École Doctorale Galilée

Laboratoire Analyse Géométrie et Applications

Thèse de doctorat

Discipline : Mathématiques

présentée par

Mattia CAVICCHI

Weights of the boundary motive of some Shimura varieties

(Poids du motif bord de certaines variétés de Shimura)

---

dirigée par Jörg Wildeshaus

Soutenue le 5 juillet 2019 devant le jury composé de :

M. Giuseppe Ancona	Université de Strasbourg	examinateur
M. Charles De Clercq	Université Paris XIII	examinateur
M. Marc Levine	Universität Duisburg-Essen	rapporteur
M. Luca Migliorini	Università di Bologna	examinateur
M. Rutger Noot	Université de Strasbourg	rapporteur
M. Jacques Tilouine	Université Paris XIII	examinateur
M. Alberto Vezzani	Université Paris XIII	examinateur
M. Jörg Wildeshaus	Université Paris XIII	directeur

Université Paris 13
École doctorale Galilée
LAGA - Laboratoire Analyse Géométrie et Applications.
99 avenue Jean-Baptiste Clément
93430 Villetaneuse, France.

Mattia Cavicchi
LAGA, Bureau D304
cavicchi@math-univ.paris13.fr

“Dalla muta distesa delle cose deve partire un segno, un richiamo, un ammicco: una cosa si stacca dalle altre con l’intenzione di significare qualcosa... che cosa? se stessa, una cosa è contenta d’essere guardata dalle altre cose solo quando è convinta di significare se stessa e nient’altro, in mezzo alle cose che significano se stesse e nient’altro.”

Italo Calvino,
Palomar, Einaudi, Torino, 1983, p. 117

“We have to look for power sources here, and distribution networks we were never taught, routes of power our teachers never imagined, or were encouraged to avoid... we have to find meters whose scales are unknown in the world, draw our own schematics, getting feedback, making connections, reducing the error, trying to learn the real function... zeroing in on what incalculable plot? Up here, on the surface, coal-tars, hydrogenation, synthesis were always phony, dummy functions to hide the real, the planetary mission yes perhaps centuries in the unrolling... this ruinous plant, waiting for its Kabbalists and new alchemists to discover the Key, teach the mysteries to others...”

Thomas Pynchon,
Gravity’s Rainbow, Viking Press, New York, 1973, p. 521

“Le nouveau ne se produit jamais par simple interpolation de l’ancien ; les informations s’ajoutaient aux informations comme poignées de sable, prédéfinies dans leur nature par le cadre conceptuel délimitant le champs des expériences...”

Michel Houellebecq,
Les particules élémentaires, Flammarion, Paris, 1998, p. 279

“Boy, you’re gonna carry that weight
Carry that weight a long time
Boy, you’re gonna carry that weight
Carry that weight a long time”

The Beatles, *Carry That Weight*
from *Abbey Road*, Apple Records, 1969

Poids du motif bord de certaines variétés de Shimura

Résumé. Pour S une variété de Shimura associée à un groupe réductif G, nous étudions la filtration par le poids dans la cohomologie des variations de structure de Hodge $\mu_H(V)$ et des faisceaux ℓ-adiques $\mu_\ell(V)$ sur S construits à partir des représentations algébriques de G, avec le but de définir des motifs pour les représentations automorphes de G.

Dans les deux premiers chapitres nous rappelons les théories utilisées et nous y ajoutons des compléments. Dans le premier, nous faisons un survol des rélations entre cohomologie des variétés de Shimura, représentations automorphes et théorie des poids, tandis que dans le deuxième nous introduisons les motifs de Chow et de Beilinson rélatifs sur les variétés de Shimura de type PEL, ainsi que les applications de la théorie des structures des poids dans ce contexte. En particulier, nous étudions en detail l'action de l'algèbre de Hecke au niveau des motifs.

Dans les deux derniers chapitres, nous nous concentrons sur le cas du groupe $G = \mathrm{Res}_{F|\mathbb{Q}}\mathrm{GSp}_{4,F}$, où F est un corps de nombres totalement réel, et sur les variétés de Shimura S qui lui sont associées, les variétés de Shimura de Hilbert-Siegel de genre 2. Dans le troisième chapitre, nous menons une étude approfondie de la dégénérescence des faisceaux $\mu_\ell(V)$ au bord de la compactification de Baily-Borel de S. Nous parvenons à décrire les poids en terme d'un invariant de la représentation V, appelé corang. Nous en déduisons une caractérisation complète des représentations V telles que la dégénérescence de $\mu_\ell(V)$ évite les poids 0 et 1, une classe qui s'avère être très large.

Dans le quatrième chapitre, étant donnée une représentation V de G qui vérifie la condition susmentionnée, nous définissons des motifs attachés aux représentations automorphes apparaissant dans la cohomologie du faisceau $\mu_\ell(V)$. Nous étudions donc les propriétés de ces motifs.

Weights of the boundary motive of some Shimura varieties

Abstract. Given a Shimura variety S associated to a reductive group G, we study the weight filtration in the cohomology of variations of Hodge structure $\mu_H(V)$ and ℓ-adic sheaves $\mu_\ell(V)$ on S coming from algebraic representations V of G, with the aim of constructing motives for automorphic representations of G.

In the first two chapters we review the theories that we use and we give some complements to them. In the first one we summarize the relationship between cohomology of Shimura varieties, automorphic representations and weights, whereas in the second one we recall relative Chow and Beilinson motives over PEL Shimura varieties and the applications of the theory of weight structures to this setting. In particular, we study in detail the action of the Hecke algebra at the level of motives.

In the last two chapters we concentrate on the case of the group $G = \mathrm{Res}_{F|\mathbb{Q}}\mathrm{GSp}_{4,F}$, for F a totally real number field, and to the associated Shimura varieties S (genus 2 Hilbert-Siegel varieties). In the third chapter, we study in detail the weight filtration on the degeneration of the sheaves $\mu_\ell(V)$ along the boundary of the Baily-Borel compactification of S. We are able to describe the weights in terms of an invariant of the representation V, called corank. From this, we deduce a complete characterization of the representations V such that the degeneration of $\mu_\ell(V)$ avoids the weights 0 and 1, and we find that they form a quite large class.

In the fourth chapter, given such a representation V, we define motives for those automorphic representations of G which appear in the cohomology of $\mu_\ell(V)$. We then study the properties of such motives.

REMERCIEMENTS

Je ne peux que commencer par remercier mon directeur de thèse, Jörg Wildeshaus. Je l'ai contacté au début de mon Master 2 à Paris VII, quand je venais d'arriver en France. Il a cru en moi, un étudiant naïf et un peu perdu, qui commençait tout juste à apprendre la théorie des schémas - et à se débrouiller dans la vie parisienne. Il a accepté de partager avec moi sa vision extrêmement profonde et originelle des mathématiques, ainsi que, le temps venu, de me proposer comme sujet de thèse l'étude d'un problème naturellement issu de sa recherche. Merci, Jörg, pour m'avoir donné l'opportunité de travailler sur ces mathématiques fascinantes, pour toutes les discussions qu'on a eues, pour les mondes que tu m'as ouverts. Et sur le plan strictement personnel, merci pour le support que tu as su me donner avec le bon mot prononcé au bon moment, et pour l'approche lucide, cohérente et aussi ironique que tu m'as transmise tout au long de ces quatre années de doctorat ; merci pour m'avoir appris à donner le juste *poids* aux choses. La (mauvaise) blague est voulue, mais il s'agit de bien plus que d'une blague.

Je suis très honoré d'avoir comme rapporteurs de cette thèse deux mathématiciens du calibre de Marc Levine et Rutger Noot. Je les remercie beaucoup d'avoir accepté de lire et de commenter ce manuscrit.

Pendant ces années au LAGA, l'ambiance offerte par l'équipe AGAA a été très accueillante et fructueuse pour mon développement en tant que mathématicien. Je tiens à remercier Jacques Tilouine pour sa gentillesse en répondant à mes questions, pour l'organisation du groupe de travail sur le papier de Galatius-Venkatesh, et pour son soutien au cours de cette dernière année : sa présence dans le jury me fait un grand plaisir. Je suis également ravi de la participation de Alberto Vezzani, avec qui j'ai pu discuter plusieurs fois, notamment pendant cette dernière année, et que je remercie aussi pour son énergie et sympathie en tant qu'organisateur des séminaires hebdomadaires de l'équipe. Ceux-ci ont représenté pour moi des moments très importants pour élargir ma vision de la géométrie arithmétique, ainsi que pour avoir des échanges intéressantes. Je profite de l'occasion pour remercier l'autre organisateur Cédric Pepin, aussi pour les conversations qu'on a eues, et tou(te)s les autres membres de l'équipe, en particulier Pascal Boyer, Farid Mokrane, Jaclyn Lang.

Entre toutes les raisons pour lesquelles je suis reconnaissant à Giuseppe Ancona, je pourrais citer le fait qu'il est pour moi un point de référence pour ses travaux sur le motifs des schéma abéliens, sur lesquels une partie importante de cette thèse se fonde ; je pourrais souligner qu'il a été organisateur d'innombrables conférences auxquelles j'ai participé, et qu'il m'a invité à Strasbourg pour présenter les résultats de cette thèse ; je pourrais, enfin, rappeler tous les conseils précieux, de nature mathématique ou pas, qu'il a su me donner à chaque rencontre, pendant mon doctorat. Je dirais seulement qu'il ne s'agit pas que d'un mathématicien brillant, mais aussi d'une personne d'une générosité exceptionnelle. Je me réjouis de sa présence dans le jury.

Parmi les conférences organisées par Giuseppe sur une île de la Normandie désormais légendaire, il y a en particulier l'une des meilleurs écoles d'été à la quelle j'aie eu le privilège d'assister, dont le programme avait été conçu par Luca Migliorini. Ceci m'a donné l'opportunité de faire sa connaissance et de me familiariser avec ses magnifiques travaux sur la conjecture P=W. Je le remercie énormément d'avoir accepté d'être partie du jury. Vue l'importance, dans cette thèse, de la t-structure perverse, j'espère qu'il puisse aujourd'hui me pardonner pour ne pas avoir eu le temps d'en parler, lors d'un exposé où j'ai introduit le formalisme de six foncteurs face à des experts qui le connaissaient bien mieux que moi.

Je suis aussi très heureux de voir Charles De Clercq dans la commission.

Je tiens à remercier d'autres mathématiciens que j'ai rencontré et qui ont contribué à enrichir mes connaissances et mon expérience de thésard : je pense à J. Fasel et A. Asok (pour l'école de K-théorie à L.A., où j'ai pu parler de cohomologie motivique quand je venais à peine de la découvrir), M. Maculan et J. Fresàn (co-organisateurs, entre autres, des conférences sur l'île susmentionnée), F. Déglise (merci pour l'école d'A1-homotopie à Freiburg), F. Lemma (dont les travaux sur les variétés de Siegel m'ont illuminé), B. Klingler (merci pour ses cours exceptionnels et sa disponibilité à discuter), N. Bergeron (qui a repondu à mes question très vagues), M. Dimitrov (qui m'a permis de présenter mes résultats à Lille), D. Petersen (organisateur d'une splendide conférence à Stockholm, où j'ai eu la chance d'assister à des cours fantastiques de R. Hain et F. Brown).

En parlant de voyages et de conférences, je veux exprimer ma gratitude au laboratoire LAGA et à l' École Doctorale Galilée pour m'avoir toujours fourni les moyens de me déplacer, ainsi qu'au travail administratif impeccable de Yolande, d'Isabelle qui vient de partir et de Frédéric qui vient de la remplacer.

J'ai eu la chance d'être accompagné dans ce long parcours par plusieurs collègues doctorant(e)s (parmi eux, quelqu'un(e) est déjà lancé(e) vers des nouvelles aventures). Ma première pensée est pour Guillaume, mon grand-frère de thèse. Nos conversations régulières pendant mes deux premières années de thèse ont été absolument cruciales pour m'orienter dans mon travail (et dans mon doctorat en général).

Je veux aussi remercier Emiliano, Diego, Victoria, Leonardo, Salim, Vincent, Kieu Hieu, Xiaoyu pour les groupes de travail, Xiaozong pour l'invitation au séminaire Math-Jeunes, Arthur et Matthieu pour l'invitation au séminaire RéGA.

L'ambiance qu'on a su créer parmi les doctorant(e)s du LAGA a été extrêmement agréable. Au délà de Kieu Hieu et Xiaoyu que j'ai déjà cités, merci donc à Bruno, Irene, Didier, Giuseppe, Pierre, Tom, Carlos, Tarek, Nicolas, Daniel, Thuy, Marta, Ayoub, Oussama, Ana, Safaa, Nelly, Mona, Ghina, Hugo, Bart, Jorge... Une mention spéciale est due à Annalaura, organisatrice du séminaire des doctorants pendant ma première année et co-organisatrice avec moi pendant la deuxième, personne douée d'une positivité et d'une énergie remarquables, qui investissent tous ceux qui ont l'occasion de la connaître. Et je remercie aussi particulièrement ceux qui ont pris en charge le séminaire à leur tour, Delphin et Jean Michel. Ce dernier a été une des premières personnes que j'ai connues sur Paris, pendant mon Master 2, et dont l'étonnant pluri-linguisme franco-italo-espagnol m'a permis à l'époque de m'expérimenter sans peur avec le français. Ça a été un plaisir de le retrouver avec moi à Paris 13, et de discuter avec lui autour des sujets les plus divers, de la théorie de l'homotopie aux gilets jaunes.

En effet, ma vie à Paris pendant ces cinq années a été rendue riche et mouvementée par un grand nombre d'autres personnes. Ce serait impossible de les citer toutes, mais au même temps je ne vais pas renoncer à en mentionner ici explicitement quelques unes. Je prends donc ce risque, en choisissant de commencer par ceux que j'ai rencontré grâce aux mathématiques, pour passer ensuite aux autres ; si quelqu'un(e) ne se retrouve pas dans ces lignes, il/elle ne devrait pourtant pas penser que je l'ai oublié(e) ou que je ne lui accorde pas de considération...

Merci Juan Pablo, pour tout ce qu'on a partagé, qu'il s'agisse de réflexions sur la philosophie des sciences, ou des concerts, à la Gare, à la Philharmonie, ou au Primavera Sound à Barcelona, où d'ailleurs tu m'as fait rencontrer un autre personnage exceptionnel : merci Ricardo, pour toute la musique qu'on a fait ensemble. Merci d'avoir déplacé ton piano jusqu'au O'Mok@ Bar...

Merci Alexandre, de ta gentillesse, de ta présence au cours de ces années, tant pour des discussions dans le salles de la BU de Jussieu, il y a tellement longtemps, quant sur une terrasse à Santa Caterina dello Ionio Marina, en dormant sous les étoiles...

Merci Carlo, pour le pique-nique à la Cité Universitaire où tout à commencé, pour les débats interminables sur la politique et la réligion, pour la promenade autour du pendule de Foucault, en attendant de jouer "*Gli Uccelli*" avec toi et de te revoir dans un club de Berlin... Et une pensée à Andrea et Filippo que j'ai connu grâce à toi...

Merci Leandro, pour les soirées à Saint Ouen qui ont été le modèle pour une quantité infinie d'autres soirées, pour nos déjeuners où on a mélangé Lacan et l'attente pour la fin du capitalisme qui est en train d'anéantir la planète, pour le soutien que tu m'as offert en beaucoup de moments... Et merci de m'avoir amené à chanter en espagnol (ou portuguais !), en jouant avec toi, avec les frères Daniel et Andrés, avec le légendaire Zaca Zuli...

Merci Romain, voisin dans une ville et dans un quartier où il paraît impossible de construire des relations avec ses voisins. Merci de ton altruisme, merci de m'avoir introduit dans l'équipe de volley-ball où j'ai passé des moments parmi les meilleurs des deux dernières années (et merci à tous ceux/celles qui nous ont entraînés, ainsi qu'à ceux/celles qui se sont entraîné(e)s et qui ont joué avec nous...)

Merci *a palate* à Olivier, à tes photos vintage, aux bons moments sur le canal, dans les parcs, dans des maisons d'inconnus... Aussi avec Charlotte, ou avec le magnifique couple formée par Tommaso et Angela : merci pour les soirs sur la terrasse de Les Lilas, ou en dansant comme des *zampi*...

Merci à Luca, mathématicien extraordinaire et personne enthousiaste, à la créativité polyédrique, que j'ai croisé à plusieurs reprises, pas du tout à cause des maths, mais plutôt pour des motivations qui ont à voire avec la mer, le vin (le Salon des Vignerons !), la bonne bouffe... Merci de m'avoir fait connaître Gloria et Sonia, avec qui on a partagé tellement de beaux moments, dans beaucoup de villes différentes, tout en réfléchissant sur Paris...

Paris, une ville où beaucoup de gens passent et partent, en laissant quand même une trace : je pense par exemple à Claudia, Alberto, Sangwon...

Et je pense aussi à tous ceux et celles que j'ai recontré à travers Monica (à la quelle je demande d'attendre encore un petit peu, avant de se voir apparaître à plein titre) : merci à Chiara, Cecilia, Marco, Valentina (et Federico), Francesco, Brigitta, Lucia, Giulia, Valeria, Laura, pour votre affection, pour les nuits passées sur une terrasse incroyable face à la tour Eiffel, pour le concert de Alt-J, pour les apéros à l'IIC, pour le dîner avec un jambon inoubliable à Langhirano, pour la soirée dans un palace de nobles en assistant à une bizarre opéra de Stockhausen, pour un anniversaire avec beaucoup trop des paillettes (chacun(e) saura se reconnaître dans au moins un de ces moments...) Merci à l'extraordinaire Isabel (d'ailleurs, responsable des paillettes). Merci à Enrica et Marc pour leur bienveillance vers moi.

En terminant ces longues pages de remerciements, c'est naturel de penser au chemin que j'ai parcouru avant d'arriver à Paris. Je ne vais pas citer ici tous mes ami(e)s et tous mes proches en Italie : même s'il s'agit de personnes fondamentales pour moi, cela ferait vraiment trop de pages...

En revanche, je veux au moins remercier les enseignant(e)s qui m'ont formé tout au long de ma vie. J'ai eu la chance de rencontrer des personnes de valeur inestimable, auxquelles je dois beaucoup de ce que je suis aujourd'hui. En particulier, merci à Patrizia, Maria Teresa, Brunella, Lucia, Elisabetta, Angela, Paola, Franca, et en particulier aux professeures de mathématiques du Liceo Copernico à Prato, Rossella et Giovanna.

Je ne l'ai pas rencontré en tant qu'enseignant, mais je veux aussi remercier Gianni, qui m’a quand même formé à son tour ; il n’est plus parmi nous, mais je ne l’oublierai jamais. Et merci à mes enseignants de musique, Patrizio et Luigi.

Merci aux professeur(e)s de l’Università di Firenze, qui m’ont donné des bases très solides : je continue à penser à leurs cours comme parmi les meilleurs que j’ai pu suivre pendant mes études.

Un remerciement spéciale pour Giorgio Ottaviani. Il m’a permis d’explorer la géométrie algébrique dans une fac où cette discipline n’était presque pas enseignée. C’est lui qui m’a suggéré de postuler pour la bourse de la Fondation de Sciences Mathématiques, en me soutenant fortement pendant la préparation de la candidature : sans lui, mon aventure parisienne n’aurait simplement pas été possible. Il reste aujourd’hui un point de référence pour moi ; sa récente invitation à Florence, pour exposer les sujets qui ont fait l’objet de mon travail à Paris, a contribué à renforcer le lien entre le passé et le présent. Ceci m’a permis aussi de rencontrer à nouveau G. Vezzosi, qui m’a soutenu à son tour à l’époque de ma candidature, et que je tiens à remercier également.

Ma famille m’a supporté au cours de toute ma vie. Merci aux tantes Deanna et Daniela, merci à Enrico, merci à ma tante Manuela et à Roberto, à mes cousins Andrea et Roberta, sans oublier ni Emanuele, ni Nicoletta, Sandro et Nicole : ma gratitude pour vous est sans limites. Une pensée va à mes grand-parents, qui ne sont plus là ; en particulier à Nicla, qui nous a quitté tout récemment, tellement présente pendant mon enfance et mon adolescence.

J’arrive ainsi aux deux remerciements finales, qui ont une importance toute particulière.

D’une part, celui pour mes parents, Alessandro et Morena. L’amour inconditionnel que vous avez toujours montré pour moi est l’une des forces les plus radicales que j’aie jamais connues. Sans vous, sans la confiance et le respect que vous m’avez toujours accordé, sans l’énormité de choses que vous m’avez transmises, il serait impossible de me concevoir comme la personne que je suis à l’heure présente. Je ne pourrais jamais vous restituer, même avec plusieurs vies à disposition, tout ce que vous avez su me donner et que vous continuez à me donner aujourd’hui, et je suis désolé du fait que mes mots limités ne puissent pas arriver à exprimer tout ce que je sens pour vous.

D’autre part, celui pour ma compagne, Monica. Les souvenirs de ces années parisiennes (et à Front Populaire !) seront marqué pour toujours par ton amour, par l’enthousiasme, la joie de vivre, les surprises dont tu m’as rempli, par la profondeur de notre connexion, par ton intelligence, tendresse, subtilité, par la quantité d’expériences et de découvertes qu’on a vécu. Que ce soit pour Palomar ; pour ta présence à mon coté, quand je me suis réveillé et que j’ai trouvé la solution d’un énigme qui m’obsédais ; pour ta disponibilité à écouter mes doutes et à m’aider dans les moments d’urgence ; pour ta curiosité vers mon travail ; pour la conscience de moi-même et de ce que je fais, que tu m’as aidé à approfondir ; (et je pourrais continuer...) pour toutes ces raisons, dans cette thèse il y aussi une partie de toi, et si cette thèse existe, c’est aussi grâce à toi.

Contents

Chapter 1

Introduction

This thesis is concerned with the analysis of *weights* in the *cohomology of Shimura varieties*. The main motivation for carrying out this analysis springs from the desire of constructing *motives for automorphic representations*, which can be seen as *algebro-geometric* objects providing a deep connection between *arithmetic*, on one side, and entities of *analytic nature*, on the other side. Starting from the end of this chain of motivations and going backwards, one may ask: why should such a connection be interesting? Why does (algebraic) geometry have a role in proving that it exists? And why should the theory of weights come into play?

The purpose of this introduction is to briefly discuss these questions, to make explicit the specific problems addressed in this thesis, and to synthesize its contents and its main results.

1.1 From arithmetic to automorphic forms

Let $\mathbb{Q}$ be the field of rational numbers and fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. It is a very basic and important problem, yet still completely open, to find a description of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q})$, which we will denote by $G_{\mathbb{Q}}$. The word *description* is admittely vague, and actually, one would like to know more about this group than its abstract structure. So, let us agree that a good way to understand a group is through its representations, and that it makes sense to begin from *complex-valued* representations of *dimension* 1. The profinite group $G_{\mathbb{Q}}$ has a natural topology, and we want to study *continuous* such representations. Since $\mathbb{C}^{\times} = \mathrm{GL}_1(\mathbb{C})$ is abelian, these factorize through the *abelianised* absolute Galois group, so what one wants to describe are the continuous group homomorphisms

$$G_{\mathbb{Q}}^{\mathrm{ab}} \to \mathrm{GL}_1(\mathbb{C}).$$

Now, the latter are completely understood, even when we start from an arbitrary number field F and we consider its absolute Galois group G_F, thanks to *Artin's reciprocity law* - which we may call, by using a somewhat overused turn of phrase, the crowning achievement of *class field theory*, and hence of the algebraic number theory of the first half of the 20th century. In the form given to it by Chevalley in the 30's, it establishes, among many other things, the existence of a canonical, *surjective* continuous homomorphism

$$F^{\times}\backslash\mathbb{I}_F \to G_F^{\mathrm{ab}} \tag{1.1}$$

where the object $\mathbb{I}_F$ on the left is the topological group of *idèles*, in which $F^{\times}$ embeds as a discrete subgroup. It is the *restricted* direct product of the invertible elements of every completion of F, and hence, in the case of $\mathbb{Q}$, it takes into account at the same time all p-adic fields (p any prime number) and the real field $\mathbb{R}$. We call $F^\times\backslash\mathbb{I}_F$ the *idèle class group* of F.

Since the idèles are by definition the invertible elements of the *adèle ring* $\mathbb{A}_F$, we get that the 1-dimensional complex Galois representations are identified with a subset of the continuous characters

$$\mathrm{GL}_1(F)\backslash \mathrm{GL}_1(\mathbb{A}_F) \to \mathbb{C}^\times \tag{1.2}$$

namely with those with *finite image*. The key invariants associated to the two sides, the respective *L-functions*, are preserved by this correspondence, and we can consider our original problem as completely solved for n-dimensional representations with $n = 1$.

What happens for $n \geq 2$? According to Langlands' insight at the end of the 60's, the correct generalization of this picture is that irreducible (so, non-abelian) representations

$$G_F \to \mathrm{GL}_n(\mathbb{C})$$

should be classified, by replacing 1 with n in (1.2), by some subclass of functions of the form

$$\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F) \to \mathbb{C}$$

where now $\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)$ is no more a group, but only a coset space. The large body of conjectures known as the *Langlands program* is even more ambitious: roughly speaking, for any connected reductive algebraic group G, say over $\mathbb{Q}$, one hopes to prove that a certain subspace of functions

$$G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C} \tag{1.3}$$

parametrizes those Galois representations, whose image lands in a well-specified reductive subgroup of GL_n, depending on G. This correspondence, again, should preserve suitably defined L-functions. The first obvious problem one faces in pursuing such a generalization is: *which* functions should one consider in (1.3)? With the benefit of hindsight, we can sketch an (historically incorrect) heuristic approach to the conjectural answer.

The (abelianized) Galois group G_F^{ab} is canonically isomorphic to the projective limit

$$\varprojlim_{F\leq L \text{ finite abelian}} \mathrm{Gal}(L|F)$$

of its (automatically finite) quotients by open subgroups, which classify all possible *finite* abelian extensions of F. Artin's reciprocity law also implies that these quotients, through the epimorphism (1.1), are in bijection with the quotients of the idèle class group by its finite-index open subgroups (the so-called *norm groups* $N_{L|F}$), i.e. with the double quotients

$$F^\times\backslash\mathbb{I}_F/N_{L|F}$$

One could try to guess how these "finite approximations" of $GL_1(F)\backslash GL_1(\mathbb{A}_F)$ should generalize to the case of a general G and to understand which objects should be considered as "good functions" on the resulting spaces, in the hope to find "in the limit" the good functions on the whole of $G(F)\backslash G(\mathbb{A}_F)$.

For this, we have to write the above double quotients in a slightly modified form. Let us stick to the case of $\mathbb{Q}$ for simplicity and write $\mathbb{I}$ for its idèles. If $\mathbb{I}_f$ denotes the *finite idèles* (which, by introducing the *finite adèles* $\mathbb{A}_f$, we can see as the group $\mathrm{GL}_1(\mathbb{A}_f)$), we have that $\mathbb{I} = \mathbb{R}^\times \times \mathbb{I}_f$. Now, a norm group corresponding to a *totally imaginary* extension can be written as a product of $\mathbb{R}^\times_{>0}$ with an open, finite index (in particular, compact) subgroup K of $\mathbb{I}_f$, so that the double quotient acquires the form

$$\mathbb{Q}^\times\backslash(\pi_0(\mathbb{R}^\times) \times \mathrm{GL}_1(\mathbb{A}_f)/K) \tag{1.4}$$

where π_0 denotes the group of connected components. We may look at $\pi_0(\mathbb{R}^\times)$ as a homogeneous space for $\mathrm{GL}_1(\mathbb{R})$, via the obvious transitive action of the latter. Then, the generalization of the other objects appearing in (1.4) being now straightforward, it is tempting to stipulate that for a reductive group G over $\mathbb{Q}$, one should replace $\pi_0(\mathbb{R}^\times)$ by some homogeneous space X for $G(\mathbb{R})$, and the above double quotients by

$$G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K)$$

with K a compact open subgroup of $G(\mathbb{A}_f)$.

For a general reductive G over $\mathbb{Q}$, there exists a natural geometric object which may serve as the $G(\mathbb{R})$-homogeneous space X, namely the *symmetric space* associated to G. This is a Riemannian manifold, defined as the orbit space $G(\mathbb{R})/A_G K_\infty$. Here, K_∞ is a compact maximal subgroup of $G(\mathbb{R})$, and if S_G is the maximal $\mathbb{Q}$-split torus in the center of G, A_G denotes the connected component of the identity of $S_G(\mathbb{R})$. As a parallel with the situation for GL_1, it turns out that (see Chapter 2 for references):

(1) the space

$$S_K := G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K) = G(\mathbb{Q})\backslash G(\mathbb{A})/A_G K_\infty K$$

is naturally a finite disjoint union of *locally symmetric spaces*, themselves Riemannian manifolds, attached to the connected component of the identity $G(\mathbb{R})^0$;

(2) for any $K' \leq K$, there are natural maps $S_{K'} \to S_K$, such that the corresponding projective limit

$$S := \varprojlim_K S_K$$

carries a natural action of $G(\mathbb{A}_f)$, which gives rise on each S_K to the action of an algebra of operators $\mathcal{H}(G,K)$, the *Hecke algebra*;

(3) the *singular cohomology* $H^\cdot(S,\mathbb{Q})$ of S can be completely expressed in terms of a special class of $\mathbb{C}$-valued functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$, called *automorphic forms*;

(4) if $H^\cdot_c(\cdot,\mathbb{Q})$ denotes cohomology with *compact support*, then information about an especially interesting type of automorphic forms, the *cuspidal* forms, can be found, for each K, in the *interior cohomology* $H^\cdot_!(S_K,\mathbb{Q}) = \mathrm{Im}(H^\cdot_c(S_K,\mathbb{Q}) \to H^\cdot(S_K,\mathbb{Q}))$ by decomposing it in $\mathcal{H}(G,K)$-submodules.

Admitting that these objects should be the right ones to consider, we have now to construct the link with Galois representations.

1.2 Shimura varieties, weights and motives

There is a set of conditions on (G,X), verified by important families of reductive groups G together with appropriate $G(\mathbb{R})$-homogeneous spaces X, under which the coset spaces S_K's defined as above acquire the structure of complex quasi-projective algebraic varieties.

As complex-analytic objects, such S_K's are actually disjoint unions of locally symmetric spaces[1]. But one of the deep results of the theory is that they admit a *canonical model over a number field* E independent of K. These varieties over E (which we will still denote by S_K for the sake of this introduction) are called *Shimura varieties*.

The obvious consequence is that one can apply the richer methods of algebraic geometry to study these spaces and their singular cohomology. But moreover, for every prime ℓ, there is now another cohomology theory available, *étale ℓ-adic* cohomology $H^{\cdot}_{\text{ét}}(S_{K,\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$. Given the Galois action on the whole projective system S, these vector spaces are equipped by construction with Galois representations, up to an important change of perspective: they are no more *complex* representations, but ℓ-adic. In any case, $H^{\cdot}_{\text{ét}}$ still admits an action of the Hecke algebra $\mathcal{H}(G, K)$, since the elements of the latter operate by *correspondences*. Moreover, this commutes with the Galois action.

Actually, in each cohomology theory, one can consider more general *twisted coefficients* over S_K, coming from an algebraic $\mathbb{Q}$-representation V of G. The bridge between the two worlds is provided by a canonical Hecke-equivariant *comparison isomorphism*

$$H^{\cdot}(S_K, V) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq H^{\cdot}_{\text{ét}}(S_{K,\bar{\mathbb{Q}}}, V_\ell) \tag{1.5}$$

In the case of $G = GL_2$, the Shimura varieties in question are disjoint unions of (non-projective) *modular curves*, and the relevant cuspidal automorphic forms are *cusp modular forms* f of *weight* ≥ 2. Such f's have an associated L-function $L(f)$, and if f is moreover an *eigenvector* for the elements of $\mathcal{H}(G, K)$, then $L(f)$ encodes the eigenvalues of the Hecke operators on $H^{\cdot}(S_K, \mathbb{Q})$. In 1969, Deligne ([Del71a]) showed that in the Galois representation on ℓ-adic interior cohomology $H^{\cdot}_{\text{ét},!}$, the eigenvalues of *Frobenius endomorphisms*[2] are *pure* (modulo the *Weil conjectures* that he later proved himself). Moreover, through the isomorphism (1.5), an Hecke submodule corresponding to an eigenform f gives rise to a Galois representation, the eigenvalues of whose Frobenii satisfy the expected relationship with the $L(f)$. The known information on the latter can be thus used to improve our knowledge of the Galois representation; on the other hand, *purity* of the Galois representations imply strong bounds on the Hecke eigenvalues (*Ramanujan's conjecture*). This is a substantial instance of the connection between Galois representations and automorphic forms hinted to in the previous paragraph, and of its deep and spectacular consequences.

But there's more. S_K being an algebraic variety, the singular cohomology spaces $H^{\cdot}(S_K, V)$ carry themselves additional information, namely they are endowed with a *mixed Hodge structure*. Then, interior cohomology inherits a Hodge structure which is *pure*. By Deligne's theory of weights, the latter, together with the above *compatible* system of pure Galois representations (for varying ℓ, the eigenvalues of Frobenius don't depend on ℓ), and with the comparison isomorphisms, constitute precisely the kind of data arising from the cohomology of *smooth projective* varieties. It is then natural to look for an underlying geometric object, built out of such varieties, responsible for those data.

Crucially, modular curves happen to be *moduli spaces* for elliptic curves with level structures. Deligne had already constructed a canonical *smooth compactification* $\mathcal{E}$ of suitable powers E^k of the resulting universal elliptic curve E over S_K, and exploited its relation with the cohomology spaces $H^{\cdot}(S_K, V)$ to show purity. Then, in 1990, Scholl ([Sch90]) proved that in the category of *Chow motives*, one could define an idempotent endomorphism of $\mathcal{E}$, the "cohomology objects" of whose image M in the different theories

[1] But associated to the *adjoint* group of G.

[2] For all primes outside a certain finite set.

(i.e., its *realizations*) coincide with the interior cohomology seen above. Since Hecke operators extend to $\mathcal{E}$, M comes equipped with an action of the Hecke algebra, which allows to cut out of M certain subobjects $M(f)$ corresponding to single eigenforms f. The L-function $L(f)$ of f coincides now with the one naturally associated with $M(f)$, and this opens the way to studying it in the framework of *Beilinson's conjectures* ([Bei87]): they suggest a deep relationship between the *special values* of $L(f)$ and *motivic cohomology*[3] of $M(f)$.

Given the above, one wants to generalize this picture to other *non-compact*, higher-dimensional Shimura varieties. The first aim is to construct an Hecke-equivariant Chow motive which realizes to the *pure* (or *weight zero*) part of the cohomology of local systems coming from algebraic representations, the one containing interior cohomology. Here we should add that it is by now classical, in the context of the Langlands program, to speculate that (at least some subclass of the cuspidal) automorphic forms should be associated not only to Galois representations, but to motives (see e.g. [Clo90]).

Second, the *higher weights*, i.e. the genuinely *mixed* part, are of considerable interest, too. To describe the weight filtration on the cohomology of Shimura varieties is certainly significant for its own sake. But moreover, according to conjectures of Harder ([Har91]), the splitting of the extensions of pure Hodge structures in these cohomology spaces should control zeros and poles of L-functions of the more general automorphic forms appearing therein, and viceversa[4].

The origin of this thesis lies in the fact that the two problems just raised, the understanding of the higher weights and the construction of the desired Chow motives, are intimately related, as we are going to see.

To begin, suppose S_K to be a *PEL-type* Shimura variety, associated to a reductive group G, defined over the number field E. Roughly speaking, this means that it admits an interpretation as moduli space of abelian varieties with polarization, endomorphisms and level structure. Hence, over S_K there is an *universal abelian variety* A_K. Choose an irreducible representation V_λ of G of *highest weight* λ and write $\mu(V_\lambda)$, resp. $\mu_\ell(V_\lambda)$ for the corresponding local system, resp. ℓ-adic sheaf on S_K. If we want to find a Chow motive underlying the interior cohomology of $\mu_\ell(V_\lambda)$ over S_K, we face immediately a problem which stops any attempt to generalizing Scholl's approach: smooth compactifications of A_K and/or of its powers, to which the Hecke action extends, are not known. But, as for any Shimura variety, there exists a canonical compactification of S_K, although in general highly singular: its *Baily-Borel compactification* S_K^*. Denote by $j : S_K \hookrightarrow S_K^*$ the associated open immersion and by $i : \partial S_K \hookrightarrow S_K^*$ the closed immersion of the boundary. We write j_* for the *total* derived direct image functor. Then, the complex $i^* j_* \mu_\ell(V_\lambda)$ is an object in the "derived category" of ℓ-adic sheaves over ∂S_K^*, which in our setting admits an intrinsic notion of *weights* (cfr. Rmk. 3.3.2.12.(1)). The theory of Wildeshaus gives a criterion on this complex which, once verified, produces the sought-for motive. We will give a much more precise form of this theorem, and also a glimpse of the ideas on which it is based, in Section 3.3.2. For the moment, let us state it in the following version:

Theorem. *([Wil19a]) If $i^* j_* \mu_\ell(V_\lambda)$* avoids weights[5] *0 and 1, then there exists a canonical Chow motive M_0 over E with an action of the Hecke algebra, whose realizations in any*

[3]Here we are glossing over the fact that $M(f)$ is defined as an *homological* motive, and only expected, but not known, to lift to a Chow motive.

[4]A vision which fits with the one of Beilinson's, if one considers the conjectural interpretation of motivic cohomology as a space of extensions in the still hypotetical *abelian* category of *mixed motives*.

[5]This *weight avoidance* has to be understood in the following sense. Denote by $\mathcal{H}^n$ the *perverse* co-

cohomological degree n coincide with $H^n_!(S_K(\mathbb{C}), \mu(V_\lambda))$ and $H^n_{\text{ét},!}(S_{K,\bar{\mathbb{Q}}}, \mu_\ell(V_\lambda))$.

This result was the starting point for this thesis, whose contents we are now going to present.

1.3 Principal results and structure of the thesis

The main problem studied in the present work is the following: which families of Shimura varieties, and of representations of the underlying group G, do verify the criterion given at the end of the previous paragraph? Let us call *degeneration at the boundary* the complex $i^* j_* \mu_\ell(V_\lambda)$, and let us say that weight α *appears* in it if it doesn't avoid such a weight. Notice that, because of its *autoduality* properties, weight 0 appears in the degeneration at the boundary if and only if weight 1 appears. We also mention that the higher weights in $H^\cdot(S_K, \mu_\ell(V_\lambda))$, as recalled in Subsection 2.3.2, are determined by *boundary cohomology*, which is the hypercohomology of the complex $i^* j_* \mu_\ell(V_\lambda)$. This fournishes the connection between the two problems raised in the previous paragraph.

Our attention has concentrated on the case of the group $G = \mathrm{Res}_{F|\mathbb{Q}} \mathrm{GSp}_{4,F}$, where F is a totally real number field of some degree d over $\mathbb{Q}$, whose set of complex (hence, real) embeddings will be denoted by I_F. The associated Shimura varieties, the *Hilbert-Siegel varieties of genus 2*, are of dimension $3d$ and are moduli spaces of polarized abelian varieties of dimension $2d$ with real multiplication (and additional structures).

Let us explain the reasons for this choice. The group G can be seen as the simultaneous generalization of the reductive $\mathbb{Q}$-groups $\mathrm{Res}_{F|\mathbb{Q}} \mathrm{GL}_{2,F}$ and of the group GSp_4. The Shimura varieties corresponding to the former are called *Hilbert modular varieties* (dimension d, parametrizing polarized abelian varieties of dimension d with real multiplication) and the ones corresponding to the latter are called *Siegel threefolds* (dimension 3, parametrizing polarized abelian surfaces)[6]. Irreducible representations V_λ of $\mathrm{Res}_{F|\mathbb{Q}} \mathrm{GL}_{2,F}$ are parametrized by vectors $\lambda = (k_\sigma)_{\sigma \in I_F}$ of non-negative integers; λ is called *regular* if $k_\sigma > 0$ for every σ. On the other hand, irreducible representations V_λ of GSp_4 are parametrized by couples (k_1, k_2) of integers such that $k_1 \geq k_2 \geq 0$, and are called *regular* if $k_1 > k_2 > 0$.

In the context of our problem, the following is known:

Theorem. *([Wil12], cfr. [Wil19a, Example 5.5]) Over Hilbert modular varieties, weights 0 and 1 appear in $i^* j_* \mu_\ell(V_\lambda)$ if and only if λ is identically equal to 0.*

Theorem. *([Wil19b], cfr. [Wil19a, Rmk. 5.11 (a)]) Over Siegel threefolds, weights 0 and 1 appear in $i^* j_* \mu_\ell(V_\lambda)$ if and only if λ is irregular.*

Notice that for Hilbert modular varieties, saying that λ is the trivial character (in particular a *parallel* character, i.e. such that each of its components is equal to each other) is equivalent to saying that it is irregular for each σ, a condition under which we

homology objects of the complex $\mathcal{F}$, by Gr_k their graded quotients of weight k, and by w the weight of the *pure object* $\mu_\ell(V_\lambda)$ (see Rmk. 2.1.3.8.(2)). Then the sentence *the complex $\mathcal{F}$ avoids weights $\alpha, \ldots, \beta$ for integers $\alpha \leq \beta$* is equivalent to: *for each integer n, the objects $\mathrm{Gr}_{w+n+\alpha} \mathcal{H}^n \mathcal{F}, \ldots, \mathrm{Gr}_{w+n+\beta} \mathcal{H}^n \mathcal{F}$ are trivial.*

[6] For the purposes of this introduction, we will ignore the fact that the center of the groups under consideration has to be modified in order to make the theory work properly. Moreover, since the results that we will state only depend on the restriction of the representations to the *derived subgroup* G, we will parametrize the weights as if we were dealing with G^{der} instead of G.

may call it *completely irregular*. It follows from the above theorems that, in both this case and the case of Siegel threefolds, one can construct Hecke-equivariant Chow motives realizing to interior cohomology for *most* λ: *regularity* of λ is sufficient, but not always necessary, for the avoidance of weights. Together with *Picard modular varieties* ([Wil15], [Clo17], cfr. [Wil19a, Rmk. 5.8]), for which a similar characterization can be proven, these were the only known families of Shimura varieties giving rise to a construction of the Chow motives with the desired properties.

Given the previous examples, the group $\mathrm{Res}_{F|\mathbb{Q}}\,\mathrm{GSp}_{4,F}$ appears as the natural following case to be investigated (we will come back later to this point, when discussing the content of Chapter 4). Our contribution has been to find a criterion which completely characterizes the presence of the weights 0 and 1 in the degeneration at the boundary, in the case of this group.

Here, the highest weight is of the form $\lambda = (k_1, k_2) := ((k_{1,\sigma})_{\sigma\in I_F}, (k_{2,\sigma})_{\sigma\in I_F})$, with $k_{1,\sigma} \geq k_{2,\sigma} \geq 0$ for every σ. It is called *regular* at σ if the inequality is strict at σ, and *completely irregular* if for every σ, it is not regular at σ. For $i = 1, 2$ we call k_i *parallel* if there exists an integer κ such that $k_{i,\sigma} = \kappa$ for every σ, and in this case we write $k_i = \underline{\kappa}$. Then, our criterion subsumes the ones for $\mathrm{Res}_{F|\mathbb{Q}}\,\mathrm{GL}_{2,F}$ and for GSp_4, in the following way:

Theorem A. *(Cor. 4.3.1.3) Over genus 2 Hilbert-Siegel varieties, weights 0 and 1 appear in $i^* j_* \mu_\ell(V_\lambda)$ if and only if λ is completely irregular and k_2 is parallel.*

As a consequence of this result, we have a large number of representations V_λ for which we can construct Chow motives realizing to interior cohomology of V_λ.

This theorem actually follows from the main result of this thesis, which consists in a detailed description of what one could call the *limit weights* in the "weight filtration" of the degeneration at the boundary. In fact, the main interest of this description lies in finding the correct invariant of the representation V_λ controlling the weights in question. We refer to Def. 4.3.1.1 for the actual definition of this invariant, called the *corank* of λ. We express our main result in the following imprecise form, calling a character $\lambda = (k_1, k_2)$ *impair* if it holds that $k_1 = \underline{\kappa_1}$ and $k_2 = \underline{\kappa_2}$ with κ_1 and κ_2 of different parity:

Theorem B. *(Thm. 4.3.1) If λ is not impair, the minimal weight $k \geq 1$ and the maximal weight $k \leq 0$ appearing in $i^* j_* \mu_\ell(V_\lambda)$ can be explicitly described in terms of $k_1 - k_2$, k_2 and of the corank of λ.*

The importance of the corank, as we will argue in Subsection 2.3.4, stems from the fact that, when suitably generalized, it controls the existence of non-trivial non-cuspidal automorphic forms on PEL-Shimura varieties of symplectic or unitary type. In the case of genus 2 Hilbert-Siegel varieties, non-trivial, non-cuspidal automorphic forms can exist only if the corank is at least 1. Now, Theorem A can be reformulated as saying that *weights 0 and 1 appear in the degeneration if and only if λ is completely irregular of corank at least 1.* The conjunction of these facts - whose proofs are independent the one of the other - may hint to a a path towards the understanding of the contribution of automorphic forms to the weight filtration in the cohomology of Shimura varieties, a subject which is still mysterious and largely unexplored (cfr. for example [OS90, 7.5]). Notice that all the assertions that we have formulated for the ℓ-adic side could be repeated word by word on the Hodge side, because the underlying computations are exactly the same (Rmk. 4.2.2.4).

Overview.

In order to give an idea of the techniques used to obtain the previous results, and also to cite the other facts that we have proven along the way, we pass to give an outline of this thesis. We have striven to introduce the material in an organic way, putting in the right context the objects and the theories that we use, whenever possible. Moreover, we wanted to present various results from the literature, often scattered among different sources, in a form adapted to our needs, and to give complements to them. This is the role of Chapters 2 and 3, which contain little of original, except for the exposition and some specific parts pointed out below. These chapters are in fact preliminary in nature, and mainly set the stage for Chapter 4. The latter, the heart of the thesis, is instead completely devoted to the proof of the main results which we have discussed above. Lastly, Chapter 5 analyses the Chow motives that we can construct as a consequence of those results and of Wildeshaus' theory. These last two chapters can be seen as providing a detailed study of an example of the general theory reviewed in the first two, and constitute the main content of the paper [Cav19].

We give now a more precise overview of each Chapter. Chapter 2 is meant to introduce our basic objects of interest, variations of Hodge structure and ℓ-adic sheaves over Shimura varieties, to illustrate the interplay between various structures appearing in the cohomology of such local systems, and to explain the role of different compactifications in their comprehension. We spend some time in defining the action of the Hecke operators, preparing the *motivic* definition of the Hecke algebra in the following chapter, and in expliciting the relation of cohomology with automorphic forms. On the one hand, this allows us to introduce L^2-*cohomology*; on the other, it is useful in view of the last chapter (see below). We give the structure of the Baily-Borel compactification, which will be crucially used in Chapter 4, and we recall some facts on its *intersection cohomology*, which we need to consider because the Chow motive realizing to interior cohomology is actually constructed - under the weight avoidance hypothesis - as an *intermediate extension*. Thanks to intersection cohomology, we can prove the only original result of this part (Thm. 2.3.2), which is basically a consequence of Zucker's conjecture, of Hodge theory and of an automorphic observation on L^2-cohomology, but seems to be absent from the literature. We see it as giving information on why weights 0 and 1 in the boundary should be related to *regularity* of the highest weight λ, a condition which may be conjectured to be *sufficient* for their absence (cfr. Rmk. 3.3.2.13). We finish the chapter by briefly discussing *toroidal compactifications*: they allow for the definition of *automorphic bundles* and hence for the definition of *cusp forms* on general Shimura varieties, which will play a central role in proving the *only if* part in Theorem A. This also permits to motivate the definition of corank, linking it to automorphic forms.

In Chapter 3 we switch to the motivic language. We begin by a rapid review of the notions that we need from the theories of Beilinson and relative Chow motives, before recalling the fundamental theorem of Ancona providing the relative motives over PEL Shimura varieties, which are the basis for all later constructions. Actually, we need to analyze Ancona's result quite in detail, in order to move on to the original part of this chapter, i.e. Subsection 3.2.2. There, we show that the correspondences defining Hecke operators actually act as an *algebra* on the relevant motives, something which wasn't obvious from the literature. One can dispose of this result for the construction of the motives we are interested in, but it seemed to us that filling this gap could be important for future applications (cfr. Remark 5.2.0.5). The chapter ends by discussing the fundamental language of *weight structures*, clarifying why the existence of the desired Chow motives follows from the weight avoidance, and formulating in our context the criterion found by Wildeshaus to check the latter. Namely, this criterion tells that we have to consider separately the restriction of the complex $i^* j_* \mu_\ell(V_\lambda)$ to each stratum of the boundary, and to look at the weights of the perverse cohomology sheaves of such restrictions.

This is the subject of Chapter 4. We refer to its introduction for an overview of the proof of Theorem B (and hence of Theorem A) contained therein, and of the intermediate results that are needed to complete it. Here we content ourselves with saying the following: Pink's theorem 4.2.1 gives an expression for the restriction to a single stratum of the *classical* cohomology sheaves of the degeneration at the boundary, and for their weight-graded objects, in terms of cohomology of unipotent groups and of cohomology of arithmetic groups. In turn, the latter involves a genuinely arithmetic component, and another one related to cohomology of free abelian groups, arising as groups of units of the field F. Compared to the list of previously studied Shimura varieties given before, Hilbert-Siegel varieties are the first ones, for which the phenomena related to these three aspects make their appearance *simultaneously*. An ulterior motivation for their choice came from the latter observation, as well as from the consideration of the structure of the boundary: a disjoint union of two types of strata, i.e. cusps and Hilbert modular varieties of dimension equal to the degree d of the field F (for Siegel threefolds, one has cusps and modular curves; for Hilbert modular varieties, only cusps). Then, this chapter is mainly concerned with computing the mentionned cohomologies, keeping track of their mutual interrelations and of the resulting combinatorics, and meaningfully interpreting the outcome (also in terms of automorphic forms), before relating the classical sheaves with their perverse counterparts - the ones we are really interested in. The action of the groups of units on the stalks of the degeneration turns out to be crucial, and to admit a geometric interpretation that deserves further attention (cfr. Rmk. 4.30).

The final Chapter 5 draws the consequences of the above. Having proven the weight avoidance for non completely irregular or corank 0 characters λ, we can apply the theory explained at the end of Chapter 3 and construct motives realizing to interior cohomology with coefficients in the corresponding representations V_λ, which turns out to coincide with *intersection cohomology*. We describe the properties of these motives and we exploit the action of the Hecke algebra to cut out *homological* submotives corresponding to irreducible Hecke submodules. The material on automorphic representations summarized in Chapter 2 explains the semisimplicity of the action of the Hecke algebra on interior cohomology, and the relation between the motives that we have constructed and the known information about automorphic representations of $\mathrm{Res}_{F|\mathbb{Q}} \mathrm{GSp}_{4,F}$, contained in Flicker's book [Fli05].

Conventions and notations.

The symbols $\mathbb{A}$, resp. $\mathbb{A}_f$ will denote the ring of adèles, resp. finite adèles.
An empty entry in a ring-valued matrix will mean that the corresponding coefficient is zero. If R and B are modules over a ring A, we will often denote $R \otimes_A B$ by R_B.
We will use the symbol $\pi_0(X)$ for the set of connected components of a topological space X. If G is an algebraic group over $\mathbb{Q}$, we will use the notation $G(\mathbb{R})^0$ for the topological connected component of the identity.
If $f : T \to S$ is a morphism of schemes and X is a S-scheme, the notation $X \times_{S,f} T$ will be used to specify that the fiber product has been taken along f: this can be shortened to $(X_T)_f$ and furthermore to X_T (if the choice of S and/or f is evident from the context). When $T = \operatorname{Spec} R$ we will often denote X_T by X_R.
An *algebraic variety* over a field k will always mean a separate, finite type scheme over k. An *algebraic group* over k will mean a finite type scheme over k with the structure of a group scheme. For an extension $k'|k$ of fields, the symbol $\operatorname{Res}_{k'|k}$ will denote Weil restriction from k' to k. We will denote by $\mathbb{S}$ the Deligne torus, i.e. the algebraic group over $\mathbb{R}$ defined by $\operatorname{Res}_{\mathbb{C}|\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$.
If $\mathcal{C}$ is a category, then $\operatorname{Gr}_{\mathbb{Z}}\mathcal{C}$ will denote its category of graded objects. A sub-category $\mathcal{B}$ of an additive category $\mathcal{A}$ is *dense* if for any object B of $\mathcal{B}$, any direct factor of B formed in $\mathcal{A}$ belongs already to $\mathcal{B}$. The pseudo-Abelian completion of an additive category $\mathcal{A}$ is denoted by $\mathcal{A}^\natural$. If $\mathcal{A}$ is an abelian category, $D^b(\mathcal{A})$ will denote its bounded derived category. The notion of a $\otimes$-*category* over a commutative ring is defined as in [And04, 2.2.2]. Idem for *rigid* $\otimes$-categories.
The *constructible category* over a complex analytic space S is the full subcategory of the bounded derived category of sheaves consisting of complexes with constructible cohomology objects. We will call such complexes *constructible complexes*.

Chapter 2

Weights in the cohomology of Shimura varieties

As alluded to in the introduction, the cohomology of locally symmetric spaces is naturally interpreted in terms of automorphic forms. When such spaces are algebraic varieties, additional structures appear in the cohomology: Galois representations and Hodge structures. Their relation with automorphic forms is the subject of this chapter and will form the background for later considerations.

We begin in Section 2.1 by introducing the facts that we need about local systems (coming from representations of the underlying group G) on locally symmetric spaces in the general setting, specializing next to variations of Hodge structures and ℓ-adic sheaves over Shimura varieties. In particular, we review the examples of Hilbert-Siegel varieties (but also of Hilbert modular varieties) that we will study in detail in Chapter 4.

In Section 2.2 we start tracing the connection with automorphic representations and forms. In particular, we define the Galois representations (2.27) associated to cuspidal representations, which should arise from *motives*.

We pass then to introduce the point of view of *weights*, which is the crucial one in this thesis (Section 2.3). Cuspidal representations lie in a subspace of the *weight zero* part of the cohomology, the *interior cohomology*. The Hodge structures associated to these automorphic representations are then defined in (2.32); these are the counterpart of the Galois representations cited above. When the Shimura varieties are not compact, which is our case of interest, the rest of the cohomology is captured by *boundary cohomology*. The study of the latter, also motivated by the desire of understanding the *higher weights*, naturally begins from a choice of a *compactification*.

We conclude the chapter by considering two types of these. The canonical one, the *Baily-Borel* compactification, will be used in detail in Chapter 4: here we give the necessary general information on the structure of its boundary, stratified by "smaller" Shimura varieties. The knowledge of its *intersection cohomology* provided by *Zucker's conjecture* implies an observation on how *regularity* of the G-representations under consideration influences the weights in the cohomology (Thm. 2.3.2), a theme which will surface again in Chapter 4.

The second type of compactification, actually a whole family, is given by *toroidal compactifications*. We explain their role in the general picture and we collect some facts that we need, both for motivation and for later use.

2.1 Local systems on locally symmetric spaces and Shimura varieties

2.1.1 Locally symmetric spaces

We start by putting the description of Shimura varieties, or better of their spaces of complex-valued points, in the broader context of *locally symmetric spaces*, of which we will encounter *non-algebraic* examples later.

Fix, for the rest of this section, a connected $\mathbb{Q}$-reductive group G and a maximal compact subgroup K_∞ of $G(\mathbb{R})$, define S_G as the maximal $\mathbb{Q}$-split torus inside the center $Z(G)$, and define the commutative Lie group A_G as

$$A_G := S_G(\mathbb{R})^0. \tag{2.1}$$

Definition 2.1.1.1. *The* symmetric space associated to G *is defined as* $D := G(\mathbb{R})/A_G K_\infty$.

Remark 2.1.1.2. The space D has a canonical structure of Riemannian manifold, independent of the choice of K_∞, and is actually a *symmetric space* as in the usual definition (as found for example in [Ji09, 4.7]). Moreover, it is a *non-positively curved* symmetric space (product of *non-compact type* and *flat* factors), which implies in particular that it is simply connected and even contractible (cfr. [BJ06, I.1.2]).

Example 2.1.1.3. (1) By choosing $G = \mathrm{SL}_2$, $K_\infty = \mathrm{SO}_2(\mathbb{R})$ one recovers the classical description of the complex upper-half plane $\mathfrak{H}$ as $\mathfrak{H} \simeq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$, compatibly with the classical action of $G(\mathbb{R})$ by fractional transformations; notice that we obtain the same space by choosing $G = \mathrm{PGL}_2$, $K_\infty = \mathrm{PO}_2(\mathbb{R})$. More generally, taking $G = \mathrm{Sp}_{2n}$ and $K_\infty = U_n(\mathbb{R})$ gives as symmetric space D the *Siegel upper-half space of genus* n of $n \times n$ complex symmetric matrices of positive definite imaginary part, denoted by $\mathfrak{H}_n$, equipped with the classical action of $G(\mathbb{R})$. Here the group A_G is always trivial, since the groups G are semisimple.

(2) Fix a totally real field F of degree d. The previous example can be then generalized in the following way: by taking $G = \mathrm{Res}_{F|\mathbb{Q}} \mathrm{Sp}_{2n,F}$ and $K_\infty = \mathrm{U}_n(\mathbb{R})^d$ we obtain as symmetric space D the d-fold product $\mathfrak{H}_n^d$.

(3) Let be F as before, and take $G = \mathrm{Res}_{F|\mathbb{Q}} \mathrm{GL}_{2,F}$, $K_\infty = \mathrm{O}_2(\mathbb{R})^d$. Then

$$A_G \simeq \mathbb{R}_{>0}$$

and the associated symmetric space is given by

$$D \simeq \mathfrak{h}^d \times \mathbb{R}^{d-1},$$

which, contrary to the previous examples, has in general no complex structure (as seen in the case $d = 2$, where the dimension $3d - 1$ of D is odd).

The locally symmetric spaces considered here will always arise as quotients of symmetric spaces by suitable *arithmetic* subgroups (i.e., having chosen an embedding $G \hookrightarrow \mathrm{GL}_n$, subgroups of $G(\mathbb{Q})$ which are *commensurable* with $G(\mathbb{Z})$):

Definition 2.1.1.4. *Let* Γ *be a torsion-free arithmetic subgroup of* $G(\mathbb{Q})$ *and* D *the symmetric space associated to* G*. Then the* locally symmetric space associated to Γ *is the space* $X_\Gamma := \Gamma \backslash D$.

Remark 2.1.1.5. (1) Since the hypotheses on Γ assure that it acts isometrically, properly and freely on D, such X_Γ's are actually locally symmetric spaces as in the classical definition (see for example [Ji09, 4.9]), so that they have in particular a canonical structure of Riemannian manifold. Moreover, as a consequence of Remark 2.1.1.2, Γ is identified with the fundamental group of X_Γ.

(2) Most commonly, it is useful to ask Γ to be *neat*, i.e. such that for any (equivalently, for all) faithful representation $\rho : G \to \mathrm{GL}_n(V)$, for every $g \in G(\mathbb{Q})$, the eigenvalues of $\rho(g)$ in $\mathbb{C}$ generate a torsion free subgroup of $\mathbb{C}^\times$. Every neat subgroup is torsion-free, but the converse doesn't hold. The notion of neatness has the advantage of being preserved under passage to subgroups and to homomorphic images by morphisms of algebraic groups.

The main examples of locally symmetric spaces which we will make use of are connected components of spaces of $\mathbb{C}$-valued points of Shimura varieties; examples will be given in the next subsection. The only exception is represented by the non-algebraic locally symmetric spaces that will appear in the proof of Lemma 4.3.2.11.

We recall the following general construction:

Construction 2.1.1.6. Let L be any extension of $\mathbb{Q}$ and V be a finite dimensional representation of G_L. Then V gives in particular a representation of the fundamental group Γ of X_Γ and hence defines a canonical local system $\mathbb{V}$ of L-vector spaces on X_Γ. Moreover, for every integer p, the group cohomology spaces $H^p(\Gamma, V)$ are canonically identified with the cohomology spaces $H^p(X_\Gamma, \mathbb{V})$ of X_Γ with coefficients in the local system $\mathbb{V}$. This follows from the contractibility of X.

For later use, we record here a general structural result on the cohomology groups of the local systems described above. Fix a number field L over which G splits and fix a (split) maximal torus T and Borel B of G_L. For any *root* α of T, denote by $\alpha^\vee$ the corresponding *coroot*. Irreducible representations of G_L are parametrized by those characters (or *weights*) λ of T which are *dominant*, i.e. such that for any *positive* root α (with respect to T, B), the evaluation $\langle \lambda, \alpha^\vee \rangle$ of the canonical pairing on λ and $\alpha^\vee$ is ≥ 0. The representation associated to such a dominant weight λ is said to be of *highest weight* λ. The dominant weight λ is called *regular* if, for any positive root α, $\langle \lambda, \alpha^\vee \rangle > 0$.

Theorem 2.1.1. *([LS04, Cor. 5.6]) Let V_λ be an irreducible representation of G_L of highest weight λ. Let $D = G(\mathbb{R})/A_G K_\infty$ be the symmetric space associated to G, X_Γ any of the associated locally symmetric spaces, and $\mathbb{V}_\lambda$ the local system on X_Γ associated to V_λ by Construction 2.1.1.6. Denote by* rk *the absolute rank of a Lie group and pose $l_0 := \mathrm{rk}(G(\mathbb{R})) - \mathrm{rk}(K_\infty)$. Then, if λ is regular, we have $H^p(X_\Gamma, \mathbb{V}_\lambda) = \{0\}$ for every p such that*

$$0 \leq p < \frac{1}{2} (\dim(D) - l_0)$$

Remark 2.1.1.7. The above vanishing results also follow from general theorems of Saper ([Sap05b, Thm. 5]), whose proof is contained in [Sap05a] but has still not appeared in print.

2.1.2 Shimura varieties

Keep the notations of the previous subsection. As is well known, Deligne has identified a set of axioms on G and on a chosen $G(\mathbb{R})$-conjugacy class X of morphisms $\mathbb{S} \to G_\mathbb{R}$ ([Del79, 2.1.1]; a couple (G, X) satisfying such axioms is then dubbed a *Shimura datum*) which ensure in particular that, for any compact open subgroup K of $G(\mathbb{A}_f)$, the double quotient

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K \tag{2.2}$$

has a canonical structure of (non-connected) complex analytic variety. Even more, one shows that this set is identified with the $\mathbb{C}$-valued points of a canonical quasi-projective algebraic variety S_K (the *Shimura variety* associated to (G, X) and of *level* K) defined over a number field E (the *reflex field*) independent of the choice of K.

Remark 2.1.2.1. (1) In all of this thesis we will use the extension of Deligne's formalism due to Pink, according to whose definition of a *pure Shimura datum* (G, X) ([Pin90, 2.1]), the object X is allowed to be more generally a left homogeneous space under $G(\mathbb{R})$ which admits a $G(\mathbb{R})$-equivariant map $X \to \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ with finite fibers.

(2) The link with the theory of locally symmetric spaces is the following. Fix an element $h \in X$ and define $K_\infty^h := \mathrm{Stab}_G(h)$. Then, the axioms imply that: (*i*) through the isomorphism $X \simeq G/K_\infty^h$, the space X is identified with a finite disjoint union of copies of the symmetric space D for the connected component of the identity of the derived subgroup[1] G^{der}; (*ii*) the symmetric space D is of non-compact type and endowed with a canonical complex structure, compatible with the Riemannian one (i.e. it is an *hermitian symmetric domain*). The group G acts on D through G^{ad}. Then, one shows that the double quotient (2.2) is identified with a finite disjoint union

$$\bigsqcup_i \Gamma_i \backslash D \tag{2.3}$$

where the Γ_i's are arithmetic subgroups of $G(\mathbb{Q})$. More precisely, their images in $G^{\mathrm{ad}}(\mathbb{Q})$ are *congruence* subgroups of the latter group (i.e. of the form $G^{\mathrm{ad}}(\mathbb{Q}) \cap K'$ for some compact open subgroup K' of $G^{\mathrm{ad}}(\mathbb{A}_f)$). This identifies $S_K(\mathbb{C})$ with a disjoint union of locally symmetric spaces associated to the connected component of the identity of G^{ad}.

(3) Suppose the compact open subgroup K to be *neat* ([Pin90, 0.6]), a notion which ensures that the arithmetic subgroups Γ_i in the previous point are themselves neat as defined in Remark 2.1.1.5.(2). Then, the associated Shimura variety is a *smooth* quasi-projective variety defined over the reflex field E. In the following, our K's will always be neat.

Example 2.1.2.2. (1) Take $G = \mathrm{GL}_2$ and take as X the conjugacy class of the morphism $h : \mathbb{S} \to G_{\mathbb{R}}$ defined on real points by

$$\begin{aligned} \mathbb{S}(\mathbb{R}) &\to G(\mathbb{R}) \\ x + iy &\mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \end{aligned}$$

Then, we have an isomorphism $X \simeq \mathbb{C} \setminus \mathbb{R}$ which $G(\mathbb{R})$-equivariantly identifies X with the disjoint union $\mathfrak{H}^{\pm}$ of two copies of the complex upper half plane, i.e. of two connected components, each of whose is isomorphic to the domain D of the first part of Example

[1] To emphasize the fact that these spaces come from the *derived* subgroup of G, notice that K_∞^h will in general different from the group $A_G K_\infty$ defined in the previous subsection; it is connected (whereas the latter, in general, is not) and, in general, it will contain *strictly* the connected component of the identity of $A_G K_\infty$. Also notice that this description implies that in general, K_∞^h is *not* compact.

2.1.1.3.(1). The couple (G, X) is a pure Shimura datum ([Del71c, 1.6]) and the associated Shimura varieties S_K are *modular curves* of level K; the connected components of their $\mathbb{C}$-points are locally symmetric spaces for PGL_2, henceforth called *complex analytic, connected* modular curves. By considering, for all positive integer n, the $2n \times 2n$ matrix

$$J_n := \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} \in \mathrm{GL}_{2n}(\mathbb{Q}),$$

by taking as G the group GSp_{2n} of *symplectic similitudes* of dimension $2n$, defined as the algebraic group over $\mathbb{Q}$ such that, for all $\mathbb{Q}$-algebra R, one has

$$\mathrm{GSp}_{2n}(R) := \left\{ g \in \mathrm{GL}_{2n}(R) | {}^t g J_n g = \nu(g) J_n, \nu(g) \in \mathbb{G}_m(R) \right\},$$

and by taking as X the $G(\mathbb{R})$-conjugacy class of the morphism $h : \mathbb{S} \to G_{\mathbb{R}}$ defined on real points by

$$\begin{aligned} \mathbb{S}(\mathbb{R}) &\to G(\mathbb{R}) \\ x + iy &\mapsto xI_2 + yJ_2, \end{aligned}$$

one gets that X is $G(\mathbb{R})$-equivariantly identified with the space $\mathfrak{H}_n^{\pm}$ of $n \times n$ complex symmetric matrices with definite imaginary part (positive or negative), i.e. with the disjoint union of two copies of the domain $D = \mathfrak{H}_n$ of the second part of Example 2.1.1.3.(1). The couple (G, X) is again a pure Shimura datum (*loc. cit.*) and the associated Shimura varieties S_K are the *Siegel varieties of genus* n (of level K); a similar remark as before, about the connected components of their $\mathbb{C}$-points, holds.

(2) Fix a totally real number field F of degree d over $\mathbb{Q}$ and denote by I_F its set of complex (hence, real) embeddings. In parallel with Example 2.1.1.3.(2), the previous example generalizes (first, in the genus 1 case) by choosing $G = \mathrm{Res}_{F|\mathbb{Q}} \mathrm{GL}_{2,F}$ and defining as X the $G(\mathbb{R})$-conjugacy class of the morphism $h : \mathbb{S} \to G_{\mathbb{R}}$ defined on real points by

$$\begin{aligned} \mathbb{S}(\mathbb{R}) &\to G(\mathbb{R}) \\ x + iy &\mapsto (\begin{pmatrix} x & y \\ -y & x \end{pmatrix}_{\sigma})_{\sigma \in I_F} \end{aligned}$$

with the effect that one has a $G(\mathbb{R})$-equivariant isomorphism $X \simeq (\mathfrak{H}^{\pm})^d$; the associated Shimura varieties are called *Hilbert-Blumenthal* or *Hilbert modular* varieties, with $\mathbb{C}$-points which are disjoint unions of *complex analytic, connected* ones (which can be seen as locally symmetric spaces for $\mathrm{Res}_{F|\mathbb{Q}} \mathrm{SL}_{2,F}$, when the groups Γ_i of Remark 2.1.2.1.(2) happen to be subgroups of $\mathrm{SL}_2(F)$). More generally, for any positive integer n, one takes $G = \mathrm{Res}_{F|\mathbb{Q}} \mathrm{GSp}_{2n,F}$ and defines X as the $G(\mathbb{R})$-conjugacy class of the morphism $h : \mathbb{S} \to G_{\mathbb{R}}$ defined on real points by

$$\begin{aligned} \mathbb{S}(\mathbb{R}) &\to G(\mathbb{R}) \\ x + iy &\mapsto ((xI_2 + yJ_2)_{\sigma})_{\sigma \in I_F}, \end{aligned}$$

obtaining a $G(\mathbb{R})$-equivariant isomorphism $X \simeq (\mathfrak{H}_n^{\pm})^d$ and, as associated Shimura varieties, the *Hilbert-Siegel varieties* of genus n. Actually, the Hilbert and Hilbert-Siegel varieties that we will consider will be associated to a slightly different Shimura datum, see Example 2.1.3.2.

Let us end this subsection by recalling the system of finite coverings which arises from a Shimura datum (which also exists, in its topological version, in the more general context of locally symmetric spaces). Let (G, X) be a pure Shimura datum and $K \subset G(\mathbb{A}_f)$ a open compact subgroup. For every $g \in G(\mathbb{A}_f)$, consider the holomorphic, *finite* maps (cfr. [Pin90, 3.4])

$$[\cdot g] : S_{K \cap gKg^{-1}}(\mathbb{C}) \to S_K(\mathbb{C})$$
$$G(\mathbb{Q})(x, h)K \cap gKg^{-1} \mapsto G(\mathbb{Q})(x, hg)K$$

Then (cfr. [Pin92, (3.4)]), this map algebraizes and gives rise to a map

$$[\cdot g] : S_{K \cap gKg^{-1}} \to S_K \tag{2.4}$$

which is an *étale covering* if K is neat. The canonical maps corresponding, by the case $g = 1$ of (2.4), to the choice of two compact open subgroups K, K' of $G(\mathbb{A}_f)$ such that $K' \leq K$, give rise to a projective system of complex spaces

$$\varprojlim_K S_K(\mathbb{C}) \tag{2.5}$$

endowed with a right action of $G(\mathbb{A}_f)$ (defined by the morphisms (2.4)). Thus, we get a projective system of finite type E-schemes whose projective limit

$$S := \varprojlim_K S_K \tag{2.6}$$

exists as a scheme over E, and is endowed with an action of $G(\mathbb{A}_f)$ ([Del79, 2.1.4]).

2.1.3 The canonical construction of mixed sheaves on Shimura varieties

Keep the notation of the previous subsection. For a reductive group G underlying a pure Shimura datum, we want now to refine Construction 2.1.1.6 and obtain not only *local systems* on the topological space of $\mathbb{C}$-valued points of Shimura varieties, but objects with more structure, i.e. *mixed sheaves*, in the sense of Hodge or étale ℓ-adic theory.

Convention 2.1.3.1. Let $w : \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ be the cocharacter which induces the inclusion $\mathbb{R}^\times \hookrightarrow \mathbb{C}^\times$ on real points. A representation (ρ, V) of $\mathbb{S}$ induces a (semisimple) mixed Hodge structure on the real vector space V. Coherently with [Pin90] 1.3, we will say that the subspace of V where $\rho \circ w$ acts as multiplication by t^{-k} is the subquotient of V of *weight* k.

Observe now that, if (G, X) is a Shimura datum, defined by $h : X \to \mathrm{Hom}(\mathbb{S}, G_\mathbb{R})$, then every representation $\rho : G \to \mathrm{GL}(V)$ gives rise, for all $x \in X$, to a Hodge structure on V, by applying the above observation to $\rho \circ h(x) \circ w$. By design, the axioms which define a Shimura datum (G, X) allow one to obtain from such punctual Hodge structures a *variation* of Hodge structure on X. In order to descend this on the $\mathbb{C}$-valued points of the associated Shimura varieties without additional complications, and in a functorial way, we will restrict to pure Shimura data (G, X) which satisfy the following assumption (cfr. [BW04, pag. 367]):

(+) the neutral connected component $Z(G)^0$ of the center $Z(G)$ of G is, up to isogeny, the direct product of a $\mathbb{Q}$-split torus with a torus T of compact type (i.e. $T(\mathbb{R})$ is compact) defined over $\mathbb{Q}$.

This condition ensures, for example, that the arithmetic subgroups Γ_i appearing in (2.3). are identified with the fundamental groups of the connected components $\Gamma_i \backslash D$ of the $\mathbb{C}$-valued points of the Shimura varieties under consideration ([Mil90, Proof of Prop. II.3.3]).

Example 2.1.3.2. In order to get couples (G, X) verifying the above condition (+), we slightly modify the Shimura data introduced in Example 2.1.2.2.(2), in the following way.

(1) Consider the canonical adjunction morphism $i : \mathbb{G}_m \to \mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F}$, induced by the fact that Weil restriction is right adjoint to base change. If $\tilde{G} := \mathrm{Res}_{F|\mathbb{Q}}\mathrm{GL}_{2,F}$, define G by

$$G := \mathbb{G}_m \times_{\mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F}} \tilde{G}, \tag{2.7}$$

where the fibered product has been taken with respect to the morphisms i and $\det : \tilde{G} \to \mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F}$; then define X exactly as in the first part of Example 2.1.2.2.(2), but as a $G(\mathbb{R})$-conjugacy class, with respect to the new G. The Shimura datum (G, X) verifies now the desired condition (cfr. Rmk. 3.2.1.3 for some comments on the difference with the Shimura datum utilising the "classical" G).

(2) More generally, if $\tilde{G} := \mathrm{Res}_{F|\mathbb{Q}}\mathrm{GSp}_{2n,F}$, define G as in (2.7), this time taking the fibered product with respect to the morphisms i and $\nu : \tilde{G} \to \mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F}$, which is the *multiplier morphism* coming from the definition of $\tilde{G}$; one then defines X as in the second part of Example 2.1.2.2.(2), obtaining a Shimura datum sastisfying the desired condition (this will follow from Rmk. 4.1.0.1.(2)).

Let now (G, X) be a Shimura datum satisfying condition (+), K a neat compact open subgroup of $G(\mathbb{A}_f)$, and S_K the corresponding Shimura variety. For any subfield L of $\mathbb{C}$, denote by $\mathrm{Rep}(G_L)$ the Tannakian category of algebraic representations of G in finite dimensional L-vector spaces, and by $\mathrm{LocSys}_L(S_K(\mathbb{C}))$ the category of local systems of L-vector spaces on $S_K(\mathbb{C})$. Thanks to the above considerations, Construction 2.1.1.6 generalizes immediately to non-connected analytic Shimura varieties:

Definition 2.1.3.3. *The* topological canonical construction *is the functor*

$$\mu^K_{\mathrm{top}} : \mathrm{Rep}(G_L) \to \mathrm{LocSys}_L(S_K(\mathbb{C})) \tag{2.8}$$

naturally extending construction 2.1.1.6.

Remark 2.1.3.4. For a representation V of G_L, the total space of the local system $\mu^K_{\mathrm{top}}(V)$ is given by

$$G(\mathbb{Q})\backslash V(\mathbb{R}) \times X \times G(\mathbb{A}_f)/K \tag{2.9}$$

with the obvious left action of $G(\mathbb{Q})$ on $V(\mathbb{R})$ defined by the representation V.

Then, because of the fact that we are working with Shimura varieties, we can enrich the previous functor:

Definition 2.1.3.5. *Denote by* $\mathrm{MVar}_L(S_K(\mathbb{C}))$ *the category of* graded-polarizable admissible variations of mixed L-Hodge structure[2] *over* $S_K(\mathbb{C})$. *We call* Hodge canonical construction functor *the exact tensor functor (cfr. [Wil97, Part II, Chap. 2])*

$$\mu^K_H : \mathrm{Rep}(G_L) \to \mathrm{MVar}_L(S_K(\mathbb{C})). \tag{2.10}$$

naturally enriching the topological canonical construction.

[2] If L is not a subfield of $\mathbb{R}$, this category is obtained from the analogous category over $\mathbb{Q}$ by tensoring morphisms with L, and passing to the pseudo-abelian completion. Also notice that, since G_L is reductive, every object in the essential image of this functor will actually be semisimple.

This admits an étale ℓ-adic analogue (over the E-variety S_K): under our assumptions, by a construction analogous to the one leading to the scheme introduced in (2.6), there is a pro-finite étale Galois covering

$$\varprojlim_{K' \leq K} S_{K'} \to S_K$$

of Galois group K (cfr. [Pin92, (3.8)]). Since a $\mathbb{Q}_\ell$-representation of G gives rise to a representation of K through $K \hookrightarrow G(\mathbb{A}_f) \twoheadrightarrow G(\mathbb{Q}_\ell)$, we obtain:

Definition 2.1.3.6. *Suppose that L is a finite extension of $\mathbb{Q}$. Let ℓ be a fixed prime, l a fixed prime of L above ℓ, and $\mathrm{Et}_{\ell,L}(S_K)$ the L_l-linear version of the category of* lisse ℓ-adic sheaves *over S_K. We call* ℓ-adic canonical construction functor[3] *the exact tensor functor (cfr.[Wil97, Part II, Chap. 4])*

$$\mu_\ell^K : \mathrm{Rep}(G_L) \to \mathrm{Et}_{\ell,L}(S_K) \tag{2.11}$$

defined as explained above.

But what about the analogy with *Hodge* theory, i.e. the *mixed* structure? In this sense, the correct target category of the above functor should be the following (cfr. [Wil97, Part I, Definition before Theorem 2.8]):

Definition 2.1.3.7. *The category of* mixed lisse sheaves with a weight filtration *is the full subcategory $\mathrm{Et}^M_{\ell,L}(S_K)$ of $\mathrm{Et}_{\ell,L}(S_K)$ formed by objects $\mathbb{V}$ for which there exists a finite set $S \subset \mathrm{Spec}(O_E)$ containing the primes dividing ℓ, a smooth, separated scheme $\mathfrak{X} \to \mathrm{Spec}(O_S)$ of finite type and a lisse L_l-sheaf $\mathfrak{V}$ on $\mathfrak{X}$ such that*

1. *$S_K = \mathfrak{X} \otimes_{O_S} E$, $\mathbb{V} = \mathfrak{V} \otimes_{O_S} E$;*
2. *the model $\mathfrak{V}$ of $\mathbb{V}$ admits an ascending finite filtration $(\mathbb{W}_n\mathfrak{V})_{n\in\mathbb{Z}}$ (the* weight filtration*) such that*
$$\mathrm{Gr}_n^{\mathbb{W}} \mathfrak{V} := \mathbb{W}_n\mathfrak{V}/\mathbb{W}_{n-1}\mathfrak{V}$$
is pure of weight n *for all n, i.e. for each closed point x of $\mathfrak{X}$, the eigenvalues of the action of the geometrical Frobenius on the stalk at x are pure of weight n.*

Note that the above definition is *stronger* than the definition of a *mixed ℓ-adic sheaf* in [Del80, Déf. 1.2.2] (which only asks for *some* finite filtration with pure quotients, not necessarily *ascending*) and that, contrary to the Hodge-theoretical picture, the fact that the objects obtained through the ℓ-adic canonical construction are mixed with a weight filtration is by no means clear.

Remark 2.1.3.8. (1) By results of Pink [Pin92, Proposition (5.6.2)], if the Shimura datum (G, X) is of *abelian type*, then the essential image of the ℓ-adic canonical construction lies in the subcategory of mixed sheaves with a weight filtration. We won't recall the definition of abelian type here, but we notice that all Shimura data we will make use of will be of *PEL type* (cfr. subsection 3.2.1), hence of abelian type.

(2) If (G, X) is of abelian type and $V \in \mathrm{Rep}(G_L)$, then the results of Pink in *loc. cit.* also imply that the weights of $\mu_\ell^K(V)$ as an object of $\mathrm{Et}^M_{\ell,L}(S_K)$ are identical to the weights of $\mu_H^K(V)$ as a variation of mixed Hodge structure.

[3] We will systematically abuse of notation and forget the choice of l.

(3) Suppose G to be split over L and V to be an *irreducible* representation of G_L. Then, the Hodge structure induced on V (cfr. the discussion after 2.1.3.1) is pure. If the irreducible representation V is of highest weight λ, we will often denote it by V_λ and by $w(\lambda)$ its weight as a pure Hodge structure.

Remark 2.1.3.9. (1) In the following we will sometimes abusively denote by the same symbol μ_H^K the obvious factorization of the latter functor through the category $\mathrm{Gr}_{\mathbb{Z}}\mathrm{Var}_L(S_K(\mathbb{C}))$ (where $\mathrm{Var}_L(S_K(\mathbb{C}))$ is the category of *pure* polarizable variations of L-Hodge structure). If (G, X) is a Shimura datum of abelian type, the above considerations will give us an obvious factorisation of the functor μ_ℓ^K through the category $\mathrm{Gr}_{\mathbb{Z}}\mathrm{Et}_{\ell,L}(S_K)$, still abusively denoted by μ_ℓ^K.

(2) The exact functor μ_H^K extends to a triangulated functor, denoted by the same symbol,

$$\mu_H^K : D^b(\mathrm{Rep}(G_L)) \to D^b(MHM(S_K(\mathbb{C}))_L, \tag{2.12}$$

where $D^b(MHM(S_K(\mathbb{C}))_L$ denotes the bounded derived category of *mixed Hodge modules* (with L-coefficients) over $S_K(\mathbb{C})$ ([Sai90]). Analogously, the exact functor μ_ℓ^K extends to a triangulated functor, denoted by the same symbol,

$$\mu_\ell^K : D^b(\mathrm{Rep}(G_L)) \to D^b_{c,\text{ét}}(S_K)_L, \tag{2.13}$$

where $D^b_{c,\text{ét}}(S_K)_L$ is the L_l-linear version of the bounded "derived" category of ℓ-adic constructible sheaves over S_K (cfr. [Eke90, Section 6]).

Now let $\mathbb{D}_{S_K}$ denote either the duality endofunctor on $D^b(MHM(S_K(\mathbb{C}))_L$ or, if L is a finite extension of $\mathbb{Q}$, ℓ is a prime and l a prime of L above ℓ, the ℓ-adic local duality endofunctor over S_K. Denote by μ^K the triangulated extension either of the Hodge or of the ℓ-adic canonical construction, and by $\bar{L}$ either the field L (over which G splits) or the field L_l. We end our recollections about the canonical construction with the following standard property:

Proposition 2.1.3.10. *Let (G, X) be a Shimura datum satisfying condition $(+)$, K a neat compact open subgroup of $G(\mathbb{A}_f)$, and S_K the corresponding Shimura variety (of dimension d_{S_K}). Let moreover V be an irreducible representation of G_L of weight w as a pure Hodge structure (cfr. Rmk. 2.1.3.8.(3)). Then there is a canonical isomorphism*

$$\mathbb{D}_{S_K}(\mu^K(V)) \simeq \mu^K(V)(w + d_{S_K})[2d_{S_K}].$$

Proof. Since S_K is smooth of dimension d_{S_K}, there exists an isomorphism

$$\mathbb{D}_{S_K}(\mu^K(V)) \simeq R\mathcal{H}om(\mu^K(V), \bar{L})(d_{S_K})[2d_{S_K}].$$

On the other hand, following [Pin90, Summary 1.18 (d)], the Tate Hodge structure $L(-w)$ is endowed with a natural G_L-representation, such that there exists a perfect pairing in $\mathrm{Rep}(G_L)$

$$V \otimes_L V \to L(-w)$$

Hence, we have a perfect pairing

$$\mu^K(V) \otimes_{\bar{L}} \mu^K(V) \to \bar{L}(-w)$$

of variations of Hodge structures over $S_K(\mathbb{C})$ or of lisse ℓ-adic sheaves over S_K, which gives a second isomorphism

$$\mu^K(V) \simeq R\mathcal{H}om(\mu^K(V), \bar{L})(-w).$$

So, these isomorphisms yield the desired one. □

2.2 Cohomology and automorphic representations

For all of this section, let (G, X) be a Shimura datum satisfying condition $(+)$, K a neat compact open subgroup of $G(\mathbb{A}_f)$, and S_K the corresponding Shimura variety, as in the previous subsection. Now that we have established a link between representation theory and mixed sheaves over Shimura varieties, we want to study the structures appearing in the cohomology of such sheaves, and their mutual interaction.

2.2.1 Hecke operators

The first structure we have to define is the action of *Hecke operators* in their various incarnations. For any $g \in G(\mathbb{A}_f)$, consider the diagram of (finite, étale) morphisms, as defined in (2.4),

$$\begin{array}{ccccc} & & S_{K\cap gKg^{-1}} & & \\ & \overset{[\cdot 1]}{\swarrow} & & \overset{[\cdot g]}{\searrow} & \\ S_K & & & & S_K \end{array}$$

For any subfield L of $\mathbb{C}$ (resp. L finite extension of $\mathbb{Q}$, ℓ any fixed prime), for any representation $V \in \operatorname{Rep}(G_L)$, if μ^K denotes the topological canonical construction (resp. the ℓ-adic canonical construction) defined in Subsection 2.1.3, then there are canonical isomorphisms

$$[\cdot 1]^*\mu^K(V) \xrightarrow[\simeq]{\theta_1^{-1}} \mu^{K\cap gKg^{-1}}(V) \xrightarrow[\simeq]{\theta_g} [\cdot g]^*\mu^K(V), \qquad \theta = \theta_g \circ \theta_1^{-1}$$

of local systems over $S_{K\cap gKg^{-1}}(\mathbb{C})$ (resp. of étale lisse ℓ-adic sheaves over $S_{K\cap gKg^{-1}}$). In the topological case, the isomorphism θ_g is given on the underlying total spaces (cfr. Rmk. 2.9) by

$$\begin{aligned} \mu_{\text{top}}^{K\cap gKg^{-1}}(V) &\to [\cdot g]^*\mu_{\text{top}}^K(V) \\ [v, x, h] &\mapsto ([v, x, hg], [x, h]) \end{aligned} \tag{2.14}$$

where the symbols $[\cdot,\cdot,\cdot]$ or $[\cdot,\cdot]$ denote the appropriate equivalence classes.

Thus, if $H^\cdot(S_K, \mu^K(V))$ denotes Betti cohomology of local systems over $S_K(\mathbb{C})$ (resp. étale ℓ-adic cohomology $H^\cdot_{\text{ét}}$ of étale ℓ-adic sheaves over $S_{K,\bar{\mathbb{Q}}}$), we get isomorphisms (abusively denoted with the same symbol)

$$\theta : H^\cdot(S_{K\cap gKg^{-1}}, [\cdot 1]^*\mu^K(V)) \simeq H^\cdot(S_{K\cap gKg^{-1}}, [\cdot g]^*\mu^K(V)).$$

Definition. *Let $g \in G(\mathbb{A}_f)$ and consider (with a slight abuse of notation) the canonical adjunction morphisms*

$$[\cdot 1]^* : H^\cdot(S_K, \mu^K(V)) \to H^\cdot(S_K, [\cdot 1]_*[\cdot 1]^*\mu^K(V)) \simeq H^\cdot(S_{K\cap gKg^{-1}}, [\cdot 1]^*\mu^K(V))$$

and (remembering that $[\cdot g]$ is finite)

$$[\cdot g]_* : H^\cdot(S_{K\cap gKg^{-1}}, [\cdot g]^*\mu^K(V)) \simeq H^\cdot(S_K, [\cdot g]_*[\cdot g]^*\mu^K(V)) \to H^\cdot(S_K, \mu^K(V))$$

Then, the Hecke operator $T_{K,g} \in \operatorname{End} H^\cdot(S_K, \mu^K(V))$ associated to g *is defined by*

$$T_{K,g} := [\cdot g]_* \circ \theta \circ [\cdot 1]^*. \tag{2.15}$$

Remark 2.2.1.1. For any finite extension L of $\mathbb{Q}$ and any prime l of L above any fixed prime ℓ, the action of T_g respects the comparison isomorphisms

$$H^{\cdot}(S_K(\mathbb{C}), \mu^K_{\text{top}}(V)) \otimes_L L_l \simeq H^{\cdot}_{\text{ét}}(S_{K,\bar{\mathbb{Q}}}, \mu^K_\ell(V)) \tag{2.16}$$

Then, we define:

Definition 2.2.1.2. *For any fixed L-representation V, the* Hecke algebra $\mathcal{H}(K, G(\mathbb{A}_f))$ *(associated to V) is the L-subalgebra of* $\operatorname{End} H^{\cdot}(S_K(\mathbb{C}), \mu^K_{\text{top}}(V))$ *generated by the operators $T_{K,g}$ with $g \in G(\mathbb{A}_f)$.*

By the previous remark, it is consistent to define the ℓ-adic counterpart of the Hecke algebra either as an analogous L_l-subalgebra of endomorphisms of ℓ-adic étale cohomology or as $\mathcal{H}(K, G(\mathbb{A}_f))_{L_l}$. We will encounter the motivic avatar of these algebras, defined as an algebra of correspondences, in Subsection 3.2.2.

The action of the operators $T_{K,g}$ is compatible with the transition maps of the $G(\mathbb{A}_f)$-projective system (2.5), resp. (2.6), of complex analytic spaces, resp. of finite-type E-schemes. Then, we can let $g \in G(\mathbb{A}_f)$ act through the $T_{K,g}$'s on the corresponding projective system of spaces and *sheaves*. Letting g vary, we get an action of $G(\mathbb{A}_f)$ on

$$H^{\cdot}(S, \mu(V)) := \varinjlim_K H^{\cdot}(S_K, \mu^K(V))$$

(either Betti or ℓ-adic cohomology).

Remark 2.2.1.3. Recall that a $G(\mathbb{A}_f)$-module H is called *smooth* if the stabilizer of any element in H is open in $G(\mathbb{A}_f)$. Denote by H the space $H^{\cdot}(S, \mu(V))$ (either Betti or ℓ-adic cohomology). Then, the above $G(\mathbb{A}_f)$-action makes of H an *admissible* $G(\mathbb{A}_f)$-module, i.e. a smooth $G(\mathbb{A}_f)$-module such that, for any compact open $K \leq G(\mathbb{A}_f)$, the K-invariants are finite dimensional; this follows from the fact that, for any such K,

$$H^{\cdot}(S, \mu(V))^K \simeq H^{\cdot}(S_K, \mu^K(V)). \tag{2.17}$$

canonically.

Lemma 2.2.1.4. *Let $C_c^\infty(G(\mathbb{A}_f))$ be the algebra of compactly supported, locally constant L-valued (or L_l-valued) functions on $G(\mathbb{A}_f)$ under convolution and let $C_c^\infty(G(\mathbb{A}_f)//K)$ denote its subalgebra of bi-invariant functions under K. Then, the space $H^{\cdot}(S, \mu(V))^K$ has a natural structure of $C_c^\infty(G(\mathbb{A}_f)//K)$-module.*

Proof. This results from considering the equivalence of categories given in [Car79, 1.4(d)] between smooth $G(\mathbb{A}_f)$-modules and *non-degenerate* (*loc. cit*, page 113) $C_c^\infty(G(\mathbb{A}_f))$-modules, which takes admissible modules to admissible ones, and passing to K-invariants. □

Remark 2.2.1.5. The isomorphism (2.17) intertwines the $C_c^\infty(G(\mathbb{A}_f)//K)$-module structure on the left-hand side with the $\mathcal{H}(K, G(\mathbb{A}_f))$-action on the right hand side in the following way: for any $g \in G(\mathbb{A}_f)$, the characteristic function $\mathbb{1}_{KgK}$ of the double coset KgK is transported to the operator $T_{K,g}$, compatibly with the algebra structure.

2.2.2 Automorphic representations and Galois representations

Keep the notations established in the previous subsection. Having endowed the cohomology spaces $H^\cdot(S_K, \mu^K(V))$ with the structure of $C_c^\infty(G(\mathbb{A}_f)//K)$-modules, we are now interested in decomposing them, if possible, into irreducible submodules, or in at least obtaining such a decomposition for some interesting subspace.

We start with the analytic picture. If $A^\cdot(S_K(\mathbb{C}), V)$ denotes the complex of C^∞-differential forms over $S_K(\mathbb{C})$ with coefficients in (the C^∞-vector bundle underlying) $\mu^K_{\text{top}}(V_\mathbb{C})$, then the de Rham theorem provides us with a canonical isomorphism

$$H^\cdot(S_K(\mathbb{C}), \mu^K_{\text{top}}(V_\mathbb{C})) \simeq H^\cdot(A^\cdot(S_K(\mathbb{C}), V)) \tag{2.18}$$

On the other hand, write

$$\mathcal{A} := C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K)$$

where $C^\infty(\cdot)$ means that we are taking functions induced by locally constant (resp. smooth) $\mathbb{C}$-valued functions over the non-archimedean (resp. archimedean, regarded as smooth manifolds) components of $G(\mathbb{A})$. Denote now by $\mathfrak{g}$ the complexification of the Lie algebra of $G(\mathbb{R})$ and recall the subgroup K^h_∞ of $G(\mathbb{R})$ introduced in Remark 2.1.2.1.(2). Then, the space $\mathcal{A}$ is a $(\mathfrak{g}, K^h_\infty)$-*module*[4] ([BW00, 0, 2.5]), equipped with a natural $C_c^\infty(G(\mathbb{A}_f)//K)_\mathbb{C}$-action (coming from the natural $G(\mathbb{A}_f)$-action on $C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$ and the equivalence of categories of Lemma 2.2.1.4). The same considerations hold for the space $\mathcal{A} \otimes V_\mathbb{C}$, and we can look at the complex

$$C^\cdot(\mathfrak{g}, K^h_\infty; \mathcal{A} \otimes V_\mathbb{C})$$

computing its $(\mathfrak{g}, K^h_\infty)$-*cohomology* ([BW00, I, 5.1]). Then, we have the following:

Proposition 2.2.2.1. *(adelic version of [BW00, VII, Corollary 2.7]) There is a canonical isomorphism of graded complexes*

$$A^\cdot(S_K(\mathbb{C}), V) \simeq C^\cdot(\mathfrak{g}, K^h_\infty; \mathcal{A} \otimes V_\mathbb{C})$$

inducing (through the isomorphism (2.18)) canonical isomorphisms of cohomology groups

$$H^\cdot(S_K(\mathbb{C}), \mu^K_{\text{top}}(V_\mathbb{C})) \simeq H^\cdot(\mathfrak{g}, K^h_\infty; \mathcal{A} \otimes V_\mathbb{C}) \tag{2.19}$$

commuting with the $C_c^\infty(G(\mathbb{A}_f)//K)_\mathbb{C}$-actions on both sides.

Remark 2.2.2.2. On the topological side, nothing that has been said until this point in this section has used the fact that we are working with Shimura varieties.

This general interpretation of Betti cohomology of the topological canonical construction brings us closer to the link with automorphic representations. Fix a translation-invariant measure dg on $G(\mathbb{A})$ and let $G(\mathbb{A})$ act by right translation on the space

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$$

[4] We flag the following oversimplifications in our exposition, which however will play no role: (*a*) the group K^h_∞ is in general only compact *modulo center* (cfr. Footnote 1), whereas strictly speaking $(\mathfrak{g}, K)$-modules are defined for *compact* subgroups K, but one can work in this slightly extended setting without any problems; (*b*) it is only a subspace of $\mathcal{A}$ which is a $(\mathfrak{g}, K^h_\infty)$-module, i.e. its subspace of K^h_∞-*finite* vectors. However, passing to this subspace doesn't affect $(\mathfrak{g}, K^h_\infty)$-cohomology; (*c*) let Z be the center of G. For the following results to be true as stated, $\mathcal{A}$ should be chosen more precisely as the space of functions which transform, under $Z(\mathbb{R})$-translation, via *multiplication by the inverse of the central character* of the representation $V_\mathbb{C}$.

of complex-valued functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ which are square-integrable *modulo center* with respect to dg, and which transform under the action of $Z(\mathbb{A})$ via multiplication by some character $w : Z(\mathbb{Q})\backslash Z(\mathbb{A}) \to \mathbb{C}^\times$ (here Z is the center of G). For us, an (irreducible) *automorphic representation* of G will be an irreducible subquotient of this $G(\mathbb{A})$-module. An automorphic representation π can be decomposed as $\pi = \pi_f \otimes \pi_\infty$ in its non-archimedean (or *finite*) and archimedean components; the spaces π_f^K of K-invariants are *irreducible* $C_c^\infty(G(\mathbb{A}_f)//K)_{\mathbb{C}}$-modules and the subspaces of *smooth vectors* (or *differentiable* vectors, [BW00, 0, 2.3]) in each space π_∞ give rise to $(\mathfrak{g}, K_\infty^h)$-modules, that we will still denote by π_∞. Then, the spaces $\pi_\infty \otimes V_{\mathbb{C}}$ inherit a $(\mathfrak{g}, K_\infty^h)$-module structure, too.

To see why such representations appear in our cohomology spaces, we have to introduce a new definition:

Definition 2.2.2.3. *Let $A_2^\cdot(S_K(\mathbb{C}), V)$ be the complex of $\mu_{\mathrm{top}}^K(V_{\mathbb{C}})$-valued, C^∞-differential forms ω over $S_K(\mathbb{C})$ such that both ω and $d\omega$ are square-integrable with respect to the canonical volume form on $S_K(\mathbb{C})$. The* L^2-cohomology of $S_K(\mathbb{C})$ with values in V *is by definition the cohomology $H_2^\cdot(S_K(\mathbb{C}), \mu_{\mathrm{top}}^K(V_{\mathbb{C}}))$ of the complex $A_2^\cdot(S_K(\mathbb{C}), V)$.*

Denote by $L^{2,\infty}$ the subspace of smooth vectors inside $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K)$ and consider the $(\mathfrak{g}, K_\infty^h)$-module (arising from) $L^{2,\infty} \otimes V_{\mathbb{C}}$. Then, isomorphism (2.19) generalizes to ([Bor83, Thm. 3.5])

$$H_2^\cdot(S_K(\mathbb{C}), \mu_{\mathrm{top}}^K(V_{\mathbb{C}})) \simeq H^\cdot(\mathfrak{g}, K_\infty^h; L^{2,\infty} \otimes V_{\mathbb{C}}) \tag{2.20}$$

Now recall that the $G(\mathbb{A})$-module $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ contains a maximal subspace

$$L_d^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$$

which decomposes as a *direct sum* of irreducible $G(\mathbb{A})$-subrepresentations, called the *discrete spectrum*. At finite levels, we can write such a decomposition as

$$L_d^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K) = \bigoplus_\pi (\pi_f^K \otimes \pi_\infty)^{m(\pi)}$$

where $m(\pi)$ denotes the (finite) multiplicity of an automorphic representation π appearing in the discrete spectrum. Then, *since Shimura varieties verify the necessary hypotheses*, passing to smooth vectors and applying $(\mathfrak{g}, K_\infty^h)$-cohomology yields the same result as if we had started with the whole of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K)$ and applied the same operations, i.e. the complement of the discrete spectrum (the *continuous spectrum*) doesn't contribute to $(\mathfrak{g}, K_\infty^h)$-cohomology. In other words, the following holds:

Proposition 2.2.2.4. *(consequence of [BC83, Sect. 4]) The isomorphism (2.20) and the $G(\mathbb{A}_f)$-action on the underlying complexes induce a decomposition into a* finite *direct sum*

$$H_2^\cdot(S_K(\mathbb{C}), \mu_{\mathrm{top}}^K(V_{\mathbb{C}})) \simeq \bigoplus_{\pi \ \text{discrete spectrum}} (\pi_f^K \otimes H^\cdot(\mathfrak{g}, K_\infty^h; \pi_\infty \otimes V_{\mathbb{C}}))^{m(\pi)} \tag{2.21}$$

which is called the Matsushima-Murakami decomposition *of L^2-cohomology.*

The natural arrow

$$H_2^\cdot(S_K(\mathbb{C}), \mu_{\mathrm{top}}^K(V_{\mathbb{C}})) \to H^\cdot(S_K(\mathbb{C}), \mu_{\mathrm{top}}^K(V_{\mathbb{C}})) \tag{2.22}$$

induced by the de Rham isomorphisms is in general neither injective nor surjective. But one can restrict attention to a subspace of the discrete spectrum, the one formed by cuspidal representations: an automorphic representation is called *cuspidal* if it is a *sub-representation* A of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ such that for every $\phi \in A$, for all parabolic subgroups P of G with unipotent radical U, and for every $g \in G(\mathbb{A}_f)$,

$$\int_{U(\mathbb{A}_f)} \phi(ng)dn = 0$$

Definition. *The* cuspidal cohomology *(with values in $V_{\mathbb{C}}$) is the subspace of L^2-cohomology defined by*

$$H^{\cdot}_{cusp}(S_K(\mathbb{C}), \mu^K_{\text{top}}(V_{\mathbb{C}})) := \bigoplus_{\pi \ cuspidal} (\pi^K_f \otimes H^{\cdot}(\mathfrak{g}, K^h_\infty; \pi_\infty \otimes V_{\mathbb{C}}))^{m(\pi)} \tag{2.23}$$

and given by the direct sum of those subspaces which correspond to cuspidal discrete spectrum automorphic representations.

Then, the crucial point is the following:

Proposition 2.2.2.5. *([Bor81, Cor. 5.5]) The natural arrow (2.22) induces an injection of $C^\infty_c(G(\mathbb{A}_f)//K)_{\mathbb{C}}$-modules*

$$H^{\cdot}_{cusp}(S_K(\mathbb{C}), \mu^K_{\text{top}}(V_{\mathbb{C}})) \hookrightarrow H^{\cdot}(S_K(\mathbb{C}), \mu^K_{\text{top}}(V_{\mathbb{C}})) \tag{2.24}$$

Hence, cuspidal cohomology is the sought-for subspace of ordinary cohomology whose decomposition in $C^\infty_c(G(\mathbb{A}_f)//K)_{\mathbb{C}}$-submodules can be meaningfully described.

We will see later (cfr. Eq. (2.31)) that we can be more precise about the the way in which cuspidal cohomology sits inside the ordinary one. Before addressing this point, let us say a word about rationality issues. Since there is already an action of the L-algebra $C^\infty_c(G(\mathbb{A}_f)//K)$ on the L-vector spaces $H^{\cdot}(S_K(\mathbb{C}), \mu^K_{\text{top}}(V))$, we have the following:

Lemma 2.2.2.6. *There exist a finite extension L' of L such that, for any cuspidal automorphic representation π such that π^K_f appears as a $C^\infty_c(G(\mathbb{A}_f)//K)_{\mathbb{C}}$-submodule in $H^{\cdot}(S_K(\mathbb{C}), \mu^K_{\text{top}}(V_{\mathbb{C}}))$, π^K_f admits an L'-model, still denoted by the same symbol.*

Notice that a cuspidal π verifies the above hypothesis if and only if its finite component π_f has non vanishing K-invariants and its infinite component has non-vanishing $(\mathfrak{g}, K^h_\infty)$-cohomology. Now, by writing

$$H^{\cdot}(\pi^K_f) := \text{Hom}_{C^\infty_c(G(\mathbb{A}_f)//K)_{L'}}(\pi^K_f, H^{\cdot}(S_K, \mu^K_{\text{top}}(V) \otimes L')) \tag{2.25}$$

we can define the cuspidal cohomology with L'-coefficients as

$$H^{\cdot}_{\text{cusp}}(S_K(\mathbb{C}), \mu^K_{\text{top}}(V)) := \bigoplus_{\pi \text{ cuspidal}} (\pi^K_f \otimes H^{\cdot}(\pi^K_f)) \tag{2.26}$$

This also allows us to consider the relationship with étale ℓ-adic cohomology. As seen in Subsection 2.2.1, $H^{\cdot}_{\text{ét}}(S_{K,\bar{\mathbb{Q}}}, \mu^K_\ell(V))$ is a $C^\infty_c(G(\mathbb{A}_f)//K)_{L_l}$-module; since it is also equipped with a $\text{Gal}(\bar{\mathbb{Q}}, E)$-action commuting with the Hecke action, we can look at it as a $C^\infty_c(G(\mathbb{A}_f)//K)_{L_l} \times \text{Gal}(\bar{\mathbb{Q}}, E)$-module. Let L' be the number field of the above lemma, use the same symbol l for a place of L' above ℓ, and write π^K_{f,L'_l} for $\pi^K_f \otimes_{L'} L'_l$. We can then pose the following:

Definition 2.2.2.7. *Let π be a cuspidal automorphic representation such that π_f^K appears as a $C_c^\infty(G(\mathbb{A}_f)//K)_\mathbb{C}$-submodule in $H^\cdot(S_K(\mathbb{C}), \mu^K_{\mathrm{top}}(V_\mathbb{C}))$. The* $\mathrm{Gal}(\bar{\mathbb{Q}}, E)$-representation *(over L'_l)* associated to π_f^K *is defined as*

$$H^\cdot_{\acute{e}t}(\pi_f^K) := \mathrm{Hom}_{C_c^\infty(G(\mathbb{A}_f)//K)_{L'_l}}(\pi^K_{f,L'_l}, H^\cdot_{\acute{e}t}(S_{K,\bar{\mathbb{Q}}}, \mu^K_\ell(V)) \otimes L'_l) \tag{2.27}$$

By choosing an embedding of L_l in $\mathbb{C}$ we get a $C_c^\infty(G(\mathbb{A}_f)//K)_\mathbb{C}$-equivariant comparison isomorphism

$$H^\cdot_{\text{ét}}(S_{K,\bar{\mathbb{Q}}}, \mu^K_\ell(V)) \otimes_{L_l} \mathbb{C} \simeq H^\cdot(S_K(\mathbb{C}), \mu^K_{\mathrm{top}}(V_\mathbb{C})) \tag{2.28}$$

induced by (2.16). Hence:

Corollary 2.2.2.8. *Define the* cuspidal étale cohomology *by*

$$H^\cdot_{cusp,\ \acute{e}t}(S_{K,\bar{\mathbb{Q}}}, \mu^K_\ell(V)) := \bigoplus_{\pi \ cuspidal} (\pi^K_{f,L'_l} \otimes H^\cdot_{\acute{e}t}(\pi_f^K)) \tag{2.29}$$

Then, the isomorphism (2.28) *restricts to*

$$H^\cdot_{cusp,\ \acute{e}t}(S_{K,\bar{\mathbb{Q}}}, \mu^K_\ell(V)) \otimes_{F_l} \mathbb{C} \simeq H^\cdot_{cusp}(S_K(\mathbb{C}), \mu^K_{\mathrm{top}}(V_\mathbb{C}))$$

and we have the relation

$$\dim_{L'_l} H^\cdot_{\acute{e}t}(\pi_f^K) = \sum_{\pi_\infty \ s.t. \ \pi \simeq \pi_f \otimes \pi_\infty} m(\pi_f \otimes \pi_\infty) \dim_\mathbb{C} H^\cdot(\mathfrak{g}, K^h_\infty; \pi_\infty \otimes V_\mathbb{C})$$

These considerations explain why the cohomology of Shimura varieties provides a vital link between Galois representations and (at least cuspidal) automorphic representations.

2.3 Weights and automorphic representations

The last layer of structure that we have to add on the cohomology of Shimura varieties, and the one that will be the most important for this thesis, is given by the *theory of weights*. Let us phrase it in the language of Hodge structures.

2.3.1 Weight zero and cuspidal cohomology

Take (G, X) satisfying condition $(+)$, K and an algebraic representation V of G over a number field L as in Subsection 2.1.3, and, as there, consider the Hodge canonical construction μ^K_H. Suppose V to be irreducible, of weight w as a pure Hodge structure (cfr. Rmk. 2.1.3.8.(3)). Then (for example as a byproduct of Saito's theory of *mixed Hodge modules*), in each degree n, the cohomology spaces $H^n(S_K(\mathbb{C}), \mu^K_H(V))$ are endowed with a *mixed Hodge structure* over L, of weights $\geq n + w$ (since S_K is smooth), hence, in particular, with a *weight filtration* $\mathbb{W}$.

On the other hand, the compactly-supported cohomology spaces $H^n_c(S_K(\mathbb{C}), \mu^K_H(V))$ are endowed with a mixed Hodge structure of weights $\leq n + w$, and, the canonical arrow $H^n_c \to H^n$ being a *morphism of mixed Hodge structures*, we have that, in each degree n, the so called *interior cohomology*, i.e. the subspace of H^n defined as

$$H^n_! := \mathrm{Im}(H^n_c \to H^n) \tag{2.30}$$

actually injects in the subspace $\mathrm{Gr}^{\mathbb{W}}_{n+w} H^n(S_K(\mathbb{C}), \mu^K_H(V))$, i.e. in the *lowest-weight* subquotient of H^n with respect to the weight filtration. We will often slightly improperly refer to this subspace as the *pure* or *weight zero* part of cohomology, since in this subspace, up to the contribution w coming from our representation, the deviation between weight and cohomological degree is zero.

Now, employing the cuspidal cohomology with L'-coefficients defined in Eq. (2.26), the link with the automorphic description of cohomology is given by the following consequence of Prop. 2.2.2.5:

Lemma 2.3.1.1. *The inclusion* (2.24) *refines to an inclusion*

$$H^n_{cusp} \hookrightarrow H^n_! \otimes L' \hookrightarrow (\mathrm{Gr}^{\mathbb{W}}_{n+w} H^n) \otimes L' \tag{2.31}$$

By the comparison isomorphism (2.16) and the decomposition (2.29), we see that the understanding of the weight zero part of cohomology is relevant for the study of the Galois representations associated to *cuspidal automorphic representations*. (See Rmk. 2.3.4.4.(3) for a comment on the relation with the more classical notion of Galois representations associated to *cusp forms*).

Notice that that since the Hecke algebra acts by morphisms of Hodge structure, a cuspidal representation π appearing inside the interior cohomology of S_K gives also rise to a (pure) Hodge structure:

Definition 2.3.1.2. *Let π be a cuspidal automorphic representation such that π^K_f appears in $H^\cdot(S_K, \mu^K_{\mathrm{top}}(V))$. Let L' be the number field of Lemma 2.2.2.6. The* Hodge structure *$H^\cdot(\pi^K_f)$ associated to π^K_f is the one that the spaces*

$$H^\cdot(\pi^K_f) = \mathrm{Hom}_{C^\infty_c(G(\mathbb{A}_f)//K)_{L'}}(\pi^K_f, H^\cdot(S_K, \mu^K_{\mathrm{top}}(V) \otimes L')) \tag{2.32}$$

(cfr. (2.25)*) are naturally endowed with.*

Remark 2.3.1.3. Observe that since $X = G(\mathbb{Q})\backslash G(\mathbb{A}_f)/KK^h_\infty$ is a disjoint union of hermitian symmetric domains, the spaces $H^\cdot(\mathfrak{g}, K^h_\infty; \pi_\infty \otimes V_\mathbb{C})$ appearing in the Matsushima-Murakami decomposition (2.23) of cuspidal cohomology carry an intrinsic *(p,q)*-decomposition ([BW00, II, 4]). This decomposition coincides, through the isomorphism (2.23), with the Hodge decomposition on $H^\cdot(\pi^K_f)_\mathbb{C}$ coming from the above-defined Hodge structure ([Har94, Prop. 3.3.9]).

2.3.2 Higher weights and boundary cohomology

The previous subsection raises the natural question of what interpretation (if any) one could give to the subquotients of higher weight of the cohomology of Shimura varieties with respect to the weight filtration. Indeed, as we will observe in Rmk. 2.3.2.3, the study of these subquotients will be relevant even for the understanding of the weight zero part itself.

Let us first address the problem from a geometric standpoint. Keep the notations of the preceding subsection and suppose that the $\mathbb{Q}$-rank of G^{ad} is strictly positive; in this case, it is well known that the corresponding Shimura varieties S_K are *non-projective* ([BHC62, Thm. 12.3 and Cor. 12.4]). Then, take any compactification[5] $\bar{S_K}$, write $\partial\bar{S}_K$ for the boundary of S_K in $\bar{S}_K$ and denote by $j : S_K \hookrightarrow \bar{S}_K$ (resp. $i : \partial\bar{S}_K$ the open

[5]In the category of *algebraic varieties*, even if *topological* compactifications of $S_K(\mathbb{C})$ can be very relevant.

(resp. closed) immersions of S_K and of the boundary inside the compactification. Then, the complex $i^* j_* \mu_H^K(V)$ (often called the *degeneration* of $\mu_H^K(V)$ at the boundary) is a *constructible complex* over $\partial \bar{S}_K(\mathbb{C})$, which can be naturally endowed with a structure of a bounded complex of mixed Hodge modules, and whose direct image via i fits in the canonical *boundary triangle* over $\bar{S}_K$

$$j_! \mu_H^K(V) \to j_* \mu_H^K(V) \to i_* i^* j_* \mu_H^K(V) \to j_! \mu_H^K(V)[1] \tag{2.33}$$

This gives rise to a natural long exact sequence

$$\cdots \to H_c^n(S_K(\mathbb{C}), \mu_H^K(V)) \to H^n(S_K(\mathbb{C}), \mu_H^K(V)) \to \partial H^n(S_K(\mathbb{C}), \mu_H^K(V)) \to \cdots \tag{2.34}$$

where the *boundary cohomology* spaces ∂H^n are defined as the hypercohomology

$$\partial H^n(S_K(\mathbb{C}), \mu_H^K(V)) := \mathbb{H}^n(\partial \bar{S}_K(\mathbb{C}), i^* j_* \mu_H^K(V))$$

Notice that

1. the long exact sequence (2.34) is a sequence of *mixed Hodge structures*, and
2. by proper base change, the boundary cohomology spaces $\partial H^n(S_K(\mathbb{C}), \mu_H^K(V))$ don't depend on the chosen compactification, and are hence canonically defined.

This implies, recalling the definition (2.30) of interior cohomology, that we have a canonical inclusion

$$H^n(S_K(\mathbb{C}), \mu_H^K(V)) / H_!^n(S_K(\mathbb{C}), \mu_H^K(V)) \hookrightarrow \partial H^n(S_K(\mathbb{C}), \mu_H^K(V))$$

by means of which, in particular, we can canonically see the higher-weight subquotients with respect to the weight filtration on $H^n(S_K(\mathbb{C}), \mu_H^K(V))$ as subquotients (in the category of Hodge structures) of the boundary cohomology spaces.

Moreover, the complex $i^* j_* \mu_H^K(V)$ has the following *autoduality* property:

Proposition 2.3.2.1. *There is an isomorphism*

$$i^* j_* \mu_H^K(V) \simeq i^! j_! \mu_H^K(V)[1] \tag{2.35}$$

The existence of *some* such isomorphism is formal, once one considers the dual of the boundary triangle (2.33). For later motivation, let us make explicit the following consequence on the weights of boundary cohomology:

Corollary 2.3.2.2. *Let w be the weight of the pure Hodge structure on the irreducible G-representation V and let $\beta \geq 0$ be an integer. If d_{S_K} is the complex dimension of S_K, then weight $w+n+\beta$ appears in $\partial H^n(S_K(\mathbb{C}), \mu_H^K(V))$ if and only if weight $w+m-\beta+1$ appears in $\partial H^m(S_K(\mathbb{C}), \mu_H^K(V))$, for $m = 2d_{S_K} - 1 - n$.*

Proof. Let $s : \bar{S}_K(\mathbb{C}) \to \{\cdot\}$ denote the constant morphism to a one-point space and denote by $\mathbb{D}_{\mathcal{S}}$ the duality functor on the bounded derived category of mixed Hodge modules over a complex variety $\mathcal{S}$. By (2.35),

$$\tilde{s}_* i_* i^* j_* \mu_H^K(V) \simeq \tilde{s}_* i_* i^! j_! \mu_H^K(V)[1]$$

and since, by Proposition 2.1.3.10, we have

$$\mu_H^K(V) \simeq \mathbb{D}_{S_K}(\mu_H^K(V)(w + d_{S_K})[2d_{S_K}])$$

one then gets

$$\tilde{s}_* i_* i^* j_* \mu_H^K(V) \simeq \tilde{s}_* \mathbb{D}_{\bar{S}_K}(i_* i^* j_* \mu_H^K(V)(w + d_{S_K})[2d_{S_K} - 1]) \tag{2.36}$$

Now, by taking cohomology objects over a point, we know that, for every bounded complex $\mathcal{M}$ of mixed Hodge modules over $\bar{S}_K(\mathbb{C})$ (and since s is proper),

$$H^n \tilde{s}_* (\mathbb{D}_{\bar{S}_K} \mathcal{M}) \simeq (H^{-n}(\tilde{s}_* \mathcal{M}))^\vee$$

where $(\cdot)^\vee$ denotes duality of mixed Hodge structures. Hence, Eq. (2.36) yields

$$\partial H^n(S_K(\mathbb{C}), \mu_H^K(V)) \simeq (\partial H^{2d_{S_K}-1-n}(S_K(\mathbb{C}), \mu_H^K(V))(w + d_{S_K}))^\vee$$

and the assertion on weights follows. □

We turn now to a second point of view. The following question is natural: if cuspidal cohomology contributes to weight zero, is there an automorphic interpretation of the higher weights? The best answer to date was provided by Franke, who showed ([Fra98]) that the isomorphism (2.19) can be refined: one can replace $\mathcal{A} \otimes V$ by a smaller $(\mathfrak{g}, K_\infty^h)$-submodule, the space $\mathcal{A}ut \otimes V$, where $\mathcal{A}ut$ is the space of *automorphic forms*[6] on $G(\mathbb{Q})\backslash G(\mathbb{A})/K$ ([FS98, 1.1(2)]), without altering $(\mathfrak{g}, K_\infty^h)$-cohomology, i.e., one has

$$H^\cdot(S_K(\mathbb{C}), \mu_{\mathrm{top}}^K(V_\mathbb{C})) \simeq H^\cdot(\mathfrak{g}, K_\infty^h; \mathcal{A}ut \otimes V_\mathbb{C})$$

More precisely, this isomorphism induces a decomposition

$$H^\cdot(S_K(\mathbb{C}), \mu_{\mathrm{top}}^K(V_\mathbb{C})) \simeq \bigoplus_{\{P\}} H^\cdot(\mathfrak{g}, K_\infty^h; \mathcal{A}ut(P) \otimes V_\mathbb{C}) \tag{2.37}$$

where $\{P\}$ varies over the conjugacy classes of *parabolic* $\mathbb{Q}$-subgroups of G, and $\mathcal{A}ut(P)$ denotes the subspace of automorphic forms *with cuspidal support in* $\{P\}$ ([FS98, 1.1(4)]), in such a way that $\mathcal{A}ut(G)$ coincides with the space of *cuspidal* automorphic forms. (For an observation relating our results on Hilbert-Siegel varieties to this decomposition, see Rmk. 4.3.4.7). In any case, we see that the higher weights are determined by non-cuspidal forms, and indeed the complement to cuspidal cohomology is called *Eisenstein cohomology* (see Remark 2.3.4.4.(3) for a classical instance of this decomposition and its relation with weights). Nevertheless, one should observe that non-cuspidal forms may also contribute to weight zero (see the Remark below).

All these facts lead to the following web of problems: how can one compute the cohomology of the degeneration of the canonical construction to the boundary of a well-chosen compactification and get a grasp on boundary cohomology? Supposing to be able to do this, is there a way to relate this information with the *automorphic* description coming from (2.37)? And how does all of this interact with the weight filtration? It is the determination of the latter that we take as our motivating goal.

Remark 2.3.2.3. Let us say that for an integer $k \geq 0$, *weight* k *appears* in $H^\cdot$, resp. in $\partial H^\cdot$, if there exists a n such that

$$\mathrm{Gr}^{\mathbb{W}}_{n+w+k} H^n(S_K(\mathbb{C}), \mu_H^K(V)), \text{ resp. } \mathrm{Gr}^{\mathbb{W}}_{n+w+k} \partial H^n(S_K(\mathbb{C}), \mu_H^K(V))$$

[6] The relation between these objects and the previously introduced *automorphic representations* is subtle, especially in the non-cuspidal case. We will be careful in distinguishing the two terminologies.

is non trivial. One way to begin our study of the weight filtration would be to ask: which is the first integer $k > 0$ which appears in $H^\cdot$? Of course, weight $k > 0$ appears in ordinary cohomology only if it appears in boundary cohomology.

Observe that Corollary 2.3.2.2 tells us that weight 1 appears in $\partial H^\cdot(S_K(\mathbb{C}), \mu_H^K(V))$ if and only if weight 0 appears. This is the case, for example, if there exists an n such that the complement of $H_!^n$ in $\mathrm{Gr}_{n+w} H^n(S_K(\mathbb{C}), \mu_H^K(V))$ is non-trivial. We will see that this latter condition can be characterized by automorphic means (Remark 2.3.3.6).

2.3.3 The Baily-Borel compactification and intersection cohomology

We start from the *first part* of the questions asked at the end of the previous subsection, namely by laying out the basis for the computation of boundary cohomology.

In fact, in order to do this, we have first to choose a compactification. With the notation of the previous subsection, there is a canonical choice of compactification of the Shimura variety S_K, called its *Baily-Borel compactification*, denoted by S_K^*. It is a projective variety, in general singular, defined over the reflex field E of S_K ([Pin90, Main Theorem 12.3, (a), (b)]), in which S_K embeds as an open dense sub-scheme.

If $\Gamma\backslash D$ is a connected component of $S_K(\mathbb{C})$, then its closure $(\Gamma\backslash D)^*$ in $S_K^*(\mathbb{C})$ (a connected component of $S_K^*(\mathbb{C})$) is still called the Baily-Borel compactification of $\Gamma\backslash D$, and the *canonical* nature of S_K^* comes from the following *minimality property* of $(\Gamma\backslash D)^*$ with respect to simple normal crossing compactifications:

Proposition 2.3.3.1. *([Bor72]) For any simple normal crossing compactification $\bar{S}$ of $\Gamma\backslash D$, there exists an unique morphism $\bar{S} \to (\Gamma\backslash D)^*$ extending the identity on $\Gamma\backslash D$.*

The variety S_K^* admits a stratification by locally closed strata, amongst which S_K is the only open stratum. The other ones form a stratification of the boundary $\partial S_K^* := S_K^* \backslash S_K$. Let us recall, for later use, how the stratification is obtained: if $(Q_m)_{m\in\Phi}$ is any ordering of any (finite) set of representatives of the conjugacy classes of *admissible parabolic subgroups*[7] ([Pin90, Def. 4.5]) in G, one sees from [Pin92, Section 3.6-3.7] that, for all $m \in \Phi$, there exist suitable finite sets $\mathcal{C}_m$ of $G(\mathbb{A}_f)$ such that the set of strata of ∂S_K^* is given by

$$\{S_{m,g} | m \in \Phi, g \in \mathcal{C}_m\}. \tag{2.38}$$

Here, the locally closed subscheme $S_{m,g}$ of ∂S_K^* is the image of a canonical morphism

$$i_g : S_{\pi_m(K_{m,g})} := S_{\pi_m(K_{m,g})}(G_m, X_m) \to S_K^* \tag{2.39}$$

where the compact open subgroup $\pi_m(K_{m,g})$ and the pure Shimura datum (G_m, X_m) defining the Shimura variety $S_{\pi_m(K_{m,g})}$ are given as follows: there exists a canonical normal subgroup P_m of Q_m ([Pin90, 4.7]) whose unipotent radical W_m coincides with the unipotent radical of Q_m (cfr. [Pin90, proof of Lemma 4.8]), and we denote $K_{m,g} := P_m(\mathbb{A}_f) \cap g \cdot K \cdot g^{-1}$, $\pi_m : P_m \to P_m/W_m$ the natural projection, (G_m, X_m) the pure Shimura datum obtained by quotienting by W_m any of the *rational boundary components* ([Pin90, 4.11]) associated to P_m. In particular, G_m is a reductive subgroup of the Levi component of Q_m. In the following, we will rather use the stratification of ∂S_K^* indexed by $m \in \Phi$, each of whose strata Z_m corresponds to the m-th conjugacy class of admissible parabolics of G and coincides with the disjoint union of those subschemes $S_{m,g}$ with $g \in \mathcal{C}_m$. The latter will be called *strata of ∂S_K^* contributing to Z_m*.

[7] For a parabolic Q_m to be admissible, it is in particular required that its projection to any $\mathbb{Q}$-simple factor G' of G^{ad} be either G' itself or a *maximal* proper parabolic $\mathbb{Q}$-subgroup of G'.

Moreover, by accepting a little restriction on our Shimura datum we can get a more precise description of the subschemes $S_{m,g}$. Define

$$H_{Q_{m,g}} := \mathrm{Stab}_{Q_m(\mathbb{Q})}(X_m) \cap P_m(\mathbb{A}_f) \cdot K^g, \tag{2.40}$$

$$H_{C_{m,g}} := \mathrm{Cent}_{Q_m(\mathbb{Q})}(X_m) \cap W_m(\mathbb{A}_f) \cdot K^g, \tag{2.41}$$

$$\Delta_{m,g} := H_{Q_{m,g}}/P_m(\mathbb{Q})H_{C_{m,g}}, \tag{2.42}$$

where we denoted by $\mathrm{Cent}_{Q_m(\mathbb{Q})}(X_m)$ the group of elements of $\mathrm{Stab}_{Q_m(\mathbb{Q})}(X_m)$ which induce the identity on X_m. The results we are interested in can be then resumed as follows.

Proposition 2.3.3.2. *Suppose that the Shimura datum (G, X) satisfies condition (+) from Subsection 2.1.3. Then: a) The group $\Delta_{m,g}$ is finite and acts naturally and freely on $S_{\pi_m(K_{m,g})}$.*
b) The stratum $i_g(S_{\pi_m(K_{m,g})})$ equals the quotient of $S_{\pi_m(K_{m,g})}$ by the action of $\Delta_{m,g}$ and is smooth over E.

Proof. Everything follows from [Wil17, Lemma 8.2] and its proof, remembering that K is neat by hypothesis. □

We will see explicitly in Chapter 4 what this description of the boundary gives in the cases of Hilbert and (genus 2) Hilbert-Siegel varieties (Subsection 4.3.4.1, resp. 4.2.1). Moreover, one can exploit the structure of the boundary in order to compute the cohomology of the restriction to each stratum of the complex $i^*j_*\mu_H^K(V)$, where $j : S_K \hookrightarrow S_K^*$ and $i : \partial S_K^* \hookrightarrow S_K^*$ are now the open-closed immersions corresponding to the Baily-Borel compactification and $V \in \mathrm{Rep}_L(G)$ for some number field L. We will explain this in Subsection 4.2.2.

Now, having in mind the *second* and *third part* of the questions raised at the end of the previous subsection, we want rather to introduce some object of crucial importance living on the Baily-Borel compactification, namely the *intersection complex*, and more generally the *intermediate extensions* to the Baily-Borel compactification of local systems arising from the canonical construction. This will allow us, in particular, to clarify completely the automorphic meaning of the weight-zero part of the cohomology of such local systems. For the rest of this subsection, keep the above notation for i and j. Let $\mathcal{S}$ denote one of the two varieties S_K, S_K^*, let ℓ be any prime, and consider the $\mathbb{Q}$-linear, abelian categories $\mathrm{Perv}(\mathcal{S}(\mathbb{C}))$, resp. $\mathrm{Perv}(\mathrm{Et})_\ell(\mathcal{S})$ of topological, resp. ℓ-adic *perverse sheaves* (with respect to the *middle perversity*) over $\mathcal{S}(\mathbb{C})$, resp. over $\mathcal{S}$. By choosing a subfield L of $\mathbb{C}$, one also has L-linear versions $\mathrm{Perv}_F(\mathcal{S}(\mathbb{C}))$. When L is a number field and l is a prime of L above ℓ, one has L_l-linear versions $\mathrm{Perv}(\mathrm{Et})_{\ell,L}(\mathcal{S})$. These categories are full *abelian* subcategories of the L-linear constructible category $D^b_c(\mathcal{S}(\mathbb{C}))_L$ of sheaves on $\mathcal{S}(\mathbb{C})$, resp. of its L_l-linear ℓ-adic analogue $D^b_{c,\acute{e}t}(\mathcal{S})_L$ on $\mathcal{S}$. Denote by $\mathrm{Perv}(\mathcal{S})$ any of these categories; then, in both contexts, topological and ℓ-adic, one has a canonical *intermediate extension* functor ([BBD82, Déf. 1.4.22])

$$j_{!*} : \mathrm{Perv}(S_K) \to \mathrm{Perv}(S_K^*) \tag{2.43}$$

suitably characterized, informally speaking, as providing a *minimal extension* of a perverse sheaf on S_K to a perverse sheaf on S_K^* (cfr. [BBD82, Cor. 1.4.25]). In particular, if d_{S_K}

is the complex dimension of S_K, then, denoting by $\mathbb{V}$ any (topological) local system on $S_K(\mathbb{C})$ or ℓ-adic lisse étale sheaf on S_K, the object $\mathbb{V}[d_{S_K}]$ is a perverse sheaf on S_K, and one can take its intermediate extension to S_K^*. The $\mathbb{V}$-valued *intersection complex* on S_K^* is then defined as

$$IC_{S_K*}(\mathbb{V}) := j_{!*}(\mathbb{V}[d_{S_K}])$$

and the hypercohomology groups

$$IH^{\cdot}(S_K^*, \mathbb{V}) := \mathbb{H}^{\cdot}(S_K^*, IC_{S_K*}(\mathbb{V})[-d_{S_K}])$$

are called the *intersection cohomology groups* of S_K^* with coefficients in $\mathbb{V}$. There is a comparison isomorphism between singular and étale intersection cohomology groups, analogous to (2.16).

From now on, we choose as local system $\mathbb{V}$, with the notations of the previous subsection, one of the objects $\mu^K(V)$ arising through the canonical construction functors; here V is a irreducible representation V of G_L (L a number field over which G splits), which carries a pure Hodge structure of some weight w, and μ^K is either the topological or the ℓ-adic canonical construction. The first property of the intermediate extension that we want to make use of is the following:

Proposition 2.3.3.3. *Let $D_c^b(S_K^*)$ denote either one of the above constructible categories. Write $j_{!*}\mu^K(V)$ for the object $j_{!*}(\mu^K(V)[d_{S_K}])[-d_{S_K}]$ of the category $D_c^b(S_K^*)$. Then, $j_{!*}\mu^K(V)$ fits in the following commutative diagram of exact triangles in $D_c^b(S_K^*)$:*

$$\begin{array}{ccccccc}
0 & \longrightarrow & i_*i^!j_{!*}\mu^K(V) & = & i_*i^!j_{!*}\mu^K(V) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
j_!\mu^K(V) & \longrightarrow & j_{!*}\mu^K(V) & \longrightarrow & i_*i^*j_{!*}\mu^K(V) & \longrightarrow & j_!\mu^K(V)[1] \\
\| & & \downarrow & & \downarrow & & \| \\
j_!\mu^K(V) & \longrightarrow & j_*\mu^K(V) & \longrightarrow & i_*i^*j_*\mu^K(V) & \longrightarrow & j_!\mu^K(V)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & i_*i^!j_{!*}\mu^K(V)[1] & = & i_*i^!j_{!*}\mu^K(V)[1] & \longrightarrow & 0
\end{array}$$

Proof. Everything follows formally from the axioms satisfied by the functors under consideration ([BBD82, Sect. 1.4.3.]), by taking into account that, with the convention in the statement, we have $j^*j_{!*}\mu^K(V) \simeq \mu^K(V)$. The reader can consult the completely analogous proof of [Wil13, Thm. 1.6(b)] for more details. □

Notice that the third line of the above diagram is exactly (the topological or ℓ-adic version of) the boundary triangle introduced in (2.33). Bringing into consideration the theory of *weights*, we get the following well-known consequence of the previous proposition, of which it is useful to spell out the details for later reference.[8]

Corollary 2.3.3.4. *For every integer $n \geq 0$, the image of $IH^n(S_K, \mu^K(V))$ in $H^n(S_K, \mu^K(V))$ (via the morphism induced by the arrow $j_{!*} \to j_*$ in the diagram of Proposition 2.3.3.3) coincides with $\mathrm{Gr}^{\mathbb{W}}_{n+w} H^n(S_K, \mu^K(V))$ (with the respect to the weight filtration of the mixed*

[8] In order to have an (ascending) weight filtration on the étale cohomology spaces, one has to suppose S_K to be of abelian type (cfr. Rmk. 2.1.3.8.(2)). Otherwise, we only get the result on the Hodge side, which is enough for our later purposes.

Hodge structure, resp. of the $\mathrm{Gal}(\bar{\mathbb{Q}}|E)$*-module structure, on Betti, resp.* ℓ*-adic étale cohomology).*

Proof. We prove the statement by working in the topological setting; the ℓ-adic proof is formally analogous, or given by applying the comparison isomorphisms. Then, the diagram of triangles in $D^b_c(S^*_K(\mathbb{C}))$ of Proposition 2.3.3.3 can be lifted to a diagram of triangles in the bounded derived category of mixed Hodge modules over $S^*_K(\mathbb{C})$, whose theory implies in particular that the object $j_{!*}\mu^K_H(V)$ is *pure of weight* w ([Sai90, Sect. 4.5]) and that therefore (*loc. cit.*, (4.5.2)) $i^! j_{!*}\mu^K_H(V)$ is of weights $\geq w$. The same is then true after application of the functor i_*.

Now remember that since S_K is smooth, $j_*\mu^K_H(V)$ is of weights $\geq w$. Using the fact that S^*_K is proper, by taking the long exact sequence in hypercohomology given from the second column of (the Hodge-theoretical analogue of) the diagram in Proposition 2.3.3.3, we not only see that, for every n, the object

$$H^n_{\mathrm{int}} := \mathrm{Im}(IH^n(S_K(\mathbb{C}), \mu^K_H(V)) \to H^n(S_K(\mathbb{C}), \mu^K_H(V)))$$

has to be contained in $\mathrm{Gr}^{\mathbb{W}}_{n+w} H^n := \mathrm{Gr}^{\mathbb{W}}_{n+w} H^n(S_K(\mathbb{C}), \mu^K_H(V))$, but also that the cokernel $\mathrm{Gr}^{\mathbb{W}}_{n+w} H^n / H^n_{\mathrm{int}}$ has to be trivial: in fact, it has to inject in the weight-$(n+w)$ subspace of $H^{n+1}(S^*_K(\mathbb{C}), i_* i^! j_{!*}\mu^K_H(V))$, which, by what we have said, is reduced to zero. □

Now we bring into the picture a deep link with the automorphic structure.

Theorem 2.3.1. *(Zucker's conjecture, [Loo88], [SS90]) There exists a canonical isomorphism* ZC *between the* L^2*-cohomology of* $S_K(\mathbb{C})$ *with values in* V *(cfr. Def. 2.2.2.3) and* $IH^{\cdot}(S^*_K(\mathbb{C}), \mu^K_{\mathrm{top}}(V_{\mathbb{C}}))$ *such that the diagram*

$$\begin{array}{ccc} H^{\cdot}_2(S_K(\mathbb{C}), \mu^K_{\mathrm{top}}(V_{\mathbb{C}})) & \xrightarrow{ZC} & IH^{\cdot}(S^*_K(\mathbb{C}), \mu^K_{\mathrm{top}}(V_{\mathbb{C}})) \\ \downarrow & & \downarrow \\ H^{\cdot}(S_K(\mathbb{C}), \mu^K_{\mathrm{top}}(V_{\mathbb{C}})) & = & H^{\cdot}(S_K(\mathbb{C}), \mu^K_{\mathrm{top}}(V_{\mathbb{C}})) \end{array}$$

commutes. Here, the left vertical arrow is the composition of the morphism induced by the natural arrow between the complexes $A^{\cdot}_2$ *and* $A^{\cdot}$ *of Subsection 2.2.2 and of the inverse of the isomorphism (2.18), while the right vertical arrow is induced by the canonical morphism* $j_{!*} \to j_*$ *in the diagram of Prop. 2.3.3.3.*

Together with the previous corollary, this gives:

Corollary 2.3.3.5. *Denote by* $H^{\cdot}_{(2)}(S_K(\mathbb{C}), \mu^K_{\mathrm{top}}(V_{\mathbb{C}}))$ *the image of* L^2*-cohomology in ordinary cohomology through the natural map. Then:*

(1) the $C^\infty_c(G(\mathbb{A}_f)//K)_{\mathbb{C}}$*-module* $H^{\cdot}_{(2)}(S_K(\mathbb{C}), \mu^K_{\mathrm{top}}(V_{\mathbb{C}}))$ *is semisimple;*

(2) for each integer $n \geq 0$*, we have a canonical isomorphism*

$$H^n_{(2)}(S_K(\mathbb{C}), \mu^K_{\mathrm{top}}(V_{\mathbb{C}})) \simeq \mathrm{Gr}^{\mathbb{W}}_{n+w} H^n(S_K(\mathbb{C}), \mu^K_H(V))_{\mathbb{C}}$$

Note that point (1) comes from the fact that $H^{\cdot}_{(2)}$ is by definition identified, through a $C^\infty_c(G(\mathbb{A}_f)//K)_{\mathbb{C}}$-equivariant morphism, with the quotient of a semisimple $C^\infty_c(G(\mathbb{A}_f)//K)_{\mathbb{C}}$-module (cfr. the Matsushima-Murakami decomposition ((2.21)). The above results give us a completely automorphic description of the weight-zero part of cohomology of local systems arising from the canonical construction, and moreover tell us, recalling (2.31), that we have the following inclusions:

$$H^{\cdot}_{\mathrm{cusp},\mathbb{C}} \hookrightarrow H^{\cdot}_{!,\mathbb{C}} \hookrightarrow H^{\cdot}_{(2)} \simeq (\mathrm{Gr}^{\mathbb{W}}_{\cdot+w} H^{\cdot})_{\mathbb{C}} \tag{2.44}$$

Remark 2.3.3.6. From the above inclusions, we see that there exists an n such that the complement of $H^n_!$ in $\mathrm{Gr}_{n+w} H^n(S_K(\mathbb{C}), \mu^K_H(V))$ is non-trivial if and only if there exist *discrete spectrum, non-cuspidal* automorphic representations (these are called *residual representations*) contributing non-trivially to cohomology in degree n.

The previous remark characterizes by automorphic means a situation which is natural to consider when studying weights (cfr. Remark 2.3.2.3). One could ask for *purely algebraic* conditions on the representation V_λ allowing to draw similar conclusions on the weight filtration (possibly, still through the intermediation of the automorphic theory). We conclude this subsection by the following observation, which puts together a good part of the information gathered so far.

Theorem 2.3.2. *Let V_λ be an irreducible representation of G_L of highest weight λ and let d_{S_K} be the complex dimension of the Shimura variety S_K. If λ is* regular*, then the following hold:*

(1) $H^n_!$ is non-trivial if and only if $n = d_{S_K}$, and in that case

$$H^n_! \simeq \mathrm{Gr}^{\mathbb{W}}_{n+w} H^n$$

(2) The cohomology spaces with compact support H^n_c can be non-trivial only if $n \in \{1, \dots, d_{S_K}\}$. For every $n \in \{1, \dots, d_{S_K} - 1\}$, they verify

$$H^n_c \simeq \partial H^{n-1}$$

and they are of weights $< n + w$.

(3) The cohomology spaces H^n can be non-trivial only if $n \in \{d_{S_K}, \dots, 2d_{S_K}\}$. For every $n \in \{d_{S_K} + 1, \dots, 2d_{S_K}\}$, they verify

$$H^n \simeq \partial H^n$$

and they are of weights $> n + w$.

Proof. The crucial input here is a result explicitly stated and proved in [LS04, 5.3] (but also a consequence of [Fra98, Thm. 19, II]), which says that if λ is regular, then cuspidal cohomology coincides with L^2-cohomology, hence with its image in ordinary cohomology. Thus, by (2.44) we get that, in every degree, interior cohomology coincides with the lowest weight-graded step of the weight filtration (with $\mathbb{C}$-coefficients). But then, this is necessarily already true with L-coefficients. Now, by the hypothesis on λ and Thm. 2.1.1, the spaces H^n can be non-trivial only if $n \in \{d_{S_K}, \dots, 2d_{S_K}\}$ (notice that, in the case of Shimura varieties, the constant l_0 is equal to 0). By Poincaré duality, the spaces H^n_c can in turn be non-trivial only for $n \in \{0, \dots, d_{S_K}\}$, and the long exact sequence (2.34) tells us that it is actually necessary that $n \in \{1, \dots, d_{S_K}\}$. Hence, $H^n_!$ can be non-trivial only if $n = d_{S_K}$, and we have proven (1). But the same is then true *a fortiori* for $\mathrm{Gr}^{\mathbb{W}}_{n+w} H^n$. This, together with consideration of the same long exact sequence (2.34) (of mixed Hodge structures), yield the isomorphisms and the assertions on weights in (2) and (3). □

2.3.4 Toroidal compactifications and automorphic bundles

Keep the notations of the preceding subsection. The aim of this last subsection is to explain the role of another possible choice of a family of compactifications of the Shimura variety S_K, the *toroidal compactifications*. In particular, this will allow us to explain why we see the main result of Chapter 4 as a partial step in the direction of the last two questions emerging at the end of Subsection 2.3.2.

The toroidal compactifications of S_K are a class of compactifications parameterized by so-called *K-admissible cone decompositions* [Pin90, 6.4], i.e. certain collections $\mathfrak{S}$ of *convex rational polyhedral cones*, whose associated compactification is then denoted $S_K(\mathfrak{S})$. Modulo a suitable refinement of $\mathfrak{S}$, the variety $S_K(\mathfrak{S})$, which is defined over E, can be supposed to be projective and *smooth* [Pin90, proof of Thm. 9.21], which explains the appeal of such compactifications when compared to the Baily-Borel one, although they are highly *non-canonical*. Moreover, one can assume that the divisor $\partial S_K(\mathfrak{S}) := S_K(\mathfrak{S}) \setminus S_K$ has simple normal crossings. We will always implicitly assume that the toroidal compactifications that we consider satisfy these requirements.

Remark 2.3.4.1. The minimality property of the Baily-Borel compactification (Rmk. 2.3.3.1) tells us that, for every toroidal compactification $S_K(\mathfrak{S})$ as above, we have a (proper) map $\pi_{\mathfrak{S}} : S_K(\mathfrak{S}) \to S_K^*$ extending the identity on S_K.

The second class of objects that we need to define is a certain kind of locally free coherent sheaves on S_K and on its toroidal compactifications, called *automorphic bundles*: a reference for everything we will say on this subject is [Mil90, Chap. III]. We start by recalling that the complex analytic manifold $X \simeq G(\mathbb{R})/K_\infty^h$ associated to the Shimura datum admits a canonical open embedding β in a *compact* complex analytic manifold $\check{X}(\mathbb{C})$ (respectively called the *Borel embedding* and the *compact dual* of X) in such a way that there exist a parabolic subgroup P of $G_\mathbb{C}$ for which $\check{X}(\mathbb{C}) \simeq G(\mathbb{C})/P(\mathbb{C})$, $K_\infty^h = P(\mathbb{C}) \cap G(\mathbb{R})$, $K_\infty^h \hookrightarrow P(\mathbb{C})$ identifies[9] $K_{\infty,\mathbb{C}}^h$ with a Levi subgroup of $P(\mathbb{C})$ and the diagram

$$\begin{array}{ccc} G(\mathbb{R})/K_\infty^h & \hookrightarrow & G(\mathbb{C})/P(\mathbb{C}) \\ \downarrow \wr & & \downarrow \wr \\ X & \overset{\beta}{\hookrightarrow} & \check{X}(\mathbb{C}) \end{array}$$

commutes. Moreover, the complex manifold $\check{X}(\mathbb{C})$ is identified with the space of $\mathbb{C}$-points of a canonical projective variety $\check{X}$ over the reflex field E (hence the notation), and there is an equivalence of categories

$$\mathrm{Rep}_\mathbb{C}(P) \overset{i}{\simeq} \{G_\mathbb{C} - \text{vector bundles on } \check{X}_\mathbb{C}\} \tag{2.45}$$

Hence, in particular, any representation $V_\mathbb{C}$ of $G_\mathbb{C}$, seen as a P-representation by restriction, defines a $G_\mathbb{C}$-vector bundle over $\check{X}$. One then sees that

$$\mathcal{B}(V_\mathbb{C})_K(\mathbb{C}) := G(\mathbb{Q})\backslash \beta^* i(V_\mathbb{C}) \times G(\mathbb{A}_f)/K$$

is a vector bundle over $S_K(\mathbb{C})$, which canonically algebraizes to a vector bundle $\mathcal{B}(V_\mathbb{C})_K$ on $S_{K,\mathbb{C}}$: this is called the *automorphic bundle* associated to the representation $V_\mathbb{C}$. Moreover, if L is a number field containing the reflex field and $V \in \mathrm{Rep}_L(G)$, then $\mathcal{B}(V_\mathbb{C})_K$ admits a model over $S_{K,L}$, denoted by $\mathcal{B}(V)_K$.

[9] Here we are denoting by a subscript $\mathbb{C}$ the complexification of the real Lie group K_∞^h.

Definition 2.3.4.2. *Suppose that the number field L contains the reflex field E and that G splits over L. If V_λ denotes the irreducible representation of G over L of highest weight λ, then the* automorphic bundle of type λ *over $S_{K,L}$ is $\mathcal{B}(\lambda)_K := \mathcal{B}(V_\lambda)_K$.*

Remark 2.3.4.3. (1) In the following we will make no difference between vector bundles and the associated locally free coherent sheaves of sections.

(2) The automorphic vector bundles $\mathcal{B}(V)_K(\mathbb{C})$ associated to G_F-representations V are endowed with a canonical *flat connection* such that the corresponding locally constant subsheaf of *horizontal sections* coincides precisely with the local system $\mu^K_{\mathrm{top}}(V)$.

Now, given any toroidal compactification $S_K(\mathfrak{S})$, the bundles $\mathbb{V}(V)_K$ defined above admit a *canonical extension* $\mathcal{B}(V)^{\mathrm{can}}_K$ on $S_K(\mathfrak{S})$ ([Har89]). This allows us to give the following definition, in the cases of interest to us:

Definition. *Let V_λ be as in Definition 2.3.4.2. Then:*

(1) the space of automorphic forms of type λ *and* of level K *is the space of global sections $M_{\lambda,K} := H^0(S_K(\mathfrak{S})_L, \mathcal{B}(\lambda)^{\mathrm{can}}_K)$;*

(2) the space of cusp forms of type λ *and* of level K *is the space of global sections $M^{\mathrm{cusp}}_{\lambda,K} := H^0(S_K(\mathfrak{S})_L, \mathcal{B}(\lambda)^{\mathrm{can}}_K \otimes (-D))$, where D is the divisor $\partial S_K(\mathfrak{S})$.*

These definitions can be seen to be independent of the choice of the toroidal compactification $S_K(\mathfrak{S})$ ([Har90]).

Remark 2.3.4.4. (1) Denote by D a connected component of X, so that a connected component of $S_K(\mathbb{C})$ will be of the form $\Gamma\backslash D$ for a suitable arithmetic subgroup Γ of $G^{\mathrm{ad}}(\mathbb{Q})$ (cfr. Remark 2.2.(2)). The construction of the compact dual (resp. of the toroidal compactifications, of the automorphic bundles and of their canonical extensions to such compactifications) can be (and are) of course carried out first for D (resp. for $\Gamma\backslash D$). Hence, we can define in the obvious way the spaces $M_{\lambda,\Gamma}$ of automorphic forms of type λ and level Γ, along with the spaces $M^{\mathrm{cusp}}_{\lambda,\Gamma}$ of cuspidal ones, as spaces (at least with $\mathbb{C}$-coefficients) of sections over $\Gamma\backslash D$.

(2) In the case of $G = \mathrm{GL}_2$, we have that the arithmetic subgroups Γ arising in the context of the preceding point are conjugates of the image in $\mathrm{PGL}_2(\mathbb{Q})$ of the group $K \cap \mathrm{GL}_2(\mathbb{Q})$. Assume that they can be identified with congruence subgroups of $\mathrm{SL}_2(\mathbb{Q})$. In this case, as a highest weight of SL_2, the character λ corresponds to a non-negative integer k, and the resulting spaces $M_{\lambda,\Gamma}$, resp. $M^{\mathrm{cusp}}_{\lambda,\Gamma}$, are identified with the usual spaces $M_{k+2}(\Gamma)$ of *modular forms*, resp. $S_{k+2}(\Gamma)$ of *cusp modular forms*, *of weight* $k+2$. Notice the difference between the notion of *type* and *weight*, which is the reason for adopting our (slightly non-standard) terminology. Also notice that we are already using other two notions of *weight*, the representation-theoretic and Hodge-theoretic ones, which one has to be careful to distinguish.

(3) In the setting of the preceding point, denote simply by k the (representation-theoretic) weight λ and recall the classical *Eichler-Shimura isomorphism* ([Shi71, 8.2])

$$H^1_!(Y_\Gamma, \mu^K_{\mathrm{top}}(V_{k,\mathbb{C}})) \simeq S_{k+2}(\Gamma) \oplus \overline{S_{k+2}(\Gamma)}, \tag{2.46}$$

where Y_Γ is the modular curve $\Gamma\backslash D \simeq \Gamma\backslash\mathfrak{h}$; in particular, this isomorphism intertwines the action of Hecke operators on Betti cohomology as defined in Subsection 2.2.1 with the classically-defined one on cusp forms. Besides providing an explicit identification of cusp forms with interior cohomology classes of the local system $\mu^K_{\mathrm{top}}(V_{k,\mathbb{C}})$, it can be seen that (2.46) gives a realization of the Hodge decomposition of the space $H^1_!(Y_\Gamma, \mu^K_{\mathrm{top}}(V_{k,\mathbb{C}}))$ (notice that in this case, $H^1_!(Y_\Gamma, \mu^K_H(V_k))$ carries a pure Hodge structure of weight $k+1$, cfr. Subsection 2.3.1) into a direct sum of two subspaces of type $(k+1, 0)$ (the space of cuspidal modular forms) and $(0, k+1)$ (the conjugate of the former). Also, it can be shown that $H^1_!(Y_\Gamma, \mu^K_H(V_k))$ and $\mathrm{Gr}_{k+1} H^1(Y_\Gamma, \mu^K_H(V_k))$ coincide. The only other non-trivial weight graded object, the quotient of H^1 by $H^1_!$, is then $\mathrm{Gr}_{2k+2} H^1(Y_\Gamma, \mu^K_H(V_k))$, whose complexification is identified with the space of *Eisenstein*, i.e. non-cuspidal, modular forms of level Γ and weight $k+2$.

On the other hand, the Galois representations associated by Deligne to *classical* cusp forms ([Del71c]) are then found inside the ℓ-adic étale cohomology spaces $H^1_{\text{ét},!}(Y_{\Gamma,\bar{\mathbb{Q}}}, \mu^K_\ell(V_k))$ and related to the action of Hecke operators on cusp forms through the comparison isomorphism.

(4) The decomposition of $H^1_!(Y_\Gamma, \mu^K_H(V_{k,\mathbb{C}}))$ in eigenspaces for Hecke operators induced by the isomorphism (2.46) and the decomposition of (in general, a subspace of) interior cohomology as sum of cuspidal $C^\infty_c(G(\mathbb{A}_f)//K)$-modules discussed in Subsection 2.3.1 are related by a standard recipe which associates a cuspidal automorphic representation to a (classical) cusp form ([Bum97, 3.6]). Through this, the Hodge decomposition induced on each Hecke-eigenspace by (2.46) can be seen as a special case of the intrinsic Hodge decomposition existing on the $(\mathfrak{g}, K^h_\infty)$-cohomology spaces $H^\cdot(\mathfrak{g}, K^h_\infty; \pi_\infty \otimes V_\mathbb{C})$ associated to cuspidal automorphic representations (cfr. Rmk. 2.3.1.3).

(5) The previous points emphasize, in the case $G = \mathrm{GL}_2$, a relationship between sections of automorphic bundles and cohomology of the associated local systems of horizontal sections (cfr. Remark 2.3.4.3.(2)). For general G, it is a spectral sequence coming from the so-called *BGG-complex* ([Fal83]) which encodes the (much more complicated) way in which the coherent cohomology of automorphic bundles contributes to Betti cohomology of the local systems $\mu^K_{\mathrm{top}}(V_\mathbb{C})$.

We are now prepared to end this chapter by introducing the crucial notion of *corank*. We have seen that in Subsection 2.3.3 that the boundary ∂S^*_K of the Baily-Borel compactification of our Shimura variety S_K admits a stratification of the form (with the notations of that Subsection)

$$\partial S^*_K = \bigsqcup_{m=1}^{p} \bigsqcup_{g \in \mathcal{C}_m} S_{m,g},$$

where in particular the integer p is the number of conjugacy classes of admissible $\mathbb{Q}$-parabolics in G. Coherently with the notation in 2.3.3, denote, for $m = 1, \cdots, p$, $Z_m = \bigsqcup_{g \in \mathcal{C}_i} S_{m,g}$. Then, the geometrically connected components of Z_m are of equal dimension d_m, and we can suppose that the ordering of the conjugacy classes of parabolics has been chosen so that $d_m \geq d_n$ whenever $m \leq n$.

Fix a number field L over which G splits and an irreducible representation V_λ of G_L of highest weight λ. Then, we pose the following:

Definition. *(cfr. [BR18, Def. 1.10]) Let $f \in \mathrm{M}_{\lambda,K}$ be a non-zero automorphic form of type λ and level K. The* corank *of f is the minimal integer $\mathrm{cor}(f) := q$ such that $\oplus_{g \in \mathcal{C}_{q+1}} f|_{S_{q+1,g}} = 0$ (with $\mathcal{C}_{p+1} = \varnothing$ by convention).*

Notice that this notion gives a measure, in some sense, of the *degree of cuspidality* of f: for example, $\mathrm{cor}(f) = 0 \iff f$ is cuspidal, and we could define *completely non-cuspidal forms* the f's such that $\mathrm{cor}(f) = p$.

Now, the results of [BR18] imply that, at least when the Shimura datum to which G is associated is of *PEL-type* (see Subsection 3.2.1)[10], the existence of non-zero forms of a given corank is controlled by a purely representation-theoretic invariant, the so-called *corank* of the highest weight λ. Let us define it in the case which will be of interest to us, for G equal to the modification of $\mathrm{Res}_{F|\mathbb{Q}} \mathrm{GSp}_{2n,F}$ (for F totally real of degree d) defined as in (2.7) : it is the case of genus n Hilbert-Siegel varieties, which we will later specialize to $n = 2$ and which in turn subsumes the cases of Hilbert modular varieties ($n = 1$) and Siegel threefolds ($F = \mathbb{Q}$). In order to give such a definition, notice that in this case, the highest weight λ will be of the form

$$\lambda = ((k_{i,\sigma})_{\substack{i=1,\cdots,n \\ \sigma\in I_F}}, c)$$

where I_F is the set of (real) embeddings of F and, in particular, the non-negative integers $k_{i,\sigma}$ are such that, for each σ, $k_{1,\sigma} \geq \cdots \geq k_{n,\sigma} \geq 0$; see Subsection 4.1.3 for more details in the case $n = 2$ (the general case is completely analogous). In the following, let L be a Galois closure of F.

Definition 2.3.4.5. *(cfr. [BR18, Def. 1.5.3]) Let* $\lambda = ((k_{i,\sigma})_{\substack{i=1,\cdots,n \\ \sigma\in I_F}}, c)$ *be a highest weight of* G_L*.*

1. *For each* $i \in 1, \cdots, n$*,* $k_i := (k_{i,\sigma})_{\sigma\in I_F}$ *is called* parallel *if* $k_{i,\sigma}$ *is constant on* I_F*, equal to a non-negative integer* κ*.*
2. *We say that* λ *has* corank q *(and we write* $\mathrm{cor}(\lambda) = q$*) if*

$$q = |\{1 \leq i \leq n | k_i = k_n\}|$$

and k_n *is parallel. If there is no such* q*, we say that the corank is 0.*

Now, the result that we need from [BR18] (where the role of λ, resp. $\mathrm{M}_{\lambda,K}$, is played by k, resp. $\mathrm{M}_k(\mathcal{H}, R)$) spells out as follows (for the translation between our *analytical* definition of the relevant objects and the *algebraic* definitions of *loc. cit.*, see [Lan12a]):

Theorem 2.3.3. *([BR18, Thm. 1.5.6]) If* $f \in \mathrm{M}_{\lambda,K}$ *is non-zero, then* $\mathrm{cor}(\lambda) \geq \mathrm{cor}(f)$*.*

This implies for example that, in order to have non-zero non-cuspidal forms, it is necessary for λ to be of corank ≥ 1; moreover, in order to have non-zero *completely non-cuspidal* forms, it is necessary for λ to be of corank n. In the case of Hilbert modular varieties, λ is a list of integers indexed by the embeddings σ, and hence the latter observation generalizes the well-known fact that there can't be non-zero non-cusp forms of type λ over Hilbert modular varieties, if λ is not parallel ([Fre90, Rem. I.4.8]).

The importance of this theorem for us lies in the fact that, in the genus 2 case, the main result of Chapter 4, i.e. Thm. 4.3.1, says that the corank of λ is *also* the invariant which controls the presence of higher weights in the cohomology of Hilbert-Siegel varieties. In particular, it allows to give a precise bound on the first strictly positive weights appearing in the boundary cohomology, in the direction of the problem raised in Remark

[10] More precisely, when the PEL-datum is of *symplectic* or *unitary* type.

2.3.2.3. Notice that we make no direct use of the result from [BR18], which served only as inspiration for imagining that a description of the weights in these terms should have been possible.

We think of these facts as an evidence for the possible role of the corank as a gateway between automorphic description of the cohomology and weight filtration.

Chapter 3

Motives of Shimura varieties and weight structures

For a Shimura variety S_K associated to a reductive group G as considered in Subsection 2.1.3, and for a representation V of G, the behaviour of the Hodge and ℓ-adic canonical constructions hints at the existence of an underlying algebro-geometric object over S_K, of which the local systems $\mu_H^K(V)$, $\mu_\ell^K(V)$, with their additional structures (Hodge structure, Galois representations) should be the *realizations*. At least for *PEL-type* Shimura varieties (moduli spaces for abelian varieties with a prescribed choice of polarization, endomorphisms and level structure), this is actually the case, due to the presence of an *universal abelian scheme* A_K over S_K, as we will recall in Section 3.2. In fact, $\pi : A_K \to S_K$ being proper and smooth, we have at our disposal the framework of *relative Chow motives* over S_K (to be firstly reviewed in Section 3.1), which allows, in the case of abelian schemes, to give an intrinsic geometric meaning to the corresponding relative cohomology sheaves, independent of the chosen cohomology theory. This is the source of the relative motives $\mathcal{V}$ giving rise to the objects in the image of the canonical constructions.

Nevertheless, our real aim is to find motives realizing to the Galois representations $H^\cdot_{\text{ét}}(\pi_f^K)$ and Hodge structures $H^\cdot(\pi_f^K)$ attached in Subsections 2.2.2, resp. 2.3.1 to cuspidal automorphic representations π appearing in the cohomology of the canonical constructions over S_K. If we are to build such motives out of the universal abelian scheme, we are confronted to the fact that A_K, seen (through $A_K \xrightarrow{\pi} S_K \xrightarrow{\tilde{s}} \operatorname{Spec} E$) as a variety *over the point* $\operatorname{Spec} E$ (E the reflex field of S_K), is in general not proper at all, due to non-properness of S_K. For reasons that will become clear in the following, we will also need to consider the *degeneration* of such an abelian scheme at the boundary of a suitable compactification of S_K. Thus, we need first a setting in which to treat these more general objects, i.e. the theory of *Beilinson motives* over a base, also reviewed in Subsection 3.1, equipped with its adequate *six functors formalism*. Secondly, we have seen in Subsection 2.3.1 that our Galois representations and Hodge structures are found inside interior cohomology and hence in particular, in the terminology established there, in *weight zero*. This implies that we have to extract, from the *mixed* object $\tilde{s}_*\mathcal{V}$, a Chow motive realizing to the lowest-weight subquotients of cohomology. We will prove in Subsection 3.2.2 that there is an *algebra* action of the Hecke operators on $\tilde{s}_*\mathcal{V}$, and we would like our Chow motive to carry an induced action of this algebra, whose idempotent elements will then cut out the submotives corresponding to individual automorphic representations.

It is the theory of *weight structures*, to be reviewed in the first part of Section 3.3, which provides the motivic version of the weight filtration. But the lack of functoriality of the latter forces one to look for an additional criterion, allowing to define a *canonical*, hence Hecke-equivariant, lowest-weight Chow submotive. This criterion, due to Wildeshaus, is reviewed at the end of Section 3.3. The attempt to verify it for specific families of Shimura varieties has been the motivation for the study of weights carried out in this thesis.

3.1 Review of Beilinson motives and relative Chow motives

For a varying base scheme $\mathcal{S}$, the system of $\mathbb{Q}$-linear, triangulated categories $DM_{\text{B}}(\mathcal{S})$ of *Beilinson motives* over $\mathcal{S}$ can be thought of, roughly speaking, as a universal system of triangulated categories of $\mathbb{Q}$-linear mixed (complexes of) sheaves over $\mathcal{S}$, equipped with a six functors formalism, thus realizing to any available theory of mixed (complexes of) sheaves (depending on the base scheme $\mathcal{S}$: ℓ-adic sheaves for suitable primes ℓ, mixed Hodge modules...), compatibly with this formalism. This system of categories is known to satisfy almost all of the expected properties required by the original program of Beilinson's ([Bei87]), the most notable exception being the existence of a suitable (perverse) *t-structure*, which would allow one to define an *Abelian* category of *mixed motivic sheaves* and to get far-reaching consequences (for example, over a point of characteristic zero, the validity of the so- called *standard conjectures* ([Bei12]) and hence the completion of Grothendieck's original program for the construction of a *Tannakian* category of *pure motives*).

For our purposes, it will be enough to consider as base $\mathcal{S}$ a separated, finite type $\mathbb{Q}$-scheme (in the following, we will refer to such schemes as to *base schemes*). Even in this case, we won't give the definition of the category of Beilinson motives ([CD12, Def. 14.2.1]), but only list the properties of $DM_{\text{B}}(\mathcal{S})$ which we will concretely need (the sense in which to understand the *mixed* structure will be explained in Subsection 3.3). We recall, however, that this category can be canonically identified with the category $D_{\mathbb{A}^1,\text{ét}}(\mathcal{S},\mathbb{Q})$ obtained from the derived category of étale sheaves of $\mathbb{Q}$-vector spaces on the site of smooth $\mathcal{S}$-schemes by, informally speaking, imposing the relation $Y \times_{\mathcal{S}} \mathbb{A}^1_{\mathcal{S}} \simeq Y$ for any smooth $\mathcal{S}$-scheme Y (the so-obtained category is in particular monoidal symmetric, so that it admits a tensor product) and by making the operation of tensoring with $\mathbb{P}^1_{\mathcal{S}}$ invertible (more precisely, applying the process of $\mathbb{P}^1_{\mathcal{S}}$-*stabilization*); this is the content of [CD12, Thm. 16.2.18]. One of the crucial consequences of the approach of Cisinski and Déglise is that spaces of morphisms in the resulting category have the correct relation with *K-theory* ([CD12, Cor. 14.2.14]). The list of the features of $DM_{\text{B}}(\mathcal{S})$ that we will need is the following:

Property 3.1.0.1. (1) The construction of $DM_{\text{B}}(\mathcal{S})$ can be done taking an arbitrary $\mathbb{Q}$-algebra L as ring of coefficients instead of $\mathbb{Q}$ ([CD12, 14.2.20]), yielding triangulated, L-linear categories $DM_{\text{B}}(\mathcal{S})_L$, such that the canonical functor $DM_{\text{B}}(\mathcal{S}) \otimes_{\mathbb{Q}} L \to DM_{\text{B}}(\mathcal{S})_L$ is fully faithful, and satisfying the F-linear analogues of the properties of $DM_{\text{B}}(\mathcal{S})$. In particular, such categories are *pseudo-Abelian* ([Hé11, Sect. 2.10]).

(2) The categories $DM_{\text{B}}(\mathcal{S})_L$ are monoidal symmetric, and we denote the unit of their tensor product $\otimes$ by $\mathbb{1}_{\mathcal{S}}$. For every integer i, one has objects $\mathbb{1}_{\mathcal{S}}(i)$ called *Tate twists*, and for every smooth $\mathcal{S}$-scheme X, one has a corresponding object $M_{\mathcal{S}}(X)$ in $DM_{\text{B}}(\mathcal{S})_L$, the *motive* of X [CD12, 1.1.34]; its Tate twist by i is the object $M_{\mathcal{S}}(X)(i) := M_{\mathcal{S}}(X) \otimes \mathbb{1}_{\mathcal{S}}(i)$. Then, the category $DM_{\text{B},c}(\mathcal{S})_L$ of *constructible Beilinson motives* over $\mathcal{S}$ is defined as the full, thick, triangulated subcategory of $DM_{\text{B}}(\mathcal{S})_L$ generated by the objects $M_{\mathcal{S}}(X)(i)$, for varying X and i. Over a point $\mathcal{S} = \operatorname{Spec} k$, for k, say, a finite extension of $\mathbb{Q}$, the

category $DM_{\mathrm{B},c}(k)_L$ is canonically identified ([CD12, Rmk. 11.1.14, Thm. 16.1.4]) with the F-linear variant of Voevodsky's category $DM_{gm}(k)_L$ of *geometrical motives*[1] ([Voe00]).

(3) Any morphism $f : \mathcal{T} \to \mathcal{S}$ of base schemes induces a functor $f_* : DM_{\mathrm{B}}(\mathcal{T})_L \to DM_{\mathrm{B}}(\mathcal{S})_L$ with a left adjoint $f^* : DM_{\mathrm{B}}(\mathcal{S})_L \to DM_{\mathrm{B}}(\mathcal{T})_L$ which is a *monoidal functor*. There is also a functor $f_!$ with right adjoint $f^!$, along with a natural transformation $\alpha_f : f_! \to f_*$ which is an *isomorphism* if f is *proper* [CD12, Thm. 2.2.14 (2)]. The system of such functors, together with the $\otimes$ bifunctor, which admits a right adjoint $\underline{\mathrm{Hom}}_{\mathcal{S}}$, satisfies more generally the *six functor formalism* as defined in [CD12, A.5.1]. Moreover, these functors preserve constructible Beilinson motives ([CD12, 4.2.29]).

(4) The only other part of the six functors formalism that we want to recall is the following: if f as in the preceding point and it is the base of a Cartesian diagram

$$\begin{array}{ccc} \mathcal{T}' & \xrightarrow{f'} & \mathcal{S}' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{T} & \xrightarrow{f} & \mathcal{S} \end{array}$$

of base schemes, then the exchange transformation $g^* f_! \to f'_! g'^*$ is an isomorphism; the same then holds for the adjoint exchange transformation $g'_* f'^! \to f^! g_*$.

(5) Let L be a number field. For every prime ℓ, there exists (see [CD16, Sect. 7.2]) a canonical triangulated functor $\mathcal{R}_\ell : DM_{\mathrm{B},c}(\mathcal{S})_L \to D^b_{c,\text{ét}}(\mathcal{S})_L$, compatible with the six-functor formalism, called the *ℓ-adic realization functor* (here $D^b_{c,\text{ét}}(\mathcal{S})_L$ is the category of Remark 2.1.3.9.(2)). Composition with the collection of cohomology functors, resp. perverse cohomology functors, $R^* : D^b_{c,\text{ét}}(\mathcal{S})_L \to \mathrm{Gr}_{\mathbb{Z}}\mathrm{Et}_{\ell,L}(\mathcal{S})$, resp. $\mathcal{H}^* : D^b_{c,\text{ét}}(\mathcal{S})_L \to \mathrm{Gr}_{\mathbb{Z}}\mathrm{Perv}(\mathrm{Et})_{\ell,L}(\mathcal{S})$ (where $\mathrm{Perv}(\mathrm{Et})_{\ell,L}(\mathcal{S})$ is the L_l-linear category of ℓ-adic perverse sheaves over $\mathcal{S}$ introduced in Subsection 2.3.3), gives rise to the ℓ-adic *cohomological realization*, resp. *perverse cohomological realization* functors.

Remark 3.1.0.2. Notice that an analogous functor to the ℓ-adic realization $\mathcal{R}_\ell$ introduced above, but taking values in the category of *mixed Hodge modules* over $\mathcal{S}(\mathbb{C})$ (i.e. a *Hodge realization functor* with the expected properties) doesn't exist yet in the generality that we need (namely, we will need to consider *singular* bases $\mathcal{S}$). This is why we will only work with ℓ-adic sheaves in Chapter 4, whereas the (completely analogous) computations and results there could be phrased and obtained in the Hodge setting too (cfr. Rmk. 4.2.2.4).

Suppose from this point until the end of this Section that all base schemes are quasi-projective and regular. We have now to introduce a more classical category of relative motives over $\mathcal{S}$, which recovers over $\mathrm{Spec}\, k$ the category $CHM(k)_{\mathbb{Q}}$ of *Chow motives* (with rational coefficients) originally constructed by Grothendieck. We are speaking of the category $CHM^s(\mathcal{S})$ of *smooth Chow motives over* $\mathcal{S}$, as introduced in [DM91, Sect. 1]: it is obtained in the following way. First, one takes the category $SmProj(\mathcal{S})$ of smooth, projective $\mathcal{S}$-schemes and replaces the spaces of morphisms between objects X and Y, if X is connected of relative dimension $d_{X/\mathcal{S}}$ over $\mathcal{S}$, by the (rational) Chow groups of *relative correspondences of degree* $d_{X/\mathcal{S}}$

$$\mathrm{CH}^{d_{X/\mathcal{S}}}(X \times_{\mathcal{S}} Y) := \mathrm{CH}^{d_{X/\mathcal{S}}}(X \times_{\mathcal{S}} Y, \mathbb{Q})$$

[1]Note however that realizations (see below) behave covariantly on Beilinson motives and *contravariantly* on Voevodsky motives.

with composition given in the usual way (for a first cycle on $X \times_{\mathcal{S}} Y$ and a second one on $Y \times_{\mathcal{S}} Z$, pull back the two to $X \times_{\mathcal{S}} Y \times_{\mathcal{S}} Z$, intersect there and push forward the result to $X \times_{\mathcal{S}} Z$), obtaining an additive category $Corr(\mathcal{S})$. Second, one takes the idempotent completion $CHM^{s,\, eff}(\mathcal{S})$ of $Corr(\mathcal{S})$, which comes then equipped with a canonical, contravariant faithful functor $\mathfrak{h} : SmProj(\mathcal{S}) \to CHM^{s,\, eff}(\mathcal{S})$, sending a morphism f to the class ${}^t\Gamma_f$ of the transpose of its graph; it is a pseudo-abelian category with a tensor product induced by the Cartesian product of schemes. Third, one inverts a canonical direct factor $\mathbb{L}_{\mathcal{S}}$ of $\mathfrak{h}(\mathbb{P}^1_{\mathcal{S}})$, the *Lefschetz motive*, with respect to the tensor product, obtaining the desired ($\mathbb{Q}$-linear, rigid) $\otimes$-category $CHM^s(\mathcal{S})$, in which $CHM^{s,\, eff}(\mathcal{S})$ naturally embeds; in particular, every object M of $CHM^{s,eff}(\mathcal{S})$ admits a dual $M^\vee$ in $CHM^s(\mathcal{S})$ (see *loc. cit.* for more details on each step). One can obviously redo the whole construction with coefficients in a $\mathbb{Q}$-algebra L and obtain a L-linear category $CHM^s(\mathcal{S})_L$. Notice that for a morphism of base schemes $f : \mathcal{T} \to \mathcal{S}$, the natural functor given by fiber product with $\mathcal{T}$ over $\mathcal{S}$ induces an additive functor $f^* : CHM^s(\mathcal{S})_L \to CHM^s(\mathcal{T})_L$ commuting with tensor product.

The link between smooth Chow motives and Beilinson motives comes from the fact that by [Lev09, Prop. 5.19, Cor. 6.14] there is a canonical, fully faithful embedding

$$CHM^s(\mathcal{S})_L \hookrightarrow DM_{\mathrm{B},c}(\mathcal{S})_L$$

which is a tensor functor. Notice that, over a point, this embedding recovers the fully faithful embedding $CHM(k)^{\mathrm{op}} \hookrightarrow DM_{gm}(k)$ constructed by Voevodsky in [Voe00]. For any smooth projective $p : X \to \mathcal{S}$, it sends $\mathfrak{h}(X)$ to $p_* \mathbb{1}_X$ (we may call the latter the *cohomological motive* of X). Moreover, it sends $\mathbb{L}_{\mathcal{S}}$ to $\mathbb{1}_{\mathcal{S}}(-1)[-2]$, and it is compatible with the functors f^* induced by morphisms $f : \mathcal{T} \to \mathcal{S}$ on the two sides. In particular, for any couple of objects $p : X \to \mathcal{S}$, $q : Y \to \mathcal{S}$ in $SmProj(\mathcal{S})$, with X connected of relative dimension $d_{X/\mathcal{S}}$ over $\mathcal{S}$, one has

$$\mathrm{Hom}_{DM_{\mathrm{B},c}(\mathcal{S})_L}(p_* \mathbb{1}_X, q_* \mathbb{1}_Y) = \mathrm{CH}^{d_{X/\mathcal{S}}}(X \times_{\mathcal{S}} Y)_L \tag{3.1}$$

compatibly with composition.

Remark 3.1.0.3. Let L be a number field. The constructions of [DM91, Sec. 1.8] show that on $CHM^s(\mathcal{S})_L$ one has many *realization functors* at one's disposal. As in *loc. cit.*, there exists, for any prime ℓ, an *ℓ-adic cohomological realization functor* on smooth Chow motives over $\mathcal{S}$; one sees that this equals the restriction of $R^* \circ \mathcal{R}_\ell$ (Property 3.1.0.1.(5)) to $CHM^s(\mathcal{S})_L$. Moreover, one can construct in the same way the *Hodge cohomological realization*, with values in the category $\mathrm{Gr}_{\mathbb{Z}}\mathrm{Var}_L(\mathcal{S}(\mathbb{C}))$ of Rmk. 2.1.3.9.(1)

We will say more on the link between relative Chow motives and Beilinson motives in Section 3.3.

3.2 The motivic canonical construction and the action of the Hecke algebra

3.2.1 The motivic canonical construction over PEL Shimura varieties

In order to apply the theory of Beilinson motives and of relative Chow motives to Shimura varieties, we need to introduce the class of (rational) *PEL data*. Referring to [Lan17, 5.1] for precise definitions and for all the facts recalled on this subject, what we actually need to use is that such data are given by tuples $(V, B, *, \langle\cdot,\cdot\rangle, h_0)$ where B is a finite dimensional semisimple $\mathbb{Q}$-algebra with positive involution $*$, V is a finite dimensional $\mathbb{Q}$-vector space on which B acts, $\langle\cdot,\cdot\rangle$ is a non-degenerate, alternating paring on V, and h_0 is a certain $\mathbb{R}$-algebra homomorphism $\mathbb{C} \to \mathrm{End}_{B_{\mathbb{R}}}(V_{\mathbb{R}})$. These objects are required to satisfy a series of compatibilities. Denoting by G the (reductive) $\mathbb{Q}$-algebraic group of automorphisms of V commuting with B and preserving the pairing $\langle\cdot,\cdot\rangle$ up to a scalar, and putting $X := G(\mathbb{R}) \cdot h_0$, one defines coset spaces

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

where K is an open compact subgroup of $G(\mathbb{A}_f)$. If such a (G, X) is a Shimura datum, it is called a *PEL-type* Shimura datum; then the above coset spaces are Shimura varieties S_K, defined over their reflex field E.

In the following, we will denote a PEL datum simply by $(V, B, \langle\cdot,\cdot\rangle)$, assuming as implicit the choices of the involution and of h_0.

Remark 3.2.1.1. If G is the group associated to a PEL datum $(V, B, \langle\cdot,\cdot\rangle)$, denote by $\mathbb{Q}(1)$ the 1-dimensional representation of G on which G acts by the character $g \mapsto gg^*$, and for any representation W of G, for any positive integer n, denote $W(n) := W \otimes \mathbb{Q}(1)^{\otimes n}$, $W(-n) := W \otimes (\mathbb{Q}(1)^\vee)^{\otimes n}$. Then, the pairing $\langle\cdot,\cdot\rangle$ induces a canonical morphism of G-representations

$$V \otimes V \to \mathbb{Q}(1)$$

and hence a canonical isomorphism

$$V^\vee \simeq V(-1). \tag{3.2}$$

When G underlies a Shimura datum, this coincides with a particular case of the pairing utilised in the proof of Prop. 2.1.3.10.

The above defined coset spaces can be canonically identified with (a disjoint union of connected components of) moduli spaces of complex abelian varieties A equipped, in particular, with a polarization of a specific type, with a canonical injection of a fixed order of B in $\mathrm{End}(A)$, and with a suitable *level structure*, all of this depending on the choice of K.

From now on, we will fix a PEL datum $(V, B, \langle\cdot,\cdot\rangle)$ and we will suppose that it gives rise to a Shimura datum (G, X) satisfying condition $(+)$ of 2.1.3. As usual, we will assume the compact open subgroups K that we use to be neat, in order to work with smooth (quasi-projective) Shimura varieties. The description of the complex points of the latter as moduli spaces descend to the reflex field. Hence, we will have *universal abelian schemes* $A_K \to S_K$ at our disposal; such schemes are of relative dimension over S_K equal to half the (necessarily even) dimension of V over $\mathbb{Q}$, and admit a canonical injection of $\mathbb{Q}$-algebras $B \hookrightarrow \mathrm{End}(A_K) \otimes \mathbb{Q}$, the latter denoting the algebra of endomorphisms of A_K as an abelian scheme over S_K.

Example 3.2.1.2. The Shimura data defined in Example 2.1.3.2 are all examples of PEL Shimura data satisfying condition $(+)$. In these cases, V is just the standard reprentation of the associated group G, the semisimple algebra B is just the field F and the pairing is the one tautologically preserved by definition by G; thus, when this field is a non-trivial extension of $\mathbb{Q}$, the abelian varieties parametrized by the corresponding moduli problem are equipped with *real multiplication* by a fixed order $\mathcal{O}$ of F.

We will use freely the terminology *universal* for the abelian schemes defined above, and we will not discuss whether the previous moduli spaces are *fine moduli spaces*, but as an example of the utility of condition (+), we note the following:

Remark 3.2.1.3. The objects of the more restricted class of Hilbert modular varieties, obtained in Example 2.1.3.2.(1) from a group G satisfying condition (+), can be interpreted as *fine* moduli spaces, whereas "classical" Hilbert modular varieties arising from $G = \mathrm{Res}_{F|\mathbb{Q}}\,\mathrm{GL}_{2,F}$ can not (although they admit a modular interpretation, too). Cfr. [DT04, Remarque 1.1] for details and for a discussion of advantages and disadvantages of working with the two classes of Hilbert modular varieties.

Remark 3.2.1.4. For a PEL Shimura variety S_K, we denote by $H^i(A)$ the local systems over $S_K(\mathbb{C})$, resp. ℓ-adic sheaves over S_K, defined by $R^i\pi_*\underline{\mathbb{Q}}$, resp. $R^i\pi_*\underline{\mathbb{Q}_\ell}$.

On the Hodge side, the functor H^1 induces an (anti-)equivalence of categories ([Del71b, 4.4.3])

$$\left\{ \begin{array}{c} \text{abelian schemes over } S_{K_x,\mathbb{C}} \\ \text{modulo isogeny} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{polarizable variations of } \mathbb{Q}\text{-Hodge structure} \\ \text{over } S_{K_x}(\mathbb{C}) \text{ of type } (1,0),(0,1) \end{array} \right\}$$

such that $H^1(A_K)$ is canonically identified with $\mu_K^H(V^\vee)$, where μ_H^K is the Hodge canonical construction functor (Subsection 2.1.3). In particular, the pure Hodge structure, which V is endowed with by definition of Shimura datum, is of type $(-1,0),(0,-1)$.

Let us connect the above with the theory described in Section 3.1. Since $\pi : A_K \to S_K$ is proper and smooth over the regular quasi-projective variety S_K, we can consider the object $\mathfrak{h}(A_K)$ of the category of smooth Chow motives over S_K. This motive admits a remarkable *functorial* decomposition:

Theorem 3.2.1. *([DM91, Thm. 3.1, Cor. 3.2]) Let $\pi : A \to S_K$ be an abelian scheme and let n denote the endomorphism of multiplication by n on A. Let $g = \dim_{S_K} A$. Then, for each $i \in \{0, \cdots, 2g\}$ there exist canonical idempotents $\mathfrak{p}_A^i \in \mathrm{CH}^g(A \times_{S_K} A)$ (called the* Chow-Künneth projectors*) uniquely characterized by the equation*

$${}^t\Gamma_n \circ \mathfrak{p}_A^i = n^i \mathfrak{p}_A^i = \mathfrak{p}_A^i \circ {}^t\Gamma_n \tag{3.3}$$

in $\mathrm{CH}^g(A \times_{S_K} A)$. They are such that, if $\mathfrak{h}^i(A)$ denotes the direct factor of $\mathfrak{h}(A) = \pi_*\mathbb{1}_A$ in $CHM^s(S_K)$ determined by $\mathfrak{p}_A^i$ (called the i-th Chow-Künneth component *of $\mathfrak{h}(A)$), one has*

$$\pi_*\mathbb{1}_A = \bigoplus_{i=0}^{2g} \mathfrak{h}^i(A)[-i]$$

Moreover, if $R = (R^i)_{i\in\mathbb{Z}}$ denotes the Hodge or the ℓ-adic cohomological realization (cfr. Remark 3.1.0.3), and H^i is as in Rmk. 3.2.1.4, then

$$R(\pi_*\mathbb{1}_A) = (R^i(\mathfrak{h}^i(A)))_{i=0}^{2g} = (H^i(A))_{i=0}^{2g}. \tag{3.4}$$

Remark 3.2.1.5. (1) The Chow-Künneth projectors $\mathfrak{p}_A^i$ are such that, for any i, ${}^t\mathfrak{p}_A^i = \mathfrak{p}_A^{2g-i}$ ([DM91, Remark 3, page 218]).

(2) For any homomorphism $f : A \to B$ of abelian schemes over S_K, we have

$${}^t\Gamma_f \circ \mathfrak{p}_B^i = \mathfrak{p}_A^i \circ {}^t\Gamma_f,$$

so that any such f induces a map

$$f^* : \mathfrak{h}^i(B) \to \mathfrak{h}^i(A)$$

for all i ([DM91, Prop. 3.3]).

The Chow-Künneth components of the universal abelian scheme are at the heart of the following crucial result:

Theorem 3.2.2. *([Anc15, Thm. 8.6], stated as in [Wil19a, Thm. 5.1])*
Let L be a number field and fix (G, X) a PEL-type Shimura datum satisfying condition (+) associated to a PEL datum $(V, B, \langle\cdot,\cdot\rangle)$, S_K one of the associated Shimura varieties, $A_K \to S_K$ the corresponding universal abelian scheme. Let $CHM^s(S_K)_L$ denote the category of smooth Chow motives over S_K of Section 3.1, seen as a full subcategory of $DM_{\text{Б},c}(S_K)$. Then, there exists a L-linear tensor functor

$$\tilde{\mu} : \mathrm{Rep}(G_L) \to CHM^s(S_K)_L \tag{3.5}$$

called the motivic canonical construction*, with the following properties:*

1. *The composition of $\tilde{\mu}$ with the Hodge cohomological realization, resp., for all primes ℓ, with the ℓ-adic cohomological realization (Remark 3.1.0.3), is isomorphic to μ_H^K, resp. to μ_ℓ^K (with the convention of Rmk. 2.1.3.9. (1)).*
2. *$\tilde{\mu}$ commutes with Tate twists, in the sense that for any $W \in \mathrm{Rep}(G_L)$ and any $n \in \mathbb{Z}$, with the notation of Rmk. 3.2.1.1,*

$$\tilde{\mu}(W(n)) = \tilde{\mu}(W)(n)[2n]$$

3. *The functor $\tilde{\mu}$ maps the G_L-representation V_L to the Chow motive $\mathfrak{h}^1(A_K)(1)[2]$.*

For the last point, one should keep in mind Rmk. 3.2.1.4 and the normalization concerning Tate twists explained before equation (3.1).

Remark 3.2.1.6. For every positive integer n, let $\pi_n : A_K^n \to S_K$ be the n-fold fibred product of A_K with itself over S_K. Observe that since the group G underlies a PEL Shimura datum, it will be isomorphic over $\mathbb{R}$ to a product of classical groups ([Lan17, pag. 51]); hence, the direct sum $V \oplus V^\vee$ of the standard representation V with its dual generates the Tannakian category $\mathrm{Rep}(G_L)$, by taking tensor products and direct summands. As a consequence, Theorem 3.2.2 implies that every object in the essential image of $\tilde{\mu}$ is isomorphic to a finite direct sum $\bigoplus_i M_i$, where each M_i is a direct factor of a Tate twist of a Chow motive of the form $\pi_{n_i,*}\mathbb{1}_{A_K^{n_i}}$, for suitable n_i's.

This theorem allows us to define the objects which will be the main characters in everything to follow:

Definition 3.2.1.7. *Suppose G to be split over L, and let V_λ be a irreducible L-representation of G_L of highest weight λ. The Chow motive ${}^\lambda\mathcal{V}$ over S_K is defined by*

$${}^\lambda\mathcal{V} := \tilde{\mu}(V_\lambda).$$

Remark 3.2.1.8. (1) Let $w(\lambda)$ be the weight of the pure objects $\mu_H^K(V_\lambda)$ and $\mu_\ell^K(V_\lambda)$ (cfr. Remark 2.1.3.8.(3)). Then, the Hodge, resp. ℓ-adic cohomological realizations of ${}^\lambda\mathcal{V}$ are zero in degree $\neq w(\lambda)$, and identical to $\mu_H^K(V_\lambda)$, resp. $\mu_\ell^K(V_\lambda)$, in degree $w(\lambda)$.

(2) Let d_{S_K} be the dimension of S_K. Then, (1) can be reformulated by saying that the *perverse* ℓ-adic cohomological realizations (cfr. Property 3.1.0.1.(5)) are zero in perverse degree $\neq w(\lambda) + d_{S_K}$, and identical to $\mu_\ell^K(V_\lambda)$ in perverse degree $w(\lambda) + d_{S_K}$.

(3) Let $\mathbb{D}_{\ell,S_K}$ denote the ℓ-adic local duality endofunctor over S_K. Then, since we have that

$$R_\ell({}^\lambda\mathcal{V}) = \mu_\ell^K(V_\lambda)[-w(\lambda)] \tag{3.6}$$

Proposition 2.1.3.10 implies that

$$\mathbb{D}_{\ell,S_K}(R_\ell({}^\lambda\mathcal{V})) \simeq R_\ell({}^\lambda\mathcal{V})(w(\lambda) + d_{S_K}))[2w(\lambda) + 2d_{S_K}].$$

3.2.2 The Hecke algebra

Retain the notation of the preceding subsection and fix a Shimura variety S_K of PEL-type, with underlying group G and associated PEL datum $(V, B, \langle\cdot,\cdot\rangle)$, and fix a field L of characteristic 0. If ${}^\lambda\mathcal{V}$ are the Chow motives over S_K defined in the previous Subsection and $\tilde{s} : S_K \to \operatorname{Spec} E$ is the structural morphism, we want to construct an algebra of correspondences acting on the object $\tilde{s}_*^\lambda\mathcal{V}$ of the category $DM_{\text{B},c}(E)_L$, in such a way to recover, on its cohomological realization, the action of the Hecke algebra defined in Subsection 2.2.1.

Hence, fix an element $x \in G(\mathbb{A}_f)$, define $K_x := K \cap xKx^{-1}$, and recall the two finite, étale morphisms defined in (2.4)

$$\begin{aligned} g_1 &:= [\cdot 1] : S_{K_x} \to S_K \\ g_2 &:= [\cdot x] : S_{K_x} \to S_K \end{aligned} \tag{3.7}$$

There is a compact open subgroup W of $V(\mathbb{A}_f)$ such that the complex points of the universal abelian scheme $\pi : A_K \to S_K$ over S_K can be written as

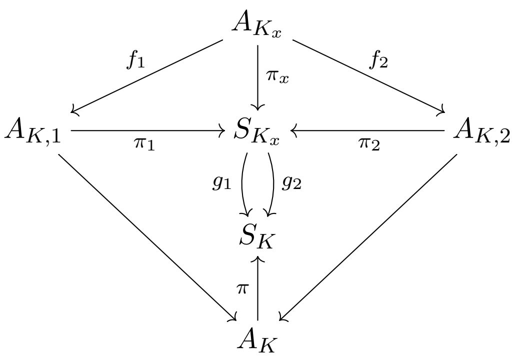
$$A_K(\mathbb{C}) = V(\mathbb{Q}) \rtimes G(\mathbb{Q}) \backslash V(\mathbb{R}) \times X \times V(\mathbb{A}_f) \rtimes G(\mathbb{A}_f)/W \rtimes K$$

where the semidirect product is defined by the standard representation of G on V. Then, seeing x as an element of $V(\mathbb{A}_f) \rtimes G(\mathbb{A}_f)$ and denoting $W_x := W \cap xWx^{-1}$, the complex points of the universal abelian scheme A_{K_x} over S_{K_x} are given by

$$A_{K_x}(\mathbb{C}) = V(\mathbb{Q}) \rtimes G(\mathbb{Q}) \backslash V(\mathbb{R}) \times X \times V(\mathbb{A}_f) \rtimes G(\mathbb{A}_f)/W_x \rtimes K_x$$

Remark 3.2.2.1. These descriptions identify A_K and A_{K_x} as *mixed Shimura varieties* ([Pin90, Def. 3.1]).

Now, for $i = 1, 2$, define the abelian schemes $A_{K,i}$ over S_{K_x} as the fiber products of A_K and S_{K_x} over S_K along the morphisms g_i. These objects and morphisms fit in the following diagram, where all subdiagrams commute and the two lower subdiagrams are cartesian:



The morphisms f_i are isogenies; they are concretely described on $\mathbb{C}$-points by (we give the example of f_2)

$$\begin{aligned} A_{K_x}(\mathbb{C}) &\to A_K(\mathbb{C}) \times_{S_K(\mathbb{C}),g_2} S_{K_x}(\mathbb{C}) \\ [(v,p),(w,h)] &\mapsto ([(v,p),(wx,hx)],[(p,h)]) \end{aligned} \tag{3.8}$$

where square brackets denote the appropriate equivalence classes.

Definition 3.2.2.2. *Let g denote the relative dimension $\dim_{A_{K,1}/S_{K_x}}$. We define the morphism*

$$\phi_x \in \mathrm{Hom}_{DM_{\mathrm{B},c}(S_{K_x})_L}(\pi_{1,*}\mathbb{1}_{A_{K,1}}, \pi_{2,*}\mathbb{1}_{A_{K,2}}) = \mathrm{CH}^g(A_{K,1} \times_{S_{K_x}} A_{K,2})_L$$

(cfr. (3.1)) as the (class of the) correspondence $\Gamma_{f_2} \circ {}^t\Gamma_{f_1}$.

Remark 3.2.2.3. Unraveling the identifications, one can see that the above morphism is the same as the one given by the following composition of adjunction morphisms associated to f_1, f_2 (cfr. Property 3.1.0.1.(3))

$$\phi_x : \pi_{1,*}\mathbb{1}_{A_{K,1}} \to \pi_{x,*}\mathbb{1}_{A_{K_x}} \to \pi_{2,*}\mathbb{1}_{A_{K,2}},$$

Remark 3.2.2.4. By Rmk. 3.2.1.5.(1) and by functoriality of the Chow-Künneth components with respect to isogenies (point (2)), we get in particular a canonical morphism

$$\phi_x^1 : \mathfrak{h}^1(A_{K,1}) \to \mathfrak{h}^1(A_{K,2}),$$

and hence, for any positive integer i, a canonical morphism

$$\phi_x^{i,1} := (\phi_x^1)^{\otimes i} : (\mathfrak{h}^1(A_{K,1}))^{\otimes i} \to (\mathfrak{h}^1(A_{K,2}))^{\otimes i}. \tag{3.9}$$

Remark 3.2.2.5. Thanks to the motivic canonical construction (Thm. 3.2.2), we can define a canonical isomorphism

$$\alpha : \mathfrak{h}^1(A) \simeq \mathfrak{h}^1(A)^\vee(-1) \tag{3.10}$$

induced by the canonical isomorphism in (3.2).

For later reference, let us give some more details on this definition. Fix a field L of characteristic zero and let the abelian scheme A be either equal to A_K or to A_{K_x}. By [Kin98, Prop. 2.2.1], the functor $\mathfrak{h}^1$ (on the category of abelian schemes over $\mathcal{S}$) induces an isomorphism of L-algebras

$$\mathrm{End}(A)^{\mathrm{op}} \otimes L \simeq \mathrm{End}_{CHM^s(\mathcal{S})_L}(\mathfrak{h}^1(A)) \tag{3.11}$$

On the other hand, reasoning as in [Anc15, Prop. 4.4], one sees that the natural arrow induces an isomorphism $B^{\mathrm{op}} \otimes L \simeq \mathrm{End}_{\mathrm{Rep}(G_F)}(V_L^\vee)$. Hence, by composing the inverse of this isomorphism with the inclusion

$$B^{\mathrm{op}} \otimes L \hookrightarrow \mathrm{End}(A)^{\mathrm{op}} \otimes L$$

and with the first isomorphism, we obtain a canonical identification of $\mathrm{End}_{\mathrm{Rep}(G_L)}(V_L^\vee)$ as a subalgebra of $\mathrm{End}_{CHM^s(\mathcal{S})_L}(\mathfrak{h}^1(A))$, which is seen to be induced by the motivic canonical construction functor of Subsection 3.2. In this way, one gets a canonical injection

$$\mathrm{Hom}_{\mathrm{Rep}(G_L)}(V_L^\vee, V_L(-1)) \hookrightarrow \mathrm{Hom}_{CHM^s(\mathcal{S})_L}(\mathfrak{h}^1(A), \mathfrak{h}^1(A)^\vee(-1))$$

which in particular associates the above defined α to the canonical isomorphism in (3.2).

In order to define the Hecke algebra, we have to look more in detail at the construction lying behind the functor $\tilde{\mu}$ of Subsection 3.2:

Definition 3.2.2.6. *[Anc15, Def. 5.2] Let the abelian scheme A be either equal to A_K or to A_{K_x}. For each positive integer i, the L-algebra $\mathcal{B}_{i,L}$ is defined*[2] *as the sub-L-algebra*

$$\mathcal{B}_{i,L} \hookrightarrow \mathrm{End}_{CHM(\mathcal{S})_L}((\mathfrak{h}^1(A))^{\otimes i})$$

generated by the following:

- *the permutation group $\mathcal{S}_i$,*
- *the ring $B^{\mathrm{op}} \otimes \mathrm{Id}_{\mathfrak{h}^1(A)}^{\otimes i-1}$ (canonically seen as a subalgebra of $\mathrm{End}_{CHM(\mathcal{S})_L}((\mathfrak{h}^1(A))^{\otimes i}$ as in Remark 3.2.2.5),*
- *the morphism $P \otimes \mathrm{Id}_{\mathfrak{h}^1(A)}^{\otimes(i-2)}$, if $i \geq 2$ (where, for $2g = \dim V$, P is the projector*

$$\frac{1}{2g}\iota \circ p \in \mathrm{End}_{CHM(\mathcal{S})_L}(\mathfrak{h}^1(A) \otimes \mathfrak{h}^1(A))$$

defined by the morphisms

$$\iota : \mathbb{L} \to \mathfrak{h}^1(A) \otimes \mathfrak{h}^1(A)$$

resp.

$$p : \mathfrak{h}^1(A) \otimes \mathfrak{h}^1(A) \to \mathbb{L}$$

corresponding to $\alpha^{-1} \circ \tau$, resp. α (see (3.10)) by adjunction - denoting by τ the canonical isomorphism switching the factors in a tensor product).

The above algebra enjoys the following fundamental property[3].

Proposition 3.2.2.7. *[Anc15, Prop. 8.5] The motivic canonical construction functor*

$$\tilde{\mu} : \mathrm{Rep}(G_L) \to CHM(\mathcal{S})_L$$

induces an isomorphism of L-algebras

$$\mathcal{B}_{i,L} \simeq \mathrm{End}_{\mathrm{Rep}(G_L)}((V^\vee)^{\otimes i}).$$

[2] Notice that we are adapting the original definition, given for general abelian schemes, to the more restricted context of *loc. cit.*, Section 8.3.

[3] Here we are in some sense reversing the natural order of reasoning, because it is exactly this property which actually allows one to *define* the motivic canonical construction functor.

Remark 3.2.2.8. Suppose that G splits over L. As observed in Remark 3.2.1.6, any object of the category $\mathrm{Rep}(G_L)$ is isomorphic to a direct factor N of

$$\bigoplus_s V_{a_s,b_s} := \bigoplus_s V^{\otimes a_s} \otimes (V^\vee)^{\otimes b_s},$$

for some choice of $(a_s)_s, (b_s)_s$. Denote by $\mathcal{N}$ the object $\tilde{\mu}(N) \in CHM(\mathcal{S})_L$. Without loss of generality, let us suppose that $\mathcal{N}$ is a direct factor of $\tilde{\mu}(V_{a,b})$, for some a, b, and use the canonical isomorphism (3.10) to identify it with a direct factor of $\tilde{\mu}(V^\vee)^{\otimes(a+b)}(a)$. By writing $i := a+b$, we get a direct factor of $\tilde{\mu}(V^\vee)^{\otimes i} = \mathfrak{h}^1(A)^{\otimes i}$, still denoted by $\mathcal{N}$, corresponding to an idempotent element e of the algebra $\mathcal{B}_{i,L}$.

In particular, for κ replacing any of the symbols K, K_x, we will denote by $\mathcal{N}_\kappa$ the corresponding object over S_κ, a direct factor of $\mathfrak{h}^1(A_\kappa)^{\otimes i}$. Here and in the following, the algebra associated to any of the A_κ will be considered the same one, by identifying its elements along the isomorphisms provided by Proposition 3.2.2.7; in particular, we will use the same symbol e in both cases.

We pass now to the construction of the Hecke algebra of correspondences we are looking for. We begin by the following lemma:

Lemma 3.2.2.9. *Let g_1, g_2 be the morphisms defined in (3.7). There exist canonical isomorphisms*

$$g_1^*(\mathfrak{h}^1(A_K)^{\otimes i}) \simeq \mathfrak{h}^1(A_{K,1})^{\otimes i}$$

and

$$\mathfrak{h}^1(A_{K,2})^{\otimes i} \simeq g_2^*(\mathfrak{h}^1(A_K)^{\otimes i})$$

Proof. By proper base change (Rmk. 3.1.0.1.(4)), we have canonical isomorphisms

$$g_1^*\pi_*\mathbb{1}_{A_K} \simeq \pi_{1,*}\mathbb{1}_{A_{K,1}} \tag{3.12}$$

and

$$\pi_{2,*}\mathbb{1}_{A_{K,2}} \simeq g_2^*\pi_*\mathbb{1}_{A_K} \tag{3.13}$$

Since the characterization (3.3) of Chow-Künneth projectors shows immediately that the Chow-Künneth components are compatible with pullback, and since the functors g_1^*, g_2^* are monoidal, these isomorphisms induce the isomorphisms in the statement. □

Then, the crucial point consists in showing that the morphisms (3.9) respect the objects $\mathcal{N}$, i.e.:

Proposition 3.2.2.10. *Let $\mathcal{N}_K$ be as in Remark 3.2.2.8. For any $x \in G(\mathbb{A}_f)$, the morphism $\phi_x^{i,1}$ in (3.9) induces a canonical morphism*

$$\tilde{\phi}_x^{\mathcal{N}} : g_1^*\mathcal{N}_K \to g_2^*\mathcal{N}_K.$$

such that the following holds: fix $\tilde{x} \in G(\mathbb{A}_f)$ of the form $\tilde{x} = x'x$ and denote by g'_j, resp. by $\tilde{g}_j$, for $j = 1, 2$, the morphisms $S_{K_{\tilde{x}}} \to S_{K_x}$, resp. the morphisms $S_{K_{\tilde{x}}} \to S_K$ defined in (3.7) as associated to x', resp to $\tilde{x}$. Denote by

$$\tilde{\phi}_{x'}^{\mathcal{N}'} : g_1'^* g_2^*\mathcal{N}_K \to g_2'^* g_2^*\mathcal{N}_K$$

the morphism defined in the same way as $\tilde{\phi}_x^{\mathcal{N}}$, by considering $g_2^*\mathcal{N}_K$ as a direct factor of $\mathfrak{h}^1(A_{K,2})^{\otimes i}$ via the isomorphism of Lemma 3.2.2.9.

Then, the diagram

$$\begin{array}{ccccc} & & g_1'^* g_2^* \mathcal{N}_K & & \\ & \overset{g'^*(\tilde{\phi}_x^{\mathcal{N}})}{\nearrow} & & \overset{\tilde{\phi}_{x'}^{\mathcal{N}'}}{\searrow} & \\ g_1'^* g_1^* \mathcal{N}_K = \tilde{g}_1^* \mathcal{N}_K & & \xrightarrow[\tilde{\phi}_{\tilde{x}}^{\mathcal{N}}]{} & & \tilde{g}_2^* \mathcal{N}_K = g_2'^* g_2^* \mathcal{N}_K \end{array}$$

commutes.

Let us record here an auxiliary lemma that will be used in the course of the proof:

Lemma 3.2.2.11. *Fix three objects U, V, W in a rigid $\otimes$-category $\mathcal{T}$ (over a field L) and a morphism $\chi : W \to V^\vee$. For any triple of objects $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ in $\mathcal{T}$, denote by*

$$adj : \mathrm{Hom}_{\mathcal{T}}(\mathcal{O}_1 \otimes \mathcal{O}_2, \mathcal{O}_3) \simeq \mathrm{Hom}_{\mathcal{T}}(\mathcal{O}_1, \mathcal{O}_2^\vee \otimes \mathcal{O}_3)$$

the canonical (and functorial) adjunction isomorphism. Then:

(1) for any morphism $\psi : U \otimes V \to W$, we have

$$(\mathrm{Id}_{V^\vee} \otimes \chi) \circ adj(\psi) = adj(\chi \circ \psi);$$

(2) for any morphism $\mu : U \otimes W^\vee \to W$, we have

$$(\chi \otimes \mathrm{Id}_W) \circ adj(\mu) = adj(\mu \circ (\mathrm{Id}_U \otimes \chi^\vee));$$

(3) for any morphism $\phi : V^\vee \to V \otimes U$, we have

$$adj^{-1}(\phi) \circ (\chi \otimes \mathrm{Id}_{V^\vee}) = adj^{-1}(\phi \circ \chi);$$

(4) for any morphism $\lambda : W \to V \otimes U$, we have

$$adj^{-1}(\lambda) \circ (\mathrm{Id}_W \otimes \chi) = adj^{-1}((\chi^\vee \otimes \mathrm{Id}_U) \circ \lambda).$$

See [AK02, Sect. 6.1] for a very general treatment of rigid $\otimes$-categories (and actually, of more general *rigid monoidal* L-categories) which allows to derive the above formulae (especially as a consequence of Eq. (6.5) in *loc. cit.*, where the isomorphism ι_{AB} coincides with our isomorphism adj^{-1} provided one sets $\mathcal{O}_1 = \mathbb{1}$, $\mathcal{O}_2 = A$, $\mathcal{O}_3 = B$ - this gives back our point (1) for this special case).

Proof. (of Proposition 3.2.2.10)
For $j = 1, 2$, the algebra $\mathcal{B}_{i,L}$ acts on $\mathfrak{h}^1(A_{K,j})^{\otimes i}$, by letting an element $\beta \in \mathcal{B}_{i,L}$ act as $g_j^*(\beta)$ conjugated by one of the isomorphisms of Lemma 3.2.2.9. With these conventions, in order to get the desired statement, it suffices to show the following commutation relation with respect to the action of the idempotent e:

$$\phi_x^{1,i} \circ e = e \circ \phi_x^{1,i}$$

(as morphisms of relative Chow motives over S_{K_x}). This boils down to showing that $\phi_x^{1,i}$ commutes with the action of any of the explicit generators of the algebra $\mathcal{B}_{i,L}$. Since commutation with elements of the symmetric group $\mathcal{S}_i$ is clear, let us turn our attention to a fixed element $b \in B$.

By Def. 3.2.2.2, commutation of $b \otimes \mathrm{Id}_{\mathfrak{h}^1(A_{K,j})}^{\otimes i-1}$ with $\phi_x^{1,i}$ follows once we prove that b commutes with ${}^t\Gamma_{f_1}$ and Γ_{f_2}. But since the f_j's are isogenies, let's say of some degree d_j, we have $\Gamma_{f_j} \circ^t \Gamma_{f_j} = d_j \cdot \mathrm{Id}_{A_{K,j}}$ for $j = 1, 2$, and b commutes with ${}^t\Gamma_{f_j}$ if and only if it commutes with Γ_{f_j}, so that we are reduced to show commutation of b (as an endomorphism of $A_{K,j}$ and A_{K_x}) with the f_j's (as morphisms of abelian schemes). This is immediate from the analytical description (3.8) of the morphisms f_j.

Finally, in order to show commutation of $P \otimes \mathrm{Id}_{\mathfrak{h}^1(A_{K,j})}^{\otimes i-2}$ with $\phi_x^{1,i}$ (which will conclude the proof), we have to prove that the P's commute with $\phi_x^{1,2}$. To this end, consider the canonical isomorphism α defined in (3.10) when $A = A_K$ and the morphisms ι, p intervening in the definition of P (again, when $A = A_K$). For $j = 1, 2$, apply g_j^* to these elements and conjugate them by the isomorphisms provided by proper base change: this yields canonical isomorphisms α_j and morphisms ι_j, p_j, whose sources and targets are clear from their construction. Then, we have to show the identity

$$\iota_2 \circ p_2 \circ (\phi_x^1 \otimes \phi_x^1) = (\phi_x^1 \otimes \phi_x^1) \circ \iota_1 \circ p_1 \tag{3.14}$$

For this, we first observe that, by defining

$$\phi_{x,a}^1 := \phi_x^1 \otimes \mathrm{Id}_{\mathfrak{h}^1(A_{K,2})}, \quad \phi_{x,b}^1 := \mathrm{Id}_{\mathfrak{h}^1(A_{K,1})} \otimes \phi_x^1$$

and

$$\phi_{x,c}^1 := \mathrm{Id}_{\mathfrak{h}^1(A_{K,2})} \otimes \phi_x^1, \quad \phi_{x,d}^1 := \phi_x^1 \otimes \mathrm{Id}_{\mathfrak{h}^1(A_{K,1})}$$

we get two factorizations

$$\phi_{x,a}^1 \circ \phi_{x,b}^1 = \phi_x^1 \otimes \phi_x^1 = \phi_{x,c}^1 \circ \phi_{x,d}^1$$

Then, we claim that the diagram

$$\begin{array}{ccc} \mathfrak{h}^1(A_{K,1}) & \xrightarrow[\simeq]{\alpha_1} & \mathfrak{h}^1(A_{K,1})^\vee(-1) \\ {\scriptstyle \phi_x^1}\downarrow & & \uparrow{\scriptstyle (\phi_x^1)^\vee(-1)} \\ \mathfrak{h}^1(A_{K,2}) & \xrightarrow[\simeq]{\alpha_2} & \mathfrak{h}^1(A_{K,2})^\vee(-1) \end{array} \tag{3.15}$$

commutes. Grant this for a moment, and employ the notation *adj* as in Lemma 3.2.2.11. Choosing $U = \mathbb{L}$, $V = \mathfrak{h}^1(A_{K,2})^\vee$, $W = \mathfrak{h}^1(A_{K,1})$ and $\chi = \phi_x^1$ in that same Lemma, we see that

$$\begin{aligned}
& \iota_2 \circ p_2 \circ (\phi_x^1 \otimes \phi_x^1) = adj(\alpha_2^{-1} \circ \tau) \circ adj^{-1}(\alpha_2) \circ \phi_{x,a}^1 \circ \phi_{x,b}^1 = \\
& \text{(by commutativity of (3.15))} \\
= \; & adj(\phi_x^1 \circ \alpha_1^{-1} \circ (\phi_x^1)^\vee(-1) \circ \tau) \circ adj^{-1}(\alpha_2) \circ \phi_{x,a}^1 \circ \phi_{x,b}^1 \quad = \\
& \text{(by Lemma 3.2.2.11 in its instance (1), choosing } \psi = \alpha_1^{-1} \circ (\phi_x^1)^\vee(-1) \circ \tau) \\
= \; & \phi_{x,c}^1 \circ adj(\alpha_1^{-1} \circ (\phi_x^1)^\vee(-1) \circ \tau) \circ adj^{-1}(\alpha_2) \circ \phi_{x,a}^1 \circ \phi_{x,b}^1 \quad = \\
& \text{(by Lemma 3.2.2.11 in its instance (2), choosing } \mu = \alpha_1^{-1} \circ \tau \\
& \text{and identifying } \tau^{-1} \circ (\phi_x^1)^\vee(-1) \circ \tau = \mathrm{Id}_{\mathbb{L}} \otimes (\phi_x^1)^\vee) \\
= \; & \phi_{x,c}^1 \circ \phi_{x,d}^1 \circ adj(\alpha_1^{-1} \circ \tau) \circ adj^{-1}(\alpha_2) \circ \phi_{x,a}^1 \circ \phi_{x,b}^1 \quad = \\
& \text{(by Lemma 3.2.2.11 in its instance (3), choosing } \phi = \alpha_2) \\
= \; & \phi_{x,c}^1 \circ \phi_{x,d}^1 \circ \iota_1 \circ adj^{-1}(\alpha_2 \circ \phi_x^1) \circ \phi_{1,b}^1 \quad = \\
& \text{(by Lemma 3.2.2.11 in its instance (4), choosing } \lambda = \alpha_2 \circ \phi_x^1) \\
= \; & \phi_{x,c}^1 \circ \phi_{x,d}^1 \circ \iota_1 \circ adj^{-1}((\phi_x^1)^\vee(-1) \circ \alpha_2 \circ \phi_x^1) \quad = \\
& \text{(by commutativity of (3.15))} \\
= \; & \phi_{x,c}^1 \circ \phi_{x,d}^1 \circ \iota_1 \circ adj^{-1}(\alpha_1) \quad = \\
= \; & (\phi_x^1 \otimes \phi_x^1) \circ \iota_1 \circ p_1.
\end{aligned}$$

So, identity (3.14) is proved, and it remains only to justify the commutativity of (3.15). This can be checked by applying a pullback and working in the category of smooth Chow motives over $S_{K_x,\mathbb{C}}$, where, as a consequence of the isomorphism (3.11) and of the equivalence of categories of Rmk. 3.2.1.4, we can equivalently show the commutativity of the corresponding diagram in the category of polarizable variations of $\mathbb{Q}$-Hodge structure on $S_{K_x}(\mathbb{C})$, obtained by realization. This, by Def. 3.2.2.2, amounts to showing the commutativity of the following:

$$\begin{array}{ccc}
H^1(A_{K,1}) & \xrightarrow[\simeq]{\alpha_1} & H^1(A_{K,1})^\vee(-1) \\
{\scriptstyle \tilde f_1}\downarrow & & \uparrow{\scriptstyle (\tilde f_1)^\vee(-1)} \\
H^1(A_{K_x}) & \xrightarrow[\simeq]{\alpha_2} & H^1(A_{K_x})^\vee(-1)
\end{array}
\qquad
\begin{array}{ccc}
H^1(A_{K_x}) & \xrightarrow[\simeq]{\alpha_1} & H^1(A_{K_x})^\vee(-1) \\
{\scriptstyle \tilde f_2}\downarrow & & \uparrow{\scriptstyle (\tilde f_2)^\vee(-1)} \\
H^1(A_{K,2}) & \xrightarrow[\simeq]{\alpha_2} & H^1(A_{K,2})^\vee(-1)
\end{array}$$

where $\tilde{f}_1$, resp. $\tilde{f}_1$ denote the morphisms of sheaves obtained from ${}^t\Gamma_{f_1}$, resp. Γ_{f_2}.

By proper base change applied to the underlying local systems, we have, for $j = 1, 2$,

$$g_j^* H^1(A_K) \overset{BC_j}{\simeq} H^1(A_{K,j})$$

Given the canonical pairing on V, the canonical construction induces a canonical pairing on the variation of Hodge structure $H_1(A_K)^\vee$ and hence, by pullback and conjugation by the proper base change isomorphisms, on the variations $H^1(A_{K,j})^\vee$. There is an analogous pairing on $H^1(A_{K,x})^\vee$. Denoting by A any of the schemes $A_{K,1}, A_{K_x}, A_{K,2}$ and by

$$\langle \cdot, \cdot \rangle : H^1(A)^\vee \times H^1(A)^\vee \to \mathbb{Q}(1)$$

the corresponding pairing, showing the commutation of the previous diagrams is equivalent to showing the identities

$$\langle \cdot, \cdot \rangle = \langle \tilde{f}_1(\cdot), \tilde{f}_1(\cdot) \rangle, \quad \langle \cdot, \cdot \rangle = \langle \tilde{f}_2(\cdot), \tilde{f}(\cdot) \rangle$$

Consider the new identities obtained from the latter by replacing $H^1(A_{K,1})$ and $\tilde{f}_1$, resp. $H^1(A_{K,2})$ and $\tilde{f}_2$, with $g_1^* H^1(A_K)$ and $\tilde{f}_1 \circ BC_1$, resp. with $g_2^* H^1(A_K)$ and $BC_2^{-1} \circ \tilde{f}_2$. We can equivalently show that these identities hold. Now, it is clear from the analytical description of the f_j's (3.8) that the morphisms $\tilde{f}_1 \circ BC_1$ and $BC_2^{-1} \circ \tilde{f}_2$ are the same as the isomorphisms of sheaves θ_1^{-1}, resp. θ_x occurring in Eq. (2.14) when defining Hecke operators topologically. Using this description, we see immediately that the required identities are true. □

Remark 3.2.2.12. The last part of the above proof shows that ϕ_x, in the appropriate sense, commutes with the canonical isomorphism (3.10). Hence our previous identification of our direct factor with a direct factor of $\tilde{\mu}(V^\vee)^{\otimes(a+b)}(a)$ (Rmk. 3.2.2.8) doesn't imply any loss of generality.

We can now complete the construction of the Hecke algebra, following [Wil17, pp. 591-592]. Fix $x \in G(\mathbb{A}_f)$ and let $\mathcal{N}_K$ be as in Remark 3.2.2.8 and $\tilde{\phi}_x^{\mathcal{N}}$ be the canonical morphism defined in Proposition 3.2.2.10. Consider moreover the adjunction morphisms $adj_1 : \mathcal{N}_K \to g_{1,*} g_1^* \mathcal{N}_K$, $adj_2 : g_{2,*} g_2^* \mathcal{N}_K = g_{2,!} g_2^! \mathcal{N}_K \to \mathcal{N}_K$, and the structure morphism $\tilde{s} : S_K \to \operatorname{Spec} E$ defined at the beginning of this Subsection. Applying $\tilde{s}_*$ to adj_1 and adj_2, one gets

$$\tilde{s}_*(adj_1) : \tilde{s}_* \mathcal{N}_K \to (\tilde{s} \circ g_1)_* g_1^* \mathcal{N}_K, \;\; \tilde{s}_*(adj_2) : (\tilde{s} \circ g_2)_* g_2^* \mathcal{N}_K \to \tilde{s}_* \mathcal{N}_K \tag{3.16}$$

and by applying $(\tilde{s} \circ g_1)_* = (\tilde{s} \circ g_2)_*$ to $\tilde{\phi}_x^{\mathcal{N}}$ one gets

$$(\tilde{s} \circ g_1)_*(\tilde{\phi}_x^{\mathcal{N}}) : (\tilde{s} \circ g_1)_* g_1^* \mathcal{N}_K \to (\tilde{s} \circ g_2)_* g_2^* \mathcal{N}_K \tag{3.17}$$

Definition 3.2.2.13. *The* Hecke correspondence *on $\tilde{s}_* \mathcal{N}_K$ associated to $x \in G(\mathbb{A}_f)$ is the morphism*

$$KxK := \tilde{s}_*(adj_2) \circ (\tilde{s} \circ g_1)_*(\tilde{\phi}_x^{\mathcal{N}}) \circ \tilde{s}_*(adj_1) : \tilde{s}_* \mathcal{N}_K \to \tilde{s}_* \mathcal{N}_K. \tag{3.18}$$

The Hecke algebra *$\mathcal{H}^{DM}(K, G(\mathbb{A}_f))$ is the subalgebra of $\operatorname{End}_{DM_{\mathrm{B},c}(E)_L}(\tilde{s}_* \mathcal{N}_K)$ generated by the elements KxK for $x \in G(\mathbb{A}_f)$.*

Remark 3.2.2.14. (1) An Hecke algebra acting on $\tilde{s}_! \mathcal{N}_K$ can be defined in an analogous way, so that the canonical map $\tilde{s}_! \mathcal{N}_K \to \tilde{s}_* \mathcal{N}_K$ is equivariant with respect with the actions.

(2) Suppose that L is a number field and take as $\mathcal{N}_K$ the motive ${}^\lambda\mathcal{V}$ corresponding to the irreducible representation V_λ of G_L of highest weight λ (Def. 3.2.1.7). For any prime ℓ and any prime l of L above ℓ, we see that by construction, the application the ℓ-adic cohomological realization (over $\operatorname{Spec} E$) to $\tilde{s}_*^\lambda \mathcal{V}$ (also remembering 3.2.1.8.(3)) transports the algebra $\mathcal{H}^{DM}(K, G(\mathbb{A}_f))$ to the algebra $\mathcal{H}(K, G(\mathbb{A}_f))_{L_l}$ acting on the spaces $H_{\text{ét}}^{\cdot - w(\lambda)}(S_{K,\bar{\mathbb{Q}}}, \mu_\ell^K(V_\lambda))$ as defined in Subsection 2.2.1.

3.3 From weight structures to the intersection motive

In this final subsection, we recall the formalism of *weight structures* and its implications for the construction of motives associated to automorphic representations.

3.3.1 Weight structures

We start by recalling the notion of *weights* on triangulated categories that we will use, introduced by Bondarko.

Definition 3.3.1.1. *([Bon10, Def. 1.1.1.]) Let $\mathcal{C}$ be a triangulated category. A* weight structure *on $\mathcal{C}$ is a pair $w = (\mathcal{C}_{w\leq 0}, \mathcal{C}_{w\geq 0})$ of full sub-categories of $\mathcal{C}$, such that, putting*

$$\mathcal{C}_{w\leq n} := \mathcal{C}_{w\leq 0}[n] \ , \ \mathcal{C}_{w\geq n} := \mathcal{C}_{w\geq 0}[n]$$

(respectively called the categories of objects of weight at most n *and* at least n*) the following conditions are satisfied.*

(1) For any object M of $\mathcal{C}_{w\leq 0}$, resp. $\mathcal{C}_{w\geq 0}$, any direct summand of M in $\mathcal{C}$ is an object of $\mathcal{C}_{w\leq 0}$, resp. $\mathcal{C}_{w\geq 0}$.

(2) There are inclusions

$$\mathcal{C}_{w\leq 0} \subset \mathcal{C}_{w\leq 1} \ , \ \mathcal{C}_{w\geq 1} \subset \mathcal{C}_{w\geq 0}$$

of full sub-categories of $\mathcal{C}$.

(3) For any pair of objects $A \in \mathcal{C}_{w\leq 0}$ and $B \in \mathcal{C}_{w\geq 1}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(A, B) = 0.$$

(4) For any object M of $\mathcal{C}$, there exists an exact triangle

$$A \to M \to B \to A[1]$$

in $\mathcal{C}$, such that $A \in \mathcal{C}_{w\leq 0}$ and $B \in \mathcal{C}_{w\geq 1}$.

Definition 3.3.1.2. *For any object M of $\mathcal{C}$, for any $n \in \mathbb{Z}$, an exact triangle*

$$M_{\leq n} \to M \to M_{\geq n+1} \to A[1] \tag{3.19}$$

in $\mathcal{C}$, with $M_{\leq n} \in \mathcal{C}_{w\leq n}$ and $M_{\geq n+1} \in \mathcal{C}_{w\geq n+1}$, is called a weight filtration *of M.*

Notice the similarity with the notion of a t-structure on a triangulated category, the crucial difference being in the use of the shifts in the definition of the subcategories $\mathcal{C}_{w\leq 0}$ and $\mathcal{C}_{w\geq 0}$ of objects *of weight* ≤ 0 and *of weight* ≥ 0. In fact, this difference has the effect that, contrary to truncations with respect to a t-structure, the "weight truncated" objects in a weight filtration are in general *not* unique up to unique isomorphism, and hence *not* functorial in general.

But one can push the analogy with t-structures further, with the following notion.

Definition 3.3.1.3. *([Bon10, Def. 1.2.1 1]) Let w be a weight structure on $\mathcal{C}$. The* heart *of w is the full additive subcategory $\mathcal{C}_{w=0}$ of $\mathcal{C}$ whose objects belong both to $\mathcal{C}_{w\leq 0}$ and to $\mathcal{C}_{w\geq 0}$.*

Here is a basic property which follows from the axioms of weights structures.

Property 3.3.1.4. ([Bon10, Prop. 1.3.3 3], stated in the shape of [Wil09, Prop. 1.4]) Let $w = (\mathcal{C}_{w\leq 0}, \mathcal{C}_{w\geq 0})$ be a weight structure on $\mathcal{C}$ and let

$$L \to M \to N \to L[1]$$

be an exact triangle in $\mathcal{C}$. If both L and N belong to $\mathcal{C}_{w\leq 0}$ (resp. if both belong to $\mathcal{C}_{w\geq 0}$) then so does M.

The following result gives the crucial link between the theory of weight structures and Beilinson motives. For the latter, we use notations and conventions fixed in Subsection 3.1: in particular, our base schemes will always be regular.

Theorem 3.3.1. *([Hé11, Thm. 3.3, thm. 3.8 (i)-(ii)]) There are canonical weight structures w on the categories $DM_{\mathrm{B},c}(\mathcal{S})$, called* motivic weight structures*, uniquely characterized by the following properties.*

(1) The objects $\mathbb{1}_{\mathcal{S}}(p)[2p]$ belong to the heart $DM_{\mathrm{B},c}(\mathcal{S})_{w=0}$ for all integers p.

*(2) For a morphism of base schemes $f : \mathcal{T} \to \mathcal{S}$, the left adjoint functors f^*, resp. $f_!$ and $f_\sharp$ (the latter for f smooth), map $DM_{\mathrm{B},c}(\mathcal{S})_{w\leq 0}$ to $DM_{\mathrm{B},c}(\mathcal{T})_{w\leq 0}$, resp. $DM_{\mathrm{B},c}(\mathcal{T})_{w\leq 0}$ to $DM_{\mathrm{B},c}(\mathcal{S})_{w\leq 0}$ (they are w-*left exact*).*

*The right adjoint functors f_*, resp. $f^!$ and f^* (the latter for f smooth), map $DM_{\mathrm{B},c}(\mathcal{T})_{w\geq 0}$ to $DM_{\mathrm{B},c}(\mathcal{S})_{w\geq 0}$, resp. $DM_{\mathrm{B},c}(\mathcal{S})_{w\geq 0}$ to $DM_{\mathrm{B},c}(\mathcal{T})_{w\geq 0}$ (they are w-*right exact*).*

Since important results of Bondarko ([Bon10, Sec. 6], relying on deep inputs from [Voe00]) identify, over a point, the heart of the weight structure on $DM_{\mathrm{B},c}(\{\bullet\})$ with the category of *Chow motives*, the following definition is very natural:

Definition 3.3.1.5. *The $\mathbb{Q}$-linear category $CHM(\mathcal{S})$ of* Chow motives over $\mathcal{S}$ *is defined as the heart $DM_{\mathrm{B},c}(\mathcal{S})_{w=0}$ of the motivic weight structure.*

But then, recall that Corti and Hanamura have defined in [CH00], for a general quasi-projective base $\mathcal{S}$, a $\mathbb{Q}$-linear, pseudo-abelian category of *Chow motives over $\mathcal{S}$*, starting from the category of projective schemes over $\mathcal{S}$ which are smooth (only) over the base field, and containing the category $CHM^s(\mathcal{S})$ of smooth Chow motives over $\mathcal{S}$ of Section 3.1 as a full subcategory. Thanks to the following result, we are allowed to denote their category by $CHM(\mathcal{S})$, too:

Theorem 3.3.2. *([Fan16]) The category of Chow motives over $\mathcal{S}$ of Corti and Hanamura is canonically identified with the heart $DM_{\mathrm{B},c}(\mathcal{S})_{w=0}$ of the motivic weight structure on $DM_{\mathrm{B},c}(\mathcal{S})$.*

This is the final link between Beilinson motives and Chow motives that we wanted to emphasize. Let us conclude this subsection by observing that over the spectrum of a field k of characteristic 0 (but see [Wil09, Cor. 1.14] for more general results) the (cohomological) motive and *motive with compact support* of a smooth variety satisfy the estimates on weights which one expects from Hodge theory:

Proposition 3.3.1.6. *([Bon10, dual of Thms. 6.2.1(1) and (2)]) Let X be smooth over k of dimension d_X, with structure morphism a. Then its cohomological motive $a_*\mathbb{1}_X \in DM_{\mathrm{B},c}(k)$ lies in*

$$DM_{\mathrm{B},c}(k)_{w\geq 0} \cap DM_{\mathrm{B},c}(k)_{w\leq d_X}$$

and its cohomological motive with compact support $a_!\mathbb{1}_X \in DM_{\mathrm{B},c}(k)$ lies in

$$DM_{\mathrm{B},c}(k)_{w\geq -d_X} \cap DM_{\mathrm{B},c}(k)_{w\leq 0}$$

3.3.2 A criterion for the existence of the intersection motive

Come back now to the setting and to the notations of Subsection 3.2.2. Hence, $\tilde{s} : S_K \to \operatorname{Spec} E$ is a PEL-type Shimura variety, associated to a group G which is split over a field L, and we have objects ${}^{\lambda}\mathcal{V}$ of the category $CHM^s(S_K)_L$ corresponding to irreducible representations of G_L of highest weight λ. How to extract the "weight-zero part" (hence, by what we said before Def. 3.3.1.5, a *Chow motive* over E) of the object $\tilde{s}_*^{\lambda}\mathcal{V} \in DM_{\text{B},c}(E)_L$, in such a way that the algebra $\mathfrak{H}^{DM}(K, G(\mathbb{A}_f))$ (Def. 3.2.2.13) still act on it?

Let us begin by explaining what could be the naive approach for constructing "the weight-zero (Chow) submotive" of the cohomological motive $a_*\mathbb{1}_X \in DM_{\text{B},c}(E)$ of a smooth variety $a : X \to E$.

Approach 1. By using axiom (4) of weight structures and applying a shift, the motive $M := a_*\mathbb{1}_X$ fits in an exact triangle

$$M_{\geq 1}[-1] \to M_{\leq 0} \to M \to M_{\geq 1}$$

where the object $M_{\geq 1}$, resp. $M_{\leq 0}$, is of weights at least 1, resp. at most 0. Now, by Prop. 3.3.1.6, M is of weights at least zero, and by axiom (2) of weight structures, this is the case for $M_{\geq 1}[-1]$ too. Thus, by property 3.3.1.4, the object $M_0 := M_{\leq 0}$ belongs to the heart of the weight structure (i.e., it is of weight zero) and hence it is a Chow motive.

Remark 3.3.2.1. The problem of Approach 1 is that we have no guarantees that the object M_0 is unique up to unique isomorphism, and in fact this is far from true: one can prove that any smooth compactification $\tilde{a} : \tilde{X} \to E$ of X provides a weight filtration of M, which identifies the corresponding object M_0 with the Chow motive $\tilde{M} := \tilde{a}_*\mathbb{1}_{\tilde{X}}$. In particular, we can't conclude that endomorphisms of M induce endomorphisms of M_0.

Recall now from Subsection 2.3.3 the Baily-Borel compactification $s : S_K^* \to \operatorname{Spec} E$ of S_K, along with the open, resp. closed immersions $j : S_K \hookrightarrow S_K^*$, $i : \partial S_K \hookrightarrow S_K^*$. Let us denote

$$\partial\tilde{s}_*^{\lambda}\mathcal{V} := s_*i_*i^*j_*\tilde{s}_*^{\lambda}\mathcal{V} \tag{3.20}$$

the *boundary motive* of $\tilde{s}_*^{\lambda}\mathcal{V}$ (an object of $DM_{\text{B},c}(E)_L$ which actually doesn't depend on the choice of the particular compactification S_K^*). There is an approach, due to Wildeshaus, that allows to quantify the obstructions to "the weight-zero part" of $\tilde{s}_*^{\lambda}\mathcal{V}$ being unique up to unique isomorphism (the exposition here follows the lines of [Wil13, Rmk. 1.20]. This approach makes use of the following:

Definition 3.3.2.2. *(cfr. [Wil09, Defs. 1.6-1.10]) Let $\mathcal{S}$ be a base scheme. Let $M \in DM_{\text{B},c}(\mathcal{S})_L$ and let α, β be integers. We say that M* avoids weights $\alpha, \dots, \beta$ *if $\alpha \leq \beta$ and there exists an exact triangle in $DM_{\text{B},c}(\mathcal{S})_L$*

$$M_{\leq \alpha-1} \to M \to M_{\geq \beta+1} \to M_{\leq \alpha-1}[1]$$

such that $M_{\leq \alpha-1}$ is of weight at most $\alpha - 1$ and $M_{\geq \beta+1}$ of weight at least $\beta + 1$.

Such a triangle is called a weight filtration *of M* avoiding weights *$\alpha, \dots, \beta$.*

Proposition 3.3.2.3. *([Wil09, Cor. 1.9]) If $M \in DM_{\text{B},c}(\mathcal{S})_L$ admits a weight filtration avoiding weights $\alpha, \dots, \beta$, then it is unique up to unique isomorphism.*

Approach 2. By the formalism of six functors in the categories $DM_{\text{B},c}$, the motive $\tilde{s}_*^{\lambda}\mathcal{V}$ fits in a (canonical) exact triangle

$$\partial\tilde{s}_*^{\lambda}\mathcal{V}[-1] \xrightarrow{a} \tilde{s}_!^{\lambda}\mathcal{V} \xrightarrow{u} \tilde{s}_*^{\lambda}\mathcal{V} \xrightarrow{b} \partial\tilde{s}_*^{\lambda}\mathcal{V} \tag{3.21}$$

which is the motivic analogue of (a shift of) the boundary triangle of (2.33). By applying the axioms of weight structures, take a weight filtration of $\partial\tilde{s}_*^\lambda\mathcal{V}$ as in Def. 3.3.1.3 (for $n = 0$) and shift it by -1: we have

$$\partial\tilde{s}_*^\lambda\mathcal{V}_{\leq 0}[-1] \to \partial\tilde{s}_*^\lambda\mathcal{V}[-1] \stackrel{\delta}{\to} \partial\tilde{s}_*^\lambda\mathcal{V}_{\geq 1}[-1] \to \partial\tilde{s}_*^\lambda\mathcal{V}_{\leq 0} \tag{3.22}$$

Consider now the morphism

$$\partial\tilde{s}_*^\lambda\mathcal{V}_{\leq 0}[-1] \to \tilde{s}_!^\lambda\mathcal{V}$$

obtained by composition from the one with source $\partial\tilde{s}_*^\lambda\mathcal{V}_{\leq 0}[-1]$ in the weight filtration (3.22) and the one with target $\tilde{s}_!^\lambda\mathcal{V}$ in the triangle (3.21), and choose a cone (M_0, pr) of such a morphism, i.e. complete it to a triangle

$$\partial\tilde{s}_*^\lambda\mathcal{V}_{\leq 0}[-1] \to \tilde{s}_!^\lambda\mathcal{V} \stackrel{pr}{\to} M_0 \to \partial\tilde{s}_*^\lambda\mathcal{V}_{\leq 0} \tag{3.23}$$

Thanks to Thm. 3.3.2, the object ${}^\lambda\mathcal{V}$, being a relative Chow motive, is of weight zero, and hence $\tilde{s}_!^\lambda\mathcal{V}$ belongs to $DM_{\mathrm{B},c}(E)_{L,w\leq 0}$, since, by Thm. 3.3.1, $\tilde{s}_!$ doesn't increase weights. Thus, Property 3.3.1.4 shows that the object M_0 belongs to $DM_{\mathrm{B},c}(E)_{L,w\leq 0}$, too, and given its construction, we could informally think to it as a "quotient" of $\tilde{s}_!^\lambda\mathcal{V}$ of weights at most zero.

But then, the *octahedral axiom* of triangulated categories says in particular that one can choose morphisms δ' and in such that

$$in \circ pr = u \ , \ \ \delta' \circ \delta = pr \circ a$$

and such that the triangle

$$\partial\tilde{s}_*^\lambda\mathcal{V}_{\geq 1}[-1] \stackrel{\delta'}{\to} M_0 \stackrel{in}{\to} \tilde{s}_*^\lambda\mathcal{V} \stackrel{\delta[1]b}{\to} \partial\tilde{s}_*^\lambda\mathcal{V}_{\geq 1} \tag{3.24}$$

is exact. Since, by axiom (2), the object $\partial\tilde{s}_*^\lambda\mathcal{V}_{\geq 1}[-1]$ is of weights at least 0, and since the same is true for $\tilde{s}_*^\lambda\mathcal{V}$ by theorem 3.3.1, Property 3.3.1.4 shows that the object M_0 belongs to $DM_{\mathrm{B},c}(E)_{L,w\geq 0}$. Since the triangle (3.24) exhibits (M_0, δ') as a shift by -1 of a cone of $\tilde{s}_*^\lambda\mathcal{V} \stackrel{\delta[1]b}{\to} \partial\tilde{s}_*^\lambda\mathcal{V}_{\geq 1}$, we see that we can also informally think to M_0 as the shift by -1 of a "quotient" of $\partial\tilde{s}_*^\lambda\mathcal{V}_{\geq 1}$ of weights at least 1.

Much more importantly, being at the same time of weights at most and least 0, M_0 is of weight zero, i.e. a *Chow motive*, through which the morphism $\tilde{s}_!^\lambda\mathcal{V} \stackrel{u}{\to} \tilde{s}_*^\lambda\mathcal{V}$ factors.

The problem now becomes *(a)* to select canonical choices of the objects $\partial\tilde{s}_*^\lambda\mathcal{V}_{\leq 0}$ and $\partial\tilde{s}_*^\lambda\mathcal{V}_{\geq 1}$, and *(b)* to make the data (M_0, pr, in) unique up to unique isomorphism (if these two points are settled, then δ, δ' will be canonically determined, too). For (*b*), start by fixing a second cone (M_0', pr') (hence equipped with *some* isomorphism $(M_0, pr) \simeq (M_0', pr')$): then, supposing to have *two* isomorphisms ι_1, ι_2 between the two cones, consideration of the diagram of exact triangles

$$\begin{array}{ccccccc}
\partial\tilde{s}_*^\lambda\mathcal{V}_{\leq 0}[-1] & \longrightarrow & \tilde{s}_!^\lambda\mathcal{V} & \stackrel{pr}{\longrightarrow} & M_0 & \stackrel{d}{\longrightarrow} & \partial\tilde{s}_*^\lambda\mathcal{V}_{\leq 0} \\
\| & & \| & & \iota_1 \downarrow\downarrow \iota_2 & & \| \\
\partial\tilde{s}_*^\lambda\mathcal{V}_{\leq 0}[-1] & \longrightarrow & \tilde{s}_!^\lambda\mathcal{V} & \stackrel{pr'}{\longrightarrow} & M_0' & \longrightarrow & \partial\tilde{s}_*^\lambda\mathcal{V}_{\leq 0}
\end{array}$$

shows that since $(\iota_1 - \iota_2) \circ pr = 0$, then $\iota_1 - \iota_2 = \gamma \circ d$ for some element γ in

$$\mathrm{Hom}_{DM_{\mathrm{B},c}(E)_L}(\partial\tilde{s}_*^\lambda\mathcal{V}_{\leq 0}, M_0').$$

If the latter space is zero, then $\iota_1 = \iota_2$, and (M_0, pr) is unique up to unique isomorphism. An analogous diagram shows that we would be sure that (M_0, in) is unique up to unique isomorphism if, for some object M_0' isomorphic to M_0, the space

$$\mathrm{Hom}_{DM_{\mathrm{B},c}(E)_L}(M_0', \partial\tilde{s}_*^{\lambda}\mathcal{V}_{\geq 1}[-1]).$$

was zero. By axiom (3) of weight structures, the two spaces would be zero if one could actually choose $\partial\tilde{s}_*^{\lambda}\mathcal{V}_{\leq 0}$ *of weights at most* -1, and $\partial\tilde{s}_*^{\lambda}\mathcal{V}_{\geq 1}$ *of weights at least* 2. Now, by Proposition 3.3.2.3, if $\partial\tilde{s}_*^{\lambda}\mathcal{V}$ *avoids weights* 0 and 1 in sense of the above definition, then not only such choices exist, but they are also unique up to unique isomorphism, thus solving problem *(a)*, too.

Remark 3.3.2.4. (1) Approach 2 leads us to the problem of determining when the boundary motive $\partial\tilde{s}_*^{\lambda}\mathcal{V}$ avoids weight 0 and 1. Since *proper* morphisms respect weights (Thm. 3.3.1), a sufficient condition for this is that, for *some* compactification $\bar{S}_K$ with open, resp. closed immersions $j : S_K \hookrightarrow \bar{S}_K$, $i : \partial\bar{S}_K \hookrightarrow \bar{S}_K$, the *relative* boundary motive $i^* j_*^{\lambda}\mathcal{V} \in DM_{\mathrm{B},c}(\partial\bar{S}_K)_L$ avoids weight 0 and 1.

(2) By the minimality property of the Baily-Borel compactification S_K^* (Prop. 2.3.3.1) and by the fact that proper morphisms respect weights, if the relative boundary motive with respect to some simple normal crossing compactification (e.g., a toroidal compactification) avoids weight 0 and 1, then the relative boundary motive with respect to the Baily-Borel compactification does. Hence, we will study the weight avoidance on the latter, i.e. we will study, given an object $M \in CHM(S_K)_L$, the motive $i^* j_* M \in DM_{\mathrm{B},c}(\partial S_K^*)_L$.

To explain that Approach 2 and the above remark do actually give rise to a construction of the Chow motive we are looking for, we start by fixing some notation.

Definition 3.3.2.5. *(1) We denote by $DM_{\mathrm{B},c}(E)_{L,w\leq 0,\neq -1}$ the full subcategory of $DM_{\mathrm{B},c}(E)_{L,w\leq 0}$ of objects avoiding weight -1, and by $DM_{\mathrm{B},c}(E)_{L,w\geq 0,\neq 1}$ the full subcategory of $DM_{\mathrm{B},c}(E)_{L,w\geq 0}$ of objects avoiding weight 1.*

(2) We denote by $CHM(S_K)_{L,\partial w\neq 0,1}$ the full subcategory of $CHM(S_K)_L$ of objects M such that $i^* j_* M$ avoids weights 0 and 1, and by $CHM(S_K^*)_{L,i^*w\leq -1,i^!w\geq 1}$ the full subcategory of objects M such that $i^* M$ is of weights at most -1 and $i^! M$ is of weights at least 1.

Then, as a first step, arguments similar to the ones outlined in Approach 2 give the following:

Proposition 3.3.2.6. *(cfr. [Wil19a, Prop. 3.3]) The inclusions*

$$\iota_- : CHM(E)_L \hookrightarrow DM_{\mathrm{B},c}(E)_{L,w\leq 0,\neq -1}$$

and

$$\iota_+ : CHM(E)_L \hookrightarrow DM_{\mathrm{B},c}(E)_{L,w\geq 0,\neq 1}$$

admit a left adjoint

$$\mathrm{Gr}_0 : DM_{\mathrm{B},c}(E)_{L,w\leq 0,\neq -1} \to CHM(E)_L$$

and a right adjoint

$$\mathrm{Gr}_0 : DM_{\mathrm{B},c}(E)_{L,w\geq 0,\neq 1} \to CHM(E)_L$$

respectively. Both adjoints map objects (and morphisms) to the term of weight zero of a weight filtration avoiding weight 1 and -1, respectively. The compositions $\mathrm{Gr}_0 \circ \iota_-$ and $\mathrm{Gr}_0 \circ \iota_+$ both equal the identity on $CHM(E)_L$.

As a second step, the following theorem and its corollary give a source of objects to which the above functors can be applied (and much more than that).

Theorem 3.3.3. *(cfr. [Wil19a, Thm. 2.2, Def.2.4]) The restriction of j^* to*

$$CHM(S_K^*)_{L,i^*w\leq -1,i^!w\geq 1}$$

induces an equivalence of categories

$$CHM(S_K^*)_{L,i^*w\leq -1,i^!w\geq 1} \simeq CHM(S_K)_{L,\partial w\neq 0,1}$$

the composition of whose inverse with the inclusion $CHM(S_K^*)_{L,i^*w\leq -1,i^!w\geq 1} \hookrightarrow CHM(S_K^*)_L$ is denoted by

$$j_{!*} : CHM(S_K)_{L,\partial w\neq 0,1} \to CHM(S_K^*)_L$$

The functor $j_{!*}$ is such that, for every n, for every prime ℓ, the diagram

$$\begin{array}{ccc} CHM(S_K)_{L,\partial w\neq 0,1} & \xrightarrow{j_{!*}} & CHM(S_K^*)_L \\ \downarrow{\scriptstyle H^n\circ\mathcal{R}_\ell} & & \downarrow{\scriptstyle H^n\circ\mathcal{R}_\ell} \\ \mathrm{Perv}(\mathrm{Et})_{\ell,L}(S_K) & \xrightarrow{j_{!*}} & \mathrm{Perv}(\mathrm{Et})_{\ell,L}(S_K^*) \end{array}$$

commutes (where, for a base scheme $\mathcal{S}$, H^n are the perverse cohomology functors on $D^b_{c,\text{ét}}(\mathcal{S})_L$, and the lower $j_{!*}$ denotes the intermediate extension of perverse sheaves, cfr. Sec. 2.3.3).

Notice that the second part of the statement comes from [Wil19a, Rmk. 2.6 (d)] and [Wil17, Thm. 7.2 (b)].

Corollary 3.3.2.7. *(cfr. [Wil19a, Thm. 3.4])*

(1) The essential image of the restriction of the functor $\tilde{s}_!$ to the subcategory $CHM(S_K)_{L,\partial w\neq 0,1}$ is contained in $DM_{B,c}(E)_{L,w\leq 0,\neq -1}$.

(2) The essential image of the restriction of the functor $\tilde{s}_*$ to the subcategory $CHM(S_K)_{L,\partial w\neq 0,1}$ is contained in $DM_{B,c}(E)_{L,w\geq 0,\neq 1}$.

(3) Denote by the same symbols the restrictions of the above functors to $CHM(S_K)_{L,\partial w\neq 0,1}$. There are canonical isomorphisms of functors

$$\mathrm{Gr}_0 \circ \tilde{s}_! \simeq s_* j_{!*} \text{ and } s_* j_{!*} \simeq \mathrm{Gr}_0 \circ \tilde{s}_*$$

and if $m : \tilde{s}_! \to \tilde{s}_*$ is the standard natural transformation, then, denoting by the same symbol its restriction to $CHM(S_K)_{L,\partial w\neq 0,1}$, their composition is given by

$$\mathrm{Gr}_0 \circ m : \mathrm{Gr}_0 \circ \tilde{s}_! \to \mathrm{Gr}_0 \circ \tilde{s}_*$$

(which is then an isomorphism of functors on $CHM(S_K)_{L,\partial w\neq 0,1}$).

Observe the following: if we suppose ${}^\lambda\mathcal{V}$ to belong to $CHM(S_K)_{L,\partial w\neq 0,1}$, then by the six functors formalism, since by definition $j^* j_{!*}^\lambda\mathcal{V} \simeq {}^\lambda\mathcal{V}$, the object $j_{!*}^\lambda\mathcal{V}$ fits in a commutative diagram of triangles

$$
\begin{array}{ccccccc}
0 & \longrightarrow & i_*i^! j_{!*}^{\lambda}\mathcal{V} & = & i_*i^! j_{!*}^{\lambda}\mathcal{V} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
j_!^{\lambda}\mathcal{V} & \longrightarrow & j_{!*}^{\lambda}\mathcal{V} & \longrightarrow & i_*i^* j_{!*}^{\lambda}\mathcal{V} & \longrightarrow & j_!^{\lambda}\mathcal{V}[1] \\
\| & & \downarrow & & \downarrow & & \| \\
j_!^{\lambda}\mathcal{V} & \longrightarrow & j_*^{\lambda}\mathcal{V} & \longrightarrow & i_*i^* j_*^{\lambda}\mathcal{V} & \longrightarrow & j_!^{\lambda}\mathcal{V}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & i_*i^! j_{!*}^{\lambda}\mathcal{V}[1] & = & i_*i^! j_{!*}^{\lambda}\mathcal{V}[1] & \longrightarrow & 0
\end{array}
$$

which is the motivic analogue of the one of Proposition 2.3.3.3. This explains how the corollary follows from the equivalence of categories of the previous theorem.

We are now in position, provided a *weight avoidance* hypothesis on $\partial^{\lambda}\mathcal{V}$ is verified, to define a natural candidate for our sought-for Chow motive:

Definition 3.3.2.8. *If $i^*j_*^{\lambda}\mathcal{V}$ avoids weights 0 and 1 (i.e. if ${}^{\lambda}\mathcal{V}$ belongs to $CHM(S_K)_{L,\partial w\neq 0,1}$), the* intersection motive *of S_K (*relative to *S_K^* and* with coefficients in *${}^{\lambda}\mathcal{V}$) is defined as the Chow motive $s_*j_{!*}^{\lambda}\mathcal{V}$.*

Notice that, by the above theorem, we could have equivalently defined our candidate for the "weight-zero" object as $\mathrm{Gr}_0(\tilde{s}_*^{\lambda}\mathcal{V})$ (or $\mathrm{Gr}_0(\tilde{s}_!^{\lambda}\mathcal{V})$); but the existence of the *motivic intermediate extension* functor $j_{!*}$ on $CHM(S_K)_{L,\partial w\neq 0,1}$ allows for a definition which adds important information on this object, and justifies the terminology *intersection motive*. We will expand on this in Chapter 5, notably by clarifying the behaviour of the intersection motive with respect to realizations. This will permit to cut out (homological) submotives of the intersection motive corresponding to automorphic representations.

For the moment being, our task becomes to understand when $i^*j_*^{\lambda}\mathcal{V}$ avoids weights 0 and 1.

Remark 3.3.2.9. (1) One could hope to determine if $i^*j_*^{\lambda}\mathcal{V}$ avoids weight 0 and 1 by first proving that the weights 0 and 1 don't appear in its ℓ-adic realization (Property 3.1.0.1 and.(5); cfr. Rmk. 3.1.0.2) and then appealing to a *weight conservativity* result to deduce that this is already true for the motive itself. This is actually (part of) the content of the criterion that we will give below.

(2) The strategy outlined in the preceding point is consistent with the fact that, in the ℓ-adic realizations of the relative boundary motive, weight 0 appears if and only if weight 1 appears (cfr. Corollary 2.3.2.2 and its proof).

In fact, Wildeshaus has also found a powerful criterion to verify the desired weight avoidance, which, again, exploits the existence and the properties of the functor $j_{!*}$ on $CHM(S_K)_{L,\partial w\neq 0,1}$. We begin by introducing the subcategory of Beilinson motives which enters in the proof of this criterion.

Definition 3.3.2.10. *(Cfr. [Wil19b, Def. 2.1]) An object $M \in DM_{\mathrm{B},c}(\partial S_K^*)_L$ is said to be a* motive of abelian type *over ∂S_K^*, and a stratification Φ of ∂S_K^* is said to be* adapted to *M, if the following condition is verified: the motive M belongs to the strict, full, dense, F-linear triangulated subcategory $DM_{\mathrm{B},c,\Phi}^{Ab}(\partial S_K^*)_L$ generated by the images via π_* of the* $\mathfrak{S}$-constructible Tate motives *([Wil17, Def. 4.6 (a)]) over $S(\mathfrak{S})$, where*

$$\pi : S(\mathfrak{S}) \to \partial S_K^*$$

runs over the morphisms of abelian type *([Wil17, page 579]) whose target is* ∂S_K^*.

Now recall that by Subsection 2.3.3, ∂S_K^* admits a natural stratification $\partial S_K^* = \bigsqcup_{m\in\Phi} Z_m$, indexed by conjugacy classes of maximal parabolics of G and denoted itself by Φ. We denote then by $i_m : Z_m \hookrightarrow \partial S_K^*$ the corresponding closed immersions.

Proposition 3.3.2.11. *The motive* $i^* j_*^\lambda \mathcal{V} \in DM_{\text{B},c}(\partial S_K^*)_L$ *is of abelian type, and* Φ *is adapted to* $i^* j_*^\lambda \mathcal{V}$.

Proof. The proof of [Wil19b, Thm. 2.2] can be translated word by word in our setting. We include a sketch for the convenience of the reader: by Remark 3.2.1.6, ${}^\lambda\mathcal{V}$ is an object of the $\mathbb{Q}$-linear triangulated category $\pi_{n,*}DMT(\mathcal{A}_K^n)^\natural$ generated by the images via $\pi_{n,*}$ of the objects of the category of Tate motives over the fibred product $\pi_n : A_K^n \to S_K$ of the universal abelian variety over S_K. But then, since the pure Hodge structure on the vector space V in the Shimura datum underlying S_K is of type $(-1,0),(0,-1)$ (Rmk. 3.2.1.4), it is standard (cfr. [Pin90, Prop. 2.17]) to identify A_K^n with a *mixed Shimura variety*[4] M_{K_n} and the morphism π_n with a morphism induced by a morphism of *mixed Shimura data* ([Pin90, De. 2.1]). By choosing a suitable smooth *toroidal compactification* $M_{K_n}(\mathfrak{S})$ of M_{K_n} (as in [Pin90, proof of Thm. 9.21]), one has that π_n extends to a surjective, proper morphism $\pi_n : M_{K_n}(\mathfrak{S}) \to S_K^*$ ([Pin90, Sect. 6.24, Main Theorem 12.4 (b)]) which is seen to be *of abelian type* ([Wil17, Lemma 8.4]); hence, the category $\pi_{n,*}DMT_{\mathfrak{S}}(\pi_n^{-1}(\partial S_K^*))^\natural$ generated by the images via $\pi_{n,*}$ of $\mathfrak{S}$-constructible Tate motives over $\pi_n^{-1}(\partial S_K^*)$ is contained inside $DM^{Ab}_{\text{B},c,\Phi}(\partial S_K^*)_L$. Now, one sees that $i^* j_*^\lambda \mathcal{V}$ is an object of $\pi_{n,*}DMT_{\mathfrak{S}}(\pi_n^{-1}(\partial S_K^*))^\natural$. This follows formally from the fact that the categories of constructible Tate motives are well-behaved with respect to the six functors formalism ([Wil17, Cor. 4.10 (b), Rem. 4.7]). □

Given this, one can prove the weight avoidance by reduction to a *stratum-by-stratum* study of the weights (over ∂S_K^*) of the ℓ-adic realization of $i^* j_*^\lambda \mathcal{V}$.

Theorem 3.3.4. *Let* $\beta \geq 1$ *be an integer,* d_{S_K} *the dimension of* S_K*,* $w(\lambda)$ *the weight of the pure Hodge structure on* V_λ*, and fix a prime number* ℓ*. For any stratum* $\mathcal{Z}$ *of* ∂S_K^**, denote by* $\mathcal{H}^n$ *the* n*-th perverse cohomology functor on* $D^b_{c,\text{ét}}(\mathcal{Z})_L$ *and write* $j_{!*}(\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ *for*

$$\left(j_{!*}(\mathcal{R}_\ell({}^\lambda\mathcal{V})[w(\lambda)+d_{S_K}])\right)[-w(\lambda)-d_{S_K}].$$

The following assertions are then equivalent:

(1) the motive $i^* j_*^\lambda \mathcal{V}$ *avoids weights* $-\beta+1, -\beta+2, \ldots, \beta$*;*

(2) for every $m \in \Phi$*, for every* $n \in \mathbb{Z}$*, the perverse sheaves* $\mathcal{H}^n i_m^* i^* j_{!*}(\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ *are of weights* $\leq n-\beta$*.*

Proof. Since the ℓ-adic realization of the motive ${}^\lambda\mathcal{V}$ is concentrated in only one perverse degree (Remark 3.2.1.8.(2)) and is autodual (up to a twist and a shift) (Remark 3.2.1.8.(3)) everything follows from [Wil19a, Corollary (3.6)(b)], because we have proved that the motive $i^* j_*^\lambda \mathcal{V} \in DM_{\text{B},c}(\partial S_K^*)_L$ belongs to the subcategory $DM^{Ab}_{\text{B},c,\Phi}(\partial S_K^*)_L$. □

Remark 3.3.2.12. (1) In the above statement we are using the intrinsic notion of *weights* existing on those ℓ-adic perverse sheaves on ∂S_K^* which are in the image of the cohomological realization ([Bon15, Prop. 2.5.1 (II)]). Thus, the notion of *weight-graded* objects

[4] See Rmk. 3.2.2.1 for the case $n = 1$.

of such perverse sheaves makes sense, even if there is no weight structure on the ambient category inducing such weights.

(2) The key point of the above argument is that it is on the subcategory $DM^{Ab}_{\text{Б},c,\Phi}(\partial S^*_K)_L$ that one can invoke the *weight conservativity* argument alluded to in Remark 3.3.2.9 and contained in [Wil18], of which [Wil19a, Corollary (3.6)(b)] makes use. This wouldn't have worked for the motive $s_* i^* j^\lambda_* \mathcal{V}$ (over a point), since it is no more of Abelian type. This explains the necessity of employing the *relative* category $DM_{\text{Б},c}(\partial S^*_K)_L$.

Remark 3.3.2.13. This subsection gives a deep motivation for investigating the presence of the weights 0 and 1 in the boundary cohomology of local systems arising from the canonical construction, a problem that was already raised in Remark 2.3.2.3. Theorem 2.3.2 tells us that *regularity* of the representation-theoretic weight λ implies that boundary cohomology avoids weight 0 in *half of the cohomological degrees*, and (hence) weight 1 in the other half. Regularity may be conjectured to actually imply the full weight avoidance ([Wil19a, Question 5.13]). The next chapter studies a family of PEL Shimura varieties for which we can completely characterize the presence of the weights 0 and 1 in the *relative* boundary motive in terms of the regularity of λ and of its *corank* as introduced in Def. 2.3.4.5.

Chapter 4

The boundary motive of genus 2 Hilbert-Siegel varieties

This chapter contains the main results of this thesis. We will work with genus 2 Hibert-Siegel varieties S_K, i.e. the Shimura varieties defined in Ex. 2.1.3.2.(2), whose underlying group G is (a modification of) $\mathrm{Res}_{F|\mathbb{Q}} \mathrm{GSp}_{4,F}$, for F a totally real number field of degree d. Then, if ${}^{\lambda}\mathcal{V}$ is the Chow motive attached to a irreducible representation V_λ of G (Def. 3.2.1.7), our purpose is to show that we can completely characterize the absence of the weights 0 and 1 in the relative boundary motive $i^*j_*^{\lambda}\mathcal{V}$ introduced in Subsection 3.3.2. Thanks to Corollary 3.3.2.7, this will open the way to the construction of motives for automorphic representations of $\mathrm{Res}_{F|\mathbb{Q}} \mathrm{GSp}_{4,F}$, which will be studied in the next chapter.

Actually, as explained in the introduction, we will prove more: in fact, our main result, Theorem 4.3.1, will give a general description of the first positive and negative weights appearing in $i^*j_*^{\lambda}\mathcal{V}$, in the sense of Rmk. 2.3.2.3 (suitably generalized to our situation). This description will make use of the *corank* of λ (Def. 4.3.1.1).

Theorem 3.3.4 tells us that in order to understand the weights of the complex $i^*j_*^{\lambda}\mathcal{V}$, we have to take its ℓ-adic realization, restrict it to each stratum of the boundary of S_K^*, and study the weights of the *perverse* cohomology objects of each one of the resulting ℓ-adic complexes.

In Section 4.1 we collect all the facts that we need about the structure of $\mathrm{Res}_{F|\mathbb{Q}} \mathrm{GSp}_{4,F}$, of its parabolic subgroups and of its representation theory. Then, in Section 4.2, we give the necessary tools for carrying out the analysis of the weights of the perverse cohomology sheaves: (1) the precise structure of the boundary of S_K^* in our case (a disjoint union of cusps and Hilbert modular varieties of dimension d); (2) a theorem of Pink, which gives a formula for the ℓ-adic *classical* cohomology sheaves of the restriction of the degeneration to each stratum, in terms of cohomology of unipotent algebraic groups and of arithmetic groups; (3) a theorem of Kostant, which allows one to express the cohomology of unipotent groups in terms of representations of subgroups, which are attached to the Shimura data underlying the strata of the boundary; (4) some general lemmas which are useful for studying the cohomology of free abelian subgroups of arithmetic groups.

In Section 4.3 we begin by stating the main result cited above (Thm. 4.3.1) and by showing its main consequence (the characterisation of the absence of weights 0 and 1, Cor. 4.3.1.3). The rest of the section is occupied by the proof of Thm. 4.3.1, which is divided in the following steps:

- we begin by studying separately the classical cohomology sheaves of the degeneration along the 0-dimensional strata (Subsection 4.3.2) and along the strata of dimension $d = [F : \mathbb{Q}]$ (Subsection 4.3.3);

- in each of the two above cases, we decompose the cohomology of unipotent groups into a sum of irreducible representations carrying pure Hodges structures (paragraphs 4.3.2.1, 4.3.3.1). Thus, we get a list of the *possible* weights appearing;

- in each case, we use the cohomology of arithmetic groups to give restrictions on the non-triviality of the occurring spaces, and hence to give *necessary* conditions for certain weights to appear (paragraphs 4.3.2.2, 4.3.3.2). The main technical ideas here consist in exploiting, often by spectral sequence arguments, the action of suitable subgroups of units of F on the fibers of the degeneration (Lemmas 4.3.2.7, 4.3.2.10 - where the action is exploited in two "orthogonal" ways, Lemmas 4.3.3.5) and some vanishing theorems on the cohomology of locally symmetric spaces (Lemma 4.3.2.11);

- some additional work is needed (as an existence statement for suitable non-trivial Hilbert cusp modular forms, relying on recent results from [MSSYZ15], see Prop. 4.3.2.13) to show that the above conditions are also *sufficient* for some weights to appear;

- finally, we have to relate the weights of the classical cohomology sheaves to those of the perverse cohomology sheaves appearing in Thm. 3.3.4. For this, we are led to study the *double* degeneration along the cusps of the d-dimensional strata, along the same lines as before (Subsection 4.3.4), and to use the *perverse* t-structure (Subsection 4.3.5, where we complete the proof of the main theorem, i.e. we find the description of the relation between λ and the weights of the motive $i^* j_*^{\lambda}\mathcal{V}$).

Some interesting points appearing in the course of the proof are the geometric significance of the action of the groups of units (Rmk. 4.30), the parallels with the automorphic interpretation of cohomology described in Subsections 2.2.2 and 2.3.2 (Rmk. 4.3.2.14, Rmk. 4.3.4.7) and the role of some complex analytic Hilbert modular varieties, not appearing in the boundary (Rmk. 4.3.2.14, 4.3.5.8).

In all of this chapter, F will denote our fixed totally real field of degree d over $\mathbb{Q}$, and I_F its set of real embeddings (hence, of cardinality d). We fix moreover a Galois closure L of F in $\mathbb{C}$.

4.1 Preliminaries: the underlying group

Recall from Example 2.1.2.2.(1) the group GSp_{2n} of *symplectic similitudes* of dimension $2n$. It is a reductive group over $\mathbb{Q}$, whose center Z is isomorphic to $\mathbb{G}_m$ and whose derived subgroup is isomorphic to Sp_{2n}, the usual *symplectic group* over $\mathbb{Q}$ of dimension $2n$. Remember that the morphism $\nu : \mathrm{GSp}_{2n} \to \mathbb{G}_m$ entering in the definition is called the *multiplier* (or *similitude factor*).

For the rest of this chapter, fix $n = 2$. Define then the $\mathbb{Q}$-algebraic group $\tilde{G}$ by posing

$$\tilde{G} := \mathrm{Res}_{F|\mathbb{Q}}\mathrm{GSp}_{4,F}.$$

For every subfield k of $\mathbb{C}$ containing L, one has, for every k-algebra R, an isomorphism

$$F \otimes_{\mathbb{Q}} R \tilde{\to} \prod_{\sigma \in I_F} R, \; f \otimes r \mapsto (\sigma(f) \cdot r)_\sigma \tag{4.1}$$

which induces a canonical isomorphism

$$\tilde{G}_k \simeq \prod_{\sigma\in I_F} (\mathrm{GSp}_{4,k})_\sigma,$$

In the rest of this chapter, the symbol G will denote the reductive subgroup of $\tilde{G}$ defined in Example 2.1.3.2, i.e. (with the notations introduced there)

$$G := \mathbb{G}_m \times_{\mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F}} \tilde{G}, \tag{4.2}$$

Remark 4.1.0.1. (1) The isomorphism (4.1) induces, for every subfield k of $\mathbb{C}$ containing L, an isomorphism

$$G_k \simeq \mathbb{G}_m \times_{\prod_{\sigma\in I_F}(\mathbb{G}_{m,k})_\sigma, \prod \nu} \prod_{\sigma\in I_F} (\mathrm{GSp}_{4,k})_\sigma. \tag{4.3}$$

(2) The center of G is such that $Z(G) \simeq \mathbb{G}_m \times_{\mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F}, x\mapsto x^2} \mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F}$. Its neutral component is then isogenous to $\mathbb{G}_m$.

4.1.1 The structure of parabolic subgroups of G

The group $\mathrm{GSp}_{4,F}(F)$ acts on $F^{\oplus 4}$ through the natural action induced by its inclusion into $\mathrm{GL}_{4,F}(F)$.

The standard F-basis $\{e_1, e_2, e_3, e_4\}$ gives then a symplectic basis for the non-degenerated, F-bilinear alternated form defined by $J_2 \in \mathrm{GSp}_{4,F}(F)$, which we will also denote J_2. Fix as a maximal torus of $\mathrm{GSp}_{4,F}$ the standard diagonal torus $\tilde{T}$ defined on F-points by

$$\tilde{T}(F) := \{\mathrm{diag}(\alpha_1, \alpha_2, \alpha_1^{-1}\nu, \alpha_2^{-1}\nu) \mid \alpha_1, \alpha_2, \nu \in \mathbb{G}_m(F)\}, \tag{4.4}$$

along with the standard Borel $\tilde{B}$ containing it, defined on F-points as the subgroup of matrices in $\mathrm{GSp}_{4,F}(F)$ of the form

$$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & & * & * \end{pmatrix}$$

One knows that the parabolic subgroups of $\mathrm{GSp}_{4,F}(F)$ correspond bijectively to subgroups of the form $\mathrm{Stab}(V)$, for V a sub-F-vector space of $F^{\oplus 4}$ which is totally isotropic for the form J_2. The case $V = \{0\}$ corresponds to the whole group $\mathrm{GSp}_{4,F}(F)$, while $V = \langle e_1 \rangle$ gives the *Klingen parabolic*

$$\tilde{Q}_1(F) := \{\begin{pmatrix} \alpha & * & * & * \\ & a & * & b \\ & & \beta & \\ & c & * & d \end{pmatrix} | ad - bc = \alpha\beta \in \mathbb{G}_{m,F}(F)\} \cap \mathrm{GSp}_{4,F}(F)$$

and $V = \langle e_1, e_2 \rangle$ gives the *Siegel parabolic*

$$\tilde{Q}_0(F) := \{\begin{pmatrix} \alpha A & AM \\ & {}^tA^{-1} \end{pmatrix} | \alpha \in \mathbb{G}_{m,F}(F), A \in \mathrm{GL}_{2,F}(F), {}^tM = M\};$$

every other parabolic subgroup is conjugated to one of the above.

One also knows that a maximal torus, resp. a Borel, of $\tilde{G}$ are given by $\mathrm{Res}_{F|\mathbb{Q}}(\tilde{T})$, resp. $\mathrm{Res}_{F|\mathbb{Q}}(\tilde{B})$, which we will still denote by $\tilde{T}$, $\tilde{B}$ in the following; note that $\tilde{T}$ is not split over $\mathbb{Q}$. A maximal torus and a Borel containing it in G are then respectively defined by $T := \mathbb{G}_m \times_{\mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F}} \tilde{T}$ and $B := \mathbb{G}_m \times_{\mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F}} \tilde{B}$.

In the same way, the standard maximal parabolics of $\tilde{G}$ corresponding to the choice $(\tilde{T}, \tilde{B})$ are exactly given, up to conjugation, by $\mathrm{Res}_{F|\mathbb{Q}}\tilde{Q}_0$, $\mathrm{Res}_{F|\mathbb{Q}}\tilde{Q}_1$, which we will still denote by $\tilde{Q}_0$, $\tilde{Q}_1$. Then, $Q_0 := \mathbb{G}_m \times_{\mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F}} \tilde{Q}_0$, $Q_1 := \mathbb{G}_m \times_{\mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F}} \tilde{Q}_1$ are the standard maximal parabolics of G with respect to (T, B), still called the *Siegel* and the *Klingen* one.

4.1.2 The Levi components of parabolic subgroups

Let W_0 and W_1 be the unipotent radicals of the groups Q_0 and Q_1 defined above. The quotients Q_i/W_i will be canonically identified with subgroups of the Q_i's, thanks to the Levi decomposition of the latter.

Fix now a subfield k of $\mathbb{C}$ which contains L. One has the following explicit description of the diagonal embedding of $Q_0/W_0(\mathbb{Q})$ into $Q_0/W_0(k)$:

$$Q_0/W_0(\mathbb{Q}) \simeq \{(\begin{pmatrix} \alpha\sigma(A) & \\ & (\sigma(A)^{-1})^t \end{pmatrix})_{\sigma \in I_F} \,|\alpha \in \mathbb{Q}^\times, A \in \mathrm{GL}_2(F)\} \hookrightarrow$$

$$\hookrightarrow Q_0/W_0(k) = \{(\begin{pmatrix} \alpha A_\sigma & \\ & (A_\sigma^{-1})^t \end{pmatrix})_{\sigma \in I_F} \,|\alpha \in k^\times, A_\sigma \in \mathrm{GL}_2(k) \text{ for every } \sigma\}$$

and of the diagonal embedding of $(Q_1/W_1)(\mathbb{Q})$ into $(Q_1/W_1)(k)$:

$$Q_1/W_1(\mathbb{Q}) \simeq \{(\begin{pmatrix} \sigma(t)\cdot(ad-bc) & & & \\ & \sigma(a) & & \sigma(b) \\ & & \sigma(t^{-1}) & \\ & \sigma(c) & & \sigma(d) \end{pmatrix})_{\sigma \in I_F} \,|t \in F^\times,$$

$$a, b, c, d \in F \text{ such that } ad - bc \in \mathbb{Q}^\times\} \hookrightarrow$$

$$\hookrightarrow Q_1/W_1(k) = \{(\begin{pmatrix} t_\sigma \cdot (a_\sigma d_\sigma - b_\sigma c_\sigma) & & & \\ & a_\sigma & & b_\sigma \\ & & t_\sigma^{-1} & \\ & c_\sigma & & d_\sigma \end{pmatrix})_{\sigma \in I_F} \,|t_\sigma \in k^\times \text{ for every } \sigma,$$

$$a_\sigma, b_\sigma, c_\sigma, d_\sigma \in k \text{ such that } a_\sigma d_\sigma - b_\sigma c_\sigma = a_{\hat{\sigma}} d_{\hat{\sigma}} - b_{\hat{\sigma}} c_{\hat{\sigma}} \in k^\times \text{ for every } \sigma, \hat{\sigma} \in I_F\}.$$

Thus, there is an isomorphism

$$Q_0/W_0 \simeq \mathbb{G}_m \times \mathrm{Res}_{F|\mathbb{Q}}\mathrm{GL}_{2,F}, \tag{4.5}$$

given on k-points by

$$(Q_0/W_0)(k) \simeq \mathbb{G}_m(k) \times (\prod_{\sigma\in I_F} (\mathrm{GL}_2(k))_\sigma)$$
$$\begin{pmatrix} \alpha A_\sigma & \\ & (A_\sigma^{-1})^t \end{pmatrix}_{\sigma\in I_F} \mapsto (\alpha, (A_\sigma)_{\sigma\in I_F}),$$

and an isomorphism

$$Q_1/W_1 \simeq (\mathrm{Res}_{F|\mathbb{Q}}\mathrm{GL}_{2,F} \times_{\mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F},\det} \mathbb{G}_m) \times \mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F} \tag{4.6}$$

given on k-points by

$$(Q_1/W_1)(k) \simeq ((\prod_{\sigma\in I_F} (\mathrm{GL}_{2,k})_\sigma) \times_{\prod_{\sigma\in I_F} (\mathbb{G}_{m,k})_\sigma} \mathbb{G}_{m,k})(k) \times \prod_{\sigma\in I_F} (\mathbb{G}_m(k))_\sigma$$
$$(\begin{pmatrix} t_\sigma\cdot(a_\sigma d_\sigma - b_\sigma c_\sigma) & & & \\ & a_\sigma & & b_\sigma \\ & & t_\sigma^{-1} & \\ & c_\sigma & & d_\sigma \end{pmatrix})_{\sigma\in I_F} \mapsto \left((\begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix})_{\sigma\in I_F},\ (t_\sigma)_{\sigma\in I_F}\right).$$

4.1.3 Characters and dominant weights

Consider our fixed Galois closure L of F. Using the isomorphism (4.3) (for $k = L$) and Eq. (4.4), we get the following description for the points of the maximal torus T_L of G_L:

$$T_L(L) = \{(\mathrm{diag}(\alpha_{1,\sigma}, \alpha_{2,\sigma}, \alpha_{1,\sigma}^{-1}\nu, \alpha_{2,\sigma}^{-1}\nu))_{\sigma\in I_F} \mid \alpha_{1,\sigma}, \alpha_{2,\sigma} \in L^*, \nu \in \mathbb{Q}^\times\}$$

This description naturally identifies T_L with a subtorus of rank $2d-1$ of the rank-$3d$ torus $\prod_{\sigma\in I_F} T_\sigma$, where each T_σ is a copy of the diagonal maximal torus of $\mathrm{GSp}_{4,L}$.

The elements λ of the group $X^*(T_L)$ of characters (or "weights") of T_L (a subgroup of $\bigoplus_{\sigma\in I_F} X^*(T_\sigma)$) are then parametrized by the $(2d+1)$-tuples of integers of the form

$$((k_{1,\sigma}, k_{2,\sigma})_{\sigma\in I_F}, c) \text{ such that } \sum_{\sigma\in I_F} (k_{1,\sigma} + k_{2,\sigma}) \equiv c \pmod 2 \tag{4.7}$$

where the character $\lambda((k_{1,\sigma}, k_{2,\sigma})_{\sigma\in I_F}, c)$ corresponding to $((k_{1,\sigma}, k_{2,\sigma})_{\sigma\in I_F}, c)$ is defined by

$$(\mathrm{diag}(\alpha_{1,\sigma}, \alpha_{2,\sigma}, \alpha_{1,\sigma}^{-1}\nu, \alpha_{2,\sigma}^{-1}\nu))_{\sigma\in I_F} \mapsto \prod_{\sigma\in I_F} \alpha_{1,\sigma}^{k_{1,\sigma}} \cdot \prod_{\sigma\in I_F} \alpha_{2,\sigma}^{k_{2,\sigma}} \cdot \nu^{\frac{1}{2}\cdot[c-\sum_{\sigma\in I_F}(k_{1,\sigma}+k_{2,\sigma})]}. \tag{4.8}$$

The *dominant* weights are the characters such that $k_{1,\sigma} \geq k_{2,\sigma} \geq 0\ \forall\sigma$. A weight is called *regular at* σ if $k_{1,\sigma} > k_{2,\sigma} > 0$ and *regular* if it is regular at σ for every σ.

4.1.4 Root system and Weyl group

The choice of (T_L, B_L) (obtained from the couple (T, B) fixed at the end of 4.1.1, by base change to our fixed Galois closure L of F) allows one to identify the set of roots $\mathfrak{r}$ of G_L with $\bigsqcup_{\sigma\in I_F} \mathfrak{r}_\sigma$, where each $\mathfrak{r}_\sigma$ is a copy of the set of roots of $\mathrm{GSp}_{4,L}$ corresponding to the diagonal torus and the standard Borel. For every fixed $\hat{\sigma} \in I_F$, $\mathfrak{r}_{\hat{\sigma}}$ contains two simple roots $\rho_{1,\hat{\sigma}}$ and $\rho_{2,\hat{\sigma}}$, which, through the inclusion of $\mathfrak{r}_{\hat{\sigma}}$ into $\mathfrak{r}$, can respectively be written $\rho_{1,\hat{\sigma}} = \rho_{1,\hat{\sigma}}((k_{1,\sigma}, k_{2,\sigma})_{\sigma\in I_F}, c)$, with

$$k_{1,\sigma} = \begin{cases} 1 & \text{if } \sigma = \hat{\sigma} \\ 0 & \text{otherwise} \end{cases}, \quad k_{2,\sigma} = \begin{cases} -1 & \text{if } \sigma = \hat{\sigma} \\ 0 & \text{otherwise} \end{cases}, \quad c = 0,$$

and $\rho_{2,\hat{\sigma}} = \rho_{2,\hat{\sigma}}((k_{1,\sigma}, k_{2,\sigma})_{\sigma\in I_F}, c)$, with

$$k_{1,\sigma} = 0 \ \ \forall\sigma, \ \ k_{2,\sigma} = \begin{cases} 2 & \text{if } \sigma = \hat{\sigma} \\ 0 & \text{otherwise} \end{cases}, \quad c = 0.$$

The Weyl group Υ of G_L is in turn isomorphic to the product $\prod_{\sigma\in I_F} \Upsilon_\sigma$, where, for every fixed $\hat{\sigma} \in I_F$, $\Upsilon_{\hat{\sigma}}$ is a copy of the Weyl group of $\mathrm{GSp}_{4,L}$. The latter is a finite group of order 8 acting on $X^*(T_{\hat{\sigma}})$, generated by two elements s_1 and s_2, whose images $s_{\rho_{1,\hat{\sigma}}}$ and $s_{\rho_{2,\hat{\sigma}}}$ through the inclusion $\Upsilon_{\hat{\sigma}}$ into Υ are characterised as follows by their action on the elements of $X^*(T_L)$: if $\lambda = \lambda((k_{1,\sigma}, k_{2,\sigma})_{\sigma\in I_F}, c)$, then $s_{\rho_{1,\hat{\sigma}}}.\lambda = \lambda((h_{1,\sigma}, h_{2,\sigma})_{\sigma\in I_F}, c)$, with

$$h_{1,\sigma} = \begin{cases} k_{2,\sigma} & \text{if } \sigma = \hat{\sigma} \\ k_{1,\sigma} & \text{otherwise} \end{cases}, \quad h_{2,\sigma} = \begin{cases} k_{1,\sigma} & \text{if } \sigma = \hat{\sigma} \\ k_{2,\sigma} & \text{otherwise} \end{cases}$$

and $s_{\rho_{2,\hat{\sigma}}}.\lambda = \lambda((h_{1,\sigma}, h_{2,\sigma})_{\sigma\in I_F}, c)$, with

$$h_{1,\sigma} = k_{1,\sigma} \ \forall\sigma \in I_F, \ \ h_{2,\sigma} = \begin{cases} -k_{2,\sigma} & \text{if } \sigma = \hat{\sigma} \\ k_{2,\sigma} & \text{otherwise} \end{cases}.$$

These descriptions mean that $s_{\rho_{1,\hat{\sigma}}}$ corresponds to the reflection associated to $\rho_{1,\hat{\sigma}}$ and that $s_{\rho_{2,\hat{\sigma}}}$ corresponds to the reflection associated to $\rho_{2,\hat{\sigma}}$.

4.1.5 Irreducible representations

Irreducible representations of a split reductive group over a field of characteristic 0 are parametrized by its dominant weights. By the description of the dominant weights of G_L given in (4.7), we see that isomorphism classes of irreducible L-representations of G_L are in bijection with the set

$$\Lambda := \{\lambda((k_{1,\sigma}, k_{2,\sigma})_{\sigma\in I_F}, c)) \mid k_{1,\sigma}, k_{2,\sigma}, c \in \mathbb{Z} \text{ and } k_{1,\sigma} \geq k_{2,\sigma} \text{ for every } \sigma, \sum_{\sigma\in I_F} (k_{1,\sigma} + k_{2,\sigma}) \equiv c \mod 2)\}.$$

4.2 Preliminaries: some tools for computing the degeneration

4.2.1 The Baily-Borel compactification of genus 2 Hilbert-Siegel varieties

Recall now the genus 2 Hilbert-Siegel varieties S_K arising from the datum (G, X) introduced in Example 2.1.3.2.(2) (for $n = 2$) and their Baily-Borel compactification S_K^* introduced in Subsection 2.3.3. Let us describe in detail the (pure) Shimura data underlying the strata of ∂S_K^*, according to the description in that Subsection.

Each admissible parabolic subgroup Q of G is conjugated to exactly one of the subgroups Q_0 (Siegel parabolic) or Q_1 (Klingen parabolic) defined in 4.1.1. Denote respectively by P_0 and P_1 the canonical normal subgroups of Q_0 and Q_1 considered in 2.3.3. Denote also by G_0, resp. G_1 their quotients by the respective unipotent radicals, and by (G_0, X_0), resp. (G_1, X_1) the associated Shimura data. An immediate generalisation to $\mathrm{Res}_{F|\mathbb{Q}}\mathrm{GSp}_{4,F}$ (and then to G) of [Pin90, 4.25] (which treats the case of GSp_4) gives us the following:

- The group G_0 is identified with the factor $\mathbb{G}_m$ inside $Q_0/W_0 \simeq \mathbb{G}_m \times \mathrm{Res}_{F|\mathbb{Q}}\mathrm{GL}_{2,F}$ (remember (4.5)). Moreover, let k be the morphism $\mathbb{S} \to G_{0,\mathbb{R}}$ which induces on real points, via the above identification,

$$\begin{aligned} \mathbb{S}(\mathbb{R}) &\to G_0(\mathbb{R}) \\ z &\mapsto \left(\begin{pmatrix} z\bar{z}\cdot I_2 & \\ & I_2 \end{pmatrix}\right)_{\sigma\in I_F} \end{aligned} \tag{4.9}$$

  and let X_0 be the set of isomorphisms between $\mathbb{Z}$ and $\mathbb{Z}(1)$. Consider the unique transitive action of $\pi_0(\mathbb{G}_m(\mathbb{R}))$ on X_0 and denote by h_0 the constant map $X_0 \to \{k\} \subset \mathrm{Hom}(\mathbb{S}, G_{0,\mathbb{R}})$. Then, the Shimura datum corresponding to G_0 is given by (G_0, X_0). Thus, G_0 contributes with 0-dimensional strata to ∂S_K^*. (Here is where we need Pink's general definition of a pure Shimura datum, cfr. Rmk. 2.1.2.1.(1)).

- The group G_1 is identified with the factor $\mathrm{Res}_{F|\mathbb{Q}}\mathrm{GL}_{2,F} \times_{\mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F}} \mathbb{G}_m$ inside

$$Q_1/W_1 \simeq (\mathrm{Res}_{F|\mathbb{Q}}\mathrm{GL}_{2,F} \times_{\mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F}} \mathbb{G}_m) \times \mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F}$$

  (remember (4.6)). Denoting by X_1 the $G_1(\mathbb{R})$-conjugacy class of the morphism

$$\begin{aligned} h_1 : \mathbb{S}(\mathbb{R}) &\to G_1(\mathbb{R}) \\ x+iy &\mapsto \left(\begin{pmatrix} x^2+y^2 & & & \\ & x & & y \\ & & 1 & \\ & -y & & x \end{pmatrix}\right)_{\sigma\in I_F} \end{aligned} \tag{4.10}$$

  the Shimura datum corresponding to G_1 is then given by (G_1, X_1). Thus, G_1 contributes with d-dimensional strata to ∂S_K^*. The description of the Shimura datum shows that these strata are in particular isomorphic to (quotients by the action of a finite group of) *Hilbert modular varieties* (cfr. Example 2.1.2.2.(2)).

By the description in (2.38), each stratum of ∂S_K^* corresponds to a Shimura datum of one of the above two types. In particular, it is either of dimension 0 (and it will be then called a *Siegel stratum*) or of dimension d (and it will be then called a *Klingen stratum*).

4.2.2 Pink's theorem

Having in mind the equivalence stated in Thm. 3.3.4, our aim is to compute the weights of certain perverse cohomology sheaves on the boundary of the Baily-Borel compactification of genus 2 Hilbert-Siegel varieties. For this, we will first study the weights of some *classical* sheaves supported on the boundary, through a theorem of Pink which also provides us with the general approach to the computation of boundary cohomology alluded to at the end of Subsection 2.3.3.

It will be useful to state this theorem in general. Hence, let $j : S_K \hookrightarrow S_K^*$ be the open immersion of a Shimura variety S_K, associated to a datum $(\mathcal{G}, X)$ and to a neat compact open subgroup $K \subset \mathcal{G}(\mathbb{A}_f)$, into its Baily-Borel compactification. Recall the finite stratification $(Z_m)_{m\in\Phi}$ of ∂S_K^* introduced in 2.3.3 and for $m \in \Phi$, denote by $i_m : Z_m \hookrightarrow \partial S_K^*$ the corresponding locally closed immersion.

In the following, with the notation of 2.3.3, denote by Z a fixed stratum $S_{m,g}$ of ∂S_K^* contributing to Z_m, and denote by $\pi_m(K_m)$ the associated compact open subgroup $\pi_m(K_{m,g})$ of $\mathcal{G}_m(\mathbb{A}_f)$ (i.e., drop the subscript g), so that Z is the quotient of a Shimura variety $S_{\pi_m(K_m)}$ by the action of the finite group $\Delta_m = H_{Q_m}/P_m(\mathbb{Q})H_{C_m}$ introduced in Eq. (2.42).

Denote again by π_m the projection $Q_m \twoheadrightarrow Q_m/W_m$, and define

$$\Gamma_m := \pi_m(H_{C_m}). \tag{4.11}$$

It is an arithmetic subgroup $Q_m/W_m(\mathbb{Q})$, which is moreover torsion-free (because K is neat).

Notation 4.2.2.1. Since Q_m/W_m is reductive, there exists a *complement* M_m of $\mathcal{G}_m$ inside Q_m/W_m, i.e. a normal subgroup M_m of Q_m/W_m which is connected and reductive and such that $Q_m/W_m \simeq \mathcal{G}_m \cdot M_m$, with $\mathcal{G}_m \cap M_m$ finite.

Remark 4.2.2.2. Since K is neat, Γ_m is torsion-free. Moreover, it is such that $\Gamma_m \cap \mathcal{G}_m(\mathbb{Q}) = \{1\}$ (cfr. [BW04, Sec. 2], where Γ_m is denoted by $\bar{H}_C$). We will then see Γ_m as a subgroup of the complement $M_m(\mathbb{Q})$ introduced above.

Denote now by μ_ℓ^K, $\mu_\ell^{\pi_m(K_m)}$ the extensions of the ℓ-adic canonical construction functors introduced in Remark 2.1.3.9.(2), and by R^n the n-th *classical*, i.e. non-perverse, cohomology functor on the category $D^b_{c.\,\text{ét}}(Z_m)_L$, for any stratum Z_m of ∂S_K^*. Here is our first main tool for the analysis of the weights:

Theorem 4.2.1. *([Pin92, Thms. (4.2.1)-(5.3.1)], stated in the shape of [BW04, Thms. 2.6-2.9])*

Let R be a subfield of $\mathbb{R}$, $\mathbb{V}^\cdot \in D^b(\mathrm{Rep}_R(\mathcal{G}))$, $m \in \Phi$ and Z a stratum of ∂S_K^* contributing to Z_m.

(1) There exists a canonical isomorphism in $D^b_{c,\text{ét}}(Z)_R$

$$i_m^* i^* j_* \mu_\ell^K(\mathbb{V}^\cdot)\Big|_Z \simeq \bigoplus_n R^n i_m^* i^* j_* \mu_\ell^K(\mathbb{V}^\cdot)\Big|_Z[-n].$$

(2) For every n, there exists a canonical and functorial isomorphism in $\mathrm{Et}_{\ell,R}(Z)$

$$R^n i_m^* i^* j_* \mu_\ell^K(\mathbb{V}^\cdot)\Big|_Z \simeq \bigoplus_{p+q=n} \mu_\ell^{\pi_m(K_m)}\left(H^p(\Gamma_m, H^q(W_{m,R}, \mathbb{V}^\cdot))\right).$$

(3) Suppose that the datum $(\mathcal{G}_m, \mathfrak{H}_m)$ is of abelian type. Then, denoting by $\mathbb{W}$ both the weight filtration in the sense of Remark 2.1.3.8.(2) and the one induced on $\mathcal{G}_{m,R}$-representations as explained in 2.1.3.1, the sheaf $R^n i_m^* i^* j_* \mu_\ell^K(\mathbb{V}^\cdot)\Big|_Z$ is the direct sum of

its weight-graded objects (in particular, it is a semisimple object) and there exist canonical and functorial isomorphisms in $\mathrm{Et}_{\ell,R}(Z)$

$$\mathrm{Gr}_k^{\mathbb{W}} R^n i_m^* i^* j_* \mu_\ell^K(\mathbb{V}^\cdot)\Big|_Z \simeq \bigoplus_{p+q=n} \mu_\ell^{\pi_m(K_m)} \left(H^p(\Gamma_m, \mathrm{Gr}_k^{\mathbb{W}} H^q(W_{m,R}, \mathbb{V}^\cdot)) \right).$$

In order to explain the above statements, some remarks are in order:

Remark 4.2.2.3. (1) Reasoning as in [BW04], before Definition 2.2, we see that the functor $\mu_\ell^{\pi_m(K_m)}$, *a priori* with values in $\mathrm{Et}_{\ell,R}(S_{\pi_m(K_m)})$, gives rise to a functor with values in $\mathrm{Et}_{\ell,R}(Z)$, still denoted by the same symbol.

(2) $(Q_m/W_m)_R$ (seen as a subgroup of $Q_{m,R}$ via the Levi decomposition) acts on $H^q(W_{m,R}, \mathbb{V}^\cdot)$ via its action on W_m and on $\mathbb{V}^\cdot$, and so it acts on $H^p(\Gamma_m, H^q(W_{m,R}, \mathbb{V}^\cdot))$. Hence, the latter space is seen as a representation of $\mathcal{G}_{m,R}$ via the inclusion $\mathcal{G}_{m,R} \subset (Q_m/W_m)_R$.

(3) The statement of point (3) contains in particular the fact that

$$\mathrm{Gr}_k^{\mathbb{W}} H^p(\Gamma_m, H^q(W_{m,R}, \mathbb{V}^\cdot)) \simeq H^p(\Gamma_m, \mathrm{Gr}_k^{\mathbb{W}} H^q(W_{m,R}, \mathbb{V}^\cdot)).$$

Remark 4.2.2.4. Our necessity of working with ℓ-adic sheaves instead of variations of Hodge structures comes from the observations in Rmk. 3.1.0.2. Anyway, the above theorem admits an analogue in the Hodge setting, i.e. [BW04, Thms. 2.6-2.9], whose conclusions are formally identical to the ones of the theorem above. Because of this and of Rmk. 2.1.3.8.(3), all our results for ℓ-adic sheaves will be automatically valid for the corresponding mixed Hodge modules.

4.2.3 Kostant's theorem

The second ingredient for the analysis of the weights is a theorem of Kostant which allows one to make explicit the $(Q_m/W_m)_R$-representations $H^q(W_{m,R}, \mathbb{V}^\cdot)$ appearing in Theorem 4.2.1.

Fix a split reductive group $\mathcal{G}$ over a field of characteristic zero, with root system $\mathfrak{r}$ and Weyl group Υ. Denote by $\mathfrak{r}^+$ the set of positive roots and fix moreover a parabolic subgroup Q with its unipotent radical W. Let $\mathfrak{w}$ be the Lie algebra of W and $\mathfrak{r}_W$ the set of roots appearing inside $\mathfrak{w}$ (necessarily positive). For every $w \in \Upsilon$, we define:

$$\mathfrak{r}^+(w) := \{\alpha \in \mathfrak{r} | w^{-1}\alpha \notin \mathfrak{r}^+\}, \tag{4.12}$$

$$l(w) := |\mathfrak{r}^+(w)|, \tag{4.13}$$

$$\Upsilon' := \{w \in \Upsilon | \mathfrak{r}^+(w) \subset \mathfrak{r}_W\}. \tag{4.14}$$

We can now state Kostant's theorem:

Theorem 4.2.2. *([Vog81, Thm. 3.2.3])*

Let V_λ be an irreducible $\mathcal{G}$-representation of highest weight λ, and let ρ be the half-sum of the positive roots of $\mathcal{G}$. Then, as (Q/W)-representations,

$$H^q(W, V_\lambda) \simeq \bigoplus_{w \in \Upsilon' | l(w) = q} U_{w.(\lambda+\rho)-\rho},$$

where U_μ denotes an irreducible (Q/W)-representation of highest weight μ.

In order to spell out the consequences of this theorem in the cases of interest to us, consider our fixed Galois closure L of the totally real field F and the group G underlying the genus $n = 2$ Hilbert-Siegel Shimura datum of Example 2.1.3.2.(2). We will apply Kostant's theorem by choosing $\mathcal{G} = G_L$ and by setting Q/W equal to, for $i = 0, 1$, the Levi components $(Q_i/W_i)_L$ of the standard parabolics $Q_{i,L}$, defined as in 4.1.2. We have seen in 4.1.4 that the root system of G_L is given by $\mathfrak{r} = \bigsqcup_{\sigma \in I_F} \mathfrak{r}_\sigma$ and that every component $\mathfrak{r}_\sigma$ contains two simple roots $\rho_{1,\sigma}$, $\rho_{2,\sigma}$; the other positive roots in such a component are then given by $\rho_{1,\sigma} + \rho_{2,\sigma}$ and $2\rho_{1,\sigma} + \rho_{2,\sigma}$.

Lemma 4.2.3.1. *(1) Let Υ be the Weyl group of G_L (cfr. 4.1.4) and denote by Υ'_m, for $m \in \{0, 1\}$, the sets defined in (4.14), corresponding to the choices $\mathcal{G} = G_L$ and $Q = Q_{m,L}$. Then, in both cases, for every $\sigma \in I_F$, there exist sets $\Upsilon'_{m,\sigma} = \{w^i_\sigma\}_{i=0,\ldots,3} \subset \Upsilon_\sigma$ such that $l(w^i_\sigma) = i$ for every $i \in \{0, \ldots, 3\}$ and that $\Upsilon'_m = \prod_{\sigma \in I_F} \Upsilon'_{m,\sigma}$.*

*(2) For $m = 0, 1$, one has $0 \leq l(w) \leq 3d$ for every $w \in \Upsilon'_m$. Moreover, for every integer $q \in \{0, \ldots, 3d\}$, there exists a bijection between the set $\{w \in \Upsilon'_m \mid l(w) = q\}$ and the set of q-*admissible decompositions *of I_F*

$$\mathcal{P}_q := \{\textit{decompositions } I_F = \bigsqcup_{i=0,\ldots,3} I^i_F \mid \sum_{i=0}^{3} i|I^i_F| = q\}. \tag{4.15}$$

Proof. (1) In the case of $(Q_0/W_0)_L$, by fixing a component $\mathfrak{r}_\sigma$ of the root system of G_L (cfr. 4.1.4), one easily sees that the positive roots which are contained in such a component and which appear in the Lie algebra of $W_{0,L}$ are given by $\{\rho_{1,\sigma} + \rho_{2,\sigma}, 2\rho_{1,\sigma} + \rho_{2,\sigma}, \rho_{2,\sigma}\}$.

Coherently with the notation of 4.1.4, denote by s_ρ the reflection, belonging to the Weyl group Υ, whose axis is orthogonal to the root ρ. By direct inspection of the action of the component Υ_σ on $\mathfrak{r}_\sigma$, we find that the elements of the sets $\Upsilon'_{0,\sigma} = \{w^i_\sigma\}_{i=0,\ldots,3}$ defined in the statement (for $m = 0$) are given by

$$\begin{aligned} w^0_\sigma &= \mathrm{id}, \\ w^1_\sigma &= s_{\rho_{2,\sigma}}, \\ w^2_\sigma &= s_{\rho_{1,\sigma}+\rho_{2,\sigma}} s_{\rho_{2,\sigma}}, \\ w^3_\sigma &= s_{\rho_{1,\sigma}+\rho_{2,\sigma}} \end{aligned}$$

and that $\Upsilon'_0 = \prod_{\sigma \in I_F} \Upsilon'_{0,\sigma}$.

In the case of $(Q_1/W_1)_L$, fix again a component $\mathfrak{r}_\sigma$ of the root system of G_L: the positive roots contained in this component which appear inside the Lie algebra of $W_{1,L}$ are given this time by $\{\rho_{1,\sigma}, \rho_{1,\sigma} + \rho_{2,\sigma}, 2\rho_{1,\sigma} + \rho_{2,\sigma}\}$.

With notations as in the previous case, we find that the elements in the sets $\Upsilon'_{1,\sigma} = \{w^i_\sigma\}_{i=0,\ldots,3} \subset \Upsilon_\sigma$ appearing in the statement (for m=1) are given by

$$\begin{aligned} w^0_\sigma &= \mathrm{id}, \\ w^1_\sigma &= s_{\rho_{1,\sigma}}, \\ w^2_\sigma &= s_{\rho_{1,\sigma}+\rho_{2,\sigma}} s_{\rho_{1,\sigma}}, \end{aligned}$$

$$w^3_\sigma = s_{2\rho_{1,\sigma}+\rho_{2,\sigma}}$$

and that $\Upsilon'_1 = \prod_{\sigma\in I_F} \Upsilon'_{1,\sigma}$.

(2) The preceding point implies that every $w = (w_\sigma)_{\sigma\in I_F} \in \Upsilon'_m$ determines a decomposition

$$I_F = \bigsqcup_{i=0,\dots,3} I^i_F$$

where $I^i_F := \{\sigma \in I_F | w_\sigma = w^i_\sigma\}$. Hence, since $l((w_\sigma)_{\sigma\in I_F}) = \sum_{\sigma\in I_F} l(w_\sigma)$, we get the desired bounds on $l(w)$. The bijection in the statement follows immediately.

□

Notation 4.2.3.2. For a an integer $q \in \{0,\dots,3d\}$, a q-admissible decomposition Ψ of I_F will be denoted by $\Psi = (I^0_F, I^1_F, I^2_F, I^3_F)$. If only one of the four subsets, say I^i_F, is non-empty, we will denote Ψ by the symbol I^i_F itself.

Fix now a irreducible G_L-representation V_λ of highest weight $\lambda = \lambda((k_{1,\sigma}, k_{2,\sigma})_{\sigma\in I_F}, c)$ (as defined in 4.1.5) and, for $m = 0, 1$, apply Theorem 4.2.2 to identify the cohomology spaces $H^q(W_{m,L}, V_\lambda)$ as $(Q_m/W_m)_L$-representations: employing the notation fixed in (4.15), we get isomorphisms

$$H^q(W_{m,L}, V_\lambda) \simeq \bigoplus_{\Psi\in\mathcal{P}_q} V^{m,q}_\Psi, \tag{4.16}$$

where each $V^{m,q}_\Psi$ is an irreducible $(Q_m/W_m)_L$-representation. With the notations of Lemma 4.2.3.1.(1), the explicit computation of $w.(\lambda+\rho)-\rho$ for $w \in \Upsilon'_m$ (as in [Lem15, Sec. 4.3]) gives us the highest weight of such irreducible representations, as stated in the following lemma:

Lemma 4.2.3.3. *(1) Adopting Notation 4.2.3.2, the highest weight of the irreducible $(Q_0/W_0)_L$-representation $V^{0,q}_\Psi$ in (4.16) is given by the restriction (along the inclusion $(Q_0/W_0)_L \subset Q_{0,L} \subset G_L$) of the character*

$$\lambda((\eta_{1,\sigma}, \eta_{2,\sigma})_{\sigma\in I_F}, c), \tag{4.17}$$

where

$$\eta_{1,\sigma} = \begin{cases} k_{1,\sigma} & \text{if } \sigma \in I^0_F \\ k_{1,\sigma} & \text{if } \sigma \in I^1_F \\ k_{2,\sigma}-1 & \text{if } \sigma \in I^2_F \\ -k_{2,\sigma}-3 & \text{if } \sigma \in I^3_F \end{cases}, \quad \eta_{2,\sigma} = \begin{cases} k_{2,\sigma} & \text{if } \sigma \in I^0_F \\ -k_{2,\sigma}-2 & \text{if } \sigma \in I^1_F \\ -k_{1,\sigma}-3 & \text{if } \sigma \in I^2_F \\ -k_{1,\sigma}-3 & \text{if } \sigma \in I^3_F \end{cases}$$

(2) Adopting Notation 4.2.3.2, the highest weight of the irreducible $(Q_1/W_1)_L$-representation $V^{1,q}_\Psi$ is the restriction (along the inclusion $(Q_1/W_1)_L \subset Q_{1,L} \subset G_L$) of the character

$$\lambda((\epsilon_{1,\sigma}, \epsilon_{2,\sigma})_{\sigma\in I_F}, c) \tag{4.18}$$

where

$$\epsilon_{1,\sigma} = \begin{cases} k_{1,\sigma} & \textit{if } \sigma \in I_F^0 \\ k_{2,\sigma} - 1 & \textit{if } \sigma \in I_F^1 \\ -k_{2,\sigma} - 3 & \textit{if } \sigma \in I_F^2 \\ -k_{1,\sigma} - 4 & \textit{if } \sigma \in I_F^3 \end{cases}, \quad \epsilon_{2,\sigma} = \begin{cases} k_{2,\sigma} & \textit{if } \sigma \in I_F^0 \\ k_{1,\sigma} + 1 & \textit{if } \sigma \in I_F^1 \\ k_{1,\sigma} + 1 & \textit{if } \sigma \in I_F^2 \\ k_{2,\sigma} & \textit{if } \sigma \in I_F^3 \end{cases}$$

4.2.4 Cohomology of groups of units

We finish this section by recalling, for the convenience of the reader, some standard arguments that will be useful in the analysis of the cohomology of arithmetic groups appearing in Theorem 4.2.1.

Lemma 4.2.4.1. *Let Γ be a free abelian group of finite rank r, acting on a finite-dimensional vector space V over a field L by L-linear automorphisms. Suppose that Γ acts through a character λ. Then, there exists an integer s such that the cohomology space $H^s(\Gamma, V)$ is non-trivial if and only if the action of Γ on V is trivial.*

In this case, $H^s(\Gamma, V)$ is non-trivial if and only if $0 \le s \le r$, and for such integers s we have (non-canonically)

$$H^s(\Gamma, V) \simeq V^{\binom{r}{s}}.$$

Proof. We proceed by induction on the rank r, denoting by V^Γ the space of invariants of the Γ-action on V and by V_Γ the space of coinvariants.

- If $r = 1$, we have $\Gamma \simeq \mathbb{Z}$ and it is then well-known that

$$H^s(\Gamma, V) = \begin{cases} V^\Gamma & \text{if } s = 0 \\ V_\Gamma & \text{if } s = 1 \\ \{0\} & \text{otherwise} \end{cases}$$

Now, choose a generator γ of Γ. Then, the space V^Γ is non-trivial if and only if there exists a non-zero element $v \in V$ such that $\lambda(n\gamma) \cdot v = v$ for every $n \in \mathbb{Z}$, which is equivalent to asking that $\lambda(\gamma) = 1$, i.e. that Γ act through the trivial character. Analogously, the space V_Γ is non-trivial if and only if Γ acts trivially. Thus, $H^s(\Gamma, V) \simeq V$ if $s = 0, 1$, and is trivial otherwise.

- Suppose the assertion to be true for free abelian groups of rank r. If $\Gamma \simeq \mathbb{Z}^{r+1}$, then choose a basis of Γ as a $\mathbb{Z}$-module and use it to define an exact sequence

$$0 \to \mathbb{Z}^r \to \Gamma \to \mathbb{Z} \to 0 \tag{4.19}$$

By the case $r = 1$, the Lyndon-Hochschild-Serre spectral sequence associated to this exact sequence, i.e.

$$E_2 = H^m(\mathbb{Z}, H^n(\mathbb{Z}^r, V)) \Rightarrow H^{m+n}(\Gamma, V) \tag{4.20}$$

has only two non-trivial columns (for $n = 0, 1$). Hence, for each $s \ge 1$, we have an exact sequence

$$0 \to H^0(\mathbb{Z}, H^s(\mathbb{Z}^r, V)) \to H^s(\Gamma, V) \to H^1(\mathbb{Z}, H^{s-1}(\mathbb{Z}^r, V)) \to 0$$

and moreover

$$H^0(\Gamma, V) \simeq H^0(\mathbb{Z}, H^0(\mathbb{Z}^r, V)).$$

By the induction hypothesis, $H^s(\Gamma, V)$ can be non-trivial only if $0 \leq s \leq r+1$. Moreover, if the action of Γ is non-trivial, the subgroup isomorphic to $\mathbb{Z}^r$ appearing in (4.19) can be chosen as acting non-trivially on V, and in this case, by induction, $H^s(\Gamma, V)$ is trivial for every s. If, on the contrary, Γ acts trivially, then the induced action of $\mathbb{Z}$ on the spaces $H^n(\mathbb{Z}^r, V)$ is again trivial, so that, by the case $r = 1$, we have

$$H^0(\Gamma, V) \simeq H^0(\mathbb{Z}, H^0(\mathbb{Z}^r, V)) \simeq H^0(\mathbb{Z}^r, V) \simeq V$$

and for every $s \in \{1, \ldots, r+1\}$,

$$\begin{aligned} &H^s(\Gamma, V) \simeq \text{ (non-canonically) } H^0(\mathbb{Z}, H^s(\mathbb{Z}^r, V)) \oplus H^1(\mathbb{Z}, H^{s-1}(\mathbb{Z}^r, V)) \simeq \\ &\text{(by the case } r = 1\text{)} \\ &\simeq H^s(\mathbb{Z}^r, V) \oplus H^{s-1}(\mathbb{Z}^r, V) \simeq \\ &\text{(by the induction hypothesis, and setting } V^{\binom{r}{r+1}} = \{0\} \text{ by convention)} \\ &\simeq V^{\binom{r}{s}} \oplus V^{\binom{r}{s-1}} \simeq V^{\binom{r+1}{s}}. \end{aligned}$$

□

The preceding lemma leads to the problem of determining the triviality of the action of some free abelian groups, which in our case will arise as subgroups of units of totally real fields. This is the object of the following lemma:

Lemma 4.2.4.2. *Let $\mathcal{O}_F$ be the ring of integers of F and fix $(0, \ldots, 0) \neq (n_1, \ldots, n_d) \in \mathbb{Z}^d$. Then,*

$$\prod_{i=1,\ldots,d} |\sigma_i(t)|^{n_i} = 1 \text{ for every } t \in \mathcal{O}_F^\times$$

if and only if $(n_1, \ldots, n_d) \in \mathbb{Z} \cdot (1, \ldots, 1)$.

Proof. Choose a base $\{\gamma_1, \ldots, \gamma_{d-1}\}$ of $\mathcal{O}_F^\times$ as $\mathbb{Z}$-module. Write $t = \prod_{0,\ldots,d-1} \gamma_i^{a_i}$, choose a d-tuple of integers $(n_1, \ldots, n_d) \neq (0, \ldots, 0)$, and define

$$\Lambda := \begin{pmatrix} \log|\sigma_1(\gamma_1)| & \ldots & \log|\sigma_d(\gamma_1)| \\ \vdots & \ldots & \vdots \\ \log|\sigma_1(\gamma_{d-1})| & \ldots & \log|\sigma_d(\gamma_{d-1})| \end{pmatrix}$$

Then, we have

$$\prod_{i=1,\ldots,d} |\sigma_i(t)|^{n_i} = 1 \ \forall\ t \iff \sum_{j=1,\ldots,d-1} \left(a_j \sum_{i=1,\ldots,d} n_i \log|\sigma_i(\gamma_j)| \right) = 0,$$

$$\forall\ (a_1, \ldots, a_{d-1}) \neq (0, \ldots, 0) \in \mathbb{Z}^{d-1}$$

$$\iff \langle \begin{pmatrix} a_1 \\ \vdots \\ a_{d-1} \end{pmatrix}, \Lambda \cdot \begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix} \rangle = 0, \forall\ (a_1, \ldots, a_{d-1}) \neq (0, \ldots, 0) \in \mathbb{Z}^{d-1}$$

(where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^{d-1}$) $\iff$ $(n_1, \ldots, n_d) \in \ker \Lambda$. But by Dirichlet's unit theorem, $\ker \Lambda = \mathbb{R} \cdot (1, \ldots, 1)$. □

Remark 4.2.4.3. (1) By choosing adapted bases, Lemma 4.2.4.2 generalises immediately to the case where $\mathcal{O}_F^\times$ is replaced by a finite-index subgroup of $\mathcal{O}_F^\times$, and furthermore to the case where it is replaced by an arithmetic subgroup of $F^\times$.

(2) Consider the *norm* morphism $N : \mathcal{O}_F^\times \to \{\pm 1\}$. As the image of a neat subgroup by a morphism is again neat, the elements of a *neat* subgroup of $\mathcal{O}_F^\times$ are of norm 1. Thus, if the neat subgroup $\Gamma_{0,Z}$ is a finite-index subgroup of $\mathcal{O}_F^\times$, we have that $\prod_{i=1,\dots,d} |\sigma_i(t)|^{n_i} = 1 \ \forall\ t \in \Gamma_{0,Z} \iff \prod_{i=1,\dots,d} \sigma_i(t)^{n_i} = 1 \ \forall\ t \in \Gamma_{0,Z}$, and, by (1), Lemma 4.2.4.2 tells us that the action of $\Gamma_{0,Z}$ on a vector space by multiplication by $\prod_{i=1,\dots,d} \sigma_i(t)^{n_i}$ is trivial if and only if $(n_1, \dots, n_d) \in \mathbb{Z} \cdot (1, \dots, 1)$. This equivalence continues to hold in the general case where $\Gamma_{0,Z}$ is a (neat) arithmetic subgroup of $F^\times$, again via (1).

4.3 The degeneration of the canonical construction at the boundary

In this section we prove our main result (Thm. 4.3.1), i.e. the description of the interval of *weight avoidance* of the motive $i^* j_*^\lambda \mathcal{V} \in DM_{\text{B},c}(\partial S_K^*)_L$ in terms of the *corank* of λ.

4.3.1 Statement of the main result

Let G be the group considered in the previous Section. In order to state our central theorem about the degeneration of the motive ${}^\lambda\mathcal{V}$ (Subsection 3.2.1) at the boundary of the Baily-Borel compactification, we reformulate the notion of *corank* introduced in Def. 2.3.4.5 (where the reader can find some motivation for it) in the more restricted context of genus 2 Hilbert-Siegel varieties. Moreover, we add some notions about the weight λ that will be crucial in the sequel.

Definition 4.3.1.1. *Let $\lambda = \lambda((k_{1,\sigma}, k_{2,\sigma})_{\sigma \in I_F}, c))$ (cfr. 4.1.5) be a weight of G_L.*

(1) $k_1 := (k_{1,\sigma})_{\sigma \in I_F}$ or $k_2 := (k_{2,\sigma})_{\sigma \in I_F}$ is called parallel *if $k_{i,\sigma}$ is constant on I_F, equal to a positive integer κ (and we write $k_i = \underline{\kappa}$). The weight λ is called κ-*Kostant parallel *if there exist a $\kappa \in \mathbb{Z}$ and a decomposition $I_F = I_F^0 \bigsqcup I_F^1$ such that*

$$\begin{cases} k_{1,\sigma} = \kappa & \forall \sigma \in I_F^0 \\ k_{2,\sigma} = \kappa + 1 & \forall \sigma \in I_F^1 \end{cases}$$

and is called Kostant parallel *if there exists a κ such that λ is κ-Kostant parallel.*

(2) We define the corank $\operatorname{cor}(\lambda)$ *of λ by*

$$\operatorname{cor}(\lambda) = \begin{cases} 0 & \textit{if } k_2 \textit{ is not parallel} \\ 1 & \textit{if } k_2 \textit{ is parallel and } k_1 \neq k_2 \\ 2 & \textit{if } k_2 \textit{ is parallel and } k_1 = k_2 \end{cases}$$

(3) λ is completely irregular *if $(k_{1,\sigma}, k_{2,\sigma})$ is irregular for every $\sigma \in I_F$.*

Assume λ to be dominant. We make some observations that may help enlightening the above definitions and their mutual relationships:

- If $\mathrm{cor}(\lambda) = 2$, then λ is completely irregular.

- If $\mathrm{cor}(\lambda) \geq 1$, then λ is κ-Kostant parallel with respect to the decomposition $I_F = I_F^1$, with $k_2 = \underline{\kappa + 1}$; this decomposition and this κ are then the only ones such that both $I_F^1 \neq \varnothing$ and λ is Kostant-parallel with respect to them.

- If $\mathrm{cor}(\lambda) = 1$ and λ is completely irregular, then necessarily $k_2 = \underline{0}$.

- If $\mathrm{cor}(\lambda) = 0$, then there are at most a κ and a decomposition $I_F = I_F^0 \bigsqcup I_F^1$ with respect to which λ is κ-Kostant parallel.

Remark 4.3.1.2. The terminology *Kostant parallel* comes from the more specific terminology in Definitions 4.3.2.5 and 4.3.3.3 (see also Remark 4.3.3.7) and expresses the fact that some linear combinations of the coordinates of the character λ are required to take a constant (*parallel*) value over certain subsets of I_F. As the computations leading to those definitions will make clear, this terminology is motivated by the fact that such linear combinations and subsets arise from Lemma 4.2.3.3, which is an application of Kostant's theorem, Thm. 4.2.2.

We can now state our main result, in the language of weight structures introduced in 3.3.1:

Theorem 4.3.1. *Let V_λ be the irreducible representation of G_L of highest weight*

$$\lambda = \lambda((k_{1,\sigma}, k_{2,\sigma})_{\sigma \in I_F}, c)),$$

S_K the genus 2 Hilbert-Siegel variety of level K corresponding to (G, X) and ${}^\lambda\mathcal{V} \in CHM(S_K)$ the Chow motive over S_K introduced in Definition 3.2.1.7. Let moreover $j : S_K \to S_K^*$, resp. $i : \partial S_K^* \to S_K^*$ denote the open, resp. closed immersion in the Baily-Borel compactification S_K^* of S_K. Then:

(1) If λ is not Kostant parallel, then the boundary motive *$i^* j_*^\lambda \mathcal{V}$ is zero.*

(2) Suppose that $\mathrm{cor}(\lambda) = 0$ *and that λ is κ-Kostant parallel. Denote $d_1 := |I_F^1|$. Then $i^* j_*^\lambda \mathcal{V}$ avoids weights $-d_1 - d\kappa + 1, \ldots, d_1 + d\kappa$ and the weights $-d_1 - d\kappa$, $d_1 + d\kappa + 1$* do appear *in $i^* j_*^\lambda \mathcal{V}$.*

(3) Suppose that $\mathrm{cor}(\lambda) = 1$, *with $k_2 = \underline{\kappa_2}$, and that k_1 is not parallel. Then $i^* j_*^\lambda \mathcal{V}$ avoids weights $-d\kappa_2 + 1, \ldots, d\kappa_2$ and the weights $-d\kappa_2$, $d\kappa_2 + 1$* do appear *in $i^* j_*^\lambda \mathcal{V}$.*

(4) Suppose that $\mathrm{cor}(\lambda) \geq 1$, *with $k_2 = \underline{\kappa_2}$, and that $k_1 = \underline{\kappa_1}$. Denote $\kappa := \min\{\kappa_1 - \kappa_2, \kappa_2\}$. Then $i^* j_*^\lambda \mathcal{V}$ avoids weights $-d\kappa + 1, \ldots, d\kappa$. The weights $-d\kappa_2$, $d\kappa_2 + 1$* do appear *in $i^* j_*^\lambda \mathcal{V}$, and if κ_1, κ_2 have the same parity*[1]*, then the weights $-d(\kappa_1 - \kappa_2)$, $d(\kappa_1 - \kappa_2) + 1$* do appear *in $i^* j_*^\lambda \mathcal{V}$.*

The proof of theorem 4.3.1 will be completed at the end of paragraph 4.3.5.2, by invoking Theorem 3.3.4 and after having employed all the tools recalled in Sections 4.1. and 4.2. Admitting this theorem for the moment, we can prove its most important corollary for the construction of the *intersection motive* (Def. 3.3.2.8), i.e. the following characterization of the absence of the weights 0 and 1:

[1] Cfr. Footnote 2 for this supplementary hypothesis.

Corollary 4.3.1.3. *The weights 0 and 1 appear in the boundary motive $i^* j_*^\lambda \mathcal{V}$ if and only if λ is completely irregular of corank ≥ 1.*

Proof. Suppose λ to be κ-Kostant parallel with respect to (I_F^0, I_F^1) (otherwise, by point (1) of the above theorem, there is nothing to do).

If $\mathrm{cor}(\lambda) = 0$, then, by point (2) of the above theorem, the weights 0 and 1 appear if and only if $d_1 = 0 = \kappa$. But $d_1 = 0$ means that $I_F^1 = \varnothing$, i.e. $I_F^0 = I_F$, and by definition of Kostant-parallelism this implies $k_1 = \underline{\kappa}$. Now, necessarily $\kappa > 0$, because otherwise $k_2 = \underline{0}$ (remember that $k_{1,\sigma} \geq k_{2,\sigma}$ for every $\sigma \in I_F$) and $\mathrm{cor}(\lambda) = 2$, a contradiction.

If $\mathrm{cor}(\lambda) = 1$, with $k_2 = \underline{\kappa_2}$, then, by point (3) and (4) of the above theorem, the weights 0 and 1 appear if and only if $\kappa_2 = 0$; observe in fact that, even if $k_1 = \underline{\kappa_1}$, we have $\kappa_1 - \kappa_2 > 0$ (otherwise $\mathrm{cor}(\lambda) = 2$, a contradiction). But if $\kappa_2 = 0$, λ is completely irregular.

If $\mathrm{cor}(\lambda) = 2$, then $k_1 = \underline{\kappa} = k_2$; this means that λ is completely irregular, and implies that, in point (4) of the above theorem, the parity condition is trivially satisfied and that $\kappa_1 - \kappa_2 = 0$, so that the weights 0 and 1 appear.

To conclude, we only have to observe that if $\mathrm{cor}(\lambda) \geq 1$ and λ is completely irregular, either $k_2 = \underline{0}$ or $k_1 = \underline{\kappa} = k_2$ (cfr. the observations after Def. 4.3.1.1). □

The rest of this section is devoted to the proof of Thm. 4.3.1, following the outline given in the introduction to this chapter.

4.3.2 The degeneration along the Siegel strata

With notation as in the statement of Thm. 4.3.1, fix a irreducible G_L-representation V_λ of highest weight $\lambda = \lambda((k_{1,\sigma}, k_{2,\sigma})_{\sigma \in I_F}, c)$: we want to employ Theorem 4.2.1 to study the degeneration of $\mu_\ell^K(V_\lambda)$ along the Siegel strata, whose underlying Shimura datum is $(G_0, \mathfrak{H}_0)$, where $G_0 \simeq \mathbb{G}_m$, as explained in 4.2.1.

Remark 4.3.2.1. From the definition of the morphism $\mathbb{S} \to G_\mathbb{R}$ underlying the Hilbert-Siegel datum (2.1.2.2.(2)), we see that an irreducible representation of G of highest weight $\lambda = \lambda((k_{1,\sigma}, k_{2,\sigma})_{\sigma \in I_F}, c))$ is such that $\mu_H^K(V_\lambda)$, resp. $\mu_\ell^K(V_\lambda)$, is a variation of Hodge structure, resp. an ℓ-adic sheaf, pure of weight $w(\lambda) := -c$ (cfr. the convention fixed in 2.1.3.1, which is extended to variations of Hodge structure in the obvious way).

4.3.2.1 Weights in the cohomology of the unipotent radical.

We start by identifying the possible weights appearing in the degeneration along the Siegel strata, i.e. in the $(Q_0/W_0)_L$-representations

$$H^q(W_{0,L}, V_\lambda) \simeq \bigoplus_{\Psi \in \mathcal{P}_q} V_\Psi^{0,q}, \tag{4.21}$$

for $q \in \{0, \ldots, 3d\}$ (cfr. (4.16)). Recall from (4.5) that we have

$$(Q_0/W_0)_L \simeq \mathbb{G}_{m,L} \times \prod_{\sigma \in I_F} (\mathrm{GL}_{2,L})_\sigma$$

Let us then compute the weight of the pure Hodge structure carried by each irreducible summand $V_\Psi^{0,q}$.

Lemma 4.3.2.2. *For every $q \in \{0, \dots, 3d\}$ and for every q-admissible decomposition Ψ as in Notation 4.2.3.2, the action of the $\mathbb{G}_{m,L}$-factor inside $(Q_0/W_0)_L$ induces on $V_\Psi^{0,q}$ a pure Hodge structure of weight*

$$w(\lambda) - [\sum_{\sigma \in I_F^0} (k_{1,\sigma} + k_{2,\sigma}) + \sum_{\sigma \in I_F^1} (k_{1,\sigma} - k_{2,\sigma} - 2) - \sum_{\sigma \in I_F^2} (k_{1,\sigma} - k_{2,\sigma} + 4) - \sum_{\sigma \in I_F^3} (k_{1,\sigma} + k_{2,\sigma} + 6)].$$

Proof. By the discussion in 4.1.2, the L-points of the $\mathbb{G}_{m,L}$-factor are identified with the subgroup

$$\{(\begin{pmatrix} \alpha I_2 & \\ & I_2 \end{pmatrix})_{\sigma \in I_F} | \alpha \in L^\times\}$$

of $Q_0/W_0(L)$. With the notation of Lemma 4.2.3.3.(1) for the highest weight of the representation $V_\Psi^{0,q}$, and recalling (4.8), we see that $\mathbb{G}_{m,L}(L)$ acts on $V_\Psi^{0,q}$ via the character

$$\alpha \mapsto \alpha^{\frac{1}{2} \cdot [c + \sum_{\sigma \in I_F} (\eta_{1,\sigma} + \eta_{2,\sigma})]}, \tag{4.22}$$

Now remember that $w(\lambda) = -c$ (Rmk. 4.3.2.1). By the convention fixed in 2.1.3.1 and the definition in (4.9) of the Shimura datum (G_0, X_0), the expression for $\eta_{1,\sigma}$ and $\eta_{2,\sigma}$ given in Lemma 4.2.3.3.(1) yields the formula in the statement. □

Notice for later use that if V is the standard 2-dimensional L-representation of $\mathrm{GL}_{2,L}$, the above computations imply that the representation obtained by restriction to the factor $\prod_{\sigma \in I_F} (\mathrm{GL}_{2,L})_\sigma$ of $(Q_0/W_0)_L$ is isomorphic to

$$\begin{aligned} &(\bigotimes_{\sigma \in I_F^0} \mathrm{Sym}^{k_{1,\sigma} - k_{2,\sigma}} \mathrm{V} \otimes \det{}^{k_{2,\sigma}}) \otimes (\bigotimes_{\sigma \in I_F^1} \mathrm{Sym}^{k_{1,\sigma} + k_{2,\sigma} + 2} \mathrm{V} \otimes \det{}^{-k_{2,\sigma} - 2}) \otimes \\ \otimes &(\bigotimes_{\sigma \in I_F^2} \mathrm{Sym}^{k_{1,\sigma} + k_{2,\sigma} + 2} \mathrm{V} \otimes \det{}^{-k_{1,\sigma} - 3}) \otimes (\bigotimes_{\sigma \in I_F^3} \mathrm{Sym}^{k_{1,\sigma} - k_{2,\sigma}} \mathrm{V} \otimes \det{}^{-k_{1,\sigma} - 3}) \end{aligned} \tag{4.23}$$

4.3.2.2 Cohomology of the arithmetic subgroup.

Consider now the arithmetic group Γ_0 of Rmk. 4.2.2.2: according to Theorem 4.2.1, and remembering (4.38), we need to identify the cohomology spaces

$$H^p(\Gamma_0, H^q(W_{0,L}, V_\lambda)) \simeq \bigoplus_{\Psi \in \mathcal{P}_q} H^p(\Gamma_0, V_\Psi^{0,q}) \tag{4.24}$$

and their weight-graded objects $\mathrm{Gr}_k^{\mathbb{W}} H^p(\Gamma_0, H^q(W_{0,L}, V_\lambda)) \simeq \bigoplus_{\Psi \in \mathcal{P}_q} H^p(\Gamma_0, \mathrm{Gr}_k^{\mathbb{W}} V_\Psi^{0,q})$ (cfr. Remark 4.2.2.3(3)). As the cohomological dimension of $W_{0,L}$ is $3d$, these spaces can be non-zero only for $q \in \{0, \dots, 3d\}$. We are now going to put further restrictions on the non-triviality of such spaces.

Construction 4.3.2.3. Γ_0 is identified with a neat (hence, torsion-free) arithmetic subgroup of

$$\mathrm{Res}_{F|\mathbb{Q}} \mathrm{GL}_{2,F}(\mathbb{Q}) = \mathrm{GL}_2(F)$$

(Remark 4.2.2.2). Let π be the projection $\mathrm{GL}_2(F) \twoheadrightarrow \mathrm{GL}_2(F)/Z(\mathrm{GL}_2(F))$ and define $\Gamma_{0,Z} := \Gamma_0 \cap Z(\mathrm{GL}_2(F))$ and $\Gamma_0' := \pi(\Gamma_0)$ (non trivial, torsion-free arithmetic subgroups of $Z(\mathrm{GL}_2(F)) \simeq F^\times$, resp. $\mathrm{PGL}_2(F)$). Then, Γ_0 can be written as an extension

$$1 \to \Gamma_{0,Z} \to \Gamma_0 \xrightarrow{\pi} \Gamma_0' \to 1, \tag{4.25}$$

and applying the Lyndon-Hochschild-Serre spectral sequence to this extension

$$E_2 = H^r(\Gamma_0', H^s(\Gamma_{0,Z}, V_\Psi^{0,q})) \Rightarrow H^{r+s}(\Gamma_0, V_\Psi^{0,q}) \tag{4.26}$$

we see that every subspace $H^p(\Gamma_0, V_\Psi^{0,q})$ is (non-canonically) isomorphic to a direct sum

$$\bigoplus_{r+s=p} U^{r,s} \tag{4.27}$$

where every $U^{r,s}$ is a subquotient of $H^r(\Gamma_0', H^s(\Gamma_{0,Z}, V_\Psi^{0,q}))$. Thus, if $H^s(\Gamma_{0,Z}, V_\Psi^{0,q})$ is zero for every s, then $H^p(\Gamma_0, V_\Psi^{0,q})$ is.

Lemma 4.2.4.1 gives necessary conditions for the non-triviality of the cohomology of a free abelian group acting on a vector space. The following lemma tells us when these conditions are verified in a specific case:

Lemma 4.3.2.4. *Let $\Gamma_{0,Z}$ be the group defined in Construction 4.3.2.3. Then, its action on on $V_\Psi^{0,q}$ is trivial if and only if there exists an integer κ such that*

$$\begin{cases} k_{1,\sigma} + k_{2,\sigma} = \kappa & \forall \sigma \in I_F^0 \\ k_{1,\sigma} - k_{2,\sigma} - 2 = \kappa & \forall \sigma \in I_F^1 \\ -(k_{1,\sigma} - k_{2,\sigma} + 4) = \kappa & \forall \sigma \in I_F^2 \\ -(k_{1,\sigma} + k_{2,\sigma} + 6) = \kappa & \forall \sigma \in I_F^2 \end{cases} \tag{4.28}$$

(remembering Notation 4.2.3.2).

Proof. By Dirichlet's unit theorem, we have that $\mathrm{Res}_{F|\mathbb{Q}} \mathbb{G}_{m,F}(\mathbb{Z}) \simeq \mathcal{O}_F^\times \simeq \mathbb{Z}^{d-1} \times \mathbb{Z}/2\mathbb{Z}$. On the other hand, the torsion-free group $\Gamma_{0,Z}$ is commensurable to $\mathcal{O}_F^\times$. The group $\Gamma_{0,Z}$ is then isomorphic to $\mathbb{Z}^{d-1}$. By choosing generators $\gamma_1, \dots, \gamma_{d-1}$, and remembering the discussion in 4.1.2, it is then identified with the subgroup

$$\{(\begin{pmatrix} \sigma(t) & & & \\ & \sigma(t) & & \\ & & \sigma(t)^{-1} & \\ & & & \sigma(t)^{-1} \end{pmatrix})_{\sigma \in I_F} | t = \gamma_1^{p_1} \dots \gamma_{d-1}^{p_{d-1}}, p_1, \dots, p_{d-1} \in \mathbb{Z}\} \hookrightarrow Q_0/W_0(L). \tag{4.29}$$

Recalling the expression for the highest weight of the representation $V_\Psi^{0,q}$ given in Lemma 4.2.3.3.(1), we see that an element $t = \gamma_1^{p_1} \dots \gamma_{d-1}^{p_{d-1}} \in \Gamma_{0,Z}$ acts on $V_\Psi^{0,q}$ via multiplication by $\prod_{\sigma \in I_F} \sigma(t)^{\eta_{1,\sigma} + \eta'_{2,\sigma}}$, i.e. by

$$\prod_{\sigma \in I_F^0} \sigma(t)^{k_{1,\sigma} + k_{2,\sigma}} \cdot \prod_{\sigma \in I_F^1} \sigma(t)^{k_{1,\sigma} - k_{2,\sigma} - 2} \cdot \prod_{\sigma \in I_F^2} \sigma(t)^{-(k_{1,\sigma} - k_{2,\sigma} + 4)} \cdot \prod_{\sigma \in I_F^3} \sigma(t)^{-(k_{1,\sigma} + k_{2,\sigma} + 6)}.$$

The condition in the statement then follows by applying Remark 4.2.4.3. □

Definition 4.3.2.5. *If λ satisfies the above condition with respect to a q-admissible decomposition Ψ and to $\kappa \in \mathbb{Z}$, we say that λ is $(\kappa, 0)$-*Kostant parallel *with respect to Ψ.*

Definition 4.3.2.6. *A q-admissible decomposition Ψ is said to be $(\lambda, 0)$-*admissible *if there exists $\kappa \in \mathbb{Z}$ such that λ is $(\kappa, 0)$-Kostant parallel with respect to Ψ. The set of q-admissible decompositions which are moreover $(\lambda, 0)$-admissible will be denoted by $\mathcal{P}_q^{(\lambda,0)}$.*

With these definitions in hand, we can prove:

Lemma 4.3.2.7. *For every $s \notin \{0, \ldots, d-1\}$, the cohomology space $H^s(\Gamma_{0,Z}, V_\Psi^{0,q})$ is trivial. For every $s \in \{0, \ldots, d-1\}$, it is non-trivial if and only if λ is $(\kappa, 0)$-Kostant parallel with respect to Ψ and one of the following two conditions holds:*

1. *$I_F = I_F^0 \sqcup I_F^1$. In this case, $q \in \{0, \ldots, d\}$ and $\mathrm{Gr}^{\mathbb{W}}_{w(\lambda)-d\kappa} V_\Psi^{0,q} \neq \{0\}$;*

2. *$I_F = I_F^2 \sqcup I_F^3$. In this case, $q \in \{2d, \ldots, 3d\}$ and $\mathrm{Gr}^{\mathbb{W}}_{w(\lambda)-d\kappa} V_\Psi^{0,q} \neq \{0\}$.*

Proof. We make first a preliminary observation: suppose that λ satisfies (4.28) for a certain q-admissible decomposition Ψ. We see then that if $I_F^0 \sqcup I_F^1$ is non empty, then $\kappa \geq -2$, and that if $I_F^2 \sqcup I_F^3$ is non-empty, then $\kappa \leq -4$. This follows from the formulae in (4.28) and from the fact that, since λ is dominant, $k_{1,\sigma} \geq k_{2,\sigma} \geq 0$ for every $\sigma \in I_F$.

Now, Lemma 4.2.4.1 says that for each $s \in \{0, \ldots, d-1\}$, $H^s(\Gamma_{0,Z}, V_\Psi^{0,q})$ is non-trivial if and only if the action of $\Gamma_{0,Z}$ is trivial, i.e. if and only if condition (4.28) is satisfied. But in this case, the previous observation says that only one of the subsets $I_F^0 \sqcup I_F^1$, $I_F^2 \sqcup I_F^3$ can be non-empty. Then, in both situations, the assertion on q comes from the definition of q-admissible decomposition (Eq. (4.15)), while the assertion on the weight appearing in $V_\Psi^{0,q}$ comes by comparing the computation of Lemma 4.3.2.2 with the formulae in (4.28). □

Remark 4.3.2.8. The above lemma, which is an essential step towards Theorem 4.3.1, implicitly makes use of the "coincidences" in the computations in Lemma 4.3.2.2 and in (4.28), i.e. of the fact that the linear combinations of coordinates of characters that appear in the two cases are the same. This can be rephrased as follows. With M_0 as in Notation 4.2.2.1, let $\iota : \mathbb{G}_{m,L} \to Z(M_0)_L$ be the composition of the adjunction embedding $\mathbb{G}_{m,L} \hookrightarrow \mathbb{G}_{m,L}^d$ and of the isomorphism $\mathbb{G}_{m,L}^d \simeq Z(M_0)_L$ deduced from the isomorphism $(Q_0/W_0)_L \simeq G_{0,L} \times M_{0,L} \simeq \mathbb{G}_{m,L} \times \prod_{\sigma \in I_F} (\mathrm{GL}_{2,L})_\sigma$. Let moreover $w : \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$, resp. $k : \mathbb{S} \to G_{0,\mathbb{R}}$ be the cocharacter defined in 2.1.3.1, resp. the morphism defining the Shimura datum corresponding to G_0. Then, for every λ, Lemma 4.3.2.2 and (4.28) show that we have

$$\lambda|_{G_{0,\mathbb{R}}} \circ k \circ w = \lambda|_{Z(M_0)_\mathbb{R}} \circ \iota_\mathbb{R}. \tag{4.30}$$

In other words, the "Hodge weight", determined by the restriction of λ to the center of the G_0-component of Q_0/W_0, equals the (*a priori* different) character obtained by restriction to the center of the M_0-component.

This is indeed a general phenomenon, as we explain now. Denote by A the maximal $\mathbb{Q}$-split torus in the center of $(Q_0/W_0) \cap G^{\mathrm{der}}$, which is a subgroup of $Z(M_0)$ isomorphic to $\mathbb{G}_m$. If ι_A is the isomorphism $A \simeq \mathbb{G}_m$ obtained in the same way as ι, then $\lambda|_{Z(M_0)_\mathbb{R}} \circ \iota_\mathbb{R} = \lambda|_{A_\mathbb{R}} \circ \iota_{A,\mathbb{R}}$. Hence, we see that (4.30) is a consequence of [LR91, Prop. 6.4]: the proof in *loc. cit.* is valid for quite general Shimura data and is based on the description of the action of A (through λ) on the fibers of the degeneration as the lifting of a *local Hecke operator*, a self-(multivalued) map defined in a neighbourhood of the stratum (in our case, of a cusp). This is seen to coincide with an action induced by the *geodesic action* of A on the *rational boundary components* of the underlying hermitian symmetric domain (in our case, $\mathfrak{H}_n^d$).

Observe now that, by the considerations in Costruction 4.3.2.3, the necessary conditions for non-triviality of the cohomology of $\Gamma_{0,Z}$ give necessary conditions for non-triviality of the cohomology of the bigger group Γ_0. Applying this, and employing Definition 4.3.2.6, the isomorphism from Theorem 4.2.1.(2) for a stratum Z of ∂S_K^* contributing to Z_0 now becomes

$$R^n i_0^* i^* j_* \mu_\ell^K(V_\lambda)\Big|_Z \simeq \bigoplus_{p+q=n} \mu_\ell^{\pi_0(K_0)}\Big(\bigoplus_{\Psi\in\mathcal{P}_q^{(\lambda,0)}} H^p(\Gamma_0, V_\Psi^{0,q})\Big). \tag{4.31}$$

We are interested in the weight-graded objects

$$\mathrm{Gr}_k^{\mathbb{W}} R^n i_0^* i^* j_* \mu_\ell^K(V_\lambda)\Big|_Z \simeq \bigoplus_{p+q=n} \mu_\ell^{\pi_0(K_0)}\Big(\bigoplus_{\Psi\in\mathcal{P}_q^{(\lambda,0)}} H^p(\Gamma_0, \mathrm{Gr}_k^{\mathbb{W}} V_\Psi^{0,q})\Big). \tag{4.32}$$

We are going to find a *second* set of necessary conditions for these objects to be non-trivial, through a *dévissage* which is "orthogonal" to the one described in Remark 4.3.2.3.

Construction 4.3.2.9. The groups $\Gamma_{0,ss} := \Gamma_0 \cap \mathrm{SL}_2(F)$, resp. $\det \Gamma_0$ are non-trivial subgroups of $\mathrm{SL}_2(F)$, resp. $F^\times$, which are again arithmetic and torsion-free. In particular, $\det \Gamma_0 \simeq \mathbb{Z}^{d-1}$ (as in the proof of Lemma 4.3.2.4). Moreover, Γ_0 can be written as an extension

$$1 \to \Gamma_{0,ss} \to \Gamma_0 \xrightarrow{\det} \det \Gamma_0 \to 1,$$

so that the Lyndon-Hochschild-Serre spectral sequence applied to this extension

$$E_2 = H^r(\det \Gamma_0, H^s(\Gamma_{0,ss}, V_\Psi^{0,q})) \Rightarrow H^{r+s}(\Gamma_0, V_\Psi^{0,q})$$

tells us that each space $H^p(\Gamma_0, V_\Psi^{0,q})$ is (non-canonically) isomorphic to a direct sum

$$\bigoplus_{r+s=p} N^{r,s}$$

where each $N^{r,s}$ is a subquotient of

$$H^r(\det \Gamma_0, H^s(\Gamma_{0,ss}, V_\Psi^{0,q})).$$

If the latter subgroup is zero for every r or $H^s(\Gamma_{0,ss}, V_\Psi^{0,q})$ is zero for every s, then $H^p(\Gamma_0, V_\Psi^{0,q})$ is.

For every integer $q \in \{0, \dots, 3d\}$, we know by (4.23) that $\det \Gamma_0$ acts on $V_\Psi^{0,q}$, and *a fortiori* on its subspace $H^0(\Gamma_{0,ss}, V_\Psi^{0,q})$, via multiplication by the character χ defined by

$$t \mapsto \prod_{\sigma\in I_F^0} \sigma(t)^{k_{2,\sigma}} \cdot \prod_{\sigma\in I_F^1} \sigma(t)^{-k_{2,\sigma}-2} \cdot \prod_{\sigma\in I_F^2} \sigma(t)^{-k_{1,\sigma}-3} \cdot \prod_{\sigma\in I_F^3} \sigma(t)^{-k_{1,\sigma}-3} \tag{4.33}$$

and the following lemma will allow us to identify the cohomology spaces corresponding to this action:

Lemma 4.3.2.10. *Let $\lambda = \lambda(k_1, k_2, c)$ be $(\kappa, 0)$-Kostant parallel with respect to a q-admissible decomposition Ψ of I_F (notation as in 4.2.3.2). Fix $s \in \{0, \dots, 3d\}$ and suppose that $H^s(\Gamma_{0,ss}, V_\Psi^{0,q})$ is non-zero. For every $r \in \{0, \dots, d-1\}$,*

$$H^r(\det \Gamma_0, H^s(\Gamma_{0,ss}, V_\Psi^{0,q})) \neq \{0\} \iff H^0(\det \Gamma_0, H^s(\Gamma_{0,ss}, V_\Psi^{0,q})) \neq \{0\}$$

$\iff$ *one of the following conditions is satisfied:*

1. *$I_F = I_F^0$ and k_2 is parallel. In this case, $q = 0$;*

2. *$I_F = I_F^1$ and k_2 is parallel. In this case, $q = d$;*

3. *$I_F = I_F^2 \sqcup I_F^3$ and such that k_1 is parallel. In this case, $q \in \{2d, \ldots, 3d\}$.*

Proof. The point is to reduce oneself to the case where the action of $\det \Gamma_0$ on the spaces $H^s(\Gamma_{0,ss}, V_\Psi^{0,q})$, for $s > 0$, remains semisimple. Now, if $[\Phi] \in H^s(\Gamma_{0,ss}, V_\Psi^{0,q})$ is the class of a s-cocycle

$$\Phi \in \mathrm{Hom}_{L[\Gamma_{0,ss}]}(L[\Gamma_{0,ss}]^{s+1}, V_\Psi^{0,q}),$$

then, for $t \in \det \Gamma_0$, the element $t.[\Phi] \in H^s(\Gamma_{0,ss}, V_\Psi^{0,q})$ equals the class of the morphism

$$t.\Phi \in \mathrm{Hom}_{L[\Gamma_{0,ss}]}(L[\Gamma_{0,ss}]^{s+1}, V_\Psi^{0,q})$$

that to every $(t_0, \ldots, t_s)$ associates $\chi(t)\Phi(\tilde{t}^{-1}(t_0, \ldots, t_s)\tilde{t})$ (where $\tilde{t}$ is any lifting of t in Γ_0, and χ is as in (4.33)).

Consider now the subgroup of Γ_0 defined in (4.29), which is free abelian, generated by $\{\gamma_1, \ldots, \gamma_{d-1}\}$. The elements $\{(\gamma_1)^2, \ldots, (\gamma_{d-1})^2\}$ generate a free abelian subgroup $\tilde{\Gamma}$ of $\det \Gamma_0$, of rank $d-1$, each of whose elements has a central lifting in Γ_0. Then, for every s, $\tilde{\Gamma}$ still acts via the character χ on $H^s(\Gamma_{0,ss}, V_\Psi^{0,q})$. We can now apply Lemma 4.2.4.1 and Remark 4.2.4.3 to $\tilde{\Gamma}$ and conclude that if $H^s(\Gamma_{0,ss}, V_\Psi^{0,q})$ is non-zero, then $H^r(\tilde{\Gamma}, H^s(\Gamma_{0,ss}, V_\Psi^{0,q})) \simeq H^0(\tilde{\Gamma}, H^s(\Gamma_{0,ss}, V_\Psi^{0,q}))^{\binom{d-1}{r}} \neq \{0\}$ if and only if (remembering the definition of χ) there exists an integer θ such that

$$\begin{cases} k_{2,\sigma} = \theta & \forall \sigma \in I_F^0 \\ -k_{2,\sigma} - 2 = \theta & \forall \sigma \in I_F^1 \\ -k_{1,\sigma} - 3 = \theta & \forall \sigma \in I_F^2 \\ -k_{1,\sigma} - 3 = \theta & \forall \sigma \in I_F^3 \end{cases}$$

Now recall that $k_{1,\sigma} \geq k_{2,\sigma} \geq 0$: the above condition is then equivalent to the one in the statement. Remember that precisely under this condition, the character χ is trivial.

In order to finish the proof, put $\mathcal{F} := \det \Gamma_0 / \tilde{\Gamma}$: it is a finite group, that we can assume non trivial, of a certain order f (otherwise, there is nothing else to do). Let $\{\phi_1 \ldots, \phi_f\}$ be a system of representatives of $\mathcal{F}$ inside $\det \Gamma_0$ and denote by e the endomorphism of multiplication by $\frac{1}{f}\sum_{i=1}^{f} \chi(\phi_i)$. By considering the Lyndon-Hochschild-Serre spectral sequence associated to this quotient and by applying [Wei94, Prop. 6.1.10], we see that

$$H^r(\det \Gamma_0, H^s(\Gamma_{0,ss}, V_\Psi^{0,q})) \simeq H^0\left(\mathcal{F}, H^r(\tilde{\Gamma}, H^s(\Gamma_{0,ss}, V_\Psi^{0,q}))\right) \simeq e \cdot H^r(\tilde{\Gamma}, H^s(\Gamma_{0,ss}, V_\Psi^{0,q})).$$

Now, if e is not the zero endomorphism, then

$$e \cdot H^r(\tilde{\Gamma}, H^s(\Gamma_{0,ss}, V_\Psi^{0,q})) \simeq H^r(\tilde{\Gamma}, H^s(\Gamma_{0,ss}, V_\Psi^{0,q}))$$

and the lemma is demonstrated. But in the case we are working in, χ is trivial, and e is just the identity morphism. ☐

The *third* and last set of necessary conditions for non-triviality of the cohomology of Γ_0 comes from general results on the cohomology of locally symmetric spaces.

Lemma 4.3.2.11. *The following statements hold.*

(1) The cohomology space $H^p(\Gamma_0, V_\Psi^{0,q})$ is trivial for every $p < 0$ and all $p > 3d-2$.

(2) If the irreducible representation $V_\Psi^{0,q}$ is non-trivial as a $\mathrm{SL}^d_{2,L}$-representation, then $H^p(\Gamma_0, V_\Psi^{0,q}) = \{0\}$ for every $0 \leq p < d$.

Proof. 1. Recall from Notation 4.2.2.1 the group $M_0 \simeq \mathrm{Res}_{F|\mathbb{Q}}\mathrm{GL}_{2,F}$, and denote by K_∞ a maximal compact subgroup of $M_0(\mathbb{R})$ (isomorphic to $\prod_{\sigma \in I_F} \mathrm{O}_2(\mathbb{R})$), by A_{M_0} the group $S(\mathbb{R})^0$ (for S the maximal $\mathbb{Q}$-split torus inside $Z(M_0)$) and by $\mathfrak{H}$ the complex upper half plane. Then, remember the *symmetric space* associated to M_0 (Example 2.1.1.3.(3)), defined by $D := M_0(\mathbb{R})/K_\infty A_{M_0} \simeq \mathfrak{H}^d \times \mathbb{R}^{d-1}$, and recall that every $V_\Psi^{0,q}$ (as a $M_{0,L}$-representation) defines a local system $\mathbb{V}_\Psi^{0,q}$ on the locally symmetric space $X_{\Gamma_0} = \Gamma_0 \backslash D$ (Def. 2.1.1.4) such that $H^p(\Gamma_0, V_\Psi^{0,q}) \simeq H^p(X_{\Gamma_0}, \mathbb{V}_\Psi^{0,q})$ for every p (see Construction 2.1.1.6). Let now $R(\cdot)$, resp. $\mathrm{r}_{\mathbb{Q}}(\cdot)$ denote the radical, resp. the $\mathbb{Q}$-rank of a $\mathbb{Q}$-algebraic group. The statement then follows from [BS73, Thm. 11.4.4], taking into account that $\dim(D) - \mathrm{r}_{\mathbb{Q}}(M_0/R(M_0)) = 3d-2$.

2. Recall the group $\Gamma_{0,ss}$ defined in Construction 4.3.2.9, recall the (complex analytic, connected) Hilbert modular variety $X_{\Gamma_{0,ss}}$ defined as $\Gamma_{0,ss}\backslash\mathfrak{H}^d$ in Example 2.1.2.2.(2), and abusively also denote by $\mathbb{V}_\Psi^{0,q}$ the local system on $X_{\Gamma_{0,ss}}$ induced by the the restriction of the representation $V_\Psi^{0,q}$ to $M_{0,L}^{\mathrm{der}} \simeq (\mathrm{Res}_{F|\mathbb{Q}}\mathrm{SL}_{2,F})_L \simeq \mathrm{SL}^d_{2,L}$. Then, for every p, we have $H^p(\Gamma_{0,ss}, V_\Psi^{0,q}) \simeq H^p(X_{\Gamma_{0,ss}}, \mathbb{V}_\Psi^{0,q})$. The statement now follows from the fact that $H^p(X_{\Gamma_{0,ss}}, \mathbb{V}_\Psi^{0,q}) = \{0\}$ for every $0 \leq p < d$ if $\mathbb{V}_\Psi^{0,q}$ is non-trivial ([MSSYZ15, Thm. 1.1(i)]) and by employing the considerations at the end of Construction 4.3.2.9.

□

4.3.2.3 Computation of weights along the Siegel strata.

We can finally describe the weights appearing in the degeneration of the canonical construction along the Siegel strata, in the cohomological degrees which we will need in the sequel:

Proposition 4.3.2.12. *Let V_λ be the irreducible L-representation of G_L of highest weight $\lambda = \lambda((k_{1,\sigma}, k_{2,\sigma})_\sigma, c)$ and Z a stratum of ∂S_K^* which contributes to Z_0. Adopt Notation 4.2.3.2 and the notation of Definition 4.3.1.1.*

(1) Let $n < 0$ or $n > 6d-2$. Then the cohomology sheaf $R^n i_0^* i^* j_* \mu_\ell^K(V_\lambda)\big|_Z$ is zero.

(2) Let $0 \leq n < d$. Then the cohomology sheaf $R^n i_0^* i^* j_* \mu_\ell^K(V_\lambda)\big|_Z$ can be non-zero only if $k_1 = \underline{\kappa_0} = k_2$. In this case,

$$R^n i_0^* i^* j_* \mu_\ell^K(V_\lambda)\Big|_Z \simeq \mu_\ell^{\pi_0(K_0)}(H^n(\Gamma_0, V^{0,0}_{I_F^0}))$$

is pure of weight $w(\lambda) - 2d\kappa_0$. If $n = 0$, then it is non-zero.

(3) Let $n \in \{d, \ldots, 2d-1\}$. Then the cohomology sheaf $R^n i_0^* i^* j_* \mu_\ell^K(V_\lambda)\big|_Z$ can be non-zero only if $k_1 = \underline{\kappa_1}$ and $k_2 = \underline{\kappa_2}$. In this case,

$$R^n i_0^* i^* j_* \mu_\ell^K(V_\lambda) Z \simeq \mu_\ell^{\pi_0(K_0)}(H^n(\Gamma_0, V^{0,0}_{I_F^0})) \tag{4.34}$$

is pure of weight $w(\lambda) - d(\kappa_1 + \kappa_2)$. If $\kappa_1 \neq \kappa_2$ and $n = d$, then it is non-zero.

(4) Let $n \in \{2d, \ldots, 3d-1\}$. Then the cohomology sheaf $R^n i_0^* i^* j_* \mu_\ell^K(V_\lambda)\big|_Z$ can be non-zero only if $k_1 = \underline{\kappa_1}$ and $k_2 = \underline{\kappa_2}$. In this case, it is isomorphic to

$$\mu_\ell^{\pi_0(K_0)}(H^n(\Gamma_0, V^{0,0}_{I_F^0})) \oplus \mu_\ell^{\pi_0(K_0)}(H^{n-d}(\Gamma_0, V^{0,d}_{I_F^1})),$$

where the first factor is isomorphic to

$$\mathrm{Gr}^{\mathbb{W}}_{w(\lambda)-d(\kappa_1+\kappa_2)} R^n i_0^* i^* j_* \mu_\ell^K(V_\lambda)\Big|_Z$$

and the second one to

$$\mathrm{Gr}^{\mathbb{W}}_{w(\lambda)+2d-d(\kappa_1-\kappa_2)} R^n i_0^* i^* j_* \mu_\ell^K(V_\lambda)\Big|_Z$$

If $n = 2d$, then the second factor is non-zero.

Proof. We begin by proving the necessary conditions for the non-vanishing of the cohomology sheaves.

(1) Clear from Remark 4.3.2.11.(1) and from the fact that the representations $V_\Psi^{0,q}$ are trivial for $q > 3d$.

(2) The isomorphisms (4.31) and Lemma 4.3.2.10 (taking into account the considerations at the end of Construction 4.3.2.9) imply that, in order to have non-zero cohomology objects in degree $0 \le n < d$, k_2 has to be parallel and that q can only take the value 0. On the other hand, adding the Kostant-parallelism conditions imposed by Lemma 4.3.2.7, we obtain that k_1 has to be parallel, too. Moreover, by hypothesis, we are in the case $p \in \{0, \ldots, d-1\}$, but in this interval, by Lemma 4.3.2.11.(2), $V^{0,0}_{I_F^0}$ can have non-trivial cohomology objects only if it is the trivial $\mathrm{SL}^d_{2,L}$-representation. Now, looking at the description in (4.23), we see that this is the case if and only if k_1 and k_2 are equal.

(3) The isomorphisms (4.31) and Lemma 4.3.2.10 (taking into account the considerations at the end of Construction 4.3.2.9) imply that, in order to have non-zero cohomology objects in degree $d \le n < 2d$, k_2 has to be parallel and that q can only take the values 0 or d. Again, the Kostant-parallelism conditions imposed by Lemma 4.3.2.7 imply that k_1 has to be parallel, too. Now if $q = d$, then $p \in \{0, \ldots, d-1\}$, and in this interval, by Lemma 4.3.2.11.(2), $V^{0,d}_{I_F^1}$ can have non-trivial cohomology objects only if it is the trivial $\mathrm{SL}^d_{2,L}$-representation; but the description in (4.23) shows that this is never the case. The only remaining possibility is then $q = 0$ and $p \in \{d, \ldots, 2d-1\}$.

(4) Arguing as above, we see that k_1 and k_2 have to be parallel, and that q can only take values in $\{0, d, 2d, \ldots, 3d\}$. The cases $q = 0$ and $q = d$ give the two summands in the statement. If $2d \le q \le 3d-1$, then the fact that $p \in \{0, \ldots, d-1\}$ and Lemma 4.3.2.11.(2) imply that the spaces $H^p(\Gamma_0, V_\Psi^{0,q})$ can give non-trivial contributions to the cohomology objects if and only if $V_\Psi^{0,q}$ is the trivial $\mathrm{SL}^d_{2,L}$-representation, which is never the case, by the description in (4.23) (remember that in this case, Ψ is of the form $(\varnothing, \varnothing, I_F^2 \neq \varnothing, I_F^3)$).

Finally, in all cases, the statements about weight-graded objects follow from Remark 4.3.2.7 and from the isomorphisms (4.32), while the non-triviality statements are consequences of the following proposition. □

Proposition 4.3.2.13. *Let λ, V_λ and Z be as in Proposition 4.3.2.12. If $k_1 = \underline{\kappa_1}$ and $k_2 = \underline{\kappa_2}$, then:*

(1) if $\kappa_1 = \kappa_2$, then the lisse ℓ-adic sheaf $\mu_\ell^{\pi_0(K_0)}(H^0(\Gamma_0, V^{0,0}_{I_F^0}))$ on Z is non-zero;

(2) if $\kappa_1 \neq \kappa_2$, then the lisse ℓ-adic sheaf $\mu_\ell^{\pi_0(K_0)}(H^d(\Gamma_0, V^{0,0}_{I_F^0}))$ on Z is non-zero;

(3) the lisse ℓ-adic sheaf $\mu_\ell^{\pi_0(K_0)}(H^d(\Gamma_0, V^{0,d}_{I_F^1}))$ on Z is non-zero. If moreover κ_1 and κ_2 have the same parity[2]*, it is locally of dimension $> h$, where $h := |\Gamma_{0,ss} \backslash \mathbb{P}^1(F)|$ is the (strictly positive) number of cusps of the (complex analytic, connected) Hilbert modular variety $X_{\Gamma_{0,ss}}$ of Example 2.1.2.2.(2).*

Proof. If $\kappa_1 = \kappa_2$, then the spectral sequence considered in Construction 4.3.2.9 shows that the space $H^0(\Gamma_0, V)$ is isomorphic to $H^0(\det \Gamma_0, H^0(\Gamma_{0,ss}, V))$, which, by the proof of Lemma 4.3.2.10, is in turn isomorphic to $H^0(\Gamma_{0,ss}, V)$. Moreover, the description in (4.23) tells us that here, V is the trivial $\mathrm{SL}^d_{2,L}$-representation, and thus, it is a 1-dimensional L-vector space. This shows point (1).

Assume then to be in one of the two following cases: either $\kappa_1 \neq \kappa_2$ and V is the irreducible $\mathrm{SL}^d_{2,L}$-representation $V^{0,0}_{I_F^0}$ (which in this case is non-trivial), or $V := V^{0,d}_{I_F^1}$ (which by the description in (4.23) is then isomorphic to $\bigotimes_{\sigma \in I_F} \mathrm{Sym}^{\kappa_1+\kappa_2+2}\mathrm{V}$, where V is the standard 2-dimensional L-representation of $\mathrm{SL}_{2,L}$, so that $V^{0,d}_{I_F^1}$ is never trivial).

In both cases, the same Remark 4.3.2.9 and Remark 4.3.2.11 show that the space $H^d(\Gamma_0, V)$ is isomorphic to $H^0(\det \Gamma_0, H^d(\Gamma_{0,ss}, V))$, which by the hypothesis on k_1 and k_2 and by the proof of Lemma 4.3.2.10 is in turn isomorphic to $H^d(\Gamma_{0,ss}, V)$. Now, for every integer $\tilde{\kappa} > 0$, [MSSYZ15, Thm. 1.1(iv)] shows that $\dim H^d(\Gamma_{0,ss}, \bigotimes_{\sigma \in I_F} \mathrm{Sym}^{\tilde{\kappa}}\mathrm{V}) = h + \delta(\Gamma_{0,ss}, \tilde{\kappa})$, where $\delta(\Gamma_{0,ss}, \tilde{\kappa})$ is a non-negative integer which depends on $\Gamma_{0,ss}$ and on $\tilde{\kappa}$. This is enough to show (2) and the first half of (3).

To finish the proof of (3), suppose that κ_1 and κ_2 have the same parity and put $\kappa_1 + \kappa_2 =: 2\kappa$. We will show that, in this case, $\delta := \delta(\Gamma_{0,ss}, 2\kappa + 2) > 0$. Actually, [MSSYZ15, Thm. 1.1 (iv)] shows that, more precisely, $\delta = h_{I_F} + \delta'$, where δ' is a certain positive integer and h_{I_F} is the dimension of the space of *cusp forms* of (parallel) weight $2\kappa + 4$, i.e. of *type* $\underline{2\kappa + 2}$, and of *level* $\Gamma_{0,ss}$ (see Rmk. 2.3.4.4.(1)). Thus, in order to conclude, it is enough to show that this dimension is strictly positive.

Let $X_{\Gamma_{0,ss}}$ be the complex analytic Hilbert modular variety associated to $\Gamma_{0,ss}$. According to [Fre90, Chap. II, Thm. 3.5], we have

$$h_{I_F} = \mathrm{vol}(X_{\Gamma_{0,ss}})(2\kappa + 3)^d + L_{cusp}, \tag{4.35}$$

where L_{cusp} is a (not necessarily positive) integer which does not depend on κ (recall that $\Gamma_{0,ss}$ is neat). Now, if d is odd, then the discussion in [Fre90, page 111] implies that $L_{cusp} = 0$, so that we obtain $h_{I_F} > 0$, as desired.

If instead d is even, let us consider a smooth *toroidal compactification* $\bar{X}_{\Gamma_{0,ss}}$ of $X_{\Gamma_{0,ss}}$ (see Subsection 2.3.4). Then, by applying the Hirzebruch-Riemann-Roch theorem to the

[2] This restriction on parity is necessary in order to apply the results from [Fre90], which in turn depend on the formulae for the dimension of certain spaces of cusp forms proved in [Shi63]. By [Shi63, Note 11, pag. 63], it is possible that these formulae could admit a suitable generalisation, such that the hypothesis on parity could be removed.

automorphic bundles of type $\underline{2\kappa+2}$ (cfr. Def. 2.3.4.2 and Rmk. 2.3.4.4.(1)) on $\bar{X}_{\Gamma_{0,ss}}$, the authors show in [MSSYZ15, Prop. 7.10] that

$$h_{I_F} = \chi(\bar{X}_{\Gamma_{0,ss}}, \mathcal{O}_{\bar{X}_{\Gamma_{0,ss}}}) + \epsilon \tag{4.36}$$

for a certain integer ϵ. Now, [Fre90, Chap. II, Thm. 4.8], implies that, if d is even, $\chi(\bar{X}_{\Gamma_{0,ss}}, \mathcal{O}_{\bar{X}_{\Gamma_{0,ss}}}) > 0$ (this quantity is in particular equal to 1 plus the dimension of the space of cusp forms of weight 2 with respect to $\Gamma_{0,ss}$) and that

$$\chi(\bar{X}_{\Gamma_{0,ss}}, \mathcal{O}_{\bar{X}_{\Gamma_{0,ss}}}) = \mathrm{vol}(X_{\Gamma_{0,ss}}) + L_{cusp} \tag{4.37}$$

(let us stress the fact that L_{cusp} is the *same* integer of equation (4.35)). By replacing the expression for $\chi(\bar{X}_{\Gamma_{0,ss}}, \mathcal{O}_{\bar{X}_{\Gamma_{0,ss}}})$ into equation (4.36), the equality between the two expressions (4.35) and (4.36) for h_{I_F} tells us that $\epsilon = \mathrm{vol}(X_{\Gamma_{0,ss}})(2\kappa+3)^d - \mathrm{vol}(X_{\Gamma_{0,ss}}) > 0$. The equation (4.36) then implies that $h_{I_F} > 0$ in this case too. □

Remark 4.3.2.14. The above proof shows that the presence of the weights we are interested in is dued to non-trivial Hilbert cuspidal forms, living on the "virtual" (i.e., not explicitly appearing in ∂S_K^*) complex analytic Hilbert modular variety $X_{\Gamma_{0,ss}}$.

4.3.3 The degeneration along the Klingen strata

Let λ, V_λ be as in Subsection 4.3.2 and let us now study, by using Theorem 4.2.1, the degeneration of $\mu_\ell^K(V_\lambda)$ along the Klingen strata. The group G_1 in their underlying Shimura datum is isomorphic to $\mathrm{Res}_{F|\mathbb{Q}}\mathrm{GL}_{2,F} \times_{\mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F},\det} \mathbb{G}_m$ (cfr. 4.2.1).

4.3.3.1 Weights in the cohomology of the unipotent radical.

As before, let us start by identifying the possible weights appearing in the degeneration along the Siegel strata, i.e. in the $(Q_1/W_1)_L$-representations

$$H^q(W_{1,L}, V_\lambda) \simeq \bigoplus_{\Psi \in \mathcal{P}_q} V_\Psi^{1,q}, \tag{4.38}$$

for $q \in \{0, \dots, 3d\}$ (cfr. (4.16)). Recall from (4.6) that

$$(Q_1/W_1)_L \simeq ((\prod_{\sigma\in I_F} (\mathrm{GL}_{2,L})_\sigma) \times_{\prod_{\sigma\in I_F}(\mathbb{G}_{m,L})_\sigma} \mathbb{G}_{m,L}) \times \prod_{\sigma\in I_F} (\mathbb{G}_{m,L})_\sigma$$

We are now going to compute the weight of the pure Hodge structure carried by each irreducible summand $V_\Psi^{1,q}$.

Lemma 4.3.3.1. *For every* $q \in \{0, \dots, 3d\}$ *and for every* q*-admissible decomposition* Ψ *as in Notation 4.2.3.2, the action of the factor isomorphic to*

$$(\prod_{\sigma\in I_F} (\mathrm{GL}_{2,L})_\sigma) \times_{\prod_{\sigma\in I_F}(\mathbb{G}_{m,L})_\sigma, \prod \det} \mathbb{G}_{m,L}$$

inside $(Q_1/W_1)_L$ *induces on* $V_\Psi^{1,q}$ *a pure Hodge structure of weight*

$$w(\lambda) - [\sum_{\sigma\in I_F^0} k_{1,\sigma} + \sum_{\sigma\in I_F^1} (k_{2,\sigma} - 1) - \sum_{\sigma\in I_F^2} (k_{2,\sigma} + 3) - \sum_{\sigma\in I_F^3} (k_{1,\sigma} + 4)]. \tag{4.39}$$

Proof. By the discussion in 4.1.2, the L-points of the factor of $(Q_1/W_1)_L$ that we are considering are identified with the subgroup

$$\{(\begin{pmatrix} \rho & & & \\ & \tau_\sigma & & \\ & & 1 & \\ & & & \tau_\sigma^{-1}\rho \end{pmatrix})_{\sigma\in I_F} \mid \rho \in L^\times, \tau_\sigma \in L^\times \text{ for every } \sigma \in I_F\}$$

of $Q_1/W_1(L)$. By the convention fixed in 2.1.3.1 and the definition in (4.10) of the Shimura datum (G_1, X_1), the expression given in Lemma 4.2.3.3.(2) for the highest weight of the representation $V_\Psi^{1,q}$ yields the formula in the statement (recalling (4.8) and Rmk. 4.3.2.1). □

4.3.3.2 Cohomology of the arithmetic subgroup.

Consider now the arithmetic group Γ_1 of Rmk. 4.2.2.2, which is identified with a torsion-free arithmetic subgroup of $\mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F}(\mathbb{Q}) = F^\times$ (cfr. Remark 4.2.2.2). We need to identify the cohomology spaces

$$H^p(\Gamma_1, H^q(W_{1,L}, V_\lambda)) \simeq \bigoplus_{\Psi\in\mathcal{P}_q} H^p(\Gamma_1, V_\Psi^{1,q}).$$

Reasoning as in the proof of Lemma 4.3.2.4, we obtain:

Lemma 4.3.3.2. *The group Γ_1 is isomorphic to $\mathbb{Z}^{d-1}$, and for every $q \in \{0, \dots, 3d\}$, its action on $V_\Psi^{1,q}$ is trivial if and only if there exists an integer κ such that*

$$\begin{cases} k_{1,\sigma} = \kappa & \forall\sigma \in I_F^0 \\ k_{2,\sigma} - 1 = \kappa & \forall\sigma \in I_F^1 \\ -(k_{2,\sigma} + 3) = \kappa & \forall\sigma \in I_F^2 \\ -(k_{1,\sigma} + 4) = \kappa & \forall\sigma \in I_F^2 \end{cases} \tag{4.40}$$

(remembering Notation 4.2.3.2).

Hence, we are led to pose the following:

Definition 4.3.3.3. *We say that λ is* $(\kappa, 1)$-Kostant parallel *with respect to a q-admissible decomposition Ψ if λ satisfies condition (4.40) with respect to $\kappa \in \mathbb{Z}$.*

Definition 4.3.3.4. *A q-admissible decomposition Ψ is said to be* $(\lambda, 1)$-admissible *if there exists $\kappa \in \mathbb{Z}$ such that λ is $(\kappa, 1)$-Kostant parallel with respect to Ψ. The set of q-admissible decompositions which are moreover $(\lambda, 1)$-admissible will be denoted by $\mathcal{P}_q^{(\lambda,1)}$.*

Then, the proof of the following lemma is completely analoguous to the proof of Lemma 4.3.2.7:

Lemma 4.3.3.5. *For every $s \notin \{0, \dots, d-1\}$, the cohomology space $H^s(\Gamma_1, V_\Psi^{1,q})$ is trivial. For every $s \in \{0, \dots, d-1\}$, it is non-trivial if and only if λ is $(\kappa, 1)$-Kostant parallel with respect to Ψ and one of the following two conditions holds:*

(1) $I_F = I_F^0 \sqcup I_F^1$. In this case, $q \in \{0, \dots, d\}$ and $\mathrm{Gr}^{\mathbb{W}}_{w(\lambda)-d\kappa} V_\Psi^{1,q} \neq \{0\}$;

(2) $I_F = I_F^2 \sqcup I_F^3$. In this case, $q \in \{2d, \dots, 3d\}$ and $\mathrm{Gr}^{\mathbb{W}}_{w(\lambda)-d\kappa} V_\Psi^{1,q} \neq \{0\}$.

Remark 4.3.3.6. An essential ingredient for the proof of the above lemma is again the same phenomenon of Rmk. 4.3.2.8.

Remembering Definition 4.3.3.4, the isomorphism in 4.2.1.(2) for a stratum Z' of ∂S_K^* contributing to Z_1 becomes now

$$R^n i_1^* i^* j_* \mu_\ell^K(V_\lambda)\Big|_{Z'} \simeq \bigoplus_{p+q=n} \mu_\ell^{\pi_1(K_1)}\Big(\bigoplus_{\Psi \in \mathcal{P}_q^{(\lambda,1)}} H^p(\Gamma_1, V_\Psi^{1,q})\Big) \tag{4.41}$$

and we want to study the weight-graded objects

$$\mathrm{Gr}_k^{\mathbb{W}} R^n i_1^* i^* j_* \mu_\ell^K(V_\lambda)\Big|_{Z'} \simeq \bigoplus_{p+q=n} \mu_\ell^{\pi_1(K_1)}\Big(\bigoplus_{\Psi \in \mathcal{P}_q^{(\lambda,1)}} H^p(\Gamma_1, \mathrm{Gr}_k^{\mathbb{W}} V_\Psi^{1,q})\Big). \tag{4.42}$$

4.3.3.3 Computation of weights along the Klingen strata.

In order to describe the weights appearing in the degeneration of the canonical construction along the Klingen strata (in the cohomological degrees which will be needed in the sequel), we just need a last preliminary remark:

Remark 4.3.3.7. 1. Suppose the dominant weight $\lambda = \lambda((k_{1,\sigma}, k_{2,\sigma})_{\sigma \in I_F}, c)$ of G_L to be $(\kappa, 1)$-Kostant parallel with respect to a decomposition Ψ of the form $(I_F^0, I_F^1, \varnothing, \varnothing)$. Then, by the definition of Kostant-parallelism and the hypothesis on λ, we see that such κ and Ψ are necessarily unique, except if $k_1 = \underline{\kappa_1}$ and $k_2 = \underline{\kappa_2}$. In this last case, there exist exactly two pairs $(\kappa, (I_F^0, I_F^1))$ such that λ is κ-Kostant-parallel with respect to (I_F^0, I_F^1), i.e. (κ_1, I_F^0) and $(\kappa_2 - 1, I_F^1)$.

2. The condition on λ of being $(\kappa, 1)$-Kostant parallel with respect to a decomposition Ψ of the form $(I_F^0, I_F^1, \varnothing, \varnothing)$ coincides with the condition of being κ-Kostant parallel introduced in Definition 4.3.1.1; hence, we will adopt this terminology in the following. Moreover, by the preceding point, whenever we suppose λ to be κ-Kostant parallel with respect to a decomposition such that $I_F^0 \neq \varnothing$, resp. $I_F^1 \neq \varnothing$, then I_F^0, resp. I_F^1, is uniquely determined by λ.

Then, remembering the above Remark, by employing Lemma 4.3.3.5 and reasoning along the same lines of the proof of Prop. 4.3.2.12, we deduce the following:

Proposition 4.3.3.8. *Let V_λ be the irreducible L-representation of G_L of highest weight $\lambda = \lambda((k_{1,\sigma}, k_{2,\sigma})_\sigma, c)$ and Z' be a stratum of ∂S_K^* contributing to Z_1.*

(1) Let $n < 0$ or $n > 4d - 1$. Then, $R^n i_1^* i^* j_* \mu_\ell^K(V_\lambda)\Big|_{Z'}$ is zero.

(2) Let $n \in \{0, \dots, d-1\}$. Then the ℓ-adic sheaf $R^n i_1^* i^* j_* \mu_\ell^K(V_\lambda)\Big|_{Z'}$ on Z' is non-zero if and only if the following hold:

- *λ is κ-Kostant parallel and $I_F^0 \neq \varnothing$;*
- *posing $d_1 := |I_F^1| \in \{0, \dots, d-1\}$, we have $n \geq d_1$.*

In this case,

$$R^n i_1^* i^* j_* \mu_\ell^K(V_\lambda)\Big|_{Z'} \simeq \mu_\ell^{\pi_1(K_1)}(H^{n-d_1}(\Gamma_1, V_\Psi^{1,d_1}))$$

and it is pure of weight $w(\lambda) - d\kappa$.

(3) Let $n \in \{d, \dots, 2d-1\}$. Then the ℓ-adic sheaf $R^n i_1^* i^* j_* \mu_\ell^K(V_\lambda)\big|_{Z'}$ on Z' is non-zero if and only if the following hold:

- *λ is κ-Kostant parallel and $I_F^1 \neq \varnothing$;*
- *posing $d_1 := |I_F^1| \in \{1, \dots, d\}$, we have $n \leq d-1+d_1$.*

In this case,

$$R^n i_1^* i^* j_* \mu_\ell^K(V_\lambda)\big|_{Z'} \simeq \mu_\ell^{\pi_1(K_1)}(H^{n-d_1}(\Gamma_1, V_\Psi^{1,d_1}))$$

and, denoting $\kappa_2 := \kappa + 1$, it is pure of weight $w(\lambda) + d - d\kappa_2$.

(4) Let $n \in \{2d, \dots, 3d-1\}$. Then the ℓ-adic sheaf $R^n i_1^* i^* j_* \mu_\ell^K(V_\lambda)\big|_{Z'}$ on Z' is non-zero if and only if λ is $(\kappa, 1)$-Kostant parallel with respect to a Ψ such that $I_F = I_F^2 \sqcup I_F^3$ and I_F^2 is non-empty. In this case, posing $d_3 := |I_F^3| \in \{0, \dots, d-1\}$,

$$R^n i_1^* i^* j_* \mu_\ell^K(V_\lambda)\big|_{Z'} \simeq \mu_\ell^{\pi_1(K_1)}(H^{n-2d-d_3}(\Gamma_1, V_\Psi^{1,2d+d_3}))$$

and, denoting $\kappa_3 = -\kappa - 3$, it is pure of weight $w(\lambda) + 3d + d\kappa_3$.

4.3.4 The double degeneration along the cusps of the Klingen strata

Keep the notation of Thm. 4.3.1. In order to study the weights of the motive $i^* j_*^\lambda \mathcal{V}$, the study of the degeneration of the canonical construction to each stratum of ∂S_K^* will not be enough: in Lemma 4.3.5.5, we will also need to consider a *double* degeneration, the one of mixed sheaves on the Klingen strata, *already obtained by degeneration*, along the boundary of the closure in ∂S_K^* of the Klingen strata themselves.

By paragraph 4.2.1, every stratum Z' of ∂S_K^* contributing to Z_1 (as defined in paragraph 4.2.2) is (a smooth quotient by the action of a finite group of) a Hilbert modular variety $S_{\pi_1(K_1)}$ of dimension d. Remember from the same paragraph 4.2.1 that the Shimura datum underlying $S_{\pi_1(K_1)}$ corresponds to the algebraic group $G_1 \simeq \mathrm{Res}_{F|\mathbb{Q}}\mathrm{GL}_{2,F} \times_{\mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_m,\det} \mathbb{G}_m$, whose L-points are identified, up to conjugation, with

$$G_1(L) = \{ (\begin{pmatrix} \rho & & & \\ & a_\sigma & & b_\sigma \\ & & 1 & \\ & c_\sigma & & d_\sigma \end{pmatrix})_{\sigma \in I_F} \, | a_\sigma, b_\sigma, c_\sigma, d_\sigma \in L, \rho \in L^\times,$$
$$\text{such that } \rho = a_\sigma d_\sigma - b_\sigma c_\sigma \text{ for every } \sigma \in I_F \} =$$
$$= \{ (A_\sigma)_{\sigma \in I_F} \in \prod_{\sigma \in I_F} \mathrm{GL}_{2,L}(L) \text{ such that } \det(A_\sigma) = \det(A_{\hat\sigma}) \ \forall \ \sigma, \hat\sigma \in I_F \}.$$

The boundary $\partial S_{\pi_1(K_1)}^*$ of the Baily-Borel compactification $S_{\pi_1(K_1)}^*$ of $S_{\pi_1(K_1)}$ is 0-dimensional: it is in fact a finite disjoint union of strata (called *cusps*), obtained as Shimura varieties coming from the group $\mathbb{G}_m$. Fix such a stratum Z'', corresponding up to conjugation, in the formalism of 2.3.3, to the standard Borel subgroup of G_1 (denoted by Q_2 for the sake of coherence with the notations in the sequel): it is a representative of the unique $G_1(\mathbb{Q})$-conjugacy class of standard maximal parabolics of G_1. Its unipotent radical will be denoted by W_2. The Levi component of Q_2 is a torus T_1 isomorphic to

$$\mathbb{G}_m \times \mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_{m,F} \tag{4.43}$$

via the isomorphism defined on L-points by

$$T_1(L) \simeq \mathbb{G}_m(L) \times \prod_{\sigma \in I_F} \mathbb{G}_m(L)_\sigma$$
$$\begin{pmatrix} \beta & & & \\ & \beta u_\sigma & & \\ & & 1 & \\ & & & u_\sigma^{-1} \end{pmatrix})_{\sigma \in I_F} \mapsto (\beta, (u_\sigma)_{\sigma \in I_F}).$$

Then, the Shimura datum (G_2, X_2) underlying Z'' is such that the L-points of the group $G_2 \simeq \mathbb{G}_m$ are identified with

$$G_2(L) = \{(\begin{pmatrix} \beta \cdot I_2 & \\ & I_2 \end{pmatrix})_{\sigma \in I_F} | \beta \in L^\times\} \hookrightarrow T_1(L)$$

and X_2 is defined exactly as X_0 in (4.9) (cfr. [Pin90, Example 12.21]).

4.3.4.1 The degeneration of the canonical construction along the cusps of Hilbert modular varieties.

Let j' be the open immersion of $S_{\pi_1(K_1)}$ in $S^*_{\pi_1(K_1)}$ and adopt the notations of paragraph 4.2.2, by replacing j with j' and K with $\pi_1(K_1)$. The stratification Φ of $\partial S^*_{\pi_1(K_1)}$ is then formed by only one element, called Z_2. Denote by $i_2 : Z_2 \hookrightarrow S^*_{\pi_1(K_1)}$ the closed immersion complementary to j'. Let us consider a stratum Z'' contributing to Z_2 and let us spell out, thanks to Theorem 4.2.2, the conclusions of Theorem 4.2.1, applied this time to $\mu_\ell^{\pi_1(K_1)}(U_\chi)$, where U_χ is a irreducible L-representation of $G_{1,L}$.

Such a representation is determined by its highest weight $\chi = \chi((h_\sigma)_{\sigma \in I_F}, g)$, where $h_\sigma \in \mathbb{Z}, h_\sigma \geq 0\ \forall \sigma \in I_F$, $g \in \mathbb{Z}$ (we will write $\mathbf{h}$ for the vector $(h_\sigma)_{\sigma \in I_F}$). This character is defined on the points of the maximal torus $T_{1,L}$ of $G_{1,L}$ by

$$(\begin{pmatrix} \beta & & & \\ & \beta u_\sigma & & \\ & & 1 & \\ & & & u_\sigma^{-1} \end{pmatrix})_{\sigma \in I_F} \mapsto \prod_{\sigma \in I_F} u_\sigma^{h_\sigma} \cdot \beta^g.$$

The vector $\mathbf{h}$ is called *parallel* if there exists an integer h such that $h_\sigma = h$ for every $\sigma \in I_F$. In that case, we will write $\mathbf{h}= \underline{h}$.

Remark 4.3.4.1. Notice that, with these conventions, the restriction to $T_{1,L}$ of the character $\lambda((k_{1,\sigma}, k_{2,\sigma})_\sigma, c)$ defined in 4.8 is given by

$$\chi((k_{2,\sigma})_\sigma, \frac{1}{2} \cdot [c + \sum_\sigma (k_{1,\sigma} + k_{2,\sigma})]). \tag{4.44}$$

Using the notations fixed in the beginning of this subsection, we have an identification

$$(Q_2/W_2)_L \simeq T_{1,L},$$

so that, by Theorem 4.2.2, the cohomology spaces $H^q(W_{2,L}, U_\chi)$ are identified with representations of the group $T_{1,L} \simeq \mathbb{G}_{m,L} \times \prod_{\sigma \in I_F} \mathbb{G}_{m,L}$. Let us determine the weight of the pure Hodge structure carried by each irreducible factor of these representations.

Lemma 4.3.4.2. *Let* $\chi = \chi((h_\sigma)_{\sigma\in I_F}, g)$ *as above. Then, the spaces* $H^q(W_{2,L}, U_\chi)$ *are non-trivial if and only if* $q \in \{0, \ldots, d\}$*. For each* $q \in \{0, \ldots, d\}$*, letting* I *run over the subsets of* I_F *of cardinality* q*, they are direct sums of pure Hodge structures of weight*

$$-2g + 2\sum_{\sigma\in I}(h_\sigma + 1). \tag{4.45}$$

Proof. We begin by making explicit the data which are needed in order to apply Theorem 4.2.2. By choosing $(T_{1,L}, Q_{2,L})$ as a maximal torus and a Borel of $G_{1,L}$, we can identify the set of roots $\mathfrak{r}$ of $G_{1,L}$ with $\bigsqcup_{\sigma\in I_F} \mathfrak{r}_\sigma$, where each $\mathfrak{r}_\sigma$ is a copy of the set of roots of $\mathrm{GL}_{2,L}$ corresponding to the obvious choice of maximal torus and Borel. For each fixed $\hat{\sigma} \in I_F$, $\mathfrak{r}_{\hat{\sigma}}$ contains only one simple root $\rho_{\hat{\sigma}}$, which, through the inclusion of $\mathfrak{r}_{\hat{\sigma}}$ inside $\mathfrak{r}$, acquires the expression $\rho_{\hat{\sigma}} = \rho_{\hat{\sigma}}((h_\sigma)_{\sigma\in I_F}, g)$, where

$$h_\sigma = \begin{cases} 2 & \text{if } \sigma = \hat{\sigma} \\ 0 & \text{otherwise} \end{cases}, \ g = 1.$$

The Weyl group Υ of $G_{1,L}$ is in turn isomorphic to the product $\prod_{\sigma\in I_F} \Upsilon_\sigma$, where, for each fixed $\hat{\sigma} \in I_F$, $\Upsilon_{\hat{\sigma}}$ is a copy of the Weyl group of $\mathrm{GL}_{2,L}$. The latter is a finite group of order 2, the image of whose only non-trivial element through the inclusion of $\Upsilon_{\hat{\sigma}}$ in Υ is given by the element of $\tau_{\hat{\sigma}}$ which acts on $X^*(T_{1,L})$ in the following way: if $\chi = \chi((h_\sigma)_{\sigma\in I_F}, g)$, then $\tau_{\hat{\sigma}}.\chi = \chi((\ell_\sigma)_{\sigma\in I_F}, f)$, where

$$\ell_\sigma = \begin{cases} -h_\sigma & \text{if } \sigma = \hat{\sigma} \\ h_\sigma & \text{otherwise} \end{cases}, \ f = g - h_{\hat{\sigma}}.$$

By employing the notations of 4.2.3, it is now clear that, with respect to the only parabolic of $G_{1,L}$ (up to conjugation), i.e. $Q_{2,L}$, we have $\Upsilon' = \Upsilon$, and that, if $w = (w_\sigma)_{\sigma\in I_F} \in \Upsilon' \simeq \prod_{\sigma\in I_F} \Upsilon_\sigma$, we have $\ell(w) = \sharp\{\sigma \in I_F | w_\sigma = \tau_\sigma\}$.

The explicit computation of $w.(\chi + \rho) - \rho$ (for $w \in \Upsilon'$) and Theorem 4.2.2 now give the isomorphisms

$$H^q(W_{2,L}, U_\chi) \simeq \bigoplus_{I\subset I_F \text{ s.t. } |I|=q} U_I^q, \tag{4.46}$$

where the U_I^q's are 1-dimensional L-vector spaces on which $\mathbb{G}_{m,L} \times \prod_{\sigma\in I_F} \mathbb{G}_{m,L}$ acts via the character

$$\chi'((l_\sigma)_{\sigma\in I_F}, g')$$

defined by

$$l_\sigma = \begin{cases} h_\sigma & \text{if } \sigma \notin I \\ -h_\sigma - 2 & \text{if } \sigma \in I \end{cases}, \ g' = g - \sum_{\sigma\in I}(h_\sigma + 1).$$

To obtain the statement, it is now sufficient to remember that the Hodge structure on each U_I^q is induced by the action of the real points of the factor $\mathbb{G}_m$ of T_1, corresponding to the Shimura datum (G_2, X_2) defined in 4.3.4, and to employ the convention fixed in 2.1.3.1. □

Consider now the group Γ_2 of Rmk. 4.2.2.2, which is a torsion-free arithmetic subgroup of $\mathrm{Res}_{F|\mathbb{Q}}\mathbb{G}_m(\mathbb{Q})$, i.e. of $F^\times$ (cfr. the isomorphism (4.43) and Remark 4.2.2.2). By the same argument as in the proof of Lemma 4.3.2.4, it is isomorphic to $\mathbb{Z}^{d-1}$. We need to study the cohomology spaces

$$H^p(\Gamma_2, H^q(W_{2,L}, U_\chi)) \simeq \bigoplus_{I\subset I_F \text{ s.t. } |I|=q} H^p(\Gamma_2, U_I^q).$$

By choosing generators $\omega_1,\dots,\omega_{d-1}$, Γ_2 is identified with the subgroup

$$\{(\begin{pmatrix} 1 & & & \\ & \sigma(t) & & \\ & & 1 & \\ & & & \sigma(t^{-1}) \end{pmatrix})_{\sigma\in I_F} \mid t = \omega_1^{p_1}\dots\omega_{d-1}^{p_{d-1}}, p_1,\dots,p_{d-1}\in\mathbb{Z}\} \hookrightarrow T_{1,L}(L), \quad (4.47)$$

and an element $t\in\Gamma_2$ acts on U^I by multiplication by $\prod_{\sigma\notin I}\sigma(\omega)^{h_\sigma}\cdot\prod_{\sigma\in I}\sigma(\omega)^{-h_\sigma-2}$. By reasoning as in the proof of Lemma 4.3.2.7, and employing Lemma 4.3.4.2, we get:

Lemma 4.3.4.3. *The cohomology space $H^p(\Gamma_2, U_I^q)$ is non-zero if and only if $\mathbf{h} = \underline{h}$ and one of the following conditions is satisfied:*

(1) $I=\varnothing$. In this case, $q=0$ and the Hodge structure on $U_I^0 = H^0(W_{2,L}, U_\chi)$ is pure of weight $-2g$;

(2) $I=I_F$. In this case, $q=d$ and the Hodge structure on $U_I^d = H^d(W_{2,L}, U_\chi)$ is pure of weight $-2g+2d+2dh$.

The isomorphism of Theorem 4.2.1.(2) for a stratum Z'' contributing to $\partial S^*_{\pi_1(K_1)}$ now becomes

$$R^n(i_2)^* j'_* \mu_\ell^{\pi_1(K_1)}(U_\chi)\Big|_{Z''} \simeq \bigoplus_{p+q=n}\bigoplus_{I\subset I_F \text{ s. t. } |I|=q} \mu_\ell^{\pi_2(K_2)}(H^p(\Gamma_2, U_I^q)). \quad (4.48)$$

The computation of the weights of these cohomology objects is then a direct consequence of 4.3.4.3:

Proposition 4.3.4.4. *Let U_χ be the irreducible representation of $G_{1,L}$ of highest weight $\chi=\chi(\mathbf{h},g)$ and Z'' a stratum contributing to $\partial S^*_{\pi_1(K_1)}$. Then:*

(1) Let $n<0$ or $n>2d-1$. Then $R^n(i_2)^* j'_* \mu_\ell^{\pi_1(K_1)}(U_\chi)\Big|_{Z''}$ is zero.

(2) Let $n\in\{0,\dots,d-1\}$. Then $R^n(i_2)^* j'_* \mu_\ell^{\pi_1(K_1)}(U_\chi)\Big|_{Z''}$ is non-zero if and only if $\mathbf{h}$ is parallel. In this case, it is isomorphic to $\mu_\ell^{\pi_2(K_2)}(H^n(\Gamma_2, U_I^0))$, and pure of weight $-2g$.

(3) Let $n\in\{d,\dots,2d-1\}$. Then $R^n(i_2)^* j'_* \mu_\ell^{\pi_1(K_1)}(U_\chi)\Big|_{Z''}$ is non-zero if and only if $\mathbf{h}=\underline{h}$. In this case, it is isomorphic to $\mu_\ell^{\pi_2(K_2)}(H^{n-d}(\Gamma_2, U_I^d))$, and pure of weight $-2g+2d+2dh$.

Remark 4.3.4.5. The results of Proposition 4.3.4.4 had already been obtained in [Wil12, Thm. 3.5], by slightly different considerations, when the representations U_χ are such that $g=0$.

4.3.4.2 The double degeneration.

Let $\lambda = \lambda((k_{1,\sigma}, k_{2,\sigma})_\sigma, c)$, V_λ and S_K be as in Subsection 4.3.3, and let Z' and $S_{\pi_1(K_1)}$ be as in paragraph 4.3.4.1.

By (4.41) and Theorem (4.2.1).(1)-(2), we have the following isomorphism in the derived category:

$$i_1^* i^* j_* \mu_\ell^K(V_\lambda)\Big|_{Z'} \simeq \bigoplus_m \left[\bigoplus_{p+q=m} \mu_\ell^{\pi_1(K_1)}(V^{p,q}) \right][-m], \tag{4.49}$$

where

$$V^{p,q} := \bigoplus_{\Psi \in \mathcal{P}_q^{(\lambda,1)}} H^p(\Gamma_1, V_\Psi^{1,q}). \tag{4.50}$$

In this latter direct sum, every factor is, by restriction, a certain power of an irreducible representation of $G_{1,L}$, whose dominant weight is the one prescribed by Remark (4.3.4.1) applied to the character $\lambda((\epsilon_{1,\sigma}, \epsilon_{2,\sigma})_{\sigma \in I_F}, c)$ defined in (4.18).

Recall that the functor $\mu_\ell^{\pi_1(K_1)}$ used in the isomorphism (4.49), with values in $\mathrm{Et}_{\ell,R}(Z')$, is deduced from the canonical construction functor, which takes values in $\mathrm{Et}_{\ell,R}(S_{\pi_1(K_1)})$ (Remark 4.2.2.3.(1)). In order to study the degeneration of $\mu_\ell^{\pi_1(K_1)}(V^{p,q})$ along the points in the closure of Z' in ∂S_K^*, we will rather consider the sheaves on $S_{\pi_1(K_1)}$, denoted by the same symbol, which are obtained by interpreting this time $\mu_\ell^{\pi_1(K_1)}$ as the canonical construction functor.

Let us now apply Theorem 4.2.1.(2) to $\mu_\ell^{\pi_1(K_1)}(V^{p,q})[-m]$ and to $S_{\pi_1(K_1)}$, by posing $p+q=m$ and by adopting the notations of 4.3.4.1, for a stratum $\partial S_{\pi_1(K_1)}$, in order to study the weights of the objects

$$R^{n-m}(i_2)^* j'_* \mu_\ell^{\pi_1(K_1)}(V^{p,q})\Big|_{Z''}.$$

We will only need this for $m \in \{2d, \dots, 3d-1\}$.

Proposition 4.3.4.6. *Fix two positive integers p and q such that $p+q \in \{2d, \dots, 3d-1\}$ and let $V^{p,q}$ be the L-representation of $G_{1,L}$ defined in (4.50), deduced from the irreducible L-representation V_λ of G_L of highest weight $\lambda = \lambda((k_{1,\sigma}, k_{2,\sigma})_{\sigma \in I_F}, c)$. Let Z'' be a stratum contributing to $\partial S_{\pi_1(K_1)}$. Then:*

(i) if $m' \in \{0, \dots, d-1\}$, the ℓ-adic sheaf $R^{m'}(i_2)^* j'_* \mu_\ell^{\pi_1(K_1)}(V^{p,q})\Big|_{Z''}$ on Z'' is non-zero if and only if $k_1 = \underline{\kappa_1}$ and $k_2 = \underline{\kappa_2}$. In this case, it is pure of weight $w(\lambda) + 2d - d(\kappa_1 - \kappa_2)$;

(ii) if $m' \in \{d, \dots, 2d-1\}$, the ℓ-adic sheaf $R^{m'}(i_2)^* j'_* \mu_\ell^{\pi_1(K_1)}(V^{p,q})\Big|_{Z''}$ on Z'' is non-zero if and only if $k_1 = \underline{\kappa_1}$ and $k_2 = \underline{\kappa_2}$. In this case, it is pure of weight $w(\lambda) + 6d + d(\kappa_1 + \kappa_2)$;

(iii) if $m' \notin \{0, \dots, 2d-1\}$, then the ℓ-adic sheaf $R^{m'}(i_2)^* j'_* \mu_\ell^{\pi_1(K_1)}(V^{p,q})\Big|_{Z''}$ on Z'' is zero.

Proof. Lemma 4.3.3.5 implies that in order to have non-trivial cohomology objects, we must have $p \in \{0, \dots, d-1\}$; hence, by hypothesis, we have $q \in \{d+1, \dots, 3d-1\}$, and the same lemma then implies that $V^{p,q}$ is non-zero if and only if there exists a q-admissible decomposition $\Psi = (I_F^2 \neq \varnothing, I_F^3)$ of I_F and an integer ι_1 such that

$$\begin{cases} k_{2,\sigma} = \iota_1 & \forall \sigma \in I_F^2 \\ k_{1,\sigma} = \iota_1 - 1 & \forall \sigma \in I_F^3 \end{cases}$$

In this case, if $\mathcal{P}'^{(\lambda,d)}_q$ is the set of such decompositions, we have

$$V^{p,q} = \bigoplus_{\Psi \in \mathcal{P}'^{(\lambda,d)}_q} H^p(\Gamma_1, V^{1,q}_\Psi)$$

The highest weight of the action of $G_{1,L}$ on $H^p(\Gamma_1, V^{1,q}_\Psi)$ is then the restriction to $T_{1,L}$ of the character $\lambda((\epsilon_{1,\sigma}, \epsilon_{2,\sigma})_{\sigma \in I_F}, c)$ defined in (4.18), where

$$\epsilon_{1,\sigma} = \begin{cases} -k_{2,\sigma} - 3 & \text{if } \sigma \in I_F^2 \\ -k_{1,\sigma} - 4 & \text{if } \sigma \in I_F^3 \end{cases}, \quad \epsilon_{2,\sigma} = \begin{cases} k_{1,\sigma} + 1 & \text{if } \sigma \in I_F^2 \\ k_{2,\sigma} & \text{if } \sigma \in I_F^3 \end{cases}$$

By Remark 4.3.4.1, this restriction, as a character of the maximal torus $T_{1,L}$ of $G_{1,L}$, has the form $\chi((\epsilon_{2,\sigma})_\sigma, \frac{1}{2} \cdot [c + \sum_{\sigma \in I_F} (\epsilon_{1,\sigma} + \epsilon_{2,\sigma})])$.

Now, by Proposition 4.3.4.4, $R^{m'}(i_2)^* j'_* \mu_\ell^{\pi_1(K_1)}(V^{p,q})\Big|_{Z''}$ is non-zero if and only if $m' \in \{0, \dots, 2d-1\}$ and $\epsilon_{2,\sigma}$ is constant on I_F, say equal to an integer ι_2. This means that

$$\begin{cases} k_{1,\sigma} = \iota_2 - 1 & \forall \sigma \in I_F^2 \\ k_{2,\sigma} = \iota_2 & \forall \sigma \in I_F^3 \end{cases}$$

Thus, the fact that $k_{1,\sigma} \geq k_{2,\sigma}$ for every $\sigma \in I_F$ and that $I_F^2 \neq \varnothing$ imply that the sheaves we are interested in are non-zero if and only if $I_F = I_F^2$ and $k_1 = \underline{\iota_2 - 1}$, $k_2 = \underline{\iota_1}$. Denoting $\kappa_1 := \iota_2 - 1$ and $\kappa_2 := \iota_1$, we get the constants in the statement.

In order to conclude, it is now enough to apply again Proposition 4.3.4.4, by observing that if $m' \in \{0, \dots, d-1\}$ then the highest weight of the action of $G_{1,L}$ on $H^p(\Gamma_1, V^{1,q}_\Psi)$ is the character

$$((u_\sigma)_{\sigma \in I_F}, \beta) \mapsto \prod_{\sigma \in I_F} u_\sigma^{\kappa_1 + 1} \cdot \beta^{\frac{1}{2} \cdot [c - 2d + d(\kappa_1 - \kappa_2)]} \tag{4.51}$$

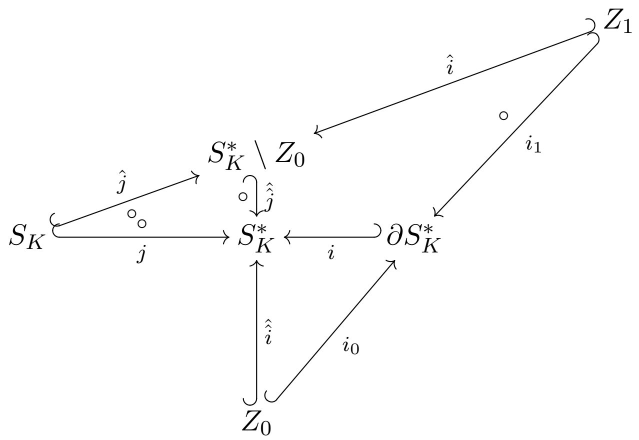
(and analogously for the case $m' \in \{d, \dots, 2d-1\}$). □

Remark 4.3.4.7. The above proof shows that the *double degeneration* $i_2^* j'_* i_1^* j_* \mu_\ell^K(V_\lambda)$ of $\mu_\ell(V_\lambda)$ along the 0-dimensional strata of ∂S_K^* is non-trivial only if the vectors k_1 and k_2 in the character λ are both parallel. This condition is then necessary for the non-triviality of the hypercohomology of $i_2^* j'_* i_1^* j_* \mu_\ell^K(V_\lambda)$. Even if we won't make this precise here, one could interpret the contribution of the latter hypercohomology to boundary cohomology of $\mu_\ell^K(V_\lambda)$ in terms of Franke's automorphic decomposition (2.37): i.e., this contribution corresponds to the summand given by the conjugacy class of the *minimal* standard parabolic subgroup B of G, which is not "seen" by the Baily-Borel compactification. Our result is then compatible with [GG13, Prop. 2.1], where it is *proved* that the summand coming from B in Franke's decomposition is actually trivial, unless k_1 and k_2 are parallel.

4.3.5 Weight avoidance

In this section, we employ the notations of 4.3.1 and 4.2.2. Our aim is to use the results of the preceding paragraphs in order to prove Theorem 4.3.1, thanks to the criterion given by Theorem 3.3.4. Thus, we have to relate, for $m \in \{0, 1\}$, the weights of the objects $\mathcal{H}^n i_m^* i^* j_{!*}(\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ to the weights of the objects $R^n i_m^* i^* j_*(\mathcal{R}_\ell({}^\lambda\mathcal{V}))$, which are now known.

In the following, the symbols $\tau_{\mathcal{Z}}^{\gtrless \cdot}$ will denote the truncation functors with respect to the perverse t-structure on $\mathcal{Z}$. We will need to keep in mind the following diagram, denoted by (D) (analogous to the one arising when studying Siegel threefolds, see [Wil19b, p. 21]):



In this diagram, arrows on the same line denote complementary immersions: the ones with "∘" are open, the others are closed.

Remark 4.3.5.1. Since the immersions $\hat{i}$ and $\hat{j}$, resp. $\hat{\hat{i}}$ and $\hat{\hat{j}}$, are complementary, [BBD82, Prop. 1.4.23] tells us that for every perverse sheaf $\mathcal{F}'$ on S_K we have an exact triangle

$$\hat{i}_*\tau^{\geqslant 0}_{Z_1}\hat{i}^*\hat{j}_*\mathcal{F}'[-1] \to \hat{j}_{!*}\mathcal{F}' \to \hat{j}_*\mathcal{F}' \to \hat{i}_*\tau^{\geqslant 0}_{Z_1}\hat{i}^*\hat{j}_*\mathcal{F}' \tag{4.52}$$

resp., for every perverse sheaf $\mathcal{F}''$ on $S_K^* \setminus Z_0$ we have an exact triangle

$$\hat{\hat{i}}_*\tau^{\geqslant 0}_{Z_0}\hat{\hat{i}}^*\hat{\hat{j}}_*\mathcal{F}''[-1] \to \hat{\hat{j}}_{!*}\mathcal{F}'' \to \hat{\hat{j}}_*\mathcal{F}'' \to \hat{\hat{i}}_*\tau^{\geqslant 0}_{Z_0}\hat{\hat{i}}^*\hat{\hat{j}}_*\mathcal{F}'' \tag{4.53}$$

4.3.5.1 Weight avoidance on the Siegel strata.

Let us begin by studying the weight avoidance on the Siegel strata, by employing Propositions 4.3.2.12 and 4.3.4.6.

Lemma 4.3.5.2. *(1) For every $n \geq w(\lambda) + 3d$, $\mathcal{H}^n(i_0^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ is zero.*

(2) For every $n \leq w(\lambda) + 3d - 1$, there are exact sequences

$$\mathcal{H}^{n-1}(i_0^* i_{1,*}\tau^{\geqslant w(\lambda)+3d}_{Z_1} i_1^* i^* j_*\mathcal{R}_\ell({}^\lambda\mathcal{V})) \to \mathcal{H}^n(i_0^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V})) \to \mathcal{H}^n(i_0^* i^* j_*\mathcal{R}_\ell({}^\lambda\mathcal{V})) \tag{4.54}$$

fitting in a long exact sequence.

(3) For every $n \leq w(\lambda) + 2d - 1$, the perverse sheaf $\mathcal{H}^n(i_0^* i_{1,*}\tau^{\geqslant w(\lambda)+3d}_{Z_1} i_1^* i^* j_*\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ is zero.

Proof. (Analogous to [Wil19b, Rmk. 2.7 (a)-(b)-(c)-(d)]) With the notations of diagram (D), we have $j_{!*} = \hat{\hat{j}}_{!*}\hat{j}_{!*}$ ([BBD82, Cor. 1.4.24]), so that, by the commutativity of that diagram, the functor that we are interested in, i.e. the functor $i_0^* i^* j_{!*}$ on perverse sheaves on S_K, verifies

$$i_0^* i^* j_{!*} \simeq \hat{\hat{i}}^*\hat{\hat{j}}_{!*}\hat{j}_{!*}$$

Now, for every perverse sheaf $\mathcal{F}'$ on S_K, triangle (4.53) applied to $\mathcal{F}'' = \hat{j}_{!*}\mathcal{F}'$ tells us that

$$\hat{\hat{i}}^*\hat{\hat{j}}_{!*}\hat{j}_{!*}\mathcal{F}' \simeq \tau^{\leqslant -1}_{Z_0}\hat{\hat{i}}^*\hat{\hat{j}}_*\hat{j}_{!*}\mathcal{F}' \tag{4.55}$$

On the other hand, application of the functor $\hat{\hat{i}}^*\hat{\hat{j}}_*$ to the triangle (4.52) and the commutativity of diagram (D) show that for such a $\mathcal{F}'$, the complex $\hat{\hat{i}}^*\hat{\hat{j}}_{!*}\hat{j}_{!*}\mathcal{F}'$ is inserted into a triangle

$$i_0^* i_{1,*}\tau_{Z_1}^{\geqslant 0}\hat{i}^*\hat{j}_*\mathcal{F}'[-1] \to \hat{\hat{i}}^*\hat{\hat{j}}_{!*}\hat{j}_{!*}\mathcal{F}' \to i_0^* i^* j_*\mathcal{F}' \to i_0^* i_{1,*}\tau_{Z_1}^{\geqslant 0}\hat{i}^*\hat{j}_*\mathcal{F}' \qquad (4.56)$$

Now choose $\mathcal{F}' = \mathcal{R}_\ell({}^\lambda\mathcal{V})[w(\lambda)+3d]$ (remember Rmk. 3.2.1.8.(2)) and remember the conventions of Thm. 3.3.4 for the symbol $j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V})$. Equation (4.55) gives point (1), and, employing the commutativity of diagram (D), the triangle (4.56) gives point (2).

To conclude, observe that by Theorem 4.2.1.(2), the classical cohomology objects of $i_1^* i^* j_*\mu_\ell(V_\lambda)$ are all lisse, hence concentrated in perverse degree equal to their classical degree plus $\dim Z_1 = d$. It follows that the classical cohomology objects of $i_1^* i^* j_*\mathcal{R}_\ell({}^\lambda\mathcal{V})$ are concentrated in perverse degree equal to their classical degree plus $w(\lambda)+d$ (remember Rmk. 3.2.1.8.(1)). Hence, perverse truncation of $i_1^* i^* j_*\mathcal{R}_\ell({}^\lambda\mathcal{V})$ above degree $w(\lambda)+3d$ gives a complex concentrated in classical degrees $\geq w(\lambda)+2d$. The same is true after application of $i_0^* i_{1,*}$. Since Z_0 is 0-dimensional, the perverse cohomology objects of the resulting complex are the same as the classical cohomology objects, which by the previous reasoning vanish in degrees $\leq w(\lambda)+2d-1$. This yields point (3). □

Corollary 4.3.5.3. *For every $n \leq w(\lambda)+2d-1$, we have*

$$\mathcal{H}^n(i_0^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V})) \simeq \mathcal{H}^n(i_0^* i^* j_*\mathcal{R}_\ell({}^\lambda\mathcal{V}))$$

The above lemma and corollary tell us the following: if $n \leq w(\lambda)+2d-1$, the weights of the perverse sheaf $\mathcal{H}^n(i_0^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ are the same as the weights of $\mathcal{H}^n(i_0^* i^* j_*\mathcal{R}_\ell({}^\lambda\mathcal{V}))$, which have already been computed. Moreover, there is nothing to do for $n \geq w(\lambda)+3d$. It remains to study the interval $[w(\lambda)+2d, w(\lambda)+3d-1]$.

Remark 4.3.5.4. Each stratum Z' of ∂S_K^* contributing to Z_1 is the quotient of a Hilbert modular variety $S_{K,Z'}$ by the action of a finite group; let $S_{K,Z'}^*$ be its Baily-Borel compactification. If $\bar{Z}_1$ is the closure of Z_1 in ∂S_K^* and

$$Z_1^* := \bigsqcup_{\substack{Z' \text{ stratum of } \partial S_K^* \\ \text{contributing to } Z_1}} S_{K,Z'}^*, \qquad (4.57)$$

then there exists a surjective, finite morphism

$$q : Z_1^* \to \bar{Z}_1 \qquad (4.58)$$

whose restriction to each $S_{K,Z'}$ is the quotient morphism from $S_{K,Z'}$ to Z' (cfr. [Pin90, Main Thm. 12.3 (c), Sec. 7.6]).

Thanks to the above Remark, we can now compute the weights of

$$\mathcal{H}^n(i_0^* i_{1,*}\tau_{Z_1}^{\geqslant w(\lambda)+3d} i_1^* i^* j_*\mathcal{R}_\ell({}^\lambda\mathcal{V}))$$

in the degrees we are interested in.

Lemma 4.3.5.5. *If $n \in [w(\lambda)+2d, w(\lambda)+3d-1]$, then $\mathcal{H}^n(i_0^* i_{1,*}\tau_{Z_1}^{\geqslant w(\lambda)+3d} i_1^* i^* j_*\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ is non-zero if and only if $k_1 = \underline{\kappa_1}$ and $k_2 = \underline{\kappa_2}$. In this case, it is pure of weight $w(\lambda)+2d-d(\kappa_1-\kappa_2)$.*

Proof. Recall that, with respect to the classical t-structure, $\mathcal{R}_\ell({}^\lambda\mathcal{V}) = \mu_\ell^K(V_\lambda)[-w(\lambda)]$. Then, for each stratum Z' contributing to Z_1, we have, by Theorem 4.2.1.(1),

$$i_1^* i^* j_* \mathcal{R}_\ell({}^\lambda\mathcal{V})_{|Z'} \simeq \bigoplus_k R^k i_1^* i^* j_* \mu_\ell^K(V_\lambda)[-w(\lambda)-k]_{|Z'}.$$

Moreover, as we observed in the proof of Lemma 4.3.5.2, Theorem 4.2.1.(2) implies that the objects $R^{h-w(\lambda)} i_1^* i^* j_* \mu_\ell^K(V_\lambda)$ are all lisse, and since Z_1 is of dimension d, perverse truncation above degree $w(\lambda)+3d$ equals classical truncation above degree $w(\lambda)+2d$. Thus, by fixing a stratum Z contributing to Z_0, we obtain

$$\mathcal{H}^n(i_0^* i_{1,*} \tau_{Z_1}^{\geqslant w(\lambda)+3d} i_1^* i^* j_* \mathcal{R}_\ell({}^\lambda\mathcal{V}))_{|Z} \simeq \bigoplus_{Z'} \left(\mathcal{H}^n i_0^* i_{1,*}(\bigoplus_{k\geq 2d} R^k i_1^* i^* j_* \mu_\ell^K(V_\lambda)[-w(\lambda)-k]_{|Z'}) \right)_{|Z}, \tag{4.59}$$

where the direct sum runs over all strata Z' contributing to Z_1 and containing Z in their closure. Fix now such a stratum: as in (4.49) and (4.50), we get

$$\bigoplus_{k\geq 2d} R^k i_1^* i^* j_* \mu_\ell^K(V_\lambda)[-w(\lambda)-k]_{|Z'} \simeq \bigoplus_{k\geq 2d} (\bigoplus_{p+q=k} \mu_\ell^{\pi_1(K_1)}(V^{p,q}))[-w(\lambda)-k], \tag{4.60}$$

and as a consequence, by taking into account the fact that Z is of dimension 0,

$$\left(\mathcal{H}^n i_0^* i_{1,*}(\bigoplus_{k\geq 2d} R^k i_1^* i^* j_* \mu_\ell^K(V_\lambda)[-w(\lambda)-k]_{|Z'})) \right)_{|Z} \simeq \bigoplus_{k\geq 2d} \bigoplus_{p+q=k} R^{n-w(\lambda)-k} i_0^* i_{1,*} \mu_\ell^{\pi_1(K_1)}(V^{p,q})\Big|_Z \tag{4.61}$$

Now, let us adopt the notations of Remark 4.3.5.4, and extend the notations of 4.3.4.1 in the following way: j' will denote the open immersion of the union of the $S_{K,Z'}$'s in the union of the $S^*_{K,Z'}$'s, while i_2 will denote the complementary closed immersion of the union of the $\partial S^*_{K,Z'}$'s in the union of the $S^*_{K,Z'}$'s. By restriction to Z', we get, by proper base change, the relation

$$i_0^* i_{1,*} \mu_\ell^{\pi_1(K_1)} \simeq q_* i_2^* j'_* \mu_\ell^{\pi_1(K_1)}, \tag{4.62}$$

where on the left, resp. right hand side, we have interpreted the functor $\mu_\ell^{\pi_1(K_1)}$ as a functor with values in $\mathrm{Et}_{\ell,R}(Z')$, resp. in $\mathrm{Et}_{\ell,R}(S_{\pi_1(K_1)})$. Denote now by $\partial_{Z'}$ the stratum of $S_{K,Z'}$ such that $q(\partial_{Z'}) = Z$ (such a stratum is unique, because two rational boundary components (cfr. 2.3.3) are conjugated by $G_1(\mathbb{Q})$ if and only if they are conjugated by $G(\mathbb{Q})$, by [Pin90, Rmk. at page 91, (iii)]). Since the morphism q is finite, we deduce that, for every k, p, q,

$$R^{n-w(\lambda)-k} i_0^* i_{1,*} \mu_\ell^{\pi_1(K_1)}(V^{p,q})\Big|_Z \simeq q_*(R^{n-w(\lambda)-k} i_2^* j'_* \mu_\ell^{\pi_1(K_1)}(V^{p,q})\Big|_{\partial_{Z'}}). \tag{4.63}$$

Now, the functor q_* preserves weights, because the morphism q is finite. Thus, the isomorphisms (4.59)-(4.63) allow us to deduce the weights of $\mathcal{H}^n(i_0^* i_{1,*} \tau_{Z_1}^{\geqslant w(\lambda)+3d} i_1^* i^* j_* \mathcal{R}_\ell({}^\lambda\mathcal{V}))$ from the weights of the sheaf $R^{n-w(\lambda)-k} i_2^* j'_* \mu_\ell^{\pi_1(K_1)}(V^{p,q})\Big|_{Z''}$. But since $n \in [w(\lambda)+2d, w(\lambda)+3d-1]$ and $k \geq 2d$, then, by Proposition 4.3.4.6, the objects which appear as summands in the right hand side of (4.61) are non-zero only for indices $n - w(\lambda) - k \in \{0, \dots d-1\}$. We can then conclude by Proposition 4.3.4.6.(i). ∎

We now dispose of all the necessary information in order to determine an interval of weight avoidance on the Siegel strata:

Proposition 4.3.5.6. *The perverse cohomology sheaf $\mathcal{H}^n(i_0^* i^* j_{!*} \mathcal{R}_\ell({}^\lambda\mathcal{V}))$ can be non-zero only if $k_1 = \underline{\kappa_1}$, $k_2 = \underline{\kappa_2}$ and $n \in \{w(\lambda), \ldots, w(\lambda)+3d-1\}$. In this case, $\mathcal{H}^n(i_0^* i^* j_{!*} \mathcal{R}_\ell({}^\lambda\mathcal{V}))$ is of weight $\leq n - d(\kappa_1 - \kappa_2)$ for each $n \in \mathbb{Z}$.*

If κ_1 and κ_2 have the same parity, then the weight-graded object of weight $w(\lambda) + 2d - d(\kappa_1 - \kappa_2)$ of the perverse sheaf $\mathcal{H}^{w(\lambda)+2d}(i_0^* i^* j_{!*} \mathcal{R}_\ell({}^\lambda\mathcal{V}))$ is non-zero.

Proof. If $n < w(\lambda)+2d$, then $\mathcal{H}^n(i_0^* i^* j_{!*} \mathcal{R}_\ell({}^\lambda\mathcal{V})) \simeq \mathcal{H}^n(i_0^* i^* j_* \mathcal{R}_\ell({}^\lambda\mathcal{V}))$, by Corollary 4.3.5.3. Now, Z_0 is of dimension 0; hence,

$$\mathcal{H}^n(i_0^* i^* j_* \mathcal{R}_\ell({}^\lambda\mathcal{V})) \simeq R^n(i_0^* i^* j_* \mathcal{R}_\ell({}^\lambda\mathcal{V})) = R^{n-w(\lambda)} i_0^* i^* j_* \mu_\ell^K(V). \tag{4.64}$$

Thus, the perverse sheaf $\mathcal{H}^n(i_0^* i^* j_{!*} \mathcal{R}_\ell({}^\lambda\mathcal{V}))$ is zero for every $n < w(\lambda)$. If k_1 and k_2 are not parallel, then Proposition 4.3.2.12 tells us that $\mathcal{H}^n(i_0^* i^* j_{!*} \mathcal{R}_\ell({}^\lambda\mathcal{V}))$ is zero for every $w(\lambda) \leq n < w(\lambda)+2d$. If instead $k_1 = \underline{\kappa_1}$, $k_2 = \underline{\kappa_2}$, then the same proposition tells us that for every $w(\lambda) \leq n < w(\lambda) + 2d$, $\mathcal{H}^n(i_0^* i^* j_{!*} \mathcal{R}_\ell({}^\lambda\mathcal{V}))$ is of weight $\leq w(\lambda) - d(\kappa_1 + \kappa_2) \leq n - d(\kappa_1 - \kappa_2)$.

Let now $n \geq w(\lambda)+2d$. If $n \geq w(\lambda)+3d$, Lemma 4.3.5.2 implies that $\mathcal{H}^n(i_0^* i^* j_{!*} \mathcal{R}_\ell({}^\lambda\mathcal{V}))$ is zero. Assume then $n \in \{w(\lambda) + 2d, \ldots, w(\lambda) + 3d - 1\}$. In this case, by reasoning as in (4.64) and by applying again Proposition 4.3.2.12, along with Lemma 4.3.5.5, we also see, by the exact sequence (4.54), that if k_1 and k_2 are not parallel, then $\mathcal{H}^n(i_0^* i^* j_{!*} \mathcal{R}_\ell({}^\lambda\mathcal{V}))$ is zero for every $n \in \{w(\lambda) + 2d, \ldots, w(\lambda) + 3d - 1\}$. If instead $k_1 = \underline{\kappa_1}$, $k_2 = \underline{\kappa_2}$, then we see in same way that the weights that can appear in $\mathcal{H}^n(i_0^* i^* j_{!*} \mathcal{R}_\ell({}^\lambda\mathcal{V}))$ are of the form $w(\lambda) - d(\kappa_1 + \kappa_2)$ or $w(\lambda) + 2d - d(\kappa_1 - \kappa_2)$. In any case, we get weights $\leq n - d(\kappa_1 - \kappa_2)$.

To see that if κ_1 and κ_2 have the same parity, then weight $w(\lambda) + 2d - d(\kappa_1 - \kappa_2)$ does appear in the perverse sheaf $\mathcal{H}^{w(\lambda)+2d}(i_0^* i^* j_{!*} \mathcal{R}_\ell({}^\lambda\mathcal{V}))$, notice that the long exact sequence (4.54) gives a short exact sequence

$$0 \longrightarrow \mathcal{H}^{w(\lambda)+2d}(i_0^* i^* j_{!*} \mathcal{R}_\ell({}^\lambda\mathcal{V})) \longrightarrow \mathcal{H}^{w(\lambda)+2d}(i_0^* i^* j_* \mathcal{R}_\ell({}^\lambda\mathcal{V})) \xrightarrow{ad} \mathcal{H}^{w(\lambda)+2d}(i_0^* i_{1,*} \tau_{Z_1}^{\geq w(\lambda)+3d} i_1^* i^* j_* \mathcal{R}_\ell({}^\lambda\mathcal{V}))$$

so that $\mathcal{H}^{w(\lambda)+2d}(i_0^* i^* j_{!*} \mathcal{R}_\ell({}^\lambda\mathcal{V}))$ is identified with the kernel of the arrow ad. Proposition 4.3.2.13 shows that if κ_1 and κ_2 have the same parity, then $\mathcal{H}^{w(\lambda)+2d}(i_0^* i^* j_* \mathcal{R}_\ell({}^\lambda\mathcal{V}))$ contains a direct factor, which is pure of weight $w(\lambda) + 2d - d(\kappa_1 - \kappa_2)$ and locally of dimension $> h$, where $h := |\Gamma_{0,ss} \backslash \mathbb{P}^1(F)|$ is the (strictly positive) number of cusps of the Hilbert modular variety $X_{\Gamma_{0,ss}}$. In order to conclude, it is then enough to show that locally, the kernel of ad has non-trivial intersection with this sub-object. The isomorphisms (4.59)-(4.63) in the proof of Lemma 4.3.5.5 show that, if we let the index Z' run over all strata Z' contributing to Z_1 and containing Z in their closure, and if the finite morphism $q : Z_1^* \to \bar{Z}$ is the one of Remark 4.3.5.4, then, above a stratum Z of ∂S_K^* contributing to Z_0, the arrow ad has the form

$$R^{2d} i_0^* i^* j_* \mu_\ell^K(V_\lambda)\Big|_Z \to q_* \bigoplus_{Z'} (R^0 i_2^* j'_* \mu_\ell^{\pi_1(K_1)}(V^{p,q})\Big|_{\partial_{Z'}}) \tag{4.65}$$

Moreover, we know that, for each fixed Z', we have

$$R^0 i_2^* j'_* \mu_\ell^{\pi_1(K_1)}(V^{p,q})\Big|_{\partial_{Z'}} \simeq \mu_\ell^{\pi_2(K_2)}(H^0(\Gamma_2, U_I^0))$$

where $H^0(\Gamma_2, U_I^0)$ is a 1-dimensional L-vector space (cfr. the proof of Lemma 4.3.4.2).

We are then reduced to show that, locally, the dimension of the target of ad is strictly smaller than the dimension of the source (remember that by Lemma 4.3.5.5, the target is pure of weight $w(\lambda) + 2d - d(\kappa_1 - \kappa_2)$). But this is true, thanks to the following proposition. □

Proposition 4.3.5.7. *Let Z, q and h be as in the proof of Proposition 4.3.5.6.*

Then, above $Z \subset Z_0$, the number of points in the geometrical fibers of $q|_{\partial Z_1^*} : \partial Z_1^* \to Z_0$ is $\leq h$.

Proof. We translate word by word in our context the part of the proof of [Wil19b, Prop. 2.9] at pages 27-28.

Our statement can be proven on $\mathbb{C}$-points. Let us employ the notation Q_{01} for the intersection of the two standard maximal parabolics Q_0 and Q_1 inside G, and $(G_m, X_m) := (P_m, \mathfrak{X}_m)/W_m)$ (where $m = 0, 1$) for the Shimura data contributing to ∂S_K^* (cfr. 4.2.1), seen as quotients of *rational boundary components* associated to P_m (cfr. 2.3.3). Then, by [Pin90, Sec. 6.3], the adelic description of the morphism obtained from q by analytification, i.e. from the diagram

$$\begin{array}{ccc} Z_1^* & \xleftarrow[i_2]{} & \partial Z_1^* \\ \downarrow q & & \downarrow q \\ S_K^* & \xleftarrow[i_0]{} & Z_0 \end{array}$$

(the notation Z_1^* is as in Rmk. 4.3.5.4) is the following:

$$\begin{array}{ccc} Q_1(\mathbb{Q})\backslash(X_1^* \times G(\mathbb{A}_f)/K) & \xleftarrow[i_2]{} & Q_{01}(\mathbb{Q})\backslash(X_0 \times G(\mathbb{A}_f)/K) \\ \downarrow q & & \downarrow q \\ G(\mathbb{Q})\backslash(X^* \times G(\mathbb{A}_f)/K) & \xleftarrow[i_0]{} & Q_0(\mathbb{Q})\backslash(X_0 \times G(\mathbb{A}_f)/K) \end{array}$$

where all maps are induced by the obvious inclusions of groups and spaces. Here we have used G and K in the upper right corner because two rational boundary components of (G_1, X_1) are conjugated under $G_1(\mathbb{Q})$ if and only if they are conjugate under $G(\mathbb{Q})$ ([Pin90, (iii) of Remark on p. 91]), and we have used $Q_m(\mathbb{Q})$ in the upper left and lower right corner $(m = 0, 1)$ because in each case, the full group $Q_m(\mathbb{Q})$ stabilizes $\mathfrak{X}_m$.

Now, Z is a subscheme of Z_0 obtained as the image of a Shimura variety associated to (G_0, X_0) under a certain morphism i_g, for a certain $g \in G(\mathbb{A}_f)$, as defined in Eq. (2.39). The adelic description of i_g in [Pin90, Sect. 6.3] tells us that with the identifications in the above diagram, a point $z \in Z(\mathbb{C})$ is represented by a class $[h_0, p_{0g}]$ in

$$Q_0(\mathbb{Q})\backslash(X_0 \times G(\mathbb{A}_f)/K)$$

of a pair of the form $(h_0, p_{0g}g)$, for a certain $h_0 \in X_0$ and $p_0 \in P_0(\mathbb{A}_f)$. Now, if we define

$$Q_0^+(\mathbb{Q}) := \{q_0 \in Q_0(\mathbb{Q}),\ \nu(q_0) > 0\}$$

then this group coincides with the group of elements which act as the identity on h_0 and more in general on the whole of X_0. Observe that, if we represent the element q_0 of

$Q_0(\mathbb{Q})$ as in the matrix-form description of the latter group given at the end of 4.1.1, then $\nu(q_0) > 0$ corresponds to $\alpha > 0$. Then, putting

$$H'_{C,0} := Q_0^+(\mathbb{Q}) := Q_0^+(\mathbb{Q}) \cap p_0 g K g^{-1} p_0^{-1}$$

a computation shows that the map

$$\begin{aligned} Q_{01}(\mathbb{Q})\backslash Q_0(\mathbb{Q})/H'_{C,0} &\to q^{-1}(z) \\ [q_0] &\mapsto q_0[h_0, p_0 g] = [q_0 h_0, q_0 p_0 g] \end{aligned}$$

is well-defined and bijective. Thus, we have identified our fibre, and we have to interpret its description on the left.

For this, observe that by strong approximation,

$$W_0(\mathbb{Q}) \cdot H'_{C,0} = Q_0^+ \cap W_0(\mathbb{A}_f) \cdot p_0 g K g^{-1} p_0^{-1}$$

But modulo W_0, elements in P_0 and Q_0 commute with each other (because of the isomorphism (4.5)) and so

$$W_0(\mathbb{Q}) \cdot H'_{C,0} = Q_0^+ \cap W_0(\mathbb{A}_f) \cdot g K g^{-1}$$

This description shows that the group $W_0(\mathbb{Q}) \cdot H'_{C,0}$ coincides, by definition, with the group $H_{C,0}$ of Eq. (2.41). Hence, under the projection $\pi_0 \to Q_0/W_0$, we have the equality $\pi_0(H'_{C,0}) = \pi_0(H_{C,0})$. This implies that reduction modulo P_0 in Q_0 (which we will denote by a vertical bar) induces a bijection

$$Q_{01}(\mathbb{Q})\backslash Q_0(\mathbb{Q})/H'_{C,0} \simeq \overline{Q_{01}(\mathbb{Q})}\backslash\overline{Q_0(\mathbb{Q})}/\overline{H'_{C,0}} = \overline{Q_{01}(\mathbb{Q})}\backslash\overline{Q_0(\mathbb{Q})}/\overline{H_{C,0}}$$

Now $\overline{Q_0(\mathbb{Q})} = \mathrm{GL}_2(F)$, and $\overline{Q_{01}(\mathbb{Q})}$ equals $B_2(F)$, where B_2 is the standard Borel subgroup of GL_2. Moreover, $\overline{H_{C,0}}$ equals by definition the group Γ_0 of Eq. (4.11). This means that our fiber is identified with $B_2(F)\backslash \mathrm{GL}_2(F)/\Gamma_0$. Now, this set is exactly the set of cusps of Γ_0, whose cardinality is $\leq$ the cardinality of the set of cusps of $\Gamma_{0,ss}$. □

Remark 4.3.5.8. By the preceding proof, the number of d-dimensional strata in ∂S_K^* which contain a fixed cusp is related to the number of cusps in the Baily-Borel compactification of a (complex analytic, connected) *virtual* Hilbert modular variety, which does not appear in ∂S_K^*, i.e. $X_{\Gamma_{0,ss}}$. Cfr. [Wil19b, Rmk. 2.10 (c)] for an analogous remark.

4.3.5.2 Weight avoidance on the Klingen strata and proof of the main theorem.

Let us now study the weight avoidance on the Klingen strata, by means of Proposition 4.3.3.8.

Remark 4.3.5.9. Using again the commutativity of diagram (D) and the fact that $j_{!*} = \hat{\hat{j}}_{!*}\hat{j}_{!*}$, we see that, by triangle (4.52),

$$i_1^* i^* j_{!*}(\mathcal{R}_\ell({}^\lambda\mathcal{V})) \simeq \tau_{Z_1}^{t\leqslant w(\lambda)+3d-1} i_1^* i^* j_* \mathcal{R}_\ell({}^\lambda\mathcal{V}), \tag{4.66}$$

Thanks to the latter remark, and remembering Remark 4.3.3.7, we are ready to determine the interval of weight avoidance on the Klingen strata.

Proposition 4.3.5.10. *The perverse cohomology sheaf* $\mathcal{H}^n(i_1^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ *can be non-zero only if* λ *is* κ*-Kostant parallel and if* $n \in \{w(\lambda)+d, \dots, w(\lambda)+3d-1\}$.

In this case, let $\kappa_2 := \kappa + 1$. *Then:*

(1) if $I_F^0 \neq \varnothing$, *denote* $d_1 := |I_F^1| \in \{0, \dots, d-1\}$. *Then, the perverse sheaf* $\mathcal{H}^n(i_1^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ *is of weight* $\leq n - d_1 - d\kappa$ *for every* $n \in \{w(\lambda)+d, \dots, w(\lambda)+2d-1\}$, *and the perverse sheaf* $\mathcal{H}^{w(\lambda)+d+d_1}(i_1^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ *is non-zero and pure of weight* $w(\lambda)+d-d\kappa$.

Otherwise, $\mathcal{H}^n(i_1^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ *is zero for every* $n \in \{w(\lambda)+d, \dots, w(\lambda)+2d-1\}$;

(2) if $I_F^1 \neq \varnothing$, *then the perverse sheaf* $\mathcal{H}^n(i_1^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ *is of weight* $\leq n - d\kappa_2$ *for every* $n \in \{w(\lambda)+2d, \dots, w(\lambda)+3d-1\}$, *and the perverse sheaf* $\mathcal{H}^{w(\lambda)+2d}(i_1^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ *is non-zero and pure of weight* $w(\lambda)+2d-d\kappa_2$.

Otherwise, $\mathcal{H}^n(i_1^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ *is zero for every* $n \in \{w(\lambda)+2d, \dots, w(\lambda)+3d-1\}$.

Proof. By Remark 4.3.5.9,

$$\mathcal{H}^n(i_1^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V})) \simeq \mathcal{H}^n i_1^* i^* j_*\mathcal{R}_\ell({}^\lambda\mathcal{V}))$$

for every $n \leq w(\lambda)+3d-1$, and $\mathcal{H}^n(i_1^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ is zero for every $n \geq w(\lambda)+3d$. Moreover,

$$\mathcal{H}^n i_1^* i^* j_*(\mathcal{R}_\ell({}^\lambda\mathcal{V})) = (R^{n-w(\lambda)-d} i_1^* i^* j_* \mu_\ell(V_\lambda))[d].$$

Then, by applying Proposition 4.3.3.8, we see the following facts.

If $n < w(\lambda)+d$, then $\mathcal{H}^n(i_1^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ is zero.

If $n \in \{w(\lambda)+d, \dots, w(\lambda)+2d-1\}$, then $\mathcal{H}^n(i_1^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ is non-zero if and only if λ is κ-Kostant parallel and $I_F^0 \neq \varnothing$, and if $n \geq w(\lambda)+d+d_1$, where $d_1 = |I_F^1| \in \{0, \dots, d-1\}$. In this case, it is pure of weight $w(\lambda)+d-d\kappa$, in particular of weight $\leq n - d_1 - d\kappa$.

If $n \in \{w(\lambda)+2d, \dots, w(\lambda)+3d-1\}$, then $\mathcal{H}^n(i_1^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ is non-zero if and only if λ is κ-Kostant parallel and $I_F^1 \neq \varnothing$, and if $n \leq w(\lambda)+2d+d_1-1$, where $d_1 = |I_F^1| \in \{1, \dots, d\}$. In this case, it is pure of weight $w(\lambda)+2d-d\kappa_2$, in particular of weight $\leq n - d\kappa_2$. □

Corollary 4.3.5.11. *Suppose* λ *to be* κ*-Kostant parallel and fix* $n \in \{w(\lambda)+d, \dots, w(\lambda)+3d-1\}$. *Let* d_1 *and* κ_2 *be as in the previous proposition, and recall the notation* $\mathrm{cor}(\lambda)$ *from Def. 4.3.1.1. Then:*

(1) if $\mathrm{cor}(\lambda) = 0$, *then* $\mathcal{H}^n(i_1^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ *is of weights* $\leq n - d_1 - d\kappa$, *and the weight* $n - d_1 - d\kappa$ does appear *when* $n = w(\lambda)+d+d_1$;

(2) if $\mathrm{cor}(\lambda) \geq 1$, *then* $\mathcal{H}^n(i_1^* i^* j_{!*}\mathcal{R}_\ell({}^\lambda\mathcal{V}))$ *is of weights* $\leq n - d\kappa_2$, *and the weight* $n - d\kappa_2$ does appear *when* $n = w(\lambda)+2d$.

Proof. Everything follows from the previous proposition, by observing that:

1. if $\mathrm{cor}(\lambda) = 0$, then $I_F^0 \neq \varnothing$, and either $I_F^1 = \varnothing$ (in which case only the weights in the first point of the previous proposition can contribute) or $I_F^1 \neq \varnothing$ (in which case, by definition of Kostant parallelism, $n - d\kappa_2 = n - d - d\kappa \leq n - d_1 - d\kappa$);

2. if $\mathrm{cor}(\lambda) \geq 1$, then either k_1 is not parallel (in which case, since λ is dominant, λ can't be Kostant parallel with $I_F^0 \neq \varnothing$, and only the weights in the second point of the previous proposition can contribute) or $k_1 = \underline{\kappa_1}$ (in which case $\kappa_1 \geq \kappa_2$ and the first point of the previous proposition gives weights $\leq n - d\kappa_1 \leq n - d\kappa_2$).

□

We now have all the necessary ingredients for the proof of Theorem 4.3.1.

Proof. (of Theorem (4.3.1))

We only have to apply the criterion 3.3.4.(2) and use Proposition 4.3.5.6 and Corollary 4.3.5.11. □

Chapter 5

Motives for automorphic representations of $\mathrm{Res}_{F|\mathbb{Q}}\,\mathrm{GSp}_{4,F}$

In this chapter we study the properties of the *intersection motive* of genus 2 Hilbert-Siegel varieties (with coefficients in suitable irreducible representations V_λ), whose existence follows from Thm. 4.3.1 and Wildeshaus' theory, and the implications for the construction of motives associated to automorphic representations.

5.1 Properties of the intersection motive

Adopt the notation of Subsection 4.3.1 and assume from now on that λ is either not completely irregular or of corank 0. Then, the weight avoidance proved in Corollary 4.3.1.3 allows us to apply the theory of Section 3.3.2 and to employ Definition 3.3.2.8 to define the *intersection motive* of a genus 2 Hilbert-Siegel variety S_K *relative to* S_K^* *with coefficients in* ${}^\lambda\mathcal{V}$. It is the object $s_* j_{!*}^\lambda \mathcal{V}$ of the category $CHM(\mathbb{Q})_L$ and it will be simply called *intersection motive*.

Let us spell out its main properties. For doing so, if λ satisfies in addition the hypotheses of point (1) or (2) or (3), resp. (4), of Theorem 4.3.1, put $\beta := d\kappa$, resp. $\beta := \min\{d\kappa_1, d(\kappa_1 - \kappa_2)\}$ (with notations as in the Theorem). The general theory then implies the following:

Corollary 5.1.0.1. *Let s and β be as above, and let $\tilde{s}$ be the structural morphism of S_K.*

(1) The motive $\tilde{s}_!^\lambda \mathcal{V} \in DM_{\mathrm{B},c}(\mathbb{Q})_L$ avoids weights $-\beta, -\beta+1, \ldots, -1$, and the motive $\tilde{s}_*^\lambda \mathcal{V} \in DM_{\mathrm{B},c}(\mathbb{Q})_L$ avoids weights $1, 2, \ldots, \beta$. More precisely, the exact triangles

$$s_* i_* i^* j_{!*}^\lambda \mathcal{V}[-1] \to \tilde{s}_!^\lambda \mathcal{V} \to s_* j_{!*}^\lambda \mathcal{V} \to s_* i_* i^* j_{!*}^\lambda \mathcal{V}$$

and

$$s_* j_{!*}^\lambda \mathcal{V} \to \tilde{s}_*^\lambda \mathcal{V} \to d_* i_* i^! j_{!*}^\lambda \mathcal{V}[1] \to s_* j_{!*}^\lambda \mathcal{V}[1]$$

are weight filtrations of $\tilde{s}_!^\lambda \mathcal{V}$, resp. of $\tilde{s}_*^\lambda \mathcal{V}$, which avoid weights $-\beta, -\beta+1, \ldots, -1$, resp. $1, 2, \ldots, \beta$.

(2) The intersection motive $s_* j_{!*}^\lambda \mathcal{V}$ is functorial with respect to $\tilde{s}_!^\lambda \mathcal{V}$ and to $\tilde{s}_*^\lambda \mathcal{V}$. In particular, every endomorphism of $\tilde{s}_!^\lambda \mathcal{V}$ or $\tilde{s}_*^\lambda \mathcal{V}$ induces an endomorphism of $s_* j_{!*}^\lambda \mathcal{V}$.

(3) If $\tilde{s}_!^\lambda\mathcal{V} \to N \to \tilde{s}_*^\lambda\mathcal{V}$ is a factorisation of $\tilde{s}_!^\lambda\mathcal{V} \to \tilde{s}_*^\lambda\mathcal{V}$ through a Chow motive $N \in CHM(\mathbb{Q})_L$, then the intersection motive $s_*j_{!*}^\lambda\mathcal{V}$ is canonically identified with a direct factor of N, with a canonical direct complement.

Proof. Since Theorem 4.3.1 tells us that, under our hypotheses on λ, $i^*j_*^\lambda\mathcal{V}$ avoids weights 0 and 1, the first two points follow from Corollary 3.3.2.7 and from the diagram of triangles considered after that corollary. The third point is [Wil19a, Thm. 2.6]. □

Fix now an integer N such that, as in Remark 3.2.1.6, ${}^\lambda\mathcal{V}$ is a direct factor of a Tate twist of $\pi_{N,*}\mathbb{1}_{A_K^N}$, where $\pi_N : A_K^N \to S_K$ denotes the N-fold fibred product of the universal abelian variety A_K with itself over S_K. The property stated in Corollary 5.1.0.1.(2) has important consequences for the *Hecke algebra* $\mathfrak{H}^{DM}(K, G(\mathbb{A}_f))$ associated to the open compact subgroup K (Def. 3.2.2.13), acting on the object $\tilde{s}_*^\lambda\mathcal{V}$. In fact, corollary 5.1.0.1.(2) gives us immediately the following consequence:

Corollary 5.1.0.2. *The algebra $\mathfrak{H}^{DM}(K, G(\mathbb{A}_f))$ acts naturally on the intersection motive $s_*j_{!*}^\lambda\mathcal{V}$.*

It is also useful to explicitly formulate the property stated in Corollary 5.1.0.1.(3) in a specific context:

Corollary 5.1.0.3. *Let $\tilde{A}_K^N$ be a smooth compactification of A_K^N. Then, the intersection motive $s_*j_{!*}^\lambda\mathcal{V}$ is canonically identified with a direct factor of a Tate twist of $a_*\mathbb{1}_{\tilde{A}_K^N}$ (where a is the structural morphism of $\tilde{A}_K^N$ towards* Spec $\mathbb{Q}$*), with a canonical direct complement.*

This corollary has important consequences for the realizations of $s_*j_{!*}^\lambda\mathcal{V}$.

Corollary 5.1.0.4. *Let $\mathcal{O}$ be the order of F prescribed by the PEL datum corresponding to S_K (cfr. Example 3.2.1.2), D the* discriminant *of $\mathcal{O}$ as defined in [Lan13, Def. 1.1.1.6], and N the* level *of K. Let p be a prime which does not divide $D \cdot N$. Then:*

(1) the p-adic realization of $s_*j_{!*}^\lambda\mathcal{V}$ is crystalline, and if ℓ is a prime different from p, the ℓ-adic realization of $s_*j_{!*}^\lambda\mathcal{V}$ is unramified at p;

(2) consider on the one hand the action of Frobenius ϕ on the ϕ-filtered module associated to the (crystalline) p-adic realization of $s_*j_{!*}^\lambda\mathcal{V}$, and on the other hand the action of a geometrical Frobenius at p on the ℓ-adic realization of $s_*j_{!*}^\lambda\mathcal{V}$ (unramified at p). Then, the characteristic polynomials of the two actions coincide.

Proof. (1) By [Wil09, Thm. 4.14], and with the notations of the preceding corollary, the existence of a smooth compactification of A_K^N with good reduction properties is enough to get the conclusion. Now, we have at our disposal the very general results of [Lan12b] on the existence of smooth projective integral models of smooth compactifications of *PEL-type Kuga-Sato families*: namely, Thm. 2.15 of *loc. cit.* (by taking into account Definition 1.6 of *loc. cit.* and [Lan13, Prop. 1.4.4.3]) implies that there exists a smooth compactification of A_K^N with good reduction at every prime p which does not divide $D \cdot N$. Thus, we can invoke [Wil09, Thm. 4.14] to conclude.

(2) We argue exactly as in [Wil19b, Cor. 1.13], in order to use [KM74, Thm. 2.2] and conclude.

□

In order to end this list of properties of $s_* j^\lambda_{!*}\mathcal{V}$, we recall that the reason for the name of the *intersection motive* is the behaviour of its realizations (recall that we are supposing that λ is either not completely irregular or of corank 0):

Corollary 5.1.0.5. *(1) For μ^K equal to the Hodge or ℓ-adic canonical construction, write $j_{!*}\mu^K$ for the complex*

$$j_{!*}(\mu^K(V_\lambda)[3d])[-3d]$$

Then, for every $n \in \mathbb{Z}$, the natural maps

$$H^n(S^*_K(\mathbb{C}), j_{!*}\mu^K_H(V_\lambda)) \to H^n(S_K(\mathbb{C}), \mu^K_H(V_\lambda))$$

(between cohomology spaces of Hodge modules) and

$$H^n((S^*_K) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, j_{!*}\mu^K_\ell(V_\lambda)) \to H^n((S_K) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu^K_\ell(V_\lambda))$$

(between cohomology spaces of ℓ-adic perverse sheaves) are injective, and dually, the natural maps

$$H^n_c(S_K(\mathbb{C}), \mu^K_H(V_\lambda)) \to H^n_c(S^*_K(\mathbb{C}), j_{!*}\mu^K_H(V_\lambda))$$

and

$$H^n_c((S_K) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu^K_\ell(V_\lambda)) \to H^n_c((S^*_K) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, j_{!*}\mu^K_\ell(V_\lambda))$$

are surjective. Consequently, the natural maps from intersection cohomology of S_K towards interior cohomology (with coefficients in $\mu^K_H(V_\lambda)$, resp. $\mu^K_\ell(V_\lambda)$) are isomorphisms.

(2) The Hodge realization, resp. ℓ-adic realization of the intersection motive $s_* j^\lambda_{!*}\mathcal{V} \in CHM(\mathbb{Q})_L$ is identified with interior cohomology

$$H^*_!(S_K(\mathbb{C}), \mathcal{R}_H({}^\lambda\mathcal{V}))$$

resp.

$$H^*_!((S_K)_{\times_{\mathbb{Q}}}\bar{\mathbb{Q}}, \mathcal{R}_\ell({}^\lambda\mathcal{V})).$$

Proof. We have proven that, under our standing hypotheses on λ, the complex of ℓ-adic sheaves $i^* j_* \mu^K_\ell(V_\lambda)$ avoids weights $-\beta, \dots, \beta+1$, and a completely analogous proof shows that this is also the case for the complex of mixed Hodge modules $i^* j_* \mu^K_H(V_\lambda)$ (cfr. Rmk. 4.2.2.4). In particular, it avoids weights 0 and 1. Now, the equivalence of categories given in Thm. 3.3.3, works whenever we have categories with weight structures *compatible with gluing* (i.e. such that the six functors satisfy the conclusions of Thm. 3.3.1.(2)), as is the case for the bounded derived categories of mixed Hodge modules. Hence, the latter weight avoidance is equivalent to saying that the complex $i^! j_{!*}\mu^K_H(V_\lambda)$ is of weights at least 1 and that the complex $i^* j_{!*}\mu^K_H(V_\lambda)$ is of weights at most -1. Then, consider the diagram of triangles of Prop. 2.3.3.3. On the one hand, absence of weight 0 in $i^* j_* \mu^K_H(V_\lambda)$ shows that in each degree n, interior cohomology is identified with the lowest weight-graded step $\mathrm{Gr}_{n+w(\lambda)}$ of H^n. On the other hand, the fact that $i^! j_{!*}\mu^K_H(V_\lambda)$ is of weights at least 1 implies that the arrow from intersection cohomology to $\mathrm{Gr}_{n+w(\lambda)}$ is not only *surjective* (Corollary 2.3.3.4), but also *injective*. This shows point (1).

Point (2) follows from (1) and from the fact that, by Thm. 3.3.3, the realizations of the intersection motive are identified with intersection cohomology. □

Remark 5.1.0.6. (1) The vanishing results of Thm. 2.1.1 imply that, if λ is regular, the spaces $H^n(S_K(\mathbb{C}), \mu_H^K(V_\lambda))$, and so (by comparison) $H^n(S_K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu_\ell^K(V_\lambda))$ are zero for $n < 3d = \dim S_K$. Dually, we get $H_c^n(S_K(\mathbb{C}), \mu_H^K(V_\lambda)) = 0$ and $H_c^n(S_K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu_\ell^K(V_\lambda)) = 0$ for $n > 3d$. As a consequence, if λ is regular, then the interior cohomology spaces $H_!^n(S_K(\mathbb{C}), \mu_H^K(V_\lambda))$ and $H_!^n(S_K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu_\ell^K(V_\lambda))$ are zero in degrees different from $n = 3d$.

(2) Corollary 5.1.0.5.(2) and the preceding point imply that, if λ is regular, the Hodge realization of the intersection motive $s_* j_{!*}^\lambda \mathcal{V} \in CHM(\mathbb{Q})_L$ is given by

$$H_!^{3d}(S_K(\mathbb{C}), \mu_H^K(V_\lambda))[-(w(\lambda) + 3d)],$$

and that its ℓ-adic realization is given by

$$H_!^{3d}(S_K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu_\ell^K(V_\lambda))[-(w(\lambda) + 3d)].$$

5.2 Homological motives for automorphic representations

Keep the notations of the preceding subsection and assume moreover that λ is regular. In this last part, following [Wil19b, Sec. 3], we would like to exploit the action of the algebra $\mathfrak{H}^{DM}(K, G(\mathbb{A}_f))$ on the intersection motive $s_* j_{!*}^\lambda \mathcal{V} \in CHM(\mathbb{Q})_L$ (cfr. Corollary 5.1.0.2) to cut out certain *homological* sub-motives thereof. Recall from Rmk. 3.2.2.14 that $\mathfrak{H}^{DM}(K, G(\mathbb{A}_f))$ acts on $H_!^{3d}(S_K(\mathbb{C}), \mu_H^K(V_\lambda))$ and on $H_!^{3d}(S_K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu_\ell^K(V_\lambda))$ through the algebra $\mathfrak{H}(K, G(\mathbb{A}_f))$ of Subsection 2.2.1. For brevity, denote the latter algebra by $\mathfrak{H}$.

Remembering that the action of $\mathfrak{H}$ is nothing but the action of the algebra $C_c^\infty(G(\mathbb{A}_f)//K)$ (Rmk. 2.2.1.5), the following is the consequence of Cor. 2.3.3.5.(1) and of the inclusions (2.44) (cfr. also [Har, Sect. 8.1.7, page 253]):

Theorem 5.2.1. *For every extension L' of L, the $\mathfrak{H}^{DM}(K, G(\mathbb{A}_f)) \otimes_L L'$-module*

$$H_!^{3d}(S_K(\mathbb{C}), \mu_H^K(V_\lambda)) \otimes L'$$

is semisimple.

Corollary 5.2.0.1. *Denote by $R(\mathfrak{H})$ the image of $\mathfrak{H}$ in the endomorphism algebra of $H_!^{3d}(S_K(\mathbb{C}), \mu_H^K(V_\lambda))$. For every extension L' of L, the algebra $R(\mathfrak{H}) \otimes_L L'$ is semisimple.*

In particular, isomorphism classes of simple right $R(\mathfrak{H}) \otimes_L L'$-modules are in bijection with isomorphism classes of minimal right ideals. Now, by fixing L', and one of these minimal right ideals Y_{π_f} of $R(\mathfrak{H}) \otimes_L L'$, there exists an idempotent $e_{\pi_f} \in R(\mathfrak{H}) \otimes_L L'$ which generates Y_{π_f}. The following definitions are motivated by (2.32) and (2.27):

Definition 5.2.0.2. *(1) The* Hodge structure $W(\pi_f)$ associated to Y_{π_f} *is defined by*

$$W(\pi_f) := \mathrm{Hom}_{R(\mathfrak{H}) \otimes_L L'}(Y_{\pi_f}, H_!^{3d}(S_K(\mathbb{C}), \mu_H^K(V_\lambda)) \otimes L').$$

(2) Let L' be a finite extension. For every prime number ℓ, and for every prime l of L' above ℓ, the Galois module $W(\pi_f)_\ell$ associated to Y_{π_f} *is defined by*

$$W(\pi_f)_\ell := \mathrm{Hom}_{R(\mathfrak{H}) \otimes_L L'_l}(Y_{\pi_f}, H_!^{3d}(S_K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu_\ell^K(V_\lambda)) \otimes L'_l).$$

The following proposition tells us how to define the desired motives:

Proposition 5.2.0.3. *(cfr. [Wil19b, Prop. 3.4])) There are canonical isomorphisms of Hodge structures, resp. of Galois modules*

$$W(\pi_f) \simeq H_!^{3d}(S_K(\mathbb{C}), \mu_H^K(V_\lambda) \otimes L') \cdot e_{\pi_f},$$

resp.

$$W(\pi_f)_\ell \simeq H_!^{3d}(S_K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mu_\ell^K(V_\lambda) \otimes L'_l) \cdot e_{\pi_f}.$$

Proof. The proof is standard, and identical in the two cases. Let us give it in the case of Hodge structures. By mapping an element g of

$$\mathrm{Hom}_{R(\mathfrak{H})\otimes_L L'}(R(\mathfrak{H}) \otimes_L L', H_!^{3d}(S_K(\mathbb{C}), \mu_H^K(V_\lambda)) \otimes L')$$

to the image of $1_{R(\mathfrak{H})\otimes_L L'}$ through g, we identify the above space of morphisms with

$$H_!^{3d}(S_K(\mathbb{C}), \mu_H^K(V_\lambda)) \otimes L'$$

But inside that space, the sub-Hodge structure $W(\pi_f)$ corresponds exactly to those morphisms g vanishing on $1_{R(\mathfrak{H})\otimes_L L'} - e_{\pi_f}$. □

Since we do not know if e_{π_f} lifts to an idempotent element of $\mathfrak{H}^{DM}(K, G(\mathbb{A}_f)) \otimes_L L'$, we are forced to consider its action on the *homological* (or Grothendieck) motive which underlies the intersection motive $s_* j_{!*}^\lambda \mathcal{V} \in CHM(\mathbb{Q})_L$. Denote then by $s_* j_{!*}^\lambda \mathcal{V}'$ this homological motive, and define, thanks to Corollary 5.1.0.2:

Definition 5.2.0.4. *The* (homological) motive corresponding to Y_{π_f} *is defined by* $\mathcal{W}(\pi_f) := s_* j_{!*}^\lambda \mathcal{V}' \cdot e_{\pi_f}$.

Remark 5.2.0.5. The above motives could have been defined *without* knowing that there is an *algebra* action of the Hecke endomorphisms on the intersection motive, because we only need the existence of an algebra action on its homological counterpart (a fact which holds by contruction). Nonetheless, it is expected that the idempotent (modulo homological equivalence) element e_{π_f} lifts to an idempotent modulo rational equivalence (and hence that the motives $\mathcal{W}(\pi_f)$ may be lifted to Chow motives). In order to even formulate this expectation - let alone trying to prove its validity - one needs the Hecke algebra action on the *Chow* motive $s_* j_{!*}^\lambda \mathcal{V}$.

We finish by making explicit the properties of the latter motive which follow from the preceding constructions:

Theorem 5.2.2. *The realizations of the motive $\mathcal{W}(\pi_f)$ are concentrated in cohomological degree $w(\lambda) + 3d$, where in particular the Hodge realization equals $W(\pi_f)$, and the ℓ-adic realizations equal $W(\pi_f)_\ell$, for every prime ℓ.*

Proof. Follows from the construction of $\mathcal{W}(\pi_f)$ and Remark 5.1.0.6.(2) (remember that we are supposing λ to be regular). □

Corollary 5.2.0.6. *Let p be a prime number which does not divide the integer $D \cdot N$ from Corollary 5.1.0.4, and ℓ a prime different from p. Then:*

(1) the p-adic realization of $\mathcal{W}(\pi_f)$ is crystalline, and the ℓ-adic realization of $\mathcal{W}(\pi_f)$ is unramified at p;

(2) consider on the one hand the action of the Frobenius ϕ on the ϕ-filtered module associated to the p-adic (crystalline) realization of $\mathcal{W}(\pi_f)$, and on the other hand the action of a geometrical Frobenius in p on the ℓ-adic realization of $\mathcal{W}(\pi_f)$ (unramified at p). Then, the characteristic polynomials of the two actions coincide.

Proof. 1. Follows from Corollary 5.1.0.4.(1), by taking into account the fact that $\mathcal{W}(\pi_f)$ is a direct factor of $s_* j^{\lambda}_{!*}\mathcal{V}'$.

2. We can argue as in Corollary 5.1.0.4.(2) to apply [KM74, Thm. 2.2] and conclude. □

Remark 5.2.0.7. (1) Suppose that λ is a regular weight of G whose restriction to the center is trivial and that the image of K along the natural projection $\mathrm{GSp}_4 \to \mathrm{PGSp}_4$ is still a compact open subgroup of $\mathrm{PGSp}_4(\mathbb{A}_f)$. Then, the ℓ-adic realizations $W(\pi_f)_\ell$ of the motive $\mathcal{W}(\pi_f)$ coincide with the Galois modules $H^*_c(\pi_{Hf})$ associated to suitable automorphic representations of G in [Fli05, Part 2, Chap. I.2, Thm. 2] (remember that by Thm. 2.3.2, under the regularity assumption, cuspidal and intersection cohomology coincide). There, the precise relation between eigenvalues of (suitable) Frobenii and Hecke eigenvalues is given. The existence of the motive $\mathcal{W}(\pi_f)$ then adds to the description in *loc. cit.* the information about the behaviour at p of the Galois module $W(\pi_f)_p$, which has been obtained in Corollary 5.2.0.6 (when p doesn't divide the integer $D \cdot N$).

(2) Keep the assumptions of the preceding point and let l be a place of E above the prime ℓ. The Galois modules $W(\pi_f)_\ell$ are then of dimension 4^d or $\frac{1}{2} \cdot 4^d$ over L'_l ([Fli05, Part 2, Chap. I.2, Thm. 2.(1),(4)]). One can expect that, in the case of a (Hilbert-Siegel) *eigenform* f, the motives $\mathcal{W}(\pi_f)$ over $\mathbb{Q}$ can be written as tensor products over L' of rank-4 motives over F, whose L-function has the correct relation with the L-function of f. However, there are no known methods for constructing motives with such properties. It is the same problem which arises for motives corresponding to Hilbert modular forms, when cut out inside Kuga-Sato varieties over Hilbert modular varieties, cfr. for example [Har94, 5.2].

Bibliography

[AK02] Y. André and B. Kahn. Nilpotence, radicaux et structures monoïdales. *Rend. Sem. Mat. Univ. Padova*, (108):107–291, 2002.

[Anc15] G. Ancona. Décomposition de motifs abéliens. *manuscripta math.*, (146):307–328, 2015.

[And04] Y. André. *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*. Number 17 in Panoramas et Synthèses. Soc. Math. France, 2004.

[BBD82] A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, number 100 in Astérisque, pages 5–171. Soc. Math. France, 1982.

[BC83] A. Borel and W. Casselman. L^2-cohomology of locally symmetric manifolds of finite volume. *Duke Math. J.*, (50):625–647, 1983.

[Bei87] A. Beilinson. Height pairing between algebraic cycles. In *K-theory, arithmetic and geometry (Moscow, 1984-1986)*, number 1289 in Lect. Notes in Math., pages 27–41. Springer, 1987.

[Bei12] A. Beilinson. Remarks on Grothendieck's standard conjectures. In *Regulators*, number 571 in Contemp. Math., pages 25–32. Amer. Math. Soc., 2012.

[BHC62] A. Borel and Harish-Chandra. Arithmetic subgroups of algebraic groups. *Ann. of Math.*, 75(3):485–535, 1962.

[BJ06] A. Borel and L. Ji. *Compactifications of symmetric and locally symmetric spaces*. Mathematics: Theory & Applications. Birkhäuser Boston, 2006.

[Bon10] M. V. Bondarko. Weight structures vs. t-structures: weight filtrations, spectral sequences, and complexes (for motives and in general). *J. K-Theory*, (6):387–504, 2010.

[Bon15] M. V. Bondarko. Mixed motivic sheaves (and weights for them) exist if 'ordinary' mixed motives do. *Compositio Math.*, (151):917–956, 2015.

[Bor72] A. Borel. Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem. *J. Differential Geometry*, (6):543–560, 1972.

[Bor81] A. Borel. Stable real cohomology of arithmetic groups II. In *Manifolds and Lie groups, papers in honor of Y. Matsushima*, Progress. in Math. Birkähuser Boston, 1981.

[Bor83] A. Borel. Regularization theorems in Lie algebra cohomology. Applications. *Duke Math, J,*, (3):605–623, 1983.

[BR18] R. Brasca and G. Rosso. Eigenvarieties for non-cuspidal forms over certain PEL Shimura varieties. *preprint*, 2016 (version of June 19, 2018). 33 pages, available on arXiv.org at the address https://arxiv.org/abs/1605.05065.

[BS73] A. Borel and J.P. Serre. Corners and arithmetic groups. *Comment. Math. Helv.*, (48):436–491, 1973.

[Bum97] D. Bump. *Automorphic forms and representations.* Number 55 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1997.

[BW00] A. Borel and N. Wallach. *Continuous cohomology, discrete subgroups and representations of reductive groups. Second edition.* Number 67 in Mathematical Surveys and Monographs. American Mathematical Society, 2000.

[BW04] J. I. Burgos and J. Wildeshaus. Hodge modules on Shimura varieties and their higher direct images in the Baily-Borel compactification. *Ann. Sci. Ec. Norm. Sup. (4)*, 37(3):363–413, 2004.

[Car79] P. Cartier. Representations of p-adic groups: a survey. In A. Borel and W. Casselman, editors, *Automorphic forms, representations and L-functions, Part 1*, number 33 in Proc. Sympos. Pure Math., pages 111–155. Amer. Math. Soc., 1979.

[Cav19] M. Cavicchi. On the boundary and intersection motives of genus 2 Hilbert-Siegel varieties. *to appear in Documenta Mathematica*, 2019. available on arXiv.org at the address https://arxiv.org/abs/1805.00440.

[CD12] D.-C. Cisinski and F. Déglise. Triangulated categories of mixed motives. *preprint*, 2009 (version of December 18, 2012). 279 pages, available on arXiv.org at the address http://arxiv.org/abs/0912.2110.

[CD16] D.-C. Cisinski and F. Déglise. Étale motives. *Compositio Math.*, (152):556–666, 2016.

[CH00] A. Corti and M. Hanamura. Motivic decomposition and intersection Chow groups. i. *Duke Math. J.*, 103(3):459–522, 2000.

[Clo90] L. Clozel. Motifs et formes automorphes: applications du principe de fonctorialité. In L. Clozel and J. Milne, editors, *Automorphic forms, Shimura varieties, and L-functions (Ann Arbor, MI, 1988), Vol. I*, number 10 in Perspect. Math., pages 77–159. Academic Press, 1990.

[Clo17] G. Cloître. *Sur le motif intérieur de certaines variétés de Shimura: le cas des variétés de Picard.* PhD thesis, Université Paris 13, Sorbonne Paris Cité, 2017.

[Del71a] P. Deligne. Formes modulaires et représentations ℓ-adiques. In *Séminaire Bourbaki, Vol. 1968/69*, number 175 in Lecture Notes in Math., chapter Exp. No. 355, pages 139–172. Springer, 1971.

[Del71b] P. Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.

[Del71c] P. Deligne. Travaux de Shimura. In *Séminaire Bourbaki, Vol. 1970/71*, number 244 in Lecture Notes in Math., chapter Exp. No. 389, pages 123–165. Springer, 1971.

[Del79] P. Deligne. Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. In A. Borel and W. Casselman, editors, *Automorphic forms, representations and L-functions, Part 2*, number 33 in Proc. Sympos. Pure Math. Amer. Math. Soc., 1979.

[Del80] P. Deligne. La conjecture de Weil. II. *Inst. Haut. Études Sci. Publ. Math.*, (52):137–252, 1980.

[DM91] C. Deninger and J. Murre. Motivic decomposition of abelian schemes and the Fourier transform. *J. Reine Angew. Math*, (422):201–219, 1991.

[DT04] M. Dimitrov and J. Tilouine. Variétés et formes modulaires de Hilbert arithmétiques pour $\Gamma_1(\mathfrak{c}, \mathfrak{n})$. In *Geometric aspects of Dwork theory*, pages 555–614. Walter De Gruyter, 2004.

[Eke90] T. Ekedahl. On the adic formalism. In P. Cartier et al., editor, *The Grothendieck Festschrift, Vol. II*, number 87 in Prog. in Math., pages 197–218. Birkähuser Boston, 1990.

[Fal83] G. Faltings. On the cohomology of locally symmetric Hermitian spaces. In *Paul Dubreil and Marie-Paule Malliavin algebra seminar, 35th year (Paris, 1982)*, number 1029 in Lect. Notes in Math., pages 55–98. Springer, 1983.

[Fan16] J. Fangzhou. Borel-Moore motivic homology and weight structure on mixed motives. *Mathematische Zeitschrift*, (283):1–35, 2016.

[Fli05] Y. Flicker. *Automorphic forms and Shimura varieties of* PGSp(2). World Scientific Publishing Co. Pte. Ltd., 2005.

[Fra98] J. Franke. Harmonic analysis in weighted L^2 spaces. *Ann. Sci. Ec. Norm. Sup. (4)*, 31(2):181–279, 1998.

[Fre90] E. Freitag. *Hilbert modular forms.* Springer-Verlag, 1990.

[FS98] J. Franke and J. Schwermer. A decomposition of spaces of automorphic forms, and the Eisenstein cohomology of arithmetic groups. *Math. Ann.*, (4):765–790, 1998.

[GG13] N. Grbac and H. Grobner. The residual Eisenstein cohomology of Sp_4 over a totally real number field. *Trans. Amer. Math. Soc.*, 365(10):5199–5235, 2013.

[Har] G. Harder. *Cohomology of arithmetic groups.* in preparation, available at the address http://www.math.uni-bonn.de/people/harder/Manuscripts/buch/Volume-III-Jan-2019.pdf.

[Har89] M. Harris. Functorial properties of toroidal compactifications of locally symmetric varieties. *Proc. London Math. Soc. (3)*, 59(1):1–22, 1989.

[Har90] M. Harris. Automorphic forms of $\bar{\partial}$-cohomology type as coherent cohomology classes. *J. Differential Geometry*, 31(1):1–63, 1990.

[Har91] G. Harder. Eisenstein cohomology of arithmetic groups and its applications to number theory. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, page 779–790. Math. Soc. Japan, 1991.

[Har94] M. Harris. Hodge-de Rham structures and periods of automorphic forms. In *Motives (Seattle, WA, 1991)*, volume 2 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., 1994.

[Hé11] D. Hébert. Structures de poids à la Bondarko sur les motifs de beilinson. *Compositio Math.*, (147):1447–1462, 2011.

[Ji09] L. Ji. *Arithmetic groups and their generalizations.* Number 43 in AMS/IP Studies in Advanced Mathematics. American Mathematical Society, International Press, 2009.

[Kin98] G. Kings. Higher regulators, Hilbert modular surfaces, and special values of L-functions. *Duke Math. J.*, 92(1):61–127, 1998.

[KM74] M. Katz and W. Messing. Some consequences of the Riemann hypothesis for varieties over finite fields. *Invent. Math.*, (23):73–77, 1974.

[Lan12a] K.-W. Lan. Comparison between analytic and algebraic constructions of toroidal compactifications of PEL-type Shimura varieties. *J. Reine Angew. Math.*, 2012.

[Lan12b] K.-W. Lan. Toroidal compactifications of PEL-type Kuga families. *Algebra Number Theory*, (5):885–966, 2012.

[Lan13] K.-W. Lan. *Arithmetic compactifications of PEL-type Shimura varieties.* Number 36 in London Mathematical Society Monographs Series. Princeton University Press, 2013.

[Lan17] K.-W. Lan. An example-based introduction to Shimura varieties. In *Proceedings of the ETHZ Summer School on Motives and Complex Multiplication (to appear)*. preprint (2017). available on the author's webpage at http://www-users.math.umn.edu/ kwlan/articles/intro-sh-ex.pdf.

[Lem15] F. Lemma. On higher regulators of Siegel threefolds I: the vanishing on the boundary. *Asian J. Math*, (19):83–120, 2015.

[Lev09] M. Levine. Smooth motives. In *Motives and algebraic cycles*, number 56 in Fields Inst. Commun., pages 175–231. Amer. Math. Soc., 2009.

[Loo88] E. Looijenga. L^2-cohomology of locally symmetric varieties. *Compositio Math.*, 67(1):3–20, 1988.

[LR91] E. Looijenga and M. Rapoport. Weights in the local cohomology of a Baily-Borel compactification. In *Complex geometry and Lie theory (Sundance, UT, 1989)*, number 53 in Proc. Sympos. Pure Math. Amer. Math. Soc., 1991.

[LS04] J.-S. Li and J. Schwermer. On the Eisenstein cohomology of arithmetic groups. *Duke Math. J.*, 123(1):141–169, 2004.

[Mil90] J. Milne. Canonical models of (mixed) Shimura varieties and automorphic vector bundles. In *Automorphic forms, Shimura varieties and L-functions (Ann Arbor, MI, 1988), Vol. I*, volume II of *Perspect. Math.*, pages 283–414. Academic Press, 1990.

[MSSYZ15] S. Müller-Stach, M. Sheng, X. Ye, and K. Zuo. On the cohomology groups of local systems over Hilbert modular varieties via Higgs bundles. *Amer. J. Math.*, (137):1–35, 2015.

[OS90] T. Oda and J. Schwermer. Mixed Hodge structures and automorphic forms for Siegel modular varieties of degree two. *Math. Ann.*, (286):481–509, 1990.

[Pin90] R. Pink. *Arithmetical compactifications of mixed Shimura varieties*. PhD thesis, Univ. Bonn, 1990.

[Pin92] R. Pink. On ℓ-adic sheaves on Shimura varieties and their higher direct images in the Baily-Borel compactification. *Math. Ann.*, (292):197–240, 1992.

[Sai90] M. Saito. Mixed Hodge modules. *Publ. Res. Inst. Math. Sci.*, (2):221–333, 1990.

[Sap05a] L. Saper. L-modules and micro-support. *preprint*, 2001 (version of January 9, 2005). available on arXiv at the address https://arxiv.org/abs/math/0112251.

[Sap05b] L. Saper. $\mathcal{L}$-modules and the conjectures of Rapoport-Goresky-MacPherson. In *Automorphic forms. I*, number 298 in Astérisque. Soc. Math. France, 2005.

[Sch90] A. J. Scholl. Motives for modular forms. *Invent. Math.*, (100):419–430, 1990.

[Shi63] H. Shimizu. On discontinuous groups operating on the product of the upper half planes. *Ann. of Math.*, 2(77):33–71, 1963.

[Shi71] G. Shimura. *Introduction to the arithmetic theory of automorphic functions*. Princeton University Press, 1971.

[SS90] L. Saper and M. Stern. L^2-cohomology of algebraic varieties. *Ann. of Math.*, 132(1):1–69, 1990.

[Voe00] V. Voevodsky. Triangulated categories of motives. In *Cycles, transfers, and motivic homology theories*, number 143 in Annals of Mathematics Studies. Princeton University Press, 2000.

[Vog81] D. Vogan. *Representations of real reductive groups*. Birkhäuser, 1981.

[Wei94] C. Weibel. *An introduction to homological algebra*. Number 38 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.

[Wil97] J. Wildeshaus. *Realizations of Polylogarithms*. Number 1650 in Lect. Notes Math. Springer-Verlag, 1997.

[Wil09] J. Wildeshaus. Chow motives without projectivity. *Compositio Math.*, (145):1196–1226, 2009.

[Wil12] J. Wildeshaus. On the interior motive of certain Shimura varieties: the case of Hilbert-Blumenthal varieties. *Int. Math. Res. Notices*, (2012):2321–2355, 2012.

[Wil13] J. Wildeshaus. Boundary motive, relative motives and extensions of motives. In M. Levine, J. Wildeshaus, and B. Kahn, editors, *Autour des motifs - École d'été franco-asiatique de géométrie algébrique et de théorie des nombres, Vol. II*, number 41 in Panoramas et Synthèses, pages 143–185. Soc. Math. France, 2013.

[Wil15] J. Wildeshaus. On the interior motive of certain Shimura varieties: the case of Picard surfaces. *manuscripta math.*, (148):351–377, 2015.

[Wil17] J. Wildeshaus. Intermediate extensions of Chow motives of abelian type. *Adv. Math.*, (305):515–600, 2017.

[Wil18] J. Wildeshaus. Weights and conservativity. *Algebr. Geom.*, 5(6):686–702, 2018.

[Wil19a] J. Wildeshaus. Chow motives without projectivity, II. *to appear in Int. Math. Res. Not.*, 2019. available on arXiv.org at the address https://arxiv.org/abs/1705.10502.

[Wil19b] J. Wildeshaus. On the intersection motive of certain Shimura varieties: the case of Siegel threefolds. *to appear in Annals of K-Theory*, 2019. available on arXiv.org at the address https://arxiv.org/abs/1706.02743.