UNIVERSITÉ PARIS 13

THÈSE

en mathématiques appliquées

Modélisation, analyse et simulation numérique d'une coque poroélastique et son interaction avec un fluide

présentée publiquement par Dang Phuoc Nhat

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Titre: **Modélisation, analyse et simulation numérique d'une coque poroélastique et son interaction avec un fluide**

Résumé

Nous proposons dans cette thèse deux modèles de coques poroélastiques. Il s'agit des modèles Biot-Naghdi et Biot-Koiter. Leurs dérivations sont basées sur les hypothèses de Reissner-Mindlin et celles de Kirchhoff-Love respectivement et où la coque poreuse est saturée.

Nous démontrons en utilisant la méthode de Galerkin et le théorème de Banach-Nečas-Babuška que ces modèles sont bien posés, c'est-à-dire qu'ils admettent des solutions uniques dans des espaces fonctionnels appropriés.

Nous proposons également un modèle décrivant l'interaction entre un fluide incompressible et une coque poreuse et élastique. C'est le modèle Stokes-Biot-Naghdi. Ici, la structure étant poreuse, les conditions de glissement à l'interface fluide/coque ne sont pas standards. Nous considérons alors les conditions de Beavers-Joseph-Saffman à l'interface sous forme faible en introduisant un mutiplicateur de Lagrange. Nous prouvons ensuite que le modèle obtenu est bien posé grâce à la méthode de Galerkin et la théorie des équations algébro-différentielles.

Enfin, nous proposons un algorithme de découplage pour résoudre et simuler numériquement la solution du modèle Biot-Naghdi.

Mots-clés: Coque poroélastique, modèle Biot, modèle de Naghdi, modèle de Koiter, Interaction fluide-structure.

Title: **Modelling, analysis and numerical simulation of a poroelastic shell and its interaction with a fluid**

Abstract

This thesis is devoted to the study of poroelastic thin shells. For derivation of poroelastic shell equations, we make use of Reissner-Mindlin assumptions for Biot-Naghdi's model and Kirchhoff-Love assumptions for Biot-Koiter's model. We prove the well-posedness of the obtained models by Galerkin semi-discrete method and Banach-Nečas-Babuška theorem as well. Moreover, we establish the strong formulation for Biot-Naghdi poroelastic shell model. Then, we derive the fluid-structure interaction between Stokes incompressible flow and Biot-Naghdi poroelastic shell structure where the non standard slip boundary conditions of Beavers-Joseph-Saffman type on the interface are considered. A Lagrange multiplier method is employed to impose weakly these conditions. We assume that the boundaries and the interface between the fluid and the poroelastic material are fixed. The proof proceeds by constructing a semi-discrete finite element Galerkin approximations and for the existence of the solution we adopt the theory of differential-algebraic equations. Finally, we simulate the Biot-Naghdi poroelastic shell model by FreeFem++.

Keywords: Poroelastic structure, shell, fluid-structure interaction, Naghdi model, Koiter model.

Contents

Introduction

Shells and their assemblies are a part of the huge variety of elastic structure of the biggest interest for the contemporary engineering: bodyworks, shells of ships, fuselages, wings of plane, towers of cooling, etc.

Shell structure are ubiquitous in nature because they are light but “strong”. Their geometry allows forces to be balanced by tensile strains rather bending. Some examples of shells are given below.

– Shells in nature:

(a) Clam shell

(b) Egg shell

– Manmade shells

(a) CDG Airport Terminal 2E: Basket-handle profile shell with $W^{2,\infty}$-regularity

(b) Bodywork

There exist at least two different families of linear models for thin elastic shells: the one of Reissner, which relies on the theory of Cosserat surfaces, (Cosserat and Cosserat-[39]), and the

Kirchhoff-Love type theories. This second approach is based on the famous Kirchhoff-Love's hypothesis, which states that the norms on the reference midsurface are transformed into norms on the deformed midsurface and that the distance between one point and the midsurface remains constant throughout the deformation of the shell.

Considering those hypothesis, Koiter [55] proposed a two-dimensional mathematical model for linearly elastic thin shells where the unknown of the problem is the displacement field of the points on the midsurface of shell. An approximation for the displacement field across the thickness of shell depends merely on the knowledge of this displacement via the Kirchhoff-Love's hypothesis [6].

A theorem of existence and uniqueness for the Koiter's model was established firstly by Bernadou and Ciarlet [7], who proposed a particularly technical evidence based on results of Rougée [79]. By application of a lemma on the distributions in H^{-1}, whose gradient is as well in H^{-1} of J.L. Lions, Ciarlet and Miara [32] gived a simpler demonstration of the existence and uniqueness of the same model. At last, Bernadou, Ciarlet and Miarra [8] provided some improvements of this last evidence. It is important to note that, in all of these demonstrations, the map defining the shell's midsurface is assumed at least in $\mathcal{C}^3$.

During the same period, Naghdi [68, 70] has presented a model which takes into account the effects of transverse shearing, while respecting the hypothesis of plane constraint and preservation of the distance between one point and the midsurface in the course of deformation. Under these hypothesis and the determination of displacement of points on the midsurface as well as the rotation of unique normal vector of this surface, the Naghdi's model permits to have an approximation of displacement field through the shell thickness.

The mathematical analysis of Naghdi's model is done the first time by Coutris [40] then improved by Ciarlet and Miara [32]. See as well Bernadou, Ciarlet and Miara [8] where the unknowns are the covariant of contravariant components of the displacement and the rotation. For vector formulations, Blouza[18] and Blouza - Le Dret [22] proposed a framework in which they consider shell with little regularity.

Mechanical problems, including the porous elastic body saturated with a fluid, appears in a diversity of subjects. The mathematic modelling and the numerical analysis of tridimensional elastic bodies problems are now controlled. Nevertheless, in the cases of thin structures like membranes, plates and shells, the numerical methods adapted well to the tridimensional cases fail, because of the small thickness. Therefore, it is natural for thinking about deriving from the tridimensional modelling to the bidimensional modelling which works on the midsurface of the shell for decreasing the cost of computation.

Many authors have focused on the limit behavior of the tridimensional equations when the thickness of the porous shell goes to zero. We can for example quote the works of Mikelić and Tambača [64, 65] where they proposed the limit model of poroelastic flexural and membrane shell model.

We also cite, in the generally poroelastic shell case, the work of Ljulj and Tambača[60], in which they derived an iterative method for solving a poroelastic shell model of Naghdi's type. In their model, the unknowns are given by their local basis components.

In the present work, we are not concerned with the asymptotic models. Our purpose is to derive two models from the equilibrium equations of tridimensional elasticity where the Reissner-Mindlin and Kirchhoff-Love assumptions are used in the case where the thickness is small enough with respect to other characteristics of the shell. Our framework here is a free local basis formulation for the displacement and considered shells are with little regularity midsurface.

Elasticity is the tendency of solid materials to return to their original shape after that forces are applied on them, when the forces are removed, the structure returns to its reference configuration and size. In tridimensional elasticity, the undeformed body occupies a region M. Under loading, a point $\boldsymbol{x} \in M$ moves to $\boldsymbol{x} + \boldsymbol{U}(\boldsymbol{x})$. Let $\widetilde{\Gamma}_0 \cup \widetilde{\Gamma}_1$ be a partition of the boundary ∂M with $\text{meas}(\Gamma_0) > 0$. Then the equilibrium equations of tridimensional elasticity are:

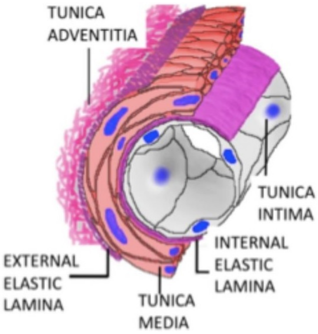
$$\begin{aligned} -\text{div}\boldsymbol{\sigma}(\boldsymbol{U}) &= \boldsymbol{f} && \text{in } M \text{ (force balance)},\\ \boldsymbol{\sigma}(\boldsymbol{U}) &= H : \varepsilon(\boldsymbol{U}) && \text{in } M \text{ (constitutive)},\\ \boldsymbol{U} &= 0 && \text{on } \widetilde{\Gamma}_0 \text{ (clamping)},\\ \boldsymbol{\sigma} \cdot \boldsymbol{n} &= h && \text{on } \widetilde{\Gamma}_1 \text{ (force balance)}. \end{aligned}$$

where $\boldsymbol{\sigma}(\boldsymbol{U})$ is the stress tensor and $\varepsilon(\boldsymbol{U}) = \dfrac{1}{2}(\nabla \boldsymbol{U} + (\nabla \boldsymbol{U})^T)$ is the strain tensor.

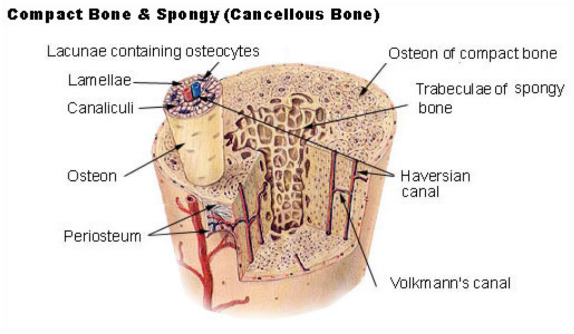
Poroelastic structure is the elastic structure having pores which contain a fluid. Then a gradient of the pressure is added to the applied force to deform the structure.

As Verruijt wrote in [88], soft soils such as sand and clay consist of small particles and the pore space between the particles often is fulfilled with water. In soil mechanics, this is denoted as a saturated or partially saturated porous medium. The deformation of such porous media depends upon the stiffness of the porous material, and upon the behavior of the fluid in the pores. If the permeability of the material is small, the deformations may be considerably retarded by the viscous behavior of the fluid in the pores. The simultaneous deformation of the porous material and the flow of the pore fluid are the subject of the consolidation theory, often denoted as *poroelasticity*. The poroelasticity theory was developed by Biot ([10], [11], [12], [13], [14] and [15]) several decades ago and it has been studied extensively since.

Poroelastic phenomena are interesting in numerous applications as geomechanics, groundsurface water flow, reservoir compaction and surface subsidence, seabed-wave interaction problem, etc. Two examples of poroelastic structures are given below.



(a) Arterial wall



(b) Bone anatomy

The poroelastic shells and their interactions with fluids are part of a wide variety of issues and a largest interest for contemporary engineering (ship hulls, reservoirs, ...) and biomedical problems (heart, bones, intestine skin, ...). As stated in [30], we point out that simultaneous with the development of poroelasticity, literature dealing with thermoelasticity has evolved. Here it is assumed that there is a coupling between the thermal diffusion equations and the equations of mechanical equilibrium. The complete mathematical analogy of the poroelastic and thermoelastic problems was noted by Biot [16]. A comprehensive treatment from this latter point of view is given by Nowacki [72].

Here we introduce the Biot model, as in [47], the constitutive equation for the Cauchy stress tensor $\tilde{\boldsymbol{\sigma}}$ in terms of the displacement $\boldsymbol{U}$ and fluid pressure p is

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\boldsymbol{U}) - \alpha p \mathbf{I}, \tag{1}$$

where $\mathbf{I}$ is the identity tensor, $\boldsymbol{\sigma}(\boldsymbol{U})$ is the stress tensor, expressing the Hooke's law:

$$\boldsymbol{\sigma}(\boldsymbol{U}) = \lambda(\text{div } \boldsymbol{U})\mathbf{I} + 2\mu\varepsilon(\boldsymbol{U}), \tag{2}$$

where $\varepsilon(\boldsymbol{U}) = \dfrac{1}{2}(\nabla \boldsymbol{U} + (\nabla \boldsymbol{U})^T)$ is the strain tensor, $\lambda \geq 0$ (dilation moduli) and $\mu > 0$ (shear moduli) are the Lamé coefficients, and $\alpha \in]0,1[$ is the Biot-Willis constant, which is usually around one. The flux of the fluid $\boldsymbol{v}_f$ is governed by Darcy's law in porous media

$$\boldsymbol{v}_f = -\frac{\kappa}{\eta}\nabla p, \tag{3}$$

where $\eta > 0$ is the fluid viscosity and fluid density assumed to be constant and κ is the permeability of porous medium.
The equation of mass conservation is

$$\frac{\partial \zeta}{\partial t} = -\text{div}\boldsymbol{v}_f + g,$$

where g is a volumetric fluid source term and ζ is the fluid content of the medium; ζ related to the fluid pressure p and material volume div $\boldsymbol{U}$ by

$$\zeta = c_0 p + \alpha \text{div } \boldsymbol{U} \tag{4}$$

where $c_0 \geq 0$ is the constrained specific storage coefficient, that is assumed to be constant. As explained by Phillips and Wheeler in [75], $c_0 = 0$ may lead to locking, whatever the value of the Lamé coefficient λ. Although in practical situation, c_0 can vanish, we do not consider this possibility here and therefore we suppose that $c_0 > 0$. With (3) and (4), the equation of mass conservation reads

$$\frac{\partial}{\partial t}(c_0 p + \alpha \mathrm{div}\, \boldsymbol{U}) - \frac{\kappa}{\eta}\mathrm{div}(\nabla p) = g. \tag{5}$$

Finally, the balance of linear momentum is derived by making a quasi-static assumption, namely by assuming that the material deformation is much slower than the flow rate, and hence the second-time derivative of the displacement (i.e. the acceleration) is zero. Denoting by $\boldsymbol{f}$ the body force, this yields

$$-\mathrm{div}\, \tilde{\boldsymbol{\sigma}} = \boldsymbol{f}. \tag{6}$$

Thus, replacing the constitutive relation (1) into (6) we obtain

$$-\nabla \cdot \sigma\big(\boldsymbol{U}\big) + \alpha \nabla p = \boldsymbol{f} \text{ in } M \times]0, T[.$$

Collecting the above equations, we have the following system of equation a.e. in $M \times]0, T[$

$$-\nabla \cdot \sigma\big(\boldsymbol{U}\big) + \alpha \nabla p = \boldsymbol{f}, \tag{7a}$$

$$\frac{\partial}{\partial t}(c_0 p + \alpha \mathrm{div} \boldsymbol{U}) - \frac{\kappa}{\eta}(\nabla p - \rho_f \boldsymbol{q}) = g. \tag{7b}$$

The coupling first order terms in the system have the following meaning: the term ∇p in the first equation results from the additional stress in the medium coming from the fluid pressure, the term div $\boldsymbol{U}$ in the second equation represents the additional fluid content due to local volume change. The Biot system should be supplemented with relevant boundary and initial conditions that have clear physical meaning.

Afterwards, we sketch out our results in this thesis. In Chapter 1, the purpose is the derivation the weak coupled formulation of shell model of Koiter type and Biot model, and the proof of its well-posedness. More precisely, we state the derivation in the following theorem,

Theorem

If $\boldsymbol{U}$ and p are a solution of the strong formulation (7), such that $\boldsymbol{U} = \boldsymbol{u} - z(\partial_\alpha \boldsymbol{u} \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha$ a Kirchhoff-Love displacement, then $\boldsymbol{u}$ and p belong, respectively, to $\mathcal{V}_K$ and $\mathcal{W}_K$ solving the following weak formulation:

$$\mathcal{A}_1^K\big(\boldsymbol{u};\boldsymbol{v}\big) + \mathcal{B}_1^K\big(p;\boldsymbol{v}\big) = \mathcal{L}_1^K(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathcal{X}_K, \tag{8a}$$

$$\mathcal{A}_2^K(p;q) + \mathcal{B}_2^N\big(\boldsymbol{u};q\big) = \mathcal{L}_2^K(q) \quad \forall q \in H_0^1(\Omega), \tag{8b}$$

$$p(0) = p_0 \text{ in } \Omega, \tag{8c}$$

where

$$\mathcal{A}_1^K(\boldsymbol{u};\boldsymbol{v}) = \int_\omega e\, a^{\alpha\beta\rho\sigma}\big[\gamma_{\alpha\beta}(\boldsymbol{u})\gamma_{\rho\sigma}(\boldsymbol{v}) + \frac{e^2}{12}\Upsilon_{\alpha\beta}(\boldsymbol{u})\Upsilon_{\rho\sigma}(\boldsymbol{v})\big]\sqrt{a}\, d\boldsymbol{x},$$

$$\mathcal{B}_1^K(p;\boldsymbol{v}) = -\alpha \int_\Omega p \operatorname{div} \boldsymbol{v} \sqrt{a}\, d\boldsymbol{X} + \alpha \int_\Omega p \operatorname{div} \big(z(\partial_\alpha \boldsymbol{v} \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)\sqrt{a}\, d\boldsymbol{X},$$

$$\mathcal{L}_1^K(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \big(\boldsymbol{v} - z(\partial_\alpha \boldsymbol{v} \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)\sqrt{a}\, d\boldsymbol{X},$$

$$\mathcal{A}_2^K(p;q) = c_0 \int_\Omega p' q \sqrt{a}\, d\boldsymbol{X} + \frac{\kappa}{\eta}\int_\Omega \nabla p \cdot \nabla q \sqrt{a}\, d\boldsymbol{X},$$

$$\mathcal{B}_2^K(\boldsymbol{u};q) = -\alpha \int_\Omega (\operatorname{div} \boldsymbol{u}') q \sqrt{a}\, d\boldsymbol{X} + \alpha \int_\Omega \operatorname{div} \big(z(\partial_\alpha \boldsymbol{u}' \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha\big) q \sqrt{a}\, d\boldsymbol{X} \text{ and}$$

$$\mathcal{L}_2^K(q) = \int_\Omega g q \sqrt{a}\, d\boldsymbol{X}.$$

In the process of deriving the weak form, we face difficulty of writing the poroelastic structure model in term of shell symbols (metric tensor $\gamma_{\alpha\beta}$, curvature tensor $\Upsilon_{\alpha\beta}$). In order to solving this problem, we use the structure displacement $\boldsymbol{U}$ belonging to the space of Kirchhoff-Love displacements $\mathcal{V}_{KL} = \{\boldsymbol{V} \in H^1(\Omega, \mathbb{R}^3);\ \varepsilon_{i3}(\boldsymbol{V}) = 0\}$.
Following, we introduce the theorem of the well-posedness,

Theorem

Let $p_0 \in H^1(\Omega)$, $\boldsymbol{f} \in H^1(0,T;L^2(\Omega,\mathbb{R}^3))$ and $g \in L^2(\Omega\times]0,T[)$. Then the problem (8) has a unique solution. The pressure p belongs to $L^\infty\big(0,T,H^1(\Omega)\big) \cap H^1\big(0,T;L^2(\Omega)\big)$.

We are not able to prove the existence and uniqueness of the displacement $\boldsymbol{U}$ and the pressure p at the same time. Therefore, we firstly turn out the well-posedness of $\boldsymbol{U}$ in the weak form of constitutive equation (7a) by Banach-Nečas-Babuška theorem with a given p. Then proving the well-posedness of p in the weak form of mass conservation equation (7b) by making use of the semi-discrete Galerkin method and the theory of initial value problem for linear systems.

In Chapter 2, we implement analogous to Chapter 1 but with another two-dimensional linear theory shell theory, Naghdi. Moreover, we obtain the strong formulation of Naghdi-Biot coupled model which we use for establishing the fluid-structure interaction between incompressible flow and poroelastic shell structure in Chapter 3. More precisely, we derive the weak coupled formulation of shell model of Naghdi type and Biot model:

Theorem

If $\boldsymbol{U}$ and p are a solution of the strong formulation (7), such that $\boldsymbol{U} = \boldsymbol{u} + z\boldsymbol{r}$, a Reissner-Mindlin displacement, then $(\boldsymbol{u}, \boldsymbol{r})$ and p belong, respectively, to $\mathcal{V}_N$ and $\mathcal{W}_N$ solving the following weak formulation:

$$\mathcal{A}_1^N\big((\boldsymbol{u},\boldsymbol{r});(\boldsymbol{v},\boldsymbol{s})\big) + \mathcal{B}_1^N\big(p;(\boldsymbol{v},\boldsymbol{s})\big) = \mathcal{L}_1^N(\boldsymbol{v}) \quad \forall(\boldsymbol{v},\boldsymbol{s}) \in \mathcal{X}_N, \tag{9a}$$

$$\mathcal{A}_2^N(p;q) + \mathcal{B}_2^N\big((\boldsymbol{u},\boldsymbol{r});q\big) = \mathcal{L}_2^N(q) \quad \forall q \in H_0^1(\Omega), \tag{9b}$$

$$p(0) = p_0 \text{ in } \Omega, \tag{9c}$$

where

$$\mathcal{A}_1^N\big((\boldsymbol{u},\boldsymbol{r});(\boldsymbol{v},\boldsymbol{s})\big) = \int_\omega e\, a^{\alpha\beta\rho\sigma}\left[\gamma_{\rho\sigma}(\boldsymbol{u})\gamma_{\alpha\beta}(\boldsymbol{v}) + \frac{e^2}{12}\chi_{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s})\right]\sqrt{a}\; d\boldsymbol{x}$$
$$+ 4\mu\int_\omega e\, a^{\alpha\beta}\delta_{\alpha 3}(\boldsymbol{u},\boldsymbol{r})\delta_{\beta 3}(\boldsymbol{v},\boldsymbol{s})\sqrt{a}\;\; d\boldsymbol{x},$$

$$\mathcal{B}_1^N\big(p;(\boldsymbol{v},\boldsymbol{s})\big) = -\alpha\int_\Omega p\mathrm{div}\boldsymbol{v}\sqrt{a}\; d\boldsymbol{X} - \alpha\int_\Omega p\mathrm{div}(z\boldsymbol{s})\;\sqrt{a}\;\; d\boldsymbol{X},$$

$$\mathcal{L}_1^N(\boldsymbol{v},\boldsymbol{s}) = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}+z\boldsymbol{s})\;\sqrt{a}\; d\boldsymbol{X},\;\; \mathcal{A}_2^N(p;q) = c_0\int_\Omega p' q\sqrt{a}\, d\boldsymbol{X} + \frac{\kappa}{\eta}\int_\Omega \nabla p\cdot\nabla q\;\sqrt{a}\, d\boldsymbol{X},$$

$$\mathcal{B}_2^N\big((\boldsymbol{u},\boldsymbol{r});q\big) = -\alpha\int_\Omega (\mathrm{div}\boldsymbol{u}')q\;\sqrt{a}\, d\boldsymbol{X} - \alpha\int_\Omega \mathrm{div}(z\boldsymbol{r}')q\;\sqrt{a}\, d\boldsymbol{X} \text{ and}$$

$$\mathcal{L}_2^N(q) = \int_\omega gq\sqrt{a}\, d\boldsymbol{X}.$$

In order to cracking the trouble of deriving the weak form, the Reissner-Mindlin displacements space $\mathcal{V}_{RM} = \big\{\boldsymbol{U} \in H^1(\Omega;\mathbb{R}^3), \varepsilon_{33}(\boldsymbol{U}) = 0\big\}$ is chosen.
We continuously prove the existence and uniqueness of the equation system (9) in the following theorem

Theorem

Let $p_0 \in H^1(\Omega)$, $\boldsymbol{f} \in H^1(0,T;L^2(\Omega,\mathbb{R}^3))$ and $g \in L^2(\Omega\times]0,T[)$. Then the problem (9) is well posed and its solution p belongs to $L^\infty\big(0,T,H^1(\Omega)\big) \cap H^1\big(0,T;L^2(\Omega)\big)$.

We firstly establish the well-posedness of $\boldsymbol{U}$ in the weak form of constitutive equation (7a) by Banach-Nečas-Babuška theorem with a given p as well (see Lemma 30). The well-posedness of p

in the weak form of mass conservation equation (7b) is also obtained by making use of the semi-discrete Galerkin method and the theory of initial value problem for linear systems. At last, we proceed the strong formulation of Naghdi-Biot coupled model by using the contravariant components of the stress resultant $n^{\rho\sigma}$, of the stress couple $m^{\rho\sigma}$ and of the transverse shear force t^{β}. We state this result in the following theorem,

Theorem

Let $(\boldsymbol{f}, g)$ in $L^2\big(0,T;L^2(\Omega,\mathbb{R}^3)\big)\times L^2(\Omega\times]0,T[)$, the system of partial differential equations of the Biot-Naghdi shell model is

$$\text{Let } (\boldsymbol{f}, g) \text{ in } L^2\big(0,T;L^2(\Omega,\mathbb{R}^3)\big)\times L^2(\Omega\times]0,T[),\ \text{we have a.e. in } \Omega\times(0,T),$$

$$\begin{cases} -\partial_\rho\Big(\big[n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3\big]\sqrt{a}\Big) + \alpha e\nabla\left(p\sqrt{a}\right) & = e\boldsymbol{f}\sqrt{a}, \\ \big[-m^{\sigma\rho}_{|\sigma}(\boldsymbol{u},\boldsymbol{r}) + t^\rho(\boldsymbol{u},\boldsymbol{r})\big]\boldsymbol{a}_\rho\sqrt{a} + \alpha ez\nabla(p\sqrt{a}) & = ze\boldsymbol{f}\sqrt{a}, \\ c_0p'\sqrt{a} + \alpha\,\mathrm{div}(\boldsymbol{u}' + z\boldsymbol{r}')\sqrt{a} - \dfrac{\kappa}{\eta}\mathrm{div}(\nabla p\sqrt{a}) & = g\sqrt{a}, \end{cases}$$

with the boundary conditions

$$\begin{cases} \boldsymbol{u} = \boldsymbol{r} & = \boldsymbol{0} \text{ on } \Gamma_0, \\ p & = 0 \text{ on } \Gamma_0, \\ \Big(\big[n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3\big]\sqrt{a}\Big)n_\rho - \alpha ep\sqrt{a}\,\boldsymbol{n} & = \boldsymbol{0} \text{ on } \Gamma_1, \\ m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_\rho\sqrt{a}\,n_\sigma - \alpha ezp\sqrt{a}\,\boldsymbol{n} & = \boldsymbol{0} \text{ on } \Gamma_1, \\ \nabla p\cdot\boldsymbol{n}\sqrt{a} & = 0 \text{ on } \Gamma_1. \end{cases}$$

We also derive the Biot-Naghdi shell model in the case where $\partial\omega = \gamma_0$,

Theorem

Assume that $\partial w = \gamma_0$, so that

$$\mathcal{X}_N = H^1_0(\omega;\mathbb{R}^3)\times H^1_0(\omega)^2 \ \text{ and } \ \mathcal{W}_N = L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega)).$$

If the solution$(u,r)\in\mathcal{V}_N$ and $p\in\mathcal{W}_N$ of the corresponding problem (2.22) of Theorem 9 is smooth enough, it also satisfies the boundary value problem:

$$\begin{aligned} -\partial_\rho((n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3)\sqrt{a}) + \alpha\nabla(p\sqrt{a}) &= \boldsymbol{f}\sqrt{a},\ \text{ in } \Omega\times]0,T[\\ (-m^{\sigma\rho}_{|\sigma}(\boldsymbol{u},\boldsymbol{r}) + t^\rho(\boldsymbol{u},\boldsymbol{r}))\boldsymbol{a}_\rho\sqrt{a} + \alpha z\nabla(p\sqrt{a}) &= z\boldsymbol{f}\sqrt{a},\ \text{ in } \Omega\times]0,T[\\ \partial_t(c_0p + \alpha\mathrm{div}(\boldsymbol{u} + z\boldsymbol{r}))\sqrt{a} - \frac{\kappa}{\eta}\mathrm{div}(\nabla p\sqrt{a}) &= g\sqrt{a},\ \text{ in } \Omega\times]0,T[\\ \boldsymbol{u} = \boldsymbol{r} &= \boldsymbol{0} \text{ on } \gamma_0, \\ p &= 0 \text{ on } \Gamma_0, \end{aligned} \tag{10}$$

In Chapter 3, we carry out the fluid-structure interaction between incompressible flow and poroelastic shell structure. We use the incompressible Stokes equations for the free fluid and the Biot-Naghdi model, derived in Theorem 11, for the poroelastic shell structure. It is equivalent to proving the following theorem

Theorem

For $\boldsymbol{f}_f \in L^\infty\big(0,T;(\mathcal{V}_f^h)'\big)$, $\boldsymbol{f}_p \in L^\infty\big(0,T;(\mathcal{X}_p^h)'\big)$, $g \in L^\infty\big(0,T;(\mathcal{W}_p^h)'\big)$ and $p_{p,h}(0) \in \mathcal{W}_p^h$, there exists a unique solution $(\boldsymbol{u}_{f,h}, \boldsymbol{\eta}_{p,h}, \boldsymbol{r}_{p,h}, \boldsymbol{u}_{p,h}, p_{f,h}, p_{p,h}, \lambda_h)$ in $L^\infty\big(0,T;\mathcal{V}_f^h\big) \times W^{1,\infty}\big(0,T;\mathcal{X}_p^h\big) \times L^\infty\big(0,T;\mathcal{S}_p^h\big) \times L^\infty\big(0,T;\mathcal{V}_p^h\big) \times L^\infty\big(0,T;\mathcal{W}_f^h\big) \times W^{1,\infty}\big(0,T;\mathcal{W}_p^h\big) \times L^\infty\big(0,T;\Lambda^h\big)$ of the weak formulation (3.29).

In process of proving the well-posedness of this model (see Theorem 13), we have trouble imposing the conditions on the interface which are mass conservation, balance of stress and the Beavers-Joseph-Saffman conditions. Therefore, the Lagrange multiplier method is employed to impose weakly this condition. We assume that the boundaries and the interface between the fluid and the poroelastic material are fixed. The proof proceeds by constructing a semi-discrete Galerkin approximations, deriving the discrete inf-sup condition and adopting the theory of differential-algebraic equations (DAEs)[26].

Lastly, in Chapter 4, we derive the validation for the Biot poroelasticity systems by using the numerical scheme proposed by Chaabane and Rivière [37](§2). Then, we employ this scheme to simulate the Biot-Naghdi poroelastic shell model on two different domain, hyperbolic paraboloid shell and plane-cylinder $W^{2,\infty}$ shell. Our results are implemented in FreeFem++, a high level, free software, finite element package (https://freefem.org/).

Notations, surfaces geometry and definitions

Geometry of the shell midsurface

Greek indices take their values in the set $\{1, 2\}$ and Latin indices take their values in the set $\{1, 2, 3\}$. Unless otherwise specified, the convention on repeated indices and exponents is used. Let (e_1, e_2, e_3) be the canonical orthonormal basis of the Euclidean space $\mathbb{R}^3$. We note $u \cdot v$ is the inner product of $\mathbb{R}^3$, $|u| = \sqrt{u \cdot u}$ is the associated Euclidean norm and $u \wedge v$ is the vector product of u and v.
Let ω be a domain of $\mathbb{R}^2$. We consider a shell whose midsurface is given by $S = \varphi(\overline{\omega})$, where $\varphi \in W^{2,\infty}(\omega; \mathbb{R}^3)$ one-to-one mapping such that the two vectors

$$a_\alpha(x) = \partial_\alpha \varphi(x) = \frac{\partial \varphi}{\partial x_\alpha}(x),$$

are linearly independent at all $x \in \overline{\omega}$. We define the normal unit vector

$$a_3(x) = \frac{a_1(x) \wedge a_2(x)}{|a_1(x) \wedge a_2(x)|},\ x \in \bar{\omega}$$

on the midsurface at point $\varphi(x)$. Then the vectors $a_\alpha(x)$ which form the covariant basis of the tangent plane to S at $\varphi(x)$ together with the vector $a_3(x)$ (which is normal to S and has Euclidean norm one) define the covariant basis at $\varphi(x)$.

The contravariant basis a^j is defined by the relations

$$a_i(x) \cdot a^j(x) = \delta_i^j,$$

where δ_i^j is the Kronecker symbol. In particular, $a^3(x) = a_3(x)$. Note that all these vectors are of class $W^{1,\infty}$.

We now define the first and second fundamental forms or the metric and curvature tensors of the surface by their covariant components

$$\begin{cases} a_{\alpha\beta}(x) & = a_\alpha(x) \cdot a_\beta(x), \\ b_{\alpha\beta}(x) & = a_3(x) \cdot \partial_\beta a_\alpha(x) = -a_\alpha(x) \cdot \partial_\beta a_3(x). \end{cases}$$

The contravariant components of the metric tensor are given by:

$$a^{\alpha\beta}(x) = a^{\alpha}(x) \cdot a^{\beta}(x).$$

The mixed components of the curvature tensor of the surface S are defined by

$$b_{\alpha}^{\beta}(x) = a^{\beta\sigma}(x) b_{\alpha\sigma}(x).$$

The derivatives of the vectors of the covariant and contravariant bases are given by the formulas of Gauss:

$$\partial_{\alpha} a_{\beta}(x) = \Gamma_{\alpha\beta}^{\sigma}(x) a_{\sigma}(x) + b_{\alpha\beta}(x) a_3(x) \quad \text{and} \quad \partial_{\alpha} a^{\beta}(x) = -\Gamma_{\alpha\sigma}^{\beta}(x) a^{\sigma}(x) + b_{\alpha}^{\beta}(x) a_3(x),$$

and Weingarten:

$$\partial_{\alpha} a_3(x) = \partial_{\alpha} a^3(x) = -b_{\alpha\beta}(x) a^{\beta}(x) = -b_{\alpha}^{\sigma}(x) a_{\sigma}(x),$$

where $\Gamma_{\alpha\beta}^{\rho}$ are the Christofell symbols of the surface, given by:

$$\Gamma_{\alpha\beta}^{\rho}(x) = \Gamma_{\beta\alpha}^{\rho}(x) = a^{\rho}(x) \cdot \partial_{\beta} a_{\alpha}(x) = -\partial_{\beta} a^{\rho}(x) \cdot a_{\alpha}(x).$$

The relation between covariant and contravariant bases is

$$\begin{cases} a_{\alpha}(x) = a_{\alpha\beta}(x) a^{\beta}(x) \\ a^{\alpha}(x) = a^{\alpha\beta}(x) a_{\beta}(x). \end{cases}$$

The matrix $\big(a^{\alpha\beta}\big)$ is the inverse matrix of $\big(a_{\alpha\beta}\big)$. We note that the determinant

$$a(x) = det\big(a_{\alpha\beta}(x)\big) = a_{11}(x) a_{22}(x) - (a_{12}(x))^2 = |a_1(x) \wedge a_2(x)|^2, \tag{1}$$

is strictly positive on $\overline{\omega}$ and we recall that $\varphi \in W^{2,\infty}(\omega)$ and $\sqrt{a(x)} \in W^{1,\infty}(\omega)$, then there exist two constants M and δ such that

$$M \geq a(x) \geq \delta > 0 \;\; \forall x \in \overline{\omega}. \tag{2}$$

The element of area dS is given by

$$dS = \sqrt{a}\, dx.$$

Note that the element of volume dV is approximately given by

$$dV = \sqrt{a(x)}\, dxdz.$$

Let there be given two vectors field $u \in H^1(\omega, \mathbb{R}^3)$ and $r \in H^1(\omega, \mathbb{R}^3)$ such that $r \cdot a_3 = 0$. Of course, a vector can be written in terms of components relative to a local or Cartesian basis.

Hence,

$$u(x) = u_i(x) a^i(x) = u^i(x) a_i(x) = u_i^c(x) e_i,$$

where

$$u_i(x) = u(x) \cdot a_i(x), u^i(x) = u(x) \cdot a^i(x) \text{ and } u_i^c(x) = u(x) \cdot e_i,$$

are respectively the covariant, contravariant and Cartesian components of u. In the same way we write:

$$r(x) = r_\alpha(x)a^\alpha(x) = r^\alpha(x)a_\alpha(x) = r_i^c(x)e_i,$$

where

$$r_\alpha(x) = r(x) \cdot a_\alpha(x), \ r^\alpha(x) = r(x) \cdot a^\alpha(x) \text{ and } r_i^c(x) = r(x) \cdot e_i.$$

are respectively the covariant, contravariant and Cartesian components of r.

Finally, for $u_i \in H^1(\omega)$ and $r_\alpha \in H^1(\omega)$ one can check that $u_i a^i \in H^1(\omega; \mathbb{R}^3)$ and $r_\alpha a^\alpha \in H^1(\omega; \mathbb{R}^3)$ and partial derivatives $\partial_\alpha(u_i a^i) \in L^2(\omega; \mathbb{R}^3)$ and $\partial_\alpha(r_\beta a^\beta) \in L^2(\omega; \mathbb{R}^3)$ are given by:

$$\begin{aligned} \partial_\alpha(u_i(x)a^i(x)) &= (\partial_\alpha u_\beta(x) - \Gamma^\sigma_{\alpha\beta}(x)a_\sigma(x) - b_{\alpha\beta}(x)u_3(x))a^\beta(x) + (\partial_\alpha u_3(x) + b^\beta_\alpha(x)u_\beta(x))a^3(x) \\ &= (u_{\beta|\alpha}(x) - b_{\alpha\beta}(x)u_3(x))a^\beta(x) + (u_{3|\alpha}(x) + b^\beta_\alpha(x)u_\beta(x))a^3(x), \end{aligned}$$

and

$$\begin{aligned} \partial_\alpha(r_\beta(x)a^\beta(x)) &= (\partial_\alpha r_\beta(x) - \Gamma^\sigma_{\alpha\beta}(x)a_\sigma(x))a^\beta(x) + b^\beta_\alpha(x)r_\beta(x))a^3(x) \\ &= r_{\beta|\alpha}(x)a^\beta(x) + b^\beta_\alpha(x)r_\beta(x))a^3(x), \end{aligned}$$

where

$$u_{\alpha|\beta}(x) = \partial_\beta u_\alpha(x) - \Gamma^\sigma_{\alpha\beta}(x)u_\sigma(x), \ u_{3|\alpha}(x) = \partial_\alpha u_3 \text{ and } r_{\alpha|\beta}(x) = \partial_\beta r_\alpha(x) - \Gamma^\sigma_{\alpha\beta}(x)r_\sigma(x).$$

denote the first-order covariant derivatives of the vectors field $u_i a^i$ and $r_\alpha a^\alpha$.

Remark 1

Whenever no confusion should arise, we henceforth drop the explicit dependence on a particular point. For instance, the relation "$\partial_\alpha a_\beta = \Gamma^\sigma_{\alpha\beta}a_\sigma + b_{\alpha\beta}a_3$" means "$\partial_\alpha a_\beta(x) = \Gamma^\sigma_{\alpha\beta}(x)a_\sigma(x) + b_{\alpha\beta}(x)a_3(x)$" for all $x \in \bar{\omega}$; "$\partial_\alpha a^3$ is in the tangent plane" means "$\partial_\alpha a^3(x)$ is in the tangent plane to S at $\varphi(x)$ for all $x \in \bar{\omega}$"; etc.

Lebesgue and Lipschitz spaces, Sobolev spaces

Lebesgue and Lipschitz Spaces

Let Ω be a domain in $\mathbb{R}^n$ with boundary Γ. Let $\mathbb{M}(\Omega)$ be the space of scalar-valued functions on Ω that are Lebesgue-measurable. In particular, $\mathbb{M}(\Omega)$ contains functions that are piecewise continuous and, more generally, all the functions that are integrable in the Riemann sense. All the functions used in this thesis are measurable.

Definition 2

For $1 \leq q \leq \infty$, let

$$L^p(\Omega) = \{f \in \mathbb{M}(\Omega);\ \| f \|_{0,p,\Omega} < \infty\},$$

where

$$\| f \|_{0,p,\Omega} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p} \qquad \text{for } 1 \leq p < \infty,$$

and

$$\| f \|_{0,\infty,\Omega} = \underset{x \in \Omega}{\text{ess sup}} \big| f(x) \big| = \inf\{M \geq 0; \big| f(x) \big| \leq M \text{ a.e. on } \Omega\}.$$

In this thesis, we employ the notation $\| f \|_{L^p(\Omega)} = \| f \|_{0,p,\Omega}$. For $1 \leq p \leq \infty$, we denote by p' its *conjugate*, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$ with the convention that $p' = 1$ if $p = \infty$ and $p' = \infty$ if $p = 1$.

Theorem 1

(Hölder Inequality) Let $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \leq p \leq \infty$ and $fg \in L^1(\Omega)$ and

$$\int_\Omega |fg| \leq \| f \|_{L^p(\Omega)} \| g \|_{L^{p'}(\Omega)} .$$

The following corollary is a consequence of Hölder's inequality:

Corollary 3

(Interpolation inequality) Let $1 \leq p \leq q \leq \infty$ and $0 \leq \alpha \leq 1$. Let r be such that $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$. Then

$$\forall f \in L^p(\Omega) \cap L^q(\Omega), \qquad \| f \|_{L^r(\Omega)} \leq \| f \|^{\alpha}_{L^p(\Omega)} \| f \|^{1-\alpha}_{L^q(\Omega)} .$$

Theorem 2

(Riesz Representation Theorem). Let $1 \leq p < \infty$, the dual space of $L^p(\Omega)$ can be identified with $L^{p'}(\Omega)$.

From Theorem 2, we see that $L^p(\Omega)$ is reflexive if $1 < p < \infty$. However, $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive.

Theorem 3

$L^2(\Omega)$ is a Hilbert space when equipped with the scalar product

$$(f,g)_{L^2(\Omega)} = \int_\Omega fg. \tag{3}$$

Hence, we denote by $(\cdot;\cdot)_{0,\Omega}$ the scalar product in $L^2(\Omega)$ and when no confusion is possible, we denote the L^2-norm by

$$\| f \|_{0,\Omega} = \left(\int_\Omega |f|^2 \right)^{\frac{1}{2}}.$$

In $L^2(\Omega)$, the Hölder inequality becomes the *Cauchy-Schwarz inequality*:

$$\forall f, g \in L^2(\Omega), \quad (f,g)_{0,\Omega} \leq \|f\|_{0,\Omega}\|g\|_{0,\Omega}.$$

Theorem 4

(Young Inequality). Let $p, q \in \mathbb{R}^*$ be strictly positive real numbers such that:

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then for any non-negative numbers a and b,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

The Banach-Nečas-Babuška (BNB) theorem

Consider the following problem

$$\begin{cases} \text{Seek } u \in W \text{ such that} \\ a(u,v) = f(v), \quad \forall v \in V, \end{cases} \tag{B}$$

where:

(i) W and V are vector spaces equipped with norms denoted by $\|\cdot\|_W$ and $\|\cdot\|_V$, respectively. In many applications, W and V are Hilbert spaces, but a more general case where V is a reflexive Banach space and W a Banach space can be considered. Unless stated otherwise, we henceforth assume that W and V are Banach spaces and that V is reflexive. W is called the solution space, and V is called the test space.

(ii) a is a continuous bilinear form on $W \times V$, i.e., $a \in \mathcal{L}(W \times V; \mathbb{R})$; henceforth, we shall also say that a is bounded on $W \times V$.

(iii) f is a continuous linear form on V, i.e., $f \in V' = \mathcal{L}(V; \mathbb{R})$. To simplify the notation, we write $f(v)$ instead of $\langle f, v \rangle_{V',V}$.

Theorem 5

(Banach-Nečas-Babuška) Let W be a Banach space and let V be a reflexive Banach space. Let $a \in \mathcal{L}(W \times V; \mathbb{R})$ and $f \in V'$. Then, problem (B) is well-posed if and only if:

(i)

$$\exists \alpha > 0, \inf_{w \in W} \sup_{v \in V} \frac{a(w,v)}{\| w \|_W \| v \|_V} \geq \alpha \quad \text{(inf-sup condition)}.$$

(ii)

$$\forall v \in V, \big(\forall w \in W, a(w,v) = 0\big) \Longrightarrow \big(v = 0\big).$$

Proof. See (Theorem 2.6, [43]).

Theorem 6

(Lax-Milgram) Let ϕ be a bounded coercive bilinear form on a Hibert space H. For every bounded linear functional f on H, there exists a unique x_f in H such that

$$\phi(x, x_f) = f(x), \tag{4}$$

for all $x \in H$.

Sobolev Spaces

We define $\mathcal{D}(\Omega)$ to be the linear space of infinitely differentiable functions, with compact support on Ω. Then, we set

$$\mathcal{D}(\overline{\Omega}) = \{\phi|_\Omega; \phi \in \mathcal{D}(\mathbb{R}^n)\},$$

or equivalently, if $\mathcal{O}$ denotes any open subset of $\mathbb{R}^n$ such that $\overline{\Omega} \subset \mathcal{O}$,

$$\mathcal{D}(\overline{\Omega}) = \{\phi|_\Omega; \phi \in \mathcal{D}(\mathcal{O})\}.$$

Now, let $\mathcal{D}'(\overline{\Omega})$ denotes the dual space of $\mathcal{D}(\overline{\Omega})$, often called the space of distributions on Ω. We denote by $(\cdot,\cdot)_\Omega$ the duality pairing between $\mathcal{D}'(\overline{\Omega})$ and $\mathcal{D}(\overline{\Omega})$, we remark that when f is a locally integrable function, then f can be identified with a distribution by

$$(f, \phi)_\Omega = \int_\Omega f(x)\phi(x)dx \quad \forall \phi \in \mathcal{D}(\Omega).$$

In other words, $(\cdot,\cdot)_\Omega$ is an extension of the scalar product of $L^2(\Omega)$. Now, we can define the derivatives of distributions. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and set

$$|\alpha| = \sum_{i=1}^{n} \alpha_i,$$

for u in $\mathcal{D}'(\Omega)$, we define $\partial^\alpha u$ in $\mathcal{D}'(\Omega)$ by:

$$(\partial^\alpha u, \phi)_\Omega = (-1)^{|\alpha|}(u, \partial^\alpha \phi), \quad \forall \phi \in \mathcal{D}(\Omega),$$

when u is α times differentiable, $\partial^\alpha u$ coincides with the usual notion of derivative:

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

For each integer $m \geq 0$ and real p with $1 \leq p \leq \infty$, we define the Sobolev space:

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega); \partial^\alpha v \in L^p(\Omega) \ \ \forall |\alpha| \leq m\},$$

which is a Banach space for the norm

$$\| \, u \, \|_{m,p,\Omega} = \|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \int_\Omega \left|\partial^\alpha u(x)\right|^p dx \right)^{1/p} \qquad \text{for } p < \infty,$$

or

$$\| \, u \, \|_{m,\infty,\Omega} = \|u\|_{W^{m,\infty}(\Omega)} = \max_{|\alpha| \leq m} \left(\operatorname*{ess\,sup}_{x \in \Omega} \left|\partial^\alpha u(x)\right| \right) \qquad \text{for } p = \infty.$$

The space $W^{m,p}(\Omega)$ is separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$. We also provide $W^{m,p}(\Omega)$ with the following seminorm

$$|u|_{m,p,\Omega} = \left(\sum_{|\alpha| = m} \int_\Omega \left|\partial^\alpha u(x)\right|^p dx \right)^{1/p} \qquad \text{for } p < \infty.$$

When $p = 2$, $W^{m,2}(\Omega)$ is usually denoted by $H^m(\Omega)$, and if there is no ambiguity, we drop the subscript $p = 2$ when referring to its norm and seminorm. The space $H^m(\Omega)$ is a Hilbert space with the inner product:

$$(u, v)_{m,\Omega} = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u(x) \partial^\alpha v(x) dx.$$

Sometimes we also make use of the following notation

$$\|u\|_{m,\Omega} = \|u\|_{H^m(\Omega)} = \|u\|_{W^{m,2}(\Omega)} = \left(\sum_{|\alpha| \leq m} \int_\Omega \left|\partial^\alpha u(x)\right|^2 dx \right)^{1/2},$$

$$|u|_{m,\Omega} = |u|_{m,2,\Omega} = \left(\sum_{|\alpha| = m} \int_\Omega \left|\partial^\alpha u(x)\right|^2 dx \right)^{1/2}.$$

Parallel to the Sobolev spaces, we recall the familiar definition of $\mathcal{C}^m$-functions:

- $\mathcal{C}^0(\Omega)$ denotes the space of continuous functions defined in Ω,
- $\mathcal{C}^m(\Omega) = \{u \in \mathcal{C}^0(\Omega); \partial^\alpha u \in \mathcal{C}^0(\Omega) \quad \forall |\alpha| \leq m\}$.

We also recall the Sobolev embeddings :

$$W^{2,\infty}(\Omega) \hookrightarrow \mathcal{C}^1(\overline{\Omega}),$$

and

$$W^{1,\infty}(\Omega) \hookrightarrow \mathcal{C}^0(\overline{\Omega}).$$

(See Theorem 4.12 in [1] for more details).

Following the Sobolev space notations, we introduce the Korn's inequality: Let $\boldsymbol{v}$ in $H^1(\Omega)^d$, there is a constant $C \geq 0$ depending only on Ω and $\partial\Omega$, known as the Korn constant of Ω, such that,

$$(\textit{Korn's inequality}) \qquad \|\boldsymbol{v}\|_{H^1(\Omega)} \leq C(\Omega, \partial\Omega)\big(\|\varepsilon(\boldsymbol{v})\|_{L^2(\Omega)} + \|\boldsymbol{v}\|_{L^2(\partial\Omega)}\big). \tag{5}$$

As usual, for handling time-dependent problem, it is convenient to consider functions defined on a time interval (a, b) with values in a functional space, say X. More precisely, let $\|\cdot\|_X$ denote the norm of X; then for any number s, $1 \leq s \leq \infty$, we define

$$L^s(a, b; X) = \left\{ f \text{ measurable in } (a, b) : \int_a^b \|f(t)\|_X^s dt < \infty \right\},$$

and its norm is given by:

$$\|f\| L^s(a, b; X) = \left(\int_a^b \|f(t)\|_X^s dt \right)^{1/s}.$$

The space $L^s(a, b; X)$ is a Banach space if X is a Banach space, and for $s = 2$, it is a Hibert space if X is a Hilbert space. We denote derivatives with respect to time with a prime and we define for instance

$$H^1(a, b; X) = \left\{ f \in L^2(a, b; X) : f' \in L^2(a, b; X) \right\}.$$

Chapter 1

Derivation and well-posedness for Biot-Koiter poroelastic shell model

Our purpose in this chapter is the derivation of a shell model of Koiter type coupled with the Biot model (see Theorem 7). We also prove the well-posedness of the resulting equations (see Theorem 8). We use here the linearly elastic thin shell with little regularity in which the midsurfaces have curvature discontinuity.

More precisely, in the process of deriving the weak form, we face difficulty of writing the poroelastic structure model in term of shell symbols (metric tensor $\gamma_{\alpha\beta}$, curvature tensor $\Upsilon_{\alpha\beta}$). In order to solving this problem, we use the structure displacement $\boldsymbol{U}$ belonging to the space of Kirchhoff-Love displacements $\mathcal{V}_{KL} = \{\boldsymbol{V} \in H^1(\Omega, \mathbb{R}^3);\ \varepsilon_{i3}(\boldsymbol{V}) = 0\}$. For the well-posedness, we are not able to prove the existence and uniqueness of the displacement $\boldsymbol{U}$ and the pressure p at the same time. Therefore, we firstly turn out the well-posedness of $\boldsymbol{U}$ in the weak form of constitutive equation (1.37) by Banach-Nečas-Babuška theorem with a given p (see Lemma 15). Then proving the well-posedness of p in the weak form of mass conservation equation in (1.37) by making use of the semi-discrete Galerkin method and the theory of initial value problem for liner systems (see Lemma 17, 18 & 19). We begin with some definitions.

1.1 Definition of a shell

Let ω be a open, bounded and connected subset of $\mathbb{R}^2$ with Lipschitz-continuous boundary and being locally on one side of its boundary. Let S be a surface of $\mathbb{R}^3$ defined by $\overline{S} = \varphi(\overline{\omega})$, where $\varphi \in W^{2,\infty}(\omega; \mathbb{R}^3)$ is an injective mapping. A shell M of the midsurface S is given by

$$M = \left\{ \Phi(\boldsymbol{x}, z) = \varphi(\boldsymbol{x}) + z\boldsymbol{a}_3(\boldsymbol{x}), \boldsymbol{x} \in \overline{\omega} \text{ and } \frac{-e(\boldsymbol{x})}{2} \leq z \leq \frac{e(\boldsymbol{x})}{2} \right\}, \tag{1.1}$$

where $e \in L^\infty(\omega)$ be the thickness of the shell, which we assume to be such that $e(\boldsymbol{x}) \geq c > 0$ almost everywhere in ω (see Fig.1.1).

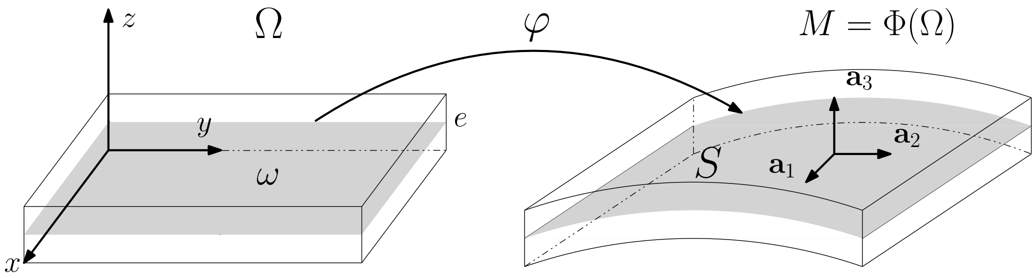


Figure 1.1: Parametrization of the shell

1.2 Koiter's approach

For obtaining his model, Koiter adapted two fundamental hypotheses of Kirchhoff-Love (see Ph. Ciarlet [35] for more details) that we recall below:

– **H1:** Any point on a normal to the midsurface remains on the normal to the deformed midsurface after the deformation has taken place, i.e.,

$$\boldsymbol{a}_3^* = \frac{\boldsymbol{a}_1^* \wedge \boldsymbol{a}_2^*}{|\boldsymbol{a}_1^* \wedge \boldsymbol{a}_2^*|}, \tag{1.2}$$

here $\boldsymbol{a}_\alpha^* = \boldsymbol{a}_\alpha + \partial_\alpha \boldsymbol{u}$ is the covariant basis of the deformed surface. We can check easily that, after linearization:

$$\boldsymbol{a}_3^*(x) = \boldsymbol{a}_3(x) - (\partial_\alpha \boldsymbol{u} \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha. \tag{1.3}$$

– **H2:** The distance between such a point and the midsurface remains constant.

Taking these two a priori assumptions into account, W.T. Koiter then shows that the displacement filed across the thickness of the shell can be completely determined from the sole knowledge of the displacement filed of the points on the midsurface S, and he identifies the two-dimensional problem, *i.e.*, posed over the two-dimensional set $\overline{\omega}$, that this displacement filed should satisfy. As in the two-dimensional theories encountered so far, the unknown is a vector field, now denoted $\boldsymbol{u} : \overline{\omega} \to \mathbb{R}^3$, whose components u_i, (resp. u_i^c) $: \overline{\omega} \to \mathbb{R}$ are the covariant (resp. Cartesian) components of the displacement field of the midsurface S. This means that $u_i(\boldsymbol{x})\boldsymbol{a}^i(\boldsymbol{x})$(resp. $u_i^c(\boldsymbol{x})\boldsymbol{e}^i$) is the displacement of the point $\varphi(\boldsymbol{x})$.

1.3 Functional spaces

Let us set the domain $\Omega = \omega\times]-e/2;e/2[$ and its boundary $\Gamma_0 = \partial\Omega = \gamma_0\times]-e/2;e/2[$, where $\gamma_0 = \partial\omega$. Let us now introduce the following functional spaces:

$$\mathcal{X}_K = \left\{ \boldsymbol{v} \in H^1(\omega, \mathbb{R}^3);\ (\partial_\alpha \boldsymbol{v} \cdot a_3)a^\alpha \in H^1(\omega, \mathbb{R}^3),\ \boldsymbol{v} = \partial_\alpha \boldsymbol{v} \cdot a_3 = 0 \text{ on } \gamma_0 \right\}, \tag{1.4}$$

$$\mathcal{V}_K = L^\infty\big(0,T;\mathcal{X}_K\big), \tag{1.5}$$

and

$$\mathcal{W}_K = \Big\{q \in L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H^1(\Omega)\big);\ q = 0 \text{ on } \Gamma_0\Big\}. \tag{1.6}$$

We equip the space $\mathcal{X}_K$ with the norm

$$\|v\|_{\mathcal{X}_K} = \Big(\|v\|^2_{H^1(\omega;\mathbb{R}^3)} + \sum_\alpha \|(\partial_\alpha v \cdot a_3)a^\alpha\|^2_{H^1(\omega,\mathbb{R}^3)}\Big)^{1/2}. \tag{1.7}$$

1.4 Kirchhoff-Love displacement

Let us introduce the Kirchhoff-Love space

$$\mathcal{V}_{KL} = \{V \in H^1(\Omega;\mathbb{R}^3);\ \varepsilon_{i3}(V) = 0\},$$

and characterize its elements. It is the object of the following lemma.

Lemma 4

A displacement field $\boldsymbol{V} \in H^1(\Omega;\mathbb{R}^3)$ is a Kirchhoff-Love displacement if and only if there exists $\boldsymbol{v} \in H^1(\omega,\mathbb{R}^3)$ such that

$$\boldsymbol{V}(x,z) = \boldsymbol{v}(x) - z\big(\partial_\alpha \boldsymbol{v}(x)\cdot a_3(x)\big)a^\alpha(x). \tag{1.8}$$

Proof. Firstly, we derive the "forward"proof. Let $\boldsymbol{V} \in \mathcal{V}_{KL}$, we have

$$0 = \varepsilon_{i3}(\boldsymbol{V}) = \frac{1}{2}(\partial_i\boldsymbol{V}\cdot g_3 + \partial_3\boldsymbol{V}\cdot g_i).$$

So one have

$$\begin{cases} \partial_3\boldsymbol{V}\cdot a_3 = 0, & (1.9)\\ \partial_\alpha\boldsymbol{V}\cdot a_3 + \partial_3\boldsymbol{V}\cdot g_\alpha = 0. & (1.10)\end{cases}$$

From (1.9), there exists $\boldsymbol{w} \in L^2(\Omega,\mathbb{R}^3)$ such that $\partial_3\boldsymbol{V} = \boldsymbol{w}\wedge a_3$. From that, we have

$$\int_0^z \partial_3\boldsymbol{V}(x,\tau)d\tau = \underbrace{\Big(\int_0^z \boldsymbol{w}(x,\tau)d\tau\Big)}_{\Theta(x,z)}\wedge a_3.$$

Hence,

$$\boldsymbol{V}(x,z) - \boldsymbol{V}(x,0) = \Theta(x,z)\wedge a_3, \text{ with } \Theta(x,0) = 0.$$

We can present $\Theta(x,z)$ as below

$$\Theta(x,z) = \underbrace{\Theta(x,0)}_{=0} + z\partial_3\Theta(x,0) + z^2\partial_3^2\Theta(x,0) + \cdots$$

Since z is belong to $\left]\frac{-e}{2};\frac{e}{2}\right[$ and e is the thickness of the shell, $z^n \approx 0$ with $n \geq 2$. Rewriting $\boldsymbol{V}$:

$$\boldsymbol{V}(x,z) = \boldsymbol{V}(x,0) + z\partial_3\Theta(x,0) \wedge a_3.$$

Let us denote $t(x,z) = z\partial_3\Theta(x,0) \in H^1(\Omega,\mathbb{R}^3)$ (note that $t(x,0)=0$) and $\boldsymbol{v}(x) = \boldsymbol{V}(x,0) \in H^1(\omega,\mathbb{R}^3)$. $\boldsymbol{V}(x,z)$ becomes

$$\boldsymbol{V}(x,z) = \boldsymbol{v}(x) + t \wedge a_3. \tag{1.11}$$

Replacing (1.11) into (1.10), we have

$$\begin{aligned} 0 &= \partial_\alpha \boldsymbol{V} \cdot a_3 + \partial_3 \boldsymbol{V} \cdot g_\alpha \\ &= \partial_\alpha \boldsymbol{V} \cdot a_3 + \underbrace{(\partial_\alpha t \wedge a_3) \cdot a_3}_{=0} + (t \wedge \partial_\alpha a_3) \cdot a_3 + (\partial_3 t \wedge a_3) \cdot g_\alpha \\ &= \partial_\alpha \boldsymbol{v} \cdot a_3 - (t \wedge a_3) \cdot \partial_\alpha a_3 + (\partial_3 t \wedge a_3) \cdot (a_\alpha + z\partial_\alpha a_3). \end{aligned} \tag{1.12}$$

Setting

$$s = \partial_\alpha \boldsymbol{v} \cdot a_3 a^\alpha \text{ and } \Theta = t \wedge a_3 \text{ where } \Theta = \Theta(x,z).$$

Hence, (1.12) becomes

$$s \cdot a_\alpha - \partial_\alpha a_3 \cdot \Theta + (a_\alpha + z\partial_\alpha a_3) \cdot \partial_3\Theta = 0.$$

Since $\partial_\alpha a_3 = -b_\alpha^\rho a_\rho$,

$$s \cdot a_\alpha + b_\alpha^\rho a_\rho \cdot \Theta + (a_\alpha - z\, b_\alpha^\rho a_\rho) \cdot \partial_3\Theta = 0,$$

and thanks to $a_\alpha = \delta_\alpha^\rho a_\rho$, we have

$$\delta_\alpha^\rho\, s \cdot a_\rho + b_\alpha^\rho a_\rho \cdot \Theta + (\delta_\alpha^\rho\, a_\rho - z\, b_\alpha^\rho a_\rho) \cdot \partial_3\Theta = 0.$$

We set $\mathrm{I} = \delta_\alpha^\rho$ and $\mathrm{II} = b_\alpha^\rho$, so

$$\mathrm{I}\, s \cdot a_\rho + \mathrm{II}\, a_\rho \cdot \Theta + (\mathrm{I}\, a_\rho - z\,\mathrm{II}\, a_\rho) \cdot \partial_3\Theta = 0.$$

It follows that

$$\left[\mathrm{I}\, r + \mathrm{II}\Theta + (\mathrm{I} - z\mathrm{II})\partial_3\Theta\right] \cdot a_\rho = 0. \tag{1.13}$$

Besides, since $s \cdot a_3 = 0$, $\Theta \cdot a_3 = 0$ and $\partial_3\Theta \cdot a_3 = (\partial_3 t \wedge a_3) \cdot a_3 = 0$, we obtain

$$\big[\mathrm{I}\, s + \mathrm{II}\Theta + (\mathrm{I} - z\mathrm{II})\partial_3\Theta\big] \cdot a_3 = 0. \tag{1.14}$$

From (1.13) and (1.14), one have

$$\begin{cases} \big[\mathrm{I}\, s + \mathrm{II}\Theta + (\mathrm{I} - z\mathrm{II})\partial_3\Theta\big] \cdot a_\rho = 0, \\ \big[\mathrm{I}\, s + \mathrm{II}\Theta + (\mathrm{I} - z\mathrm{II})\partial_3\Theta\big] \cdot a_3 = 0. \end{cases}$$

Therefore, we obtain that

$$\mathrm{I}\, s + \mathrm{II}\Theta + (\mathrm{I} - z\mathrm{II})\partial_3\Theta = 0.$$

We set $A = \mathrm{I} - z\mathrm{II}$ and $B = \mathrm{II}$, then we have

$$A\partial_3\Theta + B\Theta = -\mathrm{I}\, s. \tag{1.15}$$

Since $B = -\partial_3 A$, (1.15) is equivalent to

$$A\,\partial_3\Theta - \partial_3 A\,\Theta = -\mathrm{I}\, s.$$

Hence,

$$A^{-1}\,\partial_3\Theta - A^{-2}\partial_3 A\,\Theta = -A^{-2}\mathrm{I}\, s.$$

Finally, we have

$$\begin{cases} \partial_3(A^{-1}\Theta) = -A^{-2}\mathrm{I}\, s, \\ \Theta(x,0) = 0. \end{cases} \tag{E}$$

A particular solution of (E) is $\Theta(x,z) = -zs(x)$.
Indeed, replacing this solution in (E), we have

$$\begin{aligned} \partial_3(A^{-1}\Theta) &= \partial_3(-A^{-1}zs) \\ &= \partial_3(-A^{-1}z)s \\ &= (zA^{-2}\partial_3 A - A^{-1})s \\ &= A^{-2}(z\partial_3 A - A)s \\ &= A^{-2}(-z\mathrm{II} - \mathrm{I} + z\mathrm{II})s \\ &= (-A^{-2}\mathrm{I})s. \end{aligned}$$

Hence, we have the general solution of the homogeneous equation

$$\Theta(x,z) = -zs(x), \Theta \in H^1(\Omega, \mathbb{R}^3) \text{ with } s = \partial_\alpha \boldsymbol{v} \cdot a_3\, a^\alpha \in H^1(\omega, \mathbb{R}^3).$$

Consequently, we can write $\boldsymbol{V}(x,z) = \boldsymbol{v}(x) - z\partial_\alpha \boldsymbol{v} \cdot a_3\, a^\alpha$.

Now, we prove the reverse. Let us suppose now that there exist $\boldsymbol{v} \in H^1(\omega, \mathbb{R}^3)$ such that

$$\boldsymbol{V}(x,z) = \boldsymbol{v}(x) - z\,(\partial_\alpha \boldsymbol{v} \cdot a_3) a^\alpha,$$

is a Kirchhoff-Love displacement field and prove that $\varepsilon_{i3}(\boldsymbol{V}) = 0$. It is easy to check that:

$$\varepsilon_{33}(\boldsymbol{V}) \quad = \partial_3 \boldsymbol{V} \cdot g_3 = (\partial_\alpha v \cdot a_3) a^\alpha \cdot a_3 = 0,$$

and

$$\begin{aligned}
2\varepsilon_{\beta 3}(\boldsymbol{V}) &= \partial_\beta \boldsymbol{V} \cdot g_3 + \partial_3 \boldsymbol{V} \cdot g_\beta = \partial_\beta \boldsymbol{V} \cdot a_3 + \partial_3 \boldsymbol{V} \cdot g_\beta \\
&= \big(\partial_\beta \boldsymbol{v} - z\partial_\beta\big[(\partial_a \boldsymbol{v} \cdot a_3) a^\alpha\big]\big) \cdot a_3 - (\partial_\alpha \boldsymbol{v} \cdot a_3) a^\alpha \cdot (a_\beta + z\partial_\beta a_3) \\
&= \partial_\beta \boldsymbol{v} \cdot a_3 - (\partial_\alpha \boldsymbol{v} \cdot a_3) a^\alpha \cdot a_\beta - z(\partial_\alpha \boldsymbol{v} \cdot a_3)(\partial_\beta a^\alpha \cdot a_3 + a^\alpha \cdot \partial_\beta a_3) \\
&= \underbrace{\partial_\beta \boldsymbol{v} \cdot a_3 - (\partial_\alpha \boldsymbol{v} \cdot a_3)\delta_\alpha^\beta}_{=0} - z(\partial_\alpha \boldsymbol{v} \cdot a_3)\partial_\beta \underbrace{(a^\alpha \cdot a_3)}_{=0} = 0.
\end{aligned}$$

Therefore, $\varepsilon_{i3}(\boldsymbol{V}) = 0$. We proved Lemma 4. □

1.5 Definition of a deformed shell

After deformation, the normal unit vector $\boldsymbol{a}_3$ is deformed to a vector $\boldsymbol{a}_3^*$ which is orthonormal to the deformed surface (transverse shears are neglected). Then, the point $\Phi(\boldsymbol{x}, z)$ becomes

$$\Phi^*(\boldsymbol{x}, z) = \varphi^*(\boldsymbol{x}) + f(z)\boldsymbol{a}_3^*(\boldsymbol{x}), \tag{1.16}$$

where $\varphi^* = \varphi + \boldsymbol{u}$ the mapping defining the deformed midsurface, $\boldsymbol{u}$ being the displacement of point $\varphi(\boldsymbol{x})$ on ω. It should be mentioned that $f(z)$ is an arbitrary function of z. Thus, under the hypothesis **H1** and **H2**, the deformed shell will be defined by the chart

$$\Phi^*(x, z) = \varphi^*(x) + z(\boldsymbol{a}_3 - (\partial_\alpha \boldsymbol{u} \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha). \tag{1.17}$$

1.6 Displacement of a Koiter's shell

Under the hypothesis $\mathbf{H}_1$ and $\mathbf{H}_2$ of Kirchhoff-Love, the displacement of a point $\Phi(\boldsymbol{x}, z)$ is written by

$$\mathcal{U}\big(\Phi(\boldsymbol{x}, z)\big) = \boldsymbol{U}(\boldsymbol{x}, z) = \Phi^*(\boldsymbol{x}, z) - \Phi(\boldsymbol{x}, z).$$

This means that

$$\boldsymbol{U}(\boldsymbol{x}, z) = \boldsymbol{u}(\boldsymbol{x}) - z(\partial_\alpha \boldsymbol{u} \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha. \tag{1.18}$$

1.7 Linearized change of shell metric tensor

Let us first introduce the local bases at each point of the three-dimensional shell. The covariant basis is defined by

$$\begin{cases} \boldsymbol{g}_\alpha = \boldsymbol{g}_\alpha(\boldsymbol{x}, z) = \partial_\alpha \Phi(\boldsymbol{x}, z) = \boldsymbol{a}_\alpha(\boldsymbol{x}) + z\partial_\alpha \boldsymbol{a}_3(\boldsymbol{x}), \\ \boldsymbol{g}_3 = \boldsymbol{g}_3(\boldsymbol{x}, z) = \partial_3 \Phi(\boldsymbol{x}, z) = \boldsymbol{a}_3(\boldsymbol{x}). \end{cases}$$

The contravariant basis is defined by

$$\boldsymbol{g}_i \cdot \boldsymbol{g}^j = \delta^i_j.$$

It follows that

$$g^{\alpha 3} = g^{3\alpha} = 0 \text{ and } g^{33} = 1.$$

We recall that, given arbitrary displacement field V of a three-dimensional manifold $\Phi(\bar{\Omega})$ in $\mathbb{R}^3$, the covariant components $\varepsilon_{ij}(V)$ of the associated linearized change of metric tensor are defined by:

$$\varepsilon\big(\boldsymbol{V}\big) = \varepsilon_{ij}(V)\boldsymbol{g}^i \otimes \boldsymbol{g}^j \quad \text{with } \varepsilon_{ij}(V) = \frac{1}{2}\big(g_{ij}(V) - g_{ij}\big)^{lin}, \tag{1.19}$$

where $g_{ij}(V)$ and $g_{ij}(V)$, respectively, are the metric tensor in the configuration of reference and the deformed shell. The vectors of the deformed local basis are defined by:

$$\begin{cases} \boldsymbol{g}_\alpha(V) = \partial_\alpha \Phi^* = \boldsymbol{a}^*_\alpha - z(b^\rho_\alpha)\boldsymbol{a}^*_\rho, \\ \boldsymbol{g}_3(V) = \partial_3 \Phi^* = \boldsymbol{a}^*_3, \end{cases}$$

and an approximation of covariant components of metric tensor of the deformed shell is given by:

$$\begin{cases} g_{\alpha\beta}(V) = \boldsymbol{g}_\alpha(V) \cdot \boldsymbol{g}_\beta(V) = a^*_{\alpha\beta} - 2z b^*_{\alpha\beta} + z^2 c^*_{\alpha\beta}, \\ g_{\alpha 3}(V) = 0, \\ g_{33}(V) = 1, \end{cases}$$

where $a^*_{\alpha\beta}$, $b^*_{\alpha\beta}$ and $c^*_{\alpha\beta}$ point out the covariant components of the first, second and third fundamental forms of the deformed midsurface.

Since a surface also has a metric tensor, it is natural to likewise define the covariant components of the linearized change of metric tensor associated with any displacement field defined on it. Thus, for a given arbitrary displacement field $\boldsymbol{v}$ of the surface S the covariant components of the linearized change of metric tensor associated with this vector field is defined by:

$$\gamma_{\alpha\beta}(\boldsymbol{v}) = \frac{1}{2}(a_{\alpha\beta}(\boldsymbol{v}) - a_{\alpha\beta})^{lin} = \frac{1}{2}(\partial_\alpha \boldsymbol{v} \cdot \boldsymbol{a}_\beta + \partial_\beta \boldsymbol{v} \cdot \boldsymbol{a}_\alpha). \tag{1.20}$$

We also introduce the linearized change of curvature tensor on the shell midsurface by its covariant components:

$$\Upsilon_{\alpha\beta}(\boldsymbol{v}) = (b_{\alpha\beta}(\boldsymbol{v}) - b_{\alpha\beta})^{lin} = (\partial_{\alpha\beta} \boldsymbol{u} - \Gamma^\rho_{\alpha\beta} \partial_\rho \boldsymbol{v}) \cdot \boldsymbol{a}_3. \tag{1.21}$$

Remark 5

Note that all these quantities make sense for shell with C^1-regularity, and are easily expressed with the Cartesian coordinates of the displacement field and geometrical data. For instance, $\Upsilon_{\alpha\beta}(\boldsymbol{v})$ is a distribution of $H^{-1}(\omega)$ when $v \in H^1(\omega;\mathbb{R}^3)$ and the chart φ is in $W^{2,\infty}(\omega;\mathbb{R}^3)$.

Now, if the three-dimensional displacement V is of Kirchhoff-Love type:

$$\boldsymbol{V}(x,z) = \boldsymbol{v}(x) - z\Big(\partial_\alpha \boldsymbol{v}(x)\cdot a_3(x)\Big)a^\alpha(x),$$

then, by neglecting all the terms containing z^2, the covariant components of the associated linearized change of metric tensor (1.19) is given in terms of the change of metric and change curvature tensors on the midsurface S by:

$$\begin{cases} \varepsilon_{\alpha\beta}(\boldsymbol{V}) = \gamma_{\alpha\beta}(\boldsymbol{v}) - z\Upsilon_{\alpha\beta}(\boldsymbol{v}), \\ \varepsilon_{\alpha 3}(\boldsymbol{V}) = 0, \\ \varepsilon_{33}(\boldsymbol{V}) = 0. \end{cases} \tag{1.22}$$

Remark 6

When the normal strain $\varepsilon_{33}(\boldsymbol{V})$ and the shear strains $\varepsilon_{\alpha 3}(\boldsymbol{V})$ are small compared to the cross-sectional strains, plane strain, (1.22), is then an acceptable approximation for the shell change metric tensor (1.19).

1.8 The tensor of shell constraints

We consider a three-dimensional continuous homogeneous shell subjected to given forces. Then, the stresses and strains of this material are connected by Hooke's law, (1.23), which is only a first-order linear approximation to the real response of the elastic shell to applied forces, when they are small enough. For a given displacement field V, the stress tensor is then defined by:

$$\sigma(\boldsymbol{V}) = \lambda tr(\varepsilon(\boldsymbol{V}))Id + 2\mu\varepsilon(\boldsymbol{V}), \tag{1.23}$$

where λ and μ are the Lamé coefficients of the material ($\lambda \geq 0$ and $\mu > 0$).

We present in the lemma below the contravariant components of the tensor $\boldsymbol{\sigma}$ in term of change metric tensor and the geometrical data of the shell.

Lemma 7

Let V be a given displacement field. The stress tensor $\sigma(\boldsymbol{V})$ given by the Hooke's law (1.23) can be rewritten as

$$\sigma(\boldsymbol{V}) = \sigma^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j \quad \text{with } \sigma^{ij} = \sigma(\boldsymbol{V}) : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = E^{ijkl}\varepsilon_{kl}(\boldsymbol{V}), \tag{1.24}$$

where the notation ":" denotes the product of two tensors, $\sigma : \tau \overset{def}{=} \sigma_{ij}g^{ik}g^{jl}\tau_{kl}$ and E^{ijkl} is the elastic tensor defined by

$$E^{ijkl} = \lambda g^{ij}g^{kl} + \mu(g^{ik}g^{jl} + g^{il}g^{jk}). \tag{1.25}$$

Proof. We remark that

$$tr(\varepsilon(\boldsymbol{V}))Id : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = tr(\varepsilon(\boldsymbol{U}))tr(Id\,\boldsymbol{g}^i \otimes \boldsymbol{g}^j) = tr(\varepsilon(\boldsymbol{V}))g^{ij},$$

and

$$tr(\varepsilon(\boldsymbol{V})) = \varepsilon_{kl}(\boldsymbol{V})tr(\boldsymbol{g}^k \otimes \boldsymbol{g}^l) = \varepsilon_{kl}g^{kl}.$$

It follows that

$$\lambda tr(\varepsilon(\boldsymbol{V}))Id : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \lambda g^{ij}g^{kl}\varepsilon_{kl}(\boldsymbol{V}). \tag{1.26}$$

Let's see the second term containing μ of (1.23). If we express the tensor $\varepsilon(\boldsymbol{V})$ in the basis $\boldsymbol{g}^k \otimes \boldsymbol{g}^l$, *i.e.*,

$$\varepsilon(\boldsymbol{V}) = \big(\varepsilon(\boldsymbol{V}) : \boldsymbol{g}_k \otimes \boldsymbol{g}_l\big)\boldsymbol{g}^k \otimes \boldsymbol{g}^l = \varepsilon_{kl}(\boldsymbol{V})\boldsymbol{g}^k \otimes \boldsymbol{g}^l,$$

we obtain

$$\varepsilon(\boldsymbol{V}) : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \varepsilon_{kl}(\boldsymbol{V})\boldsymbol{g}^k \otimes \boldsymbol{g}^l : \boldsymbol{g}^i \otimes \boldsymbol{g}^j.$$

Thanks to

$$\boldsymbol{g}^k \otimes \boldsymbol{g}^l : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = tr\big((\boldsymbol{g}^l \otimes \boldsymbol{g}^k)(\boldsymbol{g}^i \otimes \boldsymbol{g}^j)\big) = g^{il}g^{kj},$$

we get

$$\varepsilon(\boldsymbol{V}) : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \varepsilon_{kl}(\boldsymbol{V})g^{il}g^{kj}.$$

Since $\varepsilon(\boldsymbol{V})$ is a symmetrical tensor,

$$\varepsilon(\boldsymbol{V}) : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \frac{1}{2}\varepsilon_{kl}(\boldsymbol{V})(g^{ik}g^{jl} + g^{il}g^{kj}).$$

It is equivalent to

$$2\mu\varepsilon(\boldsymbol{V}) : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \mu\varepsilon_{kl}(\boldsymbol{V})(g^{ik}g^{jl} + g^{il}g^{kj}). \tag{1.27}$$

Consequently, from (1.26) and (1.27), we obtain

$$\big(\lambda tr(\varepsilon(\boldsymbol{V}))Id + 2\mu\varepsilon(\boldsymbol{V})\big) : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \big(\lambda g^{ij}g^{kl} + \mu(g^{ik}g^{jl} + g^{il}g^{kj})\big)\varepsilon_{kl}(\boldsymbol{V}).$$

Replacing the symbols of the tensor of constraint σ and the elastic tensor E^{ijkl} to the above equation, we obtain

$$\sigma(\boldsymbol{V}) : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = E^{ijkl}\varepsilon_{kl}(\boldsymbol{V}).$$

□

Remark 8

Let us recall that g^{ij} are the covariant components of shell metric tensor. It should be pointed out that $g^{\alpha 3} = g^{3\alpha} = 0$ and $g^{33} = 1$. Thus we have

$$E^{3\alpha\beta\rho} = E^{333\beta} = 0, \tag{1.28}$$

and

$$E^{33\alpha\beta} = \lambda g^{\alpha\beta}, \quad E^{3\alpha 3\beta} = \mu g^{\alpha\beta} \text{ and } E^{3333} = \lambda + 2\mu. \tag{1.29}$$

Finally, in order to derive the two-dimensional Koiter's model, we state the third hypothesis of Kirchhoff-Love:
H3: The stress tensors are planar, *i.e.*, for an arbitrary displacement field V, $\sigma^{33}(\boldsymbol{V}) = 0$.
Remind that

$$\begin{aligned}\sigma^{33}(\boldsymbol{V}) &= E^{33\alpha\beta}\varepsilon_{\alpha\beta}(\boldsymbol{V}) + E^{3333}\varepsilon_{33}(\boldsymbol{V}) \\ &= \lambda g^{\alpha\beta}\varepsilon_{\alpha\beta}(\boldsymbol{V}) + (\lambda + 2\mu)\varepsilon_{33}(\boldsymbol{V}).\end{aligned}$$

So, if $\lambda \neq 0$, **H3** leads to

$$\varepsilon_{33}(\boldsymbol{V}) = -\frac{\lambda}{\lambda + 2\mu} g^{\alpha\beta}\varepsilon_{\alpha\beta}(\boldsymbol{V}).$$

We then notice that the hypothesis $\mathbf{H}_3$ is not compatible with (1.22) as soon as $\lambda \neq 0$. Hence, we correct the definition of the change of metric tensor introduced in (1.22) as following:

$$\begin{cases} \varepsilon^{\text{cor}}_{\alpha\beta}(\boldsymbol{V}) = \varepsilon_{\alpha\beta}(\boldsymbol{V}), \\ \varepsilon^{\text{cor}}_{\alpha 3}(\boldsymbol{V}) = \varepsilon_{\alpha 3}(\boldsymbol{V}), \\ \varepsilon^{\text{cor}}_{33}(\boldsymbol{V}) = -\dfrac{\lambda}{\lambda + 2\mu} g^{\alpha\beta}\varepsilon_{\alpha\beta}(\boldsymbol{V}). \end{cases}$$

1.9 Derivation of Biot-Koiter shell model

A word of caution. For notational convenience, we omit throughout this section the exponent "cor" in the expression $\varepsilon_{ij}^{\text{cor}}$.

The poroelasticity theory was developed by Biot ([10], [11], [12], [13], [14] and [15]) several decades ago and it has been studied extensively. Poroelastic phenomena are interesting in numerous applications as geomechanics, ground-surface water flow, reservoir compaction and surface subsidence, seabed-wave interaction problem, etc.

The poroelastic shells and their interactions with fluids are part of a wide variety of issues and a largest interest for contemporary engineering (ship hulls, reservoirs, ...) and biomedical problems (heart, bones, intestine skin, ...).

Our purpose here is to derive, from the equilibrium equations of three dimensional linearized poroelasticity, a two dimensional linearized model for porous shell, essentially based on Kirchhoff-Love assumptions **H1, H2** and **H3**.

Usually, in shell theory (see Ciarlet[35] for details) the unknowns are identified to their local, covariant or contravariant components of the displacement. Our framework here is a free local basis formulation for the displacement and considered shells are with little regularity midsurface, since we authorize surfaces with curvature discontinuities.

In three dimensional linear elasticity, the undeformed body occupies a region M. Under loading, a point $\boldsymbol{x} \in M$ moves to $\boldsymbol{x}+\boldsymbol{U}(\boldsymbol{x})$. Let $\widetilde{\Gamma}_0 \cup \widetilde{\Gamma}_1$ be a partition of the boundary ∂M with $\text{meas}(\widetilde{\Gamma}_0) > 0$. Then the equilibrium equations of three dimensional elasticity are:

$$\begin{cases} -\text{div}\,\boldsymbol{\sigma}(\boldsymbol{U}) = \boldsymbol{f} & \text{in } M \quad \text{(force balance)}, \\ \boldsymbol{\sigma}(\boldsymbol{U}) = H : \varepsilon(\boldsymbol{U}) & \text{in } M \quad \text{(constitutive)}, \\ \boldsymbol{U} = 0 & \text{on } \widetilde{\Gamma}_0 \quad \text{(clamping)}, \\ \boldsymbol{\sigma}\cdot\boldsymbol{n} = h & \text{on } \widetilde{\Gamma}_1 \quad \text{(force balance)}, \end{cases}$$

where $\boldsymbol{\sigma}(\boldsymbol{U})$ is the stress tensor and $\varepsilon(\boldsymbol{U}) = \dfrac{1}{2}(\nabla\boldsymbol{U} + (\nabla\boldsymbol{U})^T)$ is the strain tensor.

Let us recall the Biot model, introduced by Biot in [47], the constitutive equation for the Cauchy stress tensor $\tilde{\boldsymbol{\sigma}}$ in terms of the displacement $\boldsymbol{U}$ and fluid pressure p is

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\boldsymbol{U}) - \alpha p\mathbf{I}, \tag{1.30}$$

where $\mathbf{I}$ is the identity tensor, $\boldsymbol{\sigma}(\boldsymbol{U})$ is the stress tensor, expressing the Hooke's law:

$$\boldsymbol{\sigma}(\boldsymbol{U}) = \lambda(\text{div}\,\boldsymbol{U})\mathbf{I} + 2\mu\varepsilon(\boldsymbol{U}), \tag{1.31}$$

where $\lambda \geq 0$ (dilation moduli) and $\mu > 0$ (shear moduli) are the Lamé coefficients, and $\alpha \in]0,1[$ is the Biot-Willis constant, which is usually around one. The flux of the fluid $\boldsymbol{v}_f$ is governed by Darcy's law in porous media

$$\boldsymbol{v}_f = -\frac{\kappa}{\eta}\nabla p, \tag{1.32}$$

where $\eta > 0$ is the fluid viscosity, assumed to be constant, and κ is the permeability of porous medium.

The equation of mass conservation is

$$\frac{\partial \zeta}{\partial t} = -\text{div}\boldsymbol{v}_f + g, \tag{1.33}$$

where g is a volumetric fluid source term and ζ is the fluid content of the medium; ζ related to the fluid pressure p and material volume div $\boldsymbol{U}$ by

$$\zeta = c_0 p + \alpha \text{div}\, \boldsymbol{U}, \tag{1.34}$$

where $c_0 \geq 0$ is the constrained specific storage coefficient, that is assumed to be constant. As explained by Phillips and Wheeler in [75], $c_0 = 0$ may lead to locking, whatever the value of the Lamé coefficient λ. Although in practical situation, c_0 can vanish, we do not consider this possibility here and therefore we suppose that $c_0 > 0$. With (1.32) and (1.34), the equation of mass conservation reads

$$\frac{\partial}{\partial t}(c_0 p + \alpha \text{div}\, \boldsymbol{U}) - \frac{\kappa}{\eta}\text{div}(\nabla p) = g. \tag{1.35}$$

Finally, the balance of linear momentum is derived by making a quasi-static assumption, namely by assuming that the material deformation is much slower than the flow rate, and hence the second-time derivative of the displacement (i.e. the acceleration) is zero. Denoting by $\boldsymbol{f}$ the body force, this yields

$$-\text{div}\, \tilde{\boldsymbol{\sigma}} = \boldsymbol{f}. \tag{1.36}$$

Thus, replacing the constitutive relation (1.30) into (1.36) we obtain

$$-\nabla \cdot \sigma\big(\boldsymbol{U}\big) + \alpha \nabla p = \boldsymbol{f} \text{ in } M \times]0, T[.$$

Collecting the above equations, we have the following system of equations a.e. in $\Omega \times]0, T[$:

$$\begin{cases} -\nabla \cdot \sigma\big(\boldsymbol{U}\big) + \alpha \nabla p = \boldsymbol{f}, \text{ in } M \times]0, T[, & \text{(1.37a)} \\ \frac{\partial}{\partial t}(c_0 p + \alpha \text{div}\boldsymbol{U}) - \frac{\kappa}{\eta}\text{div}(\nabla p) = g, \text{ in } M \times]0, T[, & \text{(1.37b)} \\ \boldsymbol{U} = 0, \text{ on } \partial M \times]0, T[, & \text{(1.37c)} \\ p = 0, \text{ on } \partial M \times]0, T[, & \text{(1.37d)} \\ p(x, z, 0) = p_0(x, z), \text{ in } M. & \text{(1.37e)} \end{cases}$$

Remark 9

i) The Biot system is supplemented with relevant boundary and initial conditions that have physical meaning. The initial condition associated to second equation of (1.37) should be given as:

$$\left(c_0 p + \alpha \text{ div } \boldsymbol{U}\right)_{t=0} = c_0 p_0 + \alpha \text{ div } \boldsymbol{U_0}. \tag{1.38}$$

However, in the practice, the initial pressure is a measured data and the initial displacement $\boldsymbol{U_0}$ is obtained as the the unique solution of first equation of (1.37), where we replace p by p_0.

ii) The coupling first order terms in the system have the following meaning: The term ∇p in the first equation results from the additional stress in the medium coming from the fluid pressure, the term div $\boldsymbol{U}$ in the second equation represents the additional fluid content due to local volume change.

Remark 10

When $\boldsymbol{v} \in \mathcal{X}_K$, it is easy to check that $\partial_{\alpha\beta} v \cdot a_3$ belongs to $L^2(\omega; \mathbb{R}^3)$. Therefore, the tensor $\Upsilon_{\alpha\beta}(\boldsymbol{v})$ belongs to $L^2(\omega)$ since a_3 is in $W^{1,\infty}$.

Let $a^{\alpha\beta\rho\sigma} \in L^\infty(\omega)$ be an elasticity tensor, which we assume to satisfy the usual symmetries and to be uniformly strictly positive, *i.e.*, for all symmetric $\tau_{\alpha\beta}$ and almost all $\boldsymbol{x} \in \omega$,

$$a^{\alpha\beta\rho\sigma}(\boldsymbol{x})\tau_{\alpha\beta}\tau_{\rho\sigma} \geq c \sum_{\alpha\beta} |\tau_{\alpha\beta}|^2, \tag{1.39}$$

with $c > 0$. In the case of homogeneous, isotropic material with Lamé moduli $\mu > 0$ and $\lambda \geq 0$, we have

$$a^{\alpha\beta\rho\sigma}(\boldsymbol{x}) = \frac{2\mu\lambda}{\lambda + 2\mu} a^{\alpha\beta} a^{\rho\sigma} + \mu(a^{\alpha\rho} a^{\beta\sigma} + a^{\alpha\sigma} a^{\beta\rho}).$$

In the theorem below, we establish the weak formulation of the Koiter-Biot shell model.

Theorem 7

If $\boldsymbol{U}$ and p are a solution of the strong formulation (1.37), such that $\boldsymbol{U} = \boldsymbol{u} - z(\partial_\alpha \boldsymbol{u} \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha$ a Kirchhoff-Love displacement, then $\boldsymbol{u}$ and p belong, respectively, to $\mathcal{V}_K$ and $\mathcal{W}_K$ solve the following weak formulation:

$$\mathcal{A}_1^K\big(\boldsymbol{u};\boldsymbol{v}\big) + \mathcal{B}_1^K\big(p;\boldsymbol{v}\big) = \mathcal{L}_1^K(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathcal{X}_K, \tag{1.40a}$$

$$\mathcal{A}_2^K(p;q) + \mathcal{B}_2^N\big(\boldsymbol{u};q\big) = \mathcal{L}_2^K(q) \quad \forall q \in H_0^1(\Omega), \tag{1.40b}$$

$$p(0) = p_0 \text{ in } \Omega, \tag{1.40c}$$

where

$$\mathcal{A}_1^K(\boldsymbol{u};\boldsymbol{v}) = \int_\omega e\, a^{\alpha\beta\rho\sigma}\big[\gamma_{\alpha\beta}(\boldsymbol{u})\gamma_{\rho\sigma}(\boldsymbol{v}) + \frac{e^2}{12}\Upsilon_{\alpha\beta}(\boldsymbol{u})\Upsilon_{\rho\sigma}(\boldsymbol{v})\big]\sqrt{a}\,d\boldsymbol{x},$$

$$\mathcal{B}_1^K(p;\boldsymbol{v}) = -\alpha\int_\Omega p \,\mathrm{div}\, \boldsymbol{v}\sqrt{a}\,d\boldsymbol{X} + \alpha\int_\Omega p\ \mathrm{div}\ \big(z(\partial_\alpha \boldsymbol{v}\cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)\sqrt{a}\,d\boldsymbol{X},$$

$$\mathcal{L}_1^K(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\big(\boldsymbol{v} - z(\partial_\alpha \boldsymbol{v}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)\sqrt{a}\,d\boldsymbol{X},$$

$$\mathcal{A}_2^K(p;q) = c_0\int_\Omega p' q\sqrt{a}\,d\boldsymbol{X} + \frac{\kappa}{\eta}\int_\Omega \nabla p\cdot\nabla q\,\sqrt{a}\,d\boldsymbol{X},$$

$$\mathcal{B}_2^K(\boldsymbol{u};q) = -\alpha\int_\Omega (\mathrm{div}\ \boldsymbol{u}')q\,\sqrt{a}\,d\boldsymbol{X} + \alpha\int_\Omega \mathrm{div}\ \big(z(\partial_\alpha \boldsymbol{u}'\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)q\,\sqrt{a}\,d\boldsymbol{X} \text{ and}$$

$$\mathcal{L}_2^K(q) = \int_\Omega g q\sqrt{a}\,d\boldsymbol{X}.$$

Remark 11

We use $p(0)$ in Eq. (1.40a) to get $\boldsymbol{u}(0)$. For a given $p(0)$, the initial displacement is the unique solution of (1.40a), where we replace p by $p(0)$.

Proof. (of Theorem 7) Let U be a solution of (1.37). We then multiply the equation (1.37a) by a test function $\boldsymbol{V} \in \mathcal{V}_{KL}$ of Kirchhoff-Love type, that is to say that, by lemma 4, there exist $\boldsymbol{v} \in H^1(\omega;\mathbb{R}^3)$ such that $\boldsymbol{V} = \boldsymbol{v} - z\partial_\alpha \boldsymbol{v}\cdot\boldsymbol{a}_3\boldsymbol{a}^\alpha$. We thus obtain:

$$-\int_M \Big(\nabla\cdot\sigma(\boldsymbol{U})\Big)\cdot\boldsymbol{V}\,dV + \alpha\int_M \nabla p\cdot\boldsymbol{V}\,dV = \int_M \boldsymbol{f}\cdot\boldsymbol{V}\,dV, \quad \forall \boldsymbol{V}\in\mathcal{V}_{KL}.$$

It follows that

$$-\int_\Omega \Big(\nabla\cdot\sigma(\boldsymbol{U})\Big)\cdot\boldsymbol{V}\sqrt{a}\,d\boldsymbol{X} + \alpha\int_\Omega \nabla p\cdot\boldsymbol{V}\,\sqrt{a}\,d\boldsymbol{X} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{V}\,\sqrt{a}\,d\boldsymbol{X}.$$

Applying Green's formula and the boundary condition, we obtain

$$\underbrace{\int_\Omega \sigma(\boldsymbol{U}) : \varepsilon(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X}}_{I} + \underbrace{\alpha \int_\Omega \nabla p \cdot \boldsymbol{V}\, \sqrt{a}\, d\boldsymbol{X}}_{J} = \underbrace{\int_\Omega \boldsymbol{f} \cdot \boldsymbol{V}\, \sqrt{a}\, d\boldsymbol{X}}_{K}. \tag{1.41}$$

We then have

$$I = \int_\Omega \sigma^{\alpha\beta}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}d\boldsymbol{X} + \underbrace{\int_\Omega \sigma^{33}(\boldsymbol{U})\varepsilon_{33}(\boldsymbol{V})\sqrt{a}d\boldsymbol{X}}_{=0} + \underbrace{2\int_\Omega \sigma^{\alpha 3}(\boldsymbol{U})\varepsilon_{\alpha 3}(\boldsymbol{V})\sqrt{a}d\boldsymbol{X}}_{=0}.$$

Therefore,

$$I = \int_\Omega E^{\alpha\beta kl}\varepsilon^{cor}_{kl}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}d\boldsymbol{X}.$$

Thanks to Remark 8, it follows that:

$$\begin{aligned}
I &= \int_\Omega E^{\alpha\beta\rho\sigma}\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} + \int_\Omega E^{\alpha\beta 33}\varepsilon^{cor}_{33}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}d\boldsymbol{X} \\
&\quad + \underbrace{\int_\Omega E^{\alpha\beta 3\sigma}\varepsilon_{3\sigma}(\boldsymbol{U})\varepsilon_{\alpha 3}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X}}_{=0} + \underbrace{\int_\Omega\int_\Omega E^{\alpha\beta\rho 3}\varepsilon_{\rho 3}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X}}_{=0} \\
&= \int_\Omega E^{\alpha\beta\rho\sigma}\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} - \frac{\lambda^2}{\lambda+2\mu}\int_\Omega g^{\alpha\beta}g^{\rho\sigma}\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} \\
&= \int_\Omega \Big[\lambda g^{\alpha\beta}g^{\rho\sigma} + \mu(g^{\alpha\rho}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\rho})\Big]\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} \\
&\qquad - \frac{\lambda^2}{\lambda+2\mu}\int_\Omega g^{\alpha\beta}g^{\rho\sigma}\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} \\
&= \frac{2\mu\lambda}{\lambda+2\mu}\int_\Omega g^{\alpha\beta}g^{\rho\sigma}\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}d\boldsymbol{X} \\
&\qquad + \mu\int_\Omega (g^{\alpha\rho}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\rho})\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} \\
&= \int_\Omega A^{\alpha\beta\rho\sigma}\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\, \sqrt{a}\, d\boldsymbol{X},
\end{aligned}$$

where

$$A^{\alpha\beta\rho\sigma} = \frac{2\lambda\mu}{\lambda+2\mu}g^{\alpha\beta}g^{\rho\sigma} + \mu(g^{\alpha\rho}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\rho}). \tag{1.42}$$

Let us set the covariant components, $g^{\alpha\beta}$, of the shell metric. Suppose that

$$g_{\alpha\beta} = a_{\alpha\beta} - 2zb_{\alpha\beta}. \tag{1.43}$$

We remark that

$$g^{11} = \frac{g_{22}}{\boldsymbol{g}}, g^{22} = \frac{g_{11}}{\boldsymbol{g}} \text{ and } g^{12} = -\frac{g_{21}}{\boldsymbol{g}}.$$

One deduce an approximation of $g^{\alpha\beta}$, given by

$$g^{\alpha\beta} = a^{\alpha\beta} + 2zb^{\alpha\beta} + \cdots \tag{1.44}$$

where $b^{\alpha\beta} = a^{\alpha\rho}a^{\beta\sigma}b_{\rho\sigma}$ are the covariant components of the secondly fundamental form of surface S. If we neglect the term containing z in (1.44), (1.42) becomes

$$\begin{aligned} I &= \int_\Omega a^{\alpha\beta\rho\sigma}\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} \\ &= \int_\omega \int_{-e/2}^{e/2} a^{\alpha\beta\rho\sigma}\Big[\gamma_{\rho\sigma}(\boldsymbol{u}) - z\Upsilon_{\rho\sigma}(\boldsymbol{u})\Big]\Big[\gamma_{\alpha\beta}(\boldsymbol{v}) - z\Upsilon_{\alpha\beta}(\boldsymbol{v})\Big]\sqrt{a}\, dz\, d\boldsymbol{x} \\ &= \int_\omega e\, a^{\alpha\beta\rho\sigma}\gamma_{\rho\sigma}(\boldsymbol{u})\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}\, d\boldsymbol{x} + \frac{1}{12}\int_\omega e^3\, a^{\alpha\beta\rho\sigma}\Upsilon_{\rho\sigma}(\boldsymbol{u})\Upsilon_{\alpha\beta}(\boldsymbol{v})\sqrt{a}\, d\boldsymbol{x} \\ &= e\int_\omega a^{\alpha\beta\rho\sigma}\Big[\gamma_{\rho\sigma}(\boldsymbol{u})\gamma_{\alpha\beta}(\boldsymbol{v}) + \frac{e^2}{12}\Upsilon_{\rho\sigma}(\boldsymbol{u})\Upsilon_{\alpha\beta}(\boldsymbol{v})\Big]\sqrt{a}\, d\boldsymbol{x}, \end{aligned} \tag{1.45}$$

where

$$a^{\alpha\beta\rho\sigma} = \frac{2\mu\lambda}{\lambda + 2\mu}a^{\alpha\beta}a^{\rho\sigma} + \mu(a^{\alpha\rho}a^{\beta\sigma} + a^{\alpha\sigma}a^{\beta\rho}).$$

The second term of (1.41) becomes:

$$\begin{aligned} J &= \alpha\int_\Omega \nabla p\cdot\boldsymbol{V}\,\sqrt{a}d\boldsymbol{X} = \alpha\int_{\Gamma_0} p\,\boldsymbol{V}\cdot\boldsymbol{n}\, d\Gamma - \alpha\int_\Omega p\,\mathrm{div}\boldsymbol{V}\sqrt{a}\, d\boldsymbol{X} = -\alpha\int_\Omega p\;\mathrm{div}\boldsymbol{V}\sqrt{a}\, d\boldsymbol{X} \\ &= -\alpha\int_\Omega p\mathrm{div}\big(\boldsymbol{v} - z(\partial_\alpha\boldsymbol{v}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)\sqrt{a}\;\; d\boldsymbol{X}. \end{aligned}$$

We continue with the term K,

$$K = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{V}\,\sqrt{a}\, d\boldsymbol{X} = \int_\Omega \boldsymbol{f}\cdot\big(\boldsymbol{v} - z(\partial_\alpha\boldsymbol{v}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)\,\sqrt{a}\; d\boldsymbol{X}.$$

Consequently, (1.41) implies that

$$\mathcal{A}_1^K\big(\boldsymbol{u};\boldsymbol{v}\big) + \mathcal{B}_1^K\big(p;\boldsymbol{v}\big) = \mathcal{L}_1^K(\boldsymbol{v}). \tag{1.46}$$

Similarly, we now multiply the equation (1.37b) by $q \in H_0^1(\Omega)$. We then obtain:

$$\int_\Omega \partial_t(c_0p + \nabla\cdot\boldsymbol{U})q\,\sqrt{a}\, d\boldsymbol{X} - \frac{\kappa}{\eta}\int_\Omega \Delta p\, q\,\sqrt{a}\, d\boldsymbol{X} = \int_\Omega gq\,\sqrt{a}d\boldsymbol{X},\ \forall q \in H_0^1(\Omega). \tag{1.47}$$

For convenience, we replace $\partial_t p$ by p' and $\partial_t U$ by U'. So, for all q in $H_0^1(\Omega)$, (1.47) is equivalent to

$$\underbrace{c_0 \int_\Omega p' q \sqrt{a}\, d\boldsymbol{X} + \alpha \int_\Omega (\text{div}\, \boldsymbol{U}') q \sqrt{a}\, d\boldsymbol{X} - \frac{\kappa}{\eta} \overbrace{\int_{\Gamma_0} q(\nabla p \cdot \boldsymbol{n}) \sqrt{a}\, d\Gamma}^{=0} + \frac{\kappa}{\eta} \int_\Omega \nabla p \cdot \nabla q \sqrt{a}\, d\boldsymbol{X}}_{LS} = \underbrace{\int_\Omega gq\, d\boldsymbol{X}}_{RS}. \tag{1.48}$$

We have

$$\begin{aligned} LS &= c_0 \int_\Omega p' q \sqrt{a}\, d\boldsymbol{X} + \alpha \int_\Omega (\text{div}\, \boldsymbol{U}') q \sqrt{a}\, d\boldsymbol{X} + \frac{\kappa}{\eta} \int_\Omega \nabla p \cdot \nabla q \sqrt{a}\, d\boldsymbol{X} \\ &= \int_\Omega p' q \sqrt{a}\, d\boldsymbol{X} + \alpha \int_\Omega \text{div}\, \big(\boldsymbol{u}' - z(\partial_\alpha \boldsymbol{u}' \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha\big) q \sqrt{a}\, d\boldsymbol{X} + \frac{\kappa}{\eta} \int_\Omega \nabla p \cdot \nabla q \sqrt{a}\, d\boldsymbol{X}, \end{aligned}$$

and

$$RS = \int_\Omega gq \sqrt{a}\, d\boldsymbol{X}.$$

Then, (1.48) becomes

$$c_0 \int_\Omega p' q \sqrt{a}\, d\boldsymbol{X} + \alpha \int_\Omega \text{div}\big(\boldsymbol{u}' - z(\partial_\alpha \boldsymbol{u}' \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha\big) q \sqrt{a}\, d\boldsymbol{X} + \frac{\kappa}{\eta} \int_\Omega \nabla p \cdot \nabla q \sqrt{a}\, d\boldsymbol{X} = \int_\Omega gq \sqrt{a}\, d\boldsymbol{X},$$

or

$$\mathcal{A}_2^K(p; q) + \mathcal{B}_2^K\big(\boldsymbol{u}; q\big) = \mathcal{L}_2^K(q). \tag{1.49}$$

From (1.46) and (1.49), we obtain Theorem 7. □

Succeedingly, we derive the well-posedness of the weak problem (1.40).

1.10 Well-posedness

In this section, we prove the well-posedness of the problem (1.40) introduced in Theorem (7) and prove that its solution p belongs to $L^\infty\big(0, T, H^1(\Omega)\big) \cap H^1\big(0, T; L^2(\Omega)\big)$. For the well-posedness, we are not able to prove the existence and uniqueness of the displacement $\boldsymbol{U}$ and the pressure p at the same time. Therefore, we firstly turn out the well-posedness of $\boldsymbol{U}$ in the weak form of constitutive equation (1.37) by Banach-Nečas-Babuš theorem with a given p (see Lemma 15). Then proving the well-posedness of p in the weak form of mass conservation equation in (1.37) by making use of the semi-discrete Galerkin method and the theory of initial value problem for linear systems (see Lemma 17, 18 & 19).

Theorem 8

Let $p_0 \in H^1(\Omega)$, $\boldsymbol{f} \in H^1(0,T;L^2(\Omega,\mathbb{R}^3))$ and $g \in L^2(\Omega\times]0,T[)$. Then the problem (1.40) has a unique solution. The pressure p belongs to $L^\infty\big(0,T,H^1(\Omega)\big) \cap H^1\big(0,T;L^2(\Omega)\big)$.

The proof of Theorem 8 is comprised of two steps and each step is comprised of a series of lemmas.

Step 1. Let us begin by providing an existence and uniqueness result of the displacement $\boldsymbol{u}(p)$ for the equation (1.40a) when p is given. First of all, we recall the rigid displacement lemma in the functional framework introduced in [21].

Lemma 12

Let $\boldsymbol{v} \in H^1(\omega;\mathbb{R}^3)$ and assume that $\varphi \in W^{2,\infty}(\omega,\mathbb{R}^3)$.

1. If $\gamma_{\alpha\beta}(\boldsymbol{v}) = 0$, then there exists $\psi \in L^2(\omega;\mathbb{R}^3)$ such that $\partial_\alpha \boldsymbol{v} = \psi \wedge a_\alpha$.
2. If $\Upsilon_{\alpha\beta}(\boldsymbol{v}) = 0$, then ψ is a constant vector in $\mathbb{R}^3$ and there exists $c \in \mathbb{R}^3$ such that

$$\boldsymbol{v}(x) = c + \psi \wedge \varphi(x). \tag{1.50}$$

Proof. See Theorem 6 in[21]. □

Thanks to Lemma 12, we introduce the new norm $|||\cdot|||_K$ on $\mathcal{X}_K$,

Lemma 13

If $\boldsymbol{v} \in H^1(\omega,\mathbb{R}^3)$ and $\varphi \in W^{2,\infty}(\omega,\mathbb{R}^3)$ such that $\boldsymbol{v} = 0$ on γ_0 and $\varphi(\gamma_0)$ is not included in a straight line, then $\boldsymbol{v} = 0$ a.e. on ω. Therefore, one can check that:

$$|||v|||_K = \left(\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(v)\|^2_{L^2(\omega)} + \sum_{\alpha,\beta} \|\Upsilon_{\alpha\beta}(v)\|^2_{L^2(\omega)} \right)^{1/2} \tag{1.51}$$

is a norm on $\mathcal{X}_N$.

Proof. It is clear that this the mapping $\boldsymbol{v} \in \mathcal{X}_K \mapsto |||\boldsymbol{v}|||_{\mathcal{X}_K}$ is a semi-norm. By Lemma 12, if $\boldsymbol{v} \in \mathcal{X}_K$ is such that $|||\cdot|||_K = 0$, then there exist ψ, $c \in \mathbb{R}^3$ such that $\boldsymbol{v}(x) = \psi \wedge \varphi(x) + c$. The set of points $y \in \mathbb{R}^3$ such that $\psi \wedge y + c$ vanishes is either a straight line ($\psi \neq 0$ and $c \neq 0$), empty ($\psi = 0$ and $c \neq 0$) or the whole space ($\psi = 0$ and $c = 0$). Since $\boldsymbol{v}$ vanishes on $\partial\omega$ and $\varphi(\partial\omega)$ is not included in a straight line, it follow that $\boldsymbol{v} = 0$. □

From Lemma 13, we prove in the following lemma the equivalence between (1.7) and (1.51).

Lemma 14
The norm

$$\|\!|v|\!\|_K = \left(\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(v)\|^2_{L^2(\omega)} + \sum_{\alpha,\beta} \|\Upsilon_{\alpha\beta}(v)\|^2_{L^2(\omega)} \right)^{1/2}$$

is equivalent to the norm (1.7) on $\mathcal{X}_N$.

Proof. We prove this lemma by contradiction. Let us assume that there exists a sequence $\boldsymbol{v}_n$ in $\mathcal{X}_K$ such that

$$\|\boldsymbol{v}_n\|_{\mathcal{X}_K} = 1 \text{ and } \|\!|\boldsymbol{v}_n|\!\|_K \to 0 \text{ when } n \to +\infty. \tag{1.52}$$

By extracting a subsequence, still denoted $\boldsymbol{v}_n$, we may assume that there exists a $\boldsymbol{v} \in \mathcal{X}_K$ such that $\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}$ weakly in $H^1(\omega, \mathbb{R}^3)$ and $\partial_{\alpha\beta}\boldsymbol{v}_n \cdot \boldsymbol{a}_3 \rightharpoonup \partial_{\alpha\beta}\boldsymbol{v} \cdot \boldsymbol{a}_3$ weakly in $L^2(\omega)$. Consequently,

$$\gamma_{\alpha\beta}(\boldsymbol{v}_n) \rightharpoonup \gamma_{\alpha\beta}(\boldsymbol{v}) \text{ and } \Upsilon_{\alpha\beta}(\boldsymbol{v}_n) \rightharpoonup \Upsilon_{\alpha\beta}(\boldsymbol{v}) \text{ weakly in } L^2(\omega), \tag{1.53}$$

by expressions (1.20) and (1.21). Since hypothesis (1.52) implies that these tensors converge strongly to zero in $L^2(\omega)$, we obtain $\boldsymbol{v} = 0$ thanks to Lemma 12. Rellich's lemma now implies that $\boldsymbol{v}_n \to 0$ strongly in $L^2(\omega; \mathbb{R}^3)$.

Let us introduce the vector $(w_n)_\alpha = \boldsymbol{v}_n \cdot \boldsymbol{a}_\alpha$, which is such that $w_n \to 0$ strongly in $L^2(\omega; \mathbb{R}^2)$. Let us define $2e_{\alpha\beta}(\boldsymbol{w}) = \partial_\beta \boldsymbol{w}_\alpha + \partial_\alpha \boldsymbol{w}_\beta$. We see that, by expression (1.20)

$$e_{\alpha\beta}(\boldsymbol{w}_n) = \gamma_{\alpha\beta}(\boldsymbol{v}_n) + \frac{1}{2}\boldsymbol{v}_n \cdot (\partial_\beta \boldsymbol{a}_\alpha + \partial_\alpha \boldsymbol{a}_\beta) \to 0 \text{ strongly in } L^2(\omega), \tag{1.54}$$

since $\boldsymbol{a}_\alpha \in W^{1,\infty}(\omega; \mathbb{R}^3)$. By the two-dimensional Korn inequality (5), we deduce then that $w_n \to 0$ strongly in $H^1(\omega; \mathbb{R}^3)$. Consequently,

$$\partial_\rho \boldsymbol{v}_n \cdot \boldsymbol{a}_\alpha = \partial_\rho\big((w_n)_\alpha\big) - \boldsymbol{v}_n \cdot \partial_\rho \boldsymbol{a}_\alpha \to 0 \text{ strongly in } L^2(\omega), \tag{1.55}$$

since $\partial_\rho \boldsymbol{a}_\alpha \in L^\infty(\omega; \mathbb{R}^3)$.

Moreover, as $\boldsymbol{v}_n \rightharpoonup 0$ in $H^1(\omega; \mathbb{R}^3)$, it follows that $\partial_\rho \boldsymbol{v}_n \cdot \boldsymbol{a}_3 \rightharpoonup 0$ in $L^2(\omega)$. On the other hand, $\partial_\beta(\partial_\rho \boldsymbol{v}_n \cdot \boldsymbol{a}_3) = \partial_{\beta\rho}\boldsymbol{v}_n \cdot \boldsymbol{a}_3 + \partial_\rho \boldsymbol{v}_n \cdot \partial_\beta \boldsymbol{a}_3 \rightharpoonup 0$ in $L^2(\omega)$. Indeed, $\partial_\beta \boldsymbol{a}_3 \in L^\infty(\omega; \mathbb{R}^3)$ and we already know that $\partial_{\beta\rho}\boldsymbol{v}_n \cdot \boldsymbol{a}_3 \rightharpoonup 0$ weakly in $L^2(\omega)$. Consequently, $\partial_\rho \boldsymbol{v}_n \cdot \boldsymbol{a}_3 \rightharpoonup 0$ weakly in $H^1(\omega)$ and by Rellich's lemma

$$\partial_\rho \boldsymbol{v}_n \cdot \boldsymbol{a}_3 \to 0 \text{ strongly in } L^2(\omega). \tag{1.56}$$

We deduce from (1.52) and (1.56) that

$$\partial_{\alpha\beta}\boldsymbol{v}_n \cdot \boldsymbol{a}_3 = \Upsilon_{\alpha\beta}(\boldsymbol{v}_n) + \Gamma^\rho_{\alpha\beta}\partial_\rho \boldsymbol{v}_n \cdot \boldsymbol{a}_3 \to 0 \text{ strongly in } L^2(\omega), \tag{1.57}$$

since $\Gamma^\rho_{\alpha\beta} \in L^\infty(\omega)$, and on the other hand that

$$\partial_\rho \boldsymbol{v}_n = (\partial_\rho \boldsymbol{v}_n \cdot \boldsymbol{a}_i)\boldsymbol{a}^i \to 0 \text{ strongly in } L^2(\omega; \mathbb{R}^3), \tag{1.58}$$

by (1.55), (1.56) and since both $\boldsymbol{a}_i$ and $\boldsymbol{a}^i$ belong to $L^\infty(\omega; \mathbb{R}^3)$. Consequently, $\boldsymbol{v}_n \to 0$ strongly in $H^1(\omega; \mathbb{R}^3)$. Since by (1.57), $\partial_{\alpha\beta}\boldsymbol{v}_n \cdot \boldsymbol{a}_3 \to 0$ strongly in $L^2(\omega)$, we see that $\|\boldsymbol{v}_n\|_{\mathcal{X}_K}$ which contradicts (1.52) and proves the lemma. □

Now, we derive the main proof of the well-posedness by the following lemma,

Lemma 15

For a given p in $L^\infty\big(0,T;L^2(\Omega)\big)$ and $\boldsymbol{f}$ in $H^1(0,T;L^2(\Omega,\mathbb{R}^3))$, there exists a unique solution $\boldsymbol{u}(p)$ in $\mathcal{V}_K$ solving, for a.e. t in $]0,T[$,

$$\mathcal{A}_1^K\big(\boldsymbol{u}(p),\boldsymbol{v}\big) = \mathcal{L}_{1,p}^K(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathcal{X}_K, \tag{1.59}$$

where $\mathcal{L}_{1,p}^K(\boldsymbol{v}) = \mathcal{L}_1^K(\boldsymbol{v}) - \mathcal{B}_1^K(p;\boldsymbol{v})$.

Proof. We will prove the well-posedness by using the Banach-Nečas-Babuška theorem 5. For a.e. $t \in]0,T[$ and for all $\boldsymbol{w}$ in $\mathcal{V}_K$, we have, thanks to Remark 10, ii):

$$\begin{aligned}
\sup_{\boldsymbol{v}\in\mathcal{X}_K} \frac{\mathcal{A}_1^K\big(\boldsymbol{w};\boldsymbol{v}\big)}{\|\boldsymbol{v}\|_{\mathcal{X}_K}} &\geq \frac{\mathcal{A}_1^K\big(\boldsymbol{w};\boldsymbol{w}\big)}{\|\boldsymbol{w}\|_{\mathcal{X}_K}} \\
&\geq \frac{\min\{e,\frac{e^3}{12}\}|||\boldsymbol{w}|||_K^2}{\|\boldsymbol{w}\|_{\mathcal{X}_K}} \\
&\geq \frac{\min\{e,\frac{e^3}{12}\}C\|\boldsymbol{w}\|_{\mathcal{X}_K}^2}{\|\boldsymbol{w}\|_{\mathcal{X}_K}} \\
&= \min\{e,\frac{e^3}{12}\}C\|\boldsymbol{w}\|_{\mathcal{X}_K},
\end{aligned}$$

by using Lemma 13 and 14.

Therefore, for a.e. t in $]0,T[$ and for all $\boldsymbol{w}$ in $\mathcal{V}_K$, there exists $\theta = \min\{e,\frac{e^3}{12}\}C$ such that:

$$\sup_{\boldsymbol{v}\in\mathcal{X}_K} \frac{\mathcal{A}_1^K\big(\boldsymbol{w};\boldsymbol{v}\big)}{\|\boldsymbol{v}\|_{\mathcal{X}_K}} \geq \theta\|\boldsymbol{w}\|_{\mathcal{X}_K}. \tag{1.60}$$

On the other hand, for all $\boldsymbol{v}$ in $\mathcal{X}_K$, suppose that, for almost every $t \in]0,T[$,

$$\mathcal{A}_1^K\big(\boldsymbol{w};\boldsymbol{v}\big) = 0, \;\; \forall\, \boldsymbol{w} \in \mathcal{V}_K.$$

Hence, by the equivalence between two norms $||\cdot||_{\mathcal{X}_K}$ and $|||\cdot|||_K$, we have

$$0 = \mathcal{A}_1^K\big(\boldsymbol{v};\boldsymbol{v}\big) \geq \min\big\{c;\frac{c^3}{12}\big\}|||\boldsymbol{v}|||_K^2 \geq \min\{c;\frac{c^3}{12}\}C\|\boldsymbol{v}\|_{\mathcal{X}_K}^2 \geq 0.$$

Therefore,

$$\boldsymbol{v} = \mathbf{0}. \tag{1.61}$$

Additionally, there exist positive constants C_1, C_2, C_3, C_4 such that

$$\begin{aligned}
|\mathcal{L}^K_{1,p}(\boldsymbol{v})| &= \left|\alpha \int_\omega \overline{p}\mathrm{div}\boldsymbol{v}\sqrt{a}\, d\boldsymbol{x} - \alpha \int_\omega \left(\int_{-e/2}^{e/2} p \ \mathrm{div} \ \big(z(\partial_\alpha \boldsymbol{v}\cdot a_3)a^\alpha\big) dz \right) \sqrt{a}\, d\boldsymbol{x} \right. \\
&\qquad \left. + \int_\omega e\, \boldsymbol{f}\cdot \boldsymbol{v}\sqrt{a}\, d\boldsymbol{x} \right| \\
&\leq C_1 \|\overline{p}\|_{0,\omega}\|\mathrm{div}\boldsymbol{v}\|_{0,\omega} + C_2\|\overline{p}\|_{0,\omega}\|(\partial_\alpha \boldsymbol{v}\cdot a_3)a^\alpha\|_{H^1(\omega,\mathbb{R}^3)} + C_3\|\boldsymbol{f}\|_{0,\omega}\|\boldsymbol{v}\|_{0,\omega} \\
&\leq C_4\sqrt{2\|\overline{p}\|^2_{0,\omega} + \|\boldsymbol{f}\|^2_{0,\omega}}\sqrt{\|\mathrm{div}\boldsymbol{v}\|^2_{0,\omega} + \|(\partial_\alpha \boldsymbol{v}\cdot a_3)a^\alpha\|^2_{H^1(\omega,\mathbb{R}^3)} + \|\boldsymbol{v}\|^2_{0,\omega}},
\end{aligned}$$

by using the Cauchy-Schwarz inequality. Note that $\bar{p}(x) = \displaystyle\int_{-e/2}^{e/2} p(x,z)dz$. Therefore, with a given p, there exists a constant $C > 0$ such that

$$\left|\mathcal{L}^K_{1,p}(\boldsymbol{v})\right| \leq C\sqrt{\|\boldsymbol{v}\|^2_{H^1(\omega,\mathbb{R}^3)} + \|(\partial_\alpha \boldsymbol{v}\cdot a_3)a^\alpha\|^2_{H^1(\omega,\mathbb{R}^3)}} = C\|\boldsymbol{v}\|_{\mathcal{X}_K}. \tag{1.62}$$

Combining (1.60), (1.61) and (1.62), we can see that the bilinear form $\mathcal{A}^K_1(\cdot\,;\,\cdot)$ and the continuous linear form $\mathcal{L}^K_{1,p}(\cdot)$ satisfy the conditions of the Banach-Nečas-Babuška theorem. So, the problem (1.59) is well-posed. □

Comment. We are able to prove this theorem by using the Lax-Milgram lemma. Thanks to the inequality (1.39) and the equivalence between two norms $|||\cdot|||_K$ and $||\cdot||_{\mathcal{X}_k}$, we obtain the continuity and coercivity of $\mathcal{A}^K_1$. More precisely, by the Holder inequality and the upper bounded quality of $a^{\alpha\beta\rho\sigma}$ and $\sqrt{a}$, there exist $C_1,\ C_2$ and C_3 such that

$$\begin{aligned}
|\mathcal{A}^K_1(\boldsymbol{u},\boldsymbol{v})| &= \left| \int_\omega e\, a^{\alpha\beta\rho\sigma}\left[\gamma_{\alpha\beta}(\boldsymbol{u})\gamma_{\rho\sigma}(\boldsymbol{v}) + \frac{e^2}{12}\Upsilon_{\alpha\beta}(\boldsymbol{u})\Upsilon_{\rho\sigma}(\boldsymbol{v})\right]\sqrt{a}\, d\boldsymbol{x}\right| \\
&\leq C_1\left|\int_\omega \gamma_{\alpha\beta}(\boldsymbol{u})\gamma_{\rho\sigma}(\boldsymbol{v})\, d\boldsymbol{x}\right| + C_2\left|\int_\omega \Upsilon_{\alpha\beta}(\boldsymbol{u})\Upsilon_{\rho\sigma}(\boldsymbol{v})\, d\boldsymbol{x}\right| \\
&\leq C_1||\gamma_{\alpha\beta}(\boldsymbol{u})||_{L^2(\omega)}||\gamma_{\rho\sigma}(\boldsymbol{v})||_{L^2(\omega)} + C_2||\Upsilon_{\alpha\beta}(\boldsymbol{u})||_{L^2(\omega)}||\Upsilon_{\rho\sigma}(\boldsymbol{v})||_{L^2(\omega)} \\
&\leq C_3\left(||\gamma_{\alpha\beta}(\boldsymbol{u})||^2_{L^2(\omega)} + ||\Upsilon_{\alpha\beta}(\boldsymbol{u})||^2_{L^2(\omega)}\right)^{1/2}\left(||\gamma_{\alpha\beta}(\boldsymbol{v})||^2_{L^2(\omega)} + ||\Upsilon_{\alpha\beta}(\boldsymbol{v})||^2_{L^2(\omega)}\right)^{1/2} \\
&= C_3|||\boldsymbol{u}|||_K|||\boldsymbol{v}|||_K, \qquad \forall \boldsymbol{u},\, \boldsymbol{v}\in \mathcal{X}_K.
\end{aligned}$$

By the equivalence between two norms $|||\cdot|||_K$ and $||\cdot||_{\mathcal{X}_k}$, there exists a constant C_4 such that

$$|\mathcal{A}^K_1(\boldsymbol{u},\boldsymbol{v})| \leq C_4|||\boldsymbol{u}|||_{\mathcal{X}_K}|||\boldsymbol{v}|||_{\mathcal{X}_K}, \qquad \forall \boldsymbol{u},\, \boldsymbol{v}\in \mathcal{X}_K. \tag{1.63}$$

For the coercivity, by the inequality (1.39) and the relation of two norms $|||\cdot|||_K$ and $||\cdot||_{\mathcal{X}_k}$, there exist constants C_1', C_2' and C_3' such that

$$\begin{aligned}
|\mathcal{A}_1^K(\boldsymbol{v},\boldsymbol{v})| &= \left| \int_\omega e\, a^{\alpha\beta\rho\sigma}\left[\gamma_{\alpha\beta}(\boldsymbol{v})\gamma_{\rho\sigma}(\boldsymbol{v}) + \frac{e^2}{12}\Upsilon_{\alpha\beta}(\boldsymbol{v})\Upsilon_{\rho\sigma}(\boldsymbol{v})\right]\sqrt{a}\,d\boldsymbol{x}\right| \\
&\geq C_1'\int_\omega |\gamma_{\alpha\beta}(\boldsymbol{v})|^2 d\boldsymbol{x} + C_2'\int_\omega |\Upsilon_{\alpha\beta}(\boldsymbol{v})|^2 d\boldsymbol{x} \\
&\geq \min\{C_1';C_2'\}\int_\omega \left(|\gamma_{\alpha\beta}(\boldsymbol{v})|^2 + |\Upsilon_{\alpha\beta}(\boldsymbol{v})|^2\right)d\boldsymbol{x} \\
&= \min\{C_1';C_2'\}|||\boldsymbol{v}|||_K^2 \geq C_3'||\boldsymbol{v}||_{\mathcal{X}_K}^2, \qquad \forall \boldsymbol{v}\in\mathcal{X}_K. \qquad (1.64)
\end{aligned}$$

From (1.63), (1.64) and the Lax-Milgram Lemma, Lemma 1.59 is proved.

Remark 16

i) When the initial condition p_0 is given, the initial displacement u_0 is obtained as the unique solution of the equation (1.40a) in which we replace p by p_0.

ii) Since the mapping $p \longmapsto \boldsymbol{u}(p)$ is a continuous affine mapping from $L^2(\Omega)$ to $\mathcal{X}_K$: There exists a constant C such that

$$\forall p_1,\, p_2 \in L^2(\Omega),\ \left\|\boldsymbol{u}(p_1) - \boldsymbol{u}(p_2)\right\|_{\mathcal{X}_K} \leq C \parallel p_1 - p_2 \parallel_{L^2(\Omega)}.$$

Hence, (1.40) has the equivalent implicit formulation: Find p in $L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H^1(\Omega)\big)$ satisfying (1.40c) and for almost every t in $]0,T[$:

$$\begin{aligned}
c_0\int_\Omega p' q\sqrt{a}\,d\boldsymbol{X} + \alpha\int_\Omega \operatorname{div}\big(\boldsymbol{u}'(p) - z(\partial_\alpha\boldsymbol{u}'(p)\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)q\sqrt{a}\,d\boldsymbol{X}& \\
+\frac{\kappa}{\eta}\int_\Omega \nabla p\cdot\nabla q\sqrt{a}\,d\boldsymbol{X} = \int_\Omega gq\sqrt{a}\,d\boldsymbol{X},\quad \forall q\in H_0^1(\Omega),& \qquad (1.65)
\end{aligned}$$

with $\boldsymbol{u}(p)$ defined by (1.59).

For proving the well-posedness of the semi-discrete problem (1.70). It is convenient to split $\boldsymbol{u}(p)$ as follows:

$$\boldsymbol{u}(p) = \overline{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}(p), \qquad (1.66)$$

where $\overline{\boldsymbol{u}} \in \mathcal{X}_K$ is the unique solution of

$$\forall \boldsymbol{v}\in\mathcal{X}_K, \mathcal{A}_1^K\big(\overline{\boldsymbol{u}},\boldsymbol{v}\big) = \big(\boldsymbol{f}, \big(\boldsymbol{v} - z(\partial_\alpha\boldsymbol{v}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)\sqrt{a}\big)_\Omega,\ \text{a.e. } t\in]0,T[, \qquad (1.67)$$

and $\widetilde{\boldsymbol{u}}(p) \in \mathcal{X}_K$ is the unique solution of

$$\forall \boldsymbol{v}\in\mathcal{X}_K, \mathcal{A}_1^K\big(\widetilde{\boldsymbol{u}}(p),\boldsymbol{v}\big) = \alpha\big(p, \operatorname{div}\big(\boldsymbol{v} - z(\partial_\alpha\boldsymbol{v}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)\sqrt{a}\big)_\Omega,\ \text{a.e. } t\in]0,T[. \qquad (1.68)$$

According to the proof of Theorem 7, since the term I is in both (1.41) and (1.45), we see that

$$e\int_\omega a^{\alpha\beta\rho\sigma}\left[\gamma_{\rho\sigma}(\boldsymbol{u})\gamma_{\alpha\beta}(\boldsymbol{v})+\frac{e^2}{12}\Upsilon_{\rho\sigma}(\boldsymbol{u})\Upsilon_{\alpha\beta}(\boldsymbol{v})\right]\sqrt{a}\,d\boldsymbol{x}=\big(\sigma(\boldsymbol{U}),\varepsilon(\boldsymbol{V})\sqrt{a}\big)_\Omega.$$

It leads to

$$\mathcal{A}_1^K\big(\boldsymbol{u},\boldsymbol{v}\big)=\big(\sigma(\boldsymbol{U}),\varepsilon(\boldsymbol{V})\sqrt{a}\big)_\Omega. \tag{1.69}$$

Subsequently, we show the second step of proof of Theorem 8.

Step 2. We use the Galerkin method to construct a solution p.

Let $(\theta_n)_{n\geq 1}$ be a smooth basis of $H_0^1(\Omega)$ and let Q_k be the space spanned by $(\theta_i)_{i=1}^k$, i.e., $Q_k=\mathrm{Vect}\{\theta_1,...,\theta_k\}$. Then, our semi-discrete problem reads: Find

$$p_k(t)=\sum_{i=1}^k \pi_i(t)\theta_i\in H^1(0,T;Q_k),$$

such that

$$\begin{aligned}c_0\int_\Omega p_k'\theta_i\sqrt{a}\,d\boldsymbol{X}+\alpha\int_\Omega \mathrm{div}\,\big(\widetilde{\boldsymbol{u}}_k'-z(\partial_\alpha\widetilde{\boldsymbol{u}}_k'\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)\theta_i\sqrt{a}\,d\boldsymbol{X}+\frac{\kappa}{\eta}\int_\Omega\nabla p_k\cdot\nabla\theta_i\sqrt{a}\,d\boldsymbol{X}\\ =-\alpha\int_\Omega\mathrm{div}\big(\overline{\boldsymbol{u}}'-z(\partial_\alpha\overline{\boldsymbol{u}}'\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)\theta_i\sqrt{a}\,d\boldsymbol{X}+\int_\Omega g\theta_i\sqrt{a}\,d\boldsymbol{X}\qquad 1\leq i\leq k\,,\end{aligned}\tag{1.70a}$$

$$p_k(0)=p_{0k}, \tag{1.70b}$$

where $\overline{\boldsymbol{u}}$ is defined by (1.67), $\tilde{\boldsymbol{u}}_k=\tilde{\boldsymbol{u}}(p_k)$, i.e. $\boldsymbol{u}(p_k)=\overline{\boldsymbol{u}}+\tilde{\boldsymbol{u}}_k$ and $p_{0k}\in Q_k$ satisfies

$$\lim_{k\to\infty}\|p_{0k}-p_0\|_{H_0^1(\Omega)}=0.$$

The system (1.70) is a square linear system of k ODEs of order one in matrix form. We establish now the well-posedness of the problem (1.70). It is the object of Lemma 17.

Lemma 17

Let $p_0\in H^1(\Omega)$, $\boldsymbol{f}\in H^1(0,T;L^2(\Omega,\mathbb{R}^3))$, $g\in L^2(\Omega\times]0,T[)$. The semi-discrete problem (1.70) has exactly one solution on $[0,T]$.

Proof. Thanks to (1.68), we have

$$\mathcal{A}_1^K\left(\widetilde{\boldsymbol{u}}\big(\pi_i'(t)\theta_i\big),\boldsymbol{v}\right)=\alpha\big(\pi_i'(t)\theta_i,\mathrm{div}\boldsymbol{v}\sqrt{a}\big)_\Omega+\alpha\big(\pi_i'(t)\theta_i,\mathrm{div}\big(z(\partial_\alpha\boldsymbol{v}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)\sqrt{a}\big)_\Omega,$$

$$\mathcal{A}_1^K\left(\widetilde{\boldsymbol{u}}(\theta_i),\boldsymbol{v}\pi_i'(t)\right)=\alpha\big(\theta_i,\pi_i'(t)\mathrm{div}\boldsymbol{v}\sqrt{a}\big)_\Omega+\alpha\big(\theta_i,\pi_i'(t)\mathrm{div}\big(z(\partial_\alpha\boldsymbol{v}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)\sqrt{a}\big)_\Omega.$$

Hence, we obtain

$$\mathcal{A}_1^K\Big(\widetilde{\boldsymbol{u}}\big(\pi_i'(t)\theta_i\big),\boldsymbol{v}\Big)=\mathcal{A}_1^K\Big(\widetilde{\boldsymbol{u}}\big(\theta_i\big),\boldsymbol{v}\pi_i'(t)\Big)=\mathcal{A}_1^K\Big(\widetilde{\boldsymbol{u}}\big(\theta_i\big)\pi_i'(t),\boldsymbol{v}\Big).$$

This follows that

$$\widetilde{\boldsymbol{u}}\big(\pi_i'(t)\theta_i\big)=\widetilde{\boldsymbol{u}}\big(\theta_i\big)\pi_i'(t).$$

We write (1.70a) in matrix form and define the following vectors and matrices

$$\mathbf{P}_i=\pi_i,\ 1\le i\le k,\ \mathbf{C}_{ij}=c_0(\theta_j,\theta_i\sqrt{a})_\Omega,\ 1\le i,j\le k,$$

$$\mathbf{A}_{ij}=\alpha(\operatorname{div}\big[\widetilde{\boldsymbol{u}}(\theta_j)-z(\partial_\alpha\widetilde{\boldsymbol{u}}(\theta_j)\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\big],\theta_i\sqrt{a})_\Omega,\ 1\le i,j\le k,$$

and

$$\mathbf{D}_{ij}=\frac{\kappa}{\mu}(\nabla\theta_j,\nabla\theta_i\sqrt{a})_\Omega,\ 1\le i,j\le k.$$

With those notations, (1.70) is a square system of k linear ODEs of order one: Find $P\in[H^1(0,T)]^k$ such that

$$\begin{cases}(\mathbf{C}+\mathbf{A})\mathbf{P}'+\mathbf{D}\mathbf{P} &=\mathbf{H},\ \forall t\in]0,T[,\\ \qquad\qquad\mathbf{P}(0) &=\mathbf{P}_0,\end{cases}\tag{1.71}$$

where $\mathbf{H}$ is the vector of the right-hand side of (1.70a). Note that $\mathbf{C}$ and $\mathbf{D}$ are square and symmetric matrices. Let us prove that they are positive-definite.

For $\mathbf{Z}\in\mathbb{R}^k$, we have

$$\begin{aligned}\mathbf{Z}^T\mathbf{C}\mathbf{Z}&=c_0\sum_{i,j=1}^k(\theta_j,\theta_i\sqrt{a})_\Omega Z_iZ_j=c_0\sum_{i,j=1}^k(Z_j\theta_j,Z_i\theta_i\sqrt{a})_\Omega=c_0\sum_{i,j=1}^k\|Z_i\theta_ia^{1/4}\|^2_{L^2(\Omega)}\\&\ge c_0\sum_{i,j=1}^k\sqrt{\delta}\|Z_i\theta_i\ \|^2_{L^2(\Omega)}\ge 0.\end{aligned}$$

Since c_0 and δ are both positive constants, if $\mathbf{Z}^T\mathbf{C}\mathbf{Z}=0$ then $\sum_{i,j=1}^kZ_i\theta_i=0$. Because $\{\theta_n\}_{n\ge1}$ is a basis of $H_0^1(\Omega)$, then $Z_i=0$ for all i. It follows that $\mathbf{Z}=0$. Therefore $\mathbf{Z}^T\mathbf{C}\mathbf{Z}>0$ for all $\mathbf{Z}\neq0$ and $\mathbf{C}$ is a positive-definite matrix.

Similarly, for matrix $\mathbf{D}$ we have,

$$\begin{aligned}\mathbf{Z}^T\mathbf{D}\mathbf{Z}&=\frac{\kappa}{\mu}\sum_{i,j=1}^k(\nabla\theta_j,\nabla\theta_i\sqrt{a})_\Omega Z_iZ_j=\frac{\kappa}{\mu}\sum_{i,j=1}^k(Z_j\nabla\theta_j,Z_i\nabla\theta_i\sqrt{a})_\Omega\\&=\frac{\kappa}{\mu}\sum_{i,j=1}^k\|Z_i\nabla\theta_ia^{1/4}\|^2_{L^2(\Omega)}\ge\frac{\kappa}{\mu}\sqrt{\delta}\sum_{i,j=1}^k\|Z_i\nabla\theta_i\|^2_{L^2(\Omega)}\ge\frac{\kappa}{\mu}\sqrt{\delta}C\|Z_i\theta_i\|^2_{L^2(\Omega)}\ge0,\end{aligned}$$

thanks to Poincaré's inequality.

Since C and δ are both positive constants, if $\mathbf{Z^T D Z} = 0$ then $\sum_{i,j=1}^{k} Z_i\theta_i = 0$. Because $\{\theta_n\}_{n\geq 1}$ is a basis of $H_0^1(\Omega)$, $Z_i = 0$ for all i. It follows that $\mathbf{Z} = 0$. Therefore $\mathbf{Z^T D Z} > 0$ for all $\mathbf{Z} \neq 0$ and $\mathbf{D}$ is a positive-definite matrix.

Regarding $\mathbf{A}$, by choosing $p = \theta_i$ and $\boldsymbol{v} = \tilde{\boldsymbol{u}}(\theta_j)$ in (1.68), we have

$$\boldsymbol{A}_{i,j} = \mathcal{A}_1^K\big(\widetilde{\boldsymbol{u}}(\theta_i);\widetilde{\boldsymbol{u}}(\theta_j)\big).$$

Note that $\mathbf{A}$ is a square and symmetric matrix. Let us prove that $\mathbf{A}$ is positive. Indeed, we have

$$\begin{aligned}
\mathbf{Z^T A Z} &= \mathcal{A}_1^K\big(\widetilde{\boldsymbol{u}}(\theta_i);\widetilde{\boldsymbol{u}}(\theta_j)\big)Z_iZ_j = \Big(\sigma\big(\widetilde{\boldsymbol{u}}(\theta_i)\big),\varepsilon\big(\widetilde{\boldsymbol{v}}(\theta_j)\big)\sqrt{a}\Big)_\Omega Z_iZ_j \\
&\geq \sqrt{\delta}\Big(\sigma\big(\widetilde{\boldsymbol{u}}(\theta_i)\big),\varepsilon\big(\widetilde{\boldsymbol{v}}(\theta_j)\big)\Big)_\Omega Z_iZ_j \\
&= 2\sqrt{\delta}\mu\Big(\varepsilon\big(\widetilde{\boldsymbol{u}}(\theta_i)\big),\varepsilon\big(\widetilde{\boldsymbol{v}}(\theta_j)\big)\Big)_\Omega Z_iZ_j + \sqrt{\delta}\lambda\Big(\operatorname{div}\big(\widetilde{\boldsymbol{u}}(\theta_i)\big),\operatorname{div}\big(\widetilde{\boldsymbol{v}}(\theta_j)\big)\Big)_\Omega Z_iZ_j \\
&= 2\sqrt{\delta}\mu\Big(\sum_i\varepsilon\big(\widetilde{\boldsymbol{u}}(\theta_i)\big)Z_i,\sum_j\varepsilon\big(\widetilde{\boldsymbol{v}}(\theta_j)\big)Z_j\Big)_\Omega + \sqrt{\delta}\lambda\Big(\sum_i\operatorname{div}\big(\widetilde{\boldsymbol{u}}(\theta_i)\big)Z_i,\sum_j\operatorname{div}\big(\widetilde{\boldsymbol{v}}(\theta_j)\big)Z_j\Big)_\Omega \\
&= 2\sqrt{\delta}\mu\Big\|\sum_i\varepsilon\big(\widetilde{\boldsymbol{u}}(\theta_i)\big)Z_i\Big\|_\Omega^2 + \sqrt{\delta}\lambda\Big\|\sum_i\operatorname{div}\big(\widetilde{\boldsymbol{u}}(\theta_i)\big)Z_i\Big\|_\Omega^2 \geq 0.
\end{aligned}$$

Then, we conclude that the matrix $\mathbf{C+A}$ is square, symmetric and positive definite. Hence, (1.71) has exactly one solution on $]0,T[$ by the theory of initial value problem for linear systems ([38]-Theorem 7.4). □

Let us prove now that the sequences $(p_k)_k$ and $(\boldsymbol{u}(p_k))_k$ of semi-discrete problem (1.70) are bounded. For obtaining this result, we firstly test (1.59) with $\boldsymbol{u}'(p_k) = \boldsymbol{u}(p_k')$, (1.70a) with p_k, add two equations and integrate with respect to time on $]0,t[$. Next, we test (1.70a) with p_k', differentiating in time (1.59) written for $p = p_k$, testing the resulting equation with $\boldsymbol{u}'(p_k)$, adding the two equations and integrating with respect to time. Finally, we add all equations together and apply the Young and Hölder inequalities. It is the object of the following lemma.

Lemma 18

Let $p_0 \in H^1(\Omega)$, $\boldsymbol{f} \in H^1(0,T;L^2(\Omega,\mathbb{R}^3))$, $g \in L^2(\Omega\times]0,T[)$.

i. The solution p_k and $\boldsymbol{u}(p_k)$ of the semi-discrete problem (1.70) satisfy the following uniform bounds

$$\big|\big|\boldsymbol{u}(p_k)\big|\big|^2_{L^\infty(0,T;\mathcal{X}_K)} + \|p_k\|^2_{L^\infty\left(0,T;L^2(\Omega)\right)} + \|\nabla p_k\|^2_{L^2\left(0,T;L^2(\Omega)\right)} \leq C \text{ and} \tag{1.72a}$$

$$\big|\big|\boldsymbol{u}'(p_k)\big|\big|^2_{L^2(0,T;\mathcal{X}_K)} + \|p_k'\|^2_{L^2\left(0,T;L^2(\Omega)\right)} + \|\nabla p_k\|^2_{L^\infty\left(0,T;L^2(\Omega)\right)} \leq C, \tag{1.72b}$$

where C depends on T, f, p_0 and g.

ii. Moreover, there exists a function $\hat{p}$ in $H^1\big(0,T;L^2(\Omega)\big) \cap L^\infty\big(0,T;H^1(\Omega)\big)$ and a function $\boldsymbol{w}$ in $H^1\big(0,T;H^1_0(\Omega)\big)$ such that

$$p_k \overset{*}{\rightharpoonup} \hat{p} \text{ weakly-}\star \text{ in } L^\infty\big(0,T;H^1(\Omega)\big), \tag{1.73a}$$

$$p_k \rightharpoonup \hat{p} \text{ weakly in } H^1\big(0,T;L^2(\Omega)\big), \tag{1.73b}$$

$$\boldsymbol{u}(p_k) \rightharpoonup \boldsymbol{w} \text{ weakly in } H^1(0,T;\mathcal{X}_K). \tag{1.73c}$$

Proof. We adapt some arguments of [47]. We test (1.59) with $\boldsymbol{u}'(p_k)$, (1.70a) by p_k, add two equations and integrate with respect to time on $]0,t[$. This becomes

$$\underbrace{\int_0^t \int_\omega e\, a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}\big(\boldsymbol{u}(p_k(t))\big)\gamma_{\rho\sigma}\big(\boldsymbol{u}'(p_k(t))\big)\sqrt{a}\,d\boldsymbol{x}\,dt}_{I_1}$$

$$+\underbrace{\frac{1}{12}\int_0^t \int_\omega e^3 a^{\alpha\beta\rho\sigma}\Upsilon_{\alpha\beta}\big(\boldsymbol{u}(p_k(t))\big)\Upsilon_{\rho\sigma}\big(\boldsymbol{u}'(p_k(t))\big)\sqrt{a}\,d\boldsymbol{x}\,dt}_{I_2}$$

$$+\underbrace{\int_0^t \int_\Omega p_k'(t)p_k(t)\sqrt{a}\; d\boldsymbol{X}\,dt}_{I_3} + \frac{\kappa}{\eta}\int_0^t \int_\Omega \big|\nabla p_k(t)\big|^2\sqrt{a}\,d\boldsymbol{X}\,dt$$

$$= \underbrace{\int_0^t \int_\Omega g p_k(t)\sqrt{a}\,d\boldsymbol{X}\,dt}_{I_4} + \underbrace{\int_0^t \int_\Omega \boldsymbol{f}\cdot\Big(\boldsymbol{u}'(p_k(t)) - z\big(\partial_\alpha\boldsymbol{u}'(p_k(t))\cdot\boldsymbol{a}_3\big)\boldsymbol{a}^\alpha\Big)\sqrt{a}\; d\boldsymbol{X}dt}_{I_5}. \tag{1.74}$$

Using the symmetric property of $\gamma_{\alpha\beta}$ and applying the inequality (1.39) to I_1, there exists a positive constants c_1 such that

$$\begin{aligned}
I_1 =& \int_0^t \int_\omega e\, a^{\alpha\beta\rho\sigma} \gamma_{\alpha\beta}\big(\boldsymbol{u}(p_k(t))\big)\gamma_{\rho\sigma}\big(\boldsymbol{u'}(p_k(t))\big)\sqrt{a}\, d\boldsymbol{x}\, dt \\
=& \frac{1}{2}\int_0^t \int_\omega e\, a^{\alpha\beta\rho\sigma} \big[\gamma_{\alpha\beta}\big(\boldsymbol{u}(p_k(t))\big)\gamma_{\rho\sigma}\big(\boldsymbol{u}(p_k(t))\big)\big]' \sqrt{a}\, d\boldsymbol{x}\, dt \\
=& \frac{1}{2}\int_\omega e\, a^{\alpha\beta\rho\sigma} \int_0^t \big[\gamma_{\alpha\beta}\big(\boldsymbol{u}(p_k(t))\big)\gamma_{\rho\sigma}\big(\boldsymbol{u}(p_k(t))\big)\big]' \, dt \sqrt{a}\, d\boldsymbol{x} \\
=& \frac{1}{2}\int_\omega e\, a^{\alpha\beta\rho\sigma} \big[\gamma_{\alpha\beta}\big(\boldsymbol{u}(p_k(t))\big)\gamma_{\rho\sigma}\big(\boldsymbol{u}(p_k(t))\big) - \gamma_{\alpha\beta}\big(\boldsymbol{u}(p_{k0})\big)\gamma_{\rho\sigma}\big(\boldsymbol{u}(p_{k0})\big)\big] \sqrt{a}\, d\boldsymbol{x} \\
=& \frac{1}{2}\int_\omega e\, a^{\alpha\beta\rho\sigma} \gamma_{\alpha\beta}\big(\boldsymbol{u}(p_k(t))\big)\gamma_{\rho\sigma}\big(\boldsymbol{u}(p_k(t))\big)\sqrt{a}\, d\boldsymbol{x} \\
& \qquad - \frac{1}{2}\int_\omega e\, a^{\alpha\beta\rho\sigma} \gamma_{\alpha\beta}\big(\boldsymbol{u}(p_{k0})\big)\gamma_{\rho\sigma}\big(\boldsymbol{u}(p_{k0})\big)\sqrt{a}\, d\boldsymbol{x} \\
\geq& \frac{c_1}{2}\int_\omega \big|\gamma_{\alpha\beta}\big(\boldsymbol{u}(p_k(t))\big)\big|^2 \sqrt{a}\, d\boldsymbol{x} - \frac{1}{2}\int_\omega e\, a^{\alpha\beta\rho\sigma} \gamma_{\alpha\beta}\big(\boldsymbol{u}(p_{k0})\big)\gamma_{\rho\sigma}\big(\boldsymbol{u}(p_{k0})\big)\sqrt{a}\, d\boldsymbol{x}.
\end{aligned}$$

Similarly, thanks to the symmetric property of $\Upsilon_{\alpha\beta}$ and the inequality (1.39), there exists a positive constants c_2 such that

$$\begin{aligned}
I_2 =& \frac{1}{12}\int_0^t \int_\omega e^3\, a^{\alpha\beta\rho\sigma} \Upsilon_{\alpha\beta}\big(\boldsymbol{u}(p_k(t))\big)\Upsilon_{\rho\sigma}\big(\boldsymbol{u'}(p_k(t))\big)\sqrt{a}\, d\boldsymbol{x} dt \\
=& \frac{1}{24}\int_0^t \int_\omega e^3\, a^{\alpha\beta\rho\sigma} \big[\Upsilon_{\alpha\beta}\big(\boldsymbol{u}(p_k(t))\big)\Upsilon_{\rho\sigma}\big(\boldsymbol{u}(p_k(t))\big)\big]' \sqrt{a}\, d\boldsymbol{x} dt \\
=& \frac{1}{24}\int_\omega e^3\, a^{\alpha\beta\rho\sigma} \left(\int_0^t \big[\Upsilon_{\alpha\beta}\big(\boldsymbol{u}(p_k(t))\big)\Upsilon_{\rho\sigma}\big(\boldsymbol{u}(p_k(t))\big)\big]' dt \right) \sqrt{a}\, d\boldsymbol{x} \\
=& \frac{1}{24}\int_\omega e^3\, a^{\alpha\beta\rho\sigma} \Big[\Upsilon_{\alpha\beta}\big(\boldsymbol{u}(p_k(t))\big)\Upsilon_{\rho\sigma}\big(\boldsymbol{u}(p_k(t))\big) - \Upsilon_{\alpha\beta}\big(\boldsymbol{u}(p_{k0})(t))\big)\Upsilon_{\rho\sigma}\big(\boldsymbol{u}(p_{k0})\big) \Big] \sqrt{a}\, d\boldsymbol{x} \\
=& \frac{1}{24}\int_\omega e^3\, a^{\alpha\beta\rho\sigma} \Upsilon_{\alpha\beta}\big(\boldsymbol{u}(p_k(t))\big)\Upsilon_{\rho\sigma}\big(\boldsymbol{u}(p_k(t))\big)\sqrt{a}\, d\boldsymbol{x} \\
& \qquad - \frac{1}{24}\int_\omega e^3\, a^{\alpha\beta\rho\sigma} \Upsilon_{\alpha\beta}\big(\boldsymbol{u}(p_{k0})\big)\Upsilon_{\rho\sigma}\big(\boldsymbol{u}(p_{k0})\big)\sqrt{a}\, d\boldsymbol{x} \\
\geq& \frac{c_2}{24}\int_\omega \big|\Upsilon_{\alpha\beta}\big(\boldsymbol{u}(p_k(t))\big)\big|^2 \sqrt{a}\, d\boldsymbol{x} - \frac{1}{24}\int_\omega e^3\, a^{\alpha\beta\rho\sigma} \Upsilon_{\alpha\beta}\big(\boldsymbol{u}(p_{k0})\big)\Upsilon_{\rho\sigma}\big(\boldsymbol{u}(p_{k0})\big)\sqrt{a}\, d\boldsymbol{x}.
\end{aligned}$$

Subsequently, we make I_3 clearer,

$$\begin{aligned}
I_3 &= \int_0^t \int_\Omega p_k'(t) p_k(t) \sqrt{a}\, d\boldsymbol{X}\, dt = \int_\Omega \left(\int_0^t p_k'(t) p_k(t)\, dt \right) \sqrt{a}\, d\boldsymbol{X} \\
&= \frac{1}{2}\int_\Omega \left(p_k^2(t) - p_{k0}^2 \right) \sqrt{a}\, d\boldsymbol{X} = \frac{1}{2}\int_\Omega \big|p_k(t)\big|^2 \sqrt{a}\, d\boldsymbol{X} - \frac{1}{2}\int_\Omega \big|p_{k0}\big|^2 \sqrt{a}\, d\boldsymbol{X}.
\end{aligned}$$

Applying the Young and Hölder's inequalities to I_4, we have

$$\begin{aligned}
I_4 &= \int_0^t \int_\Omega g p_k(t) \sqrt{a}\, d\boldsymbol{X}\, dt \\
&\leq \sqrt{M} \int_0^t \int_\Omega g p_k(t)\, d\boldsymbol{X}\, dt \leq \sqrt{M} \int_0^t \|g\|_{0,\Omega} \|p_k(t)\|_{0,\Omega} dt \\
&\leq \sqrt{M} \frac{1}{2\varepsilon_1} \int_0^t \|g\|^2_{0,\Omega}\, dt + \sqrt{M} \frac{\varepsilon_1}{2} \int_0^t \|p_k(t)\|^2_{0,\Omega} dt \\
&\leq \sqrt{M} \frac{1}{2\varepsilon_1} \|g\|^2_{L^2\left(0,T;L^2(\Omega)\right)} + \sqrt{M} \frac{\varepsilon_1}{2} \int_0^t \|\nabla p_k(t)\|^2_{0,\Omega} \\
&\leq \sqrt{M} \frac{1}{2\varepsilon_1} \|g\|^2_{L^2\left(0,T;L^2(\Omega)\right)} + \sqrt{M} \frac{\varepsilon_1}{2} \|\nabla p_k\|^2_{L^2\left(0,T;L^2(\Omega)\right)}.
\end{aligned}$$

Lastly, analogous to I_4, we derive the upper bound of I_5 by the Young and Hölder inequalities

$$\begin{aligned}
I_5 &= \int_0^t \int_\Omega \boldsymbol{f} \cdot \left[\boldsymbol{u}'(p_k(t)) - z(\partial_\alpha \boldsymbol{u}'(p_k(t)) \cdot \boldsymbol{a}_3) \boldsymbol{a}^\alpha\right] \sqrt{a}\; d\boldsymbol{X} dt \\
&\leq \sqrt{M} \int_0^t \|\boldsymbol{f}\|_{0,\Omega} \left\|\boldsymbol{u}'(p_k(t)) - z(\partial_\alpha \boldsymbol{u}'(p_k(t)) \cdot \boldsymbol{a}_3) \boldsymbol{a}^\alpha\right\|_{0,\Omega}\, dt \\
&\leq \frac{\sqrt{M}}{2\varepsilon_2} \int_0^t \|\boldsymbol{f}\|^2_{0,\Omega}\, dt + \frac{\sqrt{M}\varepsilon_2}{2} \int_0^t \left[\|\boldsymbol{u}'(p_k(t))\|^2_{0,\omega} + \frac{e}{2} \|(\partial_\alpha \boldsymbol{u}'(p_k(t)) \cdot \boldsymbol{a}_3) \boldsymbol{a}^\alpha\|^2_{0,\omega}\right] dt \\
&\leq \frac{\sqrt{M}}{2\varepsilon_2} \|\boldsymbol{f}\|^2_{L^2\left(0,T;L^2(\Omega,\mathbb{R}^3)\right)} + \frac{\sqrt{M}\varepsilon_2}{2} \left\|\boldsymbol{u}'(p_k)\right\|^2_{L^2(0,T;\mathcal{X}_k)} \quad (e \ll 1).
\end{aligned}$$

where ε_1 and ε_2 are positive constants from Young inequality.

Consequently, (1.74) becomes

$$\begin{aligned}
&\frac{c_1}{2} \int_\omega \left|\gamma_{\alpha\beta}\big(\boldsymbol{u}(p_k(t))\big)\right|^2 \sqrt{a}\, dx + \frac{c_2}{24} \int_\omega \left|\Upsilon_{\alpha\beta}\big(\boldsymbol{u}(p_k(t))\big)\right|^2 \sqrt{a}\, d\boldsymbol{x} \\
&+ \frac{1}{2} \int_\Omega \left|p_k(t)\right|^2 \sqrt{a}\, d\boldsymbol{X} + \frac{\kappa}{\eta} \int_0^t \int_\Omega \left|\nabla p_k(t)\right|^2 \sqrt{a}\, d\boldsymbol{X}\, dt \\
&\leq \frac{1}{2} \int_\omega e\, a^{\alpha\beta\rho\sigma} \gamma_{\alpha\beta}\big(\boldsymbol{u}(p_{k0})\big) \gamma_{\rho\sigma}\big(\boldsymbol{u}(p_{k0})\big) \sqrt{a}\, d\boldsymbol{x} \\
&+ \frac{1}{24} \int_\omega e^3\, a^{\alpha\beta\rho\sigma} \Upsilon_{\alpha\beta}\big(\boldsymbol{u}(p_{k0})\big) \Upsilon_{\rho\sigma}\big(\boldsymbol{u}(p_{k0})\big) \sqrt{a}\, d\boldsymbol{x} + \frac{1}{2} \int_\Omega \left|p_{k0}\right|^2 \sqrt{a}\, d\boldsymbol{X} \\
&+ \sqrt{M} \frac{1}{2\varepsilon_2} \|g\|^2_{L^2\left(0,T;L^2(\Omega)\right)} + \sqrt{M} \frac{\varepsilon_2}{2} \|\nabla p_k\|^2_{L^2\left(0,T;L^2(\Omega)\right)} + \frac{\sqrt{M}}{2\varepsilon_1} \|\boldsymbol{f}\|^2_{L^2\left(0,T,L^2(\Omega,\mathbb{R}^3)\right)} \\
&+ \frac{\sqrt{M}\varepsilon_1}{2} \left\|\big(\boldsymbol{u}'(p_k), \boldsymbol{r}'(p_k)\big)\right\|^2_{L^2(0,T;\mathcal{X}_k)}.
\end{aligned}$$

Since $0 < \sqrt{\delta} \leq \sqrt{a(x)} \leq \sqrt{M} < \infty$ (see (2)-page 12),

$$c_3\sqrt{\delta}\big\|\big(\boldsymbol{u}(p_k(t))\big)\big\|^2_{\mathcal{X}_K} + \frac{\sqrt{\delta}}{2}\|p_k(t)\|^2_{0,\Omega} + \frac{\kappa\sqrt{\delta}}{\eta}\int_0^t \|\nabla p_k(t)\|^2_{0,\Omega}\,dt$$

$$\leq \frac{\sqrt{M}}{2}\int_\omega e\, a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}\big(\boldsymbol{u}(p_{k0})\big)\gamma_{\rho\sigma}\big(\boldsymbol{u}(p_{k0})\big)\,d\boldsymbol{x}$$

$$+\frac{\sqrt{M}}{24}\int_\omega e^3\, a^{\alpha\beta\rho\sigma}\Upsilon_{\alpha\beta}\big(\boldsymbol{u}(p_{k0})\big)\Upsilon_{\rho\sigma}\big(\boldsymbol{u}(p_{k0})\big)\,d\boldsymbol{x} + \frac{\sqrt{M}}{2}\int_\Omega \big|p_{k0}\big|^2\,d\boldsymbol{X}$$

$$+\sqrt{M}\frac{1}{2\varepsilon_2}\|g\|^2_{L^2\left(0,T;L^2(\Omega)\right)} + \sqrt{M}\frac{\varepsilon_2}{2}\|\nabla p_k\|^2_{L^2\left(0,T;L^2(\Omega)\right)} + \frac{\sqrt{M}}{2\varepsilon_1}\|\boldsymbol{f}\|^2_{L^2\left(0,T;L^2(\Omega,\mathbb{R}^3)\right)}$$

$$+\frac{\sqrt{M}\varepsilon_1}{2}\big\|\big(\boldsymbol{u}'(p_k)\big)\big\|^2_{L^2(0,T;\mathcal{X}_K)},$$

here $c_3 = \min\left\{\frac{c_1}{2};\frac{c_2}{24}\right\}$.
It follows to

$$\left(c_3\sqrt{\delta} - \frac{\sqrt{M}\varepsilon_1}{2}\right)\big\|\big(\boldsymbol{u}(p_k)\big)\big\|^2_{L^\infty(0,T;\mathcal{X}_K)} + \frac{\sqrt{\delta}}{2}\|p_k(t)\|^2_{L^\infty\left(0,T,L^2(\Omega)\right)}$$

$$+\left(\frac{\kappa\sqrt{\delta}}{\eta} - \sqrt{M}\frac{\varepsilon_2}{2}\right)\|\nabla p_k\|^2_{L^2\left(0,T;L^2(\Omega)\right)}$$

$$\leq \frac{\sqrt{M}}{2}\int_\omega e\, a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}\big(\boldsymbol{u}(p_{k0})\big)\gamma_{\rho\sigma}\big(\boldsymbol{u}(p_{k0})\big)\,d\boldsymbol{x} + \frac{\sqrt{M}}{24}\int_\omega e^3\, a^{\alpha\beta\rho\sigma}\Upsilon_{\alpha\beta}\big(\boldsymbol{u}(p_{k0})\big)\Upsilon_{\rho\sigma}\big(\boldsymbol{u}(p_{k0})\big)\,d\boldsymbol{x}$$

$$+\frac{\sqrt{M}}{2}\int_\Omega \big|p_{k0}\big|^2\,d\boldsymbol{X} + \sqrt{M}\frac{1}{2\varepsilon_2}\|g\|^2_{L^2\left(0,T;L^2(\Omega)\right)} + \frac{\sqrt{M}}{2\varepsilon_1}\|\boldsymbol{f}\|^2_{L^2\left(0,T;L^2(\Omega,\mathbb{R}^3)\right)}.$$

By choosing the ε_1 and ε_2 suitably, *i.e.*, $\varepsilon_1 < \frac{2c_3\sqrt{\delta}}{\sqrt{M}}$ and $\varepsilon_2 < \frac{2\kappa\sqrt{\delta}}{\eta\sqrt{M}}$, we obtain that

$$\big\|\big(\boldsymbol{u}(p_k)\big)\big\|^2_{L^\infty(0,T;\mathcal{X}_K)} + \|p_k\|^2_{L^\infty\left(0,T,L^2(\Omega)\right)} + \|\nabla p_k\|^2_{L^2\left(0,T;L^2(\Omega)\right)} \leq C_1, \tag{1.75}$$

here C_1 depends only on T, $\|\boldsymbol{f}\|_{L^2\left(0,T;L^2(\Omega,\mathbb{R}^3)\right)}$, $\|p_0\|_{L^\infty\left(0,T;L^2(\Omega)\right)}$ and $\|g\|_{L^2\left(0,T;L^2(\Omega)\right)}$.
We have just proved the inequality (1.72a). Next, we derive the inequality (1.72b) by testing (1.70a) with p_k', it leads

$$\int_\Omega |p_k'|^2\sqrt{a}\,d\boldsymbol{X} + \alpha\int_\Omega \operatorname{div}\big(\boldsymbol{u}'(p_k) - z(\partial_\alpha\boldsymbol{u}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)p_k'\sqrt{a}\,d\boldsymbol{X}$$

$$+\frac{\kappa}{\eta}\int_\Omega \nabla p_k\cdot\nabla p_k'\sqrt{a}\,d\boldsymbol{X} = \int_\Omega g p_k'\sqrt{a}\,d\boldsymbol{X}. \tag{1.76}$$

And differentiating in time (1.40b) written for $p = p_k$,

$$\int_\omega e\, a^{\alpha\beta\rho\sigma}\Big[\gamma_{\rho\sigma}(\boldsymbol{u}'(p_k))\gamma_{\alpha\beta}(\boldsymbol{v}) + \frac{e^2}{12}\Upsilon_{\rho\sigma}(\boldsymbol{u}'(p_k))\Upsilon_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s})\Big]\sqrt{a}\; d\boldsymbol{x}$$
$$- \alpha \int_\Omega p_k' \mathrm{div}\big(\boldsymbol{v} - z(\partial_\alpha \boldsymbol{v}\cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)\sqrt{a}\;\; d\boldsymbol{X} = \int_\Omega \boldsymbol{f}'\cdot\big(\boldsymbol{v} - z(\partial_\alpha \boldsymbol{v}\cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)\sqrt{a}\; d\boldsymbol{X}.$$

Testing the above achieving equation with $\boldsymbol{u}'(p_k)$,

$$\int_\omega e\, a^{\alpha\beta\rho\sigma}\Big[\gamma_{\rho\sigma}(\boldsymbol{u}'(p_k))\gamma_{\alpha\beta}(\boldsymbol{u}'(p_k)) + \frac{e^2}{12}\Upsilon_{\rho\sigma}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\Upsilon_{\alpha\beta}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\Big]\sqrt{a}\; d\boldsymbol{x}$$
$$- \alpha \int_\Omega p_k' \mathrm{div}\Big(\boldsymbol{u}'(p_k) - z(\partial_\alpha \boldsymbol{u}'(p_k)\cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha\Big)\sqrt{a}\;\; d\boldsymbol{X} = \int_\Omega \boldsymbol{f}'\cdot\big(\boldsymbol{u}' - z(\partial_\alpha \boldsymbol{u}'\cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)\sqrt{a}\; d\boldsymbol{X}. \tag{1.77}$$

Adding the two equations (1.76) and (1.77) then integrating with respect to time, we obtain

$$\underbrace{\int_0^t\int_\omega e\, a^{\alpha\beta\rho\sigma}\Big[\gamma_{\rho\sigma}(\boldsymbol{u}'(p_k))\gamma_{\alpha\beta}(\boldsymbol{u}'(p_k)) + \frac{e^2}{12}\Upsilon_{\rho\sigma}(\boldsymbol{u}'(p_k))\Upsilon_{\alpha\beta}(\boldsymbol{u}'(p_k))\Big]\sqrt{a}\; d\boldsymbol{x}dt}_{Y_1}$$
$$+ \underbrace{\int_0^t\int_\Omega |p_k'|^2\sqrt{a}\, d\boldsymbol{X}dt + \frac{\kappa}{\eta}\int_0^t\int_\Omega \nabla p_k\cdot\nabla p_k'\sqrt{a}\, d\boldsymbol{X}dt}_{Y_2}$$
$$= \underbrace{\int_0^t\int_\Omega \boldsymbol{f}'\cdot\big(\boldsymbol{u}' - z(\partial_\alpha \boldsymbol{u}'\cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha\big)\sqrt{a}\; d\boldsymbol{X}dt}_{Y_3} + \underbrace{\int_0^t\int_\Omega g p_k'\sqrt{a}\, d\boldsymbol{X}dt.}_{Y_4} \tag{1.78}$$

Applying the inequality (1.39), there are positive constants c_1', c_2', c_3' such that

$$Y_1 = \int_0^t\int_\omega e\, a^{\alpha\beta\rho\sigma}\Big[\gamma_{\rho\sigma}(\boldsymbol{u}'(p_k))\gamma_{\alpha\beta}(\boldsymbol{u}'(p_k)) + \frac{e^2}{12}\Upsilon_{\rho\sigma}(\boldsymbol{u}'(p_k))\Upsilon_{\alpha\beta}(\boldsymbol{u}'(p_k))\Big]\sqrt{a}\; d\boldsymbol{x}dt$$
$$\geq c_1'\int_0^t\int_\omega\Big[e|\gamma_{\alpha\beta}(\boldsymbol{u}'(p_k))|^2 + \frac{e^3}{12}|\Upsilon_{\alpha\beta}(\boldsymbol{u}'(p_k))|^2\Big]\sqrt{a}d\boldsymbol{x}dt \geq c_2'\sqrt{\delta}\int_0^t |||\boldsymbol{u}'(p_k)|||^2 dt$$
$$\geq c_3'\sqrt{\delta}\int_0^t \|\boldsymbol{u}'(p_k)\|^2_{\mathcal{X}_K}dt = c_3'\sqrt{\delta}\|\boldsymbol{u}'(p_k)\|^2_{L^2(0,t;\mathcal{X}_K)}.$$

Using the bounds of $\sqrt{a}$ and integration with respect to time, we carry out the lower bound of Y_2,

$$Y_2 = \int_0^t\int_\Omega |p_k'|^2\sqrt{a}\, d\boldsymbol{X}dt + \frac{\kappa}{\eta}\int_0^t\int_\Omega \nabla p_k\cdot\nabla p_k'\sqrt{a}\, d\boldsymbol{X}dt$$
$$\geq \sqrt{\delta}\int_0^t \|p_k'\|^2_{L^2(\Omega)}dt + \frac{1}{2}\int_\Omega\big(|\nabla p_k(t)|^2 - |\nabla p_{k0}|^2\big)\sqrt{a}\, d\boldsymbol{X}$$
$$\geq \sqrt{\delta}\|p_k'\|^2_{L^2\left(0,t;L^2(\Omega)\right)} + \frac{\sqrt{\delta}}{2}\|\nabla p_k\|^2_{L^2(\Omega)} - \frac{\sqrt{M}}{2}\|\nabla p_{k0}\|^2_{L^2(\Omega)}.$$

In the following, we employ the Young and Hölder inequalities to find the upper bound of Y_3,

$$\begin{aligned}
Y_3 &= \int_0^t \int_\Omega \boldsymbol{f}' \cdot \big[\boldsymbol{u}'(p_k(t)) - z(\partial_\alpha \boldsymbol{u}'(p_k(t)) \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha\big]\sqrt{a}\, d\boldsymbol{X} dt \\
&\leq \sqrt{M} \int_0^t \|\boldsymbol{f}'\|_{0,\Omega} \big\|\boldsymbol{u}'(p_k(t)) - z(\partial_\alpha \boldsymbol{u}'(p_k(t)) \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha\big\|_{0,\Omega}\, dt \\
&\leq \frac{\sqrt{M}}{2\varepsilon_1'} \int_0^t \|\boldsymbol{f}'\|^2_{0,\Omega}\, dt + \frac{\sqrt{M}\varepsilon_1'}{2} \int_0^t \left[\|\boldsymbol{u}'(p_k(t))\|^2_{0,\omega} + \frac{e}{2}\|(\partial_\alpha \boldsymbol{u}'(p_k(t)) \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha\|^2_{0,\omega}\right] dt \\
&\leq \frac{\sqrt{M}}{2\varepsilon_1'} \|\boldsymbol{f}'\|^2_{L^2\left(0,T;L^2(\Omega,\mathbb{R}^3)\right)} + \frac{\sqrt{M}\varepsilon_1'}{2}\big\|\boldsymbol{u}'(p_k)\big\|^2_{L^2(0,T;\mathcal{X}_k)} \quad (e \ll 1).
\end{aligned}$$

Similar to Y_3, we also derive the lower bound of Y_4,

$$\begin{aligned}
Y_4 &= \int_0^t \int_\Omega g p_k'(t)\sqrt{a}\, d\boldsymbol{X}\, dt \leq \sqrt{M} \int_0^t \int_\Omega g p_k'(t)\, d\boldsymbol{X}\, dt \leq \sqrt{M} \int_0^t \|g\|_{0,\Omega} \|p_k(t)\|_{0,\Omega} dt \\
&\leq \frac{\sqrt{M}}{2\varepsilon_2'} \int_0^t \|g\|^2_{0,\Omega}\, dt + \frac{\sqrt{M}\varepsilon_2'}{2} \int_0^t \|p_k'(t)\|^2_{0,\Omega} dt \leq \frac{\sqrt{M}}{2\varepsilon_2'} \|g\|^2_{L^2\left(0,T;L^2(\Omega)\right)} \\
&\qquad + \frac{\sqrt{M}\varepsilon_2'}{2} \|p_k'\|^2_{L^2\left(0,T;L^2(\Omega)\right)}.
\end{aligned}$$

where ε_1 and ε_2 are positive constants from Young inequality.
Therefore, (1.78) becomes

$$\begin{aligned}
&c_3'\sqrt{\delta} \parallel \boldsymbol{u}'(p_k) \parallel_{L^2\left(0,t;\mathcal{X}_K\right)} + \sqrt{\delta} \parallel p_k' \parallel^2_{L^2(0,t;L^2(\Omega))} + \frac{\sqrt{\delta}}{2} \parallel \nabla p_k \parallel^2_{L^2(\Omega)} \\
&\quad\leq \frac{\sqrt{M}}{2\varepsilon_1'} \|\boldsymbol{f}'\|^2_{L^2\left(0,T;L^2(\Omega,\mathbb{R}^3)\right)} + \frac{\sqrt{M}\varepsilon_1'}{2}\big\|\boldsymbol{u}'(p_k)\big\|^2_{L^2(0,T;\mathcal{X}_k)} + \frac{\sqrt{M}}{2\varepsilon_2'} \|g\|^2_{L^2\left(0,T;L^2(\Omega)\right)} \\
&\qquad + \frac{\sqrt{M}\varepsilon_2'}{2} \|p_k'\|^2_{L^2\left(0,T;L^2(\Omega)\right)} + \frac{M}{2} \parallel \nabla p_{k0} \parallel^2_{L^2(\Omega)} .
\end{aligned}$$

It implies to

$$\begin{aligned}
&\left(c_3'\sqrt{\delta} - \frac{\sqrt{M}\varepsilon_1'}{2}\right) \parallel \boldsymbol{u}'(p_k) \parallel_{L^2\left(0,t;\mathcal{X}_K\right)} + \left(\sqrt{\delta} - \frac{\sqrt{M}\varepsilon_2'}{2}\right) \parallel p_k' \parallel^2_{L^2(0,t;L^2(\Omega))} + \frac{\sqrt{\delta}}{2} \parallel \nabla p_k \parallel^2_{L^2(\Omega)} \\
&\quad\leq \frac{\sqrt{M}}{2\varepsilon_1'} \|\boldsymbol{f}'\|^2_{L^2\left(0,T;L^2(\Omega,\mathbb{R}^3)\right)} + \frac{\sqrt{M}}{2\varepsilon_2'} \|g\|^2_{L^2\left(0,T;L^2(\Omega)\right)} + \frac{M}{2} \parallel \nabla p_{k0} \parallel^2_{L^2(\Omega)} .
\end{aligned}$$

By choosing the ε_1' and ε_2' suitably, *i.e.*, $\varepsilon_1' < \dfrac{2c_3'\sqrt{\delta}}{\sqrt{M}}$ and $\varepsilon_2' < \dfrac{2\sqrt{\delta}}{\sqrt{M}}$, we obtain that

$$\big|\big|\boldsymbol{u}'(p_k)\big|\big|^2_{L^2(0,T;\mathcal{X}_K)} + \|p_k'\|^2_{L^2\left(0,T;L^2(\Omega)\right)} + \|\nabla p_k\|^2_{L^\infty\left(0,T;L^2(\Omega)\right)} \leq C_2,$$

here C_2 depends only on T, $\|\boldsymbol{f}\|_{L^2\left(0,T;L^2(\Omega,\mathbb{R}^3)\right)}$, $\|p_0\|_{L^\infty\left(0,T;L^2(\Omega)\right)}$ and $\|g\|_{L^2\left(0,T;L^2(\Omega)\right)}$. This inequality implies (1.72b) in Lemma 18. From (1.72), we have that $\|p_k\|_{L^\infty\left(0,T;H^1(\Omega)\right)}$, $\|p_k\|_{H^1\left(0,T;L^2(\Omega)\right)}$ and $\left\|\boldsymbol{u}(p_k)\right\|^2_{H^1(0,T;\mathcal{X}_K)}$ are uniformly bounded. This yields (1.73). We just finished the proof of Lemma 18. □

Lastly, we prove that the pressure solution $\hat{p}$, which is derived in Lemma 18, satisfies (1.65) and (1.40c).

Lemma 19

The solution $\hat{p}$ in Lemma 18 solves (1.65) and (1.40c), and $\boldsymbol{w} = \boldsymbol{u}(\hat{p})$.

Proof. We pass to the limit in (1.59) after having replaced p by p_k and based on (1.73c),

$$\int_\omega e\, a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(\boldsymbol{w})\gamma_{\rho\sigma}(\boldsymbol{v})\sqrt{a}\,dx + \frac{1}{12}\int_\omega e^3\, a^{\alpha\beta\rho\sigma}\Upsilon_{\alpha\beta}(\boldsymbol{w})\Upsilon_{\rho\sigma}(\boldsymbol{v})\sqrt{a}\,d\boldsymbol{x}$$
$$-\int_\Omega \hat{p}\ \mathrm{div}\ \left(\boldsymbol{v} - z(\partial_\alpha \boldsymbol{v}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\right)\sqrt{a}\ d\boldsymbol{X} = \int_\Omega \boldsymbol{f}\cdot\left(\boldsymbol{v} - z(\partial_\alpha \boldsymbol{v}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\alpha\right)\sqrt{a}\ d\boldsymbol{X}.$$

By uniqueness of (1.59), we have $\boldsymbol{w} = \boldsymbol{u}(\hat{p})$. Then passing to the limit in (1.70), we also obtain that $\hat{p}$ solves (1.65) and the continuity in time of $\hat{p}$ gives the initial condition (1.40c). □

Proof. (of Theorem 8). From Lemma 15 and Lemma 17, we can derive that the problem (1.40) is well posed. The remain of theorem is proved by Lemma 18 and Lemma 19. □

Conclusion: In this chapter, we established the derivation of a poroelastic shell model of Koiter type coupled with the Biot model. It is the Biot-Koiter poroelastic shell model. We also proved the well-posedness of the resulting equations by the theory of DAEs and Galerkin semi-discrete method. We used here the linearly elastic thin shell with little regularity in which the midsurfaces have curvature discontinuities.

Chapter 2

Derivation and well-posedness for Biot-Naghdi poroelastic shell model

In this chapter we derive, from the three dimensional elasticity equations, another two-dimensional linear model for poroelastic shell. A well known shell model, in the engineering community, as well as Koiter's model, is due to P. M. Naghdi. In Naghdi's approach, the a priori assumption of a mechanical nature about the stress inside the shell is the same as in Koiter's approach (Section 1.2), but the a priori assumption of a geometrical nature is different: The points situated on a line normal to S remain on a line and the lengths are constant along this line after the deformation has taken place, as in Koiter's approach; however, this line need no longer remain normal to the deformed midsurface. In the linearized version of this approach, there are five unknowns, the three covariant components $u_i : \overline{\omega} \to \mathbb{R}$ of the *displacement* filed $u_i \boldsymbol{a}^i$ of the midsurface S and the two covariant components $r_\alpha : \overline{\omega} \to \mathbb{R}$ of the linearized *rotation* field $r_\alpha \boldsymbol{a}^\alpha$ of the unit normal vector along S.

We also propose the strong formulation of Naghdi-Biot coupled model. The strong formation will be used later for establishing the fluid-structure interaction between incompressible flow and a poroelastic shell.

More precisely, we derive the weak coupled formulation of shell model of Naghdi type and Biot model (see Theorem 9), and its well-posedness (see Theorem 10). In order to cracking the trouble of deriving the weak form, the Reissner-Mindlin displacements space $\mathcal{V}_{RM} = \{\boldsymbol{V} \in H^1(\Omega;\mathbb{R}^3), \varepsilon_{33}(\boldsymbol{V}) = 0\}$ is chosen. For the well-posedness, we firstly prove the well-posedness of $\boldsymbol{U}$ in the weak form of constitutive equation (1.37a) by Banach-Nečas-Babuška theorem with p given as well (see Lemma 30). The well-posedness of p in the weak form of mass conservation equation (1.37b) is also obtained by making use of the sem-discrete Galerkin method and the theory of initial value problem for liner systems (see Lemma 32, 33 & 34). At last, we proceed the strong formulation of Naghdi-Biot coupled model by using the contravariant components of the stress resultant $n^{\rho\sigma}$, of the stress couple $m^{\rho\sigma}$ and of the transverse shear force t^β (see Theorem 11).

Using the same domain Ω in the previous chapter, let us consider the functional spaces:

$$\mathcal{X}_N = \{(\boldsymbol{v}, \boldsymbol{s} = (s_\alpha)) \in H^1(\omega;\mathbb{R}^3) \times H^1(\omega)^2; \boldsymbol{v} = 0, s_\alpha = 0 \text{ on } \gamma_0\}, \tag{2.1}$$
$$\mathcal{V}_N = L^\infty(0,T;\mathcal{X}_N), \tag{2.2}$$
$$\mathcal{W}_N = \{q \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)); q = 0 \text{ on } \Gamma_0\}. \tag{2.3}$$

The space $\mathcal{X}_N$ equipped with the norm

$$\|(v,s)\|_{\mathcal{X}_N} = (\|v\|^2_{H^1(\omega;\mathbb{R}^3)} + \sum_\alpha \|s_\alpha\|^2_{H^1(\omega)})^{1/2} \tag{2.4}$$

is a Hilbert space.

Let us introduce the space of Reissner-Mindlin displacements

$$\mathcal{V}_{RM} = \{\boldsymbol{V} \in H^1(\Omega;\mathbb{R}^3), \varepsilon_{33}(\boldsymbol{V}) = 0\},$$

and describe its elements. It is the object of the following lemma.

Lemma 20

A displacement field $\boldsymbol{V} \in H^1(\Omega;\mathbb{R}^3)$ is a Reissner-Mindlin displacement if and only if there exist $\boldsymbol{v} \in H^1(\omega,\mathbb{R}^3)$ and $s = (s_\alpha) \in H^1(\omega)^2$ such that

$$\boldsymbol{V}(x,z) = \boldsymbol{v}(x) + z\, s_\alpha a^\alpha.$$

Proof. Let $\boldsymbol{V}(x,z) \in \mathcal{V}_{RM}$, we have

$$0 = \varepsilon_{33}(\boldsymbol{V}) = \partial_3 \boldsymbol{V} \cdot g_3 = \partial_3 \boldsymbol{V} \cdot a_3.$$

So, there exists $\boldsymbol{w} \in L^2(\Omega,\mathbb{R}^3)$ such that $\partial_3 \boldsymbol{V} = \boldsymbol{w} \wedge a_3$. From that, we have

$$\int_0^z \partial_3 \boldsymbol{V}(x,\tau)d\tau = \underbrace{(\int_0^z \boldsymbol{w}(x,\tau)d\tau)}_{\Theta(x,z)} \wedge a_3.$$

Hence,

$$\boldsymbol{V}(x,z) - \boldsymbol{V}(x,0) = \Theta(x,z) \wedge a_3, \text{ with } \Theta(x,0) = 0.$$

We can present $\Theta(x,z)$ as below

$$\Theta(x,z) = \underbrace{\Theta(x,0)}_{=0} + z\partial_3\Theta(x,0) + z^2\partial_3^2\Theta(x,0) + \cdots$$

Since z is belong to $]\frac{-e}{2}; \frac{e}{2}[$ and e is the thickness of the shell, $z^n \approx 0$ with $n \geq 2$. Rewriting $\boldsymbol{V}$:

$$\boldsymbol{V}(x,z) = \boldsymbol{V}(x,0) + z\partial_3\Theta(x,0) \wedge a_3.$$

Setting $\boldsymbol{v} = \boldsymbol{V}(\cdot, 0) \in H^1(\omega, \mathbb{R}^3)$ and $s = \partial_3 \Theta(x, 0) \wedge a_3$, one have $\boldsymbol{V}(x, z) = \boldsymbol{v}(x) + zs(x)$.

Next, we prove that s is a function of $H^1(\omega, \mathbb{R}^3)$. We have

$$\begin{aligned}\|zs\|^2_{L^2(\Omega,\mathbb{R}^3)} &= \|\boldsymbol{V} - \boldsymbol{v}\|^2_{L^2(\Omega,\mathbb{R}^3)} \\ &\leq \|\boldsymbol{V}\|^2_{L^2(\Omega,\mathbb{R}^3)} + 2\|\boldsymbol{V}\|_{L^2(\Omega,\mathbb{R}^3)}\|\boldsymbol{v}\|_{L^2(\Omega,\mathbb{R}^3)} + \|\boldsymbol{v}\|^2_{L^2(\Omega,\mathbb{R}^3)} \\ &= \|\boldsymbol{V}\|^2_{L^2(\Omega,\mathbb{R}^3)} + 2\|\boldsymbol{V}\|_{L^2(\Omega,\mathbb{R}^3)}\|\sqrt{e}\,\boldsymbol{v}\|_{L^2(\omega,\mathbb{R}^3)} + \|\sqrt{e}\,\boldsymbol{v}\|^2_{L^2(\omega,\mathbb{R}^3)},\end{aligned}$$

and

$$\begin{aligned}\|zs\|^2_{L^2(\Omega,\mathbb{R}^3)} &= \int_\omega (\int_{-e/2}^{e/2} z^2\,dz)|s(x)|^2\,dx \\ &= \frac{1}{12}\int_\omega e^3|s(x)|^2\,dx \\ &= \frac{1}{12}\|e^{3/2}s\|^2_{L^2(\omega,\mathbb{R}^3)}.\end{aligned}$$

From that, we obtain

$$\frac{1}{12}\|e^{3/2}\,s\|^2_{L^2(\omega,\mathbb{R}^3)} \leq \|\boldsymbol{V}\|^2_{L^2(\Omega,\mathbb{R}^3)} + 2\|\boldsymbol{V}\|_{L^2(\Omega,\mathbb{R}^3)}\|e\|^{1/2}_{L^\infty(\omega)}\|\,\boldsymbol{v}\|_{L^2(\omega,\mathbb{R}^3)} + \|e\|_{L^\infty(\omega)}\|\,\boldsymbol{v}\|^2_{L^2(\omega,\mathbb{R}^3)}. \tag{2.5}$$

On the other hand, we have

$$\begin{aligned}\|\nabla(zs)\|^2_{L^2(\Omega,\mathbb{R}^3)} &= \|\nabla\boldsymbol{V} - \nabla\boldsymbol{v}\|^2_{L^2(\Omega,\mathbb{R}^3)} \\ &\leq \|\nabla\boldsymbol{V}\|^2_{L^2(\Omega,\mathbb{R}^3)} + 2\|\nabla\boldsymbol{V}\|_{L^2(\Omega,\mathbb{R}^3)}\|\nabla\boldsymbol{v}\|_{L^2(\Omega,\mathbb{R}^3)} + \|\nabla\boldsymbol{v}\|^2_{L^2(\Omega,\mathbb{R}^3)} \\ &= \|\nabla\boldsymbol{V}\|^2_{L^2(\Omega,\mathbb{R}^3)} + 2\|\nabla\boldsymbol{V}\|_{L^2(\Omega,\mathbb{R}^3)}\|\sqrt{e}\,\nabla\boldsymbol{v}\|_{L^2(\omega,\mathbb{R}^3)} + \|\sqrt{e}\,\nabla\boldsymbol{v}\|^2_{L^2(\omega,\mathbb{R}^3)},\end{aligned}$$

and

$$\begin{aligned}\|\nabla(zs)\|^2_{L^2(\Omega,\mathbb{R}^3)} &= \int_\omega\int_{-e/2}^{e/2}[z^2(|\partial_1 s(x)|^2 + |\partial_2 s(x)|^2) + |s(x)|^2]\,dz\,dx \\ &= \frac{1}{12}\int_\omega e^3\,|\nabla s(x)|^2\,dx + \int_\omega e|s(x)|^2\,dx \\ &= \frac{1}{12}\|e^{3/2}\nabla s\|^2_{L^2(\omega,\mathbb{R}^3)} + \|e^{1/2}\,s\|^2_{L^2(\omega,\mathbb{R}^3)}.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{1}{12}\|e^{3/2}\nabla s\|^2_{L^2(\omega,\mathbb{R}^3)} + \|e^{1/2}\,s\|^2_{L^2(\omega,\mathbb{R}^3)} \leq \|\nabla\boldsymbol{V}\|^2_{L^2(\Omega,\mathbb{R}^3)} \\ &+ 2\|e\|^{1/2}_{L^\infty(\omega)}\|\nabla\boldsymbol{V}\|_{L^2(\Omega,\mathbb{R}^3)}\|\nabla\boldsymbol{v}\|_{L^2(\omega,\mathbb{R}^3)} + \|e\|_{L^\infty(\omega)}\|\nabla\boldsymbol{v}\|^2_{L^2(\omega,\mathbb{R}^3)}.\end{aligned} \tag{2.6}$$

Since $\boldsymbol{V} \in H^1(\Omega, \mathbb{R}^3)$, $\boldsymbol{v} \in H^1(\omega, \mathbb{R}^3)$, $e \in L^\infty(\omega)$, $e(x) \geq c > 0$ and from (2.5) & (2.6), one can check that $s \in H^1(\omega; \mathbb{R}^3)$.

Conversly, suppose now that $V \in H^1(\Omega, \mathbb{R}^3)$ such that $V(x,z) = \boldsymbol{v}(x) + z\, s_\alpha(x) a^\alpha(x)$. Then, we have

$$\begin{aligned}
\varepsilon_{33}(\boldsymbol{V}(x,z)) &= \partial_3(\boldsymbol{V}(x,z)) \cdot a_3(x) \\
&= \partial_3(\boldsymbol{v}(x) + z s_\alpha(x) a^\alpha(x)) \cdot a_3(x) \\
&= (\underbrace{\partial_3 \boldsymbol{v}(x)}_{=0} + s_\alpha(x) a^\alpha(x)) \cdot a_3(x) \\
&= s_\alpha(x) a^\alpha(x) \cdot a_3(x) \\
&= 0.
\end{aligned}$$

Therefore, $\boldsymbol{V} \in \mathcal{V}_{RM} = \{\boldsymbol{V} \in H^1(\Omega, \mathbb{R}^3); \varepsilon_{33}(\boldsymbol{V}) = 0\}$. □

2.1 Definition of a deformed shell

Let M be a shell of the midsurface S and of the thickness e defined by (1.1) of previous chapter. After deformation, the normal unit vector $\boldsymbol{a}_3$ is deformed to a vector $\boldsymbol{a}_3^*$ which is not *a priori* orthonormal to the deformed surface. The point $\Phi(\boldsymbol{x}, z)$ becomes

$$\Phi^\star(\boldsymbol{x}, z) = \varphi^\star(\boldsymbol{x}) + z\boldsymbol{a}_3^\star(\boldsymbol{x}), \tag{2.7}$$

with

$$\boldsymbol{a}_3^*(\boldsymbol{x}) = \boldsymbol{a}_3(\boldsymbol{x}) + r_\alpha \boldsymbol{a}^\alpha \text{ and } \varphi^\star = \varphi + \boldsymbol{u}, \tag{2.8}$$

where r_α are the linearly covariant components of the field of rotations of the norm $\boldsymbol{a}_3$. In fact, $\boldsymbol{r} = \boldsymbol{a}_3^* - \boldsymbol{a}_3$.

Remark 21

Let us recall that the assumption $\mathbf{H}_2$ (of Kirchhoff-Love) which means the normal conservation property leads to the following approximation of the deformed normal:

$$\boldsymbol{a}_3^* = \boldsymbol{a}_3 - (\partial_\alpha \boldsymbol{u} \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha = \boldsymbol{a}_3 - \pi_{\alpha\beta}\psi^\alpha(\boldsymbol{u})\boldsymbol{a}^\beta,$$

where $\psi^\alpha(\boldsymbol{u})$ are the tangential contravariant components of the infinitesimal rotation field of the surface. It is given by

$$\psi(\boldsymbol{u}) = \pi^{\alpha\beta}(\partial_\beta \boldsymbol{u} \cdot \boldsymbol{a}_3)\boldsymbol{a}_\alpha + \frac{1}{2}\pi^{\alpha\beta}(\partial_\alpha \boldsymbol{u} \cdot \boldsymbol{a}_\beta)\boldsymbol{a}_3$$

with $\pi^{\alpha\alpha} = 0$ and $\pi^{12} = -\pi^{21} = 1/a$.

2.2 Displacement of a Naghdi's shell

Under the hypothesis $\mathbf{H}_1$ and $\mathbf{H}_2$ of Kirchhoff-Love, the displacement of a point $\Phi(\boldsymbol{x}, z)$ is written by

$$\mathcal{U}(\Phi(\boldsymbol{x}, z)) = U(\boldsymbol{x}, z) = \Phi^*(\boldsymbol{x}, z) - \Phi(\boldsymbol{x}, z).$$

This means that

$$\boldsymbol{U}(x, z) = \boldsymbol{u}(x) + z r_\alpha(\boldsymbol{x}) \boldsymbol{a}^\alpha(\boldsymbol{x}). \tag{2.9}$$

Note that $\boldsymbol{u}(\boldsymbol{x})$ is the displacement of midsurface and r_α are the covariant components of the rotation of the norm $\boldsymbol{a}_3$.

2.3 Linearized change of shell metric tensor

Let us first introduce the local bases at each point of the three-dimensional shell. The covariant basis is defined by

$$\begin{cases} \boldsymbol{g}_\alpha = \boldsymbol{g}_\alpha(\boldsymbol{x}, z) = \partial_\alpha \Phi(\boldsymbol{x}, z) = \boldsymbol{a}_\alpha(\boldsymbol{x}) + z \partial_\alpha \boldsymbol{a}_3(\boldsymbol{x}), \\ \boldsymbol{g}_3 = \boldsymbol{g}_3(\boldsymbol{x}, z) = \partial_3 \Phi(\boldsymbol{x}, z) = \boldsymbol{a}_3(\boldsymbol{x}). \end{cases}$$

The contravariant basis is defined by

$$\boldsymbol{g}_i \cdot \boldsymbol{g}^j = \delta^i_j.$$

It follows that

$$g^{\alpha 3} = g^{3\alpha} = 0 \text{ and } g^{33} = 1.$$

We recall that, given arbitrary displacement field V of a three-dimensional manifold $\Phi(\bar{\Omega})$ in $\mathbb{R}^3$, the covariant components $\varepsilon_{ij}(V)$ of the associated linearized change of metric tensor are defined by:

$$\varepsilon(\boldsymbol{V}) = \varepsilon_{ij}(V) \boldsymbol{g}^i \otimes \boldsymbol{g}^j \quad \text{with } \varepsilon_{ij}(V) = \frac{1}{2}(g_{ij}(V) - g_{ij})^{lin}, \tag{2.10}$$

where g_{ij} and $g_{ij}(V)$, respectively, are the metric tensor in the configuration of reference and the deformed shell. The vectors of the deformed local basis are defined by:

$$\begin{cases} \boldsymbol{g}_\alpha(V) = \partial_\alpha \Phi^* = \boldsymbol{a}^*_\alpha + z \partial_\alpha a^*_3, \\ \boldsymbol{g}_3(V) = \partial_3 \Phi^* = \boldsymbol{a}^*_3. \end{cases}$$

Obviously, $\partial_\alpha \boldsymbol{a}^*_3 \neq (b^\rho_\alpha)^* \boldsymbol{a}^*_\rho$ in this case. This is the difference between Naghdi and Koiter shell model.

An approximation of covariant components of metric tensor of the deformed shell is given by:

$$\begin{cases} g_{\alpha\beta}(V) = a^*_{\alpha\beta} + z(a^*_\alpha \cdot \partial_\beta \boldsymbol{a}^*_3 + \boldsymbol{a}^*_\beta \cdot \partial_\alpha \boldsymbol{a}^*_3), \\ g_{\alpha 3}(V) = (\boldsymbol{a}^*_\alpha + z\partial_\alpha \boldsymbol{a}^*_3) \cdot \boldsymbol{a}^*_3, \\ g_{33}(V) = \boldsymbol{a}^*_3 \cdot \boldsymbol{a}^*_3. \end{cases}$$

Let us recall that:

$$\boldsymbol{a}^*_\alpha = \boldsymbol{a}_\alpha + \partial_\alpha \boldsymbol{u} \quad \text{and} \quad \partial_\alpha \boldsymbol{a}^*_3 = \partial_\alpha \boldsymbol{a}_3 + \partial_\alpha (r_\beta \boldsymbol{a}^\beta),$$

with

$$\partial_\alpha \boldsymbol{u} = (\boldsymbol{u}_{\beta|\alpha} - b_{\alpha\beta}\boldsymbol{u}_3)a^\beta + (\partial_\alpha \boldsymbol{u}_3 + b^\beta_\alpha \boldsymbol{u}_\beta)\boldsymbol{a}_3,$$

and

$$\partial_\alpha (r_\beta \boldsymbol{a}^\beta) = r_{\beta|\alpha}\boldsymbol{a}^\beta + b^\beta_\alpha r_\beta \boldsymbol{a}_3.$$

Since a surface also has a metric tensor, it is natural to likewise define the covariant components of the linearized change of metric tensor associated with any displacement field defined on it.

We use here all notations introduced in chapter 1. We begin by introducing the expressions associated with an arbitrary displacement $\boldsymbol{u}$ and rotation $\boldsymbol{r} = r_\alpha \boldsymbol{a}^\alpha = r^\alpha \boldsymbol{a}_\alpha$ fields. Our functional framework, in this chapter, still free local basis for the displacement however the second unknown is identified with its covariant or contravariant components.

We recall the covariant components of the change of metric tensor as in the previous chapter, *i.e*,

$$\gamma_{\alpha\beta}(\boldsymbol{u}) = \frac{1}{2}(\partial_\alpha \boldsymbol{u} \cdot \boldsymbol{a}_\beta + \partial_\beta \boldsymbol{u} \cdot \boldsymbol{a}_\alpha). \tag{2.11}$$

We introduce the new expression of covariant components of the change of transverse shear tensor and the covariant components of the change of curvature tensor, respectively, are

$$\delta_{\alpha 3}(u, r) = \frac{1}{2}(\partial_\alpha \boldsymbol{u} \cdot \boldsymbol{a}_3 + \boldsymbol{r}_\alpha), \tag{2.12}$$

$$\chi_{\alpha\beta}(\boldsymbol{u}, \boldsymbol{r}) = \frac{1}{2}(r_{\alpha|\beta} + r_{\beta|\alpha}) + \frac{1}{2}(\partial_\alpha \boldsymbol{u} \cdot \partial_\beta \boldsymbol{a}_3 + \partial_\beta \boldsymbol{u} \cdot \partial_\alpha \boldsymbol{a}_3), \tag{2.13}$$

where $r_{\alpha|\beta} = \partial_\beta r_\alpha - \Gamma^\rho_{\alpha\beta} r_\rho$ and $\Gamma^\rho_{\alpha\beta}$ are the Christofell symbols of the surface.

Remark 22

i. Note that all those quantities make sense in $L^2(\omega)$ for shell with $W^{1,\infty}$-regularity such that $\boldsymbol{a}_3 \in W^{1,\infty}$ and for $(\boldsymbol{v}, s_\alpha) \in H^1(\omega; \mathbb{R}^3) \times H^1(\omega)^2$. They are easily expressed with the displacement field, the rotation components and geometrical data.

ii. Let us note that, when the transverse shears are neglected, *i.e.* $\delta_{\alpha 3}(\boldsymbol{v}, \boldsymbol{s}) = 0$, we have

$$\chi_{\alpha\beta}(\boldsymbol{v}, \boldsymbol{s}) = -\Upsilon_{\alpha\beta}(\boldsymbol{v}) \text{ a.e. on } \omega.$$

Now, if the three dimensional displacement field V is of Reissner-Mindlin type:

$$\boldsymbol{V}(x,z) = \boldsymbol{v}(x) + z s_\alpha(x)\boldsymbol{a}^\alpha(x),$$

then, by neglecting all the terms containing z^2, the covariant components of the associated linearized change of metric tensor (2.10) is given in terms of the change of metric and change curvature tensors on the midsurface S by:

$$\begin{cases} \varepsilon_{\alpha\beta}(\boldsymbol{V}) = \gamma_{\alpha\beta}(\boldsymbol{v}) + z\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s}), \\ \varepsilon_{\alpha 3}(\boldsymbol{V}) = \frac{1}{2}(\partial_\alpha \boldsymbol{u}_3 + b_\alpha^\rho \boldsymbol{u}_\rho + r_\alpha), \\ \varepsilon_{33}(\boldsymbol{V}) = 0. \end{cases} \tag{2.14}$$

Remark 23

When the normal strain $\varepsilon_{33}(\boldsymbol{V})$ is small compared to the cross-sectional strains, plane strain, (2.14), is then an acceptable approximation for the shell change metric tensor (2.10).

2.4 The tensor of shell constraints

We consider a tridimensional homogeneous isotropic elastic shell satisfying the Hooke's law

$$\sigma(\boldsymbol{V}) = \lambda tr(\varepsilon(\boldsymbol{V}))Id + 2\mu\varepsilon(\boldsymbol{V}). \tag{2.15}$$

We recall that λ and μ are the Lamé coefficients of material ($\lambda \geq 0$ and $\mu > 0$) and σ is the stress tensor.

Lemma 24

Let V be a given displacement field. The street tensor $\sigma(\boldsymbol{V})$ given by the Hooke's law (2.15) can be rewritten as

$$\sigma(\boldsymbol{V}) = \sigma^{ij} g_i \otimes g_j \quad \text{with } \sigma^{ij} = \sigma(\boldsymbol{U}) : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = E^{ijkl}\varepsilon_{kl}(\boldsymbol{V}), \tag{2.16}$$

where the notation ":" denotes the product of two tensors, $\sigma : \tau \overset{def}{=} \sigma_{ij} g^{ik} g^{jl} \tau_{kl}$ and E^{ijkl} is the elastic tensor defined by:

$$E^{ijkl} = \lambda g^{ij} g^{kl} + \mu(g^{ik} g^j l + g^{il} g^{jk}). \tag{2.17}$$

Proof. We remark that

$$tr(\varepsilon(\boldsymbol{V}))Id : g^i \otimes g^j = tr(\varepsilon(\boldsymbol{V}))tr(Id\, g^i \otimes g^j) = tr(\varepsilon(\boldsymbol{V}))g^{ij},$$

and

$$tr(\varepsilon(\boldsymbol{V})) = \varepsilon_{kl}(\boldsymbol{V}) tr(\boldsymbol{g}^k \otimes \boldsymbol{g}^l) = \varepsilon_{kl} g^{kl}.$$

It follows that

$$\lambda tr(\varepsilon(\boldsymbol{V})) Id : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \lambda g^{ij} g^{kl} \varepsilon_{kl}(\boldsymbol{V}). \tag{2.18}$$

Let's see the second term containing μ of (2.15). If we express the tensor $\varepsilon(\boldsymbol{V})$ in the basis $\boldsymbol{g}^k \otimes \boldsymbol{g}^l$, *i.e.*,

$$\varepsilon(\boldsymbol{V}) = (\varepsilon(\boldsymbol{V}) : \boldsymbol{g}_k \otimes \boldsymbol{g}_l) \boldsymbol{g}^k \otimes \boldsymbol{g}^l = \varepsilon_{kl}(\boldsymbol{V}) \boldsymbol{g}^k \otimes \boldsymbol{g}^l,$$

we obtain

$$\varepsilon(\boldsymbol{V}) : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \varepsilon_{kl}(\boldsymbol{V}) \boldsymbol{g}^k \otimes \boldsymbol{g}^l : \boldsymbol{g}^i \otimes \boldsymbol{g}^j.$$

Thanks to

$$\boldsymbol{g}^k \otimes \boldsymbol{g}^l : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = tr((\boldsymbol{g}^l \otimes \boldsymbol{g}^k)(\boldsymbol{g}^i \otimes \boldsymbol{g}^j)) = g^{il} g^{kj},$$

we get

$$\varepsilon(\boldsymbol{V}) : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \varepsilon_{kl}(\boldsymbol{V}) g^{il} g^{kj}.$$

Since $\varepsilon(\boldsymbol{V})$ is a symmetrical tensor, it follows that

$$\varepsilon(\boldsymbol{V}) : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \frac{1}{2} \varepsilon_{kl}(\boldsymbol{V})(g^{ik} g^{jl} + g^{il} g^{kj}).$$

Therefore,

$$2\mu\varepsilon(\boldsymbol{V}) : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = 2\mu\varepsilon_{kl}(\boldsymbol{V})(g^{ik} g^{jl} + g^{il} g^{kj}).$$

It is equivalent to

$$2\mu\varepsilon(\boldsymbol{V}) : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \mu\varepsilon_{kl}(\boldsymbol{V})(g^{ik} g^{jl} + g^{il} g^{kj}). \tag{2.19}$$

Consequently, from (2.18) and (2.19), we obtain

$$(\lambda tr(\varepsilon(\boldsymbol{V})) Id + 2\mu\varepsilon(\boldsymbol{V})) : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = (\lambda g^{ij} g^{kl} + \mu(g^{ik} g^{jl} + g^{il} g^{kj}))\varepsilon_{kl}(\boldsymbol{V}).$$

Replacing the symbols of the tensor of constraint σ and the elastic tensor E^{ijkl} to the above equation, we obtain

$$\sigma(\boldsymbol{V}) : \boldsymbol{g}^i \otimes \boldsymbol{g}^j = E^{ijkl} \varepsilon_{kl}(\boldsymbol{V}).$$

□

Remark 25

It should be pointed out that $g^{\alpha 3} = g^{3\alpha} = 0$ and $g^{33} = 1$. Thus we have

$$E^{3\alpha\beta\rho} = E^{333\beta} = 0, \tag{2.20}$$

and

$$E^{33\alpha\beta} = \lambda g^{\alpha\beta}, \quad E^{3\alpha 3\beta} = \mu g^{\alpha\beta} \text{ and } E^{3333} = \lambda + 2\mu. \tag{2.21}$$

Finally, in order to obtain the two-dimensional Naghdi's model, we state the third hypothesis of Kirchhoff-Love:

H3: The stress tensors are approximatively flat, *i.e.*, for an arbitrary displacement field V, $\sigma^{33}(\boldsymbol{V}) = 0$.

Remind that

$$\begin{aligned}\sigma^{33}(\boldsymbol{V}) &= E^{33\alpha\beta}\varepsilon_{\alpha\beta}(\boldsymbol{V}) + E^{3333}\varepsilon_{33}(\boldsymbol{V}) \\ &= \lambda g^{\alpha\beta}\varepsilon_{\alpha\beta}(\boldsymbol{V}) + (\lambda + 2\mu)\varepsilon_{33}(\boldsymbol{V}).\end{aligned}$$

So, if $\lambda \neq 0$, **H3** leads to

$$\varepsilon_{33}(\boldsymbol{V}) = -\frac{\lambda}{\lambda + 2\mu} g^{\alpha\beta}\varepsilon_{\alpha\beta}(\boldsymbol{V}).$$

We then notice that the hypothesis **H3** is not compatible with (2.14) as soon as $\lambda \neq 0$. Hence, we correct the definition of the change of metric tensor introduced in (2.14) as following:

$$\begin{cases}\varepsilon^{\text{cor}}_{\alpha\beta}(\boldsymbol{V}) = \varepsilon_{\alpha\beta}(\boldsymbol{V}), \\ \varepsilon^{\text{cor}}_{\alpha 3}(\boldsymbol{V}) = \varepsilon_{\alpha 3}(\boldsymbol{V}), \\ \varepsilon^{\text{cor}}_{33}(\boldsymbol{V}) = -\dfrac{\lambda}{\lambda + 2\mu} g^{\rho\sigma}\varepsilon_{\rho\sigma}(\boldsymbol{V}).\end{cases}$$

2.5 Derivation of the Biot-Naghdi shell model

A word of caution. For notational convenience, we omit throughout this section the exponent "cor" in the expression $\varepsilon^{\text{cor}}_{ij}$.

In this section, we will establish the weak formulation of the Naghdi Biot shell model. In the process of deriving the weak form, we face difficulty of writing the poroelastic structure model in term of shell symbols (metric tensor $\gamma_{\alpha\beta}$, curvature tensor $\chi_{\alpha\beta}$, transverse shear tensor $\delta_{\alpha 3}$). In order to solving this problem, we use the structure displacement $\boldsymbol{U}$ belonging to the space of Reissner-Mindlin displacements $\mathcal{V}_{RM} = \{\boldsymbol{U} \in H^1(\Omega; \mathbb{R}^3), \varepsilon_{33}(\boldsymbol{U}) = 0\}$. In further detail, we introduce the following theorem.

Theorem 9

If $\boldsymbol{U}$ and p are solution of the strong formulation (1.37a)-(1.37b), such that $\boldsymbol{U} = \boldsymbol{u} + z r_\alpha \boldsymbol{a}^\alpha$, then $(\boldsymbol{u}, \boldsymbol{r})$ and p belong, respectively, to $\mathcal{V}_N$ and $\mathcal{W}_N$ and solve the weak equations: Find $(\boldsymbol{u}, \boldsymbol{r}) \in \mathcal{V}_N$ and $p \in \mathcal{W}_N$ such that

$$\mathcal{A}_1^N((\boldsymbol{u},\boldsymbol{r});(\boldsymbol{v},\boldsymbol{s})) + \mathcal{B}_1^N(p;(\boldsymbol{v},\boldsymbol{s})) = \mathcal{L}_1^N(\boldsymbol{v}) \quad \forall(\boldsymbol{v},\boldsymbol{s}) \in \mathcal{X}_N, \tag{2.22a}$$

$$\mathcal{A}_2^N(p;q) + \mathcal{B}_2^N((\boldsymbol{u},\boldsymbol{r});q) = \mathcal{L}_2^N(q) \quad \forall q \in H_0^1(\Omega), \tag{2.22b}$$

$$p(0) = p_0 \text{ in } \Omega, \tag{2.22c}$$

where

$$\mathcal{A}_1^N((\boldsymbol{u},\boldsymbol{r});(\boldsymbol{v},\boldsymbol{s})) = \int_\omega e\, a^{\alpha\beta\rho\sigma}[\gamma_{\rho\sigma}(\boldsymbol{u})\gamma_{\alpha\beta}(\boldsymbol{v}) + \frac{e^2}{12}\chi_{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s})]\sqrt{a}\, d\boldsymbol{x}$$
$$+ 4\mu \int_\omega e\, a^{\alpha\beta}\delta_{\alpha3}(\boldsymbol{u},\boldsymbol{r})\delta_{\beta3}(\boldsymbol{v},\boldsymbol{s})\sqrt{a}\ \ d\boldsymbol{x},$$

$$\mathcal{B}_1^N(p;(\boldsymbol{v},\boldsymbol{s})) = -\alpha \int_\Omega p \mathrm{div}\boldsymbol{v}\sqrt{a}\, d\boldsymbol{X} - \alpha \int_\Omega p\mathrm{div}(z\boldsymbol{s})\ \sqrt{a}\ \ d\boldsymbol{X},$$

$$\mathcal{L}_1^N(\boldsymbol{v},\boldsymbol{s}) = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}+z\boldsymbol{s})\sqrt{a}\ d\boldsymbol{X},\ \mathcal{A}_2^N(p;q) = c_0\int_\Omega p'q\sqrt{a}\, d\boldsymbol{X} + \frac{\kappa}{\eta}\int_\Omega \nabla p\cdot\nabla q\,\sqrt{a}\, d\boldsymbol{X},$$

$$\mathcal{B}_2^N((\boldsymbol{u},\boldsymbol{r});q) = -\alpha\int_\Omega(\mathrm{div}\boldsymbol{u}')q\,\sqrt{a}\, d\boldsymbol{X} - \alpha\int_\Omega \mathrm{div}(z\boldsymbol{r}')q\,\sqrt{a}\, d\boldsymbol{X},\ \mathcal{L}_2^N(q) = \int_\omega gq\sqrt{a}\, d\boldsymbol{X}.$$

Remark 26

We use $p(0)$ in Eq. (2.22a) to get $\boldsymbol{u}(0)$ and $r_\alpha(0)$. For a given $p(0)$, the initial displacement is the unique solution of (2.22a), where we replace p by $p(0)$.

Proof. (of Theorem 9) For proving the Theorem 9, we use the structure of the displacement $\boldsymbol{U}$ belonging to the space of Reissner-Mindlin displacements $\mathcal{V}_{RM} = \{\boldsymbol{U} \in H^1(\Omega;\mathbb{R}^3), \varepsilon_{33}(\boldsymbol{U}) = 0\}$ and by Lemma 20, $\boldsymbol{U}$ can be presented by $\boldsymbol{u} + z\boldsymbol{r}$, where $(\boldsymbol{u},\boldsymbol{r})$ is the displacement-rotation defined on ω. In order to use the property $\varepsilon_{33}(\boldsymbol{V}) = 0$, we then multiply the Eq. (1.37a) by $\boldsymbol{V} = \boldsymbol{v} + z\boldsymbol{s}$ with $(\boldsymbol{v},\boldsymbol{s}) \in \mathcal{X}_K$, we thus obtain

$$-\int_M (\nabla\cdot\sigma(\boldsymbol{U}))\cdot\boldsymbol{V}\, dV + \alpha\int_M \nabla p\cdot\boldsymbol{V}\, dV = \int_M \boldsymbol{f}\cdot\boldsymbol{V}\, dV.$$

It is equivalent to write

$$-\int_\Omega (\nabla\cdot\sigma(\boldsymbol{U}))\cdot\boldsymbol{V}\sqrt{a}\, d\boldsymbol{X} + \alpha\int_\Omega \nabla p\cdot\boldsymbol{V}\sqrt{a}\, d\boldsymbol{X} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{V}\sqrt{a}\, d\boldsymbol{X}.$$

Applying the Green formula and using the boundary condition, we obtain

$$\underbrace{\int_\Omega \sigma(\boldsymbol{U})\colon \varepsilon(\boldsymbol{V})\sqrt{a}d\boldsymbol{X}}_{I} + \alpha \underbrace{\int_\Omega \nabla p \cdot \boldsymbol{V}\sqrt{a}\, d\boldsymbol{X}}_{J} = \underbrace{\int_\Omega \boldsymbol{f}\cdot \boldsymbol{V}\sqrt{a}\, d\boldsymbol{X}}_{K}. \tag{2.23}$$

We then have

$$\begin{aligned}
I &= \int_\Omega \sigma^{ij}(\boldsymbol{U})\varepsilon_{ij}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} \\
&= \int_\Omega \sigma^{\alpha\beta}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} + \underbrace{\int_\Omega \sigma^{33}(\boldsymbol{U})\varepsilon_{33}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X}}_{=0} \\
&\qquad + \int_\Omega \sigma^{\alpha 3}(\boldsymbol{U})\varepsilon_{\alpha 3}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} + \int_\Omega \sigma^{3\beta}(\boldsymbol{U})\varepsilon_{3\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} \\
&= \underbrace{\int_\Omega E^{\alpha\beta kl}\varepsilon_{kl}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X}}_{I_1} + \underbrace{\int_\Omega E^{\alpha 3kl}\varepsilon_{kl}(\boldsymbol{U})\varepsilon_{\alpha 3}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X}}_{I_2} \\
&\qquad + \underbrace{\int_\Omega E^{3\beta kl}\varepsilon_{kl}(\boldsymbol{U})\varepsilon_{3\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X}}_{I_3}.
\end{aligned} \tag{2.24}$$

From Lemma 24, one has

$$\begin{aligned}
I_1 &= \int_\Omega E^{\alpha\beta\rho\sigma}\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} + \int_\Omega E^{\alpha\beta 33}\varepsilon_{33}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} \\
&\qquad + \underbrace{\int_\Omega E^{\alpha\beta 3\sigma}\varepsilon_{3\sigma}(\boldsymbol{U})\varepsilon_{\alpha 3}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X}}_{=0} + \underbrace{\int_\Omega E^{\alpha\beta\rho 3}\varepsilon_{\rho 3}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X}}_{=0} \\
&= \int_\Omega E^{\alpha\beta\rho\sigma}\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} - \frac{\lambda^2}{\lambda+2\mu}\int_\Omega g^{\alpha\beta}g^{\rho\sigma}\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X}.
\end{aligned} \tag{2.25}$$

Similarly, we simplify I_2 by applying Lemma 24,

$$\begin{aligned}
I_2 &= \underbrace{\int_\Omega E^{\alpha 3\rho\sigma}\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha 3}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X}}_{=0} + \underbrace{\int_\Omega E^{\alpha 333}\varepsilon_{33}(\boldsymbol{U})\varepsilon_{\alpha 3}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X}}_{=0} \\
&\qquad + \int_\Omega E^{\alpha 33\sigma}\varepsilon_{3\sigma}(\boldsymbol{U})\varepsilon_{\alpha 3}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} + \int_\Omega E^{\alpha 3\rho 3}\varepsilon_{\rho 3}(\boldsymbol{U})\varepsilon_{\alpha 3}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} \\
&= \mu\int_\Omega g^{\alpha\sigma}\varepsilon_{3\sigma}(\boldsymbol{U})\varepsilon_{\alpha 3}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X} + \mu\int_\Omega g^{\alpha\rho}\varepsilon_{\rho 3}(\boldsymbol{U})\varepsilon_{\alpha 3}(\boldsymbol{V})\sqrt{a}\, d\boldsymbol{X}.
\end{aligned} \tag{2.26}$$

And the third term of (2.24) becomes

$$\begin{aligned}
I_3 = &\underbrace{\int_\Omega E^{3\beta\rho\sigma}\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{3\beta}(\boldsymbol{V})\sqrt{a}\,d\boldsymbol{X}}_{=0} + \underbrace{\int_\Omega E^{3\beta 33}\varepsilon_{33}(\boldsymbol{U})\varepsilon_{3\beta}(\boldsymbol{V})\sqrt{a}\,d\boldsymbol{X}}_{=0} \\
&+ \int_\Omega E^{3\beta 3\sigma}\varepsilon_{3\sigma}(\boldsymbol{U})\varepsilon_{3\beta}(\boldsymbol{V})\sqrt{a}\,d\boldsymbol{X} + \int_\Omega E^{3\beta\rho 3}\varepsilon_{\rho 3}(\boldsymbol{U})\varepsilon_{3\beta}(\boldsymbol{V})\sqrt{a}\,d\boldsymbol{X} \\
= &\mu\int_\Omega g^{\beta\sigma}\varepsilon_{3\sigma}(\boldsymbol{U})\varepsilon_{3\beta}(\boldsymbol{V})\sqrt{a}\,d\boldsymbol{X} + \mu\int_\Omega g^{\beta\rho}\varepsilon_{\rho 3}(\boldsymbol{U})\varepsilon_{3\beta}(\boldsymbol{V})\sqrt{a}\,d\boldsymbol{X}. \qquad (2.27)
\end{aligned}$$

Therefore, from (2.25), (2.26) and (2.27), we obtain that,

$$\begin{aligned}
I &= I_1 + I_2 + I_3 \\
&= \int_\Omega [\lambda g^{\alpha\beta}g^{\rho\sigma} + \mu(g^{\alpha\rho}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\rho})]\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\,d\boldsymbol{X} \\
&\quad - \frac{\lambda^2}{\lambda+2\mu}\int_\Omega g^{\alpha\beta}g^{\rho\sigma}\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\,d\boldsymbol{X} + 4\mu\int_\Omega g^{\alpha\beta}\varepsilon_{3\alpha}(\boldsymbol{U})\varepsilon_{\beta 3}(\boldsymbol{V})\sqrt{a}\,d\boldsymbol{X} \\
&= \int_\Omega [\frac{2\lambda\mu}{\lambda+2\mu}g^{\alpha\beta}g^{\rho\sigma} + \mu(g^{\alpha\rho}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\rho})]\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\,d\boldsymbol{X} \\
&\quad + 4\mu\int_\Omega g^{\alpha\beta}\varepsilon_{3\alpha}(\boldsymbol{U})\varepsilon_{\beta 3}(\boldsymbol{V})\sqrt{a}\,d\boldsymbol{X} \\
&= \int_\Omega A^{\alpha\beta\rho\sigma}\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}\,d\boldsymbol{X} + 4\mu\int_\Omega g^{\alpha\beta}\varepsilon_{3\alpha}(\boldsymbol{U})\varepsilon_{\beta 3}(\boldsymbol{V})\sqrt{a}\,d\boldsymbol{X}, \qquad (2.28)
\end{aligned}$$

where

$$A^{\alpha\beta\rho\sigma} = \frac{2\lambda\mu}{\lambda+2\mu}g^{\alpha\beta}g^{\rho\sigma} + \mu(g^{\alpha\rho}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\rho}). \qquad (2.29)$$

As in the previous chapter, we neglect term z in the expression of $g^{\alpha\beta}$ and then (2.28) becomes

$$\begin{aligned}
I &= \int_\Omega a^{\alpha\beta\rho\sigma}\varepsilon_{\rho\sigma}(\boldsymbol{U})\varepsilon_{\alpha\beta}(\boldsymbol{V})\sqrt{a}d\boldsymbol{X} + 4\mu\int_\Omega g^{\alpha\beta}\varepsilon_{3\alpha}(\boldsymbol{U})\varepsilon_{\beta 3}(\boldsymbol{V})\sqrt{a}d\boldsymbol{X} \\
&= \int_\Omega a^{\alpha\beta\rho\sigma}[\gamma_{\rho\sigma}(\boldsymbol{u}) + z\chi_{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})][\gamma_{\alpha\beta}(\boldsymbol{v} + z\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s}))]\sqrt{a}d\boldsymbol{X} \\
&\quad + 4\mu\int_\Omega a^{\alpha\beta}\delta_{\alpha 3}(\boldsymbol{u},\boldsymbol{r})\delta_{\beta 3}(\boldsymbol{v},\boldsymbol{s})\sqrt{a}\;d\boldsymbol{X} \\
&= e\int_\omega a^{\alpha\beta\rho\sigma}[\gamma_{\rho\sigma}(\boldsymbol{u})\gamma_{\alpha\beta}(\boldsymbol{v}) + \frac{e^2}{12}\chi_{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s})]\sqrt{a}\;d\boldsymbol{x} \\
&\quad + 4\mu e\int_\omega a^{\alpha\beta}\delta_{\alpha 3}(\boldsymbol{u},\boldsymbol{r})\delta_{\beta 3}(\boldsymbol{v},\boldsymbol{s})\sqrt{a}\;\; d\boldsymbol{x}, \qquad (2.30)
\end{aligned}$$

where

$$a^{\alpha\beta\rho\sigma}(\boldsymbol{x}) = \frac{2\mu\lambda}{\lambda+2\mu}a^{\alpha\beta}a^{\rho\sigma} + \mu(a^{\alpha\rho}a^{\beta\sigma} + a^{\alpha\sigma}a^{\beta\rho}).$$

On the otherwise, the second term of (2.23) implies that

$$J = \alpha \int_\Omega \nabla p \cdot \boldsymbol{V} \sqrt{a}\, d\boldsymbol{X} = \alpha \int_{\Gamma_0} p \boldsymbol{V} \cdot \boldsymbol{n} \sqrt{a}\, d\Gamma - \alpha \int_\Omega p\, \mathrm{div} \boldsymbol{V} \sqrt{a}\, d\boldsymbol{X}$$
$$= -\alpha \int_\Omega p\, \mathrm{div} \boldsymbol{v} \sqrt{a}\, d\boldsymbol{X} - \alpha \int_\Omega p \mathrm{div}(z\boldsymbol{s}) \sqrt{a}\, d\boldsymbol{X}.$$

We continue with the term K,

$$K = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{V} \sqrt{a}\, d\boldsymbol{X} = \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} + z\boldsymbol{s}) \sqrt{a}\, d\boldsymbol{X}.$$

Consequently, (2.23) becomes

$$\int_\omega e\, a^{\alpha\beta\rho\sigma} [\gamma_{\rho\sigma}(\boldsymbol{u})\gamma_{\alpha\beta}(\boldsymbol{v}) + \frac{e^2}{12} \chi_{\rho\sigma}(\boldsymbol{u}, \boldsymbol{r}) \chi_{\alpha\beta}(\boldsymbol{v}, \boldsymbol{s})] \sqrt{a}\, d\boldsymbol{x} + 4\mu \int_\omega e\, a^{\alpha\beta} \delta_{\alpha 3}(\boldsymbol{u}, \boldsymbol{r}) \delta_{\beta 3}(\boldsymbol{v}, \boldsymbol{s}) \sqrt{a}\, d\boldsymbol{x}$$
$$- \alpha \int_\Omega p\, \mathrm{div}(\boldsymbol{v} + z\boldsymbol{s}) \sqrt{a}\, d\boldsymbol{X} = \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} + z\boldsymbol{s}) \sqrt{a}\, d\boldsymbol{X} \quad \forall (\boldsymbol{v}, \boldsymbol{s}) \in \mathcal{X}_N,$$

or

$$\mathcal{A}_1^N((\boldsymbol{u}, \boldsymbol{r}); (\boldsymbol{v}, \boldsymbol{s})) + \mathcal{B}_1^N(p; (\boldsymbol{v}, \boldsymbol{s})) = \mathcal{L}_1^N(\boldsymbol{v}, \boldsymbol{s}). \tag{2.31}$$

Similarly, we get the weak form of (1.37b) by multiplying by q in $H_0^1(\Omega)$,

$$\int_\Omega \partial_t (c_0 p + \nabla \cdot \boldsymbol{U}) q \sqrt{a}\, d\boldsymbol{X} - \frac{\kappa}{\eta} \int_\Omega (\Delta p) q \sqrt{a}\, d\boldsymbol{X} = \int_\Omega g q \sqrt{a}\, d\boldsymbol{X},\ \forall q \in H_0^1(\Omega). \tag{2.32}$$

For convenience, we replace $\partial_t p$ by p' and $\partial_t \boldsymbol{U}$ by $\boldsymbol{U}'$. So, for all q in $H_0^1(\Omega)$, (2.32) is equivalent to

$$\underbrace{\int_\Omega c_0 p' q\, d\boldsymbol{X} + \alpha \int_\Omega (\mathrm{div}\, \boldsymbol{U}') q\, d\boldsymbol{X} - \frac{\kappa}{\eta} \overbrace{\int_{\Gamma_0} q (\nabla p \cdot \overrightarrow{\boldsymbol{n}})\, d\Gamma}^{=0} + \frac{\kappa}{\eta} \int_\Omega \nabla p \cdot \nabla q\, d\boldsymbol{X}}_{LS} = \underbrace{\int_\Omega g q\, d\boldsymbol{X},}_{RS}\ .$$

We have

$$LS = \int_\Omega p' q \sqrt{a}\, d\boldsymbol{X} + \alpha \int_\Omega (\mathrm{div}\, \boldsymbol{U}') q \sqrt{a}\, d\boldsymbol{X} + \frac{\kappa}{\eta} \int_\Omega \nabla p \cdot \nabla q \sqrt{a}\, d\boldsymbol{X}$$
$$= \int_\Omega p' q \sqrt{a}\, d\boldsymbol{X} + \alpha \int_\Omega \mathrm{div}\, (\boldsymbol{u}' + z\boldsymbol{r}') q \sqrt{a}\, d\boldsymbol{X} + \frac{\kappa}{\eta} \int_\Omega \nabla p \cdot \nabla q \sqrt{a}\, d\boldsymbol{X},$$

and

$$RS = \int_\Omega g q \sqrt{a}\, d\boldsymbol{X}.$$

Then, (2.32) becomes

$$c_0 \int_\Omega p' q \sqrt{a}\, d\boldsymbol{X} + \alpha \int_\Omega \mathrm{div}\, (\boldsymbol{u}' + z\boldsymbol{r}') \sqrt{a}\, d\boldsymbol{X} + \frac{\kappa}{\eta} \int_\Omega \nabla p \cdot \nabla q \sqrt{a}\, d\boldsymbol{X} = \int_\Omega g q \sqrt{a}\, d\boldsymbol{X},$$

or

$$\mathcal{A}_2^N(p; q) + \mathcal{B}_2^N((\boldsymbol{u}, \boldsymbol{r}); q) = \mathcal{L}_2^N(q). \tag{2.33}$$

From (2.31) and (2.33), we can obtain Theorem (9). □

Succeedingly, we derive the well-posedness of the weak problem (2.22).

2.6 Well-posedness

In this section, we will prove the well-posedness of the problem (2.22) introduced in theorem (9) and prove that its solution p belongs to $L^\infty(0,T,H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$. For the well-posedness (see Theorem 10), we also face the difficulty as in the previous chapter. Therefore, we firstly prove the existence and uniqueness of $\boldsymbol{U}$ of the weak form of constitutive equation (1.37a) by Banach-Nečas-Babuška theorem when p is given (see Lemma 30). We then prove the existence and uniqueness of p in the weak form of mass conservation equation (1.37b) by making use of the semi-discrete Galerkin method and the theory of initial value problem for liner systems (see Lemma 32, 33 & 34).

Theorem 10

Let $p_0 \in H^1(\Omega)$, $\boldsymbol{f} \in H^1(0,T;L^2(\Omega,\mathbb{R}^3))$ and $g \in L^2(\Omega\times]0,T[)$. Then the problem (2.22) is well posed and its solution p belongs to $L^\infty(0,T,H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$.

The proof of Theorem 10 is comprised of two steps and each step is composed of a series of lemmas.

Step 1. Let us begin by providing an existence and uniqueness result of the displacement $\boldsymbol{u}(p)$ and rotation $\boldsymbol{r}(p)$ for the equation (2.22a) when p is given. First of all, we recall the rigid displacement lemma in the hybrid functional framework introduced in [24]. It is a version of the well known infinitesimal rigid displacement of [8] and [22] for a surface with little regularity.

Lemma 27

Let $\boldsymbol{v} \in H^1(\omega;\mathbb{R}^3)$, $s_\alpha \in H^1(\omega)$ and $\varphi \in W^{2,\infty}(\omega;\mathbb{R}^3)$.

1. If $\gamma_{\alpha\beta}(\boldsymbol{v}) = 0$, then there exist a unique $\psi \in L^2(\omega;\mathbb{R}^3)$ such that
$$\partial_\alpha \boldsymbol{v} = \psi \wedge \partial_\alpha \varphi.$$
2. If $\delta_{\alpha 3}(\boldsymbol{v},\boldsymbol{s}) = 0$, then $\partial_\alpha \boldsymbol{v}\cdot \boldsymbol{a}_3 = -s_\alpha$ belongs to $H^1(\omega)$. Furthermore $s_\alpha = -\pi_{\alpha\beta}\psi\cdot \boldsymbol{a}^\beta$, with $\pi_{11} = \pi_{22} = 0$ et $\pi_{12} = -\pi_{21} = \sqrt{a}$.
3. If in addition $\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s}) = 0$, then ψ is equal to a constant vector of $\mathbb{R}^3$ and we have:
$$v(x) = c + \psi \wedge \varphi(x). \tag{2.34}$$

Proof. We remark that if $\delta_{\alpha 3}(\boldsymbol{v},\boldsymbol{s}) = 0$, then $\partial_{\alpha\beta}\boldsymbol{s}\cdot \boldsymbol{a}_3 \in L^2(\omega)$. Indeed, $\partial_{\alpha\beta}\boldsymbol{v}\cdot\boldsymbol{a}_3 = \partial_\beta(\partial_\alpha \boldsymbol{v}\cdot \boldsymbol{a}_3 - \partial_\alpha\boldsymbol{v}\cdot\partial_\beta \boldsymbol{a}_3) \in L^2(\omega)$ since $\boldsymbol{a}_3 \in L^\infty(\omega,\mathbb{R}^3)$ and $\partial_\alpha\boldsymbol{v}\cdot\boldsymbol{a}_3 = -s_\alpha \in H^1(\omega)$. Thus, $\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s}) =$

$-(\partial_{\alpha\beta}\boldsymbol{v} - \Gamma^{\rho}_{\alpha\beta}\partial_\rho \boldsymbol{v}) \cdot \boldsymbol{a}_3 = 0$. To conclude, it is enough to use the rigid displacement lemma for the Koiter Shell introduced in Lemma 12. □

Remark 28

If $\boldsymbol{v} \in H^1(\omega;\mathbb{R}^3)$, $s_\alpha \in H^1(\omega)$ and $\varphi \in W^{2,\infty}(\omega;\mathbb{R}^3)$ such that $\boldsymbol{v} = 0$ on γ_0 and $\varphi(\gamma_0)$ is not included in a straight line, then $\boldsymbol{v} = 0$ and $s_\alpha = 0$ a.e. on ω. Therefore, one can check that:

$$|||(\boldsymbol{v},\boldsymbol{s})|||_N = (\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s})\|^2_{L^2(\omega)} + \sum_{\alpha,\beta} \|\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s})\|^2_{L^2(\omega)} + \|\delta_\alpha(\boldsymbol{v},\boldsymbol{s})\|^2_{L^2(\omega)})^{1/2} \tag{2.35}$$

is a norm on $\mathcal{X}_N$.

We prove in the following lemma the equivalence between (2.4) and (2.35).

Lemma 29

The norm

$$|||(\boldsymbol{v},\boldsymbol{s})|||_N = (\sum_{\alpha\beta} \|\gamma_{\alpha\beta}(\boldsymbol{v})\|^2_{L^2(\omega)} + \sum_{\alpha\beta} \|\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s})\|^2_{L^2(\omega)} + \sum_{\alpha} \|\delta_{\alpha 3}(\boldsymbol{v},\boldsymbol{s})\|^2_{L^2(\omega)})^{1/2},$$

is equivalent to the norm (2.4) of $\mathcal{X}_N$.

Proof. We use the standard contradiction argument. Let us assume that there exists a sequence $\boldsymbol{v}_n \in \mathcal{X}_N$ such that

$$\|(\boldsymbol{v}_n,\boldsymbol{s}_n)\|_{\mathcal{X}_N} = 1 \qquad \text{but} \qquad |||(\boldsymbol{v}_n,\boldsymbol{s}_n)|||_N \longrightarrow 0 \text{ when } n \to +\infty. \tag{2.36}$$

There exists a subsequence, still denoted $(\boldsymbol{v}_n,\boldsymbol{s}_n) \in \mathcal{X}_N$ such that

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{v},\ (s_\alpha)_n \rightharpoonup s_\alpha,\ \gamma_{\alpha\beta}(\boldsymbol{v}_n) \rightharpoonup \gamma_{\alpha\beta}(\boldsymbol{v}),\ \chi_{\alpha\beta}(\boldsymbol{v}_n,s_n) \rightharpoonup \chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s}),\ \text{ and } \delta_{\alpha 3}(\boldsymbol{v}_n,\boldsymbol{s}_n) \rightharpoonup \delta_{\alpha 3}(\boldsymbol{v},\boldsymbol{s})$$

weakly in their respective spaces. By Rellich's theorem, we have

$$(\boldsymbol{v}_n,\boldsymbol{s}_n) \longrightarrow (\boldsymbol{v},\boldsymbol{s}) \text{ strongly in } L^2(\omega;\mathbb{R}^3) \times L^2(\omega)^2.$$

Moreover, as $|||(\boldsymbol{v}_n,\boldsymbol{s}_n)|||_N \to 0$, it holds that

$$\gamma_{\alpha\beta}(\boldsymbol{v}_n) \to 0,\ \chi_{\alpha\beta}(\boldsymbol{v}_n,\boldsymbol{s}_n) \to 0,\ \text{and } \delta_{\alpha 3}(\boldsymbol{v}_n,\boldsymbol{s}_n) \to 0 \text{ strongly in } L^2(\omega). \tag{2.37}$$

Therefore, we get

$$\gamma_{\alpha\beta}(\boldsymbol{v}) = \chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s}) = \delta_{\alpha 3}(\boldsymbol{v},\boldsymbol{s}) = 0.$$

By the infinitesimal rigid displacement lemma 27 and boundary conditions, we first conclude that $\boldsymbol{v} = \psi = 0$ and thus $s_\alpha = 0$ a.e. in ω.

Let us now introduce the two-dimensional vector $(w_n)_\alpha = \boldsymbol{v}_n \cdot \boldsymbol{a}_\alpha$. We have $w_n \to 0$ strogly in $L^2(\omega;\mathbb{R}^2)$. Let us define

$$2e_{\alpha\beta}(w) = \partial_\alpha w_\beta + \partial_\beta w_\alpha.$$

It is easy to see that

$$e_{\alpha\beta}(w_n) = \gamma_{\alpha\beta}(\boldsymbol{v}_n) + \boldsymbol{v}_n \cdot \partial_\alpha \boldsymbol{a}_\beta \longrightarrow 0 \text{ strongly in } L^2(\omega).$$

Then, by (2.37) and the two-dimensional Korn inequality 5, we deduce that

$$w_n \longrightarrow 0 \text{ strongly in } H^1(\omega;\mathbb{R}^2).$$

Next, we note that

$$\partial_\rho \boldsymbol{v}_n \cdot \boldsymbol{a}_\alpha = \partial_\rho((w_n)_\alpha) - \boldsymbol{v}_n \cdot \partial_\rho \boldsymbol{a}_\alpha \longrightarrow 0 \text{ strongly in } L^2(\omega), \tag{2.38}$$

since $\partial_\rho \boldsymbol{a}_\alpha \in L^\infty(\omega)$. Moreover, as $(s_\alpha)_n \to 0$ in $L^2(\omega)$ strongly, and $\partial_\rho \boldsymbol{v}_n \cdot \boldsymbol{a}_3 = \gamma_{\rho 3}(\boldsymbol{v}_n, \boldsymbol{s}_n) - (s_\alpha)_n$, then $\partial_\rho \boldsymbol{v}_n \cdot \boldsymbol{a}_3 \to 0$ strongly in $L^2(\omega)$. We deduce that $\partial_\rho \boldsymbol{v}_n \to 0$ in $L^2(\omega;\mathbb{R}^3)$ strongly. Thus, $\boldsymbol{v}_n \to 0$ in $H^1(\omega;\mathbb{R}^3)$ strongly.

Next, since

$$e_{\alpha\beta}(\boldsymbol{s}_n) = \chi_{\alpha\beta}(\boldsymbol{v}_n, \boldsymbol{s}_n) - \frac{1}{2}(\partial_\alpha \boldsymbol{v}_n \cdot \partial_\beta \boldsymbol{a}_3 + \partial_\beta \boldsymbol{v}_n \cdot \partial_\alpha \boldsymbol{a}_3) + \Gamma^\rho_{\alpha\beta}(s_\rho)_n,$$

we see by (2.37)

$$e_{\alpha\beta}(\boldsymbol{s}_n) \longrightarrow 0 \text{ strongly in } L^2(\omega).$$

Thus, again by the two-dimensional Korn inequality, we conclude that

$$\boldsymbol{s}_n \longrightarrow 0 \text{ strongly in } H^1(\omega)^2.$$

Combining now the convergence of $\boldsymbol{v}_n$ and $\boldsymbol{s}_n$, we see that $||(\boldsymbol{v}_n, \boldsymbol{s}_n)||_{\mathcal{X}_N} \to 0$, which contradicts the hypothesis and proves the lemma. □

Now, we derive the main proof of the well-posedness by the following lemma,

Lemma 30

For a given p in $L^\infty(0,T;L^2(\Omega))$ and $\boldsymbol{f}$ in $H^1(0,T;L^2(\Omega,\mathbb{R}^3))$, there exists a unique solution $(\boldsymbol{u}(p), \boldsymbol{r}(p))$ in $\mathcal{X}_N$ solving, for a.e. t in $(0,T)$,

$$\mathcal{A}_1^N((\boldsymbol{u}(p), \boldsymbol{r}(p)); (\boldsymbol{v}, \boldsymbol{s})) = \mathcal{L}_{1,p}^N(\boldsymbol{v}, \boldsymbol{s}) \ \ \forall (\boldsymbol{v}, \boldsymbol{s}) \in \mathcal{X}_N, \tag{2.39}$$

where $\mathcal{L}_{1,p}^N(\boldsymbol{v}, \boldsymbol{s}) = \mathcal{L}_1^N(\boldsymbol{v}, \boldsymbol{s}) - \mathcal{B}_1^N(p; (\boldsymbol{v}, \boldsymbol{s}))$.

Proof. We will prove the well-posedness by using the Banach-Nečas-Babuška theorem . For all $(\boldsymbol{u}(p),\boldsymbol{r}(p))$ in $\mathcal{X}_N$, we have

$$\begin{aligned}\sup_{(\boldsymbol{v},\boldsymbol{s})\in\mathcal{X}_N}\frac{\mathcal{A}_1^N\big((\boldsymbol{u}(p),\boldsymbol{r}(p));(\boldsymbol{v},\boldsymbol{s})\big)}{\|(\boldsymbol{v},\boldsymbol{s})\|_{\mathcal{X}_N}} &\geq \frac{\mathcal{A}_1^N\big((\boldsymbol{u}(p),\boldsymbol{r}(p));(\boldsymbol{u}(p),\boldsymbol{r}(p))\big)}{\|(\boldsymbol{u}(p),\boldsymbol{r}(p))\|_{\mathcal{X}_N}}\\ &\geq \frac{\min\{c;\frac{c^3}{12};4\mu c\}|||\boldsymbol{u}(p),\boldsymbol{r}(p)|||_N^2}{\|(\boldsymbol{u}(p),\boldsymbol{r}(p))\|_{\mathcal{X}_N}}\\ &\geq \frac{\min\{c;\frac{c^3}{12};4\mu c\}C\|\boldsymbol{u}(p),\boldsymbol{r}(p)\|_{\mathcal{X}_N}^2}{\|(\boldsymbol{u}(p),\boldsymbol{r}(p))\|_{\mathcal{X}_N}}\\ &= \min\{c;\frac{c^3}{12};4\mu c\}C\|\boldsymbol{u}(p),\boldsymbol{r}(p)\|_{\mathcal{X}_N},\end{aligned}$$

by using Remark 28 and Lemma 29.

Therefore, for all $(\boldsymbol{u}(p),\boldsymbol{r}(p))$ in $\mathcal{X}_N$, there exists $\theta=\min\{c;\frac{c^3}{12};4\mu c\}C$ such that

$$\sup_{(\boldsymbol{v},\boldsymbol{s})\in\mathcal{X}_N}\frac{\mathcal{A}_1^N((\boldsymbol{u}(p),\boldsymbol{r}(p));(\boldsymbol{v},\boldsymbol{s}))}{\|(\boldsymbol{v},\boldsymbol{s})\|_{\mathcal{X}_N}}\geq\theta\|\boldsymbol{u}(p),\boldsymbol{r}(p)\|_{\mathcal{X}_N}. \tag{2.40}$$

On the other hand, for all $(\boldsymbol{v},\boldsymbol{s})$ in $\mathcal{X}_N$, suppose that

$$\mathcal{A}_1^N((\boldsymbol{u}(p),\boldsymbol{r}(p));(\boldsymbol{v},\boldsymbol{s}))=0\ \ \forall(\boldsymbol{u}(p),\boldsymbol{r}(p))\in\mathcal{X}_N.$$

Hence, we have

$$0=\mathcal{A}_1^N((\boldsymbol{v},\boldsymbol{s});(\boldsymbol{v},\boldsymbol{s}))\geq\min\{c;\frac{c^3}{12};4\mu c\}|||(\boldsymbol{v},\boldsymbol{s})|||_N^2\geq\min\{c;\frac{c^3}{12};4\mu c\}C\|(\boldsymbol{v},\boldsymbol{s})\|_{\mathcal{X}_N}^2\geq 0.$$

Therefore,

$$(\boldsymbol{v},\boldsymbol{s})\equiv\mathbf{0}. \tag{2.41}$$

Additionally, there exist positive constants C_1, C_2, C_3, C_4 such that

$$\begin{aligned}\mathcal{L}_{1,p}^N(\boldsymbol{v},\boldsymbol{s}) &= \alpha\int_\omega\overline{p}\mathrm{div}\boldsymbol{v}\sqrt{a}\,d\boldsymbol{x}+\alpha\int_\omega\Big(\int_{-e/2}^{e/2}p\mathrm{div}(z\boldsymbol{s})\,dz\Big)\sqrt{a}\ \ d\boldsymbol{x}+\int_\omega e\,\boldsymbol{f}\cdot\boldsymbol{v}\sqrt{a}\,d\boldsymbol{x}\\ &\leq C_1\|\overline{p}\|_{0,\omega}\|\mathrm{div}\boldsymbol{v}\|_{0,\omega}+C_2\|\overline{p}\|_{0,\omega}\|\boldsymbol{s}\|_{H^1(\omega)}+C_3\|\boldsymbol{f}\|_{0,\omega}\|\boldsymbol{v}\|_{0,\omega}\\ &\leq C_4\sqrt{2\|\overline{p}\|_{0,\omega}^2+\|\boldsymbol{f}\|_{0,\omega}^2}\sqrt{\|\mathrm{div}\boldsymbol{v}\|_{0,\omega}^2+\|\boldsymbol{s}\|_{H^1(\omega)}^2+\|\boldsymbol{v}\|_{0,\omega}^2},\end{aligned}$$

thanks to Cauchy-Schwarz inequality. Note that $\bar{p}(x)=\displaystyle\int_{-e/2}^{e/2}p(x,z)dz$.

Therefore, with a given p, there exists a constant $C>0$ such that

$$|\mathcal{L}_{1,p}^N(\boldsymbol{v},\boldsymbol{s})|\leq C\sqrt{\|\boldsymbol{v}\|_{H^1(\omega,\mathbb{R}^3)}^2+\|\boldsymbol{s}\|_{H^1(\omega)}^2}=C\|(\boldsymbol{v},\boldsymbol{s})\|_{\mathcal{X}_N}. \tag{2.42}$$

Combining (2.40), (2.41) and (2.42), we can see that the bilinear form $\mathcal{A}^N(\cdot\,;\,\cdot)$ and the continuous linear form $\mathcal{L}_{1,p}^N(\cdot,\cdot)$ satisfy the conditions of the Banach-Nečas-Babuška theorem. So, the problem (2.39) is well-posed. □

Comment. We are able to prove this theorem by using the Lax-Milgram lemma. Thanks to the inequality (1.39) and the equivalence between two norms $|||\cdot|||_N$ and $||\cdot||_{\mathcal{X}_N}$, we obtain the continuity and coercivity of $\mathcal{A}_1^N$. More precisely, by the Holder inequality and the upper bounded quality of $a^{\alpha\beta\rho\sigma}$ and $\sqrt{a}$, there exist $C_1,\ C_2,\ C_3$ and C_4 such that

$$
\begin{aligned}
|\mathcal{A}_1^N((\boldsymbol{u},\boldsymbol{r});(\boldsymbol{v},\boldsymbol{s}))| &= \bigg| \int_\omega e\, a^{\alpha\beta\rho\sigma}[\gamma_{\rho\sigma}(\boldsymbol{u})\gamma_{\alpha\beta}(\boldsymbol{v}) + \frac{e^2}{12}\chi_{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s})]\sqrt{a}\, d\boldsymbol{x} \\
&\qquad + 4\mu \int_\omega e\, a^{\alpha\beta}\delta_{\alpha 3}(\boldsymbol{u},\boldsymbol{r})\delta_{\beta 3}(\boldsymbol{v},\boldsymbol{s})\sqrt{a}\ \ d\boldsymbol{x}\bigg| \\
&\le C_1\bigg|\int_\omega \gamma_{\alpha\beta}(\boldsymbol{u})\gamma_{\rho\sigma}(\boldsymbol{v})\, d\boldsymbol{x}\bigg| + C_2\bigg|\int_\omega \chi_{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s})\, d\boldsymbol{x}\bigg| \\
&\qquad + C_3\bigg|\int_\omega \delta_{\alpha 3}(\boldsymbol{u},\boldsymbol{r})\delta_{\beta 3}(\boldsymbol{v},\boldsymbol{s})\, d\boldsymbol{x}\bigg| \\
&\le C_1||\gamma_{\alpha\beta}(\boldsymbol{u})||_{L^2(\omega)}||\gamma_{\rho\sigma}(\boldsymbol{v})||_{L^2(\omega)} + C_2||\chi_{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})||_{L^2(\omega)}||\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s})||_{L^2(\omega)} \\
&\qquad + C_3||\delta_{\alpha 3}(\boldsymbol{u},\boldsymbol{r})||_{L^2(\omega)}||\delta_{\beta 3}(\boldsymbol{v},\boldsymbol{s})||_{L^2(\omega)} \\
&\le C_4\Big(||\gamma_{\alpha\beta}(\boldsymbol{u})||^2_{L^2(\omega)} + ||\chi_{\alpha\beta}(\boldsymbol{u},\boldsymbol{r})||^2_{L^2(\omega)} + ||\delta_{\alpha 3}(\boldsymbol{u},\boldsymbol{r})||^2_{L^2(\omega)}\Big)^{1/2} \\
&\qquad \times \Big(||\gamma_{\alpha\beta}(\boldsymbol{v})||^2_{L^2(\omega)} + ||\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s})||^2_{L^2(\omega)} + ||\delta_{\alpha 3}(\boldsymbol{v},\boldsymbol{s})||^2_{L^2(\omega)}\Big)^{1/2} \\
&= C_4|||(\boldsymbol{u},\boldsymbol{r})|||_N|||(\boldsymbol{v},\boldsymbol{s})|||_N, \qquad \forall \boldsymbol{u},\ \boldsymbol{v} \in \mathcal{X}_N.
\end{aligned}
$$

By the equivalence between two norms $|||\cdot|||_N$ and $||\cdot||_{\mathcal{X}_N}$, there exists a constant C_5 such that

$$
|\mathcal{A}_1^N(\boldsymbol{u},\boldsymbol{v})| \le C_5|||\boldsymbol{u}|||_{\mathcal{X}_N}|||\boldsymbol{v}|||_{\mathcal{X}_N}, \qquad \forall \boldsymbol{u},\ \boldsymbol{v} \in \mathcal{X}_N. \tag{2.43}
$$

For the coercivity, by the inequality (1.39) and the relation of two norms $|||\cdot|||_N$ and $||\cdot||_{\mathcal{X}_N}$, there exist constants $C_1',\ C_2'; C_3'$ and C_4' such that

$$
\begin{aligned}
|\mathcal{A}_1^K(\boldsymbol{v},\boldsymbol{v})| &= \bigg| \int_\omega e\, a^{\alpha\beta\rho\sigma}[\gamma_{\rho\sigma}(\boldsymbol{v})\gamma_{\alpha\beta}(\boldsymbol{v}) + \frac{e^2}{12}\chi_{\rho\sigma}(\boldsymbol{v},\boldsymbol{s})\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s})]\sqrt{a}\, d\boldsymbol{x} \\
&\qquad + 4\mu \int_\omega e\, a^{\alpha\beta}\delta_{\alpha 3}(\boldsymbol{v},\boldsymbol{s})\delta_{\beta 3}(\boldsymbol{v},\boldsymbol{s})\sqrt{a}\ \ d\boldsymbol{x}\bigg| \\
&\ge C_1'\int_\omega |\gamma_{\alpha\beta}(\boldsymbol{v})|^2 d\boldsymbol{x} + C_2'\int_\omega |\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s})|^2 d\boldsymbol{x} + C_3'\int_\omega |\delta_{\alpha 3}(\boldsymbol{v},\boldsymbol{s})|^2 d\boldsymbol{x} \\
&\ge \min\{C_1'; C_2'; C_3'\}\int_\omega \big(|\gamma_{\alpha\beta}(\boldsymbol{v})|^2 + |\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s})|^2 + |\delta_{\alpha 3}(\boldsymbol{v},\boldsymbol{s})|^2\big) d\boldsymbol{x} \\
&= \min\{C_1'; C_2; C_3'\}|||(\boldsymbol{v},\boldsymbol{s})|||_N^2 \ge C_4'||(\boldsymbol{v},\boldsymbol{s})||^2_{\mathcal{X}_N}, \qquad \forall \boldsymbol{v} \in \mathcal{X}_N.
\end{aligned} \tag{2.44}
$$

From (2.43), (2.44) and the Lax-Milgram Lemma, Lemma 30 is proved.

Remark 31

Since the mapping $p \longmapsto (\boldsymbol{u}(p), \boldsymbol{r}(p))$ is a continuous affine mapping from $L^2(\Omega)$ to $\mathcal{X}_N$: There exists a constant C such that

$$\forall p_1,\, p_2 \in L^2(\Omega),\; \|(\boldsymbol{u}(p_1), \boldsymbol{r}(p_1)) - (\boldsymbol{u}(p_2), \boldsymbol{r}(p_2))\|_{\mathcal{X}_N} \leq C \parallel p_1 - p_2 \parallel_{L^2(\Omega)} .$$

Hence, (2.22) has the equivalent implicit formulation: Find p in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ satisfying (2.22c) and for a.e t in $(0,T)$

$$c_0 \int_\Omega p' q \sqrt{a}\, d\boldsymbol{X} + \alpha \int_\Omega \operatorname{div}(\boldsymbol{u}'(p) + z\boldsymbol{r}'(p)) q \sqrt{a}\, d\boldsymbol{X} + \frac{\kappa}{\eta} \int_\Omega \nabla p \cdot \nabla q \sqrt{a}\, d\boldsymbol{X} = \int_\Omega g q \sqrt{a}\, d\boldsymbol{X} \quad \forall q \in H_0^1(\Omega), \tag{2.45}$$

with $(\boldsymbol{u}(p), \boldsymbol{r}(p))$ defined by (2.39).

For proving the well-posedness of the semi-discrete problem (2.50), it is convenient to split $\boldsymbol{u}(p)$ and $\boldsymbol{r}(p)$ as follows:

$$\begin{cases} \boldsymbol{u}(p) & = \overline{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}(p), \\ \boldsymbol{r}(p) & = \overline{\boldsymbol{r}} + \widetilde{\boldsymbol{r}}(p), \end{cases} \tag{2.46}$$

where $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{r}}) \in \mathcal{X}_N$ is the unique solution of

$$\forall (\boldsymbol{v}, \boldsymbol{s}) \in \mathcal{X}_N, \mathcal{A}_1^N((\overline{\boldsymbol{u}}, \overline{\boldsymbol{r}}); (\boldsymbol{v}, \boldsymbol{s})) = (\boldsymbol{f}, (\boldsymbol{v} + z\boldsymbol{s})\sqrt{a})_\Omega, \text{ a.e. } t \in]0, T[, \tag{2.47}$$

and $(\widetilde{\boldsymbol{u}}(p), \widetilde{\boldsymbol{r}}(p)) \in \mathcal{X}_N$ is the unique solution of

$$\forall (\boldsymbol{v}, \boldsymbol{s}) \in \mathcal{X}_N, \mathcal{A}_1^N((\widetilde{\boldsymbol{u}}(p), \widetilde{\boldsymbol{r}}(p)); (\boldsymbol{v}, \boldsymbol{s})) = \alpha(p, \operatorname{div}(\boldsymbol{v} + z\boldsymbol{s})\sqrt{a})_\Omega, \text{ a.e. } t \in]0, T[. \tag{2.48}$$

According to the proof of Theorem 9, since the term I is in both (2.23) and (2.30), we see that

$$e \int_\omega a^{\alpha\beta\rho\sigma} [\gamma_{\rho\sigma}(\boldsymbol{u}) \gamma_{\alpha\beta}(\boldsymbol{v}) + \frac{e^2}{12} \chi_{\rho\sigma}(\boldsymbol{u}, \boldsymbol{r}) \chi_{\alpha\beta}(\boldsymbol{v}, \boldsymbol{s})] \sqrt{a}\; d\boldsymbol{x} + 4\mu e \int_\omega a^{\alpha\beta} \delta_{\alpha 3}(\boldsymbol{u}, \boldsymbol{r}) \delta_{\beta 3}(\boldsymbol{v}, \boldsymbol{s}) \sqrt{a}\;\; d\boldsymbol{x} = (\sigma(\boldsymbol{U}), \varepsilon(\boldsymbol{V})\sqrt{a})_\Omega.$$

It leads to

$$\mathcal{A}_1^N((\boldsymbol{u}, \boldsymbol{r}); (\boldsymbol{v}, \boldsymbol{s})) = (\sigma(\boldsymbol{U}), \varepsilon(\boldsymbol{V})\sqrt{a})_\Omega. \tag{2.49}$$

Subsequently, we consider the second step of proof of Theorem 10.

Step 2. We use the Galerkin method to construct a solution p.

Let $(\theta_n)_{n\geq 1}$ be a smooth basis of $H_0^1(\Omega)$ and let Q_k be the space spanned by $(\theta_i)_{i=1}^k$, i.e., $Q_k = \mathrm{Vect}\{\theta_1, ..., \theta_k\}$. Then, our semi-discrete problem reads: Find

$$p_k(t) = \sum_{i=1}^{k} \pi_i(t)\theta_i \in H^1(0,T;Q_k),$$

such that

$$\begin{aligned} c_0 \int_\Omega p_k' \theta_i \sqrt{a}\, d\boldsymbol{X} + \alpha \int_\Omega \mathrm{div}(\tilde{\boldsymbol{u}}_k' + z\tilde{\boldsymbol{r}}_k')\theta_i \sqrt{a}\, d\boldsymbol{X} + \frac{\kappa}{\eta}\int_\Omega \nabla p_k \cdot \nabla\theta_i \sqrt{a}\, d\boldsymbol{X} & \\ = -\alpha \int_\omega \mathrm{div}(\overline{\boldsymbol{u}}' + z\overline{\boldsymbol{r}}')\theta_i \sqrt{a}\, d\boldsymbol{X} + \int_\Omega g\theta_i \sqrt{a}\, d\boldsymbol{X} \qquad 1 \leq i \leq k\,, & \quad (2.50a) \\ p_k(0) = p_{k0}, & \quad (2.50b) \end{aligned}$$

where $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{r}})$ is defined by (2.47), $\tilde{\boldsymbol{u}}_k = \tilde{\boldsymbol{u}}(p_k)$, $\tilde{\boldsymbol{r}}_k = \tilde{\boldsymbol{r}}(p_k)$, i.e. $\boldsymbol{u}(p_k) = \overline{\boldsymbol{u}} + \tilde{\boldsymbol{u}}_k$, $\boldsymbol{r}(p_k) = \overline{\boldsymbol{r}} + \tilde{\boldsymbol{r}}_k$ and $p_{k0} \in Q_k$ satisfies

$$\lim_{k\to\infty} \|p_{k0} - p_0\|_{H^1(\Omega)} = 0.$$

Afterwards, we proceed the well-posedness of the problem (2.50) by the Lemma 32. More precisely, we present (2.50) as a square system of k linear ODEs of order one in matrix form and then using the theory of initial value problem for linear systems and the technique of solution splitting (2.46).

Lemma 32

Let $p_0 \in H^1(\Omega)$, $\boldsymbol{f} \in H^1(0,T;L^2(\Omega,\mathbb{R}^3))$, $g \in L^2(\Omega\times]0,T[)$. The semi-discrete problem (2.50) has exactly one solution on $[0,T]$.

Proof. Thanks to (2.48), we have

$$\begin{aligned} \mathcal{A}_1^N\big((\widetilde{\boldsymbol{u}}(\pi_i'(t)\theta_i), \widetilde{\boldsymbol{r}}(\pi_i'(t)\theta_i)); (\boldsymbol{v}, \boldsymbol{s})\big) &= \alpha\big(\pi_i'(t)\theta_i, \mathrm{div}\boldsymbol{v}\sqrt{a}\big)_\Omega + \alpha(\pi_i'(t)\theta_i, \mathrm{div}(z\boldsymbol{s})\sqrt{a})_\Omega, \\ \mathcal{A}_1^N\big((\widetilde{\boldsymbol{u}}(\theta_i), \widetilde{\boldsymbol{r}}(\theta_i)); (\boldsymbol{v}\pi_i'(t), \boldsymbol{s}\pi_i'(t))\big) &= \alpha\big(\theta_i, \pi_i'(t)\mathrm{div}\boldsymbol{v}\sqrt{a}\big)_\Omega + \alpha(\theta_i, \pi_i'(t)\mathrm{div}(z\boldsymbol{s})\sqrt{a})_\Omega. \end{aligned}$$

Hence, we obtain

$$\mathcal{A}_1^N((\widetilde{\boldsymbol{u}}(\pi_i'(t)\theta_i), \widetilde{\boldsymbol{r}}(\pi_i'(t)\theta_i)); (\boldsymbol{v}, \boldsymbol{s})) = \mathcal{A}_1^N((\widetilde{\boldsymbol{u}}(\theta_i), \widetilde{\boldsymbol{r}}(\theta_i)); (\boldsymbol{v}\pi_i'(t), \boldsymbol{s}\pi_i'(t))).$$

This implies that

$$\begin{aligned} \widetilde{\boldsymbol{u}}(\pi_i'(t)\theta_i) &= \widetilde{\boldsymbol{u}}(\theta_i)\pi_i'(t), \\ \widetilde{\boldsymbol{r}}(\pi_i'(t)\theta_i) &= \widetilde{\boldsymbol{r}}(\theta_i)\pi_i'(t). \end{aligned}$$

Combining the above equality and the note $(\boldsymbol{u}'(p), \boldsymbol{r}'(p)) = (\boldsymbol{u}(p'), \boldsymbol{r}(p'))$, we write (2.50) in matrix form and define the following vectors and matrices

$$\begin{aligned}
\mathbf{P}_i &= \pi_i,\ 1 \le i \le k,\\
\mathbf{C}_{i,j} &= c_0(\theta_j, \theta_i\sqrt{a})_\Omega,\ ,\ 1 \le i,j \le k,\\
\mathbf{A}_{i,j} &= \alpha(\mathrm{div}\widetilde{\boldsymbol{u}}(\theta_j) + \mathrm{div}(\widetilde{z\boldsymbol{r}}(\theta_j)), \theta_i\sqrt{a})_\Omega,\ 1 \le i,j \le k,\\
\mathbf{D}_{i,j} &= \frac{\kappa}{\mu}(\nabla\theta_j, \nabla\theta_i\sqrt{a})_\Omega,\ 1 \le i,j \le k.
\end{aligned}$$

With this notation, writing the time derivative with a prime, (2.50) is a square system of k linear ODEs of order one: Find $P \in [H^1(0,T)]^k$ such that

$$\begin{cases} (\mathbf{C}+\mathbf{A})\mathbf{P}' + \mathbf{D}\mathbf{P} &= \mathbf{H},\ \forall t \in [0,T],\\ \qquad \mathbf{P}(0) &= \mathbf{P}_0, \end{cases} \tag{2.51}$$

where $\mathbf{H}$ is the vector of the right-hand side of (2.50a). Note that $\mathbf{C}$ and $\mathbf{D}$ are square, symmetric matrices. Next, we will prove that they are positive-definite.

For $\mathbf{Z} \in \mathbb{R}^k$, we have

$$\begin{aligned}
\mathbf{Z^T C Z} &= c_0 \sum_{i,j=1}^{k} (\theta_j, \theta_i\sqrt{a})_\Omega Z_i Z_j = c_0 \sum_{i,j=1}^{k} (Z_j\theta_j, Z_i\theta_i\sqrt{a})_\Omega = c_0 \sum_{i,j=1}^{k} \|Z_i\theta_i a^{1/4}\|^2_{L^2(\Omega)}\\
&\ge c_0 \sum_{i,j=1}^{k} \sqrt{\delta}\|Z_i\theta_i\ \|^2_{L^2(\Omega)} \ge 0.
\end{aligned}$$

Since c_0 and δ are both positive constant, if $\mathbf{Z^T C Z} = 0$ then $\sum_{i,j=1}^{k} Z_i\theta_i = 0$. Because $\{\theta_n\}_{n\ge1}$ is a basis of $H_0^1(\Omega)$, $Z_i = 0$ for all i. It follows that $\mathbf{Z} = 0$. Therefore $\mathbf{Z^T C Z} > 0$ for all $\mathbf{Z} \ne 0$ and $\mathbf{C}$ is a positive-definite matrix. Similarly, for matrix $\mathbf{D}$ we have,

$$\begin{aligned}
\mathbf{Z^T D Z} &= \frac{\kappa}{\mu}\sum_{i,j=1}^{k} (\nabla\theta_j, \nabla\theta_i\sqrt{a})_\Omega Z_i Z_j = \frac{\kappa}{\mu}\sum_{i,j=1}^{k} (Z_j\nabla\theta_j, Z_i\nabla\theta_i\sqrt{a})_\Omega\\
&= \frac{\kappa}{\mu}\sum_{i,j=1}^{k} \|Z_i\nabla\theta_i a^{1/4}\|^2_{L^2(\Omega)} \ge \frac{\kappa}{\mu}\sqrt{\delta}\sum_{i,j=1}^{k} \|Z_i\nabla\theta_i\|^2_{L^2(\Omega)} \ge \frac{\kappa}{\mu}\sqrt{\delta}C\|Z_i\theta_i\|^2_{L^2(\Omega)} \ge 0,
\end{aligned}$$

by using Poincaré's inequality.

Since C and δ are both positive constant, if $\mathbf{Z^T D Z} = 0$ then $\displaystyle\sum_{i,j=1}^{k} Z_i\theta_i = 0$. Because $\{\theta_n\}_{n\ge1}$ is a basis of $H_0^1(\Omega)$, $Z_i = 0$ for all i. It follows that $\mathbf{Z} = 0$. Therefore $\mathbf{Z^T D Z} > 0$ for all $\mathbf{Z} \ne 0$ and $\mathbf{D}$ is a positive-definite matrix.

Regarding $\mathbf{A}$, by choosing $p = \theta_i$ and $(\boldsymbol{v}, \boldsymbol{s}) = (\tilde{\boldsymbol{u}}(\theta_j), \tilde{\boldsymbol{r}}(\theta_j))$ in (2.48), we consider that

$$(\boldsymbol{A})_{i,j} = \mathcal{A}_1^N[(\widetilde{\boldsymbol{u}}(\theta_i), \widetilde{\boldsymbol{r}}(\theta_i)); (\widetilde{\boldsymbol{u}}(\theta_j), \widetilde{\boldsymbol{r}}(\theta_j))].$$

Note that $\mathbf{A}$ is a square, symmetric matrix. We prove that $\mathbf{A}$ is positive by the following part. We have

$$\begin{aligned}
\mathbf{Z^T A Z} &= \mathcal{A}_1^N[(\widetilde{\boldsymbol{u}}(\theta_i), \widetilde{\boldsymbol{r}}(\theta_i)); (\widetilde{\boldsymbol{u}}(\theta_j), \widetilde{\boldsymbol{r}}(\theta_j))]Z_i Z_j = (\sigma(\widetilde{\boldsymbol{U}}(\theta_i)), \varepsilon(\widetilde{\boldsymbol{V}}(\theta_j))\sqrt{a})_\Omega Z_i Z_j \\
&\geq \sqrt{\delta}(\sigma(\widetilde{\boldsymbol{U}}(\theta_i)), \varepsilon(\widetilde{\boldsymbol{V}}(\theta_j)))_\Omega Z_i Z_j \\
&= 2\sqrt{\delta}\mu(\varepsilon(\widetilde{\boldsymbol{U}}(\theta_i)), \varepsilon(\widetilde{\boldsymbol{V}}(\theta_j)))_\Omega Z_i Z_j + \sqrt{\delta}\lambda(\operatorname{div}(\widetilde{\boldsymbol{U}}(\theta_i)), \operatorname{div}(\widetilde{\boldsymbol{V}}(\theta_j)))_\Omega Z_i Z_j \\
&= 2\sqrt{\delta}\mu(\sum_i \varepsilon(\widetilde{\boldsymbol{U}}(\theta_i))Z_i, \sum_j \varepsilon(\widetilde{\boldsymbol{V}}(\theta_j))Z_j)_\Omega + \sqrt{\delta}\lambda(\sum_i \operatorname{div}(\widetilde{\boldsymbol{U}}(\theta_i))Z_i, \sum_j \operatorname{div}(\widetilde{\boldsymbol{V}}(\theta_j))Z_j)_\Omega \\
&= 2\sqrt{\delta}\mu\| \sum_i \varepsilon(\widetilde{\boldsymbol{U}}(\theta_i))Z_i\|_\Omega^2 + \sqrt{\delta}\lambda\| \sum_i \operatorname{div}(\widetilde{\boldsymbol{U}}(\theta_i))Z_i\|_\Omega^2 \geq 0.
\end{aligned}$$

Then, we conclude that the matrix $\mathbf{C}+\mathbf{A}$ is square, symmetric, positive definite. Hence, (2.50) has exactly one solution on $[0, T]$ by the theory of initial value problem for linear systems (Theorem 7.4 in [38]). □

Let us prove now that the sequences (p_k) and $(\boldsymbol{u}(p_k), \boldsymbol{r}(p_k))$ of the semi-discrete problem (2.50) are bounded. For obtaining this result, we firstly test (2.39) with $(\boldsymbol{u}'(p_k), \boldsymbol{r}'(p_k)) = (\boldsymbol{u}(p_k'), \boldsymbol{r}(p_k'))$, (2.50a) with p_k, add two equations and integrate with respect to time on $]0, t[$ for arbitrary $t \in [0, T]$. Next, we test (2.50a) with p_k', differentiating in time (2.39) written for $p = p_k$, testing the resulting equation (2.50a) with $(\boldsymbol{u}'(p_k), \boldsymbol{r}'(p_k))$, adding the two equations and integrating with respect to time. Finally, we add all equations together and apply the Young, Hölder's inequalities and the inequality (1.39). We itemize them in the following lemma.

Lemma 33

Let $p_0 \in H^1(\Omega)$, $\boldsymbol{f} \in H^1(0, T; L^2(\Omega, \mathbb{R}^3))$, $g \in L^2(\Omega\times]0, T[)$.

i. The solution p_k and $(\boldsymbol{u}(p_k), \boldsymbol{r}(p_k))$ of the semi-discrete problem (2.50) satisfy the following uniform bounds:

$$||(\boldsymbol{u}(p_k), \boldsymbol{r}(p_k))||^2_{L^\infty(0,T;\mathcal{X}_N)} + \|p_k\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla p_k\|^2_{L^2(0,T;L^2(\Omega))} \leq C \text{ and} \tag{2.52a}$$
$$||(\boldsymbol{u}'(p_k), \boldsymbol{r}'(p_k))||^2_{L^2(0,T;\mathcal{X}_N)} + \|p_k'\|^2_{L^2(0,T;L^2(\Omega))} + \|\nabla p_k\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{2.52b}$$

where C depends only on T, $\boldsymbol{f}$, p_0 and g.

ii. Moreover, there exists a function $\hat{p}$ in $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$ and functions $\boldsymbol{w}_1, \boldsymbol{w}_2$ in $H^1(0, T; H_0^1(\Omega))$ such that

$$p_k \overset{*}{\rightharpoonup} \hat{p} \text{ weakly-} * \text{ in } L^\infty(0, T; H^1(\Omega)), \tag{2.53a}$$
$$p_k \rightharpoonup \hat{p} \text{ weakly in } H^1(0, T; L^2(\Omega)), \tag{2.53b}$$
$$(\boldsymbol{u}(p_k), \boldsymbol{r}(p_k)) \rightharpoonup (\boldsymbol{w}_1, \boldsymbol{w}_2) \text{ weakly in } H^1(0, T; \mathcal{X}_N). \tag{2.53c}$$

Proof. Following [47], we test (2.39) with $(\boldsymbol{u}'(p_k), \boldsymbol{r}'(p_k))$, (2.50) with p_k, add two equations and integrate with respect to time on $(0,t)$ for arbitrary $t \in [0,T]$. This leads to:

$$
\begin{aligned}
&\underbrace{\int_0^t \int_\omega e\, a^{\alpha\beta\rho\sigma} \gamma_{\alpha\beta}(\boldsymbol{u}(p_k(t)))\gamma_{\rho\sigma}(\boldsymbol{u}'(p_k(t)))\sqrt{a}\, d\boldsymbol{x}\, dt}_{I_1} \\
&\quad + \underbrace{\frac{1}{12}\int_0^t \int_\omega e^3 a^{\alpha\beta\rho\sigma} \chi_{\alpha\beta}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))\chi_{\rho\sigma}(\boldsymbol{u}'(p_k(t)), \boldsymbol{r}'(p_k(t)))\sqrt{a}\, d\boldsymbol{x}\, dt}_{I_2} \\
&\quad + \underbrace{4\mu \int_0^t \int_\omega e a^{\alpha\beta} \delta_{\alpha 3}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))\delta_{\beta 3}(\boldsymbol{u}'(p_k(t)), \boldsymbol{r}'(p_k(t)))\sqrt{a}\, d\boldsymbol{x}\, dt}_{I_3} \\
&\quad + \underbrace{\int_0^t \int_\Omega p_k'(t) p_k(t)\sqrt{a}\; d\boldsymbol{X}\, dt}_{I_4} + \frac{\kappa}{\eta}\int_0^t \int_\Omega |\nabla p_k(t)|^2 \sqrt{a}\, d\boldsymbol{X}\, dt \\
&= \underbrace{\int_0^t \int_\Omega g p_k(t)\sqrt{a}\, d\boldsymbol{X}\, dt}_{I_5} + \underbrace{\int_0^t \int_\Omega \boldsymbol{f} \cdot [\boldsymbol{u}'(p_k(t)) + z\boldsymbol{r}'(p_k(t))]\sqrt{a}\; d\boldsymbol{x} dt}_{I_6} .
\end{aligned}
$$

Using the symmetric property of $\gamma_{\alpha\beta}$ and applying the inequality (1.39) to I_1, there exists a positive constants c_1 such that

$$
\begin{aligned}
I_1 &= \int_0^t \int_\omega e\, a^{\alpha\beta\rho\sigma} \gamma_{\alpha\beta}(\boldsymbol{u}(p_k(t)))\gamma_{\rho\sigma}(\boldsymbol{u}'(p_k(t)))\sqrt{a}\, d\boldsymbol{x}\, dt \\
&= \frac{1}{2}\int_0^t \int_\omega e\, a^{\alpha\beta\rho\sigma} [\gamma_{\alpha\beta}(\boldsymbol{u}(p_k(t)))\gamma_{\rho\sigma}(\boldsymbol{u}(p_k(t)))]'\sqrt{a}\, d\boldsymbol{x}\, dt \\
&= \frac{1}{2}\int_\omega e\, a^{\alpha\beta\rho\sigma} \int_0^t [\gamma_{\alpha\beta}(\boldsymbol{u}(p_k(t)))\gamma_{\rho\sigma}(\boldsymbol{u}(p_k(t)))]'\, dt\, \sqrt{a}\, d\boldsymbol{x} \\
&= \frac{1}{2}\int_\omega e\, a^{\alpha\beta\rho\sigma} [\gamma_{\alpha\beta}(\boldsymbol{u}(p_k(t)))\gamma_{\rho\sigma}(\boldsymbol{u}(p_k(t))) - \gamma_{\alpha\beta}(\boldsymbol{u}(p_{k0}))\gamma_{\rho\sigma}(\boldsymbol{u}(p_{k0}))]\sqrt{a}\, d\boldsymbol{x} \\
&= \frac{1}{2}\int_\omega e\, a^{\alpha\beta\rho\sigma} \gamma_{\alpha\beta}(\boldsymbol{u}(p_k(t)))\gamma_{\rho\sigma}(\boldsymbol{u}(p_k(t)))\sqrt{a}\, d\boldsymbol{x} \\
&\qquad\qquad - \frac{1}{2}\int_\omega e\, a^{\alpha\beta\rho\sigma} \gamma_{\alpha\beta}(\boldsymbol{u}(p_{k0}))\gamma_{\rho\sigma}(\boldsymbol{u}(p_{k0})))\sqrt{a}\, d\boldsymbol{x} \\
&\geq \frac{c_1}{2}\int_\omega |\gamma_{\alpha\beta}(\boldsymbol{u}(p_k(t)))|^2 \sqrt{a}\, dx - \frac{1}{2}\int_\omega e\, a^{\alpha\beta\rho\sigma} \gamma_{\alpha\beta}(\boldsymbol{u}(p_{k0}))\gamma_{\rho\sigma}(\boldsymbol{u}(p_{k0})))\sqrt{a}\, d\boldsymbol{x}.
\end{aligned}
$$

Similarly, thanks to the symmetric property of $\chi_{\alpha\beta}$ and the inequality (1.39), there exists a positive constants c_2 such that

$$
\begin{aligned}
I_2 &= \frac{1}{12}\int_0^t \int_\omega e^3\, a^{\alpha\beta\rho\sigma} \chi_{\alpha\beta}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))\chi_{\rho\sigma}(\boldsymbol{u'}(p_k(t)), \boldsymbol{r'}(p_k(t)))\sqrt{a}\, d\boldsymbol{x}dt \\
&= \frac{1}{24}\int_0^t \int_\omega e^3\, a^{\alpha\beta\rho\sigma} [\chi_{\alpha\beta}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))\chi_{\rho\sigma}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))]'\sqrt{a}\, d\boldsymbol{x}dt \\
&= \frac{1}{24}\int_\omega e^3\, a^{\alpha\beta\rho\sigma} (\int_0^t [\chi_{\alpha\beta}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))\chi_{\rho\sigma}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))]'dt)\sqrt{a}\, d\boldsymbol{x} \\
&= \frac{1}{24}\int_\omega e^3\, a^{\alpha\beta\rho\sigma} [\chi_{\alpha\beta}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))\chi_{\rho\sigma}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t))) \\
&\qquad\qquad - \chi_{\alpha\beta}(\boldsymbol{u}(p_{k0}), \boldsymbol{r}(p_{k0}(t)))\chi_{\rho\sigma}(\boldsymbol{u}(p_{k0}), \boldsymbol{r}(p_{k0}))]\sqrt{a}\, d\boldsymbol{x} \\
&= \frac{1}{24}\int_\omega e^3\, a^{\alpha\beta\rho\sigma} \chi_{\alpha\beta}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))\chi_{\rho\sigma}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))\sqrt{a}\, d\boldsymbol{x} \\
&\qquad\qquad - \frac{1}{24}\int_\omega e^3\, a^{\alpha\beta\rho\sigma} \chi_{\alpha\beta}(\boldsymbol{u}(p_{k0}), \boldsymbol{r}(p_{k0}(t)))\chi_{\rho\sigma}(\boldsymbol{u}(p_{k0}), \boldsymbol{r}(p_{k0}))\sqrt{a}\, d\boldsymbol{x} \\
&\geq \frac{c_2}{24}\int_\omega |\chi_{\alpha\beta}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))|^2\sqrt{a}\, d\boldsymbol{x} \\
&\qquad\qquad - \frac{1}{24}\int_\omega e^3\, a^{\alpha\beta\rho\sigma} \chi_{\alpha\beta}(\boldsymbol{u}(p_{k0}), \boldsymbol{r}(p_{k0}(t)))\chi_{\rho\sigma}(\boldsymbol{u}(p_{k0}), \boldsymbol{r}(p_{k0}))\sqrt{a}\, d\boldsymbol{x}.
\end{aligned}
$$

By the symmetric property of $\chi_{\alpha\beta}$ and the inequality (1.39), there exists a positive constants c_3 such that

$$
\begin{aligned}
I_3 &= 4\mu \int_0^t \int_\omega e\, a^{\alpha\beta} \delta_{\alpha 3}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))\delta_{\beta 3}(\boldsymbol{u'}(p_k(t)), \boldsymbol{r'}(p_k(t)))\sqrt{a}\, d\boldsymbol{x}\, dt \\
&= 2M\mu \int_0^t \int_\omega e\, a^{\alpha\beta} [\delta_{\alpha 3}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))\delta_{\beta 3}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))]'\sqrt{a}\, d\boldsymbol{x}\, dt \\
&= 2\mu \int_\omega e\, a^{\alpha\beta} (\int_0^t [\delta_{\alpha 3}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))\delta_{\beta 3}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))]'\, dt)\sqrt{a}\, d\boldsymbol{x} \\
&= 2\mu \int_\omega e\, a^{\alpha\beta} [\delta_{\alpha 3}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))\delta_{\beta 3}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t))) \\
&\qquad\qquad - \delta_{\alpha 3}(\boldsymbol{u}(p_{k0}), \boldsymbol{r}(p_{k0}))\delta_{\beta 3}(\boldsymbol{u}(p_{k0}), \boldsymbol{r}(p_{k0}))]\sqrt{a}\, d\boldsymbol{x} \\
&= 2\mu \{\int_\omega e\, a^{\alpha\beta} \delta_{\alpha 3}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))\delta_{\beta 3}(\boldsymbol{u}(p_k(t)), \boldsymbol{r}(p_k(t)))\sqrt{a} d\boldsymbol{x} \\
&\qquad\qquad - \int_\omega e\, a^{\alpha\beta} \delta_{\alpha 3}(\boldsymbol{u}(p_{k0}), \boldsymbol{r}(p_{k0}))\delta_{\beta 3}(\boldsymbol{u}(p_{k0}), \boldsymbol{r}(p_{k0}))\sqrt{a}\, d\boldsymbol{x}\}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\mu\{\int_\omega e\, a^{\alpha\beta}\delta_{\alpha3}(\boldsymbol{u}(p_k(t)),\boldsymbol{r}(p_k(t)))\delta_{\beta3}(\boldsymbol{u}(p_k(t)),\boldsymbol{r}(p_k(t)))\sqrt{a}d\boldsymbol{x}\\
&\qquad\qquad - \int_\omega e\, a^{\alpha\beta}\delta_{\alpha3}(\boldsymbol{u}(p_{k0}),\boldsymbol{r}(p_{k0}))\delta_{\beta3}(\boldsymbol{u}(p_{k0}),\boldsymbol{r}(p_{k0}))\sqrt{a}\,d\boldsymbol{x}\}\\
&\geq 2\mu c_3\int_\omega |\delta_{\alpha3}(\boldsymbol{u}(p_k(t)),\boldsymbol{r}(p_k(t)))|^2\sqrt{a}d\boldsymbol{x}\\
&\qquad\qquad - 2\mu\int_\omega e\, a^{\alpha\beta}\delta_{\alpha3}(\boldsymbol{u}(p_{k0}),\boldsymbol{r}(p_{k0}))\delta_{\beta3}(\boldsymbol{u}(p_{k0}),\boldsymbol{r}(p_{k0}))\sqrt{a}\,d\boldsymbol{x}.
\end{aligned}
$$

Subsequently, we make I_4 clearer,

$$
\begin{aligned}
I_4 &= \int_0^t\int_\Omega p_k'(t)p_k(t)\sqrt{a}\,d\boldsymbol{x}\,dt = \int_\Omega(\int_0^t p_k'(t)p_k(t)\,dt)\sqrt{a}\,d\boldsymbol{X} = \frac{1}{2}\int_\Omega(p_k^2(t)-p_{k0}^2)\sqrt{a}\,d\boldsymbol{X}\\
&= \frac{1}{2}\int_\Omega |p_k(t)|^2\sqrt{a}\,d\boldsymbol{X} - \frac{1}{2}\int_\Omega |p_{k0}|^2\sqrt{a}\,d\boldsymbol{X}.
\end{aligned}
$$

Applying the Young and Hölder inequalities to I_5, we have

$$
\begin{aligned}
I_5 &= \int_0^t\int_\Omega g p_k(t)\sqrt{a}\,d\boldsymbol{X}\,dt\\
&\leq \sqrt{M}\int_0^t\int_\Omega g p_k(t)\,d\boldsymbol{X}\,dt \leq \sqrt{M}\int_0^t \|g\|_{0,\Omega}\|p_k(t)\|_{0,\Omega}\\
&\leq \sqrt{M}\frac{1}{2\varepsilon_2}\int_0^t \|g\|_{0,\Omega}^2\,dt + \sqrt{M}\frac{\varepsilon_2}{2}\int_0^t \|p_k(t)\|_{0,\Omega}^2\\
&\leq \sqrt{M}\frac{1}{2\varepsilon_2}\|g\|_{L^2(0,T;L^2(\Omega))}^2 + \sqrt{M}\frac{\varepsilon_2}{2}\int_0^t \|\nabla p_k(t)\|_{0,\Omega}^2\\
&\leq \sqrt{M}\frac{1}{2\varepsilon_2}\|g\|_{L^2(0,T;L^2(\Omega))}^2 + \sqrt{M}\frac{\varepsilon_2}{2}\|\nabla p_k\|_{L^2(0,T;L^2(\Omega))}^2.
\end{aligned}
$$

Lastly, analogous to I_5, we derive the upper bound of I_6 by the Young and Hölder inequalities

$$
\begin{aligned}
I_6 &= \int_0^t\int_\Omega \boldsymbol{f}\cdot[\boldsymbol{u}'(p_k(t)) + z\boldsymbol{r}'(p_k(t))]\sqrt{a}\;d\boldsymbol{X}dt\\
&\leq \sqrt{M}\int_0^t \|\boldsymbol{f}\|_{0,\Omega}\|\boldsymbol{u}'(p_k(t)) + z\boldsymbol{r}'(p_k(t))\|_{0,\Omega}\,dt\\
&\leq \frac{\sqrt{M}}{2\varepsilon_1}\int_0^t \|\boldsymbol{f}\|_{0,\Omega}^2\,dt + \frac{\sqrt{M}\varepsilon_1}{2}\int_0^t [\|\boldsymbol{u}'(p_k(t))\|_{0,\omega}^2 + \frac{e}{2}\|\boldsymbol{r}'(p_k(t))\|_{0,\omega}^2]dt\\
&\leq \frac{\sqrt{M}}{2\varepsilon_1}\|\boldsymbol{f}\|_{L^2(0,T;L^2(\Omega,\mathbb{R}^3))}^2 + \frac{\sqrt{M}\varepsilon_1}{2}\|(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\|_{L^2(0,T;\mathcal{X}_N)}^2 \quad (e \ll 1).
\end{aligned}
$$

where ε_1 and ε_2 are positive constants from Young's inequality.

Consequently, one gets

$$
\begin{aligned}
&\frac{c_1}{2}\int_\omega |\gamma_{\alpha\beta}(\boldsymbol{u}(p_k(t)))|^2\sqrt{a}\,d\boldsymbol{x} + \frac{c_2}{24}\int_\omega |\chi_{\alpha\beta}(\boldsymbol{u}(p_k(t)),\boldsymbol{r}(p_k(t)))|^2\sqrt{a}\,d\boldsymbol{x}\\
&\quad + 2c_3\mu\int_\omega |\delta_{\alpha 3}(\boldsymbol{u}(p_k(t)),\boldsymbol{r}(p_k(t)))|^2\sqrt{a}d\boldsymbol{x} + \frac{1}{2}\int_\Omega |p_k(t)|^2\,\sqrt{a}\,d\boldsymbol{X} + \frac{\kappa}{\eta}\int_0^t\int_\Omega |\nabla p_k(t)|^2\sqrt{a}\,d\boldsymbol{X}\,dt\\
&\le \frac{1}{2}\int_\omega e\,a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(\boldsymbol{u}(p_{k0}))\gamma_{\rho\sigma}(\boldsymbol{u}(p_{k0})))\sqrt{a}\,d\boldsymbol{x}\\
&\quad + \frac{1}{24}\int_\omega e^3\,a^{\alpha\beta\rho\sigma}\chi_{\alpha\beta}(\boldsymbol{u}(p_{k0}),\boldsymbol{r}(p_{k0}))\chi_{\rho\sigma}(\boldsymbol{u}(p_{k0}),\boldsymbol{r}(p_{k0}))\sqrt{a}\,d\boldsymbol{x}\\
&\quad + 2\mu\int_\omega e\,a^{\alpha\beta}\delta_{\alpha 3}(\boldsymbol{u}(p_{k0}),\boldsymbol{r}(p_{k0}))\delta_{\beta 3}(\boldsymbol{u}(p_{k0}),\boldsymbol{r}(p_{k0}))\sqrt{a}\,d\boldsymbol{x} + \frac{1}{2}\int_\Omega |p_{k0}|^2\,\sqrt{a}\,d\boldsymbol{X}\\
&\quad + \sqrt{M}\frac{1}{2\varepsilon_2}\|g\|^2_{L^2(0,T;L^2(\Omega))} + \sqrt{M}\frac{\varepsilon_2}{2}\|\nabla p_k\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\sqrt{M}}{2\varepsilon_1}\|\boldsymbol{f}\|^2_{L^2(0,T;L^2(\Omega,\mathbb{R}^3))}\\
&\qquad + \frac{\sqrt{M}\varepsilon_1}{2}\|(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\|^2_{L^2(0,T;\mathcal{X}_N)}.
\end{aligned}
$$

Since $0<\sqrt{\delta}\le\sqrt{a(x)}\le\sqrt{M}<\infty$ (see (2)-page 12),

$$
\begin{aligned}
&c_4\sqrt{\delta}\|(\boldsymbol{u}(p_k(t)),\boldsymbol{r}(p_k(t)))\|^2_{\mathcal{X}_N} + \frac{\sqrt{\delta}}{2}\|p_k(t)\|^2_{0,\Omega} + \frac{\kappa\sqrt{\delta}}{\eta}\int_0^t \|\nabla p_k(t)\|^2_{0,\Omega}\,dt\\
&\le \frac{\sqrt{M}}{2}\int_\omega e\,a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(\boldsymbol{u}(p_{k0}))\gamma_{\rho\sigma}(\boldsymbol{u}(p_{k0}))\,d\boldsymbol{x}\\
&\quad + \frac{\sqrt{M}}{24}\int_\omega e^3\,a^{\alpha\beta\rho\sigma}\chi_{\alpha\beta}(\boldsymbol{u}(p_{k0}),\boldsymbol{r}(p_{k0}))\chi_{\rho\sigma}(\boldsymbol{u}(p_{k0}),\boldsymbol{r}(p_{k0}))\,d\boldsymbol{x}\\
&\quad + 2\mu\sqrt{M}\int_\omega e\,a^{\alpha\beta}\delta_{\alpha 3}(\boldsymbol{u}(p_{k0}),\boldsymbol{r}(p_{k0}))\delta_{\beta 3}(\boldsymbol{u}(p_{k0}),\boldsymbol{r}(p_{k0}))\,d\boldsymbol{x} + \frac{\sqrt{M}}{2}\int_\Omega |p_{k0}|^2\,d\boldsymbol{X}\\
&\quad + \sqrt{M}\frac{1}{2\varepsilon_2}\|g\|^2_{L^2(0,T;L^2(\Omega))} + \sqrt{M}\frac{\varepsilon_2}{2}\|\nabla p_k\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\sqrt{M}}{2\varepsilon_1}\|\boldsymbol{f}\|^2_{L^2(0,T;L^2(\Omega,\mathbb{R}^3))}\\
&\qquad + \frac{\sqrt{M}\varepsilon_1}{2}\|(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\|^2_{L^\infty(0,T;\mathcal{X}_N)},
\end{aligned}
$$

here $c_4=\min\{\frac{c_1}{2};\frac{c_2}{24};2c_3\mu\}$.

It follows to

$$(c_4\sqrt{\delta}-\frac{\sqrt{M}\varepsilon_1}{2})\|(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\|^2_{L^\infty(0,T;\mathcal{X}_N)}+\frac{\sqrt{\delta}}{2}\|p_k(t)\|^2_{0,\Omega}+(\frac{\kappa\sqrt{\delta}}{\eta}-\sqrt{M}\frac{\varepsilon_2}{2})\|\nabla p_k\|^2_{L^2(0,T;L^2(\Omega))}$$
$$\leq \frac{\sqrt{M}}{2}\int_\omega e\,a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(\boldsymbol{u}(p_{k0}))\gamma_{\rho\sigma}(\boldsymbol{u}(p_{k0}))\,d\boldsymbol{x}$$
$$+\frac{\sqrt{M}}{24}\int_\omega e^3\,a^{\alpha\beta\rho\sigma}\chi_{\alpha\beta}(\boldsymbol{u}(p_{k0}),\boldsymbol{r}(p_{k0}))\chi_{\rho\sigma}(\boldsymbol{u}(p_{k0}),\boldsymbol{r}(p_{k0}))\,d\boldsymbol{x}$$
$$+2\mu\sqrt{M}\int_\omega e\,a^{\alpha\beta}\delta_{\alpha3}(\boldsymbol{u}(p_{k0}),\boldsymbol{r}(p_{k0}))\delta_{\beta3}(\boldsymbol{u}(p_{k0}),\boldsymbol{r}(p_{k0}))\,d\boldsymbol{x}+\frac{\sqrt{M}}{2}\int_\Omega|p_{k0}|^2\;d\boldsymbol{X}$$
$$+\sqrt{M}\frac{1}{2\varepsilon_2}\|g\|^2_{L^2(0,T;L^2(\Omega))}+\frac{\sqrt{M}}{2\varepsilon_1}\|\boldsymbol{f}\|^2_{L^2(0,T;L^2(\Omega,\mathbb{R}^3))}.$$

By choosing the ε_1 and ε_2 suitably, *i.e.*, $\varepsilon_1<\dfrac{2c_4\sqrt{\delta}}{\sqrt{M}}$ and $\varepsilon_2<\dfrac{2\kappa\sqrt{\delta}}{\eta\sqrt{M}}$, we obtain that

$$||(\boldsymbol{u}(p_k),\boldsymbol{r}(p_k))||^2_{L^\infty(0,T;\mathcal{X}_N)}+\|p_k\|^2_{L^\infty(0,T;L^2(\Omega))}+\|\nabla p_k\|^2_{L^2(0,T;L^2(\Omega))}<C_1, \tag{2.54}$$

here C_1 depends only on T, $\|\boldsymbol{f}\|_{L^2(0,T;L^2(\Omega,\mathbb{R}^3))}$, $\|p_0\|_{L^\infty(0,T;L^2(\Omega))}$ and $\|g\|_{L^2(0,T;L^2(\Omega))}$.

We have just proved the inequality (2.52a). Next, we derive the inequality (2.52b) by testing (2.50a) with p'_k, it leads

$$\int_\Omega|p'_k|^2\sqrt{a}\,d\boldsymbol{X}+\alpha\int_\Omega \text{div}\;(\boldsymbol{u}'(p_k)+z\boldsymbol{r}'(p_k))p'_k\sqrt{a}\,d\boldsymbol{X}$$
$$+\frac{\kappa}{\eta}\int_\Omega\nabla p_k\cdot\nabla p'_k\sqrt{a}\,d\boldsymbol{X}=\int_\Omega g p'_k\sqrt{a}\,d\boldsymbol{X}. \tag{2.55}$$

And differentiating in time (2.22a) written for $p=p_k$,

$$\int_\omega e\,a^{\alpha\beta\rho\sigma}[\gamma_{\rho\sigma}(\boldsymbol{u}'(p_k))\gamma_{\alpha\beta}(\boldsymbol{v})+\frac{e^2}{12}\chi_{\rho\sigma}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\chi_{\alpha\beta}(\boldsymbol{v},\boldsymbol{s})]\sqrt{a}\;d\boldsymbol{x}$$
$$+4\mu\int_\omega e\,a^{\alpha\beta}\delta_{\alpha3}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\delta_{\beta3}(\boldsymbol{v},\boldsymbol{s})\sqrt{a}\;\;d\boldsymbol{x}-\alpha\int_\Omega p'_k\;\text{div}(\boldsymbol{v}+z\boldsymbol{s})\sqrt{a}\;d\boldsymbol{X}$$
$$=\int_\Omega\boldsymbol{f}'\cdot(\boldsymbol{v}+z\boldsymbol{s})\sqrt{a}\;d\boldsymbol{X}.$$

Testing the above achieving equation with $(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))$,

$$\int_\omega e\,a^{\alpha\beta\rho\sigma}[\gamma_{\rho\sigma}(\boldsymbol{u}'(p_k))\gamma_{\alpha\beta}(\boldsymbol{u}'(p_k))+\frac{e^2}{12}\chi_{\rho\sigma}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\chi_{\alpha\beta}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))]\sqrt{a}\;d\boldsymbol{x}$$
$$+4\mu\int_\omega e\,a^{\alpha\beta}\delta_{\alpha3}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\delta_{\beta3}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\sqrt{a}\;\;d\boldsymbol{x}-\alpha\int_\Omega p'_k\;\text{div}(\boldsymbol{u}'(p_k)+z\boldsymbol{r}'(p_k))\sqrt{a}\;d\boldsymbol{X}$$
$$=\int_\Omega\boldsymbol{f}'\cdot(\boldsymbol{u}'(p_k)+z\boldsymbol{r}'(p_k))\sqrt{a}\;d\boldsymbol{X}. \tag{2.56}$$

Adding the two equations (2.55) and (2.56) then integrating with respect to time, we obtain

$$\underbrace{\int_0^t\int_\omega e\, a^{\alpha\beta\rho\sigma}[\gamma_{\rho\sigma}(\boldsymbol{u}'(p_k))\gamma_{\alpha\beta}(\boldsymbol{u}'(p_k)) + \frac{e^2}{12}\chi_{\rho\sigma}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\chi_{\alpha\beta}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))]\sqrt{a}\, d\boldsymbol{x}dt}_{Y_1}$$
$$+\underbrace{4\mu\int_0^t\int_\omega e\, a^{\alpha\beta}\delta_{\alpha3}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\delta_{\beta3}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\sqrt{a}\;\; d\boldsymbol{x}dt}_{Y_2} + \underbrace{\int_0^t\int_\Omega |p_k'|^2\sqrt{a}\, d\boldsymbol{X}dt}_{Y_3}$$
$$+\underbrace{\frac{\kappa}{\eta}\int_0^t\int_\Omega \nabla p_k\cdot\nabla p_k'\sqrt{a}\, d\boldsymbol{X}dt}_{Y_4} = \underbrace{\int_0^t\int_\Omega \boldsymbol{f}'\cdot(\boldsymbol{u}'(p_k)+z\boldsymbol{r}'(p_k))\sqrt{a}\, d\boldsymbol{X}dt}_{Y_5}$$
$$+\underbrace{\int_0^t\int_\Omega g p_k'\sqrt{a}\, d\boldsymbol{X}dt}_{Y_6}. \tag{2.57}$$

Applying the inequality (1.39), there are positive constants c_1', c_2', c_3' such that

$$\begin{aligned}
&Y_1+Y_2\\
&=\int_0^t\int_\omega e\, a^{\alpha\beta\rho\sigma}[\gamma_{\rho\sigma}(\boldsymbol{u}'(p_k))\gamma_{\alpha\beta}(\boldsymbol{u}'(p_k)) + \frac{e^2}{12}\chi_{\rho\sigma}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\chi_{\alpha\beta}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))]\sqrt{a}\, d\boldsymbol{x}dt\\
&\qquad + 4\mu\int_0^t\int_\omega e\, a^{\alpha\beta}\delta_{\alpha3}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\delta_{\beta3}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))\sqrt{a}\;\; d\boldsymbol{x}dt\\
&\geq c_1'\int_0^t\int_\omega [e|\gamma_{\alpha\beta}(\boldsymbol{u}'(p_k))|^2 + \frac{e^3}{12}|\chi_{\alpha\beta}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))|^2 + 4\mu e|\delta_{\alpha3}(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))|^2]\sqrt{a}d\boldsymbol{x}dt\\
&\geq c_2'\sqrt{\delta}\int_0^t |||(\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k))|||^2 dt \geq c_3'\sqrt{\delta}\int_0^t \parallel (\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k)) \parallel_{\mathcal{X}_N}^2\, dt\\
&= c_3'\sqrt{\delta}\parallel (\boldsymbol{u}'(p_k),\boldsymbol{r}'(p_k)) \parallel_{L^2(0,t;\mathcal{X}_N)}^2 .
\end{aligned}$$

Using the bounds of $\sqrt{a}$ and integration with respect to time, we carry out the lower bound of Y_3+Y_4,

$$\begin{aligned}
Y_3+Y_4 &= \int_0^t\int_\Omega |p_k'|^2\sqrt{a}\, d\boldsymbol{X}dt + \int_0^t\int_\Omega \nabla p_k\cdot\nabla p_k'\sqrt{a}\, d\boldsymbol{X}dt\\
&\geq \sqrt{\delta}\int_0^t \parallel p_k' \parallel_{L^2(\Omega)}^2\, dt + \frac{1}{2}\int_\Omega (|\nabla p_k(t)|^2 - |\nabla p_{k0}|^2)\sqrt{a}\, d\boldsymbol{X}\\
&\geq \sqrt{\delta}\parallel p_k' \parallel_{L^2(0,t;L^2(\Omega))}^2 + \frac{\sqrt{\delta}}{2}\parallel \nabla p_k \parallel_{L^2(\Omega)}^2 - \frac{\sqrt{M}}{2}\parallel \nabla p_{k0} \parallel_{L^2(\Omega)}^2 .
\end{aligned}$$

In the following, we employ the Young and Hölder inequalities to find the upper bound of Y_5,

$$\begin{aligned}
Y_5 &= \int_0^t \int_\Omega \boldsymbol{f}' \cdot [\boldsymbol{u}'(p_k) + z\boldsymbol{r}'(p_k)]\sqrt{a}\, d\boldsymbol{X} dt \\
&\leq \sqrt{M} \int_0^t \|\boldsymbol{f}'\|_{0,\Omega} \|\boldsymbol{u}'(p_k(t)) + z\boldsymbol{r}'(p_k(t))\|_{0,\Omega}\, dt \\
&\leq \frac{\sqrt{M}}{2\varepsilon_1'} \int_0^t \|\boldsymbol{f}'\|_{0,\Omega}^2\, dt + \frac{\sqrt{M}\varepsilon_1'}{2} \int_0^t [\|\boldsymbol{u}'(p_k(t))\|_{0,\omega}^2 + \frac{e}{2}\|\boldsymbol{r}'(p_k(t))\|_{0,\omega}^2] dt \\
&\leq \frac{\sqrt{M}}{2\varepsilon_1'} \|\boldsymbol{f}'\|_{L^2(0,T;L^2(\Omega,\mathbb{R}^3))}^2 + \frac{\sqrt{M}\varepsilon_1'}{2} \|(\boldsymbol{u}'(p_k), \boldsymbol{r}'(p_k))\|_{L^2(0,T;\mathcal{X}_N)}^2 \quad (e \ll 1).
\end{aligned}$$

Similar to Y_5, we also derive the lower bound of Y_6,

$$\begin{aligned}
Y_6 = &\int_0^t \int_\Omega g p_k'(t)\sqrt{a}\, d\boldsymbol{X}\, dt \leq \sqrt{M} \int_0^t \int_\Omega g p_k'(t)\, d\boldsymbol{X}\, dt \leq \sqrt{M} \int_0^t \|g\|_{0,\Omega} \|p_k(t)\|_{0,\Omega} \\
&\leq \frac{\sqrt{M}}{2\varepsilon_2'} \int_0^t \|g\|_{0,\Omega}^2\, dt + \frac{\sqrt{M}\varepsilon_2'}{2} \int_0^t \|p_k'(t)\|_{0,\Omega}^2 \leq \frac{\sqrt{M}}{2\varepsilon_2'} \|g\|_{L^2(0,T;L^2(\Omega))}^2 + \frac{\sqrt{M}\varepsilon_2'}{2} \|p_k'\|_{L^2(0,T;L^2(\Omega))}^2,
\end{aligned}$$

where ε_1' and ε_2' are positive constants from Young inequality.
Therefore, (2.57) becomes

$$\begin{aligned}
&c_3'\sqrt{\delta} \parallel (\boldsymbol{u}'(p_k), \boldsymbol{r}'(p_k)) \parallel_{L^2(0,t;\mathcal{X}_N)} + \sqrt{\delta} \parallel p_k' \parallel_{L^2(0,t;L^2(\Omega))}^2 + \frac{\sqrt{\delta}}{2} \parallel \nabla p_k \parallel_{L^2(\Omega)}^2 \\
&\leq \frac{\sqrt{M}}{2\varepsilon_1'} \parallel \boldsymbol{f}' \parallel_{L^2(0,T;L^2(\Omega,\mathbb{R}^3))} + \frac{\sqrt{M}\varepsilon_1'}{2} \parallel (\boldsymbol{u}'(p_k), \boldsymbol{r}'(p_k)) \parallel_{L^2(0,T;\mathcal{X}_N)} + \frac{\sqrt{M}}{2\varepsilon_2'} \|g\|_{L^2(0,T;L^2(\Omega))}^2 \\
&\qquad + \frac{\sqrt{M}\varepsilon_2'}{2} \|p_k'\|_{L^2(0,T;L^2(\Omega))}^2 + \frac{\sqrt{M}}{2} \parallel \nabla p_{k0} \parallel_{L^2(\Omega)}^2 .
\end{aligned}$$

It implies to

$$\begin{aligned}
&(c_3'\sqrt{\delta} - \frac{\sqrt{M}\varepsilon_1'}{2}) \parallel (\boldsymbol{u}'(p_k), \boldsymbol{r}'(p_k)) \parallel_{L^2(0,T;\mathcal{X}_N)} + (\sqrt{\delta} - \frac{\sqrt{M}\varepsilon_2'}{2}) \|p_k'\|_{L^2(0,T;L^2(\Omega))}^2 + \frac{\sqrt{\delta}}{2} \parallel \nabla p_k \parallel_{L^2(\Omega)}^2 \\
&\leq \frac{\sqrt{M}}{2\varepsilon_1'} \parallel \boldsymbol{f}' \parallel_{L^2(0,T;L^2(\Omega,\mathbb{R}^3))} + \frac{\sqrt{M}}{2\varepsilon_2'} \|g\|_{L^2(0,T;L^2(\Omega))}^2 + \frac{\sqrt{M}}{2} \parallel \nabla p_{k0} \parallel_{L^2(\Omega)}^2 .
\end{aligned}$$

By choosing the ε_1' and ε_2' suitably, *i.e.*, $\varepsilon_1' < \frac{2c_3'\sqrt{\delta}}{\sqrt{M}}$ and $\varepsilon_2' < \frac{2\sqrt{\delta}}{\sqrt{M}}$, we obtain that

$$||(\boldsymbol{u}'(p_k), \boldsymbol{r}'(p_k))||_{L^2(0,T;\mathcal{X}_N)}^2 + \|p_k'\|_{L^2(0,T;L^2(\Omega))}^2 + \|\nabla p_k\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq C_2,$$

here C_2 depends only on T, $\|\boldsymbol{f}\|_{L^2(0,T;L^2(\Omega,\mathbb{R}^3))}$, $\|p_0\|_{L^\infty(0,T;L^2(\Omega))}$ and $\|g\|_{L^2(0,T;L^2(\Omega))}$.
This inequality implies (2.52b) in Lemma 33. From (2.52), we have that $\|p_k\|_{L^\infty(0,T;H^1(\Omega))}$, $\|p_k\|_{H^1(0,T;L^2(\Omega))}$ and $\|(\boldsymbol{u}(p_k), \boldsymbol{r}(p_k))\|_{H^1(0,T;\mathcal{X}_N)}^2$ are uniformly bounded. This yields (2.53). We just finished the proof of Lemma 33. □

Lastly, we prove that the pressure solution $\hat{p}$, which is derived in Lemma 33, satisfies (2.45) and (2.22c).

Lemma 34

The limit $\hat{p}$ in Lemma 33 solves (2.45) and (2.22c).

Proof. We pass to the limit in (2.39) after having replaced p by p_k and based on (2.53c),

$$\int_\omega e\, a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(\boldsymbol{w}_1)\gamma_{\rho\sigma}(\boldsymbol{v})\sqrt{a}\,d\boldsymbol{x} + \frac{1}{12}\int_\omega e^3\, a^{\alpha\beta\rho\sigma}\chi_{\alpha\beta}(\boldsymbol{w}_1,\boldsymbol{w}_2)\chi_{\rho\sigma}(\boldsymbol{v},\boldsymbol{s})\sqrt{a}\,d\boldsymbol{x}$$
$$+ 4\mu\int_\omega e\, a^{\alpha\beta}\delta_{\alpha 3}(\boldsymbol{w}_1,\boldsymbol{w}_2)\delta_{\beta 3}(\boldsymbol{v},\boldsymbol{s})\sqrt{a}\,d\boldsymbol{x} - \int_\Omega \hat{p}\ \mathrm{div}\ (\boldsymbol{v}+z\boldsymbol{s})\sqrt{a}\ d\boldsymbol{X} = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}+z\boldsymbol{s})\sqrt{a}\ d\boldsymbol{X}.$$

By uniqueness of (2.39), we have $\boldsymbol{w}_1 = \boldsymbol{u}(\hat{p})$ and $\boldsymbol{w}_2 = \boldsymbol{r}(\hat{p})$. Then passing to the limit in (2.50), we also obtain that $\hat{p}$ solves (2.45) and the continuity in time of $\hat{p}$ gives the initial condition (2.22c). □

Proof. (of Theorem 10). From Lemma 30 and Lemma 32, we can derive that the problem (2.22) is well posed. The remain of theorem is proved by Lemma 33 and Lemma 34. □

2.7 Boundary value problem

In this section, we derive the boundary value problem that is, at least formally, equivalent to the Biot-Naghdi weak equations (2.22).
Let us consider the boundary of Ω and ω:

$$\partial\omega = \gamma_0 \cup \gamma_1 \text{ and } \partial\Omega = \Gamma_0 \cup \Gamma_1 = \gamma_0 \times (-\frac{e}{2},\frac{e}{2}) \cup \gamma_1 \times (-\frac{e}{2},\frac{e}{2}),$$

and introduce the space

$$H^1_{\Gamma_0}(\Omega) = \{q \in H^1(\Omega);\ q = 0 \text{ on } \Gamma_0\}.$$

We recall the variational formulation of the problem corresponding to the linearized Biot-Naghdi shell model. For data $(\boldsymbol{f}, g)$ in $L^2(0,T;L^2(\Omega,\mathbb{R}^3)) \times L^2(\Omega\times]0,T[)$, it reads: Find $(\boldsymbol{u},\boldsymbol{r}) \in \mathcal{V}_N$ and $p \in \mathcal{W}_N$ such that

$$\begin{cases} \mathcal{A}\big((\boldsymbol{u},\boldsymbol{r},p);(\boldsymbol{v},\boldsymbol{s})\big) & = 0 \quad \forall\,(\boldsymbol{v},\boldsymbol{s}) \in \mathcal{X}_N, \\ \mathcal{B}\big((\boldsymbol{u},\boldsymbol{r},p);q\big) & = 0 \quad \forall\, q \in H^1_{\Gamma_0}(\Omega), \end{cases} \tag{2.58}$$

where

$$\begin{aligned}\mathcal{A}[(\boldsymbol{u},\boldsymbol{r},p);(\boldsymbol{v},\boldsymbol{s})] &= \mathcal{A}_1^N[(\boldsymbol{u},\boldsymbol{r}),(\boldsymbol{v},\boldsymbol{s})] + \mathcal{B}_1^N[p,(\boldsymbol{v},\boldsymbol{s})] - \mathcal{L}_1^N(\boldsymbol{v},\boldsymbol{s}) \\ &= \int_\omega e\, a^{\alpha\beta\rho\sigma}[\gamma_{\alpha\beta}(\boldsymbol{u})\gamma_{\rho\sigma}(\boldsymbol{v}) + \frac{e^2}{12}\chi_{\alpha\beta}(\boldsymbol{u},\boldsymbol{r})\chi_{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})]\sqrt{a}\, d\boldsymbol{x} \\ &+ 4\mu\int_\omega e\, a^{\alpha\beta}\delta_{\alpha 3}(\boldsymbol{u},\boldsymbol{r})\delta_{\beta 3}(\boldsymbol{u},\boldsymbol{r})\sqrt{a}\, d\boldsymbol{x} - \alpha\int_\Omega p\,\mathrm{div}(\boldsymbol{v}+z\boldsymbol{s})\sqrt{a}\, d\boldsymbol{X} \\ &- \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}+z\boldsymbol{s})\sqrt{a}\, d\boldsymbol{X}. \end{aligned} \tag{2.59}$$

Next, we state the system of partial differential equations of the Biot-Naghdi shell model (2.58) in the case when $\partial\omega = \gamma_0 \cup \gamma_1$.

Theorem 11

If the solution$(u, r) \in \mathcal{V}_N$ and $p \in \mathcal{W}_N$ of the corresponding problem (2.22) of Theorem 9 is smooth enough, it also satisfies the boundary value problem:

$$\begin{aligned} -\partial_\rho((n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3)\sqrt{a}) + \alpha\nabla(p\sqrt{a}) &= \boldsymbol{f}\sqrt{a},\ \text{ in } \Omega\times]0,T[\\ (-m^{\sigma\rho}_{|\sigma}(\boldsymbol{u},\boldsymbol{r}) + t^\rho(\boldsymbol{u},\boldsymbol{r}))\boldsymbol{a}_\rho\sqrt{a} + \alpha z\nabla(p\sqrt{a}) &= z\boldsymbol{f}\sqrt{a},\ \text{ in } \Omega\times]0,T[\\ \partial_t(c_0 p + \alpha\mathrm{div}(\boldsymbol{u}+z\boldsymbol{r}))\sqrt{a} - \frac{\kappa}{\eta}\mathrm{div}(\nabla p\sqrt{a}) &= g\sqrt{a},\ \text{ in } \Omega\times]0,T[\\ \boldsymbol{u} = \boldsymbol{r} &= \boldsymbol{0} \text{ on } \gamma_0, \\ p &= 0 \text{ on } \Gamma_0, \\ ((n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3)\sqrt{a})n_\rho - \alpha p\sqrt{a}\,\boldsymbol{\nu} &= 0 \text{ on } \Gamma_1, \\ m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_\rho\sqrt{a}\, n_\sigma - \alpha z p\sqrt{a}\,\boldsymbol{\nu} &= 0 \text{ on } \Gamma_1, \\ \nabla p\cdot\boldsymbol{\nu}\sqrt{a} &= 0 \text{ on } \Gamma_1, \end{aligned} \tag{2.60}$$

where

$$n^{\rho\sigma}(\boldsymbol{u}) = e a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(\boldsymbol{u}), \tag{2.61a}$$

$$m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r}) = \frac{e^3}{12}a^{\alpha\beta\rho\sigma}\chi_{\alpha\beta}(\boldsymbol{u},\boldsymbol{r}), \tag{2.61b}$$

$$t^\beta(\boldsymbol{u},\boldsymbol{r}) = 4\mu e a^{\alpha\beta}\delta_{\alpha 3}(\boldsymbol{u},\boldsymbol{r}) \tag{2.61c}$$

and for an arbitrary tensor with differentiable covariant components,

$$m^{\alpha\beta}|_\beta = \partial_\beta m^{\alpha\beta} + \Gamma^\alpha_{\beta\sigma}m^{\beta\sigma} + \Gamma^\beta_{\beta\sigma}m^{\alpha\sigma}.$$

Proof. Let us remind the tensors:

$$\gamma_{\rho\sigma}(\boldsymbol{v}) = \gamma_{\sigma\rho}(\boldsymbol{v}) = \frac{1}{2}(\partial_\sigma\boldsymbol{v}\cdot\boldsymbol{a}_\rho + \partial_\rho\boldsymbol{v}\cdot\boldsymbol{a}_\sigma),$$

$$\chi_{\rho\sigma}(\boldsymbol{v},\boldsymbol{s}) = \chi_{\sigma\rho}(\boldsymbol{v},\boldsymbol{s}) = \frac{1}{2}(\boldsymbol{s}_{\rho|\sigma} + \boldsymbol{s}_{\sigma|\rho}) + \frac{1}{2}(\partial_\rho\boldsymbol{v}\cdot\partial_\sigma\boldsymbol{a}_3 + \partial_\sigma\boldsymbol{v}\cdot\partial_\rho\boldsymbol{a}_3)$$

and

$$\delta_{\beta 3}(\boldsymbol{v},\boldsymbol{s}) = \frac{1}{2}(\partial_\beta \boldsymbol{v}\cdot \boldsymbol{a}_3 + s_\beta).$$

Hence, combining with the symmetry property of $n^{\rho\sigma}$ and $m^{\rho\sigma}$ we obtain that

$$n^{\rho\sigma}(\boldsymbol{u})\gamma_{\rho\sigma}(\boldsymbol{v}) = n^{\rho\sigma}\partial_\rho \boldsymbol{v}\cdot \boldsymbol{a}_\sigma,$$

and

$$m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\chi_{\rho\sigma}(\boldsymbol{v},\boldsymbol{s}) = m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})(\boldsymbol{s}_{\rho|\sigma} + \partial_\rho \boldsymbol{v}\cdot \partial_\sigma \boldsymbol{a}_3).$$

Therefore, $\mathcal{A}_1^N[(\boldsymbol{u},\boldsymbol{r}),(\boldsymbol{v},\boldsymbol{s})]$ becomes:

$$\begin{aligned}\mathcal{A}_1^N[(\boldsymbol{u},\boldsymbol{r}),(\boldsymbol{v},\boldsymbol{s})] &= \frac{1}{e}\int_\Omega [n^{\rho\sigma}(\boldsymbol{u})\partial_\rho \boldsymbol{v}\cdot \boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\rho \boldsymbol{v}\cdot\partial_\sigma \boldsymbol{a}_3 + t^\beta(\boldsymbol{u},\boldsymbol{r})\partial_\beta \boldsymbol{v}\cdot a_3]\sqrt{a}\,d\boldsymbol{X}\\ &+ \frac{1}{e}\int_\Omega [m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\boldsymbol{s}_{\rho|\sigma} + t^\beta(\boldsymbol{u},\boldsymbol{r})s_\beta]\sqrt{a}\,d\boldsymbol{X} - \alpha\int_\Omega p\,\mathrm{div}(\boldsymbol{v}+z\boldsymbol{s})\sqrt{a}\,d\boldsymbol{X}.\end{aligned}$$

Then, we rewrite $\mathcal{A}$

$$\begin{aligned}\mathcal{A}[(\boldsymbol{u},\boldsymbol{r},p);(\boldsymbol{v},\boldsymbol{s})] &= \frac{1}{e}\int_\Omega [n^{\rho\sigma}(\boldsymbol{u})\partial_\rho \boldsymbol{v}\cdot \boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\rho \boldsymbol{v}\cdot\partial_\sigma \boldsymbol{a}_3 + t^\beta(\boldsymbol{u},\boldsymbol{r})\partial_\beta \boldsymbol{v}\cdot a_3]\sqrt{a}\,d\boldsymbol{X}\\ &+ \frac{1}{e}\int_\Omega [m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\boldsymbol{s}_{\rho|\sigma} + t^\beta(\boldsymbol{u},\boldsymbol{r})s_\beta]\sqrt{a}\,d\boldsymbol{X} - \alpha\int_\Omega p\,\mathrm{div}(\boldsymbol{v}+z\boldsymbol{s})\sqrt{a}\,d\boldsymbol{X}\\ &- \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}+z\boldsymbol{s})\sqrt{a}\,d\boldsymbol{X}.\end{aligned}$$

By changing the indices of two terms $t^\beta(\boldsymbol{u},\boldsymbol{r})\partial_\beta \boldsymbol{v}\cdot \boldsymbol{a}_3$ and $t^\beta(\boldsymbol{u},\boldsymbol{r})\boldsymbol{s}_\beta$ from β to ρ, we obtain

$$\begin{aligned}\mathcal{A}[(\boldsymbol{u},\boldsymbol{r},p);(\boldsymbol{v},\boldsymbol{s})] &= \frac{1}{e}\int_\Omega [n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma \boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3]\sqrt{a}\cdot\partial_\rho \boldsymbol{v}\,d\boldsymbol{X}\\ &+ \frac{1}{e}\int_\Omega m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\boldsymbol{s}_{\rho|\sigma}\sqrt{a}\,d\boldsymbol{X} + \frac{1}{e}\int_\Omega t^\rho(\boldsymbol{u},\boldsymbol{r})\,s_\rho\,\sqrt{a}d\boldsymbol{X}\\ &- \alpha\int_\Omega p\,\mathrm{div}(\boldsymbol{v}+z\boldsymbol{s})\sqrt{a}\,d\boldsymbol{X} - \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}+z\boldsymbol{s})\sqrt{a}\,d\boldsymbol{X}.\end{aligned}$$

From (2.58), we see that $\mathcal{A}[(\boldsymbol{u},\boldsymbol{r},p);(\boldsymbol{v},\boldsymbol{s})] = 0$ for all $(\boldsymbol{v},\boldsymbol{s})\in\mathcal{X}_N$, so one takes $(\boldsymbol{v},\boldsymbol{s}) = (\boldsymbol{v},0)$

$$\begin{aligned}\mathcal{A}[(\boldsymbol{u},\boldsymbol{r},p);(\boldsymbol{v},0)] &= \frac{1}{e}\int_\Omega [n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma \boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3]\sqrt{a}\cdot\partial_\rho \boldsymbol{v}\,d\boldsymbol{X}\\ &\qquad - \alpha\int_\Omega p\,\mathrm{div}\boldsymbol{v}\sqrt{a}\,d\boldsymbol{X} - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\sqrt{a}\,d\boldsymbol{X}.\end{aligned}$$

Applying the Green theorem to the first and second term and using the elimination of $(\boldsymbol{v},\boldsymbol{s})$ on

Γ_0, we have

$$\begin{aligned}
&\frac{1}{e}\int_{\Omega}[n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3]\sqrt{a}\cdot\partial_\rho\boldsymbol{v}\,d\boldsymbol{X}\\
&= -\frac{1}{e}\int_{\Omega}\partial_\rho([n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3]\sqrt{a})\cdot\boldsymbol{v}\,d\boldsymbol{X}\\
&+\frac{1}{e}\int_{\Gamma_0\cup\Gamma_1}([n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3]\sqrt{a})n_\rho\cdot\boldsymbol{v}\,d\Gamma\\
&= -\frac{1}{e}\int_{\Omega}\partial_\rho([n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3]\sqrt{a})\cdot\boldsymbol{v}\,d\boldsymbol{X}\\
&+\frac{1}{e}\int_{\Gamma_1}([n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3]\sqrt{a})n_\rho\cdot\boldsymbol{v}\,d\Gamma
\end{aligned}$$

and

$$\begin{aligned}
\alpha\int_{\Omega}p\,\mathrm{div}(\boldsymbol{v}+z\boldsymbol{s})\sqrt{a}\,d\boldsymbol{X} &= \alpha\int_{\Gamma_0\cup\Gamma_1}p\,(\boldsymbol{v}+z\boldsymbol{s})\cdot\boldsymbol{n}\sqrt{a}d\Gamma - \alpha\int_{\Omega}\nabla(p\sqrt{a})\cdot(\boldsymbol{v}+z\boldsymbol{s})\,d\boldsymbol{X}\\
&= \alpha\int_{\Gamma_1}p(\boldsymbol{v}+z\boldsymbol{s})\cdot\boldsymbol{n}\sqrt{a}d\Gamma - \alpha\int_{\Omega}\nabla(p\sqrt{a})\cdot(\boldsymbol{v}+z\boldsymbol{s})d\boldsymbol{X}.
\end{aligned}$$

So we get that

$$\begin{aligned}
\mathcal{A}[(\boldsymbol{u},\boldsymbol{r},p);(\boldsymbol{v},0)] &= -\frac{1}{e}\int_{\Omega}\partial_\rho([n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3]\sqrt{a})\cdot\boldsymbol{v}\,d\boldsymbol{X}\\
&+\frac{1}{e}\int_{\Gamma_1}([n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3]\sqrt{a})n_\rho\cdot\boldsymbol{v}\,d\Gamma\\
&-\alpha\int_{\Gamma_1}p\,\boldsymbol{v}\cdot\boldsymbol{n}\sqrt{a}d\Gamma + \alpha\int_{\Omega}\nabla(p\sqrt{a})\cdot\boldsymbol{v}\,d\boldsymbol{X} - \int_{\Omega}\boldsymbol{f}\cdot\boldsymbol{v}\sqrt{a}\,d\boldsymbol{X}.
\end{aligned}$$

Then we derive that:

$$\begin{cases}
-\partial_\rho([n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3]\sqrt{a}) + \alpha e\nabla\,(p\sqrt{a}) & = e\boldsymbol{f}\sqrt{a}\text{ in }\Omega,\\
([n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3]\sqrt{a})n_\rho - \alpha ep\boldsymbol{n}\sqrt{a} & = \boldsymbol{0}\text{ on }\Gamma_1.
\end{cases} \tag{2.62}$$

Similarly, taking $(\boldsymbol{v},\boldsymbol{s}) = (0,\boldsymbol{s})$, one can check that

$$\begin{aligned}
&\mathcal{B}[(\boldsymbol{u},\boldsymbol{r},p);(0,\boldsymbol{s})] = \frac{1}{e}\int_{\Omega}m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\boldsymbol{s}_{\rho|\sigma}\sqrt{a}\,d\boldsymbol{X} + \frac{1}{e}\int_{\Omega}t^\rho(\boldsymbol{u},\boldsymbol{r})\,s_\rho\sqrt{a}\,d\boldsymbol{X}\\
&-\alpha\int_{\Gamma_1}p\,(z\boldsymbol{s})\cdot\boldsymbol{n}\sqrt{a}d\Gamma + \alpha\int_{\Omega}\nabla(p\sqrt{a})\cdot(z\boldsymbol{s})\,d\boldsymbol{X}\\
&= \frac{1}{e}\int_{\Omega}m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})(\partial_\sigma s_\rho - \Gamma^\alpha_{\rho\sigma}s_\alpha)\sqrt{a}\,d\boldsymbol{X} + \frac{1}{e}\int_{\Omega}t^\rho(\boldsymbol{u},\boldsymbol{r})\,s_\rho\sqrt{a}\,d\boldsymbol{X}\\
&-\alpha\int_{\Gamma_1}p\,(z\boldsymbol{s})\cdot\boldsymbol{n}\sqrt{a}d\Gamma + \alpha\int_{\Omega}\nabla(p\sqrt{a})\cdot(z\boldsymbol{s})\,d\boldsymbol{X}
\end{aligned}$$

$$= \underbrace{\frac{1}{e}\int_\Omega m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})(\partial_\sigma s_\rho)\sqrt{a}\,d\boldsymbol{X}}_{I_1} - \underbrace{\frac{1}{e}\int_\Omega m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\Gamma^\alpha_{\rho\sigma}s_\alpha\sqrt{a}\,d\boldsymbol{X}}_{I_2} + \frac{1}{e}\int_\Omega t^\rho(\boldsymbol{u},\boldsymbol{r})\,s_\rho\sqrt{a}\,d\boldsymbol{X}$$

$$-\alpha\int_{\Gamma_1} p\,(z\boldsymbol{s})\cdot\boldsymbol{n}\sqrt{a}d\Gamma + \alpha\int_\Omega \nabla(p\sqrt{a})\cdot(z\boldsymbol{s})\,d\boldsymbol{X} - \int_\Omega \boldsymbol{f}\cdot(z\boldsymbol{s})\sqrt{a}\,d\boldsymbol{X}.$$

In detail, we make I_1 clearer

$$\begin{aligned}
I_1 &= -\frac{1}{e}\int_\Omega \partial_\sigma[m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\sqrt{a}]s_\rho\,d\boldsymbol{X} + \int_{\Gamma_0\cup\Gamma_1} n_\sigma[m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\sqrt{a}]s_\rho d\Gamma \\
&= -\frac{1}{e}\int_\Omega [\partial_\sigma m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\sqrt{a} + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma\sqrt{a}]s_\rho\,d\boldsymbol{X} + \int_{\Gamma_1} n_\sigma[m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\sqrt{a}]s_\rho d\Gamma \\
&= -\frac{1}{e}\int_\Omega [\partial_\sigma m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\sqrt{a} + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\Gamma^\alpha_{\alpha\sigma}\sqrt{a}]s_\rho\,d\boldsymbol{X} + \int_{\Gamma_1} n_\sigma[m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\sqrt{a}]s_\rho d\Gamma \\
&= -\frac{1}{e}\int_\Omega [\partial_\sigma m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r}) + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\Gamma^\alpha_{\alpha\sigma}]\sqrt{a}\,s_\rho\,d\boldsymbol{X} + \int_{\Gamma_1} n_\sigma[m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\sqrt{a}]s_\rho d\Gamma.
\end{aligned}$$

by $\partial_\rho\sqrt{a} = \Gamma^\alpha_{\alpha\rho}\sqrt{a}$.
Next, exchanging the index of I_2 between ρ and α,

$$I_2 = \frac{1}{e}\int_\Omega m^{\alpha\sigma}(\boldsymbol{u},\boldsymbol{r})\Gamma^\rho_{\alpha\sigma}\sqrt{a}\,s_\rho\,d\boldsymbol{X}.$$

Therefore

$$\begin{aligned}
&\mathcal{A}[(\boldsymbol{u},\boldsymbol{r},p);(0,\boldsymbol{s})] \\
&= I_1 - I_2 + \frac{1}{e}\int_\Omega t^\rho(\boldsymbol{u},\boldsymbol{r})\,s_\rho\sqrt{a}\,d\boldsymbol{X} - \alpha\int_{\Gamma_1} p\,(z\boldsymbol{s})\cdot\boldsymbol{n}\sqrt{a}d\Gamma + \alpha\int_\Omega \nabla(p\sqrt{a})\cdot(z\boldsymbol{s})\,d\boldsymbol{X} \\
&= -\frac{1}{e}\int_\Omega [\partial_\sigma m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r}) + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\Gamma^\alpha_{\alpha\sigma}]\sqrt{a}\,s_\rho\,d\boldsymbol{X} - \frac{1}{e}\int_\Omega m^{\alpha\sigma}(\boldsymbol{u},\boldsymbol{r})\Gamma^\rho_{\alpha\sigma}\sqrt{a}\,s_\rho\,d\boldsymbol{X} \\
&\quad + \frac{1}{e}\int_{\Gamma_1} n_\sigma[m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\sqrt{a}]s_\rho d\Gamma + \frac{1}{e}\int_\Omega t^\rho(\boldsymbol{u},\boldsymbol{r})\sqrt{a}\,s_\rho\,d\boldsymbol{X} - \alpha\int_{\Gamma_1} p\,(z\boldsymbol{s})\cdot\boldsymbol{n}\sqrt{a}d\Gamma \\
&\quad + \alpha\int_\Omega \nabla(p\sqrt{a})\cdot(z\boldsymbol{s})\,d\boldsymbol{X} \\
&= -\frac{1}{e}\int_\Omega [\partial_\sigma m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r}) + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\Gamma^\alpha_{\alpha\sigma} + m^{\alpha\sigma}(\boldsymbol{u},\boldsymbol{r})\Gamma^\rho_{\alpha\sigma}]\sqrt{a}\,s_\rho\,d\boldsymbol{X} \\
&\quad + \frac{1}{e}\int_{\Gamma_1} n_\sigma[m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\sqrt{a}]s_\rho d\Gamma + \frac{1}{e}\int_\Omega t^\rho(\boldsymbol{u},\boldsymbol{r})\sqrt{a}\,s_\rho\,d\boldsymbol{X} - \alpha\int_{\Gamma_1} p\,(z\boldsymbol{s})\cdot\boldsymbol{n}\sqrt{a}d\Gamma \\
&\quad + \alpha\int_\Omega \nabla(p\sqrt{a})\cdot(z\boldsymbol{s})\,d\boldsymbol{X} - \int_\Omega \boldsymbol{f}\cdot(z\boldsymbol{s})\sqrt{a}\,d\boldsymbol{X}.
\end{aligned}$$

Exchanging the indices of $m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\Gamma^{\alpha}_{\alpha\sigma}$ and $m^{\alpha\sigma}(\boldsymbol{u},\boldsymbol{r})\Gamma^{\rho}_{\alpha\sigma}$ between ρ and α, one has

$$\begin{aligned}
&(\partial_\sigma m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r}) + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\Gamma^{\alpha}_{\alpha\sigma} + m^{\alpha\sigma}(\boldsymbol{u},\boldsymbol{r})\Gamma^{\rho}_{\alpha\sigma})\sqrt{a}\, s_\rho \\
&= (\partial_\sigma m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r}) + m^{\rho\alpha}(\boldsymbol{u},\boldsymbol{r})\Gamma^{\sigma}_{\sigma\alpha} + m^{\alpha\sigma}(\boldsymbol{u},\boldsymbol{r})\Gamma^{\rho}_{\sigma\alpha})\sqrt{a}\, s_\rho \\
&= (\partial_\sigma m^{\sigma\rho}(\boldsymbol{u},\boldsymbol{r}) + m^{\alpha\rho}(\boldsymbol{u},\boldsymbol{r})\Gamma^{\sigma}_{\sigma\alpha} + m^{\alpha\sigma}(\boldsymbol{u},\boldsymbol{r})\Gamma^{\rho}_{\sigma\alpha})\sqrt{a}\, s_\rho \quad (m^{\rho\sigma} \text{ is symmetric}) \\
&= m^{\sigma\rho}_{|\sigma}(\boldsymbol{u},\boldsymbol{r})\sqrt{a} s_\rho.
\end{aligned}$$

Then,

$$\begin{aligned}
&\mathcal{A}[(\boldsymbol{u},\boldsymbol{r},p);(0,\boldsymbol{s})] \\
&= -\frac{1}{e}\int_\Omega m^{\sigma\rho}_{|\sigma}(\boldsymbol{u},\boldsymbol{r})\sqrt{a}\, s_\rho\, d\boldsymbol{X} + \frac{1}{e}\int_{\Gamma_1} n_\sigma m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\sqrt{a} s_\rho d\Gamma + \frac{1}{e}\int_\Omega t^\rho(\boldsymbol{u},\boldsymbol{r})\sqrt{a}\, s_\rho\, d\boldsymbol{X} \\
&\quad - \alpha\int_{\Gamma_1} p\,(z\boldsymbol{s})\cdot\boldsymbol{n}\sqrt{a} d\Gamma + \alpha\int_\Omega \nabla(p\sqrt{a})\cdot(z\boldsymbol{s})\, d\boldsymbol{X} - \int_\Omega \boldsymbol{f}\cdot(z\boldsymbol{s})\sqrt{a}\; d\boldsymbol{X} \\
&= \frac{1}{e}\int_\Omega [-m^{\sigma\rho}_{|\sigma}(\boldsymbol{u},\boldsymbol{r}) + t^\rho(\boldsymbol{u},\boldsymbol{r})]\sqrt{a}\, s_\rho\, d\boldsymbol{X} + \frac{1}{e}\int_{\Gamma_1} n_\sigma m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\sqrt{a}\, s_\rho d\Gamma \\
&\quad - \alpha\int_{\Gamma_1} p\,(z\boldsymbol{s})\cdot\boldsymbol{n}\sqrt{a} d\Gamma + \alpha\int_\Omega \nabla(p\sqrt{a})\cdot(z\boldsymbol{s})\, d\boldsymbol{X} - \int_\Omega \boldsymbol{f}\cdot(z\boldsymbol{s})\sqrt{a}\; d\boldsymbol{X} \\
&= \frac{1}{e}\int_\Omega [-m^{\sigma\rho}_{|\sigma}(\boldsymbol{u},\boldsymbol{r}) + t^\rho(\boldsymbol{u},\boldsymbol{r})]\sqrt{a}\, \boldsymbol{a}_\rho\cdot\boldsymbol{s}\, d\boldsymbol{X} + \frac{1}{e}\int_{\Gamma_1} n_\sigma m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\sqrt{a}\, \boldsymbol{a}_\rho\cdot\boldsymbol{s}\; d\Gamma \\
&\quad - \alpha\int_{\Gamma_1} p\,(z\boldsymbol{s})\cdot\boldsymbol{n}\sqrt{a} d\Gamma + \alpha\int_\Omega \nabla(p\sqrt{a})\cdot(z\boldsymbol{s})\, d\boldsymbol{X} - \int_\Omega \boldsymbol{f}\cdot(z\boldsymbol{s})\sqrt{a}\; d\boldsymbol{X} \\
&= \frac{1}{e}\int_\Omega [-m^{\sigma\rho}_{|\sigma}(\boldsymbol{u},\boldsymbol{r}) + t^\rho(\boldsymbol{u},\boldsymbol{r})]\sqrt{a}\, \boldsymbol{a}_\rho\cdot\boldsymbol{s}\, d\boldsymbol{X} + \alpha\int_\Omega \nabla(p\sqrt{a})\cdot(z\boldsymbol{s})\, d\boldsymbol{X} \\
&\quad - \int_\Omega \boldsymbol{f}\cdot(z\boldsymbol{s})\sqrt{a}\; d\boldsymbol{X} + \frac{1}{e}\int_{\Gamma_1} n_\sigma m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\sqrt{a}\, \boldsymbol{a}_\rho\cdot\boldsymbol{s}\; d\Gamma - \alpha\int_{\Gamma_1} p\,(z\boldsymbol{s})\cdot\boldsymbol{n}\sqrt{a} d\Gamma.
\end{aligned}$$

Hence we get

$$\begin{cases}
(-m^{\sigma\rho}_{|\sigma}(\boldsymbol{u},\boldsymbol{r}) +\ t^\rho(\boldsymbol{u},\boldsymbol{r}))\sqrt{a}\,\boldsymbol{a}_\rho + \alpha e z\nabla(p\sqrt{a}) & = ze\boldsymbol{f}\sqrt{a} \text{ in } \Omega, \\
m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_\rho\sqrt{a}\, n_\sigma - \alpha e z p\,\boldsymbol{n}\sqrt{a} & = \boldsymbol{0} \text{ on } \Gamma_1.
\end{cases} \tag{2.63}$$

Next, we work with the weak form

$$\mathcal{B}[(\boldsymbol{u},\boldsymbol{r},p);q] = 0 \quad \forall\, q \in H^1_{\Gamma_0}(\Omega),$$

where

$$\begin{aligned}
\mathcal{B}[(\boldsymbol{u},\boldsymbol{r},p);q] &= c_0\int_\Omega p' q\sqrt{a}\, d\boldsymbol{X} + \alpha\int_\Omega \text{div}\ (\boldsymbol{u}' + z\boldsymbol{r}')q\,\sqrt{a}\, d\boldsymbol{X} \\
&\quad + \frac{\kappa}{\eta}\int_\Omega \nabla p\cdot\nabla q\,\sqrt{a}\, d\boldsymbol{X} - \int_\Omega gq\sqrt{a}\, d\boldsymbol{X}.
\end{aligned}$$

Applying the Green theorem to the third term, we have

$$\begin{aligned}\int_\Omega \nabla p\cdot\nabla q\sqrt{a}\,d\boldsymbol{x} &= \int_{\Gamma_0\cup\Gamma_1}(\nabla p\cdot\boldsymbol{\nu})q\sqrt{a}\,d\Gamma - \int_\Omega \mathrm{div}(\nabla p\sqrt{a})q\,d\boldsymbol{X}\\ &= \int_{\Gamma_1}(\nabla p\cdot\boldsymbol{n})q\sqrt{a}\,d\Gamma - \int_\Omega \mathrm{div}(\nabla p\sqrt{a})q\,d\boldsymbol{X}.\end{aligned}$$

Then, it follows that

$$\begin{aligned}\mathcal{B}[(\boldsymbol{u},\boldsymbol{r},p);q] = &\int_\Omega (c_0\,p'\sqrt{a}+\alpha\,\mathrm{div}\,(\boldsymbol{u}'+z\boldsymbol{r}')\sqrt{a}-\frac{\kappa}{\eta}\mathrm{div}(\nabla p\sqrt{a})-g\sqrt{a})q\;d\boldsymbol{X}\\ &+\alpha\int_{\Gamma_1}(\nabla p\cdot\boldsymbol{n})q\sqrt{a}\,d\Gamma.\end{aligned}$$

Since $\mathcal{B}[(\boldsymbol{u},\boldsymbol{r},p);q]=0$ for all q in $H^1(\Omega)$,

$$\begin{cases} c_0p'\sqrt{a}+\alpha\,\mathrm{div}(\boldsymbol{u}'+z\boldsymbol{r}')\sqrt{a}-\dfrac{\kappa}{\eta}\,\mathrm{div}(\nabla p\sqrt{a}) &= g\sqrt{a}\text{ in }\Omega,\\ \nabla p\cdot\boldsymbol{n}\sqrt{a} &= 0\quad\text{on }\Gamma_1.\end{cases}\tag{2.64}$$

From (2.62), (2.63) and (2.64) we obtain the Theorem (11). □

Now, we consider the Biot-Naghdi shell model in the case where $\partial\omega=\gamma_0$.

Theorem 12

Assume that $\partial w=\gamma_0$, so that

$$\mathcal{X}_N = H_0^1(\omega;\mathbb{R}^3)\times H_0^1(\omega)^2\;\;\text{and}\;\;\mathcal{W}_N = L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega)).$$

If the solution $(u,r)\in\mathcal{V}_N$ and $p\in\mathcal{W}_N$ of the corresponding problem (2.22) of Theorem 9 is smooth enough, it also satisfies the boundary value problem:

$$\begin{aligned} -\partial_\rho((n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma+m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma\boldsymbol{a}_3+t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3)\sqrt{a})+\alpha\nabla(p\sqrt{a}) &= \boldsymbol{f}\sqrt{a},\;\text{ in }\Omega\times]0,T[\\ (-m^{\sigma\rho}_{|\sigma}(\boldsymbol{u},\boldsymbol{r})+t^\rho(\boldsymbol{u},\boldsymbol{r}))\boldsymbol{a}_\rho\sqrt{a}+\alpha z\nabla(p\sqrt{a}) &= z\boldsymbol{f}\sqrt{a},\;\text{ in }\Omega\times]0,T[\\ \partial_t(c_0p+\alpha\mathrm{div}(\boldsymbol{u}+z\boldsymbol{r}))\sqrt{a}-\frac{\kappa}{\eta}\mathrm{div}(\nabla p\sqrt{a}) &= g\sqrt{a},\;\text{ in }\Omega\times]0,T[\\ \boldsymbol{u}=\boldsymbol{r} &= \boldsymbol{0}\text{ on }\gamma_0,\\ p &= 0\text{ on }\Gamma_0,\end{aligned}\tag{2.65}$$

$$\begin{aligned}
\mathcal{A}[(\boldsymbol{u},\boldsymbol{r},p);(\boldsymbol{v},\boldsymbol{s})] =& \int_\omega [n^{\rho\sigma}(\boldsymbol{u})\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\partial_\sigma \boldsymbol{a}_3 + t^\rho(\boldsymbol{u},\boldsymbol{r})\boldsymbol{a}_3]\sqrt{a}\cdot\partial_\rho \boldsymbol{v}\, d\boldsymbol{x} \\
&+ \int_\omega m^{\rho\sigma}(\boldsymbol{u},\boldsymbol{r})\boldsymbol{s}_{\rho|\sigma}\sqrt{a}\, d\boldsymbol{x} + \int_\omega t^\rho(\boldsymbol{u},\boldsymbol{r})\, s_\rho \sqrt{a} d\boldsymbol{x} - \alpha\int_\Omega p\,\mathrm{div}\boldsymbol{v}\sqrt{a}\, d\boldsymbol{x} \\
&- \alpha\int_\Omega p\,\mathrm{div}(z\boldsymbol{s})\sqrt{a}\, d\boldsymbol{X} - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\sqrt{a}\, d\boldsymbol{X} - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{s}\sqrt{a}\, d\boldsymbol{x}, \\
\mathcal{B}[(\boldsymbol{u},\boldsymbol{r},p);q] =& c_0\int_\Omega p' q\sqrt{a}\, d\boldsymbol{x} + \alpha\int_\Omega (\mathrm{div}\ \boldsymbol{u}' + \boldsymbol{r}')q\sqrt{a}\, d\boldsymbol{x} + \alpha\int_\Omega \mathrm{div}(z\boldsymbol{r}')q\sqrt{a}\, d\boldsymbol{X} \\
&+ \frac{\kappa}{\eta}\int_\Omega \nabla p\cdot\nabla q\sqrt{a}\, d\boldsymbol{x} - \int_\omega \overline{g} q\sqrt{a}\, d\boldsymbol{x}.
\end{aligned}$$

Proof. Following the same steps in Theorem 11, we obtain Theorem 12. □

Remark 35

i. The solution regularity depends on the regularity of chart defining the shell midsurface. In the case where the chart is of class C^3, O. Iosifescu in [53] has established the H^2-regularity of the solution in a local covariant or contravariant framework.

ii. For our Hybrid Naghdi's formulation of the present work and also for Cartesian equations where the surface is globally $W^{2,\infty}$, the solution regularity is still an open problem.

iii. In a recent work [63], I. Merabet and S. Nicaise have introduced a new mixed formulation for Nagdhi's shell which is appropriate for folded surfaces and have approximated the solution of the problem using the DK method. The standard a priori error analysis of such methods uses additional regularity on the solution but in [63], I. Merabet and S. Nicaise have carried out an error analysis which only requires the regularity of the weak solution.

Conclusion: In this chapter, we established the derivation of a poroelastic shell model of Naghdi type coupled with the Biot model. It is the Biot-Naghdi poroelastic shell model. We also proved the well-posedness of the resulting equations by the theory of DAEs and Galerkin semi-discrete method. Moreover, we obtain the strong formulation of Naghdi-Biot coupled model which we use for establishing the fluid-structure interaction between incompressible flow and poroelastic shell structure in Chapter 3.

Chapter 3

Interaction between a fluid and a poroelastic shell

In this chapter, we carry out the fluid-structure interaction between incompressible flow and poroelastic shell structure. We use the incompressible Stokes for the free fluid and the Biot-Naghdi poroelastic shell model, which we derived in Theorem 11, for the poroelastic shell structure. In process of proving the well-posedness of this model (see Theorem 13), we have trouble imposing the conditions on the interface which are mass conservation, balance of stress and the Beavers-Joseph-Saffman conditions. Therefore, the Lagrange multiplier method is employed to impose weakly this condition. We assume that the boundaries and the interface between the fluid and the poroelastic material are fixed. The proof proceeds by constructing a semi-discrete Galerkin approximations, deriving the discrete inf-sup condition and adopting the theory of differential-algebraic equations (DAEs)[26].

3.1 Introduction

The interaction between a free fluid and a deformable porous medium is found a wide range of applications: ground-surface water flow, reservoir engineering and blood-vessel interactions. We have three common coupling models: Stokes-Darcy [42, 48, 57, 78, 87, 89], Stokes-Biot[2, 82] and Navier-Stokes-Biot [3, 27, 31]. In Stokes-Darcy model, the fluid flow is governed by Stokes equations and the flow in a porous medium is governed by Darcy equations. Coupling the Stokes and Darcy equations has become a very active area of research because of its potential for practical applications. Such models can be used to describe physiological phenomena like the blood motion in vessels, hydrological systems in which surface water percolates through rocks and sand, and various industrial processes involving filtration. One serious problem today is surface water and groundwater contamination resulting from leaky underground storage tanks, chemical spills, and various human activities. A model coupling the Stokes-Darcy equations with a transport equation [87] can be used to study the spread of pollution released in the water and assess the danger. J.M. Urquiza [86] considered a weak formulation of the coupled problem which allows to use classical Stokes finite elements in the fluid domain, and standard continuous piecewise polynomials in the porous medium domain. In his work, the formulation of Stokes equations is standard while a Galerkin least-squares formulation is used for a mixed form of Darcy equations, as in [2]. Unlike Stokes-Darcy model, coupling the Stokes-Biot equations adopt the Biot system [10] for fluid in the poroelastic media. An alternative partitioned approach for the coupled Stokes-Biot problem based on the Nitsche's method is studied in [29]. The resulting method is loosely coupled and non-iterative with conditional stability. Different to the method in [28], which is suitable for the pressure formulation of Darcy flow, the Nitsche's method can handle the mixed Darcy formulation. In [2], the authors focus on the the monolithic scheme for full-dimensional Stokes-Biot problem with the approximation of the continuity of normal velocity condition through the use of a Lagrange multiplier. They consider the mixed formulation for Darcy flow in the Biot system, which provides a locally mass conservation flow approximation and an accuracy Darcy velocity. The advantage of the Lagrange multiplier method is that it does not involve a penalty parameter and it can enforce the continuity of normal velocity with machine precision accuracy on matching grids. The most popular fluid-structure interaction (FSI) is Navier-Stokes-Biot coupling model. In which, the Navier-Stokes equations describes the incompressible fluid. The finite element approximation of the FSI problem [3] is involved due to the fact that both subproblems are indefinite. In this work, the authors design residual-based stabilization techniques for Biot systems, motived by the variational multiscale approach. Then, they state the monolithic Navier-Stokes-Biot system with the appropriate transmission conditions at the interface. For the solution of the coupled system, they adopt both monolithic solvers and heterogeneous domain decomposition strategies. In Navier-Stokes-Biot coupling model, the transmission conditions on the interface have a essential role. The Beavers-Joseph-Saffman (BJS) condition ([5],[54],[81]) describes the slip boundary conditions between the fluid and elastic shell.

In our work, we use the Stokes equations for the incompressible free fluid and the Biot-Naghdi shell model, which we derived in section 2.7, for the poroelastic shell structure. Mass conservation, balance of stress and the BJS condition are imposed on the interface. A Lagrange multiplier method is employed to impose weakly this condition. We assume that the boundaries and the interface between the fluid and the poroelastic material are fixed. The proof proceeds by constructing a semi-discrete Galerkin approximations and for the existence of the solution we adopt the theory of differential-algebraic equations (DAEs)[26].

3.2 Problem setting

3.2.1 Stokes-Biot-Naghdi poroelastic shell model

We consider a multiphysics model problem for a free fluid's interaction with a flow in a deformable porous media, where the simulation domain Ω in $\mathbb{R}^3$ is a union of non-overlapping polygonal region Ω_f and $\widetilde{\Omega}_p$. Here Ω_f is a free region with flow governed by the Stokes equations and $\widetilde{\Omega}_p$ is a poroelastic shell structure governed by the Biot-Naghdi system.

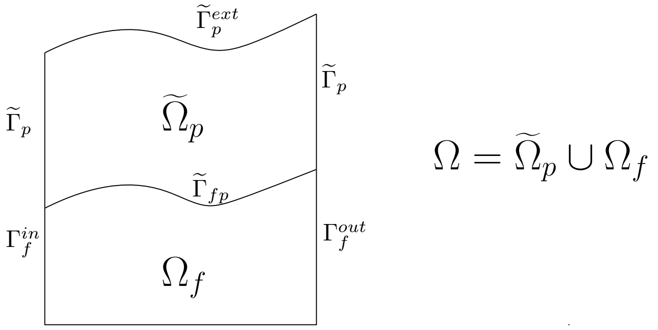


Figure 3.1: Domain Ω

Note that, for the poroelastic shell structure, we do not work directly on $\widetilde{\Omega}_p$ but on $\Omega_p = \Phi(\widetilde{\Omega}_p)$ with Φ being the chart given in 1.1. Let $(\boldsymbol{u}_f, p_f)$ be the velocity-pressure pair in Ω_f and $(\boldsymbol{U}_p, p_p)$ be the displacement-pressure pair in Ω_p and let $\boldsymbol{u}_p$ be the velocity in Ω_p. Remind that $\boldsymbol{U}_p = \boldsymbol{\eta}_p + z\boldsymbol{r}_p$ with $(\boldsymbol{\eta}_p, \boldsymbol{r}_p)$ is a displacement-rotation pair in ω_p, where $\omega_p \subset \mathbb{R}^2$ is the midsurface of Ω_p and satisfies $\varphi(\omega_p) = \widetilde{\omega}_p$. By the quality of Naghdi shell, we see that $\widetilde{\omega}_p \subset \mathbb{R}^3$ is the midsurface of $\widetilde{\Omega}_p$. More precisely,

$$\widetilde{\Omega}_p = \widetilde{\omega}_p \times (-e/2, e/2) \text{ and } \Omega_p = \omega_p \times (-e/2, e/2). \tag{3.1}$$

Let $\Gamma_f = \partial\Omega \cap \partial\Omega_f = \Gamma_f^{in} \cup \Gamma_f^{out}$ and $\widetilde{\Gamma}_p = \partial\Omega \cap \partial\widetilde{\Omega}_p = \widetilde{\Gamma}_p^0 \cup \widetilde{\Gamma}_p^1 \cup \widetilde{\Gamma}_p^{ext}$. We also denote $\Gamma_p = \partial\Omega_p = \Gamma_p^0 \cup \Gamma_p^1 \cup \Gamma_p^{ext}$.

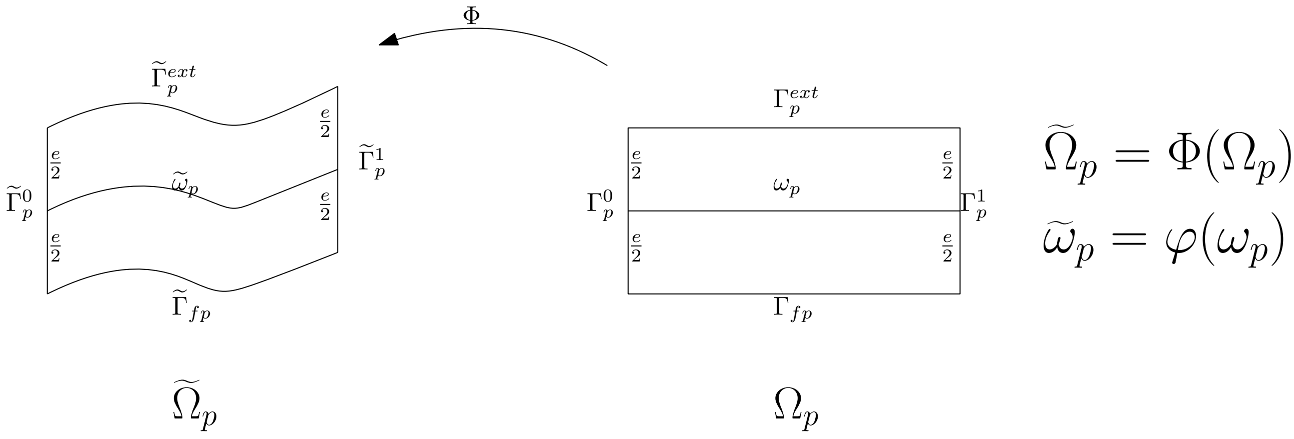


Figure 3.2: Domain $\widetilde{\Omega}_p$ and Ω_p

Here

$$\widetilde{\Gamma}_p^{ext} = \varphi^+(\Gamma_p^{ext}) \text{ with } \varphi^+(\boldsymbol{x}) = \Phi(\boldsymbol{x}, e/2) = \varphi(\boldsymbol{x}) + \frac{e}{2}\boldsymbol{a}_3(\boldsymbol{x}), \tag{3.2a}$$

$$\widetilde{\Gamma}_{fp} = \varphi^-(\Gamma_{fp}) \text{ with } \varphi^-(\boldsymbol{x}) = \Phi(\boldsymbol{x}, -e/2) = \varphi(\boldsymbol{x}) - \frac{e}{2}\boldsymbol{a}_3(\boldsymbol{x}). \tag{3.2b}$$

In the following, we derive the relationship between the area element of the midsurface in the chart φ and area elements of surfaces Γ_{fp} and Γ_p^{ext} in charts φ^+ and φ^-.

Lemma 36

Let S_ε be a surface given by $\bar{S}_\varepsilon = \varphi_\varepsilon(\bar{\omega})$ where $\varphi_\varepsilon(x) = \varphi(x) + \varepsilon \boldsymbol{a}_3(x), x \in \omega$, $\varphi \in W^{2,\infty}(\omega; \mathbb{R}^3)$ being the chart defining the shell midsurface.
We let $a_\varepsilon(x) = |\partial_1 \varphi_\varepsilon \wedge \partial_2 \varphi_\varepsilon|^2$ so that $\sqrt{a_\varepsilon}$ is the area element of the surface S_ε in the chart φ_ε. Then, if ε is small enough, the area element a_ε associated to the surface S_ε in the chart φ_ε is given by:

$$a_\varepsilon = a(1 - 4\varepsilon \mathbf{H} + \mathcal{O}(\varepsilon^2)), \tag{3.3}$$

where $\mathbf{H}$ is the mean curvature of the midsurface S and $\sqrt{a}$ is the area element of the midsurface S in the chart φ.

Proof. From the definition of $\partial_\alpha \varphi_\varepsilon$, we have:

$$\sqrt{a_\varepsilon(x)} = |\partial_1 \varphi_\varepsilon \wedge \partial_2 \varphi_\varepsilon| = |(\boldsymbol{a}_1 - \varepsilon b_1^\beta \boldsymbol{a}_\beta) \wedge (\boldsymbol{a}_2 - \varepsilon b_2^\beta \boldsymbol{a}_\beta)|.$$

It follows that:

$$\begin{aligned}\sqrt{a_\varepsilon(x)} &= |\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 - \varepsilon b_2^2(\boldsymbol{a}_1 \wedge \boldsymbol{a}_2) - \varepsilon b_1^1(\boldsymbol{a}_1 \wedge \boldsymbol{a}_2) + \varepsilon^2(b_2^1 b_2^2(\boldsymbol{a}_1 \wedge \boldsymbol{a}_2) + b_2^1 b_1^2(\boldsymbol{a}_2 \wedge \boldsymbol{a}_1))| \\ &= |\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 - \varepsilon(b_1^1 + b_2^2)(\boldsymbol{a}_1 \wedge \boldsymbol{a}_2) + \varepsilon^2(b_1^1 b_2^2 - b_2^1 b_1^2)(\boldsymbol{a}_1 \wedge \boldsymbol{a}_2)| \\ &= |(\boldsymbol{a}_1 \wedge \boldsymbol{a}_2)||(1 - \varepsilon(b_1^1 + b_2^2) + \varepsilon^2(b_1^1 b_2^2 - b_2^1 b_1^2))|.\end{aligned}$$

Recall that $\mathbf{H} = 1/2(b_1^1 + b_2^2)$ and $\mathbf{K} = b_1^1 b_2^2 - b_2^1 b_1^2$ are the mean curvature and the Gaussian curvature of the midsurface S, respectively. Then we obtain:

$$\sqrt{a_\varepsilon(\boldsymbol{x})} = |\boldsymbol{a}_1 \wedge \boldsymbol{a}_2||1 - 2\varepsilon \mathbf{H} + \varepsilon^2 \mathbf{K}|.$$

Hence, since $\mathbf{K}$ is in $W^{1,\infty}(\omega; \mathbb{R}^3)$, $a_\varepsilon(x)$ can be written

$$a_\varepsilon(\boldsymbol{x}) = |\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|^2 \big(1 - 4\varepsilon \mathbf{H} + \mathcal{O}(\varepsilon^2)\big).$$

□

Remark 37

Therefore, thanks to the previous lemma, the area element $a_\varepsilon(\boldsymbol{x})$ of the surface S_ε can be approximated by the area element of the surface S when ε is small enough. In other words one can assume that:

$$dS_\varepsilon = \sqrt{a} d\boldsymbol{x}.$$

Therefore, we have:

$$\int_{\widetilde{\Gamma}^{ext}} F(\boldsymbol{x}) d\widetilde{\Gamma} = \int_{\Gamma^{ext}} F(\boldsymbol{x}) \sqrt{a_{e/2}}\, dx = \int_{\Gamma^{ext}} F(\boldsymbol{x}) \sqrt{a}\, dx, \tag{3.4}$$

$$\int_{\widetilde{\Gamma}_{fp}} F(\boldsymbol{x}) d\widetilde{\Gamma} = \int_{\Gamma_{fp}} F(\boldsymbol{x}) \sqrt{a_{-e/2}}\, dx = \int_{\Gamma_{fp}} F(\boldsymbol{x}) \sqrt{a}\, dx. \tag{3.5}$$

Next, we introduce the Stokes and Biot-Naghdi models.

Let $\mu_f > 0$ be the fluid viscosity, let $\boldsymbol{f}_f$ be the body force term in Ω_f. Let $D(\boldsymbol{u}_f)$ and $\sigma_f(\boldsymbol{u_f}, p_f)$ denote, respectively, the strain tensor and the stress tensor,

$$D(\boldsymbol{u}_f) = \frac{1}{2}\big(\nabla \boldsymbol{u}_f + \nabla \boldsymbol{u}_f^T\big), \quad \boldsymbol{\sigma}_f(\boldsymbol{u}_f, p_f) = -p_f \mathbf{I} + 2\mu_f D(\boldsymbol{u}_f). \tag{3.6}$$

In the free fluid region Ω_f, $(\boldsymbol{u}_f, p_f)$ satisfies the incompressible Stokes equations

$$-\nabla \cdot \boldsymbol{\sigma}_f(\boldsymbol{u}_f, p_f) = \boldsymbol{f}_f \text{ in } \Omega_f \times (0,T], \tag{3.7a}$$

$$\nabla \cdot \boldsymbol{u}_f = 0 \text{ in } \Omega_f \times (0,T], \tag{3.7b}$$

where $T > 0$ is the final time.

In this section, we consider the relative velocity of the fluid within the porous structure $\boldsymbol{u}_p$ as another unknown via the Darcy's law. Therefore, we couple also the Darcy's law to the Biot-Naghdi model. More detailed, let $\boldsymbol{f}_p$ be in $L^2\big(0,T; L^2(\Omega_p, \mathbb{R}^3)$ and g be in $L^2(\Omega_p \times]0,T[)$, $\big((\boldsymbol{\eta}_p, \boldsymbol{r}_p), \boldsymbol{u}_p, p_p\big)$ satisfy the Biot-Naghdi shell model and Darcy's equation:

$$-\partial_\rho((n^{\rho\sigma}(\boldsymbol{\eta}_p)\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{\eta}_p, \boldsymbol{r}_p)\partial_\sigma \boldsymbol{a}_3 + t^\rho(\boldsymbol{\eta}_p, \boldsymbol{r}_p)\boldsymbol{a}_3)\sqrt{a}) + \alpha e \nabla\,(p_p\sqrt{a}) = e\boldsymbol{f}_p\sqrt{a} \text{ in } \Omega_p \times (0,T], \tag{3.8a}$$

$$(-m^{\sigma\rho}_{|\sigma}(\boldsymbol{\eta}_p, \boldsymbol{r}_p) + t^\rho(\boldsymbol{\eta}_p, \boldsymbol{r}_p))\boldsymbol{a}_\rho\sqrt{a} + \alpha e z \nabla(p_p\sqrt{a}) = z e \boldsymbol{f}_p\sqrt{a} \text{ in } \Omega_p \times (0,T], \tag{3.8b}$$

$$c_0 p_p'\sqrt{a} + \alpha\,\mathrm{div}(\boldsymbol{\eta}_p' + z\boldsymbol{r}_p')\sqrt{a} - \frac{\kappa}{\eta}\mathrm{div}(\boldsymbol{u}_p\sqrt{a}) = g\sqrt{a} \text{ in } \Omega_p \times (0,T], \tag{3.8c}$$

$$\frac{\mu_p}{\kappa}\boldsymbol{u}_p + \nabla p_p = 0 \text{ in } \Omega_p \times (0,T], \tag{3.8d}$$

where c_0 is the constrained specific storage coefficient, κ is the permeability of porous medium, $\mu_p > 0$ is the Lamé coefficient and $\alpha \in (0,1)$ is the Biot-Willis constant, which is usually around one.

Recall that $m^{\sigma\rho}_{|\sigma}$ is the covariant derivative of tensor defined by

$$m^{\sigma\rho}_{|\sigma}(\boldsymbol{\eta}_p, \boldsymbol{r}_p) = \partial_\sigma m^{\sigma\rho}(\boldsymbol{\eta}_p, \boldsymbol{r}_p) + m^{\alpha\rho}(\boldsymbol{\eta}_p, \boldsymbol{r}_p)\Gamma^\sigma_{\sigma\alpha} + m^{\alpha\sigma}(\boldsymbol{\eta}_p, \boldsymbol{r}_p)\Gamma^\rho_{\sigma\alpha}. \tag{3.9}$$

3.2.2 Boundary and initial conditions

The above system of equations needs to be complemented by a set of boundary and initial conditions. Let $\boldsymbol{n}_f$, $\boldsymbol{n}_p$ and $\boldsymbol{\nu}_p$ be the outward unit normal vectors to $\partial\Omega_f$, $\partial\Omega_p$ and $\partial\omega_p$, respectively. Since the boundary conditions have no significant impact on the fluid poroelastic interaction, for simplicity they are chosen such that the normal fluid stress is prescribed on inlet and outlet boundaries:

$$\boldsymbol{\sigma}^f \boldsymbol{n}_f = -p_{in}(t)\,\boldsymbol{n}_f \text{ on } \Gamma_f^{in} \times (0,T], \tag{3.10a}$$
$$\boldsymbol{\sigma}^f \boldsymbol{n}_f = 0 \text{ on } \Gamma_f^{out} \times (0,T]. \tag{3.10b}$$

On the boundary of Ω_p, we impose

$$\boldsymbol{\eta}_p = \boldsymbol{r}_p = \mathbf{0} \text{ on } \gamma_p^0 \times (0,T],\ p_p = 0 \text{ on } \Gamma_p^0 \times (0,T],\ \boldsymbol{u}_p \cdot \mathbf{n}_p = 0 \text{ on } \Gamma_p^1 \times (0,T], \tag{3.11a}$$
$$(\big[n^{\rho\sigma}(\boldsymbol{\eta}_p)\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\partial_\sigma \boldsymbol{a}_3 + t^\rho(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\boldsymbol{a}_3\big]\sqrt{a})n_\rho - \alpha e p_p \sqrt{a}\,\boldsymbol{\nu}_p = \mathbf{0} \text{ on } \Gamma_p^1 \times (0,T], \tag{3.11b}$$
$$m^{\rho\sigma}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\boldsymbol{a}_\rho\sqrt{a}\; n_\sigma - \alpha e z p_p \sqrt{a}\,\boldsymbol{n}_p = \mathbf{0} \text{ on } \Gamma_p^1 \times (0,T],\ \nabla p_p \cdot \boldsymbol{n}_p\sqrt{a} = 0 \text{ on } \Gamma_p^1 \times (0,T]. \tag{3.11c}$$

To avoid the issue with restricting the mean value of the pressure, we assume that $|\Gamma_p^0| > 0$. For proving the well-posedness of the weak formulation, we also need that $dist(\Gamma_p^1, \Gamma_{fp}) > 0$. For more details, see [46].

Later, we set the initial conditions

$$p_p(\boldsymbol{X},0) = p_{p,0}(\boldsymbol{X}) \text{ in } \Omega_p, \boldsymbol{X} = (x,z). \tag{3.12}$$

Remark 38

The initial condition $\boldsymbol{U}_p(\boldsymbol{X},0) = \boldsymbol{\eta}_p(x,0) + z\boldsymbol{r}_p(x,0)$ is computed by solving the Naghdi equations when $p_p(\boldsymbol{X},0)$ is given.

3.2.3 Interface conditions

For a discussion on the interface conditions on the fluid-poroelasticity interface Γ_{fp}, see [3, 83, 66, 67]. For mass conservation, we require that the normal fluid flux must be continuous across the interface

$$(\textit{Mass conservation}) \qquad \boldsymbol{u}_f \cdot \mathbf{n}_f + \alpha\partial_t \boldsymbol{U}_p \cdot \mathbf{n}_p + \boldsymbol{u}_p \cdot \mathbf{n}_p = 0 \quad \text{on } \Gamma_{fp} \times (0,T].$$

Note that, on the interface Γ_{fp}, $\boldsymbol{U}_p(x,-e/2) = \boldsymbol{\eta}_p(x) - \dfrac{e}{2}\boldsymbol{r}_p(x)$, we then obtain

$$\boldsymbol{u}_f \cdot \mathbf{n}_f + \alpha(\partial_t \boldsymbol{\eta}_p - \frac{e}{2}\partial_t \boldsymbol{r}_p) \cdot \mathbf{n}_p + \boldsymbol{u}_p \cdot \mathbf{n}_p = 0 \quad \text{on } \Gamma_{fp} \times (0,T]. \tag{3.13}$$

For the balance of the normal components of stress in fluid phase across Γ_{fp}, one have

$$-(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = p_p \quad \text{on } \Gamma_{fp} \times (0,T]. \tag{3.14}$$

The conservation of momentum requires that the stress of the porous medium is balanced by the stress of fluid, that is, the total stresses of the fluid and the poroelastic medium must match at the interface:

$$(\textit{Momentum conservation}\ ) \qquad \boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_p \mathbf{n}_p = 0 \quad \text{on } \Gamma_{fp} \times (0,T]. \tag{3.15}$$

Finally, the tangential component of the fluid stress (which is equal to the one of the solid phase) is supposed to be proportional to the slip rate according to the Beavers-Joseph-Saffman (BJS) condition

$$(\textit{BJS condition}) \qquad -(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \boldsymbol{\tau}_f = \mu_f \alpha_{BJS} \sqrt{\kappa^{-1}} \big(\boldsymbol{u}_f - \alpha(\partial_t \boldsymbol{\eta}_p - \frac{e}{2} \partial_t \boldsymbol{r}_p)\big) \cdot \boldsymbol{\tau}_f \tag{3.16}$$

here $\boldsymbol{\tau}_f$ is the unit tangent vector on Γ_{fp} and $\alpha_{BJS} \geq 0$ is an experimentally determined friction coefficient.

3.3 Weak formulation and Analysis

3.3.1 Weak formulation

In this section, we construct an appropriate variational formulation of Stokes-Biot-Naghdi system (3.7) and (3.8) coupled by the interface conditions in 3.2.3.

In order to find weak formulation, we introduce our functional framework.

- For the Biot-Naghdi model, we introduce

$$\mathcal{X}_p = \{\boldsymbol{\xi}_p \in H^1(\omega_p, \mathbb{R}^3) : \boldsymbol{\xi}_p = 0 \text{ on } \gamma_p^0\},\ \mathcal{S}_p = \{\boldsymbol{s}_p \in [H^1(\omega_p, \mathbb{R})]^2 : \boldsymbol{s}_p = 0 \text{ on } \gamma_p^0\},$$
$$\mathcal{V}_p = \{\boldsymbol{v}_p \in H(\text{div}, \Omega_p) : \boldsymbol{v}_p \cdot \mathbf{n}_p = 0 \text{ on } \Gamma_p^1\},\ \mathcal{W}_p = \{w_p \in H^1(\Omega_p) : w_p = 0 \text{ on } \Gamma_p^0\},$$

  where $H(\text{div}, \Omega_p)$ is the space of $[L^2(\Omega_p)]^3$-vector with divergence in $L^2(\Omega_p)$ with a norm

$$\|\boldsymbol{v}_p\|_{H(\text{div},\Omega_p)} = (\|\boldsymbol{v}_p\|^2_{L^2(\Omega_p,\mathbb{R}^3)} + \|\nabla \cdot \boldsymbol{v}_p\|^2_{L^2(\Omega_p)})^{1/2}.$$

- For the Stokes equations, we set

$$\mathcal{V}_f = \{\boldsymbol{v}_f \in [H^1(\Omega_f)]^3 :\ \boldsymbol{v}_f = 0 \text{ on } \Gamma_f\} \text{ and } \mathcal{W}_f = L^2(\Omega_f).$$

  We define the global velocity and pressure spaces:

$$\boldsymbol{V} = \{\boldsymbol{v} = (\boldsymbol{v}_f, \boldsymbol{v}_p) \in \mathcal{V}_f \times \mathcal{V}_p\} \text{ and } W = \{w = (w_f, w_p) \in \mathcal{W}_p \times \mathcal{W}_f\}$$

  equipped with the norms

$$\|\boldsymbol{v}\|_V = (\|\boldsymbol{v}_f\|^2_{H^1(\Omega_f,\mathbb{R}^3)} + \|\boldsymbol{v}_p\|^2_{H^1(\Omega_p,\mathbb{R}^3)})^{1/2},$$
$$\|w\|_W = (\|w_f\|^2_{L^2(\Omega_f,\mathbb{R}^3)} + \|w_p\|^2_{L^2(\Omega_p)})^{1/2}.$$

The weak formulation is obtained by multiplying the equations in each region by suitable test functions, integrating by parts in space and utilizing the interface and boundary conditions. Let us itemize the weak formulation in each region:

- The variational formulation of the Biot-Naghdi shell equations now reads: Given $t \in (0,T]$, find $\big(\boldsymbol{\eta}_p(t), \boldsymbol{r}_p(t), \boldsymbol{u}_p(t), p_p(t)\big)$ in $\mathcal{X}_p \times \mathcal{S}_p \times \mathcal{V}_p \times \mathcal{W}_p$ such that

$$
\begin{aligned}
&-\int_{\Omega_p} \partial_\rho\big(\big[n^{\rho\sigma}(\boldsymbol{\eta}_p)\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\partial_\sigma \boldsymbol{a}_3 + t^\rho(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\boldsymbol{a}_3\big]\sqrt{a}\big)\cdot \boldsymbol{\xi}_p \, d\boldsymbol{X} \\
&\qquad\qquad + \alpha e \int_{\Omega_p} \nabla(p_p\sqrt{a})\cdot \boldsymbol{\xi}_p \, d\boldsymbol{X} = e\int_{\Omega_p} \boldsymbol{f}_p \cdot \boldsymbol{\xi}_p \sqrt{a}\, d\boldsymbol{X}, \\
&-\int_{\Omega_p} m^{\sigma\rho}_{|\sigma}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\boldsymbol{a}_\rho \cdot \boldsymbol{s}_p\sqrt{a}\, d\boldsymbol{X} + \int_{\Omega_p} t^\rho(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\boldsymbol{a}_\rho\cdot \boldsymbol{s}_p\sqrt{a}\, d\boldsymbol{X} \\
&\qquad\qquad + \alpha e\int_{\Omega_p} z\nabla(p_p\sqrt{a})\cdot \boldsymbol{s}_p d\boldsymbol{X} = e\int_{\Omega_p} z\boldsymbol{f}_p\cdot \boldsymbol{s}_p\sqrt{a}\, d\boldsymbol{X}, \\
&\int_{\Omega_p} \partial_t\big(c_0 p_p + \alpha \mathrm{div}\,(\boldsymbol{\eta}_p + z\boldsymbol{r}_p)\big) w_p\sqrt{a}\, d\boldsymbol{X} + \int_{\Omega_p} \mathrm{div}\,(\boldsymbol{u}_p\sqrt{a}) w_p\, d\boldsymbol{X} = \int_{\Omega_p} g w_p \sqrt{a}\, d\boldsymbol{X}, \\
&\frac{\mu_p}{\kappa}\int_{\widetilde{\Omega}_p} \boldsymbol{u}_p\cdot \boldsymbol{v}_p\, d\boldsymbol{X} - \int_{\widetilde{\Omega}_p} \nabla p_p \cdot \boldsymbol{v}_p\, d\boldsymbol{X} = 0, \qquad \text{for all } \big(\boldsymbol{\xi}_p, \boldsymbol{s}_p, \boldsymbol{v}_p, w_p\big) \text{ in } \mathcal{X}_p \times \mathcal{S}_p \times \mathcal{V}_p \times \mathcal{W}_p.
\end{aligned}
$$

Applying the Green Theorem, the boundary conditions and domain transposition from $\widetilde{\Omega}_p$ to Ω_p, the weak formulation of Biot-Naghdi is written

$$
\begin{aligned}
&\int_{\omega_p} -\partial_\rho\big(\big[n^{\rho\sigma}(\boldsymbol{\eta}_p)\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\partial_\sigma \boldsymbol{a}_3 + t^\rho(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\boldsymbol{a}_3\big]\sqrt{a}\big)\cdot \boldsymbol{\xi}_p\, d\boldsymbol{x} + \alpha\int_{\Gamma_1\cup\Gamma_{fp}} p_p\sqrt{a}\,\boldsymbol{\xi}_p\cdot \mathbf{n}_p\, d\Gamma \\
&\qquad + \alpha\int_{\Gamma_{fp}} [(p_p\mathbf{n}_p)\cdot \boldsymbol{\tau}_f]\,[\boldsymbol{\xi}_p\cdot \boldsymbol{\tau}_f]\sqrt{a}\, d\Gamma - \alpha\int_{\Omega_p} p_p\sqrt{a}\,\mathrm{div}\,\boldsymbol{\xi}_p\, d\boldsymbol{X} = \int_{\Omega_p} \boldsymbol{f}_p\cdot \boldsymbol{\xi}_p\sqrt{a}\, d\boldsymbol{X}, \\
&-\int_{\omega_p} m^{\sigma\rho}_{|\sigma}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\boldsymbol{a}_\rho\cdot \boldsymbol{s}_p\sqrt{a}\, d\boldsymbol{x} + \int_{\omega_p} t^\rho(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\boldsymbol{a}_\rho\cdot \boldsymbol{s}_p\sqrt{a}\, d\boldsymbol{x} + \alpha\int_{\Gamma_1\cup\Gamma_{fp}} p_p\sqrt{a}\,(z\boldsymbol{s}_p)\cdot \mathbf{n}_p\, d\Gamma \\
&\quad + \alpha\int_{\Gamma_{fp}} [(p_p\mathbf{n}_p)\cdot \boldsymbol{\tau}_f]\,[(-\frac{e}{2}\boldsymbol{s}_p)\cdot \boldsymbol{\tau}_f]\sqrt{a}\, d\Gamma - \alpha\int_{\Omega_p} p_p\sqrt{a}\,\mathrm{div}(z\boldsymbol{s}_p)\, d\boldsymbol{X} = \int_{\Omega_p} \boldsymbol{f}_p\cdot (z\boldsymbol{s}_p)\sqrt{a} d\boldsymbol{X}, \\
&\int_{\Omega_p} \partial_t\big(c_0 p_p + \alpha\mathrm{div}(\boldsymbol{\eta}_p + z\boldsymbol{r}_p)\big) w_p\sqrt{a}\, d\boldsymbol{X} + \int_{\Omega_p} \mathrm{div}(\boldsymbol{u}_p\sqrt{a}) w_p\, d\boldsymbol{X} = \int_{\Omega_p} g w_p\sqrt{a}\, d\boldsymbol{X}, \\
&\frac{\mu_p}{\kappa}\int_{\Omega_p} \boldsymbol{u}_p\cdot \boldsymbol{v}_p\,\sqrt{a}\, d\boldsymbol{X} + \int_{\Gamma_{fp}} p_p\,\boldsymbol{v}_p\cdot \mathbf{n}_p\,\sqrt{a}\; d\Gamma - \int_{\Omega_p} p_p \mathrm{div}\,(\boldsymbol{v}_p\sqrt{a})\; d\boldsymbol{X} = 0.
\end{aligned}
$$

Adding the all four equations and applying the interface conditions, the balance of the normal components of stress (3.14) and BJS condition (3.16), the weak form of the Biot-

Naghdi becomes

$$
\begin{aligned}
&-\int_{\omega_p} \partial_\rho\big(\big[n^{\rho\sigma}(\boldsymbol{\eta}_p)\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\boldsymbol{a}_3\big]\sqrt{a}\big)\cdot\boldsymbol{\xi}_p\,d\boldsymbol{x} - \int_{\omega_p} m^{\sigma\rho}_{|\sigma}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\boldsymbol{a}_\rho\cdot\boldsymbol{s}_p\sqrt{a}\,d\boldsymbol{x}\\
&\quad+\int_{\omega_p} t^\rho(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\boldsymbol{a}_\rho\cdot\boldsymbol{s}_p\sqrt{a}\,d\boldsymbol{x} + \alpha\int_{\Gamma_p^1} p_p\sqrt{a}\,(\boldsymbol{\xi}_p + z\boldsymbol{s}_p)\cdot\mathbf{n}_p\,d\Gamma - \alpha\int_{\Omega_p} p_p\sqrt{a}\,\mathrm{div}(\boldsymbol{\xi}_p + z\boldsymbol{s}_p)\,d\boldsymbol{X}\\
&\quad+\int_{\Omega_p}\partial_t\big(c_0p_p + \alpha\mathrm{div}(\boldsymbol{\eta}_p + z\boldsymbol{r}_p)\big)w_p\sqrt{a}\,d\boldsymbol{X} + \int_{\Omega_p}\mathrm{div}\,(\boldsymbol{u}_p\sqrt{a})w_p\,d\boldsymbol{X}\\
&\quad+\frac{\mu_p}{\kappa}\int_{\Omega_p}\boldsymbol{u}_p\cdot\boldsymbol{v}_p\sqrt{a}\,d\boldsymbol{X} - \int_{\Omega_p} p_p\mathrm{div}\,(\boldsymbol{v}_p\sqrt{a})\,d\boldsymbol{X} + \int_{\Gamma_{fp}} p_p\sqrt{a}\,\big[\alpha(\boldsymbol{\xi}_p - \frac{e}{2}\boldsymbol{s}_p) + \boldsymbol{v}_p\big]\cdot\mathbf{n}_p\,d\Gamma\\
&\quad-\int_{\Gamma_{fp}}\mu_f\alpha_{BJS}\sqrt{\kappa^{-1}}\big(\big[\boldsymbol{u}_f - \alpha(\partial_t\boldsymbol{\eta}_p - \frac{e}{2}\partial_t\boldsymbol{r}_p)\big]\cdot\boldsymbol{\tau}_f\big)\,(\alpha(\boldsymbol{\xi}_p - \frac{e}{2}\boldsymbol{s}_p)\cdot\boldsymbol{\tau}_f)\sqrt{a}\,d\Gamma\\
&\qquad\qquad= \int_{\Omega_p}\boldsymbol{f}_p\cdot(\boldsymbol{\xi}_p + z\boldsymbol{s}_p)\sqrt{a}\,d\boldsymbol{X} + \int_{\Omega_p} g w_p\sqrt{a}\,d\boldsymbol{X}. \qquad (3.17)
\end{aligned}
$$

- Using the boundary conditions (3.11b)-(3.11c) and the definitions of $n^{\rho\sigma}$, $m^{\rho\sigma}$ and t^ρ, we have

$$
\begin{aligned}
&\int_\omega -\partial_\rho\big(\big[n^{\rho\sigma}(\boldsymbol{\eta}_p)\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\boldsymbol{a}_3\big]\sqrt{a}\big)\cdot\boldsymbol{\xi}_p\,d\boldsymbol{x} - \int_\omega m^{\sigma\rho}_{|\sigma}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\boldsymbol{a}_\rho\cdot\boldsymbol{s}_p\sqrt{a}\,d\boldsymbol{x}\\
&\qquad+\int_\omega t^\rho(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\boldsymbol{a}_\rho\cdot\boldsymbol{s}_p\sqrt{a}\,d\boldsymbol{x} + \alpha\int_{\Gamma_p^1} p_p\sqrt{a}\,(\boldsymbol{\xi}_p + z\boldsymbol{s}_p)\cdot\boldsymbol{n}_p\,d\Gamma\\
&=\int_\omega\big(\big[n^{\rho\sigma}(\boldsymbol{\eta}_p)\boldsymbol{a}_\sigma + m^{\rho\sigma}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\partial_\sigma\boldsymbol{a}_3 + t^\rho(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\boldsymbol{a}_3\big]\sqrt{a}\big)\cdot\partial_\rho\boldsymbol{\xi}_p\,d\boldsymbol{x}\\
&\qquad+\int_\omega m^{\sigma\rho}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)s_{\sigma|\rho}\sqrt{a}dx + \int_\omega t^\rho(\boldsymbol{\eta}_p,\boldsymbol{r}_p)s_\rho\sqrt{a}\,d\boldsymbol{x}\\
&=\int_\omega n^{\rho\sigma}(\boldsymbol{\eta}_p)\boldsymbol{a}_\sigma\cdot\partial_\rho\boldsymbol{\xi}_p\sqrt{a}\,d\boldsymbol{x} + \int_\omega m^{\rho\sigma}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)(\partial_\sigma\boldsymbol{a}_3\cdot\partial_\rho\boldsymbol{\xi}_p + s_{\sigma|\rho})\cdot\partial_\rho\boldsymbol{\xi}_p\sqrt{a}\,d\boldsymbol{x}\\
&\qquad+\int_\omega t^\rho(\boldsymbol{\eta}_p,\boldsymbol{r}_p)(\boldsymbol{a}_3\cdot\partial_\rho\boldsymbol{\xi}_p + s_\rho)\sqrt{a}\,d\boldsymbol{x}\\
&=\int_\omega n^{\rho\sigma}(\boldsymbol{\eta}_p)\gamma_{\rho\sigma}(\boldsymbol{\xi}_p)\sqrt{a}\,d\boldsymbol{x} + \int_\omega m^{\rho\sigma}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\chi_{\rho\sigma}(\boldsymbol{\xi}_p,\boldsymbol{s}_p)\sqrt{a}\,d\boldsymbol{x} + \int_\omega t^\rho(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\delta_{\rho3}(\boldsymbol{\xi}_p,\boldsymbol{s}_p)\sqrt{a}\,d\boldsymbol{x}\\
&=\int_\omega n^{\rho\sigma}(\boldsymbol{\eta}_p)\gamma_{\rho\sigma}(\boldsymbol{\xi}_p)\sqrt{a}\,d\boldsymbol{x} + \int_\omega m^{\rho\sigma}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\chi_{\rho\sigma}(\boldsymbol{\xi}_p,\boldsymbol{s}_p)\sqrt{a}\,d\boldsymbol{x} + \int_\omega t^\beta(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\delta_{\beta3}(\boldsymbol{\xi}_p,\boldsymbol{s}_p)\sqrt{a}\,d\boldsymbol{x}\\
&=\int_\omega e\,a^{\alpha\beta\rho\sigma}\big[\gamma_{\alpha\beta}(\boldsymbol{\eta}_p)\gamma_{\rho\sigma}(\boldsymbol{\xi}_p) + \frac{e^2}{12}\chi_{\alpha\beta}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\chi_{\rho\sigma}(\boldsymbol{\xi}_p,\boldsymbol{s}_p)\big]\sqrt{a}\,d\boldsymbol{x}\\
&\qquad+4\mu_p\int_\omega e\,a^{\alpha\beta}\delta_{\alpha3}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\delta_{\beta3}(\boldsymbol{\xi}_p,\boldsymbol{s}_p)\sqrt{a}\,d\boldsymbol{x}.
\end{aligned}
$$

Then (3.18) becomes

$$
\begin{aligned}
&\int_\omega e\, a^{\alpha\beta\rho\sigma}\big[\gamma_{\alpha\beta}(\boldsymbol{\eta}_p)\gamma_{\rho\sigma}(\boldsymbol{\xi}_p) + \frac{e^2}{12}\chi_{\alpha\beta}(\boldsymbol{\eta}_p, \boldsymbol{r}_p)\chi_{\rho\sigma}(\boldsymbol{\xi}_p, \boldsymbol{s}_p)\big]\sqrt{a}\, d\boldsymbol{x} \\
&+ 4\mu_p \int_\omega e\, a^{\alpha\beta}\delta_{\alpha 3}(\boldsymbol{\eta}_p, \boldsymbol{r}_p)\delta_{\beta 3}(\boldsymbol{\xi}_p, \boldsymbol{s}_p)\sqrt{a}\, d\boldsymbol{x} - \alpha \int_{\Omega_p} p_p \sqrt{a}\, \mathrm{div}(\boldsymbol{\xi}_p + z\boldsymbol{s}_p)\, d\boldsymbol{X} \\
&+ \int_{\Omega_p} \partial_t\big(c_0 p_p + \alpha \mathrm{div}(\boldsymbol{\eta}_p + z\boldsymbol{r}_p)\big) w_p \sqrt{a}\, d\boldsymbol{X} + \int_{\Omega_p} \mathrm{div}\ (\boldsymbol{u}_p\sqrt{a}) w_p\, d\boldsymbol{X} + \frac{\mu_p}{\kappa}\int_{\Omega_p} \boldsymbol{u}_p \cdot \boldsymbol{v}_p \sqrt{a}\, d\boldsymbol{X} \\
&- \int_{\Omega_p} p_p \mathrm{div}\ (\boldsymbol{v}_p\sqrt{a})\ d\boldsymbol{X} + \int_{\Gamma_{fp}} p_p\sqrt{a}\ \big[\alpha(\boldsymbol{\xi}_p - \frac{e}{2}\boldsymbol{s}_p) + \boldsymbol{v}_p\big] \cdot \mathbf{n}_p\, d\Gamma \\
&- \int_{\Gamma_{fp}} \mu_f \alpha_{BJS}\sqrt{\kappa^{-1}}\big(\big[\boldsymbol{u}_f - \alpha(\partial_t\boldsymbol{\eta}_p - \frac{e}{2}\partial_t\boldsymbol{r}_p)\big] \cdot \boldsymbol{\tau}_f\big)\ \big(\alpha(\boldsymbol{\xi}_p - \frac{e}{2}\boldsymbol{s}_p) \cdot \boldsymbol{\tau}_f\big)\sqrt{a}\, d\Gamma \\
&\qquad = \int_{\Omega_p} \boldsymbol{f}_p \cdot (\boldsymbol{\xi}_p + z\boldsymbol{s}_p)\sqrt{a}\ d\boldsymbol{X} + \int_{\Omega_p} g w_p \sqrt{a}\, d\boldsymbol{X}. \qquad (3.18)
\end{aligned}
$$

- Following, the weak form of the Stokes system reads as follow: Given $t \in (0, T]$, find $\big(\boldsymbol{u}_f(t), p_f(t)\big)$ in $\mathcal{V}_f \times \mathcal{W}_f$ for all $\big(\boldsymbol{v}_f, w_f\big)$ in $\mathcal{V}_f \times \mathcal{W}_f$.

$$
\begin{aligned}
&- \int_{\widetilde{\Gamma}_{fp}} \boldsymbol{v}_f \cdot (\boldsymbol{\sigma_f}\mathbf{n}_f)\, d\widetilde{\Gamma} - \int_{\widetilde{\Gamma}_{fp}} (\boldsymbol{v}_f \cdot \boldsymbol{\tau}_f)[(\boldsymbol{\sigma_f}\mathbf{n}_f) \cdot \boldsymbol{\tau}_f]\, d\widetilde{\Gamma} - \int_{\Gamma_f^{in} \cup \Gamma_f^{out}} \boldsymbol{v}_f \cdot (\boldsymbol{\sigma_f}\mathbf{n}_f)\, d\Gamma \\
&\qquad - \int_{\Omega_f} p_f\, \mathrm{div}\, \boldsymbol{v}_f\, d\boldsymbol{X} + 2\mu_f \int_{\Omega_f} D(\boldsymbol{u}_f) : D(\boldsymbol{v}_f)\, dX = \int_{\Omega_f} \boldsymbol{f}_f \cdot \boldsymbol{v}_f\ d\boldsymbol{X}. \\
&\int_{\Omega_f} \mathrm{div}\ \boldsymbol{u}_f\, w_f\ d\boldsymbol{X} = 0.
\end{aligned}
$$

Applying the domain transposition (3.5) to the integrations on $\widetilde{\Gamma}_{fp}$, using the boundary conditions (3.10) of $\boldsymbol{\sigma_f}\mathbf{n}_f$ on $\Gamma_f^{in} \cup \Gamma_f^{out}$ and the interface condition (3.14) and BJS (3.16), then adding two equations, one have

$$
\begin{aligned}
&- \int_{\Omega_f} p_f\, \mathrm{div}\, \boldsymbol{v}_f\, d\boldsymbol{X} + 2\mu_f \int_{\Omega_f} D(\boldsymbol{u}_f) : D(\boldsymbol{v}_f)\, dX + \int_{\Omega_f} \mathrm{div}\ \boldsymbol{u}_f\, w_f\ d\boldsymbol{X} + \int_{\Gamma_{fp}} p_p(\boldsymbol{v}_f \cdot \mathbf{n}_f)\sqrt{a}\, d\Gamma \\
&\qquad + \int_{\Gamma_{fp}} \mu_f \alpha_{BJS}\sqrt{\kappa^{-1}}\big(\big[\boldsymbol{u}_f - \alpha(\partial_t\boldsymbol{u}_p - \frac{e}{2}\partial_t\boldsymbol{r}_p)\big] \cdot \boldsymbol{\tau}_f\big)(\boldsymbol{v}_f \cdot \boldsymbol{\tau}_f)\sqrt{a}\, d\Gamma \\
&\qquad\qquad = \int_{\Omega_f} \boldsymbol{f}_f \cdot \boldsymbol{v}_f\ d\boldsymbol{X} - \int_{\Gamma_f^{in}} (\boldsymbol{v}_f \cdot \mathbf{n}_f) p_{in}(t)\, d\Gamma. \qquad (3.19)
\end{aligned}
$$

- Finally, we write the weak formulation of Stokes-Biot-Naghdi system by adding together equations (3.18) and (3.19): For $t \in (0, T]$, find $\boldsymbol{\eta}_p(t) \in \mathcal{X}_p$, $\boldsymbol{r}_p(t) \in \mathcal{S}_p$, $\boldsymbol{u}_p(t) \in \mathcal{V}_p$,

$p_p(t) \in \mathcal{W}_p$, $\boldsymbol{u}_f(t) \in \mathcal{V}_f$ and $p_f(t) \in \mathcal{W}_f$

$$
\begin{aligned}
&\int_\omega e\, a^{\alpha\beta\rho\sigma}\big[\gamma_{\alpha\beta}(\boldsymbol{\eta}_p)\gamma_{\rho\sigma}(\boldsymbol{\xi}_p) + \frac{e^2}{12}\chi_{\alpha\beta}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\chi_{\rho\sigma}(\boldsymbol{\xi}_p,\boldsymbol{s}_p)\big]\sqrt{a}\, d\boldsymbol{x} \\
&\quad + 4\mu_p \int_\omega e\, a^{\alpha\beta}\delta_{\alpha 3}(\boldsymbol{\eta}_p,\boldsymbol{r}_p)\delta_{\beta 3}(\boldsymbol{\xi}_p,\boldsymbol{s}_p)\sqrt{a}\, d\boldsymbol{x} - \alpha\int_{\Omega_p} p_p\sqrt{a}\,\mathrm{div}(\boldsymbol{\xi}_p + z\boldsymbol{s}_p)\, d\boldsymbol{X} \\
&\quad + \int_{\Omega_p} \partial_t\big(c_0 p_p + \alpha\mathrm{div}(\boldsymbol{\eta}_p + z\boldsymbol{r}_p)\big) w_p\sqrt{a}\, d\boldsymbol{X} + \int_{\Omega_p}\mathrm{div}(\boldsymbol{u}_p\sqrt{a}) w_p\, d\boldsymbol{X} + \frac{\mu_p}{\kappa}\int_{\Omega_p}\boldsymbol{u}_p\cdot\boldsymbol{v}_p\sqrt{a}\, d\boldsymbol{X} \\
&\quad - \int_{\Omega_p} p_p\mathrm{div}(\boldsymbol{v}_p\sqrt{a})\, d\boldsymbol{X} - \int_{\Omega_f} p_f\,\mathrm{div}\,\boldsymbol{v}_f\, d\boldsymbol{X} + 2\mu_f\int_{\Omega_f} D(\boldsymbol{u}_f) : D(\boldsymbol{v}_f)\, dX + \int_{\Omega_f}\mathrm{div}\,\boldsymbol{u}_f\, w_f\, d\boldsymbol{X} \\
&\quad + \int_{\Gamma_{fp}} \mu_f\alpha_{BJS}\sqrt{\kappa^{-1}}\big(\big[\boldsymbol{u}_f - \alpha\big(\partial_t\boldsymbol{\eta}_p - \frac{e}{2}\partial_t\boldsymbol{r}_p\big)\big]\cdot\boldsymbol{\tau}_f\big)\,\big(\big[\boldsymbol{v}_f - \alpha\big(\boldsymbol{\xi}_p - \frac{e}{2}\boldsymbol{s}_p\big)\big]\cdot\boldsymbol{\tau}_f\big)\sqrt{a}\, d\Gamma \\
&\quad + \int_{\Gamma_{fp}} p_p\sqrt{a}\,\big(\boldsymbol{v}_f\cdot\mathbf{n}_f + \big[\alpha\big(\boldsymbol{\xi}_p - \frac{e}{2}\boldsymbol{s}_p\big) + \boldsymbol{v}_p\big]\cdot\mathbf{n}_p\big)\, d\Gamma = \int_{\Omega_p}\boldsymbol{f}_p\cdot(\boldsymbol{\xi}_p + z\boldsymbol{s}_p)\sqrt{a}\, d\boldsymbol{X} \\
&\qquad\qquad + \int_{\Omega_p} g w_p\sqrt{a}\, d\boldsymbol{X} + \int_{\Omega_f}\boldsymbol{f}_f\cdot\boldsymbol{v}_f\, d\boldsymbol{X} - \int_{\Gamma_f^{in}}(\boldsymbol{v}_f\cdot\mathbf{n}_f) p_{in}(t)\, d\Gamma \qquad (3.20)
\end{aligned}
$$

for all $\boldsymbol{\xi}_p \in \mathcal{X}_p, \boldsymbol{s}_p \in \mathcal{S}, \boldsymbol{v}_p \in \mathcal{V}_p, w_p \in \mathcal{W}_p, \boldsymbol{v}_f \in \mathcal{V}_f, w_f \in \mathcal{W}_f$.

- We consider the interface terms

$$
I_{\Gamma_{fp}} = a_{BJS}\big(\boldsymbol{u}_f, \partial_t\boldsymbol{\eta}_p, \partial_t\boldsymbol{r}_p; \boldsymbol{v}_f, \boldsymbol{\xi}_p, \boldsymbol{s}_p\big) + b_\Gamma\big(\boldsymbol{\xi}_p, \boldsymbol{s}_p, \boldsymbol{v}_f, \boldsymbol{v}_p; p_p\big), \qquad (3.21)
$$

where

$$
\begin{aligned}
&a_{BJS}(\boldsymbol{u}_f, \partial_t\boldsymbol{\eta}_p, \partial_t\boldsymbol{r}_p; \boldsymbol{v}_f, \boldsymbol{\xi}_p, \boldsymbol{s}_p) = \\
&\qquad \int_{\Gamma_{fp}} \mu_f\alpha_{BJS}\sqrt{\kappa^{-1}}\big[\big(\boldsymbol{u}_f - \alpha(\partial_t\boldsymbol{\eta}_p - \frac{e}{2}\partial_t\boldsymbol{r})\big)\cdot\boldsymbol{\tau}_f\big]\big[\big(\boldsymbol{v}_f - \alpha(\boldsymbol{\xi}_p - \frac{e}{2}\boldsymbol{s}_p)\big)\cdot\boldsymbol{\tau}_f\big]\sqrt{a}\, d\Gamma, \\
&\qquad b_\Gamma\big(\boldsymbol{\xi}_p, \boldsymbol{s}_p, \boldsymbol{v}_f, \boldsymbol{v}_p; p_p\big) = \big(\boldsymbol{v}_f\cdot\boldsymbol{n}_f + \alpha(\boldsymbol{\xi}_p - \frac{e}{2}\boldsymbol{s}_p)\cdot\mathbf{n}_p + \boldsymbol{v}_p\cdot\mathbf{n}_p; p_p\sqrt{a}\big)_{\Gamma_{fp}}.
\end{aligned}
$$

Using the interface condition for balance of the normal components of stress (3.14), we set

$$
\lambda = -(\boldsymbol{\sigma}_f\mathbf{n}_f)\cdot\mathbf{n}_f = p_p; \quad \text{on } \Gamma_{fp}\times(0,T], \qquad (3.22)
$$

which will be used as a Lagrange multiplier to impose the mass conservation interface condition (3.13). Then, $I_{\Gamma_{fp}}$ becomes

$$
I_{\Gamma_{fp}} = a_{BJS}\big(\boldsymbol{u}_f, \partial_t\boldsymbol{\eta}_p, \partial_t\boldsymbol{r}_p; \boldsymbol{v}_f, \boldsymbol{\xi}_p, \boldsymbol{s}_p\big) + b_\Gamma\big(\boldsymbol{\xi}_p, \boldsymbol{s}_p, \boldsymbol{v}_f, \boldsymbol{v}_p; \lambda\big). \qquad (3.23)
$$

For well-posedness of b_Γ, we need that $\lambda \in \Lambda = (\mathcal{V}_p\cdot\mathbf{n}_p)'$. Here the space $\mathcal{V}_p\cdot\mathbf{n}_p$ is defined by

$$
\mathcal{V}_p\cdot\mathbf{n}_p = \{\boldsymbol{v}_p\cdot\mathbf{n}_p;\ \boldsymbol{v}_p \in \mathcal{V}_p\}. \qquad (3.24)
$$

According to the normal trace theorem, since $\boldsymbol{v}_p \in \mathcal{V}_p \subset H(\mathrm{div}, \Omega_p)$, then $\boldsymbol{v}_p \cdot \mathbf{n}_p \in H^{-1/2}(\partial\Omega_p)$. Moreover, since $\boldsymbol{v}_p \cdot \mathbf{n}_p = 0$ on Γ_p^1 and $\mathrm{dist}(\Gamma_p^0, \Gamma_{fp}) \geq s > 0$, then $\boldsymbol{v}_p \cdot \mathbf{n}_p \in H^{-1/2}(\Gamma_{fp})$ (see [46]). Then we take $\Lambda = H^{1/2}(\Gamma_{fp})$.

- The Lagrange multiplier variational formulation of Stokes/Biot-Naghdi couple is: For $t \in (0, T]$, find $\boldsymbol{\eta}_p(t) \in \mathcal{X}_p$, $\boldsymbol{r}_p(t) \in \mathcal{S}_p$, $p_p(t) \in \mathcal{W}_p$, $\boldsymbol{u}_p(t) \in \mathcal{V}_p$, $\boldsymbol{u}_f(t) \in \mathcal{V}_f$, $p_f(t) \in \mathcal{W}_f$ and $\lambda(t) \in \Lambda$ such that

$$
\begin{aligned}
&\int_\omega e\, a^{\alpha\beta\rho\sigma}\big[\gamma_{\alpha\beta}(\boldsymbol{\eta}_p)\gamma_{\rho\sigma}(\boldsymbol{\xi}_p) + \frac{e^2}{12}\chi_{\alpha\beta}(\boldsymbol{\eta}_p, \boldsymbol{r}_p)\chi_{\rho\sigma}(\boldsymbol{\xi}_p, \boldsymbol{s}_p)\big]\sqrt{a}\, d\boldsymbol{x} \\
&\quad + 4\mu_p \int_\omega e\, a^{\alpha\beta}\delta_{\alpha 3}(\boldsymbol{\eta}_p, \boldsymbol{r}_p)\delta_{\beta 3}(\boldsymbol{\xi}_p, \boldsymbol{s}_p)\sqrt{a}\, d\boldsymbol{x} - \alpha \int_{\Omega_p} p_p \sqrt{a}\, \mathrm{div}\, (\boldsymbol{\xi}_p + z\boldsymbol{s}_p)\, d\boldsymbol{X} \\
&\quad + \int_{\Omega_p} \partial_t\big(c_0 p_p + \alpha \mathrm{div}(\boldsymbol{\eta}_p + z\boldsymbol{r}_p)\big) w_p \sqrt{a}\, d\boldsymbol{X} + \int_{\Omega_p} \mathrm{div}(\boldsymbol{u}_p\sqrt{a}) w_p\, d\boldsymbol{X} + \frac{\mu_p}{\kappa}\int_{\Omega_p} \boldsymbol{u}_p \cdot \boldsymbol{v}_p \sqrt{a}\, d\boldsymbol{X} \\
&\quad - \int_{\Omega_p} p_p \mathrm{div}(\boldsymbol{v}_p\sqrt{a})\, d\boldsymbol{X} - \int_{\Omega_f} p_f\, \mathrm{div}\, \boldsymbol{v}_f\, d\boldsymbol{X} + 2\mu_f \int_{\Omega_f} D(\boldsymbol{u}_f) : D(\boldsymbol{v}_f)\, dX + \int_{\Omega_f} \mathrm{div}\, \boldsymbol{u}_f\, w_f\, d\boldsymbol{X} \\
&\quad + a_{BJS}\big(\boldsymbol{u}_f, \partial_t\boldsymbol{\eta}_p, \partial_t\boldsymbol{r}_p; \boldsymbol{v}_f, \boldsymbol{\xi}_p, \boldsymbol{s}_p\big) + b_\Gamma\big(\boldsymbol{\xi}_p, \boldsymbol{s}_p, \boldsymbol{v}_f, \boldsymbol{v}_p; \lambda\big) = \int_{\Omega_p} \boldsymbol{f}_p \cdot (\boldsymbol{\xi}_p + z\boldsymbol{s}_p)\sqrt{a}\, d\boldsymbol{X} \\
&\qquad\qquad + \int_{\Omega_p} g w_p \sqrt{a}\, d\boldsymbol{X} + \int_{\Omega_f} \boldsymbol{f}_f \cdot \boldsymbol{v}_f\, d\boldsymbol{X} - \int_{\Gamma_f^{in}} (\boldsymbol{v}_f \cdot \mathbf{n}_f) p_{in}(t)\, d\Gamma,
\end{aligned} \tag{3.25a}
$$

$$
b_\Gamma\big(\partial_t\boldsymbol{\eta}_p, \partial_t\boldsymbol{r}_p, \boldsymbol{u}_f, \boldsymbol{u}_p; \mu\big) = 0, \tag{3.25b}
$$

for all $\boldsymbol{\xi}_p \in \mathcal{X}_p$, $\boldsymbol{s}_p \in \mathcal{S}_p$, $w_p \in \mathcal{W}_p$, $\boldsymbol{v}_p \in \mathcal{V}_p$, $\boldsymbol{v}_f \in \mathcal{V}_f$, $w_f \in \mathcal{W}_f$ and $\mu \in \Lambda$.

We write (3.25) in short form,

$$
\begin{aligned}
&\mathcal{A}_1^N\big((\boldsymbol{\eta}_p, \boldsymbol{r}_p); (\boldsymbol{\xi}_p, \boldsymbol{s}_p)\big) + \alpha b_p(p_p\sqrt{a}, \boldsymbol{\xi}_p) + \alpha b_p(p_p\sqrt{a}, z\boldsymbol{s}_p) + c_0(\partial_t p_p, w_p\sqrt{a})_{\Omega_p} - \alpha b_p(w_p\sqrt{a}, \partial_t\boldsymbol{\eta}_p) \\
&\quad - \alpha b_p(w_p\sqrt{a}, z\partial_t\boldsymbol{s}_p) - b_p(w_p, \boldsymbol{u}_p\sqrt{a}) + a_p(\boldsymbol{u}_p, \boldsymbol{v}_p) + b_p(p_p, \boldsymbol{v}_p\sqrt{a}) + b_f(p_f, \boldsymbol{v}_f) + a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) \\
&\quad - b_f(w_f, \boldsymbol{u}_f) + b_\Gamma\big(\boldsymbol{\xi}_p, \boldsymbol{s}_p, \boldsymbol{v}_f, \boldsymbol{v}_p; \lambda\big) + a_{BJS}\big(\boldsymbol{u}_f, \partial_t\boldsymbol{\eta}_p, \partial_t\boldsymbol{r}_p; \boldsymbol{v}_f, \boldsymbol{\xi}_p, \boldsymbol{s}_p\big) = (g\sqrt{a}, w_p)_{\Omega_p} \\
&\qquad\qquad + \big(\boldsymbol{f}_p, (\boldsymbol{\xi}_p + z\boldsymbol{s}_p)\sqrt{a}\big)_{\Omega_p} + (\boldsymbol{f}_f, \boldsymbol{v}_f)_{\Omega_f} + b_{\Gamma^{in}}(\boldsymbol{v}_f, p_{in})
\end{aligned} \tag{3.26a}
$$

$$
b_\Gamma\big(\partial_t\boldsymbol{\eta}_p, \partial_t\boldsymbol{r}_p, \boldsymbol{u}_f, \boldsymbol{u}_p; \mu\big) = 0. \tag{3.26b}
$$

where

$$
\begin{aligned}
\mathcal{A}_1^N\big((\boldsymbol{\eta}_p, \boldsymbol{r}_p); (\boldsymbol{\xi}_p, \boldsymbol{s}_p)\big) = &\int_\omega e\, a^{\alpha\beta\rho\sigma}[\gamma_{\rho\sigma}(\boldsymbol{u}_p)\gamma_{\alpha\beta}(\boldsymbol{v}_p) + \frac{e^2}{12}\chi_{\rho\sigma}(\boldsymbol{\eta}_p, \boldsymbol{r}_p)\chi_{\alpha\beta}(\boldsymbol{\xi}_p, \boldsymbol{s}_p)]\sqrt{a}\, d\boldsymbol{x}, \\
&+ 4\mu_p \int_\omega e\, a^{\alpha\beta}\delta_{\alpha 3}(\boldsymbol{\eta}_p, \boldsymbol{r}_p)\delta_{\beta 3}(\boldsymbol{\xi}_p, \boldsymbol{s}_p)\sqrt{a}\ \ d\boldsymbol{x},
\end{aligned}
$$

$$
\begin{aligned}
&b_p(p_p, \boldsymbol{\xi}_p) = -(p_p, \mathrm{div}\, \boldsymbol{\xi}_p)_{\Omega_p}, \quad b_f(p_f, \boldsymbol{v}_f) = -(p_f, \mathrm{div}\, \boldsymbol{v}_f)_{\Omega_f}, \\
&a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) = 2\mu_f\big(D(\boldsymbol{u}_f), D(\boldsymbol{v}_f)\big)_{\Omega_f}, \quad a_p(\boldsymbol{u}_p, \boldsymbol{v}_p) = \mu_p\kappa^{-1}(\boldsymbol{u}_p, \boldsymbol{v}_p\sqrt{a})_{\Omega_p} \\
&b_{\Gamma^{in}}(\boldsymbol{v}_f, p_{in}) = \int_{\Gamma_f^{in}} p_{in}(t)\boldsymbol{n}_f \cdot \boldsymbol{v}_f\, d\Gamma.
\end{aligned}
$$

In the next section, we construct the semi-discrete formulation of (3.26) and use the DEA theory, the discrete Lagrange multiplier and inf-sup conditions for proceeding its well-posedness.

3.3.2 Analysis: Semi-discrete formulation

Let $\mathcal{T}_h^f$ and $\mathcal{T}_h^p$ be shape-regular and quasi-uniform partitions of Ω_f and Ω_p, respectively, both consisting of affine elements with maximal element diameter h. The two partitions may be non-matching at the interface Γ_{fp}. For the discretization of the fluid velocity and pressure we choose finite element spaces $\mathcal{V}_f^h \subset \mathcal{V}_f$ and $\mathcal{W}_f^h \subset \mathcal{W}_f$, which are assumed to be inf-sup stable. For the discretization of the porous medium, we choose $\mathcal{V}_p^h \subset \mathcal{V}_p$ and $\mathcal{W}_p^h \subset \mathcal{W}_p$ to be any of well-known inf-sup stable mixed finite element spaces. The global spaces are

$$\boldsymbol{V}^h = \{\boldsymbol{v}_h = (\boldsymbol{v}_{f,h}, \boldsymbol{v}_{p,h}) \in \mathcal{V}_f^h \times \mathcal{V}_p^h\} \text{ and } W^h = \{w_h = (w_{f,h}, w_{p,h}) \in \mathcal{W}_p^h \times \mathcal{W}_f^h\}. \tag{3.27}$$

To approximate the structure displacement, we make use of a conforming Lagrange finite element spaces $\mathcal{X}_p^h \subset \mathcal{X}_p$ and $\mathcal{S}_p^h \subset \mathcal{S}_p$. We set also $\mathbf{X}_p^h = \mathcal{X}_p^h \times \mathcal{S}_p^h$. Note that the finite element spaces $\mathcal{V}_{f,h}$, $\mathcal{V}_{p,h}$, $\mathcal{X}_{p,h}$ and $\mathcal{S}_{p,h}$ satisfy the homogeneous boundary conditions on external boundaries (3.11). For the discrete Lagrange multiplier space we take

$$\Lambda^h = \mathcal{V}_{p,h} \cdot \mathbf{n}_p|_{\Gamma_{fp}}. \tag{3.28}$$

Therefore, the semi-discrete continuous-in-time coupled problem is: Given $p_{p,h}(0)$, for $t \in (0,T]$, find $\boldsymbol{\eta}_{p,h}(t) \in \mathcal{X}_p^h$, $\boldsymbol{r}_{p,h}(t) \in \mathcal{S}_p^h$, $\boldsymbol{u}_{p,h}(t) \in \mathcal{V}_p^h$, $p_{p,h}(t) \in \mathcal{W}_p^h$, $\boldsymbol{u}_{f,h}(t) \in \mathcal{V}_f^h$, $p_{f,h} \in \mathcal{W}_f^h$ and $\lambda_h(t) \in \Gamma^h$ for all $\boldsymbol{\xi}_{p,h} \in \mathcal{X}_p^h$, $\boldsymbol{s}_{p,h} \in \mathcal{S}_p^h$, $\boldsymbol{v}_{p,h} \in \mathcal{V}_p^h$, $w_{p,h} \in \mathcal{W}_p^h$, $\boldsymbol{v}_f^h \in \mathcal{V}_f^h$, $w_{f,h} \in \mathcal{W}_f^h$ and $\mu_h \in \Gamma^h$

$$\begin{aligned}
&\mathcal{A}_1^N\big((\boldsymbol{\eta}_{p,h}, \boldsymbol{r}_{p,h}); (\boldsymbol{\xi}_{p,h}, \boldsymbol{s}_{p,h})\big) + \alpha b_p(p_{p,h}\sqrt{a}, \boldsymbol{\xi}_{p,h}) + \alpha b_p(p_{p,h}\sqrt{a}, z\boldsymbol{s}_{p,h}) + c_0(\partial_t p_{p,h}, w_{p,h}\sqrt{a})_{\Omega_p} \\
&\quad - \alpha b_p(w_{p,h}\sqrt{a}, \partial_t \boldsymbol{\eta}_{p,h}) - \alpha b_p(w_{p,h}\sqrt{a}, z\partial_t \boldsymbol{r}_{p,h}) - b_p(w_{p,h}, \boldsymbol{u}_{p,h}\sqrt{a}) + a_p(\boldsymbol{u}_{p,h}, \boldsymbol{v}_{p,h}) \\
&\quad + b_p(p_{p,h}, \boldsymbol{v}_{p,h}\sqrt{a}) + b_f(p_{f,h}, \boldsymbol{v}_{f,h}) + a_f(\boldsymbol{u}_{f,h}, \boldsymbol{v}_{f,h}) - b_f(w_{f,h}, \boldsymbol{u}_{f,h}) + b_\Gamma\big(\boldsymbol{\xi}_{p,h}, \boldsymbol{s}_{p,h}, \boldsymbol{v}_{f,h}, \boldsymbol{v}_{p,h}; \lambda_h\big) \\
&\quad + a_{BJS}\big(\boldsymbol{u}_{f,h}, \partial_t \boldsymbol{\eta}_{p,h}, \partial_t \boldsymbol{r}_{p,h}; \boldsymbol{v}_{f,h}, \boldsymbol{\xi}_{p,h}, \boldsymbol{s}_{p,h}\big) = (g\sqrt{a}, w_{p,h})_{\Omega_p} + \big(\boldsymbol{f}_p, (\boldsymbol{\xi}_{p,h} + z\boldsymbol{s}_{p,h})\sqrt{a}\big)_{\Omega_p} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + (\boldsymbol{f}_f, \boldsymbol{v}_{f,h})_{\Omega_f} + b_{\Gamma^{in}}(\boldsymbol{v}_{f,h}, p_{in}),
\end{aligned} \tag{3.29a}$$

$$b_\Gamma\big(\partial_t \boldsymbol{\eta}_{p,h}, \partial_t \boldsymbol{r}_{p,h}, \boldsymbol{u}_{f,h}, \boldsymbol{u}_{p,h}; \mu_h\big) = 0. \tag{3.29b}$$

We state the main result of this chapter in the following theorem,

Theorem 13

For $\boldsymbol{f}_f \in L^\infty\big(0,T;(\mathcal{V}_f^h)'\big)$, $\boldsymbol{f}_p \in L^\infty\big(0,T;(\mathcal{X}_p^h)'\big)$, $g \in L^\infty\big(0,T;(\mathcal{W}_p^h)'\big)$ and $p_{p,h}(0) \in \mathcal{W}_p^h$, there exists a unique solution $(\boldsymbol{u}_{f,h}, \boldsymbol{\eta}_{p,h}, \boldsymbol{r}_{p,h}, \boldsymbol{u}_{p,h}, p_{f,h}, p_{p,h}, \lambda_h)$ in $L^\infty\big(0,T;\mathcal{V}_f^h\big) \times W^{1,\infty}\big(0,T;\mathcal{X}_p^h\big) \times L^\infty\big(0,T;\mathcal{S}_p^h\big) \times L^\infty\big(0,T;\mathcal{V}_p^h\big) \times L^\infty\big(0,T;\mathcal{W}_f^h\big) \times W^{1,\infty}\big(0,T;\mathcal{W}_p^h\big) \times L^\infty\big(0,T;\Lambda^h\big)$ of the weak formulation (3.29).

For proving Theorem 13, we firstly derive the existence by writing (3.29) in matrix form and using the DAE theory (see Lemma 39). We then prove the uniqueness by the discrete inf-sup condition (see Lemma 43).

More detailed, let $\{\phi_{\eta_p,i}\}$, $\{\phi_{r_p,i}\}$, $\{\phi_{u_p,i}\}$, $\{\phi_{p_p,i}\}$, $\{\phi_{u_f,i}\}$, $\{\phi_{p_f,i}\}$ and $\{\phi_{\lambda,i}\}$ be bases of $\mathcal{X}_p^h$, $\mathcal{S}_p^h$, $\mathcal{V}_p^h$, $\mathcal{W}_p^h$, $\mathcal{V}_f^h$, $\mathcal{W}_f^h$ and Λ^h. We define the matrices A_{η_p}, A_{r_p}, $A_{u_f}^{BJS}$, $A_{\eta_p}^{BJS}$, $A_{r_p}^{BJS}$, A_{u_p}, A_{u_f}, B_{u_f}, B_{η_p}, B_{r_p}, B_{u_p}, $B_{u_f,\Gamma}$, $B_{\eta_p,\Gamma}$, $B_{r_p,\Gamma}$, $B_{u_p,\Gamma}$, F_{η_p}, F_{r_p}, F_{u_f}, G_{p_p}, $B_{\Gamma^{in}}$ as follow

$$
\begin{aligned}
A_{\eta_p} &= \mathcal{A}_1^N\left[(\phi_{\eta_p,j},0);(\phi_{\eta_p,i},\phi_{r_p,i})\right], A_{r_p} = \mathcal{A}_1^N\left[(0,\phi_{r_p,j});(\phi_{\eta_p,i},\phi_{r_p,i})\right],\\
A_{u_p} &= a_p(\phi_{u_p,j},\phi_{u_p,i}), A_{u_f} = a_f(\phi_{u_f,j};\phi_{u_f,i})\\
A_{u_f}^{BJS} &= a_{BJS}(\phi_{u_f,j},0,0;\phi_{u_f,i},\phi_{u_p,i},\phi_{r_p,i}),\\
A_{\eta_p}^{BJS} &= a_{BJS}(0,\phi_{\eta_p,j},0;\phi_{u_f,i},\phi_{\eta_p,i},\phi_{r_p,i}),\\
A_{r_p}^{BJS} &= a_{BJS}(0,0,\phi_{r_p,j};\phi_{u_f,i},\phi_{\eta_p,i},\phi_{r_p,i}),\\
B_{u_f} &= b_f(\phi_{p_f,j};\phi_{u_f,i}), B_{\eta_p} = b_p(\phi_{p_p,j},\phi_{\eta_p,i}),\\
B_{r_p} &= b_p(\phi_{p_p,j},z\phi_{r_p,i}), B_{u_p} = b_p(\phi_{p_p,j},\phi_{u_p,j}\sqrt{a}),\\
B_{u_f,\Gamma} &= b_\Gamma(\phi_{u_f,j},0,0,0;\phi_{\lambda,i}), B_{\eta_p,\Gamma} = b_\Gamma(0,\phi_{\eta_p,j},0,0;\phi_{\lambda,i}),\\
B_{r_p,\Gamma} &= b_\Gamma(0,0,\phi_{r_p,j},0;\phi_{\lambda,i}), B_{u_p,\Gamma} = b_\Gamma(0,0,0,\phi_{u_p,j};\phi_{\lambda,i}),\\
G_p &= (g\sqrt{a};\phi_{p_p,i})_{\Omega_p}, F_p = (\boldsymbol{f}_p\sqrt{a},\phi_{\eta_p,i}+z\phi_{r_p,i})_{\Omega_p},\\
F_{u_f} &= (\boldsymbol{f}_f\sqrt{a},\phi_{u_f,i})_{\Omega_f}, B_{\Gamma^{in}} = (p_{in},\phi_{u_f,i}).
\end{aligned}
$$

Taking in (3.26a) -(3.26b)

$$
\begin{aligned}
&\boldsymbol{u}_f(t,\boldsymbol{x}) = \sum_i u_{f,i}(t)\phi_{\boldsymbol{u}_p,i};\ \boldsymbol{\eta}_p(t,\boldsymbol{x}) = \sum_i \eta_{p,i}(t)\phi_{\eta_p,i};\ \boldsymbol{r}_p(t,\boldsymbol{x}) = \sum_i r_{p,i}(t)\phi_{\boldsymbol{r}_p,i};\\
&p_f(t,\boldsymbol{x}) = \sum_i p_{f,i}(t)\phi_{p_f,i};\ p_p(t,\boldsymbol{x}) = \sum_i p_{p,i}(t)\phi_{p_p,i};\ \lambda(t,\boldsymbol{x}) = \sum_i \lambda_i(t)\phi_{\lambda,i};\\
&\boldsymbol{u}_p(t,\boldsymbol{x}) = \sum_i u_{p,i}(t)\phi_{u_p,i},
\end{aligned}
$$

with time-dependent coefficients $\overline{\boldsymbol{u}}_f$, $\overline{\boldsymbol{u}}_p$, $\overline{\boldsymbol{\eta}}_p$, $\overline{\boldsymbol{r}}_p$, $\overline{p}_f$, $\overline{p}_p$, $\overline{\lambda}$, we get the matrix-vector system of (3.26):

$$
\begin{aligned}
&A_{\eta_p}\,\overline{\boldsymbol{\eta}}_p + A_{r_p}\,\overline{\boldsymbol{r}}_p + \alpha(B_{\eta_p}^T + B_{r_p}^T)\,\overline{p}_p + c_0M\,\partial_t\overline{p}_p - \alpha B_{\eta_p}\,\partial_t\overline{\boldsymbol{\eta}}_p - \alpha B_{r_p}\,\partial_t\overline{\boldsymbol{r}}_p - B_{u_p}\,\overline{\boldsymbol{u}}_p\\
&\quad + A_{u_p}\overline{\boldsymbol{u}}_p + B_{u_p}^T\overline{p}_p + B_{u_f}^T\,\overline{p}_f + A_{u_f}\,\overline{\boldsymbol{u}}_f - B_{u_f}\,\overline{\boldsymbol{u}}_f + (B_{u_f,\Gamma}^T + B_{\eta_p,\Gamma}^T + B_{r_p,\Gamma}^T + B_{u_p,\Gamma}^T)\,\lambda\\
&\qquad + A_{u_f}^{BJS}\,\overline{\boldsymbol{u}}_f + A_{u_p}^{BJS}\,\partial_t\overline{\boldsymbol{u}}_p + A_{r_p}^{BJS}\,\partial_t\overline{\boldsymbol{r}}_p = G_p + F_p + F_{u_f} + B_{\Gamma^{in}},\\
&B_{u_f,\Gamma}\,\overline{\boldsymbol{u}}_f + B_{\eta_p,\Gamma}\,\partial_t\overline{\boldsymbol{\eta}}_p + B_{r_p,\Gamma}\,\partial_t\overline{\boldsymbol{r}}_p + B_{u_p,\Gamma}\,\partial_t\overline{\boldsymbol{u}}_p = 0.
\end{aligned}
$$

Therefore, the above weak form can be written in the matrix form

$$EX'(t) + HX(t) = L(t) \tag{3.30}$$

where

$$H = \begin{bmatrix} A_{u_f} + A_{u_f}^{BJS} & 0 & 0 & 0 & B_{u_f}^{\mathrm{T}} & 0 & B_{u_f,\Gamma}^{\mathrm{T}} \\ 0 & A_{\eta_p} & 0 & 0 & 0 & \alpha B_{\eta_p}^T & B_{\eta_p,\Gamma}^T \\ 0 & 0 & A_{u_p} & 0 & 0 & B_{u_p}^T & B_{u_p,\Gamma}^T \\ 0 & 0 & 0 & A_p^r & 0 & B_{r_p}^T & B_{r_p,\Gamma}^T \\ -B_{u_f} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -B_{u_p} & 0 & 0 & 0 & 0 \\ -B_{u_f,\Gamma} & 0 & -B_{u_p,\Gamma} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{u_p}^{BJS} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{r_p}^{BJS} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha B_{\eta_p} & 0 & -\alpha B_{r_p} & 0 & c_0 M & 0 \\ 0 & -B_{\eta_p,\Gamma} & 0 & -B_{r_p,\Gamma} & 0 & 0 & 0 \end{bmatrix}, \ L(t) = \begin{bmatrix} F_f^u \\ F_p^u \\ B_{\Gamma^{in}} \\ 0 \\ 0 \\ G_p \\ 0 \end{bmatrix}, \ X(t) = \begin{bmatrix} \overline{\boldsymbol{u}}_f \\ \overline{\boldsymbol{\eta}}_p \\ \overline{\boldsymbol{u}}_p \\ \overline{\boldsymbol{r}}_p \\ \overline{p}_f \\ \overline{p}_p \\ \overline{\lambda} \end{bmatrix}.$$

The theory of DAEs say that, if the matrix $s\mathbf{E}{+}\mathbf{H}$ is nonsingular for some $s \neq 0$, the equation (3.30) has a solution. So the idea here is to prove that $s\mathbf{E}{+}\mathbf{H}$ is nonsingular when $s = 1$ or $\mathbf{E}{+}\mathbf{H}$ is nonsingular ou invertible. To state the desired result, we note that

$$\mathbf{E} + \mathbf{H} = \begin{bmatrix} A_{u_f}^{BJS} + A_{u_f} & 0 & 0 & 0 & B_{u_f}^{\mathrm{T}} & 0 & B_{u_f,\Gamma}^{\mathrm{T}} \\ 0 & A_{\eta_p}^{BJS} + A_{\eta_p} & 0 & 0 & 0 & \alpha B_{\eta_p}^T & B_{\eta_p,\Gamma}^T \\ 0 & 0 & A_{u_p} & 0 & 0 & B_{u_p}^T & B_{u_p,\Gamma}^T \\ 0 & 0 & 0 & A_{r_p}^{BJS} + A_{r_p} & 0 & \alpha B_{r_p}^T & B_{r_p,\Gamma}^T \\ -B_{u_f} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha B_{\eta_p} & -B_{u_p} & -\alpha B_{r_p} & 0 & c_0 M & 0 \\ -B_{u_f,\Gamma} & -B_{\eta_p,\Gamma} & -B_{u_p,\Gamma} & -B_{r_p,\Gamma} & 0 & 0 & 0 \end{bmatrix}.$$

It can be written as a block 2×2 matrix $\mathbf{E} + \mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{B}^{\mathrm{T}} \\ -\mathbf{B} & \mathbf{C} \end{bmatrix}$, where

$$\mathbf{A} = \begin{bmatrix} A_{u_f}^{BJS} + A_{u_f} & 0 & 0 & 0 \\ 0 & A_{\eta_p}^{BJS} + A_{\eta_p} & 0 & 0 \\ 0 & 0 & A_{u_p} & 0 \\ 0 & 0 & 0 & A_{r_p}^{BJS} + A_{r_p} \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} B_{u_f} & 0 & 0 & 0 \\ 0 & \alpha B_{\eta_p} & B_{u_p} & \alpha B_{r_p} \\ B_{u_f,\Gamma} & B_{\eta_p,\Gamma} & B_{u_p,\Gamma} & B_{r_p,\Gamma} \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_0 M & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Subsequently, we derive our desired results by the theory of DEAs.

Lemma 39

If $\mathbf{A}$ and $\mathbf{C}$ are positive semi-definite and $\ker(\mathbf{A}) \cap \ker(\mathbf{B}) = \ker(\mathbf{C}) \cap \ker(\mathbf{B}^T) = \{0\}$, then $\mathbf{E}+\mathbf{H}$ is invertible.

Proof. The proof is inspired by the Son-Young Yi's work [91] and the Horn-Johnson's book (page 431,[52]). For convenience, we remind here some main points.
Let $\mathbf{U} = (\mathbf{U}_1, \mathbf{U}_2)^{\mathrm{T}}$ and suppose that $(\mathbf{E}+\mathbf{H})\mathbf{U} = 0$. This follows

$$\begin{cases} \mathbf{A}\mathbf{U}_1 + \mathbf{B}^{\mathrm{T}}\mathbf{U}_2 = 0, \\ -\mathbf{B}\mathbf{U}_1 + \mathbf{C}\mathbf{U}_2 = 0. \end{cases} \tag{3.31}$$

Hence, we obtain that $\mathbf{U}_1^{\mathrm{T}}\mathbf{A}\ \mathbf{x} + \mathbf{U}_1^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{U}_2 = \mathbf{U}_1^{\mathrm{T}}\mathbf{A}\ \mathbf{x} + \mathbf{U}_2^{\mathrm{T}}\mathbf{C}\ \mathbf{y} = 0$. Since $\mathbf{A}$ and $\mathbf{C}$ are positive semidefinite, we have that if $\mathbf{U}_1^{\mathrm{T}}\mathbf{A}\ \mathbf{x} = 0$ and $\mathbf{U}_2^{\mathrm{T}}\mathbf{C}\ \mathbf{y} = 0$, then $\mathbf{A}\mathbf{x} = 0$ and $\mathbf{C}\mathbf{x} = 0$ [52] (page 431). Then $\mathbf{U}_1 \in \ker(\mathbf{A})$ and $\mathbf{U}_2 \in \ker(\mathbf{B})$. Combining with (3.31), we get $\mathbf{U}_1 \in \ker(\mathbf{A}) \cap \ker(\mathbf{B}) = \{0\}$ and $\mathbf{U}_2 \in \ker(\mathbf{B}^{\mathrm{T}}) \cap \ker(\mathbf{C}) = \{0\}$, *i.e.*, $\mathbf{U}$=$\mathbf{0}$ or $\mathbf{E}+\mathbf{H}$ is invertible. □

To prove the conditions of Lemma 39, we do not use the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ directly. We will use the bilinear forms that are associated with those matrices. We define then bilinear forms $\phi_A(.,.)$, $\phi_B(.,.)$ and $\phi_C(.,.)$ on $(\boldsymbol{V}^h \times \mathcal{X}_p^h \times \mathcal{S}_p^h) \times (\boldsymbol{V}^h \times \mathcal{X}_p^h \times \mathcal{S}_p^h)$, $(\boldsymbol{V}^h \times \mathcal{X}_p^h \times \mathcal{S}_p^h) \times (\boldsymbol{W}^h \times \Lambda^h)$ and $(\boldsymbol{W}^h \times \Lambda^h) \times (\boldsymbol{W}^h \times \Lambda^h)$, respectively:

$$\begin{aligned}
\phi_A\big((\boldsymbol{u}_h, \boldsymbol{\eta}_{p,h}, \boldsymbol{r}_{p,h}), (\boldsymbol{v}_h, \boldsymbol{\xi}_{p,h}, \boldsymbol{s}_{p,h})\big) =& A_1^N\big[(\boldsymbol{\eta}_{p,h}, \boldsymbol{r}_{p,h}); (\boldsymbol{\xi}_{p,h}, \boldsymbol{s}_{p,h})\big] + a_p(\boldsymbol{u}_{p,h}, \boldsymbol{v}_{p,h}) + a_f(\boldsymbol{u}_{f,h}, \boldsymbol{v}_{f,h}) \\
&+ a_{BJS}(\boldsymbol{u}_{f,h}, \boldsymbol{\eta}_{p,h}, \boldsymbol{r}_{p,h}; \boldsymbol{v}_{f,h}, \boldsymbol{\xi}_{p,h}, \boldsymbol{s}_{p,h}), \\
\phi_B\big((\boldsymbol{u}_h, \boldsymbol{\eta}_{p,h}, \boldsymbol{r}_{p,h}), (w_h, \mu_h)\big) =& b_f(\boldsymbol{u}_{f,h}, w_{f,h}) + b_p(\boldsymbol{u}_{p,h}, w_{p,h}) + \alpha b_p(\boldsymbol{\eta}_{p,h}, w_{p,h}) \\
&+ b_\Gamma(\boldsymbol{u}_{f,h}, \boldsymbol{\eta}_{p,h}, \boldsymbol{r}_{p,h}, \boldsymbol{u}_{p,h}; \mu), \\
\phi_C\big((p_h, \mu_h), (w_h, \lambda_h)\big) =& c_0(p_{p,h}\sqrt{a}; w_{p,h})_{\Omega_p}.
\end{aligned}$$

Note that $\boldsymbol{u}_h = (\boldsymbol{u}_{f,h}, \boldsymbol{u}_{p,h}),\ \boldsymbol{v}_h = (\boldsymbol{v}_{f,h}, \boldsymbol{v}_{p,h}) \in \boldsymbol{V}_h$ and $p_h = (p_{f,h}, p_{p,h}),\ w_h = (w_{f,h}, w_{p,h}) \in W_h$.

The first condition in Lemma 39, $\mathbf{A}$ and $\mathbf{B}$ are positive semi-definite matrices, is obtained by the two following lemmas,

Lemma 40

The bilinear ϕ_A is positive semidefinite, that is, for any $(\boldsymbol{v}_h, \boldsymbol{\xi}_{p,h}, \boldsymbol{s}_{p,h}) \in \boldsymbol{V}^h \times \mathcal{X}_p^h \times \mathcal{S}_p^h$,

$$\phi_A\big((\boldsymbol{v}_h, \boldsymbol{\xi}_{p,h}, \boldsymbol{s}_{p,h}), (\boldsymbol{v}_h, \boldsymbol{\xi}_{p,h}, \boldsymbol{s}_{p,h})\big) \geq 0. \tag{3.32}$$

Proof. The assumptions on the fluid viscosity μ_f and the material coefficients κ and μ_p imply that the bilinear forms $A_N^1(\cdot,\cdot)$, $a_f(\cdot,\cdot)$ and $a_p(\cdot,\cdot)$ are coercive. In particular, thanks to Poincaré inequality, there exist positive constants c_N, c_p, c_f such that

$$\begin{aligned}
\mathcal{A}_1^N\big((\boldsymbol{\xi}_p,\boldsymbol{s}_p);(\boldsymbol{\xi}_p,\boldsymbol{s}_p)\big) &\geq c_N\|(\boldsymbol{v}_p,\boldsymbol{s}_p)\|^2_{\mathcal{X}_N} \qquad \forall(\boldsymbol{\xi}_p,\boldsymbol{s}_p)\in\mathcal{X}_N=\mathcal{X}_p\times\mathcal{S}_p,\\
a_p(\boldsymbol{v}_p,\boldsymbol{v}_p) &\geq c_p\|\boldsymbol{v}_p\|^2_{\mathcal{V}_p} \qquad \forall\boldsymbol{v}_p\in\mathcal{V}_p,\\
a_f(\boldsymbol{v}_f,\boldsymbol{v}_f) &\geq c_f\|\boldsymbol{v}_f\|^2_{\mathcal{V}_f} \qquad \forall\boldsymbol{v}_f\in\mathcal{V}_f,\\
a_{BJS}(\boldsymbol{v}_f,\boldsymbol{\xi}_p,\boldsymbol{s}_p;\boldsymbol{v}_f,\boldsymbol{\xi}_p,\boldsymbol{s}_p) &\geq \sqrt{\delta}\mu_f\alpha_{BJS}\Big\|\kappa^{-1/4}(\boldsymbol{v}_f-\alpha(\boldsymbol{\xi}_p-\frac{e}{2}\boldsymbol{s}_p))\Big\|^2_{L^2(\Gamma_{fp})}\geq 0\ \ (a\geq\delta>0)\\
&\qquad \forall(\boldsymbol{v}_f,\boldsymbol{\xi}_p,\boldsymbol{s}_p)\in\mathcal{V}_f\times\mathcal{X}_p\times\mathcal{S}_p.
\end{aligned}$$

From the above inequalities, we get (3.32). It means that A is a positive semi-definite matrix. □

Similarly, we get the same result for $\mathbf{C}$ by the following lemma,

Lemma 41

The bilinear ϕ_C is positive semidefinite, that is, for any $(w_h,\lambda_h)\in\boldsymbol{W}^h\times\Lambda^h$,

$$\phi_C\big((w_h,\lambda_h),(w_h,\lambda_h)\big)\geq 0.$$

Proof. The proof is straightforward since $c_0\geq 0$. □

Afterwards, we introduce the kernel spaces of the above bilinear forms. Let

$$\begin{aligned}
\ker(\phi_A) &= \big\{(\boldsymbol{u}_h,\boldsymbol{\eta}_{p,h},\boldsymbol{r}_{p,h})\in\boldsymbol{V}^h\times\mathcal{X}_p^h\times\mathcal{S}_p^h:\phi_A\big((\boldsymbol{u}_h,\boldsymbol{\eta}_{p,h},\boldsymbol{r}_{p,h}),(\boldsymbol{v}_h,\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h})\big)=0,\\
&\qquad \forall(\boldsymbol{v}_h,\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h})\in\boldsymbol{V}^h\times\mathcal{X}_p^h\times\mathcal{S}_p^h\big\},\\
\ker(\phi_B) &= \big\{(\boldsymbol{u}_h,\boldsymbol{\eta}_{p,h},\boldsymbol{r}_{p,h})\in\boldsymbol{V}^h\times\mathcal{X}_p^h\times\mathcal{S}_p^h:\phi_B\big((\boldsymbol{u}_h,\boldsymbol{\eta}_{p,h},\boldsymbol{r}_{p,h}),(w_h,\mu_h)\big)=0,\\
&\qquad \forall(w_h,\mu_h)\in\boldsymbol{W}^h\times\Lambda^h\big\},\\
\ker(\phi_{B^T}) &= \big\{(w_h,\mu_h)\in\boldsymbol{W}^h\times\Lambda^h:\phi_B\big((\boldsymbol{u}_h,\boldsymbol{\eta}_{p,h},\boldsymbol{r}_{p,h}),(w_h,\mu_h)\big)=0,\\
&\qquad \forall(\boldsymbol{u}_h,\boldsymbol{\eta}_{p,h},\boldsymbol{r}_{p,h})\in\boldsymbol{V}^h\times\mathcal{X}_p^h\times\mathcal{S}_p^h\big\},\\
\ker(\phi_C) &= \big\{(p_h,\mu_h)\in\boldsymbol{W}^h\times\Lambda^h:\phi_C\big((p_h,\mu_h),(w_h,\lambda_h)\big)=0,\ \forall(w_h,\mu_h)\in\boldsymbol{W}^h\times\Lambda^h\big\}.
\end{aligned}$$

We note that $\ker(\phi_A)=\ker(\mathbf{A})$, $\ker(\phi_B)=\ker(\mathbf{B})$, $\ker(\phi_{B^T})=\ker(\mathbf{B}^T)$ and $\ker(\phi_C)=\ker(\mathbf{C})$.

We then show that the second condition in Lemma 39 is satisfied.

Lemma 42

The bilinear forms ϕ_A and ϕ_B satisfy

$$\ker(\phi_A)\cap\ker(\phi_B)=\{0\}.$$

Proof. From the coercivity of $A_1^N[\cdot;\cdot]$, $a_p(\cdot,\cdot)$, $a_f(\cdot,\cdot)$ and the non-negativity of $a_{BJS}(\cdot,\cdot)$ follow that $\ker(\phi_A)=0$. And we can easily see that $\{0\}\in\ker(\phi_B)$. So we obtain $\ker(\phi_A)\cap\ker(\phi_B)=\{0\}$. □

Now for proving that $\ker(\mathbf{B}^T)$ $=\{0\}$, we state a discrete inf-sup condition, which will be utilized to control the pressure in two region and the Lagrange multiplier. Following [46], we define a semi-norm in Λ^h

$$|\mu_h|^2_{\Lambda^h}=a_p\big(\boldsymbol{u}^*_{p,h}(\mu_h),\boldsymbol{u}^*_{p,h}(\mu_h)\big)$$

where $(\boldsymbol{u}^*_{p,h}(\mu_h),p^*_{p,h}(\mu_h))\in\mathcal{V}_p^h\times\mathcal{W}_p^h$ is the mixed finite element solution to the Darcy problem with Direchlet condition μ_h on Γ_{fp}:

$$a_p\big(\boldsymbol{u}^*_{p,h}(\mu_h),\boldsymbol{v}_{p,h}\big)+b_p(p^*_{p,h}(\mu_h),\boldsymbol{v}_{p,h}\sqrt{a})=-\langle\boldsymbol{v}_{p,h}\cdot\mathbf{n}_p,\mu_h\rangle_{\Gamma_{fp}}\quad\forall\boldsymbol{v}_{p,h}\in\mathcal{V}_p^h,$$
$$b_p\big(w_{p,h},\boldsymbol{u}^*_{p,h}(\mu_h)\sqrt{a}\big)=0\quad\forall w_{p,h}\in\mathcal{W}_p^h.$$

We equip Λ^h with the norm $\|\mu_h\|^2_{\Lambda^h}=\|\mu_h\|^2_{L^2(\Gamma_{fp})}+|\mu_h|^2_{\Lambda^h}$. This norm can be considered as a discrete version of the $H^{1/2}(\Gamma_{fp})$-norm [46]. For convenience of notation we define the composite norms

$$\|(\boldsymbol{v}_h,\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h})\|^2_{\mathbf{V}\times\mathcal{X}_p\times\mathcal{S}_p}=\|\boldsymbol{\tau}_h\|^2_{\boldsymbol{V}}+\|\boldsymbol{v}_{p,h}\|^2_{\mathcal{X}_p}+\|\boldsymbol{s}_p\|^2_{\mathcal{S}_p},$$
$$\|(w_h,\mu_h)\|^2_{\boldsymbol{W}\times\Lambda^h}=\|w_h\|^2_{\boldsymbol{W}}+\|\mu_h\|^2_{\Lambda^h},$$

and

$$\begin{aligned}b\big(\boldsymbol{v}_h,\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h};w_h\big)=&b_f(w_{f,h},\boldsymbol{v}_{f,h})+b_p(w_{p,h},\boldsymbol{v}_{p,h})+\alpha b_p(w_{p,h},\boldsymbol{\xi}_{p,h}+z\boldsymbol{s}_{p,h}),\\ b_\Gamma\big(\boldsymbol{v}_h,\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h};\mu_h\big)=&b_\Gamma(\boldsymbol{v}_{f,h},\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h},\boldsymbol{v}_{p,h};\mu_h).\end{aligned}$$

With these notations, we proceed the following discrete inf-sup condition.

Lemma 43

There exists a constant $\beta>0$ independent of h such that

$$\inf_{(w_h,\mu_h)\in W^h\times\Lambda^h}\ \sup_{(\boldsymbol{v}_h,\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h})\in\boldsymbol{V}^h\times\mathcal{X}_p^h\times\mathcal{S}_p^h}\frac{b\big(\boldsymbol{v}_h,\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h};w_h\big)+b_\Gamma\big(\boldsymbol{v}_h,\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h};\mu_h\big)}{\|(\boldsymbol{v}_h,\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h})\|_{\boldsymbol{V}\times\mathcal{X}_p\times\mathcal{S}_p}\|(w_h,\mu_h)\|_{W\times\Lambda^h}}\geq\beta. \tag{3.33}$$

Instead of proving Lemma 43 directly, we introduce the other inf-sup condition because its proof is simpler than (3.33).

Lemma 44

There exists a constant $\beta>0$ independent of h such that

$$\inf_{(w_h,\mu_h)\in\boldsymbol{W}^h\times\Lambda^h}\ \sup_{\boldsymbol{v}_h\in\boldsymbol{V}^h}\frac{b_f(w_{f,h},\boldsymbol{v}_{f,h})+b_p(w_{p,h},\boldsymbol{v}_{p,h})+\big(\boldsymbol{v}_{f,h}\cdot\boldsymbol{n}_f+\boldsymbol{v}_{p,h}\cdot\boldsymbol{n}_p;\mu_h\sqrt{a}\big)_{\Gamma_{fp}}}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}\|(w_h,\mu_h)\|_{W\times\Lambda^h}}\geq\beta. \tag{3.34}$$

Proof. The proof is based on [46]. Firstly, we define two following spaces

$$\boldsymbol{V}^h_{\circ} = \big\{(\boldsymbol{v}_{f,h}, \boldsymbol{v}_{p,h}) \in \boldsymbol{V}^h : \big(\boldsymbol{v}_{f,h} \cdot \boldsymbol{n}_f + \boldsymbol{v}_{p,h} \cdot \boldsymbol{n}_p; \mu_h \sqrt{a}\big)_{\Gamma_{fp}} = 0 \text{ for all } \mu_h \text{ in } \Lambda^h\big\}, \tag{3.35}$$

$$\boldsymbol{V}^h_{\circ\circ} = \big\{(\boldsymbol{v}_{f,h}, \boldsymbol{v}_{p,h}) \in \boldsymbol{V}^h : b_f(w_{f,h}, \boldsymbol{v}_{f,h}) + b_p(w_{p,h}, \boldsymbol{v}_{p,h}) = 0 \text{ for all } w_h \text{ in } W^h\big\}. \tag{3.36}$$

Due to the Lemmas 5.5 and 5.6 in [46], there exist two positive constants β_1 and β_2 independent of h such that

$$\inf_{w_h \in W^h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}^h} \frac{b_f(w_{f,h}, \boldsymbol{v}_{f,h}) + b_p(w_{p,h}, \boldsymbol{v}_{p,h})}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}} \|w_h\|_W} \geq \beta_1, \tag{3.37}$$

$$\inf_{\mu_h \in \Lambda^h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}^h} \frac{\big(\boldsymbol{v}_{f,h} \cdot \boldsymbol{n}_f + \boldsymbol{v}_{p,h} \cdot \boldsymbol{n}_p; \mu_h \sqrt{a}\big)_{\Gamma_{fp}}}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}} \|\mu_h\|_{\Lambda^h}} \geq \beta_2. \tag{3.38}$$

Given $(w_h, \mu_h) \in W \times \Lambda^h$, if $w_h \neq 0$, from (3.37) there exists $\overset{\circ}{\boldsymbol{v}}_h = (\overset{\circ}{\boldsymbol{v}}_{f,h}, \overset{\circ}{\boldsymbol{v}}_{p,h}) \in \boldsymbol{V}^h_{\circ}$ such that

$$\frac{b_f(w_{f,h}, \overset{\circ}{\boldsymbol{v}}_{f,h}) + b_p(w_{p,h}, \overset{\circ}{\boldsymbol{v}}_{p,h})}{\|\overset{\circ}{\boldsymbol{v}}_h\|_{\boldsymbol{V}}} \geq \beta_1 \|w_h\|_W > 0, \tag{3.39}$$

where β_1 independent of w_h. If $\mu_h \neq 0$, from (3.38) there exists $\overset{\circ\circ}{\boldsymbol{v}}_h = (\overset{\circ\circ}{\boldsymbol{v}}_{f,h}, \overset{\circ\circ}{\boldsymbol{v}}_{p,h}) \in \boldsymbol{V}^h_{\circ\circ}$ such that

$$\frac{\big(\overset{\circ\circ}{\boldsymbol{v}}_{f,h} \cdot \boldsymbol{n}_f + \overset{\circ\circ}{\boldsymbol{v}}_{p,h} \cdot \boldsymbol{n}_p; \mu_h \sqrt{a}\big)_{\Gamma_{fp}}}{\|\overset{\circ\circ}{\boldsymbol{v}}_h\|_{\boldsymbol{V}}} \geq \beta_2 \|\mu_h\|_{\Lambda^h} > 0. \tag{3.40}$$

We see that, if $w_h \neq 0$

$$\begin{aligned}
&\sup_{\boldsymbol{v}_h \in \boldsymbol{V}^h} \frac{b_f(w_{f,h}, \boldsymbol{v}_{f,h}) + b_p(w_{p,h}, \boldsymbol{v}_{p,h}) + \big(\boldsymbol{v}_{f,h} \cdot \boldsymbol{n}_f + \boldsymbol{v}_{p,h} \cdot \boldsymbol{n}_p; \mu_h \sqrt{a}\big)_{\Gamma_{fp}}}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}} \\
&\geq \frac{b_f(w_{f,h}, \overset{\circ}{\boldsymbol{v}}_{f,h}) + b_p(w_{p,h}, \overset{\circ}{\boldsymbol{v}}_{p,h}) + \big(\overset{\circ}{\boldsymbol{v}}_{f,h} \cdot \boldsymbol{n}_f + \overset{\circ}{\boldsymbol{v}}_{p,h} \cdot \boldsymbol{n}_p; \mu_h \sqrt{a}\big)_{\Gamma_{fp}}}{\|\overset{\circ}{\boldsymbol{v}}_h\|_{\boldsymbol{V}}} \\
&= \frac{b_f(w_{f,h}, \overset{\circ}{\boldsymbol{v}}_{f,h}) + b_p(w_{p,h}, \overset{\circ}{\boldsymbol{v}}_{p,h}) + 0}{\|\overset{\circ}{\boldsymbol{v}}_h\|_{\boldsymbol{V}}} \geq \beta_1 \|w_h\|_W \quad \text{by (3.39).}
\end{aligned}$$

Similarly, if $\mu_h \neq 0$,

$$\begin{aligned}
&\sup_{\boldsymbol{v}_h \in \boldsymbol{V}^h} \frac{b_f(w_{f,h}, \boldsymbol{v}_{f,h}) + b_p(w_{p,h}, \boldsymbol{v}_{p,h}) + \big(\boldsymbol{v}_{f,h} \cdot \boldsymbol{n}_f + \boldsymbol{v}_{p,h} \cdot \boldsymbol{n}_p; \mu_h \sqrt{a}\big)_{\Gamma_{fp}}}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}} \\
&\geq \frac{b_f(w_{f,h}, \overset{\circ\circ}{\boldsymbol{v}}_{f,h}) + b_p(w_{p,h}, \overset{\circ\circ}{\boldsymbol{v}}_{p,h}) + \big(\overset{\circ\circ}{\boldsymbol{v}}_{f,h} \cdot \boldsymbol{n}_f + \overset{\circ\circ}{\boldsymbol{v}}_{p,h} \cdot \boldsymbol{n}_p; \mu_h \sqrt{a}\big)_{\Gamma_{fp}}}{\|\overset{\circ\circ}{\boldsymbol{v}}_h\|_{\boldsymbol{V}}} \\
&= \frac{\big(\overset{\circ\circ}{\boldsymbol{v}}_{f,h} \cdot \boldsymbol{n}_f + \overset{\circ\circ}{\boldsymbol{v}}_{p,h} \cdot \boldsymbol{n}_p; \mu_h \sqrt{a}\big)_{\Gamma_{fp}}}{\|\overset{\circ\circ}{\boldsymbol{v}}_h\|_{\boldsymbol{V}}} \geq \beta_2 \|\mu_h\|_{\Lambda^h} \quad \text{by (3.40).}
\end{aligned}$$

Then,

$$\sup_{\boldsymbol{v}_h\in\boldsymbol{V}^h}\frac{b_f(w_{f,h},\boldsymbol{v}_{f,h})+b_p(w_{p,h},\boldsymbol{v}_{p,h})+\big(\boldsymbol{v}_{f,h}\cdot\boldsymbol{n}_f+\boldsymbol{v}_{p,h}\cdot\boldsymbol{n}_p;\mu_h\sqrt{a}\big)_{\Gamma_{fp}}}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}}$$
$$\geq\frac{\min\{\beta_1,\beta_2\}}{2}(\|w_h\|_W+\|\mu_h\|_{\Lambda^h})\geq\frac{\min\{\beta_1,\beta_2\}}{2}\|(w_h,\mu_h)\|_{W\times\Lambda^h}.$$

By choosing $\beta=\dfrac{\min\{\beta_1,\beta_2\}}{2}$, it can be written as

$$\inf_{(w_h,\mu_h)\in\boldsymbol{W}^h\times\Lambda^h}\sup_{\boldsymbol{v}_h\in\boldsymbol{V}^h}\frac{b_f(w_{f,h},\boldsymbol{v}_{f,h})+b_p(w_{p,h},\boldsymbol{v}_{p,h})+\big(\boldsymbol{v}_{f,h}\cdot\boldsymbol{n}_f+\boldsymbol{v}_{p,h}\cdot\boldsymbol{n}_p;\mu_h\sqrt{a}\big)_{\Gamma_{fp}}}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}\|(w_h,\mu_h)\|_{W\times\Lambda^h}}\geq\beta.$$

We completed the proof of Lemma 44. Subsequently, we use it for obtaining Lemma 43. □

Proof. (of Lemma 43): By simply taking $\boldsymbol{\xi}_{p,h}=0$ and $\boldsymbol{s}_{p,h}=0$ and from Lemma 44 we obtain the positive constant β satisfying (3.33). □

From the result of Lemma 44, we derive the third condition of Lemma 39 below,

Lemma 45

The bilinear forms ϕ_C and ϕ_{B^T} satisfy

$$\ker(\phi_C)\cap\ker(\phi_{B^T})=\{0\}.$$

Proof. The inf-sup condition (3.33) implies $\ker(\phi_{B^T})=\{0\}$ and we can easily see that $\{0\}\in\ker(\phi_C)$. Therefore this gives $\ker(\phi_C)\cap\ker(\phi_{B^T})=\{0\}$. □

We obtained completely the conditions in Lemma 44 and the discrete inf-sup condition. It is time to prove our main result (Theorem 13).

Proof. (of Theorem 13) The proof is inspired by Theorem 3.1 of [2].
By the DAE theory [26], if the matrix $s\mathbf{E}+\mathbf{H}$ is nonsingular for some $s\neq 0$ and the initial data is consistent, then the equation (3.30) has a solution. Lemma 42 and 45 guarantee that $s\mathbf{E}+\mathbf{H}$ with $s=1$ is invertible.
For proving the uniqueness, we assume that there are two solutions satisfying these equations (3.29) with the same initial conditions. Then their difference $(\widetilde{\boldsymbol{u}}_{f,h},\widetilde{\boldsymbol{\eta}}_{p,h},\widetilde{\boldsymbol{r}}_{p,h},\widetilde{\boldsymbol{u}}_{p,h},\widetilde{p}_{f,h},\widetilde{p}_{p,h},\widetilde{\lambda}_h)$ satisfies (3.29) with zero data. By taking

$$(\boldsymbol{v}_{f,h},\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h},\boldsymbol{v}_{p,h},w_{f,h},w_{p,h},\mu_h)=(\partial_t\widetilde{\boldsymbol{u}}_{f,h},\partial_t\widetilde{\boldsymbol{\eta}}_{p,h},\widetilde{\boldsymbol{r}}_{p,h},\widetilde{\boldsymbol{u}}_{p,h},\widetilde{p}_{f,h},\widetilde{p}_{p,h},\widetilde{\lambda}_h),\tag{3.41}$$

we obtain the energy equality

$$\mathcal{A}_1^N\big((\widetilde{\boldsymbol{\eta}}_{p,h},\widetilde{\boldsymbol{r}}_{p,h});(\partial_t\widetilde{\boldsymbol{\eta}}_{p,h},\partial_t\widetilde{\boldsymbol{r}}_{p,h})\big)+c_0(\partial_t\widetilde{p}_{p,h},\widetilde{p}_{p,h}\sqrt{a})_{\Omega_p}+a_p(\widetilde{\boldsymbol{u}}_{p,h},\widetilde{\boldsymbol{u}}_{p,h})+a_f(\widetilde{\boldsymbol{u}}_{f,h},\widetilde{\boldsymbol{u}}_{f,h}) \\ +a_{BJS}\big(\widetilde{\boldsymbol{u}}_{f,h},\partial_t\widetilde{\boldsymbol{\eta}}_{p,h},\partial_t\widetilde{\boldsymbol{r}}_{p,h};\widetilde{\boldsymbol{u}}_{f,h},\partial_t\widetilde{\boldsymbol{\eta}}_{p,h},\partial_t\widetilde{\boldsymbol{r}}_{p,h}\big)=0.$$

Because of $\int_D\varphi\partial_t\varphi=\frac{1}{2}\partial_t\|\varphi\|^2_{L^2(D)}$, we write the energy equality as

$$\frac{1}{2}\partial_t[\mathcal{A}_1^N\big((\widetilde{\boldsymbol{\eta}}_{p,h}(t),\widetilde{\boldsymbol{r}}_{p,h}(t));(\widetilde{\boldsymbol{\eta}}_{p,h}(t),\widetilde{\boldsymbol{r}}_{p,h}(t))\big)+c_0\|\widetilde{p}_{p,h}\|^2_{L^2(\Omega_p)}] \tag{3.43}$$

$$+a_p(\widetilde{\boldsymbol{u}}_{p,h},\widetilde{\boldsymbol{u}}_{p,h})+a_f(\widetilde{\boldsymbol{u}}_{f,h},\widetilde{\boldsymbol{u}}_{f,h})+|\widetilde{\boldsymbol{u}}_{f,h}-\alpha\big(\partial_t\widetilde{\boldsymbol{\eta}}_{p,h}-\frac{e}{2}\partial_t\widetilde{\boldsymbol{r}}_{p,h}\big)|^2_{a_{BJS}}=0. \tag{3.44}$$

where

$$|\widetilde{\boldsymbol{u}}_{f,h}-\alpha\big(\partial_t\widetilde{\boldsymbol{\eta}}_{p,h}-\frac{e}{2}\partial_t\widetilde{\boldsymbol{r}}_{p,h}\big)|^2_{a_{BJS}}=a_{BJS}\big(\widetilde{\boldsymbol{u}}_{f,h},\partial_t\widetilde{\boldsymbol{\eta}}_{p,h},\partial_t\widetilde{\boldsymbol{r}}_{p,h};\widetilde{\boldsymbol{u}}_{f,h},\partial_t\widetilde{\boldsymbol{\eta}}_{p,h},\partial_t\widetilde{\boldsymbol{r}}_{p,h}\big).$$

Integrating in time over $[0,t]$ for arbitrary $t\in(0,T]$, we get

$$\frac{1}{2}[\mathcal{A}_1^N\big((\widetilde{\boldsymbol{\eta}}_{p,h}(t),\widetilde{\boldsymbol{r}}_{p,h}(t));(\widetilde{\boldsymbol{\eta}}_{p,h}(t),\widetilde{\boldsymbol{r}}_{p,h}(t))\big)+c_0\|\widetilde{p}_{p,h}\|^2_{L^2(\Omega_p)}] \tag{3.45}$$

$$+\int_0^t[a_p(\widetilde{\boldsymbol{u}}_{p,h},\widetilde{\boldsymbol{u}}_{p,h})+a_f(\widetilde{\boldsymbol{u}}_{f,h},\widetilde{\boldsymbol{u}}_{f,h})+\big|\widetilde{\boldsymbol{u}}_{f,h}-\alpha\big(\partial_t\widetilde{\boldsymbol{\eta}}_{p,h}-\frac{e}{2}\partial_t\widetilde{\boldsymbol{r}}_{p,h}\big)\big|_{a_{BJS}}]ds=0. \tag{3.46}$$

Because of the coercivity of bilinear forms and $c_0\geq 0$, we derive that $\widetilde{\boldsymbol{\eta}}_{p,h}(t)=0$, $\widetilde{\boldsymbol{r}}_{p,h}(t)=0$, $\widetilde{\boldsymbol{u}}_{p,h}=0$ and $\widetilde{p}_{p,h}=0$ for all t in $[0,T]$. From the inf-sup condition (3.33) and (3.29a),

$$\begin{aligned}
&\beta\|(\widetilde{p}_h,\widetilde{\lambda}_h)\|_{W\times\Lambda^h}\\
\leq&\sup_{(\boldsymbol{v}_h,\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h})\in\boldsymbol{V}^h\times\mathcal{X}_p^h\times\mathcal{S}_p^h}\frac{b_f(\widetilde{p}_{f,h},\boldsymbol{v}_{f,h})+b_p(\widetilde{p}_{p,h},\boldsymbol{v}_{p,h})+\alpha b_p(\widetilde{p}_{p,h},\boldsymbol{\xi}_{p,h}+z\boldsymbol{s}_{p,h})+b_\Gamma\big(\boldsymbol{v}_h,\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h};\widetilde{\lambda}_h\big)}{\|(\boldsymbol{v}_h,\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h})\|_{\boldsymbol{V}\times\mathcal{X}_p\times\mathcal{S}_p}}\\
=&\sup_{(\boldsymbol{v}_h,\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h})\in\boldsymbol{V}^h\times\mathcal{X}_p^h\times\mathcal{S}_p^h}\{\big[-\mathcal{A}_1^N\big((\widetilde{\boldsymbol{\eta}}_{p,h},\widetilde{\boldsymbol{r}}_{p,h});(\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h})\big)-a_p(\widetilde{\boldsymbol{u}}_{p,h},\boldsymbol{v}_{p,h})\\
&\quad-a_f(\widetilde{\boldsymbol{u}}_{f,h},\boldsymbol{v}_{f,h})-a_{BJS}\big(\widetilde{\boldsymbol{u}}_{f,h},\partial_t\widetilde{\boldsymbol{\eta}}_{p,h},\partial_t\widetilde{\boldsymbol{r}}_{p,h};\boldsymbol{v}_{f,h},\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h}\big)\big]/\|(\boldsymbol{v}_h,\boldsymbol{\xi}_{p,h},\boldsymbol{s}_{p,h})\|_{\boldsymbol{V}\times\mathcal{X}_p\times\mathcal{S}_p}\}\\
=&\,0.
\end{aligned}$$

Hence, we conclude that $\widetilde{p}_{f,h}(t)=0$, $\widetilde{\lambda}_{p,h}(t)=0$ for all $t\in(0,T]$. Therefore the solution of (3.29) is unique. We finished the proof of Theorem 13. □

Conclusion: In this chapter, we carried out the fluid-structure interaction between incompressible flow and poroelastic shell structure. We used the Stokes equations for the incompressible free fluid and the Biot-Naghdi shell model for the poroelastic shell structure. In process of proving the well-posedness of this model (see Theorem 13), we had trouble imposing the conditions on the interface which are mass conservation, balance of stress and the Beavers-Joseph-Saffman conditions. For the existence, we adopted the theory of differential-algebraic equations (DAEs) (see Lemma 39) and for the uniqueness, we used the semi-discrete Galerkin method and the Lagrange multiplier method.

Chapter 4

Numerical simulations

In this chapter, we propose to simulate numerically the poroelastic shell models studied in the previous chapter. We follow the idea of Chaabane and Rivière [37] for the Biot poroelasticity system. Several approaches have been developed for the Biot poroelasticity system. Let us mention them:

- Implicit approach where the fully coupled system is solved;
- Loosely explicit where the mechanical response is only updated every few time steps;
- Iterative scheme where a set of equations is solved iteratively at every time step until a certain tolerance is reached;
- Sequential method where the system is completely decoupled and no iterations are needed.

In this work, we use the sequential approach based on a discontinuous Galerkin discretization.

4.1 Notation

Let us first introduce some notations. Let $\mathcal{T}_h$ be a shape-regular family of triangulations of ω. For a triangle $E \in \mathcal{T}_h$, let h_E be its diameter and set $h = \max_{E\in\mathcal{T}_h} h_E$. For any subdomain Δ of ω and for $k \geq 0$ we define by $\mathbb{P}_k(\Delta)$ the space of polynomials on Δ with degree $\leq k$. We suppose that the mesh is nondegenerate *i.e.* there exists a constant $\tau > 0$ independent of h, such that

$$\frac{h_E}{\rho_E} \leq \tau, \quad \forall E \in \mathcal{T}_h,$$

where h_E is the diameter of the element E and ρ_E is the diameter of the largest ball inside of E. Let $k_1 \geq 1$ and $k_2 \geq 1$ are two integers. We define the following finite element spaces

$$\begin{aligned} \boldsymbol{X}^h &= \{\boldsymbol{v} \in \mathcal{C}^0(\Omega)^d; \forall E \in \mathcal{T}_h, \boldsymbol{v}|_E \in \mathbb{P}_{k_1}(E)^d\}, & \boldsymbol{X}_0^h &= \{\boldsymbol{v} \in \boldsymbol{X}^h; \boldsymbol{v} = 0 \text{ on } \partial\Omega\}, \\ M^h &= \{q \in \mathcal{C}^0(\Omega); \forall E \in \mathcal{T}_h, q|_E \in \mathbb{P}_{k_2}(E)\}, & M_0^h &= \{q \in M^h;\ q = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

We endow the spaces $\boldsymbol{X}_0^h$ and M_0^h by the norms:

$$\|\boldsymbol{v}\|_X = \|\varepsilon(\boldsymbol{v})\|_{L^2(\Omega)},\ \forall \boldsymbol{v} \in \boldsymbol{X}_0^h,$$
$$\|q\|_M = \|\kappa^{1/2}\nabla q\|_{L^2(\Omega)},\ \forall q \in M_0^h.$$

4.2 Validation for the Biot poroelasticity system

In this section, we carry out the validation to the following problem

$$\partial_t(c_0 p + \alpha \mathrm{div}\boldsymbol{u}) - \kappa \mathrm{div}(\nabla p) = f \text{ in } \Omega \times [0, T], \tag{4.1}$$
$$-\nabla \cdot \sigma(\boldsymbol{u}) + \alpha \nabla p = \boldsymbol{g} \text{ in } \Omega \times [0, T]. \tag{4.2}$$

The system is completed by the following boundary and initial conditions

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{0}, & p &= 0, \quad \text{on } \partial\Omega \times [0, T], \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0, & p(0) &= p_0, \quad \text{on } \Omega. \end{aligned}$$

We define some bilinear forms:

$$\begin{aligned} a(p,q) &= (\kappa \nabla p, \nabla q)_\Omega,\ \forall p, q \in H_0^1(\Omega), \\ b(\boldsymbol{v}, q) &= -\alpha(\boldsymbol{v}, \nabla q)_\Omega,\ \forall \boldsymbol{v} \in \left(H_0^1(\Omega)\right)^d, \forall q \in H_0^1(\Omega), \\ c(\boldsymbol{u}, \boldsymbol{v}) &= \big(\sigma(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_\Omega,\ \forall \boldsymbol{u}, \boldsymbol{v} \in H_0^1(\Omega). \end{aligned}$$

We solve Eqs (4.1)-(4.2) at the discrete times $t_i = i\Delta t, i = 0, 1, \cdots, N$ where $\Delta t > 0$ denotes the time step and $t_N = T$. Thus a weak form the Biot poroelasticity system reads as:

$$(P) \begin{cases} \text{Find } (p, u) \in \mathbb{M}_0 \times \boldsymbol{X}_0 \text{ such that:} & & \\ c_0(p', q)_\Omega + a(p, q) + b(\boldsymbol{u}', q) & = (f, \boldsymbol{v})_\Omega, & \forall \boldsymbol{v} \in \boldsymbol{X}_0, \\ b(\boldsymbol{v}, q) + c(\boldsymbol{u}, \boldsymbol{v}) & = (g, q)_\Omega, & \forall q \in M_0. \end{cases}$$

4.2.1 The numerical approximation

The weak form above is approximated by a finite element approximation in space and a bacward Euler scheme in time, *i.e*:

$$(P_h) \begin{cases} \text{Find } (p_h, u_h) \in \ \text{such that:} & & \\ c_0(p_h', q_h)_\Omega + a(p_h, q_h) + b(\boldsymbol{u}_h', q_h) & = (f, \boldsymbol{v}_h)_\Omega, & \forall \boldsymbol{v}_h \in \boldsymbol{X}_0^h, \\ b(\boldsymbol{v}_h, q_h) + c(\boldsymbol{u}_h, \boldsymbol{v}_h) & = (g, q_h)_\Omega, & \forall q_h \in M_0^h. \end{cases}$$

The well-posedness of the numerical scheme problem (P_h) and the error analysis of the method are proved in [37].

4.2.2 The splitting algorithm

The decoupling approach consists of two step:

- Solve for the pressure p_h^{n+1} in the mass balance equation (4.1) by the time-lagging the displacement.
- Use p_h^{n+1} in the momentum equation (4.2) to solve for the displacement $\boldsymbol{u}_h^{n+1}$.

The functions p_h^{n+1} and $\boldsymbol{u}_h^{n+1}$ denote the value of the discrete solutions p_h and $\boldsymbol{u}_h$ at time t_{n+1} and similar to f and $\boldsymbol{g}$, *i.e.* $f^n = f(t_n)$ and $\boldsymbol{g}^n = \boldsymbol{g}(t_n)$.

Following the work of Chaabane and Rivière [37], we introduce the numerical scheme: For $1 \leq n \leq N-1$, solve the following equations:
Step 1: Given $u_h^n \in \boldsymbol{X}_0^h$ and $\boldsymbol{u}_h^{n-1} \in \boldsymbol{X}_0^h$, find $p_h^{n+1} \in M_0^h$ such that

$$\big(c_0 \frac{p_h^{n+1} - p_h^n}{\Delta t}, q\big)_\Omega + a(p_h^{n+1}, q) + b(\frac{\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}}{\Delta t}, q) = (f^{n+1}, q)_\Omega \;\; \forall q \in M_0^h. \tag{4.3}$$

Step 2: Find $\boldsymbol{u}_h^{n+1} \in \boldsymbol{X}_0^h$ such that

$$c(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}) - b(\boldsymbol{v}, p_h^{n+1}) + \gamma_1\big(\frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\Delta t}\big)_\Omega - \gamma_1\big(\frac{\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}}{\Delta t}\big)_\Omega = (\boldsymbol{g}^{n+1}, \boldsymbol{v})_\Omega, \;\; \forall \boldsymbol{v} \in \boldsymbol{X}_0^h, \tag{4.4}$$

where $\gamma_1 > 0$ is a stabilization parameter.
The issue of approximating the primary variables at the initial time step was addressed in [36] where Chaabane and Rivière showed numerically that by choosing a small initial Δt_{init}, the convergence rates do not deteriorate. We approximate p_h^1 and $\boldsymbol{u}_h^1$ in two following initial steps:
Initial step 1: Find $p_h^1 \in M_0^h$ such that

$$\big(c_0 \frac{p_h^1 - p_h^0}{\Delta t_{init}}, q\big)_\Omega + a(p_h^1, q) = (f^1, q)_\Omega, \quad \forall q \in M_0^h. \tag{4.5}$$

Initial step 2: Find $\boldsymbol{u}_h^1 \in \boldsymbol{X}_0^h$ such that

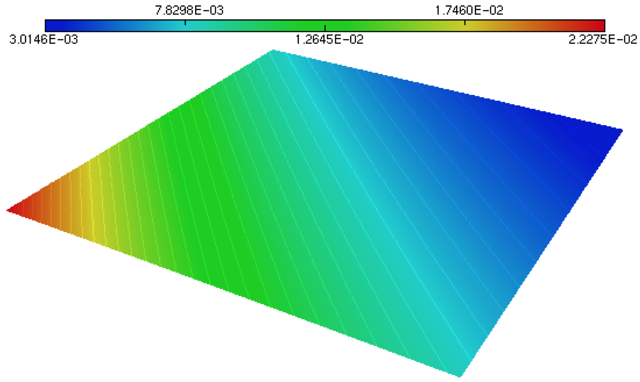
$$c(\boldsymbol{u}_h^1, \boldsymbol{v}) - b(\boldsymbol{v}, p_h^1) = (\boldsymbol{g}^1, \boldsymbol{v}) \;\; \forall \boldsymbol{v} \in \boldsymbol{X}_0^h. \tag{4.6}$$

4.2.3 Numerical tests

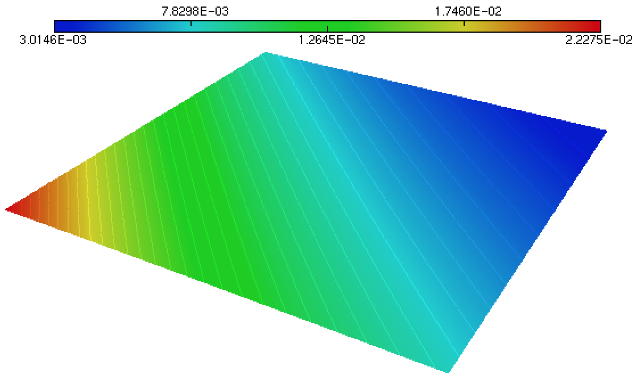
In oder to simulate the numerical approximation, we chose the domain Ω is the unit square $\Omega = (0,1) \times (0,1)$ and $\Gamma = \partial\Omega = [0,1] \times \{0;1\} \cup \{0;1\} \times [0,1]$.
The source functions and the exact solutions are given as follows:

$$\begin{aligned}
&f(x,y,t) = 30c_0 t^2 e^{(x+y)/10} + 0.1\alpha(e^t + 6ty^2) - 0.2\kappa(1+t^3)e^{(x+y)/10},\\
&g(x,y,t) = \big(\alpha(1+t^3)e^{(x+y)/10} - 0.2\mu y e^t, -0.2\mu x t^2 - 1.8yt^2 + \alpha(1+t^3)e^{(x+y)/10}\big)^T,\\
&\boldsymbol{u}(x,y,t) = 0.1\big(e^t(y^3+x), t^2(x^3+y^3)\big)^T, \; p(x,y,t) = 10e^{(x+y)/10}(1+t^3).
\end{aligned}$$

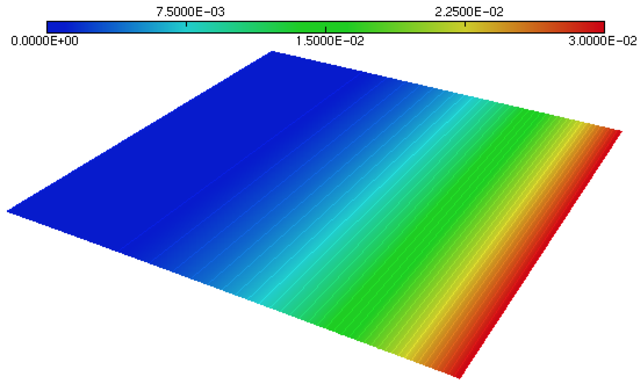
The following parameters $\kappa = 1,\ \alpha = 1,\ c_0 = 1,\ \lambda = 1,\ \mu = 1,\ \gamma = \dfrac{3\alpha^2}{\kappa}$ and initial time step $\Delta t_{int} = 10^{-6}$ are used. The figures 4.1 present the numerical and exact solution of Biot model.



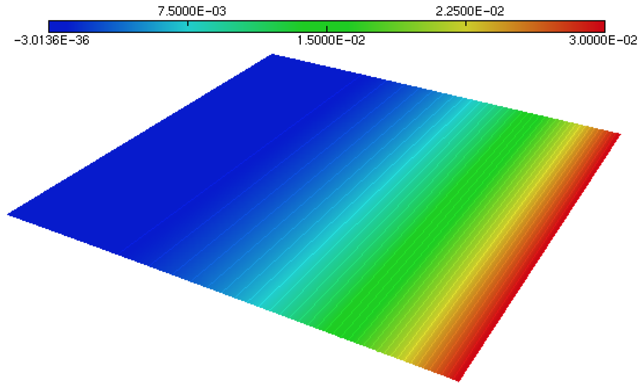
(a) Exact displacement component u_1.



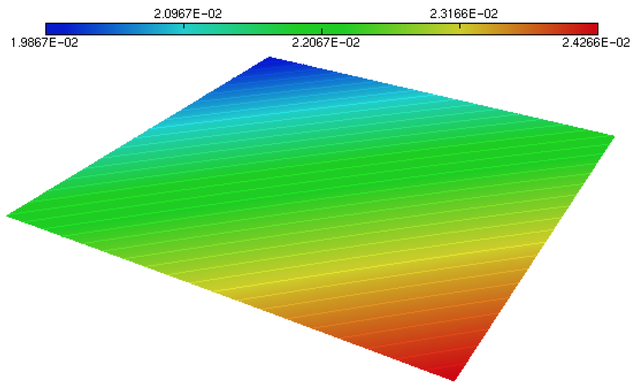
(b) Approximate displacement component u_1.



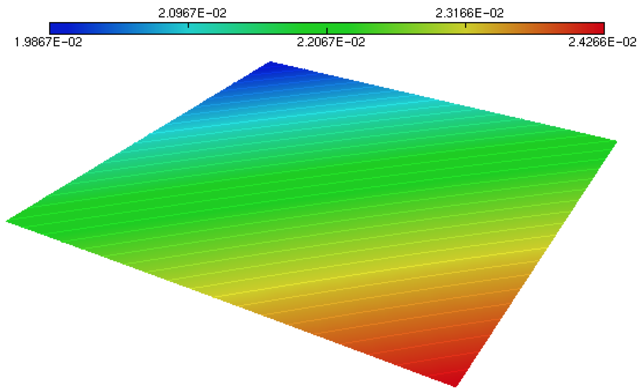
(c) Exact displacement component u_2.



(d) Approximate displacement component u_2.



(e) Exact pressure p



(f) Approximate pressure p.

Figure 4.1: Exact and approximate solution

We also obtain the tables which describes the error between the approximate solution and the exact solution.

Pressure rates when $k_1, k_2 = 1$:

h	Δt	$\|p^N - p_h^N\|_{L^2(\Omega)}$	Rate	$\|\nabla(p^N - p_h^N)\|_{L^2(\Omega)}$	Rate
1/2	1/2	3.33e - 01		1.68e+00	
1/4	1/16	1.48e - 01	1.17	6.9e - 01	1.28
1/8	1/32	4e - 02	1.89	1.84e - 01	1.91
1/16	1/128	1e - 02	2	4.68e - 02	1.98

Displacement rates when $k_1, k_2 = 1$:

h	Δt	$\|\boldsymbol{u}^N - \boldsymbol{u}_h^N\|_{L^2(\Omega)}$	Rate	$\|\nabla(\boldsymbol{u}^N - \boldsymbol{u}_h^N)\|_{L^2(\Omega)}$	Rate
1/2	1/2	1.3e - 02		7.23e - 02	
1/4	1/16	3.47e - 03	1.9	2.46e - 02	1.56
1/8	1/32	9.76e - 04	1.83	6.87e - 03	1.84
1/16	1/128	2.54e - 04	1.94	1.78e - 03	1.95

Pressure rates when $k_1, k_2 = 2$:

h	Δt	$\|p^N - p_h^N\|_{L^2(\Omega)}$	Rate	$\|\nabla(p^N - p_h^N)\|_{L^2(\Omega)}$	Rate
1/2	1/2	4.29e - 01		1.95e+00	
1/4	1/16	1.58e - 01	1.44	7.18e - 01	1.44
1/8	1/32	4.08e - 02	1.95	1.86e - 01	1.95
1/16	1/128	1e - 02	2	4.69e - 02	1.99

Displacement rates when $k_1, k_2 = 2$:

h	Δt	$\|p^N - p_h^N\|_{L^2(\Omega)}$	Rate	$\|\nabla(p^N - p_h^N)\|_{L^2(\Omega)}$	Rate
1/2	1/2	1.65e - 02		9.15e - 02	
1/4	1/16	3.98e - 03	2	2.75e - 02	1.73
1/8	1/32	1e - 03	1.99	7.1e - 03	1.95
1/16	1/128	2.57e - 04	1.96	1.79e - 03	1.99

Comment: We vary the time step and mesh size accordingly to compute the order of convergence of the method and report the results in the previous tables. We observe that the method exhibits optimal order of convergence.

4.3 Some simulations of the Biot-Naghdi Model

In this section, we apply the algorithm in the previous section to simulate the Biot-Naghdi model in Theorem 9. For the numerical part, we suppose that p does not depend on z, *i.e.*, $p = p(\boldsymbol{x})$. Hence, we simulate our model on two-dimensional domain ω. Recall that $\partial\omega = \gamma$. Using the same mesh in the previous section, we introduce the discrete space of displacements, rotations and pressures

$$\mathcal{X}_N^h = \Big\{ \big(\boldsymbol{v}, \boldsymbol{s} = (s_\alpha)\big) \in C^0(\omega, \mathbb{R}^3) \times [C^0(\omega)]^2;\ \forall E \in \mathcal{E}_h,\\ \big(\boldsymbol{v}, \boldsymbol{s} = (s_\alpha)\big)\big|_E \in \mathbb{P}_1(E, \mathbb{R}^3) \times \mathbb{P}_1(E)^2,\ \boldsymbol{v} = s_\alpha = 0 \text{ on } \gamma \Big\} \subset \mathcal{X}_N,$$

$$\mathcal{W}_N^h = \big\{ q \in C^0(\omega);\ \forall E \in \mathcal{E}_h,\ q|_E \in \mathbb{P}_1(E),\ q = 0 \text{ on } \gamma \big\} \subset \mathcal{W}_N.$$

Thus the discrete problem reads: Find $(\boldsymbol{u}_h, \boldsymbol{r}_h) \in \mathcal{X}_N^h$ and $p_h \in \mathcal{W}_N^h$ such that

$$\mathcal{A}_1^N[(\boldsymbol{u}_h, \boldsymbol{r}_h); (\boldsymbol{v}_h, \boldsymbol{s}_h)] - \alpha e\big(p_h, \mathrm{div}\boldsymbol{v}_h\sqrt{a}\big)_\omega = e\big(\boldsymbol{f}, \boldsymbol{v}_h\sqrt{a}\big)_\omega, \ \forall(\boldsymbol{v}_h, \boldsymbol{s}_h) \in \mathcal{X}_N^h, \quad (4.7a)$$

$$c_0(p_h', q_h\sqrt{a})_\omega + \alpha(\mathrm{div}\boldsymbol{u}_h', q_h\sqrt{a})_\omega + \frac{\kappa}{\eta}\big(\nabla p_h, \nabla q_h\sqrt{a}\,\big)_\omega = \big(g, q_h\sqrt{a}\big)_\omega, \ \forall q_h \in \mathcal{W}_N^h, \quad (4.7b)$$

$$p_h(0) = p_{h,0}. \quad (4.7c)$$

We solve Eqs 4.7 at the discrete times $t_i = i\Delta t, i = 0, 1, \cdots, N$ where $\Delta t > 0$ denotes the time step and $t_N = T$. The decoupling approach consists of two step:

- Solve for the pressure p_h^{n+1} in the mass balance equation (4.7b) by the time-lagging the displacement.
- Use p_h^{n+1} in the momentum equation (4.7a) to solve for the displacement $\boldsymbol{u}_h^{n+1}$.

The functions $\boldsymbol{u}_h^{n+1}$, $\boldsymbol{r}_h^{n+1}$ and p_h^{n+1} denote the value of the discrete solutions $\boldsymbol{u}_h$, $\boldsymbol{r}_h$ and p_h at time t_{n+1} and similar to f and $\boldsymbol{g}$, *i.e.* $f^n = f(t_n)$ and $\boldsymbol{g}^n = \boldsymbol{g}(t_n)$. Now, we introduce the numerical scheme: For $1 \leq n \leq N-1$, solve the following equations:
Step 1: Given $(\boldsymbol{u}_h^n, \boldsymbol{r}_h^n) \in \mathcal{X}_N^h$ and $(\boldsymbol{u}_h^{n-1}, \boldsymbol{r}_h^{n-1}) \in \mathcal{X}_N^h$, find $p_h^{n+1} \in \mathcal{W}_N^h$ such that

$$\frac{c_0}{\delta t}\left(p_h^{n+1} - p_h^n, q\right)_\omega + \frac{\kappa}{\eta}\left(\nabla p_h^{n+1}, \nabla q\right)_\omega = \left(g^{n+1}, q\right)_\omega - \frac{\alpha}{\delta t}\left(\mathrm{div}\big(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\big), q\right)_\omega.$$

Step 2: Find $(\boldsymbol{u}_h^{n+1}, \boldsymbol{r}_h^{n+1}) \in \mathcal{X}_N^h$ such that

$$\begin{aligned}\mathcal{A}_1^N[(\boldsymbol{u}_h^{n+1}, \boldsymbol{r}_h^{n+1}); (\boldsymbol{v}, \boldsymbol{s})] &+ \frac{\gamma_1}{\delta t}\big(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n, \boldsymbol{v}\big)_\omega - \frac{\gamma_1}{\delta t}\big(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{v}\big)_\omega \\ &+ \frac{\gamma_1}{\delta t}\big(\boldsymbol{r}_h^{n+1} - \boldsymbol{r}_h^n, \boldsymbol{s}\big)_\omega - \frac{\gamma_1}{\delta t}\big(\boldsymbol{r}_h^n - \boldsymbol{r}_h^{n-1}, \boldsymbol{s}\big)_\omega = \alpha e\big(p_h^{n+1}, \mathrm{div}\boldsymbol{v}\big)_\omega + e\big(\boldsymbol{f}^{n+1}, \boldsymbol{v}\big)_\omega.\end{aligned}$$

We approximate p_h^1, $(\boldsymbol{u}_h^0, \boldsymbol{r}_h^0)$ and $(\boldsymbol{u}_h^1, \boldsymbol{r}_h^1)$ in two following initial steps:
Initial step 0: Find $(\boldsymbol{u}_h^0, \boldsymbol{r}_h^0) \in \mathcal{X}_N^h$ such that

$$\mathcal{A}_1^N[(\boldsymbol{u}_h^0, \boldsymbol{r}_h^0); (\boldsymbol{v}, \boldsymbol{s})] = \alpha e\left(p_h^0, \mathrm{div}\boldsymbol{v}\right)_\omega + e\big(\boldsymbol{f}^0, \boldsymbol{v}\big)_\omega.$$

Initial step 1: Find $p_h^1 \in \mathcal{W}_0^h$ such that

$$\frac{c_0}{\delta t_{int}}\left(p_h^1 - p_h^0, q\right)_\omega + \frac{\kappa}{\eta}\left(\nabla p_h^1, \nabla q\right)_\omega = \left(g^1, q\right)_\omega.$$

Initial step 2: Find $(\boldsymbol{u}_h^1, \boldsymbol{r}_h^1) \in \mathcal{X}_N^h$ such that

$$\mathcal{A}_1^N[(\boldsymbol{u}_h^1, \boldsymbol{r}_h^1); (\boldsymbol{v}, \boldsymbol{s})] = \alpha e\left(p_h^1, \mathrm{div}\boldsymbol{v}\right)_\omega + e\big(\boldsymbol{f}^1, \boldsymbol{v}\big)_\omega.$$

From the above scheme, we now performe the numerical experiment on the finite element code FreeFem++. We first consider a hyperbolic paraboloid shell. The reference domain ω is the square

$$\omega = \{(x, y); |x| + |y| \leq \sqrt{2}b\},$$

as illustrated in [6](§1.3.3 & §2.4.2) and the chart $\boldsymbol{\varphi}$ is defined by

$$\boldsymbol{\varphi}(x, y) = \left(x, y, \frac{c}{b^2}(x^2 - y^2)\right)^T.$$

However, the symmetry properties of the problem allows us to solve it only on the traingle ω' with vertices $(0, 0)^T$, $(b, 0)^T$ and $(0, b)^T$. We choose here

$$b = 50\,\text{cm},\ c = 10\,\text{cm},\ c_0 = 1,\ \kappa = 1,\ \alpha = 0.8,\ \eta = 1, g = 0 \text{ and } \boldsymbol{f} = (0, 0, -0.01)^T.$$

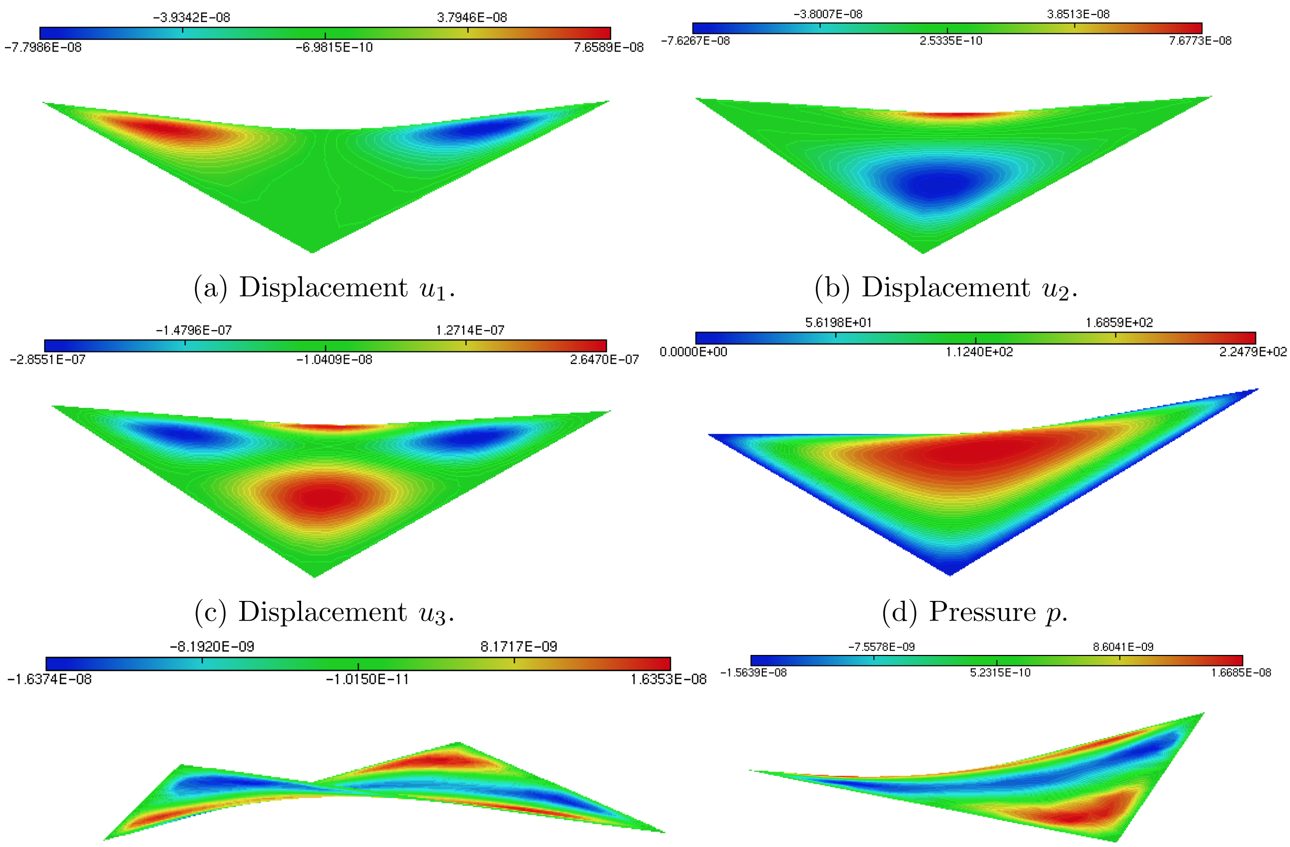
The thickness of the shell is $e = 0.8$ cm. We assume that the shell is clamped on the whole boundary, i.e.

$$\boldsymbol{u}_h = \boldsymbol{r}_h = p = 0 \text{ on } \partial\omega.$$

We refer to [23](§6.2) for the artificial conditions issued from the symmetry conditions. The mechanical data are

$$E = 2.85 \times 10^9\,\text{Pa}, \qquad \nu = 0.4.$$

Figure 4.2 presents the numerical solution of Biot-Naghdi model on the hyperbolic paraboloid shell:



(a) Displacement u_1. (b) Displacement u_2.

(c) Displacement u_3. (d) Pressure p.

(e) Rotation r_1. (f) Rotation r_2

Figure 4.2: Solution on a hyperbolic paraboloid shell

Next, we consider an other hyperbolic paraboloid shell where the boundary $\varphi(\partial\omega)$ is not included in a straight line. The reference domain ω of the midsurface is given by

$$\omega =]-R, R[\times]-R, R[,$$

and the chart $\boldsymbol{\varphi}$ is defined by

$$\boldsymbol{\varphi}(x, y) = \left(x, y, \frac{c}{b^2}(x^2 - y^2)\right)^T.$$

We take here

$$R = 50\sqrt{2}\,\text{cm},\ b = 50,\ c = 10,\ c_0 = 1,\ \kappa = 1,\ \alpha = 0.8,\ \eta = 1, g = 0 \text{ and } \boldsymbol{f} = (0, 0, -0.01)^T.$$

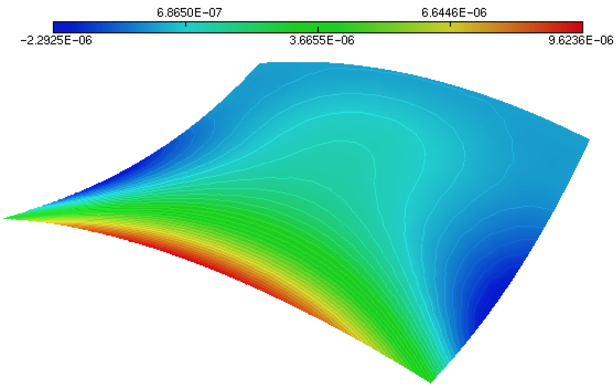
The thickness of the shell is $e = 0.8$ cm. The mechanical data are

$$E = 2.85 \times 10^9\,\text{Pa}, \qquad \nu = 0.4.$$

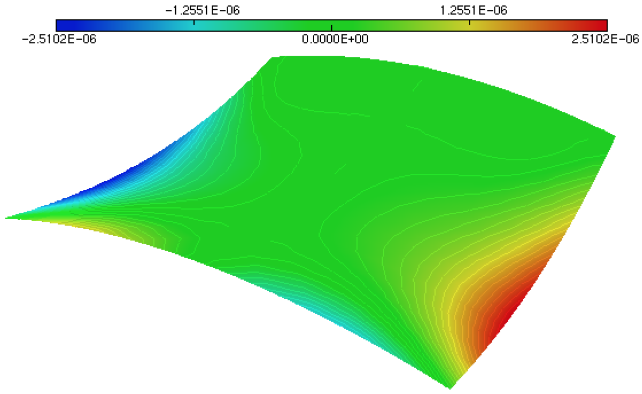
The shell is submitted to a uniform downward pressure. Concerning boundary conditions, we consider the case of hard clamping on the edge

$$\gamma_0 = \{-R\}\times]-R, R[,$$

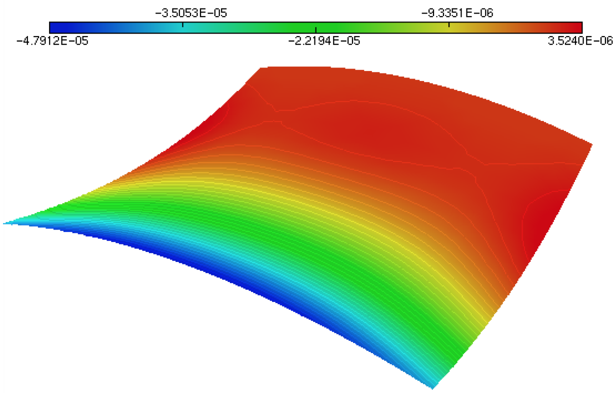
and the shell is free on its remaining edges.
Figure 4.3 presents the numerical solution of Biot-Naghdi model on the hyperbolic paraboloid shell:



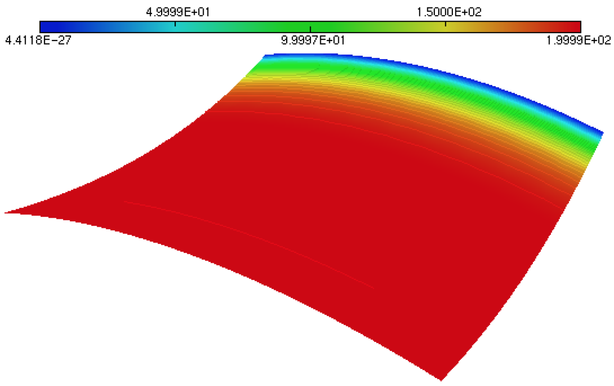
(a) Displacement u_1.



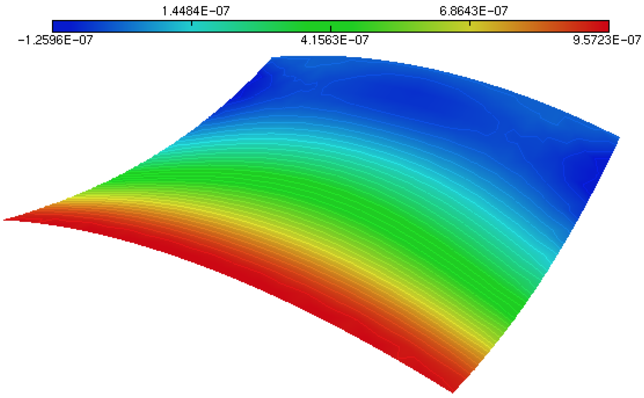
(b) Displacement u_2.



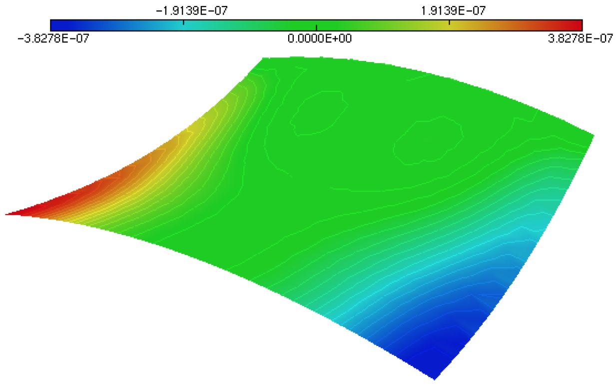
(c) Displacement u_3.



(d) Pressure p.



(e) Rotation r_1.



(f) Rotation r_2

Figure 4.3: Solution on the hyperbolic paraboloid shell

Comment: With the simulations on two different domains, we need that c_0 has to be small enough. We observe that the boundary conditions influence our solutions significantly.

Conclusion and Perspectives

In this work, we have done the derivation and the well-posedness of the weak coupled formulation between shell models and Biot model. We made of using two shell models, Naghdi and Koiter, which are respectively respect to two displacement spaces, Kirchhoff-Love and Reissner-Mindlin. In the process of proving the well-posedness, we used the Banach-Nečas-Babuška theorem, splitting method, the semi-discrete Galerkin method and the theory of initial value problem for linear systems. Afterwards, we derived the fluid-structure interaction between incompressible flow and poroelastic shell structure. We use the Stokes system for the incompressible free fluid and the Biot-Naghdi poroelastic shell model for the poroelastic shell structure. In process of proving the well-posedness of this model, we imposed the conditions on the interface which are mass conservation, balance of stress and the Beavers-Joseph-Saffman conditions. Therefore, the Lagrange multiplier method is employed to impose weakly these conditions. We assume that the boundaries and the interface between the fluid and the poroelastic material are fixed. The proof proceeds by constructing a semi-discrete Galerkin approximations, deriving the discrete inf-sup condition and adopting the theory of differential-algebraic equations (DAEs). At last, we simulated our results in FreeFem++, a high level, free software, finite element package.

There are some open problems inspired from our results we are able to develop. Firstly, the couple between Stokes and Koiter's model (Blouza, Al Alaoui, Mani – [25]) could be established similar to Biot-Naghdi model. Secondly, remind that we assumed that the domains of fluid and poroelastic material are fixed in the fluid structure interaction model, *i.e.*, the domains do not depend on time. Alternatively, we could extend our results to the time dependent domains $\Omega_* = \Omega_*(t)$. You can see some fluid structure interaction models with time dependent domain in [27, 3]. Finally, for numerical simulation, we could consider the simulations of the Stokes-Biot-Naghdi and Biot-Koiter models.

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