UNIVERSITÉ PARIS 13
Laboratoire Analyse,
Géométrie et Applications, UMR 7539

THÈSE

présentée pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ PARIS 13

Discipline : Mathématiques

présentée et soutenue publiquement par :

Elizaveta VASILEVSKAYA

le 07 juillet 2016

Open periodic waveguides. Theory and computation

JURY

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Mme	Laurence HALPERN	Université Paris 13	*directrice*
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M.	Konstantin PANKRASHKIN	Université Paris-Sud	*rapporteur*
M.	Nicolas POPOFF	Université de Bordeaux	*examinateur*
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À ma famille

CONTENTS

INTRODUCTION

Motivation

The present work deals with study of periodic media. Periodic media play an important role in applications, such as solid state physics and optics. One talks about a periodic medium when the geometry and the physical characteristics of the problem are periodic functions. The periodicity of materials is observed at the atomic scale. Arranging different materials in repeating patterns can give rise to metamaterials. The properties of such a material are mainly no more determined by the properties of the materials it is composed of but by the way they are assembled. As a consequence, a metamaterial can present properties that do not exist in nature or are difficult to obtain.

Electromagnetic bandgap metamaterials affect light propagation. This is the case of photonic crystals which are structures composed of periodically alternated dielectric regions with high and low dielectric constants. The interest of these structures lies in the existence of intervals of "forbidden" frequences, i.e. frequences for which light cannot propagate in the medium (such intervals are called band gaps). The intervals of "permitted" frequences (for which light propagation is possible) are called spectral bands.

From the mathematical point of view, the existence of band gaps is explained by the spectral properties of the underlying periodic partial differential operators. The spectra of such operators are known to have a band-gap structure (Floquet-Bloch theory [15, 40]).

At the same time, introducing a perturbation to a perfectly periodic medium can lead to appearance of "permitted" frequences inside spectral gaps (which corresponds to the appearance of isolated eigenvalues of finite multiplicity for the underlying operator). One often talks about local perturbations and linear defects (the one considered in the present work). These eigenvalues inside gaps give rise to the so-called "trapped modes" (in the case of a local perturbation) and "guided modes" (in the case of a linear defect), which are, roughly speaking, solutions of the wave equation which are localized in the neighbourhood of the perturbation (a guided mode can be seen as a wave propagating along the defect and confined in the transversal direction). Such localized solutions are particularly interesting for applications such as design of lasers, filters and waveguides (cf. [33, 34]). The present work is devoted to the study of a particular type of waveguides that will be described in detail below. The aim is to create guided modes by introducing a geometrical perturbation of a purely periodic material.

State of the art

Two questions arise naturally when studying periodic media: the existence of gaps and the existence of eigenvalues inside gaps when a perturbation is introduced. Neither of them is completely answered. In the one-dimensional case, the gaps always exist except for constant media ([9]).

Necessary conditions of existence of gaps in higher dimensions are not known (in the one-dimensional case it is well-known that a periodic operator has no gaps if and only if it is constant, see [9]). However, some examples of sufficient conditions leading to the presence of gaps have been found in [20, 21, 52, 54, 3, 28, 29, 39] and references therein. According to the Bethe-Sommerfeld conjecture, a periodic operator in higher dimensions can only have a finite number of gaps (this is not true in the one-dimensional case, where, in general, a periodic operator has infinitely many gaps). The Bethe-Sommerfeld conjecture has been completely proved for the periodic Schrödinger operator ([55, 56]), but is still partially open for Maxwell equations ([67]).

For the second question, which is the possibility of creating eigenvalues inside gaps, some examples have been given in [18, 19, 1, 41, 46]. In these works strong contrasts in the properties of the medium are required in order to ensure the existence of eigenvalues. In [48, 49] guided modes are found in periodic lattices.

In the present work we consider open periodic waveguides having the geometry of a fattened grid described in more detail in Section 0.1. As the thickness of the grid tends to zero, the domain shrinks to a graph. We use then the classical approach of asymptotic analysis (used, for instance, in [20, 54]), which consists in approximating the problem in question by a limit one posed on a graph, for which the spectrum is easier to determine.

The convergence of the spectrum of operators in thin domains to the spectrum of the corresponding limit operator defined on a graph has been studied in the literature. In [61, 47] the convergence of the eigenvalues has been established in the case of bounded domains (which implies due to the Floquet-Bloch theory the convergence of the spectrum for periodic domains). In [57] this result has been extended to much more general domains (not necessarily bounded), for which the convergence of all components of the spectrum has been proved. In our case this implies the existence of eigenvalues for ε small enough. We propose though another (a less general but more explicit) proof based on the construction of a quasimode.

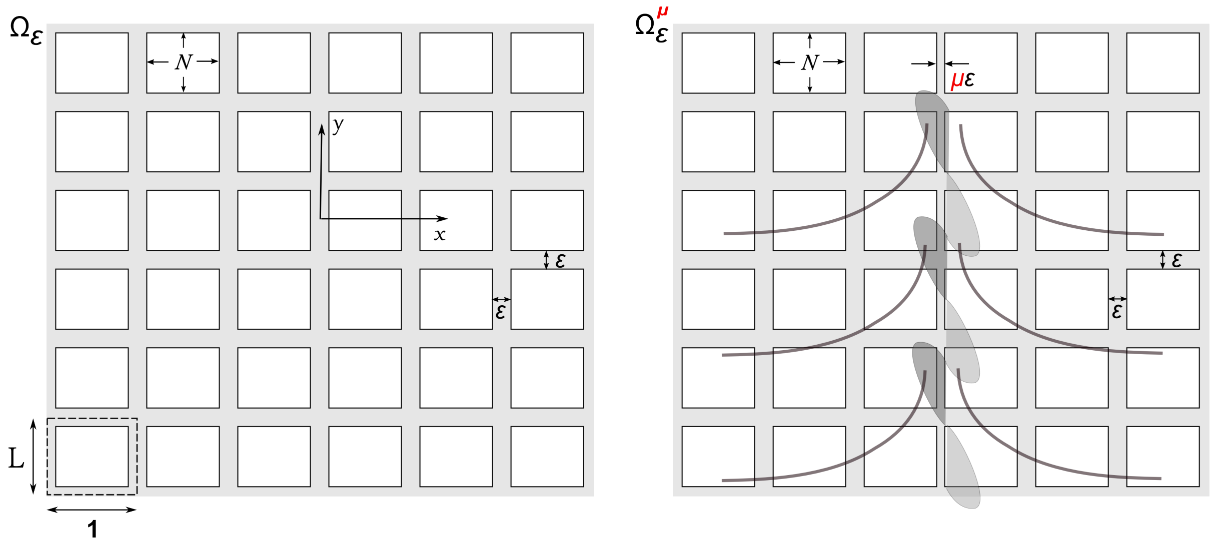
A periodic medium being infinite, this presents a difficulty for numerical study of localized modes. Several methods have been developed in order to overcome this difficulty. The most classical one is the "supercell" method which consists in truncating the computation domain far from the perturbation and solving the problem in the truncated domain with periodic boundary conditions. The localized modes being exponentially decaying, the solution of the problem in the truncated domain converges exponentially to the solution of the initial problem when the size of the domain tends to infinity ([10], [64], [62], [13]). Another method (and this is the one we use in the present work) is based on the Dirichlet-to-Neumann (DtN) approach developed in [24], [22], [23] (see also [38]). This method consists in replacing the initial eigenvalue problem in an infinite domain by a nonlinear eigenvalue problem posed in a neighbourhood of the perturbation (not necessarily big) via special DtN operators. The DtN operators depending themselves on the frequency, the obtained problem is thus nonlinear. It is solved using fix point type methods.

0.1 Description of the problem

In this present work we study the propagation of acoustic waves described by the scalar wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u, \tag{0.1.1}$$

and homogeneous Neumann boundary conditions in open waveguides having the geometry shown in figure 1. The propagation domain (in grey) is supposed homogeneous. It can be seen as $\mathbb{R}^2$ minus an infinite set of periodically spaced rectangular obstacles of size $1 \times L$. The distance between the obstacles, denoted by ε, is supposed to be small.



(a) Periodic domain

(b) Perturbed domain

Figure 1: Propagation domain (grey area)

The principle question that we are interested in is the possibility of creating of guided modes in this type of waveguides. It is known that the existence of guided modes requires the introduction of a perturbation. The perturbation that we consider is a geometric one: without changing the properties of the medium (which is still homogeneous) we introduce a linear defect in its geometry by modifying the thickness of one infinite branch of the domain from ε to $\mu\varepsilon$ with some positive coefficient μ (cf. figure 1b). It turns out that for $\mu < 1$ (i.e. when the domain shrinks) guided modes do appear. We conjecture that for $\mu > 1$ (i.e. when the domain is enlarged) there are no guided modes (at least for ε small enough).

By creating a guided mode we mean, roughly speaking, searching a solution of the equation (0.1.1) with Neumann boundary conditions which propagates along the perturbation and is confined in the transversal direction. As explained in more detail in Chapter 4, this implies the following form for the solution:

$$u(x, y, t) = e^{i\omega t} v(x, y),$$

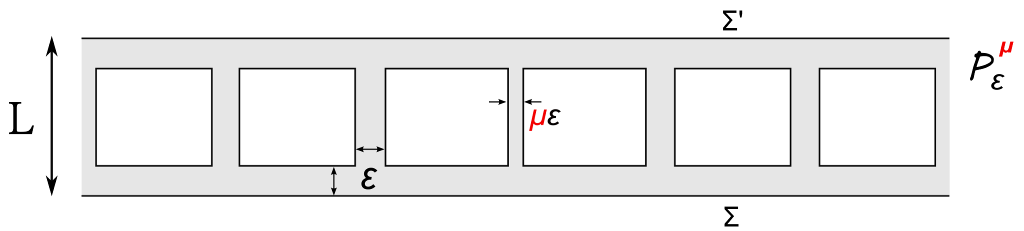
where the function v is β-quasiperiodic:

$$v(x, y + L) = e^{-i\beta} v(x, y).$$

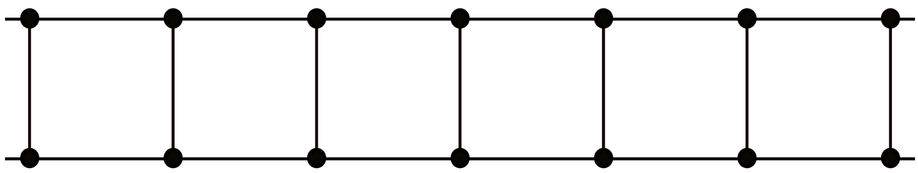
For the function v one gets then an eigenvalue problem for the Laplacian in a periodicity band $\mathcal{P}_\varepsilon^\mu$ of the domain Ω_ε^μ (cf. figure 2a) with β-quasiperiodic boundary conditions on the upper and lower parts of the boundary:

$$\text{Find} \quad v \in L_2\left(\mathcal{P}_\varepsilon^\mu\right) \quad \text{such that} \quad \begin{cases} \Delta v = -\omega^2 v, \\ v|_{\Sigma'} = e^{-i\beta}\, v|_{\Sigma}\,, \\ \partial_n v = 0 \quad \text{on the boundaries of the obstacles.} \end{cases}$$

Thus, we have to study the discrete spectrum (the so-called trapped modes) of the Lapla-



(a) The periodicity band $\mathcal{P}_\varepsilon^\mu$ ("ladder")



(b) The limit graph

cian with β-quasiperiodic conditions in the periodicity band $\mathcal{P}_\varepsilon^\mu$ that we call a "ladder". It turns out that the spectral properties of the Laplacian with β-quasiperiodic conditions in the "ladder" are very similar to the ones of the Laplacian with Neumann boundary conditions. For this reason, we consider the Neumann Laplacian in the ladder as a model problem and the first three chapters are devoted to the of study this model problem. In Chapter 4 we study the β-quasiperiodic case using the same scheme as the Neumann case.

General methodology

Both in the case of Neumann Laplacian and in the β-quasiperiodic case the method of study consists of three main steps:

1. **Identification of the (formal) limit problem.** As $\varepsilon \to 0$, the periodicity band shown in figure 2a (the "ladder") tends to a periodic graph (cf. figure 2b). Moreover, due to Neumann boundary conditions, a trapped mode in the "ladder" can be approximated by a one-dimensional function defined on the limit graph. This limit function is an eigenfunction of the limit operator defined as the second-order derivative operator on each edge of the graph completed by transmission conditions (called Kirchhoff's conditions) at the vertices of the graph ([17, 11, 47]). Let us mention that quantum graphs have been studied exhaustively in the literature: one can refer, for example, to the surveys [42, 43, 44] as well as the books [6, 58] and the bibliography therein.

2. **Computation of the spectrum of the limit problem.** We first investigate the essential spectrum using the Floquet-Bloch theory. Then, we compute the discrete spectrum using a reduction to a finite difference scheme ([2, 16]).

3. **Asymptotic analysis.** The last step is to show that the formal limit problem is indeed a good approximation of the initial one. In order to prove the existence of guided modes we need precisely the following assertion: if the limit operator has an eigenvalue λ_0 inside a gap $[a_0, b_0]$, then the non-limit operator also has an eigenvalue λ_ε inside a gap $[a_\varepsilon, b_\varepsilon]$ such that a_ε, b_ε, λ_ε are close to a_0, b_0, λ_0 respectively for ε small enough. Notice that the convergence of the spectrum of the non-limit operator to the spectrum of the limit one cannot be uniform.

Structure of the work

The present work is organized as follows.

In **Chapter 1** we study the spectrum of the limit problem for the ladder. We prove that the limit operator (more precisely, its symmetric part A_s^μ) has infinitely many gaps and one or two simple eigenvalues inside each gap if $\mu < 1$. For $\mu > 1$ the limit operator has no eigenvalues (Proposition 1.3.4 and Theorem 1.3.1). We then deduce by asymptotic analysis that for ε small enough the non-limit operator has arbitrarily many gaps (for any $k \in \mathbb{N}$ there exists $\varepsilon_k > 0$ such that for $\varepsilon < \varepsilon_k$ the operator $A^\mu_{\varepsilon,s}$ has at least k gaps and at least one or two eigenvalues in each of these gaps), cf. Theorems 1.4.1, 1.4.3. This last result concerning the eigenvalues is obtained by considering a "naive" quasimode that permits to show the convergence of the eigenvalues of the operator $A^\mu_{\varepsilon,s}$ to the ones of the operator A^μ_s at order $\sqrt{\varepsilon}$, which is not optimal (the convergence is actually linear in ε).

In **Chapter 2** we obtain a full asymptotic expansion of the eigenvalues of the operator $A^\mu_{\varepsilon,s}$ which proves at the same time the linear convergence of the eigenvalues. This is done by considering other quasimodes which are constructed using matched asymptotic expansions of the solution in the ladder. Far from the vertices the solution is modelized by one-dimensional functions defined on the graph (the so-called far field expansion). In the neighbourhood of the vertices a rescaling is done (the so-called near field expansion). Finally, both expansions are supposed to be valid in some intermediate areas called matching areas.

In **Chapter 3** the numerical approach to the problem is discussed. We first remind the Dirichlet-to-Neumann (DtN) operator method developed for numerical study of periodic media. More precisely, the initial eigenvalue problem on an unbounded domain can be reduced to a nonlinear eigenvalue problem posed on a bounded domain. Then, we give some details of implementation of this method using the P1 finite element discretization. Finally, we present numerical results for the problem in the ladder obtained using this method. We compare the results for the eigenvalues obtained via the DtN approach with the ones obtained by computing numerically the first terms of the full asymptotic expansion.

In **Chapter 4** we come back to the waveguide problem. The computations in this chapter are very similar to the ones performed in Chapter 1 for the ladder. Analogues of the results obtained for the operator $A^{\mu}_{\varepsilon,s}$ (existence of gaps and of eigenvalues inside gaps) are established in the β-quesiperiodic case for the operator $A^{\mu}_{\varepsilon}(\beta)$ (Theorems 4.1.2, 4.1.3). This shows the existence of guided modes in the case $\mu < 1$ for ε small enough. A slightly more general geometry is discussed in Section 4.1.4. Varying an additional parameter permits to influence the size of the gaps. In conclusion, numerical results for the β-quasiperiodic case are presented. A time-dependent simulation is described that shows the presence of a guided mode (in the temporal regime).

In **Chapter 5** we discuss a 3D generalization of the waveguide studied in Chapter 4. The limit problem on the corresponding graph is studied and analogues of the most part of the results obtained in the 2D case (existence of gaps and of eigenvalues inside gaps) are established, cf. Proposition 5.2.3 and Theorem 5.2.1. The equations describing the 3D case are very similar to the ones describing the 2D case. The conclusions for the non-limit operator are given in Theorems 5.3.1, 5.3.2. The principal difference between the 2D and the 3D case is that in the 3D the equations are less explicit, which makes the analysis somewhat more technical.

CHAPTER 1

TRAPPED MODES IN A LOCALLY PERTURBED PERIODIC LADDER: EXISTENCE RESULTS

1.1 Presentation of the problem

This chapter is devoted to the research of localized modes (also called trapped modes) in a ladder-like periodic domain (cf. figure 1.1). The domain Ω_ε is supposed homogeneous and consists of an infinite band of height L minus an infinite set of equispaced rectangular obstacles. The domain is 1-periodic (with respect to x). The distance between two consecutive obstacles and the distance from the obstacles to the boundary of the band is denoted by ε and is supposed to be small:

$$\Omega_\varepsilon = \left(\mathbb{R} \times \left]-\tfrac{L}{2}, \tfrac{L}{2}\right[\right) \setminus \bigcup_{j\in\mathbb{Z}} \mathcal{S}_{\varepsilon,j}, \qquad \mathcal{S}_{\varepsilon,j} = \left[j + \tfrac{\varepsilon}{2}, j + 1 - \tfrac{\varepsilon}{2}\right] \times \left[-\tfrac{L}{2} + \varepsilon, \tfrac{L}{2} - \varepsilon\right].$$

By a localized mode we mean a solution of the homogeneous scalar wave equation with

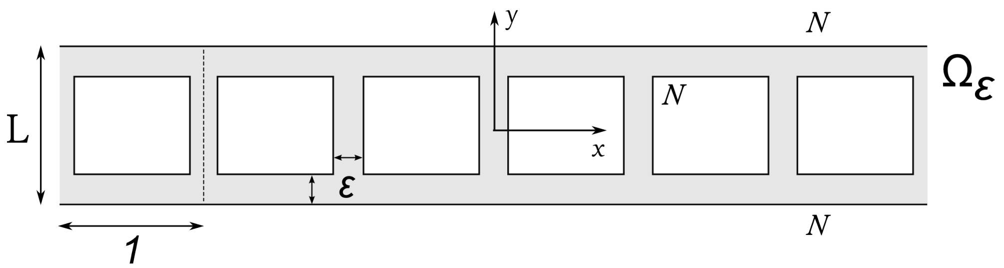


Figure 1.1: Propagation domain (in grey)

Neumann boundary conditions, i.e.,

$$\frac{\partial^2 u}{\partial t^2} = \Delta u \quad \text{in} \quad \Omega_\varepsilon, \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega_\varepsilon} = 0, \tag{1.1.1}$$

which is confined in the x-direction. More precisely, u is supposed to have the following form:

$$u(x,y,t) = v(x,y)e^{i\omega t}, \qquad v \in L_2(\Omega_\varepsilon), \tag{1.1.2}$$

where the term $e^{i\omega t}$ shows the harmonic dependence on time whereas the function v (which does not depend on the time) is in some sense confined (since it belongs to $L_2(\Omega_\varepsilon)$). Plugging (1.1.2) into (1.1.1) leads to the following problem for the function v:

$$\begin{cases} -\Delta v = \omega^2 v \quad \text{in} \quad \Omega_\varepsilon, \\ \left.\dfrac{\partial v}{\partial n}\right|_{\partial\Omega_\varepsilon} = 0. \end{cases} \tag{1.1.3}$$

Problem (1.1.3) is an eigenvalue problem posed in the unbounded domain Ω_ε. It is well-known (cf. Theorem XIII.86 in [60], volume IV) that elliptic periodic operators in $2D$ domains have no eigenvalue. In order to create eigenvalues one needs to introduce a perturbation. We will consider a local perturbation of the geometry of the domain where the width of one vertical edge is modified from ε to $\mu\varepsilon$ with some $\mu > 0$ (see figure 1.2):

$$\Omega_\varepsilon^\mu = \left(\mathbb{R} \times \left]-\tfrac{L}{2}, \tfrac{L}{2}\right[\right) \setminus \left\{\mathcal{S}_\varepsilon^{\mu,+} \cup \mathcal{S}_\varepsilon^{\mu,-}\right\},$$

$$\mathcal{S}_\varepsilon^{\mu,+} = \left(\left[\tfrac{\mu\varepsilon}{2}, 1 - \tfrac{\varepsilon}{2}\right] \cup \bigcup_{j\in\mathbb{N}^*} \left[j + \tfrac{\varepsilon}{2}, j + 1 - \tfrac{\varepsilon}{2}\right]\right) \times \left[-\tfrac{L}{2} + \varepsilon, \tfrac{L}{2} - \varepsilon\right],$$

$$\mathcal{S}_\varepsilon^{\mu,-} = \left(\left[-1 + \tfrac{\varepsilon}{2}, -\tfrac{\mu\varepsilon}{2},\right] \cup \bigcup_{j\in\mathbb{N}^*} \left[-j - 1 + \tfrac{\varepsilon}{2}, -j - \tfrac{\varepsilon}{2}\right]\right) \times \left[-\tfrac{L}{2} + \varepsilon, \tfrac{L}{2} - \varepsilon\right],$$

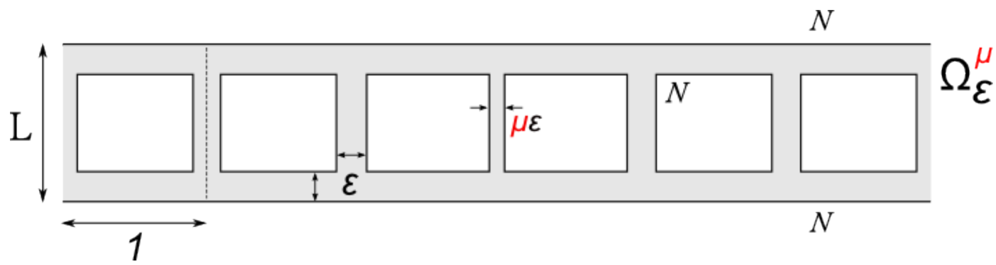


Figure 1.2: Perturbed domain ($\mu < 1$)

As we will see, such a perturbation does not change the essential spectrum of the underlying operator but it can introduce a non-empty discrete spectrum, which is exactly what we are interested in (since this discrete spectrum corresponds to trapped modes).

A precise mathematical description of the problem is given in the next section.

1.2 Mathematical formulation of the problem

Let us introduce the operator A_ε^μ in the space $L_2(\Omega_\varepsilon^\mu)$, associated with the eigenvalue problem (1.1.3) in the perturbed domain:

$$A_\varepsilon^\mu u = -\Delta u, \qquad D(A_\varepsilon^\mu) = \left\{ u \in H^1_\Delta(\Omega_\varepsilon^\mu), \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega_\varepsilon^\mu} = 0 \right\}.$$

Here

$$H^1_\Delta(\Omega^\mu_\varepsilon) = \left\{ u \in H^1(\Omega^\mu_\varepsilon), \quad \Delta u \in L_2(\Omega^\mu_\varepsilon) \right\}.$$

The operator A^μ_ε is self-adjoint and positive. We have then to study its spectrum and, more precisely, to find sufficient conditions for existence of eigenvalues.

1.2.1 Determination of the essential spectrum of A^μ_ε

To determine the essential spectrum of the operator A^μ_ε, we start by studying the perfectly periodic case ($\mu = 1$). In this case, the domain is Ω_ε (figure 1.1) and the corresponding operator A^1_ε will be denoted by A_ε.

According to the Floquet-Bloch theory, periodic elliptic operators do not have discrete spectrum and their essential spectrum has a band-gap structure [15, 60, 40]:

$$\sigma(A_\varepsilon) = \sigma_{ess}(A_\varepsilon) = \mathbb{R} \setminus \bigcup_{1 \leqslant n \leqslant N}]a_n, b_n[. \tag{1.2.1}$$

The intervals $]a_n, b_n[$ are called spectral gaps. Their number N is conjectured to be finite (Bethe-Sommerfeld, 1933, [55, 56, 67]).

The band-gap structure of the spectrum is a consequence of the following result given by the Floquet-Bloch theory:

$$\sigma(A_\varepsilon) = \bigcup_{\theta \in [-\pi, \pi]} \sigma\left(A_\varepsilon(\theta)\right), \tag{1.2.2}$$

where for any $\theta \in [-\pi, \pi]$, $A_\varepsilon(\theta)$ is the Laplace operator defined on the periodicity cell $\mathcal{C}_\varepsilon = \Omega_\varepsilon \cap \{x \in \left[-\frac{1}{2}, \frac{1}{2}\right]\}$ (cf. figure 1.3) with θ-quasiperiodic boundary conditions on the lateral boundaries: for $\theta \in [-\pi, \pi]$,

$$A_\varepsilon(\theta) \,:\, L_2(\mathcal{C}_\varepsilon) \longrightarrow L_2(\mathcal{C}_\varepsilon), \qquad A_\varepsilon(\theta) u = -\Delta u, \tag{1.2.3}$$

$$D(A_\varepsilon(\theta)) = \Big\{ u \in H^1_\Delta(\mathcal{C}_\varepsilon), \quad \partial_n u|_{\partial\mathcal{C}_\varepsilon \cap \left\{x \in]-\frac{1}{2}, \frac{1}{2}[\right\}} = 0, \tag{1.2.4}$$

$$u|_{x=\frac{1}{2}} = e^{-i\theta} u|_{x=-\frac{1}{2}}, \quad \partial_x u|_{x=\frac{1}{2}} = e^{-i\theta} \partial_x u|_{x=-\frac{1}{2}} \Big\}. \tag{1.2.5}$$

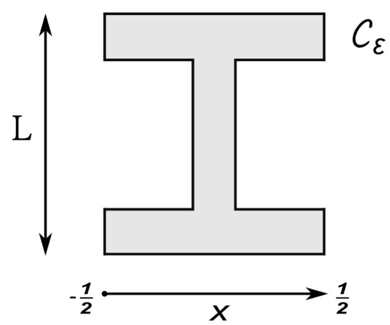


Figure 1.3: Periodicity cell

For each $\theta \in [-\pi, \pi]$ the operator $A_\varepsilon(\theta)$ is self-adjoint and positive and its resolvent is compact due to the compactness of the embedding $H^1(\mathcal{C}_\varepsilon) \subset L_2(\mathcal{C}_\varepsilon)$. Its spectrum is then a sequence of non-negative eigenvalues of finite multiplicity tending to infinity:

$$0 \leqslant \lambda_1(\varepsilon, \theta) \leqslant \lambda_2(\varepsilon, \theta) \leqslant \cdots \leqslant \lambda_n(\varepsilon, \theta) \leqslant \ldots, \qquad \lim_{n \to \infty} \lambda_n(\varepsilon, \theta) = +\infty. \tag{1.2.6}$$

In (1.2.6) the eigenvalues are repeated with their multiplicity. The functions $\theta \mapsto \lambda_n(\varepsilon, \theta)$ are called dispersion curves and are known to be continuous and non-constant (cf. Theorem XIII.86 in [60], volume IV). Thus, (1.2.2) can be rewritten as

$$\sigma(A_\varepsilon) = \bigcup_{n\in\mathbb{N}} \lambda_n\left(\varepsilon, [-\pi, \pi]\right),$$

which gives (1.2.1). The conjecture of Bethe-Sommerfeld means that for n large enough the intervals $\lambda_n\left(\varepsilon, [-\pi, \pi]\right)$ overlap. The fact that the dispersion curves are non-constant implies that no one of the intervals $\lambda_n\left(\varepsilon, [-\pi, \pi]\right)$ is reduced to a point and the operator A_ε has no eigenvalues. Finally, the dispersion curves are even: indeed, $D(A_\varepsilon(-\theta)) = \overline{D(A_\varepsilon(\theta))}$ and the operators $A_\varepsilon(\theta)$ have real coefficients. Thus, it is sufficient to consider $\theta \in [0, \pi]$ in (1.2.2).

1.2.2 The essential spectrum of the operator A^μ_ε.

It is well-known that local perturbations of the domain do not change the essential spectrum of the corresponding operator. This is due to Weyl's Theorem (see, for example, Ch.13 Vol. 4 in [60], Ch. 9 in [7], Theorem 1 in [18]). For the sake of completeness we prove this result in our case.

Proposition 1.2.1. *$\sigma_{ess}(A^\mu_\varepsilon) = \sigma_{ess}(A_\varepsilon)$.*

This is a direct consequence of the following assertion.

Lemma 1.2.1. *Let $\chi \in C^\infty(\Omega_\varepsilon)$ be a function such that*

1. *$\partial_\chi|_{\partial\Omega_\varepsilon} = 0$,*
2. *$\exists M > 0$ such that $|x| > M \;\Rightarrow\; \chi(x, y) = 1$.*

If $\{u_j\}_{j\in\mathbb{N}}$ is a singular sequence for the operator A_ε corresponding to the value λ, then there exists a subsequence of $\{\chi u_j\}_{j\in\mathbb{N}}$ which is also a singular sequence for the operator A_ε corresponding to the value λ.

Proof. By definition, the sequence $\{u_j\}_{j\in\mathbb{N}}$ has the following properties:

1. $u_j \in D(A_\varepsilon), \quad j \in \mathbb{N}$;
2. $\inf_j \|u_j\|_{L_2(\Omega_\varepsilon)} > 0$;
3. $u_j \xrightarrow{w} 0$ in $L_2(\Omega_\varepsilon)$ ($\{u_j\}$ is weakly convergent to 0 in $L_2(\Omega_\varepsilon)$);
4. $A_\varepsilon u_j - \lambda u_j \longrightarrow 0$.

Let us show that there exists a subsequence of $\{\chi u_j\}_{j\in\mathbb{N}}$ which has the same properties as well. The property 1 is verified due to the property 1 of the function χ. To obtain a lower bound for $\|\chi u_j\|_{L_2(\Omega_\varepsilon)}$ we will show that

$$\|u_j\|_{L_2(K)} \longrightarrow 0, \qquad j \longrightarrow \infty, \qquad \text{for all compact } K \subset \Omega_\varepsilon. \tag{1.2.7}$$

Indeed, from

$$(A_\varepsilon u_j - \lambda u_j, u_j)_{L_2(\Omega_\varepsilon)} = \|\nabla u_j\|_{L_2(\Omega_\varepsilon)} - \lambda \|u_j\|_{L_2(\Omega_\varepsilon)},$$

taking into account the properties 2–4 of the sequence $\{u_j\}$ we conclude that $\sup\limits_j \|u_j\|_{H^1(\Omega_\varepsilon)} < C$ for some $C > 0$. We can then extract a subsequence still denoted by $\{u_j\}$ such that

$$u_j \xrightarrow{w} 0, \qquad j \longrightarrow \infty, \qquad \text{in} \quad H^1(\Omega_\varepsilon). \tag{1.2.8}$$

Thus, (1.2.7) is verified and, consequently, $\inf\limits_j \|\chi u_j\|_{L_2(\Omega_\varepsilon)} \geqslant \inf\limits_j \|u_j\|_{L_2(\Omega_\varepsilon \cap \{|x|>M\})} > 0$. The property 3 being obvious the only thing to show is the property 4 for the sequence $\{\chi u_j\}_{j\in\mathbb{N}}$. We have:

$$\|A_\varepsilon(\chi u_j) - \lambda(\chi u_j)\|^2_{L_2(\Omega_\varepsilon)} \leqslant \|\chi(A_\varepsilon u_j - \lambda u_j)\|^2_{L_2(\Omega_\varepsilon)} + 2\|\nabla\chi\nabla u_j\|^2_{L_2(\Omega_\varepsilon)} + \|u_j\Delta\chi\|^2_{L_2(\Omega_\varepsilon)}.$$

The first and the last terms in the right-hand side tend to zero (due to the property 4 of $\{u_j\}$ and (1.2.7)). Let us estimate the second term.

$$\|\nabla\chi\nabla u_j\|^2_{L_2(\Omega_\varepsilon)} = \int\limits_{\Omega_\varepsilon} \nabla u_j \nabla \overline{u}_j |\nabla\chi|^2 d\Omega$$

$$= - \int\limits_{supp(\nabla\chi)} u_j \mathrm{div}\left(\nabla\overline{u}_j|\nabla\chi|^2\right) d\Omega + \int\limits_{\partial\Omega_\varepsilon} u_j \partial_{\mathrm{n}}\overline{u}_j |\nabla\chi|^2 d\Gamma.$$

The last term in the right-hand side vanishes due to the property 1 of $\{u_j\}$. The first term tends to 0 due to (1.2.7), (1.2.8) and the properties 2, 4 of $\{u_j\}$. □

Proof of Proposition 1.2.1. It is sufficient to take a function χ in the previous lemma which does not depend on y, vanishes in a neighbourhood of the perturbed edge and such that $\nabla\chi$ vanishes in a neighbourhood of all vertical edges. Then, it follows from Lemma 1.2.1 that any singular sequence of the operator A_ε contains a singular sequence of the operator A_ε^μ and vice versa. □

The essential spectrum of the operator $\sigma_{ess}(A_\varepsilon^\mu)$ having a band-gap structure, we will be interested in finding eigenvalues inside gaps (once the existence of gaps is established).

1.2.3 Method of study

Our analysis consists of three main steps.

- First, we find a formal limit of the problem when $\varepsilon \to 0$. It is clear that geometrically as ε goes to zero, the domains Ω_ε and Ω_ε^μ shrink to a graph. The limit problem is then associated with a second-order differential operator defined on the graph. Neumann boundary conditions on the ladder give rise to the so called Kirchhoff's conditions at the vertices of the graph. This limit operator is well-known from the works [61] and Kuchment-Zheng [47]. We will describe it more rigorously in section 1.3.1.

- The second step is an explicit calculation of the spectrum of the limit operator. The essential spectrum is determined using the Floquet-Bloch theory (by solving a set of cell problems) while the discrete spectrum of the perturbed operator is found using a reduction to a finite difference equation (section 1.3.2.2). In particular, we will see that the limit operator has infinitely many gaps. For $\mu < 1$ it also has infinitely many eigenvalues, whereas for $\mu > 1$ it has no discrete spectrum.

- Finally, we have to show that the operator on the ladder also has eigenvalues for $\mu < 1$ when ε is small enough (section 1.4). Despite the fact that this result follows from [57], we will give another proof based on the construction of a quasi-mode (an approximation of the eigenfunction). To go further, we will compute the full asymptotic expansion of the eigenvalues inside the gaps (chapter 2).

To begin with, we will decompose the operator A_ε^μ into the sum of its symmetric and antisymmetric parts. As it will become clear later, the "true model problem" for the β-quasiperiodic case is actually given by the considering only the symmetric (or only the antisymmetric) part of the operator A_ε^μ. Let us introduce the following orthogonal decomposition of the space $L_2(\Omega_\varepsilon^\mu)$:

$$L_2(\Omega_\varepsilon^\mu) = L_{2,s}(\Omega_\varepsilon^\mu) \oplus L_{2,as}(\Omega_\varepsilon^\mu),$$

where $L_{2,s}(\Omega_\varepsilon^\mu)$ and $L_{2,as}(\Omega_\varepsilon^\mu)$ are the subspaces consisting of functions respectively symmetric and antisymmetric with respect to the axis $y = 0$:

$$L_{2,s}(\Omega_\varepsilon^\mu) = \{u \in L_2(\Omega_\varepsilon^\mu) \,/\, u(x,y) = u(x,-y)\}\,,$$

$$L_{2,as}(\Omega_\varepsilon^\mu) = \{u \in L_2(\Omega_\varepsilon^\mu) \,/\, u(x,y) = -u(x,-y)\}\,.$$

Consequently, the operator A_ε^μ is decomposed into the orthogonal sum

$$A_\varepsilon^\mu = A_{\varepsilon,s}^\mu \oplus A_{\varepsilon,as}^\mu$$

with

$$A_{\varepsilon,s}^\mu = A_\varepsilon^\mu\big|_{L_{2,s}(\Omega_\varepsilon^\mu)}\,, \qquad A_{\varepsilon,as}^\mu = A_\varepsilon^\mu\big|_{L_{2,as}(\Omega_\varepsilon^\mu)}\,.$$

The spectrum of the operator A_ε^μ is then given by the union of the spectra of its symmetric and antisymmetric parts:

$$\sigma(A_\varepsilon^\mu) = \sigma(A_{\varepsilon,s}^\mu) \cup \sigma(A_{\varepsilon,as}^\mu).$$

1.3 Spectral problem on the graph

1.3.1 The operator A^μ.

As $\varepsilon \to 0$, the domain Ω_ε tends to the periodic graph G represented in figure 1.4. Let us enumerate the vertices of the graph from left to right by an integer index j, the superscripts "+" and "-" corresponding to the upper and the lower vertices respectively. The set of all the vertices of the graph is then

$$V = \left\{v_j^\pm\right\}_{j\in\mathbb{Z}}\,, \qquad v_j^\pm = \left(j, \pm\tfrac{L}{2}\right).$$

The vertical edge joining the vertices v_j^+ and v_j^- is denoted by e_j and the horizontal edge joining the vertices $v_j^\pm$ and $v_{j+1}^\pm$ is denoted by $e_{j+\frac{1}{2}}^\pm$. Thus, the set of all the edges of the graph is

$$E = \left\{e_j, e_{j+\frac{1}{2}}^\pm\right\}_{j\in\mathbb{Z}}\,, \qquad e_j = \{j\} \times \left[-\tfrac{L}{2}, \tfrac{L}{2}\right], \qquad e_{j+\frac{1}{2}} = [j, j+1] \times \left\{\pm\tfrac{L}{2}\right\}.$$

The edge corresponding to the perturbation is e_0. The set of all the edges of the graph containing the vertex v is denoted by E_v.

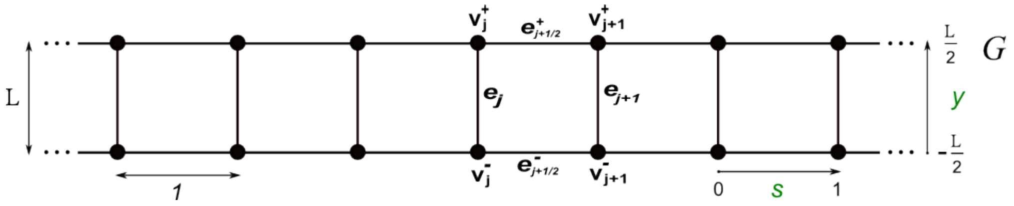


Figure 1.4: Limit graph

If u is a function defined on the graph G, we will use the following notation:

$$\mathbf{u}_j^{\pm} = u(v_j^{\pm}), \qquad u_j = u|_{e_j}, \qquad u_{j+\frac{1}{2}}^{\pm} = u|_{e_{j+\frac{1}{2}}^{\pm}}.$$

We introduce a local coordinate s at each horizontal edge $e_{j+\frac{1}{2}}^{\pm}$ of the graph that varies 0 at $v_j^{\pm}$ to 1 at $v_{j+1}^{\pm}$. In other words,

$$u_j(y) = u(j, y), \quad y \in \left[-\tfrac{L}{2}, \tfrac{L}{2}\right], \qquad u_{j+\frac{1}{2}}^{\pm}(s) = u\left(j + s, \pm\tfrac{L}{2}\right), \quad s \in [0, 1].$$

Let $w^\mu : E \to \mathbb{R}^+$ be a weight function which is equal to μ on the "perturbed edge" e_0 and to 1 on the other edges:

$$\begin{cases} w^\mu(e_0) = \mu, \\ w^\mu(e) = 1, \qquad \forall e \in E, \quad e \neq e_0. \end{cases}$$

Sometimes, to simplify the expressions, we will use the notation

$$w^\mu(j) = w_j^\mu = \begin{cases} \mu, & j = 0, \\ 1, & \forall j \in \mathbb{Z}^*. \end{cases} \tag{1.3.1}$$

Let us now introduce the following function spaces:

$$L_2^\mu(G) = \left\{ u \,/\, u \in L_2(e),\, \forall e \in E; \quad \|u\|^2_{L_2^\mu(G)} = \sum_{e \in E} w^\mu(e) \|u\|^2_{L_2(e)} < \infty \right\}, \tag{1.3.2}$$

$$\begin{aligned} H^1(G) &= \left\{ u \,/\, u \in C(G); \quad u \in H^1(e),\, \forall e \in E; \quad \|u\|^2_{H^1(G)} = \sum_{e \in E} \|u\|^2_{H^1(e)} < \infty \right\}, \\ H^2(G) &= \left\{ u \,/\, u \in C(G); \quad u \in H^2(e),\, \forall e \in E; \quad \|u\|^2_{H^2(G)} = \sum_{e \in E} \|u\|^2_{H^2(e)} < \infty \right\}. \end{aligned} \tag{1.3.3}$$

Notice that by definition functions belonging to $H^1(G)$ or $H^2(G)$ are continuous at the vertices of the graph.
We define the limit operator A^μ in $L_2^\mu(G)$ as follows:

$$(A^\mu u)|_e = -\left(u|_e\right)'', \qquad \forall e \in E, \tag{1.3.4}$$

$$D(A^\mu) = \left\{ u \in H^2(G) \,/ \sum_{e_v \in E_v} w^\mu(e_v)(u_{e_v})'_{ext}(v) = 0, \quad \forall v \in V \right\}, \tag{1.3.5}$$

where $(u_{v_e})'_{ext}(v)$ stands for the derivative of the function u at the edge e_v taken at the vertex v in the outgoing direction. The relations in (1.3.5) are called Kirchoff's conditions. We note that the perturbation is only present in the definition of the operator A^μ via the Kirchoff's condition at the perturbed edge (i.e. for the vertices $v_0^\pm$ since $w^\mu(e_0) = \mu$).

The following assertion is proved in [42], section 3.3 (we give its proof reformulated for our particular case in Appendix).

Proposition 1.3.1 (Kuchment)**.** *The operator A^μ in the space $L_2^\mu(G)$ is self-adjoint. The corresponding closed sesqui-linear form has the following form:*

$$a^\mu[f,g] = (f', g')_{L_2^\mu(G)}, \quad \forall f, g \in D[a^\mu], \qquad D[a^\mu] = H^1(G).$$

As for the ladder Ω_ε^μ, we introduce the following decomposition of the space $L_2^\mu(G)$ into the spaces of symmetric and antisymmetric functions:

$$L_2^\mu(G) = L_{2,s}^\mu(G) \oplus L_{2,as}^\mu(G),$$

$$L_{2,s}^\mu(G) = \{u \in L_2(G) \,/\, u(x,y) = u(x,-y)\},$$

$$L_{2,as}^\mu(G) = \{u \in L_2(G) \,/\, u(x,y) = -u(x,-y)\}.$$

Again, the operator A^μ can be decomposed into the orthogonal sum

$$A^\mu = A_s^\mu \oplus A_{as}^\mu,$$

with

$$A_s^\mu = A^\mu\big|_{L_{2,s}^\mu(G)}, \qquad A_{as}^\mu = A^\mu\big|_{L_{2,as}^\mu(G)},$$

which implies

$$\sigma(A^\mu) = \sigma(A_s^\mu) \cup \sigma(A_{as}^\mu).$$

Thus, it is sufficient to study the spectra of the operators A_s^μ and A_{as}^μ separately. The analysis of these two operators being analogous, we will present a detailed study of A_s^μ (section 1.3.2) and state the results for A_{as}^μ (section 1.3.3).

1.3.2 The spectrum of the operator A_s^μ.

In this section we study the spectrum of the operator A_s^μ. Using the notation introduced in the beginning of this section with $u_{j+\frac{1}{2}} = u^+_{j+\frac{1}{2}} = u^-_{j+\frac{1}{2}}$ and $\mathbf{u}_j = \mathbf{u}_j^+ = \mathbf{u}_j^-$, it can be rewritten as

$$(A_s^\mu u)|_{e_{j+\frac{1}{2}}} = -u''_{j+\frac{1}{2}}, \qquad (A_s^\mu u)|_{e_j} = -u''_j, \qquad j \in \mathbb{N}, \tag{1.3.6}$$

$$D(A_s^\mu) = \Big\{u \in H^2(G) \,/\, \quad u'_j(0) = 0, \qquad j \in \mathbb{N},$$
$$u'_{j+\frac{1}{2}}(0) + w_j^\mu u'_j\left(\tfrac{-L}{2}\right) - u'_{j-\frac{1}{2}}(1) = 0, \qquad j \in \mathbb{N}\Big\}. \tag{1.3.7}$$

1.3.2.1 Essential spectrum of A_s^μ

First of all, we reduce the study of the essential spectrum of the operator A_s^μ to the study of the periodic case ($\mu = 1$). The corresponding operator A_s^1 will be denoted by A_s. Indeed, similarly to Proposition 1.2.1, by introducing a cut-off function which vanishes in a neighbourhood of the perturbed edge of the graph, we can prove that the essential spectrum of A_s^μ coincides with the spectrum of A_s:

Proposition 1.3.2.

$$\sigma_{ess}(A_s^\mu) = \sigma_{ess}(A_s). \tag{1.3.8}$$

Computation of the spectrum of the operator A_s

As previously explained, the spectrum of the periodic operator A_s can determined using the Floquet-Bloch theory. One has then to study a set of problems posed on the periodicity cell of G. Since we consider the subspace of symmetric functions with respect to the axis $y = 0$, this permits to reduce the problem to the lower half part of the periodicity cell defined as (see figure 1.5):

$$\mathcal{C}_- = G \cap \left\{\left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \times \left[-\tfrac{L}{2}, 0\right]\right\} = \bigcup_{1 \leqslant j \leqslant 3} e_j,$$

$$e_1 = \left[-\tfrac{1}{2}, 0\right] \times \{-\tfrac{L}{2}\}, \qquad e_2 = \left[0, \tfrac{1}{2}\right] \times \{-\tfrac{L}{2}\}, \qquad e_3 = \{0\} \times \left[-\tfrac{L}{2}, 0\right].$$

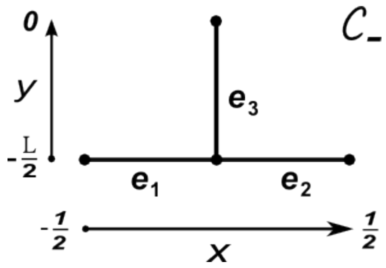


Figure 1.5: The half periodicity cell $\mathcal{C}_-$

For a function u defined on $\mathcal{C}_-$ we use the notation $u_j = u|_{e_j}$, $1 \leqslant j \leqslant 3$. Let us introduce the spaces $L_2(\mathcal{C}_-)$ and $H^2(\mathcal{C}_-)$ analogously to (1.3.2), (1.3.3):

$$L_2(\mathcal{C}_-) = \{u \,/\, u \in L_2(e_j), \;\; 1 \leqslant i \leqslant 3\},$$

$$H^2(\mathcal{C}_-) = \left\{u \,/\, u \in C(\mathcal{C}_-); \quad u \in H^2(e_j), \;\; 1 \leqslant i \leqslant 3\right\}.$$

We have then

$$\sigma(A_s) = \bigcup_{\theta \in [0,\pi]} \sigma\left(A_s(\theta)\right), \tag{1.3.9}$$

where $A_s(\theta)$ is the following operator defined in $L_2(\mathcal{C}_-)$ with θ-quasiperiodic boundary conditions:

$$A_s(\theta) u_i = -u_i'', \qquad 1 \leqslant i \leqslant 3,$$

$$D(A_s(\theta)) = \left\{u \in H^2(\mathcal{C}_-) \,/\;\; u_3'(0) = 0,\right.$$

$$-u_1'(0) + u_2'(0) + u_3'\left(-\tfrac{L}{2}\right) = 0, \tag{1.3.10}$$

$$\left. u_2\left(\tfrac{1}{2}\right) = e^{-i\theta} u_1\left(-\tfrac{1}{2}\right), \quad u_2'\left(\tfrac{1}{2}\right) = e^{-i\theta} u_1\left(-\tfrac{1}{2}\right)\right\}. \tag{1.3.11}$$

The condition $u_3'(0) = 0$ comes from the fact that the symmetric subspace with respect to the axis $y = 0$ is considered. The condition (1.3.10) is the Kirchhoff's condition and the relations (1.3.11) are the quasiperiodic conditions. For each $\theta \in [0, \pi]$ the operator $A_s(\theta)$ is self-adjoint and positive and its resolvent is compact due to the compactness of the embedding $H^1(\mathcal{C}_-) \subset L_2(\mathcal{C}_-)$. Consequently, its spectrum is a sequence of non-negative eigenvalues of finite multiplicity tending to infinity:

$$0 \leqslant \lambda_{1,s}(\theta) \leqslant \lambda_{2,s}(\theta) \leqslant \cdots \leqslant \lambda_{n,s}(\theta) \leqslant \dots, \qquad \lim_{n\to\infty} \lambda_{n,s}(\theta) = +\infty.$$

In the present case, the eigenvalues can be computed explicitly.

Proposition 1.3.3. *For $\theta \in [0, \pi]$, $\lambda^2 \in \sigma(A_s(\theta))$ if and only if λ is a solution of the equation*

$$2\cos\left(\tfrac{\lambda L}{2}\right)(\cos\lambda - \cos\theta) = \sin\lambda \sin\left(\tfrac{\lambda L}{2}\right). \tag{1.3.12}$$

Proof. If $\lambda^2 \neq 0$ is an eigenvalue of the operator $A_s(\theta)$ then the corresponding eigenfunction $u = \{u_1, u_2, u_3\}$ is of the form

$$u_1(x) = c_1 e^{i\lambda x} + d_1 e^{-i\lambda x}, \qquad x \in \left[-\tfrac{1}{2}, 0\right], \tag{1.3.13}$$

$$u_2(x) = c_2 e^{i\lambda x} + d_2 e^{-i\lambda x}, \qquad x \in \left[0, \tfrac{1}{2}\right], \tag{1.3.14}$$

$$u_3(y) = c_3 e^{i\lambda y} + d_3 e^{-i\lambda y}, \qquad y \in \left[-\tfrac{L}{2}, 0\right]. \tag{1.3.15}$$

Taking into account that $u \in D(A_s(\theta))$ we arrive at the following linear system of equations for the coefficients c_i, d_i, $1 \leqslant i \leqslant 3$:

$$c_1 + d_1 = c_2 + d_2 = c_3 e^{-\frac{i\lambda L}{2}} + d_3 e^{\frac{i\lambda L}{2}}, \tag{1.3.16}$$

$$c_3 = d_3, \tag{1.3.17}$$

$$c_2 e^{\frac{i\lambda}{2}} + d_2 e^{-\frac{i\lambda}{2}} = e^{-i\theta}\left(c_1 e^{-\frac{i\lambda}{2}} + d_1 e^{\frac{i\lambda}{2}}\right), \tag{1.3.18}$$

$$c_2 e^{\frac{i\lambda}{2}} - d_2 e^{-\frac{i\lambda}{2}} = e^{-i\theta}\left(c_1 e^{-\frac{i\lambda}{2}} - d_1 e^{\frac{i\lambda}{2}}\right), \tag{1.3.19}$$

$$-c_1 + d_1 + c_2 - d_2 + c_3 e^{-\frac{i\lambda L}{2}} - d_3 e^{\frac{i\lambda L}{2}} = 0. \tag{1.3.20}$$

The relations (1.3.16) express the continuity of the eigenfunction at the vertex $\left(0, -\frac{L}{2}\right)$. The equation (1.3.17) comes from the condition $u_3'(0) = 0$. The relations (1.3.18), (1.3.19) correspond to the quasiperiodicity conditions (1.3.11) and the equation (1.3.20) corresponds to the Kirchhoff's condition (1.3.10). Let us introduce the notation

$$\alpha = e^{i\theta}, \qquad z = e^{i\lambda}.$$

We notice that the relations (1.3.18), (1.3.19) imply

$$c_1 = c_2 \alpha z, \qquad d_1 = d_2 \alpha \overline{z}.$$

We used the fact that $z^{-1} = \overline{z}$ because the operator $A_s(\theta)$ has only real eigenvalues. Thus, the system (1.3.16)–(1.3.20) can be rewritten as follows (we have eliminated d_3, c_1 and d_1):

$$\begin{cases} c_2(1 - \alpha z) + d_2(1 - \alpha\overline{z}) = 0, \\ c_2 + d_2 - c_3\left(z^{\frac{L}{2}} + \overline{z}^{\frac{L}{2}}\right) = 0, \\ c_2(1 - \alpha z) - d_2(1 - \alpha\overline{z}) + c_3\left(\overline{z}^{\frac{L}{2}} - z^{\frac{L}{2}}\right) = 0. \end{cases} \tag{1.3.21}$$

The existence of an eigenfunction is equivalent to the condition

$$D(\lambda) := \begin{vmatrix} 1-\alpha z & 1-\alpha\overline{z} & 0 \\ 1 & 1 & -\left(z^{\frac{L}{2}}+\overline{z}^{\frac{L}{2}}\right) \\ 1-\alpha z & -(1-\alpha\overline{z}) & -z^{\frac{L}{2}}+\overline{z}^{\frac{L}{2}} \end{vmatrix} = 0.$$

One has

$$D(\lambda) = 2\alpha\Re\left(z^{L/2}\left(3z+\overline{z}-4\Re\alpha\right)\right) = 4e^{i\theta}\left(2\cos\left(\tfrac{\lambda L}{2}\right)(\cos\lambda-\cos\theta) - \sin\left(\tfrac{\lambda L}{2}\right)\sin\lambda\right),$$

which implies (1.3.12) for $\lambda \neq 0$. For $\lambda = 0$ the relations (1.3.13)–(1.3.15) are replaced by

$$\begin{aligned} u_1(x) &= c_1 + d_1 x, && x \in \left[-\tfrac{1}{2}, 0\right], \\ u_2(x) &= c_2 + d_2 x, && x \in \left[0, \tfrac{1}{2}\right], \\ u_3(y) &= c_3 + d_3 y, && y \in \left[-\tfrac{L}{2}, 0\right]. \end{aligned}$$

Using the fact that $u \in D(A_s(\theta))$ we have:

$$c_1 = c_2 = c_3, \qquad d_3 = 0, \qquad d_2 = d_1 e^{-i\theta}, \qquad d_2 = c_1(e^{-i\theta}-1), \qquad d_1 = d_2.$$

Hence, there exists a non-trivial solution if and only if $\theta = 0$. The solution is a constant function $u_i = c_1$, $1 \leqslant i \leqslant 3$. At the same time, $\lambda = 0$ is a solution of (1.3.12) if and only if $\theta = 0$ which finishes the proof. □

Remark 1.3.1. We notice that for $\cos\left(\frac{\lambda L}{2}\right) \neq 0$ the equation (1.3.12) is equivalent to the relation

$$\cos\theta = \cos\lambda - \frac{1}{2}\sin\lambda\tan\left(\tfrac{\lambda L}{2}\right). \tag{1.3.22}$$

Remark 1.3.2. One can easily see that if $L \in \mathbb{Q}$, the set $\{\lambda : \lambda^2 \in \sigma(A_s)\}$ is periodic. Indeed, this is due to the fact that both the left-hand side and the right-hand side of (1.3.12) are periodic with the same period.

In the rest of this section we will use the following notation:

$$\Sigma = \{\pi n,\ n \in \mathbb{N}\}, \qquad \Sigma_s = \left\{\tfrac{2n\pi}{L},\ n \in \mathbb{N}\right\} \qquad \Sigma_s' = \left\{\tfrac{(2n+1)\pi}{L},\ n \in \mathbb{N}\right\}.$$

Some properties of the spectrum of the operator A_s

It follows from the decomposition (1.3.9) together with Proposition 1.3.3 that $\lambda^2 \in \sigma(A_s)$ if and only if λ is a solution of the equation (1.3.12) for some $\theta \in [0, \pi]$. This permits to derive some important properties of the spectrum of the operator A_s.

Proposition 1.3.4.

1. *$\{\lambda^2,\ \lambda \in \Sigma \cup \Sigma_s\} \subset \sigma(A_s)$.*

2. *The operator A_s has infinitely many gaps whose ends tend to infinity.*

Proof.

1. If $\lambda \in \Sigma \cup \Sigma_s$, the equation (1.3.12) is obviously verified for θ such that $\cos\theta = \cos\lambda$.

2. We will actually show that for any point $\lambda \in \Sigma_s'$ there exists some neighbourhood (probably, a punctured neighbourhood) of λ^2 which is included in the resolvent set of the operator A_s. Consider $\lambda_n = \frac{(2n+1)\pi}{L}$, $n \in \mathbb{N}$. There are two possibilities:

   (i) $\lambda_n \notin \Sigma$: since $\cos\left(\frac{\lambda_n L}{2}\right) = 0$ and $\sin\lambda_n \neq 0$, the left-hand side of the equation (1.3.12) is equal to zero, whereas the right-hand side is different from zero for any $\theta \in [0,\pi]$. Consequently, there exists a gap of the operator A_s containing the point λ_n^2.

   (ii) $\lambda_n \in \Sigma$: in this case it follows from the property 1 that $\lambda_n^2 \in \sigma(A_s)$. We are going to show that the point λ_n^2 is an isolated point of the spectrum of the operator A_s, so that there exist gaps to the left and to the right of it. Setting $\lambda = \lambda_n + \delta$ in (1.3.12) we get:

$$-2\sin\left(\tfrac{\delta L}{2}\right)\left(\cos\delta - \frac{\cos\theta}{\cos\lambda_n}\right) = \sin\delta\cos\left(\tfrac{\delta L}{2}\right).$$

   If δ is small enough (but different from 0) this equation cannot be verified for any θ. Indeed, for $\delta \neq 0$ it can be rewritten as

$$\frac{\cos\theta}{\cos\lambda_n} - \cos\delta = \frac{\sin\delta}{2\sin\left(\frac{\delta L}{2}\right)}\cos\left(\tfrac{\delta L}{2}\right).$$

   The limit of the right-hand side as $\delta \to 0$ is positive (it equal to $\frac{1}{L}$), whereas the limit of the left-hand side is non-positive for any θ (since $|\cos\lambda_n| = 1$) with a uniform bound in θ for δ small enough:

$$\frac{\cos\theta}{\cos\lambda_n} - \cos\delta \leqslant 1 - \cos\delta, \qquad \forall\theta \in [0,\pi].$$

   Hence, the equation (1.3.12) has no solution for δ small enough. This proves the existence of gaps of the form $]\lambda_n^2 - l_n^-, \lambda_n^2[$, $]\lambda_n^2, \lambda_n^2 + l_n^+[$ for some $l_n^-, l_n^+ > 0$.

□

Remark 1.3.3. In the proof of Proposition 1.3.4 (2) we showed for any $\lambda \in \Sigma_s'$ the existence of a punctured neighbourhood of the point λ^2 that does not belong to the spectrum of the operator A_s. This can be seen as a result of the graph decoration described in [43]. Indeed, considering the the symmetric subspace on the graph G is equivalent to considering the lower half of G (denoted by G_-) with homogeneous Neumann conditions at the vertices $(j,0)$, cf. figure 1.6.

The graph G_-, in turn, can be seen as a decoration of the graph G_0 that consists of the vertices $\{v_j^-\}_{j\in\mathbb{Z}}$ and the edges $\{e_{j+\frac{1}{2}}^-\}_{j\in\mathbb{Z}}$, obtained by attaching a copy of a segment $G_1 = [-\frac{L}{2}, 0]$ to each its vertex. Then, according to Theorem 5 in [43], for any eigenvalue λ^2 of the problem

$$\begin{cases} u'' + \lambda^2 u = 0, \\ u'(0) = 0, \\ u\left(-\frac{L}{2}\right) = 0, \end{cases} \qquad (1.3.23)$$

there exists a punctured neighbourhood that does not belong to the spectrum of the operator A_s. The set of eigenvalues of (1.6) is exactly the set $\{\lambda^2,\ \lambda \in \Sigma_s'\}$.

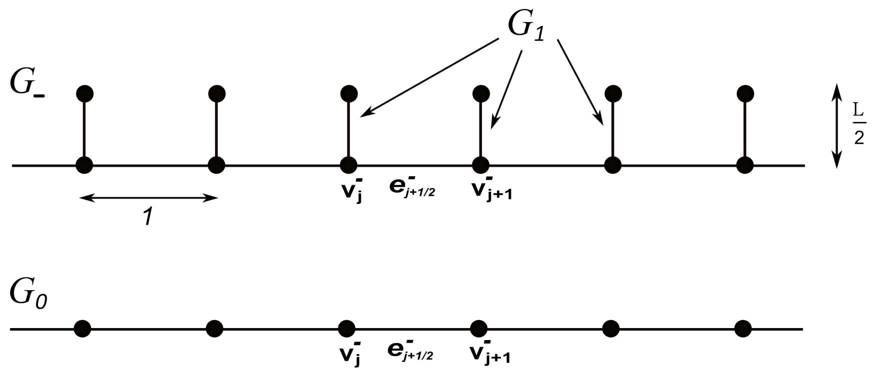


Figure 1.6: The graph G_- can be obtained by a decoration of the graph G_0: at each vertex of G_0 we attach a copy of the segment G_1.

Proposition 1.3.5. *The operator A_s has the following set of eigenvalues of infinite multiplicity which are isolated points of the spectrum:*

$$\sigma_{pp}(A_s) = \left\{\lambda^2, \ \ \lambda \in \Sigma \cup \Sigma'_s\right\}. \tag{1.3.24}$$

Proof. The point λ^2 is an eigenvalue of the operator A_s of infinity multiplicity if and only if it is an eigenvalue of the operator $A_s(\theta)$ for any $\theta \in [0, \pi]$. According to Proposition 1.3.3, this means that the equation (1.3.12) is satisfied for any $\theta \in [0, \pi]$, which is equivalent to the condition $\cos\left(\frac{\lambda L}{2}\right) = \sin\lambda = 0$ (i.e., $\lambda \in \Sigma \cup \Sigma'_s$). The fact that these points are isolated points of the spectrum is shown in the proof of Proposition 1.3.4, property 2 (case (ii)). □

Remark 1.3.4. The set (1.3.24) is non-empty if and only if $L = \frac{2m+1}{k}$ an irreducible fraction with $m \in \mathbb{N}, \ k \in \mathbb{N}^*$. In this case

$$\sigma_{pp}(A_s) = \left\{((2n+1)\pi k)^2, \ n \in \mathbb{N}\right\}.$$

Remark 1.3.5. The presence of eigenvalues of infinite multiplicity is not a common feature for elliptic periodic second-order operators in domains of $\mathbb{R}^n$. The absolute continuity of the spectrum for such operators is proved under some additional assumptions (cf., for example, [60, 65, 40, 8, 25, 45]). As explained in [43] (Section 5), the absence of pure point spectrum is related to the uniqueness of continuation property. This property fails for the graphs. Indeed, one can easily construct a compactly supported eigenfunction on a graph provided it has a loop consisting of commensurable edges (cf. figure 1.7). This is exactly what happens under the assumptions of Proposition 1.3.5: the edges of the graph G are all commensurable. Due to the periodicity of the graph, an infinity of eigenfunctions corresponding to the same eigenvalue can be constructed by a translation of one compactly supported eigenfunction. Moreover, it is shown in [43] that the eigenspace corresponding to an eigenvalue of infinite multiplicity is generated by compactly supported functions.

As we have already seen in Proposition 1.3.4, the operator A_s has infinitely many gaps which are separated by the points $\{\lambda^2, \ \ \lambda \in \Sigma \cup \Sigma_s\}$. We are going now to study in more detail the location of the gaps via a geometric interpretation of the equation (1.3.12). We will see that two types of gaps can be distinguished (Proposition 1.3.8) which will permit us to characterize the discrete spectrum of the perturbed operator (according to the type of the gap it will contain one or two eigenvalues of the perturbed operator, Theorem

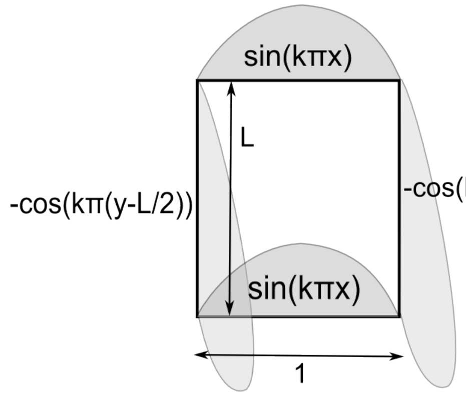


Figure 1.7: A compactly supported eigenfunction corresponding to the eigenvalue $\lambda^2 = (k\pi)^2$ for $L = 1$.

1.3.1). We start by an auxiliary statement which is an immediate consequence of the decomposition (1.3.9), Proposition 1.3.3 and Proposition 1.3.4 (1).

Proposition 1.3.6. *$\lambda^2 \in \sigma(A_s)$ if and only if one of the following possibilities holds:*

(i) $\lambda \in \Sigma \cup \Sigma_s$;

(ii) $\lambda \notin \Sigma \cup \Sigma_s$ *is a solution of the equation*

$$\phi_L(\lambda) = f_\theta(\lambda), \tag{1.3.25}$$

for some $\theta \in [0, \pi]$. *Here*

$$\phi_L(\lambda) = \frac{2}{\tan\left(\frac{\lambda L}{2}\right)}, \qquad \lambda \notin \Sigma_s, \tag{1.3.26}$$

$$f_\theta(\lambda) = \frac{\sin\lambda}{\cos\lambda - \cos\theta}, \qquad \lambda \in \{\lambda /\ \cos\lambda \neq \cos\theta\}.$$

Geometrically, the solutions of the equation (1.3.25) correspond to the abscissas of the intersections of the graph of the function $\phi_L(\lambda)$ with the one of the function $f_\theta(\lambda)$. Hence, to obtain the set described in (ii) of Proposition 1.3.6, one has to consider the union of the graphs of the functions $f_\theta(\lambda)$ for all $\theta \in [0, \pi]$. We introduce then the following set:

$$D = \bigcup_{\theta\in[0,\pi]} \{(x, f_\theta(x)) /\ x \geqslant 0,\ \ \cos x \neq \cos\theta\}.$$

Lemma 1.3.1.

$$D = \bigcup_{n\in\mathbb{N}} D_n^{\pm},$$

where

$$D_n^+ = \left\{(x,y)/\ x \in]\pi n, \pi(n+1)[,\ \ y \in \left[f^+(x), +\infty\right[\right\} \cup (\pi n, 0), \tag{1.3.27}$$

$$D_n^- = \left\{(x,y)/\ x \in]\pi n, \pi(n+1)[,\ \ y \in \left]-\infty, f^-(x)\right]\right\}, \tag{1.3.28}$$

and

$$
\begin{aligned}
f^+(x) &= \tan\left(\tfrac{x-\pi n}{2}\right), \qquad & x \in [\pi n, \pi(n+1)[,\\
f^-(x) &= -\mathrm{cotan}\left(\tfrac{x-\pi n}{2}\right), \qquad & x \in]\pi n, \pi(n+1)].
\end{aligned}
$$

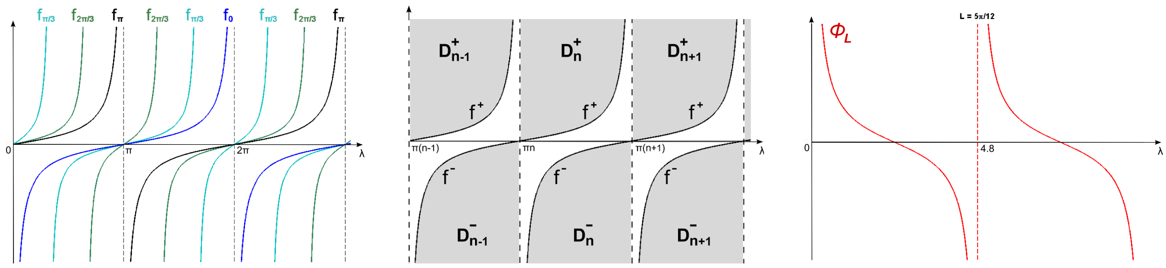
One has

$$D_n^\pm = D_0^\pm + (\pi n, 0), \qquad \forall n \in \mathbb{N}.$$

Proof. It is sufficient to notice that

$$\bigcup_{\substack{\theta\in[0,\pi]\\ \cos x\neq\cos\theta}} f_\theta(x) = \begin{cases}]-\infty, f^-(x)] \cup [f^+(x), +\infty[, & x \notin \Sigma,\\ \{0\}, & x \in \Sigma.\end{cases}$$

□



(a) The functions f_0, $f_{\pi/3}$, $f_{2\pi/3}$, f_π

(b) The domains $D_n^\pm$

(c) The function ϕ_L for $L = 5\pi/12$

Figure 1.8

The domains $D_n^\pm$ as well as the functions $f^\pm$ are shown in figure 1.8b. It is worth noticing that the functions $f^\pm$ are π-periodic, the function f^+ being right-continuous and the function f^- left-continuous. The function f^+ is strictly increasing in each interval $[\pi n, \pi(n+1)[$ and $f^+([\pi n, \pi(n+1)[) = \mathbb{R}^+$. Similarly, the function f^- is strictly increasing in each interval $]\pi n, \pi(n+1)]$ and $f^-(]\pi n, \pi(n+1)]) = \mathbb{R}^-$. As regards the function ϕ_L, it is a $2\pi/L$-periodic function defined on $\mathbb{R}^+ \setminus \Sigma_s$. In each interval $]2\pi n/L, 2\pi(n+1)/L[$, $n \in \mathbb{N}$, it is continuous and strictly decreasing and takes all the values in $\mathbb{R}$ (see an example in figure 1.8c).

As it was mentioned above, the set described in (ii) of Proposition 1.3.6 is given by the set of the abscissas of the intersections of D with the graph of the function ϕ_L. This is not exactly the image by the function $x \mapsto \sqrt{x}$ of the spectrum of the operator A_s since the discrete set $\Sigma \cup \Sigma_s$ ((i) of Proposition 1.3.6) is missing. This permits us to consider the intersection of the graph of the function ϕ_L with the domain $\overline{D}$ instead of D. Indeed, the domain $\overline{D}$ differs from D only by adding the vertical boundaries:

$$\overline{D} = \bigcup_{n\in\mathbb{N}} \overline{D_n^\pm} = D \cup \left\{\bigcup_{n\in\mathbb{N}} \{\{\pi n\} \times \mathbb{R}^+\}\right\} \cup \left\{\bigcup_{n\in\mathbb{N}^*} \{\{\pi n\} \times \mathbb{R}^-\}\right\}.$$

We have then

$$D_x := \left\{x : (x, \phi_L(\lambda)) \in \overline{D}\right\} = \{x : (x, \phi_L(\lambda)) \in D\} \cup \Sigma \setminus \Sigma_s, \tag{1.3.29}$$

which means that $\lambda \in D_x$ implies $\lambda^2 \in \sigma(A_s)$. In order to get the whole spectrum of the operator A_s, it is actually enough to add the set Σ_s, which is equivalent to taking the closure of D_x. More precisely, the following assertion holds.

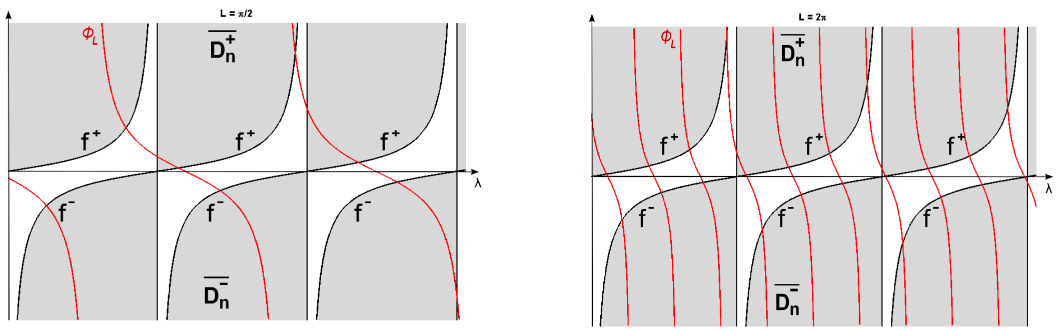
Proposition 1.3.7.

$$\lambda^2 \in \sigma(A_s) \quad \Leftrightarrow \quad \lambda \in D_x \cup \Sigma_s \quad \Leftrightarrow \quad \lambda \in \overline{D_x}.$$

Proof. The first equivalence follows immediately from Proposition 1.3.6 and (1.3.29). The inclusion $\left\{\lambda^2 : \lambda \in \overline{D_x}\right\} \subset \sigma(A_s)$ is also obvious since $\{\lambda^2 : \lambda \in D_x\} \subset \sigma(A_s)$ and the spectrum is a closed set. It remains then to prove that $\Sigma_s \subset \overline{D_x}$. Suppose that $\lambda_0 \in \Sigma_s$. Then,

$$\exists \delta_0 > 0 \quad \text{s.t.} \quad (\lambda, \phi_L(\lambda)) \in D, \quad \forall \lambda \in]\lambda_0, \lambda_0 + \delta_0[. \tag{1.3.30}$$

Indeed, for δ small enough the function f^+ is continuous in the interval $[\lambda_0, \lambda_0 + \delta]$. Consequently, for A large enough the band $]\lambda_0, \lambda_0 + \delta[\times[A, +\infty[$ is contained in D (due to (1.3.27)). However, $\lim\limits_{\lambda \to \lambda_0 + 0} \phi_L(\lambda) = +\infty$, which implies 1.3.30. This, in turn, means that $]\lambda_0, \lambda_0 + \delta_0[\subset D_x$ for some $\delta_0 > 0$. Hence, $\lambda_0 \in \overline{D_x}$. □



(a) $L = \pi/2$ (b) $L = 2\pi$

Figure 1.9: Examples of intersections of the domain $\overline{D}$ with the function ϕ_L

In the rest of this section we prove the following characterization of the gaps of the operator A_s.

Proposition 1.3.8. *An interval $]a, b[$ is a gap of the operator A_s if and only if $[\sqrt{a}, \sqrt{b}] \cap \Sigma_s = \emptyset$ and one of the following possibilities holds:*

I *There exists $n \in \mathbb{N}$ such that $\pi n < \sqrt{a} < \sqrt{b} < \pi(n+1)$, and $\phi_L(\sqrt{a}) = f^+(\sqrt{a})$, $\phi_L(\sqrt{b}) = f^-(\sqrt{b})$;*

II *(i) There exists $n \in \mathbb{N}$ such that $\pi n = \sqrt{a} < \sqrt{b} < \pi(n+1)$, and $\phi_L(\sqrt{a}) \leqslant 0$, $\phi_L(\sqrt{b}) = f^-(\sqrt{b})$;*

(ii) There exists $n \in \mathbb{N}$ such that $\pi n < \sqrt{a} < \sqrt{b} = \pi(n+1)$, and $\phi_L(\sqrt{a}) = f^+(\sqrt{a})$, $\phi_L(\sqrt{b}) \geqslant 0$.

We start by proving the following characterization of the ends of the gaps.

Lemma 1.3.2. *The point λ_0^2 is the lower end of a gap of the operator A_s if and only if one of the following possibilities holds:*

(i) $\lambda_0 \in \mathbb{R}_+ \setminus (\Sigma \cup \Sigma_s)$ *and* $\phi_L(\lambda_0) = f^+(\lambda_0)$*;*

(ii) $\lambda_0 \in \Sigma \setminus \Sigma_s$ *and* $\phi_L(\lambda_0) \leqslant 0$.

Similarly, the point λ_0^2 is the upper end of a gap of the operator A_s if and only if one of the following possibilities holds:

(iii) $\lambda_0 \in \mathbb{R}_+ \setminus (\Sigma \cup \Sigma_s)$ *and* $\phi_L(\lambda_0) = f^-(\lambda_0)$*;*

(iv) $\lambda_0 \in \Sigma \setminus \Sigma_s$ *and* $\phi_L(\lambda_0) \geqslant 0$.

Proof. Let us study the possible configurations of intersections of the function ϕ_L with the domains $\overline{D_n^\pm}$ in the interval $[\pi n, \pi(n+1)]$. We will treat separately the "regular" points (which are neither the end of the interval nor the points of discontinuity of the function ϕ_L).

- **Case 1.** Regular points: $\lambda_0 \notin \Sigma \cup \Sigma_s$

(a) The point $(\lambda_0, \phi_L(\lambda_0))$ is an interior point of the domains $\overline{D_n^\pm}$:

$$(\lambda_0, \phi_L(\lambda_0)) \in \operatorname{int}\left(\overline{D_n^\pm}\right) \;\Rightarrow\; \exists \delta > 0 \text{ s.t. } (\lambda, \phi_L(\lambda)) \in \overline{D_n^\pm},\;\; \forall \lambda \in]\lambda_0 - \delta, \lambda_0 + \delta[. \quad (1.3.31)$$

This follows immediately from the fact that the function ϕ_L is continuous at λ_0. We note that due to (1.3.31) the points such that $(\lambda_0, \phi_L(\lambda_0)) \in \operatorname{int}\left(\overline{D_n^\pm}\right)$ correspond to interior points of the spectrum (and not to the ends of gaps).

(b) The point λ_0 satisfies $\phi_L(\lambda_0) = f^+(\lambda_0)$:

$$\begin{array}{l} \lambda_0 \in]\pi n, \pi(n+1)[\\ \phi_L(\lambda_0) = f^+(\lambda_0) \end{array} \;\Rightarrow\; \exists \delta > 0 \;\text{ s.t. }\; \begin{array}{ll} (\lambda, \phi_L(\lambda)) \in \overline{D_n^+}, & \forall \lambda \in]\lambda_0 - \delta, \lambda_0], \\ (\lambda, \phi_L(\lambda)) \notin \overline{D_n^+}, & \forall \lambda \in]\lambda_0, \lambda_0 + \delta[. \end{array} \quad (1.3.32)$$

(c) The point λ_0 satisfies $\phi_L(\lambda_0) = f^-(\lambda_0)$:

$$\begin{array}{l} \lambda_0 \in]\pi n, \pi(n+1)[\\ \phi_L(\lambda_0) = f^-(\lambda_0) \end{array} \;\Rightarrow\; \exists \delta > 0 \;\text{ s.t. }\; \begin{array}{ll} (\lambda, \phi_L(\lambda)) \in \overline{D_n^-}, & \forall \lambda \in [\lambda_0, \lambda_0 + \delta[, \\ (\lambda, \phi_L(\lambda)) \notin \overline{D_n^-}, & \forall \lambda \in]\lambda_0 - \delta, \lambda_0[. \end{array} \quad (1.3.33)$$

This follows from the monotonicity of the functions $f^\pm$ defining the boundaries of the domains $\overline{D_n^\pm}$ on the one hand and from the monotonicity of the function ϕ_L from the other hand. Indeed, the point $(\lambda_0, f^+(\lambda_0))$ belongs to the boundary of the domain $\overline{D_n^+}$. Hence, the whole band $[\pi n, \lambda_0] \times [f^+(\lambda_0), \infty[$ belongs to $\overline{D_n^+}$ and the band $]\lambda_0, \pi(n+1)[\times]-\infty, f^+(\lambda_0)[$ does not intersect $\overline{D_n^+}$. Taking into account that the function ϕ_L is strictly decreasing and continuous in some neighbourhood of the point λ_0, we obtain (1.3.32) which means that λ_0^2 is the lower end of a gap. An analogous argument gives (1.3.33) which means that λ_0^2 is the upper end of a gap.

- **Case 2.** Points of discontinuity of the function ϕ_L (the set Σ_s)

  This case has been considered in the proof of Proposition 1.3.7. We showed that there exists $\delta^+ > 0$ such that $]\lambda_0, \lambda_0 + \delta^+[\subset D_x$. Using an analogous argument one easily checks that there exists $\delta^- > 0$ such that $]\lambda_0 - \delta^-, \lambda_0[\subset D_x$. Taking into account that $D_x \cup \Sigma_s = \{\lambda : \lambda^2 \in \sigma(A_s)\}$, we conclude that the points of Σ_s correspond to internal points of the spectrum (and not to the ends of gaps).

- **Case 3.** Ends of the interval

$$\begin{array}{lcl} \lambda_0 = \pi n \notin \Sigma_s & & (\lambda_0, \phi_L(\lambda_0)) \in \overline{D_n^+}, \\ \phi_L(\lambda_0) \geqslant 0 & \Rightarrow & \exists \delta > 0 \text{ s.t. } (\lambda, \phi_L(\lambda)) \notin \overline{D_n^+}, \quad \forall \lambda \in]\lambda_0 - \delta, \lambda_0[. \end{array} \tag{1.3.34}$$

$$\begin{array}{lcl} \lambda_0 = \pi n \notin \Sigma_s & & (\lambda_0, \phi_L(\lambda_0)) \in \overline{D_n^-}, \\ \phi_L(\lambda_0) \leqslant 0 & \Rightarrow & \exists \delta > 0 \text{ s.t. } (\lambda, \phi_L(\lambda)) \notin \overline{D_n^-}, \quad \forall \lambda \in]\lambda_0, \lambda_0 + \delta[. \end{array} \tag{1.3.35}$$

  These properties follow from the fact that the function ϕ_L is continuous and strictly decreasing in some neighbourhood of the point λ_0 taking into account that the rays $\{\pi n\} \times \mathbb{R}^+$ and $\{\pi n\} \times \mathbb{R}^-$ are boundaries of the domains $\overline{D_n^+}$ and $\overline{D_n^-}$ respectively.

  The properties (1.3.34), (1.3.35) imply that the point $\lambda_0^2 = (\pi n)^2$ is the upper end of a gap if $\phi_L(\lambda_0) \geqslant 0$ and the lower end of a gap if $\phi_L(\lambda_0) \leqslant 0$. Note that if $\phi_L(\lambda_0) = 0$, there is a gap to the left and to the right of this point, and $\lambda_0^2 = (\pi n)^2$ is an eigenvalue of infinite multiplicity for the operator A_s (which is in accordance with Proposition 1.3.5).

The three cases considered cover all the possible situations, which finishes the proof. □

Examples illustrating Lemma 1.3.2 are shown in figure 1.9. We can now prove Proposition 1.3.8.

Proof of Proposition 1.3.8. As it has been shown in the proof of Lemma 1.3.2, case 2, the points of Σ_s correspond to internal points of the spectrum of the operator A_s. This implies that $[\sqrt{a}, \sqrt{b}] \cap \Sigma_s = \emptyset$. Consequently, $\phi_L \in C([\sqrt{a}, \sqrt{b}])$. Since the images of the points of Σ by the function $x \mapsto x^2$ belong to the spectrum of the operator A_s (Proposition 1.3.4(1)), one necessarily has $\pi n \leqslant \sqrt{a} < \sqrt{b} \leqslant \pi(n+1)$ for some $n \in \mathbb{N}$.

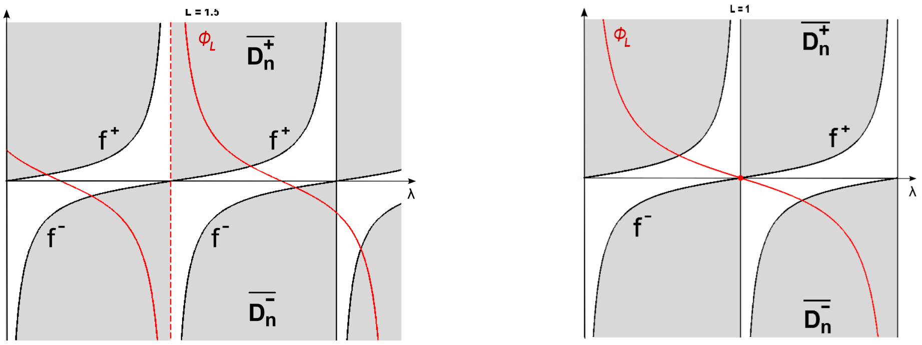
I Suppose that $\pi n < \sqrt{a} < \sqrt{b} < \pi(n+1)$. It follows from Lemma 1.3.2 that $\phi_L(\sqrt{a}) = f^+(\sqrt{a})$ and $\phi_L(\sqrt{b}) = f^-(\sqrt{b})$.

II

(i) Suppose now that $\sqrt{a} = \pi n$. Then, it follows from Lemma 1.3.2 that $\phi_L(\sqrt{a}) \leqslant 0$. The function ϕ_L being continuous and strictly decreasing in the interval $[\sqrt{a}, \sqrt{b}]$, we conclude that $\phi_L(\sqrt{b}) < 0$. Then, according to Lemma 1.3.2, $\phi_L(\sqrt{b}) = f^-(\sqrt{b})$. Consequently, $f^-(\sqrt{b}) < 0$, which implies that $\sqrt{b} < \pi(n+1)$ (since $f^-(\pi(n+1)) = 0$).

(ii) The case $\sqrt{b} = \pi(n+1)$ can be considered analogously to the previous case.

□



(a) An asymptote of the function ϕ_L (for $L = 1.5$) belongs to $\{\pi\mathbb{Z}\}$

(b) Eigenvalue of infinite multiplicity for $L = 1$

Figure 1.10: Particular cases

1.3.2.2 Discrete spectrum of the operator A_s^μ.

We now pass to the study of the discrete spectrum of the operator A_s^μ. Suppose that $\lambda^2 \neq 0$ is an eigenvalue of the operator A_s^μ. Then, an eigenfunction $u \in D(A_s^\mu)$ corresponding to λ^2 verifies the equation $u'' + \lambda^2 u = 0$ in each edge of the graph G. Hence, it has the form

$$u_{j+\frac{1}{2}}(s) = a_{j+\frac{1}{2}} \sin(\lambda s) + b_{j+\frac{1}{2}} \cos(\lambda s), \qquad s \in [0,1], \qquad \forall j \in \mathbb{Z}, \tag{1.3.36}$$

$$u_j(y) = c_j \cos(\lambda y), \qquad y \in \left[-\tfrac{L}{2}, \tfrac{L}{2}\right], \qquad \forall j \in \mathbb{Z}. \tag{1.3.37}$$

The continuity of the eigenfunction u at the vertices of the graph (which is due to the fact that $u \in D(A_s^\mu)$) implies that

$$b_{j+\frac{1}{2}} = a_{j-\frac{1}{2}} \sin\lambda + b_{j-\frac{1}{2}} \cos\lambda = c_j \cos\left(\tfrac{\lambda L}{2}\right), \qquad \forall j \in \mathbb{Z}. \tag{1.3.38}$$

The Kirchhoff's conditions (1.3.7) for the function u give

$$a_{j+\frac{1}{2}} - a_{j-\frac{1}{2}} \cos\lambda + b_{j-\frac{1}{2}} \sin\lambda + w_j^\mu c_j \sin\left(\tfrac{\lambda L}{2}\right) = 0, \qquad \forall j \in \mathbb{Z}. \tag{1.3.39}$$

Lemma 1.3.3. *If $\lambda \in \Sigma\Delta\Sigma'_s$, then λ^2 is not an eigenvalue of the operator A_s^μ for any $\mu > 0$.*

Proof.

(a) Suppose that $\lambda \in \Sigma'_s \setminus \Sigma$. Then, as $\cos\left(\frac{\lambda L}{2}\right) = 0$, the relations (1.3.38) imply that $b_{j+\frac{1}{2}} = 0$, $\forall j \in \mathbb{Z}$, and $a_{j-\frac{1}{2}} = 0$, $\forall j \in \mathbb{Z}$, since $\sin\lambda \neq 0$. Finally, taking into account that $\sin\left(\frac{\lambda L}{2}\right) \neq 0$, one finds from the relations (1.3.39) that $c_j = 0$, $\forall j \in \mathbb{Z}$. Thus, the eigenfunction u is identically zero.

(b) Suppose that $\lambda \in \Sigma \setminus \Sigma'_s$, $\lambda \neq 0$. Then, since $\sin\lambda = 0$ and $\cos\left(\frac{\lambda L}{2}\right) \neq 0$, the relations (1.3.38) imply that $|b_{j+\frac{1}{2}}| = b$, $\forall j \in \mathbb{Z}$, and $|c_j| = b\left|\cos\left(\frac{\lambda L}{2}\right)\right|^{-1}$, $\forall j \in \mathbb{Z}$. Consequently, if $b \neq 0$, then $u \notin L_2(G)$. If $b = 0$, then $b_{j+\frac{1}{2}} = 0$ and $c_j = 0$, $\forall j \in \mathbb{Z}$. In this case, one finds from the relations (1.3.39) that $|a_{j+\frac{1}{2}}| = a$, $\forall j \in \mathbb{Z}$. Again, if $a \neq 0$, then $u \notin L_2(G)$, and if $a = 0$, then the eigenfunction u is identically zero.

(c) Finally, consider the point $\lambda = 0$. Then, the relations (1.3.36)–(1.3.39) are replaced by

$$u_{j+\frac{1}{2}}(s) = a_{j+\frac{1}{2}}s + b_{j+\frac{1}{2}}, \quad s \in [0,1], \qquad u_j(y) = c_j, \quad y \in \left[-\tfrac{L}{2}, \tfrac{L}{2}\right], \qquad \forall j \in \mathbb{Z},$$
$$b_{j+\frac{1}{2}} = a_{j-\frac{1}{2}} + b_{j-\frac{1}{2}} = c_j, \qquad a_{j+\frac{1}{2}} - a_{j-\frac{1}{2}} = 0, \qquad \forall j \in \mathbb{Z}.$$

Thus, we have $a_{j+\frac{1}{2}} = a$, $\forall j \in \mathbb{Z}$, and $c_j - c_{j-1} = a$, $\forall j \in \mathbb{Z}$. If $a \neq 0$, then $u \notin L_2(G)$. If $a = 0$, then $c_j = c_{j-1} = b_{j+\frac{1}{2}} = b_{j-\frac{1}{2}} = c$, $\forall j \in \mathbb{Z}$. Again, either $u \notin L_2(G)$ (if $c \neq 0$), or $u = 0$.

□

Remark 1.3.6. Lemma 1.3.3 together with Proposition 1.3.5 imply that the set $\Sigma \cup \Sigma'_s$ can be excluded from the consideration while searching the eigenvalues of the operator A^μ_s. Indeed, the points $\lambda \in \Sigma \Delta \Sigma'_s$ do not correspond to eigenvalues whereas the set $\Sigma \cap \Sigma'_s$ corresponds to the eigenvalues of infinite multiplicity.

For $\lambda \notin \Sigma \cup \Sigma'_s$ the coefficients $\{a_{j+\frac{1}{2}}, b_{j+\frac{1}{2}}, c_j\}$ can be expressed in terms of the values $\{\mathbf{u}_j\}$ of the function u at the vertices of the graph:

$$a_{j+\frac{1}{2}} = \frac{1}{\sin\lambda}\left(\mathbf{u}_{j+1} - \cos\lambda\,\mathbf{u}_j\right), \quad b_{j+\frac{1}{2}} = \mathbf{u}_j, \quad c_j = \frac{\mathbf{u}_j}{\cos\left(\frac{\lambda L}{2}\right)}, \qquad \forall j \in \mathbb{Z}. \tag{1.3.40}$$

Then, the relations (1.3.36)–(1.3.37) take the form

$$u_{j+\frac{1}{2}}(s) = \mathbf{u}_j \frac{\sin\left(\lambda(1-s)\right)}{\sin\lambda} + \mathbf{u}_{j+1}\frac{\sin\left(\lambda s\right)}{\sin\lambda}, \qquad s \in [0,1], \qquad \forall j \in \mathbb{Z}, \tag{1.3.41}$$
$$u_j(y) = \mathbf{u}_j \frac{\cos\left(\lambda y\right)}{\cos\left(\frac{\lambda L}{2}\right)}, \qquad y \in \left[-\tfrac{L}{2}, \tfrac{L}{2}\right], \qquad \forall j \in \mathbb{Z}. \tag{1.3.42}$$

After injecting (1.3.40) into the relation (1.3.39) one obtains the following finite difference equation:

$$\mathbf{u}_{j+1} + 2g(\lambda)\mathbf{u}_j + \mathbf{u}_{j-1} = 0, \qquad j \in \mathbb{Z}^*, \tag{1.3.43}$$
$$\mathbf{u}_1 + 2g^\mu(\lambda)\mathbf{u}_0 + \mathbf{u}_{-1} = 0, \tag{1.3.44}$$

where

$$g(\lambda) = -\cos\lambda + \frac{1}{2}\sin\lambda\tan\left(\tfrac{\lambda L}{2}\right), \tag{1.3.45}$$
$$g^\mu(\lambda) = -\cos\lambda + \frac{\mu}{2}\sin\lambda\tan\left(\tfrac{\lambda L}{2}\right). \tag{1.3.46}$$

Thus, the initial problem for a differential operator on the graph reduces to a problem for a finite difference operator acting on sequences $\{\mathbf{u}_j\}_{j\in\mathbb{Z}}$. The characteristic equation associated to the system (1.3.43)–(1.3.44) is

$$r^2 + 2g(\lambda)r + 1 = 0. \tag{1.3.47}$$

We will denote by $r(\lambda)$ the solution of (1.3.47) given by the relation

$$r(\lambda) = -g(\lambda) + sign(g(\lambda))\sqrt{g^2(\lambda) - 1}. \tag{1.3.48}$$

Clearly, $|r(\lambda)| \leqslant 1$. Let us first solve the system (1.3.43) for $j < 0$ and $j > 0$. If $r(\lambda) \neq r(\lambda)^{-1}$ the general solution of (1.3.43) is

$$\mathbf{u}_j = A_+ r(\lambda)^j + B_+ r(\lambda)^{-j}, \qquad j \geqslant 0, \tag{1.3.49}$$

$$\mathbf{u}_j = A_- r(\lambda)^j + B_- r(\lambda)^{-j}, \qquad j \leqslant 0. \tag{1.3.50}$$

In the particular case when $r(\lambda) = r(\lambda)^{-1} = \pm 1$ the general solution is

$$\mathbf{u}_j = C_+ r(\lambda)^j + D_+ r(\lambda)^j j, \qquad j \geqslant 0, \tag{1.3.51}$$

$$\mathbf{u}_j = C_- r(\lambda)^j + D_- r(\lambda)^j j, \qquad j \leqslant 0. \tag{1.3.52}$$

Since an eigenfunction has to belong to $L_2(G)$, the value λ^2 can be an eigenvalue of the operator A_s^μ only if $|r(\lambda)| < 1$, which is equivalent to $|g(\lambda)| > 1$. If λ^2 is an eigenvalue of the operator A_s^μ then $B_+ = A_- = 0$. Moreover, the equations (1.3.49)–(1.3.50) for $j = 0$ imply that $A_+ = B_- = A$. Thus,

$$\mathbf{u}_j = A r(\lambda)^{|j|}, \qquad j \in \mathbb{Z}. \tag{1.3.53}$$

At this point we can remark that all the eigenvalues of the operator A_s^μ (if they exist) are simple. Finally, after injecting (1.3.53) into (1.3.44) we find

$$\lambda^2 \in \sigma_d(A_s^\mu) \quad \Leftrightarrow \quad r(\lambda) = -g^\mu(\lambda). \tag{1.3.54}$$

Taking into account (1.3.45), (1.3.46) we arrive at the following relation:

$$sign(g(\lambda))\sqrt{g^2(\lambda) - 1} = (1 - \mu)(g(\lambda) + \cos\lambda).$$

Since $|g(\lambda)| > 1$, the above relation can be rewritten as

$$\mu = F(\lambda), \tag{1.3.55}$$

$$F(\lambda) = 1 - \frac{\sqrt{g^2(\lambda) - 1}}{|g(\lambda) + \cos\lambda|}. \tag{1.3.56}$$

We arrive then at the following assertion.

Proposition 1.3.9.

$$\lambda^2 \in \sigma_d(A_s^\mu) \ \Leftrightarrow \ \lambda \ \textit{ is a solution of (1.3.55).}$$

We will now study the existence and the position of the eigenvalues of the operator A_s^μ. As it was mentioned before, if λ^2 is an eigenvalue of the operator A_s^μ of finite multiplicity, then one necessarily has $|g(\lambda)| > 1$. The following proposition establishes the relation between the absolute value of $g(\lambda)$ and the nature of the point λ^2.

Proposition 1.3.10. *For* $\lambda \notin \Sigma_s'$,

$$|g(\lambda)| \leqslant 1 \ \Leftrightarrow \ |r(\lambda)| = 1 \ \Leftrightarrow \ \lambda^2 \in \sigma(A_s). \tag{1.3.57}$$

Proof. The first equivalence follows immediately from (1.3.48). Next, we remark that $|g(\lambda)| \leqslant 1$ if and only if there exists $\theta \in [0, \pi]$ such that $g(\lambda) = -\cos\theta$. Due to (1.3.45) we get

$$\cos\lambda - \cos\theta = \frac{1}{2}\sin\lambda\tan\left(\tfrac{\lambda L}{2}\right).$$

For $\lambda \notin \Sigma \cup \Sigma_s \cup \Sigma_s'$ it is equivalent to (1.3.25). Finally, for $\lambda \in (\Sigma \cup \Sigma_s) \setminus \Sigma_s'$ we have $|g(\lambda)| = 1$ and $\lambda^2 \in \sigma(A_s)$ (cf. Proposition 1.3.6 (i)). $\square$

Remark 1.3.7. The relation $|r(\lambda)| = 1$ can be seen as the condition of existence of a generalized eigenfunction of the operator A_s^μ for the value λ^2. By a generalized eigenfunction we mean a solution of the problem $u'' + \lambda^2 u = 0$ in each edge of the graph G which is continuous and verifies the Kirchhoff's conditions and which has at most polynomial growth but does not belong to $L_2(G)$. Indeed, for $\lambda \notin \Sigma \cup \Sigma_s'$ the relations (1.3.49)–(1.3.50) or (1.3.51)–(1.3.52) completed by (1.3.44) give solutions of at most polynomial growth which do not belong to $L_2(G)$ if and only if $|r(\lambda)| = 1$. For $\lambda \in \Sigma \setminus \Sigma_s'$ one has $r(\lambda) = -g(\lambda) = \pm 1$. The existence of a generalized eigenfunction in this case has been established in the proof of Lemma 1.3.3 (b). Thus, Proposition 1.3.10 implies that in our case the existence of a generalized eigenfunction for λ^2, $\lambda \notin \Sigma_s'$, is equivalent to $\lambda^2 \in \sigma_{ess}(A_s^\mu)$. The relation between the existence of a generalized eigenfunction for λ and the fact that λ^2 belongs to the essential spectrum of the operator is known as a Schnol's theorem-type result.

It follows from Proposition 1.3.10 that if λ^2 is an eigenvalue of the operator A_s^μ of finite multiplicity, it is necessarily in a gap of the operator A_s^μ. Thus, the operator A_s^μ has no embedded eigenvalues. With the classification of the gaps in two types introduced in Proposition 1.3.8 we can state the following theorem giving the number of eigenvalues inside the gaps of the operator A_s^μ.

Theorem 1.3.1. *The operator A_s^μ has no embedded eigenvalues for any $\mu > 0$. For $0 < \mu < 1$ there exist two simple eigenvalues of the operator A_s^μ in each gap of type I and one simple eigenvalue of this operator in each gap of type II. For $\mu \geqslant 1$ the operator A_s^μ has no eigenvalues.*

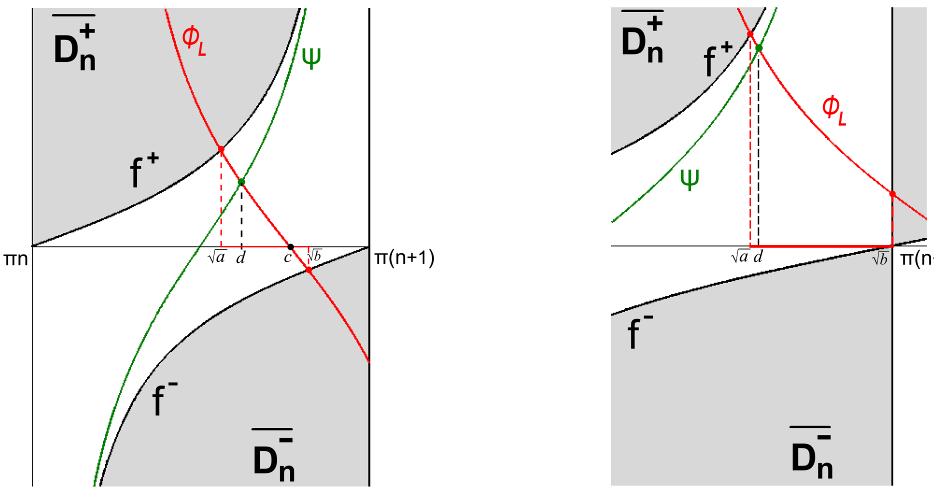
Proof. The absence of embedded eigenvalues has been discussed above. The eigenvalues of the operator A_s^μ are characterized by Proposition 1.3.9. Clearly, for $\mu \geqslant 1$ (1.3.55) has no solutions. Let us consider the case $0 < \mu < 1$. Let $]a, b[$ be a gap of the operator A_s. We will study the behaviour of the function F given by (1.3.56) inside this gap. Using (1.3.45) it can be rewritten as

$$F(\lambda) = 1 - \sqrt{1 - \phi_L(\lambda)\,(\phi_L(\lambda) - \psi(\lambda))}, \tag{1.3.58}$$

where $\psi(\lambda) = -2/\tan\lambda$ and the function ϕ_L is defined in (1.3.25). The only zeros of the function F in the interval $]\sqrt{a}, \sqrt{b}[$ are given by the zeros of ϕ_L and the zeros of $\varphi = \phi_L - \psi$. Let us investigate the variations of F in the two cases given by Proposition 1.3.8:

I The function g being continuous in $\mathbb{R} \setminus \Sigma_s'$, it follows from (1.3.57) that $|g(\sqrt{a})| = |g(\sqrt{b})| = 1$, and hence, $F(\sqrt{a}) = F(\sqrt{b}) = 1$. In view of Proposition 1.3.8, the strictly decaying function ϕ_L has exactly one zero in $]\sqrt{a}, \sqrt{b}[$ since $\phi_L(\sqrt{a}) = f^+(\sqrt{a}) > 0$, $\phi_L(\sqrt{b}) = f^-(\sqrt{b}) < 0$. We denote this zero by c. Besides, φ is strictly decreasing and one can show that $\varphi(\sqrt{a}) > 0$ and $\varphi(\sqrt{b}) < 0$. Indeed, one easily verifies that $f^-(\lambda) < \psi(\lambda) < f^+(\lambda)$, $\forall \lambda \in [\sqrt{a}, \sqrt{b}]$. It follows that φ has a unique zero (denoted by d) in $]\sqrt{a}, \sqrt{b}[$ (see figure 1.11a). As a consequence, the function $F(\lambda)$ is strictly decreasing from 1 to 0 in the interval $[\sqrt{a}, \min(c, d)]$, strictly increasing from 0 to 1 in the interval $[\max(c, d), \sqrt{b}]$ and negative in the interval $]\min(c, d), \max(c, d)[$. Thus, there exists precisely two solutions of (1.3.55) in the interval $]\sqrt{a}, \sqrt{b}[$ for $0 < \mu < 1$.

II Consider, for example, the case (ii), i.e. $\sqrt{b} = \pi(n+1)$ (the case (i) can be considered analogously). As in the case I, one has $F(\sqrt{a}) = 1$. The function ϕ_L does not change the sign in $]\sqrt{a}, \sqrt{b}[$ since $\phi_L(\sqrt{a}) > 0$, $\phi_L(\sqrt{b}) \geqslant 0$. Hence, the point c from the previous case does not exist. The point d still exists and (see figure 1.11b). In the interval $[\sqrt{a}, d]$ the function $F(\lambda)$ decreases monotonously from 0 to 1 whereas in the interval $[d, \sqrt{b}[$ it is negative. Hence, there exists a unique solution of (1.3.55) in the interval $]\sqrt{a}, \sqrt{b}[$ for $0 < \mu < 1$.



(a) Case I: both points c and d exist, which leads to two eigenvalues in such a gap

(b) Case II: the point d exists but the point c does not, which gives one eigenvalue in this gap

Figure 1.11: Illustration for two types of gaps

□

1.3.3 Results for the operator A^μ_{as}

We will now briefly describe the modifications of the previous considerations in the case of the operator A^μ_{as}. The operator corresponding to the periodic case $\mu = 1$ is denoted by A_{as}. The analogue of Proposition 1.3.3 is:

Proposition 1.3.11. *For $\theta \in [0, \pi]$, $\lambda^2 \in \sigma(A_{as}(\theta))$ if and only if λ is a solution of the equation*

$$2 \sin\left(\tfrac{\lambda L}{2}\right)(\cos\lambda - \cos\theta) = -\sin\lambda \cos\left(\tfrac{\lambda L}{2}\right), \qquad \lambda \neq 0. \tag{1.3.59}$$

For $\sin\left(\frac{\lambda L}{2}\right) \neq 0$ the equation (1.3.59) is equivalent to

$$\cos\theta = \cos\lambda - \frac{1}{2}\sin\lambda \tan\left(\tfrac{\lambda L}{2} + \tfrac{\pi}{2}\right).$$

The analogues of the properties given in Propositions 1.3.4–1.3.6 hold.

Proposition 1.3.12.

1. $\{\lambda^2, \ \lambda \in \Sigma \cup \Sigma'_s \setminus \{0\}\} \subset \sigma(A_{as})$.

2. *The operator A_{as} has infinitely many gaps whose ends tend to infinity.*

3. *The operator A_{as} has the following set of eigenvalues of infinite multiplicity which are isolated points of the spectrum:*

$$\sigma_{pp}(A_{as}) = \left\{\lambda^2,\ \ \lambda \in \Sigma \cup \Sigma_s \setminus \{0\}\right\}.$$

This set is non-empty if and only if $L = \frac{2m}{k}$ an irreducible fraction with $m \in \mathbb{N}^*$, $k \in \mathbb{N}^*$. In this case, it has the form

$$\sigma_{pp}(A_{as}) = \left\{(n\pi k)^2\,,\ n \in \mathbb{N}^*\right\}.$$

Otherwise $\sigma_{pp}(A_{as}) = \emptyset$.

4. *$\lambda^2 \in \sigma(A_{as})$ if and only if either one of the following possibilities holds:*

   (i) $\lambda \in \Sigma \cup \Sigma_s' \setminus \{0\}$;

   (ii) λ is a solution of the equation

$$-2\tan\left(\tfrac{\lambda L}{2}\right) = f_\theta(\lambda),$$

   for some $\theta \in [0, \pi]$.

Next, Proposition 1.3.2 still holds for the operator A^μ_{as}:

Proposition 1.3.13.

$$\sigma_{ess}(A^\mu_{as}) = \sigma_{ess}(A_{as}).$$

Passing to finite-difference equations for $\lambda \notin \Sigma \cup \Sigma_s$ we find again (1.3.43), (1.3.44) with $g(\lambda)$, $g^\mu(\lambda)$ replaced by $g_{as}(\lambda)$, $g^\mu_{as}(\lambda)$:

$$\begin{aligned} g_{as}(\lambda) &= -\cos\lambda + \frac{1}{2}\sin\lambda\tan\left(\tfrac{\lambda L}{2} + \tfrac{\pi}{2}\right), \\ g^\mu_{as}(\lambda) &= -\cos\lambda + \frac{\mu}{2}\sin\lambda\tan\left(\tfrac{\lambda L}{2} + \tfrac{\pi}{2}\right), \end{aligned}$$

The characteristic equation (1.3.47) as well as the characterisation (1.3.55)-(1.3.56) of the eigenvalues are still valid with $g(\lambda)$ replaced by $g_{as}(\lambda)$. The relation $|g_{as}(\lambda)| \leqslant 1 \;\Leftrightarrow\; \lambda^2 \in \sigma(A_s)$ for $\lambda \notin \Sigma_s$ analogous to Proposition 1.3.10 also holds. Thus, we can state the analogue of Theorem 1.3.1.

Theorem 1.3.2. *The operator A^μ_{as} has no embedded eigenvalues for any $\mu > 0$. For $0 < \mu < 1$ there exists either one or two simple eigenvalues of the operator A^μ_{as} in each gap of this operator. These eigenvalues are characterised as follows:*

$$\lambda^2 \in \sigma_d(A^\mu_{as}) \quad \Leftrightarrow \quad \mu = 1 - \frac{\sqrt{g^2_{as}(\lambda) - 1}}{|g_{as}(\lambda) + \cos\lambda|}.$$

For $\mu \geqslant 1$ the operator A^μ_{as} has no eigenvalues.

Remark 1.3.8. As for the operator A_s, we could give a more precise version of this theorem by distinguishing two types of gaps of the operator A_{as} analogously to Proposition 1.3.8. For $0 < \mu < 1$ the operator A_{as} has two eigenvalues in the gaps of one type and one eigenvalue in the gaps of the other type.

1.3.4 The spectrum of the operator A.

As we have seen, both of the operators A_s and A_{as} have infinitely many gaps. However, it turns out that the gaps of one operator overlap with spectral bands of the other one, so that the full operator A has no gap.

Proposition 1.3.14.

$$\sigma(A) = \mathbb{R}^+.$$

Proof. Let us suppose that there exists λ such that $\lambda^2 \notin \sigma(A)$ (of course, the same is true for some open neighbourhood of λ). Due to the characterisation of the essential spectrum (1.3.57) and its analogue for A_{as} we have

$$\begin{cases} \left| -\cos\lambda + \dfrac{1}{2}\sin\lambda\tan\left(\frac{\lambda L}{2}\right)\right| > 1, \\ \left| \cos\lambda + \dfrac{\sin\lambda}{2\tan\left(\frac{\lambda L}{2}\right)}\right| > 1. \end{cases} \tag{1.3.60}$$

Let us denote $a = \tan\left(\frac{\lambda L}{2}\right)$. Then, the system (1.3.60) can be rewritten as

$$\begin{cases} \dfrac{a^2}{4}\sin^2\lambda - a\sin\lambda\cos\lambda + \cos^2\lambda > 1, & (1.3.61) \\ \dfrac{1}{4a^2}\sin^2\lambda + \dfrac{1}{a}\sin\lambda\cos\lambda + \cos^2\lambda > 1. & (1.3.62) \end{cases}$$

Multiplying (1.3.62) by a^2 and taking the sum with (1.3.61) we obtain

$$\frac{1}{4}(1+a^2)\sin^2\lambda + (1+a^2)\cos^2\lambda > 1 + a^2,$$

which is impossible. □

We see that the eigenvalues of the operators A^μ_s, A^μ_{as} are in fact embedded eigenvalues for the operator A^μ.

1.4 Existence of eigenvalues for the non-limit operator

1.4.1 Main result

We return now to the case of the ladder. As it was mentioned above, instead of studying the full operator A^μ_ε we will study separately the operators $A^\mu_{\varepsilon,s}$, $A^\mu_{\varepsilon,as}$ for which the existence of eigenvalues inside gaps will be established. The convergence of the essential spectrum of the operator $A_{\varepsilon,s}$ (resp. $A_{\varepsilon,as}$) to the one of the operator A_s (resp. A_{as}) is known since the works [61], [47]. More precisely, the following theorem holds.

Theorem 1.4.1 (Essential spectrum). *Let $\{]a_n, b_n[,\ n \in \mathbb{N}^*\}$ be the gaps of the operator A_s (A_{as}) on the limit graph G. Then, for each $n_0 \in \mathbb{N}^*$ there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ the operator $A_{\varepsilon,s}$ ($A_{\varepsilon,as}$) has at least n_0 gaps $\{]a_{\varepsilon,n}, b_{\varepsilon,n}[,\ 1 \leqslant n \leqslant n_0\}$ such that*

$$a_{\varepsilon,n} = a_n + O(\varepsilon), \quad b_{\varepsilon,n} = b_n + O(\varepsilon), \qquad \varepsilon \to 0, \qquad 1 \leqslant n \leqslant n_0.$$

The proof of this fact is based on the Floquet-Bloch theory (which permits to reduce the study of the spectrum of a periodic operator to the study of a family of operators in a bounded domain) and the min-max principle (for studying the eigenvalues of an operator in a bounded domain). Theorem 1.4.1 guarantees the existence of gaps for the operators $A_{\varepsilon,s}$, $A_{\varepsilon,as}$ at least for ε small enough. We fix then one of these gaps and study the existence of eigenvalues inside it. Our principal result is the following.

Theorem 1.4.2 (Discrete spectrum). *Let $]a,b[$ be a gap of the operator A_s^μ (A_{as}^μ) on the limit graph G and $\lambda^{(0)} \in]a,b[$ a (simple) eigenvalue of this operator. Then there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ the operator $A_{\varepsilon,s}^\mu$ ($A_{\varepsilon,as}^\mu$) has an eigenvalue λ_ε inside a gap $]a_\varepsilon, b_\varepsilon[$ with the following asymptotic expansion at any order n:*

$$\lambda_\varepsilon = \sum_{k=0}^{n} \lambda^{(k)} \varepsilon^k + O\left(\varepsilon^{n+1}\right), \qquad \varepsilon \to 0. \tag{1.4.1}$$

We prove this theorem in Chapter 2 using Matched Asymptotic Expansions. However, we will give now a proof of a weak version of this theorem using a simpler argument.

Theorem 1.4.3 (Discrete spectrum – weak version). *Let $]a,b[$ be a gap of the operator A_s^μ (A_{as}^μ) on the limit graph G and $\lambda^{(0)} \in]a,b[$ a (simple) eigenvalue of this operator. Then there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ the operator $A_{\varepsilon,s}^\mu$ ($A_{\varepsilon,as}^\mu$) has an eigenvalue λ_ε inside a gap $]a_\varepsilon, b_\varepsilon[$ such that*

$$\lambda_\varepsilon = \lambda^{(0)} + O\left(\sqrt{\varepsilon}\right), \qquad \varepsilon \to 0.$$

Though the rate of convergence guaranteed by this theorem is not optimal, the proof given below provides an easy way to establish the existence of eigenvalues for the operators $A_{\varepsilon,s}^\mu$, $A_{\varepsilon,as}^\mu$. It also illustrates the main ideas that will be used in the proof of Theorem 1.4.2 in a simplified context.

Remark 1.4.1. In [57] O. Post proves the convergence of all the components of the spectrum of the Neumann Laplacian for a large class of graph-like manifolds (in particular, they are not necessarily compact). Applied to our case, these results imply the existence of eigenvalues of the operators $A_{\varepsilon,s}^\mu$, $A_{\varepsilon,as}^\mu$ inside gaps for ε small enough. They also guarantee that these eigenvalues are simple (which is not established in Theorem 1.4.2). The optimal rate of convergence for the eigenvalues, which is linear in ε, has also been proved in [57]. Thus, we provide an alternative proof of existence of eigenvalues of the operators $A_{\varepsilon,s}^\mu$, $A_{\varepsilon,as}^\mu$ which permits to obtain a full asymptotic expansion of the eigenvalues (1.4.1).

Combining Propositions 1.3.4 (2), 1.3.12 (2) and Theorems 1.4.1, 1.4.2, 1.3.1, 1.3.2 we can formulate the following result.

Theorem 1.4.4. *For any $n_0 \in \mathbb{N}^*$ there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ the operator $A_{\varepsilon,s}$ ($A_{\varepsilon,as}$) has at least n_0 eigenvalues. Moreover, for any $A > 0$ there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ the operator $A_{\varepsilon,s}$ ($A_{\varepsilon,as}$) has at least one eigenvalue greater than A.*

1.4.2 Method of a pseudo-mode

We will now prove Theorem 1.4.3. We consider only the operator $A_{\varepsilon,s}^\mu$, the case of the operator $A_{\varepsilon,as}^\mu$ being analogous. Our proof is based on the construction of a so-called pseudo-mode, i.e. an approximation of the eigenfunction of the operator $A_{\varepsilon,s}^\mu$. More precisely, we use Lemma A.2.1 that gives an estimate for the distance from the spectrum once such an approximation is found. In application to the operator $A^{\mu}_{\varepsilon,s}$ Lemma A.2.1 means that if there exists a function $u_\varepsilon \in H^1_s(\Omega^\mu_\varepsilon)$ (H^1_s standing for the symmetric subspace of H^1) such that

$$\left| \int\limits_{\Omega^\mu_\varepsilon} \left(\nabla u_\varepsilon \nabla v - \lambda^{(0)} u_\varepsilon v\right) d\Omega \right| \leqslant C\sqrt{\varepsilon}\|u_\varepsilon\|_{H^1(\Omega^\mu_\varepsilon)}\|v\|_{H^1(\Omega^\mu_\varepsilon)}, \qquad \forall v \in H^1_s(\Omega^\mu_\varepsilon), \tag{1.4.2}$$

then

$$dist(\sigma(A^\mu_{\varepsilon,s}), \lambda^{(0)}) \leqslant \widetilde{C}\sqrt{\varepsilon}, \tag{1.4.3}$$

with some constant $\widetilde{C}$ which does not depend on ε. According to Theorem 1.4.1, for ε small enough there exists a constant C such that $\sigma_{ess}(A^\mu_{\varepsilon,s}) \cap [a + C\varepsilon,\, b - C\varepsilon] = \emptyset$. Together with (1.4.3) it means that $\sigma_d(A^\mu_{\varepsilon,s}) \cap\, [\lambda^{(0)} - \widetilde{C}\sqrt{\varepsilon},\, \lambda^{(0)} + \widetilde{C}\sqrt{\varepsilon}] \neq \emptyset$ which proves the existence of an eigenvalue of the operator $A^\mu_{\varepsilon,s}$ in a neighbourhood of $\lambda^{(0)}$ of order $\sqrt{\varepsilon}$ for ε small enough.

1.4.3 Construction of a pseudo-mode

Let us introduce some notation for the domain Ω^μ_ε (cf. figure 1.12). We denote by $\mathcal{V}^{\varepsilon,\pm}_{j+\frac12}$ its horizontal top and bottom edges,

$$\begin{aligned} \mathcal{V}^{\varepsilon,+}_{j+\frac12} &= \left]j + \tfrac{w^\mu_j \varepsilon}{2},\, j + 1 - \tfrac{w^\mu_{j+1}\varepsilon}{2}\right[\times \left]\tfrac{L}{2} - \varepsilon,\, \tfrac{L}{2}\right[, && j \in \mathbb{Z}, \\ \mathcal{V}^{\varepsilon,-}_{j+\frac12} &= \left]j + \tfrac{w^\mu_j \varepsilon}{2},\, j + 1 - \tfrac{w^\mu_{j+1}\varepsilon}{2}\right[\times \left]-\tfrac{L}{2},\, -\tfrac{L}{2} + \varepsilon\right[, && j \in \mathbb{Z}, \end{aligned}$$

by $\mathcal{V}^\varepsilon_j$ its vertical edges,

$$\mathcal{V}^\varepsilon_j = \left]j - \tfrac{w^\mu_j \varepsilon}{2},\, j + \tfrac{w^\mu_j \varepsilon}{2}\right[\times \left]\tfrac{L}{2} - \varepsilon,\, \tfrac{L}{2} + \varepsilon\right[, \qquad j \in \mathbb{Z},$$

and by $K^{\varepsilon,\pm}_j$ the top and bottom junctions,

$$\begin{aligned} K^{\varepsilon,+}_j &= \left]j - \tfrac{w^\mu_j \varepsilon}{2},\, j + \tfrac{w^\mu_j \varepsilon}{2}\right[\times \left]\tfrac{L}{2} - \varepsilon,\, \tfrac{L}{2}\right[, && j \in \mathbb{Z}, \\ K^{\varepsilon,-}_j &= \left]j - \tfrac{w^\mu_j \varepsilon}{2},\, j + \tfrac{w^\mu_j \varepsilon}{2}\right[\times \left]-\tfrac{L}{2},\, -\tfrac{L}{2} + \varepsilon\right[, && j \in \mathbb{Z}. \end{aligned}$$

We also introduce a notation for the boundaries separating these subdomains:

$$\Gamma^{0,\varepsilon,\pm}_{j+\frac12} = \partial\mathcal{V}^{\varepsilon,\pm}_{j+\frac12} \cap \partial K^{\varepsilon,\pm}_j, \qquad \Gamma^{1,\varepsilon,\pm}_{j+\frac12} = \partial\mathcal{V}^{\varepsilon,\pm}_{j+\frac12} \cap \partial K^{\varepsilon,\pm}_{j+1}, \qquad \Gamma^{\varepsilon,\pm}_j = \partial\mathcal{V}^\varepsilon_j \cap \partial K^{\varepsilon,\pm}_j, \qquad j \in \mathbb{Z}.$$

We construct the pseudo-mode u_ε as follows. Let u_0 be the eigenfunction of the operator A^μ_s corresponding to the eigenvalue $\lambda^{(0)}$. Then, u_ε is defined on Ω^μ_ε by "fattening" u_0 (with an appropriate rescaling):

$$u_\varepsilon(x,y) = \begin{cases} u_{0,j+\frac12}(s^\varepsilon_{j+\frac12}(x)), & (x,y) \in \mathcal{V}^{\varepsilon,\pm}_{j+\frac12}, \\ u_{0,j}(t^\varepsilon(y)), & (x,y) \in \mathcal{V}^\varepsilon_j, \\ \mathbf{u}_{0,j}, & (x,y) \in K^{\varepsilon,\pm}_j. \end{cases}$$

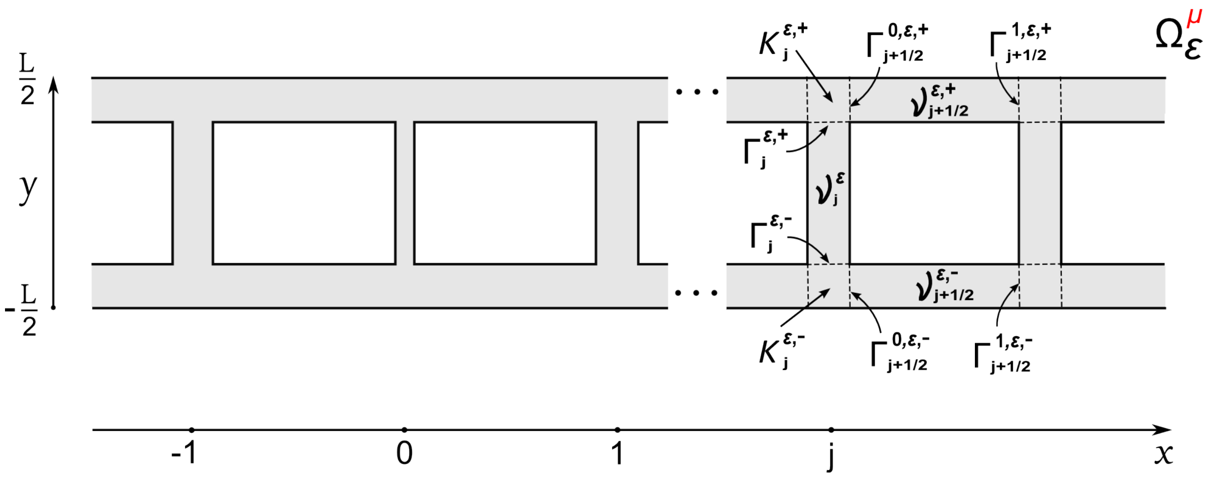


Figure 1.12: Construction of a pseudo-mode

Here $s^\varepsilon_{j+\frac12}$ and t^ε are given by the relations

$$s^\varepsilon_{j+\frac12}(x) = \frac{x - j - w^\mu_j \varepsilon/2}{1 - \left(w^\mu_j + w^\mu_{j+1}\right)\varepsilon/2}, \qquad t^\varepsilon(y) = \frac{y}{1 - 2\varepsilon/L}. \tag{1.4.4}$$

We note that due to a standard density argument (Meyers-Serrin's theorem) it is sufficient to prove (1.4.2) for any $v \in C^\infty(\Omega^\mu_\varepsilon) \cap H^1_s(\Omega^\mu_\varepsilon)$. Let us estimate the left-hand side of (1.4.2) for $v \in C^\infty(\Omega^\mu_\varepsilon) \cap H^1_s(\Omega^\mu_\varepsilon)$. Integrating by parts and using the fact that u_0 is an eigenfunction on the graph as well as the symmetry of the functions u_0 and v, we get:

$$\left|\int\limits_{\Omega^\mu_\varepsilon} \left(\nabla u_\varepsilon \nabla v - \lambda^{(0)} u_\varepsilon v\right) d\Omega\right| \leqslant 2\sum_{j\in\mathbb{Z}} \lambda^{(0)} \mathbf{u}_{0,j} \left|\int\limits_{K_j^{\varepsilon,-}} v d\Omega\right| + \lambda^{(0)} \|u_\varepsilon\|_{L_2(\Omega^\mu_\varepsilon)} \|v\|_{L_2(\Omega^\mu_\varepsilon)} O(\varepsilon)$$

$$+ 2\sum_{j\in\mathbb{Z}} \left(-u'_{0,j+\frac12}(0) \int\limits_{\Gamma^{0,\varepsilon,-}_{j+\frac12}} v(x,y)dy + u'_{0,j-\frac12}(1) \int\limits_{\Gamma^{1,\varepsilon,-}_{j-\frac12}} v(x,y)dy - u'_{0,j}\left(-\tfrac{L}{2}\right) \int\limits_{\Gamma^{\varepsilon,-}_j} v(x,y)dx \right)$$
$$\times (1 + O(\varepsilon)), \tag{1.4.5}$$

The terms in (1.4.5) containing $O(\varepsilon)$ appear because of the change of variables (1.4.4). Taking into account (1.3.53) and using Hölder's inequality, the first term in the right-hand side of (1.4.5) can be estimated as follows:

$$2\sum_{j\in\mathbb{Z}} \lambda^{(0)} \mathbf{u}_{0,j} \left|\int\limits_{K_j^{\varepsilon,-}} v d\Omega\right| \leqslant C\varepsilon \|v\|_{L_2(\Omega^\mu_\varepsilon)}. \tag{1.4.6}$$

Here and in what follows we denote by C all constants (not necessarily the same) which do not depend on ε. Next, if $M_j^{\varepsilon,-}$ is a barycentre of $K_j^{\varepsilon,-}$, then

$$\left|v(x,y) - v(M_j^{\varepsilon,-})\right| \leqslant \int\limits_{M_j^{\varepsilon,-}}^{(x,y)} |\nabla v| dt,$$

where the integral is taken along the segment joining the points (x, y) and $M_j^{\varepsilon,-}$. Due to Kirchhoff's conditions (1.3.7) verified by the function u_0 we can replace $v(x, y)$ by $v(x, y) - v(M_j^{\varepsilon,-})$ in the integrals over the boundaries in the right-hand side of (1.4.5). We have

$$\left| \int\limits_{\Gamma_j^{\varepsilon,-}} \big(v(x,y) - v(M_j^{\varepsilon,-})\big)\, dx \right| \leqslant \int\limits_{\Gamma_j^{\varepsilon,-}} \int\limits_{M_j^{\varepsilon,-}}^{(x,y)} |\nabla v| dx dt \leqslant C\varepsilon \|v\|_{H^1(K_j^{\varepsilon,-})}. \tag{1.4.7}$$

Combining (1.4.5)–(1.4.7) and taking into account (1.3.41), (1.3.42), (1.3.53) we obtain the following estimate:

$$\left| \int\limits_{\Omega_\varepsilon^\mu} \big(\nabla u_\varepsilon \nabla v - \lambda^{(0)} u_\varepsilon v\big)\, d\Omega \right| \leqslant C\varepsilon \|v\|_{H^1(\Omega_\varepsilon^\mu)}, \qquad \forall v \in C^\infty(\Omega_\varepsilon^\mu) \cap H_s^1(\Omega_\varepsilon^\mu). \tag{1.4.8}$$

Notice that by definition of u_ε one has

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon^\mu)} \geqslant C\sqrt{\varepsilon}\|u_0\|_{H^1(G)}, \qquad C > 0,$$

which together with (1.4.8) and the density argument mentioned above finishes the proof of (1.4.2) and hence, of Theorem 1.4.3.

CHAPTER 2

TRAPPED MODES IN A LOCALLY PERTURBED PERIODIC LADDER: ASYMPTOTIC EXPANSIONS OF THE EIGENVALUES

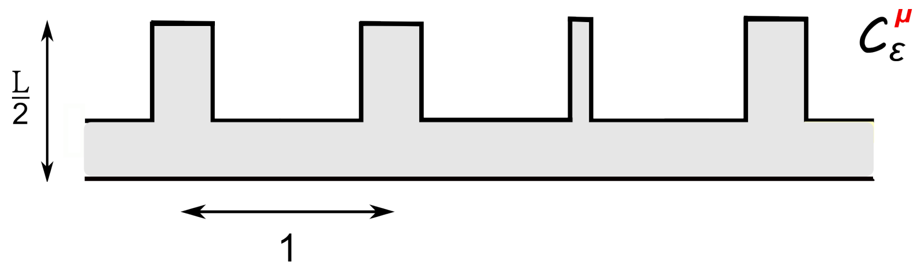
In this chapter we show how the proof of Theorem 1.4.3 can be modified in order to obtain the complete asymptotic expansion of the eigenvalue stated in Theorem 1.4.2. It will be done by constructing another pseudo-mode based on the formal asymptotic expansion of the eigenfunction of the operator $A^{\mu}_{\varepsilon,s}$ (here again, we give the proof for the operator $A^{\mu}_{\varepsilon,s}$, the proof for the operator $A^{\mu}_{\varepsilon,as}$ being analogous). To do so, we will use the matched asymptotic expansion method in the spirit of the works [63], [35], [36] (see also [66], [30], [50], [5]). The method consists in distinguishing different areas where the behaviour of the solution is different and imposing formal expansions in these areas. Then, the different expansions have to match in some intermediate zones called matching zones. This leads to matching conditions which, together with the equations satisfied in each zone, permit to determine the terms of the expansions.

The notation in this section is different from the rest of the work: the spectral parameter here is denoted by λ, contrarily to all the others chapters, where it is denoted by λ^2.

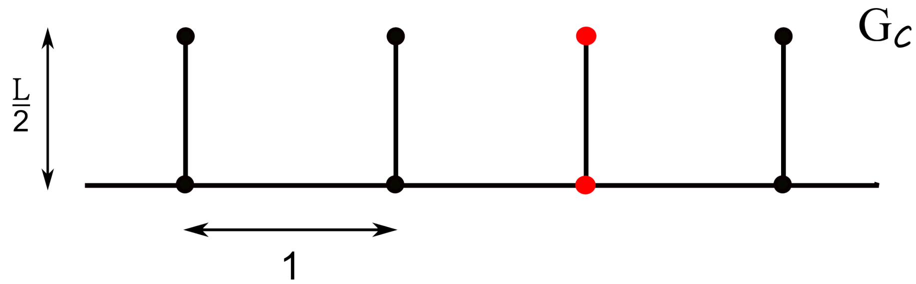
2.1 Formal expansions

Since only symmetric functions are considered when studying the operator $A^{\mu}_{\varepsilon,s}$, it is sufficient to define them on the lower half of the band $\Omega^{\mu}_{\varepsilon}$, which is a comb-shape domain that we denote by $\mathcal{C}^{\mu}_{\varepsilon}$ (see figure 2.2a):

$$\mathcal{C}^{\mu}_{\varepsilon} = \{(x, y) \in \Omega^{\mu}_{\varepsilon} \;/\; y < 0\}\,.$$



(a) The lower half of the band Ω_ε^μ: the comb-shape domain $\mathcal{C}_\varepsilon^\mu$



(b) The lower half of the graph G: the graph $G_\mathcal{C}$

Figure 2.1

Let u^ε and λ^ε be a formal solution of the eigenvalue problem

$$\begin{cases} \Delta u^\varepsilon + \lambda^\varepsilon u^\varepsilon = 0 \quad \text{in} \quad \mathcal{C}_\varepsilon^\mu, \\ \left.\dfrac{\partial u^\varepsilon}{\partial n}\right|_{\partial \mathcal{C}_\varepsilon^\mu} = 0, \end{cases} \tag{2.1.1}$$

for which the following expansions are assumed.

Far field expansion

It is valid in the regions situated far from the junctions. When ε is small, the branches of the domain are thin. For this reason, it is natural to model the solution by functions depending on the longitudinal variable only, the dependence on the transversal variable being neglected:

For $(x, y) \in \mathcal{V}^{\varepsilon,-}_{j+\frac{1}{2}} \cap \left\{x \in \left]j + \sqrt{\varepsilon},\, j + 1 - \sqrt{\varepsilon}\right[\right\}, \; j \in \mathbb{Z}$:

$$u^\varepsilon(x, y) \equiv u^\varepsilon_{j+\frac{1}{2}}(x, y) = \sum_{k \in \mathbb{N}} \varepsilon^k u^{(k)}_{j+\frac{1}{2}}(s) + o(\varepsilon^\infty), \qquad s = x - j, \tag{2.1.2}$$

For $(x, y) \in \mathcal{V}^\varepsilon_j \cap \left\{y \in \left]-\frac{L}{2} + \sqrt{\varepsilon},\, 0\right]\right\}, \; j \in \mathbb{Z}$:

$$u^\varepsilon(x, y) \equiv u^\varepsilon_j(x, y) = \sum_{k \in \mathbb{N}} \varepsilon^k u^{(k)}_j(y) + o(\varepsilon^\infty). \tag{2.1.3}$$

The functions $u^{(k)}_{j+\frac{1}{2}}$ and $u^{(k)}_j$ are defined on the edges $e_{j+\frac{1}{2}}$ and e_j of the limit graph $G_\mathcal{C}$ (which is the lower half of the graph G, cf. figure 2.2b). They are supposed smooth (this will be shown a posteriori):

$$u_{j+\frac{1}{2}}^{(k)} \in C^\infty([0,1]), \quad u_j^{(k)} \in C^\infty\left(\left[-\tfrac{L}{2},0\right]\right), \quad \left(u_j^{(k)}\right)'(0) = 0, \qquad \forall k \in \mathbb{N}, \quad \forall j \in \mathbb{Z}.$$

The set of functions $\left\{u_{j+\frac{1}{2}}^{(k)},\, u_j^{(k)}\right\}_{j\in\mathbb{Z}}$ will sometimes be denoted by $u^{(k)}$, which is a function defined on the graph.

Near field expansion

It is valid in the neighbourhood of the junctions. The solution in this region is a function of two rescaled variables X and Y:

$$\begin{gathered}\text{For } (x,y) \in \mathcal{C}_\varepsilon^\mu \cap \left\{x \in \left]j - 2\sqrt{\varepsilon},\, j + 2\sqrt{\varepsilon}\right[,\ y \in \left]-\tfrac{L}{2},\, -\tfrac{L}{2} + 2\sqrt{\varepsilon}\right]\right\}, \quad j \in \mathbb{Z}: \\ u^\varepsilon(x,y) \equiv U_j^\varepsilon(x,y) = \sum_{k\in\mathbb{N}} \varepsilon^k U_j^{(k)}(X,Y) + o(\varepsilon^\infty), \qquad X = \frac{x-j}{\varepsilon}, \qquad Y = \frac{y+L/2}{\varepsilon}.\end{gathered} \tag{2.1.4}$$

The functions $U_j^{(k)}$ are defined in the rescaled neighbourhoods of the junctions $\mathcal{J}_j$:

$$U_j^{(k)} \in H^1_{\Delta,loc}(\mathcal{J}_j), \qquad \forall j \in \mathbb{Z},$$

where

$$\mathcal{J}_j = \begin{cases} \mathcal{J}_*, & j \in \mathbb{Z}^*, \\ \mathcal{J}_0, & j = 0, \end{cases},$$

$$\mathcal{J}_* = \{\mathbb{R}\times]0,1[\} \cup \left\{\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\times\mathbb{R}\right\}, \qquad \mathcal{J}_0 = \{\mathbb{R}\times]0,1[\} \cup \left\{\left]-\tfrac{\mu}{2}, \tfrac{\mu}{2}\right[\times\mathbb{R}\right\}.$$

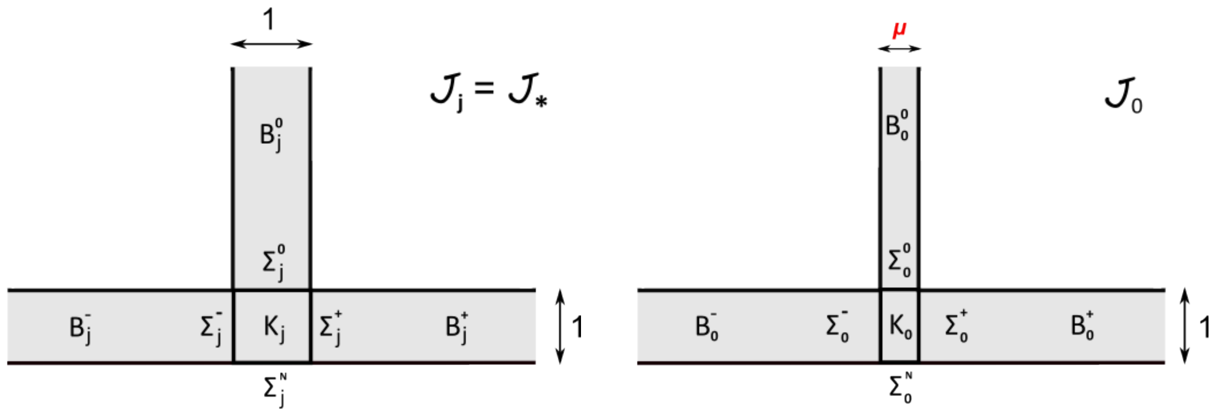
For $j \neq 0$ the domain $\mathcal{J}_j = \mathcal{J}_*$ is the unperturbed infinite junction whereas for $j = 0$ the domain $\mathcal{J}_0$ is the perturbed one (cf. figure 2.2). We will use the following notation:

$$\begin{aligned} K_j &= \left]-\tfrac{w_j^\mu}{2}, \tfrac{w_j^\mu}{2}\right[\times]0,1[, & B_j^\pm &= \mathcal{J}_j \cap \left\{\pm X > \tfrac{w_j^\mu}{2}\right\}, & B_j^0 &= \mathcal{J}_j \cap \{Y > 1\}, \\ \Sigma_j^N &= \left[-\tfrac{w_j^\mu}{2}, \tfrac{w_j^\mu}{2}\right] \times \{0\}, & \Sigma_j^\pm &= \left\{\pm\tfrac{w_j^\mu}{2}\right\} \times [0,1], & \Sigma_j^0 &= \left[-\tfrac{w_j^\mu}{2}, \tfrac{w_j^\mu}{2}\right] \times \{1\}. \end{aligned}$$

The functions $U_j^{(k)}$ are supposed to have at most polynomial growth in the infinite branches $B_j^\pm$, B_j^0 of the junctions $\mathcal{J}_j$. The set of functions $\left\{U_j^{(k)}\right\}_{j\in\mathbb{Z}}$ will sometimes be denoted by $U^{(k)}$.

Expansion for the eigenvalue:

$$\lambda^\varepsilon = \sum_{k\in\mathbb{N}} \varepsilon^k \lambda^{(k)}. \tag{2.1.5}$$



(a) The unperturbed junction $\mathcal{J}_j = \mathcal{J}_*,\ j \in \mathbb{Z}^*$. (b) The perturbed junction $\mathcal{J}_0$.

Figure 2.2

Far field and near field problems: After injecting the relations (2.1.2), (2.1.3), (2.1.5) into (2.1.1) we obtain the following problem for $u^{(k)},\ k \in \mathbb{N}$:

$$\begin{cases} \left(u^{(k)}_{j+\frac{1}{2}}\right)''(s) + \lambda^{(0)} u^{(k)}_{j+\frac{1}{2}}(s) = -\sum\limits_{m=0}^{k-1} \lambda^{(k-m)} u^{(m)}_{j+\frac{1}{2}}(s), \qquad s \in [0,1], \quad j \in \mathbb{Z}, \\ \left(u^{(k)}_{j}\right)''(y) + \lambda^{(0)} u^{(k)}_{j}(y) = -\sum\limits_{m=0}^{k-1} \lambda^{(k-m)} u^{(m)}_{j}(y), \qquad y \in \left[-\frac{L}{2}, 0\right], \quad j \in \mathbb{Z}, \\ \left(u^{(k)}_{j}\right)'(0) = 0, \qquad j \in \mathbb{Z}. \end{cases} \tag{2.1.6}$$

Analogously, after injecting (2.1.4), (2.1.5) into (2.1.1) we find a set of problems for $U^{(k)}$, $k \in \mathbb{N}$:

$$\forall j \in \mathbb{Z}, \qquad \begin{cases} \Delta U^{(k)}_j (X,Y) = -\sum\limits_{m=0}^{k-2} \lambda^{(k-m-2)} U^{(m)}_j (X,Y), \qquad (X,Y) \in \mathcal{J}_j, \\ \left.\dfrac{\partial U^{(k)}_j}{\partial n}\right|_{\partial \mathcal{J}_j} = 0. \end{cases} \tag{2.1.7}$$

Clearly, the problems (2.1.6) and (2.1.7) are not well-posed. The far field problem need to be completed by transmission conditions at the vertices of the graph and for the near field terms the behaviour at infinity has to be specified. The missing relations can be found by taking into account the matching conditions in the regions where both far field and near field expansions should hold and match.

Matching conditions: Let us define the intermediate zones (the matching areas), cf. figure 2.3:

$$\mathcal{M}^{+}_{j,\varepsilon} = \mathcal{V}^{\varepsilon,-}_{j+\frac{1}{2}} \cap \left\{ x \in \left]j + \sqrt{\varepsilon},\, j + 2\sqrt{\varepsilon}\right[\right\}, \quad \mathcal{M}^{-}_{j,\varepsilon} = \mathcal{V}^{\varepsilon,-}_{j-\frac{1}{2}} \cap \left\{ x \in \left]j - 2\sqrt{\varepsilon},\, j - \sqrt{\varepsilon}\right[\right\},$$
$$\mathcal{M}^{0}_{j,\varepsilon} = \mathcal{V}^{\varepsilon}_{j} \cap \left\{ y \in \left]-\frac{L}{2} + \sqrt{\varepsilon}, -\frac{L}{2} + 2\sqrt{\varepsilon}\right] \right\}, \qquad j \in \mathbb{Z}.$$

As follows from the way we defined the far field and the near field expansions, they should

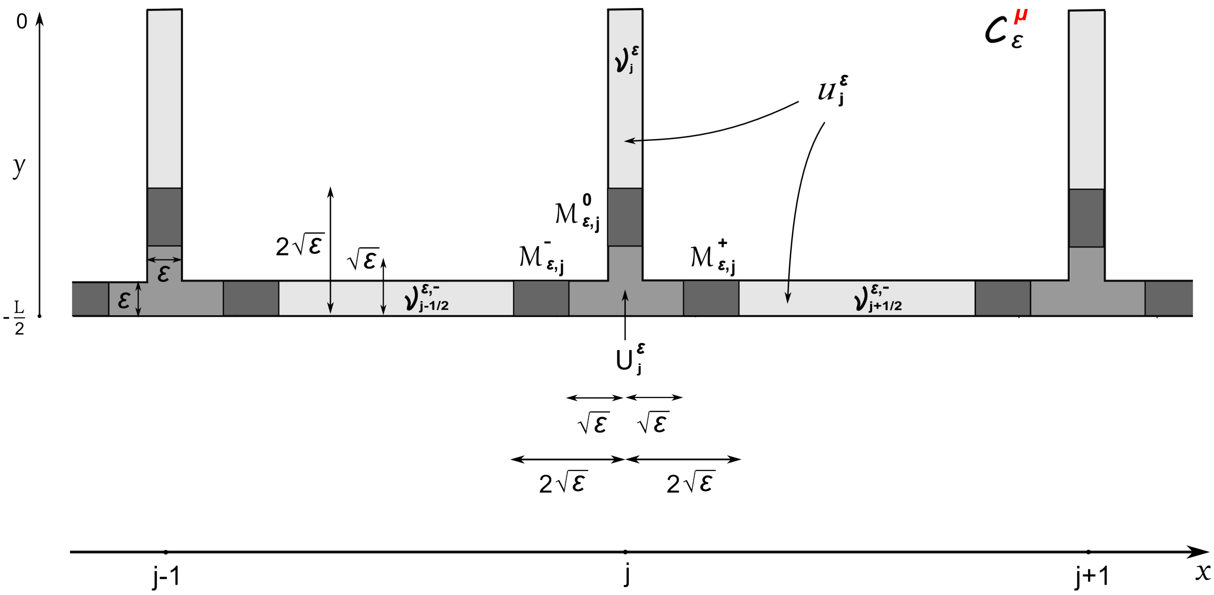


Figure 2.3: Matching areas

both hold in the matching areas $\mathcal{M}^{\pm}_{j,\varepsilon}$, $\mathcal{M}^{0}_{j,\varepsilon}$:

$$u^\varepsilon_{j+\frac{1}{2}}(x,y) = \sum_{k\in\mathbb{N}} \varepsilon^k u^{(k)}_{j+\frac{1}{2}}(x-j) + o(\varepsilon^\infty) = \sum_{k\in\mathbb{N}} \varepsilon^k U^{(k)}_j(X,Y) + o(\varepsilon^\infty), \qquad (x,y)\in\mathcal{M}^+_{j,\varepsilon}, \tag{2.1.8}$$

$$u^\varepsilon_{j-\frac{1}{2}}(x,y) = \sum_{k\in\mathbb{N}} \varepsilon^k u^{(k)}_{j-\frac{1}{2}}(x+1-j) + o(\varepsilon^\infty) = \sum_{k\in\mathbb{N}} \varepsilon^k U^{(k)}_j(X,Y) + o(\varepsilon^\infty), \quad (x,y)\in\mathcal{M}^-_{j,\varepsilon},$$

$$u^\varepsilon_{j}(x,y) = \sum_{k\in\mathbb{N}} \varepsilon^k u^{(k)}_{j}(y) + o(\varepsilon^\infty) = \sum_{k\in\mathbb{N}} \varepsilon^k U^{(k)}_j(X,Y) + o(\varepsilon^\infty), \qquad (x,y)\in\mathcal{M}^0_{j,\varepsilon}.$$

The regions $\mathcal{M}^{\pm}_{j,\varepsilon}$ correspond to $x-j\to\pm0$ and $X\to\pm\infty$. Analogously, the regions $\mathcal{M}^{0}_{j,\varepsilon}$ correspond to $y\to -L/2$ and $Y\to+\infty$. This leads us to studying the behaviour of the far field terms near the vertices of the graph and the behaviour of the near field terms at infinity.

As we will see in the following section, the far field terms have the following behaviour in the infinite branches of the junctions $\mathcal{J}_j$:

$$U^{(k)}_j(X,Y) = P^{(k)}_{j,0,\pm}(X) + \mathcal{E}^{(k)}_{j,\pm}(X,Y), \qquad (X,Y)\in B^{\pm}_j, \qquad k\in\mathbb{N}, \qquad j\in\mathbb{Z},$$
$$U^{(k)}_j(X,Y) = P^{(k)}_{j,0,0}(Y) + \mathcal{E}^{(k)}_{j,0}(X,Y), \qquad (X,Y)\in B^{0}_j, \qquad k\in\mathbb{N}, \qquad j\in\mathbb{Z},$$

where the terms $\mathcal{E}^{(k)}_{j,\pm}$, $\mathcal{E}^{(k)}_{j,0}$. Consider, for example, the region $\mathcal{M}^+_{j,\varepsilon}$. Since in this region $\mathcal{E}^{(k)}_{j,\pm}$ is exponentially decaying as $X\to\infty$ and $X=O(\varepsilon^{-1/2})$, we can rewrite (2.1.8) as

$$u^\varepsilon_{j+\frac{1}{2}}(x,y) = \sum_{k\in\mathbb{N}} \varepsilon^k P^{(k)}_{j,0,\pm}(X) + o(\varepsilon^\infty), \qquad (x,y)\in\mathcal{M}^+_{j,\varepsilon}. \tag{2.1.9}$$

On the other hand, the functions $u^{(k)}_{j+\frac{1}{2}}$ can be decomposed in Taylor series in a neighbourhood of the point $s = 0$. Then, we get from (2.1.8):

$$u^{\varepsilon}_{j+\frac{1}{2}}(x,y) = \sum_{k\in\mathbb{N}}\sum_{\ell\in\mathbb{N}} \left.\frac{d^\ell u^{(k)}_{j+\frac{1}{2}}}{ds^\ell}\right|_{s=0} \frac{\varepsilon^k (x-j)^\ell}{\ell!} + o(\varepsilon^\infty) = \sum_{k\in\mathbb{N}} \varepsilon^k \sum_{\ell=0}^{k} \left.\frac{d^\ell u^{(k-\ell)}_{j+\frac{1}{2}}}{ds^\ell}\right|_{s=0} \frac{X^\ell}{\ell!} + o(\varepsilon^\infty). \tag{2.1.10}$$

Comparing (2.1.9) with (2.1.10)and identifying the terms with the same powers of ε, we obtain the following expressions for the polynomials $P^{(k)}_{j,0,+}$:

$$P^{(k)}_{j,0,+}(X) = \sum_{\ell=0}^{k} \left.\frac{d^\ell u^{(k-\ell)}_{j+\frac{1}{2}}}{ds^\ell}\right|_{s=0} \frac{X^\ell}{\ell!}, \qquad k\in\mathbb{N}, \qquad j\in\mathbb{Z}. \tag{2.1.11}$$

2.2 Near field and far field problems

In Section 2.2.1 we obtain (without giving a rigorous argument) the problems for the near field terms posed in a bounded domain. The compatibility conditions for these problems will permit us to obtain the problem for the far field terms (this is done in Section 2.2.2). In Sections 2.2.4 we come back to the near field problems in the domains $\mathcal{J}_j$ and we show that the existence of solutions of the problems posed in the bounded domain implies the existence of near field terms satisfying the problems (2.1.7) with a specified behaviour in the infinite branches. Finally, in Section 2.2.5 we study the well-posedness of the far field problem.

2.2.1 Formal derivation of the near field problems

2.2.1.1 Near field problems in the infinite junctions

In the spirit of the works [36], [35], let us introduce the following orthonormal basis in $L_2([0,1])$ which consists of eigenfunctions of the transverse Laplacian with Neumann boundary conditions:

$$f_0(t) = 1, \qquad f_p(t) = \sqrt{2}\cos(p\pi t), \qquad p \geqslant 1,$$

Using this basis the functions $U^{(k)}_j$ in the infinite branches B^+_j can be represented as follows:

$$U^{(k)}_j(X,Y) = \sum_{p\in\mathbb{N}} U^{(k)}_{j,p,+}(X) f_p(Y), \qquad (X,Y)\in B^+_j, \quad k\in\mathbb{N}, \qquad j\in\mathbb{Z}. \tag{2.2.1}$$

After injecting (2.2.1) into (2.1.7) we get the following set of ordinary differential equations for the functions $U^{(k)}_{j,p,+}$:

$$\left(U^{(k)}_{j,p,+}\right)''(X) - \pi^2 p^2 U^{(k)}_{j,p,+}(X) = -\sum_{m=0}^{k-2} \lambda^{(m)} U^{(k-2-m)}_{j,p,+}(X), \qquad k,p\in\mathbb{N}, \quad j\in\mathbb{Z}. \tag{2.2.2}$$

Making the change of the unknown function in (2.2.2) of the form

$$U^{(k)}_{j,p,+}(X) = P^{(k)}_{j,p,+}(X) e^{-p\pi X}, \tag{2.2.3}$$

we obtain the following recurrence relation:

$$\left(P_{j,p,+}^{(k)}\right)''(X)-2p\pi\left(P_{j,p,+}^{(k)}\right)'(X)=-\sum_{m=0}^{k-2}\lambda^{(m)}P_{j,p,+}^{(k-2-m)}(X),\qquad k\in\mathbb{N},\quad p\in\mathbb{N}^*,\quad j\in\mathbb{Z}.$$

As usual, when constructing matched asymptotic expansions, for $p\neq 0$ we search solutions of at most polynomial growth, which implies

$$P_{j,p,+}^{(k)}(X)=-\int_X^\infty \widetilde{P}_{j,p,+}^{(k-2)}(X')dX'+c_{j,p,+,0}^{(k)},\qquad k\in\mathbb{N},\quad p\in\mathbb{N}^*,\quad j\in\mathbb{Z},\tag{2.2.4}$$

$$\widetilde{P}_{j,p,+}^{(k-2)}(X)=\sum_{m=0}^{k-2}\lambda^{(m)}\int_X^\infty P_{j,p,+}^{(k-2-m)}(X')e^{2p\pi(X-X')}dX'.\tag{2.2.5}$$

Hence, by induction in k, the functions $P_{j,p,+}^{(k)}$ are polynomials of degree $\left[\frac{k}{2}\right]$ for $k\in\mathbb{N}$, $p\in\mathbb{N}^*$, $j\in\mathbb{Z}$:

$$P_{j,p,+}^{(k)}(X)=\sum_{\ell=0}^{\left[\frac{k}{2}\right]}c_{j,p,+,\ell}^{(k)}X^\ell,\qquad k\in\mathbb{N},\quad p\in\mathbb{N}^*,\quad j\in\mathbb{Z}.\tag{2.2.6}$$

It follows from (2.2.4), (2.2.5) that for $\ell\neq 0$ the coefficients $c_{j,p,+,\ell}^{(k)}$ are given by the following recurrence relation:

$$c_{j,p,+,\ell}^{(k)}=\sum_{m=0}^{k-2\ell}\sum_{i=\ell-1}^{\left[\frac{k-m}{2}\right]-1}\frac{\lambda^{(m)}i!}{(2p\pi)^{i-\ell+2}\,\ell!}\,c_{j,p,+,i}^{(k-2-m)},$$
$$1\leqslant\ell\leqslant\left[\frac{k}{2}\right],\quad k\in\mathbb{N},\quad p\in\mathbb{N}^*,\quad j\in\mathbb{Z}.\tag{2.2.7}$$

The functions $U_{j,0,+}^{(k)}=P_{j,0,+}^{(k)}$ have been found in (2.1.11). Let us mention that they satisfy the recurrence relation (2.2.2) for $p=0$ due to the fact that the far field terms satisfy the equations (2.1.6). Finally, (2.2.1) can be rewritten as

$$U_j^{(k)}(X,Y)=P_{j,0,+}^{(k)}(X)+\sum_{p\in\mathbb{N}^*}P_{j,p,+}^{(k)}(X)e^{-p\pi X}f_p(Y),\qquad (X,Y)\in B_j^+,$$
$$k\in\mathbb{N},\qquad j\in\mathbb{Z},\tag{2.2.8}$$

with the polynomials $P_{j,p,+}^{(k)}$, $p\in\mathbb{N}^*$, defined in (2.2.6)–(2.2.7).
Obviously, a similar argument applies for the domains B_j^- and B_j^0, and analogues of the relations (2.2.4)–(2.2.8), (2.1.11) can be found:

$$U_j^{(k)}(X,Y)=P_{j,0,\pm}^{(k)}(X)+\sum_{p\in\mathbb{N}^*}P_{j,p,\pm}^{(k)}(\pm X)e^{\mp p\pi X}f_p(Y),\qquad (X,Y)\in B_j^\pm,\tag{2.2.9}$$

$$U_j^{(k)}(X,Y)=P_{j,0,0}^{(k)}(Y)+\frac{1}{\sqrt{w_j^\mu}}\sum_{p\in\mathbb{N}^*}P_{j,p,0}^{(k)}(Y)e^{-p\pi Y/w_j^\mu}f_p\left(\frac{X}{w_j^\mu}-\frac{1}{2}\right),\quad (X,Y)\in B_j^0.\tag{2.2.10}$$

The functions $P^{(k)}_{j,p,\delta}$ for $p \neq 0$ are polynomials of degree $\left[\frac{k}{2}\right]$ given by the relations

$$P^{(k)}_{j,p,\delta}(s) = \sum_{\ell=0}^{\left[\frac{k}{2}\right]} c^{(k)}_{j,p,\delta,\ell} s^{\ell}, \qquad \delta \in \{+,-,0\}, \quad k \in \mathbb{N}, \quad p \in \mathbb{N}^*, \quad j \in \mathbb{Z}, \tag{2.2.11}$$

$$c^{(k)}_{j,p,\pm,\ell} = \sum_{m=0}^{k-2\ell} \sum_{i=\ell-1}^{\left[\frac{k-m}{2}\right]-1} \frac{\lambda^{(m)} i!}{(2p\pi)^{i-\ell+2}\,\ell!} c^{(k-2-m)}_{j,p,\pm,i}, \qquad c^{(k)}_{j,p,0,\ell} = \sum_{m=0}^{k-2\ell} \sum_{i=\ell-1}^{\left[\frac{k-m}{2}\right]-1} \frac{(w^{\mu}_j)^{i-\ell+2} \lambda^{(m)} i!}{(2p\pi)^{i-\ell+2}\,\ell!} c^{(k-2-m)}_{j,p,0,i},$$
$$1 \leqslant \ell \leqslant \left[\tfrac{k}{2}\right], \quad k \in \mathbb{N}, \quad p \in \mathbb{N}^*, \quad j \in \mathbb{Z}. \tag{2.2.12}$$

The functions $P^{(k)}_{j,0,\delta}$, $\delta \in \{+,-,0\}$, are polynomials of degree k for k odd and $k+1$ for k even. As we have seen above (cf. (2.1.11)), they can be expressed in terms of the far field terms:

$$P^{(k)}_{j,0,+}(X) = \sum_{\ell=0}^{k} \left.\frac{d^{\ell} u^{(k-\ell)}_{j+\frac{1}{2}}}{ds^{\ell}}\right|_{s=0} \frac{X^{\ell}}{\ell!}, \qquad P^{(k)}_{j,0,-}(X) = \sum_{\ell=0}^{k} \left.\frac{d^{\ell} u^{(k-\ell)}_{j-\frac{1}{2}}}{ds^{\ell}}\right|_{s=1} \frac{X^{\ell}}{\ell!},$$
$$P^{(k)}_{j,0,0}(Y) = \sum_{\ell=0}^{k} \left.\frac{d^{\ell} u^{(k-\ell)}_{j}}{dy^{\ell}}\right|_{y=-\frac{L}{2}} \frac{Y^{\ell}}{\ell!}, \qquad k \in \mathbb{N}, \qquad j \in \mathbb{Z}. \tag{2.2.13}$$

Thus, we end up with the following problems for the near field terms:

$$\text{Find} \quad U \in H^1_{loc}(\mathcal{J}_j) \quad \text{satisfying} \quad (2.1.7) \quad \text{and} \quad (2.2.9)-(2.2.13). \tag{$\mathcal{P}^{(k)}_j$}$$

Remark 2.2.1. It is proved in [36] that a solution of (2.1.7) of at most polynomial growth admits the modal decomposition (2.2.9)–(2.2.10) with the coefficients of the polynomials $P^{(k)}_{j,p,\delta}$, $p \neq 0$, satisfying the recurrence relations (2.2.12) and the polynomials $P^{(k)}_{j,0,\delta}$ satisfying the recurrence relation (2.2.2) with $p = 0$.

2.2.1.2 Towards a bounded domain

We will now introduce some auxiliary objects that will permit us to reduce the problems $(\mathcal{P}^{(k)}_j)$ to problems posed on the bounded domains K_j. For this, we introduce the operators T (which are in fact DtN operators). Then, we formulate a problem on the domain K_j with boundary conditions containing the operators T. As we will see, the near field problems will be reduced to problems having this form. Finally, we consider two particular cases of the problem in question that we will need in the sequel.

Operators T. Let us define the following linear operators:

$$T\varphi = \sum_{p\geqslant 1} p\pi (\varphi, f_p)_{L_2} f_p, \qquad \forall \varphi \in H^{1/2}([0,1]), \tag{2.2.14}$$

$$(T_j\varphi)(X) = \sum_{p\geqslant 1} \frac{p\pi}{w^{\mu}_j} \left(\varphi, f_p\left(\tfrac{\bullet}{w^{\mu}_j} - \tfrac{1}{2}\right)\right)_{L_2} f_p\left(\tfrac{\bullet}{w^{\mu}_j} - \tfrac{1}{2}\right),$$
$$\forall \varphi \in H^{1/2}\left(\left[-\tfrac{w^{\mu}_j}{2}, \tfrac{w^{\mu}_j}{2}\right]\right), \qquad j \in \mathbb{Z}. \tag{2.2.15}$$

We will use the notation

$$\forall j \in \mathbb{Z}, \qquad T_{j,\delta} = \begin{cases} T, & \delta \in \{+,-\}, \\ T_j, & \delta = 0. \end{cases}$$

It is shown in [63] that

$$\begin{aligned} &T \in \mathcal{L}\left(H^{1/2}([0,1]), H^{-1/2}([0,1])\right), \\ &T_j \in \mathcal{L}\left(H^{1/2}\left(\left[-\tfrac{w_j^\mu}{2}, \tfrac{w_j^\mu}{2}\right]\right), H^{-1/2}\left(\left[-\tfrac{w_j^\mu}{2}, \tfrac{w_j^\mu}{2}\right]\right)\right), \qquad j \in \mathbb{Z}, \end{aligned}$$

and the operators T, T_j are positive and symmetric.

A problem in the domain K_j. Let us consider the following problem for $j \in \mathbb{Z}$:

$$\begin{cases} \Delta U(X,Y) = \Phi(X,Y), \qquad (X,Y) \in K_j, \\ \left.\dfrac{\partial U}{\partial n}\right|_{\Sigma_j^N} = 0, \\ \left.\dfrac{\partial U}{\partial n}\right|_{\Sigma_j^\delta} + T_{j,\delta}\, U|_{\Sigma_j^\delta} = g_\delta, \qquad \delta \in \{+,-,0\}. \end{cases} \tag{2.2.16}$$

The well-posedness result for the problems of this type can be found in [63] (Lemma 2.3.2). Adapted to our geometry, it implies that the following assertion holds.

Lemma 2.2.1. *For any $j \in \mathbb{Z}$, $\Phi \in L_2(K_j)$, $g_\pm \in H^{-1/2}(\Sigma_j^\pm)$, $g_0 \in H^{-1/2}(\Sigma_j^0)$ there exists a unique modulo an additive constant solution $U \in H^1(K_j)$ of the problem* (2.2.22) *if and only if the following compatibility condition is satisfied:*

$$\langle g_+, 1\rangle_{\Sigma_j^+} + \langle g_-, 1\rangle_{\Sigma_j^-} + \langle g_0, 1\rangle_{\Sigma_j^0} = \int_{K_j} \Phi. \tag{2.2.17}$$

Here $\langle g_\delta, f\rangle_{\Sigma_j^\delta}$ denotes the duality brackets for $g_\delta \in H^{-1/2}(\Sigma_j^\delta)$, $f \in H^{1/2}(\Sigma_j^\delta)$, $\delta \in \{+,-,0\}$.

The functions $W_j^\pm$. For any $j \in \mathbb{Z}$, we consider the following two problems:

$$\begin{cases} \Delta W_j^-(X,Y) = 0, \qquad (X,Y) \in K_j, \\ \left.\dfrac{\partial W_j^-}{\partial n}\right|_{\Sigma_j^N} = 0, \\ \left.\dfrac{\partial W_j^-}{\partial n}\right|_{\Sigma_{j,\delta}} + T_{j,\delta}\, W_j^-\big|_{\Sigma_{j,\delta}} = g_{j,\delta}^{(W^-)}, \qquad \delta \in \{+,-,0\}, \end{cases} \tag{2.2.18}$$

with

$$g_{j,+}^{(W^-)} = 0, \qquad g_{j,-}^{(W^-)} = 1, \qquad g_{j,0}^{(W^-)} = -\frac{1}{w_j^\mu}, \qquad j \in \mathbb{Z},$$

and

$$\begin{cases} \Delta W_j^+(X,Y) = 0, & (X,Y) \in K_j, \\ \left.\dfrac{\partial W_j^+}{\partial n}\right|_{\Sigma_j^N} = 0, & \\ \left.\dfrac{\partial W_j^+}{\partial n}\right|_{\Sigma_{j,\delta}} + T_{j,\delta}\, W_j^+\big|_{\Sigma_{j,\delta}} = g_{j,\delta}^{(W^+)}, & \delta \in \{+,-,0\}, \end{cases} \tag{2.2.19}$$

with

$$g_{j,+}^{(W^+)} = -1, \qquad g_{j,-}^{(W^+)} = 0, \qquad g_{j,0}^{(W^+)} = \frac{1}{w_j^\mu}, \qquad j \in \mathbb{Z}.$$

The problems (2.2.18), (2.2.19) are of type (2.2.16) and one easily verifies that the compatibility condition (2.2.17) is satisfied for these problems. We denote by W_j^- (resp. W_j^+) the unique (modulo an additive constant) solution in $H^1(K_j)$ of the problem (2.2.18) (resp. (2.2.19)). Note that

$$W_j^+(X,Y) = -W_j^-(-X,Y), \qquad j \in \mathbb{Z}. \tag{2.2.20}$$

Remark 2.2.2. Obviously, there are only two different functions W^-: W_0^- and W_j^-, $j \neq 0$. Similarly, there are only two different functions W^+: W_0^+ and W_j^+, $j \neq 0$. In other words,

$$W_i^\pm = W_j^\pm, \qquad \forall i,j \in \mathbb{Z}^*. \tag{2.2.21}$$

2.2.1.3 Near field problems in a bounded domain

We can now reduce the near field problems to problems set in the domains K_j. Indeed, if $U_j^{(k)}$ is a solution of (2.1.7) satisfying (2.2.8) then

$$\left.\frac{\partial U_j^{(k)}}{\partial X}\right|_{\Sigma_j^+} + T\, U_j^{(k)}\Big|_{\Sigma_j^+} = \left(P_{j,0,+}^{(k)}\right)'\left(\tfrac{w_j^\mu}{2}\right) + \sum_{p\in\mathbb{N}^*} \left(P_{j,p,+}^{(k)}\right)'\left(\tfrac{w_j^\mu}{2}\right) e^{-\frac{p\pi w_j^\mu}{2}} f_p.$$

Analogous relations can be found on the boundaries Σ_j^- and Σ_j. This yields a problem of type (2.2.16) for $U_j^{(k)}$, $j \in \mathbb{Z}$, $k \in \mathbb{N}$:

$$\begin{cases} \Delta U_j^{(k)}(X,Y) = \Phi_j^{(k-2)}(X,Y), & (X,Y) \in K_j, \\ \left.\dfrac{\partial U_j^{(k)}}{\partial n}\right|_{\Sigma_j^N} = 0, & \\ \left.\dfrac{\partial U_j^{(k)}}{\partial n}\right|_{\Sigma_j^\delta} + T_{j,\delta}\, U_j^{(k)}\Big|_{\Sigma_j^\delta} = g_{j,\delta}^{(k-1)}, & \delta \in \{+,-,0\}, \end{cases} \tag{2.2.22}$$

where

$$\Phi_j^{(k-2)} = -\sum_{m=0}^{k-2} \lambda^{(k-m-2)} U_j^{(m)}, \qquad k \geqslant 2, \qquad \Phi_j^{(-2)} = \Phi_j^{(-1)} = 0, \tag{2.2.23}$$

$$g_{j,+}^{(k-1)} = \sum_{l=0}^{k-1} \frac{\left(w_j^\mu\right)^l}{2^l l!} \frac{d^{l+1}}{ds^{l+1}} u_{j+\frac{1}{2}}^{(k-l-1)}\Big|_{s=0} + \sum_{p\in\mathbb{N}^*} P_{j,p,+}^{(k)}{}' \left(\tfrac{w_j^\mu}{2}\right) e^{-\frac{p\pi w_j^\mu}{2}} f_p, \qquad k \geqslant 1, \tag{2.2.24}$$

$$g_{j,-}^{(k-1)} = -\sum_{l=0}^{k-1} \frac{\left(-w_j^\mu\right)^l}{2^l l!} \frac{d^{l+1}}{ds^{l+1}} u_{j-\frac{1}{2}}^{(k-l-1)}\Big|_{s=1} + \sum_{p\in\mathbb{N}^*} P_{j,p,-}^{(k)}{}' \left(\tfrac{w_j^\mu}{2}\right) e^{-\frac{p\pi w_j^\mu}{2}} f_p, \quad k \geqslant 1, \tag{2.2.25}$$

$$g_{j,0}^{(k-1)} = \sum_{l=0}^{k-1} \frac{1}{l!} \frac{d^{l+1}}{dy^{l+1}} u_j^{(k-l-1)}\Big|_{y=-\frac{L}{2}} + \sum_{p\in\mathbb{N}^*} P_{j,p,0}^{(k)}{}'(1) e^{-\frac{p\pi}{w_j^\mu}} f_p \left(\tfrac{\bullet}{w_j^\mu} - \tfrac{1}{2}\right) \quad k \geqslant 1, \tag{2.2.26}$$

$$g_{j,+}^{(-1)} = g_{j,-}^{(-1)} = g_{j,0}^{(-1)} = 0. \tag{2.2.27}$$

As follows from Lemma 2.2.1, the problems (2.2.22) define the near field terms in the domains K_j modulo additive constants. In order to fix these constants, we impose the following conditions:

$$\int_{\Sigma_j^+} U_j^{(k)} = P_{j,0,+}^{(k)} \left(\tfrac{w_j^\mu}{2}\right), \qquad k \in \mathbb{N}, \qquad j \in \mathbb{Z}. \tag{2.2.28}$$

These relations are the average trace continuity conditions for the functions $U_j^{(k)}$ (cf. (2.2.8)). Taking into account the definition (2.1.11) of the polynomials $P_{j,0,+}^{(k)}$, the relations (2.2.28) can be rewritten as

$$\int_{\Sigma_j^+} U_j^{(k)} = \sum_{\ell=0}^{k} \frac{\left(w_j^\mu\right)^\ell}{2^\ell \ell!} \frac{d^\ell u_{j+\frac{1}{2}}^{(k-\ell)}}{ds^\ell}\Bigg|_{s=0}, \qquad k \in \mathbb{N}, \qquad j \in \mathbb{Z}. \tag{2.2.29}$$

Remark 2.2.3. We choose to impose the average trace continuity condition on the boundary Σ_j^+. Obviously, we could have imposed analogous conditions on the boundaries Σ_j^-, Σ_j^0 instead. However, it turns out that all these conditions are equivalent as soon as the far field terms satisfy the jump conditions that will be obtained in the next section (cf. Lemma (2.2.3)).

Finally, we introduce the following problem for the near field terms K_j:

$$\text{Find} \quad V \in H^1(K_j) \quad \text{satisfying} \quad (2.2.22) \text{ and } (2.2.29). \tag{$\widetilde{\mathcal{P}}_j^{(k)}$}$$

2.2.2 Formal derivation of the far field problem

We will now derive the problems for the far field terms. The following assertion is a direct consequence of Lemma 2.2.1.

Lemma 2.2.2. *Let $j \in \mathbb{Z}$, $k \in \mathbb{N}$. Suppose that $\Phi^{(k-1)} \in L_2(K_j)$, $g_\pm^{(k)} \in H^{-1/2}(\Sigma_j^\pm)$, $g_0^{(k)} \in H^{-1/2}(\Sigma_j^0)$. Suppose also that the following relation is satisfied:*

$$\left(u_{j+\frac{1}{2}}^{(k)}\right)'(0) - \left(u_{j-\frac{1}{2}}^{(k)}\right)'(1) + w_j^\mu \left(u_j^{(k)}\right)'\left(-\tfrac{L}{2}\right) = \Xi_j^{(k-1)}, \tag{2.2.30}$$

where

$$\Xi_j^{(k-1)} = -\sum_{\ell=1}^{k}\left(\frac{\left(w_j^\mu\right)^\ell}{2^\ell \ell!}\frac{d^{\ell+1}}{ds^{\ell+1}}u_{j+\frac12}^{(k-\ell)}\bigg|_{s=0} - \frac{\left(-w_j^\mu\right)^\ell}{2^\ell \ell!}\frac{d^{\ell+1}}{ds^{\ell+1}}u_{j-\frac12}^{(k-\ell)}\bigg|_{s=1} + w_j^\mu\frac{1}{\ell!}\frac{d^{\ell+1}}{dy^{\ell+1}}u_j^{(k-\ell)}\bigg|_{y=-\frac{L}{2}}\right) + \int_{K_j}\Phi_j^{(k-1)}, \qquad k \geqslant 1, \tag{2.2.31}$$

$\Xi_j^{(-1)} = 0.$

Then, problem $\widetilde{\mathcal{P}}_j^{(k+1)}$ *has a unique solution.*

The relation (2.2.30) gives ***non-homogeneous Kirchhoff's conditions*** for the function $u^{(k)}$ provided the functions $\left\{u^{(m)}, U^{(m)}\right\}_{m=0}^{k-1}$ are known. We need to complete the problem for $u^{(k)}$ by ***jump conditions*** at the vertices. The trick consists in performing the following integration by parts:

$$\begin{aligned}\int\limits_{K_j}\left(U_j^{(k)}\Delta W_j^- - W_j^-\Delta U_j^{(k)}\right) &= \int\limits_{\partial K_j}\left(\frac{\partial W^-{}_j}{\partial n}U_j^{(k)} - W^-{}_j\frac{\partial U_j^{(k)}}{\partial n}\right)\\ &= \int\limits_{\Sigma_j^-}U_j^{(k)} - \int\limits_{\Sigma_j^0}\frac{U_j^{(k)}}{w_j^\mu} - \sum_{\delta\in\{+,-,0\}}\left\langle g_{j,\delta}^{(k-1)}, W_j^-|_{\Sigma_j^\delta}\right\rangle_{\Sigma_j^\delta}\\ &= \sum_{\ell=0}^{k}\frac{\left(-w_j^\mu\right)^\ell}{2^\ell \ell!}\frac{d^\ell u_{j-\frac12}^{(k-\ell)}}{ds^\ell}\bigg|_{s=1} - \sum_{\ell=0}^{k}\frac{1}{\ell!}\frac{d^\ell u_j^{(k-\ell)}}{dy^\ell}\bigg|_{y=-\frac{L}{2}} - \sum_{\delta\in\{+,-,0\}}\left\langle g_{j,\delta}^{(k-1)}, W_j^-|_{\Sigma_j^\delta}\right\rangle_{\Sigma_j^\delta}.\end{aligned} \tag{2.2.32}$$

In the last equality we used the assumption that the near field terms satisfy the average trace continuity conditions of type (2.2.29) on the boundaries $\Sigma_{j,+}$, $\Sigma_{j,0}$. The relation (2.2.32) yields the following conditions on the jumps of the function $u^{(k)}$:

$$u_{j-\frac12}^{(k)}(1) - u_j^{(k)}\left(-\tfrac{L}{2}\right) = \Delta_{j,-}^{(k-1)}, \qquad j\in\mathbb{Z}, \tag{2.2.33}$$

where

$$\Delta_{j,-}^{(k-1)} = \sum_{\ell=1}^{k}\left(\frac{1}{\ell!}\frac{d^\ell u_j^{(k-\ell)}}{dy^\ell}\bigg|_{y=-\frac{L}{2}} - \frac{\left(-w_j^\mu\right)^\ell}{2^\ell \ell!}\frac{d^\ell u_{j-\frac12}^{(k-\ell)}}{ds^\ell}\bigg|_{s=1}\right) + \sum_{\delta\in\{+,-,0\}}\left\langle g_{j,\delta}^{(k-1)}, W_j^-|_{\Sigma_j^\delta}\right\rangle_{\Sigma_j^\delta} - \int_{K_j}\Phi_j^{(k-2)}W_j^-, \qquad k\geqslant 1, \tag{2.2.34}$$

$\Delta_{j,-}^{(-1)} = 0.$

In a similar way, computing $\int\limits_{K_j}\left(U_j^{(k)}\Delta W_j^+ - W_j^+\Delta U_j^{(k)}\right)$ and assuming that the average trace continuity for the near field terms on the boundaries $\Sigma_{j,0}$, $\Sigma_{j,+}$ are satisfied, we get another set of jump conditions for function $u^{(k)}$:

$$u_j^{(k)}\left(-\tfrac{L}{2}\right) - u_{j+\frac12}^{(k)}(0) = \Delta_{j,+}^{(k-1)}, \qquad j\in\mathbb{Z}, \tag{2.2.35}$$

where

$$\Delta_{j,+}^{(k-1)} = \sum_{l=1}^{k} \left(\frac{\left(w_j^{\mu}\right)^l}{2^l l!} \left. \frac{d^l u_{j+\frac{1}{2}}^{(k-l)}}{ds^l} \right|_{s=0} - \frac{1}{l!} \left. \frac{d^l u_j^{(k-l)}}{dy^l} \right|_{y=-\frac{L}{2}} \right) + \sum_{\delta\in\{+,-,0\}} \left\langle g_{j,\delta}^{(k-1)}, W_j^+|_{\Sigma_j^\delta} \right\rangle_{\Sigma_j^\delta} - \int_{K_j} \Phi_j^{(k-2)} W_j^+, \qquad k \geqslant 1, \tag{2.2.36}$$

$$\Delta_{j,+}^{(-1)} = 0.$$

Finally, combining (2.1.6), (2.2.30), (2.2.33), (2.2.35) we get up with the following set of problems for the far field terms $u^{(k)}$, $k \in \mathbb{N}$:

$$\begin{cases} \left(u_{j+\frac{1}{2}}^{(k)}\right)''(s) + \lambda^{(0)} u_{j+\frac{1}{2}}^{(k)}(s) = -\lambda^{(k)} u_{j+\frac{1}{2}}^{(0)}(s) - f_{j+\frac{1}{2}}^{(k-1)}(s), \quad s \in [0,1], & j \in \mathbb{Z}, \\ \left(u_j^{(k)}\right)''(y) + \lambda^{(0)} u_j^{(k)}(y) = -\lambda^{(k)} u_j^{(0)}(y) - f_j^{(k-1)}(y), \quad y \in \left[-\frac{L}{2}, 0\right], & j \in \mathbb{Z}, \\ \left(u_j^{(k)}\right)'(0) = 0, & j \in \mathbb{Z}, \\ u_{j-\frac{1}{2}}^{(k)}(1) - u_j^{(k)}\left(-\frac{L}{2}\right) = \Delta_{j,-}^{(k)}, \quad u_j^{(1)}\left(-\frac{L}{2}\right) - u_{j+\frac{1}{2}}^{(k)}(0) = \Delta_{j,+}^{(k)}, & j \in \mathbb{Z}, \\ \left(u_{j+\frac{1}{2}}^{(k)}\right)'(0) - \left(u_{j-\frac{1}{2}}^{(k)}\right)'(1) + w_j^{\mu}\left(u_j^{(k)}\right)'\left(-\frac{L}{2}\right) = \Xi_j^{(k-1)}, & j \in \mathbb{Z}. \end{cases} \tag{$\mathcal{P}_u^{(k)}$}$$

where

$$f_{j+\frac{1}{2}}^{(-1)} = -\lambda^{(0)} u_{j+\frac{1}{2}}^{(0)}, \qquad f_j^{(-1)} = -\lambda^{(0)} u_j^{(0)}, \qquad j \in \mathbb{Z},$$

$$f_{j+\frac{1}{2}}^{(0)} = 0, \qquad f_j^{(0)} = 0, \qquad j \in \mathbb{Z}, \tag{2.2.37}$$

$$f_{j+\frac{1}{2}}^{(k-1)} = \sum_{m=1}^{k-1} \lambda^{(k-m)} u_{j+\frac{1}{2}}^{(m)}, \qquad f_j^{(k-1)} = \sum_{m=1}^{k-1} \lambda^{(k-m)} u_j^{(m)}, \qquad j \in \mathbb{Z}, \quad k \geqslant 2. \tag{2.2.38}$$

2.2.3 Order 0

Before studying the well-posedness of the near-field and the far field problems at any order formulated in the previous sections, let us look at the order 0.

Far field problem. Putting $k = 0$ in ($\mathcal{P}_u^{(k)}$), we get

$$\begin{cases} \left(u_{j+\frac{1}{2}}^{(0)}\right)''(s) + \lambda^{(0)} u_{j+\frac{1}{2}}^{(0)}(s) = 0, \quad s \in [0,1], & j \in \mathbb{Z}, \\ \left(u_j^{(0)}\right)''(y) + \lambda^{(0)} u_j^{(0)}(y) = 0, \quad y \in \left[-\frac{L}{2}, 0\right], & j \in \mathbb{Z}, \\ \left(u_j^{(0)}\right)'(0) = 0, & j \in \mathbb{Z}, \\ u_{j-\frac{1}{2}}^{(0)}(1) = u_j^{(0)}\left(-\frac{L}{2}\right) = u_{j+\frac{1}{2}}^{(0)}(0), & j \in \mathbb{Z}, \\ \left(u_{j+\frac{1}{2}}^{(0)}\right)'(0) - \left(u_{j-\frac{1}{2}}^{(0)}\right)'(1) + w_j^{\mu}\left(u_j^{(0)}\right)'\left(-\frac{L}{2}\right) = 0, & j \in \mathbb{Z}. \end{cases} \tag{2.2.39}$$

This is exactly the eigenvalue problem for the operator A_s^{μ} (cf. (1.3.6)) which has been studied in Section 1.3.2. From now on $\lambda^{(0)}$ will stand for an eigenvalue of the operator A_s^{μ} and $u^{(0)}$ for the corresponding eigenfunction (all the eigenvalues of the operator A_s^{μ} are simple). To fix the eigenfunction in a unique way, we impose the condition $\mathbf{u}_0 = 1$.

The expressions for the eigenfunction $u^{(0)}$ have been found in (1.3.41), (1.3.42), (1.3.53), (1.3.48):

$$u^{(0)}_{j+\frac{1}{2}}(s) = \mathbf{u}_j \frac{\sin\left(\sqrt{\lambda^{(0)}}(1-s)\right)}{\sin\sqrt{\lambda^{(0)}}} + \mathbf{u}_{j+1}\frac{\sin\left(\sqrt{\lambda^{(0)}}s\right)}{\sin\sqrt{\lambda^{(0)}}}, \quad s\in[0,1], \quad \forall j\in\mathbb{Z}, \tag{2.2.40}$$

$$u^{(0)}_{j}(y) = \mathbf{u}_j \frac{\cos\left(\sqrt{\lambda^{(0)}}y\right)}{\cos\left(\sqrt{\lambda^{(0)}}L/2\right)}, \qquad y\in\left[-\tfrac{L}{2},\tfrac{L}{2}\right], \quad \forall j\in\mathbb{Z}, \tag{2.2.41}$$

where

$$\mathbf{u}_j = r^{|j|}, \qquad j\in\mathbb{Z}, \qquad r = r(\sqrt{\lambda^{(0)}}). \tag{2.2.42}$$

We remind that in this chapter the spectral parameter is denoted by λ, whereas in Chapter 1 it was denoted by λ^2. This is the reason of $\sqrt{\lambda^{(0)}}$ appearing in the expressions for the eigenfunction.

Near field problems. Consider now the near field problems $(\mathcal{P}^{(k)}_j)$ for $k=0$. The relations (2.2.11), (2.2.13) take the form

$$P^{(0)}_{j,p,\delta}(s) = c^{(0)}_{j,p,\delta,0}, \qquad \delta\in\{+,-,0\}, \quad p\in\mathbb{N}^*, \quad j\in\mathbb{Z},$$

$$P^{(0)}_{j,0,+}(X) = u^{(0)}_{j+\frac{1}{2}}(0), \qquad P^{(0)}_{j,0,-}(X) = u^{(0)}_{j-\frac{1}{2}}(1), \qquad P^{(0)}_{j,0,0}(Y) = u^{(0)}_{j}\left(-\tfrac{L}{2}\right), \qquad j\in\mathbb{Z}.$$

Taking into account that $u^{(0)}_{j+\frac{1}{2}}(0) = u^{(0)}_{j-\frac{1}{2}}(1) = u^{(0)}_{j}\left(-\frac{L}{2}\right) = \mathbf{u}_j$, we can rewrite the problems $(\mathcal{P}^{(k)}_j)$ for $k=0$ as

$$\begin{cases} \Delta U^{(0)}_j(X,Y) = 0, \qquad (X,Y)\in\mathcal{J}_j, \\ \left.\dfrac{\partial U^{(0)}_j}{\partial n}\right|_{\partial\mathcal{J}_j} = 0, \\ U^{(0)}_j(X,Y) = \mathbf{u}^{(0)}_j + \sum\limits_{p\in\mathbb{N}^*} c^{(0)}_{j,p,\pm,0} e^{\mp p\pi X} f_p(Y), \qquad (X,Y)\in B^{\pm}_j, \\ U^{(0)}_j(X,Y) = \mathbf{u}^{(0)}_j + \frac{1}{\sqrt{w^\mu_j}} \sum\limits_{p\in\mathbb{N}^*} c^{(0)}_{j,p,0,0} e^{-p\pi Y/w^\mu_j} f_p\left(\frac{X}{w^\mu_j}-\frac{1}{2}\right), \quad (X,Y)\in B^0_j. \end{cases} \tag{2.2.43}$$

Consequently, the constants

$$U^{(0)}_j(X,Y) = \mathbf{u}^{(0)}_j, \qquad (X,Y)\in\mathcal{J}_j, \qquad \forall j\in\mathbb{Z}, \tag{2.2.44}$$

are obviously solutions of (2.2.43) with

$$c^{(0)}_{j,p,\delta,0} = 0, \qquad \delta\in\{+,-,0\}, \qquad p\in\mathbb{N}^*, \qquad j\in\mathbb{Z}. \tag{2.2.45}$$

2.2.4 Well-posedness of the near field problems

In this section we suppose that the far fields up to the order k are constructed, i.e., there exist functions $\{u^{(m)},\ 0\leqslant m\leqslant k\}$ satisfying the problems $\{(\mathcal{P}^{(m)}),\ 0\leqslant m\leqslant k\}$. The near field terms are supposed constructed up to the order $k-1$, i.e., there exist functions $\left\{U^{(m)}_j,\ j\in\mathbb{Z},\ 0\leqslant m\leqslant k-1\right\}$ that solve the problems $\left\{(\mathcal{P}^{(m)}_j),\ j\in\mathbb{Z},\ 0\leqslant m\leqslant k-1\right\}$.

We also suppose that at order k the near field terms are constructed on a bounded domain, i.e., there exist functions $\left\{V_j^{(k)}\right\}_{j\in\mathbb{Z}}$ solving the problems $\left\{(\widetilde{\mathcal{P}}_j^{(k)})\right\}_{j\in\mathbb{Z}}$. We will show that under these assumptions the near filed terms at order k can be continued to the whole infinite junctions, i.e., there exist functions $\left\{U_j^{(k)}\right\}_{j\in\mathbb{Z}}$ solving the problems $\left\{(\mathcal{P}_j^{(k)})\right\}_{j\in\mathbb{Z}}$.

We start by some auxiliary assertions. The following lemma that establishes the equivalence between the average trace continuity conditions imposed on different boundaries of the domain K_j mentioned in Remark 2.2.3.

Lemma 2.2.3. *Suppose that there exist functions $\left\{u^{(m)},\ 0\leqslant m\leqslant k\right\}$ satisfying the problems $\left\{(\mathcal{P}^{(m)}),\ 0\leqslant m\leqslant k\right\}$ as well as functions $\left\{U_j^{(m)},\ j\in\mathbb{Z},\ 0\leqslant m\leqslant k-1\right\}$ satisfying the problems $\left\{(\mathcal{P}_j^{(m)}),\ j\in\mathbb{Z},\ 0\leqslant m\leqslant k-1\right\}$ for some $k\in\mathbb{N}$. Suppose also that the function V satisfies the problem (2.2.22) for some $j\in\mathbb{Z}$. Then, the following three relations are equivalent:*

$$(i)\ \int\limits_{\Sigma_j^+} V = P_{j,0,+}^{(k)}\left(\tfrac{w_j^\mu}{2}\right);\qquad (ii)\ \int\limits_{\Sigma_j^-} V = P_{j,0,-}^{(k)}\left(-\tfrac{w_j^\mu}{2}\right);\qquad (iii)\ \int\limits_{\Sigma_j^0}\frac{V}{w_j^\mu} = P_{j,0,0}^{(k)}(1)\,.$$

Proof. Let us show that the relations (ii) implies (iii). All the other implications can be shown in a similar way. If W_j^- is the function introduced in Section 2.2.1.2, then

$$\int\limits_{K_j}\left(V\Delta W_j^- - W_j^-\Delta V\right) = \int\limits_{\partial K_j}\left(\frac{\partial W^-{}_j}{\partial n}V - W^-{}_j\frac{\partial V}{\partial n}\right) = \int\limits_{\Sigma_j^-} V - \int\limits_{\Sigma_j^0}\frac{V}{w_j^\mu} - \sum_{\delta\in\{+,-,0\}}\left\langle g_{j,\delta}^{(k-1)}, W_j^-|_{\Sigma_j^\delta}\right\rangle_{\Sigma_j^\delta},$$

which implies

$$\begin{aligned}\int\limits_{\Sigma_j^0}\frac{V}{w_j^\mu} &= \int\limits_{\Sigma_j^-} V - \sum_{\delta\in\{+,-,0\}}\left\langle g_{j,\delta}^{(k-1)}, W_j^-|_{\Sigma_j^\delta}\right\rangle_{\Sigma_j^\delta} + \int_{K_j}\Phi_j^{(k-2)}W_j^- \\ &= \sum_{\ell=0}^{k}\frac{\left(-w_j^\mu\right)^\ell}{2^\ell\ell!}\left.\frac{d^\ell u_{j-\frac12}^{(k-\ell)}}{ds^\ell}\right|_{s=1} - \sum_{\delta\in\{+,-,0\}}\left\langle g_{j,\delta}^{(k-1)}, W_j^-|_{\Sigma_j^\delta}\right\rangle_{\Sigma_j^\delta} + \int_{K_j}\Phi_j^{(k-2)}W_j^-.\end{aligned}$$

In the last equality we used the assumption (ii) and the definition (2.2.13) of the polynomial $P_{j,0,-}^{(k)}$. Taking into account the definition (2.2.34) of $\Delta_{j,-}^{(k-1)}$, we get

$$\int\limits_{\Sigma_j^0}\frac{V}{w_j^\mu} = \sum_{\ell=1}^{k}\frac{1}{\ell!}\left.\frac{d^\ell u_j^{(k-\ell)}}{dy^\ell}\right|_{y=-\frac{L}{2}} - \Delta_{j,-}^{(k-1)} + u_{j-\frac12}^{(k)}(1).$$

Finally, using the fact that $u^{(k)}$ satisfies the jump condition (2.2.33), we obtain

$$\int\limits_{\Sigma_j^0}\frac{V}{w_j^\mu} = \sum_{\ell=0}^{k}\frac{1}{\ell!}\left.\frac{d^\ell u_j^{(k-\ell)}}{dy^\ell}\right|_{y=-\frac{L}{2}} = P_{j,0,0}^{(k)}(1)\,.$$

The same computation can be used to show that (iii) implies (ii). The equivalence between (i) and (iii) can be proved using an analogous argument with W_j^- replaced by W_j^+. ☐

The following assertion concerns the behaviour of the coefficients $c^{(k)}_{j,p\delta,l}$ as $p \to \infty$ assuming the existence of solutions of the near field problems $(\mathcal{P}^{(k)}_j)$.

Lemma 2.2.4. *Suppose that for some $j \in \mathbb{Z}$ there exist functions $U^{(m)}_j \in H^1_{loc}(\mathcal{J}_j)$, $0 \leqslant m \leqslant n$, which are solutions of the problems $(\mathcal{P}^{(m)}_j)$ for $0 \leqslant m \leqslant n$. Then,*

$$\sum_{p\in\mathbb{N}^*} p\left(c^{(m)}_{j,p,\pm,\ell}\right)^2 e^{-p\pi w^\mu_j} < \infty, \qquad \sum_{p\in\mathbb{N}^*} p\left(c^{(m)}_{j,p,0,\ell}\right)^2 e^{-\frac{2p\pi}{w^\mu_j}} < \infty, \quad 0 \leqslant \ell \leqslant \left[\tfrac{m}{2}\right], \quad 0 \leqslant m \leqslant n. \tag{2.2.46}$$

Proof. We will give the proof for $\delta = +$, the proof for $\delta \in \{-, 0\}$ being analogous. The proof is done by induction in m. Indeed, for $m = 0$ the result holds due to (2.2.45). Suppose that $1 \leqslant q \leqslant n$ and the relations (2.2.46) are satisfied for $0 \leqslant m \leqslant q-1$. Let us show that they are satisfied for $m = q$ as well. First, the recurrence relation for the coefficients c (2.2.12) implies that (2.2.46) is satisfied for $m = q,\ 1 \leqslant \ell \leqslant \left[\frac{q}{2}\right]$. Then, due to (2.2.9) for $m = q$ and the fact that $U^{(q)}_j \in H^1_{loc}(\mathcal{J}_j)$ we have

$$\sum_{p\in\mathbb{N}^*} P^{(q)}_{j,p,+}\left(\frac{w^\mu_j}{2}\right) e^{-\frac{p\pi w^\mu_j}{2}} f_p \in H^{1/2}\left(\Sigma^+_j\right).$$

Taking into account (2.2.11) and (2.2.46) for $m = q,\ 1 \leqslant \ell \leqslant \left[\frac{q}{2}\right]$ we conclude that

$$\sum_{p\in\mathbb{N}^*} c^{(q)}_{j,p,+,0} e^{-\frac{p\pi w^\mu_j}{2}} f_p \in H^{1/2}\left(\Sigma^+_j\right),$$

which means that (2.2.46) is also satisfied for $m = q,\ \ell = 0$. □

We will now pass to the construction of a continuation of the near field terms from a bounded domain to the infinite junction.

Proposition 2.2.1. *Suppose that there exist functions $\left\{u^{(m)},\ 0 \leqslant m \leqslant k\right\}$ satisfying the problems $\left\{(\mathcal{P}^{(m)}),\ 0 \leqslant m \leqslant k\right\}$ as well as functions $\left\{U^{(m)}_j,\ j \in \mathbb{Z},\ 0 \leqslant m \leqslant k-1\right\}$ satisfying the problems $\left\{(\mathcal{P}^{(m)}_j),\ j \in \mathbb{Z},\ 0 \leqslant m \leqslant k-1\right\}$ for some $k \in \mathbb{N}^*$. Suppose also that the function $V^{(k)}_j \in H^1(K_j)$ solves the problem $(\widetilde{\mathcal{P}}^{(k)}_j)$ for some $j \in \mathbb{Z}$. Then, there exists a function $U^{(k)}_j \in H^1_{loc}(\mathcal{J}_j)$ solving the problem $(\mathcal{P}^{(k)}_j)$ and such that $U^{(k)}_j\Big|_{K_j} = V^{(k)}_j$.*

Proof. The proof consists of several steps. First, we define functions in the bands B^δ_j that will be shown to be the extensions of V_j. These function are constructed in such a way that the have the form (2.2.8), (2.2.9), (2.2.10). In these relations everything is defined by the previous orders except from the coefficients $c^{(k)}_{j,p,\delta,0}$. These coefficients are found from the traces of V_j on the boundaries Σ^δ_j. Then, we show that the constructed functions in the bands B^δ_j belong to $H^1_{loc}(B^\delta_j)$, that their traces on Σ^δ_j coincide with those of V_j, that they solve the Laplacian equation (first line of (2.1.7)) in B^δ_j and finally that their normal derivatives on Σ^δ_j are the opposite of those of V_j and the homogeneous boundary conditions are satisfied on the other boundaries. This permits to conclude that the constructed extension solves $(\mathcal{P}^{(k)}_j)$.

Construction of an extension: Let us denote by $\varphi_{j,\delta}^{(k)}$ the traces of the function $V_j^{(k)}$ on the boundaries Σ_j^δ:

$$\varphi_{j,\delta}^{(k)} = V_j^{(k)}\Big|_{\Sigma_j^\delta}, \qquad \varphi_{j,\delta}^{(k)} \in H^{1/2}\left(\Sigma_j^\delta\right), \qquad \delta \in \{+,-,0\}. \tag{2.2.47}$$

These traces can be decomposed in Fourier series:

$$\varphi_{j,\pm}^{(k)} = \sum_{p\in\mathbb{N}} \varphi_{j,p,\pm}^{(k)} f_p, \qquad \varphi_{j,0}^{(k)} = \sum_{p\in\mathbb{N}} \varphi_{j,p,0}^{(k)} f_p\left(\frac{\bullet}{w_j^\mu} - \frac{1}{2}\right), \tag{2.2.48}$$

and due to (2.2.47) we have

$$\sum_{p\in\mathbb{N}} p\left(\varphi_{j,p,\delta}^{(k)}\right)^2 < \infty, \qquad \delta \in \{+,-,0\}. \tag{2.2.49}$$

Note that the average trace condition (2.2.29) for $V_j^{(k)}$ together with Lemma 2.2.3 which gives analogous conditions on the other boundaries, imply that

$$\varphi_{j,0,\pm}^{(k)} = P_{j,0,\pm}^{(k)}\left(\pm\frac{w_j^\mu}{2}\right), \qquad \varphi_{j,0,0}^{(k)} = P_{j,0,0}^{(k)}(1) \qquad j\in\mathbb{Z}. \tag{2.2.50}$$

We construct the functions $\widetilde{U}_{j,\delta}^{(k)}$ in the bands B_j^δ for $\delta\in\{+,-,0\}$ as follows:

$$\widetilde{U}_{j,\pm}^{(k)}(X,Y) = P_{j,0,\pm}^{(k)}(X) + \sum_{p\in\mathbb{N}^*} \xi_{j,p,\pm}^{(k)}(X,Y), \qquad (X,Y)\in B_j^\pm, \tag{2.2.51}$$

$$\widetilde{U}_{j,0}^{(k)}(X,Y) = P_{j,0,0}^{(k)}(Y) + \sum_{p\in\mathbb{N}^*} \xi_{j,p,0}^{(k)}(X,Y), \qquad (X,Y)\in B_j^0, \tag{2.2.52}$$

$$\xi_{j,p,\pm}^{(k)}(X,Y) = \left(\varphi_{j,p,\pm}^{(k)} e^{\frac{p\pi w_j^\mu}{2}} + \sum_{\ell=1}^{\left[\frac{k}{2}\right]} c_{j,p,\pm,\ell}^{(k)}\left((\pm X)^\ell - \left(\frac{w_j^\mu}{2}\right)^\ell\right)\right) e^{\mp p\pi X} f_p(Y), \quad p\in\mathbb{N}^*, \tag{2.2.53}$$

$$\xi_{j,p,0}^{(k)}(X,Y) = \left(\varphi_{j,p,0}^{(k)} e^{\frac{p\pi}{w_j^\mu}} + \frac{1}{\sqrt{w_j^\mu}} \sum_{\ell=1}^{\left[\frac{k}{2}\right]} c_{j,p,0,\ell}^{(k)}\left(Y^\ell - 1\right)\right) e^{-\frac{p\pi Y}{w_j^\mu}} f_p\left(\frac{X}{w_j^\mu} - \frac{1}{2}\right), \quad p\in\mathbb{N}^*, \tag{2.2.54}$$

where the polynomials $P_{j,0,\delta}^{(k)}$ are defined in (2.2.13) and the coefficients $c_{j,p,\delta,\ell}^{(k)}$ are defined in (2.2.12). We note that all the objects appearing in the definition of $P_{j,0,\delta}^{(k)}$ and $c_{j,p,\delta,\ell}^{(k)}$, i.e., the far field terms up to the order k and the coefficients c for the previous orders, are already constructed by assumption. We will perform the rest of the analysis for the function $\widetilde{U}_{j,+}^{(k)}$, the analysis for the functions $\widetilde{U}_{j,0}^{(k)}$, $\widetilde{U}_{j,0}^{(k)}$ being analogous.

The function $\widetilde{U}_{j,+}^{(k)}$ belongs to $H^1_{loc}\left(B_j^+\right)$: We have:

$$\left\|\xi_{j,p,+}^{(k)}\right\|^2_{H^1\left(B_j^+\right)} \leqslant C(k)\left(p^2\left(\varphi_{j,p,+}^{(k)}\right)^2 e^{p\pi w_j^\mu}\int\limits_{\frac{w_j^\mu}{2}}^{\infty} e^{-2p\pi X}dX + \sum_{\ell=1}^{\left[\frac{k}{2}\right]} p^2\left(c_{j,p,+,\ell}^{(k)}\right)^2 \int\limits_{\frac{w_j^\mu}{2}}^{\infty} X^{2\ell}e^{-2p\pi X}dX\right).$$

Using the fact that

$$\int\limits_{\frac{w_j^\mu}{2}}^{\infty} X^{2\ell}e^{-2p\pi X}dX \leqslant C(\ell)p^{-1}e^{-p\pi w_j^\mu},$$

we get

$$\left\|\xi_{j,p,+}^{(k)}\right\|^2_{H^1\left(B_j^+\right)} \leqslant C(k)\left(p\left(\varphi_{j,p,+}^{(k)}\right)^2 + \sum_{\ell=1}^{\left[\frac{k}{2}\right]} p\left(c_{j,p,+,\ell}^{(k)}\right)^2 e^{-p\pi w_j^\mu}\right).$$

As follows from Lemma 2.2.4, the inequalities (2.2.46) hold for $0 \leqslant m \leqslant k-1$. Taking into account the definition (2.2.12) of the coefficients $c_{j,p,+,\ell}^{(k)}$ as well as the relation (2.2.49), we conclude that

$$\sum_{p\in\mathbb{N}^*}\left\|\xi_{j,p}^{(k)}\right\|^2_{H^1\left(B_j^+\right)} < \infty, \tag{2.2.55}$$

and consequently, $\widetilde{U}_{j,+}^{(k)} \in H^1_{loc}\left(B_j^+\right),\ j\in\mathbb{Z}$.

Trace continuity: We can now find the trace of the function $\widetilde{U}_{j,+}^{(k)}$ on the surface Σ_j^+. It follows from the definition of the function $\widetilde{U}_{j,+}^{(k)}$ ((2.2.51)–(2.2.53)) that

$$\left.\widetilde{U}_{j,+}^{(k)}\right|_{\Sigma_j^+} = P_{j,0,+}^{(k)}\left(\frac{w_j^\mu}{2}\right) + \sum_{p\in\mathbb{N}^*}\varphi_{j,p,+}^{(k)} f_p.$$

Comparing it with the trace of the function $V_j^{(k)}$ ((2.2.47), (2.2.48), (2.2.50)), we see that

$$\left.\widetilde{U}_{j,+}^{(k)}\right|_{\Sigma_j^+} = \left.V_j^{(k)}\right|_{\Sigma_j^+}, \qquad j\in\mathbb{Z}. \tag{2.2.56}$$

The Laplacian equation for $\widetilde{U}_{j,+}^{(k)}$: Using the relation (2.2.12) between the coefficients c of the polynomials P, we find from (2.2.53)

$$\Delta\xi_{j,p,+}^{(k)} = -\sum_{m=0}^{k-2}\lambda^{(m)}P_{j,p,+}^{(k-2-m)}(X)e^{-p\pi X}f_p(Y), \qquad (X,Y)\in B_j^+, \qquad p\in\mathbb{N}^+. \tag{2.2.57}$$

Since

$$\sum_{p=1}^{N}\xi_{j,p}^{(k)} \xrightarrow[L_2\left(B_j^+\right)]{} \widetilde{U}_{j,+}^{(k)} - P_{j,0,+}^{(k)}, \qquad N\to\infty,$$

(as follows from (2.2.51), (2.2.55)), and

$$\sum_{p=1}^{N} \Delta\xi_{j,p,+}^{(k)} \xrightarrow[L_2\left(B_j^+\right)]{} -\sum_{m=0}^{k-2} \lambda^{(m)} \left(U_j^{(k-2-m)} - P_{j,0,+}^{(k-2-m)}\right), \qquad N \to \infty,$$

(as follows from (2.2.57) and the relations (2.2.9) satisfied by the functions $U_j^{(k-2-m)}$, $0 \leqslant m \leqslant k-2$), we can conclude that $\Delta\widetilde{U}_{j,+}^{(k)} \in L_2\left(B_j^+\right)$ and

$$\Delta\widetilde{U}_{j,+}^{(k)} = \left(P_{j,0,+}^{(k)}\right)'' - \sum_{m=0}^{k-2} \lambda^{(m)} \left(U_j^{(k-2-m)} - P_{j,0,+}^{(k-2-m)}\right) = -\sum_{m=0}^{k-2} \lambda^{(m)} U_j^{(k-2-m)}.$$

The last equality follows from the definition (2.2.13) of the polynomials $P_{j,0,+}^{(k)}$ as well as the fact that the far field terms verify the equations (2.1.6) (as was mentioned above, this implies that the polynomials $P_{j,0,+}^{(k)}$ satisfy the recurrence relation (2.2.2) for $p = 0$).

Continuity of the normal derivative: We have $\partial_n \widetilde{U}_{j,+}^{(k)} \in H^{-1/2}\left(\partial B_j^+\right)$ and

$$\partial_n \widetilde{U}_{j,+}^{(k)} = 0 \quad \text{on} \quad \partial B_j^+ \setminus \Sigma_j^+,$$

$$\begin{aligned} \partial_n \widetilde{U}_{j,+}^{(k)}\Big|_{\Sigma_j^+} &= -\left(P_{j,0,+}^{(k)}\right)' + \sum_{p\in\mathbb{N}^*} p\pi \varphi_{j,p,+}^{(k)} f_p - \sum_{p\in\mathbb{N}^*} \sum_{\ell=1}^{\left[\frac{k}{2}\right]} c_{j,p,+,\ell}^{(k)} \ell \left(\frac{w_j^\mu}{2}\right)^{\ell-1} e^{-\frac{p\pi w_j^\mu}{2}} f_p \\ &= T\varphi_{j,+}^{(k)} - g_{j,+}^{(k-1)}. \end{aligned}$$

The last equality follows from the definitions of the function $g_{j,+}^{(k-1)}$ and the operator T (cf. (2.2.24), (2.2.14)) as well as the definition of the polynomials P (cf. (2.2.13), (2.2.11)). Finally, taking into account the fact that $V_j^{(k)}$ satisfies the problem (2.2.22), we get

$$\partial_n \widetilde{U}_{j,+}^{(k)}\Big|_{\Sigma_j^+} = -\,\partial_n V_j^{(k)}\Big|_{\Sigma_j^+}.$$

Conclusion: The function $U_j^{(k)}$ constructed as

$$U_j^{(k)}(X,Y) = \begin{cases} V_j^{(k)}, & (X,Y) \in K_j, \\ \widetilde{U}_{j,\delta}^{(k)}, & (X,Y) \in B_j^\delta, \quad \delta \in \{+,-,0\}, \end{cases} \tag{2.2.58}$$

satisfies the problem $(\mathcal{P}_j^{(k)})$. □

Remark 2.2.4. As follows from (2.2.51)–(2.2.54), the coefficients $c_{p,j,\delta,0}^{(k)}$ are given by the following relation:

$$c_{j,p,\pm,0}^{(k)} = \varphi_{j,p,\pm}^{(k)} e^{\frac{p\pi w_j^\mu}{2}} - \sum_{\ell=1}^{\left[\frac{k}{2}\right]} c_{j,p,\pm,\ell}^{(k)} \left(\frac{w_j^\mu}{2}\right)^{\ell}, \qquad c_{j,p,0,0}^{(k)} = \varphi_{j,p,0}^{(k)} e^{\frac{p\pi}{w_j^\mu}} - \sum_{\ell=1}^{\left[\frac{k}{2}\right]} c_{j,p,0,\ell}^{(k)},$$

$$j \in \mathbb{Z}, \qquad p \in \mathbb{N}^*. \tag{2.2.59}$$

To find them, we need to know the coefficients c at the orders up to $k-1$ and the function $V_j^{(k)}$ (cf. (2.2.47)–(2.2.48)).

Remark 2.2.5. In view of Remark 2.2.1, the uniqueness of the solution of the near field problem $(\widetilde{\mathcal{P}}_j^{(k)})$ in K_j implies the uniqueness of the solution of the problem (2.2.2) of at most polynomial growth satisfying the matching conditions (2.2.13).

2.2.5 Well-posedness of the far field problem

In this section we prove the following well-posedness result for the problem $(\mathcal{P}_u^{(k)})$ with $k \geqslant 1$:

$$\begin{cases} \big(u_{j+\frac{1}{2}}^{(k)}\big)''(s) + \lambda^{(0)} u_{j+\frac{1}{2}}^{(k)}(s) = -\lambda^{(k)} u_{j+\frac{1}{2}}^{(0)}(s) - f_{j+\frac{1}{2}}^{(k-1)}(s), \quad s \in [0,1], & j \in \mathbb{Z}, \\ \big(u_j^{(k)}\big)''(y) + \lambda^{(0)} u_j^{(k)}(y) = -\lambda^{(k)} u_j^{(0)}(y) - f_j^{(k-1)}(y), \quad y \in \left[-\frac{L}{2}, 0\right], & j \in \mathbb{Z}, \\ \big(u_j^{(k)}\big)'(0) = 0, & j \in \mathbb{Z}, \\ u_{j-\frac{1}{2}}^{(k)}(1) - u_j^{(k)}\left(-\frac{L}{2}\right) = \Delta_{j,-}^{(k)}, \qquad u_j^{(1)}\left(-\frac{L}{2}\right) - u_{j+\frac{1}{2}}^{(k)}(0) = \Delta_{j,+}^{(k)}, & j \in \mathbb{Z}, \\ \big(u_{j+\frac{1}{2}}^{(k)}\big)'(0) - \big(u_{j-\frac{1}{2}}^{(k)}\big)'(1) + w_j^\mu \big(u_j^{(k)}\big)'\left(-\frac{L}{2}\right) = \Xi_j^{(k-1)}, & j \in \mathbb{Z}. \end{cases}$$

Proposition 2.2.2. *For $k \in \mathbb{N}^*$, $f^{(k-1)} \in L_2^\mu(G_{\mathcal{C}})$, $\left\{\Delta_{j,\pm}^{(k-1)}\right\}_{j\in\mathbb{Z}}, \left\{\Xi_j^{(k-1)}\right\}_{j\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$, the problem $(\mathcal{P}_u^{(k)})$ is has a solution in $L_2(G_{\mathcal{C}})$ if and only if*

$$\lambda^{(k)} = \left\|u^{(0)}\right\|_{L_2^\mu(G_{\mathcal{C}})}^{-2} \left(\sum_{j\in\mathbb{Z}} \widetilde{\Xi}_j^{(k-1)} \mathbf{u}_j^{(0)} - \left(f^{(k-1)}, u^{(0)}\right)_{L_2^\mu(G_{\mathcal{C}})} \right), \tag{2.2.60}$$

where

$$\widetilde{\Xi}_j^{(k-1)} = \Xi_j^{(k-1)} - \frac{\sqrt{\lambda^{(0)}}}{\sin\sqrt{\lambda^{(0)}}} \left(\Delta_{j+1,-}^{(k-1)} - \Delta_{j-1,+}^{(k-1)} + \cos\sqrt{\lambda^{(0)}} \left(\Delta_{j,+}^{(k-1)} - \Delta_{j,-}^{(k-1)} \right) \right), \qquad j \in \mathbb{Z}. \tag{2.2.61}$$

The solution is unique in $L_2(G_{\mathcal{C}})/span\left\{ u^{(0)}\big|_{G_{\mathcal{C}}} \right\}$.

Remark 2.2.6. As we see, the far field terms are not defined in a unique way. We will explain in the proof of Lemma 2.3.1 how we fix them. This choice is completely arbitrary and could be done in a different way. We could show by an explicit computation that the choice of the far field terms does not influence the values $\lambda^{(k)}$ defined by the relation (2.2.60). However, this will be justified a posteriori by proving Theorem 1.4.2.

To prove Proposition 2.2.2, we start with the following assertion for the problem with zero jump conditions at the vertices.

Proposition 2.2.3. *Let $\lambda^{(0)}$ be an eigenvalue of the operator A_s^μ and $u^{(0)}$ the corresponding eigenfunction. For $f \in L_2^\mu(G_{\mathcal{C}})$, $\{\Xi_j\}_{j\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ the problem*

$$\begin{cases} u''_{j+\frac{1}{2}}(s) + \lambda^{(0)} u_{j+\frac{1}{2}}(s) = f_{j+\frac{1}{2}}(s), \qquad s \in [0,1], & j \in \mathbb{Z}, \\ u''_j(y) + \lambda^{(0)} u_j(y) = f_j(y), \qquad y \in \left[-\frac{L}{2}, 0\right], & j \in \mathbb{Z}, \\ u'_j(0) = 0, & j \in \mathbb{Z}, \\ u_{j-\frac{1}{2}}(1) = u_j\left(-\frac{L}{2}\right) = u_{j+\frac{1}{2}}(0), & j \in \mathbb{Z}, \\ u'_{j+\frac{1}{2}}(0) - u'_{j-\frac{1}{2}}(1) + w_j^\mu u'_j\left(-\frac{L}{2}\right) = \Xi_j, & j \in \mathbb{Z}, \end{cases} \tag{2.2.62}$$

has a solution in $L_2(G_{\mathcal{C}})$ if and only if the following compatibility condition is verified:

$$\left(f,u^{(0)}\right)_{L_2^\mu(G_{\mathcal{C}})}+\sum_{j\in\mathbb{Z}}\Xi_j\mathbf{u}_j^{(0)}=0.$$

The solution is unique in $L_2(G_{\mathcal{C}})/span\left\{u^{(0)}\big|_{G_{\mathcal{C}}}\right\}$.

Proof. The uniqueness of the solution in $L_2(G_{\mathcal{C}})/span\left\{u^{(0)}\big|_{G_{\mathcal{C}}}\right\}$ is obvious. The proof of the existence will be analogous to the proof of Theorem 4.10 from [51]. Passing to the variational formulation of the problem (2.2.62) and considering it as a problem in the symmetric subspace $L^\mu_{2,s}(G)$ on the whole graph G, we will search a function $u\in H^1_s(G)$ such that

$$-\left(u',v'\right)_{L_2^\mu(G)}+\lambda^{(0)}(u,v)_{L_2^\mu(G)}=(f,v)_{L_2^\mu(G)}+2\sum_{j\in\mathbb{Z}}\Xi_j\mathbf{v}_j,\qquad\forall v\in H^1_s(G),\tag{2.2.63}$$

where $\mathbf{v}_j=v\left(j,-\frac{L}{2}\right)$. Due to the estimate

$$\left|-\left(u',v'\right)_{L_2^\mu(G)}+\lambda^{(0)}\left(u,v\right)_{L_2^\mu(G)}\right|\leqslant\max\left\{1,\lambda^{(0)}\right\}\|u\|_{H^1(G)}\|v\|_{H^1(G)},\qquad\forall u,v\in H^1_s(G),$$

we can then define a bounded linear self-adjoint operator $A:\,H^1_s(G)\to H^1_s(G)$ such that

$$(Au,v)_{H^1(G)}=-\left(u',v'\right)_{L_2^\mu(G)}+\lambda^{(0)}\left(u,v\right)_{L_2^\mu(G)},\qquad\forall u,v\in H^1_s(G).\tag{2.2.64}$$

In the same manner, we have the estimate

$$\left|(f,v)_{L_2^\mu(G)}+2\sum_{j\in\mathbb{Z}}\Xi_j\mathbf{v}_j\right|\leqslant\left(\|f\|_{L_2^\mu(G)}+C\left(\sum_{j\in\mathbb{Z}}\Xi_j^2\right)^{1/2}\right)\|v\|_{H^1(G)},\qquad\forall v\in H^1_s(G),$$

with some constant C depending only on the geometry of the graph. Hence, there exists a unique function $F\in H^1_s(G)$ such that

$$(F,v)_{H^1(G)}=(f,v)_{L_2^\mu(G)}+2\sum_{j\in\mathbb{Z}}\Xi_j\mathbf{v}_j,\qquad\forall v\in H^1_s(G).\tag{2.2.65}$$

Thus, combining (2.2.63), (2.2.64) and (2.2.65) we end up with the equation

$$Au=F,\tag{2.2.66}$$

which is equivalent to the problem (2.2.62). Indeed, considering in (2.2.63) functions v that vanish at the vertices of the graph, we get the first two lines of (2.2.62) and considering functions v that do not vanish at only one vertex of the graph we recover the last line (Kirchhoff's conditions). Taking $F=0$ we get then

$$KerA=span\left\{u^{(0)}\right\}.\tag{2.2.67}$$

We will now use Lemma 2.2.5 proved below which states that ImA is closed. The operator A being self-adjoint, Lemma 2.2.5 together with (2.2.67) imply that the problem (2.2.66) is solvable if and only if $\left(F,u^{(0)}\right)_{H^1(G)}=0$ (cf. Theorem 5, Chapter 3 in [7]). Together with (2.2.65) this finishes the proof (the coefficient 2 disappears when we pass to the graph $G_{\mathcal{C}}$). □

Lemma 2.2.5. *The image of the operator A defined in* (2.2.64) *is closed:*

$$ImA = \overline{ImA}.$$

Proof. Suppose that $ImA \neq \overline{ImA}$. Then, there exists a singular sequence $\{u_n\}_{n\in\mathbb{N}} \subset H^1_s(G)$ such that

(a) $\|u_n\|_{H^1(G)} = 1$,

(b) $u_n \xrightarrow{w} 0 \quad \text{in} \quad H^1_s(G)$,

(c) $\|Au_n\|_{H^1_s(G)} \longrightarrow 0$.

Let us show that there exists $n_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$\|u_n\|_{L^\mu_2(G)} \geqslant \delta, \qquad \forall n \geqslant n_0. \tag{2.2.68}$$

Indeed,

$$-\|u_n'\|^2_{L^\mu_2(G)} + \lambda^{(0)}\|u_n\|^2_{L^\mu_2(G)} = (Au_n, u_n)_{H^1(G)} \leqslant \|Au_n\|_{H^1_s(G)} \to 0, \qquad n \to \infty.$$

If there was a subsequence of $\{u_{n_k}\}_{k\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ such that $\|u_{n_k}\|_{L^\mu_2(G)} \to 0,\ k \to \infty$, it would mean that $\left\|u'_{n_k}\right\|_{L^\mu_2(G)} \to 0$, and hence, $\|u_{n_k}\|_{H^1(G)} \to 0$ which contradicts the property (a) of the sequence $\{u_n\}$. Next,

$$\left(u_n, \left(A^\mu_s - \lambda^{(0)}I\right) v\right)_{L^\mu_2(G)} = -(Au_n, v)_{H^1(G)} \qquad \forall v \in D(A^\mu_s).$$

Since $\lambda^{(0)}$ is a simple (isolated) eigenvalue of the self-adjoint operator A^μ_s in $L^\mu_{2,s}(G)$, we have $Im\left(A^\mu_s - \lambda^{(0)}I\right) = \left\{u^{(0)}\right\}^{\perp_{L^\mu_{2,s}(G)}}$ and the operator $B = \left(A^\mu_s - \lambda^{(0)}I\right)^{-1}|_{\left\{u^{(0)}\right\}^{\perp_{L^\mu_{2,s}(G)}}}$ is a continuous operator from $L^\mu_{2,s}(G)$ to $H^1_s(G)$. Hence,

$$(u_n, w)_{L^\mu_2(G)} = -(Au_n, Bw)_{H^1(G)}, \qquad \forall w \in \left\{u^{(0)}\right\}^{\perp_{L^\mu_{2,s}(G)}}. \tag{2.2.69}$$

Let us introduce the notation

$$u_n^\perp = u_n - \frac{\left(u_n, u^{(0)}\right)_{L^\mu_2(G)}}{\|u^{(0)}\|^2_{L^\mu_2(G)}} u^{(0)}, \qquad n \in \mathbb{N}. \tag{2.2.70}$$

Obviously, $u_n^\perp \in \left\{u^{(0)}\right\}^{\perp_{L^\mu_{2,s}(G)}}$. Then, applying (2.2.69) to $u_n^\perp$ we obtain:

$$\left\|u_n^\perp\right\|^2_{L^\mu_2(G)} = \left(u_n, u_n^\perp\right)_{L^\mu_2(G)} \leqslant \|Au_n\|_{H^1_s(G)} \|B\|_{L^\mu_{2,s}(G)\to H^1_s(G)} \left\|u_n^\perp\right\|_{L^\mu_2(G)}, \qquad n \in \mathbb{N}.$$

Taking into account the property (c) of the sequence $\{u_n\}$ we conclude that $\left\|u_n^\perp\right\|^2_{L^\mu_2(G)} \to 0$ as $n \to \infty$. The property (b) implies that $u_n \xrightarrow{w} 0$ in $L^\mu_{2,s}(G)$. Consequently, $\left(u_n, u^{(0)}\right)_{L^\mu_2(G)} \to 0$. Thus, it follows from the relation (2.2.70) that $\|u_n\|^2_{L^\mu_2(G)} \to 0$ which contradicts (2.2.68). □

We can now deduce Proposition 2.2.2 from Proposition 2.2.3.

Proof of Proposition 2.2.2. Let us introduce the following function $v^{(k)}$ for $k \in \mathbb{N}$:

$$v^{(k)}_{j+\frac{1}{2}}(s) = -\Delta^{(k-1)}_{j,+}\frac{\sin\left(\sqrt{\lambda^{(0)}}(1-s)\right)}{\sin\sqrt{\lambda^{(0)}}} + \Delta^{(k-1)}_{j+1,-}\frac{\sin\left(\sqrt{\lambda^{(0)}}s\right)}{\sin\sqrt{\lambda^{(0)}}}, \qquad s \in [0,1], \qquad j \in \mathbb{Z},$$
$$v^{(k)}_{j}(y) = 0, \qquad y \in \left[-\tfrac{L}{2}, \tfrac{L}{2}\right], \qquad j \in \mathbb{Z}.$$

Then, the function $\widetilde{u}^{(k)} = u^{(k)} - v^{(k)},\ k \in \mathbb{N}$, satisfies the following problem:

$$\begin{cases} \left(\widetilde{u}^{(k)}_{j+\frac{1}{2}}\right)''(s) + \lambda^{(0)}\widetilde{u}^{(k)}_{j+\frac{1}{2}}(s) = -\lambda^{(k)}u^{(0)}_{j+\frac{1}{2}}(s) - f^{(k-1)}_{j+\frac{1}{2}}(s), & s \in [0,1], \quad j \in \mathbb{Z}, \\ \left(\widetilde{u}^{(k)}_{j}\right)''(y) + \lambda^{(0)}\widetilde{u}^{(k)}_{j}(y) = -\lambda^{(k)}u^{(0)}_{j}(y) - f^{(k-1)}_{j}(y), & y \in \left[-\tfrac{L}{2}, 0\right], \quad j \in \mathbb{Z}, \\ \left(\widetilde{u}^{(k)}_{j}\right)'(0) = 0, & j \in \mathbb{Z}, \\ \widetilde{u}^{(k)}_{j-\frac{1}{2}}(1) = \widetilde{u}^{(k)}_{j}\left(-\tfrac{L}{2}\right) = \widetilde{u}^{(k)}_{j+\frac{1}{2}}(0), & j \in \mathbb{Z}, \\ \left(\widetilde{u}^{(k)}_{j+\frac{1}{2}}\right)'(0) - \left(\widetilde{u}^{(k)}_{j-\frac{1}{2}}\right)'(1) + w^{\mu}_{j}\left(\widetilde{u}^{(k)}_{j}\right)'\left(-\tfrac{L}{2}\right) = \widetilde{\Xi}^{(k-1)}_{j}, & j \in \mathbb{Z}, \end{cases} \tag{2.2.71}$$

Applying Proposition 2.2.3 to the problem (2.2.71) we finish the proof. □

We finish this section with the following lemma that establishes a symmetry property of a solution of $(\mathcal{P}^{(k)}_{u})$ provided the right-hand terms are symmetric.

Lemma 2.2.6. *Suppose that for some $k \in \mathbb{N}^*$, $f^{(k-1)} \in L^{\mu}_{2}(G_{\mathcal{C}})$, $\left\{\Delta^{(k-1)}_{j,\pm}\right\}_{j\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$, $\left\{\Xi^{(k-1)}_{j}\right\}_{j\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ and the condition (2.2.60) is satisfied. Suppose also that*

$$f^{(k-1)}_{-j-\frac{1}{2}}(1-s) = f^{(k-1)}_{j+\frac{1}{2}}(s), \quad s \in [0,1], \qquad f^{(k-1)}_{-j}(y) = f^{(k-1)}_{j}(y), \quad y \in [-\tfrac{L}{2}, 0], \qquad j \in \mathbb{Z},$$

and

$$\Delta^{(k-1)}_{j,+} = -\Delta^{(k-1)}_{-j,-}, \qquad \Delta^{(k-1)}_{j,-} = -\Delta^{(k-1)}_{-j,+}, \qquad \Xi^{(k-1)}_{j} = \Xi^{(k-1)}_{-j}, \qquad \forall j \in \mathbb{Z}.$$

If $u^{(k)}$ is a solution of $(\mathcal{P}^{(k)}_{u})$, then

$$u^{(k)}_{-j-\frac{1}{2}}(1-s) = u^{(k)}_{j+\frac{1}{2}}(s), \quad s \in [0,1], \qquad u^{(k)}_{-j}(y) = u^{(k)}_{j}(y), \quad y \in [-\tfrac{L}{2}, 0], \qquad j \in \mathbb{Z}.$$

Proof. Let us introduce the function $\widehat{u}^{(k)}$ which is obtained from $u^{(k)}$ by the reflection with respect to the axis $x = 0$:

$$\widehat{u}^{(k)}_{j+\frac{1}{2}}(s) = u^{(k)}_{-j-\frac{1}{2}}(1-s), \quad s \in [0,1], \qquad \widehat{u}^{(k)}_{j}(s) = u^{(k)}_{-j}(s), \quad y \in \left[-\tfrac{L}{2}, 0\right], \qquad j \in \mathbb{Z}.$$

Then, the function $w^{(k)}$ which is defined as the difference $w^{(k)} = u^{(k)} - \widehat{u}^{(k)}$, solves the homogeneous problem (2.2.39). Consequently, $w^{(k)} = cu^{(0)}$ for some $c \in \mathbb{R}$. On the other hand, $\mathbf{w}^{(k)}_{0} = 0$, which implies that $w^{(k)} = 0$. This finishes the proof.

□

2.3 Existence of the terms of the asymptotic expansions

In this section we will prove the existence of all the terms of the formal asymptotic expansions considered in Section 2.1 and, consequently, justify these expansions. We proceed by induction assuming that both the far field and the near field terms are constructed up to some order n and showing that they can be constructed at order $n+1$. We remind that the terms of order 0 have been constructed in Section 2.2.3.

We start with an auxiliary result which will permit us to conclude that if both the far field and the near field terms are exponentially decaying up to the order n, then the far field terms at order $n+1$ are also exponentially decaying.

Lemma 2.3.1. *Let $k \in \mathbb{N}^*$. Suppose that $f^{(k-1)} \in L_2^{\mu}(G_{\mathcal{C}})$ and $\left\{\Delta_{j,\pm}^{(k-1)}\right\}_{j\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$, $\left\{\Xi_j^{(k-1)}\right\}_{j\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$, and that the compatibility condition (2.2.60) is satisfied. Suppose also that there exists a family of polynomials $\left\{a_\ell^{(k-1)},\, b_\ell^{(k-1)},\, c_\ell^{(k-1)},\, d_\ell^{(k-1)}\right\}$ of degree $k-\ 1-\ell,\ 0 \leqslant \ell \leqslant k-1$, and three polynomials q_{Δ_+}, q_{Δ_-}, q_Σ of degree $k-1$, such that*

$$F_{j+\frac{1}{2}}(s) = r^j \sum_{\ell=0}^{k-1} s^\ell \left(a_\ell^{(k-1)}(j)\cos\left(\sqrt{\lambda_0}s\right) + b_\ell^{(k-1)}(j)\sin\left(\sqrt{\lambda_0}s\right)\right), \quad s\in[0,1], \quad j\in\mathbb{N}, \tag{2.3.1}$$

$$F_j(y) = r^j \sum_{\ell=0}^{k-1} y^\ell \left(c_\ell^{(k-1)}(j)\cos\left(\sqrt{\lambda_0}y\right) + d_\ell^{(k-1)}(j)\sin\left(\sqrt{\lambda_0}y\right)\right), \quad y\in\left[-\tfrac{L}{2},0\right], \quad j\in\mathbb{N}^*, \tag{2.3.2}$$

$$\Delta_{j,+}^{(k-1)} = r^j\, q_{\Delta_+}(j), \qquad \Delta_{j,-}^{(k-1)} = r^j\, q_{\Delta_-}(j), \qquad \Xi_j^{(k-1)} = r^j\, q_\Sigma(j), \qquad j\in\mathbb{N}^*, \tag{2.3.3}$$

where

$$F_{j+\frac{1}{2}} = -\lambda^{(k)} u_{j+\frac{1}{2}}^{(0)} - f_{j+\frac{1}{2}}^{(k-1)}, \qquad F_j = -\lambda^{(k)} u_j^{(0)} - f_j^{(k-1)}, \qquad j\in\mathbb{N}.$$

Let $u^{(k)}$ be a solution of $(\mathcal{P}_u^{(k)})$. Then, there exist polynomials $\left\{\widetilde{a}_\ell^{(k)},\, \widetilde{b}_\ell^{(k)},\, \widetilde{c}_\ell^{(k)},\, \widetilde{d}_\ell^{(k)}\right\}$ of degree $k-\ell,\ 0\leqslant \ell \leqslant k$, such that

$$u_{j+\frac{1}{2}}^{(k)}(s) = r^j \sum_{\ell=0}^{k} s^\ell \left(\widetilde{a}_\ell^{(k)}(j)\cos\left(\sqrt{\lambda_0}s\right) + \widetilde{b}_\ell^{(k)}(j)\sin\left(\sqrt{\lambda_0}s\right)\right), \qquad s\in[0,1], \qquad j\in\mathbb{N}, \tag{2.3.4}$$

$$u_j^{(k)}(y) = r^j \sum_{\ell=0}^{k} y^\ell \left(\widetilde{c}_\ell^{(k)}(j)\cos\left(\sqrt{\lambda_0}y\right) + \widetilde{d}_\ell^{(k)}(j)\sin\left(\sqrt{\lambda_0}y\right)\right), \qquad y\in\left[-\tfrac{L}{2},0\right], \quad j\in\mathbb{N}^*. \tag{2.3.5}$$

Proof. First of all, we notice that the first two lines of $(\mathcal{P}_u^{(k)})$ together with the assumptions (2.3.1), (2.3.2) for the right-hand sides imply the forms (2.3.4), (2.3.5) of the solutions $u_{j+\frac{1}{2}}^{(k)}$, $u_j^{(k)}$ with some coefficients $\left\{\widetilde{a}_\ell^{(k)}(j),\, \widetilde{b}_\ell^{(k)}(j)\ \widetilde{c}_\ell^{(k)}(j),\, \widetilde{d}_\ell^{(k)}(j)\right\}_{\ell=0}^{k}$ (not necessarily polynomial with respect to j). We will start by establishing a recurrence relations for the coefficients with $\ell>0$ which will permit us to show that they are polynomials. Then, using the transmission conditions we will get a system of equations for the coefficients with $\ell = 0$ which will yield a finite difference equation for $\widetilde{a}_0^{(k)}$. The study of this finite difference equation will show that the coefficients with $\ell = 0$ are also polynomials.

Recurrence relations for the coefficients with $\ell \geqslant 1$: In order to determine the dependence of these coefficients on j we plug the relations (2.3.4), (2.3.1) into the first line of $(\mathcal{P}_u^{(k)})$. We get then:

$$
\begin{aligned}
2(\ell+1)\sqrt{\lambda_0}\widetilde{b}_{\ell+1}^{(k)}(j) + (\ell+1)(\ell+2)\widetilde{a}_{\ell+2}^{(k)}(j) &= a_\ell^{(k-1)}(j), \qquad 0 \leqslant \ell \leqslant k-1,\\
-2(\ell+1)\sqrt{\lambda_0}\widetilde{a}_{\ell+1}^{(k)}(j) + (\ell+1)(\ell+2)\widetilde{b}_{\ell+2}^{(k)}(j) &= b_\ell^{(k-1)}(j), \qquad 0 \leqslant \ell \leqslant k-1,
\end{aligned}
$$

where, by convention, $\widetilde{a}_{k+1}^{(k)}(j) = \widetilde{b}_{k+1}^{(k)}(j) = 0$. Hence,

$$
\widetilde{b}_k^{(k)}(j) = \frac{a_{k-1}^{(k-1)}(j)}{2k\sqrt{\lambda_0}}, \qquad \widetilde{a}_k^{(k)}(j) = -\frac{b_{k-1}^{(k-1)}(j)}{2k\sqrt{\lambda_0}}, \tag{2.3.6}
$$

and the following recurrence relations define all the other coefficients except $\widetilde{a}_0^{(k)}(j)$, $\widetilde{b}_0^{(k)}(j)$ which remain undetermined:

$$
\widetilde{b}_\ell^{(k)}(j) = \frac{a_{\ell-1}^{(k-1)}(j) - \ell(\ell+1)\widetilde{a}_{\ell+1}^{(k)}(j)}{2\ell\sqrt{\lambda_0}}, \qquad \widetilde{a}_\ell^{(k)}(j) = -\frac{b_{\ell-1}^{(k-1)}(j) - \ell(\ell+1)\widetilde{b}_{\ell+1}^{(k)}(j)}{2\ell\sqrt{\lambda_0}},
$$
$$
1 \leqslant \ell \leqslant k-1. \tag{2.3.7}
$$

One can prove by induction with respect to ℓ that the coefficients $\widetilde{a}_\ell^{(k)}(j)$, $\widetilde{b}_\ell^{(k)}(j)$ are polynomials (with respect to j) of degree $k-\ell$ for $1 \leqslant \ell \leqslant k$. Indeed, taking into account the assumptions for the coefficients $\left\{a_\ell^{(k-1)}(j),\, b_\ell^{(k-1)}(j)\right\}_{\ell=0}^{k-1}$ we see from (2.3.6) that for $\ell = k$ the coefficients $\widetilde{a}_k^{(k)}(j)$, $\widetilde{b}_k^{(k)}(j)$ are constants in j. If we suppose that $\widetilde{a}_\ell^{(k)}(j)$, $\widetilde{b}_\ell^{(k)}(j)$ are polynomials (with respect to j) of degree $k-\ell$ for all ℓ such that $m \leqslant \ell \leqslant k$ for some $1 < m \leqslant k$ then the relations (2.3.7) imply that $\widetilde{a}_{m-1}^{(k)}(j)$, $\widetilde{b}_{m-1}^{(k)}(j)$ are polynomials in j of degree $k-m+1$. Thus, it remains only to determine the behaviour of the coefficients $\widetilde{a}_0^{(k)}(j)$, $\widetilde{b}_0^{(k)}(j)$ with respect to j for $j \geqslant 0$.

Repeating the same argument applied to the coefficients $\left\{\widetilde{c}_\ell^{(k)}(j),\, \widetilde{d}_\ell^{(k)}(j)\right\}_{\ell=0}^{k}$, i. e., injecting (2.3.5), (2.3.2) into the second line of $(\mathcal{P}_u^{(k)})$ we get the analogues of the relations (2.3.6), (2.3.7) with $a_\ell^{(k-1)}(j)$, $b_\ell^{(k-1)}(j)$, $\widetilde{a}_\ell^{(k)}(j)$, $\widetilde{b}_\ell^{(k)}(j)$ replaced by $c_\ell^{(k-1)}(j)$, $d_\ell^{(k-1)}(j)$, $\widetilde{c}_\ell^{(k)}(j)$, $\widetilde{d}_\ell^{(k)}(j)$ respectively. However, this time there is an additional relation coming from the third line of $(\mathcal{P}_u^{(k)})$ (Neumann boundary condition):

$$
\widetilde{c}_1^{(k)}(j) + \sqrt{\lambda_0}\widetilde{d}_0^{(k)}(j) = 0. \tag{2.3.8}
$$

Hence, in the same manner as before we see that the coefficients $\widetilde{c}_\ell^{(k)}(j)$, $\widetilde{d}_\ell^{(k)}(j)$ are polynomials with respect to j (for $j \geqslant 1$) of degree $k-\ell$ for $1 \leqslant \ell \leqslant k$ and the coefficient $\widetilde{d}_0^{(k)}(j)$ is a polynomial of degree $k-1$.

The coefficients $\widetilde{a}_0^{(k)}(j)$, $\widetilde{b}_0^{(k)}(j)$, $\widetilde{c}_0^{(k)}(j)$: Let us now establish the dependence of the coefficients $\widetilde{a}_0^{(k)}(j)$, $\widetilde{b}_0^{(k)}(j)$, $\widetilde{c}_0^{(k)}(j)$ on j using the transmission conditions (last two lines of $(\mathcal{P}_u^{(k)})$). We get for $j \geqslant 1$:

$$r^{j-1}\cos\sqrt{\lambda_0}\widetilde{a}_0^{(k)}(j-1) + r^{j-1}\sin\sqrt{\lambda_0}\widetilde{b}_0^{(k)}(j-1) - r^j\cos\left(\tfrac{\sqrt{\lambda_0}L}{2}\right)\widetilde{c}_0^{(k)}(j) = t_1^{(k)}(j), \quad (2.3.9)$$

$$r^j\cos\left(\tfrac{\sqrt{\lambda_0}L}{2}\right)\widetilde{c}_0^{(k)}(j) - r^j\widetilde{a}_0^{(k)}(j) = t_2^{(k)}(j), \quad (2.3.10)$$

$$r^j\widetilde{b}_0^{(k)}(j) + r^{j-1}\sin\sqrt{\lambda_0}\widetilde{a}_0^{(k)}(j-1) - r^{j-1}\cos\sqrt{\lambda_0}\widetilde{b}_0^{(k)}(j-1) + r^j\sin\left(\tfrac{\sqrt{\lambda_0}L}{2}\right)\widetilde{c}_0^{(k)}(j) = t_3^{(k)}(j), \quad (2.3.11)$$

where

$$t_1^{(k)}(j) = \Delta_{j,-}^{(k-1)} - r^{j-1}\sum_{\ell=1}^{k}\left(\cos\sqrt{\lambda_0}\widetilde{a}_\ell^{(k)}(j-1) + \sin\sqrt{\lambda_0}\widetilde{b}_\ell^{(k)}(j-1)\right)$$
$$+ r^j\sum_{\ell=1}^{k}\left(-\tfrac{L}{2}\right)^\ell\left(\cos\left(\tfrac{\sqrt{\lambda_0}L}{2}\right)\widetilde{c}_\ell^{(k)}(j) - \sin\left(\tfrac{\sqrt{\lambda_0}L}{2}\right)\widetilde{d}_\ell^{(k)}(j)\right) + \frac{r^j}{\sqrt{\lambda_0}}\sin\left(\tfrac{\sqrt{\lambda_0}L}{2}\right)\widetilde{c}_1^{(k)}(j),$$

$$t_2^{(k)}(j) = \Delta_{j,+}^{(k-1)} - r^j\sum_{\ell=1}^{k}\left(-\tfrac{L}{2}\right)^\ell\left(\cos\left(\tfrac{\sqrt{\lambda_0}L}{2}\right)\widetilde{c}_\ell^{(k)}(j) - \sin\left(\tfrac{\sqrt{\lambda_0}L}{2}\right)\widetilde{d}_\ell^{(k)}(j)\right)$$
$$- \frac{r^j}{\sqrt{\lambda_0}}\sin\left(\tfrac{\sqrt{\lambda_0}L}{2}\right)\widetilde{c}_1^{(k)}(j),$$

$$t_3^{(k)}(j) = r^{j-1}\sum_{\ell=1}^{k}\left(\left(\ell\frac{\cos\sqrt{\lambda_0}}{\sqrt{\lambda_0}} - \sin\sqrt{\lambda_0}\right)\widetilde{a}_\ell^{(k)}(j-1) + \left(\ell\frac{\sin\sqrt{\lambda_0}}{\sqrt{\lambda_0}} + \cos\sqrt{\lambda_0}\right)\widetilde{b}_\ell^{(k)}(j-1)\right)$$
$$- r^j\sum_{\ell=1}^{k}\left(-\tfrac{L}{2}\right)^\ell\left(\sin\left(\tfrac{\sqrt{\lambda_0}L}{2}\right) - \frac{2\ell}{L\sqrt{\lambda_0}}\cos\left(\tfrac{\sqrt{\lambda_0}L}{2}\right)\right)\widetilde{c}_\ell^{(k)}(j)$$
$$- r^j\sum_{\ell=1}^{k}\left(-\tfrac{L}{2}\right)^\ell\left(\cos\left(\tfrac{\sqrt{\lambda_0}L}{2}\right) + \frac{2\ell}{L\sqrt{\lambda_0}}\sin\left(\tfrac{\sqrt{\lambda_0}L}{2}\right)\right)\widetilde{d}_\ell^{(k)}(j)$$
$$+ \frac{\Xi_j^{(k-1)}}{\sqrt{\lambda_0}} - \frac{r^j}{\sqrt{\lambda_0}}\left(\widetilde{a}_1^{(k)}(j) - \cos\left(\tfrac{\sqrt{\lambda_0}L}{2}\right)\widetilde{c}_1^{(k)}(j)\right).$$

The expressions above imply that the right-hand terms of the equations (2.3.9)–(2.3.11) have the form

$$t_i^{(k)}(j) = r^j p_{t,i}^{(k)}(j), \qquad 1 \leqslant i \leqslant 3, \quad (2.3.12)$$

where $\left\{p_{t,i}^{(k)}\right\}_{i=1}^{3}$ are polynomials in j of degree $k-1$. The system (2.3.9)–(2.3.11) reduces to the following one:

$$\widetilde{c}_0^{(k)}(j) = \frac{1}{\cos\left(\frac{\sqrt{\lambda_0}L}{2}\right)}\left(\widetilde{a}_0^{(k)}(j) + p_{t,2}^{(k)}(j)\right), \qquad j \geqslant 1, \quad (2.3.13)$$

$$\widetilde{b}_0^{(k)}(j) = \frac{1}{\sin\sqrt{\lambda_0}}\left(r\widetilde{a}_0^{(k)}(j+1) - \widetilde{a}_0^{(k)}(j)\cos\sqrt{\lambda_0} + r\left(p_{t,1}^{(k)}(j+1) + p_{t,2}^{(k)}(j+1)\right)\right), \quad j \geqslant 0, \quad (2.3.14)$$

$$r^2\widetilde{a}_0^{(k)}(j+2)+2rg\left(\sqrt{\lambda_0}\right)\widetilde{a}_0^{(k)}(j+1)+\widetilde{a}_0^{(k)}(j)=p_a^{(k)}(j),\qquad j\geqslant 0,\tag{2.3.15}$$

where the function g is defined in (1.3.45) and

$$\begin{aligned}p_a^{(k)}(j)&=r\sin\sqrt{\lambda_0}p_{t,3}^{(k)}(j+1)-r^2\left(p_{t,1}^{(k)}(j+2)+p_{t,2}^{(k)}(j+2)\right)\\&\quad-r\sin\sqrt{\lambda_0}\tan\left(\tfrac{\sqrt{\lambda_0}L}{2}\right)p_{t,2}^{(k)}(j+1)+r\cos\sqrt{\lambda_0}\left(p_{t,1}^{(k)}(j+1)+p_{t,2}^{(k)}(j+1)\right).\end{aligned}$$

The function $p_a^{(k)}(j)$ being a polynomial in j of degree $k-1$, we introduce a notation for its coefficients:

$$p_a^{(k)}(j)=\sum_{m=0}^{k-1}\rho_m j^m.\tag{2.3.16}$$

Study of the finite difference equation (2.3.15)**:** First, recalling that r and r^{-1} are solutions of the equation (1.3.47), we find the general (real) solution of the corresponding homogeneous equation:

$$\widetilde{a}_0^{(k)}(j)=C+Dr^{-2j},\qquad C,\,D\in\mathbb{R},\qquad j\geqslant 0.$$

However, the condition $u^{(k)}\in L_2^\mu(G_\mathcal{C})$ implies that $D=0$. Next, the right-hand side of the equation (2.3.15) being a polynomial of degree $k-1$, we search a particular solution as a polynomial of degree k:

$$\widetilde{a}_0^{(k)}(j)=\sum_{m=0}^{k}\alpha_m j^m.$$

Together with (2.3.16) it gives the following system for the coefficients $\{\gamma_m\}_{m=0}^k$:

$$\sum_{i=m}^{k}C_i^m r^2\alpha_i+2g\left(\sqrt{\lambda^{(0)}}\right)r\alpha_m+\sum_{i=m}^{k}(-1)^{i-m}C_i^m r^2\alpha_i=\rho_m,\qquad 0\leqslant m\leqslant k,$$

where $\rho_k=0$. Since r is a solution of (1.3.47), the equation for $m=k$ is satisfied automatically and the other equations take the form

$$\sum_{i=m+1}^{k}C_i^m\left(\left(2^{i-m}-1\right)r^2-1\right)\alpha_i=\rho_m,\qquad 0\leqslant m\leqslant k-1,\tag{2.3.17}$$

which is a system with an upper-diagonal matrix with non-zero elements ($|r|<1$) for the coefficients $\{\alpha_m\}_{m=1}^k$. The coefficient α_0 cannot be determined, which corresponds to the general solution $\widetilde{a}_0^{(k)}(j)=C,\quad j\geqslant 0$. We can fix it by choosing, for example, $\alpha_0=0$. Hence, the coefficient $\widetilde{a}_0^{(k)}(j)$ is a polynomial in j of degree k for $j\geqslant 0$ as well as the coefficients $\widetilde{b}_0^{(k)}(j)$ for $j\geqslant 0$ and $\widetilde{c}_0^{(k)}(j)$ for $j\geqslant 1$ (cf. (2.3.13), (2.3.14)). This finishes the proof. □

We are now able to prove the existence of the terms of the asymptotic expansions (2.1.2)–(2.1.5).

Proposition 2.3.1. *There exist functions* $\left\{u^{(k)},U^{(k)}\right\}_{k\in\mathbb{N}}$ *and real numbers* $\left\{\lambda^{(k)}\right\}_{k\in\mathbb{N}}$ *having the following properties:*

1. *For any $k \in \mathbb{N}$, $u^{(k)}$ is a solution of the far field problem $(\mathcal{P}_u^{(k)})$. Moreover,*

$$u_{j+\frac{1}{2}}^{(k)} \in C^\infty([0,1]), \qquad u_j^{(k)} \in C^\infty\left(\left[-\tfrac{L}{2},0\right]\right), \qquad j \in \mathbb{Z}, \qquad k \in \mathbb{N}, \tag{2.3.18}$$

and

$$u_{-j-\frac{1}{2}}^{(k)}(1-s) = u_{j+\frac{1}{2}}^{(k)}(s), \quad s \in [0,1], \qquad u_{-j}^{(k)}(y) = u_j^{(k)}(y), \quad y \in [-\tfrac{L}{2},0], \qquad j \in \mathbb{Z}, \qquad k \in \mathbb{N}. \tag{2.3.19}$$

2. *For $k \in \mathbb{N}$, $j \in \mathbb{Z}$, $U_j^{(k)} \in H^1_{loc}(\mathcal{J}_j)$ is a solution of the problem $(\mathcal{P}_j^{(k)})$. Moreover,*

$$U_{-j}^{(k)}(-X,Y) = U_j^{(k)}(X,Y), \qquad j \in \mathbb{Z}, \qquad k \in \mathbb{N}. \tag{2.3.20}$$

3. *The number $\lambda^{(k)}$ satisfies* (2.2.60)*.*

4. *For any $k \in \mathbb{N}$ the relations* (2.3.4)*,* (2.3.5) *for the far field terms $u^{(k)}$ are satisfied. Moreover, there exist functions $\mathcal{U}_q^{(k)} \in H^1_{loc}(\mathcal{J}_*)$ such that the near field terms satisfy the following relation:*

$$U_j^{(k)} = r^j \sum_{q=0}^{k} j^q \mathcal{U}_q^{(k)}, \qquad j \in \mathbb{N}^*. \tag{2.3.21}$$

The last property means that the far field terms $u_j^{(k)}$ and the near field terms $U_j^{(k)}$ are exponentially decaying as $|j|$ tends to infinity.

Proof. First, it is clear that the the functions $u^{(0)}$, $U^{(0)}$ and the number $\lambda^{(0)}$ found in Section 2.2.3 satisfy the properties 1–4. Suppose that for some $n \geqslant 1$ functions $\left\{u^{(k)}, U^{(k)}\right\}_{k=0}^n$ and real numbers $\left\{\lambda^{(k)}\right\}_{k=0}^n$ having the required properties are constructed. Let us show the existence of functions $u^{(n+1)}$, $U^{(n+1)}$ and a real number $\lambda^{(n+1)}$ satisfying the properties 1–4 as well.

Construction of the right-hand terms of the problems $\mathcal{P}_u^{(n+1)}$ and $\mathcal{P}_j^{(n+1)}$

- The function $f^{(n)}$ is given by the relations (2.2.37), (2.2.38) with k replaced by $n+1$. Since the functions $\left\{u^{(k)}\right\}_{k=1}^n$ satisfy the property 1, one has $f^{(n)} \in L_2(G_{\mathcal{C}})$ and

$$f_{j+\frac{1}{2}}^{(n)} \in C^\infty([0,1]), \qquad f_j^{(n)} \in C^\infty\left(\left[-\tfrac{L}{2},0\right]\right), \qquad j \in \mathbb{Z}, \tag{2.3.22}$$

$$f_{-j-\frac{1}{2}}^{(n)}(1-s) = f_{j+\frac{1}{2}}^{(n)}(s), \quad s \in [0,1], \qquad f_{-j}^{(n)}(y) = f_j^{(n)}(y), \quad y \in [-\tfrac{L}{2},0], \qquad j \in \mathbb{Z}. \tag{2.3.23}$$

Moreover, the functions $\left\{u^{(k)}\right\}_{k=1}^n$ satisfying the property 4, the function $f^{(n)}$ is also of the form (2.3.4), (2.3.5). In other words, there exist polynomials $\left\{a_{f,\ell}^{(n)}, b_{f,\ell}^{(n)}, c_{f,\ell}^{(n)}, d_{f,\ell}^{(n)}\right\}$ of degree $n-\ell$, $0 \leqslant \ell \leqslant n$, such that

$$f_{j+\frac{1}{2}}^{(n)}(s) = r^j \sum_{\ell=0}^{n} s^\ell \left(a_{f,\ell}^{(n)}(j)\cos\left(\sqrt{\lambda_0}s\right) + b_{f,\ell}^{(n)}(j)\sin\left(\sqrt{\lambda_0}s\right)\right), \quad s \in [0,1], \qquad j \in \mathbb{N}, \tag{2.3.24}$$

$$f_j^{(n)}(y) = r^j \sum_{\ell=0}^{n} y^\ell \left(c_{f,\ell}^{(n)}(j)\cos\left(\sqrt{\lambda_0}y\right) + d_{f,\ell}^{(n)}(j)\sin\left(\sqrt{\lambda_0}y\right)\right), \quad y \in \left[-\tfrac{L}{2},0\right], \quad j \in \mathbb{N}^*. \tag{2.3.25}$$

- The coefficients $\left\{c^{(n+1)}_{j,p,\delta,\ell},\ j\in\mathbb{Z},\ p\in\mathbb{N}^*,\ \delta\in\{+,-,0\},\ 1\leqslant\ell\leqslant\left[\frac{n+1}{2}\right]\right\}$ are computed via the recurrence relation (2.2.12). Since the functions $\left\{U^{(k)}\right\}_{k=0}^{n}$ verify the property 2, Lemma 2.2.4 applies which gives the convergence result (2.2.46) for the coefficients c up to the order n, but also for the coefficients of order $n+1$ for $\ell\geqslant 1$:

$$\sum_{p\in\mathbb{N}^*} p\left(c^{(n+1)}_{j,p,\pm,\ell}\right)^2 e^{-p\pi w_j^\mu}<\infty,\qquad \sum_{p\in\mathbb{N}^*} p\left(c^{(n+1)}_{j,p,0,\ell}\right)^2 e^{-\frac{2p\pi}{w_j^\mu}}<\infty,\qquad 1\leqslant\ell\leqslant\left[\tfrac{n+1}{2}\right]. \tag{2.3.26}$$

Moreover, the following property can be easily shown by induction in k:

$$c^{(k)}_{-j,p,\pm,\ell}=c^{(k)}_{j,p,\mp,\ell},\qquad c^{(k)}_{-j,p,0,\ell}=c^{(k)}_{j,p,0,\ell},\qquad 0\leqslant k\leqslant n,\quad j\in\mathbb{Z},\quad p\in\mathbb{N}^*,\quad 0\leqslant\ell\leqslant\left[\tfrac{k}{2}\right],$$

and there exist real numbers $\check{c}^{(k)}_{q,p,\delta,\ell}$, $0\leqslant k\leqslant n$, $0\leqslant q\leqslant k$, $p\in\mathbb{N}^*$, $\delta\in\{+,-,0\}$, $0\leqslant\ell\leqslant\left[\frac{k}{2}\right]$, such that

$$c^{(k)}_{j,p,\delta,\ell}=r^j\sum_{q=0}^{k} j^q\check{c}^{(k)}_{q,p,\delta,\ell},\quad 0\leqslant k\leqslant n,\quad j\in\mathbb{N}^*,\quad p\in\mathbb{N}^*,\quad \delta\in\{+,-,0\},\quad 0\leqslant\ell\leqslant\left[\tfrac{k}{2}\right].$$

In the base case $k=0$ these properties obviously hold since all the coefficients are equal to zero (cf. (2.2.45)). The proof of the inductive step relies on the recurrence relation (2.2.12) for $\ell\geqslant 1$ and the relation (2.2.59) for $\ell=0$, taking into account the properties 2 and 4 of the functions $\left\{U^{(k)}\right\}_{k=0}^{n}$.

Again, due to the recurrence relation (2.2.12), the same is true for the coefficients of order $n+1$ for $\ell\geqslant 1$:

$$c^{(n+1)}_{-j,p,\pm,\ell}=c^{(n+1)}_{j,p,\mp,\ell},\qquad c^{(n+1)}_{-j,p,0,\ell}=(-1)^p c^{(n+1)}_{j,p,0,\ell},\qquad j\in\mathbb{Z},\quad p\in\mathbb{N}^*,\quad 1\leqslant\ell\leqslant\left[\tfrac{n+1}{2}\right], \tag{2.3.27}$$

and there exist real numbers $\check{c}^{(n+1)}_{q,p,\delta,\ell}$, $0\leqslant q\leqslant n+1$, $p\in\mathbb{N}^*$, $\delta\in\{+,-,0\}$, $1\leqslant\ell\leqslant\left[\frac{n+1}{2}\right]$, such that

$$c^{(n+1)}_{j,p,\delta,\ell}=r^j\sum_{q=0}^{n} j^q\check{c}^{(n+1)}_{q,p,\delta,\ell},\quad j\in\mathbb{N}^*,\quad p\in\mathbb{N}^*,\quad \delta\in\{+,-,0\},\quad 1\leqslant\ell\leqslant\left[\tfrac{n+1}{2}\right]. \tag{2.3.28}$$

- The functions $\left\{g^{(n)}_{j,\delta}\right\}_{j\in\mathbb{Z}}$, $\delta\in\{+,-,0\}$, are defined by the relations (2.2.24)–(2.2.26). The polynomials ${P^{(n+1)}_{j,p,\delta}}'$ appearing in this relations are completely defined at this point as follows since

$${P^{(k)}_{j,p,\delta}}'=\sum_{\ell=1}^{\left[\frac{k}{2}\right]} c^{(k)}_{j,p,\delta,\ell}\,\ell s^{\ell-1},\qquad \delta\in\{+,-,0\},\quad k\in\mathbb{N},\quad p\in\mathbb{N}^*,\quad j\in\mathbb{Z}.$$

The relations (2.3.26) permit to conclude that

$$g^{(n)}_{j,\delta}\in H^{-1/2}\left(\Sigma_j^{\pm}\right),\qquad \delta\in\{+,-,0\}. \tag{2.3.29}$$

Moreover, the relations (2.3.27) together with the property 1 satisfied by the functions $\left\{u^{(k)}\right\}_{k=0}^{n}$ imply that

$$g_{j,\pm}^{(n)} = g_{-j,\mp}^{(n)}, \qquad g_{j,0}^{(n)}(X) = g_{-j,0}^{(n)}(-X), \qquad j \in \mathbb{Z}, \tag{2.3.30}$$

where we identify the spaces $H^{-1/2}\left(\Sigma_j^{+}\right)$ and $H^{-1/2}\left(\Sigma_j^{-}\right)$. The relations (2.3.28) together with the property 4 satisfied by the functions $\left\{u^{(k)}\right\}_{k=0}^{n}$ imply that there exist functions $\breve{g}_{q,\delta}^{(n)} \in H^{-1/2}\left(\Sigma_j^{\delta}\right)$, $0 \leqslant q \leqslant n$, $\delta \in \{+,-,0\}$, such that

$$g_{j,\delta}^{(n)} = r^{j} \sum_{q=0}^{n} j^{q} \breve{g}_{q,\delta}^{(n)}, \qquad j \in \mathbb{N}^{*}, \qquad \delta \in \{+,-,0\}. \tag{2.3.31}$$

- The functions

$$\Phi_j^{(n-1)},\ \Phi_j^{(n)} \in H^1_{loc}(\mathcal{J}_j), \qquad j \in \mathbb{Z}, \tag{2.3.32}$$

are defined by the relation (2.2.23). From the properties 2 and 4 (symmetry and exponential decay) satisfied by the functions $\left\{U^{(k)}\right\}_{k=0}^{n}$ it follows that

$$\Phi_{-j}^{(k)}(-X,Y) = \Phi_j^{(k)}(X,Y), \qquad j \in \mathbb{Z}, \qquad 0 \leqslant k \leqslant n, \tag{2.3.33}$$

and there exist functions $\check{\Phi}_q^{(k)} \in H^1_{loc}(\mathcal{J}_*)$, $0 \leqslant k \leqslant n$, $0 \leqslant q \leqslant k$, such that

$$\Phi_j^{(k)} = r^{j} \sum_{q=0}^{k} j^{q} \check{\Phi}_q^{(k)}, \qquad \forall j \in \mathbb{N}^{*}, \qquad 0 \leqslant k \leqslant n. \tag{2.3.34}$$

- The sequence $\left\{\widetilde{\Xi}_j^{(n)}\right\}_{j\in\mathbb{Z}}$ is defined in (2.2.61) with k replaced by $n+1$, and the sequences $\left\{\Delta_{j,\pm}^{(n)}\right\}_{j\in\mathbb{Z}}$, $\left\{\Xi_j^{(n)}\right\}_{j\in\mathbb{Z}}$ are defined in (2.2.36), (2.2.34), (2.2.31). The symmetry properties of the functions $\left\{u^{(k)}\right\}_{k=0}^{n}$ (property 1), $g_{j,\delta}^{(n)}$ (2.3.30), $\Phi_j^{(n)}$, $\Phi_j^{(n-1)}$ (2.3.33) and $W_j^{\pm}$ (2.2.20) imply that

$$\Delta_{j,+}^{(n)} = -\Delta_{-j,-}^{(n)}, \qquad \Delta_{j,-}^{(n)} = -\Delta_{-j,+}^{(n)}, \qquad \Xi_j^{(n)} = \Xi_{-j}^{(n)}, \qquad j \in \mathbb{Z}. \tag{2.3.35}$$

Similarly, the decay properties of the functions $\left\{u^{(k)}\right\}_{k=0}^{n}$ (property 4), $g_{j,\delta}^{(n)}$ (2.3.31), $\Phi_j^{(n)}$, $\Phi_j^{(n-1)}$ (2.3.34) and the relations (2.2.21) for $W_j^{\pm}$ imply that there exist polynomials $q_{\Sigma}^{(n)}$, $q_{\Delta_\pm}^{(n)}$ of degree n such that

$$\Xi_j^{(n)} = r^{j}\, q_{\Sigma}^{(n)}(j), \qquad \Delta_{j,\pm}^{(n)} = r^{j}\, q_{\Delta_\pm}^{(n)}(j), \qquad j \in \mathbb{N}^{*}. \tag{2.3.36}$$

Construction of the number $\lambda^{(n+1)}$ and the far field terms $u^{(n+1)}$

Due to the decay properties of $f^{(n)}$, $\left\{\Xi_j^{(n)}\right\}_{j\in\mathbb{Z}}$, $\left\{\Delta_{j,\pm}^{(n)}\right\}_{j\in\mathbb{Z}}$ (cf. (2.3.24), (2.3.25), (2.3.23), (2.3.36), (2.3.35)), we have $f^{(n)} \in L_2(G_{\mathcal{C}})$, $\left\{\Xi_j^{(n)}\right\}_{j\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$, $\left\{\Delta_{j,\pm}^{(n)}\right\}_{j\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$. This permits to apply Proposition 2.2.2 for $k = n+1$. The number $\lambda^{(n+1)}$ is defined by the relation (2.2.60) and the function $u^{(n+1)}$ is a corresponding solution of the problem

$(\mathcal{P}_u^{(n+1)})$. To fix it in a unique way, one can, for example, impose the condition $\mathbf{u}_0^{(n+1)}$. The smoothness (2.3.18) of the functions $u_{j+\frac{1}{2}}^{(n+1)}$, $u_j^{(n+1)}$ at each vertex of the graph $G_{\mathcal{C}}$ is guaranteed by the smoothness of the functions $f_{j+\frac{1}{2}}^{(n)}$, $f_j^{(n)}$ (2.3.22). Due to the symmetry properties of $f^{(n)}$, $\left\{\Xi_j^{(n)}\right\}_{j\in\mathbb{Z}}$ and $\left\{\Delta_{j,\pm}^{(n)}\right\}_{j\in\mathbb{Z}}$ (cf. (2.3.23), (2.3.35)), Lemma 2.2.6 applies and we get the symmetry property (2.3.19) for the function $u^{(n+1)}$. Finally, to get the decay property 4 for $u^{(n+1)}$, we apply Lemma 2.3.1. Indeed, the hypothesis of the Lemma are verified due to the relations (2.3.24), (2.3.25), (2.3.36).

At this point, we can remark that the polynomials (2.2.13) satisfy the following symmetry property:

$$P_{j,0,\pm}^{(n+1)}(X) = P_{-j,0,\mp}^{(n+1)}(-X), \qquad P_{j,0,0}^{(n+1)}(Y) = P_{-j,0,0}^{(n+1)}(Y), \qquad j \in \mathbb{Z}. \tag{2.3.37}$$

Moreover, there exists real numbers $\left\{\check{p}_q^{(n+1)}\right\}_{q=0}^{n+1}$ such that

$$P_{j,0,+}^{(n+1)}\left(\tfrac{1}{2}\right) = r^j \sum_{q=0}^{n+1} \check{p}_q^{(n+1)} j^q, \qquad j \in \mathbb{N}^*.$$

Construction of the near field terms $U^{(n+1)}$

- In a bounded domain: for any $j \in \mathbb{Z}$, we start by constructing the solution $V_j^{(n+1)}$ of the problem $(\widetilde{\mathcal{P}}_j^{(n+1)})$ in a bounded domain. The existence (and uniqueness) of such a solution is guaranteed by Lemma 2.2.2. Indeed, the right-hand sides belong to the appropriate spaces (cf. (2.3.32), (2.3.29)). The compatibility condition (2.2.30) is satisfied since the function $u^{(n)}$ solves the problem $(\mathcal{P}_u^{(n)})$. Notice that the average trace condition (2.2.29) requires the knowledge of the function $u^{(n+1)}$. That is why at a given order the far field problem should be solved first.

  Let us show that the functions $\left\{V_j^{(n+1)}\right\}_{j\in\mathbb{Z}}$ verify the following symmetry property:

$$V_{-j}^{(n+1)}(-X,Y) = V_j^{(n+1)}(X,Y), \qquad j \in \mathbb{Z}. \tag{2.3.38}$$

  Indeed, if $V_{-j}^{(n+1)}$ satisfies the problem $(\widetilde{\mathcal{P}}_{-j}^{(n+1)})$, then the function $V_{-j}^{(n+1)}(-X,Y)$ satisfies the problem $(\widetilde{\mathcal{P}}_j^{(n+1)})$. This follows from the symmetry propetries of the right-hand sides $\Phi_j^{(n-1)}$, $g_{j,\delta}^{(n)}$ (cf. (2.3.33), (2.3.30)) together with Lemma 2.2.3 and the symmetry property 1 of the functions $\left\{u^{(k)}\right\}_{k=0}^{n+1}$ which ensure the equivalence of the trace average conditions in the problems $(\widetilde{\mathcal{P}}_j^{(n+1)})$ and the one satisfied by the function $V_{-j}^{(n+1)}(-X,Y)$. The solution of the problem $(\widetilde{\mathcal{P}}_j^{(n+1)})$ being unique, we get (2.3.38).

  Finally, let us show that there exist functions $\mathcal{V}_q^{(n+1)} \in H^1(K_*)$, $0 \leqslant q \leqslant n+1$, such that

$$V_j^{(n+1)} = r^j \sum_{q=0}^{n+1} j^q \mathcal{V}_q^{(n+1)}, \qquad j \in \mathbb{N}^*. \tag{2.3.39}$$

To do this, we introduce the following set of problems for $0 \leqslant q \leqslant n+1$:

$$\begin{aligned}&\text{Find}\quad V \in H^1(K_*) \quad \text{satisfying} \quad (2.2.16)\ \text{with}\ \Phi = \check{\Phi}_q^{(n-1)},\\ &g_\delta = \check{g}_{q,\delta}^{(n)}, \quad \delta \in \{+,-,0\}, \quad \text{and such that} \int\limits_{\Sigma_*^+} V = \check{p}_q^{(n+1)}.\end{aligned} \qquad (\check{\mathcal{P}}_q^{(n+1)})$$

Here $K_* = \left]-\frac{1}{2}, \frac{1}{2}\right[\times]0,1[$, $\Sigma_*^+ = \left\{\frac{1}{2}\right\} \times [0,1]$ (which are just K_j and Σ_j^+ for $j \in \mathbb{Z}^*$), and

$$\check{\Phi}_n^{(n-1)} = \check{\Phi}_{n+1}^{(n-1)} = 0, \qquad \check{g}_{n+1,\delta}^{(n)} = 0, \qquad \delta \in \{+,-,0\}. \tag{2.3.40}$$

The compatibility condition (2.2.17) is satisfied for the problem $\check{\mathcal{P}}_q^{(n+1)}$ for any $0 \leqslant q \leqslant n+1$. Indeed, we have already seen that it is satisfied for $\Phi = \Phi_j^{(n-1)}$, $g_\delta = g_{j,\delta}^{(n)}$, $\delta \in \{+,-,0\}$ with any $j \in \mathbb{N}^*$ (these are the problems for the functions $V_j^{(n+1)}$). Taking into account the relations (2.3.34), (2.3.31) completed by (2.3.40), we get

$$\sum_{q=0}^{n+1} j^q \left(\left\langle \check{g}_{q,+}^{(n)}, 1 \right\rangle_{\Sigma_j^+} + \left\langle \check{g}_{q,-}^{(n)}, 1 \right\rangle_{\Sigma_j^-} + \left\langle \check{g}_{q,0}^{(n)}, 1 \right\rangle_{\Sigma_j^0} - \int\limits_{K_j} \check{\Phi}_q^{(n-1)} \right) = 0, \qquad j \in \mathbb{N}^*.$$

This yields the compatibility condition for the problem $\check{\mathcal{P}}_q^{(n+1)}$ for any $0 \leqslant q \leqslant n+1$. Let us denote by $\mathcal{V}_q^{(n+1)}$ the unique solution of the problem $\check{\mathcal{P}}_q^{(n+1)}$, $0 \leqslant q \leqslant n+1$. Then, we get the relations (2.3.39) for the functions $\left\{V_j^{(n+1)}\right\}_{j\in\mathbb{N}^*}$.

- Extension to the unbounded domain $\mathcal{J}_j$: applying Proposition 2.2.1, we construct the near field terms $U_j^{(n+1)} \in H^1_{loc}(\mathcal{J}_j)$ that solve the problems $(\mathcal{P}_j^{(n+1)})$. We have already proved the symmetry property (2.3.38) for the functions $\left\{V_j^{(n+1)}\right\}_{j\in\mathbb{Z}}$. The symmetry property 2 for the functions $\left\{U_j^{(n+1)}\right\}_{j\in\mathbb{Z}}$ follows immediately from the symmetry property (2.3.38) of the functions $\left\{V_j^{(n+1)}\right\}_{j\in\mathbb{Z}}$. Similarly, the decay property 4 follows from (2.3.39) by taking $\mathcal{U}_q^{(n+1)}$ as the continuation to $\mathcal{J}_*$ of the function $\mathcal{V}_q^{(n+1)}$ for any $0 \leqslant q \leqslant n+1$.

□

2.4 Construction of a pseudo-mode at any order

In this section we construct an appropriate pseudo-mode that will permit to prove Theorem 1.4.2.

Let us introduce a function $\chi \in C^\infty(\mathbb{R})$ such that

$$0 \leqslant \chi_0(x) \leqslant 1, \quad \forall x \in \mathbb{R}, \qquad \chi_0(x) = \begin{cases} 0, & x \geqslant 1, \\ 1, & x \leqslant 2. \end{cases}$$

We construct the pseudo-mode at order n as follows:

$$\mathcal{U}_\varepsilon(x,y) = \begin{cases} \mathcal{U}_\varepsilon^{(1)}(x) + \mathcal{U}_\varepsilon^{(2)}(y) + \mathcal{U}_\varepsilon^{(3)}(x,y), & (x,y) \in \mathcal{C}_\varepsilon^\mu, \\ \mathcal{U}_\varepsilon(x,-y), & (x,y) \in \Omega_\varepsilon^\mu \setminus \mathcal{C}_\varepsilon^\mu, \end{cases} \tag{2.4.1}$$

where

$$\mathcal{U}_\varepsilon^{(1)}(x) = \sum_{j\in\mathbb{Z}} \left(\sum_{k=0}^{n} \varepsilon^k u_{j+\frac{1}{2}}^{(k)}(x-j) \right) \chi\left(\tfrac{x-j}{\varepsilon^\alpha}\right) \chi\left(\tfrac{j+1-x}{\varepsilon^\alpha}\right), \tag{2.4.2}$$

$$\mathcal{U}_\varepsilon^{(2)}(y) = \sum_{j\in\mathbb{Z}} \left(\sum_{k=0}^{n} \varepsilon^k u_j^{(k)}(y) \right) \chi\left(\tfrac{2y+L}{2\varepsilon^\alpha}\right), \tag{2.4.3}$$

$$\mathcal{U}_\varepsilon^{(3)}(x,y) = \sum_{j\in\mathbb{Z}} \left(\sum_{k=0}^{n} \varepsilon^k U_j^{(k)}\left(\tfrac{x-j}{\varepsilon}, \tfrac{2y+L}{2\varepsilon}\right) \right) \left(1-\chi\left(\tfrac{x-j}{\varepsilon^\alpha}\right)\right) \left(1-\chi\left(\tfrac{j-x}{\varepsilon^\alpha}\right)\right) \left(1-\chi\left(\tfrac{2y+L}{2\varepsilon^\alpha}\right)\right). \tag{2.4.4}$$

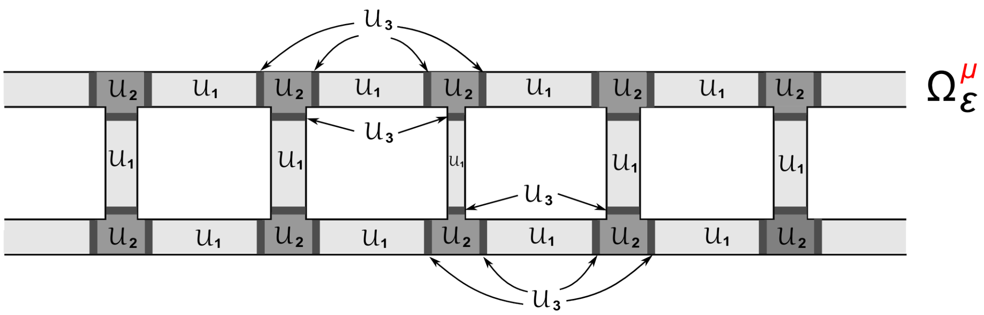


Figure 2.4: Construction of a pseudo-mode

For ε small enough we have the following estimate:

$$\|\mathcal{U}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^\mu)} \geqslant C_n \sum_{j\in\mathbb{Z}} \left(\int\limits_{\frac{1}{4}}^{\frac{3}{4}} \int\limits_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left| u_{j+\frac{1}{2}}^{(0)}(s,y) \right|^2 ds dy + \int\limits_{-\frac{w_j^\mu \varepsilon}{2}}^{\frac{w_j^\mu \varepsilon}{2}} \int\limits_{-\frac{L}{4}}^{0} \left| u_j^{(0)}(x,y) \right|^2 dx dy \right) \geqslant C_n \varepsilon. \tag{2.4.5}$$

Here and in what follows we denote by C_n all the constants that do not depend on ε. In the rest of the section we prove the following estimate.

Proposition 2.4.1. *For any* $0<\alpha<1$ *and* $n\in\mathbb{N}$ *there exist* $\varepsilon(n,\alpha)>0$ *and* $C(n,\alpha)>0$ *such that*

$$\left| \int\limits_{\Omega_\varepsilon^\mu} (\nabla\mathcal{U}_\varepsilon \nabla v - \lambda_{\varepsilon,n}\mathcal{U}_\varepsilon v)\, d\Omega \right| \leqslant C(n,\alpha)\varepsilon^{\alpha n+\frac{1}{2}} \|v\|_{H^1(\Omega_\varepsilon^\mu)}, \quad \forall v \in H_s^1(\Omega_\varepsilon^\mu), \quad 0<\varepsilon<\varepsilon(n,\alpha),$$

with

$$\lambda_{\varepsilon,n} = \sum_{k=0}^{n} \varepsilon^k \lambda^{(k)}.$$

Proof. Let us denote

$$\mathcal{I}_{\mathcal{U}_\varepsilon}(v) = \int\limits_{\mathcal{C}_\varepsilon^\mu} (\nabla\mathcal{U}_\varepsilon \nabla v - \lambda_{\varepsilon,n}\mathcal{U}_\varepsilon v)\, d\Omega = \frac{1}{2} \int\limits_{\Omega_\varepsilon^\mu} (\nabla\mathcal{U}_\varepsilon \nabla v - \lambda_{\varepsilon,n}\mathcal{U}_\varepsilon v)\, d\Omega. \tag{2.4.6}$$

After injecting (2.4.1)–(2.4.4) in (2.4.6) the value $\mathcal{I}_{\mathcal{U}_\varepsilon}(v)$ can be rewritten as

$$\mathcal{I}_{\mathcal{U}_\varepsilon}(v) = \mathcal{I}_{\mathcal{U}_\varepsilon^{(1)}}(v) + \mathcal{I}_{\mathcal{U}_\varepsilon^{(2)}}(v) + \mathcal{I}_{\mathcal{U}_\varepsilon^{(3)}}(v),$$

with

$$\begin{aligned}
\mathcal{I}_{\mathcal{U}_\varepsilon^{(1)}}(v) &= \sum_{j\in\mathbb{Z}} \int\limits_{\mathcal{C}_\varepsilon^\mu} \left(\sum_{k=0}^{n} \varepsilon^k \left(u_{j+\frac{1}{2}}^{(k)}\right)'(x-j)\right) \chi\left(\tfrac{x-j}{\varepsilon^\alpha}\right) \chi\left(\tfrac{j+1-x}{\varepsilon^\alpha}\right) \partial_x v(x,y) d\Omega \\
&\quad - \lambda_{\varepsilon,n} \sum_{j\in\mathbb{Z}} \int\limits_{\mathcal{C}_\varepsilon^\mu} \left(\sum_{k=0}^{n} \varepsilon^k u_{j+\frac{1}{2}}^{(k)}(x-j)\right) \chi\left(\tfrac{x-j}{\varepsilon^\alpha}\right) \chi\left(\tfrac{j+1-x}{\varepsilon^\alpha}\right) v(x,y) d\Omega \\
&\quad + \varepsilon^{-\alpha} \sum_{j\in\mathbb{Z}} \int\limits_{\mathcal{C}_\varepsilon^\mu} \left(\sum_{k=0}^{n} \varepsilon^k u_{j+\frac{1}{2}}^{(k)}(x-j)\right) \left(\chi'\left(\tfrac{x-j}{\varepsilon^\alpha}\right) - \chi'\left(\tfrac{j+1-x}{\varepsilon^\alpha}\right)\right) \partial_x v(x,y) d\Omega \\
&= -\sum_{j\in\mathbb{Z}} \sum_{k=n+1}^{2n} \sum_{p=k-n}^{n} \varepsilon^k \int\limits_{\mathcal{C}_\varepsilon^\mu} \lambda^{(k-p)} u_{j+\frac{1}{2}}^{(p)}(x-j) \chi\left(\tfrac{x-j}{\varepsilon^\alpha}\right) \chi\left(\tfrac{j+1-x}{\varepsilon^\alpha}\right) v(x,y) d\Omega \\
&- \varepsilon^{-\alpha} \sum_{j\in\mathbb{Z}} \int\limits_{\mathcal{C}_\varepsilon^\mu} \left(\sum_{k=0}^{n} \varepsilon^k \left(\left(u_{j+\frac{1}{2}}^{(k)}\right)'(x-j) v(x,y) - u_{j+\frac{1}{2}}^{(k)}(x-j) \partial_x v(x,y)\right)\right) \chi'\left(\tfrac{x-j}{\varepsilon^\alpha}\right) d\Omega \\
&\varepsilon^{-\alpha} \sum_{j\in\mathbb{Z}} \int\limits_{\mathcal{C}_\varepsilon^\mu} \left(\sum_{k=0}^{n} \varepsilon^k \left(\left(u_{j+\frac{1}{2}}^{(k)}\right)'(x-j) v(x,y) - u_{j+\frac{1}{2}}^{(k)}(x-j) \partial_x v(x,y)\right)\right) \chi'\left(\tfrac{j+1-x}{\varepsilon^\alpha}\right) d\Omega,
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{\mathcal{U}_\varepsilon^{(2)}}(v) &= -\sum_{j\in\mathbb{Z}} \sum_{k=n+1}^{2n} \sum_{p=k-n}^{n} \varepsilon^k \int\limits_{\mathcal{C}_\varepsilon^\mu} \lambda^{(k-p)} u_j^{(p)}(y) \chi\left(\tfrac{2y+L}{2\varepsilon^\alpha}\right) v(x,y) d\Omega \\
&\quad - \varepsilon^{-\alpha} \sum_{j\in\mathbb{Z}} \int\limits_{\mathcal{C}_\varepsilon^\mu} \left(\sum_{k=0}^{n} \varepsilon^k \left(\left(u_j^{(k)}\right)'(y) v(x,y) - u_j^{(k)}(y) \partial_y v(x,y)\right)\right) \chi'\left(\tfrac{2y+L}{2\varepsilon^\alpha}\right) d\Omega,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{I}_{\mathcal{U}_\varepsilon^{(3)}}(v) = \\
&-\sum_{j\in\mathbb{Z}} \sum_{k=n-1}^{n} \sum_{p=0}^{k} \varepsilon^k \int\limits_{\mathcal{C}_\varepsilon^\mu} \lambda^{(k-p)} U_j^{(p)}\left(\tfrac{x-j}{\varepsilon}, \tfrac{2y+L}{2\varepsilon}\right) \left(1-\chi\left(\tfrac{x-j}{\varepsilon^\alpha}\right)\right) \left(1-\chi\left(\tfrac{j-x}{\varepsilon^\alpha}\right)\right) \left(1-\chi\left(\tfrac{2y+L}{2\varepsilon^\alpha}\right)\right) v(x,y) d\Omega \\
&-\sum_{j\in\mathbb{Z}} \sum_{k=n+1}^{2n} \sum_{p=k-n}^{n} \varepsilon^k \int\limits_{\mathcal{C}_\varepsilon^\mu} \lambda^{(k-p)} U_j^{(p)}\left(\tfrac{x-j}{\varepsilon}, \tfrac{2y+L}{2\varepsilon}\right) \left(1-\chi\left(\tfrac{x-j}{\varepsilon^\alpha}\right)\right) \left(1-\chi\left(\tfrac{j-x}{\varepsilon^\alpha}\right)\right) \left(1-\chi\left(\tfrac{2y+L}{2\varepsilon^\alpha}\right)\right) v(x,y) d\Omega \\
&+\varepsilon^{-\alpha} \sum_{j\in\mathbb{Z}} \int\limits_{\mathcal{C}_\varepsilon^\mu} \left(\sum_{k=0}^{n} \varepsilon^k \left(\partial_x U_j^{(k)}\left(\tfrac{x-j}{\varepsilon}, \tfrac{2y+L}{2\varepsilon}\right) v(x,y) - U_j^{(k)}\left(\tfrac{x-j}{\varepsilon}, \tfrac{2y+L}{2\varepsilon}\right) \partial_x v(x,y)\right)\right) \chi'\left(\tfrac{x-j}{\varepsilon^\alpha}\right) d\Omega \\
&-\varepsilon^{-\alpha} \sum_{j\in\mathbb{Z}} \int\limits_{\mathcal{C}_\varepsilon^\mu} \left(\sum_{k=0}^{n} \varepsilon^k \left(\partial_x U_j^{(k)}\left(\tfrac{x-j}{\varepsilon}, \tfrac{2y+L}{2\varepsilon}\right) v(x,y) - U_j^{(k)}\left(\tfrac{x-j}{\varepsilon}, \tfrac{2y+L}{2\varepsilon}\right) \partial_x v(x,y)\right)\right) \chi'\left(\tfrac{j-x}{\varepsilon^\alpha}\right) d\Omega
\end{aligned}$$

$$+\varepsilon^{-\alpha}\sum_{j\in\mathbb{Z}}\int_{\mathcal{C}_\varepsilon^\mu}\left(\sum_{k=0}^{n}\varepsilon^k\left(\partial_y U_j^{(k)}\left(\tfrac{x-j}{\varepsilon},\tfrac{2y+L}{2\varepsilon}\right)v(x,y)-U_j^{(k)}\left(\tfrac{x-j}{\varepsilon},\tfrac{2y+L}{2\varepsilon}\right)\partial_y v(x,y)\right)\right)\chi'\left(\tfrac{2y+L}{2\varepsilon^\alpha}\right)d\Omega.$$

Summing up these expressions we can regroup the terms in the following way:

$$\mathcal{I}_{\mathcal{U}_\varepsilon}(v)=\mathcal{I}_{\varepsilon,F}(v)+\mathcal{I}_{\varepsilon,N}(v)+\mathcal{I}_{\varepsilon,M}(v),$$

where

$$\mathcal{I}_{\varepsilon,F}(v)=\mathcal{I}_{\varepsilon,F}^{(1)}(v)+\mathcal{I}_{\varepsilon,F}^{(2)}(v),$$

$$\mathcal{I}_{\varepsilon,F}^{(1)}(v)=-\sum_{j\in\mathbb{Z}}\sum_{k=n+1}^{2n}\sum_{p=k-n}^{n}\varepsilon^k\int_{\mathcal{C}_\varepsilon^\mu}\lambda^{(k-p)}u_{j+\frac{1}{2}}^{(p)}(x-j)\chi\left(\tfrac{x-j}{\varepsilon^\alpha}\right)\chi\left(\tfrac{j+1-x}{\varepsilon^\alpha}\right)v(x,y)d\Omega,$$

$$\mathcal{I}_{\varepsilon,F}^{(2)}(v)=-\sum_{j\in\mathbb{Z}}\sum_{k=n+1}^{2n}\sum_{p=k-n}^{n}\varepsilon^k\int_{\mathcal{C}_\varepsilon^\mu}\lambda^{(k-p)}u_j^{(p)}(y)\chi\left(\tfrac{2y+L}{2\varepsilon^\alpha}\right)v(x,y)d\Omega,$$

$$\mathcal{I}_{\varepsilon,N}(v)=$$

$$-\sum_{j\in\mathbb{Z}}\sum_{k=n-1}^{n}\sum_{p=0}^{k}\varepsilon^k\int_{\mathcal{C}_\varepsilon^\mu}\lambda^{(k-p)}U_j^{(p)}\left(\tfrac{x-j}{\varepsilon},\tfrac{2y+L}{2\varepsilon}\right)\left(1-\chi\left(\tfrac{x-j}{\varepsilon^\alpha}\right)\right)\left(1-\chi\left(\tfrac{j-x}{\varepsilon^\alpha}\right)\right)\left(1-\chi\left(\tfrac{2y+L}{2\varepsilon^\alpha}\right)\right)v(x,y)d\Omega$$

$$-\sum_{j\in\mathbb{Z}}\sum_{k=n+1}^{2n}\sum_{p=k-n}^{n}\varepsilon^k\int_{\mathcal{C}_\varepsilon^\mu}\lambda^{(k-p)}U_j^{(p)}\left(\tfrac{x-j}{\varepsilon},\tfrac{2y+L}{2\varepsilon}\right)\left(1-\chi\left(\tfrac{x-j}{\varepsilon^\alpha}\right)\right)\left(1-\chi\left(\tfrac{j-x}{\varepsilon^\alpha}\right)\right)\left(1-\chi\left(\tfrac{2y+L}{2\varepsilon^\alpha}\right)\right)v(x,y)d\Omega,$$

$$\mathcal{I}_{\varepsilon,M}(v)=\sum_{\delta\in\{+,-,0\}}\mathcal{I}_{\varepsilon,M}^{\delta}(v),\qquad \mathcal{I}_{\varepsilon,M}^{\delta}(v)=\mathcal{I}_{\varepsilon,M}^{\delta,1}(v)+\mathcal{I}_{\varepsilon,M}^{\delta,2}(v),\qquad \delta\in\{+,-,0\},$$

$$\mathcal{I}_{\varepsilon,M}^{+,1}(v)=-\varepsilon^{-\alpha}\sum_{j\in\mathbb{Z}}\sum_{k=0}^{n}\varepsilon^k\int_{\mathcal{C}_\varepsilon^\mu}\left(\left(u_{j+\frac{1}{2}}^{(k)}\right)'(x-j)-\partial_x U_j^{(k)}\left(\tfrac{x-j}{\varepsilon},\tfrac{2y+L}{2\varepsilon}\right)\right)\chi'\left(\tfrac{x-j}{\varepsilon^\alpha}\right)v(x,y)d\Omega,$$

$$\mathcal{I}_{\varepsilon,M}^{+,2}(v)=\varepsilon^{-\alpha}\sum_{j\in\mathbb{Z}}\sum_{k=0}^{n}\varepsilon^k\int_{\mathcal{C}_\varepsilon^\mu}\left(u_{j+\frac{1}{2}}^{(k)}(x-j)-U_j^{(k)}\left(\tfrac{x-j}{\varepsilon},\tfrac{2y+L}{2\varepsilon}\right)\right)\chi'\left(\tfrac{x-j}{\varepsilon^\alpha}\right)\partial_x v(x,y)d\Omega,$$

$$\mathcal{I}_{\varepsilon,M}^{-,1}(v)=\varepsilon^{-\alpha}\sum_{j\in\mathbb{Z}}\sum_{k=0}^{n}\varepsilon^k\int_{\mathcal{C}_\varepsilon^\mu}\left(\left(u_{j-\frac{1}{2}}^{(k)}\right)'(x+1-j)-\partial_x U_j^{(k)}\left(\tfrac{x-j}{\varepsilon},\tfrac{2y+L}{2\varepsilon}\right)\right)\chi'\left(\tfrac{j-x}{\varepsilon^\alpha}\right)v(x,y)d\Omega,$$

$$\mathcal{I}_{\varepsilon,M}^{-,2}(v)=-\varepsilon^{-\alpha}\sum_{j\in\mathbb{Z}}\sum_{k=0}^{n}\varepsilon^k\int_{\mathcal{C}_\varepsilon^\mu}\left(u_{j-\frac{1}{2}}^{(k)}(x+1-j)-U_j^{(k)}\left(\tfrac{x-j}{\varepsilon},\tfrac{2y+L}{2\varepsilon}\right)\right)\chi'\left(\tfrac{j-x}{\varepsilon^\alpha}\right)\partial_x v(x,y)d\Omega,$$

$$\mathcal{I}_{\varepsilon,M}^{0,1}(v)=-\varepsilon^{-\alpha}\sum_{j\in\mathbb{Z}}\sum_{k=0}^{n}\varepsilon^k\int_{\mathcal{C}_\varepsilon^\mu}\left(\left(u_j^{(k)}\right)'(y)-\partial_y U_j^{(k)}\left(\tfrac{x-j}{\varepsilon},\tfrac{2y+L}{2\varepsilon}\right)\right)\chi'\left(\tfrac{2y+L}{2\varepsilon^\alpha}\right)v(x,y)d\Omega,$$

$$\mathcal{I}_{\varepsilon,M}^{0,2}(v)=\varepsilon^{-\alpha}\sum_{j\in\mathbb{Z}}\sum_{k=0}^{n}\varepsilon^k\int_{\mathcal{C}_\varepsilon^\mu}\left(u_j^{(k)}(y)-U_j^{(k)}\left(\tfrac{x-j}{\varepsilon},\tfrac{2y+L}{2\varepsilon}\right)\right)\chi'\left(\tfrac{2y+L}{2\varepsilon^\alpha}\right)\partial_y v(x,y)d\Omega.$$

The term $\mathcal{I}_{\varepsilon,F}(v)$ corresponds to the regions where the far field expansion holds and the term $\mathcal{I}_{\varepsilon,N}(v)$ corresponds to the regions where the near field expansion holds. Roughly speaking, these terms measure the error between the true eigenfunction and the constructed quasimode in the corresponding regions. The term $\mathcal{I}_{\varepsilon,M}(v)$ corresponds to the regions of matching. It can be seen as a measure of the difference between the two expansions in these regions.

- Estimation of the term $\mathcal{I}_{\varepsilon,F}(v)$

  Using the Cauchy-Schwarz inequality, we get the following estimate:

$$\left|\mathcal{I}^{(1)}_{\varepsilon,F}(v)\right|$$
$$\leqslant C \sum_{k=n+1}^{2n} \sum_{p=k-n}^{n} \varepsilon^k \lambda^{(k-p)} \sum_{j\in\mathbb{Z}} \left(\int_0^1 \int_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left| u^{(p)}_{j+\frac{1}{2}}(s) \right|^2 ds dy \right)^{1/2} \left(\int_j^{j+1} \int_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} |v(x,y)|^2 dx dy \right)^{1/2}$$
$$\leqslant C \sum_{k=n+1}^{2n} \sum_{p=k-n}^{n} \varepsilon^{k+\frac{1}{2}} \lambda^{(k-p)} \left\| u^{(p)} \right\|_{L_2(G_{\mathcal{C}})} \|v\|_{L_2\left(\mathcal{C}^{\mu}_{\varepsilon}\right)} \leqslant C \varepsilon^{n+\frac{3}{2}} \|v\|_{H^1\left(\mathcal{C}^{\mu}_{\varepsilon}\right)}. \tag{2.4.7}$$

  The term $\mathcal{I}^{(2)}_{\varepsilon,F}(v)$ can be estimated in the same way.

- Estimation of the term $\mathcal{I}_{\varepsilon,N}(v)$

  In order to estimate the term $\mathcal{I}_{\varepsilon,N}(v)$ we will use Lemma A.3.1 (cf. Annexe). First, assuming that $\alpha < 1$ we can estimate $\mathcal{I}_{\varepsilon,N}(v)$ as follows:

$$|\mathcal{I}_{\varepsilon,N}(v)| \leqslant C \left(\mathcal{I}^{(0)}_{\varepsilon,N}(v) + \mathcal{I}^{(1)}_{\varepsilon,N}(v) + \mathcal{I}^{(2)}_{\varepsilon,N}(v) + \mathcal{I}^{(3)}_{\varepsilon,N}(v) \right),$$

  where

$$\mathcal{I}^{(0)}_{\varepsilon,N}(v) = \sum_{j\in\mathbb{Z}} \int_{Q^j_\varepsilon} \left| S^j_\varepsilon(x,y) v(x,y) \right| d\Omega, \qquad Q^j_\varepsilon = \left[j - w^{\mu}_j \varepsilon, j + w^{\mu}_j \varepsilon \right] \times \left[-\tfrac{L}{2}, -\tfrac{L}{2} + 2\varepsilon \right],$$

$$\mathcal{I}^{(1)}_{\varepsilon,N}(v) = \sum_{j\in\mathbb{Z}} \int_{j+\varepsilon}^{j+\varepsilon^\alpha} \int_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left| S^j_\varepsilon(x,y) v(x,y) \right| d\Omega,$$

$$\mathcal{I}^{(2)}_{\varepsilon,N}(v) = \sum_{j\in\mathbb{Z}} \int_{j-\varepsilon^\alpha}^{j-\varepsilon} \int_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left| S^j_\varepsilon(x,y) v(x,y) \right| d\Omega,$$

$$\mathcal{I}^{(3)}_{\varepsilon,N}(v) = \sum_{j\in\mathbb{Z}} \int_{j-\frac{w^{\mu}_j \varepsilon}{2}}^{j+\frac{w^{\mu}_j \varepsilon}{2}} \int_{-\frac{L}{2}+2\varepsilon}^{-\frac{L}{2}+\varepsilon^\alpha} \left| S^j_\varepsilon(x,y) v(x,y) \right| d\Omega,$$

$$S^j_\varepsilon(x,y) = \sum_{k=n-1}^{n} \sum_{p=0}^{k} \varepsilon^k \lambda^{(k-p)} U^{(p)}_j \left(\tfrac{x-j}{\varepsilon}, \tfrac{2y+L}{2\varepsilon} \right) + \sum_{k=n+1}^{2n} \sum_{p=k-n}^{n} \varepsilon^k \lambda^{(k-p)} U^{(p)}_j \left(\tfrac{x-j}{\varepsilon}, \tfrac{2y+L}{2\varepsilon} \right).$$

For $\mathcal{I}^{(0)}_{\varepsilon,N}(v)$ we have:

$$\mathcal{I}^{(0)}_{\varepsilon,N}(v) \leqslant$$
$$\sum_{j\in\mathbb{Z}}\left(\sum_{k=n-1}^{n}\sum_{p=0}^{k}\varepsilon^{k+1}\left|\lambda^{(k-p)}\right|\left\|U_j^{(p)}\right\|_{L_2(K_j)}+\sum_{k=n+1}^{2n}\sum_{p=k-n}^{n}\varepsilon^{k+1}\left|\lambda^{(k-p)}\right|\left\|U_j^{(p)}\right\|_{L_2(K_j)}\right)\|v\|_{L_2(Q^j_\varepsilon)}$$
$$\leqslant C\varepsilon^n\sum_{p=0}^{n}\left(\sum_{j\in\mathbb{Z}}\left\|U_j^{(p)}\right\|^2_{L_2(K_j)}\right)^{1/2}\left(\sum_{j\in\mathbb{Z}}\|v\|^2_{L_2(Q^j_\varepsilon)}\right)^{1/2}\leqslant C\varepsilon^{n+\frac{1}{2}}\|v\|_{H^1(\mathcal{C}^\mu_\varepsilon)}. \tag{2.4.8}$$

Here we used the estimate (A.3.2) for the function v, fact that $U_j^{(k)}\in H^1_{loc}(\mathcal{J}_j)$, $\forall j\in\mathbb{Z}$, $k\in\mathbb{N}$ as well as the exponential decay property (2.3.21) for the functions $\left\{U^{(p)}\right\}_{p=0}^{n}$.

In order to estimate the term $\mathcal{I}^{(1)}_{\varepsilon,N}(v)$ we notice that the behaviour of the functions $\left\{U^{(p)}\right\}_{p=0}^{n}$ in the band B_j^+ (cf. (2.2.9), (2.2.11), (2.2.13)) implies the following inequality:

$$\left|S^j_\varepsilon(x,y)\right|\leqslant C_n(j)\left(\sum_{k=n-1}^{n}\sum_{p=0}^{k}\varepsilon^{k+p(\alpha-1)}+\sum_{k=n+1}^{2n}\sum_{p=k-n}^{n}\varepsilon^{k+p(\alpha-1)}\right)\leqslant C_n(j)\varepsilon^{(n-1)\alpha},$$
$$(x,y)\in]j+\varepsilon,\, j+\varepsilon^\alpha[\times\left]-\tfrac{L}{2},-\tfrac{L}{2}+\varepsilon\right[.$$

Due (A.3.1) and the exponential decay of $\left\{U^{(p)}\right\}_{p=0}^{n}$ (2.3.21) (which implies the exponential decay of constants $C_n(j)$ in j) we get:

$$\mathcal{I}^{(1)}_{\varepsilon,N}(v)\leqslant C_n\varepsilon^{n\alpha+\frac{1}{2}}\|v\|_{H^1(\mathcal{C}^\mu_\varepsilon)}. \tag{2.4.9}$$

The terms $\mathcal{I}^{(2)}_{\varepsilon,N}(v)$ and $\mathcal{I}^{(3)}_{\varepsilon,N}(v)$ can be estimated analogously to $\mathcal{I}^{(1)}_{\varepsilon,N}(v)$.

- Estimation of the term $\mathcal{I}_{\varepsilon,M}(v)$

  We will estimate the term $\mathcal{I}^+_{\varepsilon,M}(v)$, the estimation of the terms $\mathcal{I}^-_{\varepsilon,M}(v)$, $\mathcal{I}^0_{\varepsilon,M}(v)$ being analogous.

  Using the relation (2.2.9) giving the behaviour of the functions $\left\{U^{(k)}\right\}_{k=0}^{n}$ in the band B_j^+, we get

$$\left|\mathcal{I}^{+,1}_{\varepsilon,M}(v)\right|$$
$$\leqslant C_n\varepsilon^{-\alpha}\left(\sum_{j\in\mathbb{N}}\int\limits_{j+\varepsilon^\alpha}^{j+2\varepsilon^\alpha}\int\limits_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon}\left|\mathcal{P}^j_\varepsilon(x)v(x,y)\right|dxdy+\sum_{j\in\mathbb{N}}\int\limits_{j+\varepsilon^\alpha}^{j+2\varepsilon^\alpha}\int\limits_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon}\left|\mathcal{E}^j_\varepsilon(x,y)v(x,y)\right|dxdy\right),$$

where

$$\mathcal{P}^j_\varepsilon(x)=\sum_{k=0}^{n}\varepsilon^k\left(u^{(k)}_{j+\frac{1}{2}}\right)'(x-j)-\sum_{k=0}^{n}\varepsilon^{k-1}\left(P^{(k)}_{j,0,+}\right)'\left(\tfrac{x-j}{\varepsilon}\right),$$
$$\mathcal{E}^j_\varepsilon(x,y)=\sum_{k=0}^{n}\varepsilon^k\partial_x\left(\sum_{p\in\mathbb{N}^*}P^{(k)}_{j,p,+}\left(\tfrac{x-j}{\varepsilon}\right)e^{-\frac{p\pi(x-j)}{\varepsilon}}f_p\left(\tfrac{2y+L}{2\varepsilon}\right)\right).$$

Taking into account the definition (2.2.13) of the polynomials $P^{(k)}_{j,0,+}$ we can rewrite $\mathcal{P}^j_\varepsilon$ as

$$\mathcal{P}^j_\varepsilon(x) = \sum_{k=0}^{n} \varepsilon^k \left(\left(u^{(k)}_{j+\frac{1}{2}}\right)'(x-j) - \sum_{\ell=0}^{n-k-1} \left.\frac{d^\ell \left(u^{(k)}_{j+\frac{1}{2}}\right)'}{ds^l}\right|_{s=0} \frac{(x-j)^\ell}{\ell!} \right).$$

Hence,

$$\left|\mathcal{P}^j_\varepsilon(x)\right| \leqslant \sum_{k=0}^{n} \varepsilon^k \left\| \frac{d^{n-k+1} u^{(k)}_{j+\frac{1}{2}}}{ds^{n-k+1}} \right\|_{L_\infty([0,1])} (2\varepsilon^\alpha)^{n-k} \leqslant C_n(j)\, \varepsilon^{\alpha n}, \qquad x \in [j+\varepsilon^\alpha,\, j+2\varepsilon^\alpha],$$

and using (A.3.1) we get

$$\sum_{j\in\mathbb{N}} \int\limits_{j+\varepsilon^\alpha}^{j+2\varepsilon^\alpha} \int\limits_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left|\mathcal{P}^j_\varepsilon(x) v(x,y)\right| dxdy \leqslant C_n \varepsilon^{\alpha(n+1)+\frac{1}{2}} \|v\|_{H^1(\mathcal{C}^\mu_\varepsilon)}.$$

Next, we obviously have the following estimate for the other term due to the decaying exponentials ($\alpha < 1$):

$$\sum_{j\in\mathbb{N}} \int\limits_{j+\varepsilon^\alpha}^{j+2\varepsilon^\alpha} \int\limits_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left|\mathcal{E}^j_\varepsilon(x,y) v(x,y)\right| dxdy \leqslant C_n(N)\varepsilon^N \|v\|_{H^1(\mathcal{C}^\mu_\varepsilon)}, \qquad \forall n \in \mathbb{N}. \tag{2.4.10}$$

Thus,

$$\left|\mathcal{I}^{+,1}_{\varepsilon,M}(v)\right| \leqslant C_n \varepsilon^{\alpha n+\frac{1}{2}} \|v\|_{H^1(\Omega^{\mu,-}_\varepsilon)}. \tag{2.4.11}$$

Finally, for $\mathcal{I}^{+,2}_{\varepsilon,M}(v)$ we get

$$\left|\mathcal{I}^{+,2}_{\varepsilon,M}(v)\right| \leqslant$$
$$C\varepsilon^{-\alpha} \left(\sum_{j\in\mathbb{N}} \int\limits_{j+\varepsilon^\alpha}^{j+2\varepsilon^\alpha} \int\limits_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left|\mathcal{T}^j_\varepsilon(x)\partial_x v(x,y)\right| dxdy + \sum_{j\in\mathbb{N}} \int\limits_{j+\varepsilon^\alpha}^{j+2\varepsilon^\alpha} \int\limits_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left|\mathcal{R}^j_\varepsilon(x,y)\partial_x v(x,y)\right| dxdy \right),$$

where

$$\mathcal{T}^j_\varepsilon(x) = \sum_{k=0}^{n} \varepsilon^k u^{(k)}_{j+\frac{1}{2}}(x-j) - \sum_{k=0}^{n} \varepsilon^k P^{(k)}_{j,0,+}\left(\tfrac{x-j}{\varepsilon}\right),$$
$$\mathcal{R}^j_\varepsilon(x,y) = \sum_{k=0}^{n} \sum_{p\in\mathbb{N}^*} \varepsilon^k P^{(k)}_{j,p,+}\left(\tfrac{x-j}{\varepsilon}\right) e^{-\frac{p\pi(x-j)}{\varepsilon}} f_p\left(\tfrac{2y+L}{2\varepsilon}\right).$$

Using the definition (2.2.13) of the polynomials $P^{(k)}_{j,0,+}$, the term $\mathcal{T}^j_\varepsilon$ can be rewritten as

$$\mathcal{T}^j_\varepsilon(x) = \sum_{k=0}^{n} \varepsilon^k \left(u^{(k)}_{j+\frac{1}{2}}(x-j) - \sum_{\ell=0}^{n-k} \left.\frac{d^\ell u^{(k)}_{j+\frac{1}{2}}}{ds^\ell}\right|_{s=0} \frac{(x-j)^\ell}{\ell!} \right),$$

which implies

$$\left|\mathcal{T}_\varepsilon^j(x)\right| \leqslant \sum_{k=0}^{n} \varepsilon^k \left\| \frac{d^{n-k+1} u_{j+\frac{1}{2}}^{(k)}}{ds^{n-k+1}} \right\|_{L_\infty([0,1])} (2\varepsilon^\alpha)^{n-k} \leqslant C(j)\, \varepsilon^{\alpha n}, \qquad x \in [j+\varepsilon^\alpha,\, j+2\varepsilon^\alpha].$$

Thus, using Cauchy-Schwartz inequality we get:

$$\sum_{j\in\mathbb{N}} \int\limits_{j+\varepsilon^\alpha}^{j+2\varepsilon^\alpha} \int\limits_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left|\mathcal{T}_\varepsilon^j(x)\partial_x v(x,y)\right| dxdy \leqslant C\varepsilon^{\alpha(n+1)+\frac{1}{2}} \|v\|_{H^1(\mathcal{C}_\varepsilon^\mu)}.$$

Analogously to (2.4.10) we can estimate the term containing the decaying exponentials as follows:

$$\sum_{j\in\mathbb{N}} \int\limits_{j+\varepsilon^\alpha}^{j+2\varepsilon^\alpha} \int\limits_{-\frac{L}{2}}^{-\frac{L}{2}+\varepsilon} \left|\mathcal{R}_\varepsilon^j(x,y)v(x,y)\right| dxdy \leqslant C_n(N)\varepsilon^N \|v\|_{H^1(\mathcal{C}_\varepsilon^\mu)}, \qquad \forall N \in \mathbb{N}.$$

Finally,

$$\left|\mathcal{I}_{\varepsilon,M}^{+,2}(v)\right| \leqslant C_n \varepsilon^{\alpha n+\frac{1}{2}} \|v\|_{H^1(\mathcal{C}_\varepsilon^\mu)}. \tag{2.4.12}$$

Summing up the estimates for all the terms (cf. (2.4.7), (2.4.8), (2.4.9), (2.4.11), (2.4.12)) we can conclude that for $0 < \alpha < 1$ there exists a constant $C(n,\alpha) > 0$ such that

$$|\mathcal{I}_{\mathcal{U}_\varepsilon}(v)| \leqslant C(n,\alpha)\varepsilon^{\alpha n+\frac{1}{2}} \|v\|_{H^1(\mathcal{C}_\varepsilon^\mu)}, \qquad \forall v \in H_s^1(\Omega_\varepsilon^\mu),$$

for ε small enough. This finishes the proof. □

Proof of Theorem 1.4.2. Proposition 2.4.1 together with the estimate (2.4.5) imply due to Lemma A.2.1 that

$$dist\left(\sigma\left(A_\varepsilon^\mu\right), \lambda_{\varepsilon,n}\right) \leqslant C(n,\alpha)\varepsilon^{\alpha n}.$$

for ε small enough. This is not exactly the estimate we need in order to prove Theorem 1.4.2. However, going up to the order $n+2$, we would get

$$dist\left(\sigma\left(A_\varepsilon^\mu\right), \lambda_{\varepsilon,n+2}\right) \leqslant C(n+2,\alpha)\varepsilon^{\alpha(n+2)} \leqslant C(n+2)\varepsilon^{n+1},$$

if α is chosen close enough to 1. On the other hand,

$$\lambda_{\varepsilon,n+2} - \lambda_{\varepsilon,n} = \lambda^{(n+1)}\varepsilon^{n+1} + \lambda^{(n+2)}\varepsilon^{n+2}.$$

Hence, due to the triangle inequality, we finally obtain

$$dist\left(\sigma\left(A_\varepsilon^\mu\right), \lambda_{\varepsilon,n}\right) \leqslant C(n)\varepsilon^{n+1}.$$

This finishes the proof of Theorem 1.4.2. □

CHAPTER 3

TRAPPED MODES IN A LOCALLY PERTURBED PERIODIC LADDER: NUMERICAL STUDY

3.1 Goals and difficulties

In the previous chapter we dealt with an eigenvalue problem in an unbounded domain Ω_ε. This requires some special methods when trying to find numerical approximations for this problem. One of the most common methods is the Supercell method ([64, 10, 62]) which consists in considering a big bounded domain with periodic boundary conditions. The solution in the truncated domain converges to the exact one exponentially if the defect mode is exponentially decaying. However, this method appears to be costly, especially when the mode is not well confined.

We apply here another method developed by S. Fliss ([24]) which is based on the construction of an appropriate Dirichlet-to-Neumann (DtN) operator. In the framework of this method the initial eigenvalue problem set in an unbounded domain is replaced by a nonlinear eigenvalue problem set in a bounded domain containing the defect (this time the domain does not have to be big). This nonlinear eigenvalue problem can be discretized by finite element method and a Newton type algorithm can be applied to solve the discrete problem.

In section 3.2 we explain how to reduce the initial problem to a problem posed in a bounded domain and in section 3.3 we show how this last problem is discretized by finite element method.

3.2 DtN operator method

Let us remind the initial problem. It consists in finding values λ_ε such that there exists a non-trivial function $u_\varepsilon \in H^1_\Delta(\Omega^\mu_\varepsilon)$ solving the problem

$$\begin{cases} -\Delta u_\varepsilon = \lambda_\varepsilon^2 u_\varepsilon \quad \text{in} \quad \Omega^\mu_\varepsilon, \\ \left.\dfrac{\partial u_\varepsilon}{\partial n}\right|_{\partial\Omega^\mu_\varepsilon} = 0. \end{cases} \tag{3.2.1}$$

To simplify the notation we will write in this section u_ε, λ_ε, keeping in mind that these values depend on μ. The idea of the DtN operator method is to find a problem which is equivalent to (3.2.1) but posed in a neighbourhood of the perturbation (the perturbed cell). The boundary conditions on the boundaries separating the perturbed cell from the rest of the domain (two half-bands) will contain the DtN operators which are defined using the corresponding problems in these half-bands.

3.2.1 Half-band problems

We denote $B^\pm_\varepsilon = \Omega^\mu_\epsilon \cap \{\pm x > 1\}$, $\Gamma^\pm_\varepsilon = \{\pm 1\} \times \left]-\frac{L}{2}, \frac{L}{2}\right[$ (cf. figure 3.1). Let us introduce

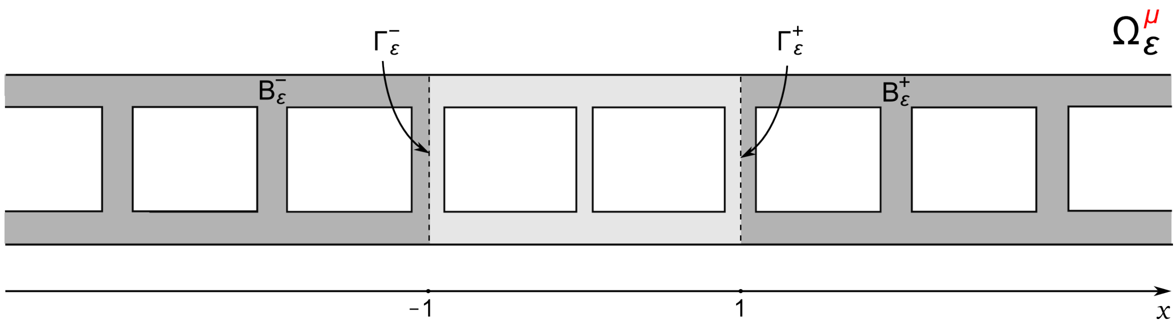


Figure 3.1: The half-bands B^+_ε and B^-_ε

the following function spaces:

$$H^1_\Delta\left(B^\pm_\varepsilon\right) = \left\{u \in H^1\left(B^\pm_\varepsilon\right), \quad \Delta u \in L_2\left(B^\pm_\varepsilon\right)\right\},$$

$$H^1_{\Delta,N}\left(B^\pm_\varepsilon\right) = \left\{u \in H^1_\Delta\left(B^\pm_\varepsilon\right), \quad \left.\frac{\partial u}{\partial n}\right|_{\partial B^\pm_\varepsilon \setminus \Gamma^\pm_\varepsilon} = 0\right\}.$$

We denote by $\gamma^\pm_0$ and $\gamma^\pm_1$ the trace maps on $\Gamma^\pm_\varepsilon$:

$$\gamma^\pm_0 \in \mathcal{L}\left(H^1\left(B^\pm_\varepsilon\right), H^{1/2}\left(\Gamma^\pm_\varepsilon\right)\right): \qquad \forall u \in H^1\left(B^\pm_\varepsilon\right), \quad \gamma^\pm_0 u = u|_{\Gamma^\pm_\varepsilon},$$

$$\gamma^\pm_1 \in \mathcal{L}\left(H^1_\Delta\left(B^\pm_\varepsilon\right), H^{-1/2}\left(\Gamma^\pm_\varepsilon\right)\right): \qquad \forall u \in H^1_\Delta\left(B^\pm_\varepsilon\right), \quad \gamma^\pm_1 u = \mp \left.\frac{\partial u}{\partial x}\right|_{\Gamma^\pm_\varepsilon}.$$

Let us consider the operators $A^\pm_\varepsilon$ defined by the relations

$$A^\pm_\varepsilon u = -\Delta u, \qquad D(A^\pm_\varepsilon) = \left\{u \in H^1_{\Delta,N}\left(B^\pm_\varepsilon\right), \quad u|_{\Gamma^\pm_\varepsilon} = 0\right\}. \tag{3.2.2}$$

Obviously, the two operators are unitarily-equivalent. Let $\mathcal{U}$ be the unitary operator from $L_2(B_\varepsilon^+)$ to $L_2(B_\varepsilon^-)$ defined on smooth functions by the relation

$$(\mathcal{U}u)(x,y) = u(-x,y), \qquad (x,y) \in B_\varepsilon^-,$$

and extended by continuity to $L_2(B_\varepsilon^+)$. Then,

$$D(A_\varepsilon^-) = \mathcal{U}D(A_\varepsilon^+), \qquad A_\varepsilon^- = \mathcal{U}A_\varepsilon^+\mathcal{U}^{-1}.$$

In fact, we do not have to distinguish between the two operators since the difference is purely geometric. Even if we still keep the symbols $\pm$ in the notation in order to point out this geometric difference, all the results will be proved for the operator A_ε^+, the analogous results for the operator A_ε^- being obvious. We will use the notation

$$d^\pm(\alpha) = \operatorname{dist}\left(\alpha^2, \sigma(A_\varepsilon^\pm)\right).$$

As before, A_ε stands for the non-perturbed operator defined in Section 1.2.1 with empty discrete spectrum (cf. (1.2.1)).

Lemma 3.2.1.

$$\sigma(A_\varepsilon^\pm) = \sigma(A_\varepsilon). \tag{3.2.3}$$

Proof. Let $\mathcal{P}$ be the operator of continuation by antisymmetry from $L_2(B_\varepsilon^+)$ to $L_2(\Omega_\varepsilon)$ defined on smooth functions as

$$(\mathcal{P}u)(x,y) = \begin{cases} u(x,y), & (x,y) \in B_\varepsilon^+, \\ -u(1-x,y), & (x,y) \in \Omega_\varepsilon \setminus B_\varepsilon^+, \end{cases}$$

and extended by continuity to $L_2(B_\varepsilon^+)$. Then, the operator $\mathcal{P}$ transforms an eigenfunction of the operator A_ε^+ into an eigenfunction of the operator A_ε and any singular sequence of the operator A_ε^+ into a singular sequence of the operator A_ε, which proves the inclusion $\sigma(A_\varepsilon^+) \subset \sigma(A_\varepsilon)$. Conversely, the operator $\widetilde{\mathcal{P}}$ from $L_2(\Omega_\varepsilon)$ to $L_2(B_\varepsilon^+)$ defined on smooth functions as

$$(\widetilde{\mathcal{P}}u)(x,y) = u(x,y) - u(1-x,y), \qquad (x,y) \in B_\varepsilon^+,$$

and extended by continuity to $L_2(\Omega_\varepsilon)$ transforms an eigenfunction of the operator A_ε into an eigenfunction of the operator A_ε^+ and any singular sequence of the operator A_ε into a singular sequence of the operator A_ε^+. This proves the inclusion $\sigma(A_\varepsilon) \subset \sigma(A_\varepsilon^+)$. □

For any $\alpha \in \mathbb{R}$, $\varphi \in H^{1/2}(\Gamma_\varepsilon^\pm)$ we consider the following half-band problems for $u_\varepsilon^\pm \in H^1_{\Delta,N}(B_\varepsilon^\pm)$:

$$\begin{cases} \Delta u_\varepsilon^\pm + \alpha^2 u_\varepsilon^\pm = 0 \ \text{ in } \ B_\varepsilon^\pm, \\ u_\varepsilon^\pm|_{\Gamma_\varepsilon^\pm} = \varphi. \end{cases} \tag{3.2.4}$$

The following statement establishes the well-posedness of these problems.

Proposition 3.2.1. *For any $\alpha^2 \notin \sigma(A_\varepsilon)$, $\varphi \in H^{1/2}(\Gamma_\varepsilon^\pm)$ the problems (3.2.4) have unique solutions. Moreover, the following estimates for the norms of the solutions hold:*

$$\|u_\varepsilon^\pm\|_{H^1(B_\varepsilon^\pm)} \leqslant \left(C_1(\alpha) + \frac{C_2(\alpha)}{d^\pm(\alpha)}\right) \|\varphi\|_{H^{1/2}(\Gamma_\varepsilon^\pm)}, \tag{3.2.5}$$

where $C_1(\alpha)$, $C_2(\alpha)$ are continuous functions of α depending only on the domains $B_\varepsilon^\pm$.

Proof. Let us denote by $\widehat{\varphi}$ a lift function belonging to $H^1\left(B_\varepsilon^+\right)$ such that $\widehat{\varphi}|_{\Gamma_\varepsilon^+} = \varphi$ and

$$\|\widehat{\varphi}\|_{H^1\left(B_\varepsilon^+\right)} \leqslant C_+ \|\varphi\|_{H^{1/2}\left(\Gamma_\varepsilon^+\right)}, \tag{3.2.6}$$

where the constant C_+ depends only on the domain B_ε^+. We also introduce the following function space:

$$H_D^1\left(B_\varepsilon^+\right) = \left\{u \in H^1\left(B_\varepsilon^+\right), \quad u|_{\Gamma_\varepsilon^+} = 0\right\},$$

which we equip with H^1-norm. Multiplying the first line of (3.2.4) by a test function $v \in H_D^1\left(B_\varepsilon^+\right)$ and changing the unknown function by

$$\widehat{u} = u_\varepsilon^+ - \widehat{\varphi}, \tag{3.2.7}$$

we get the following variational problem for $\widehat{u} \in H_D^1\left(B_\varepsilon^+\right)$:

$$(\nabla \widehat{u}, \nabla v)_{L_2\left(B_\varepsilon^+\right)} - \alpha^2 (\widehat{u}, v)_{L_2\left(B_\varepsilon^+\right)} = -(\nabla \widehat{\varphi}, \nabla v)_{L_2\left(B_\varepsilon^+\right)} + \alpha^2 (\widehat{\varphi}, v)_{L_2\left(B_\varepsilon^+\right)}, \quad \forall v \in H_D^1\left(B_\varepsilon^+\right). \tag{3.2.8}$$

The right-hand side of (3.2.8) is a continuous linear functional in v with respect to the norm H^1:

$$\left| -(\nabla \widehat{\varphi}, \nabla v)_{L_2\left(B_\varepsilon^+\right)} + \alpha^2 (\widehat{\varphi}, v)_{L_2\left(B_\varepsilon^+\right)} \right| \leqslant C_\varphi(\alpha) \|v\|_{H^1\left(B_\varepsilon^+\right)}, \qquad \forall v \in H_D^1\left(B_\varepsilon^+\right),$$

where

$$C_\varphi(\alpha) = (1+\alpha^2) \|\widehat{\varphi}\|_{H^1\left(B_\varepsilon^+\right)}. \tag{3.2.9}$$

Hence, there exists a function $f \in H_D^1\left(B_\varepsilon^+\right)$ such that

$$-(\nabla \widehat{\varphi}, \nabla v)_{L_2\left(B_\varepsilon^+\right)} + \alpha^2 (\widehat{\varphi}, v)_{L_2\left(B_\varepsilon^+\right)} = (f, v)_{H^1\left(B_\varepsilon^+\right)}, \qquad \forall v \in H_D^1\left(B_\varepsilon^+\right). \tag{3.2.10}$$

Besides, $\|f\|_{H^1\left(B_\varepsilon^+\right)} \leqslant C_\varphi(\alpha)$. Consequently, the problem (3.2.8) can be rewritten as

$$(\widehat{u}, v)_{H^1\left(B_\varepsilon^+\right)} - (1+\alpha^2) (\widehat{u}, v)_{L_2\left(B_\varepsilon^+\right)} = (f, v)_{H^1\left(B_\varepsilon^+\right)}, \qquad \forall v \in H_D^1\left(B_\varepsilon^+\right). \tag{3.2.11}$$

Due to the estimate

$$\left|(u, v)_{L_2\left(B_\varepsilon^+\right)}\right| \leqslant \|u\|_{H^1\left(B_\varepsilon^+\right)} \|v\|_{H^1\left(B_\varepsilon^+\right)}, \qquad \forall u, v \in H_D^1\left(B_\varepsilon^+\right),$$

we can introduce the bounded self-adjoint operator $K(\alpha) \in \mathcal{L}\left(H_D^1\left(B_\varepsilon^+\right)\right)$ defined by the relation

$$(\widehat{u}, v)_{H^1\left(B_\varepsilon^+\right)} - (1+\alpha^2) (\widehat{u}, v)_{L_2\left(B_\varepsilon^+\right)} = (K(\alpha)\widehat{u}, v)_{H^1\left(B_\varepsilon^+\right)}, \qquad \forall v \in H_D^1\left(B_\varepsilon^+\right). \tag{3.2.12}$$

Finally, the problem (3.2.11) takes the form

$$K(\alpha)\widehat{u} = f. \tag{3.2.13}$$

We remark that this problem is equivalent to the initial problem (3.2.4). Indeed, we have just shown that if u_ε^+ is a solution of (3.2.4), then the function $\widehat{u}$ defined in (3.2.7) is a solution of (3.2.13) with f defined in (3.2.10). Conversely, if $\widehat{u}$ is a solution of (3.2.13) with f defined in (3.2.10), then, repeating the argument in the other sense, we arrive at the variational formulation (3.2.8) which is equivalent to (3.2.4) with the change of unknown function (3.2.7). Therefore, in order to establish the well-posedness of (3.2.4) it is enough to prove the well-posedness of (3.2.13). Let us denote $d_K(\alpha) = \text{dist}\,(0, \sigma\,(K(\alpha)))$. We will use the result proved in Lemma 3.2.2 below which states that

$$d_K(\alpha) \geqslant \frac{d(\alpha)}{d(\alpha) + 1 + \alpha^2}. \tag{3.2.14}$$

Since by assumption $\alpha^2 \notin \sigma\,(A_\varepsilon)$, due to Lemma 3.2.1 we have $\alpha^2 \notin \sigma\,(A_\varepsilon^+)$. Hence, $d(\alpha) > 0$ and (3.2.14) implies that $d_K(\alpha) > 0$. This shows that the problem (3.2.13) is well-posed. Moreover, the norm of its solution can be estimated as follows:

$$\|\widehat{u}\|_{H^1(B_\varepsilon^+)} \leqslant \|K^{-1}\|_{H^1(B_\varepsilon^+) \to H^1(B_\varepsilon^+)} \|f\|_{H^1(B_\varepsilon^+)} \leqslant \frac{C_\varphi(\alpha)}{d_K(\alpha)}. \tag{3.2.15}$$

Putting together the relations (3.2.7), (3.2.6), (3.2.9), (3.2.14), (3.2.15) we get:

$$\|u_\varepsilon^+\|_{H^1(B_\varepsilon^+)} \leqslant \left(C_+ + 1 + \alpha^2 + \frac{(1+\alpha^2)^2}{d(\alpha)}\right) \|\varphi\|_{H^{1/2}(\Gamma_\varepsilon^+)}.$$

This is exactly the estimate (3.2.5) with $C_1(\alpha) = C_+ + 1 + \alpha^2$, $C_2(\alpha) = (1+\alpha^2)^2$. ☐

Remark 3.2.1. The uniqueness of solutions of the problems (3.2.4) is obvious since Lemma 3.2.1 implies that the operators $A_\varepsilon^\pm$ have no eigenvalues. The goal of the above proof is to establish the existence. However, the argument used in the proof establishes the existence and the uniqueness simultaneously (the reason for which we talk about well-posedness).

Remark 3.2.2. The proof of Proposition 3.2.1 would be more straightforward for $\varphi \in H^{3/2}\,(\Gamma_\varepsilon^\pm)$. In this case it would be unnecessary to pass to the weak formulation and the well-posedness would be directly guaranteed by the fact that $\alpha^2 \notin \sigma\,(A_\varepsilon^\pm)$. However, we need to consider $\varphi \in H^{1/2}\,(\Gamma_\varepsilon^\pm)$ since the problem will be discretized by P1 finite elements (cf. section 3.3) which implies working with H^1 functions, and, consequently, $H^{1/2}$ on the boundary.

Lemma 3.2.2. *In the notation of the proof of Proposition 3.2.1 the estimate* (3.2.14) *holds.*

Proof. If $d_K(\alpha) \geqslant 1$ the result is obvious. Suppose that $d_K(\alpha) < 1$. Then, since $\sigma(K(\alpha))$ is a closed subset of $\mathbb{R}$, there exists $\gamma \in \mathbb{R}$ such that $|\gamma| = d_k(\alpha)$ and $\gamma \in \sigma(K(\alpha))$ (in other words, either $d_K(\alpha)$ or $-d_k(\alpha)$ belongs to $\sigma(K(\alpha))$). The operator $K(\alpha)$ being self-adjoint, this means that there exists a singular sequence $\{u_n\}_{n\in\mathbb{N}} \subset H_D^1(B_\varepsilon^+)$ such that

1. $\|u_n\|_{H^1(B_\varepsilon^+)} = 1, \quad n \in \mathbb{N},$
2. $\|(K(\alpha) - \gamma I)\, u_n\|_{H^1(B_\varepsilon^+)} \to 0, \quad n \to \infty,$

where I is the identity operator in $H^1(B_\varepsilon^+)$. This implies that for any $\delta > 0$ there exists $u_\delta \in H_D^1(B_\varepsilon^+)$, $\|u_\delta\|_{H^1(B_\varepsilon^+)} = 1$, such that $\|(K(\alpha) - \gamma I)\, u_\delta\|_{H^1(B_\varepsilon^+)} \leqslant \delta$. Hence,

$$\begin{aligned} \left|((K(\alpha) - \gamma I)\, u_\delta, v)_{H^1(B_\varepsilon^+)}\right| &\leqslant \|(K(\alpha) - \gamma I)\, u_\delta\|_{H^1(B_\varepsilon^+)}\, \|v\|_{H^1(B_\varepsilon^+)} \\ &\leqslant \delta \|u_\delta\|_{H^1(B_\varepsilon^+)} \|v\|_{H^1(B_\varepsilon^+)}, \qquad \forall v \in H_D^1(B_\varepsilon^+). \end{aligned} \tag{3.2.16}$$

Using the definition (3.2.12) of the operator $K(\alpha)$, we get:

$$\left|(u_\delta, v)_{H^1(B_\varepsilon^+)} - \frac{1+\alpha^2}{1-\gamma}(u_\delta, v)_{L_2(B_\varepsilon^+)}\right| \leqslant \frac{\delta\|u_\delta\|_{H^1(B_\varepsilon^+)}\|v\|_{H^1(B_\varepsilon^+)}}{1-\gamma}, \qquad \forall v \in H_D^1(B_\varepsilon^+).$$

The sesquilinear form $a[u,v] = (u_\delta, v)_{H^1(B_\varepsilon^+)}$ defined on $D[a] = H_D^1(B_\varepsilon^+)$ is the unique sesquilinear form corresponding to the operator $A_\varepsilon^+ + I$. It obviously satisfies the estimate $a[u,u] \geqslant \|u\|^2_{L_2(B_\varepsilon^+)}$, $\forall u \in D[a]$, which permits us to apply Lemma A.2.1. Thus, we can conclude that

$$\operatorname{dist}\left(\sigma\left(A_\varepsilon^+ + I\right), \frac{1+\alpha^2}{1-\gamma}\right) < C\delta, \qquad \forall \delta > 0.$$

Since δ can be chosen arbitrarily small, one gets $\dfrac{1+\alpha^2}{1-\gamma} \in \sigma(A_\varepsilon^+ + I)$, which is equivalent to $\dfrac{\gamma+\alpha^2}{1-\gamma} \in \sigma(A_\varepsilon^+)$. This means that $d \leqslant \dfrac{|\gamma|(1+\alpha^2)}{1-\gamma} \leqslant \dfrac{d_K(\alpha)(1+\alpha^2)}{1-d_K(\alpha)}$. This yields (3.2.14). □

From now on $u_\varepsilon^\pm(\alpha,\varphi)$ will stand for the unique solutions of the problems (3.2.4) for $\alpha^2 \notin \sigma(A_\varepsilon)$, $\varphi \in H^{1/2}(\Gamma_\varepsilon^\pm)$.

3.2.2 The DtN operators $\Lambda^\pm$

We define the DtN operators $\Lambda^\pm : H^{1/2}(\Gamma_\varepsilon^\pm) \to H^{-1/2}(\Gamma_\varepsilon^\pm)$ for $\alpha^2 \notin \sigma(A_\varepsilon)$ as follows:

$$\left\langle \Lambda^\pm(\alpha)\varphi, \psi\right\rangle = \int\limits_{B_\varepsilon^\pm} \nabla u_\varepsilon^\pm(\alpha,\varphi)\nabla u_\varepsilon^\pm(\alpha,\psi) - \alpha^2 \int\limits_{B_\varepsilon^\pm} u_\varepsilon^\pm(\alpha,\varphi)u_\varepsilon^\pm(\alpha,\psi), \qquad \forall \varphi,\psi \in H^{1/2}\left(\Gamma_\varepsilon^\pm\right). \tag{3.2.17}$$

Here $u_\varepsilon^\pm(\alpha,\varphi)$ is the solution of the problem (3.2.4) and $\langle\cdot,\cdot\rangle$ stands for the duality brackets between $H^{-1/2}(\Gamma_\varepsilon^\pm)$ and $H^{1/2}(\Gamma_\varepsilon^\pm)$. In other words,

$$\Lambda^\pm(\alpha)\varphi = \left.\frac{\partial u_\varepsilon^\pm(\alpha,\varphi)}{\partial n}\right|_{\Gamma_\varepsilon^\pm}, \qquad \forall \varphi \in H^{1/2}\left(\Gamma_\varepsilon^\pm\right). \tag{3.2.18}$$

The following assertion which states the norm continuity of the DtN operators with respect to α will be used in Proposition 3.2.5 in order to show the continuity of the functions to which a Newton type algorithm will be applied.

Proposition 3.2.2. *The operators $\Lambda^\pm(\alpha)$ defined by* (3.2.17) *are continuous from $H^{1/2}(\Gamma_\varepsilon^\pm)$ to $H^{-1/2}(\Gamma_\varepsilon^\pm)$ and norm-continuous with respect to α:*

$$\left\|\Lambda^\pm(\alpha_1) - \Lambda^\pm(\alpha_2)\right\|_{H^{1/2}\left(\Gamma_\varepsilon^\pm\right)\to H^{-1/2}\left(\Gamma_\varepsilon^\pm\right)} \leqslant C\left|\alpha_1^2 - \alpha_2^2\right|. \tag{3.2.19}$$

Proof. Due to the estimate (3.2.5) one finds

$$\left|\left\langle\Lambda^\pm(\alpha)\varphi,\psi\right\rangle\right| \leqslant (1+\alpha^2)\left(C_1(\alpha) + \frac{C_2(\alpha)}{d^\pm(\alpha)}\right)^2 \|\varphi\|_{H^{1/2}\left(\Gamma_\varepsilon^\pm\right)}\|\psi\|_{H^{1/2}\left(\Gamma_\varepsilon^\pm\right)}.$$

Consequently,

$$\left\|\Lambda^\pm(\alpha)\right\|_{H^{1/2}\left(\Gamma_\varepsilon^\pm\right)\to H^{-1/2}\left(\Gamma_\varepsilon^\pm\right)} \leqslant (1+\alpha^2)\left(C_1(\alpha) + \frac{C_2(\alpha)}{d^\pm(\alpha)}\right)^2.$$

If $\alpha_1, \alpha_2 \notin \sigma(A_\varepsilon^\pm)$ then one can estimate the norm of the difference $\Lambda^\pm(\alpha_1) - \Lambda^\pm(\alpha_2)$. Let us denote $\widetilde{u}_\varepsilon^\pm(\varphi) = u_\varepsilon^\pm(\alpha_1, \varphi) - u_\varepsilon^\pm(\alpha_2, \varphi)$. Then for any $\varphi, \psi \in H^{1/2}\left(\Gamma_\varepsilon^\pm\right)$ we have:

$$\begin{aligned}\left\langle \left(\Lambda^\pm(\alpha_1) - \Lambda^\pm(\alpha_2)\right)\varphi, \psi\right\rangle &= \int\limits_{B_\varepsilon^\pm} \left(\nabla\widetilde{u}_\varepsilon^\pm(\varphi)\nabla u_\varepsilon^\pm(\alpha_1,\psi) + \nabla u_\varepsilon^\pm(\alpha_2,\varphi)\nabla\widetilde{u}_\varepsilon^\pm(\psi)\right)\\ - \alpha_1^2 \int\limits_{B_\varepsilon^\pm} &\left(\widetilde{u}_\varepsilon^\pm(\varphi)u_\varepsilon^\pm(\alpha_1,\psi) + u_\varepsilon^\pm(\alpha_2,\varphi)\widetilde{u}_\varepsilon^\pm(\psi)\right) + \left(\alpha_2^2 - \alpha_1^2\right)\int\limits_{B_\varepsilon^\pm} u_\varepsilon^\pm(\alpha_2,\varphi)u_\varepsilon^\pm(\alpha_2,\psi).\end{aligned} \tag{3.2.20}$$

Remark that $\widetilde{u}_\varepsilon^\pm(\varphi)$ solves the problem

$$A_\varepsilon^\pm\widetilde{u}_\varepsilon^\pm(\varphi) - \alpha_1^2\widetilde{u}_\varepsilon^\pm(\varphi) = \left(\alpha_1^2 - \alpha_2^2\right)u_\varepsilon^\pm(\alpha_2,\varphi). \tag{3.2.21}$$

Therefore, its L_2-norm can be estimated as follows:

$$\left\|\widetilde{u}_\varepsilon^\pm(\varphi)\right\|_{L_2(B_\varepsilon^\pm)} \leqslant \frac{|\alpha_1^2 - \alpha_2^2|\,\|u_\varepsilon^\pm(\alpha_2,\varphi)\|_{L_2(B_\varepsilon^\pm)}}{d^\pm(\alpha_1)}. \tag{3.2.22}$$

Multiplying (3.2.21) by $\widetilde{u}_\varepsilon^\pm(\varphi)$ and integrating by parts yields

$$\left\|\nabla\widetilde{u}_\varepsilon^\pm(\varphi)\right\|^2_{L_2(B_\varepsilon^\pm)} = \alpha_1^2\left\|\widetilde{u}_\varepsilon^\pm(\varphi)\right\|^2_{L_2(B_\varepsilon^\pm)} + \left(\alpha_1^2 - \alpha_2^2\right)\left(u_\varepsilon^\pm(\alpha_2,\varphi), \widetilde{u}_\varepsilon^\pm(\varphi)\right)_{L_2(B_\varepsilon^\pm)}.$$

Thus, taking into account (3.2.22), on obtains:

$$\left\|\widetilde{u}_\varepsilon^\pm(\varphi)\right\|_{H^1(B_\varepsilon^\pm)} \leqslant \left(\alpha_1^2 + 1 + d^\pm(\alpha_1)\right)^{1/2}\frac{|\alpha_1^2 - \alpha_2^2|\,\|u_\varepsilon^\pm(\alpha_2,\varphi)\|_{L_2(B_\varepsilon^\pm)}}{d^\pm(\alpha_1)}.$$

In other terms, the operator $(A_\varepsilon^\pm - \alpha_1^2)^{-1}$ is continuous from $L_2(B_\varepsilon^\pm)$ to $H^1(B_\varepsilon^\pm)$. Finally, using (3.2.5) we conclude that

$$\left\|\widetilde{u}_\varepsilon^\pm(\varphi)\right\|_{H^1(B_\varepsilon^\pm)} \leqslant C(\alpha_1,\alpha_2)\frac{|\alpha_1^2 - \alpha_2^2|\,\|\varphi\|_{H^{1/2}(\Gamma_\varepsilon^\pm)}}{d^\pm(\alpha_1)d^\pm(\alpha_2)}, \qquad \forall\varphi \in H^{1/2}\left(\Gamma_\varepsilon^\pm\right),$$

where $C(\alpha_1,\alpha_2)$ is a continuous function of α_1, α_2. Notice that it depends on $d^\pm(\alpha_1)$, $d^\pm(\alpha_2)$. Coming back to (3.2.20) we get the following estimate for any φ, $\psi \in H^{1/2}(\Gamma_\varepsilon^\pm)$:

$$\begin{aligned}&\left|\left\langle\left(\Lambda^\pm(\alpha_1) - \Lambda^\pm(\alpha_2)\right)\varphi,\psi\right\rangle\right|\\ &\qquad\leqslant C(\alpha_1,\alpha_2)\frac{|\alpha_1^2 - \alpha_2^2|\,\|\varphi\|_{H^{1/2}(\Gamma_\varepsilon^\pm)}\,\|\psi\|_{H^{1/2}(\Gamma_\varepsilon^\pm)}}{d^\pm(\alpha_1)d^\pm(\alpha_2)}\left(1 + \frac{1}{d^\pm(\alpha_1)} + \frac{1}{d^\pm(\alpha_2)}\right).\end{aligned}$$

Here $C(\alpha_1,\alpha_2)$ stands for some continuous function of α_1, α_2 (not necessarily the same as above). The obtained estimate implies that if $d^\pm(\alpha_1), d^\pm(\alpha_2) \geqslant c_0 > 0$ then the inequality (3.2.19) holds. □

We will need the following technical result.

Lemma 3.2.3 (Gårding's inequality for the DtN operators)**.**
For $\alpha^2 \notin \sigma(A_\varepsilon)$ there exist a constant C_1 and a continuous function $C_2(\alpha)$ depending only on the geometry of the domain such that

$$\langle\Lambda^\pm(\alpha)\varphi,\varphi\rangle \geqslant C_1\|\varphi\|^2_{H^{1/2}(\Gamma_\varepsilon^\pm)} - C_2(\alpha)\|\varphi\|^2_{L_2(\Gamma_\varepsilon^\pm)}, \qquad \forall\varphi \in H^{1/2}\left(\Gamma_\varepsilon^\pm\right). \tag{3.2.23}$$

Proof. From (3.2.17) we get:

$$\langle \Lambda^{\pm}(\alpha)\varphi, \varphi\rangle = \left\| u_\varepsilon^{\pm}(\alpha,\varphi)\right\|^2_{H^1\left(B_\varepsilon^{\pm}\right)} - (1+\alpha^2)\left\| u_\varepsilon^{\pm}(\alpha,\varphi)\right\|^2_{L_2\left(B_\varepsilon^{\pm}\right)}.$$

The trace applications $\gamma_0^{\pm}$ being continuous, this implies that

$$\langle \Lambda^{\pm}(\alpha)\varphi, \varphi\rangle \geqslant C_1 \left\|\varphi\right\|^2_{H^{1/2}\left(\Gamma_\varepsilon^{\pm}\right)} - (1+\alpha^2)\left\| u_\varepsilon^{\pm}(\alpha,\varphi)\right\|^2_{L_2\left(B_\varepsilon^{\pm}\right)}. \tag{3.2.24}$$

In order to estimate the L_2-norm of the solutions $u_\varepsilon^{\pm}(\alpha,\varphi)$ let us consider the following problems with $u_\varepsilon^{\pm}(\alpha,\varphi)$ being source terms:

$$A_\varepsilon^{\pm} v^{\pm} - \alpha^2 v^{\pm} = u_\varepsilon^{\pm}(\alpha,\varphi). \tag{3.2.25}$$

Since $\alpha^2 \notin \sigma\left(A_\varepsilon^{\pm}\right)$ this problems have unique solutions in $D\left(A_\varepsilon^{\pm}\right)$. Multiplying (3.2.25) by $u_\varepsilon^{\pm}(\alpha,\varphi)$ and using Green's formula we get:

$$\left\| u_\varepsilon^{\pm}(\alpha,\varphi)\right\|^2_{L_2\left(B_\varepsilon^{\pm}\right)} = -\left\langle \frac{\partial v^{\pm}}{\partial n}, \varphi\right\rangle. \tag{3.2.26}$$

The domains $B_\varepsilon^{\pm}$ are not convex, which makes it impossible to use the argument of global regularity of weak solutions. Nevertheless, it is still possible to use the local regularity near the boundaries that do not contain reentrant angles. More precisely, let $K^{\pm}$ stand for the rectangles $\left]1, 1+\frac{\varepsilon}{8}\right[\times \left]-\frac{L}{2}, \frac{L}{2}\right[$ and $\left]-1-\frac{\varepsilon}{8}, -1, \right[\times \left]-\frac{L}{2}, \frac{L}{2}\right[$ respectively. Then there exists a continuous function $C(\alpha)$ depending only on the geometry of the domain such that

$$\left\| v^{\pm}\right\|_{H^2(K^{\pm})} \leqslant C(\alpha)\left\| u_\varepsilon^{\pm}\right\|_{L_2\left(B_\varepsilon^{\pm}\right)}. \tag{3.2.27}$$

We detail the proof of this fact in Lemma A.3.3 given in Annexe. Then, the continuity of the trace applications $v \to \left.\dfrac{\partial v}{\partial n}\right|_{\Gamma_\varepsilon^{\pm}}$ defined as operators from $H^2\left(K^{\pm}\right)$ to $H^{1/2}\left(\Gamma_\varepsilon^{\pm}\right)$, implies that

$$\left\| \frac{\partial v^{\pm}}{\partial n}\right\|_{H^{1/2}\left(\Gamma_\varepsilon^{\pm}\right)} \leqslant C(\alpha)\left\| u_\varepsilon^{\pm}\right\|_{L_2\left(B_\varepsilon^{\pm}\right)}.$$

Here we denote by $C(\alpha)$ any continuous function that depends only on the geometry of the domain (without changing the notation even if its value changes). Thus, coming back to (3.2.26), we get:

$$\left\| u_\varepsilon^{\pm}(\alpha,\varphi)\right\|_{L_2\left(B_\varepsilon^{\pm}\right)} \leqslant C(\alpha)\left\|\varphi\right\|_{H^{1/2}\left(\Gamma_\varepsilon^{\pm}\right)}.$$

Injecting this estimate in (3.2.24) yields (3.2.23). □

3.2.3 The interior problem

We can now state the following problem in $\mathcal{C}_\varepsilon^{\mu}$ (which is a bounded domain corresponding to the perturbed periodicity cell, cf. figure 3.2). It consists in finding values λ_ε such that there exists a function $u_\varepsilon^0 \in H^1_\Delta(\mathcal{C}_\varepsilon^{\mu})$ solving the problem

$$\begin{cases} -\Delta u_\varepsilon^0 = \lambda_\varepsilon^2 u_\varepsilon^0 \ \text{ in } \ \mathcal{C}_\varepsilon^{\mu}, \\ \left.\dfrac{\partial u_\varepsilon^0}{\partial n}\right|_{\Gamma_\varepsilon^{\pm}} + \Lambda^{\pm}(\lambda_\varepsilon)\left. u_\varepsilon^0\right|_{\Gamma_\varepsilon^{\pm}} = 0, \\ \left.\dfrac{\partial u_\varepsilon^0}{\partial n}\right|_{\partial\mathcal{C}_\varepsilon^{\mu}\setminus\left\{\Gamma_\varepsilon^{+}\cup\Gamma_\varepsilon^{-}\right\}} = 0. \end{cases} \tag{3.2.28}$$

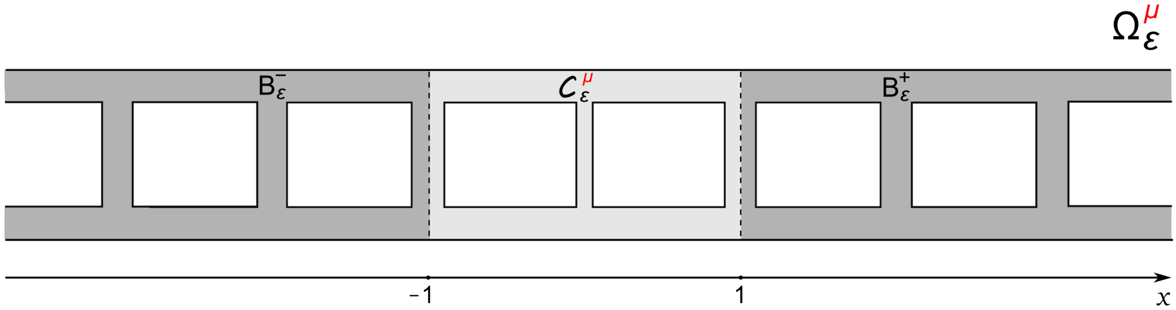


Figure 3.2: The interior domain $\mathcal{C}_\varepsilon^\mu$

Proposition 3.2.3. *If $\lambda_\varepsilon^2 \notin \sigma(A_\varepsilon)$ then the problem* (3.2.28) *is equivalent to the problem* (3.2.1).

Proof. Suppose first that u_ε solves the problem (3.2.1) for some λ_ε. Let us denote $u_\varepsilon^\pm = u_\varepsilon|_{B_\varepsilon^\pm}$, $u_\varepsilon^0 = u_\varepsilon|_{\mathcal{C}_\varepsilon^\mu}$. Then, $u_\varepsilon^\pm$ are the unique solutions of (3.2.4) for φ replaced by $u_\varepsilon|_{\Gamma_\varepsilon^\pm}$ and $\alpha = \lambda_\varepsilon$. Therefore, $\Lambda^\pm(\lambda_\varepsilon)\, u_\varepsilon^\pm|_{\Gamma_\varepsilon^\pm} = \left.\dfrac{\partial u_\varepsilon^\pm}{\partial n}\right|_{\Gamma_\varepsilon^\pm}$ by definition of the DtN operators (cf. (3.2.18)). On the other hand, since $u_\varepsilon \in H^1_\Delta(\Omega_\varepsilon^\mu)$, both its traces and the traces of its normal derivative are continuous: $u_\varepsilon^\pm|_{\Gamma_\varepsilon^\pm} = u_\varepsilon^0|_{\Gamma_\varepsilon^\pm}$, $\left.\dfrac{\partial u_\varepsilon^\pm}{\partial n}\right|_{\Gamma_\varepsilon^\pm} = -\left.\dfrac{\partial u_\varepsilon^0}{\partial n}\right|_{\Gamma_\varepsilon^\pm}$. This implies that u_ε^0 satisfies the second line of (3.2.28). It also satisfies the first and the last lines (the equation and Neumann boundary condition) since it is a restriction of u_ε which solves the initial problem (3.2.1).

Conversely, suppose that u_ε^0 is a solution of (3.2.28). Let us denote by $u_\varepsilon^\pm$ the unique solutions of (3.2.4) with α replaced by λ_ε and φ replaced by $u_\varepsilon^0|_{\Gamma_\varepsilon^\pm}$. Then, the function constructed as

$$u_\varepsilon(x,y) = \begin{cases} u_\varepsilon^\pm(x,y), & (x,y) \in B_\varepsilon^\pm, \\ u_\varepsilon^0(x,y), & (x,y) \in \mathcal{C}_\varepsilon^\mu, \end{cases}$$

solves (3.2.1) for $\alpha = \lambda_\varepsilon$. Indeed, its traces on $\Gamma_\varepsilon^\pm$ are continuous by definition of $u_\varepsilon^\pm$. The continuity of the normal derivative across $\Gamma_\varepsilon^\pm$ is guaranteed by definition of the DtN operators. Hence, u_ε constructed in such a way appears to be a function in $H^1_\Delta(\mathcal{C}_\varepsilon^\mu)$. It satisfies the equation in first line of (3.2.1) since the same equation is satisfied by u_ε^0, $u_\varepsilon^\pm$. Finally, it satisfies Neumann boundary conditions on $\partial\Omega_\varepsilon^\mu$, which follows from Neumann boundary conditions for u_ε^0, $u_\varepsilon^\pm$ on the respective boundaries. □

We see that the initial eigenvalue problem (3.2.1) posed in the (unbounded) domain Ω_ε^μ can be replaced by an equivalent problem (3.2.28) posed in the (bounded) domain $\mathcal{C}_\varepsilon^\mu$. However, this problem is a nonlinear one since the DtN operators appearing in the boundary conditions on $\Gamma_\varepsilon^\pm$ depend themselves on the spectral parameter.

Let us now study the nonlinear problem (3.2.28). For $\alpha^2 \notin \sigma(A_\varepsilon)$ we introduce the operator $A_\varepsilon^0(\alpha)$ defined as follows:

$$\begin{aligned} &A_\varepsilon^0(\alpha)u = -\Delta u, \\ &D\left(A_\varepsilon^0(\alpha)\right) = \left\{ u \in H^1_\Delta\left(\mathcal{C}_\varepsilon^\mu\right), \quad \left.\frac{\partial u}{\partial n}\right|_{\Gamma_\varepsilon^\pm} + \Lambda(\alpha)^\pm\, u|_{\Gamma_\varepsilon^\pm} = 0, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\mathcal{C}_\varepsilon^\mu \setminus \{\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-\}} = 0 \right\}. \end{aligned} \tag{3.2.29}$$

We will show some important properties of this operator.

Proposition 3.2.4. *For $\alpha^2 \notin \sigma(A_\varepsilon)$ the operator $A^0_\varepsilon(\alpha)$ defined in* (3.2.29) *is self-adjoint and bounded from below. Moreover, its resolvent is compact.*

Proof. The fact that the operator $A^0_\varepsilon(\alpha)$ is self-adjoint follows directly from the fact that the DtN operators are symmetric. Let us show that it is bounded from below. Indeed, for any $u \in D\left(A^0_\varepsilon(\alpha)\right)$ one has:

$$\begin{aligned}\left(A^0_\varepsilon(\alpha)u, u\right)_{L_2\left(\mathcal{C}^\mu_\varepsilon\right)} &= \|\nabla u\|^2_{L_2\left(\mathcal{C}^\mu_\varepsilon\right)} + \left\langle \Lambda^+(\alpha)\, u|_{\Gamma^+_\varepsilon}, u|_{\Gamma^+_\varepsilon}\right\rangle + \left\langle \Lambda^-(\alpha)\, u|_{\Gamma^-_\varepsilon}, u|_{\Gamma^-_\varepsilon}\right\rangle \\ &\geqslant \|u\|^2_{H^1\left(\mathcal{C}^\mu_\varepsilon\right)} - \|u\|^2_{L_2\left(\mathcal{C}^\mu_\varepsilon\right)} - C_2(\alpha)\left(\left\| u|_{\Gamma^+_\varepsilon}\right\|^2_{L_2\left(\Gamma^+_\varepsilon\right)} + \left\| u|_{\Gamma^-_\varepsilon}\right\|^2_{L_2\left(\Gamma^-_\varepsilon\right)}\right),\end{aligned}$$

where $C_2(\alpha)$ is the constant from Lemma 3.2.3. The L_2-norms of the traces of u on $\Gamma^\pm_\varepsilon$ can be estimated using Lemma A.3.2 (Annexe) applied to the rectangles $\left]-1, -1+\frac{\varepsilon}{4}\right[\times \left]-\frac{L}{2}, \frac{L}{2}\right[$ and $\left]1-\frac{\varepsilon}{4}, 1\right[\times \left]-\frac{L}{2}, \frac{L}{2}\right[$ respectively. Then, for $\delta > 0$ we get:

$$\left(A^0_\varepsilon(\alpha)u, u\right)_{L_2\left(\mathcal{C}^\mu_\varepsilon\right)} \geqslant \left(1 - C_2(\alpha)\delta\right)\|u\|^2_{H^1\left(\mathcal{C}^\mu_\varepsilon\right)} - \left(1 + C_2(\alpha)C(\delta)\right)\|u\|^2_{L_2\left(\mathcal{C}^\mu_\varepsilon\right)}, \quad \forall u \in D\left(A^0_\varepsilon(\alpha)\right).$$

Choosing $\delta < 1/C_2(\alpha)$ one obtains the boundedness from below for the operator $A^0_\varepsilon(\alpha)$. In fact, we got even a stronger inequality than just a lower bound. More precisely, if we fix δ in an appropriate way and put $C(\alpha) = 1 - C_2(\alpha)\delta$, $m(\alpha) = 1 + C_2(\alpha)C(\delta)$, then

$$\left(A^0_\varepsilon(\alpha)u, u\right)_{L_2\left(\mathcal{C}^\mu_\varepsilon\right)} \geqslant C(\alpha)\|u\|^2_{H^1\left(\mathcal{C}^\mu_\varepsilon\right)} - m(\alpha)\|u\|^2_{L_2\left(\mathcal{C}^\mu_\varepsilon\right)}, \qquad \forall u \in D\left(A^0_\varepsilon(\alpha)\right), \quad (3.2.30)$$

where $C(\alpha)$, $m(\alpha) > 0$ are continuous functions of α. It is now easy to see that the resolvent of the operator $A^0_\varepsilon(\alpha)$ is compact. Indeed, let f be a function in $L_2\left(\mathcal{C}^\mu_\varepsilon\right)$ and $u = \left(A^0_\varepsilon(\alpha) + m(\alpha)I\right)^{-1} f$. Then, the estimate (3.2.30) implies that

$$C(\alpha)\|u\|^2_{H^1\left(\mathcal{C}^\mu_\varepsilon\right)} \leqslant (f, u)_{L_2\left(\mathcal{C}^\mu_\varepsilon\right)}.$$

This, in turn, implies that the resolvent $\left(A^0_\varepsilon(\alpha) + m(\alpha)I\right)^{-1}$ is a continuous operator from $L_2\left(\mathcal{C}^\mu_\varepsilon\right)$ to $H^1\left(\mathcal{C}^\mu_\varepsilon\right)$. Due to the compactness of the embedding $H^1\left(\mathcal{C}^\mu_\varepsilon\right) \subset L_2\left(\mathcal{C}^\mu_\varepsilon\right)$ we conclude that the resolvent of the operator $A^0_\varepsilon(\alpha)$ is a compact operator in $L_2\left(\mathcal{C}^\mu_\varepsilon\right)$. □

It follows from the previous theorem that for $\alpha^2 \notin \sigma(A_\varepsilon)$ the spectrum of the operator $A^0_\varepsilon(\alpha)$ is discrete and consists of a sequence of eigenvalues of finite multiplicity tending to infinity:

$$\varkappa_1(\alpha) \leqslant \varkappa_2(\alpha) \leqslant \cdots \leqslant \varkappa_n(\alpha) \leqslant \ldots, \qquad \varkappa_n(\alpha) \xrightarrow[n\to\infty]{} +\infty.$$

Let us consider the following positively defined operator:

$$\widetilde{A}^0_\varepsilon(\alpha) = A^0_\varepsilon(\alpha) + m(\alpha)I.$$

Due to (3.2.30), the boundedness of the DtN operators and the boundedness of the trace operators from $H^1(\mathcal{C}^\mu_\varepsilon)$ to $H^{1/2}(\Gamma^\pm_\varepsilon)$ one has:

$$C(\alpha)\|u\|^2_{H^1(\mathcal{C}^\mu_\varepsilon)} \leqslant \left(\widetilde{A}^0_\varepsilon(\alpha)u, u\right)_{L_2\left(\mathcal{C}^\mu_\varepsilon\right)} \leqslant \widetilde{C}(\alpha)\|u\|^2_{H^1(\mathcal{C}^\mu_\varepsilon)}, \qquad \forall u \in D\left(A^0_\varepsilon(\alpha)\right),$$

with some constant $\widetilde{C}(\alpha)$. This implies that the sesquilinear form $\widetilde{a}^0_\varepsilon(\alpha)$ corresponding to the operator $\widetilde{A}^0_\varepsilon(\alpha)$ is defined on $H^1(\mathcal{C}^\mu_\varepsilon)$ and the norm defined by this form is equivalent to the H^1-norm:

$$D[\widetilde{a}^0_\varepsilon(\alpha)] = H^1(\mathcal{C}^\mu_\varepsilon),$$
$$\widetilde{a}^0_\varepsilon(\alpha)[u,v] = (\nabla u, \nabla v)_{L_2(\mathcal{C}^\mu_\varepsilon)} + \sum_{\delta\in\{+,-\}} \left\langle \Lambda^\delta(\alpha)\, u|_{\Gamma^\delta_\varepsilon}, v|_{\Gamma^\delta_\varepsilon}\right\rangle + m(\alpha)(u,v)_{L_2(\mathcal{C}^\mu_\varepsilon)},$$

$$C(\alpha)\,\|u\|^2_{H^1(\mathcal{C}^\mu_\varepsilon)} \leqslant \widetilde{a}^0_\varepsilon(\alpha)[u,u] \leqslant \widetilde{C}(\alpha)\,\|u\|^2_{H^1(\mathcal{C}^\mu_\varepsilon)}, \qquad \forall u \in H^1(\mathcal{C}^\mu_\varepsilon). \tag{3.2.31}$$

This permits to characterize the eigenvalues $\{\widetilde{\varkappa}_n(\alpha)\}_{n\in\mathbb{N}}$ of the operator $\widetilde{A}^0_\varepsilon(\alpha)$ using the min-max principle as follows:

$$\widetilde{\varkappa}_n(\alpha) = \inf_{\substack{\mathcal{M}\subset H^1(\mathcal{C}^\mu_\varepsilon)\\ \dim\mathcal{M}=n}} \sup_{\substack{u\in\mathcal{M}\\ \|u\|_{L_2(\mathcal{C}^\mu_\varepsilon)}=1}} \widetilde{a}^0_\varepsilon(\alpha)[u,u]. \tag{3.2.32}$$

The following assertion states the continuity of these eigenvalues and, as a consequence, of the eigenvalues $\varkappa_n(\alpha)$, with respect to the parameter α.

Proposition 3.2.5. *The functions $\varkappa_n(\alpha)$ are continuous for $\alpha^2 \notin \sigma(A_\varepsilon)$, $n \in \mathbb{N}$.*

Proof. First, we notice that the spectrum of the operator $A^0_\varepsilon(\alpha)$ can be obtained by a translation by $m(\alpha)$ of the spectrum of the operator $\widetilde{A}^0_\varepsilon(\alpha)$. Hence, the function $m(\alpha)$ being continuous, it is enough to prove the continuity of the functions $\widetilde{\varkappa}_n(\alpha)$. For this we will use (3.2.32). Let us consider the difference $\widetilde{a}^0_\varepsilon(\alpha_1)[u,u] - \widetilde{a}^0_\varepsilon(\alpha_2)[u,u]$ for $\alpha_1^2, \alpha_2^2 \notin \sigma(A_\varepsilon)$ and $u \in H^1(\mathcal{C}^\mu_\varepsilon)$ such that $\|u\|_{L_2(\mathcal{C}^\mu_\varepsilon)} = 1$. One gets:

$$\widetilde{a}^0_\varepsilon(\alpha_1)[u,u]-\widetilde{a}^0_\varepsilon(\alpha_2)[u,u] = \sum_{\delta\in\{+,-\}} \left\langle\left(\Lambda^\delta(\alpha_1) - \Lambda^\delta(\alpha_2)\right)\varphi^\delta, \varphi^\delta\right\rangle + (m(\alpha_1) - m(\alpha_2))\,\|u\|^2_{L_2(\mathcal{C}^\mu_\varepsilon)},$$

where $\varphi^\pm = u|_{\Gamma^\pm_\varepsilon}$. Hence, the estimate (3.2.19) implies that if $d^\pm(\alpha_1), d^\pm(\alpha_2) \geqslant c_0 > 0$ then

$$\left|\widetilde{a}^0_\varepsilon(\alpha_1)[u,u] - \widetilde{a}^0_\varepsilon(\alpha_2)[u,u]\right| \leqslant \left(C\left|\alpha_1^2 - \alpha_2^2\right| + |m(\alpha_1) - m(\alpha_2)|\right)\|u\|^2_{H^1(\mathcal{C}^\mu_\varepsilon)}.$$

The H^1-norm of u can be estimated using (3.2.31) with $\alpha = \alpha_2$, which yields

$$\widetilde{a}^0_\varepsilon(\alpha_2)[u,u](1 - f(\alpha_1,\alpha_2)) \leqslant \widetilde{a}^0_\varepsilon(\alpha_1)[u,u] \leqslant \widetilde{a}^0_\varepsilon(\alpha_2)[u,u](1 + f(\alpha_1,\alpha_2)), \qquad \forall u \in H^1(\mathcal{C}^\mu_\varepsilon), \tag{3.2.33}$$

where $f(\alpha_1,\alpha_2) = \left(C\,|\alpha_1^2 - \alpha_2^2| + |m(\alpha_1) - m(\alpha_2)|\right)/C(\alpha_2)$ and $C(\alpha_2)$ is the constant in the left-hand side of (3.2.31) which is strictly positive. This implies that the function f is continuous and vanishes on the diagonal: $f(\alpha_1,\alpha_2) \to 0$ when $\alpha_1 \to \alpha_2$. Taking the upper bound of all the terms in (3.2.33) over all unit vectors belonging to some n-dimensional subspace of $H^1(\mathcal{C}^\mu_\varepsilon)$ and then the lower bound over all such subspaces we get:

$$\widetilde{\varkappa}_n(\alpha_2)(1 - f(\alpha_1,\alpha_2)) \leqslant \widetilde{\varkappa}_n(\alpha_1) \leqslant \widetilde{\varkappa}_n(\alpha_2)(1 + f(\alpha_1,\alpha_2)), \qquad \forall n \in \mathbb{N}.$$

This together with the properties of the function f mentioned above proves the continuity of the functions $\widetilde{\varkappa}_n(\alpha)$, $n \in \mathbb{N}$. □

The following theorem is an immediate consequence of Proposition 3.2.3.

Theorem 3.2.1. *A number λ_ε^2 is an eigenvalue of the operator A_ε^μ if and only if λ_ε is a solution of the equation*

$$\alpha^2 = \varkappa_n(\alpha), \tag{3.2.34}$$

for some $n \in \mathbb{N}$.

The functions $\varkappa_n$ being continuous, it's the equation (3.2.34) that we solve numerically using a Newton type algorithm.

3.2.4 Practical construction of the operators $\Lambda^\pm(\alpha)$

Let us now give a procedure of construction of the DtN operators $\Lambda^\pm(\alpha)$ that avoids solving the problems (3.2.4) posed in unbounded domains and consequently can be used in numerical computations.

3.2.4.1 The local DtN operators

We introduce a periodicity cell $\mathcal{C}_\varepsilon$ such that the band B_ε^+ is a union of translated periodicity cells (fig. 3.3):

$$\overline{B_\varepsilon^+} = \bigcup_{n\in\mathbb{N}} \overline{\mathcal{C}_{\varepsilon,n}}, \qquad \mathcal{C}_{\varepsilon,n} = \mathcal{C}_\varepsilon + (n,0), \qquad \mathcal{C}_\varepsilon = B_\varepsilon^+ \cap \{1 < x < 2\}.$$

We also denote the vertical boundaries of the cells $\mathcal{C}_{\varepsilon,n}$ by $\Gamma_{\varepsilon,n} = \{n+1\} \times \left]-\frac{L}{2}, \frac{L}{2}\right[$, $n \in \mathbb{N}$.

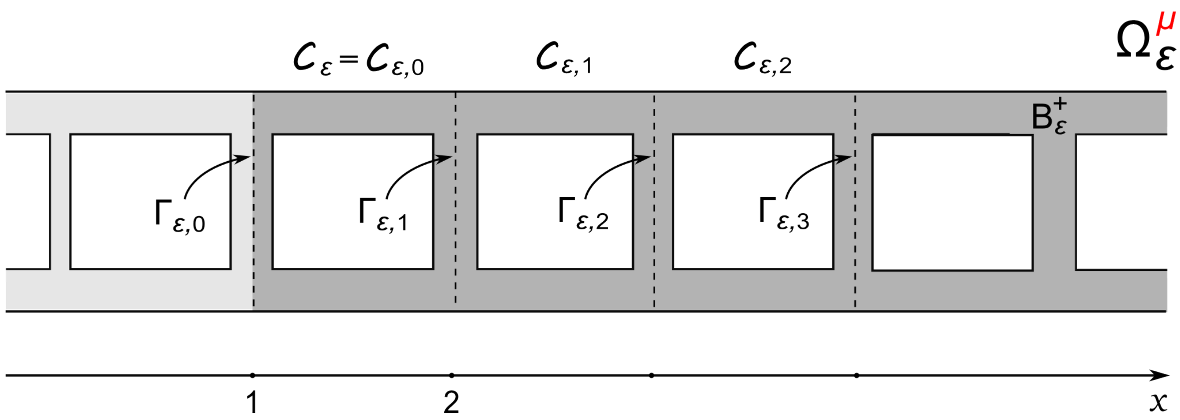


Figure 3.3: The cells $\mathcal{C}_{\varepsilon,n}$ and the boundaries $\Gamma_{\varepsilon,n}$

Function spaces on $\mathcal{C}_{\varepsilon,n}$ and $\Gamma_{\varepsilon,n}$ for different n will be often identified. Let us consider the following cell problems for $\alpha^2 \notin \sigma(A_\varepsilon)$, $\delta \in \{0,1\}$:

$$\begin{cases} -\Delta e_\delta - \alpha^2 e_\delta = 0 \ \text{ in } \ \mathcal{C}_\varepsilon, \\ \left.\dfrac{\partial e_\delta}{\partial n}\right|_{\partial\mathcal{C}_\varepsilon \setminus \{\Gamma_{\varepsilon,0}\cup\Gamma_{\varepsilon,1}\}} = 0, \\ e_\delta|_{\Gamma_{\varepsilon,\delta}} = \varphi, \qquad e_\delta|_{\Gamma_{\varepsilon,1-\delta}} = 0. \end{cases} \tag{3.2.35}$$

Proposition 3.2.6. *For any $\alpha^2 \notin \sigma_D$ and $\varphi \in H^{1/2}(\Gamma_{\varepsilon,\delta})$ the problems* (3.2.35) *have unique solutions in $H^1_\Delta(\mathcal{C}_\varepsilon)$, where σ_D is a countable set.*

Proof. We will give the proof for $\delta = 0$, the proof for $\delta = 1$ being analogous. The argument is very close to the one used in the proof of Proposition 3.2.1. We repeat it for the sake of completeness. Let us $\widehat{\varphi}$ be a function in $H^1(\mathcal{C}_\varepsilon)$ such that $\widehat{\varphi}|_{\Gamma_{\varepsilon,0}} = \varphi$, $\widehat{\varphi}|_{\Gamma_{\varepsilon,1}} = 0$ and $\|\widehat{\varphi}\|_{H^1(\mathcal{C}_\varepsilon)} \leqslant C\|\varphi\|_{H^{1/2}(\Gamma_{\varepsilon,0})}$, the constant C depending only on the domain $\mathcal{C}_\varepsilon$. After multiplication of the first line of (3.2.35) by a test function $v \in H^1_D(\mathcal{C}_\varepsilon)$, where

$$H^1_D(\mathcal{C}_\varepsilon) = \left\{u \in H^1(\mathcal{C}_\varepsilon), \quad u|_{\Gamma_{\varepsilon,0}} = 0, \quad u|_{\Gamma_{\varepsilon,1}} = 0\right\},$$

and changing the unknown function by $\widehat{e}_0 = e_0 - \widehat{\varphi}$, we get the following variational problem for $\widehat{e}_0 \in H^1_D(\mathcal{C}_\varepsilon)$:

$$(\nabla\widehat{e}_0, \nabla v)_{L_2(\mathcal{C}_\varepsilon)} - \alpha^2(\widehat{e}_0, v)_{L_2(\mathcal{C}_\varepsilon)} = -(\nabla\widehat{\varphi}, \nabla v)_{L_2(\mathcal{C}_\varepsilon)} + \alpha^2(\widehat{\varphi}, v)_{L_2(\mathcal{C}_\varepsilon)}, \quad \forall v \in H^1_D(\mathcal{C}_\varepsilon). \tag{3.2.36}$$

This problem can be reduced to the form

$$\left(I - (1+\alpha^2)P\right)\widehat{e}_0 = f, \tag{3.2.37}$$

where the bounded self-adjoint operator $P \in \mathcal{L}(H^1_D(\mathcal{C}_\varepsilon))$ is defined by the relation

$$(u, v)_{L_2(\mathcal{C}_\varepsilon)} = (Pu, v)_{H^1(\mathcal{C}_\varepsilon)}, \qquad \forall u, v \in H^1_D(\mathcal{C}_\varepsilon), \tag{3.2.38}$$

and $f \in H^1_D(\mathcal{C}_\varepsilon)$ is defined by the relation

$$-(\nabla\widehat{\varphi}, \nabla v)_{L_2(\mathcal{C}_\varepsilon)} + \alpha^2(\widehat{\varphi}, v)_{L_2(\mathcal{C}_\varepsilon)} = (f, v)_{H^1(\mathcal{C}_\varepsilon)}, \qquad \forall v \in H^1_D(\mathcal{C}_\varepsilon).$$

The operator P is compact. Indeed, it is bounded as an operator from $L_2(\mathcal{C}_\varepsilon)$ to $H^1(\mathcal{C}_\varepsilon)$ since taking in (3.2.38) $v = Pu$ one has:

$$\|Pu\|^2_{H^1(\mathcal{C}_\varepsilon)} = (u, Pu)_{L_2(\mathcal{C}_\varepsilon)} \leqslant \|Pu\|_{H^1(\mathcal{C}_\varepsilon)}\|u\|_{L_2(\mathcal{C}_\varepsilon)}, \qquad \forall u \in H^1_D(\mathcal{C}_\varepsilon).$$

Due to the compactness of the embedding $H^1(\mathcal{C}_\varepsilon) \subset L_2(\mathcal{C}_\varepsilon)$ one concludes that the operator $P \in \mathcal{L}(H^1_D(\mathcal{C}_\varepsilon))$ is compact. Hence, its spectrum $\sigma(P)$ consists of eigenvalues of finite multiplicity that can only accumulate to 0. The problem (3.2.37) is then well-posed if and only if $\alpha^2 \notin \{1/\gamma - 1,\ \gamma \in \sigma(P)\}$, which is a discrete set accumulating only to infinity that we denote by σ_D. □

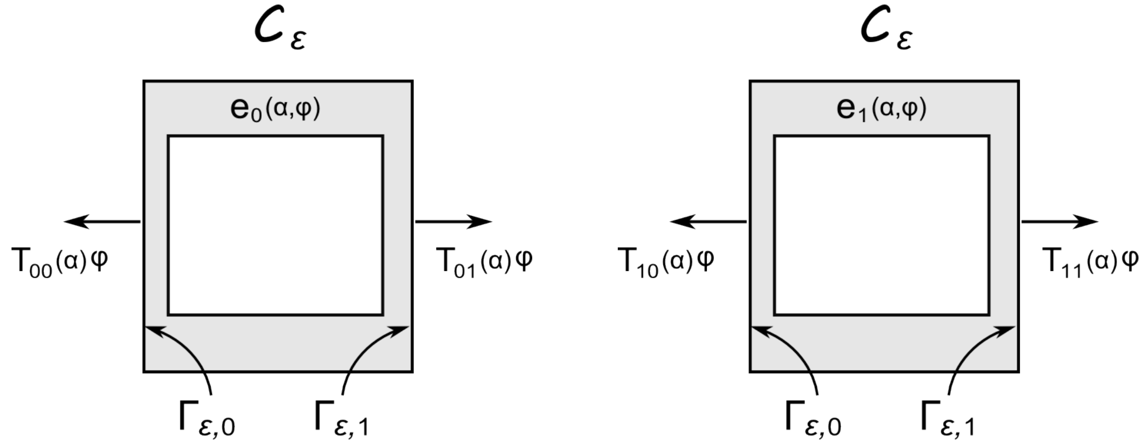
The unique solutions of the problems (3.2.35) for $\alpha^2 \notin \sigma_D$, $\varphi \in H^{1/2}(\Gamma_{\varepsilon,\delta})$ will be denoted by $e_\delta(\alpha, \varphi)$, $\delta \in \{0, 1\}$. Notice that repeating the same argument as the one used in the proof of Proposition 3.2.1 one can see that

$$\|e_\delta(\alpha, \varphi)\|_{H^1(\mathcal{C}_\varepsilon)} \leqslant C(\alpha)\|\varphi\|_{H^{1/2}(\Gamma_{\varepsilon,\delta})}, \qquad \forall \varphi \in H^{1/2}(\Gamma_{\varepsilon,\delta}), \quad \delta \in \{0, 1\}, \tag{3.2.39}$$

where the function $C(\alpha)$ is continuous and depends only on the geometry of the domain. We can now introduce the local DtN operators $T_{\gamma\delta}(\alpha) \in \mathcal{L}\left(H^{1/2}(\Gamma_{\varepsilon,\gamma}), H^{-1/2}(\Gamma_{\varepsilon,\delta})\right)$, $\gamma, \delta \in \{0, 1\}$, as follows:

$$T_{\gamma\delta}(\alpha)\varphi = \left.\frac{\partial e_\gamma(\alpha, \varphi)}{\partial n}\right|_{\Gamma_{\varepsilon,\delta}}, \qquad \gamma, \delta \in \{0, 1\}, \tag{3.2.40}$$

where the normal derivative is taken along the outside normal to the domain $\mathcal{C}_\varepsilon$.



(a) The operators $T_{00}(\alpha)$, $T_{01}(\alpha)$ (b) The operators $T_{10}(\alpha)$, $T_{11}(\alpha)$

Figure 3.4: The local DtN operators

Lemma 3.2.4. *The operators $T_{\gamma\delta}(\alpha)$ are symmetric for any $\gamma, \delta \in \{0,1\}$, $\alpha^2 \notin \sigma_D$.*

Proof. Indeed, we have

$$\langle T_{\gamma\delta}(\alpha)\varphi, \psi\rangle = \int\limits_{\mathcal{C}_\varepsilon} \Delta e_\gamma(\alpha,\varphi) e_\delta(\alpha,\psi) - \int\limits_{\mathcal{C}_\varepsilon} e_\gamma(\alpha,\varphi)\Delta e_\delta(\alpha,\psi) + \int\limits_{\Gamma_{\varepsilon,\gamma}} e_\gamma(\alpha,\varphi)\frac{\partial e_\delta(\alpha,\psi)}{\partial n}$$
$$= \langle T_{\delta\gamma}(\alpha)\psi, \varphi\rangle .$$

Since in our case of a symmetric cell the operators T_{01} and T_{10} coincide, this finishes the proof. □

3.2.4.2 The propagation operator

Let us introduce the propagation operator $P(\alpha)$ for $\alpha^2 \notin \sigma(A_\varepsilon)$ as follows:

$$P(\alpha) \in \mathcal{L}\left(H^{1/2}(\Gamma_{\varepsilon,0}), H^{1/2}(\Gamma_{\varepsilon,1})\right), \qquad P(\alpha)\varphi = u_\varepsilon^+(\alpha,\varphi)\big|_{\Gamma_{\varepsilon,1}}. \tag{3.2.41}$$

The spectral radius of the propagation operator will play an important role in the sequel. It will be denoted by $\rho(P(\alpha))$.

Lemma 3.2.5. *For $\alpha^2 \notin \sigma(A_\varepsilon)$ the operator $P(\alpha)$ is compact. Moreover, $\rho(P(\alpha)) < 1$.*

Proof. Let φ be a function in $H^{1/2}(\Gamma_{\varepsilon,0})$. We will use Lemma 3.2.6 proved below, according to which $\|P(\alpha)u_\varepsilon^+(\alpha,\varphi)\|_{H^1(\Gamma_{\varepsilon,1})} \leqslant C(\alpha)\|\varphi\|_{H^{1/2}(\Gamma_{\varepsilon,0})}$. Recalling the definition (3.2.41) of the operator $P(\alpha)$ and using the compactness of the embedding $H^1(\Gamma_{\varepsilon,1}) \subset H^{1/2}(\Gamma_{\varepsilon,1})$ we conclude that the propagation operator is compact. Consequently, its spectrum is a sequence of isolated eigenvalues of finite multiplicity with the only possible accumulation point at 0 (in this case $0 \in \sigma_c(P(\alpha))$). If λ_1 is the eigenvalue of $P(\alpha)$ with the biggest absolute value then $|\lambda_1| = \rho(P(\alpha))$. Suppose that $|\lambda_1| \geqslant 1$ and $\varphi_1 \in H^{1/2}(\Gamma_{\varepsilon,1})$ is a corresponding eigenfunction. Then, $u_\varepsilon^+(\alpha,\varphi_1)|_{\Gamma_{\varepsilon,n}} = P(\alpha)^n\varphi_1 = \lambda_1^n\varphi_1$, $\forall n \in \mathbb{N}$. On the other hand, due to the uniqueness of solutions of (3.2.4) and the translation invariance of the domain B_ε^+ we have

$$u_\varepsilon^+(\alpha,\varphi_1)\big|_{B_{\varepsilon,n}} = \mathcal{T}_n u_\varepsilon^+(\alpha,\lambda_1^n\varphi_1) = \lambda_1^n\mathcal{T}_n u_\varepsilon^+(\alpha,\varphi_1), \qquad \forall n \in \mathbb{N}, \tag{3.2.42}$$

where

$$B_{\varepsilon,n} = B_\varepsilon^+ \cap \{X > n+1\}, \qquad n \in \mathbb{N},$$

and $\mathcal{T}_n$ is the translation operator from $L_2(B_\varepsilon^+)$ to $L_2(B_{\varepsilon,n})$ defined on smooth functions by the relation

$$(\mathcal{T}_n u)(x,y) = u(x-n,y), \qquad (x,y) \in B_{\varepsilon,n},$$

and extended by continuity to $L_2(B_\varepsilon^+)$. It follows from (3.2.42) that

$$\left\| u_\varepsilon^+(\alpha,\varphi_1)\big|_{B_{\varepsilon,n}} \right\|_{L_2(B_{\varepsilon,n})} = |\lambda_1^n| \left\| u_\varepsilon^+(\alpha,\varphi_1)\right\|_{L_2\left(B_\varepsilon^+\right)} \geqslant \left\| u_\varepsilon^+(\alpha,\varphi_1)\right\|_{L_2\left(B_\varepsilon^+\right)}, \qquad \forall n \in \mathbb{N}.$$

However, $u_\varepsilon^+(\alpha,\varphi_1) \in L_2(B_\varepsilon^+)$ which implies $\left\| u_\varepsilon^+(\alpha,\varphi_1)|_{B_{\varepsilon,n}} \right\|_{L_2(B_{\varepsilon,n})} \xrightarrow[n\to\infty]{} 0$. This is possible only if $u_\varepsilon^+(\alpha,\varphi_1)$ identically zero which contradicts the fact φ_1 is an eigenfunction of the operator $P(\alpha)$. □

The following Lemma used in the proof of Lemma 3.2.5 states in fact the interior regularity of the solutions $u_\varepsilon^\pm$ of (3.2.4). For the sake of completeness we will give the argument in the case of our geometry that permits to apply the regularity result (Theorems 2.3.7 and 2.4.3 from [27]).

Lemma 3.2.6. *For any $\alpha^2 \notin \sigma(A_\varepsilon)$ and $\varphi \in H^{1/2}(\Gamma_{\varepsilon,0})$ one has $P(\alpha)u_\varepsilon^+(\alpha,\varphi) \in H^1(\Gamma_{\varepsilon,1})$. Moreover, there exists a continuous function $C(\alpha)$ depending only on the geometry of the domain such that*

$$\left\| P(\alpha)u_\varepsilon^+(\alpha,\varphi)\right\|_{H^1(\Gamma_{\varepsilon,1})} \leqslant C(\alpha)\|\varphi\|_{H^{1/2}(\Gamma_{\varepsilon,0})}, \qquad \forall \varphi \in H^{1/2}(\Gamma_{\varepsilon,0}).$$

Proof. The argument is very close to the one used in the proof of Lemma A.3.3. Let us introduce a cut-off function $\chi \in C^\infty(\mathbb{R})$ such that

$$\begin{cases} \chi(x) = 1, & x \in \left[2-\frac{\varepsilon}{8}, 2+\frac{\varepsilon}{8}\right], \\ \chi(x) = 0, & x \in \left]-\infty,\, 2-\frac{\varepsilon}{4}\right] \cup \left[2+\frac{\varepsilon}{4},\, \infty\right[, \\ 0 \leqslant \chi(x) \leqslant 1, & \forall x \in \mathbb{R}. \end{cases}$$

Let $\mathcal{K}_\varepsilon$ stand for the rectangle $\left]2-\frac{\varepsilon}{4}, 2+\frac{\varepsilon}{4}\right[\times\left]-\frac{L}{2}, \frac{L}{2}\right[$. Then, the function $\widetilde{u} = \chi u_\varepsilon^+(\alpha,\varphi)$ solves the following problem in $\mathcal{K}_\varepsilon$:

$$\begin{cases} -\Delta\widetilde{u} = \alpha^2 u_\varepsilon^+(\alpha,\varphi)\chi - u_\varepsilon^+(\alpha,\varphi)\Delta\chi - 2\nabla u_\varepsilon^+(\alpha,\varphi)\nabla\chi \quad \text{in} \quad \mathcal{K}_\varepsilon, \\ \widetilde{u}|_{\left\{2-\frac{\varepsilon}{4}\right\}\times\left]-\frac{L}{2},\frac{L}{2}\right[} = 0, \quad \widetilde{u}|_{\left\{2+\frac{\varepsilon}{4}\right\}\times\left]-\frac{L}{2},\frac{L}{2}\right[} = 0, \\ \left.\dfrac{\partial\widetilde{u}}{\partial n}\right|_{\left]2-\frac{\varepsilon}{4},2+\frac{\varepsilon}{4}\right[\times\left\{\frac{L}{2}\right\}} = 0, \quad \left.\dfrac{\partial\widetilde{u}}{\partial n}\right|_{\left]2-\frac{\varepsilon}{4},2+\frac{\varepsilon}{4}\right[\times\left\{-\frac{L}{2}\right\}} = 0. \end{cases}$$

Then, applying the regularity result ([27]), we conclude that $\widetilde{u} \in H^2(\mathcal{K}_\varepsilon)$ and

$$\|\widetilde{u}\|_{H^2(\mathcal{K}_\varepsilon)} \leqslant C\left\|\alpha^2 u_\varepsilon^+(\alpha,\varphi)\chi - u_\varepsilon^+(\alpha,\varphi)\Delta\chi - 2\nabla u_\varepsilon^+(\alpha,\varphi)\nabla\chi\right\|_{L_2(\mathcal{K}_\varepsilon)} \leqslant C(\alpha)\|\varphi\|_{H^{1/2}(\Gamma_{\varepsilon,0})},$$

where we took into account (3.2.5). Then, using the continuity of the trace application from $H^2(K)$ to $H^1(\partial K)$ in a Lipschitz domain $K = \left]2, 2+\frac{\varepsilon}{4}\right[\times\left]-\frac{L}{2}, \frac{L}{2}\right[$ (cf. [14]) we get

$$\left\| u|_{\Gamma_{\varepsilon,1}}\right\|_{H^1(\Gamma_{\varepsilon,1})} = \left\|\widetilde{u}|_{\Gamma_{\varepsilon,1}}\right\|_{H^1(\Gamma_{\varepsilon,1})} \leqslant \|\widetilde{u}\|_{H^2(K)} \leqslant C(\alpha)\|\varphi\|_{H^{1/2}(\Gamma_{\varepsilon,0})}.$$

□

Remark 3.2.3. The propagation operator $P(\alpha)$ is injective.

Proof. Indeed, suppose that $P(\alpha)\varphi_0 = 0$ for some $\varphi_0 \in H^{1/2}(\Gamma_{\varepsilon,0})$. Then, $u_\varepsilon^+(\alpha,\varphi)|_{B_{\varepsilon,1}}$ solves the problem (3.2.4) in $B_{\varepsilon,1}$ with $\varphi = 0$, which implies that $u_\varepsilon^+(\alpha,\varphi)|_{B_{\varepsilon,1}} = 0$. From the unique continuation property it follows that $u_\varepsilon^+(\alpha,\varphi) = 0$ and, consequently, $\varphi_0 = 0$. □

3.2.4.3 Another characterization of the propagation operator. Riccati equation

Due to the translation invariance of the domain B_ε^+ one has

$$u_\varepsilon^+(\alpha,\varphi)\big|_{\Gamma_{\varepsilon,n}} = (P(\alpha))^n\,\varphi, \qquad \forall n \in \mathbb{N}, \qquad \forall \varphi \in H^{1/2}(\Gamma_{\varepsilon,0}).$$

Then, the restriction of $u_\varepsilon^+(\alpha,\varphi)$ to the cell $\mathcal{C}_{\varepsilon,n}$ can be computed as follows:

$$u_\varepsilon^+(\alpha,\varphi)\big|_{\mathcal{C}_{\varepsilon,n}} = e_0\left(\alpha, (P(\alpha))^{n-1}\varphi\right) + e_1\left(\alpha, (P(\alpha))^n\varphi\right), \qquad \forall n \in \mathbb{N}^*, \quad \forall \varphi \in H^{1/2}(\Gamma_{\varepsilon,0}).$$

The continuity of the normal derivative of $u_\varepsilon^+(\alpha,\varphi)$ across $\Gamma_{\varepsilon,1}$ yields, in view of (3.2.40), that

$$T_{10}(\alpha)\,(P(\alpha))^2 + (T_{00}(\alpha) + T_{11}(\alpha))\,P(\alpha) + T_{01}(\alpha) = 0. \tag{3.2.43}$$

This relation can be seen as the stationary Riccati equation for $P(\alpha)$. It turns out that it can be used to determine the propagation operator $P(\alpha)$ without solving the problems (3.2.4) once the local DtN operators are constructed.

Proposition 3.2.7. *For $\alpha^2 \notin \sigma(A_\varepsilon) \cup \sigma_D$ the propagation operator $P(\alpha)$ is the unique solution of the Riccati equation* (3.2.43) *in the set of operators in $H^{1/2}(\Gamma_{\varepsilon,0})$ with spectral radius smaller than* 1.

Proof. Suppose that there exists another compact operator $\widetilde{P}$ in $H^{1/2}(\Gamma_{\varepsilon,0})$ with spectral radius smaller than 1 satisfying (3.2.43). Then, for any $\varphi \in H^{1/2}(\Gamma_{\varepsilon,0})$ one can construct the following function $u \in L_2(B_\varepsilon^+)$:

$$u|_{\mathcal{C}_{\varepsilon,n}} = e_0\left(\alpha, \widetilde{P}^{n-1}\varphi\right) + e_1\left(\alpha, \widetilde{P}^n\varphi\right), \qquad \forall n \in \mathbb{N}^*. \tag{3.2.44}$$

By construction, its trace on the boundary $\Gamma_{\varepsilon,0}$ is φ and u is continuous across the boundaries $\Gamma_{\varepsilon,n}$: $\left(u|_{\mathcal{C}_{\varepsilon,n}}\right)\Big|_{\Gamma_{\varepsilon,n}} = \left(u|_{\mathcal{C}_{\varepsilon,n+1}}\right)\Big|_{\Gamma_{\varepsilon,n}} = \widetilde{P}^n\varphi$, $\forall n \in \mathbb{N}^*$. Consequently, $u \in H^1_{loc}(B_\varepsilon^+)$. Moreover, its normal derivative is also continuous. Indeed,

$$\left(\frac{\partial}{\partial n}\left(u|_{\mathcal{C}_{\varepsilon,n}}\right)\right)\Bigg|_{\Gamma_{\varepsilon,n}} = T_{01}(\alpha)\widetilde{P}^{n-1}\varphi + T_{11}(\alpha)\widetilde{P}^n\varphi,$$
$$\left(\frac{\partial}{\partial n}\left(u|_{\mathcal{C}_{\varepsilon,n+1}}\right)\right)\Bigg|_{\Gamma_{\varepsilon,n}} = T_{00}(\alpha)\widetilde{P}^n\varphi + T_{10}(\alpha)\widetilde{P}^{n+1}\varphi = -T_{01}(\alpha)\widetilde{P}^{n-1}\varphi - T_{11}(\alpha)\widetilde{P}^n\varphi,$$

where we used the fact that the operator $\widetilde{P}$ solves the equation (3.2.43). Hence,

$$\left(\frac{\partial}{\partial n}\left(u|_{\mathcal{C}_{\varepsilon,n}}\right)\right)\Bigg|_{\Gamma_{\varepsilon,n}} = -\left(\frac{\partial}{\partial n}\left(u|_{\mathcal{C}_{\varepsilon,n+1}}\right)\right)\Bigg|_{\Gamma_{\varepsilon,n}}, \qquad \forall n \in \mathbb{N}^*,$$

which implies that $u \in H^1_{\Delta,loc}(B_\varepsilon^+)$. Let us show that in fact $u \in H^1_\Delta(B_\varepsilon^+)$. Since $\rho(\widetilde{P}) = \lim_{n\to+\infty} \|\widetilde{P}^n\|^{1/n}_{\mathcal{L}\left(H^{1/2}(\Gamma_{\varepsilon,0})\right)} < 1$, there exists $N \in \mathbb{N}$ and $0 < c_0 < 1$ such that

$$\|\widetilde{P}^n\|_{\mathcal{L}\left(H^{1/2}(\Gamma_{\varepsilon,0})\right)} < c_0^n, \qquad \forall n \geqslant N.$$

Therefore,

$$\sum_{n=N+1}^{\infty} \left\| u|_{\mathcal{C}_{\varepsilon,n}} \right\|^2_{H^1(\mathcal{C}_{\varepsilon,n})} \leqslant C(\alpha)\|\varphi\|_{H^{1/2}(\Gamma_{\varepsilon,0})} \sum_{n=N+1}^{\infty} c_0^{2n} < \infty,$$

where we used (3.2.44) and (3.2.39). This proves that $u \in H^1_\Delta(B_\varepsilon^+)$ and $-\Delta u - \alpha^2 u = 0$ in B_ε^+ since. Finally, it means that u solves (3.2.4) in B_ε^+, i.e. $u = u_\varepsilon^+(\alpha,\varphi)$. Then, from (3.2.44) for $n = 1$ it follows that $u_\varepsilon^+(\alpha,\varphi)|_{\Gamma_{\varepsilon,1}} = \widetilde{P}\varphi$ for any $\varphi \in H^{1/2}(\Gamma_{\varepsilon,0})$. Comparing this with the definition of the propagation operator (3.2.41) one conclude that $\widetilde{P} = P(\alpha)$. □

Proposition 3.2.7 permits to determine the propagation operator by solving Riccati equation (3.2.43) instead of solving the problems (3.2.4). It is now easy to construct the DtN operators $\Lambda^\pm(\alpha)$. Indeed, comparing the definition of the DtN operators (3.2.18) with the definition of the local DtN operators (3.2.40) one finds that

$$\Lambda^+(\alpha) = T_{00}(\alpha) + T_{10}(\alpha)P(\alpha). \tag{3.2.45}$$

Thus, we do not need any more to consider unbounded domains in order to solve the eigenvalue problem (3.2.1).

Remark 3.2.4. The spectral radius of the propagation operator can be used to characterize the essential spectrum of the operator A_ε as well. It turns out that

$$\alpha^2 \notin \sigma(A_\varepsilon) \;\Leftrightarrow\; \rho\left(P(\alpha)\right) < 1, \qquad \alpha^2 \in \sigma(A_\varepsilon) \;\Leftrightarrow\; \rho\left(P(\alpha)\right) = 1.$$

This property can be used in the numerical computation of the essential spectrum of the operator A_ε^μ, but we will mostly use dispersion curves method as discussed below.

3.3 Discretization by a conform finite element method

From a practical point of view, the numerical method used consists in solving the nonlinear equation

$$\alpha^2 = \varkappa_n^h(\alpha) \tag{3.3.1}$$

by a Newton type algorithm. The equation (3.3.1) corresponds to the equation (3.2.34) where the eigenvalues $\varkappa_n$ of the operator A_ε^μ are replaced by those of its approximation $A_\varepsilon^{\mu,h}$ discretized by a standard finite element method. The principal difficulty in the construction of the operator $A_\varepsilon^{\mu,h}$ consists in the discretization of the DtN operators $\Lambda^\pm(\alpha)$ that we detail in Section 3.3.1. In the sequel we explain how to apply a Newton type algorithm to the equation (3.3.1). Finally, we briefly describe how to compute numerically the essential spectrum of the operator A_ε^μ.

3.3.1 Discretization of the operator A_ε^μ

In this section we will describe in detail how we perform the discretization of the problem (3.2.28) without giving the numerical analysis which can be found in [22]. The first step is the construction of the DtN operators using the relation (3.2.45). Thus, the local DtN operators $T_{00}(\alpha)$, $T_{10}(\alpha)$ as well as the propagation operator $P(\alpha)$ have to be constructed. The construction of the operators $T_{00}(\alpha)$, $T_{10}(\alpha)$ requires the resolution of cell problems that are solved using a standard P_1 Lagrange finite element method.

3.3.1.1 Cell problems

Let $\mathcal{T}_h = \{T_\ell\}_{\ell=1}^L$ be a triangular mesh of the domain $\mathcal{C}_\varepsilon$. The set of all the vertices of the mesh $\mathcal{T}_h$ will be denoted by $\mathcal{M}_h$: $\mathcal{M}_h = \{M_j\}_{j=1}^N$. Let us introduce the following function space:

$$V_h = \left\{ v_h \in C\left(\overline{\mathcal{C}_\varepsilon}\right), \quad v_h|_{T_\ell} \text{ is linear}, \ 1 \leqslant \ell \leqslant L \right\}. \tag{3.3.2}$$

Clearly, V_h is a subspace of $H^1(\mathcal{C}_\varepsilon)$ of dimension N. As usual, we consider the basis $\mathcal{B}_h = \{w_j\}_{j=1}^N$ in V_h, which consists of continuous piecewise linear functions w_j, such that

$$w_j(M_i) = \delta_{ij}, \qquad 1 \leqslant i, j \leqslant N. \tag{3.3.3}$$

We will use the space V_h (and its analogue for the domain $\mathcal{C}_\varepsilon^\mu$) to construct internal approximations (also called Galerkin approximations) of all variational problems that we have to solve in our algorithm. The spaces of the traces of the functions in V_h on the boundaries $\Gamma_{\varepsilon,0}$, $\Gamma_{\varepsilon,1}$ will be denoted by

$$S_h^0 = Span \left\{ w_j|_{\Gamma_{\varepsilon,0}}, \ 1 \leqslant j \leqslant N \right\}, \qquad S_h^1 = Span \left\{ w_j|_{\Gamma_{\varepsilon,1}}, \ 1 \leqslant j \leqslant N \right\}.$$

To simplify the practical implementation of the DtN operator, we make the assumption that the meshes of the boundaries $\Gamma_{\varepsilon,0}$, $\Gamma_{\varepsilon,1}$ coincide. Moreover, if N_J is the number of vertices in the mesh of $\Gamma_{\varepsilon,0}$, then we suppose that the vertices situated on the boundary $\Gamma_{\varepsilon,0}$ are those with the index $1 \leqslant j \leqslant N_J$ (going from up to down) and the vertices situated on the boundary $\Gamma_{\varepsilon,1}$ are those with the index $N_J + 1 \leqslant j \leqslant 2N_J$ (going from up to down as well), cf. figure 3.5. For the functions w_j corresponding to the vertices situated on the boundaries $\Gamma_{\varepsilon,0}$, $\Gamma_{\varepsilon,1}$ their traces on these boundaries will be denoted by

$$\varphi_i^0 = w_i|_{\Gamma_{\varepsilon,0}}, \qquad \varphi_i^1 = w_{N_J+i}|_{\Gamma_{\varepsilon,1}}, \qquad 1 \leqslant i \leqslant N_J.$$

Notice that all the other traces (of the functions w_j with $1 \leqslant j \leqslant N_J$ on $\Gamma_{\varepsilon,1}$, of the functions w_j with $N_J + 1 \leqslant j \leqslant 2N_J$ on $\Gamma_{\varepsilon,0}$ and of the functions w_j with $j \geqslant 2N_J + 1$ on both boundaries) are zero. Hence,

$$S_h^0 = Span \left\{ \varphi_i^0, \ 1 \leqslant i \leqslant N_J \right\}, \qquad S_h^1 = Span \left\{ \varphi_i^1, \ 1 \leqslant i \leqslant N_J \right\}.$$

Let us start by constructing the solutions of the discretized problem (3.2.35) for $\varphi \in S_h^0 \cup S_h^1$. Recall the variational formulation of the problem (3.2.35) (cf. (3.2.36)):

$$\left(\nabla \widehat{e}_\delta(\alpha,\varphi), \nabla v\right)_{L_2(\mathcal{C}_\varepsilon)} - \alpha^2 \left(\widehat{e}_\delta(\alpha,\varphi), v\right)_{L_2(\mathcal{C}_\varepsilon)} = -\left(\nabla u_\delta(\varphi), \nabla v\right)_{L_2(\mathcal{C}_\varepsilon)} + \alpha^2 \left(u_\delta(\varphi), v\right)_{L_2(\mathcal{C}_\varepsilon)}, \quad \forall v \in H_D^1(\mathcal{C}_\varepsilon), \tag{3.3.4}$$

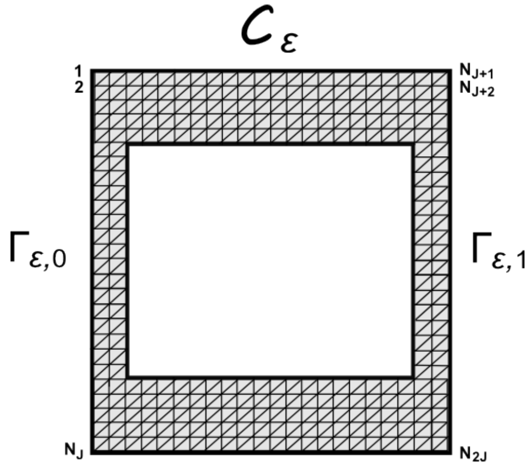


Figure 3.5: The mesh $\mathcal{T}_h$

where $\widehat{e}_\delta(\alpha,\varphi) = e_\delta(\alpha,\varphi) - u_\delta(\varphi) \in H^1_D(\mathcal{C}_\varepsilon)$ and $u_\delta(\varphi)$ is a lift function in $H^1(\mathcal{C}_\varepsilon)$ such that

$$u_\delta(\varphi)|_{\Gamma\varepsilon,\delta} = \varphi, \qquad u_\delta(\varphi)|_{\Gamma\varepsilon,1-\delta} = 0. \tag{3.3.5}$$

Discretizing of (3.3.4) consists in considering its restriction to the space $V_h^D = V_h \cap H^1_D(\mathcal{C}_\varepsilon)$. Thus, for $\varphi_h \in S_h^\delta$, $\delta \in \{0,1\}$, we search $\widehat{e}^h_\delta(\alpha,\varphi_h) \in V_h^D$ such that

$$\begin{aligned}\left(\nabla\widehat{e}^h_\delta(\alpha,\varphi_h),\nabla v\right)_{L_2(\mathcal{C}_\varepsilon)} &- \alpha^2\left(\widehat{e}^h_\delta(\alpha,\varphi_h),v\right)_{L_2(\mathcal{C}_\varepsilon)} \\ &= -\left(\nabla u_\delta(\varphi_h),\nabla v\right)_{L_2(\mathcal{C}_\varepsilon)} + \alpha^2\left(u_\delta(\varphi_h),v\right)_{L_2(\mathcal{C}_\varepsilon)}, \qquad \forall v \in V_h^D. \end{aligned} \tag{3.3.6}$$

If $\varphi_h = \sum\limits_{i=1}^{N_J} c_j\varphi_i^\delta$, then by linearity $\widehat{e}^h_\delta(\alpha,\varphi_h) = \sum\limits_{i=1}^{N_J} c_j\widehat{e}^h_{\delta,i}(\alpha)$, where $\widehat{e}^h_{\delta,i}(\alpha) = \widehat{e}^h_\delta\left(\alpha,\varphi_i^\delta\right)$, $1 \leqslant i \leqslant N_J$, $\delta \in \{0,1\}$. Remarking that one can choose $u_\delta\left(\varphi_i^\delta\right) = w_{i+\delta N_J}$. Therefore, it is enough to compute the $2N_J$ functions $\widehat{e}^h_{\delta,i}(\alpha)$, $\delta \in \{0,1\}$, $1 \leqslant i \leqslant N_J$ solutions to the following system of linear equations:

$$\begin{aligned}\left(\nabla\widehat{e}^h_{\delta,i}(\alpha),\nabla w_j\right)_{L_2(\mathcal{C}_\varepsilon)} - \alpha^2\left(\widehat{e}^h_{\delta,i}(\alpha),w_j\right)_{L_2(\mathcal{C}_\varepsilon)} &= -\left(\nabla w_{i+\delta N_J},\nabla w_j\right)_{L_2(\mathcal{C}_\varepsilon)} + \alpha^2\left(w_{i+\delta N_J},w_j\right)_{L_2(\mathcal{C}_\varepsilon)}, \\ &1 \leqslant j \leqslant N, \qquad j \notin \mathcal{J}_0 \cup \mathcal{J}_1. \end{aligned} \tag{3.3.7}$$

Let us decompose the functions $\widehat{e}^h_{\delta,i}(\alpha)$ in the basis $\mathcal{B}_h$: $\widehat{e}^h_{\delta,i}(\alpha) = \sum\limits_{j=1}^{N} \widehat{E}^i_{\delta,j}(\alpha)w_j$. Clearly, $\widehat{E}^i_{\delta,j}(\alpha) = 0$ for $j \geqslant 2N_J + 1$, $1 \leqslant i \leqslant N_J$, $\delta \in \{0,1\}$. Then, the problems (3.3.7) take the form

$$\left(K^D(\alpha) - \alpha^2 M^D(\alpha)\right)\widehat{E}^i_\delta(\alpha) = F^i_\delta(\alpha), \tag{3.3.8}$$

where

$$K^D_{i,j} = \begin{cases} K_{i,j}, & i,j \geqslant 2N_J+1, \\ \delta_{i,j}, & 1 \leqslant i,j \leqslant 2N_J, \\ 0, & \text{otherwise,} \end{cases} \qquad M^D_{i,j} = \begin{cases} M_{i,j}, & i,j \geqslant 2N_J+1, \\ \delta_{i,j}, & 1 \leqslant i,j \leqslant 2N_J, \\ 0, & \text{otherwise,} \end{cases}$$

$K = (K_{i,j})$ and $M = (M_{i,j})$ are the rigidity and the mass matrices respectively defined as

$$K_{i,j} = \left(\nabla w_i,\nabla w_j\right)_{L_2(\mathcal{C}_\varepsilon)}, \qquad M_{i,j} = \left(w_i,w_j\right)_{L_2(\mathcal{C}_\varepsilon)}, \qquad 1 \leqslant i,j \leqslant N, \tag{3.3.9}$$

and the right-hand side $F^i_\delta(\alpha)$ is the $(N \times 1)$-vector defined as

$$F^i_{\delta,j}(\alpha) = \begin{cases} -\left(K_{i+\delta N_J,j} - \alpha^2 M_{i+\delta N_J,j}\right), & j \geqslant 2N_J + 1, \\ 0, & 1 \leqslant j \leqslant 2N_J. \end{cases}$$

Finally, the vector $E^i_\delta(\alpha)$ of the coordinates of the solution $e_\delta(\alpha, \varphi^\delta_i)$ in the basis $\mathcal{B}_h$ is

$$E^i_\delta(\alpha) = \widehat{E}^i_\delta(\alpha) + I^i_\delta, \qquad 1 \leqslant i \leqslant N_J, \qquad \delta \in 0, 1,$$

where I^i_δ is the vector with the coordinates $I^i_{\delta,i+\delta N_J} = 1$ and $I^i_{\delta,j} = 0$ for $j \neq i + \delta N_J$.

3.3.1.2 The local DtN operators

Recalling the definition (3.2.40) and using the fact that $e_\gamma(\alpha, \varphi)$ solves the problem (3.2.35) with δ replaced by γ one easily gets

$$\langle T_{\gamma\delta}(\alpha)\varphi, \psi\rangle = \left(\nabla e_\gamma(\alpha, \varphi), \nabla u_\delta(\psi)\right)_{L_2(\mathcal{C}_\varepsilon)} - \alpha^2 \left(e_\gamma(\alpha, \varphi), u_\delta(\psi)\right)_{L_2(\mathcal{C}_\varepsilon)}, \quad \forall \varphi \in H^{1/2}(\Gamma_{\varepsilon,\gamma}), \qquad \forall \psi \in H^{1/2}(\Gamma_{\varepsilon,\delta}), \quad (3.3.10)$$

where $u_\delta(\psi)$ is a function introduced in (3.3.5). We now define the approximated local DtN operators $T^h_{\gamma\delta}(\alpha)$ as $(N_J \times N_J)$-matrices the (i, j) matrix elements of which approach the matrix elements of the operator $T_{\gamma\delta}(\alpha)$ computed on the functions from S^0_h, S^1_h:

$$T^h_{\gamma\delta}(\alpha)(i, j) \simeq \langle T_{\gamma\delta}(\alpha)\varphi^\gamma_i, \varphi^\delta_j\rangle, \qquad 1 \leqslant i, j \leqslant N_J.$$

More precisely, they are obtained if one replaces in (3.3.10) φ and ψ by φ^γ_i and φ^δ_j respectively, $e_\gamma(\alpha, \varphi)$ by $e^h_{\gamma,i}(\alpha)$ and $u_\delta(\psi)$ by $w_{J_\delta(j)}$:

$$\begin{aligned} T^h_{\gamma\delta}(\alpha)(i, j) &= \left(\nabla e^h_{\gamma,i}(\alpha), \nabla w_{j+\delta N_J}\right)_{L_2(\mathcal{C}_\varepsilon)} - \alpha^2 \left(e^h_{\gamma,i}(\alpha), w_{j+\delta N_J}\right)_{L_2(\mathcal{C}_\varepsilon)} \\ &= \left(\left(K(\alpha) - \alpha^2 M(\alpha)\right) E^i_\gamma(\alpha)\right)(j + \delta N_J), \qquad 1 \leqslant i, j \leqslant N_J, \qquad \gamma, \delta \in \{0, 1\}. \end{aligned}$$

The next step is now to construct the approximated propagation operator $P^h(\alpha)$ using the discretised analogue of the Riccati equation (3.2.43).

3.3.1.3 The propagation operator

We search the approximated propagation operator $P^h(\alpha) \in M_{N_J}(\mathbb{R})$, which is the propagation operator corresponding to the discretized problem, as the solution of the equation

$$T^h_{10}(\alpha)\left(P^h(\alpha)\right)^2 + \left(T^h_{00}(\alpha) + T^h_{11}(\alpha)\right)P^h(\alpha) + T^h_{01}(\alpha) = 0,$$

completed by the condition

$$\rho\left(P^h(\alpha)\right) < 1. \qquad (3.3.11)$$

As described in [22], this equation is solved using Newton's method in Banach spaces (in our case the space $M_{N_J}(\mathbb{R})$). Let F be the following mapping in $M_{N_J}(\mathbb{R})$:

$$F : X \mapsto T^h_{10}(\alpha)X^2 + \left(T^h_{00}(\alpha) + T^h_{11}(\alpha)\right)X + T^h_{01}(\alpha).$$

Then, given the n-th iteration X_n the $(n+1)$-st iteration is found from the relation

$$F'(X_n)\left(X_n - X_{n+1}\right) = F(X_n), \qquad (3.3.12)$$

where $F'(X_n)$ is the Fréchet derivative of the mapping computed F at X_n. It is defined as follows:

$$F'(X) \;:\; h \;\mapsto\; T^h_{10}(\alpha)\,(Xh + hX) + \left(T^h_{00}(\alpha) + T^h_{11}(\alpha)\right) h, \qquad \forall X, h \in M_{N_J}(\mathbb{R}).$$

Combining this with (3.3.12) we finally obtain the equation for $\Delta_{n+1} = X_{n+1} - X_n$:

$$T^h_{10}(\alpha)\,(X_n \Delta X_{n+1} + \Delta_{n+1} X_n) + \left(T^h_{00}(\alpha) + T^h_{11}(\alpha)\right) \Delta_{n+1} = -F(X_n).$$

This, in turn, can be rewritten as

$$A\Delta_{n+1}B^T + C\Delta_{n+1}D^T = E,$$

where

$$\begin{aligned} A &= \left(T^h_{00}(\alpha) + T^h_{11}(\alpha)\right)^{-1} T^h_{10}(\alpha), \qquad B = X_n^T, \qquad C = AB^T + I, \\ D &= I, \qquad E = -\left(B^T + A\left(B^T\right)^2 + \left(T^h_{00}(\alpha) + T^h_{11}(\alpha)\right)^{-1} T^h_{01}(\alpha)\right). \end{aligned}$$

This equation is solved using the method described in [26]. The constraint (3.3.11) is taken into account by projecting of the obtained matrix at each iteration on the set of matrices with the spectral radius smaller than 1. More precisely, if $\rho\,(X_{n+1}) > 1$, then it is replaced by $X_{n+1}/\rho\,(X_{n+1})$.

3.3.1.4 The DtN operators and the interior problem

The approximated DtN operators $\Lambda^{+,h}(\alpha) = \Lambda^{-,h}(\alpha) = \Lambda^h(\alpha)$ are now obtained using the discrete analogue of the relation (3.2.45):

$$\Lambda^h(\alpha) = T^h_{00}(\alpha) + T^h_{10}(\alpha) P^h(\alpha).$$

The discretised version of the interior problem (3.2.28) is obtained from the weak formulation by replacing the DtN operators $\Lambda^{\pm}(\alpha) = \Lambda(\alpha)$ by their discrete analogue $\Lambda^h(\alpha)$. We denote by V^μ_h the analogue of the space (3.3.2) for the interior domain $\mathcal{C}^\mu_\varepsilon$:

$$V^\mu_h = \left\{ v_h \in C(\overline{\mathcal{C}^\mu_\varepsilon}), \quad v_h|_{T^\mu_\ell} \text{ is linear}, \;\; 1 \leqslant \ell \leqslant L^\mu \right\},$$

where L^μ is the number of triangles T^μ_ℓ in the mesh $\mathcal{T}^\mu_h$ of $\mathcal{C}^\mu_\varepsilon$. The set of the vertices of this mesh is denoted by $\mathcal{M}^\mu_h = \left\{M^\mu_j\right\}_{j=1}^{N^\mu}$ and the basis $\mathcal{B}^\mu_h = \left\{w^\mu_j\right\}_{j=1}^{N^\mu}$ is constructed analogously to (3.3.3). To be able to match this mesh with the mesh $\mathcal{T}_h$ from the right we need the number of vertices in the mesh of the boundary Γ^+_ε to be equal to the number of vertices in the mesh of Γ_0 (which is N_J). Similarly, the number of vertices in the mesh Γ^-_ε should be the same as the number of vertices in the mesh of Γ_1 (which is also N_J). We suppose for simplicity that the vertices that belong to the boundary Γ^-_ε are those with the index $1 \leqslant N_J$ (enumerated from up to down) and the vertices that belong to the boundary Γ^+_ε are those with the index $N_J + 1 \leqslant 2N_J$ (enumerated from up to down as well). Then, we have to solve the following problem:

$$\left(\nabla u^{0,h}_\varepsilon, \nabla v\right) + \left\langle \Lambda^h\left(\lambda^h_\varepsilon\right) u^{0,h}_\varepsilon, v \right\rangle\big|_{\Gamma^+_\varepsilon} + \left\langle \Lambda^h\left(\lambda^h_\varepsilon\right) u^{0,h}_\varepsilon, v \right\rangle\big|_{\Gamma^-_\varepsilon} = \left(\lambda^h_\varepsilon\right)^2 \left(u^{0,h}_\varepsilon, v\right), \qquad \forall v \in V^\mu_h.$$

If $U_\varepsilon^{0,h}$ is the vector of coordinates of $u_\varepsilon^{0,h}$ in the basis $\mathcal{B}_h^\mu$, then this problem can be rewritten in a matrix form as

$$\left(K^\mu + L\left(\lambda_\varepsilon^h\right)\right) U_\varepsilon^{0,h} = \left(\lambda_\varepsilon^h\right)^2 M^\mu U_\varepsilon^{0,h}, \tag{3.3.13}$$

where $L(\alpha)$ is the $(N^\mu \times N^\mu)$-matrix such that

$$L_{i,j}(\alpha) = \begin{cases} \left(\Lambda^h(\alpha)\right)_{(J^+)^{-1}(i),(J^+)^{-1}(j)}, & i,j \in \mathcal{J}^+, \\ \left(\Lambda^h(\alpha)\right)_{(J^-)^{-1}(i),(J^-)^{-1}(j)}, & i,j \in \mathcal{J}^-, \\ 0, & \text{otherwise,} \end{cases}$$

and K^μ, M^μ are the analogues of the matrices (3.3.9) for the mesh $\mathcal{T}_h^\mu$. The problem (3.3.13) is a nonlinear generalized eigenvalue problem (the nonlinearity is contained in the dependence of the matrix L on the spectral parameter). It is solved using the false position method described below (cf. for, example, [59]).

3.3.1.5 False position method

Let $\left]a_\varepsilon^h, b_\varepsilon^h\right[$ be an approximation of a gap of the operator A_ε^μ obtained as described in section 3.3.2. In order to solve the problem (3.3.13) we introduce the functions $\varkappa_n(\alpha)$, $n \in \mathbb{N}$, for $\alpha^2 \in \left]a_\varepsilon^h, b_\varepsilon^h\right[$ that correspond to n-th biggest generalized eigenvalue of the problem

$$\left(K^\mu + L(\alpha)\right) U = \varkappa^h(\alpha) M^\mu U. \tag{3.3.14}$$

Thus, λ_ε^h is a solution of the nonlinear equation

$$\varkappa_n^h(\alpha) = \alpha^2$$

for some $n \in \mathbb{N}$. Let $\varkappa^*(\alpha)$ be the nearest to α^2 eigenvalue:

$$\varkappa^{h,*}(\alpha) = \varkappa_n^h(\alpha), \qquad \left|\varkappa_n^h(\alpha) - \alpha^2\right| = \min_{j\in\mathbb{N}} \left|\varkappa_j^h(\alpha) - \alpha^2\right|.$$

Then, the problem is reduced to searching the roots of the function

$$f(\alpha) = \varkappa^{h,*}(\alpha) - \alpha^2. \tag{3.3.15}$$

This function is piecewise continuous on the interval $\left]a_\varepsilon^h, b_\varepsilon^h\right[$. In order to find its roots we shall first determine empirically the intervals where it is continuous, monotone and has a unique root and then apply the false position method on each of this intervals (it is impossible to apply the standard Newton's algorithm since we do not have analytical expressions for the derivative of the function $\varkappa^{h,*}(\alpha)$). Let $[\xi, \eta]$ be such an interval. Then, we initialize ξ_1 by ξ, η_1 by η and put at n-th iteration

$$x_n = \frac{\xi_n f(\eta_n) - \eta_n f(\xi_n)}{f(\eta_n) - f(\xi_n)}, \tag{3.3.16}$$

$$\xi_{n+1} = \xi_n, \quad \eta_{n+1} = x_n, \qquad \text{if} \qquad f(\xi_n) f(x_n) < 0, \tag{3.3.17}$$

$$\xi_{n+1} = x_n, \quad \eta_{n+1} = \eta_n, \qquad \text{if} \qquad f(\eta_n) f(x_n) < 0. \tag{3.3.18}$$

The limit of the sequence $\{x_n\}$ is the value λ_ε^h which gives an approximation of λ_ε.
The initial interval $[\xi, \eta]$ can be chosen for example by plotting the function $f(\alpha)$. We can also note that we expect to find one or two such intervals depending on the type of the corresponding gap of the operator A^μ on the graph (cf. Theorem 1.3.1) at least for ε small enough. The knowledge of the eigenvalues of the operator A^μ also gives an idea about the location of the ones of the operator A_ε^μ and can help to choose the intervals in question.

3.3.2 The essential spectrum

There are two ways of determining the essential spectrum of the operator A_ε^μ. The first one is based on the computation of the spectral radius of the propagation operator (cf. Remark 3.2.4). Discretizing the interval of interest and computing the spectral radius of the propagation operator at each point of the grid one finds if this point belongs to the essential spectrum or not. However, this method requires the computation of the propagation operator, and, consequently, the solution of the Riccati equation, for each value of α. For this reason we privilege a more standard method which is based on the Floquet-Bloch decomposition and the determination of dispersion curves. As it was mentioned in Section 1.2.1, the essential spectrum of the operator A_ε^μ can be decomposed as

$$\sigma_{ess}\left(A_\varepsilon^\mu\right) = \bigcup_{n\in\mathbb{N}} \lambda_n(\varepsilon, [0,\pi]),$$

where $\lambda_n(\varepsilon,\theta)$ is the n-th biggest eigenvalue of the operator $A_\varepsilon(\theta)$ defined in (1.2.3). Recall that for fixed ε and n it is a continuous function of θ. Hence, the above characterisation of the spectrum can be rewritten as

$$\sigma_{ess}\left(A_\varepsilon^\mu\right) = \bigcup_{n\in\mathbb{N}} \left[\min_{\theta\in[0,\pi]} \lambda_n(\varepsilon,\theta),\ \max_{\theta\in[0,\pi]} \lambda_n(\varepsilon,\theta)\right].$$

Thus, it is sufficient to discretize the interval $[0,\pi]$ by the points $0 < \theta_1 < \theta_2 \ldots \theta_K < \pi$ and compute the first Q eigenvalues of the operator $A_\varepsilon^h(\theta_i)$ (obtained after a discretization of the operator $A_\varepsilon(\theta_i)$) for each of this points. This will yield an approximation of the beginning of the essential spectrum of the operator A_ε^μ:

$$\sigma_{ess}^{K,Q}\left(A_\varepsilon^\mu\right) = \bigcup_{1\leqslant n\leqslant Q} \left[\min_{1\leqslant i\leqslant K} \lambda_n(\varepsilon,\theta_i),\ \max_{1\leqslant i\leqslant K} \lambda_n(\varepsilon,\theta_i)\right]. \tag{3.3.19}$$

In order to find approximations of the eigenvalues of the operator $A_\varepsilon(\theta)$ for some $\theta\in[0,\pi]$ we use the mesh $\mathcal{T}_h$ described in section 3.3.1.1. However, we shall now take into account the θ-quasiperiodic boundary conditions on the boundaries $\Gamma_{\varepsilon,0}$, $\Gamma_{\varepsilon,1}$. It can be expressed by the fact that the space V_h should be replaced by its θ-quasiperiodic subspace

$$V_h^\theta = \left\{ v_h \in V_h, \quad v_h\big|_{\Gamma_{\varepsilon,1}} = e^{-i\theta}\, v_h\big|_{\Gamma_{\varepsilon,0}} \right\}.$$

Then, the weak formulation for the eigenvalue problem for the operator $A_\varepsilon(\theta)$ implies solving the following discretised problem for $\lambda_\varepsilon^h(\theta)$ and $u\in V_h^\theta$:

$$(\nabla u, \nabla v)_{L_2(\mathcal{C}_\varepsilon)} = \lambda_\varepsilon^h(\theta)(u,v)_{L_2(\mathcal{C}_\varepsilon)}, \qquad \forall v\in V_h^\theta.$$

From a practical point of view, the space V_h^θ is obtained from V_h by considering the basis $\mathcal{B}_h^\theta = \{\widetilde{w}_j,\ j \in [1, N_J] \cup [2N_J+1, N]\}$ in the space V_h^θ, where

$$\widetilde{w}_j = \begin{cases} w_j, & j \geqslant 2N_J + 1, \\ w_j + e^{-i\theta} w_{j+N_J}, & 1 \leqslant j \leqslant N_J. \end{cases}$$

Decomposing u in this basis as $u = \sum_{j=1}^{N_J} U_j \widetilde{w}_j + \sum_{j=2N_J+1}^{N} U_j \widetilde{w}_j$ and denoting by U the $(N - N_J) \times 1$ vector of coordinates U_j (where the coordinates with $N_J + 1 \leqslant j \leqslant 2N_J$ are excluded), we end up with the following generalised eigenvalue problem:

$$\widetilde{K}U = \lambda_\varepsilon^h(\theta)\widetilde{M}U. \tag{3.3.20}$$

Here $\widetilde{K}$ and $\widetilde{M}$ are $(N - N_J) \times (N - N_J)$ with matrix elements defined as follows:

$$\widetilde{K}_{i,j} = (\nabla\widetilde{w}_i, \nabla\widetilde{w}_j)_{L_2(\mathcal{C}_\varepsilon)}, \quad \widetilde{M}_{i,j} = (\widetilde{w}_i, \widetilde{w}_j)_{L_2(\mathcal{C}_\varepsilon)}, \qquad j \in [1, N_J] \cup [2N_J+1, N].$$

Once the matrices K, M are known, the matrices $\widetilde{K}$, $\widetilde{M}$ can be constructed using the relations

$$\widetilde{K}_{i,j} = \begin{cases} K_{i,j}, & i, j \geqslant 2N_J + 1 \\ K_{i,j} + e^{-i\theta} K_{i+N_J,j}, & 1 \leqslant i \leqslant N_J, \quad j \geqslant 2N_J + 1, \\ K_{i,j} + e^{i\theta} K_{i,j+N_J}, & i \geqslant 2N_J + 1, \quad 1 \leqslant j \leqslant N_J, \\ K_{i,j} + e^{-i\theta} K_{i+N_J,j} + e^{i\theta} K_{i,j+N_J} + K_{i+N_J,j+N_J}, & 1 \leqslant i, j \leqslant N_J, \end{cases}$$

and its analogue for the matrix $\widetilde{M}$. Now the generalized eigenvalue problem (3.3.20) can be solved for each θ_i, $1 \leqslant i \leqslant K$ which yields an approximation of the beginning of the essential spectrum of the operator A_ε^μ due to (3.3.19). We remark however one drawback of this method compared to the method based on the computation of the spectral radius of the propagation operator: we do not know a priori how many spectral bands we have to compute in order to cover the interval of frequences we are interested in. Here again the knowledge of the spectrum of the operator A on the graph can give an approximate idea, at least for ε small enough.

3.3.3 Summary of the algorithm

Let us now resume the algorithm of computation of the eigenvalues of the operator A^μ_ε and give some details of implementation in Matlab.

I **Essential spectrum**

- The function `CalculSpectreEssentiel.m` computes an approximation of the beginning of the essential spectrum of the operator A^μ_ε using the first N eigenvalues $\lambda_n(\varepsilon, \theta)$.
- The function `Calc_spec_ess_R.m` computes an approximation of the essential spectrum of the operator A^μ_ε on the interval $[a_0^2, b_0^2]$ using the spectral radius of the propagation operator.

Using one of these methods one finds approximations of gaps of the operator A^μ_ε. To be precise, the result provided by these functions refers to the square root of the spectral parameter. In order to pass to the spectral parameter, we should take the square of the obtained result. Let us fix one of the approximated gaps, that we denote by $]a^2, b^2[$, and look for eigenvalues inside it.

II **Discrete spectrum**

(a) Let us start by plotting the function $f(\alpha)$ defined in (3.3.15) on the interval $]a, b[$. This step can be skipped if an interval $[\xi, \eta]$ described in Section 3.3.1.5 can be found in another way.

Construction of the matrices K, M and K^μ, M^μ ← `KMA.m`
for α = a : pas : b ← `dessin_eig.m`
compute $\varkappa^*(\alpha)$ ← `compute_eig.m` which consists in
- construction of the matrices $T^h_{\gamma\delta}(\alpha)$ ← `DtN_eig.m`
  - construction of the matrices $A(\alpha)$ and $A^D(\alpha)$
  - construction of the matrices $E_0(\alpha)$, $E_1(\alpha)$
  - construction of the matrices $T^h_{00}(\alpha)$, $T^h_{01}(\alpha)$, $T^h_{10}(\alpha)$, $T^h_{11}(\alpha)$
- computation of the matrix $P^h(\alpha)$ ← `newton1.m` (solving the Riccati equation)
- construction of the matrices $L^\pm(\alpha)$
- solving the eigenvalue problem (3.3.14) ← Matlab function `eigs.m`

end

(b) Once an interval $[\xi, \eta]$ is found, the false position method is applied on this interval.

while (abs(f(x_n)) > tol) or (abs(x_n-x_{n-1}) > tol) ← `quasinewton.m`
compute x_{n+1} (cf. (3.3.16)) ← using `compute_eig.m`
compute ξ_{n+2}, η_{n+2} (cf. (3.3.17)–(3.3.18))
end

3.4 Numerical results

In this section we present numerical results obtained with a Matlab code which implements the algorithm described above.

3.4.1 Validation of the computation of the local DtN operators

We take as a test geometry the square with a unit edge shown in figure 3.6.

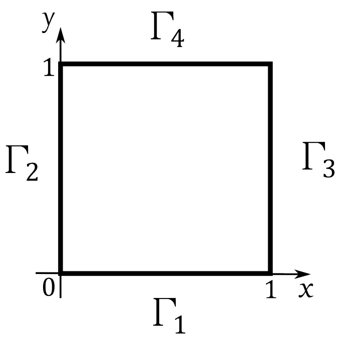


Figure 3.6: Test geometry: a unit square

The first test problem is the following:

$$\begin{cases} \Delta u = 0, \\ u|_{\Gamma_2} = 1, \\ u|_{\Gamma_3} = 0, \\ \partial_n u|_{\Gamma_1 \cup \Gamma_4} = 0. \end{cases} \tag{3.4.1}$$

Its exact solution is $u_{ex}(x, y) = 1 - x$. We compute the matrix $E_0(0)$ with the help of the function `DtN.m`. It follows from the definition of the matrix $E_0(0)$ that the sum of its columns gives an approximation of u_{ex}. It is shown in figure 3.7. In this case there is no error due to the finite element approximation since the exact solution is linear.

The second test problem is

$$\begin{cases} -\Delta u = \frac{\pi^2}{4} u, \\ u|_{\Gamma_2} = \cos\left(\frac{\pi y}{2}\right), \\ u|_{\Gamma_3} = \cos\left(\frac{\pi y}{2}\right), \\ u|_{\Gamma_4} = 0, \\ \partial_n u|_{\Gamma_1} = 0. \end{cases} \tag{3.4.2}$$

Its exact solution is $u_{ex}(x, y) = \cos\left(\frac{\pi y}{2}\right)$. Let U_{ex} be the vector of values of u_{ex} at the vertices of the grid $\mathcal{T}_h$:

$$U_{ex}(j) = \cos\left(\frac{\pi y(M_j)}{2}\right), \qquad 1 \leqslant j \leqslant N.$$

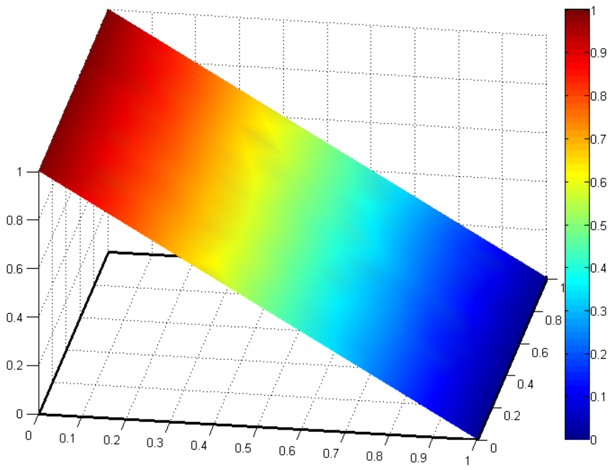


Figure 3.7: Solution of the problem (3.4.1) obtained with the function `DtN.m` using an unstructured grid with $h = 0.1$

We compute the matrices $E_0\left(\frac{\pi}{2}\right)$, $E_1\left(\frac{\pi}{2}\right)$ with the help of the function `DtN.m`. This yields the following approximation of U_{ex}:

$$U = \sum_{i=1}^{N_y} \left(E_0^i\left(\tfrac{\pi}{2}\right) \cos\left(\tfrac{\pi y(M_{J_0(i)})}{2}\right) + E_1^i\left(\tfrac{\pi}{2}\right) \cos\left(\tfrac{\pi y(M_{J_1(i)})}{2}\right)\right),$$

where E_0^i, E_1^i are the i-th columns of the matrices E_0, E_1 respectively. The solution U is represented in figure 3.8a for an unstructured grid with $h = 0.1$. We study the L_2 and H^1 errors given by the relations

$$err\_L2 = \left((U - U_{ex})^t M (U - U_{ex})\right)^{1/2}, \qquad err\_H1 = \left((U - U_{ex})^t K (U - U_{ex})\right)^{1/2}.$$

In figure 3.8b the dependence of these errors on h is represented in a logarithmic scale. As we see, when $h \to 0$, the slopes are in agreement with the theoretic ones: 2 for the L_2 error and 1 for the H^1 error.

Next, we validate the computation of the operators $T_{\gamma\delta}$, $\gamma, \delta \in \{0, 1\}$ considering the test problems

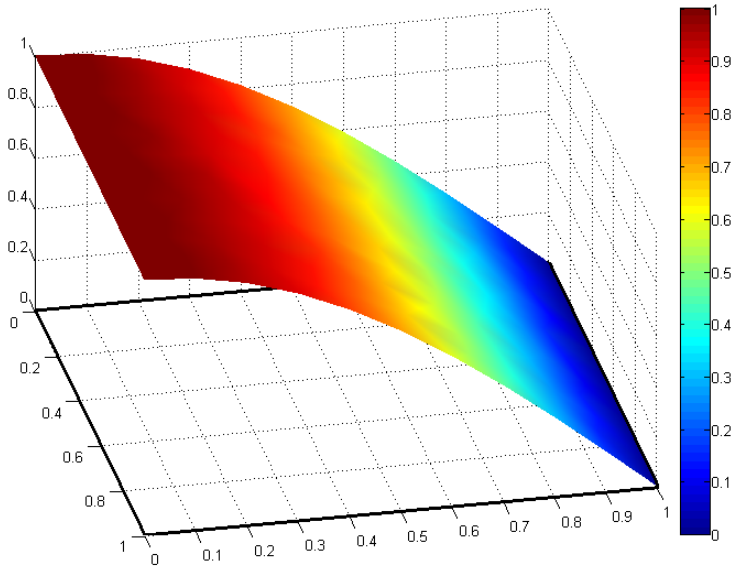
$$\begin{cases} -\Delta u_0 = \omega^2 u_0, \\ u_0|_{\Gamma_2} = \cos(k\pi y), \\ u_0|_{\Gamma_3} = 0, \\ \partial_n u_0|_{\Gamma_1 \cup \Gamma_4} = 0, \end{cases} \qquad \begin{cases} -\Delta u_1 = \omega^2 u_1, \\ u_1|_{\Gamma_2} = 0, \\ u_1|_{\Gamma_3} = \cos(k\pi y), \\ \partial_n u_1|_{\Gamma_1 \cup \Gamma_4} = 0, \end{cases} \tag{3.4.3}$$

for $k \in \mathbb{N}$. The exact expressions for $T_{\gamma\delta}(\omega)\varphi_k$, $f_k(y) = \cos(k\pi y)$, are

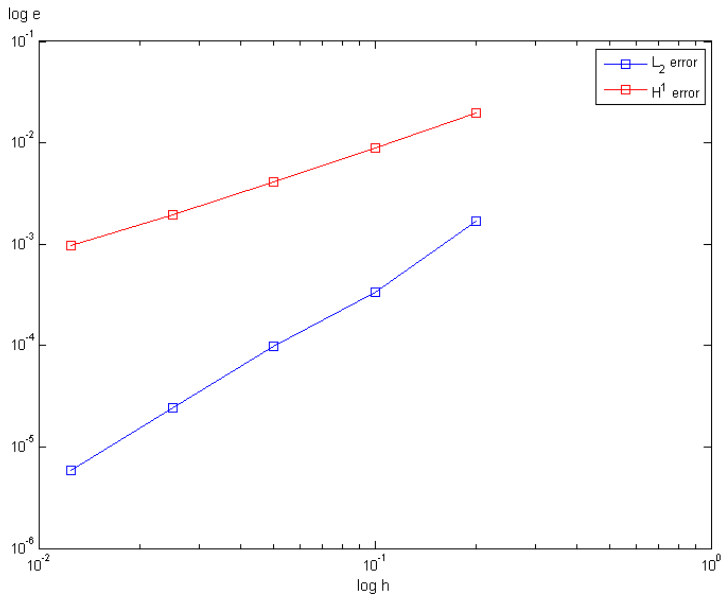
$$T_{00}(\omega) f_k = T_{11}(\omega) f_k = \begin{cases} \dfrac{\alpha}{\tan\alpha} \cos(k\pi y), & \omega > k\pi, \\ \cos(k\pi y), & \omega = k\pi, \\ |\alpha| \dfrac{1 + e^{2|\alpha|}}{e^{2|\alpha|} - 1} \cos(k\pi y), & \omega < k\pi, \end{cases}$$

$$T_{01}(\omega)f_k = T_{10}(\omega)f_k = \begin{cases} -\dfrac{\alpha}{\sin\alpha}\cos(k\pi y), & \omega > k\pi, \\ -\cos(k\pi y), & \omega = k\pi, \\ -2|\alpha|\dfrac{e^{|\alpha|}}{e^{2|\alpha|}-1}\cos(k\pi y), & \omega < k\pi, \end{cases}$$

where $\alpha = \sqrt{(k\pi)^2 - \omega^2}$.



(a) Solution U obtained for an unstructured grid with $h = 0.1$



(b) L_2 and H^1 error for different values of h

Figure 3.8: Numerical resolution of the test problem (3.4.2) using the function `DtN.m`

We compute the L_2 error

$$error\_T_{\gamma\delta} = \left\|T^h_{\gamma\delta}(\omega) f_k - T_{\gamma\delta}(\omega) f^h_k\right\|_{L_2(\Gamma_{\delta+2})}$$
$$\simeq \left(\left(T^h_{\gamma\delta}(\omega) F_k - S^{(\delta)} Y^k_{\gamma\delta}(\omega)\right)^t S^{(\delta)} \left(T^h_{\gamma\delta}(\omega) F_k - S^{(\delta)} Y^k_{\gamma\delta}(\omega)\right)\right)^{1/2}. \quad (3.4.4)$$

Here $T^h_{\gamma\delta}(\omega)$ are the approximated local DtN operators computed with the function `DtN.m`, F_k is the vector of values of the function f_k at the vertices $\mathcal{M}^\gamma_h$, $Y^k_{\gamma\delta}(\omega)$ is the vector of values of $T_{\gamma\delta}(\omega) f_k$ at the vertices $\mathcal{M}^\delta_h$ and $S^{(\delta)}$ are the surface mass matrices for the boundaries $\Gamma_{\delta+2}$ defined as follows:

$$S^{(\delta)}_{i,j} = \int\limits_{\Gamma_{\delta+2}} \varphi^\delta_i \varphi^\delta_j, \qquad 1 \leqslant i, j \leqslant N_J, \qquad \delta \in \{0, 1\}.$$

In figure 3.9 we represent the dependence of the error (3.4.4) on h for $\omega = 5$, $k = 2$. The slope in a logarithmic scale is 2.

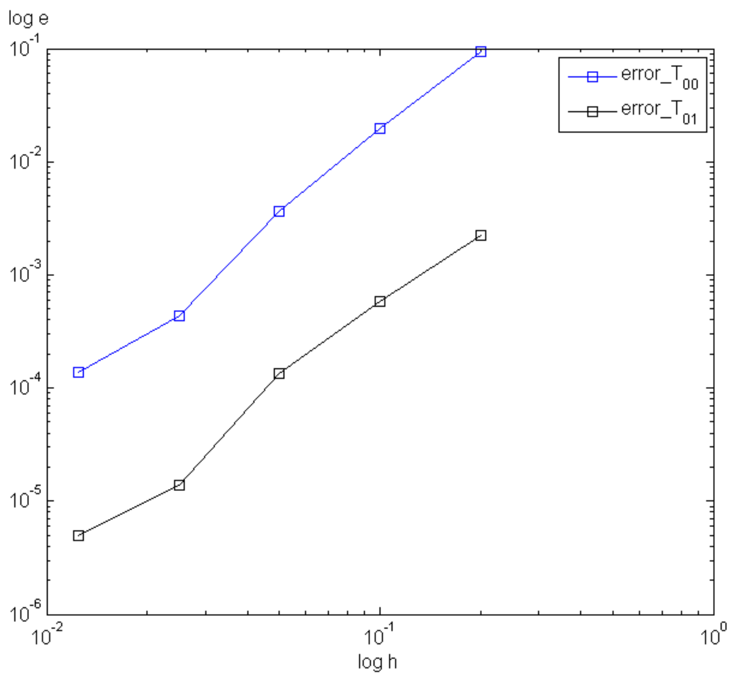
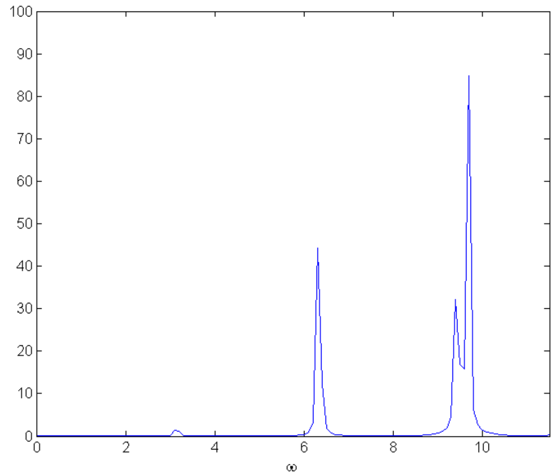


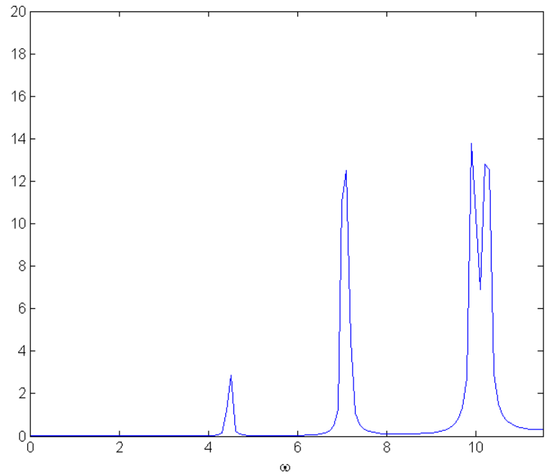
Figure 3.9: Dependence of the error (3.4.4) on h

In figure 3.10 we represent the dependence of the error on ω for fixed k and h. We can see that it has singularities at the points ω that belong to the set $\left\{\pi\sqrt{m^2 + n^2},\ m \in \mathbb{N},\ n \in \mathbb{N}^*\right\}$. For these points ω^2 is an eigenvalue of the problem

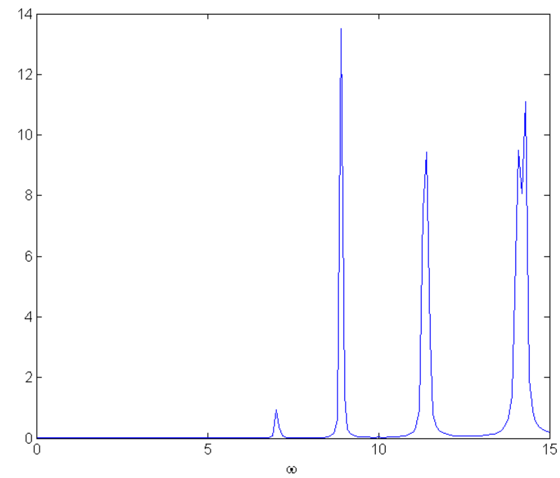
$$\begin{cases} -\Delta u = \omega^2 u, \\ u|_{\Gamma_{2,3}} = 0, \\ \partial_n u|_{\Gamma_1 \cup \Gamma_4} = 0. \end{cases} \quad (3.4.5)$$



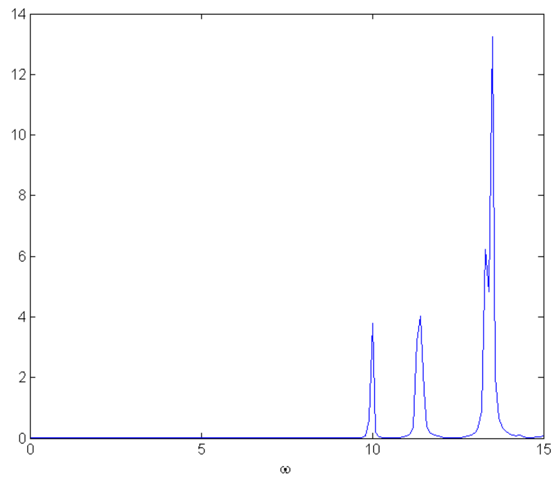
(a) $error\_T_{01}$ for $k = 0$, $h = 0.1$



(b) $error\_T_{00}$ for $k = 1$, $h = 0.1$



(c) $error\_T_{11}$ for $k = 2$, $h = 0.05$



(d) $error\_T_{10}$ for $k = 3$, $h = 0.05$

Figure 3.10: Dependence of the error (3.4.4) on ω: as expected, singularities are observed at the points corresponding to the eigenvalues of the problem (3.4.5)

3.4.2 Numerical computation of the essential spectrum

We now compute the essential spectrum of the operator $A_{\varepsilon,s}$ for $L = 2$ using the function `CalculSpectreEssentiel.m`. In figure 3.11 we represent the essential spectrum computed for different values of ε using the first 5 spectral bands. The intervals in blue correspond to the values λ such that λ^2 belongs to the spectral bands and the intervals in white separating them correspond to the values λ such that the λ^2 belongs to the spectral gaps. In what follows we mean by representing the spectrum the representation of the parameter λ (and not λ^2 which is the spectral parameter). For $\varepsilon = 0$ the spectrum of the limit operator A_s is represented.

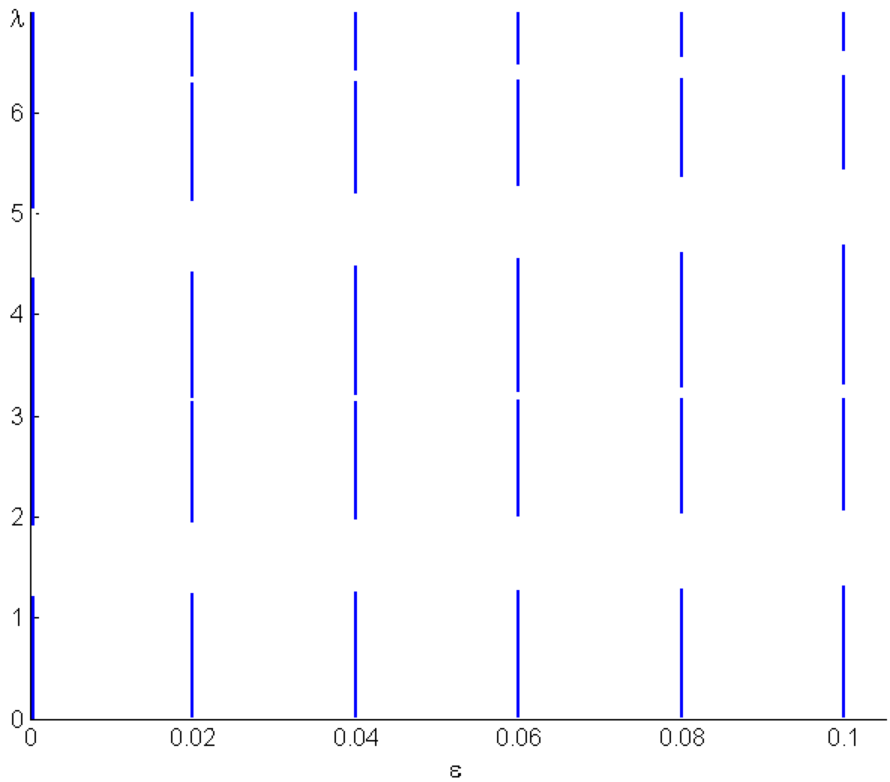


Figure 3.11: Dependence of the essential spectrum of the operator A^s_ε on ε: the first 4 gaps. For each value of ε an unstructured grid has been used with h chosen in such a way that the error due to the discretization is very small compared to the effect due to ε: $h = 0.00125$ for $\varepsilon = 0.02$ and $\varepsilon = 0.04$, $h = 0.0025$ for $\varepsilon = 0.06$, $\varepsilon = 0.08$, $\varepsilon = 0.1$.

One can see that the spectrum of the operator $A_{\varepsilon,s}$ is very close to the spectrum of the limit operator for small values of ε (more precisely, the convergence is linear as it is predicted by the theory, cf. figure 3.12). However, we can notice a phenomenon that has not been studied theoretically in Chapter 1: opening of a gap near the values $\{\pi\mathbb{N}^*\}$. These are the points where the dispersion curves for the limit operator A_s touch (cf. figure 3.14). As shown in figure 3.13, the size of these gaps is also linear in h.

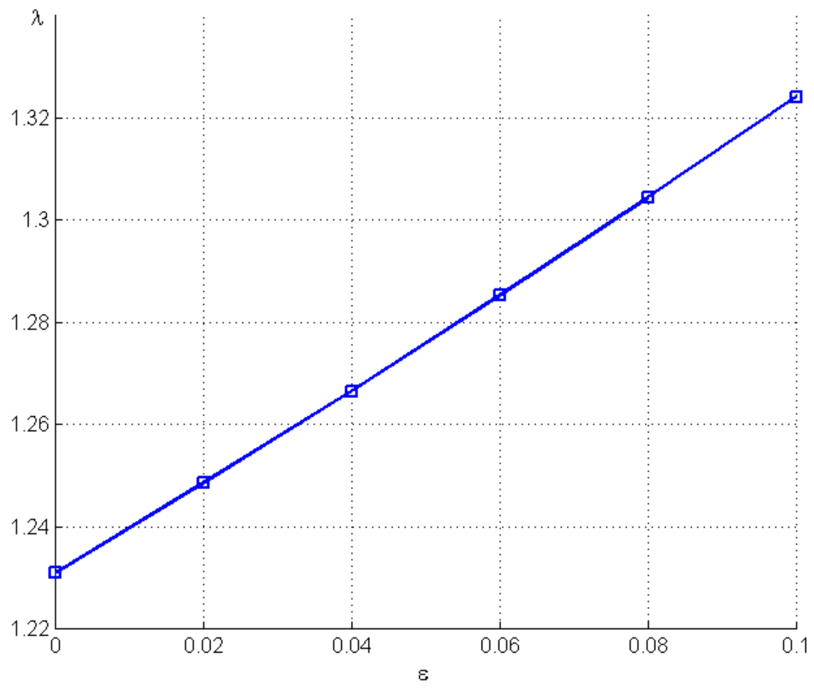


Figure 3.12: Dependence of the lower end of the first gap on ε: linear convergence to the lower end of the first gap of the limit operator A_s.

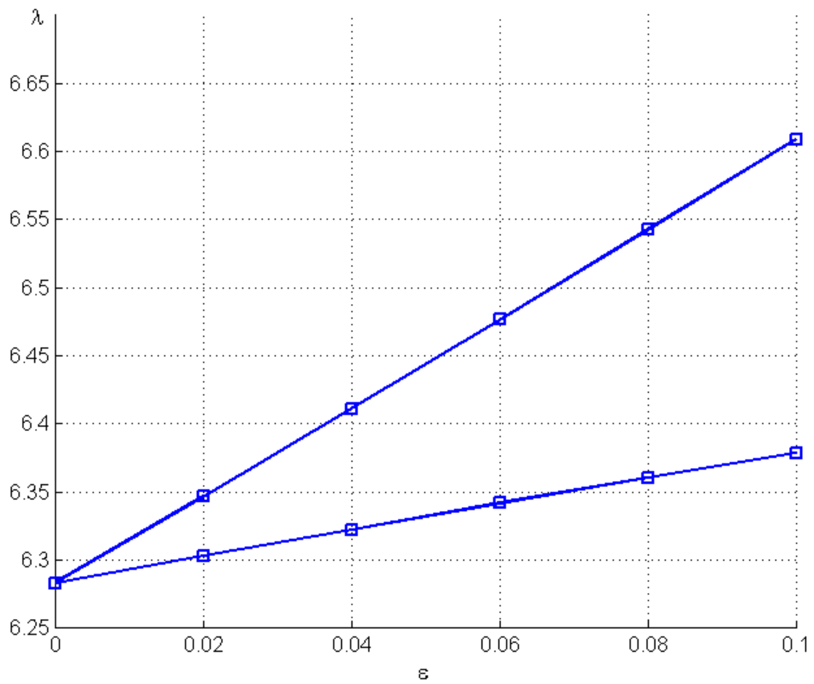
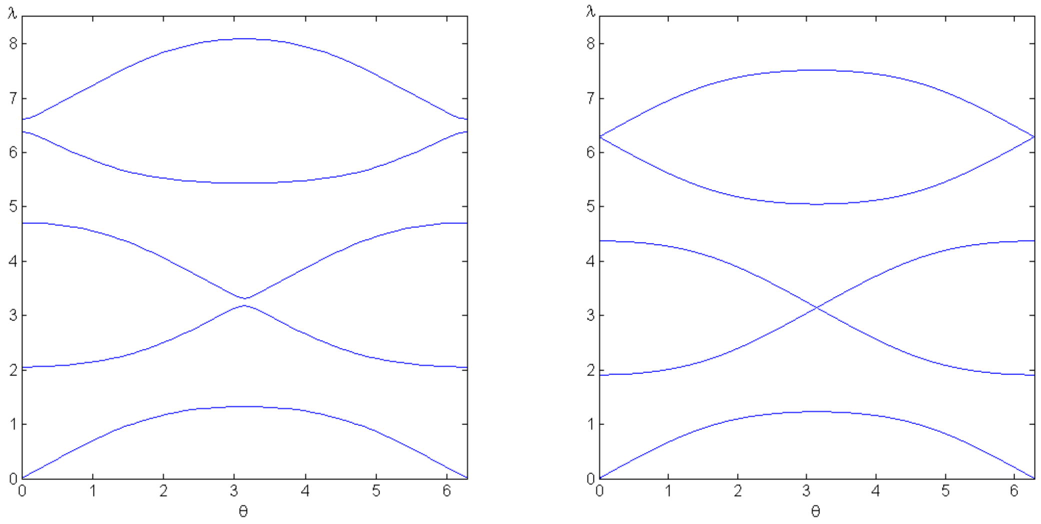


Figure 3.13: Opening of a gap in the neighbourhood of the point 2π.



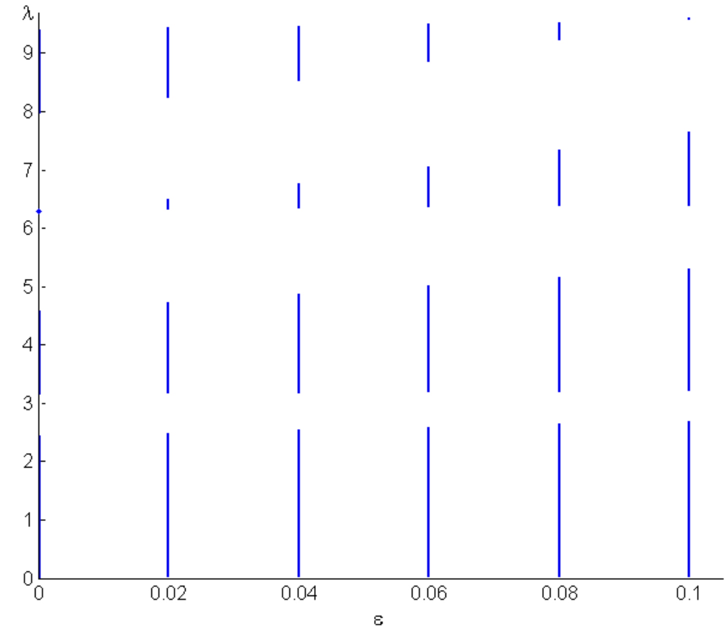
(a) $\varepsilon = 0.1$ (b) graph

Figure 3.14: Dispersion curves in the case $L = 2$ for the graph and the 2D domain with $\varepsilon = 0.1$: the points $\{\lambda = \pi n,\ n \in \mathbb{N}^*\}$ where two spectral bands for the graph touch become gaps for the 2D domain.

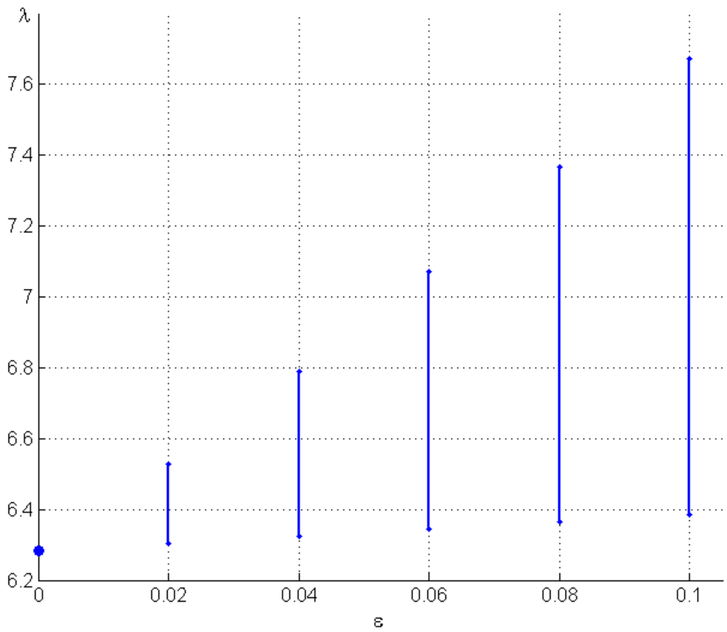
Another phenomenon concerns the eigenvalues of infinite multiplicity for the limit operator. Consider the case $L = 0.5$. Then, the operator A_s has the following set of eigenvalues of infinite multiplicity: $\sigma_{pp} = \{\lambda^2,\ \lambda = 2(2n+1)\pi,\ n \in \mathbb{N}\}$ (cf. property 4 of Proposition 1.3.4). An eigenvalue of infinite multiplicity can only become a spectral band in the 2D case with ε small enough. Indeed, as shown in [57], the dimension of the spectral projector on any interval is preserved for ε small enough. On the other hand, a periodic 2D operator cannot have eigenvalues. Thus, the only possibility is that the operator $A_{\varepsilon,s}$ has a small spectral band in a neighbourhood of an eigenvalue of infinite multiplicity. This situation is shown in figure 3.15.

3.4.3 Numerical computation of the discrete spectrum

We now present the results for the discrete spectrum of the operator $A^{\mu}_{\varepsilon,s}$. In figure 3.16 the function $f(\alpha)$ defined in (3.3.15) is represented in the first gap of the operator $A^{\mu}_{\varepsilon,s}$ in the case $L = 2$, $\mu = 0.25$, $\varepsilon = 0.1$. We see from this graph that there are exactly 2 roots of the function f in the gap which correspond to two eigenvalues. The interval $[\xi, \eta]$ can be chosen, for instance, as $[1.4, 1.5]$ for the first eigenvalue and $[1.9, 2]$ for the second one. The eigenvalues computed for different values of ε using the function `quasinewton.m` are represented in figure 3.17. In figure 3.18 we show the convergence of the eigenvalues in the first gap which is linear as predicted by the theory. In figure 3.19 the eigenfunction corresponding to the first eigenvalue of the operator $A^{\mu}_{\varepsilon,s}$ is represented.
In figure 3.20 we represent the dependence of the eigenvalues on $\mu \in]0, 1[$. As it is natural to expect, the smaller μ is (so, the stronger the perturbation is), the better the eigenvalues are separated from the essential spectrum. When μ is close to 1, the computation becomes more costly since the distance between the eigenvalue and the essential spectrum is very small.



(a) Essential spectrum for $L = 0.5$



(b) Zoom at the neighbourhood of the point $\lambda = 2\pi$.

Figure 3.15: Case $L = 0.5$: the size of the spectral band that appears in a neighbourhood of the point $\lambda = 2\pi$ (which is an eigenvalue of infinite multiplicity of the limit operator A_s) is linear in ε.

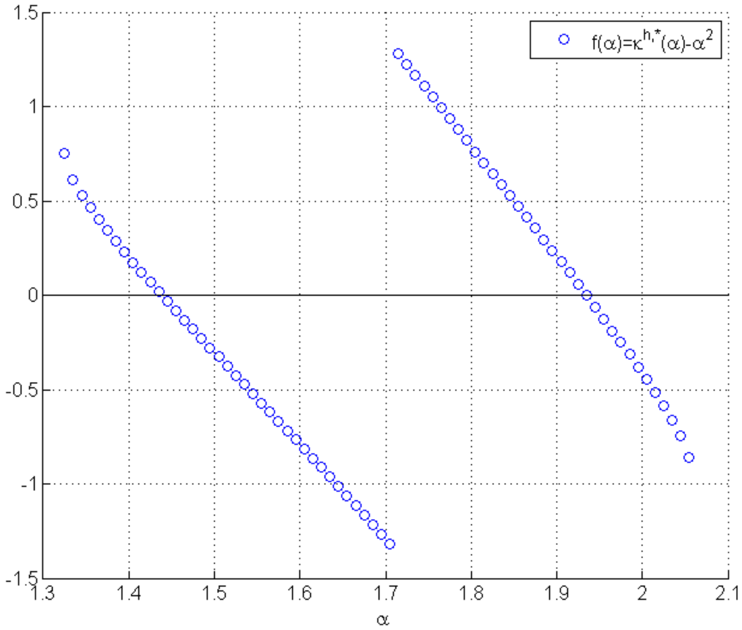


Figure 3.16: Function $f(\alpha) = \varkappa^{h,*}(\alpha) - \alpha^2$ in the first gap of the operator $A^{\mu}_{\varepsilon,s}$ for $L = 2$, $\mu = 0.25$, $\varepsilon = 0.1$.

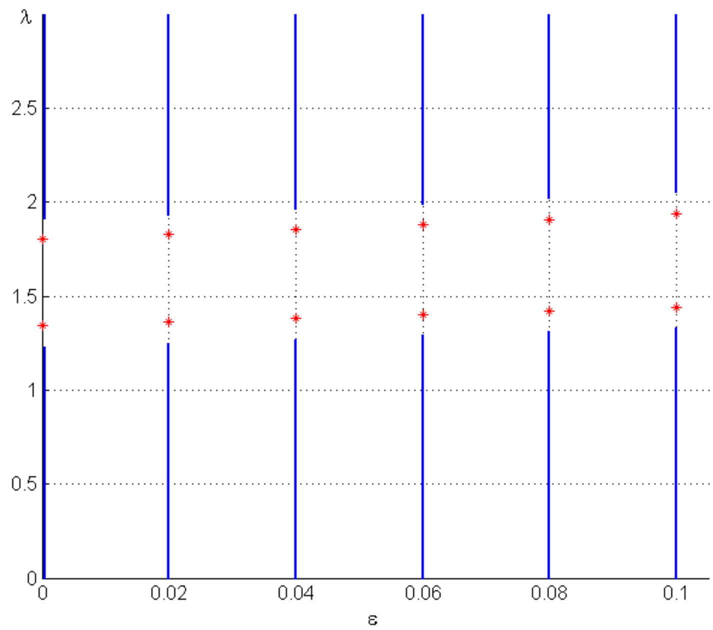


Figure 3.17: Eigenvalues of the operator $A^{\mu}_{\varepsilon,s}$ for $L = 2$, $\mu = 0.25$ (red asterisks) computed with the function `quasinewton.m`. The values for $\varepsilon = 0$ correspond to the limit operator A^{μ}_{s}.

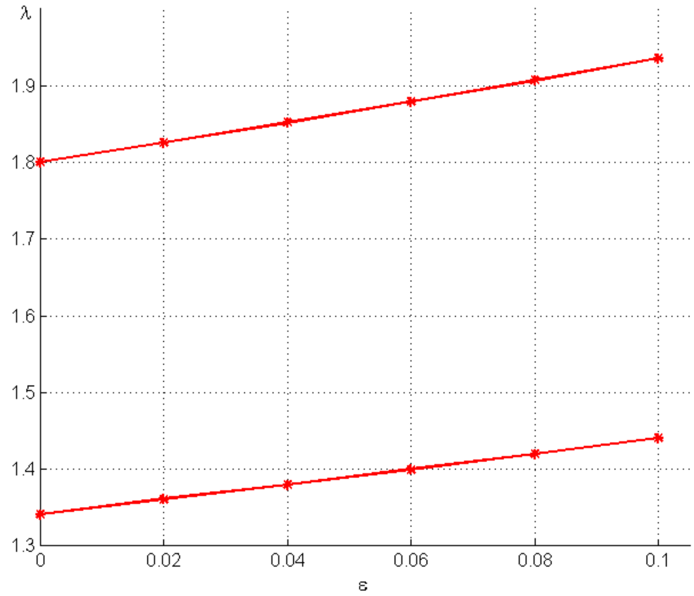
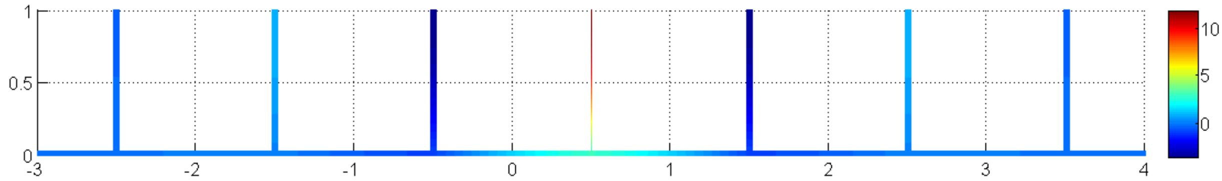
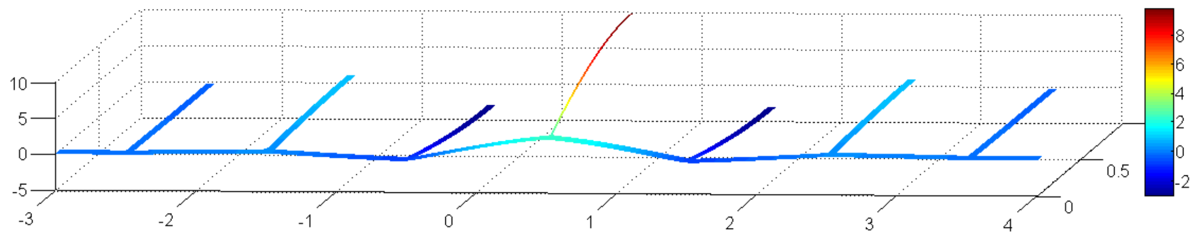


Figure 3.18: Linear convergence of the eigenvalues of the operator $A^{\mu}_{\varepsilon,s}$ in the first gap for $L = 2$, $\mu = 0.25$ as $\varepsilon \to 0$.



(a) $\varepsilon = 0.04$, $\lambda_1 \approx 1.38$



(b) $\varepsilon = 0.06$, $\lambda_1 \approx 1.40$

Figure 3.19: Eigenfunction corresponding to the first eigenvalue of the operator $A^{\mu}_{\varepsilon,s}$ for $L = 2$, $\mu = 0.25$.

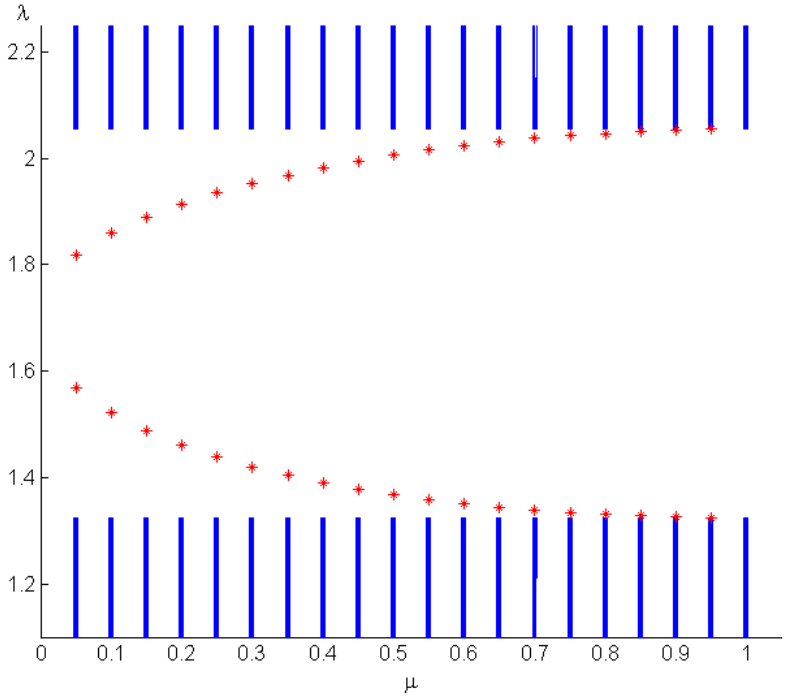
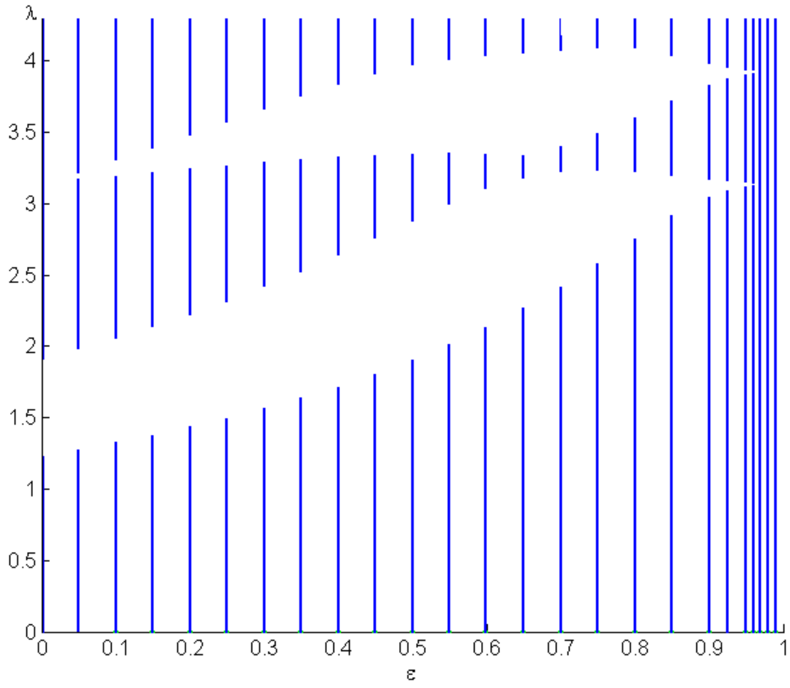


Figure 3.20: Dependence of the eigenvalues in the first gap on μ for $L = 2$, $\varepsilon = 0.1$.

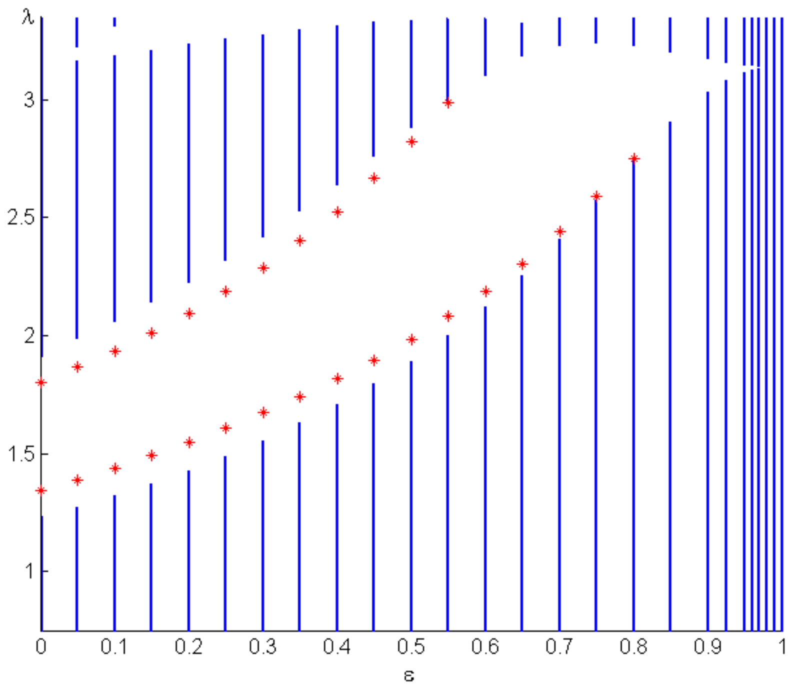
Another question for which no theoretical answer has been given for the moment is what happens for larger values of ε when the spectrum of the operator $A^{\mu}_{\varepsilon,s}$ is not close to the spectrum of the limit operator. In particular, a gap that exists for small values of ε does it still exist for any value of ε (until the obstacles disappear)? The eigenvalues inside the gaps that exist for the limit operator do they exist for any value of ε or do they immerse into the essential spectrum before the gap disappears?

In the cases that we tested numerically we saw that the gaps were present for any value of ε for which the obstacles are present, i.e. for $\varepsilon \in]0, \min\{1, L/2\}[$. In figure 3.21a we show the dependence of the first two gaps on ε in the case $L = 2$. Of course, when ε is close to 1 the computation becomes very costly since the size of the gaps is very small. For any discretization taken we obtain reliable results only up to a certain value of ε that is smaller than 1. However, the existence of the gaps at least up to this value (0.9 in the particular case represented in figure 3.21a) permits to conjecture that they exist for any $\varepsilon < 1$.

The behaviour of the eigenvalues is even more unclear. In figure 3.21b we represent the eigenvalues in the first gap of the operator $A^{\mu}_{\varepsilon,s}$ for $L = 2$, $\mu = 0.25$. It seems that the eigenvalues immerse into the essential spectrum for some values of $\varepsilon < 1$ (the second eigenvalue seems to disappear between $\varepsilon = 0.55$ and $\varepsilon = 0.6$). Again, the analysis becomes costly when the eigenvalues approach the essential spectrum. For this reason we cannot distinguish between the case when the eigenvalues do not exist any more and the case when they exist but are very close to the essential spectrum. For the second eigenvalue the computation with a mesh of size $h = 0.0025$ gave the following results: for $\varepsilon = 0.55$ the upper edge of the first gap $b_1 \approx 2.995$, the second eigenvalue $\lambda_2 \approx 2.987$, for $\varepsilon = 0.6$ $b_1 \approx 3.1$, the second eigenvalue was not found.



(a) Dependence of the first two gaps of the operator $A_{\varepsilon,s}$ on ε for $L = 2$. The gaps are present for any $\varepsilon < 1$.



(b) Eigenvalues of the operator $A_{\varepsilon,s}$ in the first gap for $L = 2$, $\mu = 0.25$ seem to immerse into the essential spectrum at some $\varepsilon < 1$.

Figure 3.21: Behaviour of the spectrum for large values of ε.

Comparison with the eigenvalues found from using asymptotic development

We will now compare the results obtained for the eigenvalues using the DtN operator method described in this chapter with the results obtained from the asymptotic expansion of the eigenvalue found in Chapter 2. We remind the algorithm of computation of the first n terms of the asymptotic expansion.

Algorithm of computation of $\lambda^{(n)}$

I Initialization:

- $\lambda^{(0)}$ is an eigenvalue of the limit operator A_s^μ. It is computed by solving the equation (1.3.55) (we remind that according to the notation of Chapter 2 λ should be replaced by $\sqrt{\lambda}$):

$$1 - \frac{\sqrt{g^2(\sqrt{\lambda}) - 1}}{|g(\sqrt{\lambda}) + \cos\sqrt{\lambda}|} = \mu.$$

- $u^{(0)}$ is the corresponding eigenfunction on the graph given by the relations (2.2.40)–(2.2.42). We represent it in the form (2.3.4)–(2.3.5):

$$\widetilde{a}_0^{(0)}(j) = 1, \qquad \widetilde{b}_0^{(0)}(j) = \frac{1}{\sin\sqrt{\lambda^{(0)}}}\left(r - \cos\sqrt{\lambda^{(0)}}\right), \qquad j \in \mathbb{N},$$

$$\widetilde{c}_0^{(0)}(j) = \frac{1}{\cos\left(\frac{\sqrt{\lambda^{(0)}}L}{2}\right)}, \qquad \widetilde{d}_0^{(0)}(j) = 0, \qquad j \in \mathbb{N}^*.$$

- $U_j^{(0)} = \mathbf{u}_j^{(0)}, \quad \forall j \in \mathbb{Z}$. We represent it in the form (2.3.21):

$$\mathcal{U}_0^{(0)} = 1, \qquad j \in \mathbb{N}^*.$$

II Computation of the functions $W_j^\pm$ solving the problems (2.2.18), (2.2.19) : P_1 finite elements. There are 4 problems to solve: (2.2.18) and (2.2.19) for $j = 0$. and for $j \neq 0$. The solutions of these problems are unique modulo a constant. In order to fix the missing constants, we add the condition

$$\int_{\Sigma_j^+} W^\pm = 0.$$

III For $k = 1 : n$

$\left\{\lambda^{(\ell)}, u^{(\ell)}, U^{(\ell)}\right\}_{\ell=1}^{k-1}$ are known

1) Computation of the right-hand sides of the far field problem for $u^{(k)}$ and the near field problem for $U^{(k)}$ (see the proof of Proposition 2.3.1):

- The function $f^{(k-1)}$ in the form (2.3.24)–(2.3.25):
  - Construction of the polynomials $\left\{a_{f,\ell}^{(k-1)},\, b_{f,\ell}^{(k-1)},\, c_{f,\ell}^{(k-1)},\, d_{f,\ell}^{(k-1)}\right\}, \ 0 \leqslant \ell \leqslant k-1$;
  - Construction of $f_0^{(k-1)}$.

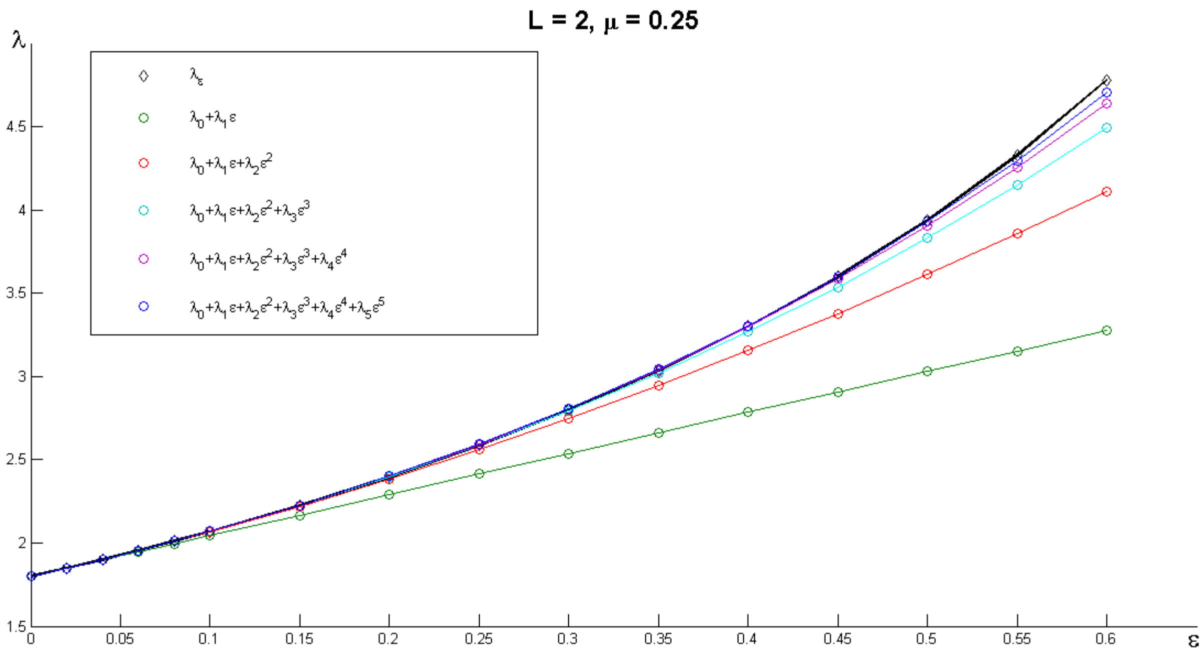
- The coefficients $\left\{c^{(k)}_{j,p,\delta,\ell},\ j \in \mathbb{Z},\ \delta \in \{+,-,0\},\ 1 \leqslant \ell \leqslant \left[\frac{k}{2}\right],\ 1 \leqslant p \leqslant N\right\}$ in the form (2.3.28) with some N corresponding to the truncation of the infinite series in (2.2.24)–(2.2.26):
  - Construction of the coefficients $\check{c}^{(k)}_{q,p,\delta,\ell}$, $0 \leqslant q \leqslant k$, $\delta \in \{+,-,0\}$, $1 \leqslant \ell \leqslant \left[\frac{k}{2}\right]$, $1 \leqslant p \leqslant N$;
  - Construction of $c^{(k)}_{0,p,\delta,\ell}$, $\delta \in \{+,-,0\}$, $1 \leqslant \ell \leqslant \left[\frac{k}{2}\right]$, $1 \leqslant p \leqslant N$.
- The functions $\left\{g^{(k-1)}_{j,\delta}\right\}_{j\in\mathbb{Z}}$, $\delta \in \{+,-,0\}$, in the form (2.3.31):
  - Construction of the functions $\left\{\check{g}^{(k-1)}_{q,\delta},\ 0 \leqslant q \leqslant k-1,\ \delta \in \{+,-,0\}\right\}$;
  - Construction of $\left\{g^{(k-1)}_{0,\delta}\right\}$, $\delta \in \{+,-,0\}$.
- The functions $\left\{\Phi^{(k-1)}_{j}\right\}_{j\in\mathbb{Z}}$, in the form (2.3.34):
  - Construction of the functions $\check{\Phi}^{(k-1)}_{q}$, $0 \leqslant q \leqslant k-1$;
  - Construction of $\Phi^{(k-1)}_{0}$.
- The sequences $\left\{\Xi^{(k-1)}_{j}\right\}_{j\in\mathbb{Z}}$, $\left\{\Delta^{(k-1)}_{j,\pm}\right\}_{j\in\mathbb{Z}}$ in the form (2.3.36):
  - Construction of the polynomials $q^{(k-1)}_{\Sigma}$, $q^{(k-1)}_{\Delta_\pm}$;
  - Construction of $\Xi^{(k-1)}_{0}$, $\Delta^{(k-1)}_{0,\pm}$.

2) $\lambda^{(k)}$ is given by the relation (2.2.60):

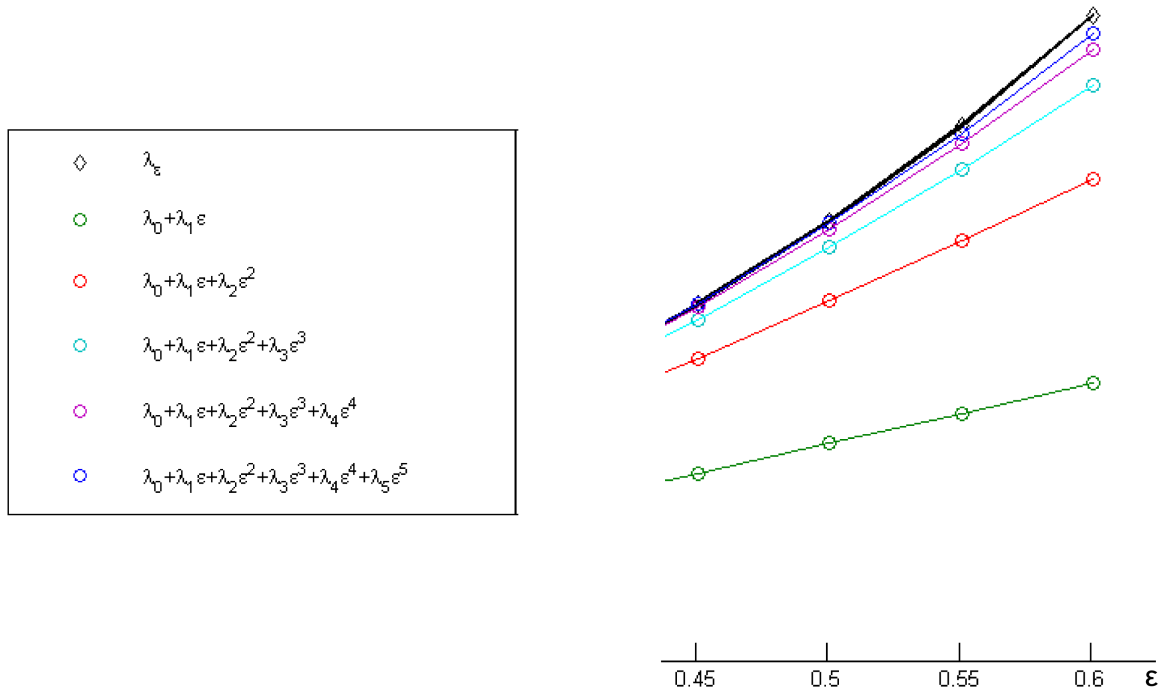
$$\lambda^{(k)} = \left\|u^{(0)}\right\|^{-2}_{L^\mu_2(G_\mathcal{C})} \left(\sum_{j\in\mathbb{Z}} \widetilde{\Xi}^{(k-1)}_{j} \mathbf{u}^{(0)}_{j} - \left(f^{(k-1)}, u^{(0)}\right)_{L^\mu_2(G_\mathcal{C})} \right).$$

3) Resolution of the far field problem for $u^{(k)}$:
We search $u^{(k)}$ in the form (2.3.4)–(2.3.5). As explained in the proof of Lemma 2.3.1, the problem reduces to solving a linear system (2.3.17) of size $k \times k$. This permits to find $u^{(k)}$ at all edges of the graph $G_\mathcal{C}$ except at the edge e_0 (since in (2.3.5) $j \geqslant 1$). To determine $u^{(k)}_0$, one can use the transmission conditions at the vertex v^-_0. Let us note that there are two transmission conditions at the vertex v^-_0: the jump condition (which is the same to the right and to the left of the vertex due to the symmetry of the problem with respect to the axis $y = 0$, see Lemma 2.2.6) and the Kirchhoff's condition to determine one missing parameter $\widetilde{c}^{(k)}_0$ (since $\widetilde{d}^{(k)}_0$ is defined by the relation (2.3.8)). One can use any of these two conditions, and the other one will be automatically satisfied since the far field problem has a solution (the other condition is in fact equivalent to the compatibility condition (2.2.60)).
4) Resolution of the near field problem for $U^{(k)}$:
We search $U^{(k)}$ in the form (2.3.21). Thus, there are $k+2$ problems to solve using P_1 finite elements: $k+1$ problems for the functions $\left\{\mathcal{U}^{(k)}_q\right\}^{k}_{q=0}$ (which permits to find all the near field terms except $U^{(k)}_0$) and the problem for $U^{(k)}_0$.

In figure 3.22a we compare the results for the first eigenvalue λ_ε in the first gap obtained using the DtN operator method and the asymptotic expansion of the eigenvalue.



(a) Dependence on ε of the eigenvalues computed with the DtN method and of the first several terms of the asymptotic expansion.



(b) Zoom on the interval $[0.45, 0.6]$

Figure 3.22: Comparison of the numerical approximations of the first eigenvalue in the first gap of the operator $A^{\mu}_{\varepsilon,s}$ computed with the DtN operator method and with the asymptotic expansion of the eigenvalue for $L = 2$, $\mu = 0.25$, $\varepsilon \in [0, 0.6]$. The black curve is obtained using the DtN operator method and the coloured curves correspond to the first terms of the asymptotic expansion of the eigenvalue, $\sum_{k=0}^{n} \lambda^{(k)}\varepsilon^k$, for $1 \leqslant k \leqslant 5$.

CHAPTER 4

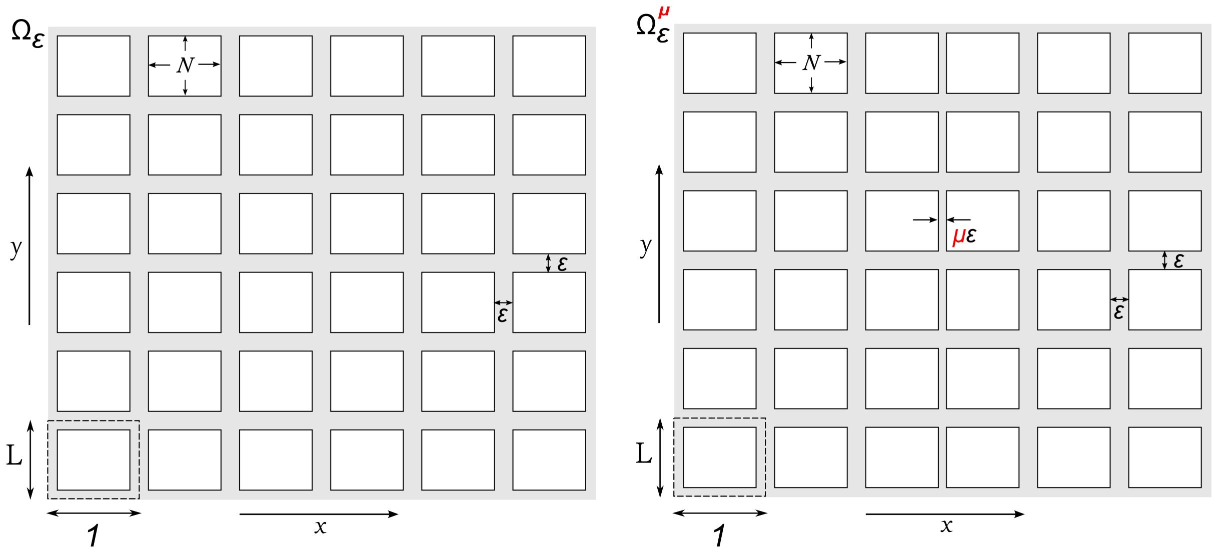
GUIDED MODES IN OPEN PERIODIC LINEIC WAVEGUIDES: THE 2D CASE

In this chapter we study the problem creating of guided modes in the domain Ω_ε^μ represented in figure 4.1b. This domain, supposed homogeneous, can be seen as a perturbation of the 2D periodic domain Ω_ε (represented in figure 4.1a) defined by

$$\Omega_\varepsilon = \mathbb{R}^2 \setminus \mathcal{S}_\varepsilon,$$

$$\mathcal{S}_\varepsilon = \bigcup_{(j,k)\in\mathbb{Z}^2} \left[\tfrac{\varepsilon}{2}, 1-\tfrac{\varepsilon}{2}\right] \times \left[-\tfrac{L}{2}+\tfrac{\varepsilon}{2}, \tfrac{L}{2}-\tfrac{\varepsilon}{2}\right] + (j, kL), \qquad \varepsilon > 0, \quad L > 0,$$

ε being a small parameter. The domain Ω_ε is $\mathbb{R}^2$ minus an infinite set of rectangular



(a) The periodic domain Ω_ε (b) The perturbed domain Ω_ε^μ

Figure 4.1: The purely periodic domain Ω_ε and the perturbed domain Ω_ε^μ.

obstacles (of size $(1-\varepsilon)\times(L-\varepsilon)$) placed periodically with the period 1 in the x-direction and the period L in the y-direction. The distance between two consecutive obstacles is ε. The domain Ω_ε^μ is obtained from Ω_ε by changing the distance between two neighbour columns of obstacles from ε to $\mu\varepsilon$ with $\mu > 0$ and simultaneously modifying the width of the obstacles of these two columns from $1-\varepsilon$ to $1-(1+\mu)\varepsilon/2$ (the total width of the two columns is then preserved):

$$\Omega_\varepsilon^\mu = \mathbb{R}^2 \setminus \left\{\mathcal{S}_\varepsilon^{\mu,+} \cup \mathcal{S}_\varepsilon^{\mu,-}\right\},$$

$$\mathcal{S}_\varepsilon^{\mu,+} = \bigcup_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \left[\tfrac{w_j^\mu \varepsilon}{2}, 1-\tfrac{\varepsilon}{2}\right] \times \left[-\tfrac{L}{2}+\tfrac{\varepsilon}{2}, \tfrac{L}{2}-\tfrac{\varepsilon}{2}\right] + (j, kL),$$
$$\mathcal{S}_\varepsilon^{\mu,-} = \bigcup_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \left[-1+\tfrac{\varepsilon}{2}, -\tfrac{w_j^\mu \varepsilon}{2}\right] \times \left[-\tfrac{L}{2}+\tfrac{\varepsilon}{2}, \tfrac{L}{2}-\tfrac{\varepsilon}{2}\right] - (j, kL),$$

where w_j^μ is the weight function defined in (1.3.1). Neumann boundary conditions are imposed on the boundaries of the obstacles.

Our goal is to prove the existence of guided modes. By a guided mode we mean, roughly speaking, a solution of the scalar wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \Delta u \quad \text{in} \quad \Omega_\varepsilon^\mu, \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega_\varepsilon^\mu} = 0, \tag{4.0.1}$$

which propagates along the perturbation (in the y-direction) and is confined in the transversal direction (x-direction). Thus, it should have the form of a plane wave propagating in the y-direction multiplied by some function $v(x,y)$ periodic in the y-direction with period L:

$$u(x,y,t) = e^{i(\omega t - \beta y/L)} v(x,y), \qquad v(x,y) = v(x, y+L). \tag{4.0.2}$$

To express the fact that the solution is confined in x-direction we impose the condition

$$v \in L_2\left(\mathcal{B}_\varepsilon^\mu\right),$$

where $\mathcal{B}_\varepsilon^\mu$ is a periodicity band of the domain Ω_ε^μ (cf. figure 4.2) which can be formally described by the expression

$$\mathcal{B}_\varepsilon^\mu = \left\{\mathbb{R} \times \left]-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}\right[\right\} \cup \left\{\bigcup_{j\in\mathbb{N}} \left]j - \tfrac{w_j^\mu \varepsilon}{2}, j + \tfrac{w_j^\mu \varepsilon}{2}\right[\times \left]-\tfrac{L}{2}, \tfrac{L}{2}\right[\right\}.$$

Remark 4.0.1. Notice that the choice of the periodicity band is obviously not unique. In particular, we could have taken a periodicity band that has a ladder shape. However, the periodicity cell of the band $\mathcal{B}_\varepsilon^\mu$ which has the form of a cross turns out to be more convenient for numerical simulations.

One can separate in (4.0.2) the harmonic factor in time to get

$$u(x,y,t) = e^{i\omega t}\widetilde{v}(x,y), \qquad \widetilde{v}(x,y) = e^{-i\beta y/L} v(x,y), \tag{4.0.3}$$

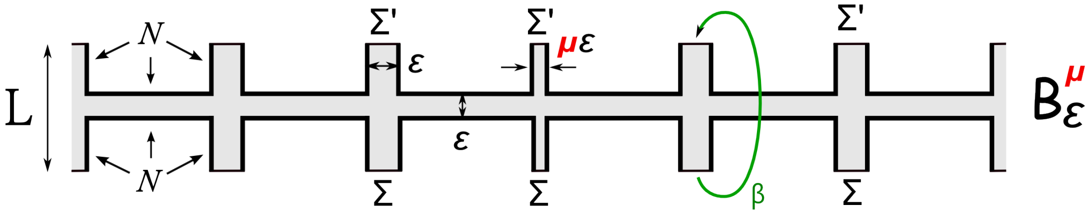


Figure 4.2: Periodicity band (grey area)

where the function $\widetilde{v}$ is β-quasiperiodic:

$$\widetilde{v}(x, y + L) = e^{-i\beta}\widetilde{v}(x, y).$$

Then, after plugging (4.0.3) into the equation (4.0.1), one gets the following problem for the function $\widetilde{v} \in L_2(\mathcal{B}_\varepsilon^\mu)$:

$$\begin{cases} \Delta\widetilde{v} = -\omega^2\widetilde{v}, \\ \widetilde{v}|_{\Sigma'} = e^{-i\beta}\,\widetilde{v}|_{\Sigma}\,, \\ \left.\dfrac{\partial\widetilde{v}}{\partial n}\right|_{\partial\mathcal{B}_\varepsilon^\mu\setminus\{\Sigma\cup\Sigma'\}} = 0. \end{cases}$$

Here

$$\Sigma = \bigcup_{j\in\mathbb{N}} \left[j - \tfrac{w_j^\mu\varepsilon}{2}, j + \tfrac{w_j^\mu\varepsilon}{2}\right] \times \left\{-\tfrac{L}{2}\right\}, \qquad \Sigma' = \bigcup_{j\in\mathbb{N}} \left[j - \tfrac{w_j^\mu\varepsilon}{2}, j + \tfrac{w_j^\mu\varepsilon}{2}\right] \times \left\{\tfrac{L}{2}\right\}.$$

Thus, one ends up with an eigenvalue problem for the Laplacian in the periodicity band $\mathcal{B}_\varepsilon^\mu$ with β-quasiperiodic boundary conditions on the upper and lower parts of the boundary for $\beta \in [0, \pi]$. More precisely, one has to study the spectrum of the following operator:

$$A_\varepsilon^\mu(\beta) : L_2(\mathcal{B}_\varepsilon^\mu) \to L_2(\mathcal{B}_\varepsilon^\mu), \qquad A_\varepsilon^\mu(\beta)v = -\Delta v,$$

$$D(A_\varepsilon^\mu(\beta)) = \left\{ v \in H_\Delta^1(\mathcal{B}_\varepsilon^\mu), \quad \left.\frac{\partial v}{\partial n}\right|_{\partial\mathcal{B}_\varepsilon^\mu\setminus\{\Sigma\cup\Sigma'\}} = 0, \right.$$
$$\left. v|_{\Sigma'} = e^{-i\beta}v|_{\Sigma}, \quad \left.\frac{\partial v}{\partial y}\right|_{\Sigma'} = e^{-i\beta}\left.\frac{\partial v}{\partial y}\right|_{\Sigma} \right\}.$$

We will show (cf. Theorems 4.1.2, 4.1.1 and 4.1.3) that for any $\beta \in [0, \pi]$, for any $k \in \mathbb{N}$ there exists $\varepsilon_k > 0$ such that for $\varepsilon < \varepsilon_k$ the operator $A_\varepsilon^\mu(\beta)$ has at least k gaps and at least one or two eigenvalues in each of these gaps.

As for the ladder, the study is based on asymptotic analysis: as $\varepsilon \to 0$, the domain $\mathcal{B}_\varepsilon^\mu$ shrinks to a graph and the spectrum of the limit operator defined on this graph can be computed explicitly. The essential spectrum of the limit operator is discussed in Section 4.1.1 and the discrete spectrum in Section 4.1.2. This will permit to derive existence results for the operator $A_\varepsilon^\mu(\beta)$ in Section 4.1.3. In Section 4.2 we present numerical results for the operator A_ε^μ.

4.1 Existence results

When $\varepsilon \to 0$ the domain $\mathcal{B}_\varepsilon^\mu$ shrinks to the graph $G_{\mathcal{B}} = \bigcap_{\varepsilon>0} \mathcal{B}_\varepsilon^\mu$ shown in figure 4.3.

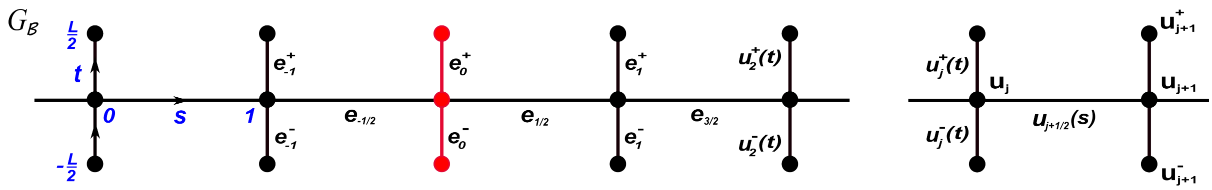


Figure 4.3: The limit graph $G_{\mathcal{B}}$

We will use the following notation. We enumerate the vertices by the index $j \in \mathbb{Z}$, $j = 0$ corresponding to the perturbed vertex. The value of the function u at the j-th middle vertex is denoted by u_j, its value at the upper vertex by u_j^+ and at the lower vertex by u_j^-. The horizontal edge joining the vertices j and $j+1$ is denoted by $e_{j+\frac{1}{2}}$ and the function on it is denoted by $u_{j+\frac{1}{2}}(s)$, the local variable s taking values in $[0,1]$. The vertical edges above and below the vertex j are denoted by e_j^+ and e_j^- respectively and the function on this edges is denoted by $u_j^+(t)$ and $u_j^-(t)$ respectively, the local variable t taking values in $\left[-\frac{L}{2}, \frac{L}{2}\right]$ ($t = 0$ at the middle vertex). We denote by $\mathcal{E}_{G_{\mathcal{B}}}$ the set of the edges of $G_{\mathcal{B}}$.

Similarly to the case of the ladder, the appropriate functional spaces are

$$L_2^\mu\left(G_{\mathcal{B}}\right) = \left\{u : \ \ \|u\|^2_{L_2^\mu(G_{\mathcal{B}})} < \infty\right\}, \qquad H^2\left(G_{\mathcal{B}}\right) = \left\{u \in C\left(G_{\mathcal{B}}\right) : \ \ \|u\|^2_{H^2(G_{\mathcal{B}})} < \infty\right\},$$

$$\|u\|^2_{L_2^\mu(G_{\mathcal{B}})} = \sum_{j\in\mathbb{Z}} \left(w_j^\mu \|u_j^+\|^2_{L_2(e_j^+)} + w_j^\mu \|u_j^-\|^2_{L_2(e_j^-)} + \|u_{j+\frac{1}{2}}\|^2_{L_2\left(e_{j+\frac{1}{2}}\right)} \right),$$

$$\|u\|^2_{H^2(G_{\mathcal{B}})} = \sum_{j\in\mathbb{Z}} \left(\|u_j^+\|^2_{H^2(e_j^+)} + \|u_j^-\|^2_{H^2(e_j^+)} + \|u_{j+\frac{1}{2}}\|^2_{H^2\left(e_{j+\frac{1}{2}}\right)} \right).$$

The limit operator $A^\mu(\beta) : L_2^\mu\left(G_{\mathcal{B}}\right) \to L_2^\mu\left(G_{\mathcal{B}}\right)$ is now defined as

$$\left(A^\mu(\beta)u\right)\big|_e = -\left(u|_e\right)'', \qquad \forall e \in \mathcal{E}_{G_{\mathcal{B}}},$$

$$D\left(A^\mu(\beta)\right) = \Big\{u \in H^2\left(G_{\mathcal{B}}\right) : \tag{4.1.1}$$

$$\mathrm{u}_j^+ = e^{-i\beta}\mathrm{u}_j^-, \quad (u_j^+)'\left(\tfrac{L}{2}\right) = e^{-i\beta}(u_j^-)'\left(-\tfrac{L}{2}\right), \qquad \forall j \in \mathbb{Z}, \tag{4.1.2}$$

$$u'_{j+\frac{1}{2}}(0) - u'_{j-\frac{1}{2}}(1) + w_j^\mu(u_j^+)'(0) - w_j^\mu(u_j^-)'(0) = 0, \quad \forall j \in \mathbb{Z}\Big\}. \tag{4.1.3}$$

The relations (4.1.2) express the quasiperiodicity and the relations (4.1.3) are the Kirchhoff's conditions at the middle vertices. Notice that by definition of $H^2\left(G_{\mathcal{B}}\right)$ the condition (4.1.1) implies the continuity at the middle vertices.

As previously, we start with the explicit computation of the essential spectrum of the operator $A(\beta)$ which turns out to have infinitely many gaps for any β. The computations in this section will be very similar to the ones done for the ladder in Section 1.3. We will still repeat it in detail for the seek of completeness. Then we prove that the perturbed operator has one or two eigenvalues in each gap.

4.1.1 The essential spectrum of the limit operator

As usual, due to a compact perturbation argument the study of the essential spectrum of the operator $A^{\mu}(\beta)$ is reduced to the study of the spectrum of the non-perturbed operator $A^{1}(\beta)$ (corresponding to the case $\mu = 1$) that will be denoted by $A(\beta)$.

Proposition 4.1.1. *For any $\mu > 0,\ \beta \in [0, \pi]$*

$$\sigma_{ess}\left(A^{\mu}(\beta)\right) = \sigma_{ess}\left(A(\beta)\right).$$

For this reason, we start by determining the (essential) spectrum of the operator $A(\beta)$.

4.1.1.1 Computation of the spectrum of the non-perturbed limit operator

According to the Floquet-Bloch theory, in order to determine the spectrum of the operator $A(\beta)$ one has to study a set of problems on the periodicity cell $\mathcal{C}_{\mathcal{B}}$ that consists of four edges: $I_1 = \left[-\frac{1}{2}, 0\right] \times 0$, $I_2 = \left[0, \frac{1}{2}\right] \times 0$, $I_3 = 0 \times \left[-\frac{L}{2}, 0\right]$, $I_3 = 0 \times \left[0, \frac{L}{2}\right]$. A function on the cell is defined by its four restrictions on the edges of the cell: $u = \{u_i\}_{i=1}^{4}$ (cf. figure 4.4).

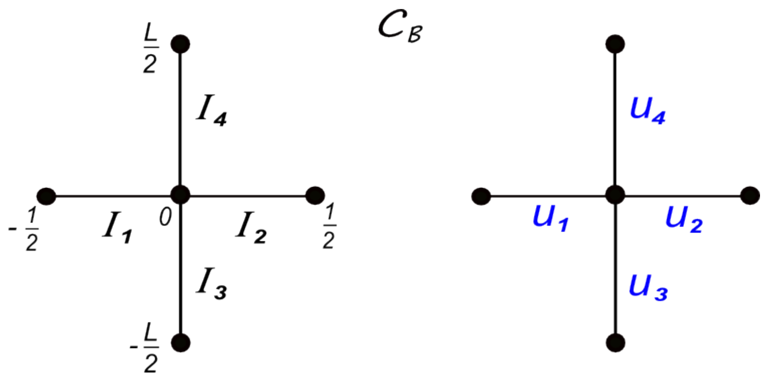


Figure 4.4: Periodicity cell $\mathcal{C}_{\mathcal{B}}$

The corresponding functional spaces on the cell are defined as

$$L_2\left(\mathcal{C}_{\mathcal{B}}\right) = \left\{ u : \sum_{i=1}^{4} \|u_i\|^2_{L_2(I_i)} < \infty \right\}, \quad H^2\left(\mathcal{C}_{\mathcal{B}}\right) = \left\{ u \in C\left(\mathcal{C}_{\mathcal{B}}\right) : \sum_{i=1}^{4} \|u_i\|^2_{H^2(I_i)} < \infty \right\}.$$

We have to determine the spectrum of the operator family $A_{\beta}(\theta) : L_2\left(\mathcal{C}_{\mathcal{B}}\right) \to L_2\left(\mathcal{C}_{\mathcal{B}}\right)$, $\theta \in [0, \pi]$, defined as

$$A_{\beta}(\theta)u = -u'',$$

$$D\left(A_{\beta}(\theta)\right) = \Big\{ u \in H^2\left(\mathcal{C}_{\mathcal{B}}\right) : \quad u_4\left(\tfrac{L}{2}\right) = e^{-i\beta} u_3\left(-\tfrac{L}{2}\right), \quad u_4'\left(\tfrac{L}{2}\right) = e^{-i\beta} u_3'\left(-\tfrac{L}{2}\right), \quad (4.1.4)$$

$$u_2\left(\tfrac{1}{2}\right) = e^{-i\theta} u_1\left(-\tfrac{1}{2}\right), \quad u_2'\left(\tfrac{1}{2}\right) = e^{-i\theta} u_1'\left(-\tfrac{1}{2}\right), \quad (4.1.5)$$

$$u_2'(0) - u_1'(0) + u_4'(0) - u_3'(0) = 0 \Big\}. \quad (4.1.6)$$

Proposition 4.1.2. *For $\theta, \beta \in [0,\pi]$, $\lambda^2 \in \sigma\left(A_\beta(\theta)\right) \setminus \{0\}$ if and only if $\lambda > 0$ is a solution of the equation*

$$\sin\lambda\left(\cos\beta - \cos\left(\lambda L\right)\right) = \sin\left(\lambda L\right)\left(\cos\lambda - \cos\theta\right). \tag{4.1.7}$$

The point $\lambda^2 = 0$ belongs to $\sigma(A_\beta(\theta))$ if and only if $\theta = \beta = 0$.

Proof. Searching the eigenvalues of the operator $A_\beta(\theta)$ implies solving the ordinary differential equation

$$u'' + \lambda^2 u = 0 \tag{4.1.8}$$

on each edge of the cell $\mathcal{C}_\mathcal{B}$. We first consider the case $\lambda \neq 0$. We have:

$$u_1(x) = c_1 e^{i\lambda x} + d_1 e^{-i\lambda x}, \qquad x \in \left[-\tfrac{1}{2}, 0\right], \tag{4.1.9}$$
$$u_2(x) = c_2 e^{i\lambda x} + d_2 e^{-i\lambda x}, \qquad x \in \left[0, \tfrac{1}{2}\right], \tag{4.1.10}$$
$$u_3(y) = c_3 e^{i\lambda y} + d_3 e^{-i\lambda y}, \qquad y \in \left[-\tfrac{L}{2}, 0\right], \tag{4.1.11}$$
$$u_4(y) = c_4 e^{i\lambda y} + d_4 e^{-i\lambda y}, \qquad y \in \left[0, \tfrac{1}{2}\right]. \tag{4.1.12}$$

The continuity of the eigenfunction at the vertex $x = 0$, $y = 0$ implies the relations

$$c_1 + d_1 = c_2 + d_2 = c_3 + d_3 = c_4 + d_4. \tag{4.1.13}$$

The quasi-periodicity conditions (4.1.4), (4.1.5) lead to the following relations:

$$c_2 e^{i\lambda/2} + d_2 e^{-i\lambda/2} = e^{-i\theta}\left(c_1 e^{-i\lambda/2} + d_1 e^{i\lambda/2}\right), \tag{4.1.14}$$
$$c_2 e^{i\lambda/2} - d_2 e^{-i\lambda/2} = e^{-i\theta}\left(c_1 e^{-i\lambda/2} - d_1 e^{i\lambda/2}\right), \tag{4.1.15}$$
$$c_4 e^{i\lambda L/2} + d_4 e^{-i\lambda L/2} = e^{-i\beta}\left(c_3 e^{-i\lambda L/2} + d_3 e^{i\lambda L/2}\right), \tag{4.1.16}$$
$$c_4 e^{i\lambda L/2} - d_4 e^{-i\lambda L/2} = e^{-i\beta}\left(c_3 e^{-i\lambda L/2} - d_3 e^{i\lambda L/2}\right). \tag{4.1.17}$$

Finally, we take into account the Kirchhoff's condition (4.1.6) which can be rewritten as

$$c_2 - d_2 - c_1 + d_1 + c_4 - d_4 - c_3 + d_3 = 0. \tag{4.1.18}$$

Let us denote

$$\alpha = e^{i\theta}, \qquad \gamma = e^{i\beta}, \qquad z = e^{i\lambda}.$$

Then, the relations (4.1.14), (4.1.15) imply that

$$c_1 = \alpha z c_2, \qquad d_1 = \alpha \overline{z} d_2. \tag{4.1.19}$$

Analogously, the relations (4.1.16), (4.1.21) imply

$$c_3 = \gamma z^L c_4, \qquad d_3 = \gamma \overline{z}^L d_4. \tag{4.1.20}$$

Taking into account (4.1.13), we get:

$$c_2\left(\alpha z - 1\right) = d_2\left(1 - \alpha\overline{z}\right), \tag{4.1.21}$$
$$c_4\left(\gamma z^L - 1\right) = d_4\left(1 - \gamma\overline{z}^L\right). \tag{4.1.22}$$

Combining the relations (4.1.13), (4.1.18), (4.1.21), (4.1.22) we arrive at the following linear system of four equations for the unknowns c_2, d_2, c_4, d_4:

$$\begin{cases} c_2 + d_2 - c_4 - d_4 = 0, \\ c_2(\alpha z - 1) + d_2(\alpha \overline{z} - 1) = 0, \\ c_4\left(\gamma z^L - 1\right) + d_4\left(\gamma \overline{z}^L - 1\right) = 0, \\ c_2\left(1 - \alpha z\right) + d_2\left(\alpha \overline{z} - 1\right) + c_4\left(1 - \gamma z^L\right) + d_4\left(\gamma \overline{z}^L - 1\right) = 0. \end{cases}$$

The criterion of existence of a non-trivial solution is:

$$D(\lambda) = 0, \qquad D(\lambda) = \begin{vmatrix} 1 & 1 & -1 & -1 \\ \alpha z - 1 & \alpha \overline{z} - 1 & 0 & 0 \\ 0 & 0 & \gamma z^L - 1 & \gamma \overline{z}^L - 1 \\ 1 - \alpha z & \alpha \overline{z} - 1 & 1 - \gamma z^L & \gamma \overline{z}^L - 1 \end{vmatrix}.$$

Let us compute the determinant $D(\lambda)$. We have

$$\begin{aligned} D(\lambda) = -(\alpha z - 1) &\begin{vmatrix} 1 & -1 & -1 \\ 0 & \gamma z^L - 1 & \gamma \overline{z}^L - 1 \\ \alpha \overline{z} - 1 & 1 - \gamma z^L & \gamma \overline{z}^L - 1 \end{vmatrix} + (\alpha \overline{z} - 1)\begin{vmatrix} 1 & -1 & -1 \\ 0 & \gamma z^L - 1 & \gamma \overline{z}^L - 1 \\ 1 - \alpha z & 1 - \gamma z^L & \gamma \overline{z}^L - 1 \end{vmatrix} \\ &= (1 - \alpha z)\left(2 + 2\gamma^2 + \alpha\gamma\overline{z}\left(z^L - \overline{z}^L\right) - \gamma\left(3z^L + \overline{z}^L\right)\right) \\ &\qquad + (\alpha \overline{z} - 1)\left(2 + 2\gamma^2 + \alpha\gamma z\left(\overline{z}^L - z^L\right) - \gamma\left(3\overline{z}^L + z^L\right)\right), \end{aligned}$$

$$\begin{aligned} \frac{D(\lambda)}{\gamma} &= (1 - \alpha z)\left(4\Re\gamma + 2i\alpha\overline{z}\Im z^L - \left(3z^L + \overline{z}^L\right)\right) + (\alpha \overline{z} - 1)\left(4\Re\gamma - 2i\alpha z\Im z^L - \left(3\overline{z}^L + z^L\right)\right) \\ &= 8i\alpha\Re\gamma\Im\overline{z} + 4i\Im\overline{z}^L + 8i\alpha\Im z^{L+1} + 4i\alpha^2\Im\overline{z}^L, \end{aligned}$$

$$\frac{D(\lambda)}{4i\alpha\gamma} = 2\Re\gamma\Im\overline{z} + \overline{\alpha}\Im\overline{z}^L + 2\Im z^{L+1} + \alpha\Im\overline{z}^L = 2\left(\Re\gamma\Im\overline{z} + \Re\alpha\Im\overline{z}^L + \Im z^{L+1}\right).$$

Consequently, the condition $D(\lambda) = 0$ is equivalent to the relation

$$\Re\gamma\Im\overline{z} + \Re\alpha\Im\overline{z}^L + \Im z^{L+1} = 0,$$

which leads to the relation (4.1.7) for $\lambda \neq 0$.

If $\lambda = 0$, instead of the relations (4.1.9)–(4.1.12) we have

$$\begin{aligned} u_1(x) &= c_1 + d_1 x, & x &\in \left[-\tfrac{1}{2}, 0\right], \\ u_2(x) &= c_2 + d_2 x, & x &\in \left[0, \tfrac{1}{2}\right], \\ u_3(y) &= c_3 + d_3 y, & y &\in \left[-\tfrac{L}{2}, 0\right], \\ u_4(y) &= c_4 + d_4 y, & y &\in \left[0, \tfrac{1}{2}\right]. \end{aligned}$$

The continuity at the vertex $x = 0$, $y = 0$ implies that $c_i = c$, $1 \leqslant i \leqslant 4$. From the quasiperiodic conditions it follows that

$$d_2 = e^{-i\theta} d_1, \quad d_4 = e^{-i\beta} d_3, \quad d_1 = c\left(1 - e^{i\theta}\right), \quad d_3 L = c\left(1 - e^{i\beta}\right). \tag{4.1.23}$$

Finally, Kirchhoff's condition gives

$$-d_1 + d_2 - d_3 + d_4 = 0. \tag{4.1.24}$$

After injecting (4.1.23) in (4.1.24) we get

$$c\left(\cos\theta - 1 + \frac{\cos\beta - 1}{L}\right) = 0.$$

Hence, there exists a non-trivial solution if and only if $\theta = \beta = 0$. This finishes the proof. □

Remark 4.1.1. One can notice that if $L \in \mathbb{Q}$, then the set $\{\lambda : \lambda^2 \in \sigma(A_\beta(\theta))\}$ is periodic. Indeed, this is due to the fact that both the left-hand side and the right-hand side of (4.1.7) are periodic with the same period.

We can now find the spectrum of the operator $A(\beta)$ due to the decomposition

$$\sigma(A(\beta)) = \bigcup_{\theta\in[0,\pi]} \sigma\left(A_\beta(\theta)\right). \tag{4.1.25}$$

Thus, the point λ^2 (different from zero) belongs to the spectrum of the operator $A(\beta)$ if and only if there exists a value of θ such that the relation (4.1.7) is verified.

In the rest of this section we will use the following notation:

$$\Sigma = \{\pi n,\ n \in \mathbb{N}\}, \qquad \Sigma^* = \Sigma \setminus \{0\}, \qquad \Sigma_\beta = \left\{\pm\tfrac{\beta}{L} + \tfrac{2\pi n}{L},\ \ n \in \mathbb{N}\right\}, \qquad \Sigma_\beta^* = \Sigma_\beta \setminus \{0\}.$$

Proposition 4.1.3.

1. *$\{\lambda^2,\ \ \lambda \in \Sigma^* \cup \Sigma_\beta\} \subset \sigma(A(\beta))$.*

2. *For any $\beta \in [0,\pi]$, the operator $A(\beta)$ has infinitely many gaps whose ends tend to infinity.*

Proof.

1. For $\lambda \neq 0$, $\lambda \in \Sigma^* \cup \Sigma_\beta$ the equation (4.1.7) is obviously verified for $\cos\theta = \cos\lambda$. The point $\lambda = 0$ belongs to $\Sigma^* \cup \Sigma_\beta$ if and only if $\beta = 0$. At the same time, according to Proposition 4.1.2, $0 \in \sigma(A(\beta))$ if and only if $\beta = 0$.

2. Let us consider two cases:

a) $\beta \notin \{0,\pi\}$:
Consider $\lambda_n = \frac{\pi n}{L}$, $n \in \mathbb{N}^*$. There are two possibilities:

(i) $\lambda_n \notin \Sigma^*$: then, the right-hand side of the equation (4.1.7) equals to zero, whereas the left-hand side is different from zero for any $\theta \in [0,\pi]$. Consequently, there exists a gap of the operator $A(\beta)$ containing the point λ_n^2.

(ii) $\lambda_n \in \Sigma^*$: in this case it follows from the property 1 that $\lambda_n^2 \in \sigma(A(\beta))$. We are going to show that the point λ_n^2 is an isolated point of the spectrum of the operator $A(\beta)$, so that there exist gaps to the left and to the right of it. Setting $\lambda = \lambda_n + \delta$ in (4.1.7) we get:

$$\sin\delta \frac{\cos\beta}{\cos(\lambda_n L)} + \sin(\delta L)\frac{\cos\theta}{\cos\lambda_n} = \sin(\delta(L+1)). \tag{4.1.26}$$

If δ is small enough (but different from 0) this equation cannot be verified for any θ. Indeed, for $\delta \neq 0$ it can be rewritten as

$$\frac{\cos\beta}{\cos(\lambda_n L)} - \cos(\delta L) = \frac{\sin(\delta L)}{\sin\delta}\left(\cos\delta - \frac{\cos\theta}{\cos\lambda_n}\right).$$

Since $|\cos\beta| < 1$ and $|\cos\lambda_n| = |\cos(\lambda_n L)| = 1$, the limit of the left-hand side as $\delta \to 0$ is negative, whereas the limit of the right-hand side is non-negative for any θ with a uniform bound in θ for δ small enough:

$$\frac{\sin(\delta L)}{\sin\delta}\left(\cos\delta - \frac{\cos\theta}{\cos\lambda_n}\right) \geqslant \frac{\sin(\delta L)}{\sin\delta}(\cos\delta - 1), \qquad \forall\theta \in [0,\pi].$$

Hence, the equation (4.1.26) does not have solutions for δ small enough. This proves the existence of gaps of the form $]\lambda_n^2 - l_n^-, \lambda_n^2[$, $]\lambda_n^2, \lambda_n^2 + l_n^+[$ for some $l_n^-, l_n^+ > 0$.

Thus, we see that for any $n \in \mathbb{N}^*$ there exists a gap of the operator $A(\beta)$ of the form $]\lambda_n^2 - l_n^-, \lambda_n^2 + l_n^+[$, $]\lambda_n^2 - l_n^-, \lambda_n^2[$ or $]\lambda_n^2, \lambda_n^2 + l_n^+[$ with $l_n^-, l_n^+ > 0$. We also know from the property 1 that there exists an infinite sequence of points tending to infinity and belonging to the spectrum of the operator $A(\beta)$. This proves the existence of an infinity of gaps of the operator $A(\beta)$ for $\beta \notin \{0,\pi\}$.

b) $\beta \in \{0,\pi\}$:

For $\beta = 0$ the equation (4.1.7) takes the form

$$\sin\left(\tfrac{\lambda L}{2}\right)\cos\left(\tfrac{\lambda L}{2}\right)(\cos\lambda - \cos\theta) = \sin\lambda\sin^2\left(\tfrac{\lambda L}{2}\right), \tag{4.1.27}$$

and for $\beta = \pi$ it takes the form

$$\sin\left(\tfrac{\lambda L}{2}\right)\cos\left(\tfrac{\lambda L}{2}\right)(\cos\lambda - \cos\theta) = -\sin\lambda\cos^2\left(\tfrac{\lambda L}{2}\right). \tag{4.1.28}$$

It can be shown that the operators $A(0)$ and $A(\pi)$ have gaps that contain some neighbourhoods (or deleted neighbourhoods) of the points $\left\{\lambda^2,\ \cos\left(\tfrac{\lambda L}{2}\right) = 0\right\}$ and $\left\{\lambda^2,\ \sin\left(\tfrac{\lambda L}{2}\right) = 0\right\}$ respectively. Indeed, in the neighbourhoods of these points the relations (4.1.27) and (4.1.28) can be rewritten as

$$\cos\left(\tfrac{\lambda L}{2}\right)(\cos\lambda - \cos\theta) = \sin\lambda\sin\left(\tfrac{\lambda L}{2}\right), \qquad (\beta = 0), \tag{4.1.29}$$

$$\sin\left(\tfrac{\lambda L}{2}\right)(\cos\lambda - \cos\theta) = -\sin\lambda\cos\left(\tfrac{\lambda L}{2}\right), \qquad (\beta = \pi). \tag{4.1.30}$$

Comparing these relations with the relations (1.3.12), (1.3.59) characterizing the spectra of the operators A_s and A_{as} respectively, we see that they coincide up to a factor 2. This factor being unimportant in the proofs of Propositions 1.3.4 (2), 1.3.12 (2), the same argument (together with the property 1) shows the existence of an infinity of gaps for the operators $A(0)$, $A(\pi)$.

□

We will come back to this similarity of the relations (4.1.29), (1.3.12) and (4.1.30), (1.3.59) which implies some relation between the symmetric case and the 0-quasiperiodic one, as well as between the antisymmetric case and the π-quasiperiodic one (see Remark 4.1.6).

Proposition 4.1.4. *The operator $A(\beta)$ has the following set of eigenvalues of infinite multiplicity:*

$$\sigma_{pp}(A(\beta)) = \Sigma_{is}(\beta) \cup \Sigma_{emb}(\beta),$$

where

$$\Sigma_{is}(\beta) = \begin{cases} \{\lambda^2 /\ \lambda \in \Sigma^*,\ \ \sin(\lambda L) = 0\}, & \beta \in]0,\pi[, \\ \{\lambda^2 /\ \lambda \in \Sigma^*,\ \ \cos(\lambda L) = -1\}, & \beta = 0, \\ \{\lambda^2 /\ \lambda \in \Sigma^*,\ \ \cos(\lambda L) = 1\}, & \beta = \pi, \end{cases} \tag{4.1.31}$$

and

$$\Sigma_{emb}(\beta) = \begin{cases} \emptyset, & \beta \in]0,\pi[, \\ \{\lambda^2 /\ \lambda \neq 0,\ \ \cos(\lambda L) = 1\}, & \beta = 0, \\ \{\lambda^2 /\ \lambda \neq 0,\ \ \cos(\lambda L) = -1\}, & \beta = \pi. \end{cases} \tag{4.1.32}$$

The eigenvalues of the set $\Sigma_{is}(\beta)$ are isolated points of the spectrum whereas the eigenvalues of the set $\Sigma_{emb}(\beta)$ are embedded (interior points of the spectrum).

Proof. The point λ^2 is an eigenvalue of the operator $A(\beta)$ of infinity multiplicity if and only if it is an eigenvalue of the operator $A(\beta(\theta))$ for any $\theta \in [0,\pi]$. It follows from Proposition 4.1.2 that 0 is not an eigenvalue of infinite multiplicity for any $\beta \in [0,\pi]$. For $\lambda \neq 0$ it means that the equation (4.1.7) is satisfied for any $\theta \in [0,\pi]$. This is possible in two cases:

(a) $\sin\lambda = \sin(\lambda L) = 0$ and $\cos(\lambda L) \neq \cos\beta$;

(b) $\cos(\lambda L) = \cos\beta$ and $\sin(\lambda L) = 0$.

In both cases (and only in this cases) the left-hand side and the right-hand side of (4.1.7) equal to zero simultaneously and independently of $\theta \in [0,\pi]$. The case (a) leads to the set $\Sigma_{is}(\beta)$. We have already shown in the proof of Proposition 4.1.3 (2) that these points are isolated points of the spectrum. The case (b) leads to the set $\Sigma_{emb}(\beta)$. Let us show that the points of this set are interior points of the spectrum. Consider $\beta \in \{0,\pi\}$ and λ_0 such that $\cos(\lambda_0 L) = \cos\beta = \pm 1$. Then, the equation (4.1.7) for $\lambda = \lambda_0 + \delta$ takes the form

$$\cos\theta = \cos\lambda - \sin\lambda \tan\left(\tfrac{\delta L}{2}\right).$$

It is clear that if $|\cos\lambda_0| < 1$ then this equation admits a solution when δ is small enough (since the absolute value of the right-hand side is smaller than 1 for δ small enough). If $|\cos\lambda_0| = 1$, then the last equation can be rewritten as

$$\cos\theta = \cos\lambda_0 \left(\cos\delta - \sin\delta \tan\left(\tfrac{\delta L}{2}\right)\right),$$

and again it admits a solution when δ is small enough. This shows that the points of the set $\Sigma_{emb}(\beta)$ are interior points of the spectrum. □

The following assertion is an immediate consequence of the decomposition (4.1.25), Proposition 4.1.2 and property 1 of Proposition 4.1.3.

Proposition 4.1.5. $\lambda^2 \in \sigma(A(\beta))$ *if and only if one of the following possibilities holds:*

(i) $\lambda \in \Sigma^* \cup \Sigma_\beta$*;*

(ii) $\lambda \in \mathbb{R}_+ \setminus (\Sigma \cup \Sigma_\beta)$ *is a solution of the equation*

$$\phi_{L,\beta}(\lambda) = f_\theta(\lambda), \tag{4.1.33}$$

for some $\theta \in [0, \pi]$*. Here*

$$\phi_{L,\beta}(\lambda) = \frac{\sin(\lambda L)}{\cos\beta - \cos(\lambda L)}, \qquad \lambda \notin \Sigma_\beta,$$

$$f_\theta(\lambda) = \frac{\sin\lambda}{\cos\lambda - \cos\theta}, \qquad \lambda \in \{\lambda /\ \cos\lambda \neq \cos\theta\}.$$

As in the case of the ladder (cf. Proposition 1.3.6) a simple geometric interpretation can be given to Proposition 4.1.5. Again, the spectrum is obtained as the image by the function $x \mapsto x^2$ of the closure of the projection on the axis of positive abscissas of the intersection of the the domain $\overline{D}$ with graph of the function $\phi_{L,\beta}$:

$$\lambda^2 \in \sigma(A(\beta)) \quad \Leftrightarrow \quad \lambda \in \overline{D_x \setminus \{0\}},$$

where

$$D_x = \left\{x : (x, \phi_L(\lambda)) \in \overline{D}\right\}, \qquad D = \bigcup_{n \in \mathbb{N}} D_n^\pm,$$

$$\begin{aligned} D_n^+ &= \left\{(x,y)/\ x \in]\pi n, \pi(n+1)[,\ \ y \in \left[f^+(x), +\infty\right[\right\} \cup (\pi n, 0), \\ D_n^- &= \left\{(x,y)/\ x \in]\pi n, \pi(n+1)[,\ \ y \in \left]-\infty, f^-(x)\right]\right\}, \end{aligned}$$

$$\begin{aligned} f^+(x) &= \tan\left(\tfrac{x - \pi n}{2}\right), && x \in [\pi n, \pi(n+1)[, \\ f^-(x) &= -\text{cotan}\left(\tfrac{x - \pi n}{2}\right), && x \in]\pi n, \pi(n+1)]. \end{aligned}$$

An example illustrating this geometric interpretation is given in figure 4.5.

Similarly to Proposition 1.3.8, two types of gaps can be distinguished.

Proposition 4.1.6. *An interval* $]a, b[$ *is a gap of the operator* $A(\beta)$ *for* $\beta \in [0, \pi]$ *if and only if* $[a, b] \cap \Sigma_\beta = \emptyset$ *and one of the following possibilities holds:*

I *There exists* $n \in \mathbb{N}$ *such that* $\pi n < \sqrt{a} < \sqrt{b} < \pi(n+1)$*, and* $\phi_{L,\beta}(\sqrt{a}) = f^+(\sqrt{a})$*,* $\phi_{L,\beta}(\sqrt{b}) = f^-(\sqrt{b})$*;*

II *(i) There exists* $n \in \mathbb{N}$ *such that* $\pi n = \sqrt{a} < \sqrt{b} < \pi(n+1)$*, and* $\phi_{L,\beta}(\sqrt{a}) \leqslant 0$*,* $\phi_{L,\beta}(\sqrt{b}) = f^-(\sqrt{b})$*;*

(ii) There exists $n \in \mathbb{N}$ *such that* $\pi n < \sqrt{a} < \sqrt{b} = \pi(n+1)$*, and* $\phi_{L,\beta}(\sqrt{a}) = f^+(\sqrt{a})$*,* $\phi_{L,\beta}(\sqrt{b}) \geqslant 0$*.*

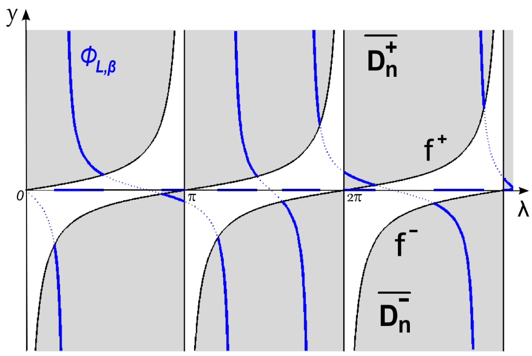


Figure 4.5: The images of the spectral bands by the function $x \mapsto \sqrt{x}$ are given by the closure of the projection on the axis of positive abscissas of the intersections of the function $\phi_{L,\beta}$ with the domains $\overline{D}_n^+$, $\overline{D}_n^-$.

Remark 4.1.2. If $\beta \in]0,\pi]$, then $\sigma_* = \min \sigma(A(\beta)) > 0$. According to Proposition 4.1.6, the interval $]0, \sigma_*[$ is considered as a gap of type II (it satisfies the condition $II(i)$). Strictly speaking, this is false: σ_* is the bottom of the essential spectrum. Being aware of this inaccuracy, we will still call the interval $]0, \sigma_*[$ a gap. This will permit us to study the number of eigenvalues of the perturbed operator inside this interval without considering it as a particular case (cf. Theorem 4.1.1).

The proof of Proposition 4.1.6 is based on the following lemma (which is an analogue of Lemma 1.3.2 for the ladder).

Lemma 4.1.1. *The point λ_0^2 is the lower end of a gap of the operator $A(\beta)$ if and only if one of the following possibilities holds:*

(i) $\lambda_0 \in \mathbb{R}_+ \setminus (\Sigma \cup \Sigma_s)$ *and* $\phi_{L,\beta}(\lambda_0) = f^+(\lambda_0)$*;*

(ii) $\lambda_0 \in \Sigma \setminus \Sigma_\beta$ *and* $\phi_{L,\beta}(\lambda_0) \leqslant 0$.

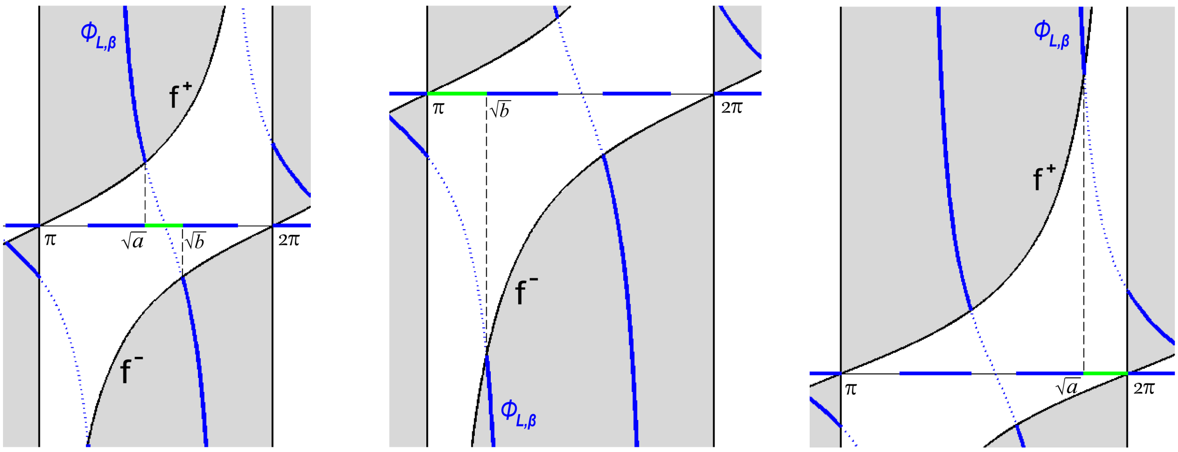
Similarly, the point λ_0^2 is the upper end of a gap of the operator $A(\beta)$ if and only if one of the following possibilities holds:

(iii) $\lambda_0 \in \mathbb{R}_+ \setminus (\Sigma \cup \Sigma_s)$ *and* $\phi_{L,\beta}(\lambda_0) = f^-(\lambda_0)$*;*

(iv) $\lambda_0 \in \Sigma^* \setminus \Sigma_\beta$ *and* $\phi_{L,\beta}(\lambda_0) \geqslant 0$.

Without giving a detailed proof (which is an obvious modification of the proof of Lemma 1.3.2) we mention that it is based on the properties of the function $\phi_{L,\beta}$ which has in general the same behaviour as the function ϕ_L. More precisely, it is a $\frac{2\pi}{L}$-periodic function defined on $\mathbb{R} \setminus \Sigma_\beta$. In each interval $\left]-\frac{\beta}{L} + \frac{2\pi n}{L}, \frac{\beta}{L} + \frac{2\pi n}{L}\right[$, $\left]\frac{\beta}{L} + \frac{2\pi n}{L}, -\frac{\beta}{L} + \frac{2\pi(n+1)}{L}\right[$, $n \in \mathbb{Z}$, it is continuous and strictly decreasing and takes all the values in $\mathbb{R}$ (cf. figure 4.5).

Remark 4.1.3. If $\beta \in]0,\pi]$, then according to Lemma 4.1.1, 0 is the lower end of a gap since it satisfies *(ii)* : $0 \in \Sigma \backslash \Sigma_\beta$ and $\phi_{L,\beta}(0) = 0$. Similarly, $\sigma_* = \min \sigma(A(\beta))$ is the upper end of a gap since the cases *(iii)*, *(iv)* characterize all the points λ such that $]\lambda^2 - \delta, \lambda^2] \cap \sigma(A(\beta)) = \{\lambda^2\}$ for some $\delta > 0$ (cf. Remark 4.1.2).



(a) Gap of type I ($\pi < \sqrt{a} < \sqrt{b} < 2\pi$)

(b) Gap of type $II(i)$ ($\pi = \sqrt{a} < \sqrt{b} < 2\pi$)

(c) Gap of type $II(ii)$ ($\pi < \sqrt{a} < \sqrt{b} = 2\pi$)

Figure 4.6: Illustration for Proposition 4.1.6: types of gaps.

4.1.2 The discrete spectrum of the limit operator

We now come back to the perturbed operator $A^\mu(\beta)$. We are interested in determining its discrete spectrum. If λ^2 is an eigenvalue of the operator $A^\mu(\beta)$, then the corresponding eigenfunction $u \in D\left(A^\mu(\beta)\right)$ solves the equation $u'' + \lambda^2 u = 0$ on each edge of the graph $G_\mathcal{B}$.

Case $\lambda \neq 0$

In this case the eigenfunction u it has the form

$$u_{j+\frac{1}{2}}(s) = a_{j+\frac{1}{2}} \sin(\lambda s) + b_{j+\frac{1}{2}} \cos(\lambda s), \qquad s \in [0,1], \qquad j \in \mathbb{Z}, \tag{4.1.34}$$
$$u_j^+(y) = c_j^+ \sin(\lambda y) + d_j^+ \cos(\lambda y), \qquad y \in \left[0, \tfrac{L}{2}\right], \qquad j \in \mathbb{Z}, \tag{4.1.35}$$
$$u_j^-(y) = c_j^- \sin(\lambda y) + d_j^- \cos(\lambda y), \qquad y \in \left[-\tfrac{L}{2}, 0\right], \qquad j \in \mathbb{Z}. \tag{4.1.36}$$

The continuity of an eigenfunction at the vertices $(j,0)$ implies

$$b_{j+\frac{1}{2}} = a_{j-\frac{1}{2}} \sin\lambda + b_{j-\frac{1}{2}} \cos\lambda = d_j^+ = d_j^-, \qquad j \in \mathbb{Z}. \tag{4.1.37}$$

Thus, we will denote $d_j := d_j^+ = d_j^-$, $j \in \mathbb{Z}$. After plugging (4.1.35), (4.1.36) into the quasiperiodicity conditions (4.1.2), we get

$$\left(c_j^+ + e^{-i\beta} c_j^-\right) \sin\left(\tfrac{\lambda L}{2}\right) + d_j \cos\left(\tfrac{\lambda L}{2}\right)\left(1 - e^{-i\beta}\right) = 0, \qquad j \in \mathbb{Z}, \tag{4.1.38}$$
$$\left(c_j^+ - e^{-i\beta} c_j^-\right) \cos\left(\tfrac{\lambda L}{2}\right) - d_j \sin\left(\tfrac{\lambda L}{2}\right)\left(1 + e^{-i\beta}\right) = 0, \qquad j \in \mathbb{Z}. \tag{4.1.39}$$

Finally, from the Kirchhoff's conditions (4.1.3) we get

$$w_j^\mu \left(c_j^+ - c_j^-\right) + a_{j+\frac{1}{2}} - a_{j-\frac{1}{2}} \cos\lambda + d_{j-1} \sin\lambda = 0, \qquad j \in \mathbb{Z}. \tag{4.1.40}$$

Let us denote

$$\Sigma_L = \left\{ \tfrac{\pi n}{L},\ n \in \mathbb{N}^* \right\}.$$

We will show that the set $\Sigma^* \cup \Sigma_L$ can be excluded from the consideration while searching the eigenvalues of the operator $A^\mu(\beta)$. Indeed, due to Proposition 4.1.4 we know that the sets $\Sigma^* \cap \Sigma_L$ for $\beta \in]0,\pi[$ and Σ_L for $\beta \in \{0,\pi\}$ correspond to eigenvalues of infinite multiplicity. The following Lemma states that the points of $\Sigma^* \cup \Sigma_L$ which are not covered by these cases do not correspond to eigenvalues of the operator $A^\mu(\beta)$.

Lemma 4.1.2. *Suppose that one of the following assumptions holds:*

(i) $\beta \in [0,\pi], \quad \lambda \in \Sigma^* \setminus \Sigma_L$;

(ii) $\beta \in]0,\pi[, \quad \lambda \in \Sigma_L \setminus \Sigma^*$.

Then, λ^2 *is not an eigenvalue of the operator* $A^\mu(\beta)$ *for any* $\mu > 0$.

Proof.

(i) Since $\sin(\lambda L) \neq 0$, one can derive from (4.1.38)–(4.1.39) the following relations:

$$c_j^+ = \alpha^+ d_j, \quad \alpha^+ = \frac{1}{2}\left(\tan\left(\tfrac{\lambda L}{2}\right)\left(1+e^{-i\beta}\right) + \frac{e^{-i\beta}-1}{\tan\left(\frac{\lambda L}{2}\right)}\right), \quad \forall j \in \mathbb{Z}, \tag{4.1.41}$$

$$c_j^- = \alpha^- d_j, \quad \alpha^- = \frac{1}{2}\left(-\tan\left(\tfrac{\lambda L}{2}\right)\left(1+e^{i\beta}\right) + \frac{1-e^{i\beta}}{\tan\left(\frac{\lambda L}{2}\right)}\right), \quad \forall j \in \mathbb{Z}. \tag{4.1.42}$$

We have then

$$u_j^+(y) = d_j\left(\alpha^+ \sin(\lambda y) + \cos(\lambda y)\right), \qquad y \in \left[0, \tfrac{L}{2}\right], \qquad j \in \mathbb{Z},$$
$$u_j^-(y) = d_j\left(\alpha^- \sin(\lambda y) + \cos(\lambda y)\right), \qquad y \in \left[-\tfrac{L}{2}, 0\right], \qquad j \in \mathbb{Z}.$$

On the other hand, it follows from (4.1.37) that $|b_{j+\frac{1}{2}}| = |d_j| = d, \ \forall j \in \mathbb{Z}$. Consequently, one necessarily has $d = 0$ (otherwise $u \notin L_2(G_\mathcal{B})$). Thus, $b_{j+\frac{1}{2}} = d_j = c_j^\pm = 0$, $\forall j \in \mathbb{Z}$. Finally, from the relation (4.1.40) we get $a_{j+\frac{1}{2}} = a_{j-\frac{1}{2}} \cos\lambda, \ \forall j \in \mathbb{Z}$, which implies $|a_{j+\frac{1}{2}}| = a, \ \forall j \in \mathbb{Z}$. If $a \neq 0$ then $u \notin L_2(G_\mathcal{B})$. Otherwise $u = 0$.

(ii) If $\sin\left(\frac{\lambda L}{2}\right) = 0$, then it follows from (4.1.38) that $d_j = 0, \ \forall j \in \mathbb{Z}$ (since $1 - e^{-i\beta} \neq 0$). Similarly, if $\cos\left(\frac{\lambda L}{2}\right) = 0$, then it follows from (4.1.39) that $d_j = 0, \ \forall j \in \mathbb{Z}$ (since $1 + e^{-i\beta} \neq 0$). Consequently, the relations (4.1.37) imply that $b_{j+\frac{1}{2}} = a_{j+\frac{1}{2}} = 0$, $\forall j \in \mathbb{Z}$ (since $\sin\lambda \neq 0$). Next, it follows from (4.1.40) that $c_j^\pm = c_j, \ \forall j \in \mathbb{Z}$. If $\sin\left(\frac{\lambda L}{2}\right) = 0$, then (4.1.39) implies $c_j\left(1 - e^{-i\beta}\right) = 0, \ \forall j \in \mathbb{Z}$. If $\cos\left(\frac{\lambda L}{2}\right) = 0$, then (4.1.38) implies $c_j\left(1 + e^{-i\beta}\right) = 0, \ \forall j \in \mathbb{Z}$. In both cases we get $c_j = 0, \ \forall j \in \mathbb{Z}$, and, consequently, $u = 0$.

□

For $\lambda \notin \Sigma \cup \Sigma_L$ one can express the coefficients $\left\{a_{j+\frac{1}{2}}, b_{j+\frac{1}{2}}, c_j^\pm, d_j\right\}$ in terms of the values $\{\mathbf{u}_j\}$:

$$a_{j+\frac{1}{2}} = \frac{1}{\sin\lambda}\left(\mathbf{u}_{j+1} - \mathbf{u}_j \cos\lambda\right), \quad b_{j+\frac{1}{2}} = d_j = \mathbf{u}_j, \quad c_j^\pm = \alpha^\pm \mathbf{u}_j, \qquad j \in \mathbb{Z}, \tag{4.1.43}$$

where $\alpha^{\pm}$ are defined in (4.1.41)–(4.1.42). Then, the relations (4.1.34)–(4.1.36) can be rewritten as

$$u_{j+\frac{1}{2}}(s) = \mathbf{u}_j \frac{\sin \lambda(1-s)}{\sin \lambda} x + \mathbf{u}_{j+1} \frac{\sin(\lambda s)}{\sin \lambda}, \qquad s \in [0,1], \qquad \forall j \in \mathbb{Z}, \tag{4.1.44}$$

$$u_j^+(y) = \mathbf{u}_j \frac{\sin\left(\lambda\left(\frac{L}{2}-y\right)\right)}{\sin\left(\frac{\lambda L}{2}\right)} + \mathbf{u}_j^+ \frac{\sin(\lambda y)}{\sin\left(\frac{\lambda L}{2}\right)}, \qquad y \in \left[0, \tfrac{L}{2}\right], \qquad \forall j \in \mathbb{Z}, \tag{4.1.45}$$

$$u_j^-(y) = \mathbf{u}_j \frac{\sin\left(\lambda\left(\frac{L}{2}+y\right)\right)}{\sin\left(\frac{\lambda L}{2}\right)} - \mathbf{u}_j^- \frac{\sin(\lambda y)}{\sin\left(\frac{\lambda L}{2}\right)}, \qquad y \in \left[-\tfrac{L}{2}, 0\right], \qquad \forall j \in \mathbb{Z}. \tag{4.1.46}$$

After plugging (4.1.43) into the relation (4.1.40) we end up with the following finite difference equation:

$$\mathbf{u}_{j+1} + 2g_\beta(\lambda)\mathbf{u}_j + \mathbf{u}_{j-1} = 0, \qquad \forall j \in \mathbb{Z}^*, \tag{4.1.47}$$

$$\mathbf{u}_1 + 2g_\beta^\mu(\lambda)\mathbf{u}_0 + \mathbf{u}_{-1} = 0. \tag{4.1.48}$$

Here

$$g_\beta(\lambda) = -\cos\lambda + \frac{\sin\lambda(\cos\beta - \cos(\lambda L))}{\sin(\lambda L)}, \qquad \lambda \notin \Sigma \cup \Sigma_L,$$

$$g_\beta^\mu(\lambda) = -\cos\lambda + \mu\frac{\sin\lambda(\cos\beta - \cos(\lambda L))}{\sin(\lambda L)}, \qquad \lambda \notin \Sigma \cup \Sigma_L.$$

Case $\lambda = 0$

If 0 is an eigenvalue of the operator $A^\mu(\beta)$, then the corresponding eigenfunction has the form

$$u_{j+\frac{1}{2}}(s) = \mathbf{u}_j(1-s) + \mathbf{u}_{j+1}s, \qquad s \in [0,1], \qquad \forall j \in \mathbb{Z}, \tag{4.1.49}$$

$$u_j^+(y) = \mathbf{u}_j\left(1 - \frac{2y}{L}\right) + \frac{2\mathbf{u}_j^+}{L}y, \qquad y \in \left[0, \tfrac{L}{2}\right], \qquad \forall j \in \mathbb{Z}, \tag{4.1.50}$$

$$u_j^-(y) = \mathbf{u}_j\left(1 + \frac{2y}{L}\right) - \frac{2\mathbf{u}_j^-}{L}y, \qquad y \in \left[-\tfrac{L}{2}, 0\right], \qquad \forall j \in \mathbb{Z}. \tag{4.1.51}$$

From the quasiperiodicity conditions (4.1.2) we find

$$\mathbf{u}_j^+ = \frac{1+e^{-i\beta}}{2}\mathbf{u}_j, \qquad \mathbf{u}_j^- = \frac{1+e^{i\beta}}{2}\mathbf{u}_j, \qquad j \in \mathbb{Z}.$$

Taking into account the Kirchhoff's conditions (4.1.3), we get a finite difference equation again:

$$\mathbf{u}_{j+1} - 2\left(1 + \tfrac{w_j^\mu}{L}(1-\cos\beta)\right)\mathbf{u}_j + \mathbf{u}_{j-1} = 0, \qquad j \in \mathbb{Z}. \tag{4.1.52}$$

Finally, combining the cases $\lambda \notin \Sigma \cup \Sigma_L$ and $\lambda = 0$, we get the finite difference equation (4.1.47)–(4.1.48) with the function g_β defined as

$$g_\beta(\lambda) = \begin{cases} -\cos\lambda + \dfrac{\sin\lambda(\cos\beta - \cos(\lambda L))}{\sin(\lambda L)}, & \lambda \notin \Sigma \cup \Sigma_L, \\ \dfrac{\cos\beta - 1}{L} - 1, & \lambda = 0, \end{cases} \tag{4.1.53}$$

$$g^{\mu}_{\beta}(\lambda) = \mu g_{\beta}(\lambda) + (\mu - 1), \qquad \lambda \notin \Sigma^* \cup \Sigma_L.$$

Consider the characteristic equation associated with the system (4.1.47):

$$r^2 + 2g_{\beta}(\lambda)r + 1 = 0, \tag{4.1.54}$$

It has a solution such that $|r_{\beta}(\lambda)| \leqslant 1$:

$$r_{\beta}(\lambda) = -g_{\beta}(\lambda) + sign\,(g_{\beta}(\lambda))\sqrt{g^2_{\beta}(\lambda) - 1}. \tag{4.1.55}$$

The following proposition, analogous to Proposition 1.3.10, gives the relation between the absolute value of $g_{\beta}(\lambda)$ and the nature of the point λ^2.

Proposition 4.1.7. *For* $\lambda \in \mathbb{R}_+ \setminus \Sigma_L$,

$$|g_{\beta}(\lambda)| \leqslant 1 \quad \Leftrightarrow \quad |r_{\beta}(\lambda)| = 1 \quad \Leftrightarrow \quad \lambda^2 \in \sigma(A(\beta)). \tag{4.1.56}$$

Proof. The first equivalence follows immediately from (4.1.55). Next, $|g_{\beta}(\lambda)| \leqslant 1$ if and only if there exists $\theta \in [0, \pi]$ such that $g_{\beta}(\lambda) = -\cos\theta$. Taking into account the definition (4.1.53) of the function g_{β}, we get

$$\cos\lambda - \cos\theta = \frac{\sin\lambda(\cos\beta - \cos(\lambda L))}{\sin(\lambda L)}, \qquad \lambda \notin \Sigma \cup \Sigma_L. \tag{4.1.57}$$

The relation (4.1.57) is equivalent to (4.1.33) for $\lambda \in \mathbb{R}_+ \setminus (\Sigma \cup \Sigma_{\beta} \cup \Sigma_L)$. If $\lambda \in \left\{\Sigma^* \cup \Sigma^*_{\beta}\right\} \setminus \Sigma_L$, then we have $|g_{\beta}(\lambda)| \leqslant 1$ and $\lambda^2 \in \sigma(A(\beta))$ (cf. Proposition 4.1.5 (i)). Finally, $0 \in \sigma(A(\beta))$ if and only if $\beta = 0$. At the same time, $|g_{\beta}(0)| \leqslant 1$ if and only if $\beta = 0$. □

The argument used in Section 1.3.2.2 applies in the present case to show that

$$\lambda^2 \in \sigma_d\left(A^{\mu}(\beta)\right) \quad \Leftrightarrow \quad r_{\beta}(\lambda) = -g^{\mu}_{\beta}(\lambda),$$

and the corresponding eigenfunction is

$$\mathbf{u}_j = \mathbf{u}_0 r^{|j|}_{\beta}(\lambda), \qquad \forall j \in \mathbb{Z}.$$

With the classification of the gaps in two types introduced in Proposition 4.1.6 we can state the following theorem.

Theorem 4.1.1. *The operator $A^{\mu}(\beta)$ has no embedded eigenvalues of finite multiplicity for any $\mu > 0$, $\beta \in [0, \pi]$. For any $0 < \mu < 1$, $\beta \in [0, \pi]$ there exist two simple eigenvalues of the operator $A^{\mu}(\beta)$ in each gap of this operator of type I and one simple eigenvalue in each gap of type II. These eigenvalues are characterised as follows:*

$$\lambda^2 \in \sigma_d\left(A^{\mu}(\beta)\right) \quad \Leftrightarrow \quad \lambda \textit{ is a solution of the equation } \quad \mu = F_{\beta}(\lambda),$$

where

$$F_{\beta}(\lambda) = 1 - \frac{\sqrt{g^2_{\beta}(\lambda) - 1}}{|g_{\beta}(\lambda) + \cos\lambda|}.$$

For $\mu \geqslant 1$ the operator $A^{\mu}(\beta)$ has no eigenvalues.

The proof is an obvious modification of the one of Theorem (1.3.1), the relation (1.3.58) being replaced by the relation

$$F_\beta(\lambda) = 1 - \sqrt{1 - \phi_{L,\beta}(\lambda)\left(\phi_{L,\beta}(\lambda) - \psi(\lambda)\right)}.$$

Remark 4.1.4. For $\beta \in]0,\pi]$, the interval $]0,\sigma_*[$, $\sigma_* = \min \sigma(A(\beta))$, is considered as a gap of type II (cf. Remark 4.1.2). Thus, the operator $A^\mu(\beta)$ has one simple eigenvalue below the essential spectrum if $\beta \in]0,\pi]$.

4.1.3 Results for the operator $A^\mu_\varepsilon(\beta)$

We can now give the analogues of Theorems 1.4.1, 1.4.3 for the operator $A^\mu_\varepsilon(\beta)$.

Let $A_\varepsilon(\beta)$ be the non-perturbed operator defined as $A^\mu_\varepsilon(\beta)$ with $\mu = 1$, acting in $L_2(\mathcal{B}_\varepsilon)$, where $\mathcal{B}_\varepsilon = \mathcal{B}^1_\varepsilon$ is the unperturbed periodicity band.

Theorem 4.1.2 (Essential spectrum)**.** *Let $\{]a_n(\beta), b_n(\beta)[,\ n \in \mathbb{N}^*\}$ be the gaps of the operator $A(\beta)$ on the limit graph $G_\mathcal{B}$ for $\beta \in [0,\pi]$. Then, for each $n_0 \in \mathbb{N}^*$ there exists $\varepsilon_0(\beta) > 0$ such that if $\varepsilon < \varepsilon_0(\beta)$ the operator $A_\varepsilon(\beta)$ has at least n_0 gaps $\{]a_{\varepsilon,n}(\beta), b_{\varepsilon,n}(\beta)[\}_{n=1}^{n_0}$ such that*

$$a_{\varepsilon,n}(\beta) = a_n(\beta) + O(\varepsilon), \quad b_{\varepsilon,n}(\beta) = b_n(\beta) + O(\varepsilon), \qquad \varepsilon \to 0, \qquad 1 \leqslant n \leqslant n_0.$$

As in the case of the ladder, the proof of this theorem is based on the reduction to the periodicity cell due to Floquet-Bloch theory and the min-max principle for the bounded domain. Next, using a standard compact perturbation argument (an analogue of Proposition 1.2.1, the proof being entirely similar), we know that

$$\sigma_{ess}(A^\mu_\varepsilon(\beta)) = \sigma_{ess}(A_\varepsilon(\beta)).$$

Theorem 4.1.3 (Discrete spectrum – weak version)**.** *Let $]a(\beta), b(\beta)[$ be a gap of the operator $A^\mu(\beta)$ on the limit graph $G_\mathcal{B}$ for $\beta \in [0,\pi]$ and $\lambda^{(0)}(\beta) \in\ ]a(\beta), b(\beta)[$ a (simple) eigenvalue of this operator. Then there exists $\varepsilon_0(\beta) > 0$ such that if $\varepsilon < \varepsilon_0(\beta)$ the operator $A^\mu_\varepsilon(\beta)$ has an eigenvalue $\lambda_\varepsilon(\beta)$ inside a gap $]a_\varepsilon(\beta), b_\varepsilon(\beta)[$ such that:*

$$\lambda_\varepsilon(\beta) = \lambda^{(0)}(\beta) + O\left(\sqrt{\varepsilon}\right), \qquad \varepsilon \to 0.$$

This theorem can be proved using the same argument as in the case of the ladder based on construction of an appropriate pseudo-mode obtained as an extrapolation of the eigenfunction corresponding to the eigenvalue $\lambda^{(0)}(\beta)$.

Remark 4.1.5. An analogue of Theorem 1.4.2 should also be possible to derive using matched asymptotic expansions with obvious minor modifications in the proof. This would give the optimal order of convergence ε for the eigenvalue as well as its complete asymptotic expansion:

$$\lambda_\varepsilon(\beta) = \sum_{k=0}^{n} \lambda^{(k)}(\beta)\varepsilon^k + O\left(\varepsilon^{n+1}\right), \qquad \varepsilon \to 0.$$

4.1.4 A more general geometry

We can also consider a slightly more general geometry, where the thickness of the horizontal edges is different from the thickness of the vertical edges. In other words, the distance between the obstacles is ε in the x-direction and $\nu\varepsilon$ in the y-direction for some $\nu > 0$ (cf. figure 4.7).

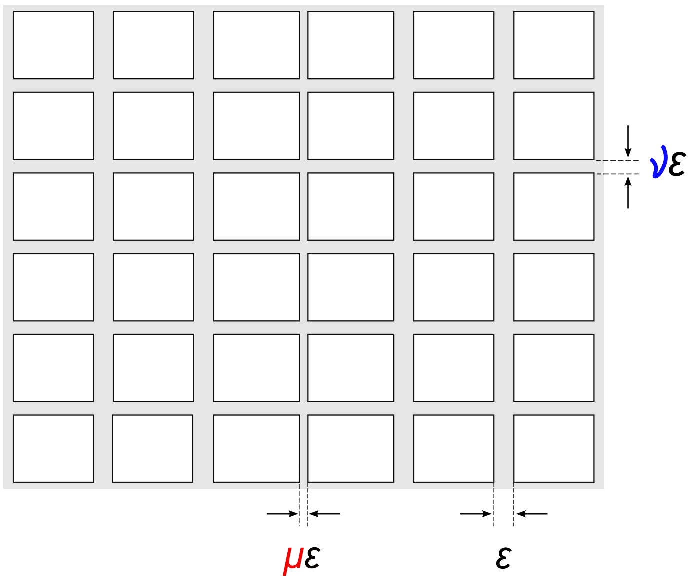


Figure 4.7: A more general domain: the thickness of the non-perturbed vertical edges is ε whereas the thickness of the horizontal edges is $\nu\varepsilon$.

This does not change the results qualitatively. Indeed, the only modification in the definition of the limit operator is the appearance of a factor ν in Kirchhoff's conditions. More precisely, the operator $A^\mu_\nu(\beta)$ is defined in the space

$$L_2^\mu(G_\mathcal{B}) = \left\{u : \ \|u\|^2_{L_2^\mu(G_\mathcal{B})} < \infty\right\},$$

$$\|u\|^2_{L_2^\mu(G_\mathcal{B})} = \sum_{j\in\mathbb{Z}} \left(w_j^\mu \|u_j^+\|^2_{L_2(e_j^+)} + w_j^\mu \|u_j^-\|^2_{L_2(e_j^-)} + \nu \|u_{j+\frac{1}{2}}\|^2_{L_2\left(e_{j+\frac{1}{2}}\right)} \right),$$

as

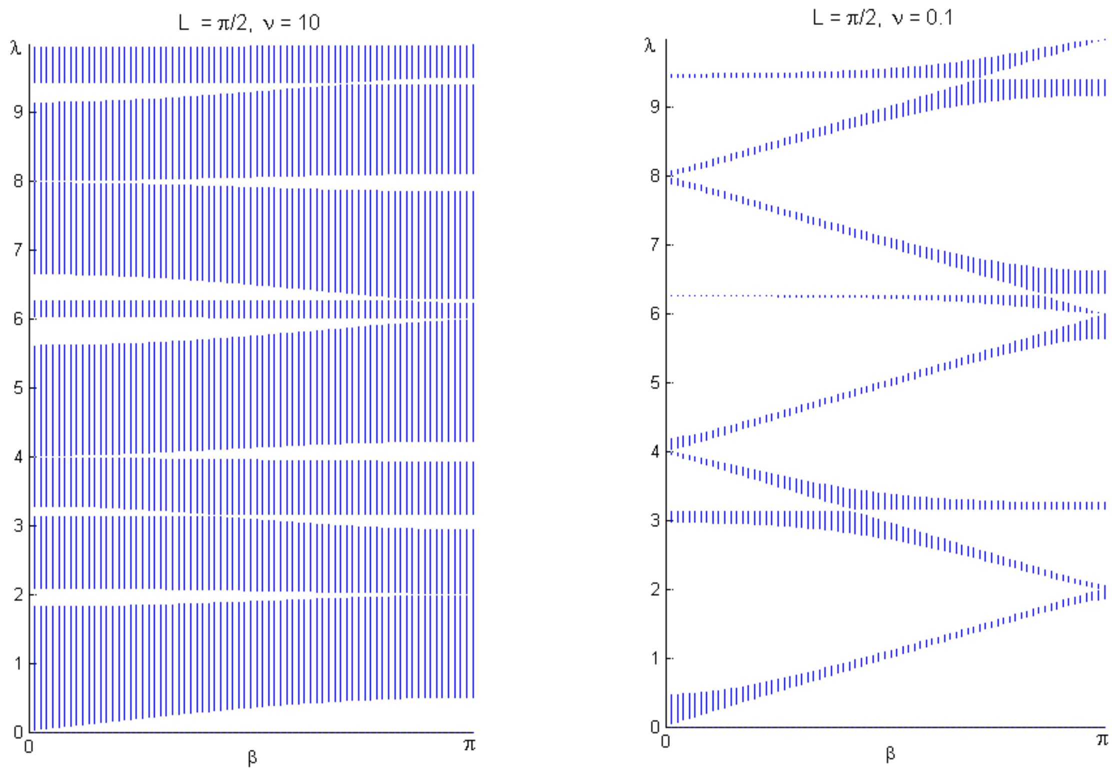
$$\left.(A^\mu_\nu(\beta)u)\right|_e = -(u|_e)'', \qquad e \in \mathcal{E}_{G_\mathcal{B}},$$

$$\begin{aligned} D\left(A^\mu_\nu(\beta)\right) = \Big\{ & u \in H^2\left(G_\mathcal{B}\right) : \\ & \mathrm{u}_j^+ = e^{-i\beta}\mathrm{u}_j^-, \quad (u_j^+)'\left(\tfrac{L}{2}\right) = e^{-i\beta}(u_j^-)'\left(-\tfrac{L}{2}\right), \qquad \forall j \in \mathbb{Z}, \\ & \nu u'_{j+\frac{1}{2}}(0) - \nu u'_{j-\frac{1}{2}}(1) + w_j^\mu (u_j^+)'(0) - w_j^\mu (u_j^-)'(0) = 0, \qquad \forall j \in \mathbb{Z} \Big\}. \end{aligned}$$

This results in the same characterization of the spectrum of the operator $A_\nu(\beta)$ (corresponding to the non-perturbed case $\mu = 1$) as the one given in Proposition 4.1.3 (5) with the equation (4.1.33) replaced by

$$\nu\phi_{L,\beta}(\lambda) = f_\theta(\lambda). \tag{4.1.58}$$

All the other statements of Proposition 4.1.3 hold without any change. This means that the spectrum of the $A_\nu(\beta)$ for $\nu \neq 1$, does not change qualitatively from the case $\nu = 1$. Nevertheless, the parameter ν influences the size of the gaps. Clearly, the size of the image of a gap by the function $x \mapsto \sqrt{x}$ can never exceed π (since $\{\lambda^2,\ \lambda \in \{\pi\mathbb{N}^*\}\} \subset \sigma\left(A_\nu(\beta)\right)$, cf. Proposition 4.1.3 (2)). However, choosing a bigger value of ν leads to decreasing of this size whereas choosing a smaller value of ν leads to increasing of this size. An example is given in figure 4.8. Roughly speaking, for ν big there is a lot of spectrum with small gaps inside it and for small ν there are big gaps separated by small pieces of spectrum.



(a) $\nu = 10$ (b) $\nu = 0.1$

Figure 4.8: Influence of the parameter ν on the size of the gaps for $L = \frac{\pi}{2}$: for smaller values of ν the gaps are bigger and the spectral bands are smaller.

Similarly, the results about the discrete spectrum when a perturbation is introduced hold true for this more general geometry. The equation characterization for the eigenvalues given in Theorem 4.1.1 is be replaced by

$$\mu = F_\beta^\nu(\lambda), \qquad F_\beta^\nu(\lambda) = 1 - \sqrt{1 - \nu\phi_{L,\beta}(\lambda)\left(\nu\phi_{L,\beta(}\lambda) - \psi(\lambda)\right)}.$$

Obviously, the conclusion about the number of eigenvalues inside the gaps does not change: there are still precisely one or two eigenvalues inside each gap depending on its type.

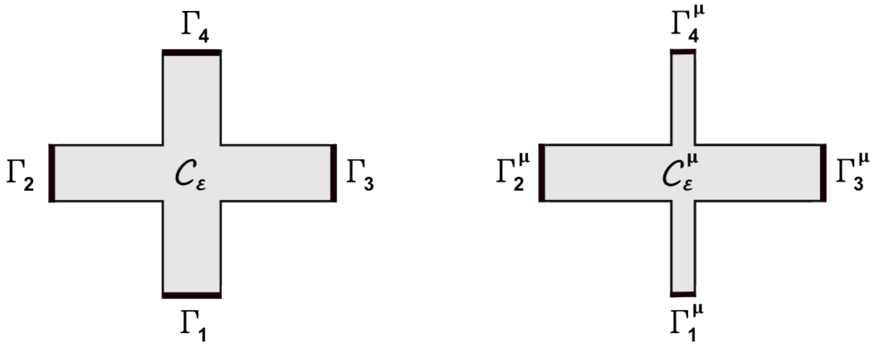
Remark 4.1.6. We see that the ladder considered in Chapter 1 is a periodicity band for the domain shown in figure 4.7 with $\nu = 2$. We remember that the equations describing the spectra of the operators $A(0)$, $A(\pi)$ respectively, differed from the ones corresponding to the operators A_s, A_{as} by a coefficient 2 (cf. (4.1.29), (4.1.30), (1.3.12), (1.3.59)). On the other hand, for $\nu = 2$ the spectra of the operators $A_2(0)$, $A_2(\pi)$ are described by the equation $2\phi_{L,\beta}(\lambda) = f_\theta(\lambda)$ (cf. (4.1.58)). This leads precisely to the relations (1.3.12), (1.3.59). However, the relations (4.1.29), (4.1.30) are not exactly the ones describing the spectra of the operators $A(0)$, $A(\pi)$. Indeed, while obtaining them from the "true" ones, (4.1.27) and (4.1.28), we performed a division by $\sin\left(\frac{\lambda L}{2}\right)$ and $\cos\left(\frac{\lambda L}{2}\right)$ respectively. Thus, we excluded from consideration the sets $\left\{\lambda^2, \sin\left(\frac{\lambda L}{2}\right) = 0\right\}$ for $\beta = 0$ and $\left\{\lambda^2, \cos\left(\frac{\lambda L}{2}\right) = 0\right\}$ for $\beta = \pi$ which belong to the spectra of the operators $A(0)$ and $A(\pi)$ respectively. They also belong to the spectra of the operators A_s and A_{as} respectively, as shown in Propositions 1.3.4 (1), 1.3.12 (1). Thus, we can finally conclude that

$$\sigma(A_2(0)) = \sigma(A_s), \qquad \sigma(A_2(\pi)) = \sigma(A_{as}).$$

To resume, we see that considering the symmetric (resp. antisymmetric) part of the operator with Neumann boundary conditions and the 0-quasiperiodic (resp. π-quasiperiodic) operator leads to the same limit problems on the graph.

4.2 Numerical approach

The numerical method described in Section 3.2 applies to the β-quasiperiodic case with minor modifications (the detailed description of the method in the β-quasiperiodic case can be found in [24]). Since the domain in question is now $\mathcal{B}_\varepsilon^\mu$ (cf. figure 4.2), the periodicity cell and the interior domains are the ones shown in figure 4.9. Since β-



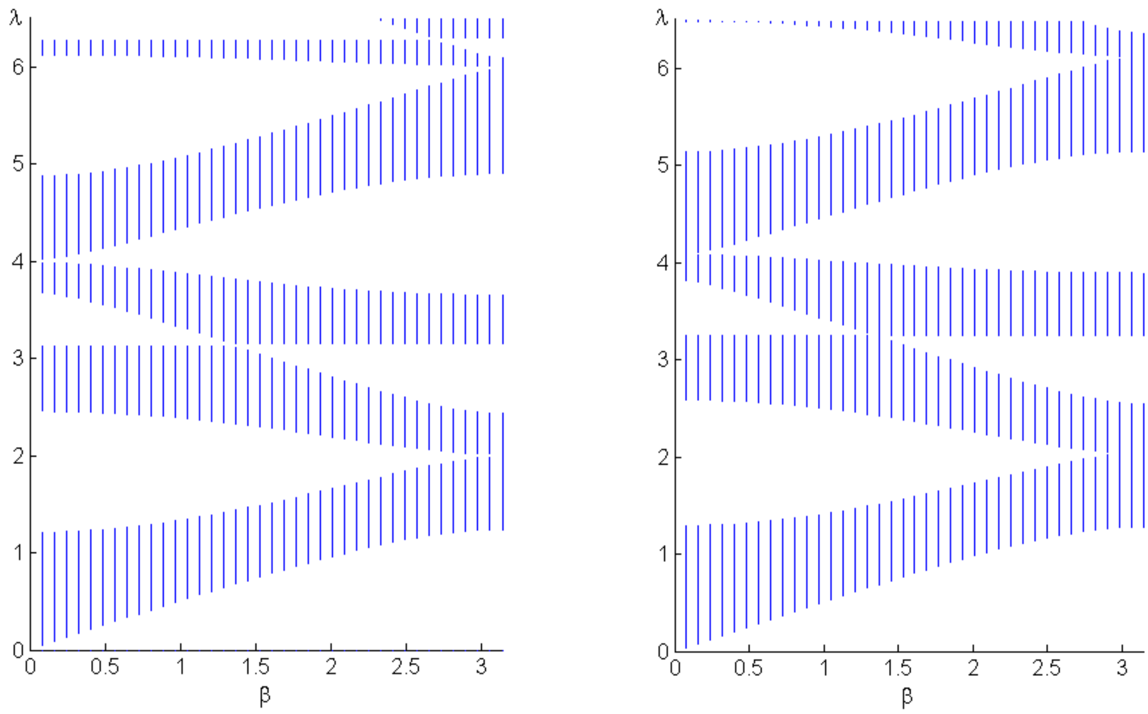
(a) Periodic cell (b) Perturbed cell

Figure 4.9

quasiperiodic conditions are imposed on the boundaries Γ_1, Γ_4 (resp. Γ_1^μ, Γ_4^μ), in our numerical method the meshes of these two boundaries have to be the same, and the corresponding vertices of the two meshes are coupled. This reduces the number of degrees of freedom in the corresponding functional space.

4.2.1 Numerical results

In this section we give some numerical results for the operator $A_\varepsilon^\mu(\beta)$. For a fixed β they are very similar to those shown in Section 3.4. For this reason we will concentrate on the dependence on β. In figure 4.10 we show the dependence of the essential spectrum of the operator $A_\varepsilon(\beta)$ as well as the limit operator $A(\beta)$ with respect to β for $L = \frac{\pi}{2}$, $\varepsilon = 0.1$. For the limit operator it follows from Lemma 4.1.1 that any point $\lambda \in \pi\mathbb{N}^*$ (in the range represented in figure 4.10 there is only the point $\lambda = \pi$) corresponds to the lower end of a gap for some values of β and to the upper end of a gap for the others. At the point β_0 where the "transition" happens it corresponds to an interior point of the spectrum. This transition point is the point of discontinuity of the function $\phi_{L,\beta}$, i.e. the point given by the relation $\cos\beta_0 = \cos(\lambda L)$. It is interesting to notice that according to the numerical results it seems that the same property holds for the non-limit operator as well. In other words, there is a point λ which corresponds to the lower end of a gap for some values of β and to the upper end of a gap for the others. However, it is no more π (in the example given in figure 4.10 $\lambda \approx 3.24$). In figure 4.11 the dependence on β of the eigenvalues

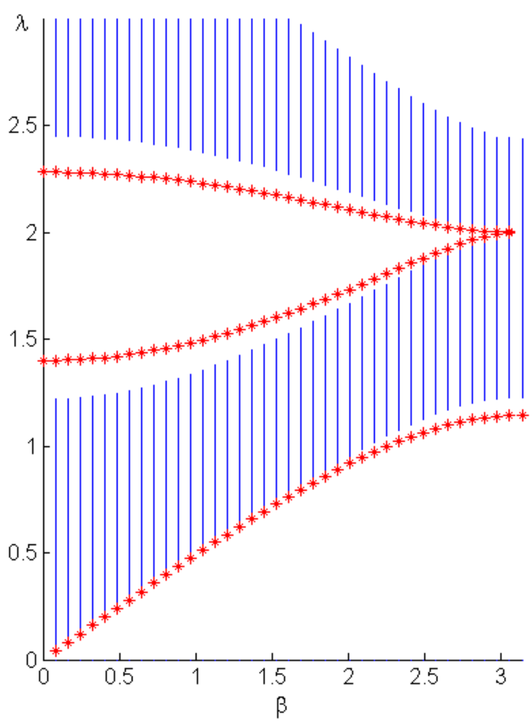


(a) The limit operator on the graph, $A(\beta)$

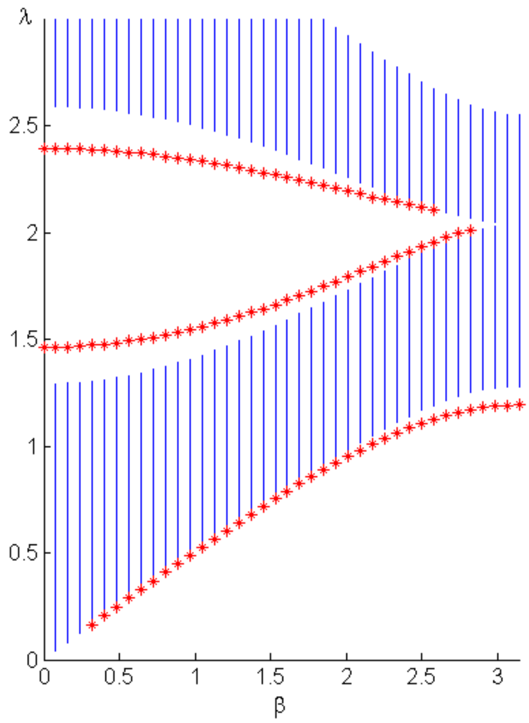
(b) The operator $A_\varepsilon(\beta)$ for $\varepsilon = 0.1$

Figure 4.10: Dependence of the essential spectrum on β for $L = \frac{\pi}{2}$. The first 5 spectral bands are represented.

in the first gap and below the essential spectrum is shown for $L = \pi/2$, $\mu = 0.25$ in the case of the graph and for the $2D$ domain with $\varepsilon = 0.1$. One can remark that the eigenvalues below the essential spectrum are very close to it. In the first gap they are better separated from the essential spectrum if β is not close to π. Again, the question of existence of eigenvalues for any $\beta \in [0, \pi]$ is not answered since the computations become costly when the eigenvalue approaches the essential spectrum.



(a) The limit operator on the graph, $A^\mu(\beta)$



(b) The operator $A^\mu_\varepsilon(\beta)$ for $\varepsilon = 0.1$

Figure 4.11: Dependence of the eigenvalues in the first gap and below the essential spectrum on β for $L = \frac{\pi}{2}$, $\mu = 0.25$.

Time-dependent simulations

As it was discussed in the beginning of this chapter, an eigenvalue ω^2 of the operator $A^\mu_\varepsilon(\beta)$ corresponds to a guided mode propagating the the y direction with the speed $\frac{\omega L}{\beta}$. We would like then to see such guided modes in a time-dependent simulation. The idea is to put a time-harmonic source with the frequency ω at some point of the perturbed line which would give rise to the corresponding guided mode.

Two kinds of simulations have been preformed. The first one is based on a "naive" Matlab finite elements code where a finite difference scheme in time is implemented. The infinite domain Ω^μ_ε is replaced by a finite one, $\widetilde{\Omega}^\mu_\varepsilon$, without putting any special boundary conditions that would take into account the infinite nature of the domain. For this reason, one obtains an approximation of the solution as long as the wave does not reach the boundary of the domain. The problem in question is

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \Delta u + f_\omega(x,y,t) \quad \text{in} \quad \widetilde{\Omega}^\mu_\varepsilon, \\ u\big|_{t=0} = 0, \qquad \dfrac{\partial u}{\partial t}\Big|_{t=0} = 0, \\ \dfrac{\partial u}{\partial n}\Big|_{\partial\widetilde{\Omega}^\mu_\varepsilon} = 0, \end{cases}$$

where

$$f_\omega(x, y, t) = \sin(\omega t)\delta(x_0, y_0).$$

For the discretization, we use a finite element method in space and an explicit finite difference scheme in time.

In the second set of simulations we used a much more sophisticated C++ code by Julien Coatléven which takes into account the infinite nature of the propagation domain (for more details see [12]). In figure 4.12 is represented the solution obtained with this code for $\omega = 2.25$ at $t = 70$.

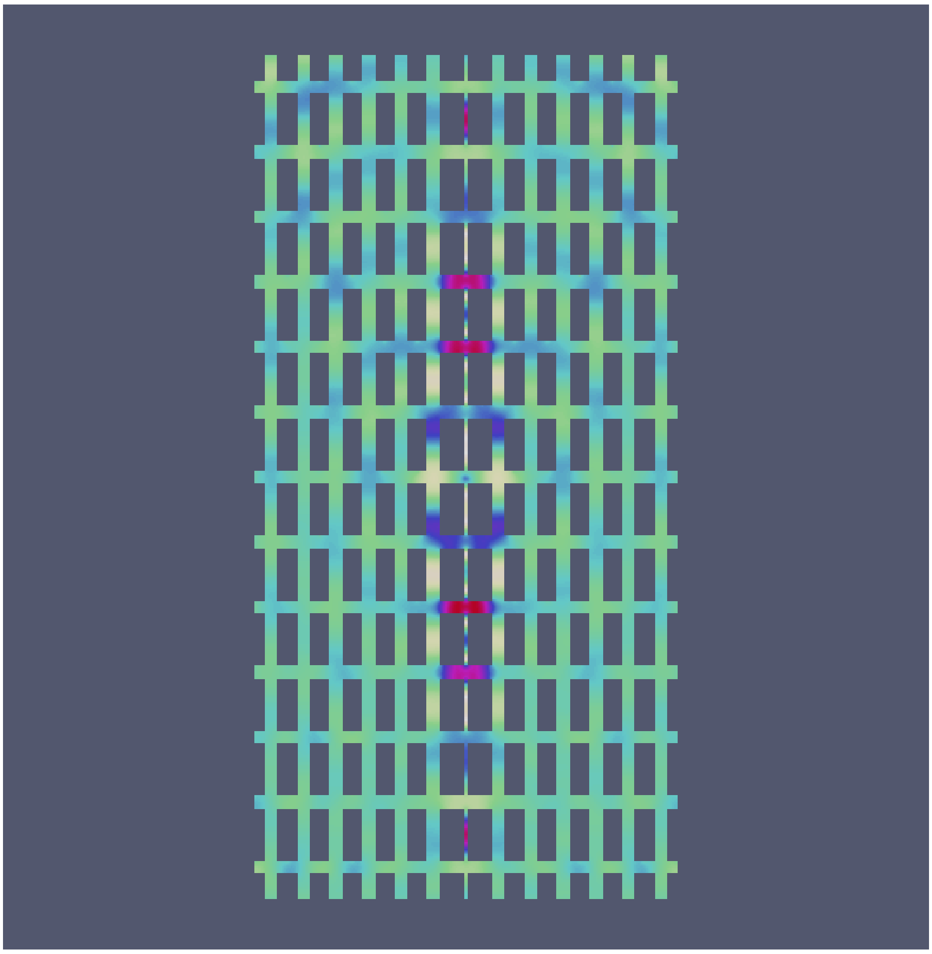


Figure 4.12: Solution in the time-dependent simulation performed using the C++ code (Julien Coatléven) for $L = 2$, $\mu = 0.25$, $\omega = 2.25$, $t = 70$. A guided mode propagating along the linear defect seems to appear.

CHAPTER 5

GUIDED MODES IN OPEN PERIODIC LINEIC WAVEGUIDES: THE 3D CASE

5.1 Geometry and statement of the problem

Propagation domain

In this chapter we consider a 3D generalization of the plane waveguide considered in the previous chapters. By a 3D generalization we mean a domain obtained by fattening a 3D infinite primitive orthorhombic lattice (which is periodic in three orthogonal directions). In each direction the straight lines of the lattice are replaced by parallel infinite pipes of a constant cross-section of size of order ε which is going to be small. The cross-sections are not necessarily the same for the three directions. More precisely, let L_x, L_y, L_z be positive numbers (the periods of the domain in the x, y and z directions respectively). Let ω_x, ω_y, ω_z be bounded domains of $\mathbb{R}^2$ containing the origin (the normalized cross sections of the pipes parallel to the axes x, y and z respectively). Then, the propagation domain Ω_ε is defined as follows:

$$\Omega_\varepsilon = \bigcup_{(k,\ell)\in\mathbb{Z}^2} \left\{\mathcal{P}^x_{\varepsilon,k,\ell} \cup \mathcal{P}^y_{\varepsilon,k,\ell} \cup \mathcal{P}^z_{\varepsilon,k,\ell}\right\},$$

$$\mathcal{P}^\gamma_{\varepsilon,k,\ell} = \left\{\left(\tfrac{\alpha}{\varepsilon}, \tfrac{\beta}{\varepsilon}\right) \in \omega_\gamma + (kL_\alpha, \ell L_\beta)\right\}, \qquad (\alpha,\beta,\gamma) \in \{(x,y,z),(y,z,x),(z,x,y)\}.$$

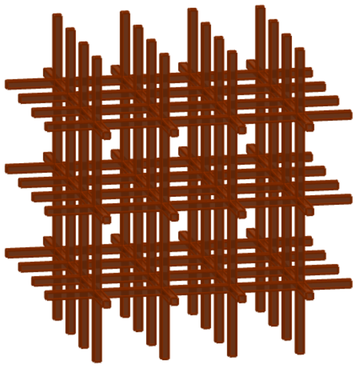
Some examples of such domains are shown in figure 5.1.

A linear defect is introduced to the structure by changing the thickness of one pipe (for instance, parallel to the z direction): the characteristic size of its cross section is set to be $\sqrt{\mu}\varepsilon$ instead of ε. The perturbed domain Ω^μ_ε is defined by the following relation:

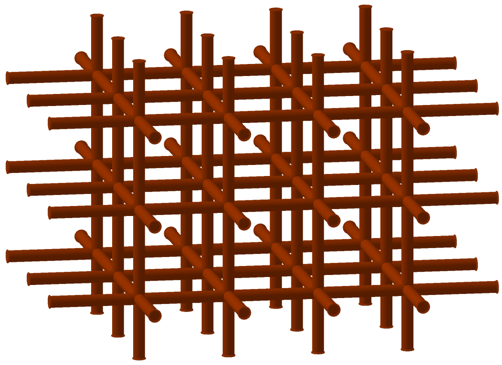
$$\Omega^\mu_\varepsilon = \mathcal{P}^x_{\varepsilon,0,0} \cup \mathcal{P}^y_{\varepsilon,0,0} \cup \mathcal{P}^{z,\mu}_{\varepsilon,0,0} \cup \bigcup_{(k,\ell)\in\mathbb{Z}^2\setminus(0,0)} \left\{\mathcal{P}^x_{\varepsilon,k,\ell} \cup \mathcal{P}^y_{\varepsilon,k,\ell} \cup \mathcal{P}^z_{\varepsilon,k,\ell}\right\},$$

$$\mathcal{P}^{z,\mu}_{\varepsilon,0,0} = \left\{\left(\frac{x}{\sqrt{\mu}\varepsilon}, \frac{y}{\sqrt{\mu}\varepsilon}\right) \in \omega_z\right\}.$$

The limit problem for $\varepsilon \to 0$ will be posed on a 3D graph and its spectrum will be easy to analyse. For ε small enough the spectrum of the non-limit operator will be approximated by the spectrum of the limit operator. In this chapter we will study essentially the limit operator. In Section 5.3 we mention the results for the non-limit operator that follow from the works [61], [47], [57] (see also [31, 32, 4] for an example of application for a Maxwell problem). We do not perform the asymptotic analysis which would permit to obtain a full asymptotic expansion for the eigenvalues. We mention that the approach used in the 2D case should be possible to apply in the 3D case as well with appropriate modifications.



(a) Rectangular cross section



(b) Circular cross section

Figure 5.1: Propagation domain

Guided modes

As in the 2D, case we are interested in guided modes, i.e. solutions of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u \quad \text{in} \quad \Omega_\varepsilon^\mu, \tag{5.1.1}$$

that propagate along the defect and stay confined in the transversal directions. Neumann boundary conditions are imposed on the boundaries of the obstacles. More precisely, we search solutions of the form

$$u(x, y, z, t) = v(x, y, z) e^{i(\omega t - \beta z / L_z)}, \tag{5.1.2}$$

where v is a L_z-periodic function in z-direction confined in the periodicity band $B_\varepsilon^\mu = \Omega_\varepsilon^\mu \times \left]-\frac{L_z}{2}, \frac{L_z}{2}\right[$:

$$v(x, y, z + L_z) = v(x, y, z), \qquad v \in L_2\left(B_\varepsilon^\mu\right).$$

Clearly, we could choose any other periodicity band, but we will consider the symmetric one. After injecting (5.1.2) in the wave equation (5.1.1) , taking into account Neumann boundary conditions on the boundaries of the obstacles,

$$\left.\frac{\partial u}{\partial n}\right|_{\partial \Omega_\varepsilon^\mu} = 0,$$

and introducing the function

$$\widetilde{v} = v(x, y, z) e^{-i\beta z / L_z},$$

we end up with the following problem posed in the periodicity band:

$$\begin{cases} -\Delta\widetilde{v} = \omega^2\widetilde{v} \quad \text{in} \;\; B^\mu_\varepsilon, \\ \widetilde{v} \in L_2\left(B^\mu_\varepsilon\right), \\ \left.\dfrac{\partial\widetilde{v}}{\partial n}\right|_{\partial B^\mu_\varepsilon \setminus \{\Sigma\cup\Sigma'\}} = 0, \\ \widetilde{v} \quad \text{is} \;\; \beta-\text{quasiperiodic (with respect to } z). \end{cases}$$

This is an eigenvalue problem for the operator $A^\mu_\varepsilon(\beta) : L_2\left(B^\mu_\varepsilon\right) \to L_2\left(B^\mu_\varepsilon\right)$ defined as

$$A^\mu_\varepsilon(\beta) = -\Delta u,$$
$$D\left(A^\mu_\varepsilon(\beta)\right) = \left\{ u \in H^1_\Delta\left(B^\mu_\varepsilon\right), \quad \left.\frac{\partial u}{\partial n}\right|_{\partial B^\mu_\varepsilon \setminus \{\Sigma\cup\Sigma'\}} = 0, \quad u|_{\Sigma'} = e^{-i\beta}\, u|_{\Sigma}, \quad \left.\frac{\partial u}{\partial z}\right|_{\Sigma'} = e^{-i\beta} \left.\frac{\partial u}{\partial z}\right|_{\Sigma} \right\}.$$

In the next section we describe the limit operator and compute its spectrum. The general strategy is analogous to the one used in the 2D case. We start by studying the non-perturbed operator using the Floquet-Bloch theory (Section 5.2.1). This leads to the equation (5.2.4) for the eigenvalues of the operator in the periodicity cell which is very similar to the one found in the 2D case. Here again we show that the limit operator has infinitely many gaps whose ends tend to infinity (Proposition 5.2.3). Next, we study the discrete spectrum of the perturbed operator (Section 5.2.2). As in the 2D case, the problem reduces the to study of a finite difference equation (Lemma 5.2.4). However, this time the equation is two-dimensional and cannot be solved via the associated characteristic equation as before. To study this equation we apply the discrete Fourier transform. We end up with the characterisation of the eigenvalues given in Proposition 5.2.9. This characterisation is more difficult to analyse than the one obtained in the 2D case. In Section 5.2.2.4 we prove the existence of at least one or two eigenvalues in each gap according to its type but contrarily to the 2D case we are not able to find the exact number of eigenvalues in each gap.

5.2 The limit operator

The limit operator is defined on the graph $G = \bigcup\limits_{\varepsilon>0} \Omega_\varepsilon \cap \left\{ z \in \left[-\frac{L_z}{2}, \frac{L_z}{2}\right]\right\}$ shown in figure 5.2. Let us introduce some notation for this graph. The vertices in the plane $z = 0$ are enumerated by two indices $k, \ell \in \mathbb{Z}$ in such a way that the coordinates of the vertex $v_{k,\ell}$ are $(kL_x, \ell L_y, 0)$. The vertex $v_{0,0}$ corresponds to the perturbed line. We denote by $v^\pm_{k,\ell}$ the vertices with the coordinates $(kL_x, \ell L_y, \pm L_z/2)$. The edge joining the vertices $v_{k,\ell}$ and $v_{k+1,\ell}$ is denoted by $e_{k+\frac{1}{2},\ell}$, the one joining the vertices $v_{k,\ell}$ and $v_{k,\ell+1}$ is denoted by $e_{k,\ell+\frac{1}{2}}$ and the edges joining the vertex $v_{k,\ell}$ with the vertices $v^\pm_{k,\ell}$ are denoted by $e^\pm_{k,\ell}$. Let u be a function on G. Then, its value at the vertex $v_{k,\ell}$ is denoted by $\mathbf{u}_{k,\ell}$ and its values at the vertices $v^\pm_{k,\ell}$ by $\mathbf{u}^\pm_{k,\ell}$. The restrictions of the function u at the edge $e_{k+\frac{1}{2},\ell}$ is denoted $u_{k+\frac{1}{2},\ell}(s)$, where the local variable s takes values in $[0, L_x]$. Similarly, its restriction at the edge $e_{k,\ell+\frac{1}{2}}$ is denoted by $u_{k,\ell+\frac{1}{2}}(t)$, where the local variable t takes values in $[0, L_y]$. Finally, the restrictions of u at the edges $e^\pm_{k,\ell}$ are denoted by $u^\pm_{k,\ell}(z)$.

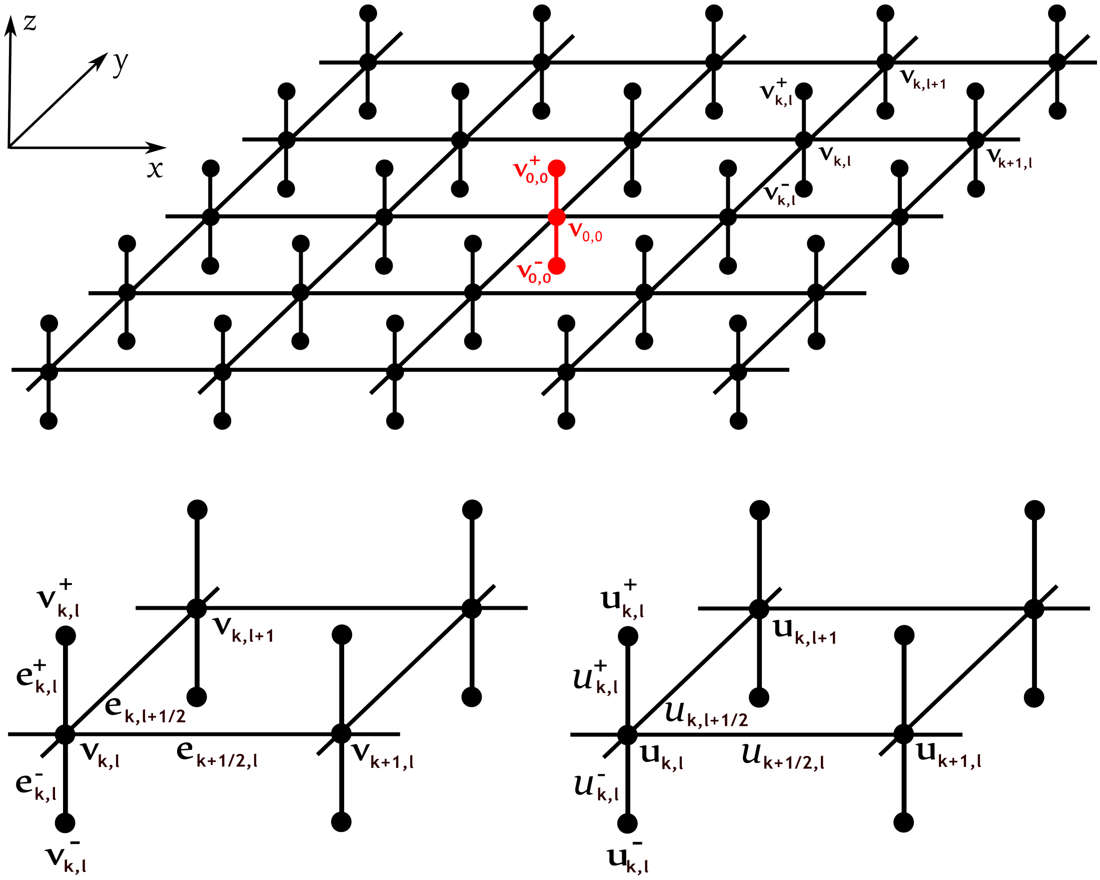


Figure 5.2: Graph G

The following function spaces are introduced analogously to the 2D case:

$$L_2^\mu(G) = \left\{ u : \|u\|^2_{L_2^\mu(G)} < \infty \right\}, \qquad H^2(G) = \left\{ u \in C(G) : \ \|u\|^2_{H^2(G)} < \infty \right\},$$

$$\|u\|^2_{L_2^\mu(G)} = \sum_{k,\ell\in\mathbb{Z}} \left(w^\mu_{k,\ell} \left(\|u^+_{k,\ell}\|^2_{L_2(e^+_{k,\ell})} + \|u^-_{k,\ell}\|^2_{L_2(e^-_{k,\ell})} \right) + \|u_{k+\frac12,\ell}\|^2_{L_2(e_{k+\frac12,\ell})} + \|u_{k,\ell+\frac12}\|^2_{L_2(e_{k,\ell+\frac12})} \right),$$

$$\|u\|^2_{H^2(G)} = \sum_{k,\ell\in\mathbb{Z}} \left(\|u^+_{k,\ell}\|^2_{H^2(e^+_{k,\ell})} + \|u^-_{k,\ell}\|^2_{H^2(e^-_{k,\ell})} + \|u_{k+\frac12,\ell}\|^2_{H^2(e_{k+\frac12,\ell})} + \|u_{k,\ell+\frac12}\|^2_{H^2(e_{k,\ell+\frac12})} \right).$$

Here w^μ is the weight defined as

$$w^\mu_{k,\ell} = \begin{cases} \mu, & k = \ell = 0, \\ 1, & \text{otherwise.} \end{cases}$$

The limit operator $A^\mu(\beta) : L_2(G) \to L_2(G)$ is then defined as follows:

$$A^\mu(\beta) u = -u'',$$

$$D\left(A^\mu(\beta)\right) = \Big\{ u \in H^2(G) : \ \forall k, \ell \in \mathbb{Z}, \ \ \mathrm{u}^+_{k,\ell} = e^{-i\beta} \mathrm{u}^-_{k,\ell}, \ \ (u^+_{k,\ell})'\left(\tfrac{L_z}{2}\right) = e^{-i\beta} (u^-_{k,\ell})'\left(-\tfrac{L_z}{2}\right), \tag{5.2.1}$$

$$u'_{k+\frac12,\ell}(0) - u'_{k-\frac12,\ell}(L_x) + u'_{k,\ell+\frac12}(0) - u'_{k,\ell-\frac12}(L_y) + w^\mu_{k,\ell}\left((u^+_{k,\ell})'(0) - (u^-_{k,\ell})'(0)\right) = 0 \Big\}. \tag{5.2.2}$$

The relations (5.2.2) are Kirchhoff's conditions. As in the 2D case, they mean that the weighted sum of outgoing derivatives is zero at each vertex of the graph G. As usually, we will start by studying the non-perturbed operator $A(\beta)$ that corresponds to the case $\mu = 1$.

Proposition 5.2.1.

$$\sigma_{ess}\left(A^{\mu}(\beta)\right) = \sigma(A(\beta)).$$

The proof of this proposition is an obvious modification of the one for the 2D case if the 1D characteristic function is replaced by a 2D one (cf. Proposition 1.2.1). The periodic operator $A(\beta)$ has only essential spectrum and we compute it in the next section.

5.2.1 Computation of the essential spectrum

According to Floquet-Bloch theory in order to study the spectrum of the operator $A(\beta)$ we have to study the operators $A_\beta(k_x, k_y)$ defined on the periodicity cell

$$\mathcal{C} = G \cup \left\{(x,y) \in \left[-\tfrac{L_x}{2}, \tfrac{L_x}{2}\right] \times \left[-\tfrac{L_y}{2}, \tfrac{L_y}{2}\right]\right\},$$

shown in figure 5.3, with (k_x, k_y)-quasiperiodicity conditions:

$$\begin{aligned}
&A_\beta(k_x,k_y) : L_2(\mathcal{C}) \to L_2(\mathcal{C}), \qquad A_\beta(k_x,k_y)u = -u'', \\
&D\left(A_\beta(k_x,k_y)\right) = \Big\{u \in H^2(\mathcal{C}) : \quad u_6\left(\tfrac{L_z}{2}\right) = e^{-i\beta} u_6\left(-\tfrac{L_z}{2}\right), \qquad u_6'\left(\tfrac{L_z}{2}\right) = e^{-i\beta} u_5'\left(-\tfrac{L_z}{2}\right), \\
&\qquad u_2\left(\tfrac{L_x}{2}\right) = e^{-ik_x} u_1\left(-\tfrac{L_x}{2}\right), \quad u_2'\left(\tfrac{L_x}{2}\right) = e^{-ik_x} u_1'\left(-\tfrac{L_x}{2}\right), \\
&\qquad u_4\left(\tfrac{L_y}{2}\right) = e^{-ik_y} u_3\left(-\tfrac{L_y}{2}\right), \quad u_4'\left(\tfrac{L_y}{2}\right) = e^{-ik_y} u_3'\left(-\tfrac{L_y}{2}\right), \\
&\qquad u_2'(0) - u_1'(0) + u_4'(0) - u_3'(0) + u_6'(0) - u_5'(0) = 0\Big\}. \qquad (5.2.3)
\end{aligned}$$

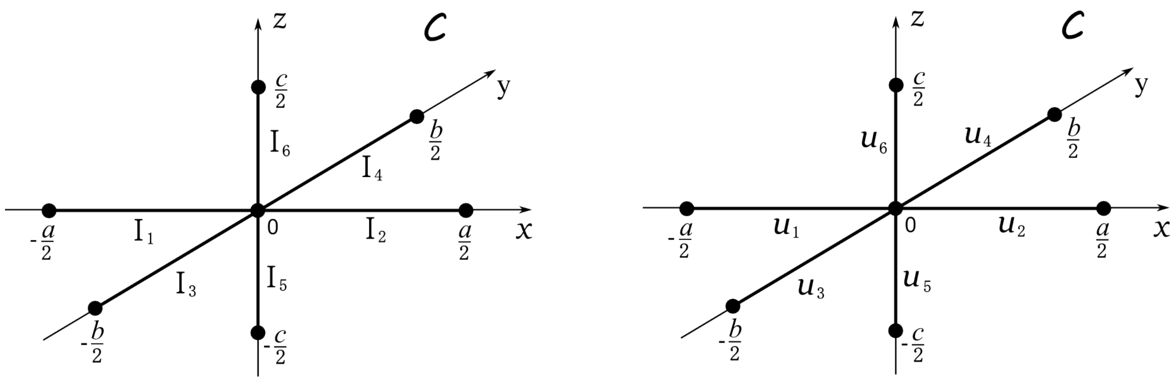


Figure 5.3: Periodicity cell $\mathcal{C}$

The function spaces $L_2(\mathcal{C})$, $H^2(\mathcal{C})$ are defined in a standard way, i.e.

$$L_2(\mathcal{C}) = \left\{u : \sum_{i=1}^{6} \|u_i\|^2_{L^2(I_i)} < \infty\right\}, \qquad H^2(\mathcal{C}) = \left\{u \in C(\mathcal{C}) : \sum_{i=1}^{6} \|u_i\|^2_{H^2(I_i)} < \infty\right\}.$$

Proposition 5.2.2. *For* $(k_x, k_y) \in [0,\pi]^2$, $\lambda^2 \in \sigma(A_\beta(k_x,k_y)) \setminus \{0\}$ *if and only if* λ *is a solution of the equation*

$$\sin(\lambda L_y)\sin(\lambda L_z)(\cos(\lambda L_x) - \cos k_x) + \sin(\lambda L_z)\sin(\lambda L_x)(\cos(\lambda L_y) - \cos k_y) \\ + \sin(\lambda L_x)\sin(\lambda L_y)(\cos(\lambda L_z) - \cos\beta) = 0. \quad (5.2.4)$$

The point $\lambda = 0$ *belongs to* $\sigma(A_\beta(k_x,k_y))$ *if and only if* $k_x = k_y = \beta = 0$.

Remark 5.2.1. One can notice that the equation (5.2.4) has a similar structure as the one found in the 2D case (cf. Proposition 4.1.2).

Proof. Let us start with the case $\lambda \neq 0$. If λ^2 is an eigenvalue of the operator $A_\beta(k_x,k_y)$, then the corresponding eigenfunction $u \in D(A_\beta(k_x,k_y))$ solves the equation

$$u'' + \lambda^2 u = 0$$

on each edge of the periodicity cell $\mathcal{C}$. We have then

$$\begin{aligned}
u_1(x) &= c_1 e^{i\lambda x} + d_1 e^{-i\lambda x}, && x \in \left[-\tfrac{L_x}{2}, 0\right],\\
u_2(x) &= c_2 e^{i\lambda x} + d_2 e^{-i\lambda x}, && x \in \left[0, \tfrac{L_x}{2}\right],\\
u_3(y) &= c_3 e^{i\lambda y} + d_3 e^{-i\lambda y}, && y \in \left[-\tfrac{L_y}{2}, 0\right],\\
u_4(y) &= c_4 e^{i\lambda y} + d_4 e^{-i\lambda y}, && y \in \left[0, \tfrac{L_y}{2}\right],\\
u_5(z) &= c_5 e^{i\lambda z} + d_5 e^{-i\lambda z}, && z \in \left[-\tfrac{L_z}{2}, 0\right],\\
u_6(z) &= c_6 e^{i\lambda z} + d_6 e^{-i\lambda z}, && z \in \left[0, \tfrac{L_z}{2}\right].
\end{aligned}$$

From the continuity of the eigenfunction at the central vertex we get

$$c_1 + d_1 = c_2 + d_2 = c_3 + d_3 = c_4 + d_4 = c_5 + d_5 = c_6 + d_6. \quad (5.2.5)$$

The quasiperiodic conditions imply that

$$\left(c_1 e^{-\frac{i\lambda L_x}{2}} + d_1 e^{\frac{i\lambda L_x}{2}}\right) e^{-ik_x} = c_2 e^{\frac{i\lambda L_x}{2}} + d_2 e^{-\frac{i\lambda L_x}{2}}, \quad (5.2.6)$$

$$\left(c_1 e^{-\frac{i\lambda L_x}{2}} - d_1 e^{\frac{i\lambda L_x}{2}}\right) e^{-ik_x} = c_2 e^{\frac{i\lambda L_x}{2}} - d_2 e^{-\frac{i\lambda L_x}{2}}, \quad (5.2.7)$$

$$\left(c_3 e^{-\frac{i\lambda L_y}{2}} + d_3 e^{\frac{i\lambda L_y}{2}}\right) e^{-ik_y} = c_4 e^{\frac{i\lambda L_y}{2}} + d_4 e^{-\frac{i\lambda L_y}{2}}, \quad (5.2.8)$$

$$\left(c_3 e^{-\frac{i\lambda L_y}{2}} - d_3 e^{\frac{i\lambda L_y}{2}}\right) e^{-ik_y} = c_4 e^{\frac{i\lambda L_y}{2}} - d_4 e^{-\frac{i\lambda L_y}{2}}, \quad (5.2.9)$$

$$\left(c_5 e^{-\frac{i\lambda L_z}{2}} + d_5 e^{\frac{i\lambda L_z}{2}}\right) e^{-i\beta} = c_6 e^{\frac{i\lambda L_z}{2}} + d_6 e^{-\frac{i\lambda L_z}{2}}, \quad (5.2.10)$$

$$\left(c_5 e^{-\frac{i\lambda L_z}{2}} - d_5 e^{\frac{i\lambda L_z}{2}}\right) e^{-i\beta} = c_6 e^{\frac{i\lambda L_z}{2}} - d_6 e^{-\frac{i\lambda L_z}{2}}. \quad (5.2.11)$$

Finally, Kirchhoff's condition yields

$$-c_1 + d_1 + c_2 - d_2 - c_3 + d_3 + c_4 - d_4 - c_5 + d_5 + c_6 - d_6 = 0. \quad (5.2.12)$$

From the equations (5.2.6)–(5.2.11) one finds:

$$c_1 = e^{i\lambda L_x + ik_x} c_2, \qquad c_3 = e^{i\lambda L_y + ik_y} c_4, \qquad c_5 = e^{i\lambda L_z + i\beta} c_6, \tag{5.2.13}$$

$$d_1 = e^{-i\lambda L_x + ik_x} d_2, \qquad d_3 = e^{-i\lambda L_y + ik_y} d_4, \qquad d_5 = e^{-i\lambda L_z + i\beta} d_6. \tag{5.2.14}$$

Due to the relations (5.2.13), (5.2.14) the system (5.2.5)–(5.2.12) reduces to the following one:

$$c_2 + d_2 = c_4 + d_4 = c_6 + d_6, \tag{5.2.15}$$

$$c_2\left(1 - e^{i\lambda L_x + ik_x}\right) + d_2\left(1 - e^{-i\lambda L_x + ik_x}\right) = 0, \tag{5.2.16}$$

$$c_4\left(1 - e^{i\lambda L_y + ik_y}\right) + d_4\left(1 - e^{-i\lambda L_y + ik_y}\right) = 0, \tag{5.2.17}$$

$$c_6\left(1 - e^{i\lambda L_z + i\beta}\right) + d_6\left(1 - e^{-i\lambda L_z + i\beta}\right) = 0, \tag{5.2.18}$$

$$c_2\left(1 - e^{i\lambda L_x + ik_x}\right) + c_4\left(1 - e^{i\lambda L_y + ik_y}\right) + c_6\left(1 - e^{i\lambda L_z + i\beta}\right) = 0. \tag{5.2.19}$$

Thus, the system (5.2.15)–(5.2.19) has a non-trivial solution if and only if

$$D(\lambda) = 0, \tag{5.2.20}$$

where

$$D(\lambda) = \begin{vmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 1 - e^{i(\lambda L_x + k_x)} & 1 - e^{i(k_x - \lambda L_x)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 - e^{i(\lambda L_y + k_y)} & 1 - e^{i(k_y - \lambda L_y)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 - e^{i(\lambda L_z + \beta)} & 1 - e^{i(\beta - \lambda L_z)} \\ 1 - e^{i(\lambda L_x + k_x)} & 0 & 1 - e^{i(\lambda L_y + k_y)} & 0 & 1 - e^{i(\lambda L_z + \beta)} & 0 \end{vmatrix}. \tag{5.2.21}$$

Computing $D(\lambda)$ leads to the condition (5.2.4) (see Lemma A.4.1 in Annexe for the details of the computation).

If $\lambda = 0$, then the eigenfunction has to be linear at each edge of the graph:

$$\begin{aligned} u_1(x) &= c_1 + d_1 x, & x &\in \left[-\tfrac{L_x}{2}, 0\right], \\ u_2(x) &= c_2 + d_2 x, & x &\in \left[0, \tfrac{L_x}{2}\right], \\ u_3(y) &= c_3 + d_3 y, & y &\in \left[-\tfrac{L_y}{2}, 0\right], \\ u_4(y) &= c_4 + d_4 y, & y &\in \left[0, \tfrac{L_y}{2}\right], \\ u_5(z) &= c_5 + d_5 z, & z &\in \left[-\tfrac{L_z}{2}, 0\right], \\ u_6(z) &= c_6 + d_6 z, & z &\in \left[0, \tfrac{L_z}{2}\right]. \end{aligned}$$

The continuity condition implies that $c_i = c_0$, $1 \leqslant i \leqslant 6$. From the quasiperiodic conditions it follows that

$$d_2 = e^{-ik_x} d_1, \qquad d_1 = \frac{c_0}{L_x}\left(1 - e^{ik_x}\right), \tag{5.2.22}$$

$$d_4 = e^{-ik_y} d_3, \qquad d_3 = \frac{c_0}{L_y}\left(1 - e^{ik_y}\right), \tag{5.2.23}$$

$$d_6 = e^{-ik_y} d_5, \qquad d_5 = \frac{c_0}{L_z}\left(1 - e^{i\beta}\right). \tag{5.2.24}$$

Finally, Kirchhoff's condition (5.2.3) gives

$$-d_1 + d_2 - d_3 + d_4 - d_5 + d_6 = 0. \tag{5.2.25}$$

After injecting (5.2.22)–(5.2.24) in (5.2.25) we get

$$c_0 \left(\frac{\cos k_x - 1}{L_x} + \frac{\cos k_y - 1}{L_y} + \frac{\cos \beta - 1}{L_z} \right) = 0.$$

Hence, there exists a non-trivial solution if and only if $k_x = k_y = \beta = 0$ (since otherwise the quantity inside the brackets is strictly negative). This finishes the proof. □

Remark 5.2.2. Similarly to the 2D case, one can notice that if L_x, L_y and L_z are commensurable, then the set $\{\lambda : \lambda^2 \in \sigma(A_\beta(k_x, k_y))\}$ is periodic. Indeed, in this case the left-hand side of the equation (5.2.4) is a periodic function.

The spectrum of the operator $A(\beta)$ can now be determined due to the decomposition

$$\sigma(A(\beta)) = \bigcup_{k_x, k_y \in [0,\pi]} \sigma(A_\beta(k_x, k_y)). \tag{5.2.26}$$

Thus, the point λ^2 (different from zero) belongs to the spectrum of the operator $A(\beta)$ if and only if there exists a couple (k_x, k_y) such that the relation (5.2.4) is satisfied. Let us introduce some notation that we will use throughout the chapter:

$$\begin{aligned}
\Sigma_x &= \{\lambda \geqslant 0 : \sin(\lambda L_x) = 0\}, & \Sigma_x^* &= \Sigma_x \setminus \{0\}, \\
\Sigma_y &= \{\lambda \geqslant 0 : \sin(\lambda L_y) = 0\}, & \Sigma_y^* &= \Sigma_y \setminus \{0\}, \\
\Sigma_z &= \{\lambda \geqslant 0 : \sin(\lambda L_z) = 0\}, & \Sigma_z^* &= \Sigma_z \setminus \{0\},
\end{aligned}$$

$$\Sigma = \Sigma_x \cup \Sigma_y \cup \Sigma_z, \qquad \Sigma^* = \Sigma \setminus \{0\},$$

$$\Sigma_z(\beta) = \{\lambda \geqslant 0 : \cos(\lambda L_z) = \cos\beta\}, \qquad \Sigma_z^*(\beta) = \Sigma_z(\beta) \setminus \{0\},$$

$$\widetilde{\Sigma}_z(\beta) = \begin{cases} \Sigma_z = \{\lambda \geqslant 0 : \sin(\lambda L_z) = 0\}, & \beta \in]0,\pi[, \\ \Sigma_z \setminus \Sigma_z(\beta) = \{\lambda \geqslant 0 : \cos(\lambda L_z) = -\cos\beta\}, & \beta \in \{0,\pi\}, \end{cases} \tag{5.2.27}$$

$$\widetilde{\Sigma}_z^*(\beta) = \widetilde{\Sigma}_z(\beta) \setminus \{0\}. \tag{5.2.28}$$

Proposition 5.2.3.

1. *$\left\{\lambda^2 : \lambda \in \Sigma_x^* \cup \Sigma_y^* \cup \Sigma_z(\beta)\right\} \subset \sigma(A(\beta))$.*
2. *For any $\beta \in [0,\pi]$, the operator $A(\beta)$ has infinitely many gaps whose ends tend to infinity.*

Proof.

1. For $\lambda \neq 0$, $\lambda \in \Sigma_x^* \cup \Sigma_y^* \cup \Sigma_z(\beta)$ the equation (5.2.4) is obviously verified for $\cos k_x = \cos(\lambda L_x)$, $\cos k_y = \cos(\lambda L_y)$. The point $\lambda = 0$ belongs to the set $\Sigma_x^* \cup \Sigma_y^* \cup \Sigma_z(\beta)$ if and only if $\beta = 0$. At the same time, according to Proposition 5.2.2, $0 \in \sigma(A(\beta))$ if and only if $\beta = 0$.

2. We will prove the existence of a gap for the operator $A(\beta)$ in a (deleted) neighbourhood of each point of the set $\left\{\lambda^2,\ \lambda \in \widetilde{\Sigma}_z^*(\beta)\right\}$. More precisely, we will show that for any $\lambda \in \widetilde{\Sigma}_z^*(\beta)$ there exist positive numbers l^+ and l^- such that $]\lambda^2 - l^-, \lambda^2[\cap \sigma(A(\beta)) = \emptyset$ and $]\lambda^2, \lambda^2 + l^+[\cap \sigma(A(\beta)) = \emptyset$.

   Let $\lambda_0 \in \widetilde{\Sigma}_z^*(\beta)$. Then,
$$\frac{\cos\beta}{\cos(\lambda_0 L_z)} < 1. \tag{5.2.29}$$
   Let us rewrite the relation (5.2.4) in a neighbourhood of λ_0 by putting $\lambda = \lambda_0 + \delta$:
$$\begin{aligned}\sin(\delta L_z)\left(\sin(\lambda L_x + \lambda L_y) - \cos k_x \sin(\lambda L_y) - \cos k_y \sin(\lambda L_x)\right)\\ = \sin(\lambda L_x)\sin(\lambda L_y)\left(\frac{\cos\beta}{\cos(\lambda_0 L_z)} - \cos(\delta L_z)\right).\end{aligned} \tag{5.2.30}$$

   (i) $\lambda_0 \notin \Sigma_x \cup \Sigma_y$: the equation (5.2.30) has no solution for δ small enough. Indeed, due to the inequality (5.2.29), its right-hand side tends to a non-zero limit as $\delta \to 0$, whereas its left-hand side tends to zero uniformly in $(k_x, k_y) \in [0,\pi]^2$. This proves the existence of a gap of the operator $A(\beta)$ containing the point λ_0^2.

   (ii) $\lambda_0 \in \Sigma_x^* \setminus \Sigma_y^*$ or $\lambda_0 \in \Sigma_y^* \setminus \Sigma_x^*$: we will consider the case $\lambda_0 \in \Sigma_x^* \setminus \Sigma_y^*$, the case $\lambda_0 \in \Sigma_y^* \setminus \Sigma_x^*$ can be considered in a similar way. For $\delta \neq 0$ small enough the relation (5.2.30) can be rewritten as
$$\begin{aligned}\sin(\delta L_x)\frac{\cos(\lambda L_y) - \cos k_y}{\sin(\lambda L_y)} + \cos(\delta L_x) - \frac{\cos k_x}{\cos(\lambda_0 L_x)}\\ = \frac{\sin(\delta L_x)}{\sin(\delta L_z)}\left(\frac{\cos\beta}{\cos(\lambda_0 L_z)} - \cos(\delta L_z)\right).\end{aligned} \tag{5.2.31}$$
   Taking into account that $|\cos(\lambda_0 L_x)| = 1$, when δ is small enough, the left-hand side of this relation can be bounded from below uniformly in $(k_x, k_y) \in [0,\pi]^2$ by a continuous function in δ that tends to zero as $\delta \to 0$:
$$\begin{aligned}\sin(\delta L_x)\frac{\cos(\lambda L_y) - \cos k_y}{\sin(\lambda L_y)} + \cos(\delta L_x) - \frac{\cos k_x}{\cos(\lambda_0 L_x)}\\ \geqslant \cos(\delta L_x) - 1 - \sin(\delta L_x)\frac{|\cos(\lambda L_y)| + 1}{|\sin(\lambda L_y)|} \xrightarrow[\delta\to 0]{} 0, \qquad \forall (k_x, k_y) \in [0,\pi]^2.\end{aligned}$$
   The limit of the right-hand side of (5.2.31) when $\delta \to 0$ is
$$\frac{\sin(\delta L_x)}{\sin(\delta L_z)}\left(\frac{\cos\beta}{\cos(\lambda_0 L_z)} - \cos(\delta L_z)\right) \xrightarrow[\delta\to 0]{} \frac{L_x}{L_z}\left(\frac{\cos\beta}{\cos(\lambda_0 L_z)} - 1\right) < 0,$$
   which is strictly negative due to the inequality (5.2.29). Hence, the equation (5.2.31) has no solution for $\delta \neq 0$ small enough.

   (iii) $\lambda_0 \in \Sigma_x^* \cap \Sigma_y^*$: the relation (5.2.30) can be rewritten as
$$\begin{aligned}\frac{\sin(\delta L_z)}{\sin(\delta L_y)}\left(\cos(\delta L_y) - \frac{\cos k_y}{\cos(\lambda_0 L_y)}\right) + \frac{\sin(\delta L_z)}{\sin(\delta L_x)}\left(\cos(\delta L_x) - \frac{\cos k_x}{\cos(\lambda_0 L_x)}\right)\\ = \frac{\cos\beta}{\cos(\lambda_0 L_z)} - \cos(\delta L_z).\end{aligned} \tag{5.2.32}$$

Again, since $|\cos(\lambda_0 L_x)| = |\cos(\lambda_0 L_y)| = 1$, when δ is small enough, the left-hand side of this relation can be bounded from below uniformly in $(k_x, k_y) \in [0,\pi]^2$ by a continuous function in δ that tends to zero as $\delta \to 0$:

$$\frac{\sin(\delta L_z)}{\sin(\delta L_y)}\left(\cos(\delta L_y) - \frac{\cos k_y}{\cos(\lambda_0 L_y)}\right) + \frac{\sin(\delta L_z)}{\sin(\delta L_x)}\left(\cos(\delta L_x) - \frac{\cos k_x}{\cos(\lambda_0 L_x)}\right)$$
$$\geqslant \frac{L_z}{L_y}(\cos(\delta L_y) - 1) + \frac{L_z}{L_x}(\cos(\delta L_x) - 1) \xrightarrow[\delta\to 0]{} 0, \qquad \forall (k_x, k_y) \in [0,\pi]^2.$$

The limit of the right-hand side of (5.2.32) as $\delta \to 0$ is $\cos\beta / \cos(\lambda_0 L_z) - 1 < 0$. Consequently, the equation (5.2.32) has no solution for δ small enough.

□

The following assertion gives a description of the set of eigenvalues of infinite multiplicity of the operator $A(\beta)$. We give its proof in Appendix.

Proposition 5.2.4. *The operator $A(\beta)$ has the following set of eigenvalues of infinite multiplicity:*

$$\sigma_{pp}(A(\beta)) = \left\{\lambda^2 : \ \lambda \in \Sigma_{is}(\beta) \cup \Sigma_{emb}(\beta)\right\},$$

where

$$\Sigma_{is}(\beta) = \left(\Sigma_x^* \cup \Sigma_y^*\right) \cap \widetilde{\Sigma}_z^*(\beta), \tag{5.2.33}$$

$$\Sigma_{emb}(\beta) = \begin{cases} \left(\Sigma_x^* \cap \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta), & \beta \in]0,\pi[, \\ \left(\left(\Sigma_x^* \cap \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta)\right) \cup \Sigma_z^*(\beta), & \beta \in \{0,\pi\}. \end{cases}$$

The eigenvalues of the set $\{\lambda^2 : \lambda \in \Sigma_{is}(\beta)\}$ are isolated points of the spectrum whereas the eigenvalues of the set $\{\lambda^2 : \lambda \in \Sigma_{emb}(\beta)\}$ are not isolated points of the spectrum.

The following proposition gives a criterion for λ^2 to be a point of the spectrum of the operator $A(\beta)$.

Proposition 5.2.5. *$\lambda^2 \in \sigma(A(\beta))$ if and only if one of the following possibilities holds:*

(i) $\lambda \in \Sigma_x^* \cup \Sigma_y^*$;

(ii) $\lambda \in \mathbb{R}_+ \setminus \left(\Sigma_x^* \cup \Sigma_y^*\right)$ *is a solution of the equation*

$$\phi_{L_z,\beta}(\lambda) = f_{L_x,L_y,k_x,k_y}(\lambda). \tag{5.2.34}$$

for some $(k_x, k_y) \in [0,\pi]^2$. Here

$$f_{L_x,L_y,k_x,k_y}(\lambda) = \begin{cases} f_{L_x,k_x}(\lambda) + f_{L_y,k_y}(\lambda), & \lambda \in \mathbb{R}_+^* \setminus \left(\Sigma_x^* \cup \Sigma_y^*\right), \\ 0, & (k_x, k_y, \lambda) = (0,0,0), \end{cases} \tag{5.2.35}$$

$$f_{L,k}(\lambda) = \frac{\cos k - \cos(\lambda L)}{\sin(\lambda L)}, \qquad \sin(\lambda L) \neq 0, \tag{5.2.36}$$

$$\phi_{L_z,\beta}(\lambda) = \begin{cases} \dfrac{\cos(\lambda L_z) - \cos\beta}{\sin(\lambda L_z)}, & \lambda \in \mathbb{R}^+ \setminus \Sigma_z, \\ 0, & \lambda \in \Sigma_z(\beta). \end{cases} \tag{5.2.37}$$

Proof. This is a consequence of the decomposition (5.2.26), Proposition 5.2.2 and property 1 of Proposition 5.2.3. Indeed, the equation (5.2.34) is equivalent to the equation (5.2.4) for $\lambda \notin \Sigma$. The case $\lambda \in \Sigma_x^* \cup \Sigma_y^*$ is treated directly: the corresponding points belong to the spectrum of the operator $A(\beta)$ for any $\beta \in [0,\pi]$. If $\lambda \in \Sigma_z \setminus \left(\Sigma_x^* \cup \Sigma_y^*\right)$, then $\lambda^2 \in \sigma(A(\beta))$ if and only if $\lambda \in \Sigma_z(\beta)$ (and hence, $\beta \in \{0,\pi\}$). Notice that this is also valid for $\lambda = 0$. As well, if $\lambda \in \Sigma_z$, then the function $\phi_{L_z,\beta}$ is defined at λ if and only if $\lambda \in \Sigma_z(\beta)$, and its value is 0. In this case, the equation (5.2.34) is satisfied for $\cos k_x = \cos(\lambda L_x)$, $\cos k_y = \cos(\lambda L_y)$. □

Remark 5.2.3. Proposition 5.2.5 is an analogue of Proposition 4.1.5 obtained in the 2D case. However, we use here the inverses of the functions $\phi_{L,\beta}$, f_θ used in the 2D case. This choice is done in order to have f_{L_x,L_y,k_x,k_y} given by a sum $f_{L_x,L_y,k_x,k_y} = f_{L_x,k_x} + f_{L_y,k_y}$ and not by a some of inverses of functions. Indeed, this form will be easier to represent geometrically.

As in the 2D case two types of gaps can be distinguished for the operator $A(\beta)$:

Proposition 5.2.6. *An interval* $]a,b[$ *is a gap of the operator* $A(\beta)$ *for* $\beta \in [0,\pi]$ *if and only if* $0 \leqslant a < b$, $[\sqrt{a},\sqrt{b}] \cap \Sigma_z(\beta) = \emptyset$ *and one of the following possibilities holds:*

I $\{\sqrt{a},\sqrt{b}\} \cap \left(\Sigma_x \cup \Sigma_y \cup \widetilde{\Sigma}_z(\beta)\right) = \emptyset$, $\phi_{L_z,\beta}(\sqrt{a}) = f^-_{L_x,L_y}(\sqrt{a})$, $\phi_{L_z,\beta}(\sqrt{b}) = f^+_{L_x,L_y}(\sqrt{b})$;

II *(i)* $\sqrt{a} \in \left(\Sigma_x^* \cup \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta)$, $\phi_{L_z,\beta}(\sqrt{a}) > f^+_{L_x,L_y}(\sqrt{a})$,
$\sqrt{b} \notin \Sigma_x \cup \Sigma_y \cup \widetilde{\Sigma}_z(\beta)$, $\phi_{L_z,\beta}(\sqrt{b}) = f^+_{L_x,L_y}(\sqrt{b})$;

(i') $\sqrt{a} \in (\Sigma_x \cup \Sigma_y) \cap \widetilde{\Sigma}_z(\beta)$, $\sqrt{b} \notin \Sigma_x \cup \Sigma_y \cup \widetilde{\Sigma}_z(\beta)$, $\phi_{L_z,\beta}(\sqrt{b}) = f^+_{L_x,L_y}(\sqrt{b})$;

(ii) $\sqrt{a} \notin \Sigma_x \cup \Sigma_y \cup \widetilde{\Sigma}_z(\beta)$, $\phi_{L_z,\beta}(\sqrt{a}) = f^-_{L_x,L_y}(\sqrt{a})$,
$\sqrt{b} \in \left(\Sigma_x^* \cup \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta)$, $\phi_{L_z,\beta}(\sqrt{b}) < f^-_{L_x,L_y}(\sqrt{b})$;

(ii') $\sqrt{a} \notin \Sigma_x \cup \Sigma_y \cup \widetilde{\Sigma}_z(\beta)$, $\phi_{L_z,\beta}(\sqrt{a}) = f^-_{L_x,L_y}(\sqrt{a})$, $\sqrt{b} \in \left(\Sigma_x^* \cup \Sigma_y^*\right) \cap \widetilde{\Sigma}_z^*(\beta)$.

Remark 5.2.4. For $\beta \in]0,\pi]$, the bottom of the spectrum of the operator $\sigma_* = \min \sigma(A(\beta))$ is strictly positive. As in the 2D case (cf. Remark 4.1.3) we will call the interval $]0,\sigma_*[$ a gap even if it is not quite correct. As follows from Proposition 5.2.6, it is a gap of type II (it satisfies the condition $II\,(i')$).

The rest of this section is devoted to the proof of this Proposition 5.2.6. We start with a geometric interpretation of Proposition 5.2.5 which is similar to the one given in the 2D case. Now we have to consider the abscissas of the intersections of the graph of the function $\phi_{L_z,\beta}$ with the ones of the functions f_{L_x,L_y,k_x,k_y}. Let us introduce the union of the graphs of the functions f_{L_x,L_y,k_x,k_y} for $(k_x,k_y) \in [0,\pi]^2$:

$$D_{L_x,L_y} = \bigcup_{(k_x,k_y)\in[0,\pi]^2} \left\{\left(\lambda, f_{L_x,L_y,k_x,k_y}(\lambda)\right) /\ \lambda \in \mathcal{D}(f_{L_x,L_y,k_x,k_y})\right\},$$

where $\mathcal{D}(f)$ stands for the domain of the function f.

Lemma 5.2.1.

$$D_{L_x,L_y} = \left\{(\lambda,p) /\ \lambda \in \mathbb{R}_+^* \setminus \left(\Sigma_x^* \cup \Sigma_y^*\right),\ p \in \left[f^-_{L_x,L_y}(\lambda),\ f^+_{L_x,L_y}(\lambda)\right]\right\} \cup \{(0,0)\},$$

where

$$f^+_{L_x,L_y}(\lambda) = f^+(\lambda L_x) + f^+(\lambda L_y), \qquad f^-_{L_x,L_y}(\lambda) = f^-(\lambda L_x) + f^-(\lambda L_y), \tag{5.2.38}$$

,

$$f^+(\lambda) = \tan\left(\tfrac{\lambda-\pi n}{2}\right), \qquad \lambda \in [\pi n, \pi(n+1)[, \qquad n \in \mathbb{N}, \tag{5.2.39}$$

$$f^-(\lambda) = -\text{cotan}\left(\tfrac{\lambda-\pi n}{2}\right), \qquad \lambda \in]\pi n, \pi(n+1)], \qquad n \in \mathbb{N}. \tag{5.2.40}$$

Proof. If $\sin(\lambda L) \neq 0$, then the function $f_{L,k}$ defined in (5.2.36) is continuous in $k_x \in [0,\pi]$ and

$$\bigcup_{k_x\in[0,\pi]} f_{L,k}(\lambda) = \left[f^-(\lambda L),\, f^+(\lambda L)\right].$$

Hence, for $\lambda \notin \Sigma_x \cup \Sigma_y$ one has

$$\bigcup_{(k_x,k_y)\in[0,\pi]^2} f_{L_x,L_y,k_x,k_y}(\lambda) = \bigcup_{(k_x,k_y)\in[0,\pi]^2} \left(f_{L_x,k_x}(\lambda) + f_{L_y,k_y}(\lambda)\right) = \left[f^-_{L_x,L_y}(\lambda),\, f^+_{L_x,L_y}(\lambda)\right].$$

Finally, $0 \in \mathcal{D}(f_{L_x,L_y,k_x,k_y})$ if and only if $(k_x,k_y) = (0,0)$. In this case, $f_{L_x,L_y,0,0}(0) = 0$. □

The set D_{L_x,L_y} is shown in figure 5.4. Let us mention that the functions $f^{\pm}_{L_x,L_y}$ are continuous in $\mathbb{R}^*_+ \setminus (\Sigma_x \cup \Sigma_y)$ and strictly increasing in each interval of continuity (between two neighbour points of the set $\Sigma_x \cup \Sigma_y$). Moreover, the function $f^+_{L_x,L_y}$ is right-continuous and the function $f^-_{L_x,L_y}$ is left-continuous and

$$f^-_{L_x,L_y}(\lambda) \geqslant 0, \quad \forall \lambda \in \mathbb{R}_+, \qquad f^+_{L_x,L_y}(\lambda) > 0, \quad \forall \lambda \in \mathbb{R}^*_+ \setminus \left(\Sigma^*_x \cup \Sigma^*_y\right), \tag{5.2.41}$$

$$f^-_{L_x,L_y}(\lambda) \leqslant 0, \quad \forall \lambda \in \mathbb{R}^*_+, \qquad f^+_{L_x,L_y}(\lambda) < 0, \quad \forall \lambda \in \mathbb{R}^*_+ \setminus \left(\Sigma^*_x \cup \Sigma^*_y\right). \tag{5.2.42}$$

In order to get a geometric interpretation of the spectrum of the operator $A(\beta)$, we need to consider the abscissas of the intersections of the set D_{L_x,L_y} with the graph of the function $\phi_{L_z,\beta}$ (cf. figure 5.5). This will correspond to the points described in (*ii*) of Proposition 5.2.5. In order to get all the points λ such that $\lambda^2 \in \sigma(A(\beta))$, we have to include the set $\Sigma^*_x \cup \Sigma^*_y$. Thus, the spectrum of the operator $A(\beta)$ can be characterized as follows:

$$\sigma(A(\beta)) = \left\{\lambda^2 /\ \lambda \geqslant 0,\ \phi_{L_z,\beta}(\lambda) \in D_{L_x,L_y}\right\} \cup \left\{\lambda^2 /\ \lambda \in \Sigma^*_x \cup \Sigma^*_y\right\}.$$

This permits to give the following description of the gaps of the operator $A(\beta)$.

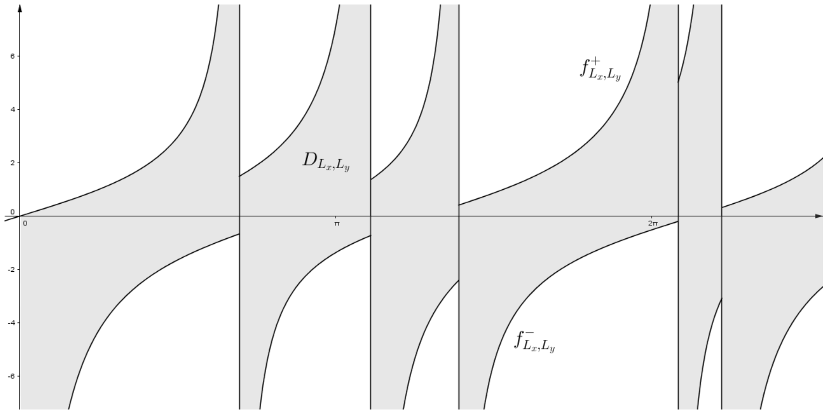
Lemma 5.2.2. *The point λ_0^2 is the lower end of a gap of the operator $A(\beta)$ if and only if one of the following possibilities holds:*

(i) $\lambda_0 \in \mathbb{R}^*_+ \setminus \widetilde{\Sigma}^*_z(\beta)$ *and* $\phi_{L_z,\beta}(\lambda_0) = f^-_{L_x,L_y}(\lambda_0) < 0$;

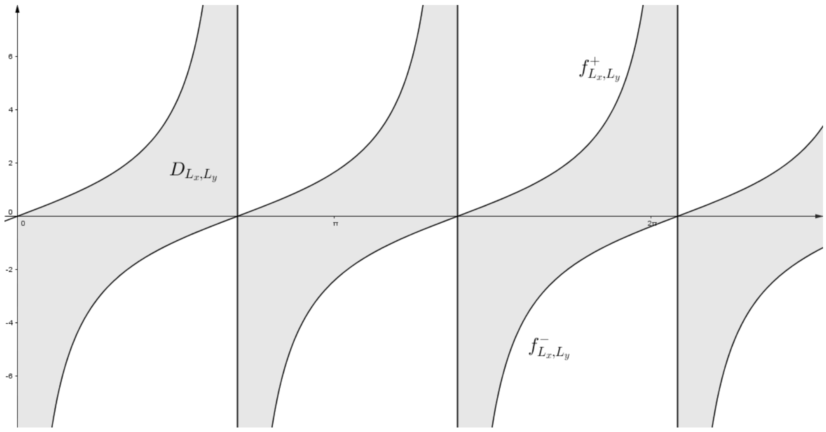
(ii) $\lambda_0 \in \left(\Sigma^*_x \cup \Sigma^*_y\right) \setminus \widetilde{\Sigma}^*_z(\beta)$ *and* $\phi_{L_z,\beta}(\lambda_0) > f^+_{L_x,L_y}(\lambda_0) \geqslant 0$;

(iii) $\lambda_0 \in (\Sigma_x \cup \Sigma_y) \cap \widetilde{\Sigma}_z(\beta)$.

Similarly, the point λ_0^2 is the upper end of a gap of the operator $A(\beta)$ if and only if one of the following possibilities holds:



(a) $L_x = 1.44$, $L_y = 0.9$



(b) $L_x = L_y = 1.44$

Figure 5.4: The set D_{L_x,L_y}. The vertical lines correspond to the points of the set $\Sigma_x \cup \Sigma_y$.

(iv) $\lambda_0 \in \mathbb{R}^*_+ \backslash \widetilde{\Sigma}^*_z(\beta)$ *and* $\phi_{L_z,\beta}(\lambda_0) = f^+_{L_x,L_y}(\lambda_0) > 0$*;*

(v) $\lambda_0 \in \left(\Sigma^*_x \cup \Sigma^*_y\right) \backslash \widetilde{\Sigma}^*_z(\beta)$ *and* $\phi_{L_z,\beta}(\lambda_0) < f^-_{L_x,L_y}(\lambda_0) \leqslant 0$*;*

(vi) $\lambda_0 \in \left(\Sigma^*_x \cup \Sigma^*_y\right) \cap \widetilde{\Sigma}^*_z(\beta)$.

Proof. The proof is very similar to the one of Lemme 1.3.2. We repeat it with appropriate modifications for the sake of completeness. Let λ_-, λ_+ be two neighbour points of the set $\Sigma_x \cup \Sigma_y$. We will study the possible configurations of the intersections of the function $\phi_{L_z,\beta}$ with the domain D_{L_x,L_y} in the interval $[\lambda_-, \lambda_+]$. Notice that $\phi_{L_z,\beta}$ is a $2\pi/L_z$-periodic function defined on $\mathbb{R}^+ \setminus \widetilde{\Sigma}_z(\beta)$. In each interval of continuity it is strictly decreasing and takes all the values in $\mathbb{R}$. We will consider separately the internal points of the interval $[\lambda_-, \lambda_+]$ and its ends.

- **Case 1.** Internal points points: $\lambda_0 \in]\lambda_-, \lambda_+[$

(a) $\lambda_0 \notin \widetilde{\Sigma}^*_z(\beta)$ and the point $(\lambda_0, \phi_{L_z,\beta}(\lambda_0))$ is an interior point of the domain D_{L_x,L_y}:

$$(\lambda_0, \phi_{L_z,\beta}(\lambda_0)) \in \text{int}\left(D_{L_x,L_y}\right) \Rightarrow \exists \delta > 0 \quad \text{s.t.} \quad (\lambda, \phi_{L_z,\beta}(\lambda)) \in D_{L_x,L_y}, \quad \forall \lambda : |\lambda - \lambda_0| < \delta. \tag{5.2.43}$$

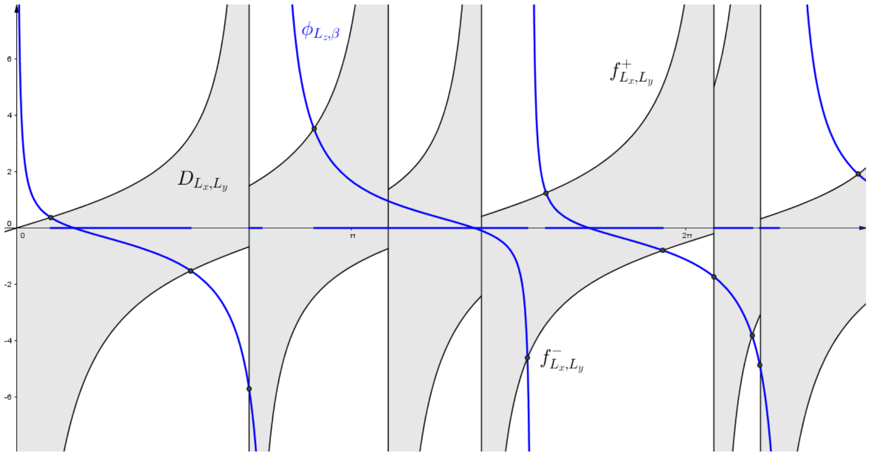


Figure 5.5: Example: $L_x = 1.44$, $L_y = 0.9$, $L_z = 1.3$, $\beta = 0.7$. The images of the spectral bands of the operator $A(\beta)$ by the function $x \mapsto \sqrt{x}$ are given by the projections on the axis of abscissas of the intersections of the set D_{L_x,L_y} with the graph of the function $\phi_{L_z,\beta}$.

Hence, the points such that $(\lambda_0, \phi_{L_z,\beta}(\lambda_0)) \in \operatorname{int}\left(D_{L_x,L_y}\right)$ correspond to the interior points of the spectrum (and not to the ends of gaps).

(b) $\lambda_0 \notin \widetilde{\Sigma}^*_z(\beta)$ and $\phi_{L_z,\beta}(\lambda_0) = f^+_{L_x,L_y}(\lambda_0)$:
(In this case, $f^+_{L_x,L_y}(\lambda_0) > 0$ due to (5.2.41)).

$$\begin{array}{l}\lambda_0 \in]\lambda_-, \lambda_+[\\ \phi_{L_z,\beta}(\lambda_0) = f^+_{L_x,L_y}(\lambda_0)\end{array} \quad \Rightarrow \quad \exists \delta > 0 \;\; \text{s.t.} \;\; \begin{array}{ll}(\lambda, \phi_{L_z,\beta}(\lambda)) \notin D_{L_x,L_y}, & \forall \lambda \in]\lambda_0 - \delta, \lambda_0[, \\ (\lambda, \phi_{L_z,\beta}(\lambda)) \in D_{L_x,L_y}, & \forall \lambda \in [\lambda_0, \lambda_0 + \delta[.\end{array} \tag{5.2.44}$$

(c) $\lambda_0 \notin \widetilde{\Sigma}^*_z(\beta)$ and $\phi_{L_z,\beta}(\lambda_0) = f^-_{L_x,L_y}(\lambda_0)$:
(In this case, $f^-_{L_x,L_y}(\lambda_0) < 0$ due to (5.2.42)).

$$\begin{array}{l}\lambda_0 \in]\lambda_-, \lambda_+[\\ \phi_{L_z,\beta}(\lambda_0) = f^-_{L_x,L_y}(\lambda_0)\end{array} \quad \Rightarrow \quad \exists \delta > 0 \;\; \text{s.t.} \;\; \begin{array}{ll}(\lambda, \phi_{L_z,\beta}(\lambda)) \notin D_{L_x,L_y}, & \forall \lambda \in]\lambda_0, \lambda_0 + \delta[, \\ (\lambda, \phi_{L_z,\beta}(\lambda)) \in D_{L_x,L_y}, & \forall \lambda \in [\lambda_0 - \delta, \lambda_0[.\end{array} \tag{5.2.45}$$

This follows from the fact that the functions $f^\pm_{L_x,L_y}$ defining the boundaries of the domain D_{L_x,L_y} are strictly increasing and the function $\phi_{L_z,\beta}$ is strictly decreasing. The relation (5.2.44) means that λ_0^2 is the lower end of a gap and the relation (5.2.45) means that λ_0^2 is the upper end of a gap.

(d) Points of discontinuity of the function $\phi_{L_z,\beta}$:

$$\lambda_0 \in \widetilde{\Sigma}^*_z(\beta) \cap]\lambda_-, \lambda_+[\quad \Rightarrow \quad \exists \delta > 0 \text{ s.t. } (\lambda, \phi_{L_z,\beta}(\lambda)) \notin D_{L_x,L_y}, \;\; \forall \lambda : \; 0 < |\lambda - \lambda_0| < \delta. \tag{5.2.46}$$

This follows from the continuity of the functions $f^\pm_{L_x,L_y}$ inside the interval $]\lambda_-, \lambda_+[$ and the behaviour of the function $\phi_{L_z,\beta}$ in a neighbourhood of its points of discontinuity.

We see that the interior points of the interval $[\lambda_-, \lambda_+]$ are ends of a gap only in the cases (b) and (c).

- **Case 2.** Ends of the interval: $\lambda_0 \in \Sigma_x^* \cup \Sigma_y^*$
  (In this case, $\lambda_0 \in \sigma(A(\beta))$ according to Proposition 5.2.5).

$$\begin{array}{l}\lambda_0 \in \left(\Sigma_x^* \cup \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta) \\ \phi_{L_z,\beta}(\lambda_0) > f^+_{L_x,L_y}(\lambda_0)\end{array} \Rightarrow \exists \delta > 0 \text{ s.t. } \begin{array}{ll}(\lambda, \phi_{L_z,\beta}(\lambda)) \notin D_{L_x,L_y}, & \forall \lambda \in]\lambda_0, \lambda_0 + \delta[, \\ (\lambda, \phi_{L_z,\beta}(\lambda)) \in D_{L_x,L_y}, & \forall \lambda \in]\lambda_0 - \delta, \lambda_0[.\end{array} \tag{5.2.47}$$

$$\begin{array}{l}\lambda_0 \in \left(\Sigma_x^* \cup \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta) \\ \phi_{L_z,\beta}(\lambda_0) < f^-_{L_x,L_y}(\lambda_0)\end{array} \Rightarrow \exists \delta > 0 \text{ s.t. } \begin{array}{ll}(\lambda, \phi_{L_z,\beta}(\lambda)) \notin D_{L_x,L_y}, & \forall \lambda \in]\lambda_0 - \delta, \lambda_0[, \\ (\lambda, \phi_{L_z,\beta}(\lambda)) \in D_{L_x,L_y}, & \forall \lambda \in]\lambda_0, \lambda_0 + \delta[.\end{array} \tag{5.2.48}$$

$$\begin{array}{l}\lambda_0 \in \left(\Sigma_x^* \cup \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta) \\ \phi_{L_z,\beta}(\lambda_0) \in \left[f^-_{L_x,L_y}(\lambda_0), f^+_{L_x,L_y}(\lambda_0)\right]\end{array} \Rightarrow \exists \delta > 0 \text{ s.t. } (\lambda, \phi_{L_z,\beta}(\lambda)) \in D_{L_x,L_y}, \quad \forall \lambda : 0 < |\lambda - \lambda_0| < \delta. \tag{5.2.49}$$

$$\lambda_0 \in \left(\Sigma_x^* \cup \Sigma_y^*\right) \cap \widetilde{\Sigma}_z^*(\beta) \Rightarrow \exists \delta > 0 \text{ s.t. } (\lambda, \phi_{L_z,\beta}(\lambda)) \notin D_{L_x,L_y}, \ \forall \lambda : 0 < |\lambda - \lambda_0| < \delta. \tag{5.2.50}$$

  Thus, the cases (5.2.47), (5.2.48), (5.2.50) correspond to ends of a gap. The case (5.2.50) corresponds to an isolated point of the spectrum which is an eigenvalue of infinite multiplicity (cf. Proposition 5.2.4).

- **Case 3.** $\lambda_0 = 0$

(a) $\beta = 0$: in this case $0 \in \sigma(A(\beta))$ and $\phi_{L_z,0}(0) = 0$. Consequently, $\phi_{L_z,0}(0) \in \left[f^-_{L_x,L_y}(0), f^+_{L_x,L_y}(0)\right]$, and similarly to (5.2.49) there exists $\delta > 0$ such that $(\lambda, \phi_{L_z,\beta}(\lambda)) \in D_{L_x,L_y}$, $\forall \lambda : 0 < \lambda - \lambda_0 < \delta$. Thus, 0 is not an end of a gap. It is not included in any of the conditions (i)–(vi) (it is not included in (iii) since $0 \notin \widetilde{\Sigma}_z(0)$).

(b) $\beta \in]0, \pi]$: in this case $0 \notin \sigma(A(\beta))$ and we call it the lower end of the gap $]0, \sigma_*[$, $\sigma_* = \min \sigma(A(\beta))$. For this reason it is included in the case (iii) (since $0 \in \widetilde{\Sigma}_z(\beta)$ for $\beta \in]0, \pi]$).

□

Corollary 5.2.1. *If $\lambda_0 \in \Sigma_z^*(\beta)$, then λ_0^2 is an internal point of $\sigma(A(\beta))$.*

Proof. It follows from Proposition 5.2.3 that $\lambda_0^2 \in \sigma(A(\beta))$. Let us show that it is not an end of a gap. Indeed, $\lambda_0 \in \Sigma_z^*(\beta)$ implies that $\phi_{L_z,\beta}(\lambda_0) = 0$. This is not compatible with any of the possibilities (i), (ii), (iv), (v) of Lemma 5.2.2. Since $\Sigma_z^*(\beta) \cap \widetilde{\Sigma}_z(\beta) = \emptyset$, it is not compatible with (iii) and (vi) neither. □

Proof of Proposition 5.2.6.

(a) Suppose first that $]a, b[$ is a gap of the operator $A(\beta)$. Then, one necessarily has $0 \leqslant a < b$, and it follows from Corollary 5.2.1 that $[\sqrt{a}, \sqrt{b}] \cap \Sigma_z^*(\beta) = \emptyset$. Moreover, $0 \in \Sigma_z(\beta)$ if and only if $\beta = 0$. Consequently, if $\beta \in]0, \pi]$, then $[\sqrt{a}, \sqrt{b}] \cap \Sigma_z(\beta) = \emptyset$. For $\beta = 0$, we have $a \neq 0$ since 0 is not the lower end of a gap, cf. Lemma 5.2.2. So, we have $[\sqrt{a}, \sqrt{b}] \cap \Sigma_z(\beta) = \emptyset$ again.

In other words,

$$\phi_{L_z,\beta}(\lambda) \neq 0, \qquad \forall \lambda \in [\sqrt{a}, \sqrt{b}] \setminus \widetilde{\Sigma}_z(\beta). \tag{5.2.51}$$

I Suppose that $\{\sqrt{a}, \sqrt{b}\} \cap \left(\Sigma_x \cup \Sigma_y \cup \widetilde{\Sigma}_z(\beta)\right) = \emptyset$. Then, due to Lemma 5.2.2 we have $\phi_{L_z,\beta}(\sqrt{a}) = f^-_{L_x,L_y}(\sqrt{a})$, $\phi_{L_z,\beta}(\sqrt{b}) = f^+_{L_x,L_y}(\sqrt{b})$.

II

(i-i') Suppose now that $\sqrt{a} \in \Sigma_x \cup \Sigma_y$. It follows from Lemma 5.2.2 that either $\sqrt{a} \in \widetilde{\Sigma}_z(\beta)$ (case (i')) or $\sqrt{a} \in \left(\Sigma_x^* \cup \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta)$ and $\phi_{L_z,\beta}(\sqrt{a}) > f^+_{L_x,L_y}(\sqrt{a}) \geqslant 0$ (case (i)). In both cases there exists $\delta > 0$ such that $\phi_{L_z,\beta}(\lambda) > 0$ for $\lambda \in]\sqrt{a}, \sqrt{a} + \delta[$. This implies that

$$]\sqrt{a}, \sqrt{b}] \cap \widetilde{\Sigma}_z(\beta) = \emptyset \quad \text{and} \quad \phi_{L_z,\beta}(\lambda) > 0, \quad \forall \lambda \in]\sqrt{a}, \sqrt{b}].$$

Indeed, in the opposite case the function $\phi_{L_z,\beta}$ is not continuous in $]\sqrt{a}, \sqrt{b}]$ (if there exists a point $\lambda_0 \in]\sqrt{a}, \sqrt{b}]$ such that $\phi_{L_z,\beta}(\lambda_0) < 0$ and $\phi_{L_z,\beta}$ is continuous in $]\sqrt{a}, \sqrt{b}]$, then $]\sqrt{a}, \sqrt{b}] \cap \Sigma_z(\beta) \notin \emptyset$, which, as was shown above, is impossible). Let λ' be the closest to $\sqrt{a}$ point of discontinuity of $\phi_{L_z,\beta}$ in $]\sqrt{a}, \sqrt{b}]$: $\lambda' \in]\sqrt{a}, \sqrt{b}] \cap \widetilde{\Sigma}_z(\beta)$. Taking into account that $\lim\limits_{\lambda \to \lambda' - 0} \phi_{L_z,\beta}(\lambda) = -\infty$, we conclude that there exists a point $\lambda'' \in]\sqrt{a}, \lambda'[$ such that $\phi_{L_z,\beta}(\lambda'') = 0$, which contradicts (5.2.51). This shows that $\sqrt{b} \notin \widetilde{\Sigma}_z(\beta)$ and $\phi_{L_z,\beta}(\sqrt{b}) > 0$, which, due to Lemma 5.2.2, implies that $\sqrt{b} \notin \Sigma_x \cup \Sigma_y$ and $\phi_{L_z,\beta}(\sqrt{b}) = f^+_{L_x,L_y}(\sqrt{b})$.

(ii-ii') The case $\sqrt{b} \in \Sigma_x^* \cup \Sigma_y^*$ can be considered analogously to the previous case.

(b) Suppose now that $0 \leqslant a < b$ and $[a, b] \cap \Sigma_z(\beta) = \emptyset$. We will show that if one of the conditions I, II holds, then $]a, b[$ is a gap of the operator $A(\beta)$.

I It follows from Lemma 5.2.2 that a is the lower end of a gap and b is the upper end of a gap. Suppose that there exists $c \in]a, b[$ such that $c \in \sigma(A(\beta))$. Then, Proposition 5.2.5 implies that $\phi_{L_z,\beta}(\sqrt{c}) \in \left[f^-_{L_x,L_y}(\sqrt{c}), f^+_{L_x,L_y}(\sqrt{c})\right]$. The functions $f^\pm_{L_x,L_y}$ being strictly increasing in $[\sqrt{a}, \sqrt{b}]$, we get $\phi_{L_z,\beta}(\sqrt{a}) < \phi_{L_z,\beta}(\sqrt{c}) < \phi_{L_z,\beta}(\sqrt{b})$. The function $\phi_{L_z,\beta}$ is, in turn, strictly decreasing in its intervals of continuity. Hence, there exist $\lambda' \in]\sqrt{a}, \sqrt{c}[$ and $\lambda'' \in]\sqrt{c}, \sqrt{b}[$ such that $\lambda', \lambda'' \in \widetilde{\Sigma}_z(\beta)$. Since in each interval of continuity the function $\phi_{L_z,\beta}$ takes all values in $\mathbb{R}$, there exists $\lambda_0 \in]\lambda', \lambda''[$ such that $\phi_{L_z,\beta}(\lambda_0) = 0$ which contradicts the assumption $[a, b] \cap \Sigma_z(\beta) = \emptyset$.

II

(i-i') It follows from Lemma 5.2.2 that b is the upper end of a gap. Let us denote by a' its lower end. Since $a \in \sigma(A(\beta))$ (cf. Proposition 5.2.5), one necessarily has $a \leqslant a'$. Suppose that $a < a'$. Then, it follows from Lemma 5.2.2 that $\sqrt{a'} \notin \widetilde{\Sigma}_z(\beta)$ and $\phi_{L_z,\beta}(\sqrt{a'}) = f^-_{L_x,L_y}(\sqrt{a'}) < 0$. On the other hand, both in the case (i) and in the case (i') there exists $\delta > 0$ such that $\phi_{L_z,\beta}(\lambda) > 0$ for $\lambda \in]\sqrt{a}, \sqrt{a} + \delta[$. Consequently, there exists $\lambda' \in]\sqrt{a}, \sqrt{a'}[$ such that $\phi_{L_z,\beta}(\lambda') = 0$, which contradicts the assumption $[a, b] \cap \Sigma_z(\beta) = \emptyset$.

(ii-ii') This case can be considered analogously to the previous one.

□

Let us mention another possible characterization of the two types of gaps. It will be used in the next section while discussing the number of eigenvalues of the perturbed operator in the gaps.

Lemma 5.2.3. *Let* $]a,b[$ *be a gap of the operator* $A(\beta)$*. Then, it is a gap of type I if and only if there exists exactly one point c such that* $c\in]a,b[$ *such that* $\sqrt{c}\in\widetilde{\Sigma}_z(\beta)$*. It is a gap of type II if and only if* $]\sqrt{a},\sqrt{b}[\cap\widetilde{\Sigma}_z(\beta)=\emptyset$*.*

Proof. First, let us mention that the intersection $]\sqrt{a},\sqrt{b}[\cap\widetilde{\Sigma}_z(\beta)$ is either reduced to one point or empty. Indeed, if there exist two distinct points $\lambda',\lambda''\in]\sqrt{a},\sqrt{b}[\cap\widetilde{\Sigma}_z(\beta)$, then as we have seen in the proof of Proposition 5.2.6 (b) I, there exists $\lambda_0\in]\lambda',\lambda''[$ such that $\phi_{L_z,\beta}(\lambda_0)=0$, which contradicts the assumption that $]a,b[$ is a gap of the operator $A(\beta)$.

I Let $]a,b[$ be a gap of the operator $A(\beta)$ of type I. In this case $\phi_{L_z,\beta}(\sqrt{a})<\phi_{L_z,\beta}(\sqrt{b})$. Consequently, $]\sqrt{a},\sqrt{b}[\cap\widetilde{\Sigma}_z(\beta)\neq\emptyset$. As it was mentioned above, this intersection is necessarily reduced to one point.

II Let $]a,b[$ be a gap of the operator $A(\beta)$ of type II. Then, there exists $\delta>0$ such that $\phi_{L_z,\beta}(\lambda')\phi_{L_z,\beta}(\lambda'')>0$ for any $(\lambda',\lambda'')\in]\sqrt{a},\sqrt{a}+\delta[\times]\sqrt{b}-\delta,\sqrt{b}[$. Suppose that there exists $\sqrt{c}\in]\sqrt{a},\sqrt{b}[\cap\widetilde{\Sigma}_z(\beta)$. Then, at least in one of the intervals $]\sqrt{a},\sqrt{c}[$, $]\sqrt{c},\sqrt{b}[$ there exists λ_0 such that $\phi_{L_z,\beta}(\lambda_0)=0$, which contradicts the assumption that $]a,b[$ is a gap of the operator $A(\beta)$.

□

Examples illustrating Proposition 5.2.6, Lemma 5.2.3 and Proposition 5.2.4 are given in figures 5.7, 5.6.

5.2.2 Computation of the discrete spectrum

5.2.2.1 Reduction to a finite difference equation

Let us now determine the discrete spectrum of the operator $A^\mu(\beta)$. If λ^2 is an eigenvalue of this operator, then the corresponding eigenfunction $u\in D\left(A^\mu(\beta)\right)$ solves the equation $u''+\lambda^2u=0$ on each edge of the graph G.

Case $\lambda>0$

In this case the eigenfunction u it has the form

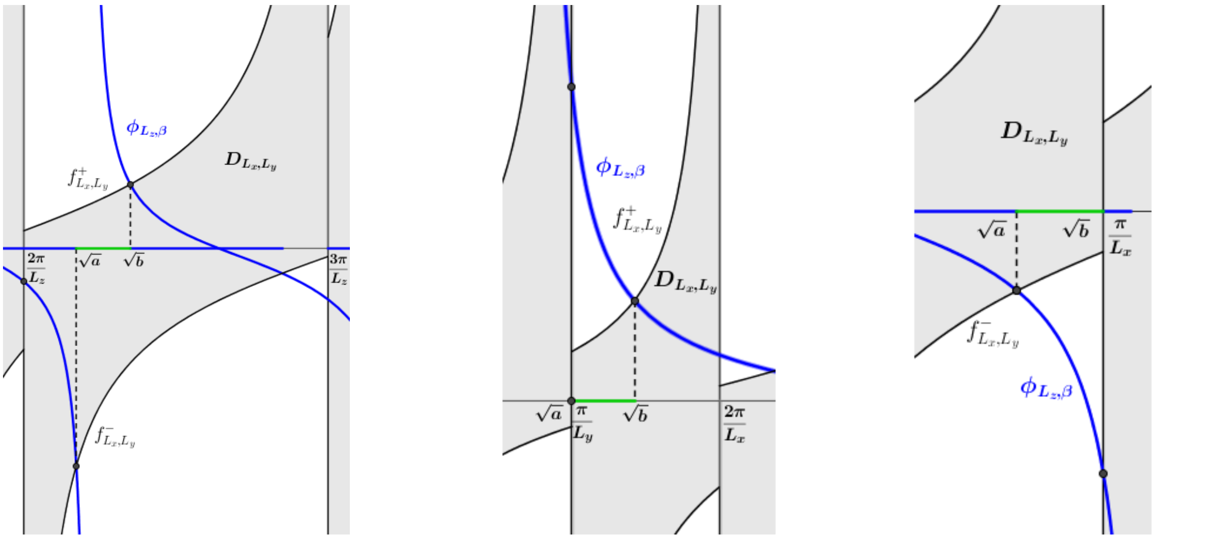
$$u_{k+\frac12,\ell}(s)=a_{k+\frac12,\ell}\sin(\lambda s)+b_{k+\frac12,\ell}\cos(\lambda s),\quad s\in[0,L_x],\quad (k,\ell)\in\mathbb{Z}^2,\tag{5.2.52}$$
$$u_{k,\ell+\frac12}(t)=a_{k,\ell+\frac12}\sin(\lambda t)+b_{k,\ell+\frac12}\cos(\lambda t),\quad t\in[0,L_y],\quad (k,\ell)\in\mathbb{Z}^2,\tag{5.2.53}$$
$$u^+_{k,\ell}(z)=c^+_{k,\ell}\sin(\lambda z)+d^+_{k,\ell}\cos(\lambda z),\quad z\in\left[0,\tfrac{L_z}{2}\right],\quad (k,\ell)\in\mathbb{Z}^2,\tag{5.2.54}$$
$$u^-_{k,\ell}(z)=c^-_{k,\ell}\sin(\lambda z)+d^-_{k,\ell}\cos(\lambda z),\quad z\in\left[-\tfrac{L_z}{2},0\right],\quad (k,\ell)\in\mathbb{Z}^2.\tag{5.2.55}$$

The continuity of the eigenfunction u at the vertices $v_{k,l}$ of the graph G implies that

$$\begin{aligned}b_{k+\frac12,\ell}=b_{k,\ell+\frac12}&=a_{k-\frac12,\ell}\sin(\lambda L_x)+b_{k-\frac12,\ell}\cos(\lambda L_x)\\&=a_{k,\ell-\frac12}\sin(\lambda L_y)+b_{k,\ell-\frac12}\cos(\lambda L_y)=d^+_{k,\ell}=d^-_{k,\ell}=:d_{k,\ell},\quad (k,l)\in\mathbb{Z}^2.\end{aligned}\tag{5.2.56}$$

After plugging (5.2.54), (5.2.55) into the quasiperiodicity conditions (5.2.1), we get

$$\left(c^+_{k,l}+e^{-i\beta}c^-_{k,l}\right)\sin\left(\tfrac{\lambda L_z}{2}\right)+d_{k,l}\cos\left(\tfrac{\lambda L_z}{2}\right)\left(1-e^{-i\beta}\right)=0,\quad (k,l)\in\mathbb{Z}^2,\tag{5.2.57}$$
$$\left(c^+_{k,l}-e^{-i\beta}c^-_{k,l}\right)\cos\left(\tfrac{\lambda L_z}{2}\right)-d_{k,l}\sin\left(\tfrac{\lambda L_z}{2}\right)\left(1+e^{-i\beta}\right)=0,\quad (k,l)\in\mathbb{Z}^2.\tag{5.2.58}$$



(a) The interval $]a, b[$ is a gap of type I

(b) The interval $]a, b[$ is a gap of type $II(i)$

(c) The interval $]a, b[$ is a gap of type $II(ii)$

Figure 5.6: Types of gaps

(a) Gap of type I: $\phi_{L_z,\beta}(\sqrt{a}) = f^-_{L_x,L_y}(\sqrt{a}) < 0$, $\phi_{L_z,\beta}(\sqrt{b}) = f^+_{L_x,L_y}(\sqrt{b}) > 0$, and there is one point of discontinuity of the function $\phi_{L_z,\beta}$ in $]\sqrt{a}, \sqrt{b}[$.
(b) Gap of type $II(i)$: $\sqrt{a} \in \Sigma^*_y$, $\phi_{L_z,\beta}(\sqrt{a}) > f^+_{L_x,L_y}(\sqrt{a})$, $\phi_{L_z,\beta}(\sqrt{b}) = f^+_{L_x,L_y}(\sqrt{b}) > 0$, and the function $\phi_{L_z,\beta}$ is continuous in $]\sqrt{a}, \sqrt{b}[$.
(c) Gap of type $II(ii)$: $\phi_{L_z,\beta}(\sqrt{a}) = f^-_{L_x,L_y}(\sqrt{b}) < 0$, $\sqrt{b} \in \Sigma^*_x$, $\phi_{L_z,\beta}(\sqrt{b}) < f^-_{L_x,L_y}(\sqrt{b})$, and the function $\phi_{L_z,\beta}$ is continuous in $]\sqrt{a}, \sqrt{b}[$.

From Kirchhoff's conditions (5.2.2) we find

$$w^\mu_{k,l}\left(c^+_{k,l} - c^-_{k,l}\right) + a_{k+\frac{1}{2},\ell} + a_{k,\ell+\frac{1}{2}} - a_{k-\frac{1}{2},\ell}\cos(\lambda L_x) - a_{k,\ell-\frac{1}{2}}\cos(\lambda L_y) \\ + d_{k-1,\ell}\sin(\lambda L_x) + d_{k,\ell-1}\sin(\lambda L_y) = 0, \qquad (k,l) \in \mathbb{Z}^2. \quad (5.2.59)$$

The following assertion permits to exclude from the consideration the set Σ^* while searching the discrete spectrum of the operator $A^\mu(\beta)$. Its proof is given in Appendix.

Proposition 5.2.7. *If $\lambda \in \Sigma^*$, then λ^2 is not an eigenvalue of finite multiplicity of the operator $A^\mu(\beta)$ for any $\beta \in [0, \pi]$, $\mu > 0$.*

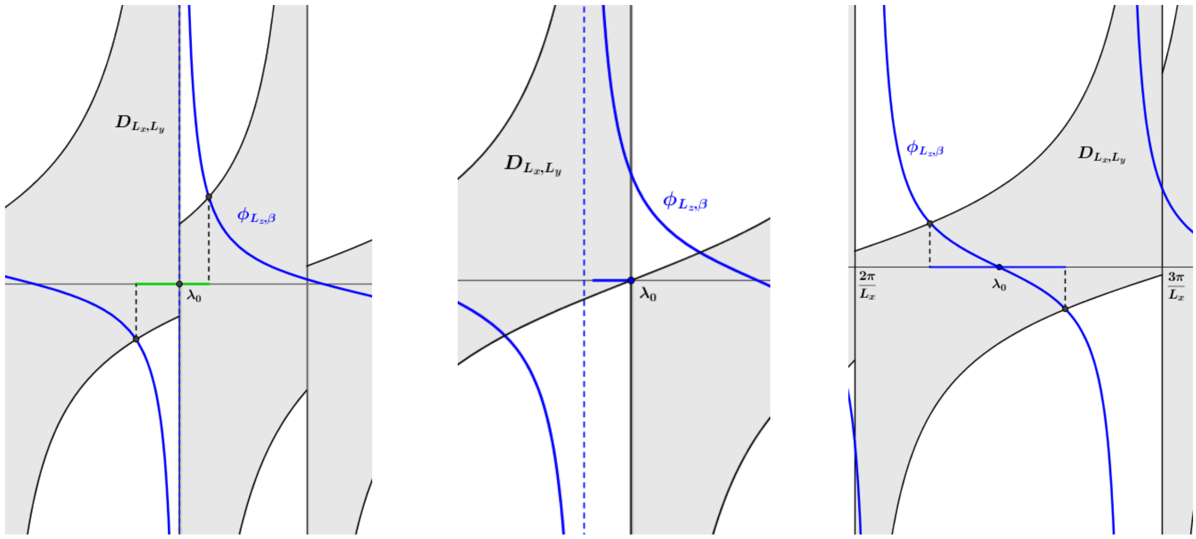
If $\lambda \notin \Sigma$, then the coefficients $\left\{a_{k+\frac{1}{2},\ell}, b_{k+\frac{1}{2},\ell}, a_{k,\ell+\frac{1}{2}}, b_{k,\ell+\frac{1}{2}}, c^\pm_{k,\ell}, d_{k,\ell}\right\}$ can be expressed in terms of the values $\{\mathbf{u}_{k,l}\}$:

$$a_{k+\frac{1}{2},\ell} = \frac{1}{\sin(\lambda L_x)}\left(\mathbf{u}_{k+1,\ell} - \mathbf{u}_{k,\ell}\cos(\lambda L_x)\right), \quad a_{k,\ell+\frac{1}{2}} = \frac{1}{\sin(\lambda L_y)}\left(\mathbf{u}_{k+1,\ell} - \mathbf{u}_{k,\ell}\cos(\lambda L_y)\right), \quad (5.2.60)$$

$$b_{k+\frac{1}{2},\ell} = b_{k,\ell+\frac{1}{2}} = d_{k,\ell} = \mathbf{u}_{k,\ell}, \qquad c^\pm_{k,l} = \alpha^\pm \mathbf{u}_{k,\ell}, \qquad (k,\ell) \in \mathbb{Z}^2. \quad (5.2.61)$$

where

$$\alpha^+ = \frac{1}{2}\left(\tan\left(\tfrac{\lambda L_z}{2}\right)\left(1 + e^{-i\beta}\right) + \frac{e^{-i\beta} - 1}{\tan\left(\frac{\lambda L_z}{2}\right)}\right), \quad \alpha^- = \frac{1}{2}\left(\frac{1 - e^{i\beta}}{\tan\left(\frac{\lambda L_z}{2}\right)} - \tan\left(\tfrac{\lambda L_z}{2}\right)\left(1 + e^{i\beta}\right)\right).$$



(a) λ_0^2 is an isolated eigenvalue of infinite multiplicity: $\lambda_0 \in \Sigma_y^* \cap \widetilde{\Sigma}_z^*(\beta)$. The gap to the left of λ_0^2 is a gap of type $II(ii')$ and the gap to the right of λ_0^2 is a gap of type $II(i')$

(b) λ_0^2 is an embedded eigenvalue of infinite multiplicity: $\lambda_0 \in \left(\Sigma_x^* \cap \Sigma_y^*\right) \cap \widetilde{\Sigma}_z^*(\beta)$

(c) λ_0^2 is an embedded eigenvalue of infinite multiplicity: $\beta = 0$ and $\lambda_0 \in \Sigma_z^*(\beta)$

Figure 5.7: Eigenvalues of infinite multiplicity

Then, the relations (5.2.52)–(5.2.55) can be rewritten as

$$u_{k+\frac{1}{2},\ell}(s) = \mathbf{u}_{k,\ell}\frac{\sin\left(\lambda(L_x - s)\right)}{\sin\left(\lambda L_x\right)} + \mathbf{u}_{k+1,\ell}\frac{\sin\left(\lambda s\right)}{\sin\left(\lambda L_x\right)}, \quad s \in [0, L_x], \qquad k, \ell \in \mathbb{Z}, \quad (5.2.62)$$

$$u_{k,\ell+\frac{1}{2}}(t) = \mathbf{u}_{k,\ell}\frac{\sin\left(\lambda(L_y - t)\right)}{\sin\left(\lambda L_y\right)} + \mathbf{u}_{k,l+1}\frac{\sin\left(\lambda t\right)}{\sin\left(\lambda L_y\right)}, \quad t \in [0, L_y], \qquad k, \ell \in \mathbb{Z}, \quad (5.2.63)$$

$$u^+_{k,\ell}(z) = \mathbf{u}_{k,\ell}\frac{\sin\left(\lambda\left(\frac{L_z}{2} - z\right)\right)}{\sin\left(\frac{\lambda L_z}{2}\right)} + \mathbf{u}^+_{k,\ell}\frac{\sin\left(\lambda z\right)}{\sin\left(\frac{\lambda L_z}{2}\right)}, \quad z \in \left[0, \tfrac{L_z}{2}\right], \qquad k, \ell \in \mathbb{Z}, \quad (5.2.64)$$

$$u^-_{k,\ell}(z) = \mathbf{u}_{k,\ell}\frac{\sin\left(\lambda\left(\frac{L_z}{2} + z\right)\right)}{\sin\left(\frac{\lambda L_z}{2}\right)} - \mathbf{u}^-_{k,\ell}\frac{\sin\left(\lambda z\right)}{\sin\left(\frac{\lambda L_z}{2}\right)}, \quad z \in \left[-\tfrac{L_z}{2}, 0\right], \qquad k, \ell \in \mathbb{Z}, \quad (5.2.65)$$

$$\mathbf{u}^+_{k,\ell} = \frac{1 + e^{-i\beta}}{2\cos\left(\frac{\lambda L_z}{2}\right)}\mathbf{u}_{k,\ell}, \qquad \mathbf{u}^-_{k,\ell} = \frac{1 + e^{i\beta}}{2\cos\left(\frac{\lambda L_z}{2}\right)}\mathbf{u}_{k,\ell}, \qquad (k,\ell) \in \mathbb{Z}^2. \qquad (5.2.66)$$

After plugging (5.2.60)–(5.2.61) into (5.2.59), we get

$$\frac{\mathbf{u}_{k+1,\ell} + \mathbf{u}_{k-1,\ell}}{\sin\left(\lambda L_x\right)} + \frac{\mathbf{u}_{k,\ell+1} + \mathbf{u}_{k,\ell-1}}{\sin\left(\lambda L_y\right)} - 2g_\beta(\lambda)\mathbf{u}_{k,l} = 0, \qquad (0,0) \neq (k,\ell) \in \mathbb{Z}^2, \quad (5.2.67)$$

$$\frac{\mathbf{u}_{k+1,\ell} + \mathbf{u}_{k-1,\ell}}{\sin\left(\lambda L_x\right)} + \frac{\mathbf{u}_{k,\ell+1} + \mathbf{u}_{k,\ell-1}}{\sin\left(\lambda L_y\right)} - 2g_\beta(\lambda)\mathbf{u}_{0,0} = 2(\mu - 1)\phi_{L_z,\beta}(\lambda)\mathbf{u}_{0,0}, \qquad (5.2.68)$$

where

$$g_\beta(\lambda) = \frac{1}{\tan\left(\lambda L_x\right)} + \frac{1}{\tan\left(\lambda L_y\right)} + \phi_{L_z,\beta}(\lambda), \qquad \lambda \notin \Sigma, \qquad (5.2.69)$$

and the function $\phi_{L_z,\beta}$ is defined in (5.2.37). Thus, we replaced the initial continuous problem for $u \in L_2(G)$ by a discrete problem for $\{\mathbf{u}_{k,l}\}_{(k,l)\in\mathbb{Z}^2} \in \ell_2(\mathbb{Z}^2)$.

Case $\lambda = 0$

If 0 is an eigenvalue of the operator $A^\mu(\beta)$, then the corresponding eigenfunction has the form

$$u_{k+\frac{1}{2},\ell}(s) = \mathbf{u}_{k,\ell}(1-s) + \mathbf{u}_{k+1,\ell}s, \qquad s \in [0, L_x], \qquad (k,\ell) \in \mathbb{Z}^2, \tag{5.2.70}$$

$$u_{k,\ell+\frac{1}{2}}(t) = \mathbf{u}_{k,\ell}(1-t) + \mathbf{u}_{k,\ell+1}t, \qquad t \in [0, L_y], \qquad (k,\ell) \in \mathbb{Z}^2, \tag{5.2.71}$$

$$u^+_{k,\ell}(z) = \mathbf{u}_{k,\ell}\left(1 - \frac{2z}{L_z}\right) + \frac{2\mathbf{u}^+_{k,\ell}}{L_z}z, \qquad z \in \left[0, \tfrac{L_z}{2}\right], \qquad (k,\ell) \in \mathbb{Z}^2, \tag{5.2.72}$$

$$u^-_{k,\ell}(z) = \mathbf{u}_{k,\ell}\left(1 + \frac{2z}{L_z}\right) - \frac{2\mathbf{u}^-_{k,\ell}}{L_z}z, \qquad z \in \left[-\tfrac{L_z}{2}, 0\right], \qquad (k,\ell) \in \mathbb{Z}^2. \tag{5.2.73}$$

The quasiperiodicity conditions (5.2.1) imply that

$$\mathbf{u}^+_{k,\ell} = \frac{1+e^{-i\beta}}{2}\mathbf{u}_{k,\ell}, \qquad \mathbf{u}^-_{k,\ell} = \frac{1+e^{i\beta}}{2}\mathbf{u}_{k,\ell}, \qquad (k,\ell) \in \mathbb{Z}^2. \tag{5.2.74}$$

After plugging (5.2.70)–(5.2.74) into the Kirchhoff's conditions (5.2.2) we find

$$\frac{\mathbf{u}_{k+1,\ell} + \mathbf{u}_{k-1,\ell}}{L_x} + \frac{\mathbf{u}_{k,\ell+1} + \mathbf{u}_{k,\ell-1}}{L_y} - 2\left(\frac{1}{L_x} + \frac{1}{L_y} + \frac{1-\cos\beta}{L_z}\right)\mathbf{u}_{k,\ell} = 0, \qquad (k,\ell) \neq (0,0), \tag{5.2.75}$$

$$\frac{\mathbf{u}_{k+1,\ell} + \mathbf{u}_{k-1,\ell}}{L_x} + \frac{\mathbf{u}_{k,\ell+1} + \mathbf{u}_{k,\ell-1}}{L_y} - 2\left(\frac{1}{L_x} + \frac{1}{L_y} + \frac{1-\cos\beta}{L_z}\right)\mathbf{u}_{k,\ell} = 2(\mu-1)\frac{1-\cos\beta}{L_z}\mathbf{u}_{0,0}. \tag{5.2.76}$$

We end up with the following assertion.

Lemma 5.2.4. *If λ^2 is an eigenvalue of finite multiplicity of the operator $A^\mu(\beta)$ and u is an eigenfunction associated with λ^2 then $\{\mathbf{u}_{k,l}\}_{(k,l)\in\mathbb{Z}^2} \in \ell_2(\mathbb{Z}^2)$ and the sequence $\{\boldsymbol{u}_{k,l}\}_{(k,l)\in\mathbb{Z}^2}$ satisfies the finite difference equation* (5.2.67)–(5.2.68) *for $\lambda \neq 0$ and* (5.2.75)–(5.2.76) *for $\lambda = 0$. Conversely, if a sequence $\{\boldsymbol{u}_{k,l}\}_{(k,l)\in\mathbb{Z}^2} \in \ell_2(\mathbb{Z}^2)$ satisfies* (5.2.67)–(5.2.68) *(resp.* (5.2.75)–(5.2.76)*) then λ^2 (resp. 0) is an eigenvalue of finite multiplicity of the operator $A^\mu(\beta)$ and the function u defined by the relations* (5.2.62)–(5.2.66) *(resp.*(5.2.70)–(5.2.74)*) is a corresponding eigenfunction.*

5.2.2.2 Absence of embedded eigenvalues

Let us apply to the sequence $\{\mathbf{u}_{k,l}\}$ the discrete Fourier transform $\mathcal{F} : \ell_2(\mathbb{Z}^2) \to L_2([0,2\pi]^2)$ defined as

$$\mathcal{F}(\{\mathbf{v}_{k,l}\}) = \widehat{\mathbf{v}}, \qquad \widehat{\mathbf{v}}(\xi,\eta) = \sum_{(k,\ell)\in\mathbb{Z}^2} e^{i(k\xi+l\eta)}\mathbf{v}_{k,\ell}, \qquad (\xi,\eta) \in [0,2\pi]^2.$$

Then, for $\widehat{\mathbf{u}}$ we get the following equation:

For $\lambda \neq 0$,

$$\left(f_{L_x,L_y,\xi,\eta}(\lambda) - \phi_{L_z,\beta}(\lambda)\right) \widehat{\mathbf{u}}(\xi,\eta) = (\mu - 1)\phi_{L_z,\beta}(\lambda)\mathbf{u}_{0,0}, \quad \forall(\xi,\eta) \in [0,2\pi]^2, \tag{5.2.77}$$

where the function $f_{L_x,L_y,\xi,\eta}$ is defined in (5.2.35).
For $\lambda = 0$,

$$\left(\frac{\cos\xi - 1}{L_x} + \frac{\cos\eta - 1}{L_y} - \frac{1-\cos\beta}{L_z}\right) \widehat{\mathbf{u}}(\xi,\eta) = (\mu - 1)\frac{1-\cos\beta}{L_z}\mathbf{u}_{0,0}, \quad \forall(\xi,\eta) \in [0,2\pi]^2. \tag{5.2.78}$$

We can now prove the absence of embedded eigenvalues for the operator $A^\mu(\beta)$.

Proposition 5.2.8. *If λ^2 is an eigenvalue of finite multiplicity of the operator $A^\mu(\beta)$ for some $\beta \in [0,\pi]$, $\mu > 0$, then $\lambda^2 \notin \sigma(A(\beta))$.*

Proof.

(a) $\lambda > 0$: suppose that $\lambda^2 \in \sigma(A(\beta))$ is an eigenvalue of finite multiplicity of the operator $A^\mu(\beta)$. As follows from Proposition 5.2.7, $\lambda \in \mathbb{R}_+^* \setminus \Sigma^*$. According to Proposition 5.2.5, there exist $(k_x,k_y) \in [0,\pi]^2$ such that $\phi_{L_z,\beta}(\lambda) = f_{L_x,L_y,k_x,k_y}(\lambda)$. On the other hand, if u is an eigenfunction corresponding to λ^2, then the equation (5.2.77) is satisfied. Its right-hand side does not depend on (ξ,η) and its left-hand side is zero for $(\xi,\eta) = (k_x,k_y)$. Thus,

$$\left(f_{L_x,L_y,\xi,\eta}(\lambda) - \phi_{L_z,\beta}(\lambda)\right) \widehat{\mathbf{u}}(\xi,\eta) = 0, \qquad \forall(\xi,\eta) \in [0,2\pi]^2.$$

Consequently, $\widehat{\mathbf{u}}(\xi,\eta) = 0$ for any $(\xi,\eta) \in [0,2\pi]^2$ such that $f_{L_x,L_y,\xi,\eta}(\lambda) \neq f_{L_x,L_y,k_x,k_y}(\lambda)$. The set $\{(\xi,\eta) \in [0,2\pi]^2/\, f_{L_x,L_y,\xi,\eta}(\lambda) \neq f_{L_x,L_y,k_x,k_y}(\lambda)\}$ being a full-mesure set, we conclude that $\widehat{\mathbf{u}} = 0$ in $L_2\left([0,2\pi]^2\right)$. Hence, $u = 0$ and λ^2 is not an eigenvalue of the operator $A^\mu(\beta)$.

(b) $\lambda = 0$: suppose that $0 \in \sigma(A(\beta))$. Then, one necessarily has $\beta = 0$. If u is an eigenfunction corresponding to 0, then its discrete Fourier transform satisfies the equation (5.2.78) with $\beta = 0$:

$$\left(\frac{\cos\xi - 1}{L_x} + \frac{\cos\eta - 1}{L_y}\right) \widehat{\mathbf{u}}(\xi,\eta) = 0, \qquad \forall(\xi,\eta) \in [0,2\pi]^2.$$

Similarly to the case $\lambda \neq 0$, we find $\widehat{\mathbf{u}}(\xi,\eta) = 0$ for any $(\xi,\eta) \in [0,2\pi] \setminus (0,0)$, which implies that $u = 0$ and 0 is not an eigenvalue of the operator $A(0)$.

□

5.2.2.3 Obtaining a characteristic equation for the eigenvalues

From now on we will consider λ such that $\lambda^2 \notin \sigma(A(\beta))$. Then, it follows from Proposition 5.2.3 (1) and Proposition 5.2.5 that

$$\phi_{L_z,\beta}(\lambda) \neq 0, \qquad f_{L_x,L_y,\xi,\eta}(\lambda) - \phi_{L_z,\beta}(\lambda) \neq 0, \qquad \forall(\xi,\eta) \in [0,2\pi]^2. \tag{5.2.79}$$

Consequently, for $\lambda \in \mathbb{R}_+ \setminus \Sigma^*,\ \lambda^2 \notin \sigma\left(A(\beta)\right)$ the relations (5.2.77), (5.2.78) impliy

$$\widehat{\mathbf{u}}(\xi,\eta) = \frac{(1-\mu)\mathbf{u}_{0,0}}{1-\varphi_\beta(\xi,\eta,\lambda)}, \qquad \forall(\xi,\eta)\in[0,2\pi]^2,$$

where

$$\varphi_\beta(\xi,\eta,\lambda) = \begin{cases} \dfrac{f_{L_x,L_y,\xi,\eta}(\lambda)}{\phi_{c,\beta}(\lambda)}, & \lambda\in\mathbb{R}_+\setminus\left(\Sigma\cup\Sigma_z(\beta)\right), \\ \dfrac{L_z}{1-\cos\beta}\left(\dfrac{\cos\xi-1}{L_x}+\dfrac{\cos\eta-1}{L_y}\right), & \beta\in]0,\pi], \quad \lambda=0. \end{cases} \tag{5.2.80}$$

Notice that for $\beta\in]0,\pi]$ the function φ_β is continuous at 0. Finally, applying the inverse Fourier transform, we get

$$\mathbf{u}_{k,\ell} = \frac{(1-\mu)\mathbf{u}_{0,0}}{4\pi^2}\iint\limits_{[0,2\pi]^2}\frac{1}{1-\varphi_\beta(\xi,\eta,\lambda)}e^{-i(k\xi+\ell\eta)}d\xi d\eta, \qquad (k,\ell)\in\mathbb{Z}^2. \tag{5.2.81}$$

We can now give a criterion of existence of an eigenfunction.

Proposition 5.2.9. *λ^2 is an eigenvalue of finite multiplicity of the operator $A^\mu(\beta)$ if and only if $\lambda\in\mathbb{R}_+\setminus\Sigma^*,\ \lambda^2\notin\sigma(A(\beta))$ and the following relation is satisfied:*

$$(1-\mu)\mathcal{I}_\beta(\lambda) = 1, \tag{5.2.82}$$

where

$$I_\beta(\lambda) = \frac{1}{\pi^2}\iint\limits_{[0,\pi]^2}\frac{1}{1-\varphi_\beta(\xi,\eta,\lambda)}d\xi d\eta, \tag{5.2.83}$$

and the function φ_β is defined in (5.2.80).

Proof. If λ^2 is an eigenvalue of finite multiplicity of the operator $A^\mu(\beta)$, then it follows from Propositions 5.2.7 and 5.2.8 that $\lambda\in\mathbb{R}_+\setminus\Sigma^*$ and $\lambda^2\notin\sigma(A(\beta))$. Let u be an eigenfunction corresponding to the eigenvalue λ^2. Then, the relation (5.2.81) is satisfied. For $k=0,\ \ell=0$ it gives

$$1 = (1-\mu)\left(\frac{1}{4\pi^2}\iint\limits_{[0,2\pi]^2}\frac{1}{1-\varphi_\beta(\xi,\eta,\lambda)}e^{-i(k\xi+\ell\eta)}d\xi d\eta\right) = (1-\mu)I_\beta(\lambda).$$

The last equality is due to the symmetry of φ_β:

$$\varphi_\beta(\xi,\eta,\lambda) = \varphi_\beta(2\pi-\xi,\eta,\lambda) = \varphi_\beta(\xi,2\pi-\eta,\lambda), \qquad \forall(\xi,\eta)\in\mathbb{R}^2, \quad \lambda\in\mathbb{R}_+\setminus\left(\Sigma^*\cup\Sigma(\beta)\right). \tag{5.2.84}$$

Conversely, suppose that $\lambda\in\mathbb{R}_+\setminus\Sigma^*$, $\lambda^2\notin\sigma(A(\beta))$ and the relation (5.2.82) is satisfied. Then, $\lambda\notin\Sigma(\beta)$ and one can define $\{\mathbf{u}_{k,\ell}\}_{(k,\ell)\in\mathbb{Z}^2}$ by the relation (5.2.81). Since $\varphi_\beta(\xi,\eta,\lambda)\neq 1$, $\forall(\xi,\eta)\in[0,\pi]^2$, one has $\{\mathbf{u}_{k,\ell}\}_{(k,\ell)\in\mathbb{Z}^2}\in\ell_2(\mathbb{Z}^2)$ and the relations (5.2.62)–(5.2.66) (or (5.2.70)–(5.2.74) for $\lambda=0$) define an eigenfunction of the operator $A^\mu(\beta)$ corresponding to the eigenvalue λ^2. □

5.2.2.4 Existence of eigenvalues

We will now study the number of eigenvalues in the gaps of the operator $A^\mu(\beta)$. With the classification of gaps introduced in Proposition 5.2.6 we can prove the following assertion.

Theorem 5.2.1. *For any $0 < \mu < 1$, $\beta \in [0, \pi]$ there exist at least two simple eigenvalues of the operator $A^\mu(\beta)$ in each gap of type I and at least one simple eigenvalue in each gap of type II. These eigenvalues are characterized as follows:*

$$\lambda^2 \in \sigma_d \left(A^\mu(\beta)\right) \quad \Leftrightarrow \quad \lambda \textit{ is a solution of } (5.2.82).$$

For $\mu \geqslant 1$ the operator $A^\mu(\beta)$ has no eigenvalues.

Remark 5.2.5. Since in our terminology the interval $]0, \min \sigma(A(\beta))[$ for $\beta \in]0, \pi]$ is called a gap of type II (cf. Remark 5.2.4), Theorem 5.2.1 implies that the operator $A^\mu(\beta)$ has one simpleeigenvalue below the essential spectrum for $\beta \in]0, \pi]$, $0 < \mu < 1$.

The rest of the section is devoted to the proof of Theorem 5.2.1. The following auxiliary assertion shows that for $\mu \geqslant 1$ the operator $A^\mu(\beta)$ has no discrete spectrum.

Lemma 5.2.5. *For any $\beta \in [0, \pi]$, if $\lambda \in \mathbb{R}_+ \setminus \Sigma^*$ and $\lambda^2 \notin \sigma\left(A(\beta)\right)$, then $\varphi_\beta(\xi, \eta, \lambda) < 1$, $\forall (\xi, \eta) \in [0, \pi]^2$.*

Proof. If $\lambda^2 \notin \sigma(A(\beta))$, then $\phi_{L_z,\beta}(\lambda) \in \mathbb{R} \setminus \left[f^-_{L_x,L_y}(\lambda), f^+_{L_x,L_y}(\lambda)\right]$. Taking into account that $f^-_{L_x,L_y}(\lambda) < 0$, $f^+_{L_x,L_y}(\lambda) > 0$, we get the result for $\lambda > 0$. Finally, for $\lambda = 0$, $\beta \in]0, \pi]$, it follows from (5.2.80) that $\varphi_\beta(\xi, \eta, \lambda) \leqslant 0$, $\forall (\xi, \eta) \in [0, \pi]^2$. □

Corollary 5.2.2. *For any $\beta \in [0, \pi]$, $\mu \geqslant 1$, the operator $A^\mu(\beta)$ has no eigenvalues of finite multiplicity.*

Proof. As follows from Lemma 5.2.5, if $\lambda^2 \in \sigma_d(A^\mu(\beta))$, then $I_\beta(\lambda) > 0$. The relation (5.2.82) implies that $\mu < 1$. □

Remark 5.2.6. Lemma 5.2.5 permits to rewrite the relation (5.2.82) as

$$\mu = 1 - F_\beta(\lambda), \tag{5.2.85}$$

with

$$F_\beta(\lambda) = I_\beta(\lambda)^{-1}.$$

We will now study the behaviour of the function F_β inside the gaps of the operator $A^\mu(\beta)$ in order to determine the possible number of eigenvalues of the operator $A^\mu(\beta)$ in each gap. As follows from Lemma 5.2.5, the function F_β is continuous inside the gaps except at the points of discontinuity of the function $\phi_{L_z,\beta}$. In Lemmas 5.2.6 and 5.2.7 we describe the behaviour of F_β near the ends of the gap and in Lemma 5.2.8 we study its behaviour near the points of discontinuity of the function ϕ_{β,L_z}.

Lemma 5.2.6. *If $\lambda_+ \in \mathbb{R}_+ \setminus \left(\Sigma_x \cup \Sigma_y \cup \widetilde{\Sigma}_z(\beta)\right)$ and $\phi_{L_z,\beta}(\lambda_+) = f^+_{L_x,L_y}(\lambda_+)$, then $F_\beta(\lambda) \underset{\lambda \to \lambda_+ - 0}{\longrightarrow} 0$. Similarly, if $\lambda_- \in \mathbb{R}_+ \setminus \left(\Sigma_x \cup \Sigma_y \cup \widetilde{\Sigma}_z(\beta)\right)$ and $\phi_{L_z,\beta}(\lambda_-) = f^-_{L_x,L_y}(\lambda_+)$, then $F_\beta(\lambda) \underset{\lambda \to \lambda_- + 0}{\longrightarrow} 0$.*

Proof. We will prove the statement for λ_+ only, the proof for λ_- being analogous. Let ξ_0, η_0 be the values of ξ, η such that $f_{L_x,L_y,\xi_0,\eta_0}(\lambda_+) = f^+_{L_x,L_y}(\lambda_+)$ (obviously, $\xi_0 \in \{0,\pi\}$ and $\eta_0 \in \{0,\pi\}$). Then, due to the symmetry of the function φ_β (cf. (5.2.84)), one has

$$I_\beta(\lambda) = \frac{1}{4\pi^2} \iint\limits_{\substack{|\xi-\xi_0|\leqslant\pi \\ |\eta-\eta_0|\leqslant\pi}} \frac{1}{1-\varphi_\beta(\xi,\eta,\lambda)} d\xi d\eta \geqslant \frac{1}{4\pi^2} \iint\limits_{\sqrt{\xi'^2+\eta'^2}\leqslant\pi} \frac{1}{1-\varphi_\beta(\xi_0+\xi',\eta_0+\eta',\lambda)} d\xi' d\eta'.$$

In the last inequality we made a change of variables $\xi = \xi_0 + \xi'$, $\eta = \eta_0 + \eta'$. We have then

$$I_\beta(\lambda) \geqslant \frac{1}{4\pi^2} I_{\beta,r}(\lambda), \qquad \forall r < \pi, \qquad \lambda \notin \Sigma, \quad \lambda^2 \notin \sigma\left(A(\beta)\right), \tag{5.2.86}$$

where

$$I_{\beta,r}(\lambda) = \iint\limits_{D_r} \frac{1}{1-\varphi_\beta(\xi_0+\xi,\eta_0+\eta,\lambda)} d\xi d\eta,$$

and $D_r = \left\{(\xi,\eta) \,/\, \sqrt{\xi^2+\eta^2} \leqslant r\right\}$. It follows from the assumptions of the lemma that λ_+^2 is the upper end of a gap (it can also be the bottom of the essential spectrum (cf. Lemma 5.2.2 and Remark 5.2.4). Thus, we can find $\delta > 0$ such that $\lambda^2 \notin \sigma_{ess}\left(A^\mu(\beta)\right)$, $\forall\lambda \in]\lambda_+ - \delta, \lambda_+[$. From now on we suppose that $\lambda^2 \notin \sigma_{ess}\left(A^\mu(\beta)\right)$ and we study the behaviour of the integral $I_{\beta,r}(\lambda)$ as $\lambda \to \lambda_+ - 0$. Notice that the function φ_β is smooth in some neighbourhood U of the point $(\xi_0, \eta_0, \lambda_+)$. Indeed, by assumption $\lambda_+ \notin \Sigma_x \cup \Sigma_y \cup \widetilde{\Sigma}_z(\beta)$ which guarantees the continuity of the functions $f_{L_x,L_y,\xi,\eta}$ and $\phi_{L_z,\beta}$. Moreover, $\phi_{L_z,\beta}(\lambda_+) \neq 0$ since $f^+_{L_x,L_y}(\lambda_+) > 0$. Let ρ and δ be positive numbers such that $\varphi_\beta \in C^\infty(\overline{U}_{\rho,\delta})$, where

$$U_{\rho,\delta} = \left\{(\xi-\xi_0)^2 + (\eta-\eta_0)^2 < \rho^2\right\} \times]\lambda_+ - \delta, \lambda_+].$$

Let us write down the Taylor series of the function φ_β in $U_{\rho,\delta}$:

$$\begin{aligned} &\varphi_\beta(\xi,\eta,\lambda) = 1 + \partial_\lambda\varphi_\beta(\xi_0,\eta_0,\lambda_+)(\lambda-\lambda_+) + \partial^2_{\xi\xi}\varphi_\beta(\xi_0,\eta_0,\lambda_+)(\xi-\xi_0)^2 \\ &+ \partial^2_{\eta\eta}\varphi_\beta(\xi_0,\eta_0,\lambda_+)(\eta-\eta_0)^2 + O\left((\lambda-\lambda_+)^2\right) + O\left((\xi-\xi_0)^4\right) + O\left((\eta-\eta_0)^4\right). \end{aligned} \tag{5.2.87}$$

We took into account the relations

$$\varphi_\beta(\xi_0+\xi,\eta,\lambda) = \varphi_\beta(\xi_0-\xi,\eta,\lambda), \qquad \varphi_\beta(\xi,\eta_0+\eta,\lambda) = \varphi_\beta(\xi,\eta_0-\eta,\lambda),$$

which lead to the absence of the terms with odd powers of $(\xi-\xi_0)$, $(\eta-\eta_0)$ in (5.2.87). It follows from (5.2.87) that there exists a constant $C_{\rho,\delta} > 0$ such that

$$\varphi_\beta(\xi,\eta,\lambda) \geqslant 1 - C_{\rho,\delta}\left(|\lambda-\lambda_+| + (\xi-\xi_0)^2 + (\eta-\eta_0)^2\right), \qquad \forall(\xi,\eta,\lambda) \in U_{\rho,\delta}.$$

Consequently,

$$I_{\beta,\rho}(\lambda) \geqslant \iint\limits_{D_\rho} \frac{C^{-1}_{\rho,\delta}}{|\lambda-\lambda_+| + \xi^2 + \eta^2} d\xi d\eta = \pi C^{-1}_{\rho,\delta} \ln\left(1 + \frac{\rho^2}{|\lambda-\lambda_+|}\right),$$

which shows that $I_{\beta,\rho}(\lambda) \underset{\lambda\to\lambda_+-0}{\longrightarrow} +\infty$. Taking into account (5.2.86) and the definition of the function F_β we conclude that $F_\beta(\lambda) \underset{\lambda\to\lambda_+-0}{\longrightarrow} 0$. □

Lemma 5.2.7. *If* $\lambda_- \in \left(\Sigma_x^* \cup \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta)$ *and* $\phi_{L_z,\beta}(\lambda_-) > f^+_{L_x,L_y}(\lambda_-)$ *or* $\lambda_- \in (\Sigma_x \cup \Sigma_y) \cap \widetilde{\Sigma}_z(\beta)$, *then* $\lim\limits_{\lambda\to\lambda_-+0} F_\beta(\lambda) \geqslant 1$. *Similarly, if* $\lambda_+ \in \left(\Sigma_x^* \cup \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta)$ *and* $\phi_{L_z,\beta}(\lambda_+) < f^-_{L_x,L_y}(\lambda_+)$ *or* $\lambda_+ \in \left(\Sigma_x^* \cup \Sigma_y^*\right) \cap \widetilde{\Sigma}_z^*(\beta)$, *then* $\lim\limits_{\lambda\to\lambda_+-0} F_\beta(\lambda) \geqslant 1$.

Proof. We give the proof for the case of λ_- only, the proof for λ_+ being similar. We will suppose that $\lambda_- \in \Sigma_x$ (the case $\lambda_- \in \Sigma_y$ can be considered analogously). It follows from Lemma 5.2.2 that λ_- is the lower end of a gap (or $\lambda_- = 0$ and 0 is not the bottom of the spectrum of $A(\beta)$). Thus, there exists $\delta > 0$ such that $\lambda^2 \notin \sigma(A(\beta))$, $\forall \lambda \in]\lambda_-, \lambda_- + \delta[$.

(i) Consider first the case $\lambda_- \in \left(\Sigma_x^* \cup \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta)$, $\phi_{L_z,\beta}(\lambda_-) > f^+_{L_x,L_y}(\lambda_-) \geqslant 0$. Let us study the behaviour of $I(\lambda)$ as $\lambda \to \lambda_- + 0$. Since $\varphi_\beta(\xi,\eta,\lambda) \underset{\lambda\to\lambda_-+0}{\longrightarrow} -\infty$ for any $\xi \in [0,\pi]$ such that $\cos\xi \neq \cos(\lambda_- L_x)$ and any $\eta \in [0,\pi]$, we have

$$\frac{1}{1-\varphi_\beta(\xi,\eta,\lambda)} \underset{\lambda\to\lambda_-+0}{\longrightarrow} 0^+, \qquad (\xi,\eta) \in [0,\pi]^2 \setminus \{(\xi,\eta)\,/\cos\xi = \cos(\lambda_- L_x)\}.$$

We can then apply the dominated convergence theorem. Indeed, there exists $\delta > 0$ such that $\phi_{L_z,\beta}(\lambda) > f^+_{L_x,L_y}(\lambda)$ for any $\lambda \in [\lambda_-, \lambda_- + \delta]$. Then,

$$\varphi_\beta(\xi,\eta,\lambda) < \frac{f^+_{L_x,L_y}(\lambda)}{\phi_{L_z,\beta}(\lambda)} \leqslant \frac{f^+_{L_x,L_y}(\lambda_-+\delta)}{\phi_{L_z,\beta}(\lambda_-+\delta)} < 1, \qquad (\xi,\eta,\lambda) \in [0,\pi]^2 \times [\lambda_-, \lambda_-+\delta]. \tag{5.2.88}$$

Here we used the monotonicity and positivity of the functions $f^+_{L_x,L_y}$ and $\phi_{L_z,\beta}$ in $[\lambda_-, \lambda_- + \delta]$. Putting

$$C = \frac{f^+_{L_x,L_y}(\lambda_-+\delta)}{\phi_{L_z,\beta}(\lambda_-+\delta)} < 1,$$

we get

$$\frac{1}{1-\varphi_\beta(\xi,\eta,\lambda)} \leqslant \frac{1}{1-C}, \qquad (\xi,\eta,\lambda) \in [0,\pi]^2 \times [\lambda_-, \lambda_-+\delta].$$

Consequently, the dominated convergence theorem applies and $I_\beta(\lambda) \underset{\lambda\to\lambda_-+0}{\longrightarrow} +0^+$, which implies $F_\beta(\lambda) \underset{\lambda\to\lambda_-+0}{\longrightarrow} +\infty$.

(ii) Suppose now that $\lambda_- \in (\Sigma_x \cup \Sigma_y) \cap \widetilde{\Sigma}_z(\beta)$. Then, if $\lambda_- \in \Sigma_x \setminus \Sigma_y$, we obtain

$$\varphi_\beta(\xi,\eta,\lambda) \underset{\lambda\to\lambda_-+0}{\longrightarrow} \frac{L_z\left(\frac{\cos\xi}{\cos(\lambda_- L_x)} - 1\right)}{L_x\left(1 - \frac{\cos\beta}{\cos(\lambda_- L_z)}\right)}, \qquad (\xi,\eta) \in [0,\pi]^2.$$

Using the relations $|\cos(\lambda_- L_x)| = 1$, $|\cos(\lambda_- L_z)| = 1$ and $\cos\beta \neq \cos(\lambda_- L_z)$ (since $\lambda_- \in \widetilde{\Sigma}_z(\beta)$), we get

$$\frac{\cos\xi}{\cos(\lambda_- L_x)} - 1 \leqslant 0, \qquad 1 - \frac{\cos\beta}{\cos(\lambda_- L_z)} > 0.$$

Consequently, $\lim\limits_{\lambda\to\lambda_-+0} \varphi_\beta(\xi,\eta,\lambda) \leqslant 0$ for any $(\xi,\eta) \in [0,\pi]^2$.

Similarly, if $\lambda_- \in \Sigma_x \cap \Sigma_y$, we obtain

$$\varphi_\beta(\xi,\eta,\lambda) \underset{\lambda\to\lambda_-+0}{\longrightarrow} \frac{L_z\left(\frac{\cos\xi}{\cos(\lambda_- L_x)}-1\right)}{L_x\left(1-\frac{\cos\beta}{\cos(\lambda_- L_z)}\right)} + \frac{L_z\left(\frac{\cos\eta}{\cos(\lambda_- L_x)}-1\right)}{L_y\left(1-\frac{\cos\beta}{\cos(\lambda_- L_z)}\right)} \leqslant 0 \qquad (\xi,\eta)\in[0,\pi]^2.$$

The dominated convergence theorem can be applied again since the estimate (5.2.88) is still valid. We have $\lim\limits_{\lambda\to\lambda_-+0} I_\beta(\lambda) \leqslant 1$, which implies $\lim\limits_{\lambda\to\lambda_-+0} F_\beta(\lambda) \geqslant 1$.

□

Lemma 5.2.8. *If $\lambda_0 \in \widetilde{\Sigma}_z(\beta) \setminus (\Sigma_x \cup \Sigma_y)$, then* $\lim\limits_{\lambda\to\lambda_0} F_\beta(\lambda) = 1$.

Proof. This is obvious since $\varphi_\beta(\xi,\eta,\lambda) \underset{\lambda\to\lambda_0}{\longrightarrow} 0$ uniformly for $(\xi,\eta)\in[0,\pi]^2$. □

We can now finish the proof of Theorem 5.2.1. We will use the classification of gaps given in Lemma 5.2.3, according to which the image of each gap of type I by the function $x \mapsto \sqrt{x}$ contains exactly one point of the set $\widetilde{\Sigma}_z(\beta)$ (point of discontinuity of the function $\phi_{L_z,\beta}$), whereas the image of each gap of type II does not contain such points.

Proof of Theorem 5.2.1. The result for $\mu \geqslant 1$ is given in Corollary 5.2.2.

(a) Let $]a,b[$ be a gap of type I. Then, according to Proposition 5.2.6, $\sqrt{a}$ and $\sqrt{b}$ satisfy the hypothesis of Lemma 5.2.6 and $\lim\limits_{\lambda\to\sqrt{a}^+}(1-F_\beta(\lambda)) = \lim\limits_{\lambda\to\sqrt{b}^-}(1-F_\beta(\lambda)) = 1$. Next, it follows from Lemma 5.2.3 that there exists $c\in]a,b[$ such that $\sqrt{c}$ satisfies the hypothesis of Lemma 5.2.8, which implies $\lim\limits_{\lambda\to\sqrt{c}}(1-F_\beta(\lambda)) = 0$. Moreover, the point $\sqrt{c}$ being the unique point of discontinuity of the function $\phi_{L_z,\beta}$ in $]\sqrt{a},\sqrt{b}[$, the function F_β is continuous in $]\sqrt{a},\sqrt{c}[$ and $]\sqrt{c},\sqrt{b}[$. Consequently, for $0<\mu<1$ there exists at least one solution of the equation (5.2.85) in each of the intervals $]\sqrt{a},\sqrt{c}[$ and $]\sqrt{c},\sqrt{b}[$, which gives at least two eigenvalues of the operator $A^\mu(\beta)$ in the gap $]a,b[$.

(b) Let $]a,b[$ be a gap of type II. In this case, it follows from Proposition 5.2.6 that one of the points $\sqrt{a}$, $\sqrt{b}$ satisfies the hypothesis of Lemma 5.2.6 whereas the other one satisfies the hypothesis of Lemma 5.2.7. Suppose that $\sqrt{b}$ satisfies the hypothesis of Lemma 5.2.6 and $\sqrt{a}$ satisfies the hypothesis of Lemma 5.2.7 (the opposite case can be considered similarly). Then, $\lim\limits_{\lambda\to\sqrt{b}^-}(1-F_\beta(\lambda)) = 1$ and $\lim\limits_{\lambda\to\sqrt{a}^+}(1-F_\beta(\lambda)) \leqslant 0$. On the other hand, Lemma 5.2.3 implies that the function F_β is continuous in $]\sqrt{a},\sqrt{b}[$. This proves that for any $0<\mu<1$ there exists at least one solution of the equation (5.2.85) in $]\sqrt{a},\sqrt{b}[$, which gives at least one eigenvalue of the operator $A^\mu(\beta)$ in the gap $]a,b[$.

Finally, the eigenvalues of the operator $A^\mu(\beta)$ are simple since the corresponding eigenfunctions satisfy the relation (5.2.81), which defines an eigenfunction corresponding to a given eigenvalue in a unique way. □

Remark 5.2.7. Similarly to the 2D case a characterization of the essential spectrum of the operator $A^\mu(\beta)$ can be given in terms of the absolute value of the function g_β. Indeed, comparing the equation (5.2.34) with the definition (5.2.69) of the function g_β we conclude that

$$\text{for} \quad \lambda \notin \Sigma, \qquad \lambda^2 \in \sigma_{ess}\left(A^\mu(\beta)\right) \quad \Leftrightarrow \quad |g_\beta| \leqslant \frac{1}{|\sin(\lambda L_x)|} + \frac{1}{|\sin(\lambda L_y)|}.$$

Notice that if we consider the non-perturbed case $\mu = 1$, the equation for the Fourier transform $\widehat{\mathbf{u}}$ takes the form

$$\left(\frac{\cos\xi}{\sin(\lambda L_x)} + \frac{\cos\eta}{\sin(\lambda L_y)} - g_\beta(\lambda)\right)\widehat{\mathbf{u}}(\xi,\eta) = 0.$$

Consequently, for $\lambda \notin \Sigma$, there exists a non-zero solution if and only if

$$\exists(\xi,\eta) \in [0,\pi]^2 \quad \text{s.t.} \quad \frac{\cos\xi}{\sin(\lambda L_x)} + \frac{\cos\eta}{\sin(\lambda L_y)} - g_\beta(\lambda) = 0,$$

i.e., if and only if $\lambda^2 \in \sigma\left(A(\beta)\right)$. This means that for $\lambda \notin \Sigma$ there exists a generalized eigenfunction of the operator $A(\beta)$ (a solution of the equation $A(\beta)u = \lambda^2 u$ of at most polynomial growth) if and only if $\lambda^2 \in \sigma\left(A(\beta)\right)$.

5.3 The operator $A_\varepsilon^\mu(\beta)$

As in the 2D case, the results of the works [61], [47], [57] can be applied to get the following assertions.

Theorem 5.3.1 (Essential spectrum). *Let $\{]a_n(\beta), b_n(\beta)[,\ n \in \mathbb{N}^*\}$ be the gaps of the operator $A(\beta)$ on the limit graph G for $\beta \in [0,\pi]$. Then, for each $n_0 \in \mathbb{N}^*$ there exists $\varepsilon_0(\beta) > 0$ such that if $\varepsilon < \varepsilon_0(\beta)$ the operator $A_\varepsilon(\beta)$ has at least n_0 gaps $\{]a_{\varepsilon,n}(\beta), b_{\varepsilon,n}(\beta)[\}_{n=1}^{n_0}$ such that*

$$a_{\varepsilon,n}(\beta) = a_n(\beta) + O(\varepsilon), \quad b_{\varepsilon,n}(\beta) = b_n(\beta) + O(\varepsilon), \qquad \varepsilon \to 0, \qquad 1 \leqslant n \leqslant n_0.$$

Theorem 5.3.2 (Discrete spectrum). *Let $]a(\beta), b(\beta)[$ be a gap of the operator $A^\mu(\beta)$ on the limit graph G for $\beta \in [0,\pi]$ and $\lambda^{(0)}(\beta) \in\ ]a(\beta), b(\beta)[$ a (simple) eigenvalue of this operator. Then there exists $\varepsilon_0(\beta) > 0$ such that if $\varepsilon < \varepsilon_0(\beta)$ the operator $A_\varepsilon^\mu(\beta)$ has an eigenvalue $\lambda_\varepsilon(\beta)$ inside a gap $]a_\varepsilon(\beta), b_\varepsilon(\beta)[$ such that:*

$$\lambda_\varepsilon(\beta) = \lambda^{(0)}(\beta) + O(\varepsilon), \qquad \varepsilon \to 0.$$

We do not construct the full asymptotic expansion of the eigenvalue here, which should be possible to do using the approach described in Chapter 2. The weak version of Theorem 5.3.2 with a suboptimal rate of convergence should also be easy to obtain by constructing a "naive" pseudo-mode and adapting the argument used in Section 1.4.3 to the 3D geometry.

APPENDIX A

A.1 Self-adjointness of the operator A^{μ}

Proof of Proposition 1.3.1. Let us show that the operator A^{μ} defined in (1.3.4)–(1.3.5) is self-adjoint. Let us show first that it is symmetric. Indeed, if $f, g \in D(A^{\mu})$ then

$$(A^{\mu}f, g) = \sum_{e\in E} \int_e -w^{\mu}(e) f'' \overline{g} d\mathbf{x} = \sum_{e\in E} \left(\int_e -w^{\mu}(e) f \overline{g}'' d\mathbf{x} - w^{\mu}(e)(f'\overline{g})|_{v_{e1}}^{v_{e2}} + w^{\mu}(e)(f\overline{g}')|_{v_{e1}}^{v_{e2}} \right)$$

$$= (f, A^{\mu}g) + \sum_{v\in V} \left(\sum_{e_v\in E_v} (w^{\mu}(e_v)(f_{e_v})'_{ext}(v)\overline{g}(v) - f(v) w^{\mu}(e_v)(\overline{g_{e_v}})'_{ext}(v)) \right) = (f, A^{\mu}g).$$

To prove that the operator A^{μ} we will show that $D\big((A^{\mu})^*\big) \subset D(A^{\mu})$. Let $f \in D\big((A^{\mu})^*\big)$. by definition there exists $h_f (= (A^{\mu})^* f) \in L_2^{\mu}(G)$ such that

$$(f, A^{\mu}g) = (h_f, g), \quad \forall g \in D(A^{\mu}), \tag{A.1.1}$$

and consequently,

$$\sum_{e\in E} \int_e -w^{\mu}(e) f \overline{g}'' d\mathbf{x} = \sum_{e\in E} \int_e w^{\mu}(e) h_f \overline{g} d\mathbf{x}, \quad \forall g \in D(A^{\mu}). \tag{A.1.2}$$

The set $D(A^{\mu})$ obviously contains the functions $C_0^{\infty}(e)$ for any e. The relation (A.1.2) applied to functions $g \in C_0^{\infty}(e)$ implies that $-f''|_e = h_f|_e$. Thus,

$$-f'' = h_f, \quad f'' \in L_2^{\mu}(G). \tag{A.1.3}$$

Hence,

$$f \in H^2(e), \quad \forall e \in E; \qquad \sum_{e\in E} \|f\|^2_{H^2(e)} < \infty; \qquad \big(A^{\mu}\big)^* f = -f''. \tag{A.1.4}$$

We have to verify the continuity of the function f at the vertices of the graph G and the Kirchhoff's conditions. Suppose that $g \in D(A^{\mu})$ vanishes outside some neighbourhood of a given vertex v_0. The relation (A.1.2) yields

$$\sum_{i=1}^{3} \int_{e^i_{v_0}} -w^{\mu}\big(e^i_{v_0}\big) f \overline{g}'' d\mathbf{x} = \sum_{i=1}^{3} \int_{e^i_{v_0}} -w^{\mu}\big(e^i_{v_0}\big) f'' \overline{g} d\mathbf{x}. \tag{A.1.5}$$

On the other hand, integrating by parts, we get:

$$\sum_{i=1}^{3}\int_{e_{v_0}^i} -w^\mu\big(e_{v_0}^i\big)f\overline{g}''d\mathbf{x} = \sum_{i=1}^{3}\int_{e_{v_0}^i} -w^\mu\big(e_{v_0}^i\big)f''\overline{g}d\mathbf{x} + \sum_{i=1}^{3} w^\mu\big(e_{v_0}^i\big)(f_{e_{v_0}^i})'_{ext}(v_0)\bar{g}(v_0)$$
$$- \sum_{i=1}^{3} w^\mu\big(e_{v_0}^i\big)f_{e_{v_0}^i}(v_0)(\overline{g_{e_{v_0}^i}})'_{ext}(v_0),$$

which implies

$$\left(\sum_{i=1}^{3} w^\mu\big(e_{v_0}^i\big)(f_{e_{v_0}^i})'_{ext}(v_0)\right)\bar{g}(v_0) - \sum_{i=1}^{3} w^\mu\big(e_{v_0}^i\big)f_{e_{v_0}^i}(v_0)(\overline{g_{e_{v_0}^i}})'_{ext}(v_0) = 0. \tag{A.1.6}$$

Choosing a function g such that it is equal to a non-zero constant in a neighbourhood of the vertex v_0, we get

$$\sum_{i=1}^{3} w^\mu\big(e_{v_0}^i\big)(f_{e_{v_0}^i})'_{ext}(v_0) = 0. \tag{A.1.7}$$

Let us now choose a function $g \in D(A^\mu)$ such that

$$g(v_0) = 0, \quad (g_{e_{v_0}^i})'_{ext}(v_0) = \frac{1}{w^\mu\big(e_{v_0}^i\big)}, \quad (g_{e_{v_0}^j})'_{ext}(v_0) = -\frac{1}{w^\mu\big(e_{v_0}^j\big)}, \quad (g_{e_{v_0}^k})'_{ext}(v_0) = 0,$$

for some permutation $\{i,j,k\} = \{1,2,3\}$. From the relation (A.1.6) applied to the function g we obtain

$$f_{e_{v_0}^i}(v_0) = f_{e_{v_0}^j}(v_0). \tag{A.1.8}$$

This proves the continuity of the function f at the vertex v_0. Thus, we conclude that $D\big((A^\mu)^*\big) \subset D(A^\mu)$ and consequently, $A^\mu = (A^\mu)^*$. □

A.2 Quasi-modes method

In this section we shall prove Lemma A.2.1 that provides a result existence of eigenvalues for the operator $A^\mu_{\varepsilon,s}$ ($A^\mu_{\varepsilon,as}$). It relies on a pseudo-mode method. The result may be found in [53] Lemma 4, but, for the sake of completeness we give a proof.

Let H be a Hilbert space, A a self-adjoint positive definite operator:

$$\exists\alpha > 0: \quad (Au,u) \geqslant \alpha\|u\|^2, \qquad \forall u \in D(A).$$

Let a be the closed positive definite sesqui-linear form which corresponds to the operator A:

$$D[a] = D(A^{1/2}), \qquad a[u,u] \geqslant \alpha\|u\|^2, \qquad \forall u \in D[a].$$

We denote by $|\cdot|_a$ the norm in the space $D[a]$ corresponding to the scalar product

$$\langle u,v\rangle_a = a[u,v], \qquad \forall u,v \in D[a].$$

Lemma A.2.1. *Suppose that $\lambda > \alpha$. If there exists $u \in D[a]$ such that*

$$|a[u,v] - \lambda(u,v)| \leqslant \varepsilon |u|_a |v|_a, \qquad \forall v \in D[a], \qquad \varepsilon < (\lambda+1)^{-1}, \tag{A.2.1}$$

then

$$dist(\sigma(A), \lambda) \leqslant C\varepsilon, \qquad C = \lambda + 1.$$

Proof. Let us define the operator $B : D[a] \to D[a]$ as follows:

$$a[Bf, v] = (f, v), \qquad \forall v \in D[a].$$

We notice that this implies

$$Bf \in D(A), \qquad ABf = f, \qquad \forall f \in D[a]. \tag{A.2.2}$$

We denote by $|\cdot|_a$ the norm associated with the sesqui-linear form a: $|f|_a^2 = a[f,f]$.
Let us prove the following assertion.

Proposition A.2.1.

$$\lambda \in \sigma(A) \quad <=> \quad \frac{1}{\lambda} \in \sigma(B), \qquad \forall \lambda > 0.$$

Proof.

Case of eigenvalues:

1. Let λ be an eigenvalue of the operator A: $Af = \lambda f$. On a

$$(f,v) = \lambda^{-1}(Af, v) = a[\lambda^{-1} f, v], \qquad \forall v \in D[a].$$

By definition of the operator B this implies $Bf = \lambda^{-1} f$.
2. Let λ^{-1} be an eigenvalue of the operator B: there exists $f \in D[a]$ such that $Bf = \lambda^{-1} f$. From (A.2.2) it follows that $f \in D(A)$ and $Af = \lambda ABf = \lambda f$.

Case of continuous spectrum:

1. Suppose that $\lambda \in \sigma_c(A)$. Then, there exist a singular sequence $\{u_n\}_{n\in\mathbb{N}} \subset D(A)$ such that

a) $\inf\limits_n \|u_n\| > 0$,

b) $u_n \xrightarrow{w} 0$ in H,

c) $\|(A-\lambda)u_n\| \longrightarrow 0$.

Let us show that $\{u_n\}$ is also singular sequence for the operator B in the space $D[a]$ equipped with the norm $|\cdot|_a$. Thus, the property

a) $\inf\limits_n |u_n|_a > 0$

is obviously verified. Let us prove the weak convergence of the sequence $\{u_n\}$ in the space $D[a]$:

$$a[u_n, v] = (Au_n, v) = ((A - \lambda)u_n, v) + \lambda(u_n, v) \longrightarrow 0, \qquad \forall v \in D[a].$$

We used the properties (b), (c) of the sequence $\{u_n\}$. Consequently,

b) $u_n \overset{w}{\longrightarrow} 0$ in $D[a]$.

The only thing to show now is that $|(B - \lambda^{-1})u_n|_a \longrightarrow 0$. Indeed,

$$\begin{aligned}\big|Bu_n - \lambda^{-1}u_n\big|_a &= a[(B - \lambda^{-1})u_n, (B - \lambda^{-1})u_n] = (A(B - \lambda^{-1})u_n, (B - \lambda^{-1})u_n) \\ &= \lambda^{-1}(\lambda u_n - Au_n, (B - \lambda^{-1})u_n) \leqslant C\|Au_n - \lambda u_n\| \longrightarrow 0.\end{aligned}$$

Thus,

c) $|(B - \lambda^{-1})u_n|_a \longrightarrow 0$,

and $\{u_n\}$ is a singular sequence for the operator B at the point λ^{-1}.

2. Suppose that $\lambda^{-1} \in \sigma_c(B)$. Then, there exists a singular sequence $\{f_n\}_{n\in\mathbb{N}} \subset D[a]$ such that

a) $\inf\limits_n |f_n|_a > 0$,

b) $f_n \overset{w}{\longrightarrow} 0$ in $D[a]$,

c) $|(B - \lambda^{-1})f_n|_a \longrightarrow 0$.

Let us show that there exists a singular sequence for the operator A. We put $u_n = Bf_n \in D(A)$. From the properties (a) and (c) of the sequence $\{f_n\}$ it follows that $\inf\limits_n |u_n|_a > 0$. We have:

$$|u_n|_a^2 = (A^{1/2}Bf_n, A^{1/2}u_n) = (ABf_n, u_n) = (f_n, u_n).$$

Therefore,

$$\inf_n \|u_n\| \geqslant \inf_n \frac{|u_n|_a^2}{\|f_n\|} > 0.$$

Thus, the property

a) $\inf\limits_n \|u_n\| > 0$

is verified. The sequence $\{u_n\}$ being bounded, we can extract a subsequence which converges weakly to some element $h \in H$. We keep the same notation $\{u_n\}$ for the subsequence:

$$(u_n, w) \longrightarrow (h, w), \quad \forall w \in H. \tag{A.2.3}$$

For $w \in D[a]$ we have:

$$(u_n, w) = a[Bf_n, w] \longrightarrow 0, \qquad \forall w \in D[a], \tag{A.2.4}$$

where we used the property (b) of the sequence $\{f_n\}$. Thus, the relations (A.2.3) and (A.2.4) imply that

$$(h, w) = 0, \qquad \forall w \in D[a].$$

Since $D[a]$ is dense in H we conclude that $h = 0$ and hence

b) $u_n \xrightarrow{w} 0$ in H.

The last thing to verify is the property (c). We have:

$$\|(A-\lambda)u_n\| = \|f_n - \lambda B f_n\| \leqslant \alpha^{-1/2}|f_n - \lambda B f_n|_a \longrightarrow 0.$$

The property

c) $\|(A-\lambda)u_n\| \longrightarrow 0$

is verified which proves that $\{u_n\}$ is a singular sequence for the operator A at the point λ. □

We can now finish the proof of Lemme A.2.1. Suppose that $u \in D[a]$ is such that (A.2.1) is verified. We have:

$$|a[u,v]-\lambda(u,v)| = |a[u,v]-\lambda(ABu,v)| = |a[u,v]-\lambda[Bu,v]| = |a[u-\lambda Bu,v]| \leqslant \varepsilon|u|_a|v|_a,$$

and hence,

$$\left|(B-\lambda^{-1})u\right|_a \leqslant \frac{\varepsilon}{\lambda}|u|_a.$$

The last relation implies that

$$dist(\sigma(B),\lambda^{-1}) \leqslant \frac{\varepsilon}{\lambda},$$

and for the operator A we obtain

$$dist(\sigma(A),\lambda) \leqslant \frac{\lambda\varepsilon}{1-\varepsilon} < (\lambda+1)\varepsilon,$$

where we took into account the relation $\varepsilon(1+\lambda) < 1$. □

A.3 Some auxiliary assertions

The following assertion is very similar to Lemma 3.10 in [37].

Lemma A.3.1. *For each $\alpha > 0$ there exists $\varepsilon_0 > 0$ and a constants $C_1(\alpha)$, $C_2(\alpha)$ which does not depend on ε such that*

$$\|v\|_{L_1(]0,\varepsilon^\alpha[\times]0,\varepsilon[)} \leqslant C_1(\alpha)\varepsilon^{\alpha+1/2}\|v\|_{H^1(]0,1[\times]0,\varepsilon[)}, \qquad \forall v \in H^1(]0,1[\times]0,\varepsilon[), \quad \forall\varepsilon < \varepsilon_0, \tag{A.3.1}$$

$$\|v\|_{L_2(]0,\varepsilon^\alpha[\times]0,\varepsilon[)} \leqslant C_2(\alpha)\varepsilon^{\alpha/2}\|v\|_{H^1(]0,1[\times]0,\varepsilon[)}, \qquad \forall v \in H^1(]0,1[\times]0,\varepsilon[), \quad \forall\varepsilon < \varepsilon_0. \tag{A.3.2}$$

Proof. We will prove the estimate (A.3.1), the proof of the estimate (A.3.2) being analogous. Due to the density of $C^\infty([0,1]\times[0,\varepsilon])$ in $H^1(]0,1[\times]0,\varepsilon[)$ it is sufficient to show (A.3.1) for functions $v \in C^\infty([0,1]\times[0,\varepsilon])$. Let us introduce a function $\psi_\alpha \in C^\infty([0,1]\times[0,\varepsilon])$ such that

$$\psi_\alpha(x,y) = 1, \quad (x,y) \in [0,\varepsilon^\alpha]\times[0,\varepsilon], \qquad \psi_\alpha(x,y) = 0, \quad x \in \left[\tfrac{1}{2},1\right]\times[0,\varepsilon].$$

Then, $\|v\|_{L_1(]0,\varepsilon^\alpha[\times]0,\varepsilon[)} = \|v\psi_\alpha\|_{L_1(]0,\varepsilon^\alpha[\times]0,\varepsilon[)}$. We have

$$(v\psi_\alpha)(x,y) = -\int_x^1 \partial_x (v\psi_\alpha)(x',y)dx'.$$

Hence,

$$|(v\psi_\alpha)(x,y)| \leqslant \int_0^1 |\nabla (v\psi_\alpha)(x',y)|\, dx'.$$

Consequently,

$$\begin{aligned}\|v\|_{L_1(]0,\varepsilon^\alpha[\times]0,\varepsilon[)} &\leqslant \int_0^{\varepsilon^\alpha} \left(\int_0^1 \int_0^\varepsilon |\nabla (v\psi_\alpha)(x',y)|\, dx'dy \right) dx \\ &\leqslant \varepsilon^{\alpha+1/2} \left(\int_0^1 \int_0^\varepsilon |\nabla (v\psi_\alpha)(x',y)|^2\, dx'dy \right)^{1/2} \leqslant C(\alpha)\varepsilon^{\alpha+1/2} \|v\|_{H^1(]0,1[\times]0,\varepsilon[)}.\end{aligned}$$

□

Lemma A.3.2. *Let K be the rectangle $]0,a[\times]0,b[$ for some $a,b>0$ and Γ its boundary $\{0\}\times[0,b]$. Then, for any $\delta>0$ there exists a constant $C(\delta)$ such that*

$$\|v\|^2_{L_2(\Gamma)} \leqslant \delta\|v\|^2_{H^1(K)} + C(\delta)\|v\|^2_{L_2(K)}, \qquad \forall v \in H^1(K). \tag{A.3.3}$$

Proof. Applying the same density argument as in the previous Lemma, as well as the fact that the traces of a convergent sequence in $H^1(K)$ converge in $L_2(\Gamma)$ we only need to prove (A.3.3) for functions in $C^\infty(K)$. Thus, for $v \in C^\infty(K)$ we have:

$$v(0,y) = v(x,y) - \int_0^x \partial_x v(\widetilde{x},y)d\widetilde{x}, \qquad (x,y)\in K,$$

and

$$|v(0,y)| \leqslant |v(x,y)| + \int_0^x |\partial_x v(\widetilde{x},y)|\, d\widetilde{x} \leqslant |v(x,y)| + \left(x\int_0^x |\partial_x v(\widetilde{x},y)|^2\, d\widetilde{x} \right)^{1/2},$$

where we used Cauchy-Schwarz inequality. Hence,

$$|v(0,y)|^2 \leqslant C\left(|v(x,y)|^2 + x\int_0^a |\nabla v(\widetilde{x},y)|^2\, d\widetilde{x} \right)$$

Integrating the last inequality over $]0,\delta[\times]0,b[$ gives:

$$\delta\|v\|^2_{L_2(\Gamma)} \leqslant C\left(\|v\|^2_{L_2(K)} + \frac{\delta^2}{2}\|v\|^2_{H^1(K)} \right)$$

Dividing by δ yields (A.3.3). □

Lemma A.3.3. *Let $v^\pm$ be the solutions of the problems* (3.2.25) *for $\alpha^2 \notin \sigma(A_\varepsilon)$, $\varphi \in H^{1/2}(\Gamma_\varepsilon^\pm)$. Then, $v^\pm \in H^2(K^\pm)$ and the estimate* (3.2.27) *holds.*

Proof. Let us introduce a cut-off function $\chi \in C^\infty\left(\left[\frac{1}{2}, \infty\right[\right)$ such that

$$\begin{cases} \chi(x) = 1, & x \in \left[\frac{1}{2}, \frac{1}{2} + \frac{\varepsilon}{8}\right], \\ \chi(x) = 0, & x \in \left[\frac{1}{2} + \frac{\varepsilon}{4}, \infty\right[, \\ 0 \leqslant \chi(x) \leqslant 1, & \forall x \in \left[\frac{1}{2}, \infty\right[. \end{cases}$$

Then, the function $\widetilde{v} = \chi v^+$ solves the following problem in $K_\varepsilon^+ = \left]\frac{1}{2}, \frac{1}{2} + \frac{\varepsilon}{4}\right[\times \left]-\frac{L}{2}, \frac{L}{2}\right[$:

$$\begin{cases} -\Delta\widetilde{v} = \alpha^2 v^+ \chi - v^+ \Delta\chi - 2\nabla v^+ \nabla\chi + u_\varepsilon^+(\alpha, \varphi) \quad \text{in} \quad K_\varepsilon^+, \\ \widetilde{v}|_{\Gamma_\varepsilon^+} = 0, \qquad \left.\dfrac{\partial \widetilde{v}}{\partial n}\right|_{\partial K_\varepsilon^+ \setminus \Gamma_\varepsilon^+} = 0, \end{cases}$$

where we used the fact that $v^+ \in D(A_\varepsilon^+)$, cf. (3.2.2). Then, applying the regularity result ([]), we conclude that $\widetilde{v} \in H^2(K_\varepsilon^+)$ and

$$\|\widetilde{v}\|_{H^2(K_\varepsilon^+)} \leqslant C \left\|\alpha^2 v^+ \chi - v^+ \Delta\chi - 2\nabla v^+ \nabla\chi + u_\varepsilon^+(\alpha, \varphi)\right\|_{L_2(K_\varepsilon^+)} \leqslant C(\alpha)\|\varphi\|_{H^{1/2}(\Gamma_{\varepsilon,0})},$$

where $C(\alpha)$ is a continuous function depending only on the geometry of the domain. We used the continuity of the operator $\left(A_\varepsilon^+ - \alpha^2\right)^{-1} : L_2(B_\varepsilon^+) \to H^1(B_\varepsilon^+)$ to estimate $\|\widetilde{v}\|_{H^1(K_\varepsilon^+)}$ as well as (3.2.5). Finally, since $\widetilde{v}|_{K^+} = v|_{K^+}$, we get (3.2.27). □

A.4 Technical results associated with Chapter 5

Lemma A.4.1. *The condition $D(\lambda) = 0$ with $D(\lambda)$ defined in* (5.2.21) *is equivalent to the relation* (5.2.4).

Proof. Let us compute the determinant $D(\lambda)$. First, we subtract the second column from the first, the fourth from the third and the sixth from the fifth. Then, we multiply the third line by e^{-ik_x}, the fourth one by e^{-ik_y} and the fifth one by e^{-ik_z}. We get:

$$D(\lambda) = e^{ik_x} e^{ik_y} e^{i\beta}$$

$$\times \begin{vmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -2i\sin(\lambda L_x) & e^{-ik_x} - e^{-i\lambda L_x} & 0 & 0 & 0 & 0 \\ 0 & 0 & -2i\sin(\lambda L_y) & e^{-ik_y} - e^{-i\lambda L_y} & 0 & 0 \\ 0 & 0 & 0 & 0 & -2i\sin(\lambda L_z) & e^{-i\beta} - e^{-i\lambda L_z} \\ 1 - e^{i(\lambda L_x + k_x)} & 0 & 1 - e^{i(\lambda L_y + k_y)} & 0 & 1 - e^{i(\lambda L_z + \beta)} & 0 \end{vmatrix}.$$

Developing the determinant with respect to the first line, we get:

$$D(\lambda) = e^{ik_x} e^{ik_y} e^{i\beta} \left(D_1(\lambda) + D_2(\lambda)\right), \tag{A.4.1}$$

where

$$D_1(\lambda) = -\begin{vmatrix} 0 & 0 & 0 & 0 & -1 \\ -2i\sin(\lambda L_x) & 0 & 0 & 0 & 0 \\ 0 & -2i\sin(\lambda L_y) & e^{-ik_y} - e^{-i\lambda L_y} & 0 & 0 \\ 0 & 0 & 0 & -2i\sin(\lambda L_z) & e^{-i\beta} - e^{-i\lambda L_z} \\ 1 - e^{i(\lambda L_x + k_x)} & 1 - e^{i(\lambda L_y + k_y)} & 0 & 1 - e^{i(\lambda L_z + \beta)} & 0 \end{vmatrix},$$

$$D_2(\lambda) = \begin{vmatrix} 0 & 1 & 0 & 0 & -1 \\ -2i\sin(\lambda L_x) & e^{-ik_x} - e^{-i\lambda L_x} & 0 & 0 & 0 \\ 0 & 0 & -2i\sin(\lambda L_y) & 0 & 0 \\ 0 & 0 & 0 & -2i\sin(\lambda L_z) & e^{-i\beta} - e^{-i\lambda L_z} \\ 1 - e^{i(\lambda L_x + k_x)} & 0 & 1 - e^{i(\lambda L_y + k_y)} & 1 - e^{i(\lambda L_z + \beta)} & 0 \end{vmatrix}.$$

The determinant $D_1(\lambda)$ can be computed directly:

$$\begin{aligned} D_1(\lambda) &= -4\sin(\lambda L_x)\sin(\lambda L_z)\left(1 - e^{i(\lambda L_y + k_y)}\right)\left(e^{-ik_y} - e^{-i\lambda L_y}\right) \\ &= -8\sin(\lambda L_x)\sin(\lambda L_z)\left(\cos k_y - \cos(\lambda L_y)\right). \end{aligned} \tag{A.4.2}$$

To compute determinant $D_2(\lambda)$, we decompose it with respect to the first line. We get:

$$D_2(\lambda) = D_{2,1}(\lambda) + D_{2,2}(\lambda),$$

$$D_{2,1}(\lambda) = -\begin{vmatrix} -2i\sin(\lambda L_x) & 0 & 0 & 0 \\ 0 & -2i\sin(\lambda L_y) & 0 & 0 \\ 0 & 0 & -2i\sin(\lambda L_z) & e^{-i\beta} - e^{-i\lambda L_z} \\ 1 - e^{i(\lambda L_x + k_x)} & 1 - e^{i(\lambda L_y + k_y)} & 1 - e^{i(\lambda L_z + \beta)} & 0 \end{vmatrix},$$

$$D_{2,2}(\lambda) = -\begin{vmatrix} -2i\sin(\lambda L_x) & e^{-ik_x} - e^{-i\lambda L_x} & 0 & 0 \\ 0 & 0 & -2i\sin(\lambda L_y) & 0 \\ 0 & 0 & 0 & -2i\sin(\lambda L_z) \\ 1 - e^{i(\lambda L_x + k_x)} & 0 & 1 - e^{i(\lambda L_y + k_y)} & 1 - e^{i(\lambda L_z + \beta)} \end{vmatrix}.$$

We find then

$$\begin{aligned} D_{2,1}(\lambda) &= -4\sin(\lambda L_x)\sin(\lambda L_y)\left(1 - e^{i(\lambda L_z + k_z)}\right)\left(e^{-i\beta} - e^{-i\lambda L_z}\right) \\ &= -8\sin(\lambda L_x)\sin(\lambda L_y)\left(\cos\beta - \cos(\lambda L_z)\right), \end{aligned} \tag{A.4.3}$$

$$\begin{aligned} D_{2,2}(\lambda) &= -4\sin(\lambda L_y)\sin(\lambda L_y)\left(1 - e^{i(\lambda L_x + k_x)}\right)\left(e^{-ik_x} - e^{-i\lambda L_x}\right) \\ &= -8\sin(\lambda L_y)\sin(\lambda L_y)\left(\cos k_x - \cos(\lambda L_x)\right). \end{aligned} \tag{A.4.4}$$

Combining the relations (5.2.20), (A.4.1)–(A.4.4), we find the condition (5.2.4). □

Proof of Proposition 5.2.4.
The point λ^2 is an eigenvalue of infinite multiplicity of the operator $A(\beta)$ if and only if it is an eigenvalue of the operator $A_\beta(k_x, k_y)$ for any $(k_x, k_y) \in [0, \pi]^2$. It follows from Proposition 5.2.2 that $\lambda^2 \in \sigma_{pp}(A(\beta))$ if and only if $\lambda > 0$ and the equation (5.2.4) is satisfied for any $(k_x, k_y) \in [0, \pi]^2$. The first term of in the left-hand side of (5.2.4) only depends on k_x, the second term depends only on k_y and the third term is constant. Consequently, $\lambda^2 \in \sigma_{pp}(A(\beta))$ if and only if $\lambda > 0$ and all the three terms are identically zero. If $\lambda > 0$, the condition for the first term to be identically zero is $\lambda \in \Sigma_y^* \cup \Sigma_z^*$, the condition for the second term to be identically zero is $\lambda \in \Sigma_x^* \cup \Sigma_z^*$ and the condition for the third term to be identically zero is $\lambda \in \Sigma_x^* \cup \Sigma_y^* \cup \Sigma_z^*(\beta)$. Thus,

$$\lambda^2 \in \sigma_{pp}(A(\beta)) \quad \Leftrightarrow \quad \lambda \in \Pi,$$

where

$$\Pi = \left(\Sigma_y^* \cup \Sigma_z^*\right) \cap \left(\Sigma_x^* \cup \Sigma_z^*\right) \cap \left(\Sigma_x^* \cup \Sigma_y^* \cup \Sigma_z^*(\beta)\right).$$

Since $\widetilde{\Sigma}_z^*(\beta) \subset \Sigma_z^*$ (cf. (5.2.27)–(5.2.28)), we have $\Sigma_{is}(\beta) \subset \left(\Sigma_x^* \cup \Sigma_y^*\right) \cap \Sigma_z^* \subset \Pi$. Moreover, we have seen in the proof of Proposition 5.2.3 (2) that for any point $\lambda \in \Sigma_{is}(\beta)$ there exist positive numbers $l^{\pm}$ such that $]\lambda^2 - l^-, \lambda^2[\cap \sigma(A(\beta)) = \emptyset$, $]\lambda^2, \lambda^2 + l^+[\cap \sigma(A(\beta)) = \emptyset$. Consequently, the set $\Sigma_{is}(\beta)$ corresponds to isolated points of the spectrum. Let us determine $\Pi \setminus \Sigma_{is}(\beta)$. We distinguish two cases.

(i) $\beta \in]0, \pi[$: in this case $\widetilde{\Sigma}_z^*(\beta) = \Sigma_z^*$ and the set $\Pi \setminus \Sigma_{is}(\beta)$ is

$$
\begin{aligned}
\Pi \setminus \Sigma_{is}(\beta) &= \overline{\Sigma_{is}(\beta)} \cap \Pi = \overline{(\Sigma_x^* \cap \Sigma_z^*)} \cap \overline{\left(\Sigma_y^* \cap \Sigma_z^*\right)} \cap (\Sigma_x^* \cup \Sigma_z^*) \cap \left(\Sigma_y^* \cup \Sigma_z^*\right) \cap \left(\Sigma_x^* \cup \Sigma_y^* \cup \Sigma_z^*(\beta)\right) \\
&= \left((\Sigma_x^* \setminus \Sigma_z^*) \cup (\Sigma_z^* \setminus \Sigma_x^*)\right) \cap \left(\left(\Sigma_y^* \setminus \Sigma_z^*\right) \cup \left(\Sigma_z^* \setminus \Sigma_y^*\right)\right) \cap \left(\Sigma_x^* \cup \Sigma_y^* \cup \Sigma_z^*(\beta)\right) \\
&= \left(\left(\left(\Sigma_x^* \cap \Sigma_y^*\right) \setminus \Sigma_z^*\right) \cup \left(\Sigma_z^* \setminus \left(\Sigma_x^* \cup \Sigma_y^*\right)\right)\right) \cap \left(\Sigma_x^* \cup \Sigma_y^* \cup \Sigma_z^*(\beta)\right) \\
&= \left(\left(\left(\Sigma_x^* \cap \Sigma_y^*\right) \setminus \Sigma_z^*\right) \cap \left(\Sigma_x^* \cup \Sigma_y^* \cup \Sigma_z^*(\beta)\right)\right) \cup \left(\left(\Sigma_z^* \setminus \left(\Sigma_x^* \cup \Sigma_y^*\right)\right) \cap \left(\Sigma_x^* \cup \Sigma_y^* \cup \Sigma_z^*(\beta)\right)\right) \\
&= \left(\left(\Sigma_x^* \cap \Sigma_y^*\right) \setminus \Sigma_z^*\right) \cup \left(\left(\Sigma_z^* \cap \Sigma_z^*(\beta)\right) \setminus \left(\Sigma_x^* \cup \Sigma_y^*\right)\right) = \left(\Sigma_x^* \cap \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta).
\end{aligned}
$$

The last equality is due to the fact that for $\beta \in]0, \pi[$, $\Sigma_z^* \cap \Sigma_z^*(\beta) = \emptyset$. Let us show that if $\lambda_0 \in \Pi \setminus \Sigma_{is}(\beta)$, then λ_0^2 is not an isolated point of $\sigma(A(\beta))$. The equation (5.2.4) with $\lambda = \lambda_0 + \delta$ can be rewritten as

$$
\frac{\cos k_x}{\cos(\lambda_0 L_x)} = \cos(\delta L_x) + \sin(\delta L_x) \frac{\cos(\lambda L_z) - \cos\beta}{\sin(\lambda L_z)} + \frac{\sin(\delta L_x)}{\sin(\delta L_y)} \left(\cos(\delta L_y) - \frac{\cos k_y}{\cos(\lambda_0 L_y)}\right). \tag{A.4.5}
$$

If $\lambda_0 \notin \Sigma_z^*(\beta)$, then for δ small enough

$$
\cos(\delta L_x) + \sin(\delta L_x) \frac{\cos(\lambda L_z) - \cos\beta}{\sin(\lambda L_z)} = 1 + \delta L_x \frac{\cos(\lambda_0 L_z) - \cos\beta}{\sin(\lambda_0 L_z)} + O(\delta^2).
$$

The quantity $(\cos(\lambda_0 L_z) - \cos\beta) / \sin(\lambda_0 L_z)$ being different from zero (so either strictly positive or strictly negative), there exists $\varepsilon > 0$ such that either for $\delta \in]0, \varepsilon[$ or for $\delta \in]-\varepsilon, 0[$ one has

$$
\left|\cos(\delta L_x) + \sin(\delta L_x) \frac{\cos(\lambda L_z) - \cos\beta}{\sin(\lambda L_z)}\right| < 1. \tag{A.4.6}
$$

Choosing, for example, $\cos k_y = \cos(\delta L_y)\cos(\lambda_0 L_y)$, one can find a k_x such that (k_x, k_y) solve the equation (A.4.5).

If $\lambda_0 \in \Sigma_z^*(\beta)$, then $\cos(\lambda_0 L_z) = \cos\beta$, and then then for δ small enough

$$
\cos(\delta L_x) + \sin(\delta L_x) \frac{\cos(\lambda L_z) - \cos\beta}{\sin(\lambda L_z)} = 1 - \delta^2 \left(\frac{L_x^2}{2} + L_x L_z\right) + O(\delta^3).
$$

Hence, the inequality (A.4.6) holds again for δ small enough and a solution (k_x, k_y) of the equation (A.4.5) can be found as in the previous case.

(ii) $\beta \in \{0, \pi\}$: in this case $\Sigma_z^*(\beta) \subset \Sigma_z^*$. Consequently, $\Sigma_z^*(\beta) \subset \Pi$. Taking into account that $\Sigma_z^*(\beta) \cap \widetilde{\Sigma}_z^*(\beta) = \emptyset$, we get $\Sigma_z^*(\beta) \cap \Sigma_{is}(\beta) = \emptyset$. Thus,

$$
\Pi \setminus \Sigma_{is}(\beta) = (\Pi \setminus (\Sigma_{is}(\beta) \cup \Sigma_z^*(\beta))) \cup \Sigma_z^*(\beta). \tag{A.4.7}
$$

Let us compute $\Pi \setminus (\Sigma_{is}(\beta) \cup \Sigma_z^*(\beta))$:

$$
\Pi \setminus (\Sigma_{is}(\beta) \cup \Sigma_z^*(\beta)) = \overline{\Sigma_{is}(\beta)} \cap \Pi \cap \overline{\Sigma_z^*(\beta)}.
$$

It follows from (5.2.33) that

$$\Sigma_{is}(\beta) = \left(\Sigma_x^* \cap \widetilde{\Sigma}_z^*(\beta)\right) \cup \left(\Sigma_y^* \cap \widetilde{\Sigma}_z^*(\beta)\right), \qquad \overline{\Sigma_{is}(\beta)} = \left(\overline{\Sigma_x^* \cap \widetilde{\Sigma}_z^*(\beta)}\right) \cap \left(\overline{\Sigma_y^* \cap \widetilde{\Sigma}_z^*(\beta)}\right).$$

Then,

$$\begin{aligned}&\Pi \setminus (\Sigma_{is}(\beta) \cup \Sigma_z^*(\beta))\\ &= \left(\overline{\Sigma_x^* \cap \widetilde{\Sigma}_z^*(\beta)}\right) \cap \left(\overline{\Sigma_y^* \cap \widetilde{\Sigma}_z^*(\beta)}\right) \cap (\Sigma_x^* \cup \Sigma_z^*) \cap \left(\Sigma_y^* \cup \Sigma_z^*\right) \cap \left(\Sigma_x^* \cup \Sigma_y^* \cup \Sigma_z^*(\beta)\right) \cap \overline{\Sigma_z^*(\beta)}.\end{aligned}$$

If $\beta \in \{0, \pi\}$, then $\Sigma_z^* \cap \overline{\Sigma_z^*(\beta)} = \widetilde{\Sigma}_z^*(\beta)$. We have then

$$\begin{aligned}&\Pi \setminus (\Sigma_{is}(\beta) \cup \Sigma_z^*(\beta))\\ &= \left(\overline{\Sigma_x^* \cap \widetilde{\Sigma}_z^*(\beta)}\right) \cap \left(\overline{\Sigma_y^* \cap \widetilde{\Sigma}_z^*(\beta)}\right) \cap \left(\Sigma_x^* \cup \widetilde{\Sigma}_z^*(\beta)\right) \cap \left(\Sigma_y^* \cup \widetilde{\Sigma}_z^*(\beta)\right) \cap \left(\Sigma_x^* \cup \Sigma_y^*\right) \cap \overline{\Sigma_z^*(\beta)}\\ &= \left(\left(\Sigma_x^* \setminus \widetilde{\Sigma}_z^*(\beta)\right) \cup \left(\widetilde{\Sigma}_z^*(\beta) \setminus \Sigma_x^*\right)\right) \cap \left(\left(\Sigma_y^* \setminus \widetilde{\Sigma}_z^*(\beta)\right) \cup \left(\widetilde{\Sigma}_z^*(\beta) \setminus \Sigma_y^*\right)\right) \cap \left(\Sigma_x^* \cup \Sigma_y^*\right) \cap \overline{\Sigma_z^*(\beta)}\\ &\quad = \left(\left(\left(\Sigma_x^* \cap \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta)\right) \cup \left(\widetilde{\Sigma}_z^*(\beta) \setminus \left(\Sigma_x^* \cup \Sigma_y^*\right)\right)\right) \cap \left(\Sigma_x^* \cup \Sigma_y^*\right) \cap \overline{\Sigma_z^*(\beta)}\\ &\qquad = \left(\left(\Sigma_x^* \cap \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta)\right) \cap \overline{\Sigma_z^*(\beta)}.\end{aligned}$$

Together with the relation (A.4.7) this implies that

$$\begin{aligned}\Pi \setminus \Sigma_{is}(\beta) &= \left(\left(\left(\Sigma_x^* \cap \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta)\right) \cap \overline{\Sigma_z^*(\beta)}\right) \cup \Sigma_z^*(\beta)\\ &= \left(\left(\left(\Sigma_x^* \cap \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta)\right) \setminus \Sigma_z^*(\beta)\right) \cup \Sigma_z^*(\beta) = \left(\left(\Sigma_x^* \cap \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta)\right) \cup \Sigma_z^*(\beta).\end{aligned}$$

It can also be rewritten as

$$\Pi \setminus \Sigma_{is}(\beta) = \left(\left(\Sigma_x^* \cap \Sigma_y^*\right) \setminus \Sigma_z^*\right) \cup \left(\Sigma_x^* \cap \Sigma_y^* \cap \Sigma_z^*(\beta)\right) \cup \left(\Sigma_z^*(\beta) \setminus \left(\Sigma_x^* \cap \Sigma_y^*\right)\right).$$

Let us show that the points of this set do not correspond to isolated points of the spectrum.

(a) $\lambda_0 \in \left(\Sigma_x^* \cap \Sigma_y^*\right) \setminus \Sigma_z^*$: this case can be treated in the same way as the case (i).

(b) $\lambda_0 \in \Sigma_x^* \cap \Sigma_y^* \cap \Sigma_z^*(\beta)$: in this case, taking into account that $\cos(\lambda_0 L_z) = \cos\beta$, the equation (5.2.4) with $\lambda = \lambda_0 + \delta$ can be rewritten as

$$\cos(\delta L_x) - \frac{\cos k_x}{\cos(\lambda_0 L_x)} + \frac{\sin(\delta L_x)}{\sin(\delta L_y)}\left(\cos(\delta L_y) - \frac{\cos k_y}{\cos(\lambda_0 L_y)}\right) + \frac{\sin(\delta L_x)}{\sin(\delta L_z)}\left(\cos(\delta L_z) - 1\right) = 0.$$

If $\delta \neq 0$ is small enough, then, choosing, for example, $\cos k_y = \cos(\lambda_0 L_y)\cos(\delta L_y)$, one can find a k_x such that

$$\frac{\cos k_x}{\cos(\lambda_0 L_x)} = \cos(\delta L_x) + \frac{\sin(\delta L_x)}{\sin(\delta L_z)}\left(\cos(\delta L_z) - 1\right).$$

Indeed, the right-hand side of this relation is of absolute value smaller than 1 for δ small enough. The couple (k_x, k_y) is then a solution of (5.2.4).

(c) $\lambda_0 \in \Sigma_z^*(\beta) \setminus \left(\Sigma_x^* \cap \Sigma_y^*\right)$: suppose that $\lambda_0 \notin \Sigma_y^*$ (the case $\lambda_0 \notin \Sigma_x^*$ can be considered analogously). Then, taking into account that $\cos(\lambda_0 L_z) = \cos\beta$, the equation (5.2.4) with $\lambda = \lambda_0 + \delta$ can be rewritten as

$$\cos(\lambda L_x) - \cos k_x + \frac{\sin(\lambda L_x)}{\sin(\lambda L_y)}\left(\cos(\lambda L_y) - \cos k_y\right) + \sin(\lambda L_x)\frac{\cos(\delta L_z) - 1}{\sin(\delta L_z)} = 0.$$

If δ is small enough, then choosing, for example, $\cos k_y = \cos(\lambda L_y)$, a k_x can be found such that

$$\cos k_x = \cos(\lambda L_x) + \sin(\lambda L_x)\frac{\cos(\delta L_z) - 1}{\sin(\delta L_z)}$$

Indeed, if $\lambda_0 \notin \Sigma_x^*$, then

$$\cos(\lambda L_x) + \sin(\lambda L_x)\frac{\cos(\delta L_z) - 1}{\sin(\delta L_z)} = \cos(\lambda_0 L_x) + O(\delta),$$

and the absolute value of this expression is smaller that 1 for δ small enough. If $\lambda_0 \in \Sigma_x^*$, then

$$\cos(\lambda L_x) + \sin(\lambda L_x)\frac{\cos(\delta L_z) - 1}{\sin(\delta L_z)} = \cos(\lambda_0 L_x)\left(1 - \delta^2\frac{L_x^2 + L_x L_z}{2}\right) + O(\delta^3),$$

and its absolute value is also smaller that 1 for δ small enough. Thus, we proved that the equation (5.2.4) has a solution when δ is small enough.

□

Proof of Proposition 5.2.7.
Some of the points of the set Σ^* correspond to eigenvalues on infinite multiplicity of the operator $A(\beta)$. In the following lemma we identify the subset of points of Σ^* that do not correspond to eigenvalues on infinite multiplicity.

Lemma A.4.2. *For any $\beta \in [0, \pi]$,*

$$\Sigma^* \setminus (\Sigma_{is}(\beta) \cup \Sigma_{emb}(\beta)) = \left(\Sigma_x^* \setminus \left(\Sigma_y^* \cup \Sigma_z^*\right)\right) \cup \left(\Sigma_y^* \setminus (\Sigma_x^* \cup \Sigma_z^*)\right) \cup \left(\widetilde{\Sigma}_z^*(\beta) \setminus \left(\Sigma_x^* \cup \Sigma_y^*\right)\right).$$

Proof.

(i) $\beta \in]0, \pi[$: in this case $\widetilde{\Sigma}_z^*(\beta) = \Sigma_z^*$. Hence,

$$\begin{aligned}\Sigma^* \setminus (\Sigma_{is}(\beta) \cup \Sigma_{emb}(\beta)) &= \left(\Sigma_x^* \cup \Sigma_y^* \cup \Sigma_z^*\right) \setminus \left((\Sigma_x^* \cap \Sigma_z^*) \cup \left(\Sigma_y^* \cap \Sigma_z^*\right) \cup \left(\left(\Sigma_x^* \cap \Sigma_y^*\right) \setminus \Sigma_z^*\right)\right) \\ &= \left(\Sigma_x^* \cup \Sigma_y^* \cup \Sigma_z^*\right) \setminus \left((\Sigma_x^* \cap \Sigma_z^*) \cup \left(\Sigma_y^* \cap \Sigma_z^*\right) \cup \left(\Sigma_x^* \cap \Sigma_y^*\right)\right),\end{aligned}$$

and the result follows.

(ii) $\beta \in \{0, \pi\}$: in this case $\Sigma_z^* = \Sigma_z(\beta)^* \cup \widetilde{\Sigma}_z^*(\beta)$ and $\Sigma_z^*(\beta) \cap \widetilde{\Sigma}_z^*(\beta) = \emptyset$. We have:

$$\begin{aligned}&\Sigma^* \setminus (\Sigma_{is}(\beta) \cup \Sigma_{emb}(\beta)) \\ &= (\Sigma_x^* \setminus (\Sigma_{is}(\beta) \cup \Sigma_{emb}(\beta))) \cup \left(\Sigma_y^* \setminus (\Sigma_{is}(\beta) \cup \Sigma_{emb}(\beta))\right) \cup (\Sigma_z^* \setminus (\Sigma_{is}(\beta) \cup \Sigma_{emb}(\beta))).\end{aligned}$$

Taking into account that

$$\Sigma_{is}(\beta) \cup \Sigma_{emb}(\beta) = \left(\Sigma_x^* \cap \widetilde{\Sigma}_z^*(\beta)\right) \cup \left(\Sigma_y^* \cap \widetilde{\Sigma}_z^*(\beta)\right) \cup \left(\left(\Sigma_x^* \cap \Sigma_y^*\right) \setminus \widetilde{\Sigma}_z^*(\beta)\right) \cup \Sigma_z^*(\beta),$$

we find the result.

□

We come back to the proof of Proposition 5.2.7.

We have to prove that if $\lambda \in \Sigma^* \setminus (\Sigma_{is}(\beta) \cup \Sigma_{emb}(\beta))$, then λ^2 is not an eigenvalue of the operator $A^\mu(\beta)$. According to Lemma A.4.2, there are 3 possible cases.

(i) $\lambda \in \Sigma_x^* \setminus \left(\Sigma_y^* \cup \Sigma_z^*\right)$: since $\sin(\lambda L_z) \neq 0$, we have $\sin\left(\frac{\lambda L_z}{2}\right) \neq 0$, $\cos\left(\frac{\lambda L_z}{2}\right) \neq 0$ and from the relations (5.2.57)–(5.2.58) we find

$$c_{k,l}^+ = \alpha^+ d_{k,l}, \quad \alpha^+ = \frac{1}{2}\left(\tan\left(\tfrac{\lambda L_z}{2}\right)\left(1+e^{-i\beta}\right) + \frac{e^{-i\beta}-1}{\tan\left(\frac{\lambda L_z}{2}\right)}\right), \quad (k,l) \in \mathbb{Z}^2, \quad \text{(A.4.8)}$$

$$c_{k,l}^- = \alpha^- d_{k,l}, \quad \alpha^- = \frac{1}{2}\left(\frac{1-e^{i\beta}}{\tan\left(\frac{\lambda L_z}{2}\right)} - \tan\left(\tfrac{\lambda L_z}{2}\right)\left(1+e^{+i\beta}\right)\right), \quad (k,l) \in \mathbb{Z}^2. \quad \text{(A.4.9)}$$

We have then

$$u_{k,l}^+(z) = d_{k,l}\left(\alpha^+ \sin(\lambda z) + \cos(\lambda z)\right), \qquad z \in \left[0, \tfrac{L_z}{2}\right], \qquad (k,l) \in \mathbb{Z}^2,$$
$$u_{k,l}^-(z) = d_{k,l}\left(\alpha^- \sin(\lambda z) + \cos(\lambda z)\right), \qquad z \in \left[-\tfrac{L_z}{2}, 0\right], \qquad (k,l) \in \mathbb{Z}^2.$$

Taking into account that $\sin(\lambda L_x) = 0$, we get from the relation (5.2.56)

$$|b_{k+\frac{1}{2},\ell}| = |b_{k-\frac{1}{2},\ell}| = |d_{k,\ell}| =: d_l, \qquad \forall (k,\ell) \in \mathbb{Z}^2.$$

Thus, one necessarily has $d_\ell = 0$, $\forall \ell \in \mathbb{Z}$ (otherwise $u \notin L_2(G)$). Consequently, $b_{k+\frac{1}{2},\ell} = b_{k,\ell+\frac{1}{2}} = a_{k,\ell+\frac{1}{2}} = c_{k,\ell}^\pm = d_{k,\ell} = 0$, $\forall (k,\ell) \in \mathbb{Z}^2$. Then, from the relation (5.2.59) we get $a_{k+\frac{1}{2},\ell} = a_{k-\frac{1}{2},\ell}\cos(\lambda L_y)$, $\forall (k,\ell) \in \mathbb{Z}^2$, which implies that $|a_{k+\frac{1}{2},\ell}| = a_\ell$, $\forall (k,\ell) \in \mathbb{Z}^2$. If $a_\ell \neq$ for some ℓ, then $u \notin L_2(G)$. Hence, $a_\ell = 0$, $\forall \ell \in \mathbb{Z}$, and $u = 0$. This proves that λ is not an eigenvalue of the operator $A^\mu(\beta)$.

(ii) $\lambda \in \Sigma_y^* \setminus (\Sigma_x^* \cup \Sigma_z^*)$: this case is treated analogously to the previous one.

(iii) $\lambda \in \widetilde{\Sigma}_z^*(\beta) \setminus \left(\Sigma_x^* \cup \Sigma_y^*\right)$: since $\sin(\lambda L_z) = 0$, two cases are possible here.

(a) $\sin\left(\frac{\lambda L_z}{2}\right) = 0$: then, $\cos(\lambda L_z) = 1$ and $\beta \neq 0$ (since $\lambda \in \widetilde{\Sigma}_z^*(\beta)$). The relations (5.2.57)–(5.2.58) imply that $d_{k,l} = 0$, $c_{k,l}^+ = e^{-i\beta} c_{k,l}^-$, $\forall (k,l) \in \mathbb{Z}^2$. Then, from (5.2.56) we get $b_{k+\frac{1}{2},\ell} = b_{k,\ell+\frac{1}{2}} = 0$, $\forall (k,l) \in \mathbb{Z}^2$, and since $\sin(\lambda L_x) \neq 0$, $\sin(\lambda L_y) \neq 0$, we also have $a_{k+\frac{1}{2},\ell} = a_{k,\ell+\frac{1}{2}} = 0$, $\forall (k,l) \in \mathbb{Z}^2$. Finally, we find from the relation (5.2.59) that $c_{k,l}^+ = c_{k,l}^-$, $\forall (k,l) \in \mathbb{Z}^2$, which implies that $c_{k,l}^+ = c_{k,l}^- = 0$, $\forall (k,l) \in \mathbb{Z}^2$ since $e^{i\beta} \neq 1$. Thus, $u = 0$ and λ is not an eigenvalue of the operator $A(\beta)$.

(b) $\cos\left(\frac{\lambda L_z}{2}\right) = 0$: then, $\cos(\lambda L_z) = -1$ and $\beta \neq \pi$ (since $\lambda \in \widetilde{\Sigma}_z^*(\beta)$). The same argument as in the previous case applies, with the only change that now we have $c_{k,l}^+ = -e^{-i\beta} c_{k,l}^-$, $\forall (k,l) \in \mathbb{Z}^2$ and $e^{-i\beta} \neq -1$.

□

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Résumé

Cette thèse porte sur la propagation des ondes acoustiques dans des milieux périodiques. Ces milieux ont des propriétés remarquables car le spectre associée à l'opérateur d'ondes dans ces milieux a une structure de bandes : il existe des plages de fréquences dans lesquelles les ondes monochromatiques ne se propagent pas. Plus intéressant encore, en introduisant des défauts linéiques dans ce type de milieux, on peut créer des modes guidés à l'intérieur de ces bandes de fréquences interdites. Dans ce manuscrit nous montrons qu'il est possible de créer de tels modes guidés dans le cas de milieux périodiques particuliers de type quadrillage : plus précisément, le domaine périodique considéré est constitué du plan $\mathbb{R}^2$ privé d'un ensemble infini d'obstacles rectangulaires régulièrement espacés (d'une distance ε) dans deux directions orthogonales du plan, que l'on perturbe localement en diminuant la distance entre deux colonnes d'obstacles. Les résultats sont ensuite étendus au cas $3D$.

Ce travail comporte un aspect théorique et un aspect numérique. Du point de vue théorique l'analyse repose sur le fait que, comme ε est petit, le spectre de l'opérateur associé à notre problème est "proche" du spectre d'un problème posé sur le graphe obtenu comme la limite géométrique du domaine quand ε tend vers 0. Or, pour le graphe limite, il est possible de calculer explicitement le spectre. Ensuite, en utilisant des méthodes d'analyse asymptotique on étudie le spectre de l'opérateur non-limite. On illustre les résultats théoriques par des résultats numériques obtenus à l'aide d'une méthode numérique spécialement dédiée aux milieux périodiques : cette dernière est basée sur la réduction du problème de valeurs propres initial (linéaire) posé dans un domaine non-borné à un problème non-linéaire posé dans un domaine borné (en utilisant l'opérateur de Dirichlet-to-Neumann exact).

Abstract

The present work deals with propagation of acoustic waves in periodic media. These media have particularly interesting properties since the spectrum associated with the underlying wave operator in such media has a band-gap structure: there exist intervals of frequences for which monochromatic waves do not propagate. Moreover, by introducing linear defects in this kind of media, one can create guided modes inside the bands of forbidden frequences. In this work we show that it is possible to create such guided modes in the case of particular periodic media of grid type: more precisely, the periodic domain in question is $\mathbb{R}^2$ minus an infinite set of rectangular obstacles periodically spaced in two orthogonal directions (the distance between two neighbour obstacles being ε), which is locally perturbed by diminishing the distance between two columns of obstacles. The results are extended to the $3D$ case.

This work has a theoretical and a numerical aspect. From the theoretical point of view the analysis is based on the fact that, ε being small, the spectrum of the operator associated with our problem is "close" to the spectrum of a problem posed on a graph which is a geometric limit of the domain as ε tends to 0. However, for the limit graph the spectrum can be computed explicitly. Then, we study the spectrum of the non-limit operator using asymptotic analysis. Theoretical results are illustrated by numerical computations obtained with a numerical method developed for study of periodic media: this method is based on the reduction of the initial (linear) eigenvalue problem posed in an unbounded domain to a non-linear problem posed in a bounded domain (using the exact Dirichlet-to-Neumann operator).