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Problèmes d'existence globale pour les équations d'évolution non-linéaires critiques à données petites et analyse semi-classique

Global existence problems for non-linear critical evolution equations with small initial data and semi-classical analysis

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Problèmes d'éxistence globale pour des équations d'évolution non-linéaires critiques à données petites et analyse semi-classique

Résumé. Cette thèse est consacrée à l'étude de l'éxistence globale de solutions pour des équations de Klein-Gordon – ou des systèmes ondes-Klein-Gordon – quasi-linéaires critiques, à données petites, régulières, décroissantes à l'infini, en dimension un ou deux d'éspace. On étudie d'abord ce problème pour des équations de Klein-Gordon à non-linéarité cubique en dimension un, pour lesquelles il est connu qu'il y a existence globale des solutions lorsque la non-linéarité vérifie une condition de structure et les données initiales sont petites et à support compact. Nous prouvons que ce résultat est vrai aussi lorsque les données initiales ne sont pas localisées en espace mais décroissent faiblement à l'infini, en combinant la méthode des champs de vecteurs de Klainerman avec une analyse micro-locale semi-classique de la solution. La deuxième et principale contribution à la thèse s'attache à l'étude de l'existence globale de solutions pour un système modèle ondes-Klein-Gordon quadratique, quasi-linéaire, en dimension deux, toujours pour des données initiales petites régulières à décroissance modérée à l'infini, les non-linéarités étant données en termes de «formes nulles ». Notre but est d'obtenir des estimations d'énergie sur la solution sur laquelle agissent des champs de Klainerman, et des estimations de décroissance uniforme optimales, dans une version para-différentielle. Nous prouvons les secondes par une réduction du système d'équations aux dérivées partielles du départ à un système d'équations ordinaires, stratégie qui pourrait nous emmener, dans le futur, à traiter le cas de non-linéarités plus générales.

Mots Clés. Existence globale de petites solutions, équations dispersives, équations de Klein-Gordon, systèmes ondes-Klein-Gordon, champs de vecteurs de Klainerman, formes normales, analyse micro-locale semi-classique, structure nulle.

Global existence problem for non-linear critical evolution equations with small initial data and semi-classical analysis.

Abstact. In this thesis we study the problem of global existence of solutions to critical quasi-linear Klein-Gordon equations – or to critical quasi-linear coupled wave-Klien-Gordon systems – when initial data are small, smooth, decaying at infinity, in space dimension one or two. We first study this problem for Klein-Gordon equations with cubic non-linearities in space dimension one. It is known that, under a suitable structure condition on the non-linearity, the global well-posedness of the solution is ensured when initial data are small and compactly supported. We prove that this result holds true even when initial data are not localized in space but only mildly decaying at infinity, by combining the Klainerman vector fields' method with a semi-classical micro-local analysis of the solution. The second and main contribution to the thesis concerns the study of the global existence of solutions to a quadratic quasi-linear wave-Klein-Gordon system in space dimension two, again when initial data are small smooth and mildly decaying at infinity. We consider the case of a model non-linearity, expressed in terms of "null forms". Our aim is to obtain some energy estimates on the solution when some Klainerman vector fields are acting on it, and sharp uniform estimates. The former ones are recovered making systematically use of normal forms' arguments for quasi-linear equations, in their para-differential version. We derive the latter ones by deducing a system of ordinary differential equations from the starting partial differential system, this strategy maying leading us in the future to treat the case of the most general non-linearities.

Keywords. Global existence of small solutions, dispersive equations, Klein-Gordon equations, wave-Klein-Gordon systems, Klainerman vector fields, normal forms, semi-classical micro-local analysis, null structure.

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Introduction

Le but de cette thèse est d'apporter des contributions à l'étude de l'existence de solutions globales pour des équations de Klein-Gordon – ou des systèmes couplés ondes-Klein-Gordon – quasi-linéaires, à données petites, régulières, décroissantes à l'infini.

Les premiers articles concernant ces questions remontent aux années 1970 en ce qui concerne l'équation des ondes. Nous renvoyons au chapitre 6 de la monographie d'Hörmander [12] pour une discussion complète de ces travaux précurseurs, et aux références bibliographiques qui y sont données. On peut d'ores et déjà remarquer qu'une dichotomie apparaît naturellement, entre problèmes sous-critiques et critiques (ou sur-critiques). Considérons en effet un opérateur différentiel elliptique linéaire d'ordre deux P (qui pour nous sera soit $-\Delta$ ou $-\Delta + m^2$ avec $m > 0$) sur l'espace euclidien $\mathbb{R}^d$, et intéressons-nous à une équation d'évolution du type

$$(\partial_t^2 + P)w = N(w, \partial w, \partial^2 w) \tag{1}$$

où $N(\cdot)$ est une non-linéarité C^∞, nulle au moins à l'ordre $p \geq 2$ en zéro, qui est de plus affine en $\partial^2 w$, de manière que (1) soit une équation hyperbolique quasi-linéaire. Nous nous intéressons uniquement au cas où les données initiales $w|_{t=0}, \partial_t w|_{t=0}$ sont très régulières et présentent une certaine décroissance (à préciser) lorsque x tend vers l'infini. Dans ce cas, le caractère dispersif de l'équation fait que les solutions de l'équation linéaire décroissent en norme L^∞ en $t^{-\kappa}$ lorsque t tend vers l'infini, avec un taux $\kappa > 0$ dépendant de P et de l'équation ($\kappa = (d-1)/2$ si $P = -\Delta$ et $\kappa = d/2$ si $P = -\Delta + m^2$). Si l'on écrit alors formellement la non-linéarité sous la forme $V(w, \partial w)\partial^2 w$, avec $V(\cdot)$ potentiel non-linéaire nul à l'ordre $p-1$ en zéro, on constate que, si l'on conjecture que les solutions du problème non-linéaire décroîtront comme les solutions de l'équation linéaire, le potentiel $V(\cdot)$ aura une norme uniforme en $O(t^{-\kappa(p-1)})$ lorsque t tend vers l'infini. Cela conduit à distinguer un cas sous-critique, $(p-1)\kappa > 1$, pour lequel le norme L^∞ de V est intégrable en temps, d'un cas critique ou sur-critique $(p-1)\kappa \leq 1$. Le cas sous-critique est le plus facile en ce qui concerne l'existence globale, puisqu'on espère que toute non-linéarité donnera alors lieu à des solutions globales, à partir du moment où les données initiales sont assez petites dans un espace de fonctions régulières et décroissantes. Par exemple, pour l'équation des ondes avec non-linéarité quadratique, c'est le cas en dimension supérieure ou égale à quatre (cf. [12] et les références qui y sont données). Par contre, en dimension trois pour cette même équation, il existe des non-linéarités pour lesquelles les solutions peuvent exploser en temps fini, et l'existence globale n'est vraie que sous une condition de structure sur le non-linéarité (cf. [21] et [12] pour les résultats ultérieurs). En ce qui concerne l'équation de Klein-Gordon, l'analyse ci-dessus semblerait indiquer que la limite entre cas critique et cas sous-critique, pour une non-linéarité quadratique, est atteinte en dimension deux d'espace. En fait, il se trouve que la méthode des formes normales de Shatah [33] permet de réduire le cas d'une non-linéarité quadratique pour cette équation à celle d'une non-linéarité cubique. Le seul cas critique apparaît donc comme celui de la dimension un. Nous renvoyons à la première partie de cette thèse pour les références bibliographiques concernant les résultats connus en dimension supérieure ou égale à deux pour l'équation de Klein-Gordon.

Notre but ici est d'étudier deux cas critiques, lorsque les données considérées sont petites, régulières, et n'ont qu'une décroissance modérée à l'infini.

La première partie de cette thèse, que nous décrivons plus en détail dans la première section ci-dessous, est consacrée à l'étude de l'existence globale à données petites régulières pour l'équation de Klein-Gordon à non-linéarité cubique en dimension un, lorsqu'on ne fait que des hypothèses de faible décroissance à l'infini pour les données.

La seconde, et principale contribution que nous présentons, s'attache à l'étude de l'existence globale de solutions pour des systèmes ondes-Klein-Gordon en dimension deux d'espace, toujours pour des données petites régulières à décroissance modérée à l'infini. Plusieurs travaux récents ont été consacrés à ces questions au cours des années récentes. L'un des premiers remonte à Georgiev [11]. Il observe que la méthode des champs de vecteurs de Klainerman doit être adaptée pour pouvoir traiter simultanément des équations d'ondes massives et sans masse, à cause du fait que le champ de vecteur $S = t\partial_t + x\cdot\nabla_x$ ne commute pas avec l'opérateur de Klein-Gordon. Il introduit ainsi une *condition de structure forte* pour des non-linéarités semi-linéaires, qui assure l'existence globale des solutions. Une telle condition a été ultérieurement affaiblie par Katayama [19] afin d'inclure la condition nulle de Klainerman [22]. Le résultat qu'il obtient peut, par conséquent, s'appliquer à d'autres systèmes physiques, notamment aux équations de Dirac-Klein-Gordon, Dirac-Proca, et encore Zakharov-Klein-Gordon. Plus tard, ce sujet a été étudié aussi par LeFloch-Ma [26] et Wang [35] comme modèle pour l'équation d'Einstein-Klein-Gordon complète

$$\text{(E-KG)} \qquad \begin{cases} Ric_{\alpha\beta} = \mathbf{D}_\alpha\psi\mathbf{D}_\beta\psi + \frac{1}{2}\psi^2 g_{\alpha\beta} \\ \Box_g\psi = \psi \end{cases}$$

Ces auteurs prouvent l'existence globale de solutions lorsque la non-linéarité est quasi-linéaire quadratique, satisfaisant des conditions de structure, et les données initiales sont petites, lisses et à support compact, en utilisant la méthode dite du *feuilletage par des hyperboloïdes* introduite dans [26]. La stabilité globale pour (E-KG) a ensuite été prouvée par LeFloch-Ma [25, 23] dans le cas de données initiales qui coïncident avec une solution de Schwarzschild en dehors d'un compact (cf. aussi [34]). Récemment, Ionescu-Pausader [17] ont prouvé un résultat de régularité globale et de diffusion modifiée dans le cas de données initiales lisses et convenablement décroissantes à l'infini. Le système quadratique qu'ils étudient est le suivant

$$\begin{cases} -\Box u = A^{\alpha\beta}\partial_\alpha v\partial_\beta v + Dv^2 \\ -(\Box + 1)v = uB^{\alpha\beta}\partial_\alpha\partial_b v \end{cases}$$

avec $A^{\alpha\beta}, B^{\alpha\beta}, D$ constantes réelles. Ce système garde la même structure linéaire de (E-KG) en jauge hyperbolique, mais fait apparaitre seulement des non-linéarités quadratiques impliquant le champ scalaire massif v (non-linéarités semi-linéaires dans l'équation des ondes, quasi-linéaires dans celle de Klein-Gordon). De plus, la non-linéarité considérée n'a pas de structure nulle, mais plutôt une certaine structure résonante. Leur méthode, qui combine des estimations d'énergie contrôlant la régularité de la solution en espace et dans des normes de Sobolev grandes, avec une analyse de Fourier, des arguments de formes normales et d'analyse de résonances, permet de prouver des estimations dispersives et la décroissance de la solution dans certaines normes à faible régularité. Les seuls résultats que nous connaissons aujourd'hui en dimension 2 d'espace sont dus à Ma, qui considère le cas de données initiales à support compact. Dans [31], il adapte la méthode de feuilletage par des hyperboloïdes, précédemment citée, aux systèmes ondes-Klein-Gordon en dimension d'espace-temps 2+1, en la combinant ensuite avec un argument de formes normales pour traiter certaines non-linéarités quadratiques quasi-linéaires (voir [30]). Plus récemment, il prouve le même résultat de stabilité globale aussi dans le cas de certaines interactions quadratiques semi-linéaires ([29]).

La plupart des travaux que nous venons de citer concernent soit des cas sous-critiques, soit des cas critiques, avec données initiales très décroissantes à l'infini. Nous nous sommes donc posé le problème suivant : quels résultats d'existence globale peut-on obtenir pour un système ondes-Klein-Gordon en dimension deux d'espace, avec une non-linéarité quasi-linéaire quadratique, et des données petites régulières, mais n'ayant qu'une décroissance modérée en espace ? Comme rappelé plus haut, l'équation de Klein-Gordon à données quadratiques n'est pas vraiment critique en dimension deux, puisqu'elle peut être réduite à une équation cubique. Par contre l'équation des ondes générale est sur-critique, et n'admet en général de solutions globales que si la non-linéarité vérifie deux conditions nulles (voir les travaux d'Alinhac [3, 4]). Un objectif de long terme serait donc de déterminer, pour un système formé d'une équation de Klein-Gordon quasi-linéaire, couplée à une équation des ondes quasi-linéaire, les conditions nulles optimales que doivent vérifier les non-linéarités afin que des données petites, peu régulières et décroissantes à l'infini, donnent lieu à des solutions globales. Nous nous contentons dans cette thèse de faire un premier pas dans un tel programme, en étudiant un modèle d'interaction, pour lequel nous prouvons un tel résultat d'existence globale. Nous décrivons nos résultats dans la deuxième section ci-dessous.

1 Équation de Klein-Gordon quasi-linéaire en dimension un

Comme nous l'avons noté ci-dessus, en dimension $d = 1$ d'espace, une équation de la forme (1) avec non-linéarité quadratique ou cubique, est sur-critique ou critique. De fait, il existe des exemples de non-linéarités pour lesquelles des données petites, régulières, décroissantes à l'infini donnent lieu à des solutions explosant en temps fini (cf. Keel et Tao [20] et Yordanov [36]). Delort [7] a construit des solutions approchées du problème de Cauchy pour l'équation

$$(\partial_t^2 - \partial_x^2 + 1)u = P(u, \partial_t\partial_x u, \partial_x^2 u; \partial_t u, \partial_x u) \tag{1}$$

où P est un polynôme, nul à l'ordre deux à l'origine, dont la dépendance en les dérivées secondes est affine. Cela lui a permis de dégager un *condition nulle* portant sur les termes quadratiques et cubiques de la non-linéarité, dont on conjecture qu'elle est nécessaire et suffisante pour que le problème soit globalement bien posé lorsque les données sont petites, régulières et ont une certaine décroissance à l'infini. La suffisance de cette condition a été établie dans [10, 9] pour des données qui sont de plus *à support compact*, cette restriction étant liée à la méthode utilisée. De plus, le comportement asymptotique des solutions, permettant de mettre en évidence une propriété de diffusion modifiée, a également été établie dans ces articles. Le but de la première partie de cette thèse est d'étendre, dans le cas des non-linéarités cubiques, ces résultats à des données dont la décroissance à l'infini est essentiellement en $O(|x|^{-1})$. Plus précisément, nous allons prouver le résultat suivant :

Théorème 1. *Supposons que la non-linéarité P de* (1) *vérifie la condition nulle (nous renvoyons à la première partie de la thèse pour l'expression explicite de cette dernière). Il existe alors $s > 0$, $\sigma > 0$, $\varepsilon_0 \in]0, 1[$, tels que pour tout couple $(u_0, u_1) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})$, à valeurs réelles, vérifiant*

$$\|u_0\|_{H^{s+1}} + \|u_1\|_{H^s} + \|xu_0\|_{H^2} + \|xu_1\|_{H^1} \leq 1, \tag{2}$$

pour tout $\varepsilon \in]0, \varepsilon_0[$*, l'équation* (1) *avec les données initiales* $u|_{t=1} = u_0, \partial_t u|_{t=1} = u_1$*, ait une unique solution globale* $u \in C^0([1,+\infty[, H^{s+1}(\mathbb{R})) \cap C^1([1,+\infty[, H^s(\mathbb{R}))$*. De plus, il existe une famille à un paramètre de fonctions continues* $(a_\varepsilon)_\varepsilon$*, uniformément bornées, supportées dans* $[-1,1]$*, telles que la solution globale ait le comportement asymptotique*

$$u(t,x) = \mathrm{Re}\left[\frac{\varepsilon}{\sqrt{t}} a_\varepsilon\Big(\frac{x}{t}\Big) \exp\left[it\varphi\Big(\frac{x}{t}\Big) + i\varepsilon^2 \Big|a_\varepsilon\Big(\frac{x}{t}\Big)\Big|^2 \Phi_1\Big(\frac{x}{t}\Big)\log t\right]\right] + \frac{\varepsilon}{t^{\frac{1}{2}+\sigma}} r(t,x), \tag{3}$$

où $\varphi(x) = \sqrt{1-x^2}$*,* $r(t,x)$ *est uniformément borné dans* $L^2 \cap L^\infty$ *et* Φ_1 *est une fonction qui se calcule explicitement à partir des coefficients de la non-linéarité* P*.*

Notre approche se distingue de [10, 9] en ce sens que nous n'utilisons pas un passage en coordonnées hyperboliques (équivalent pour notre problème de dimension un à la méthode de feuilletage par des hyperboloïdes utilisée en particulier dans les travaux récents de LeFloch et Ma [24, 28, 27] en dimension supérieure). Expliquons notre stratégie sur le modèle semi-linéaire suivant

$$\Big(D_t - \sqrt{1+D_x^2}\Big)w = \alpha w^3 + \beta|w|^2 w + \gamma|w|^2\bar{w} + \delta\bar{w}^3, \tag{4}$$

où w est une inconnue à valeurs complexes, $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $D_x = \frac{1}{i}\frac{\partial}{\partial x}$ et $\alpha, \beta, \gamma, \delta$ sont dans $\mathbb{C}$. Introduisons le champ de Klainerman $Z = t\partial_x + x\partial_t$. Il est classique que, pour obtenir un résultat d'existence globales, pour des données de taille $\varepsilon \ll 1$ dans un espace de fonctions régulières, décroissantes comme $|x|^{-1}$ à l'infini dans L^2, il suffit d'obtenir des estimations du type

$$\begin{aligned} \|w(t,\cdot)\|_{L^\infty} &\le A\varepsilon t^{-\frac{1}{2}} \\ \|w(t,\cdot)\|_{H^s} &\le B\varepsilon t^\sigma \\ \|Zw(t,\cdot)\|_{L^2} &\le B\varepsilon t^\sigma, \end{aligned} \tag{5}$$

où A et B sont des constantes, $s \gg 1$, et $\sigma > 0$ est petit. Nous établissons ces inégalités par induction, sous l'hypothèse que les constantes A, B ont été fixées au départ assez grandes, en supposant (5) vérifié sur un certain intervalle de temps $[1, T]$, et en prouvant que, si ε est assez petit, ces inégalités valent en fait sur l'intervalle considéré en remplaçant A (resp. B) par $A/2$ (resp. $B/2$).

Pour le modèle (4), la démonstration des estimations Sobolev pour w est conséquence immédiate des inégalités d'énergie, et du fait que la première des estimations a priori (5) entraîne que la norme H^s du terme source dans (4) est $O\big(\frac{\varepsilon^2}{t}\|w(t,\cdot)\|_{H^s}\big)$. L'estimation L^2 de Zw se traite de même. Par contre, l'inégalité L^∞ dans (5) ne peut se déduire par estimations de Klainerman-Sobolev des bornes Sobolev de w et L^2 de Zw, puisque celles-ci ne sont pas uniformes lorsque le temps tend vers l'infini. Pour les propager, ainsi que pour obtenir le comportement asymptotique de nos solutions, nous utilisons une méthode inspirée d'Alazard-Delort [1, 2], Ifrim-Tataru [15], Delort [10]. Nous écrivons d'abord une version semi-classique de l'équation pour l'inconnue v définie à partir de w par $w(t,x) = t^{-1/2}v(t, x/t)$, de telle manière que la borne L^∞ cherchée en $t^{-\frac{1}{2}}$ pour w équivaille à une estimation uniforme de $\|v(t,\cdot)\|_{L^\infty}$. Si nous introduisons la constante de Planck $h = \frac{1}{t}$ et la quantification semi-classique d'un symbole a par

$$\mathrm{Op}_h^w(a)v = \frac{1}{2\pi h}\int e^{i(x-y)\frac{\xi}{h}} a\Big(\frac{x+y}{2}, \xi\Big) v(y)\, dy d\xi, \tag{6}$$

on constate que v résout l'équation

$$(D_t - \mathrm{Op}_h^w(\lambda(x,\xi)))v = h(\alpha v^3 + \beta|v|^2 v + \gamma|v|^2\bar{v} + \delta\bar{v}^3) \tag{7}$$

où $\lambda(x,\xi) = x\xi + \sqrt{1+\xi^2}$. Par ailleurs, on déduit de l'équation vérifiée par w et des estimations L^2 de Zw que, si l'on définit $\mathcal{L}$ par $\mathcal{L} = \frac{1}{h}\mathrm{Op}_h^w\big(\partial_\xi \lambda(x,\xi)\big)$, alors $\|\mathcal{L}v\|_{L^2}$ est $O(B\varepsilon h^{-\sigma})$. Soit Λ la sous-variété lagrangienne de $T^*\mathbb{R}$ donnée pas $\Lambda = \{(x,\xi); \partial_\xi\lambda(x,\xi) = 0\}$. Alors Λ est un graphe qui se projette sur l'intervalle $]-1,1[$ et s'écrit donc $\Lambda = \{(x, d\varphi); x \in]-1,1[\}$, avec $\varphi(x) = \sqrt{1-x^2}$. Nous décomposons alors $v = v_\Lambda + v_{\Lambda^c}$, où v_Λ est obtenue par microlocalisation de v sur un voisinage d'ordre $\sqrt{h}$ de Λ, et v_{Λ^c} est microlocalement supportée hors d'un tel voisinage. Grâce à cela, ce dernier terme peut essentiellement s'écrire $v_{\Lambda^c} \sim h^{1/2}\mathrm{Op}_h^w(b)\mathcal{L}v$, où b est un symbole qui est $O\Big(\Big\langle \frac{\partial_\xi\lambda(x,\xi)}{\sqrt{h}}\Big\rangle^{-1}\Big)$, et une estimation de Sobolev semi-classique permet de montrer que $\|v_{\Lambda^c}\|_{L^\infty} = O(h^{\frac{1}{4}-0})$, ce qui est mieux que l'estimation uniforme souhaitée pour v. La contribution principale est donc v_Λ. Pour l'étudier, on écrit d'abord à partir de (7) l'équation vérifiée par cette fonction, qui, grâce aux bonnes propriétés de commutation entre la troncature microlocale permettant de définir v_Λ et l'opérateur linéaire, est essentiellement (7) dans laquelle v est remplacée par v_Λ, modulo un terme de reste au membre de droite en $O(h^{1+\sigma})$ pour un $\sigma > 0$. On développe alors le symbole de la partie linéaire sur Λ, i.e. on écrit

$$\lambda(x,\xi) = \lambda(x, d\varphi(x)) + O\big((\xi - d\varphi(x))^2\big). \tag{8}$$

L'action de l'opérateur de symbole le dernier terme de (8) sur v_Λ est essentiellement de la forme $h^{\frac{3}{2}}\mathcal{L}v_\Lambda$, ce qui permet, par une nouvelle estimation de Sobolev semi-classique combinée aux estimations a priori de $\mathcal{L}v_\Lambda$ dans L^2, de la majorer dans L^∞ par $O(h^{1+\sigma})$. On déduit donc de (7) une *équation différentielle ordinaire* de la forme

$$\big(D_t - \lambda(x, d\varphi(x))\big)v_\Lambda = t^{-1}(\alpha v_\Lambda^3 + \beta|v_\Lambda|^2 v_\Lambda + \gamma|v_\Lambda|^2\bar{v_\Lambda} + \delta\bar{v_\Lambda}^3) + O(t^{-1-\sigma}). \tag{9}$$

Il reste à voir que cette équation a des solutions globales bornées lorsque la donnée est assez petite, et à obtenir le comportement asymptotique de celles-ci. On prouve cela par une méthode de formes normales habituelle, lorsque le coefficient β est *réel*. Cela permet de propager des estimation uniformes pour v_Λ donc pour v, donc des estimations L^∞ optimales en $O(t^{-1/2})$ pour w, et conclut la preuve des majorations de type (5) dans le cas du modèle (4).

Bien entendu, pour la véritable équation (1), la méthode précédente est nettement plus délicate à mettre en œuvre. En particulier, les estimations de type L^∞ dans (5) doivent également faire intervenir un certain nombre de dérivées de la fonction. Par ailleurs, l'équation semi-classique analogue à (7) fait également intervenir l'action d'opérateurs (pseudo-)différentiels sur les facteurs de la non-linéarité, et ceux-ci doivent être également réduits à des opérateurs de multiplication locaux par développement de leur symbole sur Λ. En outre, le fait que la phase $\varphi(x) = \sqrt{1-x^2}$ soit singulière au bord de son domaine de définition $]-1,1[$ nécessite un traitement adéquat. Nous renvoyons à la première partie de la thèse pour ces détails. Précisons simplement ici comment la "condition nulle" que nous supposons sur la non-linéarité intervient. Nous avons vu sur le modèle que pour montrer que (9) admet des solutions globales bornées, nous avons besoin de l'hypothèse que β est réel. Lorsque nous partons de l'équation générale (1), nous obtenons après réduction une équation différentielle ordinaire de la forme (9) dans laquelle le coefficient β se calcule explicitement à partir de la non-linéarité P de (1). Le fait que β soit réel est alors équivalent au fait que P vérifie la condition nulle.

Nous avons indiqué au début de cette section que notre méthode pourrait également s'appliquer au cas où P dans (1) contient des termes quadratiques, quitte à procéder dans une première étape à l'élimination de ces termes dans l'équation aux dérivée partielles par une méthode de forme normales "à la Shatah" [33]. Nous avons préféré nous limiter à l'équation purement cubique pour nous épargner cette étape technique, qui est indépendante du reste du raisonnement. Dans la deuxième partie de cette thèse, les méthodes de formes normales sur les équations aux dérivées partielles joueront toutefois un rôle crucial. C'est ce que nous allons décrire dans la section suivante de cette introduction.

2 Système couplé ondes-Klein-Gordon

Le but de cette deuxième partie de la thèse est d'étudier l'existence globale pour un système couplé ondes-Klein-Gordon, avec données petites, régulières, et à décroissance modérée à l'infini, en dimension deux d'espace. Comme déjà indiqué, nous ne pouvons espérer obtenir l'existence de solutions globales pour toute non-linéarité. En effet, l'équation des ondes scalaire en dimension deux peut avoir des solutions explosives, si l'on ne fait pas une hypothèse convenable de condition nulle. Par ailleurs, un système comme celui que nous allons considérer n'est hyperbolique que sous des conditions de compatibilité entre les équations qui le constituent. En dimension trois d'espace, pour des systèmes couplés ondes-Klein-Gordon, LeFloch et Ma déterminent les hypothèses optimales que doit vérifier le couplage afin d'obtenir des solutions globales [24, 28, 27].

Il est donc naturel de se poser la question analogue en dimension deux d'espace, pour laquelle on peut s'attendre à des conditions plus complexes, en raison du moindre effet dispersif de l'équation des ondes libre. A la lumière de notre résultat concernant Klein-Gordon en dimension un décrit dans la section précédente, pour lequel la condition nulle à supposer se dévoile sur l'équation différentielle réduite que nous déduisons de l'équation aux dérivées partielles de départ, on pourrait s'assigner le programme suivant : Partant d'un système général ondes-Klein-Gordon, déduire de celui-ci un système d'équations différentielles ordinaires et d'équations de transport, analogue à (7), dont l'analyse révèlerait la condition optimale sur les non-linéarités assurant l'existence globale. Un tel objectif n'est atteignable qu'à long terme, aussi nous sommes-nous limités dans cette thèse à la considération d'un modèle dans lequel le couplage entre les deux équations se fait à l'aide d'une "forme nulle" au membre de droite de chacune de celles-ci. Notre but sera d'obtenir des estimations d'énergie sur la solution sur laquelle agissent des champs de Klainerman, et des estimations de décroissance uniforme optimales. Nous prouverons celles-ci par une réduction du système d'équations aux dérivées partielles à un système d'équations différentielles, stratégie qui pourrait, dans de futur travaux, nous amener à aborder le cas de systèmes plus généraux.

Nous décrivons notre résultat et les principales étapes de sa preuve dans les sous-sections suivantes.

2.1 Modèle étudié et théorème principal

Nous considérons le couplage quasi-linéaire quadratique entre une équation des ondes et une équation de Klein-Gordon, donné par le modèle suivant :

$$\begin{aligned} (\partial_t^2 - \Delta)u(t,x) &= Q_0(v, \partial_1 v) \\ (\partial_t^2 - \Delta + 1)v(t,x) &= Q_0(v, \partial_1 u) \end{aligned} \tag{1}$$

où les deux fonctions inconnues u, v sont définies sur $I \times \mathbb{R}^2$, avec I intervalle de $\mathbb{R}$, et où Q_0 est la "forme nulle"

$$Q_0(v,w) = (\partial_t v)(\partial_t w) - (\nabla_x v)(\nabla_x w). \tag{2}$$

Nous allons étudier le problème d'évolution (1) sur l'intervalle $I = [1, +\infty[$ (plutôt que sur $[0, +\infty[$, uniquement pour simplifier certaines notations), en nous donnant à l'instant $t = 1$ des données initiales

$$(3) \qquad (u,v)(1,x) = \varepsilon(u_0,v_0),\ (\partial_t u, \partial_t v)(1,x) = \varepsilon(u_1,v_1)$$

où $(\nabla_x u_0, u_1)$ est dans la boule unité de $H^n(\mathbb{R}^2,\mathbb{R}) \times H^n(\mathbb{R}^2,\mathbb{R})$ et (v_0,v_1) dans la boule unité de $H^{n+1}(\mathbb{R}^2,\mathbb{R}) \times H^n(\mathbb{R}^2,\mathbb{R})$ pour un n assez grand, et où de plus l'inégalité

$$(4) \qquad \sum_{1\le|\alpha|\le 3} \left(\|x^\alpha \nabla_x u_0\|_{H^{|\alpha|}} + \|x^\alpha v_0\|_{H^{|\alpha|+1}} + \|x^\alpha u_1\|_{H^{|\alpha|}} + \|x^\alpha v_1\|_{H^{|\alpha|}} \right) \le 1,$$

est vérifiée. Nous supposons donc que les données initiales sont très régulières, et qu'elles ont une décroissance modérée en espace, donnée par la condition (4). Notre principal résultat affirme alors :

Théorème 2. *Il existe $\varepsilon_0 > 0$ tel que pour tout $\varepsilon \in]0, \varepsilon_0[$, le système (1) avec des données vérifiant (3) et (4) admette une unique solution définie sur $[1, +\infty[$, avec $\partial_{t,x} u$ continue à valeurs H^n et $(v, \partial_t v)$ continue à valeurs $H^{n+1} \times H^n$.*

La preuve du théorème précédent nous donnera en outre des bornes convenables pour les normes des solutions.

Nous ré-exprimons tout d'abord le système en fonction des inconnues

$$(5) \qquad u_\pm = (D_t \pm |D_x|)u,\ v_\pm = (D_t \pm \langle D_x \rangle)v,$$

où $D_{t,x} = \frac{1}{i}\partial_{t,x}$, et nous introduisons les champs de Klainerman ayant de bonnes propriétés de commutation à la fois à l'opérateur des ondes et à celui de Klein-Gordon, à savoir

$$(6) \qquad \Omega = x_1\partial_2 - x_2\partial_1,\ Z_j = x_j\partial_t + t\partial_j,\ j = 1, 2.$$

Nous désignerons par la suite par $\mathcal{Z} = \{\Gamma_1, \dots, \Gamma_5\}$ la collection des trois champs précédents et des deux dérivées en espace, et si $I = (i_1, \dots, i_p)$ est une élément de $\{1, \dots, 5\}^p$, par $\Gamma^I w$ la fonction obtenue en faisant agir successivement les champs $\Gamma_{i_1}, \dots, \Gamma_{i_p}$ sur w. Nous posons alors

$$(7) \qquad u_\pm^I = (D_t \pm |D_x|)\Gamma^I u,\ v_\pm^I = (D_t \pm \langle D_x \rangle)\Gamma^I v$$

et nous introduisons les énergies suivantes

$$E_0(t; u_\pm, v_\pm) = \int_{\mathbb{R}^2} \left(|u_+(t,x)|^2 + |u_-(t,x)|^2 + |v_+(t,x)|^2 + |v_-(t,x)|^2 \right) dx$$

puis pour $n \ge 3$

$$(8) \qquad E_n(t; u_\pm, v_\pm) = \sum_{|\alpha|\le n} E_0(t; D_x^\alpha u_\pm, D_x^\alpha v_\pm),$$

qui contrôle la régularité H^n de $u_\pm, v_\pm$ et enfin, pour tout entier k compris entre zéro et deux,

$$(9) \qquad E_3^k(t; u_\pm, v_\pm) = \sum_{\substack{|\alpha|+|I|\le 3 \\ |I|\le 3-k}} E_0(t; D_x^\alpha u_\pm^I, D_x^\alpha v_\pm^I)$$

qui prend en compte la décroissance de $u_\pm, v_\pm$ et d'au plus trois de leurs dérivées en espace à l'infini en x. Par les résultats d'existence locale, une estimation a priori uniforme de E_n sur un certain intervalle de temps suffit à s'assurer du prolongement de la solution sur cet intervalle. La preuve du résultat d'existence locale est ainsi ramenée à celle de l'énoncé suivant, dans lequel nous désignons par $\mathrm{R} = (\mathrm{R}_1, \mathrm{R}_2)$ les transformées de Riesz :

Théorème 3. *Soient K_1, K_2 deux constantes strictement plus grandes que 1. Il existe des entiers $n \gg \rho \gg 1$, $\varepsilon_0 \in]0,1[$, des réels $0 < \delta \ll \delta_2 \ll \delta_1 \ll \delta_0 \ll 1$ et deux constantes assez grandes A, B telles que, si sur un certain intervalle $[1,T]$, les fonctions $u_\pm, v_\pm$ définies par (5) à partir d'une solution de (1) vérifient les estimations a priori*

$$
\begin{aligned}
&\|\langle D_x\rangle^{\rho+1}u_\pm(t,\cdot)\|_{L^\infty} + \|\langle D_x\rangle^{\rho+1}\mathrm{R}u_\pm(t,\cdot)\|_{L^\infty} \le A\varepsilon t^{-\frac{1}{2}}\\
&\|\langle D_x\rangle^{\rho}v_\pm\|_{L^\infty} \le A\varepsilon t^{-1}\\
&E_n(t;u_\pm,v_\pm) \le B^2\varepsilon^2 t^{2\delta}\\
&E_3^k(t;u_\pm,v_\pm) \le B^2\varepsilon^2 t^{2\delta_{3-k}},\ 0\le k\le 2,
\end{aligned}
\tag{10}
$$

pour tout t dans [1,T], alors, en fait, sur le même intervalle $[1,T]$, on a

$$
\begin{aligned}
&\|\langle D_x\rangle^{\rho+1}u_\pm(t,\cdot)\|_{L^\infty} + \|\langle D_x\rangle^{\rho+1}\mathrm{R}u_\pm(t,\cdot)\|_{L^\infty} \le \frac{A}{K_1}\varepsilon t^{-\frac{1}{2}}\\
&\|\langle D_x\rangle^{\rho}v_\pm\|_{L^\infty} \le \frac{A}{K_1}\varepsilon t^{-1}\\
&E_n(t;u_\pm,v_\pm) \le \frac{B^2}{K_2^2}\varepsilon^2 t^{2\delta}\\
&E_3^k(t;u_\pm,v_\pm) \le \frac{B^2}{K_2^2}\varepsilon^2 t^{2\delta_{3-k}},\ 0\le k\le 2.
\end{aligned}
\tag{11}
$$

La démonstration du théorème consiste d'une part à prouver, à l'aide d'inégalités d'énergie que (10) implique que les deux dernières inégalités de (11) sont vraies. Ensuite, on montre par réduction à des équations différentielles ordinaires ou des équations de transport que (10) implique les deux premières inégalités de (11). Nous décrivons les méthodes utilisées pour réaliser ces deux étapes dans les sous-sections suivantes.

2.2 Inégalités d'énergie I. Paralinéarisation et symétrisation

Nous pouvons réécrire le système (1) en faisant agir dessus une famille de champs Γ^I, puis en passant aux inconnues (7). Nous obtenons alors un nouveau système ayant la structure suivante

$$
\begin{aligned}
(D_t \mp |D_x|)u_\pm^I &= NL_{\mathrm{w}}(v_\pm^I, v_\pm^I)\\
(D_t \mp |D_x|)v_\pm^I &= NL_{\mathrm{kg}}(v_\pm^I, u_\pm^I)
\end{aligned}
\tag{12}
$$

où les non-linéarités, pour l'expression explicite desquelles nous renvoyons au membre de droite de (2.1.2), sont des quantités bilinéaires en leurs arguments respectifs, présentant une structure de forme nulle. La première étape consiste à réécrire ce système sous forme paradifférentielle.

Rappelons l'idée de base du calcul paradifférentiel : si u et v sont deux distributions ayant un minimum de régularité, leur produit peut se décomposer en une somme de trois termes

$$uv = T_u v + T_v u + R(u,v) \tag{13}$$

qui correspondent aux transformées de Fourier inverses des trois termes du membre de droite dans l'expression

$$
\begin{aligned}
\widehat{uv}(\xi) = &\int \chi(\xi-\eta,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)\,d\eta + \int \chi(\eta,\xi-\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)\,d\eta\\
&+ \int \big(1-\chi(\xi-\eta,\eta)-\chi(\eta,\xi-\eta)\big)\hat{u}(\xi-\eta)\hat{v}(\eta)\,d\eta,
\end{aligned}
$$

$\chi(\xi,\eta)$ désignant une fonction C^∞, supportée pour $|\xi| < (1+|\eta|)/10$, égale à un sur le domaine $|\xi| < (1+|\eta|)/100$ par exemple. Nous réécrirons (13) sous la forme

$$uv = \mathrm{Op}^{\mathrm{B}}(u)v + \mathrm{Op}^{\mathrm{B}}_{\mathrm{R}}(u)v \tag{14}$$

où $\mathrm{Op}^{\mathrm{B}}(u)v = T_u v$. A partir du moment où u est assez régulière, ce terme a la même régularité que v, alors que $\mathrm{Op}^{\mathrm{B}}_{\mathrm{R}}(u)v$ a lui la même régularité que u. Plus généralement, si l'on considère une expression bilinéaire $q(u,v)$, dans laquelle des opérateurs (pseudo)-différentiels agissent sur les deux arguments, elle peut de même se décomposer en

$$\mathrm{Op}^{\mathrm{B}}(a(u,\cdot))v + \mathrm{Op}^{\mathrm{B}}_{\mathrm{R}}(a(u,\cdot))v$$

dans laquelle le premier opérateur a le même ordre que celui agissant sur v dans l'expression de q. Cela permet de réécrire un système de la forme (12) sous la forme suivante. Notons

$$U^I = \begin{bmatrix} u_+^I \\ 0 \\ u_-^I \\ 0 \end{bmatrix}, \; V^I = \begin{bmatrix} 0 \\ v_+^I \\ 0 \\ v_-^I \end{bmatrix}, \; W^I = U^I + V^I.$$

Alors (12) entraîne que W^I est solution d'un système paradifférentiel de la forme

$$\begin{aligned} D_t W^I &= A(D)W^I + \mathrm{Op}^{\mathrm{B}}(A'(V,\eta))W^I + \mathrm{Op}^{\mathrm{B}}(C'(W^I,\eta))V + \mathrm{Op}^{\mathrm{B}}_{\mathrm{R}}(A'(V,\eta))W^I \\ &+ \mathrm{Op}^{\mathrm{B}}(A''(V^I,\eta))U + \mathrm{Op}^{\mathrm{B}}(C''(U,\eta))V^I + \mathrm{Op}^{\mathrm{B}}_{\mathrm{R}}(A''(V^I,\eta))U + Q_0^I(V,W), \end{aligned} \tag{15}$$

où nous avons utilisé les notations suivantes :

• L'opérateur $A(D)$ est un multiplicateur de Fourier associé à une matrice diagonale, à coefficients réels et d'ordre un.

• La contribution quasi-linéaire dans le membre de droite de (15) est donnée par l'expression $\mathrm{Op}^{\mathrm{B}}(A'(V,\eta))W^I$. En effet, A' est une matrice de symboles d'ordre un. L'opérateur associé agit sur W^I, terme qui porte le maximum de dérivées possibles en termes du nombre de champs de vecteurs agissant dessus. La norme L^2 de $\mathrm{Op}^{\mathrm{B}}(A'(V,\eta))W^I$ ne peut donc se contrôler que par $\|W^I\|_{H^1}$, cas typique d'une contribution quasi-linéaire.

• L'expression $\mathrm{Op}^{\mathrm{B}}(C''(U,\eta))V^I$ est elle semi-linéaire, car C'' est une matrice d'opérateurs d'ordre zéro, dont l'action sur V^I ne perd donc aucune dérivée. Il en est de même pour la contribution $Q_0^I(V,W)$, qui est une expression quadratique, ayant une structure nulle, et qui ne fait apparaître qu'un nombre de dérivées égal au plus à $|I|$. Les termes en $\mathrm{Op}^{\mathrm{B}}_{\mathrm{R}}(A'(V,\eta))W^I$ peuvent également être considérés comme semi-linéaires, car leur norme L^2 s'estime à partie de la norme L^2 de W^I.

• Le termes en $\mathrm{Op}^{\mathrm{B}}(C'(W^I,\eta))V, \mathrm{Op}^{\mathrm{B}}(A''(V^I,\eta))U, \mathrm{Op}^{\mathrm{B}}_{\mathrm{R}}(A''(V^I,\eta))U$ méritent un commentaire particulier. Ils ont en effet une structure semi-linéaire, puisque leur norme L^2 peut s'estimer à partir de la norme L^2 de U, V. Toutefois, une telle majoration se fait au prix d'une borne sur les "coefficients" W^I ou V^I dans L^∞. Or on ne dispose dans les estimations a priori (10) que d'un contrôle L^∞ que pour essentiellement ρ dérivées de W, alors que W^I peut contenir soit $n \gg \rho$ dérivées en espace, soit des dérivées par les champs de Klainerman, pour lesquelles nous ne disposons pas de bornes L^∞. Par conséquent, de tels termes devront être estimés en fonction de la norme L^2 des coefficients W^I, V^I et d'une norme de type L^∞ de V, U et devront donc faire l'objet d'une étude séparée par rapport aux autres.

Le système (15) ne peut faire l'objet d'une inégalité d'énergie sans traitement préalable. Il se trouve en effet que le symbole principal du membre de droite est donné par une matrice à coefficients réel qui *n'est pas symétrique*. L'étape préliminaire de l'étude, menée à bien dan la section 2.1, consiste donc à effectuer un changement d'inconnues qui symétrise cette matrice. Désignant par W_s^I la nouvelle inconnue obtenue après cette opération de symétrisation, on est ramené à une nouvelle équation, de la forme

$$(16)\quad D_t W_s^I = A(D)W_s^I + \mathrm{Op}^{\mathrm{B}}(\tilde{A}_1(V,\eta))W_s^I + \mathrm{Op}^{\mathrm{B}}_{\mathrm{R}}(A''(V^I,\eta))U + \mathrm{Op}^{\mathrm{B}}_{\mathrm{R}}(\tilde{A}''(V^I,\eta))U + \mathrm{Op}^{\mathrm{B}}(C''(U,\eta))V_s^I + Q_0^I(V,W) + \mathfrak{R}(U,V),$$

dans laquelle $\tilde{A}_1$ est un nouveau symbole d'ordre un, donné par une matrice *symétrique* à coefficients réels, et $\mathfrak{R}$ est un terme de reste. Pour prouver les deux dernières inégalités de (11), on aurait besoin de pouvoir estimer, en utilisant les estimations a priori (10), la partie imaginaire du produit scalaire L^2 de W_s^I et du membre de droite de (16) par $\frac{C\varepsilon}{t}\|W^I\|_{L^2}$. Grâce au fait que l'opérateur $\mathrm{Op}^{\mathrm{B}}(\tilde{A}_1(V,\eta))$ est autoadjoint d'ordre un, et que la norme L^∞ de V est $O(\varepsilon/t)$, une telle borne peut être obtenue pour la contribution de ce terme là. Il en est de même pour les autres termes dans (16) sauf ceux en $\mathrm{Op}^{\mathrm{B}}(C''(U,\eta))V_s^I + \mathrm{Op}^{\mathrm{B}}_{\mathrm{R}}(A''(V^I,\eta))U + \mathrm{Op}^{\mathrm{B}}_{\mathrm{R}}(\tilde{A}''(V^I,\eta))U$. En effet, la norme L^2 de ces trois termes ne peut être estimée qu'en faisant apparaître la norme L^∞ sur U et la norme L^2 sur V_s^I, V^I (puisqu'on de dispose pas de bornes a priori sur $\|V^I\|_{L^\infty}$ pour $|I|$ grand). Or $\|U(t,\cdot)\|_{L^\infty}$ est en $t^{-1/2}$, décroissance très insuffisante pour permettre d'obtenir les estimations d'énergie cherchées. Nous devons donc effectuer une forme normale pour éliminer ces termes quadratiques et les remplacer par des termes cubiques.

2.3 Inégalités d'énergie II. Formes normales

Rappelons que la méthode des formes normales pour des équations dispersives, introduite par Shatah dans [33], peut se résumer de la manière suivante : considérons par exemple une équation d'évolution semi-linéaire de la forme

$$(D_t - p(D_x))u = u^2.$$

Soit $B(u,u)$ la forme bilinéaire donnée par

$$\widehat{B(u,u)}(\xi) = \int \big(p(\xi) - p(\xi-\eta) - p(\eta)\big)^{-1}\hat{u}(\xi-\eta)\hat{u}(\eta)\,d\eta.$$

Alors, si on dispose d'une minoration de la forme

$$|p(\xi) - p(\xi-\eta) - p(\eta)| \geq \min(|\xi-\eta|, |\eta|)^{-N_0}$$

pour un certain $N_0 \geq 0$, B est bien définie, envoie $H^s \times H^s$ dans H^s pour s assez grand, et

$$(D_t - p(D_x))(u - B(u,u)) = O(u^3),$$

ce qui ramène donc à une équation cubique pour la nouvelle inconnue $v = u - B(u,u)$. Dans le cas qui nous occupe, l'équation est quasi-linéaire, et une application brutale de la méthode donnerait un terme correctif $B(u,u)$ qui ne serait plus borné de H^s dans H^s, mais ferait apparaître la perte d'une dérivée. Toutefois, des formes normales pour des équations quasi-linéaires ont été utilisées par divers auteurs au cours des années récentes. Nous citons [32, 6, 5, 8] pour les équations de Klein-Gordon quasi-linéaires et [14, 13, 18, 2, 16] pour des équations issues de la mécanique des fluides. L'idée essentielle consiste à remarquer qu'il n'est pas nécessaire d'éliminer toutes les contributions quadratiques, mais uniquement celles qui vont effectivement contribuer à l'inégalité d'énergie. Dans la sous-section 2.2.1 nous effectuons donc une telle forme normale pour traiter le terme en $\mathrm{Op}^{\mathrm{B}}(C''(U,\eta))V_s^I$ dans (16). Pour cela, nous déterminons une matrice de symboles d'ordre zéro $E(U;\eta)$, linéaire en U, telle que, si nous posons

$$\widetilde{W}_s^I = (I_4 + \mathrm{Op}^{\mathrm{B}}(E(U;\cdot)))W_s^I, \tag{17}$$

nous obtenons une nouvelle inconnue, dont la norme L^2 est comparable à celle de W_s^I, et qui est solution d'une équation de la forme

(18)
$$\begin{aligned}(D_t - A(D))\widetilde{W}_s^I = \mathrm{Op}^{\mathrm{B}}\Big(\big(I_4 + E_0^d(U;\eta)\big)\tilde{A}_1(V,\eta)\big(I_4 + F_0^d(U;\eta)\big)\Big)W_s^I + \mathrm{Op}_{\mathrm{R}}^{\mathrm{B}}(A''(V^I,\eta))U \\ + \mathrm{Op}_{\mathrm{R}}^{\mathrm{B}}(\tilde{A}''(V^I,\eta))U + Q_0^I(V,W) + T_{-N}(U)W_s^I + \mathfrak{R}'(U,V),\end{aligned}$$

où E_0^d, F_0^d sont des matrices diagonales de symboles d'ordre zéro, $T_{-N}(U)$ est un opérateur régularisant gagnant N dérivées, linéaire en U, et $\mathfrak{R}'$ un nouveau reste. On définit alors à partir des normes L^2 des $\widetilde{W}_s^I$ de nouvelles énergies $\widetilde{E}_n(t;W), \widetilde{E}_3^k(t;W)$, équivalentes aux énergies de (8). Par construction, les dérivées en temps de ces énergies s'expriment à partir de quantités qui ne font plus intervenir le premier terme du membre de droite de (18). Par contre, les termes en $\mathrm{Op}^{\mathrm{B}}(A''(V^I,\eta))U, \mathrm{Op}_{\mathrm{R}}^{\mathrm{B}}(\tilde{A}''(V^I,\eta))U, T_{-N}(U)W_s^I$ de cette équation fournissent toujours une contribution à la dérivée de l'énergie, dont la décroissance en temps, données par $\|U(t,\cdot)\|_{L^\infty}$ n'est pas suffisante, ainsi que nous l'avons déjà fait remarquer.

Nous sommes donc conduits à effectuer une deuxième étape de formes normales, afin d'éliminer ces dernières contributions. Nous avons toutefois désormais l'avantage de pouvoir perdre quelques dérivées sur U dans le processus, puisque cette fonction ne porte aucune dérivée dans expressions que nous souhaitons éliminer. Ce deuxième pas de formes normales, qui fait l'objet de la sous-section 2.2.2, est donc de type semi-linéaire : nous construisons des correcteurs quadratiques de l'inconnue (ou cubiques de l'énergie), qui sont des perturbations négligeables dans les normes considérés de l'inconnue (resp. de l'énergie) que nous avons définie à l'étape précédente, et qui ne modifient la dérivée en temps de cette inconnue (resp. de cette énergie) que par des contributions cubiques (resp. quartiques), qui ont les bonnes estimations de décroissance que nous souhaitons faire apparaître.

Une fois cette deuxième étape de formes normales terminée, il ne nous reste plus qu'à propager les estimations des énergies modifiées ainsi construites (qui sont équivalentes aux énergies initiales de (8)) afin de conclure la première partie de la preuve, à savoir que (10) implique les deux dernières inégalités de (11).

2.4 Estimations uniformes I. Formulation semi-classique

Il reste désormais à prouver, pour conclure la démonstration du théorème 3 et donc du théorème 2, que (10) implique les deux premières estimations de (11). La stratégie que nous allons suivre est très voisine de celle qui a été mise en œuvre dans la première partie de cette thèse pour l'équation de Klein-Gordon cubique en dimension 1, à savoir déduire du système pseudo-différentiel un système couplé formé d'une équation différentielle ordinaire, provenant de la composante "Klein-Gordon", et d'une équation de transport, issue de la composante "ondes". L'étude de ce dernier système fournira les estimations L^∞ nécessaires.

Nous commençons notre analyse en procédant encore à une forme normale, éliminant toutes les contributions quadratiques, à l'exception du seul terme résonant en (v_+, v_-) dans l'équation des ondes qui est traité convenablement dans la suite, et nous réduisant à des équations à non-linéarité cubique. Nous n'utilisons pas directement les formes normales obtenues dans le cadre de la preuve des inégalités d'énergie, car nos buts et contraintes sont désormais différents. En effet, nous cherchons à obtenir des estimations L^∞ pour essentiellement ρ dérivées, en disposant d'hypothèses sur des normes H^s avec $s \gg \rho$. Nous pouvons donc nous permettre de perdre quelques dérivées dans la réduction par formes normales, ce qui en particulier signifie que le fait que le système soit quasi-linéaire n'importe plus guère.

Nous définissons donc à partir des fonction u_-, v_- de (5), de nouvelles inconnues u^{NF}, v^{NF}, définies à partir des précédentes en leur rajoutant une perturbation quadratique, qui sont solutions d'équations

$$(D_t + |D_x|)u^{NF} = q_w + c_w + r_w^{NF},\ (D_t + |D_x|)v^{NF} = r_{kg}^{NF}, \tag{19}$$

les membres de droite $r_w^{NF}, c_w, r_{kg}^{NF}$ étant cubiques, et où $q_w(t,x)$ est une certaine expression bilinéaire en v_+, v_-, qui ne peut être éliminée par forme normale, mais dont la structure est telle qu'elle fournira des termes de reste dans la section à suivre. Afin de déduire de ces équations des équations différentielles ordinaires pour u^{NF}, v^{NF}, nous reformulons le problème dans un cadre semi-classique. Celui-ci est introduit dans la sous-section 1.2.2, dans laquelle nous établissons également les divers résultats techniques liés au calcul pseudo-différentiel semi-classique qui nous sont utile dans la suite de l'article. Nous indiquons seulement ici les principales notations.

Désignons par h un paramètre dans $]0,1]$ (qui dans nos applications sera l'inverse du temps $h = \frac{1}{t}$). Si $a(x,\xi)$ est un symbole sur $\mathbb{R}^2 \times \mathbb{R}^2$ (qui peut également dépendre de h) i.e. une fonction C^∞, dont les dérivées vérifient des estimations de la forme

$$|(h\partial_h)^k \partial_x^{\alpha_1} \partial_\xi^{\alpha_2} a(x,\xi)| = O\big(M(x,\xi)h^{-\delta(|\alpha_1|+|\alpha_2|)}\big), \tag{20}$$

où M est un poids fixé, vérifiant des hypothèses convenables, et δ une constante dans $[0, \frac{1}{2}]$. La quantification de Weyl semi-classique de ce symbole est donnée par

$$\mathrm{Op}_h^w(a)v = \frac{1}{(2\pi h)^2} \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{i(x-y)\cdot\frac{\xi}{h}} a\Big(\frac{x+y}{2}, \xi\Big) v(y)\, dy d\xi. \tag{21}$$

Si l'on définit

$$\widetilde{u}(t,x) = t u^{NF}(t,tx),\ \widetilde{v}(t,x) = t v^{NF}(t,tx), \tag{22}$$

on obtient que ces deux fonctions vérifient les équations

$$\begin{aligned} &(D_t - \mathrm{Op}_h^w(x\cdot\xi - |\xi|))\widetilde{u} = h^{-1}\left[q_w(t,tx) + c_w(t,tx) + r_w^{NF}(t,tx)\right] \\ &(D_t - \mathrm{Op}_h^w(x\cdot\xi - \langle\xi\rangle))\widetilde{v} = h^{-1} r_{kg}^{NF}(t,tx). \end{aligned} \tag{23}$$

Nous introduisons également les opérateurs

$$\mathcal{M}_j = \frac{1}{h}\Big(x_j|\xi| - \xi_j\Big),\ \mathcal{L}_j = \frac{1}{h}\Big(x_j - \frac{\xi_j}{\langle\xi\rangle}\Big), \tag{24}$$

dont les symboles sont donnés respectivement (à multiplication près par $|\xi|$ dans le cas du premier) par la dérivée en ξ des expressions $x\cdot\xi - |\xi|$ et $x\cdot\xi - \langle\xi\rangle$ de (23). En utilisant l'équation, on peut exprimer $\mathcal{M}_j\widetilde{u}$ (resp. $\mathcal{L}_j\widetilde{v}$) en fonction de $Z_j u^{NF}$ (resp. $Z_j v^{NF}$) et de q_w, c_w, r_w^{NF} (resp. r_{kg}^{NF}). Ces expressions étant établies au début de la section 3.2 on peut passer à l'obtention des équations locales vérifiées par $\widetilde{u}, \widetilde{v}$.

2.5 Estimations uniformes II. Équations différentielles ordinaires

Nous avons vu dans la description de la première partie de la thèse comment, en dimension un, on déduit d'une équation de Klein-Gordon non-linéaire de la forme de la seconde équation (23) une équation différentielle ordinaire. La méthode que nous utilisons ici est semblable : Nous introduisons la lagrangienne

$$\Lambda_{kg} = \Big\{(x,\xi); x - \frac{\xi}{\langle\xi\rangle} = 0\Big\} \tag{25}$$

et décomposons essentiellement $\widetilde{v}$ en une partie microlocalisée sur un voisinage d'ordre $\sqrt{h}$ de Λ_{kg}, et une partie microlocalisée hors d'un tel voisinage. La deuxième contribution peut être estimée dans L^∞ en $h^{\frac{1}{2}-0}$ fois des normes L^2 d'itérés de champs $\mathcal{L}$ agissant sur $\widetilde{v}$ (qui seront elles-mêmes contrôlées par les hypothèses L^2 du théorème 3). La contribution principale à $\widetilde{v}$ est donc fournie par la partie microlocalisée près de Λ_{kg}, soit $\widetilde{v}_{\Lambda_{kg}}$. Par commutation d'une troncature pseudo-différentielle à la seconde équation (23), on obtient pour cette dernière

$$[D_t - \mathrm{Op}_h^w(x\cdot\xi - \langle\xi\rangle)]\widetilde{v}_{\Lambda_{kg}} = h^{-1}\Gamma^{kg}[r_{kg}^{NF}(t,tx)] + \text{ reste},$$

Γ^{kg} désignant la troncature microlocale près de Λ_{kg}. Développant le symbole du membre de gauche sur Λ_{kg}, on en déduit finalement l'équation différentielle cherchée. Combinant enfin cette équation différentielle avec les estimations a priori du reste, on déduit des inégalités (10) la seconde estimation (11) (avec $\rho = 0$, le cas d'un ρ général étant traité de même, au prix de quelques difficultés techniques supplémentaires).

Nous utilisons la même stratégie pour obtenir des estimations uniformes de $\widetilde{u}$, avec toutefois une différence importante. La lagrangienne naturelle à faire intervenir est ici

$$\Lambda_w = \Big\{(x,\xi); x - \frac{\xi}{|\xi|} = 0\Big\} \tag{26}$$

qui, contrairement à Λ_{kg}, n'est pas un graphe, mais se projette sur la base selon une hypersurface. A cause de cela, le problème classique associé à la première équation (23) n'est plus une équation différentielle ordinaire, mais une équation de transport. Celle-ci est obtenue en procédant *mutatis mutandis* comme pour le cas de Klein-Gordon, au prix de quelques difficultés techniques supplémentaires liées aux petites fréquences. On décompose $\widetilde{u}$ en une contribution $\widetilde{u}_{\Lambda_w}$ microlocalisée dans un voisinage $\left|x - \frac{\xi}{|\xi|}\right| < \frac{1}{h^{\frac{1}{2}-\sigma}}$ de Λ_w (avec $\sigma > 0$ petit), et une contribution localisée hors d'un tel voisinage, qui sera convenablement contrôlée grâce aux estimations L^2 d'opérateurs itérés de la forme $\mathcal{M}_j$ agissant sur $\widetilde{u}$. Par microlocalisation de la première équation (23) près de Λ_w, on obtient une équation

$$[D_t - \mathrm{Op}_h^w(x\cdot\xi - |\xi|)]\widetilde{u}_{\Lambda_w} = \text{ terme contrôlé}$$

puis, développant le symbole $x\cdot\xi - |\xi|$ sur Λ_w, on en déduit l'équation de transport cherchée. En fait, comme il est nécessaire de tenir compte des dégénérescences qui se produisent pour les petites fréquences, la stratégie précédente doit être affinée, en procédant à une troncature dyadique en fréquence supplémentaire, et en écrivant une équation de transport pour chacune de ces fréquences avant de resommer. Une fois l'équation de transport établie, il reste à l'intégrer par la méthode des caractéristiques, afin d'obtenir les estimations de la première inégalité (11). Nous prouvons d'ailleurs des inégalités plus précises, qui donnent également le comportement quasi-optimal loin du cône d'onde (voir sous-section 3.3.2).

Une fois les estimations précédentes établies, le théorème 3 en découle, et par conséquent le théorème 2 également.

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Partie I

Existence globale et comportement asymptotique pour l'équation de Klein-Gordon quasi-linéaire en dimension un, à données de Cauchy modérément décroissantes

(À paraître dans *Bulletin de la Société mathématique de France*)

Global existence and asymptotics for quasi-linear one-dimensional Klein-Gordon equations with mildly decaying Cauchy data

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Let u be a solution to a quasi-linear Klein-Gordon equation $\Box u + u = P(u, \partial_t u, \partial_x u; \partial_t\partial_x u, \partial_x^2 u)$ in one-space dimension, where P is a homogeneous polynomial of degree three, and with smooth Cauchy data of size $\varepsilon \to 0$. It is known that, under a suitable condition on the nonlinearity, the solution is global-in-time for compactly supported Cauchy data. We prove in this paper that the result holds even when data are not compactly supported but just decaying as $\langle x\rangle^{-1}$ at infinity, combining the method of Klainerman vector fields with a semiclassical normal forms method introduced by Delort. Moreover, we get a one term asymptotic expansion for u when $t \to +\infty$.

Introduction

The goal of this paper is to prove the global existence and to study the asymptotic behaviour of the solution u of the one-dimensional nonlinear Klein-Gordon equation, when initial data are small, smooth and slightly decaying at infinity. We will consider the case of a quasi-linear cubic

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nonlinearity, namely a homogeneous polynomial P of degree 3 in $(u, \partial_t u, \partial_x u;\, \partial_t\partial_x u, \partial_x^2 u)$, affine in $(\partial_t\partial_x u, \partial_x^2 u)$, so that the initial valued problem is written as

$$
(1) \qquad \begin{cases} \Box u + u = P(u, \partial_t\partial_x u, \partial_x^2 u\,;\partial_t u, \partial_x u) \\ u(1,x) = \varepsilon u_0(x) \\ \partial_t u(1,x) = \varepsilon u_1(x) \end{cases} \qquad t \geq 1\,, x\in\mathbb{R}\,, \varepsilon \in]0,1[\,.
$$

Our main concern is to obtain results for data which have only mild decay at infinity (i.e. which are $O(|x|^{-1})$, $x \to +\infty$), while most known results for quasi-linear Klein-Gordon equations in dimension 1 are proved for compactly supported data. In order to do so, we have to develop a new approach, that relies on semiclassical analysis, and that allows to obtain for Klein-Gordon equations results of global existence making use of Klainerman vector fields and usual energy estimates, instead of L^2 estimates on the hyperbolic foliation of the interior of the light cone, as done for instance in an early work of Klainerman [24] and more recently in the paper of LeFloch, Ma [26].

We recall first the state of the art of the problem. In general, the problem in dimension 1 is critical, contrary to the problem in higher dimension which is subcritical. In fact, in space dimension d, the best time decay one can expect for the solution is $\|u(t,\cdot)\|_{L^\infty} = O(t^{-\frac{d}{2}})$: therefore, in dimension 1 the decay rate is $t^{-\frac{1}{2}}$, and for a cubic nonlinearity, depending for example only on u, one has $\|P(u)\|_{L^2} \leq Ct^{-1}\|u(t,\cdot)\|_{L^2}$, with a time factor t^{-1} just at limit of integrability. In space dimension $d \geq 3$, it is well known from works of Klainerman [24] and Shatah [33] that the analogous problem has global-in-time solutions if ε is sufficiently small. In [24], Klainerman proved it for smooth, compactly supported initial data, with nonlinearities at least quadratic, using the Lorentz invariant properties of $\Box + 1$ to derive uniform decay estimates and generalized energy estimates for solutions u to linear inhomogeneous Klein-Gordon equations. Simultaneously, in [33] Shatah proved this result for smooth and integrable initial data, extending Poincaré's theory of normal forms for ordinary differential equations to the case of nonlinear Klein-Gordon equations. For space dimension $d = 2$, in [15] Hörmander refined Klainerman's techniques to obtain new time decay estimates of solutions to linear inhomogeneous Klein-Gordon equations. He showed that, for quadratic nonlinearities, the solution exists over $[-T_\varepsilon, T_\varepsilon]$ with an existence time T_ε such that $\lim_{\varepsilon\to 0}\varepsilon \log T_\varepsilon = \infty$ (while $\lim_{\varepsilon\to 0}\varepsilon^2 T_\varepsilon = \infty$ for $d = 1$). In addition, he conjectured that $T_\varepsilon = \infty$ (while for $d = 1$, $\liminf_{\varepsilon\to 0}\varepsilon^2\log T_\varepsilon > 0$). The first conjecture has been proved by Ozawa, Tsutaya and Tsutsumi in [30] in the semi-linear case, after partial results by Georgiev, Popivanov in [10], and Kosecki in [25] (for nonlinearities verifying some "suitable null conditions"). Later, in [31] Ozawa, Tsutaya and Tsutsumi announced the extension of their proof to the quasi-linear case and studied scattering of solutions. In space dimension 1, Moriyama, Tonegawa and Tsutsumi [29] have shown that the solution exists on a time interval of length longer or equal to e^{c/ε^2}, where ε is the Cauchy data's size, with a nonlinearity vanishing at least at order three at zero, or semi-linear. They also proved that the corresponding solution asymptotically approaches the free solution of the Cauchy problem for the linear Klein-Gordon equation. The fact that in general the solution does not exist globally in time was proved by Yordanov in [35], and independently by Keel and Tao [21]. However, there exist examples of nonlinearities for which the corresponding solution is global-in-time: on one hand, if P depends only on u and not on its derivatives; on the other hand, for seven special nonlinearities considered by Moriyama in [28]. A natural question is then posed by Hörmander, in [16, 15]: can we formulate a structure condition for the nonlinearity, analogous to the null condition introduced by Christodoulou [3] and Klainerman [23] for the wave equation, which implies global existence? In [7, 6] Delort proved that, when initial data are compactly supported, one can find a *null condition*, under which global existence is ensured. This condition is likely optimal, in the sense that when the structure hypothesis is violated, he constructed in [4] approximate solutions blowing up at e^{A/ε^2}, for an explicit constant A. This suggests that also the exact solution of the problem blows up in time at e^{A/ε^2}, but this remains still unproven.

Once global existence is ensured, a natural question that arises concerns the long time behaviour of the solutions. While for $d \geq 2$ it is known that the global solution behaves like a free solution, in space dimension one, only few results were known, including for the simpler equation

$$\Box u + u = \alpha u^2 + \beta u^3 + \text{order } 4 .$$

For this equation, Georgiev and Yordanov [11] proved that, when $\alpha = 0$, the distance between the solution u and linear solutions cannot tend to 0 when $t \to \infty$, but they do not obtain an asymptotic description of the solution (except for the particular case of sine-Gordon $\Box u + \sin u = 0$, for which they use methods of "nonlinear scattering"). In [27], Lindblad and Soffer studied the scattering problem for long range nonlinearities, proving that for all prescribed asymptotic solutions there is a solution of the equation with such behavior, for some choice of initial data, and finding the complete asymptotic expansion of the solutions. In [14], a sharp asymptotic behaviour of small solutions in the quadratic, semilinear case is proved by Hayashi and Naumkin, without the condition of compact support on initial data, using the method of normal forms of Shatah. The only other cases in dimension one for which the asymptotic behaviour is known concern nonlinearities studied by Moriyama in [28], where he showed that solutions have a free asymptotic behaviour, assuming the initial data to be sufficiently small and decaying at infinity.

Some results about global existence and long time behaviour are also known for solutions to systems of coupled Klein-Gordon equations. In dimension $d = 3$, we cite the work of Germain [12], and of Ionescu, Pausader [18], for a system of coupled Klein-Gordon equations with different speeds, with a quadratic nonlinearity, respectively in the semilinear case for the former, and in the quasi-linear one for the latter. For data small, smooth and localized, they prove that a global solution exists and scatters. In dimension $d = 2$, Delort, Fang and Xue proved in [8] the global existence of solutions for a quasi-linear system of two Klein-Gordon equations, with masses m_1, m_2, $m_1 \neq 2m_2$ and $m_2 \neq 2m_1$, for small, smooth, compactly supported Cauchy data, extending the result proved by Sunagawa in [34] in the semilinear case. Moreover, they proved that the global existence holds true also in the resonant case, e.g. when $m_1 = 2m_2$, and a convenient null condition is satisfied by nonlinearities. The same result in the resonant case is also proved by Katayama, Ozawa [19], and by Kawahara, Sunagawa [20], in which the structural condition imposed on nonlinearities includes the Yukawa type interaction, which was excluded from the *null condition* in the sense of [8]. We should cite also the paper [32] by Schottdorf, where he proved global well-posedness and scattering result in the semilinear case, in dimension 2 and higher, for small H^s data, using the contraction mapping technique in U^2/V^2 based spaces. There are some results also in dimension 1. In [22], Kim shows that the solution to a system of semilinear cubic Klein-Gordon equations, verifying a suitable structure condition, and with small, non compactly supported initial data in some appropriate Sobolev space, is global-in-time and has the optimal decay $t^{-1/2}$, as t tends to infinity. We should also cite the work of Guo, Han and Zhang [13] on the global existence and the long time behaviour of the solution to the one dimensional Euler-Poisson system, under weak conditions on the initial data, and of Candy and Lindblad [2], on the one dimensional cubic Dirac equation.

In most of above mentioned papers dealing with the one dimensional scalar problem, two key tools are used: normal forms methods and/or Klainerman vector fields Z. In particular, the latter are useful since they have good properties of commutation with the linear part of the equation, and their action on the nonlinearity $ZP(u)$ may be expressed from u, Zu using Leibniz rule. This allows one to prove easily energy estimates for $Z^k u$, and then to deduce from them L^∞ bounds for u, through Klainerman-Sobolev type inequalities. However, in these papers the global existence is proved assuming small, *compactly supported* initial data. This is related to the fact that the aforementioned authors use in an essential way a change of variable in hyperbolic coordinates, that does not allow for non compactly supported Cauchy data. Our aim is to extend the result of global existence for cubic quasi-linear nonlinearities in the case of small compactly supported Cauchy data of [7, 6], to the more general framework of data with mild polynomial decay. To do that, we will combine the Klainerman vector fields' method with the one introduced by Delort in [5].

In [5], Delort develops a semiclassical normal form method to study global existence for nonlinear hyperbolic equations with small, smooth, decaying Cauchy data, in the critical regime and when the problem does not admit Klainerman vector fields. The strategy employed is to construct, through semiclassical analysis, some *pseudo-differential* operators which commute with the linear part of the considered equation, and which can replace vector fields when combined with a microlocal normal form method. Our aim here is to show that one may combine these ideas together with the use of Klainerman vector fields to obtain, in one dimension, and for nonlinearities satisfying the null condition, global existence and modified scattering.

In our paper, we prove the global existence of the solution u by a *boostrap* argument, namely by showing that we can propagate some suitable *a priori* estimates made on u. We propagate two types of estimates: some energy estimates on u, Zu, and some uniform bounds on u. To prove the propagation of energy estimates is the simplest task. We essentially write an energy inequality for a solution u of the Klein-Gordon equation in the quasi-linear case (the main reference is the book of Hörmander [15], chapter 7), and then we use the commutation property of the Klainerman vector fields Z with the linear part of the equation to derive an inequality also for Zu. Moreover, Z acts like a derivation on the nonlinearity, so the Leibniz rule holds and we can estimate ZP in term of u, Zu. Injecting *a priori* estimates in energy inequalities and choosing properly all involved constants allow us to obtain the result.

The main difficulty is to prove that the uniform estimates hold and can be propagated. Actually, as mentioned above, the one dimensional Klein-Gordon equation is critical, in the sense that the expected decay for $\|u(t,\cdot)\|^2_{L^\infty}$ is in t^{-1}, so is not integrable. A drawback of that is that one cannot prove energy estimates that would be uniform as time tends to infinity. Consequently, a Klainerman-Sobolev inequality, that would control $\|u(t,\cdot)\|_{L^\infty}$ by $t^{-1/2}$ times the L^2 norms of u, Zu, would not give the expected optimal L^∞-decay of the solution, but only a bound in $t^{-\frac{1}{2}+\sigma}$ for some positive σ, which is useless to close the bootstrap argument. The idea to overcome this difficulty is, following the approach of Delort in [5], to rewrite (1) in semiclassical coordinates, for some new unknown function v. The goal is then to deduce from the PDE satisfied by v an ODE from which one will be able to get a uniform L^∞ bound for v (which is equivalent to the optimal $t^{-1/2}$ L^∞-decay of u). Let us describe our approach for a simple model of Klein-Gordon equation. Denoting by D_t, D_x respectively $\frac{1}{i}\partial_t, \frac{1}{i}\partial_x$, we consider the following :

$$(D_t - \sqrt{1+D_x^2})u = \alpha u^3 + \beta |u|^2 u + \gamma |u|^2 \bar{u} + \delta \bar{u}^3, \tag{2}$$

where $\alpha, \beta, \gamma, \delta$ are constants, β being *real* (this last assumption reflecting the null condition on that example). Performing a semiclassical change of variables and unknowns $u(t,x) = \frac{1}{\sqrt{t}} v(t, \frac{x}{t})$, we rewrite this equation as

$$[D_t - Op_h^w(\lambda_h(x,\xi))]v = h(\alpha v^3 + \beta|v|^2 v + \gamma|v|^2\bar{v} + \delta\bar{v}^3)\,, \tag{3}$$

where $\lambda_h(x,\xi) = x\xi + \sqrt{1+\xi^2}$, the semiclassical parameter h is defined as $h := 1/t$, and the Weyl quantization of a symbol a is given by

$$Op_h^w(a)v = \frac{1}{2\pi h}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{\frac{i}{h}(x-y)\xi} a\Big(\frac{x+y}{2},\xi\Big)v(y)\,dyd\xi\,.$$

One introduces the manifold $\Lambda = \{(x,\xi)\,|\,x + \frac{\xi}{\sqrt{1+\xi^2}} = 0\}$ as in figure .1, which is the graph of the smooth function $d\varphi(x)$, where $\varphi :]-1,1[\to \mathbb{R}$ is $\varphi(x) = \sqrt{1-x^2}$.

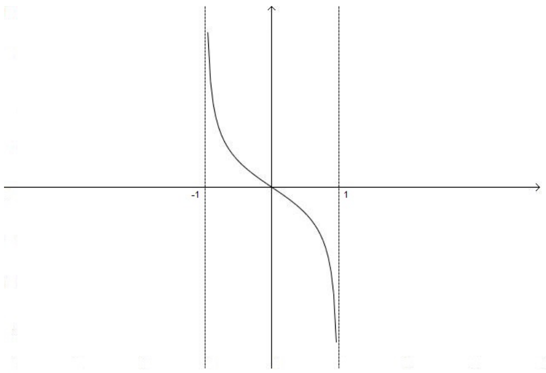

Figure .1: Λ for the Klein Gordon equation.

One can deduce an ODE from (3), developing the symbol $\lambda_h(x,\xi)$ on Λ, i.e. on $\xi = d\varphi(x)$. One obtains a first term $a(x)$ independent of ξ and a remainder, which turns out to be integrable in time as may be shown using some ideas of Ifrim-Tataru [17] and the L^2 estimates verified by v and by the action of the Klainerman vector field on v. In this way, one proves that v is solution of the equation

$$D_t v = a(x)v + h\beta|v|^2 v + \text{non characteristic terms} + \text{remainder of higher order in } h\,. \tag{4}$$

Then the idea is to eliminate *non characteristic* terms by a normal forms argument, introducing a new function f which will be finally solution of an ordinary differential equation

$$D_t f = a(x)f + h\beta|f|^2 f + \text{remainder of higher order in } h\,. \tag{5}$$

From this equation, one easily derives an uniform control L^∞ on f, and then on the starting solution u. The analysis of the above ODE provides as well a one term asymptotic expansion of the solution of equation (2) (or, more generally of the solution (1)), as proved in the last section of this paper. This expansion shows that, in general, scattering does not hold, and that one has only modified scattering. This is in contrast with higher dimensional problems for the Klein-Gordon equation where, as we already said, global solutions have at infinity the same behaviour as free solutions.

We end this introduction with few words about the case of quadratic nonlinearities, in one space dimension. In [7], Delort proves global existence and modified scattering for an equation of the form (1), where the nonlinearity may have a quadratic component, i.e. for the equation

$$\begin{cases} \Box u + u = F(u, \partial_t\partial_x u, \partial_x^2 u\,;\partial_t u, \partial_x u) \\ u(1,x) = \varepsilon u_0(x) \\ \partial_t u(1,x) = \varepsilon u_1(x) \end{cases} \qquad t \geq 1\,, x\in\mathbb{R}\,, \varepsilon \in]0,1[\,. \tag{6}$$

where

$$F(u,\partial_t\partial_x u,\partial_x^2 u\,;\partial_t u,\partial_x u)=Q(u,\partial_t\partial_x u,\partial_x^2 u\,;\partial_t u,\partial_x u)+P(u,\partial_t\partial_x u,\partial_x^2 u\,;\partial_t u,\partial_x u)$$

with Q (resp. P) homogeneous polynomial of degree 2 (resp. 3), and where one assumes a convenient null condition, that generalizes the one we impose here on the sole cubic terms. We believe that our method could be extended to that framework, providing global existence and modified scattering for (6), with small, mildly decaying initial data (instead of the compactly supported ones considered in [7]). Actually, it is well known that one may always perform a Shatah's normal form argument in order to reduce a Klein-Gordon equation with quadratic nonlinearities to a cubic one, when solutions are small. For quasi-linear equations, one should be cautious in order not to increase the number of derivatives in the nonlinearity, but this technical difficulty may be overcome using paradifferential calculus. Consequently, the case of quadratic nonlinearities can be reduced, at least in principle, to the cubic one, if one accepts to replace local cubic nonlinearities by nonlocal ones. We decided here to limit ourselves to the purely cubic case, in order to avoid the technicalities that are inherent to such reductions and keep the paper reasonably long.

1 Statement of the main results

The Cauchy problem we are considering is

$$\begin{cases}\Box u+u=P(u,\partial_t\partial_x u,\partial_x^2 u\,;\partial_t u,\partial_x u)\\ u(1,x)=\varepsilon u_0(x)\\ \partial_t u(1,x)=\varepsilon u_1(x)\end{cases}\qquad t\geq 1\,,x\in\mathbb{R} \tag{1.1}$$

where $\Box:=\partial_t^2-\partial_x^2$ is the D'Alembert operator, $\varepsilon\in]0,1[$, u_0,u_1 are smooth enough functions. P denotes a homogeneous polynomial of degree three, with real constant coefficients, affine in $(\partial_t\partial_x u,\partial_x^2 u)$. We can highlight this particular dependence on second derivatives following the approach of [7] and decomposing P as

$$P(u,\partial_t\partial_x u,\partial_x^2 u\,;\partial_t u,\partial_x u)=P'(u\,;\partial_t u,\partial_x u)+P''(u,\partial_t\partial_x u,\partial_x^2 u\,;\partial_t u,\partial_x u)\,, \tag{1.2}$$

where P',P'' are homogeneous polynomials of degree three, P'' linear in $(\partial_t\partial_x u,\partial_x^2 u)$. Moreover

$$\begin{aligned}P'(X_1;\,Y_1,Y_2)&=\sum_{k=0}^{3}i^kP'_k(X_1;-iY_1,-iY_2)\\ P''(X_1,X_2,X_3;\,Y_1,Y_2)&=\sum_{k=0}^{2}i^kP''_k(X_1,-X_2,-X_3;-iY_1,-iY_2)\end{aligned} \tag{1.3}$$

where P'_k is homogeneous of degree k in (Y_1,Y_2) and of degree $3-k$ in X_1, while P''_k is homogeneous of degree 1 in (X_2,X_3) and of degree k in (Y_1,Y_2). We denote $P_k=P'_k+P''_k$. For $x\in]-1,1[$, define

$$\begin{aligned}\omega_0(x)&:=\frac{1}{\sqrt{1-x^2}}\,,\\ \omega_1(x)&:=\frac{-x}{\sqrt{1-x^2}}\,,\end{aligned} \tag{1.4}$$

and

(1.5)
$$\Phi(x):=P'_1(1;\omega_0(x),\omega_1(x))+P''_1(1,\omega_0(x)\omega_1(x),\omega_1^2(x);\omega_0(x),\omega_1(x))+3P'_3(1;\omega_0(x),\omega_1(x))\,.$$

Definition 1.1. We say that the nonlinearity P satisfies the *null condition* if and only if $\Phi \equiv 0$.

Our goal is to prove that there is a global solution of (1.1) when ε is sufficiently small, u_0, u_1 decay rapidly enough at infinity, and when the cubic nonlinearity satisfies the *null condition.* We state the main theorem below.

Theorem 1.2 (Main Theorem)**.** *Suppose that the nonlinearity P satisfies the null condition. Then there exists an integer s sufficiently large, a positive small number σ, an $\varepsilon_0 \in]0,1[$ such that, for any real valued $(u_0,u_1) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})$ satisfying*

$$\|u_0\|_{H^{s+1}} + \|u_1\|_{H^s} + \|xu_0\|_{H^2} + \|xu_1\|_{H^1} \le 1\,, \tag{1.6}$$

for any $0 < \varepsilon < \varepsilon_0$, the problem (1.1) has an unique solution $u \in C^0([1,+\infty[;H^{s+1}) \cap C^1([1,+\infty[;H^s)$. Moreover, there exists a 1-parameter family of continuous function $a_\varepsilon : \mathbb{R} \to \mathbb{C}$, uniformly bounded and supported in $[-1,1]$, a function $(t,x) \to r(t,x)$ with values in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, bounded in $t \ge 1$, such that, for any $\varepsilon \in]0,\varepsilon_0]$, the global solution u of (1.1) has the asymptotic expansion

$$u(t,x) = \mathrm{Re}\left[\frac{\varepsilon}{\sqrt{t}} a_\varepsilon\left(\frac{x}{t}\right)\exp\left[it\varphi\left(\frac{x}{t}\right) + i\varepsilon^2\left|a_\varepsilon\left(\frac{x}{t}\right)\right|^2 \Phi_1\left(\frac{x}{t}\right)\log t\right]\right] + \frac{\varepsilon}{t^{\frac{1}{2}+\sigma}} r(t,x)\,, \tag{1.7}$$

where $\varphi(x) = \sqrt{1-x^2}$, and

$$\begin{aligned}\Phi_1(x) = \frac{1}{8}\langle\omega_0(x)\rangle^{-4}\big[&3P_0(1,\omega_0(x)\omega_1(x),\omega_1(x)^2;\,\omega_0(x),\omega_1(x)) \\ &+P_2(1,\omega_0(x)\omega_1(x),\omega_1(x)^2;\,\omega_0(x),\omega_1(x))\big]\,,\end{aligned} \tag{1.8}$$

with $\langle x\rangle = \sqrt{1+x^2}$.

We denote by Z the Klainerman vector field for the Klein-Gordon equation, that is $Z := t\partial_x + x\partial_t$, and by Γ a generic vector field in the set $\mathcal{Z} = \{Z, \partial_t, \partial_x\}$. The most remarkable properties of these vector fields are the commutation with the linear part of the equation in (1.1), namely

$$[\Box + 1, \Gamma] = 0\,, \tag{1.9}$$

and the fact that they act like a derivation on the cubic nonlinearity. We also denote by $W^{t,\rho,\infty}$ a modified Sobolev space, made by functions $t \to \psi(t,\cdot)$ defined on an interval, such that $\langle D_x\rangle^{\rho-i}D_t^i u \in L^\infty$, for $i \le 2$, with the norm

$$\|\psi(t,\cdot)\|_{W^{t,\rho,\infty}(\mathbb{R})} := \sum_{i=0}^{2} \|\langle D_x\rangle^{\rho-i}D_t^i\psi(t,\cdot)\|_{L^\infty(\mathbb{R})}\,. \tag{1.10}$$

The proof of the main theorem is based on a *bootstrap* argument. In other words, we shall prove that we are able to propagate some *a priori* estimates made on a solution u of (1.1) on some interval $[1,T]$, for some $T > 1$ fixed, as stated in the following theorem.

Theorem 1.3 (Bootstrap Theorem)**.** *There exist two integers s, ρ large enough, $s \gg \rho$, an $\varepsilon_0 \in]0,1[$ sufficiently small, and two constants $A, B > 0$ sufficiently large such that, for any $0 < \varepsilon < \varepsilon_0$, if u is a solution of (1.1) on some interval $[1,T]$, for $T > 1$ fixed, and satisfies*

$$\|u(t,\cdot)\|_{W^{t,\rho,\infty}} \le A\varepsilon t^{-\frac{1}{2}} \tag{1.11a}$$

$$\|Zu(t,\cdot)\|_{H^1} \le B\varepsilon t^\sigma\,, \qquad \|\partial_t Zu(t,\cdot)\|_{L^2} \le B\varepsilon t^\sigma \tag{1.11b}$$

$$\|u(t,\cdot)\|_{H^s} \le B\varepsilon t^\sigma\,, \qquad \|\partial_t u(t,\cdot)\|_{H^{s-1}} \le B\varepsilon t^\sigma\,, \tag{1.11c}$$

for every $t \in [1,T]$, for some $\sigma \geq 0$ small, then it verifies also

$$\|u(t,\cdot)\|_{W^{t,\rho,\infty}} \leq \frac{A}{2}\varepsilon t^{-\frac{1}{2}} \tag{1.12a}$$

$$\|Zu(t,\cdot)\|_{H^1} \leq \frac{B}{2}\varepsilon t^{\sigma}\,, \qquad \|\partial_t Zu(t,\cdot)\|_{L^2} \leq \frac{B}{2}\varepsilon t^{\sigma} \tag{1.12b}$$

$$\|u(t,\cdot)\|_{H^s} \leq \frac{B}{2}\varepsilon t^{\sigma}\,, \qquad \|\partial_t u(t,\cdot)\|_{H^{s-1}} \leq \frac{B}{2}\varepsilon t^{\sigma}\,. \tag{1.12c}$$

In section 2 we show that energy bounds (1.11b), (1.11c) can be propagated, simply recalling an energy inequality obtained by Hörmander in [15] for a solution u of a quasi-linear Klein-Gordon equation, and applying it to $\partial_x^{s-1}u$ and Zu. Sections from 3 to 5 concern instead the proof of the uniform estimate's propagation. Furthermore, in section 5 we derive also the asymptotic behaviour of the solution u.

To conclude, we can mention that we will mainly focus on not very high frequencies, for it is easier to control what happens for very large frequencies which correspond to points on Λ in figure .1 close to vertical asymptotic lines. This is justified by the fact that contributions of frequencies of the solution larger than $h^{-\beta}$, for a small positive β, have L^2 norms of order $O(h^N)$ if $s\beta \gg N$, assuming small H^s estimates on v. In this way, most of the analysis is reduced to frequencies lower than $h^{-\beta}$.

2 Generalised energy estimates

With notations introduced in the previous section, we define

$$E_0(t,u) = \big(\|\partial_t u(t,\cdot)\|^2_{L^2} + \|\partial_x u(t,\cdot)\|^2_{L^2} + \|u(t,\cdot)\|^2_{L^2}\big)^{1/2} \tag{2.1}$$

as the square root of the energy associated to the solution u of (1.1) at time t, and $E^{\Gamma}_N(t,u) = \sum_{k=0}^{N}\big(E_0(t,\Gamma^k u)^2\big)^{1/2}$, for a fixed Γ. The goal of this section is to obtain an energy inequality involving $E^{\Gamma}_N(t,u)$. In particular, since the aim is to propagate *a priori* energy bounds on u, i.e. $\|u(t,\cdot)\|_{H^s}$, $\|\partial_t u(t,\cdot)\|_{H^{s-1}}$, $\|Zu(t,\cdot)\|_{H^1}$ and $\|\partial_t Zu(t,\cdot)\|_{L^2}$, we will consider on one hand $E^{\partial_x}_{s-1}(t,u)$ where all Γ are equal to ∂_x, and on the other $E^Z_1(t,u)$ where $\Gamma = Z$. Often in what follows we will denote partial derivatives with respect to t and x respectively by ∂_0 and ∂_1.

We will use the following result, which concerns the specific energy inequality for the Klein-Gordon equation in the quasi-linear case, and which is presented here without proof (see lemma 7.4.1 in [15] for further details).

Lemma 2.1. *Let u be a solution of*

$$\Box u + u + \gamma^{01}\partial_0\partial_1 u + \gamma^{11}\partial_1^2 u + \gamma^0\partial_0 u + \gamma^1\partial_1 u = f\,, \tag{2.2}$$

where functions $\gamma^{ij} = \gamma^{ij}(t,x)$, $\gamma^j = \gamma^j(t,x)$ are smooth, such that $\sum_{i,j=0}^{1}|\gamma^{ij}| + |\gamma^j| \leq \frac{1}{2}$. Then,

$$E_0(t,u) \leq C\big[E_0(1,u) + \int_1^t \big(\|f(\tau,\cdot)\|_{L^2} + C(\tau)E_0(\tau,u)\big)d\tau\big]\,, \tag{2.3}$$

where $C(\tau) := \sum_{i,j,h=0}^{1} \sup_x \big(|\partial_h \gamma^{ij}(\tau,x)| + |\partial_h \gamma^j(\tau,x)|\big)$.

We can rewrite the equation in (1.1) in the same form as in lemma 2.1, especially highlighting the linear dependence on second derivatives,

$$\Box u + u + \gamma^{01}\partial_0\partial_1 u + \gamma^{11}\partial_1^2 u + \gamma^0\partial_0 u + \gamma^1\partial_1 u = 0\,, \tag{2.4}$$

where coefficients γ^{ij}, γ^j are homogeneous polynomials of degree two in $(u, \partial_0 u, \partial_1 u)$. Let us apply $\partial_1^{s'}$, $s' := s-1$, to this equation. If u is a solution of (2.4), then $\partial_1^{s'} u$ satisfies

$$\Box\partial_1^{s'} u + \partial_1^{s'} u + \partial_1^{s'}\big(\gamma^{01}\partial_0\partial_1 u + \gamma^{11}\partial_1^2 u + \gamma^0\partial_0 u + \gamma^1\partial_1 u\big) = 0\,, \tag{2.5}$$

and applying the Leibniz rule, we obtain that $\partial_1^{s'} u$ is solution of the equation

$$\Box\partial_1^{s'} u + \partial_1^{s'} u + \gamma^{01}\partial_0\partial_1(\partial_1^{s'} u) + \gamma^{11}\partial_1^2(\partial_1^{s'} u) + \gamma^0\partial_0(\partial_1^{s'} u) + \gamma^1\partial_1(\partial_1^{s'} u) = f^{s'}\,, \tag{2.6}$$

where $f^{s'}$ is a linear combination of terms of the form

$$\begin{aligned} &(\partial_1^{s'_1}\partial_i^{\alpha_1} u)\,(\partial_1^{s'_2}\partial_j^{\alpha_2} u)\,(\partial_1^{s'_3}\partial_{ij}^2 u)\,, \\ &(\partial_1^{s'_1}\partial_i^{\alpha_1} u)\,(\partial_1^{s'_2}\partial_j^{\alpha_2} u)\,(\partial_1^{s'_3}\partial_h u)\,, \end{aligned} \tag{2.7}$$

for $i, j, h, \alpha_1, \alpha_2 = 0, 1$, $s'_1 + s'_2 + s'_3 = s'$, $s'_3 < s'$. So taking the L^2 norm and observing that at most one index s'_j can be larger than $s'/2$, we have

$$\|f^{s'}(t,\cdot)\|_{L^2} \le \Big(\sum_{\substack{i+j=0 \\ j\le 2}}^{[\frac{s'}{2}]+2} \|\partial_x^i\partial_t^j u(t,\cdot)\|_{L^\infty}^2\Big) E_{s'}^{\partial_1}(t,u) \le \|u(t,\cdot)\|_{W^{t,\rho,\infty}}^2 E_{s'}^{\partial_1}(t,u)\,, \tag{2.8}$$

for any finite $\rho \ge [\frac{s'}{2}] + 3$. Rewriting inequality (2.3) for $\partial_1^{s'} u$, where $s' = s-1$ and $C(\tau) \le \|u(\tau,\cdot)\|_{W^{t,2,\infty}}^2$, we obtain

$$E_{s-1}^{\partial_1}(t,u) \le C\left[E_{s-1}^{\partial_1}(1,u) + \int_1^t \|u(\tau,\cdot)\|_{W^{t,\rho,\infty}}^2 E_{s-1}^{\partial_1}(\tau,u)d\tau\right]. \tag{2.9}$$

On the other hand, we want to obtain an analogous of (2.9) for $E_1^Z(t,u)$. Applying Z to (2.4), Leibniz rule and commutations, we derive that Zu is solution of the equation

$$\Box Zu + Zu + \gamma^{01}\partial_0\partial_1 Zu + \gamma^{11}\partial_1^2 Zu + \gamma^0\partial_0 Zu + \gamma^1\partial_1 Zu = f^Z\,, \tag{2.10}$$

where f^Z is linear combination of $[\gamma^{ij}\partial_{ij}^2, Z]u$ and $[\gamma^h\partial_h, Z]u$. We calculate for instance the term $[\gamma^{01}\partial_{01}^2, Z]u$ and we find that it is equal to $-(Z\gamma^{01})\partial_{01}^2 u - \gamma^{01}[\partial_{01}^2, Z]u$, that is a linear combination of

$$\begin{aligned} &(\partial_i^{\alpha_1} u)(\partial_j^{\alpha_2} Zu)(\partial_{01}^2 u)\,, \\ &(\partial_i^{\alpha_1} u)(\partial_j^{\alpha_2} u)(\partial_{hk}^2 u)\,, \end{aligned} \tag{2.11}$$

for $i, j, h, k, \alpha_1, \alpha_2 = 0, 1$. Therefore, the L^2 norm of f^Z can be estimated as follows

$$\|f^Z(t,\cdot)\|_{L^2} \le \Big(\sum_{i+j=0}^{2} \|\partial_x^i\partial_t^j u(t,\cdot)\|_{L^\infty}^2\Big) E_1^Z(t,u) \le \|u(t,\cdot)\|_{W^{t,3,\infty}}^2 E_1^Z(t,u)\,, \tag{2.12}$$

and applying lemma 2.1 for Zu, we derive

$$E^Z_1(t,u) \leq C\left[E^Z_1(1,u) + \int_1^t \|u(\tau,\cdot)\|^2_{W^{t,3,\infty}} E^Z_1(\tau,\cdot)\,d\tau\right]. \tag{2.13}$$

Remark 2.2. To make the above proof fully correct, one should check as well that the energy of Zu is actually finite at every fixed positive time. One may do that either using that the vector field Z is the infinitesimal generator of the action on the equation of a one parameter group, along the lines of appendix A.2 in [1]. Alternatively, one may instead exploit finite propagation speed, remarking that if the data are cut off on a compact set, the solution remains compactly supported at every fixed time, so that the energy of Zu is actually finite, and that the bounds we get are uniform in terms of the cut off.

Proposition 2.3 (Propagation of Energy Estimates)**.** *There exist an integer s large enough, a $\rho \geq [\frac{s-1}{2}]+3$, $\rho \ll s$, an $\varepsilon_0 \in]0,1[$ sufficiently small, a small $\sigma \geq 0$, and two constants $A, B > 0$ sufficiently large such that, for any $0<\varepsilon<\varepsilon_0$, if u is a solution of* (1.1) *on some interval $[1,T]$, for $T>1$ fixed, and satisfies*

$$\|u(t,\cdot)\|_{W^{t,\rho,\infty}} \leq A\varepsilon t^{-\frac{1}{2}}\,, \tag{2.14a}$$

$$E^{\partial_1}_{s-1}(t,u) \leq B\varepsilon t^{\sigma}\,, \tag{2.14b}$$

$$E^Z_1(t,u) \leq B\varepsilon t^{\sigma}\,, \tag{2.14c}$$

for every $t\in[1,T]$, then it verifies also

$$E^{\partial_1}_{s-1}(t,u) \leq \frac{B}{2}\varepsilon t^{\sigma}\,, \tag{2.15a}$$

$$E^Z_1(t,u) \leq \frac{B}{2}\varepsilon t^{\sigma}\,. \tag{2.15b}$$

Proof. Both estimates (2.14b) and (2.14c) can be propagated injecting *a priori* estimates (2.14) in energy inequalities (2.9) and (2.13) derived before, obtaining

$$E^{\partial_1}_{s-1}(t,u) \leq C\Big[E^{\partial_1}_{s-1}(1,u) + A^2B\varepsilon^3\int_1^t \tau^{-1+\sigma}d\tau\Big] \leq CE^{\partial_1}_{s-1}(1,u) + \frac{A^2BC\varepsilon^3}{\sigma}t^{\sigma}\,,$$
$$E^Z_1(t,u) \leq C\Big[E^Z_1(1,u) + A^2B\varepsilon^3\int_1^t \tau^{-1+\sigma}d\tau\Big] \leq CE^Z_1(1,u) + \frac{A^2BC\varepsilon^3}{\sigma}t^{\sigma}\,.$$

Then we can choose $B>0$ sufficiently large such that $CE^{\partial_1}_{s-1}(1,u)+CE^Z_1(1,u) \leq \frac{B}{4}\varepsilon$, and $\varepsilon_0>0$ sufficiently small such that $\frac{A^2C\varepsilon^2}{\sigma} \leq \frac{1}{4}$, to obtain (2.15a), (2.15b).

3 Semiclassical Pseudo-differential Operators.

As told in the introduction, in order to prove an L^∞ estimate on u and on its derivatives we need to reformulate the starting problem (1.1) in term of an ODE satisfied by a new function v obtained from u, and this will strongly use the semiclassical pseudo-differential calculus. In the following two subsections, we introduce this semiclassical environment, defining classes of symbols and operators we shall use and several useful properties, some of which are stated without proof. More details can be found in [9] and [36].

3.1 Definitions and Composition Formula

Definition 3.1. An order function on $\mathbb{R}\times\mathbb{R}$ is a smooth map from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}_+ : (x,\xi) \to M(x,\xi)$ such that there exist $N_0 \in \mathbb{N}$, $C > 0$ and for any $(x,\xi),(y,\eta) \in \mathbb{R}\times\mathbb{R}$

$$M(y,\eta) \le C\langle x-y\rangle^{N_0}\langle \xi-\eta\rangle^{N_0} M(x,\xi)\,, \tag{3.1}$$

where $\langle x\rangle = \sqrt{1+x^2}$.

Examples of order functions are $\langle x\rangle$, $\langle \xi\rangle$, $\langle x\rangle\langle \xi\rangle$.

Definition 3.2. Let M be an order function on $\mathbb{R}\times\mathbb{R}$, $\beta \ge 0$, $\delta \ge 0$. One denotes by $S_{\delta,\beta}(M)$ the space of smooth functions

$$\begin{gathered}(x,\xi,h) \to a(x,\xi,h)\\ \mathbb{R}\times\mathbb{R}\times]0,1] \to \mathbb{C}\end{gathered}$$

satisfying for any $\alpha_1,\alpha_2,k,N \in \mathbb{N}$ bounds

$$|\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}(h\partial_h)^k a(x,\xi,h)| \le CM(x,\xi)\, h^{-\delta(\alpha_1+\alpha_2)}(1+\beta h^\beta|\xi|)^{-N}\,. \tag{3.2}$$

A key role in this paper will be played by symbols a verifying (3.2) with $M(x,\xi) = \langle \frac{x+f(\xi)}{\sqrt{h}}\rangle^{-N}$, for $N \in \mathbb{N}$ and a certain smooth function $f(\xi)$. This function M is no longer an order function because of the term $h^{-\frac{1}{2}}$ but nevertheless we continue to keep the notation $a \in S_{\delta,\beta}(\langle \frac{x+f(\xi)}{\sqrt{h}}\rangle^{-N})$.

Definition 3.3. We will say that $a(x,\xi)$ is a symbol of order r if $a \in S_{\delta,\beta}(\langle\xi\rangle^r)$, for some $\delta \ge 0$, $\beta \ge 0$.

Let us observe that when $\beta > 0$, the symbol decays rapidly in $h^\beta|\xi|$, which implies the following inclusion for $r \ge 0$

$$S_{\delta,\beta}(\langle\xi\rangle^r) \subset h^{-\beta r}S_{\delta,\beta}(1)\,, \tag{3.3}$$

which will be often use in all the paper. This means that, up to a small loss in h, this type of symbols can be always considered as symbols of order zero. In the rest of the paper we will not indicate explicitly the dependence of symbols on h, referring to $a(x,\xi,h)$ simply as $a(x,\xi)$.

Definition 3.4. Let $a \in S_{\delta,\beta}(M)$ for some order function M, some $\delta \ge 0$, $\beta \ge 0$.

(i) We can define the *Weyl quantization* of a to be the operator $Op_h^w(a) = a^w(x,hD)$ acting on $u \in \mathcal{S}(\mathbb{R})$ by the formula :

$$Op_h^w(a(x,\xi))u(x) = \frac{1}{2\pi h}\int_\mathbb{R}\int_\mathbb{R} e^{\frac{i}{h}(x-y)\xi} a(\frac{x+y}{2},\xi)\, u(y)\, dyd\xi\,; \tag{3.4}$$

(ii) We define also the *standard quantization* :

$$Op_h(a(x,\xi))u(x) = \frac{1}{2\pi h}\int_\mathbb{R}\int_\mathbb{R} e^{\frac{i}{h}(x-y)\xi} a(x,\xi)\, u(y)\, dyd\xi\,. \tag{3.5}$$

It is clear from the definition that the two quantizations coincide when the symbol does not depend on x.

We introduce also a semiclassical version of Sobolev spaces, on which is more natural to consider the action of above operators.

Definition 3.5. (i) Let $\rho \in \mathbb{N}$. We define the semiclassical Sobolev space $W_h^{\rho,\infty}(\mathbb{R})$ as the space of families $(v_h)_{h\in]0,1]}$ of tempered distributions, such that $\langle hD\rangle^\rho v_h := Op_h(\langle\xi\rangle^\rho)v_h$ is a bounded family of L^∞, i.e.

$$W_h^{\rho,\infty}(\mathbb{R}) := \left\{ v_h \in \mathcal{S}'(\mathbb{R}) \,\Big|\, \sup_{h\in]0,1]} \|\langle hD\rangle^\rho v_h\|_{L^\infty(\mathbb{R})} < +\infty \right\}. \tag{3.6}$$

(ii) Let $s \in \mathbb{R}$. We define the semiclassical Sobolev space $H_h^s(\mathbb{R})$ as the space of families $(v_h)_{h\in]0,1]}$ of tempered distributions such that $\langle hD\rangle^s v_h := Op_h(\langle\xi\rangle^s)v_h$ is a bounded family of L^2, i.e.

$$H_h^s(\mathbb{R}) := \left\{ v_h \in \mathcal{S}'(\mathbb{R}) \,\Big|\, \sup_{h\in]0,1]} \int_{\mathbb{R}} (1+|h\xi|^2)^s |\hat{v}_h(\xi)|^2 \, d\xi < +\infty \right\}. \tag{3.7}$$

For future references, we write down the semiclassical Sobolev injection,

$$\|v_h\|_{W_h^{\rho,\infty}} \leq C_\theta h^{-\frac{1}{2}} \|v_h\|_{H_h^{\rho+\frac{1}{2}+\theta}}, \qquad \forall\theta > 0. \tag{3.8}$$

The following two propositions are stated without proof. They concern the adjoint and the composition of pseudo-differential operators we are considering, and a full detailed treatment is provided in chapter 7 of [9], or in chapter 4 of [36].

Proposition 3.6 (Self-Adjointness). *If a is a real symbol, its Weyl quantization is self-adjoint,*

$$\left(Op_h^w(a)\right)^* = Op_h^w(a). \tag{3.9}$$

Proposition 3.7 (Composition for Weyl quantization). *Let $a, b \in \mathcal{S}(\mathbb{R})$. Then*

$$Op_h^w(a) \circ Op_h^w(b) = Op_h^w(a\sharp b), \tag{3.10}$$

where

$$a\sharp b\,(x,\xi) := \frac{1}{(\pi h)^2} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{\frac{2i}{h}\sigma(y,\eta;\,z,\zeta)} a(x+z,\xi+\zeta) b(x+y,\xi+\eta)\, dy d\eta dz d\zeta \tag{3.11}$$

and

$$\sigma(y,\eta;\,z,\zeta) = \eta z - y\zeta.$$

It is often useful to derive an asymptotic expansion for $a\sharp b$, which allows easier computations than the integral formula (3.11). This expansion is usually obtained by applying the stationary phase argument when $a, b \in S_{\delta,\beta}(M)$, $\delta \in [0, \frac{1}{2}[$ (as shown in [36]). Here we provide an expansion at any order even when one of two symbols belongs to $S_{\frac{1}{2},\beta_1}(M)$ (still having the other in $S_{\delta,\beta_2}(M)$ for $\delta < \frac{1}{2}$, and β_1, β_2 either equal or, if not, one of them equal to zero), whose proof is based on the Taylor development of symbols a, b, and can be found in detail in the appendix.

Proposition 3.8. *Let $a \in S_{\delta_1,\beta_1}(M_1)$, $b \in S_{\delta_2,\beta_2}(M_2)$, $\delta_1, \delta_2 \in [0, \frac{1}{2}]$, $\delta_1 + \delta_2 < 1$, $\beta_1, \beta_2 \geq 0$ such that*

$$\beta_1 = \beta_2 \geq 0 \qquad or \qquad \left[\beta_1 \neq \beta_2 \; and \, \beta_i = 0, \beta_j > 0, i \neq j \in \{1,2\}\right]. \tag{3.12}$$

Then $a\sharp b \in S_{\delta,\beta}(M_1M_2)$, where $\delta = \max\{\delta_1,\delta_2\}$, $\beta = \max\{\beta_1,\beta_2\}$. Moreover,

$$a\sharp b = ab + \frac{h}{2i}\{a,b\} + \sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ 2\le|\alpha|\le k}} \left(\frac{h}{2i}\right)^{|\alpha|} \frac{(-1)^{\alpha_1}}{\alpha!}\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}a\,\partial_x^{\alpha_2}\partial_\xi^{\alpha_1}b + r_k\,, \tag{3.13}$$

where $\{a,b\} = \partial_\xi a\partial_x b - \partial_\xi b\partial_x a$, $r_k \in h^{(k+1)(1-(\delta_1+\delta_2))}S_{\delta,\beta}(M_1M_2)$ and

(3.14)

$$r_k(x,\xi) = \left(\frac{h}{2i}\right)^{k+1}\frac{k+1}{(\pi h)^2}\sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ |\alpha|=k+1}}\frac{(-1)^{\alpha_1}}{\alpha!}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z-y\zeta)}\Big\{\int_0^1 \partial_x^{\alpha_1}\partial_\xi^{\alpha_2}a(x+tz,\xi+t\zeta)(1-t)^k dt \times \partial_y^{\alpha_2}\partial_\eta^{\alpha_1}b(x+y,\xi+\eta)\Big\}\,dyd\eta dzd\zeta\,.$$

More generally, if $h^{(k+1)\delta_1}\partial^\alpha a \in S_{\delta_1,\beta_1}(M_1^{k+1})$, $h^{(k+1)\delta_2}\partial^\alpha b \in S_{\delta_2,\beta_2}(M_2^{k+1})$, for $|\alpha| = k+1$, for order functions M_1^{k+1}, M_2^{k+1}, then $r_k \in h^{(k+1)(1-(\delta_1+\delta_2))}S_{\delta,\beta}(M_1^{k+1}M_2^{k+1})$.

Remark 3.9. Observe that

$$a\sharp b = ab + \frac{h}{2i}\{a,b\} + \left(\frac{h}{2i}\right)^2\left[\frac{1}{2}\partial_x^2 a\partial_\xi^2 b + \frac{1}{2}\partial_\xi^2 a\partial_x^2 b - \partial_x\partial_\xi a\partial_x\partial_\xi b\right] + r_2^{a\sharp b}\,, \tag{3.15}$$

so the contribution of order two (and all other contributions of even order) disappears when we calculate the symbol associated to a commutator

$$a\sharp b - b\sharp a = \frac{h}{i}\{a,b\} + r_2\,, \tag{3.16}$$

where $r_2 = r_2^{a\sharp b} - r_2^{b\sharp a} \in h^{3(1-(\delta_1+\delta_2))}S_{\delta,\beta}(M_1M_2)$.

The result of proposition 3.8 is still true also when one of order functions, or both, has the form $\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-1}$, for a smooth function $f(\xi)$, $f'(\xi)$ bounded, as stated below and proved as well in the appendix.

Lemma 3.10. *Let $f(\xi)$ be a smooth function, $f'(\xi)$ bounded. Consider $a \in S_{\delta_1,\beta_1}(\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-d})$, $d \in \mathbb{N}$, and $b \in S_{\delta_2,\beta_2}(M)$, for M order function or $M(x,\xi) = \langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-l}$, $l \in \mathbb{N}$, some $\delta_1 \in [0,\frac{1}{2}]$, $\delta_2 \in [0,\frac{1}{2}[$, $\beta_1,\beta_2 \ge 0$ as in (3.12). Then $a\sharp b \in S_{\delta,\beta}(\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-d}M)$, where $\delta = \max\{\delta_1,\delta_2\}$, $\beta = \max\{\beta_1,\beta_2\}$, and the asymptotic expansion (3.13) holds, with r_k given by (3.14), $r_k \in h^{(k+1)(1-(\delta_1+\delta_2))}S_{\delta,\beta}(\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-d}M)$.*
More generally, if $h^{(k+1)\delta_1}\partial^\alpha a \in S_{\delta_1,\beta_1}(\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-d'})$ and $h^{(k+1)\delta_2}\partial^\alpha b \in S_{\delta_2,\beta_2}(M^{k+1})$, $|\alpha| = k+1$, M^{k+1} order function or $M^{k+1}(x,\xi) = \langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-l'}$, for others $d',l' \in \mathbb{N}$, then $r_k \in h^{(k+1)(1-(\delta_1+\delta_2))}S_{\delta,\beta}(\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-d'}M^{k+1})$.

3.2 Some Technical Estimates

This subsection is mostly devoted to the introduction of some technical results about symbols and operators we will often use in the entire paper, first of all continuity on Sobolev spaces. We also introduce multi-linear quantizations which will be used in the next section (and which are fully described in [5]), especially because they make our notations easier and clearer at first. Moreover, from now on we follow the notation $p(\xi) := \sqrt{1+\xi^2}$.

The first statement is about continuity on spaces $H^s_h(\mathbb{R})$, and generalises theorem 7.11 in [9]. The second statement concerns instead a result of continuity from L^2 to $W^{\rho,\infty}_h$. In the spirit of [17] for the Schrödinger equation, it allows to pass from uniform norms to the L^2 norm losing only a power $h^{-\frac{1}{4}-\sigma}$ for a small $\sigma > 0$, and not a $h^{-\frac{1}{2}}$ as for the Sobolev injection.

Proposition 3.11 (Continuity on H^s_h)**.** *Let $s \in \mathbb{R}$. Let $a \in S_{\delta,\beta}(\langle\xi\rangle^r)$, $r \in \mathbb{R}$, $\delta \in [0, \frac{1}{2}]$, $\beta \geq 0$. Then $Op^w_h(a)$ is uniformly bounded : $H^s_h(\mathbb{R}) \to H^{s-r}_h(\mathbb{R})$, and there exists a positive constant C independent of h such that*

$$\|Op^w_h(a)\|_{\mathcal{L}(H^s_h;H^{s-r}_h)} \leq C\,, \qquad \forall h \in]0,1]\,. \tag{3.17}$$

Proposition 3.12 (Continuity from L^2 to $W^{\rho,\infty}_h$)**.** *Let $\rho \in \mathbb{N}$. Let $a \in S_{\delta,\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, $\delta \in [0, \frac{1}{2}]$, $\beta > 0$. Then $Op^w_h(a)$ is bounded : $L^2(\mathbb{R}) \to W^{\rho,\infty}_h(\mathbb{R})$, and there exists a positive constant C independent of h such that*

$$\|Op^w_h(a)\|_{\mathcal{L}(L^2;W^{\rho,\infty}_h)} \leq Ch^{-\frac{1}{4}-\sigma}\,, \qquad \forall h \in]0,1]\,, \tag{3.18}$$

where $\sigma > 0$ depends linearly on β.

Proof. Firstly, remark that thanks to symbolic calculus of lemma 3.10, to estimate the $W^{k,\infty}_h$ norm of an operator whose symbol is rapidly decaying in $|h^\beta\xi|$ corresponds actually to estimate the L^∞ norm of an operator associated to another symbol (namely, $\tilde{a}(x,\xi) = \langle\xi\rangle^k a(x,\xi)$) which is still in the same class as a, up to a small loss on h, of order $h^{-k\beta}$.

From the definition of $Op^w_h(a)v$, and using thereafter integration by part, Cauchy-Schwarz inequality, and Young's inequality for convolutions, we derive what follows :

$$\begin{aligned}
&|Op^w_h(a)v| = \\
&= \left| \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(\frac{x}{\sqrt{h}}-y)\xi} a(\frac{x+\sqrt{h}y}{2}, \sqrt{h}\xi) v(\sqrt{h}y)\, dyd\xi \right| \\
&= \left| \frac{1}{(2\pi)^2\sqrt{h}}\int_{\mathbb{R}} \hat{v}(\frac{\eta}{\sqrt{h}})d\eta \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(\frac{x}{\sqrt{h}}-y)\xi+i\eta y} a(\frac{x+\sqrt{h}y}{2}, \sqrt{h}\xi)\, dyd\xi \right| \\
&= \left| \frac{1}{(2\pi)^2\sqrt{h}}\int_{\mathbb{R}} \hat{v}(\frac{\eta}{\sqrt{h}}) \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{1-i(\frac{x}{\sqrt{h}}-y)\partial_\xi}{1+(\frac{x}{\sqrt{h}}-y)^2}\right)^2 \left(\frac{1+i(\xi-\eta)\partial_y}{1+(\xi-\eta)^2}\right)^2 \left[e^{i(\frac{x}{\sqrt{h}}-y)\xi+i\eta y}\right] \right. \\
&\qquad \left. \times\, a(\frac{x+\sqrt{h}y}{2}, \sqrt{h}\xi)\, dyd\xi d\eta \right| \\
&\leq \frac{C}{\sqrt{h}}\int_{\mathbb{R}} \left|\hat{v}(\frac{\eta}{\sqrt{h}})\right| \int_{\mathbb{R}}\int_{\mathbb{R}} \langle\frac{x}{\sqrt{h}}-y\rangle^{-2}\langle\xi-\eta\rangle^{-2}\langle h^\beta\sqrt{h}\xi\rangle^{-N} \Big\langle\frac{\frac{x+\sqrt{h}y}{2}+p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1} dyd\xi d\eta \\
&\leq \frac{C}{\sqrt{h}} \left\|\hat{v}(\frac{\eta}{\sqrt{h}})\right\|_{L^2_\eta} \|\langle\eta\rangle^{-2}\|_{L^1_\eta} \left\| \int_{\mathbb{R}} \langle\frac{x}{\sqrt{h}}-y\rangle^{-2}\langle h^\beta\sqrt{h}\xi\rangle^{-N} \Big\langle\frac{\frac{x+\sqrt{h}y}{2}+p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1} dy \right\|_{L^2_\xi} \\
&\leq Ch^{-\frac{1}{4}}\|v\|_{L^2} \int_{\mathbb{R}} \langle\frac{x}{\sqrt{h}}-y\rangle^{-2} \left\| \langle h^\beta\sqrt{h}\xi\rangle^{-N} \Big\langle\frac{\frac{x+\sqrt{h}y}{2}+p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1} \right\|_{L^2_\xi} dy\,,
\end{aligned} \tag{3.19}$$

where $N > 0$ is properly chosen later. We draw attention to two facts, when we integrated by parts: in the third equality in (3.19), we use that

$$\left(\frac{1 - i(\frac{x}{\sqrt{h}} - y)\partial_\xi}{1 + (\frac{x}{\sqrt{h}} - y)^2}\right)^2 \left(\frac{1 + i(\xi - \eta)\partial_y}{1 + (\xi - \eta)^2}\right)^2 \left[e^{i(\frac{x}{\sqrt{h}} - y)\xi + i\eta y}\right] = e^{i(\frac{x}{\sqrt{h}} - y)\xi + i\eta y}$$

so, integrating by part, derivatives fall on $\langle \frac{x}{\sqrt{h}} - y\rangle^{-1}$, $\langle \xi - \eta\rangle^{-1}$, giving rise to more decreasing factors, and/or on $a\left(\frac{x+\sqrt{h}y}{2}, \sqrt{h}\xi\right)$; the symbol a belongs to $S_{\delta,\beta}(1)$ with a $\delta \le \frac{1}{2}$, but the loss of $h^{-\delta}$ is offset by the factor $\sqrt{h}$ coming from the derivation of $a(\frac{x+\sqrt{h}y}{2}, \sqrt{h}\xi)$ with respect to y and ξ.

To estimate $\|\langle h^\beta \sqrt{h}\xi\rangle^{-N} \langle \frac{\frac{x+\sqrt{h}y}{2} + p'(\sqrt{h}\xi)}{\sqrt{h}}\rangle^{-1}\|_{L^2_\xi}$ we consider a Littlewood-Paley decomposition, i.e.

$$1 = \sum_{k=0}^{+\infty} \varphi_k(\xi)\,, \tag{3.20}$$

where $\varphi_k(\xi) \in C_0^\infty(\mathbb{R})$, $supp\, \varphi_0 \subset B(0,1)$, $\varphi_k(\xi) = \varphi(2^{-k}\xi)$ and $supp\, \varphi \subset \{A^{-1} \le |\xi| \le A\}$, for a constant $A > 0$. Then,

(3.21)

$$\begin{aligned} \left\|\langle h^\beta \sqrt{h}\xi\rangle^{-N} \langle \frac{\frac{x+\sqrt{h}y}{2} + p'(\sqrt{h}\xi)}{\sqrt{h}}\rangle^{-1}\right\|^2_{L^2_\xi} &= \frac{1}{\sqrt{h}} \sum_{k\ge 0} \int_{\mathbb{R}} \langle h^\beta \xi\rangle^{-2N} \left\langle \frac{\frac{x+\sqrt{h}y}{2} + p'(\xi)}{\sqrt{h}}\right\rangle^{-2} \varphi_k(\xi) d\xi \\ &= \frac{1}{\sqrt{h}} \sum_{k\ge 0} I_k\,, \end{aligned}$$

where

$$I_0 = \int_{\mathbb{R}} \langle h^\beta \xi\rangle^{-2N} \left\langle \frac{\frac{x+\sqrt{h}y}{2} + p'(\xi)}{\sqrt{h}}\right\rangle^{-2} \varphi_0(\xi) d\xi\,, \tag{3.22}$$

and

$$\begin{aligned} I_k &= \int_{\mathbb{R}} \langle h^\beta \xi\rangle^{-2N} \left\langle \frac{\frac{x+\sqrt{h}y}{2} + p'(\xi)}{\sqrt{h}}\right\rangle^{-2} \varphi(2^{-k}\xi) d\xi \\ &= 2^k \int_{\mathbb{R}} \langle h^\beta 2^k \xi\rangle^{-2N} \left\langle \frac{\frac{x+\sqrt{h}y}{2} + p'(2^k\xi)}{\sqrt{h}}\right\rangle^{-2} \varphi(\xi) d\xi\,, \qquad k \ge 1 \\ &\le A^{2N} 2^{(-2N+1)k} h^{-2\beta N} \int_{\mathbb{R}} \left\langle \frac{\frac{x+\sqrt{h}y}{2} + p'(2^k\xi)}{\sqrt{h}}\right\rangle^{-2} \varphi(\xi) d\xi\,. \end{aligned} \tag{3.23}$$

For $k \le k_0$, for a fixed k_0, $p''(2^k\xi) \ne 0$ on the support of φ. As $\xi \to \pm\infty$ we have the expansion

$$p'(\xi) = \frac{\xi}{\sqrt{1+\xi^2}} = \pm 1 \mp \frac{1}{2\xi^2} + O(|\xi|^{-4})\,, \tag{3.24}$$

and then

$$p'(2^k\xi) = \pm 1 \mp \frac{2^{-2k}}{2\xi^2} + O(|2^k\xi|^{-4})\,. \tag{3.25}$$

For $k \geq k_0$, the function $\xi \to g_k(\xi) = 2^{2k}(\frac{x+\sqrt{h}y}{2}) + 2^{2k}p'(2^k\xi)$ is such that $|g_k'(\xi)| = |\xi|^{-3}|\widetilde{g}_k(\xi)|$, $\widetilde{g}_k(\xi) = 1 + O(2^{-2k}|\xi|^{-2})$, and $|g_k'(\xi)| \sim 1$ on the support of φ, so for every k we can perform a change of variables $z = g_k(\xi)$ in the last line of (3.23). Hence,

$$
\begin{aligned}
I_k &\leq A^{2N} 2^{(-2N+1)k} h^{-2\beta N} \int \langle \frac{z}{2^{2k}\sqrt{h}} \rangle^{-2} \varphi(g_k^{-1}(z)) dz \\
&\leq A^{2N} 2^{(-2N+3)k} h^{-2\beta N} \sqrt{h} \int \langle z \rangle^{-2} dz \\
&\leq C 2^{(-2N+3)k} h^{-2\beta N} \sqrt{h}\,,
\end{aligned}
\tag{3.26}
$$

so taking the summation of all I_k for $k \geq 0$ we deduce

$$
\left\| \langle h^\beta \sqrt{h}\xi \rangle^{-N} \Big\langle \frac{\frac{x+\sqrt{h}y}{2} + p'(\sqrt{h}\xi)}{\sqrt{h}} \Big\rangle^{-1} \right\|_{L^2_\xi} \leq C h^{-\beta N} \sum_{k \geq 0} 2^{(\frac{-2N+3}{2})k} \leq C' h^{-\beta N}\,,
\tag{3.27}
$$

if we choose $N > 0$ such that $\frac{-2N+3}{2} < 0$ (e.g. $N = 2$). Finally

$$
\| Op_h^w(a) \|_{\mathcal{L}(L^2; W_h^{\rho,\infty})} = O(h^{-\frac{1}{4}-\sigma})\,,
\tag{3.28}
$$

where $\sigma(\beta) = (N+\rho)\beta$ depends linearly on β. □

The following lemma shows that we have nice upper bounds for operators acting on v whose symbols are supported for $|\xi| \geq h^{-\beta}$, $\beta > 0$, provided that we have an a priori H^s_h estimate on v, with large enough s.

Lemma 3.13. *Let $s' \geq 0$. Let $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ in a neighbourhood of zero, e.g.*

$$
\begin{aligned}
\chi(\xi) &= 1\,, \qquad \text{for } |\xi| < C_1 \\
\chi(\xi) &= 0\,, \qquad \text{for } |\xi| > C_2\,.
\end{aligned}
\tag{3.29}
$$

Then

$$
\| Op_h((1-\chi)(h^\beta \xi)) v \|_{H^{s'}_h} \leq C h^{\beta(s-s')} \|v\|_{H^s_h}\,, \qquad \forall s > s'\,.
\tag{3.30}
$$

Proof. The result is a simple consequence of the fact that $(1-\chi)(h^\beta\xi)$ is supported for $|\xi| \geq C_1 h^{-\beta}$, because

$$
\begin{aligned}
\| Op_h((1-\chi)(h^\beta \xi)) v \|^2_{H^{s'}_h} &= \int (1+|h\xi|^2)^{s'} |(1-\chi)(h^\beta h\xi)|^2 |\hat{v}(\xi)|^2 d\xi \\
&= \int (1+|h\xi|^2)^{s} (1+|h\xi|^2)^{s'-s} |(1-\chi)(h^\beta h\xi)|^2 |\hat{v}(\xi)|^2 d\xi \\
&\leq C h^{2\beta(s-s')} \|v\|^2_{H^s_h}\,,
\end{aligned}
\tag{3.31}
$$

where the last inequality follows from an integration on $|h\xi| > C_1 h^{-\beta}$, and from the two following conditions $s'-s < 0$, $(1+|h\xi|^2)^{s'-s} \leq C h^{-2\beta(s'-s)}$. □

This result is useful when we want to reduce essentially to symbols rapidly decaying in $|h^\beta\xi|$, for example in the intention of using proposition 3.12 or when we want to pass from a symbol of a certain positive order to another one of order zero, up to small losses of order $O(h^{-\sigma})$, $\sigma > 0$

depending linearly on β. We can always split a symbol using that $1 = \chi(h^\beta\xi) + (1-\chi)(h^\beta\xi)$, and consider as remainders all contributions coming from the latter.

Define the set $\Lambda := \{(x,\xi) \in \mathbb{R}\times\mathbb{R} \,|\, x + p'(\xi) = 0\}$, i.e. the graph of the function $x \in]-1,1[\to d\varphi(x)$, $\varphi(x) = \sqrt{1-x^2}$, as drawn in picture .1. We will use the following technical lemma, whose proof can be found in lemma 1.2.6 in [5]:

Lemma 3.14. *Let $\gamma \in C_0^\infty(\mathbb{R})$. If the support of γ is sufficiently small, the two functions defined below*

(3.32)
$$e_\pm(x,\xi) = \frac{x+p'(\pm\xi)}{\xi \mp d\varphi(x)}\gamma\left(\langle\xi\rangle^2(x+p'(\pm\xi))\right) \quad \text{and} \quad \tilde{e}_\pm(x,\xi) = \frac{\xi \mp d\varphi(x)}{x+p'(\pm\xi)}\gamma\left(\langle\xi\rangle^2(x+p'(\pm\xi))\right)$$

verify estimates

$$\begin{aligned} |\partial_x^\alpha\partial_\xi^\beta e_\pm(x,\xi)| &\le C_{\alpha\beta}\langle\xi\rangle^{-3+2\alpha-\beta}, \\ |\partial_x^\alpha\partial_\xi^\beta \tilde{e}_\pm(x,\xi)| &\le C_{\alpha\beta}\langle\xi\rangle^{3+2\alpha-\beta}. \end{aligned} \tag{3.33}$$

Moreover, if suppγ is small enough, then on the support of $\gamma(\langle\xi\rangle^2(x+p'(\pm\xi)))$ one has $\langle d\varphi\rangle \sim \langle\xi\rangle$ and there is a constant $A>0$ such that, on that support

$$\begin{aligned} A^{-1}\langle\xi\rangle^{-2} &\le \pm x + 1 \le A\langle\xi\rangle^{-2}, \qquad \xi \to +\infty \\ A^{-1}\langle\xi\rangle^{-2} &\le \mp x + 1 \le A\langle\xi\rangle^{-2}, \qquad \xi \to -\infty \end{aligned} \tag{3.34}$$

Finally, as $x \to \pm 1$, for every $k \in \mathbb{N}$

$$\partial^k(d\varphi(x)) = O(\langle d\varphi\rangle^{1+2k}). \tag{3.35}$$

Lemma 3.15. *Let $\gamma \in C_0^\infty(\mathbb{R})$ such that $\gamma \equiv 1$ in a neighbourhood of zero, and define $\Gamma(x,\xi) = \gamma(\frac{x+p'(\xi)}{\sqrt{h}})$. Then $\Gamma \in S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-N})$, for all $N \ge 0$.*

Proof. Let $N \in \mathbb{N}$. Since $\gamma \in C_0^\infty(\mathbb{R})$, $p'' \in S_{0,0}(1)$, we have

$$\begin{aligned} |\Gamma(x,\xi)| &\le \|\langle x\rangle^N\gamma(x)\|_{L^\infty}\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-N}, \\ |\partial_x\Gamma(x,\xi)| &= \left|\gamma'(\frac{x+p'(\xi)}{\sqrt{h}})\frac{1}{\sqrt{h}}\right| \le h^{-\frac{1}{2}}\|\langle x\rangle^N\gamma'(x)\|_{L^\infty}\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-N}, \\ |\partial_\xi\Gamma(x,\xi)| &= \left|\gamma'(\frac{x+p'(\xi)}{\sqrt{h}})\frac{p''(\xi)}{\sqrt{h}}\right| \le h^{-\frac{1}{2}}\|p''(\xi)\|_{L^\infty}\|\langle x\rangle^N\gamma'(x)\|_{L^\infty}\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-N}, \end{aligned} \tag{3.36}$$

and going on one can prove that $|\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}\Gamma| \le C_{\alpha_1,\alpha_2,N}h^{-\frac{1}{2}(\alpha_1+\alpha_2)}\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-N}$. □

Multi-linear Operators. We briefly generalise some definitions given at the beginning of this section in order to introduce multi-linear operators. As we will consider multi-linear operators with symbols depending only on ξ and, for such symbols, in the linear case, Weyl quantization coincide with classical quantization, for simplicity we will directly talk about the Kohn-Nirenberg quantization.

Let $n \in \mathbb{N}^*$ and set $\xi = (\xi_1, \dots, \xi_n)$. An order function on $\mathbb{R} \times \mathbb{R}^n$ will be a smooth function $(x, \xi) \to M(x, \xi)$ satisfying (3.1), where $\langle \xi - \eta \rangle^{N_0}$ is replaced by

$$\prod_{i=1}^{n} \langle \xi_i - \eta_i \rangle^{N_0} .$$

Equivalently, we define the class $S_{\delta,\beta}(M, n)$, for some $\delta \geq 0$, $\beta \geq 0$ and $M(x, \xi)$ order function on $\mathbb{R} \times \mathbb{R}^n$, to be the set of smooth functions

$$\begin{gathered} (x, \xi_1, \dots, \xi_n, h) \to a(x, \xi, h) \\ \mathbb{R} \times \mathbb{R}^n \times]0, 1] \to \mathbb{C} \end{gathered}$$

satisfying the inequality (3.2), $\forall \alpha_1 \in \mathbb{N}, \alpha_2 \in \mathbb{N}^n$, $\forall k, N \in \mathbb{N}$.

Definition 3.16. Let a be a symbol in $S_{\delta,\beta}(M, n)$ for some order function M, some $\delta \geq 0$, $\beta \geq 0$.

(i) We define the n-linear operator $Op(a)$ acting on test functions $v_1, \dots, v_n$ by

$$Op(a)(v_1, \dots, v_n) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix(\xi_1 + \dots + \xi_n)} a(x, \xi_1, \dots, \xi_n) \prod_{l=1}^{n} \hat{v}_l(\xi_l) \, d\xi_1 \dots d\xi_n \, . \tag{3.37}$$

(ii) We also define the n-linear semiclassical operator $Op_h(a)$ acting on test functions $v_1, \dots, v_n$ by

$$Op_h(a)(v_1, \dots, v_n) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h} x(\xi_1 + \dots + \xi_n)} a(x, \xi_1, \dots, \xi_n) \prod_{l=1}^{n} \hat{v}_l(\xi_l) \, d\xi_1 \dots d\xi_n \, . \tag{3.38}$$

For a further need of compactness in our notations, we introduce $I = (i_1, \dots, i_n)$ a n-dimensional vector, $i_k \in \{1, -1\}$ for every $k = 1, \dots, n$. We set $|I| = n$ and define

$$w_I = (w_{i_1}, \dots, w_{i_n}) \, , \qquad w_1 = w \, , \; w_{-1} = \bar{w} \, , \tag{3.39}$$

while $m_I(\xi) \in S_{\delta,\beta}(M, n)$ will be always in what follows a symbol of the form

$$m_I(\xi) = m_1^I(\xi_1) \cdots m_n^I(\xi_n) \, . \tag{3.40}$$

Note that, when all variables ξ_j in $m_I(\xi)$ are decoupled, as in (3.40), $Op(m_I)(w_I)$ is only a compact way of writing $\prod_j Op(m_j^I) w_{i_j}$. We also warn the reader that in following sections, when we focus on a fixed general symbol $m_I(\xi)$, we will often refer to components $m_k^I(\xi_k)$ as $m_k(\xi_k)$, forgetting the superscript I in order to make notations lighter. Sometimes we will also write $m_k(\xi)$ if this makes no confusion.

4 Semiclassical Reduction to an ODE.

In this section we want to reformulate the Cauchy problem (1.1) and to deduce a new equation which can be transformed into an ODE. It is organised in three subsections. In the first one, we introduce semiclassical coordinates, rewrite the problem in this new framework and state the main theorem. The second and third sections are devoted to the proof of the main theorem. In particular, in the second one we introduce some technical lemmas we often refer to and we estimate v when it is away from Λ. In the third one, we first cut symbols in the cubic nonlinearity near Λ and away from points $x = \pm 1$, and develop them at $\xi = d\varphi(x)$, transforming multi-linear pseudo-differential operators in smooth functions of x; then, we repeat the development argument for $Op_h^w(x\xi + p(\xi))$.

4.1 Semiclassical Coordinates and Statement of the Main Result

Let u be a solution of (1.1) and set

$$(4.1)\qquad \begin{cases} w = (D_t + \sqrt{1+D_x^2})u \\ \bar{w} = -(D_t - \sqrt{1+D_x^2})u \end{cases}, \qquad \begin{cases} u = \langle D_x\rangle^{-1}(\frac{w+\bar{w}}{2}) \\ D_t u = \frac{w-\bar{w}}{2} \end{cases}.$$

With notations introduced in (1.3), the function w satisfies the following equation

$$(4.2)\qquad \begin{aligned} (D_t - \sqrt{1+D_x^2})w = &\sum_{k=0}^{3} i^k P'_k\left(\langle D_x\rangle^{-1}(\frac{w+\bar{w}}{2}); \frac{w-\bar{w}}{2}, D_x\langle D_x\rangle^{-1}(\frac{w+\bar{w}}{2})\right) \\ &+ \sum_{k=0}^{2} i^k P''_k\left(\langle D_x\rangle^{-1}(\frac{w+\bar{w}}{2}), D_x(\frac{w-\bar{w}}{2}), D_x^2\langle D_x\rangle^{-1}(\frac{w+\bar{w}}{2})\,;\right. \\ &\qquad\left.\frac{w-\bar{w}}{2}, D_x\langle D_x\rangle^{-1}(\frac{w+\bar{w}}{2})\right). \end{aligned}$$

Observe that operators which take the place of second derivatives have symbols of order one, while all other symbols are of order zero or negative (-1). Let us simplify the notation for the rest of the section by rewriting the nonlinearity in term of multi-linear pseudo-differential operators introduced in the previous section, namely as

$$(4.3)\qquad \sum_{|I|=3} Op(m_I)(w_I) + \sum_{|I|=3} Op(\widetilde{m}_I)(w_I)\,,$$

where symbols m_I, $\widetilde{m}_I$ are of the form (3.40). Moreover, m_I will denote symbols of order equal or less than zero, in the sense that all occurring symbols m^I_k are of order equal or less than zero, while in $\widetilde{m}_I$ there will be exactly one symbol of order one, thanks to the quasi-linear nature of the starting equation. Therefore (4.2) is rewritten as

$$(4.4)\qquad (D_t - \sqrt{1+D_x^2})w = \sum_{|I|=3} Op(m_I)(w_I) + \sum_{|I|=3} Op(\widetilde{m}_I)(w_I)\,,$$

and passing to the semiclassical framework by

$$(4.5)\qquad w(t,x) = \frac{1}{\sqrt{t}}v(t,\frac{x}{t})\,, \qquad h := \frac{1}{t}\,,$$

we obtain

$$(4.6)\qquad \big(D_t - Op_h^w(x\xi + p(\xi))\big)v = h\sum_{|I|=3} Op_h(m_I)(v_I) + h\sum_{|I|=3} Op_h(\widetilde{m}_I)(v_I)\,,$$

where $p(\xi) = \sqrt{1+\xi^2}$ and where we used the equality $Op_h(x\xi + p(\xi) + \frac{h}{2i}) = Op_h^w(x\xi + p(\xi))$ following from

$$\begin{aligned} Op_h^w(x\xi) &= \frac{h}{2}D_x x + \frac{h}{2}xD_x \\ &= \frac{h}{2i} + x\,hD_x = \frac{h}{2i} + Op_h(x\xi)\,. \end{aligned}$$

Furthermore, we write explicitly the nonlinearity of the equation, which will be useful hereinafter

(4.7)
$$\begin{aligned}\big(D_t - Op_h^w(x\xi + p(\xi))\big)v = h\sum_{k=0}^{3} i^k P'_k\left(\langle hD\rangle^{-1}(\frac{v+\bar v}{2}); \frac{v-\bar v}{2}, (hD)\langle hD\rangle^{-1}(\frac{v+\bar v}{2})\right)\\ + h\sum_{k=0}^{2} i^k P''_k\left(\langle hD\rangle^{-1}(\frac{v+\bar v}{2}), (hD)(\frac{v-\bar v}{2}), (hD)^2\langle hD\rangle^{-1}(\frac{v+\bar v}{2});\right.\\ \left.\frac{v-\bar v}{2}, (hD)\langle hD\rangle^{-1}(\frac{v+\bar v}{2})\right).\end{aligned}$$

Let us also define the operator $\mathcal{L}$ to be

$$\mathcal{L} := \frac{1}{h} Op_h^w(x + p'(\xi))\,. \tag{4.8}$$

The equation (4.6) represents for us the starting point to deduce an ODE satisfied by v, from which it will be easier to derive an estimate on the L^∞ norm of v. In reality, we will need more than an uniform estimate for v, namely we have to involve also a certain number of its derivatives, and then to control its $W_h^{\rho,\infty}$ norm for a fixed $\rho > 0$. With this in mind, we set $\Gamma(x,\xi) = \gamma(\frac{x+p'(\xi)}{\sqrt{h}})$, for a function $\gamma \in C_0^\infty(\mathbb{R})$, $\gamma \equiv 1$ in a neighbourhood of zero, with a small support. From lemma 3.15, $\Gamma \in S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-N})$ for every $N \in \mathbb{N}^*$, and case by case we will choose the right power we need. We consider also $\Sigma(\xi) = \langle\xi\rangle^\rho$ (in practice, at times we consider $\rho - 1 \in \mathbb{N}$, with ρ introduced for u in theorem 1.3, when we prove the bootstrap, or $\rho = -1$ when we develop asymptotics), and define

$$v^\Sigma := Op_h(\Sigma)v\,, \tag{4.9}$$

together with

$$\begin{aligned} v^\Sigma_\Lambda &:= Op_h^w(\Gamma)v^\Sigma\,,\\ v^\Sigma_{\Lambda^c} &:= Op_h^w(1-\Gamma)v^\Sigma\,, \end{aligned} \tag{4.10}$$

and symbols

$$\begin{aligned} m_I^\Sigma(\xi) &= \prod_{k=1}^{3} m_k^{I,\Sigma}(\xi_k) := \prod_{k=1}^{3} m_k^I(\xi_k)\Sigma(\xi_k)^{-1}\,,\\ \widetilde{m}_I^\Sigma(\xi) &= \prod_{k=1}^{3} \widetilde{m}_k^{I,\Sigma}(\xi_k) := \prod_{k=1}^{3} \widetilde{m}_k^I(\xi_k)\Sigma(\xi_k)^{-1}\,. \end{aligned} \tag{4.11}$$

The main result we want to prove in this section is the following:

Theorem 4.1 (Reformulation of the PDE)**.** *Suppose that we are given constants $A', B' > 0$, some $T > 1$ and a solution $v \in L^\infty([1,T]; H_h^s) \cap L^\infty([1,T]; W_h^{\rho,\infty})$ of the equation* (4.6) *(or, equivalently, of* (4.7)*), satisfying the following a priori bounds, for any $\varepsilon \in]0,1]$, $t \in [1,T]$,*

$$\|v(t,\cdot)\|_{W_h^{\rho,\infty}} \le A'\varepsilon\,, \tag{4.12}$$

$$\|\mathcal{L}v(t,\cdot)\|_{L^2} + \|v(t,\cdot)\|_{H_h^s} \le B'h^{-\sigma}\varepsilon\,, \tag{4.13}$$

for some $\sigma > 0$ small enough. Then, with preceding notations, v^{Σ}_{Λ} is solution of

(4.14)
$$\begin{aligned} D_t v^{\Sigma}_{\Lambda} &= \varphi(x)\theta_h(x) v^{\Sigma}_{\Lambda} + h\Phi^{\Sigma}_1(x)\theta_h(x)|v^{\Sigma}_{\Lambda}|^2 v^{\Sigma}_{\Lambda} \\ &\quad + hOp^w_h(\Gamma)\left[\Phi^{\Sigma}_3(x)\theta_h(x)(v^{\Sigma}_{\Lambda})^3 + \Phi^{\Sigma}_{-1}(x)\theta_h(x)|v^{\Sigma}_{\Lambda}|^2\overline{v^{\Sigma}_{\Lambda}} + \Phi^{\Sigma}_{-3}(x)\theta_h(x)(\overline{v^{\Sigma}_{\Lambda}})^3\right] + hR(v)\,, \end{aligned}$$

with $(\theta_h(x))_h$ a family of smooth functions compactly supported in $]-1,1[$, some smooth coefficients $\Phi^{\Sigma}_j(x)$, $|\Phi^{\Sigma}_j(x)| = O(h^{-\sigma'})$ on the support of θ_h, for $j \in \{3,1,-1,-3\}$ and a small $\sigma' > 0$. Moreover, $R(v)$ is a remainder verifying the following estimates

$$\|R(v)\|_{L^2} \le Ch^{\frac{1}{2}-\sigma}\left(\|\mathcal{L}v\|_{L^2} + \|v\|_{H^s_h}\right), \tag{4.15}$$

$$\|R(v)\|_{L^\infty} \le Ch^{\frac{1}{4}-\sigma}\left(\|\mathcal{L}v\|_{L^2} + \|v\|_{H^s_h}\right), \tag{4.16}$$

for a new small $\sigma \ge 0$.

Smooth coefficients $\Phi^{\Sigma}_j(x)$ in (4.14) may be explicitly calculated starting from the nonlinearity in (4.7), and in particular this will be done for $\Phi^{\Sigma}_1(x)$ at the beginning of section 5. Afterwards, we will use the notation $R_1(v)$ to refer to a remainder satisfying the following estimates:

$$\|R_1(v)\|_{H^\rho_h} \le Ch^{\frac{1}{2}-\sigma}(\|\mathcal{L}v\|_{L^2} + \|v\|_{H^s_h}), \tag{4.17}$$

$$\|R_1(v)\|_{L^\infty} \le Ch^{\frac{1}{4}-\sigma}(\|\mathcal{L}v\|_{L^2} + \|v\|_{H^s_h}), \tag{4.18}$$

for a small $\sigma \ge 0$.

4.2 Technical Results

We estimate $v^{\Sigma}_{\Lambda^c}$ as follows :

Lemma 4.2. *Let $\widetilde{\Gamma}(\xi)$ a smooth function such that $|\partial^\alpha \widetilde{\Gamma}| \lesssim \langle\xi\rangle^{-\alpha}$, χ as in lemma 3.13, $\beta > 0$. Then*

$$Op^w_h(\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}))v^{\Sigma} = Op^w_h\left(\Sigma(\xi)\chi(h^\beta\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})\right)v + R_1(v)\,, \tag{4.19}$$

where $R_1(v)$ is a remainder satisfying (4.17), (4.18).

Proof. We consider a function χ as in lemma 3.13, and we write

$$\begin{aligned} Op^w_h(\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}))v^{\Sigma} &= Op^w_h(\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}))Op^w_h(\Sigma(\xi)\chi(h^\beta\xi))v \\ &\quad + Op^w_h(\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}))Op^w_h(\Sigma(\xi)(1-\chi)(h^\beta\xi))v\,, \end{aligned} \tag{4.20}$$

for $\beta > 0$. The second term in the right hand side represents a remainder $R_1(v)$ satisfying the two inequalities of the statement just because $\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}) \in S_{\frac{1}{2},0}(1)$ (so, for instance, $\|Op^w_h(\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}))\|_{\mathcal{L}(H^{\rho+1}_h;W^{\rho,\infty}_h)} = O(h^{-\frac{1}{2}})$ by Sobolev inequality (3.8) and proposition 3.11) and $(1-\chi)(h^\beta\xi)$ is supported for $|\xi| \ge h^{-\beta}$, so that we can use essentially lemma 3.13.

On the other hand, since $|\partial^\alpha\widetilde{\Gamma}| \le \langle\xi\rangle^{-\alpha}$ and $\Sigma(\xi)\chi(h^\beta\xi) \in h^{-\sigma}S_{0,\beta}(1)$, with

$$\sigma = \begin{cases} \rho\beta & \text{if } \rho \in \mathbb{N} \\ 0 & \text{if } \rho < 0 \end{cases} \tag{4.21}$$

we use the composition formula of lemma 3.10 for the first term in the right hand side, i.e.

$$Op_h^w(\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}))Op_h^w(\Sigma(\xi)\chi(h^\beta\xi))v = Op_h^w\left(\Sigma(\xi)\chi(h^\beta\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})\right)v + Op_h(r_0)v\,, \tag{4.22}$$

where $r_0 \in h^{\frac{1}{2}-\sigma}S_{\frac{1}{2},\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$. So $Op_h(r_0)v$ satisfies inequalities (4.17), (4.18) respectively by propositions 3.11 and 3.12. □

Lemma 4.3. *Let $\widetilde{\Gamma}(\xi)$ be a smooth function such that $|\partial^\alpha\widetilde{\Gamma}| \lesssim \langle\xi\rangle^{-\alpha}$, $c(x,\xi) \in S_{\delta,\beta}(1)$, $c'(x,\xi) \in S_{\delta',0}(1)$, with $\delta,\delta' \in [0,\frac{1}{2}[$, $\beta > 0$. Then*

$$c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})\sharp(x+p'(\xi)) = c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi)) + h^{1-2\delta}r \tag{4.23}$$

with $r \in S_{\frac{1}{2},\beta}(1)$, and

$$\left\|Op_h^w\big(c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\big)Op_h^w(c')v\right\|_{L^2} \le h^{1-\sigma}(\|\mathcal{L}v\|_{L^2} + \|v\|_{H^s_h})\,, \tag{4.24}$$

$$\left\|Op_h^w\big(c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\big)Op_h^w(c')v\right\|_{L^\infty} \le h^{\frac{1}{2}-\sigma}(\|\mathcal{L}v\|_{L^2} + \|v\|_{H^s_h})\,, \tag{4.25}$$

with $\sigma = \sigma(\delta,\delta',\beta) \to 0$ as $\delta,\delta',\beta \to 0$.
Moreover, if $\widetilde{\Gamma} = \Gamma_{-1}$, with $|\partial^\alpha\Gamma_{-1}| \lesssim \langle\xi\rangle^{-1-\alpha}$, then in (4.23) $r \in S_{\frac{1}{2},\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, and the L^∞ estimate can be improved

$$\left\|Op_h^w\big(c(x,\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\big)Op_h^w(c')v\right\|_{L^\infty} \le h^{\frac{3}{4}-\sigma}(\|\mathcal{L}v\|_{L^2} + \|v\|_{H^s_h})\,. \tag{4.26}$$

Proof. The result is immediate if we use the development of proposition 3.8 at order one,

$$\begin{aligned} c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})\sharp(x+p'(\xi)) &= c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi)) \\ &+ \frac{h}{2i}\left\{c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}),(x+p'(\xi))\right\} + hr_1\,, \end{aligned} \tag{4.27}$$

where $r_1 \in h^{-2\delta}S_{\frac{1}{2},\beta}(1)$, while by direct calculation the Poisson bracket is equal to:

$$\left\{c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}),(x+p'(\xi))\right\} = \widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})(\partial_\xi c - p''\partial_x c)\,,$$

$\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})(\partial_\xi c - p''\partial_x c) \in h^{-\delta}S_{\frac{1}{2},\beta}(1)$. Therefore

$$\begin{aligned} &Op_h^w\big(c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\big)Op_h^w(c')v = \\ &= hOp_h^w\big(c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})\big)\mathcal{L}Op_h^w(c')v \\ &- \frac{h}{2i}Op_h^w\big(\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})(\partial_\xi c - p''\partial_x c) + 2i\,r_1\big)Op_h^w(c')v\,, \end{aligned} \tag{4.28}$$

and the application of proposition 3.11, along with Sobolev injection (3.8), immediately implies that the second term in the right hand side satisfies estimates (4.24), (4.25). Moreover, $[\mathcal{L}, Op_h^w(c')] = i(\partial_\xi c' - p''\partial_x c') + h^{1-2\delta'}r_1$, r_1 being a symbol in $S_{\delta',0}(1)$, hence it belongs to

$h^{-\delta'}S_{\delta',0}(1)$, and its quantization is a bounded operator from L^2 to L^2 by proposition 3.11 up to a small loss in $h^{-\delta'}$. This remark, together with $c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}) \in S_{\frac{1}{2},\beta}(1)$, $c' \in S_{\delta',0}(1)$, proposition 3.11, and Sobolev injection imply that also the first term in the right hand side verifies estimates in (4.24), (4.25). The same reasoning as above can be applied when $\widetilde{\Gamma} = \Gamma_{-1}$ with $|\partial^\alpha \Gamma_{-1}| \lesssim \langle\xi\rangle^{-1-\alpha}$. In this case, the development in (4.27) is justified by lemma 3.10. Moreover, symbols involving $c(x,\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})$ stay in $S_{\frac{1}{2},\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, so we can apply proposition 3.12, instead of Sobolev injection, to control the L^∞ norm, losing only a power $h^{-\frac{1}{4}-\sigma}$, for a small $\sigma > 0$ (and not $h^{-\frac{1}{2}}$ due to Sobolev estimate) and so deriving the improved estimate (4.26). □

Proposition 4.4 (Estimates on $v^\Sigma_{\Lambda^c}$)**.** *There exist $s \in \mathbb{N}$ and a constant $C > 0$ independent of h such that $v^\Sigma_{\Lambda^c}$ can be considered as a remainder $R(v)$ satisfying* (4.15), (4.16).

Proof. Firstly, we want to reduce to the study of the action of $Op_h^w(1-\Gamma)$ on v and not on v^Σ, so we can use lemma 4.2 with $\widetilde{\Gamma} = 1-\gamma$, obtaining

$$Op_h^w\Big((1-\gamma)(\frac{x+p'(\xi)}{\sqrt{h}})\Big)v^\Sigma = Op_h^w\Big(\Sigma(\xi)\chi(h^\beta\xi)(1-\gamma)(\frac{x+p'(\xi)}{\sqrt{h}})\Big)v + R(v)\,, \tag{4.29}$$

where $R(v)$ satisfies (4.15), (4.16). Then it remains to analyse

$$Op_h^w\left(\Sigma(\xi)\chi(h^\beta\xi)(1-\gamma)(\frac{x+p'(\xi)}{\sqrt{h}})\right)v\,.$$

We write the symbol of the operator as $\Sigma(\xi)\chi(h^\beta\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})$, with $\Gamma_{-1}(\xi) = \frac{(1-\gamma)(\xi)}{\xi}$, and we can apply the previous lemma with $c(x,\xi) = \Sigma(\xi)\chi(h^\beta\xi) \in h^{-\sigma}S_{0,\beta}(1)$, σ as in (4.21), $c'(x,\xi) \equiv 1$, to derive that it is a remainder $R(v)$ satisfying (4.15), (4.16). □

We want to apply first $Op_h^w(\Sigma(\xi))$ to (4.6). As $Op_h^w(\Sigma(\xi))$ commutes with $D_t - Op_h^w(x\xi + p(\xi))$ (because $\Sigma(D)$ commutes with $D_t - p(D)$), we obtain the equation:

$$(D_t - Op_h^w(x\xi + p(\xi)))v^\Sigma = hOp_h^w(\Sigma)\Big[\sum_{|I|=3} Op_h(m_I)(v_I) + \sum_{|I|=3} Op_h(\widetilde{m}_I)(v_I)\Big]\,. \tag{4.30}$$

Then, we apply also $Op_h^w(\Gamma)$ to (4.30), so we have to calculate its commutator with the linear part of the equation, as done in the following:

Lemma 4.5.

$$\big[D_t - Op_h^w(x\xi + p(\xi)), Op_h^w(\Gamma(x,\xi))\big] = Op_h^w(b)\,, \tag{4.31}$$

where

$$b(x,\xi) = h\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}}) + h^{\frac{3}{2}}r\,, \tag{4.32}$$

$r \in S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, *and* Γ_{-1} *satisfies* $|\partial^\alpha \Gamma_{-1}(\xi)| \lesssim \langle\xi\rangle^{-1-\alpha}$.

Proof. First we start by calculating $[D_t, Op_h^w(\Gamma)] = D_t Op_h^w(\Gamma) - Op_h^w(\Gamma) D_t$:

(4.33)
$$
\begin{aligned}
D_t Op_h^w(\Gamma) v &= \frac{1}{i}\partial_t \left[\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(x-y)\xi}\gamma(\frac{\frac{x+y}{2}+p'(h\xi)}{\sqrt{h}}) v(t,y)\, dy d\xi\right] \\
&= \frac{-h^2}{2\pi i}\partial_h \left[\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(x-y)\xi}\gamma(\frac{\frac{x+y}{2}+p'(h\xi)}{\sqrt{h}}) v(t,y)\, dy d\xi\right] \\
&= -\frac{h}{2\pi i}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(x-y)\xi}\,\gamma'(\frac{\frac{x+y}{2}+p'(h\xi)}{\sqrt{h}})\frac{p''(h\xi)h\xi}{\sqrt{h}} v(t,y)\, dy d\xi \\
&+ \frac{h}{4\pi i}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(x-y)\xi}\,\gamma'(\frac{\frac{x+y}{2}+p'(h\xi)}{\sqrt{h}})(\frac{\frac{x+y}{2}+p'(h\xi)}{\sqrt{h}}) v(t,y)\, dy d\xi \\
&+ \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(x-y)\xi}\gamma(\frac{\frac{x+y}{2}+p'(h\xi)}{\sqrt{h}}) D_t v(t,y)\, dy d\xi \\
&= ih\, Op_h^w\left(\gamma'(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{p''(\xi)\xi}{\sqrt{h}})\right) v - \frac{ih}{2}\, Op_h^w\left(\gamma'(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\right) v \\
&+ Op_h^w(\Gamma) D_t v\,.
\end{aligned}
$$

Then, using (3.14) and (3.16) we write

$$
[Op_h^w(\Gamma(x,\xi)), Op_h^w(x\xi+p(\xi))] = \frac{h}{i} Op_h^w\Big(\big\{\gamma(\frac{x+p'(\xi)}{\sqrt{h}}), x\xi+p(\xi)\big\}\Big) + r_2\,, \tag{4.34}
$$

with $r_2 \in h^{\frac{3}{2}} S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$ from lemma 3.10, since $\partial^\alpha \Gamma \in h^{-\frac{|\alpha|}{2}} S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, $\partial^\alpha(x\xi + p'(\xi)) \in S_{0,0}(1)$ for $|\alpha| = 3$. On the other hand, developing the braces in (4.34) we find

$$
\begin{aligned}
\frac{h}{i} Op_h^w\Big(\big\{\gamma(\frac{x+p'(\xi)}{\sqrt{h}}), x\xi+p(\xi)\big\}\Big) &= -ih\, Op_h^w\left(\gamma'(\frac{x+p'(\xi)}{\sqrt{h}})\frac{p''(\xi)\xi}{\sqrt{h}}\right) \\
&+ ih\, Op_h^w\left(\gamma'(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\right)\,,
\end{aligned}
$$

so when we add it to $[D_t, Op_h^w(\Gamma)]$ calculated before, we obtain the result just choosing $\Gamma_{-1}(\xi) = \frac{1}{2}\gamma'(\xi)$. ☐

We apply $Op_h^w(\Gamma)$ to equation (4.30). Using lemma 4.5, we obtain

(4.35)
$$
\begin{aligned}
(D_t - Op_h^w(x\xi + p(\xi)) v_\Lambda^\Sigma &= h\, Op_h^w(\Gamma) Op_h^w(\Sigma)\Big[\sum_{|I|=3} Op_h(m_I)(v_I) + \sum_{|I|=3} Op_h(\widetilde{m}_I)(v_I)\Big] \\
&+ h Op_h^w\left(\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\right) v^\Sigma + h^{\frac{3}{2}} Op_h^w(r) v^\Sigma\,,
\end{aligned}
$$

$r \in S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, where the second and third term in the right hand side are of the form $hR(v)$, $R(v)$ satisfying the estimates (4.15),(4.16). In fact, using lemma 4.2 with $\widetilde{\Gamma}(\xi) = \Gamma_{-1}(\xi)\xi$, and lemma 4.3, we have

(4.36)
$$
\begin{aligned}
Op_h^w\left(\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\right) v^\Sigma &= Op_h^w\left(\Sigma(\xi)\chi(h^\beta\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\right) v + R(v) \\
&= R(v)\,,
\end{aligned}
$$

while r can be split via a function χ as in lemma 3.13, with $\beta > 0$, obtaining $r(x,\xi)\chi(h^\beta\xi) \in S_{\frac{1}{2},\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$ to which we can apply proposition 3.12, and $r(x,\xi)(1-\chi)(h^\beta\xi)$ to which can be instead applied lemma 3.13. Then also $h^{\frac{3}{2}}Op_h^w(r)v^\Sigma = hR(v)$. Therefore, the equation we want to deal with becomes

(4.37)

$$(D_t - Op_h^w(x\xi + p(\xi))v_\Lambda^\Sigma = h\, Op_h^w(\Gamma)Op_h^w(\Sigma)\Big[\sum_{|I|=3} Op_h(m_I)(v_I) + \sum_{|I|=3} Op_h(\widetilde{m}_I)(v_I)\Big] + hR(v)\,,$$

with a remainder $R(v)$ which satisfies (4.15), (4.16).

4.3 Development at $\xi = d\varphi(x)$

The next step now is to develop the symbol of the *characteristic* term in the nonlinearity, i.e. the one corresponding to $I = (1,1,-1)$, at $\xi = d\varphi(x)$. This will allow us to write it from $|v_\Lambda^\Sigma|^2 v_\Lambda^\Sigma$ up to some remainders, transforming the action of pseudo-differential operators acting on it into a product of smooth functions of x. Such development is not essential on *non characteristic* terms, which will be eliminated through a normal form argument in the next section. However, some results, such as proposition 4.7 and lemma 4.8, apply suitably also to *non characteristic* terms, so we will freely make use of them to get some simplifications.

We want to prove the following result:

Proposition 4.6. *Suppose that v satisfies the a priori estimates in* (4.12), (4.13). *Then there exists a family of functions $\theta_h(x) \in C_0^\infty(]-1,1[)$, real valued, equal to one on an interval $[-1+ch^{2\beta}, 1-ch^{2\beta}]$, $\|\partial^\alpha\theta_h\|_{L^\infty} = O(h^{-2\beta\alpha})$, for a small $\beta > 0$, such that the nonlinearity in* (4.37) *can be written as*

(4.38)

$$h\Phi_1^\Sigma(x)\theta_h(x)|v_\Lambda^\Sigma|^2 v_\Lambda^\Sigma + hOp_h^w(\Gamma)\left[\Phi_3^\Sigma(x)\theta_h(x)(v_\Lambda^\Sigma)^3 + \Phi_{-1}^\Sigma(x)\theta_h(x)|v_\Lambda^\Sigma|^2\overline{v_\Lambda^\Sigma} + \Phi_{-3}^\Sigma(x)\theta_h(x)(\overline{v_\Lambda^\Sigma})^3\right]$$
$$+ hR(v)\,,$$

where $\Phi_j^\Sigma(x)$ are smooth functions of x, $|\Phi_j^\Sigma(x)| = O(h^{-\sigma})$ on the support of θ_h, for $j \in \{3,1,-1,-3\}$, and where the remainder $R(v)$ satisfies estimates (4.15), (4.16), *with $\sigma = \sigma(\beta) > 0$ small.*

Before proving this proposition, we need the following general result.

Proposition 4.7. *Let $a(x,\xi)$ be a real symbol in $S_{\delta,0}(\langle\xi\rangle^q)$, $q \in \mathbb{R}$, for some $\delta > 0$ small. There exists a family $(\theta_h(x))_h$ of C^∞ functions, real valued, supported in some interval $[-1+ch^{2\beta}, 1-ch^{2\beta}]$, for a small $\beta > 0$, with $(h\partial_h)^k\theta_h$ bounded for every k, such that*

$$Op_h^w(a)v = \theta_h(x)a(x, d\varphi(x))v + R_1(v)\,, \tag{4.39}$$

where $R_1(v)$ is a remainder satisfying estimates (4.17), (4.18), *with $\sigma = \sigma(\beta,\delta) > 0$, $\sigma \to 0$ as $\beta, \delta \to 0$. The same equality holds replacing v by v^Σ.*

Proof. In order to develop the symbol $a(x,\xi)$ at $\xi = d\varphi(x)$ we need to stay away from points $x = \pm 1$, so the idea is to truncate near Λ and to estimate different terms coming out.

First, let us consider a function $\chi \in C_0^\infty(\mathbb{R})$ as in lemma 3.13, $\beta > 0$ small. We decompose $a(x,\xi)$ as follows

$$a(x,\xi) = a(x,\xi)\chi(h^\beta\xi) + a(x,\xi)(1-\chi)(h^\beta\xi)\,. \tag{4.40}$$

It turns out from symbolic calculus, proposition 3.11, lemma 3.13 and Sobolev injection (3.8), that $Op_h^w(a(x,\xi)(1-\chi)(h^\beta\xi))v$ is of the form $R_1(v)$, if we choose $s \gg 1$ sufficiently large, so we have just to deal with $a(x,\xi)\chi(h^\beta\xi)$. Since this symbol is rapidly decaying in $|h^\beta\xi|$, we notice that, to prove that the estimate (4.17) holds for terms of interest, we can turn the H_h^ρ norm into the L^2 norm up to a small loss in h, and then simply estimate the L^2 norm of these terms. This is obvious when $\rho < 0$, for H_h^ρ injects in L^2, while for $\rho \in \mathbb{N}$ this follows using the definition 3.5 (ii), symbolic calculus, and the fact that $\langle\xi\rangle^\rho\chi(h^\beta\xi) \le h^{-\rho\beta}$. Therefore, it is sufficient for our aim to prove that terms coming out are remainders $R(v)$, in the sense of inequalities (4.15), (4.16).
Secondly, we consider a smooth compactly supported function $\gamma \in C_0^\infty(\mathbb{R})$, $\gamma \equiv 1$ in a neighbourhood of zero, with a sufficiently small support, and we split $a(x,\xi)\chi(h^\beta\xi)$ as

$$a(x,\xi)\chi(h^\beta\xi) = a(x,\xi)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) + a(x,\xi)\chi(h^\beta\xi)(1-\gamma)(\frac{x+p'(\xi)}{\sqrt{h}})\,. \tag{4.41}$$

Again, the second symbol in the right hand side gives us a remainder. In fact, since $(1-\gamma)(\xi)$ is supported for $|\xi| > \alpha_1$, we can write

$$a(x,\xi)\chi(h^\beta\xi)(1-\gamma)(\frac{x+p'(\xi)}{\sqrt{h}}) = a(x,\xi)\chi(h^\beta\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\,, \tag{4.42}$$

where $\Gamma_{-1}(\xi) = \frac{(1-\gamma)(\xi)}{\xi}$, $|\partial^\alpha\Gamma_{-1}(\xi)| \lesssim \langle\xi\rangle^{-1-\alpha}$. Lemma 4.3 with $c(x,\xi) = a(x,\xi)\chi(h^\beta\xi) \in h^{-\sigma}S_{\delta,\beta}(1)$, $\sigma \ge 0$ small (either equal to $q\beta$ for $q \ge 0$, or to 0 for $q < 0$), $c'(x,\xi) \equiv 1$, implies that $Op_h^w\Big(a(x,\xi)\chi(h^\beta\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\Big)\,v$ satisfies (4.15), (4.16).

The last remaining symbol is $a(x,\xi)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})$, which is supported in $\{(x,\xi)\in\mathbb{R}\times\mathbb{R}\,\big|\,|\xi| < C_2h^{-\beta}\,,|\frac{x+p'(\xi)}{\sqrt{h}}| < \alpha_2\}$, so x is bounded in a compact subset of $]-1,1[$ of the form $[-1+ch^{2\beta}, 1-ch^{2\beta}]$, for a suitable positive constant c. We may find a smooth function $\theta_h(x) \in C_0^\infty(]-1,1[)$, $\theta_h \equiv 1$ on $[-1+ch^{2\beta}, 1-ch^{2\beta}]$, satisfying $\|\partial^\alpha\theta_h\|_{L^\infty} = O(h^{-2\beta\alpha})$, and write

$$a(x,\xi)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) = a(x,\xi)\theta_h(x)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\,. \tag{4.43}$$

Since on the support of θ_h we are away from $x = \pm 1$, we may now develop $a(x,\xi)$ at $\xi = d\varphi(x)$,

$$\begin{aligned} a(x,\xi) &= a(x,d\varphi(x)) + \int_0^1 \partial_\xi a(x, t\xi + (1-t)d\varphi(x))dt\,(\xi - d\varphi(x)) \\ &= a(x,d\varphi(x)) + b(x,\xi)(x+p'(\xi))\,, \end{aligned} \tag{4.44}$$

where

$$b(x,\xi) = \int_0^1 \partial_\xi a(x, t\xi+(1-t)d\varphi(x))\,dt\,\frac{\xi - d\varphi(x)}{x+p'(\xi)}\,. \tag{4.45}$$

Then,

(4.46)

$$\begin{aligned} a(x,\xi)\theta_h(x)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) &= a(x,d\varphi(x))\theta_h(x) + a(x,d\varphi(x))\theta_h(x)\big[\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) - 1\big] \\ &\quad + b(x,\xi)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\,. \end{aligned}$$

Let us call I_1 and I_2 the Weyl quantizations respectively of the second and third term in the right hand side of (4.46). We want to show that they satisfy (4.15), (4.16).
First we analyse the third term in the right hand side of (4.46). We may find another smooth function $\widetilde{\gamma}$, with a small support, such that

$$\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) = \chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\widetilde{\gamma}(\langle\xi\rangle^2(x+p'(\xi)))\,. \tag{4.47}$$

From $a \in S_{\delta,0}(\langle\xi\rangle^q)$ and lemma 3.14, $B(x,\xi) := b(x,\xi)\chi(h^\beta\xi)\widetilde{\gamma}(\langle\xi\rangle^2(x+p'(\xi)))$ is an element of

$$h^{-\delta}S_{2\beta,\beta}(\langle\xi\rangle^{3+q}) \subset h^{-\sigma}S_{\delta',\beta}(1)\,,$$

for $\delta' = \max\{\delta, 2\beta\}$, $\sigma > 0$ small depending on β and δ. Moreover, $|\partial^\alpha\gamma(\xi)| \le \langle\xi\rangle^{-1-\alpha}$, so lemma 4.3 implies that $Op^w_h\big(B(x,\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\big)$ is a remainder $h^{\frac{1}{2}}R(v)$.
On the other hand, I_1 has a symbol whose support is included in the union $\{|\xi| > C_1h^{-\beta}\} \cup \{|\frac{x+p'(\xi)}{\sqrt{h}}| > \alpha_1\}$. Take $\widetilde{\chi} \in C^\infty_0(\mathbb{R})$, $\widetilde{\chi} \equiv 1$ in a neighbourhood of zero, $supp\widetilde{\chi} \subset \{|\xi| < C_1h^{-\beta}\}$, so that $\chi\widetilde{\chi} \equiv \widetilde{\chi}$. We make a further decomposition,

$$\begin{aligned}
&\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) - 1 = \\
&= \Big(\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) - 1\Big)\widetilde{\chi}(h^\beta\xi) + \Big(\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) - 1\Big)(1-\widetilde{\chi})(h^\beta\xi) \\
&= \Big(\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) - 1\Big)\widetilde{\chi}(h^\beta\xi) + \Big(\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) - 1\Big)(1-\widetilde{\chi})(h^\beta\xi)\,.
\end{aligned} \tag{4.48}$$

The first term in the right hand side is supported for $|\frac{x+p'(\xi)}{\sqrt{h}}| > \alpha_1$, so it can be written as

$$\widetilde{\chi}(h^\beta\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\,,$$

and it is a remainder by lemma 4.3. Besides, the second term in the right hand side is supported for $|\xi| > C_1h^{-\beta}$, so it is essentially an application of lemma 3.13 to show that it is a remainder $R(v)$. This shows finally that the development in (4.39) holds. For the last statement of the proposition, one can show that the same proof we did for v can be applied for v^Σ, just changing $a(x,\xi)$ into $a(x,\xi)\Sigma(\xi)$ trough lemma 4.2 when needed, and for a new $\sigma > 0$ depending also on ρ.

□

Proof of Proposition 4.6. The idea of the proof is to develop all symbols $m_I(\xi), \widetilde{m}_I(\xi)$ occurring in the cubic nonlinearity at $\xi = d\varphi(x)$ using the previous proposition. Remark that, when $i_k = -1$, $v_{i_k} = \bar{v}$ and we have

$$Op_h(m_k(\xi))\bar{v} = \overline{Op_h(m_k(-\xi))v} = \overline{Op_h(m_k(i_k\xi))v}\,, \tag{4.49}$$

so the development at $\xi = d\varphi(x)$ will give us a term $m_k(i_kd\varphi(x))v_{i_k}$, since $m_k(\xi), d\varphi(x)$ are real valued.

We first write $Op_h(m_k(\xi))v_{i_k} = Op^w_h(m_k(\xi))v_{i_k} = Op^w_h(m_k(\xi)\Sigma(\xi)^{-1})v^\Sigma_{i_k} = Op^w_h(m^\Sigma_k(\xi))v^\Sigma$ (following the notation introduced in (4.11) - remind that classical quantization coincide with the

Weyl one on symbols depending only on ξ) and then we apply proposition 4.7. From a-priori estimates (4.12), (4.13), we have $\|m^\Sigma_k(i_k d\varphi(x))\theta_h(x)v^\Sigma_{i_k}\|_{L^\infty} = O(h^{-\sigma})$, $\|m^\Sigma_k(i_k d\varphi(x))\theta_h(x)v^\Sigma_{i_k}\|_{H^\rho_h} = O(h^{-\sigma})$, for a $\sigma>0$ depending on β, so we immediately obtain that

$$Op_h(m_I)(v_I) = \prod_{k=1}^{3} m^\Sigma_k(i_k d\varphi(x))\theta_h(x)v^\Sigma_{i_k} + R_1(v),$$

$R_1(v)$ satisfying estimates (4.17), (4.18), and we can perform the passage from the term

$$\sum_{|I|=3} Op_h(m_I)(v_I) + \sum_{|I|=3} Op_h(\widetilde{m}_I)(v_I) \tag{4.50}$$

to its development

$$\sum_{|I|=3} m^\Sigma_I(d\varphi_I(x))\theta_h(x)v^\Sigma_I + \sum_{|I|=3} \widetilde{m}^\Sigma_I(d\varphi_I(x))\theta_h(x)v^\Sigma_I + R_1(v)\,. \tag{4.51}$$

The nonlinearity in (4.37) becomes

$$\begin{aligned} & h\, Op^w_h(\Gamma)Op^w_h(\Sigma(\xi))\Big[\sum_{|I|=3} m^\Sigma_I(d\varphi_I(x))\theta_h(x)v^\Sigma_I + \sum_{|I|=3} \widetilde{m}^\Sigma_I(d\varphi_I(x))\theta_h(x)v^\Sigma_I\Big] \\ & + hOp^w_h(\Gamma)Op^w_h(\Sigma(\xi))R_1(v)\,, \end{aligned} \tag{4.52}$$

where $R_1(v)$ satisfies (4.17), so that $Op^w_h(\Gamma)Op^w_h(\Sigma(\xi))R_1(v)$ is a remainder of the form $R(v)$, satisfying the estimates (4.15), (4.16), by propositions 3.11 and 3.12.
The following three lemmas allow us to conclude the proof. In particular, we underline that in lemma 4.8 we only need an L^2 estimate on what we denote $R(v)$, because we will apply to it the operator $Op^w_h(\Gamma)$, which is continuous from L^2 to L^∞ with norm $\|Op^w_h(\Gamma)\|_{\mathcal{L}(L^2;L^\infty)} = O(h^{-\frac{1}{4}-\sigma})$ by proposition 3.12. ☐

Lemma 4.8. *Let $I=(i_1,i_2,i_3)$, $i_k\in\{1,-1\}$ for $k=1,2,3$, be a fixed vector. Denote by $A(\xi)$ the function $\Sigma(\xi)\chi(h^\beta\xi)$, with χ as in lemma 3.13, $\beta>0$. Then*

$$\begin{aligned} & Op^w_h(\Sigma(\xi))\big(m^\Sigma_I(d\varphi_I(x))\theta_h(x)v^\Sigma_I\big) = A\Big(\sum_{l=1}^{3} i_l d\varphi(x)\Big)m^\Sigma_I(d\varphi_I(x))\theta_h(x)v^\Sigma_I + h^{\frac12}R(v)\,, \\ & Op^w_h(\Sigma(\xi))\big(\widetilde{m}^\Sigma_I(d\varphi_I(x))\theta_h(x)v^\Sigma_I\big) = A\Big(\sum_{l=1}^{3} i_l d\varphi(x)\Big)\widetilde{m}^\Sigma_I(d\varphi_I(x))\theta_h(x)v^\Sigma_I + h^{\frac12}R(v)\,, \end{aligned} \tag{4.53}$$

where $R(v)$ satisfies the estimate (4.15).

Proof. Before proving the result, let us make some observations: first, in all the proof we will use generically the letter σ to denote a small non-negative constant depending on β, that goes to zero when β goes to zero; the symbol $\Sigma(\xi)$ can be reduced to $\Sigma(\xi)\chi(h^\beta\xi)\in h^{-\sigma}S_{0,\beta}(1)$, σ as in (4.21), up to remainders (essentially using lemma 3.13); from the *a priori* estimates (4.12), (4.13) made on v, we have $\|m^\Sigma_I(d\varphi_I(x))\theta_h(x)v^\Sigma_I\|_{L^2} = O(h^{-\sigma})$.

Let us consider a smooth function $\tilde{\theta}_h(x)\in C^\infty_0(]-1,1[)$, such that $\tilde{\theta}_h\theta_h\equiv\theta_h$, and let us write

$$m^\Sigma_I(d\varphi_I(x))\theta_h(x)v^\Sigma_I = \tilde{\theta}_h(x)m^\Sigma_I(d\varphi_I(x))\theta_h(x)v^\Sigma_I\,.$$

We enter $\tilde{\theta}_h(x)$ in $Op_h^w(\Sigma(\xi)\chi(h^\beta\xi))$ applying symbolic calculus of proposition 3.8, to be able to develop the symbol at $\xi = \sum_{l=1}^{3} i_l d\varphi(x)$. We have

$$\Sigma(\xi)\chi(h^\beta\xi)\sharp\tilde{\theta}_h(x) = \Sigma(\xi)\chi(h^\beta\xi)\tilde{\theta}_h(x) + r_0\,, \tag{4.54}$$

with $r_0 \in h^{1-\sigma}S_{\delta,\beta}(1)$, $\delta > 0$ small, so proposition 3.11 implies that its quantization gives a remainder as in the statement, when applied to $m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma$. Now, since we are away from $x = \pm 1$, we can develop $A(\xi) = \Sigma(\xi)\chi(h^\beta\xi)$ at $\xi = \sum_{l=1}^{3} i_l d\varphi(x)$ by Taylor's formula, i.e.

$$A(\xi) = A\Big(\sum_{l=1}^{3} i_l d\varphi(x)\Big) + A'(x,\xi)(\xi - \sum_{l=1}^{3} i_l d\varphi(x))\,, \tag{4.55}$$

with

$$A'(x,\xi) = \int_0^1 A'\Big(t\xi + (1-t)\sum_{l=1}^{3} i_l d\varphi(x))\Big)\,dt\,, \tag{4.56}$$

$A'(x,\xi)\tilde{\theta}_h(x)$ belonging to $h^{-\sigma}S_{\delta,0}(1)$. To conclude the proof, we also need to show that

$$Op_h^w\Big(A'(x,\xi)\tilde{\theta}_h(x)(\xi - \sum_{l=1}^{3} i_l d\varphi(x))\Big)\,\big(m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma\big) = h^{\frac{1}{2}}R(v).$$

So let us consider a new function $\tilde{\tilde{\theta}}_h(x) \in C_0^\infty(]-1,1[)$, such that $\tilde{\tilde{\theta}}_h\tilde{\theta}_h \equiv \tilde{\theta}_h$. Since $\tilde{\theta}_h(\xi - \sum_{l=1}^{3} i_l d\varphi(x)) \in h^{-\sigma}S_{\delta,0}(\langle\xi\rangle)$, and using symbolic calculus of proposition 3.8, we write

$$A'(x,\xi)\tilde{\tilde{\theta}}_h(x)\sharp\Big(\tilde{\theta}_h(\xi - \sum_{l=1}^{3} i_l d\varphi(x))\Big) = A'(x,\xi)\tilde{\theta}_h(x)(\xi - \sum_{l=1}^{3} i_l d\varphi(x)) + r_0'\,, \tag{4.57}$$

where $r_0' \in h^{1-\sigma}S_{\delta,0}(1)$. Again proposition 3.11 and the uniform bound on v imply that $Op_h^w(r_0')(m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma)$ is a remainder $h^{\frac{1}{2}}R(v)$. We can focus on the term

$$Op_h^w\big(A'(x,\xi)\tilde{\tilde{\theta}}_h(x)\big)Op_h^w\Big(\tilde{\theta}_h(x)(\xi - \sum_{l=1}^{3} i_l d\varphi(x))\Big)(m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma)\,, \tag{4.58}$$

and we can go further, limiting ourselves to consider the action of these operators when v_I^Σ is replaced by

$$v_I^0 := \prod_{k=1}^{3} Op_h^w(\Sigma(\xi)\chi(h^\beta\xi))v_{i_k}\,, \tag{4.59}$$

again up to terms with symbols supported for $|\xi| \geq h^{-\beta}$, which are remainders from lemma 3.13. The operator $Op_h^w\Big(\tilde{\theta}_h(x)(\xi - \sum_{l=1}^{3} i_l d\varphi(x))\Big)$ has a symbol linear in ξ, so

$$\begin{aligned} Op_h^w\Big(\tilde{\theta}_h(x)(\xi - \sum_{l=1}^{3} i_l d\varphi(x))\Big) &= \frac{1}{2}hD_x\tilde{\theta}_h(x) + \frac{1}{2}\tilde{\theta}_h(x)hD_x - \tilde{\theta}_h(x)\sum_{l=1}^{3} i_l d\varphi(x) \\ &= h\frac{\tilde{\theta}_h'(x)}{2i} + \tilde{\theta}_h(x)(hD_x - \sum_{l=1}^{3} i_l d\varphi(x))\,, \end{aligned} \tag{4.60}$$

and if we choose $\tilde{\theta}_h$ such that $\tilde{\theta}_h \theta_h \equiv \theta_h$, we have that $\tilde{\theta}'_h \equiv 0$ on the support of θ_h, giving no contributions when $h\frac{\tilde{\theta}'_h(x)}{2}$ is multiplied by $m_I^{\Sigma}(d\varphi_I(x))\theta_h(x)v_I^0$. Here $(hD_x - \sum_{l=1}^{3} i_l d\varphi(x))$ acts like a derivation on v_I^0, so the Leibniz's rule holds and

$$
\begin{aligned}
&Op_h^w\Big(\tilde{\theta}_h(x)(\xi - \sum_{l=1}^{3} i_l d\varphi(x))\Big)\left(m_I^{\Sigma}(d\varphi_I(x))\theta_h(x)v_I^0\right) = \\
&= \tilde{\theta}_h(x)(hD_x - \sum_{l=1}^{3} i_l d\varphi(x))\left(m_I^{\Sigma}(d\varphi_I(x))\theta_h(x)v_I^0\right) \qquad (4.61)\\
&= \left[hD_x(m_I^{\Sigma}(d\varphi_I(x))\theta_h(x))\right]v_I^0 + m_I^{\Sigma}(d\varphi_I(x))\theta_h(x)\tilde{\theta}_h(x)(hD_x - \sum_{l=1}^{3} i_l d\varphi(x))(v_I^0)\,.
\end{aligned}
$$

Then, if for instance $v_I^0 = (v^0)^3$ (i.e. $I = (1,1,1)$, and the same idea can be applied with $|v^0|^2 v^0$, $|v^0|^2\overline{v^0}$ and $(\overline{v^0})^3$), we derive

$$
\begin{aligned}
\tilde{\theta}_h(x)(hD_x - 3d\varphi(x))(v^0)^3 &= 3(v^0)^2\tilde{\theta}_h(x)(hD_x - d\varphi(x))v^0 \qquad (4.62)\\
&= 3(v^0)^2 Op_h^w(\tilde{\theta}_h(x)(\xi - d\varphi(x)))v^0 - \frac{3}{2i}h\tilde{\theta}'_h(x)(v^0)^3\,,
\end{aligned}
$$

using the relation (4.60) in the last equality (however, the second term in the right hand side disappears when we inject (4.62) in (4.61)). Now we can re-express the first term in the right hand side from $h\mathcal{L}v^0$. In fact, up to further remainders, $Op_h^w(\tilde{\theta}_h(x)(\xi - d\varphi(x)))v^0$ can be reduced to $Op_h^w(\tilde{\theta}_h(x)\chi(h^\beta\xi)(\xi - d\varphi(x)))v^0$, and this term can be split with the help of a $\gamma \in C_0^\infty(\mathbb{R})$, $\gamma \equiv 1$ in zero, namely

(4.63)
$$
\begin{aligned}
Op_h^w\left(\tilde{\theta}_h(x)\chi(h^\beta\xi)(\xi - d\varphi(x))\right)v^0 &= Op_h^w\left(\tilde{\theta}_h(x)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(\xi - d\varphi(x))\right)v^0 \\
&\quad + Op_h^w\left(\tilde{\theta}_h(x)\chi(h^\beta\xi)(1-\gamma)(\frac{x+p'(\xi)}{\sqrt{h}})(\xi - d\varphi(x))\right)v^0\,.
\end{aligned}
$$

Both terms have an L^2 norm controlled from above by

$$
Ch^{1-\sigma}(\|\mathcal{L}v\|_{L^2} + \|v\|_{H_h^s})\,.
$$

In fact, on one hand, we can take up the observation made in (4.47), and rewrite the first term in the right hand side as

$$
Op_h^w\left(\tilde{\theta}_h(x)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\tilde{e}_+(x+p'(\xi))\right)v^0 \qquad (4.64)
$$

where $\tilde{e}_+$ is defined in (3.32). On the other hand, also the symbol associated to the second operator in the right hand side can be rewritten highlighting the factor $(x + p'(\xi))$, as follows

$$
\tilde{\theta}_h(x)\chi(h^\beta\xi)\left(\frac{\xi - d\varphi(x)}{x+p'(\xi)}\right)(1-\gamma)(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\,,
$$

with $\tilde{\theta}_h(x)\chi(h^\beta\xi)\left(\frac{\xi - d\varphi(x)}{x+p'(\xi)}\right)(1-\gamma)(\frac{x+p'(\xi)}{\sqrt{h}}) \in h^{-\sigma}S_{\frac{1}{2},\beta}(1)$. Then, to both operators we can apply lemma 4.3, for $c(x,\xi)$ respectively equal to $\tilde{\theta}_h(x)\chi(h^\beta\xi)\tilde{e}_+$ and $\tilde{\theta}_h(x)\chi(h^\beta\xi)\left(\frac{\xi - d\varphi(x)}{x+p'(\xi)}\right)$, $c'(x,\xi) = \Sigma(\xi)\chi(h^\beta\xi)$, obtaining that they satisfy (4.24). Summing all up, together with (4.58), (4.61), (4.62), the fact that $A'(x,\xi)\tilde{\tilde{\theta}}_h(x) \in h^{-\sigma}S_{\delta,0}(1)$, and propositions 3.11, we obtain the result of the lemma. ☐

From now on, we will denote by $\Phi^\Sigma_3(x), \Phi^\Sigma_1(x), \Phi^\Sigma_{-1}(x), \Phi^\Sigma_{-3}(x)$ respectively the coefficients of $(v^\Sigma)^3, |v^\Sigma|^2 v^\Sigma, |v^\Sigma|^2\overline{v^\Sigma}, (\overline{v^\Sigma})^3$. Since they come from $m^\Sigma_I(d\varphi_I(x))\theta_h(x), \widetilde{m}^\Sigma_I(d\varphi_I(x))\theta_h(x)$ which are $O(h^{-\sigma})$, for a small $\sigma>0$, they are also $O(h^{-\sigma})$.

Lemma 4.9. *With same notations as before,*

$$Op^w_h(\Gamma)(\Phi^\Sigma_1(x)\theta_h(x)|v^\Sigma|^2v^\Sigma) = \Phi^\Sigma_1(x)\theta_h(x)|v^\Sigma|^2v^\Sigma + R(v)\,, \tag{4.65}$$

where $R(v)$ satisfies estimates (4.15), *(*4.16*).*

Proof. Let us write $Op^w_h(\Gamma) = 1 - Op^w_h(1-\Gamma)$. We want to show that the action of $Op^w_h(1-\Gamma)$ on $\Phi^\Sigma_1(x)\theta_h(x)|v^\Sigma|^2v^\Sigma$ gives us a remainder $R(v)$. First, we can reduce the symbol $1-\Gamma$ to $(1-\Gamma)\chi(h^\beta\xi)$, with χ cut-off function, $\beta>0$, up to remainders from lemma 3.13. Moreover, we can consider a smooth function $\tilde{\tilde{\theta}}_h(x)\in C^\infty_0(]-1,1[)$ such that $\tilde{\tilde{\theta}}_h\theta_h\equiv\theta_h$, and use symbolic calculus to enter $\tilde{\tilde{\theta}}_h(x)$ in $Op^w_h((1-\Gamma)\chi(h^\beta\xi))$ (again up to a remainder $R(v)$). Then, we can write

$$(1-\Gamma)\chi(h^\beta\xi)\tilde{\tilde{\theta}}_h(x) = \frac{1}{\sqrt{h}}b(x,\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})\tilde{\theta}_h(x)(\xi-d\varphi(x))\,, \tag{4.66}$$

where $b(x,\xi)=\chi(h^\beta\xi)\tilde{\tilde{\theta}}_h(x)(\frac{x+p'(\xi)}{\xi-d\varphi(x)})\in h^{-\sigma}S_{\delta,\beta}(1)$, $\Gamma_{-1}(\xi)=\frac{(1-\gamma)(\xi)}{\xi}$, $\sigma,\delta>0$ small depending on β, and $\tilde{\theta}_h(x)$ a new smooth function in $C^\infty_0(]-1,1[)$, identically equal to 1 on the support of $\tilde{\tilde{\theta}}_h(x)$. Applying symbolic calculus of lemma 3.10, we derive

$$\begin{aligned}\frac{1}{\sqrt{h}}b(x,\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})\sharp\tilde{\theta}_h(x)(\xi-d\varphi(x)) &= \frac{1}{\sqrt{h}}b(x,\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})\tilde{\theta}_h(x)(\xi-d\varphi(x))\\ &+\frac{\sqrt{h}}{2i}\left\{b(x,\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}}),\tilde{\theta}_h(x)(\xi-d\varphi(x))\right\}\\ &+r_1\,,\end{aligned} \tag{4.67}$$

with $r_1\in h^{\frac12-\sigma}S_{\frac12,\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, for a new small $\sigma>0$. An explicit calculation, and the observation that $\tilde{\theta}'_h\equiv 0$ on the support of $\tilde{\tilde{\theta}}_h$, show that the Poisson bracket is equal to

$$\begin{aligned}&\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})\left[\tilde{\theta}_h(x)(-\partial_\xi b(x,\xi)d^2\varphi(x)-\partial_x b(x,\xi))\right]+\\ &+\Gamma'_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\chi(h^\beta\xi)\tilde{\theta}_h(x)\left[\frac{-d^2\varphi(x)p''(\xi)-1}{\xi-d\varphi(x)}\right]\,,\end{aligned} \tag{4.68}$$

and since $x+p'(d\varphi)=0$, we have $-d^2\varphi(x)=\frac{1}{p''(d\varphi)}$ and

$$\chi(h^\beta\xi)\tilde{\theta}_h(x)\left[\frac{-d^2\varphi(x)p''(\xi)-1}{\xi-d\varphi(x)}\right] = \frac{\chi(h^\beta\xi)\tilde{\theta}_h(x)}{p''(d\varphi(x))}\int_0^1 p'''(t\xi+(1-t)d\varphi(x))dt\in h^{-\sigma}S_{\delta,\beta}(1)\,. \tag{4.69}$$

Therefore, from $\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})$, $\Gamma'_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\in S_{\frac12,0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, other appearing symbols in (4.68) belonging to $h^{-\sigma}S_{\delta,\beta}(1)$, we can rewrite the second and third term in the right hand side of (4.67) as $h^{\frac12-\sigma}r$, with $r\in S_{\frac12,\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, so their action on $\Phi^\Sigma_1(x)\theta_h(x)|v^\Sigma|^2v^\Sigma$ gives us a remainder $R(v)$ by propositions 3.11, 3.12. In this way, we are reduce to estimate

$$\frac{1}{\sqrt{h}}Op^w_h\left(b(x,\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})\right)Op^w_h\big(\tilde{\theta}_h(x)(\xi-d\varphi(x))\big)(\Phi^\Sigma_1(x)\theta_h(x)|v^\Sigma|^2v^\Sigma)\,. \tag{4.70}$$

Taking up (4.59), (4.60), (4.61) for $I=(1,1,-1)$, we obtain that $Op_h^w\big(\tilde{\theta}_h(x)(\xi-d\varphi(x))\big)$ acts like a derivation on its argument and

$$\|Op_h^w\big(\tilde{\theta}_h(x)(\xi-d\varphi(x))\big)\Phi_1^\Sigma(x)\theta_h(x)|v^\Sigma|^2v^\Sigma\|_{L^2}\le Ch^{1-\sigma}(\|\mathcal{L}v\|_{L^2}+\|v\|_{H^s_h})\,, \tag{4.71}$$

for a new small $\sigma>0$ still depending on β, so the fact that $b(x,\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})$ belongs to $S_{\frac{1}{2},\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, along with propositions 3.11, 3.12, imply that the term in (4.70) is a remainder $R(v)$ satisfying (4.15), (4.16). This concludes the proof. □

Proposition 4.6 allows us to arrive at the following equation

$$\begin{aligned}(D_t-Op_h^w(x\xi+p(\xi))v_\Lambda^\Sigma = {}& h\Phi_1^\Sigma(x)\theta_h(x)|v^\Sigma|^2v^\Sigma+hOp_h^w(\Gamma)\left[\Phi_3^\Sigma(x)\theta_h(x)(v^\Sigma)^3\right.\\ &\left.+\Phi_{-1}^\Sigma(x)\theta_h(x)|v^\Sigma|^2\overline{v^\Sigma}+\Phi_{-3}^\Sigma(x)\theta_h(x)(\overline{v^\Sigma})^3\right]+hR(v)\,,\end{aligned} \tag{4.72}$$

which is not entirely in v_Λ^Σ, so to transform to the right equation we use the following lemma, whose proof comes directly from proposition 4.4, and this is the reason why we omit the details.

Lemma 4.10. *Under the same hypothesis as in theorem 4.1, there exists $s>0$ sufficiently large, and a constant $C>0$ independent of h, such that*

$$\|v_I^\Sigma-(v_\Lambda^\Sigma)_I\|_{L^2}\le Ch^{\frac{1}{2}-\sigma}\left(\|\mathcal{L}v\|_{L^2}+\|v\|_{H^s_h}\right)\,, \tag{4.73}$$

$$\|v_I^\Sigma-(v_\Lambda^\Sigma)_I\|_{L^\infty}\le Ch^{\frac{1}{4}-\sigma}\left(\|\mathcal{L}v\|_{L^2}+\|v\|_{H^s_h}\right)\,, \tag{4.74}$$

for a small $\sigma>0$ depending on β.

Therefore v_Λ^Σ is solution of the following equation :

$$\begin{aligned}(D_t-Op_h^w(x\xi+p(\xi))v_\Lambda^\Sigma = {}& h\Phi_1^\Sigma(x)\theta_h(x)|v_\Lambda^\Sigma|^2v_\Lambda^\Sigma+hOp_h^w(\Gamma)\left[\Phi_3^\Sigma(x)\theta_h(x)(v_\Lambda^\Sigma)^3\right.\\ &\left.+\Phi_{-1}^\Sigma(x)\theta_h(x)|v_\Lambda^\Sigma|^2\overline{v_\Lambda^\Sigma}+\Phi_{-3}^\Sigma(x)\theta_h(x)(\overline{v_\Lambda^\Sigma})^3\right]+hR(v)\,,\end{aligned} \tag{4.75}$$

$R(v)$ being a remainder satisfying estimates (4.15), (4.16).

To conclude this section, we develop $Op_h^w(x\xi+p(\xi))v_\Lambda^\Sigma$ at $\xi=d\varphi(x)$.

Proposition 4.11. *Under the same hypothesis as in theorem 4.1,*

$$Op_h^w(x\xi+p(\xi))v_\Lambda^\Sigma=\varphi(x)\theta_h(x)v_\Lambda^\Sigma+hR(v)\,, \tag{4.76}$$

where $R(v)$ satisfies the estimates in (4.15), (4.16). Then, v_Λ^Σ is solution of the following equation:

$$\begin{aligned}D_tv_\Lambda^\Sigma &= \varphi(x)\theta_h(x)v_\Lambda^\Sigma+h\Phi_1^\Sigma(x)\theta_h(x)|v_\Lambda^\Sigma|^2v_\Lambda^\Sigma\\ &+hOp_h^w(\Gamma)\left[\Phi_3^\Sigma(x)\theta_h(x)(v_\Lambda^\Sigma)^3+\Phi_{-1}^\Sigma(x)\theta_h(x)|v_\Lambda^\Sigma|^2\overline{v_\Lambda^\Sigma}+\Phi_{-3}^\Sigma(x)\theta_h(x)(\overline{v_\Lambda^\Sigma})^3\right]\\ &+hR(v)\,,\end{aligned} \tag{4.77}$$

Proof. Consider a cut-off function χ as in lemma 3.13, and $\beta>0$. One can split as follows

$$v_\Lambda^\Sigma=Op_h^w(\chi(h^\beta\xi)\Gamma(x,\xi))v^\Sigma+Op_h^w((1-\chi)(h^\beta\xi)\Gamma(x,\xi))v^\Sigma\,, \tag{4.78}$$

and easily show that $Op_h^w(x\xi+p(\xi))Op_h^w((1-\chi)(h^\beta\xi)\Gamma(x,\xi))v^\Sigma$ is a remainder of the form $hR(v)$, $R(v)$ satisfying estimates (4.15), (4.16), just using symbolic calculus and lemma 3.13.

Therefore, we have to deal with $Op_h^w(x\xi+p(\xi))Op_h^w(\chi(h^\beta\xi)\Gamma(x,\xi))v^\Sigma$. We have already observed that for (x,ξ) in the support of $\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})$, x is bounded on a compact set $[-1+ch^{2\beta},1-ch^{2\beta}]$, which allows us to consider a smooth function $\theta_h(x)\in C_0^\infty(]-1,1[)$, identically equal to one on this interval, and then on the support of $\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})$, so that:

$$\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})=\theta_h(x)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\,. \tag{4.79}$$

Moreover, the derivatives of θ_h are zero on the support of $\partial^\alpha(\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}))$, for every multi-index α. This implies that products of $\theta'_h(x)$ with $\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})$ and all its derivatives are always zero so, by lemma 3.10,

$$\theta_h(x)\sharp\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})=\theta_h(x)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})+r_\infty\,, \tag{4.80}$$

where $r_\infty\in h^NS_{\frac{1}{2},\beta}(\langle x\rangle^{-\infty})$, for N as large as we want. In this way we can factor out $\theta_h(x)$, write the equality

$$\begin{aligned}&Op_h^w(x\xi+p(\xi))Op_h^w\left(\theta_h(x)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\right)v^\Sigma=\\&=Op_h^w(x\xi+p(\xi))\theta_h(x)Op_h^w\left(\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\right)v^\Sigma+hR(v)\,,\end{aligned} \tag{4.81}$$

and return to v^Σ_Λ by

$$Op_h^w\Big(\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\Big)v^\Sigma=v^\Sigma_\Lambda-Op_h^w\Big((1-\chi(h^\beta\xi))\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\Big)v^\Sigma\,. \tag{4.82}$$

Then,

$$\begin{aligned}&Op_h^w(x\xi+p(\xi))\theta_h(x)Op_h^w\Big(\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\Big)v^\Sigma=\\&=Op_h^w(x\xi+p(\xi))\theta_h(x)v^\Sigma_\Lambda-Op_h^w(x\xi+p(\xi))\theta_h(x)Op_h^w\Big((1-\chi(h^\beta\xi))\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\Big)v^\Sigma\,,\end{aligned} \tag{4.83}$$

and one can show that the second term in the right hand side is a remainder $hR(v)$ essentially using symbolic calculus, lemma 3.13, and Sobolev injection. Symbolic calculus enables us also to put $\theta_h(x)$ in $Op_h^w(x\xi+p(\xi))$, as the following deduction shows,

$$\begin{aligned}Op_h^w(x\xi+p(\xi))\theta_h(x)v^\Sigma_\Lambda&=Op_h^w\big((x\xi+p(\xi))\theta_h(x)\big)v^\Sigma_\Lambda+\frac{h}{2i}Op_h^w\big(\theta'_h(x)(x+p'(\xi))\big)v^\Sigma_\Lambda+hR(v)\\&=Op_h^w\big((x\xi+p(\xi))\theta_h(x)\big)v^\Sigma_\Lambda+hR(v)\,,\end{aligned} \tag{4.84}$$

with $R(v)$ satisfying (4.15), (4.16), using proposition 3.11 and Sobolev injection. In the last equality, $\frac{h}{2i}Op_h^w\big(\theta'_h(x)(x+p'(\xi))\big)v^\Sigma_\Lambda$ enters in the remainder, for $\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\in S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-2})$ by lemma 3.15, $\theta'_h(x)(x+p'(\xi))\in h^{-\delta}S_{\delta,0}(1)$ for a small $\delta>0$, and using symbolic calculus. Actually, we first write

$$\frac{h}{2i}Op_h^w\big(\theta'_h(x)(x+p'(\xi))\big)v^\Sigma_\Lambda=\frac{h^{\frac{3}{2}}}{2i}Op_h^w\Big(\theta'_h(x)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\Big)v^\Sigma+h^{\frac{3}{2}}Op_h^w(r_0)v^\Sigma\,, \tag{4.85}$$

where $r_0 \in h^{-2\delta} S_{\frac{1}{2},0}(\langle \frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, and then we use lemma 4.2 with $\tilde{\Gamma}(\xi) = \gamma(\xi)\xi$, and lemma 4.3 to deduce that it is a remainder $hR(v)$.

We can now analyse $Op^w_h((x\xi + p(\xi))\theta_h(x))v^\Sigma_\Lambda$. As we are away from points $x = \pm 1$, we can develop the symbol $x\xi + p(\xi)$ at $\xi = d\varphi(x)$, and since $\partial_\xi(x\xi + p(\xi))|_{\xi = d\varphi(x)} = 0$ we have

$$\begin{aligned} x\xi + p(\xi) &= xd\varphi(x) + p(d\varphi(x)) + \int_0^1 p''(t\xi + (1-t)d\varphi(x))(1-t)dt\,(\xi - d\varphi(x))^2 \\ &= xd\varphi(x) + p(d\varphi(x)) + b(x,\xi)(x + p'(\xi))^2\,, \end{aligned} \tag{4.86}$$

where

$$b(x,\xi) = \int_0^1 p''(t\xi + (1-t)d\varphi(x))(1-t)dt \left(\frac{\xi - d\varphi(x)}{x + p'(\xi)}\right)^2 .$$

Observe that $xd\varphi(x) + p(d\varphi(x)) = \varphi(x)$. To conclude the proof, we need to show that

$$Op^w_h\left(b(x,\xi)\theta_h(x)(x + p'(\xi))^2\right) v^\Sigma_\Lambda$$

gives rise to a remainder, too. First, we may consider a function χ as in lemma 3.13, $\beta > 0$, and split $b(x,\xi)\theta_h(x)(x+p'(\xi))^2$ as the sum of $b(x,\xi)\theta_h(x)(x+p'(\xi))^2(1-\chi(h^\beta\xi)) \in h^{-\sigma}S_{\delta,0}(\langle\xi\rangle^2)$, for small $\delta, \sigma > 0$, whose quantization acts on v^Σ_Λ as a remainder because of lemma 3.13, and $b(x,\xi)\theta_h(x)(x+p'(\xi))^2\chi(h^\beta\xi)$. For $b(x,\xi)\theta_h(x)\chi(h^\beta\xi)(x+p'(\xi))^2$, we can perform a further splitting through a function $\tilde{\gamma} \in C^\infty_0(\mathbb{R})$, such that $\tilde{\gamma}\left(\langle\xi\rangle^2(x+p'(\xi))\right) \equiv 1$ on the support of $\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})$, i.e.

$$\begin{aligned} &b(x,\xi)\theta_h(x)\chi(h^\beta\xi)(x+p'(\xi))^2\tilde{\gamma}\left(\langle\xi\rangle^2(x+p'(\xi))\right) \\ &+ b(x,\xi)\theta_h(x)\chi(h^\beta\xi)(x+p'(\xi))^2(1-\tilde{\gamma})\left(\langle\xi\rangle^2(x+p'(\xi))\right) . \end{aligned} \tag{4.87}$$

As discussed before, this implies that $(1-\tilde{\gamma})(\langle\xi\rangle^2(x+p'(\xi)))$ and all its derivatives are equal to zero on that support. Since $b(x,\xi)\theta_h(x)\chi(h^\beta\xi)(x+p'(\xi))^2(1-\tilde{\gamma})\left(\langle\xi\rangle^2(x+p'(\xi))\right) \in h^{-\sigma}S_{\delta,\beta}(1)$ for $\sigma, \delta > 0$ small depending on β, one can apply symbolic calculus (up to a large enough order) to obtain

$$b(x,\xi)\theta_h(x)\chi(h^\beta\xi)(x+p'(\xi))^2(1-\tilde{\gamma})\left(\langle\xi\rangle^2(x+p'(\xi))\right)\sharp\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) = r'_\infty\,, \tag{4.88}$$

with $r'_\infty = h^N S_{\frac{1}{2},\beta}(1)$, N sufficiently large, to have

$$Op^w_h\Big(b(x,\xi)\theta_h(x)\chi(h^\beta\xi)(x+p'(\xi))^2(1-\tilde{\gamma})\left(\langle\xi\rangle^2(x+p'(\xi))\right)\Big) v^\Sigma_\Lambda = hR(v)\,.$$

On the other hand, $B(x,\xi) := b(x,\xi)\theta_h(x)\chi(h^\beta\xi)\tilde{\gamma}(\langle\xi\rangle^2(x+p'(\xi)))$ belongs to $h^{-\sigma}S_{\delta,\beta}(1)$, for $\delta \ge 2\beta$, by lemma 3.14. Using twice lemma 3.10, together with the fact that $\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) \in S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-3})$ and $B(x,\xi)(x+p'(\xi))^2 \in h^{1-\sigma}S_{\delta,\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^2)$, we derive

$$\left(B(x,\xi)(x+p'(\xi))^2\right)\sharp\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) = B(x,\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))^2 + hr_0\,, \tag{4.89}$$

and

$$\Big(B(x,\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\Big)\sharp(x+p'(\xi)) = B(x,\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))^2 + hr'_0\,, \tag{4.90}$$

where $r_0, r_0' \in h^{\frac{1}{2}-\sigma}S_{\frac{1}{2},\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$. Therefore

(4.91)
$$\big(B(x,\xi)(x+p'(\xi))^2\big)\,\sharp\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) = \Big(B(x,\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\Big)\,\sharp(x+p'(\xi))+h(r_0-r_0'),$$

and

(4.92)
$$Op_h^w(B(x,\xi)(x+p'(\xi))^2)v_\Lambda^\Sigma = hOp_h^w\Big(B(x,\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\Big)\mathcal{L}v^\Sigma + hOp_h^w(r_0-r_0')v^\Sigma,$$

so one can show that the right hand side is a remainder $hR(v)$, commutating $\mathcal{L}$ with $Op_h^w(\Sigma(\xi))$, using that $B(x,\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi)), r_0-r_0' \in h^{\frac{1}{2}-\sigma}S_{\frac{1}{2},\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, and propositions 3.11, 3.12. We finally obtain

$$Op_h^w(x\xi+p(\xi))v_\Lambda^\Sigma = \varphi(x)\theta_h(x)v_\Lambda^\Sigma + hR(v)\,, \tag{4.93}$$

and according to (4.75), v_Λ^Σ is solution of

$$\begin{aligned} D_t v_\Lambda^\Sigma &= \varphi(x)\theta_h(x)v_\Lambda^\Sigma + h\Phi_1^\Sigma(x)\theta_h(x)|v_\Lambda^\Sigma|^2 v_\Lambda^\Sigma \\ &+ hOp_h^w(\Gamma)\Big[\Phi_3^\Sigma(x)\theta_h(x)(v_\Lambda^\Sigma)^3 + \Phi_{-1}^\Sigma(x)\theta_h(x)|v_\Lambda^\Sigma|^2\overline{v_\Lambda^\Sigma} + \Phi_{-3}^\Sigma(x)\theta_h(x)(\overline{v_\Lambda^\Sigma})^3\Big] \\ &+ hR(v)\,, \end{aligned} \tag{4.94}$$

where $R(v)$ is a remainder satisfying estimates (4.15), (4.16). □

5 Study of the ODE and End of the Proof

5.1 The uniform estimate

The goal of this subsection is to the derive from the equation (4.77) an ODE for a new function f_Λ^Σ obtained from v_Λ^Σ, from which we can deduce uniform bounds for v_Λ^Σ, and for the starting function v, with a certain number $\rho \in \mathbb{N}$ of its derivatives. The idea is to get rid of contributions of *non characteristic* terms (i.e. of cubic terms different from $|v_\Lambda^\Sigma|^2 v_\Lambda^\Sigma$) by a reasoning of normal forms. This will allow us to eliminate all terms still containing pseudo-differential operators, to finally write an ODE, and to prove the required L^∞ estimate, if the *null condition* is satisfied.

In the previous section, we denoted by $\Phi_3^\Sigma(x)$, $\Phi_1^\Sigma(x)$, $\Phi_{-1}^\Sigma(x)$ and $\Phi_{-3}^\Sigma(x)$ (modulo some new smooth terms) respectively the coefficients of $(v_\Lambda^\Sigma)^3$, $|v_\Lambda^\Sigma|^2 v_\Lambda^\Sigma$, $|v_\Lambda^\Sigma|^2\overline{v_\Lambda^\Sigma}$, $(\overline{v_\Lambda^\Sigma})^3$ in the right hand side of (4.77). One can calculate them explicitly, using both the expression of the nonlinearity obtained in proposition 4.6 and its polynomial representation as in equation (4.7). In the latter, after the development at $\xi = d\varphi(x)$, we essentially replaced hD by $d\varphi(x)$ when it is applied to v_Λ^Σ, and by $-d\varphi(x)$ when it is applied to $\overline{v_\Lambda^\Sigma}$, modulus some new smooth coefficients $a_I(x) := A(\sum_{l=1}^{3} i_l d\varphi(x))\Sigma(d\varphi(x))^{-3}$, for every $I = (i_1, i_2, i_3)$ (the factor $\Sigma(d\varphi(x))^{-3}$ coming out from $m_I^\Sigma(d\varphi_I(x)) = m_I(d\varphi(x))\Sigma(d\varphi(x))^{-3}$, according to the notation introduced in (4.11), $A(\xi) = \Sigma(\xi)\chi(h^\beta\xi)$).

We are interested in particular in $\Phi_1^\Sigma(x)$ or, to be more precise, to its real part. In fact, the *null condition* introduced in definition 1.1 at the very beginning is the same as requiring for the coefficient of $|v^\Sigma_\Lambda|^2 v^\Sigma_\Lambda$ to be real, i.e. its imaginary part must be equal to zero. Since polynomials P'_k, P''_k are real as well as $d\varphi(x), \langle d\varphi(x)\rangle$, the only contribution to the imaginary part comes from P'_k, P''_k for $k=1,3$ (which have a factor i^k) and produces a multiple of the function $\Phi(x)$ defined in (1.5). Therefore, if we suppose that the nonlinearity satisfies this *null condition* (as demanded in theorem 1.2) then we find for $\Phi^\Sigma_1(x)$ that

$$\begin{aligned}\Phi^\Sigma_1(x) = \frac{1}{8} a_{(1,1,-1)}(x)\langle d\varphi\rangle^{-3}\big[&3P_0(1, d\varphi\langle d\varphi\rangle, (d\varphi)^2;\, \langle d\varphi\rangle, d\varphi)\\ &+P_2(1, d\varphi\langle d\varphi\rangle, (d\varphi)^2;\, \langle d\varphi\rangle, d\varphi)\big]\,.\end{aligned} \tag{5.1}$$

Proposition 5.1. *Suppose we are given two constants $A'', B''>0$, some $T>1$ and a $\sigma>0$ small. Let v^Σ_Λ be a solution of the equation* (4.77) *on the interval $[1,T]$, v^Σ_Λ satisfying the a priori estimates*

$$\|v^\Sigma_\Lambda(t,\cdot)\|_{L^\infty(\mathbb{R})} \le A''\varepsilon\,, \tag{5.2}$$

$$\|v^\Sigma_\Lambda(t,\cdot)\|_{L^2(\mathbb{R})} \le B''\varepsilon h^{-\sigma}\,, \tag{5.3}$$

for all $t\in[1,T]$. Let $\tilde{\theta}_h(x)\in C^\infty_0(]-1,1[)$, such that $\tilde{\theta}_h\theta_h\equiv\theta_h$, and define

$$f^\Sigma_\Lambda := v^\Sigma_\Lambda + Op^w_h(\Gamma)\left[-\frac{h}{2}\frac{\tilde{\theta}_h(x)}{\varphi(x)}\Phi^\Sigma_3(x)(v^\Sigma_\Lambda)^3 + \frac{h}{2}\frac{\tilde{\theta}_h(x)}{\varphi(x)}\Phi^\Sigma_{-1}(x)|v^\Sigma_\Lambda|^2\overline{v^\Sigma_\Lambda} + \frac{h}{4}\frac{\tilde{\theta}_h(x)}{\varphi(x)}\Phi^\Sigma_{-3}(x)(\overline{v^\Sigma_\Lambda})^3\right]. \tag{5.4}$$

Then f^Σ_Λ is well defined and it is solution of the ODE:

$$D_t f^\Sigma_\Lambda = \varphi(x)\theta_h(x) f^\Sigma_\Lambda + h\theta_h(x)\Phi^\Sigma_1(x)|f^\Sigma_\Lambda|^2 f^\Sigma_\Lambda + hR(v)\,, \tag{5.5}$$

where $R(v)$ is a remainder satisfying estimates (4.15), (4.16).

Proof. Firstly, we would like to underline that, if we suppose bounds in (4.12) and (4.13) on v, then hypothesis (5.2) and (5.3) follow immediately, because of the definition of v^Σ_Λ as $Op^w_h(\Gamma)v^\Sigma$. In fact, estimate (5.3) follows from proposition 3.11 and the *a priori* estimate (4.13), with $B''=B'$. Regarding the estimate (5.2), we can write

$$v^\Sigma_\Lambda = v^\Sigma - v^\Sigma_{\Lambda^c}\,, \tag{5.6}$$

and since $\|v^\Sigma(t,\cdot)\|_{L^\infty} = \|v(t,\cdot)\|_{W^{\rho,\infty}_h}$,

$$\begin{aligned}\|v^\Sigma_\Lambda(t,\cdot)\|_{L^\infty} &\le \|v^\Sigma(t,\cdot)\|_{L^\infty} + \|v^\Sigma_{\Lambda^c}(t,\cdot)\|_{L^\infty}\\ &= \|v(t,\cdot)\|_{W^{\rho,\infty}_h} + \|v^\Sigma_{\Lambda^c}(t,\cdot)\|_{L^\infty}\,,\end{aligned} \tag{5.7}$$

where we estimated $\|v^\Sigma_{\Lambda^c}(t,\cdot)\|_{L^\infty}$ in proposition 4.4. Therefore, using that for $\sigma>0$ sufficiently small $h^{\frac14-\sigma}\le h^{\frac18}$, we have

$$\begin{aligned}\|v^\Sigma_\Lambda(t,\cdot)\|_{L^\infty} &\le \|v(t,\cdot)\|_{W^{\rho,\infty}_h} + Ch^{\frac18}(\|\mathcal{L}v(t,\cdot)\|_{L^2} + \|v(t,\cdot)\|_{H^s_h})\\ &\le A'\varepsilon + CB'\varepsilon\, h^{\frac18-\sigma}\\ &\le A''\varepsilon\,,\end{aligned} \tag{5.8}$$

if we choose $A''>0$ sufficiently large to have $A', CB'\le\frac{A''}{2}$.

Secondly, $\varphi(x)\neq 0$ for all x in the support of $\tilde{\theta}_h$. In fact, we consider $\tilde{\theta}_h$ such that $\tilde{\theta}_h\theta_h\equiv\theta_h$, so we can suppose that its support is of the form $[-1+C'h^{2\beta},1-C'h^{2\beta}]$, for a suitable small positive constant C'. On this interval $x^2\le(1-C'h^{2\beta})^2=1+C'^2h^{4\beta}-2C'h^{2\beta}$, so

$$\varphi(x)=\sqrt{1-x^2}\ge\sqrt{C'h^{2\beta}(2-C'h^{2\beta})}\gtrsim h^\beta\,, \tag{5.9}$$

which implies that the quotient $\frac{\tilde{\theta}_h(x)}{\varphi(x)}$ is well defined and $|\frac{\tilde{\theta}_h(x)}{\varphi(x)}|\le h^{-\beta}$. Then, set

$$f^\Sigma_\Lambda:=v^\Sigma_\Lambda+Op^w_h(\Gamma)\left[h\frac{\tilde{\theta}_h(x)}{\varphi(x)}\left(k_1\Phi^\Sigma_3(x)(v^\Sigma_\Lambda)^3+k_2\Phi^\Sigma_{-1}(x)|v^\Sigma_\Lambda|^2\overline{v^\Sigma_\Lambda}+k_3\Phi^\Sigma_{-3}(x)(\overline{v^\Sigma_\Lambda})^3\right)\right]\,, \tag{5.10}$$

with $k_1,k_2,k_3\in\mathbb{R}$ to be properly chosen, and apply D_t to this expression. We have already calculated $D_tOp^w_h(\Gamma)$ in (4.33), obtaining that the commutator is

$$[D_t,Op^w_h(\Gamma)]=ih^{\frac12}Op^w_h\left(\gamma'(\frac{x+p'(\xi)}{\sqrt{h}})p''(\xi)\xi\right)-\frac{ih}{2}Op^w_h\left(\gamma'(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\right)\,, \tag{5.11}$$

where both appearing symbols belong to $S_{\frac12,0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$. The truncation of these symbols through a function $\chi(h^\beta\xi)$ as in lemma 3.13, and propositions 3.11, 3.12, together with estimates (5.2), (5.3) on v^Σ_Λ, show that the action of the commutator on brackets in (5.10) gives rise to a remainder $hR(v)$.
Denoting by $O(5)$ all terms of order 5 in $(v^\Sigma_\Lambda,\overline{v^\Sigma_\Lambda})$, and using (4.77), we can compute

$$\begin{aligned}D_tf^\Sigma_\Lambda=D_tv^\Sigma_\Lambda+Op^w_h(\Gamma)\Bigg[&k_1h\frac{\tilde{\theta}_h(x)}{\varphi(x)}\Phi^\Sigma_3(x)[3\varphi(x)\theta_h(x)(v^\Sigma_\Lambda)^3+h^2O(5)]\\&+k_2h\frac{\tilde{\theta}_h(x)}{\varphi(x)}\Phi^\Sigma_{-1}(x)[-\varphi(x)\theta_h(x)|v^\Sigma_\Lambda|^2\overline{v^\Sigma_\Lambda}+h^2O(5)]\\&+k_3h\frac{\tilde{\theta}_h(x)}{\varphi(x)}\Phi^\Sigma_{-3}(x)[-3\varphi(x)\theta_h(x)(\overline{v^\Sigma_\Lambda})^3+h^2O(5)]\Bigg]+hR(v)\,,\end{aligned} \tag{5.12}$$

where $hR(v)$ includes also terms coming out from $D_t(h\tilde{\theta}_h(x))$, and

$$\begin{aligned}D_tf^\Sigma_\Lambda&=\varphi(x)\theta_h(x)v^\Sigma_\Lambda+h\theta_h(x)\Phi^\Sigma_1(x)|v^\Sigma_\Lambda|^2v^\Sigma_\Lambda\\&+Op^w_h(\Gamma)\Big[h\theta_h(x)\Big((3k_1+1)\Phi^\Sigma_3(x)(v^\Sigma_\Lambda)^3+(-k_2+1)\Phi^\Sigma_{-1}(x)|v^\Sigma_\Lambda|^2\overline{v^\Sigma_\Lambda}\\&\qquad+(-3k_3+1)\Phi^\Sigma_{-3}(x)(\overline{v^\Sigma_\Lambda})^3\Big)\Big]+hR(v)\,,\end{aligned} \tag{5.13}$$

where $h^2O(5)$ entered in $hR(v)$ from propositions 3.11, 3.12, estimates (5.2), (5.3), and the fact that involved coefficients are $O(h^{-\sigma})$, for a small $\sigma>0$. We use again the definition of f^Σ_Λ to replace v^Σ_Λ in the linear and in the *characteristic* part. We have $h\theta_h(x)\Phi^\Sigma_1(x)|v^\Sigma_\Lambda|^2v^\Sigma_\Lambda=h\theta_h(x)\Phi^\Sigma_1(x)|f^\Sigma_\Lambda|^2f^\Sigma_\Lambda+h^2O(5)$ and

(5.14)

$$\begin{aligned}\varphi(x)\theta_h(x)v^\Sigma_\Lambda&=\varphi(x)\theta_h(x)f^\Sigma_\Lambda-\varphi(x)\theta_h(x)Op^w_h(\Gamma)\left[h\frac{\tilde{\theta}_h(x)}{\varphi(x)}\left(k_1\Phi^\Sigma_3(x)(v^\Sigma_\Lambda)^3+k_2\Phi^\Sigma_{-1}(x)|v^\Sigma_\Lambda|^2\overline{v^\Sigma_\Lambda}\right.\right.\\&\qquad\left.\left.+k_3\Phi^\Sigma_{-3}(x)(\overline{v^\Sigma_\Lambda})^3\right)\right]\\&=\varphi(x)\theta_h(x)f^\Sigma_\Lambda-Op^w_h(\Gamma)\left[h\theta_h(x)\left(k_1\Phi^\Sigma_3(x)(v^\Sigma_\Lambda)^3+k_2\Phi^\Sigma_{-1}(x)|v^\Sigma_\Lambda|^2\overline{v^\Sigma_\Lambda}\right.\right.\\&\qquad\left.\left.+k_3\Phi^\Sigma_{-3}(x)(\overline{v^\Sigma_\Lambda})^3\right)\right]+hR(v)\,,\end{aligned}$$

where the last equality is consequence of the fact that, by lemma 3.10, $[\varphi(x)\theta_h(x), Op_h^w(\Gamma)] = h^{\frac{1}{2}-\sigma}Op_h^w(r_0)$, $r_0 \in S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, $\sigma>0$ small. Again a truncation through $\chi(h^\beta\xi)$, and the application of propositions 3.11, 3.12, together with estimates on v^Σ_Λ, ensure that the contribution coming from the action of the commutator on its argument enters in the remainder. We finally obtain

$$\begin{aligned} D_t f^\Sigma_\Lambda &= \varphi(x)\theta_h(x)f^\Sigma_\Lambda + h\theta_h(x)\Phi^\Sigma_1(x)|f^\Sigma_\Lambda|^2 f^\Sigma_\Lambda \\ &+ Op^w_h(\Gamma)\Big[h\theta_h(x)\Big((2k_1+1)\Phi^\Sigma_3(x)(v^\Sigma_\Lambda)^3 + (-2k_2+1)\Phi^\Sigma_{-1}(x)|v^\Sigma_\Lambda|^2\overline{v^\Sigma_\Lambda} \\ &+(-4k_3+1)\Phi^\Sigma_{-3}(x)(\overline{v^\Sigma_\Lambda})^3\Big)\Big] + hR(v)\,, \end{aligned} \tag{5.15}$$

and we get rid of *non-characteristic* terms by requiring

$$\begin{cases} 2k_1+1 &= 0 \\ -2k_2+1 &= 0 \\ -4k_3+1 &= 0 \end{cases} \quad\Rightarrow\quad \begin{cases} k_1 = -\frac{1}{2} \\ k_2 = \frac{1}{2} \\ k_3 = \frac{1}{4}\,, \end{cases}$$

from which the statement. □

Proposition 5.2. *Let f^Σ_Λ be the function defined in* (5.4)*, solution of the ODE* (5.5) *under the a priori estimates* (5.2), (5.3)*. Then the following inequality holds :*

$$\|f^\Sigma_\Lambda(t,\cdot)\|_{L^\infty} \le \|f^\Sigma_\Lambda(1,\cdot)\|_{L^\infty} + C\int_1^t \tau^{-\frac{5}{4}+\sigma}(\|\mathcal{L}v(\tau,\cdot)\|_{L^2} + \|v(\tau,\cdot)\|_{H^s_h})\,d\tau\,, \tag{5.16}$$

for $\sigma>0$ small, and a positive constant $C>0$.

Proof. Using the equation (5.5), we can compute

$$\begin{aligned} \frac{\partial}{\partial t}|f^\Sigma_\Lambda(t,x)|^2 &= 2\Im(f^\Sigma_\Lambda\overline{D_t f^\Sigma_\Lambda})(t,x) = 2\Im(\varphi(x)\theta_h(x)|f^\Sigma_\Lambda|^2 + h\theta_h(x)\Phi^\Sigma_1(x)|f^\Sigma_\Lambda|^4 + hR(v)f^\Sigma_\Lambda)(t,x) \\ &= 2\Im(hR(v)f^\Sigma_\Lambda)(t,x) \le 2h|f^\Sigma_\Lambda(t,x)||R(v)|\,, \end{aligned} \tag{5.17}$$

from which follows an integral inequality

$$\|f^\Sigma_\Lambda(t,\cdot)\|_{L^\infty} \le \|f^\Sigma_\Lambda(1,\cdot)\|_{L^\infty} + \int_1^t \frac{\|R(v)(\tau,\cdot)\|_{L^\infty}}{\tau}\,d\tau\,. \tag{5.18}$$

Using the estimate (4.16) for $R(v)$, we obtain the result

$$\|f^\Sigma_\Lambda(t,\cdot)\|_{L^\infty} \le \|f^\Sigma_\Lambda(1,\cdot)\|_{L^\infty} + C\int_1^t \tau^{-\frac{5}{4}+\sigma}(\|\mathcal{L}v(\tau,\cdot)\|_{L^2}) + \|v(\tau,\cdot)\|_{H^s_h})\,d\tau\,. \tag{5.19}$$

□

Finally, the L^∞ estimate we found for f^Σ_Λ in the previous proposition enables us to propagate the uniform estimate on v, as showed in the following:

Proposition 5.3 (Propagation of the uniform estimate)**.** *Let v be a solution of the equation* (4.7) *on some interval $[1,T]$, $T>1$ and $\sigma>0$ small. Then, for a fixed constant $K>1$, there* *exist two constants $A', B' > 0$ sufficiently large, $\varepsilon_0 > 0$ sufficiently small, $s, \rho \in \mathbb{N}$ with $s \gg \rho$, such that, if $0 < \varepsilon < \varepsilon_0$, and v satisfies*

$$\begin{aligned} &(A.1)\ \|v(t,\cdot)\|_{W^{\rho,\infty}_h} \le A'\varepsilon\,, \\ &(B.1)\ \|v(t,\cdot)\|_{H^s_h} \le B'\varepsilon\, h^{-\sigma}\,, \\ &(B.2)\ \|\mathcal{L}v(t,\cdot)\|_{L^2} \le B'\varepsilon\, h^{-\sigma}\,, \end{aligned} \tag{5.20}$$

for every $t \in [1, T]$, then it satisfies also

$$(A.1')\ \|v(t,\cdot)\|_{W^{\rho,\infty}_h} \le \frac{A'}{K}\varepsilon\,, \qquad \forall t \in [1, T]\,. \tag{5.21}$$

Proof. The proof of the proposition comes directly from proposition 5.2 and from the equivalence between $\|v^\Sigma_\Lambda\|_{L^\infty}$ and $\|f^\Sigma_\Lambda\|_{L^\infty}$. In fact, functions $\Phi^\Sigma_j(x)$ are cubic expressions in $d\varphi(x)$ and $\langle d\varphi(x)\rangle$, so they are bounded up to a loss $h^{-\delta}$, $\delta > 0$ depending on β, on the support of $\tilde{\theta}_h(x)$, where also $\varphi(x) \gtrsim h^\beta > 0$. This implies that $|\frac{\tilde{\theta}_h(x)}{\varphi(x)}\Phi^\Sigma_j(x)| \le Ch^{-\delta}$, $j \in \{3, -1, -3\}$, with a new $\delta > 0$ depending linearly on β, so that by the definition of f^Σ_Λ, proposition 3.12 and estimates (5.2), (5.3) (which follow from (5.20), as already observed in proposition 5.1), we find

$$\frac{1}{2}\|v^\Sigma_\Lambda(t,\cdot)\|_{L^\infty} \le \|f^\Sigma_\Lambda(t,\cdot)\|_{L^\infty} \le 2\|v^\Sigma_\Lambda(t,\cdot)\|_{L^\infty}\,. \tag{5.22}$$

Furthermore, the *a priori* estimate on the $W^{\rho,\infty}_h$ norm of v extends to the L^∞ norm of v^Σ_Λ just by the decomposition

$$v^\Sigma_\Lambda = v^\Sigma - v^\Sigma_{\Lambda^c}\,, \tag{5.23}$$

and by proposition 4.4, so for example at time $t = 1$ we have

$$\begin{aligned} \|v^\Sigma_\Lambda(1,\cdot)\|_{L^\infty} &\le \|v^\Sigma(1,\cdot)\|_{L^\infty} + \|v^\Sigma_{\Lambda^c}(1,\cdot)\|_{L^\infty} \\ &\le \|v(1,\cdot)\|_{W^{\rho,\infty}_h} + C(\|\mathcal{L}v(1,\cdot)\|_{L^2} + \|v(1,\cdot)\|_{H^s_h}) \\ &\le \frac{A'}{32K}\varepsilon + CB'\varepsilon \\ &\le \frac{A'}{16K}\varepsilon\,, \end{aligned} \tag{5.24}$$

where we choose $A' > 0$ sufficiently large such that $\|v(1,\cdot)\|_{W^{\rho,\infty}_h} \le \frac{A'}{32K}\varepsilon$ and $CB' < \frac{A'}{32K}$. Therefore

$$\|f^\Sigma_\Lambda(1,\cdot)\|_{L^\infty} \le 2\|v^\Sigma_\Lambda(1,\cdot)\|_{L^\infty} \le \frac{A'}{8K}\varepsilon\,. \tag{5.25}$$

Using proposition 5.2, (5.25) and the *a priori* estimates (B.1), (B.2), we find that

$$\begin{aligned} \|f^\Sigma_\Lambda(t,\cdot)\|_{L^\infty} &\le \frac{A'}{8K}\varepsilon + CB'\varepsilon\int_1^t \tau^{-\frac{5}{4}+\sigma}\, d\tau \\ &\le \frac{A'}{8K}\varepsilon + C'B'\varepsilon \\ &\le \frac{A'}{4K}\varepsilon\,, \end{aligned} \tag{5.26}$$

where again the last inequality follows from the choice of $A' > 0$ large enough to have $C'B' < \frac{A'}{8K}$. Then we have

$$\|v^\Sigma_\Lambda(t,\cdot)\|_{L^\infty} \le \frac{A'}{2K}\varepsilon\,, \tag{5.27}$$

and

$$
\begin{aligned}
\|v^\Sigma(t,\cdot)\|_{L^\infty} &\leq \|v^\Sigma_\Lambda(t,\cdot)\|_{L^\infty} + \|v^\Sigma_{\Lambda^c}(t,\cdot)\|_{L^\infty} \\
&\leq \frac{A'}{2K}\varepsilon + CB'\varepsilon h^{\frac{1}{4}-\sigma'} \\
&\leq \frac{A'}{K}\varepsilon\,.
\end{aligned} \tag{5.28}
$$

□

5.2 Asymptotics

We want now to derive the asymptotic expansion for the function $\langle hD\rangle^{-1}v$, v being the solution of (4.7), when it exists on $[1,+\infty[$. The reader can refer to the next subsection to find the proof of the global existence of v, which implies also the global existence of the solution u of the starting problem (1.1).

Proposition 5.4. *Under the same hypothesis as theorem 4.1, with $T=+\infty$, there exists a family $(\theta_h(x))_h$ of C^∞ functions, real valued, supported in some interval $[-1+ch^{2\beta}, 1-ch^{2\beta}]$, $\theta_h\equiv 1$ on an interval of the same form, such that $(h\partial_h)^k\theta_h(x)$ is bounded for any k, and a family $(a_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ of $\mathbb{C}$-valued functions on $\mathbb{R}$, supported in $[-1,1]$, uniformly bounded, such that*

$$
\langle hD\rangle^{-1}v = \varepsilon a_\varepsilon(x)\exp\left[i\varphi(x)\int_1^t \theta_{1/\tau}(x)d\tau + i\varepsilon^2|a_\varepsilon(x)|^2\Phi_1^\Sigma(x)\int_1^t\theta_{1/\tau}(x)\frac{d\tau}{\tau}\right] + t^{-\frac14+\sigma}r(t,x)\,, \tag{5.29}
$$

where $h=\frac1t$, $\sigma>0$ is small and $\sup_{t\geq1}\|r(t,\cdot)\|_{L^2\cap L^\infty}\leq C\varepsilon$.

Proof. Let us take $\Sigma(\xi)=\langle\xi\rangle^{-1}$, so that $v^\Sigma=\langle hD\rangle^{-1}v$. Summing all prevoius results, we have obtained that under the *a priori* estimates (4.12), (4.13), the function f^Σ_Λ defined in (5.4) satisfies (5.5), with a remainder $R(v)=O_{L^\infty\cap L^2}(\varepsilon t^{-\frac14+\sigma})$, for a sufficiently small $\sigma>0$. Inequality (5.17) and the bound (4.16) show that

$$
\|f^\Sigma_\Lambda(t,\cdot)-f^\Sigma_\Lambda(t',\cdot)\|_{L^\infty}\leq C\int_{t'}^t \tau^{-\frac54+\sigma}(\|\mathcal{L}v(\tau,\cdot)\|_{L^2}+\|v(\tau,\cdot)\|_{H^s_h})\,d\tau\,.
$$

Combining with the *a priori* estimate (4.13), there is a continuous function $x\to|\tilde a(x)|$ such that $\big||f^\Sigma_\Lambda(t,x)|^2-|\tilde a(x)|^2\big| = O(\varepsilon t^{-\frac12+\sigma})$, for a new small $\sigma>0$, and replacing this new function in (5.5) we obtain the equation

$$
D_t f^\Sigma_\Lambda = \theta_h(x)\left[\varphi(x)+h\Phi^\Sigma_1(x)|\tilde a(x)|^2\right]f^\Sigma_\Lambda + h\,r(t,x)\,, \tag{5.30}
$$

for $r=O_{L^\infty\cap L^2}(\varepsilon t^{-\frac14+\sigma})$, which is a linear non homogeneous ODE for f^Σ_Λ. This implies that there is a $O(\varepsilon)$ continuous function $\tilde a$ such that

$$
f^\Sigma_\Lambda(t,x)=\tilde a(x)\exp\left[i\varphi(x)\int_1^t\theta_{1/\tau}(x)d\tau + i|\tilde a(x)|^2\Phi^\Sigma_1(x)\int_1^t\theta_{1/\tau}(x)\frac{d\tau}{\tau}\right]+t^{-\frac14+\sigma}r(t,x)\,, \tag{5.31}
$$

for a new r. Finally, using the definition of f^Σ_Λ and proposition 4.4, we have $\|f^\Sigma_\Lambda - v^\Sigma_\Lambda\|_{L^2\cap L^\infty} = O(\varepsilon t^{-\frac34+\sigma})$ and $\|v^\Sigma_\Lambda - v^\Sigma\|_{L^2\cap L^\infty} = O(\varepsilon t^{-\frac14+\sigma})$, so we can deduce from (5.31) the asymptotic expansion for $v^\Sigma=\langle hD\rangle^{-1}v$. Since (4.39) for $a\equiv1$ shows that v^Σ vanishes to main order when $x\notin[-1,1]$ and $t\to+\infty$, we get that $\tilde a(x)$ is supported for $x\in[-1,1]$, and we conclude the proof choosing $\tilde a(x)=\varepsilon a_\varepsilon(x)$ for a bounded $a_\varepsilon(x)$ as in the statement. □

5.3 End of the Proof

Proof of Theorem 1.2. Let us prove that, for small enough data, the solution of the initial Cauchy problem (1.1) is global. We show that we can propagate some convenient *a priori* estimates on u, as stated in theorem 1.3, namely we want to show that there are some integers $s \gg \rho \gg 1$, some constants $A, B > 0$ large enough, $\varepsilon_0 \in]0,1]$ and $\sigma > 0$ small enough such that, if $u \in C^0([1,T[;H^{s+1}) \cap C^1([1,T[;H^s)$ is solution of (1.1) for some $T > 1$, and satisfies

$$\begin{aligned} &\|u(t,\cdot)\|_{W^{t,\rho,\infty}} \le A\varepsilon t^{-\frac{1}{2}}\,, \\ &\|Zu(t,\cdot)\|_{H^1} \le B\varepsilon t^{\sigma}\,, \qquad \|\partial_t Zu(t,\cdot)\|_{L^2} \le B\varepsilon t^{\sigma} \\ &\|u(t,\cdot)\|_{H^s} \le B\varepsilon t^{\sigma}\,, \qquad \|\partial_t u(t,\cdot)\|_{H^{s-1}} \le B\varepsilon t^{\sigma}\,, \end{aligned}$$

for every $t \in [1,T]$, then in the same interval it verifies improved estimates,

$$\begin{aligned} &\|u(t,\cdot)\|_{W^{t,\rho,\infty}} \le \frac{A}{2}\varepsilon t^{-\frac{1}{2}}\,, \\ &\|Zu(t,\cdot)\|_{H^1} \le \frac{B}{2}\varepsilon t^{\sigma}\,, \qquad \|\partial_t Zu(t,\cdot)\|_{L^2} \le \frac{B}{2}\varepsilon t^{\sigma} \\ &\|u(t,\cdot)\|_{H^s} \le \frac{B}{2}\varepsilon t^{\sigma}\,, \qquad \|\partial_t u(t,\cdot)\|_{H^{s-1}} \le \frac{B}{2}\varepsilon t^{\sigma}\,. \end{aligned}$$

We can immediately observe that from (1.6), these bounds are verified at time $t = 1$. In theorem 2.3 in section 2, we proved that we can improve the energy bounds $\|Zu(t,\cdot)\|_{H^1}$, $\|\partial_t Zu(t,\cdot)\|_{L^2}$, $\|u(t,\cdot)\|_{H^s}$ and $\|\partial_t u(t,\cdot)\|_{H^{s-1}}$. To show that the propagation of the uniform bound $\|u(t,\cdot)\|_{W^{t,\rho,\infty}}$ holds, we passed from equation (1.1) to (4.2) at the beginning of section 4, and then we showed that the function v is solution of (4.7). The *a priori* assumptions made on u imply the following estimates on v,

$$\begin{aligned} &\|v(t,\cdot)\|_{W_h^{\rho-1,\infty}} \le C_1 A\varepsilon\,, \\ &\|\mathcal{L}v(t,\cdot)\|_{H_h^1} \le 5B\varepsilon h^{-\sigma}\,, \qquad \|v(t,\cdot)\|_{H_h^{s-1}} \le B\varepsilon h^{-\sigma}\,, \end{aligned} \tag{5.32}$$

for $h^{-1} := t$ in $[1,T]$. In fact, from (4.1), the definition (4.5) of v in semiclassical coordinates and the equation (1.1),

$$\begin{aligned} &C_2\|u(t,\cdot)\|_{W^{t,\rho,\infty}} \le t^{-\frac{1}{2}}\|v(t,\cdot)\|_{W_h^{\rho-1,\infty}} \le C_1\|u(t,\cdot)\|_{W^{t,\rho,\infty}}\,, \\ &\|v(t,\cdot)\|_{H_h^{s'}} = \|w(t,\cdot)\|_{H^{s'}} \le \|\partial_t u(t,\cdot)\|_{H^{s'}} + \|u(t,\cdot)\|_{H^{s'+1}}\,, \end{aligned}$$

for some positive constants C_1, C_2, so the first and third inequality in (5.32) are satisfied. Moreover, $\mathcal{L}v$ can be expressed in term of w, Zw, as showed below using equation (4.7),

$$\begin{aligned} \frac{1}{i}Zw(t,y) &= h^{\frac{1}{2}}\left[(1-x^2)D_x + txD_t + i\frac{x}{2}\right]v(t,x)|_{x=\frac{y}{t}} \\ &= \left(h^{\frac{1}{2}}\left[(1-x^2)D_x + tx\,Op_h^w(x\xi + p(\xi)) + i\frac{x}{2}\right]v + h^{\frac{1}{2}}x\widetilde{P}\right)|_{x=\frac{y}{t}} \\ &= \left(h^{\frac{1}{2}}\left[D_x + tx\,Op_h^w(p(\xi))\right]v + h^{\frac{1}{2}}x\widetilde{P}\right)|_{x=\frac{y}{t}}\,, \end{aligned} \tag{5.33}$$

where $\widetilde{P}$ denotes the right hand side of (4.7) multiplied by h^{-1}. Using symbolic calculus of proposition 3.8,

$$\begin{aligned} \frac{1}{i}Zw(t,y) &= \left(h^{\frac{1}{2}}\left[h^{-1}Op_h^w(xp(\xi) + \xi) - \frac{1}{2i}Op_h^w(p'(\xi))\right]v + h^{\frac{1}{2}}x\widetilde{P}\right)|_{x=\frac{y}{t}} \\ &= \left(h^{\frac{1}{2}}\left[Op_h^w(p(\xi))\mathcal{L}v - \frac{1}{i}Op_h^w(p'(\xi))v + x\widetilde{P}\right]\right)|_{x=\frac{y}{t}}\,, \end{aligned} \tag{5.34}$$

where we used that $p(\xi)=\sqrt{1+\xi^2}$, $p'(\xi)=\xi/p(\xi)$. Therefore, since $Op_h^w(p'(\xi)):H^s_h\to H^s_h$ are uniformly bounded for all $s\in\mathbb{R}$ by proposition 3.11, and from $\|v(t,\cdot)\|_{L^2}=\|w(t,\cdot)\|_{L^2}$, we derive $\|\mathcal{L}v(t,\cdot)\|_{H^1_h}\le\|Zw(t,\cdot)\|_{L^2}+\|w(t,\cdot)\|_{L^2}+\|x\widetilde{P}\|_{L^2}$, where

$$\|x\widetilde{P}(t,\cdot)\|_{L^2}\le C\|v(t,\cdot)\|^2_{W_h^{\rho-1,\infty}}(\|\mathcal{L}v(t,\cdot)\|_{L^2}+\|w(t,\cdot)\|_{H^s})\,.$$

Then, from definition (4.1),

$$\|w(t,\cdot)\|_{L^2}\le\|\partial_tu(t,\cdot)\|_{L^2}+\|u(t,\cdot)\|_{H^1}\,,$$

$$\|Zw(t,\cdot)\|_{L^2}\le\|\partial_tZu(t,\cdot)\|_{L^2}+\|Zu(t,\cdot)\|_{H^1}+\|\partial_tu(t,\cdot)\|_{L^2}+\|u(t,\cdot)\|_{H^1}\,,$$

so we can use the uniform estimate $\|v(t,\cdot)\|_{W_h^{\rho-1,\infty}}\le C_1A\varepsilon$, choose $\varepsilon_0\ll1$ small enough such that $CC_1A^2\varepsilon_0^2<\frac{1}{2}$, and use the *a priori* energy bounds on u in (1.11), to have

$$\|\mathcal{L}v(t,\cdot)\|_{H^1_h}\le2\|Zu(t,\cdot)\|_{L^2}+2\|u(t,\cdot)\|_{L^2}+\|u(t,\cdot)\|_{H^s}\le5B\varepsilon h^{-\sigma}\,.$$

Under these bounds on v, in proposition 5.3 we proved that, for $A'=C_1A$ and $B'=5B$, the uniform estimate on v can be propagated, choosing for instance $K=\frac{2C_1}{C_2}$ to obtain $\|v(t,\cdot)\|_{W_h^{\rho-1,\infty}}\le\frac{AC_2}{2}\varepsilon$, and then $\|u(t,\cdot)\|_{W^{t,\rho,\infty}}\le\frac{A}{2}\varepsilon t^{-\frac{1}{2}}$, which concludes the proof of the boostrap and of global existence.

We prove now the asymptotics. We consider $\Sigma(\xi)=\langle\xi\rangle^{\rho+1}$ and we write

$$\langle hD\rangle^{-1}v=Op_h^w(\langle\xi\rangle^{-1}\langle\xi\rangle^{-\rho-1})v^{\Sigma}\,.$$

Using proposition 4.7, we develop the symbol $\langle\xi\rangle^{-\rho-2}$ at $\xi=d\varphi(x)$,

$$Op_h^w(\langle\xi\rangle^{-\rho-2})v^{\Sigma}=\theta_h(x)\langle d\varphi(x)\rangle^{-\rho-2}v^{\Sigma}+O_{L^\infty\cap L^2}(\varepsilon h^{\frac14-\sigma})\,,$$

and using the expression obtained in (5.29), along with the uniform bound on v^{Σ}, we derive that in the limit $t\to+\infty$ the function $\tilde{a}(x)=\varepsilon a_\varepsilon(x)$ verifies

$$|\tilde{a}(x)|\le|\theta_h(x)\langle d\varphi(x)\rangle^{-\rho-2}v^{\Sigma}|+O(\varepsilon t^{-\frac14+\sigma})\overset{t\to+\infty}{\le}C\varepsilon\langle d\varphi(x)\rangle^{-\rho-2}\,.\tag{5.35}$$

For points x in $]-1,1[$ such that $\langle d\varphi(x)\rangle\ge\alpha h^{-\beta}$, for a small $\alpha>0$, we have $|\tilde{a}(x)|=O(\varepsilon h^{\beta(\rho+2)})$ and then the corresponding contribution to the right hand side of (5.29) is $O(\varepsilon t^{-\min(\beta(\rho+2),\frac14-\sigma)})$ in $L^\infty\cap L^2$.

Let us now consider points x in $]-1,1[$ such that $\langle d\varphi(x)\rangle\le\alpha h^{-\beta}$, and remind that the function $\theta_h(x)$ in (5.29) is identically equal to one on some interval $[-1+ch^{2\beta},1-ch^{2\beta}]$. We can write

$$\int_1^t\theta_{1/\tau}(x)d\tau=t-1+\int_1^\infty(\theta_{1/\tau}(x)-1)d\tau-\int_t^\infty(\theta_{1/\tau}(x)-1)d\tau\,,\tag{5.36}$$

observing that on the support of $\theta_{1/\tau}(x)-1$, $\tau<\max c^{\frac{1}{2\beta}}(1-x,x+1)^{-\frac{1}{2\beta}}$. Therefore the last integral is taken on a finite interval and since $|x\pm1|\sim\langle d\varphi(x)\rangle^{-2}$ as $x\to\mp1$ by (3.34), this implies that at the same time we have $\tau\le c\langle d\varphi(x)\rangle^{\frac{1}{\beta}}$ and $\langle d\varphi(x)\rangle^{\frac{1}{\beta}}\le\alpha t$. For $t\le\tau$ and $\alpha>0$ small, this leads to a contradiction and to the fact that the last integral in (5.36) is equal to zero. Then in (5.29) we can write

$$a_\varepsilon(x)\exp\left[i\varphi(x)\int_1^t\theta_{1/\tau}(x)d\tau\right]=a_\varepsilon(x)\exp[i\varphi(x)t+ig(x)]\,,$$

with $g(x) = \varphi(x)\left[\int_1^\infty (\theta_{1/\tau}(x) - 1)d\tau - 1\right]$, and similarly, for x satisfying $\langle d\varphi(x)\rangle \le \alpha h^{-\beta}$,

$$|a_\varepsilon(x)|^2 \Phi_1^\Sigma(x) \int_1^t \theta_{1/\tau}(x)\frac{d\tau}{\tau} = |a_\varepsilon(x)|^2 \Phi_1^\Sigma(x) \log t + \tilde{g}(x),$$

for $\tilde{g}(x) = |a_\varepsilon(x)|^2\Phi_1^\Sigma(x)\left[\int_1^\infty (\theta_{1/\tau}(x) - 1)\frac{d\tau}{\tau} - 1\right]$. Moreover, for $\langle d\varphi(x)\rangle \le \alpha h^{-\beta}$ the coefficient $a_{(1,1,-1)}(x)$ appearing in $\Phi_1^\Sigma(x)$ is equal to $\langle d\varphi(x)\rangle^{-1}$, since $\chi(h^\beta d\varphi(x))\gamma(\frac{x+p'(d\varphi(x))}{\sqrt{h}}) \equiv 1$ if α is chosen sufficiently small, which implies that $\Phi_1^\Sigma(x)$ is exactly $\Phi_1(x)$ introduced in (1.8). Modifying the function $a_\varepsilon(x)$ by a factor of modulus one, we derive from (5.29) the asymptotic behaviour for $\langle hD\rangle^{-1}v$:

$$\langle hD\rangle^{-1}v = \varepsilon a_\varepsilon(x)\exp\left[i\varphi(x)t + i(\log t)\varepsilon^2|a_\varepsilon(x)|^2\Phi_1(x)\right] + t^{-\theta}r(t,x), \tag{5.37}$$

for some $\theta > 0$ and $\|r(t,\cdot)\|_{L^\infty} = O(\varepsilon)$, and reminding the relationship between v and w in (4.5), and between w and u in (4.1), we finally obtain the asymptotics for u in (1.7).

□

Appendix

This appendix is devoted to the detailed proof of proposition 3.8 and lemma 3.10, which are technical.

Proof of Proposition 3.8. Let us expand $a(x+z, \xi+\zeta)$ at (x,ξ) with Taylor's formula :

$$\begin{aligned} a(x+z,\xi+\zeta) = a(x,\xi) &+ \sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ 1\le|\alpha|\le k}} \frac{1}{\alpha!}\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}a(x,\xi)z^{\alpha_1}\zeta^{\alpha_2} \\ &+ \sum_{\substack{\beta=(\beta_1,\beta_2)\\ |\beta|=k+1}} \frac{k+1}{\beta!}z^{\beta_1}\zeta^{\beta_2}\int_0^1 \partial_x^{\beta_1}\partial_\xi^{\beta_2}a(x+tz,\xi+t\zeta)(1-t)^k\,dt, \end{aligned}$$

and replace this development in (3.11), obtaining :

$$\begin{aligned} a\sharp b &= \frac{1}{(\pi h)^2}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z - y\zeta)}a(x,\xi)b(x+y,\xi+\eta)\,dyd\eta dzd\zeta \\ &+ \frac{1}{(\pi h)^2}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z - y\zeta)} \sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ 1\le|\alpha|\le k}} \frac{1}{\alpha!}\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}a(x,\xi)b(x+y,\xi+\eta)\,z^{\alpha_1}\zeta^{\alpha_2}\,dyd\eta dzd\zeta \\ &+ \frac{1}{(\pi h)^2}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z - y\zeta)}\Bigg\{\sum_{\substack{\beta=(\beta_1,\beta_2)\\ |\beta|=k+1}} \frac{k+1}{\beta!}z^{\beta_1}\zeta^{\beta_2}\int_0^1 \partial_x^{\beta_1}\partial_\xi^{\beta_2}a(x+tz,\xi+t\zeta)(1-t)^k\,dt\Bigg\} \\ &\qquad\qquad \times b(x+y,\xi+\eta)\,dyd\eta dzd\zeta \\ &:= I_1 + I_2 + I_3\,. \end{aligned}$$

From a direct calculation and using that the inverse Fourier transform of the complex exponential is the delta function, i.e.

$$\frac{1}{\pi h}\int_{\mathbb{R}} e^{\frac{2i}{h}XY}\,dY = \delta_0(X)\,, \tag{A}$$

we derive

$$I_1 = \frac{1}{(\pi h)^2}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z - y\zeta)} a(x,\xi)b(x+y,\xi+\eta)\,dyd\eta dzd\zeta$$
$$= a(x,\xi)\int_{\mathbb{R}^2} b(x+y,\xi+\eta)\delta_0(y)\delta_0(\eta)\,dyd\eta = a(x,\xi)b(x,\xi)\,,$$

and

$$\begin{aligned}
I_2 =& \\
&= \frac{1}{(\pi h)^2}\sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ 1\le|\alpha|\le k}} \frac{1}{\alpha!}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z - y\zeta)}\,\partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x,\xi)b(x+y,\xi+\eta)\,z^{\alpha_1}\zeta^{\alpha_2}\,dyd\eta dzd\zeta \\
&= \frac{1}{(\pi h)^2}\sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ 1\le|\alpha|\le k}} \frac{1}{\alpha!}\left(\frac{h}{2i}\right)^{|\alpha|}\int_{\mathbb{R}^4} \partial_\eta^{\alpha_1}(-\partial_y^{\alpha_2})e^{\frac{2i}{h}(\eta z - y\zeta)}\,\partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x,\xi)b(x+y,\xi+\eta)\,dyd\eta dzd\zeta \\
&= \frac{1}{(\pi h)^2}\sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ 1\le|\alpha|\le k}} \frac{(-1)^{\alpha_1}}{\alpha!}\left(\frac{h}{2i}\right)^{|\alpha|}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z - y\zeta)}\,\partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x,\xi)\,\partial_y^{\alpha_2}\partial_\eta^{\alpha_1} b(x+y,\xi+\eta)\,dyd\eta dzd\zeta \\
&= \sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ 1\le|\alpha|\le k}} \frac{(-1)^{\alpha_1}}{\alpha!}\left(\frac{h}{2i}\right)^{|\alpha|}\partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x,\xi)\,\partial_x^{\alpha_2}\partial_\xi^{\alpha_1} b(x,\xi)\,.
\end{aligned}$$

The same calculation shows that I_3 is given by

$$\begin{aligned}
I_3 = \frac{k+1}{(\pi h)^2}\left(\frac{h}{2i}\right)^{k+1}\sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ |\alpha|=k+1}} \frac{(-1)^{\alpha_1}}{\alpha!}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z - y\zeta)}\Big\{\int_0^1 \partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x+tz,\xi+t\zeta)(1-t)^k dt \\
\times\,\partial_y^{\alpha_2}\partial_\eta^{\alpha_1} b(x+y,\xi+\eta)\Big\}\,dyd\eta dzd\zeta\,,
\end{aligned}$$

and it belongs to $h^{(k+1)(1-(\delta_1+\delta_2))}S_{\delta,\beta}(M_1M_2)$ since

$$\begin{aligned}
&\frac{1}{h^2}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z - y\zeta)}\Big\{\int_0^1 \partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x+tz,\xi+t\zeta)(1-t)^k dt\partial_y^{\alpha_2}\partial_\eta^{\alpha_1} b(x+y,\xi+\eta)\Big\}\,dyd\eta dzd\zeta = \\
&= \int_{\mathbb{R}^4} e^{2i(\eta z - y\zeta)}\Big\{\int_0^1 \partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x+t\sqrt{h}z,\xi+t\sqrt{h}\zeta)(1-t)^k dt\,\partial_y^{\alpha_2}\partial_\eta^{\alpha_1} b(x+\sqrt{h}y,\xi+\sqrt{h}\eta)\Big\} \\
&\qquad dyd\eta dzd\zeta \\
&= \int_{\mathbb{R}^4}\left(\frac{1+2iy\partial_\zeta}{1+4y^2}\right)^N\left(\frac{1-2i\eta\partial_z}{1+4\eta^2}\right)^N\left(\frac{1-2iz\partial_\eta}{1+4z^2}\right)^N\left(\frac{1+2i\zeta\partial_y}{1+4\zeta^2}\right)^N e^{2i(\eta z - y\zeta)} \\
&\times\Big\{\int_0^1 \partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x+t\sqrt{h}z,\xi+t\sqrt{h}\zeta)(1-t)^k dt\,\partial_y^{\alpha_2}\partial_\eta^{\alpha_1} b(x+\sqrt{h}y,\xi+\sqrt{h}\eta)\Big\}\,dyd\eta dzd\zeta
\end{aligned}$$

so integrating by parts,

$$\begin{aligned}
&\le Ch^{-(\delta_1+\delta_2)(\alpha_1+\alpha_2)}\int_{\mathbb{R}^4}\langle y\rangle^{-N}\langle\eta\rangle^{-N}\langle z\rangle^{-N}\langle\zeta\rangle^{-N}\Big\{\int_0^1 M_1(x+t\sqrt{h}z,\xi+t\sqrt{h}\zeta)dt \\
&\qquad\times M_2(x+\sqrt{h}y,\xi+\sqrt{h}\eta)\Big\}\,dyd\eta dzd\zeta \\
&\le Ch^{-(\delta_1+\delta_2)(k+1)}\int_{\mathbb{R}^4}\langle y\rangle^{-N+N_0}\langle\eta\rangle^{-N+N_0}\langle z\rangle^{-N+N_0}\langle\zeta\rangle^{-N+N_0}\,dyd\eta dzd\zeta\,M_1(x,\xi)M_2(x,\xi) \\
&\le Ch^{-(\delta_1+\delta_2)(k+1)}M_1(x,\xi)M_2(x,\xi)\,.
\end{aligned}$$

Equivalently, one can show that $|\partial^\alpha I_3| \leq Ch^{(k+1)(1-(\delta_1+\delta_2))-\delta|\alpha|} M_1(x,\xi) M_2(x,\xi)$. The last statement of the proposition follows immediately if we replace in previous inequalities M_1 and M_2 respectively by M_1^{k+1}, M_2^{k+1}. $\square$

Proof of Lemma 3.10. The proof of the lemma is the same as the previous one if, when we calculate to which class the remainder r_k belongs, we remark that

$$\left\langle \frac{x + t\sqrt{h}z + f(\xi + t\sqrt{h}\zeta)}{\sqrt{h}} \right\rangle^{-d} = \left\langle \frac{x + f(\xi)}{\sqrt{h}} + tz + tb(\xi,\zeta)\zeta \right\rangle^{-d} \lesssim \langle tz \rangle^N \langle t\zeta \rangle^N \left\langle \frac{x + f(\xi)}{\sqrt{h}} \right\rangle^{-d}$$

$$\left\langle \frac{x + \sqrt{h}y + f(\xi + \sqrt{h}\eta)}{\sqrt{h}} \right\rangle^{-l} = \left\langle \frac{x + f(\xi)}{\sqrt{h}} + y + b'(\xi,\eta)\eta \right\rangle^{-l} \lesssim \langle y \rangle^N \langle \eta \rangle^N \left\langle \frac{x + f(\xi)}{\sqrt{h}} \right\rangle^{-l}$$

with $b(\xi,\zeta) = \int_0^1 f'(\xi + st\sqrt{h}\zeta) ds \lesssim 1$, $b'(\xi,\eta) = \int_0^1 f'(\xi + s\sqrt{h}\eta) ds \lesssim 1$, for a certain $N \in \mathbb{N}$. $\square$

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Partie II

Solutions globales d'un système couplé ondes-Klein-Gordon à données petites modérément décroissantes

Global existence of small amplitude solutions for a model quadratic quasi-linear coupled wave-Klein-Gordon system in two space dimension with mildly decaying Cauchy data

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Introduction

The result we present in this paper concerns the global existence of solutions to a quadratic quasi-linear coupled system of a wave equation and a Klein-Gordon equation in space dimension two, when initial data are small smooth and mildly decaying at infinity. We prove this result for a model non-linearity, with the aim of extend it, in the future, to the most general case. Keeping this long term objective in mind, we shall try to develop a fairly general approach, in spite of the fact that we are treating here a simple model. The Cauchy problem we consider is the following

$$\begin{cases} (\partial_t^2 - \Delta_x)u(t,x) = Q_0(v, \partial_1 v)\,, \\ (\partial_t^2 - \Delta_x + 1)v(t,x) = Q_0(v, \partial_1 u)\,, \end{cases} \qquad (t,x) \in]1,+\infty[\times\mathbb{R}^2 \tag{1}$$

with initial conditions

$$\begin{cases} (u,v)(1,x) = \varepsilon(u_0(x), v_0(x))\,, \\ (\partial_t u, \partial_t v)(1,x) = \varepsilon(u_1(x), v_1(x))\,, \end{cases} \tag{2}$$

where $\varepsilon > 0$ is a small parameter, and Q_0 is the null form:

$$Q_0(v,w) = (\partial_t v)(\partial_t w) - (\nabla_x v)\cdot(\nabla_x w)\,.$$

We also suppose that, for some $n \in \mathbb{N}$ sufficiently large, $(\nabla_x u_0, u_1)$ is in the unit ball of $H^n(\mathbb{R}^2,\mathbb{R}) \times H^n(\mathbb{R}^2,\mathbb{R})$, (v_0, v_1) in the unit ball of $H^{n+1}(\mathbb{R}^2,\mathbb{R}) \times H^n(\mathbb{R}^2,\mathbb{R})$, and that

$$\sum_{1\le|\alpha|\le 3} \left(\|x^\alpha \nabla_x u_0\|_{H^{|\alpha|}} + \|x^\alpha v_0\|_{H^{|\alpha|+1}} + \|x^\alpha u_1\|_{H^{|\alpha|}} + \|x^\alpha v_1\|_{H^{|\alpha|}}\right) \le 1. \tag{3}$$

Keywords: Global solution of coupled wave-Klein-Gordon systems, Klainerman vector fields, Normal Forms, Semiclassical Analysis. The author is supported by a PhD fellowship funded by the FSMP and the Labex MME-DII, and by For Women in Science fellowship funded by Fondation L'Oréal-UNESCO.

Some physical models, especially related to general relativity, have shown the importance of studying such systems, to which several recent works have been dedicated. Most of the results known at present concern wave-Klein-Gordon systems in space dimension 3. One of the first ones goes back to Georgiev [9]. He observed that the vector fields method developed by Klainerman was not well adapted to handle at the same time massless and massive wave equations, because of the fact that the scaling vector field $S = t\partial_t + x \cdot \nabla_x$ is not a Killing vector field for the Klein-Gordon equation. To overcome this difficulty, he adapted Klainerman's techniques, introducing a *strong null condition*, to be satisfied by semi-linear nonlinearities, that ensures global existence. In 2012, Katayama [18] showed the global existence of small amplitude solutions to coupled systems of wave and Klein-Gordon equations under certain suitable conditions on the non-linearity, that include the *null condition* of Klainerman ([19]) on self-interactions between wave components, and are weaker than the *strong null condition* of Georgiev. Consequently, the result he obtains applies also to certain other physical systems such as the Dirac-Klein-Gordon equations, the Dirac-Proca equations and the Klein-Gordon-Zakharov equations. Later, this problem was also studied by LeFloch, Ma [22] and Wang [30] as a model for the full Einstein-Klein-Gordon system (E-KG)

$$\begin{cases} Ric_{\alpha\beta} = \mathbf{D}_\alpha\psi\mathbf{D}_\beta\psi + \frac{1}{2}\psi^2 g_{\alpha\beta} \\ \square_g\psi = \psi \end{cases}$$

The authors prove global existence of the solution to the wave-Klein-Gordon system with quasi-linear quadratic non-linearities satisfying suitable conditions, when initial data are small, smooth and compactly supported, using the so-called *hyperboloidal foliation method* introduced by Le Foch, Ma in [22]. Global stability for the full (E-KG) has been then proved by LeFloch-Ma [21, 20] in the case of small smooth perturbations that agree with a Scharzschild solution outside a compact set (see also Wang [29]). In a recent paper [16], Ionescu, Pausader prove global regularity and modified scattering for small smooth initial data that decay at suitable rates at infinity, but not necessarily compactly supported. The quadratic quasi-linear problem they deal with is the following

$$\begin{cases} -\square u = A^{\alpha\beta}\partial_\alpha v\partial_\beta v + Dv^2 \\ -(\square + 1)v = uB^{\alpha\beta}\partial_\alpha\partial_b v \end{cases}$$

where $A^{\alpha\beta}, B^{\alpha\beta}, D$ are real constants. The system keeps the same linear structure as (E-KG) in harmonic gauge, but only keeps quadratic non-linearities that involve the massive scalar field v (semilinear in the wave equation, quasi-linear in the Klein-Gordon equation). Moreover, the non-linearity they consider does not present a null structure but shows a particular resonant pattern. Their result relies on a combination of energy estimates, to control high Sobolev norms and weighted norms using the admissible vector fields, and on a Fourier analysis, in connection with normal forms and analysis of resonant sets, to prove dispersive estimates and decay in suitable lower regularity norms. The only results we know about global existence of small amplitude solutions in space dimension 2 are due to Ma, who considers the case of compactly supported initial data. In [25], he adapts the hyperboloidal foliation method mentioned above to $2+1$ spacetime wave-Klein-Gordon systems, and combines it with a normal form argument to treat some quasi-linear quadratic non-linearities (see [24]). More recently, he also proved this result in the case of some semi-linear quadratic interactions ([23]).

The result we prove in this paper is the following:

Theorem 1. *There exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in]0, \varepsilon_0[$, *system* (1) *with initial data satisfying* (2), (3) *admits a unique global solution defined on* $[1, +\infty[$, *with* $\partial_{t,x}u \in C^0([1, +\infty[; H^n(\mathbb{R}^2))$ *and* $(v, \partial_t v) \in C^0([1, +\infty[; H^{n+1}(\mathbb{R}^2) \times H^n(\mathbb{R}^2))$.

We briefly discuss the strategy of the above theorem's proof. First of all, we rewrite system (1) in terms of unkowns

$$u_\pm = (D_t \pm |D_x|)\, u, \quad v_\pm = (D_t \pm \langle D_x\rangle)\, v, \tag{4}$$

where $D_{t,x} = \frac{1}{i}\partial_{t,x}$, and introduce the admissible Klainerman vector fields for this problem, i.e.

$$\Omega = x_1\partial_2 - x_2\partial_1, \quad Z_j = x_j\partial_t + t\partial_j, \quad j = 1, 2.$$

We also denote by $\mathcal{Z} = \{\Gamma_1, \dots, \Gamma_5\}$ the family made of the above vector fields together with the two derivatives in space, and if $I = (i_1, \dots, i_p)$ is an element of $\{1, \dots, 5\}^p$, $\Gamma^I w$ will be the function obtained letting vector fields $\Gamma_{i_1}, \dots, \Gamma_{i_p}$ act successively on w. We then set

$$u^I_\pm = (D_t \pm |D_x|)\,\Gamma^I u, \quad v^I_\pm = (D_t \pm \langle D_x\rangle)\,\Gamma^I v, \tag{5}$$

and introduce the following energies

$$E_0(t; u_\pm, v_\pm) = \int_{\mathbb{R}^2} \left(|u_+(t,x)|^2 + |u_-(t,x)|^2 + |v_+(t,x)|^2 + |v_-(t,x)|^2\right) dx,$$

then for $n \geq 3$,

$$E_n(t; u_\pm, v_\pm) = \sum_{|\alpha| \leq n} E_0(t; D^\alpha_x u_\pm, D^\alpha_x v_\pm),$$

which controls the H^n regularity of $u_\pm, v_\pm$ and finally, for any integer k between 0 and 2,

$$E^k_3(t; u_\pm, v_\pm) = \sum_{\substack{|\alpha|+|I| \leq 3 \\ |I| \leq 3-k}} E_0(t; D^\alpha_x u^I_\pm, D^\alpha_x v^I_\pm)$$

that takes into account the decay in space of $u_\pm, v_\pm$ and of at most three of their spatial derivatives. By a local existence argument, an a-priori uniform estimate on E_n on a certain time interval will be enough to ensure the extension of the solution to that interval. For this reason, we are led to prove a result as the following one, in which $\mathrm{R} = (\mathrm{R}_1, \mathrm{R}_2)$ denotes the Riesz transform:

Theorem 2. *Let K_1, K_2 two constants strictly bigger than 1. There exist two integers $n \gg \rho \gg 1$, $\varepsilon_0 \in]0, 1[$, some small real $0 < \delta \ll \delta_2 \ll \delta_1 \ll \delta_0 \ll 1$ and two constants A, B sufficiently large such that, if functions $u_\pm, v_\pm$ defined by* (4) *from a solution to* (1) *satisfy*

$$\begin{aligned}
&\|\langle D_x\rangle^{\rho+1} u_\pm(t,\cdot)\|_{L^\infty} + \|\langle D_x\rangle^{\rho+1} \mathrm{R} u_\pm(t,\cdot)\|_{L^\infty} \leq A\varepsilon t^{-\frac{1}{2}} \\
&\|\langle D_x\rangle^{\rho} v_\pm\|_{L^\infty} \leq A\varepsilon t^{-1} \\
&E_n(t; u_\pm, v_\pm) \leq B^2\varepsilon^2 t^{2\delta} \\
&E^k_3(t; u_\pm, v_\pm) \leq B^2\varepsilon^2 t^{2\delta_{3-k}}, \ 0 \leq k \leq 2,
\end{aligned} \tag{6}$$

for every $t \in [1, T]$, then on the same interval $[1, T]$ we have

$$\begin{aligned}
&\|\langle D_x\rangle^{\rho+1} u_\pm(t,\cdot)\|_{L^\infty} + \|\langle D_x\rangle^{\rho+1} \mathrm{R} u_\pm(t,\cdot)\|_{L^\infty} \leq \frac{A}{K_1}\varepsilon t^{-\frac{1}{2}} \\
&\|\langle D_x\rangle^{\rho} v_\pm\|_{L^\infty} \leq \frac{A}{K_1}\varepsilon t^{-1} \\
&E_n(t; u_\pm, v_\pm) \leq \frac{B^2}{K_2^2}\varepsilon^2 t^{2\delta} \\
&E^k_3(t; u_\pm, v_\pm) \leq \frac{B^2}{K_2^2}\varepsilon^2 t^{2\delta_{3-k}}, \ 0 \leq k \leq 2.
\end{aligned} \tag{7}$$

The proof of the theorem consists, on the one hand, to prove that (6) implies the latter two inequalities by means of energy inequalities. On the other, by reduction of the starting problem to a coupled system of ordinary differential equations or transport equations, we prove that (6) implies the first two inequalities in (7).

In order to recover the mentioned energy inequality that allows us to propagate the a-priori energy estimates, we rewrite system (1) by letting act on it a family Γ^I of vector fields, and then pass to unknowns (5). We obtain a new system of the form

$$\begin{aligned}(D_t \mp |D_x|)u^I_\pm &= NL_{\mathrm{w}}(v^I_\pm, v^I_\pm)\\ (D_t \mp |D_x|)v^I_\pm &= NL_{\mathrm{kg}}(v^I_\pm, u^I_\pm)\end{aligned}$$

where the non-linearities (whose explicit expression may be found in the right hand side of (2.1.2)) are bilinear quantities of their arguments. Because of the quasi-linear nature of our problem, the first step towards the derivation of the mentioned inequality is to highlight the very quasi-linear contribution to above non-linearities, and make sure that it does not lead to a loss of derivatives. For this reason, we write the above system in a vectorial fashion by introducing vectors

$$U^I = \begin{bmatrix} u^I_+ \\ 0 \\ u^I_- \\ 0 \end{bmatrix}, \ V^I = \begin{bmatrix} 0 \\ v^I_+ \\ 0 \\ v^I_- \end{bmatrix}, \ W^I = U^I + V^I,$$

we *para-linearize* the vectorial equation satisfied by W^I (using the tools introduced in subsection 1.2.1) to stress out the quasi-linear contribution to the non-linearity, and then *symmetrize* it (in the sense of subsection 2.1.3) by introducing some new unknown W^I_s comparable to W^I.

What we would need to show in order to prove the last two inequalities in (7), is that, using the estimates in (6), the derivative in time of the L^2 norm to the square of W^I_s is bounded by $\frac{C\varepsilon}{t}\|W^I\|_{L^2}$. By analysing the remaining semi-linear contributions in the symmetrized equation satisfied by W^I_s, we find out that the L^2 norm of some of those ones can only be estimated making appear the L^∞ norm on the wave factor and the L^2 norm on the Klein-Gordon one. Because of the very slow decay in time of the wave solution (the decay rate being $t^{-1/2}$, as assumed in the first inequality of (6)), we are hence very far away from the wished estimate. Consequently, the second step for the derivation of the right energy inequality consists in performing a normal form argument to get rid of those quadratic terms, and replace them with cubic ones. For that, we first use a Shatah' normal form adapted to quasi-linear equations (see subsection 2.2.1) as already used by several authors (we cite [27, 5, 4, 6] for quasi-linear Klein-Gordon equations, and [13, 12, 17, 1, 15] for quasi-linear equations arising in fluids mechanics), but also a semi-linear normal form argument to treat some other terms on which we are allowed to lose some derivatives (see subsection 2.2.2). These two normal forms' steps lead us to define some new energies $\widetilde{E}_n(t; u_\pm, v_\pm), \widetilde{E}^k_3(t; u_\pm, v_\pm)$, equivalent to the starting ones $E_n(t; u_\pm, v_\pm)$, $\widetilde{E}^k_3(t; u_\pm, v_\pm)$, and that we are able to propagate. That concludes the first part of the proof, i.e. the deduction of the latter two inequalities in (7) from (6).

The last thing that remains to prove, in order to conclude the proof of theorem 2 and hence of theorem 1, is that (6) implies the first two estimates in (7). The strategy we employ is very similar to the one developed in [28], i.e. we deduce from the starting system (1) a new coupled one made of an ordinary differential equation, coming from the "Klein-Gordon component", and of a transport equation, derived from the wave one. The study of this system will provide us with the wished L^∞ estimates.

We start our analysis by another normal form to replace almost all quadratic non-linear terms in the equations satisfied by $u_\pm, v_\pm$ with cubic ones. The only contributions that cannot be eliminated are the ones depending on (v_+, v_-), which are resonant and should be suitably treated. We do not use directly the normal forms obtained in the previous step. In fact, our aim is basically to obtain an L^∞ estimate for at most ρ derivatives of the solution, having a control on their H^s norm for $s \gg \rho$. This permits us to lose some derivatives in the normal form reduction, so the fact that the system is quasi-linear is no longer important.

We define some new unknowns u^{NF}, v^{NF} in terms of u_-, v_- by adding some quadratic perturbations, so that they are solution to

$$(D_t + |D_x|)u^{NF} = q_w + c_w + r^{NF}_w, \ (D_t + |D_x|)v^{NF} = r^{NF}_{kg}, \tag{8}$$

where $r^{NF}_w, c_w, r^{NF}_{kg}$ are cubic terms, whereas q_w is the mentioned bilinear expression in v_+, v_- that cannot be eliminated by normal forms, but whose structure will successively provide us with remainder terms. Then, if we define

$$\widetilde{u}(t,x) = tu^{NF}(t,tx), \ \widetilde{v}(t,x) = tv^{NF}(t,tx), \tag{9}$$

and introduce $h := t^{-1}$ the *semi-classical parameter*, we obtain that $\widetilde{u}, \widetilde{v}$ verify

$$\begin{aligned} (D_t - \mathrm{Op}^w_h(x\cdot\xi - |\xi|))\widetilde{u} &= h^{-1}\left[q_w(t,tx) + c_w(t,tx) + r^{NF}_w(t,tx)\right] \\ (D_t - \mathrm{Op}^w_h(x\cdot\xi - \langle\xi\rangle))\widetilde{v} &= h^{-1}r^{NF}_{kg}(t,tx) \end{aligned} \tag{10}$$

where Op^w_h is the Weyl quantization introduced, together with the semi-classical pseudo-differential calculus, in subsection 1.2.1. We also consider the following operators

$$\mathcal{M}_j = \frac{1}{h}\Big(x_j|\xi| - \xi_j\Big), \ \mathcal{L}_j = \frac{1}{h}\Big(x_j - \frac{\xi_j}{\langle\xi\rangle}\Big),$$

whose symbols are given respectively (up to the multiplication by $|\xi|$ for the former case) by the derivative with respect to ξ of symbols $x\cdot\xi - |\xi|$ and $x\cdot\xi - \langle\xi\rangle$ in (10). Using the equation, we can express $\mathcal{M}_j\widetilde{u}$ (resp. $\mathcal{L}_j\widetilde{v}$) in terms of Z_ju^{NF} (resp. Z_jv^{NF}) and of q_w, c_w, r^{NF}_w (resp. r^{NF}_{kg}).

As done in [28], we first introduce the lagrangian

$$\Lambda_{kg} = \Big\{(x,\xi) : x - \frac{\xi}{\langle\xi\rangle} = 0\Big\}$$

which is the graph of $\xi = d\phi(x)$, with $\phi(x) = \sqrt{1-|x|^2}$, and decompose $\widetilde{v}$ into the sum of a contribution micro-localised on a neighbourhood of size $\sqrt{h}$ of Λ_{kg}, and another one micro-localised out of that neighbourhood (in the spirit of [14]). The second contribution can be basically estimated in L^∞ by $h^{\frac{1}{2}-0}$ times the L^2 norm of some iterates of operator $\mathcal{L}$ acting on $\widetilde{v}$ (which are controlled by the L^2 hypothesis in theorem 2). The main contribution to $\widetilde{v}$ is then represented by $\widetilde{v}_{\Lambda_{kg}}$, which appears to be solution to

$$[D_t - \mathrm{Op}^w_h(x\cdot\xi - \langle\xi\rangle)]\widetilde{v}_{\Lambda_{kg}} = \text{ controlled terms.}$$

Developing the symbol in the above left hand side on Λ_{kg}, we finally obtain the wished ODE, which combined with the a-priori estimate of the controlled terms allows us to deduce from (6) the second estimate (7) (with $\rho = 0$, the general case being treated in the same way up to few more technicalities).

The same strategy is employed to obtain some uniform estimates on $\widetilde{u}$. We introduce the lagrangian

$$\Lambda_w = \Big\{(x,\xi) : x - \frac{\xi}{|\xi|} = 0\Big\}$$

which, differently from Λ_{kg}, is not a graph but projects on the basis as an hyper-surface. For this reason, the classical problem associated to the first equation in (10) is rather a transport equation than an ordinary differential equation. It is obtained in a similar way by decomposing $\widetilde{u}$ into two contributions: one denoted by $\widetilde{u}_{\Lambda_w}$ and micro-localised in a neighbourhood of size $h^{\frac{1}{2}-\sigma}$ (for some small $\sigma > 0$) of Λ_w; another one micro-localised away from this neighbourhood. As for the Klein-Gordon component, this latter contribution can be easily controlled thanks to the L^2 estimates that the last two inequalities in (6) infer on the iterates of $\mathcal{M}_j$ acting on $\widetilde{u}$. By micro-localisation, we derive that $\widetilde{u}_{\Lambda_w}$ satisfies

$$[D_t - \mathrm{Op}_h^w(x\cdot\xi - |\xi|)]\widetilde{u}_{\Lambda_w} = \text{ controlled terms,}$$

and by developing symbol $x\cdot\xi - |\xi|$ on Λ_w, we derive the wished transport equation. Integrating this equation by the method of characteristics, we finally obtain the first estimate in (6) and conclude the proof of theorem 2.

Chapter 1

Main Theorem and Preliminary Results

1.1 Statement of the Main Theorem

NOTATIONS: We warn the reader that, throughout the paper, we will often denote ∂_t (resp. ∂_{x_j}, $j=1,2$) by ∂_0 (resp. ∂_j, $j=1,2$), while symbol ∂ without any subscript will stand for one of three derivatives ∂_a, $a=0,1,2$. $\nabla_x f$ is the classical spatial gradient of f, $D:=\frac{1}{i}\partial$, and R_j, for $j=1,2$, denotes the Riesz operator $D_j|D_x|^{-1}$. We will also employ notation $\|\partial_{t,x}w\|$ with the meaning $\|\partial_t w\|+\|\partial_x w\|$, and $\|\mathrm{R}w\|=\sum_j\|\mathrm{R}_j w\|$.

We consider the following quadratic, quasi-linear, coupled wave/Klein-Gordon system

$$\begin{cases} (\partial_t^2-\Delta_x)u(t,x)=Q_0(v,\partial_1 v)\,, \\ (\partial_t^2-\Delta_x+1)v(t,x)=Q_0(v,\partial_1 u)\,, \end{cases} \qquad (t,x)\in]1,+\infty[\times\mathbb{R}^2 \tag{1.1.1}$$

with initial conditions

$$\begin{cases} (u,v)(1,x)=\varepsilon(u_0(x),v_0(x))\,, \\ (\partial_t u,\partial_t v)(1,x)=\varepsilon(u_1(x),v_1(x))\,, \end{cases} \tag{1.1.2}$$

where $\varepsilon>0$ is a small parameter, and Q_0 is the null form:

$$Q_0(v,w)=(\partial_t v)(\partial_t w)-(\nabla_x v)\cdot(\nabla_x w)\,. \tag{1.1.3}$$

Our aim is to prove that there is a unique solution to Cauchy problem (1.1.1)-(1.1.2) provided that ε is sufficiently small, and u_0,v_0,u_1,v_1 decay rapidly enough at infinity. The theorem we are going to demonstrate is the following:

Theorem 1.1.1 (Main Theorem). *There exist an integer n sufficiently large and $\varepsilon_0\in]0,1[$ sufficiently small such that, for any $\varepsilon\in]0,\varepsilon_0[$, any real valued u_0,v_0,u_1,v_1 satisfying:*

$$\begin{gathered} \|\nabla_x u_0\|_{H^n}+\|v_0\|_{H^{n+1}}+\|u_1\|_{H^n}+\|v_1\|_{H^n}\le 1, \\ \sum_{|\alpha|=1}^{2}\left(\|x^\alpha\nabla_x u_0\|_{H^{|\alpha|}}+\|x^\alpha v_0\|_{H^{|\alpha|+1}}+\|x^\alpha u_1\|_{H^{|\alpha|}}+\|x^\alpha v_1\|_{H^{|\alpha|}}\right)\le 1, \end{gathered} \tag{1.1.4}$$

system (1.1.1)-(1.1.2) *admits a unique global solution (u,v) with $\partial_{t,x}u\in C^0\left([1,\infty[;H^n(\mathbb{R}^2)\right)$, $v\in C^0\left([1,\infty[;H^{n+1}(\mathbb{R}^2)\right)\cap C^1\left([1,\infty[;H^n(\mathbb{R}^2)\right)$.*

The proof of the main theorem is based on the introduction of four new functions u_+, u_-, v_+, v_-, defined in terms of u, v as follows:

$$\begin{cases} u_+ := (D_t + |D_x|)u\,, \\ u_- := (D_t - |D_x|)u\,, \end{cases} \qquad \begin{cases} v_+ := (D_t + \langle D_x\rangle)v\,, \\ v_- := (D_t - \langle D_x\rangle)v\,, \end{cases} \tag{1.1.5}$$

where $D := \frac{1}{i}\partial$, and on the propagation of some a-priori estimates made on them in some interval $[1, T]$, for a fixed $T > 1$. In order to state this result, we consider the admissible Klainerman vector fields for the wave/Klein-Gordon system:

$$\Omega := x_1\partial_2 - x_2\partial_1\,, \quad Z_j := x_j\partial_t + t\partial_j\,, j = 1, 2 \tag{1.1.6}$$

and denote by Γ a generic vector field in $\mathcal{Z} = \{\Omega, Z_j, \partial_j, j = 1, 2\}$. If

$$\mathcal{Z} = \{\Gamma_1, \dots, \Gamma_5\} \tag{1.1.7}$$

is assumed ordered (e.g. with $\Gamma_1 = \Omega, \Gamma_j = Z_{j-1}$ for $j = 2, 3$, $\Gamma_j = \partial_{j-3}$ for $j = 4, 5$), then for a multi-index $I = (i_1, \dots, i_n)$, $i_j \in \{1, \dots, 5\}$ for $j = 1, \dots, n$, we define the *length* of I as $|I| := n$, and $\Gamma^I := \Gamma_{i_1} \cdots \Gamma_{i_n}$ the product of vector fields $\Gamma_{i_j} \in \mathcal{Z}$, $j = 1, \dots, n$.

Vector fields Γ have two relevant properties: they act like derivations on non-linear terms; they exactly commute with the linear part of both wave and Klein-Gordon equation. This is the reason why we exclude of our consideration the scaling vector field $S = t\partial_t + \sum_j x_j\partial_j$, which is always considered in the so-called *Klainerman vector fields' method* for the wave equation, but does not commute with the Klein-Gordon operator.

We also introduce the energy of (u_+, u_-, v_+, v_-) at time $t \geq 1$ as

$$E_0(t; u_\pm, v_\pm) := \int \left(|u_+(t,x)|^2 + |u_-(t,x)|^2 + |v_+(t,x)|^2 + |u_-(t,x)|^2\right) dx, \tag{1.1.8}$$

together with the generalized energies

$$E_n(t; u_\pm, v_\pm) := \sum_{|\alpha|\leq n} E_0(t; D^\alpha_x u_\pm, D^\alpha_x v_\pm), \quad \forall n \in \mathbb{N}, n \geq 3, \tag{1.1.9a}$$

and

$$E^k_3(t; u_\pm, v_\pm) := \sum_{\substack{|\alpha|+|I|\leq 2 \\ 0\leq|I|\leq 3-k}} E_0(t; D^\alpha_x u^I_\pm; D^\alpha_x v^I_\pm), \quad 0 \leq k \leq 2, \tag{1.1.9b}$$

where, for any multi-index I,

$$u^I_\pm := (D_t \pm |D_x|)\Gamma^I u, \quad v^I_\pm := (D_t \pm \langle D_x\rangle)\Gamma^I v. \tag{1.1.10}$$

Energy $E_n(t; u_\pm, v_\pm)$, for $n \geq 3$, is introduced with the aim of controlling the Sobolev norm H^n of $u_\pm, v_\pm$ for large values of n. The reason of dealing with $E^k_3(t; u_\pm, v_\pm)$ is, instead, to control the L^2 norm of $\Gamma^I u_\pm, \Gamma^I v_\pm$, with general $\Gamma \in \mathcal{Z}$, for any $|I| \leq 3$, and superscript k indicates that we are considering only products Γ^I containing at most $3-k$ vector fields in $\{\Omega, Z_m, m = 1, 2\}$. For instance, the L^2 norms of $\Omega^3 u_\pm, \Omega Z^2_1 v_\pm$ are bounded by $E^0_3(t; u_\pm, v_\pm)$, but not by $E^1_3(t; u_\pm, v_\pm)$, while the L^2 norms of $Z^2_1 u_\pm, \partial_2\Omega Z_2 v_\pm$ are estimated by both $E^1_3(t; u_\pm, v_\pm), E^0_3(t; u_\pm, v_\pm)$, etc. The interest of distinguishing between $k = 0, 1, 2$, is to take into account the different growth in time of the L^2 norm of such terms, linked to the number of vector fields Ω, Z_m acting on $u_\pm, v_\pm$, as emerges from a-priori estimate (1.1.11d).

Theorem 1.1.2 (Bootstrap Argument). *Let $K_1, K_2 > 1$, and $H^{\rho,\infty}$ be the space defined in 1.2.1 (iii). There exist two integers n, ρ sufficiently large, with $n \gg \rho$, some $\delta_0, \delta_1, \delta_2, \delta > 0$ small such that $\delta \ll \delta_2 \ll \delta_1 \ll \delta_0$, two constants $A, B > 0$ sufficiently large, and $\varepsilon_0 \in]0, (2A+B)^{-1}[$, such that, for any $0 < \varepsilon < \varepsilon_0$, if (u, v) is solution to (1.1.1)-(1.1.2) on some interval $[1, T]$, for a fixed $T > 1$, and $u_\pm, v_\pm$ defined in (1.1.5) satisfy:*

$$\|u_\pm(t,\cdot)\|_{H^{\rho+1,\infty}} + \|\mathrm{R}u_\pm(t,\cdot)\|_{H^{\rho+1,\infty}} \le A\varepsilon t^{-\frac{1}{2}}, \tag{1.1.11a}$$

$$\|v_\pm(t,\cdot)\|_{H^{\rho,\infty}} \le A\varepsilon t^{-1}, \tag{1.1.11b}$$

$$E_n(t; u_\pm, v_\pm)^{\frac{1}{2}} \le B\varepsilon t^{\frac{\delta}{2}}, \tag{1.1.11c}$$

$$E_3^k(t; u_\pm, v_\pm)^{\frac{1}{2}} \le B\varepsilon t^{\frac{\delta_k}{2}}, \quad \forall\, 0 \le k \le 2, \tag{1.1.11d}$$

for every $t \in [1, T]$, then in the same interval they verify also

$$\|u_\pm(t,\cdot)\|_{H^{\rho+1,\infty}} + \|\mathrm{R}u_\pm(t,\cdot)\|_{H^{\rho+1,\infty}} \le \frac{A}{K_1}\varepsilon t^{-\frac{1}{2}}, \tag{1.1.12a}$$

$$\|v_\pm(t,\cdot)\|_{H^{\rho,\infty}} \le \frac{A}{K_1}\varepsilon t^{-1}, \tag{1.1.12b}$$

$$E_n(t; u_\pm, v_\pm)^{\frac{1}{2}} \le \frac{B}{K_2}\varepsilon t^{\frac{\delta}{2}}, \tag{1.1.12c}$$

$$E_3^k(t; u_\pm, v_\pm)^{\frac{1}{2}} \le \frac{B}{K_2}\varepsilon t^{\frac{\delta_k}{2}}, \quad \forall\, 0 \le k \le 2. \tag{1.1.12d}$$

The a-priori estimates on the uniform norm of $u_\pm, \mathrm{R}u_\pm, v_\pm$, made in above theorem, translate in terms of $u_\pm, v_\pm$ the sharp decay in time we expect for the solution (u, v) to starting problem (1.1.1). Indeed, from definitions (1.1.5), it appears that

$$D_t u = \frac{u_+ + u_-}{2}, \quad D_x u = \mathrm{R}\left(\frac{u_+ - u_-}{2}\right),$$
$$D_t v = \frac{v_+ + v_-}{2}, \quad v = \langle D_x\rangle^{-1}\left(\frac{v_+ - v_-}{2}\right),$$

so (1.1.11a), (1.1.11b) imply

$$\|\partial_{t,x}u(t,\cdot)\|_{H^{\rho,\infty}} \le A\varepsilon t^{-\frac{1}{2}}, \quad \|\partial_t v(t,\cdot)\|_{H^{\rho,\infty}} + \|v(t,\cdot)\|_{H^{\rho+1,\infty}} \le A\varepsilon t^{-1}.$$

Furthermore,

$$\|\partial_t u(t,\cdot)\|_{H^n} + \|\nabla_x u(t,\cdot)\|_{H^n} + \|\partial_t v(t,\cdot)\|_{H^n} + \|\nabla_x v(t,\cdot)\|_{H^n} + \|v(t,\cdot)\|_{H^n} \le E_n(t, u_\pm, v_\pm)^{\frac{1}{2}},$$

and the propagation of energy a-priori estimate (1.1.11c) is equivalent to the propagation of the same estimate on the above Sobolev norms. This fact will imply theorem 1.1.1 thanks to a local existence argument.

Before ending this section and going into the core of the subject, we briefly remind the general definition of *null condition* for a multilinear form on $\mathbb{R}^{1+n}$ and a result by Hörmander (see [11]).

Definition 1.1.3. A k-linear form G on $\mathbb{R}^{1+n}$ is said to satisfy the *null condition* if and only if, for all $\xi \in \mathbb{R}^n, \xi = (\xi_0, \dots, \xi_n)$, such that $\xi_0^2 - \sum_{j=1}^n \xi_j^2 = 0$,

$$G(\underbrace{\xi, \dots, \xi}_{k}) = 0. \tag{1.1.13}$$

Example: The trilinear form $\xi_0^2\xi_a - \sum_{j=1,2}\xi_j^2\xi_a$ associated to $Q_0(v,\partial_a w)$, for any $a=0,1,2$, satisfies the null condition (1.1.13). It is the most common example of null form.

Lemma 1.1.4 (Hörmander [10], Lemma 6.6.5.). *Let G be a k-linear form on $\mathbb{R}^{1+n}$, $k=k_1+\dots+k_r$, with k_j positive integers, and $\Gamma\in\mathcal{Z}$. For all $u_j\in C^{k+1}(\mathbb{R}^{1+n})$, all $\alpha_j\in\mathbb{N}^{1+n}$, $|\alpha_j|=k_j$, and $u_j^{(k_j)}:=\partial^{\alpha_j}u_j$,*

$$
\begin{aligned}
\Gamma G(u_1^{(k_1)},\dots,u_r^{(k_r)}) = G((\Gamma u_1)^{(k_1)},\dots,u_r^{(k_r)}) + \dots \\
+ G(u_1^{(k_1)},\dots,(\Gamma u_r)^{(k_r)}) + G_1(u_1^{(k_1)},\dots,u_r^{(k_r)})\,,
\end{aligned} \tag{1.1.14}
$$

where G_1 satisfies the null condition.

Remark 1.1.5. Previous lemma simplifies when the multi-linear form G satisfying the null condition is $Q_0(v,\partial_a w)$, for any $a=0,1,2$. Indeed, the structure of the null form is not modified by the action of vector field Γ, in the sense that

$$
\Gamma Q_0(v,\partial_a w) = Q_0(\Gamma v,\partial_a w) + Q_0(v,\partial_a\Gamma w) + G_1(v,\partial w)\,. \tag{1.1.15}
$$

where $G_1(v,\partial w)=0$ if $\Gamma=\partial_m$, $m=1,2$, and

$$
G_1(v,\partial w) = \begin{cases} -Q_0(v,\partial_m w), & \text{if } a=0, \Gamma=Z_m, m\in\{1,2\},\\ 0, & \text{if } a=0,\Gamma=\Omega,\\ -Q_0(v,\partial_t w), & \text{if } a\neq 0,\Gamma=Z_a,\\ 0, & \text{if } a\neq 0, \Gamma=Z_m, m\in\{1,2\}\setminus\{a\},\\ (-1)^a Q_0(v,\partial_m w), & \text{with } m\in\{1,2\}\setminus\{a\},\ \text{if } a\neq 0,\Gamma=\Omega. \end{cases} \tag{1.1.16}
$$

If Γ^I contains at least $k\le |I|$ space derivatives ∂_x, then

$$
\Gamma^I Q_0(v,\partial_1 w) = \sum_{|I_1|+|I_2|=|I|} Q_0(\Gamma^{I_1}v,\partial_1\Gamma^{I_2}w) + \sum_{k\le |I_1|+|I_2|<|I|} c_{I_1,I_2} Q_0(\Gamma^{I_1}v,\partial\Gamma^{I_2}w), \tag{1.1.17}
$$

with $c_{I_1,I_2}\in\{-1,0,1\}$. In above equality, we should think of multi-index I_1 (resp. I_2) as obtained by extraction of a $|I_1|$-tuple (resp. $|I_2|$-tuple) from $I=(i_1,\dots i_n)$, in such a way that each i_j appearing in I, and corresponding to a spatial derivative (e.g. $\Gamma_{i_j}=D_m$, for $m\in\{1,2\}$), appears either in I_1 or in I_2, but not in both. For further references, we define

$$
\mathcal{I}(I) := \{(I_1,I_2) | I_1, I_2 \text{ multi-indices obtained as described above}\}\,. \tag{1.1.18}
$$

1.2 Preliminary Results

The aim of this section is to introduce most of the technical tools that will be used throughout the paper. In particular, subsections 1.2.1 and 1.2.2 are devoted to recall some definitions and results about, respectively, paradifferential and pseudo-differential calculus; subsection 1.2.3 and 1.2.4 are dedicated to the introduction of some special operators, that we will frequently use when dealing, respectively, with the wave and the Klein-Gordon component. Subsections 1.2.1, 1.2.2 barely contain proofs (we refer for that to [3], [26], [8], [31]), whereas subsections 1.2.3, 1.2.4 are much longer and richer in proofs and technicalities.

1.2.1 Paradifferential Calculus

In this subsection we recall some definitions and properties that will be useful in chapter 2. We first recall the definition of some spaces (Sobolev, Lipschitz and Hölder spaces) in space dimension $d \geq 1$, and afterwards some results concerning symbolic calculus and the action of paradifferential operators on Sobolev spaces (see for instance [26]). We warn the reader that we will use both notations $\hat{w}(\xi)$ and $\mathcal{F}_{x\mapsto\xi}w$ for the Fourier transform of a function $w = w(x)$.

Definition 1.2.1 (Spaces). (i) Let $s \in \mathbb{R}$. $H^s(\mathbb{R}^d)$ denotes the space of tempered distributions $w \in \mathcal{S}'(\mathbb{R}^d)$ such that $\hat{w} \in L^2_{loc}(\mathbb{R}^d)$ and

$$\|w\|^2_{H^s(\mathbb{R}^d)} := \frac{1}{(2\pi)^d}\int (1+|\xi|^2)^s|\hat{w}(\xi)|^2 d\xi < +\infty;$$

(ii) For $\rho \in \mathbb{N}$, $W^{\rho,\infty}(\mathbb{R}^d)$ denotes the space of distributions $w \in \mathcal{D}'(\mathbb{R}^d)$ such that $\partial_x^\alpha w \in L^\infty(\mathbb{R}^d)$, for any $\alpha \in \mathbb{N}^d$, with $|\alpha| \leq \rho$, endowed with the norm

$$\|w\|_{W^{\rho,\infty}} := \sum_{|\alpha|\leq\rho}\|\partial_x^\alpha w\|_{L^\infty};$$

(iii) For $\rho \in \mathbb{N}$, we also introduce $H^{\rho,\infty}(\mathbb{R}^d)$ as the space of tempered distributions $w \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|w\|_{H^{\rho,\infty}} := \|\langle D_x\rangle^\rho w\|_{L^\infty} < +\infty.$$

Definition 1.2.2. An operator T is said of order $\leq m \in \mathbb{R}$ if it is a bounded operator from $H^{s+m}(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$.

Definition 1.2.3 (Smooth symbols). Let $m \in \mathbb{R}$.

(i) $S^m_0(\mathbb{R}^d)$ denotes the space of functions $a(x,\eta)$ on $\mathbb{R}^d \times \mathbb{R}^d$, which are C^∞ with respect to η, and such that for all $\alpha \in \mathbb{N}^d$ there exists a constant $C_\alpha > 0$, and

$$\|\partial_\eta^\alpha a(\cdot,\eta)\|_{L^\infty} \leq C_\alpha(1+|\eta|)^{m-|\alpha|}, \qquad \forall \eta \in \mathbb{R}^d.$$

$\Sigma^m_0(\mathbb{R}^d)$ denotes the subclass of symbols $a \in S^m_0(\mathbb{R}^d)$ satisfying

(1.2.1) $$\exists \varepsilon < 1 \ : \mathcal{F}_{x\to\xi}a(\xi,\eta) = 0 \quad \text{for } |\xi| > \varepsilon(1+|\eta|).$$

S^m_0 is equipped with seminorm $M^m_0(a;n)$ given by

(1.2.2) $$M^m_0(a;n) = \sup_{|\beta|\leq n}\sup_{\eta\in\mathbb{R}^2}\left\|(1+|\eta|)^{|\beta|-m}\partial_\eta^\beta a(\cdot,\eta)\right\|_{L^\infty}.$$

(ii) More generally, for $r \geq 0$, $S^m_r(\mathbb{R}^d)$ denotes the space of symbols $a \in S^m_0(\mathbb{R}^d)$ such that for all $\alpha \in \mathbb{N}^d$ and all $\eta \in \mathbb{R}^d$, function $x \to \partial_\eta^\alpha a(x,\eta)$ belongs to $W^{r,\infty}(\mathbb{R}^d)$ if $r \in \mathbb{N}$ (resp. to $C^r(\mathbb{R}^d)$ if $r \in]0,\infty[-\mathbb{N}$), and there exists a constant $C_\alpha > 0$ such that

$$\|\partial_\eta^\alpha a(\cdot,\eta)\|_{W^{r,\infty}} \leq C_\alpha(1+|\eta|)^{m-|\alpha|}, \qquad \forall\eta \in \mathbb{R}^d, \quad \text{if } r \in \mathbb{N},$$

(resp. $\|\partial_\eta^\alpha a(\cdot,\eta)\|_{C^r} \leq C_\alpha(1+|\eta|)^{m-|\alpha|}$, $\forall\eta \in \mathbb{R}^d$, if $r \in]0,\infty[-\mathbb{N}$). $\Sigma^m_r(\mathbb{R}^d)$ denotes the subclass of symbols $a \in S^m_r(\mathbb{R}^d)$ satisfying the spectral condition (1.2.1). S^m_r is equipped with seminorm $M^m_r(a;n)$, given by

(1.2.3) $$M^m_r(a;n) = \sup_{|\beta|\leq n}\sup_{\eta\in\mathbb{R}^2}\left\|(1+|\eta|)^{|\beta|-m}\partial_\eta^\beta a(\cdot,\eta)\right\|_{W^{r,\infty}}, \quad \text{if } r \in \mathbb{N},$$

(resp. $\|\cdot\|_{W^{r,\infty}}$ replaced with $\|\cdot\|_{C^r}$, if $r \in]0,\infty[-\mathbb{N}$).

These definitions extend to matrix valued symbols $a \in S^m_r$ ($a \in \Sigma^m_r$), $m \in \mathbb{R}$, $r \geq 0$. If $a \in S^m_r$ (resp. $a \in \Sigma^m_r$), it is said *of order m*.

Definition 1.2.4. An admissible cut-off function $\psi(\xi, \eta)$ is a C^∞ function on $\mathbb{R}^d \times \mathbb{R}^d$ such that

(i) there are $0 < \varepsilon_1 < \varepsilon_2 < 1$ and

$$\begin{cases} \psi(\xi,\eta) = 1, & \text{for } |\xi| \leq \varepsilon_1(1+|\eta|) \\ \psi(\xi,\eta) = 0, & \text{for } |\xi| \geq \varepsilon_2(1+|\eta|); \end{cases} \tag{1.2.4}$$

(ii) for all $(\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d$, there is a constant $C_{\alpha,\beta} > 0$ such that

$$|\partial^\alpha_\xi \partial^\beta_\eta \psi(\xi,\eta)| \leq C_{\alpha,\beta}(1+|\eta|)^{-|\alpha|-|\beta|}, \quad \forall (\xi,\eta). \tag{1.2.5}$$

Example: If χ is a smooth cut-off function, $\chi(z) = 1$ for $|z| \leq \varepsilon_1$ and supported in the open ball $B_{\varepsilon_2}(0)$, with $0 < \varepsilon_1 < \varepsilon_2 < 1$, function $\psi(\xi,\eta) := \chi\big(\frac{\xi}{\langle\eta\rangle}\big)$ is an admissible cut-off function in the sense of definition 1.2.4. We will consider this type of admissible cut-off functions for the rest of the paper.

Definition 1.2.5. Let χ be an admissible cut-off function and $a(x,\eta) \in S^m_r$, $m \in \mathbb{R}, r \geq 0$. The *Bony quantization* (or *paradifferential quantization*) $Op^B(a(x,\eta))$ associated to symbol a, and acting on a test function w, is defined as

$$Op^B(a(x,\eta))w(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\eta}\sigma^\chi_a(x,\eta)\hat{w}(\eta)d\eta\,,$$
$$\text{with } \sigma^\chi_a(x,\eta) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x-y)\cdot\zeta}\chi\left(\frac{\zeta}{\langle\eta\rangle}\right)a(y,\eta)dyd\zeta\,.$$

The operator defined above depends on the choice of the admissible cut-off function χ. However, if $a \in S^m_r$ for some $m \in \mathbb{R}, r \geq 0$, changing χ modifies $Op^B(a)$ only by the addition of a r-smoothing operator (i.e. an operator which is bounded from H^s to H^{s+r}, see [3]), so the choice of χ will be substantially irrelevant as long as we can neglect r-smoothing operators. For this reason, we will not indicate explicitly the dependence of Op^B (resp. of σ^χ_a) on χ to keep notations as light as possible. Let us also observe that, with such a definition, the Fourier transform of $Op^B(a)w$ has a simple expression

$$\mathcal{F}_{x\to\xi}\Big(Op^B(a(x,\eta))w(x)\Big)(\xi) = \frac{1}{(2\pi)^d}\int \chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)\hat{a}_y(\xi-\eta,\eta)\hat{w}(\eta)d\eta\,, \tag{1.2.6}$$

where $\hat{a}_y(\xi,\eta) := \mathcal{F}_{y\to\xi}\big(a(y,\eta)\big)$, and the product of two functions u, v can be developed as

$$uv = Op^B(u)v + Op^B(v)u + R(u,v), \tag{1.2.7}$$

where remainder $R(u,v)$ writes on the Fourier side as

$$\widehat{R(u,v)}(\xi) = \frac{1}{(2\pi)^d}\int \left(1 - \chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right) - \chi\left(\frac{\eta}{\langle\xi-\eta\rangle}\right)\right)\widehat{u}(\xi-\eta)\widehat{v}(\eta)d\eta. \tag{1.2.8}$$

We remark that in the above integral frequencies η and $\xi - \eta$ are either bounded or equivalent, and $R(u,v) = R(v,u)$. Just to conform notations for what will follow, we introduce the operator Op^B_R associated to a symbol $a(x,\eta)$, and acting on a function w, as

$$\begin{aligned} &Op^B_R(a(x,\eta))w(x) := \frac{1}{(2\pi)^d}\int e^{ix\cdot\eta}\delta^\chi_a(x,\eta)\hat{w}(\eta)d\eta\,, \\ &\text{with } \delta^\chi_a(x,\eta) := \frac{1}{(2\pi)^d}\int e^{i(x-y)\cdot\zeta}\left(1-\chi\left(\frac{\zeta}{\langle\eta\rangle}\right)-\chi\left(\frac{\eta}{\langle\zeta\rangle}\right)\right)a(y,\eta)dyd\zeta\,. \end{aligned} \tag{1.2.9}$$

For future references, we recall the definition of the Littlewood-Paley decomposition of a function w.

Definition 1.2.6 (Littlewood-Paley decomposition)**.** Let $\chi : \mathbb{R}^2 \to [0,1]$ be a smooth, decaying, radial function, supported for $|x| \le 2 - \frac{1}{10}$, and identically equal to 1 for $|x| \le 1 + \frac{1}{10}$. Let also $\varphi(\xi) := \chi(\xi) - \chi(2\xi) \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, supported for $\frac{1}{2} < |\xi| < 2$, and $\varphi_k(\xi) := \varphi(2^{-k}\xi)$, for all $k \in \mathbb{N}^*$, with the convention that $\varphi_0 := \chi$. Then $\sum_{k\in\mathbb{N}} \varphi(2^{-k}\xi) = 1$, and for any $w \in \mathcal{S}'(\mathbb{R}^d)$

$$w = \sum_{k\in\mathbb{N}} \varphi_k(D_x)w \tag{1.2.10}$$

is the Littlewood-Paley decomposition of w.

The following proposition is a classical result about the action of para-differential operators on Sobolev spaces (see [3] for further details). Proposition 1.2.8 shows, instead, that some results of continuity over L^2 hold also for operators whose symbol $a(x,\eta)$ is not a smooth function of η, and that map $(u,v) \mapsto R(u,v)$ is continuous from $H^{4,\infty} \times L^2$ to L^2.

Proposition 1.2.7 (Action)**.** *Let $m \in \mathbb{R}$. For all $s \in \mathbb{R}$, and all $a \in S_0^m$, $Op^B(a)$ is a bounded operator from $H^{s+m}(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$. In particular,*

$$\|Op^B(a)w\|_{H^s} \lesssim M_0^m\Big(a; \Big[\frac{d}{2}\Big] + 1\Big)\|w\|_{H^{s+m}}. \tag{1.2.11}$$

Proposition 1.2.8. *(i) Let $a(x,\eta) = a_1(x)b(\eta)$, with $a_1 \in L^\infty(\mathbb{R}^2)$ and $b(\eta)$ bounded, supported in some ball centred in the origin, and such that $|\partial^\alpha b(\eta)| \lesssim_\alpha |\eta|^{-|\alpha|+1}$ for any $\alpha \in \mathbb{N}^2$ with $|\alpha| \ge 1$. Then $Op^B(a(x,\eta)) : L^2 \to L^2$ is bounded and for any $w \in L^2(\mathbb{R}^2)$*

$$\|Op^B(a(x,\eta))w\|_{L^2} \lesssim \|a_1\|_{L^\infty}\|w\|_{L^2}.$$

The same is true for $Op_R^B(a(x,\eta))$;

(ii) Map $(u,v) \in H^{4,\infty} \times L^2 \mapsto R(u,v) \in L^2$ is well defined and continuous.

Proof. As concerns (i), we have that

$$Op^B(a(x,\eta))w(x) = \int K(x-z, x-y)a_1(y)w(z)dydz$$

with

$$K(x,y) := \frac{1}{(2\pi)^4}\int e^{ix\cdot\eta + iy\cdot\zeta}\chi\Big(\frac{\zeta}{\langle\eta\rangle}\Big)b(\eta)d\eta d\zeta,$$

where χ is an admissible cut-off function. After the hypothesis on b, we have that for every $\alpha, \beta \in \mathbb{N}^2$,

$$\Big|\partial_\zeta^\beta\Big[\chi\Big(\frac{\zeta}{\langle\eta\rangle}\Big)b(\eta)\Big]\Big| \lesssim \mathbb{1}_{\{|\eta|\lesssim 1\}}|g_\beta(\zeta)|,$$
$$\Big|\partial_\eta^\alpha\partial_\zeta^\beta\Big[\chi\Big(\frac{\zeta}{\langle\eta\rangle}\Big)b(\eta)\Big]\Big| \lesssim \mathbb{1}_{\{|\eta|\lesssim 1\}}|\eta|^{-|\alpha|+1}|g_\beta(\zeta)|, \qquad |\alpha| \ge 1,$$

where functions g_β are bounded and compactly supported. Lemma A.1 (i) and corollary A.2 (i) of appendix A imply that $|K(x,y)| \lesssim |x|^{-1}\langle x\rangle^{-2}\langle y\rangle^{-3}$ for any (x,y), and statement (i) follows by an inequality such as (A.4) with $L = L^2$.

In order to prove assertion (ii), we consider a cut-off function $\psi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 in some closed ball $\overline{B_C(0)}$, for a $C \gg 1$, and decompose $R(u,v)$ as follows, using (1.2.8):

$$R(u,v) = \int K_0(x-y,y-z)u(y)v(z)dydz + \int K_1(x-y,y-z)[\langle D_x\rangle^4 u](y)v(z)dydz,$$

with

$$K_0(x,y) = \frac{1}{(2\pi)^2}\int e^{ix\cdot\xi+iy\cdot\eta}\left(1-\chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)-\chi\left(\frac{\eta}{\langle\xi-\eta\rangle}\right)\right)\psi(\eta)d\xi d\eta,$$
$$K_1(x,y) = \frac{1}{(2\pi)^2}\int e^{ix\cdot\xi+iy\cdot\eta}\left(1-\chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)-\chi\left(\frac{\eta}{\langle\xi-\eta\rangle}\right)\right)(1-\psi)(\eta)\langle\xi-\eta\rangle^{-4}d\xi d\eta.$$

Since frequencies ξ,η are both bounded on the support of $\left(1-\chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)-\chi\left(\frac{\eta}{\langle\xi-\eta\rangle}\right)\right)\psi(\eta)$, one can show, through some integration by parts, that $|K_0(x,y)| \lesssim \langle x\rangle^{-3}\langle y\rangle^{-3}$ for any (x,y), and then deduce that

$$\left\|\int K_0(x-y,y-z)u(y)v(z)dydz\right\|_{L^2(dx)} \lesssim \|u\|_{L^\infty}\|v\|_{L^2}.$$

Kernel $K_1(x,y)$ can be split using a Littlewood-Paley decomposition,

$$K_1(x,y) = \sum_{k\geq 1}\frac{1}{(2\pi)^2}\underbrace{\int e^{ix\cdot\xi+iy\cdot\eta}\left(1-\chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)-\chi\left(\frac{\eta}{\langle\xi-\eta\rangle}\right)\right)(1-\psi)(\eta)\varphi(2^{-k}\eta)\langle\xi-\eta\rangle^{-4}d\xi d\eta}_{K_{1,k}(x,y)},$$

where $\varphi \in C_0^\infty(\mathbb{R}^2\setminus\{0\})$. On the support of $\left(1-\chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)-\chi\left(\frac{\eta}{\langle\xi-\eta\rangle}\right)\right)(1-\psi)(\eta)\varphi(2^{-k}\eta)$, frequencies $\eta,\xi-\eta$ are either bounded, or equivalent of size 2^k (which implies in particular that $\langle\xi-\eta\rangle^{-4}\lesssim\langle\xi\rangle^{-3}\langle\eta\rangle^{-1}$). After a change of coordinates and some integration by parts one can show that $|K_{1,k}(x,y)|\lesssim 2^k\langle x\rangle^{-3}\langle 2^k y\rangle^{-3}$, for any $k\geq 1$, and therefore that

$$\begin{aligned}
&\left\|\int e^{i(x-y)\cdot\xi+i(y-z)\cdot\eta}K_1(x-y,y-z)[\langle D_x\rangle^4 u](y)v(z)dydz\right\|_{L^2(dx)}\\
&\lesssim\sum_{k\geq1}2^k\left\|\int\langle x-y\rangle^{-3}\langle 2^k(y-z)\rangle^{-3}|\langle D_x\rangle^4u(y)||w(z)|dydz\right\|_{L^2(dx)}\\
&\lesssim\sum_{k\geq1}2^k\int\langle y\rangle^{-3}\langle 2^kz\rangle^{-3}\|[\langle D_x\rangle^4u](\cdot-y)w(\cdot-y-z)\|_{L^2(dx)}dydz\lesssim\|u\|_{H^{4,\infty}}\|w\|_{L^2},
\end{aligned}$$

which concludes the proof of statement (ii). □

Last results of this subsection are stated without proofs. All the details can be found in chapter 6 of [26] (see theorems 6.1.1, 6.1.4, 6.2.1, 6.2.4).

Proposition 1.2.9 (Composition). *Consider $a\in S^m_r$, $b\in S^{m'}_r$, $r>0$, $m,m'\in\mathbb{R}$.*

(i) Symbol $a\sharp b := \sum_{|\alpha|<r}\frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi)D_x^\alpha b(x,\xi)$ *is well defined in* $\sum_{j<r}S^{m+m'-j}_{r-j}$;

(ii) $Op^B(a)Op^B(b)-Op^B(a\sharp b)$ *is an operator of order* $\leq m+m'-r$*, and for all* $s\in\mathbb{R}$*, there exists a constant* $C>0$ *such that, for all* $a\in S^m_r(\mathbb{R}^d)$, $b\in S^{m'}_r(\mathbb{R}^d)$*, and* $w\in H^{s+m+m'-r}(\mathbb{R}^d)$,

$$\begin{aligned}
&\|Op^B(a)Op^B(b)w-Op^B(a\sharp b)w\|_{H^s}\\
&\qquad\leq C\big(M^m_r(a;n)M^{m'}_0(b;n_0)+M^m_0(a;n)M^{m'}_r(b;n_0)\big)\|w\|_{H^{s+m+m'-r}},
\end{aligned}$$

where $n_0 = \left[\frac{d}{2}\right] + 1$, $n = n_0 + [r]_+$ *(*$[r]_+$ *denoting the smallest integer* l, $l \geq r$*). Moreover,* $Op^B(a)Op^B(b) - Op^B(a\sharp b) = \widetilde{\sigma}_r(x, D_x)$ *with*

$$\begin{aligned}\widetilde{\sigma}_r(x,\xi) &= (\sigma_a \sharp \sigma_b)(x,\xi) - \sigma_{a\sharp b}(x,\xi) \\ &+ \sum_{|\alpha|=[r]_+} \frac{1}{[r]_+!(2\pi)^d} \int e^{i(x-y)\cdot\zeta} \left(\int_0^1 \partial_\xi^\alpha \sigma_a(x, \xi + t\zeta)(1-t)^{[r]_+-1} dt \right) \theta(\zeta,\xi) D_x^\alpha \sigma_b(y,\xi) dy d\zeta\end{aligned}$$

with $\theta \equiv 1$ *in a neighbourhood of the support of* $\mathcal{F}_{y\mapsto\eta}\sigma_b(\eta,\xi)$*.*

These results extend to matrix valued symbols and operators.

Remark 1.2.10. If symbol $a(x,\xi)$ only depends on ξ, then $\sigma_a \sharp \sigma_b - \sigma_{a\sharp b} = 0$ and $\widetilde{\sigma}_r$ reduces to the only integral term. Moreover,

$$\mathcal{F}_{x\mapsto\eta}\widetilde{\sigma}_r(\eta,\xi) = \sum_{|\alpha|=[r]_+} \frac{1}{[r]_+!} \left(\int_0^1 \partial_\xi^\alpha a(\xi + t\eta)(1-t)^{[r]_+-1} dt \right) \chi\Big(\frac{\eta}{\langle\xi\rangle}\Big) \eta^\alpha \hat{b}_y(\eta,\xi), \tag{1.2.12}$$

where $\chi\big(\frac{\eta}{\langle\xi\rangle}\big)$ is the admissible cut-off function defining σ_b.

Corollary 1.2.11. *For* $d = 2$ *and all* $s \in \mathbb{R}$*, there exists a constant* $C > 0$ *such that, for* $a \in S_r^m, b \in S_r^{m'}$*,* $r \geq 1$*, and* $w \in H^{s+m+m'-1}$*,*

$$\begin{aligned}&\|Op^B(a)Op^B(b)w - Op(ab)w\|_{H^s} \\ &\qquad\qquad \leq C\big(M_1^m(a;3)M_0^{m'}(b;2) + M_0^m(a;3)M_1^{m'}(b;2)\big)\|w\|_{H^{s+m+m'-1}}.\end{aligned}$$

Proposition 1.2.12 (Adjoint)**.** *Consider* $a \in S_r^m(\mathbb{R}^d)$*, and denote by* $Op^B(a)^*$ *the adjoint operator of* $Op^B(a)$*, by* $a^*(x,\xi) = \overline{a}(x,\xi)$ *the complex conjugate of* $a(x,\xi)$*.*

(i) Symbol $b(x,\xi) := \sum_{|\alpha|<r} \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha a^*(x,\xi)$ *is well defined in* $\sum_{j<r} S_{r-j}^{m-j}$*;*

(ii) Operator $Op^B(a)^* - Op^B(b)$ *is of order* $\leq m - r$*. Precisely, for all* $s \in \mathbb{R}$ *there is a constant* $C > 0$ *such that, for all* $a \in S_r^m(\mathbb{R}^d)$ *and* $w \in H^{s+m-r}(\mathbb{R}^d)$*,*

$$\big\|Op^B(a)^*w - Op^B(b)w\big\|_{H^s} \leq CM_r^m(a;n)\|w\|_{H^{s+m-r}},$$

with $n_0 = \left[\frac{d}{2}\right] + 1$*,* $n = n_0 + [r]_+$*.*

These results extend to matrix valued symbols a*, with* $a^*(x,\xi)$ *denoting the adjoint of matrix* $a(x,\xi)$*.*

Corollary 1.2.13. *For* $d = 2$ *and all* $s \in \mathbb{R}$*, there exists a constant* $C > 0$ *such that, for* $a \in S_r^m$*,* $r \geq 1$ *and* $w \in H^{s+m-1}$*,*

$$\|Op^B(a)^*w - Op(a^*)w\|_{H^s} \leq CM_1^m(a;3)\|w\|_{H^{s+m-1}}.$$

1.2.2 Semi-classical Pseudodifferential Calculus

In this subsection we recall some definitions and results about semi-classical symbolic calculus in general space dimension $d \geq 1$, which will be used in section 3.2. We refer the reader to [8], [31] for more details.

Definition 1.2.14. An order function on $\mathbb{R}^d \times \mathbb{R}^d$ is a smooth map from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}_+$: $(x,\xi) \to M(x,\xi)$ such that there exist $N_0 \in \mathbb{N}$, $C > 0$ and for any $(x,\xi),(y,\eta) \in \mathbb{R}^d \times \mathbb{R}^d$

$$M(y,\eta) \leq C\langle x-y\rangle^{N_0}\langle \xi-\eta\rangle^{N_0} M(x,\xi)\,, \tag{1.2.13}$$

where $\langle x\rangle = \sqrt{1+|x|^2}$.

Definition 1.2.15. Let M be an order function on $\mathbb{R}^d \times \mathbb{R}^d$, $\delta \geq 0$, $\sigma \geq 0$. One denotes by $S_{\delta,\sigma}(M)$ the space of smooth functions

$$\begin{gathered}(x,\xi,h) \to a(x,\xi,h)\\ \mathbb{R}^d \times \mathbb{R}^d \times]0,1] \to \mathbb{C}\end{gathered}$$

satisfying for any $\alpha_1, \alpha_2 \in \mathbb{N}^d, k, N \in \mathbb{N}$

$$|\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}(h\partial_h)^k a(x,\xi,h)| \lesssim M(x,\xi)\, h^{-\delta(|\alpha_1|+|\alpha_2|)}(1+\sigma h^\sigma|\xi|)^{-N}\,. \tag{1.2.14}$$

A key role in this paper will be played by symbols a verifying (1.2.14) with $M(x,\xi) = \langle \frac{x+f(\xi)}{\sqrt{h}}\rangle^{-N}$, for $N \in \mathbb{N}$ and a certain smooth function $f(\xi)$. This function M is no longer an order function because of the term $h^{-\frac{1}{2}}$, but nevertheless we continue writing $a \in S_{\delta,\sigma}(\langle \frac{x+f(\xi)}{\sqrt{h}}\rangle^{-N})$.

Definition 1.2.16. In the semi-classical setting, we say that $a(x,\xi,h)$ is a symbol *of order* r if $a \in S_{\delta,\sigma}(\langle\xi\rangle^r)$, for some $\delta \geq 0$, $\sigma \geq 0$.

Let us observe that when $\sigma > 0$ the symbol decays rapidly in $h^\sigma|\xi|$, which implies the following inclusion for $r \geq 0$:

$$S_{\delta,\sigma}(\langle\xi\rangle^r) \subset h^{-\sigma r} S_{\delta,\sigma}(1).$$

This means that, up to a small loss in h, this type of symbols can be always considered as symbols of order zero. In the rest of the paper we will not indicate explicitly the dependence of symbols on h, referring to $a(x,\xi,h)$ simply as $a(x,\xi)$.

Definition 1.2.17. Let $a \in S_{\delta,\sigma}(M)$ for some order function M, some $\delta \geq 0$, $\sigma \geq 0$.

(i) We can define the *Weyl quantization* of a to be the operator $Op_h^w(a) = a^w(x,hD)$ acting on $u \in \mathcal{S}(\mathbb{R}^d)$ by the following formula:

$$Op_h^w(a(x,\xi))u(x) = \frac{1}{(2\pi h)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{i}{h}(x-y)\cdot\xi} a\Big(\frac{x+y}{2},\xi\Big)\, u(y)\, dy d\xi\,;$$

(ii) We define also the *standard quantization* of a:

$$Op_h(a(x,\xi))u(x) = \frac{1}{(2\pi h)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{i}{h}(x-y)\cdot\xi} a(x,\xi)\, u(y)\, dy d\xi\,.$$

It is clear from the definition that the two quantizations coincide when the symbol does not depend on x. We also introduce a semi-classical version of Sobolev spaces, on which the above operators act naturally.

Definition 1.2.18. (i) Let $\rho \in \mathbb{N}$. We define the semi-classical Sobolev space $H_h^{\rho,\infty}(\mathbb{R}^d)$ as the space of tempered distributions w such that $\langle hD\rangle^\rho w := Op_h(\langle\xi\rangle^\rho)w \in L^\infty$, endowed with norm

$$\|w\|_{H_h^{\rho,\infty}} = \|\langle hD\rangle^\rho w\|_{L^\infty};$$

(ii) Let $s \in \mathbb{R}$. We define the semi-classical Sobolev space $H^s_h(\mathbb{R}^d)$ as the space of tempered distributions w such that $\langle hD\rangle^s w := Op_h(\langle\xi\rangle^s)w \in L^2$, endowed with norm

$$\|w\|_{H^s} = \|\langle hD\rangle^s w\|_{L^2}.$$

For future references, we write down the semi-classical Sobolev injection in space dimension 2:

$$\|v_h\|_{H^{\rho,\infty}_h(\mathbb{R}^2)} \lesssim_\sigma h^{-1}\|v_h\|_{H^{\rho+1+\sigma}_h(\mathbb{R}^2)}\,, \qquad \forall\sigma > 0\,. \tag{1.2.15}$$

The following two propositions are stated without proof. They concern the adjoint and the composition of pseudo-differential operators, and all related details are provided in chapter 7 of [8], or in chapter 4 of [31].

Proposition 1.2.19 (Self-Adjointness)**.** *If $a(x,\xi)$ is a real symbol, its Weyl quantization is self-adjoint,*

$$\big(Op^w_h(a)\big)^* = Op^w_h(a)\,.$$

Proposition 1.2.20 (Composition for Weyl quantization)**.** *Let $a, b \in \mathcal{S}(\mathbb{R}^d)$. Then*

$$Op^w_h(a) \circ Op^w_h(b) = Op^w_h(a\sharp b)\,,$$

where

$$a\sharp b\,(x,\xi) := \frac{1}{(\pi h)^{2d}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{2i}{h}\sigma(y,\eta;\,z,\zeta)} a(x+z,\xi+\zeta)b(x+y,\xi+\eta)\,dyd\eta dz d\zeta, \tag{1.2.16}$$

and

$$\sigma(y,\eta;\,z,\zeta) = \eta\cdot z - y\cdot\zeta\,.$$

It is often useful to derive an asymptotic expansion for $a\sharp b$, as it allows easier computations than integral formula (1.2.16). This expansion is usually obtained by applying the stationary phase argument when $a, b \in S_{\delta,\sigma}(M)$, $\delta \in [0, \frac{1}{2}[$ (as shown in [31]). Here we provide an expansion at any order even when one of two symbols belongs to $S_{\frac{1}{2},\sigma_1}(M)$ (still having the other in $S_{\delta,\sigma_2}(M)$ for $\delta < \frac{1}{2}$, and σ_1, σ_2 either equal or, if not, one of them equal to zero), whose proof is based on the Taylor development of symbols a, b, and can be found in the appendix of [28] (for $d = 1$).

Proposition 1.2.21. *Let M_1, M_2 be two order functions and $a \in S_{\delta_1,\sigma_1}(M_1)$, $b \in S_{\delta_2,\sigma_2}(M_2)$, $\delta_1, \delta_2 \in [0, \frac{1}{2}]$, $\delta_1 + \delta_2 < 1$, $\sigma_1, \sigma_2 \geq 0$ such that*

$$\sigma_1 = \sigma_2 \geq 0 \qquad or \qquad \big[\sigma_1 \neq \sigma_2 \text{ and } \sigma_i = 0\,, \sigma_j > 0\,, i \neq j \in \{1,2\}\big]\,. \tag{1.2.17}$$

Then $a\sharp b \in S_{\delta,\sigma}(M_1M_2)$, where $\delta = \max\{\delta_1,\delta_2\}$, $\sigma = \max\{\sigma_1,\sigma_2\}$. Moreover,

$$a\sharp b = \sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\|\alpha|=0,\dots,N-1}} \frac{(-1)^{|\alpha_1|}}{\alpha!}\Big(\frac{h}{2i}\Big)^{|\alpha|}\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}a(x,\xi)\ \partial_x^{\alpha_2}\partial_\xi^{\alpha_1}b(x,\xi) + r_N\,, \tag{1.2.18}$$

where $r_N \in h^{N(1-(\delta_1+\delta_2))}S_{\delta,\sigma}(M_1M_2)$ and

$$\begin{aligned} r_N(x,\xi) = {} & \Big(\frac{h}{2i}\Big)^N \frac{N}{(\pi h)^{2d}}\sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\|\alpha|=N}}\frac{(-1)^{|\alpha_1|}}{\alpha!}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)} \\ & \times\Big(\int_0^1 \partial_x^{\alpha_1}\partial_\xi^{\alpha_2}a(x+tz,\xi+t\zeta)(1-t)^{N-1}dt\Big)\partial_x^{\alpha_2}\partial_\xi^{\alpha_1}b(x+y,\xi+\eta)\,dyd\eta dz d\zeta\,, \end{aligned} \tag{1.2.19}$$

or

$$(1.2.20)\quad r_N(x,\xi)=\Big(\frac{h}{2i}\Big)^N\frac{N}{(\pi h)^{2d}}\sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\|\alpha|=N}}\frac{(-1)^{|\alpha_1|}}{\alpha!}\int_{\mathbb{R}^4}e^{\frac{2i}{h}(\eta\cdot z-y\cdot\zeta)}\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}a(x+z,\xi+\zeta)$$
$$\times\Big(\int_0^1\partial_x^{\alpha_2}\partial_\xi^{\alpha_1}b(x+ty,\xi+t\eta)(1-t)^{N-1}dt\Big)\,dyd\eta dzd\zeta\,.$$

More generally, if $h^{N\delta_1}\partial^\alpha a\in S_{\delta_1,\sigma_1}(M_1^N)$, $h^{N\delta_2}\partial^\alpha b\in S_{\delta_2,\sigma_2}(M_2^N)$, *for* $|\alpha|=N$, *some order functions* M_1^N,M_2^N, *then* $r_N\in h^{N(1-(\delta_1+\delta_2))}S_{\delta,\sigma}(M_1^NM_2^N)$.

Remark 1.2.22. From previous proposition it follows that, if symbols $a\in S_{\delta_1,\sigma_1}(M_1)$, $b\in S_{\delta_2,\sigma_2}(M_2)$ are such that supp$a\cap$supp$b=\emptyset$, then $a\sharp b=O(h^\infty)$, meaning that, for every $N\in\mathbb{N}$, $a\sharp b=r_N$ with $r_N\in h^{N(1-(\delta_1+\delta_2))}S_{\delta,\sigma}(M_1M_2)$.

Remark 1.2.23. We draw reader's attention to the fact that symbol $\sharp$ is used simultaneously in Bony calculus (see proposition 1.2.9) and in Weyl semi-classical calculus (as in (1.2.18)) with two different meaning. However, we avoid to introduce different notations, as it will be clear by the context if we are dealing with the former or the latter one.

The result of proposition 1.2.21 and remark 1.2.22 are still true even when one of the two order functions, or both, has the form $\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-1}$, for a smooth function $f(\xi)$, $\nabla f(\xi)$ bounded, as stated below (see the appendix of [28]).

Lemma 1.2.24. *Let* $f(\xi):\mathbb{R}^d\to\mathbb{R}$ *be a smooth function, with* $|\nabla f(\xi)|$ *bounded. Consider* $a\in S_{\delta_1,\sigma_1}(\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-m})$, $m\in\mathbb{N}$, *and* $b\in S_{\delta_2,\sigma_2}(M)$, *for* M *order function or* $M(x,\xi)=\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-n}$, $n\in\mathbb{N}$, *some* $\delta_1\in[0,\frac{1}{2}]$, $\delta_2\in[0,\frac{1}{2}[$, $\sigma_1,\sigma_2\ge 0$ *as in* (1.2.17). *Then* $a\sharp b\in S_{\delta,\sigma}(\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-m}M)$, *where* $\delta=\max\{\delta_1,\delta_2\}$, $\sigma=\max\{\sigma_1,\sigma_2\}$, *and the asymptotic expansion* (1.2.18) *holds, with* $r_N\in h^{N(1-(\delta_1+\delta_2))}S_{\delta,\sigma}(\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-m}M)$ *given by* (1.2.19) *(or equivalently* (1.2.20)*).*
More generally, if $h^{N\delta_1}\partial^\alpha a\in S_{\delta_1,\sigma_1}(\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-m'})$ *and* $h^{N\delta_2}\partial^\alpha b\in S_{\delta_2,\sigma_2}(M^N)$, $|\alpha|=N$, M^N *order function or* $M^N(x,\xi)=\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-n'}$, *for some* $m',n'\in\mathbb{N}$, *then remainder* r_N *belongs to* $h^{N(1-(\delta_1+\delta_2))}S_{\delta,\sigma}(\langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-m'}M^N)$.

1.2.3 Semi-classical Operators for the Wave Solution: Some Estimates

From now on, we place ourselves in space dimension $d=2$. This technical subsection focuses on the introduction and the analysis of some particular operators that we will use when dealing with the wave component in the semi-classical framework (subsection 3.2.2). Lemma 1.2.25 will be often recalled when we want to prove that some operator belongs to $\mathcal{L}(L^2;L^\infty)$ and compute its norm; in propositions 1.2.27, 1.2.30 we prove the continuity of some important operators (e.g $\Gamma^{w,k}$ defined in (3.2.43)), while propositions 1.2.28, 1.2.31 are devoted to prove the continuity of some other operators often arising when we consider the quantization of symbolic integral remainders. Finally, lemmas 1.2.33 and 1.2.35 deal with the development of some special symbolic products: while 1.2.33 will be used several times throughout the paper, lemma 1.2.35 is stated explicitly on purpose to prove lemma 3.2.13.

Lemma 1.2.25. *There exists a constant* $C>0$ *such that, for any function* $A(x,\xi)$ *with* $\partial_x^\alpha\partial_\xi^\beta A\in L^2$ *for* $|\alpha|,|\beta|\le 3$, *and any function* $w\in L^2$,

$$(1.2.21)\quad \big|Op_h^w(A(x,\xi))w(x)\big|\le C\|w\|_{L^2}\int_{\mathbb{R}^2}\langle x-y\rangle^{-3}\sum_{|\alpha|,|\beta|\le 3}\Big\|\partial_y^\alpha\partial_\xi^\beta\Big[A\Big(\frac{x+y}{2},h\xi\Big)\Big]\Big\|_{L^2(d\xi)}dy.$$

Moreover, if $A(x,\xi)$ is compactly supported in x there exists a smooth function, supported in a neighbourhood of suppA, such that

$$|Op^w_h(A(x,\xi))w(x)| \le C\|w\|_{L^2}\int_{\mathbb{R}^2}\Big|\theta'\Big(\frac{x+y}{2}\Big)\Big|\sum_{|\alpha|\le 3}\Big\|\partial^\alpha_y\Big[A(\frac{x+y}{2},h\xi)\Big]\Big\|_{L^2(d\xi)}dy. \tag{1.2.22}$$

Proof. Let us prove the statement for $A\in\mathcal{S}(\mathbb{R}^2\times\mathbb{R}^2)$, $w\in\mathcal{S}(\mathbb{R}^2)$. The density of $\mathcal{S}(\mathbb{R}^2\times\mathbb{R}^2)$ into $\{A\in L^2(\mathbb{R}^2\times\mathbb{R}^2)|\partial^\alpha_x\partial^\beta_\xi A\in L^2(\mathbb{R}^2\times\mathbb{R}^2), |\alpha|,|\beta|\le 3\}$, and of $\mathcal{S}(\mathbb{R}^2)$ into L^2, will then justify the definition of $Op^w_h(A(x,\xi))w$ for A,w as in the statement, together with inequalities (1.2.21), (1.2.22).

Using integration by parts, Cauchy-Schwarz inequality, and YoungâĂŹs inequality for convolutions, we can write the following:

$$\begin{aligned}
|Op^w_h(A(x,\xi))w(x)| &= \frac{1}{(2\pi)^2}\Big|\int_{\mathbb{R}^4}e^{i(x-y)\cdot\xi}A\Big(\frac{x+y}{2},h\xi\Big)w(y)\,dyd\xi\Big|\\
&= \frac{1}{(2\pi)^4}\Big|\int_{\mathbb{R}^2}\widehat{w}(\eta)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{i(x-y)\cdot\xi+iy\cdot\eta}A\Big(\frac{x+y}{2},h\xi\Big)\,dyd\xi\,d\eta\Big|\\
&= \frac{1}{(2\pi)^4}\Big|\int_{\mathbb{R}^2}\widehat{w}(\eta)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Big(\frac{1-i(x-y)\cdot\partial_\xi}{1+|x-y|^2}\Big)^3\Big(\frac{1+i(\xi-\eta)\cdot\partial_y}{1+|\xi-\eta|^2}\Big)^3e^{i(x-y)\cdot\xi+iy\cdot\eta}\\
&\qquad\times A\Big(\frac{x+y}{2},h\xi\Big)\,dyd\xi\,d\eta\Big|\\
&\lesssim \int_{\mathbb{R}^2}|\hat{w}(\eta)|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\langle x-y\rangle^{-3}\langle\xi-\eta\rangle^{-3}\sum_{|\alpha|,|\beta|\le 3}\Big|\partial^\alpha_y\partial^\beta_\xi\Big[A\Big(\frac{x+y}{2},h\xi\Big)\Big]\Big|dyd\xi\,d\eta\\
&\lesssim \|\hat{w}\|_{L^2(d\eta)}\|\langle\eta\rangle^{-3}\|_{L^1(d\eta)}\int_{\mathbb{R}^2}\langle x-y\rangle^{-3}\sum_{|\alpha|,|\beta|\le 3}\Big\|\partial^\alpha_y\partial^\beta_\xi\Big[A(\frac{x+y}{2},h\xi)\Big]\Big\|_{L^2(d\xi)}dy\\
&\lesssim \|w\|_{L^2}\int_{\mathbb{R}^2}\langle x-y\rangle^{-3}\sum_{|\alpha|,|\beta|\le 3}\Big\|\partial^\alpha_y\partial^\beta_\xi\Big[A\Big(\frac{x+y}{2},h\xi\Big)\Big]\Big\|_{L^2(d\xi)}dy\,.
\end{aligned}$$

If symbol $A(x,\xi)$ is compactly supported in x we can consider a smooth function $\theta'\in C^\infty_0(\mathbb{R})$, identically equal to 1 on the support of $A(x,\xi)$, and write

$$\begin{aligned}
|Op^w_h(A(x,\xi))w(x)| &= \frac{1}{(2\pi)^2}\Big|\int_{\mathbb{R}^2}\widehat{w}(\eta)d\eta\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Big(\frac{1+i(\xi-\eta)\cdot\partial_y}{1+|\xi-\eta|^2}\Big)^3e^{i(x-y)\cdot\xi+iy\cdot\eta}\\
&\qquad\times A\Big(\frac{x+y}{2},h\xi\Big)\,dyd\xi\Big|\\
&\lesssim \int_{\mathbb{R}^2}|\hat{w}(\eta)|\,d\eta\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|\theta'(\frac{x+y}{2})|\langle\xi-\eta\rangle^{-3}\sum_{|\alpha|\le 3}\Big|\partial^\alpha_y\Big[A\Big(\frac{x+y}{2},h\xi\Big)\Big]\Big|dyd\xi\\
&\lesssim \|w\|_{L^2}\int_{\mathbb{R}^2}\Big|\theta'\Big(\frac{x+y}{2}\Big)\Big|\sum_{|\alpha|\le 3}\Big\|\partial^\alpha_y\Big[A\Big(\frac{x+y}{2},h\xi\Big)\Big]\Big\|_{L^2(d\xi)}dy\,.
\end{aligned}$$

□

A very important role in this subsection and in subsection 3.2.2 will be played by function $\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\psi(2^{-k}\xi)$ (or by functions strictly related to it) and its quantization, where $\gamma\in C^\infty(\mathbb{R}^2)$ is such that $|\partial^\alpha\gamma(z)|\lesssim\langle z\rangle^{-|\alpha|}$, $\psi\in C^\infty_0(\mathbb{R}^2-\{0\})$, $\sigma>0$ is a small fixed constant, and k is an integer belonging to set K, defined as

$$K:=\{k\in\mathbb{Z}\ :\ h\lesssim 2^k\lesssim h^{-\sigma}\}.$$

In various results (e.g. proposition 1.2.30), instead of having such a γ, we will need a more decaying smooth function, denoted by γ_1, such that $|\partial^\alpha \gamma_1(z)| \lesssim \langle z\rangle^{-(1+|\alpha|)}$. We introduce here some notations we will keep throughout the whole paper:

Notation 1. For any $n \in \mathbb{N}$, γ_n denotes a smooth function in $\mathbb{R}^2$ such that $|\partial^\alpha \gamma_n(z)| \lesssim_\alpha \langle z\rangle^{-(n+|\alpha|)}$, for any $\alpha \in \mathbb{N}^2$. We use the simple notation γ instead of γ_0;

Notation 2. For any integer $m \in \mathbb{Z}$, $b_m(\xi)$ will denote any function satisfying $|\partial^\beta b_m(\xi)| \lesssim_\beta |\xi|^{m-|\beta|}$, for any ξ in its domain, any $\beta \in \mathbb{N}^2$.

The following lemma is a useful reference when we need to deal with some derivatives of $\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)$.

Lemma 1.2.26. *Let us take $\sigma > 0$ sufficiently small and $n \in \mathbb{N}$. For any multi-indices $\alpha, \beta \in \mathbb{N}^2$ we have that*

$$(1.2.23)\qquad \partial^\alpha_x \partial^\beta_\xi \Big[\gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] = \sum_{k=0}^{|\beta|} h^{-(|\alpha|+k)(\frac{1}{2}-\sigma)} \gamma_{n+|\alpha|+k}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big) b_{|\alpha|+k-|\beta|}(\xi).$$

Furthermore, if $\theta = \theta(x) \in C^\infty_0(\mathbb{R}^2)$, there exists a set $\{\theta_k(x)\}_{1\le k\le|\beta|}$ of smooth compactly supported functions such that

$$(1.2.24)\qquad \theta(x)\partial^\alpha_x \partial^\beta_\xi \Big[\gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] = \sum_{k=1}^{|\beta|} h^{-(|\alpha|+k)(\frac{1}{2}-\sigma)} \gamma_{n+|\alpha|+k}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big) \theta_k(x) b_{|\alpha|+k-|\beta|}(\xi).$$

Proof. Let δ^j_i be equal to 1 if $i = j$, 0 otherwise, and $\sum'$ be a concise notation to indicate a linear combination. For $i = 1, 2$,

(1.2.25)

$$\partial_{\xi_i}\Big[\gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] = h^{-(\frac{1}{2}-\sigma)} \sum_{j=1}^2 (\partial_j\gamma_n)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)(x_j\xi_i|\xi|^{-1} - \delta^j_i)$$
$$= \sum_{j=1}^2 (\partial_j\gamma_n)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big(\frac{x_j|\xi|-\xi_j}{h^{1/2-\sigma}}\Big)\,\xi_i|\xi|^{-2} + \sum_{j=1}^2 h^{-(\frac{1}{2}-\sigma)}(\partial_j\gamma_n)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)[\xi_i\xi_j|\xi|^{-2} - \delta^j_i],$$

which can be summarized saying that

$$\partial_{\xi_i}\Big[\gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] = \sum\nolimits' \gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big) b_{-1}(\xi) + h^{-(\frac{1}{2}-\sigma)}\gamma_{n+1}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big) b_0(\xi),$$

for some new functions $\gamma_n, \gamma_{n+1}, b_0, b_{-1}$. Iterating this argument, one finds that, for all $\beta \in \mathbb{N}^2$,

$$\partial^\beta_\xi\Big[\gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] = \sum_{k=0,\dots,|\beta|}{}' h^{-k(\frac{1}{2}-\sigma)}\gamma_{n+k}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big) b_{k-|\beta|}(\xi),$$

and obtains (1.2.23) using that, for any $m \in \mathbb{N}, \alpha \in \mathbb{N}^2$,

$$(1.2.26)\qquad \partial^\alpha_x\Big[\gamma_m\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] = h^{-|\alpha|(\frac{1}{2}-\sigma)}(\partial^\alpha\gamma_m)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)|\xi|^{|\alpha|}.$$

Equality (1.2.24) is obtained replacing (1.2.25) with

$$\theta(x)\partial_{\xi_i}\Big[\gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] = h^{-(\frac{1}{2}-\sigma)} \sum_{j=1}^2 (\partial_j\gamma_n)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)(\theta(x)x_j\xi_i|\xi|^{-1} - \theta(x)\delta^j_i),$$

which means that

$$\theta(x)\partial_{\xi_i}\Big[\gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] = \sum{}' h^{-(\frac{1}{2}-\sigma)}\gamma_{n+1}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\theta_1(x)b_0(\xi),$$

where $\theta_1(x)$ is a new compactly supported function. By iteration one finds that, for any $\beta\in\mathbb{N}^2$, there is a set of $|\beta|$ compactly supported functions $\theta_k(x)$, $1\le k\le|\beta|$, such that

$$\theta(x)\partial^\beta_\xi\Big[\gamma_n\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big] = \sum_{k=1}^{|\beta|} h^{-k(\frac{1}{2}-\sigma)}\gamma_{n+k}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\theta_k(x)b_{k-|\beta|}(\xi),$$

which combined with (1.2.26) gives (1.2.24). □

Proposition 1.2.27 (Continuity on L^2)**.** *Let $\sigma>0$ be sufficiently small, $k\in K$ and $p\in\mathbb{Z}$. Let also $\psi\in C^\infty_0(\mathbb{R}^2\setminus\{0\})$ and $a(x)$ be a smooth function such that $|\partial^\alpha a(x)|\lesssim 1$. Then $Op^w_h\big(\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\psi(2^{-k}\xi)a(x)b_p(\xi)\big):L^2\to L^2$ is bounded and*

$$\Big\|Op^w_h\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)\Big\|_{\mathcal{L}(L^2)}\lesssim 2^{kp}. \tag{1.2.27}$$

Proof. Fix $k\in K$ such that $h\le 2^k\le h^{1/2-\sigma}$. If Θ_h,Θ^{-1}_h denote the operators such that $\Theta_h v(x):=v(\sqrt{h}x)$, $\Theta^{-1}_h w(x):=w(\frac{x}{\sqrt{h}})$, for any $h\in]0,1]$, then for any function $A(x,\xi)$

$$Op^w_h(A(x,\xi))v(x)=\big[\Theta_h Op^w_h(\widetilde{A}(x,\xi))\Theta^{-1}_h\big]v(x),$$

with $\widetilde{A}(x,\xi):=A(\frac{x}{\sqrt{h}},\sqrt{h}\xi)$. If $A(x,\xi)=\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\psi(2^{-k}\xi)a(x)b_p(\xi)$, then one can check, using lemma 1.2.26, that $\widetilde{A}\in 2^{kp}S_{\frac{1}{2},0}(1)$, so theorem 7.11 of [8] implies that $Op^w_h(\widetilde{A}(x,\xi)):L^2\to L^2$ is bounded with norm $O(2^{kp})$. For indices $k\in K$ such that $h^{1/2-\sigma}\le 2^k\le h^{-\sigma}$, the very symbol $A(x,\xi)$ belongs to $2^{kp}S_{\frac{1}{2},0}(1)$, and the statement follows again by theorem 7.11 of [8]. □

Proposition 1.2.28. *Let us take $\sigma>0$ sufficiently small, $k\in K$ and $p,q\in\mathbb{Z}$. Let us also consider $\psi,\widetilde{\psi}\in C^\infty_0(\mathbb{R}^2\setminus\{0\})$, $a(x),a'(x)$ smooth functions such that $|\partial^\alpha_x a|+|\partial^\alpha_x a'|\lesssim 1$, and $f\in C(\mathbb{R})$. Define*

(1.2.28)

$$\begin{aligned}I^k_{p,q}(x,\xi):=\frac{1}{(\pi h)^4}\int e^{\frac{2i}{h}(\eta\cdot z-y\cdot\zeta)}\Big[\int_0^1\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)|_{(x+tz,\xi+t\zeta)}f(t)dt\\ \times\widetilde{\psi}(2^{-k}(\xi+\eta))a'(x+y)b_q(\xi+\eta)\Big]\,dydzd\eta d\zeta.\end{aligned}$$

Then $Op^w_h(I^k_{p,q}(x,\xi)):L^2\to L^2$ is bounded and $\|Op^w_h(I^k_{p,q}(x,\xi))\|_{\mathcal{L}(L^2)}\lesssim 2^{k(p+q)}$. The same results holds for

$$\begin{aligned}J^k_{p,q}(x,\xi):=\frac{1}{(\pi h)^4}\int e^{\frac{2i}{h}(\eta\cdot z-y\cdot\zeta)}\Big[\int_0^1\widetilde{\psi}(2^{-k}(\xi+t\zeta))a'(x+tz)b_q(\xi+t\zeta)f(t)dt\\ \times\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)|_{(x+y,\xi+\eta)}\Big]\,dydzd\eta d\zeta.\end{aligned} \tag{1.2.29}$$

Proof. We show the result for operator $Op^w_h(I^k_{p,q})$, leaving the reader to check that a similar argument can be used for $Op^w_h(J^k_{p,q})$.

We distinguish between two ranges of frequencies. For indices $k \in K$ such that $h^{1/2-\sigma} \le 2^k \le h^{-\sigma}$, we observe that $I^k_{p,q}(x,\xi) \in 2^{k(p+q)}S_{\frac{1}{2},0}(1)$. Indeed, it follows from lemma 1.2.26 that $\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\psi(2^{-k}\xi)a(x)b_p(\xi) \in 2^{kp}S_{\frac{1}{2},\sigma}(1)$, while $\widetilde{\psi}(2^{-k}\xi)a'(x)b_q(\xi) \in 2^{kq}S_{\frac{1}{2}-\sigma,\sigma}(1)$, hence, performing a change of variables $y \mapsto \sqrt{h}y$, $z \mapsto \sqrt{h}z$, $\eta \mapsto \sqrt{h}\eta$, $\zeta \mapsto \sqrt{h}\zeta$, writing
(1.2.30)

$$e^{2i(\eta\cdot z-y\cdot\zeta)} = \Big(\frac{1+2iy\cdot\partial_\zeta}{1+4|y|^2}\Big)^3\Big(\frac{1-2iz\cdot\partial_\eta}{1+4|z|^2}\Big)^3\Big(\frac{1-2i\eta\cdot\partial_z}{1+4|\eta|^2}\Big)^3\Big(\frac{1+2i\zeta\cdot\partial_y}{1+4|\zeta|^2}\Big)^3 e^{2i(\eta\cdot z-y\cdot\zeta)},$$

and integrating by parts in all variables, we get that

$$\Big|I^k_{p,q}(x,\xi)\Big| \lesssim 2^{k(p+q)}\int \langle y\rangle^{-3}\langle z\rangle^{-3}\langle \eta\rangle^{-3}\langle \zeta\rangle^{-3}\, dydzd\eta d\zeta \lesssim 2^{k(p+q)},$$

without any loss in $h^{-\delta}$, due to the fact that we are considering symbols $A(x,\xi) \in S_{\delta,\sigma}(1)$, with $\delta = 0, 1/2-\sigma, 1/2$, and derivating $A(x+\sqrt{h}y, \xi+\sqrt{h}\eta)$ (or $A(x+t\sqrt{h}z, \xi+t\sqrt{h}\zeta)$) with respect to y and η (resp. with respect to z and ζ). In a similar way, one can also prove that $|\partial^\alpha_x\partial^\beta_\xi I^k_{p,q}(x,\xi)| \lesssim_{\alpha,\beta} h^{-\frac{1}{2}(|\alpha|+|\beta|)}2^{k(p+q)}$. Theorem 7.11 of [8] implies then the statement for this case.

When indices $k \in K$ are such that $h \le 2^k \le h^{1/2-\sigma}$, we consider operators Θ_h, Θ^{-1}_h defined as $\Theta_h v(x) := v(\sqrt{h}x)$, $\Theta^{-1}_h w(x) := w(\frac{x}{\sqrt{h}})$, and replace $Op^w_h(I^k_{p,q}(x,\xi))$ with $\Theta_h Op^w_h(\widetilde{I}^k_{p,q}(x,\xi))\Theta^{-1}_h$, where $\widetilde{I}^k_{p,q}(x,\xi) := I^k_{p,q}(\frac{x}{\sqrt{h}}, \sqrt{h}\xi)$. Since $\gamma\big(\frac{x|\xi|}{h^{1/2-\sigma}} - h^\sigma\xi\big)\psi(2^{-k}\sqrt{h}\xi)a(\frac{x}{\sqrt{h}})b_p(\sqrt{h}\xi) \in 2^{kp}S_{\frac{1}{2},\sigma}(1)$ and $\widetilde{\psi}(2^{-k}\sqrt{h}\xi)a'(\frac{x}{\sqrt{h}})b_q(\sqrt{h}\xi) \in 2^{kq}S_{\frac{1}{2},\sigma}(1)$, we deduce that $\widetilde{I}^k_{p,q} \in 2^{k(p+q)}S_{\frac{1}{2},0}(1)$. Theorem 7.11 of [8] implies that $Op^w_h(\widetilde{I}^k_{p,q}) : L^2 \to L^2$ is bounded, uniformly in h, and so is for $Op^w_h(I^k_{p,q})$. □

Proposition 1.2.29 (Continuity on L^p). *Let $1 \le p \le +\infty$, $\gamma \in C^\infty_0(\mathbb{R}^2)$ be radial, $\psi \in C^\infty_0(\mathbb{R}^2 \setminus \{0\})$, $a(x)$ be a smooth function such that $|\partial^\alpha a(x)| \lesssim 1$. Let also $\sigma > 0$ be small, $k \in K = \{k \in \mathbb{Z} : h \lesssim 2^k \lesssim h^{-\sigma}\}$ and $q \in \mathbb{Z}$. Then $Op^w_h\big(\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\psi(2^{-k}\xi)a(x)b_q(\xi)\big) : L^p \to L^p$ is a bounded operator such that*

$$\Big\|Op^w_h\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_q(\xi)\Big)\Big\|_{\mathcal{L}(L^p)} \lesssim 2^{kq}.$$

Proof. In order to prove the result of the statement, we need to show that the kernel $K^k(x,y)$ associated to $Op^w_h\big(\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\psi(2^{-k}\xi)a(x)b_q(\xi)\big)$, i.e.

$$K^k(x,y) := \frac{1}{(2\pi h)^2}\int e^{\frac{i}{h}(x-y)\cdot\xi}\gamma\Big(\frac{(\frac{x+y}{2})|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a\Big(\frac{x+y}{2}\Big)b_q(\xi)d\xi,$$

is such that

$$\sup_x\int |K^k(x,y)|dy \lesssim 2^{kq}, \quad \sup_y\int |K^k(x,y)|dx \lesssim 2^{kq}.$$

From the symmetry between variables x, y, it will be enough to show that one of the two above inequalities is satisfied.

In order to prove the statement, we will study the kernel associated to the operator of interest, separately, in different spatial regions, and distinguishing between indices $k \in K$ such that $2^k \le h^{1/2-\sigma}$ and $2^k > h^{1/2-\sigma}$. We hence introduce three smooth cut-off functions $\theta_s, \theta_b, \theta$, supported respectively for $|x| \le m \ll 1$, $|x| \ge M \gg 1$, and for $0 < m' \le |x| \le M' < +\infty$, for some constants m, m', M, M', and such that $\theta_s + \theta_b + \theta \equiv 1$. Denoting concisely by $A^k(x,\xi)$ the multiplier in the above kernel, we split it as follows

$$A^k(x,\xi) = A^k_s(x,\xi) + A^k_b(x,\xi) + A^k_1(x,\xi),$$

with $A^k_s(x,\xi) := A^k(x,\xi)\theta_s(x)$, $A^k_b(x,\xi) := A^k(x,\xi)\theta_b(x)$ and $A^k_1(x,\xi) := A^k(x,\xi)\theta(x)$.

<u>Case I:</u> Let us consider $k \in K$ such that $h \lesssim 2^k \le h^{1/2-\sigma}$. According to the above decomposition, we have that

$$K^k(x,y) = K^k_s(x,y) + K^k_b(x,y) + K^k_1(x,y),$$

with clear meaning of kernels K^k_s, K^k_b, K^k_1.

Let us first prove that

$$\sup_x \int |K^k_s(x,y)|dy + \sup_x \int |K^k_b(x,y)|dy \lesssim 2^{kq}. \tag{1.2.31}$$

First of all, we observe that for $|x| \ll 1$ (resp. $|x| \gg 1$), we have that $\left|\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\right| \gtrsim h^{-1/2+\sigma}|\xi|$ (resp. $\left|\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\right| \gtrsim h^{-1/2+\sigma}|\xi||x| \gtrsim h^{-1/2+\sigma}|\xi|$), and by lemma 1.2.26

$$\left|\partial^\beta_\xi \Big[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big]\right| \lesssim \sum_{j=0}^{|\beta|} h^{-j(\frac{1}{2}-\sigma)} \left\langle \frac{x|\xi|-\xi}{h^{1/2-\sigma}} \right\rangle^{-j} |b_{j-|\beta|}(\xi)| \lesssim |\xi|^{-|\beta|}. \tag{1.2.32}$$

Therefore,

$$\left|\partial^\beta_\xi A^k_s(x,2^k\xi)\right| \lesssim \sum_{|\beta_1|\le|\beta|} 2^{k|\beta|}|2^k\xi|^{-|\beta_1|}2^{-k(|\beta|-|\beta_1|)+kq}\mathbb{1}_{|\xi|\sim1} \lesssim 2^{kq}\mathbb{1}_{|\xi|\sim1}, \tag{1.2.33}$$

so making a change of coordinates $\xi \mapsto 2^k\xi$ and some integration by parts, we derive that

$$|K^k_s(x,y)| \lesssim 2^{kq}(2^kh^{-1})^2 \left\langle 2^kh^{-1}(x-y)\right\rangle^{-3},$$

for every $(x,y) \in \mathbb{R}^2\times\mathbb{R}^2$. The same argument applies to $K^k_b(x,y)$, so taking the L^1 norm, we obtain (1.2.31).

As concerns kernel $K^k_1(x,y)$, we deduce from lemma 1.2.26, the fact that $\theta_1(x)$ is supported for $|x|\sim 1$, and that $2^k \lesssim h^{1/2-\sigma}$, the following:

$$\left|\partial^\beta_\xi\Big[A^k_1\Big(\frac{x+y}{2},2^k\xi\Big)\Big]\right| \lesssim 2^{k|\beta|}\Big[2^{k(q-|\beta|)} + \sum_{j=1}^{|\beta|} h^{-j(\frac{1}{2}-\sigma)}|b_{j-|\beta|+q}(2^k\xi)|\Big] \lesssim 2^{kq}.$$

Performing a change of coordinates $\xi\mapsto 2^k\xi$, and making some integration by parts, one finds that

$$|K^k_1(x,y)| \lesssim 2^{kq}(2^kh^{-1})^2\left\langle 2^kh^{-1}(x-y)\right\rangle^{-3}, \qquad \forall (x,y),$$

and consequently that

$$\sup_x\int|K^k(x,y)|dy \lesssim 2^{kq}.$$

Summing up with (1.2.31), we deduce that

$$Op^w_h(A^k(x,\xi)) = Op^w_h(A^k_s(x,\xi)) + Op^w_h(A^k_b(x,\xi)) + Op^w_h(A^k_1(x,\xi))$$

is a bounded operator on L^p, for every $1\le p\le+\infty$, with norm $O(2^{kq})$.

<u>Case II:</u>

Let us now suppose that $k\in K$ is such that $h^{1/2-\sigma} < 2^k \le h^{-\sigma}$. If Θ_h, Θ^{-1}_h are the operators such that $\Theta_h v(x) = v(\sqrt{h}x)$, $\Theta^{-1}_h w(x) = w\big(\frac{x}{\sqrt{h}}\big)$, we have that, for any symbol $a(x,\xi)$,

$$Op^w_h(a(x,\xi)) = \Theta_h Op^w_h(\widetilde{a}(x,\xi))\Theta^{-1}_h,$$

with $\widetilde{a}(x,\xi) := a\big(\frac{x}{\sqrt{h}}, \sqrt{h}\xi\big)$. In order to prove that $Op_h^w(A_s^k(x,\xi))$ (resp. $Op_h^w(A_b^k(x,\xi))$) is a bounded operator on L^p, we reduce to prove that this holds true for $Op_h^w(\widetilde{A}_s^k(x,\xi))$ (resp. $Op_h^w(\widetilde{A}_b^k(x,\xi))$).

From (1.2.32), the fact that $\widetilde{A}_s^k(x,\xi) = A_s^k\big(\frac{x}{\sqrt{h}}, \sqrt{h}\xi\big)$, we derive that

$$\Big|\partial_\xi^\beta \widetilde{A}_s^k(x,\xi)\Big| \lesssim \sum_{|\beta_1|\le|\beta|} h^{\frac{|\beta|}{2}} |\sqrt{h}\xi|^{-|\beta_1|} 2^{-k(|\beta|-|\beta_1|)+kq} \mathbb{1}_{|\xi|\sim 2^k h^{-1/2}},$$

and hence

$$\Big|\partial_\xi^\beta \widetilde{A}_s^k(x, 2^k h^{-1/2}\xi)\Big| \lesssim \sum_{|\beta_1|\le|\beta|} 2^{k|\beta|} |2^k\xi|^{-|\beta_1|} 2^{-k(|\beta|-|\beta_1|)+kq} \mathbb{1}_{|\xi|\sim 1} \lesssim 2^{kq}\mathbb{1}_{|\xi|\sim 1},$$

for every (x,ξ). By making a change of coordinates $\xi \mapsto 2^k h^{-1/2}\xi$, some integrations by parts and using the above inequality, one can show that kernel $\widetilde{K}_s^k(x,y)$ associated to $Op_h^w(\widetilde{A}_s^k(x,\xi))$, i.e.

$$\widetilde{K}_s^k(x,y) = \frac{1}{(2\pi h)^2}\int e^{\frac{i}{h}(x-y)\cdot\xi} \widetilde{A}_s^k\Big(\frac{x+y}{2},\xi\Big)d\xi,$$

is such that

$$|\widetilde{K}_s^k(x,y)| \lesssim 2^{kq}(2^k h^{-\frac{3}{2}})^2 \left\langle 2^k h^{-\frac{3}{2}}(x-y)\right\rangle^{-3}, \qquad \forall (x,y),$$

which implies that $sup_x \int |\widetilde{K}_s^k(x,y)|dy \lesssim 2^{kq}$. The same argument, and hence the same estimate, holds for $\widetilde{K}_b^k(x,y)$.

The last thing to prove is that $Op_h^w(A_1(x,\xi)) \in \mathcal{L}(L^p)$, for every $1 \le p \le +\infty$. So let $K_1^k(x,y)$ be its associated kernel,

$$K_1^k(x,y) = \frac{1}{(2\pi h)^2}\int e^{\frac{i}{h}(x-y)\cdot\xi}\gamma\Big(\frac{(\frac{x+y}{2})|\xi| - \xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a\Big(\frac{x+y}{2}\Big)b_q(\xi)d\xi, \tag{1.2.34}$$

and assume, without loss of generality, that $\gamma(x) = \gamma(|x|^2)$. Set

$$\frac{x+y}{2} = r[\cos\theta, \sin\theta],$$

with $m' \le r \le M'$ on the support of $\theta_1\big(\frac{x+y}{2}\big)$, and for fixed r,θ let

$$\xi = \rho[\cos\theta,\sin\theta] + r\Omega[-\sin\theta,\cos\theta]. \tag{1.2.35}$$

We immediately notice that the Jacobian $[\frac{\partial(\xi_1,\xi_2)}{\partial(\rho,\Omega)}] = r \sim 1$, and that $|\xi|^2 = \rho^2 + r^2\Omega^2$. Moreover,

$$\Big|\Big(\frac{x+y}{2}\Big)|\xi| - \xi\Big|^2 = \Big[r\sqrt{\rho^2+r^2\Omega^2} - \rho\Big]^2 + r^2\Omega^2.$$

If the support of γ is of size α sufficiently small, with $0<\alpha\ll 1$, from the above equality and the fact that $|\xi|\sim 2^k$ on the support of $\psi(2^{-k}\xi)$, with $h^{1/2-\sigma} < 2^k \lesssim h^{-\sigma}$, we deduce that

$$r\Omega \le \sqrt{\alpha}h^{1/2-\sigma} \quad \text{and} \quad |\rho|\sim|\xi|\sim 2^k \quad \text{and} \quad \frac{r\Omega}{|\rho|} \le \sqrt{\alpha}$$

and consequently that

$$\alpha h^{1-2\sigma} \ge \Big[r\sqrt{\rho^2+r^2\Omega^2} - \rho\Big]^2 \gtrsim \rho^2|r-1|^2.$$

We should remark that, from the first of above inequalities, it follows that $\rho > 0$, and this condition infers the second one. Moreover

$$\alpha h^{1-2\sigma} \geq \left[r\sqrt{\rho^2+r^2\Omega^2}-\rho\right]^2 + r^2\Omega^2 = \rho^2\left[(r-1)+r\left[\sqrt{1+\frac{r^2\Omega^2}{\rho^2}}-1\right]\right]^2 + r^2\Omega^2$$
$$= \rho^2|r-1|^2 + r^2\Omega^2\left[1+a(r,\Omega,\rho)\right],$$

with $a(r,\Omega,\rho)$ bounded such that, for any $l,m,n\in\mathbb{N}$,

$$|\partial_r^l\partial_\Omega^m\partial_\rho^n a(r,\Omega,\rho)| = O(\rho^{-(m+n)}),$$

and if

$$\Gamma_h := \gamma\Big(\frac{\rho^2|r-1|^2}{h^{1-2\sigma}} + \frac{r^2\Omega^2}{h^{1-2\sigma}}\left[1+a(r,\Omega,\rho)\right]\Big)\psi(2^{-k}\sqrt{\rho^2+r^2\Omega^2})a(r,\theta)b_q(\rho),$$

from all the observations made above, along with the fact that $h^{-1/2+\sigma}\lesssim\rho^{-1}$ we deduce that for any $m,n\in\mathbb{N}$

$$|\partial_\rho^m\Gamma_h| = O(2^{kq}\rho^{-m}) \quad\text{and}\quad |\partial_\Omega^n\Gamma_h| = O(2^{kq}\rho^{-n}). \tag{1.2.36}$$

With the change of coordinates considered in (1.2.35), and setting $w:=x-y$, $e_\theta:=[\cos\theta,\sin\theta]$, kernel $K_1^k(x,y)$ transforms into

$$\frac{1}{(2\pi h)^2}\int e^{\frac{i}{h}\rho w\cdot e_\theta + \frac{i}{h}r\Omega w\cdot e_\theta^\perp}\Gamma_h\, rd\rho d\Omega,$$

and is restricted to $|\rho|\sim 2^k$, $|\Omega|\lesssim h^{1/2-\sigma}$, so by making some integrations by parts, using (1.2.36), and reminding that $|r-1|\ll 2^{-k}h^{1/2-\sigma}\ll 1$ on the support of Γ_h, we find that, for any $N\in\mathbb{N}$,

$$|K_1^k(x,y)|\lesssim h^{-\frac{3}{2}-\sigma}2^k\left\langle\frac{2^k}{h}w\cdot e_\theta\right\rangle^{-N}\left\langle\frac{2^k}{h}w\cdot e_\theta^\perp\right\rangle^{-N}\mathbb{1}_{||\frac{x+y}{2}|-1|\ll 1}.$$

Now, as $w=(x-y)$, $e_\theta=\frac{x+y}{|x+y|}$, and $|x+y|=2r\sim 1$ on the support of Γ_h, we have that $|w\cdot e_\theta|\sim||x|^2-|y|^2|$, $|w\cdot e_\theta^\perp|\sim|(x+y)(x+y)^\perp|\sim 2|x\cdot y^\perp|=2|x_1y_2-x_2y_1|$, and consequently

$$|K_1^k(x,y)|\lesssim h^{-\frac{3}{2}-\sigma}2^{k(1+q)}\left\langle\frac{2^k}{h}\big||x|^2-|y|^2\big|\right\rangle^{-N}\left\langle\frac{2^k}{h}(x_1y_2-x_2y_1)\right\rangle^{-N}\mathbb{1}_{||\frac{x+y}{2}|-1|\ll 1}.$$

Taking successively the $L^1(dy)$ norm of $K_1^k(x,y)$, and using the above estimate we find that:

- if $|x|\ll|y|$ or $|x|\gg|x|$,

$$\left\langle\frac{2^k}{h}\big||x|^2-|y|^2\big|\right\rangle^{-N}\mathbb{1}_{||\frac{x+y}{2}|-1|\ll 1}\lesssim h^{N(\frac{1}{2}+\sigma)},$$

as follows from the fact that $h2^{-k}<h^{1/2+\sigma}$, and we obtain that,

$$\sup_x\int|K_1^k(x,y)|dy\lesssim h^{-\frac{3}{2}}2^{k(1+q)}h^{N(\frac{1}{2}+\sigma)}\lesssim 1$$

by taking $N\in\mathbb{N}$ sufficiently large (e.g. $N>3$) and $\sigma>0$ small.

• if $|x|\sim|y|$, from the fact that $\big|\big|\frac{x+y}{2}\big|-1\big|\leq\sqrt{\alpha}h^{1/2-\sigma}2^{-k}$ on the support of Γ_h we deduce that $|x|\geq c>0$. Without loss of generality, we can assume that $x=\lambda e_1$ (this always being possible by making a rotation), and $|\lambda|\geq c>0$. If $w:=x+y$,

$$|x|^2-|y|^2=w\cdot(x-y)=w\cdot(2x-w)=w\cdot(2\lambda e_1-w)=2\lambda w_1-w_1^2-w_2^2,$$

and then

$$\frac{\big||x|^2-|y|^2\big|}{h}=-\frac{(w_1-\lambda)^2-\lambda^2}{h}+\Big(\frac{w_2}{\sqrt{h}}\Big)^2$$

while

$$x_1y_2-x_2y_1=\lambda w_2,$$

which implies that

$$|K_1^k(x,y)|\lesssim h^{-\frac{3}{2}-\sigma}2^{k(1+q)}\left\langle\frac{2^k}{h}\left((w_1-\lambda)^2-\lambda^2\right)\right\rangle^{-N}\left\langle\frac{2^k}{h}w_2\right\rangle^{-N}.$$

Since $\int|K_1^k(x,y)|dy=\int|K_1^k(x,y)|dw$, from the above estimate (with a fixed $N\in\mathbb{N}$ sufficiently large) this integral is bounded by 2^{kq}, when restricted to $|x|\sim|y|$. Indeed, when the integral is taken in a neighbourhood of $w_1=0$ or $w_1=2\lambda$, $(w_1-\lambda^2)-\lambda^2$ can be considered as the variable of integration, and by a change of coordinates, along with the fact that $2^{-k}<h^{-1/2+\sigma}$, one deduces that

$$\int_{U_0\cup U_{2\lambda}}|K_1^k(x,y)|dw\lesssim h^{-\frac{3}{2}-\sigma}2^{k(1+q)}h^2 2^{-2k}\lesssim 2^{kq},$$

where U_0 (resp. $U_{2\lambda}$) is a neighbourhood of $w_1=0$ (resp. of $w_1=2\lambda$); outside of $U_0\cup U_{2\lambda}$,

$$\left\langle\frac{2^k}{h}\left((w_1-\lambda)^2-\lambda^2\right)\right\rangle^{-N}\lesssim(h2^{-k})^N\langle w_1\rangle^{-N}\lesssim h^{N(\frac{1}{2}+\sigma)}\langle w_1\rangle^{-N},$$

so

$$\int_{(U_0\cup U_{2\lambda})^\complement}|K_1^k(x,y)|dw\lesssim h^{-\frac{3}{2}-\sigma}2^{k(1+q)}h2^{-k}h^{N(\frac{1}{2}+\sigma)}\lesssim 2^{kq}.$$

This finally proves that also $Op_h^w(A_1^k(x,\xi))$ is a bounded operator on L^p with norm $O(2^{kq})$. □

Let us introduce

$$\Omega_h:=x_1hD_2-x_2hD_1=Op_h^w(x_1\xi_2-x_2\xi_1)$$

the Euclidean rotation in the semi-classical setting.

Proposition 1.2.30. *Under the same assumptions as in proposition 1.2.27, with γ replaced by γ_1, we have that, for any $w\in L^2(\mathbb{R}^2)$ such that $\Omega_h w\in L^2_{loc}(\mathbb{R}^2)$,*

$$\left\|Op_h^w\Big(\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)w\right\|_{L^\infty}\lesssim 2^{kp}h^{-\frac{1}{2}-\sigma}\left(\|w\|_{L^2}+\|\theta_0\Omega_h w\|_{L^2}\right),\tag{1.2.37}$$

where θ_0 is a smooth function, supported in some annulus centred in the origin.

Proof. We prove the statement distinguishing between three spatial regions. For that, we introduce three cut-off functions: $\theta_s(x)$ supported for $|x|\leq m\ll 1$; $\theta_b(x)$ supported for $|x|\geq M'\gg 1$; $\theta(x)$ supported for $m'\leq|x|\leq M'$, for some $0<m'\ll 1, M\gg 1$, such that $\theta_s+\theta_b+\theta\equiv 1$, and we define respectively $A_s^k(x,\xi):=\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\theta_s(x)$,

$A^k_b(x,\xi) := \gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\theta_b(x)$, and $A^k(x,\xi) := \gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\theta(x)$, so that

$$\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi) = A^k_s(x,\xi) + A^k_b(x,\xi) + A^k(x,\xi).$$

The fact that $Op^w_h(A^k_s), Op^w_h(A^k_b) \in \mathcal{L}(L^2)$ and their norm is a $O(2^{kp}h^{-1/2-\sigma})$ follows from lemmas 1.2.25 and 1.2.26. Indeed, when $|x| \ll 1$ (resp. $|x| \gg 1$) we have that $\big|\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big| \gtrsim h^{-1/2+\sigma}|\xi|$ (resp. $\big|\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big| \gtrsim h^{-1/2+\sigma}|\xi||x| \gtrsim h^{-1/2+\sigma}|\xi|$), so using lemma 1.2.26 we derive that

$$\left|\partial^\alpha_x\partial^\beta_\xi\Big[\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big]\right| \lesssim \sum_{j=0}^{|\beta|} h^{-(|\alpha|+j)(\frac{1}{2}-\sigma)}\left|\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\right|^{-1-|\alpha|-j} |b_{|\alpha|+j-|\beta|}(\xi)| \lesssim h^{\frac{1}{2}-\sigma}|\xi|^{-1-|\beta|}.$$

Consequently, as $2^{-k}h \le 1$, we deduce that $|\partial^\alpha_x\partial^\beta_\xi\big[A^k_s(\frac{x+y}{2},h\xi)\big]| \lesssim 2^{kp}h^{-1/2-\sigma}|\xi|^{-1}$ for any $\alpha,\beta\in\mathbb{N}^2$. Therefore

$$\left\|\partial^\alpha_y\partial^\beta_\xi\Big[A^k_s\Big(\frac{x+y}{2},h\xi\Big)\Big]\right\|_{L^2(d\xi)} \lesssim 2^{kp}h^{-\frac{1}{2}-\sigma}\left(\int_{|\xi|\sim 2^k h^{-1}}|\xi|^{-2}d\xi\right)^{\frac{1}{2}} \lesssim 2^{kp}h^{-\frac{1}{2}-\sigma}.$$

The same holds for $A^k_b(x,\xi)$ so, injecting these estimates in inequality (1.2.21), we derive that $\|Op^w_h(A^k_s(x,\xi))w\|_{L^\infty} + \|Op^w_h(A^k_b(x,\xi))w\|_{L^\infty} \le C2^{kp}h^{-\frac{1}{2}-\sigma}\|w\|_{L^2}$.

A different analysis is needed for $Op^w_h(A^k(x,\xi))w$, since it is no longer true that $|x|\xi|-\xi| \ge C|\xi|$, for a constant $C>0$, on the support of $A^k(x,\xi)$. In this case, we exploit the fact that $A^k(x,\xi)$ is supported in an annulus, in order to localize $Op^w_h(A^k(x,\xi))w$ and then perform a change of variables. If $\theta_0\in C^\infty_0(\mathbb{R}^2\setminus\{0\})$ is a cut-off function equal to 1 on the support of θ, we have that, for any $N\in\mathbb{N}$, $A^k(x,\xi) = \theta_0(x)\sharp A^k(x,\xi) + r^k_N(x,\xi)$ by means of proposition 1.2.21, where

$$r^k_N(x,\xi) = \Big(\frac{h}{2i}\Big)^N \frac{N}{(\pi h)^4}\sum_{|\alpha|=N}\frac{(-1)^{|\alpha|}}{\alpha!}\int e^{\frac{2i}{h}(\eta\cdot z-y\cdot\zeta)}\int_0^1 \partial^\alpha_x\theta_0(x+tz)(1-t)^{N-1}dt \\ \times(\partial^\alpha_\xi A^k)(x,\xi+\eta)\,dydzd\eta d\zeta.$$

If we take N sufficiently large it turn out that the quantization of r^k_N satisfies a better estimate than (1.2.37). Indeed, using lemma 1.2.26 and integrating in $dyd\zeta$, it can be rewritten as

$$\text{(1.2.38)}\quad r^k_N(x,\xi) = \sum_{j\le N}\frac{h^{N-j(\frac{1}{2}-\sigma)}}{(\pi h)^2}\int e^{\frac{2i}{h}\eta\cdot z}\int_0^1\theta_0(x+tz)(1-t)^{N-1}dt \\ \times\gamma_{1+j}\Big(\frac{x|\xi+\eta|-(\xi+\eta)}{h^{1/2-\sigma}}\Big)\psi(2^{-k}(\xi+\eta))\theta_j(x)a(x)b_{p+j-N}(\xi+\eta)\,dzd\eta,$$

for some new functions $\theta_0,\gamma_{1+j},\psi,\theta_j,a,b_{p+j-N}$, and as it is compactly supported in x, we know by lemma 1.2.25 that for a new cut-off function (that we still call θ)

$$|Op^w_h(r^k_N(x,\xi))w| \lesssim \|w\|_{L^2}\int\Big|\theta\Big(\frac{x+y}{2}\Big)\Big|\sum_{|\alpha'|\le 3}\Big\|\partial^{\alpha'}_y\Big[r^k_N\Big(\frac{x+y}{2},h\xi\Big)\Big]\Big\|_{L^2(d\xi)}dy.$$

One can check that the action of $\partial^{\alpha'}_y$ on $r^k_N(\frac{x+y}{2},h\xi)$ makes appear factors $(h^{-1/2+\sigma}h|\xi+\eta|)^i$, for $i\le|\alpha'|$, without changing the underlining structure of the integral, and this term is bounded by $(h^{-1/2+\sigma}2^k)^i$ on the support of $\psi(2^{-k}h(\xi+\eta))$. After a change of variables $\eta\mapsto h\eta$ in (1.2.38), we use that $e^{2i\eta\cdot z}=\left(\frac{1-2i\eta\cdot\partial_z}{1+4|\eta|^2}\right)^3\left(\frac{1-2iz\cdot\partial_\eta}{1+4|z|^2}\right)^3e^{2i\eta\cdot z}$, integrate by parts, apply Young's inequality for convolutions, and fix $N>7$, in order to deduce the following chain of inequalities:

$$\begin{aligned}
&\left\|\partial_y^{\alpha'}r^k_N\Big(\frac{x+y}{2},h\xi\Big)\right\|^2_{L^2(d\xi)}\\
&\lesssim\sum_{i\le|\alpha'|,j\le N}h^{2N-2j(\frac{1}{2}-\sigma)}\big(h^{-\frac{1}{2}+\sigma}2^k\big)^{2i}2^{2k(p+j-N)}\int d\xi\left|\iint\langle z\rangle^{-3}\langle\eta\rangle^{-3}|\psi(2^{-k}h(\xi+\eta))|dzd\eta\right|^2\\
&\qquad\lesssim\sum_{i\le|\alpha'|,j\le N}h^{2N-2j(\frac{1}{2}-\sigma)}\big(h^{-\frac{1}{2}+\sigma}2^k\big)^{2i}2^{2k(p+j-N)}\int|\psi(2^{-k}h\xi)|^2d\xi\\
&\qquad\qquad\lesssim\sum_{i\le|\alpha'|,j\le N}h^{2N-2j(\frac{1}{2}-\sigma)}\big(h^{-\frac{1}{2}+\sigma}2^k\big)^{2i}2^{2k(p+j-N)}\big(h^{-1}2^k\big)^2\lesssim 2^{2kp},
\end{aligned}$$

and that $\|Op^w_h(r^k_N)\|_{\mathcal{L}(L^2;L^\infty)}\lesssim 2^{kp}$. We can then focus on the analysis of the L^∞ norm of $\theta_0(x)Op^w_h(A^k(x,\xi))w$, which is restricted to an annulus where we can perform the change of variables $x=\rho e^{i\alpha}$. In these coordinates, operator Ω_h reads as D_α so, using classical one-dimensional Sobolev injection with respect to variable α, one-dimensional semi-classical Sobolev injection with respect to variable ρ, and successively returning back to coordinates x, we deduce that

$$\begin{aligned}
\big|\theta_0(x)Op^w_h(A^k(x,\xi))w\big|&\lesssim h^{-\frac{1}{2}}\Big[\|Op^w_h(A^k)w\|_{L^2(dx)}+\|Op^w_h(\xi)Op^w_h(A^k)w\|_{L^2(dx)}\\
&\qquad+\|\Omega_h\theta_0Op^w_h(A^k)w\|_{L^2(dx)}+\|Op^w_h(\xi)\Omega_h\theta_0Op^w_h(A^k)w\|_{L^2(dx)}\Big]\\
&\lesssim 2^{kp}h^{-\frac{1}{2}-\sigma}[\|w\|_{L^2}+\|\theta_0\Omega_hw\|_{L^2}],
\end{aligned}$$

last inequality derived observing that the commutator between Ω_h and $Op^w_h(A^k)$ is a semi-classical pseudo-differential operator, whose symbol is linear combination of terms of the form

$$\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)\theta(x)b_p(\xi),$$

for some new $\gamma_1,\psi,a,\theta,b_p$, and from the fact that operators $Op^w_h(A^k(x,\xi))$, $Op^w_h(\xi A^k(x,\xi)):L^2\to L^2$ are bounded (see proposition 1.2.27), respectively with norm $O(2^{kp})$, $O(2^{k(p+1)})$, and that $2^k\le h^{-\sigma}$.

□

Proposition 1.2.31. *Under the same hypothesis as proposition 1.2.28, $Op^w_h(I^k_{p,q}(x,\xi)):L^2\to L^\infty$ is bounded and*

$$\left\|Op^w_h(I^k_{p,q}(x,\xi))\right\|_{\mathcal{L}(L^2;L^\infty)}\lesssim\sum_{i\le 6}2^{k(p+q)}(h^{-\frac{1}{2}+\sigma}2^k)^i(h^{-1}2^k). \tag{1.2.39}$$

The same result holds for $Op^w_h(J^k_{p,q})$.

Proof. As in proposition 1.2.28, we prove the statement only for $Op^w_h(I^k_{p,q})$, leaving to the reader to check that the result is true also for $Op^w_h(J^k_{p,q})$.

Let $w\in L^2$. After lemma 1.2.25, we should prove that $\left\|\partial^\alpha_y\partial^\beta_\xi\left[I^k_{p,q}(\frac{x+y}{2},h\xi)\right]\right\|_{L^2(d\xi)}$ is estimated by the right hand side of (1.2.39), for any $|\alpha|,|\beta|\le 3$. A change of variables $\eta\mapsto h\eta$, $\zeta\mapsto h\zeta$

allows us to write $I^k_{p,q}(\frac{x+y}{2},h\xi)$ as

$$\frac{1}{\pi^4}\int e^{2i(\eta\cdot z-y'\cdot\zeta)}\Big[\int_0^1\Big(\gamma\big(h^{\frac{1}{2}+\sigma}(x|\xi|-\xi)\big)\psi(2^{-k}h\xi)a(x)b_p(h\xi)\Big)|_{(\frac{x+y}{2}+tz,\xi+t\zeta)}f(t)dt \\ \times\widetilde{\psi}(2^{-k}h(\xi+\eta))a'\Big(\frac{x+y}{2}+y'\Big)b_q(h(\xi+\eta))\Big]\,dy'dzd\eta d\zeta,$$

and to observe that the action of ∂^α_y on the integrand makes appear a factor $(h^{-\frac{1}{2}+\sigma}|h(\xi+t\zeta)|)^i$, with $i\le|\alpha|$, while that of ∂^β_ξ doesn't affect the estimates of above integral, as one can check using lemma 1.2.26 and the fact that $2^{-k}h\le 1$. With this in mind, we can reduce to the study of the $L^2(d\xi)$ norm of an integral function as

$$\sum_{i\le 3}(h^{-\frac{1}{2}+\sigma}2^k)^i\int e^{2i(\eta\cdot z-y'\cdot\zeta)}\Big[\int_0^1\Big(\gamma\Big(h^{\frac{1}{2}+\sigma}(x|\xi|-\xi)\Big)\psi(2^{-k}h\xi)a(x)b_p(h\xi)\Big)|_{(\frac{x+y}{2}+tz,\xi+t\zeta)}f(t)dt \\ \times\widetilde{\psi}(2^{-k}h(\xi+\eta))a'\Big(\frac{x+y}{2}+y'\Big)b_q(h(\xi+\eta))\Big]dy'dzd\eta d\zeta,$$

for some new functions $\gamma,\psi,a,b_p,\widetilde{\psi},a',b_q$, with the same properties as their homonyms. We use that

$$e^{2i(\eta z-y'\cdot\zeta)}=\Big(\frac{1+2iy'\cdot\partial_\zeta}{1+4|y'|^2}\Big)^3\Big(\frac{1-2i\eta\cdot\partial_z}{1+4|\eta|^2}\Big)^3\Big(\frac{1-2iz\cdot\partial_\eta}{1+4|z|^2}\Big)^3\Big(\frac{1+2i\zeta\cdot\partial_{y'}}{1+4|\zeta|^2}\Big)^3e^{2i(\eta\cdot z-y'\cdot\zeta)}$$

and integration by parts to obtain the integrability in $dy'dzd\eta d\zeta$, up to new factors $(h^{-\frac{1}{2}+\sigma}|h(\xi+t\zeta)|)^j$, with $j\le 3$, coming out from the derivation of the integrand with respect to z. Then, using that functions $h^jb_{p-j}(h(\xi+t\zeta))$ (resp. $h^jb_{q-j}(h(\xi+\eta))$), $j\le 3$, appearing from the derivation of $b_p(h(\xi+t\zeta))$ with respect to ζ (resp. the derivation of $b_q(h(\xi+\eta))$ with respect to η), are such that $|h^jb_{p-j}(h(\xi+t\zeta))|\le h^j2^{k(p-j)}\lesssim 2^{kp}$ on the support of $\psi(2^{-k}h(\xi+t\zeta))$ (resp. $|h^jb_{q-j}(h(\xi+\eta))|\le 2^{kq}$ on the support of $\widetilde{\psi}(2^{-k}h(\xi+\eta))$), and the fact that $\|\int\langle\eta\rangle^{-3}|\widetilde{\psi}(2^{-k}h(\xi+\eta))|d\eta\|_{L^2(d\xi)}\le\|\widetilde{\psi}(2^{-k}h\cdot)\|_{L^2}\lesssim h^{-1}2^k$, we obtain the result of the statement. ☐

Lemma 1.2.32. *Let us take $\sigma>0$ sufficiently small, $k\in K$ and $p,q\in\mathbb{N}$. Let also $\psi,\widetilde{\psi}\in C^\infty_0(\mathbb{R}^2\setminus\{0\})$, $a(x)$ be either a smooth compactly supported function or $a\equiv 1$, and $f\in C(\mathbb{R})$. For a fixed integer $N>2(p+q)+9$, we define*

$$r^k_{N,p}(x,\xi):=\frac{h^N}{(\pi h)^4}\sum_{|\alpha|=N}\int e^{\frac{2i}{h}(\eta\cdot z-y\cdot\zeta)}\Big[\int_0^1\partial^\alpha_x\Big(\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)|_{(x+tz,\xi+t\zeta)} \\ \times f(t)dt\Big]\partial^\alpha_\xi\big(b_q(\xi)\widetilde{\psi}(2^{-k}\xi)\big)|_{(x+y,\xi+\eta)}\,dydzd\eta d\zeta, \tag{1.2.40}$$

and

$$\widetilde{r}^k_{N,p}(x,\xi):=\frac{h^N}{(\pi h)^4}\sum_{|\alpha_1|+|\alpha_2|=N}\int e^{\frac{2i}{h}(\eta\cdot z-y\cdot\zeta)}\Big[\int_0^1\partial^{\alpha_1}_x\partial^{\alpha_2}_\xi\Big(\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)|_{(x+tz,\xi+t\zeta)} \\ \times f(t)dt\Big]\partial^{\alpha_2}_x\partial^{\alpha_1}_\xi\big(x_nb_q(\xi)\widetilde{\psi}(2^{-k}\xi)\big)|_{(x+y,\xi+\eta)}\,dydzd\eta d\zeta\,. \tag{1.2.41}$$

Then

$$\|Op^w_h(r^k_{N,p})\|_{\mathcal{L}(L^2)}+\|Op^w_h(\widetilde{r}^k_{N,p})\|_{\mathcal{L}(L^2)}+\|Op^w_h(r^k_{N,p})\|_{\mathcal{L}(L^2;L^\infty)}+\|Op^w_h(\widetilde{r}^k_{N,p})\|_{\mathcal{L}(L^2;L^\infty)}\lesssim h^{p+q}. \tag{1.2.42}$$

Proof. We remind definition (1.2.28) of integral $I^k_{p,q}(x,\xi)$ for general $k\in K, p,q\in\mathbb{Z}$. After an explicit development of derivatives appearing in (1.2.40) we find that $r^k_{N,p}(x,\xi)$ may be written as

$$\sum_{j\le N} h^{N-j(\frac{1}{2}-\sigma)} I^k_{p+j,q-N}(x,\xi)$$

where γ is replaced with γ_1, and $a'\equiv 1$ in $I^k_{p+j,q-N}$. Propositions 1.2.28 and 1.2.31, combined with the fact that $h\le 2^k\le h^{-\sigma}$, imply, respectively, that

$$\begin{aligned}
&\|Op^w_h(r^k_{N,p})\|_{\mathcal{L}(L^2)} \lesssim \sum_{j\le N} h^{N-j(\frac{1}{2}-\sigma)} 2^{k(p+j+q-N)} \\
&\lesssim \sum_{\substack{j\le N\\ p+j+q\le N}} h^{N-j(\frac{1}{2}-\sigma)+p+j+q-N} + \sum_{\substack{j\le N\\ p+j+q>N}} h^{N-j(\frac{1}{2}-\sigma)-\sigma(p+j+q-N)} \lesssim h^{p+q},
\end{aligned}$$

and

$$\begin{aligned}
&\|Op^w_h(r^k_{N,p})\|_{\mathcal{L}(L^2;L^\infty)} \lesssim \sum_{i\le 6, j\le N} h^{N-j(\frac{1}{2}-\sigma)} 2^{k(p+j+q-N)} (h^{-\frac{1}{2}+\sigma}2^k)^i (h^{-1}2^k) \\
&\lesssim \sum_{\substack{i\le 6, j\le N\\ p+i+j+q\le N-1}} h^{N-1-(i+j)(\frac{1}{2}-\sigma)+p+i+j+q-N+1} + \sum_{\substack{i\le 6, j\le N\\ p+i+j+q> N-1}} h^{N-1-(i+j)(\frac{1}{2}-\sigma)-\sigma(p+i+j+q-N+1)} \\
&\lesssim h^{p+q},
\end{aligned}$$

as $N>2(p+q)+9$.

As regards (1.2.41), we first observe that index α_2 is such that $|\alpha_2|\le 1$ since $x_n b_q(\xi)\widetilde{\psi}(2^{-k}\xi)$ is linear in x_n. An explicit development of derivatives in (1.2.41), combined with lemma 1.2.26, shows that $\widetilde{r}^k_{N,p}(x,\xi)$ splits into two contributions:

$$\begin{aligned}
J_0(x,\xi) = \frac{h^N}{(\pi h)^4}\sum_{i\le N} h^{-i(\frac{1}{2}-\sigma)} \int e^{\frac{2i}{h}(\eta\cdot z-y\cdot\zeta)} \int_0^1 \Big(\gamma_{1+i}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_{p+i}(\xi)\Big)|_{(x+tz,\xi+t\zeta)} f(t)dt \\
\times (x_n+y_n) b_{q-N}(\xi+\eta)\widetilde{\psi}(2^{-k}(\xi+\eta))\, dydzd\eta d\zeta,
\end{aligned}$$

coming out when $|\alpha_2|=0$, for some new functions $a,\psi,\widetilde{\psi}$, and clear meaning for γ_i, b_{p+i}, b_{q-N};

$$\begin{aligned}
J_1(x,\xi) = \frac{h^N}{(\pi h)^4}\sum_{i\le N-1, j\le 1} h^{-(i+j)(\frac{1}{2}-\sigma)} \int e^{\frac{2i}{h}(\eta\cdot z-y\cdot\zeta)} \\
\times \int_0^1 \Big(\gamma_{1+i+j}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_{p+i+j-1}(\xi)\Big)|_{(x+tz,\xi+t\zeta)} f(t)dt \\
\times\, b_{q-N+1}(\xi+\eta)\widetilde{\psi}(2^{-k}(\xi+\eta))\, dydzd\eta d\zeta,
\end{aligned}$$

corresponding instead to $|\alpha_2|=1$, for some new other $a,\psi,\widetilde{\psi}$. One has that

$$J_1(x,\xi) = \sum_{i\le N-1, j\le 1} h^{N-(i+j)(\frac{1}{2}-\sigma)} I^k_{p+i+j-1,q-N+1}(x,\xi),$$

with γ replaced with γ_1, $a'\equiv 1$, so propositions 1.2.28 and 1.2.31, along with the fact that $N>2(p+q)+9$ imply

$$\|Op^w_h(J_1(x,\xi))\|_{\mathcal{L}(L^2)} \lesssim \sum_{i\le N-1, j\le 1} h^{N-(i+j)(\frac{1}{2}-\sigma)} 2^{k(p+i+j+q-N)} \lesssim h^{p+q},$$

$$\|Op_h^w(J_1(x,\xi))\|_{\mathcal{L}(L^2;L^\infty)} \lesssim \sum_{\substack{i\le N-1, j\le 1\\ l\le 6}} h^{N-(i+j)(\frac{1}{2}-\sigma)}2^{k(p+i+j+q-N)}(h^{-\frac{1}{2}+\sigma}2^k)^l(h^{-1}2^k) \lesssim h^{p+q}.$$

In order to derive the same estimates for $J_0(x,\xi)$, we split the sum x_n+y_n and analyse separately the two out-coming integrals, that we denote $J_{0,x}(x,\xi)$, $J_{0,y}(x,\xi)$. In the latter one, we use that $y_n e^{-\frac{2i}{h}y\cdot\zeta} = -\frac{h}{2i}\partial_{\zeta_n}e^{-\frac{2i}{h}y\cdot\zeta}$ and successively integrate by parts in $d\zeta_n$, obtaining, with the help of lemma 1.2.26, that

$$\begin{aligned}(1.2.43)\quad J_{0,y}(x,\xi) = \sum_{i\le N, j\le 1} h^{N+1-(i+j)(\frac{1}{2}-\sigma)}\int e^{\frac{2i}{h}(\eta\cdot z-y\cdot\zeta)}\\ \times \int_0^1 \Big(\gamma_{1+i+j}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_{p+i+j-1}(\xi)\Big)|_{(x+tz,\xi+t\zeta)}f(t)dt\\ \times b_{q-N}(\xi-\eta)\widetilde{\psi}(2^{-k}(\xi+\eta))\, dydzd\eta d\zeta,\end{aligned}$$

for some new functions $a,\psi,\widetilde{\psi},f$. Again by propositions 1.2.28, 1.2.31 and the fact that $h\le 2^k\le h^{-\sigma}$, $N>2(p+q)+9$, we deduce that:

$$(1.2.44a)\qquad \|Op_h^w(J_{0,y}(x,\xi))\|_{\mathcal{L}(L^2)} \lesssim \sum_{i\le N, j\le 1} h^{N+1-(i+j)(\frac{1}{2}-\sigma)}2^{k(p+i+j+q-N-1)} \lesssim h^{p+q},$$

(1.2.44b)
$$\|Op_h^w(J_{0,y}(x,\xi))\|_{\mathcal{L}(L^2;L^\infty)} \lesssim \sum_{\substack{i\le N, j\le 1\\ l\le 6}} h^{N+1-(i+j)(\frac{1}{2}-\sigma)}2^{k(p+i+j+q-N-1)}(h^{-\frac{1}{2}-\sigma}2^k)^l(h^{-1}2^k) \lesssim h^{p+q}.$$

In $J_{0,x}(x,\xi)$ we first integrate in $dyd\zeta$, and then we split the occurring integral into two other contributions, called $J_{0,x+tz}(x,\xi)$, $J_{0,tz}(x,\xi)$, by writing $x_n=(x_n+tz_n)-tz_n$. Similarly to what done above, we use that $z_n e^{\frac{2i}{h}\eta\cdot z} = \frac{h}{2i}\partial_{\eta_n}e^{\frac{2i}{h}\eta\cdot z}$ in $J_{0,tz}$, and successively integrate by parts in $d\eta_n$: as $2^{-k}h\le 1$, we obtain that $J_{0,tz}$ has an analogous form as (1.2.43), with some new $b_{q-N},\widetilde{\psi}$, and verifies (1.2.44). Finally, using that $x_n+tz_n = h^{\frac{1}{2}-\sigma}\big(\frac{(x_n+tz_n)|\xi|-\xi_n}{h^{1/2-\sigma}}\big)|\xi|^{-1}+\xi_n|\xi|^{-1}$, we derive that

$$\begin{aligned}&J_{0,x+tz}(x,\xi)\\ &= \sum_{i\le N} h^{N-(i-1)(\frac{1}{2}-\sigma)}\int e^{\frac{2i}{h}\eta\cdot z}\int \Big(\gamma_i\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_{p+i-1}(\xi)\Big)|_{(x+tz,\xi)}f(t)dt\\ &\qquad\times b_{q-N}(\xi+\eta)\widetilde{\psi}(2^{-k}(\xi+\eta))dzd\eta,\\ &+\sum_{i\le N} h^{N-i(\frac{1}{2}-\sigma)}\int e^{\frac{2i}{h}\eta\cdot z}\int \Big(\gamma_{1+i}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_{p+i}(\xi)\Big)|_{(x+tz,\xi)}f(t)dt\\ &\qquad\times b_{q-N}(\xi+\eta)\widetilde{\psi}(2^{-k}(\xi+\eta))dzd\eta,\end{aligned}$$

so by propositions 1.2.28, 1.2.31

$$\|Op_h^w(J_{0,x+tz}(x,\xi))\|_{\mathcal{L}(L^2)} \lesssim \sum_{i\le N} h^{N-i(\frac{1}{2}-\sigma)}2^{k(p+i+q-N)} \lesssim h^{p+q},$$

$$\|Op_h^w(J_{0,x+tz}(x,\xi))\|_{\mathcal{L}(L^2;L^\infty)} \lesssim \sum_{i\le N, l\le 3} h^{N-i(\frac{1}{2}-\sigma)}2^{k(p+i+q-N)}(h^{-\frac{1}{2}+\sigma}2^k)^l(h^{-1}2^k) \lesssim h^{p+q}.$$

As $\widetilde{r}^k_{N,p} = J_{0,x+tz}+J_{0,tz}+J_{0,y}+J_1$, that concludes the proof. ☐

We introduce the following operator:

$$\mathcal{M}_j := \frac{1}{h} Op_h^w(x_j|\xi| - \xi_j), \tag{1.2.45}$$

for $j = 1, 2$, and use the notation $\|\mathcal{M}^\gamma w\| = \|\mathcal{M}_1^{\gamma_1}\mathcal{M}_2^{\gamma_2} w\|$, for any $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$. We have now all the ingredients to state and prove the following two results.

Lemma 1.2.33. *Let us take $\sigma > 0$ sufficiently small, $k \in K$ and $p \in \mathbb{N}$. Let us also consider $\psi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, $a(x)$ either a smooth compactly supported function or $a \equiv 1$, and $\widetilde{a}(x)$ such that*

$$\begin{gathered}(a \equiv 1) \Rightarrow (\widetilde{a} \equiv 1),\\ (a \text{ compactly supported }) \Rightarrow [(\widetilde{a} \equiv 1) \text{ or } (\widetilde{a} \text{ compactly supported and } \widetilde{a}a \equiv a)].\end{gathered}$$

We have that

$$\begin{aligned}(1.2.46)\quad & Op_h^w\Big(\gamma_1\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_n|\xi| - \xi_n)\Big) \\ &\qquad = Op_h^w\Big(\gamma_1\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)\widetilde{a}(x)h\mathcal{M}_n + Op_h^w(r^k_p(x,\xi)),\end{aligned}$$

where

$$\big\|Op_h^w(r^k_p(x,\xi))w\big\|_{L^2} \lesssim h^{1-\beta}\|w\|_{L^2}, \tag{1.2.47a}$$

$$\big\|Op_h^w(r^k_p(x,\xi))w\big\|_{L^\infty} \lesssim h^{\frac{1}{2}-\beta}(\|w\|_{L^2} + \|\theta_0\Omega_h w\|_{L^2}), \tag{1.2.47b}$$

for some $\theta \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, and a small $\beta > 0$, $\beta \to 0$ as $\sigma \to 0$. Consequently,

$$\Big\|Op_h^w\Big(\gamma_1\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_n|\xi| - \xi_n)\Big)w\Big\|_{L^2} \lesssim h^{1-\beta}\big(\|w\|_{L^2} + \|\mathcal{M}_n w\|_{L^2}\big), \tag{1.2.48a}$$

$$\begin{aligned}(1.2.48b)\quad & \Big\|Op_h^w\Big(\gamma_1\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_n|\xi| - \xi_n)\Big)w\Big\|_{L^\infty} \\ &\qquad \lesssim h^{\frac{1}{2}-\beta}\sum_{\mu=0}^{1}\Big(\|(\theta_0\Omega_h)^\mu w\|_{L^2} + \|(\theta_0\Omega_h)^\mu\mathcal{M}_n w\|_{L^2}\Big).\end{aligned}$$

Proof. The statement of the lemma is just the result of tedious calculations and the application of propositions 1.2.27, 1.2.30 along with lemma 1.2.32.

Let $\widetilde{\psi} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ such that $\widetilde{\psi} \equiv 1$ on the support of ψ. From symbolic development's formula (1.2.18) and (1.2.19) we derive that for a fixed $N \in \mathbb{N}$, and up to negligible multiplicative constants,

(1.2.49)

$$\begin{aligned}&\Big[\gamma_1\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big]\sharp\Big[(x_n|\xi| - \xi_n)\widetilde{a}(x)\widetilde{\psi}(2^{-k}\xi)\Big]\\ &= \gamma_1\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_n|\xi| - \xi_n)\\ &+ h\Big\{\gamma_1\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi), (x_n|\xi| - \xi_n)\Big\}\\ &+ \sum_{\substack{2\le|\alpha|<N\\ |\alpha_1|+|\alpha_2|=|\alpha|}} h^{|\alpha|}\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}\Big[\gamma_1\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big]\partial_x^{\alpha_2}\partial_\xi^{\alpha_1}\big[(x_n|\xi| - \xi_n)\big] + r^k_{N,p}(x,\xi),\end{aligned}$$

with

(1.2.50)

$$r^k_{N,p}(x,\xi) = \frac{h^N}{(\pi h)^4}\sum_{|\alpha_1|+|\alpha_2|=N}\int e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)}\Big[\int_0^1 \partial_x^{\alpha_1}\partial_\xi^{\alpha_2}\Big[\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big]|_{(x+tz,\xi+t\zeta)}$$
$$\times(1-t)^{N-1}dt\Big]\partial_x^{\alpha_2}\partial_\xi^{\alpha_1}\big[(x_n|\xi|-\xi_n)\widetilde{a}(x)\widetilde{\psi}(2^{-k}\xi)\big]|_{(x+y,\xi+\eta)}\,dydzd\eta d\zeta\,.$$

If $\widetilde{a}\equiv 1$, we observe that $r^k_{N,p}$ can be decomposed into the sum of integrals of the form (1.2.40) and (1.2.41) with $q=1$, so its $\mathcal{L}(L^2)$ and $\mathcal{L}(L^2;L^\infty)$ norms are a $O(h^{1+p})$ if N is taken sufficiently large (e.g. $N>2p+11$). The same is true if functions $a,\widetilde{a}$ are compactly supported, as follows by propositions 1.2.28, 1.2.31 since, using lemma 1.2.26 and reminding definition (1.2.28) of $I^k_{p,q}$ for general $k\in K, p,q\in\mathbb{Z}$, we realize that

$$r^k_{N,p}(x,\xi) = \sum_{\substack{|\alpha_1|+|\alpha_2|=N\\ i\le|\alpha_1|,1\le j\le|\alpha_2|}} h^{N-(i+j)(\frac{1}{2}-\sigma)}I^k_{p+i+j-|\alpha_2|,1-|\alpha_1|}(x,\xi).$$

An explicit computation of the Poisson bracket in (1.2.49) shows that it is equal to

$$(1.2.51)\quad h(\partial\gamma_1)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big(\frac{x_1\xi_2-x_2\xi_1}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)$$
$$+\sum\nolimits' h\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi),$$

where in the latter contribution $\sum'$ is a concise notation to indicate a linear combination, and ψ,a,b_p are some new functions with the same features of their homonyms. After writing $(x_1\xi_2 - x_2\xi_1) = (x_1|\xi|-\xi_1)\xi_2|\xi|^{-1} - (x_2|\xi|-\xi_2)\xi_1|\xi|^{-1}$, we recognize that the quantization of (1.2.51) verifies estimates (1.2.47) thanks to propositions 1.2.27, 1.2.30 and the fact that $2^{kp}\le h^{-\sigma p}$.

Let us denote concisely by t^k_α the $|\alpha|$-order contributions in (1.2.49), for $2\le|\alpha|<N$. As factor $x_n|\xi|-\xi_n$ is affine in x_n, the length of multi-index α_2 is less or equal than 1 and, using lemma 1.2.26, t^k_α appears to be the sum of two terms: the first one, corresponding to $|\alpha_2|=0$, has the form

$$\sum_{i\le|\alpha|}{}' h^{|\alpha|-i(\frac{1}{2}-\sigma)}\gamma_{1+i}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_{p+i+1-|\alpha|}(\xi)\,x_n^\mu,$$

with $\mu=0$ or 1, for some new functions ψ,a (if $a\equiv 1$ then $\mu=0$, for the derivation $|\alpha_1|$-times with respect to x of $\gamma_1\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)$ makes appear, inter alia, a factor $|\xi|^{|\alpha_1|}$ that allows us to rewrite $\partial_\xi^{\alpha_1}(x_n|\xi|-\xi_n)$ from $(x_n|\xi|-\xi_n)+b_0(\xi)$ for some new b_0, and $\partial_z^{\alpha_1}\gamma_1(z)z_n$ is of the form $\gamma_{|\alpha_1|}(z)$); the second one, corresponding instead to $|\alpha_2|=1$, is

$$\sum_{i\le|\alpha|-1,j\le 1}{}' h^{|\alpha|-(i+j)(\frac{1}{2}-\sigma)}\gamma_{1+i+j}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_{p+i+j+1-|\alpha|}(\xi),$$

for some new other functions ψ,a. From propositions 1.2.27, 1.2.30 we then deduce that

$$(1.2.52a)\qquad \|Op_h^w(t^k_\alpha)w\|_{L^2}\lesssim (h^{\frac{|\alpha|}{2}-\beta}+h^{1+p})\|w\|_{L^2},$$

$$(1.2.52b)\qquad \|Op_h^w(t^k_\alpha)w\|_{L^\infty}\lesssim (h^{\frac{|\alpha|-1}{2}-\beta}+h^{\frac{1}{2}+p})(\|w\|_{L^2}+\|\theta\Omega_h w\|_{L^2}),$$

which concludes that

$$\Big[\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big]\,\sharp\,\Big[(x_n|\xi|-\xi_n)\widetilde{a}(x)\widetilde{\psi}(2^{-k}\xi)\Big]$$
$$=\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_n|\xi|-\xi_n)+r^k_p(x,\xi),$$

with r^k_p satisfying (1.2.47).

Finally, by symbolic calculus we have that, up to some multiplicative constants,

$$\begin{aligned}Op^w_h\big((x_n|\xi|-\xi_n)\widetilde{a}(x)\widetilde{\psi}(2^{-k}\xi)\big)&=\widetilde{a}(x)Op^w_h\big((x_n|\xi|-\xi_n)\widetilde{\psi}(2^{-k}\xi)\big)+Op^w_h(r^k(x,\xi))\\&=Op^w_h(\widetilde{\psi}(2^{-k}\xi))\widetilde{a}(x)h\mathcal{M}_n+h\widetilde{a}(x)Op^w_h((\partial\widetilde{\psi})(2^{-k}\xi)(2^{-k}|\xi|))\\&\quad+Op^w_h(\widetilde{r}^k(x,\xi))h\mathcal{M}_n+Op^w_h(r^k(x,\xi)),\end{aligned}$$

where

$$\begin{aligned}r^k(x,\xi)&=\frac{h}{(2\pi)^2}\int e^{\frac{2i}{h}\eta\cdot z}\int\partial_x\widetilde{a}(x+tz)dt\;\partial_\xi\big[(x_n|\xi|-\xi_n)\widetilde{\psi}(2^{-k}\xi)\big]|_{(x,\xi+\eta)}dzd\eta,\\ \widetilde{r}^k(x,\xi)&=\frac{h2^{-k}}{(2\pi)^2}\int e^{\frac{2i}{h}\eta\cdot z}\int\partial_x\widetilde{a}(x+tz)dt\;(\partial_\xi\widetilde{\psi})(2^{-k}(\xi+\eta))dzd\eta,\end{aligned}$$

are such that $\|Op^w_h(r^k_1)\|_{\mathcal{L}(L^2)}=O(h)$, $\|Op^w_h(\widetilde{r}^k_1)\|_{\mathcal{L}(L^2)}=O(1)$. An explicit computation shows also that $\|[\Omega_h,Op^w_h(r^k)]\|_{\mathcal{L}(L^2)}=O(h)$ and $\|[\Omega_h,Op^w_h(\widetilde{r}^k)]\|_{\mathcal{L}(L^2)}=O(1)$. Therefore, since $\widetilde{\psi}\equiv1$ on the support of ψ, $\widetilde{a}\equiv1$ on the support of a, one can use remark 1.2.22 together with propositions 1.2.28, 1.2.31, and also propositions 1.2.27, 1.2.30, to show that

$$Op^w_h\Big(\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)Op^w_h\big((x_n|\xi|-\xi_n)\widetilde{a}(x)\widetilde{\psi}(2^{-k}\xi)\big)$$
$$=Op^w_h\Big(\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big)\widetilde{a}(x)h\mathcal{M}_n+Op^w_h(r^k_p(x,\xi)),$$

for a new $Op^w_h(r^k_p(x,\xi))$ satisfying (1.2.47a). This concludes the proof of (1.2.46) and of the entire statement, applying propositions 1.2.27, 1.2.30 to the first operator in the above right hand side. ☐

Lemma 1.2.34. *Let us take $\sigma>0$ small, $k\in\mathbb{N}$, $p\in\mathbb{N}$. Let also $\gamma\in C^\infty_0(\mathbb{R}^2)$ be equal to 1 in a neighbourhood of the origin, $\psi\in C^\infty_0(\mathbb{R}^2\setminus\{0\})$, and $a\in C^\infty_0(\mathbb{R}^2)$. For any function $w\in L^2(\mathbb{R}^2)$ such that $\mathcal{M}w\in L^2(\mathbb{R}^2)$, any $m,n=1,2$, we have that*

$$Op^w_h\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_m|\xi|-\xi_m)(x_n|\xi|-\xi_n)\Big)w$$
$$=Op^w_h\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_m|\xi|-\xi_m)\Big)[h\mathcal{M}_nw]+O_{L^2}\big(h^{2-\beta}(\|w\|_{L^2}+\|\mathcal{M}w\|_{L^2})\big),$$

with $\beta>0$ small, $\beta\to0$ as $\sigma\to0$.

Proof. Let $\widetilde{\gamma}(z):=\gamma(z)z_m$, and $\widetilde{\psi}\in C^\infty_0(\mathbb{R}^2\setminus\{0\})$ be identically equal to 1 on the support of ψ. We saw in the proof of the previous lemma that the symbolic product

$$\Big[\widetilde{\gamma}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)\Big]\,\sharp[(x_n|\xi|-\xi_n)\widetilde{\psi}(2^{-k}\xi)]$$

develops as in (1.2.49), (1.2.50), with γ_1 replaced with $\widetilde{\gamma}$ and $\widetilde{a}\equiv 1$. From (1.2.51), the fact that

$$\{x_m|\xi|-\xi_m, x_n|\xi|-\xi_n\}=\begin{cases}0, & \text{if } m=n,\\ (-1)^{m+1}(x_1\xi_2-\xi_2x_1), & \text{if } m\ne n,\end{cases}$$

and that $(x_1\xi_2-\xi_2x_1)=(x_1|\xi|-\xi_1)\xi_2|\xi|^{-1}-(x_2|\xi|-\xi_2)\xi_1|\xi|^{-1}$, we derive that the first order term of the symbolic development is a linear combination of products of the form

$$h^{\frac{3}{2}}\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_j|\xi|-\xi_j),$$

for some new functions γ,ψ,a, and its quantization acting on w is a remainder as in the statement after lemma 1.2.33.

The second order term is given, up to some negligible multiplicative constants, by

$$\begin{split} h^{1+2\sigma}\sum_{|\alpha|=2}(\partial^\alpha\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a_1(x)b_{p+1}(\xi)(x_m|\xi|-\xi_m)\\ +h^{\frac{3}{2}+\sigma}\sum_{|\alpha|=1}(\partial^\alpha\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi_2(2^{-k}\xi)a_2(x)b_{p+1}(\xi)\\ +h^2{\sum}'\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi_3(2^{-k}\xi)a_3(x)b_{p+1}(\xi),\end{split}$$

for some new smooth, compactly supported, $\psi_2,\psi_3,a_1,a_2,a_3$, while for the third order one we have

$$\begin{split} h^{\frac{3}{2}+3\sigma}\sum_{|\alpha|=3}(\partial^\alpha\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a_1(x)b_{p+1}(\xi)(x_m|\xi|-\xi_m)\\ +h^2{\sum}'\gamma_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi_1(2^{-k}\xi)a_2(x)b_{p+1}(\xi),\end{split}$$

for some other ψ_1,a_1,a_2, and a new $\gamma_1\in C^\infty_0(\mathbb{R}^2)$. As the derivatives of γ vanish in a neighbourhood of the origin, when $|\alpha|=1$ we can replace $(\partial^\alpha\gamma)(z)$ with $\sum_j\gamma^j_1(z)z_j$, $\gamma^1_j(z):=(\partial^\alpha\gamma)(z)z_j|z|^{-2}$. Applying lemma 1.2.33 to sums on $|\alpha|=1,2,3$, and proposition 1.2.27 to the remaining ones, we derive that also the quantizations of the second and third order term are, when acting on w, a $O_{L^2}\big(h^{2-\beta}(\|w\|_{L^2}+\|\mathcal{M}w\|_{L^2})\big)$, for a small $\beta>0$, $\beta\to 0$ as $\sigma\to 0$.

In all the other $|\alpha|$-order terms, with $4\le|\alpha|\le N-1$, and in integral remainder $r^k_{N,p}$, we look at $\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_m|\xi|-\xi_m)$ as a symbol of the form

$$\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_{p+1}(\xi)$$

for a new $a_1\in C^\infty_0(\mathbb{R}^2)$. From (1.2.52a) and the fact that $Op^w_h(r^k_{N,p})w=O_{L^2}(h^{1+p})$ when $N>11$, we derive that also the quantizations of these terms, acting on w, are also a $O_{L^2}\big(h^{2-\beta}(\|w\|_{L^2}+\|\mathcal{M}w\|_{L^2})\big)$.

We finally obtained that

$$\begin{split} Op^w_h\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_m|\xi|-\xi_m)(x_n|\xi|-\xi_n)\Big)w\\ =Op^w_h\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)a(x)b_p(\xi)(x_m|\xi|-\xi_m)\Big)Op^w_h\big((x_n|\xi|-\xi_n)\widetilde{\psi}(2^{-k}\xi)\big)\\ +O_{L^2}\big(h^{2-\beta}(\|w\|_{L^2}+\|\mathcal{M}w\|_{L^2})\big).\end{split}$$

The conclusion of the proof comes, then, from the fact that, by symbolic calculus,

$$Op_h^w\big((x_n|\xi|-\xi_n)\widetilde{\psi}_1(2^{-k}\xi)\big) = hOp_h^w(\widetilde{\psi}_1(2^{-k}\xi))\mathcal{M}_n - \frac{h}{2i}Op_h^w\big((\partial\widetilde{\psi}_1)(2^{-k}\xi)\cdot(2^{-k}\xi)\big),$$

and by remark 1.2.22, since all derivatives of $\widetilde{\psi}$ vanish on the support of ψ. □

The following lemma is introduced especially for the proof of lemma 3.2.13. Even if quite similar to lemma 1.2.33, we are going to see that the particular structure of symbolic product in the left hand side of (1.2.53) allows for a remainder r^k_p satisfying enhanced estimate (1.2.54b) rather than (1.2.47b).

Lemma 1.2.35. *Let us take $\sigma>0$ sufficiently small, $k\in K$ and $p,q\in\mathbb{N}$. Let also $\gamma\in C^\infty_0(\mathbb{R}^2)$ such that $\gamma\equiv 1$ in a neighbourhood of the origin, $\psi,\widetilde{\psi}\in C^\infty_0(\mathbb{R}^2\setminus\{0\})$ such that $\psi\equiv 1$ on the support of $\widetilde{\psi}$, $a(x)$ be a smooth compactly supported function. Then*

$$\begin{aligned}(1.2.53)\quad \Big[(x_n|\xi|-\xi_n)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi)\Big]\sharp\Big[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)\Big]&\\ =\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi)(x_n|\xi|-\xi_n)+r^k_p(x,\xi),&\end{aligned}$$

where

$$(1.2.54a)\qquad \Big\|Op_h^w(r^k_p(x,\xi))w\Big\|_{L^2}\lesssim h^{\frac{3}{2}-\beta}(\|w\|_{L^2}+\|\mathcal{M}w\|_{L^2})+h^{1+p}\|w\|_{L^2},$$

$$(1.2.54b)\qquad \Big\|Op_h^w(r^k_p(x,\xi))w\Big\|_{L^\infty}\lesssim h^{1-\beta}\sum_{\mu=0}^{1}\Big(\|(\theta_0\Omega_h)^\mu w\|_{L^2}+\|(\theta_0\Omega_h)^\mu\mathcal{M}w\|_{L^2}\Big),$$

for some $\theta\in C^\infty_0(\mathbb{R}^2\setminus\{0\})$, and a small $\beta>0$, $\beta\to0$ as $\sigma\to0$.

Proof. Using proposition 1.2.21, for a fixed $N\in\mathbb{N}$ and up to multiplicative constants independent of h,k, we have the following symbolic development:

(1.2.55)

$$\begin{aligned}&\Big[(x_n|\xi|-\xi_n)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi)\Big]\sharp\Big[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)\Big]\\ &=\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\psi}(2^{-k})a(x)b_p(\xi)(x_n|\xi|-\xi_n)\\ &+h\Big\{(x_n|\xi|-\xi_n)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi),\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big\}\\ &+\sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\2\le|\alpha|<N}}h^{|\alpha|}\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}\Big[(x_n|\xi|-\xi_n)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi)\Big]\partial_x^{\alpha_2}\partial_\xi^{\alpha_1}\Big[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big]+r^k_{N,p}(x,\xi),\end{aligned}$$

with

$$\begin{aligned}r^k_{N,p}(x,\xi)=\frac{h^N}{(\pi h)^4}\sum_{|\alpha_1|+|\alpha_2|=N}\int e^{\frac{2i}{h}(\eta\cdot z-y\cdot\zeta)}\Big[\int_0^1\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}\big[(x_n|\xi|-\xi_n)a(x)b_p(\xi)\widetilde{\psi}(2^{-k}\xi)\big]|_{(x+tz,\xi+t\zeta)}&\\ \times(1-t)^{N-1}dt\Big]\partial_x^{\alpha_2}\partial_\xi^{\alpha_1}\Big[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)\Big]|_{(x+y,\xi+\eta)}\,dydzd\eta d\zeta.&\end{aligned}$$

For sake of simplicity, we denote by t_1^k (resp. t_α^k, $|\alpha|=2,\dots,N-1$) the Poisson brackets (resp. the $|\alpha|$-th contribution) in (1.2.55). An explicit computation of t_1^k, combined with the fact that $x_1\xi_2-x_2\xi_1=(x_1|\xi|-\xi_1)\xi_2|\xi|^{-1}-(x_2|\xi|-\xi_2)\xi_1|\xi|^{-1}$, shows that it is linear combination of terms of the form

$$h(\partial\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big(\frac{x_j|\xi|-\xi_j}{h^{1/2-\sigma}}\Big)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi),$$

for $j\in\{1,2\}$ and some new functions $\widetilde{\psi},a,b_p$, so by inequalities (1.2.48) we derive that

$$\Big\|Op_h^w(t_1^k)w\Big\|_{L^2}\lesssim h^{\frac{3}{2}-\beta}\left(\|w\|_{L^2}+\|\mathcal{M}w\|_{L^2}\right), \tag{1.2.56a}$$

$$\Big\|Op_h^w(t_1^k)w\Big\|_{L^\infty}\lesssim h^{1-\beta}\sum_{\mu=0}^{1}(\|(\theta_0\Omega_h)^\mu w\|_{L^2}+\|(\theta_0\Omega_h)^\mu\mathcal{M}w\|_{L^2}). \tag{1.2.56b}$$

The improvement of these estimates with respect to (1.2.47) is attributable to the choice of ψ identically equal to 1 on the support of $\widetilde{\psi}$. All derivatives of ψ vanish against $\widetilde{\psi}$, so in the development of t_1^k we avoid terms like $\gamma\big(\frac{x|\xi|-\xi|}{h^{1/2-\sigma}}\big)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi)(\partial\psi)(2^{-k}\xi)(2^{-k}|\xi|)$, coming out from $\{x_n|\xi|-\xi_n,\psi(2^{-k}\xi)\}\gamma\big(\frac{x|\xi|-\xi|}{h^{1/2-\sigma}}\big)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi)$, that do not enjoy estimates like (1.2.56). Using formula (1.2.24) and looking at $(x_n|\xi|-\xi_n)\widetilde{\psi}(2^{-k}\xi)a(x)b_p(\xi)$ as a linear combination of terms $\widetilde{\psi}(2^{-k}\xi)a(x)b_{p+1}(\xi)$, for some new $\widetilde{\psi},a,b_{p+1}$, we realize that, for any $2\le|\alpha|<N$,

$$t_\alpha^k=\sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha|\\1\le j\le|\alpha_1|}}h^{|\alpha|-(j+|\alpha_2|)(\frac{1}{2}-\sigma)}\gamma_{j+|\alpha_2|}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\psi}(2^{-k}\xi)a_j(x)b_{p+j+1-|\alpha_1|}(\xi),$$

for some new other $\widetilde{\psi},a_j$, with a_j compactly supported, and then that

$$\|Op_h^w(t_\alpha^k)w\|_{L^2}\lesssim\sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha|\\1\le j\le|\alpha_1|}}h^{|\alpha|-(j+|\alpha_2|)(\frac{1}{2}-\sigma)}2^{k(p+j+1-|\alpha_1|)}\|w\|_{L^2},$$

$$\begin{aligned}&\|Op_h^w(t_\alpha^k)w\|_{L^\infty}\\&\lesssim\sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha|\\1\le j\le|\alpha_1|}}h^{|\alpha|-(j+|\alpha_2|)(\frac{1}{2}-\sigma)}2^{k(p+j+1-|\alpha_1|)}h^{-\frac{1}{2}-\sigma}\sum_{\mu=0}^{1}(\|(\theta_0\Omega_h)^\mu w\|_{L^2}+\|(\theta_0\Omega_h)^\mu\mathcal{M}w\|_{L^2}),\end{aligned}$$

after propositions 1.2.27, 1.2.30. For $|\alpha|\ge3$, the above estimates imply $\|Op_h^w(t_\alpha^k)\|_{\mathcal{L}(L^2)}\lesssim h^{\frac{3}{2}-\beta}$ and $\|Op_h^w(t_\alpha^k)w\|_{L^\infty}\lesssim h^{1-\beta}\sum_{\mu=0}^{1}(\|(\theta_0\Omega_h)^\mu w\|_{L^2}+\|(\theta_0\Omega_h)^\mu\mathcal{M}w\|_{L^2})$. For $|\alpha|=2$, we exploit the fact that functions $\gamma_{j+|\alpha_2|}$ vanish in a neighbourhood of the origin, as they come from γ's derivatives, and define $\gamma^i_{j+|\alpha_2|}(z):=\gamma_{j+|\alpha_2|}(z)z_i|z|^{-2}$, $i=1,2$, so that

$$t_\alpha^k=\sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha|\\1\le j\le|\alpha_1|,i=1,2}}h^{|\alpha|-(j+|\alpha_2|)(\frac{1}{2}-\sigma)}\gamma^i_{j+|\alpha_2|}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big(\frac{x_i|\xi|-\xi_i}{h^{1/2-\sigma}}\Big)\widetilde{\psi}(2^{-k}\xi)a_j(x)b_{p+j+1-|\alpha_1|}(\xi),$$

to which we can then apply lemma 1.2.33. After inequalities (1.2.48), $Op_h^w(t_\alpha^k)$ with $|\alpha|=2$ also satisfies (1.2.56).

Finally, reminding definition (1.2.29) of $J^k_{p,q}(x,\xi)$ for general $k \in K, p,q \in \mathbb{Z}$, and developing derivatives in $r^k_{N,p}$ using lemma 1.2.26, we observe that

$$r^k_{N,p} = \sum_{\substack{|\alpha_1|+|\alpha_2|=N\\ 0\le j\le |\alpha_1|}} h^{N-(|\alpha_2|+j)(\frac{1}{2}-\sigma)} J^k_{p+1-|\alpha_2|,|\alpha_2|+j-|\alpha_1|}(x,\xi),$$

hence propositions 1.2.28 and 1.2.31 give that

$$\|Op^w_h(r^k_{N,p})\|_{\mathcal{L}(L^2)} \lesssim \sum_{\substack{|\alpha_1|+|\alpha_2|=N\\ 0\le j\le |\alpha_1|}} h^{N-(|\alpha_2|+j)(\frac{1}{2}-\sigma)} 2^{k(p+1+j-|\alpha_1|)} \lesssim h^{1+p},$$

$$\|Op^w_h(r^k_{N,p})\|_{\mathcal{L}(L^2;L^\infty)} \lesssim \sum_{\substack{|\alpha_1|+|\alpha_2|=N\\ 0\le j\le |\alpha_1|, i\le 6}} h^{N-(|\alpha_2|+j)(\frac{1}{2}-\sigma)} 2^{k(p+1+j-|\alpha_1|)} (h^{-\frac{1}{2}+\sigma}2^k)^i (h^{-1}2^k) \lesssim h^{1+p},$$

if N is chosen sufficiently large (e.g. $N > 10+2p$). We should also highlight the fact that, at the difference of (1.2.54b), (1.2.54a) does not improve (1.2.47a): if we get a $h^{\frac{3}{2}-\beta}$ factor in front of the first term in the right hand side, the second term $h^{1+p}\|w\|_{L^2}$ is just a $O(h^{1-\beta})$ in the case $p=0$, coming from $|\alpha_1|=N$, $j=|\alpha_2|=0$, $p=0$ above. □

1.2.4 Operators for the Klein-Gordon Solution: Some Estimates

This subsection is mostly devoted to the introduction of some symbols and operators, along with their properties, that we will often use in the paper when dealing with the Klein-Gordon component of the solution to starting system (1.1.1). From now on, we will use the notation $p(\xi) := \sqrt{1+|\xi|^2}$ (thus, $p'(\xi)$ denotes the gradient of $p(\xi)$, $p''(\xi) = (\partial^2_{ij}p(\xi))_{ij}$ the 2×2 Hessian matrix of $p(\xi)$).

The first statement is a general result about continuity of operators with symbols of order $r\in\mathbb{R}$ on spaces $H^s_h(\mathbb{R}^2)$, and generalises theorem 7.11 in [8]. The second statement is a result of continuity from L^2 to $H^{\rho,\infty}_h$ of a particular class of operators that will act on the Klein-Gordon component. In the spirit of [14] for the Schrödinger equation, it allows to pass from uniform norms to the L^2 norm losing only a power $h^{-\frac{1}{2}-\beta}$ for a small $\beta>0$, instead of a h^{-1} as for the semi-classical Sobolev injection. Proposition 1.2.38 is, instead, a result of uniform $L^p - L^p$ continuity of such operators, for every $1\le p\le +\infty$, and it will particularly useful in the case $p=+\infty$.

Proposition 1.2.36 (Continuity on H^s_h)**.** *Let $s\in\mathbb{R}$. Let $a\in S_{\delta,\sigma}(\langle\xi\rangle^r)$, $r\in\mathbb{R}$, $\delta\in[0,\frac{1}{2}]$, $\sigma\ge 0$. Then $Op^w_h(a)$ is uniformly bounded : $H^s_h(\mathbb{R}^2)\to H^{s-r}_h(\mathbb{R}^2)$, and there exists a positive constant C independent of h such that*

$$\|Op^w_h(a)\|_{\mathcal{L}(H^s_h;H^{s-r}_h)} \le C\,, \qquad \forall h\in]0,1]\,.$$

Proposition 1.2.37 (Continuity from L^2 to $H^{\rho,\infty}_h$)**.** *Let $\rho\in\mathbb{N}$. Let $a\in S_{\delta,\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-1})$, $\delta\in[0,\frac{1}{2}]$, $\sigma>0$. Then $Op^w_h(a)$ is bounded : $L^2(\mathbb{R}^2)\to H^{\rho,\infty}_h(\mathbb{R}^2)$, and there exists a positive constant C independent of h such that*

$$\|Op^w_h(a)\|_{\mathcal{L}(L^2;H^{\rho,\infty}_h)} \le Ch^{-\frac{1}{2}-\beta}\,, \qquad \forall h\in]0,1]\,,$$

where $\beta>0$ depends linearly on σ.

Proof. We first remark that, after definition 1.2.18 (i) of the $H^{\rho,\infty}_h$ norm,

$$\|Op^w_h(a)w\|_{H^{\rho,\infty}_h} = \|\langle hD_x\rangle^\rho Op^w_h(a)w\|_{L^\infty},$$

and that, by symbolic calculus of lemma 1.2.24, $\langle\xi\rangle^\rho\sharp a(x,\xi)$ belongs to $S_{\delta,\sigma}(\langle\xi\rangle^\rho\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-1}) \subset h^{-\rho\sigma}S_{\delta,\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-1})$. This means that estimating the $H^{\rho,\infty}_h$ norm of an operator whose symbol is rapidly decaying in $|h^\sigma\xi|$ corresponds actually to estimate the L^∞ norm of an operator associated to another symbol (namely, $\tilde{a}(x,\xi) = \langle\xi\rangle^\rho\sharp a(x,\xi)$) which is still in the same class as a, up to a small loss $h^{-\rho\sigma}$.

From definition 1.2.17 (i) of $Op^w_h(a)w$, and using a change of coordinates $y\mapsto\sqrt{h}y$, $\xi\mapsto\sqrt{h}\xi$, integration by part, Cauchy-Schwarz inequality, and Young's inequality for convolutions, we derive what follows:

(1.2.57)

$$\begin{aligned}
&|Op^w_h(a)w| = \\
&= \left|\frac{1}{(2\pi)^2}\int\int e^{i(\frac{x}{\sqrt{h}}-y)\cdot\xi}a\Big(\frac{x+\sqrt{h}y}{2},\sqrt{h}\xi\Big)w(\sqrt{h}y)\,dyd\xi\right| \\
&= \left|\frac{1}{(2\pi)^4 h}\int \hat{w}\Big(\frac{\eta}{\sqrt{h}}\Big)d\eta\int\int e^{i(\frac{x}{\sqrt{h}}-y)\cdot\xi+i\eta\cdot y}a\Big(\frac{x+\sqrt{h}y}{2},\sqrt{h}\xi\Big)\,dyd\xi\right| \\
&= \left|\frac{1}{(2\pi)^4 h}\int \hat{w}\Big(\frac{\eta}{\sqrt{h}}\Big)\int\int\left(\frac{1-i(\frac{x}{\sqrt{h}}-y)\cdot\partial_\xi}{1+|\frac{x}{\sqrt{h}}-y|^2}\right)^3\left(\frac{1+i(\xi-\eta)\cdot\partial_y}{1+|\xi-\eta|^2}\right)^3 e^{i(\frac{x}{\sqrt{h}}-y)\cdot\xi+i\eta\cdot y}\right. \\
&\qquad\left.\times\, a\Big(\frac{x+\sqrt{h}y}{2},\sqrt{h}\xi\Big)\,dyd\xi d\eta\right| \\
&\lesssim \frac{1}{h}\int\left|\hat{w}\Big(\frac{\eta}{\sqrt{h}}\Big)\right|\int\int\Big\langle\frac{x}{\sqrt{h}}-y\Big\rangle^{-3}\langle\xi-\eta\rangle^{-3}\langle h^\sigma\sqrt{h}\xi\rangle^{-N}\Big\langle\frac{\frac{x+\sqrt{h}y}{2}-p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1}dyd\xi d\eta \\
&\lesssim \frac{1}{h}\left\|\hat{w}\Big(\frac{\cdot}{\sqrt{h}}\Big)\right\|_{L^2}\|\langle\eta\rangle^{-3}\|_{L^1(\eta)}\left\|\int\Big\langle\frac{x}{\sqrt{h}}-y\Big\rangle^{-3}\langle h^\sigma\sqrt{h}\xi\rangle^{-N}\Big\langle\frac{\frac{x+\sqrt{h}y}{2}-p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1}dy\right\|_{L^2(d\xi)} \\
&\lesssim h^{-\frac{1}{2}}\|w\|_{L^2}\int\Big\langle\frac{x}{\sqrt{h}}-y\Big\rangle^{-3}\left\|\langle h^\sigma\sqrt{h}\xi\rangle^{-N}\Big\langle\frac{\frac{x+\sqrt{h}y}{2}-p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1}\right\|_{L^2(\xi)}dy\,,
\end{aligned}$$

where $N>0$ will be properly chosen later. We draw attention to two facts: in the third equality in (1.2.57), we use that

$$\left(\frac{1-i(\frac{x}{\sqrt{h}}-y)\cdot\partial_\xi}{1+(\frac{x}{\sqrt{h}}-y)^2}\right)^3\left(\frac{1+i(\xi-\eta)\cdot\partial_y}{1+(\xi-\eta)^2}\right)^3\Big[e^{i(\frac{x}{\sqrt{h}}-y)\cdot\xi+i\eta\cdot y}\Big] = e^{i(\frac{x}{\sqrt{h}}-y)\cdot\xi+i\eta\cdot y}$$

so, integrating by part, derivatives ∂_y,∂_ξ fall on $\langle\frac{x}{\sqrt{h}}-y\rangle^{-1}$, $\langle\xi-\eta\rangle^{-1}$ (giving rise to more decreasing factors) and/or on $a\left(\frac{x+\sqrt{h}y}{2},\sqrt{h}\xi\right)$; symbol a belongs to $S_{\delta,\sigma}(1)$ with $\delta\le\frac{1}{2}$, but the loss of $h^{-\delta}$ is offset by the factor $\sqrt{h}$ coming from the derivation of $a(\frac{x+\sqrt{h}y}{2},\sqrt{h}\xi)$ with respect to y and ξ.

In order to estimate $\|\langle h^\sigma\sqrt{h}\xi\rangle^{-N}\langle\frac{\frac{x+\sqrt{h}y}{2}-p'(\sqrt{h}\xi)}{\sqrt{h}}\rangle^{-1}\|_{L^2_\xi}$ we first introduce a smooth cut-off function $\chi(\frac{x+\sqrt{h}y}{2})$, with χ supported in some ball $B_C(0)$, to distinguish between the case when $\frac{x+\sqrt{h}y}{2}$

is bounded from the one where $|\frac{x+\sqrt{h}y}{2}| \to +\infty$. In the latter situation, say for $|\frac{x+\sqrt{h}y}{2}| > 2$, we have $\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\sqrt{h}\xi)}{\sqrt{h}} \rangle^{-1} \lesssim \sqrt{h}$ and

$$\Big|(1-\chi)\Big(\frac{x+\sqrt{h}y}{2}\Big)\Big|\Big\|\langle h^\sigma \sqrt{h}\xi\rangle^{-N}\Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1}\Big\|_{L^2(\xi)} \lesssim h^{-\sigma}.$$

On the other hand, when $\frac{x+\sqrt{h}y}{2}$ is bounded we consider a Littlewood-Paley decomposition and write

(1.2.58)

$$\begin{aligned}\Big\|\langle h^\sigma \sqrt{h}\xi\rangle^{-N}\Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1}\Big\|^2_{L^2(\xi)} &= h^{-1}\sum_{k\geq 0}\int \langle h^\sigma \xi\rangle^{-2N}\Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\xi)}{\sqrt{h}}\Big\rangle^{-2}\varphi_k(\xi)d\xi \\ &= h^{-1}\sum_{k\geq 0} I_k\,,\end{aligned}$$

where

$$I_0 = \int \langle h^\sigma \xi\rangle^{-2N}\Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\xi)}{\sqrt{h}}\Big\rangle^{-2}\varphi_0(\xi)d\xi\,,$$

and

$$\begin{aligned}I_k &= \int \langle h^\sigma \xi\rangle^{-2N}\Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\xi)}{\sqrt{h}}\Big\rangle^{-2}\varphi(2^{-k}\xi)d\xi \\ &= 2^{2k}\int \langle h^\sigma 2^k\xi\rangle^{-2N}\Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(2^k\xi)}{\sqrt{h}}\Big\rangle^{-2}\varphi(\xi)d\xi\,, \qquad k\geq 1 \\ &\lesssim 2^{(-2N+2)k}h^{-2\sigma N}\int \Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(2^k\xi)}{\sqrt{h}}\Big\rangle^{-2}\varphi(\xi)d\xi\,.\end{aligned} \tag{1.2.59}$$

For a fixed k_0 and any $k \leq k_0$, $|\det(p''(2^k\xi))| \geq C > 0$ on the support of φ. For $k \geq k_0$, function $\xi \to g_k(\xi) = 2^{3k}(\frac{x+\sqrt{h}y}{2}) - 2^{3k}p'(2^k\xi)$ is such that $\det(g_k'(\xi)) = \frac{2^{4k}}{(1+|2^k\xi|^2)^2}$, and $|\det(g_k'(\xi))| \sim 1$ on the support of φ. We may thus split the $d\xi$ integral in a finite number (independent of k) of integrals, computed on compact domains, on which $\xi \mapsto g_k(\xi)$ is a change of variables with jacobian of size 1. We are then reduced to estimate $2^{(-2N+2)k}h^{-2\sigma N}\int_{|z|\leq C}\langle \frac{z+g_k(\xi_0)}{2^{3k}\sqrt{h}}\rangle^{-2}dz$, where C is a positive constant and ξ_0 is in $supp\varphi$. Since we assumed that $\frac{x+\sqrt{h}y}{2}$ is bounded, $|g_k(\xi_0)| = O(2^{3k})$ and we get

$$\begin{aligned}I_k &\lesssim 2^{(-2N+2)k}h^{-2\sigma N}\int_{|z|\lesssim 2^{3k}}\langle \frac{z}{2^{3k}\sqrt{h}}\rangle^{-2}dz \\ &\lesssim 2^{(-2N+8)k}h^{-2\sigma N}h\int_{|z|\lesssim h^{-1/2}}\langle z\rangle^{-2}dz \\ &\lesssim 2^{(-2N+8)k}h^{-2\sigma N+1}\log(h^{-1})\,,\end{aligned}$$

so taking the sum of all I_k for $k \geq 0$ we deduce that

$$\Big\|\langle h^\sigma \sqrt{h}\xi\rangle^{-N}\Big\langle \frac{\frac{x+\sqrt{h}y}{2} - p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1}\Big\|_{L^2(\xi)} \lesssim h^{-\sigma N-\delta}\Big(\sum_{k\geq 0}2^{(-2N+8)k}\Big)^{\frac{1}{2}} \lesssim h^{-\sigma N-\delta}\,,$$

for $\delta > 0$ as small as we want, if we choose $N > 0$ such that $-2N + 8 < 0$ (e.g. $N = 5$). Finally

$$\|Op_h^w(a)\|_{\mathcal{L}(L^2;H_h^{\rho,\infty})} = O(h^{-\frac{1}{2}-\beta}),$$

where $\beta(\sigma) = (N+\rho)\sigma + \delta$. □

Proposition 1.2.38 (Continuity from L^p to L^p)**.** *Let $\gamma, \chi \in C_0^\infty(\mathbb{R}^2)$ be equal to 1 in a neighbourhood of the origin, with sufficiently small support, $\Sigma(\xi) = \langle\xi\rangle^\rho$ with $\rho \in \mathbb{N}$, and $\sigma > 0$. Then $Op_h^w\big(\gamma(\frac{x-p'(\xi)}{\sqrt{h}})\chi(h^\sigma\xi)\Sigma(\xi)\big) : L^p \to L^p$ is bounded, with $\mathcal{L}(L^p)$ norm bounded by $h^{-\sigma\rho-\beta}$, for a small $\beta > 0$, $\beta \to 0$ as $\sigma \to 0$, for every $1 \le p \le +\infty$.*

Proof. The first thing to observe it that, as the support of $\gamma(\frac{x-p'(\xi)}{\sqrt{h}})\chi(h^\sigma\xi)\Sigma(\xi)$ is included in $\{(x,\xi) | |\xi| \lesssim h^{-\sigma}, |x| \le 1 - ch^{2\sigma}\}$, for a small constant $c > 0$, we may find a smooth function $\theta_h(x)$, equal to 1 for $|x| \le 1 - ch^{2\sigma}$ and supported for $|x| \le 1 - c_1 h^{2\sigma}$, for some $0 < c_1 < c$, with $\|\partial^\alpha\theta_h\|_{L^\infty} = O(h^{-2|\alpha|\sigma})$ and $(h\partial_h)^k\theta_h$ bounded for every $k \in \mathbb{N}$, such that

$$\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma\xi)\Sigma(\xi) = \theta_h(x)\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma\xi)\Sigma(\xi).$$

Moreover, $\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma\xi)\Sigma(\xi)$ is localised around manifold $\Lambda_{kg} := \{(x,\xi) : x - p'(\xi) = 0\}$, which appears to be the graph of function $\xi = -d\phi(x)$, with $\phi(x) = \sqrt{1-|x|^2}$. We can therefore find a new smooth cut-off function γ_1, suitably supported, so that

$$Op_h^w\left(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma\xi)\Sigma(\xi)\theta_h(x)\right) = Op_h^w\left(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma\xi)\Sigma(\xi)\gamma_1\Big(\frac{\xi+d\phi(x)}{h^{1/2-\beta}}\Big)\theta_h(x)\right),$$

where $\beta > 0$ is a small constant, $\beta \to 0$ as $\sigma \to 0$, that takes into account the degeneracy of the equivalence between the two equations of Λ_{kg} when approaching the boundary of $supp\theta_h$. If we look at the kernel associated to above operator, denoting $\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma\xi)\Sigma(\xi)$ concisely by $A(x,\xi)$,

$$\begin{aligned}
K(x,y) &:= \frac{1}{(2\pi h)^2}\int e^{\frac{i}{h}(x-y)\cdot\xi}A\left(\frac{x+y}{2},\xi\right)\gamma_1\Big(\frac{\xi+d\phi(\frac{x+y}{2})}{h^{1/2-\beta}}\Big)\theta_h\Big(\frac{x+y}{2}\Big)d\xi \\
&= \frac{e^{-\frac{i}{h}(x-y)\cdot d\phi(\frac{x+y}{2})}}{(2\pi h)^2}\theta_h\Big(\frac{x+y}{2}\Big)\int e^{\frac{i}{h}(x-y)\cdot\xi}A\left(\frac{x+y}{2},\xi - d\phi\Big(\frac{x+y}{2}\Big)\right)\gamma_1\Big(\frac{\xi}{h^{1/2-\beta}}\Big)d\xi,
\end{aligned}$$

we observe that, since

$$\Big(\frac{x}{\sqrt{h}}\Big)^\alpha e^{\frac{i}{h}(x-y)\cdot\xi} = \Big(\frac{\sqrt{h}}{i}\Big)^{|\alpha|}\partial_\xi^\alpha e^{\frac{i}{h}(x-y)\cdot\xi}$$

and $h^{|\alpha|/2}\partial_\xi^\alpha A(\frac{x+y}{2},\xi)$ is bounded by $h^{-\sigma\rho}$, for any $\alpha \in \mathbb{N}^2$, by making some integration by parts one obtains that

$$\left|\Big(\frac{x}{\sqrt{h}}\Big)^\alpha K(x,y)\right| \lesssim h^{-2-\sigma\rho}\int_{|\xi|\lesssim h^{1/2-\beta}} d\xi \lesssim h^{-1-\sigma\rho-2\beta}, \quad \forall (x,y) \in \mathbb{R}^2\times\mathbb{R}^2.$$

This means in particular that

$$|K(x,y)| \lesssim h^{-1-\sigma\rho-2\beta}\langle\frac{x}{\sqrt{h}}\rangle^{-3}, \quad |K(x,y)| \lesssim h^{-1-\sigma\rho-2\beta}\langle\frac{y}{\sqrt{h}}\rangle^{-3}, \quad \forall (x,y)$$

implying that

$$\sup_x \int |K(x,y)|dy \lesssim h^{-\sigma\rho-2\beta}, \quad \sup_y \int |K(x,y)|dx \lesssim h^{-\sigma\rho-2\beta},$$

and its associated operator is bounded on L^p with norm $O(h^{-\sigma\rho-2\beta})$, for every $1 \le p \le +\infty$. □

The following lemma shows that we have nice upper bounds for operators whose symbol is supported for large frequencies $|\xi|\ge h^{-\sigma}$, $\sigma>0$, when acting on functions w that belong to H^s_h, for some large s. We state it in space dimension 2, but it is clear that it holds in general space dimension $d\ge 1$. This result is useful when we want to reduce to symbols rapidly decaying in $|h^\sigma\xi|$, for example in the intention of using proposition 1.2.37, or when we want to pass from a symbol of a certain positive order to another one of order zero, up to small losses of order $O(h^{-\beta})$, $\beta>0$ depending linearly on σ. We can always split a symbol using that $1=\chi(h^\sigma\xi)+(1-\chi)(h^\sigma\xi)$, for a smooth χ equal to 1 close to the origin, and consider as remainders all contributions coming from the latter.

Lemma 1.2.39. *Let $s'\ge 0$ and $\chi\in C^\infty_0(\mathbb{R}^2)$, $\chi\equiv 1$ in a neighbourhood of zero. Then*

$$\|Op^w_h((1-\chi)(h^\sigma\xi))w\|_{H^{s'}_h}\le Ch^{\sigma(s-s')}\|w\|_{H^s_h}\,,\qquad \forall s>s'\,.$$

Proof. The result is a simple consequence of the fact that $(1-\chi)(h^\sigma\xi)$ is supported for $|\xi|\gtrsim h^{-\sigma}$, because

$$\begin{aligned}\|Op^w_h((1-\chi)(h^\sigma\xi))w\|^2_{H^{s'}_h}&=\int(1+|h\xi|^2)^{s'}|(1-\chi)(h^\sigma h\xi)|^2|\hat{w}(\xi)|^2d\xi\\&=\int(1+|h\xi|^2)^{s}(1+|h\xi|^2)^{s'-s}|(1-\chi)(h^\sigma h\xi)|^2|\hat{w}(\xi)|^2d\xi\\&\le Ch^{2\sigma(s-s')}\|w\|^2_{H^s_h}\,,\end{aligned}$$

where the last inequality follows from an integration on $|h\xi|\gtrsim h^{-\sigma}$, and from the fact that $s'-s<0$, $(1+|h\xi|^2)^{s'-s}\le Ch^{-2\sigma(s'-s)}$. ☐

We introduce the following operator:

$$\mathcal{L}_j:=\frac{1}{h}Op^w_h(x-p'_j(\xi)),\quad j=1,2, \tag{1.2.60}$$

and use the notation $\|\mathcal{L}^\gamma w\|=\|\mathcal{L}^{\gamma_1}_1\mathcal{L}^{\gamma_2}_2 w\|$, for any $\gamma=(\gamma_1,\gamma_2)\in\mathbb{N}^2$.

Lemma 1.2.40. *Let $\gamma\in C^\infty_0(\mathbb{R}^2)$ be equal to 1 in a neighbourhood of the origin, $c(x,\xi)\in S_{\delta,\sigma}(1)$ with $\delta\in[0,\frac{1}{2}[$ and $\sigma>0$. Then $\gamma(\frac{x-p'(\xi)}{\sqrt{h}})c(x,\xi)$ belongs to $S_{\frac{1}{2},\sigma}(1)(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-N})$, for all $N\ge 0$.*

Proof. Straightforward. ☐

Lemma 1.2.41. *Let $n\in\mathbb{N}$ and $\gamma_n(z)$ be a smooth function such that $|\partial^\alpha\gamma_n(z)|\lesssim\langle z\rangle^{-|\alpha|-n}$ for all $\alpha\in\mathbb{N}^2$. Let also $c(x,\xi)\in S_{\delta,\sigma}(1)$, with $\delta\in[0,\frac{1}{2}[$, $\sigma>0$, be supported for $|\xi|\lesssim h^{-\sigma}$. Up to some multiplicative constants independent of h, we have the following equality:*

$$\begin{aligned}\Big[c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big]\sharp\big(x_j-p'_j(\xi)\big)&=c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\big(x_j-p'_j(\xi)\big)\\&+h\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\big[(\partial_{\xi_j}c)+(\partial_x c)\cdot(\partial_\xi p'_j)\big]+h\sum_{|\alpha|=2}(\partial^\alpha\gamma_n)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)c(x,\xi)(\partial^\alpha_\xi p'_j)(\xi)+r(x,\xi),\end{aligned}\tag{1.2.61}$$

with $r\in h^{3/2-\delta}S_{\frac{1}{2},\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-n}\big)$, and if $\chi\in C^\infty_0(\mathbb{R}^2)$ is such that $\chi(h^\sigma\xi)\equiv 1$ on the support of $c(x,\xi)$,

$$\Big\|Op^w_h\Big(c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_j-p'_j(\xi))\Big)\widetilde{v}\Big\|_{L^2}\lesssim\sum_{|\gamma|=0}^{1}h^{1-\beta}\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\gamma\widetilde{v}\|_{L^2}\,, \tag{1.2.62a}$$

(1.2.62b) $$\Big\|Op_h^w\Big(c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_j-p'_j(\xi))\Big)\widetilde{v}\Big\|_{L^\infty}\lesssim\sum_{|\gamma|=0}^{1}h^{\frac{1}{2}\delta_n-\beta}\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}^\gamma\widetilde{v}\|_{L^2}\,,$$

where $\delta_n=1$ if $n>0$, 0 otherwise, and $\beta>0$ is small, $\beta\to0$ as $\delta,\sigma\to0$.

Moreover, if $n\in\mathbb{N}^*$ and $\partial^\alpha\gamma_n$ vanishes in a neighbourhood of the origin whenever $|\alpha|\ge1$, we also have that

(1.2.63a) $$\Big\|Op_h^w\Big(c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_i-p'_i(\xi))(x_j-p'_j(\xi))\Big)\widetilde{v}\Big\|_{L^2}\lesssim \sum_{0\le|\gamma|\le2}h^{2-\beta}\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}^\gamma\widetilde{v}\|_{L^2},$$

(1.2.63b) $$\Big\|Op_h^w\Big(c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_i-p'_i(\xi))(x_j-p'_j(\xi))\Big)\widetilde{v}\Big\|_{L^\infty}\lesssim \sum_{0\le|\gamma|\le2}h^{\frac{3}{2}-\beta}\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}^\gamma\widetilde{v}\|_{L^2}.$$

Proof. As $c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\in S_{\frac{1}{2},\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-n}\big)$ and $\partial^\alpha_{x,\xi}(x_j-p'_j(\xi))\in S_{0,0}(1)$ for any $|\alpha|\ge1$, equality (1.2.61) follows from last part of lemma 1.2.24 and symbolic development (1.2.18) until order 2, after having observed that

(1.2.64) $$\Big\{c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big),x_j-p'_j(\xi)\Big\}=\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\big[(\partial_{\xi_j}c)+(\partial_xc)\cdot(\partial_\xi p'_j)\big],$$

and that, up to some multiplicative negligible,

$$\begin{aligned}h^2\sum_{|\alpha|=2}\partial^\alpha_x\Big[c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big](\partial^\alpha_\xi p'_j)(\xi)&=h\sum_{|\alpha|=2}(\partial^\alpha\gamma_n)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)c(x,\xi)(\partial^\alpha_\xi p'_j)(\xi)\\&\underbrace{+h^{\frac{3}{2}}\sum_{\substack{|\alpha|=2\\|\alpha_1|,|\alpha_2|=1}}(\partial^{\alpha_1}\gamma_n)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(\partial^{\alpha_2}_xc)(x,\xi)(\partial^\alpha_\xi p'_j)(\xi)+h^2\sum_{|\alpha|=2}\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(\partial^\alpha_xc)(x,\xi)(\partial^\alpha_\xi p'_j)(\xi)}_{\in h^{\frac{3}{2}-\delta}S_{\frac{1}{2},\sigma}\left(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-n}\right)}.\end{aligned}$$

If χ is a cut-off function as in the statement, its derivatives vanish on the support of $c(x,\xi)$, and from remark 1.2.22

(1.2.65) $$c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)=\Big[c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big]\sharp\chi(h^\sigma\xi)+r_\infty(x,\xi)$$

with $r_\infty\in h^NS_{\frac{1}{2},\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-n})$, $N\in\mathbb{N}$ as large as we want. Estimates (1.2.62) follow then as a straight consequence of (1.2.61), definition (1.2.60) of $\mathcal{L}_j$, proposition 1.2.36 and semi-classical Sobolev's injection (1.2.15) (resp. proposition 1.2.37) when $n=0$ (resp. $n>0$).

If $n\in\mathbb{N}^*$ and $\partial^\alpha\gamma_n$ vanishes in a neighbourhood of the origin whenever $|\alpha|\ge1$, we have the following equality, obtained using (1.2.61) with γ_n replaced by $\widetilde{\gamma}_{n-1}(z)=\gamma_n(z)z_i$, where

$|\partial^\alpha\widetilde{\gamma}_{n-1}(z)|\lesssim\langle z\rangle^{-|\alpha|-(n-1)}$,

$$c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_i-p'_i(\xi))(x_j-p'_j(\xi))=\Big[c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_i-p'_i(\xi))\Big]\sharp(x_j-p'_j(\xi))$$
$$-h\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_i-p'_i(\xi))\left[(\partial_{\xi_j}c)+(\partial_x c)\cdot(\partial_\xi p'_j)\right]$$
$$-h^{\frac{3}{2}}\sum_{|\alpha|=2}(\partial^\alpha\widetilde{\gamma}_{n-1})\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)c(x,\xi)(\partial^\alpha_\xi p'_j)(\xi)-\sqrt{h}r(x,\xi),$$

with $r\in h^{\frac{3}{2}-\delta}S_{\frac{1}{2},\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-(n-1)}\big)$. As $\partial^\alpha\widetilde{\gamma}_{n-1}$ vanishes in a neighbourhood of the origin for $|\alpha|=2$, we rewrite it as $\sum_{l=1}^2\widetilde{\gamma}^l_{n+2}(z)z_l$, where $\widetilde{\gamma}^l_{n+2}(z):=(\partial^\alpha\widetilde{\gamma}_{n-1})(z)z_l|z|^{-2}$ is such that $|\partial^\beta\widetilde{\gamma}^l_{n+2}(z)|\lesssim\langle z\rangle^{-|\beta|-(n+2)}$. Then, using again equality (1.2.61) for all products different from $r(x,\xi)$ in the above right hand side (with c replaced with $h^\delta[(\partial_{\xi_j}c)-(\partial_x c)\cdot(\partial_\xi p'_j)]$ in the second addend, and γ_n and c replaced, respectively, with $\widetilde{\gamma}^l_{n+2}$ and $c(\partial^\alpha_\xi p'_j)$ in the third one, $l=1,2$) we find that

$$c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_i-p'_i(\xi))(x_j-p'_j(\xi))=$$
$$\Big[c(x,\xi)\gamma_n\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big]\sharp(x_i-p'_i(\xi))\sharp(x_j-p'_j(\xi))+hr_1(x,\xi)\sharp(x_j-p'_j(\xi))-\sqrt{h}r(x,\xi),$$

for a new $r_1\in h^{-\delta}S_{\frac{1}{2},\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-n}\big)$. Estimates (1.2.63) are then obtained using (1.2.65) and propositions 1.2.36, 1.2.37). □

We will also need the following result, which is detailed in lemma 1.2.6 in [7] for the one-dimensional case.

Lemma 1.2.42. *Let $\gamma\in C^\infty_0(\mathbb{R}^2)$, and $\phi(x)=\sqrt{1-|x|^2}$. If the support of γ is sufficiently small,*

$$(x_k-p'_k(\xi))\gamma\big(\langle\xi\rangle^2(x-p'(\xi))\big)=\sum_{l=1}^2 e^k_l(x,\xi)(\xi_l+d_l\phi(\xi)),\tag{1.2.66a}$$

$$(\xi_k+d_k\phi(x))\gamma\big(\langle\xi\rangle^2(x-p'(\xi))\big)=\sum_{l=1}^2\widetilde{e}^k_l(x,\xi)(x_l-p'_l(\xi)),\tag{1.2.66b}$$

for any $k=1,2$, where functions $e^k_l(x,\xi),\widetilde{e}^k_l(x,\xi)$ are such that, for any $\alpha,\beta\in\mathbb{N}^2$,

$$|\partial^\alpha_x\partial^\beta_\xi e^k_l(x,\xi)|\lesssim_{\alpha\beta}\langle\xi\rangle^{-3+2|\alpha|-|\beta|},\tag{1.2.67a}$$

$$|\partial^\alpha_x\partial^\beta_\xi \widetilde{e}^k_l(x,\xi)|\lesssim_{\alpha\beta}\langle\xi\rangle^{3+2|\alpha|-|\beta|},\tag{1.2.67b}$$

for any $k,l=1,2$.

Chapter 2

Energy Estimates

The aim of this chapter is to derive a suitable energy inequality that would allow us to propagate the a-priori estimates we made on energies $E_n(t; u_\pm, v_\pm), E_3^k(t; u_\pm, v_\pm)$, $0 \le k \le 2$, in theorem 1.1.2, i.e. to pass from (1.1.11) to (1.1.12c), (1.1.12d). This energy inequality is deduced from the quasi-linear system solved by vector $(u^I_+, v^I_+, u^I_-, v^I_-)$, for a fixed multi-index I, in two steps: we paralinearize our system and symmetrize the quasi-linear contribution to the non-linearity in order to avoid any loss of derivatives (see section 2.1); successively, we perform two normal forms to get rid of some contributions that decay very slowly in time (see section 2.2). The first of these normal forms is performed directly on the mentioned system (subsection 2.2.1), the second one on the energy (subsection 2.2.2).

2.1 Paralinearization and Symmetrization

As briefly anticipated above, the first step towards the derivation of the right energy inequality is to make sure that the quasi-linear nature of our system does not lead to a loss of derivatives when computing the derivative in time of the energy. For that, we proceed by writing our system in a vectorial fashion and by para-linearising it, in order to highlight the very quasi-linear contribution to its non-linearity (see subsection 2.1.1). We realize that this term appears in equation (2.1.20) through a para-differential operator, whose symbol is a real *non symmetric* matrix. As we need this operator to be self-adjoint (up to an operator of order 0), we *symmetrize* equation (2.1.20) by defining a new function W^I_s in terms of W^I, that will be solution to a new equation in which the symbol of the quasi-linear contribution is a real symmetric matrix (see subsection 2.1.3). Also, we set aside subsection 2.1.2 to the estimate of the L^2 norms of the non-linear terms in the right hand side of (2.1.20).

2.1.1 Paralinearization

Let us remind definition (1.1.10). Since admissible vector fields considered in $\mathcal{Z} = \{\Omega, Z_j, \partial_j, j = 1,2\}$ exactly commute with the linear part of system (1.1.1), we deduce from remark 1.1.5 and (1.1.17) that, for any multi-index I, $(\Gamma^I u, \Gamma^I v)$ is solution to

$$\begin{cases} \left(\partial_t^2 - \Delta_x\right)\Gamma^I u = \displaystyle\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|=|I|}} Q_0(\Gamma^{I_1}v, \partial_1\Gamma^{I_2}v) + \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|<|I|}} c_{I_1,I_2}Q_0(\Gamma^{I_1}v, \partial\Gamma^{I_2}v), \\ \left(\partial_t^2 - \Delta_x + 1\right)\Gamma^I v = \displaystyle\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|=|I|}} Q_0(\Gamma^{I_1}v, \partial_1\Gamma^{I_2}u) + \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|<|I|}} c_{I_1,I_2}Q_0(\Gamma^{I_1}v, \partial\Gamma^{I_2}u), \end{cases}$$

with set $\mathcal{I}(I)$ introduced in (1.1.18), coefficients $c_{I_1,I_2}\in\{-1,0,1\}$, $c_{I_1,I_2}=1$ for $|I_1|+|I_2|=|I|$ in which case the derivative ∂ acting on $\Gamma^{I_2}v$ (resp. on $\Gamma^{I_2}u$) is equal to ∂_1, and $\partial=\partial_a$ for $a\in\{0,1,2\}$. Let us remind that, if Γ^I contains at least k ($\le|I|$) space derivatives, above summations are taken over indices I_1,I_2 such that $k\le|I_1|+|I_2|\le|I|$. Hence, introducing from (1.1.3), (1.1.5),

$$\begin{aligned}Q^{\mathrm{w}}_0(v_\pm,D_av_\pm)&:=\frac{i}{4}\Big[(v_++v_-)D_a(v_++v_-)-\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{D_xD_a}{\langle D_x\rangle}(v_+-v_-)\Big],\\ Q^{\mathrm{kg}}_0(v_\pm,D_au_\pm)&:=\frac{i}{4}\Big[(v_++v_-)D_a(u_++u_-)-\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{D_xD_a}{|D_x|}(u_+-u_-)\Big].\end{aligned}\tag{2.1.1}$$

for any $a=0,1,2$, we deduce that $(u^I_+,v^I_+,u^I_-,v^I_-)$ is solution to

$$\begin{cases}(D_t-|D_x|)u^I_+(t,x)=\displaystyle\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\|I_1|+|I_2|=|I|}}Q^{\mathrm{w}}_0(v^{I_1}_\pm,D_1v^{I_2}_\pm)+\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\|I_1|+|I_2|<|I|}}c_{I_1,I_2}Q^{\mathrm{w}}_0(v^{I_1}_\pm,Dv^{I_2}_\pm)\\ (D_t-\langle D_x\rangle)v^I_+(t,x)=\displaystyle\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\|I_1|+|I_2|=|I|}}Q^{\mathrm{kg}}_0(v^{I_1}_\pm,D_1u^{I_2}_\pm)+\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\|I_1|+|I_2|<|I|}}c_{I_1,I_2}Q^{\mathrm{kg}}_0(v^{I_1}_\pm,Du^{I_2}_\pm)\\ (D_t+|D_x|)u^I_-(t,x)=\displaystyle\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\|I_1|+|I_2|=|I|}}Q^{\mathrm{w}}_0(v^{I_1}_\pm,D_1v^{I_2}_\pm)+\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\|I_1|+|I_2|<|I|}}c_{I_1,I_2}Q^{\mathrm{w}}_0(v^{I_1}_\pm,Dv^{I_2}_\pm)\\ (D_t+\langle D_x\rangle)v^I_-(t,x)=\displaystyle\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\|I_1|+|I_2|=|I|}}Q^{\mathrm{kg}}_0(v^{I_1}_\pm,D_1u^{I_2}_\pm)+\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\|I_1|+|I_2|<|I|}}c_{I_1,I_2}Q^{\mathrm{kg}}_0(v^{I_1}_\pm,Du^{I_2}_\pm)\end{cases}\tag{2.1.2}$$

The quasi-linear structure of the above system can be emphasized by using (1.2.7) and decomposing $Q^{\mathrm{w}}_0(v_\pm,D_1v^I_\pm)$, $Q^{\mathrm{kg}}_0(v_\pm,D_1u^I_\pm)$ as follows:

$$\begin{aligned}Q^{\mathrm{w}}_0(v_\pm,D_1v^I_\pm)&=(QL)_1+(SL)_1,\\ Q^{\mathrm{kg}}_0(v_\pm,D_1u^I_\pm)&=(QL)_2+(SL)_2,\end{aligned}\tag{2.1.3}$$

with

$$\begin{aligned}(QL)_1&:=\frac{i}{4}\Big[Op^B\big((v_++v_-)\eta_1\big)(v^I_++v^I_-)-Op^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{\langle\eta\rangle}\Big)(v^I_+-v^I_-)\Big],\\ (SL)_1&:=\frac{i}{4}\Big[Op^B\big(D_1(v^I_++v^I_-)\big)(v_++v_-)-Op^B\Big(\frac{D_xD_1}{\langle D_x\rangle}(v^I_+-v^I_-)\cdot\frac{\eta}{\langle\eta\rangle}\Big)(v_+-v_-)\\ &\quad+Op^B_R\big((v_++v_-)\eta_1\big)(v^I_++v^I_-)\big)-Op^B_R\Big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{\langle\eta\rangle}\Big)(v^I_+-v^I_-)\Big],\\ (QL)_2&:=\frac{i}{4}\Big[Op^B\big((v_++v_-)\eta_1\big)(u^I_++u^I_-)-Op^B\Big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{|\eta|}\Big)(u^I_+-u^I_-)\Big],\\ (SL)_2&:=\frac{i}{4}\Big[Op^B\big(D_1(u^I_++u^I_-)\big)(v_++v_-)-Op^B\Big(\frac{D_xD_1}{|D_x|}(u^I_+-u^I_-)\cdot\frac{\eta}{\langle\eta\rangle}\Big)(v_+-v_-)\\ &\quad+Op^B_R\big((v_++v_-)\eta_1\big)(u^I_++u^I_-)-Op^B_R\Big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{|\eta|}\Big)(u^I_+-u^I_-)\Big],\end{aligned}$$

where the Bony quantization Op^B (resp. Op^B_R) has been defined in 1.2.5 (resp. in (1.2.9)). We do a similar decomposition also for the semi-linear contribution $Q^{\mathrm{kg}}_0(v^I_\pm,D_1u_\pm)$, for this term will thereafter be the object of the two normal forms mentioned at the beginning of this section:

(2.1.4)
$$\begin{aligned}Q^{\mathrm{kg}}_0(v^I_\pm, D_1u_\pm) = \frac{i}{4}&\Big[Op^B\big((v^I_+ + v^I_-)\eta_1\big)(u_+ + u_-) - Op^B\Big(\frac{D_x}{\langle D_x\rangle}(v^I_+ - v^I_-)\cdot\frac{\eta\eta_1}{|\eta|}\Big)(u_+ - u_-)\Big]\\ + \frac{i}{4}&\Big[Op^B\big(D_1(u_+ + u_-)\big)(v^I_+ + v^I_-) - Op^B\Big(\frac{D_xD_1}{|D_x|}(u_+ - u_-)\cdot\frac{\eta}{\langle\eta\rangle}\Big)(v^I_+ - v^I_-)\Big]\\ + \frac{i}{4}&\Big[Op^B_R\big((v^I_+ + v^I_-)\eta_1\big)(u_+ + u_-) - Op^B_R\Big(\frac{D_x}{\langle D_x\rangle}(v^I_+ - v^I_-)\cdot\frac{\eta\eta_1}{|\eta|}\Big)(u_+ - u_-)\Big].\end{aligned}$$

In order to handle system (2.1.2) in the most efficient way, we proceed to write it in a vectorial fashion. To this purpose, we introduce the following matrices:

$$A(\eta) = \begin{bmatrix}|\eta| & 0 & 0 & 0\\ 0 & \langle\eta\rangle & 0 & 0\\ 0 & 0 & -|\eta| & 0\\ 0 & 0 & 0 & -\langle\eta\rangle\end{bmatrix}, \quad A'(V;\eta) := \begin{bmatrix}0 & a_k\eta_1 & 0 & b_k\eta_1\\ a_0\eta_1 & 0 & b_0\eta_1 & 0\\ 0 & a_k\eta_1 & 0 & b_k\eta_1\\ a_0\eta_1 & 0 & b_0\eta_1 & 0\end{bmatrix}, \tag{2.1.5}$$

$$A''(V^I;\eta) := \begin{bmatrix}0 & 0 & 0 & 0\\ a^I_0\eta_1 & 0 & b^I_0\eta_1 & 0\\ 0 & 0 & 0 & 0\\ a^I_0\eta_1 & 0 & b^I_0\eta_1 & 0\end{bmatrix}, \tag{2.1.6}$$

$$C'(W^I;\eta) := \begin{bmatrix}0 & c^I_0 & 0 & d^I_0\\ 0 & e^I_0 & 0 & f^I_0\\ 0 & c^I_0 & 0 & d^I_0\\ 0 & e^I_0 & 0 & f^I_0\end{bmatrix}, \quad C''(U;\eta) := \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & e_0 & 0 & f_0\\ 0 & 0 & 0 & 0\\ 0 & e_0 & 0 & f_0\end{bmatrix} \tag{2.1.7}$$

where

$$\begin{cases} a_k = a_k(v_\pm;\eta) := \frac{i}{4}\big[(v_+ + v_-) - \frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta}{\langle\eta\rangle}\big]\\ b_k = b_k(v_\pm;\eta) := \frac{i}{4}\big[(v_+ + v_-) + \frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta}{\langle\eta\rangle}\big]\\ a_0 = a_0(v_\pm;\eta) := \frac{i}{4}\big[(v_+ + v_-) - \frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta}{|\eta|}\big]\\ b_0 = b_0(v_\pm;\eta) := \frac{i}{4}\big[(v_+ + v_-) + \frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta}{|\eta|}\big]\end{cases} \tag{2.1.8}$$

$$\begin{cases} c_0 = c_0(v_\pm;\eta) := \frac{i}{4}\big[D_1(v_+ + v_-) - \frac{D_xD_1}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta}{\langle\eta\rangle}\big]\\ d_0 = d_0(v_\pm;\eta) := \frac{i}{4}\big[D_1(v_+ + v_-) + \frac{D_xD_1}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta}{\langle\eta\rangle}\big]\\ e_0 = e_0(u_\pm;\eta) := \frac{i}{4}\big[D_1(u_+ + u_-) - \frac{D_xD_1}{|D_x|}(u_+ - u_-)\cdot\frac{\eta}{\langle\eta\rangle}\big]\\ f_0 = f_0(u_\pm;\eta) := \frac{i}{4}\big[D_1(u_+ + u_-) + \frac{D_xD_1}{|D_x|}(u_+ - u_-)\cdot\frac{\eta}{\langle\eta\rangle}\big]\end{cases} \tag{2.1.9}$$

$$\begin{gathered} a^I_0 = a_0(v^I_\pm;\eta), \quad b^I_0 = b_0(v^I_\pm;\eta), c^I_0 = c_0(v^I_\pm;\eta), \quad d^I_0 = d_0(v^I_\pm;\eta),\\ e^I_0 = e_0(u^I_\pm;\eta), \quad f^I_0(u^I_\pm;\eta),\end{gathered} \tag{2.1.10}$$

vectors W, U, V:

$$W := \begin{bmatrix}u_+\\ v_+\\ u_-\\ v_-\end{bmatrix}, \quad V := \begin{bmatrix}0\\ v_+\\ 0\\ v_-\end{bmatrix}, \quad U := \begin{bmatrix}u_+\\ 0\\ u_-\\ 0\end{bmatrix}, \tag{2.1.11}$$

along with W^I (resp. V^I, U^I) defined from W (resp. V, U) by replacing $u_\pm, v_\pm$ with $u^I_\pm, v^I_\pm$; and finally

$$Q^I_0(V,W) = \begin{bmatrix} \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_2|<|I|}} c_{I_1,I_2} Q^{\mathrm{w}}_0(v^{I_1}_\pm, Dv^{I_2}_\pm) \\ \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|,|I_2|<|I|}} c_{I_1,I_2} Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm) \\ \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_2|<|I|}} c_{I_1,I_2} Q^{\mathrm{w}}_0(v^{I_1}_\pm, Dv^{I_2}_\pm) \\ \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|,|I_2|<|I|}} c_{I_1,I_2} Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm) \end{bmatrix} \tag{2.1.12}$$

Let us remind that, if Γ^I contains at least $k\,(\le |I|)$ space derivatives, above summations are taken over indices I_1, I_2 such that $k \le |I_1| + |I_2| \le |I|$. The quantization Op^B (resp. Op^B_R) of a matrix $A = (a_{ij})_{1\le i,j\le n}$ is meant as a matrix of operators $Op^B(A) = (Op^B(a_{ij}))_{1\le i,j\le n}$ (resp. $Op^B_R(A) = (Op^B_R(a_{ij}))_{1\le i,j\le n}$), and for a vector $X = [x_1, \dots, x_n]$,

$$Op^B(A)X^\dagger = \begin{bmatrix} \sum_{j=1}^n Op^B(a_{1j})x_j \\ \vdots \\ \sum_{j=1}^n Op^B(a_{nj})x_j \end{bmatrix},$$

$X^\dagger$ being the transpose of X. Moreover, we will talk about the L^∞ or L^2 norm of a matrix $A = (a_{ij})$ with the meaning of evaluating A "component by component", i.e. $\|A\|_{L^2} = \big(\sum_{i,j} |a_{ij}|^2\big)^{\frac{1}{2}}$ and $\|A\|_{L^\infty} = \sup_{ij} |a_{ij}|$.

With notations introduced above, system (2.1.2) writes in the following compact fashion, which has the merit to well highlight, among all non-linear terms, the very quasi-linear contributions $(QL)_1, (QL)_2$, represented below by $Op^B(A'(V;\eta))W^I$:

$$\begin{aligned} D_t W^I &= A(D)W^I + Op^B(A'(V;\eta))W^I + Op^B(C'(W^I;\eta))V + Op^B_R(A'(V;\eta))W^I \\ &\quad + Op^B(A''(V^I;\eta))U + Op^B(C''(U;\eta))V^I + Op^B_R(A''(V^I;\eta))U + Q^I_0(V,W). \end{aligned} \tag{2.1.13}$$

Furthermore, the energies defined in (1.1.9) take the form

$$E_n(t;u_\pm,v_\pm) = \sum_{|\alpha|\le n} \|D^\alpha_x W(t,\cdot)\|_{L^2}, \quad \forall\, n\in\mathbb{N}, n\ge 3, \tag{2.1.14a}$$

$$E^k_3(t;u_\pm,v_\pm) = \sum_{\substack{|\alpha|+|I|\le 3\\ 0\le|I|\le 3-k}} \|D^\alpha_x W^I(t,\cdot)\|^2_{L^2}, \quad \forall\, 0\le k\le 2, \tag{2.1.14b}$$

and we can refer to them, respectively, as $E_n(t;W), E^k_3(t;W)$. We also notice that, since

$$[\Gamma, D_t \pm |D_x|] = \begin{cases} 0 & \text{if } \Gamma\in\{\Omega,\partial_j, j=1,2\}, \\ \mp\frac{D_m}{|D_x|}(D_t\pm|D_x|) & \text{if } \Gamma = Z_m, m=1,2, \end{cases} \tag{2.1.15a}$$

and

$$[\Gamma, D_t \pm \langle D_x\rangle] = \begin{cases} 0 & \text{if } \Gamma\in\{\Omega,\partial_j, j=1,2\}, \\ \mp\frac{D_m}{\langle D_x\rangle}(D_t\pm\langle D_x\rangle) & \text{if } \Gamma = Z_m, m=1,2, \end{cases} \tag{2.1.15b}$$

there exists a constant $C>0$ such that

$$C^{-1}\sum_{I\in\mathcal{I}_3^k}\|\Gamma^I W(t,\cdot)\|^2_{L^2}\le E_3^k(t;W)\le C\sum_{I\in\mathcal{I}_3^k}\|\Gamma^I W(t,\cdot)\|^2_{L^2}, \tag{2.1.16}$$

where

$$\mathcal{I}_3^k:=\left\{|I|\le 3:\Gamma^I=D_x^\alpha\Gamma^J \text{ with } |\alpha|+|J|=|I|, 0\le|J|\le 3-k\right\}. \tag{2.1.17}$$

For convenience, we also introduce the following set:

$$\mathcal{I}_n:=\left\{|I|\le n:\Gamma^I=D_x^\alpha \text{ with } |\alpha|=|I|\right\}. \tag{2.1.18}$$

Matrices $A(\eta), A'(V;\eta), A''(V^I;\eta)$ are of order 1, although $A'(V;\eta), A''(V^I;\eta)$ are singular at $\eta=0$ (i.e. some of their elements are singular at $\eta=0$), while $C'(W^I;\eta), C''(U;\eta)$ are of order 0. Since we will need to do some symbolic calculus on $A'(V;\eta)$, we need to isolate the mentioned singularity. For that we define
(2.1.19)

$$A_1'(V;\eta):=\begin{bmatrix}0 & a_0\eta_1 & 0 & b_0\eta_1\\ a_0\eta_1 & 0 & b_0\eta_1 & 0\\ 0 & a_0\eta_1 & 0 & b_0\eta_1\\ a_0\eta_1 & 0 & b_0\eta_1 & 0\end{bmatrix},\quad A_{-1}'(V;\eta):=\begin{bmatrix}0 & (a_k-a_0)\eta_1 & 0 & (b_k-b_0)\eta_1\\ 0&0&0&0\\ 0&0&0&0\\ 0 & (a_k-a_0)\eta_1 & 0 & (b_k-b_0)\eta_1\end{bmatrix},$$

$A_1'(V;\eta)$ being a matrix of order 1, $A_{-1}'(V;\eta)$ of order -1, both singular at $\eta=0$, and write $A_1'(V;\eta)=A_1'(V;\eta)(1-\chi)(\eta)+A_1'(V;\eta)\chi(\eta)$, where $\chi\in C_0^\infty(\mathbb{R}^2)$ is equal to 1 in the unit ball. Equation (2.1.13) hence becomes

(2.1.20)

$$\begin{aligned}D_tW^I &= A(D)W^I+Op^B(A_1'(V;\eta)(1-\chi)(\eta))W^I+Op^B(A_1'(V;\eta)\chi(\eta))W^I\\ &+Op^B(A_{-1}'(V;\eta))W^I+Op^B(C'(W^I;\eta))V+Op^B_R(A'(V;\eta))W^I+Op^B(A''(V^I;\eta))U\\ &+Op^B(C''(U;\eta))V^I+Op^B_R(A''(V^I;\eta))U+Q_0^I(V,W),\end{aligned}$$

and quasi-linear term $Op^B(A_1'(V;\eta)(1-\chi)(\eta))W^I$ is no longer singular at $\eta=0$. We observe that $A_1'(V;\eta)(1-\chi)(\eta)$ is a real matrix, since $i(v_++v_-)=2\partial_t v$, $i\frac{D_x}{\langle D_x\rangle}(v_+-v_-)=2\partial_x v$ and v is a real solution, but not symmetric. Such a lack of symmetry could lead to a loss of derivatives when writing an energy inequality for W^I, but the issue is only technical, in the sense that $A_1(V;\eta)(1-\chi)(\eta)$ can be replaced with a real, symmetric matrix, as explained in subsection 2.1.3 (see proposition 2.1.5). Before proving such result, we need to derive some L^2 estimates for the semi-linear terms in the right hand side of (2.1.20).

2.1.2 Estimates of quadratic terms

In this subsection we recover some estimates for the L^2 norm of the non-linear terms in the right hand side of equation (2.1.20).

Lemma 2.1.1. *Let I be a fixed multi-index. The following estimates hold:*

$$\left\|\left[Op^B\big(A_1'(V;\eta)\chi(\eta)\big)+Op^B\big(A_{-1}'(V;\eta)\big)\right]W^I(t,\cdot)\right\|_{L^2}\lesssim\|V(t,\cdot)\|_{H^{1,\infty}}\|W^I(t,\cdot)\|_{L^2}; \tag{2.1.21a}$$

$$\left\|Op^B(C'(W^I;\eta))V(t,\cdot)\right\|_{L^2}\lesssim\|V(t,\cdot)\|_{H^{6,\infty}}\|W^I(t,\cdot)\|_{L^2}; \tag{2.1.21b}$$

$$\|Op^B_R(A'(V;\eta))W^I(t,\cdot)\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{7,\infty}}\|W^I(t,\cdot)\|_{L^2}; \tag{2.1.21c}$$

$$\begin{aligned}&\|Op^B(A''(V^I;\eta))U(t,\cdot)\|_{L^2} + \|Op^B_R(A''(V^I;\eta))U(t,\cdot)\|_{L^2}\\ &\qquad\lesssim \big(\|R_1U(t,\cdot)\|_{H^{6,\infty}} + \|U(t,\cdot)\|_{H^{6,\infty}}\big)\|V^I(t,\cdot)\|_{L^2};\end{aligned} \tag{2.1.21d}$$

$$\|Op^B(C''(U;\eta))V^I(t,\cdot)\|_{L^2} \lesssim \big(\|R_1U(t,\cdot)\|_{H^{2,\infty}} + \|U(t,\cdot)\|_{H^{2,\infty}}\big)\|W^I(t,\cdot)\|_{L^2}; \tag{2.1.21e}$$

Proof. Inequality (2.1.21a) follows applying proposition 1.2.7 to $Op^B\left(A'_{-1}(V;\eta)(1-\chi)(\eta)\right)W^I$, whose symbol $A'_{-1}(V;\eta)(1-\chi)(\eta)$ is of order -1 and has M^{-1}_0 seminorm bounded from above by $\|V(t,\cdot)\|_{H^{1,\infty}}$, after definitions (1.2.2), (2.1.19) and (2.1.8).
Since from definition (2.1.7) of matrix $C'(W^I;\eta)$,

$$\begin{aligned}\big\|Op^B(C'(W^I;\eta))V\big\|_{L^2} &\lesssim \big\|Op^B(D_1(v^I_+ + v^I_-))v_\pm\big\|_{L^2} + \Big\|Op^B\Big(\frac{D_xD_1}{\langle D_x\rangle}(v^I_+ - v^I_-)\cdot\frac{\eta}{\langle\eta\rangle}\Big)v_\pm\Big\|_{L^2}\\ &+ \big\|D_1(u^I_+ + u^I_-))v_\pm\big\|_{L^2} + \Big\|Op^B\Big(\frac{D_xD_1}{|D_x|}(u^I_+ - u^I_-)\cdot\frac{\eta}{\langle\eta\rangle}\Big)v_\pm\Big\|_{L^2},\end{aligned}$$

we reduce to prove inequality (2.1.21b) for $Op^B\big(\frac{D_xD_1}{\langle D_x\rangle}(v^I_+ - v^I_-)\cdot\frac{\eta}{\langle\eta\rangle}\big)v_+$, the same argument being applicable to all other L^2 norms appearing in the above right hand side. Using equality (1.2.6), and considering a new admissible cut-off function χ_1, identically equal to 1 on the support of χ, we first derive that

$$\begin{aligned}&\widehat{Op^B\Big(\tfrac{D_xD_1}{\langle D_x\rangle}(v^I_+ + v^I_-)\cdot\tfrac{\eta}{\langle\eta\rangle}\Big)v_+}(\xi) = \frac{1}{(2\pi)^2}\int\chi\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)\widehat{\tfrac{D_xD_1}{\langle D_x\rangle}(v^I_+ + v^I_-)}(\xi-\eta)\cdot\widehat{\tfrac{D_x}{\langle D_x\rangle}v_+}(\eta)d\eta\\ &= \frac{1}{(2\pi)^2}\int\chi\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)\Big(\frac{\xi_1-\eta_1}{\langle\eta\rangle}\Big)\widehat{\tfrac{D_x}{\langle D_x\rangle}(v^I_+ + v^I_-)}(\xi-\eta)\cdot\widehat{D_xv_+}(\eta)d\eta\\ &= \frac{1}{(2\pi)^2}\int\chi_1\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)\widehat{\Big[\chi\Big(\tfrac{D_x}{\langle\eta\rangle}\Big)\tfrac{D_1}{\langle\eta\rangle}\tfrac{D_x}{\langle D_x\rangle}(v^I_+ + v^I_-)\Big]}(\xi-\eta)\widehat{D_xv_+}(\eta)d\eta\\ &= \widehat{Op^B\Big(\chi\Big(\tfrac{D_x}{\langle\eta\rangle}\Big)\tfrac{D_1}{\langle\eta\rangle}\tfrac{D_x}{\langle D_x\rangle}(v^I_+ + v^I_-)\Big)D_xv_+}(\xi),\end{aligned}$$

so that by decomposition (1.2.7) and the fact that $R(u,v)$ is symmetric in (u,v),

$$\begin{aligned}Op^B\Big(\frac{D_xD_1}{\langle D_x\rangle}(v^I_+ + v^I_-)\cdot\frac{\eta}{\langle\eta\rangle}\Big)v_+ &= \chi\left[\Big(\frac{D_x}{\langle\eta\rangle}\Big)\frac{D_1}{\langle\eta\rangle}\frac{D_x}{\langle D_x\rangle}(v^I_+ + v^I_-)\right]\cdot D_xv_+\\ &- \big[Op^B(D_xv_+) + Op^B_R(D_xv_+)\big]\Big[\chi\Big(\frac{D_x}{\langle\eta\rangle}\Big)\frac{D_1}{\langle\eta\rangle}\frac{D_x}{\langle D_x\rangle}(v^I_+ + v^I_-)\Big].\end{aligned}$$

Propositions 1.2.7, 1.2.8 (*ii*), and the fact that $\chi\Big(\frac{D_x}{\langle\eta\rangle}\Big)\frac{D_1}{\langle\eta\rangle}\frac{D_x}{\langle D_x\rangle}$ is an operator uniformly bounded on L^2, imply then that

$$\Big\|Op^B\Big(\frac{D_xD_1}{\langle D_x\rangle}(v^I_+ + v^I_-)\cdot\frac{\eta}{\langle\eta\rangle}\Big)v_+\Big\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{6,\infty}}\|V^I(t,\cdot)\|_{L^2}.$$

By definition (2.1.5) of $A'(V;\eta)$,

$$\begin{aligned}\big\|Op^B_R(A'(V;\eta))W^I(t,\cdot)\big\|_{L^2} &\lesssim \big\|Op^B_R(v_+ + v_-)v^I_\pm\big\|_{L^2} + \Big\|Op^B_R\Big(\frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta}{\langle\eta\rangle}\Big)v^I_\pm\Big\|_{L^2}\\ &+ \big\|Op^B_R(v_+ + v_-)u^I_\pm\big\|_{L^2} + \Big\|Op^B_R\Big(\frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot\frac{\eta}{|\eta|}\Big)u^I_\pm\Big\|_{L^2},\end{aligned}$$

so we limit ourselves to show that inequality (2.1.21c) holds for $Op^B\big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{|\eta|}\big)u^I_+$. For a smooth cut-off function ϕ equal to 1 in the unit ball we write

$$Op^B_R\left(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{|\eta|}\right)u^I_+ = Op^B_R\Big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{|\eta|}\phi(\eta)\Big)u^I_+ + Op^B_R\Big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{|\eta|}(1-\phi)(\eta)\Big)u^I_+,$$

where, by proposition 1.2.8 (i),

$$\Big\|Op^B_R\Big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{|\eta|}\phi(\eta)\Big)u^I_+\Big\|_{L^2} \lesssim \Big\|\frac{D_x}{\langle D_x\rangle}(v_+-v_-)(t,\cdot)\Big\|_{L^\infty}\|u^I_+(t,\cdot)\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{1,\infty}}\|W^I(t,\cdot)\|_{L^2},$$

while

$$Op^B_R\Big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{|\eta|}(1-\phi)(\eta)\Big)u^I_+ = \int e^{ix\cdot\xi}m(\xi,\eta)\left[\langle D_x\rangle^7(\hat{v}_+-\hat{v}_-)(\xi-\eta)\right]\hat{u}^I_+(\eta)d\xi d\eta,$$

with

$$m(\xi,\eta):=\frac{1}{(2\pi)^2}\left(1-\chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)-\chi\left(\frac{\eta}{\langle\xi-\eta\rangle}\right)\right)(1-\phi)(\eta)\frac{\xi-\eta}{\langle\xi-\eta\rangle^8}\cdot\frac{\eta\eta_1}{|\eta|}.$$

On the support of $m(\xi,\eta)$ frequencies $\xi-\eta$ and η are either bounded or equivalent so, since $|\partial^\alpha_\xi\partial^\beta_\eta m(\xi,\eta)|\lesssim\langle\xi\rangle^{-3}\langle\eta\rangle^{-3}$ for any $\alpha,\beta\in\mathbb{N}^2$, $m(\xi,\eta)$ satisfies the hypothesis of lemma A.1 (i), and by inequality (A.2a)

$$\Big\|Op^B_R\Big(\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\cdot\frac{\eta\eta_1}{|\eta|}(1-\phi)(\eta)\Big)u^I_+\Big\|_{L^2}\lesssim\|V(t,\cdot)\|_{H^{7,\infty}}\|W^I(t,\cdot)\|_{L^2}.$$

From definition (2.1.6) of $A''(V;\eta)$,

$$\big\|Op^B\big(A''(V;\eta)\big)U(t,\cdot)\big\|_{L^2}\lesssim\big\|Op^B\big((v^I_++v^I_-)\eta_1\big)u_\pm\big\|_{L^2}+\Big\|Op^B\Big(\frac{D_x}{\langle D_x\rangle}(v^I_+-v^I_-)\cdot\frac{\eta\eta_1}{|\eta|}\Big)u_\pm\Big\|_{L^2},$$

(the same inequality holds evidently when Op^B is replaced by Op^B_R) hence, as done for previous cases, we reduce to show (2.1.21d) for $Op^B\big(\frac{D_x}{\langle D_x\rangle}(v^I_+-v^I_-)\cdot\frac{\eta\eta_1}{|\eta|}\big)u_+$ (resp. for Op^B replaced with Op^B_R). Using decomposition (1.2.7) and the fact that $R(u,v)$ is symmetric in (u,v) we have that

$$Op^B\Big(\frac{D_x}{\langle D_x\rangle}(v^I_+-v^I_-)\cdot\frac{\eta\eta_1}{|\eta|}\Big)u_+ = \frac{D_x}{\langle D_x\rangle}(v^I_+-v^I_-)\cdot\frac{D_xD_1}{|D_x|}u_+ - Op^B\Big(\frac{D_xD_1}{|D_x|}u_+\cdot\frac{\eta}{\langle\eta\rangle}\Big)(v^I_+-v^I_-)-Op^B_R\Big(\frac{D_xD_1}{|D_x|}u_+\cdot\frac{\eta}{\langle\eta\rangle}\Big)(v^I_+-v^I_-),$$

and

$$Op^B_R\Big(\frac{D_x}{\langle D_x\rangle}(v^I_+-v^I_-)\cdot\frac{\eta\eta_1}{|\eta|}\Big)u_+ = Op^B_R\Big(\frac{D_xD_1}{|D_x|}u_+\cdot\frac{\eta}{\langle\eta\rangle}\Big)(v^I_+-v^I_-),$$

so a direct application of propositions 1.2.7 and 1.2.8 (ii) gives that the L^2 norm of the above right hand sides is bounded by $\big\|\frac{D_xD_1}{|D_x|}u_+\big\|_{H^{4,\infty}}\|V^I(t,\cdot)\|_{L^2}\lesssim\|R_1U(t,\cdot)\|_{H^{6,\infty}}\|V^I(t,\cdot)\|_{L^2}$, which gives inequality (2.1.21d).

Finally, from definition (2.1.7) of matrix $C''(U;\eta)$,

$$\|Op^B(C''(U;\eta))V^I\|_{L^2}\lesssim \big\|Op^B(D_1(u_++u_-))(v^I_++v^I_-)\big\|_{L^2}+\Big\|Op^B\left(\frac{D_xD_1}{|D_x|}(u_+-u_-)\cdot\frac{\eta}{\langle\eta\rangle}\right)(v^I_+-v^I_-)\Big\|_{L^2},$$

so estimate (2.1.21e) follows immediately from proposition 1.2.7. □

Lemmas 2.1.2 and 2.1.3 below are introduced with the aim of deriving an estimate of the L^2 norm of vector $Q^I_0(V,W)$ given by (2.1.12) (see corollary 2.1.4). We remind that the summations defining $Q^I_0(V,W)$ come from the action of the family Γ^I of admissible vector fields on the quadratic non-linearity $Q_0(v,\partial_1 v)$ (resp. $Q_0(v,\partial_1 u)$) in the equation satisfied by u (resp. by v) in (1.1.1), often indicated by $\Gamma^I Q^{\mathrm{w}}_0(v_\pm, D_1 v_\pm)$ (resp. $\Gamma^I Q^{\mathrm{kg}}_0(v_\pm, D_1 u_\pm)$) when dealing with functions $u_\pm, v_\pm$. According to remark 1.1.5, if $I\in\mathcal{I}_n$ and Γ^I is a product of spatial derivatives only, the action of Γ^I on $Q^{\mathrm{w}}_0(v_\pm, D_1 v_\pm)$ (resp. $Q^{\mathrm{kg}}_0(v_\pm, D_1 u_\pm)$) "distributes" entirely on its factors, meaning that

$$\Gamma^I Q^{\mathrm{w}}_0(v_\pm, D_1 v_\pm) = \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|=|I|}} Q^{\mathrm{w}}_0(v^{I_1}_\pm, D_1 v^{I_2}_\pm),$$

(the same for $\Gamma^I Q^{\mathrm{kg}}_0(v_\pm, D_1 u_\pm)$), and all coefficients c_{I_1,I_2} in the right hand side of (2.1.2) are equal to 0. On the contrary, if $I\in\mathcal{I}^k_3$ for $0\le k\le 2$, and Γ^I contains some Klainerman vector fields $\Omega, Z_m, m=1,2$, the commutation between Γ^I and the null structure gives rise to new quadratic contributions, in which the derivative D_1 is replaced with D_2, D_t. As already seen in (1.1.17), in this case we have

$$\Gamma^I Q^{\mathrm{w}}_0(v_\pm, D_1 v_\pm) = \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|=|I|}} Q^{\mathrm{w}}_0(v^{I_1}_\pm, D_1 v^{I_2}_\pm) + \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|+|I_2|<|I|}} c_{I_1,I_2} Q^{\mathrm{w}}_0(v^{I_1}_\pm, D v^{I_2}_\pm), \tag{2.1.22}$$

with some of the coefficients c_{I_1,I_2} being equal to 1 or -1, and $D\in\{D_1,D_2,D_t\}$ depending on the addend we are considering (similarly for $\Gamma^I Q^{\mathrm{kg}}_0(v_\pm, D_1 u_\pm)$). For our scopes, there will be no difference between the case $D=D_1$ and $D=D_2$, the two associated quadratic contributions enjoying the same L^2 and L^∞ estimates. When $D=D_t$, we should make use of the equation satisfied by $v^{I_2}_\pm$ (resp. by $u^{I_2}_\pm$) in system (2.1.2) to replace $Q^{\mathrm{w}}_0(v^{I_1}_\pm, D_t v^{I_2}_\pm)$ (resp. $Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D_t u^{I_2}_\pm)$) with

$$\begin{gathered} Q^{\mathrm{w}}_0(v^{I_1}_\pm, \langle D_x\rangle v^{I_2}_\pm) + Q^{\mathrm{w}}_0\left(v^{I_1}_\pm, \Gamma^{I_2} Q^{\mathrm{kg}}_0(v_\pm, D_1 u_\pm)\right), \\ \left(\text{resp. with } Q^{\mathrm{kg}}_0(v^{I_1}_\pm, |D_x| u^{I_2}_\pm) + Q^{\mathrm{kg}}_0\left(v^{I_1}_\pm, \Gamma^{I_2} Q^{\mathrm{w}}_0(v_\pm, D_1 v_\pm)\right)\right), \end{gathered} \tag{2.1.23}$$

where the left hand side quadratic terms are given by

$$\begin{gathered} Q^{\mathrm{w}}_0(v^{I_1}_\pm, \langle D_x\rangle v^{I_2}_\pm) = (v^{I_1}_+ + v^{I_1}_-)\langle D_x\rangle (v^{I_2}_+ - v^{I_2}_-) - \frac{D_x}{\langle D_x\rangle}(v^{I_1}_+ - v^{I_1}_-)\cdot D_x(v^{I_2}_+ + v^{I_2}_-), \\ \left(\text{resp. } Q^{\mathrm{kg}}_0(v^{I_1}_\pm, |D_x| u^{I_2}_\pm) = (v^{I_1}_+ + v^{I_1}_-)|D_x| (u^{I_2}_+ - u^{I_2}_-) - \frac{D_x}{\langle D_x\rangle}(v^{I_1}_+ - v^{I_1}_-)\cdot D_x(u^{I_2}_+ + u^{I_2}_-)\right), \end{gathered} \tag{2.1.24}$$

while the right hand side ones in (2.1.23) are cubic. On the Fourier side, these new quadratic contributions write as

$$\begin{gathered} \sum_{j_1,j_2\in\{+,-\}} \int j_2\left(1 - j_1 j_2 \frac{\xi-\eta}{\langle \xi-\eta\rangle}\cdot\frac{\eta}{\langle\eta\rangle}\right)\langle\eta\rangle \hat{v}^{I_1}_{j_1}(\xi-\eta)\hat{v}^{I_2}_{j_2}(\eta) d\xi d\eta, \\ \left(\text{resp. } \sum_{j_1,j_2\in\{+,-\}} \int j_2\left(1 - j_1 j_2 \frac{\xi-\eta}{\langle \xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\right)|\eta| \hat{v}^{I_1}_{j_1}(\xi-\eta)\hat{u}^{I_2}_{j_2}(\eta) d\xi d\eta\right), \end{gathered}$$

and are basically the same as the starting ones

$$\widehat{Q_0^{\mathrm{w}}(v^{I_1}_\pm, D_1 v^{I_2}_\pm)}(\xi) = \sum_{j_1,j_2\in\{+,-\}} \int \Big(1 - j_1 j_2 \frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{\langle\eta\rangle}\Big)\eta_1 \hat{v}^{I_1}_{j_1}(\xi-\eta)\hat{v}^{I_2}_{j_2}(\eta)d\xi d\eta,$$

$$\left(\text{resp. } \widehat{Q_0^{\mathrm{kg}}(v^{I_1}_\pm, D_1 u^{I_2}_\pm)}(\xi) = \sum_{j_1,j_2\in\{+,-\}} \int \Big(1 - j_1 j_2 \frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\Big)\eta_1 \hat{v}^{I_1}_{j_1}(\xi-\eta)\hat{u}^{I_2}_{j_2}(\eta)d\xi d\eta\right).$$

For this reason, as long as we can neglect the cubic terms in (2.1.23), we will not pay attention to the value of $D\in\{D_1,D_2,D_t\}$ in the second sum in the right hand side of (2.1.22). Lemma 2.1.3 is meant to show that the mentioned cubic terms are, indeed, remainders. With an abuse of notation, we introduce

$$D_3 := \begin{cases}\langle D_x\rangle, & \text{if it acts on the Klein-Gordon component,}\\ |D_x|, & \text{if it acts on the wave component,}\end{cases} \tag{2.1.25}$$

and refer, throughout this chapter, to $Q_0^{\mathrm{w}}(v^{I_1}_\pm, Dv^{I_2}_\pm)$ (resp. $Q_0^{\mathrm{kg}}(v^{I_1}_\pm, Du^{I_2}_\pm)$) for $D=D_j$, $j=1,2,3$, instead of $D\in\{D_1,D_2,D_t\}$.

Before proving lemmas 2.1.2, 2.1.3, we need to introduce a new set of indices. According to the order established in $\mathcal{Z}$ at the beginning of section 1.1 (see (1.1.7)), we define

$$\mathcal{K} := \{I=(i_1,i_2) : i_1,i_2 = 1,2,3\}, \tag{2.1.26}$$

as the set of indices I such that Γ^I is the product of two Klainerman vector fields, together with

$$\mathcal{V}^k := \{I\in\mathcal{I}^k_3 : \exists (I_1,I_2)\in\mathcal{I}(I) \text{ with } I_1\in\mathcal{K}\}, \tag{2.1.27}$$

which is evidently empty when $k=2$. We also warn the reader that, in inequality (2.1.31) with $k=2$, $E^3_3(t;W)$ stands for $E_3(t;W)$, this double notation allowing us to combine in one line all cases $k=0,1,2$.

Lemma 2.1.2. *Let I_1, I_2 be multi-indices.*

(i) Let $n\in\mathbb{N}$ and $\mathcal{J}_n := \{(I_1,I_2) | |I_1|+|I_2|\le n, |I_2|<n$ and $\Gamma^{I_1}=D^{\alpha_1}_x, \Gamma^{I_2}=D^{\alpha_2}_x\}$. Then

$$\sum_{(I_1,I_2)\in\mathcal{J}_n}\left\|Q_0^{\mathrm{w}}(v^{I_1}_\pm, D_x v^{I_2}_\pm)\right\|_{L^2} + \sum_{\substack{(I_1,I_2)\in\mathcal{J}_n\\ |I_1|\le[\frac{n}{2}]}}\left\|Q_0^{\mathrm{kg}}(v^{I_1}_\pm, D_x u^{I_2}_\pm)\right\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{[\frac{n}{2}]+2,\infty}} E_n(t,W)^{\frac12}, \tag{2.1.28}$$

$$\sum_{\substack{(I_1,I_2)\in\mathcal{J}_n\\ |I_1|>[\frac{n}{2}]}}\left\|Q_0^{\mathrm{kg}}(v^{I_1}_\pm, D_x u^{I_2}_\pm)\right\|_{L^2} \lesssim \Big(\|U(t,\cdot)\|_{H^{[\frac{n}{2}]+2,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{[\frac{n}{2}]+2,\infty}}\Big) E_n(t,W)^{\frac12}. \tag{2.1.29}$$

(ii) Let $0\le k\le 2$ and $\mathcal{J}^k_3$ be the set of couples (I_1,I_2) such that $|I_1|+|I_2|\le 3, |I_2|<3$ and $\Gamma^{I_1}\Gamma^{I_2} = D^\alpha_x\Gamma^J$ with and $|\alpha|+|J| = |I_1|+|I_2|$ and $0\le |J|\le 3-k$. There exists a constant $C>0$ such that, if we assume a-priori estimates (1.1.11a), (1.1.11b) *satisfied, and $0<\varepsilon_0<(2A+B)^{-1}$ small, for any $\chi\in C^\infty_0(\mathbb{R}^2)$ equal to 1 in a neighbourhood of the origin, and $\sigma>0$ small,*

$$\sum_{(I_1,I_2)\in\mathcal{J}^k_3} Q_0^{\mathrm{w}}(v^{I_1}_\pm, Dv^{I_2}_\pm) = \mathfrak{R}^k_3(t,x), \tag{2.1.30a}$$

$$\sum_{\substack{(I_1,I_2)\in\mathcal{J}^k_3\\ |I_1|<3}} Q_0^{\mathrm{kg}}(v^{I_1}_\pm, Du^{I_2}_\pm) = \sum_{\substack{I_1\in\mathcal{K}\\ |I_2|\le 1}} Q_0^{\mathrm{kg}}\Big(v^{I_1}_\pm, \chi(t^{-\sigma}D_x)Du^{I_2}_\pm\Big) + \mathfrak{R}^k_3(t,x), \tag{2.1.30b}$$

where

$$\|\mathfrak{R}^k_3(t,\cdot)\|_{L^2} \le C(A+B)\varepsilon t^{-1}E^k_3(t,W)^{\frac{1}{2}} + CB\varepsilon t^{-\frac{5}{4}}, \tag{2.1.31}$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$, for all $t \in [1,T]$. The same result holds with $D_x v^{I_2}_{\pm}$ (resp. $D_x u^{I_2}_{\pm}$) replaced with $\langle D_x\rangle v^{I_2}_{\pm}$ (resp. $|D_x|u^{I_2}_{\pm}$).

Proof. The proof of (i) follows straight from (2.1.1) with $a = 1, 2$, by bounding the L^2 norm of each product with the L^∞ norm of the factor indexed in $J \in \{I_1, I_2\}$ such that $|J| \le \left[\frac{|I|}{2}\right]$, times the L^2 norm of the remaining one.

The same argument (combined also with (2.1.23)) used for (i), and the fact that, by definition of $\mathcal{J}^k_3$ and of $\mathcal{I}^k_3$ in (2.1.17), $(I_1, 0), (0, I_2) \in \mathcal{J}^k_3$ if and only if $I_1, I_2 \in \mathcal{I}^k_3$, also shows that

$$\sum_{\substack{I_2\in\mathcal{I}^k_3\\|I_2|<3}} \|Q^{\mathrm{w}}_0(v_\pm, Dv^{I_2}_\pm)\|_{L^2} + \sum_{I_1\in\mathcal{I}^k_3} \|Q^{\mathrm{w}}_0(v^{I_1}_\pm, Dv_\pm)\|_{L^2} + \sum_{\substack{I_2\in\mathcal{I}^k_3\\|I_2|<3}} \|Q^{\mathrm{kg}}_0(v_\pm, Du^{I_2}_\pm)\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{2,\infty}} E^k_3(t;W)^{\frac{1}{2}}. \tag{2.1.32}$$

Moreover, for indices $(I_1, I_2) \in \mathcal{J}^k_3$ such that $|I_1|, |I_2| \ge 1$ and either Γ^{I_1}, or Γ^{I_2}, is a product of spatial derivatives only,

$$\|Q^{\mathrm{w}}_0(v^{I_1}_\pm, D_x v^{I_2}_\pm)\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{4,\infty}} E^k_3(t;W)^{\frac{1}{2}}, \tag{2.1.33}$$

and for $(I_1, I_2) \in \mathcal{J}^k_3$ such that Γ^{I_1} is a product of spatial derivatives,

$$\|Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D_x u^{I_2}_\pm)\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{3,\infty}} E^k_3(t;W)^{\frac{1}{2}}. \tag{2.1.34}$$

The remaining quadratic contributions to summations in the left hand side of (2.1.30) are, respectively: $Q^{\mathrm{w}}_0(v^{I_1}_\pm, D_x v^{I_2}_\pm)$ where both products Γ^{I_1}, Γ^{I_2} contain at least one Klainerman vector field (Γ^{I_1} containing exactly one Klainerman vector field and Γ^{I_2} containing one or two of them, and conversely); $Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D_x u^{I_2}_\pm)$ with Γ^{I_1} containing one or two Klainerman vector fields.

Let us first analyse the L^2 norm of the $Q^{\mathrm{w}}_0(v^{I_1}_\pm, Dv^{I_2}_\pm)$, for all remaining indices I_1, I_2 mentioned above, together with that of $Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm)$ for I_1 such that Γ^{I_1} contains exactly one Klainerman vector field. The underlying ides is to decompose in frequencies the Klein-Gordon factor carrying only one of those vector fields, by means of operator $\chi(t^{-\sigma}D_x)$, for some smooth cut-off function χ and $\sigma > 0$ small. Basically, the L^∞ norm of the factor truncated for large frequencies $|\xi| \gtrsim t^\sigma$ can be bounded by making appear a power of t as negative as we want, as long as we have a control on high Sobolev norms H^s of that factor. On the other hand, we make use of the sharp decay in time $O(t^{-1})$ enjoyed by the uniform norm of the Klein-Gordon component when only one vector field is acting on it and when it is localised for frequencies with moderate growth (less or equal than t^σ, see lemma B.3.21).

Therefore, by making use of corollary B.2.4 in appendix B, with $L = L^2$, $w = v$, we find that, for some $\chi \in C^\infty_0(\mathbb{R}^2)$:

• if Γ^{I_1} contains exactly one Klainerman vector field,

$$\left\|Q^{\mathrm{w}}_0(v^{I_1}_\pm, Dv^{I_2}_\pm)(t,\cdot)\right\|_{L^2} \lesssim \left\|\chi(t^{-\sigma}D_x)v^{I_1}_\pm(t,\cdot)\right\|_{H^{1,\infty}} \|v^{I_2}_\pm(t,\cdot)\|_{H^1} + t^{-N(s)}\left(\|v_\pm(t,\cdot)\|_{H^s} + \|D_t v_\pm(t,\cdot)\|_{H^s}\right)\Big(\sum_{|\mu|=0}^{1} \|x^\mu v^{I_2}_\pm(t,\cdot)\|_{H^1} + t\|v^{I_2}_\pm(t,\cdot)\|_{H^1}\Big),$$

and

$$\left\|Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm)(t,\cdot)\right\|_{L^2} \lesssim \left\|\chi(t^{-\sigma}D_x)v^{I_1}_\pm(t,\cdot)\right\|_{H^{1,\infty}} \|u^{I_2}_\pm(t,\cdot)\|_{H^1}$$
$$+ t^{-N(s)}\left(\|v_\pm(t,\cdot)\|_{H^s} + \|D_t v_\pm(t,\cdot)\|_{H^s}\right)\Big(\sum_{|\mu|=0}^{1}\|x^\mu Du^{I_2}_\pm(t,\cdot)\|_{L^2} + t\|u^{I_2}_\pm(t,\cdot)\|_{H^1}\Big);$$

• if Γ^{I_2} contains exactly one Klainerman vector field,

$$\left\|Q^{\mathrm{w}}_0(v^{I_1}_\pm, Dv^{I_2}_\pm)(t,\cdot)\right\|_{L^2} \lesssim \left\|\chi(t^{-\sigma}D_x)v^{I_2}_\pm(t,\cdot)\right\|_{H^{1,\infty}} \|v^{I_1}_\pm(t,\cdot)\|_{L^2}$$
$$+ t^{-N(s)}\left(\|v_\pm(t,\cdot)\|_{H^s} + \|D_t v_\pm(t,\cdot)\|_{H^s}\right)\Big(\sum_{|\mu|=0}^{1}\|x^\mu v^{I_1}_\pm(t,\cdot)\|_{L^2} + t\|v^{I_1}_\pm(t,\cdot)\|_{L^2}\Big),$$

where, in all above inequalities, $N(s)\ge 3$ if $s>0$ is large enough. From inequalities (B.1.5a), (B.1.6a), estimates (B.1.17), lemma B.3.21 and the boostrap assumptions (1.1.11), we derive that, for multi-indices I_1, I_2 considered in above inequalities,

$$\left\|Q^{\mathrm{w}}_0(v^{I_1}_\pm, Dv^{I_2}_\pm)(t,\cdot)\right\|_{L^2} + \left\|Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm)(t,\cdot)\right\|_{L^2} \le CB\varepsilon t^{-1}E^k_3(t;W)^{\frac{1}{2}} + CB\varepsilon t^{-\frac{5}{4}},$$

for some positive constant C, where we also used the fact that $\delta, \delta_j \ll 1$ are small, for $j=0,1,2$.
The remaining quadratic terms are $Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D_x u^{I_2}_\pm)$ with $I_1\in\mathcal{K}$ (and hence $|I_2|\le 1$). Applying corollary B.2.4 to these contributions, with $L=L^2$, $w=u$, and the same s as before, and making use of estimates (1.1.11), (B.1.17), together with inequality (B.1.5a), we see that

$$\left\|Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm)(t,\cdot)\right\|_{L^2} \lesssim \left\|Q^{\mathrm{kg}}_0\left(v^{I_1}_\pm, \chi(t^{-\sigma}D_x)Du^{I_2}_\pm\right)(t,\cdot)\right\|_{L^2}$$
$$+ t^{-3}\Big(\sum_{|\mu|=0}^{1}\|x^\mu v^{I_1}_\pm(t,\cdot)\|_{L^2} + t\|v^{I_1}(t,\cdot)\|_{L^2}\Big)\left(\|u_\pm(t,\cdot)\|_{H^s} + \|D_t u_\pm(t,\cdot)\|_{H^s}\right)$$
$$\lesssim \left\|Q^{\mathrm{kg}}_0\left(v^{I_1}_\pm, \chi(t^{-\sigma}D_x)Du^{I_2}_\pm\right)(t,\cdot)\right\|_{L^2} + CB\varepsilon t^{-\frac{5}{4}},$$

i.e. the main contribution to $Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm)$ is the one where $Du^{I_2}_\pm$ is truncated for frequencies less or equal than t^σ. Therefore, we can write

$$Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm) = Q^{\mathrm{kg}}_0\left(v^{I_1}_\pm, \chi(t^{-\sigma}D_x)Du^{I_2}_\pm\right) + \mathfrak{R}^k_3,$$

which concludes the proof of (ii). We should highlight the fact that the quadratic contribution in the above left hand side is treated differently from the previous ones, because we do not have a sharp decay $O(t^{-1})$ for $v^{I_1}_\pm$ when $I_1\in\mathcal{K}$ (neither when truncated for moderate frequencies), but only a control in $O(t^{-1+\beta'})$, for some small $\beta'>0$ (see lemma B.3.9). Moreover, the decay enjoyed by the uniform norm of $\chi(t^{-\sigma}D_x)Du^{I_2}_\pm$, appearing in the quadratic term in the above right hand side, is very weak (only $t^{-1/2+\beta'}$, see lemma B.2.10). Such terms, that contribute to the energy and decay slowly in time, will be successively eliminated by a normal form argument (see subsection 2.2.2). □

We prove in the following lemma an analogous result to that presented in lemma 2.1.2 (ii), where the space derivative D_j, for $j=1,2,3$, is replaced with D_t. We highlight the fact that the below summations are considered for multi-indices $I_1, I_2\in\mathcal{I}^k_3$ such that $|I_1|+|I_2|\le 2$. This is explained by the fact that such contributions appear when the family Γ^I of admissible vector fields commute with the null form Q_0 (the remaining products Γ^{I_1}, Γ^{I_2} acting on the arguments of Q_0 are hence such that $|I_1|+|I_2|<|I|$).

Lemma 2.1.3. *Under the same hypothesis of lemma 2.1.2 (ii),*

$$\sum_{\substack{(I_1,I_2)\in\mathcal{J}^k_3\\ |I_1|+|I_2|\le 2}} Q^{\mathrm{w}}_0(v^{I_1}_\pm, D_t v^{I_2}_\pm) = \mathfrak{R}^k_3(t,x), \tag{2.1.35a}$$

$$\sum_{\substack{(I_1,I_2)\in\mathcal{J}^k_3\\ |I_1|+|I_2|\le 2}} Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D_t u^{I_2}_\pm) = \sum_{J\in\mathcal{K}} Q^{\mathrm{kg}}_0(v^J_\pm, \chi(t^{-\sigma}D_x)|D_x|u_\pm) + \mathfrak{R}^k_3(t,x), \tag{2.1.35b}$$

with $\mathfrak{R}^k_3(t,x)$ *satisfying* (2.1.31).

Proof. The result of the statement follows using the equations satisfied by $u^{I_2}_\pm, v^{I_2}_\pm$ in system (2.1.2) with $I=I_2$, which give that

$$\sum_{\substack{(I_1,I_2)\in\mathcal{J}^k_3\\ |I_1|+|I_2|\le 2}} Q^{\mathrm{w}}_0(v^{I_1}_\pm, D_t v^{I_2}_\pm) = \sum_{\substack{(I_1,I_2)\in\mathcal{J}^k_3\\ |I_1|+|I_2|\le 2}} Q^{\mathrm{w}}_0(v^{I_1}_\pm, \langle D_x\rangle v^{I_2}_\pm) + \sum_{\substack{(I_1,I_2)\in\mathcal{J}^k_3\\ |I_1|+|I_2|\le 2}} \sum_{(J_1,J_2)\in\mathcal{I}(I_2)} c_{J_1J_2} Q^{\mathrm{w}}_0\left(v^{I_1}_\pm, Q^{\mathrm{kg}}_0(v^{J_1}_\pm, Du^{J_2}_\pm)\right) \tag{2.1.36a}$$

$$\sum_{\substack{(I_1,I_2)\in\mathcal{J}^k_3\\ |I_1|+|I_2|\le 2}} Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D_t u^{I_2}_\pm) = \sum_{\substack{(I_1,I_2)\in\mathcal{J}^k_3\\ |I_1|+|I_2|\le 2}} Q^{\mathrm{kg}}_0(v^{I_1}_\pm, |D_x| u^{I_2}_\pm) + \sum_{\substack{(I_1,I_2)\in\mathcal{J}^k_3\\ |I_1|+|I_2|\le 2}} \sum_{(J_1,J_2)\in\mathcal{I}(I_2)} c_{J_1J_2} Q^{\mathrm{kg}}_0\left(v^{I_1}_\pm, Q^{\mathrm{w}}_0(v^{J_1}_\pm, Dv^{J_2}_\pm)\right), \tag{2.1.36b}$$

with coefficients $c_{J_1J_2}\in\{-1,0,-1\}$, and where $Q^{\mathrm{w}}_0(v^{I_1}_\pm, \langle D_x\rangle v^{I_2}_\pm)$, $Q^{\mathrm{kg}}_0(v^{I_1}_\pm, |D_x| u^{I_2}_\pm)$ are given explicitly by (2.1.24). After lemma 2.1.2 (*ii*), we know that

$$\sum_{\substack{(I_1,I_2)\in\mathcal{J}^k_3\\ |I_1|+|I_2|\le 2}} \left[Q^{\mathrm{w}}_0(v^{I_1}_\pm, \langle D_x\rangle v^{I_2}_\pm) + Q^{\mathrm{kg}}_0(v^{I_1}_\pm, |D_x| u^{I_2}_\pm)\right] = \sum_{\substack{(I_1,I_2)\in\mathcal{I}^k_3\\ I_1\in\mathcal{K}, |I_2|=0}} Q^{\mathrm{kg}}_0(v^{I_1}_\pm, |D_x|u_\pm) + \mathfrak{R}^k_3,$$

with $\mathfrak{R}^k_3$ verifying (2.1.31). The only thing that remains to prove to derive the statement is that the cubic terms in the right hand side of (2.1.36) are also remainders $\mathfrak{R}^k_3$.

First of all, we should observe that, as $|I_2|\le 2$,

$$\sum_{(J_1,J_2)\in\mathcal{I}(I_2)} Q^{\mathrm{kg}}_0(v^{J_1}_\pm, Du^{J_2}_\pm) = \sum_{\substack{(J_1,J_2)\in\mathcal{I}(I_2)\\ |J_1|+|J_2|=|I_2|}} Q^{\mathrm{kg}}_0(v^{J_1}_\pm, D_1 u^{J_2}_\pm) + \sum_{\substack{(J_1,J_2)\in\mathcal{I}(I_2)\\ |J_1|+|J_2|<|I_2|}} \left[Q^{\mathrm{kg}}_0(v^{J_1}_\pm, Du_\pm) + Q^{\mathrm{kg}}_0(v_\pm, Du^{J_2}_\pm)\right], \tag{2.1.37}$$

and

$$\sum_{(J_1,J_2)\in\mathcal{I}(I_2)} Q^{\mathrm{w}}_0(v^{J_1}_\pm, Dv^{J_2}_\pm) = \sum_{\substack{(J_1,J_2)\in\mathcal{I}(I_2)\\ |J_1|+|J_2|=|I_2|}} Q^{\mathrm{w}}_0(v^{J_1}_\pm, D_1 v^{J_2}_\pm) + \sum_{\substack{(J_1,J_2)\in\mathcal{I}(I_2)\\ |J_1|+|J_2|<|I_2|}} \left[Q^{\mathrm{w}}_0(v^{J_1}_\pm, Dv_\pm) + Q^{\mathrm{w}}_0(v_\pm, Dv^{J_2}_\pm)\right]. \tag{2.1.38}$$

After lemma 2.1.2 (ii), the fact that $|I_2|\le 2$ (if $(J_1,J_2)\in\mathcal{I}$ and $J_1\in\mathcal{K}$, then $|J_2|=0$), and a-priori estimates (1.1.11), we have that

(2.1.39a)
$$\begin{split}\sum_{\substack{(J_1,J_2)\in\mathcal{I}(I_2)\\ |J_1|+|J_2|=|I_2|}}\|Q^{\mathrm{kg}}_0(v^{J_1}_\pm, D_1u^{J_2}_\pm)\|_{L^2} &\lesssim \|\mathfrak{R}^k_3(t,\cdot)\|_{L^2}+\sum_{|J|\le 2}\|Q^{\mathrm{kg}}_0(v^J_\pm, D_1\chi(t^{-\sigma}D_x)u_\pm)\|_{L^2}\\ &\lesssim \|\mathfrak{R}^k_3(t,\cdot)\|_{L^2}+t^\beta\left(\|u_\pm(t,\cdot)\|_{L^\infty}+\|\mathrm{R}_1u_\pm(t,\cdot)\|_{L^\infty}\right)E^1_3(t;W)^{\frac12}\\ &\le CB\varepsilon t^{-\frac12+\beta+\frac{\delta_1}{2}},\end{split}$$

with $\beta>0$ small, $\beta\to 0$ as $\sigma\to 0$, and moreover

(2.1.39b)
$$\begin{split}\sum_{\substack{(J_1,J_2)\in\mathcal{I}(I_2)\\ |J_1|+|J_2|<|I_2|}}&\|Q^{\mathrm{kg}}_0(v^{J_1}_\pm, Du_\pm)\|_{L^2}+\|Q^{\mathrm{kg}}_0(v_\pm, Du^{J_2}_\pm)\|_{L^2}\\ &\lesssim (\|W(t,\cdot)\|_{H^{2,\infty}}+\|D_tu_\pm(t,\cdot)\|_{H^{1,\infty}})E^2_3(t;W)^{\frac12}\le CAB\varepsilon^2t^{-\frac12+\frac{\delta_2}{2}},\end{split}$$

last estimate following from (B.1.5b) with $s=1$, (B.1.7) and a-priori estimates (1.1.11). Consequently, as the L^2 norm of cubic terms in the right hand side of (2.1.36a), for which index I_1 is such that $\Gamma^{I_1}\in\{D^\alpha_x,|\alpha|\le 2\}$, can be bounded by the L^∞ norm of the Klein-Gordon component times the L^2 norm of the remaining quadratic contribution, and is less or equal than $CAB\varepsilon^2t^{-\frac32+\beta'}$, for a small $\beta'>0$, $\beta'\to 0$ as $\sigma,\delta_0\to 0$, after (1.1.11b) and (2.1.39).

Cubic terms corresponding to $|I_1|=2$ (and hence $|I_2|=0$), are instead estimated, using (B.1.4e) and a-priori estimates, as follows:

$$\left\|Q^{\mathrm{w}}_0\left(v^{I_1}_\pm, Q^{\mathrm{kg}}_0(v_\pm,D_1u_\pm)\right)\right\|_{L^2}\lesssim\|v^{I_1}_\pm(t,\cdot)\|_{L^2}\|Q^{\mathrm{kg}}_0(v_\pm,D_1u_\pm)\|_{L^\infty}\le CB\varepsilon t^{-\frac32+\beta'},$$

for a new $\beta'>0$, small as long as σ,δ_0 are small.

Finally, cubic terms such that $\Gamma^{I_1}\in\{\Omega, Z_m, m=1,2\}$ (and $|I_2|\le 1$), are estimated by means of corollary B.2.4 in appendix B, with $L=L^2$, $w=v$. For some smooth cut-off function χ, some $\sigma>0$ small, we have that

$$\begin{split}\sum_{(J_1,J_2)\in\mathcal{I}(I_2)}&\left\|Q^{\mathrm{w}}_0\left(v^{I_1}_\pm, Q^{\mathrm{kg}}_0(v^{J_1}_\pm,Du^{J_2}_\pm)\right)(t,\cdot)\right\|_{L^2}\\ &\lesssim\sum_{(J_1,J_2)\in\mathcal{I}(I_2)}\left\|\chi(t^{-\sigma}D_x)v^{I_1}_\pm(t,\cdot)\right\|_{L^\infty}\left\|Q^{\mathrm{kg}}_0(v^{J_1}_\pm,Du^{J_2}_\pm)(t,\cdot)\right\|_{L^2}\\ &+\sum_{(J_1,J_2)\in\mathcal{I}(I_2)}t^{-N(s)}\left(\|v_\pm(t,\cdot)\|_{H^s}+\|D_tv_\pm(t,\cdot)\|_{H^s}\right)\\ &\qquad\times\Big(\sum_{|\mu|=0}^1\left\|x^\mu Q^{\mathrm{kg}}_0(v^{J_1}_\pm,Du^{J_2}_\pm)(t,\cdot)\right\|_{L^2}+t\left\|Q^{\mathrm{kg}}_0(v^{J_1}_\pm,Du^{J_2}_\pm)(t,\cdot)\right\|_{L^2}\Big),\end{split}$$

where

$$\begin{split}\sum_{(J_1,J_2)\in\mathcal{I}(I_2)}&\left\|xQ^{\mathrm{kg}}_0(v^{J_1}_\pm,Du^{J_2}_\pm)(t,\cdot)\right\|_{L^2}\\ &\lesssim\sum_{\substack{|\mu|=0,1\\|J|\le 1}}\Big[\left\|x\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\right\|_{L^\infty}\left(\|u^J_\pm(t,\cdot)\|_{H^1}+\|D_tu^J_\pm(t,\cdot)\|_{L^2}\right)\\ &\qquad+\|xv^J_\pm(t,\cdot)\|_{L^2}\left(\|\mathrm{R}^\mu u_\pm(t,\cdot)\|_{H^{2,\infty}}+\|D_t\mathrm{R}^\mu u_\pm(t,\cdot)\|_{H^{1,\infty}}\right)\Big].\end{split}$$

Therefore, choosing $s>0$ large enough to have $N(s)\ge 4$, and using estimates (2.1.39), together with (B.1.5a), (B.1.5c), (B.1.7), (B.1.10a), (B.1.10b), (B.1.17), and lemma B.3.21, we find that

$$\sum_{(J_1,J_2)\in\mathcal{I}(I_2)}\left\|Q^{\mathrm{w}}_0\left(v^{I_1}_\pm, Q^{\mathrm{kg}}_0(v^{J_1}_\pm, Du^{J_2}_\pm)\right)(t,\cdot)\right\|_{L^2}\le CB^2\varepsilon^2 t^{-\frac{3}{2}+\beta'}$$

and this cubic contribution can also be considered as a remainder $\mathfrak{R}^k_3$. A similar analysis shows that the same can be said for the cubic terms in the right hand side of (2.1.36b), and that concludes the proof of the statement. □

Corollary 2.1.4. *Let $Q^I_0(V,W)$ be the vector defined in* (2.1.12)*. There exists a constant $C>0$ such that, if we assume that a-priori estimates* (1.1.11) *are satisfied in interval $[1,T]$, for some fixed $T>1$, with $\varepsilon_0<(2A+B)^{-1}$ small:*

(i) if $I\in\mathcal{I}_n$ with $n\ge 3$:

$$\|Q^I_0(V,W)\|_{L^2}\le CA\varepsilon t^{-\frac{1}{2}+\frac{\delta}{2}}; \tag{2.1.40}$$

(ii) if $I\in\mathcal{I}^k_3$, with $0\le k\le 2$,

$$\|Q^I_0(V,W)\|_{L^2}\le C(A+B)\varepsilon t^{-\frac{1}{2}+\frac{\delta_k}{2}}. \tag{2.1.41}$$

Proof. Inequality (2.1.40) is straightforward after definition (2.1.12) (all coefficients c_{I_1,I_2} are equal to 0 when $I\in\mathcal{I}_n$), lemma 2.1.2 (i), and a-priori estimates (1.1.11a), (1.1.11b).

If $I\in\mathcal{I}^k_3$ for a fixed $0\le k\le 2$, we have by definition (2.1.12), and lemmas 2.1.2, 2.1.3, that

$$\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_2|<|I|}} Q^{\mathrm{w}}_0(v^{I_1}_\pm, Dv^{I_2}_\pm)=\mathfrak{R}^k_3(t,x),$$

with $\mathfrak{R}^k_3(t,x)$ satisfying (2.1.31) and hence bounded by the right hand side of (2.1.41), after a-priori estimates and the fact that $\delta,\delta_k\ll 1$ are small, for $0\le k\le 2$. Moreover, for some smooth $\chi\in C^\infty_0(\mathbb{R}^2)$, equal to 1 in a neighbourhood of the origin, and $\sigma>0$ small,

$$\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_2|<|I|}} Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm)=\delta_{\mathcal{V}_k}\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ I_1\in\mathcal{K},|I_2|\le 1}} Q^{\mathrm{kg}}_0\left(v^{I_1}_\pm,\chi(t^{-\sigma}D_x)Du^{I_2}_\pm\right)+\mathfrak{R}^k_3(t,x), \tag{2.1.42}$$

with sets $\mathcal{K},\mathcal{V}_k$ given, respectively, by (2.1.26), (2.1.27), $\delta_{\mathcal{V}_k}=1$ if $I\in\mathcal{V}_k$, 0 otherwise (remind that $\mathcal{V}_2$ is empty), and

$$\left\|Q^{\mathrm{kg}}_0\left(v^{I_1}_\pm,\chi(t^{-\sigma}D_x)Du^{I_2}_\pm\right)\right\|_{L^2}\lesssim\sum_{|\mu|=0}^{1}\left\|\chi(t^{-\sigma}D_x)\mathrm{R}^\mu u^{I_2}_\pm(t,\cdot)\right\|_{H^{2,\infty}}\|v^{I_1}_\pm(t,\cdot)\|_{L^2}. \tag{2.1.43}$$

Consequently, if $I\in\mathcal{I}^2_3$ then $\delta_{\mathcal{V}_2}=0$ and we immediately have, after (1.1.11d), that

$$\|Q^I_0(V,W)\|_{L^2}\lesssim\|\mathfrak{R}^k_3(t,\cdot)\|_{L^2}\le C(A+B)\varepsilon t^{-1+\frac{\delta_2}{2}}.$$

If $I\in\mathcal{I}^k_3$, for $k=0,1$, and $(I_1,I_2)\in\mathcal{I}(I)$, two situations may occur: we could have $I_1\in\mathcal{K}$ and $\Gamma^{I_2}\in\{D^\alpha_x,|\alpha|\le 1\}$, in which case product Γ^{I_1} could contain the same number of Klainerman vector fields as in Γ^I, and V^{I_1} would be at the same energy level as V^I (i.e. its L^2 being controlled by $E^k_3(t;W)^{1/2}$). In this case, for any $\rho\in\mathbb{N}$,

$$
\begin{aligned}
&\|v^{I_1}_\pm(t,\cdot)\|_{L^2}\Big(\|\chi(t^{-\sigma}D_x)u^{I_2}_\pm(t,\cdot)\|_{H^{\rho,\infty}}+\|\chi(t^{-\sigma}D_x)\mathrm{R}u^{I_2}_\pm(t,\cdot)\|_{H^{\rho,\infty}}\Big)\le A\varepsilon t^{-\frac12}E^k_3(t;W)^{\frac12}\\
&\qquad\le AB\varepsilon^2t^{-\frac12+\frac{\delta_k}{2}},
\end{aligned}
\tag{2.1.44}
$$

as follows from a-priori estimate (1.1.11a). If, instead, $(I_1,I_2)\in\mathcal{I}(I)$ with $I_1\in\mathcal{K}$ and I_2 such that $\Gamma^{I_2}\in\{\Omega,Z_m,m=1,2\}$ is a Klainerman vector field, we automatically have that Γ^I is a product of three Klainerman vector fields, and that V^{I_1} is at an energy level strictly lower than V^I (i.e. its L^2 norm is controlled by energy $E^1_3(t;W)^{1/2}$, whereas that of V^I is bounded by $E^0_3(t;W)^{\frac12}$). The small loss t^β in the uniform estimate of $\chi(t^{-\sigma}D_x)\mathrm{R}^\mu U^{I_2}$, $|\mu|=0,1$, with positive $\beta\to0$ as $\sigma\to0$ (see lemma B.2.10) is hence compensated by the fact that $\delta_1\ll\delta_0$, meaning that

$$
\begin{aligned}
&\|v^{I_1}_\pm(t,\cdot)\|_{L^2}\Big(\|\chi(t^{-\sigma}D_x)u^{I_2}_\pm(t,\cdot)\|_{H^{\rho,\infty}}+\|\chi(t^{-\sigma}D_x)\mathrm{R}u^{I_2}_\pm(t,\cdot)\|_{H^{\rho,\infty}}\Big)\\
&\qquad\le C(A+B)\varepsilon t^{-\frac12+\beta+\frac{\delta_1}{2}}E^1_3(t;W)^{\frac12}\le C(A+B)B\varepsilon^2t^{-\frac12+\frac{\delta_0}{2}},
\end{aligned}
\tag{2.1.45}
$$

last inequality following from lemma B.2.10, a-priori estimate (1.1.11d), and taking $\sigma>0$ sufficiently small so that $\beta+\delta_1<\delta_0/2$. For any $k=0,1$, we then have that

$$
\|v^{I_1}_\pm(t,\cdot)\|_{L^2}\Big(\|\chi(t^{-\sigma}D_x)u^{I_2}_\pm(t,\cdot)\|_{H^{\rho,\infty}}+\|\chi(t^{-\sigma}D_x)\mathrm{R}u^{I_2}_\pm(t,\cdot)\|_{H^{\rho,\infty}}\Big)\le C(A+B)B\varepsilon^2t^{-\frac12+\frac{\delta_k}{2}},
$$

and from (2.1.42), (2.1.43), we derive (2.1.41). ☐

2.1.3 Symmetrization

Proposition 2.1.5. *As long as $\|V(t,\cdot)\|_{H^{1,\infty}}$ is sufficiently small, there exists a real matrix $P(V;\eta)$ of order 0, and a real, symmetric matrix $\widetilde{A}_1(V;\eta)$ of order 1, vanishing at order 1 at $V=0$, such that $W^I_s:=Op^B\big(P(V;\eta)\big)W^I$ is solution to*

$$
\begin{aligned}
D_tW^I_s&=A(D)W^I_s+Op^B(\widetilde{A}_1(V;\eta))W^I_s+Op^B(A''(V^I;\eta))U\\
&\quad+Op^B(C''(U;\eta))V^I+Op^B_R(A''(V^I;\eta))U+Q^I_0(V,W)+\mathfrak{R}(U,V),
\end{aligned}
\tag{2.1.46}
$$

where $\mathfrak{R}(U,V)$ satisfies, for any $\theta\in]0,1[$,

$$
\begin{aligned}
\|\mathfrak{R}(U,V)(t,\cdot)\|_{L^2}&\lesssim\Big[\|V(t,\cdot)\|_{H^{7,\infty}}(1+\|V(t,\cdot)\|_{H^{1,\infty}})\\
&\quad+\|V(t,\cdot)\|^{1-\theta}_{H^{1,\infty}}\|V(t,\cdot)\|^\theta_{H^3}\left(\|U(t,\cdot)\|_{H^{2,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}\right)\\
&\quad+\|V(t,\cdot)\|_{L^\infty}\Big(\|U(t,\cdot)\|^{1-\theta}_{H^{2,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|^{1-\theta}_{H^{2,\infty}}\Big)\|U(t,\cdot)\|^\theta_{H^4}\Big]\|W^I(t,\cdot)\|_{L^2}\\
&\quad+\|V(t,\cdot)\|_{H^{1,\infty}}\Big(\|W(t,\cdot)\|_{H^{7,\infty}}+\|\mathrm{R}U(t,\cdot)\|_{H^{6,\infty}}\Big)\|W^I(t,\cdot)\|_{L^2}\\
&\quad+\|V(t,\cdot)\|_{H^{1,\infty}}\|Q^I_0(V,W)\|_{L^2}.
\end{aligned}
\tag{2.1.47}
$$

Moreover, for any $n,r\in\mathbb{N}$, with the notation introduced in (1.2.3),

$$
M^0_r\left(P(V;\eta)-I_4;n\right)\lesssim\|V(t,\cdot)\|_{H^{1+r,\infty}},
\tag{2.1.48}
$$

$$
M^1_r\big(\widetilde{A}_1(V;\eta);n\big)\lesssim\|V(t,\cdot)\|_{H^{1+r,\infty}},
\tag{2.1.49}
$$

and there is a constant $C = C(\|V\|_{H^{2,\infty}}) > 0$ such that

$$C^{-1}\|W^I(t,\cdot)\|_{L^2} \le \|W^I_s(t,\cdot)\|_{L^2} \le C\|W^I(t,\cdot)\|_{L^2}, \tag{2.1.50}$$

as long as $\|V(t,\cdot)|_{H^{2,\infty}}$ is small.

In order to prove proposition 2.1.5, we first need to introduce the following lemma.

Lemma 2.1.6. *Let $\alpha, \beta \in \mathbb{R}$, $L \in M_2(\mathbb{R})$ and $M_0, N(\alpha,\beta) \in M_4(\mathbb{R})$ given by*

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_0 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}, \quad N(\alpha,\beta) = \begin{bmatrix} \alpha L & \beta L \\ \alpha L & \beta L \end{bmatrix} = \begin{bmatrix} 0 & \alpha & 0 & \beta \\ \alpha & 0 & \beta & 0 \\ 0 & \alpha & 0 & \beta \\ \alpha & 0 & \beta & 0 \end{bmatrix}.$$

There exist a small $\delta > 0$ and a smooth function defined on open ball $B_\delta(0)$ of radius δ,

$$(\alpha,\beta) \in B_\delta(0) \to P(\alpha,\beta) \in Sym_4(\mathbb{R}),$$

with values in the space of real, symmetric, 4×4 matrices $Sym_4(\mathbb{R})$, such that $P(0,0) = I_4$, $P(\alpha,\beta) = I_4 + O(|\alpha|+|\beta|)$, and $P(\alpha,\beta)^{-1}\big(M_0 + N(\alpha,\beta)\big)P(\alpha,\beta)$ is symmetric for any $(\alpha,\beta) \in B_\delta(0)$. Furthermore, $P^{-1}(\alpha,\beta) = I_4 + O(|\alpha|+|\beta|)$.

Proof. Let $\mathcal{E}$ be the vector space of 2×2 matrices $B(\alpha,\beta) = \alpha I_2 + \beta L$, and $\mathcal{F}$ be the set of 4×4 matrices of the form

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$$

with $F_{ij} \in \mathcal{E}$. We look for a matrix P of the form

$$P(B) = (I_2 - B^2)^{-\frac{1}{2}} \begin{bmatrix} I_2 & -B \\ -B & I_2 \end{bmatrix}$$

with $B \in \mathcal{E}$ close to zero (so that in particular $(I_2 - B^2)^{1/2}$ is well defined). We remark that matrix $P(B)^{-1}$ has the form

$$P(B)^{-1} = (I_2 - B^2)^{-\frac{1}{2}} \begin{bmatrix} I_2 & B \\ B & I_2 \end{bmatrix},$$

and that $P(0) = P^{-1}(0) = I_4$. We consider $\Phi : \mathbb{R}^2 \times \mathcal{E} \to \mathcal{F}$ defined by $\Phi(\alpha,\beta,B) := P(B)^{-1}\big[M_0 + N(\alpha,\beta)\big]P(B) = \big(\Phi_{ij}(\alpha,\beta,B)\big)_{1\le i,j\le 2}$, where $\Phi_{ij} \in \mathcal{E}$ as $\mathcal{E}$ is a commutative sub-algebra of $M_2(\mathbb{R})$, and we define $\Psi(\alpha,\beta,B) := \Phi_{12}(\alpha,\beta,B) - \Phi^\dagger_{21}(\alpha,\beta,B)$, with $\Phi^\dagger_{21}$ denoting the transpose of Φ_{21}. We have that $\Psi(0,0,0) = 0$ and

$$D_B\Phi(0,0,0)\cdot B = \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} M_0 - M_0 \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} = 2\begin{bmatrix} 0 & -B \\ B & 0 \end{bmatrix}$$

from which follows that $D_B\Psi(0,0,0)\cdot B = -4B$, i.e. $D_B\Psi(0,0,0) = -4I$. Therefore, there exist a small $\delta > 0$ and a smooth function $(\alpha,\beta) \in B_\delta(0) \to B(\alpha,\beta) \in \mathcal{E}$, such that $B(0,0) = 0$ (which implies $P(B(0,0)) = I_4$), and $\Psi(\alpha,\beta,B(\alpha,\beta)) = 0$, $\forall(\alpha,\beta) \in B_\delta(0)$. This is equivalent to say that $\Phi(\alpha,\beta,B(\alpha,\beta))$ is symmetric, and moreover $P(B(\alpha,\beta)), P(B(\alpha,\beta))^{-1} = I_4 + O(|\alpha|+|\beta|)$. Defining $P(\alpha,\beta) := P(B(\alpha,\beta))$ concludes the proof of the statement. □

Proof of proposition 2.1.5. With notations introduced in lemma 2.1.6, and in (2.1.5), (2.1.19), $A(\eta)=\langle\eta\rangle M_0+S(\eta)$ and $A'_1(V;\eta)(1-\chi)(\eta)=\langle\eta\rangle N(\alpha,\beta)$, with

$$S(\eta)=\begin{bmatrix}|\eta|-\langle\eta\rangle & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -(|\eta|-\langle\eta\rangle) & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\text{whose elements are } O(|\eta|^{-1}), |\eta|\to+\infty,$$

and $\alpha=a_0(v_\pm;\eta)\frac{\eta_1}{\langle\eta\rangle}(1-\chi)(\eta)$, $\beta=b_0(v_\pm;\eta)\frac{\eta_1}{\langle\eta\rangle}(1-\chi)(\eta)$, a_0,b_0 defined in (2.1.8). Since $\sup_\eta\big(|\alpha|+|\beta|\big)\lesssim\|V(t,\cdot)\|_{H^{1,\infty}}$, by lemma 2.1.6 we have that, as long as $\|V(t,\cdot)\|_{H^{1,\infty}}$ is sufficiently small, there exists a real, symmetric matrix $P=P(V;\eta)$ such that $P(V;\eta)^{-1}\big[M_0+N(\alpha,\beta)\big]P(V;\eta)$ is real and symmetric. Moreover $P=I_4+Q(V;\eta), P^{-1}=I_4+Q'(V;\eta)$, where $Q(V;\eta)$, $Q'(V;\eta)$ depend smoothly on α,β (which are symbols of order 0), are null at order 1 at $V=0$, and verify, for any $n,r\in\mathbb{N}$,

$$M^0_r\left(Q(V;\eta);n\right)+M^0_r\left(Q'(V;\eta);n\right)\lesssim\|V(t,\cdot)\|_{H^{1+r,\infty}}.$$

We define

$$\widetilde{A}_1(V;\eta):=P(V;\eta)^{-1}\big[\langle\eta\rangle\big(M_0+N(\alpha,\beta)\big)\big]P(V;\eta)-\langle\eta\rangle M_0,$$

which is a matrix of order 1, and $W^I_s:=Op^B(P^{-1}(V;\eta))W^I$. From the fact that $\widetilde{A}_1(V;\eta)$ also writes as

$$\langle\eta\rangle\left[Q'(V;\eta)M_0+P^{-1}(V;\eta)M_0Q(V;\eta)+P^{-1}(V;\eta)N(\alpha,\beta)P(V;\eta)\right],$$

we see that it vanishes at order 1 at $V=0$, and is such that $M^1_r(\widetilde{A}_1(V;\eta);n)\lesssim\|V(t,\cdot)\|_{H^{1+r,\infty}}$. Moreover, from proposition 1.2.9 (*ii*) with $r=1$ it follows that

$$I=Op^B(P(V;\eta))Op^B(P^{-1}(V;\eta))+T_{-1}(V), \tag{2.1.51}$$

where $T_{-1}(V)$ is an operator of order less or equal than -1 and $\mathcal{L}(L^2)$ norm $O(\|V(t,\cdot)\|_{H^{2,\infty}})$. Therefore, $W^I=Op^B(P(V;\eta))W^I_s+T_{-1}(V)W^I$, and from proposition 1.2.7 we deduce that the L^2 norms of W^I,W^I_s are equivalent as long as $\|V(t,\cdot)\|_{H^{2,\infty}}$ is small. Using equation (2.1.20) we find that:

(2.1.52)
$$\begin{aligned}D_tW^I_s&=Op^B(P^{-1}(V;\eta))Op^B\big(A(\eta)+A'_1(V;\eta)(1-\chi)(\eta)\big)W^I\\&+Op^B(P^{-1}(V;\eta))\Big[Op^B\big(A'_1(V;\eta)\chi(\eta)\big)+Op^B\big(A'_{-1}(V;\eta)\big)\Big]W^I\\&+Op^B(P^{-1}(V;\eta))\Big[Op^B(C'(W^I;\eta))V+Op^B_R(A'(V;\eta))W^I\Big]\\&+Op^B(P^{-1}(V;\eta))\Big[Op^B(A''(V^I;\eta))U+Op^B(C''(U;\eta))V^I+Op^B_R(A''(V^I;\eta))U\Big]\\&+Op^B(P^{-1}(V;\eta))Q^I_0(V,W)+Op^B(D_tP^{-1}(V;\eta))W^I,\end{aligned}$$

where

(2.1.53)
$$\begin{aligned}&Op^B(P^{-1}(V;\eta))Op^B\big(A(\eta)+A'_1(V;\eta)(1-\chi)(\eta)\big)W^I\\&=Op^B(P^{-1}(V;\eta))Op^B\left(\langle\eta\rangle\big(M_0+N(\alpha,\beta)\big)\right)W^I+Op^B(S(\eta))W^I+Op^B(Q'(V;\eta))Op^B(S(\eta))W^I\\&=Op^B(P^{-1}(V;\eta))Op^B\left(\langle\eta\rangle\big(M_0+N(\alpha,\beta)\big)\right)Op^B(P(V;\eta))W^I_s\\&+Op^B(P^{-1}(V;\eta))Op^B\left(\langle\eta\rangle\big(M_0+N(\alpha,\beta)\big)\right)T_{-1}(V)W^I+Op^B(S(\eta))W^I_s\\&+Op^B(S(\eta))Op^B(Q(V;\eta))W^I_s+Op^B(S(\eta))T_{-1}(V)W^I+Op^B(Q'(V;\eta))Op^B(S(\eta))W^I\\&=Op^B(A(\eta)+\widetilde{A}_1(V;\eta))W^I_s+\widetilde{T}_0(V)W^I_s+\widetilde{T'}_0(V)W^I,\end{aligned}$$

where $\widetilde{T}_0(V), \widetilde{T}'_0(V)$ are also operators of order 0 and $\mathcal{L}(L^2)$ norm $O(\|V(t,\cdot)\|_{H^{2,\infty}})$. Indeed, last equality follows from the fact that, by proposition 1.2.9 (*ii*) with $r=1$, and proposition 1.2.7,

$$\begin{aligned}Op^B(P^{-1}(V;\eta))Op^B\big[\langle\eta\rangle\big(M_0+N(\alpha,\beta)\big)\big]Op^B(P(V;\eta))\\ = Op^B\big(P(V;\eta)^{-1}\big[\langle\eta\rangle\big(M_0+N(\alpha,\beta)\big)\big]P(V;\eta)\big)+\widetilde{T}_0(V),\end{aligned}$$

and $Op^B(S(\eta))Op^B(Q(V;\eta))$, $Op^B(Q'(V;\eta))Op^B(S(\eta))$ are operator of order 0, too (the former of the form $\widetilde{T}_0(V)$, the latter of the form $T_0(V)$), while $Op^B(S(\eta))T_{-1}(V)$ is of order -1 (and can be included in $T_0(V)$). After the equivalence between the L^2 norms of W^I_s, W^I, we deduce that $\widetilde{T}_0(V)W^I_s+\widetilde{T}'_0(V)W^I$ in (2.1.53) is a remainder $\mathfrak{R}(U,V)$.

All operators appearing in the second and third line of (2.1.52) are also remainders $\mathfrak{R}(U,V)$ because, from proposition 1.2.7, the fact that $M^0_0(P^{-1}(V;\eta);2)=O(1+\|V(t,\cdot)\|_{H^{1,\infty}})$, and lemma 2.1.1, their L^2 norm is bounded by $\|V(t,\cdot)\|_{H^{7,\infty}}(1+\|V(t,\cdot)\|_{H^{1,\infty}})\|W^I(t,\cdot)\|_{L^2}$. Last term in (2.1.52) also contributes to $\mathfrak{R}(U,V)$, for matrix $D_tP^{-1}(V;\eta)$ is of order 0, its $M^0_0(\cdot,2)$ seminorm is bounded by $\|D_tV(t,\cdot)\|_{H^{1,\infty}}$, and from (B.1.6b) with $s=1$, we have that, for any $\theta\in[0,1]$,

$$\begin{aligned}\|D_tV(t,\cdot)\|_{H^{1,\infty}} \lesssim \|V(t,\cdot)\|_{H^{2,\infty}}+\|V(t,\cdot)\|^{1-\theta}_{H^{1,\infty}}\|V(t,\cdot)\|^{\theta}_{H^3}\left(\|U(t,\cdot)\|_{H^{2,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}\right)\\ +\|V(t,\cdot)\|_{L^\infty}\Big(\|U(t,\cdot)\|^{1-\theta}_{H^{2,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|^{1-\theta}_{H^{2,\infty}}\Big)\|U(t,\cdot)\|^{\theta}_{H^4}.\end{aligned}$$

Finally, in remaining contributions in (2.1.52), we replace $Op^B(P^{-1}(V;\eta))$ with operator $I+Op^B(Q'(V;\eta))$, and all of the terms on which $Op^B(Q'(V;\eta))$ acts are remainders $\mathfrak{R}(U,V)$, as follows combining proposition 1.2.7, the fact that $M^0_0\left(Q'(V;\eta);2\right)=O(\|V(t,\cdot)\|_{H^{1,\infty}})$, and lemma 2.1.1. Interchanging the notation of $P(V;\eta)$ and $P^{-1}(V;\eta)$, we obtain the result of the statement. □

2.2 Normal forms and energy estimates

Before going further in writing an energy inequality for W^I_s we should make few remarks. As we previously anticipated, the L^2 norm of some of the semi-linear terms appearing in equation (2.1.46) have a very slow decay in time. It is the case of $Op^B(A''(V^I;\eta))U$, $Op^B(C''(U;\eta))V^I$ and $Op^B_R(A''(V;\eta))U$, whose L^2 norms are estimated in (2.1.21d), (2.1.21e) and depend on the uniform norms of U,R_1U, which after a-priori estimates (1.1.11a) are only a $O(t^{-1/2})$, and of some contributions in $Q^I_0(V,W)$ which, after corollary 2.1.4, are a $O_{L^2}(t^{-1/2+\beta'})$, for some small $\beta'>0$.

Nevertheless, $Op^B(A''(V^I;\eta))U$, $Op^B_R(A''(V;\eta))U$ and the mentioned contributions to $Q^I_0(V,W)$ can be easily eliminated by performing a semi-linear normal form argument in the energy inequality (see subsection 2.2.2). However, this is not the case for $Op^B(C''(U;\eta))V^I$, for which such type of argument leads to a loss of derivatives linked to the quasi-linear nature of the problem, i.e. the fact that matrix $\widetilde{A}_1(V;\eta)$ in the right hand side of (2.1.46) is of order 1. This latter contribution should instead be eliminated through normal forms directly on equation (2.1.46), which is the object of the subsection 2.2.1.

2.2.1 A first normal forms transformation and the energy inequality

First of all, let us replace $Op^B(C''(U;\eta))V^I$, in equation (2.1.46), with $Op^B(C''(U;\eta))V^I_s$ (having defined $V^I_s:=Op^B(P^{-1}(V;\eta))V^I$), up to a new remainder $\mathfrak{R}(U,V)$ that satisfies (2.1.47) thanks to estimate (2.1.21e), and deal from now on with

$$(D_t - A(D))W^I_s = Op^B(\widetilde{A}_1(V;\eta))W^I_s + Op^B(A''(V^I;\eta))U + Op^B(C''(U;\eta))V^I_s \\ + Op^B_R(A''(V^I;\eta))U + Q^I_0(V,W;\eta) + \mathfrak{R}(U,V), \tag{2.2.1}$$

for a new $\mathfrak{R}(U,V)$ satisfying (2.1.47).

We are going to prove the following result:

Proposition 2.2.1. *Let $N \in \mathbb{N}^*$. There exist three matrices of symbols, $E^0_d(U;\eta), E^{-1}_d(U;\eta)$, $E_{nd}(U;\eta)$, linear in (u_+, u_-), with $E^0_d(U;\eta)$ real, diagonal, of order 0, $E^{-1}_d(U;\eta)$ and $E_{nd}(U;\eta)$ of order -1, such that, for any $n, r \in \mathbb{N}$, any $\chi \in C^\infty_0(\mathbb{R}^2)$ equal to 1 close to the origin and supported in open ball $B_\varepsilon(0)$, with $\varepsilon > 0$ sufficiently small,*

$$M^0_r\left(E^0_d\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)U;\eta\right);n\right) \lesssim \|\mathrm{R}_1 U(t,\cdot)\|_{H^{1+r,\infty}}, \tag{2.2.2a}$$

$$M^{-1}_r\left(E^{-1}_d\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)U;\eta\right);n\right) \lesssim \|U(t,\cdot)\|_{H^{5+r,\infty}}, \tag{2.2.2b}$$

$$M^{-1}_r\left(E_{nd}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)U;\eta\right);n\right) \lesssim \|U(t,\cdot)\|_{H^{5+r,\infty}}; \tag{2.2.2c}$$

and, as long as $\|\mathrm{R}_1 U(t,\cdot)\|_{H^{2,\infty}}$ is small, there is a real diagonal matrix $F^0_d(U;\eta)$ of order 0, with

$$M^0_r\left(F^0_d\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)U;\eta\right);n\right) \lesssim \|\mathrm{R}_1 U(t,\cdot)\|_{H^{1+r,\infty}}, \tag{2.2.3}$$

so that, if one defines $\widetilde{W}^I_s := Op^B(I_4 + E(U;\eta))W^I_s$, with $E(U;\eta) := E^0_d(U;\eta) + E^{-1}_d(U;\eta) + E_{nd}(U;\eta)$, there is a positive C such that

$$C^{-1}\|W^I_s(t,\cdot)\|_{L^2} \le \|\widetilde{W}^I_s(t,\cdot)\|_{L^2} \le C\|W^I_s(t,\cdot)\|_{L^2}, \tag{2.2.4}$$

as long as $\|\mathrm{R}_1 U(t,\cdot)\|_{H^{2,\infty}} + \|U(t,\cdot)\|_{H^{5,\infty}}$ is small, and $\widetilde{W}^I_s$ is solution to

$$(D_t - A(D))\widetilde{W}^I_s = Op^B\Big((I_4 + E^0_d(U;\eta))\widetilde{A}_1(V;\eta)(I_4 + F^0_d(U;\eta))\Big)\widetilde{W}^I_s \\ + Op^B(A''(V^I;\eta))U + Op^B_R(A''(V^I;\eta))U + Q^I_0(V,W) + T_{-N}(U)W^I_s + \mathfrak{R}'(U,V). \tag{2.2.5}$$

In the above right hand side $T_{-N}(U)$ is a pseudo-differential operator of order less or equal than $-N$, $T_{-N}(U) = (\sigma_{ij}(U,D_x))_{ij}$ where symbols $\sigma_{ij}(U,\eta)_{ij}$ are such that

$$\mathcal{F}_{x\mapsto\xi}(\sigma_{ij}(U,\eta))(\xi) = \begin{cases} \sigma^+_{ij}(\xi,\eta)\hat{u}_+(\xi) + \sigma^-_{ij}(\xi,\eta)\hat{u}_-(\xi), & i,j \in \{2,4\}, \\ 0, & \textit{otherwise}, \end{cases} \tag{2.2.6a}$$

with $\sigma^\pm_{ij}(\xi,\eta)$ supported for $|\xi| \le \varepsilon\langle\eta\rangle$, for a small $\varepsilon > 0$, and for any $\alpha, \beta \in \mathbb{N}^2$

$$|\partial^\alpha_\xi \partial^\beta_\eta \sigma^\pm_{ij}(\xi,\eta)| \lesssim_{\alpha,\beta} |\xi|^{N+1-|\alpha|}\langle\eta\rangle^{-N-|\beta|}, \quad i,j \in \{2,4\}. \tag{2.2.6b}$$

Moreover, for any $s \in \mathbb{R}$,

$$\|T_{-N}(U)\|_{\mathcal{L}(H^{s-N};H^s)} \lesssim \|\mathrm{R}_1 U(t,\cdot)\|_{H^{N+2,\infty}} + \|U(t,\cdot)\|_{H^{N+6,\infty}}, \tag{2.2.7}$$

and $\mathfrak{R}'(U,V)$ *is a remainder satisfying, for any* $\theta\in]0,1[$

(2.2.8)

$$\begin{aligned}\|\mathfrak{R}'(U,V)(t,\cdot)\|_{L^2} &\lesssim (1+\|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}}+\|U(t,\cdot)\|_{H^{5,\infty}})\|\mathfrak{R}(U,V)\|_{L^2}\\ &+(\|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}}+\|U(t,\cdot)\|_{H^{5,\infty}})\Big[\|Q^I_0(V,W)\|_{L^2}\\ &+(\|\mathrm{R}U(t,\cdot)\|_{H^{6,\infty}}+\|U(t,\cdot)\|_{H^{6,\infty}})\big(1+(1+\|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}})\|V(t,\cdot)\|_{H^{2,\infty}}\big)\\ &\times(1+\|V(t,\cdot)\|_{H^{1,\infty}})\,\|W^I(t,\cdot)\|_{L^2}\Big]+\|V(t,\cdot)\|^{2-\theta}_{H^{5,\infty}}\|V(t,\cdot)\|^{\theta}_{H^7}\|W^I(t,\cdot)\|_{L^2},\end{aligned}$$

with $\mathfrak{R}(U,V)$ *verifying* (2.1.47).

Remark 2.2.2. From propositions 2.1.5, 2.2.1 it follows that, as long as $\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}$, $\|U(t,\cdot)\|_{H^{5,\infty}}$ and $\|V(t,\cdot)\|_{H^{2,\infty}}$ are small, there is a constant $C>0$ such that

$$C^{-1}\|W^I(t,\cdot)\|_{L^2}\le\|\widetilde{W}^I_s(t,\cdot)\|_{L^2}\le C\|W^I(t,\cdot)\|_{L^2}. \tag{2.2.9}$$

Introducing the following modification of the energy:

$$\widetilde{E}_n(t;W):=\sum_{|\alpha|\le n}\big\|Op^B(I_4+E(U;\eta))Op^B(P(V;\eta))D^\alpha_x W(t,\cdot)\big\|_{L^2},\quad \forall\, n\in\mathbb{N}, n\ge 3, \tag{2.2.10a}$$

$$\widetilde{E}^k_3(t;W):=\sum_{\substack{|\alpha|+|I|\le 3\\ 0\le|I|\le 3-k}}\big\|Op^B(I_4+E(U;\eta))Op^B(P(V;\eta))D^\alpha_x W^I(t,\cdot)\big\|_{L^2},\forall\, 0\le k\le 2, \tag{2.2.10b}$$

there exists a constant $C_1>0$ such that

$$\begin{aligned}C_1^{-1}E_n(t;W)\le\widetilde{E}_n(t;W)\le C_1E_n(t;W),\quad\forall\, n\ge 3,\\ C_1^{-1}E^k_3(t;W)\le\widetilde{E}^k_3(t;W)\le C_1E^k_3(t;W),\quad\forall\, 0\le k\le 2,\end{aligned} \tag{2.2.11}$$

and we can rather focus on the derivation of an energy inequality for $\widetilde{E}_n(t;W),\widetilde{E}^k_3(t;W)$.

For reasons of clarity, we split $C''(U;\eta)$ defined in (2.1.7) into the sum of the following two matrices:

$$C''_d(U;\eta)=\begin{bmatrix}0&0&0&0\\0&e_0&0&0\\0&0&0&0\\0&0&0&f_0\end{bmatrix},\quad C''_{nd}(U;\eta)=\begin{bmatrix}0&0&0&0\\0&0&0&f_0\\0&0&0&0\\0&e_0&0&0\end{bmatrix} \tag{2.2.12}$$

and proceed to eliminate $Op^B(C''_d(U;\eta))V^I_s$ and $Op^B(C''_{nd}(U;\eta))V^I_s$ in (2.2.1) separately. For compactness, we denote by $(*)$ the right hand side of equation (2.2.1). In order to get rid of $Op^B(C''_d(U;\eta))V^I_s$ (resp. $Op^B(C''_{nd}(U;\eta))V^I_s$) in (2.2.1), we seek for matrix $E_d(U;\eta)$ (resp. $E_{nd}(U;\eta)$, depending linearly on (u_+,u_-), such that $\widetilde{W}^I_s:=Op^B(I_4+E_d(U;\eta)+E_{nd}(U;\eta))W^I_s$ is solution to

(2.2.13)

$$\begin{aligned}\big(D_t-A(D)\big)\widetilde{W}^I_s&=(*)+Op^B(D_tE_d(U;\eta))W^I_s-[A(D),Op^B(E_d(U;\eta))]W^I_s+Op^B(E_d(U;\eta))(*)\\ &+Op^B(D_tE_{nd}(U;\eta))W^I_s-[A(D),Op^B(E_{nd}(U;\eta))]W^I_s+Op^B(E_{nd}(U;\eta))(*),\end{aligned}$$

and

$$\begin{aligned}Op^B(C''_d(U;\eta))V^I_s+Op^B(D_tE_d(U;\eta))W^I_s-[A(D),Op^B(E_d(U;\eta))]W^I_s&\\ =T_{-N}(U)W^I_s+\mathfrak{R}'(V,V),&\end{aligned} \tag{2.2.14}$$

$$Op^B(C''_{nd}(U;\eta))V^I_s + Op^B(D_tE_{nd}(U;\eta))W^I_s - [A(D), Op^B(E_{nd}(U;\eta))]W^I_s = T_{-N}(U)W^I_s + \mathfrak{R}'(V,V), \tag{2.2.15}$$

where $T_{-N}(U)$ is an operator of order less or equal than $-N$, for a certain $N > 0$, and $\mathfrak{R}'(V,V)$ is a new remainder satisfying a suitable L^2 estimate. The results we obtain are the following:

Lemma 2.2.3. *Let $N \in \mathbb{N}^*$. There exists a diagonal matrix $E_d(U;\eta)$ of order 0, linear in (u_+, u_-), such that*

$$Op^B(C''_{d}(U;\eta))V^I_s + Op^B(D_tE_{d}(U;\eta))W^I_s - [A(D), Op^B(E_{d}(U;\eta))]W^I_s = T_{-N}(U)W^I_s + \mathfrak{R}'(V,V), \tag{2.2.16}$$

where $\mathfrak{R}'(V,V)$ satisfies, for any $\theta \in]0,1[$,

$$\|\mathfrak{R}'(V,V)(t,\cdot)\|_{L^2} \lesssim \|V(t,\cdot)\|^{2-\theta}_{H^{5,\infty}}\|V(t,\cdot)\|^{\theta}_{H^7}\|V^I(t,\cdot)\|_{L^2}, \tag{2.2.17}$$

and $T_{-N}(U)$ is a pseudo-differential operator of order less or equal than $-N$ such that, for any $s \in \mathbb{R}$,

$$\|T_{-N}(U)\|_{\mathcal{L}(H^{s-N};H^s)} \lesssim \|\mathrm{R}_1U(t,\cdot)\|_{H^{N+2,\infty}} + \|U(t,\cdot)\|_{H^{N+6,\infty}}, \tag{2.2.18}$$

whose symbol $\sigma(U,\eta) = (\sigma_{ij}(U,\eta))_{1\le i,j\le 4}$ is such that

$$\mathcal{F}_{x\mapsto\xi}(\sigma_{ij}(U,\eta))(\xi) = \begin{cases} \sigma^+_{ii}(\xi,\eta)\hat{u}_+(\xi) + \sigma^-_{ii}(\xi,\eta)\hat{u}_-(\xi), & i = j \in \{2,4\}, \\ 0, & \textit{otherwise}, \end{cases} \tag{2.2.19a}$$

with $\sigma^\pm_{ii}(\xi,\eta)$ supported for $|\xi| \le \varepsilon\langle\eta\rangle$, for a small $\varepsilon > 0$, and verifying, for any $\alpha,\beta \in \mathbb{N}^2$,

$$|\partial^\alpha_\xi\partial^\beta_\eta\sigma^\pm_{ii}(\xi,\eta)| \lesssim_{\alpha,\beta} |\xi|^{N+1-|\alpha|}\langle\eta\rangle^{-N-|\beta|}, \quad \textit{for } i = 2,4. \tag{2.2.19b}$$

Moreover, if $\chi \in C^\infty_0(\mathbb{R}^2)$ is equal to 1 close to the origin and has a sufficiently small support,

$$E_d\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)U;\eta\right) = E^0_d\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)U;\eta\right) + E^{-1}_d\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)U;\eta\right), \tag{2.2.20}$$

the former matrix in the above right hand side being real, of order 0 and satisfying (2.2.2a)*, the latter being of order -1 and verifying* (2.2.2b).

Proof. Because of the diagonal structure of $C''_d(U;\eta)$, we look for a matrix $E_d = (e_{ij})_{1\le i,j\le 4}$ satisfying (2.2.16) such that $e_{ij} = 0$ for all i,j but $i = j \in \{2,4\}$, with symbols e_{22}, e_{44} of order 0 and linear in (u_+, u_-). If we remind that matrix $A(\eta)$, defined in (2.1.5), is of order 1, and make the ansatz that E_d is of order 0, then by symbolic calculus of proposition 1.2.9 we have that

$$-[A(D), Op^B(E_d(U;\eta))] = -\sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}Op^B\left(\partial^\alpha_\eta A(\eta)D^\alpha_x E_d(U;\eta)\right) + T_{-N}(U), \tag{2.2.21}$$

with $T_{-N}(U)$ pseudo-differential operator of order less or equal than $-N$ such that, for any $s \in \mathbb{R}$,

$$\|T_{-N}(U)\|_{\mathcal{L}(H^{s-N};H^s)} \lesssim M^1_{N+1}(A(\eta);N+3)M^0_0(E_d(U;\eta);2) + M^1_0(A(\eta);N+3)M^0_{N+1}(E_d(U;\eta);2), \tag{2.2.22}$$

and whose symbol $\sigma(U,\eta)=(\sigma_{ij}(U,\eta))_{ij}$ is such that $\sigma_{ij}(U,\eta)=0$ for all i,j but $i=j\in\{2,4\}$. Therefore, for any fixed $\chi\in C_0^\infty(\mathbb{R}^2)$ equal to 1 in $\overline{B_{\varepsilon_1}(0)}$ and supported in $B_{\varepsilon_2}(0)$, for some $0<\varepsilon_1<\varepsilon_2\ll 1$, we look for $E_d(U;\eta)$ such that

$$\chi\left(\frac{D_x}{\langle\eta\rangle}\right)\left[C''_d(U;\eta)+D_tE_d(U;\eta)-\sum_{|\alpha|=1}^N\frac{1}{\alpha!}\partial_\eta^\alpha A(\eta)D_x^\alpha E_d(U;\eta)\right]=0.$$

In fact, since E_d is required to be linear in (u_+,u_-), we should write $E_d(U;\eta)$ as $E_d(u_+,u_-;\eta)$ to then realize that, as u_+ (resp. u_-) is solution to the first (resp. to the third) equation in (2.1.2) with $|I|=0$,

$$\begin{aligned}D_tE_d(u_+,u_-;\eta)&=E_d(|D_x|u_+,-|D_x|u_-;\eta)+E_d\big(Q_0^{\mathrm{w}}(v_\pm,D_1v_\pm),Q_0^{\mathrm{w}}(v_\pm,D_1v_\pm);\eta\big),\\ D_x^\alpha E_d(u_+,u_-;\eta)&=E_d(D_x^\alpha u_+,D_x^\alpha u_-;\eta),\quad\forall\alpha\in\mathbb{N}^2,\end{aligned}$$

and rather look for $E_d(U;\eta)$ such that

$$\chi\left(\frac{D_x}{\langle\eta\rangle}\right)\left[C''_d(U;\eta)+E_d(|D_x|u_+,-|D_x|u_-;\eta)-\sum_{|\alpha|=1}^N\frac{1}{\alpha!}\partial_\eta^\alpha A(\eta)E_d(D_x^\alpha u_+,D_x^\alpha u_-;\eta)\right]=0,$$

temporarily neglecting contribution $E_d\big(Q_0^{\mathrm{w}}(v_\pm,D_1v_\pm),Q_0^{\mathrm{w}}(v_\pm,D_1v_\pm);\eta\big)$, whose quantization acting on W^I_s is going to give rise to a remainder $\mathfrak{R}'(V,V)$ satisfying (2.2.17). Recalling definition (2.1.5) of matrix $A(\eta)$, the above equation corresponds to the following:

$$\begin{cases}e_{22}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)\Big(|D_x|-\sum\limits_{|\alpha|=1}^N\frac{1}{\alpha!}\partial_\eta^\alpha(\langle\eta\rangle)D_x^\alpha\Big)u_+,-\chi\left(\frac{D_x}{\langle\eta\rangle}\right)\Big(|D_x|+\sum\limits_{|\alpha|=1}^N\frac{1}{\alpha!}\partial_\eta^\alpha(\langle\eta\rangle)D_x^\alpha\Big)u_-;\eta\right)\\ \qquad\qquad=-\chi\left(\frac{D_x}{\langle\eta\rangle}\right)e_0\\ e_{44}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)\Big(|D_x|+\sum\limits_{|\alpha|=1}^N\frac{1}{\alpha!}\partial_\eta^\alpha(\langle\eta\rangle)D_x^\alpha\Big)u_+,-\chi\left(\frac{D_x}{\langle\eta\rangle}\right)\Big(|D_x|-\sum\limits_{|\alpha|=1}^N\frac{1}{\alpha!}\partial_\eta^\alpha(\langle\eta\rangle)D_x^\alpha\Big)u_-;\eta\right)\\ \qquad\qquad=-\chi\left(\frac{D_x}{\langle\eta\rangle}\right)f_0,\end{cases}$$

with e_0,f_0 defined in (2.1.9), and if we look for e_{ii} of the form

$$e_{ii}(u_+,u_-;\eta)=\int e^{ix\cdot\xi}\alpha_{ii}(\xi,\eta)\hat{u}_+(\xi)d\xi+\int e^{ix\cdot\xi}\beta_{ii}(\xi,\eta)\hat{u}_-(\xi)d\xi,\tag{2.2.23}$$

above system imply, inter alia, that according to the definition (2.1.9) of e_0

$$\int e^{ix\cdot\xi}\chi\left(\frac{\xi}{\langle\eta\rangle}\right)\Big(|\xi|-\sum_{|\alpha|=1}^N\frac{1}{\alpha!}\partial_\eta^\alpha(\langle\eta\rangle)\xi^\alpha\Big)\alpha_{22}(\xi,\eta)\hat{u}_+(\xi)d\xi=$$
$$-\frac{i}{4}\int e^{ix\cdot\xi}\chi\left(\frac{\xi}{\langle\eta\rangle}\right)\Big(1-\frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}\Big)\xi_1\hat{u}_+(\xi)d\xi.$$

As

$$\left(1\mp\sum_{|\alpha|=1}^N\frac{1}{\alpha!}\partial_\eta^\alpha(\langle\eta\rangle)\frac{\xi^\alpha}{|\xi|}\right)=1\mp\sum_{k=1}^N\frac{1}{k!}(\xi\cdot\nabla_\eta)^k(\langle\eta\rangle),$$

and

$$(\partial_{\eta_1}\xi_1+\partial_{\eta_2}\xi_2)^k\langle\eta\rangle=\frac{|\xi|^k}{\langle\eta\rangle^{k-1}}\left(1-\left(\frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}\right)^2\right)b_k(\xi,\eta),$$

for every $2 \le k \le N$, where $b_k(\xi,\eta)$ is a polynomial of degree $k-2$ in $\frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}$, we derive that

$$(2.2.24)\qquad \left(1 \mp \sum_{|\alpha|=1}^{N} \frac{1}{\alpha!}\partial_\eta^\alpha(\langle\eta\rangle)\frac{\xi^\alpha}{|\xi|}\right) = \left(1 \mp \frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}\right)(1\mp b_\pm(\xi,\eta))$$

with $b_\pm(\xi,\eta) := \sum_{k=2}^N \frac{1}{k!}|\xi|^{k-1}\langle\eta\rangle^{-(k-1)}\big(1\pm\frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}\big)b_k(\xi,\eta)$ such that, for any $\mu,\nu\in\mathbb{N}^2$,

$$(2.2.25)\qquad |\partial_\xi^\mu\partial_\eta^\nu b_\pm(\xi,\eta)| \lesssim_{\mu,\nu} |\xi|^{1-|\mu|}\langle\eta\rangle^{-1-|\nu|},$$

and we can then choose $\alpha_{22}(\xi,\eta)$ in (2.2.23) such that, when $|\xi|\le\varepsilon_2\langle\eta\rangle$,

$$(2.2.26)\qquad \alpha_{22}(\xi,\eta) = -\frac{i}{4}\left(1-b_+(\xi,\eta)\right)^{-1}\frac{\xi_1}{|\xi|}.$$

Similarly, we choose multipliers $\beta_{22},\alpha_{44},\beta_{44}$ such that, as long as $|\xi|\le\varepsilon_2\langle\eta\rangle$,

$$\beta_{22}(\xi,\eta) = \frac{i}{4}\left(1+b_-(\xi,\eta)\right)^{-1}\frac{\xi_1}{|\xi|},$$
$$\alpha_{44}(\xi,\eta) = -\frac{i}{4}\left(1+b_-(\xi,\eta)\right)^{-1}\frac{\xi_1}{|\xi|},\quad \beta_{44}(\xi,\eta) = \frac{i}{4}\left(1-b_+(\xi,\eta)\right)^{-1}\frac{\xi_1}{|\xi|}.$$

We also observe that, since $b_\pm(\xi,\eta) = O(|\xi|\langle\eta\rangle^{-1})$, we have that $(1\pm b_j(\xi,\eta))^{-1} = 1\mp b_j(\xi,\eta) + O(|\xi|^2\langle\eta\rangle^{-2})$, $j\in\{+,-\}$, as long as $|\xi|\le\varepsilon_2\langle\eta\rangle$, and hence

$$\alpha_{22}(\xi,\eta) = -\frac{i}{4}\frac{\xi_1}{|\xi|} + \alpha_{22}^{-1}(\xi,\eta),\quad \beta_{22}(\xi,\eta) = \frac{i}{4}\frac{\xi_1}{|\xi|} + \beta_{22}^{-1}(\xi,\eta),$$
$$\alpha_{44}(\xi,\eta) = -\frac{i}{4}\frac{\xi_1}{|\xi|} + \alpha_{44}^{-1}(\xi,\eta),\quad \beta_{44}(\xi,\eta) = \frac{i}{4}\frac{\xi_1}{|\xi|} + \beta_{44}^{-1}(\xi,\eta),$$

where, for any $\mu,\nu\in\mathbb{N}^2$, $|\partial_\xi^\mu\partial_\eta^\nu\alpha_{ii}^{-1}| + |\partial_\xi^\mu\partial_\eta^\nu\beta_{ii}^{-1}| \lesssim_{\mu,\nu} |\xi|^{1-|\mu|}\langle\eta\rangle^{-1-|\nu|}$. Injecting the above α_{ii},β_{ii}, $i\in\{2,4\}$, in (2.2.23) we find that

$$e_{22}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+,\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right) = -\frac{i}{4}R_1(u_+-u_-) + e_{22}^{-1}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+,\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right),$$
$$e_{44}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+,\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right) = -\frac{i}{4}R_1(u_+-u_-) + e_{44}^{-1}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+,\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right),$$

where, for $i\in\{2,4\}$,

$$e_{ii}^{-1}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+,\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right) = \int e^{ix\cdot\xi}\chi\left(\frac{\xi}{\langle\eta\rangle}\right)\alpha_{ii}^{-1}(\xi,\eta)\hat{u}_+(\xi)d\xi + \int e^{ix\cdot\xi}\chi\left(\frac{\xi}{\langle\eta\rangle}\right)\beta_{ii}^{-1}(\xi,\eta)\hat{u}_-(\xi)d\xi.$$

After lemma A.1 (i) and above estimates for $\alpha_{ii}^{-1},\beta_{ii}^{-1}$, kernels

$$K_+^i(x,\eta) := \int e^{ix\cdot\xi}\chi\left(\frac{\xi}{\langle\eta\rangle}\right)\alpha_{ii}^{-1}(\xi,\eta)\langle\xi\rangle^{-4}d\xi,\quad K_-^i(x,\eta) := \int e^{ix\cdot\xi}\chi\left(\frac{\xi}{\langle\eta\rangle}\right)\beta_{ii}^{-1}(\xi,\eta)\langle\xi\rangle^{-4}d\xi$$

are such that, for any $\beta\in\mathbb{N}^2$, $|\partial_\eta^\beta K_\pm^i(x,\eta)| \lesssim |x|^{-1}\langle x\rangle^{-2}\langle\eta\rangle^{-1-|\beta|}$, for every (x,η), from which follows that

$$\left|\partial_\eta^\beta e_{ii}^{-1}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+,\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right)\right| \le \left|\int \partial_\eta^\beta K_+^i(x-y,\eta)[\langle D_x\rangle^4 u_+](y)dy\right| + \left|\int \partial_\eta^\beta K_-^i(x-y,\eta)[\langle D_x\rangle^4 u_-](y)dy\right| \lesssim \|U(t,\cdot)\|_{H^{4,\infty}}\langle\eta\rangle^{-1-|\beta|}$$

and e_{ii}^{-1} is a symbol of order -1, for $i = 2, 4$. Moreover, using definition (1.2.3) and the fact that space $W^{r,\infty}$ injects in $H^{r+1,\infty}$, one can check that for any $r, n \in \mathbb{N}$,

$$M_r^{-1}\left(e_{ii}^{-1}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+, \chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right);n\right) \lesssim \|U(t,\cdot)\|_{H^{5+r,\infty}},$$

and therefore that

$$M_r^{0}\left(e_{ii}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+, \chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right);n\right) \lesssim \|\mathrm{R}_1U(t,\cdot)\|_{H^{1+r,\infty}} + \|U(t,\cdot)\|_{H^{5+r,\infty}}.$$

Defining

$$E^0_d(U;\eta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{i}{4}R_1(u_+ - u_-) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{4}R_1(u_+ - u_-) \end{bmatrix}, \quad E^{-1}_d(U;\eta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & e_{22}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_{44}^{-1}, \end{bmatrix}$$

decomposition (2.2.20) and estimate (2.2.2a), (2.2.2b) hold. Consequently, as

$$E_d\big(Q^{\mathrm{w}}_0(v_\pm, D_1v_\pm), Q^{\mathrm{w}}_0(v_\pm, D_1v_\pm);\eta\big) = E^{-1}_d\big(Q^{\mathrm{w}}_0(v_\pm, D_1v_\pm), Q^{\mathrm{w}}_0(v_\pm, D_1v_\pm);\eta\big), \tag{2.2.27}$$

for any $n \in \mathbb{N}$, any $\theta \in]0,1[$, we have that

$$\begin{aligned} M^0_0\left(E_d\big(Q^{\mathrm{w}}_0(v_\pm, D_1v_\pm), Q^{\mathrm{w}}_0(v_\pm, D_1v_\pm);\eta\big);n\right) &\lesssim \|Q^{\mathrm{w}}_0(v_\pm, D_1v_\pm)\|_{H^{4,\infty}} \\ &\lesssim \|V(t,\cdot)\|^{2-\theta}_{H^{5,\infty}}\|V(t,\cdot)\|^{\theta}_{H^7}, \end{aligned}$$

last inequality obtained using (B.1.3d) with $s = 4$, and its quantization acting on V^I_s verifies (2.2.17) after proposition 1.2.7. Also, (2.2.18) is deduced from (2.2.22) while properties (2.2.19) are obtained essentially using (1.2.12). □

Lemma 2.2.4. *Let $N \in \mathbb{N}^*$. There exists a purely imaginary matrix $E_{nd}(U;\eta)$, linear in (u_+, u_-), of order -1, satisfying estimate (2.2.2c), such that*

$$\begin{aligned} Op^B(C''_{nd}(U;\eta))V^I_s + Op^B(D_tE_{nd}(U;\eta))W^I_s - [A(D), Op^B(E_{nd}(U;\eta))]W^I_s \\ = T_{-N}(U)W^I_s + \mathfrak{R}'(V,V), \end{aligned} \tag{2.2.28}$$

where $\mathfrak{R}'(V,V)$ is a remainder satisfying (2.2.17), and $T_{-N}(U)$ is a pseudo-differential operator of order less or equal than $-N$ such that, for any $s \in \mathbb{R}$,

$$\|T_{-N}(U)\|_{\mathcal{L}(H^{s-N};H^s)} \lesssim \|U(t,\cdot)\|_{H^{N+6,\infty}}. \tag{2.2.29}$$

Moreover, its symbol $\sigma(U,\eta) = (\sigma_{ij}(U,\eta))_{1\le i,j\le 4}$ is such that

$$\mathcal{F}_{x\mapsto\xi}(\sigma_{ij}(U,\eta))(\xi) = \begin{cases} \sigma^+_{ij}(\xi,\eta)\hat{u}_+(\xi) + \sigma^-_{ij}(\xi,\eta)\hat{u}_-(\xi), & (i,j) \in \{(2,4),(4,2)\}, \\ 0, & \textit{otherwise}, \end{cases} \tag{2.2.30a}$$

with $\sigma^\pm_{ij}$ supported for $|\xi| \le \varepsilon\langle\eta\rangle$, for a small $\varepsilon > 0$, and verifying, for any $\alpha, \beta \in \mathbb{N}^2$,

$$|\partial^\alpha_\xi\partial^\beta_\eta\sigma^\pm_{ij}(\xi,\eta)| \lesssim_{\alpha,\beta} |\xi|^{N+2-|\alpha|}\langle\eta\rangle^{-N-1-|\beta|}, \tag{2.2.30b}$$

for $(i,j) \in \{(2,4),(4,2)\}$.

Proof. Because of the structure of $C''_{nd}(U;\eta)$, we seek for a matrix $E_{nd}(U;\eta)$ satisfying (2.2.28), of the form $E_{nd}(U;\eta) = (e_{ij})_{1\le i,j\le 4}$ with $e_{ij} = 0$ for all i,j, except $(i,j) \in \{(2,4),(4,2)\}$. If we make the ansatz that $E_{nd}(U;\eta)$ is linear in (u_+,u_-), of order -1, and remind that $A(\eta)$ in (2.1.5) is of order 1, from symbolic calculus of proposition 1.2.9 we have that

$$\begin{aligned}-[A(D), Op^B(E_{nd}(U;\eta))] = &- Op^B(A(\eta)E_{nd}(U;\eta) - E_{nd}(U;\eta)A(\eta)) \\ &- \sum_{|\alpha|=1}^{N} \frac{1}{\alpha!} Op^B(\partial_\eta^\alpha A(\eta)\cdot D_x^\alpha E_{nd}(U;\eta)) + T_{-N}(U),\end{aligned}$$

where $T_{-N}(U) = \sigma(U,D_x)$ is a pseudo-differential operator of order less or equal than $-N$, such that, for any $s\in\mathbb{R}$,

(2.2.31)
$$\|T_{-N}(U)\|_{\mathcal{L}(H^{s-N};H^s)} \lesssim M^1_{N+1}(A(\eta);N+3)M^{-1}_0(E_{nd}(U;\eta);2) + M^1_0(A(\eta);N+3)M^{-1}_{N+1}(E_{nd}(U;\eta);2),$$

and whose symbol $\sigma(U,\eta) = (\sigma_{ij}(U,\eta))_{ij}$ is such that $\sigma_{ij}=0$ for all i,j but $(i,j)\in\{(2,4),(4,2)\}$. Hence, for any fixed $\chi\in\mathbb{R}^2$ equal to 1 in $\overline{B_{\varepsilon_1}(0)}$ and supported in $B_{\varepsilon_2}(0)$, for some $0<\varepsilon_1<\varepsilon_2\ll 1$, we look for $E_{nd}(U;\eta)$ such that

$$\begin{aligned}\chi\left(\frac{D_x}{\langle\eta\rangle}\right)\Big[C''_{nd}(U;\eta) + D_tE_{nd}(U;\eta) - A(\eta)E_{nd}(U;\eta) + E_{nd}(U;\eta)A(\eta) \\ - \sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial_\eta^\alpha A(\eta)\cdot D_x^\alpha E_{nd}(U;\eta)\Big] = 0.\end{aligned} \tag{2.2.32}$$

Furthermore, as $E_{nd}(U;\eta) = E_{nd}(u_+,u_-;\eta)$ is linear in (u_+,u_-), and u_+ (resp. u_-) is solution to the first (resp. the third) equation in (2.1.2) with $|I|=0$, we have that

$$\begin{aligned}D_tE_{nd}(u_+,u_-;\eta) &= E_{nd}(|D_x|u_+, -|D_x|u_-;\eta) + E_{nd}\big(Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm), Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm);\eta\big),\\ D_x^\alpha E_{nd}(u_+,u_-;\eta) &= E_{nd}(D_x^\alpha u_+, D_x^\alpha u_-;\eta), \quad \forall \alpha\in\mathbb{N}^2\end{aligned}$$

while

$$-A(\eta)E_{nd}(U;\eta) + E_{nd}(U;\eta)A(\eta) = \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -2\langle\eta\rangle e_{24}\\ 0 & 0 & 0 & 0\\ 0 & 2\langle\eta\rangle e_{42} & 0 & 0\end{bmatrix},$$

so we rather search for symbols e_{24}, e_{42} such that

$$\begin{cases} \chi\left(\frac{D_x}{\langle\eta\rangle}\right) e_{2,4}\left(\Big(|D_x| - \sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial^\alpha(\langle\eta\rangle)D_x^\alpha - 2\langle\eta\rangle\Big)u_+, -\Big(|D_x| + \sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial^\alpha(\langle\eta\rangle)D_x^\alpha + 2\langle\eta\rangle\Big)u_-;\eta\right) \\ \qquad\qquad = -\chi\left(\frac{D_x}{\langle\eta\rangle}\right) f_0, \\ \chi\left(\frac{D_x}{\langle\eta\rangle}\right) e_{4,2}\left(\Big(|D_x| + \sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial^\alpha(\langle\eta\rangle)D_x^\alpha + 2\langle\eta\rangle\Big)u_+, -\Big(|D_x| - \sum_{|\alpha|=1}^{N}\frac{1}{\alpha!}\partial^\alpha(\langle\eta\rangle)D_x^\alpha - 2\langle\eta\rangle\Big)u_-;\eta\right) \\ \qquad\qquad = -\chi\left(\frac{D_x}{\langle\eta\rangle}\right) e_0, \end{cases}$$

with e_0, f_0 given by (2.1.9), neglecting contribution $E_{nd}\big(Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm), Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm);\eta\big)$, whose quantization acting on W^I_s gives rise to a remainder $\mathfrak{R}'(V,V)$, as we will see at the end of the proof. We look for e_{ij} of the form

$$e_{ij}(u_+,u_-;\eta) = \int e^{ix\cdot\xi}\alpha_{ij}(\xi,\eta)\hat{u}_+(\xi)d\xi + \int e^{ix\cdot\xi}\beta_{ij}(\xi,\eta)\hat{u}_-(\xi)d\xi,$$

for $(i,j)\in\{(2,4),(4,2)\}$, thus reminding (2.2.24), (2.2.25), we choose the above multipliers such that, as long as $|\xi|\le\varepsilon_2\langle\eta\rangle$,

$$\alpha_{24}(\xi,\eta)=-\frac{i}{4}\left(1+\frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}\right)\left(\left(1-\frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}\right)(1-b_+(\xi,\eta))-2\frac{\langle\eta\rangle}{|\xi|}\right)^{-1}\frac{\xi_1}{|\xi|},$$
$$\beta_{24}(\xi,\eta)=-\frac{i}{4}\left(1-\frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}\right)\left(\left(1+\frac{\eta}{\langle\eta\rangle}\cdot\frac{\xi}{|\xi|}\right)(1+b_-(\xi,\eta))+2\frac{\langle\eta\rangle}{|\xi|}\right)^{-1}\frac{\xi_1}{|\xi|},$$
$$\alpha_{42}(\xi,\eta)=\beta_{24},\quad \beta_{42}(\xi,\eta)=\alpha_{24}(\xi,\eta).$$

One can check that, on the support of $\chi(\frac{\xi}{\langle\eta\rangle})$ and for any $\mu,\nu\in\mathbb{N}^2$, $|\partial^\mu_\xi\partial^\nu_\eta\alpha_{ij}|+|\partial^\mu_\xi\partial^\nu_\eta\beta_{ij}|\lesssim_{\mu,\nu}|\xi|^{1-|\mu|}\langle\eta\rangle^{-1-|\nu|}$, and then that, if

$$K^{ij}_+(x,\eta):=\int e^{ix\cdot\eta}\chi\Big(\frac{\xi}{\langle\eta\rangle}\Big)\alpha_{ij}(\xi,\eta)\langle\xi\rangle^{-4}d\xi,\quad K^{ij}_-(x,\eta):=\int e^{ix\cdot\eta}\chi\Big(\frac{\xi}{\langle\eta\rangle}\Big)\beta_{ij}(\xi,\eta)\langle\xi\rangle^{-4}d\xi,$$

for $(i,j)\in\{(2,4),(4,2)\}$, $|\partial^\beta_\eta K^{ij}_\pm(x,\eta)|\lesssim|x|^{-1}\langle x\rangle^{-2}\langle\eta\rangle^{-1-|\beta|}$, for any $\beta\in\mathbb{N}^2$, any $(x,\eta)\in\mathbb{R}^2\times\mathbb{R}^2$, as a consequence of lemma A.1. Therefore, for any $\beta\in\mathbb{N}^2$,

$$\left|\partial^\beta_\eta e_{ij}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+,\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right)\right|\le$$
$$\left|\int\partial^\beta_\eta K^{ij}_+(x-y,\eta)[\langle D_x\rangle^4u_+](y)dy\right|+\left|\int\partial^\beta_\eta K^{ij}_-(x-y,\eta)[\langle D_x\rangle^4u_-](y)dy\right|\lesssim\|U(t,\cdot)\|_{H^{4,\infty}}\langle\eta\rangle^{-1-|\beta|},$$

which implies that e_{24},e_{42} are symbols of order -1. Moreover, for $(i,j)\in\{(2,4),(4,2)\}$ and any $n,r\in\mathbb{N}$, one can prove that

$$M^{-1}_r\left(e_{ij}\left(\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_+,\chi\left(\frac{D_x}{\langle\eta\rangle}\right)u_-;\eta\right);n\right)\lesssim\|U(t,\cdot)\|_{H^{5+r,\infty}},$$

using definition (1.2.3) and the fact that space $W^{r,\infty}$ injects in H^{r+1} for any $r\in\mathbb{N}$. Therefore, $E_{nd}(\chi(\frac{D_x}{\langle\eta\rangle})U;\eta)$ is a matrix of order -1 satisfying (2.2.2c). Moreover, for any $s\in\mathbb{R}$ $\|T_{-N}(U)\|_{\mathcal{L}(H^{s-N};H^s)}\lesssim\|U(t,\cdot)\|_{H^{N+6,\infty}}$ after (2.2.31), and its symbol satisfies (2.2.30), as one can check using (1.2.12) and the estimates derived above for α_{ij},β_{ij}, while from (B.1.3d) with $s=4$, for any $\theta\in]0,1[$,

$$M^{-1}_0\big(E_{nd}\big(Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm),Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm);\eta\big);n\big)\lesssim\|V(t,\cdot)\|^{2-\theta}_{H^{5,\infty}}\|V(t,\cdot)\|^\theta_{H^7},$$

and its quantization acting on W^I_s is a remainder verifying (2.2.17) by proposition 1.2.7. □

Proof of Proposition 2.2.1. After lemmas 2.2.3, 2.2.4, there exist two matrices $E_d(U;\eta)$ and $E_{nd}(U;\eta)$, linear in (u_+,u_-), satisfying, respectively, equations (2.2.16) and (2.2.28), and with $E_d\big(\chi(\frac{D_x}{\langle\eta\rangle})U;\eta\big)$ decomposed as in (2.2.20), for any $\chi\in C^\infty_0(\mathbb{R}^2)$ equal to 1 close to the origin and with sufficiently small support.

If we define $\widetilde{W}^I_s:=Op^B(I_4+E(U;\eta))W^I_s$, with W^I_s solution to (2.2.1) and

$$E(U;\eta):=E_d(U;\eta)+E_{nd}(U;\eta),$$

we deduce from (2.2.13), (2.2.16), (2.2.28) that

$$\begin{aligned}(D_t-A(D))\widetilde{W}^I_s&=Op^B(\widetilde{A}_1(V;\eta))W^I_s+Op^B(A''(V^I;\eta))U+Op^B_R(A''(V^I;\eta))U\\&+Q^I_0(V,W)+\mathfrak{R}(U,V)+Op^B(E(U;\eta))\Big[Op^B(\widetilde{A}_1(V;\eta))W^I_s+Op^B(A''(V^I;\eta))U\\&+Op^B(C''(U;\eta))V^I_s+Op^B_R(A''(V^I;\eta))U+Q^I_0(V,W)+\mathfrak{R}(U,V)\Big]+T_{-N}(U)W^I_s+\mathfrak{R}'(V,V),\end{aligned}$$

where $\mathfrak{R}(U,V)$ satisfies (2.1.47), $\mathfrak{R}'(V,V)$ satisfies (2.2.17), and $T_{-N}(U)$ is a pseudo-differential operator of order less or equal than $-N$ verifying (2.2.6), (2.2.7). Contribution

$$Op^B(E(U;\eta))\Big[Op^B(A''(V^I;\eta))U + Op^B(C''(U;\eta))V^I_s + Op^B_R(A''(V^I;\eta))U \\ + Q^I_0(V,W) + \mathfrak{R}(U,V)\Big]$$

is a remainder of the form $\mathfrak{R}'(U,V)$ satisfying estimate (2.2.8), as a consequence of proposition 1.2.7, estimates (2.2.2) with $r=0$, lemma 2.1.1, and the fact that $\|V^I_s(t,\cdot)\|_{L^2} \lesssim (1+\|V(t,\cdot)\|_{H^{1,\infty}})\|V^I(t,\cdot)\|_{L^2}$, by the definition of V^I_s.

According to the definition of $E(U;\eta)$,

$$\begin{aligned} Op^B(E(U;\eta))Op^B(\widetilde{A}_1(V;\eta)) &= Op^B(E^0_d(U;\eta))Op^B(\widetilde{A}_1(V;\eta)) \\ &\quad + Op^B\big(E^{-1}_d(U;\eta) + E_{nd}(U;\eta)\big)Op^B(\widetilde{A}_1(V;\eta)), \end{aligned}$$

where, by proposition 1.2.7 and estimates (2.1.49), (2.2.2b), (2.2.2c) with $r=0$, the latter addend in above right hand side is a bounded operator on L^2, whose norm is estimated by $\|U(t,\cdot)\|_{H^{5,\infty}}\|V(t,\cdot)\|_{H^{1,\infty}}$, while the former one writes as $Op^B(E^0_d(U;\eta)\widetilde{A}_1(V;\eta)) + T_0(U,V)$, for an operator $T_0(U,V)$ of order less or equal than 0, and norm $O(\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}\|V(t,\cdot)\|_{H^{2,\infty}})$, as follows from corollary 1.2.11 and estimates (2.1.49), (2.2.2a) with $r=1$. Hence,

$$Op^B(E(U;\eta))Op^B(\widetilde{A}_1(V;\eta))W^I_s = Op^B(E^0_d(U;\eta)\widetilde{A}_1(V;\eta)) + \mathfrak{R}'(U,V),$$

for a new $\mathfrak{R}'(U,V)$ satisfying (2.2.8).

As long as $\|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}}$ is sufficiently small, matrix $I_4 + E^0_d\big(\chi(\frac{D_x}{\langle\eta\rangle})U;\eta\big)$ is invertible and $F^0_d(U;\eta) := \big[I_4 + E^0_d\big(\chi(\frac{D_x}{\langle\eta\rangle})U;\eta\big)\big]^{-1} - I_4$ is such that, for any $n,r\in\mathbb{N}$,

$$M^0_r\left(F^0_d\Big(\chi\Big(\frac{D_x}{\langle\eta\rangle}\Big)U;\eta\Big);n\right) \lesssim \|\mathrm{R}_1U(t,\cdot)\|_{H^{1+r,\infty}}.$$

Moreover, matrix $F^0_d(U;\eta)$ is real, diagonal, of order 0, and by corollary 1.2.11 with $r=1$,

$$Op^B(I_4 + F^0_d(U;\eta))Op^B(I_4 + E^0_d(U;\eta)) = Id + T_{-1}(U),$$

with $T_{-1}(U)$ of order less or equal than 0, with $\mathcal{L}(H^{s-1};H^s)$ norm bounded by $\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}$, for any $s\in\mathbb{R}$. Since $\widetilde{W^I_s} = Op^B(I_4 + E(U;\eta))W^I_s$, this implies that

$$Op^B(I_4 + F^0_d(U;\eta))\widetilde{W^I_s} = W^I_s + \widetilde{T}_{-1}(U)W^I_s,$$

with $\widetilde{T}_{-1}(U) = T_{-1}(U) + Op^B(E^{-1}_d(U;\eta) + E_{nd}(U;\eta))$ of order less or equal than -1, and $\mathcal{L}(H^{s-1};H^s)$ norm bounded by $\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}} + \|U(t,\cdot)\|_{H^{5,\infty}}$, for any $s\in\mathbb{R}$, and as long as this quantity is small, there exists a positive constant C such that (2.2.4) holds. Also,

$$\begin{aligned} &Op^B(I_4 + E^0_d(U;\eta))Op^B(\widetilde{A}_1(V;\eta))W^I_s \\ &\qquad = Op^B(I_4 + E^0_d(U;\eta))Op^B(\widetilde{A}_1(V;\eta))Op^B(I_4 + F^0_d(U;\eta))\widetilde{W^I_s} \\ &\qquad - Op^B(I_4 + E^0_d(U;\eta))Op^B(\widetilde{A}_1(V;\eta))\widetilde{T}_{-1}(U)W^I_s, \end{aligned}$$

where, from proposition 1.2.7, (2.1.49), (2.2.2a), and the estimate on the norm of $\widetilde{T}_{-1}(U)$, the L^2 norm of the latter term in the above right hand side is estimated by

$$(1 + \|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}})\|V(t,\cdot)\|_{H^{1,\infty}}\,(\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}} + \|U(t,\cdot)\|_{H^{5,\infty}})\,\|W^I_s(t,\cdot)\|_{L^2}, \tag{2.2.33}$$

and it is a remainder $\mathfrak{R}'(U,V)$, reminding that $\|W^I_s(t,\cdot)\|_{L^2}\lesssim (1+\|V(t,\cdot)\|_{H^{1,\infty}})\|W^I(t,\cdot)\|_{L^2}$ after the definition of W^I_s and estimate (2.1.48). On the other hand, by corollary 1.2.11 with $r=1$, we get that

$$\begin{split}
&Op^B(I_4+E^0_d(U;\eta))Op^B(\widetilde{A}_1(V;\eta))Op^B(I_4+F^0_d(U;\eta))\widetilde{W}^I_s\\
&\qquad = Op^B\big((I_4+E^0_d(U;\eta))\widetilde{A}_1(V;\eta)(I_4+F^0_d(U;\eta))\big)\widetilde{W}^I_s\\
&\qquad + Op^B(I_4+E^0_d(U;\eta))T_0(U,V)\widetilde{W}^I_s+\widetilde{T}_0(U,V)\widetilde{W}^I_s,
\end{split}$$

with $T_0(U,V),\widetilde{T}_0(U,V)$ operators of order less or equal than 0, and $\mathcal{L}(L^2)$ norm controlled, respectively, by $\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}\|V(t,\cdot)\|_{H^{2,\infty}}$ and (2.2.33), so the last two terms in above right hand side are also remainders $\mathfrak{R}'(U,V)$ after proposition 1.2.7 and estimate (2.2.2a). That concludes the proof of the statement. □

2.2.2 A second normal forms transformation.

In previous subsection (see proposition 2.2.1) we introduced, for every multi-index I, a new function $\widetilde{W}^I_s$ defined in terms of W^I_s (and hence in terms of W^I), solution to equation (2.2.5), in which we got rid of very slowly-decaying-in-time term $Op^w_h\big(C''(U;\eta)\big)V^I_s$ appearing in (2.1.46). That naturally led to the introduction of new energies $\widetilde{E}_n(t;W),\widetilde{E}^k_3(t;W)$ as in (2.2.10a), for integers $k,n\in\mathbb{N}$ with $n\ge 3, 0\le k\le 2$, respectively equivalent to starting $E_n(t;W),E^k_3(t;W)$ whenever some uniform norms of U,V are sufficiently small. These modified energies, however, are not the good ones we were looking for, because they do not permit to obtain the wished energy inequality with which we can propagate a-priori estimates (1.1.11c), (1.1.11d), as explained below.

For multi-indices $I\in\mathcal{I}^k_3$, for $0\le k\le 2$, this can be seen in the fact that, when computing $\partial_t\widetilde{E}^k_3(t;W)=\sum_{I\in\mathcal{I}^k_3}\langle\partial_t\widetilde{W}^I_s,\widetilde{W}^I_s\rangle$, for $0\le k\le 2$, with $\mathcal{I}^k_3$ introduced in (2.1.17), we find from equation (2.2.5) the following contribution

$$-\sum_{I\in\mathcal{I}^k_3}\Im\Big[\langle Op^B(A''(V^I;\eta))U+Op^B_R(A''(V^I;\eta))U,W^I\rangle+\langle T_{-N}(U)W^I,W^I\rangle\Big], \tag{2.2.34}$$

for which we only have, after Cauchy-Schwarz inequality, lemma 2.1.1 and a-priori estimates (1.1.11), that

$$\big|\langle[Op^B(A''(V^I;\eta))+Op^B_R(A''(V^I;\eta))]U,W^I\rangle\big|+\big|\langle T_{-N}(U)W^I,W^I\rangle\big|\lesssim\varepsilon t^{-\frac{1}{2}}E^k_3(t;W),$$

with a decay rate $t^{-1/2}$ very far away from integrability.

Moreover, from (2.2.5) we also find $-\Im[\langle Q^I_0(V,W),W^I\rangle]$ which, from Cauchy-Schwarz inequality and estimate (2.1.41), is such that

$$\big|\langle Q^I_0(V,W),W^I\rangle\big|\lesssim\varepsilon t^{-\frac{1}{2}+\frac{\delta_k}{2}}E^k_3(t;W)^{\frac{1}{2}}.$$

To be more precise, the slow decay in time of this scalar product is due to some particular quadratic term appearing in $Q^I_0(V,W)$. In fact, according to definition (2.1.12), and to (2.1.30a), (2.1.31) and (2.1.35a), we have that

$$\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\|I_2|<|I|}}\Big|\langle Q^{\mathrm{w}}_0(v^{I_1}_\pm,Dv^{I_2}_\pm),u^I_++u^I_-\rangle\Big|\lesssim\|\mathfrak{R}^k_3(t,\cdot)\|_{L^2}\|U^I(t,\cdot)\|_{L^2}\le C(A+B)\varepsilon t^{-1+\frac{\delta_k}{2}}E^k_3(t;W)^{\frac{1}{2}}, \tag{2.2.35a}$$

where last estimate is obtained from a-priori estimates (1.1.11d) (resp. (1.1.11c) if $k=2$, as $E^3_3(t;W)$ stands for $E_3(t;W)$). Also, reminding definitions (2.1.26), (2.1.27), for indices $I\notin \mathcal{V}^k$ we also have, after (2.1.30b) and (2.1.35b), that

(2.2.35b)
$$\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|,|I_2|<|I|}}\left|\left\langle Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm), v^I_+ + v^I_-\right\rangle\right| \lesssim \|\mathfrak{R}^k_3(t,\cdot)\|_{L^2}\|V^I(t,\cdot)\|_{L^2}\le C(A+B)\varepsilon t^{-1+\frac{\delta_k}{2}}E^k_3(t;W)^{\frac{1}{2}}.$$

The decay rate $O(t^{-1+\delta_k/2})$ in the right hand side of the two above inequalities, is the slowest one that allows us to propagate a-priori estimate (1.1.11d), and it gives us back exactly the slow growth in time $t^{\delta_k/2}$ enjoyed by $E^k_3(t;W)^{1/2}$, for $0\le k\le 2$.

However, for $I\in\mathcal{V}^k$ with $k=0,1$, we find from (2.1.30b) and (2.1.35b) that, for some smooth cut-off function χ and some $\sigma>0$ small,

$$\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|,|I_2|<|I|}} c_{I_1,I_2}Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm) = \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ I_1\in\mathcal{K},|I_2|\le 1}} c_{I_1,I_2}Q^{\mathrm{kg}}_0\left(v^{I_1}_\pm, \chi(t^{-\sigma}D_x)Du^{I_2}_\pm\right)+\mathfrak{R}^k_3(t,x),$$

where the derivative D in the right hand side is to be meant equal to D_j, with $j=1,2,3$ (remind that we introduced D_3, with an abuse of notation, in (2.1.25)). The L^2 norm of the first sum in the right hand side is bounded by

$$\sum_{|I_2|\le 1}\left(\|\chi(t^{-\sigma}D_x)u^{I_2}_\pm(t,\cdot)\|_{H^{2,\infty}}+\|\chi(t^{-\sigma}D_x)\mathrm{R}u^{I_2}_\pm(t,\cdot)\|_{H^{2,\infty}}\right)E^k_3(t;W)^{\frac{1}{2}},$$

and decays in time with a rate slower than $t^{-1+\delta_k/2}$, because of the very slow decay in time of the uniform norm of $\chi(t^{-\sigma}D_x)U^{I_2}, \chi(t^{-\sigma}D_x)\mathrm{R}U^{I_2}$ (see (B.2.52)). Therefore, the very contribution that has to be eliminated from $\partial_t\widetilde{E}^k_3(t;W)$, when $k=0,1$, is

$$-\sum_{I\in\mathcal{V}_k}\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ I_1\in\mathcal{K},|I_2|\le 1}} c_{I_1,I_2}\Im\left[\left\langle Q^{\mathrm{kg}}_0\left(v^{I_1}_\pm,\chi(t^{-\sigma}D_x)Du^{I_2}_\pm\right), v^I_+ + v^I_-\right\rangle\right]. \tag{2.2.36}$$

When $I\in\mathcal{I}_n$ (see definition (2.1.18)), the same contributions as in (2.2.34) appear when computing $\partial_t\widetilde{E}_n(t;W)$, for any integer $n\ge 3$, and they come along with another slow decaying term, represented by

$$-\sum_{I\in\mathcal{I}_n}\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ [\frac{|I|}{2}]<|I_1|<|I|}}\Im\left[\langle Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D_1u^{I_2}_\pm), v^I_+ + v^I_-\rangle\right], \tag{2.2.37}$$

which is estimated by $t^{-1/2}E_n(t;W)$ after Cauchy-Schwarz inequality, (2.1.29) and a-priori estimate (1.1.11a).

The aim of current subsection is, therefore, to introduce two new modified energies $\widetilde{E}^\dagger_n(t;W)$, $\widetilde{E}^{k,\dagger}_3(t;W)$, for any integer k,n, $n\ge 3$, $0\le k\le 2$, in such a way that they are equivalent, respectively, to $\widetilde{E}_n(t;W), \widetilde{E}^k_3(t;W)$ (and, then, to starting generalized energies $E_n(t;W), E^k_3(t;W)$), and such that their time derivative is suitably decaying in time. For this purpose, it is useful to remind that, after system (2.1.2), for any multi-index I vector $(\hat{u}^I_+,\hat{v}^I_+,\hat{u}^I_-,\hat{v}^I_-)$ is solution to

(2.2.38)
$$\begin{cases}(D_t-|\xi|)\hat{u}^I_+(t,\xi)=\sum_{|I_1|+|I_2|=|I|}\widehat{Q^{\mathrm{w}}_0(v^{I_1}_\pm, D_1v^{I_2}_\pm)}+\sum_{|I_1|+|I_2|<|I|}c_{I_1,I_2}\widehat{Q^{\mathrm{w}}_0(v^{I_1}_\pm, Dv^{I_2}_\pm)}\\ (D_t-\langle\xi\rangle)\hat{v}^I_+(t,\xi)=\sum_{|I_1|+|I_2|=|I|}\widehat{Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D_1u^{I_2}_\pm)}+\sum_{|I_1|+|I_2|<|I|}c_{I_1,I_2}\widehat{Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm)}\\ (D_t+|\xi|)\hat{u}^I_-(t,x)=\sum_{|I_1|+|I_2|=|I|}\widehat{Q^{\mathrm{w}}_0(v^{I_1}_\pm, D_1v^{I_2}_\pm)}+\sum_{|I_1|+|I_2|<|I|}c_{I_1,I_2}\widehat{Q^{\mathrm{w}}_0(v^{I_1}_\pm, Dv^{I_2}_\pm)}\\ (D_t+\langle\xi\rangle)\hat{v}^I_-(t,x)=\sum_{|I_1|+|I_2|=|I|}\widehat{Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D_1u^{I_2}_\pm)}+\sum_{|I_1|+|I_2|<|I|}c_{I_1,I_2}\widehat{Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm)}\end{cases}$$

with coefficients $c_{I_1,I_2}\in\{-1,0,1\}$, and indices I_1,I_2 in above right hand side such that $(I_1,I_2)\in\mathcal{I}(I)$. We proceed to write the contributions we want to get rid off under a more explicit form, focusing first on terms in (2.2.34), which are common to $\partial_t\widetilde{E}_n$ and $\partial_t\widetilde{E}^k_3$.

From definition (2.1.6) of matrix $A''(V^I,\eta)$, Plancherel's formula, (1.2.6) and the fact that $\overline{v^I_+}=-v^I_-$, we observe that

$$\begin{aligned}
&\langle Op^B(A''(V^I;\eta))U,W^I\rangle = \langle Op^B(a_0(v^I_\pm;\eta)\eta_1)u_+ + Op^B(b_0(v^I_\pm;\eta)\eta_1)u_-, v^I_+ + v^I_-\rangle\\
&= -\frac{i}{4(2\pi)^2}\int\chi\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)\Big[\widehat{(v^I_+ + v^I_-)}(\xi-\eta)\widehat{(u_+ + u_-)}(\eta) - \frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\widehat{(v^I_+ - v^I_-)}(\xi-\eta)\\
&\qquad\qquad\times\widehat{(u_+ - u_-)}(\eta)\Big]\,\eta_1\widehat{(v^I_- + v^I_-)}(-\xi)d\xi d\eta,
\end{aligned}$$

where χ denotes here a smooth function equal to 1 in $\overline{B_{\varepsilon_1}(0)}$ and supported in $B_{\varepsilon_2}(0)$, for some $0<\varepsilon_1<\varepsilon_2\ll 1$, and hence that

$$-\Im\left[\langle Op^B(A''(V^I;\eta))U,W^I\rangle\right] = \sum_{j_k\in\{+,-\}} C^I_{(j_1,j_2,j_3)}, \tag{2.2.39}$$

with

$$C^I_{(j_1,j_2,j_3)} = \frac{1}{4(2\pi)^2}\int\chi\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)\left(1-j_1j_2\frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\right)\eta_1\hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)d\xi d\eta, \tag{2.2.40}$$

for any $j_1,j_2,j_3\in\{+,-\}$. Analogously, from equality (1.2.8) we deduce that

$$-\Im\left[\langle Op^B_R(A''(V^I;\eta))U,W^I\rangle\right] = \sum_{j_k\in\{+,-\}} C^{I,R}_{(j_1,j_2,j_3)}. \tag{2.2.41}$$

with

$$\begin{aligned}
C^{I,R}_{(j_1,j_2,j_3)} = \frac{1}{4(2\pi)^2}\int\Big[1-\chi\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)-\chi\Big(\frac{\eta}{\langle\xi-\eta\rangle}\Big)\Big]\left(1-j_1j_2\frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\right)\eta_1\\
\times\hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)d\xi d\eta.
\end{aligned}\tag{2.2.42}$$

After proposition 2.2.1, we know that $T_{-N}(U)=(\sigma_{ij}(U,D_x))_{ij}$ with symbols $\sigma_{ij}(U,\eta)$ satisfying (2.2.6). Introducing $\rho:\{+,-\}\to\{2,4\}$, such that $\rho(+)=2,\rho(-)=4$, we have that

$$\begin{aligned}
\langle T_{-N}(U)W^I,W^I\rangle &= \sum_{i,j\in\{+,-\}}\langle\sigma_{\rho(i)\rho(j)}(U,D_x)v^I_j,v^I_i\rangle\\
&= -\frac{1}{(2\pi)^2}\sum_{j_k\in\{+,-\}}\int\sigma^{j_2}_{\rho(j_3),\rho(j_1)}(\eta,\xi-\eta)\hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}^I_{-j_3}(-\xi)d\xi d\eta,
\end{aligned}\tag{2.2.43}$$

with the convention that $-j_k\in\{+,-\}\setminus\{j_k\}$, and where multipliers $\sigma^{j_2}_{\rho(j_3),\rho(j_1)}(\eta,\xi-\eta)$ are supported for $|\eta|\le\varepsilon|\xi-\eta|$, and such that, for any $\alpha,\beta\in\mathbb{N}^2$,

$$\left|\partial^\alpha_\xi\partial^\beta_\eta\sigma^{j_2}_{\rho(j_3),\rho(j_1)}(\eta,\xi-\eta)\right|\lesssim_{\alpha,\beta}|\eta|^{N+1-|\beta|}\langle\xi-\eta\rangle^{-N-|\alpha|},$$

for any $(\xi,\eta)\in\mathbb{R}^2\times\mathbb{R}^2$, any $j_1,j_2,j_3\in\{+,-\}$.

Equalities (2.2.39), (2.2.41) and (2.2.43) lead us to introduce the following integrals, for any multi-index I belonging either to $\mathcal{I}_n$ or to $\mathcal{I}_3^k$, $0 \le k \le 2$, and any triplet (j_1, j_2, j_3), $j_k \in \{+,-\}$:

$$D^I_{(j_1,j_2,j_3)} := \frac{i}{4(2\pi)^2} \int \chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right) B^1_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)\, d\xi d\eta, \tag{2.2.44a}$$

$$\begin{aligned} D^{I,R}_{(j_1,j_2,j_3)} := \frac{i}{4(2\pi)^2} \int \left[1-\chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right)-\chi\left(\frac{\eta}{\langle\xi-\eta\rangle}\right)\right] B^1_{(j_1,j_2,j_3)}(\xi,\eta) \\ \times \hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi) d\xi d\eta, \end{aligned} \tag{2.2.44b}$$

$$D^{I,T_{-N}}_{(j_1,j_2,j_3)} := \mathrm{Re}\left[\frac{1}{(2\pi)^2}\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}^I_{-j_3}(-\xi)d\xi d\eta\right] \tag{2.2.44c}$$

with multipliers $B^k_{(j_1,j_2,j_3)}$, $\widetilde{\sigma}^N_{(j_1,j_2,j_3)}$ given by

$$B^k_{(j_1,j_2,j_3)}(\xi,\eta) := \frac{1}{j_1\langle\xi-\eta\rangle + j_2|\eta| + j_3\langle\xi\rangle}\left(1 - j_1 j_2 \frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\right)\eta_k, \quad k=1,2, \tag{2.2.45}$$

and

$$\widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta) := \frac{\sigma^{j_2}_{\rho(j_3),\rho(j_1)}(\eta,\xi-\eta)}{j_1\langle\xi-\eta\rangle + j_2|\eta| - j_3\langle\xi\rangle}. \tag{2.2.46}$$

It is useful to also introduce

$$B^3_{(j_1,j_2,j_3)}(\xi,\eta) := \frac{j_2}{j_1\langle\xi-\eta\rangle + j_2|\eta| + j_3\langle\xi\rangle}\left(1 - j_1 j_2 \frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\right)|\eta|, \tag{2.2.47}$$

and to refer to $B^k_{(j_1,j_2,j_3)}$ simply as $B_{(j_1,j_2,j_3)}$ when we are not interested in distinguishing between $k=1,2,3$.

Let us also observe that, for any triplet of indices (I_1, I_2, I), by (2.1.1) we have that

$$-\Im\left[\langle Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D_1 u^{I_2}_\pm), v^I_+ + v^I_-\rangle\right] = \sum_{j_k\in\{+,-\}} C^{I_1,I_2}_{(j_1,j_2,j_3)}, \tag{2.2.48a}$$

$$-\Im\left[\langle Q^{\mathrm{kg}}_0(v^{I_1}_\pm, \chi(t^{-\sigma}D_x)D u^{I_2}_\pm), v^I_+ + v^I_-\rangle\right] = \sum_{j_k\in\{+,-\}} F^{I_1,I_2}_{(j_1,j_2,j_3)}, \tag{2.2.48b}$$

where in above (2.2.48b) χ denotes a smooth cut-off function, with

$$C^{I_1,I_2}_{(j_1,j_2,j_3)} = \frac{1}{4(2\pi)^2}\int\left(1 - j_1 j_2 \frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\right)\eta_1\ \hat{v}^{I_1}_{j_1}(\xi-\eta)\hat{u}^{I_2}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)d\xi d\eta, \tag{2.2.49a}$$

(2.2.49b)

$$F^{I_1,I_2}_{(j_1,j_2,j_3)} = \frac{1}{4(2\pi)^2}\int\left(1 - j_1 j_2 \frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\right)\eta\ \hat{v}^{I_1}_{j_1}(\xi-\eta)\widehat{\chi(t^{-\sigma}D_x)u^{I_2}_{j_2}}(\eta)\hat{v}^I_{j_3}(-\xi)d\xi d\eta.$$

Note that in our notations factor η, in the multiplier defining $F^{I_1,I_2}_{(j_1,j_2,j_3)}$, corresponds to η_1 (resp. to $\eta_2, j_2|\eta|$), depending on whether $Du^{I_2}_\pm$ in the left hand side of (2.2.48b) corresponds to $D_1 u^{I_2}_\pm$ (resp. to $D_2 u^{I_2}_\pm, |D_x|u^{I_2}_\pm$) in (2.2.48b). We hence consider integrals:

$$D^{I_1,I_2}_{(j_1,j_2,j_3)} := \frac{i}{4(2\pi)^2}\int B^1_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^{I_1}_{j_1}(\xi-\eta)\hat{u}^{I_2}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)\, d\xi d\eta, \tag{2.2.50a}$$

$$G^{I_1,I_2}_{(j_1,j_2,j_3)} = \frac{i}{4(2\pi)^2}\int B_{(j_1,j_2,j_3)}(\xi,\eta)\ \hat{v}^{I_1}_{j_1}(\xi-\eta)\widehat{\chi(t^{-\sigma}D_x)u^{I_2}_{j_2}}(\eta)\hat{v}^I_{j_3}(-\xi)d\xi d\eta,, \tag{2.2.50b}$$

and finally give the following:

Definition 2.2.5. Let $n \geq 3$ and $0 \leq k \leq 2$. We define the second modification of the energy $\widetilde{E}^{\dagger}_n(t;W)$ as follows:

$$\text{(2.2.51a)}\quad \widetilde{E}^{\dagger}_n(t;W) := \widetilde{E}_n(t;W) + \sum_{\substack{I\in\mathcal{I}_n\\ j_i\in\{+,-\}}} \Big(D^{I}_{(j_1,j_2,j_3)} + D^{I,R}_{(j_1,j_2,j_3)} + D^{I,T_{-N}}_{(j_1,j_2,j_3)}\Big) + \sum_{\substack{I\in\mathcal{I}_n\\ j_j\in\{+,-\}}} \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ [\frac{|I|}{2}]<|I_1|<|I|}} D^{I_1,I_2}_{(j_1,j_2,j_3)},$$

and of $\widetilde{E}^{k,\dagger}_3(t;W)$ as

$$\text{(2.2.51b)}\quad \widetilde{E}^{k,\dagger}_3(t;W) := \widetilde{E}^{k}_3(t;W) + \sum_{\substack{I\in\mathcal{I}^k_3\\ j_i\in\{+,-\}}} \Big(D^{I}_{(j_1,j_2,j_3)} + D^{I,R}_{(j_1,j_2,j_3)} + D^{I,T_{-N}}_{(j_1,j_2,j_3)}\Big) + \delta_{k<2} \sum_{\substack{I\in\mathcal{V}_k\\ j_i\in\{+,-\}}} \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ I_1\in\mathcal{K},|I_2|\leq 1}} c_{I_1,I_2} G^{I_1,I_2}_{(j_1,j_2,j_3)},$$

with $\delta_{k<2}=1$ if $k=0,1$, 0 otherwise, and coefficients $c_{I_1,I_2}\in\{-1,0,1\}$.

In lemmas 2.2.8, 2.2.9, we will check that, with definition (2.2.51), the slow decaying contributions highlighted in (2.2.34) are replaced in $\partial_t\widetilde{E}^{\dagger}_n(t;W)$, $\partial_t\widetilde{E}^{k,\dagger}_3(t;W)$ by some new quartic terms. These latter ones are obtained from integrals (2.2.44) by replacing each factor $\hat{v}^I_{j_1}, \hat{u}_{j_2}, \hat{v}^I_{j_3}$ at a time with the non-linearity appearing in the equation that factor satisfies in (2.2.38). Lemma 2.2.10 (resp. lemma 2.2.11) shows that the same is for troublesome contributions (2.2.37) in $\partial_t\widetilde{E}^{\dagger}_n(t;W)$ (resp. for (2.2.36) in $\partial_t\widetilde{E}^{k,\dagger}_3(t;W)$). We are also going to see that, if $N\in\mathbb{N}^*$ is chosen sufficiently large (e.g. $N=18$), these quartic terms suitably decay in time, and that modified energies $\widetilde{E}^{\dagger}_n(t;W), \widetilde{E}^{k,\dagger}_3(t;W)$ are equivalent, respectively, to $E_n(t;W), E^k_3(t,W)$. We point out the fact that the normal form's step performed in previous section was necessary to avoid here some problematic quartic contributions, coming from quasi-linear terms in (2.2.38) and that could lead to some loss of derivatives.

Before proving the mentioned lemmas, we need to introduce two preliminary results, that will be useful in the proof of lemmas 2.2.8, 2.2.10.

Lemma 2.2.6. *For any $j_i\in\{+,-\}$, $i=1,2,3$, let $B^k_{(j_1,j_2,j_3)}(\xi,\eta)$ be the multiplier defined in (2.2.45) for $k=1,2$, and ψ_1,ψ_2,ψ_3 be three smooth cut-off functions such that $\psi_1(x)$ is supported for $|x|\leq c$, $\psi_2(x)$ is supported for $c'\leq|x|\leq C'$, $\psi_3(x)$ is supported for $|x|\geq C$, for some $0<c,c'\ll 1$, $C,C'\gg 1$, and $\psi_1+\psi_2+\psi_3\equiv 1$.*

(i) For any $j_1,\dots,j_5\in\{+,-\}$, $i=1,2$, and any u_1,u_2,u_3,u_4, such that $u_1\in H^{4,\infty}(\mathbb{R}^2)$, $u_2,u_4\in L^2(\mathbb{R}^2)$, $u_3\in H^{11,\infty}(\mathbb{R}^2)$ and $\mathrm{R}u_3\in H^{7,\infty}(\mathbb{R}^2)$,

$$\text{(2.2.52)}\quad \left|\int \psi_i\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big) B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{u}_1(\xi-\eta-\zeta)\hat{u}_2(\zeta)\hat{u}_3(\eta)\hat{u}_4(-\xi)\,d\xi d\eta d\zeta\right| \lesssim \|u_1\|_{H^{4,\infty}}\|u_2\|_{L^2}\left(\|u_3\|_{H^{11,\infty}}+\|\mathrm{R}u_3\|_{H^{7,\infty}}\right)\|u_4\|_{L^2};$$

(ii) For any $j_1,\dots,j_5\in\{+,-\}$, and any u_1,u_2,u_3,u_4, such that $u_1\in H^{7,\infty}(\mathbb{R}^2)$, $u_2\in H^1(\mathbb{R}^2)$,

$u_4 \in L^2(\mathbb{R}^2)$, $u_3 \in H^{4,\infty}(\mathbb{R}^2)$ and $\mathrm{R}u_3 \in L^\infty(\mathbb{R}^2)$,

(2.2.53)
$$\Big|\int \psi_3\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{u}_1(\xi-\eta-\zeta)\hat{u}_2(\zeta)\hat{u}_3(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\Big| \lesssim \|u_1\|_{H^{7,\infty}}\|u_2\|_{H^1}\left(\|u_3\|_{H^{4,\infty}}+\|\mathrm{R}u_3\|_{L^\infty}\right)\|u_4\|_{L^2}.$$

If in above integrals we consider $B^k_{(j_1,j_2,j_3)}$ for $k=3$ (see definition (2.2.47)), inequality (2.2.52) (resp. (2.2.53)) holds with $\|u_3\|_{H^{11,\infty}}+\|\mathrm{R}u_3\|_{H^{7,\infty}}$ (resp. $\|u_3\|_{H^{4,\infty}}+\|\mathrm{R}u_3\|_{L^\infty}$) replaced with $\|u_3\|_{H^{11,\infty}}$ (resp. with $\|u_3\|_{H^{4,\infty}}$).

Proof. Throughout the proof we will refer to $B^k_{(j_1,j_2,j_3)}$ simply as $B_{(j_1,j_2,j_3)}$, and rather use a superscript to define a decomposition of this multiplier (see (2.2.54)). We also adopt the notation η without subscript k, just reminding that it takes the values $\eta_1,\eta_2,j_2|\eta|$.

In order to prove the statement, we first observe that multiplier $B_{(j_1,j_2,j_3)}(\xi,\eta)$ can be rewritten as
$$B_{(j_1,j_2,j_3)}(\xi,\eta)=\frac{j_1\langle\xi-\eta\rangle+j_2|\eta|-j_3\langle\xi\rangle}{2j_1j_2\langle\xi-\eta\rangle|\eta|}\eta,$$
and that, introducing

(2.2.54)
$$\begin{aligned}B^0_{(j_1,j_2,j_3)}(\xi,\eta)&:=\frac{j_1\langle\xi-\eta\rangle+j_2|\eta|-j_3\langle\xi\rangle}{2j_1j_2\langle\xi-\eta\rangle}\phi(\eta),\\ B^1_{(j_1,j_2,j_3)}(\xi,\eta)&:=\frac{j_1\langle\xi-\eta\rangle+j_2|\eta|-j_3\langle\xi\rangle}{2j_1j_2\langle\xi-\eta\rangle|\eta|}\eta\langle\eta\rangle^{-4}(1-\phi)(\eta),\end{aligned}$$

for any smooth cut-off function ϕ, equal to 1 in a neighbourhood of the origin,

(2.2.55)
$$B_{(j_1,j_2,j_3)}(\xi,\eta)=B^0_{(j_1,j_2,j_3)}(\xi,\eta)\frac{\eta}{|\eta|}+B^1_{(j_1,j_2,j_3)}(\xi,\eta)\langle\eta\rangle^4.$$

According to above decomposition, we have that, for any $i=1,2,3$,

(2.2.56)
$$\begin{aligned}&\int \psi_i\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)B_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{u}_1(\xi-\eta-\zeta)\hat{u}_2(\zeta)\hat{u}_3(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\\ &=\int \psi_i\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)B^0_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{u}_1(\xi-\eta-\zeta)\hat{u}_2(\zeta)\widehat{\mathrm{R}u_3}(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\\ &+\int \psi_i\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)B^1_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{u}_1(\xi-\eta-\zeta)\hat{u}_2(\zeta)\widehat{\langle D_x\rangle^4 u_3}(\eta)\hat{u}_4(-\xi)\\ &\qquad d\xi d\eta d\zeta\\ &=: I^0_i+I^1_i.\end{aligned}$$

(i) The first thing we observe concerning integral I^k_1 (resp. I^k_2), for $k=0,1$, is that $|\xi-\eta|,|\xi|\lesssim\langle\eta\rangle$ on the support of $\psi_1\big(\frac{\xi-\eta}{\langle\eta\rangle}\big)$ (resp. of $\psi_2\big(\frac{\xi-\eta}{\langle\eta\rangle}\big)$), and that $|\xi|\le\langle\xi-\eta-\zeta\rangle\langle\eta\rangle$. Therefore, introducing the following multipliers
$$B^{i,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta):=\psi_i\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\langle\eta\rangle^{-7}\langle\xi-\eta-\zeta\rangle^{-4},$$
for any $j_1,\dots j_5\in\{+,-\}$, $k=0,1$, $i=1,2$, a straight computation shows that, for any $\alpha,\beta,\gamma\in\mathbb{N}^2$,

(2.2.57)
$$\begin{aligned}\left|\partial^\alpha_\xi\partial^\beta_\eta B^{i,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\right|&\lesssim\left(\mathbb{1}_{|\zeta|\lesssim1}+\mathbb{1}_{|\zeta|\gtrsim1}\langle\zeta\rangle^{-3}\right)|g_{\alpha,\beta}(\xi,\eta)|,\\ \left|\partial^\alpha_\xi\partial^\beta_\eta\partial^\gamma_\zeta B^{i,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\right|&\lesssim\left(\mathbb{1}_{|\zeta|\lesssim1}|\zeta|^{1-|\gamma|}+\mathbb{1}_{|\zeta|\gtrsim1}\langle\zeta\rangle^{-3}\right)|g_{\alpha,\beta}(\xi,\eta)|,\ |\gamma|\ge1,\end{aligned}$$

with

$$
(2.2.58)\qquad \begin{aligned}
|g_{\alpha,0}(\xi,\eta)| &\lesssim_\alpha \left(\mathbb{1}_{|\eta|\lesssim 1} + \mathbb{1}_{|\eta|\gtrsim 1}\langle\eta\rangle^{-3}\right)\langle\xi\rangle^{-3},\\
|g_{\alpha,\beta}(\xi,\eta)| &\lesssim_{\alpha,\beta} \left(\mathbb{1}_{|\eta|\lesssim 1}|\eta|^{1-|\beta|} + \mathbb{1}_{|\eta|\gtrsim 1}\langle\eta\rangle^{-3}\right)\langle\xi\rangle^{-3},\ |\beta|\ge 1,
\end{aligned}
$$

and if

$$
K^{i,k}_{(j_1,\dots,j_5)}(x,y,z) := \int e^{ix\cdot\xi+iy\cdot\eta+iz\cdot\zeta} B^{i,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\,d\xi d\eta d\zeta,
$$

by lemma A.1 (i) we first find that, for any $\alpha,\beta\in\mathbb{N}^2$,

$$
\left|\partial^\alpha_\xi\partial^\beta_\eta\int e^{iz\cdot\zeta}B^{i,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)d\zeta\right| \lesssim |z|^{-1}\langle z\rangle^{-2}|g_{\alpha,\beta}(\xi,\eta)|,
$$

and successively that

$$
\left|\partial^\alpha_\xi\int e^{iy\cdot\eta+iz\cdot\zeta}B^{i,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)d\eta d\zeta\right| \lesssim |y|^{-1}\langle y\rangle^{-2}|z|^{-1}\langle z\rangle^{-2}\langle\xi\rangle^{-3},
$$

for every $\xi\in\mathbb{R}^2$, $(y,z)\in\mathbb{R}^2\times\mathbb{R}^2$. Corollary A.2 (i) hence implies that $|K^{i,k}_{(j_1,\dots,j_5)}(x,y,z)|\lesssim \langle x\rangle^{-3}|y|^{-1}\langle y\rangle^{-2}|z|^{-1}\langle z\rangle^{-2}$, for any x,y,z. As

$$
\begin{aligned}
I^0_i &= \int B^{i,0}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\widehat{\langle D_x\rangle^4 u_1}(\xi-\eta-\zeta)\hat{u}_2(\zeta)\widehat{\langle D_x\rangle^7 \mathrm{R}u_3}(\eta)\hat{u}_4(-\xi)\,d\xi d\eta d\zeta,\\
&= \int K^{i,0}_{(j_1,\dots,j_5)}(t-x,x-z,x-y)[\langle D_x\rangle^4u_1](x)u_2(y)[\langle D_x\rangle^7\mathrm{R}u_3](z)u_4(t)dxdydzdt,
\end{aligned}
$$

$$
\begin{aligned}
I^1_i &= \int B^{i,1}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\widehat{\langle D_x\rangle^4 u_1}(\xi-\eta-\zeta)\hat{u}_2(\zeta)\widehat{\langle D_x\rangle^{11} u_3}(\eta)\hat{u}_4(-\xi)\,d\xi d\eta d\zeta\\
&= \int K^{i,1}_{(j_1,\dots,j_5)}(t-x,x-z,x-y)[\langle D_x\rangle^4u_1](x)u_2(y)[\langle D_x\rangle^{11}u_3](z)u_4(t)dxdydzdt,
\end{aligned}
$$

for any $i=1,2$, inequality (2.2.52) follows by the fact that, for any $\widetilde{u}_1,\dots\widetilde{u}_4\in L^2\cap L^\infty$, any $f,g,h\in L^1$, integrals such as

$$
(2.2.59)\qquad \int f(t-x)g(x-z)h(x-y)|\widetilde{u}_1(x)||\widetilde{u}_2(y)||\widetilde{u}_3(z)||\widetilde{u}_4(t)|dxdydzdt
$$

can be bounded from above by the product of the L^2 norm of any two functions $\widetilde{u}_k$ times the L^∞ norm of the remaining ones.

(ii) By means of the same cut-off function ϕ, introduced at the beginning of the proof, we decompose integral I^k_3, $k=0,1$, distinguishing between $|\zeta|\lesssim 1$ and $|\zeta|\gtrsim 1$. On the one hand, for any $j_1,\dots,j_5,k=0,1$, we consider

$$
B^{3,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta) := \psi_3\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)\phi(\zeta)B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\langle\xi-\eta-\zeta\rangle^{-3},
$$

and observe that, since $|\xi|\le\langle\xi-\eta-\zeta\rangle$ on the support of $\psi_3\big(\frac{\xi-\eta}{\langle\eta\rangle}\big)\phi(\zeta)$, the above multiplier satisfies estimates (2.2.57), (2.2.58), which implies, from the same argument as before, that

$$
(2.2.60)\qquad \left|J^0_3 := \int B^{3,0}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\widehat{\langle D_x\rangle^3 u_1}(\xi-\eta-\zeta)\hat{u}_2(\zeta)\widehat{\mathrm{R}u_3}(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\right| \lesssim \|u_1\|_{H^{3,\infty}}\|u_2\|_{L^2}\|\mathrm{R}u_3\|_{L^\infty}\|u_4\|_{L^2},
$$

together with

$$(2.2.61)\quad \left|J_3^1 := \int B^{3,1}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\widehat{\langle D_x\rangle^3 u_1}(\xi-\eta-\zeta)\hat{u}_2(\zeta)\widehat{\langle D_x\rangle^4 u_3}(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\right| \lesssim \|u_1\|_{H^{3,\infty}}\|u_2\|_{L^2}\|u_3\|_{H^{4,\infty}}\|u_4\|_{L^2}.$$

On the other hand, we make a further decomposition on the integral restricted to $|\zeta|\gtrsim 1$ by means of functions $\psi_i, i=1,2,3$, distinguishing between three regions: for $|\zeta|\le c\langle\xi-\eta\rangle$, for $c'\langle\xi-\eta\rangle\le|\zeta|\le C'\langle\xi-\eta\rangle$, and $|\zeta|>C\langle\xi-\eta\rangle$. For any $j_1,\dots,j_5\in\{+,-\}$, we hence introduce the following multipliers: for $i=1,3, k=0,1$,

$$\widetilde{B}^{3,i,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta) := \psi_3\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)(1-\phi)(\zeta)\psi_i\Big(\frac{\zeta}{\langle\xi-\eta\rangle}\Big) \times B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\langle\xi-\eta-\zeta\rangle^{-7};$$

for $i=2, k=0,1$,

$$(2.2.62)\quad \widetilde{B}^{3,2,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta) := \psi_3\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)(1-\phi)(\zeta)\psi_2\Big(\frac{\zeta}{\langle\xi-\eta\rangle}\Big) \times B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\langle\zeta\rangle^{-1};$$

Since $|\xi|\sim|\xi-\eta|\sim|\xi-\eta-\zeta|$ on the support of $\psi_3\big(\frac{\xi-\eta}{\langle\eta\rangle}\big)(1-\phi)(\zeta)\psi_1\big(\frac{\zeta}{\langle\xi-\eta\rangle}\big)$ (resp. $|\xi|\sim|\xi-\eta|\lesssim|\zeta|\sim|\xi-\eta-\zeta|$ on the support of $\psi_3\big(\frac{\xi-\eta}{\langle\eta\rangle}\big)(1-\phi)(\zeta)\psi_3\big(\frac{\zeta}{\langle\xi-\eta\rangle}\big)$), a straight computation shows that above multipliers verify (2.2.57), (2.2.58), from which follows that

$$(2.2.63)\quad \left|\widetilde{J}^{i,0}_3 := \int \widetilde{B}^{3,i,0}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\widehat{\langle D_x\rangle^7 u_1}(\xi-\eta-\zeta)\hat{u}_2(\zeta)\widehat{\mathrm{R}u_3}(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\right| \lesssim \|u_1\|_{H^{7,\infty}}\|u_2\|_{L^2}\|\mathrm{R}u_3\|_{L^\infty}\|u_4\|_{L^2},$$

along with

$$(2.2.64)\quad \left|\widetilde{J}^{i,1}_3 := \int \widetilde{B}^{3,i,1}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\widehat{\langle D_x\rangle^7 u_1}(\xi-\eta-\zeta)\hat{u}_2(\zeta)\widehat{\langle D_x\rangle^4 u_3}(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta\right| \lesssim \|u_1\|_{H^{7,\infty}}\|u_2\|_{L^2}\|u_3\|_{H^{4,\infty}}\|u_4\|_{L^2},$$

for $i=1,3$.

Finally, on the support of $\psi_3\big(\frac{\xi-\eta}{\langle\eta\rangle}\big)(1-\phi)(\zeta)\psi_2\big(\frac{\zeta}{\langle\xi-\eta\rangle}\big)$, we have that $|\xi|\sim|\xi-\eta|\sim|\zeta|$, and $|\xi-\eta-\zeta|\lesssim|\zeta|$, so replacing ζ with $\xi-\zeta$ by a change of coordinates, we find that, for any $\alpha,\beta,\gamma\in\mathbb{N}^2$,

$$\left|\partial^\alpha_\xi\partial^\gamma_\zeta\widetilde{B}^{3,2,k}_{(j_1,\dots,j_5)}(\xi,\eta,\xi-\zeta)\right|\lesssim_{\alpha,\gamma}\langle\eta\rangle^{-3}\langle\xi\rangle^{-|\alpha|},$$
$$\left|\partial^\alpha_\xi\partial^\beta_\eta\partial^\gamma_\zeta\widetilde{B}^{3,2,k}_{(j_1,\dots,j_5)}(\xi,\eta,\xi-\zeta)\right|\lesssim(|\eta|\langle\eta\rangle^{-1})^{1-|\beta|}\langle\eta\rangle^{-3}\langle\xi\rangle^{-|\alpha|},\ |\beta|\ge 1.$$

If we introduce a Littlewood-Paley decomposition such that

$$\widetilde{B}^{3,2,k}_{(j_1,\dots,j_5)}(\xi,\eta,\xi-\zeta)=\sum_{l\ge1}\widetilde{B}^{3,2,k}_{(j_1,\dots,j_5)}(\xi,\eta,\xi-\zeta)\varphi(2^{-l}\xi),$$

with $\varphi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, from above estimates we deduce that

$$(2.2.65)\qquad \begin{gathered}\left|\partial_\xi^\alpha \partial_\zeta^\gamma \left[\widetilde{B}^{3,2,k}_{(j_1,\dots,j_5)}(\xi,\eta,\xi-\zeta)\varphi(2^{-l}\xi)\right]\right| \lesssim_{\alpha,\gamma} \langle\eta\rangle^{-3}, \\ \left|\partial_\xi^\alpha \partial_\eta^\beta \partial_\zeta^\gamma \left[\widetilde{B}^{3,2,k}_{(j_1,\dots,j_5)}(\xi,\eta,\xi-\zeta)\varphi(2^{-l}\xi)\right]\right| \lesssim (|\eta|\langle\eta\rangle^{-1})^{1-|\beta|}\langle\eta\rangle^{-3},\ |\beta|\ge 1,\end{gathered}$$

for any $l \ge 1$, and therefore that

$$K^{k,l}_{(j_1,\dots,j_5)}(x,y,z) := \int e^{ix\cdot\xi + iy\cdot\eta + iz\cdot\zeta}\widetilde{B}^{3,2,k}_{(j_1,\dots,j_5)}(\xi,\eta,\xi-\zeta)\varphi(2^{-l}\xi)d\xi d\eta d\zeta$$

is such that

$$(2.2.66)\qquad |K^{k,l}_{(j_1,\dots,j_5)}(x,y,z)| \lesssim 2^{2l}\langle 2^l x\rangle^{-3}|y|^{-1}\langle y\rangle^{-2}\langle z\rangle^{-3}$$

for any x,y,z, any $l\ge 1$, as one can check using lemma A.1 (i) to obtain the decay in y, making a change of coordinates $\xi \mapsto 2^l\xi$, some integration by parts, and using inequalities (2.2.65). Moreover, since $|\xi| \sim |\xi-\zeta|$ on the support of $\widetilde{B}^{3,2,k}_{(j_1,\dots,j_5)}$, there are two other cut-off functions φ_1, φ_2, with suitable support, such that $\varphi(2^{-l}\xi) = \varphi(2^{-l}\xi)\varphi_1(2^{-l}\xi)\varphi_2(2^{-l}\xi-\zeta)$, for any $l\ge 1$, and if $\Delta^l_j w := \varphi_j(2^{-l}D_x)w$, we finally obtain that

$$\begin{aligned}\widetilde{J}^{2,0}_3 &:= \int \widetilde{B}^{3,2,0}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\hat{u}_1(\xi-\eta-\zeta)\widehat{\langle D_x\rangle u}_2(\zeta)\widehat{\mathrm{R}u}_3(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta \\ &= \int \widetilde{B}^{3,2,0}_{(j_1,\dots,j_5)}(\xi,\eta,\xi-\zeta)\hat{u}_1(\zeta-\eta)\widehat{\langle D_x\rangle u}_2(\xi-\zeta)\widehat{\mathrm{R}u}_3(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta \\ &= \sum_{l\ge 1}\int K^{0,l}_{(j_1,\dots,j_5)}(t-y,x-z,y-x)u_1(x)[\Delta^l_1\langle D_x\rangle u_2](y)[\mathrm{R}u_3](z)[\Delta^l_2 u_4](t)dxdydzdt,\end{aligned}$$

and by (2.2.66), together with Cauchy-Schwarz inequality, we derive that

$$(2.2.67)\qquad |\widetilde{J}^{2,0}_3| \lesssim \|u_1\|_{L^\infty}\|\mathrm{R}_1u_3\|_{L^\infty}\sum_{l\ge 1}\|\Delta^l_1\langle D_x\rangle u_2\|_{L^2}\|\Delta^l_2u_4\|_{L^2} \lesssim \|u_1\|_{L^\infty}\|u_2\|_{H^1}\|\mathrm{R}_1u_3\|_{L^\infty}\|u_4\|_{L^2}.$$

Similarly, we obtain that

$$\widetilde{J}^{2,1}_3 := \int \widetilde{B}^{3,2,1}_{(j_1,\dots,j_5)}(\xi,\eta,\xi-\zeta)\hat{u}_1(\zeta-\eta)\widehat{\langle D_x\rangle u}_2(\xi-\zeta)\widehat{\langle D_x\rangle^4 u}_3(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta$$

satisfies

$$(2.2.68)\qquad |\widetilde{J}^{2,0}_3| \lesssim \|u_1\|_{L^\infty}\|u_2\|_{H^1}\|u_3\|_{H^{4,\infty}}\|u_4\|_{L^2}.$$

The result of statement (ii) follows then from inequalities (2.2.60), (2.2.61), (2.2.63), (2.2.64), (2.2.67), (2.2.68), after recognizing that

$$\int \psi_3\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)B_{(j_1,j_2,j_3)}(\xi,\eta)\left(1-j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{u}_1(\xi-\eta-\zeta)\hat{u}_2(\zeta)\hat{u}_3(\eta)\hat{u}_4(-\xi)d\xi d\eta d\zeta = \sum_{k=0}^1 J^k_3 + \sum_{k=0}^1\sum_{i=1}^3 \widetilde{J}^{i,k}_3.$$

Finally, the same proof applies to multiplier $B^3_{(j_1,j_2,j_3)}$ introduced in (2.2.47), after observing that it decomposes as

$$j_2B^0_{(j_1,j_2,j_3)}(\xi,\eta) + \widetilde{B}^1_{(j_1,j_2,j_3)}(\xi,\eta)\langle\eta\rangle^4,$$

with the same $B^0_{(j_1,j_2,j_3)}$ as in (2.2.54), and

$$\widetilde{B}^1_{(j_1,j_2,j_3)}(\xi,\eta) := \frac{j_1\langle\xi-\eta\rangle + j_2|\eta| - j_3\langle\xi\rangle}{2j_1\langle\xi-\eta\rangle}\langle\eta\rangle^{-4}(1-\phi)(\eta).$$

The lack of factor $\eta_1|\eta|^{-1}$ against $B^0_{(j_1,j_2,j_3)}$, in comparison to decomposition (2.2.55), is the reason why inequality (2.2.52) (resp. (2.2.53)) holds with $\|u_3\|_{H^{11,\infty}} + \|\mathrm{R}u_3\|_{H^{7,\infty}}$ (resp. $\|u_3\|_{H^{4,\infty}} + \|\mathrm{R}u_3\|_{L^\infty}$) replaced with $\|u_3\|_{H^{11,\infty}}$ (resp. with $\|u_3\|_{H^{4,\infty}}$). □

Lemma 2.2.7. *For any $j_i \in \{+,-\}$, $i = 1,2,3$, let $B^k_{(j_1,j_2,j_3)}(\xi,\eta)$ be the multiplier defined in* (2.2.45) *for $k = 1,2$, and ψ_1,ψ_2,ψ_3 be three smooth cut-off functions such that $\psi_1(x)$ is supported for $|x| \le c$, $\psi_2(x)$ is supported for $c' \le |x| \le C'$, $\psi_3(x)$ is supported for $|x| \ge C$, for some $0 < c, c' \ll 1$, $C, C' \gg 1$, and $\psi_1 + \psi_2 + \psi_3 \equiv 1$.*

(i) For any $j_1,\dots,j_5 \in \{+,-\}$, $i = 1,2$, and any u_1,u_2,u_3,u_4, such that $u_1 \in H^{4,\infty}(\mathbb{R}^2)$, $u_2,u_4 \in L^2(\mathbb{R}^2)$, $u_3 \in H^{11,\infty}(\mathbb{R}^2)$ and $\mathrm{R}u_3 \in H^{7,\infty}(\mathbb{R}^2)$,

$$\left|\int \psi_i\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big) B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1 + j_4j_5\frac{\xi+\zeta}{\langle\xi+\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{u}_1(-\xi-\zeta)\hat{u}_2(\zeta)\hat{u}_3(\eta)\hat{u}_4(\xi-\eta)d\xi d\eta d\zeta\right| \\ \lesssim \|u_1\|_{H^{4,\infty}}\|u_2\|_{L^2}\left(\|u_3\|_{H^{11,\infty}} + \|\mathrm{R}u_3\|_{H^{7,\infty}}\right)\|u_4\|_{L^2}; \tag{2.2.69}$$

(ii) For any $j_1,\dots,j_5 \in \{+,-\}$, and any u_1,u_2,u_3,u_4, such that $u_1 \in H^{7,\infty}(\mathbb{R}^2)$, $u_2 \in L^2(\mathbb{R}^2)$, $u_4 \in H^1(\mathbb{R}^2)$, $u_3 \in H^{4,\infty}(\mathbb{R}^2)$ and $\mathrm{R}u_3 \in L^\infty(\mathbb{R}^2)$,

$$\left|\int \psi_3\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big) B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1 + j_4j_5\frac{\xi+\zeta}{\langle\xi+\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{u}_1(-\xi-\zeta)\hat{u}_2(\zeta)\hat{u}_3(\eta)\hat{u}_4(\xi-\eta)d\xi d\eta d\zeta\right| \\ \lesssim \|u_1\|_{H^{7,\infty}}\|u_2\|_{L^2}\left(\|u_3\|_{H^{4,\infty}} + \|\mathrm{R}u_3\|_{L^\infty}\right)\|u_4\|_{H^1}. \tag{2.2.70}$$

If in above integrals we consider $B^k_{(j_1,j_2,j_3)}$ for $k = 3$ (see definition (2.2.47)*), inequality* (2.2.52) *(resp.* (2.2.53)*) holds with $\|u_3\|_{H^{11,\infty}} + \|\mathrm{R}u_3\|_{H^{7,\infty}}$ (resp. $\|u_3\|_{H^{4,\infty}} + \|\mathrm{R}u_3\|_{L^\infty}$) replaced with $\|u_3\|_{H^{11,\infty}}$ (resp. with $\|u_3\|_{H^{4,\infty}}$).*

Proof. The proof of the statement is analogous to that of lemma 2.2.6, after a change of coordinates $-\xi \mapsto \xi - \eta$. In (2.2.70) we take the H^1 norm on u_4, instead of u_2 as done in (2.2.53), by replacing multiplier $\widetilde{B}^{3,2,k}_{(j_1,j_2,j_3)}$ in (2.2.62) with

$$\psi_3\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)(1-\phi)(\zeta)\psi_2\Big(\frac{\zeta}{\langle\xi-\eta\rangle}\Big)B^k_{(j_1,j_2,j_3)}(\xi,\eta)\left(1 - j_4j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\langle\xi\rangle^{-1}.$$

□

Lemma 2.2.8 (Analysis of quartic terms. I)**.** *Let I be a general multi-index, $C^I_{(j_1,j_2,j_3)}$, $C^{I,R}_{(j_1,j_2,j_3)}$ be the integrals defined, respectively, in* (2.2.40), (2.2.42), *and $D^I_{(j_1,j_2,j_3)}$, $D^{I,R}_{(j_1,j_2,j_3)}$ introduced, respectively, in* (2.2.44a), (2.2.44b), *for any $j_k \in \{+,-\}, k = 1,2,3$. Then*

$$\partial_t\left[D^I_{(j_1,j_2,j_3)} + D^{I,R}_{(j_1,j_2,j_3)}\right] = -C^I_{(j_1,j_2,j_3)} - C^{I,R}_{(j_1,j_2,j_3)} + \mathfrak{D}^I_{quart}, \tag{2.2.71}$$

where $\mathfrak{D}^I_{quart}$ satisfies, for any $\theta \in]0,1[$,

(2.2.72)

$$\begin{aligned}&\left|\mathfrak{D}^I_{quart}(t)\right| \\ &\lesssim \left[\|V(t,\cdot)\|^{2-(2-\theta)\theta}_{H^{5,\infty}}\|V(t,\cdot)\|^{(2-\theta)\theta}_{H^7} + \|V(t,\cdot)\|_{H^{4,\infty}}\left(\|U(t,\cdot)\|_{H^{11,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{7,\infty}}\right)\right]\|W^I(t,\cdot)\|^2_{L^2} \\ &+ \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_2|<|I|}} \|Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm)(t,\cdot)\|_{L^2}\left(\|U(t,\cdot)\|_{H^{11,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{7,\infty}}\right)\|V^I(t,\cdot)\|_{L^2}.\end{aligned}$$

Proof. Using definitions (2.2.40), (2.2.44a), (2.2.45) with $k=1$, and system (2.2.38), we find that

(2.2.73)

$$\begin{aligned}
&-4(2\pi)^2\left[\partial_t D^I_{(j_1,j_2,j_3)} + C^I_{(j_1,j_2,j_3)}\right]\\
&= \int \chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right) B^1_{(j_1,j_2,j_3)}(\xi,\eta)\left[\sum_{(I_1,I_2)\in\mathcal{I}(I)} c_{I_1,I_2}\widehat{Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm)}\right](\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)d\xi d\eta\\
&+\int \chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right) B^1_{(j_1,j_2,j_3)}(\xi,\eta)\ \hat{v}^I_{j_1}(\xi-\eta)\widehat{Q^{\mathrm{w}}_0(v_\pm, D_1 v_\pm)}(\eta)\hat{v}^I_{j_3}(-\xi)d\xi d\eta\\
&+\int \chi\left(\frac{\xi-\eta}{\langle\eta\rangle}\right) B^1_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\left[\sum_{(I_1,I_2)\in\mathcal{I}(I)} c_{I_1,I_2}\widehat{Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm)}\right](-\xi)d\xi d\eta\\
&=: S_1+S_2+S_3,
\end{aligned}$$

where coefficients $c_{I_1,I_2}\in\{-1,0,1\}$, $c_{I_1,I_2}=1$ when $|I_1|+|I_2|=|I|$, and $\chi\in C^\infty_0(\mathbb{R}^2)$ is equal to 1 close to the origin and has a sufficiently small support. All integrals in the above right hand side are quartic terms, for they involve the quadratic non-linearities of (2.2.38).

The fact that S_2 is a remainder $\mathfrak{D}^I_{\mathrm{quart}}$ follows by inequalities (A.13), (B.1.3d) with $s=7$, and the fact that

$$\|\mathrm{R}_1 Q^{\mathrm{w}}_0(v_\pm, D_1 v_\pm)\|_{H^{7,\infty}} \lesssim \|V(t,\cdot)\|^{2-(2-\theta)\theta}_{H^{10,\infty}}\|V(t,\cdot)\|^{(2-\theta)\theta}_{H^{12}}, \tag{2.2.74}$$

for any $\theta\in]0,1[$. The above inequality is justified by the fact that, for any function $w\in W^{1,\infty}\cap H^1$, $\rho\in\mathbb{N}$, and any $\theta\in]0,1[$, setting $p=\frac{2}{\theta}\in]2,\infty[$,

$$\begin{aligned}
\|\langle D_x\rangle^\rho \mathrm{R}_1 w\|_{L^\infty} \lesssim \|\langle D_x\rangle^\rho \mathrm{R}_1 w\|_{W^{1,p}} \lesssim \|\langle D_x\rangle^\rho w\|_{W^{1,p}} &\lesssim \|\langle D_x\rangle^\rho w\|^{1-\theta}_{W^{1,\infty}}\|\langle D_x\rangle^\rho w\|^\theta_{H^1}\\
&\lesssim \|\langle D_x\rangle^\rho w\|^{1-\theta}_{H^{2,\infty}}\|\langle D_x\rangle^\rho w\|^\theta_{H^1},
\end{aligned}\tag{2.2.75}$$

as a consequence of Morrey's inequality, continuity of $\mathrm{R}_1: L^p\to L^p$ for $p<+\infty$, interpolation inequality, and the injection of $W^{1,\infty}$ into $H^{2,\infty}$. This implies that

$$\|\mathrm{R}_1 Q^{\mathrm{w}}_0(v_\pm, D_1 v_\pm)\|_{H^{\rho,\infty}} \lesssim \|Q^{\mathrm{w}}_0(v_\pm, D_1 v_\pm)\|^{1-\theta}_{H^{\rho+2,\infty}}\|Q^{\mathrm{w}}_0(v_\pm, D_1 v_\pm)\|^\theta_{H^{\rho+1}}, \tag{2.2.76}$$

for any $\rho\in\mathbb{N}$, and gives (2.2.74) when $\rho=7$, after inequalities (B.1.3c) with $s=8$, (B.1.3d) with $s=9$. Therefore, for any $\theta\in]0,1[$,

$$|S_2| \lesssim \left(\|V(t,\cdot)\|^{2-\theta}_{H^{8,\infty}}\|V(t,\cdot)\|^\theta_{H^{10}} + \|V(t,\cdot)\|^{2-(2-\theta)\theta}_{H^{10,\infty}}\|V(t,\cdot)\|^{(2-\theta)\theta}_{H^{12}}\right)\|V^I(t,\cdot)\|^2_{L^2}.$$

Inequality (A.13) allows also to bound all integrals in summations S_1, S_3 corresponding to indices $(I_1,I_2)\in\mathcal{I}(I)$ with $|I_2|<|I|$. This is not the case for integrals with $I_2=I$, that contain the quasi-linear term $Q^{\mathrm{kg}}_0(v_\pm, D_1 u^I_\pm)$, because a straight application of (A.13) would give a bound at the wrong energy level $n+1$, as $\|Q^{\mathrm{kg}}_0(v_\pm, D_1 u^I_\pm)\|_{L^2}\lesssim \|V(t,\cdot)\|_{H^{1,\infty}}\|D_1 U^I(t,\cdot)\|_{L^2}$. Instead, since

$$\widehat{Q^{\mathrm{kg}}_0(v_\pm, Du^I_\pm)}(\xi) = \frac{i}{4}\sum_{j_4,j_5\in\{+,-\}}\int\Big(1-j_4 j_5\frac{\xi-\zeta}{\langle\xi-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\Big)\zeta_1\hat{v}_{j_4}(\xi-\zeta)\hat{u}^I_{j_5}(\zeta)d\zeta, \tag{2.2.77}$$

we can rather write those integrals as the sum over $j_k\in\{+,-\}, k=1,\dots 4$, of the following:

(2.2.78a)

$$\int\chi\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)B^1_{(j_1,j_2,j_3)}(\xi,\eta)\Big(1-j_4 j_5\frac{\xi-\eta-\zeta}{\langle\xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\Big)\zeta_1\hat{v}_{j_4}(\xi-\eta-\zeta)\hat{u}^I_{j_5}(\zeta)\hat{u}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)\ d\xi d\eta d\zeta,$$

(2.2.78b)

$$\int \chi\Big(\frac{\xi-\eta}{\langle\eta\rangle}\Big)B^1_{(j_1,j_2,j_3)}(\xi,\eta)\Big(1+j_4j_5\frac{\xi+\zeta}{\langle\xi+\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\Big)\zeta_1\hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}_{j_4}(-\xi-\zeta)\hat{u}^I_{j_5}(\zeta)\,d\xi d\eta d\zeta,$$

and estimate them by using, respectively, inequalities (2.2.52) and (2.2.69). We hence obtain that

$$\begin{aligned}|S_1|+|S_3| &\lesssim \|V(t,\cdot)\|_{H^{4,\infty}}\left(\|U(t,\cdot)\|_{H^{11,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{7,\infty}}\right)\|W^I(t,\cdot)\|^2_{L^2}\\ &+\sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\|I_2|<|I|}}\|Q^{\mathrm{kg}}_0(v^{I_1}_\pm,Du^{I_2}_\pm)(t,\cdot)\|_{L^2}\left(\|U(t,\cdot)\|_{H^{11,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{7,\infty}}\right)\|V^I(t,\cdot)\|_{L^2},\end{aligned}$$

and, since the same argument applies to $\partial_tD^{I,R}_{(j_1,j_2,j_3)}$, this also concludes the proof of the statement.

□

Lemma 2.2.9 (Analysis of quartic terms. II)**.** *Let I be a general multi-index, and $D^{I,T_{-N}}_{(j_1,j_2,j_3)}$ be defined as in* (2.2.44c)*, for any $j_k\in\{+,-\}, k=1,2,3$. Then,*

$$\partial_tD^{I,T_{-N}}_{(j_1,j_2,j_3)}=\Im\left[\langle T_{-N}(U)W^I,W^I\right]+\mathfrak{D}^{I,N}_{quart}, \tag{2.2.79}$$

and, if $N\geq 18$, $\mathfrak{D}^{I,N}_{quart}$satisfies, for any $\theta\in]0,1[$,

$$\begin{aligned}\left|\mathfrak{D}^{I,N}_{quart}\right| &\lesssim \|V(t,\cdot)\|^{2-\theta}_{H^{N+4,\infty}}\|V(t,\cdot)\|^\theta_{H^{N+6}}\|W^I(t,\cdot)\|^2_{L^2}\\ &+\sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\|I_2|<|I|}}\left\|Q^{\mathrm{kg}}_0(v^{I_1}_\pm,Du^{I_2}_\pm)\right\|_{L^2}\|U(t,\cdot)\|_{H^{N+3,\infty}}\|V^I(t,\cdot)\|_{L^2}.\end{aligned} \tag{2.2.80}$$

Proof. For any triplet (j_1,j_2,j_3), we compute the time derivative of $D^{I,T_{-N}}$, defined in (2.2.44c), making use of system (2.2.38). Recalling (2.2.43) and (2.2.46), we find that

(2.2.81)

$$\begin{aligned}&\partial_t\left[\sum_{j_k\in\{+,-\}}D^{I,T_{-N}}_{(j_1,j_2,j_3)}\right]-\Im[\langle T_{-N}(U)W^I,W^I\rangle]=\\ &=\mathrm{Re}\left[\frac{1}{(2\pi)^2}\int\widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\left[\sum_{(I_1,I_2)\in\mathfrak{I}(I)}c_{I_1,I_2}\widehat{Q^{\mathrm{kg}}_0(v^{I_1}_\pm,Du^{I_2}_\pm)}(\xi-\eta)\right]\hat{u}_{j_2}(\eta)\hat{v}^I_{-j_3}(-\xi)d\xi d\eta\right.\\ &\quad+\frac{1}{(2\pi)^2}\int\widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^I_{j_1}(\xi-\eta)\widehat{Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm)}(\eta)\hat{v}^I_{-j_3}(-\xi)d\xi d\eta\\ &\quad\left.+\frac{1}{(2\pi)^2}\int\widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\left[\sum_{(I_1,I_2)\in\mathfrak{I}(I)}c_{I_1,I_2}\widehat{Q^{\mathrm{kg}}_0(v^{I_1}_\pm,Du^{I_2}_\pm)}(-\xi)\right]d\xi d\eta\right]\\ &=:S^{T_{-N}}_1+S^{T_{-N}}_2+S^{T_{-N}}_3.\end{aligned}$$

After lemma A.6 and inequality (B.1.3d) with $s=N+3$, we deduce that, if $N\geq15$, for any $\theta\in]0,1[$,

$$|S^{T_{-N}}_2|\lesssim\|V(t,\cdot)\|^{2-\theta}_{H^{N+4,\infty}}\|V(t,\cdot)\|^\theta_{H^{N+6}}\|V^I(t,\cdot)\|^2_{L^2},$$

and also that each contribution in $S^{T_{-N}}_1,S^{T_{-N}}_3$ corresponding to $(I_1,I_2)\in\mathfrak{I}(I)$ with $|I_2|<|I|$, is bounded by

$$\left\|Q^{\mathrm{kg}}_0(v^{I_1}_\pm,Du^{I_2}_\pm)\right\|_{L^2}\|U(t,\cdot)\|_{H^{N+3,\infty}}\|V^I(t,\cdot)\|_{L^2}.$$

Reminding instead (2.2.77), we find that the remaining contribution in S_1^{T-N}, corresponding to $I_2 = I$, is equal to the sum over $j_1,\dots,j_5 \in \{+,-\}$ of the (imaginary part) of the following integrals:

$$\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\Big(1 - j_4 j_5 \frac{\xi-\eta-\zeta}{\langle \xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\Big)\zeta_1 \hat{v}_{j_4}(\xi-\eta-\zeta)\hat{u}^I_{j_5}(\zeta)\hat{u}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)\, d\xi d\eta d\zeta, \tag{2.2.82}$$

while that corresponding to $I_2 = I$ in S_3^{T-N} is the sum, over $j_k \in \{+,-\}, k = 1,\dots,5$, of:

$$\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\Big(1 + j_4 j_5 \frac{\xi+\zeta}{\langle \xi+\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\Big)\zeta_1 \hat{v}^I_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)\hat{v}_{j_4}(-\xi-\zeta)\hat{u}^I_{j_5}(\zeta)\, d\xi d\eta d\zeta. \tag{2.2.83}$$

Since $\widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)$ satisfies (A.17), and is supported for $|\eta| \le \varepsilon|\xi-\eta|$, for a small $0<\varepsilon\ll 1$, we rewrite above integrals, respectively, as

$$\begin{aligned}\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\langle\eta\rangle^{-N-3}\Big(1 - j_4 j_5 \frac{\xi-\eta-\zeta}{\langle \xi-\eta-\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\Big)\zeta_1 \langle \xi-\eta-\zeta\rangle^{-4} \\ \times \widehat{\langle D_x\rangle^4 v}_{j_4}(\xi-\eta-\zeta)\hat{u}^I_{j_5}(\zeta)\widehat{\langle D_x\rangle^{N+3} u}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)\, d\xi d\eta d\zeta,\end{aligned} \tag{2.2.84}$$

and

$$\begin{aligned}\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\langle\eta\rangle^{-N-7}\Big(1 + j_4 j_5 \frac{\xi+\zeta}{\langle \xi+\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\Big)\zeta_1 \langle \xi+\zeta\rangle^{-4} \\ \times \hat{v}^I_{j_1}(\xi-\eta)\widehat{\langle D_x\rangle^{N+7} u}_{j_2}(\eta)\widehat{\langle D_x\rangle^4 v}_{j_4}(-\xi-\zeta)\hat{u}^I_{j_5}(\zeta)\, d\xi d\eta d\zeta,\end{aligned} \tag{2.2.85}$$

in such a way that the corresponding multipliers, that we denote concisely by $\widetilde{\sigma}^{N,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)$, $k = 0,1$, verify, for any $\alpha,\beta,\gamma \in \mathbb{N}^2$,

$$\Big|\partial^\alpha_\xi \partial^\beta_\eta \widetilde{\sigma}^{N,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\Big| \lesssim \big(\mathbb{1}_{\{|\zeta|\le 1\}} + \mathbb{1}_{\{|\zeta|>1\}}\langle\zeta\rangle^{-3}\big)\, |g^N_{\alpha,\beta}(\xi)|,$$

$$\Big|\partial^\alpha_\xi \partial^\beta_\eta \partial^\gamma_\zeta \widetilde{\sigma}^{N,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)\Big| \lesssim \Big(\mathbb{1}_{\{|\zeta|\le 1\}}|\zeta|^{1-|\gamma|} + \mathbb{1}_{\{|\zeta|>1\}}\langle\zeta\rangle^{-3}\Big)\, |g^N_{\alpha,\beta}(\xi)|, \quad |\gamma|\ge 1,$$

with $g^N_{\alpha,\beta}(\xi,\eta)$ supported for $|\eta|\le\varepsilon|\xi-\eta|$, and such that

$$|g^N_{\alpha,\beta}(\xi,\eta)| \lesssim \langle\xi-\eta\rangle^{6-N+|\alpha|+2|\beta|}|\eta|^{N-|\beta|}\langle\eta\rangle^{-N-3}.$$

If $N\in\mathbb{N}^*$ is sufficiently large (e.g. $N\ge 18$), for any α,β of length less or equal than 3, $|g^N_{\alpha,\beta}(\xi,\eta)| \lesssim \big(\mathbb{1}_{\{|\eta|\le 1\}} + \mathbb{1}_{\{|\eta|>1\}}\langle\eta\rangle^{-3}\big)\langle\xi\rangle^{-3}$, so by lemma A.1 (*i*), together with corollary A.2 (*i*), we obtain that

$$K^{N,k}_{(j_1,\dots,j_5)}(x,y,z) := \int e^{ix\cdot\xi + iy\cdot\eta + iz\cdot\zeta}\widetilde{\sigma}^{N,k}_{(j_1,\dots,j_5)}(\xi,\eta,\zeta)d\xi d\eta d\zeta,$$

is such that $|K^{N,k}_{(j_1,\dots,j_5)}(x,y,z)| \lesssim \langle x\rangle^{-3}|y|^{-1}\langle y\rangle^{-2}|z|^{-1}\langle z\rangle^{-2}$, for any $x,y,z\in\mathbb{R}^2$, any $k=0,1$. By (2.2.84), (2.2.85), integrals (2.2.82), (2.2.83) are respectively equal to

$$\int K^{N,0}_{(j_1,\dots,j_5)}(t-x,x-z,x-y)[\langle D_x\rangle^4 v_{j_4}](x)u^I_{j_5}(y)[\langle D_x\rangle^{N+3}u_{j_2}](z)v^I_{j_3}(t)dxdydzdt,$$

and

$$\int K^{N,1}_{(j_1,\dots,j_5)}(z-x,x-y,z-t)v^I_{j_1}(x)[\langle D_x\rangle^{N+7}u_{j_2}](y)[\langle D_x\rangle^4 v_{j_4}](z)u^I_{j_5}(t)dxdydzdt,$$

and, since integrals such as (2.2.59) can be bounded from above by the product of the L^2 norm of any two functions $\widetilde{u}_k$ times the L^∞ norm of the remaining ones, they are estimated by

$$\|V(t,\cdot)\|_{H^{4,\infty}}\|U(t,\cdot)\|_{H^{N+7,\infty}}\|W^I(t,\cdot)\|^2_{L^2},$$

which concludes the proof of the statement. ☐

Lemma 2.2.10 (Analysis of quartic terms. III)**.** *Let $n \ge 3$, $I \in \mathcal{I}_n$, and $(I_1, I_2) \in \mathcal{I}(I)$ such that $[\frac{|I|}{2}] < |I_1| < |I|$. Let also $C^{I_1,I_2}_{(j_1,j_2,j_3)}$, $D^{I_1,I_2}_{(j_1,j_2,j_3)}$ be the integrals defined, respectively, in* (2.2.49a), (2.2.50a)*, for any $j_k \in \{+,-\}, k = 1,2,3$. Then*

$$\partial_t D^{I_1,I_2}_{(j_1,j_2,j_3)} = -C^{I_1,I_2}_{(j_1,j_2,j_3)} + \mathfrak{D}^{I_1,I_2}_{quart}, \tag{2.2.86}$$

where $\mathfrak{D}^{I_1,I_2}_{quart}$ satisfies, for any $\theta \in]0,1[$,

$$\left|\mathfrak{D}^{I_1,I_2}_{quart}(t)\right| \lesssim \Big[\Big(\|W(t,\cdot)\|_{H^{[\frac{n}{2}]+12,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{[\frac{n}{2}]+8,\infty}}\Big)^2 + \|V(t,\cdot)\|^{2-(2-\theta)\theta}_{H^{[\frac{n}{2}]+11,\infty}} \|V(t,\cdot)\|^{(2-\theta)\theta}_{H^{[\frac{n}{2}]+12}}\Big] E_n(t;W). \tag{2.2.87}$$

Proof. We compute the time derivative of $D^{I_1,I_2}_{(j_1,j_2,j_3)}$ making use of system (2.2.38). We remind that, after remark 1.1.5 and definition (1.1.18), if Γ^I is a product of spatial derivatives, all couples of indices (I_1, I_2) belonging to $\mathcal{I}(I)$ are such that $|I_1| + |I_2| = |I|$, and $\Gamma^{I_1}, \Gamma^{I_2}$ are also products of spatial derivatives. Therefore, all coefficients c_{I_1,I_2} appearing in the right hand side of (2.2.38) are equal to 0. By definitions (2.2.45), (2.2.49a), (2.2.50a), we find that

$$\begin{aligned}
&\partial_t D^{I_1,I_2}_{(j_1,j_2,j_3)} + C^{I_1,I_2}_{(j_1,j_2,j_3)} = \\
&- \frac{1}{4(2\pi)^2}\int B^1_{(j_1,j_2,j_3)}(\xi,\eta)\left[\sum_{(J_1,J_2)\in\mathcal{I}(I_1)} \widehat{Q^{\mathrm{kg}}_0(v^{J_1}_\pm, D_1 u^{J_2}_\pm)}(\xi-\eta)\right] \hat{u}^{I_2}_{j_2}(\eta)\hat{v}^I_{j_3}(-\xi)d\xi d\eta \\
&- \frac{1}{4(2\pi)^2}\int B^1_{(j_1,j_2,j_3)}(\xi,\eta)\ \hat{v}^{I_1}_{j_1}(\xi-\eta)\left[\sum_{(J_1,J_2)\in\mathcal{I}(I_2)} \widehat{Q^{\mathrm{w}}_0(v^{J_1}_\pm, D_1 v^{J_2}_\pm)}(\eta)\right] \hat{v}^I_{j_3}(-\xi)d\xi d\eta \\
&- \frac{1}{4(2\pi)^2}\int B^1_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^{I_1}_{j_1}(\xi-\eta)\hat{u}^{I_2}_{j_2}(\eta)\left[\sum_{(J_1,J_2)\in\mathcal{I}(I)} \widehat{Q^{\mathrm{kg}}_0(v^{J_1}_\pm, D_1 u^{J_2}_\pm)}\right](-\xi)d\xi d\eta \\
&=: S^{I_1,I_2}_1 + S^{I_1,I_2}_2 + S^{I_1,I_2}_3.
\end{aligned} \tag{2.2.88}$$

Since $|J_1| + |J_2| = |I_1| < |I| \le n$ in S^{I_1,I_2}_1, we can estimate all its contributions using inequality (A.13). Using lemma 2.1.2 (i), the fact that $|I_2| \le [\frac{n}{2}]$ by the hypothesis and, hence, that

$$\|u^{I_2}_\pm(t,\cdot)\|_{H^{7,\infty}} + \|\mathrm{R}_1 u^{I_2}_\pm(t,\cdot)\|_{H^{7,\infty}} \lesssim \|U(t,\cdot)\|_{H^{[\frac{n}{2}]+8,\infty}},$$

we hence deduce that

$$\left|S^{I_1,I_2}_1\right| \lesssim \Big(\|W(t,\cdot)\|_{H^{[\frac{n}{2}]+2}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{[\frac{n}{2}]+2,\infty}}\Big)\|U(t,\cdot)\|_{H^{[\frac{n}{2}]+8,\infty}} E_n(t;W),$$

and above estimate holds also for all integrals in S^{I_1,I_2}_3 corresponding to $|J_2| < |I|$. The same inequality (A.13), combined with (2.2.76) applied to $Q^{\mathrm{w}}_0(v^{J_1}_\pm, D_1 v^{J_2}_\pm)$, and with corollary A.4 in appendix A, gives that, for any $\theta \in]0,1[$,

$$\begin{aligned}
&|S^{I_1,I_2}_2| \\
&\lesssim \sum_{|J_1|+|J_2|=|I_2|}\left[\left\|Q^{\mathrm{w}}_0(v^{J_1}_\pm, D_1 v^{J_2}_\pm)\right\|_{H^{7,\infty}} + \left\|Q^{\mathrm{w}}_0(v^{J_1}_\pm, D_1 v^{J_2}_\pm)\right\|^{1-\theta}_{H^{9,\infty}}\left\|Q^{\mathrm{w}}_0(v^{J_1}_\pm, D_1 v^{J_2}_\pm)\right\|^{\theta}_{H^8}\right] E_n(t;W) \\
&\lesssim \|V(t,\cdot)\|^{2-(2-\theta)\theta}_{H^{[\frac{n}{2}]+11,\infty}}\|V(t,\cdot)\|^{(2-\theta)\theta}_{H^{[\frac{n}{2}]+12}} E_n(t;W).
\end{aligned}$$

Finally, the last remaining integral in $S_3^{I_1,I_2}$, corresponding to indices $J_1 = 0, J_2 = I$, can be written, using (2.2.77), as

$$\frac{-1}{4(2\pi)^2}\sum_{j_4,j_4\in\{+,-\}}\int B^1_{(j_1,j_2,j_3)}(\xi,\eta)\left(1+j_4j_5\frac{\xi+\zeta}{\langle\xi+\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{v}^{I_1}_{j_1}(\xi-\eta)\hat{u}^{I_2}_{j_2}(\eta)\hat{v}_{j_4}(-\xi-\zeta)\hat{u}^{I}_{j_5}(\zeta)\,d\xi d\eta d\zeta,$$

and is estimated, after lemma 2.2.7 and the fact that $|I_1| < |I|$, by

$$\|V(t,\cdot)\|_{H^{7,\infty}}\left(\|U(t,\cdot)\|_{H^{[\frac{n}{2}]+12,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{[\frac{n}{2}]+8,\infty}}\right)E_n(t;W),$$

which hence gives that

$$\left|S_3^{I_1,I_2}\right| \lesssim \left(\|W(t,\cdot)\|_{H^{[\frac{n}{2}]+12,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{[\frac{n}{2}]+8,\infty}}\right)^2 E_n(t;W).$$

That concludes the proof of the statement. □

Lemma 2.2.11 (Analysis of quartic terms. IV). *Let $\mathcal{V}_k$ be the set introduced in (2.1.27), $I\in\mathcal{V}_k$ for $k=0,1$, $(I_1,I_2)\in\mathcal{I}(I)$ be such that $I_1\in\mathcal{K}$ and $|I_2|\le 1$. Let also $F^{I_1,I_2}_{(j_1,j_2,j_3)}$, $G^{I_1,I_2}_{(j_1,j_2,j_3)}$ be the integrals defined in (2.2.49b), (2.2.50b), for any $j_i\in\{+,-\}, i=1,2,3$. For any triplet (j_1,j_2,j_3), we have that*

$$\partial_t G^{I_1,I_2}_{(j_1,j_2,j_3)} = -F^{I_1,I_2}_{(j_1,j_2,j_3)} + \mathfrak{G}^{I_1,I_2}_{quart}, \tag{2.2.89}$$

and there is a constant $C>0$ such that, if a-priori estimates (1.1.11) are satisfied in interval $[1,T]$ for a fixed $T>1$, with $\varepsilon_0<(2A+B)^{-1}$ small, then $\mathfrak{G}^{I_1,I_2}_{quart}$ satisfies

$$|\mathfrak{G}^{I_1,I_2}_{quart}(t)| \le C(A+B)^2\varepsilon^2 t^{-1+\frac{\delta_k}{2}}\left[E^k_3(t;W)^{\frac12}+\delta_{\mathcal{V}_0}t^{\beta+\frac{\delta_1}{2}}E^1_3(t,W)^{\frac12}+t^{-\frac14-\frac{\delta_k}{2}}\right], \tag{2.2.90}$$

with $\delta_{\mathcal{V}_0}=1$ if $I\in\mathcal{V}_0$, 0 otherwise, and $\beta>0$ as small as we want, for every $t\in[1,T]$.

Proof. First of all, we remind the reader that, as shown at the end of proof of corollary 2.1.4, for any $(I_1,I_2)\in\mathcal{I}(I)$ such that $I_1\in\mathcal{K}$, $|I_2|\le 1$,

$$\|V^{I_1}(t,\cdot)\|_{L^2}\left(\|\chi(t^{-\sigma}D_x)U^{I_2}(t,\cdot)\|_{H^{\rho,\infty}}+\|\chi(t^{-\sigma}D_x)\mathrm{R}U^{I_2}(t,\cdot)\|_{H^{\rho,\infty}}\right)\le C(A+B)B\varepsilon^2t^{-\frac12+\frac{\delta_k}{2}}, \tag{2.2.91}$$

for every $t\in[1,T]$.

For any fixed (j_1,j_2,j_3), we compute $\partial_t G^{I_1,I_2}_{(j_1,j_2,j_3)}$ recurring to system (2.2.38), along with its compact form

$$\begin{cases}(D_t\mp\langle D_x\rangle)v^I_\pm = \Gamma^I Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm),\\ (D_t\mp|D_x|)u^I_\pm=\Gamma^I Q^{\mathrm{kg}}_0(v_\pm,D_1u_\pm),\end{cases}$$

and using that $[D_t,\chi(t^{-\sigma}D_x)] = t^{-1}\chi_1(t^{-\sigma}D_x)$, with $\chi_1(\xi):=i\sigma(\partial\chi)(\xi)\cdot\xi$. We find that

$$\begin{aligned}
&-4(2\pi)^2\left[\partial_tG^{I_1,I_2}_{(j_1,j_2,j_3)}+F^{I_1,I_2}_{(j_1,j_2,j_3)}\right]\\
&=\int B_{(j_1,j_2,j_3)}(\xi,\eta)\left[\widehat{\Gamma^{I_1}Q^{\mathrm{kg}}_0(v_\pm,D_1u_\pm)}(\xi-\eta)\right]\widehat{\chi(t^{-\sigma}D_x)u^{I_2}_{j_2}}(\eta)\hat{v}^I_{j_3}(-\xi)d\xi d\eta\\
&+\int B_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^{I_1}_{j_1}(\xi-\eta)\left[\widehat{\chi(t^{-\sigma}D_x)\Gamma^{I_2}Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm)}(\eta)+t^{-1}\widehat{\chi_1(t^{-\sigma}D_x)u^{I_2}_{j_2}}(\eta)\right]\hat{v}^I_{j_3}(-\xi)d\xi d\eta\\
&+\int B_{(j_1,j_2,j_3)}(\xi,\eta)\hat{v}^{I_1}_{j_1}(\xi-\eta)\widehat{\chi(t^{-\sigma}D_x)u^{I_2}_{j_2}}(\eta)\Big[\sum_{(J_1,J_2)\in\mathcal{I}(I)}c_{J_1,J_2}\widehat{Q^{\mathrm{kg}}_0(v^{J_1}_\pm,Du^{J_2}_\pm)}(-\xi)\Big]d\xi d\eta\\
&=: S_1^{I_1,I_2}+S_2^{I_1,I_2}+S_3^{I_1,I_2},
\end{aligned}$$

with $B_{(j_1,j_2,j_3)}$ given by (2.2.45) or (2.2.47). We are going to show that quartic terms S^{I_1,I_2}_k, $k=1,2,3$, are remainders $\mathfrak{G}^{I_1,I_2}_{\text{quart}}$ satisfying (2.2.90).

Applying (A.13) to S^{I_1,I_2}_2, using (2.2.75) with $w=\Gamma^{I_2}Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm)$ and $\rho=7$, together with the fact that operators $\chi(t^{-\sigma}D_x)$, $\chi_1(t^{-\sigma}D_x)$ are bounded from L^∞ to $H^{\rho,\infty}$ for any $\rho\ge 0$, with norm $O(t^{\sigma\rho})$, and from L^2 to H^s for any $s\ge 0$, with norm $O(t^{\sigma s})$, we deduce that, for any $\theta\in]0,1[$,

$$
\begin{aligned}
(2.2.92)\quad |S^{I_1,I_2}_2| &\lesssim t^\beta\|V^{I_1}(t,\cdot)\|_{L^2}\|V^I(t,\cdot)\|_{L^2}\\
&\times\Big[\|\Gamma^{I_2}Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm)\|_{L^\infty}+\|\Gamma^{I_2}Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm)\|^{1-\theta}_{L^\infty}\|\Gamma^{I_2}Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm)\|^{\theta}_{L^2}\\
&\qquad +t^{-1}\|\chi_1(t^{-\sigma}D_x)u^{I_2}_\pm(t,\cdot)\|_{L^\infty}+t^{-1}\|\chi_1(t^{-\sigma}D_x)\mathrm{R}u^{I_2}_\pm(t,\cdot)\|_{L^\infty}\Big].
\end{aligned}
$$

Since $|I_2|\le 1$ and

$$
\Gamma^{I_2}Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm)=Q^{\mathrm{w}}_0(v^{I_2}_\pm,D_1v_\pm)+Q^{\mathrm{w}}_0(v_\pm,D_1v^{I_2}_\pm)+G^w_1(v_\pm,Dv_\pm),
$$

with $G^w_1(v_\pm,Dv_\pm)=G_1(v,\partial v)$ given by (1.1.16), by using lemma B.2.3 in appendix B, with $L=L^\infty$, when estimating the L^∞ norm of the first two quadratic terms in the above right hand side, we have that, for some new $\chi\in C^\infty_0(\mathbb{R}^2)$,

$$
\begin{aligned}
\|\Gamma^{I_2}Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm)\|_{L^\infty}&\lesssim\Big\|\chi(t^{-\sigma}D_x)v^{I_2}_\pm(t,\cdot)\Big\|_{H^{2,\infty}}\|v_\pm(t,\cdot)\|_{H^1}\\
&+t^{-N(s)}\left(\|v_\pm(t,\cdot)\|_{H^s}+\|D_tv_\pm(t,\cdot)\|_{H^s}\right)\Big(\sum_{|\mu|=0}^{1}\|x^\mu v_\pm(t,\cdot)\|_{H^1}+t\|v_\pm(t,\cdot)\|_{H^1}\Big)\\
&+\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\left(\|v_\pm(t,\cdot)\|_{H^{2,\infty}}+\|D_tv_\pm(t,\cdot)\|_{H^{1,\infty}}\right)\\
&\le CAB\varepsilon^2t^{-2},
\end{aligned}
$$

for some constant $C>0$ and some positive $\beta'\to 0$ as $\sigma,\delta_0\to 0$, as follows by picking $s>0$ sufficiently large so that $N(s)\ge 4$, and using (B.1.6a), (B.1.6b), (B.1.10a), lemma B.3.21, together with a-priori estimates.

Also, using (B.1.6a) with $s=0$ and a-priori estimates, we derive that

$$
\|\Gamma^{I_2}Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm)\|_{L^2}\lesssim\|V(t,\cdot)\|_{H^{2,\infty}}\left(\|V^{I_2}(t,\cdot)\|_{H^1}+\|D_tV(t,\cdot)\|_{L^2}\right)\le CAB\varepsilon^2t^{-1+\frac{\delta_2}{2}}.
$$

Therefore, using lemma B.2.10 and taking $\theta,\sigma>0$ sufficiently small, we deduce from (2.2.92) and the above estimates that

$$
(2.2.93)\qquad |S^{I_1,I_2}_2|\le CAB\varepsilon^2t^{-\frac{5}{4}}E^k_3(t;W)^{\frac12},
$$

for a new $C>0$.

We use inequality (A.13) also to estimate S^{I_1,I_2}_1. From (1.1.17) we have that

$$
\Gamma^{I_1}Q^{\mathrm{kg}}_0(v_\pm,D_1u_\pm)=Q^{\mathrm{kg}}_0(v^{I_1}_\pm,D_1u_\pm)+\sum_{\substack{(J_1,J_2)\in\mathcal{I}(I_1)\\|J_1|<|I_1|}}c_{J_1,J_2}Q^{\mathrm{kg}}_0(v^{J_1}_\pm,Du^{J_2}_\pm),
$$

with $c_{J_1,J_2}\in\{-1,0,1\}$, and then, from (2.1.30b), (2.1.35b), and the fact that $I_1\in\mathcal{K}$,

$$
\Gamma^{I_1}Q^{\mathrm{kg}}_0(v_\pm,D_1u_\pm)=Q^{\mathrm{kg}}_0(v^{I_1}_\pm,\chi(t^{-\sigma}D_x)D_1u_\pm)+\mathfrak{R}^k_3(t,x),
$$

with $\mathfrak{R}^k_3$ satisfying (2.1.31) and

$$\|Q^{\mathrm{kg}}_0(v^{I_1}_\pm,\chi(t^{-\sigma}D_x)D_1u_\pm)\|_{L^2}\le\left(\|U(t,\cdot)\|_{H^{2,\infty}}+\|\mathrm{R}U(t,\cdot)\|_{H^{2,\infty}}\right)\|V^{I_1}(t,\cdot)\|_{L^2}.$$

So

$$\begin{aligned}|S^{I_1,I_2}_1|&\lesssim\Big[\left(\|U(t,\cdot)\|_{H^{2,\infty}}+\|\mathrm{R}U(t,\cdot)\|_{H^{2,\infty}}\right)\|V^{I_1}(t,\cdot)\|_{L^2}+\|\mathfrak{R}^k_3(t,\cdot)\|_{L^2}\Big]\\&\times\left(\|\chi(t^{-\sigma}D_x)U^{I_2}(t,\cdot)\|_{H^{7,\infty}}+\|\chi(t^{-\sigma}D_x)\mathrm{R}U^{I_2}(t,\cdot)\|_{H^{7,\infty}}\right)\|V^I(t,\cdot)\|_{L^2}\\&\le CAB\varepsilon^2t^{-1+\frac{\delta_k}{2}}E^k_3(t;W)^{\frac12},\end{aligned}\tag{2.2.94}$$

last estimate following from (2.1.31), (2.2.91), lemma B.2.10 and priori estimates (1.1.11). As concerns S^{I_1,I_2}_3, we estimate all its contributions corresponding to $|J_2|<|I|$ again by means of (A.13). We observe that, by (2.1.30b) and (2.1.35b),

$$\sum_{(J_1,J_2)\in\mathcal{I}(I)}c_{J_1,J_2}Q^{\mathrm{kg}}_0(v^{J_1}_\pm,Du^{J_2}_\pm)=\sum_{\substack{(J_1,J_2)\in\mathcal{I}(I)\\J_1\in\mathcal{K},|J_2|\le1}}c_{J_1,J_2}Q^{\mathrm{kg}}_0(v^{J_1}_\pm,\chi(t^{-\sigma}D_x)Du^{J_2}_\pm)+\mathfrak{R}^k_3(t,x),\tag{2.2.95}$$

the set of indices on which the sum in the above right hand side is taken being non-empty since $I\in\mathcal{V}_k$, $k=0,1$. As already observed in the proof of corollary 2.1.4 (see (2.1.44), (2.1.45)),

$$\begin{aligned}&\left\|Q^{\mathrm{kg}}_0(v^{J_1}_\pm,\chi(t^{-\sigma}D_x)Du^{J_2}_\pm)\right\|_{L^2}\lesssim\sum_{|\mu|=0}^1\left\|\chi(t^{-\sigma}D_x)\mathrm{R}^\mu U^{J_2}(t,\cdot)\right\|_{H^{2,\infty}}\|V^{J_1}(t,\cdot)\|_{L^2}\\&\le\begin{cases}CA\varepsilon t^{-1}E^k_3(t;W)^{\frac12}&\text{if }\Gamma^{J_2}\in\{D^\alpha_x,|\alpha|\le1\},\\C(A+B)\varepsilon t^{-1+\beta+\frac{\delta_1}{2}}E^{k+1}_3(t;W),&\text{if }\Gamma^{I_2}\in\{\Omega,Z_m,m=1,2\}.\end{cases}\end{aligned}$$

Therefore, those integrals are bounded by

$$\begin{aligned}&\|V^{I_1}(t,\cdot)\|_{L^2}\Big(\sum_{|\mu|=0}^1\|\chi(t^{-\sigma}D_x)\mathrm{R}^\mu U^{I_2}(t,\cdot)\|_{H^{7,\infty}}\Big)\\&\qquad\times\Big[\sum_{\substack{(J_1,J_2)\in\mathcal{I}(I)\\J_1\in\mathcal{K},|J_2|\le1\\|\mu|=0,1}}\|\chi(t^{-\sigma}D_x)\mathrm{R}^\mu U^{J_2}(t,\cdot)\|_{H^{2,\infty}}\|V^{J_1}(t,\cdot)\|_{L^2}+\|\mathfrak{R}^k_3(t,\cdot)\|_{L^2}\Big],\end{aligned}$$

and hence, after (2.1.31), (2.2.91) and the above estimate, by

$$C(A+B)B\varepsilon^2t^{-1+\frac{\delta_k}{2}}\left[E^k_3(t;W)^{\frac12}+\delta_{\mathcal{V}_0}t^{\beta+\frac{\delta_1}{2}}E^1_3(t,W)^{\frac12}+t^{-\frac14-\frac{\delta_k}{2}}\right],$$

for a new constant $C>0$, and $\beta>0$ small as long as $\sigma>0$ small. Finally, last contribution to S^{I_1,I_2}_3, corresponding to $|J_1|=0,J_2=I$, in which $D=D_1$, can be rewritten, using (2.2.77), as the sum over $j_4,j_5\in\{+,-\}$ of

$$\int B^1_{(j_1,j_2,j_3)}(\xi,\eta)\left(1+j_4j_5\frac{\xi+\zeta}{\langle\xi+\zeta\rangle}\cdot\frac{\zeta}{|\zeta|}\right)\zeta_1\hat{v}_4(-\xi-\zeta)\hat{u}^I_{j_5}(\zeta)\widehat{\chi(t^{-\sigma}D_x)u^{I_2}_{j_2}}(\eta)\hat{v}^{I_1}_{j_1}(\xi-\eta)d\xi d\eta,$$

and its absolute value can be estimated, by means of lemma 2.2.7, with

$$\|V(t,\cdot)\|_{H^{7,\infty}}\Big(\sum_{|\mu|=0}^1\|\chi(t^{-\sigma}D_x)D_1\mathrm{R}^\mu U^{J_2}(t,\cdot)\|_{H^{11,\infty}}\Big)\|V^{I_1}(t,\cdot)\|_{H^1}\|U^I(t,\cdot)\|_{L^2}.$$

Using a-priori estimate (1.1.11b) and lemma B.2.10, we find that the above product is bounded by $CA(A+B)\varepsilon^2 t^{-\frac{3}{2}+\beta'}E^k_3(t;W)$, with $\beta'>0$ small as long as σ,δ_0 are small. Summing up, we obtained

$$|S^{I_1,I_2}_3|\le C(A+B)^2\varepsilon^2 t^{-1+\frac{\delta_k}{2}}\left[E^k_3(t;W)^{\frac{1}{2}}+\delta_{\mathrm{v}_0}t^{\beta+\frac{\delta_1}{2}}E^1_3(t,W)^{\frac{1}{2}}+t^{-\frac{1}{4}-\frac{\delta_k}{2}}\right]$$

and, together with (2.2.93), (2.2.94), the result of the statement. □

2.2.3 Propagation of the energy estimate

Proposition 2.2.12 (Propagation of the energy estimate). *Let us fix $K_2>0$. There exist two integers n,ρ sufficiently large, with $n\gg\rho$, two constants $A,B>0$ sufficiently large, and $\varepsilon_0\in]0,(2A+B)^{-1}[$ sufficiently small, such that, for any $0<\varepsilon<\varepsilon_0$, if (u,v) is solution to* (1.1.1)-(1.1.2) *in some interval $[1,T]$, for a fixed $T>1$, and $u_\pm,v_\pm$ defined in* (1.1.5) *satisfy a-priori estimates* (1.1.11)*, for every $t\in[1,T]$, for a small $\delta>0$, then they also verify* (1.1.12c), (1.1.12d) *on the same interval $[1,T]$.*

Proof. For any integer $k,n\in\mathbb{N}$, with $0\le k\le 2$, $n\ge 3$, let $\widetilde{E}_n(t;W)$, $\widetilde{E}^k_3(t;W)$ (resp. $\widetilde{E}^\dagger_n(t;W)$, $\widetilde{E}^{k,\dagger}_3(t;W)$) be the first (resp. the second) modified energies introduced in (2.2.10) (resp. in (2.2.51)). Let us also remind the definitions of integrals $D^I_{(j_1,j_2,j_3)}$, $D^{I,R}_{(j_1,j_2,j_3)}$, $D^{I,T_{-N}}_{(j_1,j_2,j_3)}$ in (2.2.44), of $D^{I_1,I_2}_{(j_1,j_2,j_3)}$, $G^{I_1,I_2}_{(j_1,j_2,j_3)}$ in (2.2.50), and fix $N=18$.

The first thing we observe is that, as long as estimates (1.1.11a), (1.1.11b) are satisfied, and $\rho\in\mathbb{N}$ is sufficiently large (e.g. $\rho\ge\max\{[\frac{n}{2}]+8,21\}$), there is a constant $C>0$ such that

$$C^{-1}E_n(t;W)\le\widetilde{E}^\dagger_n(t;W)\le CE_n(t;W), \tag{2.2.96a}$$

$$C^{-1}E^k_3(t;W)\le\widetilde{E}^{k,\dagger}_3(t;W)\le CE^k_3(t;W). \tag{2.2.96b}$$

Above equivalences follow from (2.2.11), a-priori estimate (1.1.11a), the fact that for a general multi-index I

$$\sum_{j_i\in\{+,-\}}\left|D^I_{(j_1,j_2,j_3)}\right|+\left|D^{I,R}_{(j_1,j_2,j_3)}\right|\lesssim\left(\|U(t,\cdot)\|_{H^{7,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{7,\infty}}\right)\|V^I(t,\cdot)\|^2_{L^2},$$

by inequality (A.13),

$$\sum_{j_k\in\{+,-\}}\left|D^{I,T_{-18}}_{(j_1,j_2,j_3)}\right|\lesssim\|U(t,\cdot)\|_{H^{21,\infty}}\|W^I(t,\cdot)\|^2_{L^2},$$

by inequality (A.18), and:

• as concerns especially (2.2.96a), from the fact that, for any $I\in\mathfrak{I}_n$, any $(I_1,I_2)\in\mathfrak{I}(I)$ with $[\frac{|I|}{2}]<|I_1|<|I|$,

$$\begin{aligned}\sum_{j_i\in\{+,-\}}\left|D^{I_1,I_2}_{(j_1,j_2,j_3)}\right|&\lesssim\left(\|U^{I_2}(t,\cdot)\|_{H^{7,\infty}}+\|\mathrm{R}_1U^{I_2}(t,\cdot)\|_{H^{7,\infty}}\right)\|V^{I_1}(t,\cdot)\|_{L^2}\|V^I(t,\cdot)\|_{L^2}\\&\lesssim\left(\|U(t,\cdot)\|_{H^{[\frac{n}{2}]+8,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{[\frac{n}{2}]+8,\infty}}\right)E_n(t;W),\end{aligned}$$

by (A.13);

• as concerns especially (2.2.96b), the fact that, for any $I \in \mathcal{V}_k$ (see definition (2.1.27)), any $(I_1, I_2) \in \mathcal{I}(I)$ with $I_1 \in \mathcal{K}$ (see (2.1.26)) and $|I_2| \le 1$,

$$
\sum_{j_i\in\{+,-\}} \Big|G^{I_1,I_2}_{(j_1,j_2,j_3)}\Big| \lesssim \sum_{|\mu|=0}^{1} \|\chi(t^{-\sigma}D_x)D\mathrm{R}^{\mu}U^{I_2}(t,\cdot)\|_{H^{7,\infty}}\|V^{I_1}(t,\cdot)\|_{L^2}\|V^{I}(t,\cdot)\|_{L^2} \le CB\varepsilon t^{-\frac{1}{2}+\frac{\delta_k}{2}}E^k_3(t;W)^{\frac{1}{2}}, \tag{2.2.97}
$$

after (A.13) and (2.2.91).

Let us now consider a general multi-index I ($I \in \mathcal{I}_n$, or $I \in \mathcal{I}^k_3$ for $0 \le k \le 2$). From equation (2.2.5) we deduce the following equality:

$$
\begin{aligned}
&\frac{1}{2}\partial_t\|\widetilde{W}^I_s(t,\cdot)\|^2_{L^2} = -\Im\Big[\langle D_t\widetilde{W}^I_s, \widetilde{W}^I_s\rangle\Big]\\
&= -\Im\Big[\langle A(D)\widetilde{W}^I_s, \widetilde{W}^I_s\rangle + \Big\langle Op^B\Big((I_4+E^0_d(U;\eta))\widetilde{A}_1(V;\eta)(I_4+F^0_d(U;\eta))\Big)\widetilde{W}^I_s, \widetilde{W}^I_s\Big\rangle\\
&\quad +\langle Op^B(A''(V^I;\eta))U + Op^B_R(A''(V^I;\eta))U, \widetilde{W}^I_s\rangle + \langle Q^I_0(V,W), \widetilde{W}^I_s\rangle\\
&\quad +\langle T_{-18}(U)W^I_s, \widetilde{W}^I_s\rangle + \langle \mathfrak{R}'(U,V), \widetilde{W}^I_s\rangle\Big],
\end{aligned} \tag{2.2.98}
$$

and immediately notice that $\Im[\langle A(D)\widetilde{W}^I_s, \widetilde{W}^I_s\rangle] = 0$ because of the fact that matrix $A(\eta)$, introduced in (2.1.5), is real and diagonal, and its quantization is a self-adjoint operator.

Since $(I_4+E^0_d(U;\eta))\widetilde{A}_1(V;\eta)(I_4+F^0_d(U;\eta))$ is a real symmetric matrix of order 1, with semi-norm

$$
M^1_1\Big((I_4+E^0_d(U;\eta))\widetilde{A}_1(V;\eta)(I_4+F^0_d(U;\eta)), 3\Big) \lesssim (1+\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}})^2\|V(t,\cdot)\|_{H^{2,\infty}},
$$

as follows by estimate (2.2.2a) on E^0_d, (2.2.3) of F^0_d, and (2.1.49) on $\widetilde{A}_1(V;\eta)$, corollary 1.2.13 and a-priori estimates (1.1.11a), (1.1.11b) imply that the second term in the right hand side of (2.2.98) reduces to $\langle T_0(U,V)\widetilde{W}^I_s, \widetilde{W}^I_s\rangle$, with $T_0(U,V)$ operator of order less or equal than 0, such that

$$
\|T_0(U,V)\|_{\mathcal{L}(L^2)} \lesssim M^1_1\Big((I_4+E^0_d(U;\eta))\widetilde{A}_1(V;\eta)(I_4+F^0_d(U;\eta)), 3\Big) \le CA\varepsilon t^{-1},
$$

and is a remainder $R(t)$, satisfying

$$
|R(t)| \le CA\varepsilon t^{-1}\|W^I(t,\cdot)\|^2_{L^2}, \tag{2.2.99}
$$

for every $t \in [1,T]$, after Cauchy-Schwarz inequality and equivalence (2.2.9) between the L^2 norms of $\widetilde{W}^I_s$ and W^I. More precisely, by the definition of $\widetilde{W}^I_s$ in proposition 2.2.1, and of W^I_s in proposition 2.1.5, we have that

$$
\begin{aligned}
\Big\|(\widetilde{W}^I_s - W^I)(t,\cdot)\Big\|_{L^2} &\le \|Op^B(P(V;\eta)-I_4)W^I\|_{L^2} + \|Op^B(E(U;\eta))W^I_s\|_{L^2}\\
&\lesssim (\|V(t,\cdot)\|_{H^{1,\infty}} + \|U(t,\cdot)\|_{H^{5,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}})\,\|W^I(t,\cdot)\|_{L^2},
\end{aligned} \tag{2.2.100}
$$

the latter inequality following from proposition 1.2.7, estimate (2.1.48), the fact that $E(U;\eta) = E^0_d(u;\eta) + E^{-1}_d(U;\eta) + E^{-1}_d(U;\eta)$ verifies, after (2.2.2),

$$
M^0_0\left(E\Big(\chi\Big(\frac{D_x}{\langle\eta\rangle}\Big)U;\eta\Big);n\right) \lesssim \|U(t,\cdot)\|_{H^{5,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}},
$$

for any admissible cut-off function χ, and equivalence (2.1.50). We can then replace third and fifth brackets in the right hand side of (2.2.98) with

$$\langle Op^B(A''(V^I;\eta))U + Op^B_R(A''(V^I;\eta))U, W^I\rangle + \langle T_{-18}(U)W^I, W^I\rangle,$$

up to some new remainders $R(t)$ satisfying (2.2.99) after Cauchy-Schwarz inequality, lemma 2.1.1, estimates (2.2.7), (2.2.100), and a-priori estimates (1.1.11a), (1.1.11b).

Consequently, equality (2.2.98) reduces to:

(2.2.101)
$$\begin{aligned}\frac{1}{2}\partial_t\|\widetilde{W}^I_s(t,\cdot)\|_{L^2} = -\Im\Big[&\langle Op^B(A''(V^I;\eta))U + Op^B_R(A''(V^I;\eta))U, W^I\rangle \\ &+ \langle Q^I_0(V,W), \widetilde{W}^I_s\rangle + \langle T_{-18}(U)W^I_s, \widetilde{W}^I_s\rangle + \langle \mathfrak{R}'(U,V), \widetilde{W}^I_s\rangle\Big] + R(t).\end{aligned}$$

We need, at this point, to distinguishing between indices $I\in\mathfrak{I}_n$ and $I\in\mathfrak{I}^k_3$, in order to analyse the behaviour of the second and fourth brakets in above right hand side, and we discuss separately about the propagation of estimates (1.1.11c) and (1.1.11d).

Propagation of a-priori estimate (1.1.11c): Let us suppose that $I\in\mathfrak{I}_n$. Using (2.2.100) and estimate (2.1.40), together with the fact that $\|W^I(t,\cdot)\|_{L^2}\le E_n(t;W)^{\frac{1}{2}}$, we find that

$$\langle Q^I_0(V,W), \widetilde{W}^I_s\rangle = \langle Q^I_0(V,W), W^I\rangle + R_n(t), \tag{2.2.102}$$

where

$$|R_n(t)| \le CA\varepsilon t^{-1+\frac{\delta}{2}} E_n(t;W)^{\frac{1}{2}}, \tag{2.2.103}$$

for a new constant $C>0$, for every $t\in[1,T]$. Reminding definition (2.1.12) of $Q^I_0(V,W)$, and the fact that coefficients c_{I_1,I_2} are all equal to 0 when $I\in\mathfrak{I}_n$, we notice that some of the contributions to the scalar product in the right hand side of (2.2.102) are also remainders $R_n(t)$. These are precisely the following ones:

$$\sum_{(I_1,I_2)\in\mathfrak{I}(I)}\langle Q^{\mathrm{w}}_0(v^{I_1}_\pm, D_1 v^{I_2}_\pm), u^I_+ + u^I_-\rangle + \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_1|\le[\frac{|I|}{2}]}}\langle Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D_1 u^{I_2}_\pm), v^I_+ + v^I_-\rangle,$$

in consequence of Cauchy-Schwarz inequality and estimates (2.1.28), (1.1.11b), (1.1.11c). Moreover, $\langle \mathfrak{R}'(U,V), \widetilde{W}^I\rangle$ is also a remainder $R_n(t)$, because of Cauchy-Schwarz, (2.2.100) and a-priori estimates (1.1.11a), (1.1.11b), along with the fact that

$$\|\mathfrak{R}'(U,V)\|_{L^2}\le CA\varepsilon t^{-1+\frac{\delta}{2}},$$

which follows choosing $\theta\ll 1$ in (2.2.8), using (2.1.40) and (1.1.11a)-(1.1.11c).

Observing that remainder $R(t)$ in (2.2.101) can be enclosed in $R_n(t)$ after (1.1.11c), we then obtain that equality (2.2.101) can be further reduced to

$$\begin{aligned}\frac{1}{2}\partial_t\|\widetilde{W}^I_s(t,\cdot)\|^2_{L^2} = -\Im\Big[&\langle Op^B(A''(V^I;\eta))U + Op^B_R(A''(V^I;\eta))U, W^I\rangle \\ &+ \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ [\frac{|I|}{2}]<|I_1|<|I|}}\langle Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D_1 u^{I_2}_\pm), v^I_+ + v^I_-\rangle + \langle T_{-18}(U)W^I, W^I\rangle\Big] + R_n(t),\end{aligned}$$

and from definition (2.2.51a), equalities (2.2.39), (2.2.41), (2.2.43) with $N=18$, and (2.2.48a), together with (2.2.71), (2.2.79) with $N=18$, (2.2.86), we deduce that

$$\frac{1}{2}\left|\partial_t \widetilde{E}^\dagger_n(t;W)\right| \lesssim |R_n(t)| + \sum_{I\in\mathcal{I}_n}\left(\left|\mathfrak{D}^I_{\text{quart}}(t)\right| + \left|\mathfrak{D}^{I,18}_{\text{quart}}(t)\right|\right) + \sum_{I\in\mathcal{I}_n}\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ [\frac{|I|}{2}]<|I_1|<|I|}}\left|\mathfrak{D}^{I_1,I_2}_{\text{quart}}(t)\right|,$$

where quartic terms $\mathfrak{D}^I_{\text{quart}}, \mathfrak{D}^{I,18}_{\text{quart}}, \mathfrak{D}^{I_1,I_2}_{\text{quart}}$ satisfy, respectively, (2.2.72), (2.2.80) with $N=18$, (2.2.87). If $\theta\in]0,1[$ is chosen sufficiently small in such inequalities, these quartic terms can also be considered as remainders $R_n(t)$ thanks to lemma 2.1.2 (i) and a-priori estimates (1.1.11), which implies that

$$\left|\partial_t \widetilde{E}^\dagger_n(t;W)\right| \le CA\varepsilon t^{-1+\frac{\delta}{2}} E_n(t;W)^{\frac{1}{2}},$$

for some new $C>0$, for every $t\in[1,T]$, and then that

$$\widetilde{E}^\dagger_n(t;W)^{\frac{1}{2}} \le \widetilde{E}^\dagger_n(1;W)^{\frac{1}{2}} + \int_1^t CA\varepsilon\tau^{-1+\frac{\delta}{2}}d\tau.$$

After equivalence (2.2.96a) and a-priori estimate (1.1.11c), we find that

$$\begin{aligned} E_n(t;W)^{\frac{1}{2}} &\le CE_n(1;W)^{\frac{1}{2}} + \int_1^t CA\varepsilon\tau^{-1+\frac{\delta}{2}}d\tau \\ &\le CE_n(1;W)^{\frac{1}{2}} + \frac{2CA\varepsilon}{\delta}t^{\frac{\delta}{2}}, \end{aligned}$$

again for a new $C>0$, and taking $B>1$ sufficiently large so that $E_n(1;W)^{\frac{1}{2}} \le \frac{B\varepsilon}{2CK_2}$ and $\frac{2CA}{\delta} < \frac{B}{2K_2}$, we finally obtain (1.1.12c).

Propagation of a-priori estimate (1.1.11d): Let us now suppose that $I\in\mathcal{I}^k_3$, for $0\le k\le 2$. After (2.1.41) and (2.2.100), we have that

$$\langle Q^I_0(V,W), \widetilde{W}^I_s\rangle = \langle Q^I_0(V,W), W^I\rangle + R^k_3(t),$$

where

$$|R^k_3(t)| \le CA^2\varepsilon^2 t^{-1+\frac{\delta_k}{2}} E^k_3(t;W)^{\frac{1}{2}}, \tag{2.2.104}$$

and moreover

$$-\Im\left[\langle Q^I_0(V,W), W^I\rangle\right] = -\delta_{\mathcal{V}_k}\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ I_1\in\mathcal{K},|I_2|\le 1}} c_{I_1,I_2}\Im\left[\left\langle Q^{\text{kg}}_0\left(v^{I_1}_\pm, \chi(t^{-\sigma}D_x)Du^{I_2}_\pm\right), v^I_+ + v^I_-\right\rangle\right] + R^k_3(t), \tag{2.2.105}$$

with $\delta_{\mathcal{V}_k}=1$ if $I\in\mathcal{V}_k$, 0 otherwise, as already seen in (2.2.35) and (2.2.36). Therefore, as $R(t)$ and $\langle\mathfrak{R}'(U,V), \widetilde{W}^I_s\rangle$ are also remainders $R^k_3(t)$, in consequence of the same argument used in the previous case, but with estimate (2.1.40) replaced with (2.1.41), we further reduce (2.2.101) to the following equality:

$$\begin{aligned} \frac{1}{2}\partial_t\|\widetilde{W}^I_s(t,\cdot)\|^2_{L^2} = &-\Im\left[\langle Op^B(A''(V^I;\eta))U + Op^B_R(A''(V^I;\eta))U, \widetilde{W}^I\rangle + \langle T_{-18}(U)W^I, W^I\rangle\right] \\ &- \delta_{\mathcal{V}_k}\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ I_1\in\mathcal{K},|I_2|\le 1}} c_{I_1,I_2}\Im\left[\left\langle Q^{\text{kg}}_0\left(v^{I_1}_\pm, \chi(t^{-\sigma}D_x)Du^{I_2}_\pm\right), v^I_+ + v^I_-\right\rangle\right] + R^k_3(t), \end{aligned}$$

and deduce from definition (2.2.51b), equalities (2.2.39), (2.2.41), (2.2.43) with $N = 18$, and (2.2.48b), together with (2.2.71), (2.2.79) with $N = 18$, and (2.2.89), that

$$\left|\partial_t \widetilde{E}_3^{k,\dagger}(t;W)\right| \lesssim |R_3^k(t)| + \sum_{I\in\mathcal{I}_3^k}\left(\left|\mathfrak{D}^I_{\text{quart}}(t)\right| + \left|\mathfrak{D}^{I,18}_{\text{quart}}(t)\right|\right) + \delta_{k<2}\sum_{\substack{I\in\mathcal{V}_k\\ j_i\in\{+,-\}}}\sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ I_1\in\mathcal{K},|I_2|\le 1}}\left|\mathfrak{G}^{I_1,I_2}_{(j_1,j_2,j_3)}\right|$$

with $\delta_{k<2} = 1$ for $k < 2$, 0 otherwise. On the one hand, quartic terms $\mathfrak{D}^I_{\text{quart}}, \mathfrak{D}^{I,18}_{\text{quart}}$ satisfy, respectively, (2.2.72) and (2.2.80) with $N = 18$, and are remainders $R_3^k(t)$ after (2.1.41) and a-priori estimates, if $\theta \ll 1$ is chosen sufficiently small; on the other hand, $\mathfrak{G}^{I_1,I_2}_{(j_1,j_2,j_3)}$ verifies estimate (2.2.90). Consequently, there is a constant $C > 0$ such that

$$\begin{aligned}\widetilde{E}_3^{k,\dagger}(t;W) \le \widetilde{E}_3^{k,\dagger}(1;W) + C(A+B)^2\varepsilon^2\int_1^t \tau^{-1+\frac{\delta_k}{2}}E_3^k(t;W)^{\frac12}d\tau \\ + \delta_{k<2}C(A+B)^2\varepsilon^2\left[\delta_{k=0}\int_1^t \tau^{-1+\frac{\delta_1}{2}}E_3^1(\tau;W)^{\frac12}d\tau + \int_1^t \tau^{-\frac54}d\tau\right],\end{aligned}$$

with $\delta_{k=0} = 1$ if $k = 0$, 0 otherwise, $\beta > 0$ as small as we want and, after equivalence (2.2.96b),

$$\begin{aligned}E_3^k(t;W) \le CE_3^k(1;W) + C(A+B)^2\varepsilon^2\int_1^t \tau^{-1+\frac{\delta_k}{2}}E_3^k(t;W)^{\frac12}d\tau \\ + \delta_{k<2}C(A+B)^2\varepsilon^2\left[\delta_{k=0}\int_1^t \tau^{-1+\frac{\delta_1}{2}}E_3^1(\tau;W)^{\frac12}d\tau + \int_1^t \tau^{-\frac54}d\tau\right],\end{aligned}$$

for a new $C > 0$. Injecting (1.1.11d) in above inequality and integrating in $d\tau$, we obtain that

$$E_3^k(t;W) \le CE_3^k(1;W) + C(A+B)^2B\varepsilon^3\left[\frac{1}{\delta_k}t^{\delta_k} + \delta_{k=0}\,\frac{2}{\delta_1}t^{\delta_1}\right],$$

and using that $\delta_1 \le \delta_0$, choosing $B > 1$ sufficiently large so that $E_3^k(1;W) \le \frac{B^2\varepsilon^2}{4CK_2^2}$ and $B \ge A$, and $\varepsilon_0 > 0$ sufficiently small so that $\varepsilon_0 < (4B)^{-1}$, we finally derive enhanced estimate (1.1.12d) and the conclusion of the proof. □

Chapter 3

Uniform Estimates

3.1 Semilinear Normal Forms

In proposition 2.2.12 of previous chapter, we proved the propagation of the energy a-priori estimates made on functions $(u_\pm, v_\pm)$, i.e. that there are some constants $A, B > 0$ large enough, $\varepsilon_0 > 0$ small, such that estimates (1.1.11) imply (1.1.12c), (1.1.12d). To conclude the proof of theorem 1.1.2, it only remains to show that estimates (1.1.11) also imply (1.1.12a), (1.1.12b). In particular, as $u_+ = -\overline{u_-}, v_+ = -\overline{u_-}$, it will be enough to prove this result only for (u_-, v_-), which is solution to the following system:

$$\begin{cases} (D_t + |D_x|)\, u_- = Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm), \\ (D_t + \langle D_x\rangle)\, v_- = Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm), \end{cases} \tag{3.1.1}$$

with $Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm), Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm)$ given by (2.1.1).

As for the simpler case of the one-dimensional Klein-Gordon equation (see [28]), the main idea is to reformulate system (3.1.1) in terms of two new functions $\widetilde{u}, \widetilde{v}$, defined from u_-, v_-, and living in a new framework (the *semi-classical framework*), and to deduce a new simpler system, made of a transport equation and of an ODE. Through this new system, we will be able to recover the required enhanced estimates (1.1.12a), (1.1.12b).

Before introducing the semi-classical framework, in which we will work for the rest of the paper, we need to replace some quadratic non-linearities in (3.1.1) with cubic ones by a normal form's argument. This is the object of the following two subsections. We highlight the fact that do not make use directly of the of the normal forms obtained in the proof of the energy inequality, because our goals and constraints are henceforth different. In fact, we want to obtain some L^∞ estimates for essentially ρ derivatives of our solution having a control on its H^s norm, for $s \gg \rho$. Therefore, we are allowed to lose some derivatives in the normal form's reduction, which means that we do not care any more about the quasi-linear nature of our problem.

3.1.1 Normal Forms for the Klein-Gordon equation

The aim of this subsection is to introduce a new unknown v^{NF}, defined as in (3.1.3) by adding some quadratic perturbations to v_-, in such a way it is solution to a half Klein-Gordon equation with a cubic non-linearity, instead of the quadratic one $Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm)$ appearing in the equation satisfied by v_- in (3.1.1). This normal form is motivated by the fact that the L^2 norm of $Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm)$ decays too slowly in time (only a $O(t^{-1+\delta/2})$), because of the fact that

$\|v_\pm(t,\cdot)\|_{H^{1,\infty}} = O(t^{-1})$ and $\|u_\pm(t,\cdot)\|_{H^1} = O(t^{\delta/2})$ by a-priori estimates (1.1.11b), (1.1.11c). This decay is not enough in view of proposition 3.2.7 (the required one being strictly faster than $t^{-3/2}$), and must be replaced with a faster one.

It is fundamental to observe that, after inequality (3.1.7b) below, with $\theta \ll 1$ small enough (e.g. $\theta < (2+\delta)^{-1}$), and a-priori estimates (1.1.11), v^{NF} and v_- are comparable in the following sense

$$\big|\|v_-(t,\cdot)\|_{H^{\rho,\infty}} - \|v^{NF}(t,\cdot)\|_{H^{\rho,\infty}}\big| \le C\varepsilon^2 t^{-1}, \tag{3.1.2}$$

where C is some positive constant. Then, bootstrap assumption (1.1.11b) implies that the new unknown v^{NF} disperses in time at the same rate t^{-1} as v_-, and a suitable propagation of the $H^{\rho,\infty}$ norm of v^{NF} will provide us with the enhanced estimate (1.1.12b).

Proposition 3.1.1. *Assume that (u,v) is solution to (1.1.1) in $[1,T]$, for a fixed $T>1$, consider (u_+,v_+,u_-,v_-) defined in (1.1.5) and solution to (2.1.2) with $|I|=0$, and remind definition (2.1.11) of vectors U,V. Let*

$$v^{NF} := v_- - \frac{i}{4(2\pi)^2}\sum_{j_1,j_2\in\{+,-\}}\int e^{ix\cdot\xi}B^1_{(j_1,j_2,+)}(\xi,\eta)\hat{v}_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)d\xi d\eta, \tag{3.1.3}$$

with $B^1_{(j_1,j_2,j_3)}(\xi,\eta)$ introduced in (2.2.45) for any $j_1,j_2,j_3\in\{+,-\}$, $k=1,2$. Then v^{NF} is solution to

$$(D_t + \langle D_x\rangle)\, v^{NF}(t,x) = r^{NF}_{kg}(t,x), \tag{3.1.4}$$

for every $t\in[1,T]$, where

$$r^{NF}_{kg}(t,x) = -\frac{i}{4(2\pi)^2}\sum_{j_1,j_2\in\{+,-\}}\int e^{ix\cdot\xi}B^1_{(j_1,j_2,+)}(\xi,\eta) \times\Big[\widehat{NL_{kg}}(\xi-\eta)\hat{u}_{j_2}(\eta) + \hat{v}_{j_1}(\xi-\eta)\widehat{NL_w}(\eta)\Big]d\xi d\eta, \tag{3.1.5}$$

satisfies

$$\|r^{NF}_{kg}(t,\cdot)\|_{L^2} \lesssim \sum_{\mu=0}^{1}\|V(t,\cdot)\|_{H^{1,\infty}}\|\mathrm{R}_1^\mu U(t,\cdot)\|_{L^\infty}\|U(t,\cdot)\|_{H^1} + \|V(t,\cdot)\|^2_{H^{2,\infty}}\|V(t,\cdot)\|_{H^2}. \tag{3.1.6a}$$

and for any $\chi\in C^\infty_0(\mathbb{R}^2)$, $\sigma>0$,

$$\|\chi(t^{-\sigma}D_x)r^{NF}_{kg}(t,\cdot)\|_{L^\infty} \lesssim \|V(t,\cdot)\|_{H^{1,\infty}}\Big(\sum_{\mu=0}^{1}\|\mathrm{R}_1^\mu U(t,\cdot)\|_{H^{2,\infty}}\Big)^2 + t^\sigma\|V(t,\cdot)\|^3_{H^{2,\infty}}. \tag{3.1.6b}$$

Moreover, for every $s,\rho\ge 0$, any $\theta\in]0,1[$,

$$\big\|(v^{NF}-v_-)(t,\cdot)\big\|_{H^s} \lesssim \sum_{\mu=0}^{1}\|V(t,\cdot)\|_{H^s}\|\mathrm{R}_1^\mu U(t,\cdot)\|_{L^\infty} + \|V(t,\cdot)\|_{L^\infty}\|U(t,\cdot)\|_{H^{s+1}}, \tag{3.1.7a}$$

$$\begin{split}\big\|(v^{NF}-v_-)(t,\cdot)\big\|_{H^{s,\infty}} \lesssim &\sum_{\mu=0}^{1}\|V(t,\cdot)\|^{1-\theta}_{H^{s,\infty}}\|V(t,\cdot)\|^{\theta}_{H^{s+2}}\|\mathrm{R}_1^\mu U(t,\cdot)\|_{L^\infty}\\ &+\sum_{\mu=0}^{1}\|V(t,\cdot)\|_{L^\infty}\|\mathrm{R}_1^\mu U(t,\cdot)\|^{1-\theta}_{H^{s+1,\infty}}\|U(t,\cdot)\|^{\theta}_{H^{s+3}}\end{split} \tag{3.1.7b}$$

(3.1.7c)

$$\left\|\Omega(v^{NF}-v_-)(t,\cdot)\right\|_{L^2}\lesssim \sum_{\mu,\nu=0}^{1}[\|\Omega^\mu V(t,\cdot)\|_{L^2}\|\mathrm{R}_1^\nu U(t,\cdot)\|_{L^\infty}+\|V(t,\cdot)\|_{L^\infty}\|\Omega^\mu U(t,\cdot)\|_{H^1}] + \|\Omega V(t,\cdot)\|_{H^2}\|U(t,\cdot)\|_{H^1}+\|V(t,\cdot)\|_{L^2}\|\Omega U(t,\cdot)\|_{H^2},$$

and

(3.1.8a)
$$\left\|\chi(t^{-\sigma}D_x)(v^{NF}-v_-)(t,\cdot)\right\|_{L^2}\lesssim t^\sigma\|V(t,\cdot)\|_{H^{1,\infty}}\|U(t,\cdot)\|_{L^2},$$

(3.1.8b)
$$\left\|\chi(t^{-\sigma}D_x)\Omega(v^{NF}-v_-)(t,\cdot)\right\|_{L^2} \lesssim t^\sigma\left[\sum_{\mu=0}^{1}\|\Omega V(t,\cdot)\|_{L^2}\|\mathrm{R}_1^\mu U(t,\cdot)\|_{L^\infty}+\|V(t,\cdot)\|_{H^{1,\infty}}\|\Omega^\mu U(t,\cdot)\|_{L^2}\right].$$

Proof. From the definition of v^{NF}, system (2.1.2) with $|I|=0$, and the fact that

(3.1.9)
$$Q_0^{\mathrm{kg}}(v_\pm,D_1u_\pm)=\frac{i}{4(2\pi)^2}\sum_{j_1,j_2\in\{+,-\}}\int e^{ix\cdot\xi}\Big(1-j_1j_2\frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{|\eta|}\Big)\eta_1\hat{v}_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)d\xi d\eta,$$

it immediately follows that v^{NF} is solution to (3.1.4), with r^{NF}_{kg} given by (3.1.5). We observe that, after formula (A.11), we have the following explicit expressions:

(3.1.10)

$$v^{NF}-v_-=-\frac{i}{8}\Big[(v_++v_-)\mathrm{R}_1(u_+-u_-)-\frac{D_1}{\langle D_x\rangle}(v_+-v_-)(u_++u_-) + D_1\big[[\langle D_x\rangle^{-1}(v_+-v_-)](u_++u_-)\big]-\langle D_x\rangle\big[[\langle D_x\rangle^{-1}(v_+-v_-)]\mathrm{R}_1(u_+-u_-)\big]\Big],$$

and

(3.1.11)
$$r^{NF}_{kg}=-\frac{i}{4}\left[NL_{kg}\,\mathrm{R}_1(u_+-u_-)-\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\,NL_w+D_1\big[\langle D_x\rangle^{-1}(v_+-v_-)\,NL_w\big]\right]$$

Inequalities (3.1.7a), (3.1.7b) are straightforward from (3.1.10) and corollary A.4 in appendix A. Inequality (3.1.7c) is also obtained from corollary A.4 and bounding the L^∞ norm of $\Omega u_\pm,\Omega v_\pm$ with their H^2 norm by means of the classical Sobolev injection, after having applied Ω to (3.1.10) and used the Leibniz rule. Finally, inequalities (3.1.8a), (3.1.8b) are also straightforward if one observes that operator $\chi(t^{-\sigma}D_x)$, with $\chi\in C_0^\infty(\mathbb{R}^2)$ and $\sigma>0$, is L^2-H^1 continuous with norm $O(t^\sigma)$.

On the other hand, after (3.1.11) and corollary A.4,

$$\|r^{NF}_{kg}(t,\cdot)\|_{L^2}\lesssim\sum_{\mu=0}^{1}\|NL_{kg}(t,\cdot)\|_{L^2}\|\mathrm{R}_1^\mu U(t,\cdot)\|_{L^\infty}+\|V(t,\cdot)\|_{L^2}\|NL_w(t,\cdot)\|_{L^\infty} + \|V(t,\cdot)\|_{L^\infty}\|NL_w(t,\cdot)\|_{H^1},$$

and

$$\|\chi(t^{-\sigma}D_x)r^{NF}_{kg}(t,\cdot)\|_{L^\infty}\lesssim\sum_{\mu=0}^{1}\|NL_{kg}(t,\cdot)\|_{L^\infty}\|\mathrm{R}_1^\mu U(t,\cdot)\|_{L^\infty}+t^\sigma\|V(t,\cdot)\|_{H^{1,\infty}}\|NL_w(t,\cdot)\|_{L^\infty},$$

then (3.1.6a) and (3.1.6b) follow by (B.1.3c) with $s=1$, (B.1.3b), (B.1.4a) and (B.1.4b). □

3.1.2 Normal Forms for the Wave Equation

We now focus on the wave equation satisfied by u_-:

$$(D_t + |D_x|)u_-(t,x) = Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm),$$

and perform a normal form argument in order to replace (a part of) the quadratic non-linearity in the above right hand side with a cubic non-local one. The fundamental reason for that is to be found in lemma 3.2.14, where we have to prove that the L^2 norm of some operator, acting on the non-linearity of the equation satisfied by u_-, decays like $O(t^{-1/2+\beta})$, for a small $\beta > 0$. That decay is obtained if the L^2 norm of the non-linearity is a $O(t^{-3/2+\beta'})$, for some new small $\beta' > 0$, which is not the case for $Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)$. This normal form can be actually performed only on contributions depending on (v_+, v_+) and (v_-, v_-), but not on the ones in (v_+, v_-), which are resonant. Nevertheless, the structure of these latter contributions allows to recover the right mentioned time decay for their L^2 norm (see lemmas 3.2.15, 3.2.16).

Thanks to inequalities (3.1.20b), (3.1.20c), and a-priori estimates (1.1.11), the new unknown u^{NF} we define in (3.1.15) below is equivalent to the former u_-, meaning that

$$\sum_{\kappa=0}^{1} \big| \|\mathrm{R}_1^\kappa u_-(t,\cdot)\|_{H^{\rho+1,\infty}} - \|\mathrm{R}_1^\kappa u^{NF}(t,\cdot)\|_{H^{\rho,\infty}} \big| \le C\varepsilon^2 t^{-1+\frac{\delta}{2}}, \tag{3.1.12}$$

Therefore, we expect for u^{NF} and $\mathrm{R}_1 u^{NF}$ to decay in the $H^{\rho+1,\infty}$ norm at the same rate $t^{-1/2}$ as $u_-, \mathrm{R}_1 u_-$, and a suitable propagation of this norm will provide us with the enhanced estimate (1.1.12a).

With this aim, we rewrite $Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)$ as follows, reminding that $v_+ = -\overline{v_-}$:

$$\begin{aligned} Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm) = &-\frac{1}{2}\Im\left[v_+\, D_1 v_- + \frac{D_x}{\langle D_x\rangle} v_+ \cdot \frac{D_x D_1}{\langle D_x\rangle} v_-\right] \\ &+ \frac{i}{4(2\pi)^2}\sum_{j\in\{+,-\}} \int e^{ix\cdot\xi}\left(1 - \frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{\langle\eta\rangle}\right)\eta_1 \hat{v}_j(\xi-\eta)\hat{v}_j(\eta)\, d\xi d\eta, \end{aligned} \tag{3.1.13}$$

and introduce, for any $j \in \{+,-\}$,

$$D_j(\xi,\eta) := \frac{\left(1 - \frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{\langle\eta\rangle}\right)\eta_1}{j\langle\xi-\eta\rangle + j\langle\eta\rangle + |\xi|}. \tag{3.1.14}$$

We warn the reader that, for seek of compactness, from now on we will often denote non-linearity $Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)$ (resp. $Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm)$) concisely by NL_w (resp. NL_{kg}).

Proposition 3.1.2. *Assume that (u,v) is solution to (1.1.1) in $[1,T]$, for a fixed $T > 1$, consider (u_+, v_+, u_-, v_-) defined in (1.1.5) and solution to (2.1.2) with $|I| = 0$, and remind definition (2.1.11) of vectors U, V, and (3.1.3) of v^{NF}. Let*

$$u^{NF} := u_- - \frac{i}{4(2\pi)^2}\sum_{j\in\{+,-\}}\int e^{ix\cdot\xi} D_j(\xi,\eta)\hat{v}_j(\xi-\eta)\hat{v}_j(\eta)\, d\xi d\eta, \tag{3.1.15}$$

with D_j defined in (3.1.14). For every $t \in [1,T]$, u^{NF} is solution to

$$(D_t + |D_x|)u^{NF}(t,x) = q_w(t,x) + c_w(t,x) + +r_w^{NF}(t,x), \tag{3.1.16}$$

where quadratic term q_w is given by

$$q_w(t,x) = \frac{1}{2}\Im\left[\overline{v^{NF}}\, D_1 v^{NF} - \overline{\frac{D_x}{\langle D_x\rangle} v^{NF}} \cdot \frac{D_x D_1}{\langle D_x\rangle} v^{NF}\right], \tag{3.1.17}$$

while cubic terms c_w, r_w^{NF} are equal, respectively, to

$$\begin{aligned} c_w(t,x) = \frac{1}{2}\Im\Big[&\overline{(v_- - v^{NF})}\, D_1 v_- + \overline{v^{NF}}\, D_1(v_- - v^{NF}) \\ &- \overline{\frac{D_x}{\langle D_x\rangle}(v_- - v^{NF})} \cdot \frac{D_x D_1}{\langle D_x\rangle} v_- - \overline{\frac{D_x}{\langle D_x\rangle} v^{NF}} \cdot \frac{D_x D_1}{\langle D_x\rangle}(v_- - v^{NF})\Big], \end{aligned} \tag{3.1.18}$$

and

$$r_w^{NF}(t,x) = -\frac{i}{4(2\pi)^2}\sum_{j\in\{+,-\}}\int e^{ix\cdot\xi} D_j(\xi,\eta)\left[\widehat{NL_{kg}}(\xi-\eta)\hat{v}_j(\eta) + \hat{v}_j(\xi-\eta)\widehat{NL_{kg}}(\eta)\right] d\xi d\eta. \tag{3.1.19}$$

For any $s,\rho \geq 0$, any $t \in [1,T]$,

$$\|u^{NF}(t,\cdot) - u_-(t,\cdot)\|_{H^s} \lesssim \|V(t,\cdot)\|_{L^\infty}\|V(t,\cdot)\|_{H^{s+15}}, \tag{3.1.20a}$$

$$\|u^{NF}(t,\cdot) - u_-(t,\cdot)\|_{H^{\rho+1,\infty}} \lesssim \|V(t,\cdot)\|_{L^\infty}\|V(t,\cdot)\|_{H^{\rho+18}}, \tag{3.1.20b}$$

$$\|\mathrm{R}_j u^{NF}(t,\cdot) - \mathrm{R}_j u_-(t,\cdot)\|_{H^{\rho+1,\infty}} \lesssim \|V(t,\cdot)\|_{L^\infty}\|V(t,\cdot)\|_{H^{\rho+8}}, \quad j=1,2. \tag{3.1.20c}$$

Moreover, for any cut-off function $\chi \in C_0^\infty(\mathbb{R}^2), \sigma > 0$, there exists some $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ such that

$$\begin{aligned} \left\|\chi(t^{-\sigma}D_x)c_w(t,\cdot)\right\|_{H^s} \lesssim\ & t^\beta \left\|\chi_1(t^{-\sigma}D_x)(v^{NF} - v_-)(t,\cdot)\right\|_{L^2}\left(\|V(t,\cdot)\|_{H^{1,\infty}} + \|v^{NF}(t,\cdot)\|_{H^{1,\infty}}\right) \\ &+ t^{-N(s)}\left\|(v^{NF} - v_-)(t,\cdot)\right\|_{H^1}\left(\|V(t,\cdot)\|_{H^s} + \|v^{NF}(t,\cdot)\|_{H^s}\right), \end{aligned} \tag{3.1.21a}$$

$$\begin{aligned} &\left\|\chi(t^{-\sigma}D_x)c_w(t,\cdot)\right\|_{H^{s,\infty}} \\ &\quad\lesssim t^\beta \left\|\chi(t^{-\sigma}D_x)\left(v^{NF} - v_-\right)(t,\cdot)\right\|_{H^{1,\infty}}\left(\|V(t,\cdot)\|_{H^{2,\infty}} + \|v^{NF}(t,\cdot)\|_{H^{1,\infty}}\right) \end{aligned} \tag{3.1.21b}$$

$$\begin{aligned} \left\|\chi(t^{-\sigma}D_x)\Omega c_w(t,\cdot)\right\|_{L^2} \lesssim\ & t^\sigma \left\|\chi_1(t^{-\sigma}D_x)\Omega(v^{NF} - v_-)(t,\cdot)\right\|_{L^2}\left(\|V(t,\cdot)\|_{H^{2,\infty}} + \|v^{NF}(t,\cdot)\|_{H^{1,\infty}}\right) \\ &+ t^{-N(s)}\left\|\Omega(v^{NF} - v_-)(t,\cdot)\right\|_{L^2}\left(\|V(t,\cdot)\|_{H^s} + \|v^{NF}(t,\cdot)\|_{H^s}\right) \\ &+ t^\sigma\left\|(v^{NF} - v_-)(t,\cdot)\right\|_{H^{1,\infty}} \times \sum_{\mu=0}^{1}\left(\|\Omega^\mu V(t,\cdot)\|_{H^1} + \|\Omega v^{NF}(t,\cdot)\|_{L^2}\right) \end{aligned} \tag{3.1.21c}$$

with $\beta > 0$ small such that $\beta \to 0$ as $\sigma \to 0$, $N(s) > 0$ as large as we want as long as $s > 0$ is large, and

$$\|\chi(t^{-\sigma}D_x)r_w^{NF}(t,\cdot)\|_{H^s} \lesssim t^\beta \|V(t,\cdot)\|^2_{H^{13,\infty}}\|U(t,\cdot)\|_{H^1}, \tag{3.1.22a}$$

$$\|\chi(t^{-\sigma}D_x)r_w^{NF}(t,\cdot)\|_{H^{\rho,\infty}} \lesssim t^\beta \|V(t,\cdot)\|^2_{H^{13,\infty}}\left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{2,\infty}}\right), \tag{3.1.22b}$$

and for any $\theta \in]0,1[$,

(3.1.22c)

$$\begin{aligned}\|\chi(t^{-\sigma}D_x)\Omega r^{NF}_w(t,\cdot)\|_{L^2} \lesssim t^\beta\Big[&\|V(t,\cdot)\|^{1-\theta}_{H^{15,\infty}}\|V(t,\cdot)\|^{\theta}_{H^{17}}\left(\|U(t,\cdot)\|_{H^{1,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}}\right)\\ &+\|V(t,\cdot)\|_{L^\infty}\left(\|U(t,\cdot)\|^{1-\theta}_{H^{16,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|^{1-\theta}_{H^{16,\infty}}\right)\|U(t,\cdot)\|^{\theta}_{H^{18}}\Big]\|\Omega V(t,\cdot)\|_{L^2}\\ &+t^\beta\Big[\|V(t,\cdot)\|_{H^{1,\infty}}\left(\|U(t,\cdot)\|_{H^1}+\|\Omega U(t,\cdot)\|_{H^1}\right)\\ &+\left(\|U(t,\cdot)\|_{H^{2,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}\right)\left(\|V(t,\cdot)\|_{L^2}+\|\Omega V(t,\cdot)\|_{L^2}\right)\Big]\|V(t,\cdot)\|_{H^{17,\infty}}.\end{aligned}$$

Proof. By the definition of u^{NF}, system (2.1.2) with $|I|=0$, (3.1.13) and (3.1.14), it follows that u^{NF} is solution to

$$(D_t+|D_x|)u^{NF}(t,x) = -\frac{1}{2}\Im\left[v_+\,D_1v_- + \frac{D_x}{\langle D_x\rangle}v_+\cdot\frac{D_xD_1}{\langle D_x\rangle}v_-\right]+r^{NF}_w(t,x),$$

with r^{NF}_w given by (3.1.19), so reminding that $v_+=-\overline{v_-}$, and replacing each occurrence of v_- in the quadratic contribution to the above right hand side, we find that u^{NF} is solution to (3.1.16).

The first part of lemma A.8, and the fact that any $H^{\rho+1,\infty}$ injects into $H^{\rho+3}$ by Sobolev inequality, immediately imply estimates (3.1.20) and

$$\|\chi(t^{-\sigma}D_x)r^{NF}_w(t,\cdot)\|_{H^s}\lesssim t^\beta\|NL_{kg}(t,\cdot)\|_{L^2}\|V(t,\cdot)\|_{H^{13,\infty}},$$

$$\|\chi(t^{-\sigma}D_x)r^{NF}_w(t,\cdot)\|_{H^{\rho,\infty}}\lesssim t^\beta\|NL_{kg}(t,\cdot)\|_{L^\infty}\|V(t,\cdot)\|_{H^{13,\infty}},$$

with $\beta>0$ small, $\beta\to0$ as $\sigma\to0$ for every $s,\rho\ge0$. Moreover, from (A.31a) we derive that

$$\begin{aligned}\|\chi(t^{-\sigma}D_x)\Omega r^{NF}_w(t,\cdot)\|_{L^2}\lesssim t^\beta\left(\|NL_{kg}(t,\cdot)\|_{L^2}+\|\Omega NL_{kg}(t,\cdot)\|_{L^2}\right)\|V(t,\cdot)\|_{H^{17,\infty}}\\ +t^\beta\|NL_{kg}(t,\cdot)\|_{H^{15,\infty}}\|\Omega V(t,\cdot)\|_{L^2},\end{aligned}$$

so estimates (3.1.22) are obtained using (B.1.4a), (B.1.4e) with $s=15$, and (B.1.4f).

Finally, inequality (3.1.21a) is obtained by using the fact that operator $\chi(t^{-\sigma}D_x)$ is continuous from L^2 to H^s with norm $O(t^{\sigma s})$, for any $s>0$, together with lemma B.2.2 in appendix B with $L=L^2$, $w=v_--v^{NF}$. Inequality (3.1.21b) is straightforward, while (3.1.21c) is deduced applying Ω to (3.1.18) and using the Leibniz rule. The L^2 norm of products in which Ω is acting on v_--v^{NF} is estimated by means of lemma B.2.2 with $L=L^2$, $w=v_--v^{NF}$, whereas the L^2 norm of the remaining products is simply estimated by taking the L^∞ norm on v_--v^{NF} times the L^2 norm of the second factor.

□

3.2 From PDEs to ODEs

In previous section we showed that, if (u_-,v_-) is solution to system (3.1.1) in some interval $[1,T]$, for a fixed $T>1$, one can define two new functions, u^{NF} as in (3.1.15) and v^{NF} as in (3.1.3), respectively comparable to u_- and v_- in the sense of (3.1.12), (3.1.2), such that (u^{NF},v^{NF}) is solution to a new cubic wave-Klein-Gordon system:

(3.2.1)
$$\begin{cases}(D_t+|D_x|)\,u^{NF}(t,x)=q_w(t,x)+c_w(t,x)+r^{NF}_w(t,x),\\ (D_t+\langle D_x\rangle)\,v^{NF}(t,x)=r^{NF}_{kg}(t,x),\end{cases}$$

for every $(t,x)\in[1,T]\times\mathbb{R}^2$, where quadratic inhomogeneous term q_w is given by (3.1.17), and cubic ones c_w, r^{NF}_w and r^{NF}_{kg} respectively by (3.1.18), (3.1.19) and (3.1.5).

As anticipated before, our aim is to deduce from (3.2.1) a system made of a transport equation and of an ODE, from which it will be possible to deduce suitable estimates on (u^{NF},v^{NF}) (and consequently on (u_-,v_-)). Thanks to (3.1.12), (3.1.2), these estimates will allow us to close the bootstrap argument and prove theorem 1.1.2.

In subsection 3.2.1 we focus on the deduction of the mentioned ODE starting from the Klein-Gordon equation satisfied by v^{NF}, while in subsection 3.2.2 we show how to derive a transport equation from the wave equation satisfied by u^{NF}. The framework in which this plan takes place is the *semi-classical framework*, introduced below.

Let us introduce the *semi-classical parameter* $h:=t^{-1}$ and two new functions:

$$\widetilde{u}(t,x):=tu^{NF}(t,tx),\qquad \widetilde{v}(t,x):=tv^{NF}(t,tx). \tag{3.2.2}$$

With notations introduced in subsection 1.2.2, a straight computation shows that $(\widetilde{u},\widetilde{v})$ satisfies the following coupled system of semi-classical pseudo-differential equations:

$$\begin{cases} \left[D_t-Op^w_h(x\cdot\xi-|\xi|)\right]\widetilde{u}(t,x)=h^{-1}\left[q_w(t,tx)+c_w(t,tx)+r^{NF}_w(t,tx)\right]\\ \left[D_t-Op^w_h(x\cdot\xi-\langle\xi\rangle)\right]\widetilde{v}(t,x)=h^{-1}r^{NF}_{kg}(t,tx).\end{cases} \tag{3.2.3}$$

Moreover, from definition (3.2.2), a-priori estimates (1.1.11a), (1.1.11b) and inequalities (3.1.12), (3.1.2), we are now led to suitably propagate the following estimates:

$$\|\widetilde{u}(t,\cdot)\|_{H^{\rho+1,\infty}_h}+\left\|Op^w_h(\xi_1|\xi|^{-1})\widetilde{u}(t,\cdot)\right\|_{H^{\rho+1,\infty}_h}\le C\varepsilon h^{-\frac{1}{2}}, \tag{3.2.4a}$$

$$\|\widetilde{v}(t,\cdot)\|_{H^{\rho,\infty}_h}\le C\varepsilon, \tag{3.2.4b}$$

for some large enough positive constant $C>0$, in order to obtain enhanced estimates (1.1.12a), (1.1.12b).

If $\mathcal{M}_j$ (resp. $\mathcal{L}_j$), $j=1,2$, is the operator introduced in (1.2.45) (resp. (1.2.60)), $\mathcal{M}_j\widetilde{u}$ (resp. $\mathcal{L}_j\widetilde{v}$) can be expressed in term of Z_ju^{NF} (resp. Z_jv^{NF}). We have the following general result:

Lemma 3.2.1. *(i) Let $w(t,x)$ be a solution to the inhomogeneous half wave equation*

$$\left[D_t+|D_x|\right]w(t,x)=f(t,x), \tag{3.2.5}$$

and $\widetilde{w}(t,x)=tw(t,tx)$. For any $j=1,2$,

$$Z_jw(t,y)=ih\left[-\mathcal{M}_j\widetilde{w}(t,x)+\frac{1}{2i}Op^w_h\left(\frac{\xi_j}{|\xi|}\right)\widetilde{w}(t,x)\right]|_{x=\frac{y}{t}}+iy_jf(t,y); \tag{3.2.6}$$

(ii) If $w(t,x)$ is solution to an inhomogeneous half Klein-Gordon equation

$$\left[D_t+\langle D_x\rangle\right]w(t,x)=f(t,x), \tag{3.2.7}$$

then

$$Z_jw(t,y)=ih\left[-Op^w_h(\langle\xi\rangle)\mathcal{L}_j\widetilde{w}(t,x)+\frac{1}{i}Op^w_h\Big(\frac{\xi_j}{\langle\xi\rangle}\Big)\widetilde{w}(t,x)\right]|_{x=\frac{y}{t}}+iy_jf(t,y). \tag{3.2.8}$$

Proof. (i) As we seen few lines above, if w is solution to half wave equation (3.2.5), $\widetilde{w}(t,x)$ satisfies

$$\left[D_t-Op^w_h(x\cdot\xi-|\xi|)\right]\widetilde{w}(t,x)=h^{-1}f(t,tx),$$

so

$$
\begin{aligned}
&Z_j w(t,y) = \\
&ih^{-1}\left[x_jD_t + Op^w_h(\xi_j - x_j x\cdot\xi) + \frac{3h}{2i}x_j\right]\left(\frac{1}{t}\widetilde{w}(t,x)\right)\Big|_{x=\frac{y}{t}} \\
&= i\left[x_jD_t + Op^w_h(\xi_j - x_j x\cdot\xi) + \frac{h}{2i}x_j\right]\widetilde{w}(t,x)\Big|_{x=\frac{y}{t}} \\
&= i\left[x_jOp^w_h(x\cdot\xi - |\xi|)\widetilde{w}(t,x) + Op^w_h(\xi_j - x_j x\cdot\xi)\widetilde{w}(t,x) + \frac{h}{2i}x_j\widetilde{u}(t,x) + h^{-1}x_jf(t,tx)\right]\Big|_{x=\frac{y}{t}} \\
&= ih\left[-\mathcal{M}_j\widetilde{w}(t,x) + \frac{1}{2i}Op^w_h\left(\frac{\xi_j}{|\xi|}\right)\widetilde{w}(t,x)\right]\Big|_{x=\frac{y}{t}} + iy_jf(t,y),
\end{aligned}
$$

for $j=1,2$. We should specify that last equality was obtained by a trivial version of symbolic calculus (1.2.18), that applies also to symbols $b(\xi)$ singular at $\xi=0$. Indeed, if symbol $a=a(x,\xi)$ is linear in x, and $b(\xi)$ is lipschitz, the development $a\sharp b$ is actually finite:

$$
a\sharp b(x,\xi) = a(x,\xi)b(\xi) - \frac{h}{2i}\partial_x a(x,\xi)\cdot\partial_\xi b(\xi).
$$

The result of (ii) follows in a similar way, using that $\widetilde{w}$ satisfies

$$
\big[D_t - Op^w_h(x\cdot\xi - \langle\xi\rangle)\big]\widetilde{w}(t,x) = h^{-1}f(t,tx).
$$

□

Since

$$
\begin{aligned}
h\mathcal{M}_j\widetilde{w}(t,x) &= \left[y_j|D_y| - tD_j + \frac{1}{2i}\frac{D_j}{|D_y|}\right]w(t,y)|_{y=tx}, \\
hOp^w_h(\langle\xi\rangle)\mathcal{L}_j\widetilde{w}(t,x) &= \left[y_j\langle D_y\rangle - tD_j - i\frac{D_j}{\langle D_y\rangle}\right]w(t,y)|_{y=tx},
\end{aligned}
$$

for any $j=1,2$, we deduce from previous lemma that, if w is solution to half wave equation (3.2.5) (resp. to half Klein-Gordon (3.2.7)),

$$
\left[y_j|D_y| - tD_j + \frac{1}{2i}\frac{D_j}{|D_y|}\right]w(t,y) = iZ_jw(t,y) + \frac{1}{2i}\frac{D_j}{|D_y|}w(t,y) + y_jf(t,y), \tag{3.2.9a}
$$

$$
\left(\text{resp. } \left[\langle D_y\rangle y_j - tD_j\right]w(t,y) = iZ_jw(t,y) - i\frac{D_j}{\langle D_y\rangle}w(t,y) + y_jf(t,y)\right). \tag{3.2.9b}
$$

Moreover, from system (3.2.3) we deduce also the following relations, for any $i=1,2$:
(3.2.10a)

$$
Z_ju^{NF}(t,y) = ih\left[-\mathcal{M}_j\widetilde{u}(t,x) + \frac{1}{2i}Op^w_h\left(\frac{\xi_j}{|\xi|}\right)\widetilde{u}(t,x)\right]\Big|_{x=\frac{y}{t}} + iy_j\left[q_w + c_w + r^{NF}_w\right](t,y),
$$

$$
Z_jv^{NF}(t,y) = ih\left[-Op^w_h(\langle\xi\rangle)\mathcal{L}_j\widetilde{v}(t,x) + \frac{1}{i}Op^w_h\Big(\frac{\xi_j}{\langle\xi\rangle}\Big)\widetilde{v}(t,x)\right]\Big|_{x=\frac{y}{t}} + iy_jr^{NF}_{kg}(t,y). \tag{3.2.10b}
$$

In view of lemma 3.2.14, it is also useful to write down the same relation between $(Z_mZ_nu)_-$ and $\mathcal{M}[t(Z_nu)_-(t,tx)]$, where $(Z_nu)_-$ is solution to

$$
\big(D_t + |D_x|\big)(Z_nu)_- = Z_nNL_w(t,x),
$$

with NL_w concisely denoting $Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)$, and

$$Z_n NL_w = Q_0^{\mathrm{w}}\big((Z_n v)_\pm, D_1 v_\pm\big) + Q_0^{\mathrm{w}}\big(v_\pm, D_1(Z_n v)_\pm\big) - \delta_n^1 Q_0^{\mathrm{w}}(v_\pm, D_t v_\pm),$$

with $\delta_n^1 = 0$ for $n=1$, 0 otherwise, as follows by (1.1.15), (1.1.16), (1.1.5) and (1.1.10). Observe that, from inequality (B.1.6a) with $s=0$,

$$\begin{aligned}(3.2.11)\quad \|Z_n NL_w(t,\cdot)\|_{L^2} &\lesssim \|Z_n V(t,\cdot)\|_{H^1}\|V(t,\cdot)\|_{H^{2,\infty}} + \big[\|V(t,\cdot)\|_{H^1} \\ &+ \|V(t,\cdot)\|_{L^2}\left(\|U(t,\cdot)\|_{H^{1,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{1,\infty}}\right) + \|V(t,\cdot)\|_{L^\infty}\|U(t,\cdot)\|_{H^1}\big]\|V(t,\cdot)\|_{H^{1,\infty}}.\end{aligned}$$

Applying equality (3.2.6) with $w = (Z_n u)_-$, commutating Z_m with $D_t - |D_x|$ (see (2.1.15a)), and considering index J such that $\Gamma^J = Z_n$, we then have

$$\begin{aligned}(3.2.12)\quad (Z_m Z_n u)_-(t,y) = ih\Big[-\mathcal{M}_m \widetilde{u}^J(t,x) + \frac{1}{2i}Op_h^w\Big(\frac{\xi_m}{|\xi|}\Big)\widetilde{u}^J(t,x)\Big]\Big|_{x=\frac{y}{t}} + iy_m Z_n NL_w(t,y) \\ - \frac{D_m}{|D_y|}(Z_n u)_-(t,y),\end{aligned}$$

with $\widetilde{u}^J(t,x) := t(Z_n u)_-(t,tx)$.

3.2.1 Derivation of the ODE and Propagation of the uniform estimate on the Klein-Gordon component

Let us firstly deal with the semi-classical Klein-Gordon equation satisfied by $\widetilde{v}$:

$$(3.2.13)\qquad \big[D_t - Op_h^w(x\cdot\xi - p(\xi))\big]\widetilde{v}(t,x) = h^{-1} r_{kg}^{NF}(t,tx),$$

where $p(\xi) = \langle\xi\rangle$, and r_{kg}^{NF} is given by (3.1.5) and satisfies (3.1.6). We remind that $p'(\xi)$ denotes the gradient of $p(\xi)$ while $p''(\xi)$ is its 2×2 Hessian matrix.

We introduce the following manifold

$$\Lambda_{kg} := \left\{(x,\xi) : x - p'(\xi) = 0\right\},$$

for some small $\sigma > 0$, which is actually the graph of function $\xi = -d\phi(x)$, with $\phi(x) = \sqrt{1-|x|^2}$, as shown in picture 3.1. The main idea to obtain an uniform-in-time control on the $H^{\rho,\infty}_h$ norm of $\widetilde{v}$ is to decompose this function into the sum of two components, one localized in a neighbourhood of Λ_{kg}, and another one localized out of this neighbourhood. Up to assume a moderate growth for the L^2 norm of $\mathcal{L}^\mu\widetilde{v}$, with $0 \le |\mu| \le 2$, the contribution localized away from Λ_{kg} shows a better decay in time $h^{1/2-0}$ than the one in (3.2.4b) (see corollary 3.2.3). Thus, the main contribution to $\widetilde{v}$ appears to be the one localized around Λ_{kg}. We are going to show that it is solution to some ODE (see proposition 3.2.6), and derive from this equation an uniform control on its $H^{\rho,\infty}_h$ norm with which we will be finally able to propagate (3.2.4b), and hence (1.1.11b) (see proposition 3.2.7). We consider a neighbourhood of size $\sqrt{h}$, in the spirit of [14].

For a fixed $\rho \in \mathbb{Z}$, let $\Sigma(\xi) := \langle\xi\rangle^\rho$ and

$$(3.2.14)\qquad \Gamma^{kg} := Op_h^w\left(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma\xi)\right),$$

for some $\gamma,\chi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 close to the origin, $\sigma > 0$ is small (e.g. $\sigma < \frac{1}{4}$). We observe that, as the support of $\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)$ is included in $\{(x,\xi)\,|\,|\xi| \lesssim h^{-\sigma}, |x| \le 1 - ch^{2\sigma}\}$, for a small constant $c>0$, we may find a smooth function $\theta_h(x)$, equal to 1 for $|x| \le 1 - ch^{2\sigma}$ and

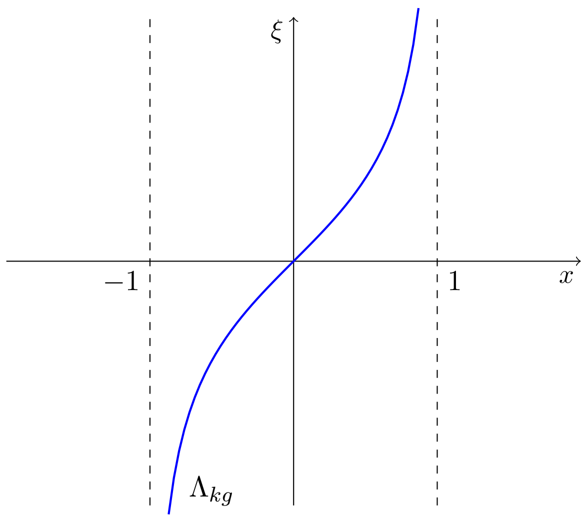

Figure 3.1: Lagrangian for the Klein-Gordon equation

supported for $|x| \le 1 - c_1 h^{2\sigma}$, for some $0 < c_1 < c$, with $\|\partial^\alpha \theta_h\|_{L^\infty} = O(h^{-2|\alpha|\sigma})$ and $(h\partial_h)^k\theta_h$ bounded for every $k \in \mathbb{N}$, such that

$$\gamma\left(\frac{x - p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma \xi) = \theta_h(x)\gamma\left(\frac{x - p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma \xi). \tag{3.2.15}$$

We also introduce the following notations:

$$\widetilde{v}^\Sigma := Op_h^w(\Sigma(\xi))\widetilde{v}, \tag{3.2.16}$$

together with

$$\widetilde{v}^\Sigma_{\Lambda_{kg}} := \Gamma^{kg}\widetilde{v}^\Sigma, \tag{3.2.17a}$$

$$\widetilde{v}^\Sigma_{\Lambda^c_{kg}} := Op_h^w\left(1 - \gamma\left(\frac{x - p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma \xi)\right)\widetilde{v}^\Sigma, \tag{3.2.17b}$$

so that $\widetilde{v}^\Sigma = \widetilde{v}^\Sigma_{\Lambda_{kg}} + \widetilde{v}^\Sigma_{\Lambda^c_{kg}}$, and remind that $\|\mathcal{L}^\gamma w\| = \|\mathcal{L}_1^{\gamma_1}\mathcal{L}_2^{\gamma_2} w\|$, for any $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$.

Lemma 3.2.2. *Let $\widetilde{\gamma} \in C^\infty(\mathbb{R}^2)$ vanish in a neighbourhood of the origin and be such that $|\partial_z^\alpha \widetilde{\gamma}(z)| \lesssim \langle z\rangle^{-|\alpha|}$, $c(x,\xi) \in S_{\delta,\sigma}(1)$ with $\delta \in [0, \frac{1}{2}]$, $\sigma > 0$, be supported for $|\xi| \lesssim h^{-\sigma}$. Then, for any $\chi \in C_0^\infty(\mathbb{R}^2)$ such that $\chi(h^\sigma\xi) \equiv 1$ on the support of $c(x,\xi)$,*

$$\left\|Op_h^w\Big(\widetilde{\gamma}\Big(\frac{x - p'(\xi)}{\sqrt{h}}\Big)c(x,\xi)\Big)w\right\|_{L^2} \lesssim \sum_{|\mu|=0}^{1} h^{\frac{1}{2}-\beta}\left\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu w\right\|_{L^2}, \tag{3.2.18a}$$

$$\left\|Op_h^w\Big(\widetilde{\gamma}\Big(\frac{x - p'(\xi)}{\sqrt{h}}\Big)c(x,\xi)\Big)w\right\|_{L^\infty} \lesssim \sum_{|\mu|=0}^{1} h^{-\beta}\left\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu w\right\|_{L^2}, \tag{3.2.18b}$$

and

$$\left\|Op_h^w\Big(\widetilde{\gamma}\Big(\frac{x - p'(\xi)}{\sqrt{h}}\Big)c(x,\xi)\Big)w\right\|_{L^2} \lesssim \sum_{|\mu|=0}^{2} h^{1-\beta}\left\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu w\right\|_{L^2}, \tag{3.2.19a}$$

$$\left\|Op_h^w\Big(\widetilde{\gamma}\Big(\frac{x - p'(\xi)}{\sqrt{h}}\Big)c(x,\xi)\Big)w\right\|_{L^\infty} \lesssim \sum_{|\mu|=0}^{2} h^{\frac{1}{2}-\beta}\left\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu w\right\|_{L^2}, \tag{3.2.19b}$$

for a small $\beta > 0$, $\beta \to 0$ as $\sigma \to 0$.

Proof. The proof of (3.2.18) (resp. of (3.2.19)) follows straightly by inequalities (1.2.62) (resp. (1.2.63)), after observing that, as $\widetilde{\gamma}$ vanishes in a neighbourhood of the origin,

$$\widetilde{\gamma}\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)c(x,\xi) = \sum_{j=1}^{2} \widetilde{\gamma}_1^j\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)c(x,\xi),$$

where $\widetilde{\gamma}_1^j(z) := \widetilde{\gamma}(z)z_j|z|^{-2}$ is such that $|\partial_z^\alpha \widetilde{\gamma}_1^j(z)| \lesssim \langle z\rangle^{-1-|\alpha|}$ (resp.

$$\widetilde{\gamma}\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)c(x,\xi) = \sum_{j=1}^{2} \widetilde{\gamma}_2\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)^2 c(x,\xi),$$

where $\widetilde{\gamma}_2(z) := \widetilde{\gamma}(z)|z|^{-2}$ is such that $|\partial_z^\alpha \widetilde{\gamma}(z)| \lesssim \langle z\rangle^{-2-|\alpha|}$). ◻

Corollary 3.2.3. *There exists $s > 0$ sufficiently large such that*

$$\Big\|\widetilde{v}^\Sigma_{\Lambda^c_{kg}}\Big\|_{L^2} \lesssim h^{1-\beta}\left(\|\widetilde{v}\|_{H^s_h} + \sum_{1\le|\mu|\le 2}\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}\|_{L^2}\right), \tag{3.2.20a}$$

$$\Big\|\widetilde{v}^\Sigma_{\Lambda^c_{kg}}\Big\|_{L^\infty} \lesssim h^{\frac{1}{2}-\beta}\left(\|\widetilde{v}\|_{H^s_h} + \sum_{1\le|\mu|\le 2}\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}\|_{L^2}\right). \tag{3.2.20b}$$

for a small positive $\beta = \beta(\sigma) \to 0$ as $\sigma \to 0$.

Proof. Since symbol $1-\gamma(\frac{x-p'(\xi)}{\sqrt{h}})\chi(h^\sigma\xi)$ is supported for $|\frac{x-p'(\xi)}{\sqrt{h}}| \ge d_1 > 0$ or $|h^\sigma\xi| \ge d_2 > 0$, for some small $d_1, d_2 > 0$, we may consider a smooth cut-off function $\widetilde{\chi}$, equal to 1 close to the origin and such that $\widetilde{\chi}\chi \equiv \widetilde{\chi}$, so that $\big(1-\gamma(\frac{x-p'(\xi)}{\sqrt{h}})\chi(h^\sigma\xi)\big)$ writes as

$$\left[1-\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\right]\widetilde{\chi}(h^\sigma\xi) + \left[1-\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi(h^\sigma\xi)\right](1-\widetilde{\chi})(h^\sigma\xi),$$

the first symbol being supported in $\{(x,\xi) | |\frac{x-p'(\xi)}{\sqrt{h}}| \ge d_1, |\xi| \lesssim h^{-\sigma}\}$, the second one for large frequencies $|\xi| \gtrsim h^{-\sigma}$.

Using lemma 1.2.24 and the fact that $\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi) \in S_{\frac{1}{2},\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-M}\big)$, for any $M \in \mathbb{N}$, we have that, for a fixed $N \in \mathbb{N}^*$,

$$\begin{aligned}\Big[1-\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big]\big(1-\widetilde{\chi}(h^\sigma\xi)\big) &= \big(1-\widetilde{\chi}(h^\sigma\xi)\big)\sharp\Big[1-\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big] \\ &\quad + \sum_{1\le j<N}\widetilde{\chi}_j(h^\sigma\xi)\sharp a_j(x,\xi) + r_N(x,\xi),\end{aligned}$$

where function $\widetilde{\chi}_j(h^\sigma\xi)$ is still supported for large frequencies $|\xi| \gtrsim h^{-\sigma}$, for every $1 \le j < N$, and up to negligible multiplicative constants,

$$a_j(x,\xi) = h^{j(\frac{1}{2}+\sigma)}\sum_{|\alpha|=j}(\partial^\alpha\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi) \in h^{j(\frac{1}{2}+\sigma)}S_{\frac{1}{2},\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-M}\big),$$

and $r_N \in h^{N(\frac{1}{2}+\sigma)}S_{\frac{1}{2},\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-M})$. Lemma 1.2.39, proposition 1.2.36, and semi-classical Sobolev injection imply that

$$\left\|Op^w_h\Big(\Big[1-\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big](1-\widetilde{\chi})(h^\sigma\xi)\Big)\widetilde{v}^\Sigma\right\|_{L^2} \lesssim h^{N(s)}\|\widetilde{v}\|_{H^s_h},$$
$$\left\|Op^w_h\Big(\Big[1-\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big](1-\widetilde{\chi})(h^\sigma\xi)\Big)\widetilde{v}^\Sigma\right\|_{L^\infty} \lesssim h^{N'(s)}\|\widetilde{v}\|_{H^s_h},$$

where $N(s), N'(s)\ge 1$ if $s>2$ is sufficiently large.

On the other hand, as function $(1-\gamma)\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)$ vanishes in a neighbourhood of the origin, and is such that $|\partial^\alpha_z(1-\gamma)(z)|\lesssim\langle z\rangle^{-|\alpha|}$, by lemma 3.2.2 and the fact that, using symbolic calculus to commute $\mathcal{L}$ with $\Sigma(\xi)$,

$$\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}^\Sigma\|_{L^2}\lesssim h^{-\nu}\sum_{|\mu_1|\le|\mu|}\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^{\gamma_1}\widetilde{v}\|_{L^2}, \tag{3.2.21}$$

with $\nu=\rho\sigma$ if $\rho\ge 0$, 0 otherwise, we have that

$$\left\|Op^w_h\Big((1-\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big)\widetilde{v}^\Sigma(t,\cdot)\right\|_{L^2}\lesssim\sum_{|\mu|\le 2}h^{1-\beta}\,\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2}\,,$$
$$\left\|Op^w_h\Big((1-\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big)\widetilde{v}^\Sigma(t,\cdot)\right\|_{L^\infty}\lesssim\sum_{|\mu|\le 2}h^{1-\beta}\,\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2}\,,$$

for a small $\beta>0$, $\beta\to 0$ as $\sigma\to 0$. $\square$

In the following lemma we show how to develop a symbol $a(x,\xi)$, associated to some operator acting on $\widetilde{v}^\Sigma_{\Lambda_{kg}}$, at $\xi=-d\phi(x)$, where $\phi(x)=\sqrt{1-|x|^2}$.

Lemma 3.2.4. *Let $a(x,\xi)$ be a real symbol in $S_{\delta,0}(\langle\xi\rangle^q)$, $q\in\mathbb{R}$, for some $\delta>0$ small. There exists a family $(\theta_h(x))_h$ of C^∞_0 functions, real valued, equal to 1 on the closed ball $\overline{B_{1-ch^{2\sigma}}(0)}$ and supported in $\overline{B_{1-c_1h^{2\sigma}}(0)}$, for some small $0<c_1<c,\sigma>0$, with $\|\partial^\alpha_x\theta_h\|_{L^\infty}=O(h^{-2|\alpha|\sigma})$ and $(h\partial_h)^k\theta_h$ bounded for every k, such that*

$$Op^w_h(a)\widetilde{v}^\Sigma_{\Lambda_{kg}}=\theta_h(x)a(x,-d\phi(x))\widetilde{v}^\Sigma_{\Lambda_{kg}}+R_1(\widetilde{v})\,, \tag{3.2.22}$$

where $R_1(\widetilde{v})$ satisfies

$$\|R_1(\widetilde{v})\|_{L^2}\lesssim h^{1-\beta}\left(\|\widetilde{v}\|_{H^s_h}+\sum_{|\gamma|=1}\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\gamma\widetilde{v}\|_{L^2}\right), \tag{3.2.23a}$$

$$\|R_1(\widetilde{v})\|_{L^\infty}\lesssim h^{\frac{1}{2}-\beta}\left(\|\widetilde{v}\|_{H^s_h}+\sum_{|\gamma|=1}\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\gamma\widetilde{v}\|_{L^2}\right), \tag{3.2.23b}$$

with $\beta=\beta(\sigma,\delta)>0$, $\beta\to 0$ as $\sigma,\delta\to 0$. Moreover, if $\partial_\xi a(x,\xi)|_{\xi=-d\phi(x)}=0$, the above estimates can be improved and $R_1(\widetilde{v})$ is rather a remainder $R_2(\widetilde{v})$, such that

$$\|R_2(\widetilde{v})\|_{L^2}\lesssim h^{2-\beta}\left(\|\widetilde{v}\|_{H^s_h}+\sum_{1\le|\gamma|\le 2}\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\gamma\widetilde{v}\|_{L^2}\right), \tag{3.2.24a}$$

$$\|R_2(\widetilde{v})\|_{L^\infty}\lesssim h^{\frac{3}{2}-\beta}\left(\|\widetilde{v}\|_{H^s_h}+\sum_{1\le|\gamma|\le 2}\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\gamma\widetilde{v}\|_{L^2}\right). \tag{3.2.24b}$$

Proof. We have already observed that the symbol associated to operator Γ^{kg} is localised in space in a closed ball $\overline{B_{1-ch^{2\sigma}}(0)}$, and that there exists a family of smooth cut-off functions $(\theta_h(x))_{h\in]0,1]}$ as in the statement, such that

$$\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)=\theta_h(x)\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi).$$

In addition, we highlight the fact that the support of any derivative of θ_h has empty intersection with that of $\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)$ and its derivatives. After remark 1.2.22, this implies that $\widetilde{v}^\Sigma_{\Lambda_{kg}}=\theta_h(x)\widetilde{v}^\Sigma_{\Lambda_{kg}}+r_\infty$, where $r_\infty\in h^N S_{\frac{1}{2},\sigma}(\langle x\rangle^{-\infty})$, and hence that $Op^w_h(a)\widetilde{v}^\Sigma_{\Lambda_{kg}}=Op^w_h(a)\theta_h(x)\widetilde{v}^\Sigma_{\Lambda_{kg}}+Op^w_h(r^a_\infty)\widetilde{v}^\Sigma_{\Lambda_{kg}}$, $r^a_\infty=a\sharp r_\infty\in h^{N-\gamma}S_{\frac{1}{2},\sigma}(\langle x\rangle^{-\infty})$ with $\gamma=q\sigma$ if $q\ge 0$, 0 otherwise. It follows at once, from proposition 1.2.36 and semi-classical Sobolev injection, that $Op^w_h(r^a_\infty)\widetilde{v}^\Sigma_{\Lambda_{kg}}$ satisfies enhanced estimates (3.2.24) if N is taken sufficiently large.

Up to negligible multiplicative constants, a further application of symbolic calculus gives also that

$$Op^w_h(a(x,\xi))\theta_h(x)\widetilde{v}^\Sigma_{\Lambda_{kg}}=Op^w_h(a(x,\xi)\theta_h(x))\widetilde{v}^\Sigma_{\Lambda_{kg}}+\sum_{|\alpha|=1}^{N-1}h^{|\alpha|}Op^w_h\big(\partial^\alpha_\xi a(x,\xi)\partial^\alpha_x\theta_h(x)\big)\widetilde{v}^\Sigma_{\Lambda_{kg}}\\+Op^w_h(r_N(x,\xi))\widetilde{v}^\Sigma_{\Lambda_{kg}},$$

where $r_N\in h^{N-\beta}S_{\delta',0}(\langle\xi\rangle^{q-N}\langle x\rangle^{-\infty})$, for a new small $\beta=\beta(\delta,2\sigma)$ and $\delta'=\max\{\delta,\sigma\}$. As before, $Op^w_h(r_N)\widetilde{v}^\Sigma_{\Lambda_{kg}}$ verifies enhanced estimates (3.2.24) if N is suitably chosen. Also, since for any $|\alpha|\ge 1$ the support of $\partial^\alpha_\xi a(x,\xi)\cdot\partial^\alpha_x\theta_h(x)$ has empty intersection with that of $\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)$, all the $|\alpha|$-order terms, with $1\le|\alpha|<N$, are of the form $Op^w_h(r_\infty)\widetilde{v}^\Sigma$, for a new $r_\infty\in h^NS_{\frac{1}{2},\sigma}(\langle x\rangle^{-\infty})$ with $N\in\mathbb{N}$ as large as we want, and are remainders $R_2(\widetilde{v})$.

Now, as symbol $a(x,\xi)\theta_h(x)$ is supported for $|x|\le 1-c_1h^{2\sigma}<1$, we are allowed to develop it at $\xi=-d\phi(x)$:

$$\begin{aligned}a(x,\xi)\theta_h(x)&=a(x,-d\phi(x))\theta_h(x)+\sum_{|\alpha|=1}\int_0^1(\partial^\alpha_\xi a)(x,t\xi+(1-t)d\phi(x))dt\,\theta_h(x)(\xi+d\phi(x))^\alpha\\&=a(x,-d\phi(x))\theta_h(x)+\sum_{j=1}^2 b_j(x,\xi)(x_j-p'_j(\xi)),\end{aligned}\tag{3.2.25}$$

with

$$b_j(x,\xi)=\sum_{|\alpha|=1}\int_0^1(\partial^\alpha_\xi a)(x,t\xi+(1-t)d\phi(x))dt\,\theta_h(x)\frac{(\xi+d\phi(x))^\alpha(x_j-p'_j(\xi))}{|x-p'(\xi)|^2},\quad j=1,2.\tag{3.2.26}$$

If $\chi_1\in C^\infty_0(\mathbb{R}^2)$ is a new cut-off function equal to 1 close to the origin, we can reduce ourselves to the analysis of symbol $b_j(x,\xi)(x_j-p'_j(\xi))\chi_1(h^\sigma\xi)$, for $b_j(x,\xi)(x_j-p'_j(\xi))(1-\chi_1)(h^\sigma\xi)$ is supported for large frequencies and its operator acting on $\widetilde{v}^\Sigma_{\Lambda_{kg}}$ is a remainder $O_{L^2\cap L^\infty}(h^N\|\widetilde{v}\|_{H^s_h})$, with $N>0$ large as long as $s>0$ is large, as one can prove using semi-classical Sobolev injection, symbolic calculus of 1.2.21, lemma 1.2.39 and proposition 1.2.36. Furthermore, considering a smooth cut-off function $\widetilde{\gamma}\in C^\infty_0(\mathbb{R}^2)$, equal to 1 close to the origin and such that $\widetilde{\gamma}\big(\langle\xi\rangle^2(x-p'(\xi))\big)\equiv 1$ on the support of $\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)$, which is possible if $\sigma<1/4$, we have that

$$b_j(x,\xi)(x_j-p'_j(\xi))\chi_1(h^\sigma\xi)=b_j(x,\xi)(x_j-p'_j(\xi))\chi_1(h^\sigma\xi)\widetilde{\gamma}\big(\langle\xi\rangle^2(x-p'(\xi))\big)\\+b_j(x,\xi)(x_j-p'_j(\xi))\chi_1(h^\sigma\xi)(1-\widetilde{\gamma})\big(\langle\xi\rangle^2(x-p'(\xi))\big).$$

Since $b_j(x,\xi)(x_j-p_j'(\xi))\chi_1(h^\sigma\xi)(1-\widetilde{\gamma})\big(\langle\xi\rangle^2(x-p'(\xi))\big)\in h^{-\beta}S_{\delta,\sigma}(1)$, for some new small $\beta,\delta>0$, and its support has empty intersection with that of $\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)$ (which instead belongs to class $S_{\frac{1}{2},0}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-M})$, for $M\in\mathbb{N}$ as large as we want),

$$\Big[b_j(x,\xi)(x_j-p_j'(\xi))\chi_1(h^\sigma\xi)(1-\widetilde{\gamma})\big(\langle\xi\rangle^2(x-p'(\xi))\big)\Big]\sharp\Big[\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)\Big]=\widetilde{r}_\infty,$$

where $\widetilde{r}_\infty\in h^NS_{\frac{1}{2},\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-M})$ with $N\in\mathbb{N}$ as large as we want, as follows from lemma 1.2.24 and remark 1.2.22. Therefore,

$$Op_h^w\left(b_j(x,\xi)(x_j-p_j'(\xi))\chi_1(h^\sigma\xi)(1-\widetilde{\gamma})\big(\langle\xi\rangle^2(x-p'(\xi))\big)\right)\widetilde{v}^\Sigma_{\Lambda_{kg}}=Op_h^w(r_\infty)\widetilde{v}^\Sigma,$$

where, as before, $Op_h^w(\widetilde{r}_\infty)\widetilde{v}^\Sigma$ is an enhanced remainder $R_2(\widetilde{v})$.
If $c(x,\xi):=b_j(x,\xi)\chi_1(h^\sigma\xi)\widetilde{\gamma}\big(\langle\xi\rangle^2(x-p'(\xi))\big)\in h^{-\beta}S_{2\sigma,\sigma}(1)$, with β depending linearly on σ, the very contribution that only enjoys estimates (3.2.23) is $Op_h^w\big(c(x,\xi)(x_j-p_j'(\xi))\big)\widetilde{v}^\Sigma_{\Lambda_{kg}}$, whose symbol is in $h^{1/2-\beta}S_{2\sigma,\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle\big)$ by lemma 1.2.42. In fact, if we assume that the support of χ_1 is sufficiently small, so that $\chi_1\chi\equiv\chi_1$ and all derivatives of χ vanish on that support, by using symbolic development (1.2.18) until a sufficiently large order N, and observing that

$$\begin{aligned}\Big\{c(x,\xi)(x_j-p_j'(\xi)),\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big\}&=\Big\{c(x,\xi),\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big\}(x_j-p_j'(\xi))\\&=\Big[(\partial_\xi c)\cdot(\partial\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)+(\partial_x c)\cdot(\partial\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)p''(\xi)\Big]\Big(\frac{x_j-p_j'(\xi)}{\sqrt{h}}\Big)\end{aligned}$$

does not lose any power $h^{-1/2}$, we derive, up to negligible constants, that

$$\begin{aligned}\Big[c(x,\xi)(x_j-p_j'(\xi))\Big]\sharp\Big[\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big]&=\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)c(x,\xi)(x_j-p_j'(\xi))\\&\quad+{\sum}'h\widetilde{\gamma}\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\widetilde{c}(x,\xi)+r_N(x,\xi),\end{aligned}$$

where $\sum'$ is a concise notation to indicate a linear combination, $\widetilde{\gamma}\in C_0^\infty(\mathbb{R}^2\backslash\{0\})$, $\widetilde{c}\in h^{-\beta}S_{\delta,\sigma}(1)$, for some new small $\beta,\delta>0$, and $r_N\in h^{N/2-\beta}S_{\frac{1}{2},\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-(M-1)}\big)$. From inequalities (1.2.62) and (3.2.21), we deduce that $Op_h^w\big(\gamma(\frac{x-p'(\xi)}{\sqrt{h}})\chi(h^\sigma\xi)c(x,\xi)(x_j-p_j'(\xi))\big)\widetilde{v}^\Sigma$ is a remainder $R_1(\widetilde{v})$ satisfying (3.2.23). The quantization of all other contributions in above right hand side, when acting on $\widetilde{v}^\Sigma$, is estimated, on the one hand, by using that $\widetilde{\gamma}(z)$ vanishes in a neighbourhood of the origin and can be rewritten as $\sum_{j=1,2}\widetilde{\gamma}_2(z)z_j^2$, with $\widetilde{\gamma}_2(z):=\widetilde{\gamma}(z)|z|^{-2}$ such that $|\partial_z^\alpha\widetilde{\gamma}_2(z)|\lesssim\langle z\rangle^{-2-|\alpha|}$. Inequalities (1.2.63), and successive commutation of $\mathcal{L}^\gamma$, $|\gamma|=1,2$, with Σ, give then that $hOp_h^w\big(\widetilde{\gamma}(\frac{x-p'(\xi)}{\sqrt{h}})\widetilde{c}(x,\xi)\big)\widetilde{v}^\Sigma$ is a remainder $R_2(\widetilde{v})$. On the other hand, as $r_N(x,\xi)\sharp\Sigma(\xi)\in h^{\frac{N}{2}-\beta-\mu}S_{\frac{1}{2},\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-(M-1)})$, with $\mu=\sigma\rho$ if $\rho\geq0$, 0 otherwise, it follows that it is a remainder just from 1.2.36, 1.2.37, fixing $N\in\mathbb{N}$ sufficiently large (e.g. $N=3$).
If symbol $a(x,\xi)$ is such that $\partial_\xi a|_{\xi=-d\phi}=0$, instead of equality (3.2.25), with b_j given by (3.2.26), we have

$$a(x,\xi)\theta_h(x)=a(x,-d\phi(x))\theta_h(x)+\sum_{j=1,2}b(x,\xi)(x_j-p_j'(\xi))^2,$$

with

$$b(x,\xi)=\sum_{|\alpha|=2}\frac{2}{\alpha!}\int_0^1(\partial_\xi^\alpha a)(t\xi-(1-t)d\phi(x))(1-t)dt\,\theta_h(x)\frac{(\xi+d\phi(x))^\alpha}{|x-p'(\xi)|^2},$$

and the same argument as before can be applied to $Op_h^w\big(b(x,\xi)\theta_h(x)(x_j-p_j'(\xi))^2\big)\widetilde{v}^\Sigma_{\Lambda_{kg}}$ to show that it reduces to

$$Op_h^w\Big(b(x,\xi)\theta_h(x)(x_j-p_j'(\xi))^2\chi_1(h^\sigma\xi)\widetilde{\gamma}\big(\langle\xi\rangle^2(x-p'(\xi))\big)\Big)\widetilde{v}^\Sigma_{\Lambda_{kg}}+R_2(\widetilde{v}),$$

with $R_2(\widetilde{v})$ satisfying (3.2.24). If

$$B(x,\xi):=b(x,\xi)\theta_h(x)\chi_1(h^\sigma\xi)\widetilde{\gamma}\big(\langle\xi\rangle^2(x-p'(\xi))\big)$$

$B(x,\xi)(x_j-p_j'(\xi))^2\in h^{-\beta}S_{\delta',\sigma}(1)$ by lemma 1.2.42, for some new small $\beta,\delta'>0$ depending on σ,δ, so using lemma 1.2.24 and symbolic development (1.2.18) until order 4, and assuming that the support of χ_1 is sufficiently small so that $\chi\chi_1\equiv\chi$, we derive that

$$\begin{aligned}\Big[B(x,\xi)(x_j-p_j'(\xi))^2\Big]\sharp\Big[\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big]&=B(x,\xi)\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(x_j-p_j'(\xi))^2\\+\frac{h}{i}\sum_{i=1}^2(\partial_i\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big(\frac{x_j-p_j'(\xi)}{\sqrt{h}}\Big)&\left[(\partial_{\xi_i}B)+\sum_j(\partial_{x_j}B)p''_{ij}(\xi)\right](x_j-p_j'(\xi))\\&+\sum_{2\le|\alpha|\le3}{}'\,h^{\frac{|\alpha|}{2}-2\delta'-\beta}\gamma_\alpha\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)B_\alpha(x,\xi)+r_4(x,\xi),\end{aligned}$$

where $\gamma_\alpha\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$, $B_\alpha(x,\xi)\in S_{\delta',\sigma}(1)$, and $r_4(x,\xi)\in h^{2-4\delta'-\beta}S_{\frac12,\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-M}\big)$. As $r_4(x,\xi)\sharp\Sigma(\xi)\in h^{2-\beta'}S_{\frac12,\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-M}\big)$, for $\beta'=2-4\delta'-\beta-\rho\sigma$ if $\rho\ge0$, $\beta'=2-4\delta'-\beta$ otherwise, it immediately follows from propositions 1.2.36, 1.2.37 that $Op_h^w(r_4)\widetilde{v}^\Sigma$ is a remainder $R_2(\widetilde{v})$. After inequalities (1.2.63) with $\gamma_n=\gamma$ and $c=B$ (resp. inequalities (1.2.62) with $\gamma_n(z)=\partial_i\gamma(z)z_j$ and $c=h^{\delta'}[(\partial_{\xi_i}B)+(\partial_xB)\cdot(\partial_\xi p_1'+\partial_\xi p_2')]\in S_{\delta',\sigma}(1)$, for $i,j=1,2$), and (3.2.21), we deduce that the quantization of the first (resp. the second) contribution in above symbolic development is a remainder $R_2(\widetilde{v})$, when acting on $\widetilde{v}^\Sigma$. Finally, as γ_α vanishes in a neighbourhood of the origin, we write

$$\gamma_\alpha\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)=\sum_{k=1}^2h^{-1}\underbrace{\gamma_\alpha\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big|\frac{x-p'(\xi)}{\sqrt{h}}\Big|^{-2}}_{\widetilde{\gamma}_\alpha\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)}\times(x_k-p_k'(\xi))^2,\quad|\alpha|=2,$$

$$\gamma_\alpha\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)=\sum_{k=1}^2h^{-\frac12}\underbrace{\gamma_\alpha\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\Big(\frac{x_k-p_k'(\xi)}{\sqrt{h}}\Big)\Big|\frac{x-p'(\xi)}{\sqrt{h}}\Big|^{-2}}_{\widetilde{\gamma}^k_\alpha\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)}\times(x_k-p_k'(\xi)),\quad|\alpha|=3$$

and obtain that the quantization of α-th order term with $|\alpha|=2$ (resp. $|\alpha|=3$) is a remainder $R_2(\widetilde{v})$, when acting on $\widetilde{v}^\Sigma$, after inequalities (1.2.63) (resp. (1.2.62)) with $\gamma_n=\widetilde{\gamma}_\alpha$ (resp. $\gamma_n=\widetilde{\gamma}^k_\alpha$, $k=1,2$) and $c=B_\alpha$. $\square$

The following two results allow us to finally derive the ODE satisfied by $\widetilde{v}^\Sigma_{\Lambda_{kg}}$.

Lemma 3.2.5. *We have that*

$$\big[D_t-Op_h^w(x\cdot\xi-p(\xi)),\Gamma^{kg}\big]=Op_h^w(b),\tag{3.2.27}$$

where

(3.2.28)
$$b(x,\xi)=-\frac{h}{2i}(\partial\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\cdot\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)-\frac{\sigma h}{i}\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(\partial\chi)(h^\sigma\xi)\cdot(h^\sigma\xi)$$
$$+\frac{i}{24}h^{\frac{3}{2}}\sum_{|\alpha|=3}(\partial^\alpha\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(\partial^\alpha_\xi p'(\xi))\chi(h^\sigma\xi)+r(x,\xi),$$

and $r\in h^{5/2}S_{\frac{1}{2},\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-N})$, *for any* $N\ge 0$. *Therefore, function* $\widetilde{v}^{\Sigma}_{\Lambda_{kg}}$ *is solution to the following equation:*

$$\big[D_t-Op^w_h(x\cdot\xi-p(\xi))\big]\widetilde{v}^{\Sigma}_{\Lambda_{kg}}=\Gamma^{kg}Op^w_h(\Sigma(\xi))\big[h^{-1}r^{NF}_{kg}(t,tx)\big]+R_2(\widetilde{v}), \tag{3.2.29}$$

with $R_2(\widetilde{v})$ *satisfying estimates* (3.2.24).

Proof. Recalling the definition (3.2.14) of Γ^{kg}, one can prove by a straight computation that

$$[D_t,\Gamma^{kg}]=\frac{h}{i}Op^w_h\Big((\partial\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\cdot\frac{p''(\xi)\xi}{\sqrt{h}}\chi(h^\sigma\xi)\Big)$$
$$+\frac{h}{2i}Op^w_h\Big((\partial\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\cdot\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big)-\frac{(1+\sigma)h}{i}Op^w_h\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(\partial\chi)(h^\sigma\xi)\cdot(h^\sigma\xi)\Big).$$

On the other hand, since the development of a commutator's symbol only contains odd-order terms, lemma 1.2.24 gives that the symbol associated to $\big[\Gamma^{kg},Op^w_h(x\cdot\xi-p(\xi))\big]$ writes as

$$\frac{h}{i}\Big\{\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi),x\cdot\xi-p(\xi)\Big\}+\frac{i}{24}h^{\frac{3}{2}}\sum_{|\alpha|=3}(\partial^\alpha\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)(\partial^\alpha_\xi p(\xi))+r_5(x,\xi)$$

with $r_5\in h^{5/2}S_{\frac{1}{2},\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-N})$, for any $N\ge 0$. Developing the above Poisson brackets, one finds that

$$\big[\Gamma^{kg},Op^w_h(x\cdot\xi-p(\xi))\big]=-\frac{h}{i}Op^w_h\Big((\partial\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\cdot\frac{p''(\xi)\xi}{\sqrt{h}}\chi(h^\sigma\xi)\Big)$$
$$-\frac{h}{i}Op^w_h\Big((\partial\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\cdot\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big)+\frac{h}{i}Op^w_h\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(\partial\chi)(h^\sigma\xi)\cdot(h^\sigma\xi)\Big)$$
$$+\frac{i}{24}h^{\frac{3}{2}}\sum_{|\alpha|=3}Op^w_h\Big((\partial^\alpha\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(\partial^\alpha_\xi p'(\xi))\chi(h^\sigma\xi)\Big)+Op^w_h(r_5(x,\xi)),$$

which summed to the previous commutator gives (3.2.28).

Last part of the statement follows applying to equation (3.2.13) operators $Op^w_h(\Sigma(\xi))$ (which commutes exactly with the linear part of the equation, evident in non semi-classical coordinates) and Γ^{kg}. Since

$$hOp^w_h\Big((\partial\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\cdot\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big)\widetilde{v}^{\Sigma}=$$
$$\sum_{k=1}^{2}Op^w_h\Big(\gamma^k\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\cdot(x-p'(\xi))(x_k-p'_k(\xi))\Big)\widetilde{v}^{\Sigma},$$

with $\gamma^k(z)=(\partial\gamma)(z)z_k|z|^{-2}$, and

$$h^{\frac{3}{2}}Op^w_h\Big((\partial^\alpha\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)(\partial^\alpha_\xi p'(\xi))\Big)=hOp^w_h\Big(\gamma^k_\alpha\Big(\frac{\xi-p'(\xi)}{\sqrt{h}}\Big)(\partial^\alpha_\xi p'(\xi))(x_k-p'_k(\xi))\Big)\widetilde{v}^{\Sigma},$$

with $\gamma^k_\alpha(z) = (\partial^\alpha\gamma)(z)z_k|z|^{-2}$, we obtain from inequalities (1.2.63) (resp. (1.2.62)) and (3.2.21) that $hOp^w_h\big((\partial\gamma)\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\cdot\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)\big)\widetilde{v}^\Sigma$ (resp. $h^{3/2}Op^w_h\big((\partial^\alpha\gamma)\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)(\partial^\alpha_\xi p'(\xi))\big)$, $|\alpha|=3$) is a remainder $R_2(\widetilde{v})$. The same holds true for $Op^w_h\big(\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)(\partial\chi)(h^\sigma\xi)\cdot(h^\sigma\xi)\big)\widetilde{v}^\Sigma$, as follows combining symbolic calculus and lemma 1.2.39, because its symbol is supported for large frequencies $|\xi|\gtrsim h^{-\sigma}$. From propositions 1.2.36, 1.2.37 it immediately follows that $Op^w_h(r_5)\widetilde{v}^\Sigma$ satisfies (3.2.24a), (3.2.24b). ☐

Proposition 3.2.6 (Deduction of the ODE)**.** *There exists a family $(\theta_h(x))_h$ of C^∞_0 functions, real valued, equal to 1 on the closed ball $\overline{B_{1-ch^{2\sigma}}(0)}$ and supported in $\overline{B_{1-c_1h^{2\sigma}}(0)}$, for some small $0<c_1<c$, $\sigma>0$, with $\|\partial^\alpha_x\theta_h\|_{L^\infty}=O(h^{-2|\alpha|\sigma})$ and $(h\partial_h)^k\theta_h$ bounded for every k, such that*

$$Op^w_h(x\cdot\xi - p(\xi))\widetilde{v}^\Sigma_{\Lambda_{kg}} = -\phi(x)\theta_h(x)\widetilde{v}^\Sigma_{\Lambda_{kg}} + R_2(\widetilde{v}), \tag{3.2.30}$$

where $\phi(x)=\sqrt{1-|x|^2}$ and $R_2(\widetilde{v})$ satisfies estimates (3.2.24)*, for a small positive $\beta=\beta(\sigma)\to 0$ as $\sigma\to 0$. Therefore, $\widetilde{v}^\Sigma_{\Lambda_{kg}}$ is solution of the following non-homogeneous ODE:*

$$D_t\widetilde{v}^\Sigma_{\Lambda_{kg}} = -\phi(x)\theta_h(x)\widetilde{v}^\Sigma_{\Lambda_{kg}} + \Gamma^{kg}Op^w_h(\Sigma(\xi))\big[h^{-1}r^{NF}_{kg}(t,tx)\big] + R_2(\widetilde{v}), \tag{3.2.31}$$

with r^{NF}_{kg} given by (3.1.5)*.*

Proof. The proof of the statement follows directly from lemma 3.2.4, if we observe that $\partial_\xi(x\cdot\xi-p(\xi))=0$ at $\xi=-d\phi(x)$ and $x\cdot(-d\phi(x))-p(-d\phi(x))=-\phi(x)$. Therefore, (3.2.30) holds and, injecting it in (3.2.29), we obtain (3.2.31). ☐

Proposition 3.2.7 (Propagation of the uniform estimate on V)**.** *Let us fix $K_1>0$. There exist two integers n,ρ sufficiently large, with $n\gg\rho$, two constants $A,B>1$ sufficiently large, and $\varepsilon_0\in]0,(2A+B)^{-1}[$ sufficiently small, such that, for any $0<\varepsilon<\varepsilon_0$, if (u,v) is solution to* (1.1.1)-(1.1.2) *in some interval $[1,T]$, for a fixed $T>1$, and $u_\pm, v_\pm$ defined in* (1.1.5) *satisfy a-priori estimates* (1.1.11)*, for every $t\in[1,T]$, for a small $\delta>0$, then it also verify* (1.1.12b) *in the same interval $[1,T]$.*

Proof. We warn the reader that, throughout the proof, we will denote by C a positive constant, and by β (resp. β') a small positive constant, such that $\beta\to 0$ as $\sigma\to 0$ (resp. $\beta'\to 0$ as $\delta,\sigma\to 0$). These constants may change line after line. We also remind that $h=1/t$.

In proposition 3.1.1, we introduced function v^{NF}, defined from v_- through (3.1.3), and proved that its $H^{\rho,\infty}$ norm differs from that of v_- by a quantity satisfying (3.1.7b), for all $\rho\in\mathbb{N}$. Hence, for $\theta\in]0,1[$ sufficiently small (e.g. $\theta<1/4$), by a-priori estimates (1.1.11a), (1.1.11b), (1.1.11c) there exists a constant $C>0$ such that

$$\|v_-(t,\cdot)\|_{H^{\rho,\infty}} \le \|v^{NF}(t,\cdot)\|_{H^{\rho,\infty}} + CA^{2-\theta}B^\theta\varepsilon^2t^{-\frac{5}{4}}. \tag{3.2.32}$$

We successively introduced $\widetilde{v}$ in (3.2.2), and decomposed it into the sum of functions $\widetilde{v}^\Sigma_{\Lambda_{kg}}$ and $\widetilde{v}^\Sigma_{\Lambda^c_{kg}}$ (see (3.2.17)). We will show in lemma B.2.14 that, for any $s\le n$,

$$\|\widetilde{v}(t,\cdot)\|_{H^s_h} + \sum_{|\gamma|=1}^{2}\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\gamma\widetilde{v}(t,\cdot)\|_{L^2} \le CB\varepsilon h^{-\beta'}, \tag{3.2.33}$$

for all $t\in[1,T]$, so inequality (3.2.20b) gives that

$$\|\widetilde{v}^\Sigma_{\Lambda^c_{kg}}(t,\cdot)\|_{L^\infty} \le CB\varepsilon h^{\frac{1}{2}-\beta'} = CB\varepsilon t^{-\frac{1}{2}+\beta'}. \tag{3.2.34}$$

On the other hand, we proved in proposition 3.2.6 that $\widetilde{v}^{\Sigma}_{\Lambda_{kg}}$ is solution to ODE (3.2.31), with r^{NF}_{kg} given by (3.1.5) and satisfying (3.1.6), and $R_2(\widetilde{v})$ verifying (3.2.24). From (3.2.33), we then have that

$$\|R_2(\widetilde{v})(t,\cdot)\|_{L^\infty} \le C\varepsilon t^{-\frac{3}{2}+\beta'}.$$

We also have that

$$\left\|\Gamma^{kg}Op^w_h(\Sigma(\xi))[tr^{NF}_{kg}(t,tx)]\right\|_{L^\infty} \le C(A+B)AB\varepsilon^3 t^{-\frac{3}{2}+\beta'}. \tag{3.2.35}$$

In fact, by symbolic calculus of lemma 1.2.24 we derive that, for a fixed $N\in\mathbb{N}$ (e.g. $N>\rho$), and up to negligible multiplicative constants,

$$\Gamma^{kg}Op^w_h(\Sigma(\xi)) = \sum_{|\alpha|=0}^{N-1} h^{\frac{|\alpha|}{2}} Op^w_h\Big((\partial^\alpha\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)(\partial^\alpha\Sigma)(\xi)\Big) + Op^w_h(r_N(x,\xi)),$$

where $r_N \in h^{\frac{N}{2}}S_{\frac{1}{2},\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-1})$. Choosing N sufficiently large, we deduce from proposition 1.2.37, the fact that $\|tw(t,t\cdot)\|_{L^2}=\|w(t,\cdot)\|_{L^2}$, inequality (3.1.6a) and a-priori estimates, that

$$\left\|Op^w_h(r_N(x,\xi))[tr^{NF}_{kg}(t,tx)]\right\|_{L^\infty} \le CA^2B\varepsilon^3t^{-2},$$

for every $t\in[1,T]$. Using, instead, proposition 1.2.38 with $p=+\infty$, and lemma B.3.5 in appendix B, we deduce that

$$\begin{aligned}\sum_{|\alpha|=0}^{N-1} h^{\frac{|\alpha|}{2}} &\left\|Op^w_h\Big((\partial^\alpha\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)(\partial^\alpha\Sigma)(\xi)\Big)Op^w_h(\chi_1(h^\sigma\xi))[tr^{NF}_{kg}(t,tx)]\right\|_{L^\infty}\\ &\lesssim t^{1+\beta}\left\|\chi(t^{-\sigma}D_x)r^{NF}_{kg}(t,\cdot)\right\|_{L^\infty} \le C(A+B)AB\varepsilon^3t^{-\frac{3}{2}+\beta'}.\end{aligned}$$

Summing up, $\Gamma^{kg}Op^w_h(\Sigma(\xi))[t^{-1}r^{NF}_{kg}(t,tx)] + R_2(\widetilde{v}) = F_{kg}(t,x)$, where

$$\|F_{kg}(t,\cdot)\|_{L^\infty} \le [C(A+B)AB\varepsilon^3 + CB\varepsilon]t^{-\frac{3}{2}+\beta'},$$

where $\beta'>0$ is small as we want as long as σ,δ_0 are small, so using equation (3.2.31) we deduce that

$$\frac{1}{2}\partial_t|\widetilde{v}^{\Sigma}_{\Lambda_{kg}}(t,x)|^2 = \Im\left(\widetilde{v}^{\Sigma}_{\Lambda_{kg}}\overline{D_t\widetilde{v}^{\Sigma}_{\Lambda_{kg}}}\right) \le |\widetilde{v}^{\Sigma}_{\Lambda_{kg}}(t,x)||F_{kg}(t,x)|, \tag{3.2.36}$$

and hence

$$\begin{aligned}\|\widetilde{v}^{\Sigma}_{\Lambda_{kg}}(t,\cdot)\|_{L^\infty} &\le \|\widetilde{v}^{\Sigma}_{\Lambda_{kg}}(1,\cdot)\|_{L^\infty} + \int_1^t \|F_{kg}(\tau,\cdot)\|_{L^\infty}d\tau\\ &\le \|\widetilde{v}^{\Sigma}_{\Lambda_{kg}}(1,\cdot)\|_{L^\infty} + CA^2(A+B)\varepsilon^3 + CB\varepsilon.\end{aligned}$$

As $\|\widetilde{v}^{\Sigma}_{\Lambda_{kg}}(1,\cdot)\|_{L^\infty} \lesssim \|\widetilde{v}(1,\cdot)\|_{L^2} \le CB\varepsilon$, as follows by proposition 1.2.37 and a-priori estimate (1.1.11c), above inequality, (3.2.34), and definition (3.2.2) of $\widetilde{v}$, give that

$$\|v^{NF}(t,\cdot)\|_{L^\infty} \le (C(A+B)AB\varepsilon^3 + CB\varepsilon)t^{-1},$$

which, injected in (3.2.32), leads finally to (1.1.12b) if we take $A>1$ sufficiently large such that $CB<\frac{A}{3K_1}$, and $\varepsilon_0>0$ sufficiently small to verify $C(A+B)B\varepsilon_0^2 + CA^{1-\theta}B^\theta\varepsilon_0 \le \frac{1}{3K_1}$. □

3.2.2 The Derivation of the Transport Equation

We now focus on the semi-classical wave equation satisfied by $\widetilde{u}$:

$$\big[D_t - Op^w_h(x\cdot\xi - |\xi|)\big]\widetilde{u}(t,x) = h^{-1}\left[q_w(t,tx) + c_w(t,tx) + r^{NF}_w(t,tx)\right], \tag{3.2.37}$$

with q_w, c_w, r^{NF}_w given, respectively, by (3.1.17), (3.1.18), (3.1.19), and on the derivation of the mentioned transport equation. As we will make use several times of proposition 1.2.30 and inequalities (1.2.48), we remind the reader, once for all, that $\theta_0(x)$ will denote a smooth radial cut-off function (often coming with operator Ω_h) and $\chi \in C^\infty_0(\mathbb{R}^2)$, suitably supported, equal to 1 in a neighbourhood of the origin.

In order to recover a sharp estimate for $\widetilde{u}$ (and consequently for u_-), we study its behaviour, separately, in different regions of the phase space $(x,\xi)\in\mathbb{R}^2\times\mathbb{R}^2$. We start by fixing $\rho\in\mathbb{Z}$, and by introducing $\Sigma(\xi) := \langle\xi\rangle^\rho$ (or $\Sigma_j(\xi) := \langle\xi\rangle^\rho\xi_j|\xi|^{-1}$, for $j=1,2$). Taking a smooth cut-off function χ_0, equal to 1 in a neighbourhood of the origin, a Littlewood-Paley decomposition, and a small $\sigma>0$, we write the following:

$$\begin{split} Op^w_h(\Sigma(\xi))\widetilde{u} = Op^w_h(\Sigma(\xi)\chi_0(h^{-1}\xi))\widetilde{u} + \sum_k Op^w_h\big(\Sigma(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)\big)\widetilde{u} \\ + Op^w_h(\Sigma(\xi)(1-\chi_0)(h^\sigma\xi))\widetilde{u}, \end{split} \tag{3.2.38}$$

observing that the above sum is actually finite, and restricted to indices $k\in K := \{k\in\mathbb{Z} : h\lesssim 2^k \lesssim h^{-\sigma}\}$. From classical Sobolev injection,

$$\|Op^w_h(\Sigma(\xi)\chi_0(h^{-1}\xi))\widetilde{u}(t,\cdot)\|_{L^\infty} = \|\Sigma(hD)\chi_0(D)\widetilde{u}(t,\cdot)\|_{L^\infty} \lesssim \|\widetilde{u}(t,\cdot)\|_{L^2}, \tag{3.2.39}$$

while

$$\|Op^w_h(\Sigma(\xi)(1-\chi_0)(h^\sigma\xi))\|_{L^\infty} \lesssim h^N\|\widetilde{u}(t,\cdot)\|_{H^s_h}, \tag{3.2.40}$$

with $N\geq 0$ if $s>0$ is sufficiently large, as follows by semi-classical Sobolev injection and lemma 1.2.39, as $(1-\chi_0)(h^\sigma\xi))$ is a smooth function supported for large frequencies $|\xi|\gtrsim h^{-\sigma}$. Remaining terms $Op^w_h\big(\Sigma(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)\big)\widetilde{u}$, localised for frequencies $|\xi|\sim 2^k$, need a sharper analysis, because a direct application of semi-classical Sobolev injection only gives that

$$\Big\|Op^w_h\big(\Sigma(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)\big)\widetilde{u}\Big\|_{L^\infty} \leq 2^k h^{-1-\mu}\|\widetilde{u}\|_{L^2},$$

with $\mu=\sigma\rho$ if $\rho\geq 0$, 0 otherwise, and factor $2^k h^{-1-\mu}$ may grow too much when $h\to 0$.

For fixed $k\in K$, $\rho\in\mathbb{Z}$, let us introduce

$$\widetilde{u}^{\Sigma,k}(t,x) := Op^w_h\big(\Sigma(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)\big)\widetilde{u}(t,x), \tag{3.2.41}$$

and observe that, from the commutation of $Op^w_h\big(\Sigma(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)\big)$ with the linear part of equation (3.2.37), we get that $\widetilde{u}^{\Sigma,k}$ is solution to

$$\begin{split} &[D_t - Op^w_h(x\cdot\xi - |\xi|)]\widetilde{u}^{\Sigma,k}(t,x) \\ &= Op^w_h\big(\Sigma(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)\big)\left[h^{-1}\left[q_w(t,tx)+c_w(t,tx)+r^{NF}_w(t,tx)\right]\right] \\ &- ih\,Op^w_h\big(\Sigma(\xi)(\partial\chi_0)(h^{-1}\xi)\cdot(h^{-1}\xi)\psi(2^{-k}\xi)\big)\widetilde{u} - i\sigma h\,Op^w_h\big(\Sigma(\xi)\psi(2^{-k}\xi)(\partial\chi_0)(h^\sigma\xi)\cdot(h^\sigma\xi)\big)\widetilde{u}. \end{split} \tag{3.2.42}$$

We also introduce operator $\Gamma^{w,k}$ as

$$\Gamma^{w,k} := Op^w_h\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)\Big), \tag{3.2.43}$$

with $\gamma \in C_0^\infty(\mathbb{R}^2)$ equal to 1 close to the origin, $\psi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ equal to 1 on $supp\varphi$, together with

(3.2.44a) $$\widetilde{u}^{\Sigma,k}_{\Lambda_w} := \Gamma^{w,k}\widetilde{u}^{\Sigma,k},$$

(3.2.44b) $$\widetilde{u}^{\Sigma,k}_{\Lambda_w^c} := Op_h^w\Big((1-\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)\Big)\widetilde{u}^{\Sigma,k},$$

so that $\widetilde{u}^{\Sigma,k} = \widetilde{u}^{\Sigma,k}_{\Lambda_w} + \widetilde{u}^{\Sigma,k}_{\Lambda_w^c}$. Analogously to the Klein-Gordon case, the above two functions are obtained by localizing $\widetilde{u}^{\Sigma,k}$ respectively in a neighbourhood of Λ_w, and outside of this neighbourhood, with manifold Λ_w given by

$$\Lambda_w := \left\{(x,\xi) : x - \frac{\xi}{|\xi|} = 0\right\},$$

(see picture 3.2). Even in this case, we consider a neighbourhood of the above manifold of size

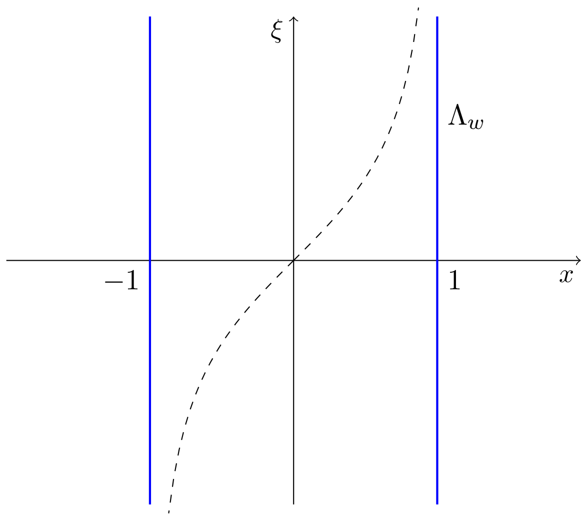

Figure 3.2: Lagrangian for the wave equation

depending on h. Up to control the L^2 norm of $(\theta_0\Omega_h)^\mu\mathcal{M}^\nu\widetilde{u}^{\Sigma,k}$ with a small negative power of h, for $\mu, |\nu| \le 1$, we find that the contribution $\widetilde{u}^{\Sigma,k}_{\Lambda_w^c}$, localized outside Λ_w, decays in the L^∞ norm as h^{-0}, faster than what expected for $\widetilde{u}$ in (3.1.12) (see proposition 3.2.8). Therefore, remaining $\widetilde{u}^{\Sigma,k}_{\Lambda_w}$ appears to be the main contribution to $\widetilde{u}^{\Sigma,k}$. We are going to show that this function is solution to a transport equation (see proposition 3.2.17, from which we will be able to derive a suitable estimate of its uniform norm, and to finally propagate (3.1.12) (see proposition 3.3.7).

Proposition 3.2.8. *There exists a constant $C > 0$ such that, for any $h \in]0,1]$, $k \in K$,*

(3.2.45a) $$\|\widetilde{u}^{\Sigma,k}_{\Lambda_w^c}(t,\cdot)\|_{L^2} \le Ch^{\frac{1}{2}-\beta}\big(\|\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2} + \|\mathcal{M}\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2}\big),$$

(3.2.45b) $$\|\widetilde{u}^{\Sigma,k}_{\Lambda_w^c}(t,\cdot)\|_{L^\infty} \le Ch^{-\beta}\sum_{\mu=0}^{1}\big(\|(\theta_0\Omega_h)^\mu\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2} + \|(\theta_0\Omega_h)^\mu\mathcal{M}\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2}\big),$$

for a small $\beta > 0$, $\beta(\sigma) \to 0$ as $\sigma \to 0$.

Proof. The proof is straightforward if one writes

$$\widetilde{u}^{\Sigma,k}_{\Lambda^c_w} = \sum_{j=1}^{2} Op^w_h\Big(\gamma^j_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big(\frac{x_j|\xi|-\xi_j}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)\Big)\widetilde{u}^{\Sigma,k},$$

where $\gamma^j_1(z) := \frac{(1-\gamma)(z)z_j}{|z|^2}$ is such that $|\partial^\alpha_z\gamma^j_1(z)| \lesssim \langle z\rangle^{-(|\alpha|+1)}$, and uses inequalities (1.2.48) with $a(x) = b_p(\xi) \equiv 1$. ☐

Lemma 3.2.9. *Let $\widetilde{\varphi} \in C^\infty_0(\mathbb{R}^2 \setminus \{0\})$ such that $\widetilde{\varphi} \equiv 1$ on suppφ, and has sufficiently small support so that $\psi\widetilde{\varphi} \equiv \psi$. Then*

$$\Big[\Gamma^{w,k}, D_t - Op^w_h\big((x\cdot\xi - |\xi|)\widetilde{\varphi}(2^{-k}\xi)\big)\Big]\, Op^w_h(\varphi(2^{-k}\xi)) = Op^w_h(b(x,\xi)), \tag{3.2.46}$$

where, for any $w \in L^2$ such that $\theta_0\Omega_h w, (\theta_0\Omega_h)^\mu \mathcal{M}w \in L^2(\mathbb{R}^2)$, for $\mu = 0,1$,

$$\|Op^w_h(b(x,\xi))w\|_{L^2} \lesssim h^{1-\beta}\left(\|w\|_{L^2} + \|\mathcal{M}w\|_{L^2}\right), \tag{3.2.47a}$$

$$\|Op^w_h(b(x,\xi))w\|_{L^\infty} \lesssim h^{1-\beta}\sum_{\mu=0}^{1}\big(\|(\theta_0\Omega_h)^\mu w\|_{L^2} + \|(\theta_0\Omega_h)^\mu\mathcal{M}w\|_{L^2}\big), \tag{3.2.47b}$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. We warn the reader that most of the terms arising from the commutator considered in the statement satisfy a better L^2 estimate than (3.2.47a), namely

$$\|\cdot\|_{L^2} \lesssim h^{\frac{3}{2}-\beta}\big(\|w\|_{L^2} + \|\mathcal{M}w\|_{L^2}\big). \tag{3.2.48}$$

The only contribution whose L^2 norm is only a $O(h\|w\|_{L^2})$ is the integral remainder called $\widetilde{r}^k_N$, appearing in symbolic development (3.2.50).

Since $\partial_t = -h^2\partial_h$, an easy computation shows that

$$\begin{aligned}[\Gamma^{w,k}, D_t] =& \Big(\frac{1}{2}+\sigma\Big)\frac{h}{i}Op^w_h\Big((\partial\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\cdot\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)\Big)\\ &+\frac{h}{i}Op^w_h\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)(\partial\psi)(2^{-k}\xi)\cdot(2^{-k}\xi)\Big).\end{aligned} \tag{3.2.49}$$

The first term in the above right hand side satisfies (3.2.48) and (3.2.47b) after inequalities (1.2.48). The same estimates hold also for the latter one when it acts on $Op^w_h(\varphi(2^{-k}\xi))w$, for the derivatives of ψ vanish on the support of $\widetilde{\varphi}$ (and then of φ) as a consequence of our assumptions. In fact, introducing a smooth function $\widetilde{\psi} \in C^\infty_0(\mathbb{R}^2\setminus\{0\})$, $\widetilde{\psi} \equiv 1$ on the support of $\partial\psi$, and using symbolic calculus, we have that, for any fixed $N \in \mathbb{N}$,

$$\begin{aligned}&Op^w_h\left(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)(\partial\psi)(2^{-k}\xi)\cdot(2^{-k}\xi)\right)Op^w_h(\varphi(2^{-k}\xi))\\ &\qquad = Op^w_h\left(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\psi}(2^{-k}\xi)(2^{-k}\xi)\right)Op^w_h\big((\partial\psi)(2^{-k}\xi)\varphi(2^{-k}\xi)\big) - Op^w_h(r^k_N),\end{aligned}$$

where the first term in the above right hand side is 0, and integral remainder r^k_N is given by

$$\begin{aligned}r^k_N = \Big(\frac{h}{2i}\Big)^N\sum_{|\alpha|=N}\frac{N(-1)^{|\alpha|}}{\alpha!(\pi h)^4}\int e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)}\int_0^1\partial^\alpha_x\Big[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)(\partial\psi)(2^{-k}\xi)\cdot(2^{-k}\xi)\Big]|_{(x+tz,\xi+t\zeta)}dt\\ \times\,\partial^\alpha_\xi\big(\widetilde{\psi}(2^{-k}\xi)\big)|_{(\xi+\eta)}\,dydzd\eta d\zeta.\end{aligned}$$

Developing explicitly the above derivatives, and reminding definition (1.2.28) of integrals $I^k_{p,q}$, for general $k \in K$, $p, q \in \mathbb{Z}$, one recognizes that, up to some multiplicative constants, r^k_N has the form

$$h^{N-N(\frac{1}{2}-\sigma)}2^{-kN}I^k_{N,0}(x,\xi),$$

with $a, a', b_q \equiv 1$, $p = N$, $\psi(2^{-k}\xi)$ replaced with $(\partial\psi)(2^{-k}\xi)(2^{-k}\xi)$. Propositions 1.2.28 and 1.2.31 imply then that $\|Op^w_h(r^k_N)\|_{\mathcal{L}(L^2)} + \|Op^w_h(r^k_N)\|_{\mathcal{L}(L^2;L^\infty)} \lesssim h$, if $N \in \mathbb{N}$ is chosen sufficiently large (e.g. $N > 9$), which implies that the $\mathcal{L}(L^2)$ and $\mathcal{L}(L^2; L^\infty)$ norms of the latter operator in (3.2.49) is $O(h^2)$.

As regards $[\Gamma^{w,k}, Op^w_h((x\cdot\xi - |\xi|)\widetilde{\varphi}(2^{-k}\xi))]$, we first remind that the symbolic development of a commutator's symbol only contains odd order terms. Consequently, its symbol has the following development, for a new fixed $N \in \mathbb{N}$ and up to multiplicative constants independent of h, k:

$$\text{(3.2.50)}\quad h\Big\{\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big), (x\cdot\xi - |\xi|)\widetilde{\varphi}(2^{-k}\xi)\Big\} \\ + \sum_{\substack{3\le|\alpha|<N\\ |\alpha|=|\alpha_1|+|\alpha_2|}} h^{|\alpha|}\partial^{\alpha_1}_x\partial^{\alpha_2}_\xi\Big[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big]\partial^{\alpha_2}_x\partial^{\alpha_1}_\xi\big[(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\big] + \widetilde{r}^k_N(x,\xi),$$

with

$$\widetilde{r}^k_N(x,\xi) = \Big(\frac{h}{2i}\Big)^N \sum_{|\alpha_1|+|\alpha_2|=N}\frac{N(-1)^{|\alpha_1|}}{\alpha!(\pi h)^4}\int e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)}\int_0^1 \partial^{\alpha_1}_x\partial^{\alpha_2}_\xi\Big[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\psi(2^{-k}\xi)\Big]|_{(x+tz,\xi+t\zeta)}dt \\ \times \partial^{\alpha_2}_x\partial^{\alpha_1}_\xi\big[(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\big]|_{(x+y,\xi+\eta)}\, dydzd\eta d\zeta\,.$$

The Poisson braket in the above sum reduces to

$$h\sum_{j,l}(\partial_j\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)(\partial_j\widetilde{\varphi})(2^{-k}\xi)\Big(\frac{x_l|\xi|-\xi_l}{h^{1/2-\sigma}}\Big)(2^{-k}\xi_l)$$

because $\big\{\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big), x\cdot\xi-|\xi|\big\} = 0$, and its quantization acting on $Op^w_h(\varphi(2^{-k}\xi))w$ satisfies (3.2.48), (3.2.47b) since $\partial\widetilde{\varphi}$ vanishes on the support of φ.

An explicit calculation of terms of order $3 \le |\alpha| < N$, with the help of lemma 1.2.26 and the fact that $|\alpha_2| \le 1$ as $(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)$ is affine in x, shows that they are linear combination of products $h^{|\alpha|-|\alpha|(\frac{1}{2}-\sigma)}\gamma_{|\alpha|}\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\widetilde{\varphi}(2^{-k}\xi)x^\nu b_1(\xi)$ and $h^{|\alpha|-(|\alpha|-1)(\frac{1}{2}-\sigma)}\widetilde{\gamma}\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\widetilde{\varphi}(2^{-k}\xi)b_0(\xi)$, with $\nu \in \mathbb{N}^2$ of length at most 1, $|\partial^\beta b_0(\xi)| \lesssim_\beta |\xi|^{-|\beta|}$, a new cut-off $\widetilde{\gamma}, \widetilde{\varphi}$, and furthermore

$$h^{|\alpha|-|\alpha|(\frac{1}{2}-\sigma)}\gamma_{|\alpha|}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\varphi}(2^{-k}\xi)x_j b_1(\xi) = h^{|\alpha|-(|\alpha|-1)(\frac{1}{2}-\sigma)}\widetilde{\gamma}^j_{|\alpha|}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\varphi}(2^{-k}\xi)b_0(\xi) \\ + h^{|\alpha|-|\alpha|(\frac{1}{2}-\sigma)}\gamma_{|\alpha|}\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\varphi}(2^{-k}\xi)\xi_j b_0(\xi),$$

for $j = 1, 2$, where $\widetilde{\gamma}^j_{|\alpha|}(z) := \gamma_{|\alpha|}(z)z_j$. From propositions 1.2.27, 1.2.30, the fact that $|\alpha| \ge 3$, and that $2^k \le h^{-\sigma}$ we deduce that the quantization of these $|\alpha|$-order terms acting on $Op^w_h(\varphi(2^{-k}\xi))w$ satisfy (3.2.48), (3.2.47b).

Finally, we notice that integral remainder $\widetilde{r}^k_N$ can be actually seen as the sum of two contributions, one of the form (1.2.40), the other like (1.2.41), with $a \equiv 1$ and $p = 1$. Lemma 1.2.32 implies then that the $\mathcal{L}(L^2)$ and $\mathcal{L}(L^2; L^\infty)$ norms of $Op^w_h(\widetilde{r}^k_N)$ are a $O(h)$, as foretold, which concludes the proof of the statement. □

Lemma 3.2.10. *Function $\widetilde{u}^{\Sigma,k}_{\Lambda_w}$ is solution to the following equation:*

(3.2.51)
$$
\begin{aligned}
&\big[D_t - Op^w_h\big((x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\big)\big]\widetilde{u}^{\Sigma,k}_{\Lambda_w}(t,x) = f^w_k(t,x)\\
&+\Gamma^{w,k}Op^w_h\big(\Sigma(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)\big)\left[h^{-1}\left[q_w(t,tx)+c_w(t,tx)+r^{NF}_w(t,tx)\right]\right]\\
&-ih\,\Gamma^{w,k}Op^w_h\big(\Sigma(\xi)(\partial\chi_0)(h^{-1}\xi)\cdot(h^{-1}\xi)\varphi(2^{-k}\xi)\big)\widetilde{u}\\
&-i\sigma h\,\Gamma^{w,k}Op^w_h\big(\Sigma(\xi)\varphi(2^{-k}\xi)(\partial\chi_0)(h^\sigma\xi))\cdot(h^\sigma\xi)\big)\widetilde{u},
\end{aligned}
$$

where $\widetilde{\varphi}\in C^\infty_0(\mathbb{R}^2\setminus\{0\})$ is equal to 1 on suppφ, and there exist two constants $C,C'>0$ such that, for any $h\in]0,1],k\in K$,

$$
\|f^w_k(t,\cdot)\|_{L^2}\le Ch^{1-\beta}\big(\|\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2}+\|\mathcal{M}\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2}\big), \tag{3.2.52a}
$$

$$
\|f^w_k(t,\cdot)\|_{L^\infty}\le C'h^{1-\beta}\sum_{\mu=0}^{1}\big(\|(\theta_0\Omega_h)^\mu\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2}+\|(\theta_0\Omega_h)^\mu\mathcal{M}\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2}\big), \tag{3.2.52b}
$$

with $\beta>0$ small, $\beta(\sigma)\to 0$ as $\sigma\to 0$.

Proof. If we consider a cut-off function $\widetilde{\varphi}\in C^\infty_0(\mathbb{R}^2\setminus\{0\})$ such that $\widetilde{\varphi}\equiv 1$ on the support of φ (φ being the truncation on $\widetilde{u}^{\Sigma,k}$'s frequencies), we have the exact equality $Op^w_h(x\cdot\xi-|\xi|)\widetilde{u}^{\Sigma,k}=Op^w_h((x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi))\widetilde{u}^{\Sigma,k}$. Moreover, if we assume that its support is sufficiently small so that $\psi\widetilde{\varphi}\equiv\widetilde{\varphi}$, and apply operator $\Gamma^{w,k}$ to equation (3.2.42), lemma 3.2.9 gives us the result of the statement. □

The transport equation we talked about at the beginning of this section will be deduced from equation (3.2.51), by suitably developing symbol $(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)$. To do that, we first need to restrict the support of that symbol to bounded values of x through the introduction of a new cut-off function $\theta(x)$. We remind that Σ' is a concise notation that we use to indicate a linear combination of a finite number of terms of the same form.

Lemma 3.2.11. *Let $\theta=\theta(x)$ be a smooth function equal to 1 for $|x|\le D_1$ and supported for $|x|\le D_2$, for any $0<D_1<D_2$. Then,*

(3.2.53)
$$
\begin{aligned}
Op^w_h\big((x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\big)=Op^w_h\big(\theta(x)(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\big)+(1-\theta)(x)Op^w_h((x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi))\\
+{\sum}'\widetilde{\theta}(x)Op^w_h(\widetilde{\varphi}_1(2^{-k}\xi))+Op^w_h(r(x,\xi)),
\end{aligned}
$$

where $\widetilde{\theta}$ is a smooth function supported for $D_1<|x|<D_2$, $\widetilde{\varphi}_1\in C^\infty_0(\mathbb{R}^2\setminus\{0\})$ and

$$
\|Op^w_h(r)\|_{\mathcal{L}(L^2)}+\|Op^w_h(r)\|_{\mathcal{L}(L^2;L^\infty)}=O(h).
$$

Therefore, $\widetilde{u}^{\Sigma,k}_{\Lambda_{kg}}$ verifies

(3.2.54)
$$
\begin{aligned}
&\Big[D_t - Op^w_h\big(\theta(x)(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\big)\Big]\widetilde{u}^{\Sigma,k}_{\Lambda_w}(t,x) = f^w_k(t,x)\\
&+(1-\theta)(x)Op^w_h((x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi))\widetilde{u}^{\Sigma,k}_{\Lambda_w}+{\sum}'\widetilde{\theta}(x)Op^w_h(\widetilde{\varphi}_1(2^{-k}\xi))\widetilde{u}^{\Sigma,k}_{\Lambda_w}\\
&+\Gamma^{w,k}Op^w_h\big(\Sigma(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)\big)\left[h^{-1}\left[q_w(t,tx)+c_w(t,tx)+r^{NF}_w(t,tx)\right]\right]\\
&-ih\,\Gamma^{w,k}Op^w_h\big(\Sigma(\xi)(\partial\chi_0)(h^{-1}\xi)\cdot(h^{-1}\xi)\varphi(2^{-k}\xi)\big)\widetilde{u}\\
&-i\sigma h\,\Gamma^{w,k}Op^w_h\big(\Sigma\varphi(2^{-k}\xi)(\partial\chi_0)(h^\sigma\xi))\cdot(h^\sigma\xi)\big)\widetilde{u},
\end{aligned}
$$

where f^w_k satisfies estimates (3.2.52).

Proof. Let $\theta(x)$ be the cut-off function of the statement. Symbol $(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)$ can be decomposed into the sum $\theta(x)(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)+(1-\theta)(x)(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)$, and by proposition 1.2.21 we have that

(3.2.55)
$$\begin{aligned}
&(1-\theta)(x)(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi) = (1-\theta)(x)\sharp\left[(x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)\right]\\
&-\frac{h}{2i}\partial_x\theta(x)\cdot\Big(x-\frac{\xi}{|\xi|}\Big)\widetilde{\varphi}(2^{-k}\xi)-\frac{h2^{-k}}{2i}(x\cdot\xi-|\xi|)\partial_x\theta(x)\cdot(\partial\widetilde{\varphi})(2^{-k}\xi)+r^k_2(x,\xi)\\
=&(1-\theta)(x)\sharp\left[\widetilde{\varphi}(2^{-k}\xi)(x\cdot\xi-|\xi|)\right]-\frac{h}{2i}\Big[\partial_x\theta(x)x\Big]\sharp\,\widetilde{\varphi}(2^{-k}\xi)+\frac{h}{2i}\partial_x\theta(x)\sharp\Big[\frac{\xi}{|\xi|}\widetilde{\varphi}(2^{-k}\xi)\Big]\\
&-\frac{h}{2i}\Big[\partial_x\theta(x)x\Big]\sharp\Big[(2^{-k}\xi)(\partial\widetilde{\varphi})(2^{-k}\xi)\Big]+\frac{h}{2i}\partial_x\theta(x)\sharp\Big[(2^{-k}|\xi|)(\partial\widetilde{\varphi})(2^{-k}\xi)\Big]+r^k_2(x,\xi)+\widetilde{r}^k_2(x,\xi),
\end{aligned}$$

where function $\partial_x\theta$ is supported for $D_1<|x|<D_2$, and $r^k_2(t,x)$ (resp. $\widetilde{r}^k_2(t,x)$) is an integral of the form
$$\frac{h^2 2^{-k}}{(\pi h)^2}\int e^{\frac{2i}{h}\eta\cdot z}\int_0^1\theta(x+tz)(1-t)^2dt\; x^\nu\widetilde{\varphi}(2^{-k}(\xi+\eta))dzd\eta,$$
with $|\nu|=0,1$ (resp. $|\nu|=0$), and some new $\theta,\widetilde{\varphi}\in C^\infty_0(\mathbb{R}^2\setminus\{0\})$. By writing x as $(x+tz)-tz$, using that $ze^{\frac{2i}{h}\eta\cdot z}=\big(\frac{h}{2i}\big)\partial_\xi e^{\frac{2i}{h}\eta\cdot z}$, and making an integration by parts, $r^k_2(t,x)$ can be rewritten as the sum over $|\nu|=0,1$, of integrals such as
$$\frac{h^2 2^{-k}(h2^{-k})^\nu}{(\pi h)^2}\int e^{\frac{2i}{h}\eta\cdot z}\int_0^1\theta(x+tz)f(t)dt\;\widetilde{\varphi}(2^{-k}(\xi+\eta))dzd\eta,$$
for some new smooth $\theta, f,\widetilde{\varphi},\theta,\varphi$ compactly supported, and one can show that, for any $\alpha,\beta\in\mathbb{N}^2$, $\big|\partial^\alpha_x\partial^\beta_\xi\left[(r^k_2+\widetilde{r}^k_2)(x,h\xi)\right]\big|\lesssim_{\alpha,\beta}h^2 2^{-k}\lesssim_{\alpha,\beta}h$. Thus $(r^k_2+\widetilde{r}^k_2)(x,h\xi)\in hS_0(1)$, which means that, by classical results on pseudo-differential operators (see for instance [11]), $Op^w_h((r^k_2+\widetilde{r}^k_2)(x,\xi))=Op^w((r^k_2+\widetilde{r}^k_2)(x,h\xi))$ is an element of $\mathcal{L}(L^2)$ with norm $O(h)$.

Furthermore, one can also show that $\|Op^w_h(r^k_2+\widetilde{r}^k_2)\|_{\mathcal{L}(L^2;L^\infty)}\lesssim h$ using lemma 1.2.25 and the fact that, by making some integrations by parts, for any multi-indices $\alpha,\beta\in\mathbb{N}^2$, and for a new $\widetilde{\varphi}\in C^\infty_0(\mathbb{R}^2\setminus\{0\})$,
$$\left\|\partial^\alpha_y\partial^\beta_\xi\left[(r^k_2+\widetilde{r}^k_2)\Big(\frac{x+y}{2},h\xi\Big)\right]\right\|_{L^2(d\xi)}\lesssim h^2 2^{-k}\left\|\int\langle\eta\rangle^{-3}|\widetilde{\varphi}(2^{-k}h(\xi+\eta))|d\eta\right\|_{L^2(d\xi)}\lesssim h.$$
These considerations, along with continuity of $\Gamma^{w,k}$ on L^2, uniformly in h and k (see 1.2.27), imply that $Op^w_h(r^k_2+\widetilde{r}^k_2)\widetilde{u}^{\Sigma,k}_{\Lambda_w}$ is a remainder f^w_k. ☐

Lemma 3.2.12. *We have that* $|\xi|-x\cdot\xi=\frac{1}{2}(1-|x|^2)x\cdot\xi+e(x,\xi)$, *with*

$$e(x,\xi)=\frac{1}{2}|\xi|\Big|x-\frac{\xi}{|\xi|}\Big|^2+\frac{1}{2}\Big(\Big(x-\frac{\xi}{|\xi|}\Big)\cdot\xi\Big)\Big(x-\frac{\xi}{|\xi|}\Big)\cdot\Big(x+\frac{\xi}{|\xi|}\Big). \tag{3.2.56}$$

Proof.
$$\begin{aligned}
|\xi|-x\xi&=\frac{1}{2}|\xi|\left|x-\frac{\xi}{|\xi|}\right|^2+\frac{1}{2}|\xi|(1-|x|^2)\\
&=\frac{1}{2}|\xi|\left|x-\frac{\xi}{|\xi|}\right|^2+\frac{1}{2}(|\xi|-x\cdot\xi)(1-|x|^2)+\frac{1}{2}(1-|x|^2)x\cdot\xi\\
&=\underbrace{\frac{1}{2}|\xi|\left|x-\frac{\xi}{|\xi|}\right|^2+\frac{1}{2}\left(\left(\frac{\xi}{|\xi|}-x\right)\cdot\xi\right)\left(\frac{\xi}{|\xi|}-x\right)\cdot\left(\frac{\xi}{|\xi|}+x\right)}_{e(x,\xi)}+\frac{1}{2}(1-|x|^2)x\cdot\xi\,.
\end{aligned}$$

Lemma 3.2.13. *Let $\gamma \in C_0^\infty(\mathbb{R}^2)$, $\theta \in C_0^\infty(\mathbb{R}^2)$, and $\widetilde{\varphi} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ such that $\widetilde{\varphi} \equiv 1$ on the support of φ ,and has a sufficiently small support so that $\psi\widetilde{\varphi} \equiv \widetilde{\varphi}$. Let also*

$$B(x,\xi) := \gamma\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\widetilde{\varphi}(2^{-k}\xi)\theta(x)\Big(x_m - \frac{\xi_m}{|\xi|}\Big), \qquad m \in \{1,2\}. \tag{3.2.57}$$

For any function $w \in L^2(\mathbb{R}^2)$ such that $\mathcal{M}w \in L^2(\mathbb{R}^2)$, any $m, n \in \{1,2\}$,

$$\Big\|Op_h^w\Big(\theta(x)\widetilde{\varphi}(2^{-k}\xi)\Big(x_m - \frac{\xi_m}{|\xi|}\Big)(x_n|\xi| - \xi_n)\Big)\Gamma^{w,k}w\Big\|_{L^2} \lesssim h^{1-\beta}\big(\|w\|_{L^2} + \|\mathcal{M}w\|_{L^2}\big), \tag{3.2.58a}$$

$$\begin{aligned}\Big\|Op_h^w\Big(\theta(x)\widetilde{\varphi}(2^{-k}\xi)\Big(x_m - \frac{\xi_m}{|\xi|}\Big)(x_n|\xi| - \xi_n)\Big)\Gamma^{w,k}w\Big\|_{L^\infty} &\lesssim \\ h^{1-\beta}\big(\|w\|_{L^2} + \|\mathcal{M}w\|_{L^2}\big) &+ h^{-\beta}\|Op_h^w\big(B(x,\xi)\xi\big)\mathcal{M}w\|_{L^2},\end{aligned} \tag{3.2.58b}$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. After lemma 1.2.35 with $p = 0$, we have that

$$Op_h^w\left(\theta(x)\widetilde{\varphi}(2^{-k}\xi)\Big(x_m - \frac{\xi_m}{|\xi|}\Big)(x_n|\xi| - \xi_n)\right)\Gamma^{w,k} = Op_h^w\left(B(x,\xi)(x_n|\xi| - \xi_n)\right) + Op_h^w(r_0^k(x,\xi)),$$

where, since $Op_h^w(r_0^k(x,\xi))$ satisfies (1.2.54), the L^2 (resp. L^∞) norm of $Op_h^w(r_0^k(x,\xi))w$ is bounded by the right hand side of (3.2.58a) (resp. (3.2.58b)). We can then focus our attention on proving that $Op_h^w\big(B(x,\xi)(x_n|\xi| - \xi_n)\big)w$ verifies the statement.

Estimate (3.2.58a) is a straight consequence of lemma 1.2.33. In order to control the L^∞ norm of $Op_h^w\big(B(x,\xi)(x_n|\xi|-\xi_n)\big)w$ and prove (3.2.58b), we start by applying classical Sobolev inequality. For that, we first consider a new cut-off function $\widetilde{\varphi}_1 \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ equal to 1 on $supp\widetilde{\varphi}$, so that its derivatives vanish against φ, and use symbolic calculus to write

$$Op_h^w\big(B(x,\xi)(x_n|\xi| - \xi_n)\big) = Op_h^w(\widetilde{\varphi}_1(2^{-k}\xi))Op_h^w\big(B(x,\xi)(x_n|\xi| - \xi_n)\big) + Op_h^w(r_{N,1}^k(x,\xi)),$$

where $r_{N,1}^k(x,\xi)$ is obtained using (1.2.20), and is analogous to integral (1.2.41) with $p = 1$, up to interchange the role of variables y and z (resp. η and ζ) and to consider $e^{\frac{2i}{h}(y\cdot\zeta - \eta\cdot\zeta)}$ instead of $e^{\frac{2i}{h}(\eta\cdot z - y\cdot\zeta)}$ (which does not affect estimate (1.2.42)). If $N \in \mathbb{N}$ is chosen sufficiently large (e.g. $N > 11$), lemma 1.2.32 implies that $\|Op_h^w(r_{N,1}^k)\|_{\mathcal{L}(L^2;L^\infty)} = O(h)$.

Since $\widetilde{\varphi}_1$ localises frequencies ξ in an annulus, classical Sobolev injection gives that

$$\begin{aligned}&\Big\|Op_h^w(\widetilde{\varphi}_1(2^{-k}\xi))Op_h^w\big(B(x,\xi)(x_n|\xi| - \xi_n)\big)w\Big\|_{L^\infty} \\ &\qquad\lesssim \big\|Op_h^w\big(B(x,\xi)(x_n|\xi| - \xi_n)\big)w\big\|_{L^2} + \big\|D_xOp_h^w\big(B(x,\xi)(x_n|\xi| - \xi_n)\big)w\big\|_{L^2},\end{aligned}$$

where, as we previously saw, the former norm in the above right hand side satisfies inequality (3.2.58a). As concerns the latter one, we remark that, thanks to the specific structure of symbol $B(x,\xi)$, its first derivative with respect to x does not lose any factor $h^{-1/2+\sigma}$ because, when ∂_x hits $\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)$,

$$\partial_x\left[\gamma\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\right]\widetilde{\varphi}(2^{-k}\xi)\theta(x)\Big(x_m - \frac{\xi_m}{|\xi|}\Big) = (\partial\gamma)\Big(\frac{x|\xi| - \xi}{h^{1/2-\sigma}}\Big)\widetilde{\varphi}(2^{-k}\xi)\theta(x)\Big(\frac{x_m|\xi| - \xi_m}{h^{1/2-\sigma}}\Big). \tag{3.2.59}$$

Consequently, by using symbolic calculus we derive that

$$D_x Op_h^w\big(B(x,\xi)(x_n|\xi|-\xi_n)\big)w = h^{-1}Op_h^w\big(B(x,\xi)(x_n|\xi|-\xi_n)\xi\big)w + \sum{}' Op_h^w\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\varphi}(2^{-k}\xi)a(x)b_0(\xi)(x_j|\xi|-\xi_j)\Big)w,$$

where $\sum'$ is a concise notation to indicate linear combinations, $j\in\{m,n\}$, $\gamma,\widetilde{\varphi},a$ are some new smooth functions, with $a(x)$ compactly supported. Again by lemma 1.2.33, the L^2 norms of latter contributions in the above right hand side are bounded by $h^{1-\beta}(\|w\|_{L^2}+\|\mathcal{M}w\|_{L^2})$.

Finally, we observe that symbol $B(x,\xi)\xi$ can be seen as

$$\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)(x_m|\xi|-\xi_m)\widetilde{\varphi}(2^{-k}\xi)\theta(x)b_0(x), \tag{3.2.60}$$

which implies, after lemma 1.2.34, that

$$h^{-1}Op_h^w\big(B(x,\xi)(x_n|\xi|-\xi_n)\xi\big)w = Op_h^w\big(B(x,\xi)\xi\big)\mathcal{M}_n w + O_{L^2}(h^{1-\beta}(\|w\|_{L^2}+\|\mathcal{M}w\|_{L^2})).$$

□

Lemma 3.2.14. *Let $e(x,\xi)$ be the symbol defined in (3.2.56), $\theta\in C_0^\infty(\mathbb{R}^2)$, and $\widetilde{\varphi}\in C_0^\infty(\mathbb{R}^2\backslash\{0\})$ with sufficiently small support so that $\psi\widetilde{\varphi}\equiv\widetilde{\varphi}$. There exists $s>0$ sufficiently large, and $\chi\in C_0^\infty(\mathbb{R}^2)$, such that*

$$\Big\|Op_h^w\Big(\theta(x)\widetilde{\varphi}(2^{-k}\xi)e(x,\xi)\Big)\widetilde{u}^{\Sigma,k}_{\Lambda_w}(t,\cdot)\Big\|_{L^2}\lesssim h^{1-\beta}\big(\|\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2}+\|\mathcal{M}\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2}\big), \tag{3.2.61a}$$

(3.2.61b)

$$\begin{aligned}
&\Big\|Op_h^w\Big(\theta(x)\widetilde{\varphi}(2^{-k}\xi)e(x,\xi)\Big)\widetilde{u}^{\Sigma,k}_{\Lambda_w}(t,\cdot)\Big\|_{L^\infty}\\
&\lesssim h^{1-\beta}\Big(\|\widetilde{u}(t,\cdot)\|_{L^2}+\|Op_h^w(\chi(h^\sigma\xi))\mathcal{M}\widetilde{u}(t,\cdot)\|_{L^2}+\sum_{|\gamma|=1}^2\|(Z^\gamma u)_-(t,\cdot)\|_{L^2}\Big)\\
&+h^{-\beta}\,[\|V(t,\cdot)\|_{H^1}+\|V(t,\cdot)\|_{L^2}(\|U(t,\cdot)\|_{H^{1,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}})+\|V(t,\cdot)\|_{H^{1,\infty}}\|U(t,\cdot)\|_{H^1}]\\
&\times(\|V(t,\cdot)\|_{H^{14,\infty}}+h\|V(t,\cdot)\|_{H^{13}})+h^{-\beta}\|Z_nV(t,\cdot)\|_{H^1}\|V(t,\cdot)\|_{H^{17,\infty}}\\
&+h^{-\frac{1}{2}-\beta}\|V(t,\cdot)\|^2_{H^{13,\infty}}\|U(t,\cdot)\|_{H^1}\\
&+h^{1-\beta}\Big(\|\widetilde{v}(t,\cdot)\|_{H^s}+\sum_{|\mu|=1}^2\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2}\Big)\|\widetilde{v}(t,\cdot)\|_{H^{1,\infty}_h}\\
&+h^{-\frac{1}{2}-\beta}\,\big\|\chi_1(t^{-\sigma}D_x)(v^{NF}-v_-)(t,\cdot)\big\|_{L^2}\,\big(\|V(t,\cdot)\|_{H^{2,\infty}}+\|v^{NF}(t,\cdot)\|_{H^{1,\infty}}\big)\\
&+h^{\frac{3}{2}}\,\big\|(v^{NF}-v_-)(t,\cdot)\big\|_{H^1}\,\big(\|V(t,\cdot)\|_{H^s}+\|v^{NF}(t,\cdot)\|_{H^s}\big)
\end{aligned}$$

with $\beta>0$ small, $\beta\to0$ as $\sigma\to0$.

Proof. Since symbol $e(x,\xi)$ writes as

$$e(x,\xi)=\frac12\sum_{m=1}^2\Big(x_m-\frac{\xi_m}{|\xi|}\Big)(x_m|\xi|-\xi_m)+\frac12\sum_{m,n=1}^2\Big(x_m-\frac{\xi_m}{|\xi|}\Big)(x_n|\xi|-\xi_n)\Big(\frac{\xi_m}{|\xi|}\frac{\xi_n}{|\xi|}+x_n\frac{\xi_m}{|\xi|}\Big),$$

it follows from lemma 3.2.13 that $Op_h^w\big(\theta(x)\widetilde{\varphi}(2^{-k}\xi)e(x,\xi)\big)\widetilde{u}^{\Sigma,k}_{\Lambda_w}$ satisfies (3.2.61a) and

$$\Big\|Op_h^w\Big(\theta(x)\widetilde{\varphi}(2^{-k}\xi)e(x,\xi)\Big)\widetilde{u}^{\Sigma,k}_{\Lambda_w}\Big\|_{L^\infty}\lesssim h^{1-\beta}\,\Big(\|\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2}+\|\mathcal{M}\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2}\Big) + h^{-\beta}\|Op_h^w\big(B(x,\xi)\xi\big)\mathcal{M}\widetilde{u}^{\Sigma,k}\|_{L^2},$$

with $B(x,\xi)$ defined in (3.2.57), and $\beta>0$ small, $\beta\to 0$ as $\sigma\to 0$.

First of all, we remind that $B(x,\xi)\xi$ can be seen as a symbol of the form (3.2.60), which implies, from proposition 1.2.27, that

$$\|Op^w_h\big(B(x,\xi)\xi\big)\|_{\mathcal{L}(L^2)}=O(h^{\frac{1}{2}-\beta}), \tag{3.2.62a}$$

but also

$$\|Op^w_h\big(B(x,\xi)\xi\big)w\|_{L^2}\lesssim h^{1-\beta}(\|w\|_{L^2}+\|\mathcal{M}w\|_{L^2}), \tag{3.2.62b}$$

using instead (1.2.48a). We also recall definition (3.2.41) of $\widetilde{u}^{\Sigma,k}$, denoting concisely by $\phi_k(\xi)$ function $\Sigma(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi)$.

Commutating operators $\mathcal{M}$ and $Op^w_h(\phi_k(\xi))$, and recalling relation (3.2.10a), we find that for any $n=1,2$,

$$\begin{aligned}\|Op^w_h\big(B(x,\xi)\xi\big)\mathcal{M}_n\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2}&\lesssim \|Op^w_h\big(B(x,\xi)\xi\big)Op^w_h(\phi_k(\xi))[t(Z_nu^{NF})(t,tx)]\|_{L^2(dx)}\\&+\Big\|Op^w_h\big(B(x,\xi)\xi\big)\Big[Op^w_h\big(\xi_n|\xi|^{-1}\varphi(2^{-k}\xi)\Sigma(\xi)\big)+Op^w_h\big(|\xi|\partial_n\phi_k(\xi)\big)\Big]\widetilde{u}(t,x)\Big\|_{L^2}\\&+\big\|Op^w_h\big(B(x,\xi)\xi\big)Op^w_h(\phi_k(\xi))\big[t(tx_n)\big[q_w(t,tx)+c_w(t,tx)+r^{NF}_w(t,tx)\big]\big]\big\|_{L^2(dx)},\end{aligned}$$

with u^{NF} defined in (3.1.15), q_w, c_w and r^{NF}_w given, respectively, by (3.1.17), (3.1.18) and (3.1.19). Evidently, the second L^2 norm in the above right hand side is estimated, after (3.2.62b) and a further commutation between $\mathcal{M}$ and, respectively, operators $Op^w_h\big(\xi_n|\xi|^{-1}\varphi(2^{-k}\xi)\Sigma(\xi)\big)$, $Op^w_h\big(|\xi|\partial_n\phi_k(\xi)\big)$, by

$$h^{1-\beta}(\|\widetilde{u}(t,\cdot)\|_{L^2}+\|Op^w_h(\chi(h^\sigma\xi))\mathcal{M}\widetilde{u}^k(t,\cdot)\|_{L^2}),$$

for some new $\chi\in C^\infty_0(\mathbb{R}^2)$, and $\beta>0$ small, $\beta\to0$ as $\sigma\to0$.

• **Estimate of $\|Op^w_h\big(B(x,\xi)\xi\big)Op^w_h(\phi_k(\xi))[t(Z_nu^{NF})(t,tx)]\|_{L^2}$:** This L^2 norm is basically estimated in terms of the L^2 norm of $(Z^\mu u)_-$, for $|\mu|\le 2$. In fact, after definition (3.1.15) and the fact that $[Z_n,D_t-|D_x|]u=\frac{D_n}{|D_x|}u_-$,

$$\begin{aligned}(Z_nu^{NF})(t,tx)&=(Z_nu)_-(t,tx)+\Big(\frac{D_n}{|D_x|}u_-\Big)(t,tx)\\&-\frac{i}{4(2\pi)^2}\sum_{j\in\{+,-\}}\Big[Z_n\int e^{iy\cdot\xi}D_j(\xi,\eta)\hat{v}_j(\xi-\eta)\hat{v}_j(\eta)d\xi d\eta\Big]\Big|_{y=tx},\end{aligned} \tag{3.2.63}$$

with D_j given by (3.1.14). On the one hand, taking a new smooth cut-off function θ_1, equal to 1 on the support of θ, using (1.2.46) with $\widetilde{a}=\theta_1$, together with (1.2.47a), proposition 1.2.27, and the continuity on L^2 of the commutator between $\mathcal{M}_n$ and $Op^w_h(\phi_k(\xi))$, with norm $O(h^{-\beta})$, we deduce that

$$\begin{aligned}&\|Op^w_h\big(B(x,\xi)\xi\big)Op^w_h(\phi_k(\xi))[t(Z_nu)_-(t,tx)]\|_{L^2}\\&\qquad\lesssim\sum_{m=1}^2 h\|\theta_1(x)Op^w_h(\phi_k(\xi))\mathcal{M}_m[t(Z_nu)_-(t,tx)]\|_{L^2}+h^{1-\beta}\|(Z_nu)_-(t,\cdot)\|_{L^2},\end{aligned}$$

where, after relation (3.2.12),

$$\begin{aligned}\|\theta_1(x)Op^w_h(\phi_k(\xi))\mathcal{M}_m[t(Z_nu)_-(t,tx)]\|_{L^2}&\lesssim\|(Z_mZ_nu)_-(t,\cdot)\|_{L^2}+\|(Z_nu)_-(t,\cdot)\|_{L^2}\\&+\Big\|\theta_1\Big(\frac{x}{t}\Big)\phi_k(D_x)\left[x_mZ_nNL_w\right](t,\cdot)\Big\|_{L^2}.\end{aligned}$$

Since

$$\theta_1\Big(\frac{x}{t}\Big)\phi_k(D_x)x_m = t\theta_{1,m}\Big(\frac{x}{t}\Big)\phi_k(D_x) + \theta_1\Big(\frac{x}{t}\Big)[\phi_k(D_x), x_m],$$

where $\theta_{1,m}(z) = \theta_1(z)z_m$ and $[\phi_k(D_x), x_m]$ is a bounded operator on L^2, with norm $O(t)$ as one can check computing its associated symbol by means of symbolic calculus, and using that $2^{-k} \lesssim h^{-1} = t$, we deduce from (3.2.11) and above inequalities that

(3.2.64)

$$\begin{aligned}
&\Big\|Op^w_h\big(B(x,\xi)\xi\big)Op^w_h(\phi_k(\xi))\Big[t(Z_n u)_-(t,tx)\Big]\Big\|_{L^2(dx)} \\
&\lesssim \sum_{|\mu|=1}^{2} h\|(Z^\mu u)_-(t,\cdot)\|_{L^2} + \|Z_nV(t,\cdot)\|_{H^1}\|V(t,\cdot)\|_{H^{2,\infty}} + \big[\|V(t,\cdot)\|_{H^1} \\
&+ \|V(t,\cdot)\|_{L^2}\left(\|U(t,\cdot)\|_{H^{1,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}}\right) + \|V(t,\cdot)\|_{L^\infty}\|U(t,\cdot)\|_{H^1}\big]\|V(t,\cdot)\|_{H^{1,\infty}}.
\end{aligned}$$

On the other hand, it is a straight consequence of (3.2.62b) and the mentioned commutation between $\mathcal{M}_n$ and $Op^w_h(\phi_k(\xi))$, that

$$\begin{aligned}
\text{(3.2.65)}\quad &\big\|Op^w_h\big(B(x,\xi)\xi\big)Op^w_h(\phi_k(\xi))[t(D_n|D_x|^{-1}u)_-(t,tx)]\big\|_{L^2} \\
&\qquad\lesssim h^{1-\beta}(\|\widetilde{u}(t,\cdot)\|_{L^2} + \|Op^w_h(\chi(h^\sigma\xi))\mathcal{M}\widetilde{u}(t,\cdot)\|_{L^2}),
\end{aligned}$$

where, as before, $\chi \in C^\infty_0(\mathbb{R}^2)$.

Finally, by symbolic calculus we have that

$$\text{(3.2.66)}\qquad Op^w_h(B(x,\xi)\xi) = Op^w_h(B(x,\xi))(hD_x) + \frac{h}{2i}Op^w_h\big(\partial_x B(x,\xi)\big),$$

where, after (3.2.59), $\partial_x B$ is of the form

$$\text{(3.2.67)}\qquad \gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\psi}(2^{-k}\xi)\theta(x)b_0(\xi),$$

for some new $\gamma, \theta \in C^\infty_0(\mathbb{R}^2)$. Consequently,

$$\begin{aligned}
&\Big\|Op^w_h(B(x,\xi)\xi)Op^w_h(\phi_k(\xi))\Big[h^{-1}Z_n\int e^{iy\cdot\xi}D_j(\xi,\eta)\hat{v}_j(\xi-\eta)\hat{v}_j(\eta)d\xi d\eta\Big]|_{y=tx}\Big\|_{L^2(dx)} \\
&\qquad\lesssim \Big\|\chi(t^{-\sigma}D_x)D_xZ_n\int e^{ix\cdot\xi}D_j(\xi,\eta)\hat{v}_j(\xi-\eta)\hat{v}_j(\eta)d\xi d\eta\Big\|_{L^2(dx)} \\
&\qquad\qquad + h\Big\|\chi(t^{-\sigma}D_x)Z_n\int e^{ix\cdot\xi}D_j(\xi,\eta)\hat{v}_j(\xi-\eta)\hat{v}_j(\eta)d\xi d\eta\Big\|_{L^2(dx)},
\end{aligned}$$

the above right hand side being bounded by

$$\begin{aligned}
&h^{-\beta}\left(\|V(t,\cdot)\|_{H^1} + \|V(t,\cdot)\|_{L^2}(\|U(t,\cdot)\|_{H^{1,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}}) + \|V(t,\cdot)\|_{L^\infty}\|U(t,\cdot)\|_{H^1}\right) \\
&\qquad\times\left(\|V(t,\cdot)\|_{H^{14,\infty}} + h\|V(t,\cdot)\|_{H^{13}}\right) + h^{-\beta}\|Z_nV(t,\cdot)\|_{L^2}\|V(t,\cdot)\|_{H^{17,\infty}},
\end{aligned}$$

with $\beta \to 0$ as $\sigma \to 0$, after inequalities (A.31b), (A.31c), and (B.1.6a) with $s = 0$.

After (3.2.63), the above estimate, together with (3.2.64), (3.2.65), gives that the L^2 norm of contribution $Op^w_h\big(B(x,\xi)\xi\big)Op^w_h(\varphi(2^{-k}\xi))[t(Z_nu^{NF})(t,tx)]$ is estimated with the right hand side of (3.2.61b).

• **Estimate of** $\|Op^w_h(B(x,\xi)\xi)\,[t(tx_n)q_w(t,tx)]\,\|_{L^2(dx)}$:

We first notice that, after definition (3.1.17) of q_w and (3.2.2) of $\widetilde{v}$,

$$tq_w(t,tx) = \widetilde{q}_w(t,x) = \frac{h}{2}\Im\Big[\overline{\widetilde{v}}\, Op_h^w(\xi_1)\widetilde{v} - \overline{Op_h^w\Big(\frac{\xi_1}{\langle\xi\rangle}\Big)\widetilde{v}}\cdot Op_h^w\Big(\frac{\xi\xi_1}{\langle\xi\rangle}\Big)\widetilde{v}\Big](t,x), \tag{3.2.68}$$

so

$$\|Op_h^w(B(x,\xi)\xi)\,[t(tx_n)q_w(t,tx)]\,\|_{L^2(dx)} = h^{-1}\|Op_h^w(B(x,\xi)\xi)\,[x_n\widetilde{q}_w(t,\cdot)]\,\|_{L^2}.$$

As $B(x,\xi)$ is compactly supported in x, by symbolic calculus we can morally reduce to the study of the L^2 norm of

$$h^{-1}Op_h^w(B(x,\xi)\xi)Op_h^w(\phi_k(\xi))\widetilde{q}_w(t,x)$$

up to a $O_{L^2}(h^{-1}\|\widetilde{q}_w\|_{L^2})$, where from (3.2.68),

$$\|\widetilde{q}_w(t,\cdot)\|_{L^2} \lesssim h\|\widetilde{v}(t,\cdot)\|_{H^{1,\infty}}\|\widetilde{v}(t,\cdot)\|_{H^1}. \tag{3.2.69}$$

Using (3.2.66), (3.2.67), together with proposition 1.2.27, we deduce that

$$h^{-1}\,\|Op_h^w(B(x,\xi)\xi)Op_h^w(\phi_k(\xi))\widetilde{q}_w(t,\cdot)\|_{L^2} \lesssim h^{-1}\|Op_h^w(\phi_k(\xi))(hD_x)\widetilde{q}_w(t,\cdot)\|_{L^2} + \|\widetilde{q}_w(t,\cdot)\|_{L^2},$$

so it follows from lemma 3.2.15 below and estimate (3.2.69) that

$$\begin{aligned} &h^{-1}\|Op_h^w(\phi_k(\xi))(hD_x)\widetilde{q}_w(t,\cdot)\|_{L^2} \\ &\lesssim h^{1-\beta}\Big(\|\widetilde{v}(t,\cdot)\|_{H^s} + \sum_{|\mu|=1}^{2}\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2}\Big)\|\widetilde{v}(t,\cdot)\|_{H^{1,\infty}}; \end{aligned} \tag{3.2.70}$$

• **Estimate of $\|Op_h^w(B(x,\xi)\xi)Op_h^w(\phi_k(\xi))(t(tx_n)c_w(t,tx)\|_{L^2(dx)}$:**

As for the previous estimate, we can reduce to the study of the L^2 norm of

$$Op_h^w(B(x,\xi)\xi)Op_h^w(\phi_k(\xi))[t^2c_w(t,tx)],$$

up to a $O_{L^2}(\|Op_h^w(\chi(h^\sigma\xi))c_w\|_{L^2})$, so using (3.2.62a), the fact that $\|tw(t,t\cdot)\|_{L^2} = \|w(t,\cdot)\|_{L^2}$, and (3.1.21a) with $s>0$ sufficiently large so that $N(s)>2$, we obtain that

$$\begin{aligned} &\|Op_h^w(B(x,\xi)\xi)Op_h^w(\phi_k(\xi))[t^2c_w(t,t\cdot)]\|_{L^2} \\ &\lesssim h^{-\frac{1}{2}-\beta}\left\|\chi_1(t^{-\sigma}D_x)(v^{NF}-v_-)(t,\cdot)\right\|_{L^2}\left(\|V(t,\cdot)\|_{H^1} + \|v^{NF}(t,\cdot)\|_{L^2}\right) \\ &+ h^{\frac{3}{2}}\left\|(v^{NF}-v_-)(t,\cdot)\right\|_{H^1}\left(\|V(t,\cdot)\|_{H^s} + \|v^{NF}(t,\cdot)\|_{H^s}\right); \end{aligned} \tag{3.2.71}$$

• **Estimate of $\|Op_h^w(B(x,\xi)\xi)Op_h^w(\phi_k(\xi))(t(tx_n)r_w^{NF}(t,tx)\|_{L^2(dx)}$:**

Analogously, from (3.1.22a) we obtain that

$$\begin{aligned} \|Op_h^w(B(x,\xi)\xi)Op_h^w(\phi_k(\xi))[t^2r_w^{NF}(t,t\cdot)]\|_{L^2} &\lesssim h^{-\frac{1}{2}-\beta}\|\chi(t^{-\sigma}D_x)r_w^{NF}(t,\cdot)\|_{L^2} \\ &\lesssim h^{-\frac{1}{2}-\beta}\|V(t,\cdot)\|^2_{H^{13,\infty}}\|U(t,\cdot)\|_{H^1}. \end{aligned} \tag{3.2.72}$$

□

Lemma 3.2.15. *Let $a_j(\xi)$ be two smooth real symbols of order $j=0,1$. Then*

$$\begin{aligned} &\left\|Op_h^w(\varphi(2^{-k}\xi))(hD_x)\left[\overline{a_0(hD_x)\widetilde{v}}\,a_1(hD_x)\widetilde{v}\right](t,\cdot)\right\|_{L^2} \\ &\lesssim h^{1-\beta}\Big(\|\widetilde{v}(t,\cdot)\|_{H_h^s} + \sum_{|\mu|=1}^{2}\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2}\Big)\|\widetilde{v}(t,\cdot)\|_{H_h^{1,\infty}}. \end{aligned} \tag{3.2.73}$$

Proof. Let us split both $\widetilde{v}$ in the left hand side of (3.2.73) into the sum $\widetilde{v}_{\Lambda_{kg}}+\widetilde{v}_{\Lambda^c_{kg}}$, with $\widetilde{v}_{\Lambda_{kg}}, \widetilde{v}_{\Lambda^c_{kg}}$ introduced in (3.2.17), with $\Sigma\equiv 1$, and remind inequality (3.2.20a) satisfied by $\widetilde{v}_{\Lambda_{kg}}$. Since

$$\|a_0(hD_x)\widetilde{v}(t,\cdot)\|_{L^\infty}+\|a_0(hD_x)\widetilde{v}_{\Lambda_{kg}}(t,\cdot)\|_{L^\infty}\lesssim h^{-\beta}\|\widetilde{v}(t,\cdot)\|_{H^{1,\infty}_h},$$

for a small $\beta>0$, $\beta\to 0$ as $\sigma\to 0$, as follows from lemma 1.2.38 with $p=+\infty$ and the uniform continuity of $a_0(hD_x)$ from $H^{1,\infty}$ to L^∞, we deduce that, for any $w_1,w_2\in\{\widetilde{v},\widetilde{v}_{\Lambda_{kg}},\widetilde{v}_{\Lambda^c_{kg}}\}$, with at least one w_j equal to $\widetilde{v}_{\Lambda^c_{kg}}$,

$$\Big\|Op^w_h(\varphi(2^{-k}\xi))(hD_x)\Big[\overline{a_0(hD_x)w_1}a_1(hD_x)w_2\Big]\Big\|_{L^2}$$
$$\lesssim h^{1-\beta}\Big(\|\widetilde{v}(t,\cdot)\|_{H^s_h}+\sum_{|\mu|=1}^{2}\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2}\Big)\|\widetilde{v}(t,\cdot)\|_{H^{1,\infty}_h}.$$

We are thus reduced to proving inequality (3.2.73) for

$$\Big\|Op^w_h(\varphi(2^{-k}\xi))(hD_x)\Big[\overline{a_0(hD_x)\widetilde{v}_{\Lambda_{kg}}}\,a_1(hD_x)\widetilde{v}_{\Lambda_{kg}}\Big](t,\cdot)\Big\|_{L^2}.$$

Furthermore, by means of proposition 3.2.4, we can replace the action of $a_j(hD_x)$, $j=1,2$, in the above L^2 norm, with the multiplication operator by a real function, up to new remainders bounded in L^2 by the right hand side of (3.2.73), for

$$a_j(hD_x)\widetilde{v}_{\Lambda_{kg}}=\theta_h(x)a_j(-d\phi(x))\widetilde{v}_{\Lambda_{kg}}+R_1(\widetilde{v}),\quad j=1,2,$$

where θ_h is a smooth cut-off function, supported in some ball $B_{1-ch^{2\sigma}}(0)$ for a small $c>0$, and such that $\|\partial^\alpha_x\theta_h\|_{L^\infty}=O(h^{-2|\alpha|\sigma})$, $\phi(x)=\sqrt{1-|x|^2}$, and $R_1(\widetilde{v})$ satisfies (3.2.23a). Now,

$$hD_x|\widetilde{v}_{\Lambda_{kg}}|^2=\big[Op^w_h(\xi+d\phi(x)\theta_h(x))\widetilde{v}_{\Lambda_{kg}}\big]\,\overline{\widetilde{v}_{\Lambda_{kg}}}-\widetilde{v}_{\Lambda_{kg}}\Big[\overline{Op^w_h(\xi+d\phi(x)\theta_h(x))\widetilde{v}_{\Lambda_{kg}}}\Big],$$

where $\xi+d\phi(x)\in h^{-2\sigma}S_{2\sigma,0}(\langle\xi\rangle)$ on the support of θ_h, and

$$\|(hD_x-d\phi(x)\theta_h(x))\widetilde{v}_{\Lambda_{kg}}(t,\cdot)\|_{L^2}\lesssim h^{1-\beta}\sum_{|\mu|=0}^{1}\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2},$$

as follows from lemma 3.2.16 below. This implies, after having applied the Leibniz rule, that

$$\big\|hD_x\big[a_0(-d\phi(x))a_1(-d\phi(x))\theta_h(x)|\widetilde{v}_{\Lambda_{kg}}|^2(t,\cdot)\big]\big\|_{L^2}$$
$$\lesssim h^{1-\beta}\Big(\|\widetilde{v}(t,\cdot)\|_{H^s_h}+\sum_{|\mu|=1}^{2}\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2}\Big)\|\widetilde{v}(t,\cdot)\|_{H^{1,\infty}_h},$$

and the conclusion of the statement. □

Lemma 3.2.16. *Let $\gamma,\chi\in C^\infty_0(\mathbb{R}^2)$ be equal to 1 in a neighbourhood of the origin, $\sigma>0$ small, $(\theta_h(x))_h$ be a family of $C^\infty_0(B_1(0))$ functions, equal to 1 on the support of $\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)$, with $\|\partial^\alpha_x\theta_h\|_{L^\infty}=O(h^{-2|\alpha|\sigma})$ and $(h\partial_h)^k\theta_h$ bounded for every k. Let also $\phi(x)=\sqrt{1-|x|^2}$. Then*

$$\Big\|Op^w_h(\xi_j+d_j\phi(x)\theta_h(x))Op^w_h\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big)\widetilde{v}(t,\cdot)\Big\|_{L^2}$$
$$\lesssim h^{1-\beta}\sum_{|\mu|=0}^{2}\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2},$$

with $\beta>0$ small, $\beta\to 0$ as $\sigma\to 0$.

Proof. By symbolic calculus of lemma 1.2.24, and the fact that $\theta_h \equiv 1$ on the support of $\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)$, we have that, for any $j=1,2$,

(3.2.74)
$$\begin{aligned}Op^w_h(\xi_j+d_j\phi(x)\theta_h(x))Op^w_h\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big)\widetilde{v} = Op^w_h\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)(\xi_j+d_j\phi(x))\Big)\widetilde{v}\\
+\frac{\sqrt{h}}{2i}Op^w_h\Big((\partial_j\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\Big)\widetilde{v}\\
-\frac{\sqrt{h}}{2i}\sum_{k,l=1}^2 Op^w_h\Big((\partial_l\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)p''_{k,l}(\xi)\partial_k(d_j\phi(x)\theta_h(x))\chi(h^\sigma\xi)\Big)\widetilde{v}\\
+\frac{h^{1+\sigma}}{2i}\sum_{k=1}^2 Op^w_h\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\partial_k(d_j\phi(x)\theta_h(x))(\partial_k\chi)(h^\sigma\xi)\Big)\widetilde{v}+Op^w_h(r_2(x,\xi))\widetilde{v},\end{aligned}$$

with $r_2\in h^{1-4\sigma}S_{\frac{1}{2},\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-1})$. On the one hand, as

$$Op^w_h\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)(\xi_j-d_j\phi(x))\Big)\widetilde{v}=\sum_{k=1}^2 Op^w_h\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)\widetilde{e}^j_k(x_k-p'_k(\xi))\Big)\widetilde{v},$$

with $\widetilde{e}^j_k$ satisfying (1.2.67b) on the support of $\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi(h^\sigma\xi)$, its L^2 norm can be estimated using (1.2.63a).

On the other hand, as $\partial\gamma$ vanishes in a neighbourhood of the origin, the L^2 norm of the second and third term in the right hand side of (3.2.74) can be estimated using (3.2.18a).

The two remaining contributions to the right hand side of (3.2.74), that already carry the right power of h, can be estimated with $h^{1-\beta}\|\widetilde{v}(t,\cdot)\|_{L^2}$ simply by proposition 1.2.36. □

We can finally state the following result:

Proposition 3.2.17 (Deduction of the transport equation)**.** *For any fixed $T>1$, $D>0$, let $\mathcal{C}^T_D := \{(t,x) : 1\le t\le T, |x|\le D\}$ be the truncated cylinder, and assume that (u,v) is solution to* (1.1.1)-(1.1.2) *in interval* $[1,T]$. *Then function $\widetilde{u}^\Sigma_{\Lambda_w}(t,x) := \sum_k \widetilde{u}^{\Sigma,k}_{\Lambda_w}(t,x)$ is solution to the following transport equation:*

$$\Big[D_t+\frac{1}{2}(1-|x|^2)x\cdot(hD_x)+\frac{h}{2i}(1-2|x|^2)\Big]\widetilde{u}^\Sigma_{\Lambda_w}(t,x)=F_w(t,x),\quad \forall(t,x)\in\mathcal{C}^T_D, \tag{3.2.75}$$

and there exists $s > 0$ sufficiently large such that

(3.2.76)

$$
\begin{aligned}
&\|F_w(t,\cdot)\|_{L^\infty} \\
&\lesssim h^{1-\beta}\Big[\|\widetilde{u}(t,\cdot)\|_{H^s_h} + \|Op^w_h(\chi(h^\sigma\xi))\mathcal{M}\widetilde{u}(t,\cdot)\|_{L^2} + \|(\theta_0\Omega_h)^\mu\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2} + \sum_{\mu=0}^{1}\|(\theta_0\Omega_h)^\mu\mathcal{M}\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2}\Big] \\
&+ h^{-\frac{1}{2}-\beta}\left\|\chi_1(t^{-\sigma}D_x)(v^{NF}-v_-)(t,\cdot)\right\|_{L^2}\left(\|V(t,\cdot)\|_{H^{2,\infty}} + \|v^{NF}(t,\cdot)\|_{H^{1,\infty}}\right) \\
&+ h^{\frac{3}{2}-\beta}\left\|(v^{NF}-v_-)(t,\cdot)\right\|_{H^1}\left(\|V(t,\cdot)\|_{H^s} + \|v^{NF}(t,\cdot)\|_{H^s}\right) \\
&+ h^{-\frac{1}{2}-\beta}\left\|\chi_1(t^{-\sigma}D_x)\Omega(v^{NF}-v_-)(t,\cdot)\right\|_{L^2}\left(\|V(t,\cdot)\|_{H^{2,\infty}} + \|v^{NF}(t,\cdot)\|_{H^{1,\infty}}\right) \\
&+ h^{\frac{3}{2}-\beta}\left\|\Omega(v^{NF}-v_-)(t,\cdot)\right\|_{L^2}\left(\|V(t,\cdot)\|_{H^s} + \|v^{NF}(t,\cdot)\|_{H^s}\right) \\
&+ h^{-\frac{1}{2}-\beta}\left\|(v^{NF}-v_-)(t,\cdot)\right\|_{H^{1,\infty}} \times \sum_{\mu=0}^{1}\left(\|\Omega^\mu V(t,\cdot)\|_{H^1} + \|\Omega v^{NF}(t,\cdot)\|_{L^2}\right) \\
&+ h^{1-\beta}\Big(\|\widetilde{v}(t,\cdot)\|_{H^s} + \sum_{|\mu|=1}^{2}\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2}\Big)\|\widetilde{v}(t,\cdot)\|_{H^{1,\infty}_h} \\
&+ h^{-\frac{1}{2}-\beta}\|V(t,\cdot)\|^2_{H^{1,\infty}}\|U(t,\cdot)\|_{H^1} \\
&+ h^{-\frac{1}{2}-\beta}\Big[\|V(t,\cdot)\|^{1-\theta}_{H^{15,\infty}}\|V(t,\cdot)\|^\theta_{H^{17}}\left(\|U(t,\cdot)\|_{H^{1,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}}\right) \\
&\quad + \|V(t,\cdot)\|_{L^\infty}\left(\|U(t,\cdot)\|^{1-\theta}_{H^{16,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|^{1-\theta}_{H^{16,\infty}}\right)\|U(t,\cdot)\|^\theta_{H^{18}}\Big]\|\Omega V(t,\cdot)\|_{L^2} \\
&+ h^{-\frac{1}{2}-\beta}\Big[\|V(t,\cdot)\|_{H^{1,\infty}}\left(\|U(t,\cdot)\|_{H^1} + \|\Omega U(t,\cdot)\|_{H^1}\right) \\
&\quad + \left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}\right)\|\Omega V(t,\cdot)\|_{L^2}\Big]\|V(t,\cdot)\|_{H^{17,\infty}} \\
&+ h^{1-\beta}\sum_{|\gamma|=1}^{2}\|(Z^\gamma u)_-(t,\cdot)\|_{L^2} + h^{-\beta}\Big[\|V(t,\cdot)\|_{H^1} + \|V(t,\cdot)\|_{L^2}(\|U(t,\cdot)\|_{H^{1,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}}) \\
&+ \|V(t,\cdot)\|_{H^{1,\infty}}\|U(t,\cdot)\|_{H^1}\Big]\left(\|V(t,\cdot)\|_{H^{14,\infty}} + h\|V(t,\cdot)\|_{H^{13}}\right) + h^{-\beta}\|Z_nV(t,\cdot)\|_{H^1}\|V(t,\cdot)\|_{H^{17,\infty}} \\
&+ h^{-\frac{1}{2}-\beta}\|V(t,\cdot)\|^2_{H^{13,\infty}}\|U(t,\cdot)\|_{H^1},
\end{aligned}
$$

for some $\chi \in C^\infty_0(\mathbb{R}^2)$, and $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. By the assumption in the statement, all that we are going to say is to be meant in time interval $[1,T]$. We also remind the reader that, by the definition of $\widetilde{u}^{\Sigma,k}_{\Lambda_w}$ in (3.2.44a) and of $\widetilde{u}^{\Sigma,k}$ in (3.2.41), the sum defining $\widetilde{u}^{\Sigma}_{\Lambda_w}$ is finite, and restricted to indices $k \in K := \{k \in \mathbb{Z} : h \lesssim 2^k \lesssim h^{-\sigma}\}$.

In lemma 3.2.11 we proved that, for any $k \in K$, any two constants $0 < D_1 < D_2$, function $\widetilde{u}^{\Sigma,k}_{\Lambda_w}$ is solution to (3.2.54), with θ (resp. $\widetilde{\theta}$) being a smooth function equal to 1 in closed ball $\overline{B_{D_1}(0)}$ (resp. supported for $D_1 < |x| < D_2$), $\widetilde{\varphi}, \widetilde{\varphi}_1 \in C^\infty_0(\mathbb{R}^2 \setminus \{0\})$, $\chi_0 \in C^\infty_0(\mathbb{R}^2)$, q_w, c_w and r^{NF}_w respectively given by (3.1.17), (3.1.18) and (3.1.19), and f^w_k verifying (3.2.52).

On the one hand, reminding (3.2.68) and using the $L^\infty - L^\infty$ continuity of operator $\Gamma^{w,k}$ with norm $O(h^{-\beta})$ (see proposition 1.2.29), together with the classical Sobolev injection, and the fact that

$$
\left\|Op^w_h(\Sigma(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi))\right\|_{L^2} = O(h^{-\mu}), \tag{3.2.77}
$$

with $\mu = \sigma\rho$ if $\rho \ge 0$, 0 otherwise, we find that

$$\begin{aligned}(3.2.78a) \quad &\left\|\Gamma^{w,k} Op_h^w(\Sigma(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi))\,[tq_w(t,tx)]\right\|_{L^\infty} \\ &\lesssim h^{-\beta}\|\widetilde{q}_w(t,\cdot)\|_{L^2} + h^{-1-\beta}\|Op_h^w(\varphi(2^{-k}\xi))(hD_x)\widetilde{q}_w(t,\cdot)\|_{L^2}\end{aligned}$$

so from (3.2.69) and(3.2.70), it is bounded by

$$(3.2.78b) \qquad h^{1-\beta}\Big(\|\widetilde{v}(t,\cdot)\|_{H^s} + \sum_{|\mu|=1}^{2}\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2}\Big)\|\widetilde{v}(t,\cdot)\|_{H_h^{1,\infty}}.$$

On the other hand, using proposition 1.2.30, estimate (3.2.77) and the fact that the commutator between $Op_h^w(\Sigma(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi))$ and Ω_h is also continuous on L^2 with norm $O(h^{-\mu})$, and equality $\|tw(t,t\cdot)\|_{L^2} = \|w(t,\cdot)\|_{L^2}$, we deduce that, from (3.1.21a), (3.1.21c) (in which we choose $s>0$ large enough to have, say, $N(s)\ge 2$),

$$\begin{aligned}&\|\Gamma^{w,k}Op_h^w\big(\Sigma(\xi)\varphi(2^{-k}\xi)\big)(h^{-1}c_w(t,tx))\|_{L^\infty} \\ &\lesssim t^{\frac12+\beta}\left\|\chi_1(t^{-\sigma}D_x)(v^{NF}-v_-)(t,\cdot)\right\|_{L^2}\left(\|V(t,\cdot)\|_{H^{2,\infty}} + \|v^{NF}(t,\cdot)\|_{H^{1,\infty}}\right) \\ &+ t^{-\frac32+\beta}\left\|(v^{NF}-v_-)(t,\cdot)\right\|_{H^1}\left(\|V(t,\cdot)\|_{H^s} + \|v^{NF}(t,\cdot)\|_{H^s}\right) \\ (3.2.79) \quad &+ t^{\frac12+\beta}\left\|\chi_1(t^{-\sigma}D_x)\Omega(v^{NF}-v_-)(t,\cdot)\right\|_{L^2}\left(\|V(t,\cdot)\|_{H^{2,\infty}} + \|v^{NF}(t,\cdot)\|_{H^{1,\infty}}\right) \\ &+ t^{-\frac32+\beta}\left\|\Omega(v^{NF}-v_-)(t,\cdot)\right\|_{L^2}\left(\|V(t,\cdot)\|_{H^s} + \|v^{NF}(t,\cdot)\|_{H^s}\right) \\ &+ t^{\frac12+\beta}\left\|(v^{NF}-v_-)(t,\cdot)\right\|_{H^{1,\infty}} \times \sum_{\mu=0}^{1}\left(\|\Omega^\mu V(t,\cdot)\|_{H^1} + \|\Omega v^{NF}(t,\cdot)\|_{L^2}\right),\end{aligned}$$

for some $\chi_1 \in C_0^\infty(\mathbb{R}^2)$, and from (3.1.22a), (3.1.22c) we get that

$$\begin{aligned}&\|\Gamma^{w,k}Op_h^w\big(\Sigma(\xi)\varphi(2^{-k}\xi)\big)(h^{-1}r_w^{NF}(t,tx))\|_{L^\infty} \lesssim t^{\frac12+\beta}\|V(t,\cdot)\|^2_{H^{1,\infty}}\|U(t,\cdot)\|_{H^1} \\ &+ t^{\frac12+\beta}\Big[\|V(t,\cdot)\|^{1-\theta}_{H^{15,\infty}}\|V(t,\cdot)\|^\theta_{H^{17}}\left(\|U(t,\cdot)\|_{H^{1,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{1,\infty}}\right) \\ (3.2.80) \quad &\quad + \|V(t,\cdot)\|_{L^\infty}\Big(\|U(t,\cdot)\|^{1-\theta}_{H^{16,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|^{1-\theta}_{H^{16,\infty}}\Big)\|U(t,\cdot)\|^\theta_{H^{18}}\Big]\|\Omega V(t,\cdot)\|_{L^2} \\ &+ t^{\frac12+\beta}\Big[\|V(t,\cdot)\|_{H^{1,\infty}}\left(\|U(t,\cdot)\|_{H^1} + \|\Omega U(t,\cdot)\|_{H^1}\right) \\ &\quad + \left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}\right)\|\Omega V(t,\cdot)\|_{L^2}\Big]\|V(t,\cdot)\|_{H^{17,\infty}},\end{aligned}$$

for a small $\beta>0$, $\beta\to 0$ as $\sigma\to 0$.

Moreover, since function $(\partial\chi_0)(h^{-1}\xi)$ is localized for frequencies ξ of size h, we have that $ih\,\Gamma^{w,k}Op_h^w\big((\partial\chi_0)(h^{-1}\xi)\cdot(h^{-1}\xi)\psi(2^{-k}\xi)\big)\widetilde{u}$ appearing in the right hand side of (3.2.54) is non-zero only for values of $k\in\mathbb{Z}$ such that $2^k\sim h$. In that case, by commutating $\Gamma^{w,k}$ with $Op_h^w\big((\partial\chi_0)(h^{-1}\xi)\cdot(h^{-1}\xi)\psi(2^{-k}\xi)\big)$, and using the classical Sobolev injection, together with proposition 1.2.27, we have that

$$(3.2.81) \qquad \left\|ih\,\Gamma^{w,k}Op_h^w\big((\partial\chi_0)(h^{-1}\xi)\cdot(h^{-1}\xi)\psi(2^{-k}\xi)\big)\widetilde{u}(t,\cdot)\right\|_{L^\infty} \lesssim h\|\widetilde{u}(t,\cdot)\|_{L^2}.$$

Since $(\partial\chi_0)(h^\sigma\xi)$ is, instead, localized for large frequencies $|\xi|\gtrsim h^{-\sigma}$, by applying the semi-classical Sobolev injection and lemma 1.2.39, we find that

$$(3.2.82) \qquad \left\|i\sigma h\,\Gamma^{w,k}Op_h^w\big(\psi(2^{-k}\xi)(\partial\chi_0)(h^\sigma\xi))\cdot(h^\sigma\xi)\big)\widetilde{u}(t,\cdot)\right\|_{L^\infty} \lesssim h^N\|\widetilde{u}(t,\cdot)\|_{H_h^s},$$

with $N = N(s) > 1$ as long as $s > 0$ is sufficiently large.

After lemma 3.2.12

$$- Op^w_h\big(\theta(x)(x\cdot\xi - |\xi|)\widetilde{\varphi}(2^{-k}\xi)\big)\widetilde{u}^{\Sigma,k}_{\Lambda_w} = \frac{1}{2}Op^w_h\big(\theta(x)(1-|x|^2)x\cdot\xi\widetilde{\varphi}(2^{-k}\xi)\big)\widetilde{u}^{\Sigma,k}_{\Lambda_w} + Op^w_h(\theta(x)e(x,\xi)\widetilde{\varphi}(2^{-k}\xi))\widetilde{u}^{\Sigma,k}_{\Lambda_w},$$

where $e(x,\xi)$ is given by (3.2.56) and latter term in the above right hand side satisfies (3.2.61). By symbolic calculus we find that

$$\begin{aligned}\frac{1}{2}&Op^w_h\big(\theta(x)(1-|x|^2)x\cdot\xi\widetilde{\varphi}(2^{-k}\xi)\big)\widetilde{u}^{\Sigma,k}_{\Lambda_w}\\ &= \theta(x)\Big[\frac{1}{2}(1-|x|^2)x\cdot(hD_x) + \frac{h}{2i}(1-2|x|^2)\Big]Op^w_h(\widetilde{\varphi}(2^{-k}\xi))\widetilde{u}^{\Sigma,k}_{\Lambda_w}\\ &+ \frac{h}{4i}(\partial\theta)(x)\cdot x(1-|x|^2)Op^w_h(\widetilde{\varphi}(2^{-k}\xi))\widetilde{u}^{\Sigma,k}_{\Lambda_w}\\ &+ {\sum}' h\theta_1(x)Op^w_h(\widetilde{\varphi}_1(2^{-k}\xi))\widetilde{u}^{\Sigma,k}_{\Lambda_w} + Op^w_h(r(x,\xi))\widetilde{u}^{\Sigma,k}_{\Lambda_w},\end{aligned}$$

with $\sum'$ being a concise notation to indicate a linear combination, $\partial\theta(x)$ supported for $|x| > D_1$, some new $\theta_1 \in C^\infty_0(\mathbb{R}^2)$, $\widetilde{\varphi}_1 \in C^\infty_0(\mathbb{R}^2\setminus\{0\})$ coming out from the derivatives of $\widetilde{\varphi}$, and $r(x,\xi)$ integral remainder of the form

$$\frac{h^N}{(\pi h)^2}\int e^{\frac{2i}{h}\eta\cdot z}\int_0^1 \theta_N(x+tz)(1-t)^{N-1}dt\ \widetilde{\varphi}_N(2^{-k}(\xi+\eta))dzd\eta,$$

for some $\theta_N \in C^\infty_0(\mathbb{R}^2)$, $\widetilde{\varphi}_N \in C^\infty_0(\mathbb{R}^2\setminus\{0\})$, and $N \in \mathbb{N}$, verifying that

$$\|Op^w_h(r(x,\xi))\|_{\mathcal{L}(L^2;L^\infty)} = O(h) \tag{3.2.83}$$

if N is taken sufficiently large. Therefore, from the $L^2 - L^2$ continuity of $\Gamma^{w,k}$ by proposition 1.2.27, and from (3.2.77),

$$\Big\|Op^w_h(r(x,\xi))\widetilde{u}^{\Sigma,k}_{\Lambda_w}(t,\cdot)\Big\|_{L^\infty} \lesssim h^{1-\mu}\|\widetilde{u}(t,\cdot)\|_{L^2}.$$

Moreover, since $\widetilde{\varphi} \equiv 1$ on the support of φ (which defines $\widetilde{u}^{\Sigma,k}$), we can replace $Op^w_h(\widetilde{\varphi}(2^{-k}\xi))\widetilde{u}^{\Sigma,k}_{\Lambda_w}$ with $\widetilde{u}^{\Sigma,k}_{\Lambda_w}$, up to some remainders $O_{L^\infty}(h^N\|\widetilde{u}\|_{L^2})$ with $N \in \mathbb{N}$ as large as we want, obtained from symbolic calculus, by commutating $Op^w_h(\widetilde{\varphi}(2^{-k}\xi))$ with $\Gamma^{w,k}$, and successively using remark 1.2.22. For the same reason, since $\widetilde{\varphi}_1$ is obtained from the derivatives of $\widetilde{\varphi}$, and hence vanishes on the support of φ, all terms $\theta_1 Op^w_h(\widetilde{\varphi}_1(2^{-k}\xi))\widetilde{u}^{\Sigma,k}_{\Lambda_w}$ are remainders $O_{L^\infty}(h^N\|\widetilde{u}\|_{L^2})$ with $N \in \mathbb{N}$ large. Therefore, we deduce that

$$\begin{aligned}- Op^w_h\big(\theta(x)(x\cdot\xi - |\xi|)\widetilde{\varphi}(2^{-k}\xi)\big)\widetilde{u}^{\Sigma,k}_{\Lambda_w} &= \theta(x)\Big[\frac{1}{2}(1-|x|^2)x\cdot(hD_x) + \frac{h}{2i}(1-2|x|^2)\Big]\widetilde{u}^{\Sigma,k}_{\Lambda_w}\\ &+ \frac{h}{4i}(\partial\theta)(x)\cdot x(1-|x|^2)\widetilde{u}^{\Sigma,k}_{\Lambda_w} + Op^w_h\big(\theta(x)\widetilde{\varphi}(2^{-k}\xi)e(x,\xi)\big)\widetilde{u}^{\Sigma,k}_{\Lambda_w}\\ &+ O_{L^\infty}(h^{1-\beta}\|\widetilde{u}(t,\cdot)\|_{L^2}),\end{aligned}$$

which implies, summed up with estimates from (3.2.78) to (3.2.82), that for any k, $\widetilde{u}^{\Sigma,k}_{\Lambda_w}$ is solution to

$$\begin{aligned}&\Big[D_t + \theta(x)\frac{1}{2}(1-|x|^2)x\cdot(hD_x) + \theta(x)\frac{h}{2i}(1-2|x|^2)\Big]\ \widetilde{u}^{\Sigma,k}_{\Lambda_w}(t,x) = F^k_w(t,x)\\ &+\Big[(1-\theta)(x)Op^w_h((x\cdot\xi-|\xi|)\widetilde{\varphi}(2^{-k}\xi)) + \widetilde{\theta}(x)Op^w_h(\widetilde{\varphi}_1(2^{-k}\xi)) - \frac{h}{4i}(\partial\theta)(x)\cdot x(1-|x|^2)\Big]\widetilde{u}^{\Sigma,k}_{\Lambda_w}(t,x)\,,\end{aligned}$$

where $F^k_w(t,x)$ satisfies (3.2.76).

Therefore, choosing $D_1 = D$, we obtain that $\widetilde{u}^{\Sigma}_{\Lambda_w}$ is solution to (3.2.75) in cylinder $\mathcal{C}^T_D$, with $F_w(t,x) := \sum_k F^k_w(t,x)$ (this sum being finite and restricted to indices $k \in K$) satisfying the same L^∞ estimate as F^k_w, up to an additional factor $h^{-\sigma}$. □

3.3 Analysis of ODEs and end of the proof

In previous section (see proposition 3.2.7) we firstly showed how to propagate a-priori uniform estimate (1.1.11b) on the Klein-Gordon component v_-, in the sense of deducing (1.1.12b) from estimates (1.1.11). We then passed to the study of the wave equation, and proved that, if (u_-, v_-) is solution to (3.1.1) in some interval $[1,T]$, a new certain function $\widetilde{u}^{\Sigma}_{\Lambda_w}$, defined from u_-, is solution to transport equation (3.2.75) in truncated cylinder $\mathcal{C}^T_D := \{(t,x) : 1 \le t \le T, D \ge 1\}$, for any $D > 0$. The aim of this section is to study such a transport equation, in order to deduce some information on the uniform norm of its solutions. This will allow us to finally propagate a-priori estimate (1.1.11a) on the wave component u_-, and to close the bootstrap argument. At the end, we will give a short proof of main theorem 1.1.1.

3.3.1 The Inhomogeneous Transport Equation

The aim of this subsection is to study the behaviour of a solution w to the following transport equation

$$\left[D_t + \frac{1}{2}(1-|x|^2)x\cdot(hD_x) - \frac{i}{2t}(1-2|x|^2)\right] w = f\,, \tag{3.3.1}$$

in a cylinder $\mathcal{C} = \{(t,x) : t \ge 1, |x| \le D\}$, for a large constant $D \gg 1$, and where $f = O_{L^\infty}(\varepsilon t^{-1+\beta})$, $\varepsilon > 0$ small, $0 \le \beta < 1/2$. We distinguish in $\mathcal{C}$ two subregions:

$$I_1 := \left\{(t,x) : t\ge 1, |x| < \Big(\frac{t}{t-1}\Big)^{\frac{1}{2}}, |x| \le D\right\}, \qquad I_2 := \left\{(t,x) : t > 1, \Big(\frac{t}{t-1}\Big)^{\frac{1}{2}} \le |x| \le D\right\},$$

and denote by $I_{1,t}, I_{2,t}$ their sections at a fixed time $t \ge 1$,

$$I_{1,t} := \left\{x : |x| < \Big(\frac{t}{t-1}\Big)^{\frac{1}{2}}, |x| \le D\right\}, \qquad I_{2,t} := \left\{x : \Big(\frac{t}{t-1}\Big)^{\frac{1}{2}} \le |x| \le D\right\}.$$

The result we prove is the following.

Proposition 3.3.1. *Let $\varepsilon > 0$ small and w be the solution to*

$$\begin{cases} \left[D_t + \frac{1}{2}(1-|x|^2)x\cdot(hD_x) - \frac{i}{2t}(1-2|x|^2)\right] w = f\,, \\ w(1,x) = \varepsilon w_0(x)\,, \end{cases} \tag{3.3.2}$$

with $f = O_{L^\infty}(\varepsilon t^{-1+\beta})$, for $0 \le \beta < 1/2$. Let us suppose that $|w_0(x)| \lesssim \langle x\rangle^{-2}$, and that $|w(t,x)| \lesssim \varepsilon t^{\beta'}$ when $|x| > D \gg 1$, for some $\beta' > 0$. Therefore,

$$|w(t,x)| \lesssim \varepsilon \|w_0\|_{L^\infty} t^{\beta''}(1+|x|)^{-\frac{1}{2}}(t^{-1} + |1-|x||)^{-\frac{1}{2}+\beta''}\,, \tag{3.3.3}$$

for every $(t,x) \in \mathcal{C}_D = \{(t,x) | t \ge 1, |x| \le D\}$, with $\beta'' = \max\{\beta, \beta'\}$.

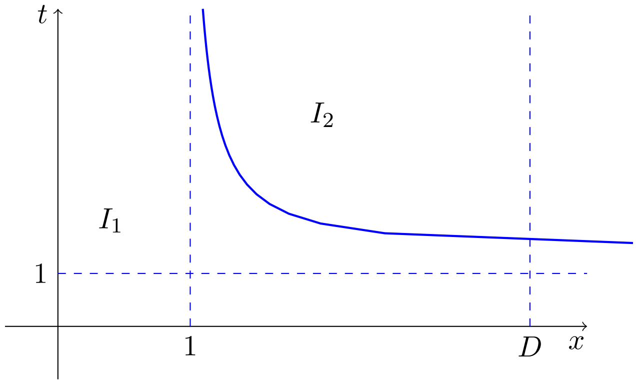

Figure 3.3: Regions I_1 and I_2 in space dimension 1

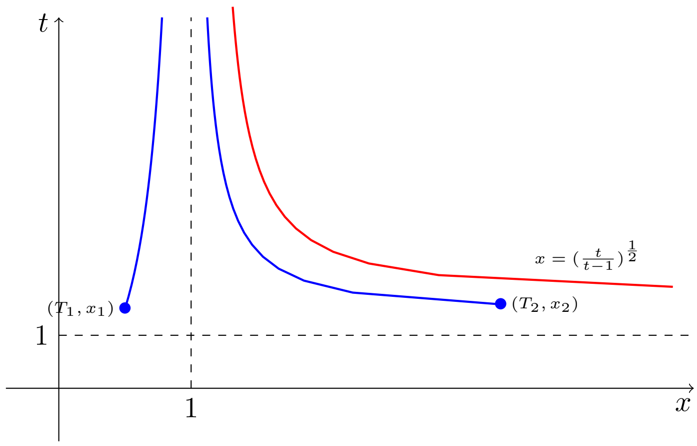

Figure 3.4: Characteristic curves of initial point $(T_i, x_i) \in I_1$, $i = 1, 2$, in space dimension 1

We observe that, if $W(t,x) = t^{-1}w(t, t^{-1}x)$, the above inequality implies that

$$|W(t,x)| \lesssim \varepsilon \|w_0\|_{L^\infty}(t+|x|)^{-\frac{1}{2}}(1+|t-|x||)^{-\frac{1}{2}+\beta''},$$

showing that the uniform norm of $W(t,\cdot)$ decays in time at a rate $t^{-1/2}$, which is enhanced to $t^{-1+\beta''}$ out of the light cone $t = |x|$.

In order to prove the result of proposition 3.3.1, we fix a time $T \geq 1$ and $x \in B_D(0)$, and look for the characteristic curve of (3.3.2) with initial point (T, x), i.e. map $t \mapsto X(t; T, x)$ solution of

$$\begin{cases} \frac{d}{dt}X(t;T,x) = \frac{1}{2t}\big(1-|X(t;T,x)|^2\big)X(t;T,x) \\ X(T;T,x) = x \end{cases} \qquad t \geq T. \tag{3.3.4}$$

Lemma 3.3.2. *Solution $X(t;T,x)$ to* (3.3.4) *writes explicitly as*

$$X(t;T,x) = \frac{\sqrt{t}x}{(T-(T-t)|x|^2)^{\frac{1}{2}}}, \tag{3.3.5}$$

and it is well defined for all $t > T(1-|x|^{-2})$. Moreover, if $t > T$ is fixed, map $x \in \mathbb{R}^2 \mapsto X(t;T,x) \in \left\{|x| < \left(\frac{t}{t-T}\right)^{\frac{1}{2}}\right\}$ is a diffeomorphism of inverse $Y(t,y) = \frac{\sqrt{T}y}{(t+(T-t)|y|^2)^{\frac{1}{2}}}$.

Proof. Multiplying equation (3.3.4) by $2X(t;T,x)$, we deduce that $|X(t;T,x)|^2$ satisfies the equation:

$$\frac{d}{dt}|X(t;T,x)|^2 = \frac{1}{t}\big(1-|X(t;T,x)|^2\big)|X(t;T,x)|^2\,,$$

from which follows that $1-|X(t;T,x)|^2 = \frac{T(1-|x|^2)}{T-(T-t)|x|^2}$. Injecting this result in (3.3.4) and integrating in time, we obtain expression (3.3.5) and observe that the map we obtained is well defined for all $t > T(1-|x|^{-2})$.

Let now suppose to fix $t > T$. In order to prove the second part of the statement, we fix $y \in \Big\{|x| \le \big(\frac{t}{t-T}\big)^{\frac{1}{2}}\Big\}$ and look for $Y(t,y)$ such that $X(t;T,Y(t,y)) = y$. In other words,

$$y = \frac{\sqrt{t}Y(t,y)}{(T-(T-t)|Y(t,y)|^2)^{\frac{1}{2}}},$$

which implies that $Y(t,y) = \frac{\sqrt{T}y}{(t+(T-t)|y|^2)^{\frac{1}{2}}}$, and this map is well defined as long as $|y| < \big(\frac{t}{t-T}\big)^{\frac{1}{2}}$. □

Along the characteristic curve $X(t;T,x)$ function w satisfies

$$\begin{aligned}\frac{d}{dt}w\big(t,X(t;T,x)\big) &= -\frac{1}{2t}\big(1-2|X(t;T,x)|^2\big)\; w\big(t,X(t;T,x)\big) + if\big(t,X(t;T,x)\big)\\ &= -\frac{1}{2t}\frac{T-T|x|^2-t|x|^2}{T-(T-t)|x|^2}\; w\big(t,X(t;T,x)\big) + if\big(t,X(t;T,x)\big)\,,\end{aligned}$$

and then

$$\begin{aligned}&\frac{d}{dt}\Big[\Big(\exp\int_T^t \frac{1}{2\tau}\frac{T-T|x|^2-\tau|x|^2}{T-(T-\tau)|x|^2}d\tau\Big)\, w\big(t,X(t;T,x)\big)\Big]\\ &\qquad = i\Big(\exp\int_T^t \frac{1}{2\tau}\frac{T-T|x|^2-\tau|x|^2}{T-(T-\tau)|x|^2}d\tau\Big)\, f\big(t,X(t;T,x)\big)\,.\end{aligned} \tag{3.3.6}$$

Lemma 3.3.3.

$$\exp\int_T^t \frac{1}{2\tau}\frac{T-T|x|^2-\tau|x|^2}{T-(T-\tau)|x|^2}\,d\tau = \Big(\frac{t}{T}\Big)^{\frac{1}{2}}\Big(\frac{T-T|x|^2+t|x|^2}{T}\Big)^{-1}. \tag{3.3.7}$$

Proof. The result follows writing

$$\frac{1}{2\tau}\frac{T-T|x|^2-\tau|x|^2}{T-(T-\tau)|x|^2} = \frac{1}{2\tau} - \frac{|x|^2}{T-T|x|^2+\tau|x|^2}\,,$$

taking the integral over $\tau \in [T,t]$ and then passing to its exponential. □

Let us first study function w in I_1, so assume that $T=1$. Integrating equality (3.3.6) over $[1,t]$, we find that

$$\begin{aligned}&\Big(\exp\int_1^t \frac{1}{2\tau}\frac{1-|x|^2-\tau|x|^2}{1-(1-\tau)|x|^2}d\tau\Big)\, w\big(t,X(t;1,x)\big)\\ &\qquad = w(1,x) + i\int_1^t\Big(\exp\int_1^s \frac{1}{2\tau}\frac{1-|x|^2-s|x|^2}{1-(1-s)|x|^2}ds\Big)\, f\big(s,X(s;1,x)\big)ds\,,\end{aligned} \tag{3.3.8}$$

so using (3.3.7) and the fact that $f = O_{L^\infty}(\varepsilon t^{-1+\beta})$, we obtain that

$$
(3.3.9)\quad \big|w(t, X(t;1,x))\big| \leq t^{-\frac{1}{2}}(1-|x|^2+t|x|^2)|w(1,x)| \\
+ C\varepsilon t^{-\frac{1}{2}}(1-|x|^2+t|x|^2)\int_1^t \frac{ds}{(1-|x|^2+s|x|^2)s^{\frac{1}{2}-\beta}}.
$$

Lemma 3.3.4.

$$
(3.3.10)\qquad \int_1^t \frac{ds}{(1-|x|^2+s|x|^2)s^{\frac{1}{2}-\beta}} \lesssim \frac{t^{\frac{1}{2}+\beta}}{(1+\sqrt{t}|x|)^{1+2\beta}}(1+|x|)^{-1+2\beta+\beta'},
$$

for all $t \geq 1$, $0 \leq \beta < 1/2$, $\beta' > 0$ *small.*

Proof. For $\sqrt{t}|x| \leq 1$, we have that

$$
\int_1^t \frac{ds}{(1-|x|^2+s|x|^2)s^{\frac{1}{2}-\beta}} \lesssim t^{\frac{1}{2}+\beta} \lesssim \frac{t^{\frac{1}{2}+\beta}}{(1+\sqrt{t}|x|)^{1+2\beta}}(1+|x|)^{-1+2\beta+\beta'}.
$$

Suppose then that $\sqrt{t}|x| > 1$. For $t \leq 2$,

$$
\int_1^t \frac{ds}{(1-|x|^2+s|x|^2)s^{\frac{1}{2}-\beta}} \lesssim (1+|x|)^{-2}\log(1+|x|^2)
$$

and $|x|^{-2}\log(1+|x|^2)\frac{(1+\sqrt{t}|x|)^{1+2\beta}}{t^{\frac{1}{2}+\beta}} \lesssim (1+|x|)^{-1+2\beta}\log(1+|x|^2)$, so we immediately derive inequality (3.3.10). If $t \geq 2$ we can split the integral as follows

$$
\int_1^t \frac{ds}{(1-|x|^2+s|x|^2)s^{\frac{1}{2}-\beta}} = \int_1^2 \frac{ds}{(1-|x|^2+s|x|^2)s^{\frac{1}{2}-\beta}} + \int_2^t \frac{ds}{(1-|x|^2+s|x|^2)s^{\frac{1}{2}-\beta}},
$$

where the first integral is bounded from the right hand side of (3.3.10). The second one is less or equal than $\int_1^{t-1} \frac{ds}{(1+s|x|^2)s^{\frac{1}{2}-\beta}}$, so if $|x| \geq 1$ it follows that

$$
\int_1^{t-1} \frac{ds}{(1+s|x|^2)s^{\frac{1}{2}-\beta}} \leq |x|^{-2}\int_1^{t-1}\frac{ds}{s^{\frac{3}{2}-\beta}} \lesssim (1+|x|)^{-2},
$$

for all $0 \leq \beta < 1/2$, and since $\frac{(1+\sqrt{t}|x|)^{1+2\beta}}{t^{\frac{1}{2}+\beta}} \leq (1+|x|)^{1+2\beta}$, this implies the right bound of the statement. If $|x| < 1$, a change of variables gives us

$$
\int_1^{t-1} \frac{ds}{(1+|x|^2 s)s^{\frac{1}{2}-\beta}} = |x|^{-1-2\beta}\int_{|x|^2}^{(t-1)|x|^2} \frac{ds}{(1+s)s^{\frac{1}{2}-\beta}} \lesssim |x|^{-1-2\beta}\frac{(t|x|^2)^{\frac{1}{2}+\beta}}{(1+t|x|^2)^{\frac{1}{2}+\beta}} \leq \frac{t^{\frac{1}{2}+\beta}}{(1+t|x|^2)^{\frac{1}{2}+\beta}}.
$$

□

If initial condition $w_0(x)$ is sufficiently decaying in space, e.g. $|w_0(x)| \lesssim \langle x\rangle^{-2}$, we deduce from inequalities (3.3.9) and (3.3.10) the following bound for w along the characteristic curve $X(t;1,x)$:

$$
(3.3.11)\qquad \big|w(t, X(t;1,x))\big| \lesssim \varepsilon\|w_0\|_{L^\infty} t^\beta (1+\sqrt{t}|x|)^{1-2\beta}(1+|x|)^{-1+2\beta+\beta'},
$$

$0 \leq \beta < 1/2, \beta' > 0$ small.

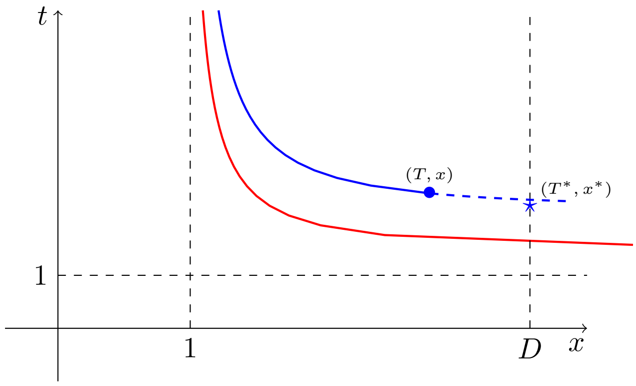

Figure 3.5: Characteristic curve of initial point $(T, x) \in I_2$

Proposition 3.3.5. *Let w be the solution to transport equation (3.3.2), with $\|f(t,\cdot)\|_{L^\infty} \lesssim \varepsilon t^{-1+\beta}$ and initial condition $|w_0(x)| \lesssim \langle x\rangle^{-2}$, $\forall x \in \mathbb{R}^2$. Then*

$$|w(t,x)| \lesssim \varepsilon t^{\beta}\big[t^{-1} + |1-|x||\big]^{-\frac{1}{2}+\beta}, \tag{3.3.12}$$

for every $(t,x) \in I_1 = \{(t,x) : t \geq 1, |x| < \big(\frac{t}{t-1}\big)^{\frac{1}{2}}, |x| \leq D\}$, and $0 \leq \beta < 1/2$.

Proof. In lemma 3.3.2, we proved that, fixed $t > T = 1$, $x \in \mathbb{R}^2 \mapsto X(t;1,x) \in \big\{x : |x| < (\frac{t}{t-1})^{\frac{1}{2}}\big\}$ is a diffeomorphism with inverse $Y(t,y) = y(t + (1-t)|y|^2)^{-1/2}$, so from inequality (3.3.11) we deduce that, for any y such that $|y| < \big(\frac{t}{t-1}\big)^{\frac{1}{2}}$,

$$|w(t,y)| \lesssim \varepsilon t^{\beta}\big(1 + \sqrt{t}|Y(t,y)|\big)^{1-2\beta}\big(1 + |Y(t,y)|\big)^{-1+2\beta+\beta'}.$$

In particular, as $t(1-|y|^2) + |y|^2) \sim t|1-|y|^2| + |y|^2$ when $|y| < \big(\frac{t}{t-1}\big)^{\frac{1}{2}}$ and $t \geq t_0 > 1$, and $t|1-|y|^2| + |y|^2 \sim t|1-|y|| + |y|$ when $|y| \leq D$, we find for those values of (t,y) that

$$|w(t,y)| \lesssim \varepsilon t^{\beta}\left(1 + \frac{\sqrt{t}|y|}{(t|1-|y|| + |y|)^{\frac{1}{2}}}\right)^{1-2\beta} \lesssim \varepsilon t^{\beta}\big[t^{-1} + |1-|y||\big]^{-\frac{1}{2}+\beta},$$

simply using that $(1 + |Y(t,y)|)^{-1+2\beta+\beta'} \leq 1$. Moreover, for $t \to 1$ and $|y| \leq D$,

$$|w(t,y)| \lesssim \varepsilon \lesssim \varepsilon t^{\beta}\big[t^{-1} + |1-|y||\big]^{-\frac{1}{2}+\beta}.$$

□

Proposition 3.3.6. *Let $\varepsilon > 0$ small and w be the solution to transport equation (3.3.2), with $\|f(t,\cdot)\|_{L^\infty} \lesssim \varepsilon t^{-1+\beta}$, and suppose that $|w(t,x)| \lesssim \varepsilon t^{\beta'}$ for $|x| \geq D$, $\beta, \beta' > 0$ small. Then*

$$|w(t,x)| \lesssim \varepsilon t^{\beta''}(|x|^2 - 1)^{\beta'' - \frac{1}{2}},$$

for every $(t,x) \in I_2 = \{(t,x) : t > 1, \big(\frac{t}{t-1}\big)^{\frac{1}{2}} \leq |x| \leq D\}$, with $0 \leq \beta < 1/2$, $\beta'' = \max\{\beta, \beta'\}$.

Proof. Fixed a point $(T,x) \in I_2$, we consider the characteristic equation with initial point (T,x), $X(t;T,x) = \frac{\sqrt{t}x}{(T-(T-t)|x|^2)^{\frac{1}{2}}}$, and observe that there exists a time T^*, $1 < T^* < T$, such that

$X(t;T,x)$ hits the boundary $|y| = D$ at $t = T^*$. In other words, $t = T^*$ is the first time when $X(t;T,x)$ enters in the region $\{(t,x) : t \ge 1, |x| \le D\}$, to never leave it again (for function $t \mapsto |X(t;T,x)|$ is strictly decreasing), and time T^* can be expressed in terms of T as

$$T^* = \frac{D^2}{D^2-1}(1-|x|^{-2})T < T\,. \tag{3.3.13}$$

Integrating expression (3.3.6) over $[T^*,T]$ and using (3.3.7), we find that

$$\begin{aligned} w(T,x) = \Big(\frac{T^*}{T}\Big)^{\frac{1}{2}}\Big(\frac{T-T(1-|x|^{-2})}{T^*-T(1-|x|^{-2})}\Big)w(T^*, X(T^*;T,x)) \\ + i\int_{T^*}^T \Big(\frac{t}{T}\Big)^{\frac{1}{2}}\Big(\frac{T-T(1-|x|^{-2})}{t-T(1-|x|^{-2})}\Big)f\big(t, X(t;T,x)\big)dt\,. \end{aligned}$$

From (3.3.13), $T^*-T(1-|x|^{-2}) = \frac{1}{D^2-1}(1-|x|^{-2})T$, $\frac{T^*}{T} = \frac{D^2}{D^2-1}(1-|x|^{-2})$, and if we knew that $|w(t,x)| \lesssim \varepsilon t^{\beta'}$ whenever $|x| \ge D$, for some $\beta' > 0$, we could control the first contribution in right hand side of previous equality by $C\varepsilon(|x|^2-1)^{-\frac{1}{2}}(T^*)^{\beta'}$, for a constant $C > 0$. In the integral term, $|f(t, X(t;T,x))| \lesssim \varepsilon t^{-1+\beta}$ by hypothesis, thus

$$\begin{aligned} \Big|\int_{T^*}^T \Big(\frac{t}{T}\Big)^{\frac{1}{2}}\Big(\frac{T-T(1-|x|^{-2})}{t-T(1-|x|^{-2})}\Big)f\big(t, X(t;T,x)\big)\ dt\Big| &\lesssim \varepsilon T^{\frac{1}{2}}\int_{T^*}^T \big(t-T(1-|x|^{-2})\big)^{-1}t^{-\frac{1}{2}+\beta}dt \\ &= \varepsilon T^{\frac{1}{2}}\int_{T^*}^T \big(t-T^*+c(1-|x|^{-2})T\big)^{-1}t^{-\frac{1}{2}+\beta}dt \\ &\le \varepsilon T^{\frac{1}{2}}\int_0^{T-T^*}\frac{dt}{\big(t+c(1-|x|^{-2})T\big)t^{\frac{1}{2}-\beta}} \\ &\lesssim \varepsilon T^{\frac{1}{2}}\big((1-|x|^{-2})T\big)^{\beta-\frac{1}{2}} = \varepsilon T^{\beta}(1-|x|^{-2})^{\beta-\frac{1}{2}}\,, \end{aligned}$$

for $c = \frac{1}{D^2-1}$. □

3.3.2 Propagation of the uniform estimate on the wave component

Proposition 3.3.7 (Propagation of the a-priori estimate on U, RU)**.** *Let us fix $K_1 > 0$. There exist two integers n, ρ sufficiently large, with $n \gg \rho$, two constants $A, B > 1$ sufficiently large, and $\varepsilon_0 \in]0,1[$ sufficiently small, such that, for any $0 < \varepsilon < \varepsilon_0$, if (u,v) is solution to* (1.1.1)-(1.1.2) *in some interval* $[1,T]$*, for a fixed $T > 1$, and $u_\pm, v_\pm$ defined in* (1.1.5) *satisfy a-priori estimates* (1.1.11)*, for every $t \in [1,T]$, for some small $0 < \delta \ll \delta_2 \ll \delta_1 \ll \delta_0$, then it also verify* (1.1.12a) *in the same interval* $[1,T]$.

Proof. We warn the reader that, throughout the proof, C, β, β' will denote some positive constants that may change line after line, such that $\beta \to 0$ as $\sigma \to 0$ (resp. $\beta' \to 0$ as $\delta, \sigma \to 0$). We also remind that $h = 1/t$.

In proposition 3.1.2 we introduced function u^{NF}, defined from u_- through (3.1.15), and showed that its $H^{\rho+1,\infty}$ norm (resp. the $H^{\rho+1,\infty}$ norm of $\mathrm{R}_j u^{NF}$) differs from that of u_- (resp. of $\mathrm{R}_j u_-$) by a quantity satisfying (3.1.20b) (resp. (3.1.20c)). If n is sufficiently large with respect to ρ (at least $n \geq \rho + 18$), a-priori estimates (1.1.11b), (1.1.11c) give that

$$
\begin{aligned}
(3.3.14) \quad \|u_-(t,\cdot)\|_{H^{\rho+1,\infty}} + \sum_{j=1}^{2} \|\mathrm{R}_j u_-(t,\cdot)\|_{H^{\rho+1,\infty}} \\
\leq \|u^{NF}(t,\cdot)\|_{H^{\rho+1,\infty}} + \sum_{j=1}^{2} \|\mathrm{R}_j u^{NF}(t,\cdot)\|_{H^{\rho,\infty}} + 2AB\varepsilon^2 t^{-1+\frac{\delta}{2}},
\end{aligned}
$$

for every $t \in [1,T]$. We successively considered $\widetilde{u}(t,x) := t\widetilde{u}^{NF}(t,tx)$, and decompose it as in (3.2.38), with $\Sigma(\xi) = \langle\xi\rangle^\rho$ or $\Sigma_j(\xi) = \langle\xi\rangle^\rho \xi_j |\xi|^{-1}$, $j = 1,2$, showing that it satisfies (3.2.39) (resp. (3.2.40)) when restricted to small frequencies $|\xi| \lesssim t^{-1}$ (resp. large frequencies $|\xi| \gtrsim t^\sigma$). So we focused on $\widetilde{u}^{\Sigma,k}$, defined in (3.2.41), and localized for frequencies supported in an annulus of size 2^k, with $k \in K = \{k \in \mathbb{Z} : h \lesssim 2^k \lesssim h^{-\sigma}\}$, and further split it into the sum of functions $\widetilde{u}^{\Sigma,k}_{\Lambda_w}$, $\widetilde{u}^{\Sigma,k}_{\Lambda_w^c}$ (see (3.2.44)).

On one hand, from inequality (3.2.45b) and lemma B.2.1 we deduce that

$$
\|\widetilde{u}^{\Sigma,k}_{\Lambda^c_w}(t,\cdot)\|_{L^\infty} \leq C\varepsilon t^{\beta'},
$$

for every $t \in [1,T]$.

On the other hand, we proved in proposition 3.2.17 that, for any $D > 0$, and any (t,x) in truncated cylinder $\mathcal{C}^T_D = \{(t,x) : 1 \leq t \leq T, |x| \leq D\}$, $\widetilde{u}^{\Sigma}_{\Lambda_w}(t,x) := \sum_k \widetilde{u}^{\Sigma,k}_{\Lambda_w}(t,x)$ is solution to inhomogeneous transport equation (3.2.75), with inhomogeneous term $F_w(t,x)$ satisfying (3.2.76), and hence such that $\|F_w(t,\cdot)\|_{L^\infty} \leq C\varepsilon t^{-1+\beta'}$ in time interval $[1,T]$, after lemmas B.2.1, B.2.14 below, and a-priori estimates (1.1.11). We notice that $|\widetilde{u}^{\Sigma}_{\Lambda_w}(1,x)| \lesssim \varepsilon\langle x\rangle^{-2}$ for every $x \in \mathbb{R}^2$, as a consequence of the fact that $\varepsilon^{-1}\langle x\rangle \widetilde{u}^{\Sigma}_{\Lambda_w} \in L^2$ (if not, $\|\langle\cdot\rangle^{-1}\|_{L^2} \leq \varepsilon^{-1}\|\langle\cdot\rangle\widetilde{u}^{\Sigma}_{\Lambda_w}(1,\cdot)\|_{L^2}$), because

$$
\|\widetilde{u}^{\Sigma}_{\Lambda_w}(1,\cdot)\|_{L^2} + \|x\widetilde{u}^{\Sigma}_{\Lambda_w}(1,\cdot)\|_{L^2} \lesssim \|\widetilde{u}(1,\cdot)\|_{L^2} + \|Op^w_h(\chi(h^\sigma\xi))\mathcal{M}\widetilde{u}(1,\cdot)\|_{L^2} \leq C\varepsilon,
$$

by definition (1.2.45) of $\mathcal{M}$, symbolic calculus, and proposition 1.2.36. Moreover, if $D \gg 1$ is sufficiently large,

$$
\begin{aligned}
(3.3.15) \quad |\mathbb{1}_{|x|\geq D}\widetilde{u}^{\Sigma}_{\Lambda_w}(t,x)| \leq C\frac{\log|x|}{|x|} h^{-\beta}\big(\|Op^w_h(\chi(h^\sigma\xi))\widetilde{u}(t,\cdot)\|_{L^2} + \|Op^w_h(\chi(h^\sigma\xi))\mathcal{M}\widetilde{u}(t,\cdot)\|_{L^2}\big) \\
\leq C\varepsilon\frac{\log|x|}{|x|} t^{\beta'},
\end{aligned}
$$

as follows from lemmas 3.3.9 and B.2.1, and then proposition 3.3.1 implies that

$$
|\widetilde{u}^{\Sigma}_{\Lambda_w}(t,x)| \lesssim \|\widetilde{u}^{\Sigma}_{\Lambda_w}(1,\cdot)\|_{L^\infty} t^{\beta'}(1+|x|)^{-\frac{1}{2}}\big(t^{-1} + |1-|x||\big)^{-\frac{1}{2}+\beta'}, \ \forall (t,x) \in \mathcal{C}^T_D,
$$

with $\|\widetilde{u}^{\Sigma}_{\Lambda_w}(1,\cdot)\|_{L^\infty} \leq C\varepsilon$.

Summing up,

$$
|\widetilde{u}^{\Sigma}(t,x)| \leq C\varepsilon\left[\mathbb{1}_{\mathcal{C}^T_D} t^{\beta'}(1+|x|)^{-\frac{1}{2}}\big(t^{-1}+|1-|x||\big)^{-\frac{1}{2}+\beta'}\right] + C\varepsilon t^{\beta'},
$$

for every $(t,x) \in [1,T]\times\mathbb{R}^2$, $\mathbb{1}_{\mathcal{C}^T_D}$ denoting the characteristic function of cylinder $\mathcal{C}^T_D$, and this means, returning back to function u^{NF} via (3.2.2), that

$$
\begin{aligned}
(3.3.16) \quad \big|\langle D_x\rangle^\rho u^{NF}(t,x)\big| + \sum_{j=1}^{2}\big|\langle D_x\rangle^\rho R_j u^{NF}(t,x)\big| \leq \\
C\varepsilon\left[\mathbb{1}_{\{|x|\leq Dt\}}(t+|x|)^{-\frac{1}{2}}(1+|t-|x||)^{-\frac{1}{2}+\beta'}\right] + C\varepsilon t^{-1+\beta'}.
\end{aligned}
$$

Finally, reminding definition 1.2.1 (*iii*) of space $H^{\rho,\infty}$, injecting the above inequality in (3.3.14), and choosing $A > 1$ sufficiently large such that $C < \frac{A}{3K_1}$, $\varepsilon_0 > 0$ sufficiently small so that $CB\varepsilon_0 < (3K_1)^{-1}$, we deduce enhanced estimate (1.1.12a). $\square$

Remark 3.3.8. Beside the propagation of estimate (1.1.11a), by combining inequalities (3.3.14) and (3.3.16), together with (1.1.5), we also deduce the following inequality

$$|\partial_t u(t,x)| + |\nabla_x u(t,x)| \le C\varepsilon \left[\mathbb{1}_{\{|x|\le Dt\}}(t+|x|)^{-\frac{1}{2}}(1+|t-|x||)^{-\frac{1}{2}+\beta'}\right] + C\varepsilon t^{-1+\beta'},$$

which shows the optimal decay in time $t^{-1+\beta'}$ enjoyed by the wave solution u out of the light cone $|x| = t$.

Lemma 3.3.9. *Let $\chi \in C_0^\infty(\mathbb{R}^2)$ be equal to 1 in a neighbourhood of the origin, and $\sigma > 0$ be small. Let also $\varphi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. There exists a constant $C > 0$, such that for every $h \in]0,1[$, $R \gg 1$, any function $w(t,x)$ such that $w(t,\cdot), Op_h^w(\chi(h^\sigma\xi))\mathcal{M}w(t,\cdot) \in L^2(\mathbb{R}^2)$,*

(3.3.17)
$$\left\|\varphi\Big(\frac{\cdot}{R}\Big)Op_h^w(\chi(h^\sigma\xi))w(t,\cdot)\right\|_{L^\infty} \le CR^{-1}(\log R + |\log h|)\sum_{|\gamma|=0}^{1}\|Op_h^w(\chi(h^\sigma\xi))\mathcal{M}^\gamma w(t,\cdot)\|_{L^2}.$$

Proof. Let us fix $R \gg 1$ and, for seek of compactness, denote $Op_h^w(\chi(h^\sigma\xi))w$ by w^χ. For a new smooth cut-off function χ_1, equal to 1 on the support of χ, we have that

$$\varphi\Big(\frac{x}{R}\Big)Op_h^w(\chi(h^\sigma\xi))w = Op_h^w(\chi_1(h^\sigma\xi))\Big[\varphi\Big(\frac{x}{R}\Big)w^\chi\Big] + \Big[\varphi\Big(\frac{x}{R}\Big), Op_h^w(\chi_1(h^\sigma\xi))\Big]w^\chi,$$

where the symbol associated to above commutator is given by

$$r_R(x,\xi) = -\frac{h^{1+\sigma}R^{-1}}{i(\pi h)^2}\int e^{\frac{2i}{h}\eta\cdot z}\left[\int_0^1(\partial\varphi)\Big(\frac{x+tz}{R}\Big)dt\right](\partial\chi_1)(h^\sigma(\xi+\eta))dzd\eta,$$

as follows from (1.2.19) and integration in variables y,ζ. Since $(\partial\chi_1)(h^\sigma\xi)$ is supported for frequencies $|\xi| \le h^{-\sigma}$, and $R^{-1}, h^{1+\sigma} \le 1$, by making a change of coordinates $\eta/h \mapsto \eta$, and using that $e^{2i\eta\cdot z} = \big(\frac{1-2i\eta\cdot\partial_z}{1+4|\eta|^2}\big)\big(\frac{1-2iz\cdot\partial_\eta}{1+4|z|^2}\big)e^{2i\eta\cdot z}$, together with some integration by parts, one can check that $\|\partial_y^\alpha\partial_\xi^\beta\big[r_R(\frac{x+y}{2},h\xi)\big]\|_{L^2(d\xi)} \lesssim R^{-1}$, for any $\alpha,\beta \in \mathbb{N}^2$, and hence obtain that

$$\|Op_h^w(r_R^k(x,\xi))w^\chi(t,\cdot)\|_{L^\infty} \lesssim R^{-1}\|w^\chi(t,\cdot)\|_{L^2},$$

after lemma 1.2.25.

Successively, taking a Littlewood-Paley decomposition such that

$$\chi_1(h^\sigma\xi) \equiv \left[\phi\Big(\frac{R}{h}\xi\Big) + \sum_{hR^{-1}\le 2^j\le h^{-\sigma}}(1-\phi)\Big(\frac{R}{h}\xi\Big)\psi(2^{-j}\xi)\right]\chi_1(h^\sigma\xi),$$

with $\phi \in C_0^\infty(\mathbb{R}^2)$, $\phi \equiv 1$ close to the origin, $\psi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, we derive that

$$\begin{aligned}(3.3.18)\quad \left\|Op_h^w(\chi_1(h^\sigma\xi))\Big[\varphi\Big(\frac{x}{R}\Big)w^\chi\Big](t,\cdot)\right\|_{L^\infty} &\lesssim \left\|Op_h^w\Big(\phi\Big(\frac{R}{h}\xi\Big)\chi_1(h^\sigma\xi)\Big)\Big[\varphi\Big(\frac{x}{R}\Big)w^\chi\Big](t,\cdot)\right\|_{L^\infty}\\ &+ \sum_{hR^{-1}\le j\le h^{-\sigma}}\left\|Op_h^w\Big((1-\phi)\Big(\frac{R}{h}\xi\Big)\psi(2^{-j}\xi)\chi_1(h^\sigma\xi)\Big)\Big[\varphi\Big(\frac{x}{R}\Big)w^\chi\Big](t,\cdot)\right\|_{L^\infty},\end{aligned}$$

and immediately notice that

$$\Big\|Op^w_h\Big(\phi\Big(\frac{R}{h}\xi\Big)\chi_1(h^\sigma\xi)\Big)\Big[\varphi\Big(\frac{x}{R}\Big)w^\chi\Big](t,\cdot)\Big\|_{L^\infty} = \Big\|\phi(RD_x)Op^w_h(\chi_1(h^\sigma\xi))\Big[\varphi\Big(\frac{x}{R}\Big)w^\chi\Big](t,\cdot)\Big\|_{L^\infty} \lesssim R^{-1}\|w^\chi(t,\cdot)\|_{L^2}, \tag{3.3.19}$$

just by classical Sobolev injection and uniform continuity of $Op^w_h(\chi_1(h^\sigma\xi))$ on L^2.
On the other hand, introducing operators Θ_R, Θ_R^{-1}, where $\Theta_R u(x) := u(Rx)$, $\Theta_R^{-1}u(x) := u(\frac{x}{R})$, we have the following equality,

$$Op^w_h\Big((1-\phi)\Big(\frac{R}{h}\xi\Big)\psi(2^{-j}\xi)\chi_1(h^\sigma\xi)\Big)\Big[\varphi\Big(\frac{x}{R}\Big)w^\chi\Big] = \Big[\Theta_R^{-1}Op^w_{h_{Rj}}\Big((1-\phi)\Big(\frac{\xi}{h_{Rj}}\Big)\psi(\xi)\chi_1(h^\sigma 2^j\xi)\Big)\varphi(x)\Theta_R\Big]w^\chi, \tag{3.3.20}$$

with $h_{Rj} := \frac{h}{R2^j} \le 1$, and by h_{Rj}-symbolic calculus (that is proposition 1.2.21 with h replaced by h_{Rj}),

$$Op^w_{h_{Rj}}\Big((1-\phi)\Big(\frac{\xi}{h_{Rj}}\Big)\psi(\xi)\chi_1(h^\sigma 2^j\xi)\Big)\varphi(x) = Op^w_{h_{Rj}}\Big((1-\phi)\Big(\frac{\xi}{h_{Rj}}\Big)\psi(\xi)\chi_1(h^\sigma 2^j\xi)\varphi(x)\Big) + Op^w_{h_{Rj}}(r(x,\xi)),$$

with

$$r(x,\xi) = \frac{h_{Rj}}{2i(\pi h_{Rj})^2}\int e^{-\frac{2i}{h_{Rj}}y\cdot\zeta}\Big[\int_0^1 \partial_\xi\Big[(1-\phi)\Big(\frac{\xi}{h_{Rj}}\Big)\psi(\xi)\chi_1(h^\sigma 2^j\xi)\Big]\big|_{(\xi+t\zeta)}dt\Big]\,(\partial\varphi)(x+y)dyd\zeta.$$

Similarly as before, one can prove that $\big\|\partial_x^\alpha\partial_\xi^\beta\big[r(\frac{x+y}{2},h\xi)\big]\big\|_{L^2(d\xi)} \lesssim 1$, observing that no negative power of h_{Rj} appears in the right hand side of this inequality, for the product of $\psi(\xi)$ with any derivative of $(1-\phi)(\frac{\xi}{h_{Rj}})$ is supported for $h_{Rj}\sim|\xi|\sim 1$, and hence that operator $Op^w_{h_{Rj}}(r(x,\xi))$ is uniformly bounded from L^2 to L^∞, thanks to lemma 1.2.25. Consequently,

$$\big\|Op^w_{h_{Rj}}(r(x,\xi))\Theta_R w^\chi(t,\cdot)\big\|_{L^\infty} \lesssim \|\Theta_R w^\chi(t,\cdot)\|_{L^2} \lesssim R^{-1}\|w^\chi(t,\cdot)\|_{L^2} \lesssim R^{-1}\|w^\chi(t,\cdot)\|_{L^2}\,.$$

Symbol $(1-\phi)(\frac{\xi}{h_{Rj}})\psi(\xi)\chi_1(h^\sigma 2^j\xi)\varphi(x)$ is supported for $|x|\sim|\xi|\sim 1$, so we can write it as

$$(1-\phi)\Big(\frac{\xi}{h_{Rj}}\Big)\psi(\xi)\chi_1(h^\sigma 2^j\xi)\varphi(x) = \sum_{l=1}^2 \underbrace{\frac{(1-\phi)(\frac{\xi}{h_{Rj}})\psi(\xi)\chi_1(h^\sigma 2^j\xi)\varphi(x)(Rx_l|2^j\xi|-2^j\xi_l)}{|Rx|2^j\xi|-2^j\xi|^2}}_{a_l(x,\xi)}\big(Rx_l|2^j\xi|-2^j\xi_l\big),$$

with $a_l(x,\xi)\in R^{-1}2^{-j}S_{0,0}(1)$ as long as $R\gg 1$, and by h_{Rj}-symbolic calculus

$$(1-\phi)\Big(\frac{\xi}{h_{Rj}}\Big)\psi(\xi)\chi_1(h^\sigma 2^j\xi)\varphi(x) = \sum_{l=1}^2 a_l(x,\xi)\sharp\Big[(Rx_l|2^j\xi|-2^j\xi_l)\widetilde{\psi}(\xi)\Big] + r_{Rj}(x,\xi),$$

with $\widetilde{\psi}\in C^\infty_0(\mathbb{R}^2\setminus\{0\})$ such that $\widetilde{\psi}\psi\equiv\psi$, and $r_{Rj}\in h_{Rj}S_{0,0}(1)$. From semi-classical Sobolev injection it follows that

$$\|Op^w_{h_{Rj}}(r_{Rj}(x,\xi))\Theta_R w^\chi(t,\cdot)\|_{L^\infty} \lesssim \|\Theta_R w^\chi(t,\cdot)\|_{L^2} \le R^{-1}\|w^\chi(t,\cdot)\|_{L^2},$$

while

(3.3.21)

$$\begin{aligned}
&Op^w_{h_{Rj}}(a_l(x,\xi))Op^w_{h_{Rj}}\big((Rx_l|2^j\xi|-2^j\xi)\widetilde{\psi}(\xi)\big)\Theta_R w^\chi\\
&= Op^w_{h_{Rj}}(a_l(x,\xi))\Theta_R\Big[Op^w_h\big((x_l|\xi|-\xi)\widetilde{\psi}(2^{-j}\xi)\big)w^\chi\Big]\\
&= Op^w_{h_{Rj}}(a_l(x,\xi))\Theta_R\Big[Op^w_h(\widetilde{\psi}(2^{-j}\xi))Op^w_h(x_l|\xi|-\xi)w^\chi-\frac{h}{2i}Op^w_h((2^{-j}\xi)\cdot(\partial\widetilde{\psi})(2^{-j}\xi))w^\chi\Big].
\end{aligned}$$

Last thing to do to conclude the proof is to study continuity of operator $Op^w_{h_{Rj}}(a_l(x,\xi))$.

Lemma 3.3.10. *Operator $Op^w_{h_{Rj}}(a_l(x,\xi)) : L^2 \to L^\infty$ is bounded, with norm*

$$\Big\|Op^w_{h_{Rj}}(a_l(x,\xi))\Big\|_{\mathcal{L}(L^2;L^\infty)} \lesssim h^{-1}.$$

Proof. The result comes straightly from lemma 1.2.25. Indeed, since symbol $a_l(x,\xi)$ is compactly supported in x, there is a smooth cut-off function $\varphi_1 \in C^\infty_0(\mathbb{R}^2\setminus\{0\})$, $\varphi_1\varphi\equiv\varphi$, such that

$$\Big|Op^w_{h_{Rj}}(a_l(x,\xi))w\Big| \lesssim \|w\|_{L^2(dx)}\int\Big|\varphi_1\Big(\frac{x+y}{2}\Big)\Big|\sum_{|\alpha|\le 3}\Big\|\partial^\alpha_y\Big[a_l\Big(\frac{x+y}{2},h_{R_j}\xi\Big)\Big]\Big\|_{L^2(d\xi)}dy\,,$$

and, for $|\alpha|\le 3$,

$$\begin{aligned}
&\Big\|\partial^\alpha_y\Big[a_l\Big(\frac{x+y}{2},h_{R_j}\xi\Big)\Big]\Big\|_{L^2(d\xi)}\\
&\quad\lesssim \frac{R}{h}\Bigg\|\partial^\alpha_y\Bigg[\frac{(1-\phi)(\xi)\psi(h_{Rj}\xi)\chi_1(h_{Rj}h^\sigma 2^j\xi)\varphi_1(\frac{x+y}{2})}{|R(\frac{x+y}{2})|\xi|-\xi|^2}\Big(R\Big(\frac{x_l+y_l}{2}\Big)|\xi|-\xi_l\Big)\Bigg]\Bigg\|_{L^2(d\xi)}\\
&\quad\lesssim \frac{|\widetilde{\varphi}(\frac{x+y}{2})|}{h}\Big(\int\frac{|\psi(h_{Rj}\xi)|^2}{|\xi|^2}d\xi\Big)^{\frac{1}{2}}\lesssim\frac{|\widetilde{\varphi}(\frac{x+y}{2})|}{h},
\end{aligned}$$

where $\widetilde{\varphi}\in C^\infty_0(\mathbb{R}^2\setminus\{0\})$. □

Finally, summing up all formulas from (3.3.20) to (3.3.21), and using lemma 3.3.10, we obtain that

$$\Big\|Op^w_h\Big((1-\phi)\Big(\frac{R}{h}\xi\Big)\psi(2^{-j}\xi)\chi_1(h^\sigma\xi)\Big)\Big[\varphi\Big(\frac{x}{R}\Big)w^\chi(t,\cdot)\Big]\Big\|_{L^\infty}\lesssim R^{-1}(\|w^\chi(t,\cdot)\|_{L^2}+\|\mathcal{M}w^\chi(t,\cdot)\|_{L^2}),$$

for any index $j\in\mathbb{Z}$ such that $hR^{-1}\le 2^j\le h^{-\sigma}$, so injecting the above inequality, together with (3.3.19), in (3.3.18), and using that $[\mathcal{M},Op^w_h(\chi(h^\sigma\xi))] = iOp^w_h((\partial\chi)(h^\sigma\xi)(h^\sigma|\xi|))$ is uniformly continuous on L^2, we deduce (3.3.17) (the loss in $\log R+|\log h|$ arising from the fact that we bounded a sum over indices j, with $\log h-\log R\lesssim j\lesssim\log(h^{-1})$). □

3.3.3 Proof of the main theorems

Proof of theorem 1.1.2. Straightforward after propositions 2.2.12, 3.2.7, 3.3.7. □

Proof of theorem 1.1.1. Let us prove that, for small enough data satisfying (1.1.4), Cauchy problem (1.1.1)-(1.1.2) has a unique global solution. This result follows by a local existence argument, after having proved that there exist two integers $n\gg\rho\gg 1$, two constants $A',B'>0$ sufficiently large, $\varepsilon_0 > 0$ sufficiently small, and $0 < \delta \ll \delta_2 \ll \delta_1 \ll \delta_0$ small, such that, for any $0 < \varepsilon < \varepsilon_0$, if (u,v) is solution to (1.1.1)-(1.1.2) in $[1,T]\times\mathbb{R}^2$, for some $T>1$, with $\partial_{t,x}u \in C^0([1,T];H^n(\mathbb{R}^2))$, $v \in C^0([1,T];H^{n+1}(\mathbb{R}^2)) \cap C^1([1,T];H^n(\mathbb{R}^2))$, and satisfies

(3.3.22a)

$$\|\partial_t u(t,\cdot)\|_{H^{\rho+1,\infty}} + \|\nabla_x u(t,\cdot)\|_{H^{\rho+1,\infty}} + \||D_x|u(t,\cdot)\|_{H^{\rho+1,\infty}} + \sum_{j=1}^{2}\|\mathrm{R}_j\partial_t u(t,\cdot)\|_{H^{\rho+1,\infty}} \le A'\varepsilon t^{-\frac{1}{2}},$$

(3.3.22b)
$$\|\partial_t v(t,\cdot)\|_{H^{\rho,\infty}} + \|v(t,\cdot)\|_{H^{\rho+1,\infty}} \le A'\varepsilon t^{-1},$$

(3.3.22c)
$$\|\partial_t u(t,\cdot)\|_{H^n} + \|\nabla_x u(t,\cdot)\|_{H^n} + \|\partial_t v(t,\cdot)\|_{H^n} + \|\nabla_x v(t,\cdot)\|_{H^n} + \|v(t,\cdot)\|_{H^n} \le B'\varepsilon t^{\frac{\delta}{2}},$$

(3.3.22d)
$$\sum_{|I|=k}\big[\|\partial_t\Gamma^I u(t,\cdot)\|_{L^2} + \|\nabla_x\Gamma^I u(t,\cdot)\|_{L^2} + \|\partial_t\Gamma^I v(t,\cdot)\|_{L^2} + \|\nabla_x\Gamma^I v(t,\cdot)\|_{L^2}$$
$$+\|\Gamma^I v(t,\cdot)\|_{L^2}\big] \le B'\varepsilon t^{\frac{\delta_{3-k}}{2}}, \quad 1\le k\le 3,$$

for every $t\in[1,T]$, then in the same interval it satisfies

(3.3.23a)

$$\|\partial_t u(t,\cdot)\|_{H^{\rho+1,\infty}} + \|\nabla_x u(t,\cdot)\|_{H^{\rho+1,\infty}} + \||D_x|u(t,\cdot)\|_{H^{\rho+1,\infty}} + \sum_{j=1}^{2}\|\mathrm{R}_j\partial_t u(t,\cdot)\|_{H^{\rho+1,\infty}} \le \frac{A'}{2}\varepsilon t^{-\frac{1}{2}},$$

(3.3.23b)
$$\|\partial_t v(t,\cdot)\|_{H^{\rho,\infty}} + \|v(t,\cdot)\|_{H^{\rho+1,\infty}} \le \frac{A'}{2}\varepsilon t^{-1},$$

(3.3.23c)
$$\|\partial_t u(t,\cdot)\|_{H^n} + \|\nabla_x u(t,\cdot)\|_{H^n} + \|\partial_t v(t,\cdot)\|_{H^n} + \|\nabla_x v(t,\cdot)\|_{H^n} + \|v(t,\cdot)\|_{H^n} \le \frac{B'}{2}\varepsilon t^{\frac{\delta}{2}},$$

(3.3.23d)

(3.3.23e)
$$\sum_{|I|=k}\big[\|\partial_t\Gamma^I u(t,\cdot)\|_{L^2} + \|\nabla_x\Gamma^I u(t,\cdot)\|_{L^2} + \|\partial_t\Gamma^I v(t,\cdot)\|_{L^2} + \|\nabla_x\Gamma^I v(t,\cdot)\|_{L^2}$$
$$+\|\Gamma^I v(t,\cdot)\|_{L^2}\big] \le \frac{B'}{2}\varepsilon t^{\frac{\delta_{3-k}}{2}}, \quad 1\le k\le 3.$$

We remind that, if $I=(i_1,\dots,i_n)$, with $i_j\in\{1,\dots,5\}$, is a multi-index of length $|I|=n$, $\Gamma^I=\Gamma_{i_1}\cdots\Gamma_{i_n}$ is a product of vector fields in family $\mathcal{Z}=\{\Omega, Z_j, \partial_j | j=1,2\}$.

We can immediately observe that the above bounds are verified at time $t=1$ after (1.1.4) and Sobolev injection. By definition (1.1.5), we also notice that

(3.3.24a)
$$\|u_\pm(t,\cdot)\|_{H^{\rho+1,\infty}} + \sum_{j=1}^{2}\|\mathrm{R}_j u_\pm(t,\cdot)\|_{H^{\rho+1,\infty}} \le 2\|\partial_t u(t,\cdot)\|_{H^{\rho+1,\infty}} + 2\||D_x|u(t,\cdot)\|_{H^{\rho+1,\infty}}$$
$$+ 2\sum_{j=1}^{2}\left(\|\partial_j u(t,\cdot)\|_{H^{\rho+1,\infty}} + \|\mathrm{R}_j\partial_t u(t,\cdot)\|_{H^{\rho+1,\infty}}\right),$$

(3.3.24b)
$$\|v_\pm(t,\cdot)\|_{H^{\rho,\infty}} \le 2\|\partial_t v(t,\cdot)\|_{H^{\rho,\infty}} + 2\|v(t,\cdot)\|_{H^{\rho+1,\infty}},$$

and, conversely,

(3.3.25a)

$$\|\partial_t u(t,\cdot)\|_{H^{\rho+1,\infty}} + \||D_x|u(t,\cdot)\|_{H^{\rho+1,\infty}} + \sum_{j=1}^{2}\left(\|\partial_j u(t,\cdot)\|_{H^{\rho+1,\infty}} + \|\mathrm{R}_j\partial_t u(t,\cdot)\|_{H^{\rho+1,\infty}}\right) \le$$
$$\|u_+(t,\cdot)\|_{H^{\rho+1,\infty}} + \|u_-(t,\cdot)\|_{H^{\rho+1,\infty}} + \sum_{j=1}^{2}\left(\|\mathrm{R}_j u_+(t,\cdot)\|_{H^{\rho+1,\infty}} + \|\mathrm{R}_j u_-(t,\cdot)\|_{H^{\rho+1,\infty}}\right),$$

(3.3.25b) $$\|\partial_t v(t,\cdot)\|_{H^{\rho,\infty}} + \|v(t,\cdot)\|_{H^{\rho+1,\infty}} \le \|v_+(t,\cdot)\|_{H^{\rho,\infty}} + \|v_-(t,\cdot)\|_{H^{\rho,\infty}}.$$

Moreover, reminding definition (1.1.9) of generalized energies $E_n(t;u_\pm,v_\pm)$, $E^k_3(t;u_\pm,v_\pm)$, for $n\ge 3$ and $0\le k\le 2$, and of set $\mathcal{I}^k_3$ in (2.1.17), there is a constant $C>0$ such that

(3.3.26a) $$C^{-1}E_n(t;u_\pm,v_\pm) \le \big[\|\partial_t u(t,\cdot)\|^2_{H^n} + \|\nabla_x u(t,\cdot)\|^2_{H^n} + \|\partial_t v(t,\cdot)\|^2_{H^n} + \|\nabla_x v(t,\cdot)\|^2_{H^n} + \|v(t,\cdot)\|^2_{H^n}\big] \le CE_n(t;u_\pm,v_\pm),$$

and for any $0\le k\le 2$,

(3.3.26b) $$C^{-1}E^k_3(t;u_\pm,v_\pm) \le \sum_{I\in\mathcal{I}^k_3}\big[\|\partial_t\Gamma^I u(t,\cdot)\|^2_{L^2} + \|\nabla_x\Gamma^I u(t,\cdot)\|^2_{L^2} + \|\partial_t\Gamma^I v(t,\cdot)\|^2_{L^2} + \|\nabla_x\Gamma^I v(t,\cdot)\|^2_{L^2} + \|\Gamma^I v(t,\cdot)\|^2_{L^2}\big] \le CE^k_3(t;u_\pm,v_\pm).$$

Therefore, after (3.3.24), (3.3.26), and (3.3.22), we deduce that estimates (1.1.11) are satisfied with $A=2A'$, $B=C_1B'$, for some new $C_1>0$, so choosing for instance $K_1=4$ and K_2 sufficiently large, theorem 1.1.2 and inequalities (3.3.25), (3.3.26) imply (3.3.23). □

Appendix A

Lemma A.1. *(i) Let $d \in \mathbb{N}^*$ and $a(\xi,\eta) : \mathbb{R}^2 \times \mathbb{R}^d \to \mathbb{C}$ such that, for any $\beta \in \mathbb{N}^d$ there exists a function $g_\beta(\eta)$ and*

$$
\begin{aligned}
|\partial_\eta^\beta a(\xi,\eta)| &\lesssim_\beta \langle\xi\rangle^{-3}|g_\beta(\eta)|,\\
|\partial_\xi^\alpha\partial_\eta^\beta a(\xi,\eta)| &\lesssim_{\alpha,\beta} (|\xi|\langle\xi\rangle^{-1})^{1-|\alpha|}\langle\xi\rangle^{-3}|g_\beta(\eta)|, \quad 1\le|\alpha|\le 4.
\end{aligned}
$$

Let also

$$
K(x,\eta) := \int e^{ix\cdot\xi}a(\xi,\eta)d\xi. \tag{A.1}
$$

Then for any $\beta \in \mathbb{N}^d$ $|\partial_\eta^\beta K(x,\eta)| \lesssim |x|^{-1}\langle x\rangle^{-2}|g_\beta(\eta)|$, for every $(x,\eta) \in \mathbb{R}^2\times\mathbb{R}^d$. The same result holds true if $|\partial_\xi^\alpha\partial_\eta^\beta a(\xi,\eta)| \lesssim_\alpha |f_\alpha(\xi)||g_\beta(\eta)|$, with $f_\alpha \in L^1(\mathbb{R}^2)$ for any $|\alpha|\le 3$;

(ii) If $a(\xi,\eta)$ only satisfies $|\partial_\xi^\alpha\partial_\eta^\beta a(\xi,\eta)| \lesssim_\alpha (\delta_{|\xi|\le 1}|\xi|^{-|\alpha|} + \delta_{|\xi|>1}\langle\xi\rangle^{-3})|g_\beta(\eta)|$ for any $\alpha\in\mathbb{N}^2$, $|\alpha|\le 3$, any $\beta\in\mathbb{N}^d$, then $|\partial_\eta^\beta K(x,\eta)| \lesssim \langle x\rangle^{-2}|g_\beta(\eta)|$.

Proof. *(i)* We consider a cut-off function $\phi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 in the unit ball, split $K(x,\eta) = K_0(x,\eta) + K_1(x,\eta)$, with

$$
K_0(x,\eta) := \int e^{ix\cdot\xi}a(\xi,\eta)\phi(\xi)d\xi, \quad K_1(x,\eta) := \int e^{ix\cdot\xi}a(\xi,\eta)(1-\phi)(\xi)d\xi,
$$

and fix $\beta\in\mathbb{N}^d$. By the hypothesis on $a(\xi,\eta)$, we have that $|\partial_\xi^\alpha\partial_\eta^\beta a(\xi,\eta)| \lesssim_{\alpha,\beta} \langle\xi\rangle^{-3}|g_\beta(\eta)|$ on the support of $(1-\phi)(\xi)$, for any $|\alpha|\le 4$, thus integrating by parts and using such inequality we deduce that $|\partial_\eta^\beta K_1(x,\eta)| \lesssim \langle x\rangle^{-4}|g_\beta(\eta)|$ for any $\beta\in\mathbb{N}^d$.

On the other hand, after an integration by parts we find that

$$
x\partial_\eta^\beta K_0(x,\eta) = \int e^{ix\cdot\xi}a_1^\beta(\xi,\eta)d\xi,
$$

where $a_1^\beta(\xi,\eta)$ is supported for $|\xi|\le 1$ and is such that, for any $|\alpha|\le 3$, $|\partial_\xi^\alpha a_1^\beta(\xi,\eta)| \lesssim |\xi|^{-|\alpha|}|g_\beta(\eta)|$ for every (ξ,η). We immediately have that $|x\partial_\eta^\beta K_0(x,\eta)| \lesssim_\beta |g_\beta(\eta)|$, for every (x,η), and furthermore $|x^\alpha x\partial_\eta^\beta K_0(x,\eta)| \lesssim_\alpha |g_\beta(\eta)|$ for any $|\alpha|\le 3$. This certainly holds when $|x|\le 1$. When $|x|>1$, we prove it taking a Littlewood-Paley decomposition

$$
\phi(\xi) = \phi(\xi)\left[\varphi_0(2^{-L_0}\xi) + \sum_{k=L_0+1}^{0}\varphi(2^{-k}\xi)\right],
$$

with $supp\varphi_0 \subset B_1(0)$, $\varphi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, and $L_0 < 0$ is such that $2^{L_0} \sim |x|^{-1}$, and splitting $x\partial_\eta^\beta K_0(x,\eta)$ into $K_0^\beta(x,\eta) + \sum_{k=L_0+1}^0 K_k^\beta(x,\eta)$, with

$$K_0^\beta(x,\eta) := \int e^{ix\cdot\xi} a_1^\beta(\xi,\eta)\varphi_0(2^{-L_0}\xi)d\xi, \quad K_k^\beta(x,\eta) := \int e^{ix\cdot\xi} a_1^\beta(\xi,\eta)\varphi_k(2^{-k}\xi)d\xi.$$

Performing a change of coordinates and some integrations by parts, we observe that $|K_0^\beta(x,\eta)| \lesssim 2^{2L_0}|g_\beta(\eta)|$, $|K_k^\beta(x,\eta)| \lesssim 2^{2k}\langle 2^k x\rangle^{-3}|g_\beta(\eta)|$, for any $L_0+1 \le k \le 0$, and from these inequalities we deduce that $|x\partial_\eta^\beta K_0(x,\eta)| \lesssim 2^{2L_0}|g_\beta(\eta)| \sim |x|^{-2}|g_\beta(\eta)|$.

Last part of statement (i) follows by the fact that, integrating by parts,

$$\left|x^\alpha \partial_\eta^\beta K(x,\eta)\right| \lesssim \int |f_\alpha(\xi)||g_\beta(\eta)|d\xi \lesssim_\alpha |g_\beta(\eta)|,$$

for any $|\alpha| \le 3, \beta \in \mathbb{N}^d$, which implies that $|\partial_\eta^\beta K(x,\eta)| \lesssim \langle x\rangle^{-3}|g_\beta(\eta)| \lesssim |x|^{-1}\langle x\rangle^{-2}|g_\beta(\eta)|$, for any $(x,\eta) \in \mathbb{R}^2 \times \mathbb{R}^d$.

(ii) The result follows splitting $K(x,\eta)$ into the sum of previously defined $K_0(x,\eta), K_1(x,\eta)$, and making for $\partial_\eta^\beta K_0(x,\eta)$ the same decomposition and analysis as we did for $x\partial_\eta^\beta K_0(x,\eta)$ in the proof of (i). □

Corollary A.2. *(i) Let $d \in \mathbb{N}^*$, $N \in \mathbb{N}$, and $a(\xi,\eta)$ as in lemma A.1 (i). If $g_\beta \in L^1(\mathbb{R}^d)$ for every $|\beta| \le N$, then*

$$\left|\int e^{ix\cdot\xi+iy\cdot\eta} a(\xi,\eta)d\xi d\eta\right| \lesssim |x|^{-1}\langle x\rangle^{-2}\langle y\rangle^{-N}, \quad \forall (x,y) \in \mathbb{R}^2 \times \mathbb{R}^d.$$

Moreover, if $d = 2$ and $N = 3$, for any $u, v \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$

$$\left\|\int e^{ix\cdot\xi} a(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{L^2(dx)} \lesssim \|u\|_{L^2}\|v\|_{L^\infty} \;(or \;\lesssim \|u\|_{L^\infty}\|v\|_{L^2}), \tag{A.2a}$$

and

$$\left\|\int e^{ix\cdot\xi} a(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{L^\infty(dx)} \lesssim \|u\|_{L^\infty}\|v\|_{L^\infty}. \tag{A.2b}$$

(ii) If $a(\xi,\eta)$ is a function as in lemma A.1 (ii), and $g_\beta \in L^1(\mathbb{R}^d)$ for every $|\beta| \le N$, then

$$\left|\int e^{ix\cdot\xi+iy\cdot\eta} a(\xi,\eta)d\xi d\eta\right| \lesssim \langle x\rangle^{-2}\langle y\rangle^{-N}, \quad \forall (x,y) \in \mathbb{R}^2 \times \mathbb{R}^d.$$

Moreover, if $d = 2, N = 3$, for any $u, v \in L^2(\mathbb{R}^2)$

$$\left\|\int e^{ix\cdot\xi} a(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{L^2(dx)} \lesssim \|u\|_{L^2}\|v\|_{L^2}, \tag{A.3a}$$

while if $u \in L^2(\mathbb{R}^2), v \in L^\infty(\mathbb{R}^2)$,

$$\left\|\int e^{ix\cdot\xi} a(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{L^\infty(dx)} \lesssim \|u\|_{L^2}\|v\|_{L^\infty}. \tag{A.3b}$$

Proof. Let $\widetilde{K}(x,y) := \int e^{iy\cdot\eta}K(x,\eta)d\eta$, with $K(x,\eta)$ introduced in (A.1). Then

$$\widetilde{K}(x,y) = \int e^{ix\cdot\xi+iy\cdot\eta}a(\xi,\eta)d\xi d\eta,$$

and, both for (i) and (ii), the first part of the statement is a straight consequence of lemma A.1 and integration by parts.

If $d=2, N=3$, and $a(\xi,\eta)$ is as in lemma A.1 (i), inequality (A.2a) from the following equality

$$\int e^{ix\cdot\xi}a(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\eta = \int \widetilde{K}(x-y,y-z)u(y)v(z)dydx,$$

and the fact that

(A.4)
$$\begin{aligned}\left\|\int \widetilde{K}(x-y,y-z)\widetilde{u}(y)\widetilde{v}(z)dydz\right\|_{L(dx)} &\lesssim \left\|\int |x-y|^{-1}\langle x-y\rangle^{-2}\langle y-z\rangle^{-3}\widetilde{u}(y)\widetilde{v}(z)dydz\right\|_{L(dx)}\\ &\lesssim \int |y|^{-1}\langle y\rangle^{-2}\langle z\rangle^{-3}\|\widetilde{u}(\cdot-y)\widetilde{v}(\cdot-y-z)\|_{L(dx)}dydz\\ &\lesssim \|\widetilde{u}\|_{L^\infty}\|\widetilde{v}\|_{L}\ (\text{or } \lesssim \|\widetilde{u}\|_{L}\|\widetilde{v}\|_{L^\infty}),\end{aligned}$$

with $L=L^2$ or $L=L^\infty$.

If $a(\xi,\eta)$ is instead as in lemma A.1 (ii), then inequalities (A.3) follows from the fact that

$$\begin{aligned}&\left\|\int \widetilde{K}(x-y,y-z)u(y)v(z)dydz\right\|_{L^2(dx)} \lesssim \left\|\int \langle x-y\rangle^{-2}\langle y-z\rangle^{-3}|u(y)||v(z)|dydz\right\|_{L^2(dx)}\\ &\lesssim \int \langle y-z\rangle^{-3}|u(y)||v(z)|dydz \lesssim \int |v(z)|\left(\int \langle y-z\rangle^{-3}dy\right)^{\frac{1}{2}}\left(\int \langle y-z\rangle^{-3}|u(y)|^2dy\right)^{\frac{1}{2}}dz\\ &\lesssim \|v\|_{L^2}\left(\int \langle y-z\rangle^{-3}|u(y)|^2dydz\right)^{\frac{1}{2}} \lesssim \|u\|_{L^2}\|v\|_{L^2},\end{aligned}$$

and

$$\begin{aligned}\left\|\int \widetilde{K}(x-y,y-z)u(y)v(z)dydz\right\|_{L^\infty(dx)} &\lesssim \left\|\int \langle x-y\rangle^{-2}\langle y-z\rangle^{-3}|u(y)||v(z)|dydz\right\|_{L^\infty(dx)}\\ &\lesssim \|v\|_{L^\infty}\left\|\int \langle x-y\rangle^{-2}|u(y)|dy\right\|_{L^\infty(dx)} \lesssim \|u\|_{L^2}\|v\|_{L^\infty}.\end{aligned}$$

□

Lemma A.3 (Sobolev norm of a product)**.** *Let $s\in\mathbb{N}^*$. For any $u,v\in H^s(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$,*

(A.5)
$$\|uv\|_{H^s} \lesssim \|u\|_{H^s}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^s};$$

for any $u,v\in H^{s,\infty}(\mathbb{R}^2)\cap H^{s+2}(\mathbb{R}^2)$, any $\theta\in]0,1]$,

(A.6)
$$\|uv\|_{H^{s,\infty}} \lesssim \|u\|^{1-\theta}_{H^{s,\infty}}\|u\|^{\theta}_{H^{s+2}}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|^{1-\theta}_{H^{s,\infty}}\|v\|^{\theta}_{H^{s+2}}.$$

Proof. Inequality (A.5) is a classical result (see, for instance, [2]).

In order to deduce (A.6), we decompose the product uv as follows:

(A.7)
$$uv = T_uv + T_vu + R(u,v),$$

where $T_u v$ is the para-product of u, v, defined by

$$T_u v := S_{-3} u S_0 v + \sum_{k\geq 1} S_{k-3} u \Delta_k v,$$

where $S_k = \chi(2^{-k}D_x)$, $\chi(\xi) = 1$ for $|\xi| \leq 1/2$, $\chi \in C_0^\infty(\mathbb{R}^2)$ such $\chi(\xi) = 0$ for $|\xi| \geq 1$, $\Delta_k = S_k - S_{k-1}$ for $k \geq 1$ (with the convention that $\Delta_0 = S_0$), and $R(u,v) = \sum_k \Delta_k u \widetilde{\Delta}_k v$, with $\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. Since

$$T_u v = \sum_{j\geq 0} \Delta_j (T_u v) = \sum_{\substack{j,k \\ |j-k|\leq N_0}} \Delta_j [S_{k-3} u \Delta_k v],$$

for a certain $N_0 \in \mathbb{N}$, by the definition of $H^{s,\infty}$ in 1.2.1 (*iii*), and the fact that $\|\Delta_k v\|_{L^\infty} \leq 2^k \|\Delta_k v\|_{L^2}$, we deduce that, for any fixed $\theta \in]0,1[$,

(A.8)

$$\begin{aligned}
&\|T_u v\|_{H^{s,\infty}} = \|\langle D_x\rangle^s T_u v\|_{H^{s,\infty}}\|_{L^\infty} \leq \sum_{\substack{j,k \\ |j=k|\leq N_0}} 2^{js} \|\Delta_j [S_{k-3} u \Delta_k v]\|_{L^\infty} \\
&\leq \sum_{\substack{j,k \\ |j=k|\leq N_0}} 2^{js} \|S_{k-3} u\|_{L^\infty} \|\Delta_k v\|_{L^\infty} \leq \sum_{\substack{j,k \\ |j=k|\leq N_0}} 2^{js} \|u\|_{L^\infty} (2^{-ks} \|\Delta_k \langle D_x\rangle^s v\|_{L^\infty})^{1-\theta} (2^k \|\Delta_k v\|_{L^2})^\theta \\
&\lesssim \sum_{\substack{j,k \\ |j=k|\leq N_0}} 2^{js-ks(1-\theta)+k\theta-k(s+2)\theta} \|u\|_{L^\infty} \|\Delta_k \langle D_x\rangle^s v\|^{1-\theta}_{L^\infty} \|\Delta_k \langle D_x\rangle^{s+2} v\|^\theta_{L^2} \\
&\lesssim \|u\|_{L^\infty} \|v\|^{1-\theta}_{H^{s,\infty}} \|v\|^\theta_{H^{s+2}}.
\end{aligned}$$

Similarly,

$$\|T_v u\|_{H^{s,\infty}} + \|R(u,v)\|_{H^{s,\infty}} \lesssim \|u\|^{1-\theta}_{H^{s,\infty}} \|u\|^\theta_{H^{s+2}} \|v\|_{L^\infty}.$$

□

Corollary A.4. *Let $s \in \mathbb{N}^*$, $a_1(\xi) \in S_0^{m_1}(\mathbb{R}^2)$, $a_2(\xi) \in S_0^{m_2}(\mathbb{R}^2)$, for some $m_1, m_2 \geq 0$. For any $u \in H^{s+m_1}(\mathbb{R}^2) \cap H^{m_1,\infty}(\mathbb{R}^2)$, $v \in H^{s+m_2}(\mathbb{R}^2) \cap H^{m_2,\infty}(\mathbb{R}^2)$,*

(A.9) $$\|[a_1(D_x)u]\,[a_2(D_x)v]\|_{H^s} \lesssim \|u\|_{H^{s+m_1}} \|v\|_{H^{m_2,\infty}} + \|u\|_{H^{m_1,\infty}} \|v\|_{H^{s+m_2}};$$

for any $u \in H^{s+m_1,\infty}(\mathbb{R}^2) \cap H^{s+m_1+2}(\mathbb{R}^2)$, $v \in H^{s+m_2,\infty}(\mathbb{R}^2) \cap H^{s+m_2+2}(\mathbb{R}^2)$, any $\theta \in]0,1[$,

(A.10) $$\begin{aligned}&\|[a_1(D_x)u]\,[a_2(D_x)v]\|_{H^{s,\infty}} \\ &\qquad\lesssim \|u\|^{1-\theta}_{H^{s+m_1,\infty}} \|u\|^\theta_{H^{s+m_1+2}} \|v\|_{H^{m_2,\infty}} + \|u\|_{H^{m_1,\infty}} \|v\|^{1-\theta}_{H^{s+m_2,\infty}} \|v\|^\theta_{H^{s+m_2+2}}.\end{aligned}$$

Proof. The result of the statement follows writing $[a_1(D_x)u]\,[a_2(D_x)v]$ in terms of para-products, as in (A.7), using that $T_{a_1(D)u}(a_2(D)v)$, $T_{a_2(D)v}(a_1(D)u)$ and remainder $R\big(a_1(D)u, a_2(D)v\big)$ can be written from $\widetilde{u} = \langle D_x\rangle^{m_1} u$, $\widetilde{v} = \langle D_x\rangle^{m_2} v$, as done below for the former of these terms,

$$\begin{aligned}T_{a_1(D)u}(a_2(D)v) &= [S_{-3} a_1(D)\langle D_x\rangle^{-m_1}\widetilde{u}][S_0 a_2(D)\langle D_x\rangle^{-m_2}\widetilde{v}] \\ &\quad+ \sum_k [S_{k-3} a_1(D)\langle D_x\rangle^{-m_1}\widetilde{u}][\Delta_k a_2(D)\langle D_x\rangle^{-m_2}\widetilde{v}],\end{aligned}$$

and observing that, since $a_1(\xi)\langle\xi\rangle^{-m_1} \in S_0^0(\mathbb{R}^2)$ (resp. $a_2(\xi)\langle\xi\rangle^{-m_2} \in S_0^0(\mathbb{R}^2)$), operators $S_k a_j(D)\langle D_x\rangle^{-m_j}$, $\Delta_k a_j(D)\langle D_x\rangle^{-m_j}$, for $j = 1, 2$, have the same spectrum (i.e. the support of the Fourier transform), respectively, of S_k, Δ_k (up to a negligible constant). □

In the following lemma we prove a result of continuity for a trilinear integral operator defined by multiplier $B^k_{(j_1,j_2,j_3)}(\xi,\eta)$ given by (2.2.45) (resp. by (2.2.47)) for any $k=1,2$ (resp. $k=3$), any $j_1,j_2,j_3\in\{+,-\}$. It is useful to observe that, as from (2.2.45),

$$B^k_{(j_1,j_2,j_3)}(\xi,\eta)=\frac{j_1\langle\xi-\eta\rangle+j_2|\eta|-j_3\langle\xi\rangle}{2j_1j_2\langle\xi-\eta\rangle|\eta|}\eta_k,\quad k=1,2,$$

while from (2.2.47),

$$B^3_{(j_1,j_2,j_3)}(\xi,\eta)=j_1\,\frac{j_1\langle\xi-\eta\rangle+j_2|\eta|-j_3\langle\xi\rangle}{2\langle\xi-\eta\rangle},$$

we have that, for $k=1,2$,

$$\text{(A.11)}\quad \frac{1}{(2\pi)^2}\int e^{ix\cdot\xi}B^k_{(j_1,j_2,j_3)}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta=\frac{j_2}{2}(u\mathrm{R}_kv)(x)-\frac{j_1}{2}\Big[\Big(\frac{D_1}{\langle D_x\rangle}u\Big)v\Big](x)$$
$$+\frac{j_1}{2}D_1\left[(\langle D_x\rangle^{-1}u)v\right](x)-\frac{j_3}{2j_1j_2}\langle D_x\rangle[(\langle D_x\rangle^{-1}u)(\mathrm{R}_kv)](x),$$

and for $k=3$,

$$\text{(A.12)}\quad \frac{1}{(2\pi)^2}\int e^{ix\cdot\xi}B^3_{(j_1,j_2,j_3)}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta=\frac{1}{2}(uv)(x)+\frac{j_1j_2}{2}[(\langle D_x\rangle^{-1}u)|D_x|v](x)$$
$$-\frac{j_1j_3}{2}\langle D_x\rangle\left[(\langle D_x\rangle^{-1}u)v\right](x).$$

Lemma A.5. *Let $s,\rho\in\mathbb{N}$ and $B^k_{(j_1,j_2,j_3)}(\xi,\eta)$ be defined in* (2.2.45) *(resp. in* (2.2.47)*), for $k=1,2$ (resp. $k=3$), any $j_1,j_2,j_3\in\{+,-\}$. Let also $\delta_k=1$ if $k\in\{1,2\}$, $\delta_k=0$ if $k=3$. For any $u,w\in L^2(\mathbb{R}^2), v\in H^{2,\infty}(\mathbb{R}^2)$ such that $\delta_k\mathrm{R}_kv\in H^{2,\infty}(\mathbb{R}^2)$,*

$$\text{(A.13)}\quad \left|\int B^k_{(j_1,j_2,j_3)}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)\hat{w}(-\xi)d\xi d\eta\right|\lesssim\|u\|_{L^2}\left(\|v\|_{H^{7,\infty}}+\delta_k\|\mathrm{R}_kv\|_{H^{7,\infty}}\right)\|w\|_{L^2}.$$

Proof. First of all, we observe that for $k=1,2$,

(A.14a)
$$\int B^k_{(j_1,j_2,j_3)}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)\hat{w}(-\xi)d\xi d\eta=\frac{j_2}{2}\int\widehat{u(\mathrm{R}_kv)}(\xi)\hat{w}(-\xi)d\xi-\frac{j_1}{2}\int\widehat{\big[\big(\tfrac{D_x}{\langle D_x\rangle}u\big)v\big]}(\xi)\hat{w}(-\xi)d\xi$$
$$+\frac{j_1}{2}\int\frac{\xi_1}{\langle\xi-\eta\rangle}\hat{u}(\xi-\eta)\hat{v}(\eta)\hat{w}(-\xi)d\xi d\eta-\frac{j_3}{2j_1j_2}\int\frac{\langle\xi\rangle}{\langle\xi-\eta\rangle}\hat{u}(\xi-\eta)\widehat{\mathrm{R}_kv}(\eta)\hat{w}(-\xi)d\xi d\eta,$$

while for $k=3$,

$$\text{(A.14b)}\quad \int B^3_{(j_1,j_2,j_3)}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)\hat{w}(-\xi)d\xi d\eta=\frac{1}{2}\int\widehat{uv}(\xi)\hat{w}(-\xi)d\xi$$
$$+\frac{j_1j_2}{2}\int\widehat{[(\langle D_x\rangle^{-1}u)|D_x|v]}(\xi)\hat{w}(-\xi)d\xi-\frac{j_1j_3}{2}\int\frac{\langle\xi\rangle}{\langle\xi-\eta\rangle}\hat{u}(\xi-\eta)\hat{v}(\eta)\hat{w}(-\xi)d\xi$$

First two terms in the above right hand sides satisfy inequality (A.13) just by Hölder inequality, while for last two contributions in (A.14a), and the latter one in (A.14b), it follows by proving that the mentioned inequality is satisfied by

$$\int a(\xi,\eta)\hat{u}_1(\xi-\eta)\hat{u}_2(\eta)\hat{u}_3(-\xi)d\xi d\eta,$$

with $a(\xi,\eta)=\xi_1\langle\xi-\eta\rangle^{-1}$, or $a(\xi,\eta)=\langle\xi\rangle\langle\xi-\eta\rangle^{-1}$, and some general functions $u_1,u_3\in L^2(\mathbb{R}^2), u_2\in L^\infty(\mathbb{R}^2)$. Taking a Littlewood-Paley decomposition, we split the above integral as follows

$$\text{(A.15)}\quad \sum_{k,l\geq 0}\int a(\xi,\eta)\varphi_k(\xi)\varphi_l(\eta)\hat{u}_1(\xi-\eta)\hat{u}_2(\eta)\hat{u}_3(-\xi)d\xi d\eta,$$

with $\varphi_0 \in C_0^\infty(\mathbb{R}^2)$, $\varphi_k(\zeta)=\varphi(2^{-k}\zeta)$ and $\varphi\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$, for any $k\in\mathbb{N}^*$, and immediately observe that, since frequencies ξ,η are bounded on the support of $\varphi_0(\xi)\varphi_0(\eta)$, kernel

$$K_0(x,y):=\int e^{ix\cdot\xi+iy\cdot\eta}a(\xi,\eta)\varphi_0(\xi)\varphi_0(\eta)d\xi d\eta$$

is such that $|K_0(x,y)|\lesssim\langle x\rangle^{-3}\langle y\rangle^{-3}$ for any (x,y), after lemma A.1 (i) and corollary A.2 (i). Therefore

$$\begin{aligned}&\left|\int a(\xi,\eta)\varphi_0(\xi)\varphi_0(\eta)\hat{u}_1(\xi-\eta)\hat{u}_2(\eta)\hat{u}_3(-\xi)d\xi d\eta\right|\\ =&\left|\int K_0(z-x,x-y)u_1(x)u_2(y)u_3(z)dxdydz\right|\\ \lesssim&\int\langle z-x\rangle^{-3}\langle x-y\rangle^{-3}|u_1(x)||u_2(y)||u_3(z)|dxdydz\\ \lesssim&\|u_2\|_{L^\infty}\int\langle x\rangle^{-3}|u_1(z-x)||u_3(z)|dxdz\lesssim\|u_1\|_{L^2}\|u_2\|_{L^\infty}\|u_3\|_{L^2},\end{aligned}$$

last inequality obtained by Hölder inequality.

For indices $l>k+N_0\geq 0$, for a suitably large integer $N_0>1$ (resp. $|l-k|\leq N_0$), we have that $|\xi|<|\eta|\sim|\xi-\eta|$ (resp. $|\xi|\sim|\eta|$) on the support of $\varphi_k(\xi)\varphi_l(\eta)$, so if we define $a_{l>k+N_0}(\xi,\eta):=a(\xi,\eta)\langle\eta\rangle^{-1}$ (resp. $a_{|l-k|\leq N_0}(\xi,\eta):=a(\xi,\eta)\langle\eta\rangle^{-7}$), it is a computation to check that $|\partial_\xi^\alpha\partial_\eta^\beta[a_{l>k+N_0}(2^k\xi,2^l\eta)]|\lesssim 2^{-l}$ (same for $a_{|l-k|\leq N_0}(\xi,\eta)$) for any $\alpha,\beta\in\mathbb{N}^2$ such that $|\alpha|,|\beta|\leq 3$. Hence, its associated kernel $K_{l>k+N_0}(x,y)$ (resp. $K_{|l-k|\leq N_0}(x,y)$) is such that $|K_{l>k+N_0}(x,y)|\lesssim 2^{2k}2^l\langle 2^kx\rangle^{-3}\langle 2^ly\rangle^{-3}$ for any (x,y) (same for $K_{|l-k|\leq N_0}(x,y)$), as follows after a change of coordinates and some integrations by parts, and then, for any $l>k+N_0$ (resp. $|l-k|\geq N_0$)

$$\begin{aligned}&\left|\int a(\xi,\eta)\varphi_k(\xi)\varphi_l(\eta)\hat{u}_1(\xi-\eta)\hat{u}_2(\eta)\hat{u}_3(-\xi)d\xi d\eta\right|\\ =&\left|\int K_{l>k+N_0}(z-x,x-y)u_1(x)[\langle D_x\rangle u_2](y)u_3(z)dxdydz\right|\\ \lesssim&2^{2k}2^l\left|\int\langle 2^k(z-x)\rangle^{-3}\langle 2^l(x-y)\rangle^{-3}|u_1(x)||\langle D_x\rangle u_2(y)||u_3(z)|dxdydz\right|\\ \lesssim&2^{-\frac{k}{2}}2^{-\frac{l}{2}}\|u_1\|_{L^2}\|u_2\|_{H^{1,\infty}}\|u_3\|_{L^2},\end{aligned}$$

(resp.

$$\begin{aligned}&\left|\int a(\xi,\eta)\varphi_l(\xi)\varphi_l(\eta)\hat{u}_1(\xi-\eta)\hat{u}_2(\eta)\hat{u}_3(-\xi)d\xi d\eta\right|\\ =&\left|\int K_{|l-k|\leq N_0}(z-x,x-y)u_1(x)[\langle D_x\rangle^7u_2](y)u_3(z)dxdydz\right|\\ \lesssim&2^{3l}\left|\int\langle 2^l(z-x)\rangle^{-3}\langle 2^l(x-y)\rangle^{-3}|u_1(x)||\langle D_x\rangle^7u_2(y)||u_3(z)|dxdydz\right|\\ \lesssim&2^{-l}\|u_1\|_{L^2}\|u_2\|_{H^{7,\infty}}\|u_3\|_{L^2}\Big).\end{aligned}$$

Finally, for positive indices k such that $k>l-N_0$, we observe that frequencies ξ and $\xi-\eta$ are equivalent, of size 2^k, so if we take $a_{k>l-N_0}(\xi,\eta)=a_{l>k+N_0}(\xi,\eta)$ (and associated kernel $K_{k>l-N_0}(x,y)=K_{l>k+N_0}(x,y)$), and introduce two new smooth cut-off function $\varphi^1,\varphi^2\in C_0^\infty(\mathbb{R}^2)$, equal to 1 on the support of φ, together with operators $\Delta_k^1:=\varphi^1(2^{-k}D_x),\Delta_k^2:=\varphi^2(2^{-k}D_x)$, we deduce that

$$\begin{aligned}&\left|\int a(\xi,\eta)\varphi_k(\xi)\varphi_l(\eta)\hat{u}_1(\xi-\eta)\hat{u}_2(\eta)\hat{u}_3(-\xi)d\xi d\eta\right|\\ =&\left|\int K_{k>l-N_0}(z-x,x-y)[\Delta_k^1u_1](x)[\langle D_x\rangle u_2](y)[\Delta_k^2u_3](z)dxdydz\right|\\ \lesssim&2^{2k}2^l\left|\int\langle 2^k(z-x)\rangle^{-3}\langle 2^l(x-y)\rangle^{-3}|[\Delta_k^1u_1](x)||\langle D_x\rangle u_2(y)||[\Delta_k^2u_3](z)|dxdydz\right|\\ \lesssim&2^{-l}\|\Delta_k^1u_1\|_{L^2}\|u_2\|_{H^{1,\infty}}\|\Delta_k^2u_3\|_{L^2}.\end{aligned}$$

Combining decomposition (A.15), above inequality and Cauchy-Schwarz inequality, we then obtain that

$$\left|\int a(\xi,\eta)\hat{u}_1(\xi-\eta)\hat{u}_2(\eta)\hat{u}_3(-\xi)d\xi d\eta\right| \lesssim \|u_1\|_{L^2}\|u_2\|_{H^{7,\infty}}\|u_3\|_{L^2}, \tag{A.16}$$

and the conclusion of the proof. □

Lemma A.6. *For any $j_1, j_2, j_3 \in \{+,-\}$, any $N \in \mathbb{N}^*$, let $\sigma^N_{(j_1,j_2,j_3)}(\xi,\eta)$ be a function supported for $|\xi| \le \varepsilon\langle\eta\rangle$, for a small $\varepsilon > 0$, and such that, for any $\alpha,\beta \in \mathbb{N}^2$, $|\partial^\alpha_\xi\partial^\beta_\eta\sigma^N_{(j_1,j_2,j_3)}(\xi,\eta)| \lesssim |\xi|^{N+1-|\alpha|}\langle\eta\rangle^{-N-|\beta|}$. If $\widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta) := \frac{\sigma^N_{(j_1,j_2,j_3)}(\eta,\xi-\eta)}{j_1\langle\xi-\eta\rangle+j_2|\eta|-j_3\langle\xi\rangle}$, then for any $\alpha,\beta \in \mathbb{N}^2$,*

$$\left|\partial^\alpha_\xi\partial^\beta_\eta\widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\right| \lesssim_{\alpha,\beta} \langle\xi-\eta\rangle^{2-N+|\alpha|+2|\beta|}|\eta|^{N-|\beta|}, \tag{A.17}$$

and moreover, if $N \ge 15$, for any $u, w \in L^2(\mathbb{R}^2)$, $v \in H^{N+3,\infty}(\mathbb{R}^2)$

$$\left|\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)\hat{w}(-\xi)d\xi d\eta\right| \lesssim \|u\|_{L^2}\|v\|_{H^{N+3,\infty}}\|w\|_{L^2}. \tag{A.18}$$

Proof. Let us write $\widetilde{\sigma}^N_{(j_1,j_2,j_3)}$ under the following form:

$$\widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta) = \frac{j_1\langle\xi-\eta\rangle + j_2|\eta| + j_3\langle\xi\rangle}{2j_1j_2\langle\xi-\eta\rangle|\eta| - 2(\xi-\eta)\cdot\eta}\sigma^N_{(j_1,j_2,j_3)}(\eta,\xi-\eta).$$

First of all we observe that denominator $j_1j_2\langle\xi-\eta\rangle|\eta| - (\xi-\eta)\cdot\eta$ is bounded in absolute value from below by $c|\eta|$ if $|\xi-\eta|$ is bounded, and by $|\eta|\langle\xi-\eta\rangle^{-1}$ if $|\xi-\eta| \to +\infty$, which hence implies that $[j_1j_2\langle\xi-\eta\rangle|\eta| - (\xi-\eta)\cdot\eta]^{-1} \lesssim \langle\xi-\eta\rangle|\eta|^{-1}$ for any $(\xi,\eta) \in \mathbb{R}^2\times\mathbb{R}^2$. Moreover, an explicit calculation shows that for any multi-indices $\alpha,\beta \in \mathbb{N}^2$ of positive length, and up to negligible multiplicative constants,

$$\begin{aligned}&\left|\partial^\alpha_\xi\left[(j_1j_2\langle\xi-\eta\rangle|\eta| - (\xi-\eta)\cdot\eta)^{-1}\right]\right| \\ &\qquad\lesssim \sum_{1\le|\alpha_1|\le|\alpha|} |j_1j_2\langle\xi-\eta\rangle|\eta| - (\xi-\eta)\cdot\eta|^{-1-|\alpha_1|}\,|\eta|^{|\alpha_1|}\langle\xi-\eta\rangle^{-(|\alpha|-|\alpha_1|)},\end{aligned}$$

$$\begin{aligned}&\left|\partial^\beta_\eta\left[(j_1j_2\langle\xi-\eta\rangle|\eta| - (\xi-\eta)\cdot\eta)^{-1}\right]\right| \\ &\qquad\lesssim \sum_{0\le|\beta_1|<|\beta|} |j_1j_2\langle\xi-\eta\rangle|\eta| - (\xi-\eta)\cdot\eta|^{-1-(|\beta|-|\beta_1|)} \sum_{\substack{i+j=|\beta|-2|\beta_1|\\ i,j\le|\beta|-|\beta_1|}} \langle\xi-\eta\rangle^i|\eta|^j.\end{aligned}$$

Combining the above information, we can deduce that, on the support of $\sigma^N_{(j_1,j_2,j_3)}(\eta,\xi-\eta)$ (i.e. for $|\eta| \le \varepsilon|\xi-\eta|$) and for any $\alpha,\beta \in \mathbb{N}^2$,

$$\left|\partial^\alpha_\xi\partial^\beta_\eta\left[(j_1j_2\langle\xi-\eta\rangle|\eta| - (\xi-\eta)\cdot\eta)^{-1}\right]\right| \lesssim_{\alpha,\beta} \langle\xi-\eta\rangle^{1+|\alpha|+2|\beta|}|\eta|^{-1-|\beta|},$$

and therefore that

$$\left|\partial^\alpha_\xi\partial^\beta_\eta\left[\frac{j_1\langle\xi-\eta\rangle + j_2|\eta| + j_3\langle\xi\rangle}{2j_1j_2\langle\xi-\eta\rangle|\eta| - 2(\xi-\eta)\cdot\eta}\right]\right| \lesssim_{\alpha,\beta} \langle\xi-\eta\rangle^{2+|\alpha|+2|\beta|}|\eta|^{-1-|\beta|} + \langle\xi\rangle\langle\xi-\eta\rangle^{1+|\alpha|+2|\beta|}|\eta|^{-1-|\beta|},$$

which summed up with the fact $|\partial^\alpha_\xi\partial^\beta_\eta[\sigma^N_{(j_1,j_2,j_3)}(\eta,\xi-\eta)]| \lesssim_{\alpha,\beta} \langle\xi-\eta\rangle^{-N-|\alpha|}|\eta|^{N+1-|\beta|}$, gives the first part of the statement.

Let us now suppose that $N \geq 15$, and take $\chi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 in a neighbourhood of the origin. We have that

$$\begin{aligned}\int \widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)\hat{w}(-\xi)d\xi d\eta &= \int K^N_0(z-x,x-y)u(x)v(y)w(z)dxdydz\\ &+\int K^N_1(z-x,x-y)u(x)[\langle D_x\rangle^{N+3}v](y)w(z)dxdydz,\end{aligned}$$

with

$$\begin{aligned}K^N_0(x,y) &:= \int e^{ix\cdot\xi+iy\cdot\eta}\widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\chi(\eta)d\xi d\eta\\ K^N_1(x,y) &:= \int e^{ix\cdot\xi+iy\cdot\eta}\widetilde{\sigma}^N_{(j_1,j_2,j_3)}(\xi,\eta)\langle\eta\rangle^{-N-3}(1-\chi)(\eta)d\xi d\eta,\end{aligned}$$

where above multipliers, that we denote by $\widetilde{\sigma}^{N,k}_{(j_1,j_2,j_3)}(\xi,\eta)$ with $k=0,1$, are such that, for any $\alpha,\beta\in\mathbb{N}^2$ of length less or equal than 3, $|\partial^\alpha_\xi\partial^\beta_\eta\widetilde{\sigma}^{N,k}_{(j_1,j_2,j_3)}(\xi,\eta)| \lesssim_{\alpha,\beta} \langle\xi\rangle^{-3}\langle\eta\rangle^{-3}$, as follows from (A.17) and the fact that they are supported for $|\eta|\lesssim\varepsilon|\xi-\eta|$. We deduce by integration by parts that $|K^N_k(x,y)|\lesssim\langle x\rangle^{-3}\langle y\rangle^{-3}$ for any $(x,y)\in\mathbb{R}^2\times\mathbb{R}^2$, $k=0,1$, and then obtain the last part of the statement using that, for any $\widetilde{u},\widetilde{w}\in L^2, \widetilde{v}\in L^\infty$,

$$\begin{aligned}\int \langle z-x\rangle^{-3}\langle x-y\rangle^{-3}|\widetilde{u}(x)||\widetilde{v}(y)||\widetilde{w}(z)|dxdydz &\lesssim \|v\|_{L^\infty}\int\langle z\rangle^{-3}|\widetilde{u}(x)||\widetilde{w}(z-x)|dxdz\\ &\lesssim \|u\|_{L^2}\|v\|_{L^\infty}\|w\|_{L^2}.\end{aligned}$$

□

In the following lemma we derive some results on the Sobolev continuity of the bilinear integral operator

$$(u,v)\mapsto \int e^{ix\cdot\xi}D_{(j_1,j_2)}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta,$$

with $D_{(j_1,j_2)}$ defined in (3.1.14), and we warn the reader that we do not get advantage of factor $(1-\frac{\xi-\eta}{\langle\xi-\eta\rangle}\cdot\frac{\eta}{\langle\eta\rangle})$ in $D_{(j_1,j_2)}(\xi,\eta)$ when deriving the estimates mentioned below. Our choice is motivated by the fact that that continuity does not depend on the null structure of the nonlinearity $Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm)$.

Lemma A.7. *Let $\rho\in\mathbb{N}$ and $D(\xi,\eta)$ a function satisfying, for any multi-indices $\alpha,\beta\in\mathbb{N}^2$, the following:*

(i) if $|\xi|\lesssim 1$,

$$\begin{aligned}|\partial^\beta_\eta D(\xi,\eta)| &\lesssim_\beta \langle\eta\rangle^{\rho+|\beta|},\\ |\partial^\alpha_\xi\partial^\beta_\eta D(\xi,\eta)| &\lesssim_{\alpha,\beta} \langle\eta\rangle^{\rho+|\alpha|+|\beta|}+\sum_{|\alpha_1|+|\alpha_2|=|\alpha|}|\xi|^{-|\alpha_1|+1}\langle\eta\rangle^{\rho+|\alpha_2|+|\beta|},\quad |\alpha|\geq 1;\end{aligned}$$

(ii) for $|\xi|\gtrsim 1$, $|\eta|\lesssim\langle\xi-\eta\rangle$,

$$|\partial^\alpha_\xi\partial^\beta_\eta D(\xi,\eta)|\lesssim_{\alpha,\beta}\langle\xi-\eta\rangle^{\rho+|\alpha|+|\beta|};$$

(iii) for $|\xi|\gtrsim 1$, $|\eta|\gtrsim\langle\xi-\eta\rangle$:

$$|\partial^\alpha_\xi\partial^\beta_\eta D(\xi,\eta)|\lesssim_{\alpha,\beta}\langle\eta\rangle^{\rho+|\alpha|+|\beta|}.$$

Then for any $s \geq 0$, any $u, v \in H^{s+\rho+13}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ (resp. $u, v \in H^{s+\rho+13,\infty}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$)

$$\text{(A.19a)} \quad \left\| \int e^{ix\cdot\xi} D(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta) d\xi d\eta \right\|_{H^s(dx)} \lesssim \|u\|_{H^{s+\rho+13}} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{H^{s+\rho+13}}$$
$$\left(or \ \lesssim \|u\|_{H^{s+\rho+13,\infty}} \|v\|_{L^2} + \|u\|_{L^2} \|v\|_{H^{s+\rho+13,\infty}}\right),$$

and for any $u, v \in H^{s+\rho+13,\infty}(\mathbb{R}^2)$

(A.19b)
$$\left\| \int e^{ix\cdot\xi} D(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta) d\xi d\eta \right\|_{H^{s,\infty}(dx)} \lesssim \|u\|_{H^{s+\rho+13,\infty}} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{H^{s+\rho+13,\infty}}.$$

Furthermore, if $\phi \in C_0^\infty(\mathbb{R}^2)$, $t \geq 1$, $\sigma > 0$ small, there exists $\delta > 0$ depending linearly on σ, such that

$$\left\| \phi(t^{-\sigma}D_x) \int e^{ix\cdot\xi} D(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta) d\xi d\eta \right\|_{H^s(dx)} \lesssim t^\delta \|u\|_{H^{\rho+13}} \|v\|_{L^\infty}$$
$$\text{(A.20a)} \qquad \left(or \ \lesssim t^\delta \|u\|_{H^{\rho+13,\infty}} \|v\|_{L^2}\right)$$
$$\left(or \ \lesssim t^\delta \|u\|_{L^\infty} \|v\|_{H^{\rho+13}}\right),$$
$$\left(or \ \lesssim t^\delta \|u\|_{L^2} \|v\|_{H^{\rho+13,\infty}}\right),$$

$$\text{(A.20b)} \quad \left\| \phi(t^{-\sigma}D_x) \int e^{ix\cdot\xi} D(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta) d\xi d\eta \right\|_{H^{s,\infty}} \lesssim t^\delta \|u\|_{H^{\rho+13,\infty}} \|v\|_{L^\infty}$$
$$\left(or \ \lesssim t^\delta \|u\|_{L^\infty} \|v\|_{H^{\rho+13,\infty}}\right).$$

Finally, if $D(\xi,\eta) = \widetilde{D}(\xi,\eta)$ satisfies, for any $\alpha, \beta \in \mathbb{N}^2$, (ii), (iii) when $|\xi| \gtrsim 1$, together with:
$(\widetilde{i})$ if $|\xi| \lesssim 1$

$$|\partial_\xi^\alpha \partial_\eta^\beta \widetilde{D}(\xi,\eta)| \lesssim_{\alpha,\beta} \langle\eta\rangle^{\rho+|\alpha|+|\beta|} + \sum_{|\alpha_1|+|\alpha_2|=|\alpha|} |\xi|^{-|\alpha_1|+1} \langle\eta\rangle^{\rho+|\alpha_2|+|\beta|},$$

then, for any $u, v \in H^{s+\rho+13}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$,

(A.21a)
$$\left\| \int e^{ix\cdot\xi} \widetilde{D}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta) d\xi d\eta \right\|_{H^s(dx)} \lesssim \|u\|_{H^{\rho+10}} \|v\|_{L^2} + \|u\|_{H^{s+\rho+13}} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{H^{s+\rho+13}}$$
$$\left(or \ \lesssim \|u\|_{L^2} \|v\|_{H^{\rho+10}} + \|u\|_{H^{s+\rho+13,\infty}} \|v\|_{L^2} + \|u\|_{L^2} \|v\|_{H^{s+\rho+13,\infty}}\right),$$

and for any $u, v \in H^{s+\rho+13,\infty}(\mathbb{R}^2)$, with $u \in H^{\rho+10}(\mathbb{R}^2)$ (or $u \in L^2(\mathbb{R}^2)$),

$$\text{(A.21b)} \quad \left\| \int e^{ix\cdot\xi} \widetilde{D}(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta) d\xi d\eta \right\|_{H^{s,\infty}(dx)} \lesssim$$
$$\|u\|_{H^{\rho+10}} \|v\|_{L^\infty} + \|u\|_{H^{s+\rho+13,\infty}} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{H^{s+\rho+13,\infty}}$$
$$\left(or \ \lesssim \|u\|_{L^2} \|v\|_{H^{\rho+10,\infty}} + \|u\|_{H^{s+\rho+13,\infty}} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{H^{s+\rho+13,\infty}}\right).$$

Proof. After definition 1.2.1 (i) of space H^s (resp. (iii) of $H^{s,\infty}$), we should prove that the L^2 norm (resp. the L^∞) norm of

$$\text{(A.22)} \qquad \int e^{ix\cdot\xi} D^s(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta) d\xi d\eta,$$

with $D^s(\xi,\eta) := D(\xi,\eta)\langle\xi\rangle^s$, satisfies inequalities (A.19a) and (A.20a) (resp. (A.19b) and (A.20b)). Let us first take $\chi \in C_0^\infty(\mathbb{R}^2)$ equal to 1 in a neighbourhood of the origin and split the above integral, distinguishing between bounded and unbounded frequencies ξ, as

$$\int e^{ix\cdot\xi}D^s(\xi,\eta)\chi(\xi)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta + \int e^{ix\cdot\xi}D^s(\xi,\eta)(1-\chi)(\xi)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta. \tag{A.23}$$

On the support of $\chi(\xi)$ frequencies $\xi-\eta,\eta$ are either bounded or equivalent, thus if $a^s_0(\xi,\eta) := D^s(\xi,\eta)\chi(\xi)\langle\xi-\eta\rangle^{-\rho-10}$ (or also $a^s_0(\xi,\eta) := D^s(\xi,\eta)\chi(\xi)\langle\eta\rangle^{-\rho-10}$), this multiplier satisfies the hypothesis of lemma A.1 (i) with $g_\beta(\eta) = \langle\eta\rangle^{-3}$ for any $|\beta|\le 3$, after hypothesis (i) on $D(\xi,\eta)$, and corollary A.2 (i) implies that, for $L = L^2$ or L^∞,

(A.24a)

$$\begin{aligned}\left\|\int e^{ix\cdot\xi}D^s(\xi,\eta)\chi(\xi)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{L(dx)} &= \left\|\int e^{ix\cdot\xi}a^s_0(\xi,\eta)\widehat{\langle D_x\rangle^{\rho+10}u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{L(dx)}\\ &\lesssim \|\langle D_x\rangle^{\rho+10}u\|_L\|v\|_{L^\infty}(\text{or } \|\langle D_x\rangle^{\rho+10}u\|_{L^\infty}\|v\|_L),\end{aligned}$$

or

(A.24b)

$$\begin{aligned}\left\|\int e^{ix\cdot\xi}D^s(\xi,\eta)\chi(\xi)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{L(dx)} &= \left\|\int e^{ix\cdot\xi}a^s_0(\xi,\eta)\hat{u}(\xi-\eta)\widehat{\langle D_x\rangle^{\rho+10}v}(\eta)d\xi d\eta\right\|_{L(dx)}\\ &\lesssim \|u\|_{L^\infty}\|\langle D_x\rangle^{\rho+10}v\|_L(\text{or } \|u\|_L\|\langle D_x\rangle^{\rho+10}v\|_{L^\infty}).\end{aligned}$$

Successively, we consider a Littlewood-Paley decomposition in order to write

$$\begin{aligned}&\int e^{ix\cdot\xi}D^s(\xi,\eta)(1-\chi)(\xi)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\\ &\qquad= \sum_{k\ge1,l\ge0}\int e^{ix\cdot\xi}D^s(\xi,\eta)(1-\chi)(\xi)\varphi_k(\xi)\varphi_l(\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta,\end{aligned} \tag{A.25}$$

where $\varphi_0\in C_0^\infty(\mathbb{R}^2)$, $\varphi_k(\zeta) = \varphi(2^{-k}\zeta)$ with $\varphi\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$, for any $k\in\mathbb{N}^*$.

On the support of $\varphi_k(\xi)\varphi_l(\eta)$, with $k > l+N_0$ and $N_0\in\mathbb{N}^*$ sufficiently large, we have that $|\eta| < |\xi-\eta|$ and $|\xi-\eta|\sim|\xi|\sim 2^k$. If $a^s_{k>l+N_0}(\xi,\eta) := D^s(\xi,\eta)\varphi_k(\xi)\varphi_l(\eta)\langle\xi-\eta\rangle^{-s-\rho-13}$, by hypothesis (ii) we deduce that, for any $\alpha,\beta\in\mathbb{N}^2$ of length less or equal than 3, $|\partial^\alpha_\xi\partial^\beta_\eta[a^s_{k>l+N_0}(2^k\xi,2^l\eta)]|\lesssim 2^{-k}$, and kernel $K^s_{k>l+N_0}(x,y)$ defined as follows

$$K^s_{k>l+N_0}(x,y) := \int e^{ix\cdot\xi+iy\cdot\eta}a^s_{k>l+N_0}(\xi,\eta)d\xi d\eta = 2^{2k}2^{2l}\int e^{i2^kx\cdot\xi+i2^ly\cdot\eta}a^s_{k>l+N_0}(2^k\xi,2^l\eta)d\xi d\eta,$$

verifies that $|K^sk>l+N_0(x,y)|\lesssim 2^k2^{2l}\langle 2^kx\rangle^{-3}\langle 2^ly\rangle^{-3}$ for any $(x,y)\in\mathbb{R}^2\times\mathbb{R}^2$, as one can check doing some integration by parts. Therefore, for $L = L^2$ or L^∞,

$$\begin{aligned}&\left\|\int e^{ix\cdot\xi}D^s(\xi,\eta)\varphi_k(\xi)\varphi_l(\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{L(dx)}\\ &=\left\|\int K^s_{k>l+N_0}(x-y,y-z)[\langle D_x\rangle^{s+\rho+13}u](y)v(z)dydz\right\|_{L(dx)}\\ &\lesssim 2^k2^{2l}\left\|\int\langle 2^k(x-y)\rangle^{-3}\langle 2^l(y-z)\rangle^{-3}|\langle D_x\rangle^{s+\rho+13}u(y)||v(z)|dydz\right\|_{L(dx)}\\ &\lesssim 2^k2^{2l}\int\langle 2^ky\rangle^{-3}\langle 2^lz\rangle^{-3}\|[\langle D_x\rangle^{s+\rho+13}u](\cdot-y)v(\cdot-y-z)\|_{L(dx)}dydz\\ &\lesssim 2^{-\frac{k}{2}}2^{-\frac{l}{2}}\|\langle D_x\rangle^{s+\rho+13}u\|_L\|v\|_{L^\infty}\ (\text{ or } 2^{-\frac{k}{2}}2^{-\frac{l}{2}}\|\langle D_x\rangle^{s+\rho+13}u\|_{L^\infty}\|v\|_L).\end{aligned} \tag{A.26}$$

Analogously, for $1 \le k \le l + N_0$, we have that $|\xi-\eta| \lesssim |\eta|$ on the support of $\varphi_k(\xi)\varphi_l(\eta)$, thus in this case we define $a^s_{k\le l+N_0}(\xi,\eta) := D^s(\xi,\eta)\varphi_k(\xi)\varphi_l(\eta)\langle\eta\rangle^{-s-\rho-13}$, which satisfies, for any multi-indices α,β of length less or equal than 3, $|\partial^\alpha_\xi\partial^\beta_\eta[a^s_{k\le l+N_0}(2^k\xi,2^l\eta)]| \lesssim_{\alpha,\beta} 2^{-l}$, after hypothesis (iii). We therefore deduce that the associated kernel $K^s_{k\le l+N_0}(x,y)$ is such that $|K^s_{k\le l+N_0}(x,y)| \lesssim 2^{2k}2^l\langle 2^kx\rangle^{-3}\langle 2^ly\rangle^{-3}$ for any (x,y), and

$$\begin{aligned}\tag{A.27} &\left\| \int e^{ix\cdot\xi} D^s(\xi,\eta)\varphi_k(\xi)\varphi_l(\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{L(dx)} \\ &\lesssim 2^{-\frac{k}{2}}2^{-\frac{l}{2}}\|u\|_{L^\infty}\|\langle D_x\rangle^{s+\rho+13}v\|_L \text{ (or } 2^{-\frac{k}{2}}2^{-\frac{l}{2}}\|u\|_L\|\langle D_x\rangle^{s+\rho+13}v\|_{L^\infty}).\end{aligned}$$

Combining inequalities (A.24), (A.26), (A.27) with $L = L^2$ (resp. $L = L^\infty$), and taking the sum over $k \ge 1, l \ge 0$, we deduce inequality (A.19a) (resp. (A.19b)).

In order to derive inequalities (A.20), we first observe that, up to a factor $t^{s\sigma}$, we can reduce to study the L^2 and L^∞ norm of (A.22) with $s = 0$ and $D(\xi,\eta)$ multiplied by $\phi(t^{-\sigma}\xi)$. Here we use again decomposition (A.23), (A.25), and only need to modify some of multipliers defined above, depending on if we want derivatives falling entirely on u, or entirely on v. In fact, in order to prove the first two inequalities in (A.20a), and the first one in (A.20b), we first observe that for any $k > l+N_0$, $|\eta| < |\xi| \sim |\xi-\eta|$ on the support of $\varphi_k(\xi)\varphi_l(\eta)$; for $|k-l| \le N_0$, $|\eta| \sim |\xi|$ on the mentioned support; while for any $l > k+N_0$, $|\eta| \sim |\xi-\eta|$. So if $a^\phi_{l\le k+N_0}(\xi,\eta) := D(\xi,\eta)\chi(t^{-\sigma}\xi)\varphi_k(\xi)\varphi_l(\eta)$, and $a^\phi_{l>k+N_0}(\xi,\eta) := D(\xi,\eta)\chi(t^{-\sigma}\xi)\varphi_k(\xi)\varphi_l(\eta)\langle\xi-\eta\rangle^{-\rho-13}$, we deduce from hypothesis $(ii)-(iii)$ on $D(\xi,\eta)$, and the fact that $|\xi| \lesssim t^\sigma$ on the support of $\phi(t^{-\sigma}\xi)$, that, for any $\alpha,\beta \in \mathbb{N}^2$ of length less or equal than 3, $|\partial^\alpha_\xi\partial^\beta_\eta[a^\phi_{l\le k+N_0}(2^k\xi,2^l\eta)]| \lesssim t^\delta 2^{-k}$, for a $\delta > 0$ depending linearly on σ, while $|\partial^\alpha_\xi\partial^\beta_\eta[a^\phi_{l>k+N_0}(2^k\xi,2^l\eta)]| \lesssim 2^{-l}$. Kernel $K^\phi_{l\le k+N_0}(x,y)$ (resp. $K^\phi_{l>k+N_0}(x,y)$), associated to $a^\phi_{l\le k+N_0}$ (resp. to $a^\phi_{l>k+N_0}$), verifies $|K^\phi_{l\le k+N_0}(x,y)| \lesssim t^\delta 2^k2^{2l}\langle 2^kx\rangle^{-3}\langle 2^ly\rangle^{-3}$ (resp. $|K^\phi_{l>k+N_0}(x,y)| \lesssim 2^{2k}2^l\langle 2^kx\rangle^{-3}\langle 2^ly\rangle^{-3}$), and then for any $l \le k+N_0$

$$\begin{aligned}\tag{A.28} &\left\| \int e^{ix\cdot\xi} D(\xi,\eta)\phi(t^\sigma\xi)\varphi_k(\xi)\varphi_l(\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{L(dx)} \\ &= \left\|\int K^\phi_{l\le k+N_0}(x-y,y-z)u(y)v(z)dydz\right\|_{L(dx)} \lesssim t^\delta 2^{-\frac{k}{2}}2^{-\frac{l}{2}}\|u\|_L\|v\|_{L^\infty} \\ &\qquad (\text{or } \lesssim t^\delta 2^{-\frac{k}{2}}2^{-\frac{l}{2}}\|u\|_{L^\infty}\|v\|_L),\end{aligned}$$

(resp. for $l > k+N_0$

$$\begin{aligned}&\left\| \int e^{ix\cdot\xi} D(\xi,\eta)\phi(t^\sigma\xi)\varphi_l(\xi)\varphi_l(\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{L(dx)} \\ &= \left\|\int K^\phi_{l>k+N_0}(x-y,y-z)[\langle D_x\rangle^{\rho+13}u](y)v(z)dydz\right\|_{L(dx)} \lesssim 2^{-\frac{k}{2}}2^{-\frac{l}{2}}\|\langle D_x\rangle^{\rho+13}u\|_L\|v\|_{L^\infty} \\ &\qquad \Big(\text{or } \lesssim 2^{-\frac{k}{2}}2^{-\frac{l}{2}}\|\langle D_x\rangle^{\rho+13}u\|_{L^\infty}\|v\|_L)\Big).\end{aligned}$$

Combining these two inequalities with (A.24a), and taking the sum over $k \ge 1, l \ge 0$, we obtain the wished estimates.

Last two inequalities in (A.20a), and last one in (A.20b), are instead obtained combining (A.24b) with (A.27) (that evidently holds for $D^s(\xi,\eta)$ replaced with $D(\xi,\eta)\phi(t^\sigma\xi)$) and (A.28).

Finally, last part of the statement follows from the same argument of above, with the only difference that, after $(\widetilde{i})$, multiplier $\widetilde{a}^s_0(\xi,\eta) := \widetilde{D}(\xi,\eta)\chi(\xi)\langle\eta\rangle^{-\rho-10}$ satisfies the hypothesis of lemma A.1 (ii), with $|g_\beta(\eta)| \lesssim \langle \eta \rangle^{-3}$ for any $|\beta| \le 3$, and

$$\left\| \int e^{ix\cdot\xi} \widetilde{D}^s(\xi,\eta)\chi(\xi)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{L(dx)} \\ = \left\| \int e^{ix\cdot\xi} \widetilde{a}^s_0(\xi,\eta)\widehat{\langle D_x\rangle^{\rho+10}u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{L(dx)} \lesssim \|\langle D_x\rangle^{\rho+10}u\|_{L^2}\|v\|_L,$$

or

$$\left\| \int e^{ix\cdot\xi} \widetilde{D}^s(\xi,\eta)\chi(\xi)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{L(dx)} \\ = \left\| \int e^{ix\cdot\xi} \widetilde{a}^s_0(\xi,\eta)\hat{u}(\xi-\eta)\widehat{\langle D_x\rangle^{\rho+10}v}(\eta)d\xi d\eta \right\|_{L(dx)} \lesssim \|u\|_{L^2}\|\langle D_x\rangle^{\rho+10}v\|_L,$$

with $L = L^2$ or L^∞, by corollary A.2 (ii). □

Lemma A.8. *Let $j \in \{+,-\}$, $\phi \in C^\infty_0(\mathbb{R}^2)$, $t \ge 1, \sigma > 0$, and $D_j(\xi,\eta)$ be defined as in* (3.1.14). *For any $s \ge 0$, $i = 1,2$, $D_j(\xi,\eta)$ and $\frac{\xi_i}{|\xi|}D_j(\xi,\eta)$ satisfy inequalities* (A.19), (A.20) *with $\rho = 2$, and*

(A.29a)
$$\left\| \int e^{ix\cdot\xi} \partial_\xi D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{H^s(dx)} \lesssim \|u\|_{H^{13}}\|v\|_{L^2} + \|u\|_{H^{s+16}}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^{s+16}} \\ (\textit{resp.} \;\lesssim \|u\|_{H^{13}}\|v\|_{L^2} + \|u\|_{H^{s+16,\infty}}\|v\|_{L^2} + \|u\|_{L^2}\|v\|_{H^{s+16,\infty}}),$$

(A.29b)
$$\left\| \int e^{ix\cdot\xi} \partial_\xi D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{H^{s,\infty}(dx)} \\ \lesssim \|u\|_{H^{13}}\|v\|_{L^\infty} + \|u\|_{H^{s+16,\infty}}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^{s+16,\infty}},$$

together with

(A.30a)
$$\left\| \phi(t^{-\sigma}D_x) \int e^{ix\cdot\xi} \partial_\xi D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{H^s(dx)} \lesssim t^\delta \|u\|_{H^{13}}\left(\|v\|_{L^2} + \|v\|_{L^\infty}\right) \\ (\textit{or} \;\lesssim t^\delta \|u\|_{L^2}\left(\|v\|_{H^{10}} + \|v\|_{H^{13,\infty}}\right)),$$

(A.30b)
$$\left\| \phi(t^{-\sigma}D_x) \int e^{ix\cdot\xi} \partial_\xi D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{H^{s,\infty}(dx)} \lesssim t^\delta \left(\|u\|_{H^{13}} + \|u\|_{H^{16,\infty}}\right)\|v\|_{L^\infty} \\ (\textit{or} \;\lesssim t^\delta \left(\|u\|_{L^2} + \|u\|_{L^\infty}\right)\|v\|_{H^{16,\infty}}).$$

Moreover, if $\Omega = x_1\partial_2 - x_2\partial_1$, and $Z_n = x_n\partial_t + t\partial_n$,

(A.31a)
$$\left\| \chi(t^{-\sigma}D_x)\Omega \int e^{ix\cdot\xi} D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{L^2(dx)} \\ \lesssim t^\delta \left[\left(\|u\|_{L^2} + \|\Omega u\|_{L^2}\right)\|v\|_{H^{17,\infty}} + \|u\|_{H^{15,\infty}}\|\Omega v\|_{L^2}\right],$$

(A.31b)
$$\left\| \chi(t^{-\sigma}D_x)Z_n \int e^{ix\cdot\xi} D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta \right\|_{L^2(dx)} \\ \lesssim t^\delta \left[\|\partial_t u\|_{L^2}\|v\|_{H^{13}} + \|u\|_{H^{13}}\|\partial_t v\|_{L^2} + \|Z_n u\|_{L^2}\|v\|_{H^{15,\infty}} + \|u\|_{H^{15,\infty}}\|Z_n v\|_{L^2}\right],$$

(A.31c) $$\left\|\chi(t^{-\sigma}D_x)D_jZ_n\int e^{ix\cdot\xi}D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta\right\|_{L^2(dx)}$$
$$\lesssim t^{\delta}\left[\|\partial_t u\|_{L^2}\|v\|_{H^{14,\infty}}+\|u\|_{H^{14,\infty}}\|\partial_t v\|_{L^2}+\|Z_nu\|_{L^2}\|v\|_{H^{17,\infty}}+\|u\|_{H^{17,\infty}}\|Z_nv\|_{L^2}\right],$$

for $\delta>0$ depending linearly on σ.

Proof. The statement follows essentially from the observation that, for $j\in\{+,-\}$, functions $D_j(\xi,\eta)$ and $[(\xi_i\partial_{\xi_j})^{k_1}(\eta_i\partial_{\eta_j})^{k_2}D_j](\xi,\eta)$, satisfy inequalities $(i)-(iii)$ in lemma A.7 respectively with $\rho=2$ and $\rho=2+2(k_1+k_2)$, while $\partial_\xi D_j(\xi,\eta)$ satisfies $\widetilde{(i)},(ii),(iii)$ with $\rho=3$. Indeed, we first remark that, for every ξ,η, denominator $1+\langle\xi-\eta\rangle\langle\eta\rangle-(\xi-\eta)\cdot\eta$ is bounded from below by a positive constant; secondly, deriving that denominator gives rise to losses in $\langle\xi-\eta\rangle,\langle\eta\rangle$:

$$\partial_{\xi_k}(1+\langle\xi-\eta\rangle\langle\eta\rangle-(\xi-\eta)\cdot\eta)=\frac{\xi_k-\eta_k}{\langle\xi-\eta\rangle}\langle\eta\rangle+\eta_k,$$
$$\partial_{\eta_k}(1+\langle\xi-\eta\rangle\langle\eta\rangle-(\xi-\eta)\cdot\eta)=\frac{\xi_k-\eta_k}{\langle\xi-\eta\rangle}\langle\eta\rangle+\langle\xi-\eta\rangle\frac{\eta_k}{\langle\eta\rangle}+\eta_k-(\xi_k-\eta_k).$$

For $|\xi|\lesssim 1$, we have that $\langle\xi-\eta\rangle\lesssim\langle\eta\rangle$, and after previous remarks

$$\left|\partial^\alpha_\xi\partial^\beta_\eta\left[\frac{j\langle\xi-\eta\rangle+j\langle\eta\rangle}{1+\langle\xi-\eta\rangle\langle\eta\rangle-(\xi-\eta)\cdot\eta}\eta_1\right]\right|\lesssim_{\alpha,\beta}\langle\eta\rangle^{2+|\alpha|+|\beta|},$$

for any α,β, while

$$\left|\partial^\beta_\eta\left[\frac{|\xi|}{1+\langle\xi-\eta\rangle\langle\eta\rangle-(\xi-\eta)\cdot\eta}\eta_1\right]\right|\lesssim_\beta\langle\eta\rangle^{1+|\beta|},$$
$$\left|\partial^\alpha_\xi\partial^\beta_\eta\left[\frac{|\xi|}{1+\langle\xi-\eta\rangle\langle\eta\rangle-(\xi-\eta)\cdot\eta}\eta_1\right]\right|\lesssim_{\alpha,\beta}\sum_{|\alpha_1|+|\alpha_2|=|\alpha|}|\xi|^{-|\alpha_1|+1}\langle\eta\rangle^{1+|\alpha_2|+|\beta|},\quad |\alpha|\ge 1.$$

For $|\xi|\gtrsim 1$ and $|\eta|\lesssim\langle\xi-\eta\rangle$ (resp. $|\eta|\gtrsim\langle\xi-\eta\rangle$) we have that $|\xi|\lesssim|\xi-\eta|$ (resp.$|\xi|\lesssim|\eta|$), so each time a derivative hits the denominator of $D_j(\xi,\eta)$ we loose a factor $\langle\xi-\eta\rangle$ (resp. $\langle\eta\rangle$).

These observations immediately imply that inequalities (A.19) hold when $D=D_j$ and $\rho=2$, and inequalities (A.29), while inequalities (A.31) follow by the further remark that, after some integration by parts,

$$\Omega\int e^{ix\cdot\xi}D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta$$
$$=\sum_{k_1+k_2+k_3+k_4=1}\int e^{ix\cdot\xi}[(\xi_1\partial_{\xi_2}-\xi_2\partial_{\xi_1})^{k_1}(\eta_1\partial_{\eta_2}-\eta_2\partial_{\eta_1})^{k_2}D_j](\xi,\eta)\widehat{\Omega^{k_3}u}(\xi-\eta)\widehat{\Omega^{k_4}v}(\eta)d\xi d\eta,$$

$$Z_n\int e^{ix\cdot\xi}D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta$$
$$=\int e^{ix\cdot\xi}[\partial_{\xi_n}D_j](\xi,\eta)D_t\big[\hat{u}(\xi-\eta)\hat{v}(\eta)\big]d\xi d\eta+\int e^{ix\cdot\xi}[\partial_{\eta_n}D_j](\xi,\eta)\hat{u}(\xi-\eta)\widehat{D_tv}(\eta)d\xi d\eta$$
$$+\int e^{ix\cdot\xi}D_j(\xi,\eta)\widehat{Z_nu}(\xi-\eta)\hat{v}(\eta)d\xi d\eta+\int e^{ix\cdot\xi}D_j(\xi,\eta)\hat{u}(\xi-\eta)\widehat{Z_nv}(\eta)d\xi d\eta,$$

and, for $\delta^n_j=1$ if $j=n$, 0 otherwise,

$$D_jZ_n\int e^{ix\cdot\xi}D_j(\xi,\eta)\hat{u}(\xi-\eta)\hat{v}(\eta)d\xi d\eta=\delta^n_j\int e^{ix\cdot\xi}D_j(\xi,\eta)D_t\left[\hat{u}(\xi-\eta)\hat{v}(\eta)\right]d\xi d\eta$$
$$+\int e^{ix\cdot\xi}\partial_{\xi_n}[\xi_jD_j](\xi,\eta)D_t\big[\hat{u}(\xi-\eta)\hat{v}(\eta)\big]d\xi d\eta+\int e^{ix\cdot\xi}\partial_{\eta_n}[\xi_jD_j](\xi,\eta)\hat{u}(\xi-\eta)\widehat{D_tv}(\eta)d\xi d\eta$$
$$+\int e^{ix\cdot\xi}\xi_jD_j(\xi,\eta)\widehat{Z_nu}(\xi-\eta)\hat{v}(\eta)d\xi d\eta+\int e^{ix\cdot\xi}\xi_jD_j(\xi,\eta)\hat{u}(\xi-\eta)\widehat{Z_nv}(\eta)d\xi d\eta.$$

□

Appendix B

The aim of this chapter is to show how, from the bootstrap assumptions (1.1.11), it is possible to derive a moderate growth in time for the L^2 norm of $\mathcal{L}^\mu \widetilde{v}$, with $0 \le |\mu| \le 2$, and of $\Omega_h^\mu \mathcal{M}^\nu \widetilde{u}^{\Sigma,k}$, with $\mu, |\nu| = 0, 1$, that are used in propositions 3.2.7 and 3.3.7. Moreover, we also prove in lemma B.3.21 a sharp decay estimate for the uniform norm of the Klein-Gordon solution when one Klainerman vector field is acting on it (and when considered for frequencies less or equal than t^σ, with $\sigma > 0$ small).

B.1 Some preliminary lemmas

In the current section we list, on the one hand, some inequalities concerning the H^s and $H^{s,\infty}$ norm of the quadratic non-linearities $Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm), Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm)$ (lemmas B.1.1, B.1.2), as they are very frequently recalled in the second part of the paper; on the other hand, we introduce some preliminary small results that will be useful in sections B.2, B.3.

For seek of compactness, we denote $Q_0^{\mathrm{w}}(v_\pm, D_1 v_\pm)$ (resp. $Q_0^{\mathrm{kg}}(v_\pm, D_1 u_\pm)$) by NL_w (resp. NL_{kg}), i.e.

$$NL_w := \frac{i}{4}\left[(v_+ + v_-)D_1(v_+ + v_-) - \frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot \frac{D_x D_1}{\langle D_x\rangle}(v_+ - v_-)\right], \tag{B.1.1a}$$

$$NL_{kg} := \frac{i}{4}\left[(v_+ + v_-)D_1(u_+ + u_-) - \frac{D_x}{\langle D_x\rangle}(v_+ - v_-)\cdot \frac{D_x D_1}{|D_x|}(u_+ - u_-)\right]. \tag{B.1.1b}$$

We recall the result of lemma 1.2.39, that can be also stated in the classical setting:

$$\|(1-\chi)(t^{-\sigma}D_x)w\|_{H^{s'}} \le C t^{-\sigma(s-s')}\|w\|_{H^s}, \quad \forall s > s'. \tag{B.1.2}$$

It is also useful to remind, in view of upcoming lemmas, that the L^2 norm of $(\Gamma^I u)_\pm, (\Gamma^I v)_\pm$ is estimated with:

$E_n(t;W)^{\frac{1}{2}}$, whenever $|I| \le n$ and Γ^I is a product of spatial derivatives;
$E_3^k(t;W)^{\frac{1}{2}}$, whenever $|I| \le 3$ and at most $3-k$ vector fields in Γ^I belong to $\{\Omega, Z_m, m = 1,2\}$.

As assumed in (1.1.11c), (1.1.11d), such energies are supposed to have a moderate growth in time, and a hierarchy is established among them in the sense that

$$0 < \delta \ll \delta_2 \ll \delta_1 \ll \delta_0 \ll 1.$$

We warn the reader that this hierarchy is often implicitly used throughout this chapter.

Lemma B.1.1. *For any $s \geq 0$, any $\theta \in]0,1[$, NL_w satisfies the following inequalities:*

$$\|NL_w(t,\cdot)\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{1,\infty}}\|V(t,\cdot)\|_{H^1}, \tag{B.1.3a}$$

$$\|NL_w(t,\cdot)\|_{L^\infty} \lesssim \|V(t,\cdot)\|^2_{H^{2,\infty}}, \tag{B.1.3b}$$

$$\|NL_w(t,\cdot)\|_{H^s} \lesssim \|V(t,\cdot)\|_{H^{s+1}}\|V(t,\cdot)\|_{H^{1,\infty}}, \tag{B.1.3c}$$

$$\|NL_w(t,\cdot)\|_{H^{s,\infty}} \lesssim \|V(t,\cdot)\|^{2-\theta}_{H^{s+1,\infty}}\|V(t,\cdot)\|^{\theta}_{H^{s+3}}, \tag{B.1.3d}$$

$$\|\Omega NL_w(t,\cdot)\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{2,\infty}}\left(\|V(t,\cdot)\|_{L^2} + \|\Omega V(t,\cdot)\|_{H^1}\right), \tag{B.1.3e}$$

while for NL_{kg} we have that:

$$\|NL_{kg}(t,\cdot)\|_{L^2} \lesssim \|V(t,\cdot)\|_{H^{1,\infty}}\|U(t,\cdot)\|_{H^1}, \tag{B.1.4a}$$

$$\|NL_{kg}(t,\cdot)\|_{L^\infty} \lesssim \|V(t,\cdot)\|_{H^{1,\infty}}\left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{2,\infty}}\right), \tag{B.1.4b}$$

(B.1.4c)
$$\|NL_{kg}(t,\cdot)\|_{H^s} \lesssim \|V(t,\cdot)\|_{H^s}\left(\|U(t,\cdot)\|_{H^{1,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{1,\infty}}\right) + \|V(t,\cdot)\|_{L^\infty}\|U(t,\cdot)\|_{H^{s+1}},$$

(B.1.4d)

$$\begin{aligned}\|NL_{kg}(t,\cdot)\|_{H^{s,\infty}} \lesssim\ & \|V(t,\cdot)\|^{1-\theta}_{H^{s,\infty}}\|V(t,\cdot)\|^{\theta}_{H^{s+2}}\left(\|U(t,\cdot)\|_{H^{1,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{1,\infty}}\right)\\ &+ \|V(t,\cdot)\|_{L^\infty}\Big(\|U(t,\cdot)\|^{1-\theta}_{H^{s+1,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|^{1-\theta}_{H^{s+1,\infty}}\Big)\|U(t,\cdot)\|^{\theta}_{H^{s+3}},\end{aligned} \tag{B.1.4e}$$

and

$$\begin{aligned}\|\Omega NL_{kg}(t,\cdot)\|_{L^2} \lesssim\ & \left(\|V(t,\cdot)\|_{L^2} + \|\Omega V(t,\cdot)\|_{L^2}\right)\left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{2,\infty}}\right)\\ &+ \|V(t,\cdot)\|_{H^{1,\infty}}\|\Omega U(t,\cdot)\|_{H^1}.\end{aligned} \tag{B.1.4f}$$

Proof. Inequalities (B.1.3a), (B.1.3b), (B.1.4a), and (B.1.4b) are straightforward. The same is for (B.1.3e), (B.1.4f), after commutation of Ω with the operators appearing in (2.1.1). All other inequalities in the statement are rather derived using corollary A.4. ☐

Lemma B.1.2. *For any $s \geq 0$, any $\theta \in]0,1[$,*

$$\|D_t U(t,\cdot)\|_{H^s} \lesssim \|U(t,\cdot)\|_{H^{s+1}} + \|V(t,\cdot)\|_{H^{s+1}}\|V(t,\cdot)\|_{H^{1,\infty}}, \tag{B.1.5a}$$

$$\|D_t U(t,\cdot)\|_{H^{s,\infty}} \lesssim \|U(t,\cdot)\|_{H^{s+2,\infty}} + \|V(t,\cdot)\|^{2-\theta}_{H^{s+1,\infty}}\|V(t,\cdot)\|^{\theta}_{H^{s+3}}, \tag{B.1.5b}$$

$$\|D_t \mathrm{R}_1 U(t,\cdot)\|_{H^{s,\infty}} \lesssim \|\mathrm{R}_1 U(t,\cdot)\|_{H^{s+1,\infty}} + \|V(t,\cdot)\|_{H^{s+3}}\|V(t,\cdot)\|_{H^{1,\infty}}, \tag{B.1.5c}$$

$$\|D_t \Omega U(t,\cdot)\|_{L^2} \leq \|\Omega U(t,\cdot)\|_{H^1} + \|V(t,\cdot)\|_{H^{2,\infty}}\left(\|V(t,\cdot)\|_{L^2} + \|\Omega V(t,\cdot)\|_{H^1}\right), \tag{B.1.5d}$$

and

$$\begin{aligned}\|D_t V(t,\cdot)\|_{H^s} \lesssim\ & \|V(t,\cdot)\|_{H^{s+1}} + \|V(t,\cdot)\|_{H^s}\left(\|U(t,\cdot)\|_{H^{1,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{1,\infty}}\right)\\ &+ \|V(t,\cdot)\|_{L^\infty}\|U(t,\cdot)\|_{H^{s+1}},\end{aligned} \tag{B.1.6a}$$

(B.1.6b)
$$\begin{aligned}\|D_t V(t,\cdot)\|_{H^{s,\infty}} \lesssim\ & \|V(t,\cdot)\|_{H^{s+1,\infty}} + \|V(t,\cdot)\|^{1-\theta}_{H^{s,\infty}}\|V(t,\cdot)\|^{\theta}_{H^{s+1}}\left(\|U(t,\cdot)\|_{H^{1,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{1,\infty}}\right)\\ &+ \|V(t,\cdot)\|_{L^\infty}\left(\|U(t,\cdot)\|^{1-\theta}_{H^{s+1,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|^{1-\theta}_{H^{s+1,\infty}}\right)\|U(t,\cdot)\|^{\theta}_{H^{s+3}},\end{aligned}$$

(B.1.6c)
$$\begin{aligned}\|D_t \Omega V(t,\cdot)\|_{L^2} \leq\ & \|\Omega V(t,\cdot)\|_{H^1} + \left(\|V(t,\cdot)\|_{L^2} + \|\Omega V(t,\cdot)\|_{L^2}\right)\left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{2,\infty}}\right)\\ &+ \|V(t,\cdot)\|_{H^{1,\infty}}\|\Omega U(t,\cdot)\|_{H^1}.\end{aligned}$$

Proof. Straight consequence of the fact that (u_+, v_+, u_-, v_-) is solution to system (3.1.1) and previous lemma. Here, (B.1.5c) is derived using that

$$\|\mathrm{R}_1 NL_w(t,\cdot)\|_{H^{s,\infty}} \lesssim \|NL_w(t,\cdot)\|_{H^{s+2}},$$

after classical Sobolev injection and continuity of $\mathrm{R}_1 : H^s \to H^s$, for any $s \geq 0$. □

Lemma B.1.3. *Let $|I| = 1$ be such that $\Gamma^I \in \{\Omega, Z_m, m = 1, 2\}$. Then*

(B.1.7)
$$\|D_t U^I(t,\cdot)\|_{L^2} \lesssim \|U^I(t,\cdot)\|_{H^1} + \|V(t,\cdot)\|_{H^{2,\infty}} \Big[\|V^I(t,\cdot)\|_{H^1} + \|V(t,\cdot)\|_{H^1}\Big(1 + \sum_{\mu=0}^{1} \|\mathrm{R}_1^\mu U(t,\cdot))\|_{H^{1,\infty}}\Big) + \|V(t,\cdot)\|_{L^\infty}\|U(t,\cdot)\|_{H^1}\Big],$$

(B.1.8)
$$\|D_t V^I(t,\cdot)\|_{L^2} \lesssim \|V^I(t,\cdot)\|_{H^1} + \sum_{\mu=0}^{1} \|\mathrm{R}_1^\mu U(t,\cdot)\|_{H^{2,\infty}} \|V^I(t,\cdot)\|_{L^2} + \|V(t,\cdot)\|_{H^{1,\infty}} \left(\|U^I(t,\cdot)\|_{H^1} + \|U(t,\cdot)\|_{H^1} + \|V(t,\cdot)\|_{H^{1,\infty}}\|V(t,\cdot)\|_{H^1}\right).$$

Proof. The result of the statement follows using the equation satisfied, respectively, by $u^I_\pm$ and $v^I_\pm$, together with (B.1.5a), (B.1.6a) with $s = 0$. In fact, by (1.1.15) with $|I| = 1$,

$$D_t u^I_\pm = \pm|D_x|u^I_\pm + Q^{\mathrm{w}}_0(v^I_\pm, D_1 v_\pm) + Q^{\mathrm{w}}_0(v_\pm, D_1 v^I_\pm) + G^{\mathrm{w}}_1(v_\pm, Dv_\pm),$$
$$D_t v^I_\pm = \pm\langle D_x\rangle v^I_\pm + Q^{\mathrm{kg}}_0(v^I_\pm, D_1 u_\pm) + Q^{\mathrm{kg}}_0(v_\pm, D_1 u^I_\pm) + G^{\mathrm{kg}}_1(v_\pm, Du_\pm),$$

with $G^{\mathrm{w}}_1(v, \partial v_\pm) = G_1(v, \partial v)$ and $G^{\mathrm{kg}}_1(v_\pm, Du_\pm) = G_1(v, \partial u)$, G_1 given by (1.1.16), and one estimates the L^2 norm of the first two quadratic terms in above equations with the L^2 norm of factors indexed in I, times the L^∞ norm of the remaining one. The L^2 norm of the latter quadratic terms can be, instead, bounded by taking the L^2 norm of one of the two factors, times the L^∞ norm of the remaining one, indifferently. We choose here to consider the L^2 norm of factors $Du_\pm, Dv_\pm$, and use (B.1.5a), (B.1.6a) if the derivative D is a time derivative. □

It is useful to remind that, if $w(t,x)$ is solution to inhomogeneous half wave equation (3.2.5), then after (3.2.9a),

(B.1.9a)
$$\begin{aligned} x_j D_k w(t,x) &= \frac{D_k}{|D_x|}\left[x_j|D_x| - tD_j + \frac{1}{2i}\frac{D_j}{|D_x|}\right] w(t,x) + t\frac{D_jD_k}{|D_x|}w(t,x) - \frac{1}{2i}\frac{D_j}{|D_x|}w(t,x) \\ &\quad - \frac{1}{i}Op\Big(\partial_j\Big(\frac{\xi_k}{|\xi)}\Big)|\xi|\Big)w(t,x) \\ &= i\frac{D_k}{|D_x|}Z_j w(t,x) + \frac{D_k}{|D_x|}[x_j f(t,x)] + t\frac{D_jD_k}{|D_x|}w(t,x). \end{aligned}$$

Analogously, if $w(t,x)$ is solution to inhomogeneous half Klein-Gordon (3.2.7), from (3.2.9b) we have that

(B.1.9b)
$$\begin{aligned} x_j w(t,x) &= \langle D_x\rangle^{-1}\left[\langle D_x\rangle x_j - tD_j\right] w(t,x) + tD_j\langle D_x\rangle^{-1} w(t,x) \\ &= i\langle D_x\rangle^{-1} Z_j w(t,x) - iD_j\langle D_x\rangle^{-2}w(t,x) + \langle D_x\rangle^{-1}[x_j f(t,x)] + tD_j\langle D_x\rangle^{-1}w(t,x). \end{aligned}$$

We also remind the reader about equivalence (2.1.16), so we won't particularly care if we are dealing with $\Gamma^I u_\pm, \Gamma^I v_\pm$, instead of $(\Gamma^I u)_\pm, (\Gamma^I v)_\pm$, when we bound the L^2 norm of those terms with the energy defined in (1.1.9).

Lemma B.1.4. *There exists a positive constant $C>0$ such that, if a-priori estimates (1.1.11) are satisfied in some interval $[1,T]$, for a fixed $T>1$, with $\varepsilon_0<(2A+B)^{-1}$ small, then*

$$\|x_j v_\pm(t,\cdot)\|_{H^1}\le CB\varepsilon t^{1+\frac{\delta}{2}}, \tag{B.1.10a}$$

$$\|x_j v_\pm(t,\cdot)\|_{H^{1,\infty}}+\left\|x_j\frac{D_x}{\langle D_x\rangle}v_\pm(t,\cdot)\right\|_{H^{1,\infty}}\le C(A+B)\varepsilon t^{\frac{\delta_2}{2}}, \tag{B.1.10b}$$

for every $t\in[1,T]$, every $j=1,2$. Moreover,

$$\|x_j D_x u_\pm(t,\cdot)\|_{L^2}\le CB\varepsilon t^{1+\frac{\delta}{2}}, \tag{B.1.11}$$

for every $j=1,2$, $t\in[1,T]$.

Proof. We warn the reader that, throughout the proof, C will denote a positive constant that may change line after line. As $v_+=-\overline{v_-}$, it is enough to prove the statement for v_-.

Since v_- is solution to equation (3.2.7) with $f=NL_{kg}$, from (B.1.9b) it immediately follows that

$$\|x_j v_-(t,\cdot)\|_{H^1}\lesssim\|Z_j v_-(t,\cdot)\|_{L^2}+t\|v_-(t,\cdot)\|_{H^1}+\|x_j NL_{kg}(t,\cdot)\|_{L^2}, \tag{B.1.12a}$$

along with

$$\|x_j v_-(t,\cdot)\|_{H^{1,\infty}}\le\|Z_j v_-(t,\cdot)\|_{H^2}+t\|v_-(t,\cdot)\|_{H^{2,\infty}}+\|x_j NL_{kg}(t,\cdot)\|_{L^\infty}, \tag{B.1.12b}$$

derived by using the classical Sobolev injection. Notice that the above inequality holds also for the $H^{1,\infty}$ norm of $x_j D_x\langle D_x\rangle^{-1}v_-$. As

$$\begin{split}x_j NL_{kg}=[x_j(v_+-v_-)]D_1(u_++u_-)-\frac{D_x}{\langle D_x\rangle}[x_j(v_+-v_-)]\cdot\frac{D_xD_1}{|D_x|}(u_+-u_-)\\-\left[x_j,\frac{D_x}{\langle D_x\rangle}\right](v_+-v_-)\cdot\frac{D_xD_1}{|D_x|}(u_+-u_-).\end{split}$$

we derive that

$$\|x_j NL_{kg}(t,\cdot)\|_{L^2}\lesssim\|x_j v_-(t,\cdot)\|_{L^2}\left(\|U(t,\cdot)\|_{H^{2,\infty}}+\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}\right)+\|V(t,\cdot)\|_{L^2}\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}} \tag{B.1.13a}$$

and, without commutating x_j to $D_x\langle D_x\rangle^{-1}$,

$$\|x_j NL_{kg}(t,\cdot)\|_{L^\infty}\lesssim\left(\|x_j v_-(t,\cdot)\|_{L^\infty}+\left\|x_j\frac{D_x}{\langle D_x\rangle}v_-(t,\cdot)\right\|_{L^\infty}\right)\sum_{\mu=0}^{1}\|\mathrm{R}_1^\mu U(t,\cdot)\|_{H^{2,\infty}}. \tag{B.1.13b}$$

Thus, if $\varepsilon_0>0$ is assumed sufficiently small to verify $\varepsilon_0<(2A)^{-1}$, by injecting (B.1.13a) (resp. (B.1.13b)) into (B.1.12a) (resp. in (B.1.12b)), and using a-priori estimates (1.1.11), we obtain that

$$\begin{split}\|x_j v_-(t,\cdot)\|_{H^1}&\le C\left[E_3^2(t;W)^{\frac12}+tE_3(t;W)^{\frac12}\right]+\|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}E_0(t;W)^{\frac12}\\&\le CB\varepsilon t^{1+\frac{\delta}{2}},\end{split}$$

(resp.

$$\|x_j v_-(t,\cdot)\|_{H^{1,\infty}}+\left\|x_j\frac{D_x}{\langle D_x\rangle}v_-(t,\cdot)\right\|_{H^{1,\infty}}\le CE_3^2(t;W)^{\frac12}+t\|V(t,\cdot)\|_{H^{2,\infty}}\le C(A+B)\varepsilon t^{\frac{\delta_2}{2}}\Big),$$

and the conclusion of the proof of (B.1.10).

Analogously, from (B.1.9a) with $w=u_-$ and $f=NL_w$,

$$\|x_jD_ku_-(t,\cdot)\|_{L^2}\lesssim \|Z_ju_\pm(t,\cdot)\|_{L^2}+t\|u_\pm(t,\cdot)\|_{L^2}+\|x_jNL_w(t,\cdot)\|_{L^2}\le CB\varepsilon t^{1+\frac{\delta}{2}},$$

as follows (1.1.11c), (1.1.11d), (B.1.10b) and the fact that

$$\|x_jNL_w(t,\cdot)\|_{L^2}\lesssim \sum_{|\mu|=0}^{1}\Big\|x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)v_\pm(t,\cdot)\Big\|_{L^\infty}\|v_\pm(t,\cdot)\|_{H^1}. \tag{B.1.14}$$

□

Corollary B.1.5. *There exists a constant $C>0$ such that, under the same hypothesis as in lemma B.1.4,*

$$\|x_jNL_{kg}(t,\cdot)\|_{L^2}\le C(A+B)B\varepsilon^2t^{\frac{\delta+\delta_2}{2}}, \tag{B.1.15a}$$

$$\|x_jNL_{kg}(t,\cdot)\|_{L^\infty}\le C(A+B)B\varepsilon^2t^{-\frac{1}{2}+\frac{\delta_2}{2}}, \tag{B.1.15b}$$

and

$$\|x_jNL_w(t,\cdot)\|_{L^2}\le C(A+B)B\varepsilon^2t^{\frac{\delta+\delta_2}{2}}, \tag{B.1.16a}$$

$$\|x_jNL_w(t,\cdot)\|_{L^\infty}\le C(A+B)B\varepsilon^2t^{-1+\frac{\delta_2}{2}}, \tag{B.1.16b}$$

for every $t\in[1,T]$, $j=1,2$.

Proof. From

$$\|x_jNL_{kg}(t,\cdot)\|_{L^2}\lesssim\sum_{\mu=0}^{1}\big\|x_j(D_x\langle D_x\rangle^{-1})^\mu v_\pm(t,\cdot)\big\|_{L^\infty}\|u_\pm(t,\cdot)\|_{H^1},$$

and (B.1.13b), together with (B.1.14) and

$$\|x_jNL_w(t,\cdot)\|_{L^\infty}\lesssim\sum_{\mu=0}^{1}\big\|x_j(D_x\langle D_x\rangle^{-1})^\mu v_\pm(t,\cdot)\big\|_{L^\infty}\|v_\pm(t,\cdot)\|_{H^{2,\infty}},$$

we immediately derive the estimates of the statement, using (B.1.10b) and a-priori estimates. □

Lemma B.1.6. *There exists a positive constant $C>0$ such that, under the same hypothesis as in lemma B.1.4, for any multi-index I of length k, with $1\le k\le 2$, any $j=1,2$,*

$$\|x_j(\Gamma^Iv)_\pm(t,\cdot)\|_{H^1}+\|x_jD_x(\Gamma^Iu)_\pm(t,\cdot)\|_{L^2}\le CB\varepsilon t^{1+\frac{\delta_{3-k}}{2}}, \tag{B.1.17}$$

for every $t\in[1,T]$.

Proof. We warn the reader that, throughout the proof, C will denote a positive constant, that may change line after line. As $\Gamma^Iw_+=-\overline{\Gamma^Iw_-}$, for any I and $w=v,u$, it is enough to prove the statement for Γ^Iv_-, Γ^Iu_-.

From (B.1.9a), (B.1.9b), together with the fact that, for any multi-index I, $(\Gamma^Iv)_-$, $(\Gamma^Iu)_-$ are solution, respectively, to

$$[D_t+\langle D_x\rangle](\Gamma^Iv)_-(t,x)=\Gamma^INL_{kg}, \tag{B.1.18a}$$

and

$$[D_t + \langle D_x\rangle](\Gamma^I u)_-(t,x) = \Gamma^I NL_w, \tag{B.1.18b}$$

we derive that

$$\|x_j(\Gamma^I v)_-(t,\cdot)\|_{H^1} \le \|Z_j(\Gamma^I v)_-(t,\cdot)\|_{L^2} + t\|(\Gamma^I v)_-(t,\cdot)\|_{L^2} + \|x_j\Gamma^I NL_{kg}(t,\cdot)\|_{L^2}, \tag{B.1.19a}$$

together with

$$\|x_j D_k(\Gamma^I u)_-(t,\cdot)\|_{L^2} \le \|Z_j(\Gamma^I u)_-(t,\cdot)\|_{L^2} + t\|(\Gamma^I u)_-(t,\cdot)\|_{L^2} + \|x_j\Gamma^I NL_w(t,\cdot)\|_{L^2}, \tag{B.1.19b}$$

for any $j,k = 1,2$. Therefore, the quantities that need to be estimated to prove the statement are the L^2 norms of $x_j\Gamma^I NL_{kg}$, $x_j\Gamma^I NL_w$, for $1 \le |I| \le 2$.
We first prove (B.1.17) for $|I| = 1$ and $\Gamma^I = \Gamma \in \mathcal{Z}$, with $\mathcal{Z}$ given by (1.1.7), reminding that, from (1.1.15),

$$\Gamma NL_{kg} = Q_0^{\mathrm{kg}}\big((\Gamma v)_\pm, D_1 u_\pm\big) + Q_0^{\mathrm{kg}}\big(v_\pm, D_1(\Gamma u)_\pm\big) + G_1^{\mathrm{kg}}\big(v_\pm, Du_\pm\big), \tag{B.1.20a}$$

along with

$$\Gamma NL_w = Q_0^{\mathrm{w}}\big((\Gamma v)_\pm, D_1 v_\pm\big) + Q_0^{\mathrm{w}}\big(v_\pm, D_1(\Gamma v)_\pm\big) + G_1^{\mathrm{w}}\big(v_\pm, Dv_\pm\big), \tag{B.1.20b}$$

with $G_1^{\mathrm{kg}}\big(v_\pm, Du_\pm\big) = G_1(v,\partial u)$, $G_1^{\mathrm{w}}\big(v_\pm, Dv_\pm\big) = G_1(v,\partial v)$, and G_1 given by (1.1.16).
By multiplying x_j against the Klein-Gordon component in each product of ΓNL_{kg}, we find that

$$\begin{aligned}\|x_j\Gamma NL_{kg}(t,\cdot)\|_{L^2} &\lesssim \sum_{\mu=0}^{1}\|x_j^\mu(\Gamma v)_-(t,\cdot)\|_{L^2}\left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 U(t,\cdot)\|_{H^{2,\infty}}\right)\\ &+ \sum_{|\mu|=0}^{1}\left\|x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\right\|_{L^\infty}\left(\|(\Gamma u)_\pm(t,\cdot)\|_{H^1} + \|u_\pm(t,\cdot)\|_{H^1} + \|D_t u_\pm(t,\cdot)\|_{L^2}\right),\end{aligned} \tag{B.1.21}$$

which injected, together with (B.1.5a) with $s=0$, (B.1.10b), and a-priori estimates (1.1.11), into (B.1.19a) with $\Gamma^I = \Gamma$, gives that

$$\|x_j(\Gamma v)_-(t,\cdot)\|_{H^1} \le CB\varepsilon t^{1+\frac{\delta_2}{2}}.$$

Similarly, combining the above estimate together with (B.1.6a) with $s=0$, (B.1.10b) and a-priori estimates, we derive that

$$\begin{aligned}\|x_j\Gamma NL_w(t,\cdot)\|_{L^2} &\lesssim \sum_{\mu=0}^{1}\|x_j^\mu(\Gamma v)_-(t,\cdot)\|_{L^2}\|v_\pm(t,\cdot)\|_{H^{2,\infty}}\\ &+ \sum_{\mu,|\nu|=0}^{1}\left\|x_j^\mu\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu v_\pm(t,\cdot)\right\|_{H^{1,\infty}}\left(\|(\Gamma v)_\pm(t,\cdot)\|_{H^1} + \|v_\pm(t,\cdot)\|_{H^1} + \|D_t v_\pm(t,\cdot)\|_{L^2}\right)\\ &\le C(A+B)B\varepsilon^2 t^{\delta_2}.\end{aligned} \tag{B.1.22}$$

Plugging the above inequality in (B.1.19b) for $\Gamma^I = \Gamma$, and using again a-priori estimates, we deduce that

$$\|x_j D_k(\Gamma u)_-(t,\cdot)\|_{L^2} \le CB\varepsilon t^{1+\frac{\delta_2}{2}},$$

and conclude the proof of (B.1.17) when $|I|=1$.

When $|I|=2$, we observe that, from (1.1.17),

$$\Gamma^I NL_{kg} = Q^{\mathrm{kg}}_0(v^I_\pm, D_1 u_\pm) + Q^{\mathrm{kg}}_0(v_\pm, D_1 u^I_\pm) + \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_1|=|I_2|=1}} Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D_1 u^{I_2}_\pm) + \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_1|+|I_2|<1}} c_{I_1,I_2} Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D u^{I_2}_\pm), \tag{B.1.23}$$

and

$$\Gamma^I NL_w = Q^{\mathrm{w}}_0(v^I_\pm, D_1 v_\pm) + Q^{\mathrm{w}}_0(v_\pm, D_1 v^I_\pm) + \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_1|=|I_2|=1}} Q^{\mathrm{w}}_0(v^{I_1}_\pm, D_1 v^{I_2}_\pm) + \sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\ |I_1|+|I_2|<2}} c_{I_1,I_2} Q^{\mathrm{w}}_0(v^{I_1}_\pm, D v^{I_2}_\pm),$$

with $c_{I_1,I_2}\in\{-1,0,1\}$. For the term indexed in I_1, I_2 such that $|I_1|=|I_2|=1$, we can use the Sobolev injection to write the following:

$$\left\| x_j Q^{\mathrm{kg}}_0(v^{I_1}_\pm, D_1 u^{I_2}_\pm)\right\|_{L^2} \lesssim \sum_{\mu=0}^1 \|v^{I_1}_\pm(t,\cdot)\|_{H^2}\|x^\mu_j D_1 u^{I_2}_\pm(t,\cdot)\|_{L^2}, \tag{B.1.24}$$

and then derive that

$$\begin{aligned}
\|x_j\Gamma^I NL_{kg}\|_{L^2} &\lesssim \sum_{\mu=0}^1 \|\mathrm{R}^\mu_1 u_\pm(t,\cdot)\|_{H^{2,\infty}} \sum_{\substack{|J|\le 2\\ \nu=0,1}} \|x^\nu_j(\Gamma^J v)_-(t,\cdot)\|_{L^2}\\
&+ \sum_{|\mu|=0}^1 \left\| x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\right\|_{L^\infty}\Big[\|u^I_\pm(t,\cdot)\|_{H^1} + \sum_{|J|<2}\big(\|u^J_\pm(t,\cdot)\|_{H^1} + \|D_t u^J_\pm(t,\cdot)\|_{L^2}\big)\Big]\\
&+ \sum_{\substack{|I_1|=|I_2|=1\\ \mu=0,1}} \|v^{I_1}_\pm(t,\cdot)\|_{H^2}\|x^\mu_j D_1 u^{I_2}_\pm(t,\cdot)\|_{L^2}.
\end{aligned}$$

As before, injecting the above inequality into (B.1.19a), using a-priori estimates (1.1.11) and the fact that $\varepsilon_0<(2A)^{-1}$, together with (B.1.10b), (B.1.5a) with $s=0$, (B.1.7), and (B.1.17) with $k=1$, we obtain that

$$\|x_j(\Gamma^I v)_-(t,\cdot)\|_{H^1} \le CB\varepsilon t^{1+\frac{\delta_1}{2}}. \tag{B.1.25}$$

Analogously, as

$$\begin{aligned}
\|x_j\Gamma^I NL_w\|_{L^2} &\lesssim \sum_{\substack{|J|\le 2\\ \mu=0,1}} \|x^\mu_j(\Gamma^J v)_\pm(t,\cdot)\|_{L^2}\|v_\pm(t,\cdot)\|_{H^{2,\infty}}\\
&+ \sum_{|\mu|=0}^1 \left\| x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu v_\pm(t,\cdot)\right\|_{H^{1,\infty}}\Bigg(\sum_{|J|\le 2}\|(\Gamma^J v)_\pm(t,\cdot)\|_{H^1} + \|v_\pm(t,\cdot)\|_{H^1} + \|D_t v_\pm(t,\cdot)\|_{L^2}\Bigg)\\
&+ \sum_{\substack{|I_1|=|I_2|=1\\ \mu=0,1}} \|(\Gamma^{I_1} v)_\pm(t,\cdot)\|_{H^2}\|x^\mu_j(\Gamma^{I_2} v)_\pm(t,\cdot)\|_{L^2},
\end{aligned}$$

from (B.1.6a) with $s = 0$, (B.1.10b), (B.1.17) with $|I| = 1$, (B.1.25) and a-priori estimates (1.1.11), we deduce

$$\|x_j D_k(\Gamma^I u)_-(t,\cdot)\|_{L^2} \lesssim CB\varepsilon t^{1+\frac{\delta_1}{2}},$$

and hence conclude the proof of inequality (B.1.17) also for the case $|I| = 2$. □

Corollary B.1.7. *There exists a positive constant $C > 0$ such that, under the same hypothesis as in lemma B.1.4, for any $\Gamma \in \mathcal{Z}$, with $\mathcal{Z}$ given by* (1.1.7),

$$\|x_j \Gamma NL_{kg}(t,\cdot)\|_{L^2} \le C(A+B)B\varepsilon^2 t^{\frac{1}{2}+\frac{\delta_2}{2}}, \tag{B.1.26a}$$

$$\|x_j \Gamma NL_{w}(t,\cdot)\|_{L^2} \le C(A+B)B\varepsilon^2 t^{\delta_2}, \tag{B.1.26b}$$

for every $t \in [1,T]$.

Proof. Estimate (B.1.26a) follows straightly from (B.1.21), (B.1.5a) with $s = 0$, and estimates (1.1.11), (B.1.10b), and (B.1.17) with $k = 1$, while (B.1.26b) has already been proved in (B.1.22). □

Lemma B.1.8. *There exists a constant $C > 0$ such that, under the same hypothesis as in lemma B.1.4,*

$$\|x_i x_j v_\pm(t,\cdot)\|_{L^2} \le CB\varepsilon t^{2+\frac{\delta_2}{2}}, \tag{B.1.27a}$$

$$\|x_i x_j v_\pm(t,\cdot)\|_{L^\infty} + \left\|x_j x_k \frac{D_x}{\langle D_x\rangle} v_\pm(t,\cdot)\right\|_{L^\infty} \le C(A+B)\varepsilon t^{1+\frac{\delta_2}{2}}. \tag{B.1.27b}$$

for every $i,j = 1,2$, every $t \in [1,T]$.

Moreover, for any $\Gamma \in \mathcal{Z}$, with $\mathcal{Z}$ given by (1.1.7),

$$\|x_i x_j (\Gamma v)_\pm(t,\cdot)\|_{L^2} \le CB\varepsilon t^{2+\frac{\delta_2}{2}}, \tag{B.1.28}$$

for every $t \in [1,T]$.

Proof. The proof of the statement follows from the fact that, by multiplying (B.1.9b) by x_i, making some commutations, and using that

$$\|x_i x_j NL_{kg}(t,\cdot)\|_{L^2} \lesssim \sum_{\mu_1,\mu_2=0}^{1} \|x_i^{\mu_1} x_j^{\mu_2} v_-(t,\cdot)\|_{L^2} \left(\|u_\pm(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 u_\pm(t,\cdot)\|_{H^{2,\infty}}\right),$$

together with

$$\begin{aligned}&\|x_i x_j NL_{kg}(t,\cdot)\|_{L^\infty} \\ &\quad\lesssim \left(\|x_i x_j v_-(t,\cdot)\|_{L^\infty} + \left\|x_j x_k \frac{D_x}{\langle D_x\rangle} v_-(t,\cdot)\right\|_{L^\infty}\right)\left(\|u_\pm(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 u_\pm(t,\cdot)\|_{H^{2,\infty}}\right),\end{aligned}$$

we derive that

$$\begin{aligned}\|x_i x_j v_-(t,\cdot)\|_{L^2} &\lesssim \sum_{\mu=0}^{1}\left(\|x_i^{\mu}(Z_j v)_-(t,\cdot)\|_{L^2} + t\|x_i^{\mu} v_-(t,\cdot)\|_{L^2}\right) \\ &+ \sum_{\mu_1,\mu_2=0}^{1} \|x_i^{\mu_1} x_j^{\mu_2} v_-(t,\cdot)\|_{L^2} \left(\|u_\pm(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 u_\pm(t,\cdot)\|_{H^{2,\infty}}\right),\end{aligned} \tag{B.1.29}$$

and

$$\|x_ix_jv_-(t,\cdot)\|_{L^\infty} \lesssim \sum_{\mu=0}^{1} (\|x_i^\mu (Z_jv)_-(t,\cdot)\|_{H^1} + t\|x_i^\mu v_-(t,\cdot)\|_{H^{1,\infty}})$$
$$+ \sum_{\mu=0}^{1}\left(\|x_i^\mu x_j v_-(t,\cdot)\|_{L^\infty} + \left\|x_i^\mu x_j \frac{D_x}{\langle D_x\rangle}v_-(t,\cdot)\right\|_{L^\infty}\right)(\|u_\pm(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 u_\pm(t,\cdot)\|_{H^{2,\infty}}),$$

(the above inequality holding true also for the uniform norm of $x_ix_jD_x\langle D_x\rangle^{-1}v_-$), obtained by using that operator $\langle D_x\rangle^{-1}$ is bounded from H^1 to L^∞. As $\varepsilon_0>0$ verifies that $\varepsilon_0<(2A)^{-1}$, inequality (B.1.10a), (B.1.17) with $\Gamma = Z_j$, and a-priori estimates (1.1.11) imply that

$$\sum_{j,k=1}^{2}\|x_ix_jv_-(t,\cdot)\|_{L^2} \lesssim CB\varepsilon t^{2+\frac{\delta_2}{2}},$$

while, from (B.1.10b), (B.1.17) with $k=1$ and a-priori estimates,

$$\|x_ix_jv_-(t,\cdot)\|_{L^\infty} + \left\|x_ix_j\frac{D_x}{\langle D_x\rangle}v_-(t,\cdot)\right\|_{L^\infty} \le C(A+B)\varepsilon t^{1+\frac{\delta_2}{2}}.$$

As $v_+ = -\overline{v_-}$, that implies the first part of the statement.

Analogously, using (B.1.9b) with $w=(\Gamma v)_-$, and multiplying that relation by x_i, we find that

(B.1.30)
$$\|x_ix_j(\Gamma v)_-(t,\cdot)\|_{L^2} \lesssim \sum_{\mu=0}^{1}[\|x_i^\mu Z_j(\Gamma v)_-(t,\cdot)\|_{L^2} + t\|x_i^\mu(\Gamma v)_-(t,\cdot)\|_{L^2} + \|x_i^\mu x_j\Gamma NL_{kg}(t,\cdot)\|_{L^2}],$$

where after (B.1.17), (B.1.26a) and a-priori estimates,

$$\sum_{\mu=0}^{1}[\|x_i^\mu Z_j(\Gamma v)_-(t,\cdot)\|_{L^2} + t\|x_i^\mu(\Gamma v)_-(t,\cdot)\|_{L^2}] + \|x_j\Gamma NL_{kg}(t,\cdot)\|_{L^2} \le CB\varepsilon t^{2+\frac{\delta_2}{2}}. \tag{B.1.31}$$

By multiplying both x_i, x_j against each Klein-Gordon factor in ΓNL_{kg}, given by (B.1.20a), we derive that

$$\|x_ix_j\Gamma NL_{kg}(t,\cdot)\|_{L^2} \lesssim \sum_{\mu_1,\mu_2\nu=0}^{1}\left\|x_i^{\mu_1}x_j^{\mu_2}(\Gamma v)_-(t,\cdot)\right\|_{L^2}\|\mathrm{R}_1^\nu u_\pm(t,\cdot)\|_{H^{2,\infty}}$$
$$+\sum_{\mu=0}^{1}\|x_ix_j(D_x\langle D_x\rangle^{-1})^\mu v_\pm(t,\cdot)\|_{L^\infty}(\|(\Gamma u)_\pm(t,\cdot)\|_{H^1} + \|u_\pm(t,\cdot)\|_{H^1} + \|D_tu_\pm(t,\cdot)\|_{L^2}),$$

so by (B.1.5a) with $s=0$, (B.1.17), (B.1.27b), a-priori estimates and the fact that $\varepsilon_0<(2A)^{-1}$,

$$\|x_ix_j\Gamma NL_{kg}(t,\cdot)\|_{L^2} \le \frac{1}{2}\|x_ix_j(\Gamma v)_-(t,\cdot)\|_{L^2} + C(A+B)B\varepsilon^2t^{1+\delta_2},$$

which injected in (B.1.30), together with (B.1.31), implies (B.1.28). ☐

Corollary B.1.9. *There exists a constant $C>0$ such that, under the same hypothesis as in lemma B.1.4,*

$$\|x_ix_jNL_{kg}(t,\cdot)\|_{L^2} + \|x_ix_jNL_w(t,\cdot)\|_{L^2} \le C(A+B)B\varepsilon^2t^{1+\frac{\delta+\delta_2}{2}}, \tag{B.1.32}$$

for every $i,j=1,2$, every $t\in[1,T]$.

Proof. Straightforward after (1.1.12c), (B.1.27b) and the following inequality

$$\|x_i x_j NL_{kg}(t,\cdot)\|_{L^2} + \|x_i x_j NL_w(t,\cdot)\|_{L^2} \lesssim \sum_{|\mu|=0}^{1} \Big\| x_i x_j \Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu} v_{\pm}(t,\cdot)\Big\|_{L^\infty} \left(\|u_{\pm}(t,\cdot)\|_{H^1} + \|v_{\pm}(t,\cdot)\|_{H^1}\right).$$

□

Corollary B.1.10. *There exists a constant $C>0$ such that, under the same hypothesis as in lemma B.1.4,*

$$\|x_i x_j x_k v_{\pm}(t,\cdot)\|_{L^2} \le CB\varepsilon t^{3+\frac{\delta_2}{2}}, \tag{B.1.33}$$

for any $i,j,k=1,2$, every $t\in[1,T]$.

Proof. The proof is a straight consequence of (B.1.9b), (B.1.10a), (B.1.27a), (B.1.17), (B.1.28), a-priori estimate and inequality

$$\begin{aligned}\|x_i x_j x_k v_-(t,\cdot)\|_{L^2} \lesssim & \sum_{\mu_1,\mu_2=0}^{1} \Big(\|x_i^{\mu_1} x_j^{\mu_2} (Z_k v)_-(t,\cdot)\|_{L^2} + t\|x_i^{\mu_1}x_j^{\mu_2} v_-(t,\cdot)\|_{L^2}\Big) \\ &+ \sum_{\mu_1,\mu_2,\mu_3=0}^{1} \|x_i^{\mu_1}x_j^{\mu_2}x_k^{\mu_3} v_-(t,\cdot)\|_{L^2}\left(\|u_{\pm}(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1 u_{\pm}(t,\cdot)\|_{H^{2,\infty}}\right),\end{aligned}$$

analogous to (B.1.29) with two factors x. □

B.2 First range of estimates

In this section, we show how the a-priori estimates (1.1.11) imply a moderate growth in time for the H^s norm of $\widetilde{u}^{\Sigma,k}$ introduced in (3.2.41), and for the L^2 norm of this function when some of the semi-classical operators $\Omega_h, \mathcal{M}$ act on it (see lemma B.2.1). We also show, with much more effort, that we have the same type of control for the H^s norm of $\widetilde{v}$ and the L^2 norm of $\mathcal{L}\widetilde{v}$ (see lemma B.2.14.

Lemma B.2.1. *Let $\widetilde{u}, \widetilde{u}^{\Sigma,k}$ be defined, respectively, in* (3.2.2) *and* (3.2.41)*, and $s\le n-15$. There exists a constant $C>0$ such that, under the same hypothesis as in lemma B.1.4, for any $\theta_0,\chi\in C_0^\infty(\mathbb{R}^2)$,*

$$\|\widetilde{u}(t,\cdot)\|_{H^s_h} + \|\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{H^s_h} \le CB\varepsilon t^{\frac{\delta}{2}+\kappa}, \tag{B.2.1a}$$

$$\|\Omega_h \widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2} \le CB\varepsilon t^{\frac{\delta_2}{2}+\kappa}, \tag{B.2.1b}$$

$$\sum_{|\mu|=1}\Big(\|Op_h^w(\chi(h^\sigma\xi))\mathcal{M}^\mu \widetilde{u}(t,\cdot)\|_{L^2} + \|\mathcal{M}^\mu \widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2}\Big) \le C(A+B)\varepsilon t^{\frac{\delta_2}{2}+\kappa}, \tag{B.2.1c}$$

$$\sum_{|\mu|=1} \|\theta_0(x)\Omega_h\mathcal{M}^\mu \widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2} \le CB\varepsilon t^{\frac{\delta_1}{2}+\kappa}, \tag{B.2.1d}$$

for every $t\in[1,T]$, with $\kappa=\sigma\rho$ if $\rho\ge 0$, 0 otherwise.

Proof. We warn the reader that, throughout the proof, C will denote a positive constant that may change line after line, and $\beta > 0$ is small as long as σ is small. We will also use the following concise notation, reminding that $h = t^{-1}$:

$$\psi_k(\xi) := \Sigma(\xi)(1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_0(h^\sigma\xi),$$

and observe that

$$\|Op_h^w(\psi_k(\xi))\|_{\mathcal{L}(L^2)} = O(h^{-\kappa}), \tag{B.2.2}$$

with $\kappa = \sigma\rho$ if $\rho \geq 0$, 0 otherwise.

It is straightforward to check that the H^s_h norm of $\widetilde{u}$ is bounded by energy $E_n(t;W)^{\frac{1}{2}}$, whenever $n \geq s+15$, after definitions (3.2.2), (3.1.15), inequality (3.1.20a), and a-priori estimate (1.1.11b). The same is true for $\widetilde{u}^{\Sigma,k}$ (up to a factor t^κ) after (B.2.2). A-priori estimate (1.1.11c) implies hence (B.2.1a).

We notice that the remaining estimates of the statement can be proven for $\widetilde{u}^{\Sigma,k}$ replaced with $\widetilde{u}$, by commutating $Op_h^w(\psi_k(\xi))$ with $\mathcal{M}$ (the commutator with Ω_h being zero if φ, χ_0 are supposed to be radial), and using (B.2.2). More precisely, we have that

$$\|\Omega_h\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2} \lesssim h^{-\kappa}\|Op_h^w(\chi_0(h^\sigma\xi))\Omega_h\widetilde{u}(t,\cdot)\|_{L^2},$$
$$\|\mathcal{M}\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2} \lesssim h^{-\kappa}\sum_{|\nu|=0}^{1}\|Op_h^w(\chi(h^\sigma\xi))\mathcal{M}^\nu\widetilde{u}(t,\cdot)\|_{L^2},$$

for a new smooth cut-off function χ, and

$$\|\theta_0(x)\Omega_h\mathcal{M}\widetilde{u}^{\Sigma,k}(t,\cdot)\|_{L^2} \lesssim \|\theta_0(x)Op_h^w(\psi_k(\xi))\Omega_h\mathcal{M}\widetilde{u}(t,\cdot)\|_{L^2} + h^{-\kappa}\sum_{\mu=0}^{1}\|Op_h^w(\chi(h^\sigma\xi))\Omega_h^\mu\widetilde{u}(t,\cdot)\|_{L^2}.$$

What we need to show is that, for any $\chi \in C_0^\infty(\mathbb{R}^2)$,

$$\|Op_h^w(\chi(h^\sigma\xi))\Omega_h\widetilde{u}(t,\cdot)\|_{L^2} \leq CB\varepsilon t^{\frac{\delta_2}{2}}, \tag{B.2.3a}$$
$$\|Op_h^w(\chi(h^\sigma\xi))\mathcal{M}\widetilde{u}(t,\cdot)\|_{L^2} \leq C(A+B)\varepsilon t^{\frac{\delta_2}{2}}, \tag{B.2.3b}$$
$$\|\theta_0(x)Op_h^w(\psi_k(\xi))\Omega_h\mathcal{M}\widetilde{u}(t,\cdot)\|_{L^2} \leq CB\varepsilon t^{\frac{\delta_1}{2}}. \tag{B.2.3c}$$

Estimate (B.2.3a) follows from (3.2.2), (3.1.15), inequality (A.31a) with $u = v = v_\pm$, and a-priori estimates (1.1.11), which give the following:

$$\begin{aligned}\|Op_h^w(\chi(h^\sigma\xi))\Omega_h\widetilde{u}(t,\cdot)\|_{L^2} &\lesssim \|(\Omega u)_-(t,\cdot)\|_{L^2} + \|\chi(t^{-\sigma}D_x)\Omega(u^{NF}-u_-)(t,\cdot)\|_{L^2}\\ &\lesssim \|\Omega U(t,\cdot)\|_{L^2} + t^\beta\left(\|V(t,\cdot)\|_{L^2} + \|\Omega V(t,\cdot)\|_{L^2}\right)\|V(t,\cdot)\|_{H^{17,\infty}}\\ &\leq C(1+A\varepsilon t^{-1+\beta})E_3^2(t;W)^{\frac{1}{2}} \leq CB\varepsilon t^{\frac{\delta_2}{2}},\end{aligned}$$

for every $t \in [1,T]$.

As concerns (B.2.3b), from relation (3.2.10a) and definition (3.1.15) of u^{NF}, we deduce that

(B.2.4)
$$\begin{aligned}&\|Op_h^w(\chi(h^\sigma\xi))\mathcal{M}_n\widetilde{u}(t,\cdot)\|_{L^2} \lesssim \|Z_nU(t,\cdot)\|_{L^2} + \|\chi(t^{-\sigma}D_x)Z_n(u^{NF}-u_-)(t,\cdot)\|_{L^2}\\ &+\|\widetilde{u}(t,\cdot)\|_{L^2} + \|Op_h^w(\chi(h^\sigma\xi))[t(tx_j)[q_w+c_w](t,tx)]\|_{L^2(dx)} + \|\chi(t^{-\sigma}D_x)(x_nr_w^{NF})(t,\cdot)\|_{L^2},\end{aligned}$$

with q_w, c_w and r_w^{NF} given, respectively, by (3.1.17), (3.1.18) and (3.1.19). We first notice that, after inequality (A.31b) with $u = v = v_\pm$, (B.1.6a) with $s = 0$, a-priori estimates, and the fact that $A\varepsilon_0 \le 1$,

$$\begin{aligned}(\text{B.2.5})\quad & \|\chi(t^{-\sigma}D_x)Z_n(u^{NF}-u_-)(t,\cdot)\|_{L^2} \lesssim \\ & \qquad t^{\beta}\left(\|D_tV(t,\cdot)\|_{L^2}\|V(t,\cdot)\|_{H^{13}} + \|V(t,\cdot)\|_{H^{15,\infty}}\|Z_nV(t,\cdot)\|_{L^2}\right) \le CB\varepsilon t^{\beta+\delta}.\end{aligned}$$

Let us also observe that, from (3.1.17), (3.1.18), we have that (3.2.68)

$$\begin{aligned} q_w(t,x)+c_w(t,x) &= \frac{1}{2}\Im\left[\overline{v_-}\,D_1v_- - \overline{\frac{D_x}{\langle D_x\rangle}v_-}\cdot\frac{D_xD_1}{\langle D_x\rangle}v_-\right](t,x) \\ &= \frac{h^2}{2}\Im\left[\overline{\widetilde{V}}\,Op_h^w(\xi_1)\widetilde{V} - \overline{Op_h^w\Big(\frac{\xi_1}{\langle\xi\rangle}\Big)\widetilde{V}}\cdot Op_h^w\Big(\frac{\xi\xi_1}{\langle\xi\rangle}\Big)\widetilde{V}\right]\Big(t,\frac{x}{t}\Big),\end{aligned}\tag{B.2.6}$$

having introduced $\widetilde{V}(t,x) := tv_-(t,tx)$, which is such that, for every $s,\rho\ge 0$,

$$\|\widetilde{V}(t,\cdot)\|_{H^s_h} = \|v_-(t,\cdot)\|_{H^s},\quad \|\widetilde{V}(t,\cdot)\|_{H^{\rho,\infty}_h} = t\|v_-(t,\cdot)\|_{H^{\rho,\infty}},$$

and

$$\begin{aligned}\|\mathcal{L}_j\widetilde{V}(t,\cdot)\|_{H^1_h} &\lesssim \|Z_jv_-(t,\cdot)\|_{L^2} + \|v_-(t,\cdot)\|_{L^2} \\ &\quad + \left(\|x_jv_\pm(t,\cdot)\|_{L^\infty} + \Big\|x_j\frac{D_x}{\langle D_x\rangle}v_\pm(t,\cdot)\Big\|_{L^\infty}\right)\|U(t,\cdot)\|_{H^1},\end{aligned}\tag{B.2.7}$$

as follows by (3.2.8) with $w = v_-$ and $f = NL_{kg}$. Using (B.2.6) along with the definition of $\mathcal{L}_j$ in (1.2.60), we derive that

$$\begin{aligned} t(tx_j)[q_w+c_w](t,tx) &= \frac{1}{2}\Im\Big[\overline{\widetilde{V}}Op_h^w(\xi_1)(h\mathcal{L}_j\widetilde{V}) + \overline{\widetilde{V}}Op_h^w\Big(\frac{\xi_1\xi_j}{\langle\xi\rangle}\Big)\widetilde{V} + \overline{\widetilde{V}}[x_j,Op_h^w(\xi_1)]\widetilde{V} \\ & - \overline{Op_h^w\Big(\frac{\xi}{\langle\xi\rangle}\Big)\widetilde{V}}\cdot Op_h^w\Big(\frac{\xi\xi_1}{\langle\xi\rangle}\Big)(h\mathcal{L}_j\widetilde{V}) - \overline{Op_h^w\Big(\frac{\xi}{\langle\xi\rangle}\Big)\widetilde{V}}\cdot Op_h^w\Big(\frac{\xi\xi_1\xi_j}{\langle\xi\rangle^2}\Big)\widetilde{V} \\ & - \overline{Op_h^w\Big(\frac{\xi}{\langle\xi\rangle}\Big)\widetilde{V}}\cdot\Big[x_j,Op_h^w\Big(\frac{\xi\xi_1}{\langle\xi\rangle}\Big)\Big]\widetilde{V}\Big](t,x),\end{aligned}\tag{B.2.8}$$

so after a-priori estimates (1.1.11) and (B.1.10b),

$$\begin{aligned}\|Op_h^w(\chi(h^\sigma\xi))[t(tx_j)[q_w+c_w](t,\cdot)]\|_{L^2} &\lesssim \left[\|\widetilde{V}(t,\cdot)\|_{H^1_h} + h\|\mathcal{L}_j\widetilde{V}(t,\cdot)\|_{H^1_h}\right]\|\widetilde{V}(t,\cdot)\|_{H^{1,\infty}_h} \\ &\le CA(A+B)\varepsilon^2t^{\frac{\delta}{2}}.\end{aligned}\tag{B.2.9}$$

Moreover, from (3.1.19), the fact that $x_je^{ix\cdot\xi} = D_{\xi_j}e^{ix\cdot\xi}$, integration by parts, and inequalities (A.20a) with $\rho = 2$ (after the first part of lemma A.8), (A.30a),

(B.2.10)

$$\begin{aligned}&\|\chi(t^{-\sigma}D_x)(x_nr_w^{NF})(t,\cdot)\|_{L^2} \\ &\quad\lesssim t^{\beta}\left[\|x_nv_-(t,\cdot)\|_{L^\infty}\|NL_{kg}(t,\cdot)\|_{H^{15}} + \|V(t,\cdot)\|_{H^{15}}\|x_nNL_{kg}(t,\cdot)\|_{L^\infty}\right. \\ &\quad\left. + \|NL_{kg}(t,\cdot)\|_{L^2}\left(\|V(t,\cdot)\|_{H^{13}} + \|V(t,\cdot)\|_{H^{13,\infty}}\right) + \|V(t,\cdot)\|_{H^{13}}\|NL_{kg}(t,\cdot)\|_{L^\infty}\right] \\ &\quad\le CB\varepsilon t^{\frac{\delta_2}{2}},\end{aligned}$$

last estimate following from (B.1.10b), (B.1.13b), inequalities (B.1.4a), (B.1.4b), (B.1.4c) with $s = 15$, and a-priori estimates (1.1.11). Consequently, from (B.2.4), (B.2.5), (B.2.9), (B.2.10), (B.2.1a) and a-priori estimate (1.1.11d) with $k=2$, along with the fact that $A\varepsilon_0\le 1$, we get (B.2.3b).

In order to prove (B.2.3c), we apply $\theta_0\big(\frac{x}{t}\big)\psi_k(D_x)\Omega$ to both sides of (3.2.10a). We find that

(B.2.11)
$$\begin{aligned}
&\|\theta_0(x)Op^w_h(\psi_k(\xi))\Omega_h\mathcal{M}_n\widetilde{u}(t,\cdot)\|_{L^2}\lesssim \|\Omega Z_nU(t,\cdot)\|_{L^2}\\
&+\Big\|\theta_0\Big(\frac{x}{t}\Big)\psi_k(D_x)\Omega Z_n(u^{NF}-u_-)(t,\cdot)\Big\|_{L^2}+\sum_{\mu=0}^{1}\|Op^w_h(\chi_0(h^\sigma\xi)\Omega_h^\mu\widetilde{u}(t,\cdot)\|_{L^2}\\
&+\|\theta_0(x)Op^w_h(\psi_k(\xi))\Omega_h[t(tx_j)(q_w+c_w)(t,tx)]\|_{L^2}+\big\|\theta_0(x)Op^w_h(\psi_k(\xi))\Omega_h[t(tx_n)r^{NF}_w](t,tx)]\big\|_{L^2(dx)}.
\end{aligned}$$

The first norm in above right hand side is bounded by $E_3^1(t;W)^{\frac{1}{2}}$, while the third one is estimated by (B.2.1a), (B.2.3a). In order to estimate the second one, we first commute Z_n to Ω ($[\Omega,Z_1]=-Z_2$ and $[\Omega,Z_2]=Z_1$), and use that

$$\theta_0\Big(\frac{x}{t}\Big)\psi_k(D_x)Z_j=\Big[t\theta_0^j\Big(\frac{x}{t}\Big)\psi_k(D_x)+\theta_0\Big(\frac{x}{t}\Big)[\psi_k(D_x),x_j]\Big]\partial_t+t\theta_0\Big(\frac{x}{t}\Big)\psi_k(D_x)\partial_j,$$

with $\theta_0^j(z):=\theta_0(z)z_j$, and commutator $[\psi_k(D_x),x_j]$ being bounded on L^2, with norm $O(t)$, and symbol still supported for moderate frequencies $|\xi|\lesssim t^{-\sigma}$. Therefore,

$$\begin{aligned}
\Big\|\theta_0\Big(\frac{x}{t}\Big)\psi_k(D_x)\Omega Z_n(u^{NF}-u_-)(t,\cdot)\Big\|_{L^2}&\lesssim t\,\big\|\chi(t^{-\sigma}D_x)\partial_{t,x}(u^{NF}-u_-)(t,\cdot)\big\|_{L^2}\\
&\quad+t\,\big\|\chi(t^{-\sigma}D_x)\partial_{t,x}\Omega(u^{NF}-u_-)(t,\cdot)\big\|_{L^2},
\end{aligned}$$

for a new $\chi\in C_0^\infty(\mathbb{R}^2)$, so using (A.20a) with $\rho=2$ (because of first part of lemma A.8), and (A.31a), both considered with $u=\partial_{t,x}v_\pm, v=v_\pm$, and $u=v_\pm, v=\partial_{t,x}v_\pm$, we obtain that the above right hand side is estimated by

$$\begin{aligned}
t^{1+\beta}\,&[(\|\partial_{t,x}V(t,\cdot)\|_{L^2}+\|\Omega\partial_{t,x}V(t,\cdot)\|_{L^2})\,\|V(t,\cdot)\|_{H^{17,\infty}}\\
&+(\|V(t,\cdot)\|_{L^2}+\|\Omega V(t,\cdot)\|_{L^2})\,\|\partial_{t,x}V(t,\cdot)\|_{H^{17,\infty}}].
\end{aligned}$$

From (B.1.6a) and (B.1.6b) with $s=0$, along with (B.1.6c) and a-priori estimates, we deduce that

$$\Big\|\theta_0\Big(\frac{x}{t}\Big)\psi_k(D_x)\Omega Z_n(u^{NF}-u_-)(t,\cdot)\Big\|_{L^2}\le CB\varepsilon t^{\beta+\frac{\delta_2}{2}}. \tag{B.2.12}$$

As concerns, instead, the estimate of the fourth L^2 norm in the right hand side of (B.2.11), we recall (B.2.8) and apply the Leibniz rule, obtaining, from the uniform continuity of operator $\theta_0(x)Op^w_h(\psi_k(\xi))$ on L^2, that

$$\begin{aligned}
\|\theta_0(x)Op^w_h(\psi_k(\xi))\Omega_h[t(tx_j)[q_w+c_w](t,tx)]\|_{L^2}&\lesssim\sum_{\mu=0}^{1}\|\widetilde{V}(t,\cdot)\|_{H^{2,\infty}_h}\|\Omega_h^\mu\widetilde{V}(t,\cdot)\|_{H^1_h}\\
&+\sum_{\mu=0}^{1}h\|\widetilde{V}(t,\cdot)\|_{H^{1,\infty}_h}\|\Omega^\mu_h\mathcal{L}_j\widetilde{V}(t,\cdot)\|_{H^1}+h\|\Omega_h\widetilde{V}(t,\cdot)\|_{L^\infty}\|\mathcal{L}_j\widetilde{V}(t,\cdot)\|_{H^1_h}.
\end{aligned}\tag{B.2.13}$$

We immediately observe that, from the semi-classical Sobolev injection, (B.2.7), (B.1.10b), the fact that $\|\Omega_h\widetilde{V}(t,\cdot)\|_{H^s_h}=\|\Omega v_-(t,\cdot)\|_{H^s}$ for any $s\ge 0$, and a-priori estimates,

$$h\|\Omega_h\widetilde{V}(t,\cdot)\|_{L^\infty}\|\mathcal{L}_j\widetilde{V}(t,\cdot)\|_{H^1_h}\lesssim\|\Omega\widetilde{V}(t,\cdot)\|_{H^2_h}\|\mathcal{L}_j\widetilde{V}(t,\cdot)\|_{H^1_h}\le CB\varepsilon t^{\frac{3\delta_2}{2}}. \tag{B.2.14}$$

Again from (3.2.8) with $w = v_-$ and $f = NL_{kg}$, we find that

$$\|\Omega_h \mathcal{L}_j \widetilde{V}(t,\cdot)\|_{L^2} \lesssim \|\Omega Z_j v_-(t,\cdot)\|_{L^2} + \sum_{\mu=0}^{1} \|\Omega^\mu v_-(t,\cdot)\|_{L^2} + \|\Omega\left(x_j NL_{kg}\right)\|_{L^2},$$

where, after making a commutation between Ω and x_j, and using (B.1.15a), (B.1.26a) with $\Gamma = \Omega$,

$$\|\Omega\left(x_j NL_{kg}\right)\|_{L^2} \le C(A+B)B\varepsilon^2 t^{\frac{1}{2}+\frac{\delta_2}{2}}.$$

Therefore,

$$h\|\widetilde{V}(t,\cdot)\|_{H^{1,\infty}_h} \|\Omega_h \mathcal{L}_j \widetilde{V}(t,\cdot)\|_{L^2} \le CAB(A+B)\varepsilon^3 t^{-\frac{1}{2}+\frac{\delta_2}{2}},$$

which combined with (B.2.13), (B.2.14) and a-priori estimates, gives that

$$\|\theta_0(x) Op^w_h(\psi_k(\xi))\Omega_h[t(tx_j)[q_w + c_w](t,tx)]\|_{L^2} \le CB\varepsilon t^{\frac{3\delta_2}{2}}. \tag{B.2.15}$$

We estimate the latter L^2 norm in (B.2.11) recalling definition (3.1.19) of r^{NF}_w, commutating Ω and x_n, and using that

$$\theta_0(x) Op^w_h(\psi_k(\xi)) x_n = \theta^n_0(x) Op^w_h(\psi_k(\xi)) + \theta_0(x)[Op^w_h(\psi_k(\xi)), x_n],$$

where

$$[Op^w_h(\psi_k(\xi)), x_n] = -ih Op^w_h(\partial_n \psi_k(\xi))$$

is uniformly bounded on L^2. We derive that, for some $\chi \in C^\infty_0(\mathbb{R}^2)$,

$$\left\|\theta_0(x) Op^w_h(\psi_k(\xi))\Omega_h[t(tx_n) r^{NF}_w](t,tx)]\right\|_{L^2(dx)} \lesssim \sum_{\mu=0}^{1} t\|\chi(t^{-\sigma}D_x)\Omega^\mu r^{NF}_w(t,\cdot)\|_{L^2} \le CB\varepsilon$$

after (3.1.22a), (3.1.22c) with $\theta \ll 1$ small, and a-priori estimates (1.1.11). Combining (B.2.11), (B.2.12), (B.2.15) and above estimate, and assuming $3\delta_2 \le \delta_1$, we finally obtain (B.2.3c) and the conclusion of the proof. ☐

In the following lemma we explain how we estimate the L^2 or the L^∞ norm of products supported for moderate frequencies $|\xi| \lesssim t^\sigma$, when we have a control on some high Sobolev norm of, at least, all factors but one. This type of estimate will be frequently used in most of the results that follow.

Lemma B.2.2. *Let $n \in \mathbb{N}$, $n \ge 2$, and $w_1, \dots, w_n$ such that $w_1 \in L^2(\mathbb{R}^2)$, $w_2, \dots, w_n \in L^\infty(\mathbb{R}^2) \cap H^s(\mathbb{R}^2)$, for some large positive s. Let also $\chi \in C^\infty_0(\mathbb{R}^2)$. There exists some $\chi_1 \in C^\infty_0(\mathbb{R}^2)$, equal to 1 on the support of χ, such that*

$$\left\|\chi(t^{-\sigma}D_x)\left[w_1 \dots w_n\right]\right\|_L \lesssim \left\|\left[\chi_1(t^{-\sigma}D_x) w_1\right] \prod_{j=2}^{n} \chi(t^{-\sigma}D_x) w_j\right\|_L + t^{-N(s)}\|w_1\|_{L^2} \sum_{j=2}^{n} \prod_{k\neq j} \|w_k\|_{L^\infty}\|w_j\|_{H^s},$$

with $L = L^2$ or $L = L^\infty$, and $N(s)$ as large as we want as long as $s > 0$ is large.

Proof. The idea of the proof is to decompose each factor w_j, for $j=2,\dots,n$ into

$$\chi(t^{-\sigma}D_x)w_j + (1-\chi)(t^{-\sigma}D_x)w_j,$$

and to estimate the L^2 norm of product

$$\chi(t^{-\sigma}D_x)\left[w_1\prod_{\substack{k=2,\dots,n\\ k\neq j}}\widetilde{w}_k\left[(1-\chi)(t^{-\sigma}D_x)w_j\right]\right], \tag{B.2.16}$$

where $\widetilde{w}_k$ is either w_k or $\chi(t^{-\sigma}D_x)w_k$, with the L^2 norm of w_1 times the L^∞ norm of all remaining factors, reminding that $\chi(t^{-\sigma}D_x)$ is uniformly bounded on L^∞, and that by Sobolev injection and (B.1.2),

$$\left\|(1-\chi)(t^{-\sigma}D_x)w_j\right\|_{L^\infty}\lesssim t^{-N(s)}\|w_j\|_{H^s}, \tag{B.2.17}$$

with $N(s)$ as large as we want as long as $s>0$ is large. The L^∞ norm of (B.2.16) is estimated in the same way, using the L^2-L^∞ continuity of operator $\chi(t^{-\sigma}D_x)$ acting on the entire product.

Then, when all factors w_j, for $j=2,\dots,n$, are truncated for frequencies less or equal than t^σ, the fact that the entire product is also restricted to this range of frequencies infers the same localization also for w_1, and that concludes the result of the lemma. ☐

Lemma B.2.3. *Let $n\in\mathbb{N}^*$ and some functions $w_1,\dots,w_n$ be given, $w\in\{u,v\}$, $\chi\in C_0^\infty(\mathbb{R}^2)$ and $\sigma>0$. Let also $I=(i_1,\dots,i_p)$ be such that $\Gamma^I=\Gamma_{i_1}\cdots\Gamma_{i_p}$ is a family of Klainerman vector fields, i.e $\Gamma_{i_j}\in\{\Omega,Z_m,m=1,2\}$, for all $j=1,\dots,p$. Then, for $|\mu|=0,1$,*

$$\begin{aligned}\left\|D^\mu_x(\Gamma^I w)_\pm w_1\dots w_n\right\|_L &\lesssim \left\|\left[\chi(t^{-\sigma}D_x)D^\mu_x(\Gamma^I w)_\pm\right]\prod_{j=1}^n w_j\right\|_L\\ &+t^{-N(s)}\Big(\sum_{a+|\alpha|=1}^p\|\partial^\alpha_x\partial^a_t w_\pm\|_{H^s}\Big)\Big(\sum_{b+|\beta|=0}^p\|t^bx^\beta w_1\|_{L^2}\Big)\prod_{j=2}^n\|w_j\|_{L^\infty}\end{aligned} \tag{B.2.18}$$

for $L=L^2$ or $L=L^\infty$, where $N(s)\in\mathbb{N}$ is as large as we want as long as $s>0$ is large, and where the second product in the above right hand side has to be meant equal to 1 if $n=2$.

Proof. The idea behind the result of the statement is to truncate factor $D^\mu_x(\Gamma^I w)_\pm$ in frequencies by writing

$$D^\mu_x(\Gamma^I w)_\pm=\chi(t^{-\sigma}D_x)D^\mu_x(\Gamma^I w)_\pm+(1-\chi)(t^{-\sigma}D_x)D^\mu_x(\Gamma^I w)_\pm.$$

When $L=L^\infty$ and $D^\mu_x(\Gamma^I w)_\pm$ is supported for large frequencies $|\xi|\gtrsim t^\sigma$, we first use the L^2-L^∞ continuity of operator $\chi(t^{-\sigma}D_x)$ acting on the entire product, with norm $O(t^\sigma)$, to bring us to estimate the L^2 norm of that product (up to a factor t^σ). Then, when dealing with

$$\left[(1-\chi)(t^{-\sigma}D_x)D^\mu_x(\Gamma^I w)_\pm\right]w_1\dots w_n,$$

we first commute Γ^I with $D^\mu_x(D_t\pm|D_x|)$ if $w=u$ (resp. with $D^\mu_x(D_t\pm\langle D_x\rangle)$ if $w=v$), and then write each below Γ^J as a linear combination of derivations $t^bx^\beta\partial^a_t\partial^\alpha_x$, for $a+|\alpha|=|J|$, $b+|\beta|\le|J|$, so that

$$D^\mu_x(\Gamma^I w)_\pm=\sum_{|J|\le|I|}{}'\,\Gamma^J D^\mu w_\pm=\sum_{\substack{a+|\alpha|\le|I|\\ b+|\beta|\le|I|}}{}'\,t^bx^\beta\partial^a_t\partial^\alpha_x D^\mu w_\pm,$$

$\sum'$ being a concise notation to indicate a linear combination. Up to a commutation with operator $(1-\chi)(t^{-\sigma}D_x)$, all these factors t^bx^β can be discharged, say, on w_1. Finally, we bound the L^2 norm of this product by taking the L^2 norm of $t^bx^\beta w_1$, the L^∞ norm of the remaining factors, and using the classical Sobolev injection together with inequality (B.1.2) to control the L^∞ norm of $\partial_t^b\partial_x^\beta D^\mu w_\pm$, obtaining the second contribution to the right hand side of (B.2.18). ☐

Corollary B.2.4. *Under the same hypothesis as in lemma B.2.3,*

$$\|D^\mu_x(\Omega w)_\pm w_1\dots w_n\|_L \lesssim \Big\|\big[\chi(t^{-\sigma}D_x)D^\mu_x(\Omega w)_\pm\big]\prod_{j=1}^n w_j\Big\|_L + t^{-N(s)}\|w_\pm(t,\cdot)\|_{H^s}\Big(\sum_{|\mu|=0}^1\|x^\mu w_1\|_{L^2}\Big)\prod_{j=2}^n\|w_j\|_{L^\infty}, \tag{B.2.19}$$

and, for $m=1,2$,

$$\|D^\mu_x(Z_m w)_\pm w_1\dots w_n\|_L \lesssim \Big\|\big[\chi(t^{-\sigma}D_x)D^\mu_x(Z_m w)_\pm\big]\prod_{j=1}^n w_j\Big\|_L + t^{-N(s)}\left(\|w_\pm(t,\cdot)\|_{H^s}+\|D_tw_\pm(t,\cdot)\|_{L^2}\right)\Big(\sum_{\mu=0}^1\|x^\mu_m w_1\|_{L^2}+t\|w_1\|_{L^2}\Big)\prod_{j=2}^n\|w_j\|_{L^\infty}. \tag{B.2.20}$$

Remark B.2.5. The same decomposition in frequencies made on $D^\mu_x(\Gamma^I w)_\pm$ in the proof of the lemma B.2.3, can be eventually repeated for the remaining factors $w_1,\dots,w_n$, obtaining that

$$\begin{aligned}\big\|D^\mu_x(\Gamma^I w)_\pm w_1\dots w_n\big\|_L &\lesssim \Big\|\big[\chi(t^{-\sigma}D_x)D^\mu_x(\Gamma^I w)_\pm\big]\prod_{j=1}^n[\chi(t^{-\sigma}D_x)w_j]\Big\|_L \\ &+ t^{-N(s)}\Big(\sum_{a+|\alpha|=1}^p\|\partial^\alpha_x\partial^a_t w_\pm\|_{H^s}\Big)\Big(\sum_{b+|\beta|=0}^p\|t^bx^\beta w_1\|_{L^2}\Big)\prod_{j=2}^n\|w_j\|_{L^\infty} \\ &+ t^{-N(s)}\|D^\mu_x(\Gamma^I w)_\pm\|_{L^2}\sum_{j=1}^n\prod_{k\neq j}\|w_k\|_{L^\infty}\|w_j\|_{H^s}.\end{aligned} \tag{B.2.21}$$

Moreover, if product $D^\mu_x(\Gamma^I w)_\pm w_1\dots w_n$ is truncated for frequencies $|\xi|\lesssim t^\sigma$, we can choose a $j_0\in\{2,\dots,n\}$ and restrict the last sum in the right hand side of (B.2.21) to the set of indices $j\in\{2,\dots,n\}$ such that $j\neq j_0$. This is due to the fact that a product such as $\chi(t^{-\sigma}D_x)\left[w_1\prod_{j=2}^n\chi_1(t^{-\sigma}D_x)w_j\right]$, with $\chi,\chi_1\in C^\infty_0(\mathbb{R}^2)$, vanishes if w_1 is supported for large frequencies $|\xi|\gg t^\sigma$, which means that there exists some $\chi_2\in C^\infty_0(\mathbb{R}^2)$ such that

$$\chi(t^{-\sigma}D_x)\left[w_1\prod_{j=2}^n\chi_1(t^{-\sigma}D_x)w_j\right] = \chi(t^{-\sigma}D_x)\left[\chi_2(t^{-\sigma}D_x)w_1\prod_{j=2}^n\chi_1(t^{-\sigma}D_x)w_j.\right]$$

This reasoning allows to avoid to consider the H^s norm, for large s, of some factor w_{j_0} for which we could not have such a control.

Lemma B.2.6. *For any $\chi\in C^\infty_0(\mathbb{R}^2)$, any $\sigma>0$ small, if $\widetilde{w}(t,x):=tw(t,tx)$ then*

$$\big\|\chi(t^{-\sigma}D_x)w(t,\cdot)\big\|_{L^\infty}\lesssim t^{-1+\beta}\sum_{|\mu|=0}^1\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{w}(t,\cdot)\|_{L^2}\,, \tag{B.2.22}$$

with $\beta>0$ small, $\beta\to0$ as $\sigma\to0$.

Proof. Since

$$\chi(t^{-\sigma}D_x)w(t,y) = t^{-1}Op_h^w(\chi(h^\sigma\xi))\widetilde{w}(t,x)|_{x=\frac{y}{t}},$$

the goal is to prove that

$$\|Op_h^w(\chi(h^\sigma\xi))\widetilde{w}(t,\cdot)\|_{L^\infty} \lesssim h^{-\beta}\sum_{|\mu|=0}^{1}\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{w}(t,\cdot)\|_{L^2}, \tag{B.2.23}$$

with $h = 1/t$, for a small $\beta > 0$, $\beta \to 0$ as $\sigma \to 0$.

Let $w^\chi := Op_h^w(\chi(h^\sigma\xi))\widetilde{w}$ and $\chi_1 \in C_0^\infty(\mathbb{R}^2)$, equal to 1 on the support of χ, so that

$$Op_h^w(\chi(h^\sigma\xi))\widetilde{w} = Op_h^w(\chi_1(h^\sigma\xi))\widetilde{w}^\chi.$$

For a $\gamma \in C_0^\infty(\mathbb{R}^2)$, equal to 1 in a neighbourhood of the origin, and with sufficiently small support, we consider the following decomposition:

$$Op_h^w\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi + Op_h^w\Big((1-\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi,$$

and immediately observe that, from inequality (3.2.18b),

$$\Big\|Op_h^w\Big((1-\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi(t,\cdot)\Big\|_{L^\infty} \lesssim h^{-\beta}\sum_{|\mu|=0}^{1}\|Op_h^w(\chi_1(h^\sigma\xi))\mathcal{L}^\mu\widetilde{w}^\chi(t,\cdot)\|_{L^2}.$$

As we remarked at the beginning of subsection 3.2.1 (see (3.2.15)), there exists a family of smooth functions $\theta_h(x)$, equal to 1 for $|x|\le 1-ch^{2\sigma}$ and supported for $|x|\le 1-c_1h^{2\sigma}$, for some $0<c_1<c$, with $\|\partial^\alpha\theta_h\|_{L^\infty} = O(h^{-2|\alpha|\sigma})$ and $(h\partial_h)^k\theta_h$ bounded for every $k\in\mathbb{N}$, such that

$$\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi) = \theta_h(x)\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi),$$

and by symbolic calculus and remark 1.2.22,

$$Op_h^w\Big(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi = \theta_h(x)Op_h^w\Big(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi + Op_h^w(r_\infty(x,\xi))\widetilde{w}^\chi,$$

with $r_\infty \in h^N S_{\frac{1}{2},\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-1})$, $N\in\mathbb{N}$ as large as we want. It is enough to take $N=1$ to have that $\|Op_h^w(r_\infty)\widetilde{w}^\chi(t,\cdot)\|_{L^\infty} \le h^{-\beta}\|\widetilde{w}^\chi(t,\cdot)\|_{L^2}$ by proposition 1.2.37. We can also replace $Op_h^w\big(\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi_1(h^\sigma\xi)\big)$ with $Op_h^w(\chi_2(h^{\sigma_1}\xi))Op_h^w\big(\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi_1(h^\sigma\xi)\big)$, for a new cut-off $\chi_2\in C_0^\infty(\mathbb{R}^2)$ equal to 1 on the support of χ_1, and a new small $\sigma_1>\sigma$, modulo an operator of the form $Op_h^w(r_\infty)$. As function $\phi(x) := \sqrt{1-|x|^2}$ is well defined on the support of θ_h, we are allowed to to write the following:

$$\begin{aligned}
&\Big\|\theta_h(x)Op_h^w(\chi_2(h^{\sigma_1}\xi))Op_h^w\Big(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi(t,\cdot)\Big\|_{L^\infty}\\
&= \Big\|e^{\frac{i}{h}\phi}\theta_h(x)Op_h^w(\chi_2(h^{\sigma_1}\xi))Op_h^w\Big(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi(t,\cdot)\Big\|_{L^\infty}\\
&\lesssim \Big\|Op_h^w(\chi_2(h^{\sigma_1}\xi))\Big[e^{\frac{i}{h}\phi}\theta_h(x)Op_h^w\Big(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi\Big]\Big\|_{L^\infty} + \|Op_h^w(r_\infty)\widetilde{w}^\chi(t,\cdot)\|_{L^\infty},
\end{aligned}$$

for a new $r_\infty \in h^N S_{\frac{1}{2},\sigma}\big(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-1}\big)$. This latter r_∞ comes out from the commutation between $e^{\frac{i}{h}\phi}\theta_h(x)$ and $Op_h^w(\chi_2(h^{\sigma_1}\xi))$, whose symbol is computed using (1.2.18) until a large enough order M. We notice that, as $\sigma_1 > \sigma$, at each order of the mentioned asymptotic development we gain a factor $h^{|\alpha|(\sigma_1-\sigma)}$; moreover, those terms write in terms of derivatives of χ_2, and hence vanish on the support of χ_1. By proposition 1.2.21 and remark 1.2.22, we then deduce that the composition of the mentioned commutator with $Op_h^w\big(\gamma\big(\frac{x-p'(\xi)}{\sqrt{h}}\big)\chi_1(h^\sigma\xi)\big)$ is an operator of symbol r_∞, with N as large as we want.

By classical Sobolev injection and symbolic calculus, we find that

$$\begin{aligned}
&\left\|Op_h^w(\chi_2(h^{\sigma_1}\xi))\left[e^{\frac{i}{h}\phi}\theta_h(x)Op_h^w\Big(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi\right]\right\|_{L^\infty}\\
&\lesssim |\log h|\left[\|\widetilde{w}^\chi(t,\cdot)\|_{L^2}+\sum_{j=1}^2\left\|D_j\left[e^{\frac{i}{h}\phi}\theta_h(x)Op_h^w\Big(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi\right]\right\|_{L^2}\right]\\
&\lesssim |\log h|\left[\|\widetilde{w}^\chi(t,\cdot)\|_{L^2}+\sum_{j=1}^2 h^{-1}\left\|Op_h^w\big((\xi_j+d_j\phi(x))\theta_h(x)\big)Op_h^w\Big(\gamma\left(\frac{x-p'(\xi)}{\sqrt{h}}\right)\chi_1(h^\sigma\xi)\Big)\widetilde{w}^\chi\right\|_{L^2}\right]\\
&\lesssim |\log h|\left[\|\widetilde{w}^\chi(t,\cdot)\|_{L^2}+h^{-\beta}\sum_{|\mu|=0}^1\|Op_h^w(\chi_1(h^\sigma\xi))\mathcal{L}^\mu\widetilde{w}^\chi(t,\cdot)\|_{L^2}\right],
\end{aligned}$$

last inequality following from lemma 3.2.16.

Finally, commutating $\mathcal{L}$ with $Op_h^w(\chi(h^\sigma\xi))$ defining $\widetilde{w}^\chi$, and reminding that $\chi_1\equiv 1$ on the support of χ, we obtain

$$\|Op_h^w(\chi(h^\sigma\xi))\widetilde{w}^\chi(t,\cdot)\|_{L^\infty}\lesssim h^{-\beta}\sum_{|\mu|=0}^1\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{w}(t,\cdot)\|_{L^2},$$

for every $t\in[1,T]$, and hence (B.2.23). □

Lemma B.2.7. *Let I be a multi-index of length j, with $j=1,2$, and*

(B.2.24)
$$v^{I,NF}(t,x):=(\Gamma^I v)_-(t,x)-\frac{i}{4(2\pi)^2}\sum_{j_1,j_2\in\{+,-\}}\int e^{ix\cdot\xi}B^1_{(j_1,j_2,+)}(\xi,\eta)\widehat{v^I_{j_1}}(\xi-\eta)\hat{u}_{j_2}(\eta)d\xi d\eta,$$

with $B^1_{(j_1,j_2,+)}$ given by (2.2.45) with $j_3=+$ and $k=1$. Then there exists a constant $C>0$ such that, if a-priori estimates (1.1.11) are satisfied in some interval $[1,T]$, for a fixed $T>1$, with $\varepsilon_0<(2A+B)^{-1}$ small, for any $\chi\in C_0^\infty(\mathbb{R}^2)$, $\sigma>0$ small,

$$\left\|\chi(t^{-\sigma}D_x)\left(v^{I,NF}-(\Gamma^I v)_-\right)(t,\cdot)\right\|_{L^\infty}\le\frac{1}{2}\left\|\chi(t^{-\sigma}D_x)(\Gamma^I v)_-(t,\cdot)\right\|_{L^\infty}+CB\varepsilon t^{-1},\tag{B.2.25}$$

for every $t\in[1,T]$. Moreover,

$$\left\|\chi(t^{-\sigma}D_x)Z_m\left(v^{I,NF}-(\Gamma^I v)_-\right)(t,\cdot)\right\|_{L^2}\le C(A+B)B\varepsilon^2t^{2\sigma+\frac{\delta_{3-j}+\delta_2}{2}},\tag{B.2.26}$$

for every $m=1,2$, $t\in[1,T]$.

Proof. First of all, we observe that after (B.2.24), (A.11) and (1.1.10) (notice that $\mathrm{R}_1(u_+-u_-)=2D_1u$), we have an explicit expression for the difference between $v^{I,NF}$ and $(\Gamma^I v)_-$

(B.2.27)
$$v^{I,NF}-(\Gamma^I v)_-=-\frac{i}{2}\left[(D_t\Gamma^I v)(D_1u)-(D_1\Gamma^I v)(D_tu)+D_1[(\Gamma^I v)D_tu]-\langle D_x\rangle[(\Gamma^I v)D_1u]\right].$$

From the above equality together with (1.1.5), (1.1.10), and lemma B.2.2 with $L = L^\infty$ and $w_1 = (\Gamma^I v)_\pm$, we deduce that there exists some $\chi_1 \in C_0^\infty(\mathbb{R}^2)$, equal to 1 on the support of χ, such that

(B.2.28)
$$\big\|\chi(t^{-\sigma}D_x)(v^{I,NF} - v^I_-)(t,\cdot)\big\|_{L^\infty} \lesssim t^\sigma \sum_{\mu=0}^{1} \big\|[\chi_1(t^{-\sigma}D_x)(\Gamma^I v)_\pm(t,\cdot)]\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm(t,\cdot)\big\|_{L^\infty} + t^{-N(s)+\sigma}\|(\Gamma^I v)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s},$$

where the second addend in the above right hand side is estimated with $B^2\varepsilon^2 t^{-3/2}$ after a-priori energy estimates, if $s>0$ is taken sufficiently large so that $N(s)\ge 2$.

What we actually want to do is to truncate $(\Gamma^I v)_\pm$, in the first norm in the above right hand side, by means of the same operator $\chi(t^{-\sigma}D_x)$ appearing in the left hand side. For that, we proceed to decompose $\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm$ as follows:

$$\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm = \chi(t^{\kappa}D_x)\mathrm{R}_1^\mu u_\pm + (1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm, \tag{B.2.29}$$

for some $\kappa\ge 1$ that we will choose later, noticing that, as $\chi(t^\kappa \xi)$ is supported for very small frequencies $|\xi|\lesssim t^{-\kappa}$,
$$\|\chi(t^\kappa D_x)\mathrm{R}_1^\mu u_\pm(t,\cdot)\|_{L^\infty} \lesssim t^{-\kappa}\|u_\pm(t,\cdot)\|_{L^2}.$$
Consequently, using the L^2-L^∞ continuity of $\chi_1(t^{-\sigma}D_x)$ with norm $O(t^{2\sigma})$, along with (1.1.11c), (1.1.11d), for $\mu=0,1$ we have that

$$\big\|[\chi_1(t^{-\sigma}D_x)(\Gamma^I v)_\pm(t,\cdot)]\chi(t^{\kappa}D_x)\mathrm{R}_1^\mu u_\pm(t,\cdot)\big\|_{L^\infty} \lesssim t^{2\sigma-\kappa}\|(\Gamma^I v)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{L^2} \le CB\varepsilon t^{-\kappa+2\sigma+\frac{\delta_{3-j}+\delta}{2}},$$

so choosing $\kappa = 1+2\sigma+\frac{\delta+\delta_1}{2}$, we deduce from (B.2.29) that

$$\begin{aligned}&\big\|[\chi_1(t^{-\sigma}D_x)(\Gamma^I v)_\pm(t,\cdot)]\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm(t,\cdot)\big\|_{L^\infty}\\ &\quad\lesssim \big\|[\chi_1(t^{-\sigma}D_x)(\Gamma^I v)_\pm(t,\cdot)](1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm(t,\cdot)\big\|_{L^\infty} + CB\varepsilon t^{-1}.\end{aligned}\tag{B.2.30}$$

At this point, we decompose $(\Gamma^I v)_\pm$ in frequencies using the wished operator $\chi(t^{-\sigma}D_x)$. In order to estimate the L^∞ norm of

$$[(1-\chi)(t^{-\sigma}D_x)\chi_1(t^{-\sigma}D_x)(\Gamma^I v)_\pm](1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm,$$

we first commute Γ^I to operator $D_t \pm \langle D_x\rangle$, and successively look at it as a linear combination of derivations of the form $x^\alpha t^a \partial_x^\alpha \partial_t^b$, with $1\le|\alpha|+a\le 2$, $1\le|\beta|+b\le 2$. Commutating x^α to $(1-\chi)(t^{-\sigma}D_x)\chi_1(t^{-\sigma}D_x)$ and multiplying it against the wave factor, and successively combining the classical Sobolev injection with inequality (B.1.2), we find that

$$\begin{aligned}&\big\|[(1-\chi)(t^{-\sigma}D_x)\chi_1(t^{-\sigma}D_x)(\Gamma^I v)_\pm(t,\cdot)](1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm(t,\cdot)\big\|_{L^\infty}\\ &\quad\lesssim t^{-N(s)}\left(\|v_\pm(t,\cdot)\|_{H^s} + \|D_t v_\pm(t,\cdot)\|_{H^s} + \|D_t^2 v_\pm(t,\cdot)\|_{H^s}\right)\\ &\qquad\times \sum_{\substack{1\le|\alpha|+a\le 2\\ |\mu|=0,1}} \big\|x^\alpha t^a(1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm\big\|_{L^\infty}.\end{aligned}\tag{B.2.31}$$

Using system (2.1.2) with $|I|=0$ and a-priori estimates, we check that

$$\|v_\pm(t,\cdot)\|_{H^s} + \|D_t v_\pm(t,\cdot)\|_{H^s} + \|D_t^2 v_\pm(t,\cdot)\|_{H^s} \le CB\varepsilon t^{\frac{\delta}{2}},$$

and also that

$$t^a \left\|(1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm\right\|_{L^\infty} \lesssim t^{a+2\sigma}\|u_\pm(t,\cdot)\|_{L^2} \le CB\varepsilon t^{a+2\sigma+\frac{\delta}{2}}.$$

On the other hand, for $|\alpha|=1,2$,

$$\left\|x^\alpha(1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm\right\|_{L^\infty} \le CB\varepsilon t^{|\alpha|+|\alpha|\kappa+\frac{\delta}{2}}. \tag{B.2.32}$$

In fact, when $|\alpha|=1$ this latter inequality is deduced by commutating x^α with operator $(1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)$, where

$$[x_n,(1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)] = -it^\kappa(\partial_n\chi)(t^\kappa D_x) + it^{-\sigma}(\partial_n\chi)(t^{-\sigma}D_x), \quad n=1,2,$$

is bounded from L^2 to L^∞ with $O(1)$ norm, and by using (1.1.11d), (B.1.16a) together with

$$\begin{aligned}&\left\|(1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)\left[x^\alpha\mathrm{R}_1^\mu u_\pm\right](t,\cdot)\right\|_{L^\infty}\\ &\qquad\lesssim t^\kappa\left[\sum_{|\mu|=1}\|Z^\mu u_\pm(t,\cdot)\|_{L^2} + t\|u_\pm(t,\cdot)\|_{H^1} + \|xNL_w(t,\cdot)\|_{L^2}\right]\end{aligned}$$

which is consequence of the following equality, with $\widetilde{\chi}(\xi) := (1-\chi)(\xi)|\xi|^{-1}$,

$$\begin{aligned}&(1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)x_n\mathrm{R}_1^\mu\\ &= t^\kappa\widetilde{\chi}_1(t^\kappa D_x)\chi(t^{-\sigma}D_x)x_n|D_x|\mathrm{R}_1^\mu + t^\kappa\widetilde{\chi}_1(t^\kappa D_x)\chi(t^{-\sigma}D_x)[|D_x|,x]\mathrm{R}_1^\mu\\ &= t^\kappa\widetilde{\chi}_1(t^\kappa D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu\left[x_n|D_x| - tD_n + \frac{1}{2i}\frac{D_n}{|D_x|}\right]\\ &+ t^\kappa\widetilde{\chi}_1(t^\kappa D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu\left[tD_n - \frac{1}{2i}\frac{D_n}{|D_x|}\right] + \delta_{\mu=1}it^\kappa\widetilde{\chi}_1(t^\kappa D_x)\chi(t^{-\sigma}D_x)Op(|\xi|\partial_n(\xi_1|\xi|^{-1}))\\ &- it^\kappa\widetilde{\chi}_1(t^\kappa D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_n\mathrm{R}_1^\mu,\end{aligned}$$

and of relation (3.2.9a) with $w=u_\pm$. When $|\alpha|=2$, we also commutate x^α with $(1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)$ (this commutator being now bounded from L^2 to L^∞ with norm $O(t^\kappa)$), and derive an analogous relation to the one of above by considering function $\widetilde{\chi}_2(\xi) := (1-\chi)(\xi)|\xi|^{-2}$ instead of $\widetilde{\chi}_1$, making some commutation, and expressing each occurrence of $x_n|D_x|$, for $n=1,2$, in terms of $x_n|D_x| - tD_n + \frac{1}{2i}\frac{D_n}{|D_x|}$ in order to make use of relation (3.2.9a). We end up with an inequality as

$$\begin{aligned}&\left\|(1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)\left[x^\alpha\mathrm{R}_1^\mu u_\pm\right](t,\cdot)\right\|_{L^\infty}\\ &\quad\lesssim t^{2\kappa}\left[\sum_{|\mu|=2}\|Z^\mu u_\pm(t,\cdot)\|_{L^2} + \sum_{|\mu|\le 1}t^{2-|\mu|}\|Z^\mu u_\pm(t,\cdot)\|_{H^1} + \sum_{|\mu|=1}^{2}\|x^\mu NL_w(t,\cdot)\|_{L^2}\right]\\ &\quad\le CB\varepsilon t^{2+2\kappa+\frac{\delta}{2}},\end{aligned}$$

last estimate following from a-priori estimates, (B.1.16a) and (B.1.32).

Therefore, choosing $s>0$ in (B.2.31) sufficiently large so that $N(s)\ge 6$, from above estimates and the fact that $\kappa = 1+2\sigma+\frac{\delta+\delta_1}{2}$, with σ,δ,δ_1 small, we deduce that

$$\left\|[(1-\chi)(t^{-\sigma}D_x)\chi_1(t^{-\sigma}D_x)(\Gamma^I v)_\pm(t,\cdot)](1-\chi)(t^\kappa D_x)\chi(t^{-\sigma}D_x)\mathrm{R}_1^\mu u_\pm(t,\cdot)\right\|_{L^\infty} \le CB\varepsilon t^{-\frac{3}{2}}, \tag{B.2.33}$$

so from (B.2.28), (B.2.30),(B.2.33), and the uniform continuity on L^∞ of $\chi(t^{-\sigma}D_x)$,

$$\left\|\chi(t^{-\sigma}D_x)\left(v^{I,NF} - v^I_-\right)(t,\cdot)\right\|_{L^\infty} \lesssim \sum_{\mu=0}^{1} t^\sigma\|\chi(t^{-\sigma}D_x)(\Gamma^I v)_\pm(t,\cdot)\|_{L^\infty}\|\mathrm{R}_1^\mu u_\pm(t,\cdot)\|_{L^\infty} + CB\varepsilon t^{-1}.$$

As σ is small and $\varepsilon_0 < (2A)^{-1}$, from (1.1.11a) we then obtain (B.2.25).

In order to estimate the L^2 norm of $\chi(t^{-\sigma}D_x)Z_m(v^{I,NF} - v^I_-)$, for $m = 1, 2$, and prove (B.2.26), we first apply the Leibniz rule and, since

$$[Z_m, D_t] = -D_m, \quad [Z_m, D_1] = -\delta^1_m D_t, \quad [Z_m, \langle D_x\rangle] = -D_m\langle D_x\rangle^{-1}D_t, \tag{B.2.34}$$

we find that

(B.2.35)

$$\begin{aligned}
&2iZ_m(v^{I,NF} - v^I_-)\\
&\quad= (D_tZ_m\Gamma^I v)(D_1u) - (D_1Z_m\Gamma^I v)(D_tu) + D_1[(Z_m\Gamma^I v)(D_tu)] - \langle D_x\rangle[(Z_m\Gamma^I v)(D_1u)]\\
&\quad+ (D_t\Gamma^I v)(D_1Z_mu) - (D_1\Gamma^I v)(D_tZ_mu) + D_1[(\Gamma^I v)(D_tZ_mu)] - \langle D_x\rangle[(\Gamma^I v)(D_1Z_mu)]\\
&\quad- (D_m\Gamma^I v)(D_1u) + \delta^1_m(D_t\Gamma^I v)(D_tu) - \delta^1_m D_t[(\Gamma^I v)(D_tu)] + \frac{D_m}{\langle D_x\rangle}D_t[(\Gamma^I v)(D_1u)]\\
&\quad- \delta^1_m(D_t\Gamma^I v)(D_tu) + (D_1\Gamma^I v)(D_mu) - \delta^1_m D_1[(\Gamma^I v)(D_tu)] + \delta^1_m\langle D_x\rangle[(\Gamma^I v)(D_tu)].
\end{aligned}$$

The L^2 norm of all products (when truncated for frequencies $|\xi| \lesssim t^\sigma$) in the above second, fourth and fifth line, i.e. those in which Z_m is not acting on the wave component u, is estimated by

$$\begin{aligned}\sum_{\mu=0}^{1} t^\sigma \left(\|(Z_m\Gamma^I v)_\pm(t,\cdot)\|_{L^2} + \|(\Gamma^I v)_\pm(t,\cdot)\|_{L^2}\right)\left(\|\mathrm{R}^\mu_1 u_\pm(t,\cdot)\|_{L^\infty} + \|D_tu_\pm(t,\cdot)\|_{L^\infty}\right)\\
\le CAB\varepsilon^2 t^{-\frac{1}{2}+\frac{\delta_0}{2}+\sigma},\end{aligned} \tag{B.2.36}$$

after inequality (B.1.5b) with $s = 0$ and a-priori estimates. The L^2 norm of products in the second above line are, instead, estimated by using (B.2.20) with $L = L^2$, $w = u$ and $s > 0$ sufficiently large so that $N(s) \ge 2$, combined with remark B.2.5. It is hence bounded by

$$\begin{aligned}
&t^\sigma \left\|\chi(t^{-\sigma}D_x)(\Gamma^I v)_\pm(t,\cdot)\right\|_{L^\infty} \|(Z_mu)_\pm(t,\cdot)\|_{L^2}\\
&+ t^{-2}\Big(\sum_{|\mu|=0}^{1} \|x^\mu(\Gamma^I v)_\pm(t,\cdot)\|_{L^2} + t\|(\Gamma^I v)_\pm(t,\cdot)\|_{L^2}\Big)\left(\|u_\pm(t,\cdot)\|_{H^s} + \|D_tu_\pm(t,\cdot)\|_{H^s}\right)\\
&\le CB^2\varepsilon^2 t^{2\sigma+\frac{\delta_{3-j}+\delta_2}{2}},
\end{aligned}$$

where the latter estimate is obtained using the fact that $\chi(t^{-\sigma}D_x)$ is a bounded operator from L^2 to L^∞ with norm $O(t^\sigma)$, together with (B.1.5a), (B.1.17) and a-priori estimates. That concludes, together with (B.2.36), the proof of (B.2.26) and of the statement. ☐

In the following lemma we provide a first estimate on the uniform norm of the Klein-Gordon component when one Klainerman vector is acting on it, and when it is localised for frequencies less or equal than t^σ, for a small $\sigma > 0$. It is not the sharpest one, and will be refined at the end of this chapter (see lemma B.3.21).

Lemma B.2.8. *There exists a constant $C > 0$ such that, under the same assumption as in lemma B.2.7, for any $\rho \in \mathbb{N}$, $\chi \in C^\infty_0(\mathbb{R}^2)$, equal to 1 in a neighbourhood of the origin, and $\sigma > 0$ small,*

$$\sum_{|I|=1} \|\chi(t^{-\sigma}D_x)V^I(t,\cdot)\|_{H^{\rho,\infty}} \le CB\varepsilon t^{-1+\beta+\frac{\delta_1}{2}}, \tag{B.2.37}$$

for a small $\beta > 0$, $\beta \to 0$ as $\sigma \to 0$.

Proof. Since $\chi(t^{-\sigma}D_x)$ is a bounded operator from L^∞ to $H^{\rho,\infty}$ with norm $O(t^{\sigma\rho})$, for any $\rho\in\mathbb{N}$, we can reduce to prove the statement for the L^∞ norm of $\chi(t^{-\sigma}D_x)V^I$, up to a loss $t^{\sigma\rho}$. Moreover, as inequality (B.2.37) is automatically satisfied when Γ is a spatial derivative, after a-priori estimate (1.1.11b) and the fact that operator $\chi(t^{-\sigma}D_x)$ is uniformly bounded on L^∞, for the rest of the proof we will suppose that $\Gamma\in\{\Omega, Z_j, j=1,2\}$ is a Klainerman vector field. We also warn the reader that, throughout the proof, C,β will always denote some positive constant, that may change line after line, with $\beta\to 0$ as $\sigma\to 0$.

Instead of proving the result of the statement directly on $\chi(t^{-\sigma}D_x)v^I_\pm$, we do it for $\chi(t^{-\sigma}D_x)v^{I,NF}$ where $v^{I,NF}$ has been introduced in (B.2.24) and is considered here for $|I|=1$ and $\Gamma^I=\Gamma$. In fact, by (B.2.25),

$$\left\|\chi(t^{-\sigma}D_x)v^I_-(t,\cdot)\right\|_{L^\infty}\le 2\left\|\chi(t^{-\sigma}D_x)v^{I,NF}(t,\cdot)\right\|_{L^\infty}+CB\varepsilon t^{-1}, \tag{B.2.38}$$

and the advantage of dealing with this new function is related to the fact that it is solution to a half Klein-Gordon equation with a more suitable non-linearity (see (B.2.39)) than the equation satisfied by v^I_-.

In fact, it is a computation to show that, from definition (B.2.24),

$$[D_t+\langle D_x\rangle]v^{I,NF}(t,x)=NL^{I,NF}_{kg} \tag{B.2.39}$$

where

$$NL^{I,NF}_{kg}=r^{I,NF}_{kg}(t,x)+Q^{\mathrm{kg}}_0(v_\pm, D_1u^I_\pm)+G^{\mathrm{kg}}_1(v_\pm, Du_\pm), \tag{B.2.40}$$

and

$$\begin{split} r^{I,NF}_{kg}(t,x)=&-\frac{i}{4(2\pi)^2}\\ &\times\sum_{j_1,j_2\in\{+,-\}}\int e^{ix\cdot\xi}B^1_{(j_1,j_2,+)}(\xi,\eta)\left[\widehat{NL^I_{kg}}(\xi-\eta)\hat{u}_{j_2}(\eta)-\widehat{v^I_{j_1}}(\xi-\eta)\widehat{NL_w}(\eta)\right]d\xi d\eta, \end{split} \tag{B.2.41}$$

with $B^1_{(j_1,j_2,+)}$ given by (2.2.45) when $j_3=+$ and $k=1$, and $NL^I_{kg}=\Gamma^I NL_{kg}=\Gamma NL_{kg}$. After (B.2.24), (A.11), it appears that $r^{I,NF}_{kg}$ has the following nice explicit expression

$$r^{I,NF}_{kg}=-\frac{i}{2}\left[NL^I_{kg}D_1u-(D_1\Gamma^Iv)NL_w+D_1[(\Gamma^Iv)NL_w]\right]. \tag{B.2.42}$$

Applying lemma B.2.6 and relation (3.2.8) with $w=v^{I,NF}$, and reminding that $\|tw(t,t\cdot)\|_{L^2}=\|w(t,\cdot)\|_{L^2}$, we find the following inequality

$$\begin{split} &\left\|\chi(t^{-\sigma}D_x)v^{I,NF}(t,\cdot)\right\|_{L^\infty}\\ &\lesssim t^{-1+\beta}\sum_{|\mu|=0}^{1}\left\|\chi(t^{-\sigma}D_x)Z^\mu v^{I,NF}(t,\cdot)\right\|_{L^2}+\sum_{j=1}^{2}t^{-1+\beta}\left\|\chi(t^{-\sigma}D_x)x_jNL^{I,NF}_{kg}(t,\cdot)\right\|_{L^2}. \end{split} \tag{B.2.43}$$

From equality (B.2.27), along with (1.1.5), (1.1.10), and a-priori estimates (1.1.11a), (1.1.11d), we immediately see that

$$\begin{split} \left\|\chi(t^{-\sigma}D_x)(v^{I,NF}-v^I_-)(t,\cdot)\right\|_{L^2}&\lesssim t^\sigma\|v^I_\pm(t,\cdot)\|_{L^2}\left(\|u_\pm(t,\cdot)\|_{L^\infty}+\|\mathrm{R}_1u_\pm(t,\cdot)\|_{L^\infty}\right)\\ &\le CAB\varepsilon^2t^{-\frac{1}{2}+\frac{\delta_2}{2}+\sigma}, \end{split} \tag{B.2.44}$$

and therefore, as $\sigma, \delta_2 \ll 1$ are small, that

$$\begin{aligned}\left\|\chi(t^{-\sigma}D_x)v^{I,NF}(t,\cdot)\right\|_{L^2} &\le \left\|\chi(t^{-\sigma}D_x)v^I_-(t,\cdot)\right\|_{L^2} + \left\|\chi(t^{-\sigma}D_x)(v^{I,NF}-v^I_-)(t,\cdot)\right\|_{L^2}\\ &\le CB\varepsilon t^{\frac{\delta_2}{2}}.\end{aligned} \tag{B.2.45}$$

Moreover, from (B.2.26) and a-priori estimate (1.1.11d),

$$\left\|\chi(t^{-\sigma}D_x)Z_m v^{I,NF}(t,\cdot)\right\|_{L^2} \le CB\varepsilon t^{\frac{\delta_1}{2}}, \tag{B.2.46}$$

for every $m = 1,2$, $t \in [1,T]$.

Finally, from (B.2.42), (1.1.5), (1.1.10), (B.1.10b), (B.1.26a) and a-priori estimates, we derive that

$$\begin{aligned}&\left\|\chi(t^{-\sigma}D_x)\left[x_j r^{I,NF}_{kg}\right](t,\cdot)\right\|_{L^2} \lesssim \|x_j NL^I_{kg}(t,\cdot)\|_{L^2}\left(\|u_\pm(t,\cdot)\|_{L^\infty} + \|\mathrm{R}_1 u_\pm(t,\cdot)\|_{L^\infty}\right)\\ &+\sum_{\mu=0}^{1} t^\sigma\left(\|x^\mu_j v_\pm(t,\cdot)\|_{L^\infty} + \left\|x^\mu_j\frac{D_x}{\langle D_x\rangle}v_\pm(t,\cdot)\right\|_{L^\infty}\right)\|v^I_\pm(t,\cdot)\|_{L^2}\|v_\pm(t,\cdot)\|_{H^{2,\infty}}\\ &\le C(A+B)B\varepsilon^2 t^{\frac{\delta_2}{2}},\end{aligned} \tag{B.2.47}$$

while from (B.1.5a) with $s=0$, (B.1.10b) and a-priori estimates

$$\begin{aligned}&\left\|\chi(t^{-\sigma}D_x)\left[x_j Q^{\mathrm{kg}}_0\left(v_\pm, D_1 u^I_\pm\right)\right](t,\cdot)\right\|_{L^2} + \left\|\chi(t^{-\sigma}D_x)\left[x_j G^{\mathrm{kg}}_1\left(v_\pm, Du_\pm\right)\right](t,\cdot)\right\|_{L^2}\\ &\lesssim \left(\|x_j v_\pm(t,\cdot)\|_{L^\infty} + \left\|x_j\frac{D_x}{\langle D_x\rangle}v_\pm(t,\cdot)\right\|_{L^\infty}\right)\left(\|u^I_\pm(t,\cdot)\|_{H^1} + \|D_t u_\pm(t,\cdot)\|_{L^2}\right)\\ &\le C(A+B)B\varepsilon t^{\delta_2}.\end{aligned}$$

Therefore, from (B.2.40) we deduce that

$$\|\chi(t^{-\sigma}D_x)x_j NL^{I,NF}_{kg}(t,\cdot)\|_{L^2} \le C(A+B)B\varepsilon^2 t^{\delta_2}, \tag{B.2.48}$$

so injecting (B.2.45), (B.2.46), (B.2.48) into (B.2.43), and summing it up with (B.2.38), we obtain the result of the statement. ☐

As done for the Klein-Gordon component in the above lemma, we derive an estimate also for the uniform norm of the wave component with a Klainerman vector field acting on it, when supported for moderate frequencies less or equal than t^σ (see lemma B.2.10). We first need the following result.

Lemma B.2.9. *Let $\Gamma \in \mathcal{Z}$, with $\mathcal{Z}$ given by* (1.1.7)*, and $\widetilde{u}^J(t,x) := t(\Gamma u)_-(t,tx)$. There exists a constant $C>0$ such that, under the same hypothesis as in lemma B.2.7, for any $\theta_0, \chi \in C^\infty_0(\mathbb{R}^2)$, $\sigma > 0$,*

$$\|\widetilde{u}^J(t,\cdot)\|_{L^2} \le CB\varepsilon t^{\frac{\delta_2}{2}}, \tag{B.2.49a}$$

$$\|\Omega_h\widetilde{u}^J(t,\cdot)\|_{L^2} \le CB\varepsilon t^{\frac{\delta_1}{2}}, \tag{B.2.49b}$$

$$\left\|\mathcal{M}\widetilde{u}^J(t,\cdot)\right\|_{L^2} \le CB\varepsilon t^{\frac{\delta_1}{2}}, \tag{B.2.49c}$$

$$\left\|\theta_0 Op^w_h(\chi(h^\sigma\xi))\Omega_h\mathcal{M}\widetilde{u}^J(t,\cdot)\right\|_{L^2} \le CB\varepsilon t^{\frac{\delta_0}{2}}. \tag{B.2.49d}$$

Proof. We warn the reader that, throughout the proof, C will denote a positive constant that may change line after line. We also recall that

$$[D_t + \langle D_x\rangle](\Gamma u)_-(t,x) = \Gamma NL_w(t,x).$$

Estimates (B.2.49a) and (B.2.49b) are straightforward after (1.1.11d) and the fact that

$$\|\widetilde{u}^J(t,\cdot)\|_{L^2} = \|(\Gamma u)_-(t,\cdot)\|_{L^2}, \quad \|\Omega_h \widetilde{u}^J(t,\cdot)\|_{L^2} = \|(\Omega\Gamma u)_-(t,\cdot)\|_{L^2}.$$

From (3.2.6) with $w = (\Gamma u)_-$ and $f = \Gamma NL_w$, estimates (1.1.11d), (B.1.26b), along with the fact that $\delta_2 \ll \delta_1$ (e.g. $2\delta_2 \le \delta_1$), and $(A+B)\varepsilon_0 < 1$, we obtain (B.2.49c).

By (3.2.6) we also derive that, for any $n = 1,2$,

$$\text{(B.2.50)} \quad \begin{aligned} \big\|\theta_0 Op^w_h(\chi(h^\sigma\xi))\Omega_h \mathcal{M}_n \widetilde{u}^J(t,\cdot)\big\|_{L^2} \lesssim \|\Omega Z_n(\Gamma u)_-(t,\cdot)\|_{L^2} + \sum_{\mu=0}^{1} \|\Omega^\mu(\Gamma u)_-(t,\cdot)\|_{L^2} \\ + \Big\|\theta_0\Big(\frac{x}{t}\Big)\chi(t^{-\sigma}D_x)\Omega[x_n \Gamma NL_w]\Big\|_{L^2}. \end{aligned}$$

The first two norms in the above right hand side are controlled by $E^0_3(t;W)^{1/2}$, and are hence bounded by $CB\varepsilon t^{\frac{\delta_0}{2}}$. By commutating x_n with $\chi(t^{-\sigma}D_x)\Omega$, and using that $\theta_0\big(\frac{x}{t}\big)x_n = t\theta^n_0\big(\frac{x}{t}\big)$, with $\theta^n_0(z) := \theta_0(z)z_n$, we deduce that

$$\Big\|\theta_0\Big(\frac{x}{t}\Big)\chi(t^{-\sigma}D_x)\Omega[x_n\Gamma NL_w]\Big\|_{L^2} \lesssim t\sum_{\mu=0}^{1} \|\chi(t^{-\sigma}D_x)\Omega^\mu \Gamma NL_w\|_{L^2}.$$

On the one hand, from (B.1.20b),

$$\text{(B.2.51)} \quad t\|\Gamma NL_w\|_{L^2} \lesssim t\|v_\pm(t,\cdot)\|_{H^{2,\infty}} \left(\|(\Gamma v)_\pm(t,\cdot)\|_{H^1} + \|v_\pm(t,\cdot)\|_{H^1} + \|D_t v_\pm(t,\cdot)\|_{L^2}\right) \lesssim CB\varepsilon t^{\frac{\delta_2}{2}},$$

as follows from (B.1.6a) with $s=0$ and a-priori estimates.

On the other hand, when computing $\Omega\Gamma NL_w$, among the out-coming quadratic terms we find

$$Q^{\mathrm{w}}_0((\Omega v)_\pm, D_1(\Gamma v)_\pm) \quad \text{and} \quad Q^{\mathrm{w}}_0((\Gamma v)_\pm, D_1(\Omega v)_\pm),$$

and we estimate their L^2 norm (when truncated for frequencies less or equal than t^σ) by means of (B.2.19), together with remark B.2.5, with $L = L^2$ and $s > 0$ large enough to have $N(s) \ge 3$. From (B.1.17), (B.2.37) and a-priori estimates, we obtain that

$$\begin{aligned} &\big\|\chi(t^{-\sigma}D_x)Q^{\mathrm{w}}_0((\Omega v)_\pm, D_1(\Gamma v)_\pm)\big\|_{L^2} + \big\|\chi(t^{-\sigma}D_x)Q^{\mathrm{w}}_0((\Gamma v)_\pm, D_1(\Omega v)_\pm)\big\|_{L^2} \\ &\lesssim t^\sigma \|\chi(t^{-\sigma}D_x)(\Omega v)_\pm(t,\cdot)\|_{L^\infty}\|(\Gamma v)_\pm(t,\cdot)\|_{H^1} + \sum_{|\mu|=0}^{1} t^{-3}\|v_\pm(t,\cdot)\|_{H^s}\|x^\mu(\Gamma v)_\pm(t,\cdot)\|_{H^1} \\ &\le CB^2\varepsilon^2 t^{-1+\beta+\frac{\delta_1+\delta_2}{2}}, \end{aligned}$$

with $\beta > 0$ small such that $\beta \to 0$ as $\sigma \to 0$.

All remaining quadratic contributions to $\Omega\Gamma NL_w$ are estimated with

$$\begin{aligned} \|(\Omega\Gamma v)_\pm(t,\cdot)\|_{H^1}\|v_\pm(t,\cdot)\|_{H^{2,\infty}} + \|(\Omega v)_\pm(t,\cdot)\|_{L^2}\left(\|v_\pm(t,\cdot)\|_{H^{1,\infty}} + \|D_t v_\pm(t,\cdot)\|_{L^\infty}\right) \\ + \|v_\pm(t,\cdot)\|_{H^{1,\infty}}\left(\|(\Omega v)_\pm(t,\cdot)\|_{H^1} + \|D_t(\Omega v)_\pm(t,\cdot)\|_{L^2}\right), \end{aligned}$$

and are hence bounded by $C(A+B)B\varepsilon^2 t^{-1+\frac{\delta_1}{2}}$, after (B.1.6b), (B.1.6c) and the a-priori estimates. This finally implies that

$$t\left\|\chi(t^{-\sigma}D_x)\Omega\Gamma NL_w\right\|_{L^2}\le C(A+B)B\varepsilon^2 t^{\beta+\frac{\delta_1+\delta_2}{2}},$$

which, together with (B.2.51) and the fact that $\beta+\frac{\delta_1+\delta_2}{2}\le\frac{\delta_0}{2}$, as $\delta_2\ll\delta_1\ll\delta_0$ and $\beta>0$ is as small as we want (provided that σ is small) gives

$$\left\|\theta_0\Big(\frac{x}{t}\Big)\chi(t^{-\sigma}D_x)\Omega[x_n\Gamma NL_w]\right\|_{L^2}\le CB\varepsilon t^{\frac{\delta_0}{2}},$$

and concludes the proof of the statement when injected in (B.2.50). □

Lemma B.2.10. *There exists a constant $C>0$ such that, under the same hypothesis as in lemma B.2.7, for any $\rho\in\mathbb{N}$, $\chi\in C_0^\infty(\mathbb{R}^2)$ equal to 1 in a neighbourhood of the origin, and $\sigma>0$ small,*

$$\sum_{|J|=1}\sum_{|\mu|=0}^{1}\|\chi(t^{-\sigma}D_x)\mathrm{R}^\mu U^J(t,\cdot)\|_{H^{\rho,\infty}}\le C(A+B)\varepsilon t^{-\frac{1}{2}+\beta+\frac{\delta_1}{2}},\tag{B.2.52}$$

for a small $\beta>0$, $\beta\to 0$ as $\sigma\to 0$.

Proof. Since $\chi(t^{-\sigma}D_x)$ is a bounded operator from L^∞ to $H^{\rho,\infty}$, with norm $O(t^{\sigma\rho})$, for any $\rho\in\mathbb{N}$, we can reduce to prove the statement for $\|\chi(t^{-\sigma}D_x)\mathrm{R}^\mu U^J(t,\cdot)\|_{L^\infty}$, for any $|J|=1$, $|\mu|=0,1$, up to a small loss $t^{\sigma\rho}$ in the right hand side of (B.2.52).

This estimate is automatically satisfied when J is such that $\Gamma^J=D_x$, as a consequence of a-priori estimate (1.1.11a). We therefore assume that Γ^J is one of the Klainerman vector fields Ω, Z_m, $m=1,2$.

Introducing $\widetilde{u}^J(t,x):=tu^J_-(t,tx)$, so that $\|\widetilde{u}^J(t,\cdot)\|_{L^2}=\|u^J(t,\cdot)\|_{L^2}$, passing to the semiclassical setting ($t\mapsto t$, $x\mapsto\frac{x}{t}$, and $h:=1/t$), and reminding that $u^J_+=-\overline{u^J_-}$, inequality (B.2.52) becomes

$$\sum_{|\mu|=0}^{1}\left\|Op^w_h\Big(\chi(h^\sigma\xi)(\xi|\xi|^{-1})^\mu\Big)\widetilde{u}^J_-(t,\cdot)\right\|_{L^\infty}\le C(A+B)\varepsilon h^{-\frac{1}{2}-\beta-\frac{\delta_1}{2}}.\tag{B.2.53}$$

We consider a Littlewood-Paley decomposition such that

$$\chi(h^\sigma\xi)=\widetilde{\chi}(h^{-1}\xi)+\sum_k(1-\widetilde{\chi})(h^{-1}\xi)\psi(2^{-k}\xi)\chi(h^\sigma\xi),\tag{B.2.54}$$

for some suitably supported $\widetilde{\chi}\in C_0^\infty(\mathbb{R}^2)$, $\psi\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$, and immediately observe that the above sum is restricted to indices k such that $h\lesssim 2^k\lesssim h^{-\sigma}$. By the classical Sobolev injection, the uniform continuity of $Op^w_h(\xi|\xi|^{-1})$ on L^2, and a-priori estimate (1.1.11d), we derive that

$$\begin{aligned}\left\|Op^w_h\big(\widetilde{\chi}(h^{-1}\xi)(\xi|\xi|^{-1})^\mu\big)\widetilde{u}^J(t,\cdot)\right\|_{L^\infty}&=\|\chi(D_x)Op^w_h((\xi|\xi|^{-1})^\mu)\widetilde{u}^J(t,\cdot)\|_{L^\infty}\\&\lesssim\|u^J_-(t,\cdot)\|_{L^2}\le CB\varepsilon h^{-\frac{\delta_2}{2}},\end{aligned}\tag{B.2.55}$$

for any $|\mu|\le 1$, every $t\in[1,T]$.

If we concisely denote by $\phi_k(\xi)$ the k-th addend in decomposition (B.2.54), and introduce two smooth cut-off functions χ_0, γ, with χ_0 radial and equal to 1 on the support of ϕ_k, and γ with sufficiently small support, we can write

$$Op_h^w\left(\phi_k(\xi)(\xi|\xi|^{-1})^\mu\right)\widetilde{u}^J = Op_h^w\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)(\xi|\xi|^{-1})^\mu\Big)Op_h^w(\chi_0(h^\sigma\xi))\widetilde{u}^J + Op_h^w\Big((1-\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)(\xi|\xi|^{-1})^\mu\Big)Op_h^w(\chi_0(h^\sigma\xi))\widetilde{u}^J.$$

On the one hand, after proposition 1.2.30, the fact that $2^k \le h^{-\sigma}$, and a-priori estimates (1.1.11d), we have that, for any $|\mu|\le 1$,

$$\Big\|Op_h^w\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)(\xi|\xi|^{-1})^\mu\Big)Op_h^w(\chi_0(h^\sigma\xi))\widetilde{u}^J(t,\cdot)\Big\|_{L^\infty} \lesssim h^{-\frac12-\beta}\left(\|Op_h^w(\chi_0(h^\sigma\xi))\widetilde{u}^J(t,\cdot)\|_{L^2}+\|\theta_0\Omega_h Op_h^w(\chi_0(h^\sigma\xi))\widetilde{u}^J(t,\cdot)\|_{L^2}\right) \lesssim h^{-\frac12-\beta}\left(\|u_-^J(t,\cdot)\|_{L^2}+\|\Omega u_-^J(t,\cdot)\|_{L^2}\right)\le CB\varepsilon h^{-\frac12-\beta-\frac{\delta_1}{2}}, \tag{B.2.56}$$

with $\beta>0$ small, $\beta\to 0$ as $\sigma\to 0$.

On the other hand, using that $(1-\gamma)(z)=\gamma_1^j(z)z_j$, where $\gamma_1^j(z):=(1-\gamma)(z)z_j|z|^{-2}$ is such that $|\partial_z^\alpha\gamma_1^j(z)|\le\langle z\rangle^{-1-|\alpha|}$, we derive from (1.2.48b), the commutation between $\mathcal{M}$ with $Op_h^w(\chi_0(h^\sigma\xi))$, and lemma B.2.9, that

$$\Big\|Op_h^w\Big((1-\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)(\xi|\xi|^{-1})^\mu\Big)Op_h^w(\chi_0(h^\sigma\xi))\widetilde{u}^J\Big\|_{L^\infty} \lesssim h^{-\beta}\sum_{\gamma,|\nu|=0}^{1}\|(\theta_0\Omega_h)^\gamma\mathcal{M}^\nu Op_h^w(\chi_0(h^\sigma\xi))\widetilde{u}^J(t,\cdot)\|_{L^2}\le CB\varepsilon t^{\beta+\frac{\delta_0}{2}},$$

for some small $\beta>0$, $\beta\to 0$ as $\sigma\to 0$. Combining this estimate with (B.2.56), we deduce that

$$\|Op_h^w\left(\phi_k(\xi)(\xi|\xi|^{-1})^\mu\right)\widetilde{u}^J(t,\cdot)\|_{L^\infty}\le C(A+B)\varepsilon h^{-\frac12-\beta-\frac{\delta_1}{2}},$$

for any $|\mu|\le 1$, and hence (B.2.53) after (B.2.54), (B.2.55), up to a further loss $|\log h|$, as a consequence of the fact that the sum in (B.2.54) is finite, taken over indices k such that $\log h\lesssim k\lesssim \log h^{-1}$. □

Lemma B.2.11. *There exists a positive constant $C>0$ such that, under the same hypothesis as in lemma B.2.7, for any $\chi\in C_0^\infty(\mathbb{R}^2)$ equal to 1 in a neighbourhood of the origin, and $\sigma>0$,*

$$\sum_{|\mu|=0}^{1}\Big\|\chi(t^{-\sigma}D_x)\Big[x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu(\Gamma v)_\pm(t,\cdot)\Big]\Big\|_{L^\infty}\le CB\varepsilon t^{\beta+\frac{\delta_1}{2}}, \tag{B.2.57}$$

for every $t\in[1,T]$, with $\beta>0$ small, $\beta\to 0$ as $\sigma\to 0$.

Proof. We warn the reader that, throughout the proof, C,β will denote two positive constants, that may change line after line, with $\beta\to 0$ as $\sigma\to 0$. As $\Gamma v_+ = -\overline{\Gamma v_-}$, it is enough to prove the statement for Γv_-.

First of all, we observe that by (B.1.9b), with $w=(\Gamma v)_-$ and $f=\Gamma NL_{kg}$, along with the classical Sobolev injection,

$$\sum_{|\mu|=0}^{1}\Big\|x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu(\Gamma v)_-(t,\cdot)\Big\|_{L^\infty}\lesssim\|Z_j(\Gamma v)_-(t,\cdot)\|_{H^1}+t\|(\Gamma v)_-(t,\cdot)\|_{H^2}+\sum_{\mu=0}^{1}\|x_j^\mu\Gamma NL_{kg}(t,\cdot)\|_{L^\infty}, \tag{B.2.58}$$

where, from (B.1.20a) and corollary B.2.4, with $L=L^\infty$ and $s>0$ large enough so that $N(s)\geq 3$,

(B.2.59)
$$
\begin{aligned}
&\|\Gamma NL_{kg}(t,\cdot)\|_{L^\infty}\\
&\lesssim \sum_{\mu=0}^{1}\big(\|\chi(t^{-\sigma}D_x)(\Gamma v)_\pm(t,\cdot)\|_{H^{1,\infty}}\|\mathrm{R}_1^\mu u_\pm(t,\cdot)\|_{H^{2,\infty}}+\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\left\|\chi(t^{-\sigma}D_x)(\Gamma u)_\pm(t,\cdot)\right\|_{H^{2,\infty}}\big)\\
&+\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\times\sum_{|\mu|=0}^{1}\big(\|\mathrm{R}^\mu u_\pm(t,\cdot)\|_{H^{2,\infty}}+\|D_t\mathrm{R}^\mu u_\pm(t,\cdot)\|_{H^{1,\infty}}\big)\\
&+t^{-3}\left(\|v_\pm(t,\cdot)\|_{H^s}+\|D_tv_\pm(t,\cdot)\|_{H^s}\right)\Big(\sum_{|\mu|=0}^{1}\|x^\mu D_1u_\pm(t,\cdot)\|_{L^2}+t\|u_\pm(t,\cdot)\|_{L^2}\Big)\\
&+t^{-3}\Big(\sum_{|\mu|=0}^{1}\|x^\mu v_\pm(t,\cdot)\|_{L^2}+t\|v_\pm(t,\cdot)\|_{L^2}\Big)\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_tu_\pm(t,\cdot)\|_{H^s}\right)\\
&\leq CAB\varepsilon^2t^{-\frac{3}{2}+\beta+\frac{\delta_1}{2}},
\end{aligned}
$$

last estimate derived from (B.1.5a), (B.1.5b), (B.1.5c), (B.1.6a), (B.1.10a), (B.1.11), (B.2.37), (B.2.52) and a-priori estimates. Moreover, as

$$
\left\|x_jQ_0^{\mathrm{kg}}\big((\Gamma v)_\pm,D_1u_\pm\big)\right\|_{L^\infty}\lesssim\sum_{|\mu|,\nu=0}^{1}\left\|x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu(\Gamma v)_-(t,\cdot)\right\|_{L^\infty}\|\mathrm{R}_1^\nu u_\pm(t,\cdot)\|_{H^{2,\infty}},\tag{B.2.60}
$$

(B.2.61)
$$
\left\|x_jG_1^{\mathrm{kg}}(v_\pm,Du_\pm)\right\|_{L^\infty}\lesssim\sum_{|\mu|=0}^{1}\left\|x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\right\|_{L^\infty}\left(\|u_\pm(t,\cdot)\|_{H^{2,\infty}}+\|D_tu_\pm(t,\cdot)\|_{H^{1,\infty}}\right),
$$

and by corollary B.2.4 with $L=L^\infty$, $w=u_\pm$, and $s>0$ large enough so that $N(s)\geq 3$,

$$
\begin{aligned}
\left\|x_jQ_0^{\mathrm{kg}}(v_\pm,D_1(\Gamma u)_\pm)\right\|_{L^\infty}&\lesssim\sum_{|\mu|=0}^{1}\left\|x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\right\|_{L^\infty}\|\chi(t^{-\sigma}D_x)(\Gamma u)_\pm(t,\cdot)\|_{H^{2,\infty}}\\
&+t^{-3}\sum_{|\mu|,\nu=0}^{1}\Big(\left\|x^\mu x_j^\nu v_\pm(t,\cdot)\right\|_{L^2}+t\|x_j^\nu v_\pm(t,\cdot)\|_{L^2}\Big)\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_tu_\pm(t,\cdot)\|_{H^s}\right),
\end{aligned}\tag{B.2.62}
$$

we derive from (B.1.20a) that
(B.2.63)
$$
\|x_j\Gamma NL_{kg}(t,\cdot)\|_{L^\infty}\leq CA\varepsilon t^{-\frac{1}{2}}\sum_{|\mu|,\nu=0}^{1}\left\|x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu(\Gamma v)_-(t,\cdot)\right\|_{L^\infty}+C(A+B)B\varepsilon^2t^{-\frac{1}{2}+\beta+\frac{\delta_1+\delta_2}{2}},
$$

as follows after (B.1.5a), (B.1.5b) with $s=1$, (B.1.10a), (B.1.10b),(B.1.27a), (B.2.52) and a-priori estimates. By injecting the above inequality into (B.2.58), and using the fact that $\varepsilon_0<(2CA)^{-1}$, we initially obtain that

$$
\|x_j(\Gamma v)_-(t,\cdot)\|_{L^\infty}+\left\|x_j\frac{D_x}{\langle D_x\rangle}(\Gamma v)_-(t,\cdot)\right\|_{L^\infty}\leq CB\varepsilon t^{1+\frac{\delta_2}{2}}.\tag{B.2.64}
$$

Now, by taking any smooth cut-off function χ, and using again (B.1.9b), instead of (B.2.58) we find that

(B.2.65)
$$\sum_{|\mu|=0}^{1}\Big\|\chi(t^{-\sigma}D_x)\Big[x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}(\Gamma v)_-(t,\cdot)\Big]\Big\|_{L^\infty}\lesssim \|Z_j(\Gamma v)_-(t,\cdot)\|_{H^1}+t\|\chi(t^{-\sigma}D_x)(\Gamma v)_-(t,\cdot)\|_{L^\infty}$$
$$+\sum_{\mu=0}^{1}\Big\|\chi(t^{-\sigma}D_x)\big[x_j^{\mu}\Gamma NL_{kg}(t,\cdot)\big]\Big\|_{L^\infty},$$

where now

$$\big\|\chi(t^{-\sigma}D_x)\big[x_j\Gamma NL_{kg}(t,\cdot)\big]\big\|_{L^\infty}\lesssim \|x_j\Gamma NL_{kg}(t,\cdot)\|_{L^\infty}\le C(A+B)B\varepsilon^2 t^{\frac{1}{2}+\frac{\delta_2}{2}},$$

after injecting (B.2.64) into (B.2.63). Therefore, using also (B.2.59), lemma B.2.8 and a-priori estimate (1.1.11d) with $k=2$, we find that

$$\big\|\chi(t^{-\sigma}D_x)\big[x_j(\Gamma v)_-(t,\cdot)\big]\big\|_{L^\infty}+\Big\|\chi(t^{-\sigma}D_x)\Big[x_j\frac{D_x}{\langle D_x\rangle}(\Gamma v)_-(t,\cdot)\Big]\Big\|_{L^\infty}\le CB\varepsilon t^{\frac{1}{2}+\frac{\delta_2}{2}}, \tag{B.2.66}$$

for any $\chi\in C_0^\infty(\mathbb{R}^2)$.

Finally, if instead of estimating the uniform norm of the first quadratic term in the right hand side of (B.1.20a) as in (B.2.60), we make use of lemma B.2.2 with $L=L^\infty$, $w_1=x(\Gamma v)_\pm$, and $s>0$ such that $N(s)\ge 2$, we would find that, for some $\chi_1\in C_0^\infty(\mathbb{R}^2)$,

$$\Big\|\chi(t^{-\sigma}D_x)x_jQ_0^{\mathrm{kg}}((\Gamma v)_\pm,D_1u_\pm)\Big\|_{L^\infty}$$
$$\lesssim\sum_{|\mu|,\nu=0}^{1}\Big\|\chi_1(t^{-\sigma}D_x)\Big[x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}(\Gamma v)_-(t,\cdot)\Big]\Big\|_{L^\infty}\|\chi(t^{-\sigma}D_x)\mathrm{R}_1^{\nu}u_\pm(t,\cdot)\|_{H^{2,\infty}}$$
$$+\sum_{\mu=0}^{1}t^{-2}\Big\|x_j^{\mu}(\Gamma v)_\pm(t,\cdot)\Big\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}.$$

Then, combining the above inequality with (B.2.61), (B.2.62), together with (B.1.17), (B.2.66), and all the other inequality to which we already referred before, from (B.1.20a) we find that

$$\big\|\chi(t^{-\sigma}D_x)\big[x_j\Gamma NL_{kg}(t,\cdot)\big]\big\|_{L^\infty}\le C(A+B)\varepsilon^2t^{\delta_2},$$

which injected into (B.2.65) finally implies, together with (1.1.11d) with $k=2$, lemma B.2.8, and (B.2.59), the wished estimate (B.2.57). □

Making use of lemmas B.2.8, B.2.11 estimate (B.2.47) can be improved of a factor $t^{-\frac{1}{2}}$. This improvement, that will be useful to derive (B.3.77), is showed in the following corollary.

Corollary B.2.12. *Let I be a multi-index of length 1, and $r^{I,NF}_{kg}$ be defined by* (B.2.41) *and having the explicit expression* (B.2.42). *There exists a constant $C>0$ such that, under the same assumption as in lemma B.2.7, for any $\rho\in\mathbb{N}$, $\chi\in C_0^\infty(\mathbb{R}^2)$, equal to 1 in a neighbourhood of the origin, $\sigma>0$ small, $j=1,2$,*

$$\Big\|\chi(t^{-\sigma}D_x)\Big[x_jr^{I,NF}_{kg}\Big](t,\cdot)\Big\|_{L^2}\le C(A+B)AB\varepsilon^3t^{-\frac{1}{2}+\beta+\frac{\delta+\delta_1}{2}}, \tag{B.2.67}$$

for every $t\in[1,T]$, with $\beta>0$ small, $\beta\to 0$ as $\sigma\to 0$.

Proof. Let us consider the cubic contribution $x_j NL^I_{kg}(D_1u)$ to $r^{I,NF}_{kg}$. Reminding (1.1.5), and applying lemma B.2.2 with $L=L^2$ and $s>0$ sufficiently large so that $N(s)\ge 2$, together with (B.1.26a) and a-priori estimates, we derive that there is some $\chi_1\in C^\infty_0(\mathbb{R}^2)$ such that

$$\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_jNL^I_{kg}(D_1u)\right](t,\cdot)\right\|_{L^2}\\
&\lesssim \left\|\chi_1(t^{-\sigma}D_x)\left[x_jNL^I_{kg}\right](t,\cdot)\right\|_{L^2}\|\mathrm{R}_1u_\pm(t,\cdot)\|_{L^\infty}+t^{-2}\|x_jNL^I_{kg}(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}\\
&\le CA\varepsilon t^{-\frac12}\left\|\chi_1(t^{-\sigma}D_x)\left[x_jNL^I_{kg}\right](t,\cdot)\right\|_{L^2}+C(A+B)B\varepsilon^2t^{-1},
\end{aligned}\tag{B.2.68}$$

Then, recalling (B.1.20a) and using again lemma B.2.2 with $L=L^2$, $w_1=(\Gamma v)_\pm$, and s large as before, in order to estimate the contribution coming from the first quadratic term in the right hand side of (B.1.20a), we find that there is a new $\chi_2\in C^\infty_0(\mathbb{R}^2)$ such that

$$\begin{aligned}
&\left\|\chi_1(t^{-\sigma}D_x)\left[x_jNL^I_{kg}\right](t,\cdot)\right\|_{L^2}\\
&\lesssim\sum_{|\mu|=0}^1\left\|\chi_2(t^{-\sigma}D_x)\Big[x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu(\Gamma v)_\pm\Big](t,\cdot)\right\|_{L^\infty}\|u_\pm(t,\cdot)\|_{H^1}+t^{-2}\|x_j(\Gamma v)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}\\
&+\sum_{|\mu|=0}^1\left\|x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\right\|_{L^\infty}\left(\|(\Gamma u)_\pm(t,\cdot)\|_{H^1}+\|u_\pm(t,\cdot)\|_{H^1}+\|D_tu_\pm(t,\cdot)\|_{L^2}\right)\\
&\le C(A+B)B\varepsilon^2t^{\beta+\frac{\delta+\delta_1}{2}},
\end{aligned}$$

where the latter estimate is obtained from (B.1.5a) with $s=0$, (B.1.10b), (B.1.17) with $k=1$, (B.2.57) and a-priori estimates. This implies, combined with (B.2.68), that

$$\left\|\chi(t^{-\sigma}D_x)\left[x_jNL^I_{kg}(D_1u)\right](t,\cdot)\right\|_{L^2}\le C(A+B)AB\varepsilon^3t^{-\frac12+\beta+\frac{\delta+\delta_1}{2}},$$

and from (B.2.42), (B.1.10b) and a-priori estimates,

$$\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_jr^{I,NF}_{kg}\right](t,\cdot)\right\|_{L^2}\lesssim\left\|\chi(t^{-\sigma}D_x)\left[x_jNL^I_{kg}(D_1u)\right](t,\cdot)\right\|_{L^2}\\
&+\sum_{\mu=0}^1t^\sigma\left(\|x^\mu_jv_\pm(t,\cdot)\|_{L^\infty}+\left\|x^\mu_j\frac{D_x}{\langle D_x\rangle}v_\pm(t,\cdot)\right\|_{L^\infty}\right)\|v^I_\pm(t,\cdot)\|_{L^2}\|v_\pm(t,\cdot)\|_{H^{2,\infty}}\\
&\le C(A+B)AB\varepsilon^3t^{-\frac12+\beta+\frac{\delta+\delta_1}{2}},
\end{aligned}$$

which concludes the proof of the statement. □

Lemma B.2.13. *Let I be a multi-index of length 2. There exists a constant $C>0$ such that, under the same hypothesis as in lemma B.2.7,*

$$\left\|x_j\Gamma^INL_{kg}(t,\cdot)\right\|_{L^2}\le C(A+B)B\varepsilon^2t^{\frac12+\beta+\frac{\delta_1+\delta_2}{2}},\tag{B.2.69}$$

with $\beta>0$ small, $\beta\to0$ as $\sigma\to0$, for every $j=1,2$, $t\in[1,T]$.

Proof. We remind the reader about (B.1.23). Instead of using (B.1.24), obtained by Sobolev injection, we apply corollary B.2.4 with $L=L^2$, $w=u$, and $s>0$ sufficiently large so that $N(s)\ge3$, and we exploit the fact that we have an estimate of the L^∞ norm of $D_1u^{I_2}$ when truncated for frequencies less or equal than t^σ (see lemma B.2.1). So, for $(I_1,I_2)\in\mathcal{I}(I)$ such

$|I_1| = |I_2| = 1$, we deduce that

$$\begin{aligned}
&\left\|x_j Q_0^{\mathrm{kg}}\left(v_\pm^{I_1}, D_1 u_\pm^{I_2}\right)\right\|_{L^2} \lesssim \sum_{\mu=0}^1 \left\|x_j^\mu v_\pm^{I_1}(t,\cdot)\right\|_{L^2} \left\|\chi(t^{-\sigma}D_x) u_\pm^{I_2}(t,\cdot)\right\|_{H^{2,\infty}} \\
&+ t^{-3}\left(\|u_\pm(t,\cdot)\|_{H^s} + \|D_t u_\pm(t,\cdot)\|_{H^s}\right)\left[\sum_{|\mu|=0}^2 \|x^\mu v_\pm^{I_1}(t,\cdot)\|_{L^2} + \sum_{|\mu|=0}^1 t\|x^\mu v_\pm^{I_1}(t,\cdot)\|_{L^2}\right] \\
&\le C(A+B)B\varepsilon^2 t^{\frac12+\beta+\frac{\delta_1+\delta_2}{2}},
\end{aligned}$$

last estimate following from lemma B.2.10 and (B.1.17) with $k=1$, together with (B.1.5a), (B.1.17) with $k=1$, (B.1.28), a-priori estimates, and the fact that $\delta_1, \delta_2 \ll 1$ are small. Consequently, from the following inequality

$$\begin{aligned}
&\|x_j \Gamma^I NL_{kg}\|_{L^2} \lesssim \sum_{\mu=0}^1 \|\mathrm{R}_1^\mu u_\pm(t,\cdot)\|_{H^{2,\infty}} \sum_{\substack{|J|\le 2\\ \mu=0,1}} \|x_j^\mu (\Gamma^J v)_-(t,\cdot)\|_{L^2} \\
&+ \sum_{|\mu|=0}^1 \left\|x_j \Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\right\|_{L^\infty} \Big[\|u_\pm^I(t,\cdot)\|_{H^1} + \sum_{|J|<2}\left(\|u_\pm^J(t,\cdot)\|_{H^1} + \|D_t u_\pm^J(t,\cdot)\|_{L^2}\right)\Big] \\
&+ \sum_{|I_1|=|I_2|=1} \left\|x_j Q_0^{\mathrm{kg}}\left(v_\pm^{I_1}, D_1 u_\pm^{I_2}\right)(t,\cdot)\right\|_{L^2},
\end{aligned}$$

together with (B.1.10b), (B.1.5a) with $s=0$, (B.1.7), and (B.1.17) with $k=1$, we finally derive (B.2.69). □

Lemma B.2.14. *Let us fix $s \in \mathbb{N}$. There exists a constant $C > 0$ such that, if we assume that a-priori estimates (1.1.11) are satisfied in some interval $[1,T]$, for a fixed $T > 1$, with $n \ge s+2$, then we have, for any $\chi \in C_0^\infty(\mathbb{R}^2)$ and $\sigma > 0$ small,*

$$\|\widetilde{v}(t,\cdot)\|_{H_h^s} \le CB\varepsilon t^{\frac{\delta}{2}}, \tag{B.2.70a}$$

$$\sum_{|\mu|=1} \|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}^\mu \widetilde{v}(t,\cdot)\|_{L^2} \le CB\varepsilon t^{\frac{\delta_2}{2}}, \tag{B.2.70b}$$

for every $t \in [1,T]$.

Proof. We warn the reader that, throughout the proof, C, β will denote two positive constants that may change line after line, and $\beta > 0$ is small as long as σ is small.

It is straightforward to check that the H_h^s norm of $\widetilde{v}$ is bounded by energy $E_n(t;W)^{\frac12}$, whenever $n \ge s+2$, after definitions (3.2.2), (3.1.3), inequality (3.1.7a), and a-priori estimates (1.1.11a), (1.1.11b).

In order to prove (B.2.70b), we first use relation (3.2.10b) and definition (3.1.3) to write that

$$\begin{aligned}
\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}_m \widetilde{v}(t,\cdot)\|_{L^2} &\lesssim \|Z_m V(t,\cdot)\|_{L^2} + \|\chi(t^{-\sigma}D_x)Z_m(v^{NF} - v_-)(t,\cdot)\|_{L^2} \\
&\quad + \|\widetilde{v}(t,\cdot)\|_{L^2} + \|\chi(t^{-\sigma}D_x)[x_m r_{kg}^{NF}](t,\cdot)\|_{L^2},
\end{aligned} \tag{B.2.71}$$

with r_{kg}^{NF} given by (3.1.5). Using (1.1.5), we can rewrite (3.1.10) and (3.1.11) similarly to (B.2.27), (B.2.42), as:

$$v^{NF} - v_- = -\frac{i}{2}\left[(D_t v)(D_1 u) - (D_1 v)(D_t u) + D_1[v D_t u] - \langle D_x\rangle[v D_1 u]\right], \tag{B.2.72}$$

and

$$r_{kg}^{NF} = -\frac{i}{2}\left[NL_{kg}D_1u - (D_1v)NL_w + D_1(vNL_w)\right]. \tag{B.2.73}$$

Reminding (1.1.5), and combining (B.2.73) with (B.1.10b) and a-priori estimates (1.1.11), we deduce that

(B.2.74)

$$\begin{aligned}\|\chi(t^{-\sigma}D_x)(x_m r_{kg}^{NF})(t,\cdot)\|_{L^2} &\lesssim t^\sigma\left(\|x_n v_-(t,\cdot)\|_{L^\infty} + \left\|x_n\frac{D_x}{\langle D_x\rangle}v_-(t,\cdot)\right\|_{L^\infty}\right)\\ &\times\Big[\left(\|U(t,\cdot)\|_{H^{2,\infty}} + \|\mathrm{R}_1U(t,\cdot)\|_{H^{2,\infty}}\right)\|U(t,\cdot)\|_{L^2} + \|V(t,\cdot)\|_{H^{2,\infty}}\|V(t,\cdot)\|_{L^2}\Big]\\ &+\|V(t,\cdot)\|^2_{H^{1,\infty}}\|V(t,\cdot)\|_{H^1} \le C(A+B)AB\varepsilon^3 t^{-\frac{1}{2}+\sigma+\frac{(\delta+\delta_2)}{2}}.\end{aligned}$$

Similarly to (B.2.35),

(B.2.75)

$$\begin{aligned}2iZ_m(v^{NF}-v_-) &= (D_tZ_mv)(D_1u) - (D_1Z_mv)(D_tu) + D_1[(Z_mv)(D_tu)] - \langle D_x\rangle[(Z_mv)(D_1u)]\\ &+ (D_tv)(D_1Z_mu) - (D_1v)(D_tZ_mu) + D_1[v(D_tZ_mu)] - \langle D_x\rangle[v(D_1Z_mu)]\\ &- (D_mv)(D_1u) + \delta^1_m(D_tv)(D_tu) - \delta^1_m D_t[v(D_tu)] + \frac{D_m}{\langle D_x\rangle}D_t[v(D_1u)]\\ &- \delta^1_m(D_tv)(D_tu) + (D_1v)(D_mu) - \delta^1_m D_1[v(D_tu)] + \delta^1_m\langle D_x\rangle[v(D_tu)],\end{aligned}$$

so bounding the L^2 norm of all products in the first line of above equality (when truncated for frequencies less or equal than t^σ) by means of lemma B.2.2, and all the others with the L^∞ norm of the Klein-Gordon factor times the L^2 norm of the wave one, we derive that

$$\begin{aligned}&\left\|\chi(t^{-\sigma}D_x)Z_m(v^{NF}-v_-)(t,\cdot)\right\|_{L^2}\\ &\quad\lesssim t^\sigma\left\|\chi_1(t^{-\sigma}D_x)(Z_mv)_\pm(t,\cdot)\right\|_{L^\infty}\|u_\pm(t,\cdot)\|_{L^2} + t^{-N(s)}\|(Z_mv)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}\\ &\qquad+ t^\sigma\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\left(\|(Z_mu)_\pm(t,\cdot)\|_{L^2} + \|u_\pm(t,\cdot)\|_{L^2} + \|D_tu_\pm(t,\cdot)\|_{L^2}\right),\end{aligned}$$

for some $\chi_1 \in C_0^\infty(\mathbb{R}^2)$. Consequently, picking $s>0$ sufficiently large such that $N(s)\ge 1$, and using (B.1.5a), lemma B.2.8 and a-priori estimates, we obtain that

$$\|\chi(t^{-\sigma}D_x)Z_m(v^{NF}-v_-)(t,\cdot)\|_{L^2} \le C(A+B)B\varepsilon^2 t^{-1+\beta+\frac{\delta+\delta_1}{2}}, \tag{B.2.76}$$

which plugged into (B.2.71), along with (B.2.74), (B.2.70a) and (1.1.11d), gives (B.2.70b). □

B.3 Second range of estimates and the sharp decay of the Klein-Gordon solution with a vector field

This subsection is focused on the derivation of a control in $O(t^{\beta'})$, for a small $\beta'>0$, of the L^2 norm of $\mathcal{L}^\mu\widetilde{v}$, with $|\mu|=2$ (see lemma B.3.7), and on the proof of a sharp decay estimate for the uniform norm of $(\Gamma v)_\pm$, where Γ is an admissible vector field in $\mathcal{Z}$ (see (1.1.7)) and when this term is truncated for frequencies less or equal than t^σ, for some small $\sigma>0$ (lemma B.3.21).

In order to prove lemma B.3.7, we need to introduce the following technical results.

Lemma B.3.1. *Let us consider v^{NF} introduced in* (3.1.3)*, $v^{I,NF}$ as in* (B.2.24) *with $|I| = 1$ and $\Gamma^I = Z_n$, and remind equalities* (B.2.72) *and* (B.2.27)*. There exists a constant $C > 0$ such that, if we assume that a-priori estimates* (1.1.11) *are satisfied in some interval $[1, T]$, for a fixed $T > 1$, with $n \geq s + 2$, then we have, for any $\chi \in C_0^\infty(\mathbb{R}^2)$ and $\sigma > 0$ small,*

$$\big\|\chi(t^{-\sigma}D_x)\big[(Z_n v)_- - v^{I,NF}\big](t,\cdot)\big\|_{L^2} \leq C(A+B)B\varepsilon^2 t^{-1+\beta+\frac{\delta+\delta_1}{2}}, \tag{B.3.1}$$

$$\begin{aligned}\big\|\chi(t^{-\sigma}D_x)\big[x_m Z_n(v^{NF} - v_-)(t,\cdot)\big]\big\|_{L^2} + \big\|\chi(t^{-\sigma}D_x)\big[x_m\big((Z_n v)_- - v^{I,NF}\big)\big](t,\cdot)\big\|_{L^2} \\ \leq C(A+B)B\varepsilon^2 t^{\beta+\frac{\delta_1+\delta_2}{2}},\end{aligned} \tag{B.3.2}$$

for every $t \in [1, T]$. The same estimates hold true if Z_n in replaced with Ω.

Proof. We warn the reader that we will denote by C and β some positive constants, that may change line after line, with $\beta \to 0$ as $\sigma \to 0$.

First of all, we observe that the difference $(Z_n v)_- - v^{I,NF}$, whose explicit expression is given by (B.2.27) with $\Gamma = Z_n$, appears in the first line of the explicit expression (B.2.75) of $Z_n(v^{NF} - v_-)$. Therefore, as inequality (B.2.76) has been obtained analysing term by term, it implies (B.3.1). For the same reason, it will be enough to prove that

$$\big\|\chi(t^{-\sigma}D_x)\big[x_m Z_n(v^{NF} - v_-)(t,\cdot)\big]\big\|_{L^2} \leq C(A+B)B\varepsilon^2 t^{\beta+\frac{\delta_1+\delta_2}{2}},$$

to deduce (B.3.2).

One can immediately check that this estimate holds true when considering all products appearing from second to fourth line in the right hand side of (B.2.75), as follows using (B.1.5a) with $s = 0$, (B.1.10b) and a-priori estimates, since their L^2 norm (when truncated for frequencies less or equal than t^σ) is estimated with

$$t^\sigma \sum_{\mu,\nu=0}^{1} \left\|x_m^\mu\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu v_\pm(t,\cdot)\right\|_{L^\infty} \left(\|(Z_m u)_\pm(t,\cdot)\|_{L^2} + \|u_\pm(t,\cdot)\|_{L^2} + \|D_t u_\pm(t,\cdot)\|_{L^2}\right).$$

The L^2 norm of the remaining terms, i.e. those coming out from the multiplication of x_m with products in the first line of (B.2.75) (and when truncated for moderate frequencies less or equal to t^σ), can be estimated using lemma B.2.2. So by (1.1.5), (1.1.10), it is bounded by

$$\begin{aligned}\sum_{\mu,\nu=0}^{1} \left\|\chi_1(t^{-\sigma}D_x)\Big[x_m^\mu\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu (Z_m v)_\pm(t,\cdot)\Big]\right\|_{L^\infty} \|u_\pm(t,\cdot)\|_{L^2} \\ + \sum_{\mu=0}^{1} t^{-N(s)}\|x_m^\mu (Z_m v)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s},\end{aligned}$$

for some smooth cut-off function χ_1, and with $N(s) \geq 2$ if $s > 0$ is large. By (B.1.17) and (B.2.57) with $\Gamma = Z_m$, together with a-priori estimates, we derive that the above contribution is estimated with $CB^2\varepsilon^2 t^{\beta+\frac{\delta+\delta_1}{2}}$, and that concludes the proof of (B.3.2).

When Z_n is replaced with Ω, instead of referring to (B.2.75) one uses that

$$\begin{aligned}2i\Omega\left(v^{NF} - v_-\right) = {} & (D_t\Omega v)(D_1 u) - (D_1\Omega v)(D_t u) + D_1[(\Omega v)(D_t u)] - \langle D_x\rangle[(\Omega v)(D_1 u)] \\ & + (D_t v)(D_1\Omega u) - (D_1 v)(D_t\Omega u) + D_1[v(D_t\Omega u)] - \langle D_x\rangle[v(D_1\Omega u)] \\ & - (D_t v)(D_2 u) + (D_2 v)(D_t u) - D_2[v(D_t u)] + \langle D_x\rangle[v(D_2 u)]\end{aligned}$$

and applies the same argument as above to recover the wished estimates. □

Lemma B.3.2. *Let v^{NF} be defined as in (3.1.3). There exists a constant $C>0$ such that, under the same assumptions as in lemma B.3.1, for any $\chi\in C_0^\infty(\mathbb{R}^2)$, $\sigma>0$ small, $m=1,2$,*

$$\big\|Op^w_h(\chi(h^\sigma\xi))[tZ_nv^{NF}(t,tx)]\big\|_{L^2(dx)}\le CB\varepsilon t^{\frac{\delta_2}{2}}, \tag{B.3.3a}$$

$$\big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m[tZ_nv^{NF}(t,tx)]\big\|_{L^2(dx)}\le CB\varepsilon t^{\frac{\delta_1}{2}}, \tag{B.3.3b}$$

for every $t\in[1,T]$.

Proof. From the fact that $\|tw(t,t\cdot)\|_{L^2}=\|w(t,\cdot)\|_{L^2}$, equality

$$Z_nv^{NF}=Z_n(v^{NF}-v_-)+\big[(Z_nv)_- - v^{I,NF}\big]+v^{I,NF}+\frac{D_n}{\langle D_x\rangle}v^{NF}+\frac{D_n}{\langle D_x\rangle}(v_- - v^{NF}), \tag{B.3.4}$$

with $v^{I,NF}$ given by (B.2.24) with $|I|=1$ and $\Gamma^I=Z_n$, and from estimates (B.2.45),(B.2.76), (B.3.1), along with (3.1.8a), a-priori estimates and

$$\begin{aligned}&\|\chi(t^{-\sigma}D_x)D_n\langle D_x\rangle^{-1}v^{NF}(t,\cdot)\|_{L^2}\\&\quad\le\|\chi(t^{-\sigma}D_x)D_n\langle D_x\rangle^{-1}v_-(t,\cdot)\|_{L^2}+\|\chi(t^{-\sigma}D_x)D_n\langle D_x\rangle^{-1}(v_- - v^{NF})(t,\cdot)\|_{L^2}\le CB\varepsilon t^{\frac{\delta}{2}},\end{aligned}$$

we immediately have (B.3.3a).

From (B.3.4) we also derive that

$$\begin{aligned}&\big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m[tZ_nv^{NF}(t,tx)]\big\|_{L^2(dx)}\lesssim\big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\big[tZ_n(v^{NF}-v_-)(t,tx)\big]\big\|_{L^2}\\&+\big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\big[t\big((Z_nv)_- - v^{I,NF}\big)(t,tx)\big]\big\|_{L^2}+\big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\big[tv^{I,NF}(t,tx)\big]\big\|_{L^2}\\&+\big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\big[tD_n\langle D_x\rangle^{-1}v^{NF}(t,tx)\big]\big\|_{L^2}\\&+\big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\big[tD_n\langle D_x\rangle^{-1}(v_- - v^{NF})(t,tx)\big]\big\|_{L^2}.\end{aligned} \tag{B.3.5}$$

Using relation (3.2.8) with $w=v^{I,NF}$ and estimates (B.2.45), (B.2.46), (B.2.48), we observe that

$$\big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\big[tv^{I,NF}(t,tx)\big]\big\|_{L^2}\le CB\varepsilon t^{\frac{\delta_1}{2}},$$

while from (3.2.2) and (B.2.70b)

$$\big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\big[tD_n\langle D_x\rangle^{-1}v^{NF}(t,tx)\big]\big\|_{L^2}\le CB\varepsilon t^{\frac{\delta_2}{2}}.$$

All other remaining L^2 norms in the right hand side of (B.3.5) are estimated reminding definition (1.2.60) of $\mathcal{L}_m$ and using the fact that

$$\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m[tw(t,tx)]\|_{L^2}\lesssim\|\chi(t^{-\sigma}D_x)[x_mw(t,\cdot)]\|_{L^2}+t\|\chi(t^{-\sigma}D_x)D_m\langle D_x\rangle^{-1}w(t,\cdot)\|_{L^2}. \tag{B.3.6}$$

Therefore, by (B.2.76) and lemma B.3.1 we derive that

$$\begin{aligned}\big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\big[tZ_n(v^{NF}-v_-)(t,tx)\big]\big\|_{L^2}+\big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\big[t\big((Z_nv)_- - v^{I,NF}\big)(t,tx)\big]\big\|_{L^2}\\\le C(A+B)B\varepsilon^2t^{\beta+\frac{\delta+\delta_1}{2}},\end{aligned}$$

while from (3.1.8a), a-priori estimates, and the following inequality

$$\|\chi(t^{-\sigma}D_x)[x_m(v_- - v^{NF})](t,\cdot)\|_{L^2} \lesssim \sum_{\mu,\nu=0}^{1} t^\sigma \left\| x_m^\mu \Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu v_-(t,\cdot)\right\|_{L^\infty} \|u_\pm(t,\cdot)\|_{L^2} \\ \le C(A+B)B\varepsilon^2 t^{\sigma+\frac{\delta+\delta_2}{2}},$$

which follows by (B.2.72), (1.1.5), (B.1.10b), and (1.1.11b), (1.1.11c), we derive

$$\big\| Op_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\big[tD_n\langle D_x\rangle^{-1}(v_- - v^{NF})(t,tx)\big]\big\|_{L^2} \le C(A+B)B\varepsilon^2 t^{\sigma+\frac{(\delta+\delta_2)}{2}},$$

which combined with previous estimates gives (B.3.3b) and concludes the proof of the statement.

□

In the following lemma we are basically going to show that, instead of having

$$\|\widetilde{v}\widetilde{u}(t,\cdot)\|_{L^2} \lesssim \|\widetilde{v}(t,\cdot)\|_{L^\infty}\|\widetilde{u}(t,\cdot)\|_{L^2} \le CAB\varepsilon^2 h^{-\frac{\delta}{2}},$$
$$\|\widetilde{v}\widetilde{u}(t,\cdot)\|_{L^\infty} \lesssim \|\widetilde{v}(t,\cdot)\|_{L^\infty}\|\widetilde{u}(t,\cdot)\|_{L^\infty} \le CA^2\varepsilon^2 h^{-\frac{1}{2}-\frac{\delta}{2}},$$

following from (B.2.1a), (B.3.8), (B.3.9), we actually have that

$$\|\widetilde{v}\widetilde{u}(t,\cdot)\|_{L^2} \lesssim \|\widetilde{v}(t,\cdot)\|_{L^\infty}\|\widetilde{u}(t,\cdot)\|_{L^2} \le CAB\varepsilon^2 h^{\frac{1}{2}-\frac{\delta}{2}},$$
$$\|\widetilde{v}\widetilde{u}(t,\cdot)\|_{L^\infty} \lesssim \|\widetilde{v}(t,\cdot)\|_{L^\infty}\|\widetilde{u}(t,\cdot)\|_{L^\infty} \le CA^2\varepsilon^2 h^{-\frac{\delta}{2}},$$

which enhance the former ones of a factor $h^{\frac{1}{2}}$ (i.e. of $t^{-1/2}$). The reason for these enhanced estimates is to be found in the fact that the main contribution to the Klein Gordon component $\widetilde{v}$ is around the lagrangian Λ_{kg}, with

$$\Lambda_{kg} = \left\{(x,\xi) : x - \frac{\xi}{\langle\xi\rangle} = 0\right\},$$

while that to the wave component $\widetilde{u}$ is localised around Λ_w,

$$\Lambda_w = \left\{(x,\xi) : x - \frac{\xi}{|\xi|} = 0\right\},$$

and these two manifolds have empty intersection (see picture B.1).

Lemma B.3.3. *Let $h = t^{-1}$, $\widetilde{u},\widetilde{v}$ be defined in* (3.2.2)*, $a_0(\xi) \in S_{0,0}(1)$, and $b_1(\xi) = \xi_j$ or $b_1(\xi) = \xi_j\xi_k|\xi|^{-1}$, with $j,k \in \{1,2\}$. There exists a constant $C > 0$ such that, under the same assumptions as in lemma B.3.1, for any $\chi,\chi_1 \in C_0^\infty(\mathbb{R}^2)$, $\sigma > 0$, we have that*

(B.3.7a) $$\big\|[Op_h^w(\chi(h^\sigma\xi)a_0(\xi))\widetilde{v}(t,\cdot)][Op_h^w(\chi_1(h^\sigma\xi)b_1(\xi))\widetilde{u}(t,\cdot)]\big\|_{L^2} \le C(A+B)B\varepsilon^2 h^{\frac{1}{2}-\beta-\frac{\delta+\delta_1}{2}},$$

(B.3.7b) $$\big\|[Op_h^w(\chi(h^\sigma\xi)a_0(\xi))\widetilde{v}(t,\cdot)][Op_h^w(\chi_1(h^\sigma\xi)b_1(\xi))\widetilde{u}(t,\cdot)]\big\|_{L^\infty} \le C(A+B)B\varepsilon^2 h^{-\beta-\frac{\delta+\delta_1}{2}},$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. Before entering in the details of the proof, we warn the reader that C and β denote two positive constants, that may change line after line, with $\beta \to 0$ as $\sigma \to 0$. Also, we will denote by

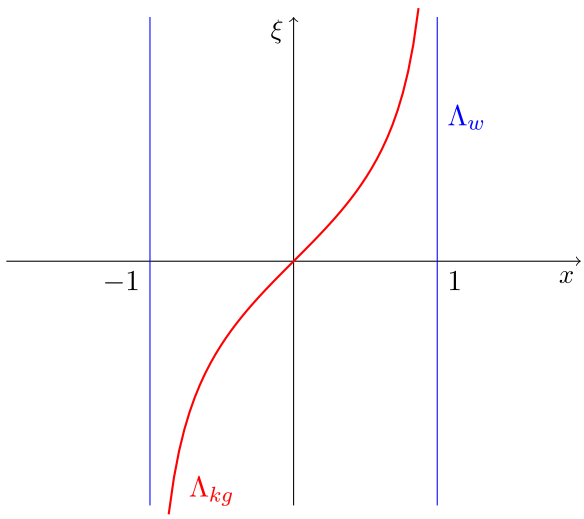

Figure B.1: Manifolds Λ_{kg} and Λ_w.

$R(t,x)$ any contribution, in what follows, that satisfies inequalities (B.3.7), and by χ_2 a smooth cut-off function, identically equal to 1 on the support of χ_1, so that

$$Op_h^w(\chi_1(h^\sigma\xi))\widetilde{u} = Op_h^w(\chi_1(h^\sigma\xi))Op_h^w(\chi_2(h^\sigma\xi))\widetilde{u},$$

assuming that, at any time, $\widetilde{u}$ can be replaced with $Op_h^w(\chi_2(h^\sigma\xi))\widetilde{u}$. Finally, we remind that from (3.2.2), (3.1.15), (3.1.20b), (3.1.20c), and a-priori estimates,

$$\|\widetilde{u}(t,\cdot)\|_{H^{\rho+1,\infty}_h} + \sum_{|\mu|=1}\|Op_h^w\big((\xi|\xi|^{-1})^\mu\big)\widetilde{u}(t,\cdot)\|_{H^{\rho+1,\infty}} \le CA\varepsilon h^{-\frac{1}{2}}, \tag{B.3.8}$$

while by (3.1.3), (3.1.7b) (for a small $\theta \ll 1$) and a-priori estimates,

$$\|\widetilde{v}(t,\cdot)\|_{H^{\rho,\infty}_h} \le CA\varepsilon, \tag{B.3.9}$$

for every $t \in [1,T]$.

First of all, we introduce $\gamma \in C_0^\infty(\mathbb{R}^2)$, equal to 1 in a neighbourhood of the origin and with sufficiently small support, and define

$$\widetilde{v}_{\Lambda_{kg}}(t,x) := Op_h^w\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)a_0(\xi)\Big)\widetilde{v}(t,x),$$
$$\widetilde{v}_{\Lambda^c_{kg}}(t,x) := Op_h^w\Big((1-\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)a_0(\xi)\Big)\widetilde{v}(t,x),$$

where $p(\xi) := \langle\xi\rangle$, so that

$$Op_h^w(\chi(h^\sigma\xi)a_0(\xi))\widetilde{v} = \widetilde{v}_{\Lambda_{kg}} + \widetilde{v}_{\Lambda^c_{kg}}.$$

We observe that, by proposition 1.2.38 with $p=+\infty$ and (B.3.9),

$$\|\widetilde{v}_{\Lambda_{kg}}(t,\cdot)\|_{L^\infty} \le CA\varepsilon h^{-\beta}, \tag{B.3.10a}$$

and that

$$\|\widetilde{v}_{\Lambda^c_{kg}}(t,\cdot)\|_{L^\infty} \le CB\varepsilon h^{\frac{1}{2}-\beta-\frac{\delta_1}{2}}. \tag{B.3.10b}$$

In fact, if we write

$$(1-\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)a_0(\xi) = \sum_{j=1}^2 \gamma^j_1\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)a_0(\xi)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)$$

with $\gamma^j_1(z) := (1-\gamma)(z)z_j|z|^{-1}$ such that $|\partial^\alpha_z\gamma^j_1(z)| \lesssim \langle z\rangle^{-|\alpha|}$, and use (1.2.61) with $c(x,\xi) = \chi(h^\sigma\xi)a_0(\xi)$, we deduce the following inequality

(B.3.11)

$$\begin{aligned}
\|\widetilde{v}_{\Lambda^c_{kg}}(t,\cdot)\|_{L^\infty} \lesssim & \sum_{j=1}^2 \sqrt{h}\Big\|Op^w_h\Big(\gamma^j_1\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)a_0(\xi)\Big)\mathcal{L}_j\widetilde{v}(t,\cdot)\Big\|_{L^\infty} \\
& + \sum_{j=1}^2 \sqrt{h}\Big\|Op^w_h\Big(\gamma^j_1\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\partial_j\big(\chi(h^\sigma\xi)a_0(\xi)\big)\Big)\widetilde{v}(t,\cdot)\Big\|_{L^\infty} \\
& + \sum_{j=1}^2\sum_{|\alpha|=2} \sqrt{h}\Big\|Op^w_h\Big((\partial^\alpha\gamma^j_1)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)a_0(\xi)(\partial^\alpha_\xi p')(\xi)\Big)\widetilde{v}(t,\cdot)\Big\|_{L^\infty} \\
& + \|Op^w_h(r(x,\xi))\widetilde{v}(t,\cdot)\|_{L^\infty},
\end{aligned}$$

with $r \in h^{1-\beta}S_{\frac{1}{2},\sigma}(\langle\frac{x-p'(\xi)}{\sqrt{h}}\rangle^{-1})$. As γ^j_1 and its derivatives vanish in a neighbourhood of the origin, we can use (3.2.18b) and successively derive, from lemma B.2.14, that the second and third norm in the above right hand side are bounded by the right hand side of (B.3.10b). The same estimate holds for the latter L^∞ norm of above, just by proposition 1.2.37 and (B.2.70a). As concerns the first norm in the right hand side of (B.3.11), it satisfies the mentioned estimate, as one can check using relation (3.2.8), followed by (3.2.18b), lemmas B.2.14, B.3.2 and (B.2.74). Consequently, from (B.2.1a), (B.3.8), and (B.3.10b), we obtain that

$$Op^w_h(\chi(h^\sigma\xi)a_0(\xi))\widetilde{v}\, Op^w_h(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u} = \widetilde{v}_{\Lambda_{kg}}Op^w_h(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u} + R(t,x).$$

On the other hand,

$$Op^w_h(\chi_1(h^\sigma\xi)b_1(\xi))\widetilde{u} = Op^w_h(\chi_0(h^{-1}\xi)b_1(\xi))\widetilde{u} + \sum_k Op^w_h\big((1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi_1(h^\sigma\xi)b_1(\xi)\big)\widetilde{u},$$

for some suitably supported $\chi_0 \in C^\infty_0(\mathbb{R}^2)$, $\varphi \in C^\infty_0(\mathbb{R}^2\setminus\{0\})$. From proposition 1.2.36 and the classical Sobolev injection, we have that

$$\big\|Op^w_h(\chi_0(h^{-1}\xi)b_1(\xi))\widetilde{u}(t,\cdot)\big\|_{L^2} + \big\|Op^w_h(\chi_0(h^{-1}\xi)b_1(\xi))\widetilde{u}(t,\cdot)\big\|_{L^\infty} \lesssim h\|\widetilde{u}(t,\cdot)\|_{L^2},$$

so after (B.3.10a) and (B.2.1a) we also derive that

$$\widetilde{v}_{\Lambda_{kg}}Op^w_h(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u} = \sum_k \widetilde{v}_{\Lambda_{kg}}Op^w_h(\phi_k(\xi)b_1(\xi))\widetilde{u} + R(t,x), \tag{B.3.12}$$

with $\phi_k(\xi) := (1-\chi_0)(h^{-1}\xi)\varphi(2^{-k}\xi)\chi(h^\sigma\xi)$. We further decompose $Op^w_h(\phi_k(\xi)b_1(\xi))\widetilde{u}$ by defining

$$\begin{aligned}
\widetilde{u}^k_{\Lambda_w}(t,x) &:= Op^w_h\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\Big)\widetilde{u}(t,x), \\
\widetilde{u}^k_{\Lambda^c_w}(t,x) &:= Op^w_h\Big((1-\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\Big)\widetilde{u}(t,x).
\end{aligned}$$

Since

$$\left\|\widetilde{u}^k_{\Lambda^c_w}(t,\cdot)\right\|_{L^2} \lesssim h^{\frac{1}{2}-\beta}\left[\|\widetilde{u}(t,\cdot)\|_{L^2} + \sum_{\mu,|\nu|=0}^{1}\|(\theta_0\Omega_h)^\mu \mathcal{M}^\nu Op^w_h(\chi_2(h^\sigma\xi))\widetilde{u}(t,\cdot)\|_{L^2}\right] \\ \leq CB\varepsilon h^{\frac{1}{2}-\beta-\frac{\delta_1}{2}},$$

and

$$\left\|\widetilde{u}^k_{\Lambda^c_w}(t,\cdot)\right\|_{L^\infty} \lesssim h^{-\beta}\left[\|\widetilde{u}(t,\cdot)\|_{L^2} + \sum_{\mu,|\nu|=0}^{1}\|(\theta_0\Omega_h)^\mu \mathcal{M}^\nu Op^w_h(\chi_2(h^\sigma\xi))\widetilde{u}(t,\cdot)\|_{L^2}\right] \leq CB\varepsilon h^{-\beta-\frac{\delta_1}{2}},$$

as follows by writing

$$(1-\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi) = \sum_{j=1}^{2}\gamma^j_1\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big(\frac{x_j|\xi|-\xi_j}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi),$$

with $\gamma^j_1(z) := (1-\gamma)(z)z_j|z|^{-2}$, and using (1.2.48) with $a\equiv 1$, $p=1$, together with lemma B.2.1, we derive that $\widetilde{v}_{\Lambda_{kg}}\widetilde{u}^k_{\Lambda^c_w}$ is a remainder $R(t,x)$, too, from (B.3.10a).

Now, reminding (3.2.15) and the fact that, by symbolic calculus,

$$\Gamma^{kg} = \Gamma^{kg}\theta_h(x) + Op^w_h(r_\infty(t,x)), \tag{B.3.13}$$

with $\|Op^w_h(r_\infty(t,x))\|_{\mathcal{L}(L^2;L^\infty)} = O(h^N)$, $N\in\mathbb{N}$ as large as we want, we can actually replace $\widetilde{v}_{\Lambda_{kg}}\widetilde{u}^k_{\Lambda_w}$ with $\theta_h\widetilde{v}_{\Lambda_{kg}}\widetilde{u}^k_{\Lambda_w}$, up to a new $R(t,x)$. As $\theta_h(x)$ is supported for $|x|\leq 1-ch^{2\sigma}$, for a small constant $c>0$,

$$\theta_h(x)\widetilde{v}_{\Lambda_{kg}}\widetilde{u}^k_{\Lambda_w} = \frac{\theta_h(x)}{|x|^2-1}\widetilde{v}_{\Lambda_{kg}}(|x|^2-1)\widetilde{u}^k_{\Lambda_w},$$

where $|\theta_h(x)(|x|^2-1)^{-1}| \lesssim h^{-2\sigma}$, so after proposition 1.2.36 and (B.2.70a), together with (B.3.10a),

$$\left\|\theta_h(x)\widetilde{v}_{\Lambda_{kg}}(t,\cdot)\widetilde{u}^k_{\Lambda_w}(t,\cdot)\right\|_{L^2} \leq CB\varepsilon h^{-\frac{\delta}{2}-\beta}\|\theta_h(x)(|x|^2-1)\widetilde{u}^k_{\Lambda_w}(t,\cdot)\|_{L^\infty}, \tag{B.3.14a}$$

$$\left\|\theta_h(x)\widetilde{v}_{\Lambda_{kg}}(t,\cdot)\widetilde{u}^k_{\Lambda_w}(t,\cdot)\right\|_{L^\infty} \leq CA\varepsilon h^{-\beta}\|\theta_h(x)(|x|^2-1)\widetilde{u}^k_{\Lambda_w}(t,\cdot)\|_{L^\infty}. \tag{B.3.14b}$$

By symbolic calculus of proposition 1.2.21, $\theta_h(x)(|x|^2-1)\widetilde{u}^k_{\Lambda_w}$ can be written in terms of $h\mathcal{M}\widetilde{u}$. In fact, for a fixed $N\in\mathbb{N}$ and up to some negligible multiplicative constants, we have that

(B.3.15)

$$\begin{aligned}\left[\theta_h(x)(|x|^2-1)\right]\sharp\left[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\right] &= \gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\theta_h(x)(|x|^2-1)\\ &+ h\left\{\theta_h(x)(|x|^2-1), \gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\right\}\\ &+ \sum_{|\alpha|=2}^{N-1} h^{|\alpha|}\partial^\alpha_x\left[\theta_h(x)(|x|^2-1)\right]\partial^\alpha_\xi\left[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\right] + r_N(x,\xi),\end{aligned}$$

with

$$\begin{aligned} r_N(x,\xi) = \frac{h^N}{(\pi h)^4}\sum_{|\alpha|=N}\int e^{\frac{2i}{h}(\eta\cdot z-y\cdot\zeta)}\int_0^1 \partial^\alpha_x[\theta_h(x)(|x|^2-1)]|_{(x+tz)}(1-t)^{N-1}dt\\ \times\partial^\alpha_\xi\left[\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\right]|_{(x+y,\xi+\eta)}dydzd\eta d\zeta.\end{aligned} \tag{B.3.16}$$

On the one hand, as

$$|x|^2 - 1 = x\cdot x - \frac{\xi\cdot\xi}{|\xi|^2} = (x|\xi|-\xi)\cdot\frac{x}{|\xi|} + (x|\xi|-\xi)\cdot\frac{\xi}{|\xi|^2},$$

the first term in the right hand side of (B.3.15) appears to be as linear combination of products of the form $\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\phi_k(\xi)a(x)b_0(\xi)(x_j|\xi|-\xi_j)$, for some smooth compactly supported function $a(x)$, and $b_0(\xi)$ such that $|\partial^\alpha b_0(\xi)|\lesssim|\xi|^{-|\alpha|}$, so from (1.2.48b) and lemma B.2.1, we deduce that

$$\text{(B.3.17a)}\qquad \Big\|Op_h^w\Big(\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\theta_h(x)(|x|^2-1)\Big)\widetilde{u}(t,\cdot)\Big\|_{L^\infty}\le CB\varepsilon h^{\frac{1}{2}-\beta-\frac{\delta_1}{2}}.$$

An explicit computation shows that

$$\begin{aligned}&h\Big\{\theta_h(x)(|x|^2-1),\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\Big\}\\&\quad=\sum_i h^{\frac{1}{2}+\sigma}\partial_{x_i}[\theta_h(x)(|x|^2-1)]\sum_j(\partial_j\gamma)\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\Big(x_j\frac{\xi_i}{|\xi|}-\delta_i^j\Big)\phi_k(\xi)b_1(\xi)\\&\qquad\qquad+h\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\partial_x[\theta_h(x)(|x|^2-1)]\partial_\xi(\phi_k(\xi)b_1(\xi)),\end{aligned}$$

with $\delta_i^j=1$ if $i=j$, 0 otherwise. As the first contribution to the above right hand side is still supported for $|x|<1-ch^{2\sigma}$, we can make appear a new factor $|x|^2-1$ in front of it (up to a loss in $h^{-2\sigma}$), and rewrite it as a linear combination of terms $h^{\frac{1}{2}-\sigma}\gamma_1\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\phi_k(\xi)a(x)b_0(\xi)(x_j|\xi|-\xi_j)$, for a new $\gamma_1\in C_0^\infty(\mathbb{R}^2)$, and some new $a(x),b_0(\xi)$ with the same properties as the ones we considered before. On the other hand, as $\partial_\xi(\phi_k(\xi)b_1(\xi))$ is uniformly bounded and supported for frequencies $|\xi|\sim 2^k$, the second term in the above right hand side writes as linear combination of products $h\gamma\big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\big)\phi^1_k(\xi)a(x)b_0(\xi)$, for some new $\phi^1_k\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$. Inequality (1.2.48b), proposition 1.2.30, and lemma B.2.1, give then that

$$\text{(B.3.18)}\qquad \Big\|hOp_h^w\Big(\Big\{\theta_h(x)(|x|^2-1),\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\phi_k(\xi)b_1(\xi)\Big\}\Big)\widetilde{u}(t,\cdot)\Big\|_{L^\infty}\le CB\varepsilon h^{\frac{1}{2}-\beta-\frac{\delta_1}{2}}.$$

Using (1.2.24), we find that the $|\alpha|$-order terms, with $2\le|\alpha|\le N-1$, are given by

$$\begin{aligned}&h^{|\alpha|}\gamma\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\partial_x^\alpha[\theta_h(x)(|x|^2-1)]\partial_\xi^\alpha(\phi_k(\xi)b_1(\xi))\\&\qquad+\sum_{\substack{|\beta_1|+|\beta_2|=|\alpha|\\|\beta_1|\ge1}}\sum_{j=1}^{|\beta_1|}h^{|\alpha|-j(\frac{1}{2}-\sigma)}\gamma_j\Big(\frac{x|\xi|-\xi}{h^{1/2-\sigma}}\Big)\widetilde{\theta}_j(x)b_{j-|\beta_1|}(\xi)\partial_\xi^{\beta_2}(\phi_k(\xi)b_1(\xi)),\end{aligned}$$

for some $\gamma_j,\widetilde{\theta}_j\in C_0^\infty(\mathbb{R}^2)$. Since $|\alpha|\ge2$ and $|\partial_\xi^\mu(\phi_k(\xi)b_1(\xi))|\lesssim 2^{-k(|\mu|-1)}$, for any $\mu\in\mathbb{N}^2$, by proposition 1.2.30 and lemma B.2.1, we obtain that the action of their quantization on $\widetilde{u}$ is estimated in the uniform norm by

$$\text{(B.3.19)}\qquad \Big[h^{|\alpha|-\frac{1}{2}-\beta}2^{-k(|\alpha|-1)}+\sum_{1\le j\le|\alpha|}h^{|\alpha|-j(\frac{1}{2}-\sigma)}2^{k(j+1-|\alpha|)}h^{-\frac{1}{2}-\beta}\Big]\times\Big[\|\widetilde{u}(t,\cdot)\|_{L^2}+\sum_{\mu,|\nu|=0}^{1}\|(\theta\Omega_h)^\mu Op_h^w(\chi_1(h^\sigma\xi)\widetilde{u}(t,\cdot)\|_{L^2}\Big]\le CB\varepsilon h^{\frac{1}{2}-\beta-\frac{\delta_1}{2}}.$$

Finally, integrating in $dyd\zeta$ and using (1.2.23), together with the fact that $|\partial_\xi^\mu(\phi_k(\xi)b_1(\xi))| \lesssim 2^{-k(|\mu|-1)}$, we find from (B.3.16) that $r_N(x,\xi)$ can be written as

$$\sum_{j\le N} h^{N-j(\frac{1}{2}-\sigma)}\frac{1}{(\pi h)^2}\int e^{\frac{2i}{h}\eta\cdot z}\int_0^1 \theta_N(x+tz)(1-t)^{N-1}dt \\ \times \gamma_j\Big(\frac{x|\xi+\eta|-(\xi+\eta)}{h^{1/2-\sigma}}\Big)\phi^j_k(\xi+\eta)b_{j+1-N}(\xi+\eta)dzd\eta,$$

for some new smooth compactly supported $\theta_N, \gamma_j, \phi^j_k$, and it follows from proposition 1.2.31 that the quantization of the above integral is a bounded operator from L^2 to L^∞, with norm controlled by

$$\sum_{\substack{j\le N\\ i\le 6}} h^{N-j(\frac{1}{2}-\sigma)}2^{k(1+j-N)}(h^{-\frac{1}{2}+\sigma}2^k)^i(h^{-1}2^k)\lesssim h,$$

if N is sufficiently large (e.g. $N\ge 10$). Consequently,

$$\|Op^w_h(r_N(x,\xi))\widetilde{u}(t,\cdot)\|_{L^\infty}\lesssim h\|\widetilde{u}^k(t,\cdot)\|_{L^2}\le CB\varepsilon h^{1-\frac{\delta}{2}}, \tag{B.3.20}$$

which, summed up with formulas from (B.3.15) to (B.3.19), gives that

$$\|\theta_h(x)(|x|^2-1)\widetilde{u}_{\Lambda_w}(t,\cdot)\|_{L^\infty}\lesssim CB\varepsilon h^{\frac{1}{2}-\beta-\frac{\delta_1}{2}}.$$

Therefore, from (B.3.14), $\theta_h(x)\widetilde{v}_{\Lambda_{kg}}\widetilde{u}^k_{\Lambda_w}$ is a remainder $R(t,x)$, and that finally concludes the proof of the statement. ☐

We state next that a result similar to lemma B.3.3 holds true if $\widetilde{u}$ is replaced with

$$\widetilde{u}^J(t,x) := t(\Gamma u)_-(t,tx) \tag{B.3.21}$$

with $\Gamma\in\{\Omega, Z_m, m=1,2\}$ being a Klainerman vector field.

Lemma B.3.4. *Let $h=t^{-1}$, $\widetilde{v}$ be defined in (3.2.2), $\widetilde{u}^J$ as in (B.3.21), $a_0(\xi)\in S_{0,0}(1)$, and $b_1(\xi)=\xi_j$ or $b_1(\xi)=\xi_j\xi_k|\xi|^{-1}$, with $j,k\in\{1,2\}$. There exists a constant $C>0$ such that, under the same assumptions as in lemma B.3.1, for any $\chi,\chi_1\in C^\infty_0(\mathbb{R}^2)$, $\sigma>0$, we have that*

$$\big\|[Op^w_h(\chi(h^\sigma\xi)a_0(\xi))\widetilde{v}(t,\cdot)][Op^w_h(\chi_1(h^\sigma\xi)b_1(\xi))\widetilde{u}^J(t,\cdot)]\big\|_{L^2}\le C(A+B)B\varepsilon^2h^{\frac{1}{2}-\beta'}, \tag{B.3.22a}$$

$$\big\|[Op^w_h(\chi(h^\sigma\xi)a_0(\xi))\widetilde{v}(t,\cdot)][Op^w_h(\chi_1(h^\sigma\xi)b_1(\xi))\widetilde{u}^J(t,\cdot)]\big\|_{L^\infty}\le C(A+B)B\varepsilon^2h^{-\beta'}, \tag{B.3.22b}$$

with $\beta'>0$ small, $\beta\to 0$ as $\sigma,\delta_0\to 0$.

Proof. The proof of this result is analogous to that of lemma B.3.3 except that, instead of referring to (B.3.8), we should use that

$$\|Op^w_h(\chi(h^\sigma\xi))\widetilde{u}^J(t,\cdot)\|_{H^{\rho+1,\infty}_h}+\sum_{|\mu|=1}\|Op^w_h\big(\chi(h^\sigma\xi))(\xi|\xi|^{-1})^\mu\big)\widetilde{u}^J(t,\cdot)\|_{H^{\rho+1,\infty}}\le CA\varepsilon h^{-\frac{1}{2}-\beta-\frac{\delta_1}{2}}, \tag{B.3.23}$$

which follows from (B.2.52) in classical coordinates, and to lemma B.2.9 instead of lemma B.2.1. ☐

Lemma B.3.5. *Let $a_0(\xi) \in S_{0,0}(1)$, $b_1(\xi) \in \{\xi_j, \xi_j\xi_k|\xi|^{-1}, |\xi|, j,k=1,2\}$, $b_0(\xi) \in \{1, \xi_j|\xi|^{-1}, j = 1,2\}$. There exists a constant $C>0$ such that, under the same assumptions as in lemma B.3.1, for any $\chi \in C_0^\infty(\mathbb{R}^2)$, $\sigma>0$ small,*

$$\left\| \chi(t^{-\sigma}D_x)\big[[a_0(D_x)v_-][b_1(D_x)u_-]b_0(D_x)u_-\big](t,\cdot)\right\|_{L^\infty} \le C(A+B)AB\varepsilon^3 t^{-\frac{5}{2}+\beta+\frac{\delta+\delta_1}{2}}, \tag{B.3.24}$$

for every $t\in[1,T]$, with $\beta>0$ small, $\beta\to 0$ as $\sigma\to 0$. Moreover,

$$\left\|\chi(t^{-\sigma}D_x)r^{NF}_{kg}(t,\cdot)\right\|_{L^\infty} \le C(A+B)AB\varepsilon^3 t^{-\frac{5}{2}+\beta+\frac{\delta+\delta_1}{2}}, \tag{B.3.25}$$

where r^{NF}_{kg} is given by (B.2.73).

Proof. We warn the reader that we denote by C and β two positive constants, that may change line after line during this proof, with $\beta\to 0$ as $\sigma \to 0$.

From lemma B.2.2, with $L=L^\infty$, and a-priori estimates, we can reduce ourselves to estimate the L^∞ norm of product in the left hand side of (B.3.24) when all its factors are supported for frequencies less or equal than t^σ, up to remainders satisfying the inequality of the statement. Moreover, since

$$\left\|\chi(t^{-\sigma}D_x)a_0(D_x)[v^{NF}-v_-](t,\cdot)\right\|_{L^\infty} \le CA^2\varepsilon^2 t^{-\frac{3}{2}+\sigma}, \tag{B.3.26a}$$

and

$$\left\|\chi(t^{-\sigma}D_x)b_1(D_x)[u^{NF}-u_-](t,\cdot)\right\|_{L^\infty} \le CA^2\varepsilon^2 t^{-2+\beta}, \tag{B.3.26b}$$

as follows, respectively, by (B.2.72) and (3.1.15), (A.20b) with $\rho=2$ (as consequence of lemma A.8), together with a-priori estimates, we can also suppose v_- (resp. u_-) be replaced with v^{NF} (resp. u^{NF}), up to some new $O_{L^\infty}(\varepsilon^3 t^{-\frac{5}{2}+\beta})$, with $\beta>0$ small, $\beta\to 0$ as $\sigma\to 0$.

This reduces us to prove that

$$\begin{aligned}\left\|[\chi(t^{-\sigma}D_x)a_0(D_x)v^{NF}][\chi(t^{-\sigma}D_x)b_1(D_x)u^{NF}][\chi(t^{-\sigma}D_x)b_0(D_x)u_-]\right\|_{L^\infty}&\\ \le C(A+B)AB\varepsilon^3 t^{-\frac{5}{2}+\beta+\frac{\delta+\delta_1}{2}},&\end{aligned}$$

or rather, using (1.1.11a), to show that

$$\left\|[\chi(t^{-\sigma}D_x)a_0(D_x)v^{NF}][\chi(t^{-\sigma}D_x)b_1(D_x)u^{NF}]\right\|_{L^\infty} \le C(A+B)B\varepsilon^2 t^{-2+\beta+\frac{\delta+\delta_1}{2}}.$$

But writing the above product in the semi-classical setting, and reminding definition (3.2.2), one can immediately check that this estimate is satisfied thanks to (B.3.7b), and that concludes the proof of (B.3.24).

Finally, (B.3.25) follows from (3.1.11), the fact that

$$\left\|\chi(t^{-\sigma}D_x)\left[-\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\,NL_w + D_1\big[\langle D_x\rangle^{-1}(v_+-v_-)\,NL_w\right](t,\cdot)\right\|_{L^\infty} \le CA^3\varepsilon^3 t^{-3+\sigma},$$

for every $t\in[1,T]$, which is consequence of (B.1.3b), and a-priori estimate (1.1.11b), and from the observation that the remaining contributions to r^{NF}_{kg} are products of the form

$$[a_0(D_x)v_-][b_1(D_x)u_-]\mathrm{R}_1 u_-,$$

with $a_0(\xi)$ equal to 1 or to $\xi_j\langle\xi\rangle^{-1}$, and $b_1(\xi)$ equal to ξ_1 or to $\xi_j\xi_1|\xi|^{-1}$, for $j=1,2$. □

Lemma B.3.6. *Under the same assumptions as in lemma B.3.5,*

(B.3.27a) $$\big\|\chi(t^{-\sigma}D_x)\big[x_n[a_0(D_x)v_-][b_1(D_x)u_-]b_0(D_x)u_-\big](t,\cdot)\big\|_{L^2}\le C(A+B)^2B\varepsilon^3t^{-1+\beta+\frac{\delta_1}{2}},$$

(B.3.27b) $$\big\|\chi(t^{-\sigma}D_x)\big[x_mx_n[a_0(D_x)v_-][b_1(D_x)u_-]b_0(D_x)u_-\big](t,\cdot)\big\|_{L^2}\le C(A+B)^2B\varepsilon^3t^{\beta+\frac{\delta_1}{2}},$$

for every $t\in[1,T]$, $m,n=1,2$, with $\beta>0$ small, $\beta\to 0$ as $\sigma\to 0$. Moreover,

(B.3.28a) $$\big\|\chi(t^{-\sigma}D_x)\big[x_nr^{NF}_{kg}(t,\cdot)\big]\big\|_{L^2}\le C(A+B)^2B\varepsilon^3t^{-1+\beta+\frac{\delta+\delta_1}{2}},$$

(B.3.28b) $$\big\|\chi(t^{-\sigma}D_x)\big[x_mx_nr^{NF}_{kg}(t,\cdot)\big]\big\|_{L^2}\le C(A+B)^2B\varepsilon^3t^{\beta+\frac{\delta+\delta_1}{2}}.$$

Proof. We warn the reader that we will denote by C,β two positive constants, that may change line after line, with $\beta\to 0$ as $\sigma\to 0$. We also denote by $R(t,x)$ any contribution verifying

(B.3.29a) $$\big\|\chi(t^{-\sigma}D_x)\big[x_nR(t,\cdot)\big]\big\|_{L^2}\le C(A+B)^2B\varepsilon^3t^{-1+\beta+\frac{\delta+\delta_1}{2}},$$

(B.3.29b) $$\big\|\chi(t^{-\sigma}D_x)\big[x_mx_nR(t,\cdot)\big]\big\|_{L^2}\le C(A+B)^2B\varepsilon^3t^{\beta+\frac{\delta+\delta_1}{2}}.$$

Let us first notice that, after (B.1.3b), (B.1.10a), (B.1.27a) along with a-priori estimates, we have that

$$\begin{aligned}\Big\|\chi(t^{-\sigma}D_x)\Big[-x_n\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\,NL_w+x_nD_1\big[\langle D_x\rangle^{-1}(v_+-v_-)\,NL_w\big]&(t,\cdot)\Big\|_{L^2}\\ \lesssim t^\sigma\sum_{\mu=0}^1\|x_n^\mu v_\pm(t,\cdot)\|_{L^2}\,\|NL_w(t,\cdot)\|_{L^\infty}&\le CA^2B\varepsilon^3t^{-1+\sigma+\frac{\delta}{2}},\end{aligned}$$

and

$$\begin{aligned}\Big\|\chi(t^{-\sigma}D_x)\Big[-x_mx_n\frac{D_x}{\langle D_x\rangle}(v_+-v_-)\,NL_w+x_mx_nD_1\big[\langle D_x\rangle^{-1}(v_+-v_-)\,NL_w\big]&(t,\cdot)\Big\|_{L^2}\\ \lesssim t^\sigma\sum_{\mu_1,\mu_2=0}^1\|x_m^{\mu_1}x_n^{\mu_2}v_\pm(t,\cdot)\|_{L^2}\,\|NL_w(t,\cdot)\|_{L^\infty}&\le C(A+B)AB\varepsilon^3t^{\sigma+\frac{\delta}{2}}.\end{aligned}$$

Therefore, since from (3.1.11) and (B.1.1b) the remaining contributions to r^{NF}_{kg} are of the form

$$[a_0(D_x)v_-][b_1(D_x)u_-]\mathrm{R}_1u_-,$$

with $a_0(\xi)$ equal to 1 or to $\xi_j\langle\xi\rangle^{-1}$, and $b_1(\xi)$ equal to ξ_1 or to $\xi_j\xi_1|\xi|^{-1}$, for $j=1,2$, estimates (B.3.28) will follow from (B.3.27).

After lemma B.2.2 with $L=L^2$, $w_1=x_na_0(D_x)v_-$ (resp. $w_1=x_mx_na_0(D_x)v_-$), and $s>0$ sufficiently large so that $N(s)>2$, together with estimates (B.1.10a), (resp. (B.1.27a)) and (1.1.11a), (1.1.11c), we can suppose all above factors truncated for frequencies less or equal than t^σ, up to remainders $R(t,x)$ satisfying (B.3.29a) (resp. (B.3.29b)). Let us also observe that, from (B.1.10b), (B.3.26b) and (1.1.11c),

$$\begin{aligned}\Big\|\chi(t^{-\sigma}D_x)\Big[[\chi_1(t^{-\sigma}D_x)[x_na_0(D_x)v_-]][\chi(t^{-\sigma}D_x)b_1(D_x)(u^{NF}-u_-)][\chi(t^{-\sigma}D_x)b_0(D_x)u_-]\Big](t,\cdot)(t,\cdot)\Big\|_{L^2}\\ \lesssim\sum_{|\mu|=0}^1\Big\|x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\Big\|_{L^\infty}\|\chi(t^{-\sigma}D_x)b_1(D_x)(u^{NF}-u_-)\|_{L^\infty}\|u_\pm(t,\cdot)\|_{L^2}\\ \le C(A+B)A^2B\varepsilon^4t^{-2+\beta+\frac{\delta+\delta_2}{2}},\end{aligned}$$

and from (B.1.27b)

$$\Big\|\chi(t^{-\sigma}D_x)\Big[[\chi_1(t^{-\sigma}D_x)[x_m x_n a_0(D_x)v_-][\chi(t^{-\sigma}D_x)b_1(D_x)(u^{NF}-u_-)][\chi(t^{-\sigma}D_x)b_0(D_x)u_-]\Big](t,\cdot)(t,\cdot)\Big\|_{L^2}$$
$$\lesssim \sum_{|\mu|,|\nu|=0}^{1} t^{\beta}\Big\|x_m x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu} v_{\pm}(t,\cdot)\Big\|_{L^\infty}\|\chi(t^{-\sigma}D_x)b_1(D_x)(u^{NF}-u_-)\|_{L^2}\|\mathrm{R}_1^{\nu}u_{\pm}(t,\cdot)\|_{L^\infty}$$
$$\le C(A+B)A^2B\varepsilon^4 t^{-1+\beta+\frac{\delta+\delta_2}{2}},$$

so, up to a new remainder $R(t,x)$, we actually replace u_- by u^{NF}.

Furthermore, we can substitute $\chi_1(t^{-\sigma}D_x)[x_m^{\mu}x_n a_0(D_x)v_-]$, for $\mu=0,1$, respectively with $\chi(t^{-\sigma}D_x)[x_m^{\mu}x_n a_0(D_x)v_-^{NF}]$, again up to some $R(t,x)$ satisfying (B.3.29), in consequence of a-priori estimate (1.1.11a), the fact that

$$\|u^{NF}(t,\cdot)\|_{L^2}\le CB\varepsilon t^{\frac{\delta}{2}}, \tag{B.3.30}$$

(see (B.2.1a) in semi-classical coordinates), and the following inequalities

$$\big\|\chi_1(t^{-\sigma}D_x)\left[x_n a_0(D_x)(v^{NF}-v_-)\right](t,\cdot)\big\|_{L^\infty}$$
$$\lesssim \sum_{\mu,\nu,\kappa=0}^{1} t^{\sigma}\Big\|x_n^{\mu}\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\nu}v_{\pm}(t,\cdot)\Big\|_{L^\infty}\|\mathrm{R}_1^{\kappa}u_{\pm}(t,\cdot)\|_{L^\infty}\le C(A+B)A\varepsilon^2 t^{-\frac{1}{2}+\sigma+\frac{\delta_2}{2}}, \tag{B.3.31a}$$

and

$$\big\|\chi_1(t^{-\sigma}D_x)\left[x_m x_n a_0(D_x)(v^{NF}-v_-)\right](t,\cdot)\big\|_{L^\infty}$$
$$\lesssim \sum_{\mu_1,\mu_2,\nu,\kappa=0}^{1}\Big\|x_m^{\mu_1}x_n^{\mu_2}\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\nu}v_{\pm}(t,\cdot)\Big\|_{L^\infty}\|\mathrm{R}_1^{\kappa}u_{\pm}(t,\cdot)\|_{L^\infty}\le C(A+B)A\varepsilon^2 t^{\frac{1}{2}+\frac{\delta_2}{2}}, \tag{B.3.31b}$$

derived from (3.1.10), (B.1.10b), (B.1.27b), (1.1.11a) and (1.1.11b).

This reduces us to prove that, for $\mu=0,1$,

$$\big\|\left[\chi_1(t^{-\sigma}D_x)[x_m^{\mu}x_n a_0(D_x)v_-^{NF}]\right]\left[\chi(t^{-\sigma}D_x)b_1(D_x)u^{NF}\right]\left[\chi(t^{-\sigma}D_x)b_0(D_x)u_-\right]\big\|_{L^2}$$
$$\le C(A+B)^2B\varepsilon^3 t^{-1+\mu+\beta+\frac{\delta_1}{2}},$$

or rather, after (1.1.11a), that

$$\big\|\left[\chi_1(t^{-\sigma}D_x)[x_m^{\mu}x_n a_0(D_x)v_-^{NF}]\right]\left[\chi(t^{-\sigma}D_x)b_1(D_x)u^{NF}\right]\big\|_{L^2}\le C(A+B)B\varepsilon^2 t^{-\frac{1}{2}+\mu+\beta+\frac{\delta+\delta_1}{2}}.$$

Passing to the semi-classical setting, with $\widetilde{u}$, $\widetilde{v}$ given by (3.2.2), this corresponds to prove that

$$\sum_{k=0}^{1}\Big\|\Big[Op_h^w(\chi_1(h^{\sigma}\xi))[x_m^k x_n Op_h^w(a_0(\xi))\widetilde{v}\Big]\left[Op_h^w(\chi(h^{\sigma}\xi)b_1(\xi))\widetilde{u}\right]\Big\|_{L^2}\le C(A+B)B\varepsilon^2 h^{\frac{1}{2}-\beta-\frac{\delta+\delta_1}{2}}. \tag{B.3.32}$$

We remind that, in this setting, we have (B.3.8), (B.3.9).

Let us notice that, from the commutation of x_n with $Op_h^w(a_0(\xi))$, and definition (1.2.60) of $\mathcal{L}$, we have

$$x_n Op_h^w(a_0(\xi))\widetilde{v}=hOp_h^w(a_0(\xi))\mathcal{L}_n\widetilde{v}+Op_h^w(a_0(\xi)\xi_n\langle\xi\rangle^{-1})\widetilde{v}-\frac{h}{2i}Op_h^w\big(\partial_{\xi_n}a_0(\xi)\big)\widetilde{v}, \tag{B.3.33}$$

while from the commutation of x_m with $Op_h^w(\chi(h^\sigma\xi)b_1(\xi))$, definition (1.2.45) of $\mathcal{M}$, and symbolic calculus,

(B.3.34)
$$x_m Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u} = hOp_h^w(\chi(h^\sigma\xi)b_1(\xi)|\xi|^{-1})\mathcal{M}_m\widetilde{u} - \frac{h}{2i}Op_h^w\big(\partial_{\xi_m}(\chi(h^\sigma\xi)b_1(\xi)|\xi|^{-1})|\xi|\big)\widetilde{u} + Op_h^w(\chi(h^\sigma\xi)b_1(\xi)\xi_m|\xi|^{-1})\widetilde{u} - \frac{h}{2i}Op_h^w\big(\partial_{\xi_m}(\chi(h^\sigma\xi)b_1(\xi))\big)\widetilde{u}.$$

Therefore, when $k=0$ in the left hand side of (B.3.32), from (B.3.33), lemma B.2.14, together with (B.3.8), we deduce that

$$\text{(B.3.35)}\quad \Big\| [Op_h^w(\chi_1(h^\sigma\xi)[x_n Op_h^w(a_0(\xi))\widetilde{v}](t,\cdot)]\, [Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}(t,\cdot)]\Big\|_{L^2} \le \big\|[Op_h^w(\chi_1(h^\sigma\xi)a_0'(\xi))\widetilde{v}(t,\cdot)][Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}(t,\cdot)]\big\|_{L^2} + CAB\varepsilon^2 h^{\frac{1}{2}-\beta-\frac{\delta_2}{2}},$$

with $a_0'(\xi) = a_0(\xi)\xi_n\langle\xi\rangle^{-1}$.
When $k=1$, we rewrite the left hand side of (B.3.32) by making use of both (B.3.33), (B.3.34) (having previously commutated x_m to $Op_h^w(\chi_1(h^\sigma\xi))$). First, we observe that from the semi-classical Sobolev injection and estimates (B.2.70b), (B.2.1c),

$$\text{(B.3.36)}\quad \big\|h^2\big[Op_h^w(\chi_1(h^\sigma\xi)a_0(\xi))\mathcal{L}_n\widetilde{v}\big]\big[Op_h^w(\chi(h^\sigma\xi)b_1(\xi)\xi_m|\xi|^{-1})\mathcal{M}_m\widetilde{u}\big]\big\|_{L^2} \lesssim h\,\|Op_h^w(\chi_1(h^\sigma\xi)a_0(\xi))\mathcal{L}_n\widetilde{v}(t,\cdot)\|_{L^2}\,\big\|Op_h^w(\chi(h^\sigma\xi)b_1(\xi)\xi_m|\xi|^{-1})\mathcal{M}_m\widetilde{u}(t,\cdot)\big\|_{L^2} \le C(A+B)B\varepsilon^2 h^{1-\delta_2-\beta}.$$

Therefore, as for any $\theta\in]0,1[$ we have that

$$\text{(B.3.37)}\quad \big\|Op_h^w(b_1(\xi)\xi_m|\xi|^{-1})\widetilde{u}(t,\cdot)\big\|_{L^\infty} = t\,\big\|b_1(D_x)D_m|D_x|^{-1}u^{NF}(t,\cdot)\big\|_{L^\infty} \lesssim t\|u^{NF}(t,\cdot)\|^{1-\theta}_{H^{3,\infty}}\|u^{NF}(t,\cdot)\|_{H^2} \le CA^{1-\theta}B^\theta\varepsilon t^{\frac{1}{2}+\frac{(1+\delta)}{2}\theta},$$

which follows similarly to (2.2.75), along with (3.1.20a), (3.1.20b) and a-priori estimates, we find from (B.3.34), (B.2.70b), (B.3.8), (B.3.37) (with $\theta\ll 1$ small) and (B.3.36) that

$$\text{(B.3.38)}\quad h\,\big\|Op_h^w(\chi_1(h^\sigma\xi)a_0(\xi))\mathcal{L}_n\widetilde{v}\,\big[x_m Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big]\big\|_{L^2} \le C(A+B)B\varepsilon^2 h^{\frac{1}{2}-\frac{\delta_2}{2}-\frac{(1+\delta)\theta}{2}}.$$

Moreover, using (B.3.9) together with (B.3.34), (B.2.1a), (B.2.1c), we also find that

$$h\,\big\|Op_h^w\big(\chi_1(h^\sigma\xi)\partial_{\xi_n}a_0(\xi)\big)\widetilde{v}\,\big[x_m Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big]\big\|_{L^2} \le C(A+B)B\varepsilon^2 h^{1-\beta-\frac{\delta_2}{2}}.$$

Summing up these two above estimates, together with (B.3.33), (B.3.34) and (B.3.35) (the contribution in the left hand side basically appears with $\chi_1(h^\sigma\xi)$ replaced with $h^\sigma(\partial_m\chi_1)(h^\sigma\xi)$, because of the commutation between x_m and $Op_h^w(\chi_1(h^\sigma\xi))$), we deduce that

(B.3.39)
$$\Big\| [Op_h^w(\chi_1(h^\sigma\xi))[x_m x_n Op_h^w(a_0(\xi))\widetilde{v}]\, [Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}]\Big\|_{L^2} \lesssim \big\|[Op_h^w(\chi_1(\xi)a_0'(\xi))\widetilde{v}][Op_h^w(\chi(h^\sigma\xi)b_1'(\xi))\widetilde{u}]\big\|_{L^2} + h^\sigma\,\big\|[Op_h^w((\partial_m\chi_1)(h^\sigma\xi)a_0'(\xi))\widetilde{v}(t,\cdot)][Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}(t,\cdot)]\big\|_{L^2} + C(A+B)B\varepsilon^2 h^{\frac{1}{2}-\beta-\frac{\delta}{2}},$$

with the same a_0', b_1 as before, and $b_1'(\xi) := b_1(\xi)\xi_m|\xi|^{-1}$.

Finally, from (B.3.7a) we derive that

(B.3.40)
$$\begin{aligned}&\big\|[Op^w_h(\chi(\xi)a_0'(\xi))\widetilde{v}][Op^w_h(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}]\big\|_{L^2} + \big\|[Op^w_h(\chi(\xi)a_0'(\xi))\widetilde{v}][Op^w_h(\chi(h^\sigma\xi)b_1'(\xi))\widetilde{u}]\big\|_{L^2}\\ &+ \big\|[Op^w_h((\partial_m\chi_1)(h^\sigma\xi)a_0'(\xi))\widetilde{v}(t,\cdot)][Op^w_h(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}(t,\cdot)]\big\|_{L^2} \le C(A+B)B\varepsilon^2 h^{\frac{1}{2}-\beta-\frac{\delta+\delta_1}{2}},\end{aligned}$$

which injected in (B.3.35), (B.3.39) gives (B.3.32), and concludes the proof of the statement. □

Lemma B.3.7. *There exists a constant $C>0$ such that, under the same hypothesis as in lemma B.2.14, for any $\chi\in C^\infty_0(\mathbb{R}^2)$ and $\sigma>0$ small,*

(B.3.41)
$$\sum_{|\mu|=2}\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{v}(t,\cdot)\|_{L^2} \le CB\varepsilon t^{\beta+\frac{\delta+\delta_1}{2}},$$

for every $t\in[1,T]$, with $\beta>0$ small, $\beta\to 0$ as $\sigma\to 0$.

Proof. From relation (3.2.10b) and the commutation between $\mathcal{L}_m$ and $Op^w_h(\langle\xi\rangle)$, we deduce that

(B.3.42)
$$\begin{aligned}\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\mathcal{L}_n\widetilde{v}(t,\cdot)\|_{H^1} \lesssim \sum_{\mu=0}^{1}\Big[&\big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu_m\big[tZ_nv^{NF}(t,tx)\big]\big\|_{L^2}\\ &+\Big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu_m Op^w_h\Big(\frac{\xi_n}{\langle\xi\rangle}\Big)\widetilde{v}(t,\cdot)\Big]\Big\|_{L^2} + \big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu_m\big[t(tx_n)r^{NF}_{kg}(t,tx)\big]\big\|_{L^2}\Big],\end{aligned}$$

so the result of the statement follows from lemmas B.2.14, B.3.2, and B.3.6 combined with the fact that

(B.3.43)
$$\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m[tw(t,tx)]\|_{L^2} \lesssim \|\chi(t^{-\sigma}D_x)[x_mw(t,\cdot)]\|_{L^2} + t\|\chi(t^{-\sigma}D_x)w(t,\cdot)\|_{L^2}.$$

□

The aim of lemma B.3.21 below is to obtain the sharp decay estimate $O(t^{-1})$ of the uniform norm of the Klein-Gordon component when some Klainerman vector fields is acting on it, and when restricted to frequencies with moderate growth t^σ, with $\sigma>0$. It makes use, and improves, the previous result of lemma B.2.8. Before proving it, we introduce some preliminary results.

Lemma B.3.8. *There exists a positive constant $C>0$ such that, under the same hypothesis as in lemma B.3.1, for any $\chi\in C^\infty_0(\mathbb{R}^2)$, $\sigma>0$ small, $n=1,2$,*

(B.3.44)
$$\sum_{\substack{|I_1|+|I_2|\le 2\\ |I_1|<2}}\Big\|\chi(t^{-\sigma}D_x)\big[x_nQ^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm)\big]\Big\|_{L^2} \le C(A+B)B\varepsilon^2 t^{\beta+\frac{\delta+\delta_2}{2}},$$

with $D=D_1$ whenever $|I_1|+|I_2|=2$, $D\in\{D_j,D_t, j=1,2\}$ otherwise, and where $\beta>0$ is small, $\beta\to 0$ as $\sigma\to 0$.

Proof. We estimate the L^2 norms in the left hand side of (B.3.44) separately.

• When $|I_1|=0$, $|I_2|=2$, we derive from (B.1.10b) and (1.1.11d) that

$$\begin{aligned}\Big\|\chi(t^{-\sigma}D_x)\big[x_nQ^{\mathrm{kg}}_0(v_\pm, D_1u^{I_2}_\pm)\big]\Big\|_{L^2} &\lesssim \sum_{|\mu|=0}^{1}\Big\|x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\Big\|_{L^\infty}\|u^{I_2}_\pm(t,\cdot)\|_{H^1}\\ &\le C(A+B)B\varepsilon t^{\frac{\delta_1+\delta_2}{2}}.\end{aligned}$$

• When $|I_1| = |I_2| = 1$, we make use of corollary B.2.4 with I_2, $w = u$, and $s > 0$ sufficiently large so that $N(s) \geq 2$, together with remark B.2.5, and derive that

$$
\begin{aligned}
&\Big\|\chi(t^{-\sigma}D_x)\left[x_n Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D_1 u_\pm^{I_2})\right]\Big\|_{L^2} \\
&\lesssim \sum_{|\mu|=0}^{1}\Big\|\chi_1(t^{-\sigma}D_x)\Big[x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu} v_\pm^{I_1}\Big](t,\cdot)\Big\|_{L^\infty}\|u_\pm^{I_2}(t,\cdot)\|_{H^1} \\
&+ \sum_{\substack{|\mu|=0,1,2\\|\nu|=0,1}} t^{-N(s)}\Big(\|x^\mu v_\pm^{I_1}(t,\cdot)\|_{L^2} + t\|x^\nu v_\pm^{I_1}(t,\cdot)\|_{L^2}\Big)\big(\|u_\pm(t,\cdot)\|_{H^s} + \|D_t u_\pm(t,\cdot)\|_{H^s}\big) \\
&\leq CB^2\varepsilon^2 t^{\frac{\delta_1+\delta_2}{2}}
\end{aligned}
\tag{B.3.45}
$$

last estimated deduced using (B.1.5a), (B.1.17), (B.1.28), (B.2.57) and (1.1.11d).

• When $|I_1| + |I_2| \leq 1$, derivative D can be equal to D_x or to D_t. Then

- If $|I_1| = 0$,

$$
\begin{aligned}
&\Big\|\chi(t^{-\sigma}D_x)\big[x_n Q_0^{\mathrm{kg}}(v_\pm, D u_\pm^{I_2})\big]\Big\|_{L^2} \\
&\lesssim \sum_{|\mu|=0}^{1}\Big\|x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu} v_\pm(t,\cdot)\Big\|_{L^\infty}\Big(\|u_\pm^{I_2}(t,\cdot)\|_{H^1} + \|D_t u_\pm^{I_2}(t,\cdot)\|_{L^2}\Big) \leq C(A+B)B\varepsilon^2 t^{\delta_2},
\end{aligned}
$$

after (B.1.7), (B.1.10b) and a-priori estimates;

- If $|I_1| = 1$, $|I_2| = 0$, after lemma B.2.2 with $L = L^2$, (B.1.5a), (B.1.17), (B.2.57) and a-priori estimates, we derive that

$$
\begin{aligned}
&\Big\|\chi(t^{-\sigma}D_x)\big[x_n Q_0^{\mathrm{kg}}(v_\pm^{I_1}, D u_\pm)\big]\Big\|_{L^2} \\
&\lesssim \sum_{|\mu|=0}^{1}\Big\|\chi_1(t^{-\sigma}D_x)\Big[x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu} v_\pm^{I_1}(t,\cdot)\Big]\Big\|_{L^\infty}\big(\|u_\pm(t,\cdot)\|_{H^1} + \|D_t u_\pm(t,\cdot)\|_{L^2}\big) \\
&+ \sum_{\mu|=0}^{1} t^{-N(s)}\Big\|x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu} v_\pm^{I_1}(t,\cdot)\Big\|_{L^2}\big(\|u_\pm(t,\cdot)\|_{H^s} + \|D_t u_\pm(t,\cdot)\|_{H^s}\big) \\
&\leq CB^2\varepsilon^2 t^{\beta+\frac{\delta+\delta_1}{2}},
\end{aligned}
\tag{B.3.46}
$$

having chosen $s > 0$ sufficiently large to have $N(s) > 1$. □

Lemma B.3.9. *There exists a positive constant $C > 0$ such that, under the same hypothesis as in lemma B.2.14, for any $\chi \in C_0^\infty(\mathbb{R}^2)$, $\sigma > 0$ small,*

$$
\sum_{|I|=2}\big\|\chi(t^{-\sigma}D_x)V^I(t,\cdot)\big\|_{H^{\rho,\infty}} \leq CB\varepsilon t^{-1+\beta+\frac{\delta_0}{2}},
\tag{B.3.47}
$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$.

Proof. Estimate (B.3.47) is evidently satisfied when $I = (i_1, i_2)$ is such Γ_{i_j} is a spatial derivative, for at least one index i_j, thanks to lemma B.2.8. We then focus on the case when $\Gamma_{i_1}, \Gamma_{i_2} \in \{\Omega, Z_m, m = 1, 2\}$ are both Klainerman vector fields, and as $v_+^I = -\overline{v_-^I}$, we reduce to prove the statement for $\chi(t^{-\sigma}D_x)v_-^I$. Moreover, from the $L^\infty - H^{\rho,\infty}$ continuity of $\chi(t^{-\sigma}D_x)$ with norm

$O(t^{\sigma\rho})$, for any $\rho \in \mathbb{N}$, we can assume the $H^{\rho,\infty}$ norm in (B.3.47) replaced with the L^∞ one, up to a loss $t^{\sigma\rho}$.

As done in lemma B.2.8, instead of proving the statement directly on $\chi(t^{-\sigma}D_x)v^I_-$ we do it for $\chi(t^{-\sigma}D_x)v^{I,NF}$, with $v^{I,NF}$ introduced in (B.2.24) and considered here with $|I| = 2$. This is justified by inequality (B.2.38), which is consequence of (B.2.25).

An explicit computation shows that, from (B.2.24), (1.1.17), and (1.1.5), (1.1.10), $v^{I,NF}$ is solution to (B.2.39), with $NL^{I,NF}_{kg}$ given here by

$$NL^{I,NF}_{kg} = r^{I,NF}_{kg}(t,x) + \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|<2}} c_{I_1,I_2} Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm), \tag{B.3.48}$$

with $c_{I_1,I_2} \in \{-1,0,1\}$, $c_{I_1,I_2} = 1$ when $|I_1| + |I_2| = 2$ (in which case derivative D corresponds to D_1), and with $r^{I,NF}_{kg}$ equal to (B.2.41) and having the explicit expression (B.2.42).

After inequality (B.2.43), estimates

$$\begin{aligned}\left\|\chi(t^{-\sigma}D_x)(v^{I,NF} - v^I_-)(t,\cdot)\right\|_{L^2} &\lesssim t^\sigma \|v^I_\pm(t,\cdot)\|_{L^2}\left(\|u_\pm(t,\cdot)\|_{L^\infty} + \|\mathrm{R}_1 u_\pm(t,\cdot)\|_{L^\infty}\right)\\ &\le CAB\varepsilon^2 t^{-\frac{1}{2}+\frac{\delta_1}{2}+\sigma},\end{aligned}$$

$$\begin{aligned}\left\|\chi(t^{-\sigma}D_x)v^{I,NF}(t,\cdot)\right\|_{L^2} &\le \left\|\chi(t^{-\sigma}D_x)v^I_-(t,\cdot)\right\|_{L^2} + \left\|\chi(t^{-\sigma}D_x)(v^{I,NF} - v^I_-)(t,\cdot)\right\|_{L^2}\\ &\le CB\varepsilon t^{\frac{\delta_1}{2}},\end{aligned} \tag{B.3.49}$$

deduced from (B.2.27) and a-priori estimates, together with (B.2.26) and (1.1.11d) with $k = 0$, the only thing we need to show in order to prove the statement is that

$$\left\|\chi(t^{-\sigma}D_x)\left[x_j NL^{I,NF}_{kg}\right](t,\cdot)\right\|_{L^2} \le C(A+B)B\varepsilon^2 t^{\beta+\frac{\delta_1+\delta_2}{2}}. \tag{B.3.50}$$

But from (B.3.48), (B.2.42) with $|I| = 2$, we have that

$$\begin{aligned}&\left\|\chi(t^{-\sigma}D_x)\left[x_j NL^{I,NF}_{kg}\right](t,\cdot)\right\|_{L^2} \lesssim \|x_j NL^I_{kg}(t,\cdot)\|_{L^2}\left(\|u_\pm(t,\cdot)\|_{L^\infty} + \|\mathrm{R}_1 u_\pm(t,\cdot)\|_{L^\infty}\right)\\ &+ \sum_{\mu=0}^1 t^\sigma\left(\|x^\mu_j v_\pm(t,\cdot)\|_{L^\infty} + \left\|x^\mu_j \frac{D_x}{\langle D_x\rangle} v_\pm(t,\cdot)\right\|_{L^\infty}\right)\|v^I_\pm(t,\cdot)\|_{L^2}\|v_\pm(t,\cdot)\|_{H^{2,\infty}}\\ &+ \sum_{\substack{(I_1,I_2)\in\mathcal{I}(I)\\ |I_1|<2}} \left\|\chi(t^{-\sigma}D_x)\left[x_j Q^{\mathrm{kg}}_0(v^{I_1}_\pm, Du^{I_2}_\pm)\right](t,\cdot)\right\|_{L^2}\end{aligned} \tag{B.3.51}$$

so (B.3.50) follows from a-priori estimates, (B.1.10b), (B.2.69) and (B.3.44). As $\delta_2 \ll \delta_1 \ll \delta_0$, that concludes the proof of the statement. ☐

Lemma B.3.10. *There exists a positive constant $C > 0$ such that, under the same hypothesis as in lemma B.3.1, for any multi-index I of length 2, any $\chi \in C^\infty_0(\mathbb{R}^2)$, $\sigma > 0$ small, and $j = 1,2$,*

$$\left\|\chi(t^{-\sigma}D_x)\left[x_j(\Gamma^I v)_\pm\right](t,\cdot)\right\|_{L^\infty} \le CB\varepsilon t^{\beta+\frac{\delta_0}{2}}, \tag{B.3.52}$$

with $\beta > 0$ small, $\beta \to 0$ as $\sigma \to 0$, for every $t \in [1,T]$.

Proof. From equality (B.1.9b) with $w = (\Gamma^I v)_-$, which is solution to equation (B.1.18a), the $L^2 - L^\infty$ continuity of operator $\chi(t^{-\sigma}D_x)\langle D_x\rangle^{-1}$ with norm $O(t^\sigma)$, and the continuity on L^∞ of $\chi(t^{-\sigma}D_x)D_x\langle D_x\rangle^{-1}$ with norm $O(t^\sigma)$, we derive that

$$\begin{aligned}\left\|\chi(t^{-\sigma}D_x)\left[x_j(\Gamma^I v)_-\right](t,\cdot)\right\|_{L^\infty} &\lesssim t^\sigma\|Z_j(\Gamma^I v)_-(t,\cdot)\|_{L^2} + t\left\|\chi(t^{-\sigma}D_x)(\Gamma^I v)_-(t,\cdot)\right\|_{L^\infty}\\ &+ t^\sigma\left\|\chi(t^{-\sigma}D_x)\left[x_j\Gamma^I NL_{kg}\right](t,\cdot)\right\|_{L^\infty}.\end{aligned} \tag{B.3.53}$$

Reminding (B.1.23) and applying lemma B.2.2 with $L = L^\infty$, we have that there is some $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ such that

(B.3.54)
$$\begin{aligned}\left\|\chi(t^{-\sigma}D_x)\left[x_j\Gamma^I NL_{kg}\right](t,\cdot)\right\|_{L^\infty} &\lesssim \sum_{\mu,\nu=0}^{1}\left\|\chi_1(t^{-\sigma}D_x)\left[x_j^\mu(\Gamma^I v)_\pm\right](t,\cdot)\right\|_{L^\infty}\|\mathrm{R}_1^\nu u_\pm(t,\cdot)\|_{H^{2,\infty}}\\ &+t^{-N(s)}\sum_{\mu=0}^{1}\left\|x_j^\mu(\Gamma^I v)_\pm(t,\cdot)\right\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}\\ &+\sum_{\substack{(I_1,I_2)\in\mathfrak{I}(I)\\|I_1|<2}}\left\|\chi(t^{-\sigma}D_x)\left[x_jQ_0^{\mathrm{kg}}\left(v_\pm^{I_1},Du_\pm^{I_2}\right)\right](t,\cdot)\right\|_{L^\infty}.\end{aligned}$$

Therefore, picking $s>0$ large so that $N(s)>1$, and using the L^2-L^∞ continuity of $\widetilde{\chi}(t^{-\sigma}D_x)$ with norm $O(t^\sigma)$ if $\widetilde{\chi}$ is a smooth cut-off function, together with the a-priori estimates, (B.1.17) with $k=2$, and (B.3.44), we find at first that

$$\left\|\chi(t^{-\sigma}D_x)\left[x_j\Gamma^I NL_{kg}\right](t,\cdot)\right\|_{L^\infty}\le CAB\varepsilon^2t^{\frac{1}{2}+\sigma+\frac{\delta_1}{2}},$$

and, from (B.3.53) and (B.3.47),

$$\left\|\chi(t^{-\sigma}D_x)\left[x_j(\Gamma^I v)_-\right](t,\cdot)\right\|_{L^\infty}\le CB\varepsilon t^{\frac{1}{2}+\sigma+\frac{\delta_1}{2}}. \tag{B.3.55}$$

The above inequality holds for any $\chi\in C_0^\infty(\mathbb{R}^2)$, so injecting it into (B.3.54), and using again a-priori estimates, (B.1.17), hte L^2-L^∞ continuity of $\chi(t^{-\sigma}D_x)$, (B.3.44), together with the fact that $\beta+(\delta+\delta_2)/2\le\delta_1/2$ as β is as small as we want as long as σ is small and $\delta,\delta_2\ll\delta_1$, we now find that

$$\left\|\chi(t^{-\sigma}D_x)\left[x_j\Gamma^I NL_{kg}\right](t,\cdot)\right\|_{L^\infty}\le C(A+B)B\varepsilon^2t^{\sigma+\frac{\delta_1}{2}}.$$

Consequently, summing up this estimate with (1.1.11d) and (B.3.47), we find (B.3.52). ☐

Lemma B.3.11. *Let $\Gamma\in\{\Omega, Z_m, m=1,2\}$ be a Klainerman vector field. There exists a positive constant C such that, under the same hypothesis as in lemma B.3.1, for any multi-index I of length 2, any $\chi\in C_0^\infty(\mathbb{R}^2)$, $\sigma>0$ small, and $i,j=1,2$,*

$$\left\|\chi(t^{-\sigma}D_x)\left[x_ix_j(\Gamma v)_\pm(t,\cdot)\right]\right\|_{L^\infty}\le CB\varepsilon t^{1+\beta+\frac{\delta_1}{2}}, \tag{B.3.56}$$

for every $t\in[1,T]$.

Proof. We prove the estimate of the statement for $(\Gamma v)_-$, using the fact that $(\Gamma v)_+=-\overline{(\Gamma v)_+}$. Multiplying x_i to relation (B.1.9b) with $w=(\Gamma v)_-$, and making some commutations, we derive the following inequality

$$\begin{aligned}&\left\|\chi(t^{-\sigma}D_x)\left[x_ix_j(\Gamma v)_\pm(t,\cdot)\right]\right\|_{L^\infty}\\ &\lesssim\sum_{\mu=0}^{1}\left[\left\|\chi(t^{-\sigma}D_x)\left[x_i^\mu Z_j(\Gamma v)_-\right](t,\cdot)\right\|_{L^\infty}+t\left\|\chi(t^{-\sigma}D_x)\left[x_i^\mu(\Gamma v)_-\right](t,\cdot)\right\|_{L^\infty}\right]\\ &+\sum_{\mu_1,\mu_2=0}^{1}\left\|\chi(t^{-\sigma}D_x)\left[x_i^{\mu_1}x_j^{\mu_2}\Gamma NL_{kg}\right](t,\cdot)\right\|_{L^\infty}\end{aligned} \tag{B.3.57}$$

where we remind the expression of ΓNL_{kg} in (B.1.20a). At first, we estimate the last contribution in the above right hand side by using the L^2-L^∞ continuity of operator $\chi(t^{-\sigma}D_x)$ with norm $O(t^\sigma)$, and write

$$
\begin{aligned}
\sum_{\mu_1,\mu_2=0}^{1} &\left\|\chi(t^{-\sigma}D_x)\left[x_i^{\mu_1}x_j^{\mu_2}\Gamma NL_{kg}\right](t,\cdot)\right\|_{L^\infty} \lesssim \sum_{\mu_1,\mu_2=0}^{1} t^\sigma\left\|\chi(t^{-\sigma}D_x)\left[x_i^{\mu_1}x_j^{\mu_2}\Gamma NL_{kg}\right](t,\cdot)\right\|_{L^2}\\
&\lesssim \sum_{\mu_1,\mu_2,\nu=0}^{1} t^\sigma\|x_i^{\mu_1}x_j^{\mu_2}(\Gamma v)_\pm(t,\cdot)\|_{L^2}\|\mathrm{R}_1^\nu u_\pm(t,\cdot)\|_{H^{2,\infty}}\\
&+\sum_{\mu_1,\mu_2,|\nu|=0}^{1} t^\sigma\left\|x_i^{\mu_1}x_j^{\mu_2}\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu v_\pm(t,\cdot)\right\|_{L^\infty}\left(\|(\Gamma u)_\pm(t,\cdot)\|_{H^1}+\|u_\pm(t,\cdot)\|_{H^1}+\|D_tu_\pm(t,\cdot)\|_{L^2}\right)\\
&\le C(A+B)B\varepsilon^2t^{\frac{3}{2}+\sigma+\frac{\delta_2}{2}},
\end{aligned}
\tag{B.3.58}
$$

where the latter estimate is obtained by (B.1.5a) with $s=0$, (B.1.10b), (B.1.17), (B.1.27b), (B.1.28), together with a-priori estimates. Injecting this estimate, along with (B.2.57) and (B.3.52), we derive that, for any smooth cut-off function χ,

$$
\left\|\chi(t^{-\sigma}D_x)\left[x_ix_j(\Gamma v)_\pm(t,\cdot)\right]\right\|_{L^\infty}\le CB\varepsilon t^{\frac{3}{2}+\sigma+\frac{\delta_2}{2}}.
\tag{B.3.59}
$$

Now, if we change the approach of bounding the L^∞ norm of $x_i^{\mu_1}x_j^{\mu_2}Q_0^{\mathrm{kg}}((\Gamma v)_-,D_1u_\pm)$, which is one of the contributions to $x_i^{\mu_1}x_j^{\mu_2}\Gamma NL_{kg}$, and make use of lemma B.2.2 with $L=L^\infty$, instead of (B.3.58) we can write

$$
\begin{aligned}
&\sum_{\mu_1,\mu_2=0}^{1}\left\|\chi(t^{-\sigma}D_x)\left[x_i^{\mu_1}x_j^{\mu_2}\Gamma NL_{kg}\right](t,\cdot)\right\|_{L^\infty}\\
&\lesssim \sum_{\mu_1,\mu_2,\nu=0}^{1}\left\|\chi_1(t^{-\sigma}D_x)\left[x_i^{\mu_1}x_j^{\mu_2}(\Gamma v)_\pm\right](t,\cdot)\right\|_{L^\infty}\|\mathrm{R}_1^\nu u_\pm(t,\cdot)\|_{H^{2,\infty}}\\
&+\sum_{\mu_1,\mu_2=0}^{1} t^{-N(s)}\|x_i^{\mu_1}x_j^{\mu_2}(\Gamma v)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}\\
&+\sum_{\mu_1,\mu_2,|\nu|=0}^{1} t^\sigma\left\|x_i^{\mu_1}x_j^{\mu_2}\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu v_\pm(t,\cdot)\right\|_{L^\infty}\left(\|(\Gamma u)_\pm(t,\cdot)\|_{H^1}+\|u_\pm(t,\cdot)\|_{H^1}+\|D_tu_\pm(t,\cdot)\|_{L^2}\right).
\end{aligned}
$$

Then, choosing $s>0$ sufficiently large so that $N(s)\ge 3$, and using again (B.1.10b), (B.1.27b), (B.1.17) with $k=1$, (B.1.28), (B.2.57) and a-priori estimates, together with (B.3.59), we find that

$$
\sum_{\mu_1,\mu_2=0}^{1}\left\|\chi(t^{-\sigma}D_x)\left[x_i^{\mu_1}x_j^{\mu_2}\Gamma NL_{kg}\right](t,\cdot)\right\|_{L^\infty}\le C(A+B)B\varepsilon^2t^{1+\sigma+\frac{\delta_2}{2}},
$$

which enhances (B.3.58) of a factor $t^{1/2}$ and implies, when combined with (B.2.57) and (B.3.52), estimate (B.3.56) and concludes the proof of the statement. ☐

Lemma B.3.12. *Let $\Gamma\in\{\Omega,Z_m,m=1,2\}$ be a Klainerman vector field, $v^{I,NF}$ be the function defined in* (B.2.24) *with $|I|=1$ and $\Gamma^I=\Gamma$, and for any $k=1,2$ (resp. $k=3$), any $j_i\in\{+,-\}$ for $i=1,2,3$, let $B^k_{(j_1,j_2,j_3)}(\xi,\eta)$ be the multiplier introduced in* (2.2.45) *(resp. in* (2.2.47)*). Let* *us consider*

(B.3.60)

$$V^{NF}_{\Gamma}(t,x) := v^{I,NF}(t,x) - \frac{i}{4(2\pi)^2}\sum_{j_1,j_2\in\{+,-\}}\int e^{ix\cdot\xi}B^1_{(j_1,j_2,+)}(\xi,\eta)\hat{v}_{j_1}(\xi-\eta)\widehat{u^J_{j_2}}(\eta)d\xi d\eta$$
$$+ \delta_\Omega\frac{i}{4(2\pi)^2}\sum_{j_1,j_2\in\{+,-\}}\int e^{ix\cdot\xi}B^2_{(j_1,j_2,+)}(\xi,\eta)\hat{v}_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)d\xi d\eta$$
$$+ \delta_{Z_1}\frac{i}{4(2\pi)^2}\sum_{j_1,j_2\in\{+,-\}}\int e^{ix\cdot\xi}B^3_{(j_1,j_2,+)}(\xi,\eta)\hat{v}_{j_1}(\xi-\eta)\hat{u}_{j_2}(\eta)d\xi d\eta,$$

where δ_Ω (resp. δ_{Z_1}) is equal to 1 if $\Gamma=\Omega$ (resp. if $\Gamma = Z_1$), 0 otherwise. Therefore, there exists a constant $C>0$ such that, under the same hypothesis as in lemma B.3.1, for any $\chi\in C^\infty_0(\mathbb{R}^2)$, $\sigma>0$,

$$\left\|\chi(t^{-\sigma}D_x)\big(V^{NF}_\Gamma - (\Gamma v)_-\big)(t,\cdot)\right\|_{L^\infty} \le C(A+B)A\varepsilon^2 t^{-\frac{5}{4}}, \tag{B.3.61}$$

for every $t\in[1,T]$. Moreover,

$$\left\||\chi(t^{-\sigma}D_x)Z_m\big(V^{NF}_\Gamma - (\Gamma v)_-\big)(t,\cdot)\right\|_{L^2} \le C(A+B)B\varepsilon^2 t^{3\sigma+\delta_2}, \tag{B.3.62}$$

for every $m=1,2$, $t\in[1,T]$.

Proof. First of all, we observe that similarly to (B.2.27),

$$\begin{aligned} V^{NF}_\Gamma - (\Gamma v)_- &= v^{I,NF} - (\Gamma v)_- \\ &- \frac{i}{2}\left[(D_t v)(D_1\Gamma u) - (D_1 v)(D_t\Gamma u) + D_1[v(D_t\Gamma u)] - \langle D_x\rangle[v(D_1\Gamma u)]\right] \\ &+ \delta_\Omega\frac{i}{2}\left[(D_t v)(D_2 u) - (D_2 v)(D_t u) + D_2[vD_t u] - \langle D_x\rangle[vD_2 u]\right] \\ &+ \delta_{Z_1}\frac{i}{2}\left[(D_t v)(D_t u) + v(|D_x|^2 u) - \langle D_x\rangle[v(D_t u)]\right]. \end{aligned} \tag{B.3.63}$$

Applying lemma B.2.2 with $L=L^\infty$, (1.1.5), (1.1.10) in order to estimate the products in the second line in the above right hand side, we find that for some new $\chi_1\in C^\infty_0(\mathbb{R}^2)$,

(B.3.64)

$$\left\|\chi(t^{-\sigma}D_x)\big(V^{NF}_\Gamma - (\Gamma v)_-\big)(t,\cdot)\right\|_{L^\infty} \lesssim \left\|\chi(t^{-\sigma}D_x)\big(v^{I,NF} - (\Gamma v)_-\big)(t,\cdot)\right\|_{L^\infty}$$
$$+ \sum_{|\mu|=0}^{1} t^\sigma\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\left\|\chi_1(t^{-\sigma}D_x)\mathrm{R}^\mu_1(\Gamma u)_\pm(t,\cdot)\right\|_{L^\infty} + t^{-N(s)}\|v_\pm(t,\cdot)\|_{H^s}\|(\Gamma u)_\pm(t,\cdot)\|_{L^2}$$
$$+ \sum_{|\mu|=0}^{1} t^\sigma\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\|\mathrm{R}^\mu_1 u_\pm(t,\cdot)\|_{H^{2,\infty}}.$$

Analogously, from (B.2.27), lemma B.2.2 with $L=L^\infty$,

(B.3.65)

$$\left\|\chi(t^{-\sigma}D_x)\big(v^{I,NF} - (\Gamma v)_-\big)(t,\cdot)\right\|_{L^\infty} \lesssim \sum_{\mu=0}^{1} t^\sigma\left\|\chi_1(t^{-\sigma}D_x)(\Gamma v)_\pm(t,\cdot)\right\|_{L^\infty}\|\mathrm{R}^\mu_1 u_\pm(t,\cdot)\|_{L^\infty}$$
$$+ t^{-N(s)}\|(\Gamma v)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s},$$

so estimate (B.3.61) is deduced by picking $s>0$ large so that $N(s)\geq 2$, and using (B.2.37) together with (B.2.52) and a-priori estimates.

In order to derive (B.3.62), we apply Z_m to (B.3.63) and use the Leibniz rule, reminding (B.2.34). We estimate the L^2 norm of products (truncated by operator $\chi(t^{-\sigma}D_x)$) in which Z_m is acting on v and Γ on u (i.e. those coming from the action of Z_m on the second line in (B.3.63)), using corollary B.2.4, together with remark B.2.5, with $L=L^2$, $w=u$, and $s>0$ such that $N(s)>1$. All the other remaining products are estimated by considering the L^∞ norm on the factor that does not contain any vector fields times the L^2 norm of the second one. We derive that

(B.3.66)
$$\begin{aligned}
&\big\|\chi(t^{-\sigma}D_x)Z_m\big(V^{NF}_\Gamma-(\Gamma v)_-\big)(t,\cdot)\big\|_{L^2}\lesssim \big\|\chi(t^{-\sigma}D_x)Z_m\big(v^{I,NF}-(\Gamma v)_-\big)(t,\cdot)\big\|_{L^2}\\
&+t^\sigma\left\|\chi_1(t^{-\sigma}D_x)(Z_m v)_\pm(t,\cdot)\right\|_{L^\infty}\|(\Gamma u)_\pm(t,\cdot)\|_{L^2}\\
&+t^{-N(s)}\Big(\sum_{|\mu|=0}^{1}\|x^\mu(Z_m v)_\pm(t,\cdot)\|_{L^2}+t\|(Z_m v)_\pm(t,\cdot)\|_{L^2}\Big)\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_t u_\pm(t,\cdot)\|_{H^s}\right)\\
&+t^\sigma\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\left(\|(Z_m\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|(\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|D_t(\Gamma u)_\pm(t,\cdot)\|_{L^2}\right)\\
&+\sum_{|\mu|=0}^{1}t^\sigma\|(\Gamma v)_\pm(t,\cdot)\|_{L^2}\|\mathrm{R}^\mu u_\pm(t,\cdot)\|_{H^{2,\infty}}\\
&+t^\sigma\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\left(\|(Z_m u)_\pm(t,\cdot)\|_{H^1}+\|u_\pm(t,\cdot)\|_{H^1}+\|D_t u_\pm(t,\cdot)\|_{L^2}\right),
\end{aligned}$$

and hence estimate (B.3.62) by (B.1.5a), (B.1.17), (B.2.37), (B.1.7), (B.2.26) with $j=1$ and a-priori estimates. ☐

Lemma B.3.13. *Let $\Gamma\in\{\Omega,Z_m,m=1,2\}$ be a Klainerman vector field, V^{NF}_Γ defined in* (B.3.60) *and*

(B.3.67)
$$\widetilde{V}^\Gamma(t,x):=tV^{NF}_\Gamma(t,tx).$$

There exists a positive constant $C>0$ such that, if we assume that a-priori estimates (1.1.11) *are satisfied in some interval $[1,T]$, for a fixed $T>1$, then we have, for any $\chi\in C^\infty_0(\mathbb{R}^2)$ and $\sigma>0$ small,*

(B.3.68a)
$$\left\|\widetilde{V}^\Gamma(t,\cdot)\right\|_{L^2}\leq CB\varepsilon t^{\frac{\delta_2}{2}},$$

(B.3.68b)
$$\sum_{|\mu|=1}\left\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{V}^\Gamma(t,\cdot)\right\|_{L^2}\leq CB\varepsilon t^{\frac{\delta_1}{2}},$$

for every $t\in[1,T]$.

Proof. Using expressions (B.3.63) and (B.2.27) with $|I|=1$, and bounding the L^2 norm of each of those products with the L^∞ norm of the (one of the) factor(s) that does not contain vector field Γ times the L^2 norms of the remaining one, we immediately derive, from a-priori estimates, that

$$\|[V^{NF}_\Gamma-(\Gamma v)_-](t,\cdot)\|_{L^2}\leq CAB\varepsilon t^{-\frac{1}{2}+\frac{\delta_2}{2}+\sigma},$$

and consequently that

(B.3.69)
$$\|V^{NF}_\Gamma(t,\cdot)\|_{L^2}\leq CB\varepsilon t^{\frac{\delta_1}{2}},$$

for every $t\in[1,T]$. Since $\|\widetilde{V}^\Gamma(t,\cdot)\|_{L^2}=\|V^{NF}_\Gamma(t,\cdot)\|_{L^2}$, this implies estimate (B.3.68a).

On the other hand, in order to derive (B.3.68b) we first observe that, from definition (B.3.60) of V^{NF}_Γ, one can check that this function is solution to

$$[D_t + \langle D_x\rangle]V^{NF}_\Gamma(t,x) = NL^{kg,c}_\Gamma(t,x) - \delta_{Z_1}Q^{\mathrm{kg}}_0\big(v_\pm, Q^{\mathrm{w}}_0(v_\pm, D_1v_\pm)\big), \tag{B.3.70}$$

with

(B.3.71)

$$\begin{aligned} NL^{kg,c}_\Gamma(t,x) &= r^{I,NF}_{kg}(t,x) \\ &- \frac{i}{4(2\pi)^2}\int e^{ix\cdot\xi}B^1_{(j_1,j_2,+)}(\xi,\eta)\left[\widehat{NL_{kg}}(\xi-\eta)\widehat{u^J_{j_2}}(\eta) - \hat{v}_{j_1}(\xi-\eta)\widehat{NL^I_w}(\eta)\right] d\xi d\eta \\ &+ \delta_\Omega\frac{i}{4(2\pi)^2}\int e^{ix\cdot\xi}B^2_{(j_1,j_2,+)}(\xi,\eta)\left[\widehat{NL_{kg}}(\xi-\eta)\hat{u}_{j_2}(\eta) - \hat{v}_{j_1}(\xi-\eta)\widehat{NL_w}(\eta)\right] d\xi d\eta \\ &+ \delta_{Z_1}\frac{i}{4(2\pi)^2}\int e^{ix\cdot\xi}B^3_{(j_1,j_2,+)}(\xi,\eta)\left[\widehat{NL_{kg}}(\xi-\eta)\hat{u}_{j_2}(\eta) - \hat{v}_{j_1}(\xi-\eta)\widehat{NL_w}(\eta)\right] d\xi d\eta, \end{aligned}$$

$r^{I,NF}_{kg}$ given by (B.2.41) (or, explicitly, by (B.2.42)) with $|I|=1$, and $NL^I_w = \Gamma NL_w$. Superscript c in $NL^{kg,c}_\Gamma$ stands for *cubic*, and wants to stress out the fact that, passing from function $(\Gamma v)_-$ to V^{NF}_Γ, we have replaced all the quadratic terms in the right hand side of (B.1.18a) (when $|I|=1$ and $\Gamma^I=\Gamma$) with cubic ones.

From relation (3.2.8) with $w = V^{NF}_\Gamma$ and equation (B.3.70), we find that

(B.3.72)

$$\begin{aligned} \left\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\widetilde{V}^\Gamma(t,\cdot)\right\|_{H^1} &\lesssim \|\chi(t^{-\sigma}D_x)Z_mV^{NF}_\Gamma(t,\cdot)\|_{L^2} + \left\|Op^w_h(\chi(h^\sigma\xi)\xi_m\langle\xi\rangle^{-1})\widetilde{V}^\Gamma(t,\cdot)\right\|_{L^2} \\ &+ \left\|\chi(t^{-\sigma}D_x)\left[x_mNL^{kg,c}_\Gamma\right](t,\cdot)\right\|_{L^2} + \delta_{Z_1}\left\|\chi(t^{-\sigma}D_x)\left[x_mQ^{\mathrm{kg}}_0\big(v_\pm, Q^{\mathrm{w}}_0(v_\pm, D_1v_\pm)\big)\right](t,\cdot)\right\|_{L^2}, \end{aligned}$$

where after (B.3.62), (1.1.11d) with $k=1$, and the fact that σ can be chosen sufficiently small so that $3\sigma+\delta_2\le\delta_1/2$, as $\delta_2\ll\delta_1$,

$$\|\chi(t^{-\sigma}D_x)Z_mV^{NF}_\Gamma(t,\cdot)\|_{L^2} \le CB\varepsilon t^{\frac{\delta_1}{2}}. \tag{B.3.73}$$

We also observe that

(B.3.74)

$$\begin{aligned} \left\|\chi(t^{-\sigma}D_x)\left[x_mQ^{\mathrm{kg}}_0\big(v_\pm, Q^{\mathrm{w}}_0(v_\pm, D_1v_\pm)\big)\right](t,\cdot)\right\|_{L^2} &\lesssim \sum_{|\mu|=0}^{1}\left\|x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\right\|_{L^\infty}\|NL_w(t,\cdot)\|_{L^2} \\ &\le C(A+B)AB\varepsilon^3t^{-1+\frac{\delta+\delta_2}{2}}, \end{aligned}$$

as follows from (B.1.3a), (B.1.10b) and a-priori estimates.

Similarly to (B.2.42), we have an explicit expression for $NL^{kg,c}_\Gamma$:

$$\begin{aligned} NL^{kg,c}_\Gamma(t,x) = r^{I,NF}_{kg}(t,x) &- \frac{i}{2}\left[NL_{kg}(D_1\Gamma u) - (D_1v)NL^I_w + D_1[vNL^I_w]\right] \\ &+ \delta_\Omega\frac{i}{2}\left[NL_{kg}(D_2u) - (D_2v)NL_w + D_2[vNL_w]\right] \\ &+ \delta_{Z_1}\left[NL_{kg}(D_tu) + (D_tv)NL_w - \langle D_x\rangle[vNL_w]\right]. \end{aligned} \tag{B.3.75}$$

Therefore, from (B.2.67), (B.1.3a), (B.1.10b), inequality

$$\|NL^I_w(t,\cdot)\|_{L^2} \lesssim \|v_\pm(t,\cdot)\|_{H^{1,\infty}}\left(\|v^I_\pm(t,\cdot)\|_{H^1} + \|v_\pm(t,\cdot)\|_{H^1} + \|D_tv_\pm(t,\cdot)\|_{L^2}\right), \tag{B.3.76}$$

(B.1.5a) with $s=0$ and a-priori estimates, we derive that

$$\begin{aligned}(\text{B.3.77})\qquad &\left\|\chi(t^{-\sigma}D_x)\left[x_j NL^{kg,c}_{\Gamma}\right](t,\cdot)\right\|_{L^2} \lesssim \left\|\chi(t^{-\sigma}D_x)\left[x_j r^{I,NF}_{kg}(t,\cdot)\right](t,\cdot)\right\|_{L^2}\\ &+\sum_{|\mu|,\nu=0}^{1}\left\|x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}v_{\pm}(t,\cdot)\right\|_{L^\infty}\|\mathrm{R}^{\nu}u_{\pm}(t,\cdot)\|_{H^{2,\infty}}\left(\|(\Gamma u)_{\pm}(t,\cdot)\|_{L^2}+\|u_{\pm}(t,\cdot)\|_{L^2}\right)\\ &+\sum_{|\mu|=0}^{1}\left\|x_j\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}v_{\pm}(t,\cdot)\right\|_{L^\infty}\left(\|NL^{I}_{w}(t,\cdot)\|_{L^2}+\|NL_{w}(t,\cdot)\|_{L^2}\right)\\ &\le C(A+B)AB\varepsilon^2 t^{-\frac{1}{2}+\beta+\frac{\delta+\delta_1}{2}}.\end{aligned}$$

By injecting the above estimate, together with (B.3.68a), (B.3.73), (B.3.74), into (B.3.72) we finally deduce (B.3.68b) and conclude the proof of the statement. ☐

Lemma B.3.14. *Let $\Gamma\in\{\Omega, Z_m, m=1,2\}$ be a Klainerman vector field, I_1 be of length 1 such that $\Gamma^{I_1}=\Gamma$, and I_2 of length 2 such that $\Gamma^{I_2}=Z_m\Gamma$, for some $m=1,2$. Let us consider $v^{I_1,NF}$ defined in* (B.2.24) *with $I=I_1$. There exists a constant $C>0$ such that, under the same hypothesis as in lemma B.3.13, for any $\chi\in C^\infty_0(\mathbb{R}^2)$, $\sigma>0$ small, $m,n=1,2$,*

$$(\text{B.3.78a})\quad \left\|\chi(t^{-\sigma}D_x)\left[Z_m\left(v^{I_1,NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}+\left\|\chi(t^{-\sigma}D_x)\left(v^{I_2,NF}-(Z_m\Gamma v)_-\right)(t,\cdot)\right\|_{L^2} \le C(A+B)B\varepsilon^2 t^{-1+\beta+\frac{\delta+\delta_1+\delta_2}{2}},$$

$$(\text{B.3.78b})\quad \left\|\chi(t^{-\sigma}D_x)\left[x_n Z_m\left(v^{I_1,NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}+\left\|\chi(t^{-\sigma}D_x)\left[x_n\left(v^{I_2,NF}-(Z_m\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2} \le C(A+B)B\varepsilon^2 t^{\beta+\frac{\delta+\delta_1+\delta_2}{2}},$$

for every $t\in[1,T]$, with $|I_2|=2$ such that $\Gamma^{I_2}=Z_m\Gamma^{I_1}$, and $\beta>0$ small such that $\beta\to 0$ as $\sigma\to 0$. Moreover, if V^{NF}_{Γ} is the function defined in (B.3.60)*, then*

$$(\text{B.3.79a})\qquad \left\|\chi(t^{-\sigma}D_x)\left[Z_m\left(V^{NF}_{\Gamma}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}\le C(A+B)B\varepsilon^2 t^{-1+\beta+\frac{\delta+\delta_1+\delta_2}{2}},$$

$$(\text{B.3.79b})\qquad \left\|\chi(t^{-\sigma}D_x)\left[x_n Z_m\left(V^{NF}_{\Gamma}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}\le C(A+B)B\varepsilon^2 t^{\beta+\frac{\delta+\delta_1+\delta_2}{2}},$$

for every $t\in[1,T]$.

Proof. We warn the reader that throughout the proof we denote by C,β two positive constants, that may change line after line, with $\beta\to 0$ as $\sigma\to 0$.

The first thing we observe is that, with the indices considered in the statement, the difference $v^{I_2,NF}-(Z_m\Gamma v)_-$ (explicitly written in (B.2.27) for $I=I_2$) appears to be equal to the first line in the right hand side of (B.2.35) for $I=I_1$. Therefore, inequalities (B.3.78a) will follow from the analysis of the terms appearing in the right hand side of (B.2.35).

Both estimates (B.3.78) follow using lemma B.2.2 with $L=L^2$ to estimate the contributions coming from the first, third and fourth line in (B.2.35), and applying lemma B.2.3 and remark B.2.5, with $L=L^2$ and $w_1=u$, in order to estimate products in the second line of (B.2.35).

Therefore, we find that there is some $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ such that

$$
\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[Z_m\left(v^{I_1,NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2} \lesssim t^\sigma \|\chi_1(t^{-\sigma}D_x)(Z_m\Gamma v)_\pm(t,\cdot)\|_{L^\infty}\|u_\pm(t,\cdot)\|_{L^2} \\
&+ t^\sigma\|\chi_1(t^{-\sigma}D_x)(\Gamma v)_\pm(t,\cdot)\|_{L^\infty}\left(\|u_\pm(t,\cdot)\|_{L^2}+\|D_t u_\pm(t,\cdot)\|_{L^2}\right) \\
&+ t^{-N(s)}\|(Z_m\Gamma v)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s} + t^{-N(s)}\|(\Gamma v)_\pm(t,\cdot)\|_{L^2}\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_t u_\pm(t,\cdot)\|_{H^s}\right) \\
&+ t^\sigma\|\chi_1(t^{-\sigma}D_x)(\Gamma v)_\pm(t,\cdot)\|_{L^\infty}\|(Z_m u)_\pm(t,\cdot)\|_{L^2} \\
&+ t^{-N(s)}\left(\sum_{\mu=0}^{1}\|x_m^\mu(\Gamma v)_\pm(t,\cdot)\|_{L^2}+t\|(\Gamma v)_\pm(t,\cdot)\|_{L^2}\right)\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_t u_\pm(t,\cdot)\|_{H^s}\right).
\end{aligned}
$$

Choosing $s > 0$ large so that $N(s) > 1$, and using (B.1.5a), (B.1.17), lemmas B.2.8, B.3.9, and a-priori estimates, we obtain (B.3.78a).
Analogously,

$$
\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_n Z_m\left(v^{I_1,NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2} \lesssim t^\sigma \|\chi_1(t^{-\sigma}D_x)\left[x_n(Z_m\Gamma v)_\pm\right](t,\cdot)\|_{L^\infty}\|u_\pm(t,\cdot)\|_{L^2} \\
&+ t^\sigma\|\chi_1(t^{-\sigma}D_x)\left[x_n(\Gamma v)_\pm\right](t,\cdot)\|_{L^\infty}\left(\|u_\pm(t,\cdot)\|_{L^2}+\|D_t u_\pm(t,\cdot)\|_{L^2}\right) \\
&+ t^{-N(s)}\|x_n(Z_m\Gamma v)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s} + t^{-N(s)}\|x_n(\Gamma v)_\pm(t,\cdot)\|_{L^2}\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_t u_\pm(t,\cdot)\|_{H^s}\right) \\
&+ t^\sigma\|\chi_1(t^{-\sigma}D_x)\left[x_n(\Gamma v)_\pm\right](t,\cdot)\|_{L^\infty}\|(Z_m u)_\pm(t,\cdot)\|_{L^2} \\
&+ t^{-N(s)}\left(\sum_{\mu=0}^{1}\|x_m^\mu x_n(\Gamma v)_\pm(t,\cdot)\|_{L^2}+t\|x_n(\Gamma v)_\pm(t,\cdot)\|_{L^2}\right)\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_t u_\pm(t,\cdot)\|_{H^s}\right),
\end{aligned}
$$

so from (B.1.5a), (B.1.17), (B.1.28), (B.2.57), (B.3.52) and a-priori estimates we derive that

$$
\left\|\chi(t^{-\sigma}D_x)\left[x_n Z_m\left(v^{I_1,NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2} \le C(A+B)B\varepsilon^2 t^{\beta+\frac{\delta+\delta_1+\delta_2}{2}},
$$

and estimate (B.3.78b) follows just by the observation that the first line in (B.2.35) corresponds to $v^{I_2,NF}-(Z_m\Gamma v)_-$ for I_2 such that $\Gamma^{I_2}=Z_m\Gamma$.

The last two estimates (B.3.79) are derived applying Z_m to equality (B.3.63), using (B.3.78) and proceeding as follows: we estimate products in which Z_m acts on v and Γ on u (i.e. those coming out from the second line of (B.3.63)) by means of corollary B.2.4 with $L = L^2$, $w = u$, and remark B.2.5; products in which Z_m is acting on v and there are no Klainerman vector fields acting on u are estimated applying lemma B.2.2 with $L = L^2$; the remaining ones are controlled by making appear the L^∞ norm on v and the L^2 norm on the wave factor. In this way we have, on the one hand,

$$
\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[Z_m\left(V_\Gamma^{NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2} \lesssim \left\|\chi(t^{-\sigma}D_x)\left[Z_m\left(v^{I_1,NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2} \\
&+ t^\sigma\left\|\chi_1(t^{-\sigma}D_x)(Z_m v)_\pm(t,\cdot)\right\|_{L^\infty}\left(\|(\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|u_\pm(t,\cdot)\|_{L^2}\right) \\
&+ t^{-N(s)}\left(\sum_{|\mu|=0}^{1}\|x^\mu(Z_m v)_\pm(t,\cdot)\|_{L^2}+t\|(Z_m v)_\pm(t,\cdot)\|_{L^2}\right)\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_t u_\pm(t,\cdot)\|_{H^s}\right) \\
&+ \|v_\pm(t,\cdot)\|_{H^{1,\infty}}\big(\|(Z_m\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|(\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|D_t(\Gamma u)_\pm(t,\cdot)\|_{L^2} \\
&\qquad +\|u_\pm(t,\cdot)\|_{L^2}+\|D_t u_\pm(t,\cdot)\|_{L^2}\big),
\end{aligned}
$$

and hence estimate (B.3.79a) by choosing $s > 0$ large so that $N(s) > 2$ and using (B.1.5a), (B.1.7), (B.1.17) with $k = 1$, (B.2.37), (B.3.78a) and a-priori estimates. On the other hand, we have

$$
\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_nZ_m\left(V^{NF}_\Gamma-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}\lesssim\left\|\chi(t^{-\sigma}D_x)\left[x_nZ_m\left(v^{I_1,NF}-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^2}\\
&+t^\sigma\left\|\chi_1(t^{-\sigma}D_x)\left[x_n(Z_mv)_\pm\right](t,\cdot)\right\|_{L^\infty}\left(\|(\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|u_\pm(t,\cdot)\|_{L^2}\right)\\
&+t^{-N(s)}\left(\sum_{|\mu|=0}^1\|x^\mu x_n(Z_mv)_\pm(t,\cdot)\|_{L^2}+t\|x_n(Z_mv)_\pm(t,\cdot)\|_{L^2}\right)\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_tu_\pm(t,\cdot)\|_{H^s}\right)\\
&+\sum_{|\mu|=0}^1\left\|x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm(t,\cdot)\right\|_{L^\infty}\big(\|(Z_m\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|(\Gamma u)_\pm(t,\cdot)\|_{L^2}+\|D_t(\Gamma u)_\pm(t,\cdot)\|_{L^2}\\
&\qquad\qquad+\|u_\pm(t,\cdot)\|_{L^2}+\|D_tu_\pm(t,\cdot)\|_{L^2}\big)\,,
\end{aligned}
$$

and estimate (B.3.79b) follows then choosing $s>0$ large so that $N(s)>1$ and using (B.1.5a), (B.1.7), (B.1.10b), (B.1.17) with $k=1$, (B.1.28), (B.2.57), (B.3.78b) and a-priori estimates. □

Lemma B.3.15. *Let $\Gamma\in\{\Omega,Z_m,m=1,2\}$ be a Klainerman vector field and V^{NF}_Γ be the function defined in* (B.3.60)*. There exists a constant $C>0$ such that, under the same hypothesis as in lemma B.3.13, for any $\chi\in C^\infty_0(\mathbb{R}^2)$, $\sigma>0$ small, $m,n=1,2$,*

$$
\left\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\left[tZ_nV^{NF}_\Gamma(t,tx)\right]\right\|_{L^2(dx)}\le CB\varepsilon t^{\frac{\delta_0}{2}}, \tag{B.3.80}
$$

for every $t\in[1,T]$.

Proof. We warn the reader that, throughout the proof, we denote by C,β two positive constants, that may change line after line, with $\beta\to0$ as $\sigma\to0$.

Let $v^{I,NF}$ be the function defined in (B.2.24), and I_1,I_2 two multi-indices of length, respectively, 1 and 2, such that $\Gamma^{I_1}=\Gamma$, $\Gamma^{I_2}=Z_n\Gamma$. We rewrite $Z_nV^{NF}_\Gamma$ as follows

$$
Z_nV^{NF}_\Gamma=Z_n\left(V^{NF}_\Gamma-(\Gamma v)_-\right)+\left[(Z_n\Gamma v)_--v^{I_2,NF}\right]+v^{I_2,NF}+\frac{D_n}{\langle D_x\rangle}v^{I_1,NF}+\frac{D_n}{\langle D_x\rangle}\left[(\Gamma v)_--v^{I_1,NF}\right]
$$

so that

$$
\begin{aligned}
&\left\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\left[tZ_nV^{NF}_\Gamma(t,tx)\right]\right\|_{L^2(dx)}\\
&\lesssim\left\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\left[tZ_n\left(V^{NF}_\Gamma-(\Gamma v)_-\right)(t,tx)\right]\right\|_{L^2(dx)}\\
&+\left\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\left[t\left[(Z_n\Gamma v)_--v^{I_2,NF}\right](t,tx)\right]\right\|_{L^2(dx)}\\
&+\left\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\left[tv^{I_2,NF}(t,tx)\right]\right\|_{L^2(dx)}+\left\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\left[t\frac{D_n}{\langle D_x\rangle}v^{I_1,NF}(t,tx)\right]\right\|_{L^2(dx)}\\
&+\left\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\left[t\frac{D_n}{\langle D_x\rangle}\left[(\Gamma v)_--v^{I_1,NF}\right](t,tx)\right]\right\|_{L^2(dx)}.
\end{aligned} \tag{B.3.81}
$$

After relation (3.2.8) with $w=v^{I_2,NF}$,

$$
\begin{aligned}
&\left\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\left[tv^{I_2,NF}(t,tx)\right]\right\|_{L^2(dx)}\lesssim\left\|\chi(t^{-\sigma}D_x)Z_m(\Gamma^{I_2}v)_-(t,\cdot)\right\|_{L^2}\\
&+\left\|\chi(t^{-\sigma}D_x)Z_m\left[v^{I_2,NF}-(\Gamma^{I_2}v)_-\right](t,\cdot)\right\|_{L^2}+\left\|\chi(t^{-\sigma}D_x)v^{I_2,NF}(t,\cdot)\right\|_{L^2}\\
&+\left\|\chi(t^{-\sigma}D_x)\left[x_mNL^{I_2,NF}_{kg}\right](t,\cdot)\right\|_{L^2}
\end{aligned}
$$

with $NL^{I_2,NF}_{kg}$ given by (B.3.48) with $I = I_2$, and we deduce from (B.2.26) with $j = 2$, (B.3.49), (B.3.50), a-priori estimate (1.1.11d) with $k = 0$, and the fact that β is as small as we want as long as σ is small, $\delta \ll \delta_2 \ll \delta_1 \ll \delta_0$, that

$$\left\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\left[tv^{I_2,NF}(t,tx)\right]\right\|_{L^2(dx)} \le CB\varepsilon t^{\frac{\delta_0}{2}}.$$

Analogously, commutating $\mathcal{L}_m$ with $Op^w_h(\xi_n\langle\xi\rangle^{-1})$, applying (3.2.8) with $w = v^{I_1,NF}$, and using inequalities (B.2.45), (B.2.46), (B.2.48), we derive that

$$\left\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\left[t\frac{D_n}{\langle D_x\rangle}v^{I_1,NF}(t,tx)\right]\right\|_{L^2(dx)} \le CB\varepsilon t^{\frac{\delta_1}{2}}.$$

The remaining norms in the right hand side of (B.3.81) are estimated by $C(A+B)B\varepsilon^2 t^{\beta+\frac{\delta+\delta_1+\delta_2}{2}}$, as follows using (B.3.6) and lemma B.3.14. Since β is as small as we want as long as σ is small, and $\delta \ll \delta_2 \ll \delta_1 \ll \delta_0$, summing up with the above two estimates, we then find the result of the statement. ☐

Lemmas B.2.8, B.3.13 and B.3.15 allow us to state the analogous result of lemma B.3.3 with $\widetilde{v}$ replaced with $\widetilde{V}^\Gamma$, introduced in (B.3.67).

Lemma B.3.16. *Let $h = t^{-1}$, $\widetilde{V}^\Gamma$ be defined in* (B.3.67)*, $\widetilde{u}$ as in* (3.2.2)*, $a_0(\xi) \in S_{0,0}(1)$, and $b_1(\xi) = \xi_j$ or $b_1(\xi) = \xi_j\xi_k|\xi|^{-1}$, with $j, k \in \{1,2\}$. There exists a constant $C > 0$ such that, under the same assumptions as in lemma B.2.14, for any $\chi, \chi_1 \in C^\infty_0(\mathbb{R}^2)$, $\sigma > 0$, we have that*

$$\left\|[Op^w_h(\chi(h^\sigma\xi)a_0(\xi))\widetilde{V}^\Gamma(t,\cdot)][Op^w_h(\chi_1(h^\sigma\xi)b_1(\xi))\widetilde{u}(t,\cdot)]\right\|_{L^2} \le C(A+B)B\varepsilon^2 h^{\frac{1}{2}-\beta'}, \tag{B.3.82a}$$

$$\left\|[Op^w_h(\chi(h^\sigma\xi)a_0(\xi))\widetilde{V}^\Gamma(t,\cdot)][Op^w_h(\chi_1(h^\sigma\xi)b_1(\xi))\widetilde{u}(t,\cdot)]\right\|_{L^\infty} \le C(A+B)B\varepsilon^2 h^{-\beta'}, \tag{B.3.82b}$$

with $\beta' > 0$ small, $\beta \to 0$ as $\sigma, \delta_0 \to 0$.

Proof. The proof of this result is similar to that of lemma B.3.3 except for few estimates, linked to the fact that we are replacing $\widetilde{v}$ with $\widetilde{V}^\Gamma$. We limit here to indicate these few differences.

Instead of referring to (B.3.9), we use the fact that, after (B.2.37) in classical coordinates,

$$\left\|Op^w_h(\chi(h^\sigma\xi))\widetilde{V}^\Gamma(t,\cdot)\right\|_{H^{\rho,\infty}} \le CB\varepsilon h^{-\beta-\frac{\delta_1}{2}}, \tag{B.3.83}$$

for some $C > 0$, $\beta > 0$ small such that $\beta \to 0$ as $\sigma \to 0$. Also, decomposing $\widetilde{V}^\Gamma$ into $\widetilde{V}^\Gamma_{\Lambda_{kg}} + \widetilde{V}^\Gamma_{\Lambda^c_{kg}}$, with

$$\widetilde{V}^\Gamma_{\Lambda_{kg}}(t,x) := Op^w_h\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)a_0(\xi)\Big)\widetilde{V}^\Gamma(t,x),$$
$$\widetilde{V}^\Gamma_{\Lambda^c_{kg}}(t,x) := Op^w_h\Big((1-\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi(h^\sigma\xi)a_0(\xi)\Big)\widetilde{V}^\Gamma(t,x),$$

and using the fact that above operators are supported for frequencies $|\xi| \lesssim h^\sigma$, together with proposition 1.2.38 with $p = +\infty$ and (B.3.83), we have that

$$\left\|\widetilde{V}^\Gamma_{\Lambda_{kg}}(t,\cdot)\right\|_{L^\infty} \le CB\varepsilon h^{-\beta-\frac{\delta_1}{2}},$$

while

$$\left\|\widetilde{V}^\Gamma_{\Lambda^c_{kg}}(t,\cdot)\right\|_{L^\infty} \le CB\varepsilon h^{\frac{1}{2}-\beta-\frac{\delta_1}{2}},$$

as follows using the analogous of (B.3.11), but combined with lemma B.3.13 (instead of B.2.14), estimates (B.3.73), (B.3.80) (instead of lemma B.3.2) and (B.3.77) (instead of (B.2.74)).

Lemma B.3.17. *Let $\Gamma \in \{\Omega, Z_m, m = 1,2\}$ be a Klainerman vector field and V^{NF}_Γ be the function defined in* (B.3.60). *There exists a constant $C>0$ such that, under the same hypothesis as in lemma B.3.13, for any $\chi \in C^\infty_0(\mathbb{R}^2)$, $\sigma>0$ small, $m,n=1,2$,*

$$\left\|\chi(t^{-\sigma}D_x)\left[x_m\left(V^{NF}_\Gamma-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^\infty} \le C(A+B)^2\varepsilon^2 t^{-\frac{1}{2}+\beta+\frac{\delta_1+\delta_2}{2}}, \tag{B.3.84a}$$

$$\left\|\chi(t^{-\sigma}D_x)\left[x_n x_m\left(V^{NF}_\Gamma-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^\infty} \le C(A+B)^2\varepsilon^2 t^{\frac{1}{2}+\beta+\frac{\delta_1+\delta_2}{2}}, \tag{B.3.84b}$$

for every $t\in[1,T]$, with $\beta>0$ small such that $\beta\to 0$ as $\sigma\to 0$.

Proof. First of all, we remind the explicit expression (B.3.63) of the difference $V^{NF}_\Gamma-(\Gamma v)_-$. We also remind (B.2.27) when $|I|=1$ such that $\Gamma^I=\Gamma$.

The idea to derive estimates (B.3.84) is to apply lemma B.2.2 with $L=L^\infty$ in order to control the contribution coming from $v^{I,NF}-(\Gamma v)_-$; to apply corollary B.2.4 and remark B.2.5, with $L=L^\infty$ in order to control the L^∞ norm of the products appearing in the second line of (B.3.63); to simply multiply x_n (together with x_m in the case of (B.3.84b)) against v in order to get a control on products in the third and fourth line of (B.3.63). More precisely, we write the following:

$$\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_m\left(V^{NF}_\Gamma-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^\infty}\\
&\lesssim \sum_{\mu=0}^{1} t^\sigma\left\|\chi_1(t^{-\sigma}D_x)\left[x_m(\Gamma v)_\pm(t,\cdot)\right]\right\|_{L^\infty}\|\mathrm{R}_1^\mu u_\pm(t,\cdot)\|_{L^\infty}+t^{-N(s)}\|x_m(\Gamma v)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}\\
&+\sum_{|\mu|=0}^{1} t^\sigma\left\|x_m\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm\right\|_{L^\infty}\|\chi_1(t^{-\sigma}D_x)(\Gamma u)_-(t,\cdot)\|_{L^\infty}\\
&+\sum_{|\mu|=0}^{2} t^{-N(s)}\|x^\mu v_\pm(t,\cdot)\|_{L^2}\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_t u_\pm(t,\cdot)\|_{H^s}\right)\\
&+\sum_{|\mu|,|\nu|=0}^{1} t^\sigma\left\|x_m\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm\right\|_{L^\infty}\|\mathrm{R}^\mu u_\pm(t,\cdot)\|_{H^{2,\infty}}
\end{aligned}$$

so estimate (B.3.84a) follows choosing $s>0$ large enough to have $N(s)\ge 2$, and using (B.1.5a), (B.1.10a), (B.1.10b), (B.1.17) with $k=1$, (B.2.57), (B.2.52) and a-priori estimates.

Analogously,

$$\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_n x_m\left(V^{NF}_\Gamma-(\Gamma v)_-\right)\right](t,\cdot)\right\|_{L^\infty}\\
&\lesssim \sum_{\mu=0}^{1} t^\sigma\left\|\chi_1(t^{-\sigma}D_x)\left[x_n x_m(\Gamma v)_\pm(t,\cdot)\right]\right\|_{L^\infty}\|\mathrm{R}_1^\mu u_\pm(t,\cdot)\|_{L^\infty}\\
&+t^{-N(s)}\|x_n x_m(\Gamma v)_\pm(t,\cdot)\|_{L^2}\|u_\pm(t,\cdot)\|_{H^s}\\
&+\sum_{|\mu|=0}^{1} t^\sigma\left\|x_n x_m\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm\right\|_{L^\infty}\|\chi_1(t^{-\sigma}D_x)(\Gamma u)_-(t,\cdot)\|_{L^\infty}\\
&+\sum_{|\mu|=0}^{3} t^{-N(s)}\|x^\mu v_\pm(t,\cdot)\|_{L^2}\left(\|u_\pm(t,\cdot)\|_{H^s}+\|D_t u_\pm(t,\cdot)\|_{H^s}\right)\\
&+\sum_{|\mu|,|\nu|=0}^{1} t^\sigma\left\|x_n x_m\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\mu v_\pm\right\|_{L^\infty}\|\mathrm{R}^\mu u_\pm(t,\cdot)\|_{H^{2,\infty}}
\end{aligned}$$

hence picking the same s as before, and using (B.1.5a), (B.1.10a), (B.1.27a), (B.1.27b), (B.1.28), (B.1.33), (B.2.52) and (B.3.56), together with a-priori estimates, we derive (B.3.84b). That concludes the proof of the statement. □

Lemma B.3.18. *Let $\Gamma \in \{\Omega, Z_m, m=1,2\}$ be a Klainerman vector field and $NL^{kg,c}_\Gamma$ be given by* (B.3.75)*. There exists a constant $C>0$ such that, under the same hypothesis as in lemma B.3.13, for any $\chi \in C^\infty_0(\mathbb{R}^2)$, $\sigma>0$ small, $m,n=1,2$,*

$$\left\|\chi(t^{-\sigma}D_x)\left[x_n NL^{kg,c}_\Gamma\right](t,\cdot)\right\|_{L^2} \le C(A+B)^2B\varepsilon^3 t^{-1+\beta'}, \tag{B.3.85a}$$

$$\left\|\chi(t^{-\sigma}D_x)\left[x_m x_n NL^{kg,c}_\Gamma\right](t,\cdot)\right\|_{L^2} \le C(A+B)^2B\varepsilon^3 t^{\beta'}, \tag{B.3.85b}$$

for every $t\in[1,T]$, with $\beta'>0$ small such that $\beta'\to 0$ as $\sigma,\delta_0\to 0$. Moreover,

$$\left\|\chi(t^{-\sigma}D_x)NL^{kg,c}_\Gamma(t,\cdot)\right\|_{L^\infty} \le C(A+B)^2B\varepsilon^3 t^{-\frac{5}{2}+\beta'}, \tag{B.3.86}$$

for every $t\in[1,T]$.

Proof. We warn the reader that we will denote by C,β,β' some positive constants that may change line after line, with $\beta\to 0$ (resp. $\beta'\to 0$) as $\sigma\to 0$ (resp. as $\sigma,\delta_0\to 0$). For a seek of compactness, we also denote by $R(t,x)$ any contribution verifying

$$\begin{aligned}\left\|\chi(t^{-\sigma}D_x)\left[x_n R(t,\cdot)\right]\right\|_{L^2} &\le C(A+B)^2B\varepsilon^3 t^{-1+\beta'},\\ \left\|\chi(t^{-\sigma}D_x)\left[x_m x_n R(t,\cdot)\right]\right\|_{L^2} &\le C(A+B)^2B\varepsilon^3 t^{\beta'},\end{aligned} \tag{B.3.87}$$

together with

$$\left\|\chi(t^{-\sigma}D_x)R(t,\cdot)\right\|_{L^\infty} \le C(A+B)^2B\varepsilon^3 t^{-\frac{5}{2}+\beta'}. \tag{B.3.88}$$

We can introduce the following notation

$$\begin{aligned} NL^{cub}_v := &-\frac{i}{2}\left[-(D_1\Gamma v)NL_w + D_1\left[(\Gamma v)NL_w\right]\right]\\ &-\frac{i}{2}\left[-(D_1 v)NL^I_w + D_1[vNL^I_w]\right] + \frac{i}{2}\delta_\Omega\left[-(D_2 v)NL_w + D_2[vNL_w]\right]\\ &+\delta_{Z_1}\left[(D_t v)NL^I_w - \langle D_x\rangle[vNL_w]\right], \end{aligned} \tag{B.3.89}$$

with $NL^I_w = \Gamma NL_w$, so that from (B.3.71)

$$NL^{kg,c}_\Gamma = \frac{i}{2}\left[NL^I_{kg}(D_1u) + NL_{kg}(D_1\Gamma u)\right] + \delta_\Omega\frac{i}{2}NL_{kg}(D_2u) + \delta_{Z_1}NL_{kg}(D_tu) + NL^{cub}_v, \tag{B.3.90}$$

with δ_Ω (resp. δ_{Z_1}) equal to 1 when $\Gamma=\Omega$ (resp. $\Gamma=Z_1$), 0 otherwise, and $NL^I_{kg}=\Gamma NL_{kg}$. Cubic contribution NL^{cub}_v satisfies, after (1.1.5), (B.1.3a), (B.1.10b), (B.3.76), (B.1.6a) with $s=0$ and a-priori estimates, the following inequality

$$\begin{aligned} &\left\|\chi(t^{-\sigma}D_x)\left[x_n NL^{cub}_v\right](t,\cdot)\right\|_{L^2}\\ &\lesssim \sum_{\mu,|\nu|=0}^1 t^\sigma\left\|x_n^\mu\Big(\frac{D_x}{\langle D_x\rangle}\Big)^\nu v_\pm(t,\cdot)\right\|_{L^\infty}\left[\|NL^I_w(t,\cdot)\|_{L^2} + \|NL_w(t,\cdot)\|_{L^2} + \|v_\pm(t,\cdot)\|_{H^{2,\infty}}\|(\Gamma^I v)_\pm(t,\cdot)\|_{L^2}\right]\\ &\le C(A+B)AB\varepsilon^3 t^{-1+\sigma+\delta_2}, \end{aligned} \tag{B.3.91}$$

and also, from the mentioned inequalities together with (B.1.27b),

$$
\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\left[x_m x_n NL^{cub}_v\right](t,\cdot)\right\|_{L^2}\\
&\lesssim \sum_{\mu_1,\mu_2,|\nu|=0}^{1} t^{\sigma}\left\|x_m^{\mu_1}x_n^{\mu_2}\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\nu}v_\pm(t,\cdot)\right\|_{L^\infty}\big[\|NL^I_w(t,\cdot)\|_{L^2}+\|NL_w(t,\cdot)\|_{L^2} \qquad \text{(B.3.92)}\\
&\qquad\qquad +\|v_\pm(t,\cdot)\|_{H^{2,\infty}}\|(\Gamma^I v)_\pm(t,\cdot)\|_{L^2}\big]\\
&\le C(A+B)AB\varepsilon^3 t^{\sigma+\delta_2}.
\end{aligned}
$$

Moreover, applying twice lemma B.2.2 with $L=L^\infty$ and $s>0$ large enough to have $N(s)\ge 2$, the first time to estimate products involving Γv and NL^I_w in (B.3.89), the second one to estimate the first two quadratic contributions to NL^I_w (see (B.1.20b)), we see that

$$
\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)NL^{cub}_v(t,\cdot)\right\|_{L^\infty}\lesssim t^{\sigma}\left\|\chi_1(t^{-\sigma}D_x)(\Gamma v)_\pm(t,\cdot)\right\|_{L^\infty}\|NL_w(t,\cdot)\|_{L^\infty}\\
&+t^{-2}\|(\Gamma v)_\pm(t,\cdot)\|_{L^2}\|NL_w(t,\cdot)\|_{H^s}+t^{\sigma}\|\chi_1(t^{-\sigma}D_x)NL^I_w(t,\cdot)\|_{L^\infty}\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\\
&+t^{-2}\|NL^I_w(t,\cdot)\|_{L^2}\|v_\pm(t,\cdot)\|_{H^s}+t^{\sigma}\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\|NL_w(t,\cdot)\|_{L^\infty}
\end{aligned}
$$

with

$$
\begin{aligned}
&\|\chi_1(t^{-\sigma}D_x)NL^I_w(t,\cdot)\|_{L^\infty}\lesssim\|\chi_2(t^{-\sigma}D_x)(\Gamma v)_\pm(t,\cdot)\|_{H^{2,\infty}}\|v_\pm(t,\cdot)\|_{H^{2,\infty}}\\
&+t^{-2}\|(\Gamma v)_\pm(t,\cdot)\|_{H^1}\|v_\pm(t,\cdot)\|_{H^s}+\|v_\pm(t,\cdot)\|_{H^{1,\infty}}\left(\|v_\pm(t,\cdot)\|_{H^{2,\infty}}+\|D_t v_\pm(t,\cdot)\|_{H^{1,\infty}}\right).
\end{aligned}
$$

From (B.1.3b), (B.1.3c), (B.1.6b) with $s=1$ and $\theta\ll 1$ small, (B.2.37), (B.3.76), together with (B.1.5a) with $s=0$ and a-priori estimates, we then obtain that

$$
\left\|\chi(t^{-\sigma}D_x)NL^{cub}_v(t,\cdot)\right\|_{L^\infty}\le CA^2B\varepsilon^3 t^{-3+\beta'},
$$

which, together with (B.3.91) and (B.3.92), implies that NL^{cub}_v is a contribution of the form $R(t,x)$. Consequently, from (B.3.90) we are left to prove that also $NL^I_{kg}(D_1u)$, $NL_{kg}(D_1\Gamma u)$, $NL_{kg}(D_2u)$ and $NL_{kg}(D_tu)$ verify (B.3.87), (B.3.88), and hence are of the form $R(t,x)$.

We immediately observe that, from (B.1.1b) and (1.1.5), products appearing in $NL_{kg}(D_2u)$ and in $NL_{kg}(D_tu)$ are of the form

$$
[a_0(D_x)v_-][b_1(D_x)u_-]b_0(D_x)u_-, \qquad \text{(B.3.93)}
$$

with $a_0(\xi)\in\{1,\xi_j\langle\xi\rangle^{-1},j=1,2\}$, $b_1(\xi)\in\{\xi_1,\xi_j\xi_1|\xi|^{-1},j=1,2\}$, and $b_0(\xi)\in\{1,\xi_2|\xi|^{-1}\}$. Therefore, lemmas B.3.5, B.3.6 imply that $NL_{kg}(D_2u)$ and $NL_{kg}(D_tu)$ are remainders $R(t,x)$. Furthermore, if we write explicitly $NL^I_{kg}=\Gamma NL_{kg}$ using (1.1.15) and the equation satisfied by $u_\pm$ in (2.1.2) with $|I|=0$,

$$
\begin{aligned}
NL^I_{kg}&=Q^{\mathrm{kg}}_0((\Gamma v)_\pm,D_1u_\pm)+Q^{\mathrm{kg}}_0(v_\pm,D_1(\Gamma u)_\pm)\\
&\qquad -\delta_\Omega Q^{\mathrm{kg}}_0(v_\pm,D_2u_\pm)-\delta_{Z_1}\Big[Q^{\mathrm{kg}}_0(v_\pm,|D_x|u_\pm)+Q^{\mathrm{kg}}_0\big(v_\pm,Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm)\big)\Big],
\end{aligned}
$$

with δ_Ω (resp. δ_{Z_1}) equal to 1 if $\Gamma=\Omega$ (resp. $\Gamma=Z_1$), 0 otherwise, we realize that from (B.3.74) and a-priori estimates,

$$
\left\|\chi(t^{-\sigma}D_x)\Big[x_nQ^{\mathrm{kg}}_0\big(v_\pm,Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm)\big)(D_1u)\Big](t,\cdot)\right\|_{L^2}\le C(A+B)A^2B\varepsilon^4 t^{-\frac{3}{2}+\frac{\delta+\delta_2}{2}},
$$

while from (B.1.27b), (B.1.3a) and the a-priori estimates,

$$
\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\Big[x_mx_nQ^{\mathrm{kg}}_0\big(v_\pm,Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm)\big)(D_1u)\Big](t,\cdot)\right\|_{L^2}\\
&\lesssim\sum_{|\mu|=0}^{1}\left\|x_mx_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}v_\pm(t,\cdot)\right\|_{L^\infty}\|NL_w(t,\cdot)\|_{L^2}\|\mathrm{R}_1u_\pm(t,\cdot)\|_{L^\infty}\le C(A+B)A^2B\varepsilon t^{-\frac{1}{2}+\frac{\delta+\delta_2}{2}}.
\end{aligned}
$$

Also, for any $\theta \in]0,1[$,

$$\left\|\chi(t^{-\sigma}D_x)\left[Q_0^{\mathrm{kg}}\big(v_\pm, Q_0^{\mathrm{w}}(v_\pm, D_1v_\pm)\big)(D_1u)\right](t,\cdot)\right\|_{L^\infty}$$
$$\lesssim \|v_\pm(t,\cdot)\|_{H^{1,\infty}}\|NL_w(t,\cdot)\|_{H^{1,\infty}}\|\mathrm{R}_1u_\pm(t,\cdot)\|_{L^\infty} \le CA^{4-\theta}B^\theta\varepsilon^4 t^{-\frac{7}{2}+\theta(1+\frac{\delta}{2})},$$

after (B.1.3d) with $s=1$ and a-priori estimates. Therefore $Q_0^{\mathrm{kg}}\big(v_\pm, Q_0^{\mathrm{w}}(v_\pm, D_1v_\pm)\big)(D_1u)$ is also a remainder $R(t,x)$.

From (2.1.1) and (1.1.5), products coming from

$$\left[-\delta_\Omega Q_0^{\mathrm{kg}}(v_\pm, D_2u_\pm) - \delta_{Z_1}Q_0^{\mathrm{kg}}(v_\pm, |D_x|u_\pm)\right](D_1u)$$

are of the form

$$[a_0(D_x)v_-]\,[b_1(D_x)u_-]\,\mathrm{R}_1u_-,$$

with the same $a_0(\xi)$ as before, and $b_1(\xi) \in \{\xi_2, \xi_2\xi_j|\xi|^{-1}, |\xi|, j=1,2\}$, so from lemmas B.3.5, B.3.6 they give rise to remainders $R(t,x)$.

Summing up, the very contributions for which we have to prove estimates (B.3.87) and (B.3.88) are the following:

$$[a_0(D_x)(\Gamma v)_-]\,[b_1(D_x)u_-]\,\mathrm{R}_1u_- \tag{B.3.94a}$$
$$[a_0(D_x)v_-]\,[b_1(D_x)(\Gamma u)_-]\,\mathrm{R}_1u_-, \tag{B.3.94b}$$

which are the remaining types of products in $NL^I_{kg}(D_1u)$, and

$$[a_0(D_x)v_-]\,[b_1(D_x)u_-]\,\mathrm{R}_1(\Gamma u)_-, \tag{B.3.94c}$$

which are the products appearing in $NL_{kg}(D_1\Gamma u)$, where a_0 is the same as above, and $b_1(\xi)$ is equal to ξ_1 or to $\xi_j\xi_1|\xi|^{-1}$, with $j=1,2$. We proceed to analyse the above products separately. The strategy to treat these terms is the same, but the lemmas and inequalities to which we refer could be different depending on the product we are considering. We explain it in details for (B.3.94a), and go faster on (B.3.94b), (B.3.94c).

• **Analysis of (B.3.94a)**:

First of all, we can assume that all factors in (B.3.94a) are supported for moderate frequencies less or equal than t^σ, up to remainders $R(t,x)$. In fact, by means of lemma B.2.2 with $L=L^2$, $w_1 = x_na_0(D_x)(\Gamma v)_-$, and $s>0$ large enough to have $N(s)>2$, together with (B.1.17) with $k=1$ and a-priori estimates, there is some $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ such that

$$\begin{aligned}
&\big\|\chi(t^{-\sigma}D_x)\big[[x_n[a_0(D_x)(\Gamma v)_-]\,[b_1(D_x)u_-]\,\mathrm{R}_1u_-\big]\big\|_{L^2}\\
&\lesssim \big\|\big[\chi_1(t^{-\sigma}D_x)[x_na_0(D_x)(\Gamma v)_-]\big]\,\big[\chi(t^{-\sigma}D_x)b_1(D_x)u_-\big]\,\big[\chi(t^{-\sigma}D_x)\mathrm{R}_1u_-\big]\big\|_{L^2}\\
&+ t^{-2}\sum_{\mu,|\nu|=0}^{1}\|x_n^\mu(\Gamma v)_-(t,\cdot)\|_{L^2}\|\mathrm{R}^\nu u_-(t,\cdot)\|_{H^{2,\infty}}\|u_-(t,\cdot)\|_{H^s}\\
&\lesssim \big\|\big[\chi_1(t^{-\sigma}D_x)[x_na_0(D_x)(\Gamma v)_-]\big]\,\big[\chi(t^{-\sigma}D_x)b_1(D_x)u_-\big]\,\big[\chi(t^{-\sigma}D_x)\mathrm{R}_1u_-\big]\big\|_{L^2}\\
&+ CAB^2\varepsilon^3t^{-\frac{3}{2}+\frac{\delta+\delta_2}{2}},
\end{aligned}$$

while from (B.1.28) and a-priori estimates

$$\begin{aligned}
&\big\|\chi(t^{-\sigma}D_x)\big[[x_mx_n[a_0(D_x)(\Gamma v)_-]\,[b_1(D_x)u_-]\,\mathrm{R}_1u_-\big]\big\|_{L^2}\\
&\lesssim \big\|\big[\chi_1(t^{-\sigma}D_x)[x_mx_na_0(D_x)(\Gamma v)_-]\big]\,\big[\chi(t^{-\sigma}D_x)b_1(D_x)u_-\big]\,\big[\chi(t^{-\sigma}D_x)\mathrm{R}_1u_-\big]\big\|_{L^2}\\
&+ CAB^2\varepsilon^3t^{\frac{\delta+\delta_2}{2}}.
\end{aligned}$$

Also, using lemma B.2.2 but with $L = L^\infty$,

$$\begin{aligned}&\big\|\chi(t^{-\sigma}D_x)\big[[a_0(D_x)(\Gamma v)_-]\,[b_1(D_x)u_-]\,\mathrm{R}_1u_-\big]\big\|_{L^\infty}\\&\lesssim \big\|\big[\chi_1(t^{-\sigma}D_x)[a_0(D_x)(\Gamma v)_-]\big]\,\big[\chi(t^{-\sigma}D_x)b_1(D_x)u_-\big]\,\big[\chi(t^{-\sigma}D_x)\mathrm{R}_1u_-\big]\big\|_{L^\infty}\\&+t^{-2}\sum_{|\mu|=0}^{1}\|(\Gamma v)_-(t,\cdot)\|_{L^2}\|\mathrm{R}^\mu u_-(t,\cdot)\|_{H^{2,\infty}}\|u_-(t,\cdot)\|_{H^s}\\&\lesssim \big\|\big[\chi_1(t^{-\sigma}D_x)[a_0(D_x)(\Gamma v)_-]\big]\,\big[\chi(t^{-\sigma}D_x)b_1(D_x)u_-\big]\,\big[\chi(t^{-\sigma}D_x)\mathrm{R}_1u_-\big]\big\|_{L^\infty}+CAB^2\varepsilon^3t^{-\frac{5}{2}+\frac{\delta+\delta_2}{2}}.\end{aligned}$$

Secondly, we can assume $b_1(D_x)u_-$ replaced with $b_1(D_x)u^{NF}$, with u^{NF} introduced in (3.1.15), again up to some terms satisfying (B.3.87) (resp. (B.3.88)), as follows using (B.1.17), (B.1.28) (resp. (B.2.37)), (B.3.26b) and (1.1.11a).

Thirdly, $a_0(D_x)(\Gamma v)_-$ can be substituted with $a_0(D_x)V^{NF}_\Gamma$, V^{NF}_Γ being defined in (B.3.60), up to remainders verifying (B.3.87) (resp. (B.3.88)) thanks to lemma B.3.17 (resp. estimate (B.3.61)), (B.3.30) (resp.

$$\|u^{NF}(t,\cdot)\|_{H^{\rho,\infty}}+\|\mathrm{R}u^{NF}(t,\cdot)\|_{H^{\rho,\infty}}\le CB\varepsilon t^{-\frac{1}{2}}, \tag{B.3.95}$$

which is the classical version of the semi-classical (B.3.8)) and (1.1.11a).

With the above manipulations, we basically reduced to prove that, for $k=0,1$,

$$\begin{aligned}\big\|[\chi_1(t^{-\sigma}D_x)[x_m^\mu x_n a_0(D_x)V^{NF}_\Gamma]\,[\chi(t^{-\sigma}D_x)b_1(D_x)u^{NF}]\chi(t^{-\sigma}D_x)\mathrm{R}_1u_-\big\|_{L^2}\\ \le C(A+B)^2B\varepsilon^3t^{-1+\mu+\beta'},\end{aligned}$$

and

$$\big\|[\chi_1(t^{-\sigma}D_x)[a_0(D_x)V^{NF}_\Gamma]\,[\chi(t^{-\sigma}D_x)b_1(D_x)u^{NF}]\chi(t^{-\sigma}D_x)\mathrm{R}_1u_-\big\|_{L^2}\le C(A+B)^2B\varepsilon^3t^{-\frac{5}{2}+\beta'}.$$

We notice that, using (1.1.11a) and passing to the semi-classical framework and unknowns, with $\widetilde{V}^\Gamma$ defined in (B.3.67) and $\widetilde{u}$ in (3.2.2), above inequalities will follow respectively from

$$\sum_{k=0}^{1}\Big\|\Big[Op_h^w(\chi_1(h^\sigma\xi))[x_m^kx_nOp_h^w(a_0(\xi))\widetilde{V}^\Gamma]\Big]\,\big[Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big](t,\cdot)\Big\|_{L^2}\le C(A+B)B\varepsilon^3h^{-\frac{1}{2}-\beta'} \tag{B.3.96}$$

and

$$\Big\|\big[Op_h^w(\chi_1(h^\sigma\xi)a_0(\xi))\widetilde{V}^\Gamma\big]\,\big[Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big](t,\cdot)\Big\|_{L^\infty}\le C(A+B)B\varepsilon^3h^{-\beta'},$$

The above L^∞ has already been proved in lemma B.3.16 (see (B.3.82b)).

Using the same argument that led us to (B.3.35) and (B.3.39) in the proof of lemma B.3.6, up to replacing $\widetilde{v}$ with $\widetilde{V}^\Gamma$ in (B.3.33), referring to lemma B.3.13 instead of B.2.14, and to estimate (B.3.83) instead of (B.3.9), we can write that

$$\begin{aligned}&\Big\|\Big[Op_h^w(\chi_1(h^\sigma\xi))[x_nOp_h^w(a_0(\xi))\widetilde{V}^\Gamma]\Big]\,\big[Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big](t,\cdot)\Big\|_{L^2}\\&\qquad\le\Big\|[Op_h^w(\chi_1(h^\sigma\xi)a_0'(\xi))\widetilde{V}^\Gamma(t,\cdot)][Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}(t,\cdot)]\Big\|_{L^2}+CAB\varepsilon^2h^{\frac{1}{2}-\beta'},\end{aligned}$$

with $a_0'(\xi)=a_0(\xi)\xi_n\langle\xi\rangle^{-1}$, together with

$$\begin{aligned}&\Big\|\Big[Op_h^w(\chi_1(h^\sigma\xi))[x_mx_nOp_h^w(a_0(\xi))\widetilde{V}^\Gamma]\Big]\,\big[Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big](t,\cdot)\Big\|_{L^2}\\&\lesssim\Big\|[Op_h^w(\chi(\xi)a_0'(\xi))\widetilde{V}^\Gamma(t,\cdot)][Op_h^w(\chi(h^\sigma\xi)b_1'(\xi))\widetilde{u}(t,\cdot)]\Big\|_{L^2}\\&+h^\sigma\Big\|[Op_h^w((\partial_m\chi_1)(h^\sigma\xi)a_0'(\xi))\widetilde{V}^\Gamma(t,\cdot)][Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}(t,\cdot)]\Big\|_{L^2}+C(A+B)B\varepsilon^2h^{\frac{1}{2}-\beta'},\end{aligned}$$

with $b'_1(\xi)=b_1(\xi)\xi_m|\xi|^{-1}$. Consequently, estimate (B.3.96) is derived from (B.3.82a), which concludes that (B.3.94a) is a remainder $R(t,x)$.

• **Analysis of** (B.3.94b):

By means of corollary B.2.4 with $L=L^2$, $w=u$, and $s>0$ sufficiently large so that $N(s)>3$, together with remark B.2.5, we can assume all factors in (B.3.94b) truncated for frequencies less or equal than t^σ, up to remainder contributions $R(t,x)$ verifying (B.3.87), (B.3.88). In fact, from (B.1.10a), (B.1.27a) and a-priori estimates, we find that

$$\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\,[x_n[a_0(D_x)v_-]\,[b_1(D_x)(\Gamma u)_-]\,\mathrm{R}_1u_-]\right\|_{L^2}\\
&\lesssim \left\|\left[\chi_1(t^{-\sigma}D_x)[x_na_0(D_x)v_-]\right]\,\left[\chi(t^{-\sigma}D_x)b_1(D_x)(\Gamma u)_-\right]\chi(t^{-\sigma}D_x)\mathrm{R}_1u_-\right\|_{L^2}\\
&+t^{-3}\sum_{|\mu_1|,\mu_2=0}^{1}\left(\|x^{\mu_1}x_n^{\mu_2}v_-(t,\cdot)\|_{L^2}+t\|x_n^{\mu_2}v_-(t,\cdot)\|_{L^2}\right)\|u_\pm(t,\cdot)\|_{H^s}\|\mathrm{R}_1u_-(t,\cdot)\|_{L^\infty}\\
&\lesssim \left\|\left[\chi_1(t^{-\sigma}D_x)[x_na_0(D_x)v_-]\right]\,\left[\chi(t^{-\sigma}D_x)b_1(D_x)(\Gamma u)_-\right]\chi(t^{-\sigma}D_x)\mathrm{R}_1u_-\right\|_{L^2}+CAB^2\varepsilon^2t^{-1+\frac{\delta+\delta_2}{2}}
\end{aligned}$$

and using also (B.1.33),

$$\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\,[x_mx_n[a_0(D_x)v_-]\,[b_1(D_x)(\Gamma u)_-]\,\mathrm{R}_1u_-]\right\|_{L^2}\\
&\lesssim \left\|\left[\chi_1(t^{-\sigma}D_x)[x_mx_na_0(D_x)v_-]\right]\,\left[\chi(t^{-\sigma}D_x)b_1(D_x)(\Gamma u)_-\right]\chi(t^{-\sigma}D_x)\mathrm{R}_1u_-\right\|_{L^2}+CAB^2\varepsilon^2t^{\frac{\delta+\delta_2}{2}}.
\end{aligned}$$

Moreover, using corollary B.2.4 with $L=L^\infty$, $w=u$, and $s>0$ such that $N(s)\geq 4$, together with (B.1.10a) and a-priori estimates, one can also check that

$$\begin{aligned}
&\left\|\chi(t^{-\sigma}D_x)\,[[a_0(D_x)v_-]\,[b_1(D_x)(\Gamma u)_-]\,\mathrm{R}_1u_-]\right\|_{L^\infty}\\
&\lesssim \left\|\left[\chi_1(t^{-\sigma}D_x)a_0(D_x)v_-\right]\,\left[\chi(t^{-\sigma}D_x)b_1(D_x)(\Gamma u)_-\right]\chi(t^{-\sigma}D_x)\mathrm{R}_1u_-\right\|_{L^\infty}+CAB^2\varepsilon^2t^{-\frac{5}{2}+\frac{\delta+\delta_2}{2}}.
\end{aligned}$$

Successively, we can assume $a_0(D_x)v_-$ replaced with $a_0(D_x)v^{NF}$, v^{NF} given by (3.1.3), up to remainders verifying (B.3.87) (resp. (B.3.88)), as follows by using (B.3.31) (resp. (B.3.26a)), (1.1.11d) with $k=2$ and (1.1.11a).

Then, using (1.1.11a) and passing to the semi-classical framework and unknowns, with $\widetilde{v}$ defined in (3.2.2) and $\widetilde{u}^I$ in (B.3.21), we should prove that the following estimates are satisfied:

$$\sum_{k=0}^{1}\left\|\left[Op_h^w(\chi_1(h^\sigma\xi))\left[x_m^kx_nOp_h^wa_0(\xi))\widetilde{v}\right]\right]\left[Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}^J\right](t,\cdot)\right\|_{L^2}\leq C(A+B)B\varepsilon^3h^{-\frac{1}{2}-\beta'},$$

along with

$$\left\|\left[Op_h^w(\chi_1(h^\sigma\xi)a_0(\xi))\widetilde{v}\right]\left[Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}^J\right](t,\cdot)\right\|_{L^\infty}\leq C(A+B)B\varepsilon^3h^{-\beta'}.$$

The latter one holds true after lemma B.3.4 (see (B.3.22b)). The former one is also consequence of this lemma (see precisely (B.3.22a)), after having observed that a similar argument to to the one that led to (B.3.35) and (B.3.39) can be applied, up to replacing $\widetilde{u}$ with $\widetilde{u}^I$ in (B.3.34), using lemma B.2.9 instead of (B.2.1a), (B.2.1c), estimate (B.3.23) instead of (B.3.8), and the fact that

$$\begin{aligned}
&\left\|Op_h^w(\chi(h^\sigma\xi)b_1(\xi)\xi_m|\xi|^{-1})\widetilde{u}^I(t,\cdot)\right\|_{L^\infty}=t\left\|\chi(t^{-\sigma}D_x)b_1(D_x)D_m|D_x|^{-1}(\Gamma u)_-(t,\cdot)\right\|_{L^\infty}\\
&\qquad\lesssim t\|\chi(t^{-\sigma}D_x)(\Gamma u)_-(t,\cdot)\|^{1-\theta}_{H^{3,\infty}}\|(\Gamma u)_-(t,\cdot)\|^\theta_{H^2}\leq C(A+B)^{1-\theta}B^\theta\varepsilon t^{\frac{1}{2}+\beta+\frac{\delta_1}{2}+\frac{(1+\delta_1+\delta_2)}{2}\theta},
\end{aligned}$$

which is the analogous of (B.3.37) (last estimate is deduced using (B.2.52) and (1.1.11d) with $k=1$). Therefore, we deduce that

$$\begin{aligned}&\big\|\big[Op_h^w(\chi_1(h^\sigma\xi))\big[x_nOp_h^w a_0(\xi))\widetilde{v}\big]\big]\ \big[Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}^J\big](t,\cdot)\big\|_{L^2}\\ &\qquad\qquad\le \big\|[Op_h^w(\chi_1(h^\sigma\xi)a_0'(\xi))\widetilde{v}(t,\cdot)][Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}^I(t,\cdot)]\big\|_{L^2}+CAB\varepsilon^2h^{\frac{1}{2}-\beta'},\end{aligned}$$

along with

$$\begin{aligned}&\big\|\big[Op_h^w(\chi_1(h^\sigma\xi))\big[x_mx_nOp_h^w a_0(\xi))\widetilde{v}\big]\big]\ \big[Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}^J\big](t,\cdot)\big\|_{L^2}\\ &\lesssim \big\|[Op_h^w(\chi(\xi)a_0'(\xi))\widetilde{v}][Op_h^w(\chi(h^\sigma\xi)b_1'(\xi))\widetilde{u}^I]\big\|_{L^2}\\ &+h^\sigma\left\|[Op_h^w((\partial_m\chi_1)(h^\sigma\xi)a_0'(\xi))\widetilde{V}^\Gamma(t,\cdot)][Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}(t,\cdot)]\right\|_{L^2}+C(A+B)B\varepsilon^2h^{\frac{1}{2}-\beta'},\end{aligned}$$

with the same a_0', b_1' as in the previous case. That concludes that (B.3.94b) also satisfies (B.3.87), (B.3.88).

• **Analysis of** (B.3.94c):

After lemma B.2.2 with $L=L^2$ (resp. $L=L^\infty$), $w_1=\mathrm{R}_1(\Gamma u)_-$, and $s>0$ large such that $N(s)>2$, together with (B.1.10a), (B.1.27a) (resp. (1.1.11b)) and a-priori estimates, we can assume all factors in (B.3.94c) localised for frequencies less or equal than t^σ, up to remainders $R(t,x)$.

Also, from (B.1.10a), (B.1.27a) (resp. (1.1.11b)), together with (B.3.26b) and (B.2.52), we can replace $b_1(D_x)u_-$ with $b_1(D_x)u^{NF}$, while from (B.3.31) (resp. (B.3.26a)), (B.3.30) (resp. (B.3.95)) and (B.2.52), we can assume $a_0(D_x)v_-$ be replaced with $a_0(D_x)v^{NF}$, up to additional contributions satisfying (B.3.87) (resp. (B.3.88)).

That reduces us to prove that, for $k=0,1$,

$$\begin{aligned}\left\|[\chi(t^{-\sigma}D_x)[x_m^kx_na_0(D_x)v^{NF}]]\,[\chi(t^{-\sigma}D_x)b_1(D_x)u^{NF}]\,\chi_1(t^{-\sigma}D_x)\mathrm{R}_1(\Gamma u)_-\right\|_{L^2}&\\ \le C(A+B)^2B\varepsilon^3t^{-1+k+\beta'}&,\end{aligned}$$

and

$$\big\|[\chi(t^{-\sigma}D_x)a_0(D_x)v^{NF}]\,[\chi(t^{-\sigma}D_x)b_1(D_x)u^{NF}]\,\chi_1(t^{-\sigma}D_x)\mathrm{R}_1(\Gamma u)_-\big\|_{L^\infty}\le C(A+B)^2B\varepsilon^3t^{-\frac{5}{2}+\beta'},$$

or equivalently, from estimate (B.2.37) and passing to the semi-classical coordinates and unknowns $\widetilde{v},\widetilde{u}$, that

$$\sum_{k=0}^{1}\left\|\Big[Op_h^w(\chi_1(h^\sigma\xi))\big[x_m^kx_nOp_h^wa_0(\xi))\widetilde{v}\big]\Big]\ \big[Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big](t,\cdot)\right\|_{L^2}\le C(A+B)B\varepsilon^2h^{\frac{1}{2}-\beta'},$$

together with

$$\big\|\big[Op_h^w(\chi_1(h^\sigma\xi)a_0(\xi))\widetilde{v}\big]\big[Op_h^w(\chi(h^\sigma\xi)b_1(\xi))\widetilde{u}\big](t,\cdot)\big\|_{L^\infty}\le C(A+B)B\varepsilon^2h^{-\beta'}.$$

The former estimate has already been proved at the end of the proof of lemma B.3.6 (see from (B.3.32) to (B.3.40)), while the latter one in lemma B.3.3 (see (B.3.7b)). That concludes that also (B.3.94c) is a remainder $R(t,x)$, and gives the result of the statement. □

Corollary B.3.19. *Let $NL_\Gamma^{kg,c}$ be given by* (B.3.75)*. There exists a constant $C>0$ such that, under the same assumptions as in lemma B.3.13, for any $\chi\in C_0^\infty(\mathbb{R}^2)$, $\sigma>0$ small, $m,n=1,2$,*

$$\left\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\left[t(tx_n)NL_\Gamma^{kg,c}(t,tx)\right]\right\|_{L^2}\le C(A+B)^2B\varepsilon^3t^{\beta'},$$

$$\left\|Op_h^w(\chi(h^\sigma\xi))\mathcal{L}_m\left[t(tx_j)Q_0^{\mathrm{kg}}\big(v_\pm,Q_0^{\mathrm{w}}(v_\pm,D_1v_\pm)\big)(t,tx)\right]\right\|_{L^2}\le C(A+B)AB\varepsilon^3t^{\frac{\delta+\delta_2}{2}},$$

for every $t\in[1,T]$, with $\beta'>0$ such that $\beta'\to0$ as $\sigma,\delta_0\to0$.

Proof. Straightforward after (B.3.43), lemma B.3.18, (B.3.74) and the fact that

$$\begin{aligned}\Big\|\chi(t^{-\sigma}D_x)\left[x_m x_n Q^{\mathrm{kg}}_0\big(v_\pm, Q^{\mathrm{w}}_0(v_\pm, D_1 v_\pm)\big)\right](t,\cdot)\Big\|_{L^2} \\ \lesssim \sum_{|\mu|=0}^{1}\Big\|x_m x_n\Big(\frac{D_x}{\langle D_x\rangle}\Big)^{\mu}v_\pm(t,\cdot)\Big\|_{L^\infty}\|NL_w(t,\cdot)\|_{L^2}\le C(A+B)AB\varepsilon^3 t^{\frac{\delta+\delta_2}{2}},\end{aligned}$$

after (B.1.3a), (B.1.27b) and a-priori estimates. ☐

Lemma B.3.20. *Let* $\widetilde{V}^\Gamma$ *be defined in* (B.3.67). *There exists some positive constant* C *such that, under the same assumptions as in lemma B.3.13, for any* $\chi\in C^\infty_0(\mathbb{R}^2)$, $\sigma>0$ *small,*

$$\sum_{|\mu|=2}\Big\|Op^w_h(\chi(h^\sigma\xi)\mathcal{L}^\mu\widetilde{V}^\Gamma(t,\cdot)\Big\|_{L^2}\le CB\varepsilon t^{\beta'}, \tag{B.3.97}$$

with $\beta'>0$ *small,* $\beta'\to 0$ *as* $\sigma,\delta_0\to 0$.

Proof. Similarly to (B.3.42), and reminding that V^{NF}_Γ is solution to (B.3.70), we have that, for any $m,n=1,2$,

$$\begin{aligned}\Big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}_m\mathcal{L}_n\widetilde{V}^\Gamma(t,\cdot)\Big\|_{H^1}\lesssim\sum_{\mu=0}^{1}\Big[\,\big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu_m\big[tZ_nV^{NF}_\Gamma(t,tx)\big]\big\|_{L^2}\\ +\Big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu_m Op^w_h\Big(\frac{\xi_n}{\langle\xi\rangle}\Big)\widetilde{V}^\Gamma(t,\cdot)\big]\Big\|_{L^2}+\Big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu_m\big[t(tx_n)NL^{kg,c}_\Gamma(t,tx)\big]\Big\|_{L^2}\\ +\Big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu_m\Big[t(tx_j)Q^{\mathrm{kg}}_0\big(v_\pm,Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm)\big)(t,tx)\Big]\Big\|_{L^2}\Big],\end{aligned}$$

so the result of the statement follows from (B.3.73), (B.3.74), (B.3.80), lemma B.3.13, estimate (B.3.77) and corollary B.3.19. ☐

Lemma B.3.21. *There exists a constant* $C>0$ *such that, under the same hypothesis as in lemma B.2.14, for any* $\chi\in C^\infty_0(\mathbb{R}^2)$, *equal to 1 in a neighbourhood of the origin, and* $\sigma>0$ *small,*

$$\sum_{|I|=1}\|\chi(t^{-\sigma}D_x)V^I(t,\cdot)\|_{L^\infty}\le CB\varepsilon t^{-1}, \tag{B.3.98}$$

for every $t\in[1,T]$, *with* $\beta>0$ *small,* $\beta\to 0$ *as* $\sigma\to 0$.

Proof. As this estimate is evidently satisfied when I is such that Γ^I is a spatial derivative, after a-priori estimate (1.1.11b), we focus on proving the statement for $\Gamma^I\in\{\Omega, Z_m, m=1,2\}$ being a Klainerman vector field. For simplicity, we refer to Γ^I simply by Γ.

Instead of proving the result of the statement directly on $(\Gamma v)_\pm$, we show that

$$\big\|V^{NF}_\Gamma(t,\cdot)\big\|_{L^\infty}\le CB\varepsilon t^{-1}, \tag{B.3.99}$$

with V^{NF}_Γ defined in (B.3.60). After (B.3.61), the above inequality evidently implies the statement. The main idea to derive the sharp decay estimate in (B.3.99) is to use the same argument that, in subsection 3.2.1, led us to the propagation of a-priori estimate (1.1.11b). Thus we are going to move to the semi-classical setting and to deduce an ODE from equation (B.3.70) satisfied by V^{NF}_Γ. The most important feature that will provide us with (B.3.99) is that the uniform norm of all involved non-linear terms is integrable in time.

Before going into the details, we remind the reader our choice to denote by C, β and β' some positive constants, that may change line after line, with $\beta \rightarrow 0$ (resp. $\beta' \rightarrow 0$) as $\sigma \rightarrow 0$ (resp. as $\sigma, \delta_0 \rightarrow 0$).

So let us consider $\widetilde{V}^\Gamma(t,x) := tV^{NF}_\Gamma(t,tx)$, operator Γ^{kg} as follows

$$\Gamma^{kg} := Op^w_h\Big(\gamma\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi_1(h^\sigma\xi)\Big),$$

with $\gamma, \chi_1 \in C^\infty_0(\mathbb{R}^2)$ such that $\gamma \equiv 1$ close to the origin, $\chi_1 \equiv 1$ on the support of χ, $p(\xi) := \langle\xi\rangle$, and

$$\begin{aligned}
&\widetilde{V}^\Gamma_{\Lambda_{kg}}(t,x) := \Gamma^{kg} Op^w_h(\chi(h^\sigma\xi))\widetilde{V}^\Gamma(t,x),\\
&\widetilde{V}^\Gamma_{\Lambda^c_{kg}}(t,x) := Op^w_h\Big((1-\gamma)\Big(\frac{x-p'(\xi)}{\sqrt{h}}\Big)\chi_1(h^\sigma\xi)\Big)Op^w_h(\chi(h^\sigma\xi))\widetilde{V}^\Gamma(t,x),
\end{aligned}$$

so that

$$Op^w_h(\chi(h^\sigma\xi))\widetilde{V}^\Gamma(t,\cdot) = \widetilde{V}^\Gamma_{\Lambda_{kg}} + \widetilde{V}^\Gamma_{\Lambda^c_{kg}}.$$

It immediately follows from inequality (3.2.19b) and lemmas B.3.13, B.3.20, that

$$\Big\|\widetilde{V}^\Gamma_{\Lambda^c_{kg}}(t,\cdot)\Big\|_{L^\infty} \lesssim \sum_{|\mu|=0}^{2} h^{\frac{1}{2}-\beta}\Big\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{V}^\Gamma(t,\cdot)\Big\|_{L^2} \le CB\varepsilon t^{-\frac{1}{2}+\beta'}. \tag{B.3.100}$$

On the other hand, an explicit computation shows that, from (B.3.70) satisfied by V^{NF}_Γ, $\widetilde{V}^\Gamma$ is solution to the following semi-classical pseudo-differential equation:

$$[D_t - Op^w_h(x\cdot\xi - \langle\xi\rangle)]\,\widetilde{V}^\Gamma(t,x) = h^{-1}NL^{kg,c}_\Gamma(t,tx) - \delta_{Z_1}h^{-1}Q^{\mathrm{kg}}_0\big(v_\pm, Q^{\mathrm{w}}_0(v_\pm, D_1v_\pm)\big)(t,tx),$$

with $NL^{kg,c}_\Gamma$ given explicitly by (B.3.75), (B.2.42). Applying successively operators $Op^w_h(\chi(h^\sigma\xi))$ and Γ^{kg} to the above equation we find, from first part of lemma 3.2.5, that $\widetilde{V}^\Gamma_{\Lambda_{kg}}$ satisfies

$$\begin{aligned}
&[D_t - Op^w_h(x\cdot\xi - \langle\xi\rangle)]\,\widetilde{V}^\Gamma_{\Lambda_{kg}}(t,x) = h^{-1}\Gamma^{kg}Op^w_h(\chi(h^\sigma\xi))\Big[NL^{kg,c}_\Gamma(t,tx)\Big]\\
&\quad - \delta_{Z_1}h^{-1}\Gamma^{kg}Op^w_h(\chi(h^\sigma\xi))\big[Q^{\mathrm{kg}}_0\big(v_\pm, Q^{\mathrm{w}}_0(v_\pm, D_1v_\pm)\big)(t,tx)\big] - Op^w_h(b(x,\xi))Op^w_h(\chi(h^\sigma\xi))\widetilde{V}^\Gamma(t,x)\\
&\quad + i\sigma h^{1+\sigma}\Gamma^{kg}Op^w_h\big((\partial\chi)(h^\sigma\xi)\cdot(h^\sigma\xi)\big)\widetilde{V}^\Gamma,
\end{aligned} \tag{B.3.101}$$

with symbol b given by (3.2.28). Since the derivatives of γ vanish in a neighbourhood of the origin, and $\partial\chi_1 \equiv 0$ on the support of χ, from inequalities (3.2.18b), (3.2.19b), together with symbolic calculus of lemma 1.2.24 and remark 1.2.22, we observe that

$$\begin{aligned}
\Big\|Op^w_h(b(x,\xi))\widetilde{V}^\Gamma(t,\cdot)\Big\|_{L^\infty} &\lesssim h^{\frac{3}{2}-\beta}\sum_{|\mu|=0}^{2}\|Op^w_h(\chi(h^\sigma\xi))\mathcal{L}^\mu\widetilde{V}^\Gamma(t,\cdot)\|_{L^2} + h^N\Big\|\widetilde{V}^\Gamma(t,\cdot)\Big\|_{L^2}\\
&\le CB\varepsilon t^{-\frac{3}{2}+\beta'},
\end{aligned}$$

where last estimate is obtained using lemmas B.3.13, B.3.20 and picking $N \ge 2$.

Moreover, reminding (B.3.13), we have that

$$\begin{aligned}
&h^{1+\sigma}\Big\|\Gamma^{kg}Op^w_h((\partial\chi(h^\sigma\xi)\cdot(h^\sigma\xi))\widetilde{V}^\Gamma(t,\cdot)\Big\|_{L^\infty}\\
&\qquad \le h^{1+\sigma}\Big\|\Gamma^{kg}\theta_h(x)Op^w_h((\partial\chi(h^\sigma\xi)\cdot(h^\sigma\xi))\widetilde{V}^\Gamma(t,\cdot)\Big\|_{L^\infty} + h^N\|\widetilde{V}^\Gamma(t,\cdot)\|_{L^2},
\end{aligned} \tag{B.3.102}$$

where $\theta_h(x)$ is a smooth cut-off function supported in closed ball $\overline{B_{1-ch^{2\sigma}}(0)}$, for some small $c>0$, and $N\in\mathbb{N}$ is as large as we want. Denoting $(\partial\chi)(\xi)\cdot\xi$ concisely by $\widetilde{\chi}(\xi)$, we observe that from proposition 1.2.38 with $p=+\infty$, together with the uniform continuity on L^∞ of operator $\widetilde{\chi}(t^{-\sigma}D_x)$, the definition of $\widetilde{V}^\Gamma$ in terms of V^{NF}_Γ, and (B.3.61),

$$h^{1+\sigma}\left\|\Gamma^{kg}\theta_h(x)Op^w_h(\widetilde{\chi}(h^\sigma\xi))\widetilde{V}^\Gamma(t,\cdot)\right\|_{L^\infty}\lesssim h^{1-\beta}\left\|\theta_h(x)Op^w_h(\widetilde{\chi}(h^\sigma\xi))\widetilde{V}^\Gamma(t,\cdot)\right\|_{L^\infty}$$
$$\le t^\beta\left\|\theta_h\Big(\frac{\cdot}{t}\Big)\widetilde{\chi}(t^{-\sigma}D_x)(\Gamma v)_-(t,\cdot)\right\|_{L^\infty}+C(A+B)B\varepsilon^2t^{-\frac{5}{4}+\beta}.$$

Using the fact that, for $\theta^j_h(z):=\theta_h(z)z_j$,

$$\theta_h\Big(\frac{x}{t}\Big)(\Omega v)_-=t\Big[\theta^1_h\Big(\frac{x}{t}\Big)\partial_2v_--\theta^2_h\Big(\frac{x}{t}\Big)\partial_1v_-\Big],$$

and

$$\theta_h\Big(\frac{x}{t}\Big)(Z_mv)_-=t\Big[\theta^m_h\Big(\frac{x}{t}\Big)\partial_tv_-+\theta_h\Big(\frac{x}{t}\Big)\partial_mv_-\Big]+\theta_h\Big(\frac{x}{t}\Big)\frac{D_m}{\langle D_x\rangle}v_-,\quad m=1,2,$$

we derive, after some commutations and up to a loss in t, that $(\Gamma v)_-$ can be expressed in terms of v_- and its derivatives, so from the classical Sobolev injection combined inequality (B.1.2) we obtain that

$$t^{-\beta}\left\|\widetilde{\chi}(t^{-\sigma}D_x)\theta_h\Big(\frac{\cdot}{t}\Big)(\Gamma v)_-(t,\cdot)\right\|_{L^\infty}\lesssim t^{-N(s)+1+\beta}\left(\|D_tv_\pm(t,\cdot)\|_{H^s}+\|v_\pm(t,\cdot)\|_{H^s}\right)$$
$$\le CB\varepsilon t^{-\frac{3}{2}},$$

last estimate following by taking $s>0$ large enough to have $N(s)\ge 3$, and using (B.1.6a) with $s=0$, together with a-priori estimates. From (B.3.102) we hence derive that

$$h^{1+\sigma}\left\|\Gamma^{kg}Op^w_h((\partial\chi(h^\sigma\xi)\cdot(h^\sigma\xi))\widetilde{V}^\Gamma(t,\cdot)\right\|_{L^\infty}\le CB\varepsilon t^{-\frac{3}{2}},$$

so the last two terms in the right hand side of equation (B.3.101) are remainders $R(t,x)$ such that

$$\|R(t,\cdot)\|_{L^\infty}\le CB\varepsilon t^{-\frac{5}{4}},\tag{B.3.103}$$

for every $t\in[1,T]$.

After proposition 1.2.38 with $p=+\infty$, estimate (B.3.86), and the fact that for any $\theta\in]0,1[$,

$$\left\|Q^{\mathrm{kg}}_0\left(v_\pm,Q^{\mathrm{w}}_0(v_\pm,D_1v_\pm)\right)\right\|_{L^\infty}\le CA^{3-\theta}B^\theta\varepsilon^3t^{-3+\theta(1+\frac{\delta}{2})},$$

as follows by (B.1.3c) with $s=1$ and a-priori estimates, we deduce (up to taking $\theta\ll 1$ small in the above inequality) that also the first two non-linear terms in the right hand side of (B.3.101) satisfy (B.3.103) and can be included into $R(t,x)$.

Therefore, $\widetilde{V}^\Gamma_{\Lambda_{kg}}$ satisfies

$$\left[D_t-Op^w_h(x\cdot\xi-\langle\xi\rangle)\right]\widetilde{V}^\Gamma_{\Lambda_{kg}}(t,x)=R(t,x),$$

and using (3.2.22) along with inequality (3.2.24b), but with $\widetilde{v}^\Sigma_{\Lambda_{kg}}$ replaced with $\widetilde{V}^\Gamma_{\Lambda_{kg}}$, together with lemmas B.3.13, B.3.20, we deduce that, for the same family of cut-off functions θ_h introduced above, $\widetilde{V}^\Gamma_{\Lambda_{kg}}$ is solution to the following ODE:

$$D_t\widetilde{V}^\Gamma_{\Lambda_{kg}}(t,x)=-\theta_h(x)\phi(x)\widetilde{V}^\Gamma_{\Lambda_{kg}}(t,x)+R(t,x),\tag{B.3.104}$$

with $\phi(x) = \sqrt{1-|x|^2}$, and where the inhomogeneous term $R(t,x)$ decays, in the uniform norm, at a rate which is integrable in time. As a consequence,

$$\|\widetilde{V}^{\Gamma}_{\Lambda_{kg}}(t,\cdot)\|_{L^\infty} \lesssim \|\widetilde{V}^{\Gamma}_{\Lambda_{kg}}(1,\cdot)\|_{L^\infty} \le CB\varepsilon,$$

which summed up with (B.3.100) implies (B.3.99), and hence the conclusion of the proof. □

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Résumé

Cette thèse est consacrée à l'étude de l'éxistence globale de solutions pour des équations de Klein-Gordon – ou des systèmes ondes-Klein-Gordon – quasi-linéaires critiques, à données petites, régulières, décroissantes à l'infini, en dimension un ou deux d'éspace. On étudie d'abord ce problème pour des équations de Klein-Gordon cubiques en dimension un, pour lesquelles il y a existence globale des solutions lorsque la non-linéarité vérifie une condition de structure et les données initiales sont petites et à support compact. Nous prouvons que ce résultat est vrai aussi lorsque les données initiales ne sont pas localisées en espace mais décroissent faiblement à l'infini, en combinant la méthode des champs de vecteurs de Klainerman avec une analyse micro-locale semi-classique de la solution. La deuxième et principale contribution à la thèse s'attache à l'étude de l'existence globale de solutions pour un système modèle ondes-Klein-Gordon quadratique, quasi-linéaire, en dimension deux, toujours pour des données initiales petites régulières à décroissance modérée à l'infini, les non-linéarités étant données en termes de «formes nulles ». Nous obtenons des estimations d'énergie sur la solution sur laquelle agissent des champs de Klainerman, et des estimations de décroissance uniforme optimales, dans une version para-différentielle. Nous prouvons les secondes par une réduction du système d'équations aux dérivées partielles du départ à un système d'équations ordinaires, stratégie qui pourrait nous emmener, dans le futur, à traiter le cas de non-linéarités plus générales.

Mots Clefs

1. Existence globale de petites solutions
2. Équations dispersives
3. Équations de Klein-Gordon
4. Systèmes ondes-Klein-Gordon
5. Champs de vecteurs de Klainerman
6. Formes normales
7. Analyse micro-locale semi-classique
8. Structure nulle

Summary

In this thesis we study the problem of global existence of solutions to critical quasi-linear Klein-Gordon equations – or to critical quasi-linear coupled wave-Klien-Gordon systems – when initial data are small, smooth, decaying at infinity, in space dimension one or two. We first study this problem for cubic Klein-Gordon equations in space dimension one. It is known that, under a suitable structure condition on the non-linearity, the global well-posedness of the solution is ensured when initial data are small and compactly supported. We prove that this result holds true even when initial data are not localized in space but only mildly decaying at infinity, by combining the Klainerman vector fields' method with a semi-classical micro-local analysis of the solution. The second and main contribution to the thesis concerns the study of the global existence of solutions to a quadratic quasi-linear wave-Klein-Gordon system in space dimension two, again when initial data are small smooth and mildly decaying at infinity. We consider the case of a model non-linearity, expressed in terms of "null forms". Our aim is to obtain some energy estimates on the solution when some Klainerman vector fields are acting on it, and sharp uniform estimates. The former ones are recovered making systematically use of normal forms' arguments for quasi-linear equations, in their para-differential version. We derive the latter ones by deducing a system of ordinary differential equations from the starting partial differential system, this strategy maying leading us in the future to treat the case of the most general non-linearities.

Keywords

1. Global existence of small solutions
2. Dispersive equations
3. Klein-Gordon equations
4. Wave-Klein-Gordon systems
5. Klainerman vector fields
6. Normal forms
7. Semi-classical micro-local analysis
8. Null structures