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Retour à l'équilibre et compacité de la résolvante pour des opérateurs de Kramers-Fokker-Planck dégénérés

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Résumé:

L'équation de Kramers-Fokker-Planck est l'équation d'évolution pour les fonctions de distribution décrivant le mouvement brownien des particules dans un champ externe. Cette équation a été dérivée et utilisée pour décrire la cinétique de la réaction chimique.

L'un des problèmes fondamentaux est d'analyser le comportement en temps grand des solutions de l'équation de Kramers-Fokker-Planck dépendant du temps (t) et de prouver que ces solutions convergent exponentiellement vers l'équilibre lorsque t tend vers $+\infty$.

Afin d'étudier le problème de retour exponentiel à l'équilibre, une méthode efficace consiste à étudier l'écart spectral, qui est réduit à l'analyse de la compacité de la résolvante de l'opérateur de Kramers-Fokker-Planck.

Dans cette direction, cette thèse porte sur la théorie spectrale des opérateurs de Kramers-Fokker-Planck dégénérés et présente des conditions suffisantes pour la compacité de la résolvante.

Cette thèse consiste principalement en trois parties bien articulées, qui ont fait l'objet de trois articles différents.

Le premier article porte sur l'analyse du cas de potentiel polynomial de degré inférieur ou égal à deux. Dans cette situation, l'opérateur de Kramers-Fokker-Planck est évidemment quadratique. Ce travail conjoint avec Francis Nier et Joe Viola a abouti à des estimations sous-elliptiques optimales avec un contrôle uniforme des inégalités par rapport aux coefficients du polynôme. Ce contrôle est dû à un calcul exact basé sur l'apparition d'une structure quaternionique.

Dans le deuxième article, nous investissons les résultats précédents et nous l'étendons, avec des corrections logarithmiques, à une classe de polynômes de degré supérieur ou égal à trois dont les modèles asymptotiques à l'infini après changement d'échelle sont de degré inférieur ou égal à 2.

Le troisième article est consacré à l'étude du cas de potentiels homogènes de degré strictement entre 2 et 6. Il étend à son tour l'estimation sous-elliptique établie dans le deuxième article dans une situation homogène sous contrôle des valeurs propres de la matrice hessienne du potentiel.

Mots-clés:

Estimations sous-elliptiques, résolvante compacte, opérateur de Kramers-Fokker-Planck, Laplacien de Witten, semi-groupe, quaternions, transformation de Bargmann.

AMS-2010: 35Q84, 35H20, 35P05, 47A10, 14P10, 20G20

Abstract:

The Kramers-Fokker-Planck equation is the evolution equation for the distribution functions describing the Brownian motion of particles in an external field. This equation was derived and used to describe reaction kinetics.

One of the basic problems is to analyze the large time behavior of solutions to the time-dependent Fokker-Planck equation and prove that these solutions converge exponentially towards the equilibrium as time t goes to $+\infty$.

In order to study the exponentially trend problem, an efficient method is to investigate the spectral gap, which is reduced to analyze the compactness of the resolvent of the Kramers-Fokker-Planck operator.

In this direction, this thesis concerns the spectral theory of Kramers-Fokker-Planck operators with degenerate potentials and provides sufficient conditions for the resolvent compactness.

The work beeing conducted in this thesis includes three main parts well articulated, which have been the subject of three different papers.

The first paper focuses on the analysis of the case of polynomial potential with degree less than three, where obviously the Kramers-Fokker-Planck operator is quadratic. This joint work with Francis Nier and Joe Viola led to optimal subelliptic estimates with a uniform control of the inequalities with respect to the polynomial's coefficients. This control is due to an exact calculus based on a quaternionic structure.

In the second paper, we invest the previous results and we extend it, with logarithm corrections, to some class of polynomials with degree greater than two whose asymptotic models at infinity, after scaling, are of degree less three.

The third paper is devoted to the study of the case of homogeneous potentials of degree strictly between 2 and 6. It extends in its turn the subelliptic estimate showed in the second article in an homogeneous situation under a control of the eigenvalues of the Hessian matrix of the potential.

Keywords:

subelliptic estimates, compact resolvent, Kramers-Fokker-Planck operator, Witten Laplacian, semigroup, quaternions, Bargmann transform.

AMS-2010: 35Q84, 35H20, 35P05, 47A10, 14P10, 20G20

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Chapitre 1

Introduction

1.1 Positionnement du problème

1.1.1 Introduction

Pour un potentiel V de classe C^∞ sur $\mathbb{R}^d$, l'opérateur de Krames-Fokker-Planck est défini sur $\mathbb{R}^{2d}$ par

$$K_V = p\partial_q - \partial_q V(q)\partial_p + \frac{1}{2}(-\Delta_p + p^2) \ .$$

Dans cette thèse, nous nous intéressons à l'étude de certaines propriétés spectrales et des critères de compacité de la résolvante pour des opérateurs de Kramers-Fokker Planck en présence de potentiels dégénérés.

Cette étude spectrale est étroitement liée, comme nous le verrons dans la Partie 1.1.1.1, à l'étude des processus stochastiques de diffusion de la physique statistique.

L'étude des équations de Kramers-Fokker-Planck, version équation aux dérivées partielles des processus stochastiques de Langevin pour le mouvement brownien, ont connu une nouvelle jeunesse ces dernières années dans la communauté mathématique.

Dans cette direction plusieurs progrès ont été faits sur le calcul analytique précis des temps de vie d'états métastables dans des situations assez générales.

Par ailleurs la théorie générale des dynamiques associées a été mieux comprise et généralisée à des cadres géométriques variés par les travaux de Hérau-Nier [HerNi], Bismut-Lebeau [BiLe], Helffer-Nier [HeNi], Eckmann-Hairer [EcHa] et Hérau-Hitrik-Sjöstrand [HHS]...

Parmi ces travaux, Helffer et Nier ont posé la question, avec une réponse partielle, de la relation entre les propriétés spectrales de l'opérateur de Kramers-Fokker-Planck (processus de Langevin en position-vitesse avec diffusion partielle en vitesse) et celles de Laplacien de Witten $\Delta_V^{(0)} = -\Delta_q + |\nabla V(q)|^2 - \Delta V(q)$ (évolution uniquement de la position avec diffusion totale).

Dans [HeNi] le lien entre les deux opérateurs $\Delta_V^{(0)}$ et K_V a été précisément décrit et a abouti à la conjecture suivante :

Conjecture 1.1.1. *L'opérateur K_V est à résolvante compacte si et seulement si $\Delta_V^{(0)}$ est à résolvante compacte.*

La partie nécessaire de cette conjecture a déjà été établie par Helffer et Nier (cf. [HeNi], Théorème 1.1]). L'implication inverse reste encore ouverte jusqu'à présent pour un potentiel général, et elle est en effet valide sous certaines conditions sur le potentiel V (cf. Partie 1.4).

Par l'un des résultats élémentaires sur les opérateurs de Schrödinger, nous voyons que le Laplacien de Witten est à résolvante compacte si

$$|\nabla V(q)|^2 - \Delta V(q) \to +\infty\,, \qquad \text{quand} \quad |q| \to +\infty\,.$$

Plus généralement (voir [Hel][HeNi][BoDaHe] par exemple), l'opérateur $\Delta_V^{(0)}$ est à résolvante compacte s'il existe un certain $t \in]0,2[$ tel que

$$t\,|\nabla V(q)|^2 - \Delta V(q) \to +\infty\,, \qquad \text{quand} \quad |q| \to +\infty\,.$$

Par ailleurs, une étude approfondie a été faite par Helffer-Nier [HeNi] suivant l'approche des groupes de Lie nilpotents de Rothschild-Stein [RoSt] et Helffer-Nourrigat [HeNo], montre que si aucun polynôme non nul d'un certain ensemble canonique contenant V n'admet un minimum local alors $\Delta_V^{(0)}$ est à résolvante compacte (cf Partie 1.3). En particulier, cette étude met en évidence l'influence du signe de V sur la compacité de la résolvante de $\Delta_V^{(0)}$. L'exemple le plus connu dans cet esprit est $V(q_1, q_2) = q_1^2 q_2^2$ en dimension 2. En effet, pour cet exemple l'opérateur $\Delta_{-V}^{(0)}$ est à résolvante compacte alors que $\Delta_{+V}^{(0)}$ ne l'est pas.

La difficulté de l'étude pour l'opérateur de Kramers-Fokker-Planck K_V vient du fait qu'il n'est ni autoadjoint, ni elliptique. De nombreux travaux ont été réalisés dans le cas polynomial où $d^\circ V \leq 2$ (voir [Hor][HiPr][Vio] [Vio1][AlVi]). Néanmoins, dès qu'un potentiel général est considéré, différents types de conditions suffisantes sur $V(q)$ ont été examinées par Hérau-Nier [HerNi], Helffer-Nier [HeNi], Villani [Vil] et Wei-Xi Li [Li]. Ces premiers résultats ne considèrent que des variantes de la situation elliptique à l'infini (pour un potentiel non dégénéré), qui ne distinguent pas le signe $\pm V(q)$. Une amélioration significative de ces travaux a récemment été apportée par Wei-Xi Li [Li2] basée sur certaines méthodes de multiplicateurs. Dans [Li2], Wei-Xi Li a montré que pour des potentiels similaires à $V(q_1, q_2) = q_1^2 q_2^2$, les résultats pour $K_{\pm V}$ étaient les mêmes que pour $\Delta_{\pm V}^{(0)}$ (voir Partie 1.4).

Dans cette voie, nous visons à développer avec cette thèse une étude systématique des opérateurs de Kramers-Fokker-Planck. Notre objectif est en particulier de donner des conditions suffisantes sur V pour que K_V soit à résolvante compacte et que ces conditions soient bien satisfaites. Une telle analyse approfondie une grande ressemblance avec les critères de compacité déjà établis pour Laplacien de Witten $\Delta_V^{(0)}$ et conforte ainsi l'idée que la conjecture 1.1.1 soit vraie (c.f Parties 2, 3, 4).

Ce problème présente ainsi de nombreux aspects du point de vue de l'analyse mathématique : ellipticité dégénérée des équations aux dérivées partielles (hypoellipticité, sous-ellipticité) ; groupe et algèbre de Lie (condition de Hörmander pour l'hypoellipticité), algèbre de Lie nilpotente (approche de Rotschild-Stein [RoSt] et hypoellipticité maximale de Helffer-Nourrigat [HeNo][HeNi][Nie]) ; extension au cadre de la géométrie riemannienne via les Laplaciens de Witten et le Laplacien hypoelliptique de Bismut ; analyse spectrale semiclassique puisque la limite de basse température (utilisée en chimie, physique ou autres sciences faisant appel au mouvement brownien) n'est rien d'autre qu'une limite semiclassique.

1.1.1.1 Lien avec l'approche probabiliste

1.1.1.1.1 Rappel sur les équations différentielles stochastiques (SDE)

Soient

$$b : \mathbb{R}^n \times \mathbb{R}_+ \longrightarrow \mathbb{R}^n \quad \text{et} \quad B : \mathbb{R}^n \times \mathbb{R}_+ \longrightarrow M_n(\mathbb{R})$$

deux fonctions où $M_n(\mathbb{R})$ désigne l'espace des matrices carrées de taille n à coefficients réels.

Considérons l'équation différentielle stochastique (SDE)

$$dX = b(X,t)\,dt + B(X,t)\,dW \ , \tag{1.1.1}$$

où $X(\cdot) \in \mathbb{R}^n$ est un processus stochastique et $dW(\cdot)$ est un "bruit blanc" de dimension n avec covariance

$$\mathbb{E}(dW(s)\,dW(t)) = \delta_0(t-s)\ .$$

Rappelons que pour une fonction lisse $(x,t) \longmapsto u(x,t)$ la formule d'Itô s'écrit

$$du(X,t) = \frac{\partial u}{\partial t}\,dt + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\,dX_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}\,dX_i\,dX_j \tag{1.1.2}$$

$$\text{avec} \quad dt^2 = 0\ , \quad dW\,dt = 0\ , \quad dW_k\,dW_l = \delta_{k,l}\,dt\ .$$

Lorsque X résout l'équation (1.1.1), en appliquant la formule d'Itô (1.1.2), on obtient

$$\begin{aligned} du(X,t) &= \frac{\partial u}{\partial t}\,dt + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\,[b_i(X,t)\,dt + (B\,dw)_i] + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial xj}(BB^t)_{ij}\,dt \\ &= \frac{\partial u}{\partial t}\,dt + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\,[b_i(X,t)\,dt + (B\,dw)_i] \\ &\qquad + \frac{1}{2}\sum_{i_1,j_1,i_2,j_2=1}^{n} \frac{\partial^2 u}{\partial x_{i_1} \partial xj_1} B_{i_1,i_2} B_{j_1,j_2}\,dW_{i_2}\,dW_{j_2} \\ &= [\partial_t u + b.\partial_x u + \frac{1}{2}\partial_x.(BB^t)\partial_x u]dt + \partial_x u.(B\,dW)\ . \end{aligned} \tag{1.1.3}$$

En particulier, si v_0 est une observable indépendante de la variable du temps t, la formule (1.1.2) mène à la relation

$$\begin{aligned} v_0(X(t)) = v_0(X(0)) &+ \int_0^t \Big(b.\partial_x v_0 + \frac{1}{2}\partial_x.(BB^t)\partial_x v_0(X(s))\Big)\,ds \\ &+ \int_0^t \partial_x v_0(X(s)).B\,dW\ . \end{aligned} \tag{1.1.4}$$

Le lien entre les équations différentielles stochastiques et les semi-groupes de diffusion est obtenu après calcul de l'espérance conditionnelle,

$$v(x_0,t) = \mathbb{E}(v_0(X,t); X(0) = x_0)\ ,$$

pour une observable

$$v(x_0,t) = v_0(x_0) + \int_0^t \Big(b.\partial_x v + \frac{1}{2}\partial_x.(BB^t)\partial_x v\Big)(x_0,s)ds + 0\ .$$

Notons que dans la dernière formule, on utilise le fait que si une fonction measurable G vérifie

$$\mathbb{E}\Big(\int_0^t G(s)^2 ds\Big) < \infty \quad \text{alors} \quad \mathbb{E}\Big(\int_0^t G(s)dW(s)\Big) = 0\ .$$

Notons

$$L = -b.\partial_x - \frac{1}{2}\partial_x.(BB^t)\partial_x\ . \tag{1.1.5}$$

On obtient alors

$$v(x_0,t) = v_0(x_0) + \int_0^t (-Lv)(s)ds = e^{-tL}v_0 \ ,$$

ou encore

$$\begin{cases} \partial_t v = -Lv = b.\partial_x v + \frac{1}{2}\partial_x.(BB^t)\partial_x v \\ v(t=0) = v_0 \ . \end{cases} \tag{1.1.6}$$

Pour plus amples détails sur les équations différentielles stochastiques voir [Ris][Nel][Eva].

1.1.1.1.2 Application aux processus de diffusion réversibles

Prenons un potentiel $V(q)$ de classe $\mathcal{C}^\infty(\mathbb{R}^d)$ et considérons

$$b(q) = -\partial_q V(q) \ , \qquad\qquad B = \sqrt{2}\,\mathrm{Id} \ . \tag{1.1.7}$$

Dans ce cas le champ de transport est un gradient et le processus de diffusion est dit alors réversible.

Avec les données (1.1.7), la SDE (1.1.1) s'écrit

$$dq = -\partial_q V(q)\,dt + \sqrt{2}\,dW \ ,$$

et le générateur du semi-groupe correspondant (voir (1.1.7)) est donné par

$$L = \partial_q V(q)\partial_q - \Delta_q = (\partial_q V(q) - \partial_q)\partial_q \ .$$

De plus la mesure de probabilité invariante vaut

$$\mu_V = \frac{e^{-V(q)}}{\int_{\mathbb{R}^n} e^{-V(q)}\,dq}\,dq \qquad \text{(à condition que } e^{-V} \text{ soit dans } L^1(\mathbb{R}^d,dq)).$$

Supposons alors que $e^{-V} \in L^1(\mathbb{R}^d,dq)$ ou encore $e^{-\frac{V}{2}} \in L^2(\mathbb{R}^d,dq)$ et écrivons

$$\begin{aligned} e^{-\frac{V}{2}}Le^{\frac{V}{2}} &= e^{-\frac{V}{2}}(\partial_q V(q) - \partial_q)\cdot(e^{\frac{V}{2}}(\partial_q + \frac{\partial_q V(q)}{2})) \\ &= -(\partial_q - \frac{1}{2}\partial_q V(q))\cdot(\partial_q + \frac{1}{2}\partial_q V(q)) \\ &= -\Delta_q + \frac{1}{4}|\partial_q V(q)|^2 - \frac{1}{2}\Delta_q V(q) \ . \end{aligned}$$

Définition 1.1.2. *Étant donné un potentiel $V \in \mathcal{C}^\infty(\mathbb{R}^d)$, le Laplacien de Witten associé à $\frac{V}{2}$ est défini par*

$$\begin{aligned} &\Delta^{(0)}_{V/2} : \mathcal{C}^\infty_0(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d,dq) \ , \\ &\Delta^{(0)}_{V/2} = -\Delta_q + \frac{1}{4}|\partial_q V(q)|^2 - \frac{1}{2}\Delta_q V(q) \ . \end{aligned}$$

Le Laplacien de Witten a été initialement introduit par E. Witten [Wit] sur une variété compacte. Étant donnée une fonction de Morse V (une fonction $\mathcal{C}^\infty$ n'ayant qu'un nombre fini de points critiques, tous non dégénérés), le Laplacien de Witten est défini sur les formes par

$$\Delta_V^{(\cdot)} = d_V \circ d_V^* + d_V^* \circ d_V \ , \qquad \text{où } \ d_V = e^{-V} \circ d \circ e^V \ .$$

Dans ce papier, on ne considère que les Laplaciens de Witten sur les 0-formes dans l'espace $\mathbb{R}^d$, et dans ce cas, il est donné par la Définition 1.1.2.

1.1.1.1.3 Application aux processus de diffusion non réversibles

Considérons maintenant le cas où $\mathbb{R}^n = \mathbb{R}^d_q \times \mathbb{R}^d_p$, $X = (q,p)$, et prenons

$$b(q,p) = \begin{pmatrix} p \\ -\partial_q V - p \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2}\,\mathrm{Id}_{\mathbb{R}^d} \end{pmatrix} \ .$$

La SDE (1.1.1) se lit alors

$$\begin{cases} dq = p\,dt \\ dp = -\partial_q V(q)\,dt - p\,dt + \sqrt{2}\,dW \ . \end{cases}$$

Le générateur du semi-groupe correspondant (cf. (1.1.7)) vaut

$$K_1 = -p\partial_q + \partial_q V(q)\partial_p + (-\partial_p + p)\partial_p$$

et son adjoint formel est

$$K_1' = p\partial_q - \partial_q V(q)\partial_p - (\partial_p + p)\partial_p \ ,$$

La mesure de probabilité invariante est donnée par

$$M(q,p) = \frac{e^{-(\frac{p^2}{2}+V(q))}}{\int_{\mathbb{R}^{2d}} e^{-(\frac{p^2}{2}+V(q))}}\,dq\,dp$$

à condition que $M(q,p) \in L^1(\mathbb{R}^{2d}, dq\,dp)$, ou de façon équivalente, que $e^{-V} \in L^1(\mathbb{R}^d, dq)$.

Supposons donc que $e^{-V} \in L^1(\mathbb{R}^d, dq)$ et écrivons

$$e^{(\frac{p^2+V(q)}{4})} K_1' e^{-(\frac{p^2+V(q)}{4})} = p\partial_q - \partial_q V(q)\partial_p + (-\partial_p + \frac{1}{2}p)(\partial_p + \frac{1}{2}p) \ .$$

Définition 1.1.3. *Étant donné un potentiel $V \in \mathcal{C}^\infty(\mathbb{R}^d)$, l'opérateur de Kramers-Fokker-Planck est défini par*

$$K_V : \mathcal{C}_0^\infty(\mathbb{R}^{2d}) \longrightarrow L^2(\mathbb{R}^{2d}, dq\, dp) \ ,$$
$$K_V = p\partial_q - \partial_q V(q)\partial_p + (-\partial_p + \frac{1}{2}p)(\partial_p + \frac{1}{2}p) \ .$$

Remarques 1.1.4. *La partie réelle de $K_{\pm V}$,*

$$\frac{1}{2}(K_{+V} + K_{-V}) = \Big(- \partial_p + \frac{1}{2}p\Big)\Big(\partial_p + \frac{1}{2}p\Big) = -\Delta_p + \frac{1}{4}|p|^2 - \frac{d}{2} \ ,$$

est le hamiltonien d'un oscillateur harmonique en p.

La partie imaginaire de K_V,

$$\tfrac{1}{2}(K_{+V} - K_{-V}) = p\partial_q - \partial_q V(q)\partial_p$$

est le champ de vecteurs hamiltonien associé à l'énergie classique $E(q,p) = \frac{p^2}{2} + V(q)$.

1.2 Propriétés spectrales et relations entre $\Delta_V^{(0)}$ et K_V

1.2.1 Propriétés spectrales de Laplacien de Witten

Commençons tout d'abord par rappeler certaines définitions élémentaires .

Définition 1.2.1. *On appelle "symbole principal" de l'opérateur différentiel $\mathfrak{D} = \sum_{|\alpha|=0}^{m} a_\alpha(x) \mathrm{D}^\alpha$ d'ordre m la fonction*

$$\sigma_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\ \xi^\alpha$$

$$\textit{(où les } a_\alpha(x) \in \mathcal{C}^\infty(\mathbb{R}^d) \textit{ et } \mathrm{D} = (D_{x_1}, ..., \mathrm{D}_{x_n}),\ \mathrm{D}_{x_j} = -i\partial_{x_j}) \ .$$

Définition 1.2.2. *L'opérateur différentiel $\mathfrak{D}$ est dit "elliptique" au point $x \in \Omega$ si et seulement si*

$$\forall \xi \in \mathbb{R}^d \setminus \{0\} \ , \quad \sigma_m(x,\xi) \neq \ 0 \ .$$

L'opérateur $\mathfrak{D}$ est dit "elliptique" dans Ω s'il est elliptique pour tout point $x \in \Omega$.

Remarque 1.2.3. *Le Laplacien de Witten* $\Delta_V^{(0)} = -\Delta_q + |\partial_q V(q)|^2 - \Delta V(q)$ *est un opérateur elliptique dans* $\mathbb{R}^d$*. Clairement, son symbole principal vérifie*

$$\forall \xi \in \mathbb{R}^d \setminus \{0\} \ , \quad \sigma_2(q,\xi) = \|\xi\|^2 \neq \ 0 \ .$$

Il est important de noter que l'ellipticité de $\Delta_V^{(0)}$ en tant qu'opérateur différentiel donne de la régularité et des informations locales, mais ne dit rien sur des estimations globales tant qu'on n'a pas fait d'hypothèses sur le comportement de $V(q)$ et de ses dérivées à l'infini.

1.2.1.1 Application du théorème de Simader

Définition 1.2.4. *Un opérateur symétrique (A,D(A)) est dit essentiellement autoadjoint si sa fermeture est autoadjointe.*

On rappele ci-après quelques critères et propriétés de base des opérateurs autoadjoints respectivement essentiellement autoadjoints (on peut par exemple regarder [ReSi] pages 256-257).

Proposition 1.2.5. *Soit T un opérateur symétrique défini sur un espace de Hilbert $\mathcal{H}$. Les assertions suivantes sont équivalentes.*
(i) T est autoadjoint.
(ii) $\mathrm{Ker}(T^* \pm i) = 0$.
(iii) $\mathrm{Ran}(T \pm i)$*est dense dans* $\mathcal{H}$.

Proposition 1.2.6. *Soit T un opérateur symétrique défini sur un espace de Hilbert $\mathcal{H}$. Les assertions suivantes sont équivalentes.*
(i) T est essentiellement autoadjoint.
(ii) T est fermé et $\mathrm{Ker}(T^* \pm i) = 0$.
(iii) $\mathrm{Ran}(T \pm i)$ *est dense dans* $\mathcal{H}$.

Preuve. On applique la proposition 1.2.5 à $\overline{A} = A^*$ et on utilise le fait que A est symétrique alors $\overline{\mathrm{Ran}(A \pm i)} = \mathrm{Ran}(\overline{A} \pm i)$

□

Remarque 1.2.7. *Si de plus T est positif, on a* $\overline{\mathrm{Ran}(T)} \subset \overline{\mathrm{Ran}(T \pm i)}$ $\Big($*soit* $u \in \mathrm{Ker}(T \pm i)$ *alors il existe* $u \in D(T)$ *tel que* $Tu = \pm iu$*. Par suite* $\langle Tu, u\rangle = \langle \pm iu, u\rangle = \pm i\|u\|^2$*. Or T est positif donc $u = 0$ d'où* $u \in \mathrm{Ker}(T)$ $\Big)$.

Théorème 1.2.1 (Théorème de Simader)**.** *Un opérateur de Schrödinger positif* $-\Delta + W(x)$ *avec* $W \in \mathcal{C}^\infty(\mathbb{R}^d)$ *est essentiellement autoadjoint sur* $\mathcal{C}_0^\infty(\mathbb{R}^d)$.

Nous renvoyons par exemple le lecteur à [Hel] (voir Théorème 6.6.2) pour une preuve du théorème 1.2.1.

Le Laplacien de Witten $\Delta_V^{(0)}$, étant positif par construction, est essentiellement autoadjoint sur $\mathcal{C}_0^\infty(\mathbb{R}^d)$. Autrement dit, il admet une unique extension autoadjointe et sa fermeture autoadjointe a pour domaine

$$D(\Delta_V^{(0)}) = \left\{ u \in L^2(\mathbb{R}^d) \ , \ \ \Delta_V^{(0)} u \in L^2(\mathbb{R}^d) \right\} \ .$$

Lemme 1.2.8. *Le noyau de Laplacien de Witten* $\Delta_V^{(0)}$ *avec domaine*

$$D(\Delta_V^{(0)}) = \left\{ u \in L^2(\mathbb{R}^d) \ , \ \ \Delta_V^{(0)} u \in L^2(\mathbb{R}^d) \right\} \ ,$$

est donné par

$$\mathrm{Ker}(\Delta_V^{(0)}) = \left\{ u \in D(\Delta_V^{(0)}) : \ d(e^V u) = 0 \ \ \textit{dans} \ \ \mathcal{D}'(\mathbb{R}^d; \mathbb{R}^d) \right\} = \left\{ \begin{array}{ll} \mathbb{C}e^{-V} & \textit{si} \quad e^{-V} \in L^2(\mathbb{R}^d) \\ 0 & \textit{sinon.} \end{array} \right. .$$

Démonstration. Pour tout $u \in D(\Delta_V^{(0)})$,

$$\langle u, \Delta_V^{(0)} u \rangle = \|d_V u\|^2_{L^2(\mathbb{R}^d)} \quad \text{où} \quad d_V = \partial_q + \partial_q V(q) \ .$$

D'autre part, pour tout $u \in L^2(\mathbb{R}^d)$

$$\begin{aligned} d(e^V u) = \sum_{i=1}^d \partial_{q_i}(e^V u) \, dq_i &= \sum_{i=1}^d e^V (\partial_{q_i} + \partial_{q_i} V(q)) u \, dq_i \\ &= e^V (\partial_q + \partial_q V(q)) u \, dq = e^V (d_V u) \, dq \ . \end{aligned}$$

Il en résulte alors l'équivalence

$$d(e^V u) = 0 \ \ \text{dans} \ \ \mathcal{D}'(\mathbb{R}^d; \mathbb{R}^d) \Leftrightarrow d_V u = 0 \ \ \text{dans} \ \ \mathcal{D}'(\mathbb{R}^d; \mathbb{R}^d) \ .$$

Si u est dans $\mathrm{Ker}(\Delta_V^{(0)})$ alors u est dans le domaine $D(\Delta_V^{(0)})$ tel que $\Delta_V^{(0)} u = 0$. Ainsi on obtient

$$0 = \langle u, \Delta_V^{(0)} u \rangle^2_{L^2} = \|d_V u\|^2_{L^2} \ ,$$

ce qui donne $d_V u = 0$. Réciproquement, si $d_V u = 0$ au sens des distributions pour un certain u dans le domaine de $\Delta_V^{(0)}$, alors comme $\Delta_V^{(0)} = d_V^* d_V$ dans $\mathcal{D}'(\mathbb{R}^d; \mathbb{R}^d)$, on a directement $d_V u = 0$. □

1.2.2 Propriétés spectrales de l'opérateur de KFP

Dans cette sous-section, on donne une version hypoelliptique du théorème 1.2.1.

Définitions 1.2.9. *Soit A un opérateur non borné dans un espace de Hilbert $\mathcal{H}$ avec domaine $D(A)$.*

∗ *On dit que A est accrétif si*

$$\mathrm{Re}\langle u, Au\rangle \geq 0 \ , \qquad \forall\, u \in D(A) \ .$$

∗ *Un opérateur accrétif A est accrétif maximal s'il n'admet aucune extension accrétive B avec domaine strictement plus grand (ou encore s'il est fermé, accrétif et $1 \in \rho(-A)$).*
∗ *Un opérateur accrétif A est essentiellement accrétif maximal si sa fermeture est accrétive maximale.*

Lemme 1.2.10. *Soit (A,D(A)) un opérateur défini sur un espace de Hilbert $\mathcal{H}$ avec domaine dense dans $\mathcal{H}$.*

Si A est accrétif alors il est fermable, sa fermeture $\bar{A}$ est accrétive et pour tout $\lambda \in \mathbb{C}$, $\mathrm{Ran}(\lambda + A)$ est dense dans $\mathrm{Ran}(\lambda + \bar{A})$.

Lemme 1.2.11. *Soit (A,D(A)) un opérateur défini sur un espace de Hilbert $\mathcal{H}$ avec domaine dense dans $\mathcal{H}$.*
1) Si A est fermé et accrétif, alors $I + A$ est injectif et $\mathrm{Ran}(I + A)$ est dense dans $\mathcal{H}$.
2) Si A est accrétif maximal, alors $(0, +\infty) \in \rho(-A)$ et pour tout $\lambda > 0$, $\lambda(\lambda I + A)^{-1}$ est un opérateur de contraction sur $\mathcal{H}$.
3) On suppose que A est accrétif alors on a l'équivalence suivante :

$$\bar{A} \text{ est accrétif maximal} \Leftrightarrow \text{il existe } \lambda > 0 \text{ tel que } \mathrm{Ran}(\lambda I + A) \text{ est dense dans } \mathcal{H}.$$

Pour les preuves des lemmes 1.2.10 et 1.2.11 et pour plus de résultats sur la notion d'accrétivité maximale on peut regarder les livres de Dautray-Lions [DaLi] (Volume 5, Chapitre XVII) et [Dav].

Théorème 1.2.2. *Soit $V(q) \in \mathcal{C}^\infty(\mathbb{R}^d)$, l'opérateur de Kramers-Fokker-Planck défini sur $C_0^\infty(\mathbb{R}^{2d})$ par*

$$K_V := -\Delta_p + \frac{1}{4}|p|^2 - \frac{d}{2} + X_0, \quad \text{où} \quad X_0 = p\partial_q - \nabla V(q)\partial_p$$

est essentiellement accrétif maximal.

Pour prouver ce dernier résultat on montre que le rang de $K_V + (\frac{d}{2}+1)I$ est dense dans $L^2(\mathbb{R}^{2d})$ et on utilise ainsi le lemme 1.2.11 pour conclure. La démonstration du théorème 1.2.2 est faite en détail dans [HeNi] (voir Proposition 5.5).

Remarque 1.2.12. *Comme K_V défini au départ avec domaine $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ est essentiellement accrétif maximal, le domaine de sa fermeture est donné par*

$$D(K_V) = \left\{ u \in L^2(\mathbb{R}^{2d}) \ , \ K_V u \in L^2(\mathbb{R}^{2d}) \right\} \ .$$

1.2.3 De Kramers-Fokker-Planck au Laplacien de Witten

Dans [HeNi] (cf. Proposition 5.19), Helffer-Nier ont établi les relations suivantes entre l'opérateur de Kramers-Fokker-Planck K_V et le Laplacien de Witten $\Delta_V^{(0)}$.

Proposition 1.2.13. *Soit $V \in \mathcal{C}^\infty(\mathbb{R}^d)$.*
i) Si l'opérateur K_V est à résolvante compacte alors $\Delta_V^{(0)}$ est aussi à résolvante compacte.
ii) Si 0 est dans le spectre essentiel de $\Delta_V^{(0)}$ alors 0 est de même dans le spectre essentiel de $K_{\pm V}$.

Démonstration. Supposons que $(1+\Delta_V^{(0)})^{-1}$ n'est pas compact. Alors il existe une suite orthonormée $(u_k)_{k\in\mathbb{N}}$ dans le domaine de $\Delta_V^{(0)}$ tel que

$$\langle u_k, \Delta_V^{(0)} u_k \rangle = \|d_V u_k\|^2_{L^2(\mathbb{R}^d)}$$

est bornée (où $d_V = \partial_q + \partial_q V(q)$).

Considérons, la suite définie par

$$U_k(q,p) = u_k(q)(2\pi)^{-d/4} e^{-p^2/4} \ .$$

Cette suite est orthonormée dans $L^2(\mathbb{R}^{2d})$ et satisfait pour tout $k \in \mathbb{N}$,

$$K_V U_k = (2\pi)^{-d/4} (d_V u_k(q)) p e^{-p^2/4} \quad \text{dans} \quad \mathcal{D}'(\mathbb{R}^{2d}) \ .$$

D'oú $U_k \in D(K_V) = \left\{ u \in L^2(\mathbb{R}^{2d}) \ , \ K_V u \in L^2(\mathbb{R}^{2d}) \right\}$ (voir Remarque 1.2.12). De plus il existe une constante $c > 0$ telle que pour tout $k \in \mathbb{N}$,

$$\|K_V U_k\|_{L^2(\mathbb{R}^{2d})} \le c \|d_V u_k\|_{L^2(\mathbb{R}^d)} \ .$$

Ainsi la suite $(U_k)_{k\in\mathbb{N}}$ est une suite orthonormée tel que $\|K_V U_k\|_{L^2}$ est uniformément bornée. Si K_V avait une résolvante compacte, nous pourrions extraire une sous suite de Cauchy U_k. Cela implique immédiatement que u_k devrait être une suite de Cauchy dans $L^2(\mathbb{R}^d)$. Mais ceci est en contradiction avec le fait que u_k est une suite orthonormée.

Pour ii), rappelons tout d'abord que $\lambda \in \mathbb{C}$ appartient au spectre essentiel d'un opérateur $(A, D(A) \subset L^2(\mathbb{R}^n))$ s'il existe une suite othonormée $(u_k)_{k\in\mathbb{N}}$ dans $L^2(\mathbb{R}^n)$, avec $u_k \in D(A)$, telle que $\lim_{k\to+\infty} \|(A-\lambda)u_k\| = 0$. Cette condition est suffisante en général et c'est une condition nécessaire et suffisante si l'opérateur A est autoadjoint. Supposons que $0 \in \sigma_{ess}(\Delta_V^{(0)})$. Il existe alors une suite orthonormée $(u_k)_{k\in\mathbb{N}}$ dans $L^2(\mathbb{R}^d)$, avec $u_k \in D(\Delta_V^{(0)})$ tel que $\lim_{k\to\infty} \|\Delta_V^{(0)} u_k\|_{L^2(\mathbb{R}^d)} = 0$.

En conséquence, la suite $U_k(q,p) = u_k(q)(2\pi)^{-d/4}e^{-p^2/4}$ est une suite orthonormée dans $L^2(\mathbb{R}^{2d})$, qui vérifie $U_k \in D(K_V)$ et

$$\|K_V U_k\|_{L^2(\mathbb{R}^{2d})} \leq c\|d_V u_k\|_{L^2(\mathbb{R}^d)} \underset{k\to+\infty}{\to} 0 \ .$$

□

1.3 Critères de compacité de Laplacien de Witten (Résultats connus)

Lemme 1.3.1. *(Formule de Localisation IMS)*
Soit $\sum_{j\in\mathbb{N}} \psi_j(q)^2 \equiv 1$ *une partition de l'unité sur* $\mathbb{R}^d$. *On a*

$$-\Delta_q = \sum_{j\in\mathbb{N}} \psi_j(-\Delta_q)\psi_j - |\partial_q \psi_j|^2(q) \ .$$

Démonstration. Pour tout $u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned} -\Delta_q(\psi_j^2 u) &= -\psi_j \Delta_q(\psi_j)u - \psi_j u \Delta_q \psi_j - 2\nabla\psi_j \nabla(\psi_j u) \\ &= -\psi_j \Delta_q(\psi_j)u - \frac{1}{2}u(\Delta_q \psi_j^2 - 2|\nabla\psi_j|^2) - 2|\nabla\psi_j|^2 u - 2\psi_j \nabla\psi_j \nabla u \\ &= -\psi_j \Delta_q(\psi_j)u - \frac{1}{2}u\Delta_q \psi_j^2 - |\nabla\psi_j|^2 u - \nabla\psi_j^2 \nabla u \ . \end{aligned}$$

Or $\sum_{j\in\mathbb{N}} \psi_j(q)^2 \equiv 1$, donc

$$\sum_{j\in\mathbb{N}} (-\Delta_q)(\psi_j^2 u) = \sum_{j\in\mathbb{N}} -\psi_j \Delta_q(\psi_j)u - |\nabla\psi_j|^2 u \ .$$

□

Proposition 1.3.2. *On considère le Laplacien de Witten $\Delta_V^{(0)}$ avec domaine*

$$D(\Delta_V^{(0)}) = \left\{ u \in L^2(\mathbb{R}^d)\ ,\ \Delta_V^{(0)} u \in L^2(\mathbb{R}^d) \right\}\ .$$

S'il existe une application continue $R : \mathbb{R}^d \longrightarrow \mathbb{R}$ telle que pour tout u dans $D(\Delta_V^{(0)})$

$$\|R(q)u\|^2_{L^2(\mathbb{R}^d)} \leq \langle u, \Delta_V^{(0)} u\rangle_{L^2(\mathbb{R}^d)} + \|u\|^2_{L^2(\mathbb{R}^d)}\ , \tag{1.3.1}$$

avec $\lim\limits_{|q|\longrightarrow+\infty} R(q) = +\infty$, alors $\Delta_V^{(0)}$ est à résolvante compacte.

Démonstration. Supposons par contradiction que $(1+\Delta_V^{(0)})^{-1}$ n'est pas compact. Il existe alors une suite $u_n \in D(\Delta_V^{(0)})$ tel que

$$\begin{cases} \|u_n\|_{L^2(\mathbb{R}^d)} = 1 \\ u_n \rightharpoonup 0 \\ \|(\Delta_V^{(0)} + 1)u_n\|_{L^2(\mathbb{R}^d)} \leq c \quad \text{pour n assez grand.} \end{cases}$$

Il s'ensuit que pour tout entier naturel n assez grand,

$$\langle u_n, \Delta_V^{(0)} u_n\rangle \leq \langle u_n, (\Delta_V^{(0)} + 1)u_n\rangle \leq \|u_n\|_{L^2(\mathbb{R}^d)} \|(\Delta_V^{(0)} + 1)u_n\|_{L^2(\mathbb{R}^d)} \leq c\ . \tag{1.3.2}$$

Considérons maintenant une partition de l'unité $\mathcal{C}^\infty$, $\psi_1(q)^2 + \psi_2(q)^2 = 1$, telle que ψ_1 soit à support compact (qui sera fixé à la fin). En particulier $\nabla\psi_1$ et $\nabla\psi_2$ sont $\mathcal{C}^\infty$ à support compact. En utilisant la formule de localisation IMS (voir Lemme 1.3.1) on obtient

$$\langle u_n, \Delta_V^{(0)} u_n\rangle = \langle \psi_1 u_n, \Delta_V^{(0)} \psi_1 u_n\rangle + \langle \psi_2 u_n, \Delta_V^{(0)} \psi_2 u_n\rangle - \sum_{j=1}^{2} \langle u_n, |\nabla\psi_j|^2 u_n\rangle\ . \tag{1.3.3}$$

Comme pour $j \in \{1,2\}$, les fonctions $\nabla\psi_j$ sont $\mathcal{C}^\infty$ à support compact, il existe une constante $c' > 0$ tel que

$$\sum_{j=1}^{2} \langle u_n, |\nabla\psi_j|^2 u_n\rangle \leq c' \|u_n\|^2_{L^2(\mathbb{R}^d)} = c'\ . \tag{1.3.4}$$

Compte-tenu de (1.3.2), (1.3.3) et (1.3.4) on obtient

$$c \geq \langle u_n, \Delta_V^{(0)} u_n\rangle \geq \langle \psi_1 u_n, \Delta_V^{(0)} \psi_1 u_n\rangle + \langle \psi_2 u_n, \Delta_V^{(0)} \psi_2 u_n\rangle - c'\ .$$

On en déduit alors qu'il existe une constante $C > 0$ tel que

$$C \geq \langle \psi_1 u_n, \Delta_V^{(0)} \psi_1 u_n\rangle + \langle \psi_2 u_n, \Delta_V^{(0)} \psi_2 u_n\rangle + \|u_n\|^2_{L^2(\mathbb{R}^d)}\ . \tag{1.3.5}$$

D'autre part en utilisant l'hypothèse (1.3.1),

$$\langle \psi_2 u_n, \Delta_V^{(0)} \psi_2 u_n \rangle + \|\psi_2 u_n\|^2_{L^2(\mathbb{R}^d)} \geq \|R(q)\psi_2 u_n\|^2_{L^2(\mathbb{R}^d)} \,. \tag{1.3.6}$$

Ainsi (1.3.5) et (1.3.6) donnent

$$C \geq \langle \psi_1 u_n, \Delta_V^{(0)} \psi_1 u_n \rangle + \|R(q)\psi_2 u_n\|^2_{L^2(\mathbb{R}^d)} + \|\psi_1 u_n\|^2_{L^2(\mathbb{R}^d)} \,.$$

Soit $\epsilon > 0$, en choisissant ψ_1 et ψ_2 tel que $R(q) \geq \frac{2C}{\epsilon}$ dans le support de ψ_2, on obtient ainsi

$$C \geq \langle \psi_1 u_n, \Delta_V^{(0)} \psi_1 u_n \rangle + \frac{2C}{\epsilon} \|\psi_2 u_n\|^2_{L^2(\mathbb{R}^d)} + \|\psi_1 u_n\|^2_{L^2(\mathbb{R}^d)} \,.$$

Par suite

$$\begin{cases} \|\psi_2 u_n\|^2_{L^2(\mathbb{R}^d)} \leq \frac{\epsilon}{2} \\ \psi_1 u_n \in H^1(\mathbb{R}^d) \end{cases} \quad \text{car} \quad \langle \psi_1 u_n, \Delta_V^{(0)} \psi_1 u_n \rangle = \|(\partial_q + \partial_q V)(\psi_1 u_n)\|^2_{L^2(\mathbb{R}^d)} \leq C \,.$$

Comme $\psi_1 u_n \in H^1(\mathbb{R}^d)$ et ψ_1 est à support compact, on peut extraire de $\psi_1 u_n$ une sous suite u_{n_k} vérifiant $\psi_1 u_{n_k} \underset{n \longrightarrow +\infty}{\overset{L^2}{\longrightarrow}} v$. Or par hypothèse $u_n \rightharpoonup 0$ donc $v = 0$.

Ainsi pour tout $\epsilon > 0$,

$$\|u_{n_k}\|^2_{L^2(\mathbb{R}^d)} = \|\psi_1 u_{n_k}\|^2_{L^2(\mathbb{R}^d)} + \|\psi_2 u_{n_k}\|^2_{L^2(\mathbb{R}^d)} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \,,$$

ce qui est absurde car $\|u_n\|_{L^2(\mathbb{R}^d)} = 1$. □

D'après la proposition 1.3.2, si

$$|\nabla V(q)|^2 - \Delta V(q) \longrightarrow +\infty \quad \text{quand} \quad |q| \longrightarrow +\infty \,,$$

alors $\Delta_V^{(0)}$ est à résolvante compacte. Une combinaison avec un argument de crochet (voir [BoDaHe][Hel][HeNi]), montre que c'est encore vrai sous la condition plus faible qu'il existe existe $t \in [1, 2[$ tel que

$$t|\nabla V(q)|^2 - \Delta V(q) \longrightarrow +\infty \quad \text{quand} \quad |q| \longrightarrow +\infty \,.$$

Un autre résultat connu (voir Corollaire 5.10 dans [HeNi]) dit que si le potentiel $V \in \mathcal{C}^\infty(\mathbb{R}^d)$ satisfait l'hypothèse 1 suivante alors $\Delta_V^{(0)}$ est à résolvante compacte.

Hypothèse 1. *Pour tout* $|\alpha| \geq 1$, *il existe une constante* $C_\alpha \geq 1$ *tel que*

$$|\partial_q^\alpha V(q)| \leq C_\alpha \langle \partial_q V(q) \rangle \,.$$

Il existe des constantes $C \geq 1, M \geq 1$ *tels que*

$$C^{-1} \langle q \rangle^{\frac{1}{M}} \leq \langle \partial_q V(q) \rangle \leq C \langle q \rangle^M \,.$$

Remarque 1.3.3. *L'hypothèse 1 implique que la hessienne* Hess $V(q)$ *est contrôlée par le gradient* $\partial_q V(q)$ *à l'infini.*

1.3.1 Premiers exemples et observations

Lemme 1.3.4. *Étant donnés deux potentiels $V_1, V_2 \in \mathcal{C}^\infty(\mathbb{R}^d)$, on a l'inégalité suivante (dans le sens des formes quadratiques sur $\mathcal{C}_0^\infty(\mathbb{R}^d)$)*

$$\Delta^{(0)}_{V_1+V_2} \geq 2\partial_q V_1(q)\cdot\partial_q V_2(q) + |\partial_q V_2(q)|^2 - \Delta V_2(q)\ .$$

Démonstration. Soit $u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\langle \Delta^{(0)}_{V_1+V_2} u, u\rangle &= \langle(-\Delta_q + |\partial_q(V_1+V_2)|^2 - \Delta(V_1+V_2))u, u\rangle \\ &= \langle(-\Delta_q + |\partial_q V_1|^2 - \Delta V_1)u + (|\partial_q V_2|^2 + 2\partial_q V_1(q)\partial_q V_2(q) - \Delta V_2)u, u\rangle \\ &= \langle \Delta^{(0)}_{V_1} u, u\rangle + \langle(|\partial_q V_2|^2 + 2\partial_q V_1\partial_q V_2 - \Delta V_2)u, u\rangle \\ &\geq \langle(|\partial_q V_2|^2 + 2\partial_q V_1\partial_q V_2 - \Delta V_2)u, u\rangle\ .\end{aligned}$$

□

Exemple 1 : Dans $L^2(\mathbb{R}^2)$, on considère $V = V_1 + V_2$ où

$$V_1(q_1,q_2) = q_1^2 q_2^2\ ,\quad V_2(q_1,q_2) = (q_1^2+q_2^2)^{\frac{1+\delta}{2}} \quad \text{pour} \quad |q|\geq 1\ ,\ 0<\delta<1\ .$$

En passant aux coordonnées polaires,

$$(q_1,q_2) = (r\cos\theta, r\sin\theta)\ ,\ \ r\in\mathbb{R}_+^*,\ \theta\in[-\pi,\pi)\ ,$$

on obtient

$$V_1 = r^4\varphi(\theta)\ ,\quad \varphi(\theta) = \frac{\sin^2(2\theta)}{4}\ ,\quad V_2 = r^{1+\delta} \quad \text{avec}\ \ r\geq 1\ .$$

Par suite

$$\Delta V_2 = (1+\delta)\delta r^{\delta-2} \leq (1+\delta)\delta := c\ .$$

Ainsi, d'après le lemme 1.3.4

$$\Delta^{(0)}_V \geq 8(1+\delta)r^{3+\delta}\varphi(\theta) + (1+\delta)^2 r^{2\delta} - c \geq (1+\delta)^2 r^{2\delta} - c\ .$$

Comme $\lim\limits_{r\longrightarrow+\infty}(1+\delta)^2 r^{2\delta} - c = +\infty$, on conclut d'après la proposition 1.3.2 que $\Delta^{(0)}_V$ est à résolvante compacte.

Remarquons que dans la direction $\theta = 0$, la hessienne de V croît comme r^2 alors que le gradient de V est borné par $r^\delta \leq r$. Donc c'est un exemple d'un potentiel pour lequel $\Delta^{(0)}_V$

est à résolvante compacte alors que l'hypothèse 1 n'est pas satisfaite.
Exemple 2 : On considére $V = -V_1 = -q_1^2 q_2^2$

$$\Delta^{(0)}_{-V_1} = -\Delta_q + |\partial_q V_1(q)|^2 - \Delta V_1(q) \geq -\Delta V_1(q) = 2(q_2^2 + q_1^2) \underset{|q|\to+\infty}{\to} +\infty \quad .$$

D'oú $\Delta^{(0)}_V$ est à résolvante compacte.
Exemple 3 : Dans le cas du potentiel $V = +V_1 = q_1^2 q_2^2$, on montre que $0 \in \sigma_{ess}(\Delta^{(0)}_V)$, ce qui implique que la résolvante de $\Delta^{(0)}_V$ n'est pas compacte. En effet, pour $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ on a

$$\begin{aligned}\langle \chi e^{-V}, \Delta^{(0)}_V(\chi e^{-V})\rangle &= \langle \chi e^{-V}, -(\partial_q - \partial_q V)(\partial_q + \partial_q V)\chi e^{-V}\rangle \\ &= \|(\partial_q + \partial_q V)(\chi e^{-V})\|^2_{L^2(\mathbb{R}^d)} \\ &= \|(\partial_q \chi)e^{-V}\|^2_{L^2(\mathbb{R}^d)} \quad .\end{aligned}$$

On prend $\chi_n(q) = \psi(2^{-n}r)\psi_0(\theta), n \in \mathbb{N}$, où ψ_0 est à support compact autour de $\theta = 0$ avec $\psi_0 \equiv 1$ dans un petit voisinage et supp $\psi \subset (\frac{3}{4}, \frac{3}{2})$ avec $\psi \equiv 1$ autour de 1.

On pose $u_n = \chi_n e^{-V}$. Cette suite est orthogonale et vérifie

$$\begin{aligned}\frac{\langle u_n, \Delta^{(0)}_V u_n\rangle}{\|u_n\|^2_{L^2(\mathbb{R}^d)}} &= \frac{\|(\partial_x \chi_n)e^{-V}\|^2_{L^2(\mathbb{R}^d)}}{\|\chi_n e^{-V}\|^2_{L^2(\mathbb{R}^d)}} \\ &= \frac{\int_{\frac{3}{4}}^{\frac{3}{2}}\int_{-\pi}^{\pi}\left[|\psi'|^2|\psi_0|^2 + r^{-2}|\psi|^2|\psi_0'|^2\right]e^{-2^{8n}\varphi(\theta)}r\,drd\theta}{\int_{\frac{3}{4}}^{\frac{3}{2}}\int_{-\pi}^{\pi}\left[|\psi|^2|\psi_0|^2\right]e^{-2^{8n}\varphi(\theta)}2^{2n}r\,drd\theta} \underset{n\longrightarrow+\infty}{\longrightarrow} 0 \quad .\end{aligned}$$

En vue des exemples 2 et 3, on conclut que la compacité de la résolvante de $\Delta^{(0)}_{\tau_0 V}$ dépend du signe de τ_0 (où τ_0 est un paramètre de friction).

1.3.2 Résultat de Helffer-Nier

Dans cette partie, nous rappelons les critères d'hypoellipticité maximale développés par Helffer et Nourrigat [HeNo] et son application pour Laplacien de Witten (voir [HeNi][Nie][Nou]).

1.3.2.1 Notion de micro-hypoellipticité maximale

Dans $\mathbb{R}^{d+1}_{q,t}$, on considère l'algèbre de Lie engendrée par les champs de vecteurs

$$X_j = \partial_{q_j}\ , \quad Y_j = \partial_{q_j}V(q)\partial_t\ , \qquad j = 1, \cdots, d\ . \tag{1.3.7}$$

Pour tout $\tau_0 \in \mathbb{R}^*$, on considère la représentation unitaire Π_{V,τ_0} de l'algèbre de Lie dans $L^2(\mathbb{R}^{d+1})$ donnée par

$$\Pi_{V,\tau_0}(X_j) = \partial_{q_j} \ , \quad \Pi_{V,\tau_0}(Y_j) = \partial_{q_j} V(q) i\tau_0 \ .$$

Notons

$$L_j = X_j - iY_j = \partial_{q_j} - i\partial_{q_j} V(q)\partial_t \ , \qquad j = 1, \cdots, d \ . \tag{1.3.8}$$

Ainsi on peut réécrire le Laplacien de Witten sous la forme

$$\begin{aligned} \Delta^{(0)}_{\tau_0 V} &= \sum_{j=1}^{d} \Pi_{V,\tau_0}(L_j)^* \Pi_{V,\tau_0}(L_j) \\ &= \sum_{j=1}^{d} \Pi_{V,\tau_0}(-X_j^2 - Y_j^2 + i[X_j, Y_j]) \\ &= \Pi_{V,\tau_0}(L^* L) \ . \end{aligned}$$

Définition 1.3.5. *Le système $L_1, \cdots, L_d$ défini sur $\mathbb{R}^{d+1}_{q,t}$ dans (3.2) est dit micro-hypoelliptique maximal autour de q_0 dans la direction $\tau > 0$, s'il existe un voisinage de q_0 ω_{q_0} et une constante $c > 0$ tels que l'estimation*

$$\sum_{j=1}^{d} \Big(\|\Pi_{V,\tau_0}(X_j)u\|^2_{L^2(\mathbb{R}^d)} + \|\Pi_{V,\tau_0}(Y_j)u\|^2_{L^2(\mathbb{R}^d)} \Big) \leq c \Big[\sum_{j=1}^{d} \|\Pi_{V,\tau_0}(L_j)u\|^2_{L^2(\mathbb{R}^d)} + \|u\|^2_{L^2(\mathbb{R}^d)} \Big]$$

est vraie pour tout $\tau_0 > 0$ et pour tout $u \in \mathcal{C}_0^\infty(\omega_{q_0})$.

Notation 1.3.6. *Pour $r \in \mathbb{N}$ on note E_r l'ensemble des polynômes de degré inférieur ou égal à r :*

$$E_r = \{ P \in \mathbb{R}[q_1, q_2, \cdots, q_d] \ , \quad \deg P \leq r \} \ .$$

Définition 1.3.7. *Pour un polynôme $P \in E_r$, on définit la fonction $R_P^{\geq 1} : \mathbb{R}^d \to \mathbb{R}$ par*

$$R_P^{\geq 1}(q) = \sum_{1 \leq |\alpha| \leq r} |\partial_q^\alpha P(q)|^{\frac{1}{|\alpha|}} \ . \tag{1.3.9}$$

Remarque 1.3.8. *La notion de micro-hypoellipticité maximale, nous conduit à considérer les inégalités globales*

$$\begin{aligned} &\sum_{j=1}^{d} \Big(\|\Pi_{V,\tau_0}(X_j)u\|^2_{L^2(\mathbb{R}^d)} + \|\Pi_{V,\tau_0}(Y_j)u\|^2_{L^2(\mathbb{R}^d)} \Big) \\ &\qquad \leq C_\delta \Big(\sum_{j=1}^{d} \|\Pi_{V,\tau_0}(L_j)u\|^2_{L^2(\mathbb{R}^d)} + \delta R_V^{\geq 1}(0)^2 \|u\|^2_{L^2(\mathbb{R}^d)} \Big) \end{aligned} \tag{1.3.10}$$

pour $\tau_0 > 0$ et $\delta \in \{0, 1\}$.

Définition 1.3.9. *L'estimation (1.3.10) est dite estimation maximale (avec reste quand $\delta = 1$).*

Par un changement de variable $u(\frac{q-q_0}{\lambda})$, on voit que l'estimation maximale (1.3.10) pour un potentiel $V \in E_r$ implique la même estimation pour un potentiel $q \to V(q_0 + \lambda q)$. De plus, si on note

$$C_\delta(V,\tau_0) = \sup_{\substack{u\in\mathcal{C}_0^\infty(\mathbb{R}^d)\\ \|u\|=1}} \frac{\sum_{j=1}^{d}\Big(\|\Pi_{V,\tau_0}(X_j)u\|^2_{L^2(\mathbb{R}^d)} + \|\Pi_{V,\tau_0}(Y_j)u\|^2_{L^2(\mathbb{R}^d)}\Big)}{\Big(\sum_{j=1}^{d}\|\Pi_{V,\tau_0}(L_j)u\|^2_{L^2(\mathbb{R}^d)} + \delta R_V^{\geq 1}(0)^2\|u\|^2_{L^2(\mathbb{R}^d)}\Big)}\ ,$$

l'ensemble des potentiels V pour lesquels $C_\delta(V,\tau_0) \leq C_\delta$ est fermé. Cette remarque a conduit à introduire la notion d'un "ensemble canonique".

Définition 1.3.10. *Un sous ensemble $\mathcal{L}$ de E_r est un "ensemble canonique" s'il est invariant par des opérations de changement d'origine et d'échelle c'est à dire s'il vérifie les trois propriétés suivantes :*

1) Si $P \in \mathcal{L}$ et $y \in \mathbb{R}^d$, alors le polynôme définie par

$$Q(q) = P(q+y) - P(y)\ , \quad \forall q \in \mathbb{R}^d\ ,$$

est aussi dans $\mathcal{L}$.

2) Si $P \in \mathcal{L}$ et $\lambda > 0$ alors $Q(q) = P(\lambda q) \in \mathcal{L}$.

3) $\mathcal{L}$ est un sous ensemble fermé de E_r.

Notation 1.3.11. *Soit V un potentiel dans E_r, on note $\mathcal{L}_V$ le plus petit ensemble canonique qui contient V.*

1.3.2.2 Estimation maximale et compacité de $\Delta_V^{(0)}$

Proposition 1.3.12. *Si l'estimation maximale (1.3.10) est vérifiée pour un certain τ_0 fixé alors il existe une constante $c > 0$ tel que*

$$\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^d)\ , \quad \|R_V^{\geq 1}u\|^2_{L^2(\mathbb{R}^d)} \leq c\Big(\langle u, \Delta_V^{(0)}u\rangle + \|u\|^2_{L^2(\mathbb{R}^d)}\Big)$$

où la fonction $R_V^{\geq 1}$ est définie dans (1.3.9).

Si de plus $\lim_{|q|\to+\infty} R_V^{\geq 1}(q) = +\infty$ alors le Laplacien de Witten est à résolvante compacte.

Théorème 1.3.1 (Théorème de Helffer-Nier)**.** *Pour un potentiel $V \in E_r$, si on suppose que :*

1. $\lim_{|q|\to+\infty} R_V^{\geq 1}(q) = +\infty$*,*

2. aucun polynôme non nul de l'ensemble canonique $\mathcal{L}_V \cap E_{r-1}$ n'admet un minimum local

(où $\mathcal{L}_V$ est le plut petit ensemble canonique, c'est à le plus petit sous ensemble fermé de E_r (muni de sa topologie naturelle) contenant V et invariant par des opérations de changement d'origine et d'échelle) ;

alors le Laplacien de Witten $\Delta_V^{(0)}$ a une résolvante compacte. De plus on a l'estimation avec reste $\Big((1.3.10)$ avec $\delta = 1)\Big)$.

La preuve du théorème 1.3.1 est basée sur une double récurrence assez fine sur la dimension et sur le degré du potentiel V (cf. [Nie]).

Pour appliquer le théorème 1.3.1, nous devons déterminer l'ensemble $\mathcal{L}_V \cap E_{r-1}$. Par conséquent, il faut déterminer les polynômes P d'ordre $r-1$ apparaissant comme limites :

$$P = \lim_{n\longrightarrow\infty} V(q_n + \lambda_n\, \cdot) - V(q_n)\ ,$$

pour une suite (λ_n, q_n) avec $\lambda_n \longrightarrow 0$. Les coefficients de ces polynômes doivent satisfaire

$$\lim_{n\longrightarrow+\infty} \lambda_n^{|\alpha|}(\partial_q^\alpha V)(q_n) = (\partial_q^\alpha P)(0)\ .$$

En particulier, pour un polynôme V homogène de degré r, ces polynômes sont donnés par

$$P = \lim_{n\longrightarrow+\infty}(\lambda_n^r V(\cdot + q_n) - \lambda_n^r V(q_n))\ ,$$

pour une suite (λ_n, q_n) avec $\lambda_n \longrightarrow 0$. De plus les coefficients de ces polynômes doivent satisfaire

$$\lim_{n\longrightarrow+\infty} \lambda_n^{r}(\partial_q^\alpha V)(q_n) = (\partial_q^\alpha P)(0)\ .$$

Notons que dans ce cas on a pour tout $n \in \mathbb{N}$,

$$\begin{aligned} V(q_n + \lambda_n q) - V(q_n) &= V(\lambda_n(\frac{q_n}{\lambda_n} + q)) - V(\lambda_n(\frac{q_n}{\lambda_n})) \\ &= \lambda_n^r V(\cdot + \frac{q_n}{\lambda_n}) - \lambda_n^r V(\frac{q_n}{\lambda_n}) = \lambda_n^r V(\cdot + q_n') - \lambda_n^r V(q_n')\ . \end{aligned}$$

Pour des exemples d'application voir pages 109--112 dans [HeNi].

1.4 Conditions suffisantes pour la compacité de $(1 + K_V)^{-1}$ (Résultats connus)

Proposition 1.4.1. *On considère l'opérateur de Kramers-Fokker-Planck*

$$K_V = p\partial_q - \partial_q V(q)\partial_p + \Big(- \Delta_p + \frac{1}{4}|p|^2 - \frac{d}{2}\Big) .$$

S'il existe une application continue $R : \mathbb{R}^d \longrightarrow \mathbb{R}$ tel que

$$\|R(q)u\|^2_{L^2(\mathbb{R}^{2d})} \leq \|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|u\|^2_{L^2(\mathbb{R}^{2d})} , \quad \forall u \in D(K_V)$$

avec $\lim\limits_{|q|\longrightarrow+\infty} R(q) = +\infty$*, alors K_V est à résolvante compacte.*

Démonstration. On suppose que $(K_V + 1)^{-1}$ n'est pas compact. Il existe alors une suite $u_n \in D(K_V)$ tel que

$$\begin{cases} \|u_n\|_{L^2(\mathbb{R}^{2d})} = 1 \\ u_n \rightharpoonup 0 \\ \|(K_V + 1)u_n\|_{L^2(\mathbb{R}^{2d})} \leq c \quad \text{pour n assez grand.} \end{cases}$$

Pour tout $u \in D(K_V)$, on a

$$\mathrm{Re}\langle u, (K_V + \frac{d}{2})u\rangle = \frac{1}{2}\Big(\langle u, -\Delta_p u\rangle + \langle u, \frac{p^2}{2}u\rangle\Big) ,$$

$$\begin{aligned} \mathrm{Re}\langle u, (K_V + \frac{d}{2})u\rangle &\leq \|u\|_{L^2(\mathbb{R}^{2d})}\|K_V u\|_{L^2(\mathbb{R}^{2d})} + \frac{d}{2}\|u\|^2_{L^2(\mathbb{R}^{2d})} \\ &\leq \frac{1}{2}\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + (\frac{1}{2} + \frac{d}{2})\|u\|^2_{L^2(\mathbb{R}^{2d})} . \end{aligned}$$

Alors la suite u_n vérifie

$$\begin{aligned} \langle u_n, -\Delta_p u_n\rangle + \langle u_n, \frac{p^2}{2}u_n\rangle &\leq 2d\Big(\|K_V u_n\|^2_{L^2(\mathbb{R}^{2d})} + \|u_n\|^2_{L^2(\mathbb{R}^{2d})}\Big) \\ &\leq 2d\Big(\|(K_V + 1)u_n\|^2_{L^2(\mathbb{R}^{2d})} + \|u_n\|^2_{L^2(\mathbb{R}^{2d})}\Big) \\ &\leq 2d(c^2 + 1) , \end{aligned}$$

où la deuxième inégalité est bien vérifiée car K_V est accrétif. On en déduit alors que $\partial_p u_n,\ p u_n \in L^2(\mathbb{R}^{2d})$.

D'autre part, en utilisant le fait que pour toute fonction $\psi \in \mathcal{C}^\infty(\mathbb{R}^d)$ on a

$$K_V\psi(q) - \psi K_V = p\partial_q\psi(q)$$

et en considérant une partition de l'unité $\mathcal{C}^\infty$, $\psi_1(q)^2 + \psi_2(q)^2 = 1$, avec ψ_1 à support compact (précisé plus loin), on obtient pour tout $u \in D(K_V)$,

$$\begin{aligned} \|K_V u\|^2_{L^2(\mathbb{R}^{2d})} &= \langle u, K_V^* K_V u\rangle \\ &= \langle u, K_V^*(\psi_1(q)^2 + \psi_2(q)^2) K_V u\rangle \\ &= \sum_{j=1}^{2} \Big(\|K_V(\psi_j u)\|^2_{L^2(\mathbb{R}^{2d})} - \langle u, (p\partial_q \psi_j)^2 u\rangle \Big) \ . \end{aligned}$$

Or il existe une constante $C > 0$ tel que pour tout $u \in D(K_V)$,

$$\langle u, (p\partial_q \psi_j)^2 u\rangle \leq C\|pu\|^2_{L^2(\mathbb{R}^{2d})} \leq C\Big(\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|u\|^2_{L^2(\mathbb{R}^{2d})} \Big) \ .$$

Ainsi il existe une constante $C > 0$ tel que

$$C\Big(\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|u\|^2_{L^2(\mathbb{R}^{2d})} \Big) \geq \sum_{j=1}^{2} \|K_V(\psi_j u)\|^2_{L^2(\mathbb{R}^{2d})} + \|\psi_1 u\|^2_{L^2(\mathbb{R}^{2d})} + \|\psi_2 u\|^2_{L^2(\mathbb{R}^{2d})} \ .$$

On utilise maintenant l'hypothèse

$$\|K_V(\psi_2 u)\|^2_{L^2(\mathbb{R}^{2d})} + \|\psi_2 u\|^2_{L^2(\mathbb{R}^{2d})} \geq \|R(q)\psi_2 u\|^2_{L^2(\mathbb{R}^{2d})} \ ,$$

avec l'estimation précédente et on en déduit que

$$C\Big(\|K_V u_n\|^2_{L^2(\mathbb{R}^{2d})} + \|u_n\|^2_{L^2(\mathbb{R}^{2d})} \Big) \geq \|K_V(\psi_1 u_n)\|^2_{L^2(\mathbb{R}^{2d})} + \|\psi_1 u_n\|^2_{L^2(\mathbb{R}^{2d})} + \|R(q)\psi_2 u_n\|^2_{L^2(\mathbb{R}^{2d})} \ . \tag{1.4.1}$$

Or par hypothèse, il existe une constante $c_0 > 0$ tel que

$$\|K_V u_n\|^2_{L^2(\mathbb{R}^{2d})} + \|u_n\|^2_{L^2(\mathbb{R}^{2d})} \leq c_0 \ . \tag{1.4.2}$$

En combinant (1.4.1) et (1.4.2),

$$Cc_0 \geq \|K_V(\psi_1 u_n)\|^2_{L^2(\mathbb{R}^{2d})} + \|\psi_1 u_n\|^2_{L^2(\mathbb{R}^{2d})} + \|R(q)\psi_2 u_n\|^2_{L^2(\mathbb{R}^{2d})} \ . \tag{1.4.3}$$

La norme $\|K_V \psi_1 u_n\|_{L^2(\mathbb{R}^{2d})}$ est bornée (d'après l'inégalité (1.4.3)). Or K_V est localement un opérateur hypoelliptique de type 2 suivant la terminologie de Hörmander, suivant le modèle $p\partial_q - \Delta_p + p^2$ avec $[\partial_p, p\partial_q] = \partial_q$ d'homogénéité 3, tandis que $\mathrm{supp}(\psi_1 u_n)$ est compact en q. On en déduit que $\psi_1 u_n$ est borné dans $H^{2/3}(\mathbb{R}^d_q)$.

Soit $\epsilon > 0$, on choisit ψ_1 et ψ_2 tel que $R(q) \geq \frac{2c_0 C}{\epsilon}$ dans le support de ψ_2. Ce choix implique l'inégalité suivante

$$Cc_0 \geq \|K_V(\psi_1 u_n)\|^2_{L^2(\mathbb{R}^{2d})} + \|\psi_1 u_n\|^2_{L^2(\mathbb{R}^{2d})} + \frac{2Cc_0}{\epsilon}\|\psi_2 u_n\|^2_{L^2(\mathbb{R}^{2d})} \ .$$

Ainsi $\|\psi_2 u_n\|_{L^2(\mathbb{R}^{2d})} \le \epsilon$.

Comme $\psi_1 u_n \in H^{2/3}(\mathbb{R}^d_q)$ et $\partial_p u_n, \quad p u_n$ sont bornés dans L^2, on peut extraire de $\psi_1 u_n$ une sous suite qui vérifie $\psi_1 u_{n_k} \underset{n\longrightarrow+\infty}{\overset{L^2}{\longrightarrow}} v$, or on a par hypothèse $u_n \rightharpoonup 0$ donc $v=0$.

On obtient par suite

$$\|u_{n_k}\|^2_{L^2(\mathbb{R}^{2d})} = \|\psi_1 u_{n_k}\|^2_{L^2(\mathbb{R}^{2d})} + \|\psi_2 u_{n_k}\|^2_{L^2(\mathbb{R}^{2d})} \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\ ,$$

ce qui est absurde car $\|u_n\|_{L^2(\mathbb{R}^{2d})} = 1$. □

1.4.1 Premiers Résultats

L'analyse de la compacité de la résolvante se fait à l'aide d'estimations sous-elliptiques. On peut envisager au moins deux types de méthodes : 1) à la Kohn telle quelle a été adaptée par Hérau-Nier [HerNi], Helffer-Nier [HeNi], Wei-Xi Li [Li][Li2] ou encore dans les méthodes dite hypocoercives de Villani [Vil], 2) à la Helffer-Nourrigat, comme par exemple pour traiter le Laplacien de Witten, qui met en jeu une structure de groupe de Lie nilpotent (voir[HeNo][RoSt]). La première ne donne pas les exposants optimaux contrairement à la deuxième, qui fournit de ce fait des pistes pour chercher des contre exemples. Sur des problèmes liés, on notera également le travail de Helffer-Mohamed [HeMo].

La méthode de Kohn a été ensuite utilisée par Hérau-Nier [HerNi] pour étudier le cas des potentiels $V(q)$ qui se comportent comme des fonctions homogènes à l'infini. Plus précisément, si $V(q)$ est une fonction $\mathcal{C}^\infty$ qui vérifie

Hypothèse 2. *Il existe $n \ge 1$ et pour tout $\alpha \in \mathbb{N}^d$ il existe $C_\alpha > 0$ tel que*

$$\forall q \in \mathbb{R}^d, \quad |\partial_q^\alpha V(q)| \le C_\alpha \Big(1 + \langle q \rangle^{2n - \min\{|\alpha|,2\}}\Big)\ .$$

Il existe deux constantes $C_0 = C_0(V) > 0$ et $C_1 = C_1(V) > 0$ tel que

$$\forall q \in \mathbb{R}^d, \quad \pm V(q) \ge C_0^{-1}\langle q\rangle^{2n} - C_0 \quad et \quad |\partial_q V(q)| \ge C_1^{-1}\langle q\rangle^{2n-1} - C_1\ ,$$

alors Hérau-Nier [HerNi] ont établi à l'aide d'un calcul pseudo-différentiel l'estimation hypoelliptique isotrope suivante,

$$\|\Lambda^\epsilon u\|^2_{L^2(\mathbb{R}^{2d})} \le C_V\Big(\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|u\|^2_{L^2(\mathbb{R}^{2d})}\Big)\ ,$$

où

$$\epsilon = \min(\frac{1}{4}, \frac{1}{4n-2})\ , \qquad \Lambda = \Big(1 - \Delta_q - \Delta_p + |\partial_q V(q)|^2 - \Delta V(q) + \frac{1}{2}|p|^2\Big)^{\frac{1}{2}}.$$

En développant les résultats et la démarche de Hérau-Nier, Helffer-Nier [HeNi] ont obtenu l'estimation

$$\|\Lambda^{\frac{1}{4}}u\|^2_{L^2(\mathbb{R}^{2d})} \leq C_V\Big(\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|u\|^2_{L^2(\mathbb{R}^{2d})}\Big)\,, \tag{1.4.4}$$

pour des potentiels $V \in \mathcal{C}^\infty$ plus générals vérifiant l'hypothèse suivante.

Hypothèse 3.

$$\exists C, M \geq 1, \quad \forall q \in \mathbb{R}^d, \quad C^{-1}\langle q\rangle^{\frac{1}{M}} \leq \langle \partial_q V(q)\rangle \leq C\langle q\rangle^M\,,$$

$$\forall \alpha \in \mathbb{N}^d, |\alpha| \geq 1, \exists C_\alpha \geq 1, \quad |\partial_q^\alpha V(q)| \leq C_\alpha \langle \partial_q V(q)\rangle\,.$$

Remarquons ici que l'exposant $\frac{1}{4}$ dans l'estimation (1.4.4) n'est pas optimal. Un meilleur exposant, qui semble être $\frac{2}{3}$ comme on le voit dans [RoSt], peut être obtenu par des méthodes explicites dans le cas particulier où $V(q)$ est une forme quadratique non dégénéré (voir [HeNi]). Dans [HeNi], Helffer-Nier ont étudié aussi le cas où $V(q)$ vérifie

Hypothèse 4.

$$\forall \alpha \in \mathbb{N}^d,\ |\alpha| = 2, \quad |\partial_q^\alpha V(q)| \leq C_\alpha \langle \partial_q V(q)\rangle^{1-\varrho_0} \quad \textit{avec} \quad \varrho_0 > \frac{1}{3}\,,$$

et ont obtenu alors l'estimation

$$\||\partial_q V(q)|^{\frac{2}{3}}u\|^2_{L^2(\mathbb{R}^{2d})} \leq C_V\Big(\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|u\|^2_{L^2(\mathbb{R}^{2d})}\Big)\,. \tag{1.4.5}$$

Ce résultat généralise en particulier celui du cas quadratique non dégénéré. De plus, l'exposant $\frac{2}{3}$ dans (1.4.5) est meilleur que $\frac{1}{4}$ qui apparaît dans l'estimation (1.4.4).

1.4.2 Résultats de Wei-Xi Li

Dans cette sous section on présente certains résultats récents développés par Wei-Xi Li [Li][Li2]. Ces résultats donnent des critères de compacité de la résolvante de K_V avec potentiel $V(q) \in \mathcal{C}^2(\mathbb{R}^d)$. En particulier, les critères développés dans [Li2] impliquent un contrôle des valeurs propres positives de la matrice hessienne du potentiel $V(q)$.

Le premier résultat dû à Wei-Xi Li est démontré dans [Li] sous des hypothèses d'ellipticité et sans contrôle des signes du potentiel. Il est rappelé dans le théorème suivant.

Théorème 1.4.1. *Soit $V(q) \in \mathcal{C}^2(\mathbb{R}^d)$ une fonction à valeurs réelles qui satisfait pour un $s < \frac{4}{3}$, la propriété :*

$$\forall |\alpha| = 2\,, \exists C_\alpha > 0\,, \quad |\partial_q^\alpha V(q)| \leq C_\alpha\Big(1 + |\partial_q V(q)|^2\Big)^{\frac{s}{2}}\,. \tag{1.4.6}$$

Il existe alors $C > 0$ et $\delta > 0$ tels que pour tout $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$\||\partial_q V(q)|^{\frac{2}{3}} u\|_{L^2(\mathbb{R}^{2d})} \leq C\Big(\|K_V u\|_{L^2(\mathbb{R}^{2d})} + \|u\|_{L^2(\mathbb{R}^{2d})}\Big)$$

et

$$\|(1-\Delta_q)^{\frac{\delta}{2}} u\|_{L^2(\mathbb{R}^{2d})} + \|(1-\Delta_p+p^2)^{\frac{1}{2}} u\|_{L^2(\mathbb{R}^{2d})} \leq C\Big(\|K_V u\|_{L^2(\mathbb{R}^{2d})} + \|u\|_{L^2(\mathbb{R}^{2d})}\Big) \ .$$

De plus δ est égal à $\frac{2}{3}$ si $s \leq \frac{2}{3}$, $\frac{4}{3} - s$ si $\frac{2}{3} < s \leq \frac{10}{9}$ et $\frac{2}{3} - \frac{s}{2}$ si $\frac{10}{9} < s < \frac{4}{3}$.

D'après ce théorème, si nous prenons en particulier $s = \frac{2}{3}$, nous avons l'estimation hypoelliptique optimale suivante

$$\||\partial_q V(q)|^{\frac{2}{3}} u\|_{L^2(\mathbb{R}^{2d})} + \|(1-\Delta_q)^{\frac{2}{3}} u\|_{L^2(\mathbb{R}^{2d})} \leq C(\|K_V u\|_{L^2(\mathbb{R}^{2d})} + \|u\|_{L^2(\mathbb{R}^{2d})}) \ .$$

Rappelons que le critère établi par Helffer-Nier et énoncé dans le deuxième point du théorème 1.3.1 montre l'importance du signe de $V(q)$ pour la compacité de la résolvante de Laplacien de Witten. En ce qui concerne l'opérateur de Kramers-Fokker-Planck, la conjecture de Helffer-Nier 1.1.1 suggère fortement qu'il devrait avoir la même propriété microlocale comme pour Laplacien de Witten. Nous remarquons ici que l'inconvénient de la condition (1.4.6) est qu'elle ne donne aucune information sur la dépendance du signe de $V(q)$. Cependant, Wei-Xi Li a donné dans [Li2] (Théorèmes 1.2 et 1.3) certaines conditions suffisantes pour la compacité de la résolvante de l'opérateur de Kramers-Fokker-Planck, reposant principalement sur le signe des valeurs propres de la matrice hessienne $(\partial_{q_i q_j} V)_{1\leq i,j\leq d}$. Ainsi ces progrès présentent les premiers résultats reflétant cette propriété de dépendance, qui joue un rôle important dans l'analyse de la compacité de la résolvante pour le Laplacien de Witten.

Les deux résultats suivants sont extraits de [Li2] (Théorèmes 1.2 et 1.3).

Théorème 1.4.2. *Soit $V(q) \in \mathcal{C}^2(\mathbb{R}^d)$. On note $\lambda_l(q)$, où $1 \leq l \leq d$ les valeurs propres de la matrice hessienne*

$$(\partial_{q_i q_j} V(q))_{1\leq i,j\leq d} \ .$$

À chaque $q \in \mathbb{R}^d$, on associe l'ensemble des indices noté I_q donné par

$$I_q = \{1 \leq l \leq d \ , \ \text{tel que} \ \ \lambda_l(q) > 0\} \ .$$

S'il existe une constante $c > 0$ telle que,

$$\forall\, q \in \mathbb{R}^d \ , \qquad \sum_{j\in I_q} \lambda_j(q) \leq c\langle \partial_q V(q)\rangle^{\frac{4}{3}} \ , \tag{1.4.7}$$

alors on a les résultats suivants :

(i) Il existe une constante $c>0$ telle que pour tout $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$\||\partial_q V(q)|^{\frac{1}{16}} u\|_{L^2(\mathbb{R}^{2d})} \leq c\Big(\|K_V u\|_{L^2(\mathbb{R}^{2d})} + \|u\|_{L^2(\mathbb{R}^{2d})}\Big) . \tag{1.4.8}$$

Par conséquent, si $\lim\limits_{|q|\to+\infty} |\partial_q V(q)| = +\infty$, l'opérateur de Kramers-Fokker-Planck est à résolvante compacte.

(ii) Si on suppose de plus qu'il existe $\alpha \geq 0$ telle que

$$\lim_{|q|\to+\infty} (\alpha|\partial_q V(q)|^2 - \Delta_q V(q)) = +\infty , \tag{1.4.9}$$

alors on peut trouver une constante $\widetilde{c}_\alpha > 0$ telle que pour tout $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$\||\alpha|\partial_q V(q)|^2 - \Delta_q V(q)|^{\frac{1}{80}} u\|_{L^2(\mathbb{R}^{2d})} \leq \widetilde{c}_\alpha\Big(\|K_V u\|_{L^2(\mathbb{R}^{2d})} + \|u\|_{L^2(\mathbb{R}^{2d})}\Big) .$$

Ceci implique que l'opérateur de Kramers-Fokker-Planck est à résolvante compacte.

Théorème 1.4.3. *Soit $V(q) \in \mathcal{C}^2(\mathbb{R}^d)$. On suppose qu'il existe une constante $\tau > 0$ tel que la matrice*

$$A_\tau(q) = (a_{ij}^\tau(q))_{1\leq i,j\leq d} , \qquad a_{ij}^\tau = \tau\langle\partial_q V\rangle^{\frac{4}{5}}(\partial_{q_i} V)(\partial_{q_j} V) - \partial_{q_i q_j} V + \tau\delta_{ij} ,$$

est définie positive pour tout $q \in \mathbb{R}^d$, où δ_{ij} désigne le symbole de Kronecker.

Alors il existe une constante $c > 0$ tel que pour tout $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$\||\partial_q V(q)|^{\frac{1}{20}} u\|_{L^2(\mathbb{R}^{2d})} + \sum_{1\leq i,j\leq d} \||a_{ij}^\tau(q)|^{\frac{1}{80}} u\|_{L^2(\mathbb{R}^{2d})} \leq c\Big(\|K_V u\|_{L^2(\mathbb{R}^{2d})} + \|u\|_{L^2(\mathbb{R}^{2d})}\Big) .$$

Par conséquent, si $\lim\limits_{|q|\to+\infty} \Big(|\partial_q V(q)| + \sum\limits_{1\leq i,j\leq d} |a_{ij}^\tau(q)|\Big) = +\infty$, l'opérateur de Kramers-Fokker-Planck est à résolvante compacte.

Les preuves des deux théorèmes précédents sont bien détaillées dans [Li2] et ils reposent principalement sur des méthodes de multiplicateurs.

1.4.2.1 Application sur l'exemple $V(q_1,q_2) = -\ q_1^2 q_2^2$

Dans le cas où $V(q) = -q_1^2 q_2^2$, il est bien connu (voir Proposition 10.19 dans [HeNi]) que le Laplacien de Witten $\Delta_V^{(0)}$ est à résolvante compacte, alors que 0 appartient au spectre essentiel $\Delta_{-V}^{(0)}$ et sa résolvante ne peut donc pas être compacte. En examinant maintenant le même potentiel susmentionné, on voit ci-après que les hypothèses des théorèmes 1.4.2 et 1.4.3 sont valides pour $V(q) = -q_1^2 q_2^2$ et violées par $V(q) = q_1^2 q_2^2$.

Application du Théorème 1.4.2 :

Le potentiel $V(q_1, q_2) = -q_1^2 q_2^2$ vérifie les hypothèses du théorème 1.4.2. En calculant

$$(\partial_{q_i q_j} V)_{1\le i,j\le 2} = \begin{pmatrix} -2q_2^2 & -4q_1q_2 \\ -4q_1q_2 & -2q_1^2 \end{pmatrix}, \quad |\partial_q V(q)|^2 = 4q_1^2q_2^4 + 4q_1^4q_2^2 = 4q_1^2q_2^2|q|^2 \ ,$$

la somme $\sum_{j\in I_q} \lambda_j(q)$ qu'on cherche à estimer ne contient en réalité qu'un seul terme, la seule racine positive $\lambda = -|q|^2 + \sqrt{|q|^4 + 12q_1^2q_2^2}$. En multipliant par le conjugué (en supposant $q \neq 0$) on a

$$\lambda = \frac{12q_1^2q_2^2}{|q|^2 + \sqrt{|q|^4 + 12q_1^2q_2^2}} \le \frac{12q_1^2q_2^2}{|q|^2} = 3\frac{4q_1^2q_2^2|q|^2}{|q|^4} = \frac{3}{|q|^4}|\partial_q V(q)|^2 \ .$$

Le reste est simple à faire :

Si $|q| \ge 1$, alors

$$\lambda \le \frac{3}{|q|^4}|\partial_q V(q)|^2 \le 3|\partial_q V(q)|^2 \le 3(1 + |\partial_q V(q)|^2) \ .$$

Et si $|q| < 1$, il suffit de remarquer que dans ce cas

$$\lambda = -|q|^2 + \sqrt{|q|^4 + 12q_1^2q_2^2} < |q|^2 + \sqrt{|q|^4 + 12q_1^2q_2^2} \le 1 + \sqrt{1+12} = 1 + \sqrt{13} < 5 \ .$$

Par suite

$$\lambda \le 5(1 + |\partial_q V(q)|^2) \ .$$

Donc dans tous les cas, c'est-à-dire pour tout $q \in \mathbb{R}^2$, on a

$$\lambda \le 5(1 + |\partial_q V(q)|^2) \ .$$

On peut donc prendre $c = 5$ comme constante pour l'hypothèse (1.4.7).

D'après l'item (i) du théorème 1.4.2, il existe une constante $c > 0$ tel que pour tout $u \in \mathcal{C}_0^\infty(\mathbb{R}^4)$,

$$\| |\partial_q V(q)|^{\frac{1}{16}} u\|_{L^2(\mathbb{R}^4)} \le c\Big(\|K_V u\|_{L^2(\mathbb{R}^4)} + \|u\|_{L^2(\mathbb{R}^4)}\Big) \ .$$

Cette dernière estimation nous dit rien sur la compacité de la résolvante de l'opérateur de Kramers-Fokker-Planck associé à $V(q_1, q_2) = -q_1^2q_2^2$ (car $\partial_q V(q)$ s'annule suivant les deux lignes $q_1 = 0$ et $q_2 = 0$). Dans ce cas, on utilise l'item (ii) du théorème 1.4.2.

En effet, le potentiel considéré vérifie l'hypothèse (ii) du théorème 1.4.2 : Pour tout $\alpha \ge 0$,

$$\lim_{|q|\to+\infty} (\alpha|\partial_q V(q)|^2 - \Delta_q V(q)) = \lim_{|q|\to+\infty} (4\alpha q_1^2q_2^2|q|^2 + 2|q|^2) = +\infty \ ,$$

(dans l'énoncé du théorème il suffit de trouver une seule constante $\alpha \geq 0$ qui vérifie l'hypothèse), alors on peut trouver une constante $\widetilde{c}_\alpha > 0$ tel que pour tout $u \in \mathcal{C}_0^\infty(\mathbb{R}^4)$,

$$\||\alpha|\partial_q V(q)|^2 - \Delta_q V(q)|^{\frac{1}{80}} u\|_{L^2(\mathbb{R}^4)} \leq \widetilde{c}_\alpha(\|K_V u\|_{L^2(\mathbb{R}^4)} + \|u\|_{L^2(\mathbb{R}^4)}).$$

Ceci implique que l'opérateur de Kramers-Fokker-Planck est à résolvante compacte.

Application du Théorème 1.4.3 :

Le potentiel $V(q_1, q_2) = -q_1^2 q_2^2$ vérifie l'hypothèse du théorème 1.4.3. On cherche $\tau > 0$, tel que la matrice $A_\tau(q) = (a_{ij}^\tau(q))_{1\leq i,j\leq 2}$, avec terme général

$$a_{ij}^\tau = \tau \langle \partial_q V \rangle^{\frac{4}{5}} (\partial_{q_i} V)(\partial_{q_j} V) - \partial_{q_i q_j} V + \tau \delta_{ij} \ ,$$

soit définie positive pour tout $q \in \mathbb{R}^2$. En effet, on a pour tout $\tau > 0$,

$$\operatorname{trace}(A_\tau) = \tau \langle \partial_q V \rangle^{4/5} \Big((\partial_{q_1} V)^2 + (\partial_{q_2} V)^2 \Big) - (\partial_{q_1}^2 V + \partial_{q_2}^2 V) + 2\tau > 0 \ .$$

Il faut donc prouver l'existence d'une constante $\tau > 0$ tel que $\det(A_\tau) > 0$:

On a

$$\det(A_\tau) = \tau^2 \langle \partial_q V \rangle^{14/5} + \tau \Big(-16 \langle \partial_q V \rangle^{4/5} (q_1 q_2)^4 + 2(q_1^2 + q_2^2) \Big) - 12 (q_1 q_2)^2 \ .$$

Pour q fixé le déterminant est un polynôme en τ de degré 2. Il faut alors étudier quand ce polynôme de degré 2 est strictement positif. Il suffit d'écrire une telle condition comme une inégalité avec τ et prendre un τ suffisamment grand pour que ça marche pour tout q.

Le coefficient dominant du déterminant est positif, donc si on note $b(q)$ la racine la plus grande de ce polynôme (à q fixé), une condition suffisante est d'avoir $\tau > 1 + b(q)$ pour tout q. Cette condition est bien vérifiée puisque $b(q)$ est majorée en q.

Ainsi, d'après le théorème 1.4.3 il existe une constante $c > 0$ telle que pour tout $u \in \mathcal{C}_0^\infty(\mathbb{R}^4)$,

$$\||\partial_q V(q)|^{\frac{1}{20}} u\|_{L^2(\mathbb{R}^4)} + \sum_{1\leq i,j\leq 2} \||a_{ij}^\tau(q)|^{\frac{1}{80}} u\|_{L^2(\mathbb{R}^4)} \leq c\Big(\|K_V u\|_{L^2(\mathbb{R}^4)} + \|u\|_{L^2(\mathbb{R}^4)} \Big)$$

Dans notre cas, $\lim_{|q|\to+\infty} \Big(|\partial_q V(q)| + \sum_{1\leq i,j\leq 2} |a_{ij}^\tau(q)| \Big) = +\infty$, donc d'après la proposition 1.4.1, l'opérateur de Kramers-Fokker-Planck est à résolvante compacte.

1.5 Principaux résultats de cette thèse

Le coeur de ce travail de thèse se décompose en trois parties bien articulées, qui ont fait l'objet de trois articles distincts. Rappelons que l'approche de Helffer-Nourrigat sur l'hypoellipticité maximale appliquée au Laplacien de Witten consiste à faire une double récurrence sur le degré du polynôme et la dimension de l'espace (après réduction de la dimension quand il y a des invariances par translation). Pour la récurrence sur le degré, ce sont des estimations uniformes par rapports aux coefficients pour des polynômes de degré r qui donnent des estimations qualitatives, sans contrôle des constantes par rapport aux coefficients, pour des polynômes de degré r+1. De ce point de vue le premier cas non trivial pour l'opérateur de Kramers-Fokker-Planck est le cas des potentiels de degré inférieur ou égal à 2. Des estimations sans contrôle des constantes dans les inégalités sous-elliptiques étaient connues (cf. [Hor]). Par ailleurs, un potentiel de degré inférieur ou égal à 2, donne un opérateur de Kramers-Fokker-Planck à symbole quadratique pour lequel la théorie générale (voir [Hor][HiPr][HPV2][Vio][Vio1][AlVi]) nous dit que l'on peut tout calculer très précisément. La mise en place de ce calcul pour aboutir à des inégalités sous-elliptiques optimales a été difficile à mettre en place et a fait finalement apparaître une structure quaternionique intéressante. Une fois le cas des polynômes de degré inférieur ou égal à 2 traité précisément, nous pouvons nous attaquer à des estimations sous-elliptiques avec des potentiels de degré supérieur mais dont les modèles asymptotiques à l'infini après changement d'échelle sont de degré inférieur ou égal à 2. Les corrections logarithmiques dont on ne peut pas se débarrasser dans le cas quadratique, sont fort heureusement absorbées grâce au théorème de Tarski-Seidenberg, outil très classique dans un cadre algébrique (potentiel polynomial). C'est l'objet du deuxième article.

Une fois ce travail fait quasiment tous les exemples présentés par Helffer et Nier dans [HeNi] pour comparer le Laplacien de Witten et l'opérateur de Kramers-Fokker-Planck, étaient complètement traités exceptés le cas simple de potentiels homogènes, non nécéssairement polynomiaux, de degré supérieur à 2. Ce dernier point est partiellement traité dans le troisième article, où l'homogénéité remplace les arguments plus implicites donnés par Tarski-Seidenberg et donne dans les cas qui manquaient de meilleurs résultats.

Nous donnons dans les paragraphes suivants un résumé un peu plus détaillé de chaque chapitre. Commençons par fixer quelques notations. On considère dans les énoncés des résultats qui suivent, l'opérateur de Kramers-Fokker-Planck défini sur $\mathbb{R}^{2d}$ par

$$K_V = p\partial_q - \partial_q V(q)\partial_p + \frac{1}{2}(-\Delta_p + p^2) \ ,$$

avec domaine

$$D(K_V) = \left\{u \in L^2(\mathbb{R}^{2d}) \ , \ K_V u \in L^2(\mathbb{R}^{2d})\right\} \ ,$$

où $V(q)$ est un potentiel qui ne dépend que de la variable de position $q \in \mathbb{R}^d$. On note

$$O_p = \frac{1}{2}(D_p^2 + p^2) \ , \qquad X_V = p\partial_q - \partial_q V(q)\partial_p \ ,$$

Pour tout $q \in \mathbb{R}^d$, on définit dans cette thèse

$$\mathrm{Tr}_{+,V}(q) = \sum_{\substack{\nu \in \mathrm{Spec}(\mathrm{Hess}\ V(q)) \\ \nu > 0}} \nu(q) \ , \qquad \mathrm{Tr}_{-,V}(q) \ = - \sum_{\substack{\nu \in \mathrm{Spec}(\mathrm{Hess}\ V(q)) \\ \nu \leq 0}} \nu(q) \ .$$

$$A_V(q) = \max\{(1 + \mathrm{Tr}_{+,V}(q))^{2/3}, 1 + \mathrm{Tr}_{-,V}(q)\} \ ,$$
$$B_V = \max\{\min_{q\in\mathbb{R}^d} |\nabla\, V(q)|^{4/3}\, , \frac{1 + \mathrm{Tr}_{-,V}(q)}{(\log(2 + \mathrm{Tr}_{-,V}(q)))^2}\} \ .$$

1.5.1 Énoncé des résultats du chapitre 2

Plusieurs travaux ont été consacrés à l'étude de certain opérateurs différentiels quadratiques non autoadjoints et non elliptiques dont en particulier l'opérateur de Kramers-Fokker-Planck avec polynôme $V(q)$ de degré inférieur ou égal à 2 (Cf. [Hor][Sjo][HiPr][HPV2][Vio] [Vio1][AlVi]). Nos premiers résultats dans [BNV] établissent des estimations sous-elliptiques globales pour l'opérateur de Kramers-Fokker-Planck K_V avec potentiel polynomial $V(q)$ de degré inférieur ou égal à 2. Dans nos preuves nous nous inspirons des techniques développées par Hitrik, Pravda-Starov, Viola, et Aleman dans [HiPr][Vio][Vio1] et [AlVi]. Ce qui est particulier dans nos résultats, qui sont résumés dans les théorèmes 1.5.1 et 1.5.2, est le contrôle uniforme des inégalités par rapport aux coefficients du polynôme V.

Théorème 1.5.1. *Soit $V(q)$ un polynôme de degré inférieur ou égal à 2. Il existe une constante $c > 0$, indépendante de V, telle que l'estimation sous-elliptique avec reste*

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + A_V \|u\|^2_{L^2(\mathbb{R}^{2d})} \geq c\Big(\|O_p u\|^2_{L^2(\mathbb{R}^{2d})} + \|X_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle\partial_q V(q)\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle D_q\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})}\Big) \tag{1.5.1}$$

est vraie pour tout $u \in D(K_V)$.

Théorème 1.5.2. *Soit $V(q)$ un polynôme de degré inférieur ou égal à 2. Il existe une constante $c > 0$, indépendante de V, telle que on a*

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq c\, B_V \|u\|^2_{L^2(\mathbb{R}^{2d})} \ ,$$

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{c}{1 + \frac{A_V}{B_V}}\Big(\|O_p u\|^2_{L^2(\mathbb{R}^{2d})} + \|X_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle\partial_q V(q)\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle D_q\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})}\Big)$$

pour tout $u \in D(K_V)$.

Comme première conséquence du théorème 1.5.1, si $|\partial_q V(q)|$ tend vers l'infini lorsque $|q|$ tend vers l'infini, alors K_V est à résolvante compacte.

Les deux théorèmes précédents se déduisent du résultat suivant.

Proposition 1.5.1. *Soit $V(q)$ un polynôme de degré inférieur ou égal à 2. Il existe une constante $c > 0$, indépendante de V, telle que*

$$\sum_{i=1}^{d} \| |D_{q_i}| e^{-t(K_V+\sqrt{A_V})} \|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} + \| |\partial_{q_i} V(q)| e^{-t(K_V+\sqrt{A_V})} \|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} \leq \frac{c}{t^{\frac{3}{2}}}$$

pour tout $t > 0$.

De plus,

$$\|K_V^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} \leq \int_0^{+\infty} \|e^{-tK_V}\|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} dt \leq \frac{c}{\sqrt{B_V}} \,.$$

Notons tout d'abord qu'étant donné un potentiel polynomial à valeurs réelles $V(q) = \sum_{|\alpha|\leq 2} V_\alpha q^\alpha$ défini sur $\mathbb{R}^d$ avec $d^\circ V = 2$, on peut supposer à l'aide d'un changement de variables orthogonal que

$$\text{Hess } V = \begin{pmatrix} \nu_1 & 0 & \dots & 0 \\ 0 & \nu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_d \end{pmatrix} .$$

Comme la constante V_0 n'apparaît pas dans la formule de K_V, elle peut être fixée à 0. Dans cette situation, on a les deux cas suivant :

• Si Hess V est non dégénérée, une translation en q réduit le problème à

$$V(q) = \sum_{i=1}^{d} \frac{\nu_i}{2} q_i^2 \,. \tag{1.5.2}$$

• Si Hess V est dégénérée, un bon choix d'une base orthonormale et une translation donnent :

$$V(q) = \lambda_1 q_1 + \sum_{i=2}^{d} \frac{\nu_i}{2} q_i^2 \,, \tag{1.5.3}$$

où λ_1 est défini par $|\lambda_1| = \min_{q\in\mathbb{R}^d} |\nabla\, V(q)| \geq 0$.
Pour démontrer les résultats que nous venons d'énoncer, il y a essentiellement deux types d'argument à utiliser.

• Comme les estimations de la proposition 1.5.1 sont exprimées en termes de semi-groupe, elles peuvent êtres démontrées après séparation des variables d'un polynôme de la forme

(1.5.2) ou (1.5.3). En effet, comme e^{-tK_V} est un produit de semi-groupes de contraction commutatifs par rapport à chaque variable (q_j, p_j), il suffit d'écrire

$$\sum_{i=1}^{d} \|M_i e^{-t(K_V+\sqrt{A})}\|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} \leq \sum_{i=1}^{d} \|M_i e^{-t(K_{V_i(q_i)}+\alpha_i)}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \quad M_i = |D_{q_i}| \text{ ou } M_i = |\partial_{q_i} V(q)|\,,$$

où $V_i(q_i)$ désigne le potentiel uni-dimensionnel à variable q_i, avec $V_1(q_1) = \frac{\nu_1 q_1^2}{2}$ ou $V_1(q_1) = \lambda_1 q_1$, $V_i(q_i) = \frac{\nu_i q_i^2}{2}$ pour $i \geq 2$, $\alpha_i = |\nu_i|^{1/2}$ si $\nu_i < 0$, $\alpha_i = \nu_i^{1/3}$ si $\nu_i > 0$ et $\alpha_i = 0$ si $\partial_{q_i}^2 V = 0$. La deuxième estimation de la proposition 1.5.1 est encore plus simple et elle découle d'un calcul exact de la norme du semi-groupe associé à l'opérateur K_V. Par conséquent, la proposition 1.5.1 sera le résultat de l'analyse des trois potentiels uni-dimensionnels $V(q) = \pm\frac{\nu q^2}{2}$, $\nu > 0$, et $V(q) = \lambda_1 q$, $\lambda_1 \in \mathbb{R}$.

Une fois le problème ramené en dimension 1, nos résultats et nos formules sont exprimés explicitement à l'aide des biquaternions (ou quaternion complexe).

• Le deuxième argument consiste à utiliser les résultats d'interpolation de Lunardi (voir Remarque 5.11, Théorème 5.12 et Corollaire 5.13 dans [Lun]). En effet, en combinant la première inégalité de la proposition 1.5.1 avec le fait que

$$|\mathbb{R}\mathrm{e}\langle [O_p, X_V]u, u\rangle| \leq C_\epsilon(\|\,|D_q|^{\frac{2}{3}}u\|^2 + \|\,|\partial_q V(q)|^{\frac{2}{3}}u\|^2) + \epsilon\|O_p u\|^2$$

pour tout $u \in D(K_V)$ (où $\epsilon > 0$ est suffisamment petit), on obtient l'estimation du théorème 1.5.1. Ensuite la première estimation du théorème 1.5.1 n'est qu'une réécriture de l'estimation de la résolvante donnée dans la proposition 1.5.1. En combinant l'estimation de la résolvante avec le résultat du théorème 1.5.1, on obtient ainsi l'estimation sans reste du théorème 1.5.2.

Ces résultats ont fait l'objet de l'article [BNV] en collaboration avec Francis Nier et Joe Viola et seront prouvés dans le chapitre 2.

1.5.2 Énoncé des résultats du chapitre 3

Le chapitre 3 de ce manuscript reprend le contenu de l'article [Ben]. Cet article propose une étude de l'opérateur de Kramers-Fokker-Planck K_V associé à une certaine classe de polynômes $V(q)$ de degré $r \geq 3$, dont le comportement à l'infini est quadratique. Nous donnons ici l'énoncé du résultat principal de [Ben] avec quelques éléments de sa preuve. Précisons au préalable quelques notations.

On désigne dans cette thèse par, $R_V^{\geq n}$ où $n \in \mathbb{N}^*$, la fonction définie sur $\mathbb{R}^d$ par

$$R_V^{\geq n}(q) = \sum_{n\leq|\alpha|\leq r} |\partial_q^\alpha V(q)|^{\frac{1}{|\alpha|}}\ .$$

Pour tout $\kappa > 0$, on note

$$\Sigma(\kappa) = \Big\{ q \in \mathbb{R}^d,\ |\nabla V(q)|^{\frac{4}{3}} \geq \kappa \Big(|\text{Hess } V(q)| + R_V^{\geq 3}(q)^4 + 1 \Big) \Big\} \ .$$

On introduit

Hypothèse 5. *Il existe des grandes constantes $\kappa_0, C_1 > 1$ tel que pour tout $\kappa \geq \kappa_0$ polynôme $V(q)$ satisfait les propriétés suivantes*

$$\text{Tr}_{-,V}(q) > \frac{1}{C_1} \text{Tr}_{+,V}(q), \ \ \textit{pour tout } \ q \in \mathbb{R}^d \setminus \Sigma(\kappa) \textit{ avec } |q| \geq C_1 \ , \tag{1.5.4}$$

de plus si $\mathbb{R}^d \setminus \Sigma(\kappa)$ n'est pas borné,

$$\lim_{\substack{|q| \to +\infty \\ q \in \mathbb{R}^d \setminus \Sigma(\kappa)}} \frac{R_V^{\geq 3}(q)^4}{|\text{Hess } V(q)|} = 0 \ . \tag{1.5.5}$$

On a alors le théorème

Théorème 1.5.3. *Pour tout polynôme $V(q)$ de degré r supérieur à deux vérifiant l'hypothèse 5, il existe une constante $C_V > 1$ telle que*

$$\|K_V u\|_{L^2}^2 + C_V \|u\|_{L^2}^2 \geq \frac{1}{C_V} \Big(\|L(O_p)u\|_{L^2}^2 + \|L(\langle |\nabla V(q)| \rangle^{\frac{2}{3}})u\|_{L^2}^2 + \|L(\langle |\text{Hess } V(q)| \rangle^{\frac{1}{2}})u\|_{L^2}^2 + \|L(\langle |D_q| \rangle^{\frac{2}{3}})u\|_{L^2}^2 \Big) \ , \tag{1.5.6}$$

pour tout $u \in D(K_V)$ où $L(s) = \frac{s+1}{\log(s+1)}$ pour tout $s \geq 1$.

En conséquence, si $V(q)$ vérifie l'hypothèse 5, alors l'opérateur Kramers-Fokker-Planck K_V est à résolvante compacte et il a donc un spectre discret.
Mentionnons aussi qu'on sait déjà que pour un polynôme V vérifiant l'hypothèse 5, le Laplacien de Witten $\Delta_V^{(0)}$ est à résolvante compacte car les modèles asymptotiques à l'infini sont de degré inférieur ou égale à 2 sans minimum local (Voir théorème 10.16 dans [HeNi]).

Il est à remarquer que malgré la perte logarithmique dans (1.5.6), nous sommes très proches des exposants optimaux figurant dans l'estimation (1.5.1).

Expliquons un peu l'hypothèse 5 et en particulier l'idée derrière la partition $\Sigma(\kappa) \sqcup (\mathbb{R}^d \setminus \Sigma(\kappa))$: La région $\Sigma(\kappa)$ est celle où le gradient de V domine sa hessienne et ses dérivées d'ordre supérieur. L'analyse, dans cette région, est essentiellement la même que dans les cas elliptiques traités dans [HerNi] [HeNi] et [Li]. Dans la région complémentaire $\mathbb{R}^d \setminus \Sigma(\kappa)$, c'est la hessienne de V qui domine, au voisinage de l'infini, le gradient et les dérivées d'ordre supérieur. Ici les estimations précises du modèle quadratique donné par le deuxième ordre de développement de Taylor doivent être utilisées. En effet, la démonstration du théorème 1.5.3 repose principalement sur deux techniques :

• La première consiste à construire une partition de l'unité localement finie suivant la variable de position $q \in \mathbb{R}^d$,

$$\sum_{j\in\mathbb{N}} \chi_j^2(q) = \sum_{j\in\mathbb{N}} \widetilde{\chi}_j^2\Big(R_V^{\geq 3}(q_j)(q-q_j)\Big) = 1 \tag{1.5.7}$$

où

$$\text{supp}\ \widetilde{\chi}_j \subset B(0,a) \text{ et } \ \widetilde{\chi}_j \equiv 1 \text{ dans } \quad B(0,b)$$

pour certains $q_j \in \mathbb{R}^d$ avec $0 < b < a$ indépendants de $j \in \mathbb{N}$. L'existence d'une telle partition de l'unité est une conséquence immédiate de la lenteur de la métrique $R_V^{\geq 3}(q)^2 dq^2$ ($\exists C > 1,\ \forall q, q' \in \mathbb{R}^d,\ \ R_V^{\geq 3}(q)|q-q'| \leq C^{-1} \Rightarrow \left(\frac{R_V^{\geq 3}(q)}{R_V^{\geq 3}(q')}\right)^{\pm 1} \leq C$), qu'on a bien vérifié dans le lemme A.4 de [Ben]. Ceci permet d'une façon générale, d'obtenir des estimations globales à partir d'une étude locale.

• Il est naturel de considérer une approximation quadratique pour $V(q)$, bien choisie, afin d'étendre les résultats de [BNV], déjà établis dans le cas $d°V \leq 2$. Pour un certain $\kappa > 1$ donné et pour tout entier naturel j, cette dernière approximation est donnée par

$$V_j^{(2)}(q) = \sum_{0\leq|\alpha|\leq 2} \frac{\partial_q^\alpha V(q_j')}{\alpha!}(q-q_j')^\alpha\ , \tag{1.5.8}$$

où

$$\begin{cases} q_j' = q_j & \text{si } \ \text{supp}\,\chi_j \subset \Sigma(\kappa) \\ q_j' \in (\text{supp}\ \chi_j) \cap \Big(\mathbb{R}^d \setminus \Sigma(\kappa)\Big) & \text{sinon.} \end{cases}$$

Il se trouve qu'en utilisant les deux techniques citées plus haut, il faut évidemment bien contrôler les erreurs dues d'une part à la partition de l'unité et d'autre part à l'approximation quadratique. Ceci est assuré en ajustant le paramètre κ à la fin de la preuve. De plus le passage d'une estimation pour $K_{V_j^{(2)}}$ en une pour K_V nécessite le recours au résultat suivant.

Lemme 1.5.2. *Soit V un polynôme de degré $r \geq 3$. On considère une partition de l'unité localement finie comme dans (1.5.7). Pour tout $\alpha \in \mathbb{N}^d$ avec $|\alpha| \in \{1,2\}$ et pour tout $j \in \mathbb{N}$, il existe une constante $c_{\alpha,d,r} > 0$ telle que*

$$\left|\partial_q^\alpha V(q) - \partial_q^\alpha V_j^{(2)}(q)\right| \leq c_{\alpha,d,r}\Big(R_V^{\geq 3}(q_j')\Big)^{|\alpha|} \tag{1.5.9}$$

est valide pour tout $q \in \text{supp}\ \chi_j = B(q_j, aR_V^{\geq 3}(q_j)^{-1})$.

Par conséquent, si V satisfait l'hypothèse 1, il existe une constante assez grande $\kappa_1 \geq \kappa_0$ tel que pour tout $\kappa \geq \kappa_1$ et tout $j \in \mathbb{N}$,

$$2^{-1}\left|\partial_q V_j^{(2)}(q)\right| \leq |\partial_q V(q)| \leq 2\left|\partial_q V_j^{(2)}(q)\right| \tag{1.5.10}$$

pour tout $q \in (\operatorname{supp} \chi_j) \cap \Sigma(\kappa)$ et

$$2^{-1}\left|\text{Hess } V_j^{(2)}(q)\right| \leq |\text{Hess } V(q)| \leq 2\left|\text{Hess } V_j^{(2)}(q)\right| , \tag{1.5.11}$$

pour tout $q \in (\operatorname{supp} \chi_j) \cap \left(\mathbb{R}^d \setminus \Sigma(\kappa)\right)$ avec $|q| \geq C_2(\kappa)$ où $C_2(\kappa) > 0$ est une constante assez grande qui dépend de κ.

La preuve du théorème 1.5.3 est basée en particulier sur une conséquence théorème de Tarski-Seidenberg examinée dans l'annexe 3.B. En effet, cette approche nous a permis d'absorber les corrections logarithmiques dont on ne peut pas s'affranchir dans le cas quadratique.

Si on veut étendre le résultat du théorème 1.5.3 à des classes de fonctions $V(q)$ non polynomiales, nous n'avons plus le droit d'utiliser l'argument algébrique de Tarski-Seidenberg. Par contre ce résultat reste évidement valable dans le cas où $V = V_1 + V_2$ avec V_1 est un polynôme satisfaisant l'hypothèse 5 et V_2 est une fonction dans $\mathcal{S}(\mathbb{R}^d)$.

1.5.3 Énoncé des résultats du chapitre 4

Nous verrons dans cette partie une extension du théorème 1.5.3 dans le cadre des potentiels homogènes de degré $2 < r < 6$ dans $\mathcal{C}^\infty(\mathbb{R}^d \setminus \{ 0\})$ satisfaisant l'hypothèse suivante.

Hypothèse 6.

$$\forall \ q \in \mathcal{S} := \left\{q \in \mathbb{R}^d, \ |q| = 1\right\} , \qquad \partial_q V(q) = 0 \Rightarrow \text{Tr}_{-,V}(q) > 0 . \tag{1.5.12}$$

Le résultat principal établi dans [Ben1] s'énonce comme suit.

Théorème 1.5.4. *Pour tout potentiel $V(q)$ vérifiant l'hypothèse 6, il existe une constante $C_V > 1$ telle que*

$$\|K_V u\|_{L^2}^2 + C_V\|u\|_{L^2}^2 \geq \frac{1}{C_V}\Big(\|L(O_p)u\|_{L^2}^2 + \|L(\langle\nabla V(q)\rangle^{\frac{2}{3}})u\|_{L^2}^2 + \|L(\langle\text{Hess } V(q)\rangle^{\frac{1}{2}})u\|_{L^2}^2 + \|L(\langle D_q\rangle^{\frac{2}{3}})u\|_{L^2}^2\Big) , \tag{1.5.13}$$

pour tout $u \in D(K_V)$ où $L(s) = \frac{s+1}{\log(s+1)}$ pour tout $s \geq 1$.

Soulignons tout d'abord que l'estimation sous-elliptique établie dans le théorème 1.5.4 est la même obtenue dans le cadre polynomial du théorème 1.5.3.

Corollaire 1.5.3. *L'opérateur de Kramers-Fokker-Planck K_V avec potentiel $V(q)$ vérifiant l'hypothèse 6 est à résolvante compacte.*

Il est important de noter que le résultat du corollaire n'est pas valide dans le cas d'un polynôme homogène de degré deux avec hessienne dégénérée. Dans ce cas avec hessienne dégénérée en effet, l'opérateur de Kramers-Fokker-Planck est invariant par translation dans la direction du noyau de la hessienne par calcul direct et donc ne peut être à résolvante compacte.
Mentionnons par ailleurs que le corollaire est en accord avec les résultats de Wei-Xi Li [Li][Li2] et ceux de Helffer-Nier sur Laplacien de Witten avec potentiel homogène [HeNi1].

Pour démontrer le théorème 1.5.4, nous nous appuyons sur les arguments suivant :

• Nous considérons tout d'abord une partition dyadique suivant la variable de position $q \in \mathbb{R}^d$ (voir proposition 4.2.1 du chapitre 4),

$$\sum_{j\geq -1} \chi_j^2(q) = \tilde{\chi}_{-1}^2(2|q|) + \tilde{\chi}_0^2(2|q|) + \sum_{j\geq 1} \widetilde{\chi}^2(2^{-j}|q|) = 1 \tag{1.5.14}$$

où les fonctions de troncature $\widetilde{\chi}_0, \tilde{\chi}$ et $\widetilde{\chi}_{-1}$ appartiennent respectivement à $\mathcal{C}_0^\infty(\left]\frac{3}{4},\frac{8}{3}\right[)$, $\mathcal{C}_0^\infty(\left]\frac{3}{4},\frac{8}{3}\right[)$ et $\mathcal{C}_0^\infty(\left[0,\frac{4}{3}\right[)$. Il existe alors une constante uniforme $c>0$ tel que

$$(1+4c)\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + c\|u\|^2_{L^2(\mathbb{R}^{2d})} \geq \sum_{j\geq -1} \|K_V u_j\|^2_{L^2(\mathbb{R}^{2d})} , \tag{1.5.15}$$

pour tout $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ avec $u_j := \chi_j u$.

Par homogénéité du potentiel V, on peut réécrire l'inégalité (1.5.15) comme suit

$$(1+4c)\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + c\|u\|^2_{L^2(\mathbb{R}^{2d})} \geq \sum_{j\geq -1} \|K_{j,V} v_j\|^2_{L^2(\mathbb{R}^{2d})} , \tag{1.5.16}$$

où

$$K_{j,V} = p(h^{\frac{1}{2(r-1)}}\partial_q) - h^{-\frac{1}{2}}\partial_q V(q)\partial_p + \frac{1}{2}(-\Delta_p + p^2) , \qquad h = 2^{-2(r-1)j} ,$$

$$v_j(q,p) = 2^{\frac{jd}{2}} u_j(2^j q, p) , \qquad \text{supp } v_j \subset \overline{\mathcal{C}} := \left\{ q \in \mathbb{R}^d,\ \frac{3}{4} \leq |q| \leq \frac{8}{3} \right\} .$$

• L'idée est ensuite d'introduire l'ensemble des zéros du gradient de $V(q)$,

$$K_0 := \left\{ q \in \overline{\mathcal{C}} ,\quad \partial_q V(q) = 0 \right\} ,$$

qui est compacte par continuité de l'application $q \mapsto \partial_q V(q)$ sur la couronne fixe $\overline{\mathcal{C}}$

Comme $q \mapsto \frac{\mathrm{Tr}_{-,V}(q)}{1+\mathrm{Tr}_{+,V}(q)}$ est uniformément continue sur tout voisinage compact de K_0 , il existe $\varepsilon_1 > 0$ tel que

$$d(q, K_0) \leq \epsilon_1 \Rightarrow \frac{\mathrm{Tr}_{-,V}(q)}{1+\mathrm{Tr}_{+,V}(q)} \geq \frac{\epsilon_0}{2} \,, \tag{1.5.17}$$

où $\epsilon_0 := \min\limits_{q \in K_0} \frac{\mathrm{Tr}_{-,V}(q)}{1+\mathrm{Tr}_{+,V}(q)}$.

D'autre part, par définition de K_0 et par continuité de $q \mapsto \partial_q V(q)$ sur $\overline{\mathcal{C}}$, il existe une constante $\epsilon_2 > 0$ (qui dépend de ϵ_1) tel que

$$\forall\, q \in \overline{\mathcal{C}} \;,\;\; d(q, K_0) \geq \epsilon_1 \Rightarrow |\partial_q V(q)| \geq \epsilon_2 \;. \tag{1.5.18}$$

Ces remarques nous ramènent à considérer un nouveaux découpage suivant la variable de position $q \in \mathbb{R}^d$, à savoir une partition de l'unité localement finie

$$\sum_{k \in K_h} \theta_{k,h}(q)^2 = 1$$

au voisinage de $\overline{\mathcal{C}}$, où pour tout $k \in K_h := \left\{ k \in \mathbb{Z}^d,\; |k| \leq \frac{3}{|\ln(h)|h^\nu} \right\}$,

$$q_{k,h} = |\ln(h)|h^\nu k \,, \qquad \mathrm{supp}\, \theta_{k,h} \subset B(q_{k,h}, |\ln(h)|h^\nu) \;, \qquad \theta_{k,h} \equiv 1 \;\text{ in }\; B(q_{k,h}, \frac{1}{2}|\ln(h)|h^\nu) \;,$$

avec ν est paramètre fixé tel que

$$\frac{3}{16} + \frac{1}{16(r-1)} < \nu \leq \frac{1}{8} + \frac{3}{8(r-1)} \;. \tag{1.5.19}$$

On considère ensuite l'ensemble

$$I(\epsilon_1) = \{k \in K_h \;,\;\; \mathrm{supp}\, \theta_{k,h} \subset \{q \in \mathcal{C} \;,\;\; d(q, K_0) \geq \epsilon_1\}\} \;.$$

L'idée derrière l'introduction de la deuxième partition de l'unité est de pouvoir analyser et bien contrôler les deux cas suivant, en vue d'utiliser une approximation quadratique.

Cas **1** $k \notin I(\epsilon_1)$. Dans cette situation, le support de la fonction de troncature $\theta_{k,h}$ peut intersecter l'ensemble des zéros du gradient de V.

Cas **2** $k \in I(\epsilon_1)$. Dans ce cas, le gradient de V est non nul pour tout q dans le support de $\theta_{k,h}$.

On renvoie ici le lecteur au chapitre 4 pour trouver davantage de précisions sur l'analyse des deux situations sur-mentionnées.

1.6 Conclusion et perspectives

L'étude des opérateurs de Kramers-Fokker-Planck sur divers espaces a donné lieu à de trop nombreuses publications pour qu'on puisse espérer en faire une liste exhaustive. Beaucoup de questions ont émergé, ont été résolues ou sont restées partiellement ouvertes dans des cadres géométriques variés. Parmi lesquelles, l'analyse de la conjecture de Helffer-Nier :

Conjecture 1.6.1. *L'opérateur K_V est à résolvante compacte si et seulement si $\Delta_V^{(0)}$ est à résolvante compacte.*

La partie nécessaire de cette conjecture étant déjà établie par Helffer et Nier (cf. [HeNi], Théorème 1.1]), l'objectif de cette thèse était en particulier de considérer des exemples et des situations pour lesquelles l'analyse de la compacité pour Laplacien de Witten est bien développée et voir si l'implication inverse de la conjecture reste encore vraie.

Dans l'article de Helffer et Nier [HeNi], les auteurs ont, pour la première fois, l'idée d'utiliser des critères d'hypoellipticité maximale, dont la notion est développée par Helffer et Nourrigat [HeNo][Nou], pour en déduire un critère de compacité de la résolvante pour Laplacien de Witten.

Ce dernier critère, étant basé en particulier sur une récurrence sur le degré du potentiel polynomial, l'idée de cette thèse était de développer une telle analyse systématique pour l'opérateur de Kramers-Fokker-Planck. Naturellement, notre point de départ était l'étude du cas d'un potentiel polynomial avec degré inférieur ou égal à 2. Cette étape d'étude basique, a abouti à des estimations sous-elliptiques optimales avec un contrôle uniforme des constantes. C'est l'objet de l'article [BNV].

L'apparition d'une structure quaterionique dans [BNV], qui a fortement simplifié l'étude, ouvre la voie vers des possibles pistes de recherche d'applications pour autres modèles d'opérateurs à symboles quadratiques.

Une fois le cas des polynômes de degré inférieur ou égal à 2 est soigneusement traité, nous avons poussé le degré du polynôme en considérant des potentiels de degré supérieur ou égal à 3 mais dont les modèles asymptotiques à l'infini après changement d'échelle sont de degré inférieur ou égal à 2. Dans cette situation, malgré les corrections logarithmiques apparaissant dans le cas quadratique, nous avons pu établir une estimation sous-elliptique pour K_V, très proche d'être optimale grâce essentiellement au théorème de Tarski-Seidenberg. C'est le contexte de l'article [Ben].

Avec ce dernier travail, nous avons traité quasiment tous les exemples présentés par Helffer et Nier dans [HeNi] pour comparer le Laplacien de Witten et l'opérateur de Kramers-Fokker-Planck sauf le cas de potentiels homogènes, non nécéssairement polynomiaux, de degré supérieur à 2. Ce dernier cas est partiellement traité dans [Ben1] et a abouti à des meilleurs résultats.

Après tout, la conjecture 1.6.1 de Helffer-Nier est encore loin d'être complètement résolue. Elle peut être décomposée en plusieurs questions qui n'ont reçu aucune réponse, parmi lesquelles :

1. Dans le cas d'un potentiel polynomial, est-il possible de montrer que l'opérateur de

Kramers-Fokker-Planck K_V est à résolvante compacte s'il vérifie les mêmes critères développés par Helffer-Nier dans le cas de Laplacien de Witten (cf. Théorème 10.16 dans

[HeNi]) ?

2. Peut-on étendre le résultat du théorème 1.5.4 (établi dans l'article [Ben1]) dans le cas d'un potentiel homogène de degré $r \geq 6$ dans $\mathcal{C}^\infty(\mathbb{R}^d \setminus \{0,1\})$? Que se passe t-il en particulier pour le cas polynomial $V(q) = -q_1^2(q_1^2 + q_2^2)^n$ avec $n \in \mathbb{N} \setminus \{0,1\}$?

3. L'équivalence (1.6.1) est-elle vraie pour un potentiel général $V \in \mathcal{C}^\infty(\mathbb{R}^d)$?

Ces questions ouvrent la porte à d'autres travaux de recherche et d'autres possibles pistes d'extension.

Chapitre 2

Structure quaternionique et analyse des opérateurs de KFP avec polynômes de degrés inférieur à 3 (article rédigé en anglais)

Article [BNV], rédigé en anglais, soumis pour publication.

Quaternionic structure and analysis of some Kramers-Fokker-Planck operators

Abstract

The present article is concerned with global subelliptic estimates for Kramers-Fokker-Planck operators with polynomials of degree less than or equal to two. The constants appearing in those estimates are accurately formulated in terms of the coefficients, especially when those are large.

Key words : subelliptic estimates, compact resolvent, Kramers-Fokker-Planck operator, quaternions, Bargmann transform.
MSC-2010 : 35Q84, 35H20, 35P05, 47A10, 14P10, 20G20

Contents

2.1 Introduction and main results

In this work, we consider the Kramers-Fokker-Planck operator given by

$$K_V = p\partial_q - \partial_q V(q)\partial_p + \frac{1}{2}(-\Delta_p + p^2), \quad (q,p) \in \mathbb{R}^{2d}\,, \tag{2.1.1}$$

where q denotes the space variable, p denotes the velocity variable, $x.y = \sum\limits_{j=1}^{d} x_j y_j$, $x^2 = \sum\limits_{j=1}^{d} x_j^2$ and the potential $V(q) = \sum\limits_{|\alpha|\leq 2} V_\alpha q^\alpha$ is a real-valued polynomial function on $\mathbb{R}^d$ with $d^\circ V = 2$. After making an orthogonal change of variables one may assume that its Hessian matrix is

$$\text{Hess } V = \begin{pmatrix} \nu_1 & 0 & \dots & 0 \\ 0 & \nu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_d \end{pmatrix} .$$

The constant term V_0 does not appear in K_V and can be set to 0 and we distinguish two cases :

- If Hess V is non-degenerate, a translation in q reduces the problem to

$$V(q) = \sum_{i=1}^{d} \frac{\nu_i}{2} q_i^2 \,. \tag{2.1.2}$$

- If Hess V is degenerate, a good choice of orthonormal basis and a translation give :

$$V(q) = \lambda_1 q_1 + \sum_{i=2}^{d} \frac{\nu_i}{2} q_i^2 \,, \tag{2.1.3}$$

where λ_1 is invariantly defined by $|\lambda_1| = \min\limits_{q\in\mathbb{R}^d} |\nabla\, V(q)| \geq 0$.

As established in [HeNi] (see Proposition 5.5, page 44), the non-selfadjoint operator K_V is maximal accretive when endowed with the domain $D(K_V) = \left\{u \in L^2(\mathbb{R}^{2d}), \quad K_V u \in L^2(\mathbb{R}^{2d})\right\}$. The question about the compactness of the resolvent combined with subelliptic estimates is intimately related with the return to the equilibrium or exponential decay estimates. As pointed out in [HerNi] and [HeNi], the analysis of K_V is also strongly related to the one of the Witten Laplacian $\Delta_V^{(0)} = -\Delta_q + |\nabla V(q)|^2 - \Delta V(q)$ for which maximal hypoelliptic techniques developed by

Helffer and Nourrigat in [HeNo] provide accurate criteria for general polynomial potentials $V(q)$.

Within this maximal hypoelliptic analysis of $\Delta_V^{(0)}$ there is a recurrent interplay between qualitative estimates and quantitative estimates in terms of the size of the coefficients of the polynomial $V(q)$. The general idea is that the study of the operator $\Delta_V^{(0)}$ as $q \to \infty$ when V is a degree r polynomial, is reduced to a quantitative version of subelliptic estimates for $\Delta_{\tau\widetilde{V}}^{(0)}$, where $\widetilde{V}$ belongs to some family of polynomials related to V with degree less than r, and τ is a large parameter.

"Quantitative estimates" means that we consider subelliptic estimates with a good and optimal control of the constant with respect to the parameter τ. Remember also that the compactness of the resolvent on $\Delta_V^{(0)}$ obtained by maximal hypoelliptic techniques, relies on the fact that no polynomial $\widetilde{V}$ of the family associated with V admits a local minimum. It shows in particular that the compactness of the resolvent of $\Delta_{+V}^{(0)}$ and $\Delta_{-V}^{(0)}$ differ and the first non trivial example comes with the potential $\pm V(q_1, q_2) = \pm q_1^2 q_2^2$ in $\mathbb{R}^2$. For the Kramers-Fokker-Planck operator K_V, no sufficient condition until the recent work by Wei-Xi Li [Li2] exhibited such a different behavior.

We hope to develop the same strategy for the non self-adjoint operator K_V as for the Witten Laplacian $\Delta_V^{(0)}$, namely try to get the optimal subelliptic estimates for some class of polynomial functions $V(q)$, by making use of quantitative estimates for some lower degree polynomials. The case $d^\circ V \leq 2$ for which the Weyl symbol of K_V is a polynomial of degree ≤ 2 in the variable (q, p, ξ_q, ξ_p) allows a lot of exact analytic calculations and was already deeply studied in [Hor][Sjo][HiPr][Vio][Vio2][AlVi]. Nevertheless exploiting those exact analytic expressions for the semigroup kernel or symbol (Mehler's type formulas) or for the spectrum does not solve completely the question of optimal quantitative subelliptic estimates for the non self-adjoint operator K_V. The semiclassical regime which can be handled quite accurately via symbolic calculus gives results after rescaling essentially when the transport part $p\partial_q - \partial_q V(q)\partial_p$ is small compared to the diffusive–friction part $\frac{-\Delta_p + p^2}{2}$.

Actually, we are mainly interested in the other regime where the Hamiltonian dynamics is stronger than the diffusive and friction part. The difficulty then appears clearly, because understanding the operator K_V requires the understanding of the Hamiltonian dynamics associated with $p\partial_q - \partial_q V(q)\partial_p$ which, for a general polynomial V exhibits a rich variety of phenomena, and which, for a polynomial of degree ≤ 2, already contains the three types of dynamics : a) elliptic (bounded trajectories when V is a positive definite quadratic form); b) hyperbolic (trajectories escaping exponentially quickly in time to infinity when V is a negative definite quadratic form); and c) parabolic (trajectories escaping polynomially quickly in time to infinity when V is linear).

At a more fundamental level, understanding the operator K_V when the transport term is dominant also proceeds in the same direction as Bismut's program : in [Bis1], Bismut introduced his hypoelliptic Laplacian in order to interpolate Morse theory (in the high diffusion-friction regime via the Witten Laplacian) and the topology of loop spaces (dominant transport term). The difficult part with a dominant transport term was understood only for the geodesic flow on symmetric spaces making use of the specific algebraic structure in [Bis2].

With this respect our simpler case also requires a better understanding of the underlying algebra, and it appeared that after using the general FBI-techniques the Kramers-Fokker-Planck evolution with quadratic potentials, even in dimension $d = 1$, is reduced to some linear dynamics on $\mathbb{C}^4$ which are easily computed after elucidating some quaternionic structure. In this specific case, this also completes the unfruitful attempts in [HeNi], Section 9.1, to exhibit some useful nilpotent Lie algebra structure for Kramers-Fokker-Planck operators. Actually, quaternions and Pauli matrices are related to the $\mathfrak{su}(2)$ Lie algebra, so the Lie algebra structure decomposition useful to the analysis of Kramers-Fokker-Planck operators with polynomial potentials is certainly not nilpotent.

Denoting

$$O_p = \frac{1}{2}(D_p^2 + p^2)$$

and

$$X_V = p\partial_q - \partial_q V(q)\partial_p \ ,$$

we can rewrite the Kramers-Fokker-Planck operator K_V defined in (2.1.1) as $K_V = X_V + O_p$.
In this work, we are mainly based on recent publications by Hitrik, Pravda-Starov, Viola, and Aleman [AlVi], [Vio2], and [HPV2] which deal with operators having polynomial symbols of degree less than or equal to two.

Notations :

$$\begin{aligned}
\mathrm{Tr}_+ &= \sum_{\nu_i > 0} \nu_i \ , \\
\mathrm{Tr}_- &= -\sum_{\nu_i \leq 0} \nu_i \ , \\
A &= \max\{(1+\mathrm{Tr}_+)^{2/3}, 1+\mathrm{Tr}_-\} \ , \\
B &= \max\{|\lambda_1|^{4/3}, \frac{1+\mathrm{Tr}_-}{(\log(2+\mathrm{Tr}_-))^2}\} \ .
\end{aligned}$$

The main goal of this work is the following subelliptic estimates.

Theorem 2.1.1. *Let $V(q)$ be a potential as in (2.1.2) or (2.1.3). Then there exists a constant $c > 0$ that does not depend on V such that the subelliptic estimate with a* *remainder term*

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + A\|u\|^2_{L^2(\mathbb{R}^{2d})} \geq c\Big(\|O_p u\|^2_{L^2(\mathbb{R}^{2d})} + \|X_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle \partial_q V(q)\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle D_q\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})}\Big) \tag{2.1.4}$$

holds for all $u \in D(K_V)$.

Theorem 2.1.2. *Let* $V(q)$ *as in (2.1.2) or (2.1.3). Then there is a constant* $c > 0$ *independent of the polynomial* V *so that the subelliptic estimate without a remainder*

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{c}{1+\frac{A}{B}}\Big(\|O_p u\|^2_{L^2(\mathbb{R}^{2d})} + \|X_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle \partial_q V(q)\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle D_q\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})}\Big)$$

holds for all $u \in D(K_V)$.

The two previous Theorems are both consequences of the following result.

Proposition 2.1.3. *Let* $V(q)$ *as in (2.1.2) or (2.1.3). Then there exists a constant* $c > 0$, *independent of* V, *such that*

$$\sum_{i=1}^{d}\Big(\||D_{q_i}|e^{-t(K_V+\sqrt{A})}\|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} + \||\partial_{q_i} V(q)|e^{-t(K_V+\sqrt{A})}\|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))}\Big) \leq \frac{c}{t^{\frac{3}{2}}}$$

for all $t > 0$.

Moreover,

$$\|K_V^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} \leq \int_0^{+\infty} \|e^{-tK_V}\|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} dt \leq \frac{c}{\sqrt{B}}\ .$$

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2.2 Reduction to a one-dimensional problem

Interpolation results of Lunardi (see Remark 5.11, Theorem 5.12 and Corollary 5.13 in [Lun]) show that the first inequality of Proposition 2.1.3 combined with the fact that

$$|\mathrm{Re}\langle [O_p, X_V]u, u\rangle| \leq C_\epsilon(\|\ |D_q|^{\frac{2}{3}}u\|^2 + \|\ |\partial_q V(q)|^{\frac{2}{3}}u\|^2) + \epsilon\|O_p u\|^2$$

for all $u \in D(K_V)$ (where $\epsilon > 0$ is small enough), implies the subelliptic estimates given in Theorem 2.1.1. Theorem 2.1.2 is then a consequence of Theorem 2.1.1 and the second inequality of Proposition 2.1.3.
Details are given below.

Proof of Proposition 2.1.3. Since this result is expressed in terms of the semigroup, it can be studied by a separation of variables for a potential of the form (2.1.2) or (2.1.3). Actually e^{-tK_V} is a product of commutative contraction semigroups with respect to each variable (q_j, p_j), and it suffices to write

$$\sum_{i=1}^{d} \|M_i e^{-t(K_V+\sqrt{A})}\|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} \leq \sum_{i=1}^{d} \|M_i e^{-t(K_{V_i(q_i)}+\alpha_i)}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \quad M_i = |D_{q_i}| \text{ or } M_i = |\partial_{q_i} V(q)| \,,$$

where $V_i(q_i)$ denotes the one-dimensional potential in the q_i variable, with $V_1(q_1) = \frac{\nu_1 q_1^2}{2}$ or $V_1(q_1) = \lambda_1 q_1$, $V_i(q_i) = \frac{\nu_i q_i^2}{2}$ for $i \geq 2$, $\alpha_i = |\nu_i|^{1/2}$ if $\nu_i < 0$, $\alpha_i = \nu_i^{1/3}$ if $\nu_i > 0$ and $\alpha_i = 0$ if $\partial^2_{q_i} V = 0$. The second estimate of Proposition 2.1.3 is even simpler. Hence Proposition 2.1.3 will be the result of a careful analysis of the three one-dimensional potentials $V(q) = \pm\frac{\nu q^2}{2}$, $\nu > 0$, and $V(q) = \lambda_1 q$, $\lambda_1 \in \mathbb{R}$, developed in the next sections. □

Proof of Theorem 2.1.1. In this proof we use nearly the same notations as in [Lun] (Remark 5.11, Theorem 5.12 and Corollary 5.13). Set

$$\begin{aligned} T(t) &= e^{-t(\sqrt{A}+K_V)} \,, \\ L^2 &= L^2(\mathbb{R}^{2d}) \,, \\ E &= \{u \in L^2(\mathbb{R}^{2d}), qu, \partial_q u \in L^2(\mathbb{R}^{2d})\} \end{aligned}$$

where E is equipped with the norm

$$\|u\|_E^2 = \sum_{i=1}^{d} \left(\||D_{q_i}|u\|^2_{L^2(\mathbb{R}^{2d})} + \||\partial_{q_i} V(q)|u\|^2_{L^2(\mathbb{R}^{2d})} \right) + \|u\|^2_{L^2(\mathbb{R}^{2d})} \,.$$

Applying Lemma 2.3.4 and Proposition 2.3.1, we obtain by separation of variables

$$\|T(t)\|_{\mathcal{L}(L^2,E)} \leq \frac{c}{t^{\frac{3}{2}}} \quad \text{for all} \quad t > 0 \,.$$

If $m = 3$ and $\beta = \frac{1}{2}$, then by Theorem 5.12 in [Lun], one has the following embedding of real interpolation spaces

$$\left(L^2, D\Big((\sqrt{A}+K_V)^3\Big)\right)_{\frac{\theta}{2},p} \subset \left(L^2, E\right)_{\theta,p} \tag{2.2.1}$$

for all $\theta \in (0,1),\ p \in [1,+\infty]$. In particular for $\theta = \frac{2}{3}$,

$$[L^2, E]_{\frac{2}{3}} = (L^2, E)_{\frac{2}{3},2} = \{u \in L^2,\ |D_{q_i}|^{\frac{2}{3}} u \in L^2,\ |\partial_{q_i} V(q)|^{\frac{2}{3}} u \in L^2 \text{ for all } 1 \leq i \leq d\} \ , \tag{2.2.2}$$

where the complex interpolation space $[L^2, E]_{\frac{2}{3}}$ is equipped with the norm

$$\|u\|_{[L^2,E]_{\frac{2}{3}}} = \sum_{i=1}^{d} \Big(\|\ |D_{q_i}|^{\frac{2}{3}} u\|^2_{L^2(\mathbb{R}^{2d})} + \|\ |\partial_{q_i} V(q)|^{\frac{2}{3}} u\|^2_{L^2(\mathbb{R}^{2d})} \Big) + \|u\|^2_{L^2(\mathbb{R}^{2d})} \ .$$

Moreover in view of Remark 5.11 and Corollary 5.13 in [Lun],

$$\Big(L^2, D\Big((\sqrt{A} + K_V)^3 \Big) \Big)_{\frac{1}{3},2} = D(\sqrt{A} + K_V) \tag{2.2.3}$$

(since L^2 is a Hilbert space and $(\sqrt{A} + K_V)$ is a maximal accretive operator). Thus taking into account (2.2.1), (2.2.2) and (2.2.3)

$$D(\sqrt{A} + K_V) \subset \{u \in L^2,\ |D_{q_i}|^{\frac{2}{3}} u \in L^2,\ |\partial_{q_i} V(q)|^{\frac{2}{3}} u \in L^2 \text{ for all } 1 \leq i \leq d\} \ .$$

Hence there exists a constant $c > 0$ such that

$$\sum_{i=1}^{d} \Big(\|\ |D_{q_i}|^{\frac{2}{3}} u\|^2 + \|\ |\partial_{q_i} V(q)|^{\frac{2}{3}} u\|^2_{L^2} \Big) \leq c \|(\sqrt{A} + K_V) u\|^2_{L^2} \tag{2.2.4}$$

holds for all $u \in D(K_V)$.

Write for $u \in D(K_V)$,

$$\|(\sqrt{A} + K_V) u\|^2_{L^2} = \|(\sqrt{A} + O_p) u\|^2_{L^2} + \|X_V u\|^2_{L^2} + 2\mathbb{R}\text{e}\langle [O_p, X_V] u, u\rangle \ , \tag{2.2.5}$$

so

$$\begin{aligned}
|2\mathbb{R}\text{e}\langle [O_p, X_V] u, u\rangle| &\leq \sum_{i=1}^{d} \Big| \mathbb{R}\text{e}\langle u, \Big(D_{p_i} D_{q_i} + p_i \partial_{q_i} V(q) \Big) u\rangle \Big| \\
&\leq \sum_{i=1}^{d} \Big(|\mathbb{R}\text{e}\langle u, (D_{p_i} D_{q_i}) u\rangle| + |\mathbb{R}\text{e}\langle u, p_i \partial_{q_i} V(q) u\rangle| \Big) \\
&\leq \sum_{i=1}^{d} \Big(\langle u, |p_i| |\partial_{q_i} V(q)| u\rangle + \langle u, |D_{p_i}| |D_{q_i}| u\rangle \Big) \\
&\leq \sum_{i=1}^{d} \Big(\epsilon \langle u, |p_i|^4 u\rangle + c_\epsilon \langle u, |\partial_{q_i} V(q)|^{\frac{4}{3}} u\rangle + \epsilon \langle u, |D_{p_i}|^4 u\rangle + c_\epsilon \langle u, |D_{q_i}|^{\frac{4}{3}} u\rangle \Big) \qquad (2.2.6) \\
&\leq c \Big(\epsilon \|O_p u\|^2_{L^2} + c_\epsilon \|(\sqrt{A} + K_V) u\|^2_{L^2} \Big) \ ,
\end{aligned}$$

where (2.2.6) is due to the Young inequality $ts \leq \frac{1}{4}t^4 + \frac{3}{4}s^{\frac{3}{4}}$ for all $t, s \geq 0$ and the last line is a consequence of (2.2.4).

Therefore, combining the last inequality with (2.2.5), we obtain

$$\begin{aligned}\|(\sqrt{A}+K_V)u\|_{L^2}^2 &\geq \|(\sqrt{A}+O_p)u\|_{L^2}^2 + \|X_V u\|_{L^2}^2 - c\Big[\epsilon\|O_p u\|_{L^2}^2 + c_\epsilon\|(\sqrt{A}+K_V)u\|_{L^2}^2\Big] \\ &\geq (1-c\epsilon)\|(\sqrt{A}+O_p)u\|_{L^2}^2 + \|X_V u\|_{L^2}^2 - cc_\epsilon\|(\sqrt{A}+K_V)u\|_{L^2}^2\end{aligned}$$

for all $u \in D(K_V)$.

To complete the proof, it is enough to use the above inequality with (2.2.4) and the fact that

$$2\Big(A\|u\|_{L^2}^2 + \|K_V u\|_{L^2}^2\Big) \geq \|(\sqrt{A}+K_V)u\|_{L^2}^2$$

for all $u \in D(K_V)$. □

Proof of Theorem 2.1.2. If $\mathrm{Tr}_- + |\lambda_1| \neq 0$, by Proposition 2.1.3, there exists a constant $c > 0$ such that

$$\begin{aligned}\|K_V^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} = \|\int_0^{+\infty} e^{-tK_V}dt\|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))} &\leq \int_0^{+\infty}\|e^{-tK_V}\|_{\mathcal{L}(L^2(\mathbb{R}^{2d}))}dt \\ &\leq \frac{c}{\sqrt{B}}\ .\end{aligned}$$

Consequently, for all $u \in D(K_V)$,

$$\|u\|_{L^2(\mathbb{R}^{2d})}^2 \leq \frac{c^2}{B}\|K_V u\|_{L^2(\mathbb{R}^{2d})}^2\ .$$

Combining the above inequality, along with (3.1.5), one gets immediately the global subelliptic estimates

$$\begin{aligned}\|K_V u\|_{L^2(\mathbb{R}^{2d})}^2 \geq \frac{c'}{1+\frac{A}{B}}\Big(&\|O_p u\|_{L^2(\mathbb{R}^{2d})}^2 + \|X_V u\|_{L^2(\mathbb{R}^{2d})}^2 \\ &+ \|\langle\partial_q V(q)\rangle^{2/3}u\|_{L^2(\mathbb{R}^{2d})}^2 + \|\langle D_q\rangle^{2/3}u\|_{L^2(\mathbb{R}^{2d})}^2\Big) \quad (2.2.7)\end{aligned}$$

for all $u \in D(K_V)$.

□

2.3 Subelliptic estimates with remainder for non-degenerate one-dimensional potentials

The operator K_V with a potential $V(q) = \mp\frac{\nu q^2}{2} = -e^{2i\alpha}\frac{\nu q^2}{2}$ (where $\nu > 0$ is a parameter and $\alpha \in \{0, \frac{\pi}{2}\}$), is unitarily equivalent to

$$\begin{aligned} K_{\nu,\alpha} &= \frac{1}{2}(-\partial_p^2 + p^2) + \left(e^{i\alpha}\sqrt{\nu}\right)e^{-i\alpha}\left(p\partial_q + e^{2i\alpha}q\partial_p\right) \\ &= O_p + zX_\alpha \end{aligned}$$

where $z := e^{i\alpha}\sqrt{\nu}$ and $X_\alpha := i(e^{-i\alpha}pD_q + e^{i\alpha}qD_p)$. Actually introducing the possibly complex parameter z allows us to use the same computations for both cases because they involve entire functions of $z \in \mathbb{C}$. On the other hand, some identities make sense only when $\alpha \in \left\{0, \frac{\pi}{2}\right\}$, particularly those involving O_q (the harmonic oscillator in q) or the symplectic product. Below we sum up the cases to be studied :

$V(q)$	α	z
$\frac{-\nu q^2}{2}$	0	$\sqrt{\nu}$
$\frac{+\nu q^2}{2}$	$\frac{\pi}{2}$	$i\sqrt{\nu}$

In this one dimensional case, we use the following notations :

$$\begin{aligned} &O_q = \frac{1}{2}(D_q^2 + q^2) \quad , \quad O_{e^{i\alpha}q} = \frac{1}{2}(e^{-2i\alpha}D_q^2 + e^{2i\alpha}q^2) \quad , \quad O_p = \frac{1}{2}(D_p^2 + p^2)\,, \\ &X_\alpha = i(e^{-i\alpha}pD_q + e^{i\alpha}qD_p) \quad , \quad Y_\alpha = i(e^{i\alpha}pq - e^{-i\alpha}D_qD_p)\,, \end{aligned}$$

where $\alpha \in \left\{0, \frac{\pi}{2}\right\}$ and $O_{e^{i\alpha}q} = e^{2i\alpha}O_q$ in the final applications.

The Hamilton map written as a matrix equals

$$H_Q := \begin{pmatrix} \mathbf{q}''_{\xi x} & \mathbf{q}''_{\xi\xi} \\ -\mathbf{q}''_{xx} & -\mathbf{q}''_{x\xi} \end{pmatrix},$$

where $\mathbf{q}(q, p, \xi_q, \xi_p)$ is the Weyl-symbol of the operator Q, meaning
$Q = \mathbf{q}^w(q, p, D_q, D_p) = q^w(x, D_x)\,,\ x = (q, p)$:

$$Qu(x) = \int_{\mathbb{R}^{4d}} e^{i(x-x')\cdot\xi} q\left(\frac{x + x'}{2}, \xi\right) u(x')\ \frac{d\xi}{(2\pi)^{2d}} dx'\,.$$

We use the nonstandard notation where H_Q is indexed by the operator Q instead of the symbol $\mathbf{q}$ to avoid introducing redundant notation for the symbols of each operator considered and to reserve q for the position variable $q \in \mathbb{R}^d$ throughout.

Noticing that O_p, O_q, $O_{e^{i\alpha}q}$, X_α, Y_α and $K_{\nu,\alpha}$ have quadratic symbols, the corresponding Hamilton maps are written accordingly H_{O_p}, H_{O_q}, $H_{O_{e^{i\alpha}q}}$, H_{X_α}, H_{Y_α} and $H_{K_{\nu,\alpha}}$. Let $E := H_{O_{e^{i\alpha}q}-O_p}$, $I := H_{-O_{e^{i\alpha}q}-O_p}$, $J := H_{-X_\alpha}$, and $K := H_{Y_\alpha}$ denote respectively the Hamiltonian matrices associated to the operators $O_{e^{i\alpha}q} - O_p$, $-O_{e^{i\alpha}q} - O_p$, $-X_\alpha$ and Y_α. Then one has

$$E = \begin{pmatrix} 0 & 0 & e^{-2i\alpha} & 0 \\ 0 & 0 & 0 & -1 \\ -e^{2i\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & 0 & -e^{-2i\alpha} & 0 \\ 0 & 0 & 0 & -1 \\ e^{2i\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & -ie^{-i\alpha} & 0 & 0 \\ -ie^{i\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & ie^{i\alpha} \\ 0 & 0 & ie^{-i\alpha} & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & -ie^{-i\alpha} \\ 0 & 0 & -ie^{-i\alpha} & 0 \\ 0 & -ie^{i\alpha} & 0 & 0 \\ -ie^{i\alpha} & 0 & 0 & 0 \end{pmatrix}.$$

Note that E commutes with I, J, K and $IJ = K$ with the relations

$$E^2 = I^2 = J^2 = K^2 = -1 \text{ for all } \alpha \in \mathbb{R}\,, \tag{2.3.1}$$

$$\text{and} \qquad \overline{E} = E,\ \overline{I} = I,\ \overline{J} = -e^{2i\alpha}J,\ \overline{K} = -e^{2i\alpha}K \quad \text{when } \alpha \in \{0, \pi/2\}\,. \tag{2.3.2}$$

These relations, $IJ = K$, and (2.3.1) ensure that $(1, I, J, K)$ can be considered algebraically as a basis of (bi-)quaternions. Note in particular that

$$H_{O_p} = -\frac{1}{2}(E + I) \quad , \quad H_{X_\alpha} = -J$$
$$H_{Y_\alpha} = K \quad , \quad H_{K_{\nu,\alpha}} = -\frac{1}{2}(E + I + 2zJ)$$

for all $\alpha \in \mathbb{R}$, while the relations

$$\begin{pmatrix} 0 & -\mathrm{Id}_{\mathbb{R}^2} \\ \mathrm{Id}_{\mathbb{R}^2} & 0 \end{pmatrix} = \sin(\alpha)E + \cos(\alpha)I \quad , \quad H_{O_q} = \frac{e^{2i\alpha}}{2}(E - I)$$

hold for $\alpha \in \{0, \frac{\pi}{2}\}$.

The commutation property with the matrix E can be interpreted as follows at the operator level : consider the two commutators $\left[O_p, X_\alpha\right] = iY_\alpha$ and $\left[O_{e^{i\alpha}q}, X_\alpha\right] = iY_\alpha$. Then the operator $O_{e^{i\alpha}q} - O_p$ commutes with O_p and X_α. Once this reduction is done, the quaternionic structure can be guessed as well from the operator level after computing all the commutators of O_p, $O_{e^{i\alpha}q}$, X_α and Y_α.

2.3.1 General estimate when $V(q) = \pm\frac{\nu q^2}{2}$, $\nu > 0$

Proposition 2.3.1. *Let $\nu > 0$ be a parameter and $\alpha \in \left\{0, \frac{\pi}{2}\right\}$. There exists a constant $C > 0$, independent of ν, such that*

$$\|\sqrt{\nu\, O_q}\; e^{-t(K_{\nu,\alpha}+\sqrt{\nu})}\|_{\mathcal{L}(L^2(\mathbb{R}^2)} \leq \frac{C}{t^{\frac{3}{2}}}$$

holds for all $t > 0$.

Lemma 2.3.2. *One can find a function $\delta_0(t) > 0$, specified below in* (2.3.8)(2.3.9)*, defined in $[0, +\infty[$ such that for all $\delta(t) \in [0, \delta_0(t)[$*

$$\|e^{\delta(t)O_q} e^{-tK_{\nu,\alpha}}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq 1$$

is satisfied for all $t > 0$.

Proof. The exact classical quantum correspondence, valid for $Q_j = q_j^w, j = 1, 2, 3$, when q_j are complex-valued quadratic forms, with associated Hamilton maps H_{Q_j} (alternative notation of H_{q_j} with our convention) and positive Hamilton flows $\exp H_{Q_j}$ (see [Hor][Vio]), says that

$$\exp H_{Q_1} \exp H_{Q_2} = \exp H_{Q_3} \iff e^{-iQ_1} e^{-iQ_2} = \pm e^{-iQ_3} .$$

We will determine conditions such that the canonical transformation

$$\exp H_{i\delta(t)O_q} \exp H_{-itK_{\nu,\alpha}}$$

is strictly positive in the sense defined in (2.3.5). Working from the Hamilton flow, one can therefore compute exactly ([Vio2], Proposition 4.8) a compact operator of the form e^{-iQ_2} for Q_2 quadratic such that

$$e^{-\delta(t)O_q} e^{-iQ_2} = e^{-i\delta(t)(i^{-1}O_q)} e^{-iQ_2} = \pm e^{-it(i^{-1}K_{\nu,\alpha})} = \pm e^{-tK_{\nu,\alpha}}$$

Applying this equality to the dense set of linear combinations of Hermite functions, this shows that $e^{-tK_{\nu,\alpha}}$ takes $L^2(\mathbb{R}^2)$ to the domain of $e^{\delta(t)O_q}$ with the estimate

$$\|e^{\delta(t)O_q} e^{-tK_{\nu,\alpha}}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} = \|e^{-iQ_2}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq 1.$$

We will compute

$$e^{i\delta(t)H_{O_q}} e^{-itH_{K_{\nu,\alpha}}}$$

which will be done by using biquatertionic expressions. The compactness of e^{-iQ_2}, and the fact that its norm is bounded by 1, is a consequence of the positivity condition (2.3.5) which will be checked explicitly.

Set, for all $t \geq 0$, $\kappa(t) = e^{-itH_{K_{\nu,\alpha}}}$ and $\kappa_0(\delta) = e^{i\delta(t)H_{O_q}}$, and consider the canonical transformation

$$\kappa(t) := e^{-itH_{K_{\nu,\alpha}}} = e^{i\frac{t}{2}(E+I+2zJ)}$$

for all $t \geq 0$. Let n_1 denote

$$n_1 = \sqrt{N(I+2zJ)} = \sqrt{1+4z^2} \neq 0 \text{ when } z \neq \pm\frac{i}{2}$$

such that $\hat{v} = \frac{I+2zJ}{n_1}$ satisfies $\hat{v}^2 = -1$.

Using the fact that E commutes with I and J, and the formula (2.A.1) ,

$$\begin{aligned}\kappa(t) = e^{i\frac{t}{2}E}e^{i\frac{t}{2}n_1\hat{v}} &= e^{i\frac{t}{2}E}\Big(\operatorname{ch}(\frac{tn_1}{2}) + i\ \frac{\operatorname{sh}(\frac{tn_1}{2})}{n_1}(I+2zJ)\Big)\\ &=: e^{i\frac{t}{2}E}\Big(C(t) + i\ S(t)(I+2zJ)\Big)\ . \qquad (2.3.3)\end{aligned}$$

The functions $\mathbb{R} \ni t \mapsto C(t)$ and $\mathbb{R} \ni t \mapsto S(t)$ do not depend on the choice of the square root $\sqrt{1+4z^2}$, because ch is an even function and sh an odd function. Moreover, they are real when $z \in \mathbb{R} \cup i\mathbb{R}$, which corresponds to $z = e^{i\alpha}\sqrt{\nu}$, $\alpha \in \{0, \frac{\pi}{2}\}$.

On the other hand,

$$\begin{aligned}\kappa_0(\delta) = e^{-i\delta(t)H_{O_q}} &= e^{\frac{i}{2}\delta(t)e^{2i\alpha}E}e^{-\frac{i}{2}\delta(t)e^{2i\alpha}I}\\ &= e^{\frac{i}{2}\delta(t)e^{2i\alpha}E}\Big(\operatorname{ch}(\frac{\delta(t)}{2}e^{2i\alpha}) - i\ \operatorname{sh}(\frac{\delta(t)}{2}e^{2i\alpha})I\Big)\ . \qquad (2.3.4)\end{aligned}$$

When $\sigma = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$ denotes the matrix of the symplectic form on $\mathbb{R}^{2\times 2}$, the equality $\sigma = \sin(\alpha)E + \cos(\alpha)I$ holds when $\alpha \in \{0, \frac{\pi}{2}\}$ (and only in those cases mod π). As established in [Vio2], it is possible to write $e^{\delta(t)O_q}e^{-tK_{\nu,\alpha}} = e^{-iQ_2}$ with $Q_2 = q_2^w$, with e^{-iQ_2} a compact operator, when the canonical transformation $\kappa_0\kappa$ satisfies the strict positivity condition

$$i\Big[\sigma\Big(\overline{\kappa_0\kappa z}, \kappa_0\kappa z\Big) - \sigma\Big(\overline{z}, z\Big)\Big] > 0 \text{ for all } \ z \in \mathbb{C}^4 \setminus \{0\}\ . \qquad (2.3.5)$$

This condition is equivalent to the condition that the Hermitian matrix

$$i\Big((\kappa_0\kappa)^*\sigma\kappa_0\kappa - \sigma\Big) = i\Big(\kappa^*\kappa_0^*\sigma\kappa_0\kappa - \sigma\Big) = i\Big(\kappa^*\kappa_0\sigma\kappa_0\kappa - \sigma\Big)\ ,$$

is positive definite, or equivalently that

$$\kappa_0(i\sigma)\kappa_0 - (\kappa^*)^{-1}(i\sigma)(\kappa)^{-1} = \kappa_0(\delta)(i\sigma)\kappa_0(\delta) - \kappa^*(-t)(i\sigma)\kappa(-t)$$

is positive definite.

Since E commutes with I, J and K, the spectral decomposition of E allows us to study 2-by-2 matrices instead of 4-by-4 matrices : $T_\pm^*(iE)T_\pm = \pm\mathrm{Id}$, where

$$T_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \mp ie^{\pm 2i\alpha} & 0 \\ 0 & \pm i \end{pmatrix}, \quad T_\pm^* T_\pm = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Letting

$$\widetilde{E} := T_\pm^* E T_\pm = \mp i\mathrm{Id} = \begin{pmatrix} \mp i & 0 \\ 0 & \mp i \end{pmatrix}, \quad \widetilde{I} := T_\pm^* I T_\pm = \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix},$$
$$\widetilde{J} := T_\pm^* J T_\pm = \begin{pmatrix} 0 & -ie^{-i\alpha} \\ -ie^{i\alpha} & 0 \end{pmatrix}, \quad \widetilde{K} := T_\pm^* K T_\pm = \begin{pmatrix} 0 & \pm e^{-i\alpha} \\ \mp e^{i\alpha} & 0 \end{pmatrix} ,$$

we get

$$T_\pm^* \kappa_0(\delta)(i\sigma)\kappa_0(\delta)T_\pm = e^{\pm\delta(t)e^{2i\alpha}}\Big[\pm\sin(\alpha)c(t) - s(t)\cos(\alpha) + i\Big(\cos(\alpha)c(t) \mp \sin(\alpha)s(t)\Big)\widetilde{I}\Big] , \tag{2.3.6}$$

where $c(t) = \mathrm{ch}(\delta(t)e^{2i\alpha})$ and $s(t) = \mathrm{sh}(\delta(t)e^{2i\alpha})$. Similarly,

$$\begin{aligned} T_\pm^*\kappa^*(-t)(i\sigma)\kappa(-t)T_\pm = e^{\mp t}\Big[&\pm\sin(\alpha)\Big(C^2(t) + (1-(2z)^2)S^2(t)\Big) - 2\cos(\alpha)C(t)S(t) \\ &+ i\cos(\alpha)\Big(C^2(t) + (1-(2z)^2)S^2(t)\Big)\widetilde{I} \mp 2i\sin(\alpha)C(t)S(t)\widetilde{I} \\ &+ 4zi\cos(\alpha)S^2(t)\widetilde{J} \mp 4z\sin(\alpha)S^2(t)\widetilde{K}\Big] . \end{aligned} \tag{2.3.7}$$

Taking into account (2.3.6) and (2.3.7),

$$\begin{aligned} T_\pm^*\kappa_0(\delta)(i\sigma)\kappa_0(\delta)T_\pm - T_\pm^*\kappa^*(-t)(i\sigma)\kappa(-t)T_\pm = &\pm e^{\pm\delta(t)e^{2i\alpha}}\Big(\sin(\alpha)c(t) \mp \cos(\alpha)s(t)\Big) \\ &+ e^{\mp t}\Big(\mp\sin(\alpha)(1+2S^2) + 2\cos(\alpha)CS\Big) \\ &+ i\Big[e^{\pm\delta(t)e^{2i\alpha}}\Big(\cos(\alpha)c(t) \mp \sin(\alpha)s(t)\Big) \\ &\quad - e^{\mp t}\Big(\cos(\alpha)(1+2S^2(t)) \mp 2\sin(\alpha)C(t)S(t)\Big)\Big]\widetilde{I} \\ &- 4zie^{\mp t}\cos(\alpha)S^2(t)\widetilde{J} \pm 4ze^{\mp t}\sin(\alpha)S^2(t)\widetilde{K} \\ =& e^{\mp t}\Big(a + b\widetilde{I} + c\widetilde{J} + d\widetilde{K}\Big) . \end{aligned}$$

The determinant of the Hermitian matrix $e^{\pm t}\Big(T_\pm^*\kappa_0(\delta)(i\sigma)\kappa_0(\delta)T_\pm - T_\pm^*\kappa^*(-t)(i\sigma)\kappa(-t)T_\pm\Big)$ is equal to

$$a^2+b^2+c^2+d^2 = 1 - e^{\pm t}(2 + 4S^2 - e^{\pm t}) \mp e^{\pm t}(1 - e^{\pm 2\delta(t)e^{2i\alpha}})\Big(2CS \mp (1 + 2S^2 - e^{\pm t})\Big) .$$

Let $\delta_0(t) > 0$ be the function which cancels the determinant, or equivalently for which one has, for all $t > 0$,

$$2\Big(2S^2 - (\mathrm{ch}(t) - 1)\Big) = \mp(1 - e^{\pm 2\delta(t)e^{2i\alpha}})\Big(2CS + \mathrm{sh}(t) \mp \Big(2S^2 - (\mathrm{ch}(t) - 1)\Big)\Big) \ .$$

After some computation, we find that this function is independent of the sign in the expression above and is given by

$$\delta_0(t) = \frac{e^{-2i\alpha}}{2} \ln\Big(1 - \frac{2A(t)}{2C(t)S(t) + \mathrm{sh}(t) + A(t)}\Big) \ , \tag{2.3.8}$$

$$\text{where} \qquad A(t) := \Big(2S^2(t) - (\mathrm{ch}(t) - 1)\Big) \, . \tag{2.3.9}$$

We know that, when $\delta = 0$ and $\alpha \in \{0, \frac{\pi}{2}\}$, the Hamilton flow $\kappa(t)$ is positive because $e^{-tK_{\nu,0}}$ is a compact operator (see [HeNi][HiPr]). By connectedness of the set of positive definite hermitian matrices and because the result holds for $\delta(t) = 0$, the flow $\kappa_0(\delta)\kappa(t)$ is a positive canonical transformation so long as the determinant is positive on $[0, \delta]$. Therefore $\delta(t) \in [0, \delta_0(t)[$ implies

$$\|e^{\delta(t)O_q} e^{-tK_{\nu,\alpha}}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq 1 \ ,$$

because any such compact Schrödinger evolution has norm less than 1 (see [Vio2]). □

Proof of Proposition 2.3.1. When $0 < \epsilon_0 < 1$, there exists a constant $c > 0$ independent of ν such that

$$\delta_0(t) \geq c\nu t^3$$

holds for all $0 < t \leq t_0 := \frac{\epsilon_0}{1+|z|} = \frac{\epsilon_0}{1+\sqrt{\nu}}$. This can be seen via the expansion

$$\frac{1}{2} \ln\left(1 - \frac{2A(t)}{A(t) + 2C(t)S(t) + \mathrm{sh}(t)}\right) = \frac{z^2}{12} t^3 + \mathcal{O}\left((1 + |2z|^2)^2 t^5\right) \ ,$$

which is uniform with respect to the parameter ν for all $t \in]0, t_0]$.

We write the quantity $\|\sqrt{\nu \, O_q} e^{-t(K_{\nu,\alpha} + \sqrt{\nu})}\|_{\mathcal{L}(L^2(\mathbb{R}^2))}$ in the form

$$\begin{cases} \|\sqrt{\frac{\nu}{\delta(t)}} \sqrt{\delta(t) \, O_q} \; e^{-\delta(t)O_q} e^{\delta(t)O_q} e^{-tK_{\nu,\alpha}} e^{-t\sqrt{\nu}} \|_{\mathcal{L}(L^2(\mathbb{R}^2))} & \text{if} \quad 0 < t \leq t_0 \, , \\ \|\sqrt{\nu} e^{-t\sqrt{\nu}} \sqrt{O_q} \; e^{-\delta(t_0)O_q} e^{\delta(t_0)O_q} e^{-t_0 K_{\nu,\alpha}} e^{-(t-t_0)K_{\nu,\alpha}} \|_{\mathcal{L}(L^2(\mathbb{R}^2))} & \text{if} \quad t \geq t_0 \, , \end{cases}$$

where Lemma 2.3.2 is applied with $\delta(t) = \frac{\delta_0(t)}{2}$. For both cases we get the upper bounds

$$\begin{cases} \underbrace{\sqrt{\frac{\nu}{\delta(t)}}}_{\leq \frac{\sqrt{2}}{\sqrt{c} t^{\frac{3}{2}}}} \underbrace{\|\sqrt{\delta(t) O_q} e^{-\delta(t) O_q}\|}_{\leq c} \cdot \underbrace{\|e^{\delta(t) O_q} e^{-tK_{\nu,\alpha}}\|}_{\leq 1} \cdot \underbrace{e^{-t\sqrt{\nu}}}_{\leq 1} & \text{if} \quad 0 < t \leq t_0\,, \\ (1+\sqrt{\nu})^{\frac{3}{2}} e^{-t\sqrt{\nu}} \underbrace{\|\frac{\sqrt{\nu}}{(1+\sqrt{\nu})^{\frac{3}{2}}} \sqrt{O_q} e^{-\delta(t_0) O_q}\|}_{\leq \frac{\sqrt{\nu}}{(1+\sqrt{\nu})^{\frac{3}{2}} \sqrt{\delta(t_0)}}} \cdot \underbrace{\|e^{\delta(t_0) O_q} e^{-t_0 K_{\nu,\alpha}}\|}_{\leq 1} \cdot \underbrace{\|e^{-(t-t_0) K_{\nu,\alpha}}\|}_{\leq e^{-\frac{(t-t_0)}{2}}} & \text{if} \quad t \geq t_0\,. \end{cases}$$

For the second case $t \geq t_0$, we use

$$(1+\sqrt{\nu})^{\frac{3}{2}} e^{-t(\frac{t}{2}+\sqrt{\nu})} \times \frac{\sqrt{\nu}}{(1+\sqrt{\nu})^{\frac{3}{2}} \sqrt{\delta(t_0)}} \times e^{\frac{t_0}{2}} \leq \frac{c_0}{t^{3/2}} \times \frac{\sqrt{2\nu}}{(1+\sqrt{\nu})^{3/2} \sqrt{c\nu \frac{\epsilon_0^3}{(1+\sqrt{\nu})^3}}} \times e^{\frac{\varepsilon_0}{2}} \leq \frac{c_0'}{t^{\frac{3}{2}}}\,.$$

This ends the proof of Proposition 2.3.1 and gives

$$\|\sqrt{\nu}(|D_q| + |q|) e^{-t(K_{\nu,\alpha}+\sqrt{\nu})}\|_{L^2} \leq \frac{C}{t^{\frac{3}{2}}} \tag{2.3.10}$$

for all $t > 0$.

□

2.3.2 Improved remainder, case $V(q) = \frac{\nu q^2}{2}$, $\nu \gg 1$

In this section we follow the explicit methods of Aleman and Viola in [Vio][AlVi]. Following [HSV][HPV] it makes use of an FBI transform, which in this specific case is nothing but the usual Bargmann transform

$$B_2 u(z) = \frac{1}{2^{2/2} \pi^{(3\times 2)/4}} \int_{\mathbb{R}^2} e^{-\frac{(z-y)^2 - z^2/2}{2}} u(y)\, dy$$

with $B_2 : L^2(\mathbb{R}^2, dy) \to L^2(\mathbb{C}^2; e^{-\frac{|z|^2}{2}} L(dz)) \cap \mathrm{Hol}(\mathbb{C}^2)$ unitary.

Lemma 2.3.3. *For $\nu > \frac{1}{4}$, the adjoint operator*

$$K^*_{\nu,\frac{\pi}{2}} = \frac{1}{2}(-\partial_p^2 + p^2) - \sqrt{\nu}\Big(p\partial_q - q\partial_p\Big) = O_p - \sqrt{\nu} X_{\frac{\pi}{2}}$$

is transformed via the Bargmann transform B_2 into

$$B_2 (K_{\nu,\frac{\pi}{2}})^* B_2^* = {}^t z M \partial_z \quad , \quad M = \begin{pmatrix} 0 & -\sqrt{\nu} \\ \sqrt{\nu} & 1 \end{pmatrix}\,.$$

and

$$[B_2(e^{-tK_{\nu,\frac{\pi}{2}}} u)](z) = (B_2 u)(e^{-tM} z)\,.$$

Proof. Although it may be proved by a direct computation, it is instructive as an illustration of the general method to follow the lines of [AlVi] or [Vio], Example 2.7. Remember that it is made in essentially two steps : 1) Write the operator, up to an additive constant, in the "supersymmetric" form ${}^t(D_x - A_+x)B(D_x - A_-x)$ after some real canonical transformation in $\mathbb{R}^{2d}$ (here $d = 2$) ; 2) transform the supersymmetric form into $i\,{}^t zM\zeta$ after some linear complex canonical transformation associated with an FBI-transform.
Step 1 : The two variables (q,p) are gathered in the notation $x = (q,p) \in \mathbb{R}^2$, with dual variable $\xi = (\xi_q, \xi_p) \in \mathbb{R}^2$. The hamiltonian matrix associated to $K^*_{\nu,\frac{\pi}{2}}$ is given by

$$H_{K^*_{\nu,\frac{\pi}{2}}} = \begin{pmatrix} 0 & -i\sqrt{\nu} & 0 & 0 \\ i\sqrt{\nu} & 0 & 0 & 1 \\ 0 & 0 & 0 & -i\sqrt{\nu} \\ 0 & -1 & i\sqrt{\nu} & 0 \end{pmatrix} .$$

Set $\lambda_{\epsilon_1,\epsilon_2} = \frac{\epsilon_1 i + \epsilon_2 i n_1}{2}$ the eigenvalues of $H_{K^*_{\nu,\frac{\pi}{2}}}$ with their associated eigenvectors

$${}^tX_{\epsilon_1,\epsilon_2} = \left(1, \frac{i\lambda_{\epsilon_1,\epsilon_2}}{\sqrt{\nu}}, \frac{(\lambda_{\epsilon_1,\epsilon_2})^2 - \nu}{\lambda_{\epsilon_1,\epsilon_2}}, i\frac{(\lambda_{\epsilon_1,\epsilon_2})^2 - \nu}{\sqrt{\nu}}\right) ,$$

where $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. In the case $\alpha = \frac{\pi}{2}$, one has $n_1 = \sqrt{1-4\nu} = i\sqrt{4\nu - 1}$ for $\nu > \frac{1}{4}$.

As a first step we need to determine the following two spaces :

$$\Lambda_- = \bigoplus_{\mathrm{Im}\,\lambda<0} \ker(H_{K^*_{\nu,\frac{\pi}{2}}} - \lambda I) = \left\{ \begin{pmatrix} x \\ A_-x \end{pmatrix}, \quad x \in \mathbb{C}^2 \right\}$$

and

$$\Lambda_+ = \bigoplus_{\mathrm{Im}\,\lambda>0} \ker(H_{K^*_{\nu,\frac{\pi}{2}}} - \lambda I) = \left\{ \begin{pmatrix} x \\ A_+x \end{pmatrix}, \quad x \in \mathbb{C}^2 \right\},$$

where A_+ and A_- are two matrices in $\mathbb{M}_2(\mathbb{C})$ satisfying ${}^tA_\pm = A_\pm$ and $\pm\mathrm{Im}(A_\pm) > 0$.

The matrix A_+ is given by $A_+ = B_{1+}^{-1}B_{2+}$ where

$$B_{1+} = \begin{pmatrix} 1 & \frac{-1-n_1}{2\sqrt{\nu}} \\ 1 & \frac{-1+n_1}{2\sqrt{\nu}} \end{pmatrix} \quad \text{and} \quad B_{2+} = \begin{pmatrix} i & \frac{-i-in_1}{2\sqrt{\nu}} \\ i & \frac{-i+in_1}{2\sqrt{\nu}} \end{pmatrix} ,$$

so $A_+ = i\mathrm{Id}$. Similarly, $A_- = B_{1-}^{-1}B_{2-}$ with

$$B_{1-} = \begin{pmatrix} 1 & \frac{1+n_1}{2\sqrt{\nu}} \\ 1 & \frac{1-n_1}{2\sqrt{\nu}} \end{pmatrix} \quad \text{and} \quad B_{2-} = \begin{pmatrix} -i & \frac{-i+in_1}{2\sqrt{\nu}} \\ -i & \frac{-i-in_1}{2\sqrt{\nu}} \end{pmatrix} ,$$

so $A_- = -i\mathrm{Id}$. This means, after [Vio] formula (2.3), that the real canonical transformation on $\mathbb{R}^4$ is nothing but the identity.

Hence it suffices to write $K^*_{\nu,\frac{\pi}{2}}$ in the form

$$K^*_{\nu,\frac{\pi}{2}} = {}^t(D_x - A_+x)B(D_x - A_-x) ,$$

for all $x = (q,p) \in \mathbb{R}^2$, where the matrix B is found by identification of the two sides :

$$B = \begin{pmatrix} 0 & \frac{-\sqrt{\nu}}{2} \\ \frac{\sqrt{\nu}}{2} & \frac{1}{2} \end{pmatrix} .$$

Step 2 : Once A_+ and A_- are known, the complex canonical transformation is given by

$$\kappa = \begin{pmatrix} 1 & -i \\ -(1-iA_+)^{-1}A_+ & (1-iA_+)^{-1} \end{pmatrix} ,$$

with associated quadratic phase $\varphi_{A_+} : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$

$$\varphi_{A_+}(x,y) = \frac{i(x-y)^2}{2} - \frac{1}{2}\Big(x, (1-iA_+)^{-1}A_+x\Big) = i\left[\frac{(x-y)^2}{2} - \frac{x^2}{4}\right] ,$$

which is the one entering in the definition of the associated FBI transform (which is B_2). The computation of $B_2 K^*_{\nu,\frac{\pi}{2}} B_2^*$ then comes from Egorov's theorem

$$K^*_{\nu,\frac{\pi}{2}}(\kappa^{-1}Z) = {}^t Z {}^t\kappa^{-1}\begin{pmatrix} -A_+ \\ \mathrm{Id} \end{pmatrix} B(-A_-, \mathrm{Id})\kappa^{-1}Z = i {}^t z M \zeta$$

$$\text{with} \qquad M = (1-iA_+)B = 2B = \begin{pmatrix} 0 & -\sqrt{\nu} \\ \sqrt{\nu} & 1 \end{pmatrix} .$$

The weight $e^{-2\phi(z)}L(dz)$ occurring in the range of B_2 is $\phi(z) = \frac{|z|^2}{4}$ which is coherent with the formulas (2.6) and (2.7) of [Vio], $\phi(x) = \frac{1}{4}\Big(|x|^2 - {}^t xCx\Big)$ because $C = (1-iA_+)^{-1}(1+iA_+) = 0$. ☐

Lemma 2.3.4. *There exists a constant $c > 0$ independent of $\nu > 1$, such that for all $t > 0$ and all $u \in L^2(\mathbb{R}^2)$, $u_t = e^{-t(K_{\nu,\frac{\pi}{2}}+\nu^{1/3})}u$ satisfies*

$$\frac{\nu}{2}\Big(\|u_t\|^2_{L^2(\mathbb{R}^2)} + \|D_q u_t\|^2_{L^2(\mathbb{R}^2)} + \|qu_t\|^2_{L^2(\mathbb{R}^2)}\Big) = \|\sqrt{\nu}(\frac{-\partial_q + q}{\sqrt{2}})e^{-t(K_{\nu,\frac{\pi}{2}}+\nu^{\frac{1}{3}})}u\|^2_{L^2(\mathbb{R}^2)} \leq \frac{c}{t^3}\|u\|^2_{L^2(\mathbb{R}^2)} . \tag{2.3.11}$$

Proof. Set $a_q = \frac{\partial_q + q}{\sqrt{2}}$ and $a_q^* = \frac{-\partial_q + q}{\sqrt{2}}$ so that $a_q a_q^* = a_q^* a_q + 1 = \frac{1}{2}(D_q^2 + q^2 + 1)$. The identity

$$\begin{aligned}\nu\|a_q^* e^{-t(K_{\nu,\frac{\pi}{2}}+\nu^{1/3})}u\|^2_{L^2(\mathbb{R}^2)} &= \nu\|e^{-t(K_{\nu,\frac{\pi}{2}}+\nu^{1/3})}u\|^2_{L^2(\mathbb{R}^2)} + \nu\|a_q e^{-t(K_{\nu,\frac{\pi}{2}}+\nu^{1/3})}u\|^2_{L^2} \\ &\leq \nu e^{-t\nu^{1/3}}\|u\|^2_{L^2(\mathbb{R}^2)} + \nu\|a_q e^{-t(K_{\nu,\frac{\pi}{2}}+\nu^{1/3})}u\|^2_{L^2}\end{aligned}$$

reduces the problem to that of estimating $\|\sqrt{\nu}a_q e^{-t(K_{\nu,\frac{\pi}{2}}+\nu^{1/3})}\|$. By taking the adjoint, it suffices to prove that

$$\|\sqrt{\nu}e^{-t(K^*_{\nu,\frac{\pi}{2}}+\nu^{1/3})}a_q^* f\|_{L^2(\mathbb{R}^2)} \leq \frac{c}{t^{\frac{3}{2}}}\|f\|_{L^2(\mathbb{R}^2)} \tag{2.3.12}$$

is satisfied for all $f \in L^2(\mathbb{R}^2, dqdp)$ and for all $t > 0$.

Conjugating by the Bargmann transform B_2, the creation operator $B_2 a_q^* B_2^* = B_2(\frac{-\partial_q+q}{\sqrt{2}})B_2^* = \frac{z_q}{\sqrt{2}}$ is nothing but multiplication by the complex component z_q in $\mathbb{C}^2 = \mathbb{C}_q \times \mathbb{C}_p$. The inequality (2.3.12) is therefore equivalent to

$$\|\sqrt{\nu}e^{-t(Mz\partial_z+\nu^{1/3})}z_q u\|_{H_\phi} \leq \frac{c}{t^{\frac{3}{2}}}\|u\|_{H_\phi} \tag{2.3.13}$$

for all $u \in H_\phi = L^2(\mathbb{C}^2, e^{-\frac{|z|^2}{2}}L(dz)) \cap \mathrm{Hol}(\mathbb{C}^2)$, with $\phi(z) = \frac{|z|^2}{4}$.

Let $u \in H_\phi$, setting $v(z) = z_q u(z)$, one has $e^{-tMz\partial_z}v(z) = v(e^{-tM}z)$ and it follows that

$$\begin{aligned}
\|e^{-tMz\partial_z}z_q u\|^2_{H_\phi} &= \int_{\mathbb{C}^2} |v(e^{-tM}z)|^2 |(e^{-tM}z)_q|^2 e^{-2\phi(z)}L(dz) \\
&= e^{2t\,\mathrm{Tr}\,M}\int_{\mathbb{C}^2} |v(z')|^2|z'_q|^2 e^{-\phi(z')}e^{-2[\phi(e^{tM}z')-\phi(z')]}L(dz') \ .
\end{aligned}$$

So our problem is reduced to the proof of the existence of a constant $c > 0$ that does not depend on ν such that

$$\sup_{z\in\mathbb{C}^2} |z_q|^2 e^{-\frac{1}{2}\left(|e^{tM}z|^2-|z|^2\right)}e^{-t\nu^{1/3}} \leq \frac{c}{\nu t^3}$$

for all $t > 0$.

Let us start by checking that $z \mapsto \phi(e^{tM}z) - \phi(z)$ defines a positive definite hermitian form for $t > 0$.

From the expression given in Lemma 2.3.3 , M is easily written in terms of Pauli's matrices :

$$M = \frac{1}{2}\mathrm{Id} - \frac{1}{2}\sigma_3 - i\sqrt{\nu}\sigma_2 \ ,$$
$$\text{with} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ ; \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

Recall that Pauli's matrices are involutory :

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -i\sigma_1\sigma_2\sigma_3 = \mathrm{Id} \ ,$$

and that $(\mathrm{Id}, -i\sigma_1, -i\sigma_2, -i\sigma_3)$ can be interpreted as a basis of (bi)quaternions.

Using formula (2.A.1), one has for all $t > 0$

$$e^{tM} = e^{\frac{t}{2}}\Big(C(t) + 2S(t)(-\frac{1}{2}\sigma_3 - i\sqrt{\nu}\sigma_2)\Big) .$$

From this, we compute

$$\begin{aligned}(e^{tM})^* e^{tM} &= e^t\Big(1 + 2S^2(t) - 2C(t)S(t)\sigma_3 - 4\sqrt{\nu}S^2(t)\sigma_1\Big)\\ &= e^t(a + v) ,\end{aligned}$$

with $a = 1 + 2S^2(t)$ and $v = -2C(t)S(t)\sigma_3 - 4\sqrt{\nu}s_1^2(t)\sigma_1$.

The eigenvalues of $(e^{tM})^* e^{tM}$ are given by

$$\lambda_\pm = e^t(a \pm \sqrt{-N(v)}) ,$$

where $N(v) = -\Big(2C(t)S(t)\Big)^2 - \Big(4\sqrt{\nu}S^2(t)\Big)^2 = -4S^2 - 4S^4 < 0$ owing to $(4\nu - 1)S^2 + C^2 = 1$, and where $\sqrt{-N(v)}$ is the usual square root.

In order to prove that the hermitian form $z \mapsto \phi(e^{tM}z) - \phi(z) = {}^t\bar{z}\Big((e^{tM})^*(e^{tM}) - \mathrm{Id}\Big)z$ is positive definite, it suffices to check $\lambda_- > 1$ for all $t > 0$, λ_+ being clearly strictly larger than 1 . The eigenvalue λ_- equals

$$\lambda_- = e^t(1 + (1 + S^2) - 2|S|\sqrt{1 + S^2}) = e^t e^{-2\,\mathrm{Argsh}\,|S|}$$

which is larger than 1 if and only if $\mathrm{sh}(t/2) - |S| > 0$ or $[\mathrm{sh}(t/2) - S(t)][\mathrm{sh}(t/2) + S(t)] > 0$ because $\mathrm{sh}(t/2) > 0$. This is true since both factors vanish at $t = 0$ with a positive derivative for $t > 0$ owing to $\mathrm{ch}(t/2) > 1 > \pm\cos(\frac{t}{2}\sqrt{4\nu - 1})$.
Now denote

$$r_1 = \sqrt{4\nu - 1}\ ; \quad Q_t(z) = {}^t\bar{z}\Big[(e^{tM})^*(e^{tM}) - \mathrm{Id}\Big]z \quad \text{and} \quad S_t(z_1, z_2) = {}^t\bar{z_1}\Big[(e^{tM})^*(e^{tM}) - \mathrm{Id}\Big]z_2$$

for all $z = (z_1, z_2) \in \mathbb{C} \times \mathbb{C}$. Writing $z_q = l(z)$ where l is a linear form with kernel $ker\ l = \mathbb{C}e_p$, $\Big(\mathbb{C}^2 = \mathbb{C}e_q \oplus \mathbb{C}e_p$ where $e_q = (1, 0)$ and $e_p = (0, 1)\Big)$, we construct an orthonormal basis (e'_q, e_p) for Q_t with

$$e'_q = e_q - \frac{S_t(e_p, e_q)}{S_t(e_p, e_p)}e_p = \begin{pmatrix} 1 \\ \frac{4\sqrt{\nu}\,S^2(t)}{(1-e^{-t})+2S^2(t)+2S(t)C(t)} \end{pmatrix} ,$$

where

$$\begin{cases} S_t(e_p, e_q) = -4\sqrt{\nu}\; e^t\; S^2(t) \\ \\ S_t(e_p, e_p) = e^t\Big[(1-e^{-t}) + 2S^2(t) + 2S(t)C(t)\Big] \; . \end{cases}$$

In this new basis, $z = \alpha e_q' + \beta e_p$ then $l(z) = \alpha\; l(e_q')$ and $Q_t(z) = |\alpha|^2 Q_t(e_q') + |\beta|^2 Q_t(e_p)$. This gives immediately

$$|z_q|^2 e^{\frac{-Q_t(z)}{2}} = |\alpha|^2 |l(e_q')|^2 e^{\frac{-|\alpha|^2 Q_t(e_q') - |\beta|^2 Q_t(e_p)}{2}} .$$

and then

$$\sup_{z\in\mathbb{C}^2} |z_q|^2 e^{-\frac{1}{2}\left(|e^{tM}z|^2 - |z|^2\right)} = \sup_{s\in\mathbb{R}_+} |l(e_q')|^2 e^{-\frac{sQ_t(e_q')}{2}} = \frac{2|l(e_q')|^2}{Q_t(e_q')} \sup_{\sigma\in\mathbb{R}_+} \sigma e^{-\sigma} = c_0 \frac{2|l(e_q')|^2}{Q_t(e_q')} = c_0 \frac{2}{Q_t(e_q')}$$

where $c_0 = \sup\limits_{\sigma\in\mathbb{R}_+} \sigma e^{-\sigma}$ and

$$Q_t(e_q') = S_t(e_q', e_q') = \frac{4\Big(\operatorname{sh}^2(\frac{t}{2}) - S^2(t)\Big)}{(1-e^{-t}) + 2S^2(t) + 2S(t)C(t)} \; .$$

Recall that, in the case $\alpha = \frac{\pi}{2}$ and for $\nu > \frac{1}{4}$, we define $C(t) = \cos(\frac{tr_1}{2})$ and $S(t) = \frac{\sin(\frac{tr_1}{2})}{r_1}$.

All that remains is to control the following quotient for all $t > 0$:

$$\frac{1}{Q_t(e_q')} = \frac{(1-e^{-t}) + 2S(t)\Big(S(t) + C(t)\Big)}{4\Big[\operatorname{sh}^2(\frac{t}{2}) - S^2(t)\Big]} := \frac{N}{D} \; .$$

• Starting with the case when $t \geq \frac{4}{r_1}$,

$$N = (1-e^{-t}) + 2S(t)\Big(S(t) + C(t)\Big) \leq 1 + \frac{4}{r_1} \leq 2 \; .$$

On the other hand,

$$|S(t)| \leq \frac{1}{r_1} \leq \frac{t}{4} \leq \frac{1}{2}\operatorname{sh}(\frac{t}{2}) \quad \text{implies} \quad D \geq \operatorname{sh}^2(\frac{t}{2}) \; .$$

Then

$$\frac{1}{Q_t(e_q')} \leq \frac{2}{\operatorname{sh}^2(\frac{t}{2})} \leq 2e^{-t}$$

for all $t \geq \frac{4}{r_1}$.

• Now observe that for $t \leq \frac{4}{r_1}$, one has the following two expansions :

$$\mathrm{sh}(\frac{t}{2}) + S(t) = \sum_{k=0}^{+\infty}(-1)^k(r_1^{2k} + (-1)^k)\frac{t^{2k+1}}{2^{2k+1}(2k+1)!}$$

and

$$\mathrm{sh}(\frac{t}{2}) - S(t) = \sum_{k=0}^{+\infty}(-1)^k(-r_1^{2k} + (-1)^k)\frac{t^{2k+1}}{2^{2k+1}(2k+1)!}\ .$$

Furthermore,

$$\left|\mathrm{sh}(\frac{t}{2}) + S(t) - t - \frac{r_1^2 - 1}{48}t^3\right| \leq \frac{(r_1^4+1)t^5}{2^5 \times 120}$$

which implies

$$\begin{aligned}\frac{1}{t}\Big(\mathrm{sh}(\frac{t}{2}) + S(t)\Big) &\geq 1 - \frac{(r_1^2-1)}{48}t^2 - \frac{r_1^4+1}{2^5 \times 120}t^4 \\ &\geq 1 - \frac{16}{48} - \frac{2 \times 4^4}{2^5 \times 120} \geq 1 - \frac{1}{3} - \frac{2}{15} = \frac{8}{15}\ . \end{aligned} \tag{2.3.14}$$

Similarly,

$$\begin{aligned}\left|\mathrm{sh}(\frac{t}{2}) - S(t) - \frac{r_1^2+1}{48}t^3\right| &\leq \frac{(r_1^4-1)t^5}{2^5 \times 120} = \frac{(r_1^2+1)t^3}{48} \times \frac{(r_1^2-1)t^3}{4 \times 20} \\ &\leq \frac{(r_1^2+1)t^3}{48}\frac{(r_1 t)^2}{4 \times 20} \\ &\leq \frac{(r_1^2+1)t^3}{48}\frac{1}{5}\ ,\end{aligned}$$

which gives

$$\mathrm{sh}(\frac{t}{2}) - S(t) \geq \frac{(r_1^2+1)t^3}{48} \times \frac{4}{5}\ . \tag{2.3.15}$$

Taking into account (2.3.14) and (2.3.15) we get

$$D \geq \Big(\mathrm{sh}(\frac{t}{2}) + S(t)\Big)\Big(\mathrm{sh}(\frac{t}{2}) - S(t)\Big) \geq t \times \frac{8}{15} \times \frac{(r_1^2+1)t^3}{48} \times \frac{4}{5}\ .$$

On the other hand,

$$\begin{aligned}N = (1 - e^{-t}) + 2S(t)\Big(S(t) + C(t)\Big) &= 2t + (1 - \frac{r_1^2}{6})t^3 - \frac{1 + r_1^2}{24}t^4 + \mathcal{O}(r_1^4 t^5) \\ &= t\Big(2 + (1 - \frac{r_1^2}{6})t^2 - \frac{1+r_1^2}{24}t^3 + \mathcal{O}((r_1 t)^4)\Big)\ .\end{aligned}$$

Hence $N \leq ct$ for all $t \leq \frac{4}{r_1}$ and

$$\frac{1}{Q_t(e'_q)} = \frac{N}{D} \leq \frac{c}{\nu t^3} \quad \text{for all} \quad t \leq \frac{4}{r_1} \ .$$

Thus there exists a constant $c > 0$ such that, for all $u \in H_\phi$,

$$\|e^{-tMz\partial_z} z_q u\|^2_{H_\phi} \leq \begin{cases} \frac{c}{\nu t^3}\|u\|^2_{H_\phi} & \text{for all} \quad t \leq \frac{4}{r_1} \\ ce^{-t}\|u\|^2_{H_\phi} & \text{for all} \quad t \geq \frac{4}{r_1} \end{cases}$$

which is equivalent to

$$\|e^{-tK^*_{\nu,\frac{\pi}{2}}} a^*_q v\|_{L^2} \leq \begin{cases} \frac{c}{\sqrt{\nu t^3}}\|v\|_{L^2} & \text{for all} \quad t \leq \frac{4}{r_1} \\ ce^{-t}\|v\|_{L^2} & \text{for all} \quad t \geq \frac{4}{r_1} \end{cases}$$

for all $v \in D(K_{\nu,\frac{\pi}{2}})$.

From this, we deduce that

$$\|a_q e^{-t(\nu^{\frac{1}{3}} + K_{\nu,\frac{\pi}{2}})} v\|_{L^2} \leq \begin{cases} \frac{c}{\sqrt{\nu t^3}}\|v\|_{L^2} & \text{if} \quad t \leq \frac{4}{r_1} \\ ce^{-\nu^{\frac{1}{3}} t}\|v\|_{L^2} & \text{if} \quad t \geq \frac{4}{r_1} \end{cases}$$

for every $v \in D(K_{\nu,\frac{\pi}{2}})$. When $0 < t \leq \frac{4}{r_1}$, we clearly have

$$\|\sqrt{\nu} a_q e^{-t(\nu^{\frac{1}{3}} + K_{\nu,\frac{\pi}{2}})}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq \frac{C}{t^{\frac{3}{2}}} \ .$$

When $t \geq \frac{4}{r_1}$, we obtain the same result by writing

$$c\sqrt{\nu} e^{-\nu^{\frac{1}{3}} t} = \frac{c}{t^{\frac{3}{2}}} \left(\nu^{\frac{1}{3}} t\right)^{\frac{3}{2}} e^{-\nu^{\frac{1}{3}} t}$$

and noting that the function $s^{3/2}e^{-s}$ is bounded on $[0, \infty)$. This establishes the inequality for all $t > 0$ and completes the proof of the lemma. □

2.4 Resolvent estimates when $V(q) = -\frac{\nu q^2}{2}, \nu \gg 1$

In this section, we use the same notations as in the previous one and we take $\alpha = 0$. Giving the exact norm of the semigroup $e^{-tK_{\nu,0}}$ allows us to control the resolvent of the operator $K_{\nu,0}$. When doing so, a logarithmic factor appears, with optimality up to an exponent.

Lemma 2.4.1. *For every $t \geq 0$, one has*

$$\|e^{-tK_{\nu,0}}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} = e^{-\operatorname{Argsh}\left(S(t)\right)}$$

where

$$S(t) = \frac{\operatorname{sh}(\frac{tn_1}{2})}{n_1} = \frac{\operatorname{sh}(\frac{t\sqrt{4\nu+1}}{2})}{\sqrt{4\nu+1}} \ .$$

Proof. Using (2.3.3) and (2.3.4), we directly compute that

$$\overline{(\kappa(t))^{-1}}\ \kappa(t) := \overline{e^{-itH_{K_{\nu,0}}}} e^{itH_{K_{\nu,0}}} = e^{itE}\Big(a + bI - cJ\Big)\Big(a + bI + cJ\Big) \ ,$$

with $a = C(t),\ b = iS(t)$ and $c = 2izS(t)$.

Note that $\Big(a + bI - cJ\Big)\Big(a + bI + cJ\Big) = a^2 - b^2 + c^2 + v$. Furthermore, $a^2 + b^2 + c^2 = 1$ and $(a^2 - b^2 + c^2)^2 + N(v) = 1$. It follows that
$N(v) = 1 - (a^2 - b^2 + c^2)^2 = 1 - (1 - 2b^2)^2 = 4b^2(1 - b^2)$.

Denote $\operatorname{sh}(u) = \sqrt{-b^2}$, so $\sqrt{-N(v)} = 2\operatorname{sh}(u)\operatorname{ch}(u) = \operatorname{sh}(2u)$.

The eigenvalues of $\overline{(\kappa(t))^{-1}}\ \kappa(t)$ are given by

$$\begin{aligned}
\frac{1}{\mu_1} &= e^t(a^2 - b^2 + c^2 + \sqrt{-N(v)})\\
\mu_1 &= e^{-t}(a^2 - b^2 + c^2 - \sqrt{-N(v)})\\
\frac{1}{\mu_2} &= e^t(a^2 - b^2 + c^2 - \sqrt{-N(v)})\\
\mu_2 &= e^{-t}(a^2 - b^2 + c^2 + \sqrt{-N(v)}) \ .
\end{aligned}$$

Therefore (see [Vio2] Theorem 1.3),

$$\|e^{-tK_{\nu,0}}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} = (\mu_1\frac{1}{\mu_2})^{\frac{1}{4}} = e^{-\frac{1}{2}\operatorname{Argsh}(\sqrt{-N(v)})} = e^{-Argsh(\sqrt{-b^2})} \ ,$$

where

$$-b^2 = \Big(S(t)\Big)^2 = \Big(\frac{\operatorname{sh}(\frac{tn_1}{2})}{n_1}\Big)^2 \ .$$

□

Proposition 2.4.2. *There exists some $c > 0$ such that, for all $\nu > c$,*

$$\|K_{\nu,0}^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq c\frac{\log(\nu)}{\sqrt{\nu}} \ .$$

Proof. Observing that

$$\|K_{\nu,0}^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} = \|\int_0^{+\infty} e^{-tK_{\nu,0}}dt\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq \int_0^{+\infty} \|e^{-tK_{\nu,0}}\|_{\mathcal{L}(L^2(\mathbb{R}^2))}dt \ ,$$

we aim to obtain an upper bound of the right-hand side.

Using the exact norm of the semigroup generated by $K_{\nu,0}$, we write

$$\begin{aligned}
\int_0^{+\infty} \|e^{-tK_{\nu,0}}\|_{\mathcal{L}(L^2(\mathbb{R}^2))}dt &= \int_0^{+\infty} e^{-\operatorname{Argsh}\left(\frac{\operatorname{sh}(\frac{tn_1}{2})}{n_1}\right)}dt = \int_0^{+\infty} \frac{1}{\frac{\operatorname{sh}(\frac{tn_1}{2})}{n_1} + \sqrt{1+\left(\frac{\operatorname{sh}(\frac{tn_1}{2})}{n_1}\right)^2}}dt \\
&= \int_0^{\log(\nu)} \frac{2du}{\operatorname{sh}(u) + \sqrt{n_1^2 + \operatorname{sh}^2(u)}} + \int_{\log(\nu)}^{+\infty} \frac{2du}{\operatorname{sh}(u) + \sqrt{n_1^2 + \operatorname{sh}^2(u)}} \\
&\leq 2\Big(\frac{\log(\nu)}{n_1} + \int_{\log(\nu)}^{+\infty} e^{-u}du\Big) \\
&\leq 2\Big(\frac{\log(\nu)}{n_1} + \frac{1}{\nu}\Big) \leq c\,\frac{\log(\nu)}{\sqrt{\nu}} \ .
\end{aligned}$$

This completes the proof.

□

2.4.1 Optimality with a logarithmic factor

Proposition 2.4.3. *One can find a function $u \in L^2(\mathbb{R}^2)$ such that*

$$\|K_{\nu,0}u\|_{L^2(\mathbb{R}^2)} \leq c\,\frac{\sqrt{\nu}}{\sqrt{\log(\nu)}}\|u\|_{L^2(\mathbb{R}^2)}$$

where $c > 0$ is a constant that does not depend on the parameter $\nu \gg 1$.

Proof. We recall here that

$$K_{\nu,0} = \frac{1}{2}(-\partial_p^2 + p^2) + \sqrt{\nu}\Big(p\partial_q + q\partial_p\Big) = O_p + \sqrt{\nu}X_0 \ .$$

For all $u \in D(K_{\nu,0})$,

$$\|K_{\nu,0}u\|^2_{L^2(\mathbb{R}^2)} \leq 2\Big(\|O_pu\|^2_{L^2(\mathbb{R}^2)} + \nu\|X_0u\|^2_{L^2(\mathbb{R}^2)}\Big) \ ,$$

then to prove the Proposition we will look for a function $u \in L^2(\mathbb{R}^2)$ such that

$$\frac{\|O_p u\|^2_{L^2(\mathbb{R}^2)} + \nu\|X_0 u\|^2_{L^2(\mathbb{R}^2)}}{\|u\|^2_{L^2(\mathbb{R}^2)}} \leq c\, \frac{\nu}{\log(\nu)} \ .$$

Consider the Gaussian

$$\varphi(q,p) = \frac{e^{-\frac{(q^2+p^2)}{2}}}{\sqrt{\pi}}$$

and set

$$u(q,p) = \frac{1}{L}\int_0^L e^{sX_0}\varphi ds = \frac{1}{L}\int_0^L \varphi_s(q,p) ds$$

where $\varphi_s(q,p) = e^{sX_0}\varphi(q,p)$ and $L > 0$ is a constant to be specified at the end of the proof. One has

$$\frac{d}{ds}\varphi_s = X_0(\varphi_s) = (p\partial_q + q\partial_p)\varphi_s \ .$$

Let $\Big(q(t), p(t)\Big)$ be the solution of the following system :

$$\begin{cases} \frac{d}{dt}q = p \\ \frac{d}{dt}p = q \end{cases}$$

with $(q(0), p(0)) = (q_0, p_0)$. The solution is given by

$$\begin{cases} q(t) = \mathrm{ch}(t)q_0 + \mathrm{sh}(t)p_0 \\ p(t) = \mathrm{sh}(t)q_0 + \mathrm{ch}(t)p_0 \ . \end{cases}$$

The function φ_s verifies

$$\frac{d}{ds}\Big(\varphi_s(q(-s), p(-s))\Big) = \frac{\partial}{\partial s}\varphi_s - \frac{d}{ds}q(-s)\partial_q\varphi_s - \frac{d}{ds}p(-s)\partial_p\varphi_s = 0 \ ,$$

then

$$\begin{aligned} \varphi_s(q,p) = \varphi_0(q(s), p(s)) &= \varphi\Big(\mathrm{ch}(s)q + \mathrm{sh}(s)p, \mathrm{sh}(s)q + \mathrm{ch}(s)p\Big) \\ &= \frac{1}{\sqrt{\pi}}\exp\Big(-\frac{\Big(\mathrm{ch}(s)q + \mathrm{sh}(s)p\Big)^2 + \Big(\mathrm{sh}(s)q + \mathrm{ch}(s)p\Big)^2}{2}\Big) \ . \end{aligned}$$

For all $p \in [0, +\infty]$, $\|\varphi_s\|_{L^p} = \|\varphi\|_{L^p}$. In particular, $\|\varphi_s\|_{L^2} = \|\varphi\|_{L^2} = 1$.

Let's start by calculating $\|X_0 u\|_{L^2(\mathbb{R}^2)}$:

$$X_0 u = \frac{1}{L}\int_0^L X_0 e^{sX_0}\varphi ds = \frac{1}{L}\int_0^L \frac{d}{ds}\varphi_s ds = \frac{1}{L}(\varphi_L - \varphi) \ .$$

As a result,

$$\begin{aligned}\|X_0 u\|^2_{L^2(\mathbb{R}^2)} &= \frac{1}{L^2}\|\varphi_L - \varphi\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{L^2}\Big(\underbrace{\|\varphi_L\|^2_{L^2(\mathbb{R}^2)}}_{=1} + \underbrace{\|\varphi\|^2_{L^2(\mathbb{R}^2)}}_{=1} - 2\int_{\mathbb{R}^2}\varphi_L\varphi \, dqdp\Big)\\ &= \frac{2}{L^2}\Big(1 - \int_{\mathbb{R}^2}\varphi_L\varphi \, dqdp\Big) \ .\end{aligned}$$

We directly compute that

$$\begin{aligned}\int_{\mathbb{R}^2}\varphi_L(q,p)\varphi(q,p)dqdp &= \frac{1}{\pi}\int_{\mathbb{R}^2} e^{-\frac{\left(\operatorname{ch}(L)q+\operatorname{sh}(L)p\right)^2 + \left(\operatorname{sh}(L)q+\operatorname{ch}(L)p\right)^2}{2}} e^{-\frac{q^2+p^2}{2}} dqdp\\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} e^{-\frac{1}{2}\left[2\operatorname{ch}^2(L)q^2 + 2\operatorname{ch}^2(L)p^2 + 4\operatorname{sh}(L)\operatorname{ch}(L)qp\right]} dqdp\\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} e^{-\frac{1}{2}(q,p)A\,{}^t(q,p)} dqdp = \frac{1}{\pi}\sqrt{\frac{(2\pi)^2}{det(A)}} = \frac{1}{\operatorname{ch}(L)} \ ,\end{aligned}$$

where

$$A = \begin{pmatrix} 2\operatorname{ch}^2(L) & 2\operatorname{ch}(L)\operatorname{sh}(L) \\ 2\operatorname{ch}(L)\operatorname{sh}(L) & 2\operatorname{ch}^2(L)\end{pmatrix} \ .$$

Then

$$\|X_0 u\|^2_{L^2(\mathbb{R}^2)} = \frac{2}{L^2}\Big(1 - \frac{1}{\operatorname{ch}(L)}\Big) \ . \tag{2.4.1}$$

Now, let's find a lower bound for $\|u\|^2_{L^2(\mathbb{R}^2)}$:

$$\begin{aligned}\|u\|^2_{L^2(\mathbb{R}^2)} &= \frac{1}{L^2}\int_0^L\int_0^L \operatorname{Re}\langle\varphi_{s_1}, \varphi_{s_2}\rangle_{L^2(\mathbb{R}^2)} ds_1 ds_2\\ &= \frac{2}{L^2}\int_0^L\Big[\int_{s_1}^L \operatorname{Re}\langle\varphi_{s_1}, \varphi_{s_2}\rangle_{L^2(\mathbb{R}^2)} ds_2\Big] ds_1\\ &\underset{s_2 = s_1 + s}{=} \frac{2}{L^2}\int_0^L\Big[\int_0^{L-s_1} \operatorname{Re}\langle\varphi_{s_1}, \varphi_{s_1+s}\rangle_{L^2(\mathbb{R}^2)} ds\Big] ds_1 \ .\end{aligned}$$

But

$$\begin{aligned}\mathbb{R}\mathrm{e}\langle\varphi_{s_1+s},\varphi_{s_1}\rangle_{L^2(\mathbb{R}^2)} &= \langle e^{s_1X_0}\varphi, e^{(s_1+s)X_0}\varphi\rangle_{L^2(\mathbb{R}^2)}\\ &= \langle e^{s_1X_0}\varphi, e^{sX_0}\varphi\rangle_{L^2(\mathbb{R}^2)}\\ &= \int_{\mathbb{R}^2}\varphi_s(q,p)\varphi(q,p)dqdp = \frac{1}{\mathrm{ch}(s)}\ .\end{aligned}$$

For $L>2$ we obtain

$$\begin{aligned}\|u\|^2_{L^2(\mathbb{R}^2)} &= \frac{2}{L^2}\int_0^L\Big[\int_0^{L-s_1}\frac{1}{\mathrm{ch}(s)}ds\Big]ds_1\\ &\geq \frac{2}{L^2}\int_0^{\frac{L}{2}}\Big[\int_0^{\frac{L}{2}}\frac{1}{\mathrm{ch}(s)}ds\Big]ds_1 \geq \frac{2}{L^2}\int_0^{\frac{L}{2}}\Big[\int_0^{1}\frac{1}{\mathrm{ch}(s)}ds\Big]ds_1\\ &\geq \frac{c}{L}\ .\end{aligned} \tag{2.4.2}$$

The final step is the upper bound of $\|O_pu\|^2_{L^2(\mathbb{R}^2)}$:

$$\|O_pu\|^2_{L^2(\mathbb{R}^2)} = \|O_p\Big(\frac{1}{L}\int_0^L\varphi_s(q,p)ds\Big)\|^2_{L^2(\mathbb{R}^2)} \leq \frac{1}{L^2}\int_0^L\|O_p\varphi_s\|^2_{L^2(\mathbb{R}^2)}ds\ .$$

With $O_p=\frac{1}{2}(D_p^2+p^2)$, we want to compute

$$\|O_p\varphi_s\|_{L^2(\mathbb{R}^2)} = \|e^{-sX_0}O_pe^{sX_0}\varphi_0\|_{L^2(\mathbb{R}^2)}$$

(because e^{-sX_0} is unitary and $\varphi_s=e^{sX_0}\varphi_0$).

For any $u\in L^2(\mathbb{R}^2)$, $e^{sX_0}u(q,p)=u(e^{sM}(q,p))$ where

$$e^{sM} = \begin{pmatrix}\mathrm{ch}\, s & \mathrm{sh}\, s\\ \mathrm{sh}\, s & \mathrm{ch}\, s\end{pmatrix}.$$

Egorov's theorem gives that, for any symbol $a(q,p,\xi_q,\xi_p)$,

$$e^{-sX_0}a^w(q,p,D_q,D_p)e^{sX_0} = a^w(e^{-sM}(q,p),e^{sM}(D_q,D_p)).$$

In particular, writing $O_q=\frac{1}{2}(D_q^2+q^2)$ as well,

$$\begin{aligned}e^{-sX_0}(p^2+D_p^2)e^{sX_0} &= (-\mathrm{sh}(s)q+\mathrm{ch}(s)p)^2+(\mathrm{sh}(s)D_q+\mathrm{ch}(s)D_p)^2\\ &= \mathrm{sh}^2(s)q^2-2\,\mathrm{ch}(s)\,\mathrm{sh}(s)qp+\mathrm{ch}^2(s)p^2\\ &\qquad+\mathrm{sh}^2(s)D_q^2+2\,\mathrm{ch}(s)\,\mathrm{sh}(s)D_qD_p+\mathrm{ch}^2(s)D_p^2\\ &= 2\,\mathrm{ch}^2(s)O_q+2\,\mathrm{sh}^2(s)O_p+2\,\mathrm{ch}(s)\,\mathrm{sh}(s)(D_qD_p-qp)\ .\end{aligned}$$

We have chosen φ_0 an eigenfunction of both O_p and O_q with eigenvalue $\frac{1}{2}$, and $D_qD_p\varphi_0 = -qp\varphi_0$. Therefore

$$e^{-sX_0}O_pe^{sX_0}\varphi_0 = \left(\frac{1}{2}(\operatorname{ch}^2(s) + \operatorname{sh}^2(s)) - 2\operatorname{ch}(s)\operatorname{sh}(s)qp\right)\varphi_0 \ .$$

This can be interpreted as the sum of products of the first two orthonormal Hermite functions : if

$$h_0(x) = \pi^{-1/4}e^{-x^2/2}, \quad h_1(x) = \sqrt{2}xh_0(x) \ ,$$

then $\varphi_0(q,p) = h_0(q)h_0(p)$ and

$$e^{-sX_0}O_pe^{sX_0}\varphi_0 = \frac{1}{2}(\operatorname{ch}^2(s) + \operatorname{sh}^2(s))h_0(q)h_0(p) - \operatorname{ch}(s)\operatorname{sh}(s)h_1(q)h_1(p).$$

This type of tensor product forms an orthonormal family, so by the Pythagorean relation the square of the norm can be computed as the sum of squares of the coefficients :

$$\|O_p\varphi_s\|^2_{L^2(\mathbb{R}^2)} = \|e^{-sX_0}O_pe^{sX_0}\varphi_0\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{4}(\operatorname{ch}^2(s) + \operatorname{sh}^2(s))^2 + \operatorname{ch}^2(s)\operatorname{sh}^2(s) = \frac{1}{4}\operatorname{ch}(4s) \ .$$

Thus we deduce that

$$\begin{aligned}\|O_pu\|^2_{L^2} &\leq \frac{1}{L^2}\int_0^L e^{4s}ds = \frac{1}{4L^2}(e^{4L}-1)\\ &\leq \frac{1}{L^2}e^{4L} \ . \end{aligned} \tag{2.4.3}$$

The estimates in (2.4.1) and (2.4.2) taken with (2.4.3), allow us to establish that

$$\frac{\|K_{\nu,0}u\|^2_{L^2}}{\|u\|^2_{L^2}} \leq \frac{\|O_pu\|^2_{L^2} + \nu\|X_0u\|^2_{L^2}}{\|u\|^2_{L^2}} \leq c\,\frac{e^{4L} + \nu\left(1 - \frac{1}{\operatorname{ch}(L)}\right)}{L} \ .$$

Now letting $L = \frac{\log(\nu)}{4}$, we get the desired inequality

$$\|K_{\nu,0}u\|^2_{L^2} \leq c\,\frac{\nu}{\log(\nu)}\|u\|^2_{L^2} \ .$$

□

2.5 Degenerate one-dimensional case

Lemma 2.5.1. *Let $\lambda_1 \in \mathbb{R}$ be a parameter. Consider the operator*

$$K_1 = p\partial_q - \lambda_1\partial_p + \frac{1}{2}(-\partial_p^2 + p^2 - 1)$$

with domain $D(K_1) = \{u \in L^2(\mathbb{R}^2), \quad K_1 u \in L^2(\mathbb{R}^2)\}$. *There exists a constant* $c > 0$ *such that*

$$\|(D_q^2 + \lambda_1^2)e^{-t(K_1+1)}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq \frac{c}{t^3}$$

holds for all $t > 0$.

Proof. For each ξ_q fixed, there is a metaplectic operator on $L^2(\mathbb{R}_p)$ which, via conjugation, takes $ip.\xi_q - i\lambda_1 D_p$ to $ip\sqrt{\xi_q^2 + \lambda_1^2}$ while leaving O_p invariant. Taking the direct integral of this rotation (whose angle depends on ξ_q) gives a unitary equivalence between the operator K_1 and

$$\widehat{K_1} = \frac{1}{2}\Big(2ip\sqrt{D_q^2 + \lambda_1^2} + (-\partial_p^2 + p^2 - 1)\Big).$$

We also note that $\sqrt{D_q^2 + \lambda_1^2}$ is left invariant by the rotation in the variables (p, ξ_p).

It is shown in [Vio2] that

$$\|e^{-i(t_1+it_2)P_b}\|_{\mathcal{L}(L^2(\mathbb{R}))} = \exp\left(\frac{\cos(t_1) - \mathrm{ch}(t_2)}{\mathrm{sh}(t_2)} b^2\right)$$

for all $t_1 \in \mathbb{R}$ and all $t_2 < 0$, where $P_b = \frac{1}{2}\Big(D_x^2 + x^2 - 1 + 2ibx - b^2\Big), \quad b \in \mathbb{R}$. Applying this result with $t_1 = 0$, $t_2 = -t < 0$ and $b = b(\xi_q) = \sqrt{\xi_q^2 + \lambda_1^2}$, we obtain

$$\|\sqrt{D_q^2 + \lambda_1^2} e^{-t\widehat{K_1}}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq \sup_{\xi_q \in \mathbb{R}} \|b^2 e^{-t(P_b + \frac{b^2}{2})}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} = \sup_{\xi_q \in \mathbb{R}} b^2 e^{-\frac{t}{2}b^2} e^{(\frac{\mathrm{ch}(t)-1}{\mathrm{sh}(t)})b^2} .$$

(We remark that this inequality can be strengthened to an equality by taking the tensor product of explicit optimisers for the norm of e^{-tP_b} with functions in q localized in phase space near the optimising ξ_q.)

For all $t \in [0, 1]$, denote $f_b(t) = b^2 e^{(\frac{\mathrm{ch}(t)-1}{\mathrm{sh}(t)} - \frac{t}{2})b^2}$, and $u(t) = \frac{\mathrm{ch}(t)-1}{\mathrm{sh}(t)} - \frac{t}{2} = \mathrm{th}(\frac{t}{2}) - \frac{t}{2} < 0$.

Since $\max_{x \in \mathbb{R}} xe^{-ax} = \frac{e^{-a}}{a}$ when $a > 0$, we get

$$b^2 t^3 \exp\Big(u(t)b^2\Big) \leq \frac{-t^3 e^{u(t)}}{u(t)} =: F(t) .$$

The expansion $u(t) = \mathrm{th}(\frac{t}{2}) - \frac{t}{2} = \frac{-t^3}{24} + \mathcal{O}(t^4)$ yields $\lim_{t \to 0} F(t) = 24$ and the function F is bounded on the interval $[0, 1]$. Replacing b^2 with $\xi_q^2 + \lambda_1^2$, we conclude that, for $t \in [0, 1]$,

$$\|(D_q^2 + \lambda_1^2)e^{-t(K_1+1)}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq \frac{c}{t^3} .$$

For all $t \geq 1$, just write with $t_0 = \frac{1}{2}$,

$$\|(D_q^2 + \lambda_1^2)e^{-t(K_1+1)}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq \underbrace{\|(D_q^2 + \lambda_1^2)e^{-t_0 K_1}\|_{\mathcal{L}(L^2(\mathbb{R}^2))}}_{\leq \frac{c}{t_0}} \underbrace{\|e^{-(t-t_0)K_1}\|_{\mathcal{L}(L^2(\mathbb{R}^2))}}_{\leq 1} e^{-\frac{t}{2}} \leq \frac{c}{t^3} .$$

□

2.A Biquaternions

We define a biquaternion W as follows :

$$W = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

where a, b, c, d are complex numbers and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ multiply according to the rules

$$\begin{aligned} &\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1 \\ &\mathbf{ij} = -\mathbf{ji} = \mathbf{k} \\ &\mathbf{jk} = -\mathbf{kj} = \mathbf{i} \\ &\mathbf{ki} = -\mathbf{ik} = \mathbf{j} . \end{aligned}$$

For convenience we use a vector notation for biquaternions as follows :

$$W = a + v , \quad v = b\mathbf{i} + c\mathbf{j} + d\mathbf{k} .$$

The conjugate of a biquaternion W is given by

$$\text{conj}(W) = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k} .$$

The biquaternion ring B_Q is isomorphic to the matrix ring $\mathbb{M}_2(\mathbb{C})$. This can be seen via the following map :

$$\begin{aligned} &f : B_Q \to \mathbb{M}_2(\mathbb{C}) \\ &a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mapsto M = \begin{pmatrix} a + b\mathbf{i} & c + d\mathbf{i} \\ -c + d\mathbf{i} & a - b\mathbf{i} \end{pmatrix} . \end{aligned}$$

The "norm" $N(W)$ of a biquaternion W is

$$N(W) = \text{conj}(W)W = \det(M) = a^2 + b^2 + c^2 + d^2 .$$

Note that the norm is homogeneous of degree 2 and may take complex values. In particular, a biquaternion W is invertible if and only if $N(W) \neq 0$. In this case its inverse is given by

$$\mathrm{inv}(W) = \frac{\mathrm{conj}(W)}{N(W)} \ .$$

Exponential and spectrum.

Let $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = a + v$ be a biquarternion such that $N(v) \neq 0$. In this case $\widehat{v} = \frac{v}{\sqrt{N(v)}}$ verifies $\widehat{v}^2 = -1$.

Hence write

$$\begin{aligned} e^{a+v} &= e^a e^v = e^a e^{\widehat{v}\sqrt{N(v)}} \\ &= e^a \sum_{k=0}^{+\infty} \frac{(\widehat{v}\sqrt{N(v)})^k}{k!} \\ &= e^a \Big(\sum_{k=0}^{+\infty} \frac{(-1)^k (\sqrt{N(v)})^{2k}}{(2k)!} + \sum_{k=0}^{+\infty} \frac{(-1)^k (\sqrt{N(v)})^{2k+1}}{(2k+1)!} \widehat{v} \Big) \\ &= e^a \Big(\cos(\sqrt{N(v)}) + \frac{\sin(\sqrt{N(v)})}{\sqrt{N(v)}} v \Big) \ . \end{aligned} \tag{2.A.1}$$

The above computation do not depend on the choice of $\sqrt{N(v)}$ because cos is even and sin is odd.
Finally the set of $\lambda \in \mathbb{C}$ such that $(a + v - \lambda)$ is non-invertible can be explicitly determined :

$(a + v - \lambda)$ is non-invertible if and only if $0 = N(a + v - \lambda) = (a - \lambda)^2 + N(v)$ if and only if $\lambda \in \Big\{ a \pm \sqrt{-N(v)} \Big\}$.

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Chapitre 3

Estimations sous-elliptiques globales pour des opérateurs de KFP avec polynômes de degré supérieur à 2 (article rédigé en anglais)

Article [Ben], rédigé en anglais, soumis pour publication.

Global subelliptic estimates for Kramers-Fokker-Planck operators with some class of polynomials

Abstract

In this article we study some Kramers-Fokker-Planck operators with a polynomial potential $V(q)$ of degree greater than two having quadratic limiting behavior. This work provides an accurate global subelliptic estimate for KFP operators under some conditions imposed on the potential.

Key words : subelliptic estimates, compact resolvent, Kramers-Fokker-Planck operator.
MSC-2010 : 35Q84, 35H20, 35P05, 47A10, 14P10

Contents

3.1 Introduction and main results

The Kramers-Fokker-Planck operator reads

$$K_V = p\partial_q - \partial_q V(q)\partial_p + \frac{1}{2}(-\Delta_p + p^2) \ , \qquad (q,p) \in \mathbb{R}^{2d} \ , \tag{3.1.1}$$

where q denotes the space variable, p denotes the velocity variable, $x.y = \sum_{j=1}^{d} x_j y_j$, $x^2 = \sum_{j=1}^{d} x_j^2$ and the potential $V(q) = \sum_{|\alpha|\leq r} V_\alpha q^\alpha$ is a real-valued polynomial function on $\mathbb{R}^d$ with $d^\circ V = r$.

There have been several works concerned with the operator K_V with diversified approaches. In this article we impose some kind of assumptions on the polynomial potential $V(q)$, so that the Kramers-Fokker-Planck operator K_V admits a global subelliptic estimate and has a compact resolvent. This problem is closely related to the return to the equilibrium for the Kramers-Fokker-Planck operator (see [HeNi][Nie][Nou]). As mentioned in [HerNi] and [Nie], the analysis of K_V is also strongly linked to the one of the Witten Laplacian $\Delta_V^{(0)} = -\Delta_q + |\nabla V(q)|^2 - \Delta V(q)$. This relation yielded the Helffer-Nier conjecture stated by Helffer and Nier :

$$(1 + K_V)^{-1} \text{ compact} \Leftrightarrow (1 + \Delta_V^{(0)})^{-1} \text{ compact} \ . \tag{3.1.2}$$

This conjecture has been partially solved in basic cases (see for example [HeNi], [HerNi] and [Li]), whereas for the operator $\Delta_V^{(0)}$ very general criteria of compactness work for polynomial potential $V(q)$ of arbitrary degree. These last criteria require an analysis of the degeneracies at infinity of the potential and rely on extremely sophisticated tools of hypoellipticity developed by Helffer and Nourrigat in the 1980's (see [HeNo], [Nie]). Among the the particularities of these last analyses, we mention that the compactness results obtained for degenerate potentials at infinity were not the same for $\Delta_{+V}^{(0)}$ as $\Delta_{-V}^{(0)}$. The typical example which was considered is the case $V(q_1, q_2) = q_1^2 q_2^2$ in dimension $d = 2$: The operator $\Delta_{-V}^{(0)}$ has a compact resolvent, while $\Delta_{+V}^{(0)}$ has not.

In the case of the Kramers-Fokker-Planck operator, there have been extensive works concerned with the case $d^\circ V \leq 2$ (see [Hor][HiPr][Vio][Vio1][AlVi][BNV]). Nevertheless, as far as general potential is concerned, different kind of sufficient conditions on $V(q)$ had been examined by Hérau-Nier [HerNi], Helffer-Nier [HeNi], Villani [Vil] and Wei-Xi Li [Li]. These first results considered only variants of the elliptic situation at the infinity (for non-degenerate potential), which did not distinguish the sign $\pm V(q)$. Lately a significant improvement of those works has been done by Wei-Xi Li [Li2] based on some multipliers methods. In [Li2], Wei-Xi Li showed that for potentials similar to $V(q_1, q_2) = q_1^2 q_2^2$ the results for $K_{\pm V}$ were the same as for $\Delta_{\pm V}^{(0)}$, thus comforting the idea that the conjecture (3.1.2) is true.

The ultimate goal would be to develop a complete recurrence with respect to $d^\circ V$ for the Kramers-Fokker-Planck operator like it is possible to do for the Witten Laplacian as recalled in [HeNi] (cf. Theorem 10.16 page 106) and [Nie] by following the general approach of Helffer-Nourrigat in [HeNo] and [Nou]. Although we are not able to write a complete induction, we establish here subelliptic estimates for K_V for a rather general class of polynomial potentials with criteria which distinguish clearly the sign of $V(q)$. The asymptotic behaviour of those polynomials is governed by at most quadratic parameter dependent potentials, and the global subelliptic estimates in which arise some logarithmic weights are know to be essentially optimal in the quadratic case (see [BNV]).

Denoting

$$O_p = \frac{1}{2}(D_p^2 + p^2)\ ,$$

and

$$X_V = p\partial_q - \partial_q V(q)\partial_p\ ,$$

we can rewrite the Kramers-Fokker-Planck operator K_V defined in (3.1.1) as
$K_V = X_V + O_p$.
Notations : Throughout the paper we use the notation

$$\langle\cdot\rangle = \sqrt{1 + |\cdot|^2}\ .$$

For an arbitrary polynomial $V(q)$ of degree r, we denote for all $q \in \mathbb{R}^d$

$$\mathrm{Tr}_{+,V}(q) = \sum_{\substack{\nu\in\mathrm{Spec}(\mathrm{Hess}\,V(q))\\ \nu>0}} \nu(q)\ ,$$

$$\mathrm{Tr}_{-,V}(q) = -\sum_{\substack{\nu\in\mathrm{Spec}(\mathrm{Hess}\,V(q))\\ \nu\leq 0}} \nu(q)\ .$$

Furthermore, for a polynomial $P \in E_r := \{P \in \mathbb{R}[X_1, ..., X_d],\ d^\circ P \leq r\}$ and all natural number $n \in \{1, ..., r\}$, we define the functions $R_P^{\geq n} : \mathbb{R}^d \to \mathbb{R}$ and $R_P^{=n} : \mathbb{R}^d \to \mathbb{R}$ by

$$R_P^{\geq n}(q) = \sum_{n\leq|\alpha|\leq r} \left|\partial_q^\alpha P(q)\right|^{\frac{1}{|\alpha|}}\ , \tag{3.1.3}$$

$$R_P^{=n}(q) = \sum_{|\alpha|=n} |\partial_q^\alpha P(q)|^{\frac{1}{|\alpha|}}\ . \tag{3.1.4}$$

For arbitrary real functions A and B, we make also use of the following notation

$$A \asymp B \iff \exists\, c \geq 1 : c^{-1}\,|B| \leq |A| \leq c\,|B|\ .$$

This work is essentially based on the recent publication by Ben Said, Nier, and Viola [BNV], which deals with the analysis of Kramers-Fokker-Planck operators with polynomials of degree less than 3. In this case we define the constants A_V and B_V by

$$\begin{aligned} A_V &= \max\{(1+\mathrm{Tr}_{+,V})^{2/3}, 1+\mathrm{Tr}_{-,V}\}\ , \\ B_V &= \max\{\min_{q\in\mathbb{R}^d} |\nabla\, V(q)|^{4/3}\ , \frac{1+\mathrm{Tr}_{-,V}}{(\log(2+\mathrm{Tr}_{-,V}))^2}\}\ . \end{aligned}$$

As proved in [BNV], there is a constant $c>0$ such that the following global subelliptic estimate with remainder

$$\begin{aligned} \|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + A_V\|u\|^2_{L^2(\mathbb{R}^{2d})} \geq c\Big(& \|O_p u\|^2_{L^2(\mathbb{R}^{2d})} + \|X_V u\|^2_{L^2(\mathbb{R}^{2d})} \\ & + \|\langle\partial_q V(q)\rangle^{2/3}u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle D_q\rangle^{2/3}u\|^2_{L^2(\mathbb{R}^{2d})}\Big) \end{aligned} \tag{3.1.5}$$

holds for all $u\in\mathcal{C}_0^\infty(\mathbb{R}^{2d})$. Moreover, there exists a constant $c>0$ such that

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq c\, B_V\|u\|^2_{L^2(\mathbb{R}^{2d})}\ , \tag{3.1.6}$$

holds for all $u\in\mathcal{C}_0^\infty(\mathbb{R}^{2d})$. Hence combining (3.1.5) and (3.1.6), there is a constant $c>0$ so that

$$\begin{aligned} \|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{c}{1+\frac{A_V}{B_V}}\Big(& \|O_p u\|^2_{L^2(\mathbb{R}^{2d})} + \|X_V u\|^2_{L^2(\mathbb{R}^{2d})} \\ & + \|\langle\partial_q V(q)\rangle^{2/3}u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle D_q\rangle^{2/3}u\|^2_{L^2(\mathbb{R}^{2d})}\Big) \end{aligned} \tag{3.1.7}$$

is valid for all $u\in\mathcal{C}_0^\infty(\mathbb{R}^{2d})$. The constants appearing in (3.1.5), (3.1.6) and (3.1.7) are independent of the potential V and depend only on the dimension d and the degree of the polynomial V. We recall here that for a smooth potential $V\in\mathcal{C}^\infty(\mathbb{R}^d)$, our operator K_V is essential maximal accretive when endowed with the domain $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ [HeNi] (cf. Proposition 5.5 page 44). As a result the domain of its closure is given by

$$D(K_V) = \big\{u\in L^2(\mathbb{R}^{2d}),\ K_V u\in L^2(\mathbb{R}^{2d})\big\}\ .$$

Consequently by density of $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ in $D(K_V)$ all estimates stated in this paper, which are checked with $C_0^\infty(\mathbb{R}^{2d})$ functions, can be extended to the domain of K_V.

Given a polynomial $V(q)$ with degree r greater than two, our result will require the following assumption after setting for $\kappa>0$

$$\Sigma(\kappa) = \Big\{q\in\mathbb{R}^d,\ |\nabla V(q)|^{\frac{4}{3}} \geq \kappa\Big(|\mathrm{Hess}\, V(q)| + R_V^{\geq 3}(q)^4 + 1\Big)\Big\}\ ,$$

where $|\mathrm{Hess}\, V(q)|$ is the norm of the matrix $(\partial^2_{q_i,q_j}V(q))_{1\leq i,j\leq d}$.

Assumption 1. *There exist large constants $\kappa_0, C_1 > 1$ such that for all $\kappa \geq \kappa_0$ the polynomial $V(q)$ satisfies the following properties*

$$\mathrm{Tr}_{-,V}(q) > \frac{1}{C_1}\mathrm{Tr}_{+,V}(q)\ , \quad \textit{for all } q \in \mathbb{R}^d \setminus \Sigma(\kappa) \textit{ with } |q| \geq C_1\ , \tag{3.1.8}$$

moreover if $\mathbb{R}^d \setminus \Sigma(\kappa)$ *is not bounded*

$$\lim_{\substack{|q|\to+\infty \\ q\in\mathbb{R}^d\setminus\Sigma(\kappa)}} \frac{R_V^{\geq 3}(q)^4}{|\mathrm{Hess}\, V(q)|} = 0\ . \tag{3.1.9}$$

Those assumptions and in particular the partition $\mathbb{R}^d = \Sigma(\kappa) \sqcup (\mathbb{R}^d \setminus \Sigma(\kappa))$ have a simple interpretation. The region $\Sigma(\kappa)$ is the one where the gradient dominates the Hessian and the higher order derivatives so that the analysis in this region is essentially the same as in the various elliptic cases discussed in [HerNi][HeNi] and [Li]. On the contrary, the Hessian dominates the gradient and the derivatives of higher degree in the region $\mathbb{R}^d \setminus \Sigma(\kappa)$ and the accurate estimates of the quadratic model given by the second order Taylor expansion have to be used. Finally the parameter κ will be adjusted in the end of the proof so that the main subelliptic estimates control the error terms due to partitions of unity and Taylor expansions. Distinguishing the sign of the potential arises in particular when the region $\mathbb{R}^d \setminus \Sigma(\kappa)$ is considered. Actually $\mathrm{Tr}_{+,V}$ and $\mathrm{Tr}_{-,V}$ play different roles in the accurate subelliptic estimate without remainder (3.1.7) for polynomials of degree less than 3. Tarski-Seidenberg Theorem and some of its consequences reviewed in Appendix 3.B transforms the assumption (3.1.9) into $R_V^{\geq 3}(q)^4 = O(|\mathrm{Hess}\, V(q)|^{1-\nu})$ as $|q| \to +\infty$, $q \in \mathbb{R}^d \setminus \Sigma(\kappa)$, for some $\nu > 0$ (with $|\mathrm{Hess}\, V(q)| \to +\infty$ as $|q| \to +\infty$, $q \in \mathbb{R}^d \setminus \Sigma(\kappa)$). Alternatively one could simply assume from the beginning the existence of such a $\nu > 0$. We mention here that one knows that for a potential V satisfying assumption 1, the resolvent of the Witten Laplacian $\Delta_V^{(0)}$ is compact (since the asymptotic models at infinity are of degree less than 3 without a local minimum. C.f Theorem 10.16 [HeNi]).
In Section 3.4 of this paper we provide some explicit families of polynomial potentials for which the conditions (3.1.8), (3.1.9) both hold.

Our main result is the following.

Theorem 3.1.1. *Let $V(q)$ be a polynomial of degree r greater than two verifying Assumption 1. Then there exists a strictly positive constant $C_V > 1$ (depending on V) such that*

$$\|K_V u\|_{L^2}^2 + C_V\|u\|_{L^2}^2 \geq \frac{1}{C_V}\Big(\|L(O_p)u\|_{L^2}^2 + \|L(\langle\nabla V(q)\rangle^{\frac{2}{3}})u\|_{L^2}^2 + \|L(\langle \mathrm{Hess}\, V(q)\rangle^{\frac{1}{2}})u\|_{L^2}^2 + \|L(\langle D_q\rangle^{\frac{2}{3}})u\|_{L^2}^2\Big)\ , \tag{3.1.10}$$

holds for all $u \in D(K_V)$ where $L(s) = \frac{s+1}{\log(s+1)}$ for any $s \geq 1$.

Corollary 3.1.2. *If $V(q)$ is polynomial of degree greater than two that satisfies Assumption 1, then the Kramers-Fokker-Planck operator K_V has a compact resolvent.*

Démonstration. Proof of Corollary 4.1.3
Assume $0 < \delta < 1$. Define the functions $f_\delta : \mathbb{R}^d \to \mathbb{R}$ by

$$f_\delta(q) = |\nabla V(q)|^{\frac{4}{3}(1-\delta)} + |\mathrm{Hess}\, V(q)|^{1-\delta} \ .$$

From (3.1.10) in Theorem 3.1.1 there is a constant $C_V > 1$ such that

$$\|K_V u\|^2_{L^2} + C_V \|u\|^2_{L^2} \geq \frac{1}{C_V}\Big(\langle u, f_\delta u\rangle + \|L(O_p)u\|^2_{L^2} + \|L(\langle D_q\rangle^{\frac{2}{3}})u\|^2_{L^2}\Big) \ ,$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and all $\delta \in (0,1)$. In order to prove that the operator K_V has a compact resolvent it is sufficient to show that $\lim\limits_{|q|\to+\infty} f_\delta(q) = +\infty$.

To do so, assume $A > 0$ and denote $\kappa = A^{\frac{1}{1-\delta}}$. If $q \in \Sigma(\kappa)$, one has

$$|\nabla V(q)|^{\frac{4}{3}(1-\delta)} \geq \kappa^{1-\delta} = A \ .$$

If $q \in \mathbb{R}^d \setminus \Sigma(\kappa)$ by (3.1.9) in Assumption 1, $\lim\limits_{\substack{|q|\to+\infty \\ q\in\mathbb{R}^d\setminus\Sigma(\kappa)}} |\mathrm{Hess}\, V(q)| = +\infty$. Hence there exists a constant $\eta > 0$ such that $|\mathrm{Hess}\, V(q)|^{1-\delta} \geq A$ for all $q \in \mathbb{R}^d \setminus \Sigma(\kappa)$ with $|q| \geq \eta$.

□

Remark 3.1.3. *The results of Theorem 3.1.1 and Corollary 4.1.3 can be extended in the case when $V = V_1 + V_2$ where V_1 is polynomial satisfying Assumption 1 and V_2 is a function in $\mathcal{S}(\mathbb{R}^d)$.*

3.2 Preliminary results

This work is essentially based on two main strategies. The first one consists in the use of a partition of unity which is the most important tool that allows one to pass from local to global estimates.

In this paper, given a polynomial $V(q)$ we make use of a locally finite partition of unity with respect to the position variable $q \in \mathbb{R}^d$

$$\sum_{j\in\mathbb{N}} \chi_j^2(q) = \sum_{j\in\mathbb{N}} \widetilde{\chi}_j^2\Big(R_V^{\geq 3}(q_j)(q - q_j)\Big) = 1 \tag{3.2.1}$$

where

$$\text{supp } \widetilde{\chi}_j \subset B(0,a) \text{ and } \widetilde{\chi}_j \equiv 1 \text{ in } \quad B(0,b)$$

for some $q_j \in \mathbb{R}^d$ with $0 < b < a$ independent of $j \in \mathbb{N}$. In our work we need to choose the constant a less or equal to $\min(C^{-1}, C'^{-1})$, where the constants C and C' are those in Lemma 3.A.5. Such a partition is described more precisely in Lemma 3.A.8 after taking $n = 3$. In our study introducing this partition yields to errors that are under control.

The second approach lies in the decomposition of the operator K_V onto two parts so that the first one be a Kramers-Fokker-Planck operator with polynomial potential of degree less than three. On this way, based on [BNV], we derive the result of Theorem 3.1.1.

In order to prove Theorem 3.1.1 we need the following basic lemmas.

Lemma 3.2.1. *Assume $V \in \mathbb{R}[q_1, ..., q_d]$ with degree $r \in \mathbb{N}$. Consider the Kramers-Fokker-Planck operator K_V defined as in (3.1.1). For a locally finite partition of unity namely $\sum_{j\in\mathbb{N}} \chi_j^2(q) = 1$ one has*

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} = \sum_{j\in\mathbb{N}} \Big(\|K_V(\chi_j u)\|^2_{L^2(\mathbb{R}^{2d})} - \|(p\partial_q \chi_j)u\|^2_{L^2(\mathbb{R}^{2d})} \Big) \ , \tag{3.2.2}$$

for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

In particular when the degree of V is larger than two and the cutoff functions χ_j have the form (3.2.1), there exists a constant $c_{d,r} > 0$ (depending only on the dimension d and the degree of V) so that

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq \sum_{j\in\mathbb{N}} \Big(\|K_V(\chi_j u)\|^2_{L^2(\mathbb{R}^{2d})} - c_{d,r} R_V^{\geq 3}(q_j)^2 \|p\chi_j u\|^2_{L^2(\mathbb{R}^{2d})} \Big) \ , \tag{3.2.3}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

Proof. First let V be a real-valued polynomial on $\mathbb{R}^d$ of degree $r \in \mathbb{N}$. Assume that $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. On the one hand,

$$\|K_V u\|^2_{L^2} = \sum_{j\in\mathbb{N}} \langle K_V u, \chi_j^2 K_V u\rangle = \sum_{j\in\mathbb{N}} \langle u, K_V^* \chi_j^2 K_V u\rangle \ .$$

On the other hand,

$$\sum_{j\in\mathbb{N}} \|K_V(\chi_j u)\|^2_{L^2} = \sum_{j\in\mathbb{N}} \langle u, \chi_j K_V^* K_V \chi_j u\rangle \ .$$

Putting the above equalities together

$$\|K_V u\|^2_{L^2} - \sum_{j\in\mathbb{N}} \|K_V(\chi_j u)\|^2_{L^2} = \sum_{j\in\mathbb{N}} \langle u, (K_V^* \chi_j^2 K_V - \chi_j K_V^* K_V \chi_j) u\rangle \ .$$

Using commutators, we compute

$$\begin{aligned}K_V^*\chi_j^2K_V &= K_V^*\chi_j[\,\chi_j, K_V\,] + K_V^*\chi_j K_V\chi_j\\ &= K_V^*\chi_j[\,\chi_j, K_V\,] + [\,K_V^*, \chi_j\,]K_V\chi_j + \chi_j K_V^* K_V\chi_j\\ &= K_V^*\chi_j[\,\chi_j, K_V\,] + [\,K_V^*, \chi_j\,]\Big([\,K_V, \chi_j\,] + \chi_j K_V\Big) + \chi_j K_V^* K_V\chi_j\ .\end{aligned}$$

Thus

$$K_V^*\chi_j^2K_V - \chi_j K_V^* K_V\chi_j = K_V^*\chi_j[\,\chi_j, K_V\,] + [\,K_V^*, \chi_j\,]\chi_j K_V + [\,K_V^*, \chi_j\,]\circ[\,K_V, \chi_j\,]\ .$$

Now it is easy to check the following commutation relations

$$\begin{cases}[\,\chi_j, K_V\,] = -[\,K_V, \chi_j\,] = -[\,p\partial_q, \chi_j(q)\,] = -p\partial_q\chi_j\\ [\,K_V^*, \chi_j\,] = [\,-p\partial_q, \chi_j(q)\,] = -p\partial_q\chi_j\\ [\,K_V^*, \chi_j\,]\circ[\,K_V, \chi_j\,] = -(p\partial_q\chi_j)^2\ .\end{cases}$$

Collecting the terms, we obtain

$$\begin{aligned}\sum_{j\in\mathbb{N}}(K_V^*\chi_j^2K_V - \chi_j K_V^* K_V\chi_j) &= \sum_{j\in\mathbb{N}}\Big(K_V^*\chi_j(-p\partial_q\chi_j) + (-p\partial_q\chi_j)\chi_j K_V - (p\partial_q\chi_j)^2\Big)\\ &= \sum_{j\in\mathbb{N}}\Big(K_V^*\Big(-p\partial_q(\frac{\chi_j^2}{2})\Big) - p\partial_q(\frac{\chi_j^2}{2})K_V - (p\partial_q\chi_j)^2\Big)\\ &= -\sum_{j\in\mathbb{N}}(p\partial_q\chi_j)^2\ ,\end{aligned}$$

where in the last line we make use of $\sum_{j\in\mathbb{N}}\chi_j^2(q) = 1$.

From this it follows immediately that

$$\|K_V u\|_{L^2}^2 = \sum_{j\in\mathbb{N}}\Big(\|K_V(\chi_j u)\|_{L^2}^2 - \|(p\partial_q\chi_j)u\|_{L^2}^2\Big)$$

for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

Next, suppose that the degree of V is greater than two and $\chi_j(q) = \widetilde{\chi}_j\Big(R_V^{\geq 3}(q_j)(q - q_j)\Big)$ for all indices j and any $q \in \mathbb{R}^d$ with

$$\text{supp}\ \widetilde{\chi}_j \subset B(0,a) \text{ and }\ \widetilde{\chi}_j \equiv 1 \text{ in }\ B(0,b)\ .$$

Then we can write

$$\begin{aligned}\sum_{j\in\mathbb{N}}\|(p\partial_q\chi_j)u\|^2 &= \sum_{j\in\mathbb{N}}\sum_{j'\in\mathbb{N}}\|(p\partial_q\chi_j)\chi_{j'}u\|^2\\ &\leq c_{d,r}\sum_{j\in\mathbb{N}}R_V^{\geq 3}(q_j)^2\|p\chi_j u\|^2\ ,\end{aligned}$$

where $c_{d,r}$ is a constant that depends only on the dimension d and the degree of V. Here the last inequality is due to the fact that for each index j there are finitely many j' such that $(\partial_q \chi_j)\chi_{j'}$ is nonzero. □

Before stating the following lemma, we fix and remind some notations.

Notations 3.2.2. *Let V be a polynomial of degree r larger than two. Consider a locally finite partition of unity $\sum_{j\in\mathbb{N}} \chi_j^2(q) = 1$ described as in (3.2.1).*

Set for all $\kappa > 0$

$$J(\kappa) = \Big\{ j \in \mathbb{N},\ \textit{such that}\ \operatorname{supp} \chi_j \subset \Sigma(\kappa) \Big\}\ ,$$

where we recall that

$$\Sigma(\kappa) = \Big\{ q \in \mathbb{R}^d,\ |\nabla V(q)|^{\frac{4}{3}} \geq \kappa\Big(|\operatorname{Hess} V(q)| + R_V^{\geq 3}(q)^4 + 1 \Big) \Big\}\ .$$

For a given $\kappa > 0$ and all indices $j \in \mathbb{N}$, let $V_j^{(2)}$ be the polynomial of degree less than three given by

$$V_j^{(2)}(q) = \sum_{0\leq|\alpha|\leq 2} \frac{\partial_q^\alpha V(q_j')}{\alpha!}(q - q_j')^\alpha\ , \tag{3.2.4}$$

where

$$\begin{cases} q_j' = q_j & \textit{if } j \in J(\kappa) \\ q_j' \in (\operatorname{supp} \chi_j) \cap \Big(\mathbb{R}^d \setminus \Sigma(\kappa)\Big) & \textit{otherwise.} \end{cases}$$

Lemma 3.2.3. *Assume V a polynomial of degree r larger than two. Consider a locally finite partition of unity described as in (3.2.1). For a multi-index $\alpha \in \mathbb{N}^d$ of length $|\alpha| \in \{1, 2\}$ and all $j \in \mathbb{N}$, one has*

$$\Big| \partial_q^\alpha V(q) - \partial_q^\alpha V_j^{(2)}(q) \Big| \leq c_{\alpha,d,r} \Big(R_V^{\geq 3}(q_j') \Big)^{|\alpha|} \tag{3.2.5}$$

for any $q \in \operatorname{supp} \chi_j = B(q_j, a R_V^{\geq 3}(q_j)^{-1})$, where $c_{\alpha,d,r} = \sum_{3\leq|\beta|\leq r} (a\,C)^{|\beta|-|\alpha|}$.

As a consequence, if V satisfies Assumption 1, there exists a large constant $\kappa_1 \geq \kappa_0$ so that for all $\kappa \geq \kappa_1$ and every $j \in \mathbb{N}$,

$$2^{-1} \Big| \partial_q V_j^{(2)}(q) \Big| \leq |\partial_q V(q)| \leq 2 \Big| \partial_q V_j^{(2)}(q) \Big|\ , \tag{3.2.6}$$

for every $q \in (\operatorname{supp} \chi_j) \cap \Sigma(\kappa)$ and

$$2^{-1} \Big| \operatorname{Hess} V_j^{(2)}(q) \Big| \leq |\operatorname{Hess} V(q)| \leq 2 \Big| \operatorname{Hess} V_j^{(2)}(q) \Big|\ , \tag{3.2.7}$$

for any $q \in (\operatorname{supp} \chi_j) \cap \Big(\mathbb{R}^d \setminus \Sigma(\kappa)\Big)$ with $|q| \geq C_2(\kappa)$ where $C_2(\kappa) > 0$ is a large constant that depends on κ.

Proof. Let V be a polynomial of degree r greater than two. In this proof we are going to need the following equivalence

$$R_V^{\geq 3}(q) \asymp R_V^{\geq 3}(q') \,, \tag{3.2.8}$$

satisfied for all $q, q' \in \operatorname{supp} \chi_j$ and proved in Lemma 3.A.5. That is, there is a constant $C > 1$ such that for every $q, q' \in \operatorname{supp} \chi_j$,

$$\Big(\frac{R_V^{\geq 3}(q)}{R_V^{\geq 3}(q')}\Big)^{\pm 1} \leq C \,. \tag{3.2.9}$$

Assume $\alpha \in \mathbb{N}^d$ of length $|\alpha| \in \{1, 2\}$. For every $j \in \mathbb{N}$, observe that

$$\begin{aligned} \Big|\partial_q^\alpha V(q) - \partial_q^\alpha V_j^{(2)}(q)\Big| &= \Big| \sum_{\substack{3 \leq |\beta| \leq r \\ \beta \geq \alpha}} \frac{\partial_q^\beta V(q_j')}{(\beta - \alpha)!} (q - q_j')^{\beta - \alpha} \Big| \\ &\leq \sum_{\substack{3 \leq |\beta| \leq r \\ \beta \geq \alpha}} \frac{\big|\partial_q^\beta V(q_j')\big|}{(\beta - \alpha)!} \ \big|q - q_j'\big|^{|\beta| - |\alpha|} \,, \end{aligned}$$

for any $q \in \mathbb{R}^d$. Hence regarding the equivalence (3.2.9), there exists a constant $c_{\alpha,d,r} > 0$ (depending as well on the multi-index α, the dimension d and the degree r of V) so that

$$\begin{aligned} \Big|\partial_q^\alpha V(q) - \partial_q^\alpha V_j^{(2)}(q)\Big| &\leq \sum_{\substack{3 \leq |\beta| \leq r \\ \beta \geq \alpha}} \frac{1}{(\beta - \alpha)!} \Big(R_V^{\geq 3}(q_j')\Big)^{|\beta|} \Big(a^{-1} R_V^{\geq 3}(q_j)\Big)^{-|\beta| + |\alpha|} \\ &\leq \sum_{\substack{3 \leq |\beta| \leq r \\ \beta \geq \alpha}} \frac{1}{(\beta - \alpha)!} (a\, C)^{|\beta| - |\alpha|} \Big(R_V^{\geq 3}(q_j')\Big)^{|\alpha|} \\ &\leq c_{\alpha,d,r} \Big(R_V^{\geq 3}(q_j')\Big)^{|\alpha|} \,, \end{aligned} \tag{3.2.10}$$

holds for all q in the support of χ_j, where the constant $C > 1$ is the one in (3.2.9) and $c_{\alpha,d,r} = \sum\limits_{3 \leq |\beta| \leq r} (a\, C)^{|\beta| - |\alpha|}$.

In the rest of the proof, let the polynomial $V(q)$ satisfies Assumption 1. In view of (3.2.10), we get when $|\alpha| = 1$

$$\Big|\nabla V(q) - \nabla V_j^{(2)}(q)\Big| \leq c_{1,d,r} \ R_V^{\geq 3}(q_j') \,, \tag{3.2.11}$$

for all $j \in \mathbb{N}$ and any $q \in \operatorname{supp} \chi_j$, where $c_{1,d,r} = \sum\limits_{3 \leq |\beta| \leq r} (a\, C)^{|\beta| - 1}$. By virtue of the equivalence (3.2.9), it results from (3.2.11)

$$\Big|\nabla V(q) - \nabla V_j^{(2)}(q)\Big| \leq c_{1,d,r} C \ R_V^{\geq 3}(q) \,, \tag{3.2.12}$$

for every $q \in \text{supp}\,\chi_j$. Given $\kappa \geq \kappa_0$, it follows from (3.2.12) and the definition of $\Sigma(\kappa)$ that

$$\left|\nabla V(q) - \nabla V_j^{(2)}(q)\right| \leq \frac{c_{1,d,r}C}{\kappa^{\frac{1}{4}}} |\nabla V(q)|^{\frac{1}{3}} \leq \frac{c_{1,d,r}C}{\kappa^{\frac{1}{4}}} |\nabla V(q)| \tag{3.2.13}$$

for all $q \in (\text{supp}\,\chi_j) \cap \Sigma(\kappa)$. For the above second inequality we know that $|\nabla V(q)| \geq 1$ for every $q \in (\text{supp}\,\chi_j) \cap \Sigma(\kappa)$, indeed since $q \in (\text{supp}\,\chi_j) \cap \Sigma(\kappa)$,

$$|\nabla V(q)| \geq \kappa^{\frac{3}{4}} \geq \kappa_0^{\frac{3}{4}} \geq 1 \ .$$

Taking the constant $\kappa_1 \geq \kappa_0$ such that $\frac{c_{1,d,r}C}{\kappa_1^{\frac{1}{4}}} \leq \frac{1}{2}$, we get for every $\kappa \geq \kappa_1$

$$\Big|\, |\nabla V(q)| - |\nabla V_j^{(2)}(q)| \Big| \leq |\nabla V(q) - \nabla V_j^{(2)}(q)| \leq \frac{1}{2}|\nabla V(q)| \ ,$$

for any $q \in (\text{supp}\,\chi_j) \cap \Sigma(\kappa)$. Therefore, for every $\kappa \geq \kappa_1$,

$$\frac{1}{2}|\nabla V_j^{(2)}(q)| \leq |\nabla V(q)| \leq \frac{3}{2}|\nabla V_j^{(2)}(q)|$$

holds for all $q \in (\text{supp}\,\chi_j) \cap \Sigma(\kappa)$.

On the other hand when $|\alpha| = 2$, by (3.2.10) there is a constant $c_{2,d,r} > 0$ so that for all $j \in \mathbb{N}$

$$|\partial_q^\alpha V(q) - \partial_q^\alpha V_j^{(2)}(q)| \leq c_{2,d,r} R_V^{\geq 3}(q_j')^2 \tag{3.2.14}$$

holds for every $q \in \text{supp}\,\chi_j$, where $c_{2,d,r} = \sum\limits_{3 \leq |\beta| \leq r} (a\, C)^{|\beta|-2}$.

Using the fact that $R_V^{\geq 3}(q) \geq R_V^{=r}(0)$ for every $q \in \mathbb{R}^d$, we derive from (3.2.14) that

$$|\partial_q^\alpha V(q) - \partial_q^\alpha V_j^{(2)}(q)| \leq c_{2,d,r} \frac{R_V^{\geq 3}(q_j')^4}{R_V^{=r}(0)^2} \ ,$$

for all $q \in \text{supp}\,\chi_j$.

Assuming $\kappa \geq \kappa_0$, we obtain using (3.1.9) in Assumption 1, if $|q_j'|$ is large enough

$$\Big| \sum_{|\beta|=2} |\partial_q^\beta V(q)| - \sum_{|\beta|=2} |\partial_q^\beta V_j^{(2)}(q)| \Big| \leq \sum_{|\beta|=2} |\partial_q^\beta V(q) - \partial_q^\beta V_j^{(2)}(q)| \leq \frac{1}{2}|\text{Hess}\, V(q_j')| \ ,$$

for any $q \in (\text{supp}\,\chi_j) \cap \Big(\mathbb{R}^d \setminus \Sigma(\kappa)\Big)$. In other words,

$$\frac{1}{2}|\text{Hess}\, V_j^{(2)}(q)| \leq |\text{Hess}\, V(q)| \leq \frac{3}{2}|\text{Hess}\, V_j^{(2)}(q)|$$

holds for all $q \in (\text{supp}\,\chi_j) \cap \Big(\mathbb{R}^d \setminus \Sigma(\kappa)\Big)$ with $|q| \geq C_2(\kappa)$ where $C_2(\kappa)$ is a strictly positive large constant depending on κ. □

Lemma 3.2.4. *Consider two positive operators A and B on a Hilbert space $\mathcal{H}$ such that*

$$\nu\|u\|^2 < \langle u, Au\rangle \leq \langle u, Bu\rangle$$

for all $u \in \mathcal{D}$ where $\mathcal{D}$ is dense in $D(B^{1/2})$ with $\nu > 1$. For all $\alpha_0 \in [0,1]$ and $k \in \mathbb{N}$, there exists $C_{k,\alpha_0,\nu} > 1$ such that

$$\langle u, \frac{A^{\alpha_0}}{(\log(A^{\alpha_0/2}))^k} u\rangle \leq C_{k,\alpha_0,\nu}\langle u, \frac{B^{\alpha_0}}{(\log(B^{\alpha_0/2}))^k} u\rangle \ , \tag{3.2.15}$$

for any $u \in \mathcal{D}$.

Proof. Assume that A, B are two positive operators so that

$$\nu\|u\|^2 < \langle u, Au\rangle \leq \langle u, Bu\rangle \ , \tag{3.2.16}$$

holds for all $u \in \mathcal{D}$ with $\nu > 1$. For $\alpha \in [0,1]$, the application $T \mapsto T^\alpha$ is operator monotone according to Example 6.8 in [Sim]. This provides the inequality

$$\nu^\alpha\|u\|^2 < \langle u, A^\alpha u\rangle \leq \langle u, B^\alpha u\rangle \ , \tag{3.2.17}$$

for any $u \in \mathcal{D}$ and every $\alpha \in [0,1]$, which is of course related with interpolation in Hilbert spaces.

Furthermore, for any positive operator $C \geq c\,\mathrm{Id}_{\mathcal{H}}\,,\ c > 0\,,$ with domain $D(C)$ we define its logarithm defined by the functional calculus satisfies for all $u \in D(C)$ and all $v \in \mathcal{H}$

$$\langle v, \log(C)u\rangle = \lim_{\alpha\to 0^+}\langle v, \frac{C^\alpha - 1}{\alpha} u\rangle \ . \tag{3.2.18}$$

Using (3.2.16),

$$\underbrace{\log(\nu)}_{>0}\|u\|^2 < \langle u, \log(A)u\rangle \leq \langle u, \log(B)u\rangle \ , \tag{3.2.19}$$

holds for all $u \in \mathcal{D}$. Integrating (3.2.17) with respect to α over $[0, \alpha_0]$ where $\alpha_0 \in [0,1]$ we get

$$\langle u, \frac{1}{\log(A)}(A^{\alpha_0} - I)u\rangle \leq \langle u, \frac{1}{\log(B)}(B^{\alpha_0} - I)u\rangle \ . \tag{3.2.20}$$

Furthermore by (3.2.19)

$$\langle u, \frac{1}{\log(B)} u\rangle \leq \langle u, \frac{1}{\log(A)} u\rangle < \frac{1}{\log(\nu)}\|u\|^2 \ . \tag{3.2.21}$$

Therefore from (3.2.20) and (3.2.21), for any $\alpha_0 \in [0,1]$ there exist $C_\nu, C_{1,\alpha_0,\nu} > 1$ such that

$$\langle u, \frac{A^{\alpha_0}}{\log(A)} u\rangle \le \langle u, \frac{B^{\alpha_0}}{\log(B)} u\rangle + C_\nu \|u\|^2 \le C_{1,\alpha_0,\nu} \langle u, \frac{B^{\alpha_0}}{\log(B)} u\rangle \ .$$

Once the constant $C_{k,\alpha_0,\nu} \ge 1$ is known for $k \ge 1$, the same integration with respect to $\alpha \in [0, \alpha_0]$ provides the constant $C_{k+1,\alpha_0,\nu} \ge 1$. We proved by induction on $k \in \mathbb{N}$, the existence of a constant $C_{k,\alpha_0,\nu} > 1$ such that

$$\langle u, \frac{A^{\alpha_0}}{(\log(A))^k} u\rangle \le C_{k,\alpha_0,\nu} \langle u, \frac{B^{\alpha_0}}{(\log(B))^k} u\rangle \ ,$$

or equivalently

$$\langle u, \frac{A^{\alpha_0}}{(\log(A^{\alpha_0/2}))^k} u\rangle \le C_{k,\alpha_0,\nu} \langle u, \frac{B^{\alpha_0}}{(\log(B^{\alpha_0/2}))^k} u\rangle \ .$$

□

Lemma 3.2.5. *Assume $V(q)$ a polynomial of degree r greater than two. Let $\sum_{j\in\mathbb{N}} \chi_j^2(q)$ be a locally finite partition of unity defined as in (3.2.1). For each $j \in \mathbb{N}$, choose any $q_j' \in \operatorname{supp} \chi_j$.*

There is a constant $c = c(d,r) > 1$ such that

$$\langle u, (2 - \Delta_q + R_V^{\ge 3}(q)^4)^\alpha u\rangle \le c \sum_{j\in\mathbb{N}} \langle u, \chi_j (2 - \Delta_q + R_V^{\ge 3}(q_j')^4)^\alpha \chi_j u\rangle \ , \tag{3.2.22}$$

is valid for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and any $\alpha \in [0,1]$.

As a consequence, there exists a constant $\widetilde{c} = \widetilde{c}(d,r) > 1$ so that

$$\sum_{j\in\mathbb{N}} \|L\Big((1 - \Delta_q + R_V^{\ge 3}(q_j')^4)^{\frac{1}{3}}\Big)\chi_j u\|^2 \ge \frac{1}{c} \|L\Big((1 - \Delta_q + R_V^{\ge 3}(q)^4)^{\frac{1}{3}}\Big) u\|^2 \ , \tag{3.2.23}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$, where $L(s) = \frac{s+1}{\log(s+1)}$ for all $s \ge 1$.

Proof. We first set $E_0 = L^2(\mathbb{R}^{2d})$ and $E_1 = \Big\{u \in L^2(\mathbb{R}^{2d}),\ \langle u, (2 - \Delta_q + R_V^{\ge 3}(q)^4)u\rangle < +\infty\Big\}$ endowed respectively with the norms $\|\cdot\|_{E_0} = \|\cdot\|_{L^2(\mathbb{R}^{2d})}$ and $\|\cdot\|_{E_1}$ defined as follows for all $u \in L^2(\mathbb{R}^{2d})$

$$\begin{aligned} \|u\|_{E_1}^2 &= 2\|u\|_{L^2(\mathbb{R}^{2d})}^2 + \|D_q u\|_{L^2(\mathbb{R}^{2d})}^2 + \|R_V^{\ge 3}(q)^2 u\|_{L^2(\mathbb{R}^{2d})}^2 \\ &= \|(2 - \Delta_q + R_V^{\ge 3}(q)^4)^{1/2} u\|_{L^2(\mathbb{R}^{2d})}^2 \ . \end{aligned}$$

By Theorem X.29 in [ReSi1], the operator $2-\Delta_q+R_V^{\geq 3}(q)^4$ is essentially self adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and hence E_1 corresponds to the spectrally defined subspace of $L^2(\mathbb{R}^{2d})$.

Given a partition of unity as in (3.2.1), define the linear map

$$T: E_0 \to (L^2(\mathbb{R}^{2d}))^{\mathbb{N}},\ u \mapsto (u_j)_{j\in\mathbb{N}} = (\chi_j u)_{j\in\mathbb{N}}\ ,$$

and denote $F_0 := \operatorname{Im} T$. Notice that $T: E_0 \to F_0$ is unitary. Indeed for all $u \in E_0$,

$$\|Tu\|_{F_0}^2 = \sum_{j\in\mathbb{N}} \|\chi_j u\|_{L^2}^2 = \|u\|_{L^2}^2 = \|u\|_{E_0}^2\ , \tag{3.2.24}$$

further the inverse map of T is well defined by

$$T^{-1}: F_0 \to E_0,\ (u_j)_{j\in\mathbb{N}} \mapsto u = \sum_{j\in\mathbb{N}} \chi_j u_j\ .$$

Now introduce the set

$$F_1 = \left\{ (u_j)_{j\in\mathbb{N}} \in F_0,\ \sum_{j\in\mathbb{N}} \langle u_j, (2-\Delta_q+R_V^{\geq 3}(q_j')^4)u_j\rangle < +\infty \right\}\ ,$$

with its associated norm defined for all $(u_j)_{j\in\mathbb{N}} \in F_1$ by

$$\begin{aligned}\|(u_j)_{j\in\mathbb{N}}\|_{F_1}^2 &= \sum_{j\in\mathbb{N}} \left(2\|u_j\|_{L^2(\mathbb{R}^{2d})}^2 + \|D_q u_j\|_{L^2(\mathbb{R}^{2d})}^2 + \|R_V^{\geq 3}(q_j')^2 u_j\|_{L^2(\mathbb{R}^{2d})}^2 \right) \\ &= \sum_{j\in\mathbb{N}} \|(2-\Delta_q+R_V^{\geq 3}(q_j')^4)^{1/2} u_j\|_{L^2(\mathbb{R}^{2d})}^2\ .\end{aligned}$$

Assume $u \in E_0$. For all $j \in \mathbb{N}$, let $q_j' \in \operatorname{supp} \chi_j$. Observe that

$$\begin{aligned}|\ \|Tu\|_{F_1}^2 - \|u\|_{E_1}^2| &= |\sum_{j\in\mathbb{N}} \langle u_j, (2-\Delta_q+R_V^{\geq 3}(q_j')^4)u_j\rangle - \langle u, (2-\Delta_q+R_V^{\geq 3}(q)^4)u\rangle| \\ &= |\sum_{j\in\mathbb{N}} \langle u_j, -\Delta_q u_j\rangle - \langle u, -\Delta_q u\rangle + \sum_{j\in\mathbb{N}} \langle u_j, (R_V^{\geq 3}(q_j')^4 - R_V^{\geq 3}(q)^4)u_j\rangle| \\ &\leq |\sum_{j\in\mathbb{N}} \langle u_j, -\Delta_q u_j\rangle - \langle u, -\Delta_q u\rangle| + \sum_{j\in\mathbb{N}} \langle u_j, |R_V^{\geq 3}(q_j')^4 - R_V^{\geq 3}(q)^4| u_j\rangle\ .\end{aligned} \tag{3.2.25}$$

Since we are dealing with cutoff functions satisfying $\sum_{j\in\mathbb{N}} |\nabla\chi_j|^2 \leq c\, R_V^{\geq 3}(q)^2$ and owing to the equivalence $R_V^{\geq 3}(q) \asymp R_V^{\geq 3}(q_j')$ for all $q \in \operatorname{supp} \chi_j$, it follows from (3.2.25)

$$|\ \|Tu\|_{F_1}^2 - \|u\|_{E_1}^2| \leq c_1 \sum_{j\in\mathbb{N}} \langle u_j, R_V^{\geq 3}(q_j')^4 u_j\rangle \leq c_1 \|Tu\|_{F_1}^2\ ,$$

and

$$| \, \|Tu\|_{F_1}^2 - \|u\|_{E_1}^2 | \leq c_1' \langle u, R_V^{\geq 3}(q)^4 u \rangle \leq c_1' \|u\|_{E_1}^2 \ ,$$

where c_1, c_1' are two strictly positive constants. As a result

$$\frac{1}{\sqrt{(c_1+1)}} \|u\|_{E_1} \leq \|Tu\|_{F_1} \leq \sqrt{(c_1'+1)} \|u\|_{E_1} \ . \tag{3.2.26}$$

In view of (3.2.24) and (3.2.26), we conclude by interpolation that for all $\alpha \in [0,1]$

$$T : E_\alpha \to F_\alpha,$$

verifies $\|T\|_{\mathcal{L}(E_\alpha, F_\alpha)} \leq (c_1'+1)^{\frac{\alpha}{2}}$ and $\|T^{-1}\|_{\mathcal{L}(F_\alpha, E_\alpha)} \leq (c_1+1)^{\frac{\alpha}{2}}$, where E_α and F_α are two interpolated spaces endowed respectively with the norms

$$\|u\|_{E_\alpha} = \|(2 - \Delta_q + R_V^{\geq 3}(q)^4)^{\alpha/2} u\|_{L^2(\mathbb{R}^{2d})} \ ,$$

and

$$\|(v_j)_{j\in\mathbb{N}}\|_{F_\alpha} = \sum_{j\in\mathbb{N}} \|(2 - \Delta_q + R_V^{\geq 3}(q_j')^4)^{\alpha/2} u_j\|_{L^2(\mathbb{R}^{2d})} \ .$$

Hence there is a constant $c > 0$ so that

$$\langle u, (2 - \Delta_q + R_V^{\geq 3}(q)^4)^\alpha u \rangle \leq c \sum_{j\in\mathbb{N}} \langle u_j, (2 - \Delta_q + R_V^{\geq 3}(q_j')^4)^\alpha u_j \rangle \ , \tag{3.2.27}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and any $\alpha \in [0,1]$. In order to prove (3.2.23), repeat the same process as in Lemma 3.2.4. Starting with

$$2^\alpha \|u\|^2 \leq \langle u, (2 - \Delta_q + R_V^{\geq 3}(q)^4)^\alpha u \rangle \leq c \sum_{j\in\mathbb{N}} \langle u_j, (2 - \Delta_q + R_V^{\geq 3}(q_j')^4)^\alpha u_j \rangle \ , \tag{3.2.28}$$

for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and any $\alpha \in [0,1]$, use the functional calculus in the left-hand side and the Fourier transform in the right-hand side. When integrating with respect to $\alpha \in [0, \frac{2}{3}]$ we can interchange for any fixed $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ the sum and the integral in the right-hand side of (3.2.28) since the partition of unity is locally finite. This leads to

$$\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}, \quad \langle u, \phi(1 - \Delta_q + R_V^{\geq 3}(q)^4) u \rangle \leq c' \sum_{j\in\mathbb{N}} \langle u, \phi(1 - \Delta_q + R_V^{\geq 3}(q_j')^4) u \rangle$$

with $\phi(t) = \frac{(1+t)^{1/3}}{\log((1+t)^{1/3}}$. By referring again to the functional calculus for the left-hand side and the Fourier transform for the right-hand side, the proof is finished after noticing the uniform equivalence

$$\sup_{t\in[1,+\infty)} \left(\frac{\phi(t)}{\psi(t)}\right)^{\pm 1} \leq \mu$$

when $\psi(t) = \frac{1+t^{1/3}}{\log(1+t^{1/3}}$. □

3.3 Proof of Theorem 3.1.1

In this section we present the proof of Theorem 3.1.1. In the sequel for a given polynomial $V(q)$ with degree r greater than two, we always use a locally finite partition of unity

$$\sum_{j\in\mathbb{N}} \chi_j^2(q) = \sum_{j\in\mathbb{N}} \widetilde{\chi}_j^2\Big(R_V^{\geq 3}(q_j)(q-q_j)\Big) = 1 \ ,$$

where

$$\text{supp } \widetilde{\chi}_j \subset B(0,a) \text{ and } \ \widetilde{\chi}_j \equiv 1 \text{ in } \ B(0,b)$$

for some $q_j \in \mathbb{R}^d$ with $0 < b < a$ independent of the natural numbers j, defined more specifically as in Lemma 3.A.8 with $n = 3$. As mentioned before we choose the constant a less or equal to $\min(C^{-1}, C'^{-1})$, where the constants C and C' are those in Lemma 3.A.5.

Proof. Let $V(q)$ be a polynomial with degree larger than two that satisfies Assumption 1. Assume $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. In the whole proof we denote $u_j = \chi_j u$ for all natural number j.

From Lemma 3.2.1 we get

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq \sum_{j\in\mathbb{N}} \Big(\|K_V u_j\|^2_{L^2(\mathbb{R}^{2d})} - c_{d,r} R_V^{\geq 3}(q_j)^2 \|p u_j\|^2_{L^2(\mathbb{R}^{2d})}\Big) \ . \tag{3.3.1}$$

Given $\kappa \geq \kappa_0$, set

$$J(\kappa) = \Big\{ j \in \mathbb{N}, \text{ such that } \ \text{supp } \chi_j \subset \Sigma(\kappa) \Big\} \ .$$

For all indices $j \in \mathbb{N}$, let $V_j^{(2)}$ be the polynomial of degree less than three given by

$$V_j^{(2)}(q) = \sum_{0\leq|\alpha|\leq 2} \frac{\partial_q^\alpha V(q_j')}{\alpha!}(q-q_j')^\alpha \ ,$$

where

$$\begin{cases} q_j' = q_j & \text{if } j \in J(\kappa) \\ q_j' \in (\text{supp } \chi_j) \cap \Big(\mathbb{R}^d \setminus \Sigma(\kappa)\Big) & \text{otherwise.} \end{cases}$$

We associate with each polynomial $V_j^{(2)}$ the Kramers-Fokker-Planck operator $K_{V_j^{(2)}}$. Observe that using the parallelogram law $2(\|x\|^2 + \|y\|^2) - \|x+y\|^2 = \|x-y\|^2 \geq 0$,

$$\begin{aligned} \sum_{j\in\mathbb{N}} \|K_V u_j\|^2_{L^2(\mathbb{R}^{2d})} &= \sum_{j\in\mathbb{N}} \|K_{V_j^{(2)}} u_j + (K_V - K_{V_j^{(2)}}) u_j\|^2_{L^2(\mathbb{R}^{2d})} \\ &\geq \sum_{j\in\mathbb{N}} \Big(\frac{1}{2}\|K_{V_j^{(2)}} u_j\|^2_{L^2(\mathbb{R}^{2d})} - \|(\nabla V(q) - \nabla V_j^{(2)}(q))\partial_p u_j\|^2_{L^2(\mathbb{R}^{2d})}\Big) \ . \end{aligned} \tag{3.3.2}$$

On the other hand, by (3.2.5) in Lemma 3.2.3

$$\sum_{j\in\mathbb{N}} \|(\nabla V(q) - \nabla V_j^{(2)}(q))\partial_p u_j\|^2_{L^2(\mathbb{R}^{2d})} \leq c_{1,d,r} \sum_{j\in\mathbb{N}} R_V^{\geq 3}(q_j')^2 \|\partial_p u_j\|^2_{L^2(\mathbb{R}^{2d})} \,. \tag{3.3.3}$$

Combining (3.3.1), (3.3.2) and (3.3.3) we get immediately

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq \sum_{j\in\mathbb{N}} \Big(\frac{1}{2}\|K_{V_j^{(2)}} u_j\|^2_{L^2(\mathbb{R}^{2d})} - c_{1,d,r} R_V^{\geq 3}(q_j')^2 \|\partial_p u_j\|^2_{L^2(\mathbb{R}^{2d})} - c_{d,r} R_V^{\geq 3}(q_j)^2 \|p u_j\|^2_{L^2(\mathbb{R}^{2d})}\Big) \,.$$

Therefore, making use of the equivalence (3.A.5), it follows

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq \sum_{j\in\mathbb{N}} \Big(\frac{1}{2}\|K_{V_j^{(2)}} u_j\|^2_{L^2(\mathbb{R}^{2d})} - c'_{d,r} R_V^{\geq 3}(q_j')^2 \langle u_j, O_p u_j\rangle_{L^2(\mathbb{R}^{2d})}\Big) \,, \tag{3.3.4}$$

where $c'_{d,r} = 2(c^2_{1,d,r} + c_{d,r}\, C^2)$.

Using respectively the Cauchy-Schwarz inequality and then the Cauchy inequality with epsilon (for any real numbers a, b and all $\epsilon > 0$, $ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2$),

$$\begin{aligned} c'_{d,r} R_V^{\geq 3}(q_j')^2 \langle u_j, O_p u_j\rangle &= c'_{d,r} R_V^{\geq 3}(q_j')^2 \mathrm{Re}\langle u_j, K_{V_j^{(2)}} u_j\rangle \\ &\leq c'_{d,r} R_V^{\geq 3}(q_j')^2 \|u_j\| \cdot \|K_{V_j^{(2)}} u_j\| \\ &\leq \Big(c'_{d,r} R_V^{\geq 3}(q_j')^2\Big)^2 \|u_j\|^2 + \frac{1}{4}\|K_{V_j^{(2)}} u_j\|^2 \,. \end{aligned}$$

Putting the above estimate and (3.3.4) together we obtain

$$\|K_V u\|^2 \geq \sum_{j\in\mathbb{N}} \Big(\frac{1}{4}\|K_{V_j^{(2)}} u_j\|^2 - (c'_{d,r})^2 R_V^{\geq 3}(q_j')^4 \|u_j\|^2\Big) \,. \tag{3.3.5}$$

From now on assume $\kappa \geq \kappa_1$, where $\kappa_1 \geq \kappa_0$ is introduced in Lemma 3.2.3. Remember as well that the constants $C_1, C_2(\kappa)$ are given respectively in Assumption 1 (see (3.1.8)) and Lemma 3.2.3 (see (3.2.7)). By introducing $C(\kappa) \geq \max(C_1, C_2(\kappa))$, which will be fixed later, we set for each κ,

$$I(\kappa) = \Big\{ j \in \mathbb{N}, \text{ such that } \operatorname{supp} \chi_j \subset \big\{ q \in \mathbb{R}^d, \ \ |q| \geq C(\kappa) \big\} \Big\} \,.$$

The rest of the proof is divided into three steps. The first one is devoted to the control of the terms in the the left-hand side of (3.3.5) for which $j \in I(\kappa)$ for some large $\kappa \geq \kappa_0$ to be chosen. At the end of the first step the constants $\kappa > \kappa_1$ and $C(\kappa) \geq \max(C_1, C_2(\kappa))$ will be fixed. The second step is concerned with the remaining terms for which the support of the cutoff functions χ_j are included in some closed ball $B(0, C'(\kappa))$. We finally sum up all the terms in Step 3 and refer to Lemma 3.2.5 after some elementary optimization trick in the last step.

Step 1, $\mathbf{j} \in \mathbf{I}(\kappa)$, $\kappa \geq \kappa_1$ to be fixed : As proved in [BNV], there is a constant $c > 0$ such that for all $j \in I(\kappa)$

$$\|K_{V_j^{(2)}} u_j\|^2 + A_{V_j^{(2)}} \|u_j\|^2 \geq c\Big(\|O_p u_j\|^2 + \|\langle \partial_q V_j^{(2)}(q)\rangle^{2/3} u_j\|^2 + \|\langle D_q\rangle^{2/3} u_j\|^2 \Big) , \qquad (3.3.6)$$

where

$$\begin{aligned} A_{V_j^{(2)}} &= \max\{(1 + \mathrm{Tr}_{+,V_j^{(2)}})^{2/3}, 1 + \mathrm{Tr}_{-,V_j^{(2)}}\} \\ &= \max\{(1 + \mathrm{Tr}_{+,V}(q_j'))^{2/3}, 1 + \mathrm{Tr}_{-,V}(q_j')\} . \end{aligned}$$

Hence there is a constant $C_0 > 0$ so that

$$\begin{aligned} \|K_{V_j^{(2)}} u_j\|^2 + (1 + 10C_0) t_j^4 \|u_j\|^2 \geq C_0 \Big(&\|O_p u_j\|^2 + \|\langle \partial_q V_j^{(2)}(q)\rangle^{2/3} u_j\|^2 \\ &+ \|\langle D_q\rangle^{2/3} u_j\|^2 + 10 t_j^4 \|u_j\|^2 \Big) , \qquad (3.3.7) \end{aligned}$$

where we use the notation $t_j = 2\langle \mathrm{Hess}\, V(q_j')\rangle^{1/4}$ throughout the proof.

Recall that as mentioned in [BNV], the constant c in (3.3.6) does not depend on the polynomial $V_j^{(2)}$ and then so is the constant C_0 in (3.3.7).
Now for all indices $j \in I(\kappa)$ we distinguish two cases : either $j \in J(\kappa)$ or $j \notin J(\kappa)$.

Case 1. Assume $j \in J(\kappa)$. Then taking into account the inequality (3.2.6) in Lemma 3.2.3 and using the estimate (3.3.7) we obtain

$$\begin{aligned} \|K_{V_j^{(2)}} u_j\|^2 + (1 + 10C) t_j^4 \|u_j\|^2 \geq C \Big(&\|O_p u_j\|^2 + \|\langle \partial_q V(q)\rangle^{2/3} u_j\|^2 \\ &+ \|\langle D_q\rangle^{2/3} u_j\|^2 + 10 t_j^4 \|u_j\|^2 \Big) , \qquad (3.3.8) \end{aligned}$$

Furthermore, since for all indices $j \in \mathbb{N}$ the quantity $R_V^{\geq 2}(q_j')^2$ is always greater than $|\mathrm{Hess}\, V(q_j')|$, there exists a constant $c_d > 0$ so that for every $j \in J(\kappa)$,

$$t_j^4 = 16\langle \mathrm{Hess}\, V(q_j')\rangle \leq c_d \langle R_V^{\geq 2}(q_j')^2\rangle .$$

Using the fact that the metric $R_V^{\geq 2}(q)\, dq^2$ is $R_V^{\geq 3}(q)\, dq^2$-slow (see Definition (3.A.2) and Lemma 3.A.5), it follows

$$t_j^4 \leq c_d \langle R_V^{\geq 2}(q)^2\rangle ,$$

for every $q \in \mathrm{supp}\ \chi_j$. Hence there is a constant $c_d' > 0$ (depending on the dimension d) such that

$$\begin{aligned} t_j^4 &\leq c_d \langle \Big(\sum_{|\alpha|=2} |\partial_q^\alpha V(q)|^{\frac{1}{|\alpha|}} + R_V^{\geq 3}(q) \Big)^2 \rangle \\ &\leq 3c_d \langle (\sum_{|\alpha|=2} |\partial_q^\alpha V(q)|^{\frac{1}{|\alpha|}})^2 + R_V^{\geq 3}(q)^2 \rangle \\ &\leq c_d' \langle |\mathrm{Hess}\, V(q)| + R_V^{\geq 3}(q)^2 \rangle , \end{aligned}$$

holds for any $q \in \operatorname{supp} \chi_j$. Now, since for every $q \in \mathbb{R}^d$ on has $R_V^{\geq 3}(q) \geq R_V^{=r}(0)$, we derive from the previous estimate that for any $q \in \operatorname{supp} \chi_j$,

$$\begin{aligned} t_j^4 &\leq c'_d \langle |\text{Hess } V(q)| + \frac{R_V^{\geq 3}(q)^4}{R_V^{=r}(0)^2} \rangle \\ &\leq \frac{c"_d}{\kappa} \max(1, R_V^{=r}(0)^{-2}) \langle \partial_q V(q) \rangle^{\frac{4}{3}} \ . \end{aligned} \tag{3.3.9}$$

Collecting the estimates (3.3.8) and (3.3.9), we get for $\kappa \geq \kappa_1$

$$\begin{aligned} \|K_{V_j^{(2)}} u_j\|^2 &+ (1 + 10C) \frac{c"_d}{\kappa} \max(1, R_V^{=r}(0)^{-2}) \| \langle \partial_q V(q) \rangle^{\frac{2}{3}} u_j \|^2 \\ &\geq C \Big(\|O_p u_j\|^2 + \| \langle \partial_q V(q) \rangle^{2/3} u_j \|^2 + \| \langle D_q \rangle^{2/3} u_j \|^2 + 10 t_j^4 \|u_j\|^2 \Big) \ . \end{aligned}$$

Choosing $\kappa_2 \geq \kappa_1$ so that

$$\frac{C}{2} \geq (1 + 10C) \frac{c"_d}{\kappa_2} \max(1, R_V^{=r}(0)^{-2}) \ ,$$

the following inequality

$$\|K_{V_j^{(2)}} u_j\|^2 \geq C \Big(\|O_p u_j\|^2 + \frac{1}{2} \| \langle \partial_q V(q) \rangle^{2/3} u_j \|^2 + \| \langle D_q \rangle^{2/3} u_j \|^2 + 10 t_j^4 \|u_j\|^2 \Big) \ , \tag{3.3.10}$$

holds for all $j \in J(\kappa)$ with $\kappa \geq \kappa_2$.

Since $j \in J(\kappa)$, there is a constant $c_1 > 0$ so that

$$\frac{1}{8} \langle \partial_q V(q) \rangle^{\frac{4}{3}} \geq c_1 \langle \text{Hess } V(q) \rangle \ , \tag{3.3.11}$$

holds for all $q \in \operatorname{supp} \chi_j$. In addition, using the equivalence (3.A.5) it follows

$$\frac{1}{8} \langle \partial_q V(q) \rangle^{\frac{4}{3}} \geq c_2 |\partial_q V(q)|^{\frac{4}{3}} \geq c_2 \kappa \, R_V^{\geq 3}(q)^4 \geq c'_2 \kappa \, R_V^{\geq 3}(q'_j)^4 \ , \tag{3.3.12}$$

for any $q \in \operatorname{supp} \chi_j$.

Putting (3.3.10), (3.3.11) and (3.3.12) together,

$$\begin{aligned} \|K_{V_j^{(2)}} u_j\|^2 \geq C \Big(&\|O_p u_j\|^2 + \frac{1}{4} \| \langle \partial_q V(q) \rangle^{2/3} u_j \|^2 + \| \langle D_q \rangle^{2/3} u_j \|^2 \\ &+ c_1 \| \langle \text{Hess } V(q) \rangle^{1/2} u_j \|^2 + c'_2 \kappa \, R_V^{\geq 3}(q'_j)^4 \|u_j\|^2 + 10 \|t_j^2 u_j\|^2 \Big) \ , \end{aligned} \tag{3.3.13}$$

holds for all $\kappa \geq \kappa_2$.

Case 2. Assume $j \notin J(\kappa)$, with $\kappa \geq \kappa_2 \geq \kappa_1 \geq \kappa_0$. Hence by Assumption 1 (see (3.1.8)), one has

$$\mathrm{Tr}_{-,V}(q) \neq 0 \;\text{ for all }\; q \in (\mathrm{supp}\,\chi_j) \cap (\mathbb{R}^d \setminus \Sigma(\kappa)) \;\text{ such that }\; |q| \geq C_1 \;.$$

In particular, since $|q_j'| \geq C(\kappa) \geq C_1$,

$$\mathrm{Tr}_{-,V_j^{(2)}} = \mathrm{Tr}_{-,V}(q_j') \neq 0 \;.$$

Referring again to [BNV],

$$\|K_{V_j^{(2)}} u_j\|^2 \geq c\, B_{V_j^{(2)}} \|u_j\|^2 \;,$$

where

$$\begin{aligned} B_{V_j^{(2)}} &= \max\Big(\min_{q\in\mathbb{R}^d} |\nabla V_j^{(2)}(q)|^{4/3}, \frac{1+\mathrm{Tr}_{-,V_j^{(2)}}}{(\log(2+\mathrm{Tr}_{-,V_j^2}))^2}\Big) \\ &= \max\Big(\min_{q\in\mathbb{R}^d} |\nabla V_j^{(2)}(q)|^{4/3}, \frac{1+\mathrm{Tr}_{-,V}(q_j')}{(\log(2+\mathrm{Tr}_{-,V}(q_j')))^2}\Big) \neq 0 \;. \end{aligned}$$

Hence we get in particular

$$\|K_{V_j^{(2)}} u_j\|^2 \geq c\,\frac{1+\mathrm{Tr}_{-,V}(q_j')}{(\log(2+\mathrm{Tr}_{-,V}(q_j')))^2} \|u_j\|^2 \;. \tag{3.3.14}$$

Using again condition (3.1.8) in Assumption 1, there is a constant $C_1 \geq 1$ so that

$$\frac{1}{2}\mathrm{Tr}_{-,V}(q_j') > \frac{1}{2C_1}\mathrm{Tr}_{+,V}(q_j') \;,$$

holds, which in turn implies

$$\mathrm{Tr}_{-,V}(q_j') \geq \frac{1}{2}\mathrm{Tr}_{-,V}(q_j') + \frac{1}{2C_1}\mathrm{Tr}_{+,V}(q_j') \geq \frac{1}{2C_1}(\mathrm{Tr}_{-,V}(q_j') + \mathrm{Tr}_{+,V}(q_j')) \;, \tag{3.3.15}$$

Then it follows from (3.3.14) and (3.3.15)

$$\|K_{V_j^{(2)}} u_j\|^2 \geq c' \Big\| \frac{\sqrt{1+|\mathrm{Hess}\, V(q_j')|}}{\log(2+|\mathrm{Hess}\, V(q_j')|)} u_j \Big\|^2 \;. \tag{3.3.16}$$

By Assumption 1 (see condition (3.1.9)) and (3.3.16), applying Lemma 3.B.6, there is $\delta \in (0,1)$ and a positive nondecreasing function $\Lambda_{\Sigma(\kappa)}$ on $(0,+\infty)$ such that $\Lambda_{\Sigma(\kappa)}(\varrho) \to +\infty$ as $\varrho \to +\infty$, and such that

$$\begin{aligned} \frac{1+|\mathrm{Hess}\, V(q_j')|}{(\log(2+|\mathrm{Hess}\, V(q_j')|))^2} &\geq \frac{1}{2^\delta}(1+|\mathrm{Hess}\, V(q_j')|)^{1-\delta} \\ &\geq \frac{1}{2}|\mathrm{Hess}\, V(q_j')|^{1-\delta} \\ &\geq \frac{\Lambda_{\Sigma(\kappa)}(|q_j'|)}{2} R_V^{\geq 3}(q_j')^4 \geq \frac{\Lambda_{\Sigma(\kappa)}(C(\kappa))}{2} R_V^{\geq 3}(q_j')^4 \,. \end{aligned}$$

Here Lemma 3.B.6 relying on Tarski-Seidenberg is crucial as shows the following argument

$$R^4(q) \overset{|q|\to+\infty}{\sim} \frac{H(q)}{\log(H(q))} \quad \text{and} \quad \lim_{|q|\to+\infty} H(q) = +\infty\,,$$

where $R(q)$ is a function which plays the same role as $R_V^{\geq 3}(q)$ and still satisfies $\lim_{|q|\to+\infty} \frac{R^4(q)}{H(q)} = 0$. For a non-polynomial function V, we may think of a function $R(q)$ which satisfies

$$\frac{1}{C}R(q) \leq \max_{q'\in B(q,\frac{b}{R(q)}),\ |\alpha|=3} |\partial_q^\alpha V(q)|^{\frac{1}{3}} \leq CR(q)\ ,$$

with $C > 1$ and $b > 0$ independent of q for $|q|$ large enough.[1]

Alternatively the asymptotic behaviour (3.1.9) of Assumption 1 should be replaced by something like $R(q)^4 = O(H(q)^{1-\nu})$ with $\nu > 0$ as $|q| \to +\infty$, $q \in \mathbb{R}^d \setminus \Sigma(\kappa)$ or $R(q)^4 = O(\Big(\frac{H(q)}{(\log H(q))^2}\Big))$ (with $|\text{Hess}\, V(q)| \to +\infty$ as $|q| \to +\infty$, $q \in \mathbb{R}^d \setminus \Sigma(\kappa))$.
Therefore we get from the above inequality and (3.3.16),

$$\|K_{V_j^{(2)}} u_j\|^2 \geq c' \,\frac{\Lambda_{\Sigma(\kappa)}(C(\kappa))}{2} R_V^{\geq 3}(q'_j)^4 \|u_j\|^2\ . \tag{3.3.17}$$

Next, remind that $t_j = 2\langle \text{Hess}\, V(q'_j)\rangle^{1/4}$. By (3.2.7) in Lemma 3.2.3, the equivalence

$$t_j \asymp 2\langle \text{Hess}\, V(q)\rangle^{1/4}\ , \tag{3.3.18}$$

holds for any $q \in \text{supp}\,\chi_j$ with $|q| \geq C(\kappa) \geq C_2(\kappa)$. From (3.3.7) and (3.3.18) we see that

$$\begin{aligned}\|K_{V_j^{(2)}} u_j\|^2 + (1+10C)t_j^4\|u_j\|^2 \geq C\Big(&\|O_p u_j\|^2 + \|\langle \partial_q V_j^{(2)}(q)\rangle^{2/3} u_j\|^2 \\ &+ \|\langle \text{Hess}\, V(q)\rangle^{1/2} u_j\|^2 + \|\langle D_q\rangle^{2/3} u_j\|^2 + 9t_j^4\|u_j\|^2\Big)\ .\end{aligned} \tag{3.3.19}$$

One has by (3.2.6) in Lemma 3.2.3,

$$\langle \partial_q V_j^{(2)}(q)\rangle \geq \frac{1}{2}\langle \partial_q V(q)\rangle\ , \tag{3.3.20}$$

for all $q \in (\text{supp}\,\chi_j) \cap \Sigma(\kappa)$. On the other hand, for every $q \in (\text{supp}\,\chi_j) \cap \Big(\mathbb{R}^d \setminus \Sigma(\kappa)\Big)$,

$$|\text{Hess}\, V(q)| + R_V^{\geq 3}(q)^4 + 1 \geq \frac{1}{\kappa}|\partial_q V(q)|^{\frac{4}{3}}\ . \tag{3.3.21}$$

[1] As example, we may take the function V on $\mathbb{R}^2$ equal to $\frac{r^6}{(\log r)^3}(1+\cos(\theta))$ in polar coordinates for $r > 1$.

Furthermore, it results from Assumption 1 , in particular (3.1.9), that for all $q \in (\text{supp}\ \chi_j) \cap \Big(\mathbb{R}^d \setminus \Sigma(\kappa)\Big)$,

$$2|\text{Hess}\ V(q)| + C^4 R_V^{\geq 3}(q)^4 + 1 \leq \frac{5}{2}|\text{Hess}\ V(q)|\ . \tag{3.3.22}$$

From (3.3.21) and (3.3.22) we get for every $q \in (\text{supp}\ \chi_j) \cap \Big(\mathbb{R}^d \setminus \Sigma(\kappa)\Big)$,

$$|\text{Hess}\ V(q)| \geq \frac{2}{5\kappa}|\nabla V(q)|^{\frac{4}{3}} \quad , \quad |\text{Hess}\ V(q)| \geq \frac{2}{5} \geq \frac{2}{5\kappa}\ .$$

Hence there exists a constant $c'' > 0$ such that

$$\langle \text{Hess}\ V(q)\rangle \geq \frac{c''}{\kappa}\langle \partial_q V(q)\rangle^{4/3}\ , \tag{3.3.23}$$

for any $q \in (\text{supp}\ \chi_j) \cap \Big(\mathbb{R}^d \setminus \Sigma(\kappa)\Big)$ with $|q| \geq C(\kappa) \geq C_2(\kappa)$.

The above inequality combined with (3.3.20) and (3.3.19) leads to

$$\begin{aligned}\|K_{V_j^{(2)}} u_j\|^2 + (1+10C)t_j^4\|u_j\|^2 \geq C\Big(&\|O_p u_j\|^2 + \|\langle D_q\rangle^{2/3}u_j\|^2 + 9t_j^4\|u_j\|^2 \\ &+ \min(\frac{1}{2^{4/3}}, \frac{c''}{2\kappa})\|\langle \partial_q V(q)\rangle^{2/3}u_j\|^2 + \frac{1}{2}\|\langle \text{Hess}\ V(q)\rangle^{1/2}u_j\|^2\Big)\ ,\end{aligned} \tag{3.3.24}$$

for all $\kappa \geq \kappa_2$.

Collecting the estimates (3.3.24) and (3.3.16) we get

$$\begin{aligned}(\log(t_j^4))^2\|K_{V_j^{(2)}} u_j\|^2 \geq C"\Big(&\|O_p u_j\|^2 + \|\langle D_q\rangle^{2/3}u_j\|^2 + 9t_j^4\|u_j\|^2 \\ &+ \min(\frac{1}{2^{4/3}}, \frac{c''}{2\kappa})\|\langle \partial_q V(q)\rangle^{2/3}u_j\|^2 + \frac{1}{2}\|\langle \text{Hess}\ V(q)\rangle^{1/2}u_j\|^2\Big)\ .\end{aligned} \tag{3.3.25}$$

In order to reduce the written expressions we denote

$$\Lambda_{1,j} = \frac{O_p}{\log(t_j^4)}\ , \quad \Lambda_{2,j} = \frac{\langle \text{Hess}\ V(q)\rangle^{1/2}}{\log(t_j^4)}\ , \quad \Lambda_{3,j} = \frac{\langle \partial_q V(q)\rangle^{2/3}}{\log(t_j^4)}\ , \quad \Lambda_{4,j} = \frac{t_j^2}{\log(t_j^4)}\ .$$

The estimate (3.3.25) can be rewritten as follows

$$\|K_{V_j^{(2)}} u_j\|^2 \geq C"\Big(\|\Lambda_{1,j}u_j\|^2 + \frac{1}{2}\|\Lambda_{2,j}u_j\|^2 + \min(\frac{1}{2^{4/3}}, \frac{c''}{2\kappa})\|\Lambda_{3,j}u_j\|^2 + 9\|\Lambda_{4,j}u_j\|^2 + \|\frac{\langle D_q\rangle^{2/3}}{\log(t_j^4)}u_j\|^2\Big)\ . \tag{3.3.26}$$

Using (3.3.26) and (3.3.17) we obtain

$$(1+C")\|K_{V_j^{(2)}}u_j\|^2 \geq C"\Big(\|\Lambda_{1,j}u_j\|^2 + \frac{1}{2}\|\Lambda_{2,j}u_j\|^2 + \min(\frac{1}{2},\frac{c''}{2\kappa})\|\Lambda_{3,j}u_j\|^2 + 9\|\Lambda_{4,j}u_j\|^2 + \|\frac{\langle D_q\rangle^{2/3}}{\log(t_j^4)}u_j\|^2 + \Lambda_{\Sigma(\kappa)}(C(\kappa))R_V^{\geq 3}(q_j')^4\|u_j\|^2\Big)\,.$$

Therefore in both cases, that is for all $j\in I(\kappa)$ where $\kappa\geq\kappa_2$

$$\|K_{V_j^{(2)}}u_j\|^2 \geq C^{(3)}\Big(\|\Lambda_{1,j}u_j\|^2 + \|\Lambda_{2,j}u_j\|^2 + \frac{1}{\kappa}\|\Lambda_{3,j}u_j\|^2 + \|\Lambda_{4,j}u_j\|^2 + \|\frac{\langle D_q\rangle^{2/3}}{\log(t_j^4)}u_j\|^2 + \min\Big(\kappa,\Lambda_{\Sigma(\kappa)}(C(\kappa))\Big)R_V^{\geq 3}(q_j')^4\|u_j\|^2\Big)\,. \tag{3.3.27}$$

Due to the elementary inequality $u^{4/3}+v^4\geq\frac{1}{c_0}(u^2+v^4)^{2/3}$ satisfied for all $u,v\geq 1$, we obtain for all $\kappa\geq\kappa_2$

$$\|\frac{\langle D_q\rangle^{2/3}}{\log(t_j^4)}u_j\|^2 + \frac{1}{2}\min\Big(\kappa,\Lambda_{\Sigma(\kappa)}(C(\kappa))\Big)R_V^{\geq 3}(q_j')^4\|u_j\|^2 \geq \frac{1}{c_0}\|\Lambda_{5,j}u_j\|^2\,, \tag{3.3.28}$$

where

$$\Lambda_{5,j} = \frac{(1+D_q^2+R_V^{\geq 3}(q_j')^4)^{\frac{1}{3}}}{\log(t_j^4)}\,.$$

In conclusion, we get by (3.3.27) and (3.3.28) for every $j\in I(\kappa)$ with $\kappa\geq\kappa_2$

$$\|K_{V_j^{(2)}}u_j\|^2 \geq C^{(3)}\Big(\|\Lambda_{1,j}u_j\|^2 + \|\Lambda_{2,j}u_j\|^2 + \frac{1}{\kappa}\|\Lambda_{3,j}u_j\|^2 + \|\Lambda_{4,j}u_j\|^2 + \frac{1}{c_0}\|\Lambda_{5,j}u_j\|^2 + \frac{1}{2}\min\Big(\kappa,\Lambda_{\Sigma(\kappa)}(C(\kappa))\Big)R_V^{\geq 3}(q_j')^4\|u_j\|^2\Big)\,.$$

We now fix the choice firstly of $C(\kappa)$ and secondly of κ. Because $\lim_{\varrho\to+\infty}\Lambda_{\Sigma(\kappa)}(\varrho)=+\infty$, we can choose for any $\kappa\geq\kappa_2$, $C(\kappa)\geq\max(C_1,C_2(\kappa))$ such that $\Lambda_{\Sigma(\kappa)}(C(\kappa))\geq\kappa$. We then choose $\kappa=\kappa_3\geq\kappa_2$ such that

$$\frac{C^{(3)}}{8}\min\Big(\kappa_3,\Lambda_{\Sigma(\kappa_3)}(\kappa_3)\Big) = \frac{C^{(3)}\kappa_3}{8} \geq (c_{d,r}')^2\,,$$

where $c_{d,r}'$ is the constant in (3.3.5),

$$\sum_{j\in I(\kappa_3)}\Big(\frac{1}{4}\|K_{V_j^{(2)}}u\|^2 - (c_{d,r}')^2R_V^{\geq 3}(q_j')^4\|u_j\|^2\Big) \geq \frac{C^{(3)}}{8}\sum_{j\in I(\kappa_3)}\Big(\|\Lambda_{1,j}u_j\|^2 + \|\Lambda_{2,j}u_j\|^2 + \frac{1}{\kappa_3}\|\Lambda_{3,j}u_j\|^2 + \|\Lambda_{4,j}u_j\|^2 + \frac{1}{c_0}\|\Lambda_{5,j}u_j\|^2\Big)\,. \tag{3.3.29}$$

Step 2, $\mathbf{j} \notin \mathbf{I}(\kappa_3)$: The set $\mathbb{N} \setminus I(\kappa_3)$ is now a fixed finite set and we can define

$$C^{(4)} = \max_{j\in\mathbb{N}\setminus I(\kappa_3)} \Big[A_{V_j^{(2)}} + \sup_{q\in\operatorname{supp}\chi_j} \Big(\langle \text{Hess } V(q)\rangle + \langle \partial_q V(q)\rangle^{4/3} \Big) + \frac{t_j^4}{(\log(t_j^4))^2} + (1+(c'_{d,r})^2)(1+R_V^{\geq 3}(q'_j))^4 \Big] \ .$$

From the lower bound (3.1.5) we deduce

$$\frac{1}{4}\|K_{V_j^{(2)}} u_j\| + C^{(4)}\|u_j\|^2 - (c'_{d,r})^2 R_V^{\geq 3}(q'_j)^4\|u_j\|^2 \geq \frac{c}{4}\left[\|O_p u_j\|^2 + \|\langle D_q\rangle^{2/3}u_j\|^2\right] + (1+R_V^{\geq 3}(q'_j))^4\|u_j\|^2 + \|\langle \partial_q V(q)\rangle^{2/3} u_j\|^2 + \|\langle \text{Hess } V(q)\rangle^{1/2} u_j\|^2 + \|\frac{t_j^2}{\log(t_j^4)} u_j\|^2$$

With the quantities

$$\Lambda_{1,j} = \frac{O_p}{\log(t_j^4)}\ , \quad \Lambda_{2,j} = \frac{\langle \text{Hess } V(q)\rangle^{1/2}}{\log(t_j^4)}\ , \quad \Lambda_{3,j} = \frac{\langle \partial_q V(q)\rangle^{\frac{2}{3}}}{\log(t_j^4)}\ ,$$
$$\Lambda_{4,j} = \frac{t_j^2}{\log(t_j^4)}\ , \quad \Lambda_{5,j} = \frac{(1+D_q^2 + R_V^{\geq 3}(q'_j)^4)^{\frac{1}{3}}}{\log(t_j^4)}\ .$$

where $t_j \geq 2$ we deduce

$$\sum_{j\notin I(\kappa_3)} \Big(\frac{1}{4}\|K_{V_j^{(2)}} u_j\|^2 - (c'_{d,r})^2 R_V^{\geq 3}(q'_j)^4\|u_j\|^2 + C^{(4)}\|u_j\|^2 \Big)$$
$$\geq C^{(5)} \sum_{j\notin I(\kappa_3)} \Big(\|\Lambda_{1,j}u_j\|^2 + \|\Lambda_{2,j}u_j\|^2 + \frac{1}{\kappa_3}\|\Lambda_{3,j}u_j\|^2 + \|\Lambda_{4,j}u_j\|^2 + \frac{1}{c_0}\|\Lambda_{5,j}u_j\|^2 \Big)\ , \quad (3.3.30)$$

Collecting (3.3.5), (3.3.29) and (3.3.30), there exists a positive constant $C^{(6)} \geq 1$ depending on V such that

$$\|K_V u\|_{L^2}^2 + C^{(6)}\|u\|_{L^2}^2 \geq \frac{1}{C^{(6)}} \sum_{j\in\mathbb{N}} \Big(\|\Lambda_{1,j}u_j\|^2 + \|\Lambda_{2,j}u_j\|^2 + \|\Lambda_{3,j}u_j\|^2 + \|\Lambda_{4,j}u_j\|^2 + \|\Lambda_{5,j}u_j\|^2 \Big)\ . \quad (3.3.31)$$

Step 3. In this final step, set $L(s) = \frac{s+1}{\log(s+1)}$ for all $s \geq 1$. Notice that there exists a constant $c > 0$ such that for all $x \geq 1$,

$$\inf_{t\geq 2} \frac{x}{\log(t)} + t \geq \frac{1}{c} L(x)\ .$$

In view of the above estimate,

$$\begin{aligned}\|\Lambda_{1,j}u_j\|^2 + \frac{1}{4}\|\Lambda_{4,j}u_j\|^2 &\geq \frac{1}{4}\int (\frac{\lambda^2}{(\log(t_j^4))^2} + t_j^2)\, d\mu_{u_j}(\lambda) \\ &\geq \frac{1}{8}\int (\frac{\lambda}{\log(t_j)} + t_j)^2 d\mu_{u_j}(\lambda) \\ &\geq \frac{1}{c_3}\|L(O_p)u_j\|^2 \ ,\end{aligned}$$

here we recall that $d\mu_{u_j}(\lambda) = d\langle E(\lambda)u_j, u_j\rangle$ where $E(\lambda)$ is the spectral family.

Summing over j, we obtain the first term in the desired estimation (3.1.10). Likewise for the second term

$$\|\Lambda_{3,j}u_j\|^2 + \frac{1}{4}\|\Lambda_{4,j}u_j\|^2 \geq \frac{1}{c_4}\|L(\langle\partial_q V(q)\rangle^{2/3})u_j\|^2 \ ,$$

with

$$\sum_{j\in\mathbb{N}} \|L(\langle\partial_q V(q)\rangle^{2/3})u_j\|^2 = \|L(\langle\partial_q V(q)\rangle^{2/3})u\|^2 \ .$$

To obtain the third term in (3.1.10) write similarly

$$\|\Lambda_{2,j}u_j\|^2 + \frac{1}{4}\|\Lambda_{4,j}u_j\|^2 \geq \frac{1}{c_5}\|L(\langle\text{Hess } V(q)\rangle^{1/2})u_j\|^2 \ ,$$

with

$$\sum_{j\in\mathbb{N}} \|L(\langle\text{Hess } V(q)\rangle^{1/2})u_j\|^2 = \|L(\langle\text{Hess } V(q)\rangle^{1/2})u\|^2 \ .$$

In the same way

$$\|\Lambda_{5,j}u_j\|^2 + \frac{1}{4}\|\Lambda_{4,j}u_j\|^2 \geq \frac{1}{c_6}\|L((1 + D_q^2 + R_V^{\geq 3}(q'_j)^4)^{\frac{1}{3}})u_j\|^2 \ .$$

By Lemma 3.2.5 we get

$$\sum_{j\in\mathbb{N}} \|L((1 + D_q^2 + R_V^{\geq 3}(q'_j)^4)^{\frac{1}{3}})u_j\|^2 \geq \frac{1}{c_6}\|L((1 + D_q^2 + R_V^{\geq 3}(q)^4)^{\frac{1}{3}})u\|^2 \ ,$$

To conclude, just remark that

$$\langle u, (1 + D_q^2 + R_V^{\geq 3}(q)^4)u\rangle \geq \langle u, (1 + D_q^2)u\rangle \geq \langle u, \langle D_q^2\rangle u\rangle > \|u\|^2$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$, then applying (3.2.15) in Lemma 3.2.4 with $A = (1 + D_q^2 + R_V^{\geq 3}(q)^4)$, $B = \langle D_q^2 \rangle$, $\alpha_0 = \frac{2}{3}$ and $k = 2$ we obtain

$$\|L((1 + D_q^2 + R_V^{\geq 3}(q)^4)^{\frac{1}{3}})u\|^2 \geq \|L(\langle D_q^2 \rangle^{\frac{1}{3}})u\|^2 \geq \frac{1}{c_7}\|L(\langle D_q \rangle^{2/3})u\|^2$$

for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

Finally collecting all terms, we have found $C_V \geq 1$ such that

$$\|K_V u\|_{L^2}^2 + C_V\|u\|_{L^2}^2 \geq \frac{1}{C_V}\Big(\|L(O_p)u\|_{L^2}^2 + \|L(\langle \nabla V(q) \rangle^{\frac{2}{3}})u\|_{L^2}^2 + \|L(\langle \text{Hess } V(q) \rangle^{\frac{1}{2}})u\|_{L^2}^2 + \|L(\langle D_q \rangle^{\frac{2}{3}})u\|_{L^2}^2\Big) \tag{3.3.32}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. Because $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ is dense in $D(K_V)$ endowed with the graph norm, the result extends to any $u \in D(K_V)$. □

3.4 Applications

This section is devoted to some applications of Theorem 3.1.1. In each of the following examples we examine that the Assumption 1 is well fulfilled. We recall here that one knows that for a potential V satisfying assumption 1, the resolvent of the Witten Laplacian $\Delta_V^{(0)}$ is compact (see Theorem 10.16 in [HeNi]). In the cas of the Witten Laplacian, the following examples were in particular considered in [HeNi] (C.f Proposition 10.19 and Proposition 10.21)

Example 1 : Let us consider as a first example of application the case

$$V(q_1, q_2) = -q_1^2 q_2^2, \text{ with } q = (q_1, q_2) \in \mathbb{R}^2 \ ,$$

By direct computation

$$\partial_q V(q) = \begin{pmatrix} -2q_1 q_2^2 \\ -2q_2 q_1^2 \end{pmatrix} \ , \ |\partial_q V(q)| = 2|q_1 q_2||q| \ ,$$

$$\text{Hess } V(q) = \begin{pmatrix} -2q_2^2 & -4q_1 q_2 \\ -4q_1 q_2 & -2q_1^2 \end{pmatrix} \ , \ |\text{Hess } V(q)| = 2\sqrt{|q|^4 + 6q_1^2 q_2^2} \asymp |q|^2 \ ,$$

$$R_V^{\geq 3}(q) = |4q_2|^{1/3} + |4q_1|^{1/3} + 2 \times 4^{1/4} \ .$$

It is clear that the trace of Hess $V(q)$ given by $-2|q|^2$ is negative for all $q \in \mathbb{R}^2 \setminus \{0\}$. Hence

$$\mathrm{Tr}_{-,V}(q) > \mathrm{Tr}_{+,V}(q) \quad \text{for all} \quad q \in \mathbb{R}^2\,, |q| \geq 1\,.$$

Moreover, for all $\kappa > 0$ the algebraic set $\mathbb{R}^2 \setminus \Sigma(\kappa)$ is not bounded since $(0, q_2) \in \mathbb{R}^2 \setminus \Sigma(\kappa)$ for all $q_2 \in \mathbb{R}$. Furthermore for $\kappa > 1$ chosen as we wish

$$\lim_{\substack{|q|\to+\infty \\ q\in\mathbb{R}^2\setminus\Sigma(\kappa)}} \frac{R_V^{\geq 3}(q)^4}{|\mathrm{Hess}\, V(q)|} = \lim_{\substack{|q|\to+\infty \\ q\in\mathbb{R}^2\setminus\Sigma(\kappa)}} \frac{|q|^{4/3}}{|q|^2} = 0\,,$$

since $R_V^{\geq 3}(q)^4 \leq |q|^{4/3}$ when $|q| \geq 2^3 \times 4^{3/4}$.

Below we sketch as example $\Sigma(800)$ in a blue color.

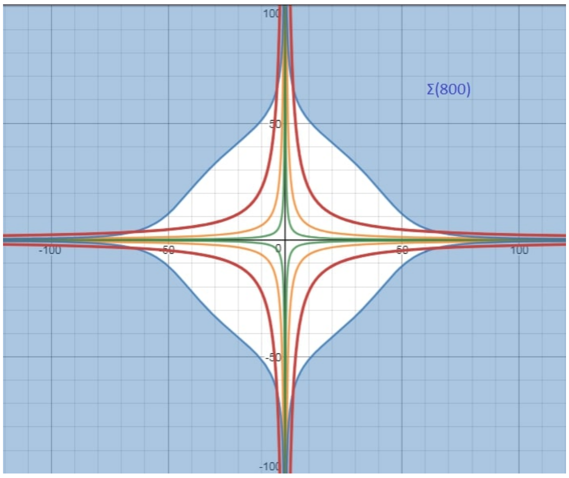

FIGURE 3.1 : Contour lines of $V(q_1, q_2) = -q_1^2 q_2^2$

The compactness of the resolvent of K_V in this case follows from Corollary 4.1.3.

Example 2 : Let $n \in \mathbb{N}$. The polynomial $V(q) = -q_1^2(q_1^2 + q_2^2)^n$ verifies the Assumption 1 only for $n = 1$.
A straightforward computation shows that

$$\partial_q V(q) = -\begin{pmatrix} 2q_1(|q|^{2n} + nq_1^2|q|^{2(n-1)}) \\ 2nq_2q_1^2|q|^{2(n-1)} \end{pmatrix}\,,$$

$$\mathrm{Hess}\, V(q) = -2|q|^{2(n-2)} \begin{pmatrix} |q|^4 + 5nq_1^2|q|^2 + 2n(n-1)q_1^4 & 2nq_1q_2|q|^2 + 2n(n-1)q_1^3q_2 \\ 2nq_1q_2|q|^2 + 2n(n-1)q_1^3q_2 & nq_1^2|q|^2 + 2n(n-1)q_1^2q_2^2 \end{pmatrix}\,.$$

Notice that the trace of Hess $V(q)$ equals

$$-2|q|^{2(n-2)}\Big(|q|^4 + 5nq_1^2|q|^2 + 2n(n-1)q_1^4 + nq_1^2|q|^2 + 2n(n-1)q_1^2q_2^2\Big) < 0\,,$$

for all $q \in \mathbb{R}^2$, $|q| \geq 1$. Hence

$$-\mathrm{Tr}_{-,V}(q) + \mathrm{Tr}_{+,V}(q) < 0, \quad \text{for any} \quad q \in \mathbb{R}^2\,, \ |q| \geq 1\ .$$

In addition for all $\kappa > 0$ the set $\mathbb{R}^2 \setminus \Sigma(\kappa)$ is not bounded since $(0, q_2) \in \mathbb{R}^2 \setminus \Sigma(\kappa)$ for all $q_2 \in \mathbb{R}$.
For q large enough $|\mathrm{Hess}\, V(q)| \asymp |q|^{2n}$ and $|D^3V(q)| \asymp |q|^{2n-1}$ then

$$\frac{R_V^{\geq 3}(q)^4}{|\mathrm{Hess}\, V(q)|} \asymp \frac{(|q|^{2n-1})^{4/3}}{|q|^{2n}}\ .$$

Hence

$$\lim_{\substack{|q|\to+\infty \\ q\in\mathbb{R}^2\setminus\Sigma(\kappa)}} \frac{R_V^{\geq 3}(q)^4}{|\mathrm{Hess}\, V(q)|} = 0 \quad \text{if and only if} \quad n < 2\ .$$

Taking as example $\kappa = 800$, we get the following shape of $\Sigma(800)$ colored in blue.

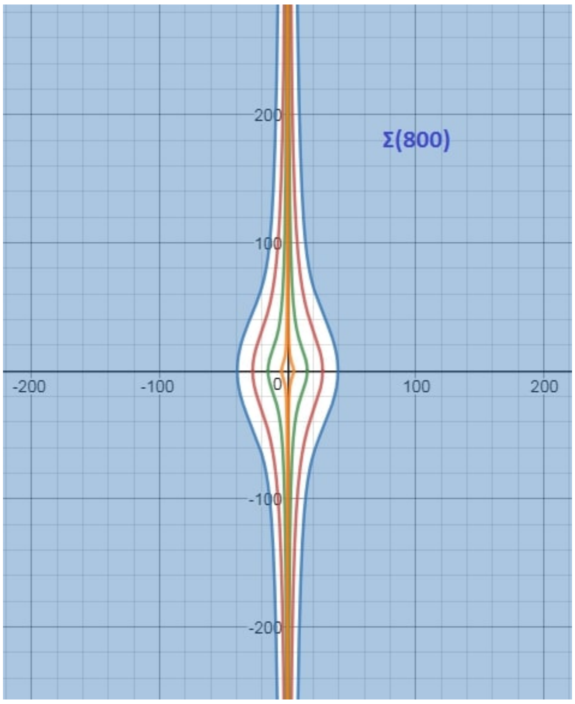

FIGURE 3.2 : Contour lines of $V(q_1, q_2) = -q_1^2(q_1^2 + q_2^2)$

In this example, the hypothesis of the Theorem 3.1.1 is satisfied only for $n = 1$. By Corollary 4.1.3, we deduce that the Kramers-Fokker-Planck operator K_V with potential $V(q) = -q_1^2(q_1^2 + q_2^2)$ has a compact resolvent.

Example 3 : For $\epsilon \in \mathbb{R} \setminus \{0, -1\}$, we consider $V(q_1, q_2) = (q_1^2 - q_2)^2 + \epsilon q_2^2$. For all $q \in \mathbb{R}^2$ one has

$$\partial_q V(q) = \begin{pmatrix} 4q_1(q_1^2 - q_2) \\ -2(q_1^2 - q_2) + 2\epsilon q_2 \end{pmatrix} \ , \ |\partial_q V(q)| = 4|q_1(q_1^2 - q_2)| + |-2(q_1^2 - q_2) + 2\epsilon q_2| \ ,$$

$$\text{Hess } V(q) = \begin{pmatrix} 12q_1^2 - 4q_2 & -4q_1 \\ -4q_1 & 2(1+\epsilon) \end{pmatrix} \ , \ |\text{Hess } V(q)| = |12q_1^2 - 4q_2| + 8|q_1| + 4|1+\epsilon| \ ,$$

$$R_V^{\geq 3}(q) = (24|q_1|)^{1/3} + 3 \times 4^{1/3} + 24^{1/4} \ .$$

In this case, we are going to show that for all $\kappa > 0$ the algebraic set $\mathbb{R}^2 \setminus \Sigma(\kappa)$ is bounded. Let $(q_1, q_2) \in \mathbb{R}^2 \setminus \Sigma(\kappa)$ then

$$\Big(|\text{Hess } V(q)| + R_V^{\geq 3}(q)^4 + 1\Big) \geq \frac{1}{\kappa} |\nabla V(q)|^{\frac{4}{3}} \ .$$

Up to a change of coordinates $X_1 = q_1, \ X_2 = q_1^2 - q_2$ the above inequality is equivalent to

$$\Big(4|2X_1^2 + X_2| + 8|X_1| + 4|1+\epsilon| + \Big((24|X_1|)^{1/3} + 3 \times 4^{1/3} + 24^{1/4}\Big)^4 + 1\Big) \geq \frac{1}{\kappa}\Big(4|X_1X_2| + |-2(1+\epsilon)X_2 + 2\epsilon X_1^2|\Big)^{\frac{4}{3}} \ .$$

Using the triangle inequality in the right-hand side and the reverse triangle inequality with the elementary inequality $(u+v)^{\frac{4}{3}} \geq u^{\frac{4}{3}} + v^{\frac{4}{3}}$ satisfied for all $u, v \geq 0$, it follows that

$$|X_1|^2 + |X_2| + |X_1| + \Big(|X_1|^{\frac{1}{3}} + c\Big)^4 \geq \frac{c'}{\kappa}\Big(\Big||2(1+\epsilon)X_2| - |2\epsilon X_1^2|\Big|^{\frac{4}{3}} + |X_1X_2|^{\frac{4}{3}}\Big) \ . \quad (3.4.1)$$

Suppose first that $|X_1| \leq 1$. Inequality (3.4.1) implies

$$|X_2| + c_1 \geq \frac{c'}{\kappa}\Big||2(1+\epsilon)X_2| - |2\epsilon X_1^2|\Big|^{\frac{4}{3}} \ . \quad (3.4.2)$$

The right-hand part in the above inequality is bounded from above by $|X_2| + c_1$ where c_1 is some positive constant. Now we distinguish two case :
Case 1 : If $\frac{1}{2}|2(1+\epsilon)X_2| \leq |2\epsilon X_1^2|$ or equivalently $|X_2| \leq |\frac{2\epsilon}{1+\epsilon}||X_1^2|$ then $|X_2| \leq |\frac{2\epsilon}{1+\epsilon}|$.
Case 2 : Otherwise, if $\frac{1}{2}|2(1+\epsilon)X_2| \geq |2\epsilon X_1^2|$, then we get

$$|X_2| + c_1 \geq \frac{c'}{\kappa}|1+\epsilon||X_2|^{4/3} \ .$$

Using the fact that $\epsilon \neq -1$, we deduce that X_2 must be also bounded.

Now if $|X_1| \geq 1$, we derive from (3.4.1) the following estimates

$$|X_1|^2 + |X_2| + c_3 \geq \frac{c_4}{\kappa}\Big||2(1+\epsilon)X_2| - |2\epsilon X_1^2|\Big|^{\frac{4}{3}} , \tag{3.4.3}$$

$$|X_1|^2 + |X_2| + c_3 \geq \frac{c_4}{\kappa}|X_1X_2|^{\frac{4}{3}} . \tag{3.4.4}$$

Here we study three cases.
• Firstly if $\frac{1}{2}|2(1+\epsilon)X_2| \geq |2\epsilon X_1^2|$ or equivalently $|X_1| \leq |\frac{1+\epsilon}{2\epsilon}||X_2|$ then (3.4.3) gives

$$(1 + |\frac{1+\epsilon}{\epsilon}|)|X_2| + c_3 \geq \frac{c_4}{\kappa}|(1+\epsilon)X_2|^{\frac{4}{3}} .$$

Since $\epsilon \neq -1$, it follows that X_2 is bounded and so is X_1.
• Now if $2|2(1+\epsilon)X_2| \leq |2\epsilon X_1^2|$ or equivalently $|X_2| \leq |\frac{\epsilon}{2(1+\epsilon)}||X_1^2|$ the estimates (3.4.3) leads to

$$(1 + |\frac{\epsilon}{2(1+\epsilon)}|)|X_1|^2 + c_3 \geq \frac{c_4}{\kappa}|\epsilon X_1|^{\frac{8}{3}} .$$

Since $\epsilon \neq 0$, it follows that X_1 is bounded and so is X_2.
• Finally if $\frac{1}{2}|2(1+\epsilon)X_2| \leq |2\epsilon X_1^2| \leq 2|2(1+\epsilon)X_2|$, then by (3.4.4)

$$(1 + |\frac{2\epsilon}{1+\epsilon}|)|X_1|^2 + c_3 \geq \frac{c_4}{\kappa}\Big(|X_1||\frac{\epsilon}{2(1+\epsilon)}|X_1^2|\Big)^{\frac{4}{3}} .$$

Hence since $\epsilon \neq 0$, X_1 is bounded and then X_2 is so.

Below we sketch as example $\Sigma(2)$ in a blue color.

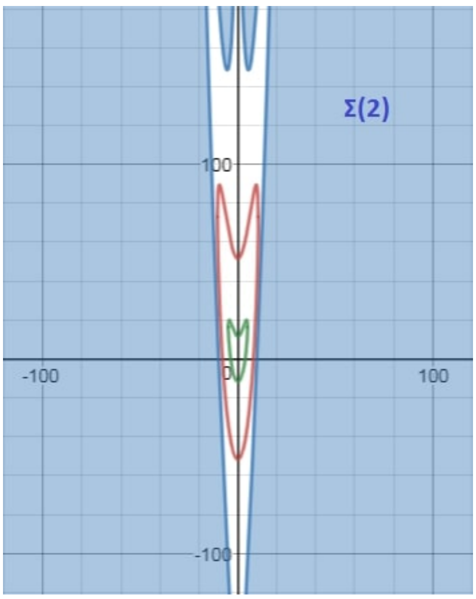

FIGURE 3.3 : Contour lines of $V(q_1, q_2) = (q_1^2 - q_2)^2 + 0.5q_2^2$.

We conclude that for $\epsilon \in \mathbb{R} \setminus \{0, -1\}$, the Assumption 1 is satisfied, and therefore by Corollary 4.1.3, K_V has a compact resolvent.

For $\epsilon = 0$, thanks to [HeNi] (see Proposition 10.21 page 111), we know that the Witten Laplacian defined by

$$\Delta_V^{(0)} = -\Delta_q + |\nabla V(q)|^2 - \Delta V(q) \ , \quad q = (x_1, x_2) \in \mathbb{R}^2$$

has no compact resolvent and then the Kramers-Fokker-Planck operator K_V has no compact resolvent.

This example was studied in the case of the Witten Laplacian operator by B.Helffer and F.Nier in their book [HeNi]. A small mistake was made in [HeNi] in Proposition 10.21. In fact the equations $l_{11} = l_{12} = l_{111} = 0$ should be replaced by $(1 + \epsilon)l_{11} = l_{12} = l_{111} = 0$. When $\epsilon = -1$, we can eventually construct a Weyl sequence for the Witten Laplacian operator in the following way. In this case the potential $V(q_1, q_2) = (q_1^2 - q_2)^2 - q_2^2$ is equal to $-2q_2q_1^2 + q_1^4$.

In order to construct a Weyl sequence for $\Delta_V^{(0)}$, it is sufficient to take $\chi(\frac{(q_2+n^2)}{n})$ where χ is a cutoff function supported in $[-1, 1]$ and then consider the sequence

$$u_n(q_1, q_2) = \chi(\frac{(q_2 + n^2)}{n}) \exp(-V(q_1, q_2)) \ .$$

The support of u_n is then included in $-n^2 - n \leq q_2 \leq -n^2 + n$. Hence the u_n's have disjoint supports for large n.

Therefore we have

$$-2n^2 \leq q_2 \leq -\frac{n^2}{2} \quad \text{and} \quad -4n^2q_1^2 - q_1^4 \leq -V(q_1, q_2) \leq -n^2q_1^2 - q_1^4 \leq -n^2q_1^2 \ .$$

As a result, we get for n large

$$\begin{aligned} \frac{\langle u_n, \Delta_V^{(0)} u_n \rangle}{\|u_n\|^2} &= \frac{\|(\partial_q + \partial_q V(q))(u_n)\|^2}{\|u_n\|^2} \\ &= \frac{\|(\partial_q \chi)e^{-V}\|^2}{\|u_n\|^2} = O(\frac{1}{n^2}) \ . \end{aligned}$$

Here to get the lower bound of the the above quantity we restrict the integral in $q_1 = O(\frac{1}{n})$. As a conclusion, for $\epsilon = -1$ the Witten Laplacian attached to $V(q_1, q_2) = q_1^2q_2^2 + \epsilon(q_1^2 + q_2^2)$ has no compact resolvent and then the Kramers-Fokker-Planck operator K_V has no compact resolvent.

3.A Slow metric, partition of unity

The purpose of this appendix is to state with references or proofs the facts concerning metrics which are needed in the article. We first remind the following definitions.

Definitions 3.A.1. *A metric g on $\mathbb{R}^m$ is called a slowly varying metric if there exists a constant $C \geq 1$ such that for all $x, y \in \mathbb{R}^m$ satisfying $g_x(x-y, x-y) \leq C^{-1}$ it follows that*

$$C^{-1} g_x(z,z) \leq g_y(z,z) \leq C g_x(z,z) \tag{3.A.1}$$

holds for all $z \in \mathbb{R}^m$.

Let g^1 and g^2 be two metrics. We say that g^1 is g^2-slow if there is a constant $c \geq 1$ such that for all $x, y \in \mathbb{R}^m$

$$g_x^2(x-y, x-y) \leq c^{-1} \Rightarrow c^{-1} g_x^1(z,z) \leq g_y^1(z,z) \leq c g_x^1(z,z) \ . \tag{3.A.2}$$

holds for all $z \in \mathbb{R}^m$.

Remark 3.A.2. *The second statement in the above definitions is a typical application of the notion of the second microlocalisation developed by Bony-Lerner (see [BoLe]).*

Remark 3.A.3. *Property 3.A.1 will be satisfied if we ask only that*

$$\exists C \geq 1, \forall x, y, z \in \mathbb{R}^m,\ g_x(x-y, x-y) \leq C^{-1} \Longrightarrow g_y(z,z) \leq C g_x(z,z) \ . \tag{3.A.3}$$

Indeed, assuming (3.A.3) gives that wherever $g_x(x-y, x-y) \leq C^{-1}$ (which is less than or equal to one since $C \geq 1$ from (3.A.3) with x=y) this implies $g_y(y-x, y-x) \leq C^{-1}$ and then $g_x(z,z) \leq C g_y(z,z)$, so that (3.A.1) is fulfilled.

Notations 3.A.4. *For $r \in \mathbb{N}$, let E_r denote the set of polynomials with degree not greater than r :*

$$E_r = \{P \in \mathbb{R}[X_1, ..., X_d],\ d^\circ P \leq r\} \ .$$

For a polynomial $P \in E_r$ of degree $r \in \mathbb{N}^*$ and for $n \in \{1, ..., r\}$, the function $R_P^{\geq n} : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$R_P^{\geq n}(q) = \sum_{n \leq |\alpha| \leq r} |\partial_q^\alpha P(q)|^{\frac{1}{|\alpha|}} \ . \tag{3.A.4}$$

In the present article we are mainly concerned with the metric $g^n = R_P^{\geq n}(q)^2\, dq^2$ where $n \in \{1, ..., r\}$ which satisfies the following properties.

Lemma 3.A.5. *Let $P \in E_r$, where $r \in \mathbb{N}^*$ is the degree of P and let n be a natural number in $\{1, ..., r\}$.*
1) The metric g^n is slow : There exists a uniform $C = C(n, r, d) \geq 1$ such that

$$R_P^{\geq n}(q)|q - q'| \leq C^{-1} \Longrightarrow \Big(\frac{R_P^{\geq n}(q)}{R_P^{\geq n}(q')}\Big)^{\pm 1} \leq C \tag{3.A.5}$$

2) The metric g^{n-1} is g^n-slow : There is a constant $C' = C'(n, r, d) \geq 1$ so that

$$R_P^{\geq n}(q)|q - q'| \leq C'^{-1} \Longrightarrow \Big(\frac{R_P^{\geq n-1}(q)}{R_P^{\geq n-1}(q')}\Big)^{\pm 1} \leq C' \tag{3.A.6}$$

Proof. The dimension d is fixed. Assume $n, r \in \mathbb{N}^*$ with $n \leq r$. The set

$$K_{n,r} := \Big\{\overline{P} \in E_r/E_{n-1},\ R_{\overline{P}}^{\geq n}(0) = R_P^{\geq n}(0) = 1\Big\}$$

is a compact set of E_r/E_{n-1}, where $\overline{P} \in E_r/E_{n-1}$ can be identified with the polynomial $\overline{P}(q) = \sum_{n\leq|\alpha|\leq r} \frac{\partial_q^\alpha P(0)}{\alpha!} q^\alpha$. For any $\varrho \geq 0$, the mapping

$$\begin{array}{ccl} K_{n,r} \times \overline{B(0,\varrho)} & \to & [0, +\infty) \\ (\overline{P}, t) & \mapsto & R_{\overline{P}}^{\geq n}(t) = \sum_{|\alpha|\geq n} |\partial_x^\alpha \overline{P}(t)|^{\frac{1}{|\alpha|}} \end{array}$$

is continuous because $s \mapsto s^\nu$ is continuous on $[0, +\infty)$ for any $\nu > 0$. On the compact set $K_{n,r} \times \overline{B(0,\varrho)}$ it admits a maximum $M_{n,r,\varrho}$ and a minimum $m_{n,r,\varrho}$ which cannot be 0. Actually $R_{\overline{P}}^{\geq n}(t_0) = 0$ means $\partial_x^\alpha P(t_0) = 0$ for all $\alpha \in \mathbb{N}^d$, $|\alpha| \geq n$, and by the Taylor expansion $\partial_x^\alpha \overline{P}(t) = 0$ for all $t \in \mathbb{R}^d$, $\alpha \in \mathbb{N}^d$, $|\alpha| \geq n$, which contradicts $R_{\overline{P}}^{\geq n}(0) = 1$. Now for a general $V \in E_r$ with $d^\circ V \geq n$, we know that for all $q \in \mathbb{R}^d$, $R_V^{\geq n}(q) \neq 0$. We thus consider for any $q \in \mathbb{R}^d$, the class $\overline{P}_q$ of $V(q + R_V^{\geq n}(q)^{-1}t)$ in E_r/E_{n-1}. It satisfies

$$R_{\overline{P}_q}^{\geq n}(t) = \sum_{n\leq|\alpha|\leq r} |\partial_t^\alpha P_q(t)|^{\frac{1}{|\alpha|}} = R_V^{\geq n}(q)^{-1} R_V^{\geq n}(q + R_V^{\geq n}(q)^{-1}t) ,$$

and in particular

$$R_{\overline{P}_q}^{\geq n}(0) = 1 \quad , \quad \overline{P}_q \in K_{n,r} .$$

Therefore we obtain for $\varrho = 1$

$$(|t| \leq 1) \Rightarrow \left(m_{n,r,1} \leq \left(\frac{R_V^{\geq n}(q + R_V^{\geq n}(q)^{-1}t)}{R_V^{\geq n}(q)}\right) \leq M_{n,r,1}\right) ,$$

which implies, with $q' = q + \frac{t}{R_V^{\geq n}(q)}$,

$$\Big(R_V^{\geq n}(q)|q - q'| \leq 1\Big) \Rightarrow \left(\frac{R_V^{\geq n}(q)}{R_V^{\geq n}(q')}\right)^{\pm 1} \leq \max\left\{M_{n,r,1}, \frac{1}{m_{n,r,1}}\right\} .$$

We conclude the proof of 1) by choosing $C(n,r,d)=\max\left\{M_{n,r,1},\frac{1}{m_{n,r,1}},1\right\}$ and by applying the more general result to $P\in E_r$ such that $d^\circ P=r$.
Let us prove 2). We still work in $K_{n,r}=\left\{\overline{P}\in E_r/E_{n-1},\ R_{\overline{P}}^{\geq n}(0)=R_P^{\geq n}(0)=1\right\}$ and with now a given $\varrho\in]0,1]$. From the proof of 1) we know that there exists $M_{n,r,1},m_{n,r,1}>0$ such that

$$m_{n,r,1}\leq R_{\overline{P}}^{\geq n}(t)\leq M_{n,r,1}$$

for all q, $|q|\leq\varrho\leq 1$ and all $\overline{P}\in K_{n,r}$.
In particular there exists a constant $\tilde{C}_{n,r}$ such that

$$\forall\overline{P}\in K_{n,r}\,,\forall t\in\overline{B(0,\varrho)}\subset\overline{B(0,1)}\,,\quad\max_{n\leq|\alpha|\leq r}\left|\partial_q^\alpha\overline{P}(t)\right|\leq\tilde{C}_{n,r}\,. \tag{3.A.7}$$

For any $P\in E_r$ in the class $\overline{P}\in E_r/E_{n-1}$, we decompose $R_P^{\geq n-1}(t)$ into

$$R_P^{\geq n-1}(t)=\sum_{|\beta|=n-1}\left|\partial_q^\beta P(t)\right|^{\frac{1}{n-1}}+R_{\overline{P}}^{\geq n}(t)\,.$$

By the Taylor expansion

$$\partial_q^\beta P(t)-\partial_q^\beta P(0)=\sum_{1\leq|\alpha'|\leq r-n+1}\frac{\partial_q^{\beta+\alpha'}P(0)}{\alpha'!}t^{\alpha'}\qquad,|\beta|=n-1\,,$$

and owing to (3.A.7), there exists a constant $C_{n,r}>0$ such that the inequality

$$\left||\partial_q^\beta P(t)|-|\partial_q^\beta P(0)|\right|\leq C_{n,r}\varrho$$

holds for all $\beta\in\mathbb{N}^d$, $|\beta|=n-1$, and all $t\in\mathbb{R}^d$, $|t|\leq\varrho\leq 1$.
The uniform continuity of $s\mapsto s^{\frac{1}{n-1}}$ on $[0,+\infty[$ now implies

$$\left|\sum_{|\beta|=n-1}\left|\partial_q^\beta P(t)\right|^{\frac{1}{n-1}}-\sum_{|\beta|=n-1}\left|\partial_q^\beta P(t)\right|^{\frac{1}{n-1}}\right|\leq\varepsilon_{n,r}(\varrho)$$

with $\lim_{\varrho\to 0}\varepsilon_{n,r}(\varrho)=0$ uniformly with respect to $P\in\overline{P}$, $\overline{P}\in K_{n,r}$ and $q\in\overline{B(0,\varrho)}\subset\overline{B(0,1)}$.
On side we write

$$\begin{aligned}R_P^{\geq n-1}(t)&\leq\sum_{|\beta|=n-1}\left|\partial_q^\beta P(t)\right|^{\frac{1}{n-1}}+M_{n,r,1}\\&\leq\sum_{|\beta|=n-1}\left|\partial_q^\beta P(0)\right|^{\frac{1}{n-1}}+\varepsilon_{n,r}(\varrho)+M_{n,r,1}\\&\leq\max(1,\varepsilon_{n,r}(\varrho)+M_{n,r,1})R_P^{\geq n-1}(0)\,.\end{aligned}$$

On the other side we have

$$\begin{aligned} R_P^{\geq n-1}(0) &\leq \sum_{|\beta|=n-1} \left|\partial_q^\beta P(0)\right|^{\frac{1}{n-1}} + M_{n,r,1} \\ &\leq \sum_{|\beta|=n-1} \left|\partial_q^\beta P(t)\right|^{\frac{1}{n-1}} + \varepsilon_{n,r}(\varrho) + M_{n,r,1} \\ &\leq \max(1, \frac{\varepsilon_{n,r}(\varrho) + M_{n,r,1}}{m_{n,r,1}}) R_P^{\geq n-1}(t)\,. \end{aligned}$$

For $\varrho_{n,r} \in]0,1]$ chosen small enough such that $\varepsilon_{n,r}(\varrho_0) \leq M_{n,r,1}$, we deduce

$$\forall \overline{P} \in K_{n,r}\,,\, \forall P \in \overline{P}\,,\, \forall t \in \overline{B(0,\varrho_{n,r})}\,, \quad \left(\frac{R_P^{\geq n-1}(t)}{R_P^{\geq n-1}(0)}\right)^{\pm 1} \leq \max(2M_{n,r,1}, \frac{2M_{n,r,1}}{m_{n,r}})\,.$$

For $V \in E_r$ such that $d^\circ V \geq n$ we apply the previous estimate to $P_q(t) = V(q + R_V^{\geq n}(q)^{-1}t)$, with $\overline{P_q} \in K_{n,r}$, which leads to

$$\left(R_V^{\geq n}(q)|q-q'| \leq \varrho_{n,r}\right) \Rightarrow \left(\left(\frac{R_V^{\geq n-1}(q)}{R_V^{\geq n-1}(q')}\right)^{\pm 1} \leq \max(2M_{n,r,1}, \frac{2M_{n,r,1}}{m_{n,r,1}})\right)\,.$$

We conclude the proof by choosing $C'(n,r,d) = \max(2M_{n,r,1}, \frac{2M_{n,r,1}}{m_{n,r,1}}, \frac{1}{\varrho_{n,r}})$ and by applying the more general result to $P \in E_r$ such that $d^\circ P = r$. □

Remark 3.A.6. *The proof of 1) gives a more general result than the slowness, namely when n, r, d are fixed : For any $\lambda > 0$ there exists $C_\lambda \geq 1$ such that*

$$(R_P^{\geq n}(q)|q-q'| \leq \lambda) \Rightarrow \left(\left(\frac{R_P^{\geq n}(q)}{R_P^{\geq n}(q')}\right)^{\pm 1} \leq C_\lambda\right)\,,$$

without assuming that $\lambda > 0$ is small. Actually it is even possible to estimate C_λ in terms of $\lambda \to \infty$ by applying (3.A.5) *to the polynomial tP , $t \in [0,1]$, with* $t^{\frac{1}{n}} R_P^{\geq n}(q) \leq R_{tP}^{\geq n}(q) \leq t^{\frac{1}{r}} R_P^{\geq n}(q)$.

The main feature of a slow varying metric is that it is possible to introduce some partitions of unity related to the metric in a way made precise in the following theorem. For more details and proof see [Hor1] (Section 1.4 page 25).

Theorem 3.A.7. *[Hor1] For any slowly varying metric g in $\mathbb{R}^m$ one can choose a sequence $x_\nu \in \mathbb{R}^m$ such that the balls*

$$B_\nu = \left\{x;\ \sqrt{g_{x_\nu}(x - x_\nu, x - x_\nu)} < 1\right\}$$

form a covering of $\mathbb{R}^m$ *for which the intersection of more than* $N = (4C^3+1)^m$ *balls* B_ν *is always empty (*C *is the constant in (3.A.1)). In addition, for any decreasing sequence* d_i *with* $\sum_j d_j = 1$ *one can choose non negative* $\phi_\nu \in \mathcal{C}_0^\infty(B_\nu)$ *with* $\sum \phi_\nu = 1$ *in* $\mathbb{R}^m$ *so that for all* k

$$|\phi_\nu^{(k)}(x; y_1, \cdots, y_k)| \leq (NCC_1)^k \sqrt{g_x(y_1,y_1)} \cdots \sqrt{g_x(y_k,y_k)}/d_1 \cdots d_k$$

where C *is the constant in (3.A.1) and* C_1 *is a constant that depends only on* m.

Regarding the above Theorem we have the following result.

Lemma 3.A.8. *Let* $P \in E_r$*, where* $r \in \mathbb{N}^*$ *is the degree of* P *and* $n \in \{1,...,r\}$*, then there exists a partition of unity* $\sum_{j\in\mathbb{N}} \Psi_j(q)^2 \equiv\ 1$ *in* $\mathbb{R}^d$ *such that :*
1) For all $q \in \mathbb{R}^d$*, the cardinality of the set* $\{j, \Psi_j(q) \neq 0\}$ *is uniformly bounded.*
2) For any natural number $j \in \mathbb{N}$,

$$\operatorname{supp} \Psi_j \subset B(q_j, aR_P^{\geq n}(q_j)^{-1}) \quad \textit{and} \quad \Psi_j \equiv 1 \ \textit{ in } B(q_j, bR_P^{\geq n}(q_j)^{-1}) \ ,$$

for some $q_j \in \mathbb{R}^d$ *with* $0 < b < a$ *independent of* $j \in \mathbb{N}$.
3) For all $\alpha \in \mathbb{N}^d \setminus \{0\}$*, there exists* $c_\alpha > 0$ *such that*

$$\sum_{j\in\mathbb{N}} |\partial_q^\alpha \Psi_j|^2 \leq c_\alpha R_P^{\geq n}(q)^{2|\alpha|} \ .$$

Moreover the constants a, b *et* c_α *can be chosen uniformly with respect to* $P \in E_r$*, once the degree* $r \in \mathbb{N}$ *and the dimension* $d \in \mathbb{N}$ *are fixed.*

3.B Around Tarski-Seidenberg theorem

In this appendix we give an application of the Tarski-Seidenberg Theorem [Hor2], which we state in the following geometric form. We first introduce a few basic concepts needed for the statement.

Definition 3.B.1. *A subset of* $\mathbb{R}^n$ *is called semi-algebraic if it is a finite union of intersections of finitely many sets defined by polynomial equations or inequalities.*

Definition 3.B.2. *Let* $A \subset \mathbb{R}^n$ *and* $B \subset \mathbb{R}^m$ *be two sub-algebraic sets. The function* $f : A \to B$ *is said to be semi-algebraic if its graph* $\Gamma_f = \{(x,y) \in A \times B;\ y = f(x)\}$ *is a semi-algebraic set of* $\mathbb{R}^n \times \mathbb{R}^m$.

Theorem 3.B.3. *[Hor2](Tarski-Seidenberg) If A is a semi-algebraic subset of $\mathbb{R}^{n+m} = \mathbb{R}^n \oplus \mathbb{R}^m$, then the projection A' of A in $\mathbb{R}^m$ is also semi-algebraic.*

Proposition 3.B.4. *[Hor2] If E is a semi-algebraic set on $\mathbb{R}^{2+n}$, and*

$$f(x) = \inf\{y \in \mathbb{R};\ \exists z \in \mathbb{R}^n, (x, y, z) \in E\}$$

is defined and finite for large positive x, then f is identically 0 for large x or else

$$f(x) = Ax^a(1 + o(1))\ , \quad x \to +\infty$$

where $A \neq 0$ and a is a rational number.

We refer to [Hor2] (see Theorem A.2.2 and Theorem A.2.5) for detailed proofs of Theorem 3.B.3 and Proposition 3.B.4.

In the final part of this section we list and recall the following notations.

Notation 3.B.5. *Let P be a real-valued polynomial on $\mathbb{R}^d$ with $d^\circ P = r$. For all natural number $n \in \{0, \cdots, r\}$ and every $q \in \mathbb{R}^d$*

$$R_P^{\geq n}(q) = \sum_{n \leq |\alpha| \leq r} |\partial_q^\alpha P(q)|^{\frac{1}{|\alpha|}}\ , \tag{3.B.1}$$

$$R_P^{=n}(q) = \sum_{|\alpha| = n} |\partial_q^\alpha P(q)|^{\frac{1}{|\alpha|}}\ . \tag{3.B.2}$$

Lemma 3.B.6. *Let $\widetilde{\Sigma}$ be an unbounded semialgebraic set and V a polynomial in $\mathbb{R}[q_1, ..., q_d]$ of degree $r \in \mathbb{N}^*$ satisfying the following assumption*

$$\lim_{\substack{|q| \to +\infty \\ q \in \widetilde{\Sigma}}} \frac{R_V^{\geq n}(q)^\alpha}{R_V^{=m}(q)^2} = 0\ , \tag{3.B.3}$$

where $\alpha \in \mathbb{Q}, n, m \in \{0, 1, \cdots, r-1\}$, $n > m$, are fixed numbers.

Then there exist $\delta \in (0, 1)$ and a positive nondecreasing function $\Lambda_{\widetilde{\Sigma}} : (0, +\infty) \to [0, +\infty)$ so that

$$\forall q \in \widetilde{\Sigma}\,,\ \forall \varrho > 0\,,\ |q| \geq \varrho\,, \quad \Lambda_{\widetilde{\Sigma}}(\varrho) R_V^{\geq n}(q)^\alpha \leq R_V^{=m}(q)^{2(1-\delta)}$$
$$\text{and} \qquad \lim_{\varrho \to +\infty} \Lambda_{\widetilde{\Sigma}}(\varrho) = +\infty\,.$$

Proof. Let V be a real-valued polynomial on $\mathbb{R}^d$ with degree $r \in \mathbb{N}^*$. Suppose that there are $\alpha \in \mathbb{Q}, n, m \in \{0, 1, \cdots, r-1\}$ such that

$$\lim_{\substack{|q|\to+\infty \\ q\in\widetilde{\Sigma}}} \frac{R_V^{\geq n}(q)^\alpha}{R_V^{=m}(q)^2} = 0 , \tag{3.B.4}$$

where $\widetilde{\Sigma}$ is a given unbounded semialgebraic set.

After setting $\tau = 2\,\mathrm{LCM}\Big(|\beta|,\ \min(n,m) \leq |\beta| \leq r\Big)$, (where the abbreviation LCM stands for least common multiple), define the functions $\widetilde{R}_V^{\geq n}$ and $\widetilde{R}_V^{=m}$, for all $q \in \mathbb{R}^d$ by

$$\widetilde{R}_V^{\geq n}(q) = \sum_{n\leq|\alpha|\leq r} |\partial_q^\alpha V(q)|^{\frac{\tau}{|\alpha|}}$$

and

$$\widetilde{R}_V^{=m}(q) = \sum_{|\alpha|=m} |\partial_q^\alpha V(q)|^{\frac{\tau}{|\alpha|}} .$$

Notice that one has the equivalences $R_V^{\geq n}(q) \asymp \Big(\widetilde{R}_V^{\geq n}(q)\Big)^{\frac{1}{\tau}}$ and $R_V^{=m}(q) \asymp \Big(\widetilde{R}_V^{=m}(q)\Big)^{\frac{1}{\tau}}$ for all $q \in \mathbb{R}^d$ where the functions $R_V^{\geq n}$ and $R_V^{=m}$ are defined respectively as in (3.B.1) and (3.B.2). Clearly the Assumption (3.B.4) is equivalent to

$$\lim_{\substack{|q|\to+\infty \\ q\in\widetilde{\Sigma}}} \frac{\widetilde{R}_V^{\geq n}(q)^\alpha}{\widetilde{R}_V^{=m}(q)^2} = 0 . \tag{3.B.5}$$

Remark here that $\widetilde{R}_V^{\geq n}(q)$ and $\widetilde{R}_V^{=m}(q)$ are polynomials in $q \in \mathbb{R}^d$ variable. Furthermore, the Assumption (3.B.5) can be written as follows

$$\widetilde{R}_V^{\geq n}(q)^\alpha \leq \epsilon(q)\widetilde{R}_V^{=m}(q)^2 ,$$

for all $q \in \widetilde{\Sigma}$ where

$$\epsilon(q) = \inf\Big\{\epsilon > 0\,,\ \epsilon\widetilde{R}_V^{=m}(q)^2 - \widetilde{R}_V^{\geq n}(q)^\alpha > 0\Big\} , \qquad \lim_{\substack{|q|\to+\infty \\ q\in\widetilde{\Sigma}}} \epsilon(q) = 0 . \tag{3.B.6}$$

Now, following the notations of Proposition 3.B.4, we introduce the set

$$E = \Big\{(q, \varrho, \epsilon) \in \mathbb{R}^{d+2} \text{ such that } \epsilon\widetilde{R}_V^{=m}(q)^2 - \widetilde{R}_V^{\geq n}(q)^\alpha > 0 \text{ and } |q|^2 \geq \varrho^2\Big\} ,$$

and the function f defined in $\mathbb{R}_+$ by

$$f(\varrho) = \inf\big\{\epsilon > 0 :\ \exists q \in \mathbb{R}^d,\ \ (q, \varrho, \epsilon) \in E\big\} . \tag{3.B.7}$$

By Tarski-Seidenberg Theorem (see Theorem 3.B.3), the function f is semialgebraic in ϱ. Moreover f is defined, finite and not identically zero. Then by Proposition 3.B.4, there exist a constant $A > 0$ and a rational number γ such that

$$f(\varrho) = A\varrho^{\gamma} + o_{\varrho\to+\infty}(\varrho^{\gamma}) \ .$$

By the definition (3.B.7) and (3.B.6), $\lim\limits_{\varrho\to+\infty} f(\varrho) = 0$ and then $\gamma < 0$. Hence for $\varrho \gg 1$, we know $f(\varrho) \leq \frac{2A}{\varrho^{|\gamma|}}$. We deduce for $|q| \gg 1$,

$$\widetilde{R}_V^{\geq n}(q)^{\alpha} \leq f(|q|)\widetilde{R}_V^{=m}(q)^2 \leq \frac{2A}{|q|^{|\gamma|}}\widetilde{R}_V^{=m}(q)^2 \tag{3.B.8}$$

and

$$\frac{|q|^{|\gamma|/2}}{2A}\widetilde{R}_V^{\geq n}(q)^{\alpha} \leq \frac{1}{|q|^{\frac{|\gamma|}{2}}}\widetilde{R}_V^{=m}(q)^2 \, . \tag{3.B.9}$$

In particular, since $\widetilde{R}_V^{\geq n}(q) \geq \widetilde{R}_V^{=r}(0) > 0$, $\widetilde{R}_V^{=m}(q)$ does not vanish for $q \in \widetilde{\Sigma}$ with $|q| \geq 1$.

On the other hand, notice

$$\forall q \in \widetilde{\Sigma}\,,\ |q| \geq 1\,, \quad \widetilde{R}_V^{=m}(q) \leq c|q|^{\tau r}\, . \tag{3.B.10}$$

The inequalities (3.B.8) and (3.B.10) lead to

$$\widetilde{R}_V^{\geq n}(q)^{\alpha} \leq C|q|^{2\tau r - |\gamma|}$$

for every $q \in \widetilde{\Sigma}$ with $|q| \geq \rho \gg 1$. Therefore since $\widetilde{R}_V^{\geq n}(q) \geq \widetilde{R}_V^{=r}(0) > 0$ we deduce $|\gamma| \leq 2\tau r$.
Using again (3.B.10) we get

$$\frac{1}{|q|^{\frac{|\gamma|}{2}}} \leq \frac{c^{\frac{|\gamma|}{2\tau r}}}{\widetilde{R}_V^{=m}(q)^{\frac{|\gamma|}{2\tau r}}} \ , \tag{3.B.11}$$

for any $q \in \widetilde{\Sigma}$ with $|q| \geq 1$.

From (3.B.9) and (3.B.11), we deduce the existence of $\varrho_0 \gg 1$ such that

$$\forall q \in \widetilde{\Sigma}\,, |q| \geq \varrho \geq \varrho_0 \gg 1\,, \quad \frac{\varrho^{|\gamma|/2}}{2A}\widetilde{R}_V^{\geq n}(q)^{\alpha} \leq \frac{|q|^{|\gamma|/2}}{2A}\widetilde{R}_V^{\geq n}(q)^{\alpha} \leq c^{\frac{|\gamma|}{2\tau r}}\,\widetilde{R}_V^{=m}(q)^{2(1-\frac{|\gamma|}{4\tau r})}\, . \tag{3.B.12}$$

We now take $\delta = \frac{|\gamma|}{4\tau r} \in (0,1)$ and

$$\widetilde{\Lambda}_{\widetilde{\Sigma}}(\varrho) = \begin{cases} \frac{\varrho^{|\gamma|/2}}{2Ac^{\frac{|\gamma|}{2\tau r}}} & \text{if} \quad \varrho \geq \varrho_0 \\ 0 & \text{otherwise}\, . \end{cases}$$

The function $\widetilde{\Lambda}_{\widetilde{\Sigma}} : (0, +\infty) \to [0, +\infty)$ is clearly positive and due to (3.B.12) it satisfies

$$\forall q \in \widetilde{\Sigma}\,,\ \forall \varrho > 0\,,\ |q| \geq \varrho\,, \quad \widetilde{\Lambda}_{\widetilde{\Sigma}}(\varrho) R_V^{\geq n}(q)^\alpha \leq\ R_V^{=m}(q)^{2(1-\delta)}$$
$$\text{and} \qquad \lim_{\varrho\to+\infty} \widetilde{\Lambda}_{\widetilde{\Sigma}}(\varrho) = +\infty\,.$$

To conclude, it is sufficient to take $\Lambda_{\widetilde{\Sigma}} : (0, +\infty) \to [0, +\infty)$ defined by

$$\Lambda_{\widetilde{\Sigma}}(\varrho) = \inf_{|q|\geq\varrho} \frac{R_V^{=m}(q)^{2(1-\delta)}}{R_V^{\geq n}(q)^\alpha}\ ,$$

which is non decreasing and larger than $\tilde{\Lambda}_{\widetilde{\Sigma}}$. □

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Chapitre 4

Opérateurs de KFP avec potentiels homogènes (article rédigé en anglais)

Article [BNV], rédigé en anglais.

Kramers-Fokker-Planck operators with homogeneous potentials

Abstract

In this article we establish a global subelliptic estimate for Kramers-Fokker-Planck operators with homogeneous potentials $V(q)$ under some conditions, involving in particular the control of the eigenvalues of the Hessian matrix of the potential. Namely, this work presents a different approach from the one in [Ben], in which the case $V(q_1, q_2) = -q_1^2(q_1^2 + q_2^2)^n$ was already treated only for $n = 1$. With this article, after the former one dealing with non homogeneous polynomial potentials, we conclude the analysis of almost all the examples of degenerate ellipticity at infinty presented in the framework of Witten Laplacian by Helffer and Nier in [HeNi]. Like in [Ben], our subelliptic lower bounds are the optimal ones up to some logarithmic correction.

Key words : subelliptic estimates, compact resolvent, Kramers-Fokker-Planck operator.
MSC-2010 : 35Q84, 35H20, 35P05, 47A10, 14P10

Contents

4.1 Introduction and main results

In this work we study the Kramers-Fokker-Planck operator

$$K_V = p\partial_q - \partial_q V(q)\partial_p + \frac{1}{2}(-\Delta_p + p^2)\ , \qquad (q,p) \in \mathbb{R}^{2d}\,, \tag{4.1.1}$$

where q denotes the space variable, p denotes the velocity variable and the potential $V(q)$ is a real-valued function defined in the whole space $\mathbb{R}^d_q$.

Setting

$$O_p = \frac{1}{2}(D_p^2 + p^2)\ , \qquad \text{and} \qquad X_V = p\partial_q - \partial_q V(q)\partial_p\ ,$$

the Kramers-Fokker-Planck operator K_V defined in (4.1.1) reads $K_V = X_V + O_p$.
We firstly list some notations used throughout the paper. We denote for an arbitrary function $V(q)$ in $\mathcal{C}^\infty(\mathbb{R}^d)$

$$\mathrm{Tr}_{+,V}(q) = \sum_{\substack{\nu \in \mathrm{Spec}(\mathrm{Hess}\ V(q)) \\ \nu > 0}} \nu(q)\,,$$

$$\mathrm{Tr}_{-,V}(q) = -\sum_{\substack{\nu \in \mathrm{Spec}(\mathrm{Hess}\ V(q)) \\ \nu \leq 0}} \nu(q)\ .$$

In particular for a polynomial V of degree less than 3, $\mathrm{Tr}_{+,V}$ and $\mathrm{Tr}_{-,V}$ are two constants. In this case we define the constants A_V and B_V by

$$A_V = \max\{(1 + \mathrm{Tr}_{+,V})^{2/3}, 1 + \mathrm{Tr}_{-,V}\}\ ,$$

$$B_V = \max\{\min_{q\in\mathbb{R}^d} |\nabla\ V(q)|^{4/3}\,, \frac{1 + \mathrm{Tr}_{-,V}}{(\log(2 + \mathrm{Tr}_{-,V}))^2}\}\ .$$

This work is principally based on the publication by Ben Said, Nier, and Viola [BNV], which concerns the study of Kramers-Fokker-Planck operators with polynomials of degree less than three. In [BNV] we proved the existence of a constant $c > 0$, independent of V, such that the following global subelliptic estimate with remainder

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + A_V\|u\|^2_{L^2(\mathbb{R}^{2d})} \geq c\Big(\|O_p u\|^2_{L^2(\mathbb{R}^{2d})} + \|X_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle\partial_q V(q)\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle D_q\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})}\Big) \tag{4.1.2}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. Furthermore, there exists a constant $c > 0$, independent of V, such that

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq c\, B_V \|u\|^2_{L^2(\mathbb{R}^{2d})}\ , \tag{4.1.3}$$

is valid for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. As a consequence collecting (4.1.3) and (4.1.2) together, there is a constant $c > 0$, independent of V, so that the global subelliptic estimates without remainder

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{c}{1 + \frac{A_V}{B_V}} \Big(\|O_p u\|^2_{L^2(\mathbb{R}^{2d})} + \|X_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle \partial_q V(q)\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle D_q\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})} \Big) \quad (4.1.4)$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. Here and throughout the paper we use the notation

$$\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}\ .$$

Moreover we remind that for an arbitrary potential $V \in \mathcal{C}^\infty(\mathbb{R}^d)$, the Kramers-Fokker-Planck operator K_V is essentially maximal accretive when endowed with the domain $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ (see Proposition 5.5, page 44 in [HeNi]). Thanks to this property we deduce that the domain of the closure of K_V is given by

$$D(K_V) = \big\{ u \in L^2(\mathbb{R}^{2d}),\ K_V u \in L^2(\mathbb{R}^{2d}) \big\}\ .$$

Resultantly, by density of $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ in the domain $D(K_V)$ all estimates written in this article, which are verified with $C_0^\infty(\mathbb{R}^{2d})$ functions, can be extended to $D(K_V)$. By relative bounded perturbation with bound less than 1, this result holds as well when $V \in \mathcal{C}^\infty(\mathbb{R} \setminus \{0\})$ is an homogeneous function of degree $r > 1$.

Our results will require the following assumption after setting

$$\mathcal{S} = \big\{ q \in \mathbb{R}^d,\ \ |q| = 1 \big\}\ . \quad (4.1.5)$$

Assumption 2. *The potential $V(q)$ is a homogeneous function of degree $2 < r < 6$ in $\mathcal{C}^\infty(\mathbb{R}^d \setminus \{\ 0\})$ and satisfies :*

$$\forall\ \ q \in \mathcal{S}\ , \qquad \partial_q V(q) = 0 \Rightarrow \mathrm{Tr}_{-,V}(q) > 0\ . \quad (4.1.6)$$

Our main result is the following.

Theorem 4.1.1. *If the potential $V(q)$ verifies Assumption 2, then there exists a strictly positive constant $C_V > 1$ (which depends on V) such that*

$$\|K_V u\|^2_{L^2} + C_V\|u\|^2_{L^2} \geq \frac{1}{C_V} \Big(\|L(O_p)u\|^2_{L^2} + \|L(\langle \nabla V(q)\rangle^{\frac{2}{3}})u\|^2_{L^2} + \|L(\langle \mathrm{Hess}\ V(q)\rangle^{\frac{1}{2}})u\|^2_{L^2} + \|L(\langle D_q\rangle^{\frac{2}{3}})u\|^2_{L^2} \Big)\ , \quad (4.1.7)$$

holds for all $u \in D(K_V)$ where $L(s) = \frac{s+1}{\log(s+1)}$ for any $s \geq 1$.

Corollary 4.1.2. *The Kramers-Fokker-Planck operator K_V with a potential $V(q)$ satisfying Assumption 2 has a compact resolvent.*

Proof. Let $0 < \delta < 1$. Define the functions $f_\delta : \mathbb{R}^d \to \mathbb{R}$ by

$$f_\delta(q) = |\nabla V(q)|^{\frac{4}{3}(1-\delta)} + |\mathrm{Hess}\, V(q)|^{1-\delta} \,.$$

As a result of (4.1.7) in Theorem 4.1.1 there is a constant $C_V > 1$ such that

$$\|K_V u\|^2_{L^2} + C_V \|u\|^2_{L^2} \geq \frac{1}{C_V}\Big(\langle u, f_\delta u\rangle + \|L(O_p)u\|^2_{L^2} + \|L(\langle D_q\rangle^{\frac{2}{3}})u\|^2_{L^2}\Big) \,,$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and all $\delta \in (0,1)$. In order to show that the operator K_V has a compact resolvent it is sufficient to prove that $\lim\limits_{|q|\to+\infty} f_\delta(q) = +\infty$. It is a matter of how different derivatives scale. Consider the unit sphere $S = \{q \in \mathbb{R}^d : |q| = 1\}$. By Assumption (4.1.6), at every point on S either $\nabla V \neq 0$ or $|\mathrm{Hess}\, V| \neq 0$. Then the function f_δ is always positive on S. By hypothesis, f_δ is continuous on S and therefore it achieves a positive minimum there, call it $m_\delta > 0$.

For any $y, |y| > 1$ there exists $\lambda > 1$ such that $y = \lambda q$ for some $q \in S$. By homogeneity,

$$V(y) = \lambda^r V\left(\frac{y}{\lambda}\right) = \lambda^r V(q)$$

and therefore, by the chain rule

$$|\nabla V(y)| = \lambda^{r-1} |\nabla V(q)|$$

and

$$|\mathrm{Hess}\; V(y)| = \lambda^{r-2} |\mathrm{Hess}\; V(q)|.$$

Adding these up,

$$|\nabla V(y)|^{\frac{4}{3}(1-\delta)} + |\mathrm{Hess}\, V(y)|^{1-\delta} \geq \lambda^{(1-\delta)\min\{\frac{4}{3}(r-1),\, r-2\}} f_\delta(q) \geq m_\delta \lambda^{(1-\delta)\min\{\frac{4}{3}(r-1),\, r-2\}}$$

which goes to infinity as $|y| = \lambda \to \infty$, since by assumption $r > 2$. □

Remark 4.1.3. *The result of Corollary does not hold in the case of homogeneous polynomial of degree 2 with degenerate Hessian. In the case with degenerate Hessian, the Kramers-Fokker-Planck operator is indeed invariant by translation in the direction of the kernel of the Hessian and then it could not have a compact resolvent.*

Remark 4.1.4. *Our results are in agreement with the results of Wei-Xi Li [Li][Li2] and those of Helffer-Nier on Witten Laplacian with homogeneous potential [HeNi1].*

4.2 Observations and first inequalities

4.2.1 Dyadic partition of unity

In this paper, we make use of a locally finite dyadic partition of unity with respect to the position variable $q \in \mathbb{R}^d$. Such a partition is described in the following Proposition. For a detailed proof, we refer to [BCD] (see page 59).

Proposition 4.2.1. *Let $\mathcal{C}$ be the shell $\left\{x \in \mathbb{R}^d,\ \frac{3}{4} < |x| < \frac{8}{3}\right\}$. There exist radial functions χ and ϕ valued in the interval $[0,1]$, belonging respectively to $\mathcal{C}_0^\infty(B(0,\frac{4}{3}))$ and to $\mathcal{C}_0^\infty(\mathcal{C})$ such that*

$$\forall x \in \mathbb{R}^d, \qquad \chi(x) + \sum_{j\geq 0} \phi(2^{-j}x) = 1\ ,$$

$$\forall x \in \mathbb{R}^d \setminus \{0\}\,, \qquad \sum_{j\in\mathbb{Z}} \phi(2^{-j}x) = 1\ .$$

Setting for all $q \in \mathbb{R}^d$,

$$\chi_{-1}(q) = \frac{\chi(2q)}{\Big(\chi^2(2q) + \sum\limits_{j'\geq 0} \phi^2(2^{-j'+1}q)\Big)^{\frac{1}{2}}} = \frac{\chi(2q)}{\Big(\chi^2(2q) + \phi^2(2q)\Big)^{\frac{1}{2}}}\ ,$$

$$\chi_0(q) = \frac{\phi(2^1 q)}{\Big(\chi^2(2q) + \sum\limits_{j'\geq 0} \phi^2(2^{-j'+1}q)\Big)^{\frac{1}{2}}} = \frac{\phi(2q)}{\Big(\chi^2(2q) + \phi^2(2q) + \phi^2(q)\Big)^{\frac{1}{2}}}$$

$$\text{and for } j \geq 1\,, \qquad \chi_j(q) = \frac{\phi(2^{-j+1}q)}{\Big(\chi^2(2q) + \sum\limits_{j'\geq 0} \phi^2(2^{-j'+1}q)\Big)^{\frac{1}{2}}} = \frac{\phi(2^{-j+1}q)}{\Big(\sum\limits_{j-1\leq j'\leq j+1} \phi^2(2^{-j'+1}q)\Big)^{\frac{1}{2}}}$$

we get a locally finite dyadic partition of unity

$$\sum_{j\geq -1} \chi_j^2(q) = \tilde{\chi}_{-1}^2(2|q|) + \tilde{\chi}_0^2(2|q|) + \sum_{j\geq 1} \tilde{\chi}^2(2^{-j+1}|q|) = 1\ , \tag{4.2.1}$$

where the cutoff functions $\widetilde{\chi}_0(r) = \frac{\phi(q)}{\left(\chi^2(q)+\phi^2(q)+\phi^2(\frac{q}{2})\right)^{\frac{1}{2}}}$, $\tilde{\chi}(r) = \frac{\phi(q)}{\left(\sum\limits_{0\leq j'\leq 2} \phi^2(2^{-j'+1}q)\right)^{\frac{1}{2}}}$ and $\widetilde{\chi}_{-1}(r) = \frac{\chi(q)}{\left(\chi^2(q)+\phi^2(q)\right)^{\frac{1}{2}}}$ when $|q| = r$, belong respectively to $\mathcal{C}_0^\infty(\left]\frac{3}{4},\frac{8}{3}\right[)$, $\mathcal{C}_0^\infty(\left]\frac{3}{4},\frac{8}{3}\right[)$ and $\mathcal{C}_0^\infty(\left[0,\frac{4}{3}\right[)$.

Lemma 4.2.2. *Let V be in $\mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\})$. Consider the Kramers-Fokker-Planck operator K_V defined as in (4.1.1). For a locally finite partition of unity $\sum_{j\geq -1} \chi_j^2(q) = 1$ one has*

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} = \sum_{j\geq -1} \left(\|K_V(\chi_j u)\|^2_{L^2(\mathbb{R}^{2d})} - \|(p\partial_q \chi_j)u\|^2_{L^2(\mathbb{R}^{2d})} \right) , \tag{4.2.2}$$

for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

In particular when the cutoff functions χ_j have the form (4.2.1), there exists an absolute constant $c > 0$ so that

$$(1+4c)\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + c\|u\|^2_{L^2(\mathbb{R}^{2d})} \geq \sum_{j\geq -1} \|K_V(\chi_j u)\|^2_{L^2(\mathbb{R}^{2d})}, \tag{4.2.3}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

Proof. The proof of the equality (4.2.2) is detailed in [Ben]. Now it remains to show the inequality (4.2.3), after considering a locally finite dyadic partition of unity (described as below Proposition 4.2.1),

$$\sum_{j\geq -1} \chi_j^2(q) = 1 , \tag{4.2.4}$$

where for all $j \in \mathbb{N}$, the cutoff functions χ_j and χ_{-1} are respectively supported in the shell $\left\{q \in \mathbb{R}^d,\ 2^j \frac{3}{4} \leq |q| \leq 2^j \frac{8}{3}\right\}$ and in the ball $B(0, 2^j \frac{3}{4})$.

Since the partition is locally finite, for each index $j \geq -1$ there are finitely many j' such that $(\partial_q \chi_j)\chi_{j'}$ is nonzero. Along these lines, there exists a uniform constant $c > 0$ so that

$$\begin{aligned} \sum_{j\geq -1} \|(p\partial_q \chi_j)u\|^2_{L^2} &= \sum_{j\geq -1} \sum_{j'\geq -1} \|(p\partial_q \chi_j)\chi_{j'} u\|^2_{L^2} \\ &\leq c \sum_{j\geq -1} \frac{1}{(2^j)^2} \|p\chi_j u\|^2_{L^2} , \end{aligned} \tag{4.2.5}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

On the other hand, for every $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$c \sum_{j\geq -1} \frac{1}{(2^j)^2} \|p\chi_j u\|^2_{L^2} \leq 4c\, \|pu\|^2_{L^2} \leq 8c \,\mathrm{Re}\, \langle u, K_V u\rangle \leq 4c\, (\|u\|^2_{L^2} + \|K_V u\|^2_{L^2}) . \tag{4.2.6}$$

Collecting the estimates (4.2.2), (4.2.5) and (4.2.6), we establish the desired inequality (4.2.3). □

4.2.2 Localisation in a fixed Shell

Lemma 4.2.3. *Let $V(q)$ be an homogeneous function in $\mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\})$ of degree r and assume $j \in \mathbb{Z}$. Given $u_j \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$, one has*

$$\|K_V u_j\|_{L^2(\mathbb{R}^{2d})} = \|K_{j,V} v_j\|_{L^2(\mathbb{R}^{2d})} ,$$

where the operator $K_{j,V}$ is defined by

$$K_{j,V} = \frac{1}{2^j} p\partial_q - (2^j)^{r-1}\partial_q V(q)\partial_p + O_p , \tag{4.2.7}$$

and $v_j(q,p) = 2^{\frac{jd}{2}} u_j(2^j q, p)$.

In particular when u_j is supported in $\left\{q \in \mathbb{R}^d,\ 2^j \frac{3}{4} \le |q| \le 2^j \frac{8}{3}\right\}$, the support of v_j is a fixed shell $\overline{\mathcal{C}} = \left\{q \in \mathbb{R}^d,\ \frac{3}{4} \le |q| \le \frac{8}{3}\right\}$.

Proof. Let $j \in \mathbb{Z}$ be an index. Assume $u_j \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and put

$$v_j(q,p) = 2^{\frac{jd}{2}} u_j(2^j q, p) . \tag{4.2.8}$$

Since V is homogeneous of degree r, its gradient $\partial_q V(q)$ is homogeneous of degree $r-1$. As follows, we can write

$$\begin{aligned} K_V u_j(q,p) &= K_V\Big(2^{\frac{-jd}{2}} v_j(2^{-j}q, p)\Big) \\ &= 2^{\frac{-jd}{2}}\Big((2^{-j}p\partial_q - (2^j)^{r-1}\partial_q V(q)\partial_p + O_p)v_j\Big)(2^{-j}q, p) . \end{aligned}$$

Notice that if

$$\text{supp}\ u_j \subset \left\{q \in \mathbb{R}^d,\ 2^j\frac{3}{4} \le |q| \le 2^j\frac{8}{3}\right\} ,$$

the cutoff functions v_j, defined in (4.2.8), are all supported in the fixed shell

$$\overline{\mathcal{C}} = \left\{q \in \mathbb{R}^d,\ \frac{3}{4} \le |q| \le \frac{8}{3}\right\} .$$

□

Remark 4.2.4. *Assume $j \in \mathbb{N}$. If we introduce a small parameter $h = 2^{-2(r-1)j}$ then the operator $K_{j,V}$, defined in (4.2.7), can be rewritten as*

$$K_{j,V} = \frac{1}{h}\Big(\sqrt{h}p(h^{\frac{1}{2}+\frac{1}{2(r-1)}}\partial_q) - \sqrt{h}\partial_q V(q)\partial_p + \frac{h}{2}(-\Delta_p + p^2)\Big) .$$

Now owing to a dilation with respect to the velocity variable p, that with $(\sqrt{h}p, \sqrt{h}\partial_p)$ *associates* $(p, h\partial_p)$*, we deduce that the operator* $K_{j,V}$ *is unitary equivalent to*

$$\widehat{K}_{j,V} = \frac{1}{h}\Big(p(h^{\frac{1}{2}+\frac{1}{2(r-1)}}\partial_q) - \partial_q V(q) h\partial_p + \frac{1}{2}(-h^2\Delta_p + p^2)\Big) \ .$$

In particular, taking $r = 2$*,*

$$\widehat{K}_{j,V} = \frac{1}{h}\Big(p(h\partial_q) - \partial_q V(q) h\partial_p + \frac{1}{2}(-h^2\Delta_p + p^2)\Big) \ ,$$

is clearly a semiclassical operator with respect to the variables q and p. However if $r > 2$*, the operator* $\widehat{K}_{j,V}$ *is semiclassical only with respect to the velocity variable p (since* $h^{\frac{1}{2}+\frac{1}{2(r-1)}} > h$*). For a polynomial* $V(q)$*, the case* $r = 2$ *corresponds to the quadratic situation. Extensive works have been done concerned with this case (see [Hor][HiPr][Vio][Vio1][AlVi][BNV]).*

4.3 Proof of the main result

In this section we present the proof of Theorem 4.1.1.

Proof. In the whole proof we denote

$$\overline{\mathcal{C}} = \left\{ q \in \mathbb{R}^d, \ \frac{3}{4} \le |q| \le \frac{8}{3} \right\} \ .$$

Assume $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and consider a locally finite dyadic partition of unity defined as in (4.2.1). By Lemma 4.2.2 (see (4.2.3)), there is a uniform constant c such that

$$(1+4c)\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + c\|u\|^2_{L^2(\mathbb{R}^{2d})} \ge \sum_{j\ge -1} \|K_V u_j\|^2_{L^2(\mathbb{R}^{2d})}. \tag{4.3.1}$$

where we denote $u_j = \chi_j u$. We obtain by Lemma 4.2.3 and the estimate (4.3.1)

$$(1+4c)\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + c\|u\|^2_{L^2(\mathbb{R}^{2d})} \ge \sum_{j\ge -1} \|K_{j,V} v_j\|^2_{L^2(\mathbb{R}^{2d})} \ , \tag{4.3.2}$$

where the operator

$$K_{j,V} = \frac{1}{2^j} p\partial_q - (2^j)^{r-1}\partial_q V(q)\partial_p + O_p \ ,$$

and $v_j(q,p) = 2^{\frac{jd}{2}} u_j(2^j q, p)$. Setting $h = 2^{-2(r-1)j}$, one has

$$K_{j,V} = p(h^{\frac{1}{2(r-1)}}\partial_q) - h^{-\frac{1}{2}}\partial_q V(q)\partial_p + \frac{1}{2}(-\Delta_p + p^2) \ .$$

In the analysis we will focus on the case when $j \geq J$, with J large enough, or equivalently $h \leq h_J$ with h_J small enough. The remaining error terms in $\sum_{j=-1}^{J}$ will be absorbed in the end by the term $C_V\|u\|^2_{L^2(\mathbb{R}^d)}$ in (4.1.7) by taking C_V large enough.

Now, fix $\nu > 0$ such that

$$\frac{3}{16} + \frac{1}{16(r-1)} < \nu \leq \frac{1}{8} + \frac{3}{8(r-1)} \ . \tag{4.3.3}$$

Such a choice is always possible when $2 < r < 6$. Taking $\nu > 0$, satisfying (4.3.3) and a function $\theta \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\sum_{k\in\mathbb{Z}^d} \theta(\cdot - k)^2 = 1$, we define for all $q \in \mathbb{R}^d$,

$$\theta_{k,h}(q) = \theta\Big(\frac{1}{|\ln(h)|h^\nu}(q - q_{k,h})\Big) \ ,$$

with $q_{k,h} = h^\nu|\ln(h)|k$. Denoting $K_h = \Big\{k \in \mathbb{Z}^d,\ |k| \leq \frac{3}{|\ln(h)|h^\nu}\Big\}$, one has

$$\sum_{k\in K_h} \theta_{k,h}(q)^2 = 1$$

in the neighborhood of $\overline{\mathcal{C}}$, where for any index $k \in K_h$,

$$\operatorname{supp} \theta_{k,h} \subset B(q_{k,h}, |\ln(h)|h^\nu) \ , \qquad \theta_{k,h} \equiv 1 \ \text{ in } \ B(q_{k,h}, \frac{1}{2}|\ln(h)|h^\nu) \ .$$

Using this partition we get through Lemma 4.2.2 (see (4.2.2)),

$$\|K_{j,V}v_j\|^2_{L^2} \geq \sum_{k\in K_h} \Big(\|K_{j,V}\theta_{k,h}v_j\|^2_{L^2} - |\ln(h)|^{-2}h^{\frac{1}{r-1}-2\nu}\|p\theta_{k,h}v_j\|^2_{L^2}\Big) \ . \tag{4.3.4}$$

In order to reduce the written expressions we denote in the whole of the proof

$$w_{k,j} = \theta_{k,h}v_j \ .$$

Taking into account (4.3.4),

$$\begin{aligned}\|K_{j,V}v_j\|^2_{L^2} &\geq \sum_{k\in K_h} \Big(\|K_{j,V}w_{k,j}\|^2_{L^2} - 2|\ln(h)|^{-2}h^{\frac{1}{r-1}-2\nu}\|w_{k,j}\|_{L^2}\|K_{j,V}w_{k,j}\|_{L^2}\Big) \\ &\geq \sum_{k\in K_h} \Big(\frac{1}{2}\|K_{j,V}w_{k,j}\|^2_{L^2} - 2(|\ln(h)|^{-2}h^{\frac{1}{r-1}-2\nu})^2\|w_{k,j}\|^2_{L^2}\Big) \ . \end{aligned} \tag{4.3.5}$$

Notice that in the last inequality we simply use respectively the fact that

$$\|pw_{k,j}\|^2_{L^2} \leq 2\mathrm{Re}\langle w_{k,j}, K_{j,V}w_{k,j}\rangle \leq 2\|w_{k,j}\|_{L^2}\|K_{j,V}w_{k,j}\|_{L^2} \ ,$$

and the Cauchy inequality. From now on, set

$$K_0 = \left\{q \in \overline{\mathcal{C}}\ , \quad \partial_q V(q) = 0\right\}\ .$$

Clearly, by continuity of the map $q \mapsto \partial_q V(q)$ on the shell $\overline{\mathcal{C}}$ (which is a compact set of $\mathbb{R}^d$), we deduce the compactness of K_0.

Since $q \mapsto \frac{\mathrm{Tr}_{-,V}(q)}{1+\mathrm{Tr}_{+,V}(q)}$ is uniformly continuous on any compact neighborhood of K_0 , there exists $\varepsilon_1 > 0$ such that

$$d(q, K_0) \leq \epsilon_1 \Rightarrow \frac{\mathrm{Tr}_{-,V}(q)}{1 + \mathrm{Tr}_{+,V}(q)} \geq \frac{\epsilon_0}{2}\ , \tag{4.3.6}$$

where $\epsilon_0 := \min\limits_{q\in K_0} \frac{\mathrm{Tr}_{-,V}(q)}{1+\mathrm{Tr}_{+,V}(q)}$.

On the other hand, in view of the definition of K_0 and by continuity of $q \mapsto \partial_q V(q)$ on $\overline{\mathcal{C}}$, there is a constant $\epsilon_2 > 0$ (that depends on ϵ_1) such that

$$\forall\ q \in \overline{\mathcal{C}}\ ,\ \ d(q, K_0) \geq \epsilon_1 \Rightarrow |\partial_q V(q)| \geq \epsilon_2\ . \tag{4.3.7}$$

Now let us introduce

$$\Sigma(\epsilon_1) = \{q \in \mathcal{C}\ ,\ \ d(q, K_0) \geq \epsilon_1\}\ ,$$

$$I(\epsilon_1) = \{k \in K_h\ ,\ \ \mathrm{supp}\ \theta_{k,h} \subset \Sigma(\epsilon_1)\}\ .$$

In order to establish a subelliptic estimate for $K_{j,V}$, we distinguish the two following cases.

Case **1** $k \notin I(\epsilon_1)$. In this case the support of the cutoff function $\theta_{k,h}$ might intersect the set of zeros of the gradient of V.

Case **2** $k \in I(\epsilon_1)$. Here the gradient of V does not vanish for all q in the support of $\theta_{k,h}$.

The idea is to use a quadratic approximating polynomial

$$V^{(2)}_{k,h}(q) = \sum_{|\alpha|\leq 2} \frac{\partial_q^\alpha V(q'_{k,h})}{\alpha!}(q - q'_{k,h})^\alpha\ .$$

near some $q'_{k,h} \in \mathrm{supp}\ \theta_{k,h}$ to write

$$\sum_{k\in K_h} \|K_{j,V} w_{k,j}\|^2_{L^2} \geq \sum_{k\in K_h} \left(\frac{1}{2}\|K_{j,V^{(2)}_{k,h}} w_{k,j}\|^2_{L^2} - \|(K_{j,V} - K_{j,V^{(2)}_{k,h}}) w_{k,j}\|^2_{L^2}\right)\ ,$$

or equivalently

$$\sum_{k\in K_h}\|K_{j,V}w_{k,j}\|_{L^2}^2 \geq \sum_{k\in K_h}\Big(\frac{1}{2}\|K_{j,V_{k,h}^{(2)}}w_{k,j}\|_{L^2}^2 - \|\frac{1}{\sqrt{h}}(\partial_q V(q)-\partial_q V_{k,h}^{(2)}(q))\partial_p w_{k,j}\|_{L^2}^2\Big)\;. \tag{4.3.8}$$

Then based on the estimates written in [BNV], which are valid for the operator $K_{V^2_{k,h}}$, we deduce a subelliptic estimate for $K_{V^2_{k,h}}$, after a careful control of the errors which appear in (4.3.5) and (4.3.8).

Notice that one has for all $q\in\mathbb{R}^d$,

$$|V(q)-V_{k,h}^{(2)}(q)| = \mathcal{O}(|q-q'_{k,h}|^3)\;. \tag{4.3.9}$$

Accordingly, for every q in the support of $w_{k,j}$,

$$\begin{aligned}|\partial_q V(q)-\partial_q V_{k,h}^{(2)}(q)| &= \mathcal{O}(|q-q'_{k,h}|^2)\\ &= \mathcal{O}(|\ln(h)|^2h^{2\nu})\;.\end{aligned} \tag{4.3.10}$$

Combining (4.3.8) and (4.3.10), there is a constant $c>0$ such that

$$\begin{aligned}\sum_{k\in K_h}\|K_{j,V}w_{k,j}\|_{L^2}^2 &\geq \sum_{k\in K_h}\Big(\frac{1}{2}\|K_{j,V_{k,h}^{(2)}}w_{k,j}\|_{L^2}^2 - c\,\frac{(|\ln(h)|^2h^{2\nu})^2}{h}\|\partial_p w_{k,j}\|_{L^2}^2\Big)\\ &\geq \sum_{k\in K_h}\Big(\frac{1}{2}\|K_{j,V_{k,h}^{(2)}}w_{k,j}\|_{L^2}^2 - 2c\,\frac{(|\ln(h)|^2h^{2\nu})^2}{h}\|w_{k,j}\|_{L^2}\|K_{j,V_{k,h}^{(2)}}w_{k,j}\|_{L^2}\Big)\\ &\geq \sum_{k\in K_h}\Big(\frac{1}{4}\|K_{j,V_{k,h}^{(2)}}w_{k,j}\|_{L^2}^2 - 16c^2\,\frac{(|\ln(h)|^2h^{2\nu})^4}{h^2}\|w_{k,j}\|_{L^2}^2\Big)\;.\end{aligned} \tag{4.3.11}$$

Putting (4.3.5) and (4.3.11) together,

$$\|K_{j,V}v_j\|^2 \geq \sum_{k\in K_h}\Big(\frac{1}{8}\|K_{j,V_{k,h}^{(2)}}w_{k,j}\|^2 - 8c^2\,\frac{(|\ln(h)|^2h^{2\nu})^4}{h^2}\|w_{k,j}\|^2 - 2(|\ln(h)|^{-2}h^{\frac{1}{r-1}-2\nu})^2\|w_{k,j}\|^2\Big)\;. \tag{4.3.12}$$

On the other hand, owing to a change of variables $q" = qh^{\frac{1}{2(r-1)}}$, one can write

$$\|K_{j,V_{k,h}^{(2)}}w_{k,j}\|_{L^2} = \|\widetilde{K}_{j,V_{k,h}^{(2)}}\widetilde{w}_{k,j}\|_{L^2}\;, \tag{4.3.13}$$

where the operator $\widetilde{K}_{j,V_{k,h}^{(2)}}$ reads

$$\widetilde{K}_{j,V_{k,h}^{(2)}} = p\partial_q - h^{-\frac{1}{2}}\partial_q V_{k,h}^{(2)}(h^{\frac{1}{2(r-1)}}q)\partial_p + \frac{1}{2}(-\Delta_p+p^2)\;,$$

and

$$w_{k,j}(q,p) = \frac{1}{h^{\frac{d}{4(r-1)}}} \widetilde{w}(\frac{q}{h^{\frac{1}{2(r-1)}}}, p) \ .$$

In the rest of the proof we denote

$$H := h^{-\frac{1}{2}} h^{\frac{1}{2(r-1)}} > 1 \ .$$

Case 1. In this situation, we use the quadratic approximation near some element $q'_{k,h} \in \text{supp}\,\theta_{k,h} \cap (\mathbb{R}^d \setminus \Sigma(\epsilon_1))$.
From now on assume $j \in \mathbb{N}$. In view of (4.3.6), $\text{Tr}_{-,V^{(2)}_{k,h}} = \text{Tr}_{-,V}(q'_{k,h}) \neq 0$. By (4.1.3),

$$\|\widetilde{K}_{j,V^{(2)}_{k,h}} \widetilde{w}_{k,j}\|^2_{L^2} \geq c \, \frac{1 + H\text{Tr}_{-,V^{(2)}_{k,h}}}{(\log(2 + H\text{Tr}_{-,V^{(2)}_{k,h}}))^2} \|\widetilde{w}_{k,j}\|^2_{L^2} \ . \tag{4.3.14}$$

In the same way

$$\|\widetilde{K}_{j,V^{(2)}_{k,h}} \widetilde{w}_{k,j}\|^2_{L^2} \geq c \, \frac{1 + H\text{Tr}_{-,V}(q'_{k,h})}{(\log(2 + H\text{Tr}_{-,V}(q'_{k,h})))^2} \|\widetilde{w}_{k,j}\|^2_{L^2} \ . \tag{4.3.15}$$

Using once more (4.3.6),

$$\text{Tr}_{-,V}(q'_{k,h}) \geq \frac{\epsilon_0}{2}(1 + \text{Tr}_{+,V}(q'_{k,h})) \ , \tag{4.3.16}$$

where we remind that $\epsilon_0 = \min\limits_{q \in K_0} \frac{\text{Tr}_{-,V}(q)}{1 + \text{Tr}_{+,V}(q)}$. Consequently

$$|\text{Hess}\, V(q'_{k,h})| \geq \text{Tr}_{-,V}(q'_{k,h}) \geq \frac{\epsilon_0}{2} \ , \tag{4.3.17}$$

and

$$\begin{aligned} \text{Tr}_{-,V}(q'_{k,h}) &\geq \frac{1}{2}\text{Tr}_{-,V}(q'_{k,h}) + \frac{\epsilon_0}{4}(1 + \text{Tr}_{+,V}(q'_{k,h})) \\ &\geq \frac{1}{2}\min(1, \frac{\epsilon_0}{2})(\text{Tr}_{-,V}(q'_{k,h}) + \text{Tr}_{+,V}(q'_{k,h})) \\ &\geq \frac{1}{2}\min(1, \frac{\epsilon_0}{2})|\text{Hess}\, V(q'_{k,h})| \ . \end{aligned} \tag{4.3.18}$$

Furthermore by continuity of the map $q \mapsto \text{Tr}_{-,V}(q)$ on the compact set $\overline{\mathcal{C}}$, there exists a constant $\epsilon_3 > 0$ such that $\text{Tr}_{-,V}(q) \leq \epsilon_3$ for all $q \in \overline{\mathcal{C}}$. Hence

$$\frac{\epsilon_0}{2} \leq \text{Tr}_{-,V}(q'_{k,h}) \leq \epsilon_3 \ . \tag{4.3.19}$$

From (4.3.15), (4.3.18) and (4.3.19), we deduce for an other suitable choice of $c > 0$,

$$\|\widetilde{K}_{j,V^{(2)}_{k,h}} \widetilde{w}_{k,j}\|^2_{L^2} \geq c \frac{H}{(\log(H))^2} \|\widetilde{w}_{k,j}\|^2_{L^2} \ .$$

It follows from the above inequality and (4.3.13), that

$$\|K_{j,V^{(2)}_{k,h}} w_{k,j}\|^2_{L^2} \geq c \frac{H}{(\log(H))^2} \|w_{k,j}\|^2_{L^2} \ . \tag{4.3.20}$$

Now using the estimate (4.3.20), we should control the errors coming from the partition of unity and the quadratic approximation. For this reason, notice that our choice of exponent ν in (4.3.3) implies

$$\begin{cases} \frac{(|\ln(h)|^2 h^{2\nu})^4}{h^2} \ll \frac{H}{(\log(H))^2} \\ (|\ln(h)|^{-2} h^{\frac{1}{r-1}-2\nu})^2 \ll \frac{H}{(\log(H))^2} \ . \end{cases}$$

As a result, collecting the estimates (4.3.12) and (4.3.20), we deduce the existence of a constant $c > 0$ such that

$$\|K_{j,V} v_j\|^2_{L^2} \geq c \sum_{k \in K_h} \|K_{j,V^{(2)}_{k,h}} w_{k,j}\|^2_{L^2} \ . \tag{4.3.21}$$

Via (4.1.2), there is a constant $c > 0$ so that

$$\begin{aligned} \|\widetilde{K}_{j,V^{(2)}_{k,h}} \widetilde{w}_{k,j}\|^2 + (1+10c) H |\mathrm{Hess}\, V(q'_{k,h})| \|\widetilde{w}_{k,j}\|^2 \geq c \Big(\|O_p \widetilde{w}_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}} \partial_q V^{(2)}_{k,h}(h^{\frac{1}{2(r-1)}} q)\rangle^{\frac{2}{3}} \widetilde{w}_{k,j}\|^2 \\ + \|\langle D_q \rangle^{\frac{2}{3}} \widetilde{w}_{k,j}\|^2 + H |\mathrm{Hess}\, V(q'_{k,h})| \|\widetilde{w}_{k,j}\|^2 \Big) \ . \end{aligned} \tag{4.3.22}$$

Hence using the inverse change of variables $q" = \frac{q}{h^{\frac{1}{2(r-1)}}}$, we obtain in view of the above estimate and (4.3.13),

$$\begin{aligned} \|K_{j,V^{(2)}_{k,h}} w_{k,j}\|^2 + (1+10c) H |\mathrm{Hess}\, V(q'_{k,h})| \|w_{k,j}\|^2 \geq c \Big(\|O_p w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}} \partial_q V^{(2)}_{k,h}(q)\rangle^{\frac{2}{3}} w_{k,j}\|^2 \\ + \|\langle h^{\frac{1}{2(r-1)}} D_q \rangle^{\frac{2}{3}} w_{k,j}\|^2 + H |\mathrm{Hess}\, V(q'_{k,h})| \|w_{k,j}\|^2 \Big) \ . \end{aligned} \tag{4.3.23}$$

Or by (4.3.18) and (4.3.19),

$$\frac{\epsilon_0}{2} \leq |\mathrm{Hess}\, V(q'_{k,h})| \leq \frac{2\epsilon_3}{\min(1, \frac{\epsilon_0}{2})} \ , \tag{4.3.24}$$

Putting (4.3.23) and (4.3.24) together, there is a constant $c > 0$ so that

$$\|K_{j,V^{(2)}_{k,h}} w_{k,j}\|^2 + H\|w_{k,j}\|^2 \geq c\Big(\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(r-1)}} D_q\rangle^{\frac{2}{3}} w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}}\partial_q V^{(2)}_{k,h}(q)\rangle^{\frac{2}{3}} w_{k,j}\|^2 + H\|w_{k,j}\|^2 + \|\langle H|\text{Hess } V(q'_{k,h})|\rangle^{\frac{1}{2}} w_{k,j}\|^2 \Big) . \tag{4.3.25}$$

On the other hand, for all $q \in \text{supp } w_{k,j}$,

$$|\text{Hess } V(q) - \text{Hess } V(q'_{k,h})| = \mathcal{O}(|q - q'_{k,h}|) = \mathcal{O}(|\ln(h)|h^{\nu}) \tag{4.3.26}$$

Therefore by (4.3.24) and (4.3.26), we obtain for every $q \in \text{supp } w_{k,j}$ and all j sufficiently large.

$$\frac{1}{2}|\text{Hess } V(q'_{k,h})| \leq |\text{Hess } V(q)| \leq \frac{3}{2}|\text{Hess } V(q'_{k,h})| . \tag{4.3.27}$$

From (4.3.25) and (4.3.37), there exists a constant $c > 0$ so that

$$\|K_{j,V^{(2)}_{k,h}} w_{k,j}\|^2 + H\|w_{k,j}\|^2 \geq c\Big(\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(r-1)}} D_q\rangle^{\frac{2}{3}} w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}}\partial_q V^{(2)}_{k,h}(q)\rangle^{\frac{2}{3}} w_{k,j}\|^2 + H\|w_{k,j}\|^2 + \|\langle H|\text{Hess } V(q)|\rangle^{\frac{1}{2}} w_{k,j}\|^2 \Big) , \tag{4.3.28}$$

is valid for all j large enough.

Furthermore, taking into account the condition (4.3.3) on ν, one has for all $q \in \text{supp } w_{k,j}$ such that $|\partial_q V(q)| \leq |\ln(h)|^2 h^{2\nu}$, and every j sufficiently large

$$\frac{1}{4}H \geq c\,|h^{-\frac{1}{2}}\partial_q V(q)|^{\frac{4}{3}} . \tag{4.3.29}$$

On the other hand, by (4.3.10), all $q \in \text{supp } w_{k,j}$ such that $|\partial_q V(q)| \geq |\ln(h)|^2 h^{2\nu}$ satisfy

$$\frac{1}{2}|\partial_q V(q)| \leq |\partial_q V^{(2)}_{k,h}(q)| \leq \frac{3}{2}|\partial_q V(q)| . \tag{4.3.30}$$

In such a way, considering (4.3.28), (4.3.29) and (4.3.30)

$$\|K_{j,V^{(2)}_{k,h}} w_{k,j}\|^2 + (2+H)\|w_{k,j}\|^2 \geq c\Big(\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(r-1)}} D_q\rangle^{\frac{2}{3}} w_{k,j}\|^2 + (2+H)\|w_{k,j}\|^2 + \|(H|\text{Hess } V(q)|)^{\frac{1}{2}} w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}}|\partial_q V(q)|\rangle^{\frac{2}{3}} w_{k,j}\|^2 \Big) . \tag{4.3.31}$$

Putting (4.3.20) and (4.3.31) together,

$$\|K_{j,V^{(2)}_{k,h}} w_{k,j}\|^2 \geq c\Big(\|\frac{O_p}{\log(2+H)} w_{k,j}\|^2 + \|\frac{\langle h^{\frac{1}{2(r-1)}} D_q\rangle^{\frac{2}{3}}}{\log(2+H)} w_{k,j}\|^2 + \|\frac{(2+H)^{\frac{1}{2}}}{\log(2+H)} w_{k,j}\|^2 + \|\frac{\langle H|\text{Hess } V(q)|\rangle^{\frac{1}{2}}}{\log(2+H)} w_{k,j}\|^2 + \|\frac{\langle h^{-\frac{1}{2}}|\partial_q V(q)|\rangle^{\frac{2}{3}}}{\log(2+H)} w_{k,j}\|^2 \Big) , \tag{4.3.32}$$

holds for all $j \geq j_0$, for some $j_0 \geq 1$ large enough.

Now let us collect the finite remaining terms for $-1 \leq j \leq j_0$. After recalling $h = 2^{-j}$ and $H = h^{-\frac{1}{2}+\frac{1}{2(r-1)}}$ we define

$$c_V^{(1)} = \max_{-1\leq j\leq j_0}\left[A_{V^{(2)}_{k,h}} + \sup_{q\in\mathrm{supp}\,(\chi_j\theta_{k,h})}\Big(\langle H|\mathrm{Hess}\ V(q)|\rangle + \langle h^{-\frac{1}{2}}\,|\partial_q V(q)|\rangle^{4/3}\Big)\right.$$
$$\left. + \frac{(2+H)}{(\log(2+H))^2} + 8c^2\,\frac{(|\ln(h)|^2 h^{2\nu})^4}{h^2} + 2(|\ln(h)|^{-2}h^{\frac{1}{r-1}-2\nu})^2\right] .$$

From the lower bound (4.1.2), we deduce the existence of a constant $c > 0$ so that

$$\frac{1}{8}\|K_{V^{(2)}_{k,h}}w_{k,j}\| + (c_V^{(1)} - 8c^2\,\frac{(|\ln(h)|^2h^{2\nu})^4}{h^2} - 2(|\ln(h)|^{-2}h^{\frac{1}{r-1}-2\nu})^2\|w_{k,j}\|^2$$
$$\geq c\Big(\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(r-1)}}D_q\rangle^{2/3}w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}}|\partial_q V(q)|\rangle^{2/3}w_{k,j}\|^2$$
$$+ \|\langle H|\mathrm{Hess}\ V(q)|\rangle^{1/2}w_{k,j}\|^2 + \|\frac{(2+H)^{\frac{1}{2}}}{\log(2+H)}w_{k,j}\|^2\Big) , \tag{4.3.33}$$

holds for all $-1 \leq j \leq j_0$.

Finally, collecting (4.3.21), (4.3.32) and (4.3.33),

$$\|K_{j,V}v_j\|^2 + c_V^{(2)}\|v_j\|^2 \geq c\sum_{k\notin I(\epsilon_1)}\Big(\|\frac{O_p}{\log(2+H)}w_{k,j}\|^2 + \|\frac{\langle h^{\frac{1}{2(r-1)}}D_q\rangle^{\frac{2}{3}}}{\log(2+H)}w_{k,j}\|^2 + \|\frac{(2+H)^{\frac{1}{2}}}{\log(2+H)}w_{k,j}\|^2$$
$$+ \|\frac{\langle H|\mathrm{Hess}\ V(q)|\rangle^{\frac{1}{2}}}{\log(2+H)}w_{k,j}\|^2 + \|\frac{\langle h^{-\frac{1}{2}}|\partial_q V(q)|\rangle^{\frac{2}{3}}}{\log(2+H)}w_{k,j}\|^2\Big) , \tag{4.3.34}$$

is valid for every $j \geq -1$.

Case 2. We consider in this case the quadratic approximating polynomial near $q_{k,h}$. Using once more [BNV] (see (4.1.2)), there is a constant $c > 0$ such that

$$\|\widetilde{K}_{j,V^{(2)}_{k,h}}\widetilde{w}_{k,j}\|^2 + H\|\widetilde{w}_{k,j}\|^2 \geq c\Big(\|O_p\widetilde{w}_{k,j}\|^2 + \|\langle D_q\rangle^{\frac{2}{3}}\widetilde{w}_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}}\partial_q V^{(2)}_{k,h}(h^{\frac{1}{2(r-1)}}q)\rangle^{\frac{2}{3}}\widetilde{w}_{k,j}\|^2\Big) . \tag{4.3.35}$$

As a consequence of (4.3.13) and (4.3.35),

$$\|K_{j,V^{(2)}_{k,h}}w_{k,j}\|^2 + H\|w_{k,j}\|^2 \geq c\Big(\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(r-1)}}D_q\rangle^{\frac{2}{3}}w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}}\partial_q V^{(2)}_{k,h}(q)\rangle^{\frac{2}{3}}w_{k,j}\|^2\Big) . \tag{4.3.36}$$

By (4.3.7) and (4.3.10),

$$\frac{1}{2}|\partial_q V(q)| \leq |\partial_q V^{(2)}_{k,h}(q)| \leq \frac{3}{2}|\partial_q V(q)| \,, \tag{4.3.37}$$

holds for all $q \in \operatorname{supp} w_{k,j}$ and any j large. Then, it follows from (4.3.37) and (4.3.36),

$$\|K_{j,V^{(2)}_{k,h}} w_{k,j}\|^2 + H\|w_{k,j}\|^2 \geq c\Big(\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(r-1)}} D_q\rangle^{\frac{2}{3}} w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}}|\partial_q V(q)|\rangle^{\frac{2}{3}} w_{k,j}\|^2\Big) \,. \tag{4.3.38}$$

Or in this case, in view of the (4.3.7), one has $|\partial_q V(q)| \geq \epsilon_2$ for all $q \in \operatorname{supp} w_{k,j}$. Hence it results from the above inequality

$$\|K_{j,V^{(2)}_{k,h}} w_{k,j}\|^2 \geq c\Big(\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(r-1)}} D_q\rangle^{\frac{2}{3}} w_{k,j}\|^2 + \|(h^{-\frac{1}{2}})^{\frac{2}{3}} w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}}|\partial_q V(q)|\rangle^{\frac{2}{3}} w_{k,j}\|^2\Big) \,. \tag{4.3.39}$$

Furthermore, by continuity of $q \mapsto |\text{Hess}\, V(q)|$ on the compact set $\overline{\mathcal{C}}$, one has for all $q \in \operatorname{supp} w_{k,j}$ and any j large

$$\frac{1}{4}(h^{-\frac{1}{2}})^{\frac{4}{3}} \geq c\, H|\text{Hess}\, V(q)| \,. \tag{4.3.40}$$

Then by the above inequality and (4.3.39), we get

$$\begin{aligned}\|K_{j,V^{(2)}_{k,h}} w_{k,j}\|^2 \geq c\Big(&\|O_p w_{k,j}\|^2 + \|\langle h^{\frac{1}{2(r-1)}} D_q\rangle^{\frac{2}{3}} w_{k,j}\|^2 + \|(h^{-\frac{1}{2}})^{\frac{2}{3}} w_{k,j}\|^2 \\ &+ \|\langle H|\text{Hess}\, V(q)|\rangle^{\frac{1}{2}} w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}}|\partial_q V(q)|\rangle^{\frac{2}{3}} w_{k,j}\|^2\Big) \,,\end{aligned} \tag{4.3.41}$$

for every $j \geq j_1$ for some $j_1 \geq 1$ large.

Now, in order to absorb the errors in (4.3.12), notice that our choice of exponent ν in (4.3.3) implies

$$\begin{cases} \frac{(|\ln(h)|^2 h^{2\nu})^4}{h^2} \ll h^{-\frac{2}{3}} \\ (|\ln(h)|^{-2} h^{\frac{1}{r-1}-2\nu})^2 \ll h^{-\frac{2}{3}} \,. \end{cases}$$

Now set

$$\begin{aligned} c_V^{(3)} = \max_{-1 \leq j \leq j_1} \Bigg[&\sup_{q \in \operatorname{supp}(\chi_j \theta_{k,h})} \Big(\langle H|\text{Hess}\, V(q)|\rangle + \langle h^{-\frac{1}{2}}\, |\partial_q V(q)|\rangle^{4/3}\Big) \\ &+ \frac{(2+H)}{(\log(2+H))^2} + 8c^2\, \frac{(|\ln(h)|^2 h^{2\nu})^4}{h^2} + 2(|\ln(h)|^{-2} h^{\frac{1}{r-1}-2\nu})^2 \Bigg] \,. \end{aligned}$$

Seeing (4.1.2), we deduce the existence of a constant $c>0$ so that

$$\begin{aligned}\frac{1}{8}\|K_{V^{(2)}_{k,h}}w_{k,j}\| + (c^{(3)}_V - 8c^2\,\frac{(|\ln(h)|^2h^{2\nu})^4}{h^2} - 2(|\ln(h)|^{-2}h^{\frac{1}{r-1}-2\nu})^2\|w_{k,j}\|^2\\ \geq c(\|O_pw_{k,j}\|^2 + \|\langle h^{\frac{1}{2(r-1)}}D_q\rangle^{2/3}w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}}|\partial_qV(q)|\rangle^{2/3}w_{k,j}\|^2\\ + \|\langle H|\mathrm{Hess}\,V(q)|\rangle^{1/2}w_{k,j}\|^2 + \|\frac{(2+H)^{\frac{1}{2}}}{\log(2+H)}w_{k,j}\|^2)\ ,\end{aligned} \tag{4.3.42}$$

holds for all $-1\leq j\leq j_1$.

Thus, combining the estimates (4.3.12), (4.3.41) and (4.3.42)

$$\begin{aligned}\|K_{j,V}v_j\|^2 + c^{(4)}_V\|v_j\|^2 \geq c\sum_{k\in I(\epsilon_1)}\Big(\|O_pw_{k,j}\|^2 + \|\langle h^{\frac{1}{2(r-1)}}D_q\rangle^{\frac{2}{3}}w_{k,j}\|^2 + \|\frac{(2+H)^{\frac{1}{2}}}{\log(2+H)}w_{k,j}\|^2\\ + \|\langle H|\mathrm{Hess}\,V(q)|\rangle^{\frac{1}{2}}w_{k,j}\|^2 + \|\langle h^{-\frac{1}{2}}|\partial_qV(q)|\rangle^{\frac{2}{3}}w_{k,j}\|^2\Big)\ ,\end{aligned} \tag{4.3.43}$$

holds for all $j\geq -1$.

In conclusion, in view of (4.3.34) and (4.3.43), there is a constant $c>0$ such that

$$\begin{aligned}\|K_{j,V}v_j\|^2 + c^{(5)}_V\|v_j\|^2 \geq c\sum_{k\in K_h}\Big(\|\frac{O_p}{\log(2+H)}w_{k,j}\|^2 + \|\frac{\langle h^{\frac{1}{2(r-1)}}D_q\rangle^{\frac{2}{3}}}{\log(2+H)}w_{k,j}\|^2 + \|\frac{(2+H)^{\frac{1}{2}}}{\log(2+H)}w_{k,j}\|^2\\ + \|\frac{\langle H|\mathrm{Hess}\,V(q)|\rangle^{\frac{1}{2}}}{\log(2+H)}w_{k,j}\|^2 + \|\frac{\langle h^{-\frac{1}{2}}|\partial_qV(q)|\rangle^{\frac{2}{3}}}{\log(2+H)}w_{k,j}\|^2\Big)\ ,\end{aligned} \tag{4.3.44}$$

holds for all $j\geq -1$.

Finally setting $L(s)=\frac{s+1}{\log(s+1)}$ for all $s\geq 1$, notice that there is a constant $c>0$ such that for all $x\geq 1$,

$$\inf_{t\geq 2}\frac{x}{\log(t)} + t \geq \frac{1}{c}L(x)\ . \tag{4.3.45}$$

After setting the quantities

$$\Lambda_{1,j} = \frac{O_p}{\log(2+H)}\ ,\quad \Lambda_{2,j} = \frac{\langle H|\mathrm{Hess}\,V(q)|\rangle^{1/2}}{\log(2+H)}\ ,\quad \Lambda_{3,j} = \frac{\langle h^{-\frac{1}{2}}|\partial_qV(q)|\rangle^{\frac{2}{3}}}{\log(2+H)}\ ,$$
$$\Lambda_{4,j} = \frac{(2+H)^{\frac{1}{2}}}{\log(2+H)}\ ,\quad \Lambda_{5,j} = \frac{\langle h^{\frac{1}{2(r-1)}}D_q\rangle)^{\frac{2}{3}}}{\log(2+H)}\ ,$$

we get through the estimate (4.3.45), for every $j \geq -1$ and $k \in K_h$,

$$\|\Lambda_{1,j} w_{k,j}\|^2_{L^2} + \frac{1}{4}\|\Lambda_{4,j} w_{k,j}\|^2_{L^2} \geq c_1 \|L(O_p) w_{k,j}\|^2_{L^2} \,,$$

$$\|\Lambda_{5,j} w_{k,j}\|^2_{L^2} + \frac{1}{4}\|\Lambda_{4,j} w_{k,j}\|^2_{L^2} \geq c_2 \|L(\langle h^{\frac{1}{2(r-1)}} D_q\rangle^{\frac{2}{3}}) w_{k,j}\|^2_{L^2} \,,$$

$$\|\Lambda_{2,j} w_{k,j}\|^2_{L^2} + \frac{1}{4}\|\Lambda_{4,j} w_{k,j}\|^2_{L^2} \geq c_3 \|L(\langle H|\mathrm{Hess}\, V(q)|\rangle^{\frac{1}{2}}) w_{k,j}\|^2_{L^2} \,,$$

$$\|\Lambda_{3,j} w_{k,j}\|^2 + \frac{1}{4}\|\Lambda_{4,j} w_{k,j}\|^2_{L^2} \geq c_4 \|L(\langle h^{-\frac{1}{2}}|\partial_q V(q)|\rangle^{\frac{2}{3}}) w_{k,j}\|^2_{L^2} \,.$$

From the above estimates and (4.3.44),

$$\|K_{j,V} v_j\|^2 + c_V^{(6)}\|v_j\|^2 \geq \; c \sum_{k\in K_h} \Big(\|L(O_p) w_{k,j}\|^2 + \|L(\langle h^{\frac{1}{2(r-1)}} D_q\rangle^{\frac{2}{3}}) w_{k,j}\|^2 + \|L(\langle H|\mathrm{Hess}\, V(q)|\rangle^{\frac{1}{2}}) w_{k,j}\|^2 + \|L(\langle h^{-\frac{1}{2}}|\partial_q V(q)|\rangle^{\frac{2}{3}}) w_{k,j}\|^2 \Big) \,. \tag{4.3.46}$$

Therefore in view of Lemma 2.5 in [Ben] conjugated by the unitary transformation of the change of scale,

$$\|K_{j,V} v_j\|^2 + c_V^{(7)}\|v_j\|^2 \geq \; c\Big(\|L(O_p) v_j\|^2 + \|L(\langle h^{\frac{1}{2(r-1)}} D_q\rangle^{\frac{2}{3}}) v_j\|^2 + \|L(\langle H|\mathrm{Hess}\, V(q)|\rangle^{\frac{1}{2}}) v_j\|^2 + \|L(\langle h^{-\frac{1}{2}}|\partial_q V(q)|\rangle^{\frac{2}{3}}) v_j\|^2 \Big) \,, \tag{4.3.47}$$

or equivalently

$$\|K_V u_j\|^2 + c_V^{(7)}\|u_j\|^2 \geq c\Big(\|L(O_p) u_j\|^2 + \|L(\langle D_q\rangle^{\frac{2}{3}}) u_j\|^2 + \|L(\langle \mathrm{Hess}\, V(q)\rangle^{\frac{1}{2}}) u_j\|^2 + \|L(\langle \partial_q V(q)\rangle^{\frac{2}{3}}) u_j\|^2 \Big) \,, \tag{4.3.48}$$

for every $j \geq -1$.

Therefore, combining the last estimate and (4.3.1), there is a constant $C_V > 1$ so that

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + C_V\|u\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{1}{C_V}\Big(\|L(O_p) u\|^2 + \|L(\langle D_q\rangle^{\frac{2}{3}}) u\|^2 + \|L(\langle \mathrm{Hess}\, V(q)\rangle^{\frac{1}{2}}) u\|^2 + \|L(\langle \partial_q V(q)\rangle^{\frac{2}{3}}) u\|^2 \Big) \tag{4.3.49}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$. □

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