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Arithmetic Statistics for Quaternion Algebras

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Foreword & Acknowledgements

The academic world, into which the PhD thesis stands as the first step, has appeared to me like an adventurous hike. It begins with the same excitement for the unknown, keeps fed by the same pleasure of discoveries, requires a strong and lasting commitment, but also hides long days of loneliness, losses and setbacks. No matter how near lies the aim nor how much have been achieved, the same endlessness of the final kilometers remains. One of the greatest challenges of this trail appears to me these days: the need to mourn the unreachable perfection and the desire to embrace the whole everything, learning to stop and finish. After all, every suffering the road could have unveiled quickly fades in front of the joy to reach the end and the strong motivation to sail for new horizons it brings.

Through more than four years of life and work, my days have been not only made of pure thoughts but also shared with many people. First and foremost I would like to acknowledge here their crucial role for me by some words, despite these remain far from what they deserve.

The appeal for this abstract adventure blossomed for about a decade, through lectures where the enthusiasm of my former teachers turned out to be a lighthouse for those long years, and a motivation to bring mathematics into life in my turn, by research and teaching, loving to discover and share. I would not have taken this path without the first fundamental steps I owe to Sébastien Moulin, Patrick Rauch, Philippe Patte, Alain Pommellet, Filippo Santambrogio, Régis de la Bretèche and Étienne Fouvry, whose advices and support led me to uncover this beautiful experience.

A PhD thesis is also a human adventure, which one of the faces, perhaps the harshest one, is loneliness. Its quiet heroes are those who everyday brought life and colors to offices and coffee rooms, those with whom I shared time and discussions, those who made it a living experience. I wish to sincerely stress how my workmates from Paris, Lausanne and Göttingen have been important to me, in particular Irène, Guillaume, Daniel, Tom, Ramon, Ian, Hunter, Andy, Dino and Jitendra whose comradeship has been considerable. The "heroic days" with Pierre and Antoine deserves a specific mention as a friendly and motivating way of study.

On the sea of formal conferences and lectures, rare possibilities of expression without pressure and tribunes for sharing what we learn and like exist, and I am happy to have taken part in the workgroup of Analytic Number Theory, organized by Étienne Fouvry then Lucile Devin in Paris, as well as the seminars organized by Harald Helfgott in Göttingen. They stayed during the four last years a great opportunity to breathe new air and renew curiosity. Those who bear the charge of organizing them serve in a crucial way the community and endure an always underestimated amount of work. While acknowledging the role of community, here should be the right place to thank the various interesting threads and sometimes exceptional contributions on MathOverflow, where the will of sharing and helping is praiseworthy. I am sincerely grateful to all of them, as well as to the participating folk for bringing them into life.

After some years of lone exploration, I discovered the joys and excitement of sharing research projects. Ian and Hunter were the first with whom a new joint adventure began in Lausanne. Both our common interests and our different cultures led to settle new exciting projects, nowadays pursued with Ian. I experienced the same enthusiasm during this last year in Göttingen, through passionate discussions with Andy and Jitendra where so many projects hatched. I would like to sincerely thank them for having made intense all those moments, both intellectually and humanly, for much more can be achieved together. I deep in my heart hope these collaborations and friendships will last for long.

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Beyond the too often closed thick doors of the academic world, friends turn to be a much stronger source of motivation than they could imagine. I never said it enough, and grasp this opportunity to underline how important they are to me, and how much they have taken part in shaping what I am today, for they are my closest friends: Adrien, Amiel, Bruno, Camille, Cécile Carrère, Cécile Houdard, Clément, Élie, Hadrien, Jill-Jênn, Justin, Laura, Luc, Marc, Maria, Michaël, Morgane, Paul Pegon, Paul Simon and Raphaël. I shared many peaceful moments and exciting discussions with Bénédicte with whom the meals in Göttingen acquired a deeper flavor; and the others rare moments of German evasion have been spent playing Go with the Göttingen club with Colin, Detlef, Gerd, Johann, Leon and Tim. The GICS and all its members have a specific status in this list, as the parallel adventure and challenging companion it has been to vulgarize science as well as climbing towards its summits.

This thesis would not have reached its term without what turned out to be a vital human presence through terrible months of sorrow and despair, when everything appeared tasteless and hopeless. Among those who shared this pain and helped me through, Luc Faucher and Farrell showed me a constant and benevolent support, leading me to begin to build a new home and life on what was no more than shattered ruins. Beyond them, many friends have shown a decisive empathy, and I dedicate without any doubt this thesis to the five heroes whose hands and hearts stayed opened and bore me during my worst days, and to whom I owe everything: Cécile, Hadrien, Justin, Luc and Maria.

A thesis also has a sneaky effect of colonizing one's life, and even if it was sometimes by will and passion, I kept in mind all along the years those who always shared part of this weight when I was at times overwhelmed, at others absent. I owe this quiet trust to my mother, father and brother for whom I never had enough time, but who I love.

At last, my thoughts fly away to Julie, the other half of our wonderful Judier, for her patience, strength and love.

in Princeton, March 1st, 2018.

To those who shone when there was only darkness,
Cécile, Hadrien, Justin, Luc & Maria.

Contents

List of Results and Conjectures

All the original results stated in the present thesis are labeled by letters instead of numbers. They are gathered in the following list for reference. The study of unitary groups and symplectic groups in Chapter 5 is a joint ongoing project with Ian Petrow, in particular the conjectural counting laws stated in conjectures E and F.

List of Abbreviations and Symbols

$\mathbf{A}$	ring of adeles of F	27
$\mathcal{A}(G)$, $\mathcal{A}(Q)$	universal family and truncated one	2, 4
$\mathcal{A}(\mathfrak{q}, \sigma_R, \delta, \Omega)$	subset of the universal family of fixed parameters	46
$A(\mathfrak{q}, \sigma_R, \delta, \Omega; \phi)$	counting measures restricted to $\mathcal{A}, \mathcal{B}(\mathfrak{q}, \sigma_R, \delta, \Omega)$	46
$\alpha_\pi(p)$	Satake parameters $\pi_\mathfrak{p}$	85
B	division quaternion algebra	30
$B(\mathfrak{q}, \sigma_R, \delta, \Omega; \phi)$	counting measures restricted to $\mathcal{B}, \mathcal{B}(\mathfrak{q}, \sigma_R, \delta, \Omega)$	47
$\mathcal{B}(\mathfrak{q}, \sigma_R, \delta, \Omega)$	subset of the universal family of fixed parameters	47
$\beta_\pi(p)$	Satake parameters $\pi_\mathfrak{p}$	85
C	main constant in the counting law	6
$c_\varepsilon(\pi)$	a priori alternative notion of conductor	2
$\mathfrak{c}(\pi_\mathfrak{p})$	multiplicative conductor	36
$c(\pi)$	analytic conductor	32
$\mathrm{d}x$, $\mathrm{d}g$	Haar measures on fields and groups	28
$\mathrm{d}\pi_v$, $\mathrm{d}\pi$	Plancherel measure on local and global duals	41
$\mathcal{D}$	set of discrete archimedean parameters	44
$D(\pi, \phi), D(\phi, Q)$	one-level density of $L(s, \pi)$, of $\mathcal{A}(Q)$	13
δ_F	constant $2(1 + [F : \mathbf{Q}])^{-1}$	4
$\underline{\delta}$	discrete archimedean parameters (δ, M)	44
$\epsilon_\rho^{\delta,\Omega}$	smoothing error in the spectral selection	53
ε_π	sign of the functional equation of $L(s, \pi)$	3
$\varepsilon_\mathfrak{d}$	selecting function for split non-archimedean places	50
$\partial_\rho B(\delta, \Omega)$	smoothing error in the identity term	61
F, F_v	number field and associated local fields	26
$F(\widehat{G_S})$	Sauvageot class of functions	39
$\mathfrak{f}(\pi_\mathfrak{p})$	additive conductor	36
$f_\rho^{\delta,\Omega,\phi}$	archimedean geometric selecting function	53
$\phi, \widehat{\phi}$	function on the Hecke algebra, its Fourier transform	41
φ_2	abbreviation for $\mathrm{id} \star \mu^2$	60

Chapter 1

On Arithmetic Statistics

Automorphic forms are central objects in modern number theory. Despite their ubiquity, they remain mysterious and their behavior is far from understood. Embedding them in wider families has a smoothing effect and leads to results on average: these are the aims of arithmetic statistics and motivates the recent interest towards automorphic forms in families. Among families, some are more natural and carry powerful results, and a particular emphasis has been granted to the universal family consisting of all the automorphic representations on a given group. This chapter is dedicated to present new results in this philosophy.

In the case of the universal family of quaternion algebras, the growth law of the truncated family with respect to a suitable notion of size is stated. Further statistics lie in the equidistribution result of the global universal family with respect to a geometrically significant measure. It leads to answering the Sato-Tate conjecture for this family, concerning the local measures. Finally, the distribution of low-lying zeros of L-functions is explored, and the density conjecture partially verified.

Other reductive groups on which these arithmetic statistics problems make sense are mentioned as an invitation to investigate universal families on a larger scale. A glimpse towards such a program of research is provided by exploring the counting law for some unitary and symplectic groups of low ranks.

Contents

1.1 Automorphic world

1.1.1 Universal family of quaternion algebras

Automorphic forms and their L-functions appear to be the central objects of modern number theory since the fertile conjectures formulated by Langlands [80] and the powerful applications of their many avatars, such as elliptic curves [35] or modular forms [108]. In spite of their ubiquity, they remain quite mysterious objects. Embedding them in larger families has a smoothing effect: singular behavior and unreachable objects lose their weights and this allows to establish regularity results concerning the family as well as results on average, supplying to the lack of pointwise knowledge on every particular object information on typical ones. This is the spirit of arithmetic statistics.

The recent years unleashed a wide enthusiasm toward the study of families of automorphic forms and their associated L-functions. Understanding what makes a family relevant for this philosophy is a critical issue. General attempts to define a suitable notion of family of automorphic forms have been made in the recent years [109, 110, 77], with a particular emphasis towards the universal family of a group, consisting of all its cuspidal automorphic representations.

Given such a family $\mathcal{F}$, the first natural question relative to it as a whole concerns its size. For infinite families, a truncation to a finite set makes sense of the problem. Assuming $\mathcal{F}_Q$ to be a finite subfamily of $\mathcal{F}$ indexed by a positive parameter Q, such that $\mathcal{F}_Q$ grow sto $\mathcal{F}$ when Q goes to infinity, the question is to determine the asymptotic behavior for the size of $\mathcal{F}_Q$.

The general linear group is the fundamental groundwork for automorphic representations, and the Langlands philosophy considers it as an ambiant group for more general reductive groups. The case of GL(2) is the first non-commutative one, yet far from totally understood. One way to explore some of its features is to consider its inner forms: they are the groups of units of quaternion algebras, groundwork of the present thesis. Let a quaternion algebra B over a number field F and introduce $G = PB^\times = Z(B)\backslash B^\times$. Consider $\mathcal{A}(G)$ the universal family of G, that is the set of all automorphic infinite-dimensional representations of in $L^2(G(F)\backslash G(\mathbf{A}))$. Following Sarnak [109], a deep understanding of $\mathcal{A}(G)$ is of fundamental importance in the theory of automorphic forms.

As an analogy and a guide for the methods, turn for a moment to a more usual setting: the one of general linear groups. The universal family $\mathcal{A}(G)$ embeds, via the Jacquet-Langlands correspondence, as a subfamily of the universal family $\mathcal{A}(\mathrm{PGL}(2))$, composed of all the cuspidal automorphic representations of PGL(2). In the latter context, even in the broader setting of cusp forms on general linear groups, Iwaniec and Sarnak [65] have defined a good notion of size, given by the analytic conductor. It is a positive real number $c(\pi)$ defined from the functional equation satisfied by the finite part L-function $L(s, \pi)$ associated to $\pi \in \mathcal{A}(\mathrm{PGL}(2))$, which takes the form

$$L(1-s,\pi) = \varepsilon_\pi X(s,\pi) L(s,\pi), \tag{1.1}$$

where ε_π is the root number of π. The completing factor $X(s,\pi)$ takes value ε_π at the central point $\frac{1}{2}$, and the additive analytic conductor is defined to be $c(\pi) = X'(\frac{1}{2},\pi)$ following the presentation of Conrey et al. [29]. Further considerations on the analytic conductor as well as discussions concerning its different avatars are carried on in Section 2.2. The function $X(s,\pi)$ involves the usual arithmetic conductor as well as archimedean gamma factors, so that the analytic conductor encapsulates the complexity of π. It allows to truncate the universal family of $\mathrm{PGL}(2)$, and hence the one of G, to a finite set [16]. The truncated universal family may then be introduced as

$$\mathcal{A}(Q) = \{\pi \in \mathcal{A}(G) \ : \ c(\pi) \leqslant Q\}, \qquad Q \geqslant 1. \tag{1.2}$$

The present work is a statistical exploration of this family in various aspects, such as asymptotic growth, equidistribution with respect to a geometric significant measure, behavior of the associated local components, and some statistics on the zeros of the associated L-functions.

1.1.2 Analogy with the height on algebraic varieties

The counting problem admits an interesting analogy with the well-known question of counting rational points of bounded height on a smooth projective variety over a number field. The absolute Weil height is the proper notion of size in this setting and is defined by

$$h_{\mathbf{P}^n}(x) = \prod_v \max_{0\leqslant i\leqslant n} |x_i|_v^{1/[F:\mathbf{Q}]}, \qquad x = (x_i)_{0\leqslant i\leqslant n} \in \mathbf{P}^n(F). \tag{1.3}$$

where the product runs over the places of F and does not depend on the choice of homogeneous coordinates. Given any projective variety V over F endowed with a fixed embedding ι into the projective space $\mathbf{P}^n(F)$, a height function on V can be defined by pulling back the Weil height on $\mathbf{P}^n(F)$, setting

$$h_V(x) = h_{\mathbf{P}^n}(\iota(x)), \qquad x \in V. \tag{1.4}$$

The most natural setting for considering such generalized questions is the one of Fano varieties, where there are precise conjectures due to Batyrev, Manin and Peyre [6, 99]. On those grounds, Northcott [95] proved the finiteness of the set of points of bounded height for the projective spaces, refined by Schanuel [112] in an asymptotic counting law.

Theorem 1 (Schanuel)**.** *There exists $C_n > 0$ such that for any $Q \geqslant 1$,*

$$\#\left\{x \in \mathbf{P}^n(F) \ : \ h(x) \leqslant Q\right\} = C_n Q^{n+1} + \begin{cases} O\left(Q \log Q\right) & \text{if } n = 1 \text{ and } F = \mathbf{Q}; \\ O\left(Q^{n-1/[F:\mathbf{Q}]}\right) & \text{otherwise.} \end{cases} \tag{1.5}$$

In recent years, Sarnak has repeatedly emphasized the analogy between the Schanuel theorem on counting rational points on projective spaces and the problem of counting automorphic cusp forms on $\mathrm{GL}(n)$, so that the natural questions on algebraic varieties carry to the theory of automorphic forms and serve as a guideline for the methods.

1.1.3 Theorem A: Counting law

Analogously to rational points on algebraic varieties, the first natural question concerning a family of automorphic forms is to determine its size. The case of quaternion algebras can be embedded in $\mathrm{GL}(2)$ so that, following the analogy with algebraic varieties, the notion of analytic conductor used in this main theorem is inspired by the procedure (1.4) for heights: given the by now standard notion of analytic conductor for $\mathrm{GL}(2)$, the analytic conductor for quaternion algebra is provided by pulling it back via the associated identity map between their dual groups. This canonical notion is consistent with the one defined by the attached functional equation. The first result of this thesis provides an asymptotic formula for the cardinality

$$N(Q) = \#\mathcal{A}(Q), \qquad Q \geqslant 1, \tag{1.6}$$

referred to by Sarnak as a Weyl-Schanuel law. Rare such results exist for the whole universal family. Petrow recently handled the problem in a fairly general fashion for automorphic forms on tori [14, 98]. The case of the universal family for $\mathrm{GL}(2)$ is handled in a recent preprint by Brumley and Milićević [17]. For quaternion algebras, the counting law is provided by the following statement.

Theorem A (Counting law for quaternion algebras)**.** *Let R be the finite set of places where B ramifies. There exists $C > 0$ such that for any $Q \geqslant 1$,*

$$N(Q) = CQ^2 + \begin{cases} O\left(Q^{1+\varepsilon}\right) & \text{if } F = \mathbf{Q} \text{ and } B \text{ totally definite;} \\ O\left(Q^{2-\delta_F}\right) & \text{if } F \neq \mathbf{Q} \text{ and } B \text{ totally definite;} \\ O\left(\frac{Q^2}{\log Q}\right) & \text{if } B \text{ is not totally definite.} \end{cases} \tag{1.7}$$

The constant $C > 0$ is defined explicitly in (1.10), and $\delta_F = 2(1 + [F : \mathbf{Q}])^{-1}$.

Remarks. The form of this asymptotic growth appeals some comments.

(i) There is a stricking similarity between the error term in Theorem A and that of the classical result of Schanuel in Theorem 1 on the number of rational points of bounded height in projective spaces. His result, when specialized to $F = \mathbf{Q}$, also has an error term that picks up an additional power of log.

(ii) The presence of a power savings error term in the totally definite case, *i.e.* when every archimedean place is ramified, is noteworthy. This feature is lost without this assumption, like the corresponding result [17] for GL(2), where only a logarithmic savings is obtained. The reason for this difference lies in the passage from smooth to sharp counting at archimedean places, see Section 3.2.5.

(iii) The assumption that B is a division quaternion algebra induces an automorphic compact quotient, hence avoiding the technical considerations due to the continuous part of the automorphic spectrum, see Section 3.1.

(iv) The center has been removed for technical purposes and to avoid to deal with the central terms in the Selberg trace formula. All the methods are expected to carry on to a setting considering the center without considerable adaptation.

The precise knowledge of the constant C unveils a lot of information, and its geometrical interpretation has considerable importance as in the conjectures of Peyre. An explicit and meaningful formulation of the constant is given below, in the context of the equidistribution properties of $\mathcal{A}(G)$, and shows striking similarities with the ones computed for algebraic varieties [22].

1.2 Equidistribution results

1.2.1 Theorem B: Equidistribution

Beyond estimating the size of the universal family lies the question of the geometrical distribution of the automorphic representations of G. A good formulation of the problem is to find a measure with respect to which the universal family equidistributes, what is carried on in this section after giving a glance at the topological and measurable structure the universal family is endowed with.

Each local unitary dual group $\widehat{G}_v$ is endowed with the Fell topology and the product $\prod_v \widehat{G}_v$ is then given the product topology. Introduce the measure μ on $\prod_v \widehat{G}_v$ that assigns to every basic open set $X = \prod_v X_v$, *i.e.* where X_v is an open set of $\widehat{G}_v$ and $X_v = \widehat{G}_v$ for all but finitely many v, the positive real number

$$\mu(X) = \int_X^\star \frac{\mathrm{d}\pi}{c(\pi)^2}, \tag{1.8}$$

where the regularized integral is defined as

$$\zeta^\star(1) \prod_v \zeta_v(1)^{-1} \int_{X_v} \frac{\mathrm{d}\pi_v}{c(\pi_v)^2}. \tag{1.9}$$

Here ζ_v is the local zeta function associated to F_v, the notation $\zeta^\star(1)$ stands for the residue of the Dedekind zeta function of F at 1, and $\mathrm{d}\pi_v$ is the Plancherel measure on $\widehat{G}_v$, introduced and normalized according to the convention in Section 2.3.2.

Remarks. This integral is not as disturbing as it seems for the following reasons.

(i) The Plancherel measure is supported on the tempered dual; since tempered representations are generic, the conductors appearing in the integral are well-defined for the sets actually arising in what follows, see Section 2.2.

(ii) It is by no mean obvious that the integral (1.8) actually converges. It is the case and Section 3.2.4 contains the explicit computations of the local factors ensuring the convergence as well as motivating the regularization.

The measure μ has finite total mass $\|\mu\|$. All the definitions are now in place to uncover the expression of the leading constant in Theorem A, namely

$$C = \frac{1}{2}\mathrm{vol}(G(F)\backslash G(\mathbf{A}))\|\mu\|, \tag{1.10}$$

where the measure giving the volume of the automorphic quotient $G(F)\backslash G(\mathbf{A})$ is normalized as in Section 2.1.1. The main result is the following one.

Theorem B (Equidistribution for quaternion algebras)**.** *The universal family of G equidistributes with respect to the measure μ, in the following sense. For every relatively quasi-compact open set X of $\prod_v \widehat{G}_v$ with boundary of measure zero,*

$$\frac{\#\{\pi \in \mathcal{A}(Q) : \pi \in X\}}{N(Q)} \longrightarrow \frac{\mu}{\|\mu\|}(X), \qquad \textit{as} \quad Q \to \infty. \tag{1.11}$$

1.2.2 Sato-Tate conjectures

Once global equidistribution results reached, the behavior of the local components at a fixed place $\mathfrak{p}$ can be investigated, following the general conjectures of Shin and Templier [118]. This is the aim of the so-called Sato-Tate conjectures, that have their own outside motivations worth reviewing.

An origin lying in elliptic curves

Pursuing the fruitful analogy with algebraic varieties gives ground to formulate statistical problems for automorphic forms. The simplest yet already rich case is the one of an elliptic curve E defined over $\mathbf{Q}$. It can be defined by the equation $y^2 = x^3 + ax + b$, assuming for the sake of this motivating groundwork that $a, b \in \mathbf{Z}$, so that its reduction E_p modulo p remains smooth for almost every p, *i.e.* E_p is an elliptic curve on $\mathbf{F}_p$.

Of great interest is to study elements of the curve $E_p(\mathbf{F}_p)$. Getting rid of the x^3 factor gives a toy model in which the number of those points would be $N_p(y^2 = ax+b) = p+1$, up to adding a point at infinity and provided a is nonzero. More generally, for an elliptic curve E, the number of its rational points modulo p can be written in the form

$$N_p(E) = \#E_p(\mathbf{F}_p) = p + 1 - a_p(E), \tag{1.12}$$

where $a_p(E)$ is the trace of the Frobenius of E at p. In 1936, Hasse proved the bound $|a_p(E)| \leqslant 2\sqrt{p}$, so that $a_p(E)/\sqrt{p}$ lies in $[-2,2]$. Rewriting it as $a_p(E) = 2\cos\theta_p(E)$ gives rise to the associated Frobenius angles $\theta_p(E)$, lying in $[0,\pi]$. The Sato-Tate problem concerns the distribution of the Frobenius traces and angles. In 1963, following the numerical works of Sato, Tate suggested the following conjecture.

Conjecture 1 (Sato-Tate for elliptic curves). *For a non CM elliptic curve E, the numbers $a_p(E)$, resp. $\theta_p(E)$, equidistribute in $[-2,2]$, resp. $[0,\pi]$, with respect to the half-circle measure*

$$\mu^{\mathrm{ST}} = \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}\,\mathrm{d}x, \qquad \textit{resp.} \quad \tilde{\mu}^{\mathrm{ST}} = \frac{2}{\pi}\sin^2\phi\,\mathrm{d}\phi. \tag{1.13}$$

The Sato-Tate conjecture for elliptic curves has been proven in 2006 by Clozel, Harris, Shepherd-Barron and Taylor [20] under the hypothesis that the j-invariant $j(E)$ is not an algebraic integer, in particular implying that E is non CM. In the exceptional case of CM curves, half of the $a_p(E)$ vanish and the limit distribution is also known.

Automorphic Sato-Tate conjecture

The work of Taylor and Wiles [126] unveiled that a semistable elliptic curve E over $\mathbf{Q}$ corresponds to a modular cusp form on GL(2) of weight 2 with integer coefficients. More generally, for modular cusp forms of weight k, the Ramanujan conjecture, known for GL(2) by the work of Deligne [33], gives the analogous of the Hasse bound and states that its Fourier coefficients $|a_p(f)|$ are bounded by $2\sqrt{p}^{k-1}$. This naturally leads to lift the Sato-Tate problem in this modular setting.

Conjecture 2 (Modular Sato-Tate). *For f a normalized non-CM modular cusp form on $SL_2(\mathbf{Z})$, the $\sqrt{p}^{-(k-1)}a_p(f)$ equidistribute in $[-2,2]$ with respect to the measure μ^{ST}.*

The aforementioned result of Clozel, Harris, Shepherd-Barron and Taylor proves that this conjecture holds in the case of modular forms of weight 2 with integer coefficients. In 2011, Barrett-Lamb, Geraghty, Harris and Taylor [5] generalized it to any weight $k \geqslant 2$.

Towards families of automorphic representations: Corollary C

Automorphic forms give rise to automorphic representations, providing a motivation to consider Sato-Tate conjectures in this even wider setting. An automorphic representation $\pi \in \mathcal{A}(\mathrm{GL}(n))$ decomposes as a restricted tensor product $\pi = \otimes_v \pi_v$ of local factors. Since only a finite number of the π_v's are ramified, for a fixed π and large enough v almost all the local component π_v can be identified with its Satake parameters

$$\pi_v \simeq (\alpha_1(\pi_v), \ldots, \alpha_n(\pi_v)) \in T_c/W, \tag{1.14}$$

where T_c is the standard complex split torus of $\mathrm{GL}(n)$ and W is its associated Weyl group. These local factors living in a priori different dual groups hence have an interpretation on a common ground, allowing to consider the Sato-Tate problem which consists in determining the distribution of the local components π_v.

Besides considering a fixed automorphic form for which information is barely reached under strong hypothesis as above, it is possible to provide results by averaging on a whole family of automorphic forms $\mathcal{F}$. This allows not only to study the varying coefficients associated to a fixed object in function of v as in the previous instances of the conjectures, what is called "horizontal" statistics, but also to fix a parameter v and study the coefficients $\alpha_i(\pi_v)$ for π varying in $\mathcal{F}$. These give rise to the "vertical" conjectures. Such families have been considered for instance in the works of Bruggeman [15], Conrey-Duke-Farmer [30], Sarnak [107] and Serre [116].

Sarnak, Shin and Templier [110] surveyed the evolution of recent definitions of general families of automorphic forms as well as their attached Sato-Tate analogues, and formulated the Sato-Tate conjecture for families.

Conjecture 3 (Sarnak-Shin-Templier). *The family $\mathcal{F}$, when ordered by the analytic conductor, is equidistributed in $\prod_v \widehat{G}_v$ with respect to a measure $\mu(\mathcal{F})$ such that*

(i) it is a probability measure supported on the tempered spectrum, so does its projections $\mu_{\mathfrak{p}}(\mathcal{F})$ on the local duals $\widehat{G}_{\mathfrak{p}}$;
(ii) the log-average over $\mathfrak{p}$ exists and defines the Sato-Tate measure, more precisely there exists a measure $\mu^{\mathrm{ST}}(\mathcal{F})$ such that

$$\frac{1}{Q}\sum_{N\mathfrak{p}\leqslant Q}\log(N\mathfrak{p})\mu_{\mathfrak{p}}(\mathcal{F})_{|T} \longrightarrow \mu^{\mathrm{ST}}(\mathcal{F}), \qquad \text{as} \quad Q\to\infty. \tag{1.15}$$

Shin and Templier [118] recently proved a precise quantitative version of this conjecture in a fairly broad setting for families of automorphic representations with discrete series at infinity. Besides, there is no result of this type when both parameters and objects vary, only horizontal or vertical conjectures have been proven so far.

Coming back to the universal family of quaternion algebras, once the global equidistribution result stated in Theorem B, the Sato-Tate conjecture questions the behavior of the projections $\mu_{\mathfrak{p}}$ of the limit measure on the local components $\widehat{G}_{\mathfrak{p}}$ when the norm of $\mathfrak{p}$ grows. On the common ground where all the representations in the support of the Plancherel measures of $G_{\mathfrak{p}}$ live, given by the tempered Satake parameters space T_c/W, the Sato-Tate question acquires a precise meaning and local representations are equidistributed with respect with the half-circle measure.

Corollary C (Sato-Tate for quaternion algebras). *For all $\phi \in C(T_c/W)$,*

$$\int_{T_c/W}\widehat{\phi}(x)\,\mathrm{d}\mu_{\mathfrak{p}}(x) \longrightarrow \int_{T_c/W}\widehat{\phi}(x)\,\mathrm{d}\mu^{\mathrm{ST}}(x), \qquad \text{as} \quad N\mathfrak{p}\longrightarrow\infty, \tag{1.16}$$

where the measures $\mu_{\mathfrak{p}}$ are explicit given by (1.9).

1.3 Low-lying zeros of L-functions

1.3.1 Importance of zeros of L-functions

The L-functions, among which the Riemann ζ function is the most celebrated representative, are ubiquitous in number theory and provide an analytic way of grasping properties of arithmetic objects. Their zeros, even if they remain mostly mysterious, carry information concerning the distribution of prime numbers, and more generally the nature of the object to which they are attached, thus justifying the tremendous efforts and interest towards the Riemann hypothesis.

Indeed, the so-called explicit formulas link distributions of the zeros to quantities of arithmetic nature. A motivation for statistical studies on zeros of L-functions is provided by Mazur [87], who notices that explicit formulas are generically of the form

$$\pi(x) = \mathrm{MT} + \mathrm{ET} + \mathrm{OT}, \tag{1.17}$$

where $\pi(x)$ is a relevant statistic on prime numbers; MT stands for the main term coming from particular zeros of the L-function; ET is a sum over trivial zeros that constitutes an error term; and OT is an oscillating term coming from the other zeros. This last term is expected to contribute as an error term, yet is often tough to estimate and requires a sufficiently precise knowledge of the behavior of the zeros in order to use compensations. For instance, assuming the Riemann hypothesis, every nontrivial zero of the Riemann zeta function lies on the critical line $\mathrm{Re}(s) = \frac{1}{2}$, improving the remainder in the prime number theorem as follows:

$$\pi(x) = \mathrm{li}(x) + O\left(xe^{-\alpha\sqrt{\log x}}\right), \qquad \text{without RH;}$$
$$\pi(x) = \mathrm{li}(x) + O\left(\sqrt{x}\log x\right), \qquad \text{with RH.}$$

Statistics on zeros of L-functions hence lead to a priori nontrivial results towards the arithmetic of the underlying objects, providing a strong motivation to their study.

1.3.2 Pair correlations

Analogy between matrices and L-functions

The theory of random matrices [88] is a glass through which understand the field of statistics on zeros of L-functions. The eigenangles of random matrices behave strikingly like these zeros and, since well more explored, will serve as a guide for the L-function world. Let $A \in M_n(F)$ be a diagonalizable unitary matrix, and consider its eigenvalues $\lambda_A^{(j)} = e^{i\theta_A^{(j)}}$ ordered such that $0 \leqslant \theta_A^{(1)} \leqslant \cdots \leqslant \theta_A^{(n)} < 2\pi$. The mean spacing between neighboring eigenangles is $\frac{2\pi}{N}$. In order to renormalize it to one, set

$$\tilde{\theta}_A^{(j)} := \frac{N}{2\pi}\theta_A^{(j)}, \qquad 1 \leqslant j \leqslant n. \tag{1.18}$$

Similarly, associate to an L-function $L(s,\pi)$ its nontrivial zeros $\rho_\pi^{(j)} = \frac{1}{2} + i\gamma_\pi^{(j)}$, with a priori $\gamma_\pi^{(j)} \in \mathbf{C}$ without assuming the Riemann hypothesis, and ordered so that the $\gamma_\pi^{(i)}$ satisfy $\cdots \leqslant \Re\gamma_\pi^{(-1)} \leqslant 0 \leqslant \Re\gamma_\pi^{(1)} \leqslant \Re\gamma_\pi^{(2)} \leqslant \cdots$. The mean spacing between neighboring zeros [65] is $m\frac{\log c(\pi)}{2\pi}$ where m is the degree of the L-function. Renormalize them to 1 introducing

$$\tilde{\gamma}_\pi^{(j)} := \frac{\log c(\pi)}{2\pi}\gamma_\pi^{(j)}, \qquad j \in \mathbf{Z}. \tag{1.19}$$

The strong similarity between both settings leads to motivate statistical questions on zeros of L-functions by existing statistical results for eigenangles of matrices.

Pair correlation for matrices

In the 50s, Wigner investigated random matrices in order to modelize atomical phenomena. These are matrices of the gaussian unitary ensemble, denoted $\mathrm{GUE}(N)$, *i.e.* the set of unitary matrices of size N with independent random gaussian coefficients. A particular way to grasp the behavior of their associated eigenangles is to study the distribution of the spacings [69] between them, given by

$$R_A[a,b] = \frac{1}{N}\left\{j \neq k \ : \ \tilde{\theta}_A^{(j)} - \tilde{\theta}_A^{(k)} \in [a,b]\right\}, \qquad A \in \mathrm{GUE}(N). \tag{1.20}$$

This statistics is called the pair correlation of the matrix A. The pair correlation of a family of matrices is naturally defined as the average of the pair correlations over the family, that is to say in the case of the whole gaussian unitary ensemble

$$R_{\mathrm{GUE}(N)}[a,b] = \int_{\mathrm{GUE}(N)} R_A[a,b]\,\mathrm{d}A. \tag{1.21}$$

Dyson determined in 1962 the correlation density of GUE in the following result.

Theorem 2 (Dyson)**.** *There is a measure r_{GUE} such that*

$$R_{\mathrm{GUE}(N)}[a,b] \underset{N\to\infty}{\longrightarrow} R_{\mathrm{GUE}}[a,b] = R_{\mathrm{GUE}}[a,b] = \int_a^b r_{\mathrm{GUE}}(x)\,\mathrm{d}x, \tag{1.22}$$

moreover $r_{\mathrm{GUE}}(x) = 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2$.

Katz and Sarnak [68] proved more generally in 1997 that the spacings between the eigenvalues of random matrices belonging to fairly general families, viz. the compact symmetric irreducible Lie group, are all governed by the GUE distribution. Introduce more precisely the classical groups $G(N)$ among the groups of unitary matrices, orthogonal matrices, or symplectic matrices of size N and independent gaussian coefficients.

Theorem 3 (Katz-Sarnak)**.** *For every family $G(N)$ of classical group, in the L^1-sense,*

$$\int_{G(N)} R_A[a,b]\,\mathrm{d}A \underset{N\to\infty}{\longrightarrow} R_{\mathrm{GUE}}[a,b], \qquad a,b\in\mathbf{R}. \tag{1.23}$$

Firk and Miller [46] gave arguments for the ubiquity of the GUE density in statistical modelisations in physics. The results summarized here suggest, following faithfully the fruitful analogy between matrices and L-functions, that the same universality holds for statistics on L-functions.

Pair correlation for L-functions

Much later, Montgomery [94] first explored the analogous distribution law of spacings between zeros of L-functions. In the particular case of the Riemann ζ function, he strikingly noticed that the pair correlation between zeros is the same than the one Dyson obtained for eigenangles of random unitary matrices in Theorem 2.

Theorem 4 (Montgomery, 1974)**.** *For $\phi\in\mathcal{S}(\mathbf{R})$ such that $\operatorname{supp}(\hat{\phi})\subseteq(-1,1)$, the pair correlation of the zeros of the Riemann zeta function is given by*

$$\frac{1}{N}\sum_{1\leqslant j\neq k\leqslant N}\phi(\tilde{\gamma}_j-\tilde{\gamma}_k) \underset{N\to\infty}{\longrightarrow} \int_{\mathbf{R}}\phi(x)r_{\mathrm{GUE}}(x)dx. \tag{1.24}$$

Many computations led by Odlyzko [96] for other L-functions then brought strong evidence that this statistical behavior of the zeros of L-functions seem to match the analogous statistics for eigenangles of random matrices in GUE, leading to expect the Montgomery result to be a general property of zeros of L-functions. This universal behavior is known as the Montgomery-Odlyzko law. Results in this direction flourished from then on, culminating with Rudnick and Sarnak [105] who proved in 1995 that the same universal distribution holds for pair correlations of a generic L-function $L(s,\pi)$ on $\mathrm{GL}(n)$. The following statement is restricted to $\mathrm{GL}(2)$ and suits the purposes of the present motivational background without having to introduce extra technical condition, yet morally holds for every general linear groups.

Theorem 5 (Rudnick-Sarnak, 1996)**.** *Let π be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbf{Q})$. Let ϕ be an even Schwartz function such that $\operatorname{supp}(\widehat{\phi})\subseteq(-1,1)$. Then*

$$\frac{1}{N}\sum_{1\leqslant j\neq k\leqslant N}\phi(\tilde{\gamma}_j-\tilde{\gamma}_k) \underset{N\to\infty}{\longrightarrow} \int_{\mathbf{R}}\phi(x)r_{\mathrm{GUE}}(x)dx. \tag{1.25}$$

The result proved by Rudnick and Sarnak is the first towards the analogous conjecture for every L-functions. Indeed, the Langlands functoriality conjectures [26] postulate that every L-function comes from an L-function attached to a cuspidal automorphic representations on a general linear group. The result of Rudnick and Sarnak positively answers these "standard" cases, even for higher level correlations.

1.3.3 One-level densities

The universality of the GUE law for pair correlations is surprising. Indeed, the previous results are disappointing for they are blind to the differences between the classical groups and it can be expected, in contrast with the Montgomery-Odlyzko law, that other statistics will be able to distinguish among them. A second disappointment with correlations is that they are blind to many modifications on the zeros, and in particular do not give any importance to zeros usually of arithmetic significance, like the central point.

One-level density for matrices

The correlation statistics considered until now are global statistics, taking into account *all* eigenangles, since they consider only the distribution of spacings between them. Katz and Sarnak broke this universality, turning their interest towards statistics concentrated on *small* eingenangles.

Definition 1. The one-level density attached to A is the distribution defined by, for ϕ be an even Schwartz function on $\mathbf{R}$ and $A \in M_n(\mathbf{R})$,

$$D(A, \phi) := \sum_{1 \leqslant j \leqslant N} \phi\left(\tilde{\theta}_A^{(j)}\right). \tag{1.26}$$

Here ϕ is a quickly decreasing test function which is no more supposed to be a function of the differences as for pair correlations. This time, large eigenangles are essentially cut off, and hence $D(A, \phi)$ is a weighted average of the small eigenangles.

Definition 2. Let ϕ be an even Schwartz function on $\mathbf{R}$. The one-level density of a family $\mathcal{F}$ endowed with a probability measure is

$$D(\mathcal{F}, \phi) = \int_{\mathcal{F}} D(A, \phi) \, \mathrm{d}A. \tag{1.27}$$

In the matrice setting, Katz and Sarnak [68] proved that the average density over a family differs depending on the group considered, breaking the embarrassing universality of GUE.

Theorem 6 (Katz-Sarnak)**.** *For the classical groups $G(N)$, for every real Schwartz function ϕ of compactly supported Fourier transform,*

$$D(G(N), \phi) \underset{N \to \infty}{\longrightarrow} \int_{\mathbf{R}} W_G(x) \phi(x) \, \mathrm{d}x, \tag{1.28}$$

where dA is a normalized Haar measure on $G(N)$, and the densities functions on $\mathbf{R}$ are defined by

$$\begin{aligned}
W_{\mathrm{U}}(x) &= 1 \\
W_{\mathrm{Sp}}(x) &= 1 - \frac{\sin 2\pi x}{2\pi x} \\
W_{\mathrm{SO(even)}}(x) &= 1 + \frac{\sin 2\pi x}{2\pi x} \\
W_{\mathrm{SO(odd)}}(x) &= 1 - \frac{\sin 2\pi x}{2\pi x} + \delta_0(x) \\
W_{\mathrm{O}}(x) &= \frac{1}{2}\left(W_{\mathrm{SO(even)}}(x) + W_{\mathrm{SO(odd)}}(x)\right) = 1 + \frac{1}{2}\delta_0(x)
\end{aligned}$$

This function W_G is the one-density function for G. The fact that the limit is no more universal but depends on the family associated to a classical group, yet also falls in finitely many cases, gives rise to the notion of type of symmetry of a family of L-functions. The computations of this limit detecting in some sense which of the classical group govern the behavior of the zeros.

Remark. The result of Katz and Sarnak holds for classical groups, yet it remains to know whether or not it remains true for more general families of groups, what would turn the classical groups as universal representant of the different symmetries governing the eigenangles.

One-level densities for families of L-functions

Following the enlightening analogy with random matrices, it can be expected that the one-level density of the zeros attached to every reasonable family of L-functions behaves as the eigenangles of classical random matrices groups, and in particular that the behavior of low-lying zeros of L-functions is modeled by the classical groups. The so-called density conjecture postulates this universality, more precisely that every reasonable family of matrices or L-functions will match one of these cases.

Definition 3. Let ϕ be an even Schwartz function on $\mathbf{R}$ and π an automorphic representation. The one-level density attached to A is

$$D(\pi, \phi) := \sum_{\gamma_\pi^{(j)}} \phi\left(\tilde{\gamma}_\pi^{(j)}\right), \tag{1.29}$$

Remark. Without assuming the Riemann hypothesis, the above definition has to be broadened, for the $\gamma_\pi^{(j)}$ need not be real. Motivated by the density conjecture, "Schwartz function on $\mathbf{R}$" has to be understood as a function of Schwartz class on $\mathbf{R}$ and of compactly supported Fourier transform. This ensures the existence of an analytic continuation to the whole $\mathbf{C}$ making sense of the expression above.

Definition 4. Let ϕ be an even Schwartz function on $\mathbf{R}$. The one-level density of a finite family $\mathcal{F}$ is

$$D(\mathcal{F},\phi)=\frac{1}{|\mathcal{F}|}\sum_{\pi\in\mathcal{F}}D(\pi,\phi). \tag{1.30}$$

The first result in this direction is given by Özlük and Snyder [127] in 1993 for L-functions attached to Dirichlet characters. Since then, a wide literature has been published concerning the statistical behavior of low-lying zeros of families of L-functions [38, 58, 64, 84, 104]. This led Katz and Sarnak [69] to formulate the so-called density conjecture stating the same universality of the types of symmetry arising for group of matrices.

Conjecture 4 (Density conjecture)**.** *Let $\mathcal{F}$ be a family of automorphic representations in the sense of Sarnak and $\mathcal{F}_Q$ a finite truncation increasing to $\mathcal{F}$ when Q grows. Then for all even Schwartz function on $\mathbf{R}$ with compactly supported Fourier transform, there is one classical group type G such that*

$$D(\mathcal{F}_Q,\phi)\underset{Q\to\infty}{\longrightarrow}\int_{\mathbf{R}}\phi(x)W_G(x)\,\mathrm{d}x. \tag{1.31}$$

The family $\mathcal{F}$ is said to have the type of symmetry of G.

Remark. For families of L-functions associated to algebraic varieties over function fields, the type of symmetry is determined by the monodromy of the family, see [68], shredding light on the reason why zeros of L-functions are governed by random matrix groups. However, no such analogue is known on number fields.

1.3.4 Theorem D: Type of symmetry

Low-lying zeros

Considering the statistics on low-lying zeros of L-functions attached to the universal family of quaternion algebras, the one-level density (1.30) of the truncated family is

$$D(\mathcal{A}(Q),\phi)=\frac{1}{N(Q)}\sum_{\pi\in\mathcal{A}(Q)}D(\pi,\phi). \tag{1.32}$$

The problem is to determine whether or not the quantity $D(\mathcal{A}(Q),\phi)$ admits a limit and unveils the associated type of symmetry according to the density conjecture. The following statement partially determines the type of symmetry of quaternion algebras and fulfilling the expectations of the density conjecture.

Theorem D. *For every even and Schwartz function ϕ with Fourier transform compactly supported in $(-2/3,2/3)$,*

$$\frac{1}{N(Q)}\sum_{\pi\in\mathcal{A}(Q)}D(\pi,\phi)\underset{Q\infty}{\longrightarrow}\widehat{\phi}(0)+\frac{1}{2}\phi(0)=\int_{\mathbf{R}}\phi(x)W_O(x)\,\mathrm{d}x. \tag{1.33}$$

In particular, the type of symmetry of inner forms of $\mathrm{PGL}(2)$ *is one of the orthogonal types of symmetry.*

An important caveat ought to be mentioned concerning the orthogonal types of symmetry. The density conjecture postulates results for Schwartz function with compactly supported Fourier transform, yet with no constraint on the support. Assuming this conjecture, proving the convergence for a narrower class of allowed Fourier supports may determine uniquely the postulated type of symmetry. However this is not the case for all supports, and an uncertainty remains in the case or supports trapped in $(-1, 1)$. Explicitly, the Plancherel formula yields

$$\int_{\mathbf{R}} \phi(x) W(x)\, \mathrm{d}x = \int_{\mathbf{R}} \widehat{\phi}(x) \widehat{W}(x)\, \mathrm{d}x, \tag{1.34}$$

and looking at the Fourier transforms of the densities, introducing η the characteristic function of $[-1, 1]$, direct computations leads to

$$\begin{aligned}
\widehat{W}_{\mathrm{U}}(x) &= \delta_0(x) \\
\widehat{W}_{\mathrm{Sp}}(x) &= \delta_0(x) - \frac{1}{2}\eta(x) \\
\widehat{W}_{\mathrm{SO(even)}}(x) &= \delta_0(x) + \frac{1}{2}\eta(x) \\
\widehat{W}_{\mathrm{SO(odd)}}(x) &= \delta_0(x) - \frac{1}{2}\eta(x) + 1 \\
\widehat{W}_{\mathrm{O}}(x) &= \delta_0(x) + \frac{1}{2}
\end{aligned}$$

Unfortunately, the three orthogonal types of symmetry, viz. $\widehat{W}_{\mathrm{O}}(x)$, $\widehat{W}_{\mathrm{SO(even)}}(x)$ and $\widehat{W}_{\mathrm{SO(odd)}}(x)$ are indistinguishable in $(-1, 1)$. Theorem D hence only partially determines the type of symmetry of the universal family of quaternion algebras. Further directions are mentioned in Chapter 4.

Non-vanishing of L-functions

Statistics on the distribution of low-lying zeros of L-functions are known to lead to results concerning vanishing at the central point, following the ideas of Iwaniec, Luo and Sarnak [64]. Introduce the proportion of automorphic representations with vanishing at the central point with order m, that is

$$p_m(Q) = \frac{1}{N(Q)} \# \left\{ \pi \in \mathcal{A}(Q) \ : \ \mathrm{ord}_{s=1/2} L(s, \pi) = m \right\}, \qquad m \in \mathbf{N}. \tag{1.35}$$

Theorem D yields densities of vanishing of the associated L-functions. Indeed, for every T_ϕ such that Theorem D holds for functions whose Fourier transform is supported in $(-T_\phi, T_\phi)$,

$$\liminf_{Q\to\infty} p_0(Q) \geqslant \frac{1}{2} - \frac{1}{T_\phi}, \tag{1.36}$$

$$\liminf_{Q\to\infty} \sum_{m\geqslant 0} m p_m(Q) \leqslant \frac{1}{2} + \frac{1}{T_\phi}. \tag{1.37}$$

Unfortunately, it yields nontrivial results on the density of non-vanishing at the central point only if T_ϕ is allowed to be large enough, namely in this case larger than two. The second density result above is interesting for every $T_\phi > 0$. This, in addition of verifying the whole density conjecture, is a strong motivation to strengthen the bounds on the support of the Fourier transform in Theorem D.

1.4 Other ground groups

Addressing arithmetic statistics problems for different groups than inner forms of GL(2) leads to determine what is critical for the use of the same counting method and what is specific to the GL(2) setting. Any result in this direction provides clues towards more general conjectures concerning both the growth rate and the form of the constant, that are fundamental in the vein of the analogous program of Batyrev, Manin and Peyre for algebraic varieties. Some unitary and symplectic groups of low ranks can be explored.

The main aim of this opening towards different settings is to identify essential assumptions, state precise conjectures and focus on the differences between these cases and the one of general linear groups, thus it is natural to use some freedom on the assumptions to make space for comments and comparisons avoid to sink in unnecessary technicalities. This also motivates to address the counting problem instead of the more general equidistribution question as it could be expected in the light of the previous sections on quaternion algebras: there is no need to say these problems are by no means irrelevant.

1.4.1 Unitary groups

Let E be a quadratic totally imaginary extension of the totally real field F, q an hermitian form on E^3, and U the unitary group associated to q, that is to say the subgroup of GL(3) preserving q. More precisely, it is the subgroup of $\mathrm{GL}(3, F)$ defined by

$$U = \left\{ g \in \mathrm{GL}(3, F) \ : \ \forall x, y \in E^3,\ q(gx, gy) = q(x, y) \right\}. \tag{1.38}$$

As for quaternion algebras, the classification of the group of points over local fields is known. Essentially, $U(F_v)$ is isomorphic to $\mathrm{GL}(3, F_v)$ at half of the non-archimedean places, and isomorphic to the unique quasi-split unitary group in three variables $U(F_v)$ at the other half of the places. There is a finite number of places, where different behavior can arise. Assume U is totally definite, that is to say isomorphic to the compact unitary group $O(3, \mathbf{R})$ at archimedean places. This in particular ensures that the automorphic quotient $U(F)\backslash U(\mathbf{A})$ is compact.

There is a notion of size on such a unitary group U, given by the same procedure as for quaternion algebras: pulling back the standard notion of analytic conductor along a suitable embedding, provided in this case by deep works of Flicker [47], in a general linear group provides an analytic conductor on U, still denoted by $c(\pi)$ where π is an automorphic representation of U. This allows to consider the truncated universal family

$$\mathcal{A}_U(Q) = \{\pi \in \mathcal{A}(U) \ : \ c(\pi) \leqslant Q\}, \qquad Q \geqslant 1. \tag{1.39}$$

Introducing the measures on the local groups and the local dual groups as normalized in Section 5.1.1, it is possible to formulate a conjecture for the counting law for the cardinality $N_U(Q)$ of the truncated universal family of U.

Conjecture E (with I. Petrow)**.** *The cardinality of the universal family of U satisfies*

$$N_U(Q) \sim \frac{1}{4}\mathrm{vol}(U(F)\backslash U(\mathbf{A})) \int_{\mathcal{A}(U)}^{\star} \frac{\mathrm{d}\pi}{c(\pi)^4} Q^4, \qquad \text{as} \quad Q \to \infty, \tag{1.40}$$

where the regularized integral is defined by

$$\int_{\mathcal{A}(U)}^{\star} \frac{\mathrm{d}\pi}{c(\pi)^4} = \zeta^{\star}(1) \prod_v \zeta_v(1)^{-1} \int_{\widehat{U}_v} \frac{\mathrm{d}\pi_v}{c(\pi_v)^4}. \tag{1.41}$$

1.4.2 Symplectic groups

The symplectic group of rank 2 is the subgroup of $\mathrm{GL}(4, F)$ defined by

$$\mathrm{GSp}(4) = \{g \in \mathrm{GL}(4, F) \ : \ \exists \lambda(g) \in \mathbf{G}_m, \ {}^t gJg = \lambda(g)J\} \quad \text{where} \quad J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Consider an inner form G of $\mathrm{GSp}(4)$ with compact automorphic quotient. These are exactly the groups of isometries of negative-definite or positive-definite hermitian forms on division quaternion algebras.

The notion of size is obtained by pulling back the analytic conductor following the procedure for quaternion algebras or unitary groups, through a functorial embedding recently provided by Gan and Takeda [49] refining the results of Roberts and Schmidt [102]. This leads to consider the truncated universal family

$$\mathcal{A}_G(Q) = \{\pi \in \mathcal{A}(G) \ : \ c(\pi) \leqslant Q\}, \qquad Q \geqslant 1. \tag{1.42}$$

Introducing the measures on the local groups and the local dual groups as normalized in Section 5.3.1, it is possible to formulate a conjecture for the main term of a counting law for the cardinality $N_G(Q)$ of the truncated universal family of G.

Conjecture F (with I. Petrow). *The cardinality of the universal family of G satisfies*

$$N_G(Q) \sim \frac{1}{3}\mathrm{vol}(G(F)\backslash G(\mathbf{A})) \int_{\mathcal{A}(G)}^{\star} \frac{\mathrm{d}\pi}{c(\pi)^3} Q^3, \qquad \textit{as} \quad Q \to \infty. \tag{1.43}$$

where the regularized integral is defined by

$$\int_{\mathcal{A}(G)}^{\star} \frac{\mathrm{d}\pi}{c(\pi)^3} = \zeta^{\star}(1) \prod_v \zeta_v(1)^{-1} \int_{\widehat{U}_v} \frac{\mathrm{d}\pi_v}{c(\pi_v)^3}. \tag{1.44}$$

Remarks. It is natural to compare the counting law provided in Theorem A and those proposed in Conjectures E or F.

(i) The growth rate of the truncated universal family of a reductive group with respect to its analytic conductor appears to be, for every example considered, its reductive rank plus one, *i.e.* its semisimple rank plus two.
(ii) The constant in the counting law for a group G appears to be of the form

$$\frac{1}{\alpha}\mathrm{vol}(G(F)\backslash G(\mathbf{A})) \int_{\mathcal{A}(G)}^{\star} \frac{\mathrm{d}\pi}{c(\pi)^{\alpha}}, \tag{1.45}$$

for a growth rate α. This is a striking similarity with the asymptotic formulas for volumes of height balls in adelic points of algebraic varieties over number fields, provided by Chambert-Loir and Tschinkel [22, Theorem 1.3], which also makes appear a zeta function of the heights.

1.5 Outline of the story

1.5.1 Universal family decomposition

Chapter 2 is mainly devoted to introducing the universal family $\mathcal{A}(G)$ for quaternion algebras, made of all its infinite-dimensional automorphic representations. This is the main groundwork on which this thesis settles and only the last chapter deals with different families. The analytic conductor $c(\pi)$ is a convenient notion of size for representations, introduced in Section 2.2 by pulling back the standard notion of conductor on GL(2). The truncated universal family is the finite set

$$\mathcal{A}(Q) = \{\pi \in \mathcal{A}(G) \ : \ c(\pi) \leqslant Q\}, \qquad Q \geqslant 0. \tag{1.46}$$

The counting law Theorem A and the equidistribution Theorem B for the universal family can be reformulated in terms of convergence of a measure ν_Q, representing the distribution of automorphic representations of the truncated family $\mathcal{A}(Q)$, to a measure ν. More precisely,

$$\nu_Q(\widehat{\phi}) = \sum_{\pi \in \mathcal{A}(Q)} \widehat{\phi}(\pi). \tag{1.47}$$

The sought convergence is reduced, by the mean of a theorem of density, to a convergence with respect to a well-behaved class of functions $\widehat{\phi} \in F(\widehat{G}_S)$ acting on a finite set of places S, as explained in Section 2.3.1. The aim then turns to proving that, as Q grows to infinity,

$$\nu_Q(\widehat{\phi}) \longrightarrow \nu(\widehat{\phi}), \qquad \widehat{\phi} \in F(\widehat{G}_S). \tag{1.48}$$

The universal family $\mathcal{A}(G)$ decomposes into harmonic subfamilies defined by adding constraints on the spectral data attached to representations, as explained in Section 2.4. This decomposition is led by the different classifications of representations that exists depending on the place, and by the different ways to grasp the conductor. Introducing R the set of places where B ramifies, the representation is decomposed as follows, and each component is to be treated in a specific way.

$$\pi = \pi_R \otimes \pi_f^R \otimes \pi_\infty^R. \tag{1.49}$$

Explicitly, the discrete spectral data consists in the arithmetic conductor $\mathfrak{q}$ of the split finite part of π, the isomorphism class of the ramified part π_R and the discrete series δ for a certain Levi subgroup, described by a certain set $\mathcal{D}$, to which the archimedean split part for π belongs. This data only partially classifies the representations in the universal family. There is a continuous set of parameters achieving the description of representations at split archimedean places π_∞^R, so that they are essentially parametrized by δ and ν, denoting $\pi_{\delta,\nu}$ the representation attached to such parameters. Continuous archimedean parameters cannot be precisely selected due to regularity constraints in the methods, hence they are allowed to vary in a restricted set of the form

$$\Omega_\delta(X) = \{\pi \in \widehat{G}_\infty^R \ : \ c(\pi) \leqslant X \text{ and } \pi \simeq \pi_{\delta,\star}\}, \qquad \delta \in \mathcal{D},\ X > 0. \tag{1.50}$$

For fixed discrete spectral data $\mathfrak{q}$, π_R and δ, consider the subfamily of automorphic representations with such spectral parameters, that is

$$\mathcal{A}(\mathfrak{q}, \sigma_R, \delta, \Omega_\delta(X)) = \{\pi \in \mathcal{A}(G) \ : \ \pi_R \simeq \sigma_R,\ \mathfrak{c}(\pi_f^R) = \mathfrak{q},\ \pi_\infty^R \in \Omega_\delta(X)\}, \tag{1.51}$$

so that the truncated universal family $\mathcal{A}(Q)$ decomposes into such subsets of restricted spectral data, when the data vary. The counting measure decomposes accordingly into

$$\nu_Q(\widehat{\phi}) = \sum_{\substack{N\mathfrak{q} \leqslant Q \\ \mathfrak{q} \wedge R,S=1}} \sum_{\substack{\sigma_R \in \widehat{G}_R \\ c(\sigma_R) \leqslant Q/N\mathfrak{q}}} \sum_{\delta \in \mathcal{D}} \sum_{\pi} \widehat{\phi}(\pi), \tag{1.52}$$

where the last sum runs over π in $\mathcal{A}(\mathfrak{q}, \sigma_R, \delta, \Omega_\delta(Q/N\mathfrak{q}c(\sigma_R)))$. This is the content of Section 2.4.2. It is then natural to consider the innermost sum running over more general sets of continuous spectral parameters Ω, namely $A(\mathfrak{q}, \sigma_R, \delta, \Omega)$. A quantity more amenable to trace formula methods has extra weights given by the spectral multiplicities, that is to say

$$B(\mathfrak{q}, \sigma_R, \delta, \Omega; \phi) = \sum_{\pi} \dim \left(\pi_f^{K_0(\mathfrak{q})}\right) \widehat{\phi}(\pi), \tag{1.53}$$

where the sum runs over π in $\mathcal{A}(\mathfrak{q}, \sigma_R, \delta, \Omega)$. The theory of local newforms for GL(2) allows to recast the counting measure ν_Q in terms of these quantities, as explained in Lemma 1, so that it is enough to study these sums over representations of fixed spectral data. More precisely,

$$\nu_Q(\widehat{\phi}) = \sum_{\substack{N\mathfrak{q} \leqslant Q \\ \mathfrak{q} \wedge R, S = 1}} \sum_{\substack{\sigma_R \in \widehat{G}_R \\ c(\sigma_R) \leqslant Q/N\mathfrak{q}}} \sum_{\delta \in \mathcal{D}} \sum_{\mathfrak{d} \mid \mathfrak{q}} \lambda_2 \left(\frac{\mathfrak{q}}{\mathfrak{d}}\right) B(\mathfrak{d}, \sigma_R, \delta, \Omega_\delta(Q/N\mathfrak{q}c(\sigma_R)); \phi). \tag{1.54}$$

1.5.2 Spectral count

The heart of the proof is to interpret the quantity $B(\mathfrak{q}, \sigma_R, \delta, \Omega; \phi)$ as a spectral side of the Selberg trace formula for a suitable test function. Lemma 4 achieves this goal up to some error terms, which precise control is fundamental.

In contrast with the discrete spectral data that can be exactly fixed by test functions admissible for the trace formulas, it is necessary to approximate the characteristic function for the continuous set of spectral parameters Ω by an admissible function. Such functions are obtained by the mean of Paley-Wiener type theorems, that provides functions blowing up outside the tempered spectrum. The better the approximation for the tempered spectral parameters, the worse this blow up on the complementary part, feature encoded in a parameter $\rho > 0$. For this reason, only the tempered part of the spectral continuous parameters Ω_{temp} is efficiently approximated by the trace formula. In other words, there is a splitting between tempered and complementary part of the archimedean spectrum, each part receiving different treatment, namely

$$B(\mathfrak{q}, \sigma_R, \delta, \Omega; \phi) = B(\mathfrak{q}, \sigma_R, \delta, \Omega_{\text{temp}}; \phi) + B(\mathfrak{q}, \sigma_R, \delta, \Omega_{\text{comp}}; \phi). \tag{1.55}$$

The smoothing step closely follows the work or Brumley and Milićević [17], and the tempered part $B(\mathfrak{q}, \sigma_R, \delta, \Omega_{\text{temp}})$ is approximated by the tempered spectral part of the trace formula applied to an explicit test function Φ, depending on the equidistribution function ϕ; the discrete spectral data $\mathfrak{q}$, σ_R, and δ; the the set of continuous parameters Ω from now on assumed to be tempered; and on the approximation parameter ρ. This is the aim of Lemma 2.70, of the form

$$B(\mathfrak{q}, \sigma_R, \delta, \Omega_{\text{temp}}) = J_{\text{temp}}(\Phi) + O(\partial_\rho B(\mathfrak{q}, \sigma_R, \delta, \Omega)), \tag{1.56}$$

where $\partial_\rho B(\mathfrak{q}, \sigma_R, \delta, \Omega)$ is an error term corresponding to the fact that the test function at archimedean split places is a smoothed version of the characteristic function of Ω. More precisely for spectral parameters farther than ρ from the boundary of Ω, thus in some sense strongly inside or strongly outside Ω, the approximation is good enough and the induced error term is of good quality. However, for spectral parameters lying in the transition zone around the boundary of Ω, the approximation is of lower quality and is the origin of this error term as well as a justification for its notation. This is precised by Lemma 3, and hence the quantity $\partial_\rho B(\mathfrak{q}, \sigma_R, \delta, \Omega)$ appears as a smoothed version of $B(\mathfrak{q}, \sigma_R, \delta, \partial_\rho\Omega)$, the number of representations with spectral parameters lying in the boundary $\partial_\rho\Omega$. This number is bounded by another tempered count, amenable to the very same methods than what follows, and hence contributing to the same error terms.

The complementary part of the spectrum is responsible for another error term. Indeed, in order to make use of the trace formula, the whole spectral part relative to Φ should be taken into account, and this one is

$$J_{\mathrm{spec}}(\Phi) = J_{\mathrm{temp}}(\Phi) + J_{\mathrm{comp}}(\Phi) + J_{\mathrm{char}}(\Phi), \tag{1.57}$$

where $J_{\mathrm{char}}(\Phi)$ corresponds to unwelcome global characters selected by the test function Φ, and where $J_{\mathrm{comp}}(\Phi)$ is the contribution coming from non-tempered representations whose continuous spectral parameters have tempered part lying in Ω. The character contribution is shown to contribute as an error term by directly estimating the number of such characters by an analogous strategy, in Lemma 12, with the Selberg trace formula replaced by the Poisson summation formula. The complementary part of the spectrum is exponentially weighted due to the behavior of the chosen test function at these places, and is bounded by a certain quantity $B_{\mathrm{comp}}(\mathfrak{q}, \sigma_R, \delta, \rho, \Omega)$ in Lemma 13.

At last, this leads to

$$B(\mathfrak{q}, \sigma_R, \delta, \Omega; \phi) = J_{\mathrm{spec}}(\Phi) + O(\partial_\rho B(\mathfrak{q}, \sigma_R, \delta, \Omega)) + O(B_{\mathrm{comp}}(\mathfrak{q}, \sigma_R, \delta, \rho, \Omega)). \tag{1.58}$$

Lemma 9 provides a bound for the complementary and smoothing terms, essentially bounding them by the counting measure for different subsets than Ω, and hence amenable to the same methods.

1.5.3 Geometric side

In Section 3.1, the Selberg trace formula comes into play and translates the spectral term appearing in (1.58) into a geometrical sum, running over $G(F)$-conjugacy classes, of weighted orbital integrals. It is of the form

$$J_{\mathrm{spec}}(\Phi) = J_{\mathrm{geom}}(\Phi). \tag{1.59}$$

The main contribution to this expansion, as often expected in applications of the trace formula, comes from the term of the geometrical expansion corresponding to the identity, leading to split the geometric side into

$$J_{\mathrm{geom}}(\Phi) = J_1(\Phi) + J_{\mathrm{ell}}(\Phi). \tag{1.60}$$

The identity term can be explicitly computed by means of the Plancherel formula and direct computations. However, there is still an error term coming from the approximation in the test function at archiemdean places, bounded similarly than the one obtained above due to the smoothing (1.56), so that

$$J_1(\Phi) = \mathrm{vol}(G(F)\backslash G(\mathbf{A}))\Phi(1) + O(\partial_\rho B(\mathfrak{q}, \sigma_R, \delta, \Omega)), \tag{1.61}$$

The critical ingredient in the estimations of this main term are sizes $\varphi(\mathfrak{q})$ of the congruence subgroups that define the conductor. The computations carried out in Section3.2 consist in summing over all the discrete spectral data, and yields the growth rate of Q^2 announced in Theorem A and the limit distribution measure announced in Theorem B.

Geometrical error terms are the ones given by nontrivial orbital integrals arising in the Selberg trace formula, encapsulated in $J_{\mathrm{ell}}(\Phi)$, which is of the form

$$J_{\mathrm{ell}}(\Phi) := \sum_{\{\gamma\}\neq 1} \mathrm{vol}\left(G_\gamma(F)\backslash G_\gamma(\mathbf{A})\right) \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} \Phi\left(x^{-1}\gamma x\right) \mathrm{d}x, \tag{1.62}$$

where the indexation runs through conjugacy classes $\{\gamma\}$ in $G(F)$. Since Φ is compactly supported and $G(F)$ is discrete, the sum is finite. The number of elements appearing in this sum depends on the parameter ρ governing the exponential type of the test function at infinity, and growth exponentially with ρ, as stated in Lemma 14. Bounds on orbital integrals are consequences of bounds on their local components, using different methods depending on the behavior of the place and the choice of the corresponding local test function, and depend essentially in $\mathfrak{q}$ as stated in Proposition 21. At last, a specific choice of ρ depending on $\mathfrak{q}$ is enough to ensure the negligibility of these error terms compared to the identity term, finishing the proof of Theorem B.

1.5.4 Further consequences

The Sato-Tate corollary, stated in Corollary C, follows from the knowledge of the equidistribution measure and known results for the spherical spectrum of GL(2). Indeed, since quaternion algebra are locally almost everywhere equal to GL(2), a result of Serre already establishes the Sato-Tate conjecture in this case when restricted to unramified representation. Ramified contribution to the equidistribution measure is shown to be negligible through explicit computations carried out in Section 3.5.

Estimating densities of low-lying zeros turns to be the main topic of Chapter 4. The explicit formula restates the problem into a question concerning sums of the

Satake parameters, rewritten as sums of associated Hecke eigenvalues of the form, for a Schwarz function ϕ,

$$D(\pi,\phi)=\widehat{\phi}(0)-\frac{2}{\log c(\pi)}\sum_{\nu=1}^{\infty}\left(\sum_{\mathfrak{p}}\left(\alpha_\pi^\nu(\mathfrak{p})+\beta_\pi^\nu(\mathfrak{p})\right)\right)\widehat{\phi}\left(\frac{\nu\log N\mathfrak{p}}{\log c(\pi)}\right)\frac{\log N\mathfrak{p}}{N\mathfrak{p}^{\nu/2}}+O\left(\frac{1}{\log c(\pi)}\right).$$

High order contributions, more precisely those corresponding to $\nu\geqslant 3$, are not a problem by bounds on Satake parameters provided by Blomer-Brumley in this setting where Ramanujan is not known to hold, and are shown not to contribute to the type of symmetry.

Lower order terms are less well controlled, even assuming the Ramanujan conjecture. A fundamental step comes from the structure of the L-functions attached to the automorphic representations of $\mathcal{A}(G)$, relating the Satake parameters $\alpha_\pi^\nu(\mathfrak{p})$ and $\beta_\pi^\nu(\mathfrak{p})$ with the Hecke eigenvalues $\lambda_\pi(\mathfrak{p}^\nu)$. The heart of the proof is to deal with the resulting sums

$$\sum_{\mathfrak{p}}\lambda_\pi(\mathfrak{p}^\nu),\qquad \nu\in\{1,2\}. \tag{1.63}$$

Here the averaging over the family is necessary, feature already underlined in the story of densities and correlations. The Selberg trace formula addresses the problem of low order terms with similar methods than for Theorem B: twisting the already built test function by the suitable Hecke opeator weights the spectral side of the trace formula by the desired Hecke eigenvalues. Since only unramified Hecke operators are considered, this allows to treat the sum (1.63) restricted to unramified representation at the chosen prime. Similar considerations than in the case of the equidistribution result leads to estimating the averaged inner sum by

$$\sum_{\substack{\pi\in\mathcal{A}(\mathfrak{d}^S,\sigma_R,\delta,\Omega)\\ \pi\text{ unramified}}}\lambda_\pi(\mathfrak{p}^\nu)=J_{\mathrm{spec}}(\Phi)+(\text{Remainder}). \tag{1.64}$$

for a suitable test function Φ, and a remainder similar to the one obtained for the equidistribution. There is no identity contribution because of the Hecke operators, the elliptic contribution is hence critical. Orbital integrals are to be precisely estimated in Section 4.4.1, and they are the seed of the limitations for the support of the Fourier transform.

Concerning the ramified part, the already stated Blomer-Brumley bound reduces the problem to the counting law for the harmonic subfamilies $\mathcal{A}(\mathfrak{q},\sigma_R,\delta,\Omega)$, already established on the way towards Theorem B, and enough to prove the ramified contributes as an error term and achieving the proof of Theorem D.

Non-vanishing results are obtained as a consequence of the type of symmetry, through Plancherel formulas and functional optimization provided for one-level densities of orthogonal type by Iwaniec, Luo and Sarnak [64] in the setting of holomorphic cusp forms on GL(2).

The conclusive Chapter 5 is devoted to openings towards two classes of other ground groups: totally definite unitary groups in three variables and inner forms of the symplectic group of degree 4 that have compact automorphic quotient. These extra assumptions are afforded in order to make these different groups amenable to the same successful methods used for quaternion algebras, and to underline new challenges arising from these new settings as well as to formulate conjecture and provide first results towards arithmetic statistics on the universal family of these groups. The main differences lie into the lack on functoriality of the conductor and the necessity to use a theory of local newform for non-split places: that was not the case for quaternion algebras since they were only finitely many.

Chapter 2

Universal Family for Quaternion Algebras

Establishing arithmetic statistics on the family of all automorphic representations of a group G requires to understand as sharply as possible the structure of the spectrum. It is necessary to order the spectrum introducing a notion of size, to be able to truncate it into a finite set and give a meaning to the sought statistics. Moreover, a choice of parameters indexing the spectrum has fundamental impact on the treatment of the problem, for it induces a decomposition of the spectrum in more or less handable subfamilies of fixed parameters. This chapter settles the groundwork, introducing the automorphic representations and providing the chosen notion of size: the analytic conductor.

The whole strategy for estimating arithmetic statistics on automorphic objects consists in interpreting the sough quantity in terms of a spectral side of the trace formula. A fundamental step is to be able select these new sets as spectral sides: suitable selecting functions are constructed in the last sections.

Contents

2.1 Odds and ends

Modern analytic number theory makes large use of methods coming from representation theory, measure theory and harmonic analysis, requiring to endow the considered algebraic objects with the relevant structures. Once done, the automorphic groundwork mentioned in the introduction is detailed in this section, in order to properly state the problems.

2.1.1 Number theoretic landscape

Number and local fields

The aim of number theory is to explore the structure and the properties of number fields, that is to say finite extensions of $\mathbf{Q}$. Consider such a number field F, and let its ring of integers be denoted by $\mathcal{O}$.

The choice of an absolute value on F allows one to embed it in the corresponding completion, endowing F with a well-behaved analytic structure. For a given absolute value v, the completion of F with respect to v is denoted F_v. It is a locally compact space, hence a local field.

Different absolute values can give rise to isomorphic completions, case in which they are said to be equivalent. In order to get rid of this redundancy, the suitable notion is the one of equivalence class of absolute values, called place and still denoted by v. The places of F are classified into finite places and archimedean places. Finite places give rise to non-archimedean local fields, and are parametrized by prime ideals of $\mathcal{O}$; the remaining ones are archimedean places, parametrized by conjugacy classes of complex embeddings. The different completions of F are indexed by its places. From now on only places will be mentioned, all the corresponding notions being defined as the ones related to any absolute value belonging to the place.

Finite places are traditionally denoted by gothic letters $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, etc. These are exclusively used for ideals of $\mathcal{O}$, and $\mathfrak{p}$ specifically for prime ideals. The norm of an ideal $\mathfrak{q}$ is denoted $N\mathfrak{q}$, it is the cardinality of the field $\mathcal{O}/\mathfrak{q}$, also called residue characteristic. For a finite place $\mathfrak{p}$, let $\varpi_\mathfrak{p}$ a uniformizer of $\mathfrak{p}$, *i.e.* a generator of $\mathfrak{p}$.

Adeles

Non-equivalent places give rise to non-isomorphic completions, so there is no way to complete F in a canonical fashion. However, F embeds in a locally compact ring taking in account all its completions. This procedure is provided by the ring of adeles. It is the breakthrough of Tate's thesis [119], allowing to enjoy properties similar to completeness without having to arbitrarily choose one place, so that no information attached to the base number field F should be lost. This opens the path to harmonic analysis on number fields.

Definition 5. The ring of adeles $\mathbf{A}$ of F is the restricted product of the completions of F with respect to their associated rings of integers. More precisely,

$$\mathbf{A} = \left\{ (x_v)_v \in \prod_v F_v \ : \ \textit{for almost every } v,\ x_v \in \mathcal{O}_v \right\}. \tag{2.1}$$

Every adele can then be written as an element of the full product $\prod_v F_v$. Similarly, some subsets X of the adeles can be decomposed as a product $\prod_v X_v$. The X_v's are called the local components of X. Given a set of places S, introduce the S-part and the prime-to-S part of a subset X, defined respectively by

$$X_S = \prod_{v \in S} X_v \qquad \text{and} \qquad X^S = \prod_{v \notin S} X_v. \tag{2.2}$$

In particular, the sets admitting a local decomposition as above decomposes as $X = X_S X^S$. For the case of a singleton $X = \{x\}$, the notation is lightened to $x = x^S x_S$. The whole idea of the ring of adeles, besides grasping all the completions at once, is to keep the possibility to enjoy the completeness of each local field. Naturally this decomposition holds as it stands for any partition of the places in more than two sets, and allows to lift a local behavior at S to a global one relative to the ring of adeles.

The ring of adeles is endowed with the restricted product topology. It is defined by the basis of open sets

$$U_S \times \prod_{v \notin S} \mathcal{O}_v, \tag{2.3}$$

where S runs through finite sets of places and U_S is an open set of F_S endowed with the product topology. This topology makes $\mathbf{A}$ a locally compact topological ring. There is a discrete and cocompact embedding of F in $\mathbf{A}$, the so-called diagonal embedding

$$\begin{aligned} F &\longrightarrow \mathbf{A} \\ x &\longmapsto (x)_v \end{aligned} \tag{2.4}$$

Remark. An important fact justifying the choice of the restricted product instead of the full product is that the product of local rings F_v would have failed to be locally compact. Moreover, since the product topology is stronger than the product one, this choice leads to strengthen density properties and are suitable to the purposes of proving equidistribution results.

Theorem 7 (Strong Approximation)**.** *Suppose that G is simply connected, in the sense that the topological space $G(\mathbf{C})$ is simply connected, and that $G'(F_S)$ is noncompact for every simple factor G' of G over F. Let K^S be a compact open subgroup of $G(\mathbf{A}^S)$. Then*

$$G(\mathbf{A}) = G(F)G(F_S)K^S. \tag{2.5}$$

Moreover, if $G'(F_S)$ is noncompact only for every simple quotient G' of G over F, then the following set of double cosets is finite:

$$G(F)\backslash G(\mathbf{A})/G(F_S)K^S. \tag{2.6}$$

This property formalizes the very spirit of the adeles as announced above: embedding F in a better world without loosing its identity. The decomposition (2.5) holds a way back to the F-points of G from properties of its adeles points, justifying the adeles setting as the ground on which modern theoretic problems are handled.

Haar measures

Besides the completeness assets provided by the adelic setting, groups and representations involved in the exploration of the automorphic world are also handled by measure theory tools. This paragraph settles the minimal setting to precisely formulate the problems and state the results, further details on measures on the dual groups are provided in Section 2.3.

Prior to any choice of measure on groups or representations, it is needed to endow underlying global and local fields with measures. A right (resp. left) Haar measure is a positive Radon measure invariant by the right action of G, that is to say $\mathrm{d}(xg) = \mathrm{d}x$ (resp. $\mathrm{d}(gx) = \mathrm{d}x$) for every g in G. It is unique up to multiplication by a scalar. A group for which left and right measures are the same is called unimodular. This is the for instance the case for compact groups, discrete groups, abelian groups, connected reductive groups or semisimple Lie groups. For a non-unimodular group with a right Haar measure $\mathrm{d}g$, the modular quasicharacter is $\delta_G(h) = \mathrm{d}(hg)/\,\mathrm{d}g$ measures the failure of G to be unimodular.

On archimedean fields, the Haar measure on $\mathbf{R}$ is the usual Lebesgue measure $\mathrm{d}x$, and the Haar measure on $\mathbf{C}$ is $2\,\mathrm{d}x \wedge \mathrm{d}y = |\,\mathrm{d}z \wedge \mathrm{d}\bar{z}|$. On non-archimedean fields, Haar measures on $F_{\mathfrak{p}}^\times$ are normalized so that its ring of integers $\mathcal{O}_{\mathfrak{p}}$ gets measure one. The ring of adeles is endowed with the product measure. On the group of units $F^\times$, choose $\mathrm{d}^\times x = \mathrm{d}x/|x|$ as Haar measure.

Turning to locally compact groups G, it is necessary to endow the group of points $G(F)$, $G(\mathbf{A})$ and $G(F_v)$ with a compatible notion of measure with the one chosen on the base local rings. Following Hahn and Getz's presentation [55] automorphically intended, let n be the dimension of G over F. There is a unique nonzero top-dimensional left-invariant differential form $\omega \in \bigwedge^n \mathfrak{g}$ up to scalar in $F^\times$. From this differential form follows through localizing a Radon measure on G_v given by

$$\begin{aligned} C_c(G_v) &\longrightarrow \mathbf{C} \\ f &\longmapsto \int_{G_v} f\, \mathrm{d}|\omega|_v \end{aligned}$$

This provides a left Haar measure on G_v since ω is already a left Haar measure on G_v. Precise construction of those measures are provided in Knapp [74] for the archimedean case and in Oesterlé [97] for the non-archimedean one. For each v, fix the Haar measure normalized so that the maximal compact subgroup K_v is given measure one. The group of adelic points is then endowed with the product measure. Since $G(F)$ embeds discretely in $G(\mathbf{A})$, these choices of measures induce a quotient measure on $G(F)\backslash G(\mathbf{A})$.

2.1.2 Automorphic world

Automorphic representations

Let G be an affine algebraic group over a global field F. The group $G(\mathbf{A})$ of adelic points is locally compact and hence admits a Haar measure. Following the discussion above, the quotient $G(F)\backslash G(\mathbf{A})$ is endowed with a Haar measure giving it finite volume if and only if $G(\mathbf{A}) = G(\mathbf{A})^1$, motivating the introduction of the automorphic quotient

$$[G] = G(F)\backslash G(\mathbf{A})^1. \tag{2.7}$$

This is the base ground on which automorphic theory takes place. The group G acts on the associated Hilbert space $L^2([G])$ by the right regular action

$$g \cdot \phi = \phi(xg), \qquad g \in G, \ \phi \in L^2([G]). \tag{2.8}$$

Definition 6. An automorphic representation of G is an irreducible unitary representation π of $G(\mathbf{A})$ that is isomorphic to a subquotient of the right regular action on $L^2([G])$. The set of all the infinite-dimensional automorphic representations of G is denoted $\mathcal{A}(G)$ and called the universal family of G.

Remarks. This definitions might seem non-standard, leading to the following remarks.

(i) This definition does not require to introduce the admissibility condition, as is the standard way to do so [52], and is more suited for efficient introductory purposes. Both notions coincide [55].
(ii) The choice of excluding finite-dimensional representations from the universal family, that is to say characters in the case of general linear groups, is made so that it embeds as a subfamily of the cuspidal automorphic representations of $\mathrm{GL}(n)$. For statistical purposes, this choice causes no trouble, for the number of characters of bounded conductor is negligible compared to the size of the universal family.

Automorphic representations are one of the most celebrated objects of modern number theory, and the present thesis addresses arithmetic statistics problems relative to them in different settings.

Tensor product theorem

Despite automorphic representations are rather mysterious objects, some of them are particularly easier to grasp. Given a maximal compact subgroup K_v of G_v, a unitary representation of G_v is unramified with respect to K_v if it admits non-zero K_v-fixed vectors, ramified otherwise.

In the same way as the ring of adeles, objects relative to local fields are expected to be easier than global objects, and there is hope that a local-global principle allows to work on local components instead of on the whole global object. Fortunately, this is the case for automorphic representations by the following structure result.

Theorem 8 (Flath). *Every automorphic representation $\pi \in \mathcal{A}(G)$ decomposes as a restricted product $\pi = \otimes_v \pi_v$ where π_v is an irreducible unitary representation of G_v and is unramified for almost every place v.*

2.1.3 Quaternion algebras

After having introduced what automorphic forms are, the time has come to choose the landscape they live on. This section is devoted to do so, presenting the quaternion algebras and their structure before turning back to the associated automorphic quotient. The settings of some unitary and symplectic groups are mentioned in Chapter 5.

First definitions

A quaternion algebra over a field F is a central simple algebra B of dimension 4 over F. Dickson proved in the early 1900s [125, Chap. IX, Theorem 1] that this definition generalizes the usual Hamiltonian quaternions and admits a familiar representation by generator and relations.

Proposition 1. *If the characteristic of F is not 2, then B admits a basis $(1, i, j, k)$ over F such that for some a and $b \in F^\times$,*

$$i^2 = a, \quad j^2 = b, \quad ij = k = -ji. \tag{2.9}$$

Such a quaternion algebra is denoted $D_F(a, b)$, and is entirely determined by a and b up to isomorphism. It can be embedded in a set of matrices over an extension of degree at most two.

Proposition 2. *If α is a root of $X^2 - a$ in an extension of F, then the following is an F-algebra embedding into $F(\alpha)$-matrices.*

$$\begin{aligned} D_F(a, b) &\longrightarrow M_2(F(\alpha)) \\ t + xi + yj + zk &\longmapsto \begin{pmatrix} t + x\alpha & b(y + z\alpha) \\ y - z\alpha & t - x\alpha \end{pmatrix} \end{aligned}$$

A detailed account of the theory of quaternion algebra is established from an algebraic point of view in Vignéras' work [121], as well as quite comprehensively covered in Voight's book [122].

Structure of quaternion algebras

Let B be a quaternion algebra over a global field F. For a given place v, denote by B_v the group of points $B(F_v)$. The local group B_v is isomorphic to either $M_2(F_v)$, case in which v is split, or to a division quaternion algebra, case in which v is ramified. For a given local field F_v, there is a unique division quaternion algebra over F_v up to isomorphism. Quaternion algebra are classified up to isomorphism by their ramification places [121].

Proposition 3. *Let B be a quaternion algebra over F, and R its set of ramification places. Then R is a finite set of even cardinality. Furthermore, it determines B up to isomorphism.*

This structure theorem states a strong local similarity with $\mathrm{GL}(2)$, for completions of a given quaternion algebra are almost everywhere the group of points of $\mathrm{GL}(2)$. This fact appeals two comments. First, while a finite number of places will need a treatment specific to the division quaternion algebras setting, most of them will borrow methods and results from the $\mathrm{GL}(2)$ setting. Second, results obtained for global quaternion algebras are expected to bear information concerning $\mathrm{GL}(2)$ automorphic forms.

From now on, consider a quaternion algebra B over F, and write R for the places of F where B is not split. Introduce $G = Z\backslash B^\times$, where Z denotes the center of $B^\times$.

Inner forms

The groups of units of quaternion algebras is not merely similar to $\mathrm{GL}(2)$ at some places, but are its inner forms. More precisely, quaternion algebras is isomorphic to $M(2)$ over an algebraic closure of F and the underlying isomorphism is given by a conjugation. Moreover, this construction exhausts all the inner forms.

Proposition 4. *Let $\overline{F}$ be an algebraic closure of F. An F-algebra A is isomorphic over $\overline{F}$ to $M_2(\overline{F})$ if and only if A is a quaternion algebra.*

Proof. The algebra A is isomorphic over $\overline{F}$ to $M_2(\overline{F})$ if and only if A is simple over F [125, IX, Coro. 2]. Moreover, the simple algebras over F are the quaternion algebras by definition. This proves quaternion algebras are forms of $M_2(F)$. They are also inner forms by the Skölem-Noether theorem. [115, III.1.4]. □

Automorphic quotient

Automorphic representations live in the automorphic quotient $G(F)\backslash G(\mathbf{A})$. In the case of division quaternion algebras, a fundamental property holds and allows the use of results unavailable for $\mathrm{GL}(2)$.

Proposition 5. *The automorphic quotient $B(F)\backslash B(\mathbf{A})$ is compact modulo the center.*

Proof. This is the Hey theorem, quoted in Voight [122, 38.4.3]. □

Jacquet-Langlands correspondence

The Jacquet-Langlands correspondence [66] provides an embedding of the representations of a division quaternion algebra into representations of GL(2). It is stated here in the centerless setting.

Theorem 9 (Local Jacquet-Langlands)**.** *Let v be a place where B ramifies. There is a bijection between irreducible smooth representations of G_v and irreducible discrete series representations of* $\mathrm{PGL}(2, F_v)$.

Theorem 10 (Global Jacquet-Langlands)**.** *There is a unique bijection between infinite dimensional automorphic representations of $G(\mathbf{A})$ and irreducible cuspidal automorphic representations of* $\mathrm{PGL}(2, \mathbf{A})$ *that is compatible with the local Jacquet-Langlands correspondence.*

2.2 Analytic conductors

Once automorphic representations introduced, it is necessary to make sense of the counting problem. In order to determine the actual size of the universal family and some sharper statistical properties, as densities or equidistribution, it is needed to truncate it to a finite set. This section explores the notion of size provided by the analytic conductor.

The analytic conductor is an intrinsic notion of size grasping the complexity of automorphic representations. There are different standard constructions, either based on representation theoretic properties or using specific invariants attached to automorphic L-functions. While some appear as more natural, others turn to suit more efficiently the purposes and methods of arithmetic statistics questions. Besides introducing some of the frequent definitions of the analytic conductor appearing in the literature, their consistency and soundness are discussed in the following paragraphs.

2.2.1 Conductor arising from functional equations

Let turn back for a moment to a more usual setting: the universal family $\mathcal{A}(G)$ embeds, via the Jacquet-Langlands correspondence, see Theorem 10, as a subfamily of the universal family $\mathcal{A}(\mathrm{PGL}(2))$, composed of all the cuspidal automorphic representations of PGL(2). In this latter context, even in the broader setting of cusp forms on general linear groups, Iwaniec and Sarnak [65] have defined a good notion of size, given by the analytic conductor. It is a real number $c(\pi)$ defined from the functional equation satisfied by the finite part L-function $L(s,\pi)$ associated to a generic $\pi \in \mathcal{A}(\mathrm{PGL}(2))$, which takes the form

$$L(1-s,\pi) = \varepsilon_\pi X(s,\pi)L(s,\pi), \tag{2.10}$$

where ε_π is the root number of π and takes value 1 or -1, since π is self-dual. The quantity $|X(s,\pi)|$ takes value 1 at the central point $\frac{1}{2}$, leading to the definition of the analytic conductor following Conrey et al. [30].

Definition 7. Let π be a generic automorphic representation on PGL(2). With the completing factor introduced in (2.10), the L-analytic conductor of π is

$$c(\pi) = \left| X'\left(\frac{1}{2},\pi\right)\right|. \tag{2.11}$$

The functional equation relates the value of an L-function at a point s with its value at the symmetric point $1-s$. The easiest case is naturally the symmetric one, *i.e.* when $X(s,\pi) = \varepsilon_\pi$, corresponding to a conductor equal to zero. This leads to interpreting $X(s,\pi)$ as a measure of the failure of $L(s,\pi)$ to be symmetric. The function $X(s,\pi)$ is built from the factors necessary to complete $L(s,\pi)$ to get a symmetrical functional equation, and involves the usual arithmetic conductor as well as archimedean gamma factors, so that the analytic conductor encapsulates the complexity of π.

It allows to truncate the universal family of PGL(2) into

$$\mathcal{A}(Q) = \{\pi \in \mathcal{A}(\mathrm{GL}(2)) \ : \ c(\pi) \leqslant Q\}, \qquad Q \geqslant 1. \tag{2.12}$$

This set is known to be discrete and finite by the work of Brumley [16]. Even if the analytic conductor is a relevant notion of size satisfying the needed finiteness property, it is a rather impenetrable quantity, as is the completing factor $X(s,\pi)$.

2.2.2 Opening the ε-factor

The definition of the analytic conductor provided above is general and efficient in its formulation, yet it is far from easily reachable in practice. As announced in the introduction, despite the finiteness of the truncated family, there is no reason for it to be more handable than the whole universal family, for this analytic conductor appears as a mysterious parameter and introducing it may seem to be an unnecessary complication. This is far from the case: the precise knowledge of the structure of the automorphic L-functions and their associated functional equations leads to a more explicit formulation of the conductor in terms of different spectral parameters, that turn out to be well suited for trace formula treatment.

The definition of the analytic conductor is summarized by Conrey, Farmer, Keating, Rubinstein and Snaith [29]. The Selberg class is an axiomatic definition of what an

L-function should be in general. In particular, there is a quantity called the ε-factor which is a way to make $L(s,\pi)$ entire, for $\pi \in \mathcal{A}(\mathrm{GL}(2))$. It is of the form

$$\varepsilon(s,\pi) = Q_\pi^s \prod_{v|\infty} \Gamma_v(s+\mu_\pi(v)). \tag{2.13}$$

where Q_π is a positive real number and v runs through the archimedean places of F. Moreover, Γ_v is equal to $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ in case of a real place, and to $\Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1)$ in case of a complex place. This ε-factor is such that the completed function

$$\xi(s,\pi) = \varepsilon(s,\pi)L(s,\pi), \qquad s \in \mathbf{C}, \tag{2.14}$$

is entire and satisfies the symmetric functional equation

$$\xi(s,\pi) = \varepsilon_\pi \xi(1-s,\pi), \qquad s \in \mathbf{C}. \tag{2.15}$$

Following the presentation of Iwaniec and Sarnak [65], the analytic conductor is defined as follows from the data of the ε-factor.

Definition 8. Let π be a generic automorphic representation on PGL(2) and introduce the associated ε-factor (2.13). The ε-conductor of π is

$$c_\varepsilon(\pi) = Q_\pi \prod_v \left(1 + |\mu_\pi(v)|\right). \tag{2.16}$$

These definitions of the analytic conductor appeal quite a few caveats concerning their compatibility. Both notions c_ε and c defined above does not coincide, as straightforward computations show. Indeed, comparing the functional equation satisfied by $\xi(s,\pi)$ with the definition of $X(s,\pi)$ in the completion (2.10), it follows that

$$X(s,\pi) = \frac{\varepsilon(1-s,\pi)}{\varepsilon(s,\pi)}. \tag{2.17}$$

The explicit definition of the gamma factors (2.13) yields

$$\log(\varepsilon)'(s) = \log Q_\pi + \log \sum_{v|\infty} \frac{\Gamma_v'}{\Gamma_v}\left(s+\mu_\pi(v)\right), \tag{2.18}$$

so that both definitions are equivalent up to the approximation of replacing the digamma function Γ'/Γ by the logarithm. This is not an equality yet a standard approximation for small values of s, in particular around the central point, provided by the Stirling formula. At last, both definitions differ by constants and normalizing factors. The so-called arithmetic part Q_π of the conductor is a well-defined and non ambiguous notion, it is always present as it stands in any definition of the analytic conductor.

However this is not the case for its archimedean part. It has to be underlined that the notion of archimedean conductor is a working definition intended to reasonably grasp the complexity of an L-function to govern some of its statistical behavior. Some attempts have recently been made, for instance in unpublished works due to Paul Nelson or Peter Humphries, to find a more canonical way to define the archimedean conductor, motivated by theories of local newforms, yet for now they fail to match the expected behavior of the analytic conductor.

The fundamental property of the conductor is the finiteness of the truncated universal family as well as its relations with invariants attached to L-functions. Hence, it allows some freedom in the choice of certain normalizing factors, for instance to ensure non-vanishing, see Chapter 4. This feature is not present in deepest problems concerning L-functions, typically subconvexity questions where the choice of the archimedean analytic conductor might be critical. There is no need to settle this debate here, and the methods borrowed from [17] still hold for any of the choices mentioned above, and for a more general class of size functions.

2.2.3 Conductor as a notion of depth

A more geometric interpretation of this notion of conductor, where the structure of the underlying group appears, would give a better computational grasp on the conductor. This is provided by the so-called theory of local newforms.

This section is devoted to $B^\times$ more than to G, for it lightens notations. This local convention makes no harm, for a representation π of $G(\mathbf{A})$ is viewed as a representation of $B^\times(\mathbf{A})$ with trivial central character. By Flath's theorem, an irreducible admissible representation of $B^\times(\mathbf{A})$ decomposes in a unique way as a restricted tensor product $\pi = \otimes_v \pi_v$ of irreducible smooth representations where almost every component π_v is unramified. It is hence natural to define first the conductor for the local components π_v. Setting $c(\pi_\mathfrak{p}) = 1$ for the finite unramified components guarantees well-definiteness of the product over all places and is required to get the consistency with the conductors defined above. The aim of the present section is to define the notion of conductor for the remaining local components.

Split local components

Proposition 6. *Let $\pi \in \mathcal{A}(G)$ and v a split place. The local component π_v is infinite-dimensional.*

Proof. Since the universal family excludes global characters, a representation π in it is generic. The Jacquet-Langlands correspondence preserves genericity, hence as shown on the diagram below, the global Jacquet-Langlands correspondence associates to it a generic representation $\mathrm{JL}(\pi)$ of $\mathrm{GL}(2)$, thus also its local components $\mathrm{JL}(\pi)_v$. These local components are also the images by the local Jacquet-Langlands correspondence

$\mathrm{JL}(\pi_v)$ of the local components of π.

$$\begin{tikzcd}
\pi \in \mathcal{A}(B^\times) \arrow[r, "\mathrm{JL}"] \arrow[d, "v"'] & \begin{array}{c}\mathrm{JL}(\pi) \in \mathcal{A}(\mathrm{GL}(2)) \\ \text{generic}\end{array} \arrow[d, "v"] \\
\pi_v \arrow[r, "\mathrm{JL}_v", "\text{id if } v \notin R"'] & \begin{array}{c}\mathrm{JL}(\pi)_v \\ \text{generic}\end{array}
\end{tikzcd}$$

At split places, the local Jacquet-Langlands correspondence is the identity, for then $B_\mathfrak{p}^\times \cong \mathrm{GL}(2, F_\mathfrak{p})$. Moreover, the correspondence is unique, thus the local components π_v, at split places, are generic hence infinite-dimensional. □

This in particular implies that local components being a character can only arise at ramified places. First of all the focus will be on split places, before turning to the ramified places by the pullback procedure to the split ones.

Non-archimedian split case

For finite split places $\mathfrak{p}$, by definition $B_\mathfrak{p} \cong M(2, F_\mathfrak{p})$ so that $B_\mathfrak{p}^\times \cong \mathrm{GL}(2, F_\mathfrak{p})$. The notion of local conductor for irreducible smooth infinite-dimensional representations of GL(2) has been introduced by Casselman [21]. Consider the sequence of compact open congruence subgroups

$$K_{0,\mathfrak{p}}(\mathfrak{p}^r) = \left\{ g \in \mathrm{GL}(2, \mathcal{O}_\mathfrak{p}) \ : \ g \equiv \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix} \bmod \mathfrak{p}^r \right\} \subseteq B_\mathfrak{p}^\times, \qquad r \geqslant 0. \tag{2.19}$$

This sequence is a filtration, *i.e.* a decreasing sequence of subgroups. Since the representations considered are smooth, the existence of fixed vectors for a small enough subgroups is guaranteed. The conductor of an irreducible admissible infinite-dimensional representation $\pi_\mathfrak{p}$ of $B_\mathfrak{p}^\times$ with trivial central character is then defined by the smallest rank for which it happens.

Definition 9. The additive conductor of $\pi_\mathfrak{p} \in \widehat{G}_\mathfrak{p}$ is

$$\mathfrak{f}(\pi_\mathfrak{p}) = \min\left\{ r \in \mathbf{N} \ : \ \pi_\mathfrak{p}^{K_{0,\mathfrak{p}}(\mathfrak{p}^r)} \neq 0 \right\}, \tag{2.20}$$

and the multiplicative and analytic conductor of $\pi_\mathfrak{p}$ are respectively defined by

$$\mathfrak{c}(\pi_\mathfrak{p}) = \mathfrak{p}^{\mathfrak{f}(\pi_\mathfrak{p})} \quad \text{and} \quad c(\pi_\mathfrak{p}) = N\mathfrak{c}(\pi_\mathfrak{p}). \tag{2.21}$$

The existence of the conductor is guaranteed by the work of Casselman [21], who also states that the growth of the dimensions of the fixed vector spaces are given by

$$\dim \pi_\mathfrak{p}^{K_{0,\mathfrak{p}}\left(\mathfrak{p}^{\mathfrak{f}(\pi_\mathfrak{p})+i}\right)} = i + 1, \qquad i \geqslant 0. \tag{2.22}$$

Global analytic conductor

Definition 10. Let π be an automorphic representation of GL(2), and write $\pi = \otimes_v \pi_v$ for its tensor product decomposition. Its global analytic conductor is defined to be

$$c(\pi_f) = \prod_v c(\pi_v). \tag{2.23}$$

This gives a well-defined notion of conductor, for the π_v are almost everywhere unramified. One of the fundamental facts of the theory of local newforms is the consistency with the definition of the arithmetic conductor, *i.e.* the depth-conductor notion defined in (2.21) and the the one coming from ε-factors (2.13) of the associated L-functions are compatible in the following way.

Proposition 7. *Let $\pi \in \mathcal{A}(\mathrm{GL}(2))$. The notions of arithmetic conductor given in* (2.23) *is compatible [66] with the one coming from L-functions* (2.11)*, that is to say*

$$\prod_{\mathfrak{p}} \mathfrak{c}(\pi_{\mathfrak{p}}) = Q_\pi. \tag{2.24}$$

Conductor of characters

Note that for now conductors have been defined only for generic representations. However, characters can arise as local component at ramified places as discussed above. Every character of $B_{\mathfrak{p}}^\times$ is a composition

$$B_{\mathfrak{p}}^\times \longrightarrow F_{\mathfrak{p}}^\times \longrightarrow \mathbf{C}, \tag{2.25}$$

where the first application is the reduced norm, and the second one a character of $F_{\mathfrak{p}}^\times$. In other words, every character of $B_{\mathfrak{p}}^\times$ is of the form $\chi_0 \circ N$ where χ_0 is a character of $F_{\mathfrak{p}}^\times$. In order to stay consistent, define the conductor of a local character at a ramified place as the conductor of its Jacquet-Langlands embedding in PGL(2). Since the character $\chi_0 \circ N$ is sent on the twisted Steinberg representation $\mathrm{St} \otimes \chi_0$, it follows

$$\mathfrak{c}(\chi_0 \circ N) = \begin{cases} \mathfrak{p} & \text{if } \chi_0 \text{ unramified;} \\ \mathfrak{c}(\chi_0)^2 & \text{if } \chi_0 \text{ ramified.} \end{cases} \tag{2.26}$$

Analytic conductor for quaternion algebras

The notion of analytic conductor defined above for automorphic representations of GL(2) extends to a definition for representations of G, viewed as automorphic representations of $B^\times$ with trivial central character. Indeed, following the pullpack procedure for heights, define for a local representation at a ramified place v its conductor as the one of its Jacquet-Langlands transfer.

Definition 11. Let $\pi_v \in \widehat{G_v}$. Its local conductor is defined by

$$c(\pi_v) = c(\mathrm{JL}(\pi_v)). \tag{2.27}$$

Since for both split and ramified places the conductor of a local representation of G_v is defined, the global conductor can be introduced.

Definition 12. Let π be an automorphic representation of G, and write $\pi = \otimes_v \pi_v$. Its global analytic conductor is defined to be

$$c(\pi) = \prod_v c(\pi_v), \tag{2.28}$$

which is well-defined, for almost every component π_v is unramified.

Remarks. The choice of defining the conductor at ramified place in this roundabout way seems far from natural and appeals for some comments.

(i) The definition of the conductor (2.11) coming from the completing factor of associated L-functions is a systematic and intrinsic way to define the conductor of a generic representation of a group G. This is provided by the Godement-Jacquet [56] construction of L-functions, making possible the definition of the ε-conductor without appealing to an embedding in $\mathrm{GL}(n)$. The Jacquet-Langlands correspondence stated in Theorem 10 preserves the notion of L-function and hence also makes the inner notion of ε-conductor for G compatible with the one defined for $\mathrm{GL}(2)$. Since the analytic conductor is by definition compatible between G and $\mathrm{GL}(2)$, this choice makes no harm compared to directly defining the conductor from the associated L-functions on G.

(ii) There is a candidate introduced by Lansky and Raghuram [82] providing a filtration of subgroups of G that are roughly of the same size as those introduced in (2.19). However, it is not known whether or not the attached notion of depth-conductor is the same.

Since the notion of size has been properly defined and motivated, it becomes possible to precisely state the problem and to consider the truncated universal family

$$\mathcal{A}(Q) = \{\pi \in \mathcal{A}(G) \ : \ c(\pi) \leqslant Q\}, \qquad Q \geqslant 1. \tag{2.29}$$

2.3 Equidistribution

The way a family localizes in a given measurable space is formalized in the notion of equidistribution with respect to a measure. It is a weak convergence against a certain class of functions. The limit measure expresses the probability of an element of the family to appear in a given region. This section is dedicated to introduce some elements of equidistribution and then restate Theorem B as a weak convergence against wider and more handable class of functions.

2.3.1 Elements of equidistribution

The whole equidistribution question consists in deciding whether the proportion of elements of a family lying in a given zone of the space tend to the measure of that zone. In the case of the universal family, for every open relatively compact set U with measure zero boundary in the unitary dual, Theorem B aims at estimating the proportion of automorphic forms lying in U, that is to say

$$\frac{\#\{\pi \in \mathcal{A}(Q) \ : \ \pi \in U\}}{N(Q)}, \qquad Q \geqslant 1. \tag{2.30}$$

This statistic question suffers from the lack of individueal knowledge of automorphic forms. The measure representing the distribution of the elements in the truncated universal family is the counting measure

$$\mu_Q = \frac{1}{N(Q)} \sum_{\pi \in \mathcal{A}(Q)} \delta_\pi, \qquad Q \geqslant 1. \tag{2.31}$$

The very same idea as the Weyl criterion for equidistribution of sequences modulo 1, or more generally Portmanteau theorems, applies to this setting and leads to restating the equidistribution (2.30) in a functional way. For a measure ν on $\widehat{G} = \prod_v \widehat{G}_v$, recall that

$$\nu(f) = \int_{\widehat{\Pi}} f(\pi)\, \mathrm{d}\nu(\pi), \tag{2.32}$$

for functions such that the expression above has a meaning. So that Theorem B states that $\mu_Q(f)$ converges to $\mu(f)$ for every f among the characteristic functions of relatively quasi-compact open sets of $\widehat{\Pi}$ with thin boundary.

The whole point in choosing a functional formulation is to enlarge the scope of the test functions involved in (2.30) to see them as part of a wider but better-behaved space of functions. Let S be a finite set of places of B. Define $F(\widehat{G}_S)$ to be the space of complex Plancherel-measurable and bounded functions on $\widehat{G}_S$ supported on a finite number of Bernstein components and whose restriction to the tempered spectrum is continuous outside a measure zero set.

Proposition 8 (Sauvageot)**.** *For every relatively compact open set U with thin boundary in $\widehat{G}_S$, the characteristic function $\mathbf{1}_U$ belongs to $F(\widehat{G}_S)$.*

A sequence $(\nu_n)_n$ of Radon positive measures on $\widehat{\Pi}$ weakly converges to a measure ν if $\nu_n(f)$ converges to $\nu(f)$ for every $f \in F(\widehat{G}_S)$ when n goes to infinity, for every finite set S of places. Since the considered characteristic functions lie in $F(\widehat{G}_S)$, weak convergence of μ_Q to μ implies Theorem B.

From now on, in order to avoid the expression of $N(Q)$, the measure considered is slightly modified to be

$$\nu_Q = \frac{1}{Q^2} \sum_{\pi \in \mathcal{A}(Q)} \delta_\pi, \qquad Q \geqslant 1. \tag{2.33}$$

This is motivated by the fact, from Theorem A, that $N(Q)$ is of asymptotical order CQ^2, so that Theorem B is equivalent to: ν_Q weakly converges to the measure

$$\nu = C\frac{\mu}{||\mu||} = \frac{1}{2}\mathrm{vol}(G(F)\backslash G(\mathbf{A}))\mu. \tag{2.34}$$

2.3.2 Plancherel formulas & Fourier transformation

Technical tools suitable to deal with the action (2.32) of spaces of functions are provided in this section. The action of automorphic representations is extended to functions on G by

$$\pi(f) = \int_G f(g)\pi(g)\,\mathrm{d}g, \qquad \pi \in \mathcal{A}(G), \quad f \in C_c(G). \tag{2.35}$$

This defines a trace class operator, and allows to define its Fourier transform

$$\widehat{f}(\pi) = \mathrm{tr}\,\pi(f), \qquad \pi \in \mathcal{A}(G), \quad f \in C_c(G). \tag{2.36}$$

Specific choices of subclasses of functions, on which the automorphic representations are considered acting on, lead to better properties of the corresponding operators, as in the following proposition.

Proposition 9. *Let K be a compact open subgroup of G and π an automorphic representation of G. For every left-K-invariant f in $C_c(G)$ and every x in the representation space V_π of π, the image $\pi(f)x$ is K-invariant.*

Proof. For every $x \in V_\pi$ and $k \in K$, the left-K-invariance of f yields

$$\begin{aligned}\pi(k)\left(\pi(f)x\right) &= \int_G f(g)\pi(kg)x\,\mathrm{d}g \\ &= \int_G f(k^{-1}g)\pi(g)x\,\mathrm{d}g \\ &= \pi(f)x\end{aligned}$$

proving the claim. □

Of particular interest are the Hecke algebras. For a finite place v, the Hecke alegbra $\mathcal{H}(G_v)$ is the convolution algebra of complex valued, locally constant and compactly supported functions on G_v. For an archimedean place v, the Hecke alegbra $\mathcal{H}(G_v)$ is the convolution algebra of complex valued, smooth and compactly supported functions on G_v. The global Hecke algebra is denoted $\mathcal{H}(G(\mathbf{A}))$ or $\mathcal{H}(G)$, and is the algebra generated by the restricted products $\phi = \prod_v \phi_v$, where ϕ_v is a function of $\mathcal{H}(G_v)$ and almost every local component $\phi_\mathfrak{p}$ equals to $\mathbf{1}_{K_\mathfrak{p}}$.

The unitary dual group $\widehat{G_v}$ is endowed with the Fell topology. The Plancherel measure associated to it is the unique positive Radon measure μ_v^{Pl} on $\widehat{G_v}$ such that the Plancherel inversion formula of Harish-Chandra holds, *i.e.*

$$\int_{\widehat{G_v}} \widehat{\phi_v}(\pi_v)\, \mathrm{d}\mu_v^{\mathrm{Pl}}(\pi_v) = \phi_v(1), \qquad \phi_v \in \mathcal{H}(G_v). \tag{2.37}$$

From now on, every integral on $\widehat{G_v}$ will be written with the convention that $\mathrm{d}\pi_v$ stands for $\mathrm{d}\mu_v^{\mathrm{Pl}}(\pi_v)$, leading to no ambiguity.

2.3.3 Sauvageot density theorem

In spite of the class $F(\widehat{G_S})$ of test functions being wider than mere characteristic functions, it has a surprusingly good analytical behavior. Indeed, the Sauvageot density theorem [111] states that any function in $F(\widehat{G_S})$ can be approximated by Fourier transforms of functions in the Hecke algebra of G_S.

Theorem 11 (Sauvageot)**.** *For every $f \in F(\widehat{G_S})$ and $\varepsilon > 0$, there exist functions ϕ and ψ in the Hecke algebra $\mathcal{H}(G_S)$ such that*

- $\forall \pi \in \widehat{G_S}, \left| f(\pi) - \widehat{\phi}(\pi) \right| \leqslant \widehat{\psi}(\pi)$
- $\mu_S^{\mathrm{Pl}}(\widehat{\psi}) \leqslant \varepsilon$

Thus, in order to prove the convergence of $\nu_Q(f)$ to $\nu(f)$ for every function f in $F(\widehat{G_S})$, it is sufficient to prove it for such Fourier transforms. Indeed, let $f \in F(\widehat{G_S})$. For $\varepsilon > 0$, Sauvageot's theorem guarantees the existence of ϕ and ψ in the Hecke algebra $\mathcal{H}(G_S)$ such that $\widehat{\phi}$ and $\widehat{\psi}$ satisfy the conditions above. Thus

$$\begin{aligned}
|\nu_Q(f) - \nu(f)| &\leqslant |\nu_Q(f) - \nu_Q(\widehat{\phi})| + |\nu_Q(\widehat{\phi}) - \nu(\widehat{\phi})| + |\nu(\widehat{\phi}) - \nu(f)| \\
&\leqslant \nu_Q(\widehat{\psi}) + |\nu_Q(\widehat{\phi}) - \nu(\widehat{\phi})| + \nu(\widehat{\psi}) \\
&\leqslant |\nu_Q(\widehat{\psi}) - \nu(\widehat{\psi})| + 2\nu(\widehat{\psi}) + |\nu_Q(\widehat{\phi}) - \nu(\widehat{\phi})|
\end{aligned}$$

From the definition of ν and the domination in the Sauvageot theorem, it follows since conductors are at lmeast one that

$$\nu(\widehat{\psi}) \ll \zeta^\star(1) \prod_v \zeta_v(1)^{-1} \int_{\widehat{G_v}} \widehat{\psi}(\pi_v) \frac{\mathrm{d}\pi_v}{c(\pi_v)^2}$$

$$\ll \prod_v \zeta_v(1)^{-1} \int_{\widehat{G}_v} \widehat{\psi}(\pi_v)\, \mathrm{d}\pi_v < \mu_S^{\mathrm{Pl}}(\widehat{\psi}) \leqslant \varepsilon$$

Thus,

$$|\nu_Q(f) - \nu(f)| \ll \varepsilon + |\nu_Q(\widehat{\psi}) - \nu(\widehat{\psi})| + |\nu_Q(\widehat{\phi}) - \nu(\widehat{\phi})|. \tag{2.38}$$

In order to prove that ν_Q weakly converges to ν, it is then sufficient to show that the second and third terms vanish for $Q \to \infty$, *i.e.* to prove the theorem for such functions $\widehat{\phi}$ and $\widehat{\psi}$. A sharper result than what is needed for Theorem B can be proven, with a precise remainder term in the case of Fourier transforms.

Theorem G. *For every finite set of places S and $\phi \in \mathcal{H}(\widehat{G}_S)$,*

$$\nu_Q\left(\widehat{\phi}\right) = \nu\left(\widehat{\phi}\right) + \begin{cases} O\left(Q^{-1+\varepsilon}\right) & \text{if } F = \mathbf{Q} \text{ and } B \text{ totally definite;} \\ O\left(Q^{-\delta_F}\right) & \text{if } F \neq \mathbf{Q} \text{ and } B \text{ totally definite;} \\ O\left(\frac{1}{\log Q}\right) & \text{if } B \text{ is not totally definite.} \end{cases}$$

Remark. The underlying constants depend on ϕ. This is not a problem since only convergences matter for Theorem B.

2.3.4 Admissible functions

Let $\phi \in \mathcal{H}(G_S)$. The action of $\widehat{\phi}(\pi)$, shortcut notation for $\widehat{\phi}(\pi_S)$, can be assumed to have a selecting effect on the spectral data. Indeed, the trace Paley-Wiener theorem of Bernstein, Deligne and Kazhdan [8] provides the fundamental properties of the Fourier transforms.

Theorem 12 (Trace Paley-Wiener). *The functions on $\widehat{G}_{\mathfrak{p}}$ lying in the image by the Fourier transform of the Hecke algebra $\mathcal{H}(G_{\mathfrak{p}})$ are the functions $\widehat{\phi}$ on $\widehat{G}\mathfrak{p}$ such that*

- *(i) for every standard Levi subgroup M of $G_{\mathfrak{p}}$ and every irreducible representation σ of M, the function $\psi \mapsto \widehat{\phi}(\mathrm{ind}_{G_{\mathfrak{p}}}^M(\psi\sigma))$ is a regular function on the complex algebraic variety $\psi(M)$ composed of the unramified characters of M;*
- *(ii) there exists an open compact subgroup K of $G_{\mathfrak{p}}$ dominating ϕ, i.e. such that ϕ is nonzero only on representations having non trivial K-fixed space π^K.*

It follows that Fourier transforms selects representations of bounded conductor.

Proposition 10. *Let $\phi \in \mathcal{H}(G_S)$. There exists $c_\phi > 0$ such that for every generic $\pi \in \widehat{G}_S$ in the support of $\hat{\phi}$, the conductor of π is less than c_ϕ.*

Proof. Since S is a finite set of places it is sufficient to prove the result for a local component. In the case of a finite place $\mathfrak{p}$, let $\phi_{\mathfrak{p}}$ be the $\mathfrak{p}$-component of ϕ, where

$\mathfrak{p} \in S$. The property (ii) of the Trace Paley-Wiener theorem states that its Fourier transform $\widehat{\phi}_\mathfrak{p}$ is dominated by a certain open compact subgroup K of $G_\mathfrak{p}$, that is to say is supported on representations $\pi_\mathfrak{p}$ having nontrivial fixed space $\pi_\mathfrak{p}^K$. Since K is open and the sequence $(K_{0,\mathfrak{p}}(\mathfrak{p}^i))_i$ is a filtration in $G_\mathfrak{p}$, K contains a certain conjugate of a $K_{0,\mathfrak{p}}(\mathfrak{p}^r)$ and hence $\widehat{\phi}_\mathfrak{p}$ is nonzero only for representations of conductor dividing $\mathfrak{p}^r$. Since S contains only a finite number of places, this proves that $\widehat{\phi}$ selects only representations $\pi_S \in \widehat{G}_S$ with conductor dividing the product of the corresponding $\mathfrak{p}^r$. □

However, bounding the conductor is not enough for the purposes of the trace formula, thus some modifications on $\widehat{\phi}$ are necessary. However, it is far from obvious that such modified functions are still approachable by Fourier transforms. In order to select automorphic representations in the universal family through trace formula methods, it is necessary to restrict the Fourier transforms considered to the generic spectrum, for otherwise there is no notion of conductor attached to a representation. The following proposition states that it is possible, up to another approximation by density.

Proposition 11. *Let $\tilde{\phi}$ be the restriction of $\hat{\phi}$ to the generic spectrum, extended by zero elsewhere on the unitary dual. Then $\tilde{\phi}$ lies in $F(\widehat{G}_S)$.*

Proof. Recall that the Sauvageot density theorem, stated in Proposition 8 provides a criterion for functions to be approachable by Fourier transforms. All the properties of the Sauvageot class $F(\widehat{G}_S)$ obviously hold for $\tilde{\phi}$ safe possibly the condition on the discontinuity points. The proof is done by the explicit classification of the representations on PGL(2) on local fields. Since $\tilde{\phi}$ is supported on a finite number of Bernstein component, it is possible to assume it is supported on only one component with no loss of generality.

For archimedean places, the only unitary non-generic representation of $\mathrm{PGL}(2, \mathbf{R})$ is the trivial representation, and it is of zero Plancherel measure. The Steinberg representation also lies in the boundary of the generic spectrum. Since the Steinberg representation has positive Plancherel measure, it is not a discontinuity point of $\tilde{\phi}$, otherwise it would already have been a discontinuity point of $\hat{\phi}$, what is incompatible with the Sauvageot conditions. The cases of $\mathrm{PGL}(2, \mathbf{C})$ and $\mathrm{PGL}(2, F_\mathfrak{p})$ should be treated similarly. □

The conductor of a representation in the generic spectrum is well-defined. It is also necessary to restrict the functions to fixed conductors, what has a meaning since Proposition 11 allows to restrict to the generic dual. This is the meaning of the next proposition.

Proposition 12. *Let $\mathfrak{q}$ be a an integer ideal. Let $\bar{\phi}$ be the restriction of $\tilde{\phi}$ to representations of fixed conductor $\mathfrak{q}$, extended by zero elsewhere. Then $\bar{\phi}$ lies in $F(\widehat{G}_S)$.*

Proof. Since the support of $\mathfrak{q}$ contains only a finite number of places, it is enough to prove the proposition for the restriction to conductors $\mathfrak{p}^r$. This is also possible by the explicit classification of the dual of $\mathrm{PGL}(2, F_v)$. □

2.4 Universal family

The truncated universal family has to be explored further. A refined decomposition of $\mathcal{A}(Q)$ using the specific behavior at the different places is possible. The explicit notion of conductor given through the depth notion associated to the filtration (2.19) is suitable to treat the corresponding split components of an automorphic representation. Ramified and archimedean components need specific treatment yet arise in a finite number of places. This section supply the necessary toolbox, following the presentation of Finis, Lapid and Müller [43] as well as of Brumley and Milićević [17].

2.4.1 Archimedean Langlands classification

The local Langlands classification of the archimedean admissible dual [73] of GL(2) provides a recipe for constructing the admissible representations of reductive groups over archimedian local fields in terms of representations of Levi subgroups. Since unitary representations are in particular admissible, it induces a parametrization of the unitary dual of $\mathrm{GL}(2, F_\infty^R)$. Let $\mathcal{L}_\infty$ the finite set of Levi subgroups of $\mathrm{GL}(2, F_\infty^R)$ containing the diagonal torus. For such a Levi M, define $\mathcal{E}_2(M^1)$ to be the set of isomorphism classes of square integrable representations of M^1. The only nonempty cases are

- $\mathcal{E}_2(\mathrm{GL}(1, \mathbf{R})^1)$ consisting into the trivial character and the sign character;
- $\mathcal{E}_2(\mathrm{GL}(1, \mathbf{C})^1)$ composed by the characters $z^k/|z|^k$ for integers k;
- $\mathcal{E}_2(\mathrm{GL}(2, \mathbf{R})^1)$ that is the set of discrete series representations of weight $k \geqslant 2$.

Introduce the set $\mathcal{D}$ of G_∞^R-classes of conjugation of pairs $\underline{\delta} = (M, \delta)$ with $M \in \mathcal{L}_\infty$ and $\delta \in \mathcal{E}_2(M^1)$: they constitute the discrete spectral data parametrizing the archimedean sprectrum. Write $\mathfrak{h}_{M,\mathbf{C}}^\star$ for the trace-zero hyperplane of the complexified dual of the Lie algebra of M, which is a finite-dimensional $\mathbf{C}$-vector space. The spectral data $\underline{\delta} \in \mathcal{D}$, consisting of a Levi $M \in \mathcal{L}_\infty$ and a discrete series representation $\delta \in \mathcal{E}_2(M^1)$, along with $\nu \in \mathfrak{h}_{M,\mathbf{C}}^\star$, give rise to an admissible representation of G_∞^R in the following way. The unitary induction $\mathrm{Ind}_P^G(\delta \otimes e^\nu)$ from the unique parabolic subgroup P containing M is not necessarily irreducible, yet the following holds.

Proposition 13 (Archimedean Langlands classification)**.** *Let $\delta \in \mathcal{D}$ and $\nu \in \mathfrak{h}_{M,\mathbf{C}}^\star$, denote W_δ the stabilizer of δ in the Weyl group of $\mathfrak{h}_M$. There is a unique ν' in the class of ν modulo translation by W_δ such that the induction $\mathrm{Ind}_P^G(\delta \otimes e^{\nu'})$ admits a unique irreducible quotient, denoted by $\pi_{\delta,\nu}$. Moreover, every admissible irreducible representation of $\mathrm{GL}(2, F_\infty^R)$ arises uniquely in this way, up to infinitesimal equivalence.*

This construction exhausts the admissible dual of $\mathrm{GL}(2, F_\infty^R)$ up to infinitesimal equivalence. This is the archimedean Langlands classification [72, Theorem 8.54], that can be reformulated as

$$\widehat{G}_\infty^{R,1} \cong \bigsqcup_{M \in \mathcal{L}_\infty} \mathcal{E}_2(M^1) \times \mathfrak{h}_{M,\mathbf{C}}^\star / W, \tag{2.39}$$

where $\widehat{G}_\infty^{R,1}$ stands for the admissible dual of $\mathrm{GL}(2, F_\infty^R)$ up to infinitesimal equivalence. Note that $\mathcal{D}$ is a discrete set, leading to refer to $\underline{\delta}$ as the discrete archimedean spectral parameter of π, or the discrete parameter of π_∞^R, while ν in $\mathfrak{h}_{M,\mathbf{C}}^\star/W$ is called the continuous archimedean parameter of π.

2.4.2 Decomposition

In order to address the problem of the weak convergence of ν_Q to prove Theorem G, it is necessary to decompose the universal family into smaller sets with fixed spectral data, amenable to trace formula methods. Let S be a finite set of places and $\phi \in \mathcal{H}(G_S)$. The conductor of $\pi \in \mathcal{A}(G)$ splits into local conductors, and in particular it can be written

$$c(\pi) = c(\pi_R) c(\pi_\infty^R) c(\pi_{S,f}^R) N\mathfrak{c}(\pi_f^{R,S}). \tag{2.40}$$

This decomposition emphasizes the different kind of information and behavior each type of place is endowed with, and turns to be a guide for decomposing the counting measure $\nu_Q(\hat{\phi})$ of the truncated universal family. Concerning the split archimedean places, introduce the truncated archimedean split dual

$$\Omega(X) = \left\{ \pi_\infty^R \in \widehat{G}_\infty^R \ : \ c(\pi_\infty^R) \leqslant X \right\}, \qquad X > 0. \tag{2.41}$$

This set of archimedean parameters factorizes further through the precise Langlands classification recalled in Section 2.4.1, by fixing discrete spectral parameters, so that

$$\Omega(X) = \Omega_{\mathrm{comp}}(X) \sqcup \bigsqcup_{\substack{\underline{\delta} \in \mathcal{D} \\ \underline{\delta} = (M, \delta)}} \Omega_{\underline{\delta}}(X), \tag{2.42}$$

where

$$\begin{aligned} \Omega_{\underline{\delta}}(X) &= \left\{ \pi_\infty^R \in \widehat{G}_\infty^R \ : \ \exists \nu \in i\mathfrak{h}_M^\star, \ \pi_\infty^R \simeq \pi_{\delta,\nu}, \ c(\pi_\infty^R) \leqslant X \right\} \\ \Omega_{\mathrm{comp}}(X) &= \left\{ \pi_\infty^R \in \widehat{G}_\infty^R \ : \ \exists \nu \in \mathfrak{h}_{M,\mathbf{C}}^\star \backslash i\mathfrak{h}_M^\star, \ \pi_\infty^R \simeq \pi_{\star,\nu}, \ c(\pi_\infty^R) \leqslant X \right\} \end{aligned}$$

and the notation $\simeq \pi_{\star,\nu}$ stands for the existence of a $\underline{\delta} \in \mathcal{D}$ such that the representation is isomorphic to $\pi_{\delta,\nu}$. The set Ω_{comp} is called the complementary part of the archimedean spectrum, while the remaining part is the tempered part of the spectrum. This denomination is motivated by the fact that the representation $\pi_{\delta,\nu}$ is tempered if and only if ν lies in $i\mathfrak{h}_M^\star$.

Concerning the remaining places, recall that every ideal $\mathfrak{m}$ is decomposed in the form $\mathfrak{m} = \mathfrak{m}_S \mathfrak{m}^S$, where such a decomposition always means that $\mathfrak{m}^S$ is the prime-to-S part of $\mathfrak{m}$, *i.e.* is such that $\mathfrak{m}^S \wedge S = 1$, and $\mathfrak{m}_S$ if the S-part of $\mathfrak{m}$, *i.e.* satisfies $\mathrm{supp}(\mathfrak{m}_S) \subseteq S$. The same decomposition is used without further notice for the other letters. The multiplicative conductor of the finite split places lying out of R is fixed to a certain ideal $\mathfrak{q}$, and the isomorphism class of the ramified part is fixed to a certain $\sigma_R \in \widehat{G}_R$.

Recall from Proposition 12 that the function $\bar{\phi}$ is so that the conductor of the S-component to be equal to a certain $\mathfrak{q}_S$. Thus, the universal family admits the following decomposition according to (2.40) and the choices made above:

$$\mathcal{A}(Q) = \mathcal{A}_{\mathrm{comp}}(Q) \sqcup \bigsqcup_{\substack{N\mathfrak{q} \leqslant Q \\ \mathfrak{q}\wedge R = 1}} \bigsqcup_{\substack{\sigma_R \in \widehat{G}_R \\ c(\sigma_R) \leqslant Q/N\mathfrak{q}}} \bigsqcup_{\substack{\underline{\delta} \in \mathcal{D} \\ \underline{\delta} = (M,\delta)}} \mathcal{A}(\mathfrak{q}, \sigma_R, \delta, \Omega), \tag{2.43}$$

where the sets of fixed spectral data are

$$\begin{aligned} \mathcal{A}(\mathfrak{q}, \sigma_R, \delta, \Omega) &= \left\{ \pi \in \mathcal{A}(G) \ : \ \pi_R \simeq \sigma_R,\ \mathfrak{c}(\pi_f^R) = \mathfrak{q},\ \pi_\infty^R \in \Omega_{\underline{\delta}}(Q/N\mathfrak{q}c(\sigma_R)) \right\} \\ \mathcal{A}_{\mathrm{comp}}(\mathfrak{q}, \sigma_R, \Omega) &= \left\{ \pi \in \mathcal{A}(G) \ : \ \pi_R \simeq \sigma_R,\ \mathfrak{c}(\pi_f^R) = \mathfrak{q},\ \pi_\infty^R \in \Omega_{\mathrm{comp}}(Q/N\mathfrak{q}c(\sigma_R)) \right\} \end{aligned}$$

and where Ω stands for $\Omega(Q/N\mathfrak{q}c(\sigma_R))$, convention used from now on to lighten notations.

The decomposition (2.43) of the universal family is critical for it reduces the study of the whole family to harmonic families, easier to grasp in the context of trace formulas. What is critical is to having got rid of the condition of belonging to $\mathcal{A}(Q)$, decomposed in local conditions. It induces a decomposition of the counting measure as

$$\begin{aligned} \nu_Q(\widehat{\phi}) &= \frac{1}{Q^2} \sum_{\pi \in \mathcal{A}(Q)} \widehat{\phi}(\pi) \\ &= \frac{1}{Q^2} \sum_{\substack{\pi \in \mathcal{A}(G) \\ c(\pi_R)c(\pi_f^{R,S})c(\pi_{S,f}^R)c(\pi_\infty^R) \leqslant Q}} \widehat{\phi}(\pi) \\ &= \frac{1}{Q^2} \sum_{\substack{\sigma_R \in \widehat{G}_R \\ c(\sigma_R) \leqslant Q}} \sum_{\substack{N\mathfrak{q} \leqslant Q/c(\sigma_R) \\ \mathfrak{q} \wedge R = 1}} \sum_{\substack{\underline{\delta} \in \mathcal{D} \\ \underline{\delta} = (M,\delta)}} \sum_{\pi \in \mathcal{A}(\mathfrak{q}, \sigma_R, \delta, \Omega)} \widehat{\phi}(\pi) \\ &\qquad + \frac{1}{Q^2} \sum_{\substack{\sigma_R \in \widehat{G}_R \\ c(\sigma_R) \leqslant Q}} \sum_{\substack{N\mathfrak{q} \leqslant Q/c(\sigma_R) \\ \mathfrak{q} \wedge R = 1}} \sum_{\pi \in \mathcal{A}_{\mathrm{comp}}(\mathfrak{q}, \sigma_R, \Omega)} \widehat{\phi}(\pi) \end{aligned} \tag{2.44}$$

where the sum over $\mathfrak{q}$ is meant to run through ideals of O^R. The complementary part corresponds to the second sum appearing in the line above and will be dealt with later and shown to contribute as an error term. Denote $A(\mathfrak{q},\sigma_R,\delta,\Omega;\phi)$ the innermost part of the splitting in the first summation above, that is to say

$$A(\mathfrak{q},\sigma_R,\delta,\Omega;\phi)=\sum_{\pi\in\mathcal{A}(\mathfrak{q},\sigma_R,\delta,\Omega)}\widehat{\phi}(\pi). \tag{2.45}$$

2.4.3 Old and new forms

The universal family (2.29) sees no multiplicities, but the trace formula counts them. The spectral multiplicities associated to the decomposition of $L^2(G(F)\backslash G(\mathbf{A}))$, which are more suitable weights for the forthcoming computations, are given by

$$m(\pi,\mathfrak{q})=\dim\left(\pi^{\overline{K}_0(\mathfrak{q})}\right), \tag{2.46}$$

where

$$ZK_0(\mathfrak{q})=\prod_{\mathfrak{p}^r\,||\,\mathfrak{q}}Z_{\mathfrak{p}}K_{0,\mathfrak{p}}(\mathfrak{p}^r)\subseteq B^\times\left(\mathbf{A}_f^R\right), \tag{2.47}$$

and $\overline{K}_0(\mathfrak{q})$ stands for the image of $ZK_0(\mathfrak{q})$ under the natural projection $B^\times\to G$. The choice is made so that $m(\pi,\mathfrak{q})\neq 0$ is equivalent to $\mathfrak{c}(\pi_f^R)\mid\mathfrak{q}$. The analogous sum to (2.45) additionally weighted by the multiplicities is

$$B(\mathfrak{q},\sigma_R,\delta,\Omega;\phi)=\sum_{\pi\in\mathcal{B}(\mathfrak{q},\sigma_R,\delta,\Omega)}m\left(\pi^S,\mathfrak{q}^S\right)\widehat{\phi}(\pi), \tag{2.48}$$

where

$$\mathcal{B}(\mathfrak{q},\sigma_R,\delta,\Omega)=\left\{\pi\in\mathcal{A}(Q)\ :\ \pi_R\cong\sigma_R,\ c\left(\pi_f^R\right)\mid\mathfrak{q},\ \pi_\infty^R\in\Omega_{\underline{\delta}}(Q/N\mathfrak{q}c(\sigma_R))\right\}. \tag{2.49}$$

The sum defined by (2.45) counts the newforms while (2.48) counts the old ones at finite split places out of S. The relation between them lies in the following lemma.

Lemma 1. *Let $\mathfrak{q}$ prime to R, σ_R irreducible unitary representations of G_R, $\underline{\delta}\in\mathcal{D}$ and $\phi\in\mathcal{H}(G_S)$. Let $\lambda_2=\mu\star\mu$ where μ is the Möbius function. For every $Q\geqslant 1$,*

$$A(\mathfrak{q},\sigma_R,\delta,\Omega;\phi)=\sum_{\mathfrak{d}\mid\mathfrak{q}}\lambda_2\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right)B(\mathfrak{d},\sigma_R,\delta,\Omega;\phi). \tag{2.50}$$

Proof. Recall that Casselman provides the local multiplicities in (2.22): for every finite split place $\mathfrak{p}$,

$$\dim\pi_{\mathfrak{p}}^{K_0\left(\mathfrak{p}^{\mathfrak{f}(\pi_{\mathfrak{p}})+i}\right)}=i+1,\qquad i\geqslant 0. \tag{2.51}$$

From this immediately follows, after taking the product over all finite split places, that the global multiplicities are

$$m(\pi, \mathfrak{q}) = \tau_2\left(\frac{\mathfrak{q}}{\mathfrak{c}(\pi^R)}\right), \tag{2.52}$$

where $\tau_2 = 1 \star 1$ is the divisor function. Since $\left(\pi^R\right)^{\overline{K}_0(\mathfrak{q})} \neq 0$ implies $\mathfrak{c}(\pi^R) \mid \mathfrak{q}$, the sum defining $B(\mathfrak{q}, \sigma_R, \delta, \Omega; \phi)$ is eventually reduced to a sum over $\mathfrak{c}(\pi^R) \mid \mathfrak{q}$. Thus, by the precise knowledge (2.52) of the multiplicities,

$$\begin{aligned} B(\mathfrak{q}, \sigma_R, \delta, \Omega; \phi) &= \sum_{\mathfrak{d} \mid \mathfrak{q}} \sum_{\pi \in \mathcal{A}(\mathfrak{d}, \sigma_R, \delta, \Omega)} \tau_2\left(\frac{\mathfrak{q}}{\mathfrak{c}(\pi_f^R)}\right) \widehat{\phi}(\pi) \\ &= \sum_{\mathfrak{d} \mid \mathfrak{q}} \tau_2\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right) \sum_{\pi \in \mathcal{A}(\mathfrak{d}, \sigma_R, \delta, \Omega)} \widehat{\phi}(\pi) \\ &= \sum_{\mathfrak{d} \mid \mathfrak{q}} \tau_2\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right) A(\mathfrak{d}, \sigma_R, \delta, \Omega; \phi) \end{aligned} \tag{2.53}$$

so that $B = \tau_2 \star A$, with a slight abuse of notation. Hence, by Möbius inversion,

$$A(\mathfrak{q}, \sigma_R, \delta, \Omega; \phi) = \sum_{\mathfrak{d} \mid \mathfrak{q}} \lambda_2\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right) B(\mathfrak{d}, \sigma_R, \delta, \Omega; \phi), \tag{2.54}$$

achieving the proof. □

Summing over the spectral data appearing in the decomposition (2.44), the counting measure rewrites as

$$\nu_Q(\widehat{\phi}) = \frac{1}{Q^2} \sum_{\substack{\sigma_R \in \widehat{G}_R \\ c(\sigma_R) \leqslant Q}} \sum_{\substack{N\mathfrak{q} \leqslant Q/c(\sigma_R)) \\ \mathfrak{q} \wedge R = 1}} \sum_{\substack{\underline{\delta} \in \mathcal{D} \\ \underline{\delta} = (M, \delta)}} \sum_{\mathfrak{d} \mid \mathfrak{q}} \lambda_2\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right) B(\mathfrak{d}, \sigma_R, \delta, \Omega; \phi). \tag{2.55}$$

2.5 Spectral selection

The equidistribution property has been formulated as a convergence of spectral measures. The Selberg trace formula translates it as a purely geometrical quantity.

2.5.1 Selberg trace formula for compact cases

Since the automorphic quotient of G is compact by Proposition 5, the original formulation of the trace formula, due to Selberg [3] in 1956, can be used and combined with the multiplicity one theorem. If Φ is a function in the Hecke algebra $\mathcal{H}(G(\mathbb{A}))$, then

$$J_{\text{geom}}(\Phi) = J_{\text{spec}}(\Phi), \tag{2.56}$$

where the spectral and geometrical parts are as follows. The geometrical part is

$$J_{\text{geom}}(\Phi) := \sum_{\{\gamma\}} \operatorname{vol}\left(G_\gamma(F)\backslash G_\gamma(\mathbf{A})\right) \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} \Phi\left(x^{-1}\gamma x\right) \mathrm{d}x. \tag{2.57}$$

The sum runs through conjugacy classes $\{\gamma\}$ in $G(F)$. Since Φ is compactly supported and $G(F)$ is discrete, the sum is finite. However its length depends on the support of Φ what turns to be a critical difficulty for estimations, for this support depends on the spectral parameters. The inner integrals appearing in this geometric side are called the orbital integrals, defined by

$$O_\gamma(\Phi) = \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} \Phi\left(x^{-1}\gamma x\right) \mathrm{d}x. \tag{2.58}$$

The spectral part is

$$J_{\text{spec}}(\Phi) = \sum_{\pi \subseteq L^2(G(F)\backslash G(\mathbf{A}))} m(\pi)\widehat{\Phi}(\pi). \tag{2.59}$$

Here π go through the isomorphism classes of unitary irreducible subrepresentations of $G(\mathbf{A})$ in $L^2(G(F)\backslash G(\mathbf{A}))$, and recall that $\widehat{\Phi}$ is the Fourier transform of Φ, see 2.3.2.

Remark. The formulaton of the spectral part (2.59) is Selberg's original one. The weights $m(\pi)$ are the multiplicities of the π's in the discrete part of the spectral decomposition of $L^2(G(F)\backslash G(\mathbf{A}))$. The multiplicity one theorem ensures these to be less than 1, and the indexation by π actually part of $L^2(G(F)\backslash G(\mathbf{A}))$ makes them nonzero, hence equal to 1.

The admissible dual can be decomposed into tempered representations and non-tempered representation. In view of (2.42) and anticipating that the selecting function at split archimedean places behaves differently on the tempered spectrum and on the complementary one, it is natural to introduce the tempered and complementary spectral parts as

$$J_{\text{temp}}(\Phi) = \sum_{\substack{\pi \subseteq L^2(G(F)\backslash G(\mathbf{A})) \\ \pi_\infty^R \simeq \pi_{\star,\nu} \\ \nu \in \Omega_{\text{temp}}}} m(\pi)\widehat{\Phi}(\pi)$$

$$J_{\text{comp}}(\Phi) = \sum_{\substack{\pi \subseteq L^2(G(F)\backslash G(\mathbf{A})) \\ \pi_\infty^R \simeq \pi_{\star,\nu} \\ \nu \in \Omega_{\text{comp}}}} m(\pi)\widehat{\Phi}(\pi)$$

As announced in the outlook of the method, in order to have a problem amenable to the trace formula it is necessary to formulate statistics quantities on the universal family as a spectral side, hence needed to select it by the Fourier transforms of suitable test functions. The aim of the present section is to construct a function $\Phi \in \mathcal{H}(G)$ such that

$$J_{\mathrm{spec}}(\Phi) = B\left(\mathfrak{d}, \sigma_R, \delta, \Omega; \phi\right). \tag{2.60}$$

In the case of factorizable test functions $\Phi = \otimes_v \Phi_v$, the spectral side of the trace formula factorizes as well, reducing the treatment to local statement on local, and hopefully simpler, quantities.

Proposition 14 (Factorization of the spectral side). *If $\pi = \otimes_v \pi_v$ and $\Phi = \otimes_v \Phi_v$, then*

$$\widehat{\Phi}(\pi) = \prod_v \widehat{\Phi}_v(\pi_v). \tag{2.61}$$

Hence, in order to achieve the spectral selection (2.60) it is sufficient locally select the conditions appearing in the decomposition of the universal family (2.55) through Fourier transforms. The places of F fall into four categories:

- the split finite part, corresponding to $\mathfrak{p} \notin R \cup S$, where the arithmetic conductor is caught by the means of an explicit filtration, see Section 2.19;
- the split finite part in the support of the test function $\widehat{\phi}$, corresponding to $\mathfrak{p} \in S \backslash R$, whose conductor is fixed by $\widehat{\phi}$, see Proposition 12;
- the ramified part, corresponding to the finite number of $v \in R$, which deliberately remains a blackbox and is handled by fixing the representations at those places by means of matrix coefficients;
- the split archimedean part, parametrized by spectral data that are handled by selecting functions provided by Paley-Wiener theorems.

The following sections are dedicated to construct local test functions doing so, aim reached in Lemma 4.

2.5.2 Non-archimedean split places

For an ideal $\mathfrak{d}$ of $\mathcal{O}$, introduce the congruence subgroup given by the product of the corresponding local congruence subgroups in (2.19), that is to say

$$K_0(\mathfrak{d}) = \prod_{\mathfrak{p}^r || \mathfrak{d}} K_{0,\mathfrak{p}}(\mathfrak{p}^r). \tag{2.62}$$

The following result gives a test function whose Fourier transform selects the finite split conductor.

Lemma 2. *For an ideal $\mathfrak{d}$ of $\mathcal{O}$, let*

$$\varepsilon_{\mathfrak{d}} = \mathrm{vol}\left(\overline{K}_0(\mathfrak{d})\right)^{-1} \mathbf{1}_{\overline{K}_0(\mathfrak{d})}. \tag{2.63}$$

Its Fourier transform selects the multiplicity relative to $\mathfrak{d}$. *More precisely,*

$$\widehat{\varepsilon_{\mathfrak{d}}}(\pi) = m(\pi, \mathfrak{d}), \qquad \pi \in \mathcal{A}(G). \tag{2.64}$$

Proof. Let π be an automorphic representation of G. Then $\pi(\varepsilon_{\mathfrak{d}})$ is the projection of the representation space V_π on the subspace $\pi^{\mathfrak{d}}$ of the fixed vectors by $\overline{K}_0(\mathfrak{d})$ under the action of π. Indeed, every $\pi(\varepsilon_{\mathfrak{d}})v$, for v in V_π, is $\overline{K}_0(\mathfrak{d})$-invariant, for it is an averaging over the action of $\overline{K}_0(\mathfrak{d})$. For $k_0 \in \overline{K}_0(\mathfrak{d})$ and $v \in V_\pi$,

$$\begin{aligned}
\pi(k_0)\pi(\varepsilon_{\mathfrak{d}})\, v &= \mathrm{vol}\left(\overline{K}_0(\mathfrak{d})\right)^{-1} \pi(k_0) \int_{\overline{K}_0(\mathfrak{d})} \pi(k)v \,\mathrm{d}k \\
&= \mathrm{vol}\left(\overline{K}_0(\mathfrak{d})\right)^{-1} \int_{\overline{K}_0(\mathfrak{d})} \pi(k_0k)v \,\mathrm{d}k \\
&= \mathrm{vol}\left(\overline{K}_0(\mathfrak{d})\right)^{-1} \int_{\overline{K}_0(\mathfrak{d})} \pi(k)v \,\mathrm{d}k = \pi(\varepsilon_{\mathfrak{d}})\, v
\end{aligned}$$

so that its image lies in $\pi^{\mathfrak{d}}$. The action of $\pi(\varepsilon_{\mathfrak{d}})$ is also idempotent, more precisely the identity on $\pi^{\mathfrak{d}}$. Indeed, for $v_0 \in \pi^{\mathfrak{d}}$,

$$\begin{aligned}
\pi(\varepsilon_{\mathfrak{d}})\, v_0 &= \mathrm{vol}\left(\overline{K}_0(\mathfrak{d})\right)^{-1} \int_{\overline{K}_0(\mathfrak{d})} \pi(k)v_0 \,\mathrm{d}k \\
&= \mathrm{vol}\left(\overline{K}_0(\mathfrak{d})\right)^{-1} \int_{\overline{K}_0(\mathfrak{d})} v_0 \,\mathrm{d}k \\
&= v_0
\end{aligned}$$

Hence $\pi(\varepsilon_{\mathfrak{d}})$ is an idempotent endomorphism of image $\pi^{\mathfrak{d}}$, *i.e.* a projection on $\pi^{\mathfrak{d}}$. The trace of a projection is its rank, that is to say $\widehat{\varepsilon_{\mathfrak{d}}}(\pi)$ is the dimension of the fixed vector spaces $\pi^{\mathfrak{d}}$. Those are the sought multiplicities $m(\pi, \mathfrak{d})$, in particular are nonzero if and only if $\mathfrak{c}(\pi) \mid \mathfrak{d}$. □

2.5.3 Ramified places

For ramified places, less is known concerning the representations and the choice made in the decomposition (2.55) is to fix the corresponding isomorphism class. In the finite dimensional case, knowing matrix coefficients is sufficient to determine the underlying matrix. This property still holds for supercuspidal representations in the following sense.

Let σ_R be a unitary representation of G_R. A matrix coefficient associated to σ_R is a function of the form, given v and w in the space of σ_R,

$$\begin{array}{rccc} \xi_{\sigma_R}^{v,w}: & G_R & \longrightarrow & \mathbf{C} \\ & g & \longmapsto & \langle \sigma(g)v, w \rangle \end{array} \tag{2.65}$$

Matrix coefficients are continuous functions on G_R, are compactly supported since G_R is compact, and are locally constant at finite places and smooth at archimedean places.

Remark. The fact that matrix coefficients is considered only for ramified places is critical for selecting purposes. The loss of the compactness of the support for matrix coefficients in the split case, where some automorphic representations are not supercuspidal, make them fail to select the corresponding isomorphism class. Such purposes can be achieved by means of existence theorem, yet are less precise, see [75]. This is the reason why the non-totally definite case or the GL(2) case are analytically harder to deal with, see Section 2.5.4.

As for finite-dimensional matrix coefficients, orthogonality relations can be formulated and are the key to selecting a fixed representation σ_R. For instance, Knightly and Li [75, Corollary 10.26] provide the following proposition.

Proposition 15. *Let σ and π be automorphic representations of G_R, and introduce d_π the formal degree of π. Then for every unit vectors v and w in the representation space of σ,*

$$\pi\left(\xi_\sigma^{v,w}\right) w = \mathbf{1}_{\pi\simeq\sigma}\frac{\langle w, v\rangle}{d_\pi} v. \tag{2.66}$$

Taking for v a vector of norm $d_\pi^{1/2}$, it follows that $\pi\left(\xi_\sigma^{v,v}\right)$ is the orthogonal projection onto $\mathbf{C}v$ and in the meanwhile selects the π's isomorphic to σ. So that, considering its trace, the above into a result concerning Fourier transforms can be restated as follows.

Proposition 16. *Let σ and π be automorphic representations of G_R. Let v be a vector of norm one in the representation space of σ. Then,*

$$\widehat{\xi_\sigma^{v,v}}(\pi) = \mathbf{1}_{\pi\simeq\sigma}. \tag{2.67}$$

From now on, denote ξ_σ any choice of matrix coefficient as in Proposition 16.

2.5.4 Archimedean parameters

Based on the decomposition of the universal family 2.43, for a general bounded set of continuous parameters Ω in $\mathfrak{h}_{M,\mathbf{C}}^\star$, the question is to select representations lying in sets of the form

$$\mathcal{A}(\mathfrak{q}, \sigma_R, \delta, \Omega) = \left\{\pi \in \mathcal{A}(Q) \ :\ \pi_R \simeq \sigma_R,\ \mathfrak{c}(\pi_f^R) = \mathfrak{q},\ \pi_\infty^R \in \Omega_{\underline{\delta}}\right\}.$$

This is a feature of non-compact archimedean groups: their dual is no more discrete and hence admits a continuous parametrization. However, many tools in harmonic analysis on Lie groups involve narrow classes of functions among which characteristic functions of such sets $\Omega_{\underline{\delta}}$ are not, requiring a smoothing construction to get admissible functions lying nearby them. This procedure is in essence provided by the fundamental work of Duistermaat, Kolk and Varadarajan [39].

Brumley and Milićević [17] adapt this method to the automorphic setting on $\mathrm{GL}(n)$ and construct a function localizing around spectral parameters (δ, ν), where δ is a fixed archimedean discrete spectral datum and where ν is a continuous parameter running through a bounded set of parameters $\Omega_{\underline{\delta}}$. Smoothing procedures behave well on tempered parameters, leading to assume Ω to be a bounded set of tempered parameters of fixed discrete part $\underline{\delta}$, leaving the non-tempered part of Ω to be proven negligible compared to the tempered contribution.

Introduce ϕ which in this section is a function in the Hecke algebra of G_∞^R, and should be denoted ϕ_∞^R in the following ones. The aim is to find a smooth enough function for trace formula purposes approximating the characteristic function of $\Omega_{\underline{\delta}}$. Brumley and Milićević [17, Section 9] achieved this goal, constructing a function $h_\rho^{\underline{\delta},\Omega}$ of Paley-Wiener type with exponential type $\rho > 0$ as tempered spectral-localizing function. Let

$$h_\rho^{\delta,\Omega,\phi} := \widehat{\phi}\, h_\rho^{\delta,\Omega}, \qquad \delta \in \mathcal{D}. \tag{2.68}$$

A direct consequence of their result is the following lemma, where the remainder term is fundamental yet willingly hidden in order to ease the exposition. What is of critical importance are the bounds on this undisclosed error term, precisely stated in Lemma 10.

Lemma 3. *For every discrete spectral data $\underline{\delta} \in \mathcal{D}$, there is a function $\epsilon_\rho^{\delta,\Omega}$ such that for every $(M, \tau) \in \mathcal{D}$ and $\nu \in \mathfrak{h}_{M,\mathbb{C}}^\star$,*

$$\begin{aligned} (i) \qquad & h_\rho^{\delta,\Omega,\phi}(\tau,\nu) = \mathbf{1}_{\substack{\tau\in W\delta \\ \nu\in\Omega}}\, \widehat{\phi}(\tau,\nu) + \epsilon_\rho^{\delta,\Omega}(\tau,\nu) \\ (ii) \qquad & h_\rho^{\delta,\Omega,\phi}(\tau,\nu) \ll \mathbf{1}_{\substack{\tau\in W\delta \\ \mathrm{Re}(\nu)\in\Omega}}\, e^{\rho\|\mathrm{Re}\,\nu\|} \end{aligned}$$

Proof. This is just encapsulating the results of [17, Lemma 9.2] and multiplying them by $\widehat{\phi}$. ◻

Remark. From now on every error term also depends on ϕ: this is not such a matter since the Sauvageot theorem will ultimately get rid of every error term to just conclude to a convergence result, and the counting law is obtained with no ϕ added.

The same arguments used by Brumley and Milićević [17] hold with $h_\rho^{\delta,\Omega}$ replaced by $h_\rho^{\delta,\Omega,\phi}$. In particular, a version of the Paley-Wiener proven by Clozel and Delorme [24] provides a function $f_\rho^{\delta,\Omega,\phi}$ whose Fourier transform is $h_\rho^{\delta,\Omega,\phi}$.

2.5.5 Spectrum selection

The weighted counting number $B(\mathfrak{d}, \sigma_R, \delta, \Omega; \phi)$ should be written as a spectral side in the trace formula. Introduce the test function

$$\Phi_{\mathfrak{d},\pi_R,\delta,\Omega,\rho;\phi} = \prod_v \Phi_v, \tag{2.69}$$

which is built with the following local functions:

Places v	Local test function Φ_v
$\notin S, \notin R, < \infty$	$\varepsilon_{\mathfrak{d},v}$
$\notin S, \notin R, \in \infty$	$f_{\rho,v}^{\delta,\Omega}$
$\notin S, \in R$	ξ_{π_v}
$\in S, \notin R, < \infty$	ϕ_v
$\in S, \notin R, \in \infty$	$f_{\rho,v}^{\delta,\Omega,\phi}$
$\in S, \in R$	$\xi_{\pi_v}\widehat{\phi_v}(\pi_v)$

where

- ϕ_v is the local component of ϕ on G_v;
- ξ_{π_v} is a matrix coefficient for π_v;
- $\varepsilon_{\mathfrak{d}}$ is the function introduced in Lemma 2, $\varepsilon_{\mathfrak{d},v}$ its v-component;
- $f_\rho^{\delta,\Omega,\phi}$ is the function constructed Lemma 5, with $\Omega = \Omega(Q/N\mathfrak{q}\mathfrak{c}(\pi_R))$.

The sought weighted measure is barely reached by the spectral side with $\Phi_{\mathfrak{d},\pi_R,\delta,\Omega,\rho;\phi}$, as stated in the following fundamental lemma.

Lemma 4. *Let $Q \geqslant c_\phi$. Let $\mathfrak{d} \wedge R = 1$, $\pi_R \in \widehat{G}_R$, $\delta \in \mathcal{E}_2(M^1)$ for an $M \in \mathcal{L}_\infty$. Then*

$$B\left(\mathfrak{d}, \pi_R, \delta, \Omega; \phi\right) = J_{\text{temp}}\left(\Phi_{\mathfrak{d},\pi_R,\delta,\Omega,\rho;\phi}\right) + O(\Xi(\phi, \pi_R)) + O(\partial_\rho B(\mathfrak{d}, \pi_R, \delta, \Omega)), \tag{2.70}$$

where, introducing the set $X^{\text{ur}}(G)$ of unramified characters of $G(\mathbf{A})$,

$$\Xi(\phi, \pi_R) = \sum_{\substack{\chi \in X^{\text{ur}}(G) \\ \chi_R \simeq \pi_R}} m(\chi^R, \mathfrak{d})\widehat{\phi}(\chi), \tag{2.71}$$

and

$$\partial_\rho B(\mathfrak{d}, \pi_R, \delta, \Omega) = \int_{\substack{\pi \in \mathcal{A}(\mathfrak{d},\pi_R,\delta) \\ \nu \in i\mathfrak{h}_M^\star}} \tau_2\left(\frac{\mathfrak{d}}{\mathfrak{c}(\pi^R)}\right) \epsilon_\rho^{\delta,\Omega}(\tau, \nu)\, \mathrm{d}\nu, \tag{2.72}$$

where this last integral means an integration over $\pi \in \mathcal{A}(G)$ of fixed discrete spectral data $\mathfrak{d}$, π_R and δ, and with continuous parameters varying in $i\mathfrak{h}_M^\star$.

Proof. Let $\Phi = \Phi_{\mathfrak{d},\pi_R,\delta,\Omega,\rho;\phi}$. In order to determine the Fourier transform of Φ recall the splitting into local components given by Proposition 14: for every places v, w and every $a \in \mathcal{H}(G_{v,w})$, $\widehat{a_v a_w} = \widehat{a_v}\widehat{a_w}$. Thus,

$$\widehat{\Phi} = \prod_{v} \widehat{\Phi}_v = h_\rho^{\delta,\Omega,\phi} \prod_{v\in R} \widehat{\xi}_{\pi_v} \prod_{\substack{\mathfrak{p}\notin R\\ \mathfrak{p}<\infty\\ \mathfrak{p}\notin S\\ \mathfrak{p}^r \| \mathfrak{d}}} \widehat{\varepsilon}_{\mathfrak{p}^r,v} \prod_{\substack{\mathfrak{p}\notin R\\ \mathfrak{p}<\infty\\ \mathfrak{p}\in S}} \widehat{\phi}_{\mathfrak{p}}. \tag{2.73}$$

Hence only the Fourier transforms of the local components of the test function have to be determined. The finite prime-to-S split part $\varepsilon_{\mathfrak{d}}$ is shown to transform into the characteristic function of conductors dividing $\mathfrak{d}$ in Lemma 2 weighted by the corresponding multiplicities. The ramified local parts ξ_{π_v} are known to transform into the characteristic functions of the isomorphism class of π_v by Lemma 16. The transform of the archimedean split part is shown to approximate the selecting function of bounded conductors in Lemma 3, up to a smoothing error term $\epsilon_\rho^{\delta,\Omega}$. The action of the Fourier transform of Φ on the tempered part follows, and (2.73) yields, for $\sigma \in \mathcal{A}(G)$ with archimedean split parameters (τ, ν),

$$\widehat{\Phi}(\sigma) = m(\sigma^R, \mathfrak{d})\widehat{\phi}(\sigma_f)\mathbf{1}_{\substack{\sigma_R\cong\pi_R\\ \mathfrak{c}(\sigma^R)\,|\,\mathfrak{d}}} \left(\mathbf{1}_{\substack{\tau\in W\delta\\ \nu\in\Omega}} \widehat{\phi}(\tau,\nu) + \epsilon_\rho^{\delta,\Omega}(\tau,\nu)\right). \tag{2.74}$$

Nevertheless, these conditions also stand for characters: in order to not being killed by $\widehat{\Phi}$ they have to be trivial on $\overline{K}_0(\mathfrak{d})$, *i.e.* they have to be unramified since $\det(\overline{K}_0(\mathfrak{d})) = \mathcal{O}^R$. Moreover, they have to be isomorphic to π_R at ramified places. The Fourier transform of the chosen test function hence does not vanish on unramified characters, unlike awaited. The corresponding extra contribution Ξ is treated separately in Lemma 12, for characters are easier to embrace and it will be shown to contribute as an error term.

After summing over the tempered spectrum, it follows by roughly bounding $\widehat{\phi}$ in the remainder smoothing term,

$$\begin{aligned} J_{\mathrm{temp}}(\Phi) = &\sum_{\sigma\in\mathcal{A}(\mathfrak{d},\pi_R,\delta,\Omega)} \tau_2\left(\frac{\mathfrak{d}}{\mathfrak{c}(\sigma^R)}\right)\widehat{\phi}(\sigma) \\ &+ O\left(\int_{\substack{\pi\in\mathcal{A}(\mathfrak{d},\pi_R,\delta)\\ \nu\in i\mathfrak{h}_M^\star}} \tau_2\left(\frac{\mathfrak{d}}{\mathfrak{c}(\pi^R)}\right)\epsilon_\rho^{\delta,\Omega}(\tau,\nu)\,\mathrm{d}\nu\right) \\ &+ O\left(\sum_{\substack{\chi\in X^{\mathrm{ur}}(G)\\ \chi_R\simeq\pi_R}} m(\chi^R,\mathfrak{d})\widehat{\phi}(\chi)\right) \end{aligned}$$

that achieves the proof. □

Remark. This lemma gets rid of treating precisely the contribution of those characters, for it is more suitable to keep the test function as easy to handle as possible in order to ease the estimations in the geometric side below. The lack of details compared to [17] concerning the treatment of the archimedean parameters may be a break to the understanding. Recall from Lemma 3 that the error term $\epsilon_\rho^{\delta,\Omega}$ is better when ν if far from the boundary of Ω, so that it should be considered as a smooth bump function concentrating around the boundary, so that the integral (2.72) is a smoothed version of the counting number $B(\mathfrak{d}, \pi_R, \delta, \partial_\rho\Omega; \phi)$, justifying the notation.

Chapter 3

Counting Law & Equidistribution

Endowed with the decomposition and the sieving of the spectrum established in Chapter 2, the counting law and equidistribution problems appear to be amenable to trace formulas methods. The spectral side of the Selberg trace formula approximates the sought quantities, while the geometrical side is dominated by the identity contribution. Many specific estimates are gathered and proven in order to show the negligibility of the remaining terms.

The identity term in the geometric side of the trace formula yields the counting law and equidistribution results. The volumes of the congruence subgroups used to grasp the conductor are critical to determine the growth order, while the constant admits a geometric interpretation in terms of the zeta function of the analytic conductor. The error terms relative to the complementary part of the spectrum and the smoothing effect are reduced to similar counting problems. The ones coming from the elliptic part of the trace formula are estimated by a precise evaluation of their number and recent bounds on orbital integrals.

Contents

3.1 Geometric reformulation

The equidistribution property has been recast as a convergence of spectral measures in Theorem G. The Selberg trace formula restates it as a purely geometrical quantity. Indeed, recall from Lemma 4 that for an ideal $\mathfrak{d} \wedge R = 1$, a ramified part $\pi_R \in \widehat{G}$, an archimedean spectral parameter $\underline{\delta} \in \mathcal{D}$, the following relation holds between the number of old forms $B(\mathfrak{d}, \pi_R, \delta, \Omega; \phi)$ and the spectral part of the trace formula.

$$J_{\text{temp}}\left(\Phi_{\mathfrak{d},\pi_R,\delta,\rho,\Omega;\phi}\right) = B\left(\mathfrak{d}, \pi_R, \delta, \Omega; \phi\right) + O(\Xi(\phi, \pi_R)) + O(\partial_\rho B(\mathfrak{d}, \pi_R, \delta, \Omega)). \tag{3.1}$$

The trace formula stated in Section 3.1 is the equality

$$J_{\text{geom}}(\Phi) = J_{\text{spec}}(\Phi), \tag{3.2}$$

where the spectral and geometrical parts are given in (2.57) and (2.59). In order to interpret $B\left(\mathfrak{d}, \pi_R, \delta, \Omega; \phi\right)$ as a spectral side, it is then necessary to add the contribution of the complementary part of the split archimedean spectrum, namely

$$J_{\text{comp}}(\Phi) = \sum_{\substack{\pi\subseteq L^2(G(F)\backslash G(\mathbb{A})) \\ \pi_\infty^R\simeq\pi_{\star,\nu} \\ \nu\in\Omega_{\text{comp}}}} \widehat{\Phi}(\pi) \tag{3.3}$$

The road is thus cleared towards a geometrical formulation of the counting measure parts $B\left(\mathfrak{d}, \pi_R, \delta, \Omega; \phi\right)$, what is carried out in this chapter. So far, from the trace formula and the splitting of the weighted counting measure (2.55), summing the expressions (3.1) above over all the spectral data and adding the complementary part of the spectrum, it yields

$$\begin{aligned}
\nu_Q(\widehat{\phi}) =& \frac{1}{Q^2} \sum_{\substack{\sigma_R\in\widehat{G}_R \\ c(\sigma_R)\leqslant Q}} \sum_{\substack{N\mathfrak{q}\leqslant Q/c(\sigma_R) \\ \mathfrak{q}\wedge R=1}} \sum_{\substack{\underline{\delta}\in\mathcal{D} \\ \underline{\delta}=(M,\delta)}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) J_{\text{geom}}\left(\Phi_{\mathfrak{d},\pi_R,\delta,\Omega,\rho;\phi}\right) \\
&- \frac{1}{Q^2} \sum_{\substack{\sigma_R\in\widehat{G}_R \\ c(\sigma_R)\leqslant Q}} \sum_{\substack{N\mathfrak{q}\leqslant Q/c(\sigma_R) \\ \mathfrak{q}\wedge R=1}} \sum_{\substack{\underline{\delta}\in\mathcal{D} \\ \underline{\delta}=(M,\delta)}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) J_{\text{comp}}(\Phi_{\mathfrak{d},\pi_R,\delta,\Omega,\rho;\phi}) \\
&+ O\left(\frac{1}{Q^2} \sum_{\substack{\sigma_R\in\widehat{G}_R \\ c(\sigma_R)\leqslant Q}} \sum_{\substack{N\mathfrak{q}\leqslant Q/c(\sigma_R) \\ \mathfrak{q}\wedge R=1}} \sum_{\substack{\underline{\delta}\in\mathcal{D} \\ \underline{\delta}=(M,\delta)}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) \Xi(\sigma_R, \phi)\right) \\
&+ O\left(\frac{1}{Q^2} \sum_{\substack{\sigma_R\in\widehat{G}_R \\ c(\sigma_R)\leqslant Q}} \sum_{\substack{N\mathfrak{q}\leqslant Q/c(\sigma_R) \\ \mathfrak{q}\wedge R=1}} \sum_{\substack{\underline{\delta}\in\mathcal{D} \\ \underline{\delta}=(M,\delta)}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) \partial_\rho B(\mathfrak{d}, \pi_R, \delta, \Omega)\right)
\end{aligned}$$

The main contribution is carried by the first term, the remaining ones being showed below to contribute as negligible terms. Decompose the geometrical side (2.57) $J_{\text{geom}}(\Phi)$ as sum of two terms, the first one corresponding to the identity contribution, and the other being the elliptic remainder, in other terms

$$J_{\text{geom}}(\Phi) = \mathrm{vol}\left(G\left(F\right)\backslash G\left(\mathbf{A}\right)\right)\Phi(1) + J_{\text{ell}}(\Phi), \tag{3.4}$$

where the elliptic part is expressed in term of orbital integrals

$$J_{\text{ell}}\left(\Phi\right) = \sum_{\{\gamma\}\neq\{1\}} \mathrm{vol}\left(G_\gamma(F)\backslash G_\gamma(\mathbf{A})\right) \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} \Phi\left(x^{-1}\gamma x\right) \mathrm{d}x.$$

The universal family counting measure now decomposes, via (3.1) and the splitting above, as

$$\nu_Q = \mathrm{vol}\left(G\left(F\right)\backslash G\left(\mathbf{A}\right)\right)\nu_{1,Q} + \nu_{\text{ell},Q} - \nu_{\text{comp},Q} + O(\nu_{\Xi,Q}) + O(\nu_{\partial,Q}), \tag{3.5}$$

where the following measures have been introduced.

$$\nu_{1,Q}\left(\widehat{\phi}\right) = \frac{1}{Q^2} \sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q}} \sum_{\substack{N\mathfrak{q}\leqslant Q/c(\sigma_R)\\ \mathfrak{q}\wedge R=1}} \sum_{\substack{\underline{\delta}\in\mathcal{D}\\ \underline{\delta}=(M,\delta)}} \sum_{\mathfrak{d}^S\mid\mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) \Phi_{\mathfrak{d},\pi_R,\delta,\Omega,\rho;\phi}(1)$$

$$\nu_{\text{ell},Q}\left(\widehat{\phi}\right) = \frac{1}{Q^2} \sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q}} \sum_{\substack{N\mathfrak{q}\leqslant Q/c(\sigma_R)\\ \mathfrak{q}\wedge R=1}} \sum_{\substack{\underline{\delta}\in\mathcal{D}\\ \underline{\delta}=(M,\delta)}} \sum_{\mathfrak{d}^S\mid\mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) J_{\text{ell}}(\Phi_{\mathfrak{d},\pi_R,\delta,\Omega,\rho;\phi})$$

$$\nu_{\text{comp},Q}\left(\widehat{\phi}\right) = \frac{1}{Q^2} \sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q}} \sum_{\substack{N\mathfrak{q}\leqslant Q/c(\sigma_R)\\ \mathfrak{q}\wedge R=1}} \sum_{\substack{\underline{\delta}\in\mathcal{D}\\ \underline{\delta}=(M,\delta)}} \sum_{\mathfrak{d}^S\mid\mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) J_{\text{comp}}(\Phi_{\mathfrak{d},\pi_R,\delta,\Omega,\rho;\phi})$$

$$\nu_{\Xi,Q}\left(\widehat{\phi}\right) = \frac{1}{Q^2} \sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q}} \sum_{\substack{N\mathfrak{q}\leqslant Q/c(\sigma_R)\\ \mathfrak{q}\wedge R=1}} \sum_{\substack{\underline{\delta}\in\mathcal{D}\\ \underline{\delta}=(M,\delta)}} \sum_{\mathfrak{d}^S\mid\mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) \Xi(\sigma_R,\phi)$$

$$\nu_{\partial,Q}\left(\widehat{\phi}\right) = \frac{1}{Q^2} \sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q}} \sum_{\substack{N\mathfrak{q}\leqslant Q/c(\sigma_R)\\ \mathfrak{q}\wedge R=1}} \sum_{\substack{\underline{\delta}\in\mathcal{D}\\ \underline{\delta}=(M,\delta)}} \sum_{\mathfrak{d}^S\mid\mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) \partial_\rho(\mathfrak{d},\sigma_R,\delta,\Omega)$$

From now on the aim is to unveil what lies behind the innermost quantities in the sums above and estimate the contribution of each measure to the counting measure ν_Q.

3.2 Identity contribution

For a given $\phi \in \mathcal{H}(G_S)$, the main term of $\nu_Q(\widehat{\phi})$ is given by the contribution $\nu_{1,Q}(\widehat{\phi})$ of the identity, and the other terms will be shown to be negligible. This section is dedicated to the computation of this identity contribution.

Proposition 17. *The contribution of the identity is, for $\phi \in \mathcal{H}(G_S)$,*

$$\operatorname{vol}\left(G\left(F\right)\backslash G\left(\mathbf{A}\right)\right)\nu_{1,Q}\left(\widehat{\phi}\right) = \nu\left(\widehat{\phi}\right) + \begin{cases} O(Q^{-1}\log Q) & \text{if } D \text{ is totally definite and } F = \mathbf{Q} \\ O(Q^{-\delta_F}) & \text{if } D \text{ is totally definite and } F \neq \mathbf{Q} \\ O(\log^{-1} Q) & \text{if there is a split infinite place} \end{cases}$$

In particular, $\operatorname{vol}\left(G\left(F\right)\backslash G\left(\mathbf{A}\right)\right)\nu_{1,Q}$ *equidistributes with respect to* ν.

3.2.1 Test function at 1

Before summing over the spectral data, it is necessary to look at the inner part of $\nu_{1,Q}\left(\widehat{\phi}\right)$. Fix $\mathfrak{d}$ an ideal of $\mathcal{O}^R$, π_R a unitary irreducible representation of G_R and δ a discrete archimedean parameter; and let for this section $\Phi = \Phi_{\mathfrak{d},\pi_R,\delta,\Omega,\rho;\phi}$ and Ω denote $\Omega(Q/N\mathfrak{d}^S c(\sigma_R))$ for convenience. The very definition (2.69) of Φ gives

$$\Phi(1) = \varepsilon_{K_0(\mathfrak{d}^S)}(1)\phi_{S,f}^R(1)\xi_{\sigma_R}(1)\widehat{\phi}_R(\pi_R)f_\rho^{\delta,\Omega,\phi}(1). \tag{3.6}$$

Finite split places out of S

For the prime-to-S split finite part, by definition

$$\varepsilon_{\overline{K}_0(\mathfrak{d}^S)}(1) = \operatorname{vol}\left(\overline{K}_0(\mathfrak{d}^S)\right)^{-1}. \tag{3.7}$$

The volume of a cofinite subgroup depends on its index, and the indices of classical congruence subgroups are well-known [35]. Introduce $K^{R,S} = \prod_{v\notin R\cup S} K_v$. Since $Z^{R,S}$ is fully contained in $K_0^{R,S}(\mathfrak{d}^S)$ for all ideal $\mathfrak{d}^S$,

$$\left[\overline{K}^{R,S} : \overline{K}_0\left(\mathfrak{d}^S\right)\right] = \left[K^{R,S} : K_0\left(\mathfrak{d}^S\right)\right], \tag{3.8}$$

by the isomorphism theorems. So thanks to the normalizations chosen for the measures,

$$\varepsilon_{\overline{K}_0(\mathfrak{d}^S)}(1) = \left[K^{R,S} : K_0\left(\mathfrak{d}^S\right)\right] = \left(\mathrm{id} \star \mu^2\right)(\mathfrak{d}^S) =: \varphi_2(\mathfrak{d}^S). \tag{3.9}$$

Finite split places in S

For the S-split finite part, the Plancherel inversion formula (4.64) gives

$$\phi_{S,f}^R(1) = \int_{\widehat{G}_{S,f}^R} \widehat{\phi}_{S,f}^R(\pi_{S,f}^R)\,\mathrm{d}\pi_{S,f}^R. \tag{3.10}$$

Ramified places

For the ramified matrix coefficient (2.67), by the Plancherel formula (4.64) and the normalization chosen for ξ_{π_R},

$$\xi_{\pi_R}(1) = \int_{\widehat{G}_R} \mathbf{1}_{\sigma \simeq \pi_R} \, \mathrm{d}\mu_R^{\mathrm{Pl}}(\sigma) = \mu_R^{\mathrm{Pl}}(\pi_R). \tag{3.11}$$

Split archimedean places

The test function $f_\rho^{\delta,\Omega,\phi}$ at archimedean split places is not so immediate to evaluate at 1, since it is not an explicit function but provided by an existence theorem. The Plancherel formula allows to express it in terms of its Fourier transform $h_\rho^{\delta,\Omega,\phi}$ for which Lemma 3 provides information. There is an error term function due to the smoothing procedure willingly kept undisclosed and for which bounds are provided later.

Lemma 5. *For every $\delta \in \mathcal{D}$, $X > 0$, $\rho > 0$ and $\phi \in \mathcal{H}(G)$,*

$$f_\rho^{\delta,\Omega,\phi}(1) = \int_\Omega \widehat{\phi}(\pi_{\pi_{\delta,\nu}}) \, \mathrm{d}\nu + \partial_\rho B(\delta, \Omega), \tag{3.12}$$

where

$$\partial_\rho B(\delta, \Omega) = \int_{i\mathfrak{h}_M^\star} \varepsilon_\rho^{\delta,\Omega}(\pi_{\delta,\nu}) \, \mathrm{d}\nu. \tag{3.13}$$

Proof. This is the proof of [17, Lemma 11.2] into which the spectral localizing function around (δ, Ω) they build, namely $h_\rho^{\delta,\Omega}$ in their notation for reference, is replaced by $h_\rho^{\delta,\Omega,\phi}$. The Plancherel formula gives

$$f_\rho^{\delta,\Omega,\phi}(1) = \int_{i\mathfrak{h}_M^\star} h_\rho^{\delta,\Omega}(\pi_{\pi_{\delta,\nu}}) \widehat{\phi}(\pi_{\pi_{\delta,\nu}}) \, \mathrm{d}\nu. \tag{3.14}$$

Integrating the approximation of Lemma 3 yields

$$\int_{i\mathfrak{h}_M^\star} h_\rho^{\delta,\Omega}(\pi_{\delta,\Omega}) \widehat{\phi}(\pi_{\pi_{\delta,\nu}}) \, \mathrm{d}\nu = \int_\Omega \widehat{\phi}(\pi_{\pi_{\delta,\nu}}) \, \mathrm{d}\nu + \int_{i\mathfrak{h}_M^\star} \varepsilon_\rho^{\delta,\Omega}(\pi_{\delta,\nu}) \, \mathrm{d}\nu, \tag{3.15}$$

and this achieves the proof. ◻

Finally it follows a more explicit form of the value at the identity, namely

$$\begin{aligned} \Phi(1) = {} & \varphi_2(\mathfrak{d}^S) \mu_R^{\mathrm{Pl}}(\pi_R) \widehat{\phi}(\pi_R) \int_\Omega \widehat{\phi}(\pi_{\delta,\nu}) \, \mathrm{d}\nu \int_{\pi_{S,f}^R \in \widehat{G}_{S,f}^R} \widehat{\phi}(\pi_{S,f}^R) \, \mathrm{d}\pi_{S,f}^R \\ & + \varphi_2(\mathfrak{d}^S) \mu_R^{\mathrm{Pl}}(\pi_R) \widehat{\phi}(\pi_R) \partial_\rho B(\delta, \Omega) \int_{\pi_{S,f}^R \in \widehat{G}_{S,f}^R} \widehat{\phi}(\pi_{S,f}^R) \, \mathrm{d}\pi_{S,f}^R. \end{aligned} \tag{3.16}$$

The tools are now in place to, after summation of (3.16), provide a decomposition of the counting measure of the whole universal family.

3.2.2 Identity contribution splitting

Recall that ϕ fixes the S-part of the conductor, so that every S-part of ideal appearing from now on is fixed, namely the only one non killed by the action of ϕ. However, the choice made is to keep formulations in terms if ideals of the whole integer ring $\mathcal{O}$, as more convenient and helping to think about the counting law situation where S is empty. The following decomposition holds for the identity part of the counting measure.

Proposition 18. *For every $Q \geqslant 1$,*

$$\nu_{1,Q} = \nu_{1,Q}^{(p)} + \nu_{1,Q}^{(e1)} + \nu_{1,Q}^{(e2)}, \tag{3.17}$$

where $\nu_{1,Q}^{(p)}$ is the main identity term, namely

$$\begin{aligned}\nu_{1,Q}^{(p)}(\widehat{\phi}) = \frac{1}{2}\frac{\zeta^{S,R\star}(1)\zeta^{S,R}(2)}{\zeta^{S,R}(4)} \int_{\pi_{S,f}^R \in \widehat{G}_S^R} \frac{\widehat{\phi}(\pi_{S,f}^R)}{c(\pi_S^R)^2} \sum_{\substack{N\mathfrak{m}^S \leqslant Q/c(\pi_S^R) \\ \mathfrak{m}^S \wedge R=1}} \frac{\lambda_2(\mathfrak{m}^S)}{(N\mathfrak{m}^S)^2} \\ \sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{m}^S c(\pi_S^R)}} \frac{\widehat{\phi}(\pi_R)}{c(\pi_R)^2}\mu_R^{\mathrm{Pl}}(\pi_R) \sum_{\substack{\underline{\delta}\in\mathcal{D} \\ \underline{\delta}=(M,\delta)}} \int_{\Omega_\delta(Q/N\mathfrak{m}^S c(\pi_R)c(\pi_S^R))} \frac{\widehat{\phi}(\pi_{\delta,\nu})}{c(\pi_{\delta,\nu})^2}\,\mathrm{d}\nu\,\mathrm{d}\pi_{S,f}^R\end{aligned} \tag{3.18}$$

and $\nu_{1,Q}^{(e1)}(\widehat{\phi})$ is the error term due to the smoothing, namely

$$\begin{aligned}\nu_{1,Q}^{(e1)}\left(\widehat{\phi}\right) \ll \frac{1}{Q^2} \int_{\pi_{S,f}^R \in \widehat{G}_S^R} \widehat{\phi}(\pi_{S,f}^R) \\ \sum_{\substack{N\mathfrak{q} \leqslant Q/c(\pi_S^R) \\ \mathfrak{q}\wedge R=1}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right)\varphi_2(\mathfrak{d}^S) \sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{q}^S c(\pi_S^R)}} \mu_R^{\mathrm{Pl}}(\pi_R)\widehat{\phi}(\pi_R) \\ \sum_{\substack{\underline{\delta}\in\mathcal{D} \\ \underline{\delta}=(M,\delta)}} \partial_\rho B(\delta, \Omega(Q/N\mathfrak{m}^S c(\pi_R)c(\pi_S^R)))\,\mathrm{d}\pi_{S,f}^R\end{aligned} \tag{3.19}$$

and $\nu_{1,Q}^{(e2)}(\widehat{\phi})$ is an extra error term, that is

$$\nu_{1,Q}^{(e2)}(\widehat{\phi}) \ll Q^{-\delta_F+\varepsilon_F} \int_{\pi_{S,f}^R \in \widehat{G}_S^R} \frac{\widehat{\phi}(\pi_{S,f}^R)}{c(\pi_S^R)^{2-\delta_F+\varepsilon_F}} \sum_{\substack{N\mathfrak{m}^S \leqslant Q/c(\pi_S^R) \\ \mathfrak{m}^S \wedge R=1}} \frac{\lambda_2(\mathfrak{m}^S)}{(N\mathfrak{m}^S)^{2-\delta_F+\varepsilon_F}}$$
$$\sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{m}^S c(\pi_S^R)}} \frac{\widehat{\phi}(\pi_R)}{c(\pi_R)^{2-\delta_F+\varepsilon_F}} \mu_R^{\mathrm{Pl}}(\pi_R) \tag{3.20}$$
$$\sum_{\substack{\underline{\delta} \in \mathcal{D} \\ \underline{\delta}=(M,\delta)}} \int_{\Omega_\delta(Q/N\mathfrak{m}^S c(\pi_R) c(\pi_S^R))} \frac{\widehat{\phi}(\pi_{\delta,\nu})}{c(\pi_{\delta,\nu})^{2-\delta_F+\varepsilon_F}} \, d\nu \, d\pi_{S,f}^R$$

Proof. The counting measure has been decomposed in measures on harmonic subfamilies (2.55) of fixed spectral parameters. These measures have been given a geometric interpretation by the mean of the trace formula in Lemma 4, whose identity contribution (3.16) is given above. After summation of the identity contributions over the spectral data constituting the truncated universal family,

$$\nu_{1,Q}^{(1)}\left(\widehat{\phi}\right) = \frac{1}{Q^2} \int_{\pi_{S,f}^R \in G_{S,f}^R} \widehat{\phi}(\pi_{S,f}^R) \sum_{\substack{N\mathfrak{q} \leqslant Q/c(\pi_{S,f}^R) \\ \mathfrak{q} \wedge R=1}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) \varphi_2(\mathfrak{d}^S)$$
$$\sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{d} c(\pi_{S,f}^R)}} \mu_R^{\mathrm{Pl}}(\pi_R) \widehat{\phi}(\pi_R) \sum_{\substack{\underline{\delta} \in \mathcal{D} \\ \underline{\delta}=(M,\delta)}} \int_{\Omega_\delta(Q/N\mathfrak{d} c(\pi_R) \pi_{S,f}^R)} \widehat{\phi}(\pi_{\delta,\nu}) \, d\nu \, d\pi_{S,f}^R.$$

Sums of arithmetic functions on ideals of number fields can be explicitly evaluated. This motivates a permutation of sums and integrals in order to estimate the sum over the volumes $\varphi_2(\mathfrak{d}^S)$ first, so that

$$\nu_{1,Q}^{(1)}\left(\widehat{\phi}\right) = \frac{1}{Q^2} \int_{\pi_{S,f}^R \in \widehat{G}_S^R} \widehat{\phi}(\pi_{S,f}^R) \sum_{\substack{N\mathfrak{m}^S \leqslant Q/c(\pi_{S,f}^R) \\ \mathfrak{m}^S \wedge R=1}} \lambda_2(\mathfrak{m}^S) \sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{m}}} \mu_R^{\mathrm{Pl}}(\pi_R) \widehat{\phi}(\pi_R)$$
$$\sum_{\substack{\underline{\delta} \in \mathcal{D} \\ \underline{\delta}=(M,\delta)}} \int_{\Omega_\delta(Q/N\mathfrak{m} c(\pi_R))} \widehat{\phi}(\nu) \sum_{\substack{N\mathfrak{d}^S \leqslant Q/N\mathfrak{m} c(\pi_R) c(\pi_{\delta,\nu}) \\ \mathfrak{d} \wedge R=1}} \varphi_2(\mathfrak{d}^S) \, d\nu \, d\pi_{S,f}^R.$$

The following lemma estimates the innermost sum.

Lemma 6. *Let $\zeta^{S,R}$ be the prime-to-R-and-S part of the zeta function associated to F, and $\zeta^{S,R\star}(1)$ its residue at 1. For any $X > 0$,*

$$\sum_{\substack{N\mathfrak{d}^S \leqslant X \\ \mathfrak{d} \wedge R=1}} \varphi_2(\mathfrak{d}^S) = \frac{1}{2} \frac{\zeta^{S,R\star}(1)\zeta^{S,R}(2)}{\zeta^{S,R}(4)} X^2 + \begin{cases} O(X \log X) & \text{if } F = \mathbf{Q}, \\ O(X^{2-\delta_F}) & \text{otherwise} \end{cases} \tag{3.21}$$

Remark. It is possible to note a posteriori that the remainder term shown here is sharp, and it gives rise to the most significant remainder appearing in Theorem A and Theorem G, safe the one coming from the smoothing part detailed in Lemma 5 that is absent from the totally definite setting. Hence, provided the smoothing problem can be solved and thus a sharp count realized without excessive loss, the error would have power savings and will be similar to the totally definite case.

Proof. Remind that all the ideals superscrited S are prime to S. Standard estimates of the sum of ideals given by Landau [79, VII u. 143 S. n°8] lead to

$$\begin{aligned}
\sum_{\substack{N\mathfrak{d}^S \leqslant X \\ \mathfrak{d}\wedge R=1}} \varphi_2(\mathfrak{d}^S) &= \sum_{\substack{N\mathfrak{l}^S \leqslant X \\ \mathfrak{l}^S\wedge R=1}} \mu^2(\mathfrak{l}^S) \sum_{\substack{N\mathfrak{m}^S \leqslant X/N\mathfrak{l} \\ \mathfrak{m}^S\wedge R=1}} N\mathfrak{m}^S \\
&= \sum_{\substack{N\mathfrak{l}^S \leqslant X \\ \mathfrak{l}^S\wedge R=1}} \mu^2(\mathfrak{l}^S) \left[\frac{\zeta^{S,R\star}(1)}{2} \frac{X^2}{(N\mathfrak{l}^S)^2} + O\left(\left(\frac{X}{N\mathfrak{l}^S} \right)^{2-\delta_F} \right) \right] \\
&= \frac{1}{2} \zeta^{S,R\star}(1) X^2 \sum_{\substack{N\mathfrak{l}^S \leqslant X \\ \mathfrak{d}\wedge R=1}} \frac{\mu^2(\mathfrak{l}^S)}{(N\mathfrak{l}^S)^2} + O\left(X^{2-\delta_F} \sum_{\substack{N\mathfrak{l}^S \leqslant X \\ \mathfrak{l}^S\wedge R=1}} \frac{\mu^2(\mathfrak{l}^S)}{(N\mathfrak{l}^S)^{2-\delta_F}} \right) \\
&= \frac{1}{2} \frac{\zeta^{S,R\star}(1)\zeta^{S,R}(2)}{\zeta^{S,R}(4)} X^2 + \begin{cases} O(X \log X) & \text{if } F = \mathbf{Q}; \\ O(X^{2-\delta_F}) & \textit{otherwise}; \end{cases}
\end{aligned}$$

where the knowledge of the Dirichlet series associated to μ^2 yielded

$$\sum_{N(\mathfrak{m}) \leqslant X} \frac{\mu^2(\mathfrak{m})}{N\mathfrak{m}} \sim \frac{\zeta^\star(1)}{\zeta(2)} \log X = O(\log X), \tag{3.22}$$

in the case $F = \mathbf{Q}$, giving the worst remainder term. Otherwise, the sum is convergent. □

This lemma induces a splitting of $\nu_{1,Q}^{(1)}$ as $\nu_{1,Q}^{(p)} + \nu_{1,Q}^{(e2)}$ according to the principal and error parts in the lemma above and, adding the error term coming from the smoothing evaluation at the identity (3.16), achieves to prove the claim. □

3.2.3 Main part

Proposition 19. *For every $Q \geqslant 1$, the main part admits the asymptotic development*

$$\operatorname{vol}(G(F)\backslash G(\mathbf{A}))\, \nu_{1,Q}^{(p)}\left(\widehat{\phi}\right) = \nu\left(\widehat{\phi}\right) + O(Q^{-2}). \tag{3.23}$$

Proof. Recall the term $v_{1,Q}^{(p)}$ of Proposition 18, namely

$$v_{1,Q}^{(p)}(\phi) = \frac{1}{2}\frac{\zeta^{S,R\star}(1)\zeta^{S,R}(2)}{\zeta^{S,R}(4)} \int_{\pi_{S,f}^R \in \widehat{G}_S^R} \frac{\widehat{\phi}(\pi_{S,f}^R)}{c(\pi_S^R)^2}$$
$$\sum_{\substack{N\mathfrak{m}^S \leqslant Q/c(\pi_S^R) \\ \mathfrak{m}^S \wedge R=1}} \frac{\lambda_2(\mathfrak{m}^S)}{(N\mathfrak{m}^S)^2} \sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{m}^S c(\pi_S^R)}} \frac{\widehat{\phi}(\pi_R)}{c(\pi_R)^2} \mu_R^{\mathrm{Pl}}(\pi_R)$$
$$\sum_{\substack{\underline{\delta} \in \mathcal{D} \\ \underline{\delta}=(M,\delta)}} \int_{\Omega_\delta(Q/N\mathfrak{m}^S c(\pi_R) c(\pi_S^R))} \frac{\widehat{\phi}(\pi_{\delta,\nu})}{c(\pi_{\delta,\nu})^2} \,\mathrm{d}\nu \,\mathrm{d}\pi_{S,f}^R$$

The following lemmas state the convergence of the integral over archimedean parameters and of the sum over ramified parts.

Lemma 7. *For every* $\mathrm{Re}(s) > 1$*, the following sum converges as* $Q \to \infty$*.*

$$\sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q}} \frac{\mu_R^{\mathrm{Pl}}(\pi_R)}{c(\pi_R)^s}. \tag{3.24}$$

Proof. The Jacquet-Langlands correspondence states a bijection between $\widehat{G}_R$ and the discrete part of the spectrum of $\widehat{\mathrm{PGL}_2}(F_R)$, which conserves both formal degrees, which are the Plancherel measures $\mu_R^{\mathrm{Pl}}(\pi_R)$, and conductors by definition. Hence,

$$\sum_{\substack{\sigma_R \in \widehat{G}_R \\ c(\sigma_R) \leqslant Q}} \frac{\mu_R^{\mathrm{Pl}}(\sigma_R)}{c(\sigma_R)^s} \leqslant \sum_{\pi_R \in \widehat{\mathrm{PGL}_2}(F_R)^{\mathrm{disc}}} \frac{\mu_R^{\mathrm{Pl}}(\pi_R)}{c(\pi_R)^s}, \tag{3.25}$$

and that last sum is finite for $\mathrm{Re}(s) > 1$ by the case of PGL_2 by the explicit computations of [17] or by Section 3.2.4 below. Hence, it follows the sought convergence for the ramified parts, ending the proof of the lemma. □

Lemma 8. *For every* $\mathrm{Re}(s) > 1$*, the following integral converges absolutely as* $X \to \infty$*.*

$$\int_{\Omega_\delta(X)} \frac{\widehat{\phi}(\pi_{\delta,\nu})}{c(\pi_{\delta,\nu})^s} \,\mathrm{d}\nu. \tag{3.26}$$

Proof. This is Lemma 6.12 of [17] concerning the GL(2) case. □

Let $\displaystyle\int_{\widehat{G}_R} \frac{\widehat{\phi}(\pi_R)}{c(\pi_R)^s} \,\mathrm{d}\pi_R$ and $\displaystyle\int_{\widehat{G}_\infty^R} \frac{\widehat{\phi}(\pi_R)}{c(\pi_\infty^R)^s} \,\mathrm{d}\pi_\infty^R$ denote the limits in the above lemmas. The prime-to-S-and-R part of the Dirichlet series associated to λ_2 converges at 2 to $\zeta_F^{S,R}(2)^{-2}$

and makes the expression of $\nu_{1,Q}^{(p)}$ converges to

$$\nu_{1,Q}^{(p)} \longrightarrow \frac{1}{2}\frac{\zeta^{S,R\star}(1)}{\zeta^{S,R}(2)\zeta^{S,R}(4)}\int_{\widehat{G}_R}\frac{\widehat{\phi}(\pi_R)}{c(\pi_R)^2}\,\mathrm{d}\pi_R\int_{\widehat{G}_S^R}\frac{\widehat{\phi}(\pi_S^R)}{c(\pi_S^R)^2}\,\mathrm{d}\pi_S^R\int_{\widehat{G}_\infty^R}\frac{\widehat{\phi}(\pi_\infty^R)}{c(\pi_\infty^R)^2}\,\mathrm{d}\pi_\infty^R. \tag{3.27}$$

3.2.4 Geometric reformulation of the constant

Previous computations unveiled the constant

$$\frac{\zeta^{S,R\star}(1)}{\zeta^{S,R}(2)\zeta^{S,R}(4)}. \tag{3.28}$$

It is possible to give to this constant a more geometrical flavour by reformulating the special values of the zeta functions appearing in terms of volumes. This is the content of the following lemma.

Proposition 20. *For every finite set of places S,*

$$\frac{\zeta^{S,R\star}(1)}{\zeta^{S,R}(2)\zeta^{S,R}(4)} = \int_{\widehat{G}^{S,R}}^{\star}\frac{\mathrm{d}\pi^{S,R}}{c(\pi^{S,R})^2} = \zeta^{S,R\star}(1)\prod_{\mathfrak{p}\notin S\cup R}\zeta_{\mathfrak{p}}(1)^{-1}. \tag{3.29}$$

Proof. The knowledge of the volumes of congruence subgroups (3.9) gives

$$\varepsilon_{\overline{K}_{0,\mathfrak{p}}(\mathfrak{p}^r)}(1) = \mathrm{vol}\left(\overline{K}_{0,\mathfrak{p}}(\mathfrak{p}^r)\right)^{-1} = (\mathrm{id}\star\mu^2)(\mathfrak{p}^r). \tag{3.30}$$

On an other hand, this volume can be computed by the Plancherel formula. Introduce the volume of slices of the spectrum of fixed conductor

$$M_{\mathfrak{p}}(\mathfrak{p}^r) = \int_{\substack{\sigma_{\mathfrak{p}}\in\widehat{G}_{\mathfrak{p}}\\ \mathfrak{c}(\sigma_{\mathfrak{p}})=\mathfrak{p}^r}}\mathrm{d}\sigma_{\mathfrak{p}}.$$

The Plancherel inversion formula then yields

$$\begin{aligned}\varepsilon_{\overline{K}_{0,\mathfrak{p}}(\mathfrak{p}^r)}(1) &= \int_{\widehat{G}_{\mathfrak{p}}}\widehat{\varepsilon}_{\overline{K}_{0,\mathfrak{p}}(\mathfrak{p}^r)}(\pi_{\mathfrak{p}})\,\mathrm{d}\pi_{\mathfrak{p}} = \int_{\widehat{G}_{\mathfrak{p}}}\tau_2\left(\frac{\mathfrak{p}^r}{\mathfrak{c}(\pi_{\mathfrak{p}})}\right)\mathrm{d}\pi_{\mathfrak{p}}\\ &= \sum_{\mathfrak{d}\mid\mathfrak{p}^r}M_{\mathfrak{p}}(\mathfrak{d})\tau_2\left(\frac{\mathfrak{p}^r}{\mathfrak{d}}\right) = (M_{\mathfrak{p}}\star\tau_2)(\mathfrak{p}^r)\end{aligned}$$

Hence, by inversion, $M_{\mathfrak{p}} = \mathrm{id}\star\mu^2\star\lambda_2$. In particular, the local Dirichlet series associated to $M_{\mathfrak{p}}$ is given by

$$D_{\mathfrak{p}}(s) = \sum_{\substack{\mathfrak{m}=\mathfrak{p}^r\\ r\geqslant 0}}\frac{M_{\mathfrak{p}}(\mathfrak{m})}{N\mathfrak{m}^s} = \frac{\zeta_{\mathfrak{p}}(s-1)}{\zeta_{\mathfrak{p}}(s)\zeta_{\mathfrak{p}}(2s)}, \qquad \mathrm{Re}(s) > 1. \tag{3.31}$$

Evaluating it at $s = 2$, a new expression for the local special values appearing in the constant is

$$\int_{\widehat{G}_{\mathfrak{p}}} \frac{\mathrm{d}\pi_{\mathfrak{p}}}{c(\pi_{\mathfrak{p}})^2} = \frac{\zeta_{\mathfrak{p}}(1)}{\zeta_{\mathfrak{p}}(2)\zeta_{\mathfrak{p}}(4)}, \tag{3.32}$$

proving the finiteness of the local integrals defining the equidistribution measure (1.8) at the finite places, as claimed in the introduction. However, the infinite product over $\mathfrak{p} \notin R$ of these quantities unfortunately diverges, for 1 is a pole of $\zeta^{S,R}$, leading to compensate it by the residue at 1 and to introduce the regularized integral

$$\int_{\widehat{G}^{S,R}}^{\star} \frac{\mathrm{d}\pi^{S,R}}{c(\pi^{S,R})^2} = \zeta^{S,R\star}(1) \prod_{\mathfrak{p}\notin S\cup R} \zeta_{\mathfrak{p}}(1)^{-1} \int_{\widehat{G}_{\mathfrak{p}}} \frac{\mathrm{d}\pi_{\mathfrak{p}}}{c(\pi_{\mathfrak{p}})^2} = \zeta^{S,R\star}(1) \prod_{\mathfrak{p}\notin S\cup R} \frac{1}{\zeta_{\mathfrak{p}}(2)\zeta_{\mathfrak{p}}(4)}, \tag{3.33}$$

ending the proof. □

The global integral is defined to be

$$\int_{\widehat{\Pi}}^{\star} \frac{\mathrm{d}\pi}{c(\pi)^2} = \int_{\widehat{G}^{S,R}}^{\star} \frac{\mathrm{d}\pi^{S,R}}{c(\pi^{S,R})^2} \int_{\widehat{G}_{S\cup R}} \frac{\widehat{\phi}(\pi_{S\cup R})}{c(\pi_{S\cup R})^2} \mathrm{d}\pi_{S\cup R} = \zeta^{\star}(1) \prod_{v} \zeta_v(1)^{-1} \int_{\widehat{G}_v} \frac{\mathrm{d}\pi_v}{c(\pi_v)^2}. \tag{3.34}$$

It thus follows the expression (1.8) of the regularized integral, giving the desired statement and motivating the choice of both the measure μ and the constant C. The expression (3.27) then rewrites

$$\nu_{1,Q}^{(p)}(\widehat{\phi}) = \frac{1}{2} \int_{\widehat{\Pi}}^{\star} \frac{\widehat{\phi}(\pi)}{c(\pi)^2} \mathrm{d}\pi + O(Q^{-2}), \tag{3.35}$$

reaching the term of the proof of Proposition 19. □

Remark. Notice that the Sauvageot theorem is a two-edged result: it opens the path to equidistribution and allows conclusions for characteristic functions which are not of the form $\widehat{\phi}$; however it also spoils the remainder term for general functions. This error term remains only for specific functions either admissible, *i.e.* of the form $\widehat{\phi}$ for ϕ in the Hecke algebra of G, in particular for the counting problem.

3.2.5 Smoothing part

The following lemma states the negligibility of the smoothing part compared to the main one.

Lemma 9. *For every $Q \geqslant 1$,*

$$\nu_{1,Q}^{(e1)}\left(\widehat{\phi}\right) \ll \log(Q)^{-1}. \tag{3.36}$$

Proof. The term (3.19) coming from the smoothed selection of archimedean parts is

$$\nu_{1,Q}^{(e1)}\left(\widehat{\phi}\right) \ll \frac{1}{Q^2}\int_{\pi_{S,f}^R\in\widehat{G}_S^R}\widehat{\phi}(\pi_{S,f}^R) \sum_{\substack{N\mathfrak{q}\leqslant Q/c(\pi_S^R)\\ \mathfrak{q}\wedge R=1}}\sum_{\mathfrak{d}^S\mid\mathfrak{q}^S}\lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right)\varphi_2(\mathfrak{d}^S)\sum_{\substack{\pi_R\in\widehat{G}_R\\ c(\pi_R)\leqslant Q/N\mathfrak{q}^S c(\pi_S^R)}}\mu_R^{\mathrm{Pl}}(\pi_R)\widehat{\phi}(\pi_R) \sum_{\substack{\delta\in\mathcal{D}\\ \underline{\delta}=(M,\delta)}}\partial_\rho B(\delta,\Omega(Q/N\mathfrak{d}^S c(\pi_R)c(\pi_S^R)))\,\mathrm{d}\pi_{S,f}^R, \tag{3.37}$$

The following lemma is a straighforward adaptation of the work of Brumley and Milićević in which the contribution of the smoothing effect is negligible.

Lemma 10. *The contribution of the smoothing error term satisfies, for a suitable choice of ρ depending on $\mathfrak{q}$,*

$$\sum_{\substack{N\mathfrak{q}\leqslant Q\\ \mathfrak{q}\wedge R,S=1}}\sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q/N\mathfrak{q}}}\sum_{\delta\in\mathcal{D}}\sum_{\mathfrak{d}\mid\mathfrak{q}}\lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right)\phi_2(\mathfrak{d}^S)\mu_R^{\mathrm{Pl}}(\sigma_R)\partial_\rho B(\delta,\Omega)\ll\frac{Q^2}{\log(Q)}$$

Proof. This is in essence Proposition 12.1 in [17]. Their Lemma 12.2 states

$$\sum_{\substack{N\mathfrak{q}\leqslant Q\\ \mathfrak{q}\wedge R,S=1}}\sum_{\delta\in\mathcal{D}}N\mathfrak{q}\partial_\rho B(\delta,\Omega(Q/N\mathfrak{q}))\ll\frac{Q^2}{\log(Q)}. \tag{3.38}$$

Two slight modifications have to be mentioned because of the existence of ramified places and of the use of a different filtration. The centerless setting leads to use a filtration of subgroups $K_0(\mathfrak{q})$ whose indices are of order $N\mathfrak{q}$, instead of $N\mathfrak{q}^2$ in their case, justifying the presence of $N\mathfrak{q}$ in the equation above. The existence of ramified places is dealt with by plugging this estimates above in the whole sum,

$$\sum_{\substack{N\mathfrak{q}\leqslant Q\\ \mathfrak{q}\wedge R,S=1}}\sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q/N\mathfrak{q}}}\sum_{\delta\in\mathcal{D}}\sum_{\mathfrak{d}\mid\mathfrak{q}}\lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right)\phi_2(\mathfrak{q})\mu_R^{\mathrm{Pl}}(\pi_R)\partial_\rho B(\delta,\Omega(Q/N\mathfrak{q}c(\pi_R))) \ll\frac{Q^2}{\log Q}\sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q/N\mathfrak{q}}}\frac{\mu_R^{\mathrm{Pl}}(\pi_R)}{c(\pi_R)^{2-\varepsilon}}$$

and this last sum converges by Lemma 7, finishing the proof. □

Remark. Note that the worst error term in Theorems A and G comes from this part, corresponding to the smoothing of the selecting function for split archimedean places. As in the GL(2) case [17], this is where the desired power savings is lost, safe in the totally definite case when there is no involved smoothing.

3.2.6 Extra error term

Lemma 11. *For every $Q \geqslant 1$,*

$$\nu_{1,Q}^{(e2)}(\phi) \ll Q^{-\delta_F+\varepsilon_F}, \tag{3.39}$$

for all $\varepsilon_F > 0$ for the case $F = \mathbf{Q}$, and for $\varepsilon_F = 0$ otherwise.

Proof. Now turn back to treatment of the $\nu_{1,Q}^{(e2)}$ term coming from the remainder in Lemma 6. The bound that has to be refined is

$$\begin{aligned}\nu_{1,Q}^{(e2)}(\phi) \ll Q^{-\delta_F+\varepsilon_F} &\int_{\pi_{S,f}^R \in \widehat{G}_S^R} \frac{\widehat{\phi}(\pi_{S,f}^R)}{c(\pi_S^R)^{2-\delta_F+\varepsilon_F}} \\ &\sum_{\substack{N\mathfrak{m}^S \leqslant Q/c(\pi_S^R) \\ m^S \wedge R=1}} \frac{\lambda_2(\mathfrak{m}^S)}{(N\mathfrak{m}^S)^{2-\delta_F+\varepsilon_F}} \sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{m}^S c(\pi_S^R)}} \frac{\widehat{\phi}(\pi_R)}{c(\pi_R)^{2-\delta_F+\varepsilon_F}} \mu_R^{\mathrm{Pl}}(\pi_R) \\ &\sum_{\substack{\underline{\delta} \in \mathcal{D} \\ \underline{\delta}=(M,\delta)}} \int_{\Omega_\delta(Q/N\mathfrak{m}^S c(\pi_R) c(\pi_S^R))} \frac{\widehat{\phi}(\pi_{\delta,\nu})}{c(\pi_{\delta,\nu})^{2-\delta_F+\varepsilon_F}} \, d\nu \, d\pi_{S,f}^R\end{aligned}$$

The inner sums converge by Lemmas 7 and 8, since $2 - \delta_F + \varepsilon_F$ is always greater than 1. It follows a remainder term in $Q^{-\delta_F+\varepsilon_F}$. □

At last, the asymptotic development obtained in Proposition 19 and the bounds obtained in Lemmas 9 and 11 prove the equidistribution of the identity part of the counting measure with respect to ν, as stated in Proposition 17.

3.3 Spectral error terms

3.3.1 Characters contribution

Recall that the global characters contribution is given by

$$\nu_{\Xi,Q}(\widehat{\phi}) = \frac{1}{Q^2} \sum_{\substack{N\mathfrak{q} \leqslant Q \\ \mathfrak{q} \wedge R=1}} \sum_{\substack{\pi_R \in \widehat{G}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{q}}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) \Xi(\phi, \pi_R). \tag{3.40}$$

Lemma 12. *For every $\varepsilon > 0$,*

$$\nu_{\Xi,Q}(\widehat{\phi}) \ll Q^{-1+\varepsilon}. \tag{3.41}$$

Proof. Similarly to the intervention of the trace formula to make explicit the measure ν_Q, the Poisson summation formula is the main tool to count characters. The counting measure for characters can be interpreted as a spectral side, such that every non-identity terms vanishes on the geometric side. Recall that for a character π_R, since the multiplicities are all equal to one,

$$\Xi(\pi_R, \phi) = \sum_{\substack{\chi \in X^{\mathrm{ur}}(G(\mathbf{A})) \\ \chi_R \simeq \pi_R}} \widehat{\phi}(\chi). \tag{3.42}$$

As in Section 2.2.3, consider GL(2) instead of PGL(2) for simplicity, characters of PGL(2) corresponding to those of GL(2) trivial on the center. Characters on GL(2) decompose through

$$G(F_{\mathfrak{p}}) \longrightarrow F_{\mathfrak{p}}^{\times} \longrightarrow S^1, \tag{3.43}$$

where the first arrow is given by the determinant and the second by characters of $F_{\mathfrak{p}}^{\times}$. In other words, a character $\chi_{\mathfrak{p}}$ of $\mathrm{GL}(2, F_{\mathfrak{p}})$ is of the form $\chi_{0,\mathfrak{p}} \circ \det$ where $\chi_{0,\mathfrak{p}}$ is a character of $F_{\mathfrak{p}}^{\times}$.

At an archimedean place v, since the considered characters are trivial on the center, they are among the trivial one and the sign, hence have conductor 1 at those places. Archimedean characters are of the form $\mathrm{sgn}^{\varepsilon}|\det|^{it}$ for $\varepsilon = \pm 1$ and $t \in \mathbf{R}$. Similarly to the smoothing function introduced in Section 2.5.4, it is not possible to select precisely continuous parameters, it is hence necessary to supply an approximation by a localizing function. This motivates the introduction of f_v a compactly supported non-negative smooth function such that $\widehat{f_v}$ is 1 for $t = 0$, and $|\widehat{f_v}| \leqslant 1$. In particular, it vanishes unless t is small enough, say $|t| \leqslant T$.

For the arithmetic part of the conductor, the only characters not killed by the action of $\widehat{\varepsilon}_{\mathfrak{p}^r}$ are the unramified ones. Indeed, recall that

$$\det\left(K_0(\mathfrak{p}^r)\right) = \mathcal{O}_{\mathfrak{p}}^{\times}, \tag{3.44}$$

so that $\chi_{0,\mathfrak{p}}$ needs to be trivial on $\mathcal{O}_{\mathfrak{p}}^{\times}$, that is to say be unramified. Introduce, for every finite split place $\mathfrak{p}$, the characteristic function $f_{\mathfrak{p}}$ of $\mathcal{O}_{\mathfrak{p}}^{\times}$, whose Fourier transform selects unramified characters analogously to Lemma 2.70. Introduce the global test function

$$f = \prod_{\mathfrak{p} \notin R} f_{\mathfrak{p}} \prod_{v \in R} \xi_{\chi_v} \prod_{\substack{v \mid \infty \\ v \notin R}} f_v. \tag{3.45}$$

Since $\hat{f}_{\mathfrak{p}}$ is 1 on unramified characters and the archimedean $\hat{f}_v$'s are less than one , the Poisson summation formula gives

$$\Xi(\pi_R, \phi) \leqslant \sum_{\chi \in \widehat{F^{\times}}} \widehat{f}(\chi) = \frac{1}{\mathrm{vol}(F^{\times} \backslash \mathbf{A}^{\times})} \sum_{\gamma \in F^{\times}} f(\gamma). \tag{3.46}$$

Since $F^\times$ is a discrete set, choosing f_∞ with a small enough support leads to kill every $f(\gamma)$ for γ nontrivial. Hence $\Xi(\pi_R, \phi) \leqslant \mathrm{vol}(F^\times\backslash\mathbf{A}^\times)^{-1} f(1)$. It remains to evaluate $f(1) = f_{\mathfrak{d}^S}(1) f_R(1) f_\infty(1)$. For the finite split places, $f_{\mathfrak{p}}(1) = 1$, and for the ramified places, $f_R(1) = \mu_R^{\mathrm{Pl}}(\pi_R)$. For the archimedean places, the Plancherel inversion formula gives

$$f_\infty(1) = \int_{\widehat{F}_\infty^R} \widehat{f_\infty}(\chi)\,\mathrm{d}\chi \leqslant \int_{|t|\leqslant T} \mathrm{d}\chi_t \ll_T 1. \tag{3.47}$$

Finally, $\Xi(\pi_R, \phi) \ll \mu_R^{\mathrm{Pl}}(\pi_R)$. Coming back to the sum (3.40) defining $\nu_{\Xi,Q}(\widehat{\phi})$, it follows by using the rough bound $\lambda_2(\mathfrak{n}) \ll N\mathfrak{n}^\varepsilon$,

$$\begin{aligned}
\nu_{\Xi,Q}(\widehat{\phi}) &\ll \frac{1}{Q^2} \sum_{\substack{N\mathfrak{q}\leqslant Q\\ \mathfrak{q}\wedge R=1}} \sum_{\substack{\pi_R\in\widehat{G}_R\\ c(\pi_R)\leqslant Q/N\mathfrak{q}}} \sum_{\mathfrak{d}^S\,|\,\mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) \mu_R^{\mathrm{Pl}}(\pi_R)\\
&\ll \frac{1}{Q^2} \sum_{\substack{\pi_R\in\widehat{G}_R\\ c(\pi_R)\leqslant Q}} \mu_R^{\mathrm{Pl}}(\pi_R) \sum_{\substack{N\mathfrak{d}\leqslant Q/c(\pi_R)\\ \mathfrak{q}\wedge R=1}} \sum_{\substack{N\mathfrak{m}\leqslant Q/N\mathfrak{d}c(\pi_R)\\ \mathfrak{q}\wedge R=1}} N\mathfrak{m}^\varepsilon\\
&\ll Q^{-1+\varepsilon} \sum_{\substack{\pi_R\in\widehat{G}_R\\ c(\pi_R)\leqslant Q}} \frac{\mu_R^{\mathrm{Pl}}(\pi_R)}{c(\pi_R)^{1+\varepsilon}} \sum_{\substack{N\mathfrak{d}\leqslant Q/c(\pi_R)\\ \mathfrak{q}\wedge R=1}} N\mathfrak{d}^{-1-\varepsilon}\\
&\ll Q^{-1+\varepsilon}
\end{aligned}$$

and this last line is provided by the convergence of the sum over ramified representation, stated in Lemma 7, proving the result. □

3.3.2 Complementary spectrum

Lemma 4 provides an interpretation of the counting measure $B(\mathfrak{q}, \pi_R, \delta, \Omega, \phi)$ as the tempered part of the trace formula for a suitable test function, so it is necessary to consider the remaining complementary part of the spectrum. Lemma 3 states exponential control for spectral parameters lying outside the tempered subspace, so that the complementary contribution is bounded by $\partial_\rho B(\delta, \partial_\rho\Omega)$. This section is dedicated to prove the following lemma, which is an adaptation of the work of Brumley and Milićević in which the complementary spectrum is shown to contribute as an error term.

Lemma 13. *The contribution of the complementary part and the smoothing error term satisfy, for a suitable choice of ρ depending on $\mathfrak{q}$ and a certain $\theta > 0$,*

$$\sum_{\substack{N\mathfrak{q}\leqslant Q\\ \mathfrak{q}\wedge R,S=1}} \sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q/N\mathfrak{q}}} \sum_{\delta\in\mathcal{D}} \sum_{\mathfrak{d}|\mathfrak{q}} \lambda_2\left(\frac{\mathfrak{q}^s}{\mathfrak{d}^S}\right) \phi_2(\mathfrak{d}^S) \mu_R^{\mathrm{Pl}}(\pi_R) B_{\mathrm{comp}}(\delta, \Omega(Q/N\mathfrak{q}c(\pi_R))) \ll Q^{2-\theta} \tag{3.48}$$

Proof. This is in essence Proposition 12.1 in [17]. Their Lemma 12.2 yields

$$\sum_{\substack{N\mathfrak{q}\leqslant Q\\ \mathfrak{q}\wedge R,S=1}} \sum_{\delta\in\mathcal{D}} N\mathfrak{q} B_{\text{comp}}(\mathfrak{q},\delta,\Omega(Q/N\mathfrak{q})) \ll Q^{2-\theta} \tag{3.49}$$

Similarly to Lemma 10, the different congruence subgroups chosen for the centerless setting leads to input $N\mathfrak{q}$ in the sum above, instead of $N\mathfrak{q}^2$ in their case. The sum over ramified places is dealt with by appealing to Lemma 7 which ensures the convergence.

$$\begin{aligned}
&\sum_{\substack{N\mathfrak{q}\leqslant Q\\ \mathfrak{q}\wedge R,S=1}} \sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q/N\mathfrak{q}}} \sum_{\delta\in\mathcal{D}}\sum_{\mathfrak{d}|\mathfrak{q}} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) J_{\text{comp}}\left(\Phi_{\mathfrak{d},\pi_R,\delta,\Omega_\delta(Q/N\mathfrak{d}\mathfrak{c}(\pi_R))}\right)\\
&\quad\ll \sum_{\substack{N\mathfrak{q}\leqslant Q\\ \mathfrak{q}\wedge R,S=1}} \sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q/N\mathfrak{q}}} \sum_{\delta\in\mathcal{D}}\sum_{\mathfrak{d}|\mathfrak{q}} N\mathfrak{q}\mu_R^{\text{Pl}}(\pi_R) B_{\text{comp}}(\delta,\Omega(Q/N\mathfrak{d}\mathfrak{c}(\pi_R)))\\
&\quad\ll Q^{2-\theta} \sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q/N\mathfrak{q}}} \frac{\mu_R^{\text{Pl}}(\pi_R)}{c(\pi_R)^{2-\theta}}
\end{aligned}$$

and this last sum converges by Lemma 7, finishing the proof. ◻

At last, it follows from this lemma along with Lemma 12 that the extra terms in the spectral selecting Lemma 4 are negligible, so that the counting measure part $B(\mathfrak{q},\pi_R,\delta,\Omega,\phi)$ is fairly well approximated by the tempered part $J_{\text{temp}}(\Phi)$ of the spectrum. Lemma 9 states that this tempered part is approximated by the whole spectral part $J_{\text{spec}}(\Phi)$ up to an error term. Considering the development of the identity contribution to the geometrical part given in Proposition 19, it follows for every $\varepsilon > 0$,

$$\nu_Q = \operatorname{vol}\left(G\left(F\right)\backslash G\left(\mathbf{A}\right)\right)\nu_{1,Q} + \nu_{\text{ell},Q} + \begin{cases} O\left(Q^{-1}\log(Q)\right) & \text{if } B \text{ totally definite and } F=\mathbf{Q};\\ O\left(Q^{-\delta_F+\varepsilon}\right) & \text{if } B \text{ totally definite and } F\neq\mathbf{Q};\\ O\left(\log(Q)^{-1}\right) & \text{if } B \text{ not totally definite.}\end{cases} \tag{3.50}$$

3.4 Geometric error terms

The present section, aims at estimating the non-identity terms appearing in the elliptic contribution to the geometric side, by the means of bounds on orbital integrals.

3.4.1 Strategy

The contribution of the elliptic terms in the trace formula (5.24) is

$$J_{\mathrm{ell}}(\Phi) = \sum_{\{\gamma\}\neq 1} \mathrm{vol}\left(G_\gamma(F)\backslash G_\gamma(\mathbf{A})\right) \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} \Phi\left(x^{-1}\gamma x\right) \mathrm{d}x. \tag{3.51}$$

As a matter of fact, this expression generally requires to bound

- the length of the summation, provided it is finite;
- the global volumes $\mathrm{vol}\left(G_\gamma(F)\backslash G_\gamma(\mathbf{A})\right)$;
- the orbital integrals.

Since $\Phi_{\mathfrak{d},\pi_R,\delta,\Omega,\rho;\phi}$ is supported on G, hence compactly supported on a compact independent of the fixed spectral parameters, the sum is eventually finite. A crucial feature is that the support is independent of $\mathfrak{d}$, for $K_0(\mathfrak{d})$ decreases with $\mathfrak{d}$. Different operators than $\varepsilon_{\mathfrak{d}}$ might have a support increasing with $\mathfrak{d}$, case in which this dependence can be no more neglected, cf. Section 4.4 for the case of Hecke operators.

However, the size of the summation does depend on the quality of the archimedean split approximation, encoded in ρ, in a critical way. The number of conjugacy classes appearing in the sum is bounded in the following lemma [17, Proposition 13.2].

Lemma 14. *There is $c > 0$ such that the number of conjugacy classes γ of $G(F)$ for which $\mathcal{O}_\gamma(\Phi_\rho)$ is nonzero is bounded by $\exp(c\rho)$.*

Since the global volumes do not depend on ρ, it is only needed to estimate both the size of the sum and to bound the orbital integrals defined by

$$\mathcal{O}_\gamma(\Phi) = \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} \Phi(x\gamma x^{-1})\,\mathrm{d}x, \qquad \Phi \in \mathcal{H}(G_S). \tag{3.52}$$

The needed bound on orbital integrals is provided by the following result.

Proposition 21. *There is a constant $c > 0$ such that for every $\gamma \in G(\mathbf{A})$,*

$$\mathcal{O}_\gamma(\Phi) \ll_{\gamma,\varepsilon} (N\mathfrak{d}^S)^{-1+\varepsilon} \left\| f_\rho^{\delta,\Omega,\phi} \right\|_\infty. \tag{3.53}$$

The local components $\Phi_{\mathfrak{p}}$'s are locally compact and almost always equal to $\mathbf{1}_{\overline{K}_{\mathfrak{p}}}$, so that the corresponding orbital integrals are trivial by the measures normalizations. The local decomposition of orbital integrals [75] for the factorizable functions $\Phi = \otimes_v \Phi_v$ then holds.

Proposition 22 (Factorization of the geometric side). *If $\gamma = \prod_v \gamma_v$ and $\Phi = \otimes_v \Phi_v$, then*

$$\mathcal{O}_\gamma(\Phi) = \prod_v \mathcal{O}_{\gamma,v}(\Phi_v), \qquad \text{where} \quad \mathcal{O}_{\gamma,v}(\Phi_v) = \int_{G_{\gamma,v}\backslash G_v} \Phi_v\left(x_v\gamma_v x_v^{-1}\right) \mathrm{d}x_v. \tag{3.54}$$

It suffices to locally dominate those orbital integrals. The split, non-split and archimedean cases behave quite differently and require specific treatments.

3.4.2 Split orbital integrals

Almost every place is split, so precise bounds are needed in order to control the global orbital integral. Fortunately, the test functions chosen at these places are explicit and allows a precise control of the associated orbital integrals.

Non-archimedean split places

Lemma 15. *For every ideal $\mathfrak{d}^S$ of $\mathcal{O}^S$,*

$$\mathcal{O}_{\gamma^S}(\mathfrak{d}^S) \ll N\left(\mathfrak{d}^S\right)^{\varepsilon}. \tag{3.55}$$

Proof. First, consider the split case $\mathfrak{p} \notin R$. Let $\gamma_{\mathfrak{p}} \in G_{\mathfrak{p}}$. In the case of a place $\mathfrak{p} \notin S$, the local test function is $\varepsilon_{\mathfrak{p}^r}$, so that

$$\mathcal{O}_{\gamma}(\varepsilon_{\mathfrak{p}^r}) = \mathrm{vol}(K)^{-1}\mathcal{O}_{\gamma}(\mathbf{1}_K), \qquad \text{where} \qquad K = \overline{K}_0(\mathfrak{p}^r). \tag{3.56}$$

Bounds for the split orbital integrals are provided by Binder [9, Corollary 10.9] in the specific case of GL(2), and yield the following estimate depending on $\gamma_{\mathfrak{p}}$.

$$\mathcal{O}_{\gamma_{\mathfrak{p}}}(\varepsilon_{\mathfrak{p}^r}) \ll_{\varepsilon} N(\mathfrak{p}^r)^{-1+\varepsilon}\mathrm{vol}(\overline{K}_0(\mathfrak{p}^r))^{-1} \ll_{\varepsilon} N(\mathfrak{p}^r)^{\varepsilon}. \tag{3.57}$$

Otherwise, for $\mathfrak{p} \in S$, the test function is fixed to $\phi_{\mathfrak{p}}$ and hence can be roughly bounded by

$$\mathcal{O}_{\gamma_{\mathfrak{p}}}(\phi_{\mathfrak{p}}) \ll 1, \tag{3.58}$$

settling the desired estimates for finite split orbital integrals. □

Archimedean split places

Lemma 16. *The following holds:*

$$\mathcal{O}_{\gamma_\infty^R}(f) \ll \|f\|_\infty. \tag{3.59}$$

Proof. This is essentially a rough bound on the archimedean orbital integral, see [17, Proposition 14.2]. □

Global bounds for the last quantity is provided by [17, Lemma 7.3].

Lemma 17. *For every $\delta \in \mathcal{D}$ and every $X > 0$, there is $c > 0$ so that*

$$\sum_{\delta\in\mathcal{D}} \left\|f_\rho^{\delta,\Omega_\delta(X),\phi}\right\|_\infty \ll e^{c\rho}X^{2-1/[F:\mathbb{Q}]}. \tag{3.60}$$

3.4.3 Non-split orbital integrals

Ramified places are in finite number but the explicit behavior of local orbital integrals could a priori be unbounded. The following lemma settles the problem.

Lemma 18. *For every ramified place v,*

$$O_{\gamma_v}(\Phi_v) \ll_{\gamma_v} 1. \tag{3.61}$$

Proof. Archimedean and non-archimedean ramified places behave differently. Before turning to the precise study of each case, note that whatever Φ_v is ξ_{π_v} or $\xi_{\pi_v}\widehat{\phi_v}(\pi_v)$, the orbital integral is dominated by the case of the matrix coefficient ξ_{π_v}, for $\widehat{\phi_v}$ is bounded. In the ramified case, orbital integrals are characters: for a representation π_v, the "main geometric lemma" of Arthur [2] states that

$$O_{\gamma_v}(\xi_{\pi_v}) = \Theta_{\pi_v}(\gamma_v), \tag{3.62}$$

where Θ_{π_v} stands for the Harish-Chandra character associated to π_v. It is in particular sufficient to bound characters on $B_{\mathfrak{p}}^\times$ in order to get a bound for orbital integrals.

Archimedean ramified places

At archimedean places $v \mid \infty$, the algebra B_v is isomorphic to the hamiltonian quaternions, hence Θ_{π_v} is a character of PSU(2). The Weyl formula for archimedean characters gives in this case that the characters are of the form

$$\chi_{\mathrm{Sym}^k}(e_\theta) = \frac{\sin((k+1)\theta)}{\sin\theta}, \tag{3.63}$$

where the rotations $e_\theta \in SO(3)$ are the standard representatives for PSU(2). This expression is uniformly bounded in k and θ, so are the Θ_{π_v}, thus also the orbital integrals.

Non-archimedean ramified places

Concerning the non-archimedean ramified places $\mathfrak{p} \in R$, the lead is given to Shin and Templier [118], who build on the Sally-Shalika character formula [106] and the expository work of Adler, DeBacker, Sally and Spice [1], in order to give explicit computations for the characters of each supercuspidal representations of SL(2). They prove that for every supercuspidal representation $\pi_{\mathfrak{p}}$ of $\mathrm{SL}(2, F_{\mathfrak{p}})$, and for all semisimple regular element $\gamma_{\mathfrak{p}}$,

$$|\Theta_{\pi_{\mathfrak{p}}}(\gamma_{\mathfrak{p}})| \leqslant 1 + 2|D(\gamma_{\mathfrak{p}})|^{-1/2}, \tag{3.64}$$

where $D(\gamma_{\mathfrak{p}})$ is the Weyl discriminant of $\gamma_{\mathfrak{p}}$. It follows $\Theta_{\pi_{\mathfrak{p}}}(\gamma_{\mathfrak{p}}) \ll 1$. Moreover, it suffices to achieve this goal for SL_2. Indeed, Labesse and Langlands [81] establish that every irreducible admissible representation of GL(2) restricts to a direct sum of at most four irreducible admissible representations of SL(2).

Since the Jacquet-Langlands correspondence maps irreducible representations of $G_\mathfrak{p}$ to supercuspidal representations, and the image of the embedding of $G_\mathfrak{p}$ in $\mathrm{GL}_2(F_\mathfrak{p})$ is made of semisimple regular elements, the bound above apply to $B_\mathfrak{p}$.

The bounds obtained in the two cases of ramified places hence settle the proof. □

3.4.4 Final estimates

All the tools are now in place to work on the final estimates on $\nu_{Q,\mathrm{ell}}(\widehat{\phi})$ and reach the term of the proof of Theorem G.

Proposition 23. *For a finite set of places S, $\phi \in \mathcal{H}(G_S)$ and any $\varepsilon > 0$, the elliptic contribution is dominated by*

$$\nu_{\mathrm{ell},Q}(\widehat{\phi}) \ll \begin{cases} Q^{-1+\varepsilon} & \text{if } B \text{ totally definite;} \\ \log(Q)^{-1} & \text{otherwise.} \end{cases} \tag{3.65}$$

Proof. Previous estimates and local decomposition of orbital integrals lead to

$$\begin{aligned} J_{\mathrm{ell}}\left(\Phi_{\mathfrak{d},\pi_R,\delta,\Omega;\phi}\right) &= \sum_{\{\gamma\}\neq 1} \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma) \mathcal{O}_\gamma\left(\Phi_{\mathfrak{d},\pi_R,\delta,\Omega;\phi}\right) \\ &\ll (N\mathfrak{d}^S)^\varepsilon e^{c\rho} \left\| f_\rho^{\delta,\Omega_\delta(Q/N\mathfrak{q}^S c(\pi_R)),\phi} \right\|_\infty \end{aligned}$$

Going back to the estimate for $\nu_{\mathrm{ell},Q}(\widehat{\phi})$ and summing over the spectral data,

$$\begin{aligned} \nu_{\mathrm{ell},Q}(\widehat{\phi}) &\ll \frac{1}{Q^2} \sum_{\substack{N\mathfrak{q}\leqslant Q \\ \mathfrak{q}\wedge R=1}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) (N\mathfrak{d}^S)^\varepsilon \sum_{\substack{\pi_R\in\widehat{G}_R \\ c(\pi_R)\leqslant Q/N\mathfrak{q}^S}} \widehat{\phi}(\pi_R) \sum_{\substack{\delta\in\mathcal{D} \\ \underline{\delta}=(M,\delta)}} e^{c\rho} \left\| f_\rho^{\delta,\Omega_\delta(Q/N\mathfrak{q}^S c(\pi_R)),\phi} \right\|_\infty \\ &\ll Q^{-1/d} \sum_{\substack{N\mathfrak{q}\leqslant Q \\ \mathfrak{q}\wedge R=1}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \lambda_2\left(\frac{\mathfrak{q}^S}{\mathfrak{d}^S}\right) e^{c\rho} (N\mathfrak{d}^S)^{\varepsilon+1/d-2} \sum_{\substack{\pi_R\in\widehat{G}_R \\ c(\pi_R)\leqslant Q/N\mathfrak{q}^S}} \frac{\widehat{\phi}(\pi_R)}{c(\pi_R)^{2-1/d}} \end{aligned}$$

where the bound (3.60) and the elementary bound $\lambda_2(\mathfrak{n}) \ll_\varepsilon N\mathfrak{n}^\varepsilon$ have been used. Thus, since Lemma 7 and 8 ensure the convergences of the inner sum, it follows that for $\rho = \alpha \log N\mathfrak{d}^S$ where $\alpha > 0$ is small enough,

$$\nu_{\mathrm{ell},Q}(\widehat{\phi}) \ll_\varepsilon Q^{-1/d+\varepsilon},$$

achieving the proof that the main term contributing in 3.5 is the one coming from the identity as stated in Proposition 23, hence also Theorems A, B and G. Note that besides the smoothing error term, the most significant error term comes from this elliptic part when $F = \mathbf{Q}$, for then $\delta_F = 1$, and from the identity part of Proposition 17 otherwise. $\square$

3.4.5 Specific savings in the case of $F = \mathbf{Q}$

In the ramified case and when $F = \mathbf{Q}$, and for totally definite quaternion algebras, there is a similar feature than the one appearing in Schanuel's theorem: the error term picks an additional power of the logarithm. However, this feature is not caught by the previous evaluation and requires specific treatment using the Howe classification of representations at ramified places, that holds when 2 is not ramified. This classification is briefly reminded and the corresponding computations carried out to reach the final specific form of Theorem A for $F = \mathbf{Q}$.

Proposition 24. *For $Q \geqslant 1$, for every $\varepsilon > 0$,*

$$\nu_{\mathrm{ell},Q}(\widehat{\phi_S}) \ll Q^{-1+\varepsilon} \log^{|R|} Q.$$

Proof. For non-archimedean ramified places, in order to estimate the sum over the ramified representations π_R appearing in (2.70), the Howe construction provides and explicit parametrization. It is exposed in the work of Corwin, Moy and Prasad [31]: for a ramified place $\mathfrak{p} \in R$, ramified representations of $G_{\mathfrak{p}}$ correspond to admissible or subadmissible characters θ on an extension E/F of degree $m = 1$ or 2. Let π_θ denote this representation, avoiding to state the explicit correspondence between θ and π_θ, using it only as parametrization and referring to the article above for details and proofs of the properties used.

Following Schmidt [113], there is a link between conductors of characters θ, denoted $\mathfrak{f}(\theta)$, and conductors of the corresponding representations π_θ. If E/F is a trivial or quadratic extension and θ a character of $E^\times$ unfactorizable by the norm, introducing the residual degree $f = f(E/F)$ of the extension, the conductor of the associated supercuspidal irreducible representation π_θ is given by

$$\mathfrak{f}(\pi_\theta) = \frac{2}{[E:F]} f(\mathfrak{f}(\theta) - 1) + 2. \tag{3.66}$$

In particular,

- if $E = F$, $\mathfrak{f}(\pi_\theta) = 2\mathfrak{f}(\theta)$, so that $\mathfrak{c}(\pi_\theta) = \mathfrak{c}(\theta)^2$;
- if E/F is unramified, $\mathfrak{f}(\pi_\theta) = 2\mathfrak{f}(\theta)$, so that $\mathfrak{c}(\pi_\theta) = \mathfrak{c}(\theta)^2$;
- if E/F is ramified, $\mathfrak{f}(\pi_\theta) = \mathfrak{f}(\theta) + 1$, so that $\mathfrak{c}(\pi_\theta) = \mathfrak{p}\mathfrak{c}(\theta)$.

Moreover, there are only a finite number of quadratic extensions of F. Indeed, Krasner's work give the exact number of extensions of F of given degree. Here, the situation is even easier and quadratic extensions of F are parametrized by $F^\times/F^{\times 2}$. This quotient is a $\mathbf{F}_2$-vector space, knowing its cardinality is thus equivalent than knowing its dimension. For ϖ a uniformizer of F, the decomposition $F^\times \cong \varpi^{\mathbf{Z}} \times \mathcal{O}^\times$ leads to

$$F^\times/F^{\times 2} \cong \left(\varpi^{\mathbf{Z}}/\varpi^{2\mathbf{Z}}\right) \times \left(\mathcal{O}^\times/\mathcal{O}^{\times 2}\right) \cong \mathbf{F}_2 \times \left(\mathcal{O}^\times/\mathcal{O}^{\times 2}\right),$$

and this last quotient is of finite rank. At last, a parametrization of the ramified representations by characters is reached, and the relation between conductors of corresponding elements. All the tools are now in place to reach the final estimates on $\nu_{Q,\mathrm{ell}}(\widehat{\phi_S})$ and the term of the proof of Proposition 23. Previous estimates and local decomposition of orbital integrals yields

$$J_{\mathrm{ell}}\left(\Phi_{\mathfrak{d},\pi_R,\phi_S}\right) = \sum_{\{\gamma\}\neq 1} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\mathcal{O}_\gamma\left(\Phi_{\mathfrak{d},\pi_R,\phi_S}\right) \ll N(\mathfrak{d}^S)^{-1+\varepsilon}.$$

Summing over the spectral data and using the elementary bound $\lambda_2(\mathfrak{n}) \ll_\varepsilon N\mathfrak{n}^\varepsilon$, it follows

$$\nu_{Q,\mathrm{ell}}(\widehat{\phi_S}) \ll \frac{1}{Q^2} \sum_{\substack{N\mathfrak{q}\leqslant Q\\ \mathfrak{q}\wedge R=1}} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \left(\frac{N\mathfrak{q}^S}{N\mathfrak{d}^S}\right)^\varepsilon (N\mathfrak{d}^S)^{-1+\varepsilon} \sum_{\substack{\pi_R\\ c(\pi_R)\leqslant Q/N\mathfrak{q}^S}} \widehat{\phi_S}(\pi_R).$$

It is then sufficient to bound the local sums appearing on the road. This is the content of the following lemma.

Lemma 19. *For every place v,*

$$\sum_{\substack{\pi_v\in\widehat{G}_v\\ c(\pi_v)\leqslant X}} \frac{1}{c(\pi_v)^\sigma} \ll \begin{cases} X^{\frac{1}{2}-\sigma} & \text{if } \sigma < 1/2;\\ \log X & \text{if } \sigma = 1/2.\end{cases}$$

Proof. For the non-archimedian ramified places, the sum splits into the different cases of the already mentioned Howe's construction, recasting the sum into

$$\sum_{c(\pi_\mathfrak{p})\leqslant X} \frac{1}{c(\pi_\mathfrak{p})^\sigma} = \sum_{E/F} \sum_{\substack{\theta\in E^\star\\ c(\pi_\theta)\leqslant X}} \frac{1}{c(\pi_\theta)^\sigma}.$$

Reminding that by the relation (3.66) between conductors of π_θ and θ, $c(\pi_\theta) = c(\theta)$ or $c(\theta)^2$, so that in any case $c(\pi_\theta) \geqslant c(\theta)$. The number of quadratic extensions of F is finite, thus

$$\sum_{E/F} \sum_{\substack{\theta\in E^\star\\ c(\pi_\theta)\leqslant X}} \frac{1}{c(\pi_\theta)^\sigma} \ll \sum_{\substack{\pi_\theta\\ c(\theta)\leqslant X\\ E/F \text{ ramified}}} \frac{1}{c(\theta)^\sigma}.$$

Moreover, in the case of the quadratic extensions E/F, recall that if $\mathfrak{f}(\pi_\theta) = k$, then $\mathfrak{c}(\pi_\theta) = \mathfrak{p}^k$, hence $c(\pi_\theta) = N\mathfrak{c}(\pi_\theta) = p^{2k}$ where $N\mathfrak{p} = p^2$. The number of characters of E such that $\mathfrak{f}(\theta) = k$ is $\varphi(\mathfrak{p}^k) = p^{k-1}(p-1)$, so that

$$\begin{aligned}\sum_{\substack{\pi_\theta \\ c(\theta)\leqslant X}} \frac{1}{c(\theta)^\sigma} &\ll \sum_{k\leqslant \log_p(X)/2} \frac{1}{N\mathfrak{p}^{k\sigma}} \sum_{\substack{\theta \\ \mathfrak{f}(\theta)=k}} 1 \\ &\ll \sum_{k\leqslant \log_p(X)/2} p^{k(1-2\sigma)} \\ &\ll \begin{cases} X^{\frac{1}{2}-\sigma} & \text{if } \sigma < 1/2, \\ \log X & \text{if } \sigma = 1/2. \end{cases}\end{aligned}$$

The archimedian case is straightforward, for the conductors are $c(\mathrm{Sym}^k(\mathrm{SU}(2))) = 1 + k^2$, where SU(2) stands for the standard representation of SU(2). This leads to

$$\sum_{c(\pi_v)\leqslant X} \frac{1}{c(\pi_v)^\sigma} = \sum_{\substack{k\geqslant 1 \\ 1+k^2\leqslant X}} \frac{1}{(1+k^2)^\sigma} \ll \begin{cases} X^{\frac{1}{2}-\sigma} & \text{if } \sigma < \frac{1}{2}; \\ \log X & \text{if } \sigma = \frac{1}{2}; \end{cases}$$

giving the claimed result for every place. □

Finally the ultimate estimate for the elliptic contribution from (3.5) can be bounded in a different way than the previous section. Introducing $R = \{r_i\}_i$ the finite set of ramified places and breaking the sum into sums over the local ramified duals lead to

$$\begin{aligned}\nu_{\mathrm{ell},Q}(\widehat{\phi_S}) &\ll Q^{-2} \sum_{\substack{N\mathfrak{q}\leqslant Q \\ \mathfrak{q}\wedge R=1}} \left(N\mathfrak{q}^S\right)^\varepsilon \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} (N\mathfrak{d}^S)^{-2+\varepsilon} \sum_{\substack{\pi_R \\ c(\pi_R)\leqslant Q/N\mathfrak{q}^S}} 1 \\ &\ll Q^{-2} \sum_{\substack{N\mathfrak{q}\leqslant Q \\ \mathfrak{q}\wedge R=1}} \left(N\mathfrak{q}^S\right)^\varepsilon \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} (N\mathfrak{d}^S)^{-2+\varepsilon} \\ &\qquad \sum_{\substack{\pi_{r_1} \\ c(\pi_{r_1})\leqslant Q/N\mathfrak{q}^S}} \sum_{\substack{\pi_{r_2} \\ c(\pi_{r_2})\leqslant Q/N\mathfrak{q}^S c(\pi_{r_1})}} \cdots \sum_{\substack{\pi_{r_{|R|}} \\ c(\pi_{r_{|R|}})\leqslant Q/N\mathfrak{q}^S c(\pi_{r_1})\cdots c(\pi_{r_{|R|}})}} 1\end{aligned}$$

Using the estimates provided by Lemma 19 repeatedly, the first bound carries a power $\frac{1}{2}$, then the following ones only logarithms, giving

$$\nu_{\mathrm{ell},Q}(\widehat{\phi_S}) \ll Q^{-3/2} \sum_{\substack{N\mathfrak{q}\leqslant Q \\ \mathfrak{q}\wedge R=1}} \left(N\mathfrak{q}^S\right)^{-1/2+\varepsilon} \sum_{\mathfrak{d}^S \mid \mathfrak{q}^S} \left(N\mathfrak{d}^S\right)^{-2+\varepsilon} \log^{|R|-1} Q.$$

The already mentioned standard estimates on the sum of ideals then imply

$$\nu_{\mathrm{ell},Q}(\widehat{\phi}_S) \ll Q^{-1+\varepsilon} \log^{|R|} Q,$$

giving the desired error term. The most significant error term comes from this elliptic part only when $F = \mathbf{Q}$, for then $\delta_F = 1$, and from the identity part of Proposition 17 otherwise. □

3.5 Proof of Corollary C: Sato-Tate conjecture

Theorem B proves the existence of a measure ν with respect to which the universal family equidistributes. It is natural to consider the projection $\nu_{\mathfrak{p}}$ of this limit measure on the local components $\widehat{G}_{\mathfrak{p}}$. Since the $\nu_{\mathfrak{p}}$ are supported on different spaces, a suitable setting is necessary in order to make sense of the Sato-Tate problem that concerns convergence of the measures $\nu_{\mathfrak{p}}$.

The literature often treat the case of measures supported on the unramified tempered spectrum, as the instances handled by Sarnak [107] or Serre [116]. In those cases, the Satake isomorphism provides a common parametrization: if T is the standard torus of $\mathrm{SL}(2, \mathbf{C})$, the dual group of PGL_2, and W is the Weyl group of T, then the isomorphism classes of unramified tempered representations are parametrized by T_c/W where $T_c = T \cap SU(2, \mathbf{C})$ is the compact part of T. This last quotient corresponds to the half-circle, giving a common ground for all the $\widehat{G}_{\mathfrak{p}}$,independent of $\mathfrak{p}$. Even if the universal family considered does include ramified representations and the $\nu_{\mathfrak{p}}$ are supported on the whole tempered unitary dual, the contribution of the ramified part of the spectrum vanish when $\mathfrak{p}$ goes to infinity, so that asymptotically the spaces can be identified and T_c/W is a posteriori still a relevant common ground to state the Sato-Tate result.

In the case of $\mathrm{GL}(2, F_{\mathfrak{p}})$, the explicit form of the Plancherel measures have been computed by Serre [116] and are given by

$$\mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(x) = \frac{N\mathfrak{p}+1}{\pi} \frac{(1-x^2/4)^{1/2}}{(N\mathfrak{p}^{1/2}+N\mathfrak{p}^{-1/2})^2 - x^2}\,\mathrm{d}x, \tag{3.67}$$

In particular they converge, as $N\mathfrak{p}$ goes to infinity, to the Sato-Tate measure on the half-circle

$$\mathrm{d}\mu^{\mathrm{ST}}(x) = \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}\,\mathrm{d}x, \tag{3.68}$$

in the sense that for any $\widehat{\phi} \in C(T_c/W, \mathbf{C})$, when $N\mathfrak{p}$ goes to infinity,

$$\int_{T_c/W} \widehat{\phi}(\pi_{\mathfrak{p}})\,\mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}}) \longrightarrow \int_{T_c/W} \widehat{\phi}(x)\,\mathrm{d}\mu^{\mathrm{ST}}(x). \tag{3.69}$$

For $\widehat{\phi} \in C(T/W, \mathbf{C})$, let decompose the measure separating whether the representations are unramified, *i.e.* of conductor 1, or not. The measure $\nu_{\mathfrak{p}}(\widehat{\phi})$ hence splits as

$$\begin{aligned}\int_{\widehat{G}_{\mathfrak{p}}} \widehat{\phi}(\pi_{\mathfrak{p}})\, \mathrm{d}\nu_{\mathfrak{p}}(\pi_{\mathfrak{p}}) &= \int_{\widehat{G}_{\mathfrak{p}}} \frac{\widehat{\phi}(\pi_{\mathfrak{p}})}{c(\pi_{\mathfrak{p}})^2}\, \mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}}) \\ &= \int_{\widehat{G}_{\mathfrak{p}}^{\mathrm{sph}}} \widehat{\phi}(\pi_{\mathfrak{p}})\, \mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}}) + \int_{\widehat{G}_{\mathfrak{p}}^{\mathrm{ram}}} \frac{\widehat{\phi}(\pi_{\mathfrak{p}})}{c(\pi_{\mathfrak{p}})^2}\, \mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}}),\end{aligned} \tag{3.70}$$

where $\widehat{G}_{\mathfrak{p}}^{\mathrm{sph}}$ stands for the unramified, also called spherical, part of the spectrum and $\widehat{G}_{\mathfrak{p}}^{\mathrm{ram}}$ for its ramified part. For $\mathfrak{p}$ sufficiently large, $G_{\mathfrak{p}}$ is isomorphic to $\mathrm{PGL}(2, F_{\mathfrak{p}})$, so the local Plancherel measures (3.67) provide the value of the first integral of the rightmost hand side as $\mathfrak{p}$ grows, in particular they converge to the Sato-Tate measure. For the second one, dominating roughly by leaving the dependence in ϕ which is fixed gives

$$\int_{\widehat{G}_{\mathfrak{p}}^{\mathrm{ram}}} \frac{\widehat{\phi}(\pi_{\mathfrak{p}})}{c(\pi_{\mathfrak{p}})^2}\, \mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}}) \ll \int_{\widehat{G}_{\mathfrak{p}}^{\mathrm{ram}}} \frac{\mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}})}{c(\pi_{\mathfrak{p}})^2} = \int_{\widehat{G}_{\mathfrak{p}}} \frac{\mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}})}{c(\pi_{\mathfrak{p}})^2} - \int_{\widehat{G}_{\mathfrak{p}}^{\mathrm{sph}}} \mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}}). \tag{3.71}$$

By the normalization of the Plancherel measure, the second integral on the right hand side is 1. Moreover, as shown in Section 3.2.4, the first integral of the right hand side is equal to

$$\int_{\widehat{G}_{\mathfrak{p}}} \frac{\mathrm{d}\mu_{\mathfrak{p}}^{\mathrm{Pl}}(\pi_{\mathfrak{p}})}{c(\pi_{\mathfrak{p}})^2} = \frac{\zeta_{\mathfrak{p}}(1)}{\zeta_{\mathfrak{p}}(2)\zeta_{\mathfrak{p}}(4)}. \tag{3.72}$$

Since this last quantity is $1 + O(N\mathfrak{p}^{-1})$ by unfolding the definition of the Dirichlet series, it follows that the ramified part is negligible, achieving the proof of Corollary C. ☐

Chapter 4

Type of Symmetry

Once established the asymptotic cardinality of the truncated universal family, further statistical results can be investigated. A fundamental invariant attached to an automorphic representation is its associated L-function. A large bunch of information is encoded in its zeros. However, statistics on all the zeros of a family of L-functions do not seem to carry much information. Once restricted to zeros lying near the real axis, the so-called low-lying zeros, the universality is broken and the corresponding statistics appear to be governed by one of the classical groups, conjecturally modeling the symmetries of the family. The one-level density problem for quaternion algebras is studied in this chapter and the type of symmetry is partially elucidated, providing further evidence towards the density conjecture of Katz and Sarnak.

Explicit formulas are the central tool and recast statistics over zeros into question concerning sums of spectral parameters. The methods involved are based on slight modifications of the selecting functions for the harmonic families built in the previous chapters. As generally expected, what critically determines the type of symmetry of the family is the second order moment of such eigenvalues. These are interpreted as a spectral sum, making the problem amenable to trace formula methods. However a crucial difference arise, since the chosen test-function are no more uniformly compactly supported, leading to finer estimations of the geometric quantities.

Contents

4.1 Low-lying zeros of L-functions

4.1.1 One-level density

As announced in Chapter 1, once the cardinality of the universal family established, it is natural to investigate the distribution of the low-lying zeros of the associated L-functions. Recall that the nontrivial zeros of the L-function $L(s,\pi)$ associated to an automorphic representation π are denoted $\rho_\pi^{(j)} = \frac{1}{2} + i\gamma_\pi^{(j)}$, for $j \in \mathbf{Z}$, with a priori $\gamma_\pi^{(j)} \in \mathbf{C}$ without assuming the Riemann hypothesis, and ordered so that

$$\cdots \leqslant \Re\gamma_\pi^{(-1)} \leqslant 0 \leqslant \Re\gamma_\pi^{(1)} \leqslant \Re\gamma_\pi^{(2)} \leqslant \cdots . \tag{4.1}$$

Introduce the renormalized zeros defined by

$$\tilde{\gamma}_\pi^{(j)} := \frac{\log c(\pi)}{2\pi}\gamma_\pi^{(j)}, \qquad j \in \mathbf{Z}. \tag{4.2}$$

This normalization can be interpreted as fixing the mean spacing between low-lying zeros to one, what is sound assuming the Riemann hypothesis for then they all lie on the critical line. Assume from now on the Riemann hypothesis for simplicity. The density conjecture of Katz and Sarnak predicts that the one-level density of zeros of $L(s,\pi)$ defined by

$$D(\pi,\phi) = \sum_{\rho_\pi} \phi\left(\tilde{\gamma}_\pi\right), \qquad \phi \in \mathcal{S}(\mathbf{R}), \tag{4.3}$$

is governed by a classical group, defining the type of symmetry of the family. From now on, let $\mathcal{S}(\mathbf{R})$ denote the space of Schwartz functions on $\mathbf{R}$ with Fourier transform compactly supported. This in particular implies that the functions can be analytically continued to all $\mathbf{C}$, so that all what follow hold unconditionally. Sarnak, Shin and Templier introduced critical invariants [110, equation (7)] leading to conjecture the underlining symmetry on the family, based on the Sato-Tate measure associated to it. By Corollary C, this one is given by the half-circle density μ^{ST}. The three invariants are given by the associated integrals on the diagonal torus $T \cap SU(2)$. Using the standard representatives $(e^{i\theta}, e^{-i\theta})$ of conjugacy classes of elements of T, they can be computed and are equal to

$$\begin{aligned} i_1(G) &= \int_T |\mathrm{tr}\, t|^2 \, \mathrm{d}\mu^{\mathrm{ST}}(t) = 1 \\ i_2(G) &= \int_T (\mathrm{tr}\, t)^2 \, \mathrm{d}\mu^{\mathrm{ST}}(t) = 1 \\ i_3(G) &= \int_T \mathrm{tr}\,(t^2) \, \mathrm{d}\mu^{\mathrm{ST}}(t) = 1 \end{aligned}$$

Even though the first two invariants lead to already known properties, viz. the automorphic representations of $\mathcal{A}(G)$ are self-dual and cuspidal, the last one pleads for an orthogonal symmetry type. This is confirmed by the following result.

Theorem 13. *For every even and Schwartz function ϕ with Fourier transform compactly supported in $(-2/3, 2/3)$, as Q grows to ∞,*

$$\frac{1}{N(Q)} \sum_{\pi \in \mathcal{A}(Q)} D(\pi, \phi) \longrightarrow \widehat{\phi}(0) + \frac{1}{2}\phi(0) = \int_{\mathbf{R}} \phi(x) W_O(x)\, \mathrm{d}x. \tag{4.4}$$

where $W_O = 1 + \frac{1}{2}\delta_0$. In particular, this is an evidence that the type of symmetry of inner forms of PGL_2 is one of the orthogonal types of symmetry.

Remark. Unfortunately, this result does not totally determine the type of symmetry, and is consistent with a family governed by the three orthogonal symmetries O, $\mathrm{SO(even)}$ and $\mathrm{SO(odd)}$, see Section 1.3.4. This issue can be addressed either by extending the support of the Fourier transform of the test function to $\beta > 1$, or by estimating the statistics given by the 2-level density of low-lying zeros. See the end of the chapter for further remarks.

4.1.2 L-functions of automorphic representations

The needed definitions and properties of L-functions are exposed for instance by Duke, Friedlander and Iwaniec [40]. The L-function associated to $\pi = \otimes_v \pi_v \in \mathcal{A}(G)$ is of the form

$$L(s, \pi) = \prod_{\mathfrak{p}} L(s, \pi_{\mathfrak{p}}) = \sum_{N\mathfrak{q} \geqslant 1} \frac{a_\pi(\mathfrak{q})}{N\mathfrak{q}^s}, \tag{4.5}$$

where the $L(s, \pi_{\mathfrak{p}})$ are the local factors associated to the components $\pi_{\mathfrak{p}}$ at finite places $\mathfrak{p}$, and can be written as

$$L(s, \pi_{\mathfrak{p}}) = \left(1 - \alpha_\pi(\mathfrak{p}) N\mathfrak{p}^{-s}\right)^{-1} \left(1 - \beta_\pi(\mathfrak{p}) N\mathfrak{p}^{-s}\right)^{-1}, \tag{4.6}$$

where $\alpha_\pi(\mathfrak{p})$ and $\beta_\pi(\mathfrak{p})$ are complex numbers, called spectral parameters of π generalizing the usual Satake parameters for unramified representations. For archimedean places, there are also complex numbers $\alpha_\pi(v)$ and $\beta_\pi(v)$ such that the associated local factors take the form

$$L(s, \pi_v) = \Gamma_v(s - \alpha_\pi(v)) \Gamma_v(s - \beta_\pi(v)),$$

where $\Gamma_v(s)$ is defined by $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ for real places, and $\Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1)$ for complex places. The product of these archimedean L-factors is denoted $L(s, \pi_\infty)$ and called the archimedean part of the L-function. Introduce the completed L-function to be

$$\Lambda(s,\pi) = c(\pi_f)^{s/2} L(s,\pi_\infty) L(s,\pi).$$

It satisfies the functional equation

$$\Lambda(s,\pi) = \varepsilon_\pi \Lambda(1-s,\pi),$$

where ε_π is the root number of π and is among 1 and -1 since $L(s,\pi)$ is self-dual..

4.2 Relation with spectral sums

4.2.1 Explicit formula

Explicit formulas relate zeros of an L-function and prime numbers and thus are a relevant tool to handle one-level densities. The explicit formula of Rudnick and Sarnak [105, 64] is in this case particularly well-suited and their proof. It is written for the base field $\mathbf{Q}$, and carry on to the setting of more general number fields without particular modification, leading to properly rewrite the proof taking the relevant modifications into account.

Proposition 25 (Explicit formula). *For every $\phi \in \mathcal{S}(\mathbf{R})$, and every $R > 0$,*

$$\sum_{\rho_\pi^{(j)}} \phi\left(\tilde{\gamma}_\pi\right) = \widehat{\phi}(0) \frac{\log c(\pi_f)}{\log R} - 2 \sum_{\mathfrak{p}} \sum_{\nu \geqslant 1} (\alpha_\pi^\nu(\mathfrak{p}) + \beta_\pi^\nu(\mathfrak{p})) \widehat{\phi}\left(\frac{\nu \log N\mathfrak{p}}{\log R}\right) \frac{\log N\mathfrak{p}}{N\mathfrak{p}^{\nu/2} \log R} + O\left(\frac{1}{\log R}\right)$$

where the sum on the left hand side runs through the zeros of $L(s,\pi)$ according to the notation $\rho_\pi^{(j)} = \frac{1}{2} + i\gamma_\pi^{(j)}$, for j running among the integers.

Proof. The zeros of $\Lambda(s,\pi)$ weighted by their multiplicities correspond to the poles of $\Lambda'(s,\pi)/\Lambda(s,\pi)$ weighted by their residues, and the explicit formula comes from a double evaluation of the integral

$$I = \frac{1}{2i\pi} \int_{\mathrm{Re}(s)=2} \frac{\Lambda'}{\Lambda}(s,\pi)\phi(s)\,\mathrm{d}s. \tag{4.7}$$

The L-function decomposes as an Euler product $\Lambda = \prod_v L_v$, where $L_v(s,\pi)$ stands for $L(s,\pi_v)$ for simplicity. Thus, integrating its logarithmic derivative leads to

$$I = \sum_v \frac{1}{2i\pi} \int_{\mathrm{Re}(s)=2} \frac{L_v'}{L_v}(s,\pi)\phi(s)\,\mathrm{d}s. \tag{4.8}$$

Denote I_v the integrals appearing in the sum above, and first consider the finite places. Since $L_\mathfrak{p}(s,\pi)$ admits the factorization $(1-\alpha_\pi(\mathfrak{p})N\mathfrak{p}^{-s})^{-1}(1-\beta_\pi(\mathfrak{p})N\mathfrak{p}^{-s})^{-1}$, at every finite place the quotient appearing in $I_\mathfrak{p}$ can be rewritten

$$\begin{aligned}\frac{L'_\mathfrak{p}}{L_\mathfrak{p}}(s,\pi) &= -\frac{\alpha_\pi(\mathfrak{p})N\mathfrak{p}^{-s}}{1-\alpha_\pi(\mathfrak{p})N\mathfrak{p}^{-s}}\log N\mathfrak{p} - \frac{\beta_\pi(\mathfrak{p})N\mathfrak{p}^{-s}}{1-\beta_\pi(\mathfrak{p})N\mathfrak{p}^{-s}}\log N\mathfrak{p}\\ &= -\log(N\mathfrak{p})\sum_{\nu\geqslant 1}(\alpha_\pi(\mathfrak{p})^\nu+\beta_\pi(\mathfrak{p})^\nu)N\mathfrak{p}^{-\nu s}\end{aligned}$$

Let $\phi^\star$ be the $\frac{1}{2}$-shift of ϕ, that is to say $\phi^\star(x)=\phi(1/2+x)$. Since ϕ is an holomorphic function, it has no poles and thus the contour appearing in the integral (4.8) can be translated from the vertical line of abscissa 2 to the one of abscissa $\frac{1}{2}$. Introduce and develop the inverse Mellin transform $\mathcal{M}\phi$ of ϕ, using change of variables to get

$$\begin{aligned}\mathcal{M}\phi(y) &= \frac{1}{2i\pi}\int_{\mathrm{Re}(s)=1/2}\phi(s)y^{-s}\,\mathrm{d}s\\ &= \frac{y^{-1/2}}{2\pi}\int_{\mathbf{R}}\phi\left(\frac{1}{2}+ir\right)e^{-ir\log y}\,\mathrm{d}r\\ &= \frac{y^{-1/2}}{2\pi}\widehat{\phi^\star}(\log y)\end{aligned}$$

so that the finite local integrals become

$$\begin{aligned}I_\mathfrak{p} &= -\log(N\mathfrak{p})\sum_{\nu\geqslant 1}(\alpha_\pi(\mathfrak{p})^\nu+\beta_\pi(\mathfrak{p})^\nu)\mathcal{M}\phi(N\mathfrak{p}^\nu)\\ &= -\frac{1}{2\pi}\sum_{\nu\geqslant 1}(\alpha_\pi(\mathfrak{p})^\nu+\beta_\pi(\mathfrak{p})^\nu)\widehat{\phi^\star}(\nu\log N\mathfrak{p})\,\frac{\log N\mathfrak{p}}{N\mathfrak{p}^{\nu/2}}.\end{aligned}$$

If ϕ is $\frac{1}{2}$-symmetrical, then letting $\phi^\vee(s)=\phi(1-s)$ and applying the computations above,

$$\mathcal{M}\phi^\vee(y) = \frac{y^{-1/2}}{2\pi}\int_{\mathbf{R}}\phi\left(\frac{1}{2}-ir\right)e^{-ir\log y}\,\mathrm{d}r = \frac{y^{-1/2}}{2\pi}\widehat{\phi^\star}(-\log y) = \mathcal{M}\phi(y). \tag{4.9}$$

On the other hand, the Cauchy theorem allows to unfold the integral (4.7) in terms of the zeros of $\Lambda(s,\pi)$. Indeed, since ϕ is entire, the only poles of the integrated function are the zeros of Λ and the corresponding residues are their multiplicities. All these zeros lie in the vertical strip $-1<\mathrm{Re}(s)<2$, so that translating the contour through this whole band captures all the zeros and gives

$$I = \sum_{\rho_\pi}\phi(\rho_\pi) + \frac{1}{2i\pi}\int_{\mathrm{Re}(s)=-1}\frac{\Lambda'}{\Lambda}(s,\pi)\phi(s)\,\mathrm{d}s. \tag{4.10}$$

The functional equation of L is of the form $\Lambda(s,\pi) = \varepsilon_\pi \Lambda(1-s,\pi)$, thus injecting it in the integral above and changing variables, to come back to the vertical line of abscissa $\mathrm{Re}(s) = 1$, yield

$$I = \sum_{\rho_\pi} \phi(\rho_\pi) - \frac{1}{2i\pi} \int_{\mathrm{Re}(s)=2} \frac{\Lambda'}{\Lambda}(s,\pi)\phi(1-s)\,\mathrm{d}s. \tag{4.11}$$

Coming back to the definition (4.8) of I,

$$\sum_{\rho_\pi} \phi(\rho_\pi) = \frac{1}{2i\pi} \int_{\mathrm{Re}(s)=2} \frac{\Lambda'}{\Lambda}(s,\pi)(\phi(1-s) + \phi(s))\,\mathrm{d}s. \tag{4.12}$$

Finally, for an even Schwartz function ϕ_0, the function

$$\phi(s) = \phi_0\left(\frac{\log R}{2\pi}\left(s - \frac{1}{2}\right)\right), \qquad R > 0, \tag{4.13}$$

is Schwartz and $\frac{1}{2}$-symmetric, and moreover satisfies

$$\widehat{\phi^\star}(s) = \int_{\mathbf{R}} \phi\left(u + \frac{1}{2}\right) e^{isu}\,\mathrm{d}u = \frac{2\pi}{\log R} \int_{\mathbf{R}} \phi_0(u) \exp\left(isu \frac{2\pi}{\log R}\right) \mathrm{d}u$$

$$\widehat{\phi^\star}(\nu \log N\mathfrak{p}) = \frac{2\pi}{\log R} \widehat{\phi_0}\left(\frac{\nu \log N\mathfrak{p}}{\log R}\right).$$

Combining the two previous expressions of the integral, and adding the archimedean places for which the treatment is the one of Rudnick and Sarnak with no modification, directly estimated by the Stirling formula since they are reduced to digamma functions, the symmetry (4.9) of the Mellin transform gives, up to an error term of size $1/\log R$,

$$\sum_{\rho_\pi} \phi(\rho_\pi) \simeq \log c(\pi) \frac{1}{2i\pi} \int_{\mathrm{Re}(s)=2} \phi(s)\,\mathrm{d}s - 2 \sum_{\mathfrak{p}} \sum_{\nu \geqslant 1} (\alpha_\pi(\mathfrak{p})^\nu + \beta_\pi(\mathfrak{p})^\nu) \widehat{\phi^\star}(\nu \log N\mathfrak{p}) \frac{\log N\mathfrak{p}}{N\mathfrak{p}^{\nu/2}},$$

retrieving the result stated in [64]. □

The explicit formula then yields, with $R = c(\pi)$, a restatement of the one-level density in purely arithmetic terms, up to an error term. More precisely,

$$D(\pi,\phi) = \widehat{\phi}(0) - \frac{2}{\log c(\pi)} \sum_{\mathfrak{p}} \sum_{\nu=1}^{\infty} (\alpha_\pi^\nu(\mathfrak{p}) + \beta_\pi^\nu(\mathfrak{p}))\, \widehat{\phi}\left(\frac{\nu \log N\mathfrak{p}}{\log c(\pi)}\right) \frac{\log N\mathfrak{p}}{N\mathfrak{p}^{\nu/2}} + O\left(\frac{1}{\log c(\pi)}\right). \tag{4.14}$$

After switching summations, introduce the inner sum for a fixed $\nu \geqslant 1$,

$$P^{(\nu)}(\pi,\phi) = \frac{2}{\log c(\pi)} \sum_{\mathfrak{p}} (\alpha_\pi^\nu(\mathfrak{p}) + \beta_\pi^\nu(\mathfrak{p})) \widehat{\phi}\left(\frac{\nu \log N\mathfrak{p}}{\log c(\pi)}\right) \frac{\log N\mathfrak{p}}{N\mathfrak{p}^{\nu/2}}, \tag{4.15}$$

so that the one-level density decomposes as

$$D(\pi,\phi)=\widehat{\phi}(0)-\sum_{\nu\geqslant 1}P^{(\nu)}(\pi,\phi)+O\left(\frac{1}{\log c(\pi)}\right). \tag{4.16}$$

The following sections are dedicated to estimate the contribution of these $P^{(\nu)}$.

4.2.2 Embedding in families

While certain quantities in (4.16) will be shown to contribute as an error term, the largest terms, corresponding to small ν, are not easily bounded and it is necessary to exploit the extra averaging over the family.This motivating the introduction of

$$\mathcal{P}_Q^{(\nu)}(\phi)=\frac{1}{N(Q)}\sum_{\pi\in\mathcal{A}(Q)}P^{(\nu)}(\pi,\phi). \tag{4.17}$$

The key technical step here is to decompose the universal family by fixing some spectral data as in Chapter 3. Recall that the quaternion algebra considered in this chapter is totally definite, so that no split archimedean parameters δ or $\nu\in\Omega$ shall appear. Introduce for an ideal $\mathfrak{q}$ of $\mathcal{O}$ prime to R and a representation σ_R in $\widehat{G}_R$,

$$\mathcal{P}_{\mathfrak{q},\sigma_R}^{(\nu)}(\phi)=\sum_{\pi\in\mathcal{B}(\mathfrak{q},\sigma_R)}m(\pi,\mathfrak{q})P^{(\nu)}(\pi,\phi), \tag{4.18}$$

where recall that

$$\mathcal{B}(\mathfrak{q},\sigma_R)=\{\pi\in\mathcal{A}(G)\ :\ c(\pi_f^R)\mid\mathfrak{q},\ \pi_R\simeq\sigma_R\}, \tag{4.19}$$

so that according to the decomposition of the universal family (2.55), the average (4.17) can be rewritten, after sieving in order to compensate the unwelcome multiplicites introduced in (4.18) and explicitly known by the work of Casselman,

$$\mathcal{P}_Q^{(\nu)}(\phi)=\frac{1}{N(Q)}\sum_{\substack{N\mathfrak{q}\leqslant Q\\ \mathfrak{q}\wedge R=1}}\sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q/N\mathfrak{q}}}\sum_{\mathfrak{d}\mid\mathfrak{q}}\lambda_2\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right)\mathcal{P}_{\mathfrak{d},\sigma_R}^{(\nu)}(\phi). \tag{4.20}$$

Developing the sum in (4.18) and switching summations lead to

$$\mathcal{P}_{\mathfrak{q},\sigma_R}^{(\nu)}(\phi)=\sum_{\mathfrak{p}}\left(\sum_{\pi\in\mathcal{B}(\mathfrak{q},\sigma_R)}m(\pi,\mathfrak{q})\ (\alpha_\pi^\nu(\mathfrak{p})+\beta_\pi^\nu(\mathfrak{p}))\right)\widehat{\phi}\left(\frac{\nu\log N\mathfrak{p}}{\log c(\pi)}\right)\frac{2\log N\mathfrak{p}}{N\mathfrak{p}^{\nu/2}\log c(\pi)}, \tag{4.21}$$

where $c(\pi)$ stands as a shortcut notation for $N\mathfrak{q}c(\sigma_R)$, justifying its presence outside the sum over the harmonic subfamily. This convention will be steadily used in the following.

4.2.3 High order contributions

For ν large enough, it is possible to bound directly $P^{(\nu)}(\pi,\phi)$ and show that they do not contribute to the type of symmetry, using the knowledge of the cardinalities of the subfamilies of fixed spectral parameters $\mathcal{B}(\mathfrak{q},\sigma_R)$ studied in Chapter 3.

Proposition 26. *For $Q \geqslant 1$,*

$$\sum_{\nu\geqslant 3} \mathcal{P}_Q^{(\nu)}(\phi) \ll \frac{1}{\log Q}. \tag{4.22}$$

Proof. The main aim is to bound the spectral parameters $\alpha_\pi^\nu(\mathfrak{p}) + \beta_\pi^\nu(\mathfrak{p})$ in the sum (4.15). For holomorphic cusp forms, the Ramanujan conjecture holds by Deligne [34] and states that $|\alpha_\pi(\mathfrak{p}) + \beta_\pi(\mathfrak{p})| \leqslant 2$. For Maass forms, Kim and Sarnak in the case of $\mathbf{Q}$, and Blomer and Brumley [11] for general number fields, proved that

$$|\alpha_\pi(\mathfrak{p}) + \beta_\pi(\mathfrak{p})| \ll N\mathfrak{p}^{7/64}. \tag{4.23}$$

Hence for any cuspidal automorphic representation of GL(2) this last bound is valid, in particular for $\pi \in \mathcal{A}(G)$. Thus, roughly bounding $\widehat{\phi}$ by a constant,

$$\begin{aligned}\sum_{\nu\geqslant 3} P^{(\nu)}(\pi,\phi) &\ll \frac{1}{\log c(\pi)} \sum_{\mathfrak{p}} \sum_{\nu\geqslant 3} \frac{\log N\mathfrak{p}}{N\mathfrak{p}^{\nu(1/2-7/64)}} \\ &\ll \frac{1}{\log c(\pi)} \sum_{\mathfrak{p}} \frac{\log N\mathfrak{p}}{N\mathfrak{p}^{3(1/2-7/64)}} \\ &\ll \frac{1}{\log c(\pi)}\end{aligned}$$

since $3(\frac{1}{2} - \frac{7}{64}) > 1$ and then the series converges. Turning back to the sums over partial families (4.18) and using the fact that $c(\pi) \geqslant N\mathfrak{q}$,

$$\mathcal{P}_{\mathfrak{q},\sigma_R}^{(\nu)}(\phi) \ll \sum_{\pi\in\mathcal{B}(\mathfrak{q},\sigma_R)} \frac{1}{\log N\mathfrak{q}} \ll \frac{B(\mathfrak{q},\sigma_R)}{\log N\mathfrak{q}}. \tag{4.24}$$

The cardinality of the sieved family has been computed previously: introducing the volumes $\varphi_2 = \lambda_2 \star \mu^2 \star \mathrm{id}$, Lemma 4 ensures there is a remainder term $R(\mathfrak{q},\sigma_R)$ such that

$$B(\mathfrak{q},\sigma_R) = \mathrm{vol}(G(F)\backslash G(\mathbf{A}))\varphi_2(\mathfrak{q})\mu^{\mathrm{Pl}}(\sigma_R) + R(\mathfrak{q},\sigma_R), \tag{4.25}$$

where the remainder satisfies, for a certain $\theta > 0$,

$$\sum_{\substack{\mathfrak{q}\leqslant Q\\ \mathfrak{q}\wedge R=1}} \sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q/q}} R(\mathfrak{q},\sigma_R(Q/N\mathfrak{q}c(\pi_R))) \ll Q^{2-\theta}. \tag{4.26}$$

Introduce then the following dampening lemma, justifying that the logarithmic factor appearing in the denominator is enough to turn to whole sum negligible compared to the one free of this factor, that is to say the cardinality of the truncated universal family $N(Q)$.

Lemma 20. *For every positive function f for which there is an $\alpha > 0$ such that*

$$\sum_{N\mathfrak{n}\leqslant X} f(\mathfrak{n}) \sim X^{\alpha}, \tag{4.27}$$

then for every $\varepsilon > 0$,

$$\sum_{N\mathfrak{n}\leqslant X} \frac{f(\mathfrak{n})}{\log(N\mathfrak{n})^{\varepsilon}} \ll \frac{1}{\log(X)^{\varepsilon}} \sum_{N\mathfrak{n}\leqslant X} f(\mathfrak{n}). \tag{4.28}$$

Proof. The hyperbola method can be efficiently used in this setting. Cutting the sum at $X^{1/2}$ for a positive X,

$$\begin{aligned}\sum_{N\mathfrak{n}\leqslant X} \frac{f(\mathfrak{n})}{\log(N\mathfrak{n})^{\varepsilon}} &= \sum_{N\mathfrak{n}\leqslant X^{1/2}} \frac{f(\mathfrak{n})}{\log(N\mathfrak{n})^{\varepsilon}} + \sum_{X^{1/2}<N\mathfrak{n}\leqslant X} \frac{f(\mathfrak{n})}{\log(N\mathfrak{n})^{\varepsilon}} \\ &\ll \sum_{N\mathfrak{n}\leqslant X^{1/2}} f(\mathfrak{n}) + \frac{1}{\log(X)^{\varepsilon}} \sum_{X^{1/2}<N\mathfrak{n}\leqslant X} f(\mathfrak{n}) \\ &\ll \frac{1}{\log(X)^{\varepsilon}} \sum_{N\mathfrak{n}\leqslant X} f(\mathfrak{n}).\end{aligned}$$

Indeed, the asymptotic assumption (4.28) yields

$$\sum_{N\mathfrak{n}\leqslant X^{1/2}} f(\mathfrak{n}) \ll X^{\alpha/2} \ll \frac{X^{\alpha}}{\log(X)^{\varepsilon}} \ll \frac{1}{\log(X)^{\varepsilon}} \sum_{N\mathfrak{n}\leqslant X} f(\mathfrak{n}), \tag{4.29}$$

proving the lemma. ☐

So after summations of the contributions of the sieved families (4.25), Lemma 20 provides the bound

$$\mathcal{P}_Q^{(\nu)}(\phi) \ll \sum_{\substack{N\mathfrak{q}\leqslant Q \\ \mathfrak{q}\wedge R=1}} \sum_{\substack{\sigma_R\in\widehat{G}_R \\ c(\sigma_R)\leqslant Q/N\mathfrak{q}}} \frac{\varphi_2(\mathfrak{q})\mu_R^{\mathrm{Pl}}(\sigma_R)}{\log(N\mathfrak{q})} \ll \frac{1}{\log Q}, \tag{4.30}$$

proving Proposition 26. More precisely, the contribution to the line above of the error term in (4.25) is negligible by (4.26), and the result is obtained by applying the dampening lemma to the sum over $\mathfrak{q}$ and then using the convergence of the ramified part following Lemma 7. ☐

Remark. Analogously to Iwaniec, Luo and Sarnak [64] and all the literature on low-lying zeros, the high order terms are negligible, with a logarithmic savings. This bound follows from directly dominating $P^{(\nu)}(\pi, \phi)$ without use of neither the average over the family nor the sum over the primes.

4.3 Traces of Hecke operators

4.3.1 Hecke operators

For the two remaining cases $\nu = 1$ and $\nu = 2$, straightforward estimations are no more sufficient, feature already present in [64] and generalized in the axiomatic proposed by Dueñez and Miller [37]. The inner spectral sums in (4.21) are closely related to traces of Hecke operators, so that $\mathcal{P}^{(\nu)}_{\mathfrak{q},\sigma_R}(\phi)$ should be interpreted as a spectral side of a trace formula, using the selecting function constructed in Section 2.5.5. Let define the normalized Hecke operators as

$$T_{\mathfrak{p}^\nu} = N\mathfrak{p}^{-\nu/2}\mathbf{1}_{T(\mathfrak{p}^\nu)}, \qquad \text{where} \quad T(\mathfrak{p}^\nu) = \bigcup_{\substack{i+j=\nu \\ 0 \leqslant i \leqslant j}} K_\mathfrak{p} \begin{pmatrix} \mathfrak{p}^i & \\ & \mathfrak{p}^j \end{pmatrix} K_\mathfrak{p}. \tag{4.31}$$

The Hecke operator for a global ideal $\mathfrak{n}$ of $\mathcal{O}$ is defined by

$$T_\mathfrak{n} = \prod_{p^r || \mathfrak{n}} T_{\mathfrak{p}^r}. \tag{4.32}$$

One of the main appeal of Hecke operators is that they provide an explicit recipe to catch the coefficients of L-functions. Indeed, they satisfy the same induction relation, hence are equal once well normalized. This is the content of the following standard proposition.

Proposition 27. *Let $\mathfrak{n}$ be an ideal of $\mathcal{O}$ and π be an unramified representation at the places dividing $\mathfrak{n}$. Introduce $\lambda_\pi(\mathfrak{n})$ the eigenvalue of $T_\mathfrak{n}$ acting on π. Then ,*

$$a_\pi(\mathfrak{n}) = \lambda_\pi(\mathfrak{n}). \tag{4.33}$$

Moreover, for all representation ramified at one of the places dividing $\mathfrak{n}$, the only eigenvalue of $T_{\mathfrak{p}^\nu}$ acting on π is zero.

Proof. The $T_{\mathfrak{p}^n}$ satisfy [18, Prop 4.6.4] the recursive relation

$$T_{\mathfrak{p}^{n+1}} = T_\mathfrak{p} T_{\mathfrak{p}^n} - T_{\mathfrak{p}^{n-1}}, \qquad n \geqslant 0, \tag{4.34}$$

which transfers at the Hecke eigenvalues level and gives

$$\lambda_\pi(\mathfrak{p}^{n+1}) = \lambda_\pi(\mathfrak{p})\lambda_\pi(\mathfrak{p}^n) - \lambda_\pi(\mathfrak{p}^{n-1}), \qquad n \geqslant 0. \tag{4.35}$$

Recall that, by the Euler product decomposition of $L(s,\pi)$, the coefficients $a_\pi(\mathfrak{n})$ are entirely determined by their values at the prime powers $\mathfrak{p}^n$. Moreover, since the centerless setting implies a trivial central character and hence Satake parameters related by $\alpha_\pi(\mathfrak{p}) = \beta_\pi(\mathfrak{p})^{-1}$, let α be a shortened version of $\alpha_\pi(\mathfrak{p})$. The local L-factors are

$$L_\mathfrak{p}(s,\pi) = (1-\alpha N\mathfrak{p}^{-s})^{-1}(1-\alpha^{-1}N\mathfrak{p}^{-s})^{-1} = \left(\sum_{i\geqslant 0}\alpha^i N\mathfrak{p}^{-is}\right)\left(\sum_{j\geqslant 0}\alpha^{-j}N\mathfrak{p}^{-js}\right). \tag{4.36}$$

By unfolding the power series, the coefficient of $N\mathfrak{p}^{ns}$ is

$$a_\pi(\mathfrak{p}^n) = \sum_{i+j=n}\frac{\alpha^i}{\alpha^j}. \tag{4.37}$$

A straightforward computation hence leads to the recursion relation

$$a(\mathfrak{p}^{n+1}) = a(\mathfrak{p})a(\mathfrak{p}^n) - a(\mathfrak{p}^{n-1}). \tag{4.38}$$

Since this is the same relation than for the Hecke eignavelues (4.34), the two sequences are proportional. Moreover, $\mathfrak{p}$-unramified newforms are normalized so that $a_\pi(1) = \lambda_\pi(1) = 1$, leading to the equality of both sequences as claimed. The second part of the claim is straightforward, since the operators are the unramified Hecke operators, in particular are bi-$K_\mathfrak{p}$-invariants, and so project on $K_\mathfrak{p}$ fixed vectors, which are reduced to zero for $\mathfrak{p}$-ramified representations by definition. □

4.3.2 Spectral selection

Proposition 27 states that Hecke eigenvalues is a way to interpret the coefficients of an automorphic representation. Since these coefficients are related to the Satake parameters by the Euler product development (4.36), this fact provides a clue to handle the sums (4.21) involving them. In order to make this question amenable to the trace formula method, it is necessary to have a grasp on these quantities by means of Fourier transforms. This is provided by the following proposition.

Proposition 28. *Every K-spherical vector v in a representation π is an eigenvector for all Hecke operators. Moreover, for every $\nu \geqslant 0$,*

$$\widehat{T}_{\mathfrak{p}^\nu}(\pi) = \begin{cases} \lambda_\pi(\mathfrak{p}^\nu) & \text{if } \pi \text{ is unramified at } \mathfrak{p}; \\ 0 & \text{otherwise.}\end{cases} \tag{4.39}$$

Proof. Since the K-Hecke operators lie in the left and right K-invariant Hecke algebra, they project on K-fixed vector spaces by Proposition 9, so only unramified representations can have nontrivial eigenvalues. By the multiplicity one theorem, the space π^K of K-fixed vectors in V_π is at most one dimensional. Hence every operator acts on it as a scalar, that is by definition the eigenvalue $\lambda_\pi(\mathfrak{p}^\nu)$, also equals to the trace of the convolution operator $T_{\mathfrak{p}^\nu}(\pi)$. By the Proposition 27, they are equal to the coefficients of π. □

4.3.3 Sums of Hecke eigenvalues

the above section showed that Hecke operators are a tool allowing a spectral reinterpretation of the spectral parameters in terms of Hecke eigenvalues in the case of unramified representations. In order to explore the relations between the sums of Satake parameters (4.21) and sums of eigenvalues of Hecke operators, this leads to introduce the spectral sums

$$\Lambda^{(\nu),\mathrm{ur}}_{\mathfrak{q},\sigma_R}(\mathfrak{p}) = \sum_{\pi\in\mathcal{B}(\mathfrak{q},\sigma_R)^{\mathrm{ur}}} m(\pi,\mathfrak{q})\lambda_\pi(\mathfrak{p}^\nu). \tag{4.40}$$

Moreover, introduce $\mathcal{P}^{(\nu),\mathrm{ur}}_{\mathfrak{q},\sigma_R}$ to be the $\mathfrak{p}$-unramified part of $\mathcal{P}^{(\nu)}_{\mathfrak{q},\sigma_R}(\phi)$. Concerning the ramified representations at $\mathfrak{p}$, introduce

$$\Lambda^{(\nu),\mathrm{r}}_{\mathfrak{q},\sigma_R}(\mathfrak{p}) = \sum_{\pi\in\mathcal{B}(\mathfrak{q},\sigma_R)^{\mathrm{ur}}} m(\pi,\mathfrak{q})\left(\alpha_\pi^\nu(\mathfrak{p})+\beta_\pi^\nu(\mathfrak{p})\right). \tag{4.41}$$

Relation in the case $\nu = 1$

Since $\alpha_\pi(\mathfrak{p})+\beta_\pi(\mathfrak{p}) = \lambda_\pi(\mathfrak{p})$ for unramified representations, the corresponding total sum in $\mathcal{P}^{(\nu)}_{\mathfrak{q},\sigma_R}$ can be rewritten as

$$\mathcal{P}^{(1),\mathrm{ur}}_{\mathfrak{q},\sigma_R}(\phi) = \sum_{\mathfrak{p}} \Lambda^{(1)}_{\mathfrak{q},\sigma_R}(\mathfrak{p})\widehat{\phi}\left(\frac{\log N\mathfrak{p}}{\log c(\pi)}\right)\frac{2\log N\mathfrak{p}}{\sqrt{N\mathfrak{p}}\log c(\pi)}. \tag{4.42}$$

Relation in the case $\nu = 2$

By identification of the corresponding expressions of the local L-factor (4.6) follows the relation $\alpha_\pi^2(\mathfrak{p})+\beta_\pi^2(\mathfrak{p}) = \lambda_\pi(\mathfrak{p}^2)-1$ holding for unramified representations. So that the sum over primes splits as

$$\mathcal{P}^{(2),\mathrm{ur}}_{\mathfrak{q},\sigma_R}(\phi) = \sum_{\mathfrak{p}} \Lambda^{(2)}_{\mathfrak{q},\sigma_R}(\mathfrak{p})\widehat{\phi}\left(\frac{2\log N\mathfrak{p}}{\log c(\pi)}\right)\frac{2\log N\mathfrak{p}}{N\mathfrak{p}\log c(\pi)} - \sum_{\mathfrak{p}}\widehat{\phi}\left(\frac{2\log N\mathfrak{p}}{\log c(\pi)}\right)\frac{2\log N\mathfrak{p}}{N\mathfrak{p}\log c(\pi)}. \tag{4.43}$$

By the prime number theorem and integration by parts, the second sum appearing in the right hand side rewrites

$$\begin{aligned}
\sum_{\mathfrak{p}} \widehat{\phi}\left(\frac{2\log N\mathfrak{p}}{\log c(\pi)}\right)\frac{2\log N\mathfrak{p}}{N\mathfrak{p}\log c(\pi)} &= \frac{1}{\log c(\pi)}\widehat{\phi}\left(\frac{2\log c(\pi)^{T_\phi/2}}{\log c(\pi)}\right) \\
&\quad - \frac{2}{\log c(\pi)}\int_1^{c(\pi)^{T_\phi/2}}\left(\sum_{\mathfrak{p}\leqslant t}\frac{\log N\mathfrak{p}}{N\mathfrak{p}}\right)\partial_t\widehat{\phi}\left(\frac{2\log t}{\log c(\pi)}\right)\mathrm{d}t \\
&= -\frac{2}{\log c(\pi)}\int_1^{c(\pi)^{T_\phi/2}}(\log(t)+O(1))\partial_t\widehat{\phi}\left(\frac{2\log t}{\log c(\pi)}\right)\mathrm{d}t \\
&= -\frac{2}{\log c(\pi)}\int_1^{c(\pi)^{T_\phi/2}}\widehat{\phi}\left(\frac{2\log t}{\log c(\pi)}\right)\frac{\mathrm{d}t}{t} + O\left(\frac{1}{\log c(\pi)}\right) \\
&= \frac{1}{2}\phi(0) + O\left(\frac{1}{\log c(\pi)}\right),
\end{aligned}$$

where the fact that $\widehat{\phi}$ is even and compactly supported in $[-T_\phi, T_\phi]$ has been used to write

$$\int_0^{T_\phi}\widehat{\phi} = \frac{1}{2}\int_{\mathbf{R}}\widehat{\phi} = \frac{1}{2}\phi(0).$$

The expression (4.43) of $\mathcal{P}^{(2),\mathrm{ur}}_{\mathfrak{q},\sigma_R}$ is hence reduced to

$$\mathcal{P}^{(2),\mathrm{ur}}_{\mathfrak{q},\sigma_R}(\phi) = \sum_{\mathfrak{p}}\Lambda^{(2)}_{\mathfrak{q},\sigma_R}(\mathfrak{p})\widehat{\phi}\left(\frac{2\log N\mathfrak{p}}{\log c(\pi)}\right)\frac{2\log N\mathfrak{p}}{N\mathfrak{p}\log c(\pi)} - \frac{1}{2}\phi(0) + O\left(\frac{1}{\log c(\pi)}\right). \tag{4.44}$$

Remark. The extra contribution $\frac{1}{2}\phi(0)$ is crucial, and will be shown to be the only non-archimedean contribution to the type of symmetry. It is noteworthy that that it is determined by the relation between the Satake parameters and the coefficients, thus as expected the structure of the L-functions attached to the elements of the family is the essential factor determining the type of symmetry and has particularly an effect when estimating the second moment of Hecke eigenvalue as regularly emphasized by Miller. This feature is already present in many classical works on low-lying zeros [64, 37].

4.4 Low order contribution

It remains to evaluate the sums of order 1 and 2, displayed in (4.42) and (4.44). Decompose them according to whether or not the representations are ramified at $\mathfrak{p}$, introducing

$$\mathcal{P}^{(\nu),\sigma}_{\mathfrak{q},\sigma_R}(\phi) = \sum_{\mathfrak{p}}\Lambda^{(\nu),\sigma}_{\mathfrak{q},\sigma_R}(\mathfrak{p})\widehat{\phi}\left(\frac{\nu\log N\mathfrak{p}}{\log c(\pi)}\right)\frac{2\log N\mathfrak{p}}{N\mathfrak{p}^{\nu/2}\log c(\pi)}. \tag{4.45}$$

where $\sigma \in \{\mathrm{ur}, \mathrm{r}\}$ indicates if the sum defining $\Lambda^{(\nu),\sigma}_{\mathfrak{q},\sigma_R}(\mathfrak{p})$ runs through the representation unramified at $\mathfrak{p}$ or ramified at $\mathfrak{p}$ respectively, following the dichotomy introduced above. Moreover, $c(\pi)$ is from now on freely use as a shortened notation when restricted to families of fixed spectral data, at holds for $c(\pi) = N\mathfrak{q}c(\sigma_R)$.

4.4.1 Unramified part

Proposition 29. *For ϕ an even Schwartz function whose Fourier transform is compactly supported in $(-2/3, 2/3)$,*

$$\mathcal{P}_Q^{(\nu),\mathrm{ur}}(\phi) \ll \frac{1}{\log Q}. \tag{4.46}$$

Proof. For the representations unramified at $\mathfrak{p}$, the coefficients are selected by the Hecke operators as stated in Proposition 28. Write the trace formula with the selecting function of the universal family twisted by the Hecke operator (4.31), that is to say

$$\Phi^{\mathfrak{p},\nu}_{\mathfrak{q},\sigma_R} = T_{\mathfrak{p}^\nu}\Phi_{\mathfrak{q},\sigma_R}, \tag{4.47}$$

so that the test function is $T_{\mathfrak{p}^\nu}$ at the place $\mathfrak{p}$ and the other places remain unchanged compared to the previous chapter. The result obtained in Lemma 2.70 and the extra effect of the Hecke operator at the place $\mathfrak{p}$ then prove that

$$J_{\mathrm{spec}}\left(\Phi^{\mathfrak{p},\nu}_{\mathfrak{q},\sigma_R}\right) = \Lambda^{(\nu),\mathrm{ur}}_{\mathfrak{q},\sigma_R}(\mathfrak{p}) + N\mathfrak{p}^{-\nu/2}R(\mathfrak{q},\sigma_R). \tag{4.48}$$

where it is known that the remainder term satisfies

$$\sum_{\substack{\mathfrak{q}\leqslant Q\\ \mathfrak{q}\wedge R=1}} \sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q/\mathfrak{q}}} \sum_{\mathfrak{d}\mid\mathfrak{q}} \lambda_2\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right) R(\mathfrak{q},\sigma_R) \ll Q^{2-\theta}. \tag{4.49}$$

so that, up to an error term, the sought spectral sums (4.40) can be approximated by the spectral side (4.48). The Selberg trace formula states that this spectral part $J_{\mathrm{spec}}(\Phi^{\mathfrak{p},\nu}_{\mathfrak{q},\sigma_R})$ is equal to the corresponding geometrical side. This one decomposes as

$$J_{\mathrm{geom}}\left(\Phi^{\mathfrak{p},\nu}_{\mathfrak{q},\sigma_R}\right) = J_1\left(\Phi^{\mathfrak{p},\nu}_{\mathfrak{q},\sigma_R}\right) + J_{\mathrm{ell}}\left(\Phi^{\mathfrak{p},\nu}_{\mathfrak{q},\sigma_R}\right), \tag{4.50}$$

where the identity and elliptic terms are defined as

$$J_1\left(\Phi^{\mathfrak{p},\nu}_{\mathfrak{q},\sigma_R}\right) = \mathrm{vol}(G(F)\backslash G(\mathbf{A}))\Phi^{\mathfrak{p},\nu}_{\mathfrak{q},\sigma_R}(1)$$
$$J_{\mathrm{ell}}\left(\Phi^{\mathfrak{p},\nu}_{\mathfrak{q},\sigma_R}\right) = \sum_{\{\gamma\}\subset G(F)} \mathrm{vol}(G_\gamma(F)\backslash G_\gamma(\mathbf{A})) \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} \Phi^{\mathfrak{p},\nu}_{\mathfrak{q},\sigma_R}(x^{-1}\gamma x)\,\mathrm{d}x$$

Since 1 lies outside the double classes $T(\mathfrak{p}^\nu)$ defining the Hecke operator (4.31), $J_1(\Phi^{\mathfrak{p},\nu}_{\mathfrak{q},\sigma_R})$ vanishes. For the elliptic terms, in critical contrast with Section 3.4, many difficulties arise due to the presence of Hecke operators. Indeed, the Hecke double classes $T(\mathfrak{p}^\nu)$ are not uniformly contained in a compact open subgroup, as it is the case for the congruence subgroups $K_0(\mathfrak{d})$. It is hence necessary to unveil the dependence on $\mathfrak{p}$ for the bounds on the length of the sum and on the global volumes, no more only for the orbital integrals.

The orbital integral factorizes (3.54) as a product of local ones. Since $\Phi^{\mathfrak{p},\nu}_{\mathfrak{q},\sigma_R,v}$ is unchanged at any place v safe $\mathfrak{p}$ compared to Lemma 2.70, the same bounds provided by Binder [9] for the local orbital integrals hold for those places, that is to say for a finite place $\mathfrak{r}$ different from $\mathfrak{p}$,

$$\mathcal{O}_{\gamma_\mathfrak{r}}\left(\Phi^{\mathfrak{p},\nu}_{\mathfrak{q},\sigma_R,r}\right) \ll N\mathfrak{r}^\varepsilon. \tag{4.51}$$

Concentrating on the $\mathfrak{p}$-component of the test function, the associated orbital integral is precise computed by Kottwitz [86, Lemma 12.12], who show that

$$\begin{aligned}\mathcal{O}_{\gamma_\mathfrak{p}}\left(\Phi^{\mathfrak{p},\nu}_{\mathfrak{q},\sigma_R,\mathfrak{p}}\right) &= \mathcal{O}_{\gamma_\mathfrak{p}}\left(T_{\mathfrak{p}^\nu}\right)\\ &= N\mathfrak{p}^{-\nu/2}\mathcal{O}_{\gamma_\mathfrak{p}}(\mathbf{1}_{T(\mathfrak{p}^\nu)})\\ &\ll N\mathfrak{p}^{\nu/2},\end{aligned}$$

Even if very general results due to Matz and Templier [86, Lemmas 11.9, 11.11, 11.14] supply needed bound in this case, our particular $GL(2)$-setting allow to be more precise and to obtain slightly better and explicit bounds. The lemma below bound the number of conjugacy classes contributing non-trivially to the elliptic terms.

Lemma 21. *The number of contributing classes to the geometric side of the trace formula applied to the test-function $\Phi^{\mathfrak{p},\nu}_{\mathfrak{q},\sigma_R}$ is bounded by $N\mathfrak{p}^{\nu/2}$.*

Proof. The proof is an adaptation of the counting provided by Matz and Templier, with no need to use their rough bounds since the situation is more precise. Let lift the setting, considering the center and embedding in GL(2) by the Jacquet-Langlands correspondence, in order to ease the argument. Counting the $G(F)$-conjugacy classes is equivalent to counting the associated characteristic polynomials, since they determine the conjugacy classes.

Let γ be a representative of a contributing conjugacy class, that is to say such that Φ does not vanish all along the conjugacy class of γ. By definition of the test function, at all non-archimedean places γ_v is a matrix with integer entries, for either $\gamma_\mathfrak{q} \in K_{0,\mathfrak{q}}(\mathfrak{q}^r)$ for a certain r, or $\gamma_\mathfrak{p} \in T(\mathfrak{p}^\nu)$. Hence its characteristic polynomial P_γ has coefficients in all the integers rings $\mathcal{O}_\mathfrak{q}$, hence in the integer ring of F. Let write

$$P_\gamma = X^2 + a_\gamma X + b_\gamma. \tag{4.52}$$

Turning to the archimedean places, the test-function is compactly supported modulo the center. Hence, up to normalizing the determinant to one by replacing γ by

$\tilde{\gamma} = \gamma|\det\gamma|^{-1/2}$, the set of the contributing $\tilde{\gamma}$ lies in a fixed compact set, hence also the coefficients of their characteristic polynomials. Since the determinant is fixed to one, and should be equal to $b_{\tilde{\gamma}}$, only the linear coefficient remains undisclosed. This is a bounded integer, and turning back to γ it shall be bounded with respect to the archimedean normal:

$$|a_\gamma|_\infty \ll |\det\gamma|_\infty^{1/2}. \tag{4.53}$$

The fact that $\gamma_\mathfrak{p}$ lies in the Hecke double class $T(\mathfrak{p}^\nu)$ and the other $\gamma_\mathfrak{q}$ in the maximal compact subgroup $K_\mathfrak{q}$ fixes the value of the non-archimedean norms of the determinant at each place $\mathfrak{p}^\nu$ for $\mathfrak{p}$, and one for the other places. Since the determinant of γ lies in F, the product formula yields

$$|a_\gamma|_\infty \ll |\det\gamma|_\infty^{1/2} = \prod_\mathfrak{q} |\det\gamma|_\mathfrak{q}^{-1/2} = |\det\gamma|_\mathfrak{p}^{-1/2} = N\mathfrak{p}^{\nu/2}, \tag{4.54}$$

achieving the proof, for there is as many different conjucacy classes as possible linear coefficients for P_γ. □

Moreover, the global volumes have been precisely bounded by Matz [85, Section 9] in the specific case of $GL(2)$, since explicitly written as special values of L-functions, and are shown to be dominated by $N\mathfrak{p}^\varepsilon$ for all $\varepsilon > 0$. These bounds, along with the bounds on orbital integrals at other places obtained in the previous chapter, imply by (4.48) that for every $\varepsilon > 0$,

$$\Lambda_{\mathfrak{q},\sigma_R}^{(\nu),\mathrm{ur}}(\mathfrak{p}) \ll N\mathfrak{p}^{\nu+\varepsilon}(N\mathfrak{q})^\varepsilon + N\mathfrak{p}^{\nu/2+\varepsilon}R(\mathfrak{q},\sigma_R). \tag{4.55}$$

Note that the action of $\widehat{\phi}$ in the explicit formula ensures a sum over primes running until $c(\pi)^{T_\phi/\nu}$. Plugging the above bounds in (4.45), the prime number theorem implies that

$$\begin{aligned}\mathcal{P}_{\mathfrak{q},\sigma_R}^{(\nu),\sigma}(\phi) &\ll N\mathfrak{q}^\varepsilon \sum_\mathfrak{p} \widehat{\phi}\left(\frac{\nu\log N\mathfrak{p}}{\log c(\pi)}\right)\frac{\log N\mathfrak{p}}{\log c(\pi)} N\mathfrak{p}^{\nu/2+\varepsilon} \\ &\ll N\mathfrak{q}^\varepsilon \frac{c(\pi)^{3T_\phi/2+\varepsilon}}{\log c(\pi)}.\end{aligned}$$

After summation over $\mathfrak{q}$, it is negligible compared to the size of the family if

$$\sum_{\substack{N\mathfrak{q}\leqslant Q\\ \mathfrak{q}\wedge R=1}} \sum_{\substack{\sigma_R\in\widehat{G}_R\\ c(\sigma_R)\leqslant Q/N\mathfrak{q}}} \sum_{\mathfrak{d}\mid\mathfrak{q}} N\mathfrak{d}^\varepsilon \frac{c(\pi)^{3T_\phi/2+\varepsilon}}{\log c(\pi)} = o(Q^2), \tag{4.56}$$

and this happens by standard estimates for $T_\phi \leqslant 2/3 - \varepsilon$ for any $\varepsilon > 0$, giving the desired result. □

4.4.2 Ramified part

It remains to estimate the contribution of $\mathfrak{p}$-ramified representations to the spectral sum. This is the content of the following lemma.

Proposition 30. *For every $Q \geqslant 1$ and $\nu \geqslant 1$,*

$$\mathcal{P}_Q^{(\nu),\mathrm{r}}(\phi) \ll \frac{Q^2}{\log(Q)^{\nu(1/2-7/64)}} \log\log Q. \tag{4.57}$$

Proof. By definition of the conductor, π is ramified at $\mathfrak{p}$ if and only if $\mathfrak{p}$ divides the arithmetic conductor of π. Hence, using the Blomer-Brumley bound and the counting law (4.25),

$$\begin{aligned}\Lambda_{\mathfrak{q},\sigma_R}^{(\nu),\mathrm{r}}(\mathfrak{p}) &= \mathbf{1}_{\mathfrak{p}|\mathfrak{q}} \sum_{\pi\in\mathcal{A}(\mathfrak{q},\sigma_R)} (\alpha_\pi^\nu(\mathfrak{p}) + \beta_\pi^\nu(\mathfrak{p})) \\ &\ll N\mathfrak{p}^{7\nu/64}\mathbf{1}_{\mathfrak{p}|\mathfrak{q}} B(\mathfrak{q},\sigma_R) \\ &\ll N\mathfrak{p}^{7\nu/64}\mathbf{1}_{\mathfrak{p}|\mathfrak{q}} J_1\left(\Phi_{\mathfrak{q},\sigma_R}\right) + N\mathfrak{p}^{7\nu/64} R(\mathfrak{q},\sigma_R).\end{aligned}$$

That leads to, after summing over the primes,

$$\mathcal{P}_{\mathfrak{q},\sigma_R}^{(\nu),\mathrm{r}}(\phi) \ll \frac{\varphi_2(\mathfrak{q})\mu_R^{\mathrm{Pl}}(\sigma_R)}{\log(N\mathfrak{q}c(\sigma_R))} \sum_{\mathfrak{p}|\mathfrak{q}} \frac{\log N\mathfrak{p}}{N\mathfrak{p}^{\nu(1/2-7/64)}}. \tag{4.58}$$

Recall a technical lemma useful for the computations below.

Lemma 22. *For every $0 < s \leqslant 1$ and every $\mathfrak{q}$,*

$$\sum_{\mathfrak{p}|\mathfrak{q}} \frac{\log(N\mathfrak{p})}{N\mathfrak{p}^s} \ll \log(N\mathfrak{q})^{1-s} + \log\log N\mathfrak{q}. \tag{4.59}$$

Proof. This is a straightforward application of the hyperbola method. Indeed, for $Y > 0$, partial summation gives

$$\begin{aligned}\sum_{\mathfrak{p}|\mathfrak{q}} \frac{\log(N\mathfrak{p})}{N\mathfrak{p}^s} &= \sum_{\substack{\mathfrak{p}|\mathfrak{q}\\ N\mathfrak{p}\leqslant Y}} \frac{\log(N\mathfrak{p})}{N\mathfrak{p}^s} + \sum_{\substack{\mathfrak{p}|\mathfrak{q}\\ N\mathfrak{p}>Y}} \frac{\log(N\mathfrak{p})}{N\mathfrak{p}^s} \\ &\ll \sum_{N\mathfrak{p}\leqslant Y} \frac{\log(N\mathfrak{p})}{N\mathfrak{p}^s} + \sum_{\substack{\mathfrak{p}|\mathfrak{q}\\ N\mathfrak{p}>Y}} \frac{\log(N\mathfrak{p})}{N\mathfrak{p}^s} \\ &\ll \sum_{N\mathfrak{p}\leqslant Y} N\mathfrak{p}^{1-s}(\log(N\mathfrak{p}+1) - \log N\mathfrak{p}) + \frac{1}{Y^s}\sum_{\mathfrak{p}|\mathfrak{q}} \log N\mathfrak{p}\end{aligned}$$

$$\ll \max\left(Y^{1-s}, \frac{\log N\mathfrak{q}}{Y^s}\right),$$

and this quantity is optimized for $Y^{1-s} = \log(N\mathfrak{q})Y^{-s}$, *i.e.* for $Y = \log N\mathfrak{q}$, which gives the claimed statement. □

This applied to (4.58) imples that, for every $\mathfrak{q}$ and σ_R,

$$\mathcal{P}^{(\nu),\mathrm{r}}_{\mathfrak{q},\sigma_R}(\phi) \ll \frac{\varphi_2(\mathfrak{q})\mu_R^{\mathrm{Pl}}(\sigma_R)}{\log(N\mathfrak{q})^{\nu(1/2-7/64)}} + \log\log N\mathfrak{q}. \tag{4.60}$$

Applying now Lemma 20 when summing over the spectral data yields

$$\mathcal{P}^{(\nu),\mathrm{r}}_{Q}(\phi) \ll \frac{1}{\log(Q)^{\nu(1/2-7/64)}} + \log\log Q, \tag{4.61}$$

using the asymptotic size of the family provided by Theorem A. Theorem D follow from the explicit formula (4.14), the contribution of the second order terms obtained in (4.44) and the bounds of the remaining terms Propositions 29 and 30. □

4.5 Non-vanishing of L-functions

The type of symmetry (4.4) leads to further statistics on the family of L-functions associated to representations in $\mathcal{A}(G)$. Following Iwaniec, Luo and Sarnak [64], it opens the path to bounds on the density of non-vanishing at the central point. Let introduce the truncated proportion of vanishing at the central point with order m, *i.e.*

$$p_m(Q) = \frac{1}{N(Q)}\#\left\{\pi \in \mathcal{A}(Q) \ : \ \mathrm{ord}_{s=1/2}L(s,\pi) = m\right\}. \tag{4.62}$$

The very definition of those proportions gives the for every $Q > 0$,

$$\sum_{m \geqslant 0} p_m(Q) = 1. \tag{4.63}$$

The proportion of vanishing at the central point (4.62) could be reached by approximating the Dirac mass $\phi = \delta_0$ in the one-level density (4.3). The Plancherel formula restates the asymptotic one-level density obtained in Theorem D, for an admissible function ϕ, as

$$\int_{\mathbf{R}} \phi(x)W(x)\,\mathrm{d}x = \int_{\mathbf{R}} \widehat{\phi}(y)\widehat{W}(y)\,\mathrm{d}y. \tag{4.64}$$

The proportion of vanishing at the central point, counted with multiplicities, can be bounded as follows. Let ϕ be a non-negative function so that $\widehat{\phi}(0) = 1$, *i.e.* $\phi \geqslant \delta_0$. By the density result (4.4), for every $\varepsilon > 0$ and for Q sufficiently large,

$$\begin{aligned}
\sum_{m \geqslant 1} m p_m(Q) &= \frac{1}{N(Q)} \sum_{\pi} \sum_{\gamma_\pi} \delta_0(\gamma_\pi) \\
&\leqslant \frac{1}{N(Q)} \sum_{\pi} \sum_{\gamma_\pi} \phi(\gamma_\pi) \\
&\leqslant \int_{\mathbf{R}} \phi(x) W(x) \, \mathrm{d}x + \varepsilon \\
&\leqslant \int_{\mathbf{R}} \widehat{\phi}(y) \widehat{W}(y) \, \mathrm{d}y + \varepsilon,
\end{aligned}$$

so that, for every $m \geqslant 1$, $\varepsilon > 0$ and Q sufficiently large,

$$p_m(Q) \leqslant \frac{1}{m} \left(\int_{\mathbf{R}} \widehat{\phi}(y) \widehat{W}(y) \, \mathrm{d}y + \varepsilon \right). \tag{4.65}$$

This along with the relation (4.63) imply a lower bound for the non-vanishing proportion

$$\begin{aligned}
p_0(m) &\geqslant \sum_{m} p_m(Q) - \sum_{m} m p_m(Q) \\
&\geqslant 1 - \int_{\mathbf{R}} \widehat{\phi}(y) \widehat{W}(y) \, \mathrm{d}y - \varepsilon,
\end{aligned}$$

providing a family of bounds depending on the function ϕ. Iwaniec, Luo and Sarnak [64, Appendix A] constructed the optimal choice among functions supported in $[-T_\phi, T_\phi]$, and computed the corresponding value given by

$$\int_{\mathbf{R}} \widehat{\phi}(y) \widehat{W}(y) \, \mathrm{d}y = \frac{1}{T_\phi} + \frac{1}{2}, \tag{4.66}$$

providing the desired result, that is in general for a support of the Fourier transform in $(-T_\phi, T_\phi)$,

$$\begin{aligned}
\liminf_{Q \infty} p_0(Q) &\geqslant \frac{1}{2} - \frac{1}{T_\phi}, \\
\liminf_{Q \infty} \sum_{m} m p_m(Q) &\leqslant \frac{1}{2} + \frac{1}{T_\phi}.
\end{aligned}$$

Unfortunaltely, the bound on the support of the Fourier transform of $\widehat{\phi}$ by $2/3$ is too small to yield non-trivial result on non-vanishing of the associated L-functions, as well as to uniquely unveil the conjectural type of symmetry of the universal family of quaternion algebras. Determining the higher densities or similar statistics on the finer families of fixed sign in the functional equation are clues to go further in this direction.

Chapter 5

Different ground groups

Previous chapters illustrated the power and relevance of trace formulas methods in addressing arithmetic statistics questions on the universal family of the group of units of quaternion algebras. It opens the path to explore different base groups, and this chapter summarize joint works in progress with Ian Petrow, giving evidences towards analogous results for some unitary as well as symplectic groups.

Exploring arithmetic statistics problems for different groups than inner forms of GL(2) leads to unveiling necessary assumptions for dealing with universal families by means of trace formulas. Any result in this direction provides clues towards more general conjectures concerning both the growth rate and the form of the constant in the counting law for universal families, in the vein of the analogous Batyrev-Manin-Peyre program.

Contents

5.1 Unitary groups of small ranks

5.1.1 Structure of unitary groups

Let E be a quadratic totally imaginary extension of F, let q a non-degenerate hermitian form on the three dimensional vector space V over E, and U the unitary group associated to q, that is to say the subgroup of transformations in $\mathrm{GL}(3)$ preserving q. More precisely, it is the group over F defined by

$$U = \{g \in \mathrm{GL}(3, F) \ : \ \forall x, y \in V,\ q(gx, gy) = q(x, y)\}\,. \tag{5.1}$$

The classification of the local hermitian vector spaces V is as follows. For archimedean places, thus in the case where $F = \mathbf{R}$ and $E = \mathbf{C}$, there are only two non-isomorphic one-dimension hermitian vector spaces denoted V^+ and V^-, associated to the hermitian forms $q_+(x, y) = x\overline{y}$ and $q_-(x, y) = -x\overline{y}$ respectively. Every hermitian space of dimension n over E is isomorphic to a space $V^{+p} \oplus V^{-q}$ where $p + q = n$, so that the signature (p, q) determines the hermitian space up to isomorphism. The corresponding unitary groups are denoted $U(p, q)$, among which only $U(p, q)$ and $U(q, p)$ are isomorphic, and only $U(n) = U(n, 0)$ is compact.

For non-archimedean places $\mathfrak{p}$, there are two non-isomorphic n-dimensional vector spaces over E, denoted V^+ and V^-. In the case of odd dimensions n, the corresponding unitary groups are isomorphic and quasi-split. In the case of even dimension, they are not isomorphic and only one of them is quasi-split.

The classification of unitary groups over local fields is given in the following proposition [59].

Proposition 31. *Let U be the unitary group attached to a vector space V over E. At a non-archimedean place $\mathfrak{p}$, the local group $U(V)$ is either isomorphic to $\mathrm{GL}_3(F_\mathfrak{p})$, in the case where $\mathfrak{p}$ splits in E, or to a unitary group over $F_\mathfrak{p}$. At archimedean places, it is classified by its signature.*

By the Chebotarev density theorem, places where U splits arise almost half of the time, precisely with natural density equal to one half. Let R be the set of the archimedean places, S be the set of finite places where U splits and T be the set of finite places where U is isomorphic to a unitary group. As for the case of quaternion algebras, the different behavior of each type of places is a critical issue in order to handle the problem by trace formula methods.

From now on, consider a unitary group U on three variables, that is to say a vector space of dimension 3 over E. Since the dimension is odd, the previous discussion ensures that at non-archimedean places, the local group $U_\mathfrak{p} = U(F_\mathfrak{p})$ is either $GL(3, F_\mathfrak{p})$ or the unique quasi-split unitary group on $F_\mathfrak{p}$. Assume U to be totally definite, *i.e.* for all archimedean place v, $U_v \cong U(3)$ is the compact unitary group in three variables. This in particular ensures that the automorphic quotient $U(F)\backslash U(\mathbf{A})$ is compact.

Remark. This assumption is made for convenience, in order to get rid of many complications coming from the smoothing of test functions at archimedean places, as in Section 2.5.4, for the purpose of this opening towards different ground groups is to underline differences with the case of the general linear group and its inner forms, motivating a wider freedom aimed at easing notation and focusing on new features.

The normalizations of measures are taken to be the ones introduced in the works of Rogawski [103] on the unitary group in three variables. First let introduce the standard maximal compact subgroups of G_v. For a finite place, define $K_\mathfrak{p} = \mathrm{GL}_3(\mathcal{O}_\mathfrak{p}) \cap G_\mathfrak{p}$, safe when the residual characteristic of $\mathfrak{p}$ is two, case in which $K_\mathfrak{p}$ is the maximal compact subgroup described by Rogawski [103, 1.10]. For archimedean places, K_v is taken to be the whole group $U(3, \mathbf{R})$. For every place v, introduce $\mathrm{d}g_v$ the Haar measure on U_v normalized such that K_v gets measure 1, and $\mathrm{d}g$ the product measure on $U(\mathbf{A})$.

Turn now to the associated local dual groups. Denote $\mathcal{H}(U_v)$ the Hecke algebra of G_v, that is the algebra consisting of complex-valued functions on U_v, compactly supported, locally constant at finite places, smooth at archimedian ones. Let $\mathcal{H}(U(\mathbf{A}))$ be the Hecke algebra of $U(\mathbf{A})$, generated by the restricted products $\otimes_v \phi_v$ where ϕ_v lies in $\mathcal{H}(G_v)$ and is $\mathbf{1}_{K_v}$ for almost every place. The unitary dual group $\widehat{U_v}$ is endowed with its usual Fell topology and Plancherel measure associated with the measure chosen on U_v, similarly to Section 2.1.1. From now on, every integral on $\widehat{U_v}$ will be written with the convention that $\mathrm{d}\pi_v = \mathrm{d}\mu_v^{\mathrm{Pl}}(\pi_v)$, leading to no ambiguity.

Consider the universal family $\mathcal{A}(U)$ consisting of all automorphic representations of U. Note that the cuspidality usually taken as a requirement for the universal family is an empty condition since the automorphic quotient is compact.

5.1.2 Functorial lift

It is desirable to stick with the functoriality spirit that consists in thinking of $\mathrm{GL}(n)$ as an ambiant group. It would allows to both define the analytic conductor by pulling back the associated notion on $\mathrm{GL}(n)$ as in Section 2.2 and use the toolbox provdided by known results in the general linear group setting. Let $\mathrm{Gal}(E/F)$ be the Galois group, W_E be the Weil group, and introduce the relative Weil group

$$W_{E/F} = W_F/[W_E, W_E],$$

which fits into an exact sequence

$$1 \to W_E \to W_{E/F} \to \mathrm{Gal}(E/F) \to 1.$$

The Galois group $\mathrm{Gal}(E/F)$ having two elements, say 1 and σ, it follows that $W_{E/F}$ is the disjoint union

$$W_{E/F} = W_E \sqcup W_E w_\sigma,$$

for some chosen pre-image w_σ of the non-trivial Galois representation σ. The L-group [93] of U is then defined as

$$^L U = \mathrm{GL}(3, \mathbf{C}) \rtimes W_{E/F},$$

where the action of $W_{E/F}$ on $\mathrm{GL}(3, \mathbf{C})$ is defined for $w \in W_E$ by

$$w(g) = g$$

and for $ww_\sigma \in W_E w_\sigma$ by

$$ww_\sigma(g) = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} {}^t g^{-1} \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}.$$

Let $G = \mathrm{Res}_{E/F}(\mathrm{GL}(3))$. The L-group of G is [93]

$$^L G = (\mathrm{GL}_3(\mathbf{C}) \times \mathrm{GL}_3(\mathbf{C})) \rtimes W_{E/F},$$

where $w \in W_E$ acts trivially and $ww_\sigma \in W_E w_\sigma$ acts by swapping the two factors.

Let $\mu : E^\times \to \mathbf{C}^\times$ be an unramified character. There are two of these, written 1 and μ_0. Corresponding to these, there are L-homomorphisms [90]

$$BC_\mu : {}^L U \to {}^L G.$$

If μ is trivial, this isomorphism is defined by

$$BC_1(g \rtimes w) = (g, g) \rtimes w,$$

for all $w \in W_{E/F}$. If $\mu = \mu_0$, the Artin map

$$E^\times \simeq W_E,$$

allows to consider μ_0 as a character of W_E, for which we use the same notation. Then define for $w \in W_E$

$$BC_{\mu_0}(g \rtimes w) = (\mu(w)g, \mu(w)g) \rtimes w$$

and for $ww_\sigma \in W_E w_\sigma$, define

$$BC_{\mu_0}(g \rtimes ww_\sigma) = (\mu(w)g, -\mu(w)g) \rtimes ww_\sigma.$$

Deep results of Flicker [47] say that corresponding to these two maps of L-groups there exist transfer maps between the associated universal families

$$\widetilde{BC}_\mu : \mathcal{A}(U/F) \to \mathcal{A}(\mathrm{GL}(3)/E)$$

satisfying the Langlands functoriality conjectures [25]. That is to say, for each irreducible admissible representation π of $U(F)$ is associated an irreducible admissible representation of $GL(3, E)$, namely $\widetilde{BC}_\mu(\pi)$. This allows to access all of the theory and results of automorphic forms on GL(3).

Remark. It could also be possible to appeal to the functoriality of cyclic base change [4, example 1] to embed the universal family $\mathcal{A}(U)$ into automorphic forms of $\mathcal{A}(\mathrm{GL}(6, F))$, in order to consider every group on the same field. However, cuspidality is not always preserved under base change transfers, breaking with the common ground of the universal family of $\mathrm{GL}(n)$. Thus, sticking with $\mathrm{GL}(3, E)$ allows more control on what the lifts are, instead of lifting from $\mathrm{GL}(3, E)$ to $\mathrm{GL}(6, F)$ would only amplify those difficulties.

5.1.3 Analytic conductors

Functorial conductor

The existence of a transfer of the universal family of U as a subfamily of automorphic representations of GL(3) over E would allow to pull back to the unitary group setting many results already acquired on general groups, and to lift the notion of analytic conductor, analogously to the case of heights for geenral algebraic varieties and to what has been done in Section 2.2.3. Following [54], there is a three dimensional representation of the L-group of U

$$ {}^L U \hookrightarrow \mathrm{GL}(3, \mathbf{C}) = {}^L\mathrm{GL}(3). \tag{5.2} $$

The Langlands functoriality principle [26] predicts the existence of a lifting to a map of representations $\mathcal{A}(G) \to \mathcal{A}(\mathrm{GL}(3))$, obtained as follows. Introduce $\Phi(U)$ the set of Langlands parameters of U, *i.e.* representations of the Weil group W_F to ${}^L U$. The Langlands conjectures, known to be true in this case by a result due to Flicker [47], states the existence of a surjective map

$$ \mathcal{A}(U) \twoheadrightarrow \Phi(U), \tag{5.3} $$

with finite fibers. Replacing $\mathcal{A}(U)$ by the set $\mathcal{A}^L(U)$ of these fibers, called L-packets of U, this map leads to a bijection

$$ \mathcal{A}^L(U) \cong \Phi(U). \tag{5.4} $$

It follows that there is a natural embedding $\mathcal{A}^L(U) \hookrightarrow \Phi(\mathrm{GL}(3))$ by composing the L-packets parametrization (5.4) with the embedding of L-groups (5.2). By the Langlands functoriality for GL(3), the set $\Phi(\mathrm{GL}(3))$ parametrizes the automorphic representations of GL(3), so that it yields an embedding

$$ \theta : \mathcal{A}^L(U) \hookrightarrow \mathcal{A}(\mathrm{GL}(3)). \tag{5.5} $$

The universal family is an infinite one, so in order to establish statistics on it a truncation to a finite set with respect to a suitable notion of size is needed. In the setting of cusp forms on general linear groups, such a notion is provided by the analytic conductor of Iwaniec and Sarnak [65], which is defined for generic representations. This notion can be pulled back through the previous embedding (5.5) to a notion of analytic conductor on U by setting

$$c_{\mathrm{GL}}(\pi) = c_{\mathrm{GL}(3)}(\theta(\pi)). \tag{5.6}$$

Geometric conductor

The notion of conductor given in Section 5.6 is hard to reach when it comes to explicit computations, and an interpretation as a notion of depth as in Section 2.2.3 is desirable. The conductor can be defined for local components of representations, the global conductor being then defined to be the product over all places. For finite places in S, where $U_{\mathfrak{p}}$ is isomorphic to the standard setting $\mathrm{GL}(3, F_{\mathfrak{p}})$, this objective is achieved by the filtration

$$K_{1,\mathfrak{p}}^{\mathrm{GL}}(\mathfrak{p}^r) = \left\{ g \in \mathrm{GL}(3)\left(\mathcal{O}_{\mathfrak{p}}\right) \;:\; g \equiv \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ 0 & 0 & 1 \end{pmatrix} \bmod \mathfrak{p}^r \right\} \subseteq \mathrm{GL}\left(3, \mathcal{O}_{\mathfrak{p}}\right), \qquad r \geqslant 0. \tag{5.7}$$

provided by Jacquet, Piatetski-Shapiro and Shalika [67] as a straightforward generalization of the one constructed by Casselman for GL(2), introduced in Section 2.2.3.

For finite places in T, where $U_{\mathfrak{p}}$ is isomorphic to the quasi-split unitary group on $F_{\mathfrak{p}}$, Miyauchi [92] built an analogous filtration of compact open subgroups of $\mathrm{GL}(3, F_{\mathfrak{p}})$ defined by

$$K_{1,\mathfrak{p}}^{U}(\mathfrak{p}^r) = \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathfrak{p}^{-r} \\ \mathfrak{p}^r & 1+\mathfrak{p}^r & \mathcal{O}_E \\ \mathfrak{p}^r & \mathfrak{p}^r & \mathcal{O}_E \end{pmatrix} \cap U(F_{\mathfrak{p}}). \tag{5.8}$$

This choice of filtration provides suitable notions of depth and newform for representations, providing the analogous toolbox in the setting of quasi-split unitary groups, as stated in his central theorem [92, Theorem 0.3] as follows.

Theorem 14 (Miyauchi)**.** *Let $\pi_{\mathfrak{p}}$ be an irreducible generic representation of the quasi-split unitary group $U(F_{\mathfrak{p}})$. Then there is a non-negative integer r such that $\pi_{\mathfrak{p}}$ admits nonzero fixed vectors by $K_{1,\mathfrak{p}}^{U}(\mathfrak{p}^r)$.*

Both the result of Jacquet, Piatetski-Shapiro, Shalika and the one of Miyauchi ensure that there exists fixed vectors for a small enough subgroups in both cases of $\mathrm{GL}(3, F_{\mathfrak{p}})$ and $U(F_{\mathfrak{p}})$. The conductor of an irreducible admissible infinite-dimensional representation $\pi_{\mathfrak{p}}$ of $U_{\mathfrak{p}}$ is then defined by the smallest rank for which it happens, analogously to Definition 9.

Definition 13. The additive conductor of a generic representation $\pi_\mathfrak{p}$ of $U_\mathfrak{p}$ is defined as

$$\mathfrak{f}\left(\pi_\mathfrak{p}\right) = \min\left\{r \in \mathbf{N} \ : \ \pi_\mathfrak{p}^{K^X_{0,\mathfrak{p}}(\mathfrak{p}^r)} \neq 0\right\}, \tag{5.9}$$

with $X \in \{\mathrm{GL}, U\}$ according to whether $U_\mathfrak{p}$ is isomorphic to $\mathrm{GL}(3, F_\mathfrak{p})$ or $U(F_\mathfrak{p})$. The multiplicative and analytic conductor of $\pi_\mathfrak{p}$ are respectively defined by

$$\mathfrak{c}\left(\pi_\mathfrak{p}\right) = \mathfrak{p}^{\mathfrak{f}(\pi_\mathfrak{p})} \quad \text{and} \quad c\left(\pi_\mathfrak{p}\right) = N\mathfrak{c}\left(\pi_\mathfrak{p}\right). \tag{5.10}$$

Moreover, the remaining archimedean places are in finite number thus a more systematic definition of the conductor can be chosen, as arising from the functional equation satisfied by the associated L-functions, as already mentioned in Section 2.2.1, or by pulling back the notion of analytic conductor on GL(3) through the embedding (5.5). As already discussed, the analytic conductor has a working definition used in asymptotical questions, hence allowing to use some freedom of normalization in his definition for a finite number of places without any impact to the finiteness the family.

Nevertheless, Miyauchi's work does not state that this notion is compatible with the one given by the conductor coming from the ε-factor of the associated L-function defined on U. The functoriality conjecture for conductors states this compatibility.

Conjecture 5 (Functoriality of the conductor). *Both notions of the conductor pulled back from* GL(3) *and of the conductor as depth are compatible, that is to say for every generic representation π of $U_\mathfrak{p}$,*

$$c(\pi) = c_{\mathrm{GL}}(\pi). \tag{5.11}$$

Notwithstanding this lack of proven consistency, the arithmetic statistics problems keep their meaning whether the functorial conductor or the filtration conductor is used to truncate the family or not, and is motivated by the expected functoriality property. From now on, the considered conductor is the one given by the filtrations at places in T. Introduce the cardinality of the truncated universal family

$$N(Q) = |\mathcal{A}_U(Q)| = \sum_{\substack{\pi \in \mathcal{A}(U) \\ c(\pi) \leqslant Q}} 1. \tag{5.12}$$

5.1.4 Theory of local newforms

Analogously to Section 2.4.3, the universal family sees no multiplicities, but the trace formula counts them. The spectral multiplicities associated to the decomposition of $L^2(U(F)\backslash U(\mathbf{A}))$ are

$$m\left(\pi, \mathfrak{q}\right) = \dim \pi^{K_1(\mathfrak{q})}, \tag{5.13}$$

where the global congruence subgroup $K_1(\mathfrak{q})$ is defined, for an ideal $\mathfrak{q}$ of $\mathcal{O}$, by

$$K_1(\mathfrak{q}) = \prod_{\substack{\mathfrak{p} \in S \\ \mathfrak{p}^r || \mathfrak{q}}} K^{\mathrm{GL}}_{1,\mathfrak{p}}(\mathfrak{p}^r) \prod_{\substack{\mathfrak{p} \in T \\ \mathfrak{p}^r || \mathfrak{q}}} K^{U}_{1,\mathfrak{p}}(\mathfrak{p}^r). \tag{5.14}$$

According to this decomposition, the multiplicity $m(\pi, \mathfrak{q})$ can be split as the product of the multiplicities $m(\pi_S, \mathfrak{q}_S)$ and $m(\pi_T, \mathfrak{q}_T)$. In the case of finite split places in S, that is to say a group $U_\mathfrak{p}$ isomorphic to $\mathrm{GL}(3, F_\mathfrak{p})$, the multiplicities are explicitly computed by Reeder [101] and given by, for a place $\mathfrak{p}$ in S,

$$m\left(\pi_\mathfrak{p}, \mathfrak{p}^r\right) = d_3\left(\frac{\mathfrak{p}^r}{\mathfrak{c}(\pi_\mathfrak{p})}\right). \tag{5.15}$$

In the case of finite places where $U_\mathfrak{p}$ is isomorphic to the quasi-split unitary group $U(F_\mathfrak{p})$, the multiplicites associated to the filtration (5.8) is given by Miyauchi [92] and equal, for a place $\mathfrak{p}$ in T,

$$m\left(\pi_\mathfrak{p}, \mathfrak{p}^r\right) = \max\left(\left\lfloor\frac{r - \mathfrak{f}(\pi)}{2}\right\rfloor + 1, 0\right). \tag{5.16}$$

What is of fundamental importance is that these multiplicities are functions only of the level $\mathfrak{p}^r$, and not of the type of representation of $\pi_\mathfrak{p}$. Even more precisely, it is a function of $\mathfrak{q}/\mathfrak{c}(\pi)$. So that the global multiplicity (2.46) is written as

$$m(\pi, \mathfrak{q}) = \prod_{\mathfrak{p}^r || \mathfrak{q}} m(\pi_\mathfrak{p}, \mathfrak{p}^r). \tag{5.17}$$

Decomposing the cardinality of the universal family (5.12) by fixing the discrete spectral data given by the conductor relative to places in S and T, and the isomorphism class of local components relatives to places in R, leads to rewriting it into

$$N(Q) = \sum_{\substack{\pi \in \mathcal{A}(U) \\ c(\pi) \leqslant Q}} 1 = \sum_{\substack{N\mathfrak{q} \leqslant Q \\ \mathfrak{q} \wedge R = 1}} \sum_{\substack{\pi_R \in \widehat{U}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{q}}} \sum_{\substack{\sigma \in \mathcal{A}(U) \\ c(\sigma^R) = \mathfrak{q} \\ \sigma_R \simeq \pi_R}} 1, \tag{5.18}$$

so that it is enough to concentrate from now on on evaluating the inner sums

$$A(\mathfrak{q}, \pi_R) = \sum_{\substack{\sigma \in \mathcal{A}(U) \\ c(\sigma^R) = \mathfrak{q} \\ \sigma_R \simeq \pi_R}} 1. \tag{5.19}$$

A more relevant quantity is to consider the counting weighted by the multiplicities, naturally grasped by the trace formula, so that it is more suitable to introduce

$$B(\mathfrak{q}, \pi_R) = \sum_{\substack{\sigma \in \mathcal{A}(U) \\ c(\sigma^R) | \mathfrak{q} \\ \sigma_R \simeq \pi_R}} m(\pi, \mathfrak{q}) = m \star A(\mathfrak{q}, \pi_R), \tag{5.20}$$

where this abuse of notation is the same one used in Section 2.4.3. Remembering that $m(\pi, \mathfrak{q})$ is a function of $\mathfrak{q}/\mathfrak{c}(\pi)$ still written m, the notation $m \star A$ stands for

$$m \star A(\mathfrak{q}, \pi_R) = \sum_{\mathfrak{d}|\mathfrak{q}} m(\mathfrak{q}/\mathfrak{d}) A(\mathfrak{d}, \pi_R). \tag{5.21}$$

Analogously to the sieving of Section 2.4.3 getting rid of the multiplicities in the case of quaternion algebras, it is possible to state an explicit relation linking the numbers of old and new forms in the case of unitary groups. Denoting $\tilde{m}$ the inverse by convolution of the global multiplicity m, the counting problem is reduced to evaluate B, for by Möbius inversion

$$A(\mathfrak{q}, \pi_R) = \tilde{m} \star B(\mathfrak{q}, \pi_R). \tag{5.22}$$

In particular, this leads to a new expression of the cardinality of the universal family,

$$N(Q) = \sum_{\substack{N\mathfrak{q} \leqslant Q \\ \mathfrak{q} \wedge R = 1}} \sum_{\substack{\pi_R \in \widehat{U}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{q}}} \sum_{\mathfrak{d}|\mathfrak{q}} \tilde{m} \left(\frac{\mathfrak{q}}{\mathfrak{d}} \right) B(\mathfrak{q}, \pi_R). \tag{5.23}$$

5.2 Counting law for unitary groups

5.2.1 Counting law at a glance

The aim of this Section is to briefly recall the main tools and steps used in the proof of the counting law, analogously to Section 2.5, and stress on the differences with the case of quaternion algebras. Since the assumption on U to be totally definite ensures a compact automorphic quotient, combined with the multiplicity one theorem, a simpler version of the Selberg trace formula holds [3]. If Φ is a function in the Hecke algebra $\mathcal{H}(U(\mathbf{A}))$, then

$$J_{\text{geom}}(\Phi) = J_{\text{spec}}(\Phi), \tag{5.24}$$

where the spectral and geometrical parts are given by

$$J_{\text{geom}}(\Phi) = \sum_{\{\gamma\}} \text{vol}\left(U_\gamma(F) \backslash U_\gamma(\mathbf{A})\right) \int_{U_\gamma(\mathbf{A}) \backslash U(\mathbf{A})} \Phi\left(x^{-1}\gamma x\right) \mathrm{d}x$$
$$J_{\text{spec}}(\Phi) = \sum_{\pi \subseteq L^2(U(F) \backslash U(\mathbf{A}))} m(\pi) \widehat{\Phi}(\pi)$$

where π runs through the isomorphism classes of unitary irreducible subrepresentations of $U(\mathbf{A})$ in $L^2(U(F) \backslash U(\mathbf{A}))$ and $\{\gamma\}$ vary among conjugacy classes in $U(F)$.

Similarly to the construction of the test function in Section 2.5.5, it is necessary to introduce a suitable test function in order to estimate $B(\mathfrak{d}, \pi_R)$ as a spectral side of a trace formula, analogously to Lemma 4. For a given ideal $\mathfrak{d}$ and a representation π_R of U_R, introduce

- for places $\mathfrak{p}$ in S or T, define $\Phi_\mathfrak{p} = \varepsilon_{K^X_{1,\mathfrak{p}}(\mathfrak{p}^r)}$, where $\mathfrak{p}^r || \mathfrak{d}$ and $X \in \{\mathrm{GL}, U\}$ is chosen according to whether $U_\mathfrak{p}$ is isomorphic to $\mathrm{GL}(3, F_\mathfrak{p})$ or the quasi-split;
- for places v of R, define $\Phi_v = \xi_{\pi_v}$ a well-normalized matrix coefficient associated to π_v, see Section 2.5.3, underlying that matrix coefficients can be used due to the definite assumption on U, ensuring that the π_v's are supercuspidal representations.

Characteristic functions of subgroups coming from the filtrations (5.7) or (5.8) are known to provide selecting functions for conductors, as in the GL(2) case treated in Lemma 2, with the very same proof.

Lemma 23. *For an ideal $\mathfrak{d}$ of $\mathcal{O}$, let*

$$\varepsilon_\mathfrak{d} = \mathrm{vol}\,(K_1(\mathfrak{d}))^{-1}\,\mathbf{1}_{K_1(\mathfrak{d})}. \tag{5.25}$$

Its Fourier transform selects the multiplicity relative to $\mathfrak{d}$. More precisely, for every $\pi \in \mathcal{A}(U)$,

$$\widehat{\varepsilon_\mathfrak{d}}(\pi) = m(\pi, \mathfrak{d}). \tag{5.26}$$

Since places in R consist of archimedean places where U_v is isomorphic to the compact unitary group $U(3)$ by the totally definite assumption, only supercuspidal representations arise at these local places, thus matrix coefficients are known to transform into the selecting function of the associated isomorphism class of representation, see Section 2.5.3.

Lemma 24. *Let σ and π be automorphic representations of U_R. Then,*

$$\widehat{\xi_{\pi_v}}(\sigma_v) = \mathbf{1}_{\sigma_v \simeq \pi_v}. \tag{5.27}$$

No other case arises for totally definite unitary groups in three variables, so that the global test function defined as

$$\Phi_{\mathfrak{q},\pi_R} = \prod_v \Phi_v, \tag{5.28}$$

is made so that its Fourier transform selects conductors dividing $\mathfrak{q}$ and bad component fixed to π_R. Analogously to lemma 4, the following lemma holds for this function, underlining the fact that the restriction to the totally definite setting as well as considering only the counting problem lead to a far less technical statement.

Lemma 25. *Let $Q \geqslant 1$. Let $\mathfrak{d} \wedge R = 1$ and $\pi_R \in \widehat{U}_R$. The old forms number and the spectral part of the trace formula are linked by*

$$J_{\text{spec}}\left(\Phi_{\mathfrak{d},\pi_R}\right) = B\left(\mathfrak{d}, \pi_R\right) + O(\Xi(\pi_R)), \tag{5.29}$$

where, introducing the set $X^R(U)$ of characters of $U(\mathbf{A})$ unramified out of R, and

$$\Xi(\pi_R) = \sum_{\substack{\chi \in X^R(G) \\ \chi_R \simeq \pi_R}} 1. \tag{5.30}$$

Remarks. In comparison with Lemma 4, some remarks arise.

(i) there is no complementary part of the spectrum, for the totally definite assumption ensures a discrete archimedean spectrum made of supercuspidal representations, hence amenable to selection by matrix coefficient;
(ii) the totally definite assumption also avoids the smoothing selecting function, cf. Section 2.5.4. This simplifies computations to a large extent, and also gets rid of the worst error term appearing in Theorem A;
(iii) dealing with the counting problem instead of addressing the more general equidistribution questions avoids technical complications too deal with the places in the support of the distribution test function, cf. Section 2.3.4.

5.2.2 Identity contribution

The main term is expected to be the one corresponding to the identity in the geometrical side of the trace formula, see Section 3.2. Hopefully, explicit and easy to handle test functions like Φ allows to evaluate it and, similarly to Section 3.2.1,

$$\Phi(1) = \mathrm{vol}(K_1(\mathfrak{q}))^{-1}\mu_R^{\mathrm{Pl}}(\pi_R). \tag{5.31}$$

By the normalization of the Haar measure as giving value 1 to the maximal compact subgroup K, it follows that the volume $\mathrm{vol}(K_1(\mathfrak{q}))^{-1}$ equals the index $[K : K_1(\mathfrak{q})]$. Denote $\psi(\mathfrak{q})$ this index, equal to

$$\psi(\mathfrak{d}) = \prod_{\mathfrak{p}^r || \mathfrak{d}} \psi_{\mathfrak{p}}(\mathfrak{p}^r), \tag{5.32}$$

where

$$\psi_{\mathfrak{p}}(\mathfrak{p}^r) = \begin{cases} [K_{\mathfrak{p}}^{\mathrm{GL}} : K_1^{\mathrm{GL}}(\mathfrak{p}^r)] & \text{if } \mathfrak{p} \in S, \\ [K_{\mathfrak{p}}^{U} : K_1^{U}(\mathfrak{p}^r)] & \text{if } \mathfrak{p} \in T. \end{cases} \tag{5.33}$$

Following the same path leading to the evaluation of the identity contribution in Proposition 18, the summation over spectral parameters $\mathfrak{q}$ and π_R leads to

$$N_1(Q) = \sum_{\substack{N\mathfrak{q} \leqslant Q \\ \mathfrak{q} \wedge R = 1}} \sum_{\substack{\pi_R \in \widehat{U}_R \\ c(\pi_R) \leqslant Q/N\mathfrak{q}}} \sum_{\mathfrak{d} | \mathfrak{q}} \tilde{m}\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right) J_1(\Phi_{\mathfrak{d},\mathfrak{p}_R})$$

$$= \mathrm{vol}(U(F)\backslash U(\mathbf{A})) \sum_{\substack{N\mathfrak{q}\leqslant Q\\ \mathfrak{q}\wedge R=1}} \sum_{\substack{\pi_R\in\widehat{U}_R\\ c(\pi_R)\leqslant Q/N\mathfrak{q}}} \sum_{\mathfrak{d}|\mathfrak{q}} \tilde{m}\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right)\psi(\mathfrak{d})\mu_R^{\mathrm{Pl}}(\pi_R)$$

$$= \mathrm{vol}(U(F)\backslash U(\mathbf{A})) \sum_{\substack{\pi_R\in\widehat{U}_R\\ c(\pi_R)\leqslant Q}} \mu_R^{\mathrm{Pl}}(\pi_R) \sum_{\substack{N\mathfrak{m}\leqslant Q/c(\pi_R)\\ \mathfrak{m}\wedge R=1}} \tilde{m}(\mathfrak{m}) \sum_{\substack{N\mathfrak{d}\leqslant Q/c(\pi_R)N\mathfrak{m}\\ \mathfrak{d}\wedge R=1}} \psi(\mathfrak{d})$$

The innermost sum is estimated by means of standard arithmetic sums. Recall that $D(f,s)$ denotes the Dirichlet series associated to f on a domain where it converges.

Lemma 26. *For every $X>0$,*

$$\sum_{\substack{N\mathfrak{d}\leqslant X\\ \mathfrak{d}\wedge R=1}} \psi(\mathfrak{d}) = \frac{1}{4}\frac{\zeta_S^\star(1)}{\zeta_S(4)}D(\psi_T,4)Q^4 + O(Q^{4-\delta_F}). \tag{5.34}$$

Remark. It is by no mean obvious that the Dirichlet series of ψ is convergent at 4. Since it is defined as indices of explicit subgroups, there should be no difficulty in verifying this property.

Proof. Decomposing the sum over ideals, and summing first over split places in order to use the standard estimates [17, (5.2)], yields

$$\sum_{\substack{N\mathfrak{d}\leqslant X\\ \mathfrak{d}\wedge R=1}} \psi(\mathfrak{d}) = \sum_{\substack{N\mathfrak{d}_T\leqslant X\\ \mathfrak{d}_T\wedge R=1}} \psi(\mathfrak{d}_T) \sum_{\substack{N\mathfrak{d}_S\leqslant X/N\mathfrak{d}_T\\ \mathfrak{d}_S\wedge R=1}} \psi(\mathfrak{d}_S)$$

$$= \sum_{\substack{N\mathfrak{d}_T\leqslant X\\ \mathfrak{d}_T\wedge R=1}} \psi(\mathfrak{d}_T)\left(\frac{1}{4}\frac{\zeta_S^\star(1)}{\zeta_S(4)}\left(\frac{X}{N\mathfrak{d}_T}\right)^4 + O\left(\left(\frac{X}{N\mathfrak{d}_T}\right)^{4-\delta_F}\right)\right)$$

$$= \frac{1}{4}\frac{\zeta_S^\star(1)}{\zeta_S(4)}X^4 \sum_{\substack{N\mathfrak{d}_T\leqslant X\\ \mathfrak{d}_T\wedge R=1}} \frac{\psi(\mathfrak{d}_T)}{N\mathfrak{d}_T^4} + O\left(X^{4-\delta_F}\sum_{\substack{N\mathfrak{d}_T\leqslant X\\ \mathfrak{d}_T\wedge R=1}} \frac{\psi(\mathfrak{d}_T)}{N\mathfrak{d}_T^{4-\delta_F}}\right) \qquad \square$$

This allows to conclude in the computations above, with a main term given by

$$N_1^{(p)}(Q) = \frac{1}{4}\frac{\zeta_S^\star(1)}{\zeta_S(4)}\mathrm{vol}(U(F)\backslash U(\mathbf{A}))D(\psi_T,4)Q^4 \sum_{\substack{\pi_R\in\widehat{U}_R\\ c(\pi_R)\leqslant Q}} \frac{\mu_R^{\mathrm{Pl}}(\pi_R)}{c(\pi_R)^4} \sum_{\substack{N\mathfrak{m}\leqslant Q/c(\pi_R)\\ \mathfrak{m}\wedge R=1}} \frac{\tilde{m}(\mathfrak{m})}{N\mathfrak{m}^4}, \tag{5.35}$$

and an error term

$$N_1^{(e)}(Q) \ll \mathrm{vol}(U(F)\backslash U(\mathbf{A}))Q^{4-\delta_F} \sum_{\substack{\pi_R\in\widehat{U}_R\\ c(\pi_R)\leqslant Q}} \frac{\mu_R^{\mathrm{Pl}}(\pi_R)}{c(\pi_R)^{4-\delta_F}} \sum_{\substack{N\mathfrak{m}\leqslant Q/c(\pi_R)\\ \mathfrak{m}\wedge R=1}} \frac{\tilde{m}(\mathfrak{m})}{N\mathfrak{m}^{4-\delta_F}}. \tag{5.36}$$

Moreover, recall that the sums over the fixed bad places converge since their embeddings into GL(3) converge [17], assuming the functoriality conjecture for the conductor. Moreover, the inner sum running over $\mathfrak{m}$ is the value at 4 of the convergent Dirichlet series attached to $\tilde{m}$, so that

$$N_1(Q) = \frac{1}{4}\frac{\zeta_S^\star(1)}{\zeta_S(4)}D(\psi_T, 4)D(\tilde{m}, 4)\mathrm{vol}(U(F)\backslash U(\mathbf{A}))Q^4 \sum_{\substack{\pi_R \in \widehat{U}_R \\ c(\pi_R) \leqslant Q}} \frac{\mu_R^{\mathrm{Pl}}(\pi_R)}{c(\pi_R)^4} + O\left(Q^{4-\delta_F}\right). \tag{5.37}$$

Remark. The Dirichlet series of $\tilde{m}$ is far easier to grasp than the one associated to ψ. Indeed, the multiplicities are often of logarithmic order compared to the conductor, so that their abscissa of convergence is expected to be 1. In the present case, this is straightforward to check thanks to the explicit expression of these multiplicities in (5.15) and (5.16).

5.2.3 Geometric interpretation of the constant

In the light of the geometrical restatement of the constant in the case of quaternion algebras carried out in Section 3.2.4, there is a more suitable way to express the constant above that also avoids unnecessary and ad hoc computations. By definition,

$$\varepsilon_{K_{1,\mathfrak{p}}^X(\mathfrak{p}^r)}(1) = \mathrm{vol}\left(K_{1,\mathfrak{p}}^X(\mathfrak{p}^r)\right)^{-1} = \psi(\mathfrak{p}^r), \qquad r \geqslant 0. \tag{5.38}$$

On an other hand, this volume can be computed by the Plancherel formula. Introduce the volume of representations of fixed conductor

$$M_\mathfrak{p}(\mathfrak{p}^r) = \int_{\substack{\sigma_\mathfrak{p} \in \widehat{U}_\mathfrak{p} \\ \mathfrak{c}(\sigma_\mathfrak{p}) = \mathfrak{p}^r}} \mathrm{d}\sigma_\mathfrak{p}, \qquad r \geqslant 0.$$

The Plancherel inversion formula then yields, using the explicit transform of $\varepsilon_{K_{1,\mathfrak{p}}^X(\mathfrak{p}^r)}$ stated in Lemma 23 and the fundamental property that the multiplicities are functions of the level,

$$\begin{aligned}\varepsilon_{K_{1,\mathfrak{p}}^X(\mathfrak{p}^r)}(1) &= \int_{\widehat{U}_\mathfrak{p}} \widehat{\varepsilon}_{K_{1,\mathfrak{p}}^X(\mathfrak{p}^r)}(\pi_\mathfrak{p})\, \mathrm{d}\pi_\mathfrak{p} = \int_{\widehat{U}_\mathfrak{p}} m\left(\frac{\mathfrak{p}^r}{\mathfrak{c}(\pi_\mathfrak{p})}\right) \mathrm{d}\pi_\mathfrak{p} \\ &= \sum_{\mathfrak{d} \mid \mathfrak{p}^r} M_\mathfrak{p}(\mathfrak{d}) m\left(\frac{\mathfrak{p}^r}{\mathfrak{d}}\right) = (M_\mathfrak{p} \star m)(\mathfrak{p}^r)\end{aligned}$$

Hence, by Möbius inversion, $M_\mathfrak{p} = \tilde{m} \star \psi$, recalling that $\tilde{m}$ stands for the inverse by convolution of m. In particular, the local Dirichlet series associated to $M_\mathfrak{p}$ is given by

$$D\left(M_\mathfrak{p}, s\right) = \sum_{\substack{\mathfrak{m} = \mathfrak{p}^r \\ r \geqslant 0}} \frac{M_\mathfrak{p}(\mathfrak{m})}{N\mathfrak{m}^s} = D(\psi, s)D(\tilde{m}, s), \qquad \mathrm{Re}(s) > 1. \tag{5.39}$$

Evaluating it at $s = 4$, motivated by the exponent obtained in the identity contribution (5.37), a new expression for the local special values appearing in the constant is

$$D(\psi, 4)D(\tilde{m}, 4) = \int_{\widehat{U}_{\mathfrak{p}}} \frac{\mathrm{d}\pi_{\mathfrak{p}}}{c(\pi_{\mathfrak{p}})^4}, \tag{5.40}$$

proving the finiteness of the local integrals defining the constant, for the Dirichlet series are convergent. However, the infinite product over $\mathfrak{p} \notin R$ of these quantities unfortunately diverges, for 1 is a pole of ζ_F^R, leading to compensate it by the residue at 1 and to introduce the regularized integral

$$\int_{\widehat{U}^R}^{\star} \frac{\mathrm{d}\pi^R}{c(\pi^R)^4} = \zeta_F^{R\star}(1) \prod_{\mathfrak{p}\notin R} \zeta_{\mathfrak{p}}(1)^{-1} \int_{\widehat{U}_{\mathfrak{p}}} \frac{\mathrm{d}\pi_{\mathfrak{p}}}{c(\pi_{\mathfrak{p}})^4}. \tag{5.41}$$

Moreover, the factors $\zeta_{\mathfrak{p}}(1)/\zeta_{\mathfrak{p}}(4)$ coming from the computations for places in S also can be written as $D(\psi_{\mathfrak{p}}, 4)$ for $\mathfrak{p} \in S$, since $\psi_S = \mathrm{id}^3 \star \mu$, so that

$$D(\psi_{\mathfrak{p}}, s) = \frac{\zeta_{\mathfrak{p}}(s-3)}{\zeta_{\mathfrak{p}}(s)}, \qquad \mathrm{Re}(s) > 0, \quad \mathfrak{p} \in S. \tag{5.42}$$

At last, the identity contribution to the counting law rewrites

$$N_1(Q) = \frac{1}{4}\mathrm{vol}(U(F)\backslash U(\mathbf{A}))Q^4 \int_{\widehat{U}^R}^{\star} \frac{\mathrm{d}\pi^R}{c(\pi^R)^4} + O\left(Q^{4-\delta_F}\right). \tag{5.43}$$

5.2.4 Towards the extra contributions

Many error terms arise similarly to Chapter 3, and will be dealt with in a similar fashion.

Characters contribution

The contribution of characters is addressed by the same means than Section 3.3.1, and shown to be negligible by explicitly computing the number of unramified characters unduly selected by the chosen test function. This result is established through use of the Poisson summation formula, following the very same method used to establish the counting law for the universal family of quaternion algebras.

Central contribution

A notable difference compared to previous chapters is the presence of central contribution other than the identity one in the geometric side of the trace formula, for the centerless assumption is removed. Since the center of $U(F)$ is a discrete subset of $U(\mathbf{A})$, its intersection with the congruence subgroups arising in the filtration is finite, and trivial for sufficiently large level $\mathfrak{q}$, so that this contribution is reduced the identity one. This result should carefully follows the lines of the corresponding treatment in the case of $\mathrm{GL}(n)$ [17, Section 10.1].

Elliptic contribution

Estimating orbital integrals is a critical issue in exploiting the trace formula in order to get the counting law. However, a general strategy to bound orbital integrals, providing also results in the general case, is provided along the lines of work of Kottwitz [76] and Matz-Templier [86], and should be adapted to the setting of unitary groups in three variables. This aspect is expected to be the more delicate and the farther from the $\mathrm{GL}(n)$ case.

Smooth selection of the archimedean spectrum

The totally definite assumption is stated in order to avoid to deal with smoothing terms in the test function at archimedean places, cf. Section 2.5.4, and non-tempered parts of the spectrum, cf. Section 3.3.2. However, it is expected that these conditions are not necessary to carry on the computations, as for quaternion algebras or $\mathrm{GL}(2)$. Indeed, the construction of the selecting functions for the continuous spectrum of locally symmetric spaces is finely explored by Duistermaat, Kolk and Varadarajan [39] and adapted to automorphic settings in various works [86, 17], providing evidence to a possible similar treatment. Indeed, the parametrization of the archimedean spectrum by discrete data and essentially a vector space of continuous data still holds for more general reductive groups than $\mathrm{GL}(n)$, see the efficient account of Knapp [73].

Conjecture for unitary groups

These comments yield strong evidences towards a conjectural counting law similar to the one stated for quaternion algebras in Theorem A, and motivates the following.

Conjecture H (with I. Petrow)**.** *The asymptotic development of the cardinality of the universal family of a totally definite unitary group U in three variables over E is*

$$N(Q) \sim \frac{1}{4}\mathrm{vol}([U]) \int_{A(U)}^{\star} \frac{\mathrm{d}\pi}{c(\pi)^4} Q^4, \qquad \text{as} \quad Q \to \infty. \tag{5.44}$$

Remarks. Removing the assumption on automorphic compact quotient seems audacious, and no attempt has been made in this direction. However, this problem could be amenable by method similar to the $\mathrm{GL}(n)$ case [17, Section 17]. The embedding in the $\mathrm{GL}(3)$ setting may provide certain convergence results by pullback as well.

5.3 Symplectic groups of small ranks

Another group for which a suitable theory of local newforms is established is $\mathrm{GSp}(4)$. In this context, Roberts and Schmidt [102] developed all the tools needed to an application of the trace formula to handle arithmetic statistics problems. This group has the further appeal to be no form of $\mathrm{GL}(n)$, hence providing in some sense a genuine new example.

5.3.1 Inner forms of $\mathrm{GSp}(4)$

The symplectic group is the algebraic group defined by

$$\mathrm{GSp}(4) = \{g \in \mathrm{GL}(4) \ : \ {}^t gJg = \lambda(g)J, \lambda(g) \in \mathbf{G}_m\} \quad \text{where} \quad J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \tag{5.45}$$

Only some inner forms of $\mathrm{GSp}(4)$ are considered here, using the same freedom than in the case of unitary groups in order to give a first glance towards arithmetic statistics for the universal family of symplectic groups. For D a division quaternion algebra over F, let (V,q) be a D-hermitian space of D-dimension 2, and introduce its group of isometries

$$\begin{aligned} \mathrm{GU}(V,q) &= \{g \in \mathrm{GL}(V) \ : \forall x,y \in V, q(gx,gy) = q(x,y)\} \\ &= \left\{M \in \mathrm{GL}_2(D) \ : M^\star A_q M = A_q\right\} \end{aligned}$$

This construction gives all the non-split inner forms [48] of $\mathrm{GSp}(4)$. Furthermore, it is possible to characterize those with compact automorphic quotient.

Proposition 32. *Let (V,q) be a D-hermitian space of D-dimension 2. Then $\mathrm{GU}(V,q)$ has compact automorphic quotient if and only if F admits a real place at which D is ramified and q is positive definite or negative definite.*

Proof. By a criterion of Borel and Harish-Chandra [13], the adelic quotient of G is compact if and only if G modulo its center is anisotropic, *i.e.* contains no nontrivial split torus. This is equivalent to (V,q) being anisotropic, *i.e.* q admitting no nonzero isotropic vector. Over a number field, by the local-global principle for quadratic forms it is equivalent to having local anisotropy at one place. For p-adic places, q has 8 variables as a quadratic form and is therefore isotropic by a result of Waring [114]. Over real places, (V,q) is anisotropic if and only if q is positive definite or negative definite. □

This proposition provides many examples of inner forms of $\mathrm{GSp}(4)$ of compact automorphic quotient, for instance the group of isometries of the D-hermitian form $x\bar{x}+y\bar{y}$ provides an example over a totally definite quaternion algebra. For almost every places v, more precisely the split ones, this group is $G_v \simeq \mathrm{GSp}(4,F_v)$. Denote by S the finite set of places where this does not happen. At these places, Ichino and Prasanna [62, Section 2.1] recall what can happen to the underlying quaternionic hermitian spaces:

- for finite places, there is a unique hermitian space giving the unique non-trivial inner form of $\mathrm{GSp}(4)$, and it is isotropic. This happens at places where D is ramified;

- for archimedean places, the dimension and signature parametrize the possible groups, up to permutation of the signature, and only the hermitian space of signature $(n, 0)$ is anisotropic;
- the Hasse principle holds in this case, without obstruction: the quaternionic hermitian spaces over a global field are in bijection with the families of local ones.

An unfortunate fact is that not every representation of the non-trivial inner forms, that arise at a finite number of places, is supercuspidal. This forbids the use of matrix coefficients since they act as a selecting function for isomorphism classes of supercuspidal representations only. The representations of inner forms of $\mathrm{GSp}(4)$ have been classified recently by Gan and Tantoto [50]. It can be expected that a solution is to use a smoothed version of such selecting functions for the non-supercuspidal representations given by existence theorems of Paley-Wiener type [24, 75], analogously to the treatment of split archimedean places for quaternion algebras. Even though, this is a non-trivial question that ought to be handled with caution, so from now on the considered family is $\mathcal{A}_{\mathrm{sc}}(G)$ the D-supercuspidal part of the universal family of G, that is to say non-characters automorphic representations of G that are supercuspidal at places where D is ramified. For this reason, $\widehat{G}_v$ stands for the supercuspidal dual of G_v at ramified places.

5.3.2 Analytic conductor

Following the lines of the cases of quaternion algebras or unitary groups, the notion of conductor coming from L-functions is hard to reach when it comes to explicit computations, so that it is suitable to define it as a notion of depth as in Sections 2.2.3 and 5.1.3. The tensor product theorem allows to define the conductor locally. For finite places in S, where $G_\mathfrak{p}$ is isomorphic to the full symplectic group $\mathrm{GSp}(4, F_\mathfrak{p})$, Roberts and Schmidt constructed a filtration for which a theory of local newforms stands, provided by the paramodular groups

$$K(\mathfrak{p}^n) = \begin{pmatrix} \mathcal{O} & & & \mathfrak{p}^{-n} \\ \mathfrak{p}^n & & & \\ \mathfrak{p}^n & & & \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathcal{O} \end{pmatrix}, \qquad n \geqslant 0, \tag{5.46}$$

where the empty entries stand for integers in $\mathcal{O}$. The first elements $K(\mathfrak{p}^0)$ and $K(\mathfrak{p}^1)$ of this sequence are the two maximal compact subgroups of $\mathrm{GSp}(4, F_\mathfrak{p})$ up to conjugacy. Roberts and Schmidt established a comprehensive study of local newforms in this setting, stating in particular the following result [102, Generic Main Theorem 7.5.4].

Proposition 33 (Roberts-Schmidt)**.** *Let π be a generic irreducible admissible representation of* $\mathrm{GSp}(4, F_\mathfrak{p})$ *with trivial central character. Then there exists an integer n such that π admits nonzero fixed vectors by $K(\mathfrak{p}^n)$.*

As for quaternion algebras in Section 2.2 or unitary groups in Section 5.1.3, the least n for which it happens defines a notion of local depth-conductor, also called paramodular level. As for the general linear group setting, this notion of conductor matches the one coming from the ε-factor appearing in the functional equation satisfied by the L-function associated to the representation [102, Corollary 7.5.5].

Roberts and Schmidt provide in particular the volumes of these [102, Lemma 3.3.3] paramodular subgroups. Assuming the Haar measure is normalized so that $K(\mathcal{O}_\mathfrak{p})$ gets measure one, the volumes are

$$\mathrm{vol}\,(K(\mathfrak{p}^n)) = (1 + N\mathfrak{p}^{-2})N\mathfrak{p}^{2n}, \qquad n \geqslant 0. \tag{5.47}$$

The global prime-to-S paramodular subgroup is defined as the product of the corresponding paramodular subgroups at local places, that is to say

$$K(\mathfrak{q}) = \prod_{\mathfrak{p}^n || \mathfrak{q}} K(\mathfrak{p}^n). \tag{5.48}$$

The global prime-to-S volume is hence given by, for a $\mathfrak{q}$ prime to S,

$$\mathrm{vol}\,(K(\mathfrak{q})) = \prod_{\mathfrak{p}^n || \mathfrak{q}} (1 + N\mathfrak{p}^{-2})N\mathfrak{p}^{2n} = \psi(\mathfrak{q}), \tag{5.49}$$

where $\psi = \mathrm{id}^2 \star \mu$ is the generalized Dirichlet ψ-function of level 2.

5.3.3 Theory of local newforms

Decomposing the universal family of G by fixing discrete spectral data, more precisely the conductor at finite places out of S and the class of isomorphism of representations at places in S, the sought cardinality of the universal family rewrites as

$$N(Q) = \sum_{\substack{\pi \in \mathcal{A}_{\mathrm{sc}}(G) \\ c(\pi) \leqslant Q}} 1 = \sum_{\substack{N\mathfrak{q} \leqslant Q \\ \mathfrak{q} \wedge S = 1}} \sum_{\substack{\pi_S \in \widehat{G}_S \\ c(\pi_S) \leqslant Q/N\mathfrak{q}}} \sum_{\substack{\sigma \in \mathcal{A}_{\mathrm{sc}}(G) \\ c(\sigma^S) = \mathfrak{q} \\ \sigma_S \simeq \pi_S}} 1. \tag{5.50}$$

The innermost sum corresponds to the number of representations of fixed prime-to-S conductor and fixed S-part, that is

$$A(\mathfrak{q}, \pi_S) = \sum_{\substack{\sigma \in \mathcal{A}_{\mathrm{sc}}(G) \\ c(\sigma^S) = \mathfrak{q} \\ \sigma_S \simeq \pi_S}} 1. \tag{5.51}$$

As for quaternion algebras or unitary groups, the trace formula naturally weights the representations with their spectral multiplicities defined by

$$m\,(\pi, \mathfrak{q}) = \dim\ (\pi)^{K(\mathfrak{q})} = \prod_{\mathfrak{p}^n || \mathfrak{q}} m(\pi_\mathfrak{p}, \mathfrak{p}^n). \tag{5.52}$$

The key fact is that the multiplicities only depend on the paramodular level, or depth-conductor, and not on the type of the representation involved, hence allowing to sieve in order to express the newforms counting number in function the oldforms one. Introduce the number of representations in the truncated universal family weighted by the spectral multiplicities, or number of oldforms,

$$B(\mathfrak{q}, \pi_S) = \sum_{\substack{\sigma \in \mathcal{A}_{\mathrm{sc}}(G) \\ c(\sigma^S) | \mathfrak{q} \\ \sigma_S \simeq \pi_S}} m(\sigma, \mathfrak{q}). \tag{5.53}$$

The dimension of the fixed vector spaces by those paramodular subgroups are given by Roberts and Schmidt [102, Generic Oldform Theorem 7.5.6].

Proposition 34 (Roberts-Schmidt). *Let π be a generic irreducible admissible representation of $\mathrm{GSp}(4, F_\mathfrak{p})$ with trivial central character. Let $\mathfrak{f}(\pi_\mathfrak{p})$ be the paramodular level of $\pi_\mathfrak{p}$. Then,*

$$\dim \pi_\mathfrak{p}^{K(\mathfrak{p}^n)} = \left\lfloor \frac{(n - \mathfrak{f}(\pi_\mathfrak{p}) + 2)^2}{4} \right\rfloor, \qquad n \geqslant \mathfrak{f}(\pi). \tag{5.54}$$

This leads to rewriting the number of oldforms as

$$B(\mathfrak{q}, \pi_S) = \sum_{\mathfrak{d} | \mathfrak{q}} m\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right) A(\mathfrak{d}, \pi_S), \tag{5.55}$$

thus, by Möbius inversion,

$$A(\mathfrak{q}, \pi_S) = \sum_{\mathfrak{d} | \mathfrak{q}} \tilde{m}\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right) A(\mathfrak{d}, \pi_S), \tag{5.56}$$

where $\tilde{m}$ is the convolution inverse of m. This yields a reformulation of the cardinality of the universal family as

$$N(Q) = \sum_{\substack{\pi \in \mathcal{A}_{\mathrm{sc}}(G) \\ c(\pi) \leqslant Q}} 1 = \sum_{\substack{N\mathfrak{q} \leqslant Q \\ \mathfrak{q} \wedge S = 1}} \sum_{\substack{\pi_S \in \widehat{G}_S \\ c(\pi_S) \leqslant Q / N\mathfrak{q}}} \sum_{\mathfrak{d} | \mathfrak{q}} \tilde{m}\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right) B(\mathfrak{d}, \pi_S). \tag{5.57}$$

5.4 Counting law for symplectic groups

5.4.1 Counting law at a glance

The aim of the present section is the interpret the number $B(\mathfrak{q}, \pi_S)$ as a spectral side of a trace formula for a suitable test-function $\Phi = \Phi_{\mathfrak{q}, \pi_S}$. A suitable test function is obtained by taking $\Phi_{\mathfrak{p}} = \varepsilon_{K(\mathfrak{p}^n)}$ for $\mathfrak{p}^n || \mathfrak{q}$ at prime-to-S places $\mathfrak{p}$, and the matrix coefficient $\Phi_{\pi_v} = \xi_{\pi_v}$ as the S-places v. Assuming the main term comes from the identity contribution,

$$B(\mathfrak{q}, \pi_S) = J_{\text{spec}}(\Phi) = J_{\text{geom}}(\Phi) \sim J_1(\Phi), \tag{5.58}$$

where

$$J_1(\Phi) = \text{vol}(G(F)\backslash G(\mathbf{A}))\Phi(1). \tag{5.59}$$

5.4.2 Identity contribution

Analogously to the previous cases and as a general guideline for conjectures around the trace formula, the main contribution to the geometrical side is expected to be the one coming from the identity. Using the preliminaries values and normalizing the measures so that $\text{vol}(K(\mathfrak{p}^0)) = 1$,

$$\begin{aligned}\Phi(1) = \Phi_S(1)\Phi^S(1) &= \text{vol}\left(K(\mathfrak{q})\right)^{-1} \xi_{\pi_S}(1) \\ &= \psi(\mathfrak{q})\mu_S^{\text{Pl}}(\pi_S).\end{aligned}$$

Summing over the discrete spectral parameters, that is the supercuspidal S-parts and the prime-to-S conductors, introduce the identity contribution to the cardinality of the universal family,

$$\begin{aligned}\text{vol}(G(F)\backslash G(\mathbf{A}))^{-1} N_1(Q) &= \sum_{\substack{N\mathfrak{q} \leqslant Q \\ \mathfrak{q} \wedge S = 1}} \sum_{\substack{\pi_S \in \widehat{G}_S \\ c(\pi_S) \leqslant Q/N\mathfrak{q}}} \sum_{\mathfrak{d} | \mathfrak{q}} \widetilde{m}\left(\frac{\mathfrak{q}}{\mathfrak{d}}\right) \psi(\mathfrak{d}) \mu_S^{\text{Pl}}(\pi_S) \\ &= \sum_{\substack{N\mathfrak{m} \leqslant Q \\ \mathfrak{m} \wedge S = 1}} \widetilde{m}(\mathfrak{m}) \sum_{\substack{\pi_S \in \widehat{G}_S \\ c(\pi_S) \leqslant Q/N\mathfrak{m}}} \mu_S^{\text{Pl}}(\pi_S) \sum_{\substack{N\mathfrak{d} \leqslant Q/N\mathfrak{m}c(\pi_S) \\ \mathfrak{d} \wedge S = 1}} \psi(\mathfrak{d})\end{aligned}$$

Standard computations lead to estimate the innermost sum in the following lemma.

Lemma 27. *For every $X > 0$,*

$$\sum_{\substack{N\mathfrak{d} \leqslant X \\ \mathfrak{d} \wedge S = 1}} \psi(\mathfrak{d}) = \frac{1}{3} \frac{\zeta^{S\star}(1)}{\zeta^S(3)} X^3 + O(X^{3-\delta_F}). \tag{5.60}$$

Proof. Decomposing the sum over ideals, and summing first over split places in order to use estimates on sums of arithmetic functions [17, equation (5.2)] yields

$$\sum_{\substack{N\mathfrak{d}^S \leqslant X \\ \mathfrak{d}^S \wedge S = 1}} \psi(\mathfrak{d}^S) = \sum_{\substack{N\mathfrak{l}^S \leqslant X \\ \mathfrak{l}^S \wedge S = 1}} \mu(\mathfrak{l}^S) \sum_{\substack{N\mathfrak{m}^S \leqslant X/N\mathfrak{l} \\ \mathfrak{m}^S \wedge S = 1}} \left(N\mathfrak{m}^S\right)^2$$

$$
\begin{aligned}
&= \sum_{\substack{N\mathfrak{l}^S \leqslant X \\ \mathfrak{l}^S \wedge S = 1}} \mu(\mathfrak{l}^S) \left[\frac{\zeta^{S\star}(1)}{3} \frac{X^3}{(N\mathfrak{l}^S)^3} + O\left(\left(\frac{X}{N\mathfrak{l}^S} \right)^{3-\delta_F} \right) \right] \\
&= \frac{1}{3} \zeta^{S\star}(1) X^3 \sum_{\substack{N\mathfrak{l}^S \leqslant X \\ \mathfrak{d}^S \wedge S = 1}} \frac{\mu(\mathfrak{l}^S)}{(N\mathfrak{l}^S)^2} + O\left(X^{3-\delta_F} \sum_{\substack{N\mathfrak{l}^S \leqslant X \\ \mathfrak{l}^S \wedge R = 1}} \frac{\mu(\mathfrak{l}^S)}{(N\mathfrak{l}^S)^{3-\delta_F-\varepsilon}} \right) \\
&= \frac{1}{3} \frac{\zeta^{S\star}(1)}{\zeta^S(3)} X^3 + O(X^{3-\delta_F}) \quad \square
\end{aligned}
$$

Inputing these estimations into the computations for the identity term leads to a main term equal to

$$
\operatorname{vol}(G(F)\backslash G(\mathbf{A}))^{-1} N_1^{(p)}(Q) \sim \frac{1}{3} \frac{\zeta^{S,\star}(1)}{\zeta^S(3)} Q^3 \sum_{\substack{N\mathfrak{m} \leqslant Q \\ \mathfrak{m} \wedge S = 1}} \frac{\widetilde{m}(\mathfrak{m})}{\mathfrak{m}^3} \sum_{\substack{\pi_S \in \widehat{G}_S \\ c(\pi_S) \leqslant Q/N\mathfrak{m}}} \frac{\mu_S^{\mathrm{Pl}}(\pi_S)}{c(\pi_S)^3}. \tag{5.61}
$$

Provided the inner sum and the Dirichlet series associated to $\widetilde{m}$ converge, it follows

$$
\begin{aligned}
\operatorname{vol}(G(F)\backslash G(\mathbf{A}))^{-1} N_1^{(p)}(Q) &= \frac{1}{3} \frac{\zeta^{S,\star}(1)}{\zeta^S(3)} Q^3 \int_{\widehat{G}_S} \frac{\mathrm{d}\mu_S^{\mathrm{Pl}}(\pi_S)}{c(\pi_S)^3} \sum_{N\mathfrak{m} \leqslant Q} \frac{\widetilde{m}(\mathfrak{m})}{N\mathfrak{m}^3} \\
&= \frac{1}{3} \frac{\zeta^{S,\star}(1)}{\zeta^S(3)} Q^3 \int_{\widehat{G}_S} \frac{\mathrm{d}\mu_S^{\mathrm{Pl}}(\pi_S)}{c(\pi_S)^3} D(\widetilde{m}, 3)
\end{aligned}
$$

where $\tilde{m}$ denotes the inverse of the multiplicity m with respect to the convolution. The remainder is bounded by, assuming the convergence of relevant series

$$
\operatorname{vol}(G(F)\backslash G(\mathbf{A}))^{-1} N_1^{(e)}(Q) \ll \frac{1}{3} \frac{\zeta^{S,\star}(1)}{\zeta^S(3)} Q^{3-\delta_F} \int_{\widehat{G}_S} \frac{\mathrm{d}\mu_S^{\mathrm{Pl}}(\pi_S)}{\mathfrak{c}(\pi_S)^{3-\delta_F}} \sum_{\substack{N\mathfrak{m} \leqslant Q \\ \mathfrak{m} \wedge S = 1}} \frac{\widetilde{m}(\mathfrak{m})}{N\mathfrak{m}^{3-\delta_F}}. \tag{5.62}
$$

Since these inner sums converge by the precise knowledge of the multiplicities, it follows the asymptotic development of the identity contribution to the geometrical side of the trace formula applied to Φ,

$$
N_1(Q) = \frac{1}{3} \frac{\zeta^{S,\star}(1)}{\zeta^S(3)} Q^3 \int_{\widehat{G}_S} \frac{\mathrm{d}\mu_S^{\mathrm{Pl}}(\pi_S)}{c(\pi_S)^3} D(\widetilde{m}, 3) + O(Q^{3-\delta_F}). \tag{5.63}
$$

Remark. As already mentioned for the case of unitary groups, the Dirichlet series are expected to have an abscissa of convergence equal to zero, for they are, in the case considered, logarithmic quantities in the conductor.

5.4.3 Geometric interpretation of the constant

The result above remains mysterious and unsatisfactory since the quantity $D(3, \widetilde{m})$ stays unfolded. Following the geometric interpretation of the constant for quaternion algebras or unitary groups, the very definition of $\varepsilon_{K(\mathfrak{p}^r)}$ yields

$$\varepsilon_{K(\mathfrak{p}^r)}(1) = \mathrm{vol}\,(K(\mathfrak{p}^r))^{-1} = \psi(\mathfrak{p}^r), \qquad r \geqslant 0. \tag{5.64}$$

Another expression of this quantity is provided by the Plancherel formula. Introduce the volume of the representations of fixed conductor

$$M_\mathfrak{p}(\mathfrak{p}^r) = \int_{\substack{\sigma_\mathfrak{p} \in \widehat{G}_\mathfrak{p} \\ \mathfrak{c}(\sigma_\mathfrak{p}) = \mathfrak{p}^r}} \mathrm{d}\sigma_\mathfrak{p}, \qquad r \geqslant 0. \tag{5.65}$$

The explicit transform of $\varepsilon_{K(\mathfrak{p}^r)}$ is known and leads to

$$\begin{aligned}\varepsilon_{K(\mathfrak{p}^r)}(1) &= \int_{\widehat{G}_\mathfrak{p}} \widehat{\varepsilon}_{K(\mathfrak{p}^r)}(\pi_\mathfrak{p})\, \mathrm{d}\pi_\mathfrak{p} = \int_{\widehat{G}_\mathfrak{p}} m\left(\frac{\mathfrak{p}^r}{\mathfrak{c}(\pi_\mathfrak{p})}\right) \mathrm{d}\pi_\mathfrak{p} \\ &= \sum_{\mathfrak{d} \mid \mathfrak{p}^r} M_\mathfrak{p}(\mathfrak{d}) m\left(\frac{\mathfrak{p}^r}{\mathfrak{d}}\right) = (M_\mathfrak{p} \star m)(\mathfrak{p}^r)\end{aligned}$$

so that $M = \tilde{m} \star \psi$, what can be recast in the language of Dirichlet series associated to these arithmetic functions, so that $D(M, s) = D(\tilde{m}, s)D(\psi, s)$ on a common domain of convergence. It follows that, for $\mathrm{Re}(s)$ sufficiently large,

$$D(M^S, s) = D(\tilde{m}^S, s)\frac{\zeta^S(s-2)}{\zeta^S(s)}, \tag{5.66}$$

thus, using the knowledge of the growth order by the computations above, evaluating at $s = 3$ gives

$$\int_{\widehat{G}^S} \frac{\mathrm{d}\pi^S}{c(\pi^S)^3} = D(M^S, 3) = D(\tilde{m}^S, 3)\frac{\zeta^{S,\star}(1)}{\zeta^S(3)}. \tag{5.67}$$

This leads to translating the constant in more geometrical terms, and no more showing different treatment depending on the type of place considered, taking the same regularized product as for quaternion algebras or unitary groups. Finally, the identity contribution is

$$N_1(Q) = \frac{1}{3}\mathrm{vol}([G]) \int_{A_{\mathrm{sc}}(G)}^{\star} \frac{\mathrm{d}\pi}{c(\pi)^3} Q^3 + O(Q^{3-\delta_F}). \tag{5.68}$$

These computations and a similar discussion than the one held in Section 5.2.4 reveal strong evidences towards the following conjecture.

Conjecture I (with I. Petrow)**.** *The asymptotic development of the cardinality of the D-supercuspidal universal family of G is given by*

$$N(Q) \sim \frac{1}{3}\mathrm{vol}([G]) \int_{A_{\mathrm{sc}}(G)}^{\star} \frac{\mathrm{d}\pi}{c(\pi)^3} Q^3, \qquad \textit{as} \quad Q \to \infty. \tag{5.69}$$

Annexe A

Statistiques arithmétiques sur les algèbres de quaternions

Conformément à l'arrêté du 25 mai 2016 fixant le cadre national de la formation et les modalités conduisant à la délivrance du diplôme national de doctorat, la rédaction de la présente thèse doit être complétée d'un résumé substantiel en langue officielle. Ce chapitre constitue ledit résumé et consiste en la traduction condensée du premier chapitre, introduction à la thèse.

Dans le filon de l'effervescence récente autour des familles de formes automorphes, ce chapitre introduit les problèmes de statistiques arithmétiques pour la famille universelle des algèbres de quaternions. Cette exposition du cadre se poursuit par l'énoncé d'une loi de comptage pour la famille universelle tronquée par un paramètre pertinent : le conducteur analytique. Des statistiques plus précises sont fournies à travers la notion d'équirépartition de la famille universelle par rapport à une mesure à forte teneur géométrique. Ce résultat débouche sur une réponse affirmative aux conjectures de Sato-Tate, concernant l'équirépartition des composantes locales pour cette famille. Enfin, la distribution des petits zéros des fonctions L associées est étudiée, et la conjecture de densité de Katz-Sarnak partiellement vérifiée. Cela permet de déterminer le type de symétrie des algèbres de quaternions ainsi que des résultats de densité de non-annulation des fonctions L associées au point central.

Contents

A.1 Paysage automorphe

A.1.1 Famille universelle

Les formes automorphes et leurs fonctions L paraissent aujourd'hui comme des objets centraux en théorie des nombres depuis les révolutionnaires conjectures de Langlands [80] et leurs différents avatars, ainsi les courbes elliptiques [35] ou les formes modulaires [108]. En dépit de leur omniprésence, elles demeurent des objets mystérieux. Plutôt que de considérer ces objets a priori très singuliers, les considérer au sein de familles a un effet de lissage : les comportements dissonants ou inaccessibles perdent du poids et cela mène à des résultats en moyenne, en d'autres termes sur des formes automorphes typiques. Voilà l'esprit des statistiques arithmétiques.

The recent years unleashed a wide enthusiasm toward the study of families of automorphic forms and their associated L-functions. Understanding what makes a family relevant for this philosophy is a critical issue. General attempts to define a suitable notion of family of automorphic forms have been made in the recent years [109, 110, 77], with a particular emphasis towards the universal family of a group, consisting of all its cuspidal automorphic representations.

Ces dernières années ont déchaîné un enthousiasme conséquent pour l'étude des familles de formes automorphes et des fonctions L associées. Des tentatives pour définir une notion raisonnable de famille de formes automorphes ont été faites [109, 110, **?**, **?**], toutes soulignant l'importance de la famille universelle associée à un groupe, constituée de toutes ses représentations automorphes.

Étant donnée une famille $\mathcal{F}$ d'objets, une première question naturelle relative à cette famille est de déterminer sa taille. Pour les familles infinies, pourvu qu'il y ait un moyen de la tronquer en des familles finies $\mathcal{F}_Q$ l'approchant, cela peut être quantifié par la vitesse de croissance de la famille tronquée $\mathcal{F}_Q$ par rapport au paramètre de troncature Q.

Le groupe général linéaire est le terrain fondamental des représentations automorphes, et $\mathrm{GL}(2)$ en est la première instance non commutative, pourtant déjà loin d'être pleinement maîtrisée. Une manière d'explorer certaines de ses propriérés est de considérer ses formes intérieures : ce sont les groupes des unités des algèbres de quaternions. Soit une algèbre de quaternion B sur un corps de nombres F, et soit G le groupe projectif de ses unités, *i.e.* $G = Z\backslash B^\times$. Soit $\mathcal{A}(G)$ la famille universelle de G, autrement dit la collection de toutes les représentations automorphes sur $G(\mathbf{A})$ de dimension infinie. Suivant les mots de Sarnak [109], une profonde compréhension de $\mathcal{A}(G)$ est capitale en théorie des formes automorphes.

En guise d'analogie et de guide pour les méthodes, considérons un instant une situation plus usuelle : la famille universelle $\mathcal{A}(G)$ se réalise, par la correspondance de Jacquet-Langlands, comme une sous-famille de la famille universelle $\mathcal{A}(\mathrm{PGL}(2))$, constituée de toutes les représentations autormorphes cuspidales de PGL(2). Dans ce dernier contexte, et même dans le cadre plus large des formes cuspidales des groupes linéaires, Iwaniec et Sarnak [65] ont introduit une bonne notion de taille, donnée par le conducteur analytique. Il s'agit d'un réel positif $c(\pi)$ défini à partir de l'équation fonctionnelle satisfaite par la partie finie de la fonction L associée à $\pi \in \mathcal{A}(\mathrm{PGL}_2)$, qui est de la forme

$$L(1-s,\pi) = \varepsilon_\pi X_L(s) L(s,\pi), \tag{A.1}$$

où ε_π est le nombre de racines de π. Le facteur complémentaire X_L prend la valeur 1 au point central $\frac{1}{2}$, et le conducteur analytique additif est défini comme $c(\pi) = |X_L'(\frac{1}{2})|$, suivant [29]. De plus amples discussions autour du conducteur analytique et de ses différents avatars sont menées en Section 2.2.

La fonction X_L est construite à partir des facteurs nécessaires pour compléter $L(s,\pi)$ de sorte à obtenir une équation fonctionnelle symétrique, et inclut le conducteur arithmétique ainsi que les facteurs gammas, de sorte que le conducteur analytique embrasse la complexité de π. Cela permet de tronquer la famille universelle de PGL(2), partant celle de G, en un ensemble fini [16]. Il est donc possible de considérer la famille universelle tronquée

$$\mathcal{A}(Q) = \{\pi \in A(G) \ : \ c(\pi) \leqslant Q\}. \tag{A.2}$$

Cette thèse s'attache à explorer quelques propriétés de cette famille, telles sa croissance asymptotique, son équirépartition par rapport à une certaine mesure, le comportement des composantes locales, ainsi que des statistiques sur les zéros des fonctions L associées.

A.1.2 Analogie avec la hauteur sur les variétés algébriques

Le problème du comptage et de l'équirépartition admettent une intéressante analogie avec le problème plus standard de comptage des points rationnels de hauteur bornée sur une variété algébrique lisse définie sur un corps de nombres. La hauteur absolue de Weil est la bonne notion de taille dans ce contexte et est définie par

$$h_{\mathbf{P}^n}(x) = \prod_v \max_i |x_i|_v, \qquad x = (x_i)_i \in \mathbf{P}^n(F). \tag{A.3}$$

Étant donnée une variété projective V munie d'un plongement ι dans l'espace projectif $\mathbf{P}^n$, une notion de hauteur sur V est obtenue en relevant la hauteur de Weil sur $\mathbf{P}^n$:

$$h_V(x) = h_{\mathbf{P}^n}(\iota(x)), \qquad x \in V. \tag{A.4}$$

Le contexte le plus naturel pour généralier de telles questions est celui des variétés de Fano, dans lequel des conjectures précises ont été formulées par Batyrev, Manin et Peyre [99]. Dans ce filon, Schanuel a établit une loi de comptage pour les points de hauteur bornée.

Théorème 1 (Schanuel)**.** *Il existe une constante $C_n > 0$ telle que pour tout $Q \geqslant 1$,*

$$\# \{x \in \mathbf{P}^n(F) \ : \ h(x) \leqslant Q\} = C_n Q^{n+1} + \begin{cases} O(Q \log Q) & \text{si } n = 1 \text{et } F = \mathbf{Q}; \\ O\left(Q^{n-1/[F:\mathbf{Q}]}\right) & \text{sinon.} \end{cases} \tag{A.5}$$

Dans les dernières années, Sarnak a régulièrement souligné la richesse de l'analogie entre le théorème de Schanuel sur le comptage de points sur les variétés projectives et le problème de comptage des formes automorphes cuspidales sur $\mathrm{GL}(n)$. Le cas des algèbres de quaternions peut être plongé dans $\mathrm{GL}(2)$ de sorte que, suivant cette analogie, la notion de conducteur analytique utilisée pour donner un sens précis au problème de comptage et à la troncature est inspirée par la procédure pour les hauteurs : étant donné la notion désormais standard de conducteur analytique sur $\mathrm{GL}(2)$, le conducteur analytique pour les algèbres de quaternions est obtenu en le relevant aux formes automorphes sur les algèbres de quaternions par le biais de l'application associée entre leurs groupes duaux.

A.1.3 Loi de comptage

Le premier résultat de cette thèse énonce une formule asymptotique pour le cardinal

$$N(Q) = \#\mathcal{A}(Q), \tag{A.6}$$

et est baptisé loi de Weyl-Schanuel par Sarnak. Le cas de la famille universelle de $\mathrm{GL}(2)$ est considéré dans une prépublication récente par Brumley et Milićević [17]. Pour les algèbres de quaternions, la loi de comptage est énoncée dans le théorème suivant.

Théorème A (Loi de comptage)**.** *Il existe une constante $C > 0$ telle que pour tout $Q \geqslant 1$,*

$$N(Q) = CQ^2 + \begin{cases} O\left(Q^{1+\varepsilon}Q\right) & \text{si } F = \mathbf{Q} \text{ et } B \text{ totalement définie;} \\ O\left(Q^{2-\delta_F}\right) & \text{si } F \neq \mathbf{Q} \text{ et } B \text{ totalement définie;} \\ O\left(\frac{Q^2}{\log Q}\right) & \text{si } B \text{ non totalement définie.} \end{cases} \tag{A.7}$$

La constante $C > 0$ est explicitement donnée en (1.10)*, et $\delta_F = 2(1 + [F : \mathbf{Q}])^{-1}$.*

Remarques. La forme de cette loi de croissance asymptotique appelle quelques commentaires.

(i) Il y a une similarité frappante entre le terme d'erreur du Théorème A et celui du résultat classique de Schanuel sur le nombre de points rationnels de hauteur bornée sur les espaces projectifs. Son résultat, lorsqu'il est spécialisé à $F = \mathbf{Q}$, présente aussi un terme d'erreur portant une puissance du logarithme additionnelle.

(ii) La présence d'un gain par une puissance dans le cas totalement défini, c'est-à-dire lorsque toutes les places archimédiennes sont ramifiées, est notable. La propriété est perdue sans cette hypothèse, ainsi que le résultat [17] dans le cadre de GL(2), où seule un gain logarithmique est obtenu. La raison de cette différence réside dans le passage du comptage lisse de la preuve au comptage réel, voir Section 3.2.5.

(iii) L'hypothèse que B est une algèbre à division implique un quotient automorphe compact, induisant en particulier une forme plus simple de la formule des traces utilisée dans la preuve, évitant les complications techniques provenant de termes supplémentaires peu maîtrisés, à savoir les séries de Eisenstein qui constituent le spectre continu, voir Section **??**.

(iv) L'algèbre de quaternion a été quotientée par son centre pour des raisons techniques, notamment de sorte à éviter à avoir à contrôler les termes centraux dans la formule des traces de Selberg. Voir 5 pour de plus amples informations et la mention de résultats lorsque le centre est considéré.

La connaissance précise de la constante C révèle de nombreuses informations, et son interprétation géométrique a une importance considérable. Une formulation explicite et riche de sens est donnée plus bas, dans le cadre des propriétés d'équirépartition de $\mathcal{A}(G)$.

A.2 Équirépartition

A.2.1 Notion d'équirépartition

Par-delà la détermination de la taille de la famille universelle demeure la question de la répartition géométrique des représentations automorphes de $G(\mathbf{A})$. Une formulation précise du problème est de trouver une mesure par rapport à laquelle la famille universelle s'équirépartit, ce à quoi est dédiée cette section en introduisant brièvement des structures topologique et mesurable dont la famille universelle est munie.

Chaque dual unitaire local $\widehat{G_v}$ est muni de la topologie de Fell et le produit $\prod_v \widehat{G_v}$ est muni de la topologie produit. Introduisons la mesure μ sur $\prod_v \widehat{G_v}$ qui assigne à chaque ouvert de base $X = \prod_v X_v$, *i.e.* où X_v est un ouvert de $\widehat{G_v}$ et $X_v = \widehat{G_v}$ pour presque tous les v, le réel positif

$$\mu(X) = \int_X^{\star} \frac{\mathrm{d}\pi}{c(\pi)^2}, \tag{A.8}$$

où l'intégrale régularisée est définie par

$$\zeta_F^{\star}(1) \prod_v \zeta_v(1)^{-1} \int_{X_v} \frac{\mathrm{d}\pi_v}{c(\pi_v)^2}. \tag{A.9}$$

Ici, ζ_v est la fonction zêta locale associée à F_v, $\zeta_F^{\star}(1)$ est le résidu de la fonction zêta de Dedekind de F en 1, et $\mathrm{d}\pi_v$ est la mesure de Plancherel sur $\widehat{G_v}$, normalisée selon les conventions adoptées en Section 2.3.2.

Remarques. (i) La mesure de Plancherel est supportée sur le dual tempéré ; puisque celles-ci sont génériques, le conducteur d'une composante π_v du support de la mesure de Plancherel est bien définie.

(ii) Il n'est nullement évident de voir que l'intégrale définissant μ converge. C'est effectivement le cas, comme prouvé en Section 3.2.4.

La mesure μ a une masse totale finie $\|\mu\|$. Il est donc possible de donner une expression pour la constante du Théorème A, à savoir

$$C = \frac{1}{2}\mathrm{vol}(G(F)\backslash G(\mathbf{A}))\|\mu\|, \tag{A.10}$$

où la mesure donnant le volume du quotient automorphe $G(F)\backslash G(\mathbf{A})$ est normalisée comme en Section 2.1.1.

La résultat principal est le suivant.

Théorème B (Équirépartition)**.** *La famille universelle de G s'équirépartit selon la mesure μ. Plus précisément, pour tout ouvert X relativement quasi-compact de $\prod_v \widehat{G_v}$ à bord de mesure nulle, lorsque $Q \to \infty$ on a*

$$\frac{\#\{\pi \in A(Q) : \pi \in X\}}{N(Q)} \longrightarrow \frac{\mu}{\|\mu\|}(X). \tag{A.11}$$

A.2.2 Conjectures de Sato-Tate

Une origine dans les courbes elliptiques

Poursuivant la fructueuse analogie avec les variétés algébriques mène à étendre les problèmes statistiques associés aux formes automorphes. Le cas le plus simple, quoique déjà bien riche, est celui des courbes elliptiques E définies sur $\mathbf{Q}$. Une telle courbe peut être définie par une équation $E : y^2 = x^3 + ax + b$, supposant pour ce paragraphe introductif que a et b sont entiers de sorte que la réduction E_p modulo p demeure lisse, autrement dit E_p est une courbe elliptique sur F_p.

Un intérêt fondamental réside dans l'étude de la courbe $E(\mathbf{F}_p)$. Retirer le facteur x^3 de l'équation donne un modèle simplifié dans lequel il y aurait $N_p(y^2 = ax+b) = p+1$ tels points, supposant que a est non nul et ajoutant un point à l'infini. Plus généralement, pour une courbe elliptique, le nombre de ses points rationnels modulo p peut être mis sous la forme

$$N_p(E) = \#E(\mathbf{F}_p) = p + 1 - a_p(E), \tag{A.12}$$

où $a_p(E)$ est la trace du Frobenius de E en p. En 1936, la borne de Hasse énonce que $|a_p(E)| \leqslant 2\sqrt{p}$, de sorte que $a_p(E)/\sqrt{p}$ est restreint à $[-2, 2]$. Il est alors possible d'écrire

$a_p(E) = 2\cos\theta_p(E)$, introduisant les angles de Frobenius associés $\theta_p(E)$, appartenant à $[0, \pi]$. Le problème de Sato-Tate concerne la distribution des traces et angles de Frobenius. En 1936, basé sur les résultats numériques de Sato, Tate suggère la conjecture suivante.

Conjecture 1 (Sato-Tate)**.** *Pour une courbe elliptique E non CM, les $a_p(E)$ (resp. les $\theta_p(E)$) s'équirépartissent dans $[-2, 2]$ (resp. dans $[0, \pi]$) selon la mesure donnée par le demi-cercle :*

$$\mu^{\mathrm{ST}} = \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}dx, \qquad resp.\ \mu'^{,\mathrm{ST}} = \frac{2}{\pi}\sin^2\phi d\phi. \tag{A.13}$$

La conjecture de Sato-Tate pour les courbes elliptiques a été prouvée en 2006 par Clozel, Harris, Shepherd-Barron et Taylor sous l'hypothèque de le j-invariant de E n'est pas entier, ce qui implique en particulier E non CM. Dans le cas exceptionnel des courbes CM, la moitié des $a_p(E)$ s'annule et la répartition limite est connue.

Conjecture de Sato-Tate automorphe

Les travaux de Taylor et Wiles [126] ont révélé qu'une courbe elliptique correspond à une forme modulaire de poids 2 et à coefficients entiers sur $\mathrm{GL}(2, \mathbf{A})$. Plus généralement, pour les formes modulaires cuspidales de poids k, la conjecture de Ramanujan, connue pour $\mathrm{GL}(2)$ par un résultat de Deligne [33], est l'analogue de la borne de Hasse et énonce que $|a_p(f)| \leqslant 2\sqrt{p}^{k-1}$. Cela mène à une généralisation naturelle du problème de Sato-Tate dans le cadre automorphe.

Conjecture 2 (Sato-Tate automorphe)**.** *Soit f une forme modulaire cuspidale non CM sur $\mathrm{SL}(2, \mathbf{Z})$, les $p^{-\frac{k-1}{2}} a_p(f)$ s'équirépartissent sur $[-2, 2]$ selon la mesure de Sato-Tate μ^{ST}.*

Le résultat déjà mentionné de Clozel, Harris, Shepherd-Barron et Taylor [5] prouve la conjecture dans le cas des formes modulaires à coefficients entiers de poids 2. En 2011, Barrett-Lamb, Geraghty, Harris et Taylor généralisent le résultat à toute forme modulaire holomorphe non CM de poids $k \geqslant 2$.

Conjectures verticales en familles

Des généralisations naturelles existent pour la conjecture de Sato-Tate automorphe. Au-delà de ne considérer qu'une unique forme automorphe pour laquelle certains résultats existent sous de fortes hypothèses, des résultats sont accessibles pour une famille entière de formes automorphes $\mathcal{F}$. Cela permet non seulement d'étudier le comportement des coefficients associés à une forme automorphe donnée comme dans les instances précédentes de la conjecture, appelées statistiques "horizontales", mais également de fixer un paramètre p et d'étudier les coefficients $a_p(\phi)$ pour ϕ parcourant $\mathcal{F}$. Ce sont les conjectures "verticales".

Conjecture 3 (Sato-Tate verticale)**.** *Pour un p fixé, les $(a_p(\phi))_{\phi\in\mathcal{F}}$ s'équirépartissent selon une mesure μ_p, qui converge vers μ^{ST} lorsque p croît. À condition d'ajouter des poids non-uniformes aux formes automorphes, à savoir $Res_{s=1}L(s,\phi\times\overline{\phi})^{-1}$, l'équirépartition a lieu selon la mesure de Sato-Tate μ^{ST}.*

De telles familles ont été considérées dans les travaux de Bruggeman [15], Sarnak [107], Conrey-Duke-Farmer [30] et Serre [116].

Vers les représentations automorphes

Les formes automorphes donnent naissance à des représentations automorphes, menant à généraliser une fois de plus les conjectures. Sarnak, Shin et Templier [110] ont essayé d'englober les tentatives récentes de définition d'une famille de forme automorphe et de donner un sens aux conjectures de Sato-Tate dans ce contexte. Une représentation automorphe $\pi \in A(G)$ admet une décomposition en produit restreint $\pi = \otimes_v \pi_v$ de facteurs locaux. Puisque seul un nombre fini des π_v sont ramifiées, pour π fixée et un p assez grand, les composantes locales π_v peuvent être identifiées à leurs paramètres de Satake

$$\pi_v \cong (\alpha_1(v),\dots,\alpha_n(v)) \in T_c/W, \tag{A.14}$$

où T_c est un tore complexe et W est le groupe de Weyl associé. Le problème de Sato-Tate consiste à déterminer la répartition des composantes locales π_v, soit en fixant π et en faisant varier v, explorant le problème horizontal, soit en fixant v et en laissant π varier dans une famille donnée, ce qui est le problème vertical. Sarnak, Shin et Templier ont formulé précisément cette conjecture pour les familles.

Conjecture 4. *La famille $\mathcal{F}$ est équirépartie dans $\widehat{\Pi}$ selon la mesure $\mu(\mathcal{F})$ telle que*

(i) c'est une mesure de probabilité supportée sur le dual tempéré de G
(ii) la moyenne sur p pondérée logarithmiquement existe et converge vers la mesure de Sato-Tate

$$\frac{1}{x}\sum_{p\leqslant x}\log(p)\mu_p(\mathcal{F})_{|T} \longrightarrow \mu^{\mathrm{ST}}(\mathcal{F}) \tag{A.15}$$

Shin et Templier ont récemment prouvé une version quantitative précise de cette conjecture dans un contexte très général pour des familles de représentations automorphes qui sont des séries discrètes à l'infini. De plus, aucun résultat de ce type n'a été énoncé lorsque à la fois objets et paramètres varient.

Revenant à la famille universelle des algèbres de quaternions, une fois énoncé le Théorème B sur l'équirépartition, la conjecture de Sato-Tate prédit le comportement des projections $\mu_{\mathfrak{p}}$ de la mesure limite sur les composantes locales $\widehat{G_{\mathfrak{p}}}$ lors la norme de $\mathfrak{p}$ croît. Dans le cadre commun dans lequel vivent les représentations dans le support des mesures de Plancherel sur $G_{\mathfrak{p}}$, à savoir l'espace des paramètres de Satake T_c/W, l'équirépartition de Sato-Tate peut être formulée précisément.

Corollaire C. *Pour toute $\phi \in C(T_c/W)$, lorsque $N\mathfrak{p}$ croît indéfiniment,*

$$\int_{T_c/W} \widehat{\phi}(x)\, \mathrm{d}\mu_{\mathfrak{p}}(x) \longrightarrow \int_{T_c/W} \widehat{\phi}(x)\, \mathrm{d}\mu^{\mathrm{ST}}(x). \tag{A.16}$$

A.3 Petits zéros de fonctions L

A.3.1 Importance des zéros des fonctions L

Les fonctions L, parmi lesquelles l'éminente fonction zêta de Riemann, sont omniprésentes en théorie des nombres et sont un moyen analytique d'embrasser des propriétés d'objets arithmétiques. Leurs zéros, bien que demeurant essentiellement mystérieux, regorgent d'informations capitales concernant la distribution des nombres premiers, et plus généralement la nature des objets auxquels elles sont attachées. Ces informations sont notamment codées dans la localisation de ces zéros, justifiant les efforts et l'intérêt considérables envers l'hypothèse de Riemann et les problèmes associés.

En effet, les soi-disantes formules explicites tissent un lien entre les répartitions de zéros et les répartitions de quantités de nature arithmétique. Une motivation pour cet intérêt envers l'étude statistique des zéros de fonctions L est synthtétisée par Mazur [87], qui souligne que les formules explicites donnent une relation de la forme

$$\pi(x) = \mathrm{MT} + \mathrm{ET} + \mathrm{OT}, \tag{A.17}$$

où $\pi(x)$ est une statistique sur les nombres premiers, MT est un terme principal provenant de zéros particuliers de la fonction L, ET est une somme sur les zéros triviaux et contribue comme un terme d'erreur, et OT est un terme oscillant venant des autres zéros. Ce dernier terme est difficile à estimer et requiert une connaissance suffisamment précise du comportement des zéros, de sorte à pouvoir détecter des compensations. Ainsi, en admettant l'hypothèse de Riemann qui postule que tout zéro non trivial de la fonction zêta se trouve sur la droite critique $\mathrm{Re}(s) = \frac{1}{2}$, le terme d'erreur dans le théorème des nombres premiers est considérablement amélioré :

$$\begin{aligned} \pi(x) &= \mathrm{li}(x) + O\left(xe^{-\alpha\sqrt{\log x}}\right), && \text{sans RH ;} \\ \pi(x) &= \mathrm{li}(x) + O\left(\sqrt{x}\log x\right), && \text{avec RH.} \end{aligned}$$

Ainsi, les statistiques sur les zéros de fonctions L portent d'a priori non triviaux résultats concernant l'arithmétique des objets sous-jacents, constituant une forte motivation pour leur étude.

A.3.2 Corrélations

Analogie entre matrices et fonctions L

La théorie des matrices aléatoires est une lunette à travers laquelle regarder le monde des statistiques de zéros de fonctions L. En effet, les angles propres de groupes de matrices aléatoires se comportent de manière étonnamment similaire à ces zéros et, puisque leurs contrées sont bien plus explorées, constituent un guide pour le monde des fonctions L automorphes. Soit $A \in M_n(F)$ une matrice unitaire diagonalisable, et soient $\lambda_A^{(j)} = e^{i\theta_A^{(j)}}$ ses valeurs propres classées de sorte que $0 \leqslant \theta_A^{(1)} \leqslant \cdots \leqslant \theta_A^{(n)} < 2\pi$. L'espacement moyen entre deux angles successifs est de $\frac{2\pi}{N}$, motivant la renormalisation

$$\tilde{\theta}_A^{(j)} := \frac{N}{2\pi}\theta_A^{(j)}. \tag{A.18}$$

De manière analogue, on peut associer à une fonction L, soit $L(s,\pi)$, ses zéros non triviaux $\rho_\pi^{(j)} = \frac{1}{2} + i\gamma_\pi^{(j)}$, avec a priori $\gamma_\pi^{(j)} \in \mathbf{C}$ sans supposer l'hypothèse de Riemann, et classés de sorte que $\cdots \leqslant \Re\gamma_\pi^{(-1)} \leqslant 0 \leqslant \Re\gamma_\pi^{(1)} \leqslant \Re\gamma_\pi^{(2)} \leqslant \cdots$. L'espacement moyen entre les zéros successifs[65] est de $\frac{\log T}{2\pi}$. Cela motive de le renormaliser à 1 en posant

$$\tilde{\gamma_j^\pi} := \frac{\log c(\pi)}{2\pi}\gamma_j^\pi. \tag{A.19}$$

Corrélation pour les matrices

Dans les années 50, Wigner a exploré les matrices aléatoires pour modéliser les phénomènes atomiques. Les matrices considérées sont dans l'Ensemble Gaussien Unitaire, noté GUE(N), autrement dit l'ensemble des matrices unitaires de taille N avec coefficients donnés par des lois gaussiennes indépendantes. Un moyen particulier de saisir le comportement des angles propres associés est d'étudier la répartition des espacement entre eux, autrement dit

$$R_2(A)[a,b] = \frac{1}{N}\{j \neq k \ : \ \tilde{\theta}_A^{(j)} - \tilde{\theta}_A^{(j)} \in [a,b]\}, \qquad A \in GUE(N). \tag{A.20}$$

Ces statistiques sont appelées corrélations de la famille. Dyson a déterminé la corrélation de GUE dans le résultat suivant.

Théorème 2 (Dyson)**.** *Il existe une mesure $r_2(GUE)$ telle que pour la famille des matrices aléatoires de l'Ensemble Gaussien Unitaire (GUE) et pour toute $\phi \in \mathcal{S}(\mathbf{R})$ telle que* $\mathrm{supp}(\hat{\phi}) \subseteq (-2,2)$,

$$\frac{1}{N}\sum_{A\in GUE(N)}\sum_{j\neq k}\phi(\tilde{\gamma}_A^{(j)} - \tilde{\gamma}_k^{(j)}) \underset{N\infty}{\longrightarrow} \int_{\mathbf{R}}\phi(x)r_2(GUE)(x)dx. \tag{A.21}$$

De plus, $R_2(GUE)[a,b] = \int_a^b r_2(GUE)(x)dx$ où $r_2(GUE)(x) = 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2$.

Katz et Sarnak ont prouvé plus généralement en 1997 que les espacements entre les valeurs propres de matrices aléatoires appartenant à des familles bien plus générales, à savoir les groupes de Lie irréductibles symétriques compacts, aussi appelés groupes classiques, sont également régis par la densité de répartition de GUE.

Théorème 3 (Katz-Sarnak)**.** *Pour toute famille $\mathcal{F}$ de groupe classique et pour $\phi \in \mathcal{S}(\mathbf{R})$ telle que* $\operatorname{supp}(\hat{\phi}) \subseteq (-2, 2)$,

$$\frac{1}{N} \sum_{A \in \mathcal{F}(N)} \sum_{j \neq k} \phi(\tilde{\gamma}_j - \tilde{\gamma}_k) \underset{N\infty}{\longrightarrow} \int_{\mathbf{R}} \phi(x) r_2(GUE)(x) dx. \tag{A.22}$$

Firk et Miller ont donné des arguments éclairant l'omniprésence de la densité de GUE dans les modélisation statistiques en physique. Les résultats recueillis ici suggèrent, suivant fidèlement la fructueuse analogie entre matrices et fonctions L, que la même universalité a lieu pour les statistiques portant sur les zéros de fonctions L.

Corrélations pour les fonctions L

Bien plus tard, Montgomery [94] a été le premier à explorer les lois de répartitions analogues pour les espacements entre zéros de fonctions L. Dans le cas particulier de la fonction zêta de Riemann, il a noté de manière surprenante en 1972 que la corrélation par paires entre zéros est la même que celle obtenue par Dyson pour les angles propres de matrices unitaires aléatoires.

Théorème 4 (Montgomery, 1974)**.** *Pour $\phi \in \mathcal{S}(\mathbf{R})$ and* $\operatorname{supp}(\hat{\phi}) \subseteq (-1, 1)$, *on a avec les notation précédentes :*

$$\frac{1}{N} \sum_{j \neq k} \phi(\tilde{\gamma}_j - \tilde{\gamma}_k) \underset{N\infty}{\longrightarrow} \int_{\mathbf{R}} \phi(x) r_2(GUE)(x) dx. \tag{A.23}$$

De nombreux calculs menés par Odlyzko ont par la suite apporté un fort soutien à la conjecture comme quoi de nombreuses statistiques sur les zéros de fonctions L se comportent comme leurs analogues pour les angles propres de matrices aléatoires dans GUE. Ce comportement universel est connu comme la *loi de Montgomery-Odlyzko*. Ces résultats ont été dès lors été peu à peu généralisés, culminant avec Rudnick et Sarnak [105] qui prouvent en 1995 que les statistiques sur GUE régissent les corrélations de familles de fonctions L sur $\mathrm{GL}(n)$. Le théorème suivant est restreint à $\mathrm{GL}(2)$ de sorte à servir de motivation suffisante pour les problèmes analogues pour les algèbres de quaternions et évitant ainsi d'introduire des conditions techniques, bien qu'il demeure moralement valable pour $\mathrm{GL}(n)$.

Théorème 5 (Rudnick-Sarnak, 1996)**.** *Soit π une représentation automorphe cuspidale de* $\mathrm{GL}(2, \mathbf{Q})$. *Soit ϕ une fonction paire de classe Schwartz telle que* $\operatorname{supp}(\widehat{\phi}) \subseteq (-1, 1)$. *Alors*

$$\frac{1}{N} \sum_{j \neq k} \phi(\tilde{\gamma}_j - \tilde{\gamma}_k) \underset{N\infty}{\longrightarrow} \int_{\mathbf{R}} \phi(x) r_2(GUE)(x) dx. \tag{A.24}$$

Le résultat prouvé par Rudnick et Sarnak est le premier pas dans la direction des conjectures analogues générales pour les fonctions L. En effet, les conjectures de fonctorialité de Langlands [26] postulent que toute fonction L se réalise comme une fonction L attachée à une représentation automorphe cuspidale sur le groupe linéaire. Le résultat de Rudnick et Sarnak valide ces conjectures statistiques pour ces cas "standard", y compris pour les corrélations de niveau supérieur.

A.3.3 Densités et type de symétrie

1-densité pour les matrices

En dépit de l'universalité surprenante de la loi GUE, les résultats précédents sont aussi décevants en ce qu'ils ne discriminent pas entre les différents groupes classiques, ce qui serait naturel. Il peut donc être attendu, à l'inverse de la loi universelle de Montgomery-Odlyzko, que d'autres statistiques sur les zéros de fonctions L puissent être capables de les distinguer. Une seconde insatisfaction vient de ce que les corrélations sont aveugles à de nombreuses modifications sur les zéros qui sont a priori conséquentes, ainsi la translation de tous les zéros et l'annulation ou non au point central, dont l'importance est bien connue en théorie des nombres.

Les corrélations considérées jusqu'à présent sont des statistiques sur *tous* les zéros, puisqu'elles ne prennent en compte que l'espacement entre eux. Katz et Sarnak ont mis un terme à cette universalité, s'intéressant à des statistiques concentrées sur les *petits* angles.

Définition 1. Soit ϕ une fonction Schwartz paire sur $\mathbf{R}$ et $A \in M_n(\mathbf{R})$. La 1-densité de A est

$$D(A,\phi) := \sum_{\theta_A^{(j)}} \phi(\tilde{\theta}_A^{(j)}), \tag{A.25}$$

Ici, ϕ décroît rapidement mais n'est plus supposée être une fonction des différences successives, comme pour les corrélations. Cette fois, les grands angles propres sont essentiellement coupés, faisant de $D(A,\phi)$ une densité sur les petits angles. De plus, $\widehat{\phi}$ est à support compact ce qui fait que ϕ s'étend analytiquement à tout le plan complexe.

Il n'y a plus d'espoir d'obtenir des estimations asymptotiques pour ces quantités, puisque les valeurs propres ou zéros ne sont plus en quantité infinie et leurs comportements peut être fort singulier. Guidés par la philosophie de la géométrie arithmétique, l'idée est de moyenner sur toute une famille de matrices ou de fonctions L.

Définition 2. Soit ϕ une fonction Schwartz paire sur $\mathbf{R}$. La 1-densité d'une famille $\mathcal{F}$ est

$$D(\mathcal{F},\phi) = \sum_{A\in\mathcal{F}} D(A,\phi). \tag{A.26}$$

Dans le contexte des matrices, Katz et Sarnak ont prouvé que la densité moyenne sur une famille diffère en fonction du groupe. Toutefois, les statistiques révèlent une certaine universalité des groupes classiques, et semblent toujours correspondre à celles de l'un des groupes classiques.

Théorème 6 (Katz-Sarnak, [68]). *Pour les groupes classiques $G(N)$, pour toute fonction Schwartz réelle ϕ de transformée de Fourier à support compact,*

$$D(G(N), \phi) \underset{N\infty}{\longrightarrow} \int_{\mathbf{R}_+} W_G(x)\phi(x)\,\mathrm{d}x, \tag{A.27}$$

où $\mathrm{d}A$ *est la mesure de Haar normalisée sur* $G(N)$ *et la densité de distribution est donnée par*

Groupe G	*Fonction de densité* W_G
U	1
USp	$1 - \frac{\sin 2\pi x}{2\pi x}$
$\mathrm{SO}(2N)$	$1 + \frac{\sin 2\pi x}{2\pi x}$
$\mathrm{SO}(2N+1)$	$1 - \frac{\sin 2\pi x}{2\pi x} + \delta_0$
O	$1 + \frac{1}{2}\delta_0$

Cette fonction W_G est la 1-densité pour $G(N)$. Le fait que la limite n'est plus universelle mais dépend de la famille associée à un nombre fini de groupes classiques mène à définir la notion de type de symétrie d'une famille de fonctions L, la densité limite révélant en un certain sens quel groupe classique gouverne le comportement des zéros.

Remark. Le résultat de Katz et Sarnak est vrai pour les groupes classiques, il demeure toutefois conjectural pour des familles plus générales, ce qui ferait des groupes classiques les représentants universels des différents types de symétrie existant pour les 1-densités.

Un problème fondamental pour prouver l'existence du type de symétrie est la nécessité d'un contrôle suffisamment fort pour pouvoir prouver la convergence de la 1-densité pour des fonctions dont la transformée de Fourier a un support compact arbitrairement grand. En effet, plus le support est grand, plus forte est la concentration de la fonction test sur les petits zéros. Cela peut être précisément formulé par les résultats de type Paley-Wiener qui énoncent une relation inverse entre le type exponentiel d'une fonction et la taille du support de sa transformée de Fourier – situation pour laquelle la gaussienne est un cas d'école éclairant. Puisque l'espoir est d'obtenir des statistiques aussi fines que possible sur les petits zéros, il est nécessaire de concentrer la fonction autant que possible dans un petit voisinage du point central, autrement dit d'autoriser le support de la transformée de Fourier à croître autant que possible.

Explorons plus avant ce qui se cache derrière les grands supports de transformées de Fourier de ces fonctions de densité. Par la formule de Plancherel,

$$\int_{\mathbf{R}} \phi(x)W(x)\,\mathrm{d}x = \int_{\mathbf{R}} \widehat{\phi}(x)\widehat{W}(x)\,\mathrm{d}x, \tag{A.28}$$

permettant à l'influence du support de la transformée de Fourier d'apparaître clairement. Les transformées de Fourier des fonctions de densité sont données dans la table suivante.

Groupe	Transformée de W
U	δ_0
USp	$\delta_0 - \frac{1}{2}\eta$
O(2N)	$\delta_0 + \frac{1}{2}\eta$
O(2N + 1)	$\delta_0 - \frac{1}{2}\eta + 1$
O(N)	$\delta_0 + \frac{1}{2}$

où η est la fonction réelle définie par

$$\eta(x) = \begin{cases} 1 & \text{si } |x| < 1 \\ \frac{1}{2} & \text{si } |x| = 1 \\ 0 & \text{si } |x| > 1 \end{cases}$$

Une observation fondamentale est que ces fonctions sont indistinguables lorsqu'elles sont restreintes à $(-1, 1)$. L'un des principaux objectifs de la littérature existante sur les densités est dédiée à briser cette barrière et permettre des supports dépassant $(-1, 1)$ de sorte à déterminer le type de symétrie de la famille. Cela fait intervenir systématiquement des contributions "non-diagonales". Cependant, Miller a donné une reformulation de la conjecture de densité et a montré qu'il suffit de déterminer la 2-densité pour des supports arbitrairement petits pour déterminer le type de symétrie.

1-densité pour les familles de fonctions L

Poursuivant l'analogie avec les matrices aléatoires, il est naturel d'espérer que la 1-densité des zéros attachée à toute famille "raisonnable" se comporte comme celles des matrices aléatoires dans les groupes classiques, et en particulier que le comportement des petits zéros de fonctions L calque celui des petits angles propres de matrices aléatoires.

Définition 3. Soit ϕ une fonction Schwartz paire sur $\mathbf{R}$ et $A \in M_n(\mathbf{R})$. La 1-densité attachée à A est

$$D(\pi, \phi) := \sum_{\gamma_\pi^{(j)}} \phi(\tilde{\theta}_A^{(j)}), \tag{A.29}$$

Définition 4. Soit ϕ une fonction Schwartz paire sur $\mathbf{R}$. La 1-densité de la famille $\mathcal{F}$ est

$$D(\mathcal{F}, \phi) = \sum_{\pi \in \mathcal{F}} D(\pi, \phi). \tag{A.30}$$

Le premier résultat dans cette direction est donné par Ozlük and Snyder [127] en 1993. Dès lors, une vaste littérature a été consacrée à l'étude statistique des petits zéros de familles de fonctions L. Cela a mené Katz et Sarnak à conjecturer que la même universalité des types de symétrie dégagée du cas des matrices aléatoires s'applique également pour les fonctions L.

Conjecture 5 (Conjecture de densité)**.** *Soit $\mathcal{F}$ une famille de représentations automorphes dans le sens de Sarnak et $\mathcal{F}(Q)$ une troncature finie croissant vers $\mathcal{F}$ lors Q tend vers l'infini. Alors, pour toute fonction Schwartz pair sur* $\mathbf{R}$ *avec une transformée de Fourier à support compact,*

$$D(\mathcal{F}(Q), \phi) \underset{Q\infty}{\longrightarrow} \int_{\mathbf{R}} \phi(x) W_G(x)\, \mathrm{d}x. \tag{A.31}$$

La famille $\mathcal{F}$ est dite avoir le type de symétrie de G.

Remark. Pour les familles de variétés algébriques sur les corps de fonctions, le type de symétrie est déterminé par la monodromie de la famille. Cependant, aucune telle analogie n'est connue sur les corps de nombres.

A.3.4 Type de symétrie des algèbres de quaternions

Petits zéros pour les algèbres de quaternions

Considérant les statistiques sur les petits zéros des fonctions L attachées à la famille universelle des algèbres de quaternions, la 1-densité (1.30) de la famille tronquée est

$$D(\mathcal{A}(Q), \phi) = \frac{1}{N(Q)} \sum_{\pi \in \mathcal{A}(Q)} D(\pi, \phi). \tag{A.32}$$

Le problème est de déterminer si la quantité $D(\mathcal{A}(Q), \phi)$ admet une limite et l'exprimer comme une convergence de mesures vers une mesure de densité, espérant identifier l'un des types de symétrie prédits par la conjecture de densité. Le théorème suivant répond partiellement à cette question, restreignant les types de symétrie possible pour les algèbres de quaternions avec un support limité pour la transformée de Fourier.

Théorème D. *Pour une fonction ϕ Schwartz paire sur* $\mathbf{R}$ *avec une transformée de Fourier de support compact dans* $(-1, 1)$,

$$\frac{1}{N(Q)} \sum_{\pi \in \mathcal{A}(Q)} D(\pi, \phi) \underset{Q\to\infty}{\longrightarrow} \widehat{\phi}(0) + \frac{1}{2}\phi(0) = \int_{\mathbf{R}} \phi(x) W_O(x)\, \mathrm{d}x. \tag{A.33}$$

où $W_O = 1 + \frac{1}{2}\delta_0$. En particulier, le type de symétrie des algèbres de quaternions est orthogonal.

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Arithmetic Statistics for Quaternion Algebras

Abstract. Automorphic forms are central objects in modern number theory. Despite their ubiquity, they remain mysterious and their behavior is far from understood. Embedding them in wider families has a smoothing effect, allowing results on average: these are the aims of arithmetic statistics. The whole family of automorphic representations of a given reductive group, referred to as its universal family, is of fundamental importance. In the case of inner forms of GL(2), that is to say groups of units of quaternion algebras, the Selberg trace formula is a powerful method to handle it. There is a way to define a suitable notion of size, the analytic conductor, allowing to truncate the universal family to a finite one amenable to arithmetic statistics methods.

A counting law for the truncated universal family is established, with a power savings error term in the totally definite case and a geometrically meaningful constant. This Weyl's law is generalized to an equidistribution result with respect to an explicit measure, and leads to answer the Sato-Tate conjectures in this case. Statistics on low-lying zeros are provided, leading to uncover part of the type of symmetry of quaternion algebras.

Strong evidence is provided that further ground groups should be amenable as well to the same methods and conjectural counting laws are given in the case of symplectic and unitary groups of low ranks.

Keywords. Number theory, automorphic representations, quaternion algebras, families of automorphic forms, analytic conductor, Selberg trace formula, counting law, equidistribution, Sato-Tate, low-lying zeros, type of symmetry.

Mathematic Subject Classification. 11F55, 11F60, 11F66, 11F67.

Statistiques arithmétiques sur les algèbres de quaternions

Résumé. Les formes automorphes sont des objets centraux en théorie des nombres. En dépit de leur omniprésence, elles demeurent mystérieuses et leur comportement est loin d'être entièrement compris. Considérer ces formes automorphes au sein de familles a un effet régularisant, et ouvre la voie aux résultats en moyenne : voilà l'esprit des statistiques arithmétiques. La famille de toutes les représentations automorphes d'un groupe réductif donné, appelée famille universelle du groupe, est particulièrement importante. Dans le cas des formes intérieures de GL(2), autrement dit les groupes d'unités d'algèbres de quaternions, la formule des traces de Selberg est une puissante méthode d'approche. Il existe une notion de taille sur les formes automorphes, le conducteur analytique, permettant de tronquer la famille universelle en un ensemble fini pour lequel ces problèmes de statistiques arithmétiques ont un sens.

Une loi de comptage pour la famille universelle tronquée est établie, avec un terme d'erreur gagnant par une puissance dans le cas totalement défini, et une constante à forte teneur géométrique. Cette loi de Weyl est généralisée en un résultat d'équirépartition par rapport à une mesure explicite, et mène à vérifier les conjectures de Sato-Tate dans ce cadre. Des statistiques sur les petits zéros des fonctions L associées sont établies, menant à dévoiler partiellement le type de symétrie des algèbres de quaternions.

Plusieurs indices sont mentionnés laissant à croire que d'autres groupes sont abordables par les mêmes méthodes, et les lois de comptage conjecturales pour certains groupes unitaires et symplectiques de petits rangs sont énoncées.

Mots-clés. Théorie des nombres, représentations automorphes, algèbres de quaternions, familles de formes automorphes, conducteur analytique, formule des traces de Selberg, loi de comptage, équirépartition, Sato-Tate, petits zéros, type de symétrie.