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Intitulée :

# Comportement asymptotique des solutions globales pour quelques problèmes paraboliques non linéaires singuliers

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# Résumé :

Dans cette thèse, nous étudions l'équation parabolique non linéaire $\partial_t u = \Delta u + a|x|^{-\gamma}|u|^{\alpha}u,\ t>0,\ x\in\mathbb{R}^N\backslash\{0\},\ N\geq 1,\ a\in\mathbb{R},\ \alpha>0,\ 0<\gamma<\min(2,N)$ et avec une donnée initiale $u(0)=\varphi$. On établit l'existence et l'unicité locale dans $L^q(\mathbb{R}^N)$ et dans $C_0(\mathbb{R}^N)$. En particulier, la valeur $q=N\alpha/(2-\gamma)$ joue un rôle critique.

Pour $\alpha>(2-\gamma)/N$, on montre l'existence de solutions auto-similaires globales avec données initiales $\varphi(x)=\omega(x)|x|^{-(2-\gamma)/\alpha}$, où $\omega\in L^\infty(\mathbb{R}^N)$ homogène de degré 0 et $\|\omega\|_\infty$ est suffisamment petite. Nous montrons ainsi que si $\varphi(x)\sim\omega(x)|x|^{-(2-\gamma)/\alpha}$ pour $|x|$ grande, alors la solution est globale et asymptotique dans $L^\infty(\mathbb{R}^N)$ à une solution auto-similaire de l'équation non linéaire. Tandis que si $\varphi(x)\sim\omega(x)|x|^{-\sigma}$ pour des $|x|$ grandes avec $(2-\gamma)/\alpha<\sigma<N$, alors la solution est globale, mais elle est asymptotique dans $L^\infty(\mathbb{R}^N)$ à $e^{t\Delta}(\omega(x)|x|^{-\sigma})$.

L'équation avec un potentiel plus général, $\partial_t u=\Delta u+V(x)|u|^\alpha u,\ V(x)|x|^\gamma\in L^\infty(\mathbb{R}^N)$, est également étudiée. En particulier, pour des données initiales $\varphi(x)\sim\omega(x)|x|^{-(2-\gamma)/\alpha}$, $|x|$ grande, nous montrons que le comportement à grand temps est linéaire si $V$ est à support compact au voisinage de l'origine, alors qu'il est non linéaire si $V$ est à support compact au voisinage de l'infini.

Nous étudions également le système non linéaire $\partial_t u=\Delta u+a|x|^{-\gamma}|v|^{p-1}v$, $\partial_t v=\Delta v+b|x|^{-\rho}|u|^{q-1}u$, $t>0$, $x\in\mathbb{R}^N\backslash\{0\}$, $N\geq 1$, $a,b\in\mathbb{R}$, $0\leq\gamma<\min(N,2)$, $0<\rho<\min(N,2)$, $p,q>1$. Sous des conditions sur les paramètres $p,q,\gamma$ et $\rho$ nous montrons l'existence et l'unicité de solutions globales avec données initiales petites par rapport à certaines normes. En particulier, on montre l'existence de solutions auto-similaires avec donnée initiale $\Phi=(\varphi_1,\varphi_2)$, où $\varphi_1$, $\varphi_2$ sont des données initiales homogènes. Nous montrons également que certaines solutions globales sont asymptotiquement auto-similaires.

Comme deuxième objectif, nous considérons l'équation de la chaleur non linéaire

$$u_t=\Delta u+|u|^{p-1}u-|u|^{q-1}u, \tag{E}$$

avec $t\geq 0$ et $x\in\Omega$, la boule unité de $\mathbb{R}^N$, $N\geq 3$, avec des conditions aux limites de Dirichlet. Soit $h$ une solution stationnaire à symétrie radiale avec changement de signe de (E). On montre que la solution de (E) avec donnée initiale $\lambda h$ explose en temps fini si $|\lambda-1|>0$ est suffisamment petit et si $1<q<p<p_S=\frac{N+2}{N-2}$ et $p$ suffisamment proche de $p_S$. Ceci prouve que l'ensemble des données initiales pour lesquelles la solution est globale n'est pas étoilé au voisinage de 0.

**Mots clés :** Équation de la chaleur non linéaire, Équation parabolique de Hardy-Hénon, Système parabolique de Hardy-Hénon, Existence globale, Solutions auto-similaires, Comportement en temps grand, Explosion en temps fini, Solutions stationnaires avec changement de signe, Opérateur linéarisé.

## Abstract:

In this thesis, we study the nonlinear parabolic equation $\partial_t u = \Delta u + a|x|^{-\gamma}|u|^{\alpha}u,\ \ t > 0,\ x \in \mathbb{R}^N\backslash\{0\},\ N \geq 1,\ a \in \mathbb{R},\ \alpha > 0,\ 0 < \gamma < \min(2, N)$ and with initial value $u(0) = \varphi$. We establish local well-posedness in $L^q(\mathbb{R}^N)$ and in $C_0(\mathbb{R}^N)$. In particular, the value $q = N\alpha/(2-\gamma)$ plays a critical role.

For $\alpha > (2-\gamma)/N$, we show the existence of global self-similar solutions with initial values $\varphi(x) = \omega(x)|x|^{-(2-\gamma)/\alpha}$, where $\omega \in L^{\infty}(\mathbb{R}^N)$ is homogeneous of degree 0 and $\|\omega\|_{\infty}$ is sufficiently small. We then prove that if $\varphi(x) \sim \omega(x)|x|^{-(2-\gamma)/\alpha}$ for $|x|$ large, then the solution is global and is asymptotic in the $L^{\infty}$-norm to a self-similar solution of the nonlinear equation. While if $\varphi(x) \sim \omega(x)|x|^{-\sigma}$ for $|x|$ large with $(2-\gamma)/\alpha < \sigma < N$, then the solution is global but is asymptotic in the $L^{\infty}$-norm to $e^{t\Delta}(\omega(x)|x|^{-\sigma})$.

The equation with more general potential, $\partial_t u = \Delta u + V(x)|u|^{\alpha}u,\ V(x)|x|^{\gamma} \in L^{\infty}(\mathbb{R}^N)$, is also studied. In particular, for initial data $\varphi(x) \sim \omega(x)|x|^{-(2-\gamma)/\alpha}$, $|x|$ large , we show that the large time behavior is linear if $V$ is compactly supported near the origin, while it is nonlinear if $V$ is compactly supported near infinity.

we study also the nonlinear parabolic system $\partial_t u = \Delta u + a|x|^{-\gamma}|v|^{p-1}v$, $\partial_t v = \Delta v + b|x|^{-\rho}|u|^{q-1}u$, $t > 0$, $x \in \mathbb{R}^N\backslash\{0\}$, $N \geq 1$, $a, b \in \mathbb{R}$, $0 \leq \gamma < \min(N, 2)$, $0 < \rho < \min(N, 2)$, $p, q > 1$. Under conditions on the parameters $p, q, \gamma$ and $\rho$ we show the existence and uniqueness of global solutions for initial values small with respect of some norms. In particular, we show the existence of self-similar solutions with initial value $\Phi = (\varphi_1, \varphi_2)$, where $\varphi_1$, $\varphi_2$ are homogeneous initial data. We also prove that some global solutions are asymptotic for large time to self-similar solutions.

As a second objective we consider the nonlinear heat equation

$$u_t = \Delta u + |u|^{p-1}u - |u|^{q-1}u, \tag{E}$$

where $t \geq 0$ and $x \in \Omega$, the unit ball of $\mathbb{R}^N$, $N \geq 3$, with Dirichlet boundary conditions. Let $h$ be a radially symmetric, sign-changing stationary solution of (E). We prove that the solution of (E) with initial value $\lambda h$ blows up in finite time if $|\lambda - 1| > 0$ is sufficiently small and if $1 < q < p < p_S = \frac{N+2}{N-2}$ and $p$ sufficiently close to $p_S$. This proves that the set of initial data for which the solution is global is not star-shaped around 0.

**Keywords:** Nonlinear heat equation, Hardy-Hénon parabolic equation, Hardy-Hénon parabolic system, Global existence, Self-similar solutions, Large time behavior, Finite-time blow-up, Sign-changing stationary solutions, Linearized operator.

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# Chapitre 1

# Introduction et résultats principaux

## 1.1 Motivation et présentation générale des problèmes

Un nombre important de chercheurs sont intéressés par le problème de Cauchy suivant :

$$\begin{cases} u_t = \Delta u + |u|^\alpha u & \text{dans} \quad \Omega \times (0,T) \\ u_{|\partial\Omega} = 0 & \text{sur} \quad \partial\Omega \times (0,T) \\ u(0) = u_0 & \text{dans} \quad \Omega, \end{cases} \tag{1.1.1}$$

où $u = u(t,x) \in \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$, $\Omega$ est un ouvert convexe borné régulier de $\mathbb{R}^N$ ou $\Omega = \mathbb{R}^N$ et $T > 0$. (La condition de bord est remplacée par $\lim_{|x|\to\infty} u(t,x) = 0$ si $\Omega = \mathbb{R}^N$).

Cette équation de réaction-diffusion constitue un modèle pour plusieurs phénomènes physiques et chimiques.

L'équation (1.1.1) présente un intérêt mathématique puisqu'elle est le prototype des équations paraboliques. Plusieurs questions autour des solutions de cette équation se posent :

- Pour quels espaces de données initiales le problème est-il localement bien posé ?
- Pour quels espaces de données initiales le problème est-il globalement bien posé ?
- Quelle est la vitesse d'explosion des solutions non globales ?
- Que peut-on dire du comportement en temps grand des solutions globales ?
- Que peut-on dire sur la structure de l'ensemble des solutions globales ?

Rappelons que dans l'étude des équations d'évolution, les termes "globale" et "locale" font référence à l'existence de la solution pour tout $t \geq 0$ ou bien dans des intervalles de la forme $[0,T)$, respectivement.

Afin d'établir l'existence locale ou globale , on peut chercher des solutions pour l'équation intégrale suivante, qui est formellement équivalente à l'équation (1.1.1),

$$u(t) = S(t)u_0 + \int_0^t S(t-\sigma)\left(|u(\sigma)|^\alpha u(\sigma)\right) d\sigma. \tag{1.1.2}$$

$S(t)$ désigne le semi-groupe (qu'on suppose exister) dont le générateur infinitésimal est égal à $\Delta$. Notons que les solutions de (1.1.2) sont des points fixes de l'opérateur

$$\mathcal{F}_{u_0} : u(t) \to S(t)u_0 + \int_0^t S(t-\sigma)\left(|u(\sigma)|^\alpha u(\sigma)\right) d\sigma.$$

On espère trouver des points fixes pour cet opérateur $\mathcal{F}_{u_0}$ dans quelques espaces de Banach bien introduit. Remarquons que le bon choix de la donnée initiale et le choix de l'espace sont fondamentaux pour l'obtention de solutions globales.

Le problème de Cauchy (1.1.1) peut être résolu dans plusieurs espaces fonctionnels. Citons par exemple $C_0(\Omega)$, avec

- $C_0(\Omega)$ est l'espace des fonctions continues sur $\overline{\Omega}$ nulle sur $\partial\Omega$ si $\Omega$ est borné,
- $C_0(\Omega)$ est l'espace des fonctions continues sur $\mathbb{R}^N$ tendant vers 0 quand $|x| \to \infty$ si $\Omega = \mathbb{R}^N$.

On peut aussi penser à le résoudre dans l'espace de Lebesgue $L^\infty(\mathbb{R}^N)$ et d'une manière plus générale dans les espaces de Lebesgue $L^q(\mathbb{R}^N)$.

Une des premières études de l'équation (1.1.1) a été faite en 1966 par Fujita dans [20], où il a été prouvé que pour $\Omega = \mathbb{R}^N$ et pour certaines données initiales dans $L^\infty(\mathbb{R}^N)$ positives et vérifiant quelques propriétés de bornitude le problème n'admet aucune solution non-triviale positive et globale si $0 < \alpha < \alpha_F = \frac{2}{N}$, alors que si $\alpha > \alpha_F$ il existe une solution globale pour toute donnée initiale positive dominée par une Gaussienne suffisamment petite. Ce résultat montre que l'exposant $\alpha$ a un effet direct sur les propriétés de la solution. $\alpha_F$ est appelé l'exposant critique de Fujita.

En 1981, Weissler dans [51] considère des données initiales $u_0 \in L^q(\mathbb{R}^N)$, il établit l'existence d'un exposant critique $q_c = \frac{N\alpha}{2}$ tel que le problème soit bien posé dans $L^q(\mathbb{R}^N)$ si $q \geq q_c$, et mal posé si $q < q_c$ par perte d'unicité locale. En 1996, Brezis et Cazenave dans [3] ont considéré le même problème, ils prouvent l'unicité des solutions dans une classe de solutions plus large que celle dans [51].

Parmi les objectifs dans l'étude du problème de Cauchy (1.1.1) c'est de construire des solutions auto-similaires, les chercheurs sont donc obligés de traiter des données initiales singulières de type

$$u_0(x) = \omega(x)|x|^{-\frac{2}{\alpha}},$$

avec $\omega$ homogène de degré 0. Il est facile de remarquer que $|x|^{-\frac{2}{\alpha}}$ n'est dans aucun espace de Lebesgue $L^q(\mathbb{R}^N)$ et donc la théorie standard dans les espaces $L^q(\mathbb{R}^N)$ ne s'applique pas. Afin de construire des solutions auto-similaires Cazenave et Weissler dans [14] ont considéré des données initiales satisfaisant la relation suivante

$$\sup_{t>0} t^\beta \|e^{t\Delta}u_0\|_{L^r(\mathbb{R}^N)} < \infty,$$

avec

$$1 < \frac{r}{\alpha+1} < \frac{N\alpha}{2} < r,$$

et

$$\beta = \frac{1}{\alpha} - \frac{N}{2r}.$$

Ils ont prouvé l'existence de solutions globales pour l'équation intégrale associée pour des données initiales petites par rapport à la norme

$$\mathcal{N}(u_0) := \sup_{t>0} t^{\beta} \|e^{t\Delta}u_0\|_{L^r(\mathbb{R}^N)},$$

en plus, ils ont prouvé que si $\mathcal{N}(u_0)$ est suffisamment petite et $u_0(x) \sim \omega(x)|x|^{-\frac{2}{\alpha}}$ dans un sens approprié, alors la solution résultante est asymptotique quand $t \to \infty$ à une solution auto-similaire de (1.1.1).

Plusieurs chercheurs ont étudié aussi la structure géométrique de l'ensemble $\mathcal{G}$ pour le problème (1.1.1) défini par

$$\mathcal{G} = \{u_0 \in C_0(\Omega), T_{u_0} = \infty\},$$

ici $T_{u_0}$ désigne le temps maximal d'existence et $\Omega$ un domaine borné régulier de $\mathbb{R}^N$. Il ont montré que $\mathcal{G}$ n'est pas étoilé dans plusieurs circonstances :

- $N \geq 3$, $\Omega = B_1$ la boule unité de $\mathbb{R}^N$, et $\alpha < \alpha_* = \frac{4}{N-2}$ suffisamment proche de $\alpha_*$, voir [8] ;
- $N = 3$, $\Omega = B_1$ la boule unité de $\mathbb{R}^N$, et $\alpha > 0$ suffisamment proche de 0, voir [10] ;
- $N \geq 3$, $\Omega$ est un domaine général, et $\alpha < \alpha_* = \frac{4}{N-2}$ suffisamment proche de $\alpha_*$ ou $\alpha = \alpha_*$, voir [29, 31] ;
- $N = 2$, $\Omega = B_1$ la boule unité de $\mathbb{R}^N$, ou bien $\Omega$ est un domaine général et $\alpha$ suffisamment grand, voir [15, 16].

Voir [11, 12, 13, 21, 22, 28, 38] pour d'autres propriétés de l'ensemble $\mathcal{G}$ pour le problème (1.1.1).

Cette thèse se compose en deux parties indépendantes. On va établir et améliorer quelques résultats analogues aux résultats précédents établis pour l'équation (1.1.1).

Le but de la première partie est d'étudier l'équation de la chaleur singulière

$$\partial_t u = \Delta u + a|\cdot|^{-\gamma}|u|^{\alpha}u, \tag{1.1.3}$$

$u = u(t,x) \in \mathbb{R}$, $t > 0$, $x \in \mathbb{R}^N$, $N \geq 1$, $a \in \mathbb{R}$, $\alpha > 0$, $\gamma > 0$ et avec donnée initiale

$$u(0) = \varphi. \tag{1.1.4}$$

Dans ce qui suit, nous notons $\|.\|_{L^q(\mathbb{R}^N)}$ par $\|.\|_q$, $1 \leq q \leq \infty$. Pour tout $t > 0$, $e^{t\Delta}$ désigne le semi-groupe de chaleur

$$\left(e^{t\Delta}f\right)(x) = \int_{\mathbb{R}^N} G(t, x-y)f(y)dy, \tag{1.1.5}$$

où

$$G(t,x) = (4\pi t)^{-N/2}e^{-\frac{|x|^2}{4t}},\ t > 0,\ x \in \mathbb{R}^N, \tag{1.1.6}$$

et $f \in L^q(\mathbb{R}^N)$, $q \in [1,\infty)$ ou bien $f \in C_0(\mathbb{R}^N)$. Pour $f \in \mathcal{S}'(\mathbb{R}^N)$, $e^{t\Delta}f$ est défini par dualité. Une solution "mild" du problème (1.1.3) est une solution de l'équation intégrale

$$u(t) = e^{t\Delta}\varphi + a\int_0^t e^{(t-s)\Delta}\left(|\cdot|^{-\gamma}|u(s)|^{\alpha}u(s)\right)ds, \tag{1.1.7}$$

et c'est sous cette forme que nous considérons le problème (1.1.3).

Le cas $\gamma = 0$ correspond à l'équation de la chaleur non linéaire standard. Pour $\gamma < 0$ elle est connue dans la littérature comme une équation parabolique de Hénon, tandis que si $\gamma > 0$ elle est connue comme une équation parabolique de Hardy. Dans ce travail nous nous intéressons au cas $\gamma > 0$. Nous sommes intéressés par l'existence et l'unicité de solution locale pour (1.1.3) avec donnée initiale $\varphi \in L^q(\mathbb{R}^N),\ 1 \leq q < \infty$, et dans $C_0(\mathbb{R}^N)$. Nous étudions également l'existence de solutions globales, y compris des solutions auto-similaires et nous prouvons l'existence de solutions asymptotiquement auto-similaires. Les méthodes sont basées sur les articles [49, 51, 14, 43].

Nous allons étudié aussi le comportement asymptotique des solutions globales de l'équation parabolique non linéaire singulière générale

$$u(t) = e^{t\Delta}\varphi + \int_0^t e^{(t-\sigma)\Delta}\left[V(.)|u(\sigma)|^{\alpha}u(\sigma)\right]d\sigma, \tag{1.1.8}$$

$u = u(t,x) \in \mathbb{R},\ \ t > 0,\ x \in \mathbb{R}^N,\ \alpha > 0$ et $\varphi \in \mathcal{S}'(\mathbb{R}^N)$. Le potentiel $V$ vérifie

$$|V(x)| \leq C|x|^{-\gamma},\ \forall\, x \in \mathbb{R}^N \backslash \{0\}, \tag{1.1.9}$$

et satisfait l'une des hypothèses suivantes :

$(H_1)$ $V(x) = a(1 - f(x))|x|^{-\gamma}$,

$(H_2)$ $V(x) = af(x)|x|^{-\gamma}$,

où $a \in \mathbb{R}$ et $f$ tel que

$$f(x)|x|^{-\gamma} \in L^s(\mathbb{R}^N),\ \ \frac{\gamma}{N} < \frac{1}{s} < \frac{2\gamma}{N} + \frac{\alpha+1}{r} - \frac{1}{q_c},\ \frac{1}{s} < \frac{2}{N} - \frac{\alpha}{r}, \tag{1.1.10}$$

averc $r > q_c$ satisfaisant $\frac{1}{q_c} - \frac{2}{N(\alpha+1)} < \frac{1}{r} < \frac{N-\gamma}{N(\alpha+1)}$, $q_c$ est défini dans (1.2.19) au dessous. A titre d'exemple pour cette fonction $f$, nous pouvons prendre une fonction à support compact et $f \equiv 1$ près de l'origine.

Nous avons considéré ensuite le système parabolique non linéaire suivant

$$(S)\begin{cases} \partial_t u = \Delta u + a|.|^{-\gamma}|v|^{p-1}v, \\ \partial_t v = \Delta v + b|.|^{-\rho}|u|^{q-1}u, \end{cases}$$

avec donnée initiale

$$u(0,x) = \varphi_1(x),\ \ v(0,x) = \varphi_2(x), \tag{1.1.11}$$

où $u = u(t,x) \in \mathbb{R}$, $v = v(t,x) \in \mathbb{R}$, $t > 0$, $x \in \mathbb{R}^N$, $a$, $b \in \mathbb{R}$, $0 \leq \gamma < \min(N,2)$, $0 < \rho < \min(N,2)$, $p$, $q > 1$.

Une solution "mild" du système (S)-(1.1.11) est une solution du système intégral

$$\begin{cases} u(t) = e^{t\Delta}\varphi_1 + a\displaystyle\int_0^t e^{(t-\sigma)\Delta}\left(|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma)\right)d\sigma, \\ v(t) = e^{t\Delta}\varphi_2 + b\displaystyle\int_0^t e^{(t-\sigma)\Delta}\left(|.|^{-\rho}|u(\sigma)|^{q-1}u(\sigma)\right)d\sigma. \end{cases} \tag{1.1.12}$$

Nous étudions l'existence de solutions globales "mild", y compris des solutions auto-similaires pour le système semi-linéaire $(S)$. De plus, nous nous intéressons de leur comportement asymptotique.

En utilisant l'estimation clé établie par la Proposition 2.1 dans [2] nous pouvons adapter la méthode de Fujita et Kato [18, 19] et récemment utilisé dans [2, 5, 6, 14, 7, 23, 30, 42, 43, 44]. Cette méthode est basée sur un argument de contraction sur le système intégral associé (1.1.12). Précisément, nous transformons le problème de l'existence et de l'unicité des solutions globales en un problème de point fixe pour une fonction définie dans un espace de Banach approprié équipé d'une norme choisie afin d'obtenir directement le caractère global de la solution.

La deuxième partie est reservée à l'étude de l'explosion en temps fini des solutions régulières et qui prennent des valeurs positives et négatives du problème

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u - |u|^{q-1}u, \\ u_{|\partial\Omega} = 0. \end{cases} \tag{1.1.13}$$

Ici, $u = u(t,x) \in \mathbb{R},\ t \geq 0,\ x \in \Omega$, et

$$\Omega = B_1, \tag{1.1.14}$$

est la boule unité ouverte de $\mathbb{R}^N$,

$$N \geq 3. \tag{1.1.15}$$

En outre, nous considérons

$$1 < q < p < p_S, \tag{1.1.16}$$

où

$$p_S = \frac{N+2}{N-2}. \tag{1.1.17}$$

Il est bien connu que le problème (1.1.13) est localement bien posé dans $C_0(\Omega)$, où $C_0(\Omega)$ est l'espace de Banach des fonctions continues sur $\overline{\Omega}$ et nulles sur $\partial\Omega$. Plus précisément, étant donné $u_0 \in C_0(\Omega)$, il existe un temps maximal $0 < T_{u_0} \leq \infty$ et une unique function $u \in C\left([0,T_{u_0}), C_0(\Omega)\right) \cap C\left((0,T_{u_0}), C^2(\overline{\Omega})\right) \cap C^1\left((0,T_{u_0}), C_0(\Omega)\right)$ qui est une solution classique de (1.1.13) dans $(0,T_{u_0})$ et tel que $u(0) = u_0$. En outre, si $T_{u_0} < \infty$, alors $\lim_{t\uparrow T_{u_0}} \|u(t)\|_\infty = \infty$, et nous disons que $u$ explose en temps fini. En plus, si $v \in C\left([0,T), C_0(\Omega)\right) \cap C\left((0,T), C^2(\overline{\Omega})\right) \cap C^1\left((0,T), C_0(\Omega)\right)$ est une sur-solution de (1.1.13), i.e $v_t - \Delta v \geq |v|^{p-1}v - |v|^{q-1}v$, $v_{|\partial\Omega} \geq 0$ and $v(0) \geq u_0$, alors $v(t) \geq u(t)$ tant que $u$ et $v$ sont définies. La notion de sous-solution est définie avec des inégalités inversées, ce qui donne une conclusion analogue. Voir, par exemple, la Proposition 52.6 dans [40].

Nous définissons l'ensemble $\mathcal{G}$ par

$$\mathcal{G} = \left\{u_0 \in C_0(\Omega), T_{u_0} = \infty\right\}.$$

Il est intéressant d'étudier les propriétés géométriques de l'ensemble $\mathcal{G}$. Tout d'abord, nous notons que chaque solution $h$ de

$$\begin{cases} -\Delta h = |h|^{p-1}h - |h|^{q-1}h, \\ h_{|\partial\Omega} = 0, \end{cases} \tag{1.1.18}$$

est une solution stationnaire, donc globale, de (1.1.13), dont la donnée initiale est évidemment $u_0 = h$, est donc dans $\mathcal{G}$. Comme la non-linéarité $|s|^{p-1}s - |s|^{q-1}s$ satisfait les propriétés de [9, Theorem 1.1, p. 15], il s'ensuit que l'ensemble $\mathcal{G}$ n'est pas convexe. Comme $u(t) = 0$ est une solution de (1.1.13) on peut demander si $\mathcal{G}$ a la propriété la plus faible d'être étoilé autour de 0. Le but de ce travail est de montrer que $\mathcal{G}$ n'est pas étoilé. Pour cela, il faut adapter les méthodes de [8].

Dans la suite nous présentons les résultats obtenus pour chaque chapitre.

## 1.2 Etude des équations de Hardy-Hénon

Nous étudions d'abord l'existence et l'unicité locale pour l'équation intégrale (1.1.7). A notre connaissance, il n'existe qu'un résultat précédent de ce type, Wang en 1993 [48], qui travaillait dans l'espace $C_B(\mathbb{R}^N)$ de fonctions continues et bornées. Pour $N \geq 3,\ a > 0$ et $\gamma < 2$, il a prouvé l'existence locale de solutions pour (1.1.7) dans $C([0,T]; C_B(\mathbb{R}^N))$ pour tout $\varphi \in C_B(\mathbb{R}^N)$.

Dans ce travail, on montre l'existence et l'unicité locale dans $C_0(\mathbb{R}^N)$ et dans $L^q(\mathbb{R}^N)$ pour certaines valeurs de $q$. Nous exigeons également la condition $\gamma < 2$, et en fait $0 < \gamma < 2$. Tout au long du document, nous posons, pour $\alpha > 0,\ 0 < \gamma < 2$,

$$q_c = \frac{N\alpha}{2-\gamma}. \tag{1.2.19}$$

L'exposant critique $q_c$ joue un rôle crucial. Nous dirons que $q$ est sous-critique, critique ou sur-critique, selon que $q < q_c,\ q = q_c$ ou $q > q_c$. Nous avons obtenu les résultats suivants.

**Théorème 1.2.1** (Existence et unicité locale)**.** *Soient $N \geq 1$ un entier, $\alpha > 0$ et $\gamma$ tel que*

$$0 < \gamma < \min(2, N). \tag{1.2.20}$$

*Soit $q_c$ donné par* (1.2.19)*. Alors on a les résultats suivants.*

*(i) L'équation* (1.1.7) *est localement bien posée dans $C_0(\mathbb{R}^N)$. Plus précisément, étant donné $\varphi \in C_0(\mathbb{R}^N)$, alors il existe $T > 0$ et une unique solution $u \in C\big([0,T]; C_0(\mathbb{R}^N)\big)$ de* (1.1.7)*. En outre, $u$ peut être prolongée à un intervalle maximal $[0, T_{\max})$ tel que soit $T_{\max} = \infty$ ou $T_{\max} < \infty$ et $\lim_{t\to T_{\max}} \|u(t)\|_\infty = \infty$.*

*(ii) Si $q$ est tel que*

$$q > \frac{N(\alpha+1)}{N-\gamma}, \quad q > q_c \quad \text{et} \quad q < \infty,$$

*alors l'équation* (1.1.7) *est localement bien posée dans $L^q(\mathbb{R}^N)$. Plus précisément, étant donné $\varphi \in L^q(\mathbb{R}^N)$, alors il existe $T > 0$ et une unique solution $u \in C\big([0,T]; L^q(\mathbb{R}^N)\big)$ de* (1.1.7)*. En outre, $u$ peut être prolongée à un intervalle maximal $[0, T_{\max})$ tel que soit $T_{\max} = \infty$ ou $T_{\max} < \infty$ et $\lim_{t\to T_{\max}} \|u(t)\|_q = \infty$.*

*(iii) Supposons que $q \geq q_c$ avec $1 < q < \infty$. Il en résulte que* (1.1.7) *est localement bien posée dans $L^q(\mathbb{R}^N)$ comme dans la Partie (ii) sauf que l'unicité n'est garantie que pour les fonctions continues $u : [0,T] \to L^q(\mathbb{R}^N)$ vérifiant également*

*(a)* $t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}\|u(t)\|_r$ *est bornée dans* $(0,T]$*, où* $r>q$ *satisfait* $\frac{1}{q(\alpha+1)}-\frac{\gamma}{N(\alpha+1)}<\frac{1}{r}<\frac{N-\gamma}{N(\alpha+1)}$, $q>q_c$;

*(b)* $\sup_{t\in(0,T]} t^{\frac{N}{2}(\frac{1}{q_c}-\frac{1}{r})}\|u(t)\|_r$ *est suffisamment petit, où* $r>q$ *satisfait* $\frac{1}{q_c}-\frac{2}{N(\alpha+1)}<\frac{1}{r}<\frac{N-\gamma}{N(\alpha+1)}$, $q=q_c$.

*En outre, u peut être prolongée à un intervalle maximal* $[0,T_{\max})$ *tel que, dans le cas* $q>q_c$*, soit* $T_{\max}=\infty$ *ou* $T_{\max}<\infty$ *et* $\lim_{t\to T_{\max}}\|u(t)\|_q=\infty$.

*(iv) Dans tous les cas ci-dessus, sauf lorsque* $q=q_c$*, le temps d'existence minimal de la solution, noté* $T$*, dépend uniquement de* $\|\varphi\|_\infty$ *ou* $\|\varphi\|_q$ *respectivement.*

**Remarque 1.2.1.** Wang montre l'équivalence des $C_B(\mathbb{R}^N)$ solutions de l'équation intégrale (1.1.7) et des solutions faibles continues de (1.1.3)-(1.1.4), qui sont aussi des solutions au sens des distributions. Voir la remarque dans [48] juste après la Définition 2.1 de [48] dans la page 563. Les $C_0(\mathbb{R}^N)$ solutions ci-dessus sont comprises dans cette situation. Pour $t>0$ les $L^q(\mathbb{R}^N)$ solutions sont aussi des $C_0(\mathbb{R}^N)$ solutions, et ont donc la même régularité.

**Remarque 1.2.2.** Le problème (1.1.7) n'est pas bien posé pour certaines valeurs $q$ sous-critique. La preuve est basée sur l'existence d'une solution auto-similaire strictement positive à décroissance rapide [26, Theorem 1.1, p. 625]. Il résulte de ce résultat, que si $1<q<q_c$, $a>0$, $N\geq 3$ et $\frac{2-\gamma}{N}<\alpha<2\frac{2-\gamma}{N-2}$, alors il existe une solution strictement positive $u$ de (1.1.7) avec donnée initiale $\varphi=0$ dans $L^q(\mathbb{R}^N)$. Ceci montre un résultat de non-unicité dans $L^q(\mathbb{R}^N)$ et donc (1.1.7) n'est pas bien posé pour certaines valeurs $q<q_c$.

La preuve du Théorème 1.2.1 est basée sur des arguments et des résultats dans [49, 50]. Pour appliquer ces méthodes, un nouveau ingrédient clé est nécessaire. Ces solutions dépendent continûment des données initiales dans un sens approprié. De plus, si $\varphi$ appartient à deux espaces différents comme décrit dans le Théorème 1.2.1, alors les solutions résultantes dans les différents espaces coïncident, et en particulier, le temps maximal d'existence ne dépend pas de l'espace.

Récemment, nous avons amélioré la Partie (iii) dans le Théorème 1.2.1, plus précisément, nous avons obtenu le résultat suivant.

**Théorème 1.2.2.** *Sous les hypothèses du Théorème 1.2.1. Supposons que* $q\geq q_c$ *avec* $1<q<\infty$*. Il en résulte que* (1.1.7) *est localement bien posée dans* $L^q(\mathbb{R}^N)$ *sauf que l'unicité n'est garantie que pour les fonctions* $u:[0,T]\to L^q(\mathbb{R}^N)$ *vérifiant également*

*(a)* $u\in C\left([0,T],L^q(\mathbb{R}^N)\right)\cap C\left((0,T],L^r(\mathbb{R}^N)\right)$, *où* $r>q$ *satisfait* $\frac{1}{q(\alpha+1)}-\frac{\gamma}{N(\alpha+1)}<\frac{1}{r}<\frac{N-\gamma}{N(\alpha+1)}$, $q>q_c$;

*(b)* $u\in C\left([0,T],L^q(\mathbb{R}^N)\right)\cap C\left((0,T],L^r(\mathbb{R}^N)\right)$, *où* $r>q$ *satisfait* $\frac{1}{q_c}-\frac{2}{N(\alpha+1)}<\frac{1}{r}<\frac{N-\gamma}{N(\alpha+1)}$, $q=q_c$.

Pour la preuve, voir Proposition 2.3.1 et Remarque 3.4.2.

Nous passons maintenant à l'existence de solutions globales. Nous avons obtenu le résultat suivant.

**Théorème 1.2.3** (Existence globale ). *Soient $N \geq 1$ un entier, $\alpha$, $\sigma$, $\gamma > 0$. Supposons que (1.2.20) est satisfaite et que*

$$\alpha > \frac{2-\gamma}{N}, \text{ de manière équivalente } q_c > 1,$$

*où $q_c$ est donné par* (1.2.19)*. Alors on a les résultats suivants.*

*(i) Si $\varphi \in L^{q_c}(\mathbb{R}^N)$ et $\|\varphi\|_{q_c}$ est suffisamment petite, alors $T_{\max} = \infty$.*

*(ii) Si $\varphi \in C_0(\mathbb{R}^N)$ tel que $|\varphi(x)| \leq c\left(1+|x|^2\right)^{-\frac{\sigma}{2}}$, pour tous $x \in \mathbb{R}^N$ avec c suffisamment petit et*

$$\sigma > \frac{2-\gamma}{\alpha}, \tag{1.2.21}$$

*alors $T_{\max} = \infty$.*

*(iii) Soit $\varphi \in L^1_{loc}(\mathbb{R}^N)$ tel que $|\varphi(\cdot)| \leq c|\cdot|^{-\frac{2-\gamma}{\alpha}}$, pour c suffisamment petit. Alors, il existe une solution globale en temps de* (1.1.7)*, $u \in C\left((0,\infty); L^q(\mathbb{R}^N)\right)$ pour tout $q > q_c$. De plus $u(t) \to \varphi$ dans $\mathcal{S}'(\mathbb{R}^N)$ quand $t \to 0$.*

**Remarque 1.2.3.** En utilisant la solution auto-similaire construite dans [17, Proposition 3.1, p. 477] et en utilisant la méthode de [45, Corollary 1.4, p. 659], nous pouvons prouver le résultat suivant. Supposons que $N \geq 3$, $a > 0$ et $\alpha$ tel que $\dfrac{2-\gamma}{N} < \alpha < 2\dfrac{2-\gamma}{N-2}$. Il existe $C > 0$ tel que si $\varphi \in C_0(\mathbb{R}^N), \varphi > 0$ et

$$\liminf_{|x|\to\infty} |x|^{\frac{2-\gamma}{\alpha}} \varphi(x) \geq C,$$

alors $T_{\max} < \infty$.

Par des résultats d'explosion de type Fujita, la condition $\alpha > (2-\gamma)/N$ ($q_c > 1$) est optimale dans le Théorème 1.2.3. En fait, si $a > 0$ et $\alpha \leq (2-\gamma)/N$ alors les solutions de (1.1.3) avec des données initiales strictement positives explosent en temps fini. Voir [37, Theorem 1.6, p. 126]. Dans [37, Theorem 1.2, p. 125], des exemples de solutions globales pour (1.1.3) avec des données initiales strictement positives pour $q_c > 1$ et $a > 0$ sont construits. Toujours pour $a > 0$, d'autres résultats d'existence globale pour des données initiales strictement positives sont prouvés dans [48] et [26] mais seulement pour $N \geq 3$ et avec des conditions supplémentaires sur $\alpha$. Voir [26, Theorem 1.3, p. 626]. Nos résultats d'existence globale sont établis sans aucune restriction sur $a$ ou sur les données initiales (sauf une condition de petitesse). Dans [48], Wang a observé qu'en modifiant les arguments de Lee et Ni [27], si (1.1.3) a une solution globale, alors nécessairement $\liminf_{|x|\to\infty} |x|^{\frac{2-\gamma}{\alpha}} \varphi(x) < \infty$. Nos résultats dans le Théorème 1.2.3 (ii) et (iii) sont compatibles avec cette condition. Pour $\gamma = 0$, (i) est connu, voir [18, Theorem 3, p. 32].

Nous passons maintenant à l'existence de solutions auto-similaires. Dans [26, 48] l'existence de solutions auto-similaires à symétrie radiale pour $\frac{2-\gamma}{N} < \alpha < 2\frac{2-\gamma}{N-2}$ et $N \geq 3$ est établie. Ici, nous n'imposons pas la symétrie radiale des solutions auto-similaires. Nous avons obtenu le résultat suivant.

**Théorème 1.2.4** ( Solutions auto-similaires). *Soient $N \geq 1$ un entier, $\alpha$, $\gamma > 0$. Supposons que $0 < \gamma < \min(2, N)$ et*

$$\alpha > \frac{2-\gamma}{N}. \tag{1.2.22}$$

*Soit* $\varphi(x) = \omega(x)|x|^{-\frac{2-\gamma}{\alpha}}$, *où* $\omega \in L^{\infty}(\mathbb{R}^N)$ *est homogène de degré* $0$ *et* $\|\omega\|_{\infty}$ *est suffisamment petite. Alors, il existe une solution globale "mild" auto-similaire* $u_{\mathcal{S}}$ *de (1.1.3)-(1.1.4). Cette solution vérifie* $u_{\mathcal{S}}(t) \to \varphi$ *dans* $\mathcal{S}'(\mathbb{R}^N)$ *quand* $t \to 0$.

Pour le comportement asymptotique des solutions globales, nous avons le résultat suivant.

**Théorème 1.2.5** (Comportement asymptotique)**.** *Soient* $N \geq 1$ *un entier,* $\alpha,\ \sigma,\ \gamma > 0$. *Supposons que* $0 < \gamma < \min(2, N)$ *et*

$$\frac{2-\gamma}{\alpha} \leq \sigma < N. \tag{1.2.23}$$

*Soit* $\varphi \in C_0(\mathbb{R}^N)$ *tel que*

$$|\varphi(x)| \leq \frac{c}{(1+|x|^2)^{\sigma/2}},\ \forall\, x \in \mathbb{R}^N,$$

*pour* $c > 0$ *suffisamment petit, et*

$$\varphi(x) = \omega(x)|x|^{-\sigma},\ |x| \geq A,$$

*Pour une constante* $A > 0$ *et* $\omega \in L^{\infty}(\mathbb{R}^N)$, *homogène de degré* $0$, *avec* $\|\omega\|_{\infty}$ *suffisamment petite.*

*Soit* $u$ *l'unique solution globale de (1.1.7) avec donnée initiale* $\varphi$ *donnée par le Théorème 1.2.3. Soit* $u_{\mathcal{S}}$ *la solution auto-similaire globale de (1.1.7) avec donnée initiale* $\omega(x)|x|^{-\frac{2-\gamma}{\alpha}}$, *donnée par le Théorème 1.2.4. Alors on a*

*(i) Comportement non linéaire : Si* $\sigma = \frac{2-\gamma}{\alpha}$, *alors il existe* $\delta > 0$ *tel que*

$$\|u(t) - u_{\mathcal{S}}(t)\|_{\infty} \leq C t^{-\frac{2-\gamma}{2\alpha}-\delta},\ \forall\ t > 0.$$

*où* $C$ *est une constante strictement positive. En particulier, si* $\omega \not\equiv 0$, *il existe* $C_1,\ C_2$ *deux constantes strictement positives telles que pour* $t$ *grand*

$$C_1 t^{-\frac{2-\gamma}{2\alpha}} \leq \|u(t)\|_{\infty} \leq C_2 t^{-\frac{2-\gamma}{2\alpha}}.$$

*(ii) Comportement linéaire : Si* $\sigma > \frac{2-\gamma}{\alpha}$, *alors il existe* $\delta > 0$ *tel que*

$$\left\| u(t) - e^{t\Delta}\left(\omega(\cdot)|\cdot|^{-\sigma}\right)\right\|_{\infty} \leq C t^{-\frac{\sigma}{2}-\delta},\ \forall\ t > 0,$$

*où* $C$ *est une constante strictement positive. En particulier, si* $\omega \not\equiv 0$, *il existe* $C_1,\ C_2$ *deux constantes strictement positives telles que pour* $t$ *grand*

$$C_1 t^{-\frac{\sigma}{2}} \leq \|u(t)\|_{\infty} \leq C_2 t^{-\frac{\sigma}{2}}.$$

Pour prouver les théorèmes précédents, nous utilisons certains arguments de [14, 43] combinés avec l'estimation clé établie par la Proposition 2.1 dans [2].

## 1.3 Etude des systèmes de Hardy-Hénon

Dans le Chapitre 3 nous cherchons des conditions sur les paramètres $N$, $p > 1$, $q > 1$, $0 \leq \gamma < 2$ et $0 < \rho < 2$ de sorte que nous avons l'existence globale d'une certaine classe de solutions, y compris des solutions auto-similaires et le comportement asymptotiquement auto-similaire de ces solutions. Pour cela, nous définissons $k$, $\alpha_1$, $\alpha_2$, $\beta_1$ et $\beta_2$ par

$$k = \frac{(2-\gamma)q + (2-\rho)}{(2-\rho)p + (2-\gamma)}, \tag{1.3.24}$$

$$\alpha_1 = \frac{1}{2(pq-1)}[(2-\rho)p + (2-\gamma)], \tag{1.3.25}$$

$$\alpha_2 = \frac{1}{2(pq-1)}\left[(2-\gamma)q + (2-\rho)\right], \tag{1.3.26}$$

$$\beta_1 = \alpha_1 - \frac{N}{2r_1} = \frac{1}{2(pq-1)}[(2-\rho)p + (2-\gamma)] - \frac{N}{2r_1}, \quad r_1 > 1, \tag{1.3.27}$$

$$\beta_2 = \alpha_2 - \frac{N}{2r_2} = \frac{1}{2(pq-1)}[(2-\gamma)q + (2-\rho)] - \frac{N}{2r_2}, \quad r_2 > 1, \tag{1.3.28}$$

pour $r_1 = kr_2 > 1$ satisfaisant les conditions du Lemma 3.2.1.

Résumons les résultats obtenus. Tout d'abord, si l'on suppose que les conditions suivantes

$$2\alpha_1 < \min\left(N, \frac{p}{q}(N-\rho)\frac{(2-\gamma)q + (2-\rho)}{[2 + (2-\rho)p - \gamma pq]_+}\right), \tag{1.3.29}$$

et

$$2\alpha_2 < \min\left(N, \frac{q}{p}(N-\gamma)\frac{(2-\rho)p + (2-\gamma)}{[2 + (2-\gamma)q - \rho pq]_+}\right), \tag{1.3.30}$$

sont satisfaites, nous prouvons l'existence de solutions globales pour certaines données initiales $\Phi = (\varphi_1, \varphi_2)$ petites par rapport à la norme $\mathcal{N}$ définie par

$$\mathcal{N}(\Phi) := \sup_{t>0}\left[t^{\beta_1}\|e^{t\Delta}\varphi_1\|_{r_1}, t^{\beta_2}\|e^{t\Delta}\varphi_2\|_{r_2}\right], \tag{1.3.31}$$

où $\beta_1$ et $\beta_2$ sont donnés par (1.3.27). Nous prouvons également, pour $\varphi_1$ homogène de degré $-2\alpha_1$ et $\varphi_2$ homogène de degré $-2\alpha_2$, où $\alpha_1$ et $\alpha_2$ sont donnés par (1.3.25) and (1.3.26), que la donnée initiale $\Phi = (\varphi_1, \varphi_2)$ donne naissance à une solution globale auto-similaire. Ensuite, nous montrons comme dans [2] que ces solutions avec des données initiales $\Psi$ qui se comportent asymptotiquement comme $\Phi$ dans un sens approprié, quand $|x| \to \infty$, sont asymptotiquement auto-similaires dans $L^\infty(\mathbb{R}^N)$. La norme $\mathcal{N}$ donnée dans (1.3.31) est assez faible pour que la donnée initiale $\Phi = (\varphi_1, \varphi_2)$ avec des composantes homogènes ait une norme finie. Nous démontrons enfin des résultats d'unicité plus fort dans des espaces de Lebesgue avec des données initiales petite par rapport à certaines normes.

Yamauchi dans [53] a étudié le système parabolique (S). Dans [53, Theorem 2.1, p. 339] il est montré que pour certaines données initiales positives sous les conditions $\gamma < \min(N, 2)$, $\rho < \min(N, 2)$, $pq - 1 > 0$ et $\max(\alpha_1, \alpha_2) \geq \frac{N}{2}$, il n'existe aucune solution positive non triviale.

Le cas $\gamma = \rho = 0$ a été déjà couvert dans [44]. Dans le cas où $p = q$ et $\gamma = \rho > 0$, le système parabolique (1.1.12) se comporte comme une équation parabolique avec une singularité dans la non linéarité.

Fixons maintenant deux nombres $r_1 = kr_2 > 1$ satisfaisants les conditions du Lemma 3.2.1. Nous avons obtenu le résultat d'existence globale suivant.

**Théorème 1.3.1** (Existence globale et dépendance continue)**.** *Soit $N$ un entier non nul. Soient $p$, $q > 1$. Soient $0 \leq \gamma < \min(N, 2)$ et $0 < \rho < \min(N, 2)$. Soient $\alpha_1$, $\alpha_2$ définis par (1.3.25) et (1.3.26). Supposons que (1.3.29) et (1.3.30) sont satisfaites. Soient $\beta_1$, $\beta_2$ donnés par (1.3.27) et (1.3.28). Soit $M > 0$ tel que*

$$\nu = \max(M^{p-1}\nu_1, M^{q-1}\nu_2) < 1, \tag{1.3.32}$$

*où $\nu_1$ et $\nu_2$ sont deux constantes positives. Choisissons $R > 0$ tel que*

$$R + M\nu \leq M. \tag{1.3.33}$$

*Soit $\Phi = (\varphi_1, \varphi_2)$ un élément de $\mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$ tel que*

$$\mathcal{N}(\Phi) := \sup_{t>0} \left[ t^{\beta_1} \|e^{t\Delta}\varphi_1\|_{r_1}, t^{\beta_2} \|e^{t\Delta}\varphi_2\|_{r_2} \right] \leq R. \tag{1.3.34}$$

*Alors, il existe une unique solution globale $U = (u, v) \in C((0, \infty); L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N))$ du système intégral (1.1.12) tel que*

$$\sup_{t>0} \left[ t^{\beta_1} \|u(t)\|_{r_1}, t^{\beta_2} \|v(t)\|_{r_2} \right] \leq M. \tag{1.3.35}$$

*De plus,*

*(a) $\lim_{t \searrow 0} u(t) = \varphi_1$ et $\lim_{t \searrow 0} v(t) = \varphi_2$ au sens des distributions tempérées,*

*(b) $u(t) - e^{t\Delta}\varphi_1 \in C\left([0, \infty), L^{\tau_1}(\mathbb{R}^N)\right)$ pour $\tau_1$ satisfaisant $\frac{2\alpha_1}{N} < \frac{1}{\tau_1} < \frac{\gamma}{N} + \frac{p}{r_2}$,*

*(c) $v(t) - e^{t\Delta}\varphi_2 \in C\left([0, \infty), L^{\tau_2}(\mathbb{R}^N)\right)$ pour $\tau_2$ satisfaisant $\frac{2\alpha_2}{N} < \frac{1}{\tau_2} < \frac{\rho}{N} + \frac{q}{r_1}$,*

*(d) $\sup_{t>0} t^{\alpha_1 - \frac{N}{2r}} \|u(t)\|_r < \infty, \quad \forall r \in [r_1, \infty]$, et $u \in C((0, \infty), L^r(\mathbb{R}^N) \cap C_0(\mathbb{R}^N))$,*

*(e) $\sup_{t>0} t^{\alpha_2 - \frac{N}{2r}} \|v(t)\|_r < \infty, \quad \forall r \in [r_2, \infty]$, et $v \in C((0, \infty), L^r(\mathbb{R}^N) \cap C_0(\mathbb{R}^N))$.*

*En outre, si $\Phi = (\varphi_1, \varphi_2)$ et $\Psi = (\psi_1, \psi_2)$ satisfait (1.3.34), et si $U_1 = (u_1, v_1)$ et $U_2 = (u_2, v_2)$ sont respectivement les solutions du système (1.1.12) avec données initiales $\Phi$ et $\Psi$, alors*

$$\sup_{t>0} \left[ t^{\beta_1} \|u_1(t) - u_2(t)\|_{r_1}, t^{\beta_2} \|v_1(t) - v_2(t)\|_{r_2} \right] \leq (1 - \nu)^{-1} \mathcal{N}(\Phi - \Psi). \tag{1.3.36}$$

*De plus, si les données initiales $\Phi$ et $\Psi$ sont telles que*

$$\mathcal{N}_\delta(\Phi - \Psi) = \sup_{t>0} \left[ t^{\beta_1 + \delta} \|e^{t\Delta}(\varphi_1 - \psi_1)\|_{r_1}, t^{\beta_2 + \delta} \|e^{t\Delta}(\varphi_2 - \psi_2)\|_{r_2} \right] < \infty, \tag{1.3.37}$$

*pour $0 < \delta < \delta_0$, où*

$$\delta_0 = \min\{1 - \beta_1 q, 1 - \beta_2 p\}, \tag{1.3.38}$$

*alors*

$$\sup_{t>0}\left[t^{\beta_1+\delta}\|u_1(t)-u_2(t)\|_{r_1}, t^{\beta_2+\delta}\|v_1(t)-v_2(t)\|_{r_2}\right] \leq (1-\nu')^{-1}\mathcal{N}_\delta(\Phi-\Psi), \tag{1.3.39}$$

*où la constante positive $M$ est choisie assez petite.*

*Enfin, si l'on suppose aussi que $\Phi=(\varphi_1,\varphi_2)\in L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N)\times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)$ tel que*

$$\mathcal{N}'(\Phi) := \max\left[\|\varphi_1\|_{\frac{N}{2\alpha_1}}, \|\varphi_2\|_{\frac{N}{2\alpha_2}}\right] < R, \tag{1.3.40}$$

*alors la solution $U=(u,v)$ du système intégral (1.1.12) satisfait également $U\in C\left([0,\infty), L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N)\right)\times C\left([0,\infty), L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)\right)$ et*

$$\sup_{t\geq 0}\left[\|u(t)\|_{\frac{N}{2\alpha_1}}, \|v(t)\|_{\frac{N}{2\alpha_2}}\right] \leq M, \tag{1.3.41}$$

*où $M$ et $R$ sont suffisamment petites.*

Nous donnons maintenant le résultat suivant qui prouve l'existence de solutions auto-similaires.

**Théorème 1.3.2** (Solutions auto-similaires)**.** *Soi $N$ un entier non nul. Soient $p,\ q>1$. Soient $0\leq\gamma<\min(N,2)$ et $0<\rho<\min(N,2)$. Soient $\alpha_1$, $\alpha_2$ définis par (1.3.25) et (1.3.26). Supposons que (1.3.29) et (1.3.30) sont satisfaites. Soient $\varphi_1(x)=\omega_1(x)|x|^{-2\alpha_1}$, $\varphi_2(x)=\omega_2(x)|x|^{-2\alpha_2}$, où $\omega_1,\omega_2\in L^\infty(\mathbb{R}^N)$ sont homogènes de degré $0$ et $\|\omega_1\|_\infty, \|\omega_2\|_\infty$ sont suffisamment petites. Notons $\Phi=(\varphi_1,\varphi_2)$, alors il existe une solution globale auto-similaire $U_S=(u_S,v_S)$ de (1.1.12) avec donnée initiale $\Phi$. De plus $U_S(t)\to\Phi$ dans $\mathcal{S}'(\mathbb{R}^N)$ quand $t\to 0$.*

Nous passons maintenant au comportement asymptotique.

**Théorème 1.3.3** (Comportement asymptotique)**.** *Soit $N$ un entier non nul. Soient $p,\ q>1$. Soient $0\leq\gamma<\min(N,2)$ et $0<\rho<\min(N,2)$. Soient $\alpha_1$, $\alpha_2$ définis par (1.3.25) et(1.3.26). Supposons que (1.3.29) et (1.3.30) sont satisfaites. Soient $\beta_1$, $\beta_2$ donnés par (1.3.27) et (1.3.28). Définissons $\beta_1(q)$ et $\beta_2(q)$ par*

$$\beta_1(q)=\alpha_1-\frac{N}{2q},\quad \beta_2(q)=\alpha_2-\frac{N}{2q},\quad q>1. \tag{1.3.42}$$

*Soit $\Phi$ donnée par*

$$\Phi(x)=(\varphi_1(x),\varphi_2(x)) := \left(\omega_1(x)|x|^{-2\alpha_1}, \omega_2(x)|x|^{-2\alpha_2}\right)$$

*avec $\omega_1$, $\omega_2$ homogène de degré $0$, $\omega_1,\omega_2\in L^\infty(\mathbb{R}^N)$ et $\|\omega_1\|_\infty$, $\|\omega_1\|_\infty$ sont suffisamment petites. Soit*

$$U_{\mathcal{S}}(t,x)=\left(t^{-\alpha_1}u_{\mathcal{S}}(1,\frac{x}{\sqrt{t}}), t^{-\alpha_2}v_{\mathcal{S}}(1,\frac{x}{\sqrt{t}})\right)$$

*la solution auto-similaire de (1.1.12) donnée par le Théorème 1.3.2.*

*Soit $\Psi=(\psi_1,\psi_2)\in C_0(\mathbb{R}^N)\times C_0(\mathbb{R}^N)$ tel que*

$$|\psi_1(x)|\leq\frac{c}{(1+|x|^2)^{\alpha_1}},\ \forall\, x\in\mathbb{R}^N,\ \psi_1(x)=\omega_1(x)|x|^{-2\alpha_1},\ |x|\geq A,$$

$$|\psi_2(x)| \leq \frac{c}{(1+|x|^2)^{\alpha_2}},\ \forall\, x \in \mathbb{R}^N,\ \psi_2(x) = \omega_2(x)|x|^{-2\alpha_2},\ |x| \geq A,$$

*pour une constante $A > 0$, où $c$ est une petite constante strictement positive. (On prend $\|\omega_1\|_\infty$, $\|\omega_2\|_\infty$ et $c$ suffisamment petite pour que (1.3.34) soit satisfaite par $\Phi$ et $\Psi$).*

*Soit $U = (u, v)$ la solution globale de (1.1.12) avec donnée initiale $\Psi$ construite par le Théorème 1.3.1. Alors il existe $\delta > 0$ suffisamment petit tel que*

$$\|u(t) - u_{\mathcal{S}}(t)\|_{q_1} \leq C_\delta t^{-\beta_1(q_1)-\delta},\ \forall\, t > 0, \tag{1.3.43}$$

$$\|v(t) - v_{\mathcal{S}}(t)\|_{q_2} \leq C_\delta t^{-\beta_2(q_2)-\delta},\ \forall\, t > 0, \tag{1.3.44}$$

*pour tous $q_1 \in [r_1, \infty]$, $q_2 \in [r_2, \infty]$. De plus, nous avons*

$$\|t^{\alpha_1} u(t, .\sqrt{t}) - u_{\mathcal{S}}(1, .)\|_{q_1} \leq C_\delta t^{-\delta},\ \forall\, t > 0, \tag{1.3.45}$$

$$\|t^{\alpha_2} v(t, .\sqrt{t}) - v_{\mathcal{S}}(1, .)\|_{q_2} \leq C_\delta t^{-\delta},\ \forall\, t > 0, \tag{1.3.46}$$

*pour tous $q_1 \in [r_1, \infty]$, $q_2 \in [r_2, \infty]$.*

Il a été prouvé dans le Théorème 1.3.1 que pour une donnée initiale petite $\Phi = (\varphi_1, \varphi_2) \in L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)$ par rapport à la norme $\mathcal{N}'$, il existe une solution $U_\Phi = (u_\Phi, v_\Phi)$ du système intégral (1.1.12) et l'unicité est garantie uniquement parmi les fonctions continues $U : [0, \infty) \to L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)$ qui vérifient également $\sup\limits_{t>0} \left[t^{\beta_1}\|u(t)\|_{r_1}, t^{\beta_2}\|v(t)\|_{r_2}\right]$ est suffisamment petit. Notre objectif est de prouver que l'unicité est garantie pour les solutions qui appartiennent à $C\left([0, \infty), L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N)\right) \times C\left([0, \infty), L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)\right) \cap C\left((0, \infty), L^{r_1}(\mathbb{R}^N)\right) \times C\left((0, \infty), L^{r_2}(\mathbb{R}^N)\right)$, ce qui améliore le résultat de l'unicité dans les espaces de Lebesgue donné dans le Théorème 1.3.1. On utilise des arguments de type Brezis Cazenave [3]. Nous avons obtenu le résultat suivant.

**Théorème 1.3.4.** *Soit $N$ un entier non nul. Soient $p$, $q > 1$. Soient $0 \leq \gamma < \min(N, 2)$ et $0 < \rho < \min(N, 2)$. Soient $\alpha_1$, $\alpha_2$ définis par (1.3.25) et (1.3.26). Supposons que (1.3.29) et (1.3.30) sont satisfaites. Soient $\beta_1$, $\beta_2$ donnés par (1.3.27) et (1.3.28). Soient $M$, $R > 0$ suffisamment petites. Soit $\Phi = (\varphi_1, \varphi_2) \in L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)$ satisfaisant (1.3.40). Soit $U_\Phi = (u_\Phi, v_\Phi)$ la solution du système intégral (1.1.12) avec donnée initiale $\Phi$ construite par la dernière partie du Théorème 1.3.1. Soit $V = (v_1, v_2) \in C\left([0, \infty), L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N)\right) \times C\left([0, \infty), L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)\right) \cap C\left((0, \infty), L^{r_1}(\mathbb{R}^N)\right) \times C\left((0, \infty), L^{r_2}(\mathbb{R}^N)\right)$ une solution de (1.1.12) avec la même donnée initiale $\Phi$. Alors*

$$V(t) = U_\Phi(t), \quad \forall t \in [0, \infty).$$

## 1.4 Instabilité de certaines solutions globales

Le but principal du Chapitre 4 est d'établir le résultat suivant.

**Théorème 1.4.1.** *Supposons (1.1.14)-(1.1.15). Etant donné* $1 < q < p_S = \frac{N+2}{N-2}$. *Il existe* $1 < q < \underline{p} < p_S$ *avec la propriété suivante. Si* $\underline{p} < p < p_S$ *et si* $h_p \in C_0(\Omega)$ *est une solution stationnaire à symétrie radiale de (1.1.13) qui prend des valeurs positives et négatives, alors il existe* $0 < \underline{\lambda} < 1 < \overline{\lambda}$ *tel que si* $\underline{\lambda} < \lambda < \overline{\lambda}$ *et* $\lambda \neq 1$, *alors la solution classique de (1.1.13) avec la condition initiale* $u(0) = \lambda h$ *explose en temps fini.*

La première observation est qu'il existe une solution stationnaire à symétrie radiale de (1.1.13), puisque la non-linéarité $|s|^{p-1}s - |s|^{q-1}s$ satisfait les hypothèses de [32, Theorem 2 , p. 376]. Plus précisément, si l'on considère le problème :

$$\begin{cases} h'' + \frac{N-1}{r}h' + |h|^{p-1}h - |h|^{q-1}h = 0, \\ h(0) = a > 0, \quad h'(0) = 0. \end{cases} \tag{1.4.47}$$

Il est bien connu par [32] que (1.4.47) admet une solution unique $h \in C^2\left([0,\infty), \mathbb{R}\right)$, qu'on la note parfois $h_p(r,a)$ pour mettre l'accent sur la dépendance par rapport à $a$. Sous ces conditions, par le Théorème 2 dans [32] pour tout entier $m \geq 0$, il existe $a_{p,m}$ tel que

a) $h_p(1, a_{p,m}) = 0$,

b) $h_p(r, a_{p,m})$ a précisément $m$ zéros dans $(0,1)$.

En particulier, $h_p(\cdot, a_{p,m})$, considérée comme une fonction sur $\Omega = B_1$, est une solution à symétrie radiale de (1.1.18) qui change de signe précisément $m$ fois.

Maintenant, soit $h_p$ une solution non triviale de (1.1.18) et considérons l'opérateur linéarisé $F_p$ dans $L^2(\Omega)$ défini par

$$\begin{cases} D(F_p) = H^2(\Omega) \cap H_0^1(\Omega), \\ F_p u = -\Delta u - \left(p|h_p|^{p-1} - q|h_p|^{q-1}\right)u, \qquad u \in D(F_p). \end{cases} \tag{1.4.48}$$

Rappelons le résultat suivant de [9].

**Théorème 1.4.2** ([9, Corollary 2.5, p. 18])**.** *Soit* $h_p \in C_0(\Omega)$ *une solution qui prend des valeurs positives et négatives de (1.1.18). Soit* $\varphi_p$ *un vecteur propre strictement positif de l'opéateur auto-adjoint* $F_p$ *donné par (1.4.48), associé à la première valeur propre. Supposons que*

$$\int_\Omega h_p \varphi_p \neq 0.$$

*Il en résulte qu'il existe* $\epsilon > 0$ *tel que si* $0 < |1-\lambda| < \epsilon$, *alors la solution de (1.1.13) avec la donnée initiale* $u_0 = \lambda h_p$ *explose en temps fini.*

Pour montrer le Théorème 1.4.1, il suffit donc d'établir ce qui suit.

**Théorème 1.4.3.** *Supposons (1.1.14)-(1.1.15). Etant donné* $1 < q < p_S = \frac{N+2}{N-2}$. *Il existe* $1 < q < \underline{p} < p_S$ *avec la propriété suivante. Si* $\underline{p} < p < p_S$ *et si* $h_p \in C_0(\Omega)$ *est une solution stationnaire à symétrie radiale de (1.1.13) qui prend des valeurs positives et négatives, alors*

$$\int_\Omega h_p \varphi_p \neq 0.$$

*Où* $\varphi_p$ *est un vecteur propre strictement positif de l'opérateur auto-adjoint* $F_p$ *donné par (1.4.48), associé à la première valeur propre.*

La preuve du Théorème 1.4.3 est basée sur un argument de changement d'échelle. Contrairement au cas d'une seul puissance dans la non-linéarité, la fonction rééchelonnée $v_p$ définie en termes de $h_p$, où $h_p$ est une solution stationnaire à symétrie radiale de(1.1.13) ne satisfait pas la même équation différentielle satisfaite par $h_p$, ce qui rend la situation plus difficile. En outre, contrairement au cas d'une seul non-linéarité dans la puissance, il existe des solutions $v_p(r)$ qui ne tendent pas vers zéro quand $r \to \infty$.

## 1.5 Publications liées à la thèse

[1] B. BEN SLIMENE, S. TAYACHI, F.B. WEISSLER, Well-posedness, global existence and large time behavior for Hardy-Hénon parabolic equations, Nonlinear Anal., **152** (2017), pp. 116-148, (Chapitre 2).

[2] B. BEN SLIMENE, Sign-changing stationary solutions and blowup for the two power nonlinear heat equation in a ball, J. Math. Anal. Appl., **454** (2017), pp. 1067-1084, (Chapitre 4).

[3] B. BEN SLIMENE, Asymptotically self-similar global solutions for Hardy-Hénon Parabolic Systems, Submitted, (Chapitre 3).

# Chapitre 2

# Study of the Hardy-Hénon equations

## 2.1 Introduction

In this chapter we consider the singular nonlinear parabolic equation

$$\partial_t u = \Delta u + a|\cdot|^{-\gamma}|u|^\alpha u, \tag{2.1.1}$$

$u = u(t,x) \in \mathbb{R},\ \ t > 0,\ x \in \mathbb{R}^N,\ N \geq 1,\ \ a \in \mathbb{R},\ \alpha > 0,\ \gamma > 0$ and with initial value

$$u(0) = \varphi. \tag{2.1.2}$$

The case $\gamma = 0$ corresponds to the standard nonlinear heat equation. For $\gamma < 0$ it is known in the literature as a Hénon parabolic equation, while if $\gamma > 0$ it is known as a Hardy parabolic equation. In this chapter we are concerned with the case $\gamma > 0$. We are interested in the well-posedness of (2.1.1) with initial data $\varphi \in L^q(\mathbb{R}^N),\ 1 \leq q < \infty$, and in $C_0(\mathbb{R}^N)$. We also study the existence of global solutions, including self-similar solutions and prove the existence of asymptotically self-similar solutions.

In what follows, we denote $\|.\|_{L^q(\mathbb{R}^N)}$ by $\|.\|_q,\ 1 \leq q \leq \infty$. For all $t > 0$, $e^{t\Delta}$ denotes the heat semi-group

$$\left(e^{t\Delta}f\right)(x) = \int_{\mathbb{R}^N} G(t, x-y)f(y)dy, \tag{2.1.3}$$

where

$$G(t,x) = (4\pi t)^{-N/2}e^{-\frac{|x|^2}{4t}},\ t > 0,\ x \in \mathbb{R}^N, \tag{2.1.4}$$

and $f \in L^q(\mathbb{R}^N),\ q \in [1,\infty)$ or $f \in C_0(\mathbb{R}^N)$. For $f \in \mathcal{S}'(\mathbb{R}^N),\ e^{t\Delta}f$ is defined by duality. A mild solution of the problem (2.1.1)-(2.1.2) is a solution of the integral equation

$$u(t) = e^{t\Delta}\varphi + a\int_0^t e^{(t-s)\Delta}\left(|\cdot|^{-\gamma}|u(s)|^\alpha u(s)\right)ds, \tag{2.1.5}$$

and it is in this form that we consider problem (2.1.1)-(2.1.2).

We first consider local well-posedness for the integral equation (2.1.5). To our knowledge, there is only one previous result of this type, Wang [48], who works in the space $C_B(\mathbb{R}^N)$ of continuous bounded functions. For $N \geq 3,\ a > 0$ and $\gamma < 2$, he proves local existence of solutions to (2.1.5) in $C([0,T]; C_B(\mathbb{R}^N))$ for all $\varphi \in C_B(\mathbb{R}^N)$. See [48, Theorem 2.3, p. 563].

In this chapter, we prove local well-posedness in $C_0(\mathbb{R}^N)$, the space of continuous functions vanishing at infinity, and in $L^q(\mathbb{R}^N)$ for certain values of $q$. We also require the condition $\gamma < 2$, and in fact $0 < \gamma < 2$. Throughout the chapter we put, for $\alpha > 0,\ 0 < \gamma < 2$,

$$q_c = \frac{N\alpha}{2-\gamma}. \tag{2.1.6}$$

The critical exponent $q_c$ plays a crucial role in this theory. We will say that $q$ is subcritical, critical or supercritical, according to whether $q < q_c,\ q = q_c$ or $q > q_c$. We have obtained the following results.

**Theorem 2.1.1** (Local well-posedness)**.** *Let $N \geq 1$ be an integer, $\alpha > 0$ and $\gamma$ such that*

$$0 < \gamma < \min(2, N). \tag{2.1.7}$$

*Let $q_c$ be given by (2.1.6). Then we have the following.*

*(i) Equation (2.1.5) is locally well-posed in $C_0(\mathbb{R}^N)$. More precisely, given $\varphi \in C_0(\mathbb{R}^N)$, then there exist $T > 0$ and a unique solution $u \in C\big([0,T]; C_0(\mathbb{R}^N)\big)$ of (2.1.5). Moreover, $u$ can be extended to a maximal interval $[0, T_{\max})$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t \to T_{\max}} \|u(t)\|_\infty = \infty$.*

*(ii) If $q$ is such that*

$$q > \frac{N(\alpha+1)}{N-\gamma}, \quad q > q_c \quad \text{and} \quad q < \infty,$$

*then equation (2.1.5) is locally well-posed in $L^q(\mathbb{R}^N)$. More precisely, given $\varphi \in L^q(\mathbb{R}^N)$, then there exist $T > 0$ and a unique solution $u \in C\big([0,T]; L^q(\mathbb{R}^N)\big)$ of (2.1.5). Moreover, $u$ can be extended to a maximal interval $[0, T_{\max})$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t \to T_{\max}} \|u(t)\|_q = \infty$.*

*(iii) Assume that $q \geq q_c$ with $1 < q < \infty$. It follows that equation (2.1.5) is locally well-posed in $L^q(\mathbb{R}^N)$ as in Part (ii) except that uniqueness is guaranteed only among continuous functions $u : [0,T] \to L^q(\mathbb{R}^N)$ which also verify*

*(a) $t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}\|u(t)\|_r$ is bounded on $(0,T]$, where $r > q$ satisfies (2.3.6), $q > q_c$;*

*(b) $\sup_{t \in (0,T]} t^{\frac{N}{2}(\frac{1}{q_c}-\frac{1}{r})}\|u(t)\|_r$ is sufficiently small, where $r > q$ is given in (2.4.19), $q = q_c$.*

*Moreover, $u$ can be extended to a maximal interval $[0, T_{\max})$ such that, in the case $q > q_c$, either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t \to T_{\max}} \|u(t)\|_q = \infty$.*

*(iv) In all the above cases, except where $q = q_c$, the minimal existence time of the solution, denoted by $T$, depends only on $\|\varphi\|_\infty$ or $\|\varphi\|_q$ respectively.*

**Remark 2.1.1.** Wang shows the equivalence of $C_B(\mathbb{R}^N)$ solutions of the integral equation (2.1.5) and weak continuous solutions of (2.1.1)-(2.1.2), which are also distribution solutions. See the remark in [48] just after Definition 2.1 of [48] on page 563. The $C_0(\mathbb{R}^N)$ solutions above are included in this situation. For $t > 0$ the $L^q(\mathbb{R}^N)$ solutions are also $C_0(\mathbb{R}^N)$ solutions, and therefore have the same regularity. See Part (i) of Proposition 2.3.2.

**Remark 2.1.2.** Well-posedness of (2.1.5) breaks down for $q$ subcritical. The proof is based on the existence of a positive forward rapidly decaying self-similar solution [26, Theorem 1.1, p. 625]. It follows from this result, in analogy with [25], that if $1 < q < q_c$, $a > 0$, $N \geq 3$ and $\frac{2-\gamma}{N} < \alpha < 2\frac{2-\gamma}{N-2}$, then there exists a positive solution $u$ of (2.1.5) with initial data $\varphi = 0$ in $L^q(\mathbb{R}^N)$. This shows a non-uniqueness result in $L^q(\mathbb{R}^N)$ and thus the ill-posedness for some values $q < q_c$.

The proof of Theorem 2.1.1 is based on arguments and results in [49, 50]. To apply these methods, a key new ingredient is needed. See Proposition 2.2.1. As the proofs will show, these solutions depend continuously on the initial data in an appropriate sense. Moreover, if a given $\varphi$ belongs to two different spaces as described in Theorem 2.1.1, then the resulting solutions in the different spaces coincide, and in particular, the maximal existence time does not depend on the space. In addition we have the following lower estimate for the blow-up rate.

**Theorem 2.1.2** (Lower blow-up rate)**.** *Under the hypotheses of Theorem 2.1.1, let $\varphi \in C_0(\mathbb{R}^N)$, respectively $\varphi \in L^q(\mathbb{R}^N)$, $q > 1$ with $q > q_c$, and suppose that $T_{\max} < \infty$, where $T_{\max}$ is the existence time of the resulting maximal solution of* (2.1.5)*. It follows*

$$\|u(t)\|_q \geq C\,(T_{\max} - t)^{\frac{N}{2q} - \frac{2-\gamma}{2\alpha}}\,,\ \forall\, t \in [0, T_{\max}), \tag{2.1.8}$$

*where $C$ is a positive constant.*

We are unaware of any previous lower blow-up estimates. On the hand, the following upper blow-up estimate has been established by [1, Theorems 1.2 and 1.3] and [33, Theorem 1.6] in the case $q = \infty$ (with various restriction on $\alpha$):

$$\|u(t)\|_\infty \leq C\,(T_{\max} - t)^{-\frac{1}{\alpha}}\,,\ \forall\, t \in [0, T_{\max}). \tag{2.1.9}$$

Note that there is a gap between the above lower and upper estimates (2.1.8) and (2.1.9), in particular for solutions blowing up at the origin. The blow-up rate (2.1.8) can in fact be realized as shown by the existence of a backward self-similar solution in [17, Theorem A, p. 470 and Proposition 3.1, p. 477]. We believe that the techniques of [52] can be adapted to show that, under certain conditions, (2.1.8) gives an upper bound for the blow up rate. To carry out these arguments in the present context seems nontrivial.

Recently we have improved Part (iii) in Theorem 2.1.1, more precisely we have obtained the following.

**Theorem 2.1.3.** *Under hypotheses of Theorem 2.1.1. Assume that $q \geq q_c$ with $1 < q < \infty$. It follows that equation* (2.1.5) *is locally well-posed in $L^q(\mathbb{R}^N)$ except that uniqueness is guaranteed only among functions $u : [0,T] \to L^q(\mathbb{R}^N)$ which also verify*

*(a)* $u \in C\left([0,T], L^q(\mathbb{R}^N)\right) \cap C\left((0,T], L^r(\mathbb{R}^N)\right)$, *where $r > q$ satisfies* (2.3.6), $q > q_c$;

*(b)* $u \in C\left([0,T], L^q(\mathbb{R}^N)\right) \cap C\left((0,T], L^r(\mathbb{R}^N)\right)$, *where $r > q$ is given in* (2.4.19), $q = q_c$.

*Moreover, $u$ can be extended to a maximal interval $[0, T_{\max})$ such that, in the case $q > q_c$, either $T_{\max} = \infty$ or $T_{\max} < \infty$ and* $\lim\limits_{t \to T_{\max}} \|u(t)\|_q = \infty$.

For the proof see Proposition 2.3.1 and Remark 3.4.2.

We now turn to the global existence of solutions. We have obtained the following result.

**Theorem 2.1.4** (Global existence)**.** *Let $N \geq 1$ be an integer, $\alpha$, $\sigma$, $\gamma > 0$. Suppose that (2.1.7) is satisfied and*

$$\alpha > \frac{2-\gamma}{N}, \quad \textit{equivalently } q_c > 1,$$

*where $q_c$ is given by* (2.1.6)*. Then we have the following.*

*(i) If $\varphi \in L^{q_c}(\mathbb{R}^N)$ and $\|\varphi\|_{q_c}$ is sufficiently small, then $T_{\max} = \infty$.*

*(ii) If $\varphi \in C_0(\mathbb{R}^N)$ such that $|\varphi(x)| \leq c\left(1+|x|^2\right)^{-\frac{\sigma}{2}}$, for all $x \in \mathbb{R}^N$ with $c$ sufficiently small and*

$$\sigma > \frac{2-\gamma}{\alpha}, \tag{2.1.10}$$

*then $T_{\max} = \infty$.*

*(iii) Let $\varphi \in L^1_{loc}(\mathbb{R}^N)$ be such that $|\varphi(\cdot)| \leq c|\cdot|^{-\frac{2-\gamma}{\alpha}}$, for $c$ sufficiently small. Then there exists a global in time solution of* (2.1.5)*, $u \in C\left((0,\infty); L^q(\mathbb{R}^N)\right)$ for all $q > q_c$. Moreover $u(t) \to \varphi$ in $\mathcal{S}'(\mathbb{R}^N)$ as $t \to 0$.*

**Remark 2.1.3.** Using the backwards self-similar solution constructed in [17, Proposition 3.1, p. 477] and arguing as in [45, Corollary 1.4, p. 659], we can prove the following result. Suppose $N \geq 3$, $a > 0$ and $\alpha$ such that $\frac{2-\gamma}{N} < \alpha < 2\frac{2-\gamma}{N-2}$. There exists $C > 0$ such that if $\varphi \in C_0(\mathbb{R}^N), \varphi > 0$ and

$$\liminf_{|x| \to \infty} |x|^{\frac{2-\gamma}{\alpha}} \varphi(x) \geq C,$$

then $T_{\max} < \infty$.

By Fujita type blow-up results, the condition $\alpha > (2-\gamma)/N$ ($q_c > 1$) is optimal in Theorem 2.1.4. In fact, if $a > 0$ and $\alpha \leq (2-\gamma)/N$ then the solutions of (2.1.1) with positive initial data blow-up in finite time. See [37, Theorem 1.6, p. 126]. In [37, Theorem 1.2, p. 125], examples of global solutions to (2.1.1) with positive initial data for $q_c > 1$ and $a > 0$ are constructed. Still for $a > 0$, other global existence results for positive initial data are proved in [48] and [26] but only for $N \geq 3$ and under supplementary conditions on $\alpha$. See [26, Theorem 1.3, p. 626]. Our global existence results are established without any restriction on $a$ or on the initial data (other than a smallness condition).

In [48], Wang observed that, by modifying the arguments of Lee and Ni [27], if (2.1.1) has a global solution, then necessarily $\liminf_{|x|\to\infty} |x|^{\frac{2-\gamma}{\alpha}}\varphi(x) < \infty$, so that our results in Theorem 2.1.4 (ii) and (iii) are consistent with this condition. For $\gamma = 0$, (i) is known, see [18, Theorem 3, p. 32].

We now turn to the existence of forward self-similar solutions. In [26, 48] the existence of radially symmetric self-similar solutions for $\frac{2-\gamma}{N} < \alpha < 2\frac{2-\gamma}{N-2}$ and $N \geq 3$ is established. Here we do not impose radial symmetry of the self-similar solutions. We have obtained the following result.

**Theorem 2.1.5** (Self-similar solutions)**.** *Let $N \geq 1$ be an integer, $\alpha,\ \gamma > 0$. Suppose that $0 < \gamma < \min(2, N)$ and*

$$\alpha > \frac{2-\gamma}{N}. \tag{2.1.11}$$

*Let $\varphi(x) = \omega(x)|x|^{-\frac{2-\gamma}{\alpha}}$, where $\omega \in L^\infty(\mathbb{R}^N)$ is homogeneous of degree 0 and $\|\omega\|_\infty$ is sufficiently small. Then there exists a global mild self-similar solution $u_{\mathcal{S}}$ of (2.1.1)-(2.1.2). This solution verifies $u_{\mathcal{S}}(t) \to \varphi$ in $\mathcal{S}'(\mathbb{R}^N)$ as $t \to 0$.*

Concerning the asymptotic behavior of global solutions, we have the following result.

**Theorem 2.1.6** (Asymptotic Behavior)**.** *Let $N \geq 1$ be an integer, $\alpha,\ \sigma,\ \gamma > 0$. Suppose that $0 < \gamma < \min(2, N)$ and*

$$\frac{2-\gamma}{\alpha} \leq \sigma < N. \tag{2.1.12}$$

*Let $\varphi \in C_0(\mathbb{R}^N)$ be such that*

$$|\varphi(x)| \leq \frac{c}{(1+|x|^2)^{\sigma/2}},\ \forall\, x \in \mathbb{R}^N,$$

*for $c > 0$ sufficiently small, and*

$$\varphi(x) = \omega(x)|x|^{-\sigma},\ |x| \geq A,$$

*for some constant $A > 0$ and some $\omega \in L^\infty(\mathbb{R}^N)$, homogeneous of degree 0, with $\|\omega\|_\infty$ sufficiently small.*

*Let $u$ be the unique global solution of (2.1.5) with initial data $\varphi$ given by Theorem 2.1.4. Let $u_{\mathcal{S}}$ be the global self-similar solution of (2.1.5) with initial data $\omega(x)|x|^{-\frac{2-\gamma}{\alpha}}$, given by Theorem 2.1.5. Then we have the following.*

*(i) Nonlinear behavior: If $\sigma = \frac{2-\gamma}{\alpha}$, then there exists $\delta > 0$ such that*

$$\|u(t) - u_{\mathcal{S}}(t)\|_\infty \leq Ct^{-\frac{2-\gamma}{2\alpha}-\delta},\ \forall\, t > 0.$$

*where $C$ is a positive constant. In particular, if $\omega \not\equiv 0$, there exist $C_1$, $C_2$, two positive constants such that for $t$ large*

$$C_1 t^{-\frac{2-\gamma}{2\alpha}} \leq \|u(t)\|_\infty \leq C_2 t^{-\frac{2-\gamma}{2\alpha}}.$$

*(ii) Linear behavior: If $\sigma > \frac{2-\gamma}{\alpha}$, then there exists $\delta > 0$ such that*

$$\left\| u(t) - e^{t\Delta}\left(\omega(\cdot)|\cdot|^{-\sigma}\right)\right\|_\infty \leq Ct^{-\frac{\sigma}{2}-\delta},\ \forall\, t > 0,$$

*where $C$ is a positive constant. In particular, if $\omega \not\equiv 0$, there exist $C_1$, $C_2$ two positive constants such that for $t$ large*

$$C_1 t^{-\frac{\sigma}{2}} \leq \|u(t)\|_\infty \leq C_2 t^{-\frac{\sigma}{2}}.$$

For $\gamma = 0$, analogous results were obtained in [14] and [43]. For $\gamma > 0$ the only known result to our knowledge, is a decay rate of the sup norm. See [48]. To prove the previous theorems, we use some arguments from [14, 43] (see also references therein), combined with the estimates of Proposition 2.2.1.

The rest of the chapter is organized as follows. In Section 2, we establish the estimate for the heat semi-group needed to treat the singular potential in (2.1.1). See Proposition 2.2.1. In Section 3 we prove Theorem 2.1.1, except the case $q = q_c$, and we also prove Theorem 2.1.2. In Section 4, we prove Theorem 2.1.1 for $q = q_c$, Theorem 2.1.4 and Theorem 2.1.5. In Section 5 we prove the nonlinear asymptotic behavior in Theorem 2.1.6. Section 6 is devoted to the proof of the linear asymptotic behavior in Theorem 2.1.6. Finally in Section 7, we consider a more general equation where $|x|^{-\gamma}$ is replaced by a function $V(x)$. Throughout this chapter $C$ will be a positive constant which may have different values at different places.

## 2.2 A Key Estimate

In this section we prove the new estimate which is needed for the proofs of essentially all the results in this chapter. Let $e^{t\Delta}$ be the linear heat semi-group defined by: $e^{t\Delta}\varphi = G(t,.) * \varphi,\ t > 0$, where G is the heat kernel defined by (2.1.4). We recall the well-known smoothing effect of the heat semi-group on Lebesgue spaces,

$$\|e^{t\Delta}u\|_{s_2} \leq (4\pi t)^{-\frac{N}{2}(\frac{1}{s_1}-\frac{1}{s_2})}\|u\|_{s_1}, \tag{2.2.1}$$

for $1 \leq s_1 \leq s_2 \leq \infty$, $t > 0$ and $u \in L^{s_1}(\mathbb{R}^N)$.

To treat the nonlinear term in Eq. (2.1.1), which includes the factor $|\cdot|^{-\gamma}$, we establish the following estimate, analogous to (2.2.1).

**Proposition 2.2.1.** *Let $N \geq 1$ be an integer. Let $\gamma$ such that $0 < \gamma < N$. Let $q_1 \in (1, \infty]$ and $q_2 \in (1, \infty]$ satisfy*

$$0 \leq \frac{1}{q_2} < \frac{\gamma}{N} + \frac{1}{q_1} < 1.$$

*Then, for all $t > 0$, the following are bounded maps*

*(i)* $e^{t\Delta}|\cdot|^{-\gamma} : L^{q_1}(\mathbb{R}^N) \to L^{q_2}(\mathbb{R}^N),\ q_2 < \infty;$

*(ii)* $e^{t\Delta}|\cdot|^{-\gamma} : L^{q_1}(\mathbb{R}^N) \to C_0(\mathbb{R}^N),\ q_2 = \infty.$

*Furthermore, there exists a constant $C > 0$ depending on $N, \gamma, q_1$ and $q_2$ such that*

$$\|e^{t\Delta}(|.|^{-\gamma}u)\|_{q_2} \leq C t^{-\frac{N}{2}(\frac{1}{q_1}-\frac{1}{q_2})-\frac{\gamma}{2}}\|u\|_{q_1}, \quad \forall\, t > 0, \quad \forall\, u \in L^{q_1}(\mathbb{R}^N). \tag{2.2.2}$$

*Proof.* We set $m = \frac{N}{\gamma}$. Let $\epsilon, \delta > 0$ satisfy

$$\epsilon < m, \ \frac{1}{q_2} \le \frac{1}{m+\delta} + \frac{1}{q_1} \le \frac{1}{m-\epsilon} + \frac{1}{q_1} \le 1.$$

Let us consider the following decomposition

$$|.|^{-\gamma} = \psi_1 + \psi_2; \quad \psi_1 \in L^{m-\epsilon}(\mathbb{R}^N), \quad \psi_2 \in L^{m+\delta}(\mathbb{R}^N).$$

Using the Hölder inequality

$$\|\psi_1 u\|_{r_1} \le \|\psi_1\|_{m-\epsilon}\|u\|_{q_1},$$

where

$$\frac{1}{r_1} = \frac{1}{m-\epsilon} + \frac{1}{q_1}.$$

Similarly

$$\|\psi_2 u\|_{r_2} \le \|\psi_2\|_{m+\delta}\|u\|_{q_1},$$

where

$$\frac{1}{r_2} = \frac{1}{m+\delta} + \frac{1}{q_1}.$$

So that by the smoothing effect of the heat equation (2.2.1) we have

$$\begin{aligned} \|e^{\Delta}(|.|^{-\gamma}u)\|_{q_2} &\le \|e^{\Delta}(\psi_1 u)\|_{q_2} + \|e^{\Delta}(\psi_2 u)\|_{q_2} \\ &\le C\|\psi_1 u\|_{r_1} + C\|\psi_2 u\|_{r_2} \\ &\le C\Big(\|\psi_1\|_{m-\epsilon} + \|\psi_2\|_{m+\delta}\Big)\|u\|_{q_1}, \end{aligned}$$

where we used $1 \le r_1 \le r_2 \le q_2$. Therefore we obtain

$$\|e^{\Delta}(|.|^{-\gamma}u)\|_{q_2} \le C(N,\gamma,q_1,q_2)\|u\|_{q_1}. \tag{2.2.3}$$

Since $r_1 < \infty$ and $r_2 < \infty$, $e^{\Delta}(\psi_1 u)$ and $e^{\Delta}(\psi_2 u)$ are in $C_0(\mathbb{R}^N)$, and so therefore is $e^{\Delta}(|.|^{-\gamma}u)$.

We now prove (2.2.2) by a scaling argument. Given $\lambda > 0$, we define the dilation operator $D_\lambda$ by $D_\lambda\varphi(x) = \varphi(\lambda x)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$. This operator is extended by duality to $\mathcal{S}'(\mathbb{R}^N)$. Clearly we have

(i) $D_\lambda(e^{\lambda^2 t\Delta}\varphi) = e^{t\Delta}(D_\lambda\varphi)$ for all $\varphi \in \mathcal{S}'(\mathbb{R}^N)$,

(ii) $D_\lambda(D_{\frac{1}{\lambda}}\varphi) = \varphi$ for all $\varphi \in \mathcal{S}'(\mathbb{R}^N)$,

(iii) $\|D_\lambda\varphi\|_r = \lambda^{-\frac{N}{r}}\|\varphi\|_r$ for all $\varphi \in L^r(\mathbb{R}^N), r \ge 1$,

(iv) $D_\lambda(\varphi\psi) = D_\lambda\varphi D_\lambda\psi$, for all $\varphi, \ \psi, \ \varphi\psi \in L^1_{loc}(\mathbb{R}^N)$,

(v) $D_\lambda(|.|^{-\gamma}) = \lambda^{-\gamma}|.|^{-\gamma}$, for all $\gamma > 0$.

It follows that $e^{\lambda^2 t\Delta}\varphi = D_{\frac{1}{\lambda}}e^{t\Delta}D_\lambda\varphi$, and so $e^{\Delta}\varphi = D_{\sqrt{t}}e^{t\Delta}D_{\frac{1}{\sqrt{t}}}\varphi$ for all $\varphi \in \mathcal{S}'(\mathbb{R}^N)$.

Then, from (2.2.3), we have that

$$\|D_{\sqrt{t}}e^{t\Delta}D_{\frac{1}{\sqrt{t}}}(|.|^{-\gamma}u)\|_{q_2} \le C(N,\gamma,q_1,q_2)\|u\|_{q_1},$$

and so

$$t^{-\frac{N}{2q_2}}\|e^{t\Delta}D_{\frac{1}{\sqrt{t}}}(|.|^{-\gamma}u)\|_{q_2} \leq C(N,\gamma,q_1,q_2)\|u\|_{q_1}.$$

Therefore

$$t^{-\frac{N}{2q_2}}t^{\frac{\gamma}{2}}\|e^{t\Delta}(|.|^{-\gamma}D_{\frac{1}{\sqrt{t}}}u)\|_{q_2} \leq C(N,\gamma,q_1,q_2)\|u\|_{q_1}.$$

Replacing $u$ by $D_{\sqrt{t}}u$, we obtain

$$t^{-\frac{N}{2q_2}}t^{\frac{\gamma}{2}}\|e^{t\Delta}(|.|^{-\gamma}u)\|_{q_2} \leq C(N,\gamma,q_1,q_2)\|D_{\sqrt{t}}u\|_{q_1},$$

which gives

$$t^{-\frac{N}{2q_2}}t^{\frac{\gamma}{2}}\|e^{t\Delta}(|.|^{-\gamma}u)\|_{q_2} \leq C(N,\gamma,q_1,q_2)t^{-\frac{N}{2q_1}}\|u\|_{q_1}.$$

Hence

$$\|e^{t\Delta}(|.|^{-\gamma}u)\|_{q_2} \leq C(N,\gamma,q_1,q_2)t^{-\frac{N}{2}(\frac{1}{q_1}-\frac{1}{q_2})-\frac{\gamma}{2}}\|u\|_{q_1}.$$

This shows (2.2.2) and the boundedness of the maps. □

**Remark 2.2.1.** The conditions on $q_1$ and $q_2$ in the previous proposition can be expressed as

$$q_1 > \frac{N}{N-\gamma}, \quad q_2 > \frac{Nq_1}{N+\gamma q_1}.$$

Note also that for $\gamma > 0$, we may take $q_2 < q_1$, unlike the case $\gamma = 0$, where we must have $q_2 \geq q_1$.

Finally, we mention that Proposition 2.2.1 can be proved, except in the cases $q_1 = \infty$ or $q_2 = \infty$, using Hölder's inequality in weak spaces [24, p. 15] and the generalized Young's inequality [24, p. 63]. The authors thank N. Chikami for pointing this out.

## 2.3 Local well-posedness

In this section we establish the well-posedness results for Eqs. (2.1.1)-(2.1.2) in Lebesgue spaces $L^q(\mathbb{R}^N)$ and in $C_0(\mathbb{R}^N)$. We do this study via the nonlinear integral equation

$$u(t) = e^{t\Delta}\varphi + a\int_0^t e^{(t-s)\Delta}\left(|.|^{-\gamma}|u(s)|^{\alpha}u(s)\right)ds, \tag{2.3.4}$$

$t > 0$, $x \in \mathbb{R}^N$, $\alpha > 0$, $a \in \mathbb{R}$, $\gamma > 0$. Our aim is to prove Theorem 2.1.1 for $q > q_c$ and Theorem 2.1.2. The proof of Theorem 2.1.1 for $q = q_c$ is given in Section 4.

We first show that Theorem 2.1.1 Parts (i) and (ii) are immediate consequence of Theorem 1, page 279 in [49].

*Proof of Theorem 2.1.1, (i)-(ii).* Let us define the maps

$$K_t(u) = e^{t\Delta}\left(|.|^{-\gamma}|u|^\alpha u\right),\ t>0.$$

(i) Let $u\in C_0(\mathbb{R}^N)$. By Proposition 2.2.1, $K_t(u)\in C_0(\mathbb{R}^N)$. Moreover, by Proposition 2.2.1, for each $t>0$, $K_t : C_0(\mathbb{R}^N)\longrightarrow C_0(\mathbb{R}^N)$ is locally Lipschitz with

$$\begin{aligned}\|K_t(u)-K_t(v)\|_\infty &\leq Ct^{-\frac{\gamma}{2}}\,\||u|^\alpha u-|v|^\alpha v\|_\infty\\ &\leq Ct^{-\frac{\gamma}{2}}(\|u\|_\infty^\alpha+\|v\|_\infty^\alpha)\|u-v\|_\infty\\ &\leq 2CM^\alpha t^{-\frac{\gamma}{2}}\|u-v\|_\infty,\end{aligned}$$

for $\|u\|_\infty\leq M$ and $\|v\|_\infty\leq M$. We have also that $t^{-\frac{\gamma}{2}}\in L^1_{loc}(0,\infty)$, since $\gamma<2$. Obviously $t\mapsto \|K_t(0)\|_\infty=0\in L^1_{loc}(0,\infty)$, also $e^{s\Delta}K_t=K_{t+s}$ for $s,\ t>0$. Then the results of (i) follow by [49, Theorem 1, p. 279].

(ii) Similarly, by Proposition 2.2.1, for each $t>0$ and if $q>\frac{N(\alpha+1)}{N-\gamma}$, $K_t : L^q(\mathbb{R}^N)\longrightarrow L^q(\mathbb{R}^N)$ is locally Lipschitz with

$$\begin{aligned}\|K_t(u)-K_t(v)\|_q &\leq Ct^{-\frac{N}{2}(\frac{\alpha+1}{q}-\frac{1}{q})-\frac{\gamma}{2}}\,\||u|^\alpha u-|v|^\alpha v\|_{\frac{q}{\alpha+1}}\\ &\leq Ct^{-\frac{N\alpha}{2q}-\frac{\gamma}{2}}(\|u\|_q^\alpha+\|v\|_q^\alpha)\|u-v\|_q\\ &\leq 2CM^\alpha t^{-\frac{N\alpha}{2q}-\frac{\gamma}{2}}\|u-v\|_q,\end{aligned}$$

for $\|u\|_q\leq M$ and $\|v\|_q\leq M$. We have also, that $t^{-\frac{N\alpha}{2q}-\frac{\gamma}{2}}\in L^1_{loc}(0,\infty)$, since $q>q_c=\frac{N\alpha}{2-\gamma}$. Obviously $t\mapsto\|K_t(0)\|_q=0\in L^1_{loc}(0,\infty)$, also $e^{s\Delta}K_t=K_{t+s}$ for $s,\ t>0$. Then the proof of (ii) follows by [49, Theorem 1, p. 279]. □

We now consider the case where initial data are in $L^q(\mathbb{R}^N)$, $q>q_c$, where $q_c$ is given by (2.1.6). We use the method introduced in [50]. However, the abstract Theorem in [50] does not directly apply if $\gamma>0$. Thus we need to give the details of the proofs.

*Proof of Theorem 2.1.1 (iii)-(iv) when $q>q_c$.* Let $q$ be such that

$$q>q_c \quad\text{and}\quad 1<q<\infty. \tag{2.3.5}$$

We begin with the observation that, since $q>1$, there exists $r>q$ satisfying

$$\frac{1}{q(\alpha+1)}-\frac{\gamma}{N(\alpha+1)}<\frac{1}{r}<\frac{N-\gamma}{N(\alpha+1)}. \tag{2.3.6}$$

The two inequalities in (2.3.6) imply that we may apply Proposition 2.2.1 with $q_1=\frac{r}{\alpha+1}$ and $q_2=q$. Also we may apply Proposition 2.2.1 with $q_1=\frac{r}{\alpha+1}$ and $q_2=r$.

We then observe that, since $q>q_c$, we have

$$\frac{1}{q}-\frac{2}{N(\alpha+1)}<\frac{1}{q(\alpha+1)}-\frac{\gamma}{N(\alpha+1)}.$$

Hence any $r > q$ satisfying (2.3.6) verifies

$$\frac{1}{q} - \frac{2}{N(\alpha+1)} < \frac{1}{r}.$$

This last inequality implies that

$$\beta(\alpha+1) < 1,$$

where

$$\beta = \frac{N}{2q} - \frac{N}{2r}. \tag{2.3.7}$$

This estimate is crucial to the local existence argument below.

In what follows we fix a value of $r > q$ satisfying (2.3.6), and let $\beta$ be given by (2.3.7). Let $M > 0$, $\rho > 0$, $T > 0$ and $\varphi \in L^q(\mathbb{R}^N)$ be such that

$$\|\varphi\|_q < \rho, \tag{2.3.8}$$

$$\rho + KM^{\alpha+1}T^{1-\frac{N\alpha}{2q}-\frac{\gamma}{2}} \leq M \quad \text{and} \quad KM^{\alpha}T^{1-\frac{N\alpha}{2q}-\frac{\gamma}{2}} < 1, \tag{2.3.9}$$

where $K$ is a positive constant. We will show that there exists a unique solution $u$ of (2.3.4) such that $u \in C\left([0,T]; L^q(\mathbb{R}^N)\right) \cap C\left((0,T]; L^r(\mathbb{R}^N)\right)$ with

$$\sup_{t\in[0,T]} \|u(t)\|_q \leq M \text{ and } \sup_{t\in(0,T]} t^{\beta}\|u(t)\|_r \leq M.$$

The proof is based on a contraction mapping argument in the set

$$Y_M = \left\{u \in C([0,T]; L^q(\mathbb{R}^N)) \cap C((0,T]; L^r(\mathbb{R}^N)); \max[\sup_{t\in[0,T]} \|u(t)\|_q, \sup_{t\in(0,T]} t^{\beta}\|u(t)\|_r] \leq M\right\}.$$

Endowed with the metric

$$d(u,v) = \max[\sup_{t\in[0,T]} \|u(t)-v(t)\|_q, \sup_{t\in(0,T]} t^{\beta}\|u(t)-v(t)\|_r],$$

$Y_M$ is a nonempty complete metric space. Given $u \in Y_M$, we set

$$\mathcal{F}_{\varphi}(u)(t) = e^{t\Delta}\varphi + a\int_0^t e^{(t-s)\Delta}\left(|.|^{-\gamma}|u(s)|^{\alpha}u(s)\right)ds, \tag{2.3.10}$$

where $\varphi \in L^q(\mathbb{R}^N)$. We will show that $\mathcal{F}_{\varphi}$ is a strict contraction on $Y_M$.

Let $\varphi,\ \psi \in L^q(\mathbb{R}^N)$ and $u, v \in Y_M$. Using Proposition 2.2.1 with $q_1 = r/(\alpha+1),\ q_2 = q$, it follows that

$$\begin{aligned}
&\|\mathcal{F}_{\varphi}(u)(t) - \mathcal{F}_{\psi}(v)(t)\|_q \leq \\
&\|e^{t\Delta}(\varphi-\psi)\|_q + |a|\int_0^t \|e^{(t-s)\Delta}[|.|^{-\gamma}\left(|u(s)|^{\alpha}u(s) - |v(s)|^{\alpha}v(s)\right)]\|_q ds \\
&\leq \|\varphi-\psi\|_q + |a|C\int_0^t (t-s)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{q})-\frac{\gamma}{2}} \||u(s)|^{\alpha}u(s) - |v(s)|^{\alpha}v(s)\|_{\frac{r}{\alpha+1}} ds \\
&\leq \|\varphi-\psi\|_q + \left(2|a|(\alpha+1)CM^{\alpha}\int_0^t (t-s)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{q})-\frac{\gamma}{2}} s^{-\beta(\alpha+1)}ds\right) d(u,v).
\end{aligned}$$

Using the fact that $\beta = \frac{N}{2q} - \frac{N}{2r}$, we get

$$\|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_q \leq$$
$$\|\varphi - \psi\|_q + \left(2|a|(\alpha+1)CM^\alpha t^{1-\frac{\gamma}{2}-\frac{N\alpha}{2q}} \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{q})-\frac{\gamma}{2}}\, \sigma^{-\beta(\alpha+1)} d\sigma\right) d(u,v).$$

Since $r > q > \frac{N\alpha}{2-\gamma} := q_c$, it follows that

$$1 - \frac{\gamma}{2} - \frac{N\alpha}{2q} > 0,\ \frac{N}{2}\left(\frac{\alpha+1}{r} - \frac{1}{q}\right) + \frac{\gamma}{2} < \frac{N}{2}\left(\frac{\alpha+1}{r} - \frac{1}{r}\right) + \frac{\gamma}{2} = \frac{N\alpha}{2r} + \frac{\gamma}{2} < 1.$$

Using also the fact that $\beta(\alpha+1) < 1$, we get

$$\|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_q \leq \|\varphi - \psi\|_q + C_1 M^\alpha T^{1-\frac{\gamma}{2}-\frac{N\alpha}{2q}} d(u,v), \tag{2.3.11}$$

where $C_1 = 2|a|(\alpha+1)C \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{q})-\frac{\gamma}{2}}\, \sigma^{-\beta(\alpha+1)} d\sigma$, is a finite positive constant.

Similarly, using the smoothing effect of the heat semi-group (2.2.1) with $s_1 = q < s_2 = r$ and Proposition 2.2.1 with $q_1 = r/(\alpha+1)$, $q_2 = r$, we have

$$\begin{aligned}
&\|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_r \leq \\
&\|e^{t\Delta}(\varphi - \psi)\|_r + |a| \int_0^t \|e^{(t-s)\Delta}[|.|^{-\gamma}\big(|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s)\big)]\|_r ds \\
&\leq t^{-\beta}\|\varphi - \psi\|_q + |a|C \int_0^t (t-s)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{r})-\frac{\gamma}{2}}\, \||u(s)|^\alpha u(s) - |v(s)|^\alpha v(s)\|_{\frac{r}{\alpha+1}}\, ds \\
&\leq t^{-\beta}\|\varphi - \psi\|_q + \left(2|a|(\alpha+1)CM^\alpha \int_0^t (t-s)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{r})-\frac{\gamma}{2}}\, s^{-\beta(\alpha+1)} ds\right) d(u,v) \\
&\leq t^{-\beta}\|\varphi - \psi\|_q + \left(2|a|(\alpha+1)CM^\alpha \int_0^t (t-s)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}}\, s^{-\beta(\alpha+1)} ds\right) d(u,v).
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
&t^\beta \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_r \leq \\
&\|\varphi - \psi\|_q + \left(2|a|(\alpha+1)CM^\alpha t^\beta \int_0^t (t-s)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}}\, s^{-\beta(\alpha+1)} ds\right) d(u,v) \\
&\leq \|\varphi - \psi\|_q + \left(2|a|(\alpha+1)CM^\alpha t^{1-\frac{\gamma}{2}-\frac{N\alpha}{2q}} \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}}\, \sigma^{-\beta(\alpha+1)} d\sigma\right) d(u,v),
\end{aligned}$$

and so, by the conditions on $\beta$, $q$ and $r$ we have

$$t^\beta \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_r \leq \|\varphi - \psi\|_q + C_2 M^\alpha T^{1-\frac{\gamma}{2}-\frac{N\alpha}{2q}} d(u,v), \tag{2.3.12}$$

where $C_2 = 2|a|(\alpha+1)C \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}}\, \sigma^{-\beta(\alpha+1)} d\sigma$, is a finite positive constant. From (2.3.11) and (2.3.12) it follows that

$$d(\mathcal{F}_\varphi(u), \mathcal{F}_\psi(v)) \leq \|\varphi - \psi\|_q + KM^\alpha T^{1-\frac{\gamma}{2}-\frac{N\alpha}{2q}} d(u,v), \tag{2.3.13}$$

where $K = \max(C_1, C_2)$.

It is clear that if $u \in Y_M$ and since $1 - \frac{\gamma}{2} - \frac{N\alpha}{2q} > 0$ (i.e. $q > q_c$), then $\mathcal{F}_\varphi(u) \in C\left([0,T]; L^q(\mathbb{R}^N)\right) \cap C\left((0,T]; L^r(\mathbb{R}^N)\right)$. Setting $\psi = 0$ and $v = 0$ in (2.3.13) and using (2.3.8) and (2.3.9), we obtain

$$d(\mathcal{F}_\varphi(u), 0) \leq \rho + KM^{\alpha+1}T^{1-\frac{\gamma}{2}-\frac{N\alpha}{2q}} \leq M.$$

And so $\mathcal{F}_\varphi$ maps $Y_M$ into itself. Letting $\varphi = \psi$ in (2.3.13), we get

$$d(\mathcal{F}_\varphi(u), \mathcal{F}_\varphi(v)) \leq KM^{\alpha}T^{1-\frac{\gamma}{2}-\frac{N\alpha}{2q}} d(u,v).$$

Hence, using (2.3.9), it follows that $\mathcal{F}_\varphi$ is a strict contraction mapping from $Y_M$ into itself. So $\mathcal{F}_\varphi$ has a unique fixed point in $Y_M$ which is solution of (2.3.4). The proof of uniqueness for arbitrary $M$ follows by taking $T$ sufficiently small in (2.3.9). This solution can be extended to a maximal solution by well known argument. The proof of Part (iv) follows by the previous calculations. We note also that by the previous calculations, precisely (2.3.13) we have the following continuous dependence property: Let $\varphi,\ \psi \in L^q(\mathbb{R}^N)$ and let $u$ and $v$ be the solutions of (2.1.5) with initial values $\varphi$ and respectively $\psi$, with $\sup_{t\in[0,T]} \|u(t)\|_q \leq M$ and $\sup_{t\in[0,T]} \|v(t)\|_q \leq M$ for some $M > 0$. Then

$$\sup_{t\in[0,T]} \|u(t) - v(t)\|_q \leq \left(1 - KM^{\alpha}T^{1-\frac{N\alpha}{2q}-\frac{\gamma}{2}}\right)^{-1} \|\varphi - \psi\|_q, \tag{2.3.14}$$

for $t \in [0,T]$ and for some positive constant $K$. □

**Remark 2.3.1.** Using a fixed point argument on

$$Y'_M = \left\{ u \in C((0,T]; L^r(\mathbb{R}^N));\ \sup_{t\in(0,T]} t^{\beta}\|u(t)\|_r \leq M \right\},$$

endowed with the metric $d'(u,v) = \sup_{t\in(0,T]} t^{\beta}\|u(t) - v(t)\|_r$, we can prove the local existence in $Y'_M$ with initial data in $L^q(\mathbb{R}^N)$,

$$q > q_c,\ 1 \leq q < \infty. \tag{2.3.15}$$

That is we may include the case $q = 1$ if $q_c < 1$. In fact, we can choose $r > q$ satisfying:

$$\frac{1}{q} - \frac{2}{N(\alpha+1)} < \frac{1}{r} < \frac{N-\gamma}{N(\alpha+1)}. \tag{2.3.16}$$

The inequalities (2.3.16) imply that we may apply Proposition 2.2.1 with $q_1 = \frac{r}{\alpha+1}$ and $q_2 = r$, and imply also that $\beta(\alpha+1) < 1$. Hence, we may perform a fixed point argument on $Y'_M$. But it is not clear that if $q = 1$, the solution is still in $L^1(\mathbb{R}^N)$, for $t > 0$.

To see how one can choose $r > q$ satisfying (2.3.16), note first that the inequality $\frac{1}{q} - \frac{2}{N(\alpha+1)} < \frac{N-\gamma}{N(\alpha+1)}$ is equivalent to $q > \frac{N(\alpha+1)}{N+2-\gamma} \equiv q_r$. If $q_c = \frac{N\alpha}{2-\gamma} < 1$, then $q_r < 1$ and so $q \geq 1 > q_r$. If $q_c = \frac{N\alpha}{2-\gamma} \geq 1$, then $q_c \geq q_r$ and so $q > q_c \geq q_r$. The conditions (2.3.15) therefore imply $\frac{1}{q} - \frac{2}{N(\alpha+1)} < \frac{N-\gamma}{N(\alpha+1)}$, which means that there exists $r > q$ which verifies (2.3.16).

We prove now that uniqueness is guaranteed for solutions which belong to $C\left([0,T],L^q(\mathbb{R}^N)\right)\cap C\left((0,T],L^r(\mathbb{R}^N)\right)$. We will use arguments of type Brezis Cazenave [3]. We have obtained the following proposition.

**Proposition 2.3.1.** *Let $r>q$ satisfying (2.3.6), $q>q_c$ and let $T_1>0$. Let $v\in C([0,T_1];L^q(\mathbb{R}^N))\cap C((0,T_1];L^r(\mathbb{R}^N))$ be a solution of (2.1.5) with initial data $\varphi\in L^q(\mathbb{R}^N)$ and $u_\varphi\in Y_M$ the solution with initial value $\varphi$ constructed by The previous fixed point argument. Then there exists $\varepsilon>0$ such that $v(t)=u_\varphi(t)$, $\forall t\in[0,\varepsilon]$.*

*Proof.* Since $v\in C\left([0,T_1],L^q(\mathbb{R}^N)\right)$, then there exists $0<\varepsilon_1\le\min(T,T_1)$ such that

$$\|v(s)\|_q<\rho,\quad \forall s\in[0,\varepsilon_1], \tag{2.3.17}$$

($\rho>0$, $T>0$ are defined by (2.3.8) and (2.3.9)). Let us define $v_\tau$ by $v_\tau(t)=v(t+\tau)$, $\forall\tau\in(0,\frac{\varepsilon_1}{2}]$, $\forall t\in[0,\frac{\varepsilon_1}{2}]$. We have from (2.3.17) and since $t^\beta\|v_\tau(t)\|_r\underset{t\to0}{\longrightarrow}0$, $\forall\tau\in(0,\frac{\varepsilon_1}{2}]$

(a) $\|v_\tau(0)\|_q=\|v(\tau)\|_q<\rho$, $\forall\tau\in\left(0,\frac{\varepsilon_1}{2}\right]$.

(b) $\sup_{t\in[0,\frac{\varepsilon_1}{2}]}\|v_\tau(t)\|_q<\rho\le M$, $\forall\tau\in\left(0,\frac{\varepsilon_1}{2}\right]$,

(c) There exists $0<T_\tau\le\varepsilon_1$ such that $\sup_{t\in(0,\frac{T_\tau}{2}]}t^\beta\|v_\tau(t)\|_r\le M$, $\forall\tau\in(0,\frac{\varepsilon_1}{2}]$.

It follows then that $v_\tau\in Y_{M,\frac{T_\tau}{2}}$, and so by uniqueness in $Y_{M,\frac{T_\tau}{2}}$ it follows that $v_\tau(t)=u_{v_\tau(0)}(t)$, $\forall\tau\in(0,\frac{\varepsilon_1}{2}]$, $\forall t\in[0,\frac{T_\tau}{2}]$. Hence $v_\tau(t)=u_{v_\tau(0)}(t)$, $\forall\tau\in(0,\frac{\varepsilon_1}{2}]$, $\forall t\in[0,\frac{\varepsilon_1}{2}]$, where $u_{v_\tau(0)}$ is the solution of the integral equation constructed by the previous fixed point argument. By the continuous dependence relation (2.3.14), we obtain $v_\tau(t)\underset{\tau\to0}{\longrightarrow}u_\varphi(t)$, in $L^q(\mathbb{R}^N)$, $\forall t\in[0,\frac{\varepsilon_1}{2}]$. On the other hand $v_\tau(t)=v(t+\tau)\underset{\tau\to0}{\longrightarrow}v(t)$, in $L^q(\mathbb{R}^N)$, $\forall t\in[0,\frac{\varepsilon_1}{2}]$, (since $v$ is continuous in $[0,\varepsilon_1]$). Finally, we conclude by uniqueness of the limit that $v(t)=u_\varphi(t)$, $\forall t\in[0,\frac{\varepsilon_1}{2}]$.

□

We have also the following result.

**Proposition 2.3.2.** *Let $\alpha>0$ and let $\gamma$ be such that $0<\gamma<\min(2,N)$. Assume the hypotheses of Theorem 2.1.1. Let $T_{\max}(\varphi,q)$ denote the maximal existence time of the solution of (2.1.5) with initial data $\varphi\in L^q(\mathbb{R}^N)$. Then we have the following.*

*(i) If $\varphi\in L^q(\mathbb{R}^N)$, then for $t\in(0,T_{\max}(\varphi,q))$, $u(t)\in C_0(\mathbb{R}^N)$.*

*(ii) If $\varphi\in L^p(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$, $1<q<p\le\infty$ and $q>q_c$. Then $T_{\max}(\varphi,p)=T_{\max}(\varphi,q)$.*

*Proof.* (i) Let $\varphi\in L^q(\mathbb{R}^N)$, $q>q_c$ and $q>1$. Let $r>q$ and $\beta$ be as in (2.3.6) and (2.3.7). Let $p$ be such that $r<p\le\infty$. Hence $p>q$,

$$0\le\frac{1}{p}<\frac{\gamma}{N}+\frac{\alpha+1}{r}<1$$

and for $0 < T < T_{\max}(\varphi, q)$, we have

$$\begin{aligned}
\|u(t)\|_p &\leq \|e^{t\Delta}\varphi\|_p + |a|C\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})-\frac{\gamma}{2}}\|u(\sigma)\|_r^{\alpha+1}d\sigma \\
&\leq (4\pi t)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|\varphi\|_q + |a|Ct^{1-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})-\frac{\gamma}{2}-\beta(\alpha+1)}\sup_{s\in(0,T]}\Big(s^{\beta(\alpha+1)}\|u(s)\|_r^{\alpha+1}\Big)\times \\
&\quad \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})-\frac{\gamma}{2}}\sigma^{-\beta(\alpha+1)}d\sigma \\
&\leq (4\pi t)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|\varphi\|_q + |a|M^{\alpha+1}Ct^{1-\frac{\gamma}{2}-\frac{N}{2}(\frac{\alpha+1}{q}-\frac{1}{p})}\int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})-\frac{\gamma}{2}}\sigma^{-\beta(\alpha+1)}d\sigma.
\end{aligned}$$

Since $r > q > q_c$, it follows that if

$$\frac{\alpha+1}{r} - \frac{2-\gamma}{N} < \frac{1}{p} < \frac{1}{r},$$

then $u(t)$ is in $L^p(\mathbb{R}^N)$ for all $t \in \big(0, T_{\max}(\varphi, q)\big)$. The result for general $p > q$ follows by iteration. Hence $u(t)$ is in $L^\infty(\mathbb{R}^N)$, for $t \in \big(0, T_{\max}(\varphi, q)\big)$. The fact that $u(t) \in C_0(\mathbb{R}^N)$, for $t \in \big(0, T_{\max}(\varphi, q)\big)$ follows by Proposition 2.2.1.

(ii) To emphasize the dependence on $q$ let us denote the metric space $(Y_M, d)$ by $(Y_M(q), d_q)$. Let $\varphi \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$. Let $u_q$ be the solution with initial data $\varphi \in L^q(\mathbb{R}^N)$. Let $u_p$ be the solution with initial data $\varphi \in L^p(\mathbb{R}^N)$. We can show, by similar calculations that the mapping defined by (2.3.10) is a contraction on $Y_M(q) \cap Y_M(p)$ endowed with the metric $d = \max(d_q, d_p)$. This gives the existence of a unique solution in $Y_M(q) \cap Y_M(p)$. Hence the solutions $u_q = u_p$ for $t \in [0, T]$, $T$ small. We deduce also, by well known arguments, that the maximal time of existence is independent of $q$. This finishes the proof of the proposition. □

*Proof of Theorem 2.1.2.* Let $\varphi \in L^q(\mathbb{R}^N)$ be such that $T_{\max} < \infty$ and $u \in C\Big(\big[0, T_{\max}\big), L^q(\mathbb{R}^N)\Big)$ be the maximal solution of (2.1.5). Fix $s \in \big[0, T_{\max}\big)$ and let

$$w(t) = u(t+s), \quad t \in \big[0, T_{\max} - s\big),$$

with $w(0) = u(s)$. Then we have, as in the proof of Theorem 2.1.1 Part (iii),

$$\|u(s)\|_q + KM^{\alpha+1}(T_{\max} - s)^{1-\frac{N\alpha}{2q}-\frac{\gamma}{2}} > M, \quad \forall\, M > 0. \tag{2.3.18}$$

In fact, if not, there exists $M > 0$ such that

$$\|u(s)\|_q + KM^{\alpha+1}(T_{\max} - s)^{1-\frac{N\alpha}{2q}-\frac{\gamma}{2}} \leq M,$$

$w$ will be defined on $\big[0, T_{\max} - s\big]$ in particular $u(T_{\max})$ is well defined, a contradiction. Hence (2.3.18) is verified, for any $t \in \big[0, T_{\max}\big)$ fixed and for all $M > 0$. Let

$$M = 2\|u(t)\|_q.$$

From (2.3.18) we have

$$\|u(t)\|_q + K2^{\alpha+1}\|u(t)\|_q^{\alpha+1}(T_{\max}-t)^{1-\frac{N\alpha}{2q}-\frac{\gamma}{2}} > 2\|u(t)\|_q.$$

Then

$$K2^{\alpha+1}\|u(t)\|_q^{\alpha}(T_{\max}-t)^{1-\frac{N\alpha}{2q}-\frac{\gamma}{2}} > 1, \quad \forall\, t \in [0, T_{\max}).$$

Hence we derive a lower bound of the blow-up rate

$$\|u(t)\|_q \geq C(T_{\max}-t)^{\frac{N}{2q}-\frac{2-\gamma}{2\alpha}}, \quad \forall\, t \in [0, T_{\max}).$$

The proof in the case where $\varphi \in C_0(\mathbb{R}^N)$ is similar. This finishes the proof of Theorem 2.1.2. □

**Remark 2.3.2.** By the same methods used to prove Theorem 2.1.1 (i), (ii) and Proposition 2.3.2, one can show that (2.1.5) is well posed in

- $C_0^1(\mathbb{R}^N) := \left\{u \in C_0(\mathbb{R}^N),\ \nabla u \in C_0(\mathbb{R}^N)\right\}$, if $0 < \gamma < 1$,
- $W^{1,q}(\mathbb{R}^N)$, if $0 < \gamma < 1$, $q > \frac{N\alpha}{1-\gamma}$ and $q > \frac{N(\alpha+1)}{N-\gamma}$.

Moreover under some conditions on $\gamma$, $\alpha$, and $q$, the solutions constructed in Theorem 2.1.1, are in $C_0^1(\mathbb{R}^N)$ and $W^{1,q}(\mathbb{R}^N)$ for $t > 0$.

## 2.4 Global existence

In this section we prove Theorem 2.1.1 Part (iii) for $q = q_c > 1$, Theorem 2.1.4 and Theorem 2.1.5. We consider the solutions of the integral equation

$$u(t) = e^{t\Delta}\varphi + a\int_0^t e^{(t-s)\Delta}\left(|.|^{-\gamma}|u(s)|^{\alpha}u(s)\right) ds,$$

where $t > 0$, $x \in \mathbb{R}^N$, $a \in \mathbb{R}$, $0 < \gamma < \min(2, N)$ and $\alpha > (2-\gamma)/N$, i.e. $q_c = \frac{N\alpha}{2-\gamma} > 1$. Given such an $\alpha$, one can choose $r > q_c$ such that

$$\frac{1}{q_c} - \frac{2}{N(\alpha+1)} < \frac{1}{r} < \frac{N-\gamma}{N(\alpha+1)}. \tag{2.4.19}$$

This relationship is analogous to (2.3.6) with $q = q_c$. In fact

$$\frac{1}{q_c} - \frac{2}{N(\alpha+1)} = \frac{1}{q_c(\alpha+1)} - \frac{\gamma}{N(\alpha+1)}.$$

The existence of such an $r > q_c$ follows from the fact that $q_c > 1$.

While $r$ is not uniquely determined, we consider $r$ fixed and set

$$\beta = \frac{N}{2q_c} - \frac{N}{2r} = \frac{2-\gamma}{2\alpha} - \frac{N}{2r}. \tag{2.4.20}$$

One verifies that

$$\beta(\alpha+1) < 1,\ \frac{N\alpha}{2r} + \frac{\gamma}{2} < 1,\ \beta + 1 - \left(\frac{N\alpha}{2r} + \frac{\gamma}{2}\right) - \beta(\alpha+1) = 0. \tag{2.4.21}$$

We have obtained the following global existence result.

**Theorem 2.4.1** (Global existence)**.** *Let $0<\gamma<\min(2,N)$ and $\alpha>(2-\gamma)/N$. Let $r$ and $\beta$ verify* (2.4.19) *and* (2.4.20). *Suppose that $\rho>0$ and $M>0$ satisfy the inequality*

$$\rho + KM^{\alpha+1} \leq M, \tag{2.4.22}$$

*where $K=K(\alpha,N,\gamma,r)>0$ is a constant and can explicitly be computed. Let $\varphi$ be a tempered distribution such that*

$$\sup_{t>0} t^{\beta}\|e^{t\Delta}\varphi\|_r \leq \rho. \tag{2.4.23}$$

*It follows that there exists a unique global solution $u$ of* (2.3.4) *such that*

$$\sup_{t>0} t^{\beta}\|u(t)\|_r \leq M. \tag{2.4.24}$$

*Furthermore,*

*(i)* $u(t)-e^{t\Delta}\varphi \in C([0,\infty);L^s(\mathbb{R}^N))$, $\quad \frac{1}{q_c}<\frac{1}{s}<\frac{\gamma}{N}+\frac{\alpha+1}{r}$.

*(ii)* $u(t)-e^{t\Delta}\varphi \in L^\infty((0,\infty);L^s(\mathbb{R}^N))$, $\quad \frac{1}{q_c}\leq\frac{1}{s}<\frac{\gamma}{N}+\frac{\alpha+1}{r}$.

*(iii)* $\lim_{t\to 0} u(t)=\varphi$ *in the sense of distributions.*

*(iv)* $\sup_{t>0} t^{\frac{2-\gamma}{2\alpha}-\frac{N}{2q}}\|u(t)\|_q<\infty$, $\forall\, q\in[r,\infty]$.

*Moreover, let $\varphi$ and $\psi$ satisfy (2.4.23) and let $u$ and $v$ be respectively the solutions of (2.3.4) with initial values $\varphi$ and $\psi$. Then*

$$\sup_{t>0} t^{\frac{2-\gamma}{2\alpha}-\frac{N}{2q}}\|u(t)-v(t)\|_q \leq C\sup_{t>0} t^{\beta}\|e^{t\Delta}(\varphi-\psi)\|_r,\ \forall\, q\in[r,\infty]. \tag{2.4.25}$$

*If in addition, $e^{t\Delta}(\varphi-\psi)$ has the stronger decay property*

$$\sup_{t>0} t^{\beta+\delta}\|e^{t\Delta}(\varphi-\psi)\|_r<\infty, \tag{2.4.26}$$

*for some $\delta>0$ such that $\beta(\alpha+1)+\delta<1$, and with $M$ perhaps smaller, then*

$$\sup_{t>0} t^{\beta+\delta}\|u(t)-v(t)\|_r \leq C\sup_{t>0} t^{\beta+\delta}\|e^{t\Delta}(\varphi-\psi)\|_r, \tag{2.4.27}$$

*where $C>0$ is a constant.*

**Remarks 2.4.1.** *(a) If we suppose that $\varphi-\psi\in L^s(\mathbb{R}^N)$, $\frac{1}{q_c}<\frac{1}{s}<\frac{\alpha+1}{r}+\frac{\gamma}{N}$, then (2.4.26) is verified with $\delta=\frac{N}{2s}-\frac{2-\gamma}{2\alpha}>0$. By the conditions on $s$ we have $\delta<\delta_0$ where*

$$\delta_0=\frac{N(\alpha+1)}{2r}+\frac{\gamma}{2}-\frac{2-\gamma}{2\alpha}. \tag{2.4.28}$$

*Since $\beta(\alpha+1)+\delta_0=1$, it follows that (2.4.27) holds for all $\delta\in(0,\delta_0)$.*

*(b) If the hypotheses of Theorem 2.4.1 are verified on some finite interval $t\in(0,T)$, instead of $\forall\, t>0$, the conclusion still holds, but only on the interval (0,T).*

The following corollary gives the proof of Theorem 2.1.1 Part (iii) for $q = q_c > 1$. The proof of Parts (i)-(iii) below is similar to that in [42, Corollary 2.6, p. 1296], so we omit it. The proof of Part (iv) is similar to that in [39, Theorem 20.19(iii)], so we likewise omit it.

**Corollary 2.4.1.** *Suppose the hypotheses of Theorem 2.4.1 are satisfied.*

*(i) If $\varphi \in L^{q_c}(\mathbb{R}^N)$ and $\|\varphi\|_{q_c}$ is sufficiently small, then $\varphi$ satisfies (2.4.23).*

*(ii) If $\varphi \in L^{q_c}(\mathbb{R}^N)$ (without any assumption of smallness), then there exists $T > 0$, such that $\varphi$ satisfies (2.4.23), but only on $(0, T)$.*

*(iii) In the above two cases, if $u$ is the resulting solution of (2.1.5), then $u \in C([0,\infty); L^{q_c}(\mathbb{R}^N))$, respectively, $u \in C([0,T]; L^{q_c}(\mathbb{R}^N))$.*

*(iv) If $\varphi \in L^{q_c}(\mathbb{R}^N)$ and $\|\varphi\|_{q_c}$ is sufficiently small and if $u \in C([0,\infty); L^{q_c}(\mathbb{R}^N))$ is the resulting solution, then $\|u(t)\|_{q_c} \to 0$ as $t \to \infty$.*

*Proof of Theorem 2.1.1 Part (iii) for $q = q_c > 1$.* The proof follows by Theorem 2.4.1 and Corollary 2.4.1. The fact that $u : [0, T] \to L^{q_c}(\mathbb{R}^N)$ is continuous and the condition (iii) (b), imply (2.4.24) on $(0, T)$ and then is sufficient to guarantee uniqueness. □

Using now the previous results we give the proof of Theorem 2.1.4.

*Proof of Theorem 2.1.4.* Part (i) follows by Corollary 2.4.1 and Theorem 2.4.1.

(ii) By the condition on $\sigma$, $\varphi$ verifies the hypothesis of (i).

(iii) Since $\varphi \in L^1_{loc}(\mathbb{R}^N)$ and $|\varphi(\cdot)| \leq c|\cdot|^{-\frac{2-\gamma}{\alpha}}$, then $\varphi \in \mathcal{S}'(\mathbb{R}^N)$. By writing $|\cdot|^{-\frac{2-\gamma}{\alpha}} = \varphi_1 + \varphi_2$, with $\varphi_1 \in L^s(\mathbb{R}^N)$, $1 \leq s < N\alpha/(2-\gamma)$, $\varphi_2 \in L^\tau(\mathbb{R}^N)$, $\tau > q_c$ it follows by the smoothing properties of the heat semigroup that $e^{t\Delta}|\cdot|^{-\frac{2-\gamma}{\alpha}} \in L^r(\mathbb{R}^N)$, $\forall\ t > 0$ and by homogeneity, we have that $\sup\limits_{t>0} t^\beta \|e^{t\Delta}|\cdot|^{-\frac{2-\gamma}{\alpha}}\|_r < \infty$. Since $|\varphi(\cdot)| \leq c|\cdot|^{-\frac{2-\gamma}{\alpha}}$, $\varphi$ verifies (2.4.23) for $c$ sufficiently small. Then the first statement of Part (iii) follows by Theorem 2.4.1. The fact that $u \in C\left((0,\infty); L^q(\mathbb{R}^N)\right)$ for all $q > q_c$ follows by iteration as in the proof of Proposition 2.3.2 and by Theorem 2.4.1 (i). The last statement of Part (iii) follows by Theorem 2.4.1 (iii). This completes the proof of Theorem 2.1.4. □

*Proof of Theorem 2.4.1.* The proof is based on a contraction mapping argument. Let X be the set of Bochner measurable functions $u : (0,\infty) \to L^r(\mathbb{R}^N)$ such that $\sup_{t>0} t^\beta \|u(t)\|_r$ is finite. We denote by $X_M$ the set of $u \in X$ such that $\sup_{t>0} t^\beta \|u(t)\|_r \leq M$. Endowed with the metric, $d(u,v) = \sup\limits_{t>0} t^\beta \|u(t) - v(t)\|_r$, $X_M$ is a nonempty complete metric space. Consider the mapping defined by

$$\mathcal{F}_\varphi(u)(t) = e^{t\Delta}\varphi + a\int_0^t e^{(t-s)\Delta}(|.|^{-\gamma}|u(s)|^\alpha u(s))ds, \tag{2.4.29}$$

where $\varphi$ is a tempered distribution satisfying (2.4.23). We will show that $\mathcal{F}_\varphi$ is a strict contraction on $X_M$.

Let $\varphi$ and $\psi$ satisfy (2.4.23) and $u, v \in X_M$. It follows that

$$\begin{aligned} t^\beta \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_r &\leq t^\beta \|e^{t\Delta}(\varphi - \psi)\|_r + \\ & |a| t^\beta \int_0^t \|e^{(t-s)\Delta}[|.|^{-\gamma}(|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s))]\|_r ds. \end{aligned}$$

Using Proposition 2.2.1 with $(q_1, q_2) = (\frac{r}{\alpha+1}, r)$, we obtain

$$\begin{aligned} \|e^{(t-s)\Delta}\left[|.|^{-\gamma}(|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s))\right]\|_r &\leq C(t-s)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}} \||u(s)|^\alpha u(s) - |v(s)|^\alpha v(s)\|_{\frac{r}{\alpha+1}} \\ &\leq C(t-s)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}}(\alpha+1)(\|u(s)\|_r^\alpha + \|v(s)\|_r^\alpha)\|u(s) - v(s)\|_r \\ &\leq 2(\alpha+1)C(t-s)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}} s^{-\beta(\alpha+1)} M^\alpha d(u,v). \end{aligned}$$

By (2.4.21), we obtain

$$t^\beta \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_r \leq t^\beta \|e^{t\Delta}(\varphi - \psi)\|_r + KM^\alpha d(u,v), \tag{2.4.30}$$

where $K = 2|a|(\alpha+1)C \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}} \sigma^{-\beta(\alpha+1)} d\sigma$ is a finite positive constant.

Setting $\psi = 0$ and $v = 0$ in (2.4.30), we see that

$$\sup_{t>0} t^\beta \|\mathcal{F}_\varphi(u)(t)\|_r \leq \rho + KM^{\alpha+1} \leq M.$$

That is, $\mathcal{F}_\varphi$ maps $X_M$ into itself. Letting $\varphi = \psi$ in (2.4.30), we observe that

$$d(\mathcal{F}_\varphi(u), \mathcal{F}_\varphi(v)) \leq KM^\alpha d(u,v).$$

Since $KM^\alpha < 1$, we see that $\mathcal{F}_\varphi$ is a strict contraction on $X_M$, and so $\mathcal{F}_\varphi$ has a unique fixed point $u$ in $X_M$ solution of (2.3.4).

We now prove that $u(t) - e^{t\Delta}\varphi \in C([0,\infty); L^s(\mathbb{R}^N))$ for $s$ satisfying

$$\frac{2-\gamma}{N\alpha} < \frac{1}{s} < \frac{\gamma}{N} + \frac{\alpha+1}{r}. \tag{2.4.31}$$

Since continuity for $t > 0$ can be handled by well known arguments, we only give the proof at t=0. Write

$$u(t) - e^{t\Delta}\varphi = a \int_0^t e^{(t-\sigma)\Delta}(|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)) d\sigma.$$

Then for $s$ satisfying (2.4.31) and by Proposition 2.2.1 with $(q_1, q_2) = (\frac{r}{\alpha+1}, s)$, we obtain

$$\begin{aligned} \|u(t) - e^{t\Delta}\varphi\|_s &\leq |a| \int_0^t \|e^{(t-\sigma)\Delta}[|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)]\|_s d\sigma \\ &\leq |a| C \int_0^t (t-\sigma)^{\frac{N}{2s}-\frac{\gamma}{2}-\frac{N(\alpha+1)}{2r}} \|u(\sigma)\|_r^{\alpha+1} d\sigma \\ &\leq |a| C M^{\alpha+1} \int_0^t (t-\sigma)^{\frac{N}{2s}-\frac{\gamma}{2}-\frac{N(\alpha+1)}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma \\ &= |a| C M^{\alpha+1} t^{\frac{N}{2s}-\frac{\gamma}{2}-\frac{N(\alpha+1)}{2r}+1-\beta(\alpha+1)} \int_0^1 (1-\sigma)^{\frac{N}{2s}-\frac{\gamma}{2}-\frac{N(\alpha+1)}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma. \end{aligned}$$

Therefore we obtain

$$\|u(t) - e^{t\Delta}\varphi\|_s \leq |a|CM^{\alpha+1}t^{\frac{N}{2s}-\frac{2-\gamma}{2\alpha}} \int_0^1 (1-\sigma)^{\frac{N}{2s}-\frac{\gamma}{2}-\frac{N(\alpha+1)}{2r}}\sigma^{-\beta(\alpha+1)}d\sigma. \tag{2.4.32}$$

By (2.4.31), $\int_0^1 (1-\sigma)^{\frac{N}{2s}-\frac{\gamma}{2}-\frac{N(\alpha+1)}{2r}}\sigma^{-\beta(\alpha+1)}d\sigma$ is finite and that $t^{\frac{N}{2s}-\frac{2-\gamma}{2\alpha}}$ converges to zero as $t \searrow 0$. This proves the statements (i) and (iii) of Theorem 2.4.1. Statement (ii) with $s = N\alpha/(2-\gamma)$ follows from (2.4.32) which still holds if $s = N\alpha/(2-\gamma) := q_c$.

To prove the stronger decay estimate (2.4.27), we observe that, by the previous calculations we have,

$$\|u(t) - v(t)\|_r \leq \|e^{t\Delta}(\varphi-\psi)\|_r + 2|a|M^\alpha C(\alpha+1)\int_0^t (t-s)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}}s^{-\beta\alpha}\|u(s)-v(s)\|_r ds.$$

Let $\delta > 0$ be such that $\beta(\alpha+1)+\delta < 1$. For arbitrary $T > 0$, we have

$$\begin{aligned} t^{\beta+\delta}\|u(t)-v(t)\|_r &\leq t^{\beta+\delta}\|e^{t\Delta}(\varphi-\psi)\|_r + 2|a|M^\alpha C(\alpha+1)t^{\beta+\delta} \times \\ &\quad \left(\sup_{0<t\leq T} t^{\beta+\delta}\|u(t)-v(t)\|_r\right)\int_0^t (t-s)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}}s^{-\beta(\alpha+1)-\delta}ds \\ &\leq t^{\beta+\delta}\|e^{t\Delta}(\varphi-\psi)\|_r + KM^\alpha t^{\beta+\delta-\frac{N\alpha}{2r}-\frac{\gamma}{2}+1-\beta(\alpha+1)-\delta} \times \\ &\quad \left(\sup_{0<t\leq T} t^{\beta+\delta}\|u(t)-v(t)\|_r\right)\int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}}\sigma^{-\beta(\alpha+1)-\delta}d\sigma \\ &\leq t^{\beta+\delta}\|e^{t\Delta}(\varphi-\psi)\|_r + KM^\alpha\left(\sup_{0<t\leq T} t^{\beta+\delta}\|u(t)-v(t)\|_r\right) \times \\ &\quad \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}}\sigma^{-\beta(\alpha+1)-\delta}d\sigma, \ \forall\ 0<t\leq T. \end{aligned}$$

Since the constants does not depend on $T$ and finite by the hypothesis on $\delta$, the result follows.

We now prove (iv) of Theorem 2.4.1 for $q = \infty$. The result for the other values of $q$ will then follow by using the Hölder inequality and (2.4.24). We need the following lemma.

**Lemma 2.4.1.** *Let $N$ be a positive integer and $s$, $q$ be two real numbers, and suppose that*

$$\alpha+1 < s < q \leq \infty,\ 0 \leq \frac{1}{q} < \frac{\gamma}{N} + \frac{\alpha+1}{s} < 1,\ \frac{N}{2}(\frac{\alpha+1}{s} - \frac{1}{q}) < 1 - \frac{\gamma}{2}.$$

*Let $u$ be the solution of (2.3.4) with initial data $\varphi \in \mathcal{S}'(\mathbb{R}^N)$. Assume that*

$$\sup_{t>0} t^{\frac{2-\gamma}{2\alpha}-\frac{N}{2s}}\|u(t)\|_s < \infty.$$

*It follows that*

$$\sup_{t>0} t^{\frac{2-\gamma}{2\alpha}-\frac{N}{2q}}\|u(t)\|_q < \infty.$$

Assuming this lemma for the moment, we continue the proof of the theorem. Let us consider the solution of (2.3.4) constructed by the first part of Theorem 2.4.1. We have

$$\sup_{t>0} t^{\frac{2-\gamma}{2\alpha}-\frac{N}{2r}}\|u(t)\|_r < \infty,$$

for $r$ satisfying the conditions specified in (2.4.19). We use the Lemma 2.4.1 with an iterative argument as in [43]: $s_i$ will play the role of $s$, and $s_{i+1}$ will play the role of $q$, for $i = 0, 1, \cdots$. Let $s_0 = r$, and choose $s_1$ satisfy the hypothesis of Lemma 2.4.1. Then by Lemma 2.4.1 we get that

$$\sup_{t>0} t^{\frac{2-\gamma}{2\alpha}-\frac{N}{2s_1}}\|u(t)\|_{s_1} < \infty.$$

We iterate this procedure. For the next step it is clear we can choose $s_2$ so that

$$\alpha+1 < s_1 < s_2 \leq \infty,\ 0 \leq \frac{1}{s_2} < \frac{\gamma}{N} + \frac{\alpha+1}{s_1} < 1,\ \frac{N}{2}(\frac{\alpha+1}{s_1} - \frac{1}{s_2}) < 1 - \frac{\gamma}{2}.$$

Then we conclude that

$$\sup_{t>0} t^{\frac{2-\gamma}{\alpha}-\frac{N}{2s_2}}\|u(t)\|_{s_2} < \infty.$$

One can check easily that by this iterative procedure, we can reach $s_{i+1} = \infty$ for some finite i. This proves (iv) for $q = \infty$. The other cases follow by interpolation. The continuous dependence relation (2.4.25) with $q = r$, of the solution on the initial data can be easily deduced by setting in (2.4.30) $\mathcal{F}_\varphi(u) = u$ and $\mathcal{F}_\varphi(v) = v$. Formula (2.4.25) for all $q \in [r, \infty]$, can be proved by using an iterative procedure similar to the proof of (iv). In particular, one can prove a version of Lemma 2.4.1 with $u$ replaced by $u - v$. This completes the proof of Theorem 2.4.1. □

*Proof of Lemma 2.4.1.* We use similar argument as in [43]. Set

$$A = \sup_{t>0} t^{\beta(s)}\|u(t)\|_s,\ \beta(s) = \frac{2-\gamma}{2\alpha} - \frac{N}{2s}.$$

We use the integral equation (2.3.4) from $\frac{t}{2}$ to $t$:

$$u(t) = e^{\frac{t}{2}\Delta}u(t/2) + a\int_{\frac{t}{2}}^{t} e^{(t-\sigma)\Delta}(|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma))d\sigma. \tag{2.4.33}$$

It follows from the smoothing properties of the heat semigroup, with $s_2 = q$, $s_1 = s$ and (2.2.2) with

$q_2 = q$, $q_1 = s/(\alpha+1)$, that

$$\begin{aligned}
\|u(t)\|_q &\leq \|e^{\frac{t}{2}\Delta}u(t/2)\|_q + |a|\int_{\frac{t}{2}}^{t}\|e^{(t-s)\Delta}(|.|^{-\gamma}|u(\sigma)|^{\alpha}u(\sigma))\|_q d\sigma \\
&\leq Ct^{-\frac{N}{2}(\frac{1}{s}-\frac{1}{q})}\|u(t/2)\|_s + C|a|\int_{\frac{t}{2}}^{t}(t-\sigma)^{\frac{N}{2q}-\frac{\gamma}{2}-\frac{N(\alpha+1)}{2s}}\|u(\sigma)\|_s^{\alpha+1}d\sigma \\
&\leq Ct^{-\frac{N}{2}(\frac{1}{s}-\frac{1}{q})}\|u(t/2)\|_s + C|a|A^{\alpha+1}t^{\frac{N}{2q}-\frac{\gamma}{2}-\frac{N(\alpha+1)}{2s}+1-\beta(s)(\alpha+1)} \times \\
&\quad \int_{\frac{1}{2}}^{1}(1-\sigma)^{\frac{N}{2q}-\frac{\gamma}{2}-\frac{N(\alpha+1)}{2s}}\sigma^{-\beta(s)(\alpha+1)}d\sigma \\
&\leq C't^{\frac{N}{2q}-\frac{2-\gamma}{2\alpha}}A + C|a|A^{\alpha+1}t^{-\frac{1}{\alpha}+\frac{\gamma}{2\alpha}+\frac{N}{2q}}\int_{\frac{1}{2}}^{1}(1-\sigma)^{\frac{N}{2q}-\frac{\gamma}{2}-\frac{N(\alpha+1)}{2s}}\sigma^{-\beta(s)(\alpha+1)}d\sigma.
\end{aligned}$$

Therefore we obtain

$$t^{\frac{2-\gamma}{2\alpha}-\frac{N}{2q}}\|u(t)\|_q \leq C'A + C|a|A^{\alpha+1}C_1,$$

where

$$C_1 = \int_{\frac{1}{2}}^{1}(1-\sigma)^{-\frac{N}{2}\left(\frac{\alpha+1}{s}-\frac{1}{q}\right)-\frac{\gamma}{2}}\sigma^{-\beta(s)(\alpha+1)}d\sigma.$$

By the conditions imposed on the parameters in the lemma, $C_1$ is positive and finite. Thus, we get

$$\sup_{t>0} t^{\frac{2-\gamma}{2\alpha}-\frac{N}{2q}}\|u(t)\|_q \leq C(A) < \infty,$$

where $C(A)$ is a positive constant. Remark that $C(A) \to 0$ as $A \to 0$. This completes the proof of the lemma. ☐

We now give the proof of Theorem 2.1.5.

*Proof of Theorem 2.1.5.* Due to the homogeneity properties of (2.1.1)-(2.1.2), and hence (2.3.4), it is clear that the set of solutions of (2.3.4) is invariant under the transformation $u \to u_\lambda$ for all $\lambda > 0$, where

$$u_\lambda(t,x) = \lambda^{\frac{2-\gamma}{\alpha}}u(\lambda^2 t, \lambda x), \quad t>0, \quad x \in \mathbb{R}^N. \tag{2.4.34}$$

A self-similar solution is a solution such that $u_\lambda = u$ for all $\lambda > 0$. We claim that solutions given in Theorem 2.4.1 with homogeneous initial data of degree $-(2-\gamma)/\alpha$ are self-similar.

In fact, let $\varphi$ be a tempered distribution satisfying (2.4.23). Let $u$ be the solution of (2.1.5) with initial data $\varphi$ given by Theorem 2.4.1. Now define the scaling function $\varphi_\lambda$ by

$$\varphi_\lambda(x) = \lambda^{\frac{2-\gamma}{\alpha}}\varphi(\lambda x) \quad \forall \lambda>0, \quad x \in \mathbb{R}^N.$$

This makes sense for distributions by duality. Since $\sup_{t>0} t^\beta\|e^{t\Delta}\varphi_\lambda\|_r = \sup_{t>0} t^\beta\|e^{t\Delta}\varphi\|_r$ for all $\lambda > 0$, it follows that $\varphi_\lambda$ satisfies also (2.4.23) for all $\lambda > 0$. We can easily compute that the function

$u_\lambda$ given by (2.4.34) is the solution of (2.3.4) with initial data $\varphi_\lambda$. Finally, if $\varphi_\lambda = \varphi$, that is $\varphi$ homogeneous of degree $-(2-\gamma)/\alpha$, and since $\sup_{t>0} t^\beta \|u_\lambda(t)\|_r = \sup_{t>0} t^\beta \|u(t)\|_r, \ \forall\ \lambda > 0$, then $u_\lambda = u$ and thus $u$ is self-similar. Let us denote it by $u_{\mathcal{S}}$. The fact that $u_{\mathcal{S}}(t) \to \varphi$ in $\mathcal{S}'(\mathbb{R}^N)$ as $t \to 0$ follows by Theorem 2.4.1 Part (iii). □

## 2.5 Asymptotic behavior: Nonlinear case

In this section we give the proof of Theorem 2.1.6 Part (i). In fact, we will prove the following more general version. In particular the asymptotic behavior is given in $L^q(\mathbb{R}^N)$ for all $q \geq r$.

**Theorem 2.5.1** (Nonlinear behavior)**.** *Let $0 < \gamma < \min(2, N)$, $\alpha > (2-\gamma)/N$. Let $r$ and $\beta$ verify* (2.4.19) *and* (2.4.20)*. Define $\beta(q)$ by*

$$\beta(q) = \frac{2-\gamma}{2\alpha} - \frac{N}{2q}, \quad q > 1.$$

*Let $\Phi$ be given by*

$$\Phi(x) = \omega(x)|x|^{-\frac{2-\gamma}{\alpha}}$$

*with $\omega$ homogeneous of degree* 0*, $\omega \in L^\infty(S^{N-1})$ and $\|\omega\|_\infty$ is sufficiently small. Let*

$$u_{\mathcal{S}}(t,x) = t^{-\frac{2-\gamma}{2\alpha}} u_{\mathcal{S}}(1, \frac{x}{\sqrt{t}})$$

*be the self-similar solution of (2.3.4) with initial data $\Phi$ given by Theorem 2.1.5.*

*Let $\varphi \in C_0(\mathbb{R}^N)$ be such that*

$$|\varphi(x)| \leq \frac{c}{(1+|x|^2)^{\frac{2-\gamma}{2\alpha}}}, \ \forall\, x \in \mathbb{R}^N, \ \varphi(x) = \omega(x)|x|^{-\frac{2-\gamma}{\alpha}}, \ |x| \geq A,$$

*for some constant $A > 0$, where $c$ is a small positive constant. (We take $\|\omega\|_\infty$ and $c$ sufficiently small so that (2.4.23) is satisfied by $\Phi$ and $\varphi$).*

*Let $u$ be the global solution of (2.1.5) with initial data $\varphi$ constructed by Theorem 2.1.4. Then there exists $\delta > 0$ sufficiently small such that*

$$\|u(t) - u_{\mathcal{S}}(t)\|_q \leq C_\delta t^{-\beta(q)-\delta}, \ \forall\ t > 0, \tag{2.5.35}$$

*for all $q \in [r, \infty]$. Also, we have*

$$\|t^{\frac{2-\gamma}{2\alpha}} u(t, .\sqrt{t}) - u_{\mathcal{S}}(1, .)\|_q \leq C_\delta t^{-\delta}, \ \forall\ t > 0, \tag{2.5.36}$$

*for all $q \in [r, \infty]$.*

**Remark 2.5.1.** In the previous theorem, if $\delta$ is sufficiently small, then all the quantities on the right hand side of inequalities (2.5.35)-(2.5.36) converge to zero as $t \to \infty$. Also, the difference $u - u_{\mathcal{S}}$ goes to zero as $t$ goes to infinity more rapidly than each of them does separately.

*Proof.* We have that $|\Phi(x)-\varphi(x)| = 0$ for $|x| \geq A$ and $|\Phi(x)-\varphi(x)| \leq (\|\omega\|_\infty + c)\,|x|^{-\frac{2-\gamma}{\alpha}}$ for $|x| \leq A$. Then

$$|\Phi - \varphi| \leq (\|\omega\|_\infty + c)\,\varphi_1,$$

with $\varphi_1 = |\cdot|^{-\frac{2-\gamma}{\alpha}} 1_{\{|x|\leq A\}} \in L^s(\mathbb{R}^N),\ 1 \leq s < N\alpha/(2-\gamma)$. By the smoothing properties of the heat semigroup (2.2.1), we have that $e^{t\Delta}\varphi_1 \in L^r(\mathbb{R}^N)$ and

$$\sup_{t>0} t^{\beta+\delta}\|e^{t\Delta}\varphi_1\|_r < \infty, \quad \text{for} \ \ 0 < \delta < \frac{N}{2} - \frac{2-\gamma}{2\alpha}.$$

From the latter part of Theorem 2.4.1, and, in particular, formula (2.4.27), we have that

$$\sup_{t>0} t^{\beta+\delta}\|u(t) - u_{\mathcal{S}}(t)\|_r \leq C \sup_{t>0} t^{\beta+\delta}\|e^{t\Delta}(\Phi - \varphi)\|_r = C \sup_{t>0} t^{\beta+\delta}\|e^{t\Delta}\varphi_1\|_r.$$

That is

$$\sup_{t>0} t^{\beta+\delta}\|u(t) - u_{\mathcal{S}}(t)\|_r \leq \mathcal{C}, \tag{2.5.37}$$

for $\delta > 0$ sufficiently small and $\mathcal{C}$ a finite positive constant. This gives (2.5.35) directly, and (2.5.36) by a simple dilation argument for $q = r$.

We now turn to the asymptotic result in the $L^\infty$-norm. Write

$$u(t) - u_{\mathcal{S}}(t) = e^{\frac{t}{2}\Delta}\big(u(t/2) - u_{\mathcal{S}}(t/2)\big) + a\int_{\frac{t}{2}}^{t} e^{(t-\sigma)\Delta}\big[|.|^{-\gamma}\left(|u(\sigma)|^\alpha u(\sigma) - |u_{\mathcal{S}}(\sigma)|^\alpha u_{\mathcal{S}}(\sigma)\right)\big]d\sigma.$$

Let $T > 0$ be an arbitrary real number. By using the smoothing properties of the heat semi-group and Proposition 2.2.1 with $(q_1, q_2) = (\infty, \infty)$, we have that

$$\begin{aligned} t^{\frac{2-\gamma}{2\alpha}+\delta}\|u(t) - u_{\mathcal{S}}(t)\|_\infty \ &\leq \ t^{\frac{2-\gamma}{2\alpha}+\delta}\left\|e^{\frac{t}{2}\Delta}\big(u(t/2) - u_{\mathcal{S}}(t/2)\big)\right\|_\infty + |a|t^{\frac{2-\gamma}{2\alpha}+\delta} \times \\ &\int_{\frac{t}{2}}^{t}\left\|e^{(t-\sigma)\Delta}\left[|.|^{-\gamma}\big(|u(\sigma)|^\alpha u(\sigma) - |u_{\mathcal{S}}(\sigma)|^\alpha u_{\mathcal{S}}(\sigma)\big)\right]\right\|_\infty d\sigma \\ &\leq \ Ct^{\beta+\delta}\|u(t/2) - u_{\mathcal{S}}(t/2)\|_r + \\ |a|Ct^{\frac{2-\gamma}{2\alpha}+\delta} &\int_{\frac{t}{2}}^{t}(t-\sigma)^{-\frac{\gamma}{2}}\left(\|u(\sigma)\|_\infty^\alpha + \|u_{\mathcal{S}}(\sigma)\|_\infty^\alpha\right)\|u(\sigma) - u_{\mathcal{S}}(\sigma)\|_\infty d\sigma. \end{aligned}$$

Using (2.5.37) to estimate the first term and the fact that $\|u_{\mathcal{S}}(t)\|_\infty \leq Ct^{-\frac{2-\gamma}{2\alpha}}, \quad \|u(t)\|_\infty \leq Ct^{-\frac{2-\gamma}{2\alpha}}$ to estimate the last term, we get

$$\begin{aligned} t^{\frac{2-\gamma}{2\alpha}+\delta}\|u(t) - u_{\mathcal{S}}(t)\|_\infty \ &\leq \ C(\delta) + 2C^\alpha|a|C \times \\ &\left[\int_{\frac{1}{2}}^{1}(1-\sigma)^{-\frac{\gamma}{2}}\sigma^{-(\alpha+1)\frac{2-\gamma}{2\alpha}-\delta}d\sigma\right]\sup_{t\in(0,T]}\left(t^{\frac{2-\gamma}{2\alpha}+\delta}\|u(t) - u_{\mathcal{S}}(t)\|_\infty\right). \end{aligned}$$

This gives that, $\sup_{t\in(0,T]} t^{\frac{2-\gamma}{2\alpha}+\delta}\|u(t) - u_{\mathcal{S}}(t)\|_\infty \leq C'(\delta)$. Since the constant $C'(\delta)$ does not depend on $T > 0$, one can take the supremum over $(0, \infty)$. It follows that for all $\delta > 0$ sufficiently small there exists a constant $C'(\delta)$ such that

$$\|u(t) - u_{\mathcal{S}}(t)\|_\infty \leq C'(\delta)t^{-\frac{2-\gamma}{2\alpha}-\delta},$$

for all $t > 0$. This proves (2.5.35) for $r = \infty$. The general result (2.5.35) follows now by the interpolation inequality. The estimate (2.5.36) follows by a simple dilation argument. □

## 2.6 Asymptotic behavior: Linear case

In this section we give the proof of Theorem 2.1.6 Part (ii). To prove the asymptotic linear behavior, we need to establish an adequate global existence result.

### 2.6.1 More global existence results

We have the following lemma used to establish the needed global existence result. We denote, for $a \in \mathbb{R}$, $a_+$ by $a_+ := \max(a, 0)$ and $\frac{1}{a_+} = \frac{1}{a}$ if $a > 0$, and $\infty$ if $a \leq 0$.

**Lemma 2.6.1.** *Assume that $0 < \gamma < \min(2, N)$ and $\alpha > (2-\gamma)/N$. Let $\alpha_1$ be a real number such that*

$$\alpha > \alpha_1 > \frac{2-\gamma}{N}.$$

*Let $r_1$ be a real number satisfying*

$$\max\left(\frac{N(\alpha_1+1)}{N-\gamma}, \frac{N\alpha_1}{2-\gamma}\right) < r_1 < \frac{N\alpha_1(\alpha_1+1)}{(2-\gamma(\alpha_1+1))_+}, \tag{2.6.38}$$

*Let*

$$r_2 = \frac{\alpha}{\alpha_1} r_1, \tag{2.6.39}$$

$$\beta_1 = \frac{2-\gamma}{2\alpha_1} - \frac{N}{2r_1}, \tag{2.6.40}$$

$$\beta_2 = \frac{2-\gamma}{2\alpha} - \frac{N}{2r_2}. \tag{2.6.41}$$

*Define $r_{12}$ and $\beta_{12}$ by*

$$r_{12} = \frac{\alpha+1}{\alpha_1+1} r_1,\ \beta_{12} = \frac{\alpha_1+1}{\alpha+1}\beta_1. \tag{2.6.42}$$

*Then we have the following*

*(i)* $\beta_1 > 0,\ \beta_2 > 0,\ \beta_{12} > 0,$

*(ii)* $\frac{1}{r_1} < \frac{\gamma}{N} + \frac{\alpha+1}{r_{12}} < 1,\ \frac{1}{r_2} < \frac{\gamma}{N} + \frac{\alpha+1}{r_2} < 1,$

*(iii)* $\frac{N}{2}(\frac{\alpha+1}{r_{12}} - \frac{1}{r_1}) + \frac{\gamma}{2} = \frac{N\alpha}{2r_2} + \frac{\gamma}{2} < 1,$

*(iv)* $\beta_2(\alpha+1) < 1,\ \beta_{12}(\alpha+1) < 1,$

*(v)* $\beta_2 - \frac{N\alpha}{2r_2} - \frac{\gamma}{2} - \beta_2(\alpha+1) + 1 = 0,$

*(vi)* $\beta_1 - \frac{N}{2}(\frac{\alpha+1}{r_{12}} - \frac{1}{r_1}) - \frac{\gamma}{2} - \beta_{12}(\alpha+1) + 1 = 0.$

The proof of Lemma 2.6.1 is given in Section 2.8. We now give the following global existence result.

**Theorem 2.6.1.** *Let $0 < \gamma < \min(2, N)$ and $\alpha > (2-\gamma)/N$. Let $\alpha_1$ be a real number such that*

$$\alpha > \alpha_1 > \frac{2-\gamma}{N}.$$

*Let $r_1$, $r_2$, $r_{12}$, $\beta_1$ and $\beta_2$ be real numbers as in Lemma 2.6.1. Suppose further that $M > 0$ satisfies the inequality*

$$KM^{\alpha} < 1, \tag{2.6.43}$$

*where $K$ is a positive constant. Choose $R > 0$ such that*

$$R + KM^{\alpha+1} \leq M. \tag{2.6.44}$$

*Let $\varphi$ be a tempered distribution such that*

$$\max\left[\sup_{t>0} t^{\beta_1}\|e^{t\Delta}\varphi\|_{r_1}, \sup_{t>0} t^{\beta_2}\|e^{t\Delta}\varphi\|_{r_2}\right] \leq R. \tag{2.6.45}$$

*It follows that there exists a unique global solution $u$ of (2.3.4 ) such that*

$$\max\left[\sup_{t>0} t^{\beta_1}\|u(t)\|_{r_1}, \sup_{t>0} t^{\beta_2}\|u(t)\|_{r_2}\right] \leq M. \tag{2.6.46}$$

*Furthermore,*

*(i)* $u(t) - e^{t\Delta}\varphi \in C\big([0,\infty); L^s(\mathbb{R}^N)\big)$, *for $s$ satisfying* $\frac{2-\gamma}{N\alpha_1} < \frac{1}{s} < \frac{\gamma}{N} + \frac{\alpha+1}{r_{12}}$.

*(ii)* $u(t) - e^{t\Delta}\varphi \in L^{\infty}\big((0,\infty); L^s(\mathbb{R}^N)\big)$, *for $s$ satisfying* $\frac{2-\gamma}{N\alpha_1} \leq \frac{1}{s} < \frac{\gamma}{N} + \frac{\alpha+1}{r_{12}}$

*(iii)* $\lim_{t\to 0} u(t) = \varphi$ *in the sense of distributions.*

*(iv)* $\sup_{t>0} t^{\frac{2-\gamma}{2\alpha_1} - \frac{N}{2q}}\|u(t)\|_q < \infty$, $\forall q \in [r_1, \infty]$.

*(v)* $\sup_{t>0} t^{\frac{2-\gamma}{2\alpha} - \frac{N}{2q}}\|u(t)\|_q < \infty$, $\forall q \in [r_2, \infty]$.

*Moreover, let $\varphi$ and $\psi$ satisfy* (2.6.45) *and let $u$ and $v$ be respectively the solutions of (2.3.4) with initial values $\varphi$ and $\psi$ respectively. Then*

$$\max\left[\sup_{t>0} t^{\beta_1}\|u(t)-v(t)\|_{r_1}, \sup_{t>0} t^{\beta_2}\|u(t)-v(t)\|_{r_2}\right] \leq (1-KM^{\alpha})^{-1}\times \\ \max\left[\sup_{t>0} t^{\beta_1}\|e^{t\Delta}(\varphi-\psi)\|_{r_1}, \sup_{t>0} t^{\beta_2}\|e^{t\Delta}(\varphi-\psi)\|_{r_2}\right] \tag{2.6.47}$$

*and*

$$\max\left[\sup_{t>0} t^{\frac{2-\gamma}{2\alpha_1}}\|u(t)-v(t)\|_{\infty}, \sup_{t>0} t^{\frac{2-\gamma}{2\alpha}}\|u(t)-v(t)\|_{\infty}\right] \leq \\ C\max\left[\sup_{t>0} t^{\beta_1}\|e^{t\Delta}(\varphi-\psi)\|_{r_1}, \sup_{t>0} t^{\beta_2}\|e^{t\Delta}(\varphi-\psi)\|_{r_2}\right]. \tag{2.6.48}$$

*Proof.* The proof is based on a contraction mapping argument and uses some idea of [43]. Let X be the Bochner of measurable functions $u : (0,\infty) \to L^{r_1}(\mathbb{R}^N) \cap L^{r_2}(\mathbb{R}^N)$ such that

$$\max\left[\sup_{t>0} t^{\beta_1}\|u(t)\|_{r_1},\ \sup_{t>0} t^{\beta_2}\|u(t)\|_{r_2}\right] < \infty.$$

We denote by $X_M$ the set of $u \in X$ such that $\max\left[\sup_{t>0} t^{\beta_1}\|u(t)\|_{r_1}, \sup_{t>0} t^{\beta_2}\|u(t)\|_{r_2}\right] \le M$. Endowed with the metric: $d(u,v) = \max\left[\sup_{t>0} t^{\beta_1}\|u(t)-v(t)\|_{r_1}, \sup_{t>0} t^{\beta_2}\|u(t)-v(t)\|_{r_2}\right]$, $X_M$ is a nonempty complete metric space. Let $M$, $R$ be two real numbers satisfying (2.6.43)-(2.6.44). Consider the mapping defined by

$$\mathcal{F}_\varphi(u)(t) = e^{t\Delta}\varphi + a\int_0^t e^{(t-\sigma)\Delta}\big(|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\big)d\sigma, \tag{2.6.49}$$

where $\varphi$ is a tempered distribution satisfying (2.6.45). We will show that $\mathcal{F}_\varphi$ is a strict contraction on $X_M$.

Let $\varphi$ and $\psi$ satisfy (2.6.45) and $u, v \in X_M$. It follows that

$$\begin{aligned} t^{\beta_1}\|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_{r_1} &\le t^{\beta_1}\|e^{t\Delta}(\varphi-\psi)\|_{r_1} + \\ &\quad |a|t^{\beta_1}\int_0^t \|e^{(t-\sigma)\Delta}\left[|.|^{-\gamma}\big(|u(\sigma)|^\alpha u(\sigma) - |v(\sigma)|^\alpha v(\sigma)\big)\right]\|_{r_1} d\sigma. \end{aligned}$$

It follows, by Proposition 2.2.1 with $(q_1, q_2) = (\frac{r_{12}}{\alpha+1}, r_1)$ due to Lemma 2.6.1 Part (ii) and Hölder inequality that

$$\begin{aligned} \|e^{(t-\sigma)\Delta}\left[|.|^{-\gamma}\big(|u(\sigma)|^\alpha u(\sigma) - |v(\sigma)|^\alpha v(\sigma)\big)\right]\|_{r_1} &\le C(t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{r_1})-\frac{\gamma}{2}} \times \\ &\quad \||u(\sigma)|^\alpha u(\sigma) - |v(\sigma)|^\alpha v(\sigma)\|_{\frac{r_{12}}{\alpha+1}} \\ &\le C(t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{r_1})-\frac{\gamma}{2}}(\alpha+1)\Big(\|u(\sigma)\|_{r_{12}}^\alpha + \|v(\sigma)\|_{r_{12}}^\alpha\Big)\|u(\sigma)-v(\sigma)\|_{r_{12}}. \end{aligned}$$

Using the interpolation inequality

$$\|u(\sigma)\|_s \le \|u(\sigma)\|_{r_1}^\theta \|u(\sigma)\|_{r_2}^{1-\theta},\ \frac{1}{s} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}, \tag{2.6.50}$$

where $\theta = \frac{1}{\alpha+1}$ and $s = r_{12}$ along with the fact that $u$, $v$ are in $X_M$ we see that

$$\begin{aligned} \|e^{(t-\sigma)\Delta}\left[|.|^{-\gamma}\big(|u(\sigma)|^\alpha u(\sigma) - |v(\sigma)|^\alpha v(\sigma)\big)\right]\|_{r_1} &\le 2(\alpha+1)C \times \\ &\quad (t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{r_1})-\frac{\gamma}{2}}\sigma^{-\beta_{12}(\alpha+1)}M^\alpha d(u,v). \end{aligned}$$

Hence,

$$\begin{aligned} \|\int_0^t e^{(t-\sigma)\Delta}\left[|.|^{-\gamma}\big(|u(\sigma)|^\alpha u(\sigma) - |v(\sigma)|^\alpha v(\sigma)\big)\right]d\sigma\|_{r_1} &\le 2(\alpha+1)CM^\alpha \times \\ &\quad \left[\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{r_1})-\frac{\gamma}{2}}\sigma^{-\beta_{12}(\alpha+1)}d\sigma\right]d(u,v) \\ &\le 2(\alpha+1)CM^\alpha d(u,v)t^{-\beta_{12}(\alpha+1)-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{r_1})-\frac{\gamma}{2}+1} \times \\ &\quad \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{r_1})-\frac{\gamma}{2}}\sigma^{-\beta_{12}(\alpha+1)}d\sigma. \end{aligned}$$

Then, using Part (vi) of Lemma 2.6.1, we obtain

$$\begin{aligned} t^{\beta_1}\|\mathcal{F}_\varphi(u)(t)-\mathcal{F}_\psi(v)(t)\|_{r_1} &\leq t^{\beta_1}\|e^{t\Delta}(\varphi-\psi)\|_{r_1}+2|a|(\alpha+1)CM^\alpha\times \\ &\quad \left[\int_0^1(1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{r_1})-\frac{\gamma}{2}}\sigma^{-\beta_{12}(\alpha+1)}d\sigma\right]d(u,v) \\ &\leq t^{\beta_1}\|e^{t\Delta}(\varphi-\psi)\|_{r_1}+C_1M^\alpha d(u,v), \end{aligned} \tag{2.6.51}$$

where

$$C_1=2|a|(\alpha+1)C\int_0^1(1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{r_1})-\frac{\gamma}{2}}\sigma^{-\beta_{12}(\alpha+1)}d\sigma.$$

On the other hand, we have

$$\begin{aligned} t^{\beta_2}\|\mathcal{F}_\varphi(u)(t)-\mathcal{F}_\psi(v)(t)\|_{r_2} &\leq t^{\beta_2}\|e^{t\Delta}(\varphi-\psi)\|_{r_2}+|a|t^{\beta_2}\times \\ &\quad \int_0^t\|e^{(t-\sigma)\Delta}\left[|.|^{-\gamma}\big(|u(\sigma)|^\alpha u(\sigma)-|v(\sigma)|^\alpha v(\sigma)\big)\right]\|_{r_2}d\sigma. \end{aligned}$$

It follows by Proposition 2.2.1 with $(q_1,q_2)=(r_2/(\alpha+1),r_2)$ and with the fact that $u,v$ are in $X_M$ that

$$\begin{aligned} \|e^{(t-\sigma)\Delta}\left[|.|^{-\gamma}\big(|u(\sigma)|^\alpha u(\sigma)-|v(\sigma)|^\alpha v(\sigma)\big)\right]\|_{r_2} &\leq C(t-\sigma)^{-\frac{N\alpha}{2r_2}-\frac{\gamma}{2}}\times \\ &\quad \||u(\sigma)|^\alpha u(\sigma)-|v(\sigma)|^\alpha v(\sigma)\|_{\frac{r_2}{\alpha+1}} \\ &\leq C(t-\sigma)^{-\frac{N\alpha}{2r_2}-\frac{\gamma}{2}}(\alpha+1)\Big(\|u(\sigma)\|_{r_2}^\alpha+\|v(\sigma)\|_{r_2}^\alpha\Big)\|u(\sigma)-v(\sigma)\|_{r_2} \\ &\leq 2(\alpha+1)C(t-\sigma)^{-\frac{N\alpha}{2r_2}-\frac{\gamma}{2}}\sigma^{-\beta_2(\alpha+1)}M^\alpha d(u,v). \end{aligned}$$

This together with Part (v) of Lemma 2.6.1 gives

$$t^{\beta_2}\|\mathcal{F}_\varphi(u)(t)-\mathcal{F}_\psi(v)(t)\|_{r_2}\leq t^{\beta_2}\|e^{t\Delta}(\varphi-\psi)\|_{r_2}+C_2M^\alpha d(u,v), \tag{2.6.52}$$

where $C_2=2|a|(\alpha+1)C\int_0^1(1-\sigma)^{-\frac{N\alpha}{2r_2}-\frac{\gamma}{2}}\sigma^{-\beta_2(\alpha+1)}d\sigma$.

Due to Parts (iii) and (iv) of Lemma 2.6.1, $C_1$ and $C_2$ are finite positive constants. Now we get by (2.6.51) and (2.6.52) that,

$$\begin{aligned} &\max\left[\sup_{t>0}t^{\beta_1}\|\mathcal{F}_\varphi(u)(t)-\mathcal{F}_\psi(v)(t)\|_{r_1},\sup_{t>0}t^{\beta_2}\|\mathcal{F}_\varphi(u)(t)-\mathcal{F}_\psi(v)(t)\|_{r_2}\right]\leq \\ &\max\left[\sup_{t>0}t^{\beta_1}\|e^{t\Delta}(\varphi-\psi)\|_{r_1},\sup_{t>0}t^{\beta_2}\|e^{t\Delta}(\varphi-\psi)\|_{r_2}\right]+KM^\alpha d(u,v), \end{aligned} \tag{2.6.53}$$

where $K=\max(C_1,C_2)$. Setting $\psi=0$ and $v=0$, and using (2.6.45), (2.6.44) we obtain

$$\max\left[\sup_{t>0}t^{\beta_1}\|\mathcal{F}_\varphi(u)(t)\|_{r_1},\sup_{t>0}t^{\beta_2}\|\mathcal{F}_\varphi(u)(t)\|_{r_2}\right]\leq R+KM^{\alpha+1}\leq M.$$

Then $\mathcal{F}_\varphi$ maps $X_M$ into itself. Letting $\varphi=\psi$, we get

$$d\left(\mathcal{F}_\varphi(u), \mathcal{F}_\varphi(v)\right) \leq KM^\alpha d(u,v).$$

Hence inequality (2.6.43) gives that $\mathcal{F}_\varphi$ is a strict contraction mapping from $X_M$ into itself. Then $\mathcal{F}_\varphi$ has a unique fixed point $u$ in $X_M$ which is solution of (2.3.4).

We now prove that $u(t) - e^{t\Delta}\varphi \in C\big([0,\infty), L^s(\mathbb{R}^N)\big)$ for $s$ satisfying

$$\frac{2-\gamma}{N\alpha_1} < \frac{1}{s} < \frac{\gamma}{N} + \frac{\alpha+1}{r_{12}}. \tag{2.6.54}$$

First, the existence of such $s$ is insured by Part (iv) of Lemma 2.6.1. Write

$$u(t) - e^{t\Delta}\varphi = a\int_0^t e^{(t-\sigma)\Delta}\big(|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\big)d\sigma.$$

Then for $s$ satisfying (2.6.54), we obtain by Proposition 2.2.1, with $(q_1,q_2) = (\frac{r_{12}}{\alpha+1}, s)$,

$$\begin{aligned}
\|u(t) - e^{t\Delta}\varphi\|_s &\leq |a|\int_0^t \|e^{(t-\sigma)\Delta}\big(|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\big)\|_s d\sigma \\
&\leq |a|C\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{s})-\frac{\gamma}{2}}\|u(\sigma)\|_{r_{12}}^{\alpha+1}d\sigma \\
&\leq |a|CM^{\alpha+1}\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{s})-\frac{\gamma}{2}}\sigma^{-\beta_{12}(\alpha+1)}d\sigma \\
&= |a|CM^{\alpha+1}t^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{s})-\frac{\gamma}{2}-\beta_{12}(\alpha+1)+1}\times \\
&\quad \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{s})-\frac{\gamma}{2}}\sigma^{-\beta_{12}(\alpha+1)}d\sigma.
\end{aligned}$$

Therefore we obtain

$$\|u(t) - e^{t\Delta}\varphi\|_s \leq |a|CM^{\alpha+1}t^{\frac{N}{2s}-\frac{2-\gamma}{2\alpha_1}}\int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{s})-\frac{\gamma}{2}}\sigma^{-\beta_{12}(\alpha+1)}d\sigma. \tag{2.6.55}$$

Owing to (2.6.54) we can see that $\int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{s})-\frac{\gamma}{2}}\sigma^{-\beta_{12}(\alpha+1)}d\sigma$ is finite and that $t^{\frac{N}{2s}-\frac{2-\gamma}{2\alpha_1}}$ converges to zero as $t \searrow 0$. This proves the statements (i) and (iii) of Theorem 2.6.1. Statement (ii) with $s = \frac{N\alpha_1}{2-\gamma}$ follows from (2.6.55) which still holds if $s = \frac{N\alpha_1}{2-\gamma}$.

The proof of Parts (iv)-(v) for $r = \infty$ follows by iterative argument as in the proof of Theorem 2.4.1 Part (v) so we omit it. The result for the other values of $r$ will then follow by using the Hölder inequality and (2.6.46).

The continuous dependence relation (2.6.47) of the solution on the initial data can be easily deduced by (2.6.53) with $\mathcal{F}_\varphi(u) = u$, $\mathcal{F}_\psi(v) = v$. Formula (2.6.48) can be proved starting with (2.6.47) and using an iterative procedure. This completes the proof of Theorem 2.6.1. □

We now turn to establish the linear behavior.

### 2.6.2 Linear behavior

The following technical lemma will be needed in the proof of the linear asymptotic behavior.

**Lemma 2.6.2.** *Let $0 < \gamma < \min(2, N)$. Let the real numbers $\alpha_1$ and $\alpha$ be such that*

$$\alpha > \alpha_1 > \frac{2-\gamma}{N}.$$

*Let $r_1$ and $r_2$ be two real numbers as in Lemma 2.6.1. Let $\beta_1$ and $\beta_2$ be given by (2.6.40) and (2.6.41). Then there exists a real number $\delta_0 > 0$ such that, for all $0 < \delta < \delta_0$, there exists a real number*

$$0 < \theta_\delta < 1, \tag{2.6.56}$$

*with the properties that, the two real numbers $r'$ and $\beta'$ given by*

$$\frac{1}{r'} = \frac{\theta_\delta}{r_1} + \frac{1-\theta_\delta}{r_2}, \quad \beta' = \theta_\delta \beta_1 + (1-\theta_\delta)\beta_2 = \frac{(2-\gamma)\left[\theta_\delta(\alpha-\alpha_1)+\alpha_1\right]}{2\alpha_1\alpha} - \frac{N}{2r'}, \tag{2.6.57}$$

*satisfy the following conditions*

*(a)* $\frac{1}{r_1} < \frac{\gamma}{N} + \frac{\alpha+1}{r'} < 1,$

*(b)* $\beta_1 + \delta - \frac{N}{2}(\frac{\alpha+1}{r'} - \frac{1}{r_1}) - \frac{\gamma}{2} - \beta'(\alpha+1) + 1 = 0,$

*(c)* $\frac{N}{2}(\frac{\alpha+1}{r'} - \frac{1}{r_1}) + \frac{\gamma}{2} < 1, \quad \beta'(\alpha+1) < 1.$

*Moreover, the real number $\theta_\delta$ satisfies*

$$\theta_\delta = \frac{1}{\alpha+1} + \frac{2\alpha_1\alpha}{(2-\gamma)(\alpha-\alpha_1)(\alpha+1)}\delta. \tag{2.6.58}$$

The proof of the previous lemma is given in Section 2.8. We now give the asymptotic behavior result. We have the following more general version. In particular the asymptotic behavior is given in $L^q(\mathbb{R}^N)$ for all $q \geq r_1$.

**Theorem 2.6.2** (Linear behavior)**.** *Let $0 < \gamma < \min(2, N)$. Suppose that*

$$\alpha > \alpha_1 > \frac{2-\gamma}{N}.$$

*Let $r_1,\ r_2$ be two real numbers as in Lemma 2.6.1. Let $\beta_1,\ \beta_2$ be given by (2.6.40), (2.6.41) and define $\beta_1(q)$ by*

$$\beta_1(q) = \frac{2-\gamma}{2\alpha_1} - \frac{N}{2q}, \quad q > 1. \tag{2.6.59}$$

*Let $\Psi(x) = \omega(x)|x|^{-\frac{2-\gamma}{\alpha_1}}$, where $\omega \in L^\infty(S^{N-1})$ is homogeneous of degree 0. Let $\varphi \in C_0(\mathbb{R}^N)$ be such that*

$$|\varphi(x)| \leq \frac{c}{(1+|x|^2)^{\frac{2-\gamma}{2\alpha_1}}},\ \forall\, x \in \mathbb{R}^N,\ \varphi(x) = \omega(x)|x|^{-\frac{2-\gamma}{\alpha_1}},\ |x| \geq A,$$

*for some constant $A > 0$, where $c$ is a small positive constant and $\|\omega\|_\infty$ is sufficiently small.*

*Let $u$ be the solution of (2.3.4) with initial data $\varphi$, constructed by Theorem 2.6.1 and let $w$ be the self-similar solution of (2.3.4) constructed by Theorem 2.6.1 with $a = 0$, and with initial data $\Psi$. Then* *there exists $\delta_1 > 0$ such that for all $\delta$, $0 < \delta < \delta_1$, and with $M$ perhaps smaller, there exists $C_\delta > 0$ such that*

$$\|u(t) - w(t)\|_q \leq C_\delta t^{-\beta_1(q)-\delta}, \quad \forall\, t > 0, \tag{2.6.60}$$

$$\|t^{\frac{2-\gamma}{2\alpha_1}} v(t, .\sqrt{t}) - w(1, .)\|_q \leq C_\delta t^{-\delta}, \quad \forall\, t > 0, \tag{2.6.61}$$

*for all $q \in [r_1, \infty]$. In particular, if $\omega \not\equiv 0$, there exist $d_1 > 0$, $d_2 > 0$ two constants, such that*

$$d_1 t^{-\beta_1(q)} \leq \|u(t)\|_q \leq d_2 t^{-\beta_1(q)},$$

*for large time and for all $r_1 \leq q \leq \infty$.*

*Proof.* By writing $|\cdot|^{-\frac{2-\gamma}{\alpha_1}} = f_1 + f_2$, with $f_1 \in L^s(\mathbb{R}^N)$, $1 \leq s < N\alpha_1/(2-\gamma)$, $f_2 \in L^{r_1}(\mathbb{R}^N)$, it follows by the smoothing properties of the heat semigroup that $e^{t\Delta}\Psi \in L^{r_1}(\mathbb{R}^N)$ and by homogeneity, we have that $\sup_{t>0} t^{\beta_1}\|e^{t\Delta}\Psi\|_{r_1} < \infty$. Since $|\varphi(x)| \leq (c + \|\omega\|_\infty)|x|^{-\frac{2-\gamma}{\alpha_1}}$, and because $\alpha_1 < \alpha$, and by conditions on $\varphi$, we have also $|\varphi(x)| \leq (c + \|\omega\|_\infty)|x|^{-\frac{2-\gamma}{\alpha}}$, then $\sup_{t>0} t^{\beta_1}\|e^{t\Delta}\varphi\|_{r_1} < \infty$ and $\sup_{t>0} t^{\beta_2}\|e^{t\Delta}\varphi\|_{r_2} < \infty$. Hence $\varphi$ verifies (2.6.45) for $c$ and $\|\omega\|_\infty$ sufficiently small. We have also that $|\Psi(x) - \varphi(x)| = 0$ for $|x| \geq A$ and $|\Psi(x) - \varphi(x)| \leq (\|\omega\|_\infty + c)\,|x|^{-\frac{2-\gamma}{\alpha_1}}$ for $|x| \leq A$. Then

$$|\Psi - \varphi| \leq (\|\omega\|_\infty + c)\,\varphi_1,$$

with $\varphi_1 = |\cdot|^{-\frac{2-\gamma}{\alpha_1}} 1_{\{|x|\leq A\}} \in L^s(\mathbb{R}^N)$, $1 \leq s < N\alpha_1/(2-\gamma)$. By the smoothing properties of the heat semigroup (2.2.1), we have that $e^{t\Delta}\varphi_1 \in L^{r_1}(\mathbb{R}^N)$ and

$$\sup_{t>0} t^{\beta_1+\delta}\|e^{t\Delta}\varphi_1\|_{r_1} < \infty, \ \text{ for } \ 0 < \delta < \frac{N}{2} - \frac{2-\gamma}{2\alpha_1}.$$

Let $v$ be the solution of (2.3.4) with $a = 0$ and with initial data $\varphi$. We have

$$u(t) - v(t) = a\int_0^t e^{(t-\sigma)\Delta}\big(|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\big)d\sigma,$$

and so

$$\|u(t) - v(t)\|_{r_1} \leq |a|\int_0^t \|e^{(t-\sigma)\Delta}\big(|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\big)\|_{r_1} d\sigma. \tag{2.6.62}$$

Let $0 < \delta < \delta_0$, where $\delta_0$ is as in Lemma 2.6.2 and consider the two real numbers $r'$ and $\beta'$ given by (2.6.57). Then thanks to (a) of Lemma 2.6.2, we obtain

$$\|u(t) - v(t)\|_{r_1} \leq |a|C\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r'} - \frac{1}{r_1}) - \frac{\gamma}{2}}\|u(\sigma)\|_{r'}^{\alpha+1} d\sigma. \tag{2.6.63}$$

Using the fact that $u$ and $v$ belong to $X_M$ and the interpolation inequality (2.6.50) with $s = r'$ and $\theta_\delta$, given by (2.6.58), we deduce from (2.6.63) that

$$\begin{aligned} t^{\beta_1+\delta}\|u(t) - v(t)\|_{r_1} \ \leq\ & |a|CM^{\alpha+1}t^{\beta_1+\delta-\frac{N}{2}(\frac{\alpha+1}{r'}-\frac{1}{r_1})-\frac{\gamma}{2}-\beta'(\alpha+1)+1} \\ & \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r'}-\frac{1}{r_1})-\frac{\gamma}{2}}\sigma^{-\beta'(\alpha+1)} d\sigma. \end{aligned}$$

By Lemma 2.6.2 $\int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r'}-\frac{1}{r_1})-\frac{\gamma}{2}}\sigma^{-\beta'(\alpha+1)}d\sigma$ is finite. Now by (b) of Lemma 2.6.2 we deduce that

$$t^{\beta_1+\delta}\|u(t)-v(t)\|_{r_1} \quad\leq\quad C_\delta.$$

And so

$$\|u(t)-v(t)\|_{r_1} \leq C_\delta t^{-\beta_1-\delta}, \quad \forall\ t>0. \tag{2.6.64}$$

Then, we obtain

$$\|u(t)-w(t)\|_{r_1} \leq \|u(t)-v(t)\|_{r_1} + \|e^{t\Delta}(\varphi-\Psi)\|_{r_1} \leq C_\delta t^{-\beta_1-\delta} + C'_\delta t^{-\beta_1-\delta}, \tag{2.6.65}$$

where $\delta>0$ sufficiently small. Hence (2.6.65) gives (2.6.60) for $q=r_1$.

We now turn to the asymptotic result in the $L^\infty$-norm. Write

$$u(t)-w(t) = e^{\frac{t}{2}\Delta}\big(u(t/2)-w(t/2)\big) + a\int_{\frac{t}{2}}^t e^{(t-\sigma)\Delta}(|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma))d\sigma.$$

Let $\delta>0$ sufficiently small. Let $\theta_\delta$ be given by (2.6.58), where $\delta>0$ is chosen such that $\theta_\delta<1$. By using the smoothing properties of the heat semi-group and Proposition 2.2.1 with $(q_1,q_2)=(\infty,\infty)$, we have that

$$\begin{aligned} t^{\frac{2-\gamma}{2\alpha_1}+\delta}\|u(t)-w(t)\|_\infty &\leq t^{\frac{2-\gamma}{2\alpha_1}+\delta}\|e^{\frac{t}{2}\Delta}\big(u(t/2)-w(t/2)\big)\|_\infty + |a|t^{\frac{2-\gamma}{2\alpha_1}+\delta}\times \\ &\quad \int_{\frac{t}{2}}^t \|e^{(t-\sigma)\Delta}\left(|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\right)\|_\infty d\sigma \\ &\leq Ct^{\beta_1+\delta}\|u(t/2)-w(t/2)\|_{r_1} + \\ &\quad |a|Ct^{\frac{2-\gamma}{2\alpha_1}+\delta}\int_{\frac{t}{2}}^t (t-\sigma)^{-\frac{\gamma}{2}}\|u(\sigma)\|_\infty^{\alpha+1}d\sigma. \end{aligned}$$

Using the fact that, $\|u(\sigma)\|_\infty \leq C\sigma^{-[\theta_\delta\frac{2-\gamma}{2\alpha_1}+(1-\theta_\delta)\frac{2-\gamma}{2\alpha}]}$, to estimate this last term, that is $\|u(\sigma)\|_\infty^{\alpha+1}\leq C^{\alpha+1}\sigma^{-\frac{2-\gamma}{2\alpha_1}-\delta+\frac{\gamma}{2}-1}$, we get

$$t^{\frac{2-\gamma}{2\alpha_1}+\delta}\|u(t)-w(t)\|_\infty \leq C(\delta) + C^{\alpha+1}|a|C\int_{\frac{1}{2}}^1 (1-\sigma)^{-\frac{\gamma}{2}}\sigma^{-\frac{2-\gamma}{2\alpha_1}-\delta+\frac{\gamma}{2}-1}d\sigma.$$

Thus, it follows that for all $\delta>0$ sufficiently small there exists a constant $C'(\delta)$ such that

$$\|u(t)-w(t)\|_\infty \leq C'(\delta)t^{-\frac{2-\gamma}{2\alpha_1}-\delta},$$

for all $t>0$. This proves (2.6.60) for $q=\infty$. The general result (2.6.60) follows now by the Hölder inequality. The estimate (2.6.61) follows by a simple dilation argument. The proof of the theorem is now complete. □

We now give the proof of Theorem 2.1.6 for the linear case.

*Proof of Theorem 2.1.6 Part (ii).* Let $\alpha_1$ be as in Theorem 2.6.2. Put $\sigma = (2-\gamma)/\alpha_1$. Since $\alpha > \alpha_1 > (2-\gamma)/N$ then $(2-\gamma)/\alpha < \sigma < N$. Then (ii) follows by Theorem 2.6.2, precisely by (2.6.60) with $q = \infty$. □

## 2.7 General singular problem

In this section we study the general singular nonlinear parabolic equation

$$u(t) = e^{t\Delta}\varphi + \int_0^t e^{(t-\sigma)\Delta}\left[V(.)|u(\sigma)|^\alpha u(\sigma)\right] d\sigma, \tag{2.7.66}$$

$u = u(t,x) \in \mathbb{R},\ \ t > 0,\ x \in \mathbb{R}^N,\ \alpha > 0$ and $\varphi \in \mathcal{S}'(\mathbb{R}^N)$. The potential $V$ verifies

$$|V(x)| \leq C|x|^{-\gamma},\ \forall\, x \in \mathbb{R}^N\backslash\{0\}, \tag{2.7.67}$$

and satisfies one of the hypotheses:

$(H_1)$ $V(x) = a(1-f(x))|x|^{-\gamma}$,

$(H_2)$ $V(x) = af(x)|x|^{-\gamma}$,

where $a \in \mathbb{R}$ and $f$ is such that

$$f(x)|x|^{-\gamma} \in L^s(\mathbb{R}^N),\ \ \frac{\gamma}{N} < \frac{1}{s} < \frac{2\gamma}{N} + \frac{\alpha+1}{r} - \frac{1}{q_c},\ \frac{1}{s} < \frac{2}{N} - \frac{\alpha}{r}, \tag{2.7.68}$$

with $r > q_c$ satisfies (2.4.19). As an example for such function $f$ we may take a cut-off function compactly supported and $f \equiv 1$ near the origin.

It is clear that Proposition 2.2.1, the $C_0$-well-posedness, $L^q$-well-posedness and the global existence results hold as for the case $V(x) = a|x|^{-\gamma}$. In particular, we can prove similar results of global existence as in Theorem 2.1.4. Here, we are mainly concerned with the asymptotic behavior of global solutions with initial values $\varphi(x) \sim \omega(x)|x|^{-\frac{2-\gamma}{\alpha}}$ as $|x| \to \infty$, where $\omega \in L^\infty(\mathbb{R}^N)$ is homogeneous of degree 0 and $\|\omega\|_\infty$ is sufficiently small. We show that for the case $(H_1)$, we have the same asymptotic behavior as for the equation (2.1.1). While for the case $(H_2)$ the behavior is linear, in particular it is different from the case of equation (2.1.1). Precisely, we have the following result.

**Theorem 2.7.1** (General Potential). *Let $0 < \gamma < \min(2, N)$. Suppose that*

$$\alpha > \frac{2-\gamma}{N}. \tag{2.7.69}$$

*Let $\varphi \in C_0(\mathbb{R}^N)$ be such that*

$$|\varphi(x)| \leq \frac{c}{(1+|x|^2)^{\frac{2-\gamma}{2\alpha}}},\ \forall\, x \in \mathbb{R}^N,$$

*for $c > 0$ sufficiently small, and*

$$\varphi(x) = \omega(x)|x|^{-\frac{2-\gamma}{\alpha}},\ |x| \geq A,$$

*for some constant* $A > 0$ *and some* $\omega \in L^\infty(\mathbb{R}^N)$*, homogeneous of degree* 0*, with* $\|\omega\|_\infty$ *sufficiently small. Assume that* $V$ *satisfies* (2.7.67)*.*

*Let* $u$ *be the global solution of* (2.7.66) *with initial data* $\varphi$*. Let* $u_{\mathcal{S}}$ *be the global mild self-similar solution of (2.1.1)-(2.1.2) with initial data* $\Phi(x) = \omega(x)|x|^{-\frac{2-\gamma}{\alpha}}$*, given by Theorem 2.1.5. Then we have the following.*

*(i) Nonlinear behavior: If* $V(x) = a(1 - f(x))|x|^{-\gamma}$*, where* $f$ *satisfies* (2.7.68)*, then there exists* $\delta > 0$ *such that*

$$\|u(t) - u_{\mathcal{S}}(t)\|_\infty \leq Ct^{-\frac{2-\gamma}{2\alpha}-\delta},\ \forall\, t > 0.$$

*(ii) Linear behavior: If* $V(x) = af(x)|x|^{-\gamma}$*, where* $f$ *satisfies* (2.7.68)*, then there exists* $\delta > 0$ *such that*

$$\left\|u(t) - e^{t\Delta}\left(\omega(\cdot)|\cdot|^{-\frac{2-\gamma}{\alpha}}\right)\right\|_\infty \leq Ct^{-\frac{2-\gamma}{2\alpha}-\delta},\ \forall\, t > 0,$$

*where* $C$ *is a positive constant.*

One should emphasize that Eq. (2.7.66) has no self-similar structure in general, but the previous result shows that global solutions are asymptotically self-similar. In [36], Pinsky considers all positive solutions to Eq. (2.7.66) where $0 \lneq V \in C^\delta(\mathbb{R}^N)$ and that for large $|x|$ and constants $c_1, c_2 > 0$,

$$c_1|x|^{-\gamma} \leq V(x) \leq c_2|x|^{-\gamma}, \quad 0 < \gamma < \min(N, 2).$$

He shows that if $0 < \alpha < (2-\gamma)/N$, then (2.7.66) does not have global solutions, for any choice of initial data $u_0(x) \gneq 0$. This shows that the condition (2.7.69) is optimal.

*Proof of Theorem 2.7.1.* Let $r$ and $\beta$ verify (2.4.19) and (2.4.20). Define $\beta(p)$ by

$$\beta(p) = \frac{2-\gamma}{2\alpha} - \frac{N}{2p}, \quad \forall\, p > 1.$$

Let $q \geq 1$ and $\delta' > 0$ be such that

$$\frac{\gamma}{N} + \frac{\alpha+1}{r} < \frac{1}{q} < \frac{2\gamma}{N} + \frac{2(\alpha+1)}{r} - \frac{2-\gamma}{N\alpha}, \tag{2.7.70}$$

$$\delta' = \frac{N}{2}\left(\frac{1}{q} - \frac{\alpha+1}{r}\right) - \frac{\gamma}{2}, \ \ \frac{N}{2}\left(\frac{1}{q} - \frac{1}{r}\right) < 1. \tag{2.7.71}$$

Proof of Part (i). We will show the following:

$$\|u(t) - u_{\mathcal{S}}(t)\|_p \leq Ct^{-\beta(p)-\delta'},\ \forall\, p \in [r, \infty]. \tag{2.7.72}$$

We have

$$\begin{aligned} u(t) - u_{\mathcal{S}}(t) &= e^{t\Delta}(\varphi - \Phi) + \int_0^t e^{(t-\sigma)\Delta}\left[V(.)|u(\sigma)|^\alpha u(\sigma) - a|.|^{-\gamma}|u_{\mathcal{S}}(\sigma)|^\alpha u_{\mathcal{S}}(\sigma)\right] d\sigma \\ &= e^{t\Delta}(\varphi - \Phi) + a\int_0^t e^{(t-\sigma)\Delta}[|.|^{-\gamma}(|u(\sigma)|^\alpha u(\sigma) - |u_{\mathcal{S}}(\sigma)|^\alpha u_{\mathcal{S}}(\sigma))]d\sigma \\ &\qquad -a\int_0^t e^{(t-\sigma)\Delta}\left[f(.)|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\right] d\sigma. \end{aligned}$$

Then using Proposition 2.2.1 with $(q_1, q_2) = (\frac{r}{\alpha+1}, r)$ we obtain

$$
\begin{aligned}
t^{\beta+\delta'}\|u(t) - u_{\mathcal{S}}(t)\|_r &\leq t^{\beta+\delta'}\|e^{t\Delta}(\varphi - \Phi)\|_r + \\
&\quad |a|t^{\beta+\delta'}\int_0^t \left\|e^{(t-\sigma)\Delta}\left[|.|^{-\gamma}(|u(\sigma)|^\alpha u(\sigma) - |u_{\mathcal{S}}(\sigma)|^\alpha u_{\mathcal{S}}(\sigma))\right]\right\|_r d\sigma \\
&\quad +|a|t^{\beta+\delta'}\int_0^t \left\|e^{(t-\sigma)\Delta}[f(.)|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)]\right\|_r d\sigma \\
&\leq t^{\beta+\delta'}\|e^{t\Delta}(\varphi - \Phi)\|_r + Ct^{\beta+\delta'}\int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}}\left(\|u(\sigma)\|_r^\alpha + \|u_{\mathcal{S}}(\sigma)\|_r^\alpha\right)\|u(\sigma) - u_{\mathcal{S}}(\sigma)\|_r d\sigma \\
&\quad +|a|t^{\beta+\delta'}\int_0^t \left\|e^{(t-\sigma)\Delta}\left[f(.)|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\right]\right\|_r d\sigma.
\end{aligned}
$$

We begin by estimating the first term on the left hand side of the previous inequality. Since $q$ satisfies (2.7.70) it follows that $0 < \delta' < \frac{N}{2} - \frac{2-\gamma}{2\alpha}$ and so as in the proof of Theorem 2.5.1 we obtain the estimate of the first term

$$t^{\beta+\delta'}\|e^{t\Delta}(\varphi - \Phi)\|_r \leq C.$$

We now estimate of the second term. Let $T > 0$ be an arbitrary real number. Since $\sup_{t>0}\|u(t)\|_r \leq M$ and $\sup_{t>0}\|u_{\mathcal{S}}(t)\|_r \leq M$ and by using the expression of $\beta$, we have

$$
\begin{aligned}
&Ct^{\beta+\delta'}\int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}}\left(\|u(\sigma)\|_r^\alpha + \|u_{\mathcal{S}}(\sigma)\|_r^\alpha\right)\|u(\sigma) - u_{\mathcal{S}}(\sigma)\|_r d\sigma \leq \\
&\qquad 2CM^\alpha\left(t^{\beta+\delta'}\int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}}\sigma^{-\beta(\alpha+1)-\delta'}d\sigma\right)\left(\sup_{t\in(0,T]} t^{\beta+\delta'}\|u(t) - u_{\mathcal{S}}(t)\|_r\right) \\
&\qquad \leq 2CM^\alpha\left(t^{\beta+\delta'-\frac{N\alpha}{2r}-\frac{\gamma}{2}-\beta(\alpha+1)-\delta'+1}\int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}}\sigma^{-\beta(\alpha+1)-\delta'}d\sigma\right)\times \\
&\qquad\qquad \left(\sup_{t\in(0,T]} t^{\beta+\delta'}\|u(t) - u_{\mathcal{S}}(t)\|_r\right) \\
&\qquad \leq 2CM^\alpha\left(\int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}}\sigma^{-\beta(\alpha+1)-\delta'}d\sigma\right)\left(\sup_{t\in(0,T]} t^{\beta+\delta'}\|u(t) - u_{\mathcal{S}}(t)\|_r\right) \\
&\qquad \leq C'\left(\sup_{t\in(0,T]} t^{\beta+\delta'}\|u(t) - u_{\mathcal{S}}(t)\|_r\right),
\end{aligned}
$$

where $C' = 2CM^\alpha\int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}}\sigma^{-\beta(\alpha+1)-\delta'}d\sigma$. By the hypotheses on $r$, $q$ and the expressions of $\beta$ and $\delta'$, $C'$ is a finite positive constant.

We turn now to estimate the third term. Let $r'$ be such that

$$\frac{1}{q} = \frac{1}{r'} + \frac{\alpha+1}{r}. \tag{2.7.73}$$

By the assumptions on $f$ we have that $f(.)|.|^{-\gamma}$ is in $L^{r'}(\mathbb{R}^N)$. Using the Hölder inequality

$$\|f(.)|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\|_q \leq \|f(.)|.|^{-\gamma}\|_{r'}\|u(\sigma)\|_r^{\alpha+1}.$$

By the smoothing effect (2.2.1) with $(s_1, s_2) = (q, r)$,

$$\begin{aligned}\|e^{(t-\sigma)\Delta} f(.)|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\|_r &\leq C(t-\sigma)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}\|f(.)|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\|_q \\ &\leq C(t-\sigma)^{-(\frac{N\alpha}{2r}+\frac{\gamma}{2})-\delta'}\|u(\sigma)\|_r^{\alpha+1}.\end{aligned}$$

Then, by (2.4.21) and (2.7.71), we obtain

$$\begin{aligned}|a|t^{\beta+\delta'}\int_0^t \|e^{(t-\sigma)\Delta} f(.)|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\|_r d\sigma &\leq Ct^{\beta+\delta'}\int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}-\delta'}\|u(\sigma)\|_r^{\alpha+1} d\sigma \\ \leq CM^{\alpha+1}t^{\beta+\delta'-\frac{N\alpha}{2r}-\frac{\gamma}{2}-\delta'-\beta(\alpha+1)+1}&\int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}-\delta'}\sigma^{-\beta(\alpha+1)} d\sigma \\ \leq CM^{\alpha+1}\int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}-\delta'}&\sigma^{-\beta(\alpha+1)} d\sigma \leq C.\end{aligned}$$

Since the constants in the estimate of the three terms do not depend on $T$, we obtain the asymptotic behavior (2.7.72) for $p = r$.

We now turn to the asymptotic result in the $L^\infty$-norm. Write

$$\begin{aligned}u(t) - u_{\mathcal{S}}(t) &= e^{\frac{t}{2}\Delta}\big(u(t/2) - u_{\mathcal{S}}(t/2)\big) + \int_{\frac{t}{2}}^t e^{(t-\sigma)\Delta}\big[V(.)|u(\sigma)|^\alpha u(\sigma) - a|.|^{-\gamma}(|u_{\mathcal{S}}(\sigma)|^\alpha u_{\mathcal{S}}(\sigma))\big] d\sigma \\ &= e^{\frac{t}{2}\Delta}\big(u(t/2) - u_{\mathcal{S}}(t/2)\big) + a\int_{\frac{t}{2}}^t e^{(t-\sigma)\Delta}[|.|^{-\gamma}(|u(\sigma)|^\alpha u(\sigma) - |u_{\mathcal{S}}(\sigma)|^\alpha u_{\mathcal{S}}(\sigma))] d\sigma \\ &\qquad -a\int_{\frac{t}{2}}^t e^{(t-\sigma)\Delta} f(.)|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma) d\sigma.\end{aligned}$$

By using the smoothing properties of the heat semi-group, we have that

$$\begin{aligned}t^{\frac{2-\gamma}{2\alpha}+\delta'}\|u(t) - u_{\mathcal{S}}(t)\|_\infty \leq\ & Ct^{\beta+\delta'}\|u(t/2) - u_{\mathcal{S}}(t/2)\|_r + \\ & |a|t^{\frac{2-\gamma}{2\alpha}+\delta'}\int_{\frac{t}{2}}^t \|e^{(t-\sigma)\Delta}\big(|.|^{-\gamma}(|u(\sigma)|^\alpha u(\sigma) - |u_{\mathcal{S}}(\sigma)|^\alpha u_{\mathcal{S}}(\sigma))\big)\|_\infty d\sigma + \\ & |a|t^{\frac{2-\gamma}{2\alpha}+\delta'}\int_{\frac{t}{2}}^t \|e^{(t-\sigma)\Delta}\big(|.|^{-\gamma}(f(.)|u(\sigma)|^\alpha u(\sigma))\big)\|_\infty d\sigma.\end{aligned}$$

Using the fact that $\|u_{\mathcal{S}}(t)\|_\infty \leq Ct^{-\frac{2-\gamma}{2\alpha}}, \quad \|u(t)\|_\infty \leq Ct^{-\frac{2-\gamma}{2\alpha}}$ we get

$$\begin{aligned}t^{\frac{2-\gamma}{2\alpha}+\delta'}\int_{\frac{t}{2}}^t \|e^{(t-\sigma)\Delta}\big(|.|^{-\gamma}(|u(\sigma)|^\alpha u(\sigma) - |u_{\mathcal{S}}(\sigma)|^\alpha u_{\mathcal{S}}(\sigma))\big)\|_\infty d\sigma \leq\ & C\ \times \\ \left[\int_{\frac{1}{2}}^1 (1-\sigma)^{-\frac{\gamma}{2}}\sigma^{-(\alpha+1)\frac{2-\gamma}{2\alpha}-\delta'} d\sigma\right] & \sup_{t\in(0,T]}\left(t^{\frac{2-\gamma}{2\alpha}+\delta'}\|u(t) - u_{\mathcal{S}}(t)\|_\infty\right).\end{aligned}$$

We turn now to estimate the last term. Let $r'$ be given by (2.7.73). By the smoothing effect

$$\begin{aligned}\|e^{(t-\sigma)\Delta}\big(f(.)|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\big)\|_\infty &\leq C(t-\sigma)^{-\frac{N}{2r'}}\|f(.)|.|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\|_{r'} \\ &\leq C(t-\sigma)^{-\frac{N}{2r'}}\|f(.)|.|^{-\gamma}\|_{r'}\|u(\sigma)\|_\infty^{\alpha+1} \\ &\leq C'M^{\alpha+1}(t-\sigma)^{-\frac{N}{2r'}}\sigma^{-\frac{2-\gamma}{2\alpha}(\alpha+1)}.\end{aligned}$$

Since $r'$ satisfies (2.7.73) and $\delta'$ satisfies (2.7.71) we have,

$$t^{\frac{2-\gamma}{2\alpha}+\delta'}\int_{\frac{t}{2}}^{t}\|e^{(t-\sigma)\Delta}\big(|.|^{-\gamma}(f(.)|u(\sigma)|^{\alpha}u(\sigma))\big)\|_{\infty}d\sigma \leq C.$$

Finally we can conclude that, $\sup_{t\in(0,T]} t^{\frac{2-\gamma}{2\alpha}+\delta'}\|u(t)-u_{\mathcal{S}}(t)\|_{\infty} \leq C'(\delta')$. Since the constant $C'(\delta')$ does not depend on $T>0$, one can take the supremum over $(0,\infty)$. It follows that there exists a constant $C'(\delta')$ such that

$$\|u(t)-u_{\mathcal{S}}(t)\|_{\infty} \leq C'(\delta')t^{-\frac{2-\gamma}{2\alpha}-\delta'},$$

for all $t>0$. This proves (2.7.72) for $r=\infty$. The general result (2.7.72) follows now by the Hölder inequality.

Proof of Part (ii). Let $v(t)=e^{t\Delta}\left(\omega(\cdot)|\cdot|^{-\frac{2-\gamma}{\alpha}}\right)$. We will show the following:

$$\|u(t)-v(t)\|_{p} \leq Ct^{-\beta(p)-\delta'},\ \forall\, p\in[r,\infty].$$

We have

$$\begin{aligned} u(t)-v(t) &= e^{t\Delta}(\varphi-\Phi)+\int_{0}^{t}e^{(t-\sigma)\Delta}\big[V(.)|u(\sigma)|^{\alpha}u(\sigma)\big]d\sigma \\ &= e^{t\Delta}(\varphi-\Phi)+\int_{0}^{t}e^{(t-\sigma)\Delta}\big(af|.|^{-\gamma}|u(\sigma)|^{\alpha}u(\sigma)\big)d\sigma. \end{aligned}$$

Then we obtain

$$\|u(t)-v(t)\|_{r} \;\leq\; \|e^{t\Delta}(\varphi-\Phi)\|_{r}+|a|\int_{0}^{t}\|e^{(t-\sigma)\Delta}\left(f(.)|.|^{-\gamma}\right)|u(\sigma)|^{\alpha}u(\sigma)\|_{r}d\sigma.$$

The rest of the proof is similar to that of Part (i), so we omit the details. □

## 2.8 Auxiliary lemmas

The proof of Lemma 2.6.1 follows by the following two lemmas.

**Lemma 2.8.1.** *Let $N\geq 1$ be an integer. Let $\gamma$ be a real number such that $0<\gamma<\min(2,N)$. Let the real number $\alpha_1$ be such that*

$$\alpha_1 > \frac{2-\gamma}{N}.$$

*Given such $\gamma$ and $\alpha_1$ one can always choose $r_1$ such that*

*(i)* $\frac{\gamma}{N}+\frac{\alpha_1+1}{r_1}<1$,

*(ii)* $\frac{N\alpha_1}{2r_1}+\frac{\gamma}{2}<1$,

*(iii)* $(\frac{2-\gamma}{2\alpha_1}-\frac{N}{2r_1})(\alpha_1+1)<1$.

*Proof of Lemma 2.8.1.* Note that (i)-(iii) are equivalent to

(i) $\frac{N(\alpha_1+1)}{N-\gamma} < r_1$,
(ii) $\frac{N\alpha_1}{2-\gamma} < r_1$,
(iii) $r_1 < \frac{N\alpha_1(\alpha_1+1)}{2-\gamma(\alpha_1+1)}$ if $2-\gamma(\alpha_1+1) > 0$.

Since $\alpha_1$ and $\gamma$ satisfy the conditions of Lemma 2.8.1 we can easily show that if $2-\gamma(\alpha_1+1) > 0$, that is $\alpha_1 < \frac{2-\gamma}{\gamma}$, then

$$\frac{N(\alpha_1+1)}{N-\gamma} < \frac{N\alpha_1(\alpha_1+1)}{2-\gamma(\alpha_1+1)},$$

$$\frac{N\alpha_1}{2-\gamma} < \frac{N\alpha_1(\alpha_1+1)}{2-\gamma(\alpha_1+1)}.$$

And so we can choose $r_1$ satisfying Lemma 2.8.1. □

Next, we set

**Lemma 2.8.2.** *Let $N \geq 1$ be an integer. Let $\gamma$ be a real number such that $0 < \gamma < \min(2, N)$. Let the real number $\alpha$ be such that $\alpha > \dfrac{2-\gamma}{N}$. Let the real number $\alpha_1$ be such that*

$$\alpha > \alpha_1 > \frac{2-\gamma}{N}.$$

*Choose $r_1$ satisfying (2.6.38) and $r_2$ satisfying (2.6.39). It follows that*

*(i)* $r_1 < r_2$,
*(ii)* $\frac{\gamma}{N} + \frac{\alpha_1+1}{r_1} < 1$, $\frac{\gamma}{N} + \frac{\alpha+1}{r_2} < 1$,
*(iii)* $\frac{N\alpha_1}{2r_1} + \frac{\gamma}{2} < 1$, $\frac{N\alpha}{2r_2} + \frac{\gamma}{2} < 1$,
*(iv)* $(\frac{2-\gamma}{2\alpha_1} - \frac{N}{2r_1})(\alpha_1+1) < 1$, $(\frac{2-\gamma}{2\alpha} - \frac{N}{2r_2})(\alpha+1) < 1$,
*(v)* $\frac{2-\gamma}{N\alpha_1} < \frac{\gamma}{N} + \frac{1}{r_1} + \frac{\alpha}{r_2} < 1$,
*(vi)* $\frac{2-\gamma}{N\alpha} < \frac{\gamma}{N} + \frac{\alpha_1}{r_1} + \frac{1}{r_2} < 1$.

*Proof.* The proof of Lemma 2.8.2 is obvious and can be omitted. □

We now give the proof of Lemma 2.6.2.

*Proof of Lemma 2.6.2.* One verifies, using the expression for $r'$, $\beta'$ and $\beta_1$, that condition (b) is equivalent to (2.6.58). Since $\delta > 0$, one must have $\theta_\delta > \frac{1}{\alpha+1}$. Write now

$$\theta_\delta = \frac{1}{\alpha+1} + \varepsilon.$$

In the limiting case $\varepsilon = 0$, we see that $\theta_\delta = \frac{1}{\alpha+1}$, $r' = r_{12}$, $\beta' = \beta_{12}$ and $\delta = 0$; and so conditions (a), (b) and (c) are consequences of Lemma 2.6.1. Since conditions (a) and (c) are open, it is clear that they still hold for small $\varepsilon > 0$, assuming (2.6.58).

In order to get a specific bound on allowable $\varepsilon > 0$, note that (a)-(c) are equivalent to

(a') $\frac{1}{r_1} < \frac{\gamma}{N} + \frac{\alpha+1}{r_{12}} + \varepsilon(\alpha+1)(\frac{1}{r_1} - \frac{1}{r_2}) < 1$,

(b') $\delta = \varepsilon\frac{(2-\gamma)(\alpha+1)(\alpha-\alpha_1)}{2\alpha_1\alpha}$,
(c') $\frac{N\alpha}{2r_2} + \varepsilon\frac{N(\alpha+1)}{2}(\frac{1}{r_1} - \frac{1}{r_2}) + \frac{\gamma}{2} < 1, \quad \beta_{12}(\alpha+1) + \varepsilon(\alpha+1)(\beta_1 - \beta_2) < 1,$
where $r_{12}$ and $\beta_{12}$ are as in Lemma 2.6.1. We clearly have $\theta_\delta < 1$, for small $\varepsilon$, and thanks to (ii)-(iv) of Lemma 2.6.1, (a')-(c') are satisfied for

$$0 < \varepsilon < \varepsilon_{\min} = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}, \tag{2.8.1}$$

where

$$\begin{aligned}
\varepsilon_1 &= \frac{\alpha}{\alpha+1}; \\
\varepsilon_2 &= \frac{1}{\alpha+1}(1 - \frac{\gamma}{N} - \frac{\alpha+1}{r_{12}})(\frac{1}{r_1} - \frac{1}{r_2})^{-1}; \\
\varepsilon_3 &= \frac{2}{N(\alpha+1)}(\frac{2-\gamma}{2} - \frac{N\alpha}{2r_2})(\frac{1}{r_1} - \frac{1}{r_2})^{-1}; \\
\varepsilon_4 &= \frac{1}{\alpha+1}\big(1 - \beta_{12}(\alpha+1)\big)|\beta_1 - \beta_2|^{-1}.
\end{aligned}$$

That is (2.6.56), (a)-(c) are satisfied for $0 < \delta < \delta_0$, where

$$\delta_0 = \varepsilon_{\min}\frac{(2-\gamma)(\alpha+1)(\alpha-\alpha_1)}{2\alpha_1\alpha}, \tag{2.8.2}$$

and $\varepsilon_{\min}$ is given by (2.8.1). □

**Remark 2.8.1.** One can verify easily that $\frac{1}{r_{12}} = \frac{1}{\alpha+1}\frac{1}{r_1} + \frac{\alpha}{\alpha+1}\frac{1}{r_2}$, $\beta_{12} = \dfrac{1}{\alpha+1}\beta_1 + \dfrac{\alpha}{\alpha+1}\beta_2 = \dfrac{2-\gamma}{2\alpha_1}\dfrac{\alpha_1+1}{\alpha+1} - \dfrac{N}{2r_{12}}$, and $\beta_2 = \frac{\alpha_1}{\alpha}\beta_1$.

# Chapitre 3

# Study of the Hardy-Hénon Systems

## 3.1 Introduction

In this chapter we consider global in time solutions of the following nonlinear parabolic system

$$(S)\begin{cases} \partial_t u = \Delta u + a|.|^{-\gamma}|v|^{p-1}v, \\ \partial_t v = \Delta v + b|.|^{-\rho}|u|^{q-1}u, \end{cases}$$

with initial data

$$u(0,x) = \varphi_1(x), \quad v(0,x) = \varphi_2(x), \tag{3.1.1}$$

where $u = u(t,x) \in \mathbb{R}$, $v = v(t,x) \in \mathbb{R}$, $t > 0$, $x \in \mathbb{R}^N$, $a$, $b \in \mathbb{R}$, $0 \leq \gamma < \min(N,2)$, $0 < \rho < \min(N,2)$, $p$, $q > 1$.

In what follows, we denote $\|.\|_{L^r(\mathbb{R}^N)}$ by $\|.\|_r$. For $f, g : I \to \mathbb{R}$, we denote when there exists $sup_{t\in I}\,[f(t), g(t)] = \max\,[sup_{t\in I} f(t), sup_{t\in I} g(t)]$. For all $t > 0$, $e^{t\Delta}$ denotes the heat semi-group, that is

$$\left(e^{t\Delta} f\right)(x) = \int_{\mathbb{R}^N} G(t, x-y) f(y) dy,$$

where

$$G(t,x) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}},\ t > 0,\ x \in \mathbb{R}^N,$$

and $f \in L^r(\mathbb{R}^N),\ r \in [1,\infty)$ or $f \in C_0(\mathbb{R}^N)$. For $f \in \mathcal{S}'(\mathbb{R}^N),\ e^{t\Delta} f$ is defined by duality.

A mild solution of the system $(S)$-(3.1.1) is a solution of the integral system

$$\begin{cases} u(t) = e^{t\Delta}\varphi_1 + a\displaystyle\int_0^t e^{(t-\sigma)\Delta}\left(|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma)\right) d\sigma, \\ v(t) = e^{t\Delta}\varphi_2 + b\displaystyle\int_0^t e^{(t-\sigma)\Delta}\left(|.|^{-\rho}|u(\sigma)|^{q-1}u(\sigma)\right) d\sigma. \end{cases} \tag{3.1.2}$$

We investigate the existence of global solutions, including self-similar solutions for the semilinear system (3.1.2). Moreover, we are concerned with estimating the decaying rate in time of some global solutions and their asymptotic behavior.

Using the key estimate established by Proposition 2.1 in [2] we can adapt the method in Fujita and Kato [18, 19] and recently used in [2, 5, 6, 14, 7, 23, 30, 42, 43, 44].

This method is based on a contraction mapping argument on the associated integral system (3.1.2). Precisely we transform the problem of existence and uniqueness of global solutions into a problem of a fixed point for a function defined in a suitable Banach space equipped with a norm chosen so that we obtain directly the global character of the solution.

In this chapter we seek conditions for the following parameters $p$, $q$, $\gamma$ and $\rho$ such that we have the global existence of some class of solutions, including self-similar solutions and the nonlinear asymptotic self-similar behavior of these solutions. For this we define $k$, $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ by

$$k = \frac{(2-\gamma)q + (2-\rho)}{(2-\rho)p + (2-\gamma)}, \tag{3.1.3}$$

$$\alpha_1 = \frac{1}{2(pq-1)}[(2-\rho)p + (2-\gamma)], \tag{3.1.4}$$

$$\alpha_2 = \frac{1}{2(pq-1)}\left[(2-\gamma)q + (2-\rho)\right], \tag{3.1.5}$$

$$\beta_1 = \alpha_1 - \frac{N}{2r_1} = \frac{1}{2(pq-1)}[(2-\rho)p + (2-\gamma)] - \frac{N}{2r_1}, \quad r_1 > 1, \tag{3.1.6}$$

$$\beta_2 = \alpha_2 - \frac{N}{2r_2} = \frac{1}{2(pq-1)}[(2-\gamma)q + (2-\rho)] - \frac{N}{2r_2}, \quad r_2 > 1. \tag{3.1.7}$$

Note that $\alpha_1$ and $\alpha_2$ verify the following system

$$\begin{cases} 2-\gamma+2\alpha_1 = 2\alpha_2 p, \\ 2-\rho+2\alpha_2 = 2\alpha_1 q, \end{cases} \tag{3.1.8}$$

and that

$$kp > 1, \quad q > k \quad \text{and} \quad \frac{\alpha_2}{\alpha_1} = k.$$

Let us summarize the results of this chapter. First of all if we suppose that the following conditions

$$2\alpha_1 < \min\left(N, \frac{p}{q}(N-\rho)\frac{(2-\gamma)q + (2-\rho)}{[2+(2-\rho)p - \gamma pq]_+}\right), \tag{3.1.9}$$

and

$$2\alpha_2 < \min\left(N, \frac{q}{p}(N-\gamma)\frac{(2-\rho)p + (2-\gamma)}{[2+(2-\gamma)q - \rho pq]_+}\right), \tag{3.1.10}$$

are satisfied, then we prove the global existence of solutions for some initial data $\Phi = (\varphi_1, \varphi_2)$ small with respect to the norm $\mathcal{N}$ defined by

$$\mathcal{N}(\Phi) := \sup_{t>0}\left[t^{\beta_1}\|e^{t\Delta}\varphi_1\|_{r_1}, t^{\beta_2}\|e^{t\Delta}\varphi_2\|_{r_2}\right], \tag{3.1.11}$$

where $\beta_1$ and $\beta_2$ are given by (3.1.6) and (3.1.7), $r_1$ and $r_2$ are defined in Lemma 3.2.1 below. See Theorem 3.2.1 below. We also prove, for $\varphi_1$ homogeneous of degree $-2\alpha_1$ and $\varphi_2$ homogeneous of degree $-2\alpha_2$, where $\alpha_1$ and $\alpha_2$ are given by (3.1.4) and (3.1.5), that the initial data $\Phi = (\varphi_1, \varphi_2)$ gives rise to a global self-similar solution. See Theorem 3.2.2 below. Next we show as in [2], that solutions with initial data $\Psi$ which behaves asymptotically like $\Phi$ in some appropriate sense as $|x| \to \infty$, are asymptotically self-similar in the $L^\infty$-norm. See Theorem 3.2.3 below. The norm $\mathcal{N}$ given in (3.1.11) is weak enough so that initial data $\Phi = (\varphi_1, \varphi_2)$ with homogeneous components have finite norm. We prove finally stronger uniqueness results in Lebesgue spaces for initial values small with respect of some norm. See Theorem 3.4.1 below.

Yamauchi in [53] studied the parabolic system (S). In [53, Theorem 2.1, p. 339] it is shown that for some nonnegative initial values under the conditions $\gamma < \min(N, 2)$, $\rho < \min(N, 2)$, $pq - 1 > 0$ and $\max(\alpha_1, \alpha_2) \geq \frac{N}{2}$, that no nonnegative nontrivial solutions exist.

The case $\gamma = \rho = 0$ has been already covered in [44]. In the case where $p = q$ and $\gamma = \rho > 0$, the parabolic system (S) behaves like a parabolic equation with singularity in the nonlinearity. For more reading about Hardy-Hénon equations see [2, 33, 35, 48].

The rest of the chapter is organized as follows. In Section 2, we state the main results. In Section 3, we give the proofs of the main theorems. Finally, in Section 4, we give stronger uniqueness results. Throughout this chapter $C$ will be a positive constant which may have different values at different places. We denote sometimes $u(t)$ by $u(t, .)$.

## 3.2 Main results

We now state the main results of the chapter. Let $e^{t\Delta}$ be the linear heat semi-group defined by

$$(e^{t\Delta}\varphi)(x) = (G(t, .) * \varphi)(x),$$

where G is the heat kernel

$$G(t, x) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}, \quad t > 0, \quad x \in \mathbb{R}^N.$$

We recall the smoothing effect of the heat semi-group

$$\|e^{t\Delta} f\|_{s_2} \leq (4\pi t)^{-\frac{N}{2}(\frac{1}{s_1} - \frac{1}{s_2})} \|f\|_{s_1}, \tag{3.2.12}$$

for $1 \leq s_1 \leq s_2 \leq \infty$, $t > 0$ and $f \in L^{s_1}(\mathbb{R}^N)$. We recall also the following key estimate from [2]

$$\|e^{t\Delta}(|.|^{-\gamma} f)\|_{q_2} \leq C(N, \gamma, q_1, q_2) t^{-\frac{N}{2}(\frac{1}{q_1} - \frac{1}{q_2}) - \frac{\gamma}{2}} \|f\|_{q_1}, \tag{3.2.13}$$

for $0 \leq \gamma < N$, $q_1$ and $q_2$ such that $0 \leq \frac{1}{q_2} < \frac{\gamma}{N} + \frac{1}{q_1} < 1$, $t > 0$ and $f \in L^{q_1}(\mathbb{R}^N)$. We note that if $q_2 = \infty$, then $e^{t\Delta}(|.|^{-\gamma} f) \in C_0(\mathbb{R}^N)$.

We begin with the following technical lemma.

**Lemma 3.2.1** (Technical lemma)**.** *Let $N$ be a positive integer. Let $p$, $q > 1$. Let $0 \leq \gamma < \min(N, 2)$ and $0 < \rho < \min(N, 2)$. Let $k$ be given by (3.1.3). Let $\alpha_1$, $\alpha_2$ defined by (3.1.4) and (3.1.5). Suppose* *that (3.1.9) and (3.1.10) are satisfied. Let $\beta_1$, $\beta_2$ be given by (3.1.6) and (3.1.7). Then there exist $r_1 > 1$ and $r_2 > 1$ satisfying*

$$r_1 = kr_2, \tag{3.2.14}$$

*such that*

(i) $\beta_1 > 0$, $\beta_2 > 0$ *and* $\beta_2 = k\beta_1$,
(ii) $\frac{1}{r_1} < \frac{\gamma}{N} + \frac{p}{r_2} < 1$ *and* $\frac{1}{r_2} < \frac{\rho}{N} + \frac{q}{r_1} < 1$,
(iii) $\beta_2 p < 1$ *and* $\beta_1 q < 1$,
(iv) $\frac{N}{2r_1}\left(-1 + \frac{r_1}{r_2}p\right) < \frac{2-\gamma}{2}$ *and* $\frac{N}{2r_2}\left(-1 + \frac{r_2}{r_1}q\right) < \frac{2-\rho}{2}$,
(v) $\frac{1}{r_1} < \frac{2\alpha_1}{N} < \frac{\gamma}{N} + \frac{p}{r_2}$ *and* $\frac{1}{r_2} < \frac{2\alpha_2}{N} < \frac{\rho}{N} + \frac{q}{r_1}$,
(vi) $-\frac{N}{2}(\frac{p}{r_2} - \frac{1}{r_1}) - \frac{\gamma}{2} - \beta_2 p + 1 + \beta_1 = 0$ *and* $-\frac{N}{2}(\frac{q}{r_1} - \frac{1}{r_2}) - \frac{\rho}{2} - \beta_1 q + 1 + \beta_2 = 0$.

We prove this lemma in the Section 3.5.

**Theorem 3.2.1** (Global existence and continuous dependence)**.** *Let $N$ be a positive integer. Let $p$, $q > 1$. Let $0 \leq \gamma < \min(N, 2)$ and $0 < \rho < \min(N, 2)$. Let $\alpha_1$, $\alpha_2$ defined by (3.1.4) and (3.1.5). Suppose that (3.1.9) and (3.1.10) are satisfied. Let $\beta_1$, $\beta_2$ be given by (3.1.6) and (3.1.7). Let $r_1$ and $r_2$ be as in Lemma 3.2.1. Let $M > 0$ be such that*

$$\nu = \max(M^{p-1}\nu_1, M^{q-1}\nu_2) < 1, \tag{3.2.15}$$

*where $\nu_1$ and $\nu_2$ are two positive constants given by (3.3.37) and (3.3.38) below. Choose $R > 0$ such that*

$$R + M\nu \leq M. \tag{3.2.16}$$

*Let $\Phi = (\varphi_1, \varphi_2)$ be an element of $\mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$ such that*

$$\mathcal{N}(\Phi) := \sup_{t>0} \left[t^{\beta_1}\|e^{t\Delta}\varphi_1\|_{r_1}, t^{\beta_2}\|e^{t\Delta}\varphi_2\|_{r_2}\right] \leq R. \tag{3.2.17}$$

*Then there exists a unique global solution $U = (u, v) \in C((0, \infty); L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N))$ of the integral system (3.1.2) such that*

$$\sup_{t>0} \left[t^{\beta_1}\|u(t)\|_{r_1}, t^{\beta_2}\|v(t)\|_{r_2}\right] \leq M. \tag{3.2.18}$$

*Furthermore,*

(a) $\lim_{t\searrow 0} u(t) = \varphi_1$ *and* $\lim_{t\searrow 0} v(t) = \varphi_2$ *in the sense of tempered distributions,*
(b) $u(t) - e^{t\Delta}\varphi_1 \in C\left([0, \infty), L^{\tau_1}(\mathbb{R}^N)\right)$ *for $\tau_1$ satisfying* $\frac{2\alpha_1}{N} < \frac{1}{\tau_1} < \frac{\gamma}{N} + \frac{p}{r_2}$,
(c) $v(t) - e^{t\Delta}\varphi_2 \in C\left([0, \infty), L^{\tau_2}(\mathbb{R}^N)\right)$ *for $\tau_2$ satisfying* $\frac{2\alpha_2}{N} < \frac{1}{\tau_2} < \frac{\rho}{N} + \frac{q}{r_1}$,
(d) $\sup_{t>0} t^{\alpha_1 - \frac{N}{2r}}\|u(t)\|_r < \infty, \quad \forall r \in [r_1, \infty]$, *and* $u \in C((0, \infty), L^r(\mathbb{R}^N) \cap C_0(\mathbb{R}^N))$,
(e) $\sup_{t>0} t^{\alpha_2 - \frac{N}{2r}}\|v(t)\|_r < \infty, \quad \forall r \in [r_2, \infty]$, *and* $v \in C((0, \infty), L^r(\mathbb{R}^N) \cap C_0(\mathbb{R}^N))$.

*In addition, if* $\Phi = (\varphi_1, \varphi_2)$ *and* $\Psi = (\psi_1, \psi_2)$ *satisfy (3.2.17), and if* $U_1 = (u_1, v_1)$ *and* $U_2 = (u_2, v_2)$ *respectively are the solutions of the system (3.1.2) with initial values* $\Phi$ *and* $\Psi$*, then*

$$\sup_{t>0} \left[ t^{\beta_1} \|u_1(t) - u_2(t)\|_{r_1}, t^{\beta_2} \|v_1(t) - v_2(t)\|_{r_2} \right] \leq (1-\nu)^{-1} \mathcal{N}(\Phi - \Psi). \tag{3.2.19}$$

*Furthermore, if the initial data* $\Phi$ *and* $\Psi$ *are such that*

$$\mathcal{N}_\delta(\Phi - \Psi) = \sup_{t>0} \left[ t^{\beta_1+\delta} \|e^{t\Delta}(\varphi_1 - \psi_1)\|_{r_1}, t^{\beta_2+\delta} \|e^{t\Delta}(\varphi_2 - \psi_2)\|_{r_2} \right] < \infty, \tag{3.2.20}$$

*for some* $0 < \delta < \delta_0$*, where*

$$\delta_0 = \min\{1 - \beta_1 q, 1 - \beta_2 p\}. \tag{3.2.21}$$

*Then*

$$\sup_{t>0} \left[ t^{\beta_1+\delta} \|u_1(t) - u_2(t)\|_{r_1}, t^{\beta_2+\delta} \|v_1(t) - v_2(t)\|_{r_2} \right] \leq (1-\nu')^{-1} \mathcal{N}_\delta(\Phi - \Psi), \tag{3.2.22}$$

*where the positive constant* $M$ *is chosen small enough so that* $0 < \nu' < 1$*, where* $\nu'$ *is given by the relations (3.3.45)-(3.3.47) below.*

*Finally, if we suppose also that* $\Phi = (\varphi_1, \varphi_2) \in L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)$ *such that*

$$\mathcal{N}'(\Phi) := \max \left[ \|\varphi_1\|_{\frac{N}{2\alpha_1}}, \|\varphi_2\|_{\frac{N}{2\alpha_2}} \right] < R, \tag{3.2.23}$$

*then the solution* $U = (u, v)$ *of the integral system (3.1.2) satisfies also* $U \in C\left([0,\infty), L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N)\right) \times C\left([0,\infty), L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)\right)$ *and*

$$\sup_{t\geq 0} \left[ \|u(t)\|_{\frac{N}{2\alpha_1}}, \|v(t)\|_{\frac{N}{2\alpha_2}} \right] \leq M. \tag{3.2.24}$$

*Where* $M$ *and* $R$ *are sufficiently small.*

Now we give the following result which proves the existence of self-similar solutions.

**Theorem 3.2.2** (Self-similar solutions)**.** *Let* $N$ *be a positive integer. Let* $p$*,* $q > 1$*. Let* $0 \leq \gamma < \min(N, 2)$ *and* $0 < \rho < \min(N, 2)$*. Let* $\alpha_1$*,* $\alpha_2$ *defined by (3.1.4) and (3.1.5). Suppose that (3.1.9) and (3.1.10) are satisfied. Let* $\varphi_1(x) = \omega_1(x)|x|^{-2\alpha_1}$*,* $\varphi_2(x) = \omega_2(x)|x|^{-2\alpha_2}$*, where* $\omega_1, \omega_2 \in L^\infty(\mathbb{R}^N)$ *are homogeneous of degree 0 and* $\|\omega_1\|_\infty, \|\omega_2\|_\infty$ *are sufficiently small. Denote* $\Phi = (\varphi_1, \varphi_2)$*, then there exists a global self-similar solution* $U_S = (u_S, v_S)$ *of (3.1.2) with initial data* $\Phi$*. Moreover* $U_S(t) \to \Phi$ *in* $\mathcal{S}'(\mathbb{R}^N)$ *as* $t \to 0$*.*

We turn now to the asymptotic behavior.

**Theorem 3.2.3** (Asymptotic behavior)**.** *Let* $N$ *be a positive integer. Let* $p$*,* $q > 1$*. Let* $0 \leq \gamma < \min(N, 2)$ *and* $0 < \rho < \min(N, 2)$*. Let* $\alpha_1$*,* $\alpha_2$ *defined by (3.1.4) and (3.1.5). Suppose that (3.1.9) and*

*(3.1.10) are satisfied. Let $\beta_1$, $\beta_2$ be given by (3.1.6) and (3.1.7). Let $r_1$ and $r_2$ be as in Lemma 3.2.1. Define $\beta_1(q)$ and $\beta_2(q)$ by*

$$\beta_1(q) = \alpha_1 - \frac{N}{2q}, \quad \beta_2(q) = \alpha_2 - \frac{N}{2q}, \quad q > 1. \tag{3.2.25}$$

*Let $\Phi$ be given by*

$$\Phi(x) = (\varphi_1(x), \varphi_2(x)) := \big(\omega_1(x)|x|^{-2\alpha_1}, \omega_2(x)|x|^{-2\alpha_2}\big)$$

*with $\omega_1$, $\omega_2$ homogeneous of degree 0, $\omega_1, \omega_2 \in L^\infty(\mathbb{R}^N)$ and $\|\omega_1\|_\infty$, $\|\omega_1\|_\infty$ are sufficiently small. Let*

$$U_{\mathcal{S}}(t,x) = \left(t^{-\alpha_1} u_{\mathcal{S}}(1, \frac{x}{\sqrt{t}}), t^{-\alpha_2} v_{\mathcal{S}}(1, \frac{x}{\sqrt{t}})\right)$$

*be the self-similar solution of (3.1.2) given by Theorem 3.2.2.*

*Let $\Psi = (\psi_1, \psi_2) \in C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N)$ be such that*

$$|\psi_1(x)| \le \frac{c}{(1+|x|^2)^{\alpha_1}},\ \forall\, x \in \mathbb{R}^N,\ \psi_1(x) = \omega_1(x)|x|^{-2\alpha_1},\ |x| \ge A,$$

$$|\psi_2(x)| \le \frac{c}{(1+|x|^2)^{\alpha_2}},\ \forall\, x \in \mathbb{R}^N,\ \psi_2(x) = \omega_2(x)|x|^{-2\alpha_2},\ |x| \ge A,$$

*for some constant $A > 0$, where $c$ is a small positive constant. (We take $\|\omega_1\|_\infty$, $\|\omega_2\|_\infty$ and $c$ sufficiently small so that (3.2.17) is satisfied by $\Phi$ and $\Psi$).*

*Let $U = (u, v)$ be the global solution of (3.1.2) with initial data $\Psi$ constructed by Theorem 3.2.1. Then there exists $\delta > 0$ sufficiently small such that*

$$\|u(t) - u_{\mathcal{S}}(t)\|_{q_1} \le C_\delta t^{-\beta_1(q_1)-\delta},\ \forall\, t > 0, \tag{3.2.26}$$

$$\|v(t) - v_{\mathcal{S}}(t)\|_{q_2} \le C_\delta t^{-\beta_2(q_2)-\delta},\ \forall\, t > 0, \tag{3.2.27}$$

*for all $q_1 \in [r_1, \infty]$, $q_2 \in [r_2, \infty]$. Also, we have*

$$\|t^{\alpha_1} u(t, .\sqrt{t}) - u_{\mathcal{S}}(1, .)\|_{q_1} \le C_\delta t^{-\delta},\ \forall\, t > 0, \tag{3.2.28}$$

$$\|t^{\alpha_2} v(t, .\sqrt{t}) - v_{\mathcal{S}}(1, .)\|_{q_2} \le C_\delta t^{-\delta},\ \forall\, t > 0, \tag{3.2.29}$$

*for all $q_1 \in [r_1, \infty]$, $q_2 \in [r_2, \infty]$.*

To close this section we give the conditions on $p, q, \gamma, \rho$ which guarantee that the relations (3.1.9) and (3.1.10) are satisfied.

**Proposition 3.2.1.** *Let $N$ be a positive integer. Let the real numbers $p$, $q > 1$. Suppose that*

$$\max[p, q] + 1 < \frac{N}{2}(pq - 1).$$

*Then there exist $\gamma_0, \rho_0 > 0$ such that for all $0 \le \gamma < \gamma_0$, $0 < \rho < \rho_0$, (3.1.9) and (3.1.10) are satisfied.*

**Proposition 3.2.2.** *Let $N$ be a positive integer. Fix $0 < \gamma < \min(2, N)$ and $0 < \rho < \min(2, N)$. Let $p, q > 1$ such that*

$$p \geq \max\left(\frac{2-\gamma}{N} + \frac{2-\rho}{N} + 1, \frac{2-\gamma}{\rho} + \frac{2}{\rho}\right),$$

*and*

$$q \geq \max\left(\frac{2-\rho}{N} + \frac{2-\gamma}{N} + 1, \frac{2-\rho}{\gamma} + \frac{2}{\gamma}\right).$$

*Then (3.1.9) and (3.1.10) are satisfied.*

The proof of those two propositions is given in the next section.

## 3.3 Proof of main results

We look for global solutions of the system (3.1.2) via a fixed point argument. Let us denote $U = (u, v)$, $\Phi = (\varphi_1, \varphi_2)$ and

$$\mathcal{F}_\Phi(U) = (F_\Phi(U), G_\Phi(U)), \tag{3.3.30}$$

where

$$F_\Phi(U)(t) = e^{t\Delta}\varphi_1 + a \int_0^t e^{(t-\sigma)\Delta}\left(|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma)\right) d\sigma, \tag{3.3.31}$$

$$G_\Phi(U)(t) = e^{t\Delta}\varphi_2 + b \int_0^t e^{(t-\sigma)\Delta}\left(|.|^{-\rho}|u(\sigma)|^{q-1}u(\sigma)\right) d\sigma, \tag{3.3.32}$$

with $\varphi_1$ and $\varphi_2$ being two tempered distributions, $a, b \in \mathbb{R}$, $0 \leq \gamma < \min(N, 2)$, $0 < \rho < \min(N, 2)$, $p, q > 1$.

*Proof of Theorem 3.2.1.* Let $X$ be the set of continuous functions

$$\begin{array}{rcccc} U & : & (0, \infty) & \to & L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N), \\ & & t & \mapsto & (u(t), v(t)) \end{array}$$

such that

$$\|U\|_X := \sup_{t>0}\left[t^{\beta_1}\|u(t)\|_{r_1}, t^{\beta_2}\|v(t)\|_{r_2}\right] < \infty,$$

where $r_1, r_2$ are two positive real numbers satisfying conditions in Lemma 3.2.1 and $\beta_1$, $\beta_2$ are respectively given by (3.1.6) and (3.1.7).

Let $M > 0$ and define the closed ball in the Banach space $X$ by

$$X_M = \{U \in X, \|U\|_X \leq M\}.$$

$X_M$, endowed with the metric $d(U_1, U_2) = \|U_1 - U_2\|_X$, is a complete metric space.

Consider the mapping $\mathcal{F}_\Phi$ defined by (3.3.30), where $\Phi = (\varphi_1, \varphi_2) \in \mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ satisfies (3.2.17). We will show that $\mathcal{F}_\Phi = (F_\Phi, G_\Phi)$ is a strict contraction mapping on $X_M$.

Let $\Phi = (\varphi_1, \varphi_2)$ and $\Psi = (\psi_1, \psi_2)$ belong to $\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ satisfying (3.2.17). Let $U_1 = (u_1, v_1)$ and $U_2 = (u_2, v_2)$ be two elements of $X_M$. Then we have

$$t^{\beta_1}\|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{r_1} \le t^{\beta_1}\|e^{t\Delta}(\varphi_1 - \psi_1)\|_{r_1} + |a|t^{\beta_1}\int_0^t \|e^{(t-\sigma)\Delta}|.|^{-\gamma}[|v_1(\sigma)|^{p-1}v_1(\sigma) - |v_2(\sigma)|^{p-1}v_2(\sigma)]\|_{r_1} d\sigma.$$

It follows, by the key estimate (3.2.13) with $(q_1, q_2) = (\frac{r_2}{p}, r_1)$ that

$$\begin{aligned} t^{\beta_1}\|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{r_1} &\le t^{\beta_1}\|e^{t\Delta}(\varphi_1 - \psi_1)\|_{r_1} \\ &+|a|t^{\beta_1}\int_0^t C(t-\sigma)^{-\frac{N}{2}(\frac{p}{r_2}-\frac{1}{r_1})-\frac{\gamma}{2}} \left\||v_1(\sigma)|^{p-1}v_1(\sigma) - |v_2(\sigma)|^{p-1}v_2(\sigma)\right\|_{\frac{r_2}{p}} d\sigma. \end{aligned} \tag{3.3.33}$$

Using the fact that, for $r > p > 1$,

$$\||f|^{p-1}f - |g|^{p-1}g\|_{r/p} \le p(\|f\|_r^{p-1} + \|g\|_r^{p-1})\|f - g\|_r,$$

we obtain by (3.3.33) and the fact that $U_1$ and $U_2$ belongs to $X_M$, that

$$\begin{aligned} t^{\beta_1}\|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{r_1} &\le t^{\beta_1}\|e^{t\Delta}(\varphi_1 - \psi_1)\|_{r_1} + 2p|a|Ct^{\beta_1} \\ &\times\left[\int_0^t (t-\sigma)^{-\frac{N}{2r_1}(-1+\frac{r_1}{r_2}p)-\frac{\gamma}{2}}\sigma^{-\beta_2 p}M^{p-1}d\sigma\right]\|U_1 - U_2\|_X. \end{aligned}$$

It follows that

$$\begin{aligned} t^{\beta_1}\|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{r_1} &\le t^{\beta_1}\|e^{t\Delta}(\varphi_1 - \psi_1)\|_{r_1} + 2|a|CpM^{p-1}t^{\beta_1} \\ &\times\left[\int_0^t (t-\sigma)^{-\frac{N}{2r_1}(-1+\frac{r_1}{r_2}p)-\frac{\gamma}{2}}\sigma^{-\beta_2 p}d\sigma\right]\|U_1 - U_2\|_X \\ &\le t^{\beta_1}\|e^{t\Delta}(\varphi_1 - \psi_1)\|_{r_1} \\ &+2|a|CpM^{p-1}t^{-\frac{N}{2}(\frac{p}{r_2}-\frac{1}{r_1})-\frac{\gamma}{2}-\beta_2 p+1+\beta_1} \\ &\times\left[\int_0^1 (1-\sigma)^{-\frac{N}{2r_1}(-1+\frac{r_1}{r_2}p)-\frac{\gamma}{2}}\sigma^{-\beta_2 p}d\sigma\right]\|U_1 - U_2\|_X. \end{aligned} \tag{3.3.34}$$

Similarly using estimate (3.2.13) with $(q_1, q_2) = (\frac{r_1}{q}, r_2)$, we obtain an analogous estimate of $t^{\beta_2}\|G_\Phi(U_1)(t) - G_\Psi(U_2)(t)\|_{r_2}$. Thus

$$\begin{aligned} t^{\beta_2}\|G_\Phi(U_1)(t) - G_\Psi(U_2)(t)\|_{r_2} &\le t^{\beta_2}\|e^{t\Delta}(\varphi_2 - \psi_2)\|_{r_2} + 2|b|CqM^{q-1}t^{\beta_2} \\ &\times\left[\int_0^t (t-\sigma)^{-\frac{N}{2r_2}(-1+\frac{r_2}{r_1}q)-\frac{\varrho}{2}}\sigma^{-\beta_1 q}d\sigma\right]\|U_1 - U_2\|_X \\ &\le t^{\beta_2}\|e^{t\Delta}(\varphi_2 - \psi_2)\|_{r_2} \\ &+2|b|CqM^{q-1}t^{-\frac{N}{2}(\frac{q}{r_1}-\frac{1}{r_2})-\frac{\varrho}{2}-\beta_1 q+1+\beta_2} \\ &\times\left[\int_0^1 (1-\sigma)^{-\frac{N}{2r_2}(-1+\frac{r_2}{r_1}q)-\frac{\varrho}{2}}\sigma^{-\beta_1 q}d\sigma\right]\|U_1 - U_2\|_X. \end{aligned} \tag{3.3.35}$$

Now, due to Part (vi) of Lemma 3.2.1, inequalities (3.3.34) and (3.3.35) we obtain

$$\|\mathcal{F}_\Phi(U_1) - \mathcal{F}_\Psi(U_2)\|_X \le \mathcal{N}(\Phi - \Psi) + \nu\|U_1 - U_2\|_X, \tag{3.3.36}$$

where

$$\nu = \max(M^{p-1}\nu_1, M^{q-1}\nu_2),$$

with

$$\nu_1 = 2|a|Cp\int_0^1 (1-\sigma)^{-\frac{N}{2r_1}(-1+\frac{r_1}{r_2}p)-\frac{\gamma}{2}}\sigma^{-\beta_2 p}d\sigma, \tag{3.3.37}$$

$$\nu_2 = 2|b|Cq\int_0^1 (1-\sigma)^{-\frac{N}{2r_2}(-1+\frac{r_2}{r_1}q)-\frac{\rho}{2}}\sigma^{-\beta_1 q}. \tag{3.3.38}$$

Finally, from Parts (iii)-(iv) of Lemma 3.2.1, we see that both quantities $\nu_1$ and $\nu_2$ are finite. Setting $\Psi = 0$ and $U_2 = 0$, the inequality (3.3.36) becomes

$$\|\mathcal{F}_\Phi(U_1)\|_X \leq \mathcal{N}(\Phi) + \nu\|U_1\|_X. \tag{3.3.39}$$

If we choose $M$ and $R$ such that (3.2.16) and (3.2.17) are satisfied then by (3.3.39), $\mathcal{F}_\Phi$ maps $X_M$ into itself. Letting $\Phi = \Psi$, we observe that (3.3.36) becomes

$$\|\mathcal{F}_\Phi(U_1) - \mathcal{F}_\Phi(U_2)\|_X \leq \nu\|U_1 - U_2\|_X.$$

Hence inequality (3.2.15) gives that $\mathcal{F}_\Phi$ is a strict contraction mapping from $X_M$ into itself. So $\mathcal{F}_\Phi$ has a unique fixed point $U = (u, v)$ in $X_M$ which is solution of (3.1.2). This achieves the proof of the existence of a unique global solution of (3.1.2) in $X_M$.

We now prove the statements (a)-(c). Let $\tau_1$ be a positive real number satisfying

$$\frac{2\alpha_1}{N} < \frac{1}{\tau_1} < \frac{\gamma}{N} + \frac{p}{r_2}, \tag{3.3.40}$$

then by (3.2.13) with $(q_1, q_2) = (\frac{r_2}{p}, \tau_1)$, we have

$$\begin{aligned}\|u(t) - e^{t\Delta}\varphi_1\|_{\tau_1} &\leq |a|\int_0^t \|e^{(t-\sigma)\Delta}\left(|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma)\right)\|_{\tau_1}d\sigma \\ &\leq |a|\int_0^t C(t-\sigma)^{-\frac{N}{2}(\frac{p}{r_2}-\frac{1}{\tau_1})-\frac{\gamma}{2}}\|v(\sigma)\|_{r_2}^p d\sigma \\ &\leq |a|CM^p t^{-\frac{N}{2}(\frac{p}{r_2}-\frac{1}{\tau_1})-\beta_2 p+1-\frac{\gamma}{2}}\int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{p}{r_2}-\frac{1}{\tau_1})-\frac{\gamma}{2}}\sigma^{-\beta_2 p}d\sigma.\end{aligned}$$

Therefore

$$\|u(t) - e^{t\Delta}\varphi_1\|_{\tau_1} \leq C_1 t^{-\frac{N}{2}(\frac{p}{r_2}-\frac{1}{\tau_1})-\beta_2 p+\frac{2-\gamma}{2}}, \tag{3.3.41}$$

where

$$C_1 = |a|CM^p\int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{p}{r_2}-\frac{1}{\tau_1})-\frac{\gamma}{2}}\sigma^{-\beta_2 p}d\sigma,$$

is a positive constant. Owing to (3.3.40) and Part (iii) of Lemma 3.2.1, the constant $C_1$ is finite. Similarly using (3.2.13) with $(q_1, q_2) = (\frac{r_1}{q}, \tau_2)$, we obtain for $\tau_2$ satisfying

$$\frac{2\alpha_2}{N} < \frac{1}{\tau_2} < \frac{\rho}{N} + \frac{q}{r_1}, \tag{3.3.42}$$

the following inequality

$$\|v(t) - e^{t\Delta}\varphi_2\|_{r_2} \leq C_2 t^{-\frac{N}{2}(\frac{q}{r_1}-\frac{1}{r_2})-\beta_1 q+\frac{2-\rho}{2}}, \tag{3.3.43}$$

where $C_2$ is a positive constant given by

$$C_2 = |b|CM^q \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{q}{r_1}-\frac{1}{r_2})-\frac{\rho}{2}} \sigma^{-\beta_1 q} d\sigma,$$

which is finite by (3.3.42) and Part (iii) of Lemma 3.2.1.
Owing to the conditions (3.3.40) and (3.3.42), the right hand sides of (3.3.41) and (3.3.43) converges to zero as $t \searrow 0$. This proves statements (a)-(c) of Theorem 3.2.1.

Finally, the continuous dependence relation (3.2.19) of Theorem 3.2.1 follows by considering $\mathcal{F}_\Phi(U_1) = U_1$ and $F_\Psi(U_2) = U_2$ in the inequality (3.3.36).

Now, if in addition $\Phi$ and $\Psi$ satisfy (3.2.20), then following the same steps as above but with the norm

$$\|U = (u,v)\|_{X,\delta} = \sup_{t>0}\left[t^{\beta_1+\delta}\|u(t)\|_{r_1}, t^{\beta_2+\delta}\|v(t)\|_{r_2}\right],$$

we obtain by the key estimate (3.2.13) with $(q_1,q_2) = (\frac{r_2}{p}, r_1)$, the fact that $U_1$ and $U_2$ belongs to $X_M$ and the estimate $\|v_1(\sigma) - v_2(\sigma)\|_{r_2} \leq \sigma^{-\beta_2-\delta}\|U_1 - U_2\|_{X,\delta}$

$$\begin{aligned}
t^{\beta_1+\delta}\|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{r_1} &\leq t^{\beta_1+\delta}\|e^{t\Delta}(\varphi_1-\psi_1)\|_{r_1} + |a|t^{\beta_1+\delta} \\
&\quad\times \int_0^t C(t-\sigma)^{-\frac{N}{2}(\frac{p}{r_2}-\frac{1}{r_1})-\frac{\gamma}{2}} \left\||v_1(\sigma)|^{p-1}v_1(\sigma) - |v_2(\sigma)|^{p-1}v_2(\sigma)\right\|_{\frac{r_2}{p}} d\sigma \\
&\leq t^{\beta_1+\delta}\|e^{t\Delta}(\varphi_1-\psi_1)\|_{r_1} + 2|a|CpM^{p-1}t^{\beta_1+\delta} \\
&\quad\times \left[\int_0^t (t-\sigma)^{-\frac{N}{2r_1}(-1+\frac{r_1}{r_2}p)-\frac{\gamma}{2}}\sigma^{-\beta_2 p-\delta}d\sigma\right]\|U_1-U_2\|_{X,\delta} \\
&\leq t^{\beta_1+\delta}\|e^{t\Delta}(\varphi_1-\psi_1)\|_{r_1} \\
&\quad +2|a|CpM^{p-1}t^{-\frac{N}{2}(\frac{p}{r_2}-\frac{1}{r_1})-\frac{\gamma}{2}-\beta_2 p+1+\beta_1} \\
&\quad\times \left[\int_0^1 (1-\sigma)^{-\frac{N}{2r_1}(-1+\frac{r_1}{r_2}p)-\frac{\gamma}{2}}\sigma^{-\beta_2 p-\delta}d\sigma\right]\|U_1-U_2\|_{X,\delta}.
\end{aligned}$$

We obtain also

$$\begin{aligned}
t^{\beta_2+\delta}\|G_\Phi(U_1)(t) - G_\Psi(U_2)(t)\|_{r_2} &\leq t^{\beta_2+\delta}\|e^{t\Delta}(\varphi_2-\psi_2)\|_{r_2} \\
&\quad +2|b|CqM^{q-1}t^{-\frac{N}{2}(\frac{q}{r_1}-\frac{1}{r_2})-\frac{\rho}{2}-\beta_1 q+1+\beta_2} \\
&\quad\times\left[\int_0^1 (1-\sigma)^{-\frac{N}{2r_2}(-1+\frac{r_2}{r_1}q)-\frac{\rho}{2}}\sigma^{-\beta_1 q-\delta}d\sigma\right]\|U_1-U_2\|_{X,\delta}.
\end{aligned}$$

Then

$$\|\mathcal{F}_\Phi(U_1) - \mathcal{F}_\Psi(U_2)\|_{X,\delta} \leq \mathcal{N}_\delta(\Phi-\Psi) + \nu'\|U_1-U_2\|_{X,\delta}, \tag{3.3.44}$$

where

$$\nu' = \max(M^{p-1}\nu_1', M^{q-1}\nu_2'), \tag{3.3.45}$$

with

$$\nu_1' = 2|a|Cp\int_0^1 (1-\sigma)^{-\frac{N}{2r_1}(-1+\frac{r_1}{r_2}p)-\frac{\gamma}{2}}\sigma^{-\beta_2 p-\delta}d\sigma, \tag{3.3.46}$$

$$\nu_2' = 2|b|Cq\int_0^1 (1-\sigma)^{-\frac{N}{2r_2}(-1+\frac{r_2}{r_1}q)-\frac{\varrho}{2}}\sigma^{-\beta_1 q-\delta}. \tag{3.3.47}$$

Since $\mathcal{F}_\Phi(U_1) = U_1$ and $\mathcal{F}_\Psi(U_2) = U_2$, then (3.3.44) becomes

$$\sup_{t>0}\left[t^{\beta_1+\delta}\|u_1(t)-u_2(t)\|_{r_1}, t^{\beta_2+\delta}\|v_1(t)-v_2(t)\|_{r_2}\right] \leq (1-\nu')^{-1}\mathcal{N}_\delta(\Phi-\Psi).$$

Now, since $0<\delta<\delta_0$ with $\delta_0$ given by (3.2.21), $\nu_1'$ and $\nu_2'$ are finite. Thus, (3.2.22) holds by choosing $\nu'<1$ (this choice is possible for $M$ small enough), where $\nu'$ is given by (3.3.45)-(3.3.47).

We now prove statements (d)-(e) of Theorem 3.2.1 for $r=\infty$, we use some arguments of [43]. Let us consider two real numbers $r$ and $r'$ such that $r = kr'$ and

$$\begin{array}{ll} 1 < r_1 < r \leq \infty & 1 < r_2 < r' \leq \infty, \\ 0 < \frac{N}{2}\left(\frac{p}{r_2} - \frac{1}{r}\right) < \frac{2-\gamma}{2}, & 0 < \frac{N}{2}\left(\frac{q}{r_1} - \frac{1}{r'}\right) < \frac{2-\varrho}{2}. \end{array} \tag{3.3.48}$$

Remark that such a choice is possible owing to Lemma 3.5.1. Write now,

$$u(t) = e^{\frac{t}{2}\Delta}u(t/2) + a\int_{\frac{t}{2}}^t e^{(t-\sigma)\Delta}\left(|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma)\right)d\sigma.$$

Then by using the smoothing properties of the heat semigroup (3.2.12), the estimate (3.2.13) with $(q_1,q_2) = (\frac{r_2}{p}, r)$, (3.3.48) and the estimate (3.2.18), we obtain

$$\begin{aligned} t^{\alpha_1-\frac{N}{2r}}\|u(t)\|_r &\leq C\sup_{t>0}\left[t^{\alpha_1-\frac{N}{2r_1}}\|u(t)\|_{r_1}\right] \\ &\qquad + |a|t^{\alpha_1-\frac{N}{2r}}\int_{\frac{t}{2}}^t \|e^{(t-\sigma)\Delta}\left(|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma)\right)\|_r d\sigma \\ &\leq CM + Ct^{\alpha_1-\frac{N}{2r}}\int_{\frac{t}{2}}^t (t-\sigma)^{-\frac{N}{2}(\frac{p}{r_2}-\frac{1}{r})-\frac{\gamma}{2}}\|v(\sigma)\|_{r_2}^p d\sigma \\ &\leq CM + CM^p t^{\alpha_1-\frac{N}{2r}}\int_{\frac{t}{2}}^t (t-\sigma)^{-\frac{N}{2}(\frac{p}{r_2}-\frac{1}{r})-\frac{\gamma}{2}}\sigma^{-\beta_2 p}d\sigma \\ &\leq CM + CM^p\int_{\frac{1}{2}}^1 (1-\sigma)^{-\frac{N}{2}(\frac{p}{r_2}-\frac{1}{r})-\frac{\gamma}{2}}\sigma^{-\beta_2 p}d\sigma, \end{aligned}$$

which leads to

$$\sup_{t>0}\left[t^{\alpha_1-\frac{N}{2r}}\|u(t)\|_r\right] \leq C(M) < \infty.$$

Analogously, we obtain the following estimate on the second component $v$:

$$\sup_{t>0}\left[t^{\alpha_2-\frac{N}{2r'}}\|v(t)\|_{r'}\right] \leq C(M) < \infty.$$

We iterate this procedure, for the next step we replace in (3.3.48) $r_1$ by $r$, $r_2$ by $r'$ and we consider two real numbers $s_2$ and $s_2'$ such that $s_2 = ks_2'$ and

$$\begin{aligned} &1 < r < s_2 \leq \infty && 1 < r' < s_2' \leq \infty, \\ &0 < \frac{N}{2}\left(\frac{p}{r'} - \frac{1}{s_2}\right) < \frac{2-\gamma}{2}, && 0 < \frac{N}{2}\left(\frac{q}{r} - \frac{1}{s_2'}\right) < \frac{2-\rho}{2}. \end{aligned}$$

We obtain

$$\sup_{t>0}\left[t^{\alpha_1 - \frac{N}{2s_2}}\|u(t)\|_{s_2}, t^{\alpha_2 - \frac{N}{2s_2'}}\|v(t)\|_{s_2'}\right] \leq C(M) < \infty.$$

We therefore construct two sequences $(s_i)_i$ and $(s_i')_i$ with $s_0 = r_1$, $s_0' = r_2$, $s_1 = r$, $s_1' = r'$ and such that $s_i = ks_i'$, $\forall i = 0, 1, 2, ...$ and

$$\begin{aligned} &1 < s_i < s_{i+1} \leq \infty, && 1 < s_i' < s_{i+1}' \leq \infty, \\ &0 < \frac{N}{2}\left(\frac{p}{s_i'} - \frac{1}{s_{i+1}}\right) < \frac{2-\gamma}{2}, && 0 < \frac{N}{2}\left(\frac{q}{s_i} - \frac{1}{s_{i+1}'}\right) < \frac{2-\rho}{2}. \end{aligned}$$

We prove that

$$\sup_{t>0}\left[t^{\alpha_1 - \frac{N}{2s_i}}\|u(t)\|_{s_i}, t^{\alpha_2 - \frac{N}{2s_i'}}\|v(t)\|_{s_i'}\right] \leq C(M) < \infty, \quad \forall i = 0, 1, 2, ...$$

Now by Lemma 3.5.1, one can choose the sequences $(s_i)_i$ and $(s_i')_i$ such that they reach $\infty$ for some finite $i$. We finally obtain

$$\sup_{t>0}\left[t^{\alpha_1}\|u(t)\|_\infty, t^{\alpha_2}\|v(t)\|_\infty\right] \leq C(M) < \infty,$$

with $C(M) \searrow 0$ as $M \searrow 0$.

Finally, if in addition $\Phi$ satisfies (3.2.23), the fact that the solution $U = (u, v)$ of the integral system (3.1.2) with initial value $\Phi$ belongs to $C\left([0,\infty), L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N)\right) \times C\left([0,\infty), L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)\right)$ and the proof of the affirmation (3.2.24) are based on a contraction mapping argument in the set

$$\begin{aligned} Y_M = \Big\{ & U = (u,v) \in C\left([0,\infty), L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N)\right) \times C\left([0,\infty), L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)\right) \cap C\left((0,\infty), L^{r_1}(\mathbb{R}^N)\right) \\ & \times C\left((0,\infty), L^{r_2}(\mathbb{R}^N)\right); \max\left[\sup_{t\geq 0}[\|u(t)\|_{\frac{N}{2\alpha_1}}, \|v(t)\|_{\frac{N}{2\alpha_2}}], \sup_{t>0}[t^{\beta_1}\|u(t)\|_{r_1}, t^{\beta_2}\|v(t)\|_{r_2}]\right] \leq M \Big\}. \end{aligned}$$

Endowed with the metric

$$\begin{aligned} d(U_1, U_2) := d\left((u_1, v_1), (u_2, v_2)\right) = \max\Big[ & \sup_{t\geq 0}[\|u_1(t) - u_2(t)\|_{\frac{N}{2\alpha_1}}, \|v_1(t) - v_2(t)\|_{\frac{N}{2\alpha_2}}], \\ & \sup_{t>0}[t^{\beta_1}\|u_1(t) - u_2(t)\|_{r_1}, t^{\beta_2}\|v_1(t) - v_2(t)\|_{r_2}]\Big], \end{aligned}$$

$Y_M$ is a nonempty complete metric space.

Consider the mapping $\mathcal{F}_\Phi$ defined by (3.3.31)-(3.3.32), where $\Phi = (\varphi_1, \varphi_2) \in L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)$ satisfies (3.2.23). We will show that $\mathcal{F}_\Phi = (F_\Phi, G_\Phi)$ is a strict contraction mapping on $Y_M$.

Let $\Phi = (\varphi_1, \varphi_2)$ and $\Psi = (\psi_1, \psi_2)$ belong to $L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)$ satisfying (3.2.23). Let $U_1 = (u_1, v_1)$ and $U_2 = (u_2, v_2)$ be two elements of $Y_M$. Then we have

$$\|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{\frac{N}{2\alpha_1}} \leq \|e^{t\Delta}(\varphi_1 - \psi_1)\|_{\frac{N}{2\alpha_1}} + |a| \int_0^t \|e^{(t-\sigma)\Delta}|.|^{-\gamma}[|v_1(\sigma)|^{p-1}v_1(\sigma) - |v_2(\sigma)|^{p-1}v_2(\sigma)]\|_{\frac{N}{2\alpha_1}} d\sigma.$$

It follows, by the key estimate (3.2.13) with $(q_1, q_2) = (\frac{r_2}{p}, \frac{N}{2\alpha_1})$ that

$$\begin{aligned} \|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{\frac{N}{2\alpha_1}} &\leq \|\varphi_1 - \psi_1\|_{\frac{N}{2\alpha_1}} \\ +|a| \int_0^t C(t-\sigma)^{-\frac{N}{2}(\frac{p}{r_2} - \frac{2\alpha_1}{N}) - \frac{\gamma}{2}} &\left\| |v_1(\sigma)|^{p-1}v_1(\sigma) - |v_2(\sigma)|^{p-1}v_2(\sigma) \right\|_{\frac{r_2}{p}} d\sigma, \end{aligned} \tag{3.3.49}$$

we obtain by (3.3.49) and the fact that $U_1$ and $U_2$ belongs to $Y_M$, that

$$\|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{\frac{N}{2\alpha_1}} \leq \|\varphi_1 - \psi_1\|_{\frac{N}{2\alpha_1}} + |a|C \times \left[ \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{p}{r_2} - \frac{2\alpha_1}{N}) - \frac{\gamma}{2}} 2p\sigma^{-\beta_2 p} M^{p-1} d\sigma \right] d(U_1, U_2),$$

it follows that

$$\begin{aligned} \|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{\frac{N}{2\alpha_1}} &\leq \|\varphi_1 - \psi_1\|_{\frac{N}{2\alpha_1}} + 2|a|CpM^{p-1} \\ &\quad \times \left[ \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{p}{r_2} - \frac{2\alpha_1}{N}) - \frac{\gamma}{2}} \sigma^{-\beta_2 p} d\sigma \right] d(U_1, U_2) \\ &\leq \|\varphi_1 - \psi_1\|_{\frac{N}{2\alpha_1}} \\ &\quad + 2|a|CpM^{p-1} t^{-\frac{N}{2}(\frac{p}{r_2} - \frac{2\alpha_1}{N}) - \frac{\gamma}{2} - \beta_2 p + 1} \\ &\quad \times \left[ \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{p}{r_2} - \frac{2\alpha_1}{N}) - \frac{\gamma}{2}} \sigma^{-\beta_2 p} d\sigma \right] d(U_1, U_2). \end{aligned}$$

Owing to (3.1.7), we get

$$\begin{aligned} \|F_\Phi(U_1)(t) - F_\Psi(U_2)(t)\|_{\frac{N}{2\alpha_1}} &\leq \|\varphi_1 - \psi_1\|_{\frac{N}{2\alpha_1}} + 2|a|CpM^{p-1} t^{\alpha_1 - p\alpha_2 + \frac{2-\gamma}{2}} \\ &\quad \times \left[ \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{p}{r_2} - \frac{2\alpha_1}{N}) - \frac{\gamma}{2}} \sigma^{-\beta_2 p} d\sigma \right] d(U_1, U_2). \end{aligned}$$

Since $\alpha_1$, $\alpha_2$ satisfy (3.1.4) and (3.1.5), using the fact that $r_1 > \frac{N}{2\alpha_1}$ and due to Part (iv) of Lemma 3.2.1, it follows that

$$\alpha_1 - p\alpha_2 + \frac{2-\gamma}{2} = 0, \quad \frac{N}{2}(\frac{p}{r_2} - \frac{2\alpha_1}{N}) + \frac{\gamma}{2} < \frac{N}{2}(\frac{p}{r_2} - \frac{1}{r_1}) + \frac{\gamma}{2} < 1.$$

Using also the fact that $\beta_2 p < 1$, we get

$$\|F_\Phi(U_1)(t) - F_\Psi(U_2)(t\|_{\frac{N}{2\alpha_1}} \leq \mathcal{N}'(\Phi - \Psi) + M^{p-1}\nu_1'' d(U_1, U_2), \tag{3.3.50}$$

with $\nu_1''$ is a finite positive constant defined by

$$\nu_1'' = 2|a|Cp \times \left[ \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{p}{r_2}-\frac{2\alpha_1}{N})-\frac{\gamma}{2}} \sigma^{-\beta_2 p} d\sigma \right].$$

Similarly, we get

$$\|G_\Phi(U_1)(t) - G_\Psi(U_2)(t\|_{\frac{N}{2\alpha_2}} \quad \leq \quad \mathcal{N}'(\Phi-\Psi) + M^{q-1}\nu_2'' d(U_1,U_2), \tag{3.3.51}$$

with $\nu_2''$ is a finite positive constant defined by

$$\nu_2'' = 2|b|Cq \times \left[ \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{q}{r_1}-\frac{2\alpha_2}{N})-\frac{\rho}{2}} \sigma^{-\beta_1 q} d\sigma \right].$$

Owing to (3.3.50) and (3.3.51) we get

$$\sup_{t\geq 0}\left[ \|F_\Phi(U_1)(t) - F_\Psi(U_2)(t\|_{\frac{N}{2\alpha_1}}, \|G_\Phi(U_1)(t) - G_\Psi(U_2)(t\|_{\frac{N}{2\alpha_2}} \right] \leq \mathcal{N}'(\Phi-\Psi) + \nu'' d(U_1,U_2), \tag{3.3.52}$$

where

$$\nu'' = \max(M^{p-1}\nu_1'', M^{q-1}\nu_2'').$$

We can conclude now from (3.3.36) and (3.3.52) and from the estimate $\mathcal{N}(\Phi-\Psi) \leq \mathcal{N}'(\Phi-\Psi)$, that

$$d(\mathcal{F}_\Phi(U_1), \mathcal{F}_\Psi(U_2)) \leq \mathcal{N}'(\Phi-\Psi) + \max(\nu,\nu'')d(U_1,U_2). \tag{3.3.53}$$

It is clear that if $U \in Y_M$, then $\mathcal{F}_\Phi(U) \in C\left([0,\infty), L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N)\right) \times C\left([0,\infty), L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)\right) \cap C\left((0,\infty), L^{r_1}(\mathbb{R}^N)\right) \times C\left((0,\infty), L^{r_2}(\mathbb{R}^N)\right)$. Hence, by choosing $M$ and $R$ such that

$$R + M\max(\nu,\nu'') \leq M, \tag{3.3.54}$$

it follows that $\mathcal{F}_\Phi$ is a strict contraction from $Y_M$ into itself. So $\mathcal{F}_\varphi$ has a unique fixed point in $Y_M$ which is solution of (3.1.2).

Remark finally when the initial data $\Phi$ belongs to $L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)$ with respect to the norm $\mathcal{N}'$, that the condition (3.2.17) is satisfied, since $\mathcal{N}(\Phi) \leq \mathcal{N}'(\Phi)$. We note also that by the previous calculations, precisely (3.3.53) we have the following continuous dependence property: Let $\Phi = (\varphi_1,\varphi_2),\ \Psi = (\psi_1,\psi_2) \in L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)$ and let $U_\Phi = (u_\Phi, v_\Phi)$ and $U_\Psi = (u_\Psi, v_\Psi)$ be the solutions of (3.1.2) with initial values $\Phi$ and respectively $\Psi$, with $\sup_{t\geq 0}\left[\|u_\Phi(t)\|_{\frac{N}{2\alpha_1}}, \|v_\Phi(t)\|_{\frac{N}{2\alpha_2}}\right] \leq M$ and $\sup_{t\geq 0}\left[\|u_\Psi(t)\|_{\frac{N}{2\alpha_1}}, \|v_\Psi(t)\|_{\frac{N}{2\alpha_2}}\right] \leq M$. Then

$$\begin{aligned}\sup_{t\geq 0}\left[\|u_\Phi(t)-u_\Psi(t)\|_{\frac{N}{2\alpha_1}}, \|v_\Phi(t)-v_\Psi(t)\|_{\frac{N}{2\alpha_2}}\right] \quad &\leq \quad (1-K)^{-1} \\ &\times \max\left[\|\varphi_1-\psi_1\|_{\frac{N}{2\alpha_1}}, \|\varphi_2-\psi_2\|_{\frac{N}{2\alpha_2}}\right],\end{aligned} \tag{3.3.55}$$

for some positive constant $K = \max(\nu,\nu'')$. This finishes the proof of Theorem 3.2.1.

□

Let us define the scaling operator $d_\lambda$ by

$$[d_\lambda \varphi](x) = \varphi(\lambda x).$$

It follows that

$$e^{t\Delta} d_\lambda = d_\lambda e^{\lambda^2 t\Delta}, \forall \lambda > 0.$$

*Proof of Theorem 3.2.2.* We now construct self-similar solution with initial data $\Phi$. We adapt the method used in [2]. Let us define $\Phi_\lambda$, for $\lambda > 0$, by

$$\Phi_\lambda(x) := \left(\lambda^{2\alpha_1}\varphi_1(\lambda x), \lambda^{2\alpha_2}\varphi_2(\lambda x)\right).$$

It is clear that $\Phi_\lambda$ satisfies

$$\Phi_\lambda(x) \quad = \quad \Phi(x), \forall \quad \lambda > 0.$$

Let $U$ be the solution of the integral system (3.1.2) with initial data $\Phi$ constructed by Theorem 3.2.1 ( remark that $\mathcal{N}(\Phi) < \infty$, since $r_1$ satisfies Parts (i)-(ii) of Lemma 3.5.1 below and by homogeneity, also $\mathcal{N}(\Phi)$ is sufficiently small since $\|\omega_1\|_\infty$ and $\|\omega_2\|_\infty$ are sufficiently small). That is $U$ belong to $X_M$. We want to prove that $U_\lambda = U$, $\forall \lambda > 0$, where $U_\lambda(t,x) := (u_\lambda(t,x), v_\lambda(t,x))$, $\forall \lambda > 0$, with

$$u_\lambda(t,x) = \lambda^{2\alpha_1} u(\lambda^2 t, \lambda x),$$

and

$$v_\lambda(t,x) = \lambda^{2\alpha_2} v(\lambda^2 t, \lambda x).$$

To do this it suffice to prove that $U_\lambda$ is also a solution of (3.1.2) with the same initial data $\Phi_\lambda = \Phi$ and that $U_\lambda$ belong to $X_M$. On one hand due the homogeneity properties of the system (3.1.2), if $U = (u,v)$ solves this system, then the scaled function solve it also. In fact

$$\begin{aligned}
d_\lambda u(\lambda^2 t) &= d_\lambda e^{\lambda^2 t\Delta}\varphi_1 + a\int_0^{\lambda^2 t} d_\lambda e^{(\lambda^2 t-\sigma)\Delta}\left(|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma)\right) d\sigma \\
&= e^{t\Delta} d_\lambda\varphi_1 + a\int_0^{\lambda^2 t} e^{(t-\frac{\sigma}{\lambda^2})\Delta}\left(d_\lambda(|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma))\right) d\sigma \\
&= e^{t\Delta} d_\lambda\varphi_1 + a\int_0^{\lambda^2 t} \lambda^{-\gamma} e^{(t-\frac{\sigma}{\lambda^2})\Delta}\left(|.|^{-\gamma}|d_\lambda v(\sigma)|^{p-1}d_\lambda v(\sigma)\right) d\sigma \\
&= e^{t\Delta} d_\lambda\varphi_1 + a\int_0^{t} \lambda^{2-\gamma} e^{(t-\sigma)\Delta}\left(|.|^{-\gamma}|d_\lambda v(\lambda^2\sigma)|^{p-1}d_\lambda v(\lambda^2\sigma)\right) d\sigma.
\end{aligned}$$

Hence by (3.1.8), we get

$$\begin{aligned}
\lambda^{2\alpha_1} d_\lambda u(\lambda^2 t) &= e^{t\Delta} d_\lambda(\lambda^{2\alpha_1}\varphi_1) + a\int_0^{t} e^{(t-\sigma)\Delta}\left(|.|^{-\gamma}\lambda^{2-\gamma+2\alpha_1}|d_\lambda v(\lambda^2\sigma)|^{p-1}d_\lambda v(\lambda^2\sigma)\right) d\sigma \\
&= e^{t\Delta} d_\lambda(\lambda^{2\alpha_1}\varphi_1) + a\int_0^{t} e^{(t-\sigma)\Delta}\left(|.|^{-\gamma}|\lambda^{2\alpha_2}d_\lambda v(\lambda^2\sigma)|^{p-1}\lambda^{2\alpha_2}d_\lambda v(\lambda^2\sigma)\right) d\sigma,
\end{aligned}$$

we conclude finally that

$$u_\lambda(t) = e^{t\Delta}\varphi_1 + a\int_0^t e^{(t-\sigma)\Delta}\left(|.|^{-\gamma}|v_\lambda(\sigma)|^{p-1}v_\lambda(\sigma)\right)d\sigma. \tag{3.3.56}$$

Similarly we obtain

$$v_\lambda(t) = e^{t\Delta}\varphi_2 + b\int_0^t e^{(t-\sigma)\Delta}\left(|.|^{-\rho}|u_\lambda(\sigma)|^{p-1}u_\lambda(\sigma)\right)d\sigma. \tag{3.3.57}$$

The affirmation follows from (3.3.56)-(3.3.57). On the other hand we have

$$\begin{aligned}\|u_\lambda(t)\|_{r_1} &= \lambda^{2\alpha_1}\|d_\lambda u(\lambda^2 t)\|_{r_1}\\ &= \lambda^{2\alpha_1}\lambda^{-\frac{N}{r_1}}\|u(\lambda^2 t)\|_{r_1}\\ &= (\lambda^2)^{\beta_1}\|u(\lambda^2 t)\|_{r_1}.\end{aligned}$$

Hence

$$\begin{aligned}\sup_{t>0} t^{\beta_1}\|u_\lambda(t)\|_{r_1} &= \sup_{\lambda^2 t>0}(\lambda^2 t)^{\beta_1}\|u(\lambda^2 t)\|_{r_1}\\ &= \sup_{t>0} t^{\beta_1}\|u(t)\|_{r_1},\end{aligned}$$

similarly $\sup_{t>0} t^{\beta_2}\|v_\lambda(t)\|_{r_2} = \sup_{t>0} t^{\beta_2}\|v(t)\|_{r_2}$. It follows so that $\|U_\lambda\|_X = \|U\|_X$. Then by uniqueness in $X_M$, we have $U_\lambda = U$ and thus $U$ is self-similar. Let us denote it by $U_S$. The fact that $U_S(t) \to \Phi$ in $\mathcal{S}'(\mathbb{R}^N)$ as $t \to 0$ follows by statement (c) in Theorem 3.2.1. ☐

*Proof of Theorem 3.2.3.* The proof is similar to the one of Theorem 5.1 in [2], we simply indicate that

(i) $\sup_{t>0} t^{\beta_1+\delta}\|e^{t\Delta}(\varphi_1 - \psi_1)\|_{r_1} < \infty$, for $0 < \delta < \frac{N}{2} - \alpha_1$.

(ii) $\sup_{t>0} t^{\beta_2+\delta}\|e^{t\Delta}(\varphi_2 - \psi_2)\|_{r_2} < \infty$, for $0 < \delta < \frac{N}{2} - \alpha_2$.

By the formula (3.2.22), we have that

$$\sup_{t>0}\left[t^{\beta_1+\delta}\|u(t) - u_{\mathcal{S}}(t)\|_{r_1}, t^{\beta_2+\delta}\|v(t) - v_{\mathcal{S}}(t)\|_{r_2}\right] \le C\mathcal{N}_\delta(\Phi - \Psi).$$

That is

$$\sup_{t>0}\left[t^{\beta_1+\delta}\|u(t) - u_{\mathcal{S}}(t)\|_{r_1}, t^{\beta_2+\delta}\|v(t) - v_{\mathcal{S}}(t)\|_{r_2}\right] \le \mathcal{C},$$

for $\delta > 0$ sufficiently small and $\mathcal{C}$ a finite positive constant. This gives (3.2.26)-(3.2.27) directly for $q_1 = r_1$ and $q_2 = r_2$.

We now turn to prove the asymptotic result in the $L^\infty$-norm. Write

$$u(t) - u_{\mathcal{S}}(t) = e^{\frac{t}{2}\Delta}\big(u(t/2) - u_{\mathcal{S}}(t/2)\big) + a\int_{\frac{t}{2}}^t e^{(t-\sigma)\Delta}\left[|.|^{-\gamma}\left(|v(\sigma)|^{p-1}v(\sigma) - |v_{\mathcal{S}}(\sigma)|^{p-1}v_{\mathcal{S}}(\sigma)\right)\right]d\sigma,$$

$$v(t)-v_{\mathcal{S}}(t)=e^{\frac{t}{2}\Delta}\big(v(t/2)-v_{\mathcal{S}}(t/2)\big)+ \\ b\int_{\frac{t}{2}}^{t} e^{(t-\sigma)\Delta}\left[|.|^{-\gamma}\left(|u(\sigma)|^{q-1}u(\sigma)-|u_{\mathcal{S}}(\sigma)|^{q-1}u_{\mathcal{S}}(\sigma)\right)\right]d\sigma.$$

Let $T>0$ be an arbitrary real number. By using the smoothing properties of the heat semi-group with $(s_1,s_2)=(r_1,\infty)$ and the estimate (3.2.13) with $(q_1,q_2)=(\infty,\infty)$, it follows that

$$\begin{aligned} t^{\alpha_1+\delta}\|u(t)-u_{\mathcal{S}}(t)\|_\infty &\leq t^{\alpha_1+\delta}\|e^{\frac{t}{2}\Delta}\big(u(t/2)-u_{\mathcal{S}}(t/2)\big)\|_\infty+|a|t^{\alpha_1+\delta}\times \\ &\quad \int_{\frac{t}{2}}^{t}\|e^{(t-\sigma)\Delta}\left[|.|^{-\gamma}\big(|v(\sigma)|^{p-1}v(\sigma)-|v_{\mathcal{S}}(\sigma)|^{p-1}v_{\mathcal{S}}(\sigma)\big)\right]\|_\infty d\sigma \\ &\leq Ct^{\beta_1+\delta}\|u(t/2)-u_{\mathcal{S}}(t/2)\|_{r_1}+|a|Ct^{\alpha_1+\delta}\times \\ &\quad \int_{\frac{t}{2}}^{t}(t-\sigma)^{-\frac{\gamma}{2}}\big(\|v(\sigma)\|_\infty^{p-1}+\|v_{\mathcal{S}}(\sigma)\|_\infty^{p-1}\big)\|v(\sigma)-v_{\mathcal{S}}(\sigma)\|_\infty d\sigma. \end{aligned}$$

Using (3.2.22) to estimate the first term and the fact that $\|v_{\mathcal{S}}(t)\|_\infty\leq Ct^{-\alpha_2}$, $\|v(t)\|_\infty\leq Ct^{-\alpha_2}$ to estimate the last term, we get

$$\begin{aligned} t^{\alpha_1+\delta}\|u(t)-u_{\mathcal{S}}(t)\|_\infty &\leq C(\delta)+|a|C\times \\ &\quad \left[\int_{\frac{1}{2}}^{1}(1-\sigma)^{-\frac{\gamma}{2}}\sigma^{-\alpha_2 p-\delta}d\sigma\right]\sup_{t\in(0,T]}\left(t^{\alpha_2+\delta}\|v(t)-v_{\mathcal{S}}(t)\|_\infty\right). \end{aligned}$$

Which leads to

$$t^{\alpha_1+\delta}\|u(t)-u_{\mathcal{S}}(t)\|_\infty\leq C(\delta)+C\sup_{t\in(0,T]}\left[t^{\alpha_1+\delta}\|u(t)-u_{\mathcal{S}}(t)\|_\infty,t^{\alpha_2+\delta}\|v(t)-v_{\mathcal{S}}(t)\|_\infty\right]. \tag{3.3.58}$$

Similarly we have

$$t^{\alpha_2+\delta}\|v(t)-v_{\mathcal{S}}(t)\|_\infty\leq C(\delta)+C\sup_{t\in(0,T]}\left[t^{\alpha_1+\delta}\|u(t)-u_{\mathcal{S}}(t)\|_\infty,t^{\alpha_2+\delta}\|v(t)-v_{\mathcal{S}}(t)\|_\infty\right]. \tag{3.3.59}$$

Using (3.3.58) and (3.3.59) we obtain

$$\sup_{t\in(0,T]}\left[t^{\alpha_1+\delta}\|u(t)-u_{\mathcal{S}}(t)\|_\infty,t^{\alpha_2+\delta}\|v(t)-v_{\mathcal{S}}(t)\|_\infty\right]\leq C'(\delta).$$

Since the constant $C'(\delta)$ does not depend on $T>0$, one can take the supremum over $(0,\infty)$. This prove (3.2.26)-(3.2.27) for $r_1=\infty$ and $r_2=\infty$. Using the interpolation inequality

$$\|u(t)-u_{\mathcal{S}}(t)\|_{q_1}\leq\|u(t)-u_{\mathcal{S}}(t)\|_{r_1}^{\mu_1}\|u(t)-u_{\mathcal{S}}(t)\|_\infty^{1-\mu_1},$$

where

$$\frac{1}{q_1}=\frac{\mu_1}{r_1}+\frac{1-\mu_1}{\infty}=\frac{\mu_1}{r_1}.$$

We get

$$\begin{aligned} \|u(t)-u_{\mathcal{S}}(t)\|_{q_1} &\leq \|u(t)-u_{\mathcal{S}}(t)\|_{r_1}^{\mu_1}\|u(t)-u_{\mathcal{S}}(t)\|_\infty^{1-\mu_1} \\ &\leq Ct^{\mu_1[-\beta_1(r_1)-\delta]+(1-\mu_1)[-\beta_1(\infty)-\delta]} \\ &= Ct^{-\beta_1(q_1)-\delta}. \end{aligned}$$

We have also

$$\|v(t) - v_{\mathcal{S}}(t)\|_{q_2} \leq Ct^{-\beta_2(q_2)-\delta}.$$

Hence we obtain the results (3.2.26)-(3.2.27) in general case. The estimate (3.2.28)-(3.2.29) follows by a simple dilation argument. We prove just the first estimate (3.2.28), the proof of the second estimate is similar. We have

$$\begin{aligned}
\|u(t) - u_{\mathcal{S}}(t)\|_{q_1} &= \|u(t,.) - t^{-\alpha_1} u_{\mathcal{S}}(1, \frac{\cdot}{\sqrt{t}})\|_{q_1} \\
&= \|d_{\frac{1}{\sqrt{t}}} u(t, .\sqrt{t}) - t^{-\alpha_1} d_{\frac{1}{\sqrt{t}}} u_{\mathcal{S}}(1,.)\|_{q_1} \\
&= \|d_{\frac{1}{\sqrt{t}}} [u(t, .\sqrt{t}) - t^{-\alpha_1} u_{\mathcal{S}}(1,.)]\|_{q_1} \\
&= (\frac{1}{\sqrt{t}})^{-\frac{N}{q_1}} \|u(t, .\sqrt{t}) - t^{-\alpha_1} u_{\mathcal{S}}(1,.)\|_{q_1}.
\end{aligned}$$

Then by using inequality (3.2.26) and relation (3.2.25), we get (3.2.28). □

*Proof of Proposition 3.2.1.* If $\gamma = 0$ and $\rho = 0$, then (3.1.9) and (3.1.10) are verified. Since these are strict inequalities, they must hold for small $\gamma \geq 0$ and $\rho > 0$. This finishes the proof of the proposition. □

*Proof of Proposition 3.2.2.* Let $\alpha_1$ and $\alpha_2$ defined by (3.1.4) and (3.1.5) respectively. Under the conditions

$$q \geq \frac{2-\rho}{\gamma} + \frac{2}{\gamma},$$

and

$$p \geq \frac{2-\gamma}{\rho} + \frac{2}{\rho},$$

we have that conditions (3.1.9) and (3.1.10) are equivalent to the conditions $2\alpha_1 < N$ and $2\alpha_2 < N$. Now, since $q \geq \frac{2-\rho}{N} + \frac{2-\gamma}{N} + 1$, we see that $2\alpha_1 < N$ and since $p \geq \frac{2-\gamma}{N} + \frac{2-\rho}{N} + 1$, we obtain that $2\alpha_2 < N$. This finishes the proof of the proposition. □

## 3.4 Stronger uniqueness results

It has been proved in Theorem 3.2.1 that for small initial data $\Phi = (\varphi_1, \varphi_2) \in L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)$ with respect of the norm $\mathcal{N}'$, there exists a solution $U_\Phi = (u_\Phi, v_\Phi)$ of the integral system (3.1.2) and uniqueness is guaranteed only among continuous functions $U : [0,\infty) \to L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)$ which also verify $\sup_{t>0} \left[t^{\beta_1}\|u(t)\|_{r_1}, t^{\beta_2}\|v(t)\|_{r_2}\right]$ is sufficiently small. Our aim in this section is to prove that uniqueness is guaranteed for solutions which belong to $C\left([0,\infty), L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N)\right) \times C\left([0,\infty), L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)\right) \cap C\left((0,\infty), L^{r_1}(\mathbb{R}^N)\right) \times C\left((0,\infty), L^{r_2}(\mathbb{R}^N)\right)$, which improves the result of uniqueness in Lebesgue spaces given in Theorem 3.2.1. We will use arguments of type Brezis Cazenave [3]. We have obtained the following result.

**Theorem 3.4.1.** *Let $N$ be a positive integer. Let $p$, $q > 1$. Let $0 \le \gamma < \min(N,2)$ and $0 < \rho < \min(N,2)$. Let $\alpha_1$, $\alpha_2$ defined by (3.1.4) and (3.1.5). Suppose that (3.1.9) and (3.1.10) are satisfied. Let $\beta_1$, $\beta_2$ be given by (3.1.6) and (3.1.7). Let $r_1$ and $r_2$ be as in Lemma 3.2.1. Let $M$, $R > 0$ be such that (3.3.54) is satisfied. Let $\Phi = (\varphi_1, \varphi_2) \in L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)$ satisfying (3.2.23). Let $U_\Phi = (u_\Phi, v_\Phi) \in Y_M$ be the solution of the integral system (3.1.2) with initial data $\Phi$ constructed by Theorem 3.2.1. Let $V = (v_1, v_2) \in C\left([0,\infty), L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N)\right) \times C\left([0,\infty), L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)\right) \cap C\left((0,\infty), L^{r_1}(\mathbb{R}^N)\right) \times C\left((0,\infty), L^{r_2}(\mathbb{R}^N)\right)$ be a solution of (3.1.2) with the same initial data $\Phi$. Then*

$$V(t) = U_\Phi(t), \quad \forall t \in [0,\infty).$$

The proof of this theorem relies on the following two lemmas.

**Lemma 3.4.1.** *Let $N$ be a positive integer. Let $p$, $q > 1$. Let $0 \le \gamma < \min(N,2)$ and $0 < \rho < \min(N,2)$. Let $\alpha_1$, $\alpha_2$ defined by (3.1.4) and (3.1.5). Suppose that (3.1.9) and (3.1.10) are satisfied. Let $\beta_1$, $\beta_2$ be given by (3.1.6) and (3.1.7). Let $r_1$ and $r_2$ be as in Lemma 3.2.1. Let $M$, $R > 0$ be such that (3.3.54) is satisfied. Let $\Phi = (\varphi_1, \varphi_2) \in L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)$ satisfying (3.2.23). Let $U_\Phi = (u_\Phi, v_\Phi)$ be the solution of the integral system (3.1.2) with initial data $\Phi$ constructed by Theorem 3.2.1. Then for all $T > 0$, there exists a unique solution $U_{\Phi,T} = U_\Phi \in Y_{M,T}$ of (3.1.2) with initial data $\Phi$, where*

$$\begin{aligned} Y_{M,T} = \Big\{ & U = (u,v) \in C\left([0,T), L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N)\right) \times C\left([0,T), L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)\right) \cap C\left((0,T), L^{r_1}(\mathbb{R}^N)\right) \times \\ & C\left((0,T), L^{r_2}(\mathbb{R}^N)\right); \max\Big[\sup_{t\in[0,T)} [\|u(t)\|_{\frac{N}{2\alpha_1}}, \|v(t)\|_{\frac{N}{2\alpha_2}}], \sup_{t\in(0,T)} [t^{\beta_1}\|u(t)\|_{r_1}, t^{\beta_2}\|v(t)\|_{r_2}]\Big] \le M \Big\}. \end{aligned}$$

*Proof.* The existence of the unique solution $U_{\Phi,T}$ of (3.1.2) with initial data $\Phi$ follows by a fixed point argument in $Y_{M,T}$. Let $U_\Phi \in Y_M$ the solution of (3.1.2) with initial data $\Phi$. Owing to the fact that $U_\Phi \in Y_M \subset Y_{M,T}$ and by uniqueness in $Y_{M,T}$, we obtain $U_{\Phi,T} = U_\Phi$. □

**Lemma 3.4.2.** *Let $N$ be a positive integer. Let $p$, $q > 1$. Let $0 \le \gamma < \min(N,2)$ and $0 < \rho < \min(N,2)$. Let $\alpha_1$, $\alpha_2$ defined by (3.1.4) and (3.1.5). Suppose that (3.1.9) and (3.1.10) are satisfied. Let $M$, $R > 0$ be such that (3.3.54) is satisfied. Let $\Phi = (\varphi_1, \varphi_2) \in L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)$ satisfying (3.2.23). Let $U_\Phi = (u_\Phi, v_\Phi)$ be the solution of the integral system (3.1.2) with initial data $\Phi$ constructed by Theorem 3.2.1. Let $(\Phi_\tau) = ((\varphi_{1,\tau}, \varphi_{2,\tau}))$ be a family of functions satisfying (3.2.23) such that*

$$\Phi_\tau \underset{\tau\to 0}{\longrightarrow} \Phi, \quad \text{in } L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N).$$

*Then the family of solutions $(U_{\Phi_\tau}) = ((u_{\Phi_\tau}, v_{\Phi_\tau}))$ of the integral system (3.1.2) verify*

$$U_{\Phi_\tau}(t) \underset{\tau\to 0}{\longrightarrow} U_\Phi(t), \quad \textit{in} \quad L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N), \forall t \in [0,\infty).$$

*Proof.* By continuous dependance (3.3.55) in $Y_M$, it follows that

$$\begin{aligned}\max\left[\|u_{\Phi_\tau}(t) - u_\Phi(t)\|_{\frac{N}{2\alpha_1}}, \|v_{\Phi_\tau}(t) - v_\Phi(t)\|_{\frac{N}{2\alpha_2}}\right] \quad &\leq \quad (1-K)^{-1} \\ &\times \max\left[\|\varphi_{1,\tau} - \varphi_1\|_{\frac{N}{2\alpha_1}}, \|\varphi_{2,\tau} - \varphi_2\|_{\frac{N}{2\alpha_2}}\right], \quad \forall t \in [0,\infty).\end{aligned}$$

By letting $\tau \to 0$, we obtain the result. □

*Proof of Theorem 3.4.1.* Since $V = (v_1, v_2) \in C\left([0,\infty), L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N)\right) \times C\left([0,\infty), L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)\right)$, then there exists $\varepsilon_1 > 0$ such that

$$\mathcal{N}'(V(s)) = \max\left[\|v_1(s)\|_{\frac{N}{2\alpha_1}}, \|v_2(s)\|_{\frac{N}{2\alpha_2}}\right] < R, \quad \forall s \in [0,\varepsilon_1]. \tag{3.4.60}$$

Let us define $V_\tau = (v_{1,\tau}, v_{2,\tau})$ by $V_\tau(t) = V(t+\tau)$, $\forall \tau \in (0, \frac{\varepsilon_1}{2}]$, $\forall t \in [0, \frac{\varepsilon_1}{2}]$. We have from (3.4.60) and since $\left(t^{\beta_1}\|v_{1,\tau}(t)\|_{r_1}, t^{\beta_2}\|v_{2,\tau}(t)\|_{r_2}\right) \to (0,0)$ as $t \to 0$, $\forall \tau \in (0, \frac{\varepsilon_1}{2}]$

(a) $\max\left[\|v_{1,\tau}(0)\|_{\frac{N}{2\alpha_1}}, \|v_{2,\tau}(0)\|_{\frac{N}{2\alpha_2}}\right] = \max\left[\|v_1(\tau)\|_{\frac{N}{2\alpha_1}}, \|v_2(\tau)\|_{\frac{N}{2\alpha_2}}\right] < R$, $\forall \tau \in \left(0, \frac{\varepsilon_1}{2}\right]$,

(b) $\sup\limits_{t\in[0,\frac{\varepsilon_1}{2}]}\left[\|v_{1,\tau}(t)\|_{\frac{N}{2\alpha_1}}, \|v_{2,\tau}(t)\|_{\frac{N}{2\alpha_2}}\right] < R \leq M$, $\forall \tau \in \left(0, \frac{\varepsilon_1}{2}\right]$,

(c) There exists $0 < T_\tau \leq \varepsilon_1$ such that $\sup\limits_{t\in(0,\frac{T_\tau}{2}]}\left[t^{\beta_1}\|v_{1,\tau}(t)\|_{r_1}, t^{\beta_2}\|v_{2,\tau}(t)\|_{r_2}\right] \leq M$, $\forall \tau \in (0, \frac{\varepsilon_1}{2}]$.

It follows then that $V_\tau \in Y_{M,\frac{T_\tau}{2}}$, using now Lemma 3.4.1 we deduce that $V_\tau(t) = U_{V_\tau(0)}(t)$, $\forall \tau \in (0, \frac{\varepsilon_1}{2}]$, $\forall t \in [0, \frac{T_\tau}{2}]$, where $U_{V_\tau(0)}$ is the solution of the integral system (3.1.2) with initial data $V_\tau(0)$ constructed by Theorem 3.2.1. Hence $V_\tau(t) = U_{V_\tau(0)}(t)$, $\forall \tau \in (0, \frac{\varepsilon_1}{2}]$, $\forall t \in [0,\infty)$. By Lemma 3.4.2, we obtain $V_\tau(t) \underset{\tau\to 0}{\longrightarrow} U_\Phi(t)$, in $L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N, )$ $\forall t \in [0,\infty)$. On the other hand $V_\tau(t) = V(t+\tau) \underset{\tau\to 0}{\longrightarrow} V(t)$, in $L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)$, $\forall t \in [0,\infty)$, (since $V$ is continuous in $[0,\infty)$). Finally, we conclude by uniqueness of the limit that $V(t) = U_\Phi(t)$, $\forall t \in [0,\infty)$. □

Consider now the integral equation

$$u(t) = e^{t\Delta}\varphi + a\int_0^t e^{(t-s)\Delta}\left(|.|^{-\gamma}|u(s)|^{p-1}u(s)\right)ds, \tag{3.4.61}$$

where $u = u(t,x) \in \mathbb{R}$, $t > 0$, $x \in \mathbb{R}^N$, $a \in \mathbb{R}$, $0 < \gamma < \min(N,2)$ and $p > 1$. Set

$$q_c = \frac{N(p-1)}{2-\gamma}. \tag{3.4.62}$$

Suppose that

$$\frac{N(p-1)}{2-\gamma} > 1, \quad (\text{i.e. } q_c > 1). \tag{3.4.63}$$

By choosing $\gamma = \rho$, $p = q$ and $r_1 = r$ in Lemma 3.5.1, using the fact that $\frac{1}{q_c} - \frac{2}{Np} = \frac{2+(2-\gamma)p-\gamma p^2}{Np(p^2-1)}$ and the equivalence $q_c > 1 \Leftrightarrow$ (3.5.1), it follows that there exists $r > q_c$ satisfying

$$\frac{1}{q_c} - \frac{2}{Np} < \frac{1}{r} < \frac{N-\gamma}{Np} \tag{3.4.64}$$

**Corollary 3.4.1.** *Let $N$ be a positive integer. Suppose that $p > 1$. Let $0 < \gamma < \min(N, 2)$. let $q_c$ defined by (3.4.62). Suppose that (3.4.63) is satisfied. Let $r > q_c$ satisfying (3.4.64). Let $\varphi \in L^{q_c}(\mathbb{R}^N)$ sufficiently small. Then there exists a global solution of the integral equation (3.4.61), which is unique in the class of functions $u \in C\left([0,\infty), L^{q_c}(\mathbb{R}^N)\right) \cap C\left((0,\infty), L^r(\mathbb{R}^N)\right)$.*

*Proof.* Let $N$ be a positive integer. Suppose that $p = q > 1$. Suppose that $\gamma = \rho$ with $0 < \gamma < \min(N, 2)$. Let $\alpha_1 = \alpha_2$ defined by (3.1.4). Suppose that (3.5.1) is satisfied. Let $\beta_1$, $\beta_2$ be given by (3.1.6) and (3.1.7). Let $r_1 = r_2$ be as in Lemma 3.2.1. Let $M$, $R > 0$ be such that (3.3.54) is satisfied. Let $\Phi = (\varphi_1, \varphi_1) \in L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N) \times L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)$ satisfying (3.2.23). Let $U_\Phi = (u_\Phi, u_\Phi) \in Y_M$ be the solution of the integral system (3.1.2) with initial data $\Phi$ constructed by Theorem 3.2.1. Let $V = (v_1, v_1) \in C\left([0,\infty), L^{\frac{N}{2\alpha_1}}(\mathbb{R}^N)\right) \times C\left([0,\infty), L^{\frac{N}{2\alpha_2}}(\mathbb{R}^N)\right) \cap C\left((0,\infty), L^{r_1}(\mathbb{R}^N)\right) \times C\left((0,\infty), L^{r_2}(\mathbb{R}^N)\right)$ be a solution of (3.1.2) with the same initial data $\Phi$. Then by Theorem 3.4.1

$$V(t) = U_\Phi(t), \quad \forall t \in [0,\infty).$$

This finishes the proof. □

**Remark 3.4.1.** The previous corollary improves the class of uniqueness for the scalar Hardy-Hénon parabolic equations given by Theorem 1.1 (iii)-(b) in [2].

**Remark 3.4.2.** Using the same steps as above we can improve the class of uniqueness for local solutions of the scalar Hardy-Hénon parabolic equations in the critical case.

**Remark 3.4.3.** Using the same steps as above we prove that for initial data $\Phi = (\varphi_1, \varphi_2) \in L^{q_1}(\mathbb{R}^N) \times L^{q_2}(\mathbb{R}^N)$ such that $\frac{N}{2\alpha_1} < q_1 < r_1$ and $\frac{N}{2\alpha_2} < q_2 < r_2$, there exists a local solution $U_\Phi = (u_\Phi, v_\Phi)$ of the integral system (3.1.2) and uniqueness is guaranteed in the class of solutions which belong to $C\left([0,T], L^{q_1}(\mathbb{R}^N)\right) \times C\left([0,T], L^{q_2}(\mathbb{R}^N)\right) \cap C\left((0,T], L^{r_1}(\mathbb{R}^N)\right) \times C\left((0,T], L^{r_2}(\mathbb{R}^N)\right)$, for any fixed $0 < T < T_{\max}$, where $T_{\max}$ is the maximal existence time. This improves the result of uniqueness in Lebesgue spaces given by Theorem 1.1 (iii)-(a) in [2].

## 3.5 Auxiliary lemmas

Let us state the following result which will be needed in the proof of the technical lemma.

**Lemma 3.5.1.** *Let $N$ be a positive integer. Let $p$, $q > 1$. Let $0 \le \gamma < \min(N, 2)$ and $0 < \rho < \min(N, 2)$. Let $k$ given by (3.1.3). Suppose that (3.1.9) and (3.1.10) are satisfied. Then there exists a real number $r_1$ satisfying the conditions*

*(i)* $N\frac{pq-1}{(2-\rho)p+(2-\gamma)} < r_1$,
*(ii)* $Nk\frac{pq-1}{(2-\gamma)q+(2-\rho)} < r_1$,
*(iii)* $\frac{N}{N-\gamma}kp < r_1$,
*(iv)* $\frac{N}{N-\rho}q < r_1$,
*(v)* $\frac{N}{2-\gamma}(kp-1) < r_1$,
*(vi)* $\frac{N}{2-\rho}(q-k) < r_1$,
*(vii)* $r_1 < Nk\frac{p(pq-1)}{[2+(2-\rho)p-\gamma pq]_+}$,
*(viii)* $r_1 < N\frac{q(pq-1)}{[2+(2-\gamma)q-\rho pq]_+}$.

*Proof.* We will treat the cases where $2+(2-\rho)p-\gamma pq > 0$ and $2+(2-\gamma)q-\rho pq > 0$, the other cases are simple. One can easily see that $r_1$ exists if and only if the left-hand sides of inequalities (i)-(vi) are less than the right-hand sides of inequalities (vii) and (viii). Since $pq-1>0$ we verify easily

(i) $N\frac{pq-1}{(2-\rho)p+(2-\gamma)} < Nk\frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$,
(ii) $Nk\frac{pq-1}{(2-\gamma)q+(2-\rho)} < Nk\frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$,
(iii) $\frac{N}{2-\gamma}(kp-1) < Nk\frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$,
(iv) $\frac{N}{2-\rho}(q-k) < Nk\frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$,
(v) $N\frac{pq-1}{(2-\rho)p+(2-\gamma)} < N\frac{q(pq-1)}{2+(2-\gamma)q-\rho pq}$,
(vi) $Nk\frac{pq-1}{(2-\gamma)q+(2-\rho)} < N\frac{q(pq-1)}{2+(2-\gamma)q-\rho pq}$,
(vii) $\frac{N}{2-\gamma}(kp-1) < N\frac{q(pq-1)}{2+(2-\gamma)q-\rho pq}$,
(viii) $\frac{N}{2-\rho}(q-k) < N\frac{q(pq-1)}{2+(2-\gamma)q-\rho pq}$.

Condition $2\alpha_1 < N$ implies that $\frac{N}{N-\gamma}kp < Nk\frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$, condition $2\alpha_1 < \frac{p}{q}(N-\rho)\frac{(2-\gamma)q+(2-\rho)}{2+(2-\rho)p-\gamma pq}$ implies that $\frac{N}{N-\rho}q < Nk\frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$, condition $2\alpha_2 < N$ implies that $\frac{N}{N-\rho}q < N\frac{q(pq-1)}{2+(2-\gamma)q-\rho pq}$ and finally condition $2\alpha_2 < \frac{q}{p}(N-\gamma)\frac{(2-\rho)p+(2-\gamma)}{2+(2-\gamma)q-\rho pq}$ implies that $\frac{N}{N-\gamma}kp < N\frac{q(pq-1)}{2+(2-\gamma)q-\rho pq}$. This finishes the proof of the lemma. ☐

*Proof of Lemma 3.2.1.* Owing to relation (3.2.14) and Lemma 3.5.1, the proof of Lemma 3.2.1 is simple and can be omitted. ☐

**Remark 3.5.1.** In the case where $\gamma = \rho$ and $p = q$ it suffice to change the hypotheses (3.1.9) and (3.1.10) by the hypothesis

$$2\alpha_1 < N. \tag{3.5.1}$$

# Chapitre 4

# Instability of some global solutions

## 4.1 Introduction

This chapter studies finite-time blowup of sign-changing, regular solutions of the initial value problem

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u - |u|^{q-1}u, \\ u_{|\partial\Omega} = 0. \end{cases} \tag{4.1.1}$$

Here, $u = u(t,x) \in \mathbb{R},\ t \geq 0,\ x \in \Omega$, and

$$\Omega = B_1, \tag{4.1.2}$$

is the (open) unit ball of $\mathbb{R}^N$,

$$N \geq 3. \tag{4.1.3}$$

Furthermore, we consider

$$1 < q < p < p_S, \tag{4.1.4}$$

where

$$p_S = \frac{N+2}{N-2}. \tag{4.1.5}$$

It is well known that the initial value problem (4.1.1) is locally well-posed in $C_0(\Omega)$, where $C_0(\Omega)$ is the Banach space of continuous functions on $\overline{\Omega}$ that vanish on $\partial\Omega$, with the sup norm. More precisely, given $u_0 \in C_0(\Omega)$, there exists a maximal time $0 < T_{u_0} \leq \infty$ and a unique function $u \in C\left([0,T_{u_0}), C_0(\Omega)\right) \cap C\left((0,T_{u_0}), C^2(\overline{\Omega})\right) \cap C^1\left((0,T_{u_0}), C_0(\Omega)\right)$ which is a classical solution of (4.1.1) on $(0,T_{u_0})$ and such that $u(0) = u_0$. Furthermore if $T_{u_0} < \infty$, then $\lim\limits_{t\uparrow T_{u_0}} \|u(t)\|_\infty = \infty$, and we say that $u$ blows up in finite time. In addition, if $v \in C\left([0,T), C_0(\Omega)\right) \cap C\left((0,T), C^2(\overline{\Omega})\right) \cap C^1\left((0,T), C_0(\Omega)\right)$ is a supersolution of (4.1.1), i.e. $v_t - \Delta v \geq |v|^{p-1}v - |v|^{q-1}v$, $v_{|\partial\Omega} \geq 0$ and $v(0) \geq u_0$, then $v(t) \geq u(t)$ as long as both $u$ and $v$ are defined. The notion of subsolution is defined with reversed inequalities, yielding the analogous conclusion. See, for example Proposition 52.6 in [40].

We define the set $\mathcal{G}$ by

$$\mathcal{G} = \left\{u_0 \in C_0(\Omega), T_{u_0} = \infty\right\}.$$

It is interesting to study the geometrical properties of the set $\mathcal{G}$. First of all we note that every solution $h$ of

$$\begin{cases} -\Delta h = |h|^{p-1}h - |h|^{q-1}h, \\ h_{|\partial\Omega} = 0, \end{cases} \tag{4.1.6}$$

is a stationary, hence global, solution of (4.1.1), whose initial value is of course $u_0 = h$, and so is in $\mathcal{G}$. Since the nonlinearity $|s|^{p-1}s - |s|^{q-1}s$ satisfies the properties of [9, Theorem 1.1, p. 15], it follows that the set $\mathcal{G}$ is not convex. As $u(t) = 0$ is a solution of (4.1.1) one can ask if $\mathcal{G}$ has the weaker property of being star-shaped around 0. The aim of this chapter is to prove that $\mathcal{G}$ is not star-shaped.

This result is already well-known in the case of a single power nonlinearity

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, \\ u_{|\partial\Omega} = 0. \end{cases} \tag{4.1.7}$$

In particular, it is proved in [8] that if $h$ is a radially symmetric, sign-changing stationary solution of the problem (4.1.7), with $\Omega = B_1$, then the solution of (4.1.7) with initial value $\lambda h$ blows up in finite time if $|\lambda - 1| > 0$ is sufficiently small and if $p$ is subcritical and sufficiently close to $p_S = \frac{N+2}{N-2}$. More precisely, there exists $1 < \underline{p} < p_S = \frac{N+2}{N-2}$ such that if $\underline{p} < p < p_S$ and if $h \in C_0(\Omega)$ is a radially symmetric, sign-changing stationary solution of (4.1.7), then there exists $\varepsilon > 0$ such that if $0 < |\lambda - 1| < \epsilon$, then the classical solution of (4.1.7) with the initial condition $u(0) = \lambda h$ blows up in finite time. In particular, $\mathcal{G}$, for the problem (4.1.7), is not star-shaped.

The fact that $h$ changes sign is fundamental in this affirmation. In fact in the case where $h > 0$ it follows from the comparison principle of the heat equation that if $0 < \lambda \leq 1$, then the solution is global and if $\lambda > 1$, then $u$ blows up in finite time. For an elementary proof of the case $\lambda > 1$, see Theorem 17.8 in [40]. We remark, as was done in [8], that if $h$ changes sign, then $h$ and $\lambda h$ are not comparable if $\lambda \neq 1$.

In addition to the result in [8], it is known that $\mathcal{G}$ for the problem (4.1.7) is not star-shaped in several other circumstances:

- $N = 3$, $\Omega = B_1$ and $p > 1$ sufficiently near to 1, see [10] ;
- $N \geq 3$, $\Omega$ is a general domain and $p < p_S$ sufficiently near to $p_S$ or $p = p_S$, see [29, 31] ;
- $N = 2$, $\Omega = B_1$ or $\Omega$ is a general domain and $p$ sufficiently large, see [15, 16].

See [11, 12, 13, 21, 22, 28, 38] for other properties of the set $\mathcal{G}$ for the problem (4.1.7).

We now turn to problem (4.1.1), and we recall the following explosion criterion, see [8, Proposition B.1, p. 447].

**Proposition 4.1.1** ( [8, Proposition B.1, p. 447])**.** *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Let $g \in C^1(\mathbb{R}, \mathbb{R})$ satisfy $g(0) = 0$,*

$$s^2 g'(s) \geq (1 + \epsilon) s g(s), \tag{4.1.8}$$

*and*

$$|g(s)| \leq C(1 + |s|^{\beta}), \tag{4.1.9}$$

*for all* $s \in \mathbb{R}$, *where* $\epsilon > 0$ *and* $1 \leq \beta < \frac{N+2}{N-2}$. *Let* $\psi \in C_0(\Omega)$ *be a solution of the equation*

$$\begin{cases} -\Delta\psi = g(\psi), \\ \psi_{|\partial\Omega} = 0. \end{cases} \tag{4.1.10}$$

*Let* $u_0 \in C_0(\Omega)$ *and let* $u \in C\left([0, T_{u_0}); C_0(\Omega)\right)$ *be the maximal solution of*

$$\begin{cases} u_t = \Delta u + g(u), \\ u_{|\partial\Omega} = 0, \end{cases} \tag{4.1.11}$$

*with the initial condition* $u(0) = u_0$. *If* $\psi^+ \neq 0$ *and* $u_0 \geq \psi$, $u_0 \neq \psi$, *then* $u$ *blows up in finite time. Similarly if* $\psi^- \neq 0$ *and* $u_0 \leq \psi$, $u_0 \neq \psi$, *then* $u$ *blows up in finite time.*

**Remark 4.1.1.** Note that if $1 < q < p < p_S$, then $g(s) = |s|^{p-1}s - |s|^{q-1}s$ satisfies (4.1.8) with $\epsilon = q - 1$ and (4.1.9) with C sufficiently large and $\beta = p$.

It is immediate that if $h$ is a positive solution of (4.1.6) with $1 < q < p < p_S$, and if $u$ is the solution of (4.1.1) with initial value $u(0) = \lambda h$, then for $0 < \lambda \leq 1$, $u$ is global (by the comparison principle) and if $\lambda > 1$, then $u$ blows up in finite time (by Proposition 4.1.1).

The question remains as to whether or not the result in [8], cited above, concerning sign-changing solutions to (4.1.7) also carries over to sign-changing solutions of (4.1.1).

The point of view in this chapter is to fix a value of $q$ with

$$1 < q < p_S, \tag{4.1.12}$$

and then consider all $p$ with

$$q < p < p_S. \tag{4.1.13}$$

In fact we will ultimately consider what happens as $p \to p_S$. The main purpose of this chapter is to establish the following result.

**Theorem 4.1.1.** *Assume (4.1.2)-(4.1.3). Given* $1 < q < p_S = \frac{N+2}{N-2}$. *It follows that there exists* $1 < q < \underline{p} < p_S$ *with the following property. If* $\underline{p} < p < p_S$ *and if* $h_p \in C_0(\Omega)$ *is a radially symmetric stationary solution of (4.1.1) which takes both positive and negative values, then there exist* $0 < \underline{\lambda} < 1 < \overline{\lambda}$ *such that if* $\underline{\lambda} < \lambda < \overline{\lambda}$ *and* $\lambda \neq 1$, *then the classical solution of (4.1.1) with the initial condition* $u(0) = \lambda h$ *blows up in finite time.*

The first observation is that there does exist a radially symmetric, sign-changing stationary solution of (4.1.1), since the nonlinearity $|s|^{p-1}s - |s|^{q-1}s$ satisfies the hypothesis of [32, Theorem 2 , p. 376]. More precisely, if we consider the problem:

$$\begin{cases} h'' + \frac{N-1}{r}h' + |h|^{p-1}h - |h|^{q-1}h = 0, \\ h(0) = a > 0, \quad h'(0) = 0. \end{cases} \tag{4.1.14}$$

It is well-known by [32] that (4.1.14) admits a unique solution $h \in C^2([0,\infty),\mathbb{R})$, which we denote sometimes by $h_p(r,a)$ to emphasize the dependence on $a$. Recall that we are fixing a value of $q$ satisfying (4.1.12) and letting $p$ vary in the interval (4.1.13). Under these conditions, by Theorem 2 in [32] for all integer $m \geq 0$, there exists $a_{p,m}$ such that

a) $h_p(1, a_{p,m}) = 0$,

b) $h_p(r, a_{p,m})$ has precisely $m$ zeros in $(0,1)$.

In particular, $h_p(\cdot, a_{p,m})$, considered as a function on $\Omega = B_1$, is a radially symmetric solution of (4.1.6) which changes sign precisely $m$ times.

Now, let $h_p$ be any nontrivial solution of (4.1.6) and consider the linearized operator $F_p$ on $L^2(\Omega)$ defined by

$$\begin{cases} D(F_p) = H^2(\Omega) \cap H_0^1(\Omega), \\ F_p u = -\Delta u - \left(p|h_p|^{p-1} - q|h_p|^{q-1}\right) u, \qquad u \in D(F_p). \end{cases} \tag{4.1.15}$$

We recall the following result from [9].

**Theorem 4.1.2** ([9, Corollary 2.5, p. 18])**.** *Let $h_p \in C_0(\Omega)$ be a sign-changing solution of (4.1.6). Let $\varphi_p$ be a positive eigenvector of the self-adjoint operator $F_p$ given by (4.1.15), corresponding to the first eigenvalue. Suppose that*

$$\int_\Omega h_p \varphi_p \neq 0.$$

*It follows that there exists $\epsilon > 0$ such that if $0 < |1-\lambda| < \epsilon$, then the solution of (4.1.1) with the initial value $u_0 = \lambda h_p$ blows up in finite time.*

To prove Theorem 4.1.1, it thus suffices to establish the following.

**Theorem 4.1.3.** *Assume (4.1.2)-(4.1.3). Given $1 < q < p_S = \frac{N+2}{N-2}$. It follows that there exists $1 < q < \underline{p} < p_S$ with the following property. If $\underline{p} < p < p_S$ and if $h_p \in C_0(\Omega)$ is a radially symmetric stationary solution of (4.1.1) which takes both positive and negative values, then*

$$\int_\Omega h_p \varphi_p \neq 0.$$

*There $\varphi_p$ is a positive eigenvector of the self-adjoint operator $F_p$ given by (4.1.15), corresponding to the first eigenvalue.*

The proof of Theorem 4.1.3 is based on rescaling argument. Contrary to the case of single power nonlinearity, a rescaled function $v_p$ defined by (4.2.19) below in terms of $h_p$, where $h_p$ is a radially symmetric stationary solution of (4.1.1) doesn't satisfy the same differential equation satisfied by $h_p$, which make the situation more difficult. Also, unlike the case of the single power nonlinearity, there exist some solutions $v_p(r)$ of the problem (4.2.20) below which do not tend to zero as $r \to \infty$.

The rest of the chapter is devoted to proving Theorem 4.1.3, which as already noted, implies Theorem 4.1.1 when combined with Theorem 4.1.2. Our basic approach follows that in [8]. However because of the differences just noted between the single power and the two power cases, many of the arguments in [8] do not immediately apply for the current situation.

**Remark 4.1.2.** The results in this chapter are equally valid for

$$u_t = \Delta u + |u|^{p-1}u - c|u|^{q-1}u,$$

for any $c > 0$. The case where $c < 0$ is not as clear, since in that case, the proof of Proposition 4.2.1 below is no longer valid.

## 4.2 Stationary solutions

The proof of Theorem 4.1.3 exploits strongly the radial symmetry of the stationary solutions. By abuse of notation we will use the same letter, for example $h$, to denote a radially symmetric function $h : \mathbb{R}^N \to \mathbb{R}$, and the corresponding function $h : [0, \infty) \to \mathbb{R}$ such that, $h(x) = h(|x|)$, $\forall x \in \mathbb{R}^N$. Throughout this chapter, we will use this convention without further comment.

Any radially symmetric solution $h_p \in C_0(\Omega)$ of (4.1.6) satisfies the ODE

$$\begin{cases} h_p'' + \frac{N-1}{r}h_p' + |h_p|^{p-1}h_p - |h_p|^{q-1}h_p = 0, \\ h_p'(0) = h_p(1) = 0. \end{cases} \tag{4.2.16}$$

Since $h_p \neq 0$, it follows by uniqueness for the ODE (4.2.16) that $h_p(0) \neq 0$. Therefore, since if $u$ satisfies (4.1.1) then $-u$ satisfies the same problem, it suffices to prove Theorem 4.1.3 under the additional assumption

$$h_p(0) > 0. \tag{4.2.17}$$

In the rest of this chapter we set

$$h_p(0) = a_p > 0.$$

Clearly $h_p(r) = h_p(r, a_p)$, where $h_p(., a_p)$ is the solution of (4.1.14) with $a = a_p$. We let $\lambda_p > 0$ be such that

$$\lambda_p^{\frac{2}{p-1}} = a_p, \tag{4.2.18}$$

also we define

$$v_p(r) = \lambda_p^{-\frac{2}{p-1}} h_p\left(\frac{r}{\lambda_p}, \lambda_p^{\frac{2}{p-1}}\right). \tag{4.2.19}$$

A simple calculation shows that $v_p$ satisfies

$$\begin{cases} v'' + \frac{N-1}{r}v' + |v|^{p-1}v - \lambda_p^{-\frac{2}{p-1}(p-q)}|v|^{q-1}v = 0, \\ v(0) = 1, \quad v'(0) = 0. \end{cases} \tag{4.2.20}$$

As such, $v_p$ may be considered as a function $[0, \infty) \to \mathbb{R}$. It is known by [32, Lemma 1, p. 371] that $a_p \geq 1$. In fact, if $0 < a_p \leq \left(\frac{p+1}{q+1}\right)^{\frac{1}{p-q}}$ then $h_p(r, a_p) > 0$ for all $r > 0$. Thus,

$$\lambda_p \geq 1. \tag{4.2.21}$$

We have also

$$v_p(\lambda_p) = 0. \tag{4.2.22}$$

**Proposition 4.2.1.** *Let $\lambda_p$ be defined in (4.2.18), then*

$$\lambda_p \underset{p\to p_S}{\longrightarrow} \infty. \tag{4.2.23}$$

*Proof.* Suppose to the contrary that $\lambda_p \nrightarrow \infty$ as $p \to p_S$. It follows that there exists a subsequence $(p_k)$ such that $p_k \underset{k\to\infty}{\longrightarrow} p_S$ and

$$\lambda_{p_k} \underset{k\to\infty}{\longrightarrow} \overline{\lambda}, \tag{4.2.24}$$

where $1 \le \overline{\lambda} < \infty$, by (4.2.21). By continuous dependence it follows that

$$v_{p_k} \underset{k\to\infty}{\longrightarrow} \overline{v}, \tag{4.2.25}$$

uniformly on all compact intervals $[0, M] \subset [0, \infty)$, where $\overline{v}$ satisfies

$$\begin{cases} \overline{v}'' + \frac{N-1}{r}\overline{v}' + |\overline{v}|^{p_S-1}\overline{v} - \overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)}|\overline{v}|^{q-1}\overline{v} = 0, \\ \overline{v}(0) = 1, \quad \overline{v}'(0) = 0. \end{cases} \tag{4.2.26}$$

It follows from (4.2.22), (4.2.24) and (4.2.25) that

$$\overline{v}(\overline{\lambda}) = 0. \tag{4.2.27}$$

And so $\overline{v}$ satisfies the equation

$$\begin{cases} -\Delta\overline{v} = |\overline{v}|^{p_S-1}\overline{v} - \overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)}|\overline{v}|^{q-1}\overline{v}, \\ \overline{v}_{|\partial B(0,\overline{\lambda})} = 0. \end{cases} \tag{4.2.28}$$

If we apply the Pohozaev identity as was done in [4, Remark 1.2, p. 442], and if we set $g(u) = |u|^{p_S-1}u - \overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)}|u|^{q-1}u$ and $G(u) = \frac{|u|^{p_S+1}}{p_S+1} - \overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)}\frac{|u|^{q+1}}{q+1}$, we obtain

$$\begin{aligned} \frac{2-N}{2}\int_{B(0,\overline{\lambda})} g(\overline{v})\overline{v} + N\int_{B(0,\overline{\lambda})} G(\overline{v}) &= \overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)}\left[\frac{N-2}{2} - \frac{N}{q+1}\right]\int_{B(0,\overline{\lambda})} |\overline{v}|^{q+1} \\ &= \frac{1}{2}\int_{\partial B(0,\overline{\lambda})} (x.\nu)\left(\frac{\partial\overline{v}}{\partial\nu}\right)^2 \ge 0. \end{aligned} \tag{4.2.29}$$

From (4.2.29), one can conclude that

$$0 \le \left(\frac{N-2}{2} - \frac{N}{q+1}\right)\overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)}\|\overline{v}\|^{q+1}_{L^{q+1}(B(0,\overline{\lambda}))}. \tag{4.2.30}$$

Since $q < p_S$ inequality (4.2.30) is possible only if $\overline{v} = 0$, which contradicts $\overline{v}(0) = 1$. □

Let now $w_p$ be the solution of

$$\begin{cases} w'' + \frac{N-1}{r}w' + |w|^{p-1}w = 0, \\ w(0) = 1, \quad w'(0) = 0. \end{cases} \tag{4.2.31}$$

It is well-known and easy to verify that $w_{p_S}$ given by

$$w_{p_S}(r) = \left(1 + \frac{1}{N(N-2)}r^2\right)^{-\frac{N-2}{2}} \tag{4.2.32}$$

is the solution of (4.2.31) with $p = p_S$.

**Proposition 4.2.2.** *Let $v_p$ be defined by (4.2.19) and $w_{p_S}$ by (4.2.32), then*

$$v_p \underset{p \to p_S}{\longrightarrow} w_{p_S}, \tag{4.2.33}$$

*uniformly on bounded sets of* $[0, \infty)$.

*Proof.* By Proposition 4.2.1 $\lambda_p \to \infty$ as $p \to p_S$, and so by continuous dependence we can conclude that

$$v_p \underset{p \to p_S}{\longrightarrow} w_{p_S},$$

uniformly on bounded sets of $[0, \infty)$. □

**Proposition 4.2.3.** *Given $1 < q < p_S$ and $0 < \eta < p_S - q$. There exist $M, C > 0$ such that for all $p \in [q + \eta, p_S)$ and $r \geq 0$,*

$$|v_p(r)| \leq M \quad \text{and} \quad |v_p'(r)| \leq C. \tag{4.2.34}$$

*Proof.* Let $1 < q < p_S$ and $0 < \eta < p_S - q$. Note first that by (4.2.20)

$$\left[\frac{1}{2}v_p'(r)^2 + \frac{1}{p+1}|v_p(r)|^{p+1} - \frac{1}{q+1}\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p(r)|^{q+1}\right]' = -\frac{N-1}{r}|v_p'(r)|^2, \tag{4.2.35}$$

so that

$$\begin{aligned} \frac{1}{2}v_p'(r)^2 + \frac{1}{p+1}|v_p(r)|^{p+1} - \frac{1}{q+1}\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p(r)|^{q+1} &\leq \frac{1}{p+1} - \frac{1}{q+1}\lambda_p^{-\frac{2}{p-1}(p-q)} \\ &\leq \frac{1}{p+1}. \end{aligned} \tag{4.2.36}$$

Now since $\lambda_p$ satisfies (4.2.21), it follows from (4.2.36) that

$$\frac{1}{p+1}|v_p(r)|^{p+1} - \frac{1}{q+1}|v_p(r)|^{q+1} \leq \frac{1}{p+1}. \tag{4.2.37}$$

Suppose by contradiction that, there exist $(p_n) \subset [q + \eta, p_S)$ and $(r_n) \subset [0, \infty)$ such that

$$|v_{p_n}(r_n)| \underset{n \to \infty}{\longrightarrow} \infty.$$

Since $(p_n)$ is bounded we can suppose that $p_n \to p_* \in [q + \eta, p_S]$, we apply now inequality (4.2.37), which we note as

$$|v_p(r)|^{p+1}\left(\frac{1}{p+1} - \frac{1}{q+1}|v_p(r)|^{q-p}\right) \leq \frac{1}{p+1},$$

with $p = p_n$, $r = r_n$. By letting $n \to \infty$, it follows that

$$\infty \leq \frac{1}{p_* + 1},$$

which is absurd. It follows so that there exists $M > 0$, such that for all $p \in [q + \eta, p_S)$ and $r \geq 0$,

$$|v_p(r)| \leq M. \tag{4.2.38}$$

We turn now to prove the second assertion. It follows from (4.2.36), $\lambda_p \geq 1$, (4.2.38) and $p > q$ that

$$\begin{aligned} \frac{1}{2}v_p'(r)^2 &\leq \frac{1}{p+1} + \frac{1}{q+1}|v_p(r)|^{q+1} \\ &\leq \frac{1}{q+1} + \frac{1}{q+1}M^{q+1}, \quad \forall p \in [q+\eta, p_S), \quad \forall r \geq 0, \end{aligned}$$

so that

$$|v_p'(r)| \leq \sqrt{\frac{2}{q+1}}\sqrt{1+M^{q+1}}, \quad \forall p \in [q+\eta, p_S), \quad \forall r \geq 0.$$

□

The following lemma is one of the key points which differ from the calculations in [8]. Compare Lemma 3.3 in [8]. Indeed, Lemma 3.3 in [8] cannot be true in the present context since not all solutions $v_p$ of (4.2.20) tend to 0 as $r \to \infty$. We do obtain, however, a similar estimate, valid only for $r \leq \lambda_p$.

**Lemma 4.2.1.** *Given $1 < q < p_S$ and $0 < \eta < p_S - q$. There exists a constant $\gamma = \gamma(N, q)$ such that*

$$\frac{1}{2}|v_p'(r)|^2 + \frac{1}{p+1}|v_p(r)|^{p+1} \leq \gamma\left[\frac{1}{r+1} + \frac{1}{(r+1)^{\frac{2}{p_S-1}\eta}}\right], \tag{4.2.39}$$

*for all $p \in [q+\eta, p_S)$ and for all $0 \leq r \leq \lambda_p$.*

*Proof.* Fix $1 < q < p_S$ and $0 < \eta < p_S - q$. Let $r$ be such that $1 \leq r \leq \lambda_p$ and $p \in [q+\eta, p_S)$. Define now

$$F(r) = \frac{1}{2}v_p'(r)^2 + \frac{1}{p+1}|v_p(r)|^{p+1} - \frac{1}{q+1}\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p(r)|^{q+1} + \frac{1}{r}v_p(r)v_p'(r). \tag{4.2.40}$$

It follows from (4.2.35) and (4.2.20) that

$$\begin{aligned} F'(r) &= -\frac{N-1}{r}v_p'(r)^2 - \frac{1}{r^2}v_p(r)v_p'(r) + \frac{1}{r}v_p'(r)^2 + \frac{1}{r}v_p(r)v_p''(r) \\ &= -\frac{N-2}{r}v_p'(r)^2 - \frac{1}{r^2}v_p(r)v_p'(r) + \frac{1}{r}v_p(r)v_p''(r) \\ &= -\frac{N-2}{r}v_p'(r)^2 - \frac{1}{r^2}v_p(r)v_p'(r) + \\ &\qquad \frac{1}{r}v_p(r)\left[-\frac{N-1}{r}v_p'(r) - |v_p(r)|^{p-1}v_p(r) + \lambda_p^{-\frac{2}{p-1}(p-q)}|v_p(r)|^{q-1}v_p(r)\right]. \end{aligned}$$

From (4.2.34), (4.1.3), the fact that $1 \leq r \leq \lambda_p$, $1 < q < p$, Young's inequality (applied twice) and denoting $\alpha := \frac{p-q}{p+1} + \frac{2}{p-1}(p-q)$, one can find the estimate

$$
\begin{aligned}
F'(r) + \frac{1}{r}F(r) &= -\frac{2N-5}{2r}v_p'(r)^2 - \frac{p}{(p+1)r}|v_p(r)|^{p+1} - \frac{N-1}{r^2}v_p(r)v_p'(r) \\
&\quad + \lambda_p^{-\frac{2}{p-1}(p-q)}\frac{q}{(q+1)r}|v_p(r)|^{q+1} \\
&\le -\frac{2N-5}{2r}v_p'(r)^2 - \frac{p}{(p+1)r}|v_p(r)|^{p+1} + \frac{1}{2}\left[\frac{(N-1)^2}{r^3}v_p(r)^2 + \frac{1}{r}v_p'(r)^2\right] \\
&\quad + \frac{q}{q+1}|v_p(r)|^{q+1}r^{-\frac{q+1}{p+1}}r^{-\alpha} \\
&\le \frac{(N-1)^2}{2r^3}v_p(r)^2 - \frac{p}{(p+1)r}|v_p(r)|^{p+1} \\
&\quad + \frac{q}{(p+1)r}|v_p(r)|^{p+1} + \frac{q(p-q)}{(q+1)(p+1)}r^{-\alpha\frac{p+1}{p-q}} \\
&\le \frac{(N-1)^2}{2r^3}M^2 + \frac{q(p_S-q)}{(q+1)^2}r^{-\alpha\frac{p+1}{p-q}}.
\end{aligned}
$$

Now since $\alpha\frac{p+1}{p-q} = 1 + 2\frac{p+1}{p-1} \ge 3$, we obtain that for $1 \le r \le \lambda_p$

$$F'(r) + \frac{1}{r}F(r) \le Ar^{-3}.$$

One can conclude now for all $s \in [1, \lambda_p]$, for all $p \in [q+\eta, p_S)$ that

$$\frac{d}{ds}\left(sF(s)\right) = sF'(s) + F(s) \le As^{-2}. \tag{4.2.41}$$

Integration of (4.2.41) on $[1, r]$ gives

$$rF(r) - F(1) \le A\left(-\frac{1}{r} + 1\right).$$

We can affirm for $r \in [1, \lambda_p]$ that

$$F(r) \le B\frac{1}{r}. \tag{4.2.42}$$

Using also (4.2.40), (4.2.42), (4.2.34), $p \in [q+\eta, p_S)$ and the fact that $1 \le r \le \lambda_p$, it follows that

$$
\begin{aligned}
\frac{1}{2}v_p'(r)^2 + \frac{1}{p+1}|v_p(r)|^{p+1} &\le \frac{1}{q+1}\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p(r)|^{q+1} - \frac{1}{r}v_p(r)v_p'(r) + B\frac{1}{r} \\
&\le \frac{1}{q+1}\frac{1}{r^{\frac{2}{p_S-1}\eta}}M^{q+1} + \frac{M.C}{r} + B\frac{1}{r}.
\end{aligned}
$$

Finally, using (4.2.34) one can conclude that there exists $\gamma > 0$ such that

$$\frac{1}{2}v_p'(r)^2 + \frac{1}{p+1}|v_p(r)|^{p+1} \le \gamma\left[\frac{1}{r+1} + \frac{1}{(r+1)^{\frac{2}{p_S-1}\eta}}\right],$$

for all $0 \le r \le \lambda_p$. □

We set

$$\widetilde{v}_p(r) = \begin{cases} v_p(r) & \text{if } \ 0 \le r \le \lambda_p, \\ 0 & \text{if } \ r > \lambda_p. \end{cases} \tag{4.2.43}$$

**Corollary 4.2.1.** *Given* $1 < q < p_S$ *and* $0 < \eta < p_S - q$*. There exists a decreasing function* $j : [0,\infty) \to [0,\infty)$ *satisfying* $j(r) \underset{r\to\infty}{\longrightarrow} 0$ *such that*

$$|\widetilde{v}_p(r)| \le j(r), \quad \forall r \ge 0, \quad \forall p \in [q+\eta, p_S). \tag{4.2.44}$$

**Proposition 4.2.4.** $\|\widetilde{v}_p - w_{p_S}\|_{L^\infty(\mathbb{R}^N)} \to 0,$ *as* $p \to p_S$*.*

*Proof.* Fix $1 < q < p_S$, $0 < \eta < p_S - q$ and $R \ge 0$. Let $p \in [q+\eta, p_S)$, on the one hand it follows from (4.2.44) and (4.2.32) that

$$\begin{aligned} |\widetilde{v}_p(r) - w_{p_S}(r)| &\le |\widetilde{v}_p(r)| + w_{p_S}(r) \\ &\le j(r) + w_{p_S}(r) \\ &\le j(R) + w_{p_S}(R), \quad \forall r \ge R. \end{aligned}$$

It follows that

$$\sup_{r\ge R} |\widetilde{v}_p(r) - w_{p_S}(r)| \le j(R) + w_{p_S}(R) \underset{R\to\infty}{\longrightarrow} 0.$$

Thus, there exists $R_0 \ge 0$ such that

$$\sup_{r\ge R_0} |\widetilde{v}_p(r) - w_{p_S}(r)| \le \frac{\varepsilon}{2}. \tag{4.2.45}$$

On the other hand, since $\lambda_p \underset{p\to p_S}{\longrightarrow} \infty$, by choosing $p_0$ sufficiently close to $p_S$, we can assume that $R_0 \le \lambda_p$ for $p_0 \le p < p_S$. It follows from (4.2.33 ) that there exists $p_0 \le \overline{p} < p_S$ such that if $\overline{p} < p < p_S$ then

$$\sup_{r\in[0,R_0]} |\widetilde{v}_p(r) - w_{p_S}(r)| = \sup_{r\in[0,R_0]} |v_p(r) - w_{p_S}(r)| \le \frac{\varepsilon}{2}. \tag{4.2.46}$$

One can conclude from (4.2.45) and(4.2.46). □

## 4.3 The linearized operator

We consider now the self-adjoint operator $F_p$ defined on $L^2(\Omega)$ by

$$\begin{cases} D(F_p) = H^2(\Omega) \cap H_0^1(\Omega), \\ F_p u = -\Delta u - \left(p|h_p|^{p-1} - q|h_p|^{q-1}\right) u, \quad \forall u \in D(H_p). \end{cases} \tag{4.3.47}$$

We denote by

$$\theta_p = \theta_p\left(F_p\right), \tag{4.3.48}$$

its first eigenvalue and by $\varphi_p$ the corresponding eigenvector, i.e.

$$F_p\varphi_p = -\Delta\varphi_p - (p|h_p|^{p-1} - q|h_p|^{q-1})\varphi_p = \theta_p\varphi_p, \tag{4.3.49}$$

where we require

$$\varphi_p > 0, \quad \|\varphi_p\|_{L^2(\Omega)} = 1. \tag{4.3.50}$$

Since $\varphi_p$ is radially symmetric, it satisfies the ODE

$$\varphi_p'' + \frac{N-1}{r}\varphi_p' + \left(p|h_p|^{p-1} - q|h_p|^{q-1}\right)\varphi_p + \theta_p\varphi_p = 0. \tag{4.3.51}$$

In order to transform the operator $F_p$ into another operator we introduce $l_p \in \mathbb{R}$ and $\psi_p$, a positive, spherically symmetric function on $\Omega_p$ defined by

$$\theta_p = \lambda_p^2 l_p, \quad \varphi_p(x) = \lambda_p^{\frac{N}{2}}\psi_p(\lambda_p x), \tag{4.3.52}$$

where

$$\Omega_p = B(0, \lambda_p). \tag{4.3.53}$$

It follows from (4.3.51), (4.2.19) and (4.3.52) that $\psi_p$ satisfies the equation

$$\begin{cases} -\Delta\psi_p - \left[p|v_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p|^{q-1}\right]\psi_p = l_p\psi_p \quad \text{in } \Omega_p, \\ \psi_p = 0 \quad \text{on } \partial\Omega_p, \end{cases} \tag{4.3.54}$$

and that

$$\int_\Omega h_p\varphi_p = \lambda_p^{\frac{2}{p-1}-\frac{N}{2}}\int_{\Omega_p} v_p\psi_p, \tag{4.3.55}$$

and

$$\psi_p > 0, \quad \|\psi_p\|_{L^2(\Omega_p)} = 1. \tag{4.3.56}$$

We have also that $l_p$ is the first eigenvalue associated to the eigenvector $\psi_p$ of the self-adjoint operator $L_p$ defined on $L^2(\Omega_p)$ by

$$\begin{cases} D(L_p) = H^2(\Omega_p) \cap H_0^1(\Omega_p), \\ L_p u = -\Delta u - \left[p|v_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p|^{q-1}\right]u, \quad \forall u \in D(L_p). \end{cases} \tag{4.3.57}$$

Given $0 < p < p_S$, we set

$$J_p(w) = \int_{\Omega_p}|\nabla w|^2 - \int_{\Omega_p}\left[p|v_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p|^{q-1}\right]w^2, \tag{4.3.58}$$

for all $w \in H_0^1(\Omega_p)$, so that

$$l_p = \inf\left\{J_p(u), u \in H_0^1(\Omega_p), \|u\|_{L^2(\Omega_p)} = 1\right\}. \tag{4.3.59}$$

Also we define the self-adjoint operator $L_*$ on $L^2(\mathbb{R}^N)$ by

$$\begin{cases} D(L_*) = H^2(\mathbb{R}^N), \\ L_* u = -\Delta u - p_S w_{p_S}^{p_S-1}u, \quad \forall u \in D(L_*), \end{cases} \tag{4.3.60}$$

where $w_{p_S}$ is given by (4.2.32). We set

$$\lambda_* = \inf\left\{J_*(u), u \in H^1(\mathbb{R}^N), \|u\|_{L^2(\mathbb{R}^N)} = 1\right\}, \tag{4.3.61}$$

where

$$J_*(w) = \int_{\mathbb{R}^N} |\nabla w|^2 - p_S \int_{\mathbb{R}^N} w_{p_S}^{p_S-1} w^2, \tag{4.3.62}$$

for all $w \in H^1(\mathbb{R}^N)$. We recall now the following proposition from [8].

**Proposition 4.3.1** ( [8, Proposition 3.4, p. 439]). *If $L_*$ is defined by (4.3.60) and $\lambda_*$ is defined by (4.3.61), then the following properties hold.*

*(i) $\lambda_* < 0$ and $\lambda_*$ is an eigenvalue of $L_*$.*

*(ii) There exists a unique eigenvector $\psi_*$ of $L_*$ corresponding to the eigenvalue $\lambda_*$ which is positive, radially decreasing with $\|\psi_*\|_{L^2(\mathbb{R}^N)} = 1$.*

*(iii) If $(u_n)_{n\geq 1} \subset H^1(\mathbb{R}^N)$ is a minimizing sequence of (4.3.61) and $u_n \geq 0$, then $u_n \to \psi_*$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$.*

We set

$$\widetilde{\psi}_p(x) = \begin{cases} \psi_p(x) & \text{if } 0 \leq |x| < \lambda_p, \\ 0 & \text{if } |x| \geq \lambda_p, \end{cases} \tag{4.3.63}$$

for all $1 < p < p_S$, so that

$$\widetilde{\psi}_p \in H^1(\mathbb{R}^N), \quad \|\widetilde{\psi}_p\|_{L^2(\mathbb{R}^N)} = 1, \quad \widetilde{\psi}_p \geq 0. \tag{4.3.64}$$

**Lemma 4.3.1.** *Let $\psi \in H^1(\mathbb{R}^N)$ such that $\|\psi\|_{L^2(\mathbb{R}^N)} = 1$. Consider a smooth radial cut-off function $\eta : \mathbb{R}^N \to [0,1]$ such that $\eta(r) = 1$ for $r \leq \frac{1}{2}$ and $\eta(r) = 0$ for $r \geq 1$. Set*

$$k_\lambda(r) = \eta\left(\frac{r}{\lambda}\right)\psi(r), \tag{4.3.65}$$

*and*

$$u_\lambda = \frac{k_\lambda}{\|k_\lambda\|_{L^2(\mathbb{R}^N)}}. \tag{4.3.66}$$

*Then $u_\lambda \in H_0^1(\Omega_\lambda)$ and*

$$\|u_\lambda - \psi\|_{H^1(\mathbb{R}^N)} \underset{\lambda\to\infty}{\longrightarrow} 0. \tag{4.3.67}$$

*There $\Omega_\lambda = B(0,\lambda)$.*

*Proof.* This follows by standard arguments, using the observation that $\|k_\lambda\|_{L^2(\mathbb{R}^N)} \underset{\lambda\to\infty}{\longrightarrow} 1$. □

**Lemma 4.3.2.** *Let $l_p$ be defined by (4.3.52), then*

$$l_p \to \lambda_* \quad \text{as} \quad p \to p_S.$$

*Proof.* We first use $\widetilde{\psi}_p$ as a test function in (4.3.61). It follows from (4.3.64) that

$$\begin{aligned}\lambda_* \leq J_*(\widetilde{\psi}_p) &= J_p(\widetilde{\psi}_p) + \int_{\mathbb{R}^N} \left[p|\widetilde{v}_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\widetilde{v}_p|^{q-1} - p_S w_{p_S}^{p_S-1}\right]\widetilde{\psi}_p^2 \\ &= l_p + \int_{\mathbb{R}^N} \left[p|\widetilde{v}_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\widetilde{v}_p|^{q-1} - p_S w_{p_S}^{p_S-1}\right]\widetilde{\psi}_p^2 \\ &= l_p + \int_{\mathbb{R}^N} \left[p|\widetilde{v}_p|^{p-1} - p_S w_{p_S}^{p_S-1}\right]\widetilde{\psi}_p^2 - q\lambda_p^{-\frac{2}{p-1}(p-q)} \int_{\mathbb{R}^N} |\widetilde{v}_p|^{q-1}\widetilde{\psi}_p^2. \qquad (4.3.68)\end{aligned}$$

It follows from (4.3.68), Proposition 4.2.3 and (4.3.64) that

$$\lambda_* - l_p \leq \left\|p|\widetilde{v}_p|^{p-1} - p_S w_{p_S}^{p_S-1}\right\|_{L^\infty(\mathbb{R}^N)} + qM^{q-1}\lambda_p^{-\frac{2}{p-1}(p-q)}, \quad \forall p \in [q+\eta, p_S). \qquad (4.3.69)$$

One can conclude now by applying Proposition 4.2.4 and Proposition 4.2.1 that

$$\limsup_{p\to p_S}(\lambda_* - l_p) \leq 0. \qquad (4.3.70)$$

Next, we would like to use $\psi_*$ as a test function in (4.3.59), but $\psi_* \notin H_0^1(\Omega_p)$. Thus, we need to approximate $\psi_*$ by a sequence in $H_0^1(\Omega_p)$. Consider a smooth radial cut-off function $\eta : \mathbb{R}^N \to [0,1]$ such that $\eta(r) = 1$ for $r \leq \frac{1}{2}$ and $\eta(r) = 0$ for $r \geq 1$. Setting

$$k_p(r) = \eta\left(\frac{r}{\lambda_p}\right)\psi_*(r), \qquad (4.3.71)$$

and

$$u_p = \frac{k_p}{\|k_p\|_{L^2(\mathbb{R}^N)}}, \qquad (4.3.72)$$

it follows from Lemma 4.3.1 (since $\lambda_p \to \infty$ as $p \to p_S$) that

$$\|u_p - \psi_*\|_{H^1(\mathbb{R}^N)} \underset{p\to p_S}{\longrightarrow} 0. \qquad (4.3.73)$$

Moreover $u_p \in H_0^1(\Omega_p)$, so that

$$l_p \leq J_p(u_p) = \lambda_* - J_*(\psi_*) + J_*(u_p) - J_*(u_p) + J_p(u_p). \qquad (4.3.74)$$

On the one hand we have by Proposition 4.2.4 and Proposition 4.2.1 that

$$\begin{aligned}|J_*(u_p) - J_p(u_p)| &= \left|\int_{\mathbb{R}^N} \left[p|\widetilde{v}_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\widetilde{v}_p|^{q-1} - p_S w_{p_S}^{p_S-1}\right]{u_p}^2\right| \\ &\leq \int_{\mathbb{R}^N} \left|p|\widetilde{v}_p|^{p-1} - p_S w_{p_S}^{p_S-1}\right|{u_p}^2 + q\lambda_p^{-\frac{2}{p-1}(p-q)}\int_{\mathbb{R}^N} |\widetilde{v}_p|^{q-1}{u_p}^2 \\ &\leq \left\|p|\widetilde{v}_p|^{p-1} - p_S w_{p_S}^{p_S-1}\right\|_{L^\infty(\mathbb{R}^N)} + qM^{q-1}\lambda_p^{-\frac{2}{p-1}(p-q)} \underset{p\to p_S}{\longrightarrow} 0. \qquad (4.3.75)\end{aligned}$$

On the other hand, using the fact that $|J_*(\psi_*) - J_*(u_p)| \leq \left|\|\nabla u_p\|^2_{L^2(\mathbb{R}^N)} - \|\nabla \psi_*\|^2_{L^2(\mathbb{R}^N)}\right| + p_S \int_{\mathbb{R}^N} |u_p^2 - \psi_*^2|$, it easily follows from (4.3.73) and the dominated convergence theorem that

$$|J_*(\psi_*) - J_*(u_p)| \underset{p\to p_S}{\longrightarrow} 0. \qquad (4.3.76)$$

We deduce from (4.3.75) and (4.3.76) that

$$-\liminf_{p\to p_S}(\lambda_* - l_p) = \limsup_{p\to p_S}(l_p - \lambda_*) \le 0. \tag{4.3.77}$$

We can confirm so by (4.3.70) and (4.3.77) that

$$\liminf_{p\to p_S}(\lambda_* - l_p) = \limsup_{p\to p_S}(\lambda_* - l_p) = \lim_{p\to p_S}(\lambda_* - l_p) = 0.$$

The result follows now. □

**Lemma 4.3.3.** *Given $q \in (1, p_S)$ and $\eta \in (0, p_S - q)$. There exists $C > 0$ such that*

$$|\widetilde{\psi}_p(r)| + |\psi_*(r)| \le C\frac{1}{r^{\frac{N-1}{2}}} \le C, \tag{4.3.78}$$

*for all $r \ge 1$ and $q + \eta \le p < p_S$.*

*Proof.* Fix $q \in (1, p_S)$ and $\eta \in (0, p_S - q)$. We affirm first that

$$l_p < 0 \quad \text{for all} \quad p \in (q, p_S). \tag{4.3.79}$$

In fact, since $l_p$ satisfies (4.3.52), it suffices to prove that $\theta_p < 0$. We have on the one hand since $p > q$

$$\begin{aligned}\left(\int_{\Omega_p} h_p^2\right)\theta_p &\le \int_{\Omega_p}|\nabla h_p|^2 - \int_{\Omega_p}(p|h_p|^{p-1} - q|h_p|^{q-1})h_p^2 \\ &\le \int_{\Omega_p}|\nabla h_p|^2 - q\int_{\Omega_p}(|h_p|^{p+1} - |h_p|^{q+1}).\end{aligned} \tag{4.3.80}$$

On the other hand since $h_p$ satisfies (4.2.16) it follows that

$$\int_{\Omega_p}|\nabla h_p|^2 = \int_{\Omega_p}\left(|h_p|^{p+1} - |h_p|^{q+1}\right). \tag{4.3.81}$$

It follows from (4.3.80) and (4.3.81) since $q > 1$ that $\theta_p < 0$.

We complete now our proof. Since $l_p < 0$, we deduce from (4.3.59) and Proposition 4.2.3 that

$$\|\nabla\widetilde{\psi}_p\|^2_{L^2(\mathbb{R}^N)} \le p_S(M^{q-1} + M^{p_S-1}), \quad \forall p \in [q+\eta, p_S). \tag{4.3.82}$$

By (4.3.64), (4.3.82) and Strauss' radial lemma [46] that

$$|\widetilde{\psi}_p(r)| \le c\sqrt{1 + p_S\left(M^{q-1} + M^{p_S-1}\right)}\frac{1}{r^{\frac{N-1}{2}}}, \tag{4.3.83}$$

for all $r \ge 1$.

A similar argument applies to $\psi_*$ which completes the proof. □

**Lemma 4.3.4.** *Given $q \in (1, p_S)$ and $\eta \in (0, p_S - q)$. There exist $R$, $C > 0$, $\theta > 0$ and $q + \eta \leq p_0 < p_S$ such that*

$$|\widetilde{\psi}_p(r)| + |\psi_*(r)| \leq Ce^{-\theta r}, \tag{4.3.84}$$

*for all $r \geq R$ and $p_0 \leq p < p_S$.*

*Proof.* Fix $q \in (1, p_S)$ and $\eta \in (0, p_S - q)$. We start first by showing that there exist $R$, $C > 0$, $\theta > 0$ and $q + \eta \leq p_0 < p_S$ such that

$$|\psi_p(r)| \leq Ce^{-\theta r}, \tag{4.3.85}$$

for all $r \geq R$ (with $r \leq \lambda_p$) and $p_0 \leq p < p_S$. It follows from (4.3.54) that $\psi_p$ satisfies

$$-\psi_p''(r) - \frac{N-1}{r}\psi_p'(r) - \left\{ \left[ p|\widetilde{v}_p(r)|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\widetilde{v}_p(r)|^{q-1} \right] + l_p \right\} \psi_p(r) = 0, \tag{4.3.86}$$

for all $0 \leq r < \lambda_p$. We would like to use a method of energy in equation (4.3.86), but the term $-\Big[p|\widetilde{v}_p(r)|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\widetilde{v}_p(r)|^{q-1}\Big] - l_p$ is difficult to handle so we may estimate it. On the one hand, since $|p|\widetilde{v}_p(r)|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\widetilde{v}_p(r)|^{q-1}| \leq L(r) \underset{r\to\infty}{\longrightarrow} 0$ by Corollary 4.2.1 and the fact that $p \in [q + \eta, p_S)$, it follows that there exists $R > 0$, such that for all $r \geq R$

$$-\left[ p|\widetilde{v}_p(r)|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\widetilde{v}_p(r)|^{q-1} \right] \geq \frac{\lambda_*}{4}. \tag{4.3.87}$$

On the other hand, since $-l_p \to -\lambda_*$ as $p \to p_S$ by Lemma 4.3.2, it follows that there exists $p_0 \in [q + \eta, p_S)$ such that for all $p_0 \leq p < p_S$

$$-l_p \geq -\frac{3}{4}\lambda_*. \tag{4.3.88}$$

Finally one can conclude from (4.3.87) and (4.3.88) that there exist $R > 0$ and $q + \eta \leq p_0 < p_S$ such that

$$-\left[ p|\widetilde{v}_p(r)|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\widetilde{v}_p(r)|^{q-1} \right] - l_p \geq -\frac{\lambda_*}{2} > 0, \tag{4.3.89}$$

for all $p_0 \leq p < p_S$ and all $r \geq R$. By choosing $p_0$ possibly larger, we also may assume that $\lambda_p > R$ for $p_0 \leq p < p_S$. Since $\psi_p \geq 0$, we deduce from (4.3.86) and (4.3.89) that

$$\psi_p'' + \frac{N-1}{r}\psi_p' \geq -\frac{\lambda_*}{2}\psi_p, \tag{4.3.90}$$

for all $R \leq r \leq \lambda_p$. We now claim that

$$\psi_p'(r) < 0, \tag{4.3.91}$$

for all $p_0 \leq p < p_S$ and all $R < r < \lambda_p$. We argue by contradiction and suppose that $\psi_p'(r_p) \geq 0$ for some $p_0 \leq p < p_S$ and some $R < r_p < \lambda_p$. Since $\psi_p(\lambda_p) = 0$, there exists $r_p \leq r_p' < \lambda_p$ such that $\psi_p'(r_p') = 0$ and $\psi_p''(r_p') \leq 0$. This is impossible by (4.3.90) since $\lambda* < 0$. Multiplying (4.3.90) by $\psi_p' < 0$, see that

$$\psi_p''\psi_p' + \frac{N-1}{r}\psi_p'\psi_p' \leq -\frac{\lambda_*}{2}\psi_p\psi_p',$$

which implies

$$\left(\psi_p'^2 + \frac{\lambda_*}{2}\psi_p^2\right)' \leq 0, \tag{4.3.92}$$

for $R \leq r \leq \lambda_p$. It follows from (4.3.92) that

$$\left[\psi_p'^2 + \frac{\lambda_*}{2}\psi_p^2\right](r) \geq \psi_p'(\lambda_p)^2 \geq 0,$$

for $R < r < \lambda_p$. Since $\psi_p > 0$ and $\psi_p' < 0$, we obtain that $\psi_p' + \sqrt{-\frac{\lambda_*}{2}}\psi_p \leq 0$ for $R < r < \lambda_p$, so that

$$\psi_p(r) \leq \psi_p(R)e^{\sqrt{-\frac{\lambda_*}{2}}R}e^{-\sqrt{-\frac{\lambda_*}{2}}r},$$

for $R < r < \lambda_p$. By choosing $R \geq 1$ we have $\psi_p(R) \leq C$ by Lemma 4.3.3 . The exponential decay follows. As remarked in [8], the proof for $\psi_*$ is similar. This completes the proof. □

**Lemma 4.3.5.** *$\widetilde{\psi}_p$ and $\psi_* \in L^1(\mathbb{R}^N)$. Moreover, $\|\widetilde{\psi}_p - \psi_*\|_{L^1(\mathbb{R}^N)} \to 0$ as $p \to p_S$.*

*Proof.* The proof is similar to the proof of Lemma 3.7 in [8]. □

*Proof of Theorem 4.1.3 .* Fix $q \in (1, p_S)$ and $0 < \eta < p_S - q$. Let $h_p \in C_0(\Omega)$ be a radially symmetric, sign-changing stationary solution of (4.1.1). Let $\varphi_p$ be the positive eigenvector normalized in $L^2(\Omega)$ of the self-adjoint operator $F_p$ given by (4.3.47), corresponding to the first eigenvalue. We have from Proposition 4.2.3

$$\begin{aligned}\left|\int_{\Omega_p} v_p\psi_p - \int_{\mathbb{R}^N} w_{p_S}\psi_*\right| &= \left|\int_{\mathbb{R}^N} \widetilde{v}_p\widetilde{\psi}_p - \int_{\mathbb{R}^N} w_{p_S}\psi_*\right| \\ &\leq \left|\int_{\mathbb{R}^N} \widetilde{v}_p\left(\widetilde{\psi}_p - \psi_*\right)\right| + \left|\int_{\mathbb{R}^N} \left(\widetilde{v}_p - w_{p_S}\right)\psi_*\right| \\ &\leq M\left\|\widetilde{\psi}_p - \psi_*\right\|_{L^1(\mathbb{R}^N)} + \|\psi_*\|_{L^1(\mathbb{R}^N)}\|\widetilde{v}_p - w_{p_S}\|_{L^\infty(\mathbb{R}^N)},\end{aligned}$$

$\forall p \in [q+\eta, p_S)$. It follows so by Lemma 4.3.5, Proposition 4.2.4 that

$$\int_\Omega v_p\varphi_p \underset{p\to p_S}{\longrightarrow} \int_{\mathbb{R}^N} w_{p_S}\psi_* > 0.$$

We can now conclude from (4.3.55)that there exists $1 < q < \underline{p} < p_S$ such that if $\underline{p} < p < p_S$, then

$$\int_\Omega h_p\varphi_p > 0.$$

This finishes the proof. □

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