistent sensible preorder theory. We will make an extensive use of this maximality in this thesis.

The following characterization of $\sqsubseteq_{\mathcal{H}^*}$ comes from [Hyl76] (see also [Bar84] and [RP04]).

Definition 1.3.3. Let $M, N \in \Lambda$. We write $M \sqsubseteq^k_{\mathcal{H}^*} N$ if and only if either $k = 0$, or M is unsolvable, or $k > 0$ and

$$M \;=_\beta\; \lambda x_1 \dots x_{n_1}.y\, M_1 \cdots M_{m_1} \qquad\qquad N \;=_\beta\; \lambda x_1 \dots x_{n_2}.y\, N_1 \cdots N_{m_2}$$

where $n_1 - m_1 = n_2 - m_2$ and if, say, $m_1 \leqslant m_2$ (hence $n_1 \leqslant n_2$ and there exists $p \geqslant 0$ such that $n_2 = n_1 + p$ and $m_2 = m_1 + p$) then:

- either y is free or $y = x_j$ for some $j \leqslant n_1$;
- $M_i \sqsubseteq^{k-1}_{\mathcal{H}^*} N_i$ for all $i \in \{1, \dots, m_1\}$ and $x_{n_1+j} \sqsubseteq^{k-1}_{\mathcal{H}^*} N_{m_1+j}$ for all $j \in \{1, \dots, p\}$.

(The case $m_1 \geqslant m_2$ is symmetrical.)

Proposition 1.3.4. *Let $M, N \in \Lambda$. We have $M \sqsubseteq_{\mathcal{H}^*} N$ if and only if $M \sqsubseteq^k_{\mathcal{H}^*} N$ for all $k \in \mathbb{N}$.*

For convenience, we reformulate Proposition 1.3.4 also in terms of approximants.

Definition 1.3.5. Let $a, b \in \mathcal{N} - \{\bot\}$. Since elements of $\mathcal{N}$ are considered up to α-conversion, it makes sense to write them as $a = \lambda x_1 \dots x_n.x\, a_1 \cdots a_m$ and $b = \lambda x_1 \dots x_{n'}.y\, b_1 \cdots b_{m'}$ for some $n, m, n', m' \in \mathbb{N}$. We write $a \sim b$ if and only if $x = y$, $n - m = n' - m'$ and either x is free or $x = x_j$ for a certain $j \in \big\{1, \dots, \min(n, n')\big\}$. When $a \sim b$ we make use of the following notation:

- if $m' \leqslant m$ we set $b_{m'+i} := x_{n'+i}$ for all $i \in \{1, \dots, m-m'\}$;
- if $m \leqslant m'$ we set $a_{m+i} := x_{m+i}$ for all $i \in \{1, \dots, m'-m\}$.

Example 1.3.6. We have $\lambda xyz.y\, y\, \bot \sim \lambda xy.y\, \mathbf{I}$ since they have the same head variable y and $3-2 = 2-1$. And if we call $\lambda x_1x_2x_3.x_2\, a_1\, a_2 := \lambda xyz.y\, y\, \bot$ and $\lambda x_1x_2.x_2\, b_1 := \lambda xy.y\, \mathbf{I}$ then according to the notation set above $\lambda x_1x_2x_3.x_2\, b_1 b_2 := \lambda xyz.y\, \mathbf{I}\, z$.

Notice that $a \sim b$ is not an inductive notion, in the sense that in Definition 1.3.5 nothing is asked concerning the relation between the a_i's and the b_i's. We are going to do that in the following definition, which is moreover an *order* refinement of the idea.

Definition 1.3.7. Let $a, b \in \mathcal{N}$. We write $a \precsim b$ if and only if either $a = \bot$, or $a \sim b$ and for all $i \in \{1, \dots, \max(m, m')\}$ we have $a_i \precsim b_i$.

Definition 1.3.8. Let $a \in \mathcal{N}$ and $T \in \Lambda^{\mathcal{B}}$. We write $a \precsim T$ if and only if there exists $b \in T^*$ such that $a \precsim b$.

One can easily check that Definition 1.3.8 is consistent with Definition 1.3.7 whenever $T = b \in \mathcal{N}$. Finally, we can reformulate Proposition 1.3.4 as follows.

Proposition 1.3.9. *Let* $M, N \in \Lambda$. *Then* $M \sqsubseteq_{\mathcal{H}^*} N$ *if and only if* $a \precsim BT(N)$ *for all* $a \in BT(M)^*$.

Morris's Observability

Taking hnf's as observables is the most common choice. Such observational equivalence is by far the most investigated in the literature [Hyl76, Wad76, GFH99, Mano9, Bre16], at least concerning the untyped setting (for typed versions of the system, most notably PCF [Mil77, Plo77, HO00, AJM00], one has some notion of *values* as observables). But in this thesis we focus on a different observability.

Let $\mathcal{O}$ be the set of λ-terms in β-nf. Then $\sqsubseteq^{\mathcal{O}}$ is called *Morris's observational preorder* and denoted by $\sqsubseteq_{\mathcal{H}^+}$. More explicitly

$$M \sqsubseteq_{\mathcal{H}^+} N \quad \text{if and only if} \quad \text{for all } C[-] \ \Big(C[M] \text{ is } \beta\text{-normalizable } \Rightarrow C[N] \text{ is } \beta\text{-normalizable}\Big).$$

The induced equivalence $=^{\mathcal{O}}$ is called *Morris's observational equivalence* and denoted by $=_{\mathcal{H}^+}$.

It is easy to check that $\sqsubseteq_{\mathcal{H}^+}$ and $\mathcal{H}^+$ are consistent, extensional and sensible. Notice also that in the characterization of $\sqsubseteq_{\mathcal{H}^+}$ here above one can replace the β-reduction with the $\beta\eta$-reduction, namely

$$M \sqsubseteq_{\mathcal{H}^+} N \quad \text{if and only if} \quad \text{for all } C[-] \ \Big(C[M] \text{ is } \beta\eta\text{-normalizable } \Rightarrow C[N] \text{ is } \beta\eta\text{-normalizable}\Big).$$

Even in this case we can focus our attention on head contexts.

Lemma 1.3.10 (Context Lemma)**.** *Let* $M, N \in \Lambda$. *Then* $M \sqsubseteq_{\mathcal{H}^+} N$ *if and only if for all head context* $E[-]$ *whenever* $E[M]$ *has a* β*-nf then* $E[N]$ *has a* β*-nf.*

Corollary 1.3.11. *Let* $M, N \in \Lambda^0$. *Then* $M \sqsubseteq_{\mathcal{H}^*} N$ *if and only if for all applicative and closed context* $\vec{X}$ *whenever* $M\vec{X}$ *has a* β*-nf then* $N\vec{X}$ *has a* β*-nf.*

In [Bar84, Ch.16] the λ-theory $\mathcal{H}^+$ is denoted by $\mathcal{T}_{NF}$ and called *Morris's extensional theory*. That name aims to distinguish $\mathcal{H}^+$ from the original *Morris's theory* [Mor68], which is defined likewise but further requires C[M] and C[N] to reduce to the *same* β-nf. This original version of $\mathcal{H}^+$ is clearly not extensional (it does not equate y and $\lambda x.yx$). As far as we know, it was the very first instance of an observational equivalence ever introduced in the literature. Anyway, it will not be considered in this thesis.

1.4 Extensional Böhm trees

Studying observational equivalences is not an easy task, especially because of the quantification over all possible contexts appearing in their definition. This is the reason why since the 70's several attempts have been made to find alternative characterizations of $\mathcal{H}^*$ and $\mathcal{H}^+$. We recall in this section those that use Böhm-like trees.

In the paper [Böh68] from 1968 Böhm presented his famous theorem (see [Bar84, § 10.4]): if two β-normalizable λ-terms M and N have different βη-nf's then they can be separated, i.e. there exists a context $C[-]$ such that $C[M]$ has a β-nf whereas $C[N]$ does not. On the other hand, if M and N have β-nf's that differ only for some η-conversions then they cannot be separated (take for instance $M = x$ and $N = \lambda y.xy$). Stated in contrapositive form, Böhm's theorem says that *as far as we consider only* β*-normalizable terms* $\mathcal{H}^+$ corresponds to $\boldsymbol{\lambda}\eta$, and in fact also to $\mathcal{B}\eta$ (for such terms being β-convertible means having the same Böhm tree). Can we extend this approach — looking at Böhm trees up to η-conversion — *to all* λ*-terms* in order to fully characterize $\mathcal{H}^+$ or other observational equivalences? The answer is yes, provided we have some appropriate notion of η-reduction for generic Böhm-like trees.

There exist two distinct such notions. Both of them allow to reduce at once infinitely many η-redexes, each occurring at a certain position on the tree. This is reasonable, as Böhm-like trees are infinitary objects. The difference lies in what one considers to be an η-redex (in a certain position on the tree). Take for instance $\mathrm{BT}(\mathbf{J})$ in Figure 1. It looks like an *infinitely deep* η-expansion of $\lambda x.x$. Shall we allow $\mathrm{BT}(\mathbf{J})$ to η-reduce to $\mathbf{I}$? Or must we only reduce *finitely deep* η-extensions? Both options are of interest: the first approach provides a model of $\mathcal{H}^*$; the more restrictive choice gives a model of $\mathcal{H}^+$.

We start by defining the reduction that takes infinitely deep η-expansions into account. The definition is the same as [Bar84, Definition 10.2.10], where it is denoted by $B \geqslant_\eta A$. As a matter of fact, what we are going to define is not a reduction rule in a formal rewriting system, but rather an order on $\Lambda^{\mathcal{B}}$. Nevertheless, we prefer to denote it by $B \twoheadrightarrow_\eta A$, just to stay close to the underlying intuition. In fact $\twoheadrightarrow_\eta$ can be defined as an actual reduction in an infinitary rewriting system, as recently shown by Severi and de Vries in [SdV16].

Remember that, given a Böhm-like tree $A : \mathbb{N}^* \rightharpoonup \Sigma \times \mathbb{N}$, for convenience we use $A(\sigma)$ instead of $(\pi_1 \circ A)(\sigma)$. For instance, if A is the tree $\perp$, i.e. formally $\mathrm{dom}(A) = \{\varepsilon\}$ and $A(\varepsilon) = (\perp, 0)$, then with an abuse of language we directly write $A(\varepsilon) = \perp$.

Definition 1.4.1. Let $A \in \Lambda^{\mathcal{B}}$ and $T \in \mathbb{T}$. We say that T *extends* A if and only if $\mathrm{dom}(A) \subseteq \mathrm{dom}(T)$, and whenever $A(\varphi) = \perp$ then $T(\varphi) = 0$ (namely φ is a terminal node in T).

Definition 1.4.2. Let $A \in \Lambda^{\mathcal{B}}$ and $T \in \mathbb{T}$ such that T extends A. The Böhm-like tree $(A\,;T)$ is defined on $\varphi \in \mathbb{N}^*$ as follows.

1. If $\varphi \in \mathrm{dom}(A)$ and $A(\varphi) = \bot$ (hence $T(\varphi) = 0$ by Definition 1.4.1), we set
$$(A\,;T)(\varphi) \;:=\; \bot\,.$$
2. If $\varphi \in \mathrm{dom}(A)$, $A(\varphi) = \lambda\vec{x}.x$ and the number of children of the node φ in A is m
$$(A\,;T)(\varphi) \;:=\; \lambda\vec{x}\,y_0^{\varphi}\ldots y_{T(\varphi)-m-1}^{\varphi}.x\,.$$
Notice that in particular if $T(\varphi) = m$ then $(A\,;T)(\varphi) := A(\varphi)$.
3. If $\varphi = \varphi'.m+i \in \mathrm{dom}(T) - \mathrm{dom}(A)$, $\varphi' \in \mathrm{dom}(A)$ and m is the number of children of the node φ' in A, then we set
$$(A\,;T)(\varphi) \;:=\; \lambda y_0^{\varphi}\ldots y_{T(\varphi)-1}^{\varphi}.y_i^{\varphi'}\,.$$
4. If $\varphi = \varphi'.i \in \mathrm{dom}(T) - \mathrm{dom}(A)$ and $\varphi' \notin \mathrm{dom}(A)$ we set
$$(A\,;T)(\varphi) \;:=\; \lambda y_0^{\varphi}\ldots y_{T(\varphi)-1}^{\varphi}.y_i^{\varphi'}\,.$$
5. If $\varphi \notin \mathrm{dom}(T)$ then also $\varphi \notin \mathrm{dom}(A\,;T)$, i.e. $(A\,;T)$ is undefined.

Notice that 1-5 above are all the cases that one must take into account, since T extends A.

Remark 1.4.3. Let $A \in \Lambda^{\mathcal{B}}$ and $T \in \mathbb{T}$ such that T extends A. Then $\lceil (A\,;T) \rceil = T$.

Definition 1.4.4. Let $A, B \in \Lambda^{\mathcal{B}}$. We say that B is an *infinite* η*-expansion* of A, denoted by $B \twoheadrightarrow_{\eta} A$, if and only if there exists $S \in \mathbb{T}$ such that $B = (A\,;S)$. Notice that in such a case in fact $\lceil B \rceil = S$, by Remark 1.4.3.

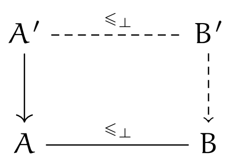
Example 1.4.5. Let $T : \mathbb{N}^* \rightharpoonup \mathbb{N}$ send $\langle \overbrace{0, 0, \ldots, 0}^{n \text{ times}} \rangle \mapsto 1$ for every $n \in \mathbb{N}$. Then $T \in \mathbb{T}$ and T extends $\mathbf{I}$. It easy to realize that $\mathrm{BT}(\mathbf{J}) = (\mathbf{I}\,;T)$. Hence $\mathrm{BT}(\mathbf{J}) \twoheadrightarrow_{\eta} \mathbf{I}$.

We will make use of the following two lemmas.

Lemma 1.4.6. *Let* $\lambda x_1 \ldots x_n.x\,B_1 \cdots B_m \twoheadrightarrow_{\eta} \lambda x_1 \ldots x_{n'}.x\,A_1 \cdots A_{m'}$. *Then* $B_i \twoheadrightarrow_{\eta} A_i$ *for all* $i \in \{1, \ldots, m'\}$ *and* $B_{m'+i} \twoheadrightarrow_{\eta} x_{n'+i}$ *for all* $i \in \{1, \ldots, m - m'\}$.

Proof. Straightforward. □

Lemma 1.4.7 ([Bar84, Lemma 10.2.14 (ii)]). *Let* $A, A', B \in \Lambda^{\mathcal{B}}$ *such that* $A' \twoheadrightarrow_{\eta} A \leqslant_{\bot} B$. *Then there is* $B' \in \Lambda^{\mathcal{B}}$ *such that* $A' \leqslant_{\bot} B' \twoheadrightarrow_{\eta} B$. *Diagrammatically*



where $\rightarrow$ *denotes* $\twoheadrightarrow_{\eta}$ *and the dashed lines are the ones given by thesis.*

As already mentioned $\twoheadrightarrow_{\eta}$ provides a model of $\mathcal{H}^*$, and in fact even of $\sqsubseteq_{\mathcal{H}^*}$.

Theorem 1.4.8 ([Bar84, Theorem 19.2.9]). *Let* $M, N \in \Lambda$.

1. $M \sqsubseteq_{\mathcal{H}^*} N$ *if and only if there are* $A, B \in \Lambda^{\mathcal{B}}$ *such that* $BT(M) \,{}_{\eta}\!\twoheadleftarrow A \leqslant_{\perp} B \twoheadrightarrow_{\eta} BT(N)$.
2. $M =_{\mathcal{H}^*} N$ *if and only if there is* $A \in \Lambda^{\mathcal{B}}$ *such that* $BT(M) \,{}_{\eta}\!\twoheadleftarrow A \twoheadrightarrow_{\eta} BT(N)$.

The notion that performs only finitely deep η-reductions can now be defined as a specific case of $B \twoheadrightarrow_{\eta} A$.

Definition 1.4.9. Let $A, B \in \Lambda^{\mathcal{B}}$. We say that B is a *finitely deep (infinite)* η*-expansion* of A, denoted by $B \twoheadrightarrow_{\eta}^{\text{fin}} A$, if and only if $B \twoheadrightarrow_{\eta} A$ and for all $\varphi \in \mathrm{dom}(A)$ whenever $\{ \psi \in \mathrm{dom}(A) \mid \psi \succ \varphi \} = \emptyset$ then the set $\{ \psi \in \mathrm{dom}(B) \mid \psi \succ \varphi \}$ is finite.

One can prove an analogue of Theorem 1.4.8(2) for $\mathcal{H}^+$, that is $M =_{\mathcal{H}^+} N$ if and only if there is $A \in \Lambda^{\mathcal{B}}$ such that $BT(M) \,{}^{\text{fin}}_{\eta}\!\twoheadleftarrow A \twoheadrightarrow_{\eta}^{\text{fin}} BT(N)$. The analogue for $\sqsubseteq_{\mathcal{H}^+}$ is not true.

Example 1.4.10. In Figure 2 we have two Böhm trees. The symbol $\eta^n(x)$ in a certain position σ on $BT(P)$ denotes the fact that the subtree $BT(P)_{\sigma}$ is an η-expansion of depth n, namely $\lambda x_1.x\, BT(P)_{\sigma\langle 0\rangle}(\lambda x_2.x_1(\lambda x_3.x_2(\cdots(\lambda x_n.x_{n-1}x_n)\cdots)))$. The λ-terms P and Q exist by [Bar84, Theorem 10.1.23]. We have $BT(P) \twoheadrightarrow_{\eta}^{\text{fin}} BT(Q)$. So $P =_{\mathcal{H}^+} Q$.

Anyway, for $\sqsubseteq_{\mathcal{H}^+}$ and $\mathcal{H}^+$ we will use a different characterization by Hyland and Lévy. It has a more set-theoretical flavor, since it makes use of finite approximants.

Definition 1.4.11. Let $M \in \mathbb{N}$. The *extensional Böhm tree* of M is the set

$$BT^e(M) \;:=\; \bigcup_{M' \twoheadrightarrow_{\eta} M} \mathrm{nf}_{\eta} BT(M')^* \;=\; \left\{ \mathrm{nf}_{\eta}(a) \mid a \in BT(M')^* \text{ and } M' \twoheadrightarrow_{\eta} M \right\}.$$

Notice that, despite the name, $BT^e(M)$ is not actually a Böhm-like tree.

Theorem 1.4.12 ([Hyl75]). *Let* $M, N \in \Lambda$.

1. $M \sqsubseteq_{\mathcal{H}^+} N$ *if and only if* $BT^e(M) \subseteq BT^e(N)$.
2. $M =_{\mathcal{H}^+} N$ *if and only if* $BT^e(M) = BT^e(N)$.

A sketch of the proof of Theorem 1.4.12 is in Hyland's original paper [Hyl75]. The reader may want to see [RP04] for a cleaner proof.

Example 1.4.13. It is easy to realize that $BT^e(\mathbf{J}) \subseteq BT^e(\mathbf{I})$. So by Theorem 1.4.12(1) we get $\mathbf{J} \sqsubseteq_{\mathcal{H}^+} \mathbf{I}$. On the other hand, $BT^e(\mathbf{I}) \not\subseteq BT^e(\mathbf{J})$, since $\mathbf{I} \in BT^e(\mathbf{I}) - BT^e(\mathbf{J})$. As a matter of fact $\mathbf{I} \not\sqsubseteq_{\mathcal{H}^+} \mathbf{J}$ (taken $C[\,] := [\,]$ we have $C[\mathbf{I}] = \mathbf{I}$ with a β-nf and $C[\mathbf{J}] = \mathbf{J}$ without).

There is a way to obtain a characterization of $\sqsubseteq_{\mathcal{H}^*}$ and $\mathcal{H}^*$ starting from $BT^e(-)$.

Definition 1.4.14. Let $a, b \in \mathcal{N}$. Then $a \leqslant^e b$ if and only if there exist $a', b' \in \mathcal{N}$ such that $a \,{}_{\eta}\!\twoheadleftarrow a' \leqslant_{\perp} b' \twoheadrightarrow_{\eta} b$.

Definition 1.4.15. The *Nakajima tree* of $M \in \Lambda$ is $NT(M) := BT^e(M) \cup \left\{ \sup_{\leqslant^e} BT^e(M) \right\}$.

Theorem 1.4.16 ([Nak75]). *Let* $M, N \in \Lambda$.

1. $M \sqsubseteq_{\mathcal{H}^*} N$ *if and only if* $\mathrm{NT}(M) \subseteq \mathrm{NT}(N)$.
2. $M =_{\mathcal{H}^*} N$ *if and only if* $\mathrm{NT}(M) = \mathrm{NT}(N)$.

Example 1.4.17. One can prove that $\mathbf{I}$ is the only element of $\mathrm{BT}^e(\mathbf{I}) - \mathrm{BT}^e(\mathbf{J})$. Moreover $\sup_{\leqslant^e} \mathrm{BT}^e(\mathbf{J}) = \mathbf{I} = \sup_{\leqslant^e} \mathrm{BT}^e(\mathbf{I})$. Hence $\mathrm{NT}(\mathbf{J}) = \mathrm{BT}^e(\mathbf{J}) \cup \{\mathbf{I}\} = \mathrm{BT}^e(\mathbf{I}) = \mathrm{NT}(\mathbf{I})$.

Definition 1.4.14 is not the original version of Nakajima's trees [Nak75], but rather a reformulation in terms of finite approximants provided by Lévy [Lev76]. Anyway, we will not use this model of $\mathcal{H}^*$ in the thesis.

There is yet another way to define the extensional Böhm trees, as given in [vBBDCdV02]. This alternative definition is not exactly equivalent to $\mathrm{BT}^e(-)$. In fact, it only provides a model of $\mathcal{H}^+$, not of $\sqsubseteq_{\mathcal{H}^+}$. It is based on a coinductive definition of what can be seen as the normal form of a Böhm-like tree w.r.t. the notion $\twoheadrightarrow_\eta^{\mathrm{fin}}$ discussed above.

Definition 1.4.18. Let $A \in \Lambda^{\mathcal{B}}$. The η-*normal form of* A, denoted by $\eta(A)$, is defined coinductively as follows: $\eta(\bot) = \bot$ and

$$\eta\big(\lambda x_1 \dots x_n.y\, A_1 \cdots A_m\big) = \begin{cases} \eta\big(\lambda x_1 \dots x_n.y\, A_1 \cdots A_{m-1}\big) & \text{if (\#) below holds} \\ \lambda x_1 \dots x_n.y\, \eta(A_1) \cdots \eta(A_m) & \text{otherwise} \end{cases}$$

where Condition (#) is: $x_n \notin \mathrm{fv}(y\, A_1 \cdots A_{m-1})$, $A_m \in \mathcal{N}$ (i.e. A_m is finite) and $A_m \twoheadrightarrow_\eta x_n$.

Definition 1.4.19. Let $M \in \Lambda$. The *Böhm* η-*tree of* M is $\mathrm{BT}^\eta(M) := \eta\big(\mathrm{BT}(M)\big)$.

Example 1.4.20. Examples of Böhm η-trees are: $\mathrm{BT}^\eta(\mathbf{J}) = \mathrm{BT}(\mathbf{J})$, $\mathrm{BT}^\eta(\lambda y.xyy) = \lambda y.xyy$, $\mathrm{BT}^\eta(\lambda x y_1 y_2.x(\lambda z_1.y_1(\lambda z_2.z_1(\lambda z_3.z_2 z_3))y_2) = \mathrm{BT}^\eta(\mathbf{I}) = \mathbf{I}$, $\mathrm{BT}^\eta(\lambda y.x\bot y) = x\bot$.

Theorem 1.4.21 ([vBBDCdV02]). *For* $M, N \in \Lambda$, $M =_{\mathcal{H}^+} N$ *if and only if* $\mathrm{BT}^\eta(M) = \mathrm{BT}^\eta(N)$.

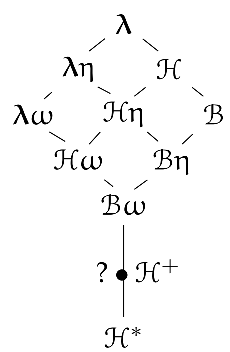
On the other hand, $\mathrm{BT}^\eta(M) \leqslant_\bot \mathrm{BT}^\eta(N)$ is not equivalent to $\mathrm{BT}^e(M) \subseteq \mathrm{BT}^e(N)$ (i.e. to $M \sqsubseteq_{\mathcal{H}^+} N$). E.g. $\mathrm{BT}^e(x\bot) \subseteq \mathrm{BT}^e(\lambda y.xyy)$ but $\mathrm{BT}^\eta(x\bot) = x\bot \not\leqslant_\bot \lambda y.xyy = \mathrm{BT}^\eta(\lambda y.xyy)$.

1.5 THE LATTICE OF λ-THEORIES

The set of all λ-theories, ordered by inclusion, forms a complete lattice. The meet of two λ-theories is their intersection. The join is the least λ-theory that includes their union. The minimum element of the lattice is $\boldsymbol{\lambda}$, whereas the maximum is the inconsistent λ-theory. The lattice of λ-theories is still largely unexplored, see [LS04] as a reference.

The picture on the right, where $\mathcal{T}$ is above $\mathcal{T}'$ if $\mathcal{T} \subset \mathcal{T}'$, is taken from [Bar84, Theorem 17.4.16] and shows some facts about the λ-theories under consideration.

The counterexample showing that $\lambda\eta \subset \lambda\omega$ is based on Plotkin's terms [Bar84, Definition 17.3.26]. Since these terms are unsolvable, they become useless when considering sensible λ-theories.



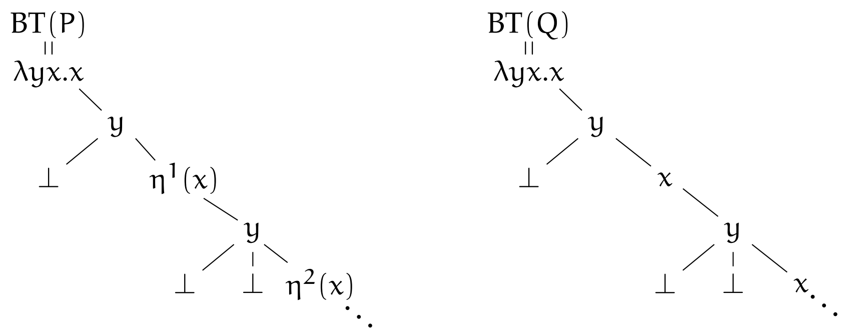


Figure 2: Example of finitely deep η-reduction: $\mathrm{BT}(P) \twoheadrightarrow_{\eta}^{\mathrm{fin}} \mathrm{BT}(Q)$

Take the two λ-terms P, Q in Example 1.4.10, whose Böhm trees are depicted in Figure 2. As we know, they satisfy $P =_{\mathcal{H}^+} Q$. The existence of P and Q entails that $\mathcal{B}\eta \subset \mathcal{H}^+$. Indeed, when $M \to_\eta N$ then $\mathrm{BT}(M)$ can be obtained from $\mathrm{BT}(N)$ by performing an η-expansion of at most depth 1 at every position (see [Bar84, Lemma 16.4.3]). As a consequence, $M =_{\mathcal{B}\eta} N$ entails that $\mathrm{BT}(M)$ can be obtained from $\mathrm{BT}(N)$ by performing possibly infinite many η-expansions, but with a bound on the depth of all of them. Clearly this not the case of P and Q, since at every level $2n$ of $\mathrm{BT}(P)$ there is an η-expansion of depth n. So $P \neq_{\mathcal{B}\eta} Q$.

Perhaps more surprisingly, P and Q can also be used to prove that $\mathcal{B}\eta \subset \mathcal{B}\omega$, since $P =_{\mathcal{B}\omega} Q$ holds. The argument is due to Barendregt, see [Bar84, Lemma 16.4.4]. Recall the following basic fact: for every $M \in \Lambda^0$ there exists $k \geqslant 0$ such that $M\Omega \cdots \Omega$, where the application to Ω is repeated k times, becomes unsolvable (see [Bar84, Lemma 17.4.4]). By inspecting Figure 2, we notice that in $\mathrm{BT}(P)$ the variable y is applied to an increasing number of Ω's (represented by $\perp$). So, when substituting some $M \in \Lambda^0$ for y in $\mathrm{BT}(Py)$, there will be a level k of the tree where $M\Omega \cdots \Omega$ become $\perp$, thus cutting $\mathrm{BT}(PM)$ at level k. The same reasoning can be done for $\mathrm{BT}(QM)$. Therefore $\mathrm{BT}(PM)$ and $\mathrm{BT}(QM)$ only differ because of finitely many η-expansions, which gives $PM =_{\mathcal{B}\eta} QM$. Since $\mathcal{B}\eta \subseteq \mathcal{B}\omega$, we have $PM =_{\mathcal{B}\omega} QM$. As this is true for all $M \in \Lambda^0$, by the ω-rule holding in $\mathcal{B}\omega$ we get $P =_{\mathcal{B}\omega} Q$.

About the relation between $\mathcal{B}\omega$ and $\mathcal{H}^+$, Sallé conjectured that $\mathcal{B}\omega \subset \mathcal{H}^+$ [Bar84, §17.4].

The fact that $\mathcal{H}^*$ satisfies the ω-rule is clearly a consequence of its maximality. However, there are several direct proofs: see [Bar84, §17.2] for a syntactic demonstration and [Wad76] for a semantic one. The longstanding open question whether $\mathcal{H}^+$ satisfies the ω-rule will be answered positively in Theorem 5.4.3.

1.6 LINEAR RESOURCE CALCULUS AND TAYLOR EXPANSION

All models of the untyped λ-calculus studied in this thesis are *resource-sensitive*, in the sense that they represent explicitly the consumption of resources by λ-terms along the process of β-reduction. For their technical development we derive much benefit from a reformulation of the untyped λ-calculus where the resource-sensitiveness is integrated in the syntax from the beginning. It is the *linear* fragment of Ehrhard's *resource calculus* [ER06a], handled here with the syntax proposed by Tranquilli in [Tra11].

The linear resource calculus

The set Λ^r of *(linear) resource terms* and the set Λ^b of *bags* are defined by the grammars

$$\Lambda^r: \quad s,t ::= x \mid \lambda x.t \mid t\,\mathbf{b} \qquad\qquad \Lambda^b: \quad \mathbf{b} ::= [s_1,\dots,s_n] \quad \text{where } n \geqslant 0. \tag{3}$$

Notice that in applications $t\mathbf{b}$ resource terms are in functional position, whereas bags are in argument position and represent unordered lists of resource terms. Intuitively, in a term $t[s_1,\dots,s_n]$ each s_i is a linear resource, meaning that t cannot duplicate nor erase it. We will deal with bags as if they were multisets presented in multiplicative notation: $b_1 \cdot b_2$ is the multiset union of b_1 and b_2. Clearly, the neutral element of this multiplication is the empty multiset $[\,]$. We write $[s^k]$ for the bag $[s,\dots,s]$ containing k copies of s.

The α-equivalence and the set $\mathrm{fv}(t)$ of free variables of t are defined as done for the ordinary λ-calculus in § 1.2. Resource terms and bags are considered up to α-equivalence, and we also apply the Variable Convention seen in § 1.2.

As a syntactic sugar, we extend all the constructors of the grammars (3) as pointwise operations on (possibly infinite) sets of resource terms or bags. That is, given $\mathbb{T} \subseteq \Lambda^r$ and $\mathbb{B}, \mathbb{B}' \subseteq \Lambda^b$ we use the following notations: $\lambda x.\mathbb{T} := \{\lambda x.t \mid t \in \mathbb{T}\}$, $\mathbb{T}\mathbb{B} := \{t\mathbf{b} \mid t \in \mathbb{T}, \mathbf{b} \in \mathbb{B}\}$, $[\mathbb{T}] := \{[t] \mid t \in \mathbb{T}\}$ and $\mathbb{B} \cdot \mathbb{B}' := \{\mathbf{b} \cdot \mathbf{b}' \mid \mathbf{b} \in \mathbb{B},\ \mathbf{b}' \in \mathbb{B}'\}$. For convenience, we also write $\mathbb{T}\mathbf{b}$ for $\mathbb{T}\{\mathbf{b}\}$ and $\mathbb{B} \cdot \mathbf{b}$ for $\mathbb{B} \cdot \{\mathbf{b}\}$. Observe that $\lambda x.\emptyset = \emptyset$, $t\emptyset = \emptyset$, $\emptyset\mathbf{b} = \emptyset$, $[\emptyset] = \emptyset$ and $\emptyset \cdot \mathbf{b} = \emptyset$. So the empty set $\emptyset$ annihilates any resource term or bag.

Given a relation $\to_R \subseteq \Lambda^r \times \mathcal{P}_f(\Lambda^r)$ its *context closure* is the least relation in $\mathcal{P}_f(\Lambda^r) \times \mathcal{P}_f(\Lambda^r)$ such that, when $t \to_R \mathbb{T}$, we have

$$\lambda x.t \to_R \lambda x.\mathbb{T}, \quad t b \to_R \mathbb{T} b, \quad s([t] \cdot b) \to_R s([\mathbb{T}] \cdot b), \quad \{t\} \cup \mathbb{S} \to_R \mathbb{T} \cup \mathbb{S}.$$

We say that $t \in \Lambda^r$ *is in* R*-normal form* if there is no $\mathbb{T}$ such that $t \to_R \mathbb{T}$. When $\to_R$ is confluent, $\mathrm{nf}_R(t) \in \mathcal{P}_f(\Lambda^r)$ denotes the unique R-normal form of t, if it exists.

The *degree of* x *in* t, denoted by $\deg_x(t)$, is the number of free occurrences of x in t. A β*-redex* is a resource term of the form $(\lambda x.t)[s_1,\dots,s_k]$ and its *contractum* is a finite set of resource terms: when $\deg_x(t) = k$, it is the set of all possible resource terms obtained by linearly replacing each free occurrence of x in t by exactly one of the s_i's; otherwise, when $\deg_x(t) \neq k$, it is just $\emptyset$. Formally, we define $\to_\beta$ as the context closure of:

$$(\lambda x.t)[s_1,\dots,s_k] \to_\beta \begin{cases} \left\{t\left\{s_{p(1)}/x_1,\dots,s_{p(k)}/x_k\right\} \mid p \in \mathfrak{S}_k\right\} & \text{if } \deg_x(t) = k, \\ \emptyset & \text{otherwise} \end{cases}$$

where $\mathfrak{S}_k$ is the group of permutations of $\{1,\dots,k\}$ and $x_1,\dots,x_n$ is a fixed arbitrary enumeration of the free occurrences of x in t. Note that the β-reduction is strongly normalizing on $\mathcal{P}_f(\Lambda^r)$, since whenever $t \to_\beta \mathbb{T}$ the size of t is strictly bigger than the size of each resource term in $\mathbb{T}$. Moreover, the β-reduction is weakly confluent, and therefore confluent by Newman's lemma.

There is no evident notion of η-reduction on $\mathcal{P}_f(\Lambda^r)$. We will deal with this issue in § 4.5.

Taylor expansion

The *Taylor expansion* of a λ-term, as defined in [ER03, ER08], is a translation developing every $M \in \Lambda$ as an infinite series of resource applications with rational coefficients. For our purpose it is enough to consider a simplified version $\mathcal{T}(-) : \Lambda \to \mathcal{P}(\Lambda^r)$ corresponding to the support of the actual Taylor expansion, namely the set of those resource terms appearing in the series with a non-zero coefficient. In other words, here a Taylor expansion will be a possibly infinite set of resource λ-terms.

Definition 1.6.1. Let $M \in \Lambda$. The *Taylor expansion of* M is a set $\mathcal{T}(M) \subseteq \Lambda^r$ defined by

$$\mathcal{T}(x) \;:=\; \{x\}, \qquad \mathcal{T}(\lambda x.M) \;:=\; \lambda x.\mathcal{T}(M), \qquad \mathcal{T}(MN) \;:=\; \mathcal{T}(M)\,\mathcal{M}_f\big(\mathcal{T}(N)\big).$$

The Taylor expansion is extended to finite approximants in $\mathcal{N}$ by adding the clause $\mathcal{T}(\bot) := \emptyset$ and to Böhm-like trees A by setting $\mathcal{T}(A) := \bigcup_{a\in A^*}\mathcal{T}(a)$.

Examples 1.6.2. Here are the Taylor expansions of some λ-terms:

$$\begin{array}{rcl}
\mathcal{T}(\mathbf{I}) & = & \{\mathbf{I}\} \\
\mathcal{T}(\lambda x.xx) & = & \{\lambda x.x[x^n] \mid n \in \mathbb{N}\}, \\
\mathcal{T}(\lambda y.xyy) & = & \{\lambda y.x[y^n][y^k] \mid n,k \in \mathbb{N}\} \\
\mathcal{T}(\boldsymbol{\Omega}) & = & \{(\lambda x.x[x^{n_0}])[\lambda x.x[x^{n_1}],\dots,\lambda x.x[x^{n_k}]] \mid k, n_0,\dots,n_k \in \mathbb{N}\}, \\
\mathcal{T}(\mathbf{Y}) & = & \Big\{\lambda f.\big(\lambda x.f[x[x^{n_1}],\dots,x[x^{n_k}]]\big)\Big[\lambda x.f[x[x^{n_{11}}],\dots,x[x^{n_{1k_1}}]],\dots, \\
 & & \qquad \lambda x.f[x[x^{n_{h1}}],\dots,x[x^{n_{hk_h}}]]\Big] \mid k, n_i, h, n_{ij} \in \mathbb{N}\Big\}, \\
\mathcal{T}(\mathbf{J}) & = & \Big\{t\Big[\lambda zxy.x[z[y^{n_{11}}],\dots,z[y^{n_{1k_1}}]],\dots, \\
 & & \quad \lambda zxy.x[z[y^{n_{h1}}],\dots,z[y^{n_{hk_h}}]]\Big] \mid t \in \mathcal{T}(\boldsymbol{\Theta}),\ h, k_i, n_{i,j} \in \mathbb{N}\Big\}.
\end{array}$$

From the examples above it is clear that if a λ-term M has a β-redex, then there are resource terms $t \in \mathcal{T}(M)$ having β-redexes too. However each t has a unique β-nf and we can always compute $\mathrm{nf}_\beta(\mathcal{T}(M)) = \bigcup_{t\in\mathcal{T}(M)}\mathrm{nf}_\beta(t)$. For instance: $\mathcal{T}(\mathbf{I})$, $\mathcal{T}(\lambda x.xx)$ and $\mathcal{T}(\lambda y.xyy)$ are already β-normal, whereas $\mathrm{nf}_\beta\big(\mathcal{T}(\boldsymbol{\Omega})\big) = \emptyset$.

We will make use of the following lemma, whose proof is simple.

Lemma 1.6.3. *Let* $a \in \mathcal{N}$ *and* $M \in \Lambda$. *Then* $\mathcal{T}(a) \subseteq \mathcal{T}(\mathrm{BT}(M))$ *entails* $a \in \mathrm{BT}(M)^*$.

There is a strong relationship between the Böhm tree of a λ-term and its Taylor expansion, as clarified by the following theorem.

Theorem 1.6.4 ([ER06a])**.** *Let* $M, N \in \Lambda$. *Then* $\mathrm{nf}_\beta\big(\mathcal{T}(M)\big) = \mathcal{T}\big(\mathrm{BT}(M)\big)$.

Corollary 1.6.5. *Let* $M, N \in \Lambda$. *Then* $\mathrm{BT}(M) = \mathrm{BT}(N)$ *if and only if* $\mathrm{nf}_\beta\big(\mathcal{T}(M)\big) = \mathrm{nf}_\beta\big(\mathcal{T}(N)\big)$.

Theorem 1.6.4 will play an important role in this thesis. It can be read as a kind of commutation: performing the Taylor expansion of $M \in \Lambda$ and then β-normalizing is equivalent to β-normalizing in the first place (something represented by taking the Böhm tree) and then doing the Taylor expansion. Here some examples of applications of Theorem 1.6.4:

$$\begin{array}{rcl}
\mathrm{nf}_\beta\big(\mathcal{T}(\mathbf{Y})\big) & = & \big\{\lambda f.f[\,],\ \lambda f.f\big[(f[\,])^n\big],\ \lambda f.f\big[f\big[(f[\,])^{n_1}\big],\dots,f\big[(f[\,])^{n_k}\big]\big],\dots\big\} \\
\mathrm{nf}_\beta\big(\mathcal{T}(\mathbf{J})\big) & = & \big\{\lambda xz_0.x[\,],\ \lambda xz_0.x[(\lambda z_1.z_0[\,])^n],\dots\big\}
\end{array}$$

The (full) resource calculus extends the one presented here above with a more general notion of bag $[s_1, \dots, s_n, t_1^!, \dots, t_m^!]$, where resources of the form $t^!$ are considered as non-linear (duplicable or erasable when substituted), and a more general β-reduction accomodating this idea. The calculus has been further studied in [PT09, PRDR10, MP11]. It shares some similarities with Boudol's λ-calculus with multiplicities [Bou93, BCL99]. The syntax of the resource calculus can be reformulated in a way very close to the common differentiation in ordinary calculus. In such a case, it goes by the name of *differential λ-calculus*, as developed by Ehrhard and Regnier in [ER03, ER06b, ER08]. (See for instance [Man12] for the formal translation between the resource calculus and the differential λ-calculus.)

2 RELATIONAL GRAPH MODELS

INTRODUCTION

In Scott's continuous semantics [Sco72] programs are interpreted as functions. Also, the interpretation of data types relies on some order. The kind of denotational semantics that we use here, called *relational semantics*, does not have neither of these features.

Relational semantics interprets types as *sets*, and λ-terms as *binary relations*. A program p of type $A \to B$ is interpreted as a relation $[\![p]\!]$ between finite multisets of A and elements of B (by convenience we are calling A the set that interprets a given type A). Intuitively

$$([a_1, \ldots, a_n], b) \quad \in \quad [\![p]\!] \tag{4}$$

means that one among the possible executions of p can produce a piece of output b by consuming *exactly* the resources $a_1, \ldots, a_n$. Let us point out two relevant features of this semantics.

1. Finite multisets provide a *resource-sensitive* interpretation of inputs of programs: if m is the multiplicity of a_i in $[a_1, \ldots, a_n]$, then the piece of information a_i is used m times in the specific execution represented by (4).

2. Such semantics carries some notion of *non-determinism*, since programs are interpreted as relations rather than functions. According to (4) the resources $a_1, \ldots, a_n$ can produce the output b. But they may also produce some other output c, since one can have $([a_1, \ldots, a_n], c) \in [\![p]\!]$ too.

This semantics is an offspring of the strict connection between λ-calculus and linear logic [Gir87]. The category **Rel** of sets and relations is suitable to model LL if one assumes the operator $\mathcal{M}_f$ of finite multisets as the comonad interpreting $!$. Then the cartesian closed category **MRel** arising from the co-Kleisli construction of $\mathcal{M}_f$ on **Rel** is a model of the simply typed λ-calculus. A *relational model* of the untyped λ-calculus is a reflexive object in **MRel**.

Here we introduce the notion of *relational graph model* (rgm for short). Rgm's form a proper subclass of the class of relational models. They are a resource-sensitive reformulation of the graph models *à la* Plotkin-Scott-Engeler [Plo72, Eng81, Lon83, Plo93, Sch91]. A graph model is a specific kind of reflexive object in the cartesian closed category of cpo's and Scott-continuous functions. Precisely, one of the form $(\mathcal{P}(D), \subseteq)$ for a given infinite set D in which $\mathcal{P}_f(D) \times D$ is injected. The idea behind rgm's is to replace $\mathcal{P}_f(D)$ with $\mathcal{M}_f(D)$. With this choice, not the set $\mathcal{M}_f(D)$, but rather D *itself* turns out to be a reflexive object in **MRel**.

Rgm's are the main subject of this work. We investigate them by employing a *type-theoretical* approach. Indeed, we formalize the interpretation of λ-terms in each specific rgm by means of a corresponding *type assignment system*. A peculiar feature of such a system is a *non-idempotent* operation defined on types, corresponding intuitively to the union of multisets. We denote this operation as an *intersection*, in accordance to a long standing tradition [BDS13, Part III]. From this perspective, rgm's remind one of the role played by *filter models* in the context of Scott's semantics.

PLAN OF THE CHAPTER In § 2.1 we collect the generic technicalities concerning **MRel**. In § 2.2 we introduce the rgm's. The type systems associated with them are presented in § 2.3. The usage of these systems to interpret λ-terms is given in § 2.4. In § 2.5 we show that rgm's are also models of the linear resource calculus. We exploit this fact in § 2.6 to prove the crucial *approximation theorem* that holds for every rgm.

2.1 RELATIONAL SEMANTICS

Relational semantics of linear logic and the λ-calculus, first conceived by Girard in [Gir88], has been mainly developed by Ehrhard and coauthors [BEM07, BEM12, Ehr12]. We recall here its main ingredients, referring to § 1.2 for the categorical notions that we mention.

In the category **Rel** objects are sets. Given two objects A and B we define $\mathbf{Rel}(A, B) := \mathcal{P}(A \times B)$. In other words, morphisms in **Rel** are binary relations between sets. The composition in **Rel** is the usual composition of binary relations, i.e. for any $f \subseteq A \times B$ and $g \subseteq B \times C$ we have $g \circ f := \{(a, c) \in A \times C \mid (a, b) \in f \text{ and } (b, c) \in g \text{ for some } b \in B\}$. The inversion of a morphism $f : A \to B$ is just its inverse relation $f^{-1} := \{(b, a) \in B \times A \mid (a, b) \in f\}$.

Rel is a $*$-autonomous category. The tensor and its closure are both given by the cartesian product of sets $A \otimes B := A \multimap B := A \times B$. However, notice that the definition differs on morphisms f and g, since $- \multimap -$ is contravariant in the first argument. So actually $f \otimes g := f \times g$ whereas $f \multimap g := f^{-1} \times g$. The unit of the adjuction between $\otimes$ and $\multimap$ is $\mathrm{ev}_{A,B} := \{(((a, b), a), b) \mid a \in A \text{ and } b \in B\}$. The unit of the tensor is the generic singleton $1 := \{*\}$. The same goes for the dualizing object, namely $\perp := \{*\}$. For every object A we have $A^{\perp} = A$ up to isomorphism, whereas for every $f : A \to B$ we have $f^{\perp} = f^{-1}$. **Rel** has the disjoint unions of sets $A \uplus B$ as finite products and the empty set $\emptyset$ as final object.

See § 1.1 for the notations on multisets that we are going to use hereafter.

One can see $\mathcal{M}_f(-)$ as an endofunctor on **Rel** by setting for all $f \subseteq A \times B$

$$\mathcal{M}_f(f) \; := \; \Big\{ \big([a_1, \ldots, a_n] \, , \, [b_1, \ldots, b_n] \big) \; \Big| \; n \in \mathbb{N} \text{ and } (a_i, b_i) \in f \text{ for all } i \Big\} .$$

The functor $\mathcal{M}_f(-)$ is a monoidal comonad on **Rel**. Its counit and comultiplication are

$$\mathsf{der}_A \; := \; \{([a] \, , a) \mid a \in A\} ,$$

$$\mathsf{dig}_A \; := \; \Big\{ (m_1 + \cdots + m_n \, , \, [m_1, \cdots , m_n]) \; \Big| \; n \in \mathbb{N} \text{ and } m_i \in \mathcal{M}_f(A) \text{ for all } i \Big\} .$$

The Seely isomorphism $m^0 : 1 \to !\top$ is just $m^0 := \{(*, [\,])\} \subseteq \{*\} \times \{[\,]\}$. The Seely isomorphism $m^2_{A,B} : !A \otimes !B \to !(A \,\&\, B)$, namely $m^2_{A,B} \subseteq (\mathcal{M}_f(A) \times \mathcal{M}_f(B)) \times \mathcal{M}_f(A \uplus B)$, is

$$m^2_{A,B} \; := \; \Big\{ \Big(\big([a_1, \ldots a_n] \, , \, [b_1, \ldots b_m] \big) \, , \, [(1, a_1), \ldots , (1, a_n), (2, b_1), \ldots , (2, b_m)] \Big) \Big\}^{n,m \in \mathbb{N}}_{a_i \in A, b_j \in B} .$$

All this makes **Rel** a Seely category. So its co-Kleisli **MRel** $:= \mathsf{Kl}_{\mathcal{M}_f}(\mathbf{Rel})$ is a cartesian closed category. The objects of **MRel** are sets. A morphism $f \in \mathbf{MRel}(A, B)$ is any relation between $\mathcal{M}_f(A)$ and B, in other words $\mathbf{MRel}(A, B) = \mathcal{P}(\mathcal{M}_f(A) \times B)$. The composition of $f \in \mathbf{MRel}(A, B)$ and $g \in \mathbf{MRel}(B, C)$ is characterized as follows:

$$g \circ f = \Big\{ \big(\sum_{i=1}^{n} m_i \,,\, x \big) \;\big|\; n \in \mathbb{N} \text{ and } \exists y_1, \ldots, y_n \text{ such that } (m_i, y_i) \in f,\ \big([y_1, \ldots, y_n], x\big) \in g \Big\}.$$

The identity of A is the relation $\mathrm{id}_A = \big\{ ([x], x) \mid x \in A \big\}$. The product is the disjoint union $A \uplus B$ and the exponential object $A \Rightarrow B$ is $\mathcal{M}_f(A) \times B$.

As it is customary in relational semantics, when dealing with an arrow coming out of a product $A \,\&\, B$, i.e. with a relation coming out of $\mathcal{M}_f(A \uplus B)$, we silently compose it with Seely's isomorphism $m^2_{A,B}$, so to see it as a relation coming out of $\mathcal{M}_f(A) \times \mathcal{M}_f(B)$.

Every function $f : A \to B$ can be sent to a morphism $f^\dagger \in \mathbf{MRel}(A, B)$ just by setting $f^\dagger := \big\{ \big([x], f(x)\big) \mid x \in A \big\}$. Then the following result is easy to prove.

Lemma 2.1.1. *Let the function $f : A \to B$ be bijective. Then $f^\dagger \in \mathbf{MRel}(A, B)$ is an isomorphism, with inverse $\big(f^{-1}\big)^\dagger \in \mathbf{MRel}(B, A)$.*

We call *relational model* of the untyped λ-calculus any reflexive object in **MRel**. This makes sense even despite the fact that the **MRel** has *not enough points*, as clarified in [BEM07]. We will briefly recall this issue in § 2.4.

2.2 RELATIONAL GRAPH MODELS

We define a class of relational models of the untyped λ-calculus. As we will see in Chapter 3, this class contains the relational model introduced by Hyland and others in [HNPR06], and up to isomorphism also the relational model of Bucciarelli, Ehrhard and Manzonetto defined in [BEM07] and further studied in [BEM12].

Definition 2.2.1. A *relational graph model* (rgm, for short) is a pair $\mathcal{D} = (D, i)$ consisting of an infinite set D and an injection $i : \mathcal{M}_f(D) \times D \to D$.

An rgm $\mathcal{D} = (D, i)$ is called *extensional* (ergm, for short) whenever i is bijective.

Proposition 2.2.2. *Let $\mathcal{D} = (D, i)$ be an rgm. Then $\big(D, i^\dagger, (i^{-1})^\dagger\big)$ is a reflexive object in* **MRel**. *If moreover $\mathcal{D}$ is an ergm then $\big(D, i^\dagger, (i^{-1})^\dagger\big)$ is an extensional reflexive object in* **MRel**.

Proof. The binary relation $i^\dagger \subseteq \mathcal{M}_f\big(\mathcal{M}_f(D) \times D\big) \times D$ is given by

$$i^\dagger \;=\; \Big\{ \big([p], d\big) \in \mathcal{M}_f\big(\mathcal{M}_f(D) \times D\big) \times D \mid i(p) = d \Big\}.$$

The binary relation $(i^{-1})^\dagger \subseteq \mathcal{M}_f(D) \times (\mathcal{M}_f(D) \times D)$ is given by

$$(i^{-1})^\dagger \;=\; \Big\{ \big([d], p\big) \in \mathcal{M}_f(D) \times (\mathcal{M}_f(D) \times D) \mid i^{-1}(d) = p \Big\}.$$

So we have

$$
\begin{aligned}
(i^{-1})^\dagger \circ i^\dagger &= \Big\{ \big(\uplus_{i=1}^{n} m_i, p \big) \in \mathcal{M}_f\big(\mathcal{M}_f(D)\times D\big) \times \big(\mathcal{M}_f(D)\times D\big) \mid n \in \mathbb{N},\ (m_i, d_i) \in i^\dagger \\
&\qquad\qquad \text{for all } i \in \{1, \ldots n\},\ \text{and } \big([d_1,\ldots,d_n], p\big) \in (i^{-1})^\dagger \Big\} \\
&= \Big\{ \big([p_1], p\big) \in \mathcal{M}_f\big(\mathcal{M}_f(D)\times D\big) \times \big(\mathcal{M}_f(D)\times D\big) \mid \big([p_1], d_1\big) \in i^\dagger \text{ and} \\
&\qquad\qquad \big([d_1], p\big) \in (i^{-1})^\dagger \Big\} \\
&= \Big\{ \big([p_1], p\big) \in \mathcal{M}_f\big(\mathcal{M}_f(D)\times D\big) \times \big(\mathcal{M}_f(D)\times D\big) \mid i(p_1) = d_1 \text{ and } i^{-1}(d_1) = p \Big\} \\
&= \Big\{ \big([p], p\big) \in \mathcal{M}_f\big(\mathcal{M}_f(D)\times D\big) \times \big(\mathcal{M}_f(D)\times D\big) \mid p \in \mathrm{dom}(i) \Big\} \\
&= \Big\{ \big([p], p\big) \in \mathcal{M}_f\big(\mathcal{M}_f(D)\times D\big) \times \big(\mathcal{M}_f(D)\times D\big) \mid p \in \mathcal{M}_f(D)\times D \Big\} \\
&= id_{D\Rightarrow D}
\end{aligned}
$$

which means that $(D \Rightarrow D) \lhd D$ in the category **MRel**.
If moreover i is a surjective function, i.e. $\mathrm{dom}\big(i^{-1}\big) = D$, then

$$
\begin{aligned}
i^\dagger \circ (i^{-1})^\dagger &= \Big\{ \big(\uplus_{i=1}^{n} m_i, d \big) \in \mathcal{M}_f(D)\times D \mid n \in \mathbb{N},\ (m_i, p_i) \in (i^{-1})^\dagger \text{ for all } i \in \{1, \ldots n\}, \\
&\qquad\qquad \text{and } \big([p_1,\ldots,p_n], d\big) \in i^\dagger \Big\} \\
&= \Big\{ \big([d_1], d\big) \in \mathcal{M}_f(D)\times D \mid \big([d_1], p_1\big) \in (i^{-1})^\dagger \text{ and } \big([p_1], d\big) \in i^\dagger \Big\} \\
&= \Big\{ \big([d_1], d\big) \in \mathcal{M}_f(D)\times D \mid i^{-1}(d_1) = p \text{ and } i(p) = d \Big\} \\
&= \Big\{ \big([d], d\big) \in \mathcal{M}_f(D)\times D \mid d \in \mathrm{dom}\big(i^{-1}\big) \Big\} \\
&= \Big\{ \big([d], d\big) \in \mathcal{M}_f(D)\times D \mid d \in D \Big\} \\
&= id_D
\end{aligned}
$$

so that $(D \Rightarrow D) \simeq D$ in the ccc **MRel**. □

According to Proposition 2.2.2 an rgm always provides a categorical model, which is in particular extensional in the case of an ergm. In fact, from now on we have no serious reason to distinguish between (D, i) and $\big(D, i^\dagger, (i^{-1})^\dagger\big)$. Hence, with an abuse of language we say that an rgm *is* a model of the untyped λ-calculus.

Remark 2.2.3. Since every isomorphism $f \in \mathbf{MRel}(X, X)$ has the form $f = \big\{([\alpha], i(\alpha)) \mid \alpha \in X\big\}$ for some bijective map i, the class of ergm's coincides with the one of extensional relational models, meaning by that *all* extensional reflexive objects in **MRel**.

Theorem 2.2.4. *Let* $M \in \Lambda$ *and* $\mathrm{fv}(M) \subseteq \{x_1, \ldots, x_n\}$. *The interpretation of* M *in* $\mathcal{D}$ *w.r.t.* $\vec{x}$ *is the relation* $[\![M]\!]_{\mathcal{D}}^{\vec{x}} \subseteq \mathcal{M}_f(D)^n \times D$ *given inductively as follows.*

1. $[\![x_i]\!]_{\mathcal{D}}^{\vec{x}} = \Big\{ \Big(([\,], \ldots, [\,], [\sigma], [\,], \ldots, [\,]), \sigma \Big) \mid \sigma \in D \Big\}$*, where* $[\sigma]$ *stands in* i*-th position.*

2. $[\![\lambda x.M]\!]_{\mathcal{D}}^{\vec{x}} = \Big\{ \big(\vec{m}\,,\, i(m, \sigma)\big) \mid \big((\vec{m}, m), \sigma\big) \in [\![M]\!]_{\mathcal{D}}^{\vec{x},x} \Big\}$*, where* $x \notin \vec{x}$ *by* α*-conversion.*

3. $[\![MN]\!]_{\mathcal{D}}^{\vec{x}} = \Big\{ \big((\vec{m}_0 + \cdots + \vec{m}_k), \sigma\big) \mid k \in \mathbb{N}$ *and* $\exists\, \sigma_1, \ldots, \sigma_k \in \mathcal{D}$ *such that* $\big(\vec{m}_0, \iota([\sigma_1, \ldots, \sigma_k], \sigma)\big) \in [\![M]\!]_{\mathcal{D}}^{\vec{x}}$ *and* $(\vec{m}_\ell, \sigma_\ell) \in [\![N]\!]_{\mathcal{D}}^{\vec{x}}$ *for all* $1 \leqslant \ell \leqslant k \Big\}$.

Proof. A straightforward application of Definition 1.2.2. Remark however that we are also composing with Seely's isomorphism m^2, following the custom recalled in § 2.1. □

Rgm's can be built from a possibly finite amount of information, as formalized here below. We will exploit such a practical benefit all over the rest of this work. The idea was pioneered by Longo in [Lon83] for graph models. See Berline's article [Ber00] on the subject.

Definition 2.2.5. A *partial pair* $\mathcal{A} = (A, j)$ consists of a non-empty set A that does not contain any pair and a partial injection $j : \mathcal{M}_f(A) \times A \rightharpoonup A$.

A partial pair $\mathcal{A}$ is *extensional* when j is a bijection between $\mathrm{dom}(j)$ and A.

Definition 2.2.6. Let $\mathcal{A} = (A, j)$ be a partial pair. The *(free) completion* $\overline{\mathcal{A}}$ of a $\mathcal{A}$ is the pair $(\overline{A}, \bar{j})$ defined as follows.

1. By induction on $n \in \mathbb{N}$ we define
   - $A_0 := A$,
   - $A_{n+1} := \big(\big(\mathcal{M}_f(A_n) \times A_n\big) - \mathrm{dom}(j)\big) \cup A$

   and then we set
   $$\overline{A} := \bigcup_{n \in \mathbb{N}} A_n .$$

2. The total function $\bar{j} : \mathcal{M}_f(\overline{A}) \times \overline{A} \to \overline{A}$ is defined as
   $$\bar{j}(m, \alpha) := \begin{cases} j(m, \alpha) & \text{if } (m, \alpha) \in \mathrm{dom}(j) \\ (m, \alpha) & \text{otherwise.} \end{cases}$$

One can think $\overline{A}$ as a solution of the set-theoretical equation $X = \big(\mathcal{M}_f(X) \times X\big) \cup A$ in the unknown X. More precisely, the intuitive reading of Definition 2.2.6 is the following: Clause 1 says that $\overline{A}$ is the least set X obtained by adding recursively to the basic set A all elements of $\mathcal{M}_f(X) \times X$ *except* for those that are already in $\mathrm{dom}(j)$; the reason for such an exception is given by Clause 2, which specifies that $(m, \alpha) \in \mathrm{dom}(j)$ is intended to be already represented in $\overline{A}$ by the object $j(m, \alpha)$.

Lemma 2.2.7. *Let* (A, j) *be a partial pair and* $(\overline{A}, \bar{j})$ *its completion. Then* $\bar{j}$ *extends* j*, namely* $\mathrm{dom}(j) \subseteq \mathrm{dom}(\bar{j})$ *and* $\bar{j}(x) = j(x)$ *for all* $x \in \mathrm{dom}(j)$*. In particular* $\mathrm{rng}(j) \subseteq \mathrm{rng}(\bar{j})$.

Proof. Since $A \subseteq \overline{A}$ we have $\mathrm{dom}(j) = \mathcal{M}_f(A) \times A \subseteq \mathcal{M}_f(\overline{A}) \times \overline{A} = \mathrm{dom}\,\bar{j}$. The rest is by definition of $\bar{j}$. □

Proposition 2.2.8. *If* $\mathcal{A}$ *is a partial pair, then* $\overline{\mathcal{A}}$ *is an* rgm*. If* $\mathcal{A}$ *is an extensional partial pair, then* $\overline{\mathcal{A}}$ *is an* ergm.

Proof. Let (m, α) and (m', α') be distinct elements of $\mathcal{M}_f(\overline{A}) \times \overline{A}$.

If $(m, \alpha), (m', \alpha') \in \mathrm{dom}(j)$ then $\bar{j}(m, \alpha) = j(m, \alpha) \neq j(m', \alpha') = \bar{j}(m', \alpha')$, as j is injective.

If $(m, \alpha), (m', \alpha') \notin \mathrm{dom}(j)$ then trivially $\bar{j}(m, \alpha) = (m, \alpha) \neq (m', \alpha') = \bar{j}(m', \alpha')$.

If $(m, \alpha) \in \mathrm{dom}(j)$ and $(m', \alpha') \notin \mathrm{dom}(j)$ then $\bar{j}(m, \alpha) \neq (m', \alpha') = \bar{j}(m', \alpha')$ since the subset $\mathrm{rng}(j)$ of A does not contain any pair by Definition 2.2.5. So $\bar{j}$ is an injection.

Finally, let j be surjective, which means that $A = \mathrm{rng}(j)$, and consider $d \in \overline{A}$. We want to prove that $d \in \mathrm{rng}(\bar{j})$.

If $d \in A_0 = A = \mathrm{rng}(j)$ then $d \in \mathrm{rng}(\bar{j})$ by Lemma 2.2.7.

If there exists $n \in \mathbb{N}$ such that $y \in A_{n+1} - A$ then

$$d \in \big(\mathcal{M}_f(A_n) \times A_n\big) - \mathrm{dom}(j) \subseteq \big(\mathcal{M}_f(\overline{A}) \times \overline{A}\big) - \mathrm{dom}(j) = \mathrm{dom}(\bar{j}) - \mathrm{dom}(j).$$

Therefore by Definition 2.2.6(2) we get $d = \bar{j}(d) \in \mathrm{rng}(\bar{j})$. □

Definition 2.2.9. Let $\mathcal{D} = (D, i)$ be an rgm. We call *atoms* of $\mathcal{D}$ the elements of the set $\mathrm{At}_{\mathcal{D}} := D - (\mathcal{M}_f(D) \times D)$.

Proposition 2.2.10. *Let $\mathcal{A} = (A, i)$ be a partial pair. Then $\mathrm{At}_{\overline{\mathcal{A}}} = A$.*

Proof. Since A does not contain any pair we have $A \subseteq \overline{A} - \big(\mathcal{M}_f(\overline{A}) \times \overline{A}\big) = \mathrm{At}_{\overline{\mathcal{A}}}$.

The inclusion $\mathrm{At}_{\overline{\mathcal{A}}} \subseteq A$ holds because the completion only adds to the basic set A new elements intended to be in $\mathcal{M}_f(\overline{A}) \times \overline{A}$: formally $A_{n+1} - A = \big(\mathcal{M}_f(A_n) \times A_n\big) - \mathrm{dom}(j) \subseteq \mathcal{M}_f(\overline{A}) \times \overline{A}$ for all $n \in \mathbb{N}$, hence $\overline{A} - A \subseteq \mathcal{M}_f(\overline{A}) \times \overline{A}$. So $\mathrm{At}_{\overline{\mathcal{A}}} = \overline{A} - \big(\mathcal{M}_f(\overline{A}) \times \overline{A}\big) \subseteq A$. □

A final technical observation. We asked that the underlying set A of a partial pair (A, j) does not contain any pair. But when proving that $\overline{(A, j)}$ is an rgm (Proposition 2.2.8) we only used the fact that $\mathrm{rng}(i)$ does not contain pairs. In fact, we could define partial pairs (A, j) allowing elements of $A - \mathrm{rng}(j)$ to be pairs. But this formalization would entail some unpleasant consequences, in particular the need for a less intuitive definition of *atoms* allowing also certain pairs to be considered as atoms. We have nothing to gain by doing that.

2.3 NON-IDEMPOTENT INTERSECTION TYPES

We study rgm's using a notion of *non-idempotent intersection types*. Instead of the standard interpretation provided by the reflexive object (and described in Theorem 2.2.4), we use an intersection type assignment system to interpret λ-terms. Such an approach is not a novelty in denotational semantics, being typical of filter models [BDS13, Part III] (see also [RP04]) and Krivine's models [Kri90]. We just adapt it to the context of relational semantics. Also, this approach is not mandatory. For instance, the content of Chapter 5 of this thesis was presented in [BMPR16] using the interpretation shown in Theorem 2.2.4.

Definition 2.3.1. Let $\mathcal{D}$ be an rgm. The set $\mathsf{T}_{\mathcal{D}}$ of *types* for $\mathcal{D}$ and the set $\mathsf{I}_{\mathcal{D}}$ of *intersections* for $\mathcal{D}$ are mutually defined by the following two grammars

$$\mathsf{T}_{\mathcal{D}}: \quad \sigma ::= \alpha \mid \mu \to \sigma \qquad\qquad \mathsf{I}_{\mathcal{D}}: \quad \mu ::= \omega \mid \sigma \mid \mu \wedge \mu$$

where

- $\alpha \in \mathsf{At}_{\mathcal{D}}$;
- the operation $\wedge$ is commutative and associative;
- ω is a neutral element of $\wedge$, i.e., for all $\mu \in \mathsf{I}_{\mathcal{D}}$ we set $\mu \wedge \omega := \mu$.

The intersection ω is called *empty intersection*.

Remarks. Here some observations to understand the definition above.

- Types are unary intersections, whereas in general intersections are not types. Indeed, non-unary intersections may only appear on the left-hand side of the operator $\to$.
- The intersection is *not idempotent*, i.e. for all $\sigma \in \mathsf{T}_{\mathcal{D}}$

$$\sigma \wedge \sigma \neq \sigma .$$

  This differs from the traditional use of intersection types [BDS13, Part III].

Notations. We will usually use (possibly with some subscript or superscript):

- the Greek letters μ and ν to denote generic intersections;
- the Greek letters $\sigma, \tau, \gamma, \delta$ for those intersections which are in particular types;
- the Greek letters α and β for those types which are in particular atoms.

Since $\wedge$ is associative, we can write

$$\wedge_{i=1}^{n} \sigma_i := \begin{cases} \sigma_1 \wedge \cdots \wedge \sigma_n & \text{if } n \geqslant 1\,, \\ \omega & \text{if } n = 0\,. \end{cases}$$

By convention $\wedge$ takes precedence over the constructor $\to$, that is

$$\wedge_{i=1}^{n} \sigma_i \to \sigma := \left(\wedge_{i=1}^{n} \sigma_i\right) \to \sigma .$$

For all $\mu \in \mathsf{I}_{\mathcal{D}}$ and for all $\sigma \in \mathsf{T}_{\mathcal{D}}$ the expression

$$\sigma \in \mu$$

means that σ is one of the types occurring in the intersection μ (up to associativity), i.e. $\sigma \in \mu$ if and only if $\mu = \wedge_{i=1}^{n} \sigma_n$ and there exists $i \in \{1, \dots, n\}$ such that $\sigma = \sigma_i$.
For all $\mu, \nu \in \mathsf{I}_{\mathcal{D}}$ we write

$$\mu - \nu$$

for the intersection obtained from μ by erasing all types $\sigma \in \nu$.

The notation $\sigma \in \mu$ does not stand for a set-theoretical membership, but it is not ambiguous. In fact, it relies on the following intuition: for any rgm $\mathcal{D} = (D, i)$, one can think of intersections in $\mathsf{I}_{\mathcal{D}}$ as multisets in $\mathcal{M}_f(D)$, and types in $\mathsf{T}_{\mathcal{D}}$ as elements of $\mathsf{At}_D \cup (\mathcal{M}_f(D) \times D)$. The set D can then be recovered as $\mathsf{T}_{\mathcal{D}} / \simeq^{\mathcal{D}}$, for a certain equivalence $\simeq^{\mathcal{D}}$ generated by the injection i. Here below the formalization of this idea.

Definition 2.3.2. Let $\mathcal{D} = (D, i)$ be an rgm. The function $(-)^{\diamond} : \mathsf{T}_{\mathcal{D}} \to D$, together with its auxiliary function $(-)^{\diamond} : \mathsf{I}_{\mathcal{D}} \to \mathcal{M}_f(D)$, is defined by the following induction on $\sigma \in \mathsf{T}_{\mathcal{D}}$:

- $\alpha^{\diamond} := \alpha$ for all $\alpha \in \mathsf{At}_{\mathcal{D}}$;
- $(\mu \to \tau)^{\diamond} := i(\mu^{\diamond}, \tau^{\diamond})$ for all $\mu \in \mathsf{I}_{\mathcal{D}}$ and for all $\tau \in \mathsf{T}_{\mathcal{D}}$, where

$$(\sigma_1 \wedge \cdots \wedge \sigma_n)^{\diamond} := \left[\sigma_1^{\diamond}, \ldots, \sigma_n^{\diamond}\right] \text{ for all } n \in \mathbb{N} \text{ and for all } \sigma_1, \ldots, \sigma_n \in \mathsf{T}_{\mathcal{D}}\,.$$

Definition 2.3.3. Let $\mathcal{D}$ be an rgm. The relation $\simeq^{\mathcal{D}} \subseteq \mathsf{T}_{\mathcal{D}} \times \mathsf{T}_{\mathcal{D}}$ is defined as:

$$\sigma \simeq^{\mathcal{D}} \tau \quad \text{if and only if} \quad \sigma^{\diamond} = \tau^{\diamond}.$$

We also extend the relation $\simeq^{\mathcal{D}}$ to intersections, in the sense that for all $\mu, \nu \in \mathsf{I}_{\mathcal{D}}$

$$\mu \simeq^{\mathcal{D}} \nu \quad \text{if and only if} \quad \mu^{\diamond} = \nu^{\diamond}.$$

When there is no ambiguity concerning $\mathcal{D}$ we write $\simeq^{\mathcal{D}}$ simply as $\simeq$.

Without loss of generality, from now on we suppose that D contains at most pairs in $\mathcal{M}_f(D) \times D$. In this way, the following function $\mathsf{depth} : D \to \mathbb{N}$ is well defined:

$$\mathsf{depth}(d) := \begin{cases} 0 & \text{if } d \in \mathsf{At}_{\mathcal{D}}\,, \\ \max_{i=1}^{n+1} \mathsf{depth}(d_i) + 1 & \text{if } d = \left([d_1, \ldots, d_n], d_{n+1}\right) \in \mathcal{M}_f(D) \times D\,. \end{cases}$$

Lemma 2.3.4. *Let $\mathcal{D}$ be an rgm. The function $(-)^{\diamond} : \mathsf{T}_{\mathcal{D}} \to D$ is surjective.*

Proof. Let $d \in D$. We prove that $d = \sigma^{\diamond}$ for some $\sigma \in \mathsf{T}_{\mathcal{D}}$. We proceed by induction on $\mathsf{depth}(d)$.
Case $\mathsf{depth}(d) = 0$. In this case $d \in \mathsf{At}_{\mathcal{D}}$, hence $d^{\diamond} = d$ by Definition 2.3.2.
Case $\mathsf{depth}(d) > 0$. We have $d = \left([d_1 \ldots, d_n], d_{n+1}\right)$ for some $n \in \mathbb{N}$. By IH there exist $\sigma_1, \ldots, \sigma_{n+1} \in \mathsf{T}_{\mathcal{D}}$ such that $\sigma_i{}^{\diamond} = d_i$ for all $i \in \{1, \ldots, n+1\}$. So

$$\left(\textstyle\bigwedge_{i=1}^{n} \sigma_i \to \sigma_{n+1}\right)^{\diamond} = \left(\left[\sigma_1^{\diamond}, \ldots, \sigma_n^{\diamond}\right], \sigma_{n+1}^{\diamond}\right) = \left([d_1, \ldots, d_n], d_{n+1}\right) = d$$

which concludes the proof. □

Remark 2.3.5. Clearly the equivalence $\simeq^{\mathcal{D}}$ is a congruence on $\mathsf{T}_{\mathcal{D}}$ w.r.t. $\wedge$ and $\to$, meaning that if $\mu \simeq^{\mathcal{D}} \mu'$, $\nu \simeq^{\mathcal{D}} \nu'$ and $\sigma \simeq^{\mathcal{D}} \sigma'$ then also $\mu \wedge \nu \simeq^{\mathcal{D}} \mu' \wedge \nu'$ and $\mu \to \sigma \simeq^{\mathcal{D}} \mu' \to \sigma'$.

Definition 2.3.6. Let $\mathcal{D}$ be an rgm. An *environment* for $\mathcal{D}$ is a map $\Gamma : \mathrm{Var} \to \mathsf{I}_{\mathcal{D}}$ such that $\mathrm{supp}(\Gamma) := \{\, x \in \mathrm{Var} \mid \Gamma(x) \neq \omega \,\}$ is finite. The set of all environments for $\mathcal{D}$ is called $\mathsf{Env}_{\mathcal{D}}$.

Notations. Let $\Gamma \in \mathsf{Env}_{\mathcal{D}}$ such that $\mathrm{supp}(\Gamma) = \{\, x_1, \ldots, x_n \,\}$ and $\Gamma(x_i) = \mu_i$ for all $i \in \{1, \ldots, n\}$. Then we denote Γ by

$$x_1 : \mu_1 \;,\; \ldots \;,\; x_n : \mu_n\,.$$

Accordingly, the environment mapping all variables to ω is just omitted.

$$\frac{}{x : \sigma \vdash^{\mathcal{D}} x : \sigma}\ \mathtt{var} \qquad \frac{\Gamma, x : \mu \vdash^{\mathcal{D}} M : \sigma}{\Gamma \vdash^{\mathcal{D}} \lambda x.M : \mu \to \sigma}\ \mathtt{lam} \qquad \frac{\Gamma \vdash^{\mathcal{D}} M : \tau \quad \sigma \simeq^{\mathcal{D}} \tau}{\Gamma \vdash^{\mathcal{D}} M : \sigma}\ \mathtt{eq}$$

$$\frac{\Gamma_0 \vdash^{\mathcal{D}} M : \wedge_{i=1}^{n} \sigma_i \to \sigma \qquad \Gamma_i \vdash^{\mathcal{D}} N : \sigma_i \quad \text{for all } i \in \{1, \ldots, n\}}{\Gamma_0 \wedge \left(\bigwedge_{i=1}^{n} \Gamma_i \right) \vdash^{\mathcal{D}} MN : \sigma}\ \mathtt{app}$$

Figure 3: The intersection type systems for Λ and $\mathcal{N}$.

Definition 2.3.7. Consider $\Gamma, \Delta \in \mathsf{Env}_{\mathcal{D}}$.

- The environment $\Gamma \wedge \Delta \in \mathsf{Env}_{\mathcal{D}}$ is defined as

$$\Gamma \wedge \Delta : \quad x \in \mathrm{Var} \ \mapsto\ \Gamma(x) \wedge \Delta(x) \in \mathsf{I}_{\mathcal{D}}.$$

- The environment $\Gamma - \Delta \in \mathsf{Env}_{\mathcal{D}}$ is defined as

$$\Gamma - \Delta : \quad x \in \mathrm{Var} \ \mapsto\ \Gamma(x) - \Delta(x) \in \mathsf{I}_{\mathcal{D}}.$$

- The equivalence $\simeq^{\mathcal{D}}$ is extended to environments as:

$$\Gamma \simeq^{\mathcal{D}} \Delta \quad \text{if and only if} \quad \Gamma(x) \simeq^{\mathcal{D}} \Delta(x) \ \text{ for all } x \in \mathrm{Var}.$$

Definition 2.3.8. Let $\mathcal{D}$ be an rgm. The *type assignment system* $\vdash^{\mathcal{D}}$ *for* Λ *and* $\mathcal{N}$ *associated to* $\mathcal{D}$ is given in Fig. 3. When $\mathcal{D}$ is clear from the context we simply write $\vdash$ instead of $\vdash^{\mathcal{D}}$.

Remark 2.3.9. The natural number n appearing in Rule app in Fig. 3 can be 0. So, given $M \in \Lambda$ and $a \in \mathcal{N}$ we have the inference rules

$$\frac{\Gamma \vdash^{\mathcal{D}} M : \omega \to \sigma}{\Gamma \vdash^{\mathcal{D}} MN : \sigma} \qquad \frac{\Gamma \vdash^{\mathcal{D}} a : \omega \to \sigma}{\Gamma \vdash^{\mathcal{D}} ab : \sigma} \tag{5}$$

whatever $N \in \Lambda$ and $b \in \mathcal{N}$ are. For example, even if $\mathbf{\Omega}$ is not typable in the system associated to any rgm, one can always derive

$$\cfrac{\cfrac{x : \omega \to \sigma \vdash x : \omega \to \sigma}{x : \omega \to \sigma \vdash x\,\mathbf{\Omega} :\to \sigma}}{\vdash \lambda x.x\,\mathbf{\Omega} : (\omega \to \sigma) \to \sigma}$$

for every $\sigma \in \mathsf{T}_{\mathcal{D}}$. In a way ω plays the role of *universal type*, a common concept in traditional intersection type theory [BDS13, Part III].

Lemma 2.3.10. *Let $\mathcal{D}$ be an rgm. If* $\Gamma \vdash^{\mathcal{D}} M : \sigma$ *is derivable then* $\mathrm{supp}(\Gamma) \subseteq \mathrm{fv}(M)$.

Proof. By a straightforward induction on the derivation of the sequent. □

The inclusion in Lemma 2.3.10 can be strict. For instance we can derive $x : \omega \to \sigma \vdash x\,y : \sigma$ where $\mathrm{supp}(x : \omega \to \sigma) = \{x\} \subset \mathrm{fv}(xy)$, for any σ. More generally, one should realize that whenever $\mathrm{supp}(\Gamma) \subset \mathrm{fv}(M)$ then along the derivation of $\Gamma \vdash M : \sigma$ some subterm N of M comes *untyped* as in (5). To be more precise, there must be a subterm PN of M such that P is typed with some $\omega \to \tau$ by a subtree of the derivation, hence its argument N comes untyped in a following instance of Rule `app`. As a consequence, any variable in $\mathrm{fv}(N)$ can possibly not receive a multiset $\mu \neq \omega$ in the environment Γ. Still it can be free in M.

Definition 2.3.11. Let $\mathcal{D}$ be an rgm. A family $\{\Gamma_i\}_{i \in \mathcal{J}}$ of environments for $\mathcal{D}$ is a *decomposition* of $\Gamma \in \mathsf{Env}_{\mathcal{D}}$ whenever $\Gamma = \wedge_{i \in \mathcal{J}} \Gamma_i$.

The following result is essential throughout our investigation.

Lemma 2.3.12 (Inversion)**.** *Let $\mathcal{D}$ be an rgm. Let $M, N \in \Lambda \cup \mathcal{N}$.*

1. *If $\Gamma \vdash^{\mathcal{D}} x : \sigma$ is derivable then there exists $\tau \in \mathsf{T}_{\mathcal{D}}$ such that $\Gamma = x : \tau$ and $\tau \simeq^{\mathcal{D}} \sigma$.*
2. *The sequent $\Gamma \vdash^{\mathcal{D}} \lambda x.M : \sigma$ is derivable if and only if there exist $\tau \in \mathsf{T}_{\mathcal{D}}$ and $\mu \in \mathsf{I}_{\mathcal{D}}$ such that $\Gamma, x : \mu \vdash^{\mathcal{D}} M : \tau$ is derivable and $\mu \to \tau \simeq^{\mathcal{D}} \sigma$;*
3. *If $\Gamma \vdash^{\mathcal{D}} MN : \sigma$ is derivable then for some $n \geqslant 0$ there exist $\sigma_1, \ldots, \sigma_n \in \mathsf{T}_{\mathcal{D}}$ and a decomposition $\{\Gamma_i\}_{i=0}^{n}$ of Γ such that the sequents $\Gamma_0 \vdash^{\mathcal{D}} M : \wedge_{i=1}^{n} \sigma_i \to \sigma$ and $\Gamma_i \vdash^{\mathcal{D}} N : \sigma_i$ for all $i \in \{1, \ldots, n\}$ are derivable.*

Proof. (1) We proceed by induction on the derivation of $\Gamma \vdash x : \sigma$. Such a derivation must terminate either by an application of Rule `var` or by an application of Rule `eq`.

In the former case $\Gamma = x : \sigma$, and we are done.

In the latter case there is a derivation of $\Gamma \vdash x : \tau$ for some $\tau \simeq \sigma$. By IH then we have $\Gamma = x : \gamma$ for some $\gamma \simeq \tau \simeq \sigma$.

(2) We proceed by induction on the derivation of $\Gamma \vdash \lambda x.M : \sigma$. Such a derivation must terminate either by an application of Rule `lam` or by an application of Rule `eq`.

In the former case $\sigma = \mu \to \tau$ and the sequent $\Gamma, x : \mu \vdash M : \tau$ is derivable.

In the latter case there is a derivation of $\Gamma \vdash \lambda x.M : \tau$ for some $\tau \simeq \sigma$. By IH then the sequent $\Gamma, x : \mu \vdash M : \gamma$ is derivable for some $\mu \to \gamma \simeq \tau \simeq \sigma$.

(3) We proceed by induction on the derivation of $\Gamma \vdash MN : \sigma$. Such a derivation must terminate either by an application of Rule `app` or by an application of Rule `eq`.

In the former case the thesis is clearly true for the definition of Rule `app` itself.

In the latter case there is a derivation of $\Gamma \vdash MN : \tau$ for some $\tau \simeq \sigma$. By IH there exists a decomposition $\{\Gamma_i\}_{i=0}^{n}$ of Γ such that the sequents $\Gamma_0 \vdash M : \wedge_{i=1}^{n} \sigma_i \to \tau$ and $\Gamma_i \vdash N : \sigma_i$ for all $i \in \{1, \ldots, n\}$ are derivable. Then we can derive

$$\dfrac{\dfrac{\Gamma_0 \vdash M : \wedge_{i=1}^{n} \sigma_i \to \tau \qquad \wedge_{i=1}^{n} \sigma_i \to \tau \simeq \wedge_{i=1}^{n} \sigma_i \to \tau}{\Gamma_0 \vdash M : \wedge_{i=1}^{n} \sigma_i \to \sigma} \qquad \Gamma_i \vdash N : \sigma_i \quad \text{for all } i \in \{1, \ldots, n\}}{\Gamma \vdash MN : \sigma}$$

which completes the proof. □

We do not claim sole authorship on the intersection type systems given in Definition 2.3.8. In fact, they first appeared in the article [PPRDR15] by Ronchi, Paolino and Piccolo. Ronchi and coauthors share with us the interest in these systems as models of the untyped λ-calculus. But the way we exploit them to interpret λ-terms (the topic of the next section) differs from theirs. As a matter of fact, the interpretation that they use is more adherent to the traditional notion of Hindley-Longo syntactical λ-model, as given in [HL80] and [Bar84, § 5.3]. Whereas ours is a bit less conventional.

Finally, we mention that an instance of these type systems — precisely the one where $\simeq^{\mathcal{D}}$ is just the equality of types — was studied by de Carvalho in [dC09]. In particular de Carvalho already recognized that specific type system to correspond to a relational model of the untyped λ-calculus. Incidentally we will have a closer look at this instance in Chapter 3.

2.4 THE TYPE-INTERPRETATION

Definition 2.4.1. Let $\mathcal{D}$ be an rgm. Let $M \in \Lambda \cup \mathcal{N}$. The *type-interpretation* of M in $\mathcal{D}$ is defined as

$$[\![M]\!]^{\mathcal{D}} := \left\{ (\Gamma, \sigma) \in \mathsf{Env}_{\mathcal{D}} \times \mathsf{T}_{\mathcal{D}} \mid \Gamma \vdash^{\mathcal{D}} M : \sigma \right\}.$$

Firstly, we show that the type-interpretation $[\![-]\!]^{\mathcal{D}}$ is equivalent to the traditional interpretation $[\![-]\!]^{-}_{\mathcal{D}}$ provided by the reflexive object and described in Theorem 2.2.4.

Theorem 2.4.2 (Type-semantics Theorem). *Let* $M \in \Lambda$ *and* $\mathrm{fv}(M) \subseteq \{x_1, \dots, x_n\}$. *Then*

1. $[\![M]\!]^{\mathcal{D}} = \left\{ (\Gamma, \sigma) \in \mathsf{Env}_{\mathcal{D}} \times \mathsf{T}_{\mathcal{D}} \mid \left(\Gamma(x_1)^{\diamond}, \dots, \Gamma(x_n)^{\diamond}, \sigma^{\diamond}\right) \in [\![M]\!]^{\vec{x}}_{\mathcal{D}} \right\}$,
2. $[\![M]\!]^{\vec{x}}_{\mathcal{D}} = \left\{ \left(\Gamma(x_1)^{\diamond}, \dots, \Gamma(x_n)^{\diamond}, \sigma^{\diamond}\right) \in \mathcal{M}_f(D)^n \times D \mid (\Gamma, \sigma) \in [\![M]\!]^{\mathcal{D}} \right\}$.

Proof. (1) We must prove that $(\Gamma, \sigma) \in [\![M]\!]^{\mathcal{D}}$ if and only if $\left(\Gamma(x_1)^{\diamond}, \dots, \Gamma(x_n)^{\diamond}, \sigma^{\diamond}\right) \in [\![M]\!]^{\vec{x}}_{\mathcal{D}}$.

We proceed by induction on M.

Case $M = x_i$**.** By Definition 2.3.8 and Lemma 2.3.12(1) the sequent $\Gamma \vdash x_i : \sigma$ is derivable if and only if $\Gamma = x_i : \tau$ for some $\tau \in \mathsf{T}_{\mathcal{D}}$ such that $\tau \simeq \sigma$, i.e. $\tau^{\diamond} = \sigma^{\diamond}$. By Lemma 2.2.4(1) this is equivalent to

$$\begin{aligned} \left(\Gamma(x_1)^{\diamond}, \dots, \Gamma(x_n)^{\diamond}, \sigma^{\diamond}\right) &= ([\,], \dots, [\,], [\tau^{\diamond}], [\,], \dots, [\,], \sigma^{\diamond}) \\ &= ([\,], \dots, [\,], [\sigma^{\diamond}], [\,], \dots, [\,], \sigma^{\diamond}) \in [\![x_i]\!]^{\vec{x}}_{\mathcal{D}}. \end{aligned}$$

Case $M = \lambda x.P$**.** By Definition 2.3.8 and Lemma 2.3.12(2) the sequent $\Gamma \vdash \lambda x.P : \sigma$ is derivable if and only if $\Gamma, x : \mu \vdash P : \tau$ is derivable for some $\mu \to \tau \simeq \sigma$. By IH this is equivalent to $\left(\Gamma(x_1)^{\diamond}, \dots, \Gamma(x_n)^{\diamond}, \mu^{\diamond}, \sigma^{\diamond}\right) \in [\![P]\!]^{\vec{x},x}_{\mathcal{D}}$. By Lemma 2.2.4(2) this fact is equivalent to $\left(\Gamma(x_1)^{\diamond}, \dots, \Gamma(x_n)^{\diamond}, i(\mu^{\diamond}, \sigma^{\diamond})\right) \in [\![\lambda x.P]\!]^{\vec{x}}_{\mathcal{D}}$. This proves the thesis as $i(\mu^{\diamond}, \sigma^{\diamond}) = (\mu \to \sigma)^{\diamond}$.
Case $M = PQ$**.** By Definition 2.3.8 and Lemma 2.3.12(3) the sequent $\Gamma \vdash PQ : \sigma$ is derivable if and only if there exist $\sigma_1, \dots, \sigma_k \in \mathsf{T}_{\mathcal{D}}$ and a decomposition $\{\Gamma_i\}_{i=0}^{k}$ of Γ such that

$$\Gamma_0 \vdash P : \textstyle\bigwedge_{i=1}^{k} \sigma_i \to \sigma \quad \text{and} \quad \Gamma_i \vdash Q : \sigma_i \ \text{ for all } i \in \{1, \dots, k\} \quad \text{are derivable.} \tag{6}$$

By IH then (6) is equivalent to

$$\Big(\Gamma_0(x_1)^\diamond, \ldots, \Gamma_0(x_n)^\diamond, \big(\wedge_{i=1}^k \sigma_i \to \sigma\big)^\diamond\Big) \in [\![P]\!]_{\mathcal{D}}^{\vec{x}}$$

and

$$\big(\Gamma_i(x_1)^\diamond, \ldots, \Gamma_i(x_n)^\diamond, \sigma_i^\diamond\big) \in [\![Q]\!]_{\mathcal{D}}^{\vec{x}} \quad \text{for all } i \in \{1, \ldots, k\}.$$

Since $\big(\wedge_{i=1}^k \sigma_i \to \sigma\big)^\diamond = \big([\sigma_1^\diamond, \ldots, \sigma_k^\diamond], \sigma^\diamond\big)$, by Lemma 2.2.4(2) such a fact is equivalent to $\big(\Gamma(x_1)^\diamond, \ldots, \Gamma(x_n)^\diamond, \sigma^\diamond\big) \in [\![PQ]\!]_{\mathcal{D}}^{\vec{x}}$.

(2) Firstly, we notice that for every $(m_1, \ldots, m_n, d) \in \mathcal{M}_f(D)^n \times D$ there exists $(\Gamma, \sigma) \in \mathsf{Env}_{\mathcal{D}} \times T_{\mathcal{D}}$ such that $(m_1, \ldots, m_n, d) = \big(\Gamma(x_1)^\diamond, \ldots, \Gamma(x_n)^\diamond, \sigma^\diamond\big)$. Indeed, as $(-)^\diamond$ is surjective by Lemma 2.3.4, there exist $\mu_1, \ldots, \mu_n \in I_{\mathcal{D}}$ and $\sigma \in T_{\mathcal{D}}$ such that $(m_1, \ldots, m_n, d) = (\mu_1^\diamond, \ldots, \mu_n^\diamond, \sigma^\diamond)$. So one just takes $\Gamma := x_1 : \mu_1, \ldots, x_n : \mu_n$.

Such observation justifies the first of the following two equalities:

$$\begin{aligned} [\![M]\!]_{\mathcal{D}}^{\vec{x}} &= \big\{ \big(\Gamma^\diamond(x_1), \ldots, \Gamma^\diamond(x_n), \sigma^\diamond\big) \in [\![M]\!]_{\mathcal{D}}^{\vec{x}} \mid (\Gamma, \sigma) \in \mathsf{Env}_{\mathcal{D}} \times T_{\mathcal{D}} \big\} \\ &= \big\{ \big(\Gamma^\diamond(x_1), \ldots, \Gamma^\diamond(x_n), \sigma^\diamond\big) \in \mathcal{M}_f(D)^n \times D \mid (\Gamma, \sigma) \in [\![M]\!]^{\mathcal{D}} \big\} \end{aligned}$$

where the last equality is given by (1). □

Corollary 2.4.3. *Let* $M, N \in \Lambda$ *and* $\mathrm{fv}(MN) \subseteq \{x_1, \ldots, x_n\}$*. Then* $[\![M]\!]^{\mathcal{D}} \subseteq [\![N]\!]^{\mathcal{D}}$ *if and only if* $[\![M]\!]_{\mathcal{D}}^{\vec{x}} \subseteq [\![N]\!]_{\mathcal{D}}^{\vec{x}}$.

A few words are maybe useful to contextualize our definition of $[\![-]\!]^{\mathcal{D}}$ in the more usual scenario of denotational semantics and intersection type theory. In general, the categorical interpretation of a λ-term M in a reflexive object $\mathcal{D}$ gives a morphism $[\![M]\!]_{\mathcal{D}}^{\vec{x}} : D^{\vec{x}} \to D$ such that $\mathrm{Th}(\mathcal{D})$ is a λ-theory. If the category is well-pointed, like in the case of Scott's continuous semantics, then it is equivalent to interpret M as a *point* of D through a *valuation* $\rho : \mathrm{Var} \to D$, namely as an arrow $[\![M]\!]_\rho : \top \to D$ from the terminal object $\top$ depending on ρ, see [Bar84, §5.5]. For this reason, in the context of Scott models it has become standard to consider the interpretation of M as an *element* of the domain. When the model is translated into a type system (as in filter models), this interpretation becomes *the set of the types of* M, as done in [Roc82, CDZ87, Ber00, RP04, BDS13]. As shown by Koymans in [Koy82], when the category is not well-pointed, points are no more suitable for interpreting λ-terms, since the induced equality is not a λ-theory as a consequence of the failure of the ξ-rule, which is $M = N \Rightarrow \lambda x.M = \lambda x.N$. In the algebraic terminology, the set of points gives a λ-*algebra* which is not a λ-*model* [Bar84, §5.2]. Nevertheless, in [BEM07] Bucciarelli and others showed that a λ-model can *still* be constructed from a reflexive object $\mathcal{D}$ of a non-well pointed category, by considering the set of $\mathbf{C}_f(D^{\mathrm{Var}}, D)$ of *finitary* morphisms from D^{Var} to D (a technical notion) and valuations $\rho : \mathrm{Var} \to \mathbf{C}_f(D^{\mathrm{Var}}, D)$. For instance, this is the approach followed in [PPRDR15]. However, in [Man08] Manzonetto remarked that the use of valuations in this context becomes redundant since $[\![M]\!]_\rho = [\![N]\!]_\rho$ exactly when they are equal under the valuation $x \mapsto \pi_x^{\mathrm{Var}}$ sending x to the corresponding projection. By applying this fact to the type-theoretical interpretation given in [PPRDR15], one can recover our type-theoretical interpretation, i.e. Definition 2.4.1.

By Corollary 2.4.3 even the type-interpretation $[\![-]\!]^{\mathcal{D}}$ provides a model of the untyped λ-calculus. In particular for every given rgm:

I. the type-interpretation is invariant under β-reduction and β-expansion;

II. when the rgm is extensional the type-interpretation is invariant under η-conversion;

III. the inclusion between type-interpretations of λ-terms defines a preorder theory;

IV. the equality between type-interpretations of λ-terms defines a λ-theory.

Points I-IV can now be considered certain. Nevertheless, in the rest of this section we prove each of them more directly, in a way that is completely independent of the interpretation $[\![-]\!]^-_{\mathcal{D}}$ given by the reflexive object. We find this exercise of interest in itself, and a good workout in the use of the type system. By the way, we will refine Point II, by revealing that the invariance under η-reduction holds in every rgm, whereas only the η-expansion actually requires an ergm.

Let us start with a basic lemma, which states the invariance under application of contexts.

Lemma 2.4.4 (Contextuality). *Let $\mathcal{D}$ be an rgm. Let $M, N \in \Lambda \cup \mathcal{N}$ and $C[\,]$ a context. Whenever $[\![M]\!]^{\mathcal{D}} \subseteq [\![N]\!]^{\mathcal{D}}$ then $[\![C[M]]\!]^{\mathcal{D}} \subseteq [\![C[N]]\!]^{\mathcal{D}}$.*

In particular $[\![M]\!]^{\mathcal{D}} = [\![N]\!]^{\mathcal{D}}$ entails $[\![C[M]]\!]^{\mathcal{D}} = [\![C[N]]\!]^{\mathcal{D}}$.

Proof. We proceed by induction on the structure of $C[\,]$.

Case $C[\,] = [\,]$. Trivial.

Case $C[\,] = C'[\,]Q$. Consider $(\Gamma, \sigma) \in \mathsf{Env}_{\mathcal{D}} \times \mathsf{T}_{\mathcal{D}}$ such that $\Gamma \vdash C'[M]Q : \sigma$ can be derived. By Lemma 2.3.12(3) we have a decomposition $\{\Gamma_i\}_{i=0}^n$ of Γ and some $\sigma_1, \ldots, \sigma_n \in \mathsf{T}_{\mathcal{D}}$ such that $\Gamma_0 \vdash C'[M] : \bigwedge_{i=1}^n \sigma_i \to \sigma$ and $\Gamma_i \vdash Q : \sigma_i$ for all $i \in \{1, \ldots, n\}$ are derivable. Then by IH $\Gamma_0 \vdash C'[N] : \bigwedge_{i=1}^n \sigma_i \to \sigma$ is derivable. Hence by app we also get $\Gamma \vdash C'[N]Q : \sigma$.

Case $C[\,] = PC'[\,]$. Consider $(\Gamma, \sigma) \in \mathsf{Env}_{\mathcal{D}} \times \mathsf{T}_{\mathcal{D}}$ such that $\Gamma \vdash PC'[M] : \sigma$ can be derived. By Lemma 2.3.12(3) we have a decomposition $\{\Gamma_i\}_{i=0}^n$ of Γ and some $\sigma_1, \ldots, \sigma_n \in \mathsf{T}_{\mathcal{D}}$ such that $\Gamma_0 \vdash P : \bigwedge_{i=1}^n \sigma_i \to \sigma$ and $\Gamma_i \vdash C'[M] : \sigma_i$ for all $i \in \{1, \ldots, n\}$ are derivable. Then by IH the sequents $\Gamma_i \vdash C'[N] : \sigma_i$ for all i are derivable. Hence by app we also get $\Gamma \vdash C'[N]Q : \sigma$.

Case $C[\,] = \lambda x.C'[\,]$. Let $\Gamma \vdash \lambda x.C'[M] : \sigma$ be derivable. By Lemma 2.3.12(2) the sequent $\Gamma, x : \mu \vdash C'[M] : \tau$ for $\mu \to \tau \simeq \sigma$ is derivable. By IH then $\Gamma, x : \mu \vdash C'[N] : \tau$ for $\mu \to \tau \simeq \sigma$ is derivable. From this we get $\Gamma \vdash \lambda x.C'[N] : \sigma$ by Rules lam and eq. □

It is now simple to prove Points III-IV above.

Definition 2.4.5. Let $\mathcal{D}$ be an rgm.

- The *preorder theory induced by* $\mathcal{D}$ is defined as

$$\mathrm{Th}_{\sqsubseteq}(\mathcal{D}) \;:=\; \{\, (M, N) \in \Lambda \times \Lambda \mid [\![M]\!]^{\mathcal{D}} \subseteq [\![N]\!]^{\mathcal{D}} \,\}.$$

  Whenever $(M, N) \in \mathrm{Th}_{\sqsubseteq}(\mathcal{D})$ we write $M \sqsubseteq_{\mathcal{D}} N$.

- The *λ-theory induced by* $\mathcal{D}$ is defined as

$$\mathrm{Th}(\mathcal{D}) \;:=\; \{\, (M, N) \in \Lambda \times \Lambda \mid [\![M]\!]^{\mathcal{D}} = [\![N]\!]^{\mathcal{D}} \,\}.$$

  Whenever $(M, N) \in \mathrm{Th}(\mathcal{D})$ we write $M =_{\mathcal{D}} N$.

Theorem 2.4.6. *Let $\mathcal{D}$ be an rgm. Then* $\mathrm{Th}_{\sqsubseteq}(\mathcal{D})$ *is a preorder theory and* $\mathrm{Th}(\mathcal{D})$ *is a λ-theory.*

Proof. The reflexivity and the transitivity of both $\sqsubseteq_{\mathcal{D}}$ and $=_{\mathcal{D}}$ are trivial, as well as the symmetry of $=_{\mathcal{D}}$.

Let us show that $\sqsubseteq_{\mathcal{D}}$ is a congruence w.r.t. λ-abstraction and application. (Then the same thing immediately follows also for $=_{\mathcal{D}}$.)

Let $M \sqsubseteq_{\mathcal{D}} N$, i.e. $[\![M]\!] \subseteq [\![N]\!]$. By applying Lemma 2.4.4 to M and N with the context $C[\,] = \lambda x.[\,]$ we get $[\![\lambda x.M]\!] \subseteq [\![\lambda x.N]\!]$, i.e. $\lambda x.M \sqsubseteq_{\mathcal{D}} \lambda x.N$.

Let $M \sqsubseteq_{\mathcal{D}} N$ and $P \sqsubseteq_{\mathcal{D}} Q$, i.e. $[\![M]\!] \subseteq [\![N]\!]$ and $[\![P]\!] \subseteq [\![Q]\!]$. By applying Lemma 2.4.4 to M and N with the context $C[\,] = [\,]P$ we get $[\![MP]\!] \subseteq [\![NP]\!]$. By applying Lemma 2.4.4 to P and Q with the context $C[\,] = N[\,]$ we get $[\![NP]\!] \subseteq [\![NQ]\!]$. So we have $[\![MP]\!] \subseteq [\![NP]\!] \subseteq [\![NQ]\!]$, hence $MP \sqsubseteq_{\mathcal{D}} NQ$. □

Let us prove Point I. We show the invariance of $[\![-]\!]^{\mathcal{D}}$ under β-reduction (corresponding to the so-called *subject reduction* property of the type assignment system $\vdash^{\mathcal{D}}$), under β-expansion (corresponding to the *subject expansion* property of the type system), hence under β-convertibility (the so-called *soundness* of the semantics). A preliminary lemma stating the invariance for substitution is needed.

Lemma 2.4.7 (Substitution). *Let $\mathcal{D}$ be an rgm. Let $M, N \in \Lambda \cup \mathcal{N}$.*

1. *If the sequents $\Gamma, x : \bigwedge_{i=1}^{n}\gamma_i \vdash^{\mathcal{D}} M : \sigma$ and $\Gamma_i \vdash^{\mathcal{D}} N : \gamma_i$ for all $i \in \{1, \ldots, n\}$ are derivable then the sequent $\Gamma \wedge \bigwedge_{i=1}^{n}\Gamma_i \vdash^{\mathcal{D}} M\{N/x\} : \sigma$ is derivable.*

2. *If the sequent $\Gamma \vdash^{\mathcal{D}} M\{N/x\} : \sigma$ is derivable then there exist $\gamma_1, \ldots, \gamma_n \in T_{\mathcal{D}}$ and a decomposition $\{\Gamma_i\}_{i=0}^{n}$ of Γ such that $\Gamma_0, x : \bigwedge_{i=1}^{n}\gamma_i \vdash^{\mathcal{D}} M : \sigma$ and $\Gamma_i \vdash^{\mathcal{D}} N : \gamma_i$ for all $i \in \{1, \ldots, n\}$ are derivable.*

Proof. (1) By induction on the structure of the λ-term M.

Case $M = \bot$**.** This case does not need to be considered, as $\bot$ cannot be typed.

Case $M = y \neq x$**.** By hypothesis $\Gamma, x : \bigwedge_{i=1}^{n}\gamma_i \vdash y : \sigma$ is derivable. By Lemma 2.3.12(1) we have $\Gamma, x : \bigwedge_{i=1}^{n}\gamma_i = y : \tau$ for a type $\tau \simeq \sigma$. So $n = 0$ and $\Gamma = y : \tau$. Hence the sequent $\Gamma \wedge \bigwedge_{i=1}^{n}\Gamma_i \vdash M\{N/x\} : \sigma$ is nothing but $\Gamma \vdash M\{N/x\} : \sigma$. Since $M\{N/x\} = y\{N/x\} = y$, such sequent is $y : \tau \vdash y : \sigma$, which is derivable by Rules `var` and `eq`.

Case $M = x$**.** By hypothesis $\Gamma, x : \bigwedge_{i=1}^{n}\gamma_i \vdash x : \sigma$ is derivable. By Lemma 2.3.12(1) we have $\Gamma, x : \bigwedge_{i=1}^{n}\gamma_i = x : \tau$ for a type $\tau \simeq \sigma$. So Γ is the empty environment, $n = 1$ and $\bigwedge_{i=1}^{n}\gamma_i = \gamma_1 = \tau$. Hence the sequent $\Gamma \wedge \bigwedge_{i=1}^{n}\Gamma_i \vdash M\{N/x\} : \sigma$ is nothing but $\Gamma_1 \vdash M\{N/x\} : \sigma$. Since $M\{N/x\} = x\{N/x\} = N$, such sequent is $\Gamma_1 \vdash N : \sigma$. It is derivable using Rule `eq`, because $\Gamma_1 \vdash N : \gamma_1$ is derivable by hypothesis and $\gamma_1 = \tau \simeq \sigma$.

Case $M = \lambda y.P$**.** By Lemma 2.3.12(2) the sequent $\Gamma, y : \mu, x : \bigwedge_{i=1}^{n}\gamma_i \vdash P : \tau$ is derivable for some $\mu \to \tau \simeq \sigma$. By IH we can derive

$$(\Gamma, y : \mu) \wedge \bigwedge_{i=1}^{n}\Gamma_i \vdash P\{N/x\} : \tau\,. \tag{7}$$

For all $i \in \{1,\dots,n\}$ Lemma 2.3.10 gives $\mathrm{supp}(\Gamma_i) \subseteq \mathrm{fv}(N)$. By the Variable Convention $y \notin \mathrm{fv}(N)$. Therefore $y \notin \mathrm{supp}(\Gamma_i)$ for all $i \in \{1,\dots,n\}$. So the sequent (7) is in fact $\Gamma \wedge \bigwedge_{i=1}^{n}\Gamma_i, y:\mu \vdash P\{N/x\}:\tau$. In the end we can derive

$$\dfrac{\dfrac{\Gamma \wedge \bigwedge_{i=1}^{n}\Gamma_i, y:\mu \vdash P\{N/x\}:\tau}{\Gamma \wedge \bigwedge_{i=1}^{n}\Gamma_i \vdash \lambda y.(P\{N/x\}):\mu \to \tau}}{\Gamma \wedge \bigwedge_{i=1}^{n}\Gamma_i \vdash (\lambda y.P)\{N/x\}:\sigma}$$

Case $M = PQ$**.** By Lemma 2.3.12(3) there are $k \in \mathbb{N}$, a partition $\{\Delta_j\}_{j=0}^{k}$ of Γ, a decomposition $\{I_j\}_{j=0}^{k}$ of the set $\{1,\dots,n\}$ and $\sigma_1,\dots,\sigma_k \in T_{\mathcal{D}}$ such that the following are derivable: $\Delta_0, x : \bigwedge_{i\in I_0}\gamma_i \vdash P : \bigwedge_{j=1}^{k}\sigma_j \to \sigma$ and $\Delta_j, x : \bigwedge_{i\in I_j}\gamma_i \vdash Q : \sigma_j$ for all $j \in \{1,\dots,k\}$.

By IH then $\Delta_0 \wedge \bigwedge_{i\in I_0}\Gamma_i \vdash P\{N/x\} : \bigwedge_{j=1}^{k}\sigma_j \to \sigma$ and $\Delta_j \wedge \bigwedge_{i\in I_j}\Gamma_i \vdash Q\{N/x\} : \sigma_j$ for all $j \in \{1,\dots,k\}$ are derivable. By applying Rule `app` we derive

$$\bigwedge_{j=0}^{k}\Delta_j \wedge \bigwedge_{j=0}^{k} \bigwedge_{i\in I_j} \Gamma_i \vdash P\{N/x\}Q\{N/x\} : \sigma\,. \tag{8}$$

Since $\bigwedge_{j=0}^{k}\Delta_j = \Gamma$, $\bigwedge_{j=0}^{k} \bigwedge_{i\in I_j} \Gamma_i = \bigwedge_{i=1}^{n}\Gamma_i$ and $M\{N/x\} = PQ\{N/x\} = P\{N/x\}Q\{N/x\}$, the sequent (8) is $\Gamma \wedge \bigwedge_{i=1}^{n}\Gamma_i \vdash M\{N/x\} : \sigma$, whose derivability was to be proved.

(2) By induction on the structure of the λ-term M.

Case $M = \bot$**.** This case must not be considered, as $\bot$ cannot be typed.

Case $M = y \neq x$**.** As $M\{N/x\} = y\{N/x\} = y$, by hypothesis the sequent $\Gamma \vdash y : \sigma$ is derivable. Then setting $n := 0$ and $\Gamma_0 := \Gamma$ the decomposition $\{\Gamma_0\}$ of Γ proves the result.

Case $M = x$**.** As $M\{N/x\} = x\{N/x\} = N$, by hypothesis the sequent $\Gamma \vdash N : \sigma$ is derivable. We set $n := 1$ and we take Γ_0 to be the empty environment and $\Gamma_1 := \Gamma$. Then the decomposition $\{\Gamma_0, \Gamma_1\}$ of Γ proves the result.

Case $M = \lambda y.P$**.** Since $(\lambda y.P)\{N/x\} = \lambda y.P\{N/x\}$, by Lemma 2.3.12(2) there is a derivation of $\Gamma, y : \mu \vdash P\{N/x\} : \tau$ for some type $\mu \to \tau \simeq \sigma$.

By IH there are $\gamma_1,\dots,\gamma_n \in T_{\mathcal{D}}$ and a decomposition $\{\Gamma_i\}_{i=0}^{n}$ of $\Gamma, y : \mu$ such that

$$\Gamma_0, x : \bigwedge_{i=1}^{n}\gamma_i \vdash P : \sigma \tag{9}$$

and

$$\Gamma_i \vdash N : \gamma_i \tag{10}$$

for all $i \in \{1,\dots,n\}$ are derivable.

From (10) Lemma 2.3.10 gives $\mathrm{supp}(\Gamma_i) \subseteq \mathrm{fv}(N)$ for all $i \in \{1,\dots,n\}$ f. By the Variable Conventiony $\notin \mathrm{fv}(N)$. Therefore $y \notin \mathrm{supp}(\Gamma_i)$ for all $i \in \{1,\dots,n\}$. So

$$\Gamma_0(y) \quad = \quad (\Gamma, y : \mu)(y) \quad = \quad \mu\,.$$

Then from the sequent (9) by Rules `lam` and `eq` we derive

$$\Gamma_0 - (y : \mu), x : \bigwedge_{i=1}^{n}\gamma_i \ \vdash\ \lambda y.P : \sigma\,.$$

We are done, because $\{\Gamma_0 - (y : \mu)\} \cup \{\Gamma_i\}_{i=1}^{n}$ is a decomposition of Γ.

Case $M = PQ$. We have $M\{N/x\} = PQ\{N/x\} = P\{N/x\}Q\{N/x\}$.

By Lemma 2.3.12(3) there are $k \in \mathbb{N}$, a decomposition $\{\Delta_i\}_{i=0}^{k}$ of Γ and $\sigma_1, \ldots, \sigma_k \in T_{\mathcal{D}}$ such that the following are derivable:

$$\Delta_0 \vdash P\{N/x\} : \wedge_{i=1}^{k} \sigma_i \to \sigma \tag{11}$$

and for all $i \in \{1, \ldots, k\}$

$$\Delta_i \vdash Q\{N/x\} : \sigma_i \, . \tag{12}$$

From (11) by IH we have $n_0 \in \mathbb{N}$, a decomposition $\{\Delta_{0j}\}_{j=0}^{n_0}$ of Δ_0 and types $\gamma_{01}, \ldots, \gamma_{0n_0}$ such that

$$\Delta_{00}\, ,\, x : \wedge_{j=1}^{n_0} \gamma_{0j} \ \vdash \ P : \wedge_{i=1}^{k} \sigma_i \to \sigma \tag{13}$$

and for all $j \in \{1, \ldots, n_0\}$

$$\Delta_{0j} \ \vdash \ N : \gamma_{0j} \tag{14}$$

are derivable.

Let $i \in \{1, \ldots, k\}$. From (12) by IH we have $n_i \in \mathbb{N}$, a decomposition $\{\Delta_{ij}\}_{j=0}^{n_i}$ of Δ_i and types $\gamma_{i1}, \ldots, \gamma_{in_i}$ such that

$$\Delta_{i0}\, ,\, x : \wedge_{j=1}^{n_i} \gamma_{ij} \ \vdash \ Q : \sigma_i \tag{15}$$

and for all $j \in \{1, \ldots, n_i\}$

$$\Delta_{ij} \ \vdash \ N : \gamma_{ij} \tag{16}$$

are derivable.

From (13) and (15) by Rule app we derive

$$\wedge_{i=0}^{k} \Delta_{i0}\, ,\, x : \wedge_{i=0}^{k} \wedge_{j=1}^{n_i} \gamma_{ij} \ \vdash \ PQ : \sigma \tag{17}$$

Notice that $\{\Delta_{ij} \mid i \in \{0, \ldots, k\} \text{ and } j \in \{1, \ldots, n_i\}\}$ is a decomposition of Γ, as the union of decompositions of $\Delta_0, \ldots, \Delta_k$, which in turn form a decomposition of Γ. Such a decomposition then proves the result, because of the derivability of (14), (16) and (17). $\square$

Lemma 2.4.8 (Subject reduction). *Let $\mathcal{D}$ be an rgm. Let $M, N \in \Lambda \cup \mathcal{N}$ such that $M \twoheadrightarrow_\beta N$. If $\Gamma \vdash^{\mathcal{D}} M : \sigma$ is derivable then $\Gamma \vdash^{\mathcal{D}} N : \sigma$ is derivable.*

Proof. We proceed by induction on M.

Case $M = \bot$. This case does not need to be considered, as $\bot$ cannot be typed.

Case $M = x$. In such a case $N = x = M$, so there is nothing to prove.

Case $M = \lambda x.P$. Then $N = \lambda x.Q$ where $P \twoheadrightarrow_\beta Q$. By Lemma 2.3.12(2) the sequent $\Gamma, x : \mu \vdash P : \tau$ is derivable for some $\mu \to \tau \simeq \sigma$. By IH we get the derivability of $\Gamma, x : \mu \vdash Q : \tau$. Hence by applying Rules lam and eq we derive $\Gamma \vdash \lambda x.Q : \sigma$.

Case $M = PQ$ **and** $N = P'Q'$ **where** $P \twoheadrightarrow_\beta P'$ **and** $Q \twoheadrightarrow_\beta Q'$. By Lemma2.3.12(3) there are a decomposition $\{\Gamma_i\}_{i=0}^{n}$ of Γ and types $\sigma_1, \ldots, \sigma_n$ such that $\Gamma_0 \vdash P : \wedge_{i=1}^{n} \sigma_i \to \sigma$

and $\Gamma_i \vdash Q : \sigma_i$ for all $i \in \{1, \dots, n\}$ are derivable. By IH then $\Gamma_0 \vdash P' : \bigwedge_{i=1}^{n} \sigma_i \to \sigma$ and $\Gamma_i \vdash Q' : \sigma_i$ for all $i \in \{1, \dots, n\}$ are derivable. By means of Rule app we derive $\Gamma \vdash P'Q' : \sigma$.
Case $M = (\lambda x.P)Q$ **and** $N = P'\{Q'/x\}$ **where** $P \twoheadrightarrow_\beta P'$ **and** $Q \twoheadrightarrow_\beta Q'$**.** By Lemma 2.3.12(3) there are a decomposition $\{\Gamma_i\}_{i=0}^{n}$ of Γ and $\sigma_1, \dots, \sigma_n \in T_{\mathcal{D}}$ such that $\Gamma_0 \vdash \lambda x.P : \bigwedge_{i=1}^{n} \sigma_i \to \sigma$ and $\Gamma_i \vdash Q : \sigma_i$ for all $i \in \{1, \dots, n\}$ are derivable. Applying then Lemma 2.3.12(2) we get the derivability of $\Gamma_0, x : \bigwedge_{i=1}^{n} \tau_i \vdash P : \tau$ where $\bigwedge_{i=1}^{n} \tau_i \to \tau \simeq \bigwedge_{i=1}^{n} \sigma_i \to \sigma$. By IH the sequents $\Gamma_0, x : \bigwedge_{i=1}^{n} \tau_i \ \vdash \ P' : \tau$ and $\Gamma_i \vdash Q' : \sigma_i$ for all $i \in \{1, \dots, n\}$ are derivable. Hence Rule eq gives us the derivability of

$$\Gamma_0, x : \bigwedge_{i=1}^{n} \tau_i \ \vdash \ P' : \sigma \tag{18}$$

and

$$\Gamma_i \ \vdash \ Q' : \tau_i \qquad \text{for all } i \in \{1, \dots, n\}\,. \tag{19}$$

By (18) and (19), Lemma 2.4.7(1) assures that $\bigwedge_{i=0}^{n} \Gamma_i \vdash P'\{Q'/x\} : \tau$ is derivable. □

An alternative way to prove the subject reduction is the following. Firstly, one can start by proving the statement for the case $M \to_\beta N$. Then the result is obviously generalized to the case $M \twoheadrightarrow_\beta N$, formally by an induction on the number of one step-reductions $\to_\beta$ that are in $M \twoheadrightarrow_\beta N$, starting by base 1. Now, to prove the statement for $M \to_\beta N$ one can distinguish two cases:

- the case where $M = (\lambda x.P)Q$ and $N = P\{Q/x\}$, which is basically proved as the analogous seen in the proof above (except for the fact that there is no use of an IH);
- the case where $M = C[(\lambda x.P)Q]$ and $N = C[P\{Q/x\}]$, which immediately follows from the previous one by contextuality of the interpretation, i.e. Lemma 2.4.4.

Choosing between this way of formalizing the proof and the one used is nothing more than a matter of taste. For instance, we use such an alternative style in the proof of the subject reduction for the η-rule (Lemma 2.4.11).

Lemma 2.4.9 (Subject expansion)**.** *Let $\mathcal{D}$ be an rgm. Let $M, N \in \Lambda \cup \mathcal{N}$ such that $M \twoheadrightarrow_\beta N$. If $\Gamma \vdash^{\mathcal{D}} N : \sigma$ is derivable then $\Gamma \vdash^{\mathcal{D}} M : \sigma$ is derivable.*

Proof. We proceed by induction on M.
Case $M = \bot$**.** This case does not need to be considered, as $\bot$ cannot be typed.
Case $M = x$**.** In such a case $N = x = M$, so there is nothing to prove.
Case $M = \lambda x.P$**.** Then $N = \lambda x.Q$ where $P \twoheadrightarrow_\beta Q$. By Lemma 2.3.12(2) the sequent $\Gamma, x : \mu \vdash Q : \tau$ is derivable for some $\mu \to \tau \simeq \sigma$. By IH we get the derivability of $\Gamma, x : \mu \vdash P : \tau$. Hence by applying Rules lam and eq we derive $\Gamma \vdash \lambda x.P : \sigma$.
Case $M = PQ$ **and** $N = P'Q'$ **where** $P \twoheadrightarrow_\beta P'$ **and** $Q \twoheadrightarrow_\beta Q'$**.** By Lemma2.3.12(3) there are a decomposition $\{\Gamma_i\}_{i=0}^{n}$ of Γ and types $\sigma_1, \dots, \sigma_n$ such that $\Gamma_0 \vdash P' : \bigwedge_{i=1}^{n} \sigma_i \to \sigma$ and $\Gamma_i \vdash Q' : \sigma_i$ for all $i \in \{1, \dots, n\}$ are derivable. By IH then $\Gamma_0 \vdash P : \bigwedge_{i=1}^{n} \sigma_i \to \sigma$ and $\Gamma_i \vdash Q : \sigma_i$ for all $i \in \{1, \dots, n\}$ are derivable. By means of Rule app we derive $\Gamma \vdash PQ : \sigma$.
Case $M = (\lambda x.P)Q$ **and** $N = P'\{Q'/x\}$ **where** $P \twoheadrightarrow_\beta P'$ **and** $Q \twoheadrightarrow_\beta Q'$**.** By Lemma 2.4.7(2) there are a decomposition $\{\Gamma_i\}_{i=0}^{n}$ of Γ and $\sigma_1, \dots, \sigma_n \in T_{\mathcal{D}}$ such that $\Gamma, x : \bigwedge_{i=1}^{n} \gamma_i \vdash P' : \sigma$

and $\Gamma_i \vdash Q' : \gamma_i$ for all $i \in \{1, \dots, n\}$ are derivable. Hence by IH the sequents $\Gamma, x : \bigwedge_{i=1}^{n} \gamma_i \vdash P : \sigma$ and $\Gamma_i \vdash Q : \gamma_i$ for all $i \in \{1, \dots, n\}$ are derivable. Then we can infer

$$\dfrac{\dfrac{\Gamma, x : \bigwedge_{i=1}^{n} \gamma_i \vdash P : \sigma}{\Gamma \vdash \lambda x.P : \bigwedge_{i=1}^{n} \gamma_i \to \sigma} \qquad \Gamma_i \vdash Q : \gamma_i \ \text{ for all } i \in \{1, \dots, n\}}{\Gamma \vdash (\lambda x.P)Q : \sigma}$$

and we are done. □

Theorem 2.4.10 (Soundness for β-conversion)**.** *Let $\mathcal{D}$ be an rgm. Let $M, N \in \Lambda \cup \mathcal{N}$ such that $M =_\beta N$. Then $[\![M]\!]^{\mathcal{D}} = [\![N]\!]^{\mathcal{D}}$.*

Proof. By hypothesis for some $k \in \mathbb{N}$ there is a finite sequence $(M_i)_{i=0}^{k}$ of λ-terms such that $M_0 = M$, $M_k = N$ and for all $i \in \{0, \dots, k-1\}$ either $M_i \to_\beta M_{i+1}$ or $M_{i+1} \to_\beta M_i$.

The result is proved by the following induction on k.

Base: $k = 0$. As $M = M_0 = N$, the thesis trivially holds.

Step: $k \geqslant 1$. The IH is applied to the sequence $(M_i)_{i=1}^{k}$ so to get $[\![M_1]\!] = [\![M_k]\!]$.

Let us see that $[\![M_0]\!] = [\![M_1]\!]$. In case $M_0 \to_\beta M_1$ we have $[\![M_0]\!] \subseteq [\![M_1]\!]$ by Lemma 2.4.8 and $[\![M_1]\!] \subseteq [\![M_0]\!]$ by Lemma 2.4.9. So $[\![M_0]\!] = [\![M_1]\!]$. The case $M_1 \to_\beta M_0$ is dual.

We can conclude that $[\![M]\!] = [\![M_0]\!] = [\![M_1]\!] = [\![M_k]\!] = [\![N]\!]$. □

We now move to Point II, namely the invariance under η-conversion.

Lemma 2.4.11 (η-subject reduction)**.** *Let $\mathcal{D}$ be an rgm. Let $M, N \in \Lambda \cup \mathcal{N}$ such that $M \twoheadrightarrow_\eta N$. If $\Gamma \vdash^{\mathcal{D}} M : \sigma$ is derivable then $\Gamma \vdash^{\mathcal{D}} N : \sigma$ is derivable.*

Proof. We prove the statement for the case $M \to_\eta N$. Then the result is obviously generalized to the case $M \twoheadrightarrow_\eta N$ (formally by induction on the number of steps $\to_\eta$ in $\twoheadrightarrow_\eta$).

Case $M = \lambda x.Nx$ **for** $x \notin \mathrm{fv}(N)$**.** Let $\Gamma \vdash \lambda x.Nx : \sigma$ be derivable. By Lemma 2.3.12(2) we can derive $\Gamma, x : \mu \vdash Nx : \tau$ for some $\mu \to \tau \simeq \sigma$. By Lemma 2.3.12(3) there exist types $\tau_1, \dots, \tau_n$ and a decomposition $\{\Gamma_i\}_{i=0}^{n}$ of the environment $\Gamma, x : \mu$ such that

$$\Gamma_0 \vdash N : \bigwedge_{i=1}^{n} \tau_i \to \tau \tag{20}$$

and $\Gamma_i \vdash x : \tau_i$ for all $i \in \{1, \dots, n\}$ are derivable. Then for all $i \in \{1, \dots, n\}$ Lemma 2.3.12(1) gives $\Gamma_i = x : \gamma_i$ for $\gamma_i \simeq \tau_i$. Hence $\bigwedge_{i=1}^{n} \Gamma_i = x : \bigwedge_{i=1}^{n} \gamma_i$.

Since $\mathrm{supp}(\Gamma_0) \subseteq \mathrm{fv}(N)$ by Lemma 2.3.10 and $x \notin \mathrm{fv}(N)$, we have $\Gamma_0(x) = \omega$. Thus

$$\bigwedge_{i=1}^{n} \tau_i \;\simeq\; \bigwedge_{i=1}^{n} \gamma_i \;=\; \left(\bigwedge_{i=1}^{n} \Gamma_i\right)(x) \;=\; \left(\bigwedge_{i=0}^{n} \Gamma_i\right)(x) \;=\; (\Gamma, x : \mu)(x) \;=\; \mu\,.$$

Therefore $\bigwedge_{i=1}^{n} \tau_i \to \tau \simeq \mu \to \tau \simeq \sigma$. And from the fact that $\bigwedge_{i=1}^{n} \Gamma_i = x : \bigwedge_{i=1}^{n} \gamma_i = x : \mu$ we also get that $\Gamma_0 = \Gamma$. Finally from (20) by Rule eq we derive $\Gamma \vdash N : \sigma$.

Case $M = C[\lambda x.N'x]$ **for** $x \notin \mathrm{fv}(N')$ **and** $N = C[N']$**.** By the case above and Lemma 2.4.11. □

Lemma 2.4.12 (η-subject expansion)**.** *Let $\mathcal{D}$ be an ergm. Let $M, N \in \Lambda \cup \mathcal{N}$ such that $M \twoheadrightarrow_\eta N$. If $\Gamma \vdash^{\mathcal{D}} N : \sigma$ is derivable then $\Gamma \vdash^{\mathcal{D}} M : \sigma$ is derivable.*

$$\dfrac{}{x:\sigma \vdash^{\mathcal{D}} x:\sigma}\ \texttt{var} \qquad \dfrac{\Gamma, x:\mu \vdash^{\mathcal{D}} t:\sigma}{\Gamma \vdash^{\mathcal{D}} \lambda x.t:\mu\to\sigma}\ \texttt{lam} \qquad \dfrac{\Gamma \vdash^{\mathcal{D}} t:\tau \qquad \sigma \simeq^{\mathcal{D}} \tau}{\Gamma \vdash^{\mathcal{D}} t:\sigma}\ \texttt{eq}$$

$$\dfrac{\Gamma_0 \vdash^{\mathcal{D}} t:\bigwedge_{i=1}^{n}\sigma_i\to\sigma \qquad \Gamma_i \vdash^{\mathcal{D}} s_i:\sigma_i \quad \text{for all } i\in\{1,\ldots,n\}}{\Gamma_0 \wedge \left(\bigwedge_{i=1}^{n}\Gamma_i\right) \vdash^{\mathcal{D}} t\left[s_1,\ldots,s_n\right]:\sigma}\ \texttt{app}$$

Figure 4: The intersection type systems for Λ^r.

Proof. We prove the statement for the case $M \to_\eta N$. Then the result is obviously generalized to the case $M \twoheadrightarrow_\eta N$ (formally by induction on the number of steps $\to_\eta$ in $\twoheadrightarrow_\eta$).
Case $M = \lambda x.Nx$ **for** $x \notin fv(N)$**.** Since $i : \mathcal{M}_f(\mathcal{D}) \times \mathcal{D} \to \mathcal{D}$ is surjective, there exists $(m, d) \in \mathcal{M}_f(\mathcal{D}) \times \mathcal{D}$ such that $i(m, d) = \sigma^\diamond$. By Lemma 2.3.4 we have $\mu \in I_{\mathcal{D}}$ such that $\mu^\diamond = m$ and $\tau \in T_{\mathcal{D}}$ such that $\tau^\diamond = d$. Hence $\sigma^\diamond = i(m, d) = i(\mu^\diamond, \tau^\diamond) = (\mu \to \tau)^\diamond$, i.e. $\sigma \simeq \mu \to \tau$. In the end, if $\mu = \bigwedge_{i=1}^{n} \gamma_i$ we can derive

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash N:\sigma}{\Gamma \vdash N:\mu\to\tau} \quad x:\gamma_i \vdash x:\gamma_i \text{ for all } i\in\{1,\ldots,n\}}{\Gamma, x:\mu \vdash Nx:\tau}}{\Gamma \vdash \lambda x.Nx:\mu\to\tau}$$

Case $M = C[\lambda x.N'x]$ **for** $x \notin fv(N')$ **and** $N = C[N']$**.** By the case above and Lemma 2.4.11. □

Theorem 2.4.13 (Soundness for η-conversion)**.** *Let $\mathcal{D}$ be an ergm. Let $M, N \in \Lambda \cup \mathcal{N}$ such that $M =_\eta N$. Then $[\![M]\!]^{\mathcal{D}} = [\![N]\!]^{\mathcal{D}}$.*

Proof. Just like the proof of Theorem 2.4.10, only replacing the reduction $\to_\beta$ with $\to_\eta$, and Lemmas 2.4.8 - 2.4.9 with Lemmas 2.4.11 - 2.4.12 respectively. □

2.5 INTERPRETING THE LINEAR RESOURCE CALCULUS

Rgm's are also models of the linear resource calculus presented in § 1.6. One way to prove this is to show that the corresponding reflexive objects are *linear* in the cartesian closed *differential* category **MRel**, in the sense of [Man12, § 4]. Here we do not follow that line. We exploit directly the type assignment discipline instead.

Definition 2.5.1. Let $\mathcal{D}$ be an rgm. The *type assignment system* $\vdash^{\mathcal{D}}$ *for* Λ^r *associated with* $\mathcal{D}$ is given in Fig. 4. When $\mathcal{D}$ is clear from the context we simply write $\vdash$ instead of $\vdash^{\mathcal{D}}$.

Remark 2.5.2. In Remark 2.3.9 we saw that from $\Gamma \vdash M : \omega \to \sigma$ one can deduce $\Gamma \vdash MN : \sigma$ for no matter what $N \in \Lambda$. This is not exactly the case for Λ^r. Of course, also the natural number n appearing in Rule app in Fig. 4 can be 0. But the premises of this version of app require a perfect matching between the number n of types σ_i that compose the intersection

$\bigwedge_{i=1}^{n}\sigma_i$ and the number n of elements of the bag $[s_1,\dots,s_n]$ that we put in argument position. So for $t\in\Lambda^r$ all we can derive is

$$\frac{\Gamma\vdash^{\mathcal{D}} t:\omega\to\sigma}{\Gamma\vdash^{\mathcal{D}} t[\,]\;:\sigma}$$

and not $\Gamma\vdash^{\mathcal{D}} t\,\mathbf{b}\;:\sigma$ for every $\mathbf{b}\in\mathcal{M}_f(\Lambda^r)$.

Lemma 2.5.3 (Inversion)**.** *Let $\mathcal{D}$ be an rgm. Let $t\in\Lambda^r$ and $\mathbf{b}\in\mathcal{M}_f(\Lambda^r)$.*

1. *If $\Gamma\vdash^{\mathcal{D}} x:\sigma$ is derivable then there exists $\tau\in\mathsf{T}_{\mathcal{D}}$ such that $\Gamma=x:\tau$ and $\tau\simeq^{\mathcal{D}}\sigma$.*
2. *The sequent $\Gamma\vdash^{\mathcal{D}}\lambda x.t\colon\sigma$ is derivable if and only if there exist $\tau\in\mathsf{T}_{\mathcal{D}}$ and $\mu\in\mathsf{I}_{\mathcal{D}}$ such that $\Gamma,x:\mu\vdash^{\mathcal{D}} t:\tau$ is derivable and $\mu\to\tau\simeq^{\mathcal{D}}\sigma$;*
3. *If $\Gamma\vdash^{\mathcal{D}} t[s_1,\dots,s_n]:\sigma$ is derivable then there exist $\sigma_1,\dots,\sigma_n\in\mathsf{T}_{\mathcal{D}}$ and a decomposition $\{\Gamma_i\}_{i=0}^{n}$ of Γ such that the sequents $\Gamma_0\vdash^{\mathcal{D}} t:\bigwedge_{i=1}^{n}\sigma_i\to\sigma$ and $\Gamma_i\vdash^{\mathcal{D}} s_i:\sigma_i$ for all $i\in\{1,\dots,n\}$ are derivable.*

Proof. Similar to the proof of Lemma 2.3.12. □

Definition 2.5.4. Let $\mathcal{D}$ be an rgm. Let $t\in\Lambda^r$. The *type-interpretation* of t in $\mathcal{D}$ is defined as

$$[\![t]\!]^{\mathcal{D}} := \Big\{(\Gamma,\sigma)\in\mathsf{Env}_{\mathcal{D}}\times\mathsf{T}_{\mathcal{D}}\mid\Gamma\vdash^{\mathcal{D}} t:\sigma\Big\}.$$

An analogue of Lemma 2.3.10 holds also for resource terms, as we state here below.

Lemma 2.5.5. *Let $\mathcal{D}$ be an rgm and $t\in\Lambda^r$. If $\Gamma\vdash^{\mathcal{D}} t:\sigma$ is derivable then* $\mathrm{supp}(\Gamma)\subseteq\mathrm{fv}(t)$.

We prove that this type-interpretation is invariant under β-reduction (subject reduction property) and under β-expansion (subject expansion property), hence it provides a sound model of the linear resource calculus. As is customary, we start with an appropriate substitution lemma.

Lemma 2.5.6 (Substitution for Λ^r)**.** *Let $\mathcal{D}$ be an rgm. Let $t\in\Lambda^r$.*

1. *If the sequents $\Gamma,x:\bigwedge_{i=1}^{n}\gamma_i\vdash^{\mathcal{D}} t\colon\sigma$ and $\Gamma_i\vdash^{\mathcal{D}} s_i:\gamma_i$ for all $i\in\{1,\dots,n\}$ are derivable then $n=\deg_t(x)$ and, given an enumeration $x_1,\dots,x_n$ of all occurrences of x in t, there exists $p\in\mathfrak{S}_n$ such that $\Gamma\wedge\bigwedge_{i=1}^{n}\Gamma_i\vdash^{\mathcal{D}} t\big\{s_{p(1)}/x_1,\dots,s_{p(n)}/x_n\big\}:\sigma$ is derivable.*
2. *Let $n=\deg_t(x)$ and consider an enumeration $x_1,\dots,x_n$ of all occurrences of x in t. If the sequent $\Gamma\vdash^{\mathcal{D}} t\big\{s_1/x_1,\dots,s_n/x_n\big\}:\sigma$ is derivable then there are $\gamma_1,\dots,\gamma_n\in\mathsf{T}_{\mathcal{D}}$ and a decomposition $\{\Gamma_i\}_{i=0}^{n}$ of Γ such that the sequents $\Gamma,x:\bigwedge_{i=1}^{n}\gamma_i\vdash^{\mathcal{D}} t\colon\sigma$ and $\Gamma_i\vdash^{\mathcal{D}} s_i:\gamma_i$ for all $i\in\{1,\dots,n\}$ are derivable.*

Proof. (1) By induction on the structure of t.

Case $t = y \neq x$**.** By Lemma 2.5.3.(1) we have $\Gamma, x : \bigwedge_{i=1}^{n} \gamma_i = y : \tau$ for some $\tau \simeq \sigma$. Hence $\bigwedge_{i=1}^{n} \gamma_i = \omega$, that is $n = 0 = \deg_y(x)$.

By hypothesis $\Gamma \vdash y : \sigma$ is derivable, and we are done. (Notice that $\mathfrak{S}_0 = \{\emptyset\}$, so formally $\emptyset$ is the permutation that we pick here.)

Case $t = x$**.** By hypothesis $\Gamma, x : \bigwedge_{i=1}^{n} \gamma_i \vdash x : \sigma$ is derivable. By Lemma 2.5.3.(1) we have $\Gamma, x : \bigwedge_{i=1}^{n} \gamma_i = x : \tau$ for a type $\tau \simeq \sigma$. So Γ is the empty environment and $\bigwedge_{i=1}^{n} \gamma_i = \gamma_1 = \tau$. In particular $n = 1 = \deg_x(x)$.

Obviously we take as p the only element of $\mathfrak{S}_1$, i.e. $1 \mapsto 1$. We must prove the derivability of $\Gamma_1 \vdash x\{s_1/x_1\} : \tau$, which is the sequent $\Gamma_1 \vdash s_1 : \tau$. Since $\Gamma_1 \vdash s_1 : \sigma$ is derivable by hypothesis and $\tau \simeq \sigma$, by applying Rule eq we are done.

Case $t = \lambda y.s$**.** By Lemma 2.5.3.(2) the sequent $\Gamma, y : \mu, x : \bigwedge_{i=1}^{n} \gamma_i \vdash s : \tau$ is derivable for some $\mu \to \tau \simeq \sigma$. By IH we have $n = \deg_s(x) = \deg_{\lambda y.s}(x)$ and there exists $p \in \mathfrak{S}_n$ such that

$$(\Gamma, y : \mu) \wedge \bigwedge_{i=1}^{n} \Gamma_i \vdash s\{s_{p(1)}/x_1, \dots, s_{p(n)}/x_n\} : \tau \,. \tag{21}$$

is derivable. For all $i \in \{1, \dots, n\}$ Lemma 2.5.5 gives $\mathrm{supp}(\Gamma_i) \subseteq \mathrm{fv}(s_i)$. By the Variable Convention $y \notin \mathrm{fv}(s_i)$. Therefore $y \notin \mathrm{supp}(\Gamma_i)$ for all $i \in \{1, \dots, n\}$. So the sequent (21) is in fact $\Gamma \wedge \bigwedge_{i=1}^{n} \Gamma_i, y : \mu \vdash s\{s_{p(1)}/x_1, \dots, s_{p(n)}/x_n\} : \tau$. So at last we derive

$$\cfrac{\cfrac{\Gamma \wedge \bigwedge_{i=1}^{n} \Gamma_i, y : \mu \vdash s\{s_{p(1)}/x_1, \dots, s_{p(n)}/x_n\} : \tau}{\Gamma \wedge \bigwedge_{i=1}^{n} \Gamma_i \vdash \lambda y.(s\{s_{p(1)}/x_1, \dots, s_{p(n)}/x_n\}) : \mu \to \tau}}{\Gamma \wedge \bigwedge_{i=1}^{n} \Gamma_i \vdash (\lambda y.s)\{s_{p(1)}/x_1, \dots, s_{p(n)}/x_n\} : \sigma}$$

Case $t = s[r_1, \dots, r_k]$ **for some** $k \in \mathbb{N}$**.** By Lemma 2.5.3(3) there are a decomposition $\{\Delta_j\}_{j=0}^{k}$ of Γ, a decomposition $\{I_j\}_{j=0}^{k}$ of the set $\{1, \dots, n\}$ and $\sigma_1, \dots, \sigma_k \in \mathrm{T}_{\mathcal{D}}$ such that the following are derivable: $\Delta_0, x : \bigwedge_{i \in I_0} \gamma_i \vdash s : \bigwedge_{j=1}^{k} \sigma_j \to \sigma$ and $\Delta_j, x : \bigwedge_{i \in I_j} \gamma_i \vdash r_j : \sigma_j$ for all $j \in \{1, \dots, k\}$. For all $j \in \{0, \dots, k\}$ we apply the IH so to get:

- $\deg_s(x) = |I_0|$;
- $p_0 \in \mathfrak{S}_{|I_0|}$ such that $\Delta_0 \wedge \bigwedge_{i \in I_0} \Gamma_i \vdash s\{s_{p_0(1)}/x_1, \dots, s_{p_0(|I_0|)}/x_{|I_0|}\} : \bigwedge_{j=1}^{k} \sigma_j \to \sigma$ is derivable;
- $\deg_{r_j}(x) = |I_j|$ for all $j \in \{1, \dots, k\}$;
- $p_j \in \mathfrak{S}_{|I_j|}$ such that $\Delta_j \wedge \bigwedge_{i \in I_j} \Gamma_i \vdash r_j\{s_{p_j(1)}/x_1, \dots, s_{p_j(|I_j|)}/x_{|I_j|}\} : \sigma_j$ is derivable, for all $j \in \{1, \dots, k\}$.

Firstly, we get

$$\deg_{s[r_1, \dots, r_k]}(x) \;=\; \deg_{s_j}(x) + \sum_{j=1}^{k} \deg_{r_j}(x) \;=\; \sum_{j=0}^{k} |I_j| \;=\; n \,.$$

By applying Rule app we derive

$$\bigwedge_{j=0}^{k} \Delta_j \wedge \bigwedge_{j=0}^{k} \bigwedge_{i \in I_j} \Gamma_i \vdash s\{s_{p_0(i)}/x_i\}_{i=1}^{|I_0|} \left[r_j\{s_{p_j(i)}/x_i\}_{i=1}^{|I_j|} \right]_{j=1}^{k} : \sigma \,. \tag{22}$$

We define

$$\begin{array}{rccc} p: & \{1,\dots,n\} & \longrightarrow & \{1,\dots,n\} \\ & m \in I_j & \mapsto & p_j(m) \end{array}$$

Such p is well defined as a function, because $\{I_j\}_{j=0}^k$ is a decomposition $\{1,\dots,n\}$. Moreover $p \in \mathfrak{S}_n$, since all the p_j's are bijective. It is evident that

$$\begin{aligned} s\{s_{p_0(i)}/x_i\}_{i=1}^{|I_0|}\left[r_j\{s_{p_j(i)}/x_i\}_{i=1}^{|I_j|}\right]_{j=1}^k &= s\{s_{p(i)}/x_i\}_{i=1}^{|I_0|}\left[r_j\{s_{p(i)}/x_i\}_{i=1}^{|I_j|}\right]_{j=1}^k \\ &= s[r_1,\dots,r_k]\{s_{p(i)}/x_i\}_{i=1}^n \end{aligned}$$

Since in addition $\wedge_{j=0}^k \Delta_j = \Gamma$ and $\wedge_{j=0}^k \wedge_{i \in I_j} \Gamma_i = \wedge_{i=1}^n \Gamma_i$, the sequent (22) is exactly $\Gamma \wedge \wedge_{i=1}^n \Gamma_i \vdash s[r_1,\dots,r_k]\{s_{p(1)}/x_1,\dots,s_{p(n)}/x_n\} : \sigma$, whose derivability was to be proved.

(2) By induction on the structure of the resource term t.

Case $t = y \neq x$. In such a case $n = 0$, hence $t\{s_1/x_1,\dots,s_n/x_n\} = t = y$. So by hypothesis $\Gamma \vdash y : \sigma$ is derivable. Then setting $\Gamma_0 := \Gamma$ the decomposition $\{\Gamma_0\}$ of Γ proves the result.

Case $t = x$. In this case $n = 1$ and $t\{s_1/x_1\} = s_1$. Hence by hypothesis $\Gamma \vdash s_1 : \sigma$ is derivable. We set Γ_0 to be the empty environment, $\Gamma_1 := \Gamma$ and $\gamma_1 := \sigma$. The sequent $\Gamma_0, x : \gamma_1 \vdash x : \sigma$ is nothing but $x : \sigma \vdash x : \sigma$, which is trivially derivable. So the decomposition $\{\Gamma_0, \Gamma_1\}$ of Γ proves the result.

Case $t = \lambda y.s$. Since $(\lambda y.s)\{s_1/x_1,\dots,s_n/x_n\} = \lambda y.s\{s_1/x_1,\dots,s_n/x_n\}$, by Lemma 2.5.3.(2) there is a derivation of $\Gamma, y : \mu \vdash s\{s_1/x_1,\dots,s_n/x_n\} : \tau$ for some $\mu \to \tau \simeq \sigma$.

By IH there are $\gamma_1,\dots,\gamma_n \in T_{\mathcal{D}}$ and a decomposition $\{\Gamma_i\}_{i=0}^n$ of $\Gamma, y : \mu$ such that

$$\Gamma_0, x : \wedge_{i=1}^n \gamma_i \vdash s : \sigma \tag{23}$$

and

$$\Gamma_i \vdash s_i : \gamma_i$$

for all $i \in \{1,\dots,n\}$ are derivable.

For all $i \in \{1,\dots,n\}$ Lemma 2.5.5 gives $\mathrm{supp}(\Gamma_i) \subseteq \mathrm{fv}(s_i)$. By the Variable Convention $y \notin \mathrm{fv}(s_i)$. Therefore $y \notin \mathrm{supp}(\Gamma_i)$ for all $i \in \{1,\dots,n\}$. So

$$\Gamma_0(y) = (\Gamma, y : \mu)(y) = \mu\,.$$

Then from the sequent (23) by Rules lam and eq we derive $\Gamma_0 - (y : \mu), x : \wedge_{i=1}^n \gamma_i \vdash \lambda y.P : \sigma$. We are done, because $\{\Gamma_0 - (y : \mu)\} \cup \{\Gamma_i\}_{i=1}^n$ is a decomposition of Γ.

Case $t = s[r_1,\dots,r_k]$ **for some** $k \in \mathbb{N}$. We have a decomposition $\{I_i\}_{i=0}^k$ of the set $\{1,\dots,n\}$ such that $s\{s_1/x_1,\dots,s_n/x_n\} = s\{s_j/x_j\}_{j \in I_0}\left[r_1\{s_j/x_j\}_{j \in I_1},\dots, r_k\{s_j/x_j\}_{j \in I_k}\right]$.

By Lemma 2.5.3.(3) there is a decomposition $\{\Delta_i\}_{i=0}^k$ of Γ and $\sigma_1,\dots,\sigma_k \in T_{\mathcal{D}}$ such that the following sequents are derivable:

$$\Delta_0 \vdash s\{s_j/x_j\}_{j \in I_0} : \wedge_{i=1}^k \sigma_i \to \sigma \tag{24}$$

and for all $i \in \{1,\dots,k\}$

$$\Delta_i \vdash r_i\{s_j/x_j\}_{j \in I_i} : \sigma_i\,. \tag{25}$$

From (24) by IH we have a decomposition $\{\Delta_{0j}\}_{j=0}^{|I_0|}$ of Δ_0 and types $\gamma_{01}, \ldots, \gamma_{0n_0}$ such that

$$\Delta_{00}\,,\, x : \bigwedge_{j=1}^{|I_0|}\gamma_{0j} \;\vdash\; s : \bigwedge_{i=1}^{k}\sigma_i \to \sigma \tag{26}$$

and for all $j \in \{1, \ldots, |I_0|\}$

$$\Delta_{0j} \;\vdash\; s_j : \gamma_{0j} \tag{27}$$

are derivable.

Let $i \in \{1, \ldots, k\}$. From (25) by IH we have a decomposition $\{\Delta_{ij}\}_{j=0}^{|I_i|}$ of Δ_i and types $\gamma_{i1}, \ldots, \gamma_{in_i}$ such that

$$\Delta_{i0}\,,\, x : \bigwedge_{j=1}^{|I_i|}\gamma_{ij} \;\vdash\; r_i : \sigma_i \tag{28}$$

and for all $j \in \{1, \ldots, |I_i|\}$

$$\Delta_{ij} \;\vdash\; s_j : \gamma_{ij} \tag{29}$$

are derivable.

From (26) and (28) by Rule app we derive

$$\bigwedge_{i=0}^{k}\Delta_{i0}\,,\, x : \bigwedge_{i=0}^{k}\bigwedge_{j=1}^{|I_i|}\gamma_{ij} \;\vdash\; s\,[r_1, \ldots, r_k] : \sigma \tag{30}$$

Notice that $\{\Delta_{ij} \mid i \in \{0, \ldots, k\} \text{ and } j \in \{1, \ldots, |I_i|\}\}$ is a decomposition of Γ, as the union of decompositions of elements of a decomposition of Γ. Such a decomposition then proves the result, because of the derivability of (27), (29) and (30). □

Lemma 2.5.7 (Subject reduction for Λ^r)**.** *Let $\mathcal{D}$ be an rgm. Let $t \in \Lambda^r$ and $\mathbb{T} \in \mathcal{P}_f(\Lambda^r)$ such that $t \twoheadrightarrow_\beta \mathbb{T}$. If $\Gamma \vdash^{\mathcal{D}} t : \sigma$ is derivable then there exists $t' \in \mathbb{T}$ such that $\Gamma \vdash^{\mathcal{D}} t' : \sigma$ is derivable.*

Proof. We prove the statement for the case $t \to_\beta \mathbb{T}$. Then the result is obviously generalized to the case $t \twoheadrightarrow_\beta \mathbb{T}$ (formally by induction on the number of steps $\to_\beta$).

The proof is by induction on t.

Case $t = x$**.** In such a case $\mathbb{T} = \{x\}$, so taking $t' := x$ the thesis is proved.

Case $t = \lambda x.s$**.** Then $\mathbb{T} = \lambda x.\mathbb{S}$ where $s \to_\beta \mathbb{S}$. By Lemma 2.5.3.(2) the sequent $\Gamma, x : \mu \vdash s : \tau$ is derivable for some $\mu \to \tau \simeq \sigma$. By IH there exists $s' \in \mathbb{S}$ such that $\Gamma, x : \mu \vdash s' : \tau$ is derivable, hence by applying lam and eq also $\Gamma \vdash \lambda x.s' : \sigma$ is derivable. We can take $t' := \lambda x.s' \in \lambda x.\mathbb{S}$.

Case $t = s\mathbf{b}$ **and** $\mathbb{T} = \mathbb{S}\mathbf{b}$ **where** $s \to_\beta \mathbb{S}$**.** Let $\mathbf{b} = [s_1, \ldots, s_n]$. By Lemma 2.5.3(3) we have a decomposition $\{\Gamma_i\}_{i=0}^{n}$ of Γ and types $\sigma_1, \ldots, \sigma_n$ such that $\Gamma_0 \vdash s : \bigwedge_{i=1}^{n}\sigma_i \to \sigma$ and $\Gamma_i \vdash s_i : \sigma_i$ for all $i \in \{1, \ldots, n\}$ are derivable. By IH there exists $s' \in \mathbb{S}$ such that $\Gamma_0 \vdash s' : \bigwedge_{i=1}^{n}\sigma_i \to \sigma$ is derivable. So by app we derive $\Gamma \vdash s'\mathbf{b} : \sigma$. Clearly we take $t' := s'\mathbf{b} \in \mathbb{S}\mathbf{b}$.

Case $t = s\big([r] \cdot \mathbf{b}\big)$ **and** $\mathbb{T} = s\big([\mathbb{R}] \cdot \mathbf{b}\big)$ **where** $r \to_\beta \mathbb{R}$**.** Let $\mathbf{b} = [s_1, \ldots, s_n]$. By Lemma 2.5.3(3) we have a decomposition $\Delta \cup \{\Gamma_i\}_{i=0}^{n}$ of Γ and $\tau, \sigma_1, \ldots, \sigma_n \in T_{\mathcal{D}}$ such that the sequents $\Gamma_0 \vdash s : \tau \wedge \bigwedge_{i=1}^{n}\sigma_i \to \sigma$, $\Delta \vdash r : \tau$ and $\Gamma_i \vdash s_i : \sigma_i$ for all $i \in \{1, \ldots, n\}$ are derivable. By IH

there exists $r' \in \mathbb{R}$ such that $\Delta \vdash r' : \tau$ is derivable. So by app we derive $\Gamma \vdash s\big([r'] \cdot \mathbf{b}\big) : \sigma$. In the end we take $t' := s\big([r'] \cdot \mathbf{b}\big) \in s\big([\mathbb{R}] \cdot \mathbf{b}\big)$ and we are done.
Case $t = (\lambda x.s)[s_1, \dots, s_n]$**.** By Lemma 2.5.3(3) there are a decomposition $\{\Gamma_i\}_{i=0}^n$ of Γ and $\sigma_1, \dots, \sigma_n \in T_{\mathcal{D}}$ such that $\Gamma_0 \vdash \lambda x.s : \bigwedge_{i=1}^n \sigma_i \to \sigma$ and $\Gamma_i \vdash s_i : \sigma_i$ for all $i \in \{1, \dots, n\}$ are derivable. Applying then Lemma 2.5.3(2) we get the derivability of $\Gamma_0, x : \bigwedge_{i=1}^n \tau_i \vdash s : \tau$ where $\bigwedge_{i=1}^n \tau_i \to \tau \simeq \bigwedge_{i=1}^n \sigma_i \to \sigma$. By Rule eq we derive

$$\Gamma_0, x : \textstyle\bigwedge_{i=1}^n \tau_i \vdash s : \sigma \tag{31}$$

and

$$\Gamma_i \vdash s_i : \tau_i \qquad \text{for all } \ i \in \{1, \dots, n\}\,. \tag{32}$$

By (31) and (32), we can apply Lemma 2.5.6(1) so to get the equality $\deg_s(x) = n$ and the existence of $p \in \mathfrak{S}_n$ making $\Gamma_0 \wedge \bigwedge_{i=1}^n \Gamma_i \vdash s\left\{s_{p(1)}/x_1, \dots, s_{p(n)}/x_n\right\} : \sigma$ derivable.
Since $\deg_s(x) = n$, by definition of $t \to_\beta \mathbb{T}$ we have $\mathbb{T} = \bigcup_{p \in \mathfrak{S}_n} s\left\{s_{p(1)}/x_1, \dots, s_{p(n)}/x_n\right\}$. So eventually $t' := s\left\{s_{p(1)}/x_1, \dots, s_{p(n)}/x_n\right\} \in \mathbb{T}$ proves the thesis. □

Lemma 2.5.8 (Subject expansion for Λ^r)**.** *Let $\mathcal{D}$ be an rgm. Let $t \in \Lambda^r$ and $\mathbb{T} \in \mathcal{P}_f(\Lambda^r)$ such that $t \twoheadrightarrow_\beta \mathbb{T}$. If there exists $t' \in \mathbb{T}$ such that $\Gamma \vdash^{\mathcal{D}} t' : \sigma$ is derivable then $\Gamma \vdash^{\mathcal{D}} t : \sigma$ is derivable.*

Proof. We prove the statement for the case $t \to_\beta \mathbb{T} \neq \emptyset$. Then the result is obviously generalized to the case $t \twoheadrightarrow_\beta \mathbb{T} \neq \emptyset$ (formally by induction on the number of steps $\to_\beta$).
The proof is by induction on t.
Case $t = x$**.** In such a case $\mathbb{T} = \{x\}$, so $t' = x = t$ and there is nothing to prove.
Case $t = \lambda x.s$**.** Then $\mathbb{T} = \lambda x.\mathbb{S}$ where $s \to_\beta \mathbb{S}$. Then $t' = \lambda x.s'$ for some $s' \in \mathbb{S}$. By Lemma 2.5.3.(2) the sequent $\Gamma, x : \mu \vdash s' : \tau$ is derivable for some $\mu \to \tau \simeq \sigma$. By IH then $\Gamma, x : \mu \vdash s : \tau$ is derivable. By lam and eq we derive $\Gamma \vdash \lambda x.s : \sigma$.
Case $t = s\mathbf{b}$ **and** $\mathbb{T} = \mathbb{S}\mathbf{b}$ **where** $s \to_\beta \mathbb{S}$**.** We have $t' = s'\mathbf{b}$ for some $s' \in \mathbb{S}$. Let $\mathbf{b} = [s_1, \dots, s_n]$. By Lemma 2.5.3(3) we have a decomposition $\{\Gamma_i\}_{i=0}^n$ of Γ and types $\sigma_1, \dots, \sigma_n$ such that $\Gamma_0 \vdash s' : \bigwedge_{i=1}^n \sigma_i \to \sigma$ and $\Gamma_i \vdash s_i : \sigma_i$ for all $i \in \{1, \dots, n\}$ are derivable. By IH $\Gamma_0 \vdash s : \bigwedge_{i=1}^n \sigma_i \to \sigma$ is derivable. So by Rule app we derive $\Gamma \vdash s\mathbf{b} : \sigma$.
Case $t = s\big([r] \cdot \mathbf{b}\big)$ **and** $\mathbb{T} = s\big([\mathbb{R}] \cdot \mathbf{b}\big)$ **where** $r \to_\beta \mathbb{R}$**.** In such a case $t' = s\big([r'] \cdot \mathbf{b}\big)$ for some $r' \in \mathbb{R}$. Let $\mathbf{b} = [s_1, \dots, s_n]$. By Lemma 2.5.3(3) we have a decomposition $\Delta \cup \{\Gamma_i\}_{i=0}^n$ of Γ and $\tau, \sigma_1, \dots, \sigma_n \in T_{\mathcal{D}}$ such that the sequents $\Gamma_0 \vdash s : \tau \wedge \bigwedge_{i=1}^n \sigma_i \to \sigma$, $\Delta \vdash r' : \tau$ and $\Gamma_i \vdash s_i : \sigma_i$ for all $i \in \{1, \dots, n\}$ are derivable. By IH also $\Delta \vdash r : \tau$ is derivable. So by Rule app we can derive $\Gamma \vdash s\big([r] \cdot \mathbf{b}\big) : \sigma$.
Case $t = (\lambda x.s)[s_1, \dots, s_n]$ **and** $\mathbb{T} = \bigcup_{p \in \mathfrak{S}_n} s\left\{s_{p(1)}/x_1, \dots, s_{p(n)}/x_n\right\}$**.** In such a case $t' = s\left\{s_{p(1)}/x_1, \dots, s_{p(n)}/x_n\right\}$ for some $p \in \mathfrak{S}_n$. By Lemma 2.5.6(2) there are $\gamma_1, \dots, \gamma_n \in T_{\mathcal{D}}$ and a decomposition $\{\Gamma_i\}_{i=0}^n$ of Γ such that $\Gamma, x : \bigwedge_{i=1}^n \gamma_i \vdash s : \sigma$ and $\Gamma_i \vdash s_{p(i)} : \gamma_i$ for all $i \in \{1, \dots, n\}$ are derivable. So one can derive

$$\cfrac{\cfrac{\Gamma, x : \bigwedge_{i=1}^n \gamma_i \vdash s : \sigma}{\Gamma \vdash \lambda x.s : \bigwedge_{i=1}^n \gamma_i \to \sigma} \qquad \Gamma_i \vdash s_{p(i)} : \gamma_i \ \text{ for all } i \in \{1, \dots, n\}}{\Gamma \vdash (\lambda x.s)\left[s_{p(1)}, \dots, s_{p(n)}\right] : \sigma}$$

which was to proved, since $(\lambda x.s)\left[s_{p(1)}, \dots, s_{p(n)}\right] = t$. □

Theorem 2.5.9 (One-step soundness for the linear resource calculus)**.** *Let* $\mathcal{D}$ *be an rgm. Let* $\mathbb{U}, \mathbb{V} \in \mathcal{P}_f(\Lambda^r)$ *such that* $\mathbb{U} \to_\beta \mathbb{V}$*. Then* $[\![\mathbb{U}]\!]^{\mathcal{D}} = [\![\mathbb{V}]\!]^{\mathcal{D}}$*.*

Proof. The hypothesis $\mathbb{U} \to_\beta \mathbb{V}$ means that $\mathbb{V} = \bigcup_{t \in \mathbb{U}} \mathbb{V}_t$ where $t \to_\beta \mathbb{V}_t$ for all $t \in \mathbb{U}$.

Let $(\Gamma, \sigma) \in [\![\mathbb{U}]\!]$, i.e. $\Gamma \vdash t : \sigma$ derivable for some $t \in \mathbb{U}$. By Lemma 2.5.7 there is $t' \in \mathbb{V}_t \subseteq \mathbb{V}$ such that $\Gamma \vdash t : \sigma$ is derivable. So $[\![\mathbb{U}]\!] \subseteq [\![\mathbb{V}]\!]$.

Let $(\Gamma, \sigma) \in [\![\mathbb{V}]\!]$, i.e. $\Gamma \vdash t' : \sigma$ derivable for some $t' \in \mathbb{V}$. By hypothesis there exists $t \in \mathbb{U}$ such that $t' \in \mathbb{V}_t$ and $t \to_\beta \mathbb{V}_t$. By Lemma 2.5.8 then $\Gamma \vdash t : \sigma$ is derivable. So $[\![\mathbb{V}]\!] \subseteq [\![\mathbb{U}]\!]$. □

Corollary 2.5.10 (Soundness for the linear resource calculus)**.** *Let* $\mathcal{D}$ *be an rgm. Let* $\mathbb{U}, \mathbb{V} \in \mathcal{P}_f(\Lambda^r)$ *such that* $\mathbb{U} =_\beta \mathbb{V}$*. Then* $[\![\mathbb{U}]\!]^{\mathcal{D}} = [\![\mathbb{V}]\!]^{\mathcal{D}}$*.*

Proof. For some $k \in \mathbb{N}$ there is a finite sequence $(\mathbb{U}_i)_{i=0}^{k}$ of elements of $\mathcal{P}_f(\Lambda^r)$ such that $\mathbb{U}_0 = M$, $\mathbb{U}_k = \mathbb{V}$ and for all $i \in \{0, \dots, k-1\}$ either $\mathbb{U}_i \to_\beta \mathbb{U}_{i+1}$ or $\mathbb{U}_{i+1} \to_\beta \mathbb{U}_i$.

The result is proved by the following induction on k.

Base: $k = 0$. As $\mathbb{U} = \mathbb{U}_0 = \mathbb{V}$, the thesis trivially holds.

Step: $k \geqslant 1$. The IH is applied to the sequence $(\mathbb{U}_i)_{i=1}^{k}$ so to get $[\![\mathbb{U}_1]\!] = [\![\mathbb{U}_k]\!]$.

Either $\mathbb{U}_0 \to_\beta \mathbb{U}_1$ or $\mathbb{U}_1 \to_\beta \mathbb{U}_0$. In either case $[\![\mathbb{U}_0]\!] = [\![\mathbb{U}_1]\!]$ by Theorem 2.5.9.

In the end we have $[\![\mathbb{U}]\!] = [\![\mathbb{U}_0]\!] = [\![\mathbb{U}_1]\!] = [\![\mathbb{U}_k]\!] = [\![\mathbb{V}]\!]$. □

2.6 Approximation theorems: from Taylor to Böhm

The Böhm tree of a λ-term M can be considered as the *possibly infinite* β-normal form of M (with the unsolvable subterms sent to $\bot$). Accordingly, the elements of $\mathrm{BT}(M)^*$ should be thought of as *finite approximations of the reduction of* M. From this perspective, it is reasonable to expect from a model to satisfy

$$[\![M]\!] \quad = \bigcup_{a \in \mathrm{BT}(M)^*} [\![a]\!] \qquad \text{for all } M \in \Lambda\,. \tag{33}$$

Although not always true, this is indeed often the case. A property of such a kind is known as *Approximation Theorem* for the given model. If at hand, Property (33) plays a key role in the study of the model: it stands as a bridge from the possibly infinite nature of (β-reduction of) λ-terms to the finitary realm of finite Böhm-like trees.

Proving an Approximation Theorem is generally not an easy task. Sometimes it can be shown by means of some *ad hoc* indexed refinement of β-reduction, as first done by Wadsworth in [Wad78]. As an alternative, the method of *reducibility candidates à la Tait-Girard* [Tai67, Gir72, GLT89] is the most widespread proof technique for the purpose. This technique is ubiquitous in logic and the theory of programming (where it goes by many names, such as *logical relations* [Rey83], *saturated sets* [Kri90], *realizability interpretations* [Kri09], *stable sets in Kripke structures* [BDS13]). But it is much complicated. Also, it must be cleverly adapted to each single model.

Here these complications are avoided thanks to the following crucial fact, which holds for every rgm $\mathcal{D}$:

$$[\![M]\!]^{\mathcal{D}} \quad = \bigcup_{t \in \mathcal{T}(M)} [\![t]\!]^{\mathcal{D}} \qquad \text{for all } M \in \Lambda\,. \tag{34}$$

We call this result *Taylor Approximation Theorem*, as opposed to (33), which we rather call *Böhm Approximation Theorem* from now on. The Taylor Approximation Theorem has great advatages. Firstly, both inclusions in (34) are proved by a straightforward induction, not by reducibility candidates. Moreover, the Böhm Approximation Theorem easily follows from the Taylor one by Theorem 1.6.4 of Ehrhard and Regnier. Finally, another great benefit of this approach lies in its generality, since the method works for *all* rgm's.

In fact (34) is the main reason why the linear resource calculus and the notion of Taylor expansion are taken into account in this work.

Definition 2.6.1. Let $\mathcal{D}$ be an rgm. Let $A \in \Lambda^{\mathcal{B}}$. The *type-interpretation* of A in $\mathcal{D}$ is

$$\llbracket A \rrbracket^{\mathcal{D}} \quad := \quad \bigcup_{a \in A^*} \llbracket a \rrbracket^{\mathcal{D}}.$$

In particular, for all $M \in \Lambda$ we have $\llbracket BT(M) \rrbracket^{\mathcal{D}} \; = \; \bigcup_{a \in BT(M)^*} \llbracket a \rrbracket^{\mathcal{D}}$.

Every λ-term M in β-normal form can be seen as a finite Böhm-like tree. So both Definition 2.4.1 (type-interpretation of terms) and Definition 2.6.1 (type-interpretation of trees) apply to M. Of course, we are using the same symbol $\llbracket M \rrbracket^{\mathcal{D}}$ for both because they coincide. Indeed, Theorem 2.6.5 below states that $\llbracket M \rrbracket^{\mathcal{D}}$ in the sense of Definition 2.4.1 is equal to $\llbracket BT(M) \rrbracket^{\mathcal{D}}$, where the latter is $\llbracket M \rrbracket^{\mathcal{D}}$ in the sense of Definition 2.6.1 whenever the β-normal form M is seen as a tree.

Definition 2.6.2. Let $\mathcal{D}$ be an rgm. Let X be in Λ or in $\Lambda^{\mathcal{B}}$. The *type-interpretation* in $\mathcal{D}$ of the Taylor expansion $\mathcal{T}(X)$ is defined as

$$\llbracket \mathcal{T}(X) \rrbracket^{\mathcal{D}} \quad := \quad \bigcup_{t \in \mathcal{T}(X)} \llbracket t \rrbracket^{\mathcal{D}}.$$

For a Böhm-like tree $A \in \Lambda^{\mathcal{B}}$ we have in particular

$$\llbracket \mathcal{T}(A) \rrbracket^{\mathcal{D}} \quad = \quad \bigcup_{t \in \mathcal{T}(A)} \llbracket t \rrbracket^{\mathcal{D}} \quad = \quad \bigcup_{a \in A^*} \; \bigcup_{t \in \mathcal{T}(a)} \llbracket t \rrbracket^{\mathcal{D}} \quad = \quad \bigcup_{a \in A^*} \llbracket \mathcal{T}(a) \rrbracket^{\mathcal{D}}. \tag{35}$$

Theorem 2.6.3 (Taylor Approximation Theorem)**.** *Let $\mathcal{D}$ be an rgm. Let M be in Λ or in $\mathcal{N}$. The sequent $\Gamma \vdash^{\mathcal{D}} M : \sigma$ is derivable if and only if there exists $t \in \mathcal{T}(M)$ such that $\Gamma \vdash^{\mathcal{D}} t : \sigma$ is derivable. In other words $\llbracket M \rrbracket^{\mathcal{D}} = \llbracket \mathcal{T}(M) \rrbracket^{\mathcal{D}}$.*

Proof. ($\Rightarrow$) The proof is by induction on the derivation of $\Gamma \vdash M : \sigma$. We proceed therefore by a case analysis on the last rule applied in the derivation.

Case `var`. If Rule `var` is the last (and only, actually) rule applied, then $\Gamma \vdash M : \sigma$ has the form $x : \sigma \vdash x : \sigma$. This case is trivial since $\mathcal{T}(x) = \{x\}$.

Case `lam`. If Rule `lam` is the last rule applied, then $\Gamma \vdash M : \sigma$ has the form $\Gamma \vdash \lambda x.P : \mu \to \tau$ and the sequent $\Gamma, x : \mu \vdash P : \tau$ is derivable. By IH there exists $t' \in \mathcal{T}(P)$ such that the sequent $\Gamma, x : \mu \vdash t' : \tau$ is derivable. By Rule `lam` we derive $\Gamma \vdash \lambda x.t' : \mu \to \tau$.

Case `app`. If `app` is the last rule applied the derivable sequent $\Gamma \vdash M : \sigma$ has the form $\Gamma_0 \wedge (\bigwedge_{i=1}^{n} \Gamma_i) \vdash PQ : \sigma$, for some $n \geqslant 0$, with sequents $\Gamma_0 \vdash P : \bigwedge_{i=1}^{n} \sigma_i \to \sigma$ and $\Gamma_i \vdash Q : \sigma_i$

for all $i \in \{1, \ldots, n\}$ derivable. By IH, there exists $s \in \mathcal{T}(P)$ such that $\Gamma_0 \vdash s : \bigwedge_{i=1}^{n} \sigma_i \to \sigma$ is derivable, and for all $i \in \{1, \ldots, n\}$ there exists $t_i \in \mathcal{T}(Q)$ such that $\Gamma_i \vdash t_i : \sigma_i$ is derivable. Therefore we can derive

$$\frac{\Gamma_0 \vdash s : \bigwedge_{i=1}^{n} \sigma_i \to \sigma \quad \Gamma_i \vdash t_i : \sigma_i \quad \forall i \in \{1, \ldots, n\}}{\Gamma \vdash s[t_1, \ldots, t_n] : \sigma} \text{ app}$$

So there is $s\,[t_1, \ldots, t_n] \in \mathcal{T}(PQ)$ such that $\Gamma \vdash s[t_1, \ldots, t_n] : \sigma$ is derivable.
Case eq. Let $\Gamma \vdash M : \sigma$ be derived by an instance of Rule eq from a derivation of a sequent $\Gamma \vdash M : \tau$ such that $\tau \simeq \sigma$. By IH there exists $t \in \mathcal{T}(M)$ such that $\Gamma \vdash t : \tau$ has a derivation. By applying eq to such derivation we derive $\Gamma \vdash t : \sigma$. □

Proof. ($\Leftarrow$) Let $t \in \mathcal{T}(M)$ such that $\Gamma \vdash t : \sigma$ has a derivation. We proceed by induction on this derivation. At that purpose we make a case analysis on the last Rule applied therein.
Case var. The derivable sequent $\Gamma \vdash t : \sigma$ has the form $x : \sigma \vdash x : \sigma$ and $x \in \mathcal{T}(M)$. Since $x \in \mathcal{T}(M)$ if and only if $M = x$ by definition of the Taylor expansion, the thesis is trivially proved.
Case lam. The derivable sequent $\Gamma \vdash t : \sigma$ has the form $\Gamma \vdash \lambda x.t' : \mu \to \tau$ and the sequent $\Gamma, x : \mu \vdash t' : \tau$ is derivable. By definition of Taylor expansion $\lambda x.t \in \mathcal{T}(M)$ entails $M = \lambda x.P$ for some $P \in \Lambda$ such that $t' \in \mathcal{T}(P)$. By IH the sequent $\Gamma, x : \mu \vdash P : \tau$ is derivable. By Rule lam we derive $\Gamma \vdash \lambda x.P : \mu \to \tau$.
Case app. The derivable sequent $\Gamma \vdash t : \sigma$ has the form $\Gamma_0 \wedge (\bigwedge_{i=1}^{n} \Gamma_i) \vdash s[t_1, \ldots, t_n] : \sigma$, for some $n \geqslant 0$, where sequents $\Gamma_0 \vdash t : \bigwedge_{i=1}^{n} \sigma_i \to \sigma$ and $\Gamma_i \vdash s_i : \sigma_i$ for all $i \in \{1, \ldots, n\}$ are derivable. Since $s[t_1, \ldots, t_n] \in \mathcal{T}(M)$, by definition of Taylor expansion $M = PQ$ for some $P, Q \in \Lambda$ such that $s \in \mathcal{T}(P)$ and $t_1, \ldots, t_n \in \mathcal{T}(Q)$. By IH there exist derivations of $\Gamma_0 \vdash P : \bigwedge_{i=1}^{n} \sigma_i \to \sigma$ and $\Gamma_i \vdash Q : \sigma_i$ for all $i \in \{1, \ldots, n\}$. Therefore we derive

$$\frac{\Gamma_0 \vdash P : \bigwedge_{i=1}^{n} \sigma_i \to \sigma \quad \Gamma_i \vdash Q : \sigma_i \quad \forall i \in \{1, \ldots, n\}}{\Gamma \vdash PQ : \sigma} \text{ app}$$

Case eq. Let $\Gamma \vdash t : \sigma$ be derived by an instance of Rule eq from a derivation of a sequent $\Gamma \vdash t : \tau$ such that $\tau \simeq \sigma$. Since $t \in \mathcal{T}(M)$ by IH the sequent $\Gamma \vdash M : \tau$ has a derivation. By applying Rule eq to such derivation we derive $\Gamma \vdash M : \sigma$. □

Corollary 2.6.4 (Taylor Approximation Theorem for Böhm-like trees)**.** *Let $\mathcal{D}$ be an rgm and $A \in \Lambda^{\mathcal{B}}$. Then $[\![A]\!]^{\mathcal{D}} = [\![\mathcal{T}(A)]\!]^{\mathcal{D}}$.*

Proof. Applying in the order Definition 2.6.1, Theorem 2.6.3 and (35) we get

$$[\![A]\!] \quad = \quad \bigcup_{a \in A^*} [\![a]\!] \quad = \quad \bigcup_{a \in A^*} [\![\mathcal{T}(a)]\!] \quad = \quad [\![\mathcal{T}(A)]\!]$$

as was to be proved. □

Theorem 2.6.5 (Böhm Approximation Theorem)**.** *Let $\mathcal{D}$ be an rgm and $M \in \Lambda$. The sequent $\Gamma \vdash^{\mathcal{D}} M : \sigma$ is derivable if and only if there exists $a \in BT(M)^*$ such that $\Gamma \vdash^{\mathcal{D}} a : \sigma$ is derivable. In other words $[\![M]\!]^{\mathcal{D}} = [\![BT(M)]\!]^{\mathcal{D}}$.*

Proof. We have the following chain of equivalences

$$\begin{array}{llll} \Gamma \vdash M : \sigma & \text{if and only if} & \exists\, t \in \mathcal{T}(M) \text{ such that } \Gamma \vdash t : \sigma & \text{(by Theorem 2.6.3)} \\ & \text{if and only if} & \exists\, t \in \mathrm{nf}_\beta \mathcal{T}(M) \text{ such that } \Gamma \vdash t : \sigma & \text{(by Corollary 2.5.10)} \\ & \text{if and only if} & \exists\, t \in \mathcal{T}(\mathrm{BT}(M)) \text{ such that } \Gamma \vdash t : \sigma & \text{(by Theorem 1.6.4)} \\ & \text{if and only if} & \exists\, a \in \mathrm{BT}(M)^* \text{ such that } \Gamma \vdash a : \sigma & \text{(by Corollary 2.6.4)} \end{array}$$

which completes the proof. Notice that the second step of this chain of equivalences relies on the fact that β-normalization is strongly normalizing on Λ^r. □

We conclude this chapter with some general facts concerning the preorder theory and the λ-theory of every rgm's.

Theorem 2.6.6. *Let $\mathcal{D}$ be an rgm.*

1. $\sqsubseteq_\mathcal{B}$ *is included in* $\mathrm{Th}_\sqsubseteq(\mathcal{D})$, *namely* $\mathrm{BT}(M) \leqslant_\perp \mathrm{BT}(N)$ *implies* $M \sqsubseteq_\mathcal{D} N$ *for all* $M, N \in \Lambda$.
2. $\mathcal{B} \subseteq \mathrm{Th}(\mathcal{D})$, *namely* $\mathrm{BT}(M) = \mathrm{BT}(N)$ *implies* $M =_\mathcal{D} N$ *for all* $M, N \in \Lambda$.

Proof. (1) Let $\mathrm{BT}(M) \leqslant_\perp \mathrm{BT}(N)$. Since this is equivalent to $\mathrm{BT}(M)^* \subseteq \mathrm{BT}(N)^*$, by Theorem 2.6.5 we have $[\![M]\!] = [\![\mathrm{BT}(M)]\!] = \bigcup_{a \in \mathrm{BT}(M)^*} [\![a]\!] \subseteq \bigcup_{a \in \mathrm{BT}(N)^*} [\![a]\!] = [\![\mathrm{BT}(N)]\!] = [\![N]\!]$.

(2) It follows immediately from 1. □

Theorem 2.6.7. *Let $\mathcal{D}$ be an rgm. Then $[\![M]\!]^\mathcal{D} = \emptyset$ for all unsolvable $M \in \Lambda$. In particular the theory $\mathrm{Th}(\mathcal{D})$ is sensible.*

Proof. Let M be unsolvable. By Theorem 2.6.5 we have $[\![M]\!] = \bigcup_{a \in \perp^*} [\![a]\!] = [\![\perp]\!] = \emptyset$. □

Corollary 2.6.8. *Let $\mathcal{D}$ be an rgm. Let $M, N \in \Lambda$. Then $[\![M]\!]^\mathcal{D} \subseteq [\![N]\!]^\mathcal{D}$ implies $M \sqsubseteq_{\mathcal{H}^*} N$.*

3 MINIMAL AND MAXIMAL RELATIONAL GRAPH THEORIES

INTRODUCTION

The study of the untyped λ-calculus is not restricted to the sole β-rule. In fact, one is more often interested in *λ-theories*, the equational extensions of β-convertibility defined in § 1.2. As we mentioned in that section, the lattice of λ-theories is still largely unexplored. A further refinement consists in investigating *inequational* extensions of β-convertibility, which we called *preorder theories* in § 1.2.

Studying preorder or λ-theories by pure syntactical methods can be very complicated. Denotational semantics comes then in handy. Indeed, given a certain preorder or λ-theory one can search for models inducing it. In Chapters 4 and 5 we will go in such a direction, by looking for rgm's that induce Morris's observational preorder and λ-theory.

However, one can also undertake a reverse path, which starts from purely semantic considerations rather than from some fixed theory. Just for the sake of convenience, let us call *semantics* any class of uniformly defined denotational models. Scott's continuous semantics, stable semantics, relational semantics are examples of the idea. Given a semantics, one can explore the range of preorder and λ-theories *represented* by it, i.e. induced by some of its models. Some natural questions then arise.

1. Is the given semantics *incomplete*, i.e. unable to represent all consistent λ-theories ?
2. Can the semantics represent exactly the least λ-theory $\boldsymbol{\lambda\beta}$, or the least extensional λ-theory $\boldsymbol{\lambda\beta\eta}$?
3. Can the semantics represent the minimal sensible λ-theory $\mathcal{H}$?
4. Has the semantics a minimal representable λ-theory?
5. Has the semantics a maximal representable λ-theory?

This kind of problems are usually anything but trivial, because often a given semantics can represent $2^{\aleph_0}$ distinct preorder and λ-theories and yet being incomplete. For instance, the representabilities of $\boldsymbol{\lambda\beta}$, $\boldsymbol{\lambda\beta\eta}$ and $\mathcal{H}$ are long-standing open problems. In particular Question 2 was raised by Honsell and Ronchi for Scott's semantics in [HRDR92]. Berline's article [Ber00] supplies a survey on these issues.

Now, rgm's can be considered to form a semantics on their own. So it makes sense to ask the questions above for such semantics.

Questions 1 and 2 have already been answered at the end of Chapter 2. As a matter of fact, Theorem 2.6.7 states that rgm's can only induce *sensible* λ-theories. So

1. the semantics of rgm's is incomplete, because unable to represent any λ-theory that does not equate two distinct unsolvable terms;

2. in particular, the class of rgm's cannot represent $\boldsymbol{\lambda\beta}$ and $\boldsymbol{\lambda\beta\eta}$, as these are not sensible.

In this chapter we provide answers to Questions 3-5.

As concerns Question 4, the minimal λ-theory turns out to be $\mathcal{B}$, namely the one equating λ-terms if and only if they have the same Böhm tree. Since $\mathcal{H} \subset \mathcal{B}$, we also get an answer to Questions 3: there is no rgm inducing $\mathcal{H}$.

More subtly, we focus on the minimal *preorder* theory represented by rgm's. We denote this preorder by $\sqsubseteq_r$ and we characterize it as follows:

$$M \sqsubseteq_r N \quad \text{if and only if} \quad \text{there exists } T \in \Lambda^{\mathcal{B}} \text{ such that } BT(M) \leqslant_{\perp} T \twoheadrightarrow_{\eta} BT(N)\ .$$

According to the meaning of $\twoheadrightarrow_{\eta}$ given in § 1.4, this reads as: the Böhm tree of M is an approximation of a tree T obtained from the Böhm tree of N by performing up to denumerable many η-expansions possibly of infinite depth.

The minimal preorder $\sqsubseteq_r$ is induced by an rgm $\mathcal{E}$ that we call *à la Engeler*, because conceived as a relational version of Engeler's graph model [Eng81], [Bar84, § 5.4]. The model $\mathcal{E}$ already appeared elsewhere. It was in fact the very first relational model of the untyped λ-calculus introduced in the literature, precisely by Hyland and others in [HNPR06]. Also, the corresponding intersection type system was studied by de Carvalho in [dC07, dC09].

As regards Question 5, the maximal λ-theory represented by rgm's is $\mathcal{H}^*$. In fact, there is in the literature an rgm that is fully abstract for $\mathcal{H}^*$. It was introduced by Bucciarelli, Ehrhard and Manzonetto in [BEM07] and proved to induce $\mathcal{H}^*$ in [Man09]. Here we provide another one.

PLAN OF THE CHAPTER. In § 3.1 we define the model $\mathcal{E}$. In § 3.2 we prove the equivalence between $\sqsubseteq_{\mathcal{E}}$ and $\sqsubseteq_r$, except for the more technical lemma, to which a separate section is devoted, namely § 3.3. In § 3.4 we reformulate the rgm $\mathcal{D}'$ introduced in [BEM07], as an isomorphic rgm $\mathcal{D}_*$ built by completion upon one single atom and one single basic equation.

3.1 A RELATIONAL GRAPH MODEL À LA ENGELER

In this and the next two sections we study the rgm defined below.

Definition 3.1.1. We call *rgm à la Engeler* the free completion

$$\mathcal{E} \ := \ \overline{(\,E, \emptyset\,)}$$

where E is a denumerable set $E := \{\alpha_n\}_{n \in \mathbb{N}}$ whose elements α_n are pairwise distinct and not pairs, and $\emptyset\ : \mathcal{M}_f(E) \times E \rightharpoonup E$ is the *empty* partial function.

More explicitly, $\mathcal{E}$ is the rgm $(\,\overline{E}, \overline{\emptyset}\,)$, where $\overline{E}$ is the union $\bigcup_{n \in \mathbb{N}} E_n$ of the sequence defined by $E_0 = E$ and $E_{n+1} = (\,\mathcal{M}_f(E_n) \times E_n\,) \cup E$, whereas $\overline{\emptyset}$ is the inclusion map of $\mathcal{M}_f(\overline{E}) \times \overline{E}$ into $\overline{E}$, meaning that

$$\overline{\emptyset}\,(m, x) \quad = \quad (m, x) \qquad \text{for all } \ (m, x) \ \in \ \mathcal{M}_f(\overline{E}) \times \overline{E}\ . \tag{36}$$

Lemma 3.1.2. *The equivalence $\simeq^{\mathcal{E}}$ is the equality on types in $T_{\mathcal{E}}$. In other words, for all $\sigma, \tau \in T_{\mathcal{E}}$ we have $\sigma \simeq^{\mathcal{E}} \tau$ if and only if $\sigma = \tau$.*

Proof. By Proposition 2.2.10 we have $\mathsf{At}_{\mathcal{E}} = \mathsf{At}_{\overline{(E,\emptyset)}} = E$. By Definition 2.3.2 and (36) we get

1. $\alpha^\diamond = \alpha$ for all $\alpha \in \mathsf{At}_{\mathcal{E}}$;
2. $(\mu \to \tau)^\diamond = (\mu^\diamond, \tau^\diamond)$ for all $\mu \in \mathsf{I}_{\mathcal{E}}$ and for all $\tau \in \mathsf{T}_{\mathcal{E}}$.

We prove that $\sigma \simeq^{\mathcal{E}} \tau$ implies $\sigma = \tau$ by induction on σ. (The reverse implication is evident.)
Case $\sigma \in \mathsf{At}_{\mathcal{E}}$**, i.e.** $\sigma = \alpha \in E$. By hypothesis $\alpha \simeq^{\mathcal{E}} \tau$, that is $\alpha = \alpha^\diamond = \tau^\diamond$. Then τ cannot be an arrow type, otherwise by Point 2 above $\tau^\diamond = \alpha$ would be a pair, contradicting the fact that $\alpha \in E$. So let $\tau = \beta \in \mathsf{At}_{\mathcal{E}}$. In the end $\sigma = \alpha = \alpha^\diamond = \beta^\diamond = \beta = \tau$.
Case $\sigma = \mu \to \sigma'$. Let $\mu \to \sigma' \simeq^{\mathcal{E}} \tau$, that is $(\mu \to \sigma')^\diamond = \tau^\diamond$. By Point 2 above then $\tau^\diamond = (\mu^\diamond, \sigma'^\diamond)$. Hence $\tau \notin \mathsf{At}_{\mathcal{E}} = E$, otherwise $\tau = \tau^\diamond$ would not be a pair. So $\tau = \nu \to \tau'$ and still by Point 2 we have $(\mu^\diamond, \sigma'^\diamond) = (\mu \to \sigma')^\diamond = (\nu \to \tau')^\diamond = (\nu^\diamond, \tau'^\diamond)$. By IH from $\mu^\diamond = \nu^\diamond$ we get $\mu = \nu$, and from $\sigma'^\diamond = \tau'^\diamond$ we get $\sigma' = \tau'$. So $\sigma = \mu \to \sigma' = \nu \to \tau' = \tau$. □

Notation. All over this chapter $\vdash$ denotes the type system obtained from Definition 2.3.8 (the system in Figure 3 of Chapter 2) by throwing away Rule eq.

Lemma 3.1.2 says that Rule eq is essentially useless for deriving typings in $\vdash^{\mathcal{E}}$. Because of this, the type system $\vdash^{\mathcal{E}}$ is equivalent to the type system $\vdash$. Here the word *equivalent* has a strong sense. In fact, they are basically the same type assignment system: every derivation in $\vdash$ is also a derivation in $\vdash^{\mathcal{E}}$, whereas every derivation in $\vdash^{\mathcal{E}}$ can be reproduced in $\vdash$ just by erasing each instance of Rule eq. For instance, in $\vdash^{\mathcal{E}}$ the *dummy* rule eq allows to derive

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x:\alpha_1\wedge\alpha_2\to\alpha_3 \vdash^{\mathcal{E}} x:\alpha_1\wedge\alpha_2\to\alpha_3}\,\mathsf{var} \quad \dfrac{}{y:\alpha_1 \vdash^{\mathcal{E}} y:\alpha_1}\,\mathsf{var} \quad \dfrac{}{y:\alpha_2 \vdash^{\mathcal{E}} y:\alpha_2}\,\mathsf{var}}{x:\alpha_1\wedge\alpha_2\to\alpha_3,\, y:\alpha_1\wedge\alpha_2 \vdash^{\mathcal{E}} xy:\alpha_3}\,\mathsf{app}}{x:\alpha_1\wedge\alpha_2\to\alpha_3,\, y:\alpha_1\wedge\alpha_2 \vdash^{\mathcal{E}} xy:\alpha_3}\,\mathsf{eq}}{x:\alpha_1\wedge\alpha_2\to\alpha_3,\, y:\alpha_1\wedge\alpha_2 \vdash^{\mathcal{E}} xy:\alpha_3}\,\mathsf{eq}}{x:\alpha_1\wedge\alpha_2\to\alpha_3,\, y:\alpha_1\wedge\alpha_2 \vdash^{\mathcal{E}} xy:\alpha_3}\,\mathsf{eq}}{y:\alpha_1\wedge\alpha_2 \vdash^{\mathcal{E}} \lambda x.xy:(\alpha_1\wedge\alpha_2\to\alpha_3)\to\alpha_3}\,\mathsf{lam}$$

which is represented in the system $\vdash$ simply as

$$\dfrac{\dfrac{\dfrac{}{x:\alpha_1\wedge\alpha_2\to\alpha_3 \vdash x:\alpha_1\wedge\alpha_2\to\alpha_3}\,\mathsf{var} \quad \dfrac{}{y:\alpha_1 \vdash y:\alpha_1}\,\mathsf{var} \quad \dfrac{}{y:\alpha_2 \vdash y:\alpha_2}\,\mathsf{var}}{x:\alpha_1\wedge\alpha_2\to\alpha_3,\, y:\alpha_1\wedge\alpha_2 \vdash xy:\alpha_3}\,\mathsf{app}}{y:\alpha_1\wedge\alpha_2 \vdash \lambda x.xy:(\alpha_1\wedge\alpha_2\to\alpha_3)\to\alpha_3}\,\mathsf{lam}$$

In particular for all $M \in \Lambda$ we have

$$[\![M]\!]^{\mathcal{E}} = \left\{(\Gamma,\sigma) \in \mathsf{Env}_{\mathcal{E}} \times \mathsf{T}_{\mathcal{E}} \mid \Gamma \vdash^{\mathcal{E}} M:\sigma\right\} = \left\{(\Gamma,\sigma) \in \mathsf{Env}_{\mathcal{E}} \times \mathsf{T}_{\mathcal{E}} \mid \Gamma \vdash M:\sigma\right\}.$$

From now on we only use $\vdash$ in order to study the interpretation of λ-terms in the model $\mathcal{E}$.

As mentioned in the introduction of this chapter, the model $\mathcal{E}$ appeared for the first time in a paper by Hyland and others [HNPR06] . We name it *rgm à la Engeler* in analogy to the traditional Engeler's graph model $\mathbb{E}$, introduced in [Eng81]. As a matter of fact, $\mathbb{E}$ can be defined as the free completion, in the sense of graph models, over the empty partial pair

$\emptyset : \mathcal{P}_f(A) \times A \rightharpoonup A$ for any non-empty set A, as Longo showed in [Lon83] (for a presentation more similar to ours see Berline's article [Ber06], where $\mathbb{E}$ is called $\mathcal{E}$).

The intersection type system $\vdash$ was introduced by de Carvalho in [dC07, dC09], where it went by the name of *System R*. De Carvalho recognized its semantic status of λ-algebra coming from a reflexive object of **MRel**. Most notably, he used System R as a tool to analyse a notion of time of execution of λ-terms in Krivine's abstract machine [Kri92].

A close system was also investigated from a purely syntactic perspective by Bernadet and Lengrand in [BL13] .

3.2 THE MINIMAL PREORDER AND λ-THEORY

In this section we characterize $Th_{\sqsubseteq}(\mathcal{E})$ and $Th(\mathcal{E})$ (with the proof of the most technical lemma actually postponed to the next section). Moreover, we prove them to be respectively the minimal preorder theory and the minimal λ-theory represented by the class of rgm's.

Before investigating $Th_{\sqsubseteq}(\mathcal{E})$, let us take a quick look at the preorders induced by some related denotational models.

Two of them are Engeler's graph model $\mathbb{E}$, introduced in [Eng81], and Scott-Plotkin graph model $\mathcal{P}_\omega$ [Bar84, § 18.1 & 19.1]. Typically $\mathbb{E}$ is presented as a *simplification* of $\mathcal{P}_\omega$, like in the standard reference [Bar84, § 5.4]. The main reason is that they both induce the λ-theory $\mathcal{B}$. But this is not relevant from our more refined perspective (studying the *preorder theories* induced by models, not just their λ-theories), since they have different preorder theories. The model $\mathbb{E}$ induces $\sqsubseteq_{\mathcal{B}}$, as proved by Longo in [Lon83]. On the other hand, the model $\mathcal{P}_\omega$ has the following preorder, as established in [Bar84, Th. 19.1.19]: for all $M, N \in \Lambda$

$$M \sqsubseteq_{\mathcal{P}_\omega} N \quad \text{if and only if} \quad \text{there is } A \in \Lambda^{\mathcal{B}} \text{ such that } BT(M)\ {}_{\eta}\!\twoheadleftarrow A \leqslant_{\perp} BT(N)\,. \tag{37}$$

In [Roc82] Ronchi studied a filter model $\mathcal{M}$ where the order between types is the equality. One can see a certain analogy with $\mathcal{E}$, the rgm where the equivalence between types is the equality. The model $\mathcal{M}$ induces the following preorder theory: for all $M, N \in \Lambda$

$$M \sqsubseteq_{\mathcal{M}} N \quad \text{if and only if} \quad \text{there exists } A \in \Lambda^{\mathcal{B}} \text{ such that } BT(M) \leqslant_{\perp} A \twoheadrightarrow_{\eta} BT(N)\,. \tag{38}$$

In some way this preorder is symmetrical to the one in (37). Indeed $\sqsubseteq_{\mathcal{P}_\omega}$ possibly requires to infinitely η-expand the Böhm tree of the λ-term *on the left hand side* of the relation, whereas $\sqsubseteq_{\mathcal{M}}$ asks for the same *on the right hand side*. Here *infinitely* means that the η-expansions may have infinite depth.

The rgm $\mathcal{E}$ can be considered as a *relational* version of $\mathbb{E}$, but at the same time also as a *relational/non-idempotent* rephrasing of $\mathcal{M}$. So it is natural to wonder if their preorder theories share some similarities. We show that $\mathcal{E}$ behaves like $\mathcal{M}$, namely it induces the preorder in (38). Such a preorder is denoted by $\sqsubseteq_r$ from now on, as formally defined below.

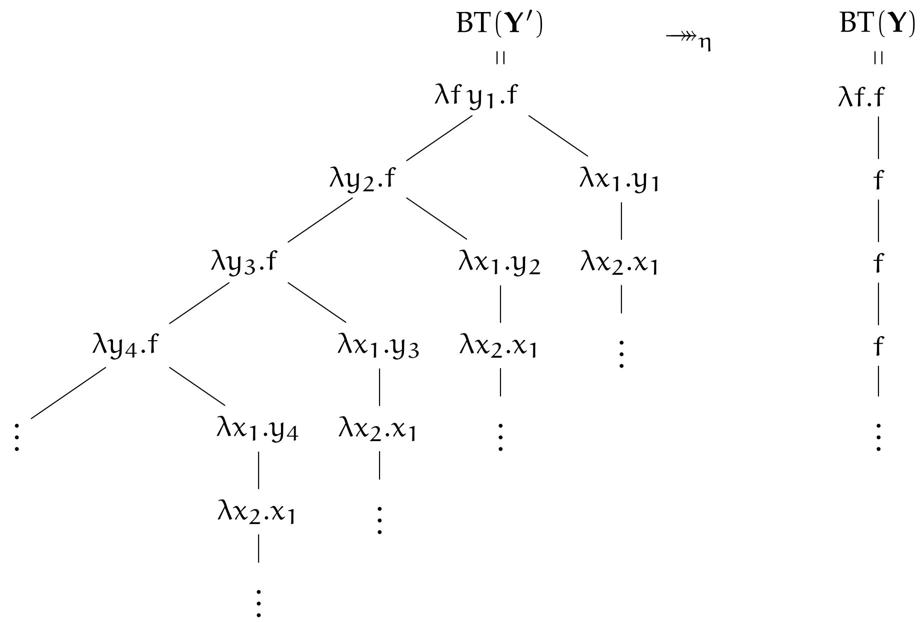
Definition 3.2.1. Let $A, B \in \Lambda^{\mathcal{B}}$. Then $A \leqslant_r B$ if and only if there exists $B' \in \Lambda^{\mathcal{B}}$ such that $A \leqslant_{\perp} B' \twoheadrightarrow_{\eta} R$.

Remark 3.2.2. Let $a, b \in \mathcal{N}$. Then $a \leqslant_r b$ if and only if there exists $b' \in \mathcal{N}$ such that $a \leqslant_{\perp} b' \twoheadrightarrow_{\eta} b$.

Definition 3.2.3. Let $M, N \in \Lambda$. Then $M \sqsubseteq_r N$ if and only if $\mathrm{BT}(M) \leqslant_r \mathrm{BT}(N)$.

Examples 3.2.4.

- For all $M, N \in \Lambda$ such that $\mathrm{BT}(M) \leqslant_\perp \mathrm{BT}(N)$ obviously $M \sqsubseteq_r N$.
- $\lambda x.y\,\mathbf{\Omega} \sqsubseteq_r y$, since $\mathrm{BT}(\lambda x.y\,\mathbf{\Omega}) = \lambda x.y\,\perp \leqslant_\perp \lambda x.y\,x \twoheadrightarrow_\eta y = \mathrm{BT}(y)$.
- Consider the λ-terms P and Q defined in Example 1.4.10. Then $P \sqsubseteq_r Q$.
- $\mathbf{J} \sqsubseteq_r \mathbf{I}$, because $\mathrm{BT}(\mathbf{J}) \twoheadrightarrow\!\!\!\!\twoheadrightarrow_\eta \mathrm{BT}(\mathbf{I})$.
- Let $\mathbf{Y'} := \lambda f.\big(\lambda xy.f(xx)(\mathbf{J}y)\big)\big(\lambda xy.f(xx)(\mathbf{J}y)\big)$. Then $\mathbf{Y'} \sqsubseteq_r \mathbf{Y}$ since



Our proof of the equality between $\sqsubseteq_\mathcal{E}$ and $\sqsubseteq_r$ relies on the type system $\vdash$. However we have given a definition of $\sqsubseteq_r$ that uses Böhm-like trees. Now, Böhm-like trees cannot be typed, since they are possibly infinite objects. But *finite* approximants can, as we know from the previous chapter. That is why now we reformulate $\sqsubseteq_r$ also in terms of elements of $\mathcal{N}$.

Theorem 3.2.5. *Let* $M, N \in \Lambda$. *The following two statements are equivalent:*

1. $M \sqsubseteq_r N$
2. *for all* $a \in \mathrm{BT}(M)^*$ *there exists* $b \in \mathrm{BT}(N)^*$ *such that* $a \leqslant_r b$.

Proof. ($1 \Rightarrow 2$) Let $a \in \mathrm{BT}(M)^*$. We seek $b \in \mathrm{BT}(N)^*$ and $b' \in \mathcal{N}$ such that $a \leqslant_\perp b' \twoheadrightarrow_\eta b$.

We proceed by induction on a.

If $a = \perp$ we take $b := b' := \perp$ and we are done.

Let $a = \lambda x_1 \ldots x_n.x\, a_1 \cdots a_m$ for some $n, m \in \mathbb{N}$. Since $a \in \mathrm{BT}(M)^*$ we have $\mathrm{BT}(M) = \lambda x_1 \ldots x_n.x\,\mathrm{BT}(M_1) \cdots \mathrm{BT}(M_m)$ for $M_1, \ldots, M_m \in \Lambda$ such that $a_i \in \mathrm{BT}(M_i)^*$ for all i.

By hypothesis 1 there exists $B = \lambda x_1 \ldots x_n.x\, B_1 \cdots B_m \in \Lambda^{\mathcal{B}}$ such that $BT(M) \leqslant_\perp B$ and

$$B \twoheadrightarrow_\eta BT(N)\,. \tag{39}$$

In particular from (39) we get $BT(N) = \lambda x_1 \ldots x_{n'}.x\, BT(N_1) \cdots BT(N_{m'})$ for some $n' \leqslant n$, $m' \leqslant m$ and $N_1, \ldots, N_{m'} \in \Lambda$.

Let $i \in \{1, \ldots, m'\}$. From (39) we get $B_i \twoheadrightarrow_\eta BT(N_i)$ by Lemma 1.4.6 . Then by IH there are $b'_i \in \mathcal{N}$ and $b_i \in BT(N_i)^*$ such that $a_i \leqslant_\perp b'_i \twoheadrightarrow_\eta b_i$.

Let $i \in \{1, \ldots, m - m'\}$. From (39) we get $B_{m'+i} \twoheadrightarrow_\eta x_{n'+i} = BT(x_{n'+i})$ by Lemma 1.4.6. Then the IH gives an approximant $b'_{m'+i} \in \mathcal{N}$ such that $a_{m'+i} \leqslant_\perp b'_{m'+i} \twoheadrightarrow_\eta x_{n'+i}$.

In the end $b := \lambda x_1 \ldots x_{n'}.x\, b_1 \cdots b_{m'}$ and $b' := \lambda x_1 \ldots x_n.x\, b'_1 \cdots b'_m$ give $a \leqslant_\perp b' \twoheadrightarrow_\eta b$.

($2 \Rightarrow 1$) We are looking for a Böhm-like tree B such that $BT(M) \leqslant_\perp B \twoheadrightarrow_\eta BT(N)$.

Take $B := \big(BT(N)\,;\, \lceil BT(M) \rceil\big)$. All we need to prove is $BT(M) \leqslant_\perp B$. As $\mathrm{dom}(BT(M)) = \mathrm{dom}(B)$ by definition of B, it is sufficient to prove that for all $\varphi \in \mathrm{dom}(BT(M))$ such that $BT(M)(\varphi) \neq \perp$ we have $BT(M)(\varphi) = B(\varphi)$. So consider such a φ.

We take an approximant $a \in BT(M)^*$ such that $\varphi \in \mathrm{dom}(a)$ and $BT(M)(\varphi) = a(\varphi)$. By hypothesis 2 there are $b \in BT(N)^*$ and $b' \in \mathcal{N}$ such that $a \leqslant_\perp b' \twoheadrightarrow_\eta b$. Since $a(\varphi) = BT(M)(\varphi) \neq \perp$ and $a \leqslant_\perp b'$, surely

$$b'(\varphi') = a(\varphi') = BT(M)(\varphi') \neq \perp \quad \text{for all } \varphi' \preceq \varphi\,. \tag{40}$$

As moreover $b' \twoheadrightarrow_\eta b$, from (40) we get

$$b(\varphi') = BT(N)(\varphi') \neq \perp \quad \text{for all } \varphi' \preceq \varphi \text{ such that } \varphi' \in \mathrm{dom}(b)\,. \tag{41}$$

Because of (40), all we need to prove is $b'(\varphi) = B(\varphi)$. We perform a case analysis on the various clauses in Definition 1.4.1 applied to $B = \big(BT(N)\,;\, \lceil BT(M) \rceil\big)$.

The case $\varphi \in \mathrm{dom}(BT(N))$ with $BT(N)(\varphi) = \perp$ must not be taken into account, for in such a case we would have $b(\varphi) = \perp$, contradicting (41).

For convenience let us denote $T := \lceil BT(M) \rceil$.

Case $\varphi \in \mathrm{dom}(BT(N))$ **and** $BT(N)(\varphi) = \lambda\vec{x}.x$**.** Let m be the number of children of the node φ in $BT(N)$. Then by Definition 1.4.1(2) we have $B(\varphi) = \lambda\vec{x}\, y_0^\varphi \ldots y_{T(\varphi)-m-1}^\varphi.x$.

By (41) we get $b(\varphi) = BT(N)(\varphi) = \lambda\vec{x}.x$. As $b' \twoheadrightarrow_\eta b$ then $b'(\varphi) = \lambda\vec{x}\, z_0 \ldots z_{T(\varphi)-m-1}.x$, where the number of those λ-abstracted variables z_i is exactly $T(\varphi) - m$ because (40) says that $b'(\varphi) = BT(M)(\varphi)$, hence the number of children of the node $b'(\varphi)$ is supposed to be $\lceil BT(M) \rceil(\varphi) = T(\varphi)$. In the end up to α-conversion $b'(\varphi) = BT(N)(\varphi)$.

Case $\varphi = \varphi'.m+i \in \mathrm{dom}(T) - \mathrm{dom}(BT(N))$ **and** $\varphi' \in \mathrm{dom}(BT(N))$ **where** m **is the number of children of the node** φ' **in** $BT(N)$**.** By Definition 1.4.1(3) we have $B(\varphi) := \lambda y_0^\varphi \ldots y_{T(\varphi)-1}^\varphi.y_i^{\varphi'}$, where $y_i^{\varphi'}$ is the i-th variable of η-expansion that is λ-abstracted at the father node $B(\varphi')$.

By (41) we have $b(\varphi') = BT(N)(\varphi')$. Since $b' \twoheadrightarrow_\eta b$ then $b'(\varphi) := \lambda z_0 \ldots z_{T(\varphi)-1}.z$, where z is the i-th variable of η-expansion λ-abstracted at the node $b'(\varphi')$. Here in particular the number of those λ-abstracted z_i's is $T(\varphi)$ because (40) says that $b'(\varphi) = BT(M)(\varphi)$, hence the number of children of the node $b'(\varphi)$ is supposed to be $\lceil BT(M) \rceil(\varphi) = T(\varphi)$.

It is clear that up to α-conversion $b'(\varphi) = BT(N)(\varphi)$.

Case $\varphi = \varphi'.i \in \mathrm{dom}(T) - \mathrm{dom}(BT(N))$ **and** $\varphi' \notin \mathrm{dom}(BT(N))$. By Definition 1.4.1(4) we have $B(\varphi) = \lambda y_0^{\varphi} \ldots y_{T(\varphi)-1}^{\varphi}.y_i^{\varphi'}$.

As $b' \twoheadrightarrow_\eta b$ and $\varphi' \notin \mathrm{dom}(b)$, for certain $b'(\varphi') = \lambda u_0 \ldots u_{T(\varphi')-1}.u$ and $b'(\varphi) = \lambda v_0 \ldots v_{T(\varphi)-1}.u_i$, where in particular

- the number of those u_i's is $T(\varphi')$ because (40) says that $b'(\varphi') = BT(M)(\varphi')$, hence the number of children of the node $b'(\varphi')$ is $\lceil BT(M) \rceil(\varphi') = T(\varphi')$.
- the number of those v_i's is $T(\varphi)$ because (40) says that $b'(\varphi) = BT(M)(\varphi)$, hence the number of children of the node $b'(\varphi)$ is $\lceil BT(M) \rceil(\varphi) = T(\varphi)$.

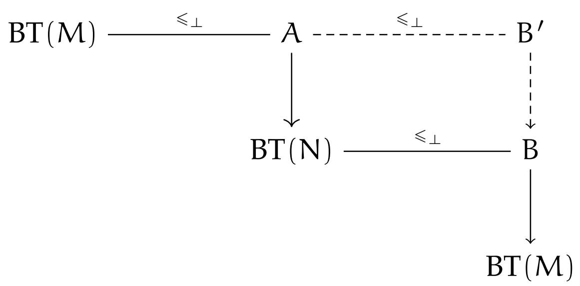
In the end up to α-conversion $b'(\varphi) = BT(N)(\varphi)$. $\square$

We prove that the equivalence generated by $\sqsubseteq_r$ is the equality of Böhm trees.

Lemma 3.2.6. *Let* $M, N \in \Lambda$. *Then* $M \sqsubseteq_r N$ *and* $N \sqsubseteq_r M$ *if and only if* $BT(M) = BT(N)$.

Proof. The right-to-left implication is trivial. Let us prove the other one.

By hypothesis there are $A, B \in \Lambda^{\mathcal{B}}$ such that $BT(M) \leqslant_\perp A \twoheadrightarrow_\eta BT(N) \leqslant_\perp B \twoheadrightarrow_\eta BT(M)$. By Lemma 1.4.7 there exists $B' \in \Lambda^{\mathcal{B}}$ such that $A \leqslant_\perp B' \twoheadrightarrow_\eta B$. Diagrammatically, using the notation in the statement of Lemma 1.4.7, we have the following situation



(42)

The fact that $BT(M) \leqslant_\perp B'$ implies $\lceil B' \rceil = \lceil BT(M) \rceil$. From $B' \twoheadrightarrow_\eta BT(M)$ we get then $B' = (BT(M) ; \lceil B' \rceil) = (BT(M) ; \lceil BT(M) \rceil) = BT(M)$. It is then clear that all the trees in (42) are in fact $BT(M)$. In particular $BT(N) = BT(M)$. $\square$

Lemma 3.2.7. *Let* $\mathcal{D}$ *be an rgm. Let* $M, N \in \Lambda$. *Then* $M \sqsubseteq_r N$ *implies* $M \sqsubseteq_{\mathcal{D}} N$.

Proof. By Theorem 3.2.5 for every $a \in BT(M)^*$ there exists $b \in BT(N)^*$ such that $a \leqslant_r b$. This means that $a \leqslant_\perp b' \twoheadrightarrow_\eta b$ for a certain $b' \in \mathcal{N}$. For every rgm $\mathcal{D}$ we have then $[\![a]\!] \subseteq [\![b']\!] \subseteq [\![b]\!]$, the last inclusion in particular given by Lemma 2.4.11 (η-subject reduction). In the end, using Theorem 2.6.5 (Böhm Approximation), we get

$$[\![M]\!] = \bigcup_{a \in BT(M)^*} [\![a]\!] \subseteq \bigcup_{b \in BT(N)^*} [\![b]\!] = [\![N]\!]$$

that is $M \sqsubseteq_{\mathcal{D}} N$. $\square$

Lemma 3.2.7 has the following relevant consequence. As soon as one finds an rgm inducing a preorder *included* in $\sqsubseteq_r$, two things can be concluded at once:

- such rgm induces *exactly* the preorder $\sqsubseteq_r$, because its preorder also includes $\sqsubseteq_r$;
- the preorder $\sqsubseteq_r$ is actually the *minimal* one represented by the class of rgm's, because no rgm can induce a strictly smaller preorder.

Now $\mathcal{E}$ is such an rgm. So basically what we need to show is the implication

$$M \sqsubseteq_{\mathcal{E}} N \quad \Longrightarrow \quad M \sqsubseteq_r N \qquad \text{for all } M, N \in \Lambda .$$

A fact that greatly facilitates the proof is that $M \sqsubseteq_{\mathcal{E}} N$ implies $M \sqsubseteq_{\mathcal{H}^*} N$ by Corollary 2.6.8, hence $a \precsim BT(N)$ for all $a \in BT(M)^*$ by Proposition 1.3.9.

The whole § 3.3 below is devoted to the proof of the following technical lemma.

Lemma 3.2.8. *Let* $B \in \Lambda^{\mathcal{B}}$ *and* $a \in \mathcal{N}$ *such that* $[\![a]\!]^{\mathcal{E}} \subseteq \bigcup_{b \in B^*} [\![b]\!]^{\mathcal{E}}$ *and* $a \precsim B$. *Then there exists* $b \in B^*$ *such that* $a \leqslant_r b$.

Lemma 3.2.9. *Let* $M, N \in \Lambda$. *Then* $M \sqsubseteq_{\mathcal{E}} N$ *implies* $M \sqsubseteq_r N$.

Proof. We prove that $M \sqsubseteq_r N$ using the characterization 2 given in Theorem 3.2.5. So let us consider an $a \in BT(M)^*$ and show that there exists $b \in BT(N)^*$ such that $a \leqslant_r b$.

By Theorem 2.6.5 (Böhm Approximation) and the hypothesis $M \sqsubseteq_{\mathcal{E}} N$ we get

$$\bigcup_{a \in BT(M)^*} [\![a]\!]^{\mathcal{E}} = [\![M]\!]^{\mathcal{E}} \subseteq [\![N]\!]^{\mathcal{E}} = \bigcup_{b \in BT(N)^*} [\![b]\!]^{\mathcal{E}} .$$

In particular

$$[\![a]\!]^{\mathcal{E}} \subseteq \bigcup_{b \in BT(N)^*} [\![b]\!]^{\mathcal{E}} . \tag{43}$$

By Corollary 2.6.8, the hypothesis $[\![M]\!]^{\mathcal{E}} \subseteq [\![N]\!]^{\mathcal{E}}$ entails $M \sqsubseteq_{H^*} N$. So by Proposition 1.3.9

$$a \precsim BT(N) . \tag{44}$$

As (43) and (44) hold, Lemma 3.2.8 gives an approximant $b \in BT(N)^*$ such that $a \leqslant_r b$. □

Theorem 3.2.10. 1. *The preorder* $Th_{\sqsubseteq}(\mathcal{E})$ *is* $\sqsubseteq_r$, *namely* $[\![M]\!]^{\mathcal{E}} \subseteq [\![N]\!]^{\mathcal{E}}$ *if and only if* $M \sqsubseteq_r N$.

2. *The* λ*-theory* $Th(\mathcal{E})$ *is* $\mathcal{B}$, *that is* $[\![M]\!]^{\mathcal{E}} = [\![N]\!]^{\mathcal{E}}$ *if and only if* $BT(M) = BT(N)$.

Proof. (1) The left-to-right implication is Lemma 3.2.9. The other is given by Lemma 3.2.7.

(2) By Point 1 we have $[\![M]\!]^{\mathcal{E}} = [\![N]\!]^{\mathcal{E}}$ if and only if $M \sqsubseteq_r N$ and $N \sqsubseteq_r M$. This is equivalent to $BT(M) = BT(N)$ because of Lemma 3.2.6. □

The result here below provides and answer to Question 4 in the introduction.

Corollary 3.2.11. 1. *The preorder* $\sqsubseteq_r$ *is the minimal preorder theory induced by any rgm.*

2. *The* λ*-theory* $\mathcal{B}$ *is the minimal* λ*-theory induced by any rgm.*

Proof. (1) By Theorem 3.2.10(1) there is an rgm that induces $\sqsubseteq_r$. By Lemma 3.2.7 there cannot be one inducing a preorder strictly lower than $\sqsubseteq_r$.

(2) By Theorem 3.2.10(2) there is an rgm that induces $\mathcal{B}$. By Theorem 2.6.6(2) there cannot be one inducing a λ-theory strictly lower than $\mathcal{B}$. □

It is worth noticing that, even when proved to exist, the minimal preorder or λ-theory represented by a given semantics is not always cleanly characterized as in Corollary 3.2.11. For example, Bucciarelli and Salibra proved the existence of a minimal λ-theory represented by graph models in [BS08], but they did not provide a characterization of this λ-theory.

Corollary 3.2.12. *There is no rgm* $\mathcal{D}$ *such that* $\mathrm{Th}_{\sqsubseteq}(\mathcal{D})$ *is* $\sqsubseteq_{\mathcal{B}}$.

Proof. By Corollary 3.2.11(1), since $\sqsubseteq_{\mathcal{B}}$ is strictly included in $\sqsubseteq_r$ (see Examples 3.2.4). □

According to Corollary 3.2.12, from the point of view of the preorder theories there is no rgm that corresponds to Engeler's $\mathbb{E}$, i.e. the graph model inducing $\sqsubseteq_{\mathcal{B}}$ (nor to Plotkin's T_ω [Plo78], another model known to induce the preorder $\sqsubseteq_{\mathcal{B}}$, as proved in [BL80]).

We can also answer Question 3 in the introduction. A long-standing open problem in denotational semantics is whether there exists a model of the untyped λ-calculus inducing the minimal sensible λ-theory, called $\mathcal{H}$ in § 1.2. Since $\mathcal{H} \subset \mathcal{B}$, as a consequence of Corollary 3.2.11(2) the class of rgm's does not help us to solve the problem.

Corollary 3.2.13. *There is no rgm* $\mathcal{D}$ *such that* $\mathrm{Th}(\mathcal{D})$ *is the minimal sensible λ-theory* $\mathcal{H}$.

3.3 Semantic separation of approximants through r-separators

This section is devoted to the proof of Lemma 3.2.8. It is proved in contrapositive form: given an approximant a and a Böhm-like tree B such that $a \precsim B$ and $a \not\leq_r b$ for all $b \in B^*$, we show that $[\![a]\!]^{\mathcal{E}} \not\subseteq \bigcup_{b\in B^*}[\![b]\!]^{\mathcal{E}}$. This means to find a pair $(\Gamma, \sigma) \in \mathrm{Env}_{\mathcal{E}} \times T_{\mathcal{E}}$ such that $\Gamma \vdash a : \sigma$ is derivable whereas for all $b \in B^*$ the sequent $\Gamma \vdash b : \sigma$ is not, written $\Gamma \nvdash b : \sigma$.

Let us give the intuition behind the next definition. We are in the following situation. On one side, by hypothesis $a \precsim B$, i.e. there is $b \in B^*$ such that $a \precsim b$. This means that

$$\text{for every position } \varphi \in \mathrm{dom}(a) \cap \mathrm{dom}(b) \text{ such that } a_\varphi \neq \bot \text{ we have } a_\varphi \sim b_\varphi\,. \quad \text{(fact 1)}$$

On the other side $a \not\leq_r b$, which means that

$$\text{however we } \eta\text{-expand } b \text{ into some } b' \text{ we have } a \not\leq_\bot b'\,. \quad \text{(fact 2)}$$

One can realize that (fact 1) and (fact 2) are compatible with each other only if there is a position φ such that $a(\varphi) \neq \bot$ and *the node* $b(\varphi)$ *has a number of λ-abstraction strictly greater than the one of* $a(\varphi)$. For instance, consider $a, b \in \mathcal{N}$ depicted in Figure 5. In this example such a position is $\varphi := \langle 0, 0\rangle$. Indeed $a_\varphi = u \sim \lambda v.uv = b_\varphi$, but at the same time $b(\varphi) = \lambda v.u$ has one λ-abstraction more than $a(\varphi) = u$. Definition 3.3.1 here below captures this idea.

Figure 5: Example of finite approximants with an r-separator.

Definition 3.3.1. Let $a \in \mathcal{N}$ and $B \in \Lambda^{\mathcal{B}}$. Let $\varphi \in \mathbb{N}^*$ such that $\varphi \in \mathrm{dom}(a) \cap \mathrm{dom}(B)$ and $a_\varphi = \lambda x_1 \dots x_n.x\, a_1 \cdots a_m$ for some $n, m \in \mathbb{N}$. The sequence φ is an *r-separator for* a *from* B if and only if there exists $b \in B^*$ such that

$$b_\varphi \;=\; \lambda x_1 \dots x_{n'}.x\, b_{\varphi.1} \cdots b_{\varphi.m'} \quad \text{where} \quad m - n \;=\; m' - n' \quad \text{and} \quad n' > n\,.$$

The r-separator φ is *minimal* if it has minimal length $|\varphi|$ among r-separators for a from B.

Remark 3.3.2. Clearly if a position φ is an r-separator for a from B then $a \neq \bot$ and $B \neq \bot$.

Lemma 3.3.3. *Let* $x \in \mathrm{Var}$ *and* $a \in \mathcal{N}$ *such that* $a \precsim x$. *Then* $a \leqslant_r x$.

Proof. We prove the existence of some $e \in \mathcal{N}$ such that $a \leqslant_\bot e \twoheadrightarrow_\eta x$ by induction on a.

If $a = \bot$ it is enough to take $e := x$.

Let $a = \lambda x_1 \dots x_n.x\, a_1 \cdots a_m$ for some $n, m \in \mathbb{N}$. Since $a \precsim x$ then $n - m = 0$ and $a_i \precsim x_i$ for all $i \in \{1, \dots, n\}$. By IH for all $i \in \{1, \dots, n\}$ there exists $e_i \in \mathcal{N}$ such that $a_i \leqslant_\bot e_i \twoheadrightarrow_\eta x_i$. We take $e := \lambda x_1 \dots x_n.x\, e_1 \cdots e_n$. So we get

$$\begin{aligned} a &= \lambda x_1 \dots x_n.x\, a_1 \cdots a_n \\ &\leqslant_\bot \lambda x_1 \dots x_n.x\, e_1 \cdots e_n \\ &\twoheadrightarrow_\eta \lambda x_1 \dots x_n.x\, x_1 \cdots x_n \quad \twoheadrightarrow_\eta \quad x \end{aligned}$$

which was to be proved. □

Lemma 3.3.4. *Let* $a \in \mathcal{N}$ *and* $B \in \Lambda^{\mathcal{B}}$ *such that* $a \precsim B$ *and* $a \not\leqslant_r b$ *for all* $b \in B^*$. *Then there exists an r-separator for* a *from* B.

Proof. We proceed by induction on a. Remark that $a \neq \bot$, otherwise $a \leqslant_r b$ for all $b \in B^*$. So there are $x \in \mathrm{Var}$ and $n, m \in \mathbb{N}$ such that

$$a \;=\; \lambda x_1 \dots x_n.x\, a_1 \cdots a_m \tag{45}$$

for some $a_1, \dots, a_m \in \mathcal{N}$.

By hypothesis $a \precsim B$, i.e. $a \precsim b$ for some $b \in B^*$. Consequently $B \neq \bot$, as $a \not\precsim \bot$. So there are $y \in \mathrm{Var}$ and $n', m' \in \mathbb{N}$ such that $B = \lambda x_1 \dots x_{n'}.y\, B_1 \cdots B_{m'}$ for some $B_1, \dots, B_{m'} \in \Lambda^{\mathcal{B}}$.

By hypothesis $a \precsim B$, i.e. there exists $b = \lambda x_1 \ldots x_{n'}.x\, b_1 \cdots b_{m'} \in B^*$ such that $a \precsim b$. Thus in particular $y = x$ and

$$n' - m' \;=\; n - m\,. \tag{46}$$

Two cases must be considered.
Case $n' > n$. In such a case ε is an r-separator for a from B, and we are done.
Case $n \geqslant n'$ (hence $m \geqslant m'$). Firstly, we prove that there exists $i \in \{1, \ldots, m'\}$ such that $a_i \not\leqslant_r c$ for all $c \in B_i^*$. By contradiction, let us suppose that for all $i \in \{1, \ldots, m'\}$ there exists $c_i \in B_i^*$ such that $a_i \leqslant_r c_i$, i.e.

$$a_i \;\leqslant_\perp\; d_i \;\twoheadrightarrow_\eta\; c_i \tag{47}$$

for some $d_i \in \mathcal{N}$. Let

$$c \;:=\; \lambda x_1 \ldots x_{n'}.x\, c_1 \cdots c_{m'} \;\in\; B^*. \tag{48}$$

Let $i \in \{1, \ldots, m - m'\}$. Since $a \precsim b$, we have $a_{m'+i} \precsim x_{n'+i}$. Therefore $a_{m'+i} \leqslant_r x_{n'+i}$ by Lemma 3.3.3. This means that there exists $e_i \in \mathcal{N}$ such that

$$a_{m'+i} \;\leqslant_\perp\; e_i \;\twoheadrightarrow_\eta\; x_{n'+i}\,. \tag{49}$$

In the end we have

$$\begin{array}{rlll} a & = & \lambda x_1 \ldots x_n\, x_{n'+1} \ldots x_{n'}.x\, a_1 \cdots a_{m'}\, a_{m'+1} \cdots a_m & \text{by (45)} \\ & \leqslant_\perp & \lambda x_1 \ldots x_{n'}\, x_{n'+1} \ldots x_{n'}.x\, d_1 \cdots d_{m'}\, e_1 \cdots e_{m-m'} & \text{by (47) and (49)} \\ & \twoheadrightarrow_\eta & \lambda x_1 \ldots x_{n'}\, x_{n'+1} \ldots x_{n'}.x\, c_1 \cdots c_{m'}\, x_{n'+1} \cdots x_{n'+(m-m')} & \text{by (47) and (49)} \\ & = & \lambda x_1 \ldots x_{n'}\, x_{n'+1} \ldots x_{n'}.x\, c_1 \cdots c_{m'}\, x_{n'+1} \cdots x_n & \text{by (46)} \\ & \twoheadrightarrow_\eta & \lambda x_1 \ldots x_{n'}.x\, c_1 \cdots c_{m'} & \text{by } \eta\text{-rule} \\ & = & c & \text{by (48)} \end{array}$$

Hence $a \leqslant_r c$. But $c \in B^*$, so by hypothesis $a \not\leqslant_r c$, and we are in contradiction.
We have proved the existence of an $i \in \{1, \ldots, m'\}$ such that $a_i \not\leqslant_r c$ for all $c \in B_i^*$. Moreover from $a \precsim b$ we get $a_i \precsim b_i$. So the IH can be applied to a_i and B_i. There is an r-separator $\varphi \in \mathbb{N}^*$ for a_i from B_i. Then clearly $\langle i-1 \rangle\, \varphi$ is an r-separator for a from B. □

Remember from Definition 3.1.1 that the atoms of $\mathcal{E}$ are named $\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots$

Definition 3.3.5. Let $\sigma \in T_\mathcal{E}$. Then σ *terminates in* α_n whenever $\sigma = \mu_1 \to \cdots \to \mu_k \to \alpha_n$ for some $k \in \mathbb{N}$ and $\mu_1, \ldots, \mu_k \in I_\mathcal{E}$.

Notation. For any $h \in \mathbb{N}$ and $\sigma \in T_\mathcal{E}$ we write $\omega^h \to \sigma$ for the type $\overbrace{\omega \to \ldots \to \omega}^{h \text{ times}} \to \sigma$.

Lemma 3.3.6. *Let $a \in \mathcal{N}$ and $B \in \Lambda^\mathcal{B}$ such that $a \precsim B$. Suppose there is an r-separator for a from B. Then there is $(\Gamma, \sigma) \in \mathsf{Env}_\mathcal{E} \times T_\mathcal{E}$ such that $\Gamma \vdash a : \sigma$ is derivable and $\Gamma \nvdash b : \sigma$ for all $b \in B^*$.*

Proof. Let φ be a minimal r-separator for a from B. We do an induction loading, by proving: *there exists $(\Gamma, \sigma) \in \mathsf{Env}_\mathcal{E} \times T_\mathcal{E}$ such that*

1. $\Gamma \vdash a : \sigma$ *is derivable,*

2. $\Gamma \nvdash b : \sigma$ *for all* $b \in B^*$*,*

3. *the type* σ *terminates in* $\alpha_{|\varphi|}$ *,*

4. *for all* $x \in \mathsf{Var}$ *and for all* $\gamma \in \Gamma(x)$ *the type* γ *terminates in* α_t *for a natural number* $t \leqslant |\varphi|$ *.*

Remember from Remark 3.3.2 that $a \neq \bot$ and $B \neq \bot$. Let $a = \lambda x_1 \ldots x_n.x\, a_1 \cdots a_m$ for some $n, m \in \mathbb{N}$. We proceed by induction on $|\varphi|$.

Base. Let $|\varphi| = 0$, i.e. $\varphi = \varepsilon$. Since $a = a_\varepsilon$, by Definition 3.3.1 there is $b \in B^*$ such that

$$b \;=\; b_\varepsilon \;=\; \lambda x_1 \ldots x_{n'}.x\, b_1 \cdots b_{m'} \tag{50}$$

where $n' - m' = n - m$ and $n' > n$. More than this: in fact, *every* $b \in B^* - \{\bot\}$ has the form (50) for some appropriate approximants b_i's.

Consider the deduction

$$\cfrac{\overline{x : \omega^m \to \alpha_0 \vdash x : \omega^m \to \alpha_0}}{x : \omega^m \to \alpha_0 \vdash x\, a_1 \ldots a_m : \alpha_0} \tag{51}$$

We distinguish two cases, depending on whether x is free or not in a.

- If $x \neq x_i$ for all $i \in \{1, \ldots, n\}$, then from (51) we derive $x : \omega^m \to \alpha_0 \vdash a : \omega^n \to \alpha_0$. In such a case we set $\Gamma := x : \omega^m \to \alpha_0$ and $\sigma := \omega^n \to \alpha_0$.

- If there exists $k \in \{1, \ldots, n\}$ such that $x = x_k$ we derive from (51) the sequent $\vdash a : \omega^{k-1} \to (\omega^m \to \alpha_0) \to \omega^{n-k} \to \alpha_0$. In this case we take Γ to be the empty environment and $\sigma := \omega^{k-1} \to (\omega^m \to \alpha_0) \to \omega^{n-k} \to \alpha_0$.

In both cases the pair (Γ, σ) clearly satisfies Conditions 1, 3 and 4.

Let us check Condition 2. Obviously $\Gamma \nvdash \bot : \sigma$. As regards every $b \in B^* - \{\bot\}$, such a b has the form (50). Then $\Gamma \nvdash b : \sigma$, since the number n of arrows in σ is strictly lower than the number n' of λ-abstractions appearing in (50).

Step. Let $|\varphi| > 0$, i.e. $\varphi = \langle k \rangle \psi$ for some $\psi \in \mathbb{N}^*$ and some $k \in \mathbb{N}$.

Since $a \precsim B$, there is $b \in B^*$ such that $a \precsim b$, namely

$$b \;=\; \lambda x_1 \ldots x_{n'}.x\, b_1 \ldots b_{m'} \tag{52}$$

with $n' - m' = n - m$ and $a_i \precsim b_i$ for all $i \in \{1, \ldots, \max(m, m')\}$. In fact, *every* $b \in B^* - \{\bot\}$ has the form (52) for some appropriate approximants b_i's.

The path ε has length $|\varepsilon| = 0 < |\varphi|$. By minimality of φ as an r-separator, ε is not an r-separator from a to B. Hence $n' \leqslant n$ (and $m' \leqslant m$).

Notice that $k \leqslant \min(m, m') = m'$, since $\varphi \in \mathrm{dom}(a) \cap \mathrm{dom}(B)$ by Definition 3.3.1. The fact that $\varphi = \langle k \rangle \psi$ is a minimal r-separator for a from $B = \lambda x_1 \ldots x_{n'}.x\, B_1 \ldots B_{m'}$ entails that ψ is a minimal r-separator for a_k from B_k. Also, $a_k \precsim B_k$, because $a_k \precsim b_k$. As $|\psi| < |\varphi|$ the IH can be applied to a_k and B_k. Therefore there exists $(\widetilde{\Gamma}, \widetilde{\sigma}) \in \mathsf{Env}_\mathcal{E} \times \mathsf{T}_\mathcal{E}$ such that

i. $\widetilde{\Gamma} \vdash a_k : \widetilde{\sigma}$ is derivable,

ii. $\widetilde{\Gamma} \nvdash c : \widetilde{\sigma}$ for all $c \in B_k^*$,

iii. the type $\widetilde{\sigma}$ terminates in $\alpha_{|\psi|}$,

iv. for all $x \in \mathrm{Var}$ and for all $\gamma \in \widetilde{\Gamma}(x)$ the type γ terminates in α_t for a $t \leqslant |\psi|$.

We type a as follows:

$$\dfrac{\dfrac{\overline{x : \omega^{k-1} \to \widetilde{\sigma} \to \omega^{m-k} \to \alpha_{|\varphi|} \vdash x : \omega^{k-1} \to \widetilde{\sigma} \to \omega^{m-k} \to \alpha_{|\varphi|}} \qquad \widetilde{\Gamma} \vdash a_k : \widetilde{\sigma}}{\widetilde{\Gamma} \wedge (x : \omega^{k-1} \to \widetilde{\sigma} \to \omega^{m-k} \to \alpha_{|\varphi|}) \vdash x\, a_1 \ldots a_m : \alpha_{|\varphi|}}}{\left(\widetilde{\Gamma} \wedge (x : \omega^{k-1} \to \widetilde{\sigma} \to \omega^{m-k} \to \alpha_{|\varphi|})\right) - (x_1 : \mu_1, \ldots, x_n : \mu_n) \vdash a : \mu_1 \to \cdots \to \mu_n \to \alpha_{|\varphi|}}$$

where for all $i \in \{1, \ldots, n\}$ we are setting $\mu_i := \left(\widetilde{\Gamma} \wedge (x : \omega^{k-1} \to \widetilde{\sigma} \to \omega^{m-k} \to \alpha_{|\varphi|})\right)(x_i)$. We take $\sigma := \mu_1 \to \cdots \to \mu_n \to \alpha_{|\varphi|}$ and

$$\Gamma \;:=\; \left(\widetilde{\Gamma} \wedge (x : \omega^{k-1} \to \widetilde{\sigma} \to \omega^{m-k} \to \alpha_{|\varphi|})\right) - (x_1 : \mu_1, \ldots, x_n : \mu_n).$$

It is immediate to check that the pair (Γ, σ) satisfies Conditions 1, 3 and 4.

Let us prove Condition 2.

By way of contradiction suppose there is $b \in B^*$ such that $\Gamma \vdash b : \sigma$. The case $b = \bot$ is impossible, since $\bot$ is not typable. So $b \neq \bot$. Remember that in such a case b has the form displayed in (52). Since $n' \leqslant n$, by Lemma 2.3.12(2) a deduction of $\Gamma \vdash b : \sigma$ can only come from a deduction of the sequent

$$\Delta \;\vdash\; x\, b_1 \ldots b_{m'} : \mu_{n'+1} \to \cdots \to \mu_n \to \alpha_{|\varphi|} \tag{53}$$

where

$$\Delta \;:=\; \left(\, \widetilde{\Gamma} \wedge (x : \omega^{k-1} \to \widetilde{\sigma} \to \omega^{m-k} \to \alpha_{|\varphi|})\, \right) - (x_{n'+1} : \mu_{n'+1}, \ldots, x_n : \mu_n)\,. \tag{54}$$

(Remark that in (54) we write $:=$ rather than $\simeq$ because in $\mathcal{E}$ the equivalence of types is just the equality of types.)

Notice that, as $a \precsim B$, the variable x is either none of the variables in $\{x_1, \ldots, x_{\max(n,n')}\} = \{x_1, \ldots, x_n\}$ or it is in $\{x_1, \ldots, x_{\min(n,n')}\} = \{x_1, \ldots, x_{n'}\}$.

We keep on backtracking a deduction by means of the Inversion Lemma. In particular by Lemma 2.3.12(3) in any deduction of the sequent (53) the type of the variable x in head position must terminate in $\alpha_{|\varphi|}$. Such type cannot be one in $\widetilde{\Gamma}(x)$, because all types in there terminate in α_t for $t \leqslant |\psi| < |\varphi|$ by Point iv of the IH. So, looking at (54) one realizes that such a type can only be $\omega^{k-1} \to \widetilde{\sigma} \to \omega^{m-k} \to \alpha_{|\varphi|}$.

From $n' - m' = n - m$ and $k \leqslant m'$ one gets $n - n' = m - m' \leqslant m - k$. Hence

$$\begin{aligned} \omega^{k-1} \to \widetilde{\sigma} \to \omega^{m-k} \to \alpha_{|\varphi|} &= \omega^{k-1} \to \widetilde{\sigma} \to \omega^{m-k-(m-m')} \to \omega^{m-m'} \to \alpha_{|\varphi|} \\ &= \omega^{k-1} \to \widetilde{\sigma} \to \omega^{m-k-(m-m')} \to \omega^{n-n'} \to \alpha_{|\varphi|}\,. \end{aligned} \tag{55}$$

The fact that the haed variable x is typed with the type in (55) implies that each of those $n - n'$ intersections μ_i's in (53) must be ω. This assures us that actually

$$\Delta \ = \ \widetilde{\Gamma} \wedge \left(x : \omega^{k-1} \to \widetilde{\sigma} \to \omega^{m-k} \to \alpha_{|\varphi|}\right) .$$

In the end the only possible deduction of (53) is

$$\frac{x : \omega^{k-1} \to \widetilde{\sigma} \to \omega^{m-k} \to \alpha_{|\varphi|} \vdash x : \omega^{k-1} \to \widetilde{\sigma} \to \omega^{m-k} \to \alpha_{|\varphi|} \qquad \widetilde{\Gamma} \vdash b_i : \widetilde{\sigma}}{\widetilde{\Gamma} \wedge \left(x : \omega^{k-1} \to \widetilde{\sigma} \to \omega^{m-k} \to \alpha_{|\varphi|}\right) \vdash x\, b_1 \ldots b_{m'} : \omega^{n-n'} \to \alpha_{|\varphi|}}$$

whereas $\widetilde{\Gamma} \nvdash b_i : \widetilde{\sigma}$ by Point ii of the IH. This contradiction proves Condition 2. □

We can finally prove Lemma 3.2.8 used in the previous section.

Proof of Lemma 3.2.8. Let $B \in \Lambda^{\mathcal{B}}$ and $a \in \mathcal{N}$ such that and $a \precsim B$. Let us suppose that $a \not\leq_r b$ for all $b \in B^*$. We must prove that $[\![a]\!]^{\mathcal{E}} \nsubseteq \bigcup_{b \in B^*} [\![b]\!]^{\mathcal{E}}$. By Lemma 3.3.4 there exists an r-separator for a from T. By Lemma 3.3.6 we have $(\Gamma, \sigma) \in \mathrm{Env}_{\mathcal{E}} \times T_{\mathcal{E}}$ such that $(\Gamma, \sigma) \in [\![a]\!]^{\mathcal{E}}$ whereas $(\Gamma, \sigma) \notin [\![b]\!]^{\mathcal{E}}$ for all $b \in B^*$. □

3.4 A FULLY ABSTRACT RELATIONAL GRAPH MODEL FOR $\mathcal{H}^*$

Beside $\mathcal{E}$, the only other rgm previously appeared in literature was introduced by Bucciarelli, Ehrhard and Manzonetto in [BEM07]. It was further studied in [Man09, BEM12] and also analyzed from the linear logic perspective in [Ehr12]. The model went by the name of $\mathcal{D}$ in those articles, but here we prefer to rename it $\mathcal{D}'$.

Notation. Let X be a set.

- We denote by $\mathcal{M}_f(X)^{(\omega)}$ the set of all sequences of elements of $\mathcal{M}_f(X)$ equal to the empty multiset $[\,]$ from a certain point on. In other words, an element of $\mathcal{M}_f(X)^{(\omega)}$ is a sequence $(m_n)_{n \in \mathbb{N}}$ where $m_n \in \mathcal{M}_f(X)$ and there exists $k \in \mathbb{N}$ such that $m_n = [\,]$ for all $n \geqslant k$.
- We call $\circledast$ the sequence constantly equal to the empty multiset, i.e. $([\,], [\,], \ldots, [\,], \ldots)$. Obviously $\circledast \in \mathcal{M}_f(X)^{(\omega)}$ for every set X.
- For all $s \in \mathcal{M}_f(X)^{(\omega)}$ and for all $m \in \mathcal{M}_f(X)$ we denote by $m :: s$ the sequence obtained from s by adding m as its first element. More explicitly, $m :: (m_n)_{n \in \mathbb{N}}$ is the function mapping $0 \mapsto m$ and $n+1 \mapsto m_n$ for all $n \in \mathbb{N}$. We assume that the operator $::$ associates to the right. In particular each $(m_n)_{n \in \mathbb{N}} \in \mathcal{M}_f(X)^{(\omega)}$ has the form $m_1 :: m_2 :: \cdots :: m_p :: \circledast$ for some $p \in \mathbb{N}$.

Definition 3.4.1 ([BEM07]). The triple $\mathcal{D}' = (D', \mathtt{Abs}', \mathtt{App}')$ consists of the following.

- Firstly, by induction on $n \in \mathbb{N}$ we define:
  - $D'_0 := \emptyset$,
  - $D'_{n+1} := \mathcal{M}_f(D'_n)^{(\omega)}$.

Then we set

$$D' \ := \ \bigcup_{n \in \mathbb{N}} D'_n \, .$$

- $\mathtt{Abs}' \ := \ \Big\{ \big([(m, s)] \, , \, m :: s \big) \ \mid \ m \in \mathcal{M}_f(D') \text{ and } s \in D' \Big\} \ \in \ \mathbf{MRel}\big(\mathcal{M}_f(D') \times D' \, , \, D'\big) \, .$
- $\mathtt{App}' \ := \ \Big\{ \big([m :: s] \, , \, (m, s) \big) \ \mid \ m \in \mathcal{M}_f(D') \text{ and } s \in D' \Big\} \ \in \ \mathbf{MRel}\big(D' \, , \, \mathcal{M}_f(D') \times D'\big) \, .$

Notice that $D'_1 = \{\circledast\}$. (In fact we could have started the sequence $(D_n)_n$ directly from $\{\circledast\}$, instead of $\emptyset$.) So intuitively D' is the hierarchy of elements of $\mathcal{M}_f(D')^{(\omega)}$ built up starting from $\circledast$. It may be worth remarking that D'_2 is basically the set of the finite sequences of natural numbers, if one thinks every natural number n represented as $[\underbrace{\circledast, \ldots, \circledast}_{n \text{ times}}]$.

It is straightforward to check $\mathtt{App}' \circ \mathtt{Abs}' = \big\{ \big([(m, s)] \, , \, (m, s) \big) \ \mid \ m \in \mathcal{M}_f(D'), s \in D' \big\} = \mathrm{id}_{\mathcal{M}_f(D') \times D'}$ and $\mathtt{Abs}' \circ \mathtt{App}' = \big\{ \big([m :: s] \, , \, m :: s \big) \mid \ m \in \mathcal{M}_f(D'), s \in D' \big\} = \big\{ \big([s] \, , \, s \big) \mid \ s \in D' \big\} = \mathrm{id}_{D'}$, hence $\mathcal{D}'$ is an extensional reflexive object in **MRel**.

It is even easiest to realize that $\mathcal{D}'$ is an ergm, if one see it as the pair $\big(D' \, , \, - :: - \big)$.

Theorem 3.4.2 ([Mano09]). *The model $\mathcal{D}'$ is fully abstract for $\sqsubseteq_{\mathcal{H}^*}$, i.e. its induced preorder theory is $\sqsubseteq_{\mathcal{H}^*}$. In particular it is fully abstract for $\mathcal{H}^*$, meaning that its induced λ-theory is $\mathcal{H}^*$.*

Corollary 3.4.3. *The λ-theory $\mathcal{H}^*$ is the maximal λ-theory induced by any rgm.*

Proof. By Theorem 3.4.2 the theory $\mathcal{H}^*$ is induced by some rgm. By Theorem 2.6.7 any other λ-theory induced by an rgm is sensible, so it cannot properly include the maximal sensible λ-theory $\mathcal{H}^*$. □

Despite being an rgm, the model $\mathcal{D}'$ does not look like one defined by completion of a partial pair, i.e. the kind of rgm whose basic equations are easy to write down and study. In the rest of this section we reformulate $\mathcal{D}'$ as such an rgm $\mathcal{D}_*$, built by completion upon one single atom $*$ and one single basic equation $* \simeq \omega \to *$.

Definition 3.4.4. We call $\mathcal{D}_*$ the rgm obtained as the completion

$$\mathcal{D}_* \ := \ \overline{(\{*\}, j)}$$

where $*$ is not a pair and $j : \mathcal{M}_f\big(\{*\}\big) \times \{*\} \rightharpoonup \{*\}$ is defined just by $j\,\big([\,], *\big) := *$.

Lemma 3.4.5. *The rgm $\mathcal{D}_*$ is extensional.*

Proof. By Proposition 2.2.8, since the partial pair $(\{*\}, j)$ is extensional, i.e. j is surjective. □

Lemma 3.4.6. *In $\mathcal{D}_*$ the equivalence $* \simeq \omega \to *$ holds.*

Proof. By Definition 2.3.3 we have $* \simeq \omega \to *$ if and only if $*^\diamond = (\omega \to *)^\diamond$, which is true since $* = *^\diamond = (\omega \to *)^\diamond = \bar{j}([\,], *^\diamond) = \bar{j}([\,], *) = j([\,], *) = *$. □

So more informally, $\mathcal{D}_*$ is the ergm built upon one single atom $*$ and the only basic equational identification

$$* \;\simeq\; \omega \to * \,.$$

It is then clear that for all $k \in \mathbb{N}$ one has

$$* \;\simeq\; \overbrace{\omega \to \cdots \to \omega}^{k \text{ times}} \to * \,,$$

as formalized in the next lemma.

Proposition 3.4.7. *Let* $\sigma \in \mathsf{T}_{\mathcal{D}_*}$. *Then* $\gamma \simeq *$ *if and only if* γ *is generated by the grammar*

$$\gamma \;::=\; * \mid \omega \to \gamma \quad .$$

In particular $\mu \to \sigma \simeq *$ *implies that* $\mu = \omega$ *and* $\sigma \simeq *$.

Proof. The right-to-left implication is given by a trivial induction on the grammar:

- the type $*$ is equivalent to itself;
- if $\gamma \simeq *$ by IH, then $\omega \to \gamma \;\simeq\; \omega \to * \;\simeq\; *$ (by Lemma 3.4.6).

The left-to-right implication is given by the following induction on γ.

- Since $\mathsf{At}_{\mathcal{D}_*} = \{*\}$ by Proposition 2.2.10, whenever $\gamma \simeq *$ is an atom we have $\gamma = *$.
- Let $\gamma = \mu \to \sigma \;\simeq\; *$. This means that $\bar{j}(\mu^\diamond, \sigma^\diamond) = (\mu \to \sigma)^\diamond = *^\diamond = *$. Since $\bar{j}$ is surjective by Lemma 3.4.5 and $\bar{j}([\,], *) = j([\,], *) = *$ by Definition 4.4.1, the pair $(\mu^\diamond, \sigma^\diamond)$ must be $([\,], *)$. So $\mu = \omega$ and $\sigma \simeq *$, which was to be proved. □

By Proposition 2.2.2 the reflexive object associated with $\mathcal{D}_*$ is $\left(\overline{\{*\}}\,,\, \bar{j}^{\dagger},\, (\bar{j}^{-1})^{\dagger}\right)$. For convenience we rename it as $\big(D\,,\, \mathtt{Abs}\,,\, \mathtt{App}\big)$, i.e. $D := \overline{\{*\}}$, $\mathtt{Abs} := \bar{j}^{\dagger}$ and $\mathtt{App} := (\bar{j}^{-1})^{\dagger}$.

In the rest of the section we prove the equivalence between $\mathcal{D}'$ and $\mathcal{D}_*$. At this purpose, remember the notion of isomorphism between reflexive objects in a cartesian closed category, given in Definition 1.2.3, and the sufficient condition for extensional reflexive objects provided by Lemma 1.2.5.

Definition 3.4.8. The function $f : D \to D'$ is defined by induction on the rank of elements of D as follows:

- $f(*) := \circledast$,
- $f\big([d_1, \dots, d_n], d\big) := \big[f(d_1), \dots, f(d_n)\big] :: f(d)$.

Remember that $([\,], *) \notin D$, since $([\,], *) \in \mathrm{dom}(j)$. So we are avoiding the eventuality $f([\,], *) = [\,] :: f(*) = [\,] :: \circledast = \circledast = f(*)$. This idea is behind the proof of the injectivity of f.

Lemma 3.4.9. *The function* $f : D \to D'$ *is injective.*

Proof. Let $d, d' \in D$ such that $d \neq d'$. Let k (respectively k') be the minimal natural number such that $d \in D_k$ (respectively $d' \in D'_k$). We show $f(d) \neq f(d')$ by induction on $n := k + k'$.

Notice that $n > 0$, for if $n = k = k' = 0$ then $d, d' \in D_0 = \{*\}$, against the hypothesis $d \neq d'$.
Case $n = 1$. One between k and k' is 0 and the other is 1. Say, $k = 0$ and $k' = 1$. Then $d \in D_0 = \{*\}$, namely $d = *$. Whereas $d' \in D_1$ implies that $d' = ([*, \dots, *], *)$. In particular that $[*, \dots, *]$ has multiplicity $\ell > 0$, because $([\,], *) \notin D$ by Definition 3.4.4. So we get

$$f(d') = f([\underbrace{*, \dots, *}_{\ell \text{ times}}], *) = [\underbrace{f(*), \dots, f(*)}_{\ell \text{ times}}] :: f(*) = [\underbrace{\circledast, \dots, \circledast}_{\ell \text{ times}}] :: \circledast \neq [\,] :: \circledast = \circledast = f(*) = f(d).$$

Case $n > 1$. At least one between k and k' must be greater than 0. Say, $k > 0$. The fact that $d \notin D_{k-1}$ for some $k - 1 \geqslant 0$ assures that $d \neq *$. So $d = (m, e)$ for $m \in \mathcal{M}_f(D_{k-1})$ and $e \in D_{k-1}$. As concerns d', there are two eventualities. If it has the form $d' = (m', e')$ then $f(d') = f(m') :: f(e')$. If $d' = *$ then $f(d') = f(*) = \circledast = [\,] :: f(*)$. In the latter case let us set $e' := *$. So it is clear that in both eventualities we can write $f(d') = \widetilde{m} :: f(e')$ for a certain $\widetilde{m} \in \mathcal{M}_f(D)$ and some $e' \in D_{k'-1}$. Since $k - 1 + k' - 1 < k + k' = n$ we can apply the IH to e and e', so to get $f(e) \neq f(e')$. Therefore $f(d) = f(m) :: f(e) \neq \widetilde{m} :: f(e') = f(d')$. □

Lemma 3.4.10. *The function* $f : D \to D'$ *is surjective.*

Proof. Take $d' \in D'$. Let n be the minimal natural number such that $d' \in D'_n$. Notice that $n > 0$, as $D'_0 = \emptyset$. We prove the existence of $d \in D$ such that $f(d) = d'$ by induction on n.
Case $n = 1$. We have $d' \in D_1 = \{\circledast\}$. So we take $d := *$, since $f(*) = \circledast$.
Case $n > 1$. Let $d' = m_1 :: m_2 :: \cdots :: m_p :: \circledast \in D'_n$ with $m_p \neq [\,]$. Notice that $p > 0$ because $d' \neq \circledast$, since $\circledast \in D'_1$ whereas the minimal n such that $d' \in D'_n$ is not 1. In particular let $m_i = [d'_{i1}, \dots, d'_{ik_i}]$ for all $i \in \{1, \dots, p\}$.

For all $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, k_i\}$ let us call n_{ij} the minimal natural number such that $d'_{ij} \in D'_{n_{ij}}$. It is clear that $n_{ij} \leqslant n - 1$. So by IH there exists $d_{ij} \in D$ such that $f(d_{ij}) = d'_{ij}$. We set $d := ([d_{11}, \dots, d_{1k_1}], ([d_{21}, \dots, d_{2k_2}], \cdots ([d_{p1}, \dots, d_{pk_p}], *) \cdots))$.

The fact that $k_p > 0$ assures that $d \in D$ (remember that $([\,], *)$ is not an element of D by the free completion given in Definition 3.4.4). By Definition 3.4.8 finally $f(d) = [f(d_{11}), \dots, f(d_{1k_1})] :: \cdots :: [f(d_{p1}), \dots, f(d_{pk_p})] :: f(*) = m_1 :: \cdots :: m_p :: \circledast = d'$. □

Corollary 3.4.11. *The morphism* $f^\dagger \in \mathbf{MRel}(D, D')$ *is an isomorphism in* **MRel**.

Proof. The function $f : D \to D'$ is a bijection by Lemmas 3.4.9 and 3.4.10. Then the relation $f^\dagger \subseteq \mathcal{M}_f(D) \times D'$ is an isomorphism in **MRel** because of Lemma 2.1.1. □

Lemma 3.4.12. *The morphism* $f^\dagger \in \mathbf{MRel}(D, D')$ *makes*

$$\begin{array}{ccc} D \Rightarrow D & \xrightarrow{\mathsf{Abs}} & D \\ \big\downarrow {\scriptstyle [f^{\dagger-1} \Rightarrow f^\dagger]} & & \big\downarrow {\scriptstyle f^\dagger} \\ [D' \Rightarrow D'] & \xrightarrow{\mathsf{Abs}'} & D' \end{array}$$

commute.

Proof. Remember from § 1.2 that given a Seely category $\mathcal{S}$ and two morphisms in its co-Kleisli $g \in \mathsf{Kl}_!(\mathcal{S})^{op}(A, A') = \mathsf{Kl}_!(\mathcal{S})(A', A) = \mathcal{S}(!A', A)$ and $h \in \mathsf{Kl}_!(\mathcal{S})(B, B') = \mathcal{S}(!B, B')$, then $g \Rightarrow h$ is the composition of arrows in $\mathcal{S}$

$$!(!A \multimap B) \xrightarrow{\mathsf{der}_{!A \multimap B}} !A \multimap B \xrightarrow{\mathsf{prom}_{A,B}} !A \multimap !B \xrightarrow{!g \multimap h} !!A' \multimap B' \xrightarrow{\mathsf{der}_{!A'} \multimap B'} !A' \multimap B'.$$

We are interested in the case where $\mathcal{S} = \mathbf{Rel}$, hence $\mathsf{Kl}_!(\mathcal{S}) = \mathsf{Kl}_{\mathcal{M}_f}(\mathbf{Rel}) = \mathbf{MRel}$, the arrow $g \in \mathsf{Kl}_!(\mathcal{S})^{\mathsf{op}}(A, A')$ is $(f^\dagger)^{-1} \in \mathbf{MRel}^{\mathsf{op}}(D, D') = \mathbf{MRel}(D', D) = \mathcal{P}(\mathcal{M}_f(D') \times D)$ and the arrow $h \in \mathsf{Kl}_!(\mathcal{S})(B, B')$ is $f^\dagger \in \mathbf{MRel}(D, D') = \mathcal{P}(\mathcal{M}_f(D) \times D')$.

We have $f^\dagger = \Big\{([d], d') \mid f(d) = d'\Big\}$ and $(f^\dagger)^{-1} = \Big\{([d'], d) \mid f(d) = d'\Big\}$. Hence

$$\begin{aligned}\mathcal{M}_f\Big((f^\dagger)^{-1}\Big) &= \Big\{\Big(\Big[[d'_1], \dots, [d'_n]\Big], [d_1, \dots, d_n]\Big) \mid n \in \mathbb{N} \text{ and } f(d_i) = d'_i \text{ for all } i\Big\} \\ &\subseteq \mathcal{M}_f(\mathcal{M}_f(D')) \times \mathcal{M}_f(D)\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_f\Big((f^\dagger)^{-1}\Big) \multimap f^\dagger &= \mathcal{M}_f\Big((f^\dagger)^{-1}\Big)^{\perp} \parr f^\dagger = \mathcal{M}_f\Big((f^\dagger)^{-1}\Big)^{-1} \times f^\dagger = \\ \Big\{\Big([d_1, \dots, d_n], \Big[[d'_1], \dots, [d'_n]\Big]\Big) &\mid n \in \mathbb{N} \text{ and } f(d_i) = d'_i \text{ for all } i\Big\} \times f^\dagger = \\ \Big\{\Big(([d_1, \dots, d_n], [d]), \Big(\Big[[d'_1], \dots, [d'_n]\Big], d'\Big)\Big) &\mid n \in \mathbb{N}, f(d) = d', f(d_i) = d'_i \text{ for all } i\Big\} = \\ \Big\{\Big(([d_1, \dots, d_n], [d]), \Big(\Big[[f(d_1)], \dots, [f(d_n)]\Big], f(d)\Big)\Big) &\mid n \in \mathbb{N}, d, d_1, \dots, d_n \in D\Big\} \subseteq \\ \Big(\mathcal{M}_f(D) \times \mathcal{M}_f(D)\Big) &\times \Big(\mathcal{M}_f(\mathcal{M}_f(D')) \times D'\Big) \quad .\end{aligned}$$

By pre-composing with $\mathsf{prom}_{D,D} \circ \mathsf{der}_{!D \multimap D}$ and post-composing with $\mathsf{der}_{!D'} \multimap D'$ we get

$$\Big[f^{\dagger^{-1}} \Rightarrow f^\dagger\Big] = \Big\{\Big(\Big[([d_1, \dots, d_n], d)\Big], ([f(d_1), \dots, f(d_n)], f(d))\Big) \mid n \in \mathbb{N}, d, d_i \in D\Big\}.$$

Remembering how $\mathtt{Abs}'$ is given in Definition 3.4.1, the lower side of our diagram is

$$\mathtt{Abs}' \circ \Big[f^{\dagger^{-1}} \Rightarrow f^\dagger\Big] = \Big\{\Big(\Big[([d_1, \dots, d_n], d)\Big], [f(d_1), \dots, f(d_n)] :: f(d)\Big) \mid n \in \mathbb{N}, d, d_i \in D\Big\}. \tag{56}$$

As concerns the upper side of the diagram, we have

$$\mathtt{Abs} := \bar{\jmath}^\dagger = \Big\{\Big([(m, d)], \bar{\jmath}(m, d)\Big) \mid m \in \mathcal{M}_f(D) \text{ and } d \in D\Big\},$$

from which we get

$$f^\dagger \circ \mathtt{Abs} = \Big\{\Big(\Big[([d_1, \dots, d_n], d)\Big], f\left(\bar{\jmath}([d_1, \dots, d_n], d)\right)\Big) \mid n \in \mathbb{N}, d, d_i \in D\Big\}. \tag{57}$$

Our thesis is the equality between (56) and (57). Clearly it is enough to show that for every $n \in \mathbb{N}$ and $d, d_1, \dots, d_n \in D$ we have $f\left(\bar{\jmath}([d_1, \dots, d_n], d)\right) = [f(d_1), \dots, f(d_n)] :: f(d)$.

We distinguish two cases.

Case $n = 0$ and $d = *$. In this case $\bar{\jmath}([d_1, \dots, d_n], d) = \bar{\jmath}([\,], *) = *$ by Definition 3.4.4. Then using Definition 3.4.8 we obtain

$$f(\bar{\jmath}([d_1, \dots, d_n], d)) = f(*) = \circledast = [\,] :: \circledast = [\,] :: f(*) = [f(d_1), \dots, f(d_n)] :: f(d).$$

Case $n > 0$ or $d \neq *$. In this case $\bar{\jmath}([d_1, \dots, d_n], d) = ([d_1, \dots, d_n], d)$ by Definition 3.4.4. So by Definition 3.4.8 we get

$$f\left(\bar{\jmath}([d_1, \dots, d_n], d)\right) = f([d_1, \dots, d_n], d) = [f(d_1), \dots, f(d_n)] :: f(d)$$

which completes the proof. □

Proposition 3.4.13. *The morphism* $f^\dagger \in \mathbf{MRel}(D, D')$ *is an isomorphism of reflexive objects between (the reflexive object associated to)* $\mathcal{D}_*$ *and* $\mathcal{D}'$*.*

Proof. The fact that $f^\dagger$ is an isomorphism in **MRel** is given by Corollary 3.4.11. Moreover $\mathcal{D}_*$ is extensional by Lemma 3.4.5 and the diagram in the statement of Lemma 3.4.12 commutes. Hence $f^\dagger$ satisfies the hypothesis of Lemma 1.2.5. □

Theorem 3.4.14. *The rgm* $\mathcal{D}_*$ *is fully abstract for* $\mathcal{H}^*$*, namely* $\mathrm{Th}(\mathcal{D}_*) = \mathcal{H}^*$*.*

Proof. By Proposition 3.4.13 and Theorem 1.2.4 we have $[\![M]\!]^{\vec{x}}_{\mathcal{D}_*} = [\![N]\!]^{\vec{x}}_{\mathcal{D}_*}$ if and only if $[\![M]\!]^{\vec{x}}_{\mathcal{D}'} = [\![N]\!]^{\vec{x}}_{\mathcal{D}'}$. So $\mathrm{Th}(\mathcal{D}_*) = \mathrm{Th}(\mathcal{D}')$, which is $\mathcal{H}^*$ by Theorem 3.4.2. □

4 THE FULL ABSTRACTION PROBLEM FOR MORRIS'S PREORDER

INTRODUCTION

Looking for characterizations of observational equivalences can be a complicated task, no matter what mathematical framework one has at hand. Relational semantics is not immune to this problem. Consider for instance the relational model $\mathcal{D}'$ defined in § 3.4. It is fully abstract for $\mathcal{H}^*$, as proved in [Man09]. However, the proof of this fact relies on realizability candidates, the tricky technique briefly recalled at the beginning of § 2.6.

In this and the next chapter we search for characterizations of Morris's observability, namely the preorder theory that we called $\sqsubseteq_{\mathcal{H}^+}$ in § 1.3.

In order to be fully abstract for Morris's preorder, an rgm must satisfy

$$[\![M]\!] \subseteq [\![N]\!] \iff \text{for all contexts } C[\,] \text{ if } C[M] \text{ has a } \beta\text{-nf then } C[N] \text{ has a } \beta\text{-nf.} \qquad (58)$$

The right-to-left implication is true for all *extensional* rgm's. We prove this by exploiting the notion of extensional Böhm tree $BT^e(-)$, which characterizes $\sqsubseteq_{\mathcal{H}^+}$ as recalled in § 1.4.

The hard work is to find conditions on ergm's giving the left-to-right implication in (58). In this thesis we address the problem in two different ways. These two *modus operandi* are not equally powerful. The stronger one — strong enough to catch *all* rgm's inducing Morris's preorder theory — will be the subject of Chapter 5. In this chapter we use a more conventional, but still interesting approach to the issue. Here is the idea in a nutshell. On the one hand, $\sqsubseteq_{\mathcal{H}^+}$ concerns β-*normalization*. On the other hand, we deal with $[\![-]\!]$ by means of a *type system*. It seems then reasonable to look for rgm's in which the notion of β-normalizability can be characterized in terms of derivable sequents of the system. More precisely, we find an infinite class of ergm's $\mathcal{D}$, which we call *uniformly bottomless* ergm's, satisfying for all $M \in \Lambda$ something like this:

$$M \text{ has a } \beta\text{-nf} \iff \text{there is } (\Gamma, \sigma) \in \mathsf{Env}_{\mathcal{D}} \times T_{\mathcal{D}} \text{ such that } \Gamma \vdash^{\mathcal{D}} M : \sigma \text{ and } P(\Gamma, \sigma)$$

where $P(-,-)$ is a certain property that only mentions typings. Roughly, the property $P(\Gamma, \sigma)$ states a certain constraint on the occurrence of the empty intersection ω in Γ and σ. In the end, assuming the hypothesis $[\![M]\!] \subseteq [\![N]\!]$, one can get

$$\begin{aligned} C[M] \text{ has a } \beta\text{-nf} &\iff \text{there is } (\Gamma, \sigma) \text{ such that } \Gamma \vdash C[M] : \sigma \text{ and } P(\Gamma, \sigma) \\ &\implies \text{there is } (\Gamma, \sigma) \text{ such that } \Gamma \vdash C[N] : \sigma \text{ and } P(\Gamma, \sigma) \qquad (59) \\ &\iff C[N] \text{ has a } \beta\text{-nf} \end{aligned}$$

where (59) is given by $[\![C[M]]\!] \subseteq [\![C[N]]\!]$, an obvious consequence of $[\![M]\!] \subseteq [\![N]\!]$.

So all uniformly bottomless ergm's satisfy (58), i.e. they induce Morris's preorder theory. The simplest one among them is denoted by $\mathcal{D}_\star$ and turns out to be built up from a single atom $\star$ and the basic equation $\star \to \star \simeq \star$.

The *(categorical) semantic* approach is not the only one possible when trying to reformulate $\mathcal{H}^+$. At the end of this chapter we show what could be called a *syntactic* characterization of $\mathcal{H}^+$. It relies on an *ad hoc* version of the Taylor expansion (§ 1.6) taking η-conversion into account, and therefore called *extensional Taylor expansion*.

PLAN OF THE CHAPTER. In § 4.1 we define the uniformly bottomless ergm's. In § 4.2 we provide some examples. In § 4.3 we show that these ergm's solve the full abstraction problem for Morris's preorder theory. In § 4.4 we take a look at the example of $\mathcal{D}_\star$, with a particular attention to the role that the atom $\star$ plays in it. In § 4.5 we present the extensional Taylor expansion of λ-terms, providing another model of Morris's equivalence.

4.1 UNIFORMLY BOTTOMLESS RELATIONAL GRAPH MODELS

We need to formulate a certain notion of *occurrence* of the empty intersection ω in the *unfolding* of types. Roughly, the fact that when a type σ is *unfolded* by means of $\simeq$ so to have a certain number $k+1$ of arrows

$$\sigma \;\simeq\; \mu_0 \to \cdots \to \mu_k \to \sigma' \tag{60}$$

then one of those μ_i's is ω, or otherwise ω can be found by keeping unfolding deeper and deeper some type inside one of those μ_i's. The idea is formalized in Definition 4.1.1 below. This is done in fact in a slightly more general way than the informal description above: firstly, the value called k in (60) will vary *level by level* of the unfolding; secondly, the definition will also carry a notion of *polarization* of the occurrence of ω.

Notice that an unfolding like in (60) is always possible if the rgm is extensional. In fact, for convenience all over this chapter we will always consider ergm's, even when the hypothesis of extensionality is not strictly necessary.

Definition 4.1.1. Let $\mathcal{D}$ be an ergm, $f : \mathbb{N} \to \mathbb{N}$ and $\sigma \in T_{\mathcal{D}}$. We define at the same time the meaning of $\omega \in^- \mathtt{UNF}^n_f(\sigma)$ and $\omega \in^+ \mathtt{UNF}^n_f(\sigma)$ by mutual induction on $n \in \mathbb{N}$.

- We write $\omega \in^- \mathtt{UNF}^n_f(\sigma)$ if and only if whenever $\sigma \simeq \mu_0 \to \cdots \to \mu_{f(n)} \to \sigma'$ there exists $i \in \{0, \dots, f(n)\}$ such that either $\mu_i = \omega$ or there is $\tau \in \mu_i$ such that $\omega \in^+ \mathtt{UNF}^{n+1}_f(\tau)$.
- We write $\omega \in^+ \mathtt{UNF}^n_f(\sigma)$ if and only if whenever $\sigma \simeq \mu_0 \to \cdots \to \mu_{f(n)} \to \sigma'$ there exist $i \in \{0, \dots, f(n)\}$ and $\tau \in \mu_i$ such that $\omega \in^- \mathtt{UNF}^{n+1}_f(\tau)$.

When $\omega \in^- \mathtt{UNF}^0_f(\sigma)$ we say that ω *f-occurs negatively in* σ, denoted by $\omega \in^-_f \sigma$. Similarly, whenever $\omega \in^+ \mathtt{UNF}^0_f(\sigma)$ then ω *f-occurs positively in* σ, denoted by $\omega \in^+_f \sigma$.

For any $\mu \in I_{\mathcal{D}}$ we say that ω *f-occurs negatively in* μ, denoted by $\omega \in^-_f \mu$, if there exists $\sigma \in \mu$ such that $\omega \in^-_f \sigma$. Similarly one defines $\omega \in^+_f \mu$.

Examples 4.1.2. Here are a couple of simple examples.

- In the rgm $\mathcal{D}_*$ (Definition 3.4.4) we have $* \simeq \omega^k \to *$ for all $k \in \mathbb{N}$, by Proposition 3.4.7. It is then clear that $\omega \in^-_f *$ for every $f : \mathbb{N} \to \mathbb{N}$. If $\sigma := \omega \to \omega \to * \to *$ then $\omega \in^+_f \sigma$ for every $f : \mathbb{N} \to \mathbb{N}$ such that $f(0) = 2$, since $\sigma \simeq \omega \to \omega \to (\omega \to *) \to *$. On the other hand, $\omega \in^-_f \sigma$ for every $f : \mathbb{N} \to \mathbb{N}$ such that $f(0) \neq 2$.

- Consider the rgm $\overline{(\{\alpha,\beta,\gamma\}, j)}$ with the partial map j defined by $j([\,],\alpha) := \beta$ and $j([\beta,\beta,\gamma],\gamma) := \alpha$. Since $\beta \simeq \omega \to \alpha \simeq \omega \to \beta \wedge \beta \wedge \gamma \to \gamma$ we have $\omega \in_f^- \beta$ for every $f : \mathbb{N} \to \mathbb{N}$ such that $f(0) = 0$. Also $\omega \in_f^+ \beta$ for every $f : \mathbb{N} \to \mathbb{N}$ such that $f(0) = 1$ and $f(1) = 0$. Since $\alpha \simeq \beta \wedge \beta \wedge \gamma \to \gamma$ we can then state that $\omega \in_f^- \alpha$ for every $f : \mathbb{N} \to \mathbb{N}$ such that $f(0) = 0$, $f(1) = 1$ and $f(2) = 0$.

Notations. We employ the following helpful notations.

- By the expression $\omega \in^p \texttt{UNF}_f^n(\sigma)$ we mean *no matter which between* $\omega \in^+ \texttt{UNF}_f^n(\sigma)$ *and* $\omega \in^- \texttt{UNF}_f^n(\sigma)$. In other words, that apex p (standing for *polarity*) is always intended to be an element of the set $\{+,-\}$.

- When $f : \mathbb{N} \to \mathbb{N}$ is a constant function, i.e. there is $k \in \mathbb{N}$ such that $f(n) = k$ for all $n \in \mathbb{N}$, we prefer to replace the pedex f with k in all the expression in Definition 4.1.1, namely writing $\omega \in^p \texttt{UNF}_k^n(\sigma)$, $\omega \in_k^p \sigma$, and so on.

Definition 4.1.1 is consistent, in the sense that the truth value associated to the expression $\omega \in^p \texttt{UNF}_f^n(\sigma)$ is unique, because independent of how σ is unfolded using $\simeq^{\mathcal{D}}$. In fact, not only $\omega \in^p \texttt{UNF}_f^n(-)$ is a function on $T_{\mathcal{D}}$, but it can also be seen as a function on $T_{\mathcal{D}}/\simeq^{\mathcal{D}}$. This is formalized in Lemma 4.1.3 below.

Lemma 4.1.3. *Let* $f : \mathbb{N} \to \mathbb{N}$. *Let* $\mathcal{D}$ *be an* ergm *and* $\sigma, \gamma \in T_{\mathcal{D}}$ *such that* $\sigma \simeq \gamma$. *Then* $\omega \in_f^p \sigma$ *if and only if* $\omega \in_f^p \gamma$.

Lemma 4.1.4. *Let* $f, g : \mathbb{N} \to \mathbb{N}$ *such that* $g(n) \leqslant f(n)$ *for all* $n \in \mathbb{N}$. *Let* $\mathcal{D}$ *be an* ergm *and* $\sigma \in T_{\mathcal{D}}$ *such that* $\omega \in_g^p \sigma$. *Then* $\omega \in_f^p \sigma$.

Corollary 4.1.5. *Let* $\mathcal{D}$ *be an* ergm, $\sigma \in T_{\mathcal{D}}$ *and* $k \in \mathbb{N}$. *If* $\omega \in_k^p \sigma$ *then* $\omega \in_{k+1}^p \sigma$.

Remark 4.1.6. One has $\omega \in^p \texttt{UNF}_k^n(\sigma)$ if and only if $\omega \in^p \texttt{UNF}_k^{n+1}(\sigma)$.

Definition 4.1.7. Let $\mathcal{D}$ be an ergm and $f : \mathbb{N} \to \mathbb{N}$. A type $\sigma \in T_{\mathcal{D}}$ is f-*bottomless* if and only if $\omega \notin_f^+ \sigma$ and $\omega \notin_f^- \sigma$. In particular if f is a constant function k we say that σ is k-*bottomless*.

We can now present the main notion of this chapter.

Definition 4.1.8. An ergm $\mathcal{D}$ is *uniformly bottomless* if and only if for all $k \in \mathbb{N}$ there exists a k- bottomless type $\sigma_k \in T_{\mathcal{D}}$.

In order to understand the concrete examples of uniformly bottomless ergm's presented in the next section, it may be useful to take into account also the following notion.

Definition 4.1.9. An ergm $\mathcal{D}$ is *strongly bottomless* if and only if for all $f : \mathbb{N} \to \mathbb{N}$ there exists an f-bottomless type $\sigma \in T_{\mathcal{D}}$.

Proposition 4.1.10. *A strongly bottomless* ergm *is uniformly bottomless.*

Proof. Among all functions $f : \mathbb{N} \to \mathbb{N}$ there are in particular all the constant ones. □

Let us see some examples of uniformly bottomless rgm's.

Example 4.2.1. We call $\mathcal{F}$ the free completion

$$\mathcal{F} \;:=\; \overline{(\,F, j\,)}$$

of the partial pair (F, j) defined as follows. The elements of the denumerable family

$$F \;:=\; \{\,*\,\} \cup \left\{\,\beta_n^k\,\right\}_{n \in \mathbb{N},\, k \leqslant n}$$

are pairwise distinct and are not pairs. The partial function $j : \mathcal{M}_f(F) \times F \rightharpoonup F$ maps

$$([\;], *) \quad\mapsto\quad * \;,$$

for all $n \in \mathbb{N}$ and $0 < k \leqslant n$

$$([\beta_n^n], \beta_n^{k-1}) \quad\mapsto\quad \beta_n^k \;,$$

for all $n \in \mathbb{N}$

$$([\beta_n^n], *) \quad\mapsto\quad \beta_n^0 \;,$$

and is undefined on any other $(m, a) \in \mathcal{M}_f(F) \times F$.

The rgm $\mathcal{F}$ is extensional, since j is surjective. Notice that the total injection $\bar{j}$ maps

$$\begin{array}{rcl} ([\;], *) & \mapsto & * \\ ([\beta_0^0], *) & \mapsto & \beta_0^0 \\ ([\beta_1^1], ([\beta_1^1], *)) & \mapsto & \beta_1^1 \\ & \vdots & \\ \overbrace{([\beta_n^n], (([\beta_n^n], (\cdots(([\beta_n^n]}^{n+1 \text{ times}}, *)\cdots))) & \mapsto & \beta_n^n \\ & \vdots & \end{array}$$

By convenience let us rename $\alpha_n := \beta_n^n$ for all $n \in \mathbb{N}$. Then one can think of $\mathcal{F}$ as the ergm relying on the basic equational identifications

$$* \;\simeq\; \omega \to *$$

and for all $n \in \mathbb{N}$

$$\alpha_n \;\simeq\; \overbrace{\alpha_n \to \cdots \to \alpha_n}^{n+1 \text{ times}} \to * \,.$$

Clearly $\mathcal{F}$ is uniformly bottomless, since for all $k \in \mathbb{N}$ the atom α_k is k-bottomless.

Example 4.2.2. We call $\mathcal{G}$ the completion

$$\mathcal{G} \;:=\; \overline{(\,G, j\,)}$$

of the partial pair (G, j) defined as follows. The $2^{\aleph_0}$ elements of the family

$$G \;:=\; \{\,*\,\} \cup \left\{\beta_f^{n,k}\right\}_{f \in \mathbb{N}^{\mathbb{N}}}^{n \in \mathbb{N},\, k \leqslant f(n)}$$

are pairwise distinct and are not pairs. The partial function $j : \mathcal{M}_f(G) \times G \rightharpoonup G$ maps

$$([\,], *) \quad\mapsto\quad * \;,$$

for all $f : \mathbb{N} \to \mathbb{N}$, for all $n \in \mathbb{N}$ and for all $0 < k \leqslant f(n)$

$$\left(\left[\beta_f^{n+1, f(n+1)}\right], \beta_f^{n, k-1}\right) \quad\mapsto\quad \beta_f^{n,k} \;,$$

for all $f : \mathbb{N} \to \mathbb{N}$ and for all $n \in \mathbb{N}$

$$\left(\left[\beta_f^{n+1, f(n+1)}\right], *\right) \quad\mapsto\quad \beta_f^{n,0} \;,$$

and is undefined on any other $(m, a) \in \mathcal{M}_f(G) \times G$.

The rgm $\mathcal{G}$ is extensional, since j is surjective.

Notice that for all $f : \mathbb{N} \to \mathbb{N}$ and for all $n \in \mathbb{N}$ the total injection $\bar{j}$ maps

$$\overbrace{\left(\left[\beta_f^{n+1, f(n+1)}\right], \left(\left[\beta_f^{n+1, f(n+1)}\right], \cdots \left(\left[\beta_f^{n+1, f(n+1)}\right]\right.\right.\right.}^{f(n)+1 \text{ times}}\left.\left.\left., *\right) \cdots\right)\right) \quad\mapsto\quad \beta_f^{n, f(n)} \,.$$

By convenience let us rename $\alpha_f^n := \beta_f^{n, f(n)}$ for all $n \in \mathbb{N}$. Then one can think of $\mathcal{G}$ as the ergm relying on the basic equational identifications

$$* \;\simeq\; \omega \to *$$

and for every $f : \mathbb{N} \to \mathbb{N}$

$$\alpha_f^0 \;\simeq\; \overbrace{\alpha_f^1 \to \cdots \to \alpha_f^1}^{f(0)+1 \text{ times}} \to *$$

$$\alpha_f^1 \;\simeq\; \overbrace{\alpha_f^2 \to \cdots \to \alpha_f^2}^{f(1)+1 \text{ times}} \to *$$

$$\vdots$$

$$\alpha_f^n \;\simeq\; \overbrace{\alpha_f^{n+1} \to \cdots \to \alpha_f^{n+1}}^{f(n)+1 \text{ times}} \to *$$

$$\vdots$$

Clearly $\mathcal{G}$ is strongly bottomless, since for all $f : \mathbb{N} \to \mathbb{N}$ the atom α_f^0 is f-bottomless. Hence $\mathcal{G}$ is uniformly bottomless by Proposition 4.1.10.

Example 4.2.3. Let $\mathcal{G}^{\mathsf{rec}}$ be defined just like the rgm $\mathcal{G}$ above, but considering only *recursive* functions from $\mathbb{N}$ to $\mathbb{N}$. In other words, we repeat the construction given in Example 4.2.2 but restricting to the set of $\aleph_0$ atoms

$$G^{\mathsf{rec}} \quad := \quad \{\,*\,\} \cup \left\{\alpha^n_f\right\}^{n \in \mathbb{N}}_{f \in \mathbb{N}^{\mathbb{N}},\, f \text{ computable}} .$$

Of course $\mathcal{G}^{\mathsf{rec}}$ is not strongly bottomless, like $\mathcal{G}$ is. Nevertheless, it is uniformly bottomless. Indeed, all constant functions are computable, so for all $k \in \mathbb{N}$ we get a k-bottomless type $\alpha^0_f \in T_{\mathcal{G}^{\mathsf{rec}}}$ by taking $f(n) := k$ for all $n \in \mathbb{N}$.

Remark 4.2.4. Let the ergm $\mathcal{D}$ be embedded into the ergm $\mathcal{D}'$, in the sense that there exists an injective homomorphism of rgm's $f : \mathcal{D} \to \mathcal{D}'$. If $\mathcal{D}$ is uniformly bottomless then also $\mathcal{D}'$ is. The reason is that for all $k \in \mathbb{N}$ the fact that a certain type $\sigma_k \in T_{\mathcal{D}}$ is k-bottomless in $\mathcal{D}$ implies that $f(\sigma_k) \in T_{\mathcal{D}'}$ is k-bottomless in $\mathcal{D}'$.

Considering for instance the rgm's in Examples 4.2.1-4.2.3, it is easy to realize that $\mathcal{F}$ is embedded into $\mathcal{G}^{\mathsf{rec}}$, which in turn is embedded into $\mathcal{G}$.

4.3 FULL ABSTRACTION FOR MORRIS'S PREORDER THEORY

In this section we prove a theorem of *full abstraction*: every uniformly bottomless ergm $\mathcal{D}$ induces Morris's preorder theory $\sqsubseteq_{\mathcal{H}^+}$.

The core of the proof is Lemma 4.3.1, which exploits typings to characterize the λ-terms having β-nf. This lemma is analogous to a classical result [BDS13, Theorem 17B.15(i)] that characterizes β-normalizable λ-terms by typability in the intersection type systems associated with certain filter models.

Let us give some intuition to Lemma 4.3.1. As explained in the introduction, we need some property $P(-,-)$ giving for all $M \in \Lambda$

$$M \text{ has a } \beta\text{-nf} \quad \Longleftrightarrow \quad \text{there is } (\Gamma, \sigma) \in \mathsf{Env}_{\mathcal{D}} \times T_{\mathcal{D}} \text{ such that } \Gamma \vdash^{\mathcal{D}} M : \sigma \text{ and } P(\Gamma, \sigma) .$$

Remember that by Böhm Approximation (Theorem 2.6.5) typing M is equivalent to typing some $a \in BT(M)^*$. Now, suppose that M does not have a β-nf. This means that every $a \in BT(M)^*$ must contain $\perp$. Of course, those occurrences of $\perp$ must appear in argument positions in a. So, when trying to type a, some variable x in a gets a type of the form

$$\mu \to \cdots \to \omega \to \cdots \to \tau , \tag{61}$$

or at least this must be the case *up to type equivalence* $\simeq^{\mathcal{D}}$. Notice that (61) corresponds to the basic case of the common idea of *negative occurrence* of ω. Such a negative occurrence of ω remains fixed in the environment Γ of the final sequent $\Gamma \vdash a : \sigma$ if x is not λ-abstracted in a; or it becomes a *positive* occurrence in σ, in case x gets λ-abstracted in a.

As an instance, consider $M = \lambda x.y\,\mathbf{\Omega I}(xy\mathbf{\Omega})$. Up to type equivalence $\simeq^{\mathcal{D}}$, any derivable sequent $\Gamma \vdash \lambda x.y\,\mathbf{\Omega I}(xy\mathbf{\Omega}) : \sigma$ has:

- a negative occurrence of ω in $\Gamma(y)$, because the free variable y must receive a type of the form $\omega \to \mu \to \nu \to \tau$ by Rule var (when in head position, y takes three arguments, namely $\mathbf{\Omega}$, $\mathbf{I}$ and $xy\mathbf{\Omega}$, and the first of them is an unsolvable);

- a positive occurrence of ω in σ, because the λ-abstracted variable x must receive a type of the form $\mu' \to \omega \to \tau'$ by Rule var (when in head position, x takes two arguments, namely y and $\mathbf{\Omega}$, and the second of them is an unsolvable).

It seems that we could have here some very rough characterization of non-β-normalizability of a λ-term M: *for every derivable typing $\Gamma \vdash M : \sigma$ the intersection ω occurs positively in σ or negatively in Γ, at least up to type equivalence*. So its negation should now look like a very rough characterization of β-normalizability of M: *there exists a typing $\Gamma \vdash M : \sigma$ such that ω does not occur positively in σ nor negatively in Γ, at least up to type equivalence*.

This intuitive idea can be further refined. Since a β-normal form has an upper bound $k \in \mathbb{N}$ on the number of arguments of all its head variables, one does not even need to talk of (positive or negative) occurrences of ω, but rather formalize the idea of (positive or negative) occurrence of ω *up to* k *arguments*. Which is exactly what we did with the notion that we denoted by $\omega \in_k^p \sigma$.

In the end, what we actually obtain is

$$M \text{ has a } \beta\text{-nf} \iff \begin{array}{l}\text{for all } k \in \mathbb{N} \text{ big enough there exists (in correspondence with } k) \\ \text{a pair } (\Gamma, \sigma) \in \mathsf{Env}_{\mathcal{D}} \times \mathsf{T}_{\mathcal{D}} \text{ such that } \Gamma \vdash^{\mathcal{D}} M : \sigma \text{ and } P(\Gamma, \sigma, k)\end{array}$$

where the property $P(-,-,-)$ is given by

$$P(\Gamma, \sigma, k) \iff \omega \notin_k^+ \sigma \text{ and } \omega \notin_k^- \Gamma(x) \text{ for all } x \in \mathrm{Var}\,.$$

This whole informal reasoning has one major issue. We are always talking of derivable sequents *up to type equivalence* $\simeq^{\mathcal{D}}$. But in general $\simeq^{\mathcal{D}}$ can mess up the polarities of the occurrences of ω in the typing.

For instance, consider the rgm's defined in Examples 4.2.1-4.2.3. They all have a type $*$ such that $* \simeq \omega \to *$. Hence at any moment in a typing derivation we can turn (via Rule eq) the type $*$, which satisfies $\omega \notin_f^- *$ for every $f : \mathbb{N} \to \mathbb{N}$, into another one $\omega \to *$ that on the opposite satisfies $\omega \in_f^- \omega \to *$ for every $f : \mathbb{N} \to \mathbb{N}$ (and *vice versa*).

So actually the informal reasoning above makes sense only if $\mathcal{D}$ has some *stability of polarities modulo* $\simeq^{\mathcal{D}}$. That is exactly the purpose of our notion of uniformly bottomless ergm: for all $k \in \mathbb{N}$ such a stability is provided *up to* k *arguments* by a k-bottomless type σ_k.

Lemma 4.3.1. *Let $\mathcal{D}$ be a uniformly bottomless* ergm. *Let $M \in \Lambda$. The following facts are equivalent.*

1. M *has a β-normal form.*
2. *There exists $a \in BT(M)^*$ that does not contain* $\perp$.
3. *There exists $t \in nf_\beta\mathcal{T}(M)$ that does not contain the empty bag* $[\,]$.
4. *For some $t \in nf_\beta\mathcal{T}(M)$ there exists $\tilde{n} \in \mathbb{N}$ such that for all $k \geqslant \tilde{n}$ there is $(\Gamma, \sigma) \in \mathsf{Env}_{\mathcal{D}} \times \mathsf{T}_{\mathcal{D}}$ satisfying:*
   - $\Gamma \vdash^{\mathcal{D}} t : \sigma$ *is derivable,*
   - $\omega \notin_k^+ \sigma$,

- *$\omega \notin^-_k \Gamma(x)$ for all $x \in \mathsf{Var}$.*

5. *There exists $\tilde{n} \in \mathbb{N}$ such that for all $k \geqslant \tilde{n}$ there is $(\Gamma, \sigma) \in \mathsf{Env}_{\mathcal{D}} \times \mathsf{T}_{\mathcal{D}}$ satisfying:*
   - *$\Gamma \vdash^{\mathcal{D}} M : \sigma$ is derivable,*
   - *$\omega \notin^+_k \sigma$,*
   - *$\omega \notin^-_k \Gamma(x)$ for all $x \in \mathsf{Var}$.*

(Remark: in Conditions 4 and 5 the pair (Γ, σ) depends on k. We do not write something like (Γ_K, σ_k) just to keep the notation the less heavy as possible.)

Proof of (1 $\Rightarrow$ 2). If N is the β-nf of M, then N itself is a finite Böhm-like tree not containing $\perp$ and such that $N \in \mathrm{BT}(M)^*$. □

Proof of (2 $\Rightarrow$ 1). For all $a \in \mathrm{BT}(M)^*$ there is $M' \in \Lambda$ such that $M =_\beta M'$ and $\mathrm{BT}(M') = a$. In particular in case a is finite and does not contain $\perp$ the corresponding M' is in β-nf, so it is the β-nf of M. □

Proof of (2 $\iff$ 3). By Definition 1.6.1 Statement 2 is equivalent to the existence of a resource term $t \in \mathcal{T}(\mathrm{BT}(M))$ that does not contain $[\,]$. By Theorem 1.6.4 the set $\mathcal{T}(\mathrm{BT}(M))$ is nothing but $t \in \mathrm{nf}_\beta \mathcal{T}(M)$. □

Proof of (4 $\iff$ 5). By Theorem 2.6.3, namely Taylor Approximation Theorem, we have

$$[\![M]\!] \quad = \quad [\![\mathcal{T}(M)]\!] \ .$$

Since $\mathcal{T}(M) =_\beta \mathrm{nf}_\beta \mathcal{T}(M)$, by Corollary 2.5.10, i.e. the soundness of the linear resource calculus, we have

$$[\![\mathcal{T}(M)]\!] \quad = \quad [\![\mathrm{nf}_\beta \mathcal{T}(M)]\!] \ .$$

By transitivity

$$[\![M]\!] \quad = \quad [\![\mathrm{nf}_\beta \mathcal{T}(M)]\!] \ .$$

In other words, a sequent $\Gamma \vdash M : \sigma$ is derivable if and only if there is some $t \in \mathrm{nf}_\beta \mathcal{T}(M)$ such that $\Gamma \vdash t : \sigma$ is derivable. (Notice that this fact is true in general, even in case M is unsolvable, hence $\mathrm{nf}_\beta \mathcal{T}(M) = \emptyset$.)

It is then clear that Condition 4 is equivalent to Condition 5. Indeed, all the other properties stated in 4 and 5 only concern Γ, σ and k, independently of M or t. And these properties are the same in 4 and 5, so there is nothing to prove. □

Proof of (3 $\Rightarrow$ 4). For all $k \in \mathbb{N}$, let $\sigma_k \in \mathsf{T}_{\mathcal{D}}$ be a fixed k-bottomless type.
(By the way, we remark that the proof of (3 $\Rightarrow$ 4) is the only one requiring the hypothesis of existence of such kind of types.)

We prove that the implication holds more generally for any β-normal form t that does not contain $[\,]$, regardless the fact that $t \in \mathrm{nf}_\beta \mathcal{T}(M)$.

We proceed by structural induction on t.

Case $t = y\,\mathbf{b_1}\cdots\mathbf{b_p}$ for some $p \geqslant 0$. Since $[\,]$ does not occur in t by hypothesis, for all $i \in \{1,\dots,p\}$ we have $\mathbf{b_i} = [s_{i1},\dots,s_{i\,n_i}]$ for some $n_i > 0$, and moreover $[\,]$ does not occur in s_{ij} for all $j \in \{1,\dots,n_i\}$.

By IH Condition 4 holds for all s_{ij}. Let $\widetilde{n}_{ij} \in \mathbb{N}$ be the natural number corresponding with s_{ij} as in 4. In order to prove that Condition 4 holds for t, we set

$$\widetilde{n} \;:=\; \max_{i,j} \widetilde{n}_{ij} \,.$$

Let $k \geqslant \widetilde{n}$. For all $i \in \{1,\dots,p\}$ and $j \in \{1,\dots,n_i\}$ we have $k \geqslant \widetilde{n} \geqslant \widetilde{n}_{ij}$, so there exist Γ_{ij} and τ_{ij} such that $\omega \notin^-_k \Gamma_{ij}(x)$ for all $x \in \mathrm{Var}$, $\omega \notin^+_k \tau_{ij}$, and $\Gamma_{ij} \vdash s_{ij} : \tau_{ij}$ is derivable.

We define $\mu_i := \bigwedge_{j=1}^{n_i} \tau_{ij}$ for all $i \in \{1,\dots,p\}$. Notice that for all $i \in \{1,\dots,p\}$

$$\omega \;\notin^+_k\; \mu_i \,. \tag{62}$$

Also, setting

$$\Gamma_0 \;:=\; (y : \mu_1 \to \cdots \to \mu_p \to \sigma_k) \tag{63}$$

and

$$\Gamma \;:=\; \Gamma_0 \wedge \left(\textstyle\bigwedge_{i=1}^{p} \bigwedge_{j=1}^{n_i} \Gamma_{ij}\right), \tag{64}$$

we derive

$$\frac{\Gamma_0 \vdash y : \mu_1 \to \cdots \to \mu_p \to \sigma_k \quad \Gamma_{ij} \vdash s_{ij} : \tau_{ij} \quad i \in \{1,\dots,p\},\ j \in \{1,\dots,n_i\}}{\Gamma \vdash y\,\mathbf{b_1}\cdots\mathbf{b_p} : \sigma_k}$$

Let us show that the pair $(\Gamma, \sigma_k) \in \mathsf{Env}_{\mathcal{D}} \times \mathsf{T}_{\mathcal{D}}$ is the one that we are seeking in correspondence with k.

As σ_k is k-bottomless, $\omega \;\notin^+_k\; \sigma_k$ holds by Definition 4.1.7. So we are left to show that $\omega \notin^-_k \Gamma(x)$ for all $x \in \mathrm{Var}$.

Firstly, let us prove that

$$\omega \;\notin^-_k\; \mu_1 \to \cdots \to \mu_p \to \sigma_k \,. \tag{65}$$

According to Definition 4.1.1, the expression (65) means that whenever

$$\mu_1 \to \cdots \to \mu_p \to \sigma_k \;\simeq\; \nu_0 \to \cdots \to \nu_k \to \delta$$

then for all $i \in \{0,\dots,k\}$ we have $\nu_i \neq \omega$ and $\omega \notin^+ \mathtt{UNF}^{\,1}_{\,k}(\tau)$ for all $\tau \in \nu_i$.

Since $f^{\geqslant 1} = f$ when f is a constant function, by Remark 4.1.6 we have $\omega \notin^+ \mathtt{UNF}^{\,1}_{\,k}(\tau)$ if and only if $\omega \notin^+ \mathtt{UNF}^{\,0}_{\,k}(\tau)$. So proving Condition (65) is equivalent to prove what follows:

whenever

$$\mu_1 \to \cdots \to \mu_p \to \sigma_k \;\simeq\; \nu_0 \to \cdots \to \nu_k \to \delta \tag{66}$$

then for all $i \in \{0,\dots,k\}$ we have $\nu_i \neq \omega$ and $\omega \notin^+_k \tau$ for all $\tau \in \nu_i$ (i.e. $\omega \notin^+_k \nu_i$).

Now, when (66) holds then for all $i \in \{0,\dots,p-1\}$ we have

$$\nu_i \simeq \mu_{i+1} \,. \tag{67}$$

By Lemma 4.1.3 from (62) and (67) we get $\omega \notin_k^+ \nu_i$ for all $i \in \{0, \dots, p-1\}$.

Moreover (67) implies $\nu_i \neq \omega$ for all $i \in \{0, \dots, p-1\}$, because $\mu_{i+1} \neq \omega$, as an intersection of $n_{i+1} > 0$ types.

By (66) we also have

$$\sigma_k \quad \simeq \quad \nu_p \to \cdots \to \nu_{p+k} \to \delta \tag{68}$$

As σ_k is k-bottomless, $\omega \notin_k^- \sigma_k$ holds by Definition 4.1.7. By Definition 4.1.1 this means that for all $i \in \{p, \dots, p+k\}$ we have $\nu_i \neq \omega$ and $\omega \notin_k^+ \nu_i$.

In the end we have proved something even stronger than what was to be proved, namely: whenever

$$\mu_1 \to \cdots \to \mu_p \to \sigma_k \quad \simeq \quad \nu_0 \to \cdots \to \nu_k \to \cdots \to \nu_{p+k} \to \delta$$

then for all $i \in \{0, \dots, p+k\}$ we have $\nu_i \neq \omega$ and $\omega \notin_k^+ \nu_i$. And we needed to prove this only for all $i \in \{0, \dots, k\}$ in order to get (65). A fortiori (65) holds.

From (63) and (65) we get that $\omega \notin_k^- \Gamma_0(x)$ for all $x \in \mathrm{Var}$. By IH for all i and j we also have $\omega \notin_k^- \Gamma_{ij}(x)$ for all $x \in \mathrm{Var}$. So from (64) we conclude that $\omega \notin_k^- \Gamma(x)$ for all $x \in \mathrm{Var}$.

In the end t validates Condition 4.

Case $t = \lambda y.t'$**.** By hypothesis $[\,]$ does not occur in $\lambda y.t'$. Therefore $[\,]$ does not occur in t'. So by IH Condition 4 holds for t'. Let $\tilde{n}$ be the natural number in correspondence with t' given by Condition 4.

In order to show that also $\lambda y.t'$ satisfies Condition 4, in correspondence with such term we take $\tilde{n}$ itself.

Let $k \geqslant \tilde{n}$. By Condition 4 for t', there are $(\Gamma, y : \mu) \in \mathsf{Env}_{\mathcal{D}}$ and $\sigma \in \mathsf{T}_{\mathcal{D}}$ such that $\Gamma, y : \mu \vdash t' : \sigma$ is derivable, $\omega \notin_k^- \Gamma(x)$ for all $x \in \mathrm{Var}$, $\omega \notin_k^- \mu$ and $\omega \notin_k^+ \sigma$. We infer

$$\frac{\Gamma, y : \mu \vdash t' : \sigma}{\Gamma \vdash \lambda y.t' : \mu \to \sigma}$$

Let us show that the pair $(\Gamma, \mu \to \sigma) \in \mathsf{Env}_{\mathcal{D}} \times \mathsf{T}_{\mathcal{D}}$ is the one that we are looking for in correspondence with k. All we need to prove is

$$\omega \ \notin_k^+ \ \mu \to \sigma\,. \tag{69}$$

By Definition 4.1.1 this means that whenever $\mu \to \sigma \simeq \mu_0 \to \cdots \to \mu_k \to \sigma'$ then for all $i \in \{0, \dots, k\}$ and for all $\tau \in \mu_i$ we have $\omega \notin^- \mathsf{UNF}_k^1(\tau)$. Since $f^{\geqslant 1} = f$ when f is a constant function, by Remark 4.1.6 we have $\omega \notin^- \mathsf{UNF}_k^1(\tau)$ if and only if $\omega \notin^- \mathsf{UNF}_k^0(\tau)$, i.e. $\omega \notin_k^- \tau$. So to prove (69) is equivalent to prove the following:

whenever $\mu \to \sigma \simeq \mu_0 \to \cdots \to \mu_k \to \sigma'$ then for all $i \in \{0, \dots, k\}$ and for all $\tau \in \mu_i$ we have $\omega \notin_k^- \tau$. This is true, as in fact something even stronger holds. Indeed, by $\omega \notin_k^- \mu$ and $\omega \notin_k^+ \sigma$ given by IH, we have the following:

whenever $\mu \to \sigma \simeq \mu_0 \to \cdots \to \mu_{k+1} \to \sigma'$ then for all $i \in \{0, \dots, k+1\}$ and $\tau \in \mu_i$ we have $\omega \notin_k^- \tau$. And we needed to prove this only for all $i \in \{0, \dots, k\}$ in order to get (69). A fortiori (69) holds.

Proof of (4 ⇒ 3). We define the function $\mathrm{arg} : \mathrm{nf}_\beta(\Lambda^r) \to \mathbb{N}$ by taking as $\mathrm{arg}(t) \in \mathbb{N}$ the maximum number of consecutive applications occurring in t. As t is in normal form, this is equivalent to say that $\mathrm{arg}(t)$ is the maximum number of arguments of any variable in t. In other words, it is the maximum number of children of any node in the tree structure of t. For example, $\mathrm{arg}(x) = 0$ and $\mathrm{arg}\big(\lambda x.y[x,x,x]\,[\lambda vz.z[v][y,v]]\,[x]\,[x,y,x[x]]\big) = 4$.

We replace Hypothesis 4 with the following equivalent version:

IV. *for some* $t \in \mathrm{nf}_\beta\mathcal{T}(M)$ *there exists* $\tilde{n} \geqslant \mathrm{arg}(t)$ *such that for all* $k \geqslant \tilde{n}$ *there is* $(\Gamma,\sigma) \in \mathsf{Env}_\mathcal{D} \times \mathsf{T}_\mathcal{D}$ *satisfying:*

- $\Gamma \vdash^{\mathcal{D}} t : \sigma$ *is derivable,*
- $\omega \notin^+_k \sigma$ *,*
- $\omega \notin^-_k \Gamma(x)$ *for all* $x \in \mathrm{Var}$ *.*

Notice that not only trivially (IV ⇒ 4), but the implication (4 ⇒ IV) is also true. Indeed, given a natural number $\tilde{n}$ in correspondence with t as in Condition 4, one simply takes $\max\big(n, \mathrm{arg}(t)\big)$ as the one in correspondence with t that validates Condition IV.

We prove (IV ⇒ 3) proceeding by induction on t.

Case $t = y\,\mathbf{b_1}\cdots\mathbf{b_p}$ **for some** $p \geqslant 0$**, where for all** $i \in \{1,\ldots,p\}$ **we have** $\mathbf{b_i} = [s_{i1},\ldots,s_{in_i}]$ **for some** $n_i \geqslant 0$**.**

We want to apply the IH to every s_{ij}. At this purpose, we prove that Condition IV holds for each s_{ij} when taking as $\tilde{n}$ exactly the same $\tilde{n}$ given by hypothesis in correspondence with t. Such a candidate makes sense, as $\mathrm{arg}(s_{ij}) \leqslant \mathrm{arg}(t) \leqslant \tilde{n}$.

Let $k \geqslant \tilde{n}$. In correspondence with k we have an environment Γ and a type σ satisfying all the properties in IV. In particular, from the derivation of $\Gamma \vdash y\,\mathbf{b_1}\cdots\mathbf{b_p} : \sigma$ we get by Lemma 2.3.12 the derivability of

$$\frac{\Gamma_0 \vdash y : \mu_1 \to \cdots \to \mu_p \to \sigma \qquad \Gamma_{ij} \vdash s_{ij} : \tau_{ij} \qquad \text{for all } i \in \{1,\ldots,p\},\ j \in \{1,\ldots,n_i\}}{\Gamma \vdash y\,\mathbf{b_1}\cdots\mathbf{b_p} : \sigma}$$

where $\mu_i := \bigwedge_{j=1}^{n_i}\tau_{ij}$ for all $i \in \{1,\ldots,p\}$, $\Gamma_0 = (y : \delta)$ with $\delta \simeq \mu_1 \to \cdots \to \mu_p \to \sigma$ and $\Gamma := \Gamma_0 \wedge \big(\bigwedge_{i=1}^{p}\bigwedge_{j=1}^{n_i}\Gamma_{ij}\big)$. We show that for every s_{ij} the environment Γ_{ij} and the type τ_{ij} are the ones that we are seeking in correspondence with k.

By hypothesis for all $x \in \mathrm{Var}$ we have $\omega \notin^-_k \Gamma(x)$, that is $\omega \notin^-_k \Gamma_0(x) \wedge \big(\bigwedge_{i=1}^{p}\bigwedge_{j=1}^{n_i}\Gamma_{ij}(x)\big)$. So $\omega \notin^-_k \Gamma_{ij}(x)$ for all $x \in \mathrm{Var}$. Also from that, $\omega \notin^-_k \Gamma_0(y)$, i.e. $\omega \notin^-_k \delta$. By Lemma 4.1.3, this entails

$$\omega \notin^-_k \mu_1 \to \cdots \to \mu_p \to \sigma\,. \tag{70}$$

We have

$$p \leqslant \mathrm{arg}(s_{ij}) \leqslant \mathrm{arg}(t) \leqslant \tilde{n} \leqslant k\,. \tag{71}$$

(We remark that (71) is the only reason why we introduced $\mathrm{arg}(-)$ in this proof.)

So from (70), (71) and Definition 4.1.1 for all $i \in \{1,\ldots,p\}$ we get:

1. $\mu_i \neq \omega$;

2. $\omega \notin^{-} \mathsf{UNF}^{1}_{k}(\tau_{ij})$ for all $j \in \{1,\dots,n_i\}$.

Remembering that $f^{\geqslant 1} = f$ when f is a constant function, by Remark 4.1.6 the expression $\omega \notin^{-} \mathsf{UNF}^{1}_{k}(\tau_{ij})$ in Point 2 is equivalent to $\omega \notin^{-} \mathsf{UNF}^{0}_{k}(\tau_{ij})$, which is $\omega \notin^{-}_{k} \tau_{ij}$. This completes the proof of the fact that IV holds for all s_{ij}.

So for all $i \in \{1,\dots,p\}$ and $j \in \{1,\dots,n_i\}$ by IH the term s_{ij} does not contain $[\,]$.

Moreover from Point 1 for all $i \in \{1,\dots,p\}$ we get $n_i > 0$, hence $\mathbf{b_i} \neq [\,]$.

So $t = y\,\mathbf{b_1}\cdots\mathbf{b_p}$ does not does not contain $[\,]$.

Case $t = \lambda y.t'$**.** By hypothesis in correspondence with $\lambda y.t'$ there exists $\tilde{n} \geqslant \arg(\lambda y.t')$ satisfying Condition IV.

We want to apply the IH to t'. At this purpose, we prove that Condition IV holds for t' when taking as $\tilde{n}$ exactly the same $\tilde{n}$ given by hypothesis in correspondence with $\lambda y.t'$. Such a candidate makes sense, as $\tilde{n} \geqslant \arg(\lambda y.t') = \arg(t')$.

Let $k \geqslant \tilde{n}$. Since $k+1 > k \geqslant \tilde{n}$, by Condition IV there is $(\Gamma,\sigma) \in \mathsf{Env}_{\mathcal{D}} \times \mathsf{T}_{\mathcal{D}}$ such that $\Gamma \vdash \lambda y.t' : \sigma$ is derivable, $\omega \notin^{+}_{k+1} \sigma$ and $\omega \notin^{-}_{k+1} \Gamma(x)$ for all $x \in \mathrm{Var}$.

By Lemma 2.3.12 we get the derivability of $\Gamma, y : \mu \vdash t' : \sigma'$ with $\sigma \simeq \mu \to \sigma'$. We show that the environment $(\Gamma, y : \mu)$ and the type σ' are the two that we are seeking in correspondence with k.

Firstly, we prove that $\omega \notin^{+}_{k} \sigma$. Let $\sigma' \simeq \mu_0 \to \cdots \to \mu_k \to \sigma''$. Then

$$\sigma \;\simeq\; \mu \to \sigma' \;\simeq\; \mu \to \mu_0 \to \ldots \to \mu_k \to \sigma''\,. \tag{72}$$

Since $\omega \notin^{+}_{k+1} \sigma$, from (72) we get

1. $\mu \neq \omega$,
2. $\mu_i \neq \omega$ for all $i \in \{1,\dots,k\}$,
3. $\omega \notin^{-}_{k+1} \mu$,
4. $\omega \notin^{-}_{k+1} \mu_i$ for all $i \in \{1,\dots,k\}$.

By Corollary 4.1.5 Point 4 implies $\omega \notin^{-}_{k} \mu_i$ for all $i \in \{1,\dots,k\}$. This fact together with Point 2 proves that $\omega \notin^{+}_{k} \sigma$.

We now prove that $\omega \notin^{-}_{k} (\Gamma, y : \mu)(x)$ for all $x \in \mathrm{Var}$.

Since $\omega \notin^{-}_{k+1} \Gamma(x)$ for all $x \in \mathrm{Var}$, by Corollary 4.1.5 we have $\omega \notin^{-}_{k} \Gamma(x)$ for all $x \in \mathrm{Var}$. Which means that $\omega \notin^{-}_{k} (\Gamma, y : \mu)(x)$ for all $x \in \mathrm{Var} - \{y\}$.

We are left to show that $\omega \notin^{-}_{k} (\Gamma, y : \mu)(y)$, i.e. $\omega \notin^{-}_{k} \mu$. Since $\sigma \simeq \mu \to \sigma'$ and $k+1 > 1$, the fact that $\omega \notin^{+}_{k+1} \sigma$ assures that $\mu \neq \omega$ and $\omega \notin^{-}_{k+1} \mu$. The latter expression implies $\omega \notin^{-}_{k} \mu$ by Corollary 4.1.5.

This completes the proof of the fact that Condition IV holds for t'. Then the IH assures that t' does not contain $[\,]$. Hence $\lambda x.t'$ does not contain $[\,]$. □

Theorem 4.3.2. *Let $\mathcal{D}$ be a uniformly bottomless* ergm*. For all $M, N \in \Lambda$ the following statements are equivalent.*

1. $[\![M]\!]^{\mathcal{D}} \subseteq [\![N]\!]^{\mathcal{D}}$.

2. $M \sqsubseteq_{\mathcal{H}^+} N$.

3. $BT^e(M) \subseteq BT^e(N)$.

(In particular the hypothesis of uniformly bottomlessness is needed only for the implication 1 ⇒ 2.)

Proof. (1 ⇒ 2) Consider a context $C[-]$ such that $C[M]$ has a β-normal form. By Lemma 4.3.1 (specifically the implication (1 ⇒ 5) stated therein) there is a natural number $\tilde{n} \in \mathbb{N}$ such that for all $k \geqslant \tilde{n}$ there exists $(\Gamma, \sigma) \in [\![C[M]]\!]$ (in correspondence with k) satisfying $\omega \notin^+_k \sigma$ and $\omega \notin^-_k \Gamma(x)$ for all $x \in \mathrm{Var}$.

By hypothesis $[\![M]\!] \subseteq [\![N]\!]$. As $[\![-]\!]^{\mathcal{D}}$ is contextual (Lemma 2.4.4) we get $[\![C[M]]\!] \subseteq [\![C[N]]\!]$. Therefore all those (Γ, σ) mentioned above (in correspondence with a give $k \geqslant \tilde{n}$) belong also to $[\![C[N]]\!]$. By applying Lemma 4.3.1 again (specifically the implication (5 ⇒ 1) stated therein), we conclude that $C[N]$ has a β-normal form.

(2 ⟺ 3) This equivalence is Theorem 1.4.12(1).

(3 ⇒ 1) By hypothesis the rgm $\mathcal{D}$ is extensional. Hence

$$\begin{array}{rcll} [\![M]\!] & = & \bigcup_{M' \twoheadrightarrow_\eta M} [\![M']\!] & \text{by Theorem 2.4.13 (η-soundness)} \\ & = & \bigcup_{M' \twoheadrightarrow_\eta M} [\![BT(M')]\!] & \text{by Theorem 2.6.5 (Böhm Approximation)} \\ & = & \bigcup_{M' \twoheadrightarrow_\eta M} [\![nf_\eta BT(M')]\!] & \text{by Theorem 2.4.13 (η-soundness)} \\ & = & [\![BT^e(M)]\!] & \text{by definition of } BT^e(M). \end{array}$$

So the hypothesis $BT^e(M) \subseteq BT^e(N)$ implies $[\![M]\!] = [\![BT^e(M)]\!] \subseteq [\![BT^e(N)]\!] = [\![N]\!]$. □

Corollary 4.3.3. *A uniformly bottomless ergm $\mathcal{D}$ is fully abstract for Morris's preorder theory, i.e.* $Th_{\sqsubseteq}(\mathcal{D})$ *is* $\sqsubseteq_{\mathcal{H}^+}$. *In particular it is fully abstract for Morris's λ-theory, namely* $Th(\mathcal{D}) = \mathcal{H}^+$.

Corollary 4.3.4. *The rgm's $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{G}^{rec}$ defined in Examples 4.2.1-4.2.3 are fully abstract for Morris's preorder theory and λ-theory.*

In [MR14] we have already given a sufficient condition to make an ergm fully abstract for Morris's preorder theory. That condition was less refined than the notion of uniformly bottomless ergm. We conclude this section by discussing this fact.

Firstly, in [MR14] we formalized what does it mean for ω to *occur positively or negatively* in a type σ, as follows.

Notation. Given a certain polarity $p \in \{+, -\}$ we denote by $\backsim p$ the other polarity. In other words $\backsim p \in \{+, -\} - \{p\}$.

Definition 4.3.5. Let $\mathcal{D}$ be an ergm and $\sigma \in T_{\mathcal{D}}$. The two expressions $\omega \in^p \sigma$ are defined for both polarities by mutual induction as follows.

- $\omega \in^- \omega \to \sigma$ for any $\sigma \in T_{\mathcal{D}}$.
- If $\omega \in^p \sigma$ then $\omega \in^p \mu \to \sigma$ for any $\mu \in I_{\mathcal{D}}$.

- If $\omega \in^{p} \sigma$ then $\omega \in^{\smallfrown p} \sigma \wedge \mu \to \tau$ for any $\tau \in T_{\mathcal{D}}$ and $\mu \in I_{\mathcal{D}}$.

Notice that the notion above speaks of occurrences of ω, but *not* up to equivalence of types. This is why we had to introduce a notion of *preservation of ω-polarity* up to equivalence of types.

Definition 4.3.6. An ergm $\mathcal{D}$ *preserves ω-polarities* if and only if for all $\sigma, \tau \in T_{\mathcal{D}}$ the hypothesis $\omega \in^{p} \sigma$ and $\sigma \simeq \tau$ imply $\omega \in^{p} \tau$.

Finally we showed that an ergm satisfying such a property induces Morris's preorder.

Theorem 4.3.7 ([MR14, Corollary 4.6])**.** *Every ergm $\mathcal{D}$ preserving ω-polarities is fully abstract for Morris's preorder $\sqsubseteq_{\mathcal{H}^+}$.*

Let us see how this theorem is related to what has been done so far in this chapter.

Firstly, the following proposition clarifies the link between the notion denoted by $\omega \in^{p} \sigma$ and the more refined one written as $\omega \in^{p}_{f} \sigma$. We omit the proof, but the statement should seem quite evident to the reader.

Proposition 4.3.8. *Let $\mathcal{D}$ be an ergm and $\sigma \in T_{\mathcal{D}}$. There exists $\psi \in [\sigma]_{\simeq}$ such that $\omega \in^{p} \psi$ if and only if there exists $f : \mathbb{N} \to \mathbb{N}$ such that $\omega \in^{p}_{f} \sigma$.*

The result of full abstraction provided by Theorem 4.3.7 turns out to be encompassed by the one in Corollary 4.3.3. In other words, the class of uniformly bottomless ergm's contains all ergm's preserving ω-polarity. This is given by the result here below.

Theorem 4.3.9. *Let $\mathcal{D}$ be an ergm. Then the following chain of implications holds, meaning that* $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$.

1. *$\mathcal{D}$ preserves ω-polarities.*
2. *There exists $\sigma \in T_{\mathcal{D}}$ such that $\omega \notin^{+} \psi$ and $\omega \notin^{-} \psi$ for all $\psi \in [\sigma]_{\simeq}$.*
3. *$\mathcal{D}$ is strongly bottomless.*
4. *$\mathcal{D}$ is uniformly bottomless.*

Proof. ($1 \Rightarrow 2$) An atom $\alpha \in At_{\mathcal{D}}$ satisfies $\omega \notin^{+} \alpha$ and $\omega \notin^{-} \alpha$ by definition. Then by preservation of ω-polarity whenever $\psi \simeq \alpha$ we also have $\omega \notin^{+} \psi$ and $\omega \notin^{-} \psi$.
($2 \Rightarrow 3$) By Proposition 4.3.8 the hypothesis 2 implies that $\omega \notin^{p}_{f} \sigma$ for all $f : \mathbb{N} \to \mathbb{N}$. Hence for every $f : \mathbb{N} \to \mathbb{N}$ the type σ is an f-bottomless type.
($3 \Rightarrow 4$) This is Proposition 4.1.10. □

The rgm's $\mathcal{F}$ and $\mathcal{G}^{\mathsf{rec}}$, defined respectively in Examples 4.2.1 and 4.2.3, satisfy Condition 4 in the chain of Theorem 4.3.9, but not Conditions 1-3. The rgm $\mathcal{G}$ defined in Examples 4.2.3 satisfies Conditions 3 and 4, but not Conditions 1 and 2. In the next section we study an rgm satisfying all four Conditions 1-4.

In this section we focus on an rgm fully abstract for Morris that strikes for its simplicity.

Definition 4.4.1. We call $\mathcal{D}_\star$ the rgm obtained as the free completion

$$\mathcal{D}_\star \;:=\; \overline{(\{\star\}, j)}$$

where $\star$ is not a pair and the partial injection $j : \mathcal{M}_f(\{\star\}) \times \{\star\} \rightharpoonup \{\star\}$ maps $j([\star], \star) := \star$.

Lemma 4.4.2. *The rgm $\mathcal{D}_\star$ is extensional.*

Proof. By Proposition 2.2.8, since the partial pair $(\{\star\}, j)$ is extensional, i.e. j is surjective. □

Lemma 4.4.3. *In $\mathcal{D}_\star$ the equivalence $\star \simeq \star \to \star$ holds.*

Proof. By Definition 2.3.3 we have $\star \simeq \star \to \star$ if and only if $\star = \star^\diamond = (\star \to \star)^\diamond = \bar{j}([\star^\diamond], \star^\diamond) = \bar{j}([\star], \star) = j([\star], \star) = \star$. □

So more informally, $\mathcal{D}_\star$ is the ergm built upon one single atom $\star$ and the only basic equational identification

$$\star \;\simeq\; \star \to \star .$$

It is then clear that for all $k \in \mathbb{N}$ one has

$$\star \;\simeq\; \overbrace{\star \to \cdots \to \star}^{k \text{ times}} \to \star ,$$

as formalized in the next lemma.

Lemma 4.4.4. *Let $\sigma \in \mathsf{T}_{\mathcal{D}_\star}$. Then $\gamma \simeq \star$ if and only if γ is generated by the following grammar:*

$$\gamma \;::=\; \star \mid \gamma \to \gamma \quad .$$

In particular $\mu \to \sigma \simeq \star$ entails that $\sigma \simeq \star$ and the intersection μ is one single type $\tau \simeq \star$.

Proof. The right-to-left implication is given by a trivial induction on the grammar:

- the type $\star$ is equivalent to itself;
- if $\gamma_1 \simeq \star$ and $\gamma_2 \simeq \star$ by IH, then $\gamma_1 \to \gamma_2 \simeq \star \to \star \simeq \star$ (by Lemma 4.4.3).

The left-to-right implication is given by the following induction on γ.

- Since $\mathrm{At}_{\mathcal{D}_\star} = \{\star\}$ by Proposition 2.2.10, whenever $\gamma \simeq \star$ is an atom we have $\gamma = \star$.
- Let $\gamma = \mu \to \sigma \simeq \star$. This means that $\bar{j}(\mu^\diamond, \sigma^\diamond) = (\mu \to \sigma)^\diamond = \star^\diamond = \star$. Since $\bar{j}$ is surjective by Lemma 4.4.2 and $\bar{j}([\star], \star) = j([\star], \star) = \star$ by Definition 4.4.1, the pair $(\mu^\diamond, \sigma^\diamond)$ must be $([\star], \star)$. So $\mu \simeq \star$ and $\sigma \simeq \star$, which was to be proved. □

Proposition 4.4.5. *The ergm $\mathcal{D}_\star$ is uniformly bottomless.*

Proof. For all $k \in \mathbb{N}$ we have $\omega \not\in^+_k \star$ and $\omega \not\in^-_k \star$, as a consequence of Lemma 4.4.4. Therefore $\star$ is a k-bottomless type. □

Corollary 4.4.6. *The model $\mathcal{D}_\star$ is fully abstract for Morris's preorder $\sqsubseteq_{\mathcal{H}^+}$ and λ-theory $\mathcal{H}^+$.*

Proof. By Proposition 4.4.5 and Theorem 4.3.3. □

Remark 4.4.7. One may also observe that $\mathcal{D}_\star$ is an ergm preserving ω-polarity, in the sense of Definition 4.3.6. So in fact $\mathcal{D}_\star$ satisfies all Conditions 1-4 in the chain of Theorem 4.3.9.

So far the filter model $\mathcal{D}_{cdz}$ of Coppo, Dezani and Zacchi [CDZ87] was the only denotational model known to induce Morris's observability. The construction of $\mathcal{D}_{cdz}$ relies on two atomic types and, as any filter model, on a subtyping preorder $\leqslant$. In comparison to that, the fact that $\mathcal{D}_\star$ only requires one atom and relies on an equivalence of types can be regarded as a valuable simplification.

We show two interesting properties of the atom $\star$.

Firstly, the type $\star$ provides a characterization of the λ-terms having a *linear* normal form, according to the notion of linearity below.

Definition 4.4.8. A λ-term $M \in \Lambda$ is called *linear* if and only if

- every $y \in fv(M)$ occurs once in M;
- for every subterm $\lambda x.N$ of M the bound variable x occurs once in N.

We recall that in general $supp(\Gamma) \subseteq fv(N)$, by Lemma 2.3.10.

Lemma 4.4.9. *Let $N \in \Lambda$ be in β-normal form. Let $(\Gamma, \sigma) \in Env_{\mathcal{D}_\star} \times T_{\mathcal{D}_\star}$ such that*

- *the sequent $\Gamma \vdash^{\mathcal{D}_\star} N : \sigma$ is derivable,*
- *$\Gamma(x) \simeq \star$ for all $x \in supp(\Gamma)$,*
- *$\sigma \simeq \star$.*

Then N is linear and $supp(\Gamma) = fv(N)$.

Proof. We proceed by structural induction on N.

Case $N = y\, N_1 \cdots N_k$ **for some** $k \in \mathbb{N}$. By Lemma 2.3.12(1),(3) the derivability of $\Gamma \vdash N : \sigma$ entails the derivability of

$$\frac{\Gamma_0 \vdash y : \mu_1 \to \cdots \to \mu_k \to \sigma \qquad \Gamma_{ij} \vdash N_{ij} : \tau_{ij} \quad i \in \{1, \ldots, k\},\ j \in \{1, \ldots, n_i\}}{\Gamma \vdash y\, N_1 \cdots N_k : \sigma}$$

where $\Gamma = \Gamma_0 \wedge (\wedge_{i=1}^{k} \wedge_{j=1}^{n_i} \Gamma_{ij})$, $\mu_i := \wedge_{j=1}^{n_i} \tau_{ij}$ for all $i \in \{1, \ldots, k\}$ and $\Gamma_0 = (y : \gamma)$ for some $\gamma \in T_{\mathcal{D}_\star}$ such that $\gamma \simeq \mu_1 \to \cdots \to \mu_k \to \sigma$.

In particular $\mu_1 \to \cdots \to \mu_k \to \sigma \simeq \gamma = \Gamma_0(y) \subseteq \Gamma(y)$. By hypothesis $\Gamma(y) \simeq \star$. So $\mu_1 \to \cdots \to \mu_k \to \sigma = \star$. By Lemma 4.4.4 this implies that for all $i \in \{1, \ldots, k\}$ we have $n_i = 1$ and the intersection μ_i is composed of a single type $\tau_i \simeq \star$.

In the end what we actually have is for all $i \in \{1, \ldots, k\}$ the derivability of the sequent $\Gamma_i \vdash N_i : \tau_i$ for some $\tau_i \simeq \star$ and for some $\Gamma_i \in Env_{\mathcal{D}_\star}$ such that

$$\Gamma \;=\; \Gamma_0 \wedge (\wedge_{i=1}^{k} \Gamma_i)\,. \tag{73}$$

Also notice that N_i is in β-nf, as a subterm of the β-normal λ-term N. In order to apply the IH to N_i we are only left to prove that that

$$\Gamma_i(x) \;\simeq\; \star \qquad \text{for all } x \in \mathrm{supp}(\Gamma_i)\,. \tag{74}$$

We have $\mathrm{supp}(\Gamma_i) \subseteq \mathrm{supp}(\Gamma)$ and $\Gamma_i(x) \subseteq \Gamma(x)$ by (73), and $\Gamma(x) \simeq \star$ by hypothesis. Therefore (74) is true.

By IH each N_i is linear and

$$\mathrm{supp}(\Gamma_i) \;=\; \mathrm{fv}(N_i). \tag{75}$$

Let us prove the linearity of N. We have to show that

1. $y \notin \bigcup_{i=1}^{k} \mathrm{fv}(N_i)$,
2. $\mathrm{fv}(N_i) \cap \mathrm{fv}(N_j) = \emptyset$ for all $i, j \in \{1, \ldots, k\}$ such that $i \neq j$.

By (73) and the fact that $\Gamma(y)$ is the unitary intersection $\Gamma_0(y)$, we have

$$y \;\notin\; \mathrm{supp}\big(\wedge_{i=1}^{k} \Gamma_i\big) \;=\; \bigcup_{i=1}^{k} \mathrm{supp}(\Gamma_i) \;=\; \bigcup_{i=1}^{k} \mathrm{fv}(N_i)\,,$$

where the last equality holds by (75). So 1 is proved.

Consider $i, j \in \{1, \ldots, k\}$ such that $i \neq j$. By way of contradiction let $x \in \mathrm{fv}(N_i) \cap \mathrm{fv}(N_j) = \mathrm{supp}(\Gamma_i) \cap \mathrm{supp}(\Gamma_j)$, where the last equality holds by (75). Then by (74) we get $\Gamma_i(x) \simeq \star$ and $\Gamma_j(x) \simeq \star$. Hence $\star \wedge \star \subseteq \Gamma(x)$, against the hypothesis that $\Gamma(x) \simeq \star$. So 2 is proved, and with that the linearity of N.

Finally let us see that $\mathrm{supp}(\Gamma) = \mathrm{fv}(N)$. It is a simple consequence of (75), as

$$\mathrm{supp}(\Gamma) \;=\; \bigcup_{i=0}^{k} \mathrm{supp}(\Gamma_i) \;=\; \{y\} \cup \bigcup_{i=1}^{k} \mathrm{supp}(\Gamma_i) \;=\; \{y\} \cup \bigcup_{i=1}^{k} \mathrm{fv}(N_i) \;=\; \mathrm{fv}(N)\,.$$

Case $N = \lambda x.N'$. By Lemma 2.3.12(2) the derivability of $\Gamma \vdash \lambda x.N' : \sigma$ implies the one of

$$\Gamma, x : \mu \;\vdash\; N' : \tau \tag{76}$$

for some $\mu \in \mathsf{I}_{\mathcal{D}_\star}$ and $\tau \in \mathsf{T}_{\mathcal{D}_\star}$ such that $\mu \to \tau \simeq \sigma$. By hypothesis $\sigma \simeq \star$. Hence by transitivity of $\simeq$ we get that $\mu \to \tau \simeq \star$. By Lemma 4.4.4 this implies that $\tau \simeq \star$ and the intersection μ is composed of a single type $\gamma \simeq \star$. Also notice that N' is in β-nf, as a subterm of the β-nf N. So we can apply the IH to N'. We get that N' is linear and $\mathrm{supp}(\Gamma, x : \mu) = \mathrm{fv}(N')$.

As a consequence of the latter fact, we have

$$\mathrm{supp}(\Gamma) \;=\; \mathrm{supp}(\Gamma, x : \mu) - \{x\} \;=\; \mathrm{fv}\big(N'\big) - \{x\} \;=\; \mathrm{fv}\big(\lambda x.N'\big)\,,$$

as it was to be proved.

We are left to show that $\lambda x.N'$ is linear. Since N' is linear by IH, all we need to prove is that x occurs exactly once in N'. Of course, it cannot occur more than once, because that would contradict the fact that N' is linear. Moreover $x \in \mathrm{supp}(\Gamma, x : \mu) = \mathrm{fv}(N')$. So x occurs at least once in N'. □

Theorem 4.4.10. *Let* $M \in \Lambda$*. Then* M *has a linear* β*-normal form if and only if one can derive* $\Gamma \vdash^{\mathcal{D}_\star} M : \sigma$ *with* $\sigma \simeq \star$ *and* $\Gamma(x) \simeq \star$ *for all* $x \in \mathrm{supp}(\Gamma)$*.*

Proof. ($\Leftarrow$) Let $\Gamma \vdash M : \sigma$ be derivable for some $\sigma \simeq \star$ and $\Gamma(x) \simeq \star$ for all $x \in \mathrm{supp}(\Gamma)$. For all $k \in \mathbb{N}$ we have $\omega \not\in^+_k \star$ and $\omega \not\in^-_k \star$. By Lemma 4.1.3 then $\omega \not\in^+_k \sigma$ and $\omega \not\in^-_k \Gamma(x)$ for all $x \in \mathrm{supp}(\Gamma)$. So Condition 5 in the statement of Lemma 4.3.1 holds for M. The rgm $\mathcal{D}_\star$ is uniformly bottomless, so Lemma 4.3.1 can be applied. Then the implication ($5 \Rightarrow 1$) of such lemma assures that M has a β-nf. By subject reduction (Lemma 2.4.8) the sequent $\Gamma \vdash \mathrm{nf}_\beta(M) : \sigma$ is derivable. Finally, by Lemma 4.4.9 we conclude that $\mathrm{nf}_\beta(M)$ is linear.

($\Rightarrow$) Suppose that $M \in \Lambda$ has a linear β-nf. Let $\Gamma \in \mathsf{Env}_\mathcal{D}$ be defined as

$$\Gamma(x) \quad := \quad \begin{cases} \star & \text{if } x \in \mathrm{fv}(\mathrm{nf}_\beta(M)), \\ \omega & \text{otherwise.} \end{cases} \tag{77}$$

We prove that $\Gamma \vdash \mathrm{nf}_\beta(M) : \star$ is derivable. After that, one can conclude by subject expansion (Lemma 2.4.9) that $\Gamma \vdash M : \star$ is derivable. We proceed by induction on $\mathrm{nf}_\beta(M)$.

Case $\mathrm{nf}_\beta(M) = y\, N_1 \cdots N_k$ **for some** $k \in \mathbb{N}$**.** For all $i \in \{1, \ldots, k\}$ let $\Gamma_i \in \mathsf{Env}_\mathcal{D}$ be

$$\Gamma_i(x) \quad := \quad \begin{cases} \star & \text{if } x \in \mathrm{fv}(N_i), \\ \omega & \text{otherwise.} \end{cases}$$

As N_i is a linear β-nf, by IH the sequent $\Gamma_i \vdash N_i : \star$ is derivable. Then we can derive

$$\dfrac{\dfrac{\dfrac{}{y : \star \vdash y : \star} \qquad \star \simeq \overbrace{\star \to \cdots \to \star}^{k \text{ times}} \to \star}{y : \star \vdash y :\simeq \underbrace{\star \to \cdots \to \star}_{k \text{ times}} \to \star}\,\mathsf{eq} \qquad \Gamma_i \vdash N_i : \star \quad \text{for all } i \in \{1, \ldots, k\}}{(y : \star) \wedge \left(\bigwedge_{i=1}^k \Gamma_i\right) \vdash y N_1 \cdots N_k : \star}$$

To conclude, notice that $(y : \star) \wedge \left(\bigwedge_{i=1}^k \Gamma_i\right)$ is exactly the environment Γ as defined in (77).

Case $\mathrm{nf}_\beta(M) = \lambda x.N$. By Definition 4.4.8, the linearity of $\lambda x.N$ implies that x occurs free in N. Hence $\mathrm{fv}(N) = \mathrm{fv}(\lambda x.N) \cup \{x\}$. Moreover obviously N is a linear β-normal term. Then by IH we have the derivability of $\Gamma, x : \star \vdash N' : \star$, where

$$\Gamma(x) \quad := \quad \begin{cases} \star & \text{if } x \in \mathrm{fv}(\lambda x.N), \\ \omega & \text{otherwise.} \end{cases}$$

By deriving

$$\dfrac{\dfrac{\Gamma, x : \star \vdash N' : \star}{\Gamma \vdash \lambda x.N : \star \to \star} \qquad \star \simeq \star \to \star}{\Gamma \vdash \lambda x.N : \star}\,\mathsf{eq}$$

we get the thesis. ∎

We conclude this section by showing that in $\mathcal{D}_\star$ the type $\star$ *separates* **I** from its infinite η-expansion **J**. This remark is a preamble to a key result of the next chapter.

The following lemma is easy to check.

Lemma 4.4.11. *Let* $x \in \mathrm{Var}$. *Then* $\mathrm{BT}(\mathbf{J}\,x)^* = \{\perp\} \cup \{\lambda y.x\,b \mid y \neq x \text{ and } b \in \mathrm{BT}(\mathbf{J}\,y)^*\}$.

Lemma 4.4.12. *Let* $\sigma \in \mathsf{T}_{\mathcal{D}_\star}$ *such that* $\sigma \simeq \star$. *Let* $x \in \mathrm{Var}$. *Then* $(x : \sigma,\ \sigma) \notin [\![\mathbf{J}\,x]\!]^{\mathcal{D}_\star}$.

Proof. By Theorem 2.6.5 (i.e. Böhm Approximation) we have $(x : \sigma,\ \sigma) \notin [\![\mathbf{J}\,x]\!]$ if and only if $(x : \sigma,\ \sigma) \notin [\![a]\!]$ for every $a \in \mathrm{BT}(\mathbf{J}\,x)^*$. We prove the right hand side of this equivalence by induction on a. According to Lemma 4.4.11 there are only two cases to consider.
Case $a = \perp$. As $[\![a]\!] = [\![\perp]\!] = \emptyset$, clearly $\sigma \notin [\![a]\!]$.
Case $a = \lambda y.xb$ for $y \neq x$ and $b \in \mathrm{BT}(\mathbf{J}\,y)$. By way of contradiction we suppose that $x : \sigma \vdash \lambda y.xb : \sigma$ is derivable. By Lemma 2.3.12(2) this is equivalent to the derivability of $x : \sigma, y : \mu \vdash xb : \tau$ for some $\mu \in \mathsf{I}_{\mathcal{D}_\star}$ and $\tau \in \mathsf{T}_{\mathcal{D}_\star}$ such that $\mu \to \tau \simeq \sigma$. Since $\sigma \simeq \star$ by hypothesis, we get $\mu \to \tau \simeq \star$. Hence by Lemma 4.4.4 the intersection μ is a single type $\gamma \simeq \star$ and $\tau \simeq \star$. By Lemma 2.3.12(3) we get the derivability of

$$\frac{\Gamma_0 \vdash x : \nu \to \tau \qquad \Gamma_i \vdash b : \delta_i \quad \text{for all } i \in \{1, \ldots, n\}}{x : \sigma,\ y : \gamma \vdash xb : \tau} \tag{78}$$

where $\nu := \bigwedge_{i=1}^{n} \delta_i$ for some $n \in \mathbb{N}$ and $\bigwedge_{i=0}^{n} \Gamma_i = (x : \sigma,\ y : \gamma)$. By Lemma 2.3.12(1) then $\Gamma_0 = x : \sigma'$ for a type $\sigma' \simeq \nu \to \tau$. So we have $(x : \sigma') \wedge \bigwedge_{i=1}^{n} \Gamma_i = (x : \sigma,\ y : \gamma)$. This implies that $\sigma' = \sigma$ and $\bigwedge_{i=1}^{n} \Gamma_i = (y : \gamma)$. Hence $n = 1$ and $\nu = \delta_1 = \gamma$. In the end (78) is

$$\frac{x : \sigma \vdash x : \nu \to \tau \qquad y : \gamma \vdash b : \gamma}{x : \sigma,\ y : \gamma \vdash xb : \tau}$$

This is a contradiction, because $b \in \mathrm{BT}(\mathbf{J}\,y)$, so by IH one cannot derive $y : \gamma \vdash b : \gamma$. □

Proposition 4.4.13. *Let* $\sigma \in \mathsf{T}_{\mathcal{D}_\star}$ *such that* $\sigma \simeq \star$. *Then* $\sigma \in [\![\mathbf{I}]\!]^{\mathcal{D}_\star} - [\![\mathbf{J}]\!]^{\mathcal{D}_\star}$.

Proof. We have $\sigma \simeq \star \simeq \star \to \star \simeq \sigma \to \sigma \in [\![\mathbf{I}]\!]$. Therefore $\sigma \in [\![\mathbf{I}]\!]$.

By way of contradiction let $\sigma \in [\![\mathbf{J}]\!]$. By Theorem 2.4.10 this is equivalent to $\sigma \in [\![\lambda x.\mathbf{J}\,x]\!]$, since $\mathbf{J} =_\beta \lambda x.\mathbf{J}\,x$. In other words, the sequent $\vdash \lambda x.\mathbf{J}\,x : \sigma$ is derivable. By Lemma 2.3.12(2) this is equivalent to the derivability of $x : \mu \vdash \mathbf{J}\,x : \tau$ for some $\mu \in \mathsf{I}_{\mathcal{D}_\star}$ and $\tau \in \mathsf{T}_{\mathcal{D}_\star}$ such that $\mu \to \tau \simeq \sigma$. Since $\sigma \simeq \star$ by hypothesis, we get $\mu \to \tau \simeq \star$. Hence by Lemma 4.4.4 the intersection μ is a single type $\gamma \simeq \star$ and $\tau \simeq \star$. So we can derive

$$\frac{x : \gamma \vdash \mathbf{J}\,x : \tau}{x : \gamma \vdash \mathbf{J}\,x : \gamma}\ \mathsf{eq}$$

In the end $((x : \gamma),\ \gamma) \in [\![\mathbf{J}\,x]\!]$ contradicts Lemma 4.4.12. □

4.5 A syntactic model of Morris's theory: extensional Taylor expansion

In this section we take a break from relational semantics and study a model of $\mathcal{H}^+$ of a very different kind. In § 1.4 we recalled two notions of *extensional Böhm tree* of a λ-term M, namely $\mathrm{BT}^e(M)$ and $\mathrm{BT}^\eta(M)$. Both provide a *syntactic* characterization of Morris's equivalence. Here we present yet another such characterization. But rather than relying on the approximation *à la Böhm*, this new characterization exploits the Taylor expansion of Ehrhard and Regnier recalled in § 1.6 (and already used in alternative to the notion of Böhm tree in § 2.6, when proving the Approximation Theorems for rgm's).

We introduce the notion of *extensional Taylor expansion* $\mathcal{T}^\eta(M)$ of a λ-term M and prove that $\mathcal{T}^\eta(M) = \mathcal{T}(\mathrm{BT}^\eta(M))$ (Theorem 4.5.32). This result is deliberately conceived as an analogue of Theorem 1.6.4, the key result by Ehrhard and Regnier stating that

$$\mathrm{nf}_\beta(\mathcal{T}(M)) \quad = \quad \mathcal{T}(\mathrm{BT}(M))\ . \tag{79}$$

By the way, (79) is not just a source of inspiration: we actually use it to achieve our result.

As a byproduct, we obtain a new syntactic characterization of $\mathcal{H}^+$, which is $M =_{\mathcal{H}^+} N$ if and only if $\mathcal{T}^\eta(M) = \mathcal{T}^\eta(N)$ (Corollary 4.5.34). For technical reasons, on this occasion we prefer to use $\mathrm{BT}^\eta(-)$ instead of $\mathrm{BT}^e(-)$. As a consequence, $\mathcal{T}^\eta(-)$ is not a model of Morris's preorder $\sqsubseteq_{\mathcal{H}^+}$. (See § 1.4 for the reason why $\mathrm{BT}^\eta(-)$ is not refined enough to get Morris's observational theory also inequationally.)

Notation. By convenience, all over this section we denote the empty multiset $[\ ]$ by the symbol $\mathbf{1}$. Notice that such a notation is consistent with the fact that $[\ ]$ is the neutral element of the union of multisets.

Let us give an overview of the situation that we are dealing with. In order to obtain an analogue of (79) in the extensional setting, we want the extensional Taylor expansion of M to be the η-*normal form of* $\mathrm{nf}_\beta\mathcal{T}(M)$, just like $\mathrm{BT}^\eta(M)$ is the η-normal form of $\mathrm{BT}(M)$. But defining an η-reduction on $\mathcal{P}(\mathrm{nf}_\beta(\Lambda^r))$ is not an easy task. As a first attempt, one may consider the *naïve* definition

$$\to_\eta \ := \ \bigcup_{k\in\mathbb{N}} (\to_{\eta_k}) \qquad \text{where} \ \ \lambda x.t\,[x^k] \to_{\eta_k} t \ \ \text{whenever} \ \ x \notin \mathrm{fv}(t)$$

and then extend it pointwise to sets of resource terms. This correctly reduces $\mathcal{T}(\lambda y.xy) = \{\lambda y.x[y^k] \mid k \in \mathbb{N}\}$ to the set $\{x\}$, which is what one would expect. But the fact that $\lambda y.x\,\mathbf{1} \to_{\eta_0} x$ is a problem. Indeed $\lambda y.x\,\mathbf{1}$ also belongs to $\mathcal{T}(\lambda y.x\boldsymbol{\Omega})$, so this Taylor expansion would η-reduce to $\{x, \lambda y.x\,\mathbf{1}\}$, whereas $x \notin \mathcal{T}(\mathrm{nf}_\eta(\lambda y.x\boldsymbol{\Omega})) = \{\lambda y.x\,\mathbf{1}\}$. Similarly, $\lambda y.x\mathbf{1}[y]$ as an element of $\mathcal{T}(\lambda y.xzy)$ is supposed to η-reduce to $x\,\mathbf{1}$, whereas as an element of $\mathcal{T}(\lambda y.xyy)$ it should be considered already η-normal.

These examples reveal that, while the β-reduction of $\mathcal{T}(M)$ can be performed *locally* by reducing each term individually, the η-reduction of $\mathrm{nf}_\beta\mathcal{T}(M)$ must be a *global* operation, that considers the whole set of terms before deciding whether a term should reduce or not.

Here is how we handle the issue. We divide the computing of the η-normal form of the set $\mathrm{nf}_\beta\mathcal{T}(M)$ into two phases.

- We first define a *labeling* $\mathcal{L}(-)$ on $\mathrm{nf}_\beta\mathcal{T}(M)$, as a global operation annotating on each empty bag $\mathbf{1}$ occurring in each $t \in \mathrm{nf}_\beta\mathcal{T}(M)$ the following piece of information:
  - whether that $\mathbf{1}$ *comes from* a finite η-expansion of some variable (for instance $\lambda y.x\,\mathbf{1} \in \mathcal{T}(\lambda y.x(\lambda z.yz))$ should be labeled as something like $\lambda y.x\,\mathbf{1}_{\eta(y)}$, meaning that $\mathbf{1}$ comes from an η-expansion of the variable y);
  - the set of free variables that were *forgotten* by taking $\mathbf{1}$ in the Taylor expansion (for instance $\lambda y.x\,\mathbf{1}[y] \in \mathcal{T}(\lambda y.xyy)$ should be labeled as $\lambda y.x\,\mathbf{1}^y[y]$).

- Then we define a local reduction $\to_{\eta^\ell}$ on elements of $\mathcal{L}(\mathrm{nf}_\beta\mathcal{T}(M))$ that exploits the extra information annotated by $\mathcal{L}(-)$ to perform the η-reduction *only when it is safe*.

The labeling $\mathcal{L}(-)$ relies on a certain homogeneity in the structure of the resource terms belonging to $\mathrm{nf}_\beta\mathcal{T}(M)$. As shown in [BHP13], this homogeneity is captured by the following *definedness relation* $\preceq$ between β-normal resource terms.

Definition 4.5.1 ([BHP13, Def. 9])**.** The relation $\preceq$ is the smallest subset of $\mathrm{nf}_\beta(\Lambda^r) \times \mathrm{nf}_\beta(\Lambda^r)$ satisfying the rules

$$\lambda x_1 \dots x_n.y \preceq \lambda x_1 \dots x_n.y \qquad \frac{t \preceq t' \quad \mathbf{b} \preceq \mathbf{b}'}{t\mathbf{b} \preceq t'\mathbf{b}'} \qquad \mathbf{1} \preceq \mathbf{b} \qquad \frac{\exists t' \in \mathbf{b}'\ \forall t \in \mathbf{b},\ t \preceq t'}{\mathbf{b} \preceq \mathbf{b}'}$$

Remark 4.5.2. The relation $\preceq$ is not a preorder, since it is transitive but not reflexive. For instance, $x[y\mathbf{1}[y], y[y]\mathbf{1}] \not\preceq x[y\mathbf{1}[y], y[y]\mathbf{1}]$, because $y\mathbf{1}[y] \not\preceq y[y]\mathbf{1}$ and $y[y]\mathbf{1} \not\preceq y\mathbf{1}[y]$. See [BHP13] for more properties of this relation.

The well-known notion of ideal used hereafter is recalled in § 1.1.

Proposition 4.5.3 ([BHP13, Lemma 12])**.** *Let $\mathcal{S}$ be an ideal of $(\mathrm{nf}_\beta(\Lambda^r), \preceq)$. Then $\mathcal{S}$ has one of the following forms: $\{x\}$ for some $x \in \mathrm{Var}$, or $\lambda x.\mathbb{T}$ for some ideal $\mathbb{T}$, or $\mathbb{T}\mathbb{B}$ for some ideal $\mathbb{T}$ and some set of bags $\mathbb{B}$ such that $\bigcup\mathbb{B}$ is an ideal.*

The following key definition is sound precisely because of Proposition 4.5.3.

Definition 4.5.4. Let $\mathcal{S}$ be an ideal of $(\mathrm{nf}_\beta(\Lambda^r), \preceq)$ and $t \in \mathcal{S}$. The *labeled (β-normal resource) term* $\mathcal{L}(t, \mathcal{S})$ is given by the following induction on t (and accordingly on the structure of $\mathcal{S}$, as given by Proposition 4.5.3).

$$\mathcal{L}(x, \{x\}) := x, \qquad \mathcal{L}(\lambda x.t, \lambda x.\mathbb{T}) := \lambda x.\mathcal{L}(t, \mathbb{T}), \qquad \mathcal{L}(t\mathbf{b}, \mathbb{T}\mathbb{B}) := \mathcal{L}(t, \mathbb{T})\mathcal{L}(\mathbf{b}, \mathbb{B}),$$

$$\mathcal{L}([t_1, \dots, t_k], \mathbb{B}) := [\mathcal{L}(t_1, \textstyle\bigcup\mathbb{B}), \dots, \mathcal{L}(t_k, \bigcup\mathbb{B})] \quad \text{for } k > 0$$

$$\mathcal{L}(\mathbf{1}, \mathbb{B}) := \begin{cases} \mathbf{1}^{x}_{\eta(x)} & \text{if there exists } t' \in \bigcup\mathbb{B} \text{ such that } t' \twoheadrightarrow_{\eta'} x \\ \mathbf{1}^{\mathrm{fv}(\mathbb{B})} & \text{otherwise} \end{cases} \qquad (\bullet)$$

where the reduction $\to_{\eta'}$ appearing in Condition $(\bullet)$ is defined as

$$\lambda x.t[x^{k+1}] \to_{\eta'} t \qquad \text{whenever } x \notin \mathrm{fv}(t) \text{ and } k \in \mathbb{N}.$$

We also set

$$\mathcal{L}(\mathcal{S}) := \{\mathcal{L}(t, \mathcal{S}) \mid t \in \mathcal{S}\}.$$

The labeling $\mathcal{L}(-)$ can be applied to $\mathrm{nf}_\beta\mathcal{T}(M)$ thanks to the following result.

Proposition 4.5.5. *[BHP13, Lemma 23] Let $M \in \Lambda$. Then $\mathrm{nf}_\beta\mathcal{T}(M)$ is an ideal of $(\mathrm{nf}_\beta(\Lambda^r), \preceq)$.*

Remark 4.5.6. Actually the definition of $\mathcal{L}(t, \mathcal{S})$ will be only used when $\mathcal{S} = \mathrm{nf}_\beta\mathcal{T}(M)$ for some $M \in \Lambda$. Under this hypothesis the case $\mathcal{L}(\mathbf{1}, \mathbb{B})$ is applied when $\mathbb{B} = \mathcal{M}_f(\mathcal{T}(N))$ for some β-normal $N \in \Lambda$, hence $\bigcup\mathbb{B} = \mathcal{T}(N)$. Then Condition $(\bullet)$ becomes there is $t \in \mathcal{T}(N)$ such that $t \twoheadrightarrow_{\eta'} x'$

which holds exactly when $N \twoheadrightarrow_{\eta} x$.

Examples 4.5.7. Here are a couple of examples of labeling of resource terms.

- Let $t = \lambda y.x\mathbf{1}\mathbf{1}$ and $\mathcal{S} = \mathrm{nf}_\beta \mathcal{T}(\lambda y.x\mathbf{\Omega}y) = \{\lambda y.x\mathbf{1}[y^n] \mid n \in \mathbb{N}\}$. Then we have $\mathcal{L}(t,\mathcal{S}) = \lambda y.\mathcal{L}(x,\{x\})\,\mathcal{L}(\mathbf{1},\{\mathbf{1}\})\,\mathcal{L}(\mathbf{1},\{[y^n] \mid n \in \mathbb{N}\}) = \lambda y.x\,\mathbf{1}^{\emptyset}\mathbf{1}^{y}_{\eta(y)}$.
- Let $t = \lambda y.x\mathbf{1}\mathbf{1}$ and $\mathcal{S} = \mathrm{nf}_\beta \mathcal{T}(\lambda y.xyy) = \{\lambda y.x[y^n][y^m] \mid n,m \in \mathbb{N}\}$. We have $\mathcal{L}(t,\mathcal{S}) = \lambda y.\mathcal{L}(x,\{x\})\,\mathcal{L}(\mathbf{1},\{[y^n] \mid n \in \mathbb{N}\})\,\mathcal{L}(\mathbf{1},\{[y^m] \mid m \in \mathbb{N}\}) = \lambda y.x\mathbf{1}^{y}_{\eta(y)}\mathbf{1}^{y}_{\eta(y)}$. More generally

$$\begin{aligned}\mathcal{L}(\mathcal{T}(\lambda y.xyy)) &= \left\{\lambda y.x\mathbf{1}^{y}_{\eta(y)}\mathbf{1}^{y}_{\eta(y)}\right\} \cup \left\{\lambda y.x\mathbf{1}^{y}_{\eta(y)}[y^{m+1}] \mid m \in \mathbb{N}\right\} \cup \\ &\quad \left\{\lambda y.x[y^{n+1}]\mathbf{1}^{y}_{\eta(y)} \mid n \in \mathbb{N}\right\} \cup \left\{\lambda y.x[y^{n+1}][y^{m+1}] \mid n,m \in \mathbb{N}\right\}.\end{aligned}$$

Definition 4.5.8. Let t be a labeled resource term. The set $\widetilde{\mathrm{fv}}(t)$ of the *free variables* of t is defined like the usual definition of $\mathrm{fv}(-)$ for resource terms, but with the addition of the clauses $\widetilde{\mathrm{fv}}(\mathbf{1}^{x}_{\eta(x)}) := \{x\}$ and $\widetilde{\mathrm{fv}}(\mathbf{1}^{V}) := V$.

Remark 4.5.9. Let $T \in \Lambda^{\mathcal{B}}$. Then $x \in \mathrm{fv}(T)$ if and only if $x \in \widetilde{\mathrm{fv}}(t)$ for every $t \in \mathcal{L}(\mathcal{T}(T))$.

Definition 4.5.10. The reduction $\to_{\eta^\ell}$ on labeled β-normal resource terms is given by

$$\lambda x.t\,\mathbf{1}^{x}_{\eta(x)} \to_{\eta^\ell} t \quad \text{whenever } x \notin \widetilde{\mathrm{fv}}(t)$$

$$\lambda x.t[x^{n+1}] \to_{\eta^\ell} t \quad \text{whenever } x \notin \widetilde{\mathrm{fv}}(t) \text{ and } n \in \mathbb{N}.$$

Examples 4.5.11. We have $\mathcal{L}(\lambda y.x\mathbf{1}[y], \mathrm{nf}_\beta \mathcal{T}(\lambda y.xzy)) = \lambda y.x\mathbf{1}^{z}_{\eta(z)}[y] \to_{\eta^\ell} x\mathbf{1}^{z}_{\eta(z)}$. On the contrary $\mathcal{L}(\lambda y.x\mathbf{1}[y], \mathrm{nf}_\beta \mathcal{T}(\lambda y.xyy)) = \lambda y.x\mathbf{1}^{y}_{\eta(y)}y$ is already in η^ℓ-normal form.

Proposition 4.5.12. *The reduction $\to_{\eta^\ell}$ is strongly normalizing and confluent.*

Proof. The reduction $\to_{\eta^\ell}$ is strongly normalizing since the size of the term decreases. It is moreover weakly confluent, and therefore confluent by Newman's lemma. □

Notation. Given a labeled term t, we write $\ulcorner t \urcorner$ for the resource term obtained from t by erasing all its labels.

Definition 4.5.13. Let $M \in \Lambda$. The *extensional Taylor expansion* of M is given by

$$\mathcal{T}^{\eta}(M) := \ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathrm{nf}_\beta \mathcal{T}(M)) \urcorner$$

Remark 4.5.14. In the definition above, the β-reduction and the η^ℓ-reduction are separated, and performed in that specific order, because the reduction $\beta \cup \eta^\ell$ is not confluent: for instance $\lambda x.\mathbf{I}\,[x,x] \to_{\eta^\ell} \mathbf{I}$ whereas $\lambda x.\mathbf{I}\,[x,x] \to_\beta \emptyset$.

We need some other technical tools. Remember that our aim is to prove the equality $\mathcal{T}^{\eta}(M) = \mathcal{T}(\mathrm{BT}^{\eta}(M))$, i.e. $\ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathrm{nf}_\beta \mathcal{T}(M))\urcorner = \mathcal{T}(\mathrm{BT}^{\eta}(M))$. By (79), it is enough to show that $\ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathcal{T}(\mathrm{BT}(M)))\urcorner = \mathcal{T}(\mathrm{BT}^{\eta}(M))$. The difficulty lies in that, on the one hand, $\mathrm{BT}^{\eta}(M)$ is $\eta(\mathrm{BT}(M))$, something defined coinductively on $\mathrm{BT}(M)$, whereas on the other hand the η^ℓ-reduction works on a set of (labeled) resource terms coming from $\mathrm{BT}(M)^*$, which is a set of finite approximants, not a coinductively defined object. In order to fill this gap, as an intermediate step we recharacterize $\eta(\mathrm{BT}(M))$ in the style of $\ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(-)\urcorner$. Basically we *mimick* on sets of finite approximants what we have done so far in this section for sets of resource terms. In particular, even for sets of finite approximants we want the η-reduction to act like a global operation; therefore, we introduce a labeling $\mathcal{E}(-)$ on ideals of $(\mathcal{N}, \leqslant_\perp)$ in the spirit of Definition 4.5.4.

Notation. Given $\mathbb{M} \subseteq \mathcal{N}$, we denote by $\mathbb{M}{\downarrow}$ its downward closure w.r.t. $\leqslant_\perp$, that is the set $\{a \in \mathcal{N} \mid \text{ there exists } b \in \mathbb{M} \text{ such that } a \leqslant_\perp b\}$.

We adopt for sets $\mathbb{M} \in \mathcal{P}(\mathcal{N})$ the same syntactic sugar that we used for $\mathcal{P}(\Lambda^r)$ since § 1.6, by extending all the constructors of the grammar of $\mathcal{N}$ as pointwise operations on $\mathcal{P}(\mathcal{N})$. For instance, $\mathbb{M}\mathbb{N}$ stands for $\{ab \mid a \in \mathbb{M} \text{ and } b \in \mathbb{N}\}$. As another example, the ideal $\mathrm{BT}(\mathbf{J}x)^*$ can be written as $\{\lambda z_0.x(\mathrm{BT}(\mathbf{J}z_0)^*)\}{\downarrow} = \lambda z_0.x(\mathrm{BT}(\mathbf{J}z_0)^*) \cup \{\perp\}$.

Remark 4.5.15. When $\mathbb{M}$ is an ideal of $(\mathcal{N}, \leqslant_\perp)$ then $\mathbb{M} = \mathbb{M}{\downarrow}$ and all its elements have a similar syntactic structure, except for $\perp$.

Definition 4.5.16. Let $\mathbb{M}$ be an ideal of $(\mathcal{N}, \leqslant_\perp)$ and $a \in \mathbb{M}$. We define the *labeled finite approximant* $\mathcal{E}(a, \mathbb{M})$ by induction on a as follows.

$$\mathcal{E}(x, \{x\}{\downarrow}) := x, \qquad \mathcal{E}(\lambda x.a, (\lambda x.\mathbb{M}){\downarrow}) := \lambda x.\mathcal{E}(a, \mathbb{M}{\downarrow}),$$

$$\mathcal{E}(ac, (\mathbb{M}\mathbb{N}){\downarrow}) := \mathcal{E}(a, \mathbb{M}{\downarrow})\mathcal{E}(c, \mathbb{N}),$$

$$\mathcal{E}(\perp, \mathbb{M}) := \begin{cases} \perp^{x}_{\eta(x)} & \text{if there exists a } \perp\text{-free } a \in \mathbb{M} \text{ such that } a \twoheadrightarrow_\eta x, \quad (\circ) \\ \perp^{\mathrm{fv}(\mathbb{M})} & \text{otherwise.} \end{cases}$$

We extend the definition to $\mathbb{M}$ by setting

$$\mathcal{E}(\mathbb{M}) := \{\mathcal{E}(a, \mathbb{M}) \mid a \in \mathbb{M}\}.$$

If a is a labeled approximants we call $\ulcorner a \urcorner$ the term obtained from a by erasing all its labels.

Notice that in the case $(\mathbb{M}\mathbb{N}){\downarrow}$ of Definition 4.5.16 the set $\mathbb{N}$ is already downward closed. This is the reason why it is not necessary to write $\mathcal{E}(ac, (\mathbb{M}\mathbb{N}){\downarrow}) := \mathcal{E}(a, \mathbb{M}{\downarrow})\mathcal{E}(c, \mathbb{N}{\downarrow})$.

Remark 4.5.17. For every $M \in \Lambda$ the set $\mathrm{BT}(M)^*$ is an ideal of $(\mathcal{N}, \leqslant_\perp)$. In fact, the definition of $\mathcal{E}(a, \mathbb{M})$ will be only used when $\mathbb{M}$ is some $\mathrm{BT}(M)^*$. Under this hypothesis the case $\mathcal{E}(\perp, \mathbb{M})$ is only applied when $\mathbb{M} = \mathrm{BT}(N)^*$ for some $N \in \Lambda$. Then Condition $(\circ)$ is simply equivalent to $N \twoheadrightarrow_\eta x$.

Definition 4.5.18. Let a be a labeled finite approximant. The set $\widetilde{\mathrm{fv}}(a)$ of the *free variables* of a is defined like the usual definition of $\mathrm{fv}(-)$ for finite approximants, but with the addition of the clauses $\widetilde{\mathrm{fv}}(\perp^{x}_{\eta(x)}) := \{x\}$ and $\widetilde{\mathrm{fv}}(\perp^{V}) := V$.

Remark 4.5.19. Let $T \in \Lambda^{\mathcal{B}}$. Then $x \in \mathrm{fv}(T)$ if and only if $x \in \widetilde{\mathrm{fv}}(t)$ for every $t \in \mathcal{E}(T^*)$.

Definition 4.5.20. The reduction $\to_{\eta^e}$ on labeled finite approximants is given by the rules

$$\lambda x.a\bot^x_{\eta(x)} \to_{\eta^e} a \quad \text{if } x \notin \widetilde{\mathrm{fv}}(a), \qquad\qquad \lambda x.ax \to_{\eta^e} a \quad \text{if } x \notin \widetilde{\mathrm{fv}}(a).$$

Proposition 4.5.21. *The reduction $\to_{\eta^e}$ is strongly normalizing and confluent.*

Proof. The reduction $\to_{\eta^e}$ is strongly normalizing since the size of the term decreases. It is moreover weakly confluent, and therefore confluent by Newman's lemma. □

After some technical lemmas, we show that the η^e-reduction on $\mathcal{E}(\mathrm{BT}(M))$ computes exactly the finite approximants of the coinductively defined tree $\mathrm{BT}^{\eta}(M)$ (Proposition 4.5.26).

Lemma 4.5.22. *Let $x \in \mathrm{Var}$ $M \in \Lambda$ such that $M \twoheadrightarrow_\eta x$. For all $a \in M^*$ either $\mathcal{E}(a, M^*) = \bot^x_{\eta(x)}$ or $\mathcal{E}(a, M^*) \twoheadrightarrow_{\eta^e} x$.*

Proof. We have $M = \lambda x_1 \ldots x_n.x\, N_1 \cdots N_n$ with $N_i \twoheadrightarrow_\beta x_i$ and $x \notin \mathrm{fv}(x\, N_1 \cdots N_{i-1})$ for all $i \in \{1, \ldots, n\}$. Consider $a \in M^*$. We proceed by induction on a.
Case $a = \bot$. In such a case $\mathcal{E}(a, M^*) = \mathcal{E}(\bot, M^*) = \bot^x_{\eta(x)}$ by Definition 4.5.16, since there is $M \in M^*$ such that $M \to_\eta x$ (M is β-normal, so it can be seen as a finite approximant).
Case $a = \lambda x_1 \ldots x_n.x\, a_1 \cdots a_n$ **with** $a_i \in N_i^*$ **for all** $i \in \{1, \ldots, n\}$. By IH for all $i \in \{1, \ldots, n\}$ either $\mathcal{E}(a_i, N_i^*) \twoheadrightarrow_{\eta^e} x_i$ or $\mathcal{E}(a_i, N_i^*) = \bot^{x_i}_{\eta(x_i)}$. So by Definition 4.5.16 and Definition 4.5.20 $\mathcal{E}(a, M^*) = \lambda x_1 \ldots x_n.x\, \mathcal{E}(a_1, N_1^*) \cdots \mathcal{E}(a_n, N_n^*) \twoheadrightarrow_{\eta^e} x$. □

Notation. Given two sets of terms $\mathbb{X}, \mathbb{Y}$ and a reduction $\to_r$ we write $\mathbb{X} \Rrightarrow_r \mathbb{Y}$ if and only if

- for all $t_1 \in \mathbb{X}$ there is $t_2 \in \mathbb{Y}$ such that $t_1 \twoheadrightarrow_r t_2$,
- for all $t_2 \in \mathbb{Y}$ there is $t_1 \in \mathbb{X}$ such that $t_1 \twoheadrightarrow_r t_2$.

Lemma 4.5.23. *Let $T = \lambda\vec{x}y.z\, T_1 \cdots T_{k+1}$ be a Böhm like tree such that T_{k+1} is finite, $T_{k+1} \twoheadrightarrow_\eta y$ and $y \notin \mathrm{fv}(z\, T_1 \cdots T_k)$. Then $\mathcal{E}(T^*) \Rrightarrow_{\eta^e} \mathcal{E}\big((\lambda\vec{x}.zT_1^* \cdots T_k^*){\downarrow}\big)$.*

Proof. Firstly we prove that given $a \in T^*$ there exists $a' \in (\lambda\vec{x}.zT_1^* \cdots T_k^*){\downarrow}$ such that $\mathcal{E}(a, T^*) \twoheadrightarrow_{\eta^e} \mathcal{E}\big(a', (\lambda\vec{x}.zT_1^* \cdots T_k^*){\downarrow}\big)$. We split into cases depending on a.
Case $a = \bot$. We have $\mathcal{E}(a, T^*) = \mathcal{E}(\bot, T^*) = \mathcal{E}\big(\bot, (\lambda\vec{x}y.zT_1^* \cdots T_k^*T_{k+1}^*){\downarrow}\big)$. As by hypothesis T_{k+1} is finite, $T_{k+1} \twoheadrightarrow_\eta y$ and $y \notin \mathrm{fv}(z\, T_1 \cdots T_k)$, we have $T = \lambda\vec{x}y.z\, T_1 \cdots T_k T_{k+1} \to_\eta \lambda\vec{x}y.z\, T_1 \cdots T_k$. It is then clear that there is a $\bot$-free $c_1 \in T^*$ such that $c_1 \twoheadrightarrow_\eta z$ if and only if there exists a $\bot$-free $c_2 \in (\lambda\vec{x}.zT_1^* \cdots T_k^*){\downarrow}$ such that $c_2 \twoheadrightarrow_\eta z$. Therefore by Definition 4.5.16 $\mathcal{E}\big(\bot, (\lambda\vec{x}y.zT_1^* \cdots T_k^*T_{k+1}^*){\downarrow}\big) = \mathcal{E}\big(\bot, (\lambda\vec{x}.zT_1^* \cdots T_k^*){\downarrow}\big)$. In the end we assume $a' := \bot$.
Case $a = \lambda\vec{x}y.z\, a_1 \cdots a_{k+1}$ **with** $a_i \in T_i^*$ **for all** $i \in \{1, \ldots, k+1\}$. By Definition 4.5.16 $\mathcal{E}(a, T^*) = \lambda\vec{x}y.z\, \mathcal{E}(a_1, T_1^*) \cdots \mathcal{E}(a_k, T_k^*)\mathcal{E}(a_{k+1}, T_{k+1}^*)$. By hypothesis $T_{k+1} \twoheadrightarrow_\eta y$, hence by Lemma 4.5.23 either $\mathcal{E}(a_{k+1}, T_{k+1}) = \bot^y_{\eta(y)}$ or $\mathcal{E}(a_{k+1}, T_{k+1}) \twoheadrightarrow_{\eta^e} y$. By Remark 4.5.19 the fact that $y \notin \mathrm{fv}(z\, T_1 \cdots T_k)$ implies $y \notin \widetilde{\mathrm{fv}}\big(z\, \mathcal{E}(a_1, T_1^*) \cdots \mathcal{E}(a_k, T_k^*)\big)$. So in both cases we get $\mathcal{E}(a, T^*) \twoheadrightarrow_{\eta^e} \lambda\vec{x}.z\, \mathcal{E}(a_1, T_1^*) \cdots \mathcal{E}(a_k, T_k^*) = \mathcal{E}\big(\lambda\vec{x}.z\, a_1 \cdots a_k, (\lambda\vec{x}.zT_1^* \cdots T_k^*){\downarrow}\big)$. We assume $a' := \lambda\vec{x}.z\, a_1 \cdots a_k$ and we are done.

Secondly, we prove that for every $a' \in (\lambda\vec{x}.zT_1^* \cdots T_k^*){\downarrow}$ there exists $a \in T^*$ such that $\mathcal{E}(a, T^*) \twoheadrightarrow_{\eta^e} \mathcal{E}\big(a', (\lambda\vec{x}.zT_1^* \cdots T_k^*){\downarrow}\big)$. Again, we split into cases depending on a'.

Case $a' = \perp$. It is enough to take $a := \perp$ and repeat the argument in **Case** $a = \perp$ above.
Case $a' = \lambda\vec{x}.za'_1 \cdots a'_k$ **with** $a'_i \in T^*_i$ **for all** $i \in \{1, \ldots, k\}$. Obviously $\perp \in T^*_{k+1}$, so it makes sense to assume $a := \lambda\vec{x}y.za'_1 \cdots a'_k \perp \in T^*$. Since by hypothesis T_{k+1} is finite and $T_{k+1} \twoheadrightarrow_\eta y$, we have

$$\mathcal{E}(\perp, T^*_{k+1}) \quad = \quad \perp^y_{\eta(y)} \,. \tag{80}$$

Moreover by Remark 4.5.19 from the hypothesis $y \notin \mathrm{fv}(z\,T_1 \cdots T_k)$ we get

$$y \quad \notin \quad \mathrm{fv}\big(z\,\mathcal{E}(a'_1, T^*_1) \cdots \mathcal{E}(a'_k, T^*_k)\big) \,. \tag{81}$$

We then have

$$\begin{array}{rcll}
\mathcal{E}(a, T^*) & = & \lambda\vec{x}y.z\mathcal{E}(a'_1, T^*_1) \cdots \mathcal{E}(a'_k, T^*_k)\mathcal{E}(a'_{k+1}, T^*_{k+1}) & \text{by Def. 4.5.16} \\
& = & \lambda\vec{x}y.z\mathcal{E}(a'_1, T^*_1) \cdots \mathcal{E}(a'_k, T^*_k)\perp^y_{\eta(y)} & \text{by (80)} \\
& \to_{\eta^e} & \lambda\vec{x}.z\mathcal{E}(a'_1, T^*_1) \cdots \mathcal{E}(a'_k, T^*_k) & \text{by (81)} \\
& = & \mathcal{E}(a', \lambda\vec{x}.zT^*_1 \cdots T^*_k). & \text{by Def. 4.5.16}
\end{array}$$

and we are done. □

Corollary 4.5.24. *Let* $T = \lambda\vec{x}\,y.z\,T_1 \cdots T_{k+1}$ *be a Böhm like tree such that* T_{k+1} *is finite,* $T_{k+1} \twoheadrightarrow_\eta y$ *and* $y \notin \mathrm{fv}(z\,T_1 \cdots T_k)$. *Then* $\mathrm{nf}_{\eta^e}\big(\mathcal{E}(T^*)\big) = \mathrm{nf}_{\eta^e}\big(\mathcal{E}\big((\lambda\vec{x}.zT^*_1 \cdots T^*_k)\!\downarrow\big)\big)$.

Proof. Let $t \in \mathrm{nf}_{\eta^e}\big(\mathcal{E}(T^*)\big)$, namely let $t_1 \in \mathcal{E}(T^*)$ be such that t is the η^e-nf of t_1. By Lemma 4.5.23 there is $t_2 \in \mathcal{E}\big((\lambda\vec{x}.zT^*_1 \cdots T^*_k)\!\downarrow\big)$ such that $t_1 \twoheadrightarrow_{\eta^e} t_2$. Hence, by confluence of $\twoheadrightarrow_{\eta^e}$, the term t is also the η^e-nf of t_2. So $t \in \mathrm{nf}_{\eta^e}\big(\mathcal{E}\big((\lambda\vec{x}.zT^*_1 \cdots T^*_k)\!\downarrow\big)\big)$. This proves that $\mathrm{nf}_{\eta^e}\big(\mathcal{E}(T^*)\big) \subseteq \mathrm{nf}_{\eta^e}\big(\mathcal{E}\big((\lambda\vec{x}.zT^*_1 \cdots T^*_k)\!\downarrow\big)\big)$.

Let $t \in \mathrm{nf}_{\eta^e}\big(\mathcal{E}\big((\lambda\vec{x}.zT^*_1 \cdots T^*_k)\!\downarrow\big)\big)$, i.e. let $t_2 \in \mathcal{E}\big((\lambda\vec{x}.zT^*_1 \cdots T^*_k)\!\downarrow\big)$ be such that t is the η^e-nf of t_2. By Lemma 4.5.23 there is $t_1 \in \mathcal{E}(T^*)$ such that $t_1 \twoheadrightarrow_{\eta^e} t_2$. Then clearly t is also the η^e-nf of t_1. So $t \in \mathrm{nf}_{\eta^e}\big(\mathcal{E}(T^*)\big)$. Eventually $\mathrm{nf}_{\eta^e}\big(\mathcal{E}(T^*)\big) \supseteq \mathrm{nf}_{\eta^e}\big(\mathcal{E}\big((\lambda\vec{x}.zT^*_1 \cdots T^*_k)\!\downarrow\big)\big)$. □

Lemma 4.5.25. *Let* $T \in \Lambda^{\mathcal{B}}$. *Then* $\ulcorner \mathrm{nf}_{\eta^e}(\mathcal{E}(T^*)) \urcorner = \eta(T)^*$.

Proof. We proceed by coinduction on T.

If $T = \perp$ then $\ulcorner \mathrm{nf}_{\eta^e}(\mathcal{E}(T^*)) \urcorner = \ulcorner \mathrm{nf}_{\eta^e}(\mathcal{E}(\{\perp\})) \urcorner = \{\ulcorner \mathcal{E}(\perp, \perp) \urcorner\} = \{\ulcorner \perp^{\emptyset} \urcorner\} = \{\perp\} = \eta(T)^*$.

Otherwise, the Böhm-like tree T can be written in a unique way as

$$T \quad = \quad \lambda x_1 \ldots x_n y_1 \ldots y_m.z\,T_1 \cdots T_k\,T'_1 \cdots T'_m$$

for some $n, m, k \in \mathbb{N}$ such that:

1. T'_i is finite, $T'_i \twoheadrightarrow_\eta y_i$ and $y_i \notin \mathrm{fv}\big(z\,T_1 \cdots T_k T'_1 \cdots T'_{i-1}\big)$, for all $i \in \{1, \ldots, m\}$;
2. T_k is infinite, or T_k is finite but does not η-reduce to x_n, or $x_n \in \mathrm{fv}(z\,T_1 \cdots T_k)$.

The following equalities hold:

$$
\begin{aligned}
\ulcorner \mathrm{nf}_{\eta^e}(\mathcal{E}(T^*))\urcorner &= \ulcorner \mathrm{nf}_{\eta^e}(\mathcal{E}(\lambda\vec{x}.z\,T_1^*\cdots T_k^*)\downarrow)\urcorner && \text{by 1 and Cor. 4.5.24}\\
&= \ulcorner \mathrm{nf}_{\eta^e}\big(\lambda\vec{x}.z\,\mathcal{E}(T_1^*)\cdots\mathcal{E}(T_n^*)) \\
&\quad \cup \{\mathcal{E}(\bot,(\lambda\vec{x}.z\,T_1^*\cdots T_k^*)\downarrow)\}\urcorner && \text{by Def. 4.5.16}\\
&= \ulcorner \lambda\vec{x}.z\,\mathrm{nf}_{\eta^e}(\mathcal{E}(T_1^*))\cdots \mathrm{nf}_{\eta^e}(\mathcal{E}(T_k^*)) \\
&\quad \cup \{\mathrm{nf}_{\eta^e}(\mathcal{E}(\bot,(\lambda\vec{x}.z\,T_1^*\cdots T_k^*)\downarrow))\}\urcorner && \text{by def. of } \mathrm{nf}_\eta(-)\\
&= \lambda\vec{x}.z\ulcorner \mathrm{nf}_{\eta^e}(\mathcal{E}(T_1^*))\urcorner\cdots\ulcorner \mathrm{nf}_{\eta^e}(\mathcal{E}(T_k^*))\urcorner \\
&\quad \cup \left\{\ulcorner \mathrm{nf}_{\eta^e}(\mathcal{E}(\bot,(\lambda\vec{x}.z\,T_1^*\cdots T_k^*)\downarrow))\urcorner\right\} && \text{by def. of } \ulcorner - \urcorner\\
&= \lambda\vec{x}.z\,\eta(T_1)^*\cdots\eta(T_k)^* \\
&\quad \cup \left\{\ulcorner \mathrm{nf}_{\eta^e}(\mathcal{E}(\bot,(\lambda\vec{x}.z\,T_1^*\cdots T_k^*)\downarrow))\urcorner\right\} && \text{by coIH}\\
&= \lambda\vec{x}.z\,\eta(T_1)^*\cdots\eta(T_k)^* \\
&\quad \cup \left\{\ulcorner \mathrm{nf}_{\eta^e}(\bot^{\mathrm{fv}(\lambda\vec{x}.z\,T_1^*\cdots T_k^*)})\urcorner\right\} && \text{by 2 and Def. 4.5.16}\\
&= \lambda\vec{x}.z\,\eta(T_1)^*\cdots\eta(T_k)^* \cup \{\bot\} && \text{by def. of } \ulcorner - \urcorner \text{ and } \mathrm{nf}_\eta(-)\\
&= \eta(T)^* && \text{by def. of } \eta(-)
\end{aligned}
$$

and we are done. □

Proposition 4.5.26. *Let* $M \in \Lambda$. *Then* $\ulcorner \mathrm{nf}_{\eta^e}\mathcal{E}(\mathrm{BT}(M)^*)\urcorner = \mathrm{BT}^\eta(M)^*$.

Proof. Since $\mathrm{BT}^\eta(M) := \eta\big(\mathrm{BT}(M)\big)$, the result follows directly from Lemma 4.5.25. □

Now that all technical tools are in place, we are finally able to prove that the extensional Taylor expansion of a λ-term M is equal to the Taylor expansion of $\mathrm{BT}^\eta(M)$ (Theorem 4.5.32).

The key passage is a sort of *commutation* between the η^ℓ-normalization and the Taylor expansion. In fact, it is at this purpose that we have introduced also the η^e-reduction: Proposition 4.5.31 below states that performing the Taylor expansion and then η^ℓ-normalizing is equivalent to η^e-normalizing in the first place and then doing the Taylor expansion.

Lemma 4.5.27. *Let* $M \in \Lambda$ *such that* $M \twoheadrightarrow_\eta x$. *Then for all* $t \in \mathcal{T}(M)$ *we have* $\mathcal{L}(t, \mathcal{T}(M)) \twoheadrightarrow_{\eta^\ell} x$.

Proof. By hypothesis $M = \lambda x_1 \ldots x_n.x\,M_1 \cdots M_n$, for some $n \in \mathbb{N}$, with $M_i \twoheadrightarrow_\eta x_i$ and $x_i \notin \mathrm{fv}(x\,M_1 \cdots M_{i-1})$ for all $i \in \{1, \ldots, n\}$. We proceed by induction on $t \in \mathcal{T}(M)$.

We have $t = \lambda x_1 \ldots x_n.x\,\mathbf{b_1} \cdots \mathbf{b_n}$ where $\mathbf{b_i} \in \mathcal{M}_f(\mathcal{T}(M_i))$ for every $i \in \{1, \ldots, n\}$.

If $n = 0$ then $\mathcal{L}(t, \mathcal{T}(M)) = \mathcal{L}(x, \{x\}) = x$ by Definition 4.5.4. So the thesis is proved.

If $n > 0$ then by Definition 4.5.4

$$\mathcal{L}(t, \mathcal{T}(M)) = \lambda x_1 \ldots x_n.x\,\mathcal{L}\big(\mathbf{b_1}, \mathcal{M}_f(\mathcal{T}(M_1))\big) \cdots \mathcal{L}\big(\mathbf{b_n}, \mathcal{M}_f(\mathcal{T}(M_n))\big) .$$

Suppose $\mathbf{b_n} = [t_1, \ldots, t_k]$ for some $k \in \mathbb{N}$. Remember that $t_j \in \mathcal{T}(M_n)$ for all $j \in \{1, \ldots, k\}$.

Let us see that

$$\mathcal{L}\big(t, \mathcal{T}(M_n)\big) \twoheadrightarrow_{\eta^\ell} \lambda x_1 \ldots x_{n-1}.x\,\mathcal{L}\big(\mathbf{b_1}, \mathcal{M}_f(\mathcal{T}(M_1))\big) \cdots \mathcal{L}\big(\mathbf{b_{n-1}}, \mathcal{M}_f(\mathcal{T}(M_{n-1}))\big). \quad (82)$$

At this purpose we distinguish two cases, depending on k.

If $k = 0$ then by Definition 4.5.4

$$\mathcal{L}\big(\mathbf{b_n}, \mathcal{M}_f(\mathcal{T}(M_n))\big) \quad = \quad \mathcal{L}\big(1, \mathcal{M}_f(\mathcal{T}(M_n))\big) \quad = \quad \mathbf{1}^{x_n}_{\eta(x_n)} \tag{83}$$

because $M_n \twoheadrightarrow_\eta x_n$ implies that there is $s \in \bigcup \mathcal{M}_f(\mathcal{T}(M_n)) = \mathcal{T}(M_n)$ such that $s \twoheadrightarrow_{\eta'} x_n$. Therefore

$$\begin{aligned}\mathcal{L}\big(t, \mathcal{T}(M_n)\big) &= \lambda x_1 \ldots x_n.x\mathcal{L}\big(\mathbf{b_1}, \mathcal{M}_f(\mathcal{T}(M_1))\big) \cdots \mathcal{L}\big(\mathbf{b_{n-1}}, \mathcal{M}_f(\mathcal{T}(M_{n-1}))\big)\mathbf{1}^{x_n}_{\eta(x_n)} && \text{by (83)}\\ &\twoheadrightarrow_{\eta^\ell} \lambda x_1 \ldots x_{n-1}.x\mathcal{L}\big(\mathbf{b_1}, \mathcal{M}_f(\mathcal{T}(M_1))\big) \cdots \mathcal{L}\big(\mathbf{b_{n-1}}, \mathcal{M}_f(\mathcal{T}(M_{n-1}))\big) && \text{by Def. 4.5.20}\end{aligned}$$

So (82) is proved.

Let us consider the case $k > 0$. Since $M_n \twoheadrightarrow_\eta x_n$, for every $j \in \{1, \ldots, k\}$ the IH can be applied to $t_j \in \mathcal{T}(M_n)$, so to get $\mathcal{L}\big(t_j, \mathcal{T}(M_n)\big) \twoheadrightarrow_{\eta^\ell} x_n$. Therefore

$$\begin{aligned}\mathcal{L}\big(t, \mathcal{T}(M_n)\big) &\twoheadrightarrow_{\eta^\ell} \lambda x_1 \ldots x_n.x\,\mathcal{L}\big(\mathbf{b_1}, \mathcal{M}_f(\mathcal{T}(M_1))\big) \cdots \mathcal{L}\big(\mathbf{b_{n-1}}, \mathcal{M}_f(\mathcal{T}(M_{n-1}))\big)\big[x_n^k\big]\\ &\twoheadrightarrow_{\eta^\ell} \lambda x_1 \ldots x_{n-1}.x\,\mathcal{L}\big(\mathbf{b_1}, \mathcal{M}_f(\mathcal{T}(M_1))\big) \cdots \mathcal{L}\big(\mathbf{b_{n-1}}, \mathcal{M}_f(\mathcal{T}(M_{n-1}))\big)\end{aligned}$$

So even in this case (82) is proved.

By iterating on the bags $\mathbf{b_{n-1}}, \ldots, \mathbf{b_1}$ the reasoning done above for $\mathbf{b_n}$ ultimately one concludes that $\mathcal{L}(t, \mathcal{T}(M)) \twoheadrightarrow_{\eta^\ell} x$. □

Lemma 4.5.28. *Let $T = \lambda\vec{x}y.zT_1 \cdots T_{k+1}$ be a Böhm like tree such that T_{k+1} is finite, $T_{k+1} \twoheadrightarrow_\eta y$ and $y \notin \mathrm{fv}(zT_1 \cdots T_k)$. Then $\mathcal{L}\big(\mathcal{T}(T)\big) \Rightarrow_{\eta^\ell} \mathcal{L}\big(\mathcal{T}(\lambda\vec{x}.zT_1 \cdots T_k)\big)$.*

Proof. We first take $t \in \mathcal{T}(T)$, namely $t = \lambda\vec{x}y.z\,\mathbf{b_1} \cdots \mathbf{b_{k+1}}$ with $\mathbf{b_i} \in \mathcal{M}_f(\mathcal{T}(T_i))$ for all $i \in \{1, \ldots, k+1\}$, and show that $\mathcal{L}\big(t, \mathcal{T}(T)\big) \twoheadrightarrow_{\eta^\ell} \mathcal{L}\big(t', \mathcal{T}(\lambda\vec{x}.zT_1 \cdots T_k)\big)$ holds when we take $t' := \lambda\vec{x}.z\,\mathbf{b_1} \cdots \mathbf{b_k} \in \mathcal{L}\big(\mathcal{T}(\lambda\vec{x}.zT_1 \cdots T_k)\big)$. By Definition 4.5.4 we have

$$\mathcal{L}(t, \mathcal{T}(T)) \quad = \quad \lambda\vec{x}y.z\,\mathcal{L}\big(\mathbf{b_1}, \mathcal{M}_f(\mathcal{T}(T_1))\big) \cdots \mathcal{L}\big(\mathbf{b_{k+1}}, \mathcal{M}_f(\mathcal{T}(T_{k+1}))\big)\,. \tag{84}$$

By Remark 4.5.9 we have that $y \notin \mathrm{fv}(zT_1 \cdots T_k)$ implies

$$y \;\notin\; \widetilde{\mathrm{fv}}\big(z\,\mathcal{L}\big(\mathbf{b_1}, \mathcal{M}_f(\mathcal{T}(T_1))\big) \;\cdots\; \mathcal{L}\big(\mathbf{b_k}, \mathcal{M}_f(\mathcal{T}(T_k))\big)\big)\,. \tag{85}$$

Suppose that $\mathbf{b_{k+1}} = [t_1, \ldots, t_n]$ for some $n \in \mathbb{N}$. We split into cases depending on n.

Case $n = 0$. The finite Böhm-like tree T_{k+1} is also $\bot$-free, because $T_{k+1} \twoheadrightarrow_\eta y$. Therefore there exist $s \in \mathcal{T}(T_{k+1}) = \bigcup \mathcal{M}_f(\mathcal{T}(T_{k+1}))$ without empty bags such that $s \twoheadrightarrow_{\eta'} y$. Hence

$$\mathcal{L}\big(\mathbf{b_{k+1}}, \mathcal{M}_f(\mathcal{T}(T_{k+1}))\big) \quad = \quad \mathcal{L}\big(\mathbf{1}, \mathcal{M}_f(\mathcal{T}(T_{k+1}))\big) \quad = \quad \mathbf{1}^{y}_{\eta(y)}\,. \tag{86}$$

We have

$$\begin{aligned}\mathcal{L}\big(t, \mathcal{T}(T)\big) &= \lambda\vec{x}y.z\,\mathcal{L}\big(\mathbf{b_1}, \mathcal{M}_f(\mathcal{T}(T_1))\big) \cdots \mathcal{L}\big(\mathbf{b_{k+1}}, \mathcal{M}_f(\mathcal{T}(T_{k+1}))\big) && \text{by (84)}\\ &= \lambda\vec{x}y.z\,\mathcal{L}\big(\mathbf{b_1}, \mathcal{M}_f(\mathcal{T}(T_1))\big) \cdots \mathcal{L}\big(\mathbf{b_k}, \mathcal{M}_f(\mathcal{T}(T_k))\big)\,\mathbf{1}^{y}_{\eta(y)} && \text{by (86)}\\ &\rightarrow_{\eta^\ell} \lambda\vec{x}.z\,\mathcal{L}(\mathbf{b_1}, \mathcal{M}_f(\mathcal{T}(T_1))) \cdots \mathcal{L}(\mathbf{b_k}, \mathcal{M}_f(\mathcal{T}(T_k))) && \text{by (85)}\\ &= \mathcal{L}\big(\lambda\vec{x}.z\mathbf{b_1} \cdots \mathbf{b_k}, \mathcal{T}(\lambda\vec{x}y.zT_1 \cdots T_k)\big) && \text{by Def. 4.5.4}\end{aligned}$$

so we are done.

Case $n > 0$. In this case $t_i \in \mathcal{T}(T_{k+1})$ for $i \in \{1, \ldots, n\}$ and by Definition 4.5.4

$$\mathcal{L}\big(\mathbf{b_{k+1}}, \mathcal{M}_f(\mathcal{T}(T_{k+1}))\big) \;=\; \big[\mathcal{L}(t_1, \mathcal{T}(T_{k+1})), \ldots, \mathcal{L}(t_n, \mathcal{T}(T_{k+1}))\big]\,. \tag{87}$$

Since $T_{k+1} \twoheadrightarrow_\eta y$, by Lemma 4.5.27 then

$$\mathcal{L}(t_i, \mathcal{T}(T_{k+1})) \;\twoheadrightarrow_{\eta^\ell}\; y \qquad \text{for every } i \in \{1, \ldots, n\}\,. \tag{88}$$

Therefore

$$\begin{array}{rlr}
\mathcal{L}\big(t, \mathcal{T}(T)\big) & = \lambda\vec{x}y.z\,\mathcal{L}\big(\mathbf{b_1}, \mathcal{M}_f(\mathcal{T}(T_1))\big) \cdots \mathcal{L}\big(\mathbf{b_{k+1}}, \mathcal{M}_f(\mathcal{T}(T_{k+1}))\big) & \text{by Def. 4.5.4} \\
& \twoheadrightarrow_{\eta^\ell} \lambda\vec{x}y.z\,\mathcal{L}\big(\mathbf{b_1}, \mathcal{M}_f(\mathcal{T}(T_1))\big) \cdots \mathcal{L}\big(\mathbf{b_k}, \mathcal{M}_f(\mathcal{T}(T_k))\big)\,[y^n] & \text{by (87) and (88)} \\
& \to_{\eta^\ell} \lambda\vec{x}.z\,\mathcal{L}\big(\mathbf{b_1}, \mathcal{M}_f(\mathcal{T}(T_1))\big) \cdots \mathcal{L}\big(\mathbf{b_k}, \mathcal{M}_f(\mathcal{T}(T_k))\big) & \text{by Def. 4.5.10}
\end{array}$$

as it was to be proved.

Secondly, we must show that for every $t' \in \mathcal{T}(\lambda\vec{x}.z\,T_1 \cdots T_k)$, namely for every t' of the form $\lambda\vec{x}.z\,\mathbf{b_1} \cdots \mathbf{b_k}$ with $\mathbf{b_i} \in \mathcal{M}_f(\mathcal{T}(T_i))$ for all $i \in \{1, \ldots, k\}$, there exists $t \in \mathcal{T}(T)$ such that $\mathcal{L}\big(t, \mathcal{T}(T)\big) \twoheadrightarrow_{\eta^\ell} \mathcal{L}\big(t', \mathcal{T}(\lambda\vec{x}.zT_1 \cdots T_k)\big)$. At this purpose it is enough to assume $t := \lambda\vec{x}y.z\,\mathbf{b_1} \cdots \mathbf{b_k}\,\mathbf{1}$. The proof then proceeds exactly like **Case** $n = 0$ above. □

Corollary 4.5.29. *Let* $T = \lambda\vec{x}\,y.z\,T_1 \cdots T_{k+1}$ *be a Böhm like tree such that* T_{k+1} *is finite,* $T_{k+1} \twoheadrightarrow_\eta y$ *and* $y \notin \mathrm{fv}(z\,T_1 \cdots T_k)$*. Then* $\mathrm{nf}_{\eta^\ell}\big(\mathcal{L}\big(\mathcal{T}(T)\big)\big) = \mathrm{nf}_{\eta^\ell}\big(\mathcal{L}\big(\mathcal{T}(\lambda\vec{x}.zT_1 \cdots T_k)\big)\big)$.

Proof. Let $t \in \mathrm{nf}_{\eta^\ell}\big(\mathcal{T}(\mathcal{L}(T))\big)$, namely let $t_1 \in \mathcal{T}(\mathcal{L}(T))$ be such that t is the η^ℓ-nf of t_1. By Lemma 4.5.28 there is $t_2 \in \mathcal{L}\big(\mathcal{T}((\lambda\vec{x}.zT_1 \cdots T_k){\downarrow})\big)$ such that $t_1 \twoheadrightarrow_{\eta^\ell} t_2$. Hence, by confluence of $\twoheadrightarrow_{\eta^\ell}$, the term t is also the η^ℓ-nf of t_2. So $t \in \mathrm{nf}_{\eta^\ell}\big(\mathcal{L}\big(\mathcal{T}((\lambda\vec{x}.zT_1 \cdots T_k){\downarrow})\big)\big)$. This proves that $\mathrm{nf}_{\eta^\ell}\big(\mathcal{T}(\mathcal{L}(T))\big) \subseteq \mathrm{nf}_{\eta^\ell}\big(\mathcal{L}\big(\mathcal{T}((\lambda\vec{x}.zT_1 \cdots T_k){\downarrow})\big)\big)$.

Let $t \in \mathrm{nf}_{\eta^\ell}\big(\mathcal{L}\big(\mathcal{T}((\lambda\vec{x}.zT_1 \cdots T_k){\downarrow})\big)\big)$, i.e. let $t_2 \in \mathcal{L}\big(\mathcal{T}((\lambda\vec{x}.zT_1 \cdots T_k){\downarrow})\big)$ be such that t is the η^ℓ-nf of t_2. By Lemma 4.5.28 there is $t_1 \in \mathcal{T}(\mathcal{L}(T))$ such that $t_1 \twoheadrightarrow_{\eta^\ell} t_2$. Then clearly t is the η^ℓ-nf of t_1. So $t \in \mathrm{nf}_{\eta^\ell}\big(\mathcal{T}(\mathcal{L}(T))\big)$ and $\mathrm{nf}_{\eta^\ell}\big(\mathcal{T}(\mathcal{L}(T))\big) \supseteq \mathrm{nf}_{\eta^\ell}\big(\mathcal{L}\big(\mathcal{T}((\lambda\vec{x}.zT_1 \cdots T_k){\downarrow})\big)\big)$. □

Lemma 4.5.30. *Let* $T \in \Lambda^{\mathcal{B}}$*. Then* $\ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathcal{T}(T)) \urcorner = \mathcal{T}(\ulcorner \mathrm{nf}_{\eta^e}\mathcal{E}(T^*) \urcorner)$.

Proof. We proceed by coinduction on T.

In case $T = \perp$ we have

$$\ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathcal{T}(\perp)) \urcorner = \ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\emptyset) \urcorner = \emptyset = \mathcal{T}(\perp) = \mathcal{T}(\ulcorner \mathrm{nf}_{\eta^e}\mathcal{E}(\perp, \perp^*) \urcorner) = \mathcal{T}(\ulcorner \mathrm{nf}_{\eta^e}\mathcal{E}(\perp^*) \urcorner)\,.$$

so the thesis is proved.

Otherwise, the Böhm-like tree T can be written in a unique way as

$$T \;=\; \lambda x_1 \ldots x_n y_1 \ldots y_m.z\,T_1 \cdots T_k\,T'_1 \cdots T'_m$$

for some $n, m, k \in \mathbb{N}$ such that:

1. T'_i is finite, $T'_i \twoheadrightarrow_\eta y_i$ and $y_i \notin \mathrm{fv}\big(z\,T_1 \cdots T_k T'_1 \cdots T'_{i-1}\big)$, for all $i \in \{1, \ldots, m\}$;
2. T_k is infinite, or T_k is finite but does not η-reduce to x_n, or $x_n \in \mathrm{fv}(z\,T_1 \cdots T_k)$.

The following equalities hold:

$$
\begin{array}{ll}
\ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathcal{T}(T))\urcorner = \ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathcal{T}(\lambda\vec{x}.z\,T_1\cdots T_k))\urcorner & \text{by Cor. 4.5.29.}\\
= \ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\lambda\vec{x}.z\,\mathcal{M}_f(\mathcal{T}(T_1))\cdots\mathcal{M}_f(\mathcal{T}(T_k)))\urcorner & \text{by def. of } \mathcal{T}(-)\\
= \ulcorner \mathrm{nf}_{\eta^\ell}(\lambda\vec{x}.z\,\mathcal{M}_f(\mathcal{L}(\mathcal{T}(T_1)))\cdots\mathcal{M}_f(\mathcal{L}(\mathcal{T}(T_k))))\urcorner & \text{by def. of } \mathcal{L}(-)\\
= \ulcorner \lambda\vec{x}.z\,\mathcal{M}_f(\mathrm{nf}_{\eta^\ell}(\mathcal{L}(\mathcal{T}(T_1))))\cdots\mathcal{M}_f(\mathrm{nf}_{\eta^\ell}(\mathcal{L}(\mathcal{T}(T_k))))\urcorner & \text{by Def. 4.5.10}\\
= \lambda\vec{x}.z\,\mathcal{M}_f(\ulcorner \mathrm{nf}_{\eta^\ell}(\mathcal{L}(\mathcal{T}(T_1)))\urcorner)\cdots\mathcal{M}_f(\ulcorner \mathrm{nf}_{\eta^\ell}(\mathcal{L}(\mathcal{T}(T_k)))\urcorner) & \text{by def. of } \ulcorner - \urcorner\\
= \lambda\vec{x}.z\,\mathcal{M}_f(\mathcal{T}(\ulcorner \mathrm{nf}_{\eta^e}(\mathcal{E}(T_1^*))\urcorner))\cdots\mathcal{M}_f(\mathcal{T}(\ulcorner \mathrm{nf}_{\eta^e}(\mathcal{E}(T_k^*))\urcorner)) & \text{by coIH}\\
= \mathcal{T}\big(\lambda\vec{x}.z\,\mathrm{nf}_{\eta^e}(\ulcorner \mathcal{E}(T_1^*)\urcorner)\cdots\mathrm{nf}_{\eta^e}(\ulcorner \mathcal{E}(T_k^*)\urcorner)\big) & \text{by def. of } \mathcal{T}(-)\\
= \mathcal{T}\big(\ulcorner \lambda\vec{x}.z\,\mathrm{nf}_{\eta^e}(\mathcal{E}(T_1^*))\cdots\mathrm{nf}_{\eta^e}(\mathcal{E}(T_k^*))\urcorner\big) & \text{by def. of } \ulcorner - \urcorner\\
= \mathcal{T}\big(\ulcorner \lambda\vec{x}.z\,\mathrm{nf}_{\eta^e}(\mathcal{E}(T_1^*))\cdots\mathrm{nf}_{\eta^e}(\mathcal{E}(T_k^*))\urcorner\big) \cup \mathcal{T}(\bot) & \text{since } \mathcal{T}(\bot) = \emptyset\\
= \mathcal{T}\big(\ulcorner \lambda\vec{x}.z\,\mathrm{nf}_{\eta^e}(\mathcal{E}(T_1^*))\cdots\mathrm{nf}_{\eta^e}(\mathcal{E}(T_k^*))\urcorner\big) \cup \mathcal{T}\big(\ulcorner \mathcal{E}(\bot,\lambda\vec{x}.z\,T_1^*\cdots T_k^*)\urcorner\big) & \text{by Def. 4.5.16}\\
= \mathcal{T}\big(\ulcorner \lambda\vec{x}.z\,\mathrm{nf}_{\eta^e}(\mathcal{E}(T_1^*))\cdots\mathrm{nf}_{\eta^e}(\mathcal{E}(T_k^*))\urcorner \cup \{\ulcorner \mathcal{E}(\bot,\lambda\vec{x}.z\,T_1^*\cdots T_k^*)\urcorner\}\big) & \text{by def. of } \mathcal{T}(-)\\
= \mathcal{T}\big(\ulcorner \mathrm{nf}_{\eta^e}(\lambda\vec{x}.z\,\mathcal{E}(T_1^*)\cdots\mathcal{E}(T_k^*))\urcorner \cup \{\ulcorner \mathcal{E}(\bot,\lambda\vec{x}.z\,T_1^*\cdots T_k^*)\urcorner\}\big) & \text{by Def. 4.5.20}\\
= \mathcal{T}\big(\ulcorner \mathrm{nf}_{\eta^e}\mathcal{E}((\lambda x_1\ldots x_n.zT_1\cdots T_k)\downarrow)\urcorner\big) & \text{by Def. 4.5.16}\\
= \mathcal{T}(\ulcorner \mathrm{nf}_{\eta^e}\mathcal{E}(T^*)\urcorner) & \text{by 1 and Cor. 4.5.29.}
\end{array}
$$

and we are done. □

Proposition 4.5.31. *Let* $M \in \Lambda$*. Then* $\ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathcal{T}(\mathrm{BT}(M)))\urcorner = \mathcal{T}(\ulcorner \mathrm{nf}_{\eta^e}\mathcal{E}(\mathrm{BT}(M)^*)\urcorner)$.

Proof. We just apply Lemma 4.5.30 for $T = \mathrm{BT}(M)$. □

We can finally prove the main result of the section.

Theorem 4.5.32. *Let* λ*-term* M*. Then* $\mathcal{T}^\eta(M) = \mathcal{T}(\mathrm{BT}^\eta(M))$.

Proof. We have the following chain of equalities

$$
\begin{array}{rcll}
\mathcal{T}^\eta(M) & = & \ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathrm{nf}_\beta\mathcal{T}(M))\urcorner & \text{by Def. 4.5.13}\\
& = & \ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathcal{T}(\mathrm{BT}(M)))\urcorner & \text{by Theor. 1.6.4}\\
& = & \mathcal{T}(\ulcorner \mathrm{nf}_{\eta^e}\mathcal{E}(\mathrm{BT}(M)^*)\urcorner) & \text{by Prop. 4.5.31}\\
& = & \mathcal{T}(\mathrm{BT}^\eta(M)^*) & \text{by Prop. 4.5.26}
\end{array}
$$

so we are done.

Corollary 4.5.33. *Let* $M, N \in \Lambda$*. Then* $\mathrm{BT}^\eta(M)^* \subseteq \mathrm{BT}^\eta(N)^*$ *if and only if* $\mathcal{T}^\eta(M) \subseteq \mathcal{T}^\eta(N)$.

Proof. ($\Rightarrow$) Using Theorem 4.5.32, the definition of the Taylor expansion for Böhm-like trees and the hypothesis we get

$$\mathcal{T}^\eta(M) = \mathcal{T}(\mathrm{BT}^\eta(M)) = \bigcup_{a\in\mathrm{BT}^\eta(M)^*}\mathcal{T}(a) \subseteq \bigcup_{a\in\mathrm{BT}^\eta(N)^*}\mathcal{T}(a) = \mathcal{T}(\mathrm{BT}^\eta(N)) = \mathcal{T}^\eta(N)\,.$$

($\Leftarrow$) Using Theorem 4.5.32, the definition of the Taylor expansion for Böhm-like trees and the hypothesis we get

$$\bigcup_{a \in \mathrm{BT}^{\eta}(M)^*} \mathcal{T}(a) = \mathcal{T}(\mathrm{BT}^{\eta}(M)) = \mathcal{T}^{\eta}(M) \subseteq \mathcal{T}^{\eta}(N) = \mathcal{T}(\mathrm{BT}^{\eta}(N)) = \bigcup_{a \in \mathrm{BT}^{\eta}(N)^*} \mathcal{T}(a)\,.$$

So given any $a \in \mathrm{BT}^{\eta}(M)^*$ we have $\mathcal{T}(a) \subseteq \mathcal{T}(\mathrm{BT}^{\eta}(N))$. By Lemma 1.6.3 we can conclude that $a \in \mathrm{BT}^{\eta}(N)^*$. □

Corollary 4.5.34. *Let* $M, N \in \Lambda$. *Then* $M =_{\mathcal{H}^+} N$ *if and only if* $\mathcal{T}^{\eta}(M) = \mathcal{T}^{\eta}(N)$.

Proof. As seen in § 1.4 we have $M =_{\mathcal{H}^+} N$ if and only if $\mathrm{BT}^{\eta}(M) = \mathrm{BT}^{\eta}(N)$, which is $\mathrm{BT}^{\eta}(M)^* = \mathrm{BT}^{\eta}(N)^*$. This is equivalent to $\mathcal{T}^{\eta}(M) = \mathcal{T}^{\eta}(N)$ by Corollary 4.5.33. □

5 CHARACTERIZING MORRIS'S THEORY: λ-KÖNIG RELATIONAL GRAPH MODELS

A key aim of this thesis is to find models of Morris's observational theory. The notion of uniformly bottomless ergm, seen in Chapter 4, provides us with plenty of such models (actually infinitely many of them). However, that notion is only a *sufficient* condition for the full abstraction. In this final chapter we achieve a much more complicated goal. We give an *exhaustive* answer to the problem within our semantics: we find *necessary and sufficient* conditions on rgm's to be fully abstract for Morris's theory. (Actually, these conditions will extend immediately to all relational models.)

Results of full abstraction are rarely as exhaustive as this, at least in the context of the untyped λ-calculus. As a matter of fact, in the literature there is only one other characterization of an observational theory in a given class of models. Such result concerns the observational theory with head normal forms as observables, namely $\mathcal{H}^*$. In [Bre16] (first published as [Bre14]) Breuvart presented a necessary and sufficient condition on Krivine's models to induce $\sqsubseteq_{\mathcal{H}^*}$. Krivine's models [Kri90] are a (large) subclass of Scott's continuous semantics, which is not the one that we use here. Nevertheless, there are some informal similarities between Krivine's models and rgm's, specifically the fact that both semantics can be handled using intersection types. So it should not be surprising if we state that Breuvart's theorem was of inspiration for the one that we provide here.

Here the full abstraction problem is considered *inequationally*, that is w.r.t. $\sqsubseteq_{\mathcal{H}^+}$. However, at the end of the process we will also find out that solving the problem inequationally is equivalent to solving it *equationally*. So there is no harm if we refer just to the equational theory $\mathcal{H}^+$ in this introduction. We have already seen in Chapter 4 that all ergm's validate *at least* the equations of $\mathcal{H}^+$, as a consequence of the fact that the extensional Böhm trees form a syntactic model of $\mathcal{H}^+$. Once again the difficult part is to find a condition guaranteeing that an ergm does not equate *more* than that. So we need to analyze the equations in $\mathcal{H}^* - \mathcal{H}^+$ (remember that the λ-theory of any rgm is sensible, hence included in the maximal sensible λ-theory $\mathcal{H}^*$). The purpose is to *avoid* these equations: whenever $(M, N) \in \mathcal{H}^* - \mathcal{H}^+$ we wish to *separate* M and N in the model, namely we want $[\![M]\!] \neq [\![N]\!]$.

We show that if two λ-terms M, N are equal in $\mathcal{H}^*$, but not in $\mathcal{H}^+$, then their Böhm trees are similar — meaning that their nodes can be equated by some η-conversions — but with the following relevant fact: there must exist a position σ where they differ because of an *infinitely deep* η-expansion. In other words, $BT(M)$ and $BT(N)$ are equal up to some conversions $\twoheadrightarrow_\eta$, and in particular in at least a position σ the infinite η-expansion cannot be finitely deep, i.e. $BT(N)_\sigma \twoheadrightarrow_\eta BT(M)_\sigma$ but $BT(N)_\sigma \not\twoheadrightarrow_\eta^{\mathrm{fin}} BT(M)_\sigma$. Such a position is called *Morris separator* for M and N. One may notice here a certain analogy with the notion of r-separator introduced in § 3.3: in that case the existence of an r-separator characterized the fact that $(M, N) \in \mathcal{H}^* - (=_r)$; here the existence of a Morris separator will characterize the fact that $(M, N) \in \mathcal{H}^* - \mathcal{H}^+$.

Thanks to an *ad hoc* refined version of the Böhm-out technique, we prove that it is always possible to extract such a difference existing in position σ by means of a suitable context. Precisely, we define a context $C[-]$ such that $\mathrm{BT}(C[M]) = \mathbf{I}$ whereas $\mathrm{BT}(C[N]) \twoheadrightarrow_\eta \mathbf{I}$ and $\mathrm{BT}(C[N]) \not\twoheadrightarrow^{\mathsf{fin}}_\eta \mathbf{I}$, i.e. $\mathrm{BT}(C[N])$ is an infinitely deep η-expansion of $\mathbf{I}$. This allows us to reduce the quest for a fully abstract ergm of $\mathcal{H}^+$ to another problem: *separating in an* ergm *the* λ*-term* $\mathbf{I}$ *from all its infinitely deep* η*-expansions*.

For this purpose, we introduce the notion of λ*-König model*. Let us explain the rough idea. For every infinite computable tree T let us call $B_T \in \Lambda^{\mathcal{B}}$ the infinite η-expansion of $\mathbf{I}$ with underlying tree T. One should realize here that these B_T's are exactly *all* infinitely deep η-expansions of $\mathbf{I}$. Intuitively an rgm is λ*-König* when every infinite computable tree T has an infinite branch (which always exists by König's lemma) *witnessed* by some type σ_T of the model. The definition of such a *witness* is given by a refined version of the unfolding of types exploited in Chapter 4, and it assures that $\sigma_T \to \sigma_T \notin [\![B_T]\!]$. Since on the other hand $\sigma_T \to \sigma_T \in [\![\mathbf{I}]\!]$, the separation is achieved.

In the end the main result of this chapter, arguably of the thesis, states that an rgm is fully abstract for $\mathcal{H}^+$ if and only if it is extensional and λ-König (Theorem 5.6.2). This actually characterizes the full abstraction for $\mathcal{H}^+$ in the whole relational semantics (Corollary 5.6.3).

As a byproduct of our version of the Böhm-out, we get another purely syntactic result. We show that $\mathcal{H}^+$ satisfies the ω-rule, the property of extensionality recalled in § 1.2 which is stronger than the η-rule, as explained in § 1.5. This gives a positive answer to a long-standing open problem (cf. [Bar84, § 17.4]).

PLAN OF THE CHAPTER. In order to approach gradually to the desired necessary and sufficient condition for the full abstraction, in § 5.1 we make some heuristic considerations inspired by the weaker condition already found in Chapter 4. In § 5.2 we associate with every infinite recursive tree T a closed λ-term $\mathbf{J}_T$ such that $\mathrm{BT}(\mathbf{J}_T)$ is the (only) infinitely deep η-expansion of $\mathbf{I}$ whose underlying naked tree is T (namely, $\mathrm{BT}(\mathbf{J}_T)$ is the Böhm-like tree called B_T in this informal introduction). In § 5.3 we introduce the notion of Morris separator, and show how to perform a Böhm-out whenever such a separator exists. In § 5.4 we use the Böhm-out Lemma to prove that the λ-theory $\mathcal{H}^+$ satisfies the ω-rule. In § 5.5 we introduce the notion of witness for an infinite computable tree T and prove that the set of witnesses for T is exactly the difference $[\![\mathbf{I}]\!] - [\![\mathbf{J}_T]\!]$. In § 5.6 we give the notion of λ-König rgm. Then we use the Böhm-out Lemma and the characterization of $[\![\mathbf{I}]\!] - [\![\mathbf{J}_T]\!]$ mentioned above to conclude that an rgm induces the preorder $\sqsubseteq_{\mathcal{H}^+}$ if and only if it is extensional and λ-König. We also provide examples of λ-König ergm (some which are not already contained in the class of fully abstract models from the previous chapter).

5.1 BY WAY OF A PREAMBLE: TWO CANDIDATE FULLY ABSTRACT MODELS?

This section features mainly informal considerations. Readers only interested in the formal results can skip to § 5.2.

To introduce the main idea of this chapter, it may help to start from what we have achieved so far, i.e. the notion of uniformly bottomlessness developed in § 4.1-4.3. For every

$k \in \mathbb{N}$ it provides the model with some type σ_k satisfying the following property: when σ_k is *unfolded* as

$$
\begin{array}{rcl}
\sigma_k & \simeq & \mu_0^0 \to \cdots \to \mu_k^0 \to \tau_k \\
\sigma_k^1 & \simeq & \mu_0^1 \to \cdots \to \mu_k^1 \to \tau_k^1 \\
& \vdots & \\
\sigma_k^n & \simeq & \mu_0^n \to \cdots \to \mu_k^n \to \tau_k^n \\
& \vdots &
\end{array}
$$

with $\sigma_k^{n+1} \in \bigcup_{i=1}^{k} \mu_i^n$ for all $n \in \mathbb{N}$, then $\mu_i^n \neq \omega$ for all $n \in \mathbb{N}$ and for all $i \in \{0, \dots, k\}$. As explained in § 4.3, this property helps us to characterize β-normalizability within the model. Even if in this chapter we *do not* exploit β-normalizability to achieve the full abstraction, nevertheless a question arise: do we really need *all* those intersections μ_i^n to be non-empty? Can we refine the notion by requiring *as few non-empty* μ_i^n *as possible* along the unfoldling?

For instance, one may ask a model to have for all $f : \mathbb{N} \to \mathbb{N}$ a type σ_f that can be infinitely unfolded as

$$
\begin{array}{rcl}
\sigma_f & \simeq & \mu_0^0 \to \cdots \to \mu_{f(0)}^0 \to \tau_k \\
\sigma_f^1 & \simeq & \mu_0^1 \to \cdots \to \mu_{f(1)}^1 \to \tau_k^1 \\
& \vdots & \\
\sigma_f^n & \simeq & \mu_0^n \to \cdots \to \mu_{f(n)}^n \to \tau_k^n \\
& \vdots &
\end{array}
$$

with $\sigma_f^{n+1} \in \mu_{f(n)}^n$ for all $n \in \mathbb{N}$. Does this condition provide *enough control* on the proliferation of ω to assure a clear separation in the model (i.e. using typings) between what is β-normalizable and what is not? We could also weaken this condition by requiring the existence of such a σ_f not for every $f : \mathbb{N} \to \mathbb{N}$, but just for every *computable* $f : \mathbb{N} \to \mathbb{N}$. What about this weaker alternative?

Here are two concrete examples of rgm's satisfying these two conditions.

Example 5.1.1. We call $\mathcal{L}$ the free completion

$$\mathcal{L} \;:=\; \overline{(\,L\,,\,j\,)}$$

of the partial pair $(\,L\,,\,j\,)$ defined as follows. The $2^{\aleph_0}$ elements of the family

$$L \;:=\; \{\,*\,\} \cup \left\{\beta_f^{n,k}\right\}_{f \in \mathbb{N}^{\mathbb{N}}}^{n \in \mathbb{N},\, k \leqslant f(n)}$$

are pairwise distinct and are not pairs. The partial function $j : \mathcal{M}_f(L) \times L \rightharpoonup L$ maps

$$([\,],*) \;\mapsto\; * \;,$$

for all $f : \mathbb{N} \to \mathbb{N}$, for all $n \in \mathbb{N}$ and for all $0 < k \leqslant f(n)$

$$\left([\,],\beta_f^{n,k-1}\right) \;\mapsto\; \beta_f^{n,k} \;,$$

for all $f : \mathbb{N} \to \mathbb{N}$ and for all $n \in \mathbb{N}$

$$\left(\left[\beta_f^{n+1, f(n+1)}\right], *\right) \quad \mapsto \quad \beta_f^{n,0} \ ,$$

and is undefined on any other $(m, a) \in \mathcal{M}_f(L) \times L$.

The rgm $\mathcal{L}$ is extensional, since j is surjective.

In particular notice that for all $f : \mathbb{N} \to \mathbb{N}$ and for all $n \in \mathbb{N}$ the completion $\bar{j}$ maps

$$\overbrace{([\ ], ([\ ], \cdots ([\ ]}^{f(n) \text{ times}}, \left(\left[\beta_f^{n+1, f(n+1)}\right], *\right)\right) \cdots)) \quad \mapsto \quad \beta_f^{n, f(n)} .$$

By convenience let us rename $\alpha_f^n := \beta_f^{n, f(n)}$ for all $n \in \mathbb{N}$. Then one can think of $\mathcal{G}$ as the ergm relying on the basic equations

$$* \ \simeq \ \omega \to *$$

and for every $f : \mathbb{N} \to \mathbb{N}$

$$\alpha_f^0 \quad \simeq \quad \overbrace{\omega \to \cdots \to \omega}^{f(0) \text{ times}} \to \alpha_f^1 \to *$$

$$\alpha_f^1 \quad \simeq \quad \overbrace{\omega \to \cdots \to \omega}^{f(1) \text{ times}} \to \alpha_f^2 \to *$$

$$\vdots$$

$$\alpha_f^n \quad \simeq \quad \overbrace{\omega \to \cdots \to \omega}^{f(n) \text{ times}} \to \alpha_f^{n+1} \to *$$

$$\vdots$$

Example 5.1.2. Let $\mathcal{L}^{rec}$ be defined just like the rgm $\mathcal{L}$ above, but considering only *recursive* functions from $\mathbb{N}$ to $\mathbb{N}$. In other words, we repeat the construction given in Example 5.1.1 but restricting to the set of $\aleph_0$ atoms

$$L^{rec} \ := \ \{*\} \cup \left\{\beta_f^{n,k}\right\}_{f \in \mathbb{N}^{\mathbb{N}},\, f \text{ computable}}^{n \in \mathbb{N},\, k \leqslant f(n)}$$

In this case also by convenience we rename $\alpha_f^n := \beta_f^{n, f(n)}$ for all $n \in \mathbb{N}$.

Is any of these two models fully abstract for $\mathcal{H}^+$? Clearly, neither $\mathcal{L}$ nor $\mathcal{L}^{rec}$ is uniformly bottomless. So we cannot answer the question for the moment. But we will at the end of this chapter. Anyhow, $\mathcal{L}$ and $\mathcal{L}^{rec}$ seem to be good candidates. After all, in both rgm's we have $[\![\mathbf{I}]\!] \neq [\![\mathbf{J}]\!]$. Indeed it easy to realize that, for the constant function $f : n \in \mathbb{N} \mapsto 0 \in \mathbb{N}$, one has $\alpha_f \to \alpha_f \notin [\![a]\!]$ for every $a \in BT(\mathbf{J})$, hence by the Böhm Approximation Theorem $\alpha_f \to \alpha_f \in [\![\mathbf{I}]\!] - [\![\mathbf{J}]\!]$. Now, the property $[\![\mathbf{I}]\!] \neq [\![\mathbf{J}]\!]$ may not be enough to get the full abstraction, but it is a clue. In fact, more than a clue: it is a starting point towards the solution, as it will be clear in the rest of this chapter.

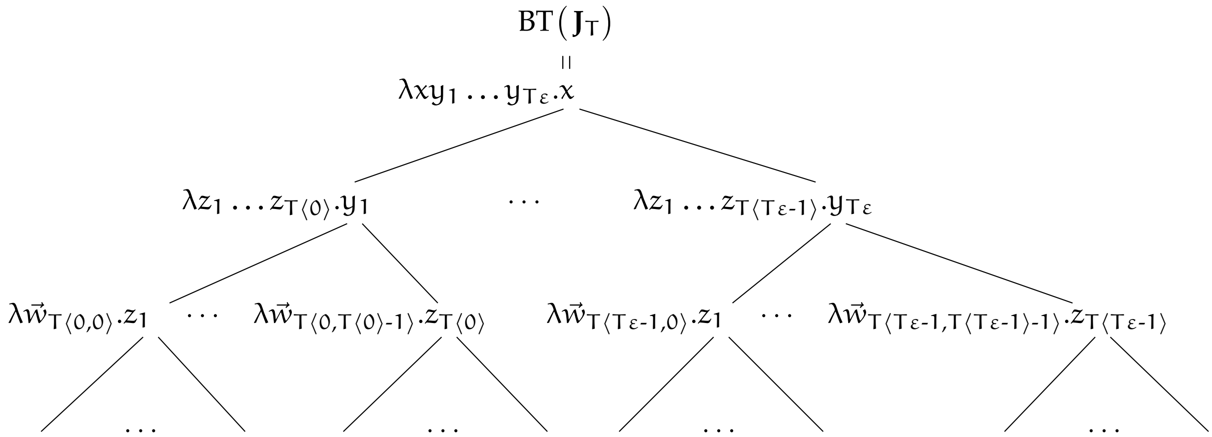


Figure 6: The Böhm tree of $\mathbf{J}_T$, an infinite η-expansion of $\mathbf{I}$ following $T \in \mathbb{T}^{\infty}_{\text{rec}}$. To lighten the notations in this figure we write $T\sigma$ rather than $T(\sigma)$ and we let $\vec{w}_n := w_1, \dots, w_n$.

5.2 INFINITE η-EXPANSIONS OF $\mathbf{I}$

So far we have made use of Wadsworth's combinator $\mathbf{J}$ [Wad76]. In particular, we have regarded its Böhm tree as a kind of paradigmatic infinite η-expansion of the identity. But $BT(\mathbf{J})$ is not the only possible such η-expansion. In fact, for each $T \in \mathbb{T}^{\infty}_{\text{rec}}$ there exists one and only one infinite η-expansion of $\mathbf{I}$ *that follows* T, meaning that T is its underlying naked tree, as in Figure 6. For every $T \in \mathbb{T}^{\infty}_{\text{rec}}$ we define here a λ-term $\mathbf{J}_T$ such that $BT(\mathbf{J}_T)$ is *the* infinite η-expansion of $\mathbf{I}$ following T.

Let us fix an effective encoding $@ : \mathbb{N}^* \to \mathbb{N}$. By that we mean that $@$ is bijective and both $@$ and $@^{-1}$ are computable functions.

Proposition 5.2.1. *The function* $\mathsf{P} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *defined by* $(@\varphi\,,\,n) \mapsto @(\varphi.n)$ *is computable.*

Proof. We describe how to compute P on any given $(m, n) \in \mathbb{N} \times \mathbb{N}$. Since $@^{-1}$ is computable, we can compute $@^{-1}m$.

In case $@^{-1}m$ is undefined we set $\mathsf{P}(m, n)$ to be undefined.

If $@^{-1}m = \varphi$ then, by computability of $@$, we can compute $@(\varphi.n)$. In the end we set $\mathsf{P}(m, n) := @(\varphi.n)$. □

By Church's thesis Proposition 5.2.1 gives the following fact.

Lemma 5.2.2. *There is* $\mathtt{Cons} \in \Lambda^0$ *such that* $\mathtt{Cons}\,\overline{@\varphi}\,\overline{n} =_\beta \overline{@(\varphi.n)}$ *for all* $\varphi \in \mathbb{N}^*$ *and* $n \in \mathbb{N}$.

Notation. From now on $\overline{\varphi}$ stands for $\overline{@\varphi}$, for all $\varphi \in \mathbb{N}^*$.

Let $\# : \Lambda \to \mathbb{N}$ be a Gödel numbering, i.e. any encoding of all λ-terms into $\mathbb{N}$.

Theorem 5.2.3. *[Bar84, Proposition 8.2.2.] Let* $(M_n)_{n\in\mathbb{N}}$ *be a sequence of closed λ-terms such that* $n \in \mathbb{N} \mapsto \#M_n \in \mathbb{N}$ *is computable. There exists* $M \in \Lambda^0$ *such that* $M\,\overline{n} =_\beta M_n$ *for all* $n \in \mathbb{N}$.

Lemma 5.2.4. *Let* $T \in \mathbb{T}_{\text{rec}}$. *Then there exists* $E^T \in \Lambda^0$ *such that for all* $\varphi \in \text{dom}(T)$

$$E^T\overline{\varphi} \;=_\beta\; \lambda r\,x\,x_1 \dots x_{T(\varphi)}.x\left(r\left(\mathtt{Cons}\,\overline{\varphi}\,\overline{0}\right)x_1\right)\cdots\left(r\left(\mathtt{Cons}\,\overline{\varphi}\,\overline{T(\varphi)-1}\right)x_{T(\varphi)}\right).$$

Proof. Let $X := @[\mathrm{dom}(T)] = \{ n \in \mathbb{N} \mid \text{there exists } \varphi \in \mathrm{dom}(T) \text{ such that } n = @\varphi \}$.

Consider the function

$$n \in X$$
$$\downarrow$$
$$\#\Big(\lambda r x x_1 \ldots x_{T(@^{-1}n)}.x\Big(r\,(\mathsf{Cons}\,\overline{n}\,\overline{0})\,x_1\Big)\cdots\Big(r\,(\mathsf{Cons}\,\overline{n}\,\overline{T(@^{-1}n)-1})\,x_{T(@^{-1}n)}\Big)\Big) \in \mathbb{N}.$$

In other words we are sending each $n \in X$ to the Gödel number of the closed λ-term

$$\lambda\, r\, x\, x_1 \ldots x_{T(\varphi)}.\, x\, \Big(r\,(\mathsf{Cons}\,\overline{\varphi}\,\overline{0})\,x_1\Big)\ \cdots\ \Big(r\,(\mathsf{Cons}\,\overline{\varphi}\,\overline{T(\varphi)-1})\,x_{T(\varphi)}\Big)\ \in \Lambda^0 \tag{89}$$

where φ is the only element of $\mathrm{dom}(T)$ such that $n = @\varphi$, hence $\overline{n} = \overline{\varphi}$.

The computability of this partial function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ follows obviously from the partial computability of T, $@^{-1}$ and $\#$.

By applying Theorem 5.2.3 to this function we get $E^T \in \Lambda^0$ such that for every $n \in X$ the λ-term $E^T\overline{n}$, namely$E^T\overline{\varphi}$, is β-corvertible to (89). □

Definition 5.2.5. Let $T \in \mathbb{T}_{\mathrm{rec}}$. We define the closed λ-terms $J^T := \mathbf{\Theta}\big(\lambda\, r\, s.\, E^T s\, r\big)$.

Lemma 5.2.6. *Let $T \in \mathbb{T}_{\mathrm{rec}}$. For every $\varphi \in \mathrm{dom}(T)$ we have*

$$J^T\overline{\varphi}\ =_\beta\ \lambda\, x\, x_1 \ldots x_{T(\varphi)}.\, x\ \big(J^T\overline{\varphi.0}\,x_1\big)\ \cdots\ \big(J^T\overline{\varphi.T(\varphi)-1}\,x_{T(\varphi)}\big)\,.$$

Proof. We have

$$\begin{aligned}
J^T\overline{\varphi} &= \mathbf{\Theta}\,(\lambda rs.\, E^T sr)\,\overline{\varphi} && \text{by Def. 5.2.5}\\
&=_\beta (\lambda rs.\, E^T sr)\,\mathbf{\Theta}\,(\lambda rs.\, E^T sr)\,\overline{\varphi} && \text{by Def. of } \mathbf{\Theta}\\
&=_\beta E^T\,\overline{\varphi}\,J^T && \text{by Def. 5.2.5}\\
&=_\beta \lambda x\, x_1 \ldots x_{T(\varphi)}.x\,\big(J^T\,(\mathsf{Cons}\,\overline{\varphi}\,\overline{0})\,x_1\big)\cdots\big(J^T\,(\mathsf{Cons}\,\overline{\varphi}\,\overline{T(\varphi)-1})\,x_{T(\varphi)}\big) && \text{by Lem. 5.2.4}\\
&=_\beta \lambda x\, x_1 \ldots x_{T(\varphi)}.x\,\big(J^T\overline{\varphi.0}\,x_1\big)\cdots\big(J^T\overline{\varphi.T(\varphi)-1}\,x_{T(\varphi)}\big) && \text{by Lem. 5.2.2}
\end{aligned}$$

which was to be proved. □

Lemma 5.2.7. *Let $T \in \mathbb{T}_{\mathrm{rec}}$. For every $\varphi, \varphi' \in \mathrm{dom}(T)$ we have* $J^T\overline{\varphi\varphi'} =_\beta J^{T_\varphi}\overline{\varphi'}$.

Proof. We proceed by coinduction on $\varphi' \in \mathbb{N}^*$.

$$\begin{aligned}
J^T\overline{\varphi\varphi'} &=_\beta \lambda x\, x_1 \ldots x_{T(\varphi\varphi')}.x\,\big(J^T\overline{\varphi\varphi'.0}\,x_1\big)\cdots\big(J^T\overline{\varphi\varphi'.T(\varphi\varphi')-1}\,x_{T(\varphi\varphi')}\big) && \text{by Lem. 5.2.6}\\
&=_\beta \lambda x\, x_1 \ldots x_{T_\varphi(\varphi')}.x\,\big(J^{T_\varphi}\overline{\varphi'.0}\,x_1\big)\cdots\big(J^{T_\varphi}\overline{\varphi'.T_\varphi(\varphi)-1}\,x_{T_\varphi(\varphi')}\big) && \text{by coIH}\\
&=_\beta J^{T_\varphi}\overline{\varphi'} && \text{by Lem. 5.2.6}
\end{aligned}$$

as we had to prove. □

Definition 5.2.8. Let $T \in \mathbb{T}_{\mathrm{rec}}$. Then $\mathbf{J}_T := J^T\overline{\varepsilon}$.

Theorem 5.2.9. *Let $T \in \mathbb{T}_{\mathrm{rec}}$. Then* $\mathbf{J}_T =_\beta \lambda\, x\, x_1 \ldots x_{T(\varepsilon)}.\, x\ \big(\mathbf{J}_{T_{\langle 0\rangle}}\ x_1\big)\cdots\big(\mathbf{J}_{T_{\langle T(\varepsilon)-1\rangle}}\ x_{T(\varepsilon)}\big)\,.$

Proof. We have

$$\begin{array}{llll} \mathbf{J}_T & := & \mathrm{J}^{T}\overline{\varepsilon} & \text{by Def. 5.2.8} \\ & =_\beta & \lambda x\, x_1 \ldots x_{T(\varepsilon)}.x\ (\mathrm{J}^{T}\overline{\langle 0\rangle}\ x_1)\ \cdots\ (\mathrm{J}^{T}\overline{\langle T(\varphi)-1\rangle}\ x_{T(\varepsilon)}) & \text{by Lem. 5.2.6} \\ & =_\beta & \lambda x\, x_1 \ldots x_{T(\varepsilon)}.x\ (\mathrm{J}^{T_{\langle 0\rangle}}\overline{\varepsilon}\ x_1)\ \cdots\ (\mathrm{J}^{T_{\langle T(\varphi)-1\rangle}}\overline{\varepsilon}\ x_{T(\varepsilon)}) & \text{by Lem. 5.2.7} \\ & =_\beta & \lambda x\, x_1 \ldots x_{T(\varepsilon)}.x\ (\mathbf{J}_{T_{\langle 0\rangle}}\ x_1) \cdots (\mathbf{J}_{T_{\langle T(\varepsilon)-1\rangle}}\ x_{T(\varepsilon)}) & \text{by Def. 5.2.8} \end{array}$$

and we are done. □

Corollary 5.2.10. *Let* $T \in \mathbb{T}_{\mathsf{rec}}$. *Then the naked tree underlying* $\mathrm{BT}(\mathbf{J}_T)$ *is* T.

Proof. We check the result by coinduction on $\mathrm{BT}(\mathbf{J}_T)$. By Theorem 5.2.9 we have

$$\mathrm{BT}(\mathbf{J}_T) \;=\; \lambda x\, x_1 \ldots x_{T(\varepsilon)}.x\ \mathrm{BT}(\mathbf{J}_{T_{\langle 0\rangle}}\ x_1)\ \cdots\ \mathrm{BT}(\mathbf{J}_{T_{\langle T(\varepsilon)-1\rangle}}\ x_{T(\varepsilon)})\ .$$

So the number of children of the root of $\lceil \mathrm{BT}(\mathbf{J}_T)\rceil$ is $T(\varepsilon)$. Also, for all $i \in \{0, \ldots, T(\varepsilon)-1\}$ we have $T_{\langle i\rangle} = \lceil \mathrm{BT}(\mathbf{J}_{T_{\langle i\rangle}})\rceil = \lceil \mathrm{BT}(\mathbf{J}_{T_{\langle i\rangle}}\ x_{i+1})\rceil$ by coIH. This proves the thesis. □

In particular given $T \in \mathbb{T}^{\infty}_{\mathsf{rec}}$ then $\mathrm{BT}(\mathbf{J}_T)$ is the tree depicted in Figure 6.

Proposition 5.2.11. *Let* $T \in \mathbb{T}_{\mathsf{rec}}$. *Then* $\mathrm{BT}(\mathbf{J}_T) \twoheadrightarrow_\eta \mathbf{I}$. *In particular whenever* $T \in \mathbb{T}^{\infty}_{\mathsf{rec}}$ *then* $\mathrm{BT}(\mathbf{J}_T) \twoheadrightarrow_\eta \mathbf{I}$ *but* $\mathrm{BT}(\mathbf{J}_T) \not\twoheadrightarrow^{\mathsf{fin}}_\eta \mathbf{I}$.

Proof. According to Definition 1.4.1 we must prove that $\mathrm{BT}(\mathbf{J}_T) = (\mathbf{I}\,;\lceil \mathrm{BT}(\mathbf{J}_T)\rceil)$. By Corollary 5.2.10 this is equivalent to $\mathrm{BT}(\mathbf{J}_T) = (\mathbf{I};T)$. In order to prove this equality we show that for every $\varphi \in \mathrm{dom}(T) = \mathrm{dom}(\mathrm{BT}(\mathbf{J}_T)) = \mathrm{dom}(\mathbf{I};T)$ we have $(\mathbf{I};T)(\varphi) = \mathrm{BT}(\mathbf{J}_T)(\varphi)$.

- Clearly $\varepsilon \in \mathrm{dom}(\mathbf{I})$, $\mathbf{I}(\varepsilon) = \lambda x.x$ and the number of children of the node ε in $\mathbf{I}$ is 0. Then by Point 2 of Definition 1.4.1 we have $(\mathbf{I};T)(\varepsilon) = \lambda x y^{\varepsilon}_0 \ldots y^{\varepsilon}_{T(\varepsilon)-1}.x = \mathrm{BT}(\mathbf{J}_T)(\varepsilon)$.
- For all $i \in \{0, \ldots, T(\varepsilon)-1\}$ we have $\langle i\rangle = \varepsilon.i \in \mathrm{dom}(T) - \mathrm{dom}(\mathbf{I})$, $\varepsilon \in \mathrm{dom}(\mathbf{I})$ and the number of children of the node ε in $\mathbf{I}$ is 0. Then by Point 3 of Definition 1.4.1 we get $(\mathbf{I};T)(\langle i\rangle) = \lambda y^{\langle i\rangle}_0 \ldots y^{\langle i\rangle}_{T(\langle i\rangle)-1}.y^{\varepsilon}_i = \mathrm{BT}(\mathbf{J}_T)(\langle i\rangle)$.
- For any $\varphi \in \mathrm{dom}(T)$ of length at least 2 we have $\varphi = \varphi'.i \in \mathrm{dom}(T) - \mathrm{dom}(\mathbf{I})$ with $\varphi' \notin \mathrm{dom}(\mathbf{I})$. So by Point 4 of Definition 1.4.1 we get $(\mathbf{I};T)(\varphi) = \lambda y^{\varphi}_0 \ldots y^{\varphi}_{T(\varphi)-1}.y^{\varphi'}_i = \mathrm{BT}(\mathbf{J}_T)(\varphi)$.

So eventually $(\mathbf{I};T) = \mathrm{BT}(\mathbf{J}_T)$. This completes the proof of $\mathrm{BT}(\mathbf{J}_T) \twoheadrightarrow_\eta \mathbf{I}$.

Finally, the fact that $\mathrm{BT}(\mathbf{J}_T) \not\twoheadrightarrow^{\mathsf{fin}}_\eta \mathbf{I}$ whenever $T \in \mathbb{T}^{\infty}_{\mathsf{rec}}$ is trivial to prove. □

Remark 5.2.12. At this point it should be clear that $\{\mathrm{BT}(\mathbf{J}_T)\}_{T\in\mathbb{T}_{\mathsf{rec}}}$ is actually the family of all λ-definable Böhm-like trees that are infinite η-expansions of $\mathbf{I}$. In particular its subfamily $\{\mathrm{BT}(\mathbf{J}_T)\}_{T\in\mathbb{T}^{\infty}_{\mathsf{rec}}}$ contains exactly what we call the *infinitely deep* η-expansions of $\mathbf{I}$.

Examples 5.2.13. Here are some examples.

- If $T \in \mathbb{T}_{\mathsf{rec}} - \mathbb{T}^{\infty}_{\mathsf{rec}}$ then $\mathrm{BT}(\mathbf{J}_T)$ is the finite η-expansion of $\mathbf{I}$ following T. For instance, by taking the partial map $T : \mathbb{N}^* \rightharpoonup \mathbb{N}$ that only sends $\varepsilon \mapsto 0$ we get $\mathbf{J}_T =_\beta \mathbf{I}$. As another example, if T maps $\varepsilon \mapsto 3$ and $\langle 2\rangle \mapsto 1$ then $\mathbf{J}_T =_\beta \lambda x y_0 y_1 y_2.x\, y_0 y_1 (\lambda z_0 y_2\, z_0)$.

- Let $T : \mathbb{N}^* \rightharpoonup \mathbb{N}$ send $\langle \overbrace{0, 0, \ldots, 0}^{n \text{ times}} \rangle \mapsto 1$ for every $n \in \mathbb{N}$. Then $T \in \mathbb{T}^{\infty}_{\text{rec}}$ and $\mathbf{J}_T = \mathbf{J}$. Pay attention: *not only* $BT(\mathbf{J}_T) = BT(\mathbf{J})$, but *actually* the two terms $\mathbf{J}_T$ and $\mathbf{J}$ are the same. In other words, our uniform definition of the function $\mathbf{J}_{\_} : T \in \mathbb{T}_{\text{rec}} \mapsto \mathbf{J}_T \in \Lambda^0$ generalizes the definition of the combinator $\mathbf{J}$ given by Wadsworth in [Wad76].

An even more explicit definition of $\mathbf{J}_T$ can be found in § 2.1 of our paper [BMPR16].

5.3 SEPARATING THE INSEPARABLE: BÖHM-OUT THROUGH MORRIS SEPARATORS

The *Böhm-out technique* [Bar84, RP04], which first appeared in [Böh68], builds a context $C[\,]$ from any two λ-terms M and N. This is done so that $C[M]$ and $C[N]$ *extract* (instances of) M_σ and N_σ, namely the subterms of M and N lying at a certain position σ on $BT(M)$ and $BT(N)$. The technique is usually used to *separate* M and N according to some notion of separation, provided that their inner structures are sufficiently different.

As explained in the introduction of the chapter, we are interested in the case where $M \sqsubseteq_{\mathcal{H}^*} N$ (hence M and N have a similar structure, meaning that in every position σ on which both $BT(M)$ and $BT(N)$ are defined the number of abstractions and applications can be matched via η-expansions), but $M \not\sqsubseteq_{\mathcal{H}^+} N$. We rephrase the traditional Böhm-out technique for such a case, so to extract $\mathbf{I}$ from one of the two λ-terms, and something with Bohm tree $BT(\mathbf{J}_T)$, for T infinite recursive, from the other (Theorem 5.3.8). Actually, rather than consider directly the hypothesis $M \not\sqsubseteq_{\mathcal{H}^+} N$, it suffices to focus first on the situation where M is β-normal, N is not β-normalizable and $M \sqsubseteq_{\mathcal{H}^*} N$. Notice that these M and N are not separable in the sense of [Bar84], from which the pun in the title of this section: we *separate the inseparable*.

We start by providing a notion of *Morris separator*, as a sequence $\sigma \in \mathbb{N}^*$ witnessing that we are in the situation described above.

Notation. We generally use for Böhm trees the same notions and notations introduced for naked trees. Except for the following convenient abuse of language: we write $\sigma \in BT(M)$ to indicate that $\sigma \in \text{dom}(BT(M))$.

Remember that for $M \in \Lambda$ and $\sigma \in BT(M)$ the subterm M_σ is defined in § 1.2.

Definition 5.3.1. Let $M, N \in \Lambda$ and $\sigma \in BT(M) \cap BT(N)$. Then σ is a *Morris separator for* M, N, written

$$\sigma : M \not\sqsubseteq_{\mathcal{H}^+} N\,,$$

if there exists $i > 0$ such that for some $p \geqslant i$ we have

$$M_\sigma =_\beta \lambda x_1 \ldots x_n.y\, M_1 \cdots M_m \qquad \text{and} \qquad N_\sigma =_\beta \lambda x_1 \ldots x_{n+p}.y\, N_1 \cdots N_{m+p}$$

where $N_{m+i} =_{\mathcal{B}} \mathbf{J}_T x_{n+i}$ for some $T \in \mathbb{T}^{\infty}_{\text{rec}}$.

Lemma 5.3.2. *Let* $M, N \in \Lambda$. *If* $\sigma : M \not\sqsubseteq_{\mathcal{H}^+} N$ *and* $\sigma = \langle k \rangle \tau$ *then* $\tau : M_{k+1} \not\sqsubseteq_{\mathcal{H}^+} N_{k+1}$.

Proof. Simply because $(M_{k+1})_\tau = M_\sigma$ and $(N_{k+1})_\tau = N_\sigma$. □

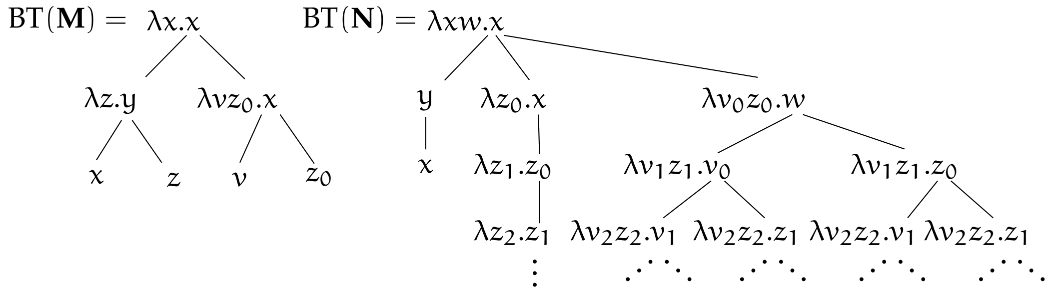


Figure 7: The Böhm trees of two λ-terms $\mathbf{M}, \mathbf{N}$ such that $\mathbf{M}$ is β-normal, $\mathbf{M} \sqsubseteq_{\mathcal{H}^*} \mathbf{N}$, but $\mathbf{M} \not\sqsubseteq_{\mathcal{H}^+} \mathbf{N}$.

Example 5.3.3. Consider two λ-terms $\mathbf{M}$ and $\mathbf{N}$ with Böhm trees as in Figure 7. There exist exactly two Morris separators for $\mathbf{M}, \mathbf{N}$. They are ε and $\langle 1, 0 \rangle$, as explained here below.

- The empty sequence $\varepsilon \in \mathrm{BT}(\mathbf{M}) \cap \mathrm{BT}(\mathbf{N})$ is a Morris separator since $\langle 2 \rangle \notin \mathrm{BT}(\mathbf{M})$, whereas $\mathbf{N}_{\langle 2 \rangle} =_{\mathcal{B}} \mathbf{J}_{T_2} w$, where T_2 is the complete binary tree (formally $T_2 : \mathbb{N}^* \rightharpoonup \mathbb{N}$ is given by $\sigma \mapsto 2$ for every sequence σ whose elements are only 0's and 1's).
- The sequence $\langle 1, 0 \rangle \in \mathrm{BT}(\mathbf{M}) \cap \mathrm{BT}(\mathbf{N})$ is a Morris separator because $\langle 1, 0, 0 \rangle \notin \mathrm{BT}(\mathbf{M})$, whereas $\mathbf{N}_{\langle 1,0,0 \rangle} =_{\mathcal{B}} \mathbf{J} z_1$.

Proposition 5.3.4. *Let* $M, N \in \Lambda$ *such that* M *is* β*-normal,* N *is not* β*-normalizable and* $M \sqsubseteq_{\mathcal{H}^*} N$. *Then there exists a position* $\sigma \in \mathrm{BT}(M) \cap \mathrm{BT}(N)$ *such that* $\sigma : M \not\sqsubseteq_{\mathcal{H}^+} N$.

Proof. Since M is β-normal, the finite tree $\mathrm{BT}(M)$ is ⊥-free. Hence, as $M \sqsubseteq_{\mathcal{H}^*} N$, also $\mathrm{BT}(N)$ is ⊥-free. But at the same time N is not β-normalizable by hypothesis. So the tree $\mathrm{BT}(N)$ must be infinite. By König's lemma there exists $f \in \Pi(\mathrm{BT}(N))$. Since $\mathrm{BT}(M)$ is finite there exists $n \in \mathbb{N}$ such that

$$\sigma := \big\langle f(0), \ldots, f(n-1) \big\rangle \in \mathrm{BT}(M) \cap \mathrm{BT}(N) \quad \text{and} \quad \big\langle f(0), \ldots, f(n) \big\rangle \notin \mathrm{BT}(M) . \tag{90}$$

(Notice that such σ is simply ε in case $n = 0$.)

Since $M \sqsubseteq_{\mathcal{H}^*} N$, by Proposition 1.3.4 and (90) there exist $m_1, n_1 \in \mathbb{N}$ and $p > 0$ such that

$$M_\sigma =_\beta \lambda x_1 \ldots x_{n_1}.y\, M_1 \cdots M_{m_1} \quad \text{and} \quad N_\sigma =_\beta \lambda x_1 \ldots x_{n_1+p}.y\, N_1 \cdots N_{m_1+p} .$$

Moreover, still by Proposition 1.3.4 if we set $j := f(n_1) + 1 - m_1$ we have $x_{n_1+j} \sqsubseteq_{\mathcal{H}^*} N_{m_1+j}$. By the characterization of $\sqsubseteq_{\mathcal{H}^*}$ in terms of η-expansions of Böhm trees, we have then $\mathrm{BT}(N_{m_1+j}) \twoheadrightarrow_\eta x_{n_1+j}$. Since $\mathrm{BT}(N_{m_1+j})$ is infinite, $\mathrm{BT}(N_{m_1+j}) \not\twoheadrightarrow_\eta x_{n_1+j}$. We conclude by Remark 5.2.12 that $N_{m_1+j} =_{\mathcal{B}} \mathbf{J}_T x_{n_1+j}$ for some $T \in \mathbb{T}^\infty_{\mathrm{rec}}$. □

The combinators known as *projectors* $\mathbf{U}^n_k$ and *tuplers* $\mathbf{P}_n$ will be used (among others recalled in § 1.2) to build the Böhm-out context. For all $n, k \in \mathbb{N}$ such that $k \geqslant n$ they are

$$\mathbf{U}^n_k := \lambda x_1 \ldots x_n.x_k \qquad \mathbf{P}_n := \lambda x_1 \ldots x_n.\lambda z.z\, x_1 \cdots x_n .$$

These combinators enjoy the following properties, whose proofs are straigthforward.

Lemma 5.3.5. *Let* $n, k \in \mathbb{N}$ *such that* $k \geqslant n$*, and let* $X_1, \ldots, X_n, Y_1, \ldots, Y_{k-n} \in \Lambda^0$. *Then*

1. $(\mathbf{P}_k X_1 \cdots X_n) Y_1 \cdots Y_{k-n} =_\beta \lambda z.z X_1 \cdots X_n Y_1 \cdots Y_{k-n}$;
2. $(\lambda z.z X_1 \cdots X_n)\, \mathbf{U}^n_i =_\beta X_i$.

Let us explain the main idea. When $\mathbf{U}^n_k$ is substituted for y in $\lambda\vec{x}.y\, M_1 \cdots M_n$, it extracts an *instance* of M_k, meaning by this, M_k possibly with some of its free variables replaced by combinators. For instance, consider the λ-term $\mathbf{N}$ whose Böhm tree is given in Figure 7. The context $[-]\mathbf{U}^3_1$ extracts from $\mathbf{N}$ the subterm yx where x is replaced by $\mathbf{U}^3_1$. The idea of the Böhm-out technique is to replace every variable along the path σ with the correct projector.

The issue is when the same variable occurs several times in σ and we must select different children in these occurrences. For example, to extract $\mathbf{N}_{\langle 1,0\rangle}$ the first occurrence of x should be replaced by $\mathbf{U}^3_2$, the second by $\mathbf{U}^1_1 = \mathbf{I}$. The problem was originally solved by Böhm in [Böh68] by first replacing the occurrences of the same variables along the path by different variables using the tupler, and then replacing each variable by the suitable projector. In the example under consideration, the context $[-]\mathbf{P}_3\mathbf{\Omega}\mathbf{U}^3_2\mathbf{U}^1_1\mathbf{\Omega}\mathbf{\Omega}\mathbf{U}^3_1$ extracts from $\mathbf{N}$ the instance of $\mathbf{N}_{\langle 1,0\rangle}$ where z_0 is replaced by $\mathbf{I}$.

Notice that finite η-differences can be destroyed during the process of Böhming out. In contrast, we show that infinitely deep η-differences can always be preserved.

Notation. Let $M \in \Lambda$. Then $M^{\sim n}$ denotes the sequence of λ-terms containing n copies of M.

Lemma 5.3.6 (Böhm-out). *Let $M, N \in \Lambda$ such that $M \sqsubseteq_{\mathcal{H}^*} N$ and let $\sigma : M \not\sqsubseteq_{\mathcal{H}^+} N$ be a Morris separator. Let $\vec{y}$ contain the variables in $\mathrm{fv}(MN)$. Then for all $k \in \mathbb{N}$ large enough there is $\vec{X} \in (\Lambda^0)^*$ such that $M\{\mathbf{P}_k/\vec{y}\}\vec{X} =_\beta \mathbf{I}$ and $N\{\mathbf{P}_k/\vec{y}\}\vec{X} =_\mathcal{B} \mathbf{J}_T$ for some $T \in \mathbb{T}^\infty_{\mathsf{rec}}$.*

Proof. We proceed by induction on σ.
Case $\sigma = \varepsilon$. Since $\varepsilon : M \not\sqsubseteq_{\mathcal{H}^+} N$, there exist $i > 0$ and $p \geqslant i$ such that

$$M =_\beta \lambda x_1 \ldots x_n.y\, M_1 \cdots M_m \quad \text{and} \quad N =_\beta \lambda x_1 \ldots x_{n+p}.yN_1 \cdots N_{m+p}$$

with

$$N_{m+i} =_\mathcal{B} \mathbf{J}_T x_{n+i} \quad \text{for some } T \in \mathbb{T}^\infty_{\mathsf{rec}} . \tag{91}$$

Since T is infinite and computable, there exists $j \in \{0, \ldots, T(\varepsilon) - 1\}$ such that $T_{\langle j\rangle}$ is still infinite and computable. Also notice that

$$(\mathbf{J}_T x_{n+i})\left\{\mathbf{U}^{T(\varepsilon)}_{j+1}/x_{n+i}\right\} =_\beta \lambda x_1 \ldots x_{T\varepsilon}.\mathbf{J}_{T_{\langle j\rangle}} . \tag{92}$$

For any $k \geqslant n + m + p$ let us set $\vec{X} := \mathbf{P}^{\sim n}_k \mathbf{U}^{T(\varepsilon)\sim p}_{j+1} \mathbf{\Omega}^{\sim k-m-p} \mathbf{U}^k_{m+i} \mathbf{I}^{\sim T(\varepsilon)}$. Notice that $\vec{X}$ depends on k (we will make this explicit using a pedex, i.e. by $\vec{X}_k$, only when needed).

We split into cases depending on whether y is free or $y = x_j$ for some $j \in \{1, \ldots, n\}$. We consider only the former case, as the latter is completely analogous. On the one side

$$\begin{array}{lll}
(\lambda x_1 \ldots x_n.y\, M_1 \cdots M_m)\{\mathbf{P}_k/\vec{y}\}\vec{X} & = & \\
(\lambda x_1 \ldots x_n.\mathbf{P}_k M'_1 \cdots M'_m)\vec{X} & =_\beta & \text{where } M'_\ell := M_\ell\{\mathbf{P}_k/\vec{y}\} \\
(\mathbf{P}_k M''_1 \cdots M''_m)\mathbf{U}^{T(\varepsilon)\sim p}_{j+1}\mathbf{\Omega}^{\sim k-m-p}\mathbf{U}^k_{m+i}\mathbf{I}^{\sim T(\varepsilon)} & =_\beta & \text{where } M''_\ell := M'_\ell\{\mathbf{P}_k/\vec{x}\} \\
(\lambda z.zM''_1 \cdots M''_m \mathbf{U}^{T(\varepsilon)\sim p}_{j+1}\mathbf{\Omega}^{\sim k-m-p})\mathbf{U}^k_{m+i}\mathbf{I}^{\sim T(\varepsilon)} & =_\beta & \text{by Lem. 5.3.5(1)} \\
\mathbf{U}^{T(\varepsilon)}_{j+1}\mathbf{I}^{\sim T(\varepsilon)} & =_\beta \ \mathbf{I} & \text{by Lem. 5.3.5(2).}
\end{array}$$

On the other side we get

$$\begin{array}{ll}
(\lambda x_1 \dots x_{n+p}.x_j N_1 \cdots N_{m+p})\{\mathbf{P}_k/\vec{y}\}\vec{X} \; = & \\
(\lambda x_1 \dots x_{n+p}.\mathbf{P}_k N'_1 \cdots N'_{m+p})\vec{X} \; =_\beta & \text{for } N'_\ell := N_\ell\{\mathbf{P}_k/\vec{y}\} \\
(\lambda x_{n+1} \dots x_{n+p}.\mathbf{P}_k N''_1 \cdots N''_{m+p})\mathbf{U}^{T(\varepsilon)}_{j+1}{}^{\sim p}\mathbf{\Omega}^{\sim k-m-p}\mathbf{U}^k_{m+i}\mathbf{I}^{\sim T(\varepsilon)} \; =_\beta & \text{for } N''_\ell := N'_\ell\{\mathbf{P}_k/x_1,\dots,x_n\} \\
(\mathbf{P}_k N'''_1 \cdots N'''_{m+p})\mathbf{\Omega}^{\sim k-m-p}\mathbf{U}^k_{m+i}\mathbf{I}^{\sim T(\varepsilon)} \; =_\beta & \text{for } N'''_\ell := N''_\ell\{\mathbf{U}^{T(\varepsilon)}_{j+1}/x_{n+1},\dots,x_{n+p}\} \\
(\lambda z.z N'''_1 \cdots N'''_{m+p}\mathbf{\Omega}^{\sim k-m-p})\mathbf{U}^k_{m+i}\mathbf{I}^{\sim T(\varepsilon)} \; =_\beta & \text{by Lem. 5.3.5(1)} \\
N'''_{m+i}\mathbf{I}^{\sim T(\varepsilon)} \; = \; N_{m+1}\left\{\mathbf{U}^{T(\varepsilon)}_{j+1}/x_{n+i}\right\}\mathbf{I}^{\sim T(\varepsilon)} \; =_{\mathcal{B}} & \text{by Lem. 5.3.5(2)} \\
(\,\mathbf{J}_T x_{n+i})\left\{\mathbf{U}^{T(\varepsilon)}_{j+1}/x_{n+i}\right\}\mathbf{I}^{\sim T(\varepsilon)} \; =_\beta & \text{by (91)} \\
\left(\lambda x_1 \dots x_{T\varepsilon}.\mathbf{J}_{T_{\langle j \rangle}}\right)\mathbf{I}^{\sim T(\varepsilon)} \; =_\beta \; \mathbf{J}_{T_{\langle j \rangle}} & \text{by (92).}
\end{array}$$

Case $\sigma = \langle i \rangle \tau$. By Lemma 1.3.4 there are $n, m, n', m' \in \mathbb{N}$ such that $n - m = n' - m'$ and

$$M \; = \; \lambda x_1 \dots x_n.y\, M_1 \cdots M_m \quad \text{and} \quad N \; = \; \lambda x_1 \dots x_{n'}.y\, N_1 \cdots N_{m'}$$

where $M_j \sqsubseteq_{\mathcal{H}^*} N_j$ for all $j \in \{1, \dots, \min(m, m')\}$ and either y is free or $y = x_j$ for some $j \in \{1, \dots, \min(n, n')\}$. Suppose that, say, $n \leqslant n'$ (the case $n' \leqslant n$ is analogous). Then we set $p := n' - n = m' - m$, so we can write $n' = n + p$ and $m' = m + p$.

Notice that $i + 1 \in \{1, \dots, \min(m, m')\}$ and by Lemma 5.3.2 we have $\tau : M_{i+1} \not\sqsubseteq_{\mathcal{H}^+} N_{i+1}$. Since moreover $M_{i+1} \sqsubseteq_{\mathcal{H}^*} N_{i+1}$ and all the free variables of $M_{i+1}N_{i+1}$ are among $\vec{y}, \vec{x}$, the IH can be applied to M_{i+1} and N_{i+1}. For any h large enough we get $\vec{Y}_h \in (\Lambda^0)^*$ such that $M_{i+1}\{\mathbf{P}_h/\vec{y}, \vec{x}\}\vec{Y}_h =_\beta \mathbf{I}$ and $N_{i+1}\{P_h/\vec{y}, \vec{x}\}\vec{Y}_h =_{\mathcal{B}} \mathbf{J}_T$ for some $T \in \mathbb{T}^\infty_{\mathsf{rec}}$. For any $k \geqslant \max(h, n + m + p)$, we set $\vec{X} := \mathbf{P}_k^{\sim n+p}\mathbf{\Omega}^{\sim k-m-p}\mathbf{U}^k_{i+1}\vec{Y}_k$.
We suppose that y is free, the other case being analogous. On the one side we have

$$\begin{array}{ll}
(\lambda x_1 \dots x_n.y\, M_1 \cdots M_m)\{\mathbf{P}_k/\vec{y}\}\vec{X} \; = & \\
(\lambda x_1 \dots x_n.\mathbf{P}_k M'_1 \cdots M'_m)\mathbf{P}_k^{\sim n+p}\mathbf{\Omega}^{\sim k-m-p}\mathbf{U}^k_{i+1}\vec{Y}_k \; =_\beta & \text{where } M'_\ell := M_\ell\{\mathbf{P}_k/\vec{y}\} \\
(\mathbf{P}_k M''_1 \cdots M''_m)\mathbf{P}_k^{\sim p}\mathbf{\Omega}^{\sim k-m-p}\mathbf{U}^k_{i+1}\vec{Y}_k \; =_\beta & \text{where } M''_\ell := M'_\ell\{\mathbf{P}_k/\vec{x}\} \\
(\lambda z.z M''_1 \cdots M''_m \mathbf{P}_k^{\sim p}\mathbf{\Omega}^{\sim k-m-p})\mathbf{U}^k_{i+1}\vec{Y}_k \; =_\beta & \text{by Lemma 5.3.5(1)} \\
M''_{i+1}\vec{Y}_k \; = \; M_{i+1}\{\mathbf{P}_k/\vec{y}, \vec{x}\}\vec{Y}_k \; =_\beta \; \mathbf{I} & \text{by Lemma 5.3.5(2) and IH.}
\end{array}$$

On the other side, we have

$$\begin{array}{ll}
(\lambda x_1 \dots x_{n+p}.y\, N_1 \cdots N_{m+p})\{\mathbf{P}_k/\vec{y}\}\vec{X} \; = & \\
(\lambda x_1 \dots x_{n+p}.\mathbf{P}_k N'_1 \cdots N'_{m+p})\mathbf{P}_k^{\sim n+p}\mathbf{\Omega}^{\sim k-m-p}\mathbf{U}^k_{i+1}\vec{Y}_k \; =_\beta & \text{where } N'_\ell := N_\ell\{\mathbf{P}_k/\vec{y}\} \\
(\mathbf{P}_k N''_1 \cdots N''_{m+p})\mathbf{\Omega}^{\sim k-m-p}\mathbf{U}^k_{i+1}\vec{Y}_k \; =_\beta & \text{where } N''_\ell := N'_\ell\{\mathbf{P}_k/\vec{x}\} \\
(\lambda z.z N''_1 \cdots N''_{m+p}\mathbf{\Omega}^{\sim k-m-p})\mathbf{U}^k_{i+1}\vec{Y}_k \; =_\beta & \text{by Lemma 5.3.5(1)} \\
N''_{i+1}\vec{Y}_k \; = \; N_{i+1}\{\mathbf{P}_k/\vec{y}, \vec{x}\}\vec{Y}_k =_{\mathcal{B}} \; \mathbf{J}_T & \text{by Lemma 5.3.5(2) and IH}
\end{array}$$

which completes the proof. □

Corollary 5.3.7 (Böhm-out). *Let $M, N \in \Lambda$ such that M is β-normal, N is not β-normalizable and $M \sqsubseteq_{\mathcal{H}^*} N$. Then there is a closed head context $C[-]$ such that $C[M] =_\beta \mathbf{I}$ and $C[N] =_{\mathcal{B}} \mathbf{J}_T$ for some $T \in \mathbb{T}^\infty_{\mathsf{rec}}$. In particular when $M, N \in \Lambda^0$ the context $C[-]$ is closed and applicative.*

Proof. Let $y_1, \ldots, y_n$ contain the variables in $\mathrm{fv}(MN)$. By Proposition 5.3.4 there is a Morris Separator $\sigma : M \not\sqsubseteq_{\mathcal{H}^+} N$. By Lemma 5.3.6 for k large enough we have $\vec{X} \in (\Lambda^0)^*$ such that $M\{\mathbf{P}_k/\vec{y}\}\vec{X} =_\beta \mathbf{I}$ and $N\{\mathbf{P}_k/\vec{y}\}\vec{X} =_\mathcal{B} \mathbf{J}_T$ for some $T \in \mathbb{T}^\infty_{\mathrm{rec}}$. The head context $C[\,] := (\lambda y_1 \ldots y_n.[\,])\, \mathbf{P}_k^{\sim n}\, \vec{X}$ proves the thesis, since

$$C[M] \;:=\; (\lambda y_1 \ldots y_n.M)\, \mathbf{P}_k^{\sim n}\, \vec{X} \;=_\beta\; M\{\mathbf{P}_k/\vec{y}\}\vec{X} \;=_\beta\; \mathbf{I}$$

and

$$C[N] \;:=\; (\lambda x_1 \ldots x_n.N)\, \mathbf{P}_k^{\sim n}\, \vec{X} \;=_\beta\; N\{\mathbf{P}_k/\vec{y}\}\vec{X} \;=_\mathcal{B}\; \mathbf{J}_T\,.$$

Clearly $C[\,]$ is closed and applicative when M and N are closed. □

Theorem 5.3.8 (Morris Separation)**.** *Let $M, N \in \Lambda$ such that $M \sqsubseteq_{\mathcal{H}^*} N$ whereas $M \not\sqsubseteq_{\mathcal{H}^+} N$. There is a closed head context $C[-]$ such that $C[M] =_\beta \mathbf{I}$ and $C[N] =_\mathcal{B} \mathbf{J}_T$ for some $T \in \mathbb{T}^\infty_{\mathrm{rec}}$. In particular when $M, N \in \Lambda^0$ the context $C[-]$ is closed and applicative.*

Proof. Since $M \not\sqsubseteq_{\mathcal{H}^+} N$, by Corollary 1.3.11 there is a closed head context $E[-]$ such that $E[M]$ has a β-nf whereas $E[N]$ does not. From $M \sqsubseteq_{\mathcal{H}^*} N$ we obtain $E[M] \sqsubseteq_{\mathcal{H}^*} E[N]$. Therefore we can apply Corollary 5.3.7 to the λ-terms $E[M]$ and $E[N]$, and get a closed head context $D[-]$ such that $D[E[M]] =_\beta \mathbf{I}$ and $D[E[N]] =_\mathcal{B} \mathbf{J}_T$ for some $T \in \mathbb{T}^\infty_{\mathrm{rec}}$. So the closed head context $C[-] := D[E[-]]$ gives us the thesis. When M, N are closed, all the contexts can be chosen closed and applicative. □

5.4 Intermezzo: $\mathcal{H}^+$ and the ω-rule

Before going on with the quest for an exhaustive solution to our full abstraction problem, we wish to show that our Böhm-out (Corollary 5.3.7) is of interest in itself, independently of the semantic problem that we are confronting here. In this brief section we use it to achieve a purely syntactic result. We prove that $\mathcal{H}^+$ satisfies the ω-rule, the strong form of extensionality recalled in § 1.2. As recalled there, for any λ-theory $\mathcal{T}$ one has $\mathcal{T} \vdash \omega$ if and only if $\mathcal{T} \vdash \omega^0$, where ω^0 denotes the restriction of ω to closed λ-terms. So we want to prove that for all $M, N \in \Lambda^0$

$$\left(MZ =_{\mathcal{H}^+} NZ \;\text{ for all } Z \in \Lambda^0\right) \quad \text{implies} \quad M =_{\mathcal{H}^+} N\,.$$

Lemma 5.4.1. *Let $M, N \in \Lambda^0$ such that M has a β-nf whereas N does not. Then, there exist $n \geqslant 1$ and closed λ-terms $Z_1, \ldots, Z_n \in \Lambda^0$ such that $M\vec{Z}$ has a β-nf whereas $N\vec{Z}$ does not.*

Proof. We distinguish two possible cases.
Case $M \sqsubseteq_{\mathcal{H}^+} N$. By Corollary 5.3.7 there exist $n \in \mathbb{N}$ and $Z_1, \ldots, Z_n \in \Lambda^0$ such that $M\vec{Z} =_\beta \mathbf{I}$ and $N\vec{Z} =_\mathcal{B} \mathbf{J}_T$ for some $T \in \mathbb{T}^\infty_{\mathrm{rec}}$. Since T is infinite and computable, there exists $j \in \{0, \ldots, T(\varepsilon) - 1\}$ such that $T_{\langle j \rangle}$ is still infinite and computable.
If $n \geqslant 1$ we are done.
If $n = 0$ just take $Z_1 := \mathbf{U}^{T(\varepsilon)}_{j+1}$ and conclude since $\mathbf{J}_T \mathbf{U}^{T(\varepsilon)}_{j+1} =_\beta \lambda x_1 \ldots x_{T\varepsilon}.\mathbf{J}_{T_{\langle j \rangle}}$.
Case $M \not\sqsubseteq_{\mathcal{H}^+} N$. In this case by Corollary 1.3.2 there are $n \in \mathbb{N}$ and $Z_1, \ldots, Z_n \in \Lambda^0$ such that $M\vec{Z} =_\beta \lambda x_1 \ldots x_m.x_i M_1 \cdots M_k$ for some $m, k \in \mathbb{N}$ and $N\vec{Z} =_\beta U$ for some unsolvable

U. Then we have $M\vec{Z}\mathbf{I}^m =_\beta \mathbf{I}$ and $N\vec{Z}\mathbf{I}^m =_\beta U'$ for some unsolvable U'. If $n+m \geqslant 1$ we are done.

If $n+m=0$ just take $Z_1 := \mathbf{I}$ and conclude since $U'\mathbf{I}$ is still unsolvable. □

Lemma 5.4.2. *Let $M, N \in \Lambda^0$. Suppose that $MZ =_{\mathcal{H}^+} NZ$ for all $Z \in \Lambda^0$. Then $M =_{\mathcal{H}^+} N$. In other words $\mathcal{H}^+ \vdash \omega^0$.*

Proof. By Corollary 1.3.11 the hypothesis states that

$$\text{for all } Z \in \Lambda^0 \text{ and for all } \vec{Y} \in \left(\Lambda^0\right)^* \quad MZ\vec{Y} \text{ has a } \beta\text{-nf if and only if } NZ\vec{Y} \text{ has a } \beta\text{-nf.} \quad (93)$$

Still by Corollary 1.3.11 the thesis is equivalent to

$$\text{for all } \vec{X} \in \left(\Lambda^0\right)^* \quad M\vec{X} \text{ has a } \beta\text{-nf if and only if } N\vec{X} \text{ has a } \beta\text{-nf.}$$

We distinguish two cases depending on the length k of $\vec{X}$ (but notice that this is *not* a proof by induction on k).
Case $k=0$. Lemma 5.4.1 is exactly the contrapositive of what we have to show here. So there is nothing more to prove.
Case $k>0$. This is given by the hypothesis (93) itself. □

As a consequence, we get our main result, which solves positively a long-standing open problem (see [Bar84, § 17.4]).

Theorem 5.4.3. *$\mathcal{H}^+$ satisfies the ω-rule.*

5.5 WITNESSING TREES

If an rgm $\mathcal{D}$ is extensional, every $\sigma \in T_{\mathcal{D}}$ is equivalent to an arrow type. So we can always try to unfold it *following* a function $f : \mathbb{N} \to \mathbb{N}$, in the following sense. Starting from $\sigma = \sigma_0$, at every level $\ell \in \mathbb{N}$ we consider $\sigma_\ell \simeq \mu_0 \to \cdots \to \mu_{f(\ell)} \to \sigma'_\ell$ and, as long as there is a $\sigma_{\ell+1} \in \mu_{f(\ell)}$, we can keep unfolding it at level $\ell+1$. Obviously there are two possibilities. If at some level ℓ we have $\mu_{f(\ell)} = \omega$, then the process is forced to stop and σ cannot be unfolded following f. If this process continues indefinitely, then we consider that σ can actually be unfolded following f. This idea is made rigorous in Definition 5.5.1 below by means of a coinduction.

Notation. Let $f : \mathbb{N} \to \mathbb{N}$. Then $f^{\geqslant k}$ denotes the function defined by

$$n \in \mathbb{N} \quad \mapsto \quad f(n+k) \in \mathbb{N}\,.$$

Definition 5.5.1. Let $\mathcal{D}$ be an rgm. Let $T \in \mathbb{T}^{\infty}_{\mathsf{rec}}$ and $f \in \Pi(T)$. Then $\sigma \in T_{\mathcal{D}}$ is a *witness for* T *following* f if and only if there are $\mu_0, \ldots, \mu_{f(0)} \in I_{\mathcal{D}}$ and $\sigma' \in T_{\mathcal{D}}$ such that

$$\sigma \;\simeq\; \mu_0 \to \cdots \to \mu_{f(0)} \to \sigma'$$

and there exists $\tau \in \mu_{f(0)}$ that is a witness for $T_{\langle f(0)\rangle}$ following $f^{\geqslant 1}$.

We say that σ is a *witness for* T whenever there exists an $f \in \Pi(T)$ such that σ is a witness for T following f.

Notice that in this definition we do not ask f to be computable, despite T being computable.

Notation. We denote by

- $W_{\mathcal{D},f}(T)$ the set of all witnesses for T following f,
- $W_{\mathcal{D}}(T)$ the set of all witnesses for T.

If the model $\mathcal{D}$ is clear from the context, we simply write $W_f(T)$ and $W(T)$.

Definition 5.5.1 is consistent, because it is independent of the arrow type $\mu_0 \to \cdots \to \mu_{f(0)} \to \sigma'$ (equivalent to σ) that we choose. This is formalized in Lemma 5.5.2 below. Actually, we could also get rid of $\simeq$ in Definition 5.5.1 and just write $\sigma = \mu_0 \to \cdots \to \mu_{f(0)} \to \sigma'$ therein. That would be enough for our purposes. Nevertheless, even with such a choice we would need Lemma 5.5.2, so it makes no great difference.

Lemma 5.5.2. *Let $\mathcal{D}$ be an rgm. Let $T \in \mathbb{T}^{\infty}_{\mathsf{rec}}$ and $f \in \Pi(T)$. If $\sigma \in W_{\mathcal{D},f}(T)$ and $\gamma \simeq \sigma$ then $\gamma \in W_{\mathcal{D},f}(T)$.*

Proof. We proceed by coinduction on T. By Definition 5.5.1 $\sigma \simeq \mu_0 \to \cdots \to \mu_{f(0)} \to \sigma'$ and there exists $\tau \in \mu_{f(0)}$ such that $\tau \in W_{f^{\geqslant 1}}(T_{\langle f(0)\rangle})$. As $\gamma \simeq \sigma \simeq \mu_0 \to \cdots \to \mu_{f(0)} \to \sigma'$, there must exist $\nu_0, \cdots, \nu_{f(0)} \in I_{\mathcal{D}}$ and $\gamma' \in T_{\mathcal{D}}$ such that $\gamma \simeq \nu_0 \to \cdots \to \nu_{f(0)} \to \gamma'$. In particular $\nu_{f(0)} \simeq \mu_{f(0)}$. Hence there is $\psi \in \nu_{f(0)}$ such that $\psi \simeq \tau$. Since $\tau \in W_{f^{\geqslant 1}}(T_{\langle f(0)\rangle})$, by coIH we have that $\psi \in W_{f^{\geqslant 1}}(T_{\langle f(0)\rangle})$. So $\gamma \in W_f(T)$. □

As explained in the introduction of the chapter, in an ergm $W_{\mathcal{D}}(T)$ is constituted by those $\sigma \in T_{\mathcal{D}}$ such that $\sigma \to \sigma \notin [\![\mathbf{J}_T]\!]$. To prove this, we first need a couple of technical lemmas.

Lemma 5.5.3. *Let $\mathcal{D}$ be an rgm and $T \in \mathbb{T}_{\mathsf{rec}}$. If $a \in \mathrm{BT}(\mathbf{J}_T x)^*$ and $\Gamma \vdash a : \sigma$ is derivable then $\Gamma = x : \gamma$ for some $\gamma \simeq \sigma$.*

Proof. By Lemma 2.3.10 $\mathrm{supp}(\Gamma) \subseteq \mathrm{fv}(a) = \{x\}$. Since x is in head position in a we have $\{x\} \subseteq \mathrm{supp}(\Gamma)$. So $\mathrm{supp}(\Gamma) = \{x\}$, namely $\Gamma = x : \mu$ for some $\mu \in I_{\mathcal{D}}$.

Clearly $a \neq \bot$, since a is typable. So by Theorem 5.2.9 we have $a = \lambda x_1 \ldots x_m.x\, a_1 \cdots a_m$ with $a_i \in \mathrm{BT}(\mathbf{J}_{T_{\langle i-1\rangle}} x_i)^*$ for all $i \in \{1, \ldots, m\}$, and in fact $m = T(0)$. From the derivability of $x : \mu \vdash \lambda x_1 \ldots x_m.x\, a_1 \cdots a_m : \sigma$ we get by Lemma 2.3.12(2) the derivability of the sequent $x : \mu, x_1 : \mu_0, \ldots, x_m : \mu_m \vdash x\, a_1 \cdots a_m : \tau$ for some $\mu_0 \to \cdots \to \mu_m \to \tau \simeq \sigma$.

By (applying m times) Lemma 2.3.12(3) we get a decomposition

$$\{\Delta\} \cup \{\Gamma_{ij} \mid i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n_i\}\}$$

of the environment $x : \mu, x_1 : \mu_1, \ldots, x_m : \mu_m$ making the following sequents derivable:

$$\Delta \vdash x : \nu_0 \to \cdots \to \nu_m \to \tau \tag{94}$$

and for all $i \in \{1, \ldots, m\}$ and for all $j \in \{1, \ldots, n_i\}$

$$\Gamma_{ij} \vdash a_i : \gamma_{ij}\,, \tag{95}$$

where $\nu_i = \bigwedge_{j=1}^{n_i} \gamma_{ij}$.

By Lemma 2.3.12(1) from (94) we get $\Delta = x : \gamma$ for a certain $\gamma \simeq \nu_0 \to \cdots \to \nu_m \to \tau$.

Since $a_i \in \mathrm{BT}(\mathbf{J}_{T_{\langle i-1\rangle}} x_i)^*$, by Lemma 5.5.3 from (95) we get $\nu_i \simeq \bigwedge_{j=1}^{n_i} \Gamma_{ij}(x_i) \simeq \mu_i$. Hence $\gamma \simeq \nu_0 \to \cdots \to \nu_m \to \tau \simeq \mu_0 \to \cdots \to \mu_m \to \tau \simeq \sigma$.

By Lemma 2.3.10 from (95) we get $\mathrm{supp}(\Gamma_{ij}) \subseteq \mathrm{fv}(a_i) \not\ni x$, hence $\Gamma_{ij}(x) = \omega$, for all i and j. In the end

$$\mu = \Gamma(x) = (\Delta \wedge \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{n_i} \Gamma_{ij})(x) = \Delta(x) \wedge \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{n_i} \Gamma_{ij}(x) = \Delta(x) = \gamma .$$

This completes the proof of the fact that $\Gamma = x : \gamma$ for some $\gamma \simeq \sigma$. □

Lemma 5.5.4. *Let $\mathcal{D}$ be an rgm. Let $T \in \mathbb{T}^{\infty}_{\mathsf{rec}}$ and $\sigma \in W_{\mathcal{D}}(T)$. For all $a \in \mathrm{BT}(\mathbf{J}_T x)^*$ we have $(x : \sigma, \sigma) \notin [\![a]\!]$. Equivalently, for all $a \in \mathrm{BT}(\mathbf{J}_T)^*$ we have $\sigma \to \sigma \notin [\![a]\!]$.*

Proof. Let $a \in \mathrm{BT}(\mathbf{J}_T x)^*$. We prove $x : \sigma \not\vdash a : \sigma$ by induction on a.

Case $a = \bot$. Trivial.

Case $a = \lambda x_1 \ldots x_m . x\, a_1 \cdots a_n$. We have $n = m = T(\varepsilon)$ by Theorem 5.2.9. Notice that $n > 0$, since the naked tree T underlying $\mathrm{BT}(\mathbf{J}_T x)$ is infinite.

By hypothesis there exists $f \in \Pi(T)$ such that $\sigma \in W_f(T)$. So $f(0) < n$ and considering

$$\sigma \simeq \mu_0 \to \cdots \to \mu_{f(0)} \to \cdots \to \mu_{n-1} \to \sigma' \tag{96}$$

there exists $\tau \in \mu_{f(0)}$ such that $\tau \in W_{f^{\geqslant 1}}(T_{\langle f(0)\rangle})$.

By way of contradiction let $x : \sigma \vdash a : \sigma$ be derivable. From (96) the sequent

$$x : \sigma \vdash \lambda x_1 \ldots x_n . x\, a_1 \cdots a_n : \mu_0 \to \cdots \to \mu_{f(0)} \to \cdots \to \mu_{n-1} \to \sigma'$$

is derivable by applying rule eq. Then by Lemma 2.3.12(2) the sequent

$$x : \sigma, x_1 : \nu_0, \ldots, x_n : \nu_{n-1} \vdash x\, a_1 \cdots a_n : \sigma''$$

is derivable for $\sigma'' \simeq \sigma'$ and $\nu_i \simeq \mu_i$ for all $i \in \{0, \ldots, n-1\}$.

By (applying n times) Lemma 2.3.12(3) we get a decomposition

$$\{\Delta\} \cup \{\Gamma_{ij} \mid i \in \{0, \ldots, n-1\} \text{ and } j \in \{1, \ldots, n_i\}\}$$

of the environment $x : \sigma, x_1 : \nu_0, \ldots, x_n : \nu_{n-1}$ making the following sequents derivable:

$$\Delta \vdash x : \nu'_0 \to \cdots \to \nu'_{f(0)} \to \cdots \to \nu'_{n-1} \to \sigma''$$

and for all $i \in \{0, \ldots, n-1\}$ and for all $j \in \{1, \ldots, n_i\}$

$$\Gamma_{ij} \vdash a_{i+1} : \gamma_{ij} , \tag{97}$$

where

$$\nu'_i = \bigwedge_{j=1}^{n_i} \gamma_{ij} \quad \text{for some } n_i \in \mathbb{N} . \tag{98}$$

Since $\tau \in \mu_{f(0)} \simeq \nu_{f(0)} \simeq \nu'_{f(0)}$, there exists $\gamma_{f(0)j} \in \nu'_{f(0)}$ such that $\gamma_{f(0)j} \simeq \tau$. In particular we get from (97) the derivability of

$$\Gamma_{f(0)j} \vdash a_{f(0)+1} : \gamma_{f(0)j} . \tag{99}$$

Since $a_{f(0)+1} \in \mathrm{BT}(\mathbf{J}_{T_{\langle f(0)\rangle}} x_{f(0)+1})^*$, by Lemma 5.5.3 and (99) we have $\Gamma_{f(0)\,j} = x_{f(0)+1} : \gamma$ for some $\gamma \simeq \gamma_{f(0)\,j}$. By applying rule eq to (99) we then derive

$$x_{f(0)+1} : \gamma \;\vdash\; a_{f(0)+1} : \gamma\,. \tag{100}$$

Since $\tau \in W_{f^{\geqslant 1}}(T_{\langle f(0)\rangle})$ and $\gamma \simeq \gamma_{f(0)\,j} \simeq \tau$, by Lemma 5.5.2 we have $\gamma \in W_{f^{\geqslant 1}}(T_{\langle f(0)\rangle})$. Therefore by IH we should have $x_{f(0)+1} : \gamma \nvdash a_{f(0)+1} : \gamma$, contradicting (100). □

Proposition 5.5.5. *For any ergm* $\mathcal{D}$ *and any tree* $T \in \mathbb{T}^{\infty}_{\mathsf{rec}}$

$$W_{\mathcal{D}}(T) \;=\; \left\{\sigma \in T_{\mathcal{D}} \mid \sigma \to \sigma \notin \left[\!\left[\mathbf{J}_T \right]\!\right]\right\}.$$

Proof. ($\subseteq$) By Lemma 5.5.4 when $\sigma \in W(T)$ we have $\sigma \to \sigma \notin \bigcup_{a \in \mathrm{BT}(\mathbf{J}_T)^*} [\![a]\!] = \left[\!\left[\mathbf{J}_T \right]\!\right]$, where the last equality is given by Theorem 2.6.5 (Böhm Approximation).

($\supseteq$) Let $n := T(\varepsilon)$. Consider $\sigma \in T_{\mathcal{D}}$ such that $x : \sigma \nvdash \mathbf{J}_T : \sigma$, which is equivalent to

$$x : \sigma \;\nvdash\; \lambda x_1 \ldots x_n . x \left(\mathbf{J}_{T_{\langle 0\rangle}}\, x_1\right) \cdots \left(\mathbf{J}_{T_{\langle n-1\rangle}}\, x_n\right) : \sigma \tag{101}$$

by Theorems 5.2.9 and 2.4.10. We coinductively construct $f \in \Pi(T)$ such that $\sigma \in W_f(T)$.

Since $\mathcal{D}$ is extensional, there exist $\mu_0, \ldots, \mu_{n-1} \in I_{\mathcal{D}}$ and $\sigma' \in T_{\mathcal{D}}$ such that $\sigma \simeq \mu_0 \to \cdots \to \mu_{n-1} \to \sigma'$. There must exist $i < n$ and $\tau \in \mu_i$ such that $x_{i+1} : \tau \nvdash \mathbf{J}_{T_{\langle i\rangle}} x_{i+1} : \tau$, because otherwise one would contradict (101) by deriving

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{x : \sigma \vdash x : \sigma}}{x : \sigma \vdash x : \mu_0 \to \cdots \to \mu_{n-1} \to \sigma'} \quad x_{i+1} : \tau \vdash \mathbf{J}_{T_{\langle i\rangle}} x_{i+1} : \tau \ \text{ for all } i \in \{0, \ldots, n-1\} \text{ for all } \tau \in \mu_i}{x : \sigma, x_1 : \mu_0, \ldots, x_n : \mu_{n-1} \;\vdash\; x \left(\mathbf{J}_{T_{\langle 0\rangle}}\, x_1\right) \cdots \left(\mathbf{J}_{T_{\langle n-1\rangle}}\, x_n\right) : \sigma'}}{x : \sigma \;\vdash\; \lambda x_1 \ldots x_n . x \left(\mathbf{J}_{T_{\langle 0\rangle}}\, x_1\right) \cdots \left(\mathbf{J}_{T_{\langle n-1\rangle}}\, x_n\right) : \mu_0 \to \cdots \to \mu_{n-1} \to \sigma'}}{x : \sigma \;\vdash\; \lambda x_1 \ldots x_n . x \left(\mathbf{J}_{T_{\langle 0\rangle}}\, x_1\right) \cdots \left(\mathbf{J}_{T_{\langle n-1\rangle}}\, x_n\right) : \sigma}$$

By coIH there exists $g \in \Pi(T_{\langle i\rangle})$ such that $\tau \in W_g(T_{\langle i\rangle})$. We define $f : \mathbb{N} \to \mathbb{N}$ by setting $f(0) := i$ and $f(n+1) := g(n)$ for all $n \in \mathbb{N}$. From $i < n = T(\varepsilon)$ and $g \in \Pi(T_{\langle i\rangle})$ we get that $f \in \Pi(T)$. From $\tau \in \mu_i$ and $\tau \in W_g(T_{\langle i\rangle})$ we obtain $\sigma \in W_f(T)$. □

At the end of § 4.4 we proved that $\sigma \in [\![\mathbf{I}]\!]^{\mathcal{D}_\star} - [\![\mathbf{J}]\!]^{\mathcal{D}_\star}$ whenever $\sigma \simeq \star$. That was a foretaste of what we have just seen: we were demonstrating that every $\sigma \simeq \star$ is a witness in $\mathcal{D}_\star$ for the tree that underlies $\mathrm{BT}(\mathbf{J})$. Indeed, one can remark the resemblance (even for what concerns the proofs) of Lemma 4.4.12 and Proposition 4.4.13 there respectively to Lemma 5.5.4 and Proposition 5.5.5 here.

5.6 λ-König relational graph models

As we prove in this section, the following notion (together with extensionality) characterizes the observability in the sense of Morris within the class of rgm's.

Definition 5.6.1. An rgm $\mathcal{D}$ is λ-*König* if and only if $W_{\mathcal{D}}(T) \neq \emptyset$ for every $T \in \mathbb{T}^{\infty}_{\mathsf{rec}}$.

The name is a clear reference to König's Lemma, which is indeed indispensable to assure the existence of the infinite branch f of T followed by the witness.

Here are the main ideas behind the proof of the characterization.

1. The 'Morris Separation' (Theorem 5.3.8) allows to reduce our specific full abstraction problem to the search for an rgm separating $\mathbf{I}$ from $\mathbf{J}_T$ for all $T \in \mathbb{T}^{\infty}_{\mathsf{rec}}$.

2. Let $\mathcal{D}$ be a λ-König ergm. Since every $T \in \mathbb{T}^{\infty}_{\mathsf{rec}}$ has a non-empty set of witnesses $W_{\mathcal{D}}(T)$ and Proposition 5.5.5 gives $W_{\mathcal{D}}(T) \subseteq \{\sigma \in T_{\mathcal{D}} \mid \sigma \to \sigma \notin [\![\mathbf{J}_T]\!]\}$, there is a type σ such that $\sigma \to \sigma \in [\![\mathbf{I}]\!] - [\![\mathbf{J}_T]\!]$. Thus, $\mathcal{D}$ separates $\mathbf{I}$ from all the $\mathbf{J}_T$'s for $T \in \mathbb{T}^{\infty}_{\mathsf{rec}}$. So $\mathcal{D}$ is inequationally fully abstract for Morris's observability for what we said in Point 1.

3. Proposition 5.5.5 provides another inclusion, i.e. $W_{\mathcal{D}}(T) \supseteq \{\sigma \in T_{\mathcal{D}} \mid \sigma \to \sigma \notin [\![\mathbf{J}_T]\!]\}$. As a consequence of this, λ-König ergm's turn out to be *all possible* fully abstract rgm's in the sense of Morris. Indeed, a model $\mathcal{D}$ inducing $\mathcal{H}^+$ must separate $\mathbf{I}$ from $\mathbf{J}_T$ for all $T \in \mathbb{T}^{\infty}_{\mathsf{rec}}$. So such a $\mathcal{D}$ makes the set $\{\sigma \in T_{\mathcal{D}} \mid \sigma \to \sigma \notin [\![\mathbf{J}_T]\!]\}$ non-empty. In the end $W_{\mathcal{D}}(T)$ is non-empty for all $T \in \mathbb{T}^{\infty}_{\mathsf{rec}}$, i.e. the model is λ-König.

4. By the way we find out that an rgm induces the preorder theory $\sqsubseteq_{\mathcal{H}^+}$ if and only if it induces the λ-theory $\mathcal{H}^+$. In other words, in this context the inequational approach to the problem is perfectly equivalent to the equational one, not a refinement of it.

Theorem 5.6.2. *Let $\mathcal{D}$ be an rgm. The following statements are equivalent.*

1. $\mathrm{Th}(\mathcal{D}) = \mathcal{H}^+$, *namely* $M =_{\mathcal{H}^+} N$ *if and only if* $[\![M]\!]^{\mathcal{D}} = [\![N]\!]^{\mathcal{D}}$ *for all* $M, N \in \Lambda$.

2. *the rgm $\mathcal{D}$ is λ-König and extensional.*

3. $\mathrm{Th}_{\sqsubseteq}(\mathcal{D})$ *is* $\sqsubseteq_{\mathcal{H}^+}$, *namely* $M \sqsubseteq_{\mathcal{H}^+} N$ *if and only if* $[\![M]\!]^{\mathcal{D}} \subseteq [\![N]\!]^{\mathcal{D}}$ *for all* $M, N \in \Lambda$.

In other words, an ergm is λ-König if and only if it is inequationally fully abstract for Morris's observability, which is the case if and only if it is equationally fully abstract for Morris's observability.

Proof. ($1 \Rightarrow 2$) Obviously $\mathcal{D}$ must be extensional since $\mathcal{H}^+$ is an extensional λ-theory. By contradiction let us suppose that $\mathcal{D}$ is not λ-König. Then there exists $T \in \mathbb{T}^{\infty}_{\mathsf{rec}}$ such that $W_{\mathcal{D}}(T) = \emptyset$. By Proposition 5.5.5 then $[\![\mathbf{I}]\!] = [\![\mathbf{J}_T]\!]$. This is impossible since $\mathbf{I} \neq_{\mathcal{H}^+} \mathbf{J}_T$.

($2 \Rightarrow 3$) The fact that $M \sqsubseteq_{\mathcal{H}^+} N$ implies $[\![M]\!] \subseteq [\![N]\!]$ is given by the implication $2 \Rightarrow 1$ in Theorem 4.3.2. (Notice that such implication is proved therein relying only on the extensionality of $\mathcal{D}$, an hypothesis that we have also here.)

We are left to show the opposite implication in 3. By the way of contradiction let us assume that $[\![M]\!] \subseteq [\![N]\!]$ but $M \not\sqsubseteq_{\mathcal{H}^+} N$.

By maximality of $\sqsubseteq_{\mathcal{H}^*}$ among the sensible preorders, $[\![M]\!] \subseteq [\![N]\!]$ implies $M \sqsubseteq_{\mathcal{H}^*} N$.

We can then apply Theorem 5.3.8 (Morris Separation) and get a context $C[-]$ such that $C[M] =_{\beta} \mathbf{I}$ and $C[N] =_{\mathcal{B}} \mathbf{J}_T$ for some $T \in \mathbb{T}^{\infty}_{\mathsf{rec}}$.

By Theorem 2.4.10 (β-Soundness) $C[M] =_{\beta} \mathbf{I}$ implies $[\![C[M]]\!] = [\![\mathbf{I}]\!]$.

Since $\mathcal{B} \subseteq \mathrm{Th}(\mathcal{D})$ by Theorem 2.6.6(2), from $C[N] =_{\mathcal{B}} \mathbf{J}_T$ we get $[\![C[N]]\!] = [\![\mathbf{J}_T]\!]$.

By Lemma 2.4.4, i.e. contextuality of $[\![-]\!]$, from $[\![M]\!] \subseteq [\![N]\!]$ we obtain

$$[\![\mathbf{I}]\!] \;=\; [\![C[M]]\!] \;\subseteq\; [\![C[N]]\!] \;=\; [\![\mathbf{J}_T]\!] \,. \tag{102}$$

By Definition 5.6.1 there exists $\sigma \in W_{\mathcal{D}}(T)$. By Proposition 5.5.5 we have $\sigma \to \sigma \notin [\![\mathbf{J}_T]\!]$. Hence $\sigma \to \sigma \notin [\![\mathbf{I}]\!]$ by (102). Clearly this is a contradiction.

(3 ⇒ 1) Trivial. □

Actually, we have characterized the full abstraction fo $\mathcal{H}^+$ in the *whole* relational semantics, not only for what concerns rgm's. This follows immediately from Remark 2.2.3.

Corollary 5.6.3. *A relational model $\mathcal{D}$ is (in)equationally fully abstract for Morris's observability if and only if it is a λ-König ergm.*

Proof. Any reflexive object $\mathcal{D}$ in **MRel** fully abstract for $\mathcal{H}^+$ must be extensional, since such is $\mathcal{H}^+$. By Remark 2.2.3 then $\mathcal{D}$ is necessarily an ergm. We then apply Theorem 5.6.2. □

In § 5.1 we have defined two ergm's $\mathcal{L}$ and $\mathcal{L}^{\text{rec}}$, wondering if they are fully abstract for Morris's observability (preorder theory or λ-theory, the distinction has become pointless). We can now answer that question: only one of them is!

Theorem 5.6.4. *The rgm $\mathcal{L}$ defined in Examples 5.1.1 is fully abstract for Morris's preorder theory and λ-theory.*

Proof. Let $T \in \mathbb{T}^{\infty}_{\text{rec}}$ and take any $f \in \Pi(T)$ (there exists at least one such f by König's Lemma). Then $\alpha_f \in W_{\mathcal{L},f}(T)$ simply by the definition of the model $\mathcal{L}$. So the ergm $\mathcal{L}$ is λ-König and Theorem 5.6.2 gives the thesis. □

It is clear that in defining $\mathcal{L}$ we have put plenty of atoms α_f not really necessary in order to make it a λ-König rgm. We can minimize that number of atoms as follows.

Example 5.6.5. For every $T \in \mathbb{T}^{\infty}_{\text{rec}}$ choose a function $f_T \in \Pi(T)$, which exists of course by König's Lemma. Let $\mathcal{L}'$ be defined just like the rgm $\mathcal{L}$, but considering only the selected functions. In other words, we repeat the construction given in Example 5.1.1 but restricting to the set of $\aleph_0$ atoms

$$L' \;:=\; \{*\} \cup \left\{\beta^n, k_f\right\}^{n \in \mathbb{N},\, k \leqslant f(n)}_{f \in \left\{f_T : \mathbb{N} \to \mathbb{N} \;\mid\; T \in \mathbb{T}^{\infty}_{\text{rec}}\right\}} \,.$$

Theorem 5.6.6. *The rgm $\mathcal{L}'$ defined in Examples 5.6.5 is fully abstract for Morris's preorder theory and λ-theory.*

Proof. Just like the proof of Theorem 5.6.4. □

On the contrary, $\mathcal{L}^{\text{rec}}$ turns out not to induce $\sqsubseteq_{\mathcal{H}^+}$. The reason is a classical result in recursion theory, known as the *failure of the recursive version of König's Lemma*: there is an infinite computable tree, sometimes called *Kleene's Tree*, with no infinite computable branches.

Theorem 5.6.7 (Kleene, [Odi89, Theorem V.5.25])**.** *There exists $T_{\text{Kleene}} \in \mathbb{T}^{\infty}_{\text{rec}}$ such that every $f \in \Pi(T_{\text{Kleene}})$ is not computable.*

Lemma 5.6.8. *Let* $T \in \mathbb{T}^{\infty}_{rec}$ *and* $f : \mathbb{N} \to \mathbb{N}$. *If* $W_{\mathcal{L}^{rec},f}(T) \neq \emptyset$ *then* $\alpha^0_f \in W_{\mathcal{L}^{rec},f}(T)$.

Proof. By hypothesis there exists $\sigma \in W_{\mathcal{L}^{rec},f}(T)$. We proceed by induction on σ.
Case $\sigma \in At_{\mathcal{L}^{rec}}$. We have $\sigma = \beta^{n,k}_h$ where $h : \mathbb{N} \to \mathbb{N}$ is a computable function, $n \in \mathbb{N}$ and $k \leqslant h(n)$. Firstly, we prove the existence of an atom of the form α^0_g such that $\sigma \simeq \alpha^0_g$. By definition of the model $\beta^{n,k}_h \simeq \omega^k \to \beta^{n,h(n)}_h \to *$. Then clearly $\beta^{n,k}_h \simeq \beta^{n,\ell(n)}_\ell = \alpha^n_\ell$ whenever ℓ is the function defined as $\ell(n) := k$ and $\ell(m) := h(m)$ for all $m \neq n$. Of course such an atom α^n_ℓ exists since ℓ is computable. Finally, it is easy to realize that $\alpha^n_\ell \simeq \alpha^0_{\ell^{\geqslant n}}$, where the latter atom exists since also $\ell^{\geqslant n}$ is computable.

So we have $\sigma \simeq \alpha^0_g$. We show that $f = g$, so that $\alpha^0_f = \alpha^0_g \simeq \sigma \in W_{\mathcal{L}^{rec},f}(T)$ implies $\alpha^0_f \in W_{\mathcal{L}^{rec},f}(T)$ by Lemma 5.5.2.

By way of contradiction suppose $f \neq g$. Let $m := \min\{n \in \mathbb{N} \mid f(n) \neq g(n)\}$. By definition of $\mathcal{L}^{rec}$ we have

$$\alpha^m_g \simeq \overbrace{\omega \to \cdots \to \omega}^{g(m) \text{ times}} \to \alpha^{m+1}_g \to \overbrace{\omega \to \cdots \to \omega}^{k \text{ times}} \to * \tag{103}$$

for every $k \in \mathbb{N}$. As $f(m) \neq g(m)$, the intersection in position $f(m)+1$ on the right hand side of (103) is not the one in position $g(m)+1$, namely α^{m+1}_g. So it must be ω. Clearly this contradicts the hypothesis $\alpha^0_g \in W_{\mathcal{L}^{rec},f}(T)$, which requires such intersection to contain a witness for the tree $T_{\langle f(0),\ldots,f(m)\rangle}$.
Case $\sigma \notin At_{\mathcal{L}^{rec}}$. Let $h \in \mathbb{N}$ be the minimal natural number giving $\sigma = \nu_0 \to \cdots \to \nu_h \to \alpha^0_g$ for some $\alpha^0_g \in At_{\mathcal{L}^{rec}}$. Let $\sigma = \nu_0 \to \cdots \to \nu_h \to \alpha^0_g \simeq \mu_0 \to \cdots \to \mu_{f(0)} \to \tau$. We distinguish two subcases.

Subcase $f(0) > h$. In this case

$$\mu_{h+1} \to \cdots \to \mu_{f(0)} \to \tau \simeq \alpha^0_g \simeq \overbrace{\omega \to \cdots \to \omega}^{g(m) \text{ times}} \to \alpha^1_g \to \overbrace{\omega \to \cdots \to \omega}^{k \text{ times}} \to *$$

for every $k \in \mathbb{N}$. Since $\mu_{f(0)}$ must contain a witness for $T_{f(0)}$, in particular $\mu_{f(0)} \simeq \alpha^1_g$. So $\alpha^1_g \in W_{\mathcal{L}^{rec},\alpha^n_g,f1}(T_{f(0)})$ by Lemma 5.5.2. This implies $\alpha^0_g \in W_{\mathcal{L}^{rec},f}(T)$. Finally, the fact that $g = f$ is proved just like we have done in the basic case $\sigma \in At_{\mathcal{L}^{rec}}$ above.

Subcase $f(0) \leqslant h$. In this case $\nu_{f(0)} \simeq \mu_{f(0)}$. By hypothesis there exists $\sigma_1 \in \mu_{f(0)}$ such that $\sigma_1 \in W_{\mathcal{L}^{rec},f^{\geqslant 1}}(T_{f(0)})$. So there is $\tau_1 \in \nu_{f(0)}$ such that $\tau_1 \simeq \sigma_1$, and $\tau_1 \in W_{\mathcal{L}^{rec},f^{\geqslant 1}}(T_{f(0)})$ by Lemma 5.5.2. Since τ_1 is a subtype of σ, the IH can be applied to it. We obtain that $\alpha^0_{f^{\geqslant 1}} \in W_{\mathcal{L}^{rec},f^{\geqslant 1}}(T_{f(0)})$. Therefore $\alpha^0_f \in W_{\mathcal{L}^{rec},f}(T)$. ☐

Corollary 5.6.9. *Let* $f : \mathbb{N} \to \mathbb{N}$ *be non-computable. Then* $W_{\mathcal{L}^{rec},f}(T) = \emptyset$.

Proof. By way of contradiction suppose there is $\sigma \in W_{\mathcal{L}^{rec},f}(T)$. By Lemma 5.6.8 then $\alpha^0_f \in W_{\mathcal{L}^{rec},f}(T)$. But α^0_f does not even exist as a type of $\mathcal{L}^{rec}$, since f is not computable. ☐

Theorem 5.6.10. *The rgm* $\mathcal{L}^{rec}$ *defined in Examples 5.1.2 is not fully abstract for Morris's preorder theory and* λ*-theory.*

Proof. Take any $f \in \Pi(T_{\text{Kleene}})$ (there exists at least one such f by König's Lemma). By Theorem 5.6.7 the function $f : \mathbb{N} \to \mathbb{N}$ is not computable. Hence $W_{\mathcal{L}^{rec},f}(T) = \emptyset$ by Corollary 5.6.9. So the ergm $\mathcal{L}^{rec}$ is not λ-König and Theorem 5.6.2 gives the thesis. ☐

A

APPENDIX

Despite being most probably mathematical folklore, we could not find any literature references for Definition 1.2.3, namely the notion of *isomorphic reflexive objects* in a cartesian closed category. Hence any formal proofs of Theorem 1.2.4 and Lemma 1.2.5 given in the preliminaries. Just for the record, we provide those proofs in this appendix.

Let us remind Definition 1.2.3 here below.

Definition A.0.11. Let $\mathcal{D} = (D, \mathtt{Abs}, \mathtt{App})$ and $\mathcal{D}' = (D', \mathtt{Abs}', \mathtt{App}')$ be reflexive objects in a given cartesian closed category. An *isomorphism of reflexive objects* $f : \mathcal{D} \to \mathcal{D}'$ is an isomorphism $f : D \to D'$ making the two diagrams

$$\begin{array}{ccccc} D \Rightarrow D & \xrightarrow{\mathtt{Abs}} & D & \xrightarrow{\mathtt{App}} & D \Rightarrow D \\ {\scriptstyle f^{-1} \Rightarrow f}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^{-1} \Rightarrow f} \\ D' \Rightarrow D' & \xrightarrow{\mathtt{Abs}'} & D' & \xrightarrow{\mathtt{App}'} & D' \Rightarrow D' \end{array} \tag{104}$$

commute.

We recall some very basic facts from category theory.

Lemma A.0.12. *A final object in a category is unique up to isomorphism. Moreover, given two final objects in the category the isomorphism from one to the other is unique.*

Proof. Let T and T' be final. Let t_X be the only morphism from an object X to T, and t'_X the only one from X to T'. Then $t_{T'} \circ t'_T : T \to T$ is the only morphism from T to T, i.e. $t_{T'} \circ t'_T = t_T = \mathrm{id}_T$. Switching T and T' we get by the same argument $t'_T \circ t_{T'} = t'_{T'} = \mathrm{id}_{T'}$. So $t'_T : T \to T'$ is an isomorphism. Moreover it is the only one possible, as T' is final. □

Lemma A.0.13. *Let $\{X_i\}_{i \in I}$ be a family of objects of a category. If existing, a product of the family is unique up to isomorphism. Moreover, considered two such products $\{\pi_i : V \to X_i\}_{i \in I}$ and $\{\pi'_i : V' \to X_i\}_{i \in I}$ there exists a unique isomorphism from one to the other. In particular, whenever their vertexes V and V' are equal then the two products are equal, namely $\pi_i = \pi'_i$ for all $i \in I$.*

Proof. The statement is an instance of Lemma A.0.12, since a product of the family $\{X_i\}_{i \in I}$ is a final object in the category of cones from any object to $\{X_i\}_{i \in I}$. In particular, let $\{\pi_i : V \to X_i\}_{i \in I}$ and $\{\pi'_i : V \to X_i\}_{i \in I}$ be two products with the same vertex V. The unique isomorphism from the former product to the other must be the identity id_V. Such isomorphism is also the arrow associated with the cone $\{\pi_i : V \to X_i\}_{i \in I}$ according to the universal property of the product $\{\pi'_i : V \to X_i\}_{i \in I}$. So $\pi'_i = \pi_i \circ \mathrm{id}_V = \pi_i$ for all $i \in I$. □

Remember that every category $\mathcal{C}$ gives a functor $\mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$ defined on morphisms $f \in \mathcal{C}^{op}(A, A') = \mathcal{C}(A', A)$ and $g \in \mathcal{C}(B, B')$ by the function

$$\begin{array}{rccc} \mathcal{C}(f,g) : & \mathcal{C}(A,B) & \to & \mathcal{C}(A',B') \\ & h & \mapsto & g \circ h \circ f \end{array}$$

Lemma A.0.14. *Let* $f : \mathcal{D} \to \mathcal{D}'$ *be an isomorphism of reflexive objects in a cartesian closed category* $\mathcal{C}$. *Then for all* $M \in \Lambda$ *and for all finite sequence of variables* $x_1, \dots, x_n$ *such that* $\mathrm{fv}(M) \subseteq \vec{x}$ *we have* $f \circ [\![M]\!]^{\vec{x}}_{\mathcal{D}} \circ \left(f^{-1}\right)^n = [\![M]\!]^{\vec{x}}_{\mathcal{D}'}$.

Proof. Let $\mathcal{D} = \left(D, \mathtt{Abs}, \mathtt{App}\right)$ and $\mathcal{D}' = \left(D', \mathtt{Abs}', \mathtt{App}'\right)$. We proceed by induction on M.

Case $M = x_i$ **for some** $i \in \{1, \dots, n\}$**.** Let us call π_i the i-th projection of the product D^n and π'_i the i-th projection of the product $(D')^n$. By Definition 1.2.2 we have $[\![x_i]\!]^{\vec{x}}_{\mathcal{D}} = \pi_i$ and $[\![x_i]\!]^{\vec{x}}_{\mathcal{D}'} = \pi'_i$. So the thesis is

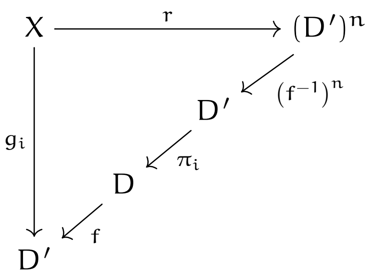
$$f \circ \pi_i \circ \left(f^{-1}\right)^n = \pi'_i \,. \tag{105}$$

We show that the cone

$$\left\{ f \circ \pi_i \circ \left(f^{-1}\right)^n : (D')^n \to D' \right\}_{i \in \{1,\dots,n\}} \tag{106}$$

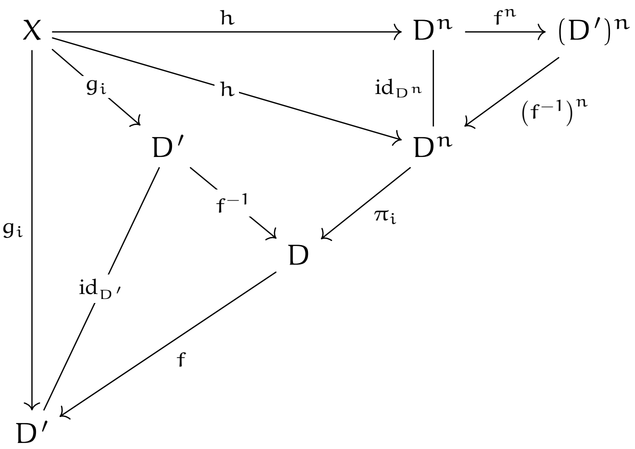
is a product of the family containing n times D'. Once we have that, since also the cone $\left\{ \pi'_i : (D')^n \to D' \right\}_{i \in \{1,\dots,n\}}$ is a product of that family and the two cones have the exact same vertex $(D')^n$, then they are equal by Lemma A.0.13, and in particular we get (105).

Proving that (106) is a product means to associate with every given family of arrows $\{g_i : X \to D'\}_{i \in \{1,\dots,n\}}$ a unique $r : X \to (D')^n$ making for every $i \in \{1, \dots, n\}$ the following diagram commute



(107)

Let h be the morphism associated with the family of arrows $\{f^{-1} \circ g_i : X \to D'\}_{i \in \{1,\dots,n\}}$ by the product $\{\pi_i : D^n \to D\}_{i\in\{1,\dots,n\}}$, namely the only h making the central diagram here below commute



Then clearly $r := f^n \circ h$ makes (107) commute.

In order to prove the unicity of r, suppose to have another $\tilde{r} : X \to (D')^n$ such that

$$f \circ \pi_i \circ \left(f^{-1}\right)^n \circ \tilde{r} \;=\; g_i$$

for all $i \in \{1,\dots,n\}$. Then

$$\pi_i \circ \left(f^{-1}\right)^n \circ \tilde{r} \;=\; f^{-1} \circ g_i\,. \tag{108}$$

Since h is the only arrow such that $\pi_i \circ h \;=\; f^{-1} \circ g_i$, from (108) we get $\left(f^{-1}\right)^n \circ \tilde{r} = h$. We conclude that $\tilde{r} = id_{(D')^n} \circ \tilde{r} = f^n \circ \left(f^{-1}\right)^n \circ \tilde{r} = f^n \circ h = r$.

Case $M = \lambda x.P$. The adjunction $- \times - \dashv - \Rightarrow -$ assures, *inter alia*, that the bijection $\Lambda_{A,B,C} : \mathcal{C}(A \times B, C) \to \mathcal{C}(A, B \Rightarrow C)$ is natural in A, B and C. In particular the diagram

$$\begin{array}{ccc}
\mathcal{C}(D^n \times D, D) & \xrightarrow{\;\Lambda_{D^n,D,D}\;} & \mathcal{C}(D^n, D \Rightarrow D) \\
\Big\downarrow{\scriptstyle \mathcal{C}((f^{-1})^n \times f^{-1}, f)} & & \Big\downarrow{\scriptstyle \mathcal{C}((f^{-1})^n, f^{-1} \Rightarrow f)} \\
\mathcal{C}(D'^n \times D', D') & \xrightarrow{\;\Lambda_{D'^n,D',D'}\;} & \mathcal{C}(D'^n, D' \Rightarrow D')
\end{array}$$

commutes. So for all $g : D^n \times D \to D$ we have

$$\left(f^{-1} \Rightarrow f\right) \circ \Lambda(g) \circ \left(f^{-1}\right)^n \;=\; \Lambda\Big(f \circ g \circ \left(\left(f^{-1}\right)^n \times f^{-1}\right)\Big) \;=\; \Lambda\Big(f \circ g \circ \left(f^{-1}\right)^{n+1}\Big)\,. \tag{109}$$

We deduce that

$$\begin{array}{rcll}
f \circ [\![\lambda x.P]\!]^{\vec{x}}_{\mathcal{D}} \circ \left(f^{-1}\right)^n & = & f \circ Abs \circ \Lambda\left([\![P]\!]^{x,\vec{x}}_{\mathcal{D}}\right) \circ \left(f^{-1}\right)^n & \text{by Definition 1.2.2} \\
& = & Abs' \circ \left(f^{-1} \Rightarrow f\right) \circ \Lambda\left([\![P]\!]^{x,\vec{x}}_{\mathcal{D}}\right) \circ \left(f^{-1}\right)^n & \text{by Definition A.0.11} \\
& = & Abs' \circ \Lambda\left(f \circ [\![P]\!]^{x,\vec{x}}_{\mathcal{D}} \circ \left(f^{-1}\right)^{n+1}\right) & \text{by (109)} \\
& = & Abs' \circ \Lambda\left([\![P]\!]^{x,\vec{x}}_{\mathcal{D}'}\right) & \text{by IH} \\
& = & [\![\lambda x.P]\!]^{\vec{x}}_{\mathcal{D}'} & \text{by Definition 1.2.2}
\end{array}$$

Case $M = PQ$**.** The adjunction $- \times - \dashv - \Rightarrow -$ assures, *inter alia*, that the bijection $\Lambda^{-1}_{A,B,C} : \mathcal{C}(A, B \Rightarrow C) \to \mathcal{C}(A \times B, C)$ is natural in A, B and C. In particular the diagram

$$
\begin{array}{ccc}
\mathcal{C}\big(D \Rightarrow D, D \Rightarrow D\big) & \xrightarrow{\Lambda^{-1}_{D\Rightarrow D,D,D}} & \mathcal{C}\big((D \Rightarrow D) \times D, D\big) \\
{\scriptstyle \mathcal{C}(D \Rightarrow D, D \Rightarrow f)} \big\downarrow & & \big\downarrow {\scriptstyle \mathcal{C}((D \Rightarrow D) \times D, f)} \\
\mathcal{C}\big(D \Rightarrow D, D \Rightarrow D'\big) & \xrightarrow{\Lambda^{-1}_{D\Rightarrow D,D,D'}} & \mathcal{C}\big((D \Rightarrow D) \times D, D'\big)
\end{array}
$$

commutes. This means that for all $g \in \mathcal{C}\big(D \Rightarrow D, D \Rightarrow D\big)$ we have

$$f \circ \Lambda^{-1}(g) \circ \mathrm{id}_{(D\Rightarrow D)\times D} \quad = \quad \Lambda^{-1}\Big((D \Rightarrow f) \circ g \circ \mathrm{id}_{D\Rightarrow D}\Big)\,. \tag{110}$$

By taking in particular as g the identity $\mathrm{id}_{D\Rightarrow D}$ we get

$$
\begin{aligned}
f \circ \Lambda^{-1}\big(\mathrm{id}_{D\Rightarrow D}\big) \quad &= \quad \Lambda^{-1}\big(D \Rightarrow f\big) \\
&= \quad \Lambda^{-1}\big(\big(f^{-1} \circ f\big) \Rightarrow \big(\mathrm{id}_{D'} \circ f\big)\big) \\
&= \quad \Lambda^{-1}\big(\big(f \Rightarrow D'\big) \circ \big(f^{-1} \Rightarrow f\big)\big)
\end{aligned} \tag{111}
$$

where the last equality depends on the fact that the bifunctor $- \Rightarrow -$ is contravariant in the first argument and covariant in the second one.

Exploiting once again the fact that $\Lambda^{-1}_{A,B,C} : \mathcal{C}(A, B \Rightarrow C) \to \mathcal{C}(A \times B, C)$ is natural in A, B and C, we get the commutativity of the following diagram

$$
\begin{array}{ccc}
\mathcal{C}\big(D \Rightarrow D, D' \Rightarrow D'\big) & \xrightarrow{\Lambda^{-1}_{D\Rightarrow D,D',D'}} & \mathcal{C}\big((D \Rightarrow D) \times D', D'\big) \\
{\scriptstyle \mathcal{C}(D \Rightarrow D, f \Rightarrow D')} \big\downarrow & & \big\downarrow {\scriptstyle \mathcal{C}((D \Rightarrow D) \times f, D')} \\
\mathcal{C}\big(D \Rightarrow D, D \Rightarrow D'\big) & \xrightarrow{\Lambda^{-1}_{D\Rightarrow D,D,D'}} & \mathcal{C}\big((D \Rightarrow D) \times D, D'\big)
\end{array}
$$

This means that for all $g \in \mathcal{C}\big(D \Rightarrow D, D' \Rightarrow D'\big)$ we have

$$\mathrm{id}_{D'} \circ \Lambda^{-1}(g) \circ \big((D \Rightarrow D) \times f\big) \quad = \quad \Lambda^{-1}\Big(\big(f \Rightarrow D'\big) \circ g \circ \mathrm{id}_{D\Rightarrow D}\Big)\,.$$

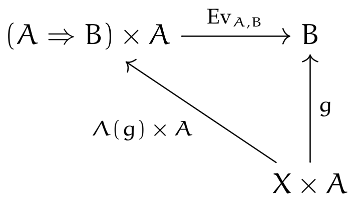
By taking in particular as g the morphism $f^{-1} \Rightarrow f$ we get

$$\Lambda^{-1}\big(f^{-1} \Rightarrow f\big) \circ \big((D \Rightarrow D) \times f\big) \quad = \quad \Lambda^{-1}\Big(\big(f \Rightarrow D'\big) \circ \big(f^{-1} \Rightarrow f\big)\Big)\,.$$

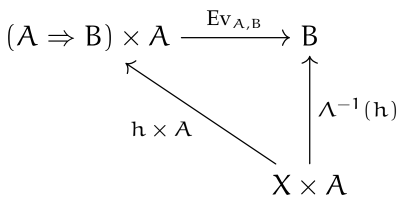
The last equality and (111) give by transitivity

$$f \circ \Lambda^{-1}\big(\mathrm{id}_{D\Rightarrow D}\big) \quad = \quad \Lambda^{-1}\big(f^{-1} \Rightarrow f\big) \circ \big((D \Rightarrow D) \times f\big)\,. \tag{112}$$

Also, remember that for all objects A, B of $\mathcal{C}$ and for every $g : X \times A \to B$ the morphism $\Lambda(g) : X \to A \Rightarrow B$ is the only one that makes the following diagram commute:



Since Λ is an isomorphism, this is equivalent to say that for all $h : X \to A \Rightarrow B$ the morphism $\Lambda^{-1}(h) : X \times A \to B$ is the only one that makes the following diagram commute:



(113)

In particular this implies that

$$\mathrm{Ev}_{A,B} \;=\; \mathrm{Ev}_{A,B} \circ (\mathrm{id}_{A\Rightarrow B} \times A) \;=\; \Lambda^{-1}(\mathrm{id}_{A\Rightarrow B})\,. \tag{114}$$

Now, we are interested in

$$\begin{aligned} f \circ [\![PQ]\!]^{\vec{x}}_{\mathcal{D}} \circ (f^{-1})^{n} &= f \circ \mathrm{Ev}_{D,D} \circ \langle\, [\![P]\!]^{\vec{x}}_{\mathcal{D}} \circ \mathrm{App}\,,\ [\![Q]\!]^{\vec{x}}_{\mathcal{D}} \,\rangle \circ (f^{-1})^{n} \\ &= f \circ \mathrm{Ev}_{D,D} \circ (\mathrm{App} \times D) \circ \langle\, [\![P]\!]^{\vec{x}}_{\mathcal{D}}\,,\ [\![Q]\!]^{\vec{x}}_{\mathcal{D}} \,\rangle \circ (f^{-1})^{n}\,. \end{aligned} \tag{115}$$

We have

$$\begin{aligned} f \circ \mathrm{Ev}_{D,D} \circ (\mathrm{App} \times D) &= f \circ \Lambda^{-1}(\mathrm{id}_{D\Rightarrow D}) \circ (\mathrm{App} \times D) && \text{by (114)} \\ &= \Lambda^{-1}(f^{-1}\!\Rightarrow f) \circ ((D \Rightarrow D) \times f) \circ (\mathrm{App} \times D) && \text{by (112)} \\ &= \Lambda^{-1}(f^{-1}\!\Rightarrow f) \circ (\mathrm{App} \times f) && \text{as } - \times - \text{ is functor} \\ &= \mathrm{Ev}_{D',D'} \circ ((f^{-1}\!\Rightarrow f) \times D') \circ (\mathrm{App} \times f) && \text{by (113)} \\ &= \mathrm{Ev}_{D',D'} \circ ((f^{-1}\!\Rightarrow f) \circ \mathrm{App}) \times (D' \circ f) && \text{as } - \times - \text{ is functor} \\ &= \mathrm{Ev}_{D',D'} \circ (\mathrm{App}' \circ f) \times (D' \circ f) && \text{by Definition A.0.11} \\ &= \mathrm{Ev}_{D',D'} \circ (\mathrm{App}' \times D') \circ (f \times f) && \text{as } - \times - \text{ is functor} \end{aligned}$$

So continuing from (115) we get

$$\begin{aligned} f \circ [\![PQ]\!]^{\vec{x}}_{\mathcal{D}} \circ (f^{-1})^{n} &= \mathrm{Ev}_{D',D'} \circ (\mathrm{App}' \times D') \circ (f \times f) \circ \langle\, [\![P]\!]^{\vec{x}}_{\mathcal{D}}\,,\ [\![Q]\!]^{\vec{x}}_{\mathcal{D}} \,\rangle \circ (f^{-1})^{n} \\ &= \mathrm{Ev}_{D',D'} \circ (\mathrm{App}' \times D') \circ \langle\, f \circ [\![P]\!]^{\vec{x}}_{\mathcal{D}}\,,\ f \circ [\![Q]\!]^{\vec{x}}_{\mathcal{D}} \,\rangle \circ (f^{-1})^{n} \\ &= \mathrm{Ev}_{D',D'} \circ (\mathrm{App}' \times D') \circ \langle\, f \circ [\![P]\!]^{\vec{x}}_{\mathcal{D}} \circ (f^{-1})^{n}\,,\ f \circ [\![Q]\!]^{\vec{x}}_{\mathcal{D}} \circ (f^{-1})^{n} \,\rangle \\ &= \mathrm{Ev}_{D',D'} \circ (\mathrm{App}' \times D') \circ \langle\, [\![P]\!]^{\vec{x}}_{\mathcal{D}'}\,,\ [\![Q]\!]^{\vec{x}}_{\mathcal{D}'} \,\rangle \\ &= [\![PQ]\!]^{\vec{x}}_{\mathcal{D}'} \end{aligned}$$

where the penultimate equality is clearly given by IH. □

We can now prove Theorem 1.2.4 from the preliminaries, restated here below.

Theorem A.0.15. *Let $\mathcal{D}$ and $\mathcal{D}'$ be isomorphic reflexive objects in a cartesian closed category. Then for all* $M, N \in \Lambda$ *and for all finite sequence of variables* $x_1, \dots, x_n$ *such that* $\mathrm{fv}(MN) \subseteq \vec{x}$ *we have* $[\![M]\!]^{\vec{x}}_{\mathcal{D}} = [\![N]\!]^{\vec{x}}_{\mathcal{D}}$ *if and only if* $[\![M]\!]^{\vec{x}}_{\mathcal{D}'} = [\![N]\!]^{\vec{x}}_{\mathcal{D}'}$.

Proof. Let $f : \mathcal{D} \to \mathcal{D}'$ be an isomorphism of reflexive objects. Let us prove the equivalence

$$[\![M]\!]^{\vec{x}}_{\mathcal{D}} = [\![N]\!]^{\vec{x}}_{\mathcal{D}} \quad\iff\quad f \circ [\![M]\!]^{\vec{x}}_{\mathcal{D}} \circ (f^{-1})^n = f \circ [\![N]\!]^{\vec{x}}_{\mathcal{D}} \circ (f^{-1})^n . \tag{116}$$

from which the thesis immediately follows by Lemma A.0.14.

The left-to-right implication of (116) is obvious. The right-to-left implication follows from the fact that f and $(f^{-1})^n$ are isomorphisms, since we have

$$\begin{array}{rcll} f \circ [\![M]\!]^{\vec{x}}_{\mathcal{D}} \circ (f^{-1})^n & = & f \circ [\![N]\!]^{\vec{x}}_{\mathcal{D}} \circ (f^{-1})^n & \Rightarrow \\ f^{-1} \circ f \circ [\![M]\!]^{\vec{x}}_{\mathcal{D}} \circ (f^{-1})^n \circ f^n & = & f^{-1} \circ f \circ [\![N]\!]^{\vec{x}}_{\mathcal{D}} \circ (f^{-1})^n \circ f^n & \Rightarrow \\ \mathrm{id}_D \circ [\![M]\!]^{\vec{x}}_{\mathcal{D}} \circ (f^{-1} \circ f)^n & = & \mathrm{id}_D \circ [\![N]\!]^{\vec{x}}_{\mathcal{D}} \circ (f^{-1} \circ f)^n & \Rightarrow \\ [\![M]\!]^{\vec{x}}_{\mathcal{D}} \circ \mathrm{id}_{D^n} & = & [\![N]\!]^{\vec{x}}_{\mathcal{D}} \circ \mathrm{id}_{D^n} & \Rightarrow \\ [\![M]\!]^{\vec{x}}_{\mathcal{D}} & = & [\![N]\!]^{\vec{x}}_{\mathcal{D}} & \end{array}$$

which concludes the proof of (116). □

We are now going to prove Lemma 1.2.5, i.e. the fact that when it comes to isomorphisms between *extensional* reflexive objects we do not need to check the commutativity of the right diagram of (104) in Definition A.0.11.

Definition A.0.16. Let $\mathcal{C}$ be a category. A morphism $f \in \mathcal{C}(A, B)$ is an *epi* if and only if for every object C and for all morphisms $g, h \in \mathcal{C}(B, C)$ whenever $g \circ f = h \circ f$ then $g = h$.

Lemma A.0.17. *Let* $(D, \mathtt{Abs}, \mathtt{App})$ *and* $(D', \mathtt{Abs}', \mathtt{App}')$ *be reflexive objects in a cartesian closed category. Let* $\mathtt{Abs}$ *be an epi. If the isomorphism* $f : D \to D'$ *makes the left diagram of (104) commute, then* f *is an isomorphism of reflexive objects.*

Proof. By Definition 1.2.1 we have $\mathtt{App} \circ \mathtt{Abs} = \mathrm{id}_{D \Rightarrow D}$ and $\mathtt{App}' \circ \mathtt{Abs}' = \mathrm{id}_{D' \Rightarrow D'}$. So the outermost diagram in (104) commutes, i.e.

$$(f^{-1} \Rightarrow f) \circ \mathtt{App} \circ \mathtt{Abs} = \mathtt{App}' \circ \mathtt{Abs}' \circ (f^{-1} \Rightarrow f) . \tag{117}$$

By hypothesis the left one also commutes, meaning that $\mathtt{Abs}' \circ (f^{-1} \Rightarrow f) = f \circ \mathtt{Abs}$. Therefore from (117) we get

$$(f^{-1} \Rightarrow f) \circ \mathtt{App} \circ \mathtt{Abs} = \mathtt{App}' \circ f \circ \mathtt{Abs} .$$

Since $\mathtt{Abs}$ is epi, the last equality entails $(f^{-1} \Rightarrow f) \circ \mathtt{App} = \mathtt{App}' \circ f$, which is the commutativity of the right diagram of (104). □

Lemma A.0.18. *Let* $(D, \mathtt{Abs}, \mathtt{App})$ *be an extensional reflexive object in a cartesian closed category. Then* $\mathtt{Abs}$ *is an epi.*

Proof. Consider two arrows $g, h : D \to C$ such that $g \circ \mathtt{Abs} = h \circ \mathtt{Abs}$, from which we get $g \circ \mathtt{Abs} \circ \mathtt{App} = h \circ \mathtt{Abs} \circ \mathtt{App}$. Since $\mathtt{Abs} \circ \mathtt{App} = \mathrm{id}_D$ by Definition 1.2.1, which means $g \circ \mathrm{id}_D = h \circ \mathrm{id}_D$, i.e. the thesis $g = h$. □

We can finally prove Lemma 1.2.5 from the preliminaries, restated here below.

Lemma A.0.19. *Let $\mathcal{D}$ and $\mathcal{D}'$ be reflexive objects in a cartesian closed category. In particular let $\mathcal{D}$ be extensional. If the isomorphism $f : D \to D'$ makes the left diagram of (104) commute, then f is an isomorphism of reflexive objects.*

Proof. By Lemma A.0.17 and Lemma A.0.18. □

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Title: Relational Graph Models and Morris's Observability

ABSTRACT

This thesis is a contribution to the study of Church's untyped λ-calculus, a term rewriting system having the β-*reduction* (the formal counterpart of the idea of *execution of programs*) as main rule. The focus is on *denotational semantics*, namely the investigation of mathematical models of the λ-calculus giving the same denotation to β-convertible λ-terms. We investigate *relational semantics*, a resource-sensitive semantics interpreting λ-terms as relations, with their inputs grouped together in multisets. We define a large class of relational models, called *relational graph models* (rgm's), and we study them in a type/proof-theoretical way, using some *non-idempotent intersection type systems*. Firstly, we find the minimal and maximal λ-theories (equational theories extending β-conversion) represented by the class. Then we use rgm's to solve the full abstraction problem for *Morris's observational λ-theory*, the contextual equivalence of programs that one gets by taking the β-normal forms as observable outputs. We solve the problem in different ways. Through a type-theoretical characterization of β-normalizability, we find infinitely many fully abstract rgm's, which we call *uniformly bottomless*. We then give an exhaustive answer to the problem, by showing that an rgm is fully abstract for Morris's observability if and only if it is extensional (a model of η-conversion) and λ-*König*. Intuitively an rgm is λ-König when every infinite computable tree has an infinite branch *witnessed* by some type of the model, where the *witnessing* is a property of non-well-foundedness on the type.

Titre: Modèles de Graphe Relationnels et Observabilité à la Morris

RÉSUMÉ

La thèse contribue à l'étude du λ-calcul non-typé de Church, un système de réécriture dont la règle principale est la β-*réduction* (formalisant l'*exécution d'un programme*). Nous nous concentrons sur la *sémantique dénotationnelle*, l'étude de modèles du λ-calcul interprétant de la même façon les λ-termes β-convertibles. On examine la *sémantique relationnelle*, une sémantique sensible aux ressources qui interprète les λ-termes comme des relations avec les entrées regroupées en multi-ensembles. Nous définissons une classe de modèles relationnels, les *modèles de graphe relationnels* (rgm's), que nous étudions avec une approche issue de la théorie des types et de la démonstrsation, par le biais de certains *systèmes de types avec intersection non-idémpotente*. D'abord, nous découvrons la plus petite et la plus grande λ-théorie (théorie equationnelle étendant la β-conversion) représentées dans la classe. Ensuite, nous utilisons les rgm's afin de résoudre le problème de l'adéquation complète pour la λ-*théorie observationnelle de Morris*, à savoir l'equivalence contextuelle de programmes que l'on obtient lorsqu'on prend les β-formes normales comme sorties observables. On résoudre le problème de différentes façons. En caractérisant la β-normalisabilité avec les types, nous découvrons une infinité de rgm's complètement adéquats, que nous appelons *uniformément sans fond*. Puis, nous résolvons le problème de façon exhaustive, en prouvant qu'un rgm est complètement adéquat pour l'observabilité de Morris si et seulement si il est extensionnel (il modèle l'η-conversion) et λ-*König*. Moralement un rgm est λ-König si tout arbre récursif infini a une branche infinie *témoignée* par un type non-bien-fondé.