UNIVERSITÉ PARIS 13

École Doctorale Galilée
Laboratoire d'Analyse, Géométrie et Applications

THÈSE DE DOCTORAT

présentée pour obtenir le grade de

DOCTEUR EN SCIENCES DE L'UNIVERSITÉ PARIS 13

Discipline : Mathématiques
par

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Formation de singularités en temps fini pour les équations aux dérivées partielles non symétriques ou non variationnelles

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Sous la direction de **Hatem ZAAG**

Soutenue à *Villetaneuse, le 28 Juin 2019*

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Remerciements

Avant d'aborder le contenu de cette thèse, je souhaiterais écrire ici quelques mots évoquant la période de sa préparation. C'est pour moi l'occasion d'exprimer ma profonde gratitude à toute personne ayant contribué de près ou de loin à l'élaboration de cet humble travail.

Je tiens en tout premier lieu à adresser mes remerciements les plus sincères à mon directeur de thèse, Monsieur Hatem Zaag. En particulier, il a dirigé mes travaux avec beaucoup de disponibilité et de bienveillance. En plus de ça, il m'a beaucoup inspiré dans la recherche. Avec son ingéniosité et son expérience, il m'a aidé à formidablement surmonter plusieurs challenges dans ma thèse. Plus important encore, il était un professeur de la vie avec ses conseils utiles et son enthousiasme. Travailler avec lui fût pour moi un grand honneur et une occasion de profiter de son expérience et de sa gentillesse.

Je suis également très reconnaissant aux rapporteurs Madame Monica Musso et Monsieur Stefan Le Coz qui ont bien voulu regarder de très près mon travail, et dont les conseils ont indéniablement amélioré le manuscrit. Ma gratitude va également à Madame Nejla Nouaili auprès de qui j'ai beaucoup appris sur l'explosion en temps fini, en particulier sur l'équation complexe de Ginzburg-Landau (CGL). Je la remercie d'avoir accepté de faire partie de mon jury de thèse. Je tiens également à remercier Madame Hajer Bahouri et Monsieur Philippe Souplet qui me font l'honneur de siéger dans mon jury.

À cette occasion, je souhaiterais ensuite exprimer ma reconnaissance aux professeurs à l'institut de Mathématiques du Vietnam qui m'ont aidé à trouver l'opportunité de faire ma thèse à l'université Paris 13. Je remercie particulièrement monsieur Ho Hai Phung, qui m'a soutenu beaucoup dans la recherche d'une bourse de thèse à Paris. À côté de ça, je voudrais remercier aussi les professeurs: Viet Dung Nguyen, Minh Tri Nguyen, Minh Chuong Nguyen, Bich Van Nguyen, Trieu Duong Pham dont les cours m'ont marqué lorsque j'étais à Hanoi.

Je ne peux pas oublier d'exprimer ma gratitude aux professeurs, collègues à l'université Paris 13. En particulier, Prof. Jean-Stéphane Dershin, Isabelle Barbotin, Jean-Philippe Dru, Yolande Jimenez et Frédéric Manikcaros.

Je tiens également à remercier vivement Van Tien Nguyen, un autre ancien étudiant de mon directeur de thèse. Il a toujours été heureux de m'aider dans mes travaux avec ses merveilleuses expériences.

Par ailleurs, je suis très reconnaissant aux amis proches comme Madame Phuong, Diep, Thi, Nhat, Son, Hoa, Hieu, Viet, Kien, Phong et tous mes amis de l'université Paris 13.

Ensuite, je veux donner ma reconnaissance et mon affection à mes camarades de promotion au Master International de Hanoi en 2014-2015: Phuong Thuy, Thi Thuy, Thieu, Tue, Thinh, Tam, Huong, Hanh, Nga, Hoang, Phuong. Mes remerciements vont également à Madame Hanh qui m'a appris à prononcer les premiers mots en français et qui m'a beaucoup aidé lorsque j'ai été à Hanoi.

Ma thèse n'aurait pas pu être complétée sans le soutien du programme INSPIRE de Sorbonne Paris Cité, financé par le programme de recherche et d'innovation Horizon 2020 de l'Union européenne, au titre de la convention de subvention Marie Skłodowska-Curie No 665850. Par ailleurs, un grand merci amical à la Fondation Sciences Mathématiques de

Paris qui a financé mes études dans le cadre de la Paris Graduate School of Mathematics en 2015-2016.

Enfin, mes plus grands remerciements vont à ma famille: ma mère, ma femme et ma petite sœur. Elles ont toujours été à mes côtés dans les moments difficiles.

L'été 2019 *à Villetaneuse*

À la mémoire de mon père

À ma mère, ma femme et ma petite sœur

Résumé

Formation de singularités en temps fini pour les équations aux dérivées partielles non symétriques ou non variationnelles

Dans le cadre de cette thèse, nous nous intéresserons à la formation de singularités en temps fini pour les équations d'évolution de type parabolique. En particulier, nous nous concentrons sur l'étude des deux phénomènes principaux suivants : *l'explosion* et *l'extinction* en temps fini. Dans cette thèse, nous considérons les équations suivantes :

$$\partial_t u = \Delta u + |u|^{p-1} u \ln^\alpha(2+u^2), p > 1, \alpha \in \mathbb{R} \text{ et } u : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R}, \tag{1}$$

$$\partial_t u = \Delta u + u^p, p > 1 \text{ et } u : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{C}, \tag{2}$$

$$\partial_t u = \Delta u + \frac{\lambda}{(1-u)^2 \left(1 + \gamma \int_\Omega \frac{1}{1-u} dx\right)^2}, u : (x,t) \in \Omega \times [0,T) \to [0,1), \tag{3}$$

où Ω est un domaine borné de classe C^2 dans $\mathbb{R}^N$ et λ, γ sont positifs.

Ces modèles se rapportent à plusieurs phénomènes naturels. En particulier, l'équation (3) modélise un système micro électro-mécanique (MEMS).

Dans ce travail, nous avons construit des solutions explosives (pour (1) et des (2)) et des solutions avec extinction pour (3). En plus de ça, nous décrivons le comportement asymptotique des solutions autour du point singulier.

Comme cadre pour notre travail, nous utilisions celui des *variables auto-similaires* qui a été introduit par Giga et Kohn dans *CPAM* 1985. Nous obtenons les résultats en utilisant une réduction en dimension finie du problème et un argument topologique qui a été notamment introduit par Bressan, Bricmont et Kupiainen ainsi que par Merle et Zaag.

Clairement, notre travail n'est pas une simple adaptation des travaux cités ci-haut. En effet, nos modèles, par leur proximité avec les applications, sortent du cadre idéal considéré dans les travaux pionniers. En particulier, l'équation (1) n'est pas invariante par changement d'échelle, alors que (2) n'admet pas de structure variationelle. Quant à (3), la présence du terme intégral (donc non-local) nous oblige à une manipulation plus délicate. En fait, nous avons atteint nos objectifs grâce à quelques nouvelles idées. Plus précisément, pour (2), nous effectuons un contrôle délicat de la solution afin qu'elle reste dans un domaine où la nonlinéarité est défine sans ambiguïté. Pour (3), nous contrôlons l'oscillation du terme non-local afin qu'il reste assez petit et nous en d'éduisons sa convergence.

Mots clés: *équation de type parabolique, équation des MEMS, explosion en temps fini, extinction en temps fini, profil à l'explosion, explosion de type I, comportement asymptotique.*

Abstract

Finite time singularity formation for non symmetric or non variational partial differential equations

In the context of this thesis, we are interested in finite time singularity formation for non symmetric or non variational partial differential equations of parabolic type. In particular, we mainly focus on the following two phenomena: *blowup* and *quenching* (touch-down) in finite time. In this thesis, we aim at studying the following equations:

$$\partial_t u = \Delta u + |u|^{p-1} u \ln^{\alpha}(2+u^2), p > 1, \alpha \in \mathbb{R} \text{ et } u : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R}, \quad (4)$$

$$\partial_t u = \Delta u + u^p, p > 1 \text{ et } u : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{C}, \quad (5)$$

$$\partial_t u = \Delta u + \frac{\lambda}{(1-u)^2 \left(1 + \gamma \int_\Omega \frac{1}{1-u} dx\right)^2}, u \colon (x,t) \in \Omega \times [0,T) \to [0,1), \quad (6)$$

where Ω is a C^2 bounded domain in $\mathbb{R}^N$ and λ, γ are positive constants.

These models are closely related to many common phenomena in nature. In particular, equation (6) is a model for Micro Electro Mechanical Systems (MEMS).

In this work, we construct blowup solutions to (4) and (5) and solutions with extinction to (6). In addition to that, we describe the asymptotic behavior of these solutions around the singular point.

We use in this thesis the framework of *similarity variables*, introduced by Giga and Kohn in CPAM 1985. We finally derive the results by using a reduction to a finite dimensional problem and a topological argument which was introduced in particular by Bressan, Bricmont and Kupiainen, and also Merle and Zaag.

Clearly, our work is not a simple adaptation of the works cited above. Indeed, our models, by their proximity to applications, are outside the ideal framework considered in pioneering works. In particular, equation (4) is not scaling-invariant, whereas (5) does not admit variational structure. As for (6), the presence of the integral term (non-local term) requires us to treat this term more delicately. In fact, we have achieved our goals thanks to some new ideas. More precisely, for (5), we carry out a delicate control of the solution so that it always stays in the domain where the nonlinearity is defined with no ambiguity. For (6), we control the oscillation of the non-local term to keep it small enough, and this allows us to deduce its convergence.

Keywords: *Parabolic equation, MEMS model, finite time blowup, touch-down phenomenon, blowup profile, type I blowup, asymptotic behavior.*

Contents

Introduction

Science is a differential equation and religion is a boundary condition

Alan Turing

I. Modeling nature by parabolic PDE

In the age of science and technology, mathematics strongly shows us its influence in our life. Particularly, there is a wide variety of phenomena which have been mathematically modeled by partial differential equations (PDE) such as: heat transfer, propagation of waves, electrodynamics, fluid dynamics, elasticity, quantum mechanics and so on. The more we understand these equations, the better we know about the corresponding phenomena.

More specifically, the class of parabolic PDE is important in modeling nature. As many authors did earlier, we are interested in this thesis in reaction-diffusion systems of the following type

$$\begin{cases} \partial_t u &= D \cdot \Delta u + F(u, \nabla u, \int_\Omega g(u)dx) \text{ in } & \Omega \times [0,T), \\ u &= 0 \qquad\qquad \text{on} & \partial\Omega \times [0,T), \\ u(.,0) &= u_0 \qquad\qquad \text{in} & \Omega, \end{cases} \tag{7}$$

where $u : (x,t) \in \Omega \times [0,T) \mapsto \mathbb{K}^M$; $u_0 : x \in \Omega \mapsto \mathbb{K}^M$; $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$; Ω is an open set of $\mathbb{R}^N$; $g : \mathbb{K}^M \to \mathbb{K}$ is continuous and $F : D_F \subset \mathbb{K}^L \to \mathbb{K}^M$ is continuous on its domain. In addition to that, we note that $\nabla u = (\partial_{x_1} u, ..., \partial_{x_n} u)$, $\Delta u = \sum_{j=1}^N \partial^2_{x_j^2} u$ and $D = (D_{i,j})_{i,j \le N}$ is a diagonal matrix of diffusion coefficients. Note that when $\Omega = \mathbb{R}^N$, there is no boundary condition in (7).

Reaction-diffusion systems are mathematical models which correspond to many physical, chemical and biological phenomena. For more details about the applications of these models, we kindly address the readers to some representative works:

- *The combustion phenomenon*: We have Bebernes and Eberly [3]; Bebernes and Kassoy [4]; Galaktionov and Vázquez [30]; Kapila [49]; Kassoy and Poland [51]; [52]; Williams [85]; Zel'dovich, Barenblatt and Librovich in [87] and their references.

- *Superconductivity phenomenon*: This is described by a mathematical physical theory, often called Ginzburg-Landau theory, named after Ginzburg and Landau, see the works by Ginzburg and Landau [38]; Aranson and Kramer in [1]; Popp et al [74]; Cross and Hohenberg [15].

- *Fluid mechanics and optics* derived from Ginzburg-Landau theory, see Levermore and Oliver [54].

- *Theory of Micro-electro-mechanical systems (MEMS) devices*: We would like to adress to Guo and Kavallaris [42]; Pelesko and Bernstein [48]; Kavallaris and Suzuki [53]; Pelesko and Triolo [73] and references therein.

- *The physical mechanism of vortex stretching, turbulent flows*: These theories have a relation to the Constantin-Lax-Majda equation, as in the works of Constantin, Lax and Majda [14]; Guo, Ninomiya, Shimojo and Yanagida [40]; Murthy [66] and references therein.

There are many other phenomena which are not presented in this text, because of lack of time and space.

II. Defining finite time singularity

In this section, we are interested in introducing the notion of finite time singularity formation in parabolic PDE. Then, we aim at considering some illustrating examples.

II.1. Mathematical treatment

When facing any submodel included in (7), we first address the issue of existence and uniqueness of solutions, or the "Cauchy problem". As a matter of fact, some of the submodels can be solved in a lot of classes of functional spaces such as: $L^p(\Omega), p \in [1, \infty]$, Sobolev spaces $W^{1,p}(\Omega)$ and so on. For more details on the Cauchy problem, we kindly refer the readers to Friedman [24]; Henry [44]; Pazy [72]; Ladyženskaja, Solonnikov and Ural'ceva [70]; Souplet and Quittner [75]. In this thesis, we mainly focus on $L^\infty(\Omega)$. Indeed, thanks to the regularity of the semi-group $e^{t\Delta}$ (see its definition and its properties in [70] and [75]), parabolic regularity and a fixed-point argument, the Cauchy problem is well-posed in $L^\infty(\Omega)$ (also in $W^{1,\infty}(\Omega)$) under some reasonable conditions on F and g in (7). Roughly speaking, we may define $T_{max} > 0$ as the maximal existence time of the solution. Then, one of the following statements holds:

(a) Either $T_{max} = +\infty$, which implies that the solution is global.

(b) Or $T_{max} < +\infty$, which implies that

$$\|u(.,t)\|_{L^\infty(\Omega)} \to +\infty \quad (\text{or } \|u(.,t)\|_{W^{1,\infty}(\Omega)} \to \infty) \quad \text{as } t \to T.$$

We call the second case finite time blowup phenomenon and T is called the blowup time of u. We may also introduce the definition of a *Blowup point*. Note that these notations follow the introduction of Friedman and McLeod [25]:

Definition 0.1 (Blowup point). *Let us consider u, a function on $\Omega \times [0,T), T > 0$ which blows up at time T. A point $a \in \bar{\Omega}$ is called a blowup point of u, if and only if there exist $\{(x_n, t_n)\}_{n\geq 1} \subset \Omega \times [0,T)$, converging to (a,T) as $n \to +\infty$, such that the following holds*

$$|u(x_n, t_n)| \to +\infty \text{ as } n \to +\infty.$$

If we work in $L^\infty(\Omega)$ with Ω bounded, then we can prove that there exists at least a blowup point. Following this, two interesting issues arise:

$a)$ *Existence*: Does a blow up solution for system (7) exist?

$b)$ *Asymptotic behavior*: Can we describe the asymptotic behavior of the solution near the blowup point?

Thus, we aim in this thesis at studying the following two main issues:

1) Construct blowup solutions to system (7) for some explicit cases.

2) Describe the asymptotic behavior of the constructed solutions near the blowup point.

II.2. Blowup examples in ODE and PDE

As we mentioned at the head of this section, we would like to take the following examples:

- ***Example 1:*** Let us consider $(p, a_0) \in \mathbb{R}^2, a_0 > 0, p > 1$ and the following Ordinary Differential Equation (ODE)

$$\begin{cases} u'(t) &= u^p(t), t > 0, \\ u(0) &= a_0. \end{cases}$$

Then, the solution is

$$u(t) = \kappa(T_0 - t)^{-\frac{1}{p-1}},$$

where $\kappa = (p-1)^{-\frac{1}{p-1}}$ and $T_0 = \frac{1}{(p-1)a_0^{p-1}} > 0$. We observe more closely that the existence time interval of that solution cannot cross T_0, because of the following fact

$$u(t) \to +\infty, \text{ as } t \to T_0.$$

We say that $u(t)$ blows up at time T_0.

- ***Example 2 (Osgood's condition):*** More generally, we consider the following ODE:

$$\begin{cases} u'(t) &= f(u(t)), t > 0, \\ u(0) &= a_0 > 0. \end{cases}$$

If f is a positive and continuous function which satisfies

$$\int_0^\infty \frac{dx}{f(x)} < +\infty,$$

then, the solution cannot be globally extended to infinity. This result was established in [71] by Osgood, as the necessary and sufficient condition so that the solution of the above equation blows up for any positive initial data.

- ***Example 3:*** We next consider the following PDE:

$$\begin{cases} \partial_t u &= \Delta u + u^p, (x,t) \in \Omega \times [0, T), \\ u(0) &= u_0(x). \end{cases} \tag{8}$$

If $u_0 \in H_0^1(\Omega), u_0 \not\equiv 0, u_0 \geq 0, \Omega$ is bounded and u_0 satisfies the following condition:

$$E[u_0] \leq 0 \quad \text{where} \quad E[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p+1}\int_\Omega u^{p+1} dx, \tag{9}$$

then, u blows up in finite time. This result was proved in [55] by Levine (see also Ball [2]).

The above-mentioned examples show us an important thing: Under some conditions and even for a small and smooth initial data, the solution to some PDE may develop singularities in some finite time $T > 0$. In particular, they may become large in the functional spaces where the PDE is considered: we say that they develop singularities in finite time. This phenomenon occurs in a variety of PDEs, including those modeling the real world. For more information on singularities phemomena, we kindly refer the readers to Horstmann [47]; Martel, Merle and Raphaël [57], Galaktionov and Vázquez [30]; Aranson and Kramer [1]; Bebernes and Eberly [3]; Bressan [6]; Constantin, Lax and Majda [14]; Cross and Hohenberg [15]; Flores, Mercado, Pelesko and Smyth [23]; Ginzburg and Landau [38]; Guo [41]; Guo and Kavallaris [42]; Pelesko and Bernstein [49]; Vázquez [77] and the references therein.

II.3. Notion of "structure" in PDE

As illustrated in Example 3 above, many blowup results take advantage of the "structure" of the PDE. Indeed, we say for example that equation (8) has a variational structure, which results in the existence of the Lyapunov functional $E[u]$ defined as in (9), crucial in deriving the above-mentioned blowup criterion.

It happens that other elements of "structure" are important in the literature, when it comes to study PDE, in particular in the context of singularity formation.

Let us introduce in the following the definitions of symmetric and variational structures in PDE, in the context of this thesis.

(*i*) *Symmetric structure:* A PDE is *symmetric* if for any solution u we have that $u(t+t_0, x), u(t, x+x_0)$ or $e^{i\theta}u(t,x)$ are also solutions.

(*ii*) *Variational structure*: Let us consider the following parabolic equation

$$\partial_t u = \Delta u + F(u), \tag{10}$$

where $u : (x,t) \in \Omega \times [0,T) \to \mathbb{R}^M$. Then, problem (10) is variational if there exists a function

$$G : \mathbb{R}^M \to \mathbb{R} \text{ such that } F = \nabla G.$$

In this case, equation (10) has the energy functional which is decreasing in time:

$$E[u] = \sum_{i=1}^{m} \int_{\Omega} \frac{|\nabla u_i|^2}{2} dx - \int_{\Omega} G(u) dx.$$

We say that $E[u]$ is a Lyapunov functional for equation (10).

Note that the notion of "Symmetric structure" and "Variational structure" holds also for other types of PDE, in particular, hyperbolic PDE. However, we don't consider them in this thesis.

II.4. Relevant questions for blowup

As in many mathematical areas, two major questions arise when we consider a given PDE. The study of blowup is no exception to that.

These are the questions one may ask when studying blowup for some given PDE:

- **Classification of general solutions**: Given a general blowup solution, can we give a full classification of *all* possible asymptotic behaviors at blowup?

- **Construction of examples of solutions**: Can we find some examples of solutions showing some specific blowup behaviors?

These two questions are related, in the sense that the "construction" may provide examples confirming some type of behavior available in the "classification".

Sometimes, as this is the case in this thesis, the "classification" may be too hard to obtain, because of the lack of structure in the PDE. In that case, the "construction" may be of great help, in the sense that its products will be the *only* examples available.

In this thesis, we precisely consider PDE lacking "structure", making the classification question out of reach. Accordingly, we will only focus on the "construction" issue, providing important examples of blowup solutions, **presenting novel and unprecedented types of behaviors**.

III. Specific difficulties in this thesis: non symmetric or non variational PDE

As we mentioned before, we treated in this thesis models with non symmetric or non variational structure. Let us explain in the following why we focus on such models in our works. It happens in fact that most of the mathematical analysis of singularities was done for "idealized" situations, where the models were simplified in order to be easily track able in mathematical tools. Indeed, having a variational structure, satisfying a maximum principle property, or enjoying a scaling invariance property do help a lot in understanding finite time singularity occurrance in PDE.

However, when simplifying some model, we may loose essential physical features, making the PDE behavior very far from reality. Therefore, this motivates us to study models that are close to the realty and are either *non-symmetric* or *non-variational* or both. As a matter of fact, we consider in this thesis some real-world situations which are far from the "idealized" situations described earlier, and we try to built new tools on order to better understand finite-time singularity formation via this modest dissertation.

As we pointed out earlier, the "classification" question is largely out of reach in this thesis, because of the lack of structure. As a consequence, we focus on the question of "construction" here.

For the sake of completeness, we will address in the following the two questions:

- The classification in the literature, for some ideal standard case
- The construction in the literature and in our work.

IV. The classification question in the literature for some ideal standard case

In this section, we address the "classification" question in the literature, for some ideal standard case of system (7), studied by many authors:

$$\partial_t u = \Delta u + |u|^{p-1} u, \tag{11}$$

where $u : (x,t) \in \Omega \times [0,T) \to \mathbb{R}, \Omega$ is a open set of $\mathbb{R}^N$ and p is assumed to satisfy the following subcritical condition

$$p \in (1,+\infty) \text{ if } N \leq 2 \text{ and } p \in \left(1, \frac{N+2}{N-2}\right) \text{ if } N \geq 3. \tag{12}$$

As one may think, this is an idealized case which is out of the scope of the thesis. Nevertheless, we choose to include information on it for the sake of historical completeness. Indeed, equation (11) is the simplest parabolic PDE showing blowup, and it has attracted a lot of attention in the last 50 years.

IV.1. The existence of the finite time blowup phenomenon

In this part, we aim at introducing some results related to the existence of finite time blowup and blowup points in particularlly. In fact, these problems have been studied by many authors such as Ball [2]; Fujita [27] and [28]; Kaplan [50]; Levine [55]; Weissler [82]. For example, Levine [55] and Ball [2] have obtained an existence by using the following Lyapunov functional defined as in (9):

$$E[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx.$$

More precisely, this is the statement (see for example Theorem 3.2 in [2]):

Let us consider Ω a bounded open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. If $u_0 \in H_0^1(\Omega), u_0 \not\equiv 0$ and $E[u_0] \leq 0$, then there exists $T_{max}(u_0) \in (0,+\infty)$ such that $u \in C([0,T_{max}), H_0^1(\Omega))$ and the following holds

$$\|u(t)\|_{L^{p+1}(\Omega)} \to +\infty \text{ as } t \to T_{max}.$$

In this case, we say that u blows up in finite time.

Next, we would like to mention some results related to the existence of blowup points. In order to get more information, we kindly refer the reader to Caffarelli and Friedman [11]; Chen and Suzuki [13]; Chen and Matano [12]; Friedman [26]; Friedman and McLeod [25]; Fujita and Chen [29] and so on. In particular, Giga and Kohn have established in [35] a criterion which allows us to conclude whether a given point is singular or not. In fact, they mainly used the following local energy functional:

$$\begin{aligned} E_{a,t}[u] &= t^{\frac{2}{p-1}-\frac{N}{2}+1}\int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\right) e^{-\frac{|x-a|^2}{4t}} dx \\ &+ t^{\frac{2}{p-1}-\frac{N}{2}}\int_\Omega \frac{1}{2(p-1)}|u|^2 e^{-\frac{|x-a|^2}{4t}} dx, \end{aligned} \tag{13}$$

where $a \in \Omega$ and $t > 0$. The following is their result (see Corollary 3.6 in [35]):

Let us consider Ω a domain which is strictly star-shaped about $a \in \bar{\Omega}$. Then, there exists $\epsilon(\Omega, p) > 0$ such that the following holds: If u is a solution of (11) which blows up at time T satisfying $E_{a,T}(u_0) < \epsilon$, then a cannot be a blowup point.

In addition to that, these authors have also proved in Corollary 4.3 in [35], another important criterion which implies whether a given point is a blowup point or not.

Let us consider Ω *a convex domain in* $\mathbb{R}^2$ *with* C^2 *boundary. Then,* $a \in \Omega$ *is a blowup point if and only if the following holds:*

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} u(a + y\sqrt{T-t}, t) = \pm\kappa, \text{ where } \kappa = (p-1)^{-\frac{1}{p-1}}, \tag{14}$$

uniformly for y *in compact sets.*

In particular, in the case where Ω is bounded, the Dirichlet condition implies that $u(.,t)|_{\partial\Omega} = 0$, for all $t < T$. Then, this rises the question whether u blows up at $\partial\Omega$ or not. As a matter of fact, we don't have the answer in the general case. However, the answer is negative for some special cases. More precisely, we have the following result (see Theorem 5.3 in [35]):

We consider Ω *a* $C^{2,\alpha}$ *domain which is strictly star-shaped about* a*, where* $a \in \partial\Omega$*. Then,* a *cannot be a blowup point.*

Furthermore, we have the situation where the solution blows up at many points in Ω. In that case, the blowup set is an interesting object to study. For example, in Theorem 5.1 of [35], the authors proved the following:

If $u_0 \in H_1(\mathbb{R}^n)$ *and* u *blows up in finite time, the blowup set is then compact.*

On the other hand, there were also many authors who have constructed special initial data u_0 so that the blowup set is explicit. For example, Merle in [60] gave a construction with k exactly given blowup points. Another example for dimension $N \geq 2$: Giga and Kohn gave the existence of a positive, radially symmetric initial data for which the blowup set is some $(N-1)$-dimensional sphere (see Corollary 5.7 in [35]).

Allowing the solution to be independent of some coordinate, we may obtain examples where the blowup set is some infinite cylinder, or parallel hyperplanes or even concentric spheres, which all come from the case of k given points or a sphere we have just mentioned above.

Apart from these two cases, no other example of blowup sets in known. For example, the question of constructing a solution for (11) blowing up on a ellipse in a 2 space dimensional remains largely open.

IV.2. Blowup asymptotic behavior and blowup profile

In this paragraph, we aim at mentioning some results about the aysmptotic behavior of the solution of equation (11) when the blowup phenomenon occurs. In order to study the asymptotic behavior, we have many ways to approach this problem. One of them is to use the so-called *self-similar variables* (note that this notation was initially used in the work of Giga and Kohn [33]):

$$w_a(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \qquad y = \frac{x-a}{\sqrt{T-t}} \quad \text{and} \quad s = -\ln(T-t). \tag{15}$$

With this transformation, the study of the blowup behavior of u reduces to the study of the asymptotic behavior of w_a as $s \to +\infty$.

From equation (11), we easily write the equation satisfied by $w(y,s)$ as follows:

$$\partial_s w_a = \Delta w_a - \frac{1}{2} y \cdot \nabla w_a - \frac{w_a}{p-1} + |w_a|^{p-1} w_a. \tag{16}$$

Note that w is defined on $\{(y,s) \in \Omega_{a,s} = e^{\frac{s}{2}}(\Omega - a) \times [-\ln T, +\infty)\}$.

From comparison techniques, we may show (at least when $\Omega = \mathbb{R}^N$) that

$$\|u(.,t)\|_{L^\infty(\Omega)} \geq \kappa (T-t)^{-\frac{1}{p-1}}, \forall t \in [0,T),$$

(see Weissler [83], Friedman and McLeod [25], Giga and Kohn [35]). Following this fact, two situations which identified in the literature named by Matano and Merle [59]:

The blowup solution u is of type I if there exists $C > 0$ such that

$$\|u(.,t)\|_{L^\infty(\Omega)} \leq C h(t), \forall t \in [0,T),$$

where $h(t)$ is the positive solution of the ODE, connected to (11). *Namely, we can explicitly write the formula of $h(t)$:*

$$h(t) = (T-t)^{-\frac{1}{p-1}}.$$

Otherwise, the solution u is called of type II.

In the context of this thesis, we only focus on *type I blowup* (of course, for other equations different from (11)). In other words, we are interested in the case where we may find the lower and upper bounds for u. This means there exist $C > 0$ such that the following holds

$$\frac{1}{C}(T-t)^{-\frac{1}{p-1}} \leq \|u(t)\|_{L^\infty(\Omega)} \leq C(T-t)^{-\frac{1}{p-1}}. \tag{17}$$

This leads the following estimates:

$$\frac{1}{C} \leq \|w_a(.,s)\|_{L^\infty(\Omega_{a,s})} \leq C. \tag{18}$$

In fact, we call the above bounds the blowup rates. The upper bound in (17) has been discovered by Mueller and Weissler [65], Weissler [82] and [84] under some conditions. In particular, Giga and Kohn have established in [34] (see also [35]) the bounds of (17) in the case where Ω is a bounded convex domain with the assumption that one of the following conditions holds:

- *Either initial data u_0 is nonnegative or p satisfies furthermore the following condition*

$$p > 1 \text{ if } N = 1 \text{ and } p \in \left(1, \frac{3N+8}{3N-4}\right) \text{ if } N \geq 2. \tag{19}$$

Later, Giga, Matsui and Sasayama have removed condition (19) (see [36] for the case $\Omega = \mathbb{R}^N$, and then, [37] for a more general smooth convex domain Ω), extending the result to all Sobolev subcritical exponent $p > 1$ as in (12). In order to overcome the challenges, the authors used the arguments on the following Lyapunov functional associated to equation (16):

$$\mathcal{E}[w_a](s) = \int_{\Omega_{a,s}} \left[\frac{1}{2}|\nabla w_a|^2 + \frac{1}{2(p-1)}|w_a|^2 - \frac{1}{p+1}|w_a|^{p+1}\right] \rho(y) dy. \tag{20}$$

where

$$\rho(y) = \frac{1}{(4\pi)^{\frac{N}{2}}} e^{-\frac{|y|^2}{4}}. \tag{21}$$

Finally, they have obtained the key integral estimate in the sense that for all $q \geq 2$ and $a \in \Omega$, there exists $R_1(N, p, q, \Omega) > 0$, independent of a such that the following estimate holds

$$\sup_{s \geq -\ln T} \int_s^{s+1} \|w_a(;s)\|^{(p+1)q}_{L^{p+1}(\Omega_{a,s} \cap B(0,R_1))} ds \leq \hat{C}, \tag{22}$$

where $\hat{C}$ depends only on N, p, q, Ω and a bound on $\mathcal{E}[w_a](0)$ as well as some norms of w_a^0, where w_a^0 is initial data of w_a. We kindly refer the reader to page 1774 in [35] for more details.

On the other hand, Merle and Zaag obtained in [64] (see also [62]) the following optimal blowup rates:

Let us consider Ω a convex bounded $C^{2,\alpha}$ domain in $\mathbb{R}^N$ and u a solution which blows up at time $T > 0$. We assume furthermore that $u_0 \in H^1(\Omega)$. Then, the following limits hold:

$$\|w_a(.,s)\|_{L^\infty(\Omega_{a,s})} \to \kappa = (p-1)^{-\frac{1}{p-1}}, \tag{23}$$

and

$$\|\nabla w_a(.,s)\|_{L^\infty(\Omega_{a,s})} + \|\Delta w_a(.,s)\|_{L^\infty(\Omega_{a,s})} \to 0,$$

as $s \to +\infty$ and for any $a \in \Omega$.

As a matter of fact, studying blowup rates is a fundamental step towards the study of the asymptotic behavior of solutions to problem (16) as we will mention below.

We now assume that u blows up at time T and at some point $a \in \Omega$. Firstly, we derive from (14) the asymptotic behavior of w_a on every compact set: for each $K > 0$

$$\sup_{|y| \leq K} |w_a(y,s) - \kappa^*| \to 0, \text{ as } s \to +\infty, \tag{24}$$

where $\kappa^* \in \{-\kappa, \kappa\}$ and κ is defined in (14). Note that $\kappa, -\kappa, 0$ are constant solutions of (16). In particular, in the case where $\Omega = \mathbb{R}^N$, they are the only stationnary solutions under condition (12) (see Giga and Kohn [33]). Concerning the blowup behavior, we kindly refer the readers to Filippas and Kohn [21]; Filippas and Liu [22]; Herrero and Velázquez [45] and [46]; and Velázquez [78] and [81].

More precisely, Giga and Kohn used in [33] some analysis in Sobolev spaces with the Gaussian weight ρ defined in (21) to derive (24), see also [21]; [22]; [33]; [45]; [46]; [78] and [81]. More importantly, Velázquez established in [79] a classification of the asymptotic behavior of solutions to problem (16) (although some of the above-mentioned authors may have considered the nonlinearity u^p instead of $|u|^{p-1}u$, all their results hold also for $|u|^{p-1}u$ with the same proof). More precisely, this is the result in [79]:

There exist an orthogonal matrix $\mathcal{O}$ of order N and an integer number $k \in \{0, ..., N-1\}$ such that one of the following statements holds (up to replacing u by $-u$ if necessary):

a)- Exponential decay: There exists $\nu > 0$ such that for all $K > 0$, we have

$$\sup_{|y|\le K} |w_a(y,s) - \kappa| \le C(K)e^{-\nu s}, \forall s \ge -\ln T. \tag{25}$$

b)-Non exponential decay: There exists $\mu > 1$ such that for all $K > 0$, we have

$$\sup_{|y|\le K} \left| w_a(y,s) - \left[\kappa + \frac{\kappa}{2ps}\left[(N-k) - \frac{1}{2}y^T \mathcal{M}_k y\right]\right]\right| = O\left(\frac{1}{s^\mu}\right), \ \text{as } s \to +\infty, \tag{26}$$

where

$$\mathcal{M}_k = \mathcal{O}^{-1}\begin{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix} \mathcal{O}, \ \text{with } N-k \text{ of } 1's \text{ digits.} \tag{27}$$

In this thesis, we are interested in case b), non exponential decay, of course, for other equations more general than (11). Note that using (26), we have the behavior of w in the set $|y| \le K$, for any $K > 0$. This fact is equivalent to the behavior of u in a small ball $|x-a| \le K\sqrt{T-t} \to 0$ as $t \to T$. The more t approaches the blowup time, the less we know about the behavior of w_a. In fact, both for u and w_a, we see that the solution becomes flat approaching a constant, and no shape arises. This is disappointing from a physical point of view.

Later, Herrero and Velázquez [45] (in the one dimensional case), Liu [56] (in the multi-dimensional case) have dealt with this challenge. More precisely, they improved the estimate in (27) by finding another expansion valid in larger domains of the form $\{|y| \le K\sqrt{s}\}$ for any $K > 0$. In addition to that, Merle and Zaag [63] have obtained later the same result with a different proof based on some compacness properties of w_a, uniformly with respect to $a \in \mathbb{R}^N$. Note that this uniform property on $a \in \mathbb{R}^N$ was not proven before. This result helped Merle and Zaag to establish in [63] the following blowup profile with respect to the variable

$$z = \frac{y}{\sqrt{s}}, \tag{28}$$

which may be called the *blowup variable*. The following is their result:

There exist $k \in \{0, 1, ..., N\}$ and an orthogonal matrix $\mathcal{O}$ such that for all $K > 0$, the following holds:

$$\sup_{|z|\le K} \left| w_a(z\sqrt{s}, s) - f_k(z)\right| \to 0 \ \text{as } s \to +\infty, \tag{29}$$

where

$$f_k(z) = \left(p - 1 + \frac{(p-1)^2}{4p} z^T \mathcal{M}_k z\right)^{-\frac{1}{p-1}}, \tag{30}$$

with $\mathcal{M}_k$ defined as in (27). Note that when $k = N$ in case b) mentioned aboved, this is a degenerate case with $\mathcal{M}_N = 0$, and in fact, we are in case a). Note also that the profile (30) is referred to as the "*intermediate*" blowup profile of w, since it is close to the solution for $s \in [s_0, +\infty)$ for some s_0 (or $t \in [T - e^{-s_0}, T)$) by (15). In fact, we will introduce later a notion of "*final*" blowup profile. In the case where $k = 0$, we would like to mention that (29) was first found numerically by Berger and Kohn in [5].

Let us now introduce the notion of final profile where $k = 0$ in (27). In fact, Herrero and Veláquez [45] (see also [78] and [80]) derived a final profile for the blowup solution. More precisely, there exists $u^*(x)$ such that $u(x,t) \to u^*(x)$ as $t \to T$, for any $x \neq a$. Moreover, we have the following

$$u^*(x) \sim \left[\frac{8p}{(p-1)^2} \frac{|\ln(x-a)|}{|x-a|^2} \right]^{\frac{1}{p-1}}, \text{ as } x \to a. \tag{31}$$

V. The construction of Type I blowup solutions

In this section, we address the question of "construction" of examples of blowup solutions for some PDE.

In fact, we rely here on some general method which we could adapt in our work to various situations, after many nontrivial adaptations.

This method was introduced by various authors, and goes back to the works of Bressan [6] and [7]; Bricmont and Kupiainen [8] and [9]; Merle and Zaag [61].

It relies on some two parts:

- The derivation of *approximate* solution, through a *formal approach*;

- The construction of an *exact* solution close to the approximate solution, through a perturbative *rigorous* argument. This part relies on a good knowledge of the special properties of the linearized operator around the approximate solution. It consists in 2 steps:

Step 1

Reduction to a finite dimensional problem, to control the negative directions of the operator

Step 2

Topological argument based on index theory, to control the nonnegative directions of the spectrum

In the context of this thesis, we call it *the finite reduction method.* As a matter of fact, this method was introduced in Merle and Zaag [61] by improving of the proof given in Bricmont and Kupiainen [9]. In particular, the finite reduction method can be resumed by two steps:

In some specific situations, the construction method provides the stability of the blowup profile under perturbations of initial data by using the interpretation of the parameters of the finite-dimensional problem in terms of the blowup time and the blowup point. In fact, the construction in [61] corresponds to case (26) where $k = 0$.

To be more specific, we will present in the following the "construction" method as it is available in the literature for the ideal case of equation (11).

The construction result is due to Bricmont and Kupiainen [8] who have constructed a nonnegative blowup solution $u(x,t)$ to (11) (see also Bricmont and Kupiainen [9], and

Bricmont, Kupiainen and Lin [10]), satisfying the following L^∞-estimate, after the change of variables (15):

$$\left\| w_a(.,s) - f_0\left(\frac{y}{\sqrt{s}}\right)\right\|_{L^\infty(\mathbb{R}^N)} \to 0 \text{ as } s \to +\infty, \tag{32}$$

where

$$f_0(z) = \left(p - 1 + \frac{(p-1)^2}{4p}|z|^2\right)^{-\frac{1}{p-1}}. \tag{33}$$

Estimate (32) yields in fact the following

$$\left\| (T-t)^{\frac{1}{p-1}} u(.,t) - f_0\left(\frac{. - a}{\sqrt{(T-t)|\ln(T-t)|}}\right)\right\|_{L^\infty(\mathbb{R}^N)} \to 0, \tag{34}$$

where u is the constructed solution of equation (11), blowing up at time T and only at a.

V.1. A formal approach to derive an approximate solution (i.e the blowup profile)

We aim at explaining in this part how the blowup profile (33) arises formally. In order to get a simple situation, we suggest to take $N = 1$, and $\Omega = \mathbb{R}$ and in the nonnegative case.

In fact, in order to get a blowup solution to (11), we will in fact construct a bounded solution to (16). Since (16) is of parabolic type, it is reasonable to work with the "blowup variable"

$$z = \frac{y}{\sqrt{s}},$$

as mentioned by Tayachi and Zaag in [76]. Following these authors and adpting an original idea by Berger and Kohn in [5], we may try to find a solution w with the following form

$$w_a(y,s) = \sum_{j=0}^{\infty} \frac{w_j(z)}{s^j}, \tag{35}$$

where functions $w_j, j \geq 0$ are assumed to be smooth and bounded. In particular, $w_0 \geq 0$ because of the assumption that w is nonnegative.

Using equation (16), (35) and gathering terms of order $\frac{1}{s^j}, j = 0, 1$, we derive the following equations

$$-\frac{1}{2} z.w_0'(z) - \frac{w_0(z)}{p-1} + w_0^p(z) = 0,$$

and

$$-\frac{1}{2} z.w_1'(z) - \frac{w_1(z)}{p-1} + p w_0^{p-1} w_1(z) + w_0''(z) + \frac{z.w_0'}{2} = 0.$$

Following for example the justification in Berger and Kohn [5] and Duong [17], we get

$$w_0(z) = \left(p - 1 + bz^2\right)^{-\frac{1}{p-1}}, \tag{36}$$

and

$$w_1(z) = \frac{(p-1)}{2p}(p-1+bz^2)^{-\frac{p}{p-1}} - \frac{(p-1)}{4p} z^2 \ln(p-1+bz^2)\left(p-1+bz^2\right)^{-\frac{p}{p-1}},$$

where

$$b = \frac{(p-1)^2}{4p}.$$

Thus, from (36), we can formally derive f_0 as the blowup profile in our construction. More precisely, we can see that for all $|y| \leq K_0\sqrt{s}$ for some $K_0 > 0$, we have

$$w_a(y,s) \sim \varphi_1(y,s) \text{ as } s \to +\infty, \tag{37}$$

where

$$\varphi_1(y,s) = w_0(z) + \frac{w_1(0)}{s} = f_0\left(\frac{y}{\sqrt{s}}\right) + \frac{\kappa}{2ps} \text{ and } f_0 \text{ defined in (33).} \tag{38}$$

Note that for $N \geq 2$, our profile will be the following

$$\varphi_N(y,s) = f_0\left(\frac{y}{\sqrt{s}}\right) + \frac{N\kappa}{2ps}. \tag{39}$$

V.2. The rigorous proof

In this paragraph, we present the perturbative rigorous method which provides the existence of a solution to equation (16) in $\mathbb{R}^N$ satisfying

$$\|w_a(.,s) - \varphi_N(.,s)\|_{L^\infty(\mathbb{R}^N)} \to 0 \text{ as } s \to +\infty, \tag{40}$$

where φ_N is defined as in (39).

Introducing

$$q(y,s) = w_a(y,s) - \varphi_N(y,s),$$

we transform the PDE (11) into the following equation satisfied by q:

$$\partial_s q = [\mathcal{L} + V(y,s)]q + B(q,y,s) + R(y,s), \tag{41}$$

where

$$\begin{aligned}
\mathcal{L} &= \Delta - \frac{1}{2}\nabla\cdot y + Id, && (42)\\
V(y,s) &= p\left[\varphi_N^{p-1}(y,s) - \frac{1}{p-1}\right], && (43)\\
B(q,s) &= |q+\varphi_N|^{p-1}(q+\varphi_N) - \varphi_N^p - p\varphi_N^{p-1}q, && (44)\\
R(y,s) &= \Delta\varphi_N(y,s) - \frac{1}{2}\nabla\varphi_N(y,s)\cdot y - \frac{\varphi_N(y,s)}{p-1} && (45)\\
&+ \varphi_N^p(y,s) - \partial_s\varphi_N(y,s).
\end{aligned}$$

As a matter of fact, our problem is reduced to the construction of a solution for equation (41) satisfying

$$\|q(.,s)\|_{L^\infty(\mathbb{R}^N)} \to 0 \text{ as } s \to +\infty. \tag{46}$$

We first note the following fact

$$\|R(s)\|_{L^\infty(\mathbb{R}^N)} \lesssim \frac{1}{s}.$$

Moreover, once q is small enough in L^∞, the term B is then formally "quadratic". This leads to the smallness of B. It remains to understand the effects of $\mathcal{L}$ and V.

(*i*) *Operator* $\mathcal{L}$*:* It is self-adjoint in $\mathcal{D}(\mathcal{L}) \subset L^2_\rho$, where

$$L^2_\rho(\mathbb{R}^N) = \{f \in L^\infty_{loc}(\mathbb{R}^N) \text{ such that} \int_{\mathbb{R}^N} |f(y)|^2 \rho(y) dy < +\infty\} \text{ with } \rho \text{ defined as in (21).}$$

On the other hand, we have

$$\operatorname{Spec} \mathcal{L} = \left\{ 1 - \frac{n}{2} \middle| \, n \in \mathbb{N} \right\},$$

Note that the largest eigenvalue is 1, and for every eigenvalue $1 - \frac{n}{2}$, we have the associated eigenspace

$$\mathcal{E}_n = \left\langle H_\alpha(y) = h_{\alpha_1}(y_1)...h_{\alpha_N}(y_N) \middle| \, |\alpha| = \sum_{i=1}^{N} \alpha_i = n \text{ and } \alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N \right\rangle,$$

where function $h_{\alpha_i}(y_j)$ is the rescaled Hermite polynomial of order α_i (see [19] and [61] for more details). In addition to that, the following set

$$\mathcal{B} = \{H_\alpha(y) \mid \alpha \in \mathbb{N}^N\},$$

makes a basis of L^2_ρ.

(*ii*) *Potential* V*:* In fact, the value of V depends on the time variable s and also on the reduced variable

$$z = \frac{y}{\sqrt{s}},$$

rather than on y itself. For that reason, its behavior will dramatically depend on the size of z. More specifically, inside the blowup region $\{|y| \leq K\sqrt{s}\}$ for some $K > 0$, we have the following estimate

$$V(s) \to 0 \text{ in } L^2_\rho(\mathbb{R}^N) \text{ as } s \to +\infty,$$

which shows that the effect of V will be a perturbation of the effect of $\mathcal{L}$, except may be on the null modes of $\mathcal{L}$, on the one hand.

On the other hand, V significantly changes the effect of $\mathcal{L}$ outside the blowup region, namely in the set $\{|y| \geq K\sqrt{s}\}$. Indeed, for each $\epsilon > 0$, there exist $K_\epsilon > 0$ and $s_\epsilon > 0$ such that

$$\sup_{\frac{|y|}{\sqrt{s}} \geq K_\epsilon, s \geq s_\epsilon} \left| V(y,s) + \frac{p}{p-1} \right| \leq \epsilon.$$

Since $-\frac{p}{p-1} < -1$ and bearing in mind that 1 is the largest eigenvalue of $\mathcal{L}$, we can see that $\mathcal{L} + V$ behaves as an operator with a fully negative spectrum.

From the above information about $\mathcal{L}$ and V, the behavior of $\mathcal{L} + V$ inside and outside the blowup region is different. Hence, this motivates us to consider the dynamics of the solution first on $\{|y| \leq K\sqrt{s}\}$, then on $\{|y| \geq K\sqrt{s}\}$. As the authors in [9] and [61] did, we introduce the following cut-off function

$$\chi(y,s) = \chi_0 \left(\frac{|y|}{K\sqrt{s}} \right),$$

where $\chi_0 \in C_0^\infty[0,+\infty)$, $\|\chi_0\|_{L^\infty} \le 1$ and

$$\chi_0(x) = \begin{cases} 1 & \text{for} \quad x \le 1, \\ 0 & \text{for} \quad x \ge 2. \end{cases}$$

Then, we decompose q as the following

$$q = \chi q + (1-\chi)q = q_b + q_e.$$

Note that $\text{supp}(q_b) \subset B(0, 2K\sqrt{s})$ and $\text{supp}(q_e) \subset \mathbb{R}^N \setminus B(0, K\sqrt{s})$. Moreover, if $q \in L^\infty(\mathbb{R}^N)$, then we have the fact that $q_b, q_e \in L^\infty(\mathbb{R}^N) \subset L^2_\rho(\mathbb{R}^N)$. Accordingly, we may expand q on the eigenfunctions of $\mathcal{L}$ as follows:

$$q_b = q_0 + q_1 \cdot y + y^T \cdot q_2 \cdot y - 2\text{Tr}(q_2) + q_-,$$

where

$$q_m = \left(\frac{\langle H_\alpha, q_b \rangle_{L^2_\rho}}{\langle H_\alpha, H_\alpha \rangle_{L^2_\rho}} \right)_{|\alpha| = m}, m \ge 0.$$

Note that q_0 is in $\mathbb{R}$, q_1 is a vector in $\mathbb{R}^N$ and q_2 is a square matrix of order N.

Finally, we write

$$q = q_b + q_e = q_0 + q_1 \cdot y + y^T \cdot q_2 \cdot y - 2\text{Tr}(q_2) + q_- + q_e. \tag{47}$$

As a conclusion to this paragraph, we recall that our goal is to construct a solution q to equation (41) satisfying (46), where q is decomposed as in (47), a decomposition well adapted to the properties of $\mathcal{L} + V$, the linearized operator of (41).

The control of q towards 0 in (46) will follow from the control of its components q_0, q_1, q_2, q_- and q_e shown in (47), two of them being infinite dimensional (q_- and q_e).

V.2.1 . Reduction to a finite dimensional problem

In this part, we show that the control of q towards 0 in (46) reduces in fact to the control of q_0 and q_1. From the fact that $(q_0, q_1)(s) \in \mathbb{R}^{1+N}$, this makes a reduction to a finite dimensional problem.

Indeed, from the definitions of q_- and q_e in (47), we get the following facts:

- For q_-: This part corresponds to the eigenvectors H_α where $|\alpha| \ge 3$. Then, we may derive from the properties of operator $\mathcal{L}+V$ that q_- is associated to the negative eigenvalues of $\mathcal{L} + V$. Hence, it is easily controllable to 0.
- For q_e: We have $supp(q_e) \subset \{|y| \ge K\sqrt{s}\}$, a region where $\mathcal{L}+V$ has a strictly negative spectrum. Hence, q_e is easily controllable to 0.

After this reduction, when q is small, we project equation (41) on $\mathcal{E}_m$, $m = 0, 1$ and 2, then we obtain the following system:

$$q_0'(s) \sim q_0(s), \tag{48}$$

$$q_1'(s) \sim \frac{1}{2} q_1(s), \tag{49}$$

$$q_2'(s) \sim -\frac{2}{s} q_2(s), \tag{50}$$

as $s \to +\infty$. From (50) and introducing $\tau = \ln(s)$, we can write

$$\partial_\tau q_2(\tau) \sim -2q_2(\tau) \text{ as } \tau \to +\infty,$$

where we still note $q_2(\tau) = q_2(s(\tau))$, This yields that $q_2(\tau)$ is associated to a strictly negative eigenvalue. Then, $q_2(\tau)$ can be controlled to 0 and $q_2(s)$ too.

The problem remains to control two components: q_0 and q_1. As a matter of fact, we see from (48) and (49) that these components are associated to strictly positive eigenvalues. So, we cannot do as we did with the others components. Finally, we have reduced problem (46) to a finite one on q_0 and q_1 for which we will find initial data $(q_0, q_1)(s_0)$ where $s_0 = -\ln T$ such that

$$(q_0, q_1)(s) \to 0 \text{ as } s \to +\infty.$$

V.2.2. A topological argument

In order to give a flavor of our argument, we will consider the following two-dimensional model problem:

$$\begin{cases} q_0' &= q_0 + q_1^2 + \frac{1}{s^2}, \\ q_1' &= \frac{1}{2}q_1 - q_0 q_1 - \frac{2}{s^2}, \end{cases} \tag{51}$$

fitted with initial data

$$(q_0, q_1)(s_0) = (d_0, d_1) \in \mathbb{R}^2,$$

where s_0 will be taken large enough.

As mentioned in the previous part, we aim at constructing initial data (d_0, d_1) such that

$$(q_0, q_1)(s) \to 0 \text{ as } s \to +\infty.$$

More precisely, we prove that there exists $(d_0, d_1) \in \mathcal{V}(s_0)$ such that

$$|q_m(s)| \le \frac{A}{s^2}, \quad \forall m = 0, 1 \text{ and } \forall s \ge s_0, \tag{52}$$

where

$$\mathcal{V}(s) \equiv \left[-\frac{A}{s^2}, \frac{A}{s^2} \right]^2,$$

and A will be taken large enough.

Indeed, by a contradiction, we assume for all $(d_0, d_1) \in \mathcal{V}(s_0)$ that (52) fails at time s, for some $s \in [s_0, +\infty)$. In that case, there exists $s_* = s_*(d_0, d_1)$ such that

$$|q_m(s)| \le \frac{A}{s^2} \quad \forall s \in [s_0, s_*] \text{ and } \forall m \in \{0, 1\},$$

and

$$|q_0(s_*)| = \frac{A}{s_*^2} \text{ or } |q_1(s_*)| = \frac{A}{s_*^2}.$$

From the ODE system (51), we derive that the flow of $|q_m(s)|$ is transverse outgoing on the curve

$$s \mapsto \frac{A}{s^2},$$

at the crossing time $s = s_*$. This implies that

$$(d_0, d_1) \mapsto s_*(d_0, d_1)$$

is continuous.

Making the change of variables

$$(d_0, d_1) = \frac{A}{s_0^2}(\nu_0, \nu_1) \text{ where } (\nu_0, \nu_1) \in [-1, 1]^2,$$

we can construct the following mapping

$$\begin{aligned} \Gamma : [-1,1]^2 &\to \partial[-1,1]^2, \\ (\nu_0, \nu_1) \quad &\mapsto \tfrac{s_*^2}{A}(q_0, q_1)(s_*), \end{aligned}$$

where $s_* = s_*(d_0, d_1)$ and $(d_0, d_1) = \frac{A}{s_0^2}(\nu_0, \nu_1)$.

From the previous analysis, we derive that Γ has the following properties:

(i) Γ is continuous

(ii) The restriction $\Gamma\,|_{\partial \mathcal{V}_0}$ is equal to the identity.

Using a consequence of Brouwer's lemma, Γ cannot exist. Thus, there is $(d_0, d_1) \in \left[-\frac{A}{s_0^2}, \frac{A}{s_0^2}\right]^2$ such that

$$\forall s \geq s_0, \forall m \in \{0, 1\}, \text{ we have } |q_m(s)| \leq \frac{A}{s^2}.$$

This was the solution for the model (51). In the PDE that we consider in this thesis, we will handle other system similar to (51). We will use the same contradiction argument and construct a similar mapping Γ which will be continuous but not necessarily equal to the identity on the boundary. However, that property will be replaced by the following

$$\deg\left(\Gamma\,\big|_{\partial[-1,1]^2}\right) \neq 0$$

(in one dimension), which will lead to a contradiction from the degree theory.

IV.3. Construction of blowup solutions to other problems

In this paragraph, we would like to mention some constructions of blowup solutions, derived by the above-mentioned construction method. In particular, we consider the following parabolic equation

$$\partial_t u = \Delta u + F(u).$$

First, we mention the work of Bressan [6] (see also [7]) with the nonlinearity $F(u) = e^u$. Then, we also have the paper by Bricmont and Kupiainen [9] with the nonlinearity $F(u) =$

$u^p, u \in \mathbb{R}^+$. Later, we have the construction of Merle and Zaag [61] with the nonlinearity $F(u) = |u|^{p-1}u$.

Next, we also mention the paper by Nguyen and Zaag [67], with a quasi-critical double source

$$F(u) = |u|^{p-1}u + \frac{\mu}{\ln^a(2+u^2)}|u|^{p-1}u, a > 1 \text{ and } \mu \in \mathbb{R}.$$

In addition to that, we also mention the cases where the nonlinearity contains gradient terms such as in the work of Ebde and Zaag [20] with

$$F(u, \nabla u) = \mu|\nabla u|^q + |u|^{p-1}u, \text{ where } 0 \leq q < q_{cri} = \frac{2p}{p+1} \text{ and } p > 1.$$

Later, Tayachi and Zaag have treated in [76] the critical case of the above problem where $q = q_{cri}$ and $p > 3$. In addition to that, we also mention the work of Ghoul, Nguyen and Zaag [32] with $F(u, \nabla u) = \alpha|\nabla u|^2 + e^u, \alpha > -1$.

Moreover, Ghoul, Nguyen and Zaag have considered some vector cases (i.e parabolic systems). For example, there is the work by Ghoul, Nguyen and Zaag [31] who treated the case of

$$F\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} |u_2|^{p-1}u_2 \\ |u_1|^{q-1}u_1 \end{pmatrix}, p, q > 1.$$

Next, we would like to mention some cases where the solution takes complex values such as the Complex Ginzburg-Landau (CGL) equation

$$\partial_t u = (1 + i\beta)\Delta u + (1 + i\delta)|u|^{p-1}u, \quad \delta, \beta \in \mathbb{R}.$$

There were some cases of CGL which have been considered earlier such as: Zaag [86] for the case where $\beta = 0$ and $\delta \in (-\delta_0, \delta_0)$ for some small $\delta_0 > 0$; Masmoudi and Zaag [58] for the following subcritical condition

$$p - (p+1)\delta\beta - \delta^2 > 0.$$

Later, Nouaili and Zaag treated in [69] a critical case of the above-mentioned relation, where $\beta = 0$ and $\delta = \pm p$. This leaves unanswered the case where

$$p - (p+1)\delta\beta - \delta^2 = 0 \text{ and } \beta \neq 0.$$

We also mention the following complex heat equation, where

$$F(u) = u^p, p > 1.$$

In fact, this model has an important role in the literature. More precisely, where $p = 2$, it has been studied by many authors in the world (see [23], [39], [42] and their references). In particular, Nouaili and Zaag have constructed a blowup solution in the case where $p = 2$. Moreover, Harada obtained in [43] the same result by using another method. However, they leaved the unanswered question for the general case where $p > 1$.

VI. Our main results in this thesis

In this section, we aim at introducing the main results in this thesis. In fact, our results focus on the construction blowup solutions for a non-homogeneous PDE, a complex valued equation, and a MEMS model of parabolic type.

VI.1. Existence of a stable blowup solution with a prescribed behavior for a non-scaling invariant semilinear heat equation

We consider here the problem of the construction of a blowup solution to the following semilinear heat equation:

$$\begin{cases} \partial_t u &= \Delta u + |u|^{p-1} u \ln^\alpha(2+u^2), \\ u(0,x) &= u_0(x) \in L^\infty(\mathbb{R}^N), \end{cases} \tag{53}$$

where $u : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R}$, $p > 1$ and $\alpha \in \mathbb{R}$. In particular, we aim at constructing a blowup solution which blows up in finite time T, only at one blowup point $a \in \mathbb{R}^N$. From the invariance of equation (53) under translations in space, a is always assumed to be the origin. The following results follow [19] (this work is an collaboration with V. T. Nguyen and H. Zaag):

Theorem 0.1 (See Theorem 1.1 in [19], page 16)**.** *There exist initial data $u_0 \in L^\infty(\mathbb{R}^N)$ such that the corresponding solution to equation* (53)*, blows up in finite time $T = T(u_0) > 0$, only at the origin. Moreover, we have*

(*i*) *For all $t \in [0,T)$, there exists a positive constant C_0 such that*

$$\left\| \psi^{-1}(t) u(.,t) - f_0\left(\frac{.}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty(\mathbb{R}^N)} \le \frac{C_0}{\sqrt{|\ln(T-t)|}}, \tag{54}$$

where $\psi(t)$ is the unique positive solution of the following ODE

$$\psi'(t) = \psi^p(t) \ln^\alpha(\psi^2(t) + 2), \quad \lim_{t\to T} \psi(t) = +\infty, \tag{55}$$

and the profile f_0 is defined by

$$f_0(z) = \left(1 + \frac{(p-1)}{4p}|z|^2 \right)^{-\frac{1}{p-1}}. \tag{56}$$

(*ii*) *There exits $u^*(x) \in C^2(\mathbb{R}^N \backslash \{0\})$ such that $u(x,t) \to u^*(x)$ as $t \to T$ uniformly on compact sets of $\mathbb{R}^N \setminus \{0\}$, where*

$$u^*(x) \sim \left[\frac{(p-1)^2|x|^2}{8p|\ln|x||} \right]^{-\frac{1}{p-1}} \left(\frac{4}{p-1} |\ln|x|| \right)^{-\frac{\alpha}{p-1}} \quad \text{as } x \to 0, \tag{57}$$

Remark 0.2. *We derive from* (*i*) *that $u(0,t) \sim \psi(t) \to +\infty$ as $t \to T$, which yields that our solution blows up in finite time T at $x = 0$. In addition to that,* (*ii*) *gives us the fact that the solution blows up only at the origin.*

Remark 0.3. *When $\alpha = 0$, (54) is the same as the standard power-like case treated in [9] and [61]. It is different if $\alpha \neq 0$. More precisely, the final profile u^* has a difference coming from the extra multiplication of the size by $|\ln|x||^{-\frac{\alpha}{p-1}}$, which shows that the nonlinear source in equation (53) has a strong effect on the dynamics of the solution in comparison with the standard case $\alpha = 0$.*

Remark 0.4. *Using the parabolic regularity, we can show that if the initial data $u_0 \in W^{2,\infty}(\mathbb{R}^n)$, then we have for $i = 0, 1, 2$,*

$$\left\| \psi^{-1}(t)(T-t)^{\frac{i}{2}} \nabla_x^i u(.,t) - (T-t)^{\frac{i}{2}} \nabla_x^i f_0 \left(\frac{\cdot}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty} \leq \frac{C}{\sqrt{|\ln(T-t)|}},$$

where f_0 is defined by (56).

Using the techniques given by Merle in [60], we can construct a blowup solution with arbitrarily given points. We would like to refer the readers to Corollary 1.6 in [19].

Next, we use the techniques of the interpretation of the parameters of the finite dimensional problem in terms of the blowup time and blowup point given in [61] to derive the stability of the solution constructed in Theorem 1.

Theorem 0.5 (See Theorem 1.7 in [19]). *Consider $\hat{u}$ the solution constructed in Theorem 0.1 and denote by $\hat{T}$ its blowup time. Then, there exists $\mathfrak{U}_0 \subset L^\infty(\mathbb{R}^N)$ a neighborhood of $\hat{u}(0)$ such that for all $u_0 \in \mathfrak{U}_0$, equation (53) with initial data u_0 has a unique solution $u(t)$ blowing up in finite time $T(u_0)$ at a single point $a(u_0)$. Moreover, the statements (i) and (ii) in Theorem 0.1 are satisfied by $u(x - a(u_0), t)$, and*

$$(T(u_0), a(u_0)) \to (\hat{T}, 0) \text{ as } \|u_0 - \hat{u}_0\|_{L^\infty(\mathbb{R}^N)} \to 0. \tag{58}$$

VI.2. Existence of a profile for the imaginary part of a blowup solution to a complex-valued semilinear heat equation

Let us consider here the following complex heat equation

$$\begin{cases} \partial_t u &= \Delta u + u^p, \\ u(x,0) &= u_0(x) \in L^\infty\left(\mathbb{R}^N, \mathbb{C}\right), \end{cases} \tag{59}$$

where $u : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{C}$ and $p > 1$.

Our goal is to construct a blowup solution to equation (59), and to describe its asymptotic behavior as we did with (53).

a) *Integer case for p*

Inspired by the works of Nouaili and Zaag in [68] (N dimensions) and Harada in [43] (1 dimension) who treated the case $p = 2$, we extended in [17] the results of [68] to arbitrary $p > 1$ which takes an integer value. Moreover, we obtained a better result than the one in [68], in the sense that we derived the profile of the imaginary part. More precisely, we have the following result:

Theorem 0.6 (See Theorem 1.1, page 6 in [17])**.** *For each $p \geq 2, p \in \mathbb{N}$ and $p_1 \in (0,1)$, there exists $T_1(p, p_1) > 0$ such that for all $T \leq T_1$, there exist initial data $u^0 = u_1^0 + iu_2^0$, such that equation* (59) *has a unique solution on $[0,T)$, satisfying the following:*

i) The solution u blows up in finite time T only at the origin. Moreover, it satisfies the following estimates

$$\left\| (T-t)^{\frac{1}{p-1}} u(.,t) - f_0 \left(\frac{.}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{\sqrt{|\ln(T-t)|}}, \tag{60}$$

and

$$\left\| (T-t)^{\frac{1}{p-1}} |\ln(T-t)| u_2(.,t) - g_0 \left(\frac{.}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{|\ln(T-t)|^{\frac{p_1}{2}}}, \tag{61}$$

where f_0 is defined in (56) *and $g_0(z)$ is defined as follows*

$$g_0(z) = \frac{|z|^2}{\left(p - 1 + \frac{(p-1)^2}{4p}|z|^2\right)^{\frac{p}{p-1}}}. \tag{62}$$

ii) There exists a complex function $u^*(x) \in C^2(\mathbb{R}^N \backslash \{0\})$ such that $u(t) \to u^* = u_1^* + iu_2^*$ as $t \to T$ uniformly on compact sets of $\mathbb{R}^N \backslash \{0\}$ and we have the following asymptotic expansions:

$$u^*(x) \sim \left[\frac{(p-1)^2 |x|^2}{8p |\ln |x||} \right]^{-\frac{1}{p-1}}, \text{ as } x \to 0. \tag{63}$$

and

$$u_2^*(x) \sim \frac{2p}{(p-1)^2} \left[\frac{(p-1)^2 |x|^2}{8p |\ln |x||} \right]^{-\frac{1}{p-1}} \frac{1}{|\ln |x||}, \text{ as } x \to 0. \tag{64}$$

Remark 0.7. *We easily derive from* (60) *that u blows up only at 0 . Note that both the real and the imaginary parts of u blow up. We also show that the singularity of u_2 is softer than u_1 because of the quantity $\frac{1}{|\ln |x||}$.*

Remark 0.8. *From the case where $p = 2$ treated by Nouaili and Zaag [68], we suspect the behavior in Theorem 0.6 to be unstable. This is due to the number of parameters in initial data. More precisely, the number of parameters used in the proof is higher than $N+1$ which is contributed from N for the blowup point and 1 for the blowup time.*

Let us mention that Theorem 0.6 naturally leaves a question: can we extend the result to the general case where $p > 1$? This question will be treated in the next section.

b) *General case for p*

In this part, we handle the case where p is not an integer number in (59). It took a long time to fine-tune and develop our method such that the result holds in general. The following is our main result (this is in fact the same statement as Theorem 0.6, if one replaces "For any integer $p \geq 2$" by "For any $p > 1$"; of course, the proof is much harder in the second case):

Theorem 0.9 (See Theorem 1.1, page 3 in [16]). *For each $p > 1$ and $p_1 \in (0,1)$, there exists $T_1(p, p_1) > 0$ such that for all $T \le T_1$, there exist initial data $u^0 = u_1^0 + iu_2^0$, such that equation* (59) *has a unique solution u on $[0,T)$ satisfying the following:*

i) The solution u blows up in finite time T only at the origin. Moreover, it satisfies the following estimates

$$\left\| (T-t)^{\frac{1}{p-1}} u(.,t) - f_0\left(\frac{\cdot}{\sqrt{(T-t)|\ln(T-t)|}}\right)\right\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{\sqrt{|\ln(T-t)|}}, \quad (65)$$

and

$$\left\| (T-t)^{\frac{1}{p-1}} |\ln(T-t)| u_2(.,t) - g_0\left(\frac{\cdot}{\sqrt{(T-t)|\ln(T-t)|}}\right)\right\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{|\ln(T-t)|^{\frac{p_1}{2}}}, \quad (66)$$

where f_0 and g_0 are defined in (56) *and* (62), *respectively.*

ii) There exists a complex function $u^*(x) \in C^2(\mathbb{R}^N \backslash \{0\})$ such that $u(t) \to u^* = u_1^* + iu_2^*$ as $t \to T$ uniformly on compact sets of $\mathbb{R}^N \backslash \{0\}$ and we have the following asymptotic expansions:

$$u^*(x) \sim \left[\frac{(p-1)^2|x|^2}{8p|\ln|x||}\right]^{-\frac{1}{p-1}}, \quad \text{as } x \to 0. \quad (67)$$

and

$$u_2^*(x) \sim \frac{2p}{(p-1)^2}\left[\frac{(p-1)^2|x|^2}{8p|\ln|x||}\right]^{-\frac{1}{p-1}} \frac{1}{|\ln|x||}, \quad \text{as } x \to 0. \quad (68)$$

VI.3. Profile of touch-down solution to a nonlocal MEMS model

In this part, we are interested in the quenching phenomenon with MEMS models. More precisely, we consider the following equation

$$\left\{ \begin{array}{rcl} \partial_t u &=& \Delta u + \dfrac{\lambda}{(1-u)^2\left(1 + \gamma \displaystyle\int_\Omega \frac{1}{1-u} dx\right)^2}, \quad x \in \Omega, t > 0, \\ u(x,t) &=& 0, x \in \partial\Omega, t > 0, \\ u(x,0) &=& u_0(x), x \in \Omega. \end{array} \right. \quad (69)$$

We construct a solution to equation (69) such that u touches down in finite time T only at one point $a \in \Omega$ (in the sense $u(a,t) \to 1$ as $t \to T$). In addition to that, we also aim at showing its asymptotic behavior in some neighborhood of the quenching point. The following are our main statements:

Theorem 0.10 (Existence of a qenching solution, see Theorem 1.1 in [18]). *Consider $\lambda > 0, \gamma > 0$ and Ω a C^2 bounded domain in $\mathbb{R}^N$ containing the origin. Then, there exist initial data $u_0 \in C^\infty(\bar{\Omega})$ such that the solution of* (69) *quenches in finite time $T = T(u_0) > 0$ only at the origin. In particular, the following holds:*

(*i*) *The intermediate profile: For all* $t \in [0, T)$

$$\left\| \frac{(T-t)^{\frac{1}{3}}}{1-u(.,t)} - \theta^* \left(3 + \frac{9}{8} \frac{|.|^2}{\sqrt{(T-t)|\ln(T-t)|}} \right)^{-\frac{1}{3}} \right\|_{L^\infty(\Omega)} \leq \frac{C}{\sqrt{|\ln(T-t)|}}, \quad (70)$$

for some $\theta^* = \theta^*(\lambda, \gamma, \Omega, T) > 0$.

(*ii*) *The final profile: There exists* $u^* \in C^2(\Omega) \cap C(\bar{\Omega})$ *such that* u *uniformly converges to* u^* *as* $t \to T$, *and*

$$1 - u^*(x) \sim \theta^* \left[\frac{9}{16} \frac{|x|^2}{|\ln |x||} \right]^{\frac{1}{3}} \text{ as } x \to 0. \quad (71)$$

In addition to that, we also proved the stability of the constructed quenching solution in Theorem 0.10 under perturbations of initial data:

Theorem 0.11 (Stability of the constructed solution, see Theorem 1.12 in [18])**.** *Let us consider* $\hat{u}$, *the solution which we constructed in Theorem 0.10 and we also define* $\hat{T}$ *as the quenching time of the solution and* $\hat{\theta}^*$ *as the coefficient in front of the profiles* (70) *and* (71). *Then, there exists a open subset* $\hat{\mathfrak{U}}_0$ *in* $C_{0,+}(\bar{\Omega})$, *containing* $\hat{u}(0)$ *such that for all initial data* $u_0 \in \hat{\mathfrak{U}}_0$, *equation* (69) *has a unique solution* u *quenching in finite time* $T(u_0)$ *at only one quenching point* $a(u_0)$. *Moreover, the asymptotic behaviors of* (70) *and* (71) *hold by replacing* $u(x - a(u_0), t)$, *and* $\hat{\theta}^*$ *by some* $\theta^*(u_0)$ *and*

$$(a(u_0), T(u_0), \theta^*(u_0)) \to (0, \hat{T}, \hat{\theta}), \text{ as } \|u_0 - \hat{u}_0\|_{C(\bar{\Omega})} \to 0.$$

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Chapter 1

Construction of a stable blowup solution with a prescribed behavior for a non-scaling invariant semilinear heat equation [1]

G. K. Duong, V. T. Nguyen and H. Zaag

Abstract: *We consider the semilinear heat equation*

$$\partial_t u = \Delta u + |u|^{p-1} u \ln^\alpha(u^2 + 2),$$

in the whole space $\mathbb{R}^N$*, where* $p > 1$ *and* $\alpha \in \mathbb{R}$*. Unlike the standard case* $\alpha = 0$*, this equation is not scaling invariant. We construct for this equation a solution which blows up in finite time* T *only at one blowup point* a*, according to the following asymptotic dynamics:*

$$u(x,t) \sim \psi(t) \left(1 + \frac{(p-1)|x-a|^2}{4p(T-t)|\ln(T-t)|}\right)^{-\frac{1}{p-1}} \quad \text{as } t \to T,$$

where $\psi(t)$ *is the unique positive solution of the ODE*

$$\psi' = \psi^p \ln^\alpha(\psi^2 + 2), \quad \lim_{t \to T} \psi(t) = +\infty.$$

The construction relies on the reduction of the problem to a finite dimensional one and a topological argument based on the index theory to get the conclusion. By the interpretation of the parameters of the finite dimensional problem in terms of the blowup time and the blowup point, we show the stability of the constructed solution with respect to perturbations in initial data. To our knowledge, this is the first successful construction for a genuinely non-scale invariant PDE of a stable blowup solution with the derivation of the blowup profile. From this point of view, we consider our result as a breakthrough.

Mathematics Subject Classification: *35K50, 35B40 (Primary); 35K55, 35K57 (Secondary).*

Keywords: *Blowup solution, Blowup profile, Stability, Semilinear heat equation, non-scaling invariant heat equation.*

[1] This work was published in *Tunisian J. Math*, vol. 1, no. 1, pp 13–45, 2019.

1.1 Introduction.

We are interested in the semilinear heat equation

$$\begin{cases} \partial_t u &= \Delta u + F(u), \\ u(0) &= u_0 \in L^\infty(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $u(t) : \mathbb{R}^N \to \mathbb{R}$, Δ stands for the Laplacian in $\mathbb{R}^N$ and

$$F(u) = |u|^{p-1} u \ln^\alpha(u^2 + 2), \quad p > 1, \quad \alpha \in \mathbb{R}. \tag{1.2}$$

By standard results, the model (1.1) is well-posed in $L^\infty(\mathbb{R}^N)$ thanks to a fixed-point argument. More precisely, there is a unique maximal solution on $[0, T)$, with $T \leq +\infty$. If $T < +\infty$, then the solution of (1.1) may develop singularities in finite time T, in the sense that

$$\|u(., t)\|_{L^\infty(\mathbb{R}^N)} \to +\infty \text{ as } t \to T.$$

In this case, T is called the blowup time of u. A given point $a \in \mathbb{R}^N$, we say that a is a blowup point of u if and only if there exists $(a_j, t_j) \to (a, T)$ as $j \to +\infty$ such that $|u(a_j, t_j)| \to +\infty$ as $j \to +\infty$.

In the special case where $\alpha = 0$, equation (1.1) becomes the standard semilinear heat equation

$$\partial_t u = \Delta u + |u|^{p-1} u. \tag{1.3}$$

As a matter of fact, equation (1.3) is invariant under the following scaling transformation

$$u \mapsto u_\lambda(x, t) := \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t). \tag{1.4}$$

An extensive literature is devoted to equation (1.3) and no review can be exhaustive. Given our interest in the construction question with a prescribed blowup behavior, we only mention previous work in this direction.

In [2], Bricmont and Kupiainen showed the existence of a solution of (1.3) such that

$$\|(T - t)^{\frac{1}{p-1}} u(a + .\sqrt{(T - t)|\ln(T - t)|}, t) - \varphi_0(.)\|_{L^\infty(\mathbb{R}^N)} \to 0, \text{ as } t \to T, \tag{1.5}$$

where

$$\varphi_0(z) = \left(p - 1 + \frac{(p-1)^2}{4p} z^2\right)^{-\frac{1}{p-1}},$$

(note that Herrero and Velázquez [9] proved the same result with a different method; note also that Bressan [1] made a similar construction in the case of an exponential nonlinearity).

Later, Merle and Zaag [13] (see also the note [12]) simplified the proof of [2] and proved the stability of the constructed solution verifying the behavior (1.5). Their method relies on the linearization of the similarity variables version around the expected profile. In that setting, the linearized operator has two positive eigenvalues, then a non-negative spectrum. In fact, they proceed in two steps:

- Reduction of an infinite dimensional problem to finite dimensional one: they show that controlling the similarity variable version around the profile reduces to the control of the components corresponding to the two positive eigenvalues.

- Then, they solve the finite dimensional problem thanks to a topological argument based on index theory.

The method of Merle and Zaag [13] has been proved to be successful in various situations. This was the case of the complex Ginzgburg-Landau equation by Masmoudi and Zaag [10] (see also Zaag [19] for an ealier work) and also for the case of a complex semilinear heat equation with no variational structure by Nouaili and Zaag [16]. We also mention the work of Tayachi and Zaag [18] (see also [17]) and the work of Ghoul, Nguyen and Zaag [6] dealing with a nonlinear heat equation with a double source depending on the solution and its gradient in a critical way. In [5], Ghoul, Nguyen and Zaag successfully adapted the method to construct a stable blowup solution for a non variational semilinear parabolic system.

In other words, the method of [13] was proved to be efficient even for the case of systems with non variational structure. However, all the previous examples enjoy a common scaling invariant property like (1.4), which seemed at first to be a strong requirement for the method. In fact, this was proved to be untrue.

In addition to that, Ebde and Zaag [3] were able to adapt the method to construct blowup solutions for the following non scaling invariant equation

$$\partial_t u = \Delta u + |u|^{p-1}u + f(u, \nabla u), \tag{1.6}$$

where

$$|f(u, \nabla u)| \leq C(1 + |u|^q + |\nabla u|^{q'}), \text{ with } q < p, q' < \frac{2p}{p+1}.$$

These conditions ensure that the perturbation $f(u, \nabla u)$ turns out to exponentially small coefficients in the similarity variables. Later, Nguyen and Zaag [15] did a more spectacular achievement by addressing the case of stronger perturbation of (1.3), namely

$$\partial_t u = \Delta u + |u|^{p-1}u + \frac{\mu |u|^{p-1}u}{\ln^a(2 + u^2)}, \tag{1.7}$$

where $\mu \in \mathbb{R}$ and $a > 0$. When moving to the similarity variables, the perturbation turns out to have a polynomial decay. Hence, when $a > 0$ is small, we are almost in the case of a critical perturbation.

In both cases addressed in [3] and [15], the equations are indeed non-scaling invariant, which shows the robustness of the method. However, since both papers proceed by perturbations around the standard case (1.3), it is as if we are still in the scaling invariant case.

In this paper, we aim at trying the approach on a genuinely non-scaling invariant case, namely equation (1.1). The following is our main result.

Theorem 1.1 (Blowup solution for equation (1.1) with a prescribed behavior)**.** *There exist initial data $u_0 \in L^\infty(\mathbb{R}^N)$ such that the corresponding solution to equation* (1.1) *blows up in finite time $T = T(u_0) > 0$, only at the origin. Moreover, we have*

(i) For all $t \in [0, T)$, there exists a positive constant C_0 such that

$$\left\| \psi^{-1}(t)u(.,t) - f_0\left(\frac{.}{\sqrt{(T-t)|\ln(T-t)|}}\right)\right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C_0}{\sqrt{|\ln(T-t)|}}, \tag{1.8}$$

where $\psi(t)$ is the unique positive solution of the following ODE

$$\psi'(t) = \psi^p(t) \ln^\alpha(\psi^2(t) + 2), \quad \lim_{t \to T} \psi(t) = +\infty, \tag{1.9}$$

(see Lemma 1.17 for the existence and uniqueness of ψ), and profile f_0 is defined by

$$f_0(z) = \left(1 + \frac{(p-1)}{4p}|z|^2\right)^{-\frac{1}{p-1}}. \tag{1.10}$$

(*ii*) *There exits $u^*(x) \in C^2(\mathbb{R}^N \backslash \{0\})$ such that $u(x,t) \to u^*(x)$ as $t \to T$ uniformly on compact sets of $\mathbb{R}^N \setminus \{0\}$, where*

$$u^*(x) \sim \left[\frac{(p-1)^2|x|^2}{8p|\ln|x||}\right]^{-\frac{1}{p-1}} \left(\frac{4|\ln|x||}{p-1}\right)^{-\frac{\alpha}{p-1}} \quad \text{as } x \to 0, \tag{1.11}$$

Remark 1.2. *From* (*i*)*, we see that $u(0,t) \sim \psi(t) \to +\infty$ as $t \to T$, which means that the solution blows up in finite time T at $x = 0$. From* (*ii*)*, we deduce that the solution blows up only at the origin.*

Remark 1.3. *Note that the behavior in* (1.8) *is almost the same as the standard case $\alpha = 0$ treated in [2] and [13]. However, the final profile u^* has a difference coming from the extra multiplication of the size $|\ln|x||^{-\frac{\alpha}{p-1}}$, which shows that the nonlinear source in equation* (1.1) *has a strong effect to the dynamic of the solution in comparison with the standard case $\alpha = 0$.*

Remark 1.4. *Item* (*ii*) *is in fact a consequence of* (1.8) *and Lemma 1.20. Therefore, the main goal of this paper is to construct for equation* (1.1) *a solution blowing up in finite time and verifying the behavior* (1.8).

Remark 1.5. *By the parabolic regularity, one can show that if initial data $u_0 \in W^{2,\infty}(\mathbb{R}^N)$, then we have for $i = 0, 1, 2$,*

$$\left\|\psi^{-1}(t)(T-t)^{\frac{i}{2}}\nabla_x^i u(.,t) - (T-t)^{\frac{i}{2}}\nabla_x^i f_0\left(\frac{.}{\sqrt{(T-t)|\ln(T-t)|}}\right)\right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{\sqrt{|\ln(T-t)|}},$$

where f_0 is defined by (1.10).

From the technique of Merle [11], we can prove the following result.

Corollary 1.6. *For arbitrary given set of m points $x_1, ..., x_m$. There exists initial data u_0 such that the solution u of* (1.1) *with initial data u_0 blows up exactly at m points $x_1, ..., x_m$. Moreover, the local behavior at each blowup point x_i is also given as in* (1.8) *by replacing x by $x - x_i$ and $L^\infty(\mathbb{R}^N)$ by $L^\infty(|x - x_i| \leq \epsilon_i)$ for some $\epsilon_i > 0$ small enough.*

As a consequence of our technique, we prove the stability of the solution constructed in Theorem 1.1 under the perturbations of initial data. In particular, we have the following result.

Theorem 1.7 (Stability of the solution constructed in Theorem 1.1)**.** *Consider $\hat{u}$ the solution constructed in Theorem 1.1 and denote by $\hat{T}$ its blowup time. Then there exists $\mathcal{U}_0 \subset L^\infty(\mathbb{R}^N)$ a neighborhood of $\hat{u}(0)$ such that for all $u_0 \in \mathcal{U}_0$, equation* (1.1) *with initial data u_0 has a unique solution $u(t)$ blowing up in finite time $T(u_0)$ at a single point $a(u_0)$. Moreover, the statements* (*i*) *and* (*ii*) *in Theorem 1.1 are satisfied by $u(x - a(u_0), t)$, and*

$$(T(u_0), a(u_0)) \to (\hat{T}, 0) \text{ as } \|u_0 - \hat{u}_0\|_{L^\infty(\mathbb{R}^N)} \to 0. \tag{1.12}$$

Remark 1.8. *We will not give the proof of Theorem 1.7 because the stability result follows from the reduction to a finite-dimensional case as in [13] with the same proof. Here we only prove the existence and refer to [13] for the stability.*

1.2 Formulation of the problem.

In this section, we first use the matched asymptotic technique to formally derive the behavior (1.8). Then, we give the formulation of the problem in order to justify the formal result.

1.2.1 A formal approach.

In this part, we follow the approach of Tayachi and Zaag [18] to formally explain how to derive the asymptotic behavior (1.8). In fact, we introduce the following self-similarity variables

$$u(x,t) = \psi(t) w(y,s), \quad y = \frac{x}{\sqrt{T-t}}, \quad s = -\ln(T-t), \tag{1.13}$$

where $\psi(t)$ is the unique positive solution of equation (1.9) and $\psi(t) \to +\infty$ as $t \to T$. Then, we see from (1.1) that $w(y,s)$ solves the following equation: for all $(y,s) \in \mathbb{R}^N \times [-\ln T, +\infty)$

$$\partial_s w = \Delta w - \frac{1}{2} y.\nabla w - h(s) w + h(s)|w|^{p-1} w \frac{\ln^\alpha(\psi_1^2 w^2 + 2)}{\ln^\alpha(\psi_1^2 + 2)}, \tag{1.14}$$

where

$$h(s) = e^{-s} \psi_1^{p-1}(s) \ln^\alpha(\psi_1^2(s) + 2), \tag{1.15}$$

and

$$\psi_1(s) = \psi(T - e^{-s}). \tag{1.16}$$

Note that $h(s)$ admits the following asymptotic behavior as $s \to +\infty$,

$$h(s) = \frac{1}{p-1}\left(1 - \frac{\alpha}{s} - \frac{\alpha^2 \ln s}{s^2}\right) + O\left(\frac{1}{s^2}\right), \tag{1.17}$$

(see item ii) in Lemma 1.21 for the proof of (1.17)). From (1.13), we see that the study of the asymptotic behavior of $u(x,t)$ as $t \to T$ is equivalent to the study of the long time behavior of $w(y,s)$ as $s \to +\infty$. In other words, the construction of the solution $u(x,t)$, which blows up in finite time T and verifies the behavior (1.8), reduces to the construction of a global solution $w(y,s)$ for equation (1.14) satisfying

$$0 < \epsilon_0 \le \limsup_{s \to +\infty} \|w(s)\|_{L^\infty(\mathbb{R}^N)} \le \frac{1}{\epsilon_0} \quad \text{for some } \epsilon_0 > 0, \tag{1.18}$$

and

$$\left\| w(y,s) - \left(1 + \frac{(p-1)y^2}{4ps}\right)^{-\frac{1}{p-1}} \right\|_{L^\infty(\mathbb{R}^N)} \to 0 \quad \text{as } s \to +\infty. \tag{1.19}$$

In the following, we will formally explain how to derive the behavior (1.19).

Inner expansion

We remark that $0, \pm 1$ are the trivial constant solutions to equation (1.14). Since we are looking for a non zero solution, let us consider the case when $w \to 1$ as $s \to +\infty$ (up to replacing w by $-w$ if necessary). We now introduce

$$w = 1 + \bar{w}, \tag{1.20}$$

then from equation (1.14), we see that $\bar{w}$ satisfies

$$\partial_s \bar{w} = \mathcal{L}(\bar{w}) + N(\bar{w}, s), \tag{1.21}$$

where

$$\mathcal{L} = \Delta - \frac{1}{2} y.\nabla + \mathrm{Id}, \tag{1.22}$$

and

$$N(\bar{w}, s) = h(s)|\bar{w}+1|^{p-1}(\bar{w}+1)\frac{\ln^\alpha(\psi_1^2(\bar{w}+1)^2+2)}{\ln^\alpha(\psi_1^2+2)} - h(s)(\bar{w}+1) - \bar{w}, \tag{1.23}$$

with $\psi_1(s)$ and $h(s)$ are defined in (1.16) and (1.15), respectively. Note that N admits the following asymptotic behavior (see Lemma 1.22 for the proof of this one):

$$N(\bar{w}, s) = \frac{p\bar{w}^2}{2} + O\left(\frac{|\bar{w}| \ln s}{s^2}\right) + O\left(\frac{|\bar{w}|^2}{s}\right) + O(|\bar{w}|^3) \quad \text{as} \quad (\bar{w}, s) \to (0, +\infty). \tag{1.24}$$

Since $\bar{w}(s) \to 0$ as $s \to +\infty$ and N is formally "quadratic" in $\bar{w}$, we see from equation (1.21) that the linear part will play the main role in the analysis of our solution. Let us recall some properties of $\mathcal{L}$. In fact, $\mathcal{L}$ is self-adjoint in $\mathcal{D}(\mathcal{L}) \subset L^2_\rho(\mathbb{R}^N)$, where $L^2_\rho(\mathbb{R}^N)$ is the weighted space associated with the weight ρ defined by

$$\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{\frac{N}{2}}},$$

and

$$\mathrm{Spec}\,\mathcal{L} = \left\{1 - \frac{m}{2}, m \in \mathbb{N}\right\}.$$

More precisely, we have

- When $N = 1$, all the eigenvalues of $\mathcal{L}$ are simple and the eigenfunction corresponding to the eigenvalue $1 - \frac{m}{2}$ is the Hermite polynomial defined by

$$h_m(y) = \sum_{j=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^j m! y^{m-2j}}{j!(m-2j)!}. \tag{1.25}$$

  In particular, we have the following orthogonality

$$\int_{\mathbb{R}} h_i h_j \rho dy = i! 2^i \delta_{i,j}, \quad \forall (i,j) \in \mathbb{N}^2.$$

- When $N \geq 2$, the eigenspace corresponding to the eigenvalue $1 - \frac{m}{2}$ is defined as follows

$$\mathcal{E}_m = \left\langle h_\beta = h_{\beta_1} \cdots h_{\beta_N}, \text{ for all } \beta \in \mathbb{N}^N, |\beta| = m, |\beta| = \beta_1 + \cdots + \beta_N \right\rangle. \tag{1.26}$$

Since the eigenfunctions of $\mathcal{L}$ is a basic of L^2_ρ, we can expand $\bar{w}$ in this basic as follows

$$\bar{w}(y,s) = \sum_{\beta \in \mathbb{N}^N} \bar{w}_\beta(s) h_\beta(y).$$

For simplicity, let us assume that $\bar{w}$ is radially symmetric in y. Since h_β with $|\beta| \geq 3$ corresponds to negative eigenvalues of $\mathcal{L}$, we may consider the solution $\bar{w}$ taking the form

$$\bar{w} = \bar{w}_0 + \bar{w}_2(s)(|y|^2 - 2N), \tag{1.27}$$

where $|\bar{w}_0(s)|$ and $|\bar{w}_2(s)|$ go to 0 as $s \to +\infty$. Injecting (1.27) and (1.24) into (1.21), then projecting equation (1.21) on the eigenspace $\mathcal{E}_m$ with $m = 0$ and $m = 2$,

$$\begin{cases} \bar{w}_0' = \bar{w}_0 + \frac{p}{2}\left(\bar{w}_0^2 + 8n\bar{w}_2^2\right) + O\left(\frac{(|\bar{w}_0| + |\bar{w}_2|)\ln s}{s^2}\right) \\ \qquad + O\left(\frac{|\bar{w}_0|^2 + |\bar{w}_2|^2}{s}\right) + O\left(|\bar{w}_0|^3 + |\bar{w}_2|^3\right), \\ \bar{w}_2' = 4p\bar{w}_2^2 + p\bar{w}_0\bar{w}_2 + O\left(\frac{(|\bar{w}_0| + |\bar{w}_2|)\ln s}{s^2}\right) \\ \qquad + O\left(\frac{|\bar{w}_0|^2 + |\bar{w}_2|^2}{s}\right) + O\left(|\bar{w}_0|^3 + |\bar{w}_2|^3\right), \end{cases} \tag{1.28}$$

as $s \to +\infty$. In addition to that, we now assume that $|\bar{w}_0(s)| \ll |\bar{w}_2(s)|$ as $s \to +\infty$, then (1.29) becomes the following

$$\begin{cases} \bar{w}_0' = \bar{w}_0 + O(|\bar{w}_2|^2) + O\left(\frac{|\bar{w}_2|\ln s}{s^2}\right), \\ \bar{w}_2' = 4p\bar{w}_2^2 + o(|\bar{w}_2|^2) + O\left(\frac{|\bar{w}_2|\ln s}{s^2}\right), \end{cases} \quad \text{as } s \to +\infty. \tag{1.29}$$

Let us consider the following cases:

- Case 1: Either $|\bar{w}_2| = O\left(\frac{\ln s}{s^2}\right)$ or $|\bar{w}_2| \ll \frac{\ln s}{s}$ as $s \to +\infty$, then the second equation in (1.29) becomes

$$\bar{w}_2' = O\left(\frac{|\bar{w}_2|\ln s}{s^2}\right) \text{ as } s \to +\infty,$$

which yields

$$\ln|\bar{w}_2| = O\left(\frac{\ln s}{s}\right) \text{ as } s \to +\infty,$$

this contradicts the assumption that $\bar{w}_2(s) \to 0$ as $s \to +\infty$.

- Case 2: $|\bar{w}_2| \gg \frac{\ln s}{s^2}$ as $s \to +\infty$, then (1.29) becomes

$$\begin{cases} \bar{w}_0' = \bar{w}_0 + O(|\bar{w}_2|^2), \\ \bar{w}_2' = 4p\bar{w}_2^2 + o(|\bar{w}_2|^2), \end{cases} \text{ as } s \to +\infty.$$

This yields

$$\begin{cases} \bar{w}_0 = O\left(\frac{1}{s^2}\right), \\ \bar{w}_2 = -\frac{1}{4ps} + o(\frac{1}{s}), \end{cases} \text{ as } s \to +\infty. \tag{1.30}$$

Substituting (1.30) into (1.29) yields

$$\begin{cases} \bar{w}_0' = O\left(\frac{1}{s^2}\right), \\ \bar{w}_2' = 4p\bar{w}_2^2 + O\left(\frac{\ln s}{s^3}\right), \end{cases} \text{ as } s \to +\infty,$$

from which we improve the error for $\bar{w}_2$ as follows

$$\begin{cases} \bar{w}_0 = O\left(\dfrac{1}{s^2}\right), \\ \bar{w}_2 = -\dfrac{1}{4ps} + O\left(\dfrac{\ln^2 s}{s^2}\right), \end{cases} \text{ as } s \to +\infty. \tag{1.31}$$

Thus, from (1.20), (1.27) and (1.31), we derive

$$w(y,s) = 1 - \frac{y^2}{4ps} + \frac{N}{2ps} + O\left(\frac{\ln^2 s}{s^2}\right) \text{ in } L^2_\rho(\mathbb{R}^N), \tag{1.32}$$

as $s \to +\infty$. Note that the asymptotic expansion (1.32) also holds for all $|y| \le K$ for some $K > 0$.

Outer expansion

The asymptotic behavior of (1.32) suggests that the blowup profile may be depend on the following variable

$$z = \frac{y}{\sqrt{s}},$$

From (1.32), let us try to search a regular solution of equation (1.14) of the form

$$w(y,s) = \phi_0(z) + \frac{N}{2ps} + o\left(\frac{1}{s}\right) \text{ in } L^\infty_{loc}(\mathbb{R}^N) \text{ as } s \to +\infty, \tag{1.33}$$

where ϕ_0 is a bounded, smooth function to be determined. From (1.32), we impose the following condition

$$\phi_0(0) = 1 \text{ and } \phi_0(z) \ge 0. \tag{1.34}$$

Since $w(y,s)$ is supposed to be bounded, we obtain from Lemma 1.23 that

$$\left| h(s)|w|^{p-1}w\frac{\ln^\alpha(\psi_1^2w^2+2)}{\ln^\alpha(\psi_1^2+2)} - \frac{|w|^{p-1}w}{p-1}\right| \leq \frac{C}{s},$$

Note also from (1.33) that

$$\left||w|^{p-1}w - |\phi_0(z)|^{p-1}\phi_0(z)\right| = O\left(\frac{1}{s}\right) \text{ in } L^\infty_{loc}(\mathbb{R}^N) \text{ as } s\to+\infty$$

Injecting (1.33) into equation (1.14) and comparing terms of order $O(1)$, we derive the following equation

$$-\frac{1}{2}z.\nabla\phi_0(z) - \frac{\phi_0(z)}{p-1} + \frac{|\phi_0|^{p-1}\phi_0(z)}{p-1} = 0, \quad \forall z\in\mathbb{R}^N. \tag{1.35}$$

Solving (1.35) with condition (1.34), we obtain

$$\phi_0(z) = \left(1+c_0|z|^2\right)^{-\frac{1}{p-1}}, \tag{1.36}$$

for some constant $c_0\geq 0$ (since we want ϕ_0 to be bounded for all $z\in\mathbb{R}^N$). From (1.33), (1.36) and a Taylor expansion, we obtain

$$w(y,s) = 1 - \frac{c_0y^2}{(p-1)s} + \frac{N}{2ps} + o\left(\frac{1}{s}\right), \quad \forall|y|\leq K \text{ as } s\to+\infty,$$

from which and the asymptotic behavior (1.32), we find that

$$c_0 = \frac{p-1}{4p}.$$

In conclusion, we have just derived the following asymptotic profile

$$w(y,s)\sim\varphi(y,s) \quad \text{as} \quad s\to+\infty, \tag{1.37}$$

where

$$\varphi(y,s) = \left(1+\frac{(p-1)y^2}{4ps}\right)^{-\frac{1}{p-1}} + \frac{N}{2ps}. \tag{1.38}$$

1.2.2 Formulation of the problem.

In this subsection, we set up the problem in order to justify the formal approach presented in the Section 1.2.1. In particular, we give a formulation to prove item (i) of Theorem 1.1. We aim at constructing for equation (1.1) a solution blowing up in finite time T only at the origin and verifying the behavior (1.8). In comparison with (1.13), our problem is reduced to the construction of a solution $w(y,s)$ for equation (1.14) defined for all $(y,s)\in\mathbb{R}^N\times[s_0,+\infty)$, $s_0=-\ln T$ and satisfying (1.19). The formal approach given in Subsection

1.2.1 (see (1.37)), we are interested in the linearization w around profile φ, defined by (1.38). Let us introduce

$$q(y,s) = w(y,s) - \varphi(y,s), \tag{1.39}$$

where φ is defined in (1.38). From (1.14), we see that q satisfies the following equation

$$\partial_s q = \mathcal{L} q + Vq + B(q) + R(y,s) + D(q,s), \tag{1.40}$$

where $\mathcal{L}$ is the linear operator defined in (1.22), and

$$V = \frac{p}{p-1}\left[\varphi^{p-1} - 1\right], \tag{1.41}$$

$$B(q) = \frac{|q+\varphi|^{p-1}(q+\varphi) - \varphi^p - p\varphi^{p-1}q}{p-1}, \tag{1.42}$$

$$R(y,s) = \Delta\varphi - \frac{1}{2} y \nabla \varphi - \frac{\varphi}{p-1} + \frac{\varphi^p}{p-1} - \partial_s \varphi, \tag{1.43}$$

and D is defined as follows

$$D(q,s) = (q+\varphi)\left(\left(h(s) - \frac{1}{p-1}\right)(|q+\varphi|^{p-1} - 1) + h(s)|q+\varphi|^{p-1}(q+\varphi)L(q+\varphi,s)\right), \tag{1.44}$$

where

$$L(v,s) = \frac{2\alpha\psi_1^2}{\ln(\psi_1^2+2)(\psi_1^2+2)}(v-1) + \frac{1}{\ln^\alpha(\psi_1^2+2)}\int_1^v f''(u)(v-u)du, \tag{1.45}$$

and $h, \psi_1(s)$ and φ being defined by (1.15), (1.16) and (1.38) respectively, and

$$f(z) = \ln^\alpha(\psi_1^2 z^2 + 2), z \in \mathbb{R}.$$

Thus, problem (1.8) is reduced to construct for equation (1.40) a solution q such that

$$\|q(.,s)\|_{L^\infty(\mathbb{R}^N)} \to 0 \text{ as } s \to +\infty.$$

Since we construct for equation (1.40) a solution q verifying $\|q(s)\|_{L^\infty} \to 0$ as $s \to +\infty$, and the fact that

$$|B(q)| \le C|q|^{\min(2,p)}, \quad \|R(s)\|_{L^\infty(\mathbb{R}^N)} + \|D(q,s)\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{s},$$

(see Lemmas 1.24, 1.25 and 1.26 for these estimates), we see that the linear part of equation (1.40) will play an important role in the analysis of the solution. The property of the linear operator $\mathcal{L}$ has been studied in previous section (see page 39), and the potential V has the following properties:

i) Perturbation of effect of $\mathcal{L}$ inside the blowup region $\{|y| \le K\sqrt{s}\}$:

$$\|V(s)\|_{L^2_\rho} \to 0 \quad \text{as} \quad s \to +\infty.$$

ii) For each $\epsilon > 0$, there exist $K_\epsilon > 0$ and $s_\epsilon > 0$ such that

$$\sup_{\frac{y}{\sqrt{s}} \ge K_\epsilon, s \ge s_\epsilon} \left| V(y,s) + \frac{p}{p-1} \right| \le \epsilon.$$

Since 1 is the biggest eigenvalue of $\mathcal{L}$, the operator $\mathcal{L} + V$ behaves as one with a fully negative spectrum outside blowup region $\{|y| \geq K\sqrt{s}\}$, which makes the control of the solution in this region easily.

Since the behavior of the potential V inside and outside the blowup region is different, we will consider the dynamics of the solution for $|y| \leq 2K\sqrt{s}$ and for $|y| \geq K\sqrt{s}$ separately for some K to be fixed large. We introduce the following function

$$\chi(y,s) = \chi_0\left(\frac{|y|}{K\sqrt{s}}\right), \tag{1.46}$$

where $\chi_0 \in C_0^\infty[0,+\infty)$, $\|\chi_0\|_{L^\infty(\mathbb{R}^N)} \leq 1$ and

$$\chi_0(x) = \begin{cases} 1 & \text{for } x \leq 1, \\ 0 & \text{for } x \geq 2, \end{cases}$$

and K is a positive constant to be fixed large later. We now decompose q by

$$q = \chi q + (1-\chi)q = q_b + q_e. \tag{1.47}$$

(Note that $\mathrm{supp}(q_b) \subset \{|y| \leq 2K\sqrt{s}\}$ and $\mathrm{supp}(q_e) \subset \{|y| \geq K\sqrt{s}\}$). Since the eigenfunctions of $\mathcal{L}$ span the whole space $L^2_\rho(\mathbb{R}^N)$, let us write

$$q_b(y,s) = q_0(s) + q_1(s)\cdot y + y^T \cdot q_2(s)\cdot y - 2\mathrm{Tr}(q_2(s)) + q_-(y,s), \tag{1.48}$$

where $q_m(s) = \big(q_\beta(s)\big)_{\beta\in\mathbb{N}^N, |\beta|=m}$ and

$$\forall \beta \in \mathbb{N}^N, \quad q_\beta(s) = \int_{\mathbb{R}^N} q_b(y,s)\tilde{h}_\beta(y)\rho dy, \quad \tilde{h}_\beta = \frac{h_\beta}{\|h_\beta\|^2_{L^2_\beta}}, \tag{1.49}$$

and

$$q_-(y,s) = \sum_{\beta\in\mathbb{N}^N, |\beta|\geq 3} q_\beta(s)h_\beta(y). \tag{1.50}$$

In particular, we denote $q_1 = (q_{1,i})_{1\leq i\leq N}$ and $q_2(s)$ is a $N\times N$ symmetric matrix defined explicitly by

$$q_2(s) = \int q_b \mathcal{M}(y)\rho dy = (q_{2,i,j})_{1\leq i,j\leq N}, \tag{1.51}$$

with

$$\mathcal{M}(y) = \left\{\frac{1}{8}y_i y_j - \frac{\delta_{i,j}}{4}\right\}_{1\leq i,j\leq N}. \tag{1.52}$$

Thus, by (1.47) and (1.48), we can write

$$q(y,s) = q_0(s) + q_1(s)\cdot y + y^T\cdot q_2(s)\cdot y - 2\mathrm{Tr}(q_2(s)) + q_-(y,s) + q_e(y,s). \tag{1.53}$$

Note that $q_m (m = 0,1,2)$ and q_- are the components of q_b, and not those of q.

1.3 Proof of the existence assuming some technical results

In this section, we shall describe the main argument behind the proof of Theorem 1.1. In order to avoid winding up with details, we shall postpone most of the technicalities involved to the next section.

According to transformations (1.13) and (1.39), the proof of item (i) of Theorem 1.1 is equivalent to showing that there exists initial data $q_0(y)$ at the time s_0 such that the corresponding solution q of equation (1.40) satisfies

$$\|q(s)\|_{L^\infty(\mathbb{R}^N)} \to 0 \text{ as } s \to +\infty.$$

In particular, we consider the following function

$$\psi_{d_0,d_1}(y) = \frac{A}{s_0^2}\,(d_0 + d_1.y)\,\chi(2y, s_0), \tag{1.54}$$

as initial data for equation (1.40), where $(d_0, d_1) \in \mathbb{R}^{1+N}$ are the parameters to be determined, $s_0 > 1$ and $A > 1$ are constants to be fixed large enough, and χ is the function defined by (1.46).

We aim at proving that there exists $(d_0, d_1) \in \mathbb{R} \times \mathbb{R}^N$ such that the solution $q(y, s) = q_{d_0,d_1}(y, s)$ of (1.40) with initial data $\psi_{d_0,d_1}(y)$ satisfies

$$\|q_{d_0,d_1}(s)\|_{L^\infty(\mathbb{R}^N)} \to 0 \text{ as } s \to +\infty.$$

More precisely, we will show that there exists $(d_0, d_1) \in \mathbb{R}\times\mathbb{R}^N$ such that solution $q_{d_0,d_1}(y, s)$ belongs to the shrinking set S_A defined as follows:

Definition 1.1 (A shrinking set to zero)**.** *For all $A \geq 1, s \geq 1$ we define $S_A(s)$ being the set of all functions $q \in L^\infty(\mathbb{R}^N)$ such that*

$$|q_0| \leq \frac{A}{s^2}, \quad |q_{1,i}| \leq \frac{A}{s^2}, \quad |q_{2,i,j}| \leq \frac{A^2 \ln^2 s}{s^2}, \quad \forall 1 \leq i, j \leq N,$$
$$\left\| \frac{q_-(y)}{1 + |y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A}{s^2}, \quad \|q_e(y)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^2}{\sqrt{s}},$$

where q_0, $q_1 = \left(q_{1,i}\right)_{1\leq i\leq N}$, $q_2 = \left(q_{2,i,j}\right)_{1\leq i,j\leq N}$, q_- and q_e are defined as in (1.53)*.*

We also denote by $\hat{S}_A(s)$ being the set

Remark 1.9. *For each $A \geq 1, s \geq 1$, we have the following estimates for all $q(s) \in S_A(s)$:*

$$|q(y, s)| \leq \frac{CA^2 \ln^2 s}{s^2}(1 + |y|^3), \quad \forall y \in \mathbb{R}^N, \tag{1.55}$$
$$\|q(s)\|_{L^\infty(\{|y|\leq 2K\sqrt{s}\})} \leq \frac{CA}{\sqrt{s}}, \tag{1.56}$$
$$\|q(s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{CA^2}{\sqrt{s}}. \tag{1.57}$$

In fact, we aim at proving the following central proposition which implies Theorem 1.1:

Proposition 1.10 (Existence of a solution trapped in $S_A(s)$)**.** *There exists $A_1 \geq 1$ such that for all $A \geq A_1$, there exists $s_1(A) \geq 1$ such that for all $s_0 \geq s_1(A)$, there exists $(d_0, d_1) \in \mathbb{R}^{1+N}$ such that the solution $q(y,s) = q_{d_0,d_1}(y,s)$ of* (1.40) *with initial data $q(y, s_0) = \psi_{d_0,d_1}(y)$ defined in* (1.54)*, satisfies*

$$q(s) \in S_A(s), \quad \forall s \in [s_0, +\infty).$$

From (1.57), we see that once Proposition 1.10 is proved, item (i) of Theorem 1.1 directly follows. In the following, we shall give all the main arguments for the proof of this proposition assuming some technical results which are left to the next section.

As for initial data at time s_0 defined as in (1.54), we have the following properties:

Proposition 1.11 (Properties of initial data (1.54))**.** *For each $A \geq 1$, there exists $s_2(A) > 1$ such that for all $s_0 \geq s_2(A)$ we have the following properties:*

i) There exists $\mathbb{D}_{A,s_0} \subset [-2; 2] \times [-2; 2]^N$ such that the mapping

$$\begin{array}{rcl} \Phi_1 : \mathbb{R}^{1+N} & \to & \mathbb{R}^{1+N}, \\ (d_0, d_1) & \mapsto & (\psi_0, \psi_1) \end{array}$$

is linear, one to one from $\mathbb{D}_{A,s_0}$ onto $\hat{S}_A(s_0)$. Moreover, we have

$$\Phi_1 \left(\partial \mathbb{D}_{A,s_0}\right) \subset \partial \hat{S}_A(s_0),$$

where $\hat{S}_A(s)$ is defined as follows:

$$\hat{S}_A(s) = \left[-\frac{A}{s^2}, \frac{A}{s^2}\right] \times \left[-\frac{A}{s^2}, \frac{A}{s^2}\right]^N. \tag{1.58}$$

ii) For all $(d_0, d_1) \in \mathbb{D}_{A,s_0}$, we have $\psi_{d_0,d_1} \in S_A(s_0)$ with strict inequalities in the sense that

$$|\psi_0| \leq \frac{A}{s_0^2}, \quad |\psi_{1,i}| \leq \frac{A}{s_0^2}, \quad |\psi_{2,i,j}| < \frac{A \ln^2 s_0}{s_0^2}, \quad \forall 1 \leq i, j \leq N,$$
$$\left\| \frac{\psi_-}{1 + |y|^3} \right\|_{L^\infty(\mathbb{R}^N)(\mathbb{R})} < \frac{A}{s_0^2}, \quad \psi_e \equiv 0.$$

where $\chi(y, s_0)$ is defined in (1.46)*, $\psi_0, (\psi_{1,i})_{1 \leq i \leq N}, (\psi_{2,i,j})_{1 \leq i,j \leq N}, \psi_-, \psi_e$ are the components of ψ_{d_0,d_1} defined as in* (1.53)*, ψ_{d_0,d_1} is defined by* (1.54)*.*

Proof. See Propositon 4.5 in Tayachi and Zaag [18] for a similar proof to this proposition. ∎

From now on, we denote by C as the universal constant which only depends on K, where K is introduced in (1.46). Let us now give the proof of Proposition 1.10 to complete the proof of item (i) of Theorem 1.1.

Proof of Proposition 1.10. We proceed into two steps to prove Proposition 1.10:

- In the first step, we reduce the problem of controlling $q(s)$ in $S_A(s)$ to the control of $(q_0, q_1)(s)$ in $\hat{S}_A(s)$, where q_0 and q_1 are the component of q corresponding to the positive modes defined as in (1.53) and $\hat{S}_A$ is defined by (1.58). This means that we reduce the problem to a finite dimensional one.

- In the second step, we argue by contradiction to solve the finite dimensional problem thanks to a topological argument.

Step 1: Reduction to a finite dimensional problem

In this step, we show through *a priori estimate* that the control of $q(s)$ in $S_A(s)$ reduces to the control of $(q_0, q_1)(s)$ in $\hat{S}_A(s)$. This mainly follows from a good understanding of the properties of the linear part $\mathcal{L} + V$ of equation (1.40). In particular, we claim the following which is the heart of our analysis.

Proposition 1.12 (Control of $q(s)$ in $S_A(s)$ by $(q_0, q_1)(s)$ in $\hat{S}_A(s)$)**.** *There exists $A_3 \geq 1$ such that for all $A \geq A_3$, there exists $s_3(A) \geq 1$ such that for all $s_0 \geq s_3(A)$, the following holds: If $q(y, s)$ is the solution of equation* (1.40) *with initial data at time s_0, given by* (1.54) *with $(d_0, d_1) \in \mathbb{D}_{A,s_0}$, and $q(s) \in S_A(s)$ for all $s \in [s_0, s_1]$ for some $s_1 \geq s_0$ and $q(s_1) \in \partial S_A(s_1)$, then:*
(i) (Reduction to a finite dimensional problem): We have $(q_0, q_1)(s_1) \in \partial \hat{S}_A(s_1)$.
(ii) (Transverse outgoing crossing): There exists $\delta_0 > 0$ such that

$$\forall \delta \in (0, \delta_0), \quad (q_0, q_1)(s_1 + \delta) \notin \hat{S}_A(s_1 + \delta).$$

Hence, $q(s_1 + \delta) \notin S_A(s_1 + \delta)$, where $\hat{S}_A$ is defined in (1.58) *and $\mathbb{D}_{A,s_0}$ is introduced in Proposition 1.11.*

Let us suppose for the moment that Proposition 1.12 holds. Then, we can take advantage of a topological argument quite similar to that already used in Merle and Zaag [13].

Step 2: A basic topological argument

From Proposition 1.12, we claim that there exists $(d_0, d_1) \in \mathbb{D}_{A,s_0}$ such that equation (1.40) with initial data given as in (1.54), has a solution

$$q_{d_0,d_1}(s) \in S_A(s), \quad \forall s \in [s_0, +\infty),$$

for suitable choice of the parameters A, K and s_0. Since, the argument is analogous as in [13], we only give the main ideas.

In fact, let us consider K, A and s_0 such that Propositions 1.11 and 1.12 hold. From Proposition 1.11, we have

$$\forall (d_0, d_1) \in \mathbb{D}_{A,s_0}, \quad q_{d_0,d_1}(y, s_0) := \psi_{d_0,d_1} \in S_A(s_0),$$

where ψ_{d_0,d_1} is defined by (1.54). As a matter of fact, $\psi_{d_0,d_1} \in L^\infty(\mathbb{R}^N)$ for all $(d_0, d_1) \in \mathbb{D}_{A,s_0}$, we then deduce from the local existence theory in $L^\infty(\mathbb{R}^N)$ that we can define for each $(d_0, d_1) \in \mathbb{D}_{A,s_0}$, a maximum time $s_*(d_0, d_1) \in [s_0, +\infty)$ such that

$$q_{d_0,d_1}(s) \in S_A(s), \quad \forall s \in [s_0, s_*(d_0, d_1)).$$

If $s_*(d_0, d_1) = +\infty$ for some $(d_0, d_1) \in \mathbb{D}_{A,s_0}$, then we have the conclusion of Proposition 1.10.

Otherwise, we argue by contradiction and assume that $s_*(d_0, d_1) < +\infty$ for all $(d_0, d_1) \in \mathbb{D}_{A,s_0}$. By continuity and the definition of s_*, we deduce that $q_{d_0,d_1}(s_*)$ is on the boundary of $S_A(s_*)$. Using item (i) in Proposition 1.12, we derive the following

$$(q_0, q_1)(s_*) \in \partial \hat{S}_A(s_*).$$

Hence, we may define the rescaled function

$$\begin{aligned} \Gamma : \ \mathbb{D}_{A,s_0} &\mapsto \partial\big([-1, 1]^{1+N}\big) \\ (d_0, d_1) &\to \frac{s_*^2}{A}(q_0, q_1)(s_*). \end{aligned}$$

From item (i) of Proposition 1.11, we see that if $(d_0, d_1) \in \partial\mathbb{D}_{A,s_0}$, then

$$q(s_0) \in S_A(s_0), \quad (q_0, q_1)(s_0) \in \partial \hat{S}_A(s_0).$$

From item (ii) of Proposition 1.12, we see that $q(s)$ must leave $S_A(s)$ at $s = s_0$, this yields that $s_*(d_0, d_1) = s_0$. Therefore, the restriction of Γ to $\partial\mathbb{D}_{A,s_0}$ is homeomorphic to the identity mapping, which is impossible thanks to index theorem, and the contradiction is obtained. This concludes the proof of Proposition 1.10 as well as item (i) of Theorem 1.1, assuming that Proposition 1.12 holds. $\square$

The proof of Theorem 1.1

As we mentioned in the above, item (i) of Theorem 1.1 follows from Proposition 1.10 and the proof of item (ii) is the following:

Proof of item (ii) of Theorem 1.1. The existence of $u^* \in C^2(\mathbb{R}^N \setminus \{0\})$ follows from the technique of Merle [4]. Here, we want to find an equivalent formation for u^* near the blowup point $x = 0$. The case $\alpha = 0$ was treated in [19]. When $\alpha \neq 0$, we follow the method of [19], and no new idea is needed. Therefore, we just sketch the main steps for the sake of completeness.

We consider $K_0 > 0$, a constant to be fixed large enough, and $|x_0| \neq 0$ small enough. Then, we introduce the following function

$$v(x_0, \xi, \tau) = \psi^{-1}(t_0(x_0))u(x, t), \tag{1.59}$$

where $(\xi, \tau) \in \mathbb{R}^N \times \Big[-\frac{t_0(x_0)}{T - t_0(x_0)}, 1\Big)$, and

$$(x, t) = \big(x_0 + \xi\sqrt{T - t_0(x_0)}, t_0(x_0) + \tau(T - t_0(x_0))\big), \tag{1.60}$$

with $t_0(x_0)$ being uniquely determined by

$$|x_0| = K_0\sqrt{(T - t_0(x_0))|\ln(T - t_0(x_0))|}. \tag{1.61}$$

From (1.8), (1.59), (1.60) and (1.61), we derive that

$$\sup_{|\xi|<2|\ln(T-t_0(x_0))|^{\frac{1}{4}}} |v(x_0,\xi,0) - \varphi_0(K_0)| \leq \frac{C}{1 + (|\ln(T - t_0(x_0))|^{\frac{1}{4}})} \to 0 \quad \text{as } x_0 \to 0,$$

where $\varphi_0(x) = \left(1 + \frac{(p-1)}{4p}|x|^2\right)^{\frac{1}{p-1}}$. As in [19], we use the continuity with respect to initial data for equation (1.1) associated to a space-localization in the ball $B(0, |\xi| < |\ln(T - t_0(x_0))|^{\frac{1}{4}})$ to derive

$$\sup_{|\xi|<|\ln(T-t_0(x_0))|^{\frac{1}{4}}, \tau\in[0,1)} |v(x_0,\xi,\tau) - \hat{v}_{K_0}(\tau)| \leq \epsilon(x_0) \to 0, \quad \text{as } x_0 \to 0, \tag{1.62}$$

where $\hat{v}_{K_0}(\tau) = \left((1-\tau) + \frac{(p-1)K_0^2}{4p}\right)^{-\frac{1}{p-1}}$.

From (1.60) and (1.62), we deduce

$$u^*(x_0) = \lim_{t\to T} u(x_0,t) = \psi(t_0(x_0)) \lim_{\tau\to 1} v(x_0,0,\tau) \sim \psi(t_0(x_0)) \left(\frac{p-1}{4p}\right)^{-\frac{1}{p-1}}. \tag{1.63}$$

Using the relation (1.61), we find that

$$T - t_0(x_0) \sim \frac{|x_0|^2}{2K_0|\ln|x_0||} \text{ and } \ln(T - t_0(x_0)) \sim 2\ln(|x_0|), \quad \text{as } x_0 \to 0, \tag{1.64}$$

The formula (1.11) then follows from Lemma 1.17, (1.63) and (1.64). This concludes the proof of Theorem 1.1, assuming that Proposition 1.12 holds. □

1.4 Proof of Proposition 1.12.

This section is devoted to the proof of Proposition 1.12, which is the heart of our analysis. We proceed into two parts. In the first part, we derive *a priori estimates* on $q(s)$ in $S_A(s)$. In the second part, we show that the new bounds are better than those defined in $S_A(s)$, except for the first two components (q_0, q_1). This means that the problem is reduced to the control of a finite dimensional function (q_0, q_1), which is the conclusion of item (i) of Proposition 1.12. Item (ii) of Proposition 1.12 is just direct consequence of the dynamics of q_0 and q_1. Let us start the first part.

1.4.1 A priori estimates on $q(s)$ in $S_A(s)$.

In this part we derive the *a priori estimates* on the components q_2, q_-, q_e which implies the conclusion of Proposition 1.12. Firstly, let us give some dynamics of $q_0, q_1 = (q_{1,i})_{1\le i\le N}$ and $q_2 = (q_{2,i,j})_{1\le i,j\le N}$. More precisely, we claim the following.

Proposition 1.13 (Dynamics of equation (1.40))**.** *There exists $A_4 \ge 1$, such that $\forall A \ge A_4$ there exists $s_4(A) \ge 1$, such that the following holds for all $s_0 \ge s_4(A)$: Assume that for all $s \in [s_0, s_1]$ for some $s_1 \ge s_0$, $q(s) \in S_A(s)$, then, we have for all $s \in [s_0, s_1]$:*

(i) (ODE satisfied by the positive and null modes)

$$\left|q_m'(s) - \left(1 - \frac{m}{2}\right) q_m(s)\right| \le \frac{C}{s^2}, \quad \forall m = 0, 1, \tag{1.65}$$

and

$$\left|q_2'(s) + \frac{2}{s} q_2(s)\right| \le \frac{C \ln s}{s^3}. \tag{1.66}$$

(ii) (Control of the negative and outer parts)

$$\left\|\frac{q_-(y,s)}{1+|y|^3}\right\|_{L^\infty} \le \frac{C}{s^2}\left((s-\sigma) + e^{-\frac{s-\sigma}{2}} A + e^{-(s-\sigma)^2} A^2\right), \tag{1.67}$$

$$\|q_e(s)\|_{L^\infty} \le \frac{C}{\sqrt{s}}\left((s-\sigma) + A^2 e^{-\frac{s-\sigma}{p}} + A e^{s-\sigma}\right). \tag{1.68}$$

Proof. We proceed in two steps:

- In the first step we project equation (1.40) to write ODEs satisfied by q_m for $m = 0, 1, 2$.

- In the second step we use the integral form of equation (1.40) and the dynamics of the linear operator $\mathcal{L} + V$ to derive a priori estimates on q_- and q_e.

Part 1: ODEs satisfying by the positive and null modes

We give the proof of (1.65) and (1.66) in this step. However, we only deal with the proof of (1.66) because the other one is the same the proof (1.65).

In fact, by formula (1.51) and equation (1.40), we write for each $1 \le i, j \le N$,

$$\left|q_{2,i,j}'(s) - \int_{\mathbb{R}^N} \left[\mathcal{L}q + Vq + B(q) + R(y,s) + D(q,s)\right] \chi \left(\frac{y_i y_j}{8} - \frac{\delta_{i,j}}{4}\right) \rho dy\right| \le C e^{-s}. \tag{1.69}$$

Using the assumption $q(s) \in S_A(s)$ for all $s \in [s_0, s_1]$, we derive the following estimates for all $s \in [s_0, s_1]$:

$$\left|\int \mathcal{L}(q) \chi \left(\frac{y_i y_j}{8} - \frac{\delta_{i,j}}{4}\right) \rho dy\right| \le \frac{C}{s^3}.$$

On the other hand, from Lemmas 1.24, 1.25 and 1.26, we have

$$\begin{aligned}
\left|\int Vq\chi\left(\frac{y_iy_j}{8}-\frac{\delta_{i,j}}{4}\right)\rho dy+\frac{2}{s}q_{2,i,j}(s)\right| &\leq \frac{CA}{s^3},\\
\left|\int B(q)\chi\left(\frac{y_iy_j}{8}-\frac{\delta_{i,j}}{4}\right)\rho dy\right| &\leq \frac{C}{s^3},\\
\left|\int R\chi\left(\frac{y_iy_j}{8}-\frac{\delta_{i,j}}{4}\right)\rho dy\right| &\leq \frac{C}{s^3},\\
\left|\int D(q,s)\chi\left(\frac{y_iy_j}{8}-\frac{\delta_{i,j}}{4}\right)\rho dy\right| &\leq \frac{C\ln s}{s^3}.
\end{aligned}$$

Gathering all these above estimates to (1.69) yields

$$\left|q_{2,i,j}'+\frac{2}{s}q_{2,i,j}\right|\leq\frac{C\ln s}{s^3}.$$

This concludes the proof of (1.66).

Part 2: Control of the negative and outer parts

We give the proof of (1.67) and (1.68) in this part. In fact, the control of q_- and q_e mainly bases on the dynamics of the linear operator $\mathcal{L}+V$. In particular, we use the following integral form of equation (1.40): for each $s\geq\sigma\geq s_0$,

$$q(s)=\mathcal{K}(s,\sigma)q(\sigma)+\int_\sigma^s\mathcal{K}(s,\tau)\left[B(q)(\tau)+R(\tau)+D(q,\tau)\right]d\tau=\sum_{i=1}^4\vartheta_i(s,\sigma),\qquad(1.70)$$

where $\{\mathcal{K}(s,\sigma)\}_{s\geq\sigma}$ is defined by

$$\begin{cases}\partial_s\mathcal{K}(s,\sigma)=(\mathcal{L}+V)\mathcal{K}(s,\sigma), & s>\sigma,\\ \mathcal{K}(\sigma,\sigma)=Id,\end{cases}\qquad(1.71)$$

and

$$\begin{aligned}
\vartheta_1(s,\sigma)&=\mathcal{K}(s,\sigma)q(\sigma),\quad \vartheta_2(s,\sigma)=\int_\sigma^s\mathcal{K}(s,\tau)B(q)(\tau)d\tau,\\
\vartheta_3(s,\sigma)&=\int_\sigma^s\mathcal{K}(s,\tau)R(.,\tau)d\tau,\quad \vartheta_4(s,\sigma)=\int_\sigma^s\mathcal{K}(s,\tau)D(q,\tau)d\tau.
\end{aligned}$$

As a matter of fact, in (1.70), it is clear to see the strong influence of the kernel $\mathcal{K}$. It is therefore convenient to recall the following result which the dynamics of the linear operator $\mathcal{K}=\mathcal{L}+V$.

Lemme 1.14 (A priori estimates of the linearized operator in the decomposition in (1.53))**.** *For all $\rho^*\geq 0$, there exists $s_5(\rho^*)\geq 1$, such that the following holds: If $\sigma\geq s_5(\rho^*)$ and $v\in L^2_\rho(\mathbb{R}^N)$ satisfying*

$$\sum_{m=0}^2|v_m|+\left\|\frac{v_-}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)}+\|v_e\|_{L^\infty(\mathbb{R}^N)}<\infty,\qquad(1.72)$$

then, $\forall s \in [\sigma, \sigma + \rho^*]$*, the function* $\theta(s) = \mathcal{K}(s,\sigma)v$ *satisfies*

$$\begin{aligned} \left\| \frac{\theta_-(y,s)}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{Ce^{s-\sigma}\left((s-\sigma)^2+1\right)}{s} \left(|v_0| + |v_1| + \sqrt{s}|v_2|\right) \\ &\quad + Ce^{-\frac{(s-\sigma)}{2}} \left\| \frac{v_-}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} + C\frac{e^{-(s-\sigma)^2}}{s^{\frac{3}{2}}} \|v_e\|_{L^\infty(\mathbb{R}^N)}, \end{aligned} \tag{1.73}$$

and

$$\|\theta_e(y,s)\|_{L^\infty(\mathbb{R}^N)} \leq Ce^{s-\sigma}\left(\sum_{l=0}^{2} s^{\frac{l}{2}}|v_l| + s^{\frac{3}{2}} \left\| \frac{v_-}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)}\right) + Ce^{-\frac{s-\sigma}{p}} \|v_e\|_{L^\infty(\mathbb{R}^N)}. \tag{1.74}$$

Proof. The proof of this result was given by Bricmont and Kupiainen [2] in one dimensional case. It was then extended in higher dimensional case in Nguyen and Zaag [14]. We kindly refer interested readers to Lemma 2.9 in [14] for a detail of the proof. □

In view of formula (1.70), we see that Lemma 1.14 plays an important role in deriving the new bounds on the components q_- and q_e. Indeed, given bounds on the components of q, $B(q)$, $D(q)$ and R, we directly apply Lemma 1.14 with $\mathcal{K}(s,\sigma)$ replaced by $\mathcal{K}(s,\tau)$ and then integrating over τ to obtain estimates on q_- and q_e. In particular, we claim the following which immediately follows (1.67) and (1.68) by addition.

Lemme 1.15. *For all* $\tilde{A} \geq 1, A \geq 1, \rho^* \geq 0$*, there exists* $s_6(A,\rho^*) \geq 1$ *such that* $\forall s_0 \geq s_6(A,\rho^*)$ *and* $q(s) \in S_A(s), \forall s \in [\sigma, \sigma+\rho^*]$ *where* $\sigma \geq s_0$*, we have the following properties:*

a) *Case* $\sigma \geq s_0$*: for all* $s \in [\sigma, \sigma+\rho^*]$*,*

i) *The linear term* $\vartheta_1(s,\sigma)$

$$\begin{aligned} \left\| \frac{(\vartheta_1(s,\sigma))_-}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} &\leq C\frac{\left(1 + e^{-\frac{s-\sigma}{2}}A + e^{-(s-\sigma)^2}A^2\right)}{s^2}, \\ \|(\vartheta_1(s,\sigma))_e\|_{L^\infty(\mathbb{R}^N)} &\leq C\frac{A^2e^{-\frac{s-\sigma}{p}} + Ae^{s-\sigma}}{s^{\frac{1}{2}}}. \end{aligned}$$

ii) *The quadratic term* $\vartheta_2(s,\sigma)$

$$\left\| \frac{(\vartheta_2(s,\sigma))_-}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C(s-\sigma)}{s^{2+\epsilon}}, \quad \|(\vartheta_2(s,\sigma))_e\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C(s-\sigma)}{s^{\frac{1}{2}+\epsilon}}.$$

where $\epsilon = \epsilon(p) > 0$*.*

iii) *The correction term* $\vartheta_3(s,\sigma)$

$$\left\| \frac{(\vartheta_3(s,\sigma))_-}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C(s-\sigma)}{s^2}, \quad \|(\vartheta_3(s,\sigma))_e\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C(s-\sigma)}{s^{\frac{3}{4}}}.$$

iv) *The nonlinear term* $\vartheta_4(s,\sigma)$

$$\left\| \frac{(\vartheta_4(s,\sigma))_-}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C(s-\sigma)}{s^2}, \quad \|(\vartheta_4(s,\sigma))_e\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C(s-\sigma)}{s^{\frac{3}{4}}}.$$

b) Case $\sigma = s_0$, we assume in addition

$$|q_m(s_0)| \le \frac{\tilde{A}}{s_0^2}, \quad |q_2(s_0)| \le \frac{\tilde{A}\ln^2 s_0}{s_0^2},$$
$$\left\| \frac{q_-(y,s_0)}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \le \frac{\tilde{A}}{s_0^2}, \quad \|q_e(s_0)\|_{L^\infty(\mathbb{R}^N)} \le \frac{\tilde{A}}{\sqrt{s_0}}.$$

Then, for all $s \in [s_0, s_0+\rho^*]$ we have a) and the following properties:

$$\left\| \frac{(\vartheta_1(s,s_0))_-}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \le \frac{C\tilde{A}}{s^2}, \quad \|(\vartheta_1(s,s_0))_e\|_{L^\infty(\mathbb{R}^N)} \le \frac{C\tilde{A}(1+e^{s-s_0})}{\sqrt{s}}.$$

Proof. The proof simply follows from the definition of S_A and Lemma 1.14.

In fact, from the fact that $q \in S_A(s)$, we derive that Lemmas 1.24 , 1.25 and 1.26 hold. Then, we obtain the following:

$$\sum_{m\in\mathbb{N}^n,|m|=0}^{2} |B(q)_m(s)| \le \frac{C}{s^3}, \quad \left\| \frac{B(q)_-(s)}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{s^{2+\epsilon}}, \quad \|B(q)_e(s)\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{s^{\frac{1}{2}+\epsilon}},$$

and

$$\sum_{m\in\mathbb{N}^n,|m|=0}^{2} |R_m(s)| \le \frac{C}{s^2}, \quad \left\| \frac{R_-(s)}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{s^{2+\frac{1}{2}}}, \quad \|R_e(s)\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{s^{\frac{3}{4}}},$$

and

$$\sum_{m\in\mathbb{N}^n,|m|=0}^{2} |D(q)_m(s)| + \left\| \frac{D(q)_-(s)}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \le \frac{C\ln s}{s^3}, \quad \|D(q)_e(s)\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{s^{\frac{3}{4}}},$$

where $\epsilon = \epsilon(p) > 0$.

We simply inject these bounds to the a priori estimates given in Lemma 1.14 to obtain the bounds on $(\vartheta_m)_-$ and $(\vartheta_m)_e$ for $m = 2,3,4$.

On the other hand, the estimates on ϑ_1 directly follow from Lemma 1.14 and the fact that $q(s) \in S_A(s)$.

Thus, we get the conclusion the proof of Lemma 1.15. □

Bearing in mind that we are in the proof of Proposition 1.13. Indded, from formula (1.70) and Lemma 1.15, estimates in (1.67) and (1.68) simply follow by addition. Thus, conclusion of Proposition 1.13 follows. □

1.4.2 Conclusion of Proposition 1.12

In this part, we give the proof of Proposition 1.12 which is considered as a consequence of the dynamics of equation (1.40) given in Proposition 1.13. Indeed, item (i) of Proposition 1.12 directly follows from the following result:

Proposition 1.16 (Control of $q(s)$ by $(q_0, q_1)(s)$ in $S_A(s)$)**.** *There exists $A_7 \geq 1$ such that $\forall A \geq A_7$, there exists $s_7(A) \geq 1$ such that for all $s_0 \geq s_7(A)$, the following holds: If we have*

a) $q(s_0) = \psi_{d_0,d_1}(y)$, where $(d_0, d_1) \in \mathbb{D}_{A,s_0}$,

b) For all $s \in [s_0, s_1]$, $q(s) \in S_A(s)$.

Then, for all $s \in [s_0, s_1]$, we have

$$\forall i, j \in \{1, \cdots, N\}, \quad |q_{2,i,j}(s)| < \frac{A^2 \ln^2 s}{s^2}, \tag{1.75}$$

$$\left\| \frac{q_-(y,s)}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A}{2s^2}, \qquad \|q_e(s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^2}{2\sqrt{s}}, \tag{1.76}$$

where $\mathbb{D}_{A,s_0}$ is introduced in Proposition 1.11 and ψ_{d_0,d_1} is defined as in (1.54)*.*

Proof. Since the proof of (1.76) is similar to the one written in [13], we only deal with the proof of (1.75) and refer the readers to Proposition 3.7 in [13] for the proof of (1.76). We argue by contradiction to prove (1.75).

Indeed, let $i, j \in \{1, \cdots, N\}$ and assume that there is $s_* \in [s_0, s_1]$ such that

$$\forall s \in [s_0, s_*), \quad |q_{2,i,j}(s)| < \frac{A^2 \ln^2(s)}{s^2} \quad \text{and} \quad |q_{2,i,j}(s_*)| = \frac{A^2 \ln^2(s_*)}{s_*^2}.$$

In addition to that, we assume that $q_{2,i,j}(s_*) > 0$ (the negative case is the same), then, we have on the one hand

$$q'_{2,i,j}(s_*) \geq \frac{d}{ds}\left(\frac{A^2 \ln^2 s}{s^2}\right)_{s=s_*} = \frac{2A^2 \ln s_*}{s_*^3} - \frac{2A^2 \ln^2 s_*}{s_*^3}.$$

On the other hand, we have from (1.66),

$$q'_{2,i,j}(s_*) \leq -\frac{2A^2 \ln^2 s_*}{s_*^3} + \frac{C \ln s_*}{s_*^3}.$$

The contradiction then follows if $2A^2 > C$. This concludes the proof of Proposition 1.16. $\square$

We now come back to the proof of item (i) of Proposition 1.12. Indeed, from Proposition 1.16, we see that if $q(s) \in \partial S_A(s_1)$, then, the first two components $(q_0, q_1)(s_1)$ must be in $\partial \hat{S}_A(s_1)$, which is the conclusion of item (i).

The proof of item (ii): Indeed, it is easy to deduce from (1.65) the following property: If $q_0(s_1) = \varepsilon_0 \frac{A}{s_1^2}$ for some $\varepsilon_0 \in \{-1, 1\}$, then, the sign of $\frac{d}{ds} q_0\,(s_1)$ is opposite to the sign of $\frac{d}{ds}\left(\frac{\varepsilon_0 A}{s^2}\right)(s_1)$

Moreover, $q_{1,i}$ has the same property as q_0, for all $i \in \{1, ..., N\}$.

Hence, $(q_0, q_1)(s)$ will actually leave $\hat{S}_A(s)$ at $s_1 \geq s_0$ for s_0 large enough. Thus concludes the proof of Proposition 1.12.

1.5 Some elementary lemmas.

In this appendix, we state and prove several technical and and straightforward results need in our paper.

Lemme 1.17. *For each $T > 0$, there exists only one positive solution of equation* (1.9). *Moreover, the solution ψ satisfies the following asymptotic:*

$$\psi(t) \sim \kappa_\alpha (T-t)^{-\frac{1}{p-1}} |\ln(T-t)|^{-\frac{\alpha}{p-1}}, \text{ as } t \to T, \tag{1.77}$$

where $\kappa_\alpha = (p-1)^{-\frac{1}{p-1}} \left(\frac{p-1}{2}\right)^{\frac{\alpha}{p-1}}$.

Proof. Let us consider the following ODE

$$\psi' = \psi^p \ln^\alpha(\psi^2 + 2), \quad \psi(0) > 0. \tag{1.78}$$

In fact, the uniqueness and local existence are derived by the Cauchy-Lipschitz property.

Let T_{max}, T_{min} be the maximum and minimum time of the existence of the positive solution, i.e. $\psi(t)$ exists for all $t \in (T_{min}, T_{max})$. We now prove that $T_{max} < +\infty$ and $T_{min} = -\infty$. By contradiction, we suppose that the solution exists on $[0, +\infty)$, we have

$$\lim_{t_1 \to +\infty} \int_0^{t_1} \frac{\psi'}{\psi^p \ln^\alpha(\psi^2 + 2)} dt = \lim_{t_1 \to +\infty} \int_0^{t_1} dt = +\infty.$$

However, we can prove that $\int_0^{t_1} \frac{\psi'}{\psi^p \ln^\alpha(\psi^2+2)} dt$ is bounded by using the fact that

$$\int_0^{+\infty} \frac{1}{t^p \ln^\alpha(t^2+2)} dt < +\infty, \text{ for all } \alpha \in \mathbb{R} \text{ and } p > 1.$$

The contradiction then follows. In particular, we can prove $T_{min} = -\infty$ by using a similar argument.

Thus, we have proved that for every solution ψ of (1.78), there exists a maximal time $T_{max} \in (0, +\infty)$ such that ψ exists on $(-\infty, T_{max})$ and

$$\psi(t) \to +\infty \text{ as } t \to T_{max}.$$

In addition to that, if ψ_1, a solution of (1.78) which blows up at T_1, then,

$$\psi(t + T_1 - T_2) \text{ blows up at } T_2.$$

Then, we can derive that for every $T > 0$, there exists ψ_T a solution of (1.78) such that

$$\psi_T(t) \to +\infty \text{ as } t \to T.$$

We now aim at proving the uniqueness. Indeed, we suppose that ψ_1, ψ_2 satisfy equation (1.78) and blow up a the same time $T > 0$. If there exists $t_* < T$ such that

$$\psi_1(t_*) \neq \psi_2(t_*).$$

By using the following fact

$$T - t = \int_{\psi(t)}^{+\infty} \frac{du}{u^p \ln^\alpha(u^2 + 2)}, \tag{1.79}$$

we deduce that

$$\int_{\psi_1(t_*)}^{\psi_2(t_*)} \frac{du}{u^p \ln^\alpha(u^2 + 2)} = 0.$$

This is impossible and we obtain the uniqueness.

Let us now prove (1.77). Using (1.79), we deduce that for all $\delta \in (0, p - 1)$, there exists t_δ such that for all $t \in (t_\delta, T)$, we have

$$\int_{\psi(t)}^{+\infty} \frac{du}{u^{p+\delta}} \leq T - t \leq \int_{\psi(t)}^{+\infty} \frac{du}{u^{p-\delta}}.$$

This follows for all $t \in (t_\delta, T)$:

$$(p - 1 + \delta)^{-\frac{1}{p-1+\delta}} (T - t)^{-\frac{1}{p-1+\delta}} \leq \psi(t) \leq (p - 1 - \delta)^{-\frac{1}{p-1-\delta}} (T - t)^{-\frac{1}{p-1-\delta}},$$

from which we have

$$\ln \psi(t) \sim -\frac{1}{p-1} \ln(T - t) \quad \text{as} \quad t \to T.$$

So, we have

$$\ln(\psi^2(t) + 2) \sim -\frac{2}{p-1} \ln(T - t) \quad \text{as} \quad t \to T.$$

Hence, we obtain

$$\psi'(t) = \psi^p(t) \ln^\alpha(\psi^2(t) + 2) \sim \psi^p \left[-\frac{2}{p-1} \ln(T - t) \right]^\alpha \quad \text{as} \quad t \to T, \tag{1.80}$$

which yields

$$\frac{\psi'}{\psi^p} \sim \left(\frac{2}{p-1} \right)^\alpha |\ln(T - t)|^\alpha \quad \text{as} \quad t \to T.$$

This implies

$$\frac{1}{p-1} \psi^{1-p}(t) \sim \left(\frac{2}{p-1} \right)^\alpha \int_t^T |\ln(T - v)|^\alpha dv \sim \left(\frac{2}{p-1} \right)^\alpha (T - t) |\ln(T - t)|^\alpha \quad \text{as} \quad t \to T,$$

which concludes the proof of (1.77). □

Lemme 1.18. *Let us consider $\alpha \in (0,1), \theta > 0$ and $0 < h < 1$. Then, the following integral*

$$I(h) = \int_h^1 (s-h)^{-\alpha} s^{-\theta} ds$$

satisfies:

i) if $\alpha + \theta > 1$, then

$$I(h) \leq \left(\frac{1}{1-\alpha} + \frac{1}{\alpha+\theta-1} \right) h^{1-\alpha-\theta}.$$

ii) If $\alpha + \theta = 1$, then

$$I(h) \leq \frac{1}{1-\alpha} + |\ln h|.$$

iii) If $\alpha + \theta < 1$, then

$$I(h) \leq \frac{1}{1-\alpha-\theta}.$$

Proof. See Lemma 2.2 of Giga and Kohn [8]. □

Lemme 1.19 (A version of Gronwall Lemma)**.** *If $y(t), r(t)$ and $q(t)$ are continuous functions defined on $[t_0, t_1]$ such that*

$$y(t) \leq y_0 + \int_{t_0}^t y(s) r(s) ds + + \int_{t_0}^t h(s) ds, \forall t \in [t_0, t_1].$$

Then,

$$y(t) \leq e^{\int_{t_0}^t r(s) ds} \left[y_0 + \int_{t_0}^t h(s) e^{-\int_{t_0}^s r(\tau) d\tau} ds \right].$$

Proof. See Lemma 2.3 of Giga and Kohn [8]. □

Lemme 1.20. *For each $T_2 < T, \delta > 0$. There exists $\epsilon = \epsilon(T, T_2, \delta, n, p) > 0$ such that for each $v(x,t)$ satisfying*

$$|\partial_t v - \Delta v| \leq C|v|^p \ln^\alpha(v^2+2), \quad \forall |x| \leq \delta, \;\; t \in (T_2, T), \delta > 0, \tag{1.81}$$

and

$$|v(x,t)| \leq \epsilon \psi(t), \quad \forall |x| \leq \delta, \quad t \in (T_2, T), \tag{1.82}$$

where $\psi(t)$ is the unique positive solution of (1.9)*. Then, $v(x,t)$ does not blow up at $(0,T)$.*

Proof. Since the argument is almost the same as in [8] treated for the case $\alpha = 0$, we only sketch the main step for the sake of completeness. Let $\phi \in C^\infty(\mathbb{R}^N), \phi = 1$ if $|x| \leq \frac{\delta}{2}, \phi = 0$ if $|x| \geq \delta$, and consider $\omega = \phi v$ satisfying

$$\partial_t \omega - \Delta \omega = f\phi + g, \tag{1.83}$$

where

$$f = \partial_t v - \Delta v \quad \text{ and } \quad g = v\Delta\phi - 2\nabla.(v\nabla\phi).$$

By using the Duhamel's formula, we write

$$\omega(t) = e^{(t-T_2)\Delta}(\omega(T_2)) + \int_{T_2}^{t} \left(e^{(t-\tau)\Delta}(\phi f) + e^{(t-\tau)\Delta}(g)\right) d\tau, \forall t \in [T_2, T), \tag{1.84}$$

where $e^{t\Delta}$ is the heat semigroup satisfying the following properties: for all $h \in L^\infty$,

$$\|e^{t\Delta}h\|_{L^\infty(\mathbb{R}^N)} \leq \|h\|_{L^\infty(\mathbb{R}^N)} \text{ and } \|e^{t\Delta}\nabla h\|_{L^\infty(\mathbb{R}^N)} \leq Ct^{-\frac{1}{2}}\|h\|_{L^\infty(\mathbb{R}^N)}, \forall t > 0.$$

The formula (1.84) then yields

$$\begin{aligned}\|\omega(t)\|_{L^\infty(\mathbb{R}^N)} \leq C + C\int_{T_2}^{t} \|\omega(\tau)\|_{L^\infty(\mathbb{R}^N)} \||v|^{p-1}\ln^\alpha(v^2+2)(\tau)\|_{L^\infty(|x|\leq\delta)} \\ + C\int_{T_2}^{t}(t-\tau)^{-\frac{1}{2}}\|v(\tau)\|_{L^\infty(|x|\leq\delta)}d\tau,\end{aligned} \tag{1.85}$$

for some constant $C = C(n, p, \phi, T, T_2, \delta) > 0$.

From (1.81), (1.82) and Lemma (1.17), we find that for all $|x| \leq \delta$, and $\tau \in [T_2, T)$,

$$|v(\tau)|^{p-1}\ln^\alpha(v^2(\tau)+2) \leq C\psi^{p-1}(\tau)\ln^\alpha(\psi^2(\tau)+2) \leq C(T-\tau)^{-1},$$

and

$$|v(\tau)| \leq C(T-\tau)^{-\frac{1}{p-1}}|\ln(T-\tau)|^{-\frac{\alpha}{p-1}}.$$

The estimate (1.85) becomes

$$\begin{aligned}\|\omega(t)\|_{L^\infty(\mathbb{R}^N)} \leq C + C\epsilon^{p-1}\int_{T_2}^{t}(T-\tau)^{-1}\|\omega(\tau)\|_{L^\infty(\mathbb{R}^N)}d\tau \\ + C\epsilon\int_{T_2}^{t}(t-\tau)^{-\frac{1}{2}}(T-\tau)^{-\frac{1}{p-1}}|\ln(T-\tau)|^{-\frac{\alpha}{p-1}}d\tau.\end{aligned} \tag{1.86}$$

In particular, we now consider $0 < \lambda \ll \frac{1}{2}$ fixed, then we have:

$$(T-\tau)^{-\frac{1}{p-1}}|\ln(T-\tau)|^{-\frac{\alpha}{p-1}} \leq C(\alpha,\lambda)(T-\tau)^{-\left(\frac{1}{p-1}+\lambda\right)}, \forall \tau \in (T_2, T).$$

Hence, we rewrite (1.86) as follows

$$\begin{aligned}\|\omega(t)\|_{L^\infty(\mathbb{R}^N)} \leq C + C\epsilon^{p-1}\int_{T_2}^{t}(T-\tau)^{-1}\|\omega(\tau)\|_{L^\infty(\mathbb{R}^N)}d\tau \\ + C\epsilon\int_{T_2}^{t}(t-\tau)^{-\frac{1}{2}}(T-\tau)^{-\left(\frac{1}{p-1}+\lambda\right)}d\tau,\end{aligned} \tag{1.87}$$

where $C(n, p, \phi, \alpha, \epsilon, \lambda, p)$. Beside that, by changing variables $s = T - \tau, h = T - t$ we have

$$\int_{T_2}^{t}(t-\tau)^{-\frac{1}{2}}(T-\tau)^{-\theta(p,\lambda)}d\tau = \int_{h}^{T-T_2}(s-h)^{-\frac{1}{2}}(s)^{-\theta(p,\lambda)}ds, \tag{1.88}$$

where $\theta(p,\lambda)=\left(\frac{1}{p-1}+\lambda\right)$.

Case 1: If $\theta(p,\lambda)<\frac{1}{2}$, by using *iii)* of Lemma 1.18, we deduce from (1.87) and (1.88) that

$$\|\omega(t)\|_{L^\infty(\mathbb{R}^N)}\leq C+C\epsilon^{p-1}\int_{T_2}^t(T-s)^{-1}\|\omega(s)\|_{L^\infty(\mathbb{R}^N)}ds,$$

Therefore, by Lemma 1.19,

$$\|\omega(t)\|_{L^\infty(\mathbb{R}^N)}\leq C(T-t)^{-C\epsilon^{p-1}}, \tag{1.89}$$

Choosing ϵ small enough such that $C\epsilon^{p-1}\leq\frac{1}{2(p-1)}$. Then, we conclude from (1.89) that

$$|v(x,t)|\leq C(T-t)^{-\frac{1}{2(p-1)}},\text{ for }|x|\leq\frac{1}{2},t\leq T. \tag{1.90}$$

By using parabolic regularity theory and the same argument as in Lemma 3.3 of [7], we can prove that (1.90) actually prevents blowup.

Case 2: $\theta(\lambda,p)=\frac{1}{2}$, it is similar to the first case, by using *ii)* of Lemma 1.18, (1.87) and (1.88) we yield

$$\|\omega(t)\|_{L^\infty(\mathbb{R}^N)}\leq C(1+|\ln(T-t)|)+C\epsilon^{p-1}\int_{T_2}^t(T-s)^{-1}\|\omega(s)\|_{L^\infty(\mathbb{R}^N)}ds.$$

However, we derive from Lemma 1.19 that

$$\|\omega(t)\|_{L^\infty(\mathbb{R}^N)}\leq C(T-t)^{-K\epsilon^{p-1}}, \tag{1.91}$$

where $C=C(n,p,\phi,T,T_2,\delta)$. We now take ϵ is small enough such that $C\epsilon^{p-1}\leq\frac{1}{2(p-1)}$, which follows (1.90).

Case 3: $\theta(\lambda,p)>\frac{1}{2}$, by using Lemmas 1.18 1.19 and arguments similar to obtain

$$|v(x,t)|\leq C(T-t)^{\frac{1}{2}-\theta(p,\lambda)},\quad\forall|x|\leq\delta,\ t\in[T_2,T).$$

Repeating the step in finite steps would end up with (1.90). This concludes the proof of Lemma 1.20. □

The following lemma gives the asymptotic behaviors of $h(s)$ ans $\psi_1(s)$ defined in (1.15) and (1.16), respectively.

Lemme 1.21. *Let $h(s)$ and $\psi_1(s)$ be defined as in* (1.15) *and* (1.16)*, respectively. Then we have*

i)

$$\frac{1}{\ln(\psi_1^2(s)+2)}=\frac{p-1}{2s}+\frac{\alpha(p-1)\ln s}{2s^2}+O\left(\frac{1}{s^2}\right),\qquad as\ s\to+\infty. \tag{1.92}$$

ii)

$$h(s)=\frac{1}{p-1}\left[1-\frac{\alpha}{s}-\frac{\alpha^2\ln s}{s^2}\right]+O\left(\frac{1}{s^2}\right),\qquad as\ s\to+\infty. \tag{1.93}$$

Proof. *i*) Consider $\psi(t)$ the unique positive solution of (1.9). We have

$$T - t = \int_{\psi(t)}^{+\infty} \frac{dx}{x^p \ln^\alpha(x^2+2)}. \tag{1.94}$$

An integration by parts yields

$$T - t = \frac{1}{\psi^{p-1}(t)\ln^\alpha(\psi^2(t)+2)}\left[\frac{1}{p-1} - \frac{2\alpha}{(p-1)^2\ln(\psi^2(t)+2)} + O\left(\frac{1}{(\ln^2(\psi^2(t)+2))}\right)\right]. \tag{1.95}$$

Let us write $\psi(t) = \psi_1(s)$ where $s = -\log(T-t)$, then we have

$$\ln(\psi_1(s)) = \frac{s}{p-1} - \frac{\alpha}{(p-1)}\ln\left(\ln(\psi_1(s))\right) + O\left(1\right), \quad \text{as } s \to +\infty, \tag{1.96}$$

from which, we deduce that

$$\ln(\psi_1(s)) = \frac{s}{p-1} - \frac{\alpha\ln(s)}{p-1} + O(1), \quad \text{as } s \to +\infty, \tag{1.97}$$

which is the conclusion of item *i*).

ii) From (1.15) and (1.95), we have

$$h(s) = \frac{1}{p-1} - \frac{2\alpha}{(p-1)^2\ln(\psi_1^2(s)+2)} + O\left(\frac{1}{\ln^2(\psi_1^2(s)+2)}\right). \tag{1.98}$$

Using (1.92) we conclude the proof of (1.93) as well as Lemma (1.21). □

Lemme 1.22. *Let N be defined as in* (1.23)*, we have*

$$N(\bar{w}, s) = \frac{p\bar{w}^2}{2} + O\left(\frac{|\bar{w}|\ln s}{s^2}\right) + O\left(\frac{|\bar{w}|^2}{s}\right) + O(|\bar{w}|^3) \quad \textit{as} \quad (\bar{w}, s) \to (0, +\infty). \tag{1.99}$$

Proof. From the definition (1.23) of N, let us write

$$N(\bar{w}, s) = N_1(\bar{w}, s) + N_2(\bar{w}, s),$$

where

$$N_1(\bar{w}, s) = h(s)\left(|\bar{w}+1|^{p-1}(\bar{w}+1) - (\bar{w}+1)\right) - \bar{w},$$
$$N_2(\bar{w}, s) = h(s)|\bar{w}+1|^{p-1}(\bar{w}+1)\left(\frac{\ln^\alpha(\psi_1^2(\bar{w}+1)^2+2)}{\ln^\alpha(\psi_1^2+2)} - 1\right).$$

From (1.93) and a Taylor expansion, we find that

$$N_1(\bar{w}, s) = \frac{p\bar{w}^2}{2} - \frac{\alpha\bar{w}}{s} + O\left(\frac{|\bar{w}|\ln s}{s^2}\right) + O\left(\frac{|\bar{w}|^2}{s}\right) + O(|\bar{w}|^3) \quad \text{as } (\bar{w}, s) \to (0, +\infty).$$

We now claim the following

$$N_2(\bar{w}, s) = \frac{\alpha\bar{w}}{s} + O\left(\frac{|\bar{w}|\ln s}{s^2}\right) + O\left(\frac{|\bar{w}|^2}{s}\right) \quad \text{as } (\bar{w}, s) \to (0, +\infty), \tag{1.100}$$

then, the proof of (1.99) simply follows by addition.

Let us now give the proof of (1.100) to complete the proof of Lemma 1.22 . We set

$$f(\bar{w}) = \ln^{\alpha}(\psi_1^2(\bar{w}+1)^2+2), \quad |\bar{w}| \leq \frac{1}{2}.$$

We apply Taylor expansion to $f(\bar{w})$ at $\bar{w}=0$ to find that

$$f(\bar{w}) = \ln^{\alpha}(\psi_1^2+2) + 2\alpha \ln^{\alpha-1}(\psi_1^2+2)\frac{\psi_1^2}{\psi_1^2+2}\bar{w} + \frac{f''(\theta)}{2}(\bar{w})^2,$$

where θ is between 0 and $\bar{w}$, and

$$\begin{aligned} f''(\theta) = {} & \alpha(\alpha-1)\ln^{\alpha-2}(\psi_1^2(\theta+1)^2+2)\left(\frac{2(\theta+1)\psi_1^2}{\psi_1^2(\theta+1)^2+2}\right)^2 \\ & + \alpha \ln^{\alpha-1}(\psi_1^2(\theta+1)^2+2)\frac{(4\psi_1 - 2\psi_1^4(\theta+1)^2)}{(\psi_1^2(\theta+1)^2+2)^2}. \end{aligned}$$

Since $|\theta| \leq \frac{1}{2}$, one can show that

$$|f''(\theta)| \leq C \ln^{\alpha-1}(\psi_1^2+2), \quad \forall |\theta| \leq \frac{1}{2}.$$

Thus, we have

$$f(\bar{w}) = \ln^{\alpha}(\psi_1^2+2) + 2\alpha\ln^{\alpha-1}(\psi_1^2+2)\bar{w} + O\left(|\bar{w}|^2 \ln^{\alpha-1}(\psi_1^2+2)\right) + O\left(\frac{|\bar{w}|\ln^{\alpha-1}(\psi_1^2+2)}{\psi_1^2}\right),$$

as $s \to +\infty$. This yields

$$\frac{\ln^{\alpha}(\psi_1^2(\bar{w}+1)^2+2)}{\ln^{\alpha}(\psi_1^2+2)} = 1 + \frac{2\alpha\bar{w}}{\ln(\psi_1^2+2)} + O\left(\frac{|\bar{w}|^2}{\ln(\psi_1^2+2)}\right) + O\left(\frac{|\bar{w}|}{\ln(\psi_1^2+2)\psi_1^2}\right),$$

as $(\bar{w}, s) \to (0, +\infty)$, from which and (1.92) we derive

$$\frac{\ln^{\alpha}(\psi_1^2(\bar{w}+1)^2+2)}{\ln^{\alpha}(\psi_1^2(s)+2)} - 1 = \frac{\alpha(p-1)\bar{w}}{s} + O\left(\frac{\ln s|\bar{w}|}{s^2}\right) + O\left(\frac{|\bar{w}|^2}{s}\right). \tag{1.101}$$

From the definition of N_2, (1.93), (1.101) and the fact that

$$|\bar{w}+1|^{p-1}(\bar{w}+1) = 1 + p\bar{w} + O(|\bar{w}|^2) \quad \text{ as } \bar{w} \to 0,$$

we conclude the proof of (1.100) as well as Lemma 1.22. □

Lemme 1.23. *For all $|z| \leq K_1$, then there exists $C(K_1)$ such that $\forall s \geq 1$, we have*

$$\left| h(s)|z|^{p-1}z\frac{\ln^{\alpha}(\psi_1^2 z^2+2)}{\ln^{\alpha}(\psi_1^2+2)} - \frac{|z|^{p-1}z}{p-1}\right| \leq \frac{C(K_1)}{s}, \tag{1.102}$$

where $h(s)$ and $\psi_1(s)$ are defined in (1.15) *and* (1.16), *respectively.*

Proof. We consider $f(z) = \ln^\alpha(\psi_1^2 z^2 + 2), \forall z \in \mathbb{R}$, then we write

$$\ln^\alpha(\psi_1^2 z^2 + 2) = \ln^\alpha(\psi_1^2 + 2) + \int_1^{|z|} f'(v)dv.$$

Recall from (1.17) that $h(s) = \frac{1}{p-1} + O(\frac{1}{s})$, we have then

$$\left| h(s)|z|^{p-1} z \frac{\ln^\alpha(\psi_1^2 z^2 + 2)}{\ln^\alpha(\psi_1^2 + 2)} - \frac{|z|^{p-1} z}{p-1} \right| \leq \frac{C|z|^p}{\ln^\alpha(\psi_1^2 + 2)} \int_1^{|z|} |f'(v)|dv + \frac{C|z|^p}{s}. \tag{1.103}$$

From item $i)$ of Lemma 1.21 that shows $\frac{1}{\ln(\psi_1^2+2)} \leq \frac{C}{s}$. Hence, it is sufficient to prove the following

$$A(z) := \frac{|z|^p}{\ln^{\alpha-1}(\psi_1^2 + 2)} \int_1^{|z|} |f'(v)|dv \leq C(K_1), \quad \forall |z| \leq K_1,$$

where

$$f'(v) = \alpha \ln^{\alpha-1}(\psi_1^2 v^2 + 2) \frac{2v\psi_1^2}{\psi_1^2 v^2 + 2}.$$

For $1 \leq |z| \leq K_1$, it is trivial to see that $|A(z)| \leq C(K_1)$. For $|z| < 1$, we consider two cases:

- Case 1: $\alpha - 1 \geq 0$, then

$$A(z) \leq 2|\alpha||z|^p \int_{|z|}^1 \frac{1}{v} dv \leq C(K_1).$$

- Case 2: $\alpha - 1 < 0$, then

$$A(z) \leq 2|\alpha||z|^p \frac{\ln^{\alpha-1}(\psi_1^2 z^2 + 2)}{\ln^{\alpha-1}(\psi_1 + 2)} \int_{|z|}^1 \frac{1}{v} dv.$$

+ If $\psi_1 z^2 \geq 1$ then

$$A(z) \leq 2|\alpha| \frac{\ln^{1-\alpha}(\psi_1^2 + 2)}{\ln^{1-\alpha}(\psi_1 + 2)} |z|^p \int_{|z|}^1 \frac{1}{v} dv \leq C(K_1).$$

+ If $\psi_1 z^2 \leq 1$ then $|z| \leq v \leq \psi_1^{-\frac{1}{2}}$ we deduce that

$$|A(z)| \leq 2|\alpha| \psi_1^{\frac{1-p}{2}} \frac{\ln^{1-\alpha}(\psi_1^2 + 2)}{\ln^{1-\alpha}(2)} |z| \int_{|z|}^1 \leq C(K_1).$$

This concludes the proof of Lemma 1.23. □

Lemme 1.24 (Control of the nonlinear term D in $S_A(s)$)**.** *For all $A \geq 1$, there exists $\sigma_3(A) \geq 1$ such that for all $s \geq \sigma_3(A), q(s) \in S_A(s)$ implies*

$$\forall |y| \leq 2K\sqrt{s}, \quad |D(q,s)| \leq C(K) \frac{\ln s (1 + |y|)^4}{s^3}, \tag{1.104}$$

and

$$\|D(q,s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{s}. \tag{1.105}$$

Proof. From the definition (1.44) of D, let us decompose

$$D(q,s) = D_1(q,s) + D_2(q,s),$$

where

$$D_1(q,s) = \left(h(s) - \frac{1}{p-1}\right)\left(|q+\varphi|^{p-1}(q+\varphi) - (q+\varphi)\right),$$
$$D_2(q,s) = h(s)|q+\varphi|^{p-1}(q+\varphi)L(q+\varphi,s),$$

and $h(s)$ admits the asymptotic behavior (1.93), L is defined in (1.45). The proof of (1.104) will follow once the following is proved: for all $|y| \le 2K\sqrt{s}$

$$\left|D_1 - \left(\frac{\alpha(|y|^2-2N)}{4ps^2} - \frac{\alpha}{s}q\right)\right| \le C\frac{(1+|y|^4)\ln s}{s^3}, \tag{1.106}$$

and

$$\left|D_2 + \left(\frac{\alpha(|y|^2-2N)}{4ps^2} - \frac{\alpha}{s}q\right)\right| \le C\frac{(1+|y|^4)\ln s}{s^3}. \tag{1.107}$$

Let us give a proof of (1.106). From the definition of $S_A(s)$, we note that if $q(s) \in S_A(s)$, then

$$\forall y \in \mathbb{R}^N, |q(y,s)| \le \frac{CA^2\ln^2 s(1+|y|^3)}{s^2}, \tag{1.108}$$

$$\|q(s)\|_{L^\infty(\mathbb{R}^N)} \le \frac{CA^2}{\sqrt{s}}. \tag{1.109}$$

From the definition (1.38) of φ and (1.109), we see that for all $|y| \le 2K\sqrt{s}$, there exists a positive constant C(K) such that

$$0 < \frac{1}{C(K)} \le (q+\varphi)(y,s) \le C(K). \tag{1.110}$$

Using Taylor expansion and the asymptotic (1.93), we write

$$D_1(q,s) = \left(-\frac{\alpha}{(p-1)s} + O\left(\frac{\ln s}{s^2}\right)\right)\left(\varphi^p - \varphi + \left(p\varphi^{p-1}-1\right)q\right) + O\left(q^2\right). \tag{1.111}$$

Using again the definition of φ and a Taylor expansion, we derive

$$\varphi^p = 1 - \frac{(|y|^2-2N)}{4s} + O\left(\frac{1+|y|^4}{s^2}\right),$$
$$\varphi = 1 - \frac{(|y|^2-2N)}{4ps} + O\left(\frac{1+|y|^4}{s^2}\right),$$
$$p\varphi^{p-1} - 1 = p - 1 - \frac{(p-1)(|y|^2-2N)}{4ps} + O\left(\frac{1+|y|^4}{s^2}\right),$$

as $s \to +\infty$. Inserting (1.108) and these estimates into (1.111) yields (1.106).

We now turn to the proof of (1.107). Recall from (1.45) the definition of L,

$$L(q+\varphi,s) = \frac{2\alpha\psi_1^2}{\ln(\psi_1^2+2)(\psi_1^2+2)}(q+\varphi-1) + \frac{1}{\ln^\alpha(\psi_1^2+2)}\int_1^{q+\varphi} f''(v)(q+\varphi-v)dt,$$

where $f(v) = \ln^\alpha(\psi_1^2 v^2 + 2), v \in \mathbb{R}$. From (1.110) and a direct computation, we estimate

$$\left|\frac{1}{\ln^\alpha(\psi_1^2+2)}\int_1^{q+\varphi} f''(v)(q+\varphi-v)dv\right| \le C(K)\frac{|q+\varphi-1|^2}{s},$$

which yields

$$\left|L(q+\varphi,s) - \frac{2\alpha\psi_1^2(q+\varphi-1)}{\ln(\psi_1^2+2)(\psi_1^2+2)}\right| \le C(K)\frac{|q+\varphi-1|^2}{s}. \tag{1.112}$$

From (1.92) and (1.112), we then have

$$\left|L(q+\varphi,s) - \frac{\alpha(p-1)(q+\varphi-1)}{s}\right| \le C(K)\left(\frac{|q+\varphi-1|^2}{s} + \frac{\ln s|q+\varphi-1|}{s^2}\right),$$

and beside that we have

$$|q+\varphi-1| \le \frac{C(1+|y|^2}{s},$$

imply that

$$\left|L(q+\varphi,s) - \frac{\alpha(p-1)(q+\varphi-1)}{s}\right| \le C(K)\frac{\ln s(1+|y|^4)}{s^3}, \tag{1.113}$$

Moreover, from definition of D_2 and (1.113) we deduce that

$$\left|D_2(q,s) - \frac{\alpha}{s}\left(\varphi^{p+1} - \varphi^p + ((p+1)\varphi^p - p\varphi^{p-1})q\right)\right| \le C\frac{(1+|y|^4)\ln s}{s^3},$$

and

$$\varphi^{p+1} - \varphi^p = -\frac{(|y|^2-2N)}{4ps} + O\left(\frac{1+|y|^4}{s^2}\right), \text{ as },$$
$$(p+1)\varphi^p - p\varphi^{p-1} = 1 - \frac{(|y|^2-2N)}{2s} + O\left(\frac{1+|y|^4}{s^2}\right), \text{ as },$$

as $s \to +\infty$ which yield (1.107).

We now give a proof to (1.105). From (1.93) and the boundedness of q and φ, we have

$$|D_1(q,s)| \le \frac{C}{s}.$$

In fact, it is sufficient to prove that for all $y \in \mathbb{R}^N$,

$$|D_2(q,s)| \le \frac{C(K)}{s},$$

Using the definition of L in (1.45), we deduce that

$$D_2(q,s) = h(s)|q+\varphi|^{p-1}(q+\varphi)\frac{\ln^\alpha(\psi_1^2 z^2+2)}{\ln^\alpha(\psi^2+2)} - h(s)|q+\varphi|^{p-1}(q+\varphi).$$

Using Lemma 1.23, we obtain the following

$$|D_2(q,s)| \le \frac{C(K)}{s}.$$

This completes the proof of Lemma 1.24. □

Lemme 1.25. *When s large enough, then we have for all $y \in \mathbb{R}^N$:*

i) Estimates on V:

$$|V(y,s)| \leq \frac{C(1+|y|^2)}{s}, \forall y \in \mathbb{R}^N,$$

and

$$V = -\frac{(|y|^2-2N)}{4s} + \tilde{V} \quad \textit{with} \quad \tilde{V} = O\left(\frac{1+|y|^4}{s^2}\right), \forall |y| \leq K\sqrt{s}.$$

ii) Estimates on R

$$|R(y,s)| \leq \frac{C}{s}, \forall y \in \mathbb{R}^n,$$

and

$$R(y,s) = \frac{c_p}{s^2} + \tilde{R}(y,s) \quad \textit{with} \quad \tilde{R} = O\left(\frac{1+|y|^4}{s^3}\right), \forall |y| \leq K\sqrt{s}.$$

Proof. The proof simply follows from Taylor expansion. We refer to Lemmas B.1 and B.5 in [19] for a similar proof. □

Lemme 1.26 (Estimates on $B(q)$)**.** *For all $A > 0$ there exists $\sigma_5(A) > 0$ such that for all $s \geq \sigma_5(A), q(s) \in S_A(s)$ implies*

$$|B(q)| \leq C|q|^2, \forall |y| \leq 2K\sqrt{s}, \tag{1.114}$$

and

$$\|B(q)\|_{L^\infty(\mathbb{R}^N)} \leq C|q|^{\bar{p}}, \tag{1.115}$$

with $\bar{p} = \min(p,2)$.

Proof. See Lemma 3.6 in [13] for a same proof of this lemma. □

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Chapter 2

Profile for the imaginary part of a blowup solution for a complex valued semilinear heat equation[1]

G. K. Duong

Abstract: *In this paper, we consider the following complex-valued semilinear heat equation*

$$\partial_t u = \Delta u + u^p, u \in \mathbb{C},$$

in the whole space $\mathbb{R}^N$*, where* $p \in \mathbb{N}, p \geq 2$*. We aim at constructing for this equation a complex solution* $u = u_1 + iu_2$*, which blows up in finite time* T *and only at one blowup point* a*, with the following asymptotic behaviors*

$$\begin{aligned} u(x,T) &\sim \left[\frac{(p-1)^2|x-a|^2}{8p|\ln|x-a||}\right]^{-\frac{1}{p-1}}, \\ u_2(x,T) &\sim \frac{2p}{(p-1)^2}\left[\frac{(p-1)^2|x-a|^2}{8p|\ln|x-a||}\right]^{-\frac{1}{p-1}}\frac{1}{|\ln|x-a||}, \text{ as } x \to a. \end{aligned}$$

Note that the imaginary part is non-zero and that it blows up also at point a*. Our method relies on two main arguments: the reduction of the problem to a finite dimensional one and a topological argument based on the index theory to get the conclusion.*

Mathematics Subject Classification: *35K55, 35K57 35K50, 35B44 (Primary); 35K50, 35B40 (Secondary).*

Keywords: *Blowup solution, Blowup profile, Semilinear complex heat equation, non variation heat equation.*

[1] This work was accepted for publication in *J. Funct. Anal* (2019).
Doi: 10.1016/j.jfa.2019.05.009

2.1 Introduction

In this work, we are interested in the following complex-valued semilinear heat equation

$$\begin{cases} \partial_t u &= \Delta u + F(u), t \in [0,T), \\ u(0) &= u_0 \in L^\infty, \end{cases} \tag{2.1}$$

where $F(u) = u^p$, and $u(t) : \mathbb{R}^N \to \mathbb{C}$, $L^\infty := L^\infty(\mathbb{R}^N, \mathbb{C})$ and $p > 1$. Though our results hold only when $p \in \mathbb{N}$ (see Theorem 2.1 below), we keep $p \in \mathbb{R}$ in the introduction, in order to broaden the discussion.

In particular, when $p = 2$, model (2.1) evidently becomes

$$\begin{cases} \partial_t u &= \Delta u + u^2, t \in [0,T), \\ u(0) &= u_0 \in L^\infty. \end{cases} \tag{2.2}$$

We remark that equation (2.2) is rigidly related to the viscous Constantin-Lax-Majda equation with a viscosity term, which is a one dimensional model for the vorticity equation in fluids. The readers can see more in some of the typical works: Constantin, Lax, Majda [2]; Guo, Ninomiya, Shimojo and Yanagida [7]; Okamoto, Sakajo and Wunsch [20]; Sakajo [21] and [22]; Schochet [23] and their references.

The local Cauchy problem for model (2.1) can be well solved (locally in time) in $L^\infty(\mathbb{R}^N)$ in the case where p is integer, by using a fixed-point argument. However, when p is not integer, the local Cauchy problem has not been sloven yet, up to our knowledge. This probably comes from the discontinuity of $F(u)$ on $\{u \in \mathbb{R}^*_-\}$.

In addition to that, let us remark that equation (2.1) has the following family of space independent solutions:

$$u_k(t) = \kappa e^{i\frac{2k\pi}{p-1}} (T-t)^{-\frac{1}{p-1}}, \text{ for any } k \in \mathbb{Z}, \tag{2.3}$$

where $\kappa = (p-1)^{-\frac{1}{p-1}}$. In particular, we have two situations:

+ If $p \in \mathbb{Q}$, this makes then a finite number of solutions.

+ If $p \notin \mathbb{Q}$, then, the following set

$$\left\{ u_k(t) \frac{(T-t)^{\frac{1}{p-1}}}{\kappa} \mid \quad k \in \mathbb{Z} \right\}, \tag{2.4}$$

is countable and dense in the unit circle of $\mathbb{C}$.

This latter case ($p \notin \mathbb{Q}$), is somehow intermediate between the case ($p \in \mathbb{Q}$) and the case of the twin PDE

$$\partial_t u = \Delta u + |u|^{p-1} u, \tag{2.5}$$

which admits the following family of space independent solutions

$$u_\theta(t) = \kappa e^{i\theta} (T-t)^{-\frac{1}{p-1}},$$

for any $\theta \in \mathbb{R}$, which turns to be infinite and covers all the unit circle, after rescaling as in (2.4). In fact, equation (2.5) is certainly much easier than equation (2.1). As a mater of fact, it reduces to the scalar case thanks to a modulation technique, as Filippas and Merle did in [5].

Since the Cauchy problem for equation (2.1) is already hard when $p \notin \mathbb{N}$, and given that we are more interested in the asymptotic blowup behavior, rather than the well-posedness issue, we will focus in our paper on the case $p \in \mathbb{N}$. In this case, from the Cauchy theory, the solution of equation (2.1) either exists globally or blows up in finite time. Let us recall that the solution $u(t) = u_1(t) + iu_2(t)$ blows up in finite time $T < +\infty$ if and only if it exists for all $t \in [0, T)$ and

$$\limsup_{t \to T} \{\|u_1(t)\|_{L^\infty(\mathbb{R}^N)} + \|u_2(t)\|_{L^\infty(\mathbb{R}^N)}\} \to +\infty.$$

If u blows up in finite time T, a point $a \in \mathbb{R}^N$ is called a blowup point if and only if there exists a sequence $\{(a_j, t_j)\} \to (a, T)$ as $j \to +\infty$ such that

$$|u_1(a_j, t_j)| + |u_2(a_j, t_j)| \to +\infty \text{ as } j \to +\infty.$$

The blowup phenomena occur for evolution equations in general, and in semilinear heat equations in particular. Accordingly, an interesting question is to construct for those equations a solution which blows up in finite time and to describe its blowup behavior. These questions are being studied by many authors in the world. Let us recall some blowup results connected to our equation:

(i) **The real case:** Bricmont and Kupiainen [1] constructed a real positive solution to (2.1) for all $p > 1$, which blows up in finite time T, only at the origin and they also gave the profile of the solution such that

$$\left\| (T-t)^{\frac{1}{p-1}} u(., t) - f_0 \left(\frac{.}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{1 + \sqrt{|\ln(T-t)|}},$$

where the profile f_0 is defined as follows

$$f_0(z) = \left(p - 1 + \frac{(p-1)^2}{4p} |z|^2 \right)^{-\frac{1}{p-1}}. \tag{2.6}$$

In addition to that, with a different method, Herrero and Velázquez in [12] obtained the same result. Later, in [15] Merle and Zaag simplified the proof of [1] and proposed the following two-step method (see also the note [14]):

- Reduction of the infinite dimensional problem to a finite dimensional one.
- Solution of the finite dimensional problem thanks to a topological argument based on Index theory.

We would like to mention that this method has been successful in various situations such as the work of Tayachi and Zaag [24], and also the works of Ghoul, Nguyen and Zaag in [9], [10] and [8]. In those papers, the considered equations were scale invariant; this property was believed to be essential for the construction. Fortunately, with the work of Ebde and Zaag [4] for the following equation

$$\partial_t u = \Delta u + |u|^{p-1}u + f(u, \nabla u),$$

where

$$|f(u, \nabla u)| \leq C(1 + |u|^q + |\nabla u|^{q'}) \text{ with } q < p, q' < \frac{2p}{p+1},$$

that belief was proved to be wrong.

Going on the same direction as [4], Nguyen and Zaag in [18], have achieved the construction with a stronger perturbation

$$\partial_t u = \Delta u + |u|^{p-1}u + \frac{\mu |u|^{p-1}u}{\ln^a(2 + u^2)},$$

where $\mu \in \mathbb{R}, a > 0$. Though the results of [4] and [18] show that the invariance under dilations of the equation in not necessary in the construction method, we might think that the construction of [4] and [18] works because the authors adopt a perturbative method around the pure power case $F(u) = |u|^{p-1}u$. If this is true with [4], it is not the case for [18]. In order to totally prove that the construction does not need the invariance by dilation, Duong, Nguyen and Zaag considered in [3], the following equation

$$\partial_t u = \Delta u + |u|^{p-1}u \ln^\alpha(2 + u^2),$$

for some where $\alpha \in \mathbb{R}$ and $p > 1$, where we have no invariance under dilation, not even for the main term on the nonlinearity. They were successful in constructing a stable blowup solution for that equation. Following the above mentioned discussion, that work has to be considered as a breakthrough.

Let us mention that a classification of the blowup behavior of (2.2) was made available by many authors such as Herrero and Velázquez in [12] and Velázquez in [25], [26], [27] (see also Zaag in [30] for some refinement). More precisely and just to stay in one space dimension for simplicity, it is proven in [12] that if u a real solution of (2.1), which blows up in finite time T and a is a given blowup point, then:

A. Either

$$\sup_{|x-a| \leq K\sqrt{(T-t)|\ln(T-t)|}} \left| (T-t)^{\frac{1}{p-1}} u(x,t) - f_0\left(\frac{x-a}{\sqrt{(T-t)|\ln(T-t)|}}\right) \right| \to 0 \text{ as } t \to T,$$

for any $K > 0$ where $f_0(z)$ is defined in (2.6).

B. Or, there exist $m \geq 2, m \in \mathbb{N}$ and $C_m > 0$ such that

$$\sup_{|x-a| \leq K(T-t)^{\frac{1}{2m}}} \left| (T-t)^{\frac{1}{p-1}} u(x,t) - f_m\left(\frac{C_m(x-a)}{(T-t)^{\frac{1}{2m}}}\right) \right| \to 0 \text{ as } t \to T,$$

for any $K > 0$, where $f_m(z) = (p - 1 + |z|^{2m})^{-\frac{1}{p-1}}$.

(ii) **The complex case:** The blowup question for the complex-valued parabolic equations has been studied intensively by many authors, in particular for the Complex Ginzburg Landau (CGL) equation

$$\partial_t u = (1 + i\beta)\Delta u + (1 + i\delta)|u|^{p-1}u. \tag{2.7}$$

This were some ealier works treated to CGL such as: Zaag [28] for the case where $\beta = 0$ and δ small enough; Masmoudi and Zaag [16] and Nouaili and Zaag [19]. More precisely, the authors in [16], generalized the result of [28] and constructed a blowup solution for (2.7) with $p - \delta^2 - \beta\delta - \beta\delta p > 0$ such that the solution satisfies the following

$$\left\| (T-t)^{\frac{1+i\delta}{p-1}} |\ln(T-t)|^{-i\mu} u(.,t) - \left(p - 1 + \frac{b_{sub}|.|^2}{(T-t)|\ln(T-t)|} \right)^{-\frac{1+i\delta}{p-1}} \right\|_{L^\infty} \leq \frac{C}{1 + \sqrt{|\ln(T-t)|}},$$

where

$$b_{sub} = \frac{(p-1)^2}{4(p - \delta^2 - \beta\delta - \beta\delta p)} > 0.$$

Then, Nouaili and Zaag in [19] has constructed for (2.7) (in case the critical where $\beta = 0$ and $p = \delta^2$) a blowup solution satisfying

$$\left\| (T-t)^{\frac{1+i\delta}{p-1}} |\ln(T-t)|^{-i\mu} u(.,t) - \kappa^{-i\delta} \left(p - 1 + \frac{b_{cri}|.|^2}{(T-t)|\ln(T-t)|^{\frac{1}{2}}} \right)^{-\frac{1+i\delta}{p-1}} \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{1 + |\ln(T-t)|^{\frac{1}{4}}},$$

with

$$b_{cri} = \frac{(p-1)^2}{8\sqrt{p(p+1)}}, \mu = \frac{\delta}{8b}.$$

As for equation (2.2), there are many works done in dimension one, such as the work of Guo, Ninomiya, Shimojo and Yanagida, who proved in [7] the following results (see Theorems 1.2, 1.3 and 1.5 in that work):

(i) *(A Fourier- based blowup criterion). We assume that the Fourier transform of initial data of* (2.2) *is real and positive, then the solution blows up in finite time.*

(ii) *(A simultaneous blowup criterion in dimension one) If the initial data* $u^0 = u_1^0 + iu_2^0$, *satisfies*

$$u_1^0 \text{ is even }, u_2^0 \text{ is odd with } u_2^0 > 0 \text{ for } x > 0.$$

Then, the fact that the blowup set is compact implies that u_1^0, u_2^0 *blow up simultaneously.*

(iii) *Assume that* $u_0 = u_1^0 + iu_2^0$ *satisfy*

$$u_1^0, u_2^0 \in C^1(\mathbb{R}^N), 0 \leq u_1^0 \leq M, u_1^0 \not\equiv M, 0 < u_2^0 \leq L,$$

$$\lim_{|x|\to+\infty} u_1^0(x) = M \text{ and } \lim_{|x|\to+\infty} u_2^0 = 0,$$

for some constant L, M. Then, the solution $u = u_1 + iu_2$ of (2.2), with initial data u^0, blows up at time $T(M)$, with $u_2(t) \not\equiv 0$. Moreover, the real part $u_1(t)$ blows up only at space infinity and $u_2(t)$ remains bounded.

Still for equation (2.2), Nouaili and Zaag constructed in [17] a complex solution $u = u_1+iu_2$, which blows up in finite time T only at the origin. Moreover, the solution satisfies the following asymptotic behavior

$$\left\|(T-t)u(.,t) - f\left(\frac{.}{\sqrt{(T-t)|\ln(T-t)|}}\right)\right\|_{L^\infty(\mathbb{R}^N)} \to 0 \text{ as } t \to T,$$

where $f(z) = \frac{1}{8+|z|^2}$ and the imaginary part satisfies the following estimate for all $K > 0$

$$\sup_{|x|\le K\sqrt{T-t}} \left|(T-t)u_2(x,t) - \frac{1}{|\ln(T-t)|^2}\sum_{j=1}^n C_j\left(\frac{x_j^2}{T-t} - 2\right)\right| \le \frac{C(K)}{|\ln(T-t)|^\alpha}, \tag{2.8}$$

for some $(C_i)_i \ne (0,...,0)$ and $2 < \alpha < 2+\eta, \eta$ small enough. Note that the real and the imaginary parts blow up simultaneously at the origin. Note also that [17] leaves unanswered the question of the derivation of the profile of the imaginary part, and this is precisely our aim in this paper, not only for equation (2.2), but also for equation (2.1) with $p \in \mathbb{N}, p \ge 2$.

Before stating our result (see Theorem 2.1 below), we would like to mention some classification results by Harada for blowup solutions of (2.2). As a matter of fact, in [11], he classified all blowup solutions of (2.2) in dimension one, under some reasonable assumption (see (2.9), (2.10)), as follows (see Theorems 1.4, 1.5 and 1.6 in that work):

Consider $u = u_1 + iu_2$ a blowup solution of (2.2) in one dimension space with blowup time T and blowup point ξ which satisfies

$$\sup_{0<t<T} (T-t)\|u(t)\|_{L^\infty(\mathbb{R})} < +\infty. \tag{2.9}$$

Assume in addition that

$$\lim_{s\to+\infty} \|w_2(s)\|_{L^2_\rho(\mathbb{R})} = 0, w_2 \not\equiv 0, \tag{2.10}$$

where ρ is defined as follows

$$\rho(y) = \frac{e^{-\frac{y^2}{4}}}{\sqrt{4\pi}}, \tag{2.11}$$

and w_2 is defined by the following change of variables (also called similarity variables):

$w_1(y,s) = (T-t)u_1(\xi + e^{-\frac{s}{2}}y, t)$ *and* $w_2(y,s) = (T-t)u_2(\xi + e^{-\frac{s}{2}}y, t)$, *where* $t = T - e^{-s}$.

Then, one of the following cases occurs

$$(C_1) \quad \left\{\begin{array}{lll} w_1 & = & 1 - \frac{c_0}{s}h_2 + O(\frac{\ln s}{s^2}) \text{ in } L^2_\rho(\mathbb{R}), \\ w_2 & = & c_2 s^{-m} e^{-\frac{(m-2)s}{2}} h_m + O\left(s^{-(m+1)}e^{-\frac{(m-2)s}{2}}\ln s\right) \text{ in } L^2_\rho(\mathbb{R}), m \ge 2. \end{array}\right.$$

$$(C_2) \quad \left\{\begin{array}{lll} u & = & 1 - c_1 e^{-(k-1)s} h_{2k} + O(e^{-\frac{(2k-1)s}{2}}) \text{ in } L^2_\rho(\mathbb{R}), \\ v & = & c_2 e^{-\frac{(m-2)s}{2}} h_m + O\left(e^{-\frac{(m-1)s}{2}}\right) \text{ in } L^2_\rho(\mathbb{R}), k \ge 2, m \ge 2k. \end{array}\right.$$

where $c_0 = \frac{1}{8}, c_1 > 0, c_2 \neq 0$ *and* $\rho(y)$ *is defined in* (2.11) *and* $h_j(y)$ *is a rescaled version of the Hermite polynomial of order* m^{th} *defined as follows:*

$$h_m(y) = \sum_{j=0}^{[\frac{m}{2}]} \frac{(-1)^j m! y^{m-2j}}{j!(m-2j)!}. \tag{2.12}$$

Besides that, Harada has also given a profile to the solutions in similarity variables:
There exist $\kappa, \sigma, c > 0$ *such that*

$$(C_1) \quad \Rightarrow \quad \left| u - \frac{1}{1 + c_0 s^{-1} h_2} \right| + \left| s^{\frac{m}{2}} e^{\frac{(m-2)s}{2}} v - \frac{c_2 s^{-\frac{m}{2}} h_m}{(1 + c_0 s^{-1} h_2)^2} \right| < c s^{-\kappa}, \tag{2.13}$$

$$\text{for} \quad |y| \leq s^{(1+\sigma)}.$$

$$(C_2) \quad \Rightarrow \quad \left| u - \frac{1}{1 + c_1 e^{-(k-1)s} h_{2k}} \right| + \left| e^{\frac{(m-2k)s}{2k}} v - \frac{c_2 e^{-\frac{(k-1)ms}{2k}} h_m}{(1 + c_1 e^{-(k-1)s} h_{2k})^2} \right|, \tag{2.14}$$

$$\text{for} \quad |y| \leq e^{\frac{(k-1+\sigma)s}{2k}}.$$

Furthermore, he also gave the final blowup profiles *The blowup profile of* $u = u_1 + iu_2$ *is given by*

$$(C_1) \quad \Rightarrow \quad \begin{cases} u_1(x,T) &= \frac{2}{c_0}\left(\frac{|\ln|x||}{x^2}\right)(1 + o(1)), \\ u_2(x,T) &= \frac{c_2}{2^{m-2}(c_0)^2}\left(\frac{x^{m-4}}{|\ln|x||^{m-2}}\right)(1 + o(1)), \end{cases}$$

$$(C_2) \quad \Rightarrow \quad \begin{cases} u(x,T) &= \frac{1+ic_1}{(c_1 - ic_2)} x^{-2k}(1 + o(1)), \\ \text{if } m &= 2k, \\ u_1(x,T) &= (c_1)^{-1} x^{-2k}(1 + o(1)) \text{ and } u_2(x,T) = \frac{c_2}{(c_1)^2} x^{m-4k}(1 + o(1)), \\ \text{if } m &> 2k. \end{cases}$$

Then, from the work of Nouaili and Zaag in [17] and Harada in [11] for equation (2.2), we derive that the imaginary part u_2 also blows up under some conditions, however, none of them was able to give a global profile (i.e. valid uniformly on $\mathbb{R}^N$, and not just on an expanding ball as in (2.13) and (2.14)) for the imaginary part. For that reason, our main motivation in this work is to give a sharp description for the profile of the imaginary part. Our work is considered as an improvement of Nouaili and Zaag in [17] in dimension N, which is valid not only for $p = 2$, but also for any $p \geq 3, p \in \mathbb{N}$. In particular, this is the first time we give the profile for the imaginary part when the solution blows up. Without loss of generality, we assume that the blowup point, $a = 0$ and the following Theorem is our result:

Theorem 2.1 (Existence of a blowup solution for (2.1) and a sharp discription of its profile)**.** *For each* $p \geq 2, p \in \mathbb{N}$ *and* $p_1 \in (0,1)$, *there exists* $T_1(p, p_1) > 0$ *such that for all* $T \leq T_1$, *there exist initial data* $u^0 = u_1^0 + iu_2^0$, *such that equation* (2.1) *has a unique solution* u *on* $[0,T)$, *satisfying the following:*

i) The solution u blows up in finite time T only at the origin. Moreover, it satisfies the following estimates

$$\left\|(T-t)^{\frac{1}{p-1}}u(.,t)-f_0\left(\frac{.}{\sqrt{(T-t)|\ln(T-t)|}}\right)\right\|_{L^\infty(\mathbb{R}^N)}\le\frac{C}{\sqrt{|\ln(T-t)|}},\qquad(2.15)$$

and

$$\left\|(T-t)^{\frac{1}{p-1}}|\ln(T-t)|u_2(.,t)-g_0\left(\frac{.}{\sqrt{(T-t)|\ln(T-t)|}}\right)\right\|_{L^\infty(\mathbb{R}^N)}\le\frac{C}{|\ln(T-t)|^{\frac{p_1}{2}}},\qquad(2.16)$$

where f_0 is defined in (2.6) and $g_0(z)$ is defined as follows

$$g_0(z)=\frac{|z|^2}{\left(p-1+\frac{(p-1)^2}{4p}|z|^2\right)^{\frac{p}{p-1}}}.\qquad(2.17)$$

ii) There exists a complex function $u^*(x)\in C^2(\mathbb{R}^N\backslash\{0\})$ such that $u(t)\to u^*=u_1^*+iu_2^*$ as $t\to T$ uniformly on compact sets of $\mathbb{R}^N\backslash\{0\}$ and we have the following asymptotic expansions:

$$u^*(x)\sim\left[\frac{(p-1)^2|x|^2}{8p|\ln|x||}\right]^{-\frac{1}{p-1}},\ \text{as } x\to 0.\qquad(2.18)$$

and

$$u_2^*(x)\sim\frac{2p}{(p-1)^2}\left[\frac{(p-1)^2|x|^2}{8p|\ln|x||}\right]^{-\frac{1}{p-1}}\frac{1}{|\ln|x||},\ \text{as } x\to 0.\qquad(2.19)$$

Remark 2.2. *The initial data u^0 is given exactly as follows*

$$u^0=u_1^0+iu_2^0,$$

where

$$\begin{aligned}u_1^0&=T^{-\frac{1}{p-1}}\left\{\left(p-1+\frac{(p-1)^2|x|^2}{4pT|\ln T|}\right)^{-\frac{1}{p-1}}+\frac{N\kappa}{2p|\ln T|}\right.\\&+\left.\frac{A}{|\ln T|^2}(d_{1,0}+d_{1,1}\cdot y)\chi_0\left(\frac{2x}{K\sqrt{T|\ln T|}}\right)\right\},\\u_2^0&=T^{-\frac{1}{p-1}}\left\{\frac{|x|^2}{T|\ln T|^2}\left(p-1+\frac{(p-1)^2|x|^2}{4pT|\ln T|}\right)^{-\frac{p}{p-1}}-\frac{2N\kappa}{(p-1)|\ln T|^2}\right.\\&+\left[\frac{A^2}{|\ln T|^{p_1+2}}(d_{2,0}+d_{2,1}\cdot y)+\frac{A^5\ln(|\ln(T)|)}{|\ln T|^{p_1+2}}\times\right.\\&\left.\left.\left(\frac{1}{2}y^{\mathfrak{T}}\cdot d_{2,2}\cdot y-Tr(d_{2,2})\right)\right]\chi_0\left(\frac{2x}{K\sqrt{T|\ln T|}}\right)\right\}.\end{aligned}$$

with $\kappa=(p-1)^{-\frac{1}{p-1}}$, K,A are positive constants fixed large enough, $d^{(1)}=(d_{1,0},d_{1,1}),d^{(2)}=(d_{2,0},d_{2,1},d_{2,2})$ are parameters we fine tune in our proof, and $\chi_0\in C_0^\infty[0,+\infty)$, $\|\chi_0\|_{L^\infty(\mathbb{R}^N)}\le 1$, supp $\chi_0\subset[0,2]$.

Remark 2.3. *We see below in* (2.23) *that the equation satisfied by of u_2 is almost 'linear' in u_2. Accordingly, we may change a little our proof to construct a solution $u_{c_0}(t) = u_{1,c_0} + iu_{2,c_0}$ with $t \in [0,T), c_0 \neq 0$, which blows up in finite time T only at the origin such that* (2.15) *and* (2.18) *hold and the following holds*

$$\left\| (T-t)^{\frac{1}{p-1}} |\ln(T-t)| u_{2,c_0}(.,t) - c_0 g_0 \left(\frac{\cdot}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{|\ln(T-t)|^{\frac{p_1}{2}}}, \tag{2.20}$$

and

$$u_2^*(x) \sim \frac{2pc_0}{(p-1)^2} \left[\frac{(p-1)^2|x|^2}{8p|\ln|x||} \right]^{-\frac{1}{p-1}} \frac{1}{|\ln|x||}, \text{ as } x \to 0, \tag{2.21}$$

Remark 2.4. *We deduce from* (*ii*) *that u blows up only at* 0. *In particular, note that both u_1 and u_2 blow up. However, the blowup speed of u_2 is softer than u_1 because of the quantity $\frac{1}{|\ln|x||}$.*

Remark 2.5. *Nouaili and Zaag constructed a blowup solution of* (2.2) *with a less explicit behavior for the imaginary part (see* (2.8)*). Here, we do better and we obtain the profile for the imaginary part in* (2.16) *and we also describe the asymptotics of the solution in the neighborhood of the blowup point in* (2.19)*. In fact, this refined behavior comes from a more involved formal approach (see Section 2.2 below), and more parameters to be fine tuned in initial data (see Definition 2.2 where we need more parameters than in Nouaili and Zaag [17], namely $d_2 \in \mathbb{R}^{N^2}$). Note also that our profile estimates in* (2.15) *and* (2.16) *are better than the estimates* (2.13) *and* (2.14) *by Harada ($m = 2$), in the sense that we have a uniform estimate for whole space $\mathbb{R}^N$, and not just for all $|y| \leq s^{1+\sigma}$ for some $\sigma > 0$. Another point: our result hold in N space dimensions, unlike the work of Harada in [11], which holds only in one space dimension.*

Remark 2.6. *As in the case $p = 2$ treated by Nouaili and Zaag [17], we suspect this behavior in Theorem 2.1 to be unstable. This is due to the fact that the number of parameters in the initial data we consider below in Definition 2.2 is higher than the dimension of the blowup parameters which is $N + 1$ (N for the blowup points and* 1 *for the blowup time).*

Besides that, we can use the technique of Merle [13] to construct a solution which blows up at arbitrary given points. More precisely, we have the following Corollary:

Corollary 2.7 (Blowing up at k distinct points)**.** *For any given points, $x_1, ..., x_k$, there exists a solution of* (2.1) *which blows up exactly at $x_1, ..., x_k$. Moreover, the local behavior at each blowup point x_j is also given by* (2.15), (2.16), (2.18), (2.19) *by replacing x by $x - x_j$ and $L^\infty(\mathbb{R}^N)$ by $L^\infty(|x - x_j| \leq \epsilon_0)$, for some $\epsilon_0 > 0$.*

This paper is organized as follows:

- In Section 2.2, we adopt a formal approach to show how the profiles we have in Theorem 2.1 appear naturally.

- In Section 2.3, we give the rigorous proof for Theorem 2.1, assuming some technical estimates.

- In Section 2.4, we prove the techical estimates assumed in Section 2.3.

2.2 Derivation of the profile (formal approach)

In this section, we aim at giving a formal approach to our problem which helps us to explain how we derive the profiles of solution of (2.1), given in Theorem (2.1), as well the asymptotic behaviors of our solution.

2.2.1 Modeling the problem

In this part, we will give some important definitions and special symbols in our work and explain then how functions f_0 ans g_0 arise as blowup profiles for equation (2.1) as stated in (2.15) and (2.16). Our aim in this section is to give solid (though formal) hints for the existence of a solution $u(t) = u_1(t) + iu_2(t)$ to equation (2.1) such that

$$\lim_{t\to T} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = +\infty, \tag{2.22}$$

and u obeys the profiles in (2.15) and (2.16), for some $T > 0$. By using equation (2.1), we deduce that u_1 and u_2 satisfy the following

$$\begin{cases} \partial_t u_1 &= \Delta u_1 + F_1(u_1, u_2), \\ \partial_t u_2 &= \Delta u_2 + F_2(u_1, u_2). \end{cases} \tag{2.23}$$

where

$$\begin{cases} F_1(u_1, u_2) &= \operatorname{Re}\left[(u_1 + iu_2)^p\right] = \sum_{j=0}^{\left[\frac{p}{2}\right]} C_p^{2j} (-1)^j u_1^{p-2j} u_2^{2j}, \\ F_2(u_1, u_2) &= \operatorname{Im}\left[(u_1 + iu_2)^p\right] = \sum_{j=0}^{\left[\frac{p-1}{2}\right]} C_p^{2j+1} (-1)^j u_1^{p-2j-1} u_2^{2j+1}, \end{cases} \tag{2.24}$$

with $\operatorname{Re}[z]$ and $\operatorname{Im}[z]$ being respectively the real and the imaginary part of z and $C_p^m = \frac{p!}{m!(p-m)!}$, for all $m \le p$.

Let us introduce *the similarity-variables*:

$$w(y, s) = (T - t)^{\frac{1}{p-1}} u(x, t), y = \frac{x}{\sqrt{T - t}}, s = -\ln(T - t) \text{ and } w = w_1 + iw_2. \tag{2.25}$$

Thanks to (2.23), we derive the system satisfied by (w_1, w_2), for all $y \in \mathbb{R}^N$ and $s \ge -\ln T$ as follows:

$$\begin{cases} \partial_s w_1 &= \Delta w_1 - \frac{1}{2} y \cdot \nabla w_1 - \frac{w_1}{p-1} + F_1(w_1, w_2), \\ \partial_s w_2 &= \Delta w_2 - \frac{1}{2} y \cdot \nabla w_2 - \frac{w_2}{p-1} + F_2(w_1, w_2). \end{cases} \tag{2.26}$$

Then note that studying the asymptotic of u as $t \to T$ is equivalent to studying the asymptotic of w in long time. In particular, we are first interested in the set of constant solutions of (2.26) (2.26), denoted by

$$\mathcal{S} = \{(0,0)\} \cup \left\{ \left(\kappa \cos\left(\frac{2k\pi}{p-1}\right), \kappa \sin\left(\frac{2k\pi}{p-1}\right) \right) \text{ where } \kappa = (p-1)^{-\frac{1}{p-1}}, k = 0, ..., p-1 \right\}.$$

From transformation (2.25), we slightly precise our goal in (2.22) by requiring in addition that

$$(w_1, w_2) \to (\kappa, 0) \text{ as } s \to +\infty.$$

Introducing $w_1 = \kappa + \bar{w}_1$, our goal is to get

$$(\bar{w}_1, w_2) \to (0,0) \text{ as } s \to +\infty.$$

From (2.26), we deduce that $\bar{w}_1, w_2$ satisfy the following system

$$\begin{cases} \partial_s \bar{w}_1 &= \mathcal{L}\bar{w}_1 + \bar{B}_1(\bar{w}_1, w_2), \\ \partial_s w_2 &= \mathcal{L} w_2 + \bar{B}_2(\bar{w}_1, w_2), \end{cases} \tag{2.27}$$

where

$$\mathcal{L} = \Delta - \frac{1}{2} y \cdot \nabla + Id, \tag{2.28}$$

$$\bar{B}_1(\bar{w}_1, w_2) = F_1(\kappa + \bar{w}_1, w_2) - \kappa^p - \frac{p}{p-1}\bar{w}_1, \tag{2.29}$$

$$\bar{B}_2(\bar{w}_1, w_2) = F_2(\kappa + \bar{w}_1, w_2) - \frac{p}{p-1} w_2, \tag{2.30}$$

and the definitions of F_1 and F_2 are given in (2.24).

It is important to study the linear operator $\mathcal{L}$ and the asymptotic behaviors of $\bar{B}_1, \bar{B}_2$ as $(\bar{w}_1, w_2) \to (0,0)$ which will appear as "quadratic" terms.

- *The properties of $\mathcal{L}$:*

We observe that operator $\mathcal{L}$ plays an important role in our analysis. In fact, $\mathcal{L}$ is self-adjoint in $\mathcal{D} \subset L^2_\rho(\mathbb{R}^N)$, where L^2_ρ is the weighted space associated with the weight ρ defined by

$$\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{\frac{N}{2}}} = \prod_{j=1}^{N} \rho_j(y_j), \text{ with } \rho_j(y_j) = \frac{e^{-\frac{|y_j|^2}{4}}}{(4\pi)^{\frac{1}{2}}}, \tag{2.31}$$

and the spectrum set of $\mathcal{L}$ is given as follows

$$\text{Spec}(\mathcal{L}) = \left\{ 1 - \frac{m}{2}, m \in \mathbb{N} \right\}.$$

Moreover, we can find eigenfunctions which correspond to each eigenvalue $1 - \frac{m}{2}, m \in \mathbb{N}$:

- The one space dimensional case: the eigenfunction corresponding to the eigenvalue $1 - \frac{m}{2}$ is h_m, the rescaled Hermite polynomial given in (2.12). In particular, we have the following orthogonality property:

$$\int_{\mathbb{R}} h_i h_j \rho dy = i! 2^i \delta_{i,j}, \quad \forall (i,j) \in \mathbb{N}^2.$$

- The higher dimensional case: $N \geq 2$, the eigenspace $\mathcal{E}_m$, corresponding to the eigenvalue $1 - \frac{m}{2}$ is defined as follows:

$$\mathcal{E}_m = \left\langle h_\beta(y) = h_{\beta_1}(y_1) ... h_{\beta_N}(y_N) \middle| |\beta| = \sum_{i=1}^{N} \beta_i = m \text{ and } \beta = (\beta_1, ..., \beta_N) \in \mathbb{N}^N \right\rangle. \tag{2.32}$$

As a matter of fact, so we can represent an arbitrary function $r \in L^2_\rho$ as follows

$$r = \sum_{\beta, \beta \in \mathbb{N}^n} r_\beta h_\beta(y),$$

where r_β is the projection of r on h_β for any $\beta \in \mathbb{N}^N$ which is defined as follows:

$$r_\beta = \mathbb{P}_\beta(r) = \int r k_\beta \rho dy, \forall \beta \in \mathbb{N}^n, \quad (2.33)$$

with

$$k_\beta(y) = \frac{h_\beta}{\|h_\beta\|^2_{L^2_\rho}}. \quad (2.34)$$

- *The asymptotic behaviors of* $\bar{B}_1(\bar{w}_1, w_2), \bar{B}_2(\bar{w}_1, w_2)$*:* The following hold:

$$\bar{B}_1(\bar{w}_1, w_2) = \frac{p}{2\kappa}\bar{w}_1^2 + O(|\bar{w}_1|^3 + |w_2|^2), \quad (2.35)$$

$$\bar{B}_2(\bar{w}_1, w_2) = \frac{p}{\kappa}\bar{w}_1 w_2 + O\left(|\bar{w}_1|^2|w_2|\right) + O\left(|w_2|^3\right), \quad (2.36)$$

as $(\bar{w}_1, w_2) \to (0, 0)$ (see Lemma 2.17 below).

2.2.2 Inner expansion

In this part, we study the asymptotic behavior of the solution in $L^2_\rho(\mathbb{R}^N)$. Moreover, for simplicity we suppose that $N = 1$, and we recall that we aim at constructing a solution of (2.27) such that $(\bar{w}_1, w_2) \to (0, 0)$. Note first that the spectrum of $\mathcal{L}$ contains two positive eigenvalues $1, \frac{1}{2}$, a neutral eigenvalue 0 and all the other ones are strictly negative. So, in the representation of the solution in $L^2_\rho(\mathbb{R})$, it is reasonable to think that the part corresponding to the negative spectrum is easily controlled. Imposing a symmetry condition on the solution with respect of y, it is reasonable to look for a solution $\bar{w}_1, w_2$ of the forms:

$$\begin{aligned} \bar{w}_1 &= \bar{w}_{1,0} h_0 + \bar{w}_{1,2} h_2, \\ w_2 &= w_{2,0} h_0 + w_{2,2} h_2. \end{aligned}$$

From the assumption that $(\bar{w}_1, w_2) \to (0, 0)$, we see that $\bar{w}_{1,0}, \bar{w}_{1,2}, w_{2,0}, w_{2,2} \to 0$ as $s \to +\infty$. We see also that we can understand the asymptotic behaviors of $\bar{w}_1$ and w_2 in L^2_ρ from the study of the asymptotic behaviors of $\bar{w}_{1,0}, \bar{w}_{1,2}, w_{2,0}$ and $w_{2,2}$. We now project equations (2.27) on h_0 and h_2. Using behaviors of $\bar{B}_1, \bar{B}_2$, given in (2.35) and (2.36), we get the following ODEs for $\bar{w}_{1,0}, \bar{w}_{1,2}, w_{2,0}, w_{2,2}$:

$$\partial_s \bar{w}_{1,0} = \bar{w}_{1,0} + \frac{p}{2\kappa}\left(\bar{w}_{1,0}^2 + 8\bar{w}_{1,2}^2\right) + O(|\bar{w}_{1,0}|^3 + |\bar{w}_{1,2}|^3) + O(|w_{2,0}|^2 + |w_{2,2}|^2), \quad (2.37)$$

$$\partial_s \bar{w}_{1,2} = \frac{p}{\kappa}\left(\bar{w}_{1,0}\bar{w}_{1,2} + 4\bar{w}_{1,2}^2\right) + O(|\bar{w}_{1,0}|^3 + |\bar{w}_{1,2}|^3) + O(|w_{2,0}|^2 + |w_{2,2}|^2), \quad (2.38)$$

$$\begin{aligned} \partial_s w_{2,0} &= w_{2,0} + \frac{p}{\kappa}\left[\bar{w}_{1,0} w_{2,0} + 8\bar{w}_{1,2} w_{2,2}\right] + O((|\bar{w}_{1,0}|^2 + |\bar{w}_{1,2}|^2)(|w_{2,0}| + |w_{2,2}|)) \quad (2.39) \\ &+ O(|w_{2,0}|^3 + |w_{2,2}|^3), \end{aligned}$$

$$\begin{aligned} \partial_s w_{2,2} &= \frac{p}{\kappa}\left[\bar{w}_{1,0} w_{2,2} + \bar{w}_{1,2} w_{2,0} + 8\bar{w}_{1,2} w_{2,2}\right] + O((|\bar{w}_{1,0}|^2 + |\bar{w}_{1,2}|^2)(|w_{2,0}| + |w_{2,2}|)) \quad (2.40) \\ &+ O(|w_{2,0}|^3 + |w_{2,2}|^3). \end{aligned}$$

Assuming that

$$\bar{w}_{1,0}, w_{2,0}, w_{2,2} \ll \bar{w}_{1,2} \text{ as } s \to +\infty, \tag{2.41}$$

we may simplify the ODE system as follows:

- *The asymptotic behavior of $\bar{w}_{1,2}$:*

  We deduce from (2.38) and (2.41) that

$$\partial_s \bar{w}_{1,2} \sim \frac{4p}{\kappa} \bar{w}_{1,2}^2 \text{ as } s \to +\infty,$$

which yields

$$\bar{w}_{1,2} = -\frac{\kappa}{4ps} + o\left(\frac{1}{s}\right), \text{ as } s \to +\infty. \tag{2.42}$$

Assuming futher that

$$\bar{w}_{1,0}, w_{2,0}, w_{2,2} \lesssim \frac{1}{s^2}, \tag{2.43}$$

we see that

$$\bar{w}_{1,2} = -\frac{\kappa}{4ps} + O\left(\frac{\ln s}{s^2}\right), \text{ as } s \to +\infty. \tag{2.44}$$

- *The asymptotic behavior of $\bar{w}_{1,0}$:*

  By using (2.37), (2.41) and the asymptotic behaviors of $\bar{w}_{1,2}$ in (2.44), we see that

$$\bar{w}_{1,0} = O\left(\frac{1}{s^2}\right) \text{ as } s \to +\infty. \tag{2.45}$$

- *The asymptotics of $w_{2,0}$ and $w_{2,2}$:* Bisides that, we derive from (2.39), (2.40) and (2.43) that

$$\begin{aligned} \partial_s w_{2,2} &= \left(-\frac{2}{s} + O\left(\frac{\ln s}{s^2}\right)\right) w_{2,2} + o\left(\frac{1}{s^3}\right), \\ \partial_s w_{2,0} &= w_{2,0} + O\left(\frac{1}{s^3}\right), \end{aligned} \tag{2.46}$$

which yields

$$\begin{aligned} w_{2,2} &= o\left(\frac{\ln s}{s^2}\right), \\ w_{2,0} &= O\left(\frac{1}{s^3}\right), \end{aligned} \tag{2.47}$$

as $s \to +\infty$. This also yields a new ODE for $w_{2,2}$:

$$\partial_s w_{2,2} = -\frac{2}{s} w_{2,2} + o\left(\frac{\ln^2 s}{s^4}\right),$$

which implies

$$w_{2,2} = O\left(\frac{1}{s^2}\right).$$

Using again (2.46), we derive a new ODE for $w_{2,2}$

$$\partial_s w_{2,2} = -\frac{2}{s} w_{2,2} + O\left(\frac{\ln s}{s^4}\right),$$

which yields

$$w_{2,2} = \frac{\tilde{c}_0}{s^2} + O\left(\frac{\ln s}{s^3}\right), \text{ for some } \tilde{c}_0 \in \mathbb{R}^*. \tag{2.48}$$

Noting that our finding (2.44), (2.45), (2.47) and (2.48) are consistent with our hypotheses in (2.41) and (2.43), we get the asymptotics of the solution w_1 and w_2 as follows:

$$w_1 = \kappa - \frac{\kappa}{4ps}(y^2 - 2) + O\left(\frac{1}{s^2}\right), \tag{2.49}$$

$$w_2 = \frac{\tilde{c}_0}{s^2}(y^2 - 2) + O\left(\frac{\ln s}{s^3}\right), \tag{2.50}$$

in $L^2_\rho(\mathbb{R})$ for some $\tilde{c}_0$ in $\mathbb{R}^*$. Using parabolic regularity, we note that the asymptotic behaviors (2.49) and (2.50) also hold for all $|y| \le K$, where K is an arbitrary positive constant.

2.2.3 Outer expansion

As Subsection 2.2.2 above, we assume that $N = 1$. We see that asymptotics (2.49) and (2.50) can not give us a shape, since they hold uniformly on compact sets, and not in larger sets. Fortunately, we observe from (2.49) and (2.50) that the profile may be based on the following variable:

$$z = \frac{y}{\sqrt{s}}. \tag{2.51}$$

This motivates us to look for solutions of the form:

$$w_1(y,s) = \sum_{j=0}^{\infty} \frac{R_{1,j}(z)}{s^j},$$

$$w_2(y,s) = \sum_{j=1}^{\infty} \frac{R_{2,j}(z)}{s^j}.$$

Using system (2.26) and gathering terms of order $\frac{1}{s^j}$ for $j = 0, ..., 2$, we obtain

$$0 = -\frac{1}{2}R'_{1,0}(z)\cdot z - \frac{R_{1,0}(z)}{p-1} + R^p_{1,0}(z), \tag{2.52}$$

$$0 = -\frac{1}{2}zR'_{1,1} - \frac{R_{1,1}}{p-1} + pR^{p-1}_{1,0}R_{1,1} + R''_{1,0} + \frac{zR'_{1,0}}{2}, \tag{2.53}$$

$$0 = -\frac{1}{2}R'_{2,1}(z)\cdot z - \frac{R_{2,1}}{p-1} + pR^{p-1}_{1,0}R_{2,1}, \tag{2.54}$$

$$\begin{aligned} 0 = & -\frac{1}{2}R'_{2,2}(z).z - \frac{R_{2,2}}{p-1} + pR^{p-1}_{1,0}R_{2,2} + R''_{2,1} + R_{2,1} \\ & + \frac{1}{2}R'_{2,1}\cdot z + p(p-1)R^{p-2}_{1,0}R_{1,1}R_{2,1}. \end{aligned} \tag{2.55}$$

We now solve the above equations:

- *The solution $R_{1,0}$:* It is easy to solve (2.52)

$$R_{1,0}(z) = (p-1+bz^2)^{-\frac{1}{p-1}}, \tag{2.56}$$

where b is an unknown constant that will be selected accordingly to our purpose.

- *The solution $R_{1,1}$:* We rewrite (2.53) under the following form:

$$\frac{1}{2}z.R'_{1,1}(z) = \left(\frac{(p-1)^2 - bz^2}{(p-1)(p-1+bz^2)}\right) R_{1,1} + F_{1,1}(z),$$

where

$$\begin{aligned} F_{1,1}(z) &= -\frac{2b}{p-1}(p-1+bz^2)^{-\frac{p}{p-1}} + \frac{4pb^2z^2}{(p-1)^2}(p-1+bz^2)^{-\frac{(2p-1)}{p-1}} \\ &- \frac{bz^2}{p-1}(p-1+bz^2)^{-\frac{p}{p-1}}. \end{aligned}$$

Thanks to the variation of constant method, we see that

$$R_{1,1} = H^{-1}(z)\left(\int \frac{2}{z}H(z)F_{1,1}(z)dz + C_1\right), \tag{2.57}$$

where

$$H(z) = \frac{(p-1+bz^2)^{\frac{p}{p-1}}}{z^2}.$$

Besides that, we have:

$$\begin{aligned} \frac{2H}{z}F_{1,1} &= -\frac{4b}{(p-1)z^3} + \frac{8pb^2}{(p-1)^2}\left(\frac{1}{z(p-1+bz^2)}\right) - \frac{2b}{(p-1)z} \\ &= -\frac{4b}{(p-1)z^3} + \frac{1}{z}\left(-\frac{2b}{p-1} + \frac{8pb^2}{(p-1)^3}\right) \\ &+ (p-1+bz^2)^{-1}\left(-\frac{8pb^3z}{(p-1)^3}\right). \end{aligned}$$

We can see that if the coefficient of $\frac{1}{z}$ is non zero, then we will have a "$\ln z$" term in the formula of $R_{1,1}$ and this makes the fact that $R_{1,1}$ would not be analytic, creating a singularity in the solution. In order to avoid this singularity, we impose that

$$-\frac{2b}{p-1} + \frac{8pb^2}{(p-1)^3} = 0,$$

which yields

$$b = \frac{(p-1)^2}{4p}. \tag{2.58}$$

Besides that, for simplicity, we assume that $C_1 = 0$. Using (2.57), we see that

$$R_{1,1} = \frac{(p-1)}{2p}(p-1+bz^2)^{-\frac{p}{p-1}} - \frac{p-1}{4p}z^2\ln(p-1+bz^2)(p-1+bz^2)^{-\frac{p}{p-1}}. \tag{2.59}$$

- *The solution $R_{2,1}$:* It is easy to solve (2.54) as follows:

$$R_{2,1}(z) = \frac{z^2}{(p-1+bz^2)^{\frac{p}{p-1}}}. \tag{2.60}$$

- *The solution $R_{2,2}$:* We rewrite (2.55) as follows

$$\frac{1}{2}z \cdot R'_{2,2}(z) = \left(\frac{(p-1)^2 - bz^2}{(p-1)(p-1+bz^2)}\right) R_{2,2}(z) + F_{2,2}(z),$$

where

$$\begin{aligned} F_{2,2}(z) &= R''_{2,1} + R_{2,1} + \frac{1}{2}R'_{2,1} \cdot z + p(p-1)R_{1,0}^{p-2}R_{1,1}R_{2,1} \\ &= 2(p-1+bz^2)^{-\frac{p}{p-1}} \\ &- \frac{10pbz^2}{p-1}(p-1+bz^2)^{-\frac{2p-1}{p-1}} + 2z^2(p-1+bz^2)^{-\frac{p}{p-1}} + \frac{(p-1)^2}{2}z^2(p-1+bz^2)^{-\frac{3p-2}{p-1}} \\ &+ \frac{4p(2p-1)b^2z^4}{(p-1)^2}(p-1+bz^2)^{-\frac{3p-2}{p-1}} - \frac{pbz^4}{p-1}(p-1+bz^2)^{-\frac{2p-1}{p-1}} \\ &- \frac{(p-1)^2}{4}z^4\ln(p-1+bz^2)(p-1+bz^2)^{-\frac{3p-2}{p-1}}. \end{aligned}$$

By using the variation of constant method, we have

$$R_{2,2}(z) = \frac{z^2}{(p-1+bz^2)^{-\frac{p}{p-1}}}\left(\int \frac{2(p-1+bz^2)^{-\frac{p}{p-1}}}{z^3}F_{2,2}(z)dz + C_2\right), \tag{2.61}$$

where

$$\begin{aligned} \frac{2(p-1+bz^2)^{-\frac{p}{p-1}}}{z^3}F_{2,2}(z) &= \frac{4}{z^3} + \left[5 - \frac{20pb}{(p-1)^2}\right]\frac{1}{z} + \frac{z}{p-1+bz^2}\left[\frac{20pb}{(p-1)^2} - b - \frac{2pb}{p-1}\right] \\ &+ \left[\frac{8p(2p-1)b^2}{(p-1)^2} - (p-1)p\right]\frac{z}{(p-1+bz^2)^2} \\ &- \frac{(p-1)^2}{2}z\ln(p-1+bz^2)(p-1+bz^2)^{-2}. \end{aligned}$$

We observe that

$$5 - \frac{20pb}{(p-1)^2} = 0, \text{ because } b = \frac{(p-1)^2}{4p}.$$

So, from (2.61) and assuming that $C_2 = 0$, we have

$$R_{2,2}(z) = -2(p-1+bz^2)^{-\frac{p}{p-1}} + H_{2,2}(z), \tag{2.62}$$

where

$$\begin{aligned} H_{2,2}(z) &= C_{2,1}(p)z^2(p-1+bz^2)^{-\frac{2p-1}{p-1}} + C_{2,3}(p)z^2\ln(p-1+bz^2)(p-1+bz^2)^{-\frac{p}{p-1}} \\ &+ C_{2,3}(p)z^2\ln(p-1+bz^2)(p-1+bz^2)^{-\frac{2p-1}{p-1}}. \end{aligned}$$

Since the outer expansion has to match the inner expansion, we will fix several constants and derive the following profiles for w_1 and w_2:

$$\left\{\begin{array}{lll} w_1(y,s) & \sim & \Phi_1(y,s), \\ w_2(y,s) & \sim & \Phi_2(y,s), \end{array}\right. \tag{2.63}$$

where

$$\Phi_1(y,s) = \left(p-1+\frac{(p-1)^2}{4p}\frac{|y|^2}{s}\right)^{-\frac{1}{p-1}} + \frac{N\kappa}{2ps}, \tag{2.64}$$

$$\Phi_2(y,s) = \frac{|y|^2}{s^2}\left(p-1+\frac{(p-1)^2}{4p}\frac{|y|^2}{s}\right)^{-\frac{p}{p-1}} - \frac{2N\kappa}{(p-1)s^2}, \tag{2.65}$$

for all $(y,s) \in \mathbb{R}^N \times (0,+\infty)$.

2.3 Existence of a blowup solution in Theorem 2.1

In Section 2.2, we adopted a formal approach on order to justify how the profiles f_0, g_0 arise as blowup profiles for equation (2.1). In this section, we give a rigorous proof to justify the existence of a solution approaching those profiles.

2.3.1 Formulation of the problem

In this section, we aim at formulating our problem in order to justify the formal approach which is given in the previous section. Introducing

$$\left\{\begin{array}{lll} w_1 & = & \Phi_1 + q_1, \\ w_2 & = & \Phi_2 + q_2, \end{array}\right. \tag{2.66}$$

where Φ_1, Φ_2 are defined in (2.64) and (2.65) respectively, then using (2.26), we see that (q_1, q_2) satisfy

$$\partial_s \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \mathcal{L}+V & 0 \\ 0 & \mathcal{L}+V \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} V_{1,1} & V_{1,2} \\ V_{2,1} & V_{2,2} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} R_1(y,s) \\ R_2(y,s) \end{pmatrix}, \tag{2.67}$$

where linear operator $\mathcal{L}$ is defined in (2.28) and:

- Potential functions $V, V_{1,1}, V_{1,2}, V_{2,1}, V_{2,2}$ are defined as follows

$$V(y,s) = p\left(\Phi_1^{p-1} - \frac{1}{p-1}\right), \tag{2.68}$$

$$V_{1,1}(y,s) = \sum_{j=1}^{\left[\frac{p}{2}\right]} C_p^{2j}(-1)^j(p-2j)\Phi_1^{p-2j-1}\Phi_2^{2j}, \tag{2.69}$$

$$V_{1,2}(y,s) = \sum_{j=0}^{\left[\frac{p}{2}\right]} C_p^{2j}(-1)^j.(2j)\Phi_1^{p-2j}\Phi_2^{2j-1}, \tag{2.70}$$

$$V_{2,1}(y,s) = \sum_{j=0}^{\left[\frac{p-1}{2}\right]} C_p^{2j+1}(-1)^j(p-2j-1)\Phi_1^{p-2j-2}\Phi_2^{2j+1}, \tag{2.71}$$

$$V_{2,2}(y,s) = \sum_{j=1}^{\left[\frac{p-1}{2}\right]} C_p^{2j+1}(-1)^j(2j+1)\Phi_1^{p-2j-1}\Phi_2^{2j}. \tag{2.72}$$

- Quadratic terms $B_1(q_1,q_2)$ and $B_2(q_1,q_2)$ are defined as follows:

$$B_1(q_1,q_2) = F_1(\Phi_1+q_1,\Phi_2+q_2) - F_1(\Phi_1,\Phi_2) - \sum_{j=0}^{\left[\frac{p}{2}\right]} C_p^{2j}(-1)^j(p-2j)\Phi_1^{p-2j-1}\Phi_2^{2j}q_1 \tag{2.73}$$

$$- \sum_{j=0}^{\left[\frac{p}{2}\right]} C_p^{2j}(-1)^j.(2j)\Phi_1^{p-2j}\Phi_2^{2j-1}q_2,$$

$$B_2(q_1,q_2) = F_2(\Phi_1+q_1,\Phi_2+q_2) - F_2(\Phi_1,\Phi_2) - \sum_{j=0}^{\left[\frac{p-1}{2}\right]} C_p^{2j+1}(-1)^j(p-2j-1)\Phi_1^{p-2j-2}\Phi_2^{2j+1}q_1$$

$$- \sum_{j=0}^{\left[\frac{p-1}{2}\right]} C_p^{2j+1}(-1)^j(2j+1)\Phi_1^{p-2j-1}\Phi_2^{2j}q_2. \tag{2.74}$$

- Rest terms $R_1(y,s), R_2(y,s)$ are defined as follows:

$$R_1(y,s) = \Delta\Phi_1 - \frac{1}{2}y\cdot\nabla\Phi_1 - \frac{\Phi_1}{p-1} + F_1(\Phi_1,\Phi_2) - \partial_s\Phi_1, \tag{2.75}$$

$$R_2(y,s) = \Delta\Phi_2 - \frac{1}{2}y\cdot\nabla\Phi_2 - \frac{\Phi_2}{p-1} + F_2(\Phi_1,\Phi_2) - \partial_s\Phi_2, \tag{2.76}$$

where F_1, F_2 are defined in (2.24).

By the linearization around Φ_1, Φ_2, our problem is reduced to constructing a solution (q_1,q_2) of system (2.67), satisfying

$$\|q_1\|_{L^\infty(\mathbb{R}^N)} + \|q_2\|_{L^\infty(\mathbb{R}^N)} \to 0 \text{ as } s \to +\infty.$$

Concerning equation (2.67), we recall that we already know some main properties of linear operator $\mathcal{L}$ (see page 77). As for potential functions $V_{j,k}$ where $j,k \in \{1,2\}$, they admit the following asymptotic behaviors

$$\sum_{j,k\leq 2} |V_{j,k}(y,s)| \leq \frac{C}{s}, \forall y \in \mathbb{R}^N, s \geq 1,$$

(see Lemma 2.18). Regarding the terms B_1, B_2, R_1, R_2, we see that whenever $|q_1|+|q_2| \leq 2$, we have

$$\begin{aligned}
|B_1(q_1,q_2)| &\leq C(q_1^2+q_2^2),\\
|B_2(q_1,q_2)| &\leq C\left(\frac{|q_1|^2}{s}+|q_1q_2|+|q_2|^2\right),\\
\|R_1(.,s)\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C}{s},\\
\|R_2(.,s)\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C}{s^2},
\end{aligned}$$

(see Lemmas 2.19 and 2.20).

In fact, the dynamics of equation (2.67) will mainly depend on the main linear operator

$$\begin{pmatrix} \mathcal{L}+V & 0 \\ 0 & \mathcal{L}+V \end{pmatrix},$$

and the effects of the other terms will be less important. For that reason, we need to understand the dynamics of $\mathcal{L}+V$. Since the spectral properties of $\mathcal{L}$ were already introduced in Section 2.2.1, we will focus here on the effect of V.

i) Effect of V inside the blowup region $\{|y| \leq K\sqrt{s}\}$ with $K > 0$ arbitrary, we have

$$V \to 0 \text{ in } L^2_\rho(|y| \leq K\sqrt{s}) \text{ as } s \to +\infty,$$

which means that the effect of V will be negligible with respect of the effect of $\mathcal{L}$, except perhaps on the null mode of $\mathcal{L}$ (see item *(ii)* of Proposition 2.13 below)

ii) Effect of V outside the blowup region: for each $\epsilon > 0$, there exist $K_\epsilon > 0$ and $s_\epsilon > 0$ such that

$$\sup_{\frac{y}{\sqrt{s}} \geq K_\epsilon, s \geq s_\epsilon} \left| V(y,s) - \left(-\frac{p}{p-1}\right)\right| \leq \epsilon.$$

Since 1 is the biggest eigenvalue of $\mathcal{L}$, the operator $\mathcal{L}+V$ behaves as one with with a fully negative spectrum outside blowup region $\{|y| \geq K_\epsilon\sqrt{s}\}$, which makes the control of the solution in this region easily.

Since the behavior of the potential V inside and outside the blowup region is different, we will consider the dynamics of the solution for $|y| \leq 2K\sqrt{s}$ and for $|y| \geq K\sqrt{s}$ separately for some K to be fixed large. For that purpose, we introduce the following cut-off function

$$\chi(y,s) = \chi_0\left(\frac{|y|}{K\sqrt{s}}\right), \tag{2.77}$$

where $\chi_0 \in C_0^\infty[0,+\infty), \|\chi_0\|_{L^\infty(\mathbb{R}^N)} \leq 1$ and

$$\chi_0(x) = \begin{cases} 1 & \text{for } x \leq 1, \\ 0 & \text{for } x \geq 2, \end{cases}$$

and K is a positive constant to be fixed large later. Hence, it is reasonable to consider separately the solution in the blowup region $\{|y| \leq 2K\sqrt{s}\}$ and in the regular region $\{|y| \geq K\sqrt{s}\}$. More precisely, let us define the following notation for all functions q in $L^\infty(\mathbb{R}^N)$:

$$q = q_b + q_e \text{ with } q_b = \chi q \text{ and } q_e = (1-\chi)q, \tag{2.78}$$

Note in particular that $\operatorname{supp}(q_b) \subset \mathbb{B}(0, 2K\sqrt{s})$ and $\operatorname{supp}(q_e) \subset \mathbb{R}^N \setminus \mathbb{B}(0, K\sqrt{s})$.

In addition to that, we also expand q_b in L^2_ρ, according to the spectrum of $\mathcal{L}$ (see Section 2.2.1 above):

$$q_b(y) = q_0 + q_1 \cdot y + \frac{1}{2} y^{\mathfrak{T}} \cdot q_2 \cdot y - \operatorname{Tr}(q_2) + q_-(y), \tag{2.79}$$

where

$$\begin{aligned} q_0 &= \int_{\mathbb{R}^N} q_b \rho(y) dy, \\ q_1 &= \frac{1}{2} \int_{\mathbb{R}^N} q_b y \rho(y) dy, \\ q_2 &= \left(\int_{\mathbb{R}^N} q_b \left(\frac{1}{4} y_j y_k - \frac{1}{2} \delta_{j,k} \right) \rho(y) dy \right)_{1 \leq j,k \leq N}, \end{aligned}$$

and $\operatorname{Tr}(q_2)$ is the trace of the matrix q_2. The reader should keep in mind that q_0, q_1, q_2 are just coordinates of q_b, not for q. Note that q_m is the projection of q_b as the eigenspace of $\mathcal{L}$ corresponding to the eigenvalue $\lambda = 1 - \frac{m}{2}$. Accordingly, q_- is the projection of q_b on the negative part of the spectrum of $\mathcal{L}$. As a consequence of (2.78) and (2.79), we see that every $q \in L^\infty(\mathbb{R}^N)$ can be decomposed into 5 components as follows:

$$q = q_b + q_e = q_0 + q_1 \cdot y + \frac{1}{2} y^{\mathfrak{T}} \cdot q_2 \cdot y - \operatorname{Tr}(q_2) + q_- + q_e. \tag{2.80}$$

2.3.2 The shrinking set

In this part, we will construct a shrinking set, such that the control of $(q_1, q_2) \to 0$, will be a consequence of the control of (q_1, q_2) in this set, where (q_1, q_2) is the solution of (2.67). The following is our definition:

Definition 2.1 (Shrinking set). *For all $A \geq 1, p_1 \in (0,1)$ and $s \geq 1$, we introduce $V_{p_1,A,}(s)$, denoted for simplicity by $V_A(s)$, as the set of all $(q_1, q_2) \in (L^\infty(\mathbb{R}^N))^2$ satisfying the following* *conditions:*

$$\begin{array}{rclcrcl}
|q_{1,0}| &\leq& \dfrac{A}{s^2} & \text{and} & |q_{2,0}| &\leq& \dfrac{A^2}{s^{p_1+2}},\\
|q_{1,j}| &\leq& \dfrac{A}{s^2} & \text{and} & |q_{2,j}| &\leq& \dfrac{A^2}{s^{p_1+2}}, \forall 1 \leq j \leq N,\\
|q_{1,j,k}| &\leq& \dfrac{A^2 \ln s}{s^2} & \text{and} & |q_{2,j,k}| &\leq& \dfrac{A^5 \ln s}{s^{p_1+2}}, \forall 1 \leq j,k \leq N,\\
\left\| \dfrac{q_{1,-}}{1+|y|^3} \right\|_{L^\infty} &\leq& \dfrac{A}{s^2} & \text{and} & \left\| \dfrac{q_{2,-}}{1+|y|^3} \right\|_{L^\infty} &\leq& \dfrac{A^2}{s^{\frac{p_1+5}{2}}},\\
\|q_{1,e}\|_{L^\infty} &\leq& \dfrac{A^2}{\sqrt{s}} & \text{and} & \|q_{2,e}\|_{L^\infty} &\leq& \dfrac{A^3}{s^{\frac{p_1+2}{2}}},
\end{array}$$

where the above components are of $q_{1,b}$ and $q_{2,b}$, respectively, decomposed as in (2.80).

In the following Lemma, we show that belonging to $V_A(s)$ implies the convergence to 0. In fact, we have a more precise statement in the following:

Lemme 2.8. *For all $A \geq 1, s \geq 1$, if we have $(q_1, q_2) \in V_A(s)$, then the following estimates hold:*

(*i*) *Estimates in $L^\infty(\mathbb{R}^N)$: $\|q_1\|_{L^\infty(\mathbb{R}^N)} \leq \frac{CA^2}{\sqrt{s}}$ and $\|q_2\|_{L^\infty(\mathbb{R}^N)} \leq \frac{CA^3}{s^{\frac{p_1+2}{2}}}$.*

(*ii*) *For all $y \in \mathbb{R}^N$, we have*

$$|q_{1,b}(y)| \leq \frac{CA^2 \ln s}{s^2}(1+|y|^3), \quad |q_{1,e}(y)| \leq \frac{CA^2}{s^2}(1+|y|^3) \text{ and } |q_1| \leq \frac{CA^2 \ln s}{s^2}(1+|y|^3),$$

and

$$|q_{2,b}(y)| \leq \frac{CA^3}{s^{\frac{p_1+5}{2}}}(1+|y|^3), \quad |q_{2,e}(y)| \leq \frac{CA^3}{s^{\frac{p_1+5}{2}}}(1+|y|^3) \text{ and } |q_2| \leq \frac{CA^3}{s^{\frac{p_1+5}{2}}}(1+|y|^3).$$

where C will henceforth be an universal constant in our proof which depends only on K, N and p_1.

Proof. We only prove the estimates of q_2. Since, the other ones for q_1 will similarly follow and have already been proved in previous papers (see for intance Proposition 4.7 in [24]).

Let us consider $A \geq 1, s \geq 1$ and $(q_1, q_2) \in V_A(s)$ and $y \in \mathbb{R}^N$. We also recall from (2.80) that

$$q_2 = q_{2,b} + q_{2,e},$$

where $\text{supp}(q_{2,b}) \subset \mathbb{B}(0, 2K\sqrt{s})$ and $\text{supp}(q_{2,e}) \subset \mathbb{R}^N \setminus \mathbb{B}(0, K\sqrt{s})$.

(*i*) From (2.79), we have

$$q_b = q_{2,0} + q_{2,1} \cdot y + \frac{1}{2} y^{\mathcal{T}} \cdot q_{2,2} \cdot y - \text{Tr}(q_{2,2}) + q_{2,-}.$$

Therefore,

$$|q_{2,b}(y)| \leq |q_{2,0}| + |q_{2,1}||y| + C \max_{1 \leq j,k \leq N} |q_{2,j,k}|(1+|y|^2) + \left\| \frac{q_{2,-}}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} (1+|y|^3) \quad (2.81)$$

Then, recalling that $\text{supp}(q_{2,b}) \subset \mathbb{B}(0, 2K\sqrt{s})$ and using Definition 2.1, we see that

$$|q_{2,b}(y)| \leq \frac{CA^3}{s^{\frac{p_1+2}{2}}}.$$

On the other hand, we also have

$$|q_{2,e}| \leq \frac{A^3}{s^{\frac{p_1+2}{2}}}.$$

So, we end-up with the following

$$\|q_2\|_{L^\infty(\mathbb{R}^N)} \leq \|q_{2,b}\|_{L^\infty(\mathbb{R}^N)} + \|q_{2,e}\|_{L^\infty(\mathbb{R}^N)} \leq \frac{CA^3}{s^{\frac{p_1+2}{2}}}.$$

Thus, this yields the conclusion.

(ii) Using (2.81) and Definition 2.1, we derive that

$$|q_{2,b}(y)| \leq \frac{CA^3}{s^{\frac{p_1+5}{2}}}(1 + |y|^3). \tag{2.82}$$

We claim that $q_{2,e}$ satisfies a similar estimate:

$$|q_{2,e}(y)| \leq \frac{CA^3}{s^{\frac{p_1+5}{2}}}(1 + |y|^3). \tag{2.83}$$

Indeed, since $\text{supp}(q_{2,e}) \subset \mathbb{R}^N \setminus \mathbb{B}(0, K\sqrt{s})$, we may assume that

$$\frac{|y|}{K\sqrt{s}} \geq 1.$$

Hence, from Definition 2.1, we write

$$|q_{2,e}(y)| \leq \frac{A^3}{s^{\frac{p_1+2}{2}}}.1 \leq \frac{A^3}{s^{\frac{p_1+2}{2}}}\frac{|y|^3}{K^3 s^{\frac{3}{2}}} \leq \frac{CA^3}{s^{\frac{p_1+5}{2}}}(1 + |y|^3),$$

and (2.83) follows. Using (2.82) and (2.83), we see that

$$|q_2| \leq |q_{2,b}| + |q_{2,e}| \leq \frac{CA^3}{s^{\frac{p_1+5}{2}}}(1 + |y|^3).$$

□

2.3.3 Initial data

In this paragraph, we suggest a class of initial data, depending on some parameters to be fine-tuned in order to get a good solution for our problem. This is initial data:

Definition 2.2 (Initial data). *For each* $A \geq 1, s_0 \geq 1, d_1 = (d_{1,0}, d_{1,1}) \in \mathbb{R} \times \mathbb{R}^N, d_2 = (d_{2,0}, d_{2,1}, d_{2,2}) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2}$, *we introduce*

$$\phi_{1,A,d_1,s_0}(y) = \frac{A}{s_0^2}(d_{1,0} + d_{1,1} \cdot y)\,\chi(2y, s_0),$$

$$\phi_{2,A,d_2,s_0}(y) = \left(\frac{A^2}{s_0^{p_1+2}}(d_{2,0} + d_{2,1} \cdot y) + \frac{A^5 \ln s_0}{s_0^{p_1+2}}\left(\frac{1}{2}y^{\mathcal{T}} \cdot d_{2,2} \cdot y - \text{Tr}(d_{2,2})\right)\right)\chi(2y, s_0).$$

Remark: Note that $d_{1,0}$ and $d_{2,0}$ are scalars, $d_{1,1}$ and $d_{2,1}$ are vectors, $d_{2,2}$ is a square matrix of order N. For simplicity, we may drop down the parameters expect s_0 and write $\phi_1(y,s_0)$ and $\phi_2(y,s_0)$.

We next claim that we can find a domain for (d_1,d_2) so that initial data belongs to $V_A(s_0)$:

Lemme 2.9 (Control of initial data to be in $V_A(s_0)$)**.** *There exists $A_1 \geq 1$ such that for all $A \geq A_1$, there exists $s_1(A) \geq 1$ such that for all $s_0 \geq s_1(A)$, if $(q_1,q_2)(s_0) = (\phi_1,\phi_2)(s_0)$ where $(\phi_1,\phi_2)(s_0)$ are defined in Definition 2.2, then, the following properties hold:*

i) There exists a set $\mathcal{D}_{A,s_0} \subset [-2,2]^{N^2+2N+2}$ such that the mapping

$$\begin{aligned}\Psi_1 : \mathbb{R}^{N^2+2N+2} &\rightarrow \mathbb{R}^{N^2+2N+2}\\ (d_1,d_2) &\mapsto (q_{1,0},(q_{1,j})_{1\leq j\leq N}, q_{2,0},(q_{2,j})_{1\leq j\leq N},(q_{2,j,k})_{1\leq j,k\leq N})(s_0)\end{aligned}$$

is linear, one to one from $\mathcal{D}_{A,s_0}$ to $\hat{V}_A(s_0)$, where

$$\hat{V}_A(s) = \left[-\frac{A}{s^2},\frac{A}{s^2}\right]^{1+N} \times \left[-\frac{A^2}{s^{p_1+2}},\frac{A^2}{s^{p_1+2}}\right]^{1+N} \times \left[-\frac{A^5\ln s}{s^{p_1+2}},\frac{A^5\ln s}{s^{p_1+2}}\right]^{N^2}. \tag{2.84}$$

Moreover,

$$\Psi_1(\partial\mathcal{D}_{A,s_0}) \subset \partial\hat{V}_A(s_0) \text{ and } deg\left(\Psi_1\left|_{\partial\mathcal{D}_{A,s_0}}\right.\right) \neq 0. \tag{2.85}$$

ii) In particular, we have $(q_1,q_2)(s_0) \in V_A(s_0)$, and

$$\begin{aligned}|q_{1,j,k}(s_0)| &\leq \frac{A^2\ln s_0}{2s_0^2}, \forall 1\leq j,k\leq N,\\ \left\|\frac{q_{1,-}(.,s_0)}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A}{2s_0^2} \quad &\text{and} \quad \left\|\frac{q_{2,-}(.,s_0)}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^2}{2s_0^{\frac{p_1+5}{2}}},\\ q_{1,e}(.,s_0) = 0 \quad &\text{and} \quad q_{2,e}(.,s_0) = 0.\end{aligned}$$

Proof. The proof is straightforword and a bit length. For that reason, the proof is omitted, and we friendly refer the reader to Proposition 4.5 in [24] for a quite similar case. □

Now, we give a key-proposition for our argument. More precisely, in the following proposition, we prove an existence of a solution of equation (2.67) trapped in the shrinking set:

Proposition 2.10 (Existence of a solution trapped in $V_A(s)$)**.** *There exists $A_2 \geq 1$ such that for all $A \geq A_2$ there exists $s_2(A) \geq 1$ such that for all $s_0 \geq s_2(A)$, there exists $(d_1,d_2) \in \mathbb{R}^{N^2+2N+2}$ such that the solution (q_1,q_2) of equation (2.67) with initial data at the time s_0, given by $(q_1,q_2)(s_0) = (\phi_1,\phi_2)(s_0)$, where $(\phi_1,\phi_2)(s_0)$ depends on (d_1,d_2) and is defined in Definition 2.2, we have then*

$$(q_1,q_2) \in V_A(s), \quad \forall s \in [s_0,+\infty).$$

The proof is divided into 2 steps:

- The first step: In this step, we reduce our problem to a finite dimensional one. In other words, we aim at proving that the control of $(q_1,q_2)(s)$ in the shrinking set $V_A(s)$ reduces to the control of the following components:

$$(q_{1,0},(q_{1,j})_{1\le j\le N},q_{2,0},(q_{2,j})_{1\le j\le N},(q_{2,j,k})_{1\le j,k\le N})(s)$$

  in $\hat{V}_A(s)$, defined as in (2.84).

- The second step: We get the conclusion of Proposition 2.10 by using a topological argument in finite dimension.

Proof. We here give proof of Proposition 2.10:

- *Step 1: Reduction to a finite dimensional problem:* Using *a priori estimates*, our problem will be reduced to the control of a finite number of components.

Proposition 2.11 (Reduction to a finite dimensional problem)**.** *There exists $A_3\ge 1$ such that for all $A\ge A_3$, there exists $s_3(A)\ge 1$ such that for all $s_0\ge s_3(A)$, the following holds: If the two following are satisfied:*

(a) If $(q_1,q_2)(s)$ a solution of equation (2.67) with initial data $(q_1,q_2)(s_0)=(\phi_1,\phi_2)(s_0)$, defined as in Definition 2.2 for some $(d_1,d_2)\in\mathcal{D}_{A,s_0}$, introduced in Lemma 2.9

(b) If we furthemore assume that $(q_1,q_2)(s)\in V_A(s)$ for all $s\in[s_0,s_1]$ for some $s_1\ge s_0$ and $(q_1,q_2)(s_1)\in\partial V_A(s_1)$.

Then, we have the following conclusions:

(i) (Reduction to finite dimensions): We have

$$(q_{1,0},(q_{1,j})_{1\le j\le N},q_{2,0},(q_{2,j})_{1\le j\le N},(q_{2,j,k})_{1\le j,k\le N})(s_1)\in\partial\hat{V}_A(s_1)$$

(ii) (Transverse outgoing crossing): There exists $\delta_0>0$ such that

$$\forall\delta\in(0,\delta_0),(q_{1,0},(q_{1,j})_{1\le j\le N},q_{2,0},(q_{2,j})_{1\le j\le N},(q_{2,j,k})_{1\le j,k\le N})(s_1+\delta)\notin\hat{V}_A(s_1+\delta),\tag{2.86}$$

which implies that $(q_1,q_2)(s_1+\delta)\notin V_A(s_1+\delta)$ for all $\delta\in(0,\delta_0)$.

This proposition makes the heart of the paper and needs many steps to be proved. For that reason, we dedicate a whole section to its proof (Section 2.4 below). Let us admit it here, and get to the conclusion of Proposition 2.10 in the second step.

- *Step 2: Conclusion of Proposition 2.10 by a topological argument.* In this step, we finish the proof of Proposition 2.10. In fact, we aim at proving the existence of parameters $(d_1,d_2)\in\mathcal{D}_{A,s_0}$ such that the solution $(q_1,q_2)(s)$ of equation (2.67) with initial data $(q_1,q_2)(s_0)=(\phi_1,\phi_2)(s_0)$, exists globally for all $s\in[s_0,+\infty)$ and satisfies

$$(q_1,q_2)(s)\in V_A(s),$$

where initial data $(\phi_1,\phi_2)(s_0)$ is introduced in Definition 2.2.

In fact, our argument is analogous to the argument of Merle and Zaag [15]. For that reason, we only give a brief proof. Let us fix K,A and s_0 such that Lemma 2.9 and

Proposition 2.11 hold. We first consider $(q_1,q_2)_{d_1,d_2}(s), s \geq s_0$ a solution of equation (2.67) with initial data $(q_1,q_2)(s_0)$ which depends on (d_1,d_2) as follows

$$(q_1,q_2)_{d_1,d_2}(s_0) = (\phi_1,\phi_2)(s_0).$$

From Lemma 2.9 and by construction of $\mathcal{D}_{A,s_0}$, we know that

$$(q_1,q_2)(s_0) \in V_A(s_0). \tag{2.87}$$

By contradiction, we assume that for all $(d_1,d_2) \in \mathcal{D}_{A,s_0}$, there exists $s_1 \in [s_0,+\infty)$ such that

$$(q_1,q_2)_{d_1,d_2}(s_1) \notin V_A(s_1).$$

Then, for each $(d_1,d_2) \in \mathcal{D}_{A,s_0}$, we can define

$$s_*(d_1,d_2) = \inf\{s_1 \geq s_0 \text{ such that } (q_1,q_2)_{d_1,d_2}(s_1) \notin V_A(s_1)\}.$$

From the fact that $(q_1,q_2)(s_1) \notin V_A(s_1)$, we deduce that $s_*(d_1,d_2) < +\infty$ for all $(d_1,d_2) \in \mathcal{D}_{A,s_0}$. Besides that, using (2.87), and the minimality of $s_*(d_1,d_2)$, the continuity of (q_1,q_2) in s and the closeness of $V_A(s)$ we derive that $(q_1,q_2)(s_*(d_1,d_2)) \in \partial V_A(s_*(d_1,d_2))$ and for all $s \in [s_0, s_*(d_1,d_2)]$,

$$(q_1,q_2)(s) \in V_A(s).$$

Therefore, from item (i) of Proposition 2.11 we see that

$$(q_{1,0},(q_{1,j})_{1\leq j\leq N}, q_{2,0},(q_{2,j})_{1\leq j\leq N},(q_{2,j,k})_{1\leq j,k\leq N})(s_*(d_1,d_2)) \in \hat{V}_A(s_*(d_1,d_2)).$$

This means that following mapping Γ is well-defined:

$$\begin{array}{rcl}\Gamma : \mathcal{D}_{A,s_0} & \to & \partial\left([-1,1]^{N^2+2N+2}\right) \\ (d_1,d_1) & \mapsto & \Gamma(d_1,d_2),\end{array}$$

where

$$\left(\frac{s_*^2}{A}(q_{1,0},(q_{1,j})_{1\leq j\leq N})(s_*), \frac{s_*^{p_1+2}}{A^2}(q_{2,0},(q_{2,j})_{1\leq j\leq N})(s_*), \frac{s_*^{p_1+2}}{A^5\ln s_*}(q_{2,j,k})_{1\leq j,k\leq N}(s_*)\right),$$

where $s_* = s_*(d_1,d_2)$. Moreover, it satisfies the two following properties:

(i) Γ is continuous from $\mathcal{D}_{A,s_0}$ to $\partial\left([-1,1]^{N^2+2N+2}\right)$. This is a consequence of item (ii) in Proposition (2.11).

(ii) The degree of the restriction $\Gamma|_{\partial\mathcal{D}_{A,s_0}}$ is non zero. Indeed, again by item (ii) in Proposition 2.11, we have

$$s^*(d_1,d_2) = s_0,$$

in this case. Applying (2.85), we get the conclusion.

In fact, such a mapping Γ can not exist by Index theorem, this is a contradiction. Thus, Proposition 2.10 follows, assuming that Proposition 2.11 (see Section 2.4 for the proof of latter) $\square$

2.3.4 The proof of Theorem 2.1

In this section, we aim at giving the proof of Theorem 2.1.

Proof. *Proof of Theorem 2.1 assuming that Proposition 2.11*

+ *The proof of item (i) of Theorem 2.1:* Using Proposition 2.10, there exist initial data $(q_1, q_2)_{d_1,d_2}(s_0) = (\phi_1, \phi_2)(s_0)$ such that the solution of equation (2.67), exists globally on $[s_0, +\infty)$ and satisfies:

$$(q_1, q_2)(s) \in V_A(s), \forall s \in [s_0, +\infty).$$

Thanks to similarity variables (2.25), (2.66) and item (i) in Lemma 2.8, we conclude that there exist initial data u^0 of the form given in Remark 2.2 with (d_1, d_2) given in Proposition 2.10 such that the solution $u(t)$ of equation (2.1) exists on $[0, T)$, where $T = e^{-s_0}$ and satisfies (2.15) and (2.16). Using these two estimates, we see that

$$u(0, t) \sim \kappa(T - t)^{-\frac{1}{p-1}} \text{ as } t \to T,$$

which means that u blows up at time T and the origin is a blowup point. It remains to prove that for all $x \neq 0$, x is not a blowup point of u. The following Lemma allows us to conclude.

Lemme 2.12 (No blow up under some threshold)**.** *For all $C_0 > 0, 0 \leq T_1 < T$ and $\sigma > 0$ small enough, there exists $\epsilon_0(C_0, T, \sigma) > 0$ such that the following holds: If $u(\xi, \tau)$ satisfies the following estimates for all $|\xi| \leq \sigma, \tau \in [T_1, T)$:*

$$|\partial_\tau u - \Delta u| \leq C_0 |u|^p,$$

and

$$|u(\xi, \tau)| \leq \epsilon_0 (1 - \tau)^{-\frac{1}{p-1}},$$

then, u does not blow up at $\xi = 0, \tau = T$.

Proof. The proof of this Lemma is processed similarly to Theorem 2.1 in [6]. Although the proof of [6] was given in the real case, it extends naturally to the complex valued case. □

We next use Lemma 2.12 to conclude that u does not blow up at $x_0 \neq 0$. Indeed, if $x_0 \neq 0$ we use (2.15) to deduce the following:

$$\sup_{|x - x_0| \leq \frac{|x_0|}{2}} (T-t)^{\frac{1}{p-1}} |u(x, t)| \leq \left| f_0 \left(\frac{\frac{|x_0|}{2}}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right| + \frac{C}{\sqrt{|\ln(T-t)|}} \to 0, \text{ as } t \to T. \tag{2.88}$$

Applying Lemma 2.12 to $u(x - x_0, t)$, with some σ small enough such that $\sigma \leq \frac{|x_0|}{2}$, and T_1 close enough to T, we see that $u(x - x_0, t)$ does not blow up at time T and $x = 0$. Hence x_0 is not a blow-up point of u. This concludes the proof of item (i) in Theorem 2.1.

+ *The proof of item (ii) of Theorem 2.1:* Here, we use the argument of Merle in [13] to deduce the existence of $u^* = u_1^* + i u_2^*$ such that $u(t) \to u^*$ as $t \to T$ uniformly on compact sets of $\mathbb{R}^N \backslash \{0\}$. In addition to that, we use the techniques in Zaag [29], Masmoudi and Zaag [16], Tayachi and Zaag [24] for the proofs of (2.18) and (2.19).

Indeed, for all $x_0 \in \mathbb{R}^N$ and $x_0 \neq 0$, we deduce from (2.15), (2.16) that not only (2.88) holds but also the following is satisfied:

$$\begin{aligned} \sup_{|x-x_0|\le \frac{|x_0|}{2}} (T-t)^{\frac{1}{p-1}}|\ln(T-t)||u_2(x,t)| &\le \left| \frac{3|x_0|^2}{2(T-t)|\ln(T-t)|} f_0^p \left(\frac{\frac{|x_0|}{2}}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right| \\ &+ \frac{C}{|\ln(T-t)|^{\frac{p_1}{2}}} \to 0, \text{ as } t \to T. \end{aligned} \quad (2.89)$$

We now consider x_0 such that $|x_0|$ is small enough, and K_0 to be fixed later. We define $t_0(x_0)$ by

$$|x_0| = K_0\sqrt{T - t_0(x_0)|\ln(T-t_0(x_0))|}. \quad (2.90)$$

Note that $t_0(x_0)$ is unique when $|x_0|$ is small enough and $t_0(x_0) \to T$ as $x_0 \to 0$. We introduce the rescaled functions $U(x_0,\xi,\tau)$ and $V_2(x_0,\xi,\tau)$ as follows:

$$U(x_0,\xi,\tau) = (T-t_0(x_0))^{\frac{1}{p-1}}\, u(x,t). \quad (2.91)$$

and

$$V_2(x_0,\xi,\tau) = |\ln(T-t_0(x_0))| U_2(x_0,\xi,\tau), \quad (2.92)$$

where $U_2(x_0,\xi,\tau)$ is defined by

$$U(x_0,\xi,\tau) = U_1(x_0,\xi,\tau) + iU_2(x_0,\xi,\tau),$$

and

$$(x,t) = \left(x_0 + \xi\sqrt{T-t_0(x_0)}, t_0(x_0) + \tau(T-t_0(x_0))\right), \text{ and } (\xi,\tau) \in \mathbb{R}^N \times \left[-\frac{t_0(x_0)}{T-t_0(x_0)}, 1\right). \quad (2.93)$$

We can see that with these notations, we derive from item (i) in Theorem 2.1 the following estimates for initial data at $\tau = 0$ of U and V_2

$$\sup_{|\xi|\le|\ln(T-t_0(x_0))|^{\frac{1}{4}}} |U(x_0,\xi,0) - f_0(K_0)| \le \frac{C}{1 + (|\ln(T-t_0(x_0))|^{\frac{1}{4}})} \to 0 \text{ as } x_0 \to 0, \quad (2.94)$$

$$\sup_{|\xi|\le|\ln(T-t_0(x_0))|^{\frac{1}{4}}} |V_2(x_0,\xi,0) - g_0(K_0)| \le \frac{C}{1 + (|\ln(T-t_0(x_0))|^{\gamma_1})} \to 0 \text{ as } x_0 \to 0. \quad (2.95)$$

where f_0 and g_0 are defined as in (2.6) and (2.17) respectively and $\gamma_1 = \min\left(\frac{1}{4}, \frac{p_1}{2}\right)$. Moreover, using equations (2.23), we derive the following equations for U, V_2: for all $\xi \in \mathbb{R}^N, \tau \in [0,1)$

$$\partial_\tau U = \Delta_\xi U + U^p, \quad (2.96)$$

$$\partial_\tau V_2 = \Delta_\xi V_2 + V_2 G_2(U_1,U_2), \quad (2.97)$$

where G is defined by

$$G(U_1,U_2)U_2 = F_2(U_1,U_2), \quad (2.98)$$

and F_2 is defined in (2.24). We note that G_2, F_2 are polynomials of U_1, U_2.

Besides that, from (2.89) and (2.96), we can apply Lemma 2.12 to U when $|\xi| \leq |\ln(T-t_0(x_0))|^{\frac{1}{4}}$ to get the following

$$\sup_{|\xi|\leq \frac{1}{2}|\ln(T-t_0(x_0))|^{\frac{1}{4}}, \tau\in[0,1)} |U(x_0,\xi,\tau)| \leq C. \tag{2.99}$$

We aim at now proving the following

$$\sup_{|\xi|\leq \frac{1}{16}|\ln(T-t_0(x_0))|^{\frac{1}{4}}, \tau\in[0,1)} |V_2(x_0,\xi,\tau)| \leq C. \tag{2.100}$$

+ *The proof for* (2.100)*:* We first use (2.99) to derive the following rough estimate:

$$\sup_{|\xi|\leq \frac{1}{2}|\ln(T-t_0(x_0))|^{\frac{1}{4}}, \tau\in[0,1)} |V_2(x_0,\xi,\tau)| \leq C|\ln(T-t_0(x_0))|. \tag{2.101}$$

We first introduce $\psi \in C_0^\infty(\mathbb{R}^N), 0 \leq \psi \leq 1, supp(\psi) \subset B(0,1), \psi = 1$ on $B(0,\frac{1}{2})$. We also define

$$\psi_1(\xi) = \psi\left(\frac{2\xi}{|\ln(T-t_0(x_0))|^{\frac{1}{4}}}\right) \text{ and } V_{2,1}(x_0,\xi,\tau) = \psi_1(\xi)V_2(x_0,\xi,\tau). \tag{2.102}$$

Then, we deduce from (2.97) an equation satisfied by $V_{2,1}$

$$\partial_\tau V_{2,1} = \Delta_\xi V_{2,1} - 2\,\text{div}(V_2\nabla\psi_1) + V_2\Delta\psi_1 + V_{2,1}G_1(U_1,U_2). \tag{2.103}$$

Hence, we can write $V_{2,1}$ with a integral equation as follows

$$V_{2,1}(\tau) = e^{\Delta\tau}(V_{2,1}(0)) + \int_0^\tau e^{(\tau-\tau')\Delta}\left(-2\,\text{div}\,(V_2\nabla\psi_1) + V_2\Delta\psi_1 + V_{2,1}G(U_1,U_2)(\tau')\right)d\tau'. \tag{2.104}$$

Besides that, using (2.99) and (2.101) and the fact that

$$|\nabla\psi_1| \leq \frac{C}{|\ln(T-t_0(x_0))|^{\frac{1}{4}}}, |\Delta\psi_1| \leq \frac{C}{|\ln(T-t_0(x_0))|^{\frac{1}{2}}},$$

we deduce that

$$\begin{aligned}
\left|\int_0^\tau e^{(\tau-\tau')\Delta}\left(-2\,\text{div}\,(V_2\nabla\psi_1)\right)d\tau'\right| &\leq C\int_0^\tau \frac{\|V_2\nabla\psi_1\|_{L^\infty}(\tau')}{\sqrt{\tau-\tau'}}d\tau' \leq C|\ln(T-t_0(x_0))|^{\frac{3}{4}},\\
\left|\int_0^\tau e^{(\tau-\tau')\Delta}\left(V_2(\tau')\Delta\psi_1\right)d\tau'\right| &\leq C\int_0^\tau \|V_2\Delta\psi_1\|_\infty(\tau')d\tau' \leq C|\ln(T-t_0(x_0))|^{\frac{1}{2}},\\
\left|\int_0^\tau e^{(\tau-\tau')\Delta}\left(V_2\psi_1 G(U_1,U_2)(\tau')\right)d\tau'\right| &\leq C\int_0^\tau \|V_{2,1}G_2(U_1,U_2)\|_{L^\infty}(\tau')d\tau'.
\end{aligned}$$

Note that $G_2(U_1,U_2)$ in the last line is bounded on $|\xi| \leq |\ln(T-t_0)|^{\frac{1}{4}}, \tau \in [0,1)$ because it is a polynomial in U_1, U_2 and (2.99) holds, then, we derive

$$\|V_{2,1}G_2(U_1,U_2)\|_{L^\infty}(\tau') \leq C\|V_{2,1}\|_{L^\infty}(\tau').$$

Hence, from (2.104) and the above estimates, we derive

$$\|V_{2,1}(\tau)\|_{L^\infty} \leq C|\ln(T-t_0(x_0))|^{\frac{3}{4}} + C\int_0^\tau \|V_{2,1}(\tau')\|_{L^\infty} d\tau'.$$

Thanks to Gronwall Lemma, we deduce that

$$\|V_{2,1}(\tau)\|_{L^\infty} \leq C|\ln(T-t_0(x_0))|^{\frac{3}{4}}, \forall \tau \in [0,1),$$

which yields

$$\sup_{|\xi|\leq \frac{1}{4}|\ln(T-t_0(x_0))|^{\frac{1}{4}}, \tau\in[0,1)} |V_2(x_0,\xi,\tau)| \leq C|\ln(T-t_0(x_0))|^{\frac{3}{4}}. \tag{2.105}$$

We apply iteratively for

$$V_{2,2}(x_0,\xi,\tau) = \psi_2(\xi)V_2(x_0,\xi,\tau) \text{ where } \psi_2(\xi) = \psi\left(\frac{4\xi}{|\ln(T-t_0(x_0))|^{\frac{1}{4}}}\right).$$

Similarly, we deduce that

$$\sup_{|\xi|\leq \frac{1}{8}|\ln(T-t_0(x_0))|^{\frac{1}{4}}, \tau\in[0,1)} |V_2(x_0,\xi,\tau)| \leq C|\ln(T-t_0(x_0))|^{\frac{1}{2}}.$$

We apply this process a finite number of steps to obtain (2.100). We now come back to our problem, and aim at proving that:

$$\sup_{|\xi|\leq \frac{1}{16}|\ln(T-t_0(x_0))|^{\frac{1}{4}}, \tau\in[0,1)} \left|U(x_0,\xi,\tau) - \hat{U}_{K_0}(\tau)\right| \leq \frac{C}{1+|\ln(T-t_0(x_0))|^{\gamma_2}}, \tag{2.106}$$

$$\sup_{|\xi|\leq \frac{1}{32}|\ln(T-t_0(x_0))|^{\frac{1}{4}}, \tau\in[0,1)} \left|V_2(x_0,\xi,\tau) - \hat{V}_{2,K_0}(\tau)\right| \leq \frac{C}{1+|\ln(T-t_0(x_0))|^{\gamma_3}}, \tag{2.107}$$

where γ_2, γ_3 are positive small enough and $(\hat{U}_{K_0}, \hat{V}_{2,K_0})(\tau)$ is the solution of the following system:

$$\partial_\tau \hat{U}_{K_0} = \hat{U}_{K_0}^p, \tag{2.108}$$

$$\partial_\tau \hat{V}_{2,K_0} = p\hat{U}_{K_0}^{p-1}\hat{V}_{2,K_0}. \tag{2.109}$$

with initial data at $\tau = 0$

$$\begin{aligned} \hat{U}_{K_0}(0) &= f_0(K_0), \\ \hat{V}_{2,K_0}(0) &= g_0(K_0). \end{aligned}$$

given by

$$\hat{U}_{K_0}(\tau) = \left((p-1)(1-\tau) + \frac{(p-1)^2K_0^2}{4p}\right)^{-\frac{1}{p-1}}, \tag{2.110}$$

$$\hat{V}_{2,K_0}(\tau) = K_0^2\left((p-1)(1-\tau) + \frac{(p-1)^2K_0^2}{4p}\right)^{-\frac{p}{p-1}}. \tag{2.111}$$

for all $\tau \in [0,1)$. The proof of (2.106) is cited to Section 5 of Tayachi and Zaag [24] and the proof of (2.107) is similar. For the reader's convenience, we give it here. Let us consider

$$\mathcal{V}_2 = V_2 - \hat{V}_{2,K_0}(\tau). \tag{2.112}$$

Then, $\mathcal{V}_2$ satisfies

$$\sup_{|\xi| \le \frac{1}{16}|\ln(T-t_0(x_0))|^{\frac{1}{4}}, \tau \in [0,1)} |\mathcal{V}_2| \le C. \tag{2.113}$$

We use (2.97) to derive an equation on $\mathcal{V}_2$ as follows:

$$\partial_\tau \mathcal{V}_2 = \Delta \mathcal{V}_2 + p\hat{U}_{K_0}^{p-1}\mathcal{V}_2 + p(U_1^{p-1} - \hat{U}_{K_0}^{p-1})V_2 + \mathcal{G}_2(x_0,\xi,\tau), \tag{2.114}$$

where

$$\mathcal{G}_2(x_0,\xi,\tau) = V_2[G_2(U_1,U_2) - pU_1^{p-1}].$$

Note that, from definition of G_2 and (2.99) we deduce that

$$\sup_{|\xi| \le \frac{1}{2}|\ln(T-t_0(x_0))|^{\frac{1}{4}}, \tau \in [0,1)} |G_2(U_1,U_2) - pU_1^{p-1}| \le C|U_2|.$$

Hence, using (2.92) and (2.100) and we derive

$$\sup_{|\xi| \le \frac{1}{16}|\ln(T-t_0(0))|^{\frac{1}{4}}, \tau \in [0,1)} |\mathcal{G}_2(x_0,\xi,\tau)| \le \frac{C}{|\ln(T-t_0(x_0))|}. \tag{2.115}$$

We also define

$$\bar{\mathcal{V}}_2 = \psi_*(\xi)\mathcal{V}_2,$$

where

$$\psi_* = \psi\left(\frac{16\xi}{|\ln(T-t_0(x_0))|^{\frac{1}{4}}}\right),$$

and ψ is the cut-off function which has been introduced above. We also note that $\nabla\psi_*, \Delta\psi_*$ satisfy the following estimates

$$\|\nabla_\xi \psi_*\|_{L^\infty} \le \frac{C}{|\ln(T-t_0(x_0))|^{\frac{1}{4}}} \text{ and } \|\Delta_\xi \psi_*\|_{L^\infty} \le \frac{C}{|\ln(T-t_0(x_0))|^{\frac{1}{2}}}. \tag{2.116}$$

In particular, $\bar{\mathcal{V}}_2$ satisfies

$$\partial_\tau \bar{\mathcal{V}}_2 = \Delta\bar{\mathcal{V}}_2 + p\hat{U}_{K_0}^{p-1}(\tau)\bar{\mathcal{V}}_2 - 2\text{ div }(\mathcal{V}_2\nabla\psi_*) + \mathcal{V}_2\Delta\psi_* + p(U_1^{p-1} - \hat{U}_{K_0}^{p-1})\psi_*V_2 + \psi_*\mathcal{G}_2, \tag{2.117}$$

By Duhamel principal, we derive the following integral equation

$$\begin{aligned} \bar{\mathcal{V}}_2(\tau) &= e^{\tau\Delta}(\bar{\mathcal{V}}_2(\tau)) + \int_0^\tau e^{(\tau-\tau')\Delta}\Big\{p\hat{U}_{K_0}^{p-1}\bar{\mathcal{V}}_2 - 2\text{ div }(\mathcal{V}_2\nabla\psi_*) &(2.118)\\ &+ \mathcal{V}_2\Delta\psi_* + p(U_1^{p-1} - \hat{U}_{K_0}^{p-1})\psi_*V_2 + \psi_*\mathcal{G}_2\Big\}(\tau')d\tau'. &(2.119)\end{aligned}$$

Besides that, we use (2.106), (2.110), (2.113), (2.116), (2.115) to derive the following estimates: for all $\tau \in [0,1)$

$$\begin{array}{rcl} |\hat{U}_{K_0}(\tau)| & \leq & C, \\ \|\mathcal{V}_2 \nabla \psi_*\|_{L^\infty}(\tau) & \leq & \dfrac{C}{|\ln(T-t_0(x_0))|^{\frac{1}{4}}}, \\ \|\mathcal{V}_2 \Delta \psi_*\|_{L^\infty}(\tau) & \leq & \dfrac{C}{|\ln(T-t_0(x_0))|^{\frac{1}{2}}}, \\ \left\| \left(U_1^{p-1} - \hat{U}_{K_0}^{p-1} \right) \psi_* \right\|_{L^\infty}(\tau) & \leq & \dfrac{C}{|\ln(T-t_0(x_0))|^{\gamma_2}}, \\ \|\mathcal{G}_2 \psi_*\|_{L^\infty(\mathbb{R}^N)} & \leq & \dfrac{C}{|\ln(T-t_0(x_0))|}. \end{array}$$

where γ_2 given in (2.106). Hence, we derive from the above estimates that: for all $\tau \in [0,1)$

$$\begin{array}{rcl} |e^{(\tau-\tau')\Delta} p \hat{U}_{K_0}^{p-1} \bar{\mathcal{V}}_2(\tau')| & \leq & C \|\bar{\mathcal{V}}_2(\tau')\|, \\ |e^{(\tau-\tau')\Delta} (\operatorname{div}(\mathcal{V}_2 \nabla \psi_*))| & \leq & C \dfrac{1}{\sqrt{\tau-\tau'}} \dfrac{1}{|\ln(T-t_0(x_0))|^{\frac{1}{4}}}, \\ |e^{(\tau-\tau')\Delta} (\mathcal{V}_2 \Delta \psi_*)| & \leq & \dfrac{C}{|\ln(T-t_0(x_0))|^{\frac{1}{2}}}, \\ |e^{(\tau-\tau')\Delta} (p(U_1^{p-1} - \hat{U}_{K_0}^{p-1}) \psi_* V_2)(\tau')| & \leq & \dfrac{C}{|\ln(T-t_0(x_0))|^{\gamma_2}}, \\ |e^{(\tau-\tau')\Delta} (\psi_* \mathcal{G}_2)(\tau')| & \leq & \dfrac{C}{|\ln(T-t_0(x_0))|}. \end{array}$$

Plugging into (2.118), we obtain

$$\|\bar{\mathcal{V}}_2(\tau)\|_{L^\infty} \leq \frac{C}{|\ln(T-t_0(x_0))|^{\gamma_3}} + C \int_0^\tau \|\bar{\mathcal{V}}_2(\tau')\|_{L^\infty} d\tau',$$

where $\gamma_3 = \min(\frac{1}{4}, \gamma_2)$. Then, thanks to Gronwall inequality, we get

$$\|\bar{\mathcal{V}}_2\|_{L^\infty} \leq \frac{C}{|\ln(T-t_0(x_0))|^{\gamma_3}}.$$

Hence, (2.107) follows . Finally, we easily find the asymptotics of u^* and u_2^* as follows, thanks to the definition of U and V_2 and to estimates (2.106) and (2.107):

$$u^*(x_0) = \lim_{t \to T} u(x_0,t) = (T-t_0(x_0))^{-\frac{1}{p-1}} \lim_{\tau \to 1} U(x_0,0,\tau) \sim (T-t_0(x_0))^{-\frac{1}{p-1}} \left(\frac{(p-1)^2}{4p} K_0^2 \right)^{-\frac{1}{p-1}}, \tag{2.120}$$

and

$$\begin{array}{rcl} u_2^*(x_0) & = & \displaystyle\lim_{t \to T} u_2(x_0,t) = \frac{(T-t_0(x_0))^{-\frac{1}{p-1}}}{|\ln(T-t_0(x_0))|} \lim_{\tau \to 1} V_2(x_0,0,\tau) \\ & \sim & \dfrac{(T-t_0(x_0))^{-\frac{1}{p-1}}}{|\ln(T-t_0(x_0))|} \left(\dfrac{(p-1)^2}{4p} \right)^{-\frac{p}{p-1}} (K_0^2)^{-\frac{1}{p-1}}. \end{array} \tag{2.121}$$

Using the relation (2.90), we find that

$$T - t_0(x_0) \sim \frac{|x_0|^2}{2K_0^2|\ln|x_0||} \text{ and } \ln(T - t_0(x_0)) \sim 2\ln(|x_0|), \quad \text{as } x_0 \to 0. \tag{2.122}$$

Plugging (2.122) into (2.120) and (2.121), we get the conclusion of item (*ii*) of Theorem 2.1. This concludes the proof of Theorem 2.1 assuming that Proposition 2.11 holds. Naturally, we need to prove this proposition on order to finish the argument. This will be done in the next section. □

2.4 The proof of Proposition 2.11

This section is devoted to the proof of Proposition 2.11, which is the heart of our analysis. We proceed into two parts. In the first part, we derive *a priori estimates* on $q(s)$ in $V_A(s)$. In the second part, we show that the new bounds are better than those defined in $V_A(s)$, except for the first components $(q_{1,0}, (q_{1,j})_{1\leq j\leq N}, q_{2,0}, (q_{2,j})_{1\leq j\leq N}, (q_{2,j,k})_{1\leq j,k\leq N})(s)$. This means that the problem is reduced to the control of these components, which is the conclusion of item (*i*) of Proposition 2.11. Item (*ii*) of Proposition 2.11 is just a direct consequence of the dynamics of these modes. Let us start the first part.

2.4.1 A priori estimates on (q_1, q_2) in $V_A(s)$.

In this subsection, we aim at proving the following proposition:

Proposition 2.13. *There exists $A_4 \geq 1$, such that for all $A \geq A_4$ there exists $s_4(A) \geq 1$ such that for all $s_0 \geq s_4(A)$ the following holds: If we assume that for all $s \in [\sigma, s_1], (q_1, q_2)(s) \in V_A(s)$ for some $s_1 \geq s_0$, then, for all $s \in [s_0, s_1]$:*

(*i*) *(ODE satisfied by the positive modes) For all $j \in \{1, ...?N\}$, we have*

$$\left|q_{1,0}'(s) - q_{1,0}(s)\right| + \left|q_{1,j}'(s) - \frac{1}{2}q_{1,j}(s)\right| \leq \frac{C}{s^2}, \forall 1 \leq j \leq N, \tag{2.123}$$

$$\left|q_{2,0}'(s) - q_{2,0}(s)\right| + \left|q_{2,j}'(s) - \frac{1}{2}q_{2,j}(s)\right| \leq \frac{C}{s^{p_1+2}}, \forall 1 \leq j \leq N. \tag{2.124}$$

(*ii*) *(ODE satisfied by the null modes) For all $j, k \in \{1, ..., N\}$, we have*

$$\left|q_{1,j,k}'(s) + \frac{2}{s}q_{1,j,k}(s)\right| \leq \frac{CA}{s^3}, \tag{2.125}$$

$$\left|q_{2,j,k}'(s) + \frac{2}{s}q_{2,j,k}(s)\right| \leq \frac{CA^2\ln s}{s^{p_1+3}}. \tag{2.126}$$

(*iii*) *(Control the negative part)*

$$\left\|\frac{q_{1,-}(.,s)}{1+|y|^3}\right\|_{L^\infty} \leq Ce^{-\frac{s-\tau}{2}}\left\|\frac{q_{1,-}(.,\tau)}{1+|y|^3}\right\|_{L^\infty} + C\frac{e^{-(s-\tau)^2}}{s^{\frac{3}{2}}}\|q_{1,e}(.,\tau)\|_{L^\infty} + \frac{C(1+s-\tau)}{s^2}, \tag{2.127}$$

$$\left\|\frac{q_{2,-}(.,s)}{1+|y|^3}\right\|_{L^\infty} \leq Ce^{-\frac{s-\tau}{2}}\left\|\frac{q_{2,-}(.,\tau)}{1+|y|^3}\right\|_{L^\infty} + C\frac{e^{-(s-\tau)^2}}{s^{\frac{3}{2}}}\|q_{2,e}(.,\tau)\|_{L^\infty} + \frac{C(1+s-\tau)}{s^{\frac{p_1+5}{2}}}. \tag{2.128}$$

(v) *(Outer part)*

$$\|q_{1,e}(.,s)\|_{L^\infty} \leq Ce^{-\frac{(s-\tau)}{p}}\|q_{1,e}(.,\tau)\|_{L^\infty} + Ce^{s-\tau}s^{\frac{3}{2}}\left\|\frac{q_{1,-}(.,\tau)}{1+|y|^3}\right\|_{L^\infty} + \frac{C(1+s-\tau)e^{s-\tau}}{\sqrt{s}}, \tag{2.129}$$

$$\|q_{2,e}(.,s)\|_{L^\infty} \leq Ce^{-\frac{(s-\tau)}{p}}\|q_{2,e}(.,\tau)\|_{L^\infty} + Ce^{s-\tau}s^{\frac{3}{2}}\left\|\frac{q_{2,-}(.,\tau)}{1+|y|^3}\right\|_{L^\infty} + \frac{C(1+s-\tau)e^{s-\tau}}{s^{\frac{p_1+2}{2}}}. \tag{2.130}$$

Proof. The proof of this Proposition is given in two steps:

+ *Step 1:* We will give a proof to items (*i*) and (*ii*) by using the projection the equations which are satisfied by q_1 and q_2.

+ *Step 2:* We will control the other components by studying the dynamics of the linear operator $\mathcal{L} + V$.

a) **Step 1:** We observe that the techniques of the proofs for (2.123), (2.124), (2.125) and (2.126) are the same. So, we only deal with the proof of (2.125). For each $j,k \in \{1,...,N\}$ by using the equation in (2.67) and the definition of $q_{1,j,k}$ we deduce that

$$\left|q'_{1,i,j}(s) - \int\left[\mathcal{L}q_1 + Vq_1 + B_1(q_1,q_2) + R_1(y,s)\right]\chi(y,s)\left(\frac{y_iy_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho dy\right| \leq Ce^{-s}, \tag{2.131}$$

if K is large enough. In addition to that, using the fact $(q_1,q_2) \in V_A(s)$ and Lemma 2.8, Lemma 2.18, Lemma 2.19, Lemma 2.20 that

$$\begin{aligned}
\left|\int \mathcal{L}(q)\chi\left(\frac{y_iy_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho dy\right| &\leq \frac{C}{s^3},\\
\left|\int Vq_1\chi\left(\frac{y_iy_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho dy + \frac{2}{s}q_{1,i,j}(s)\right| &\leq \frac{CA}{s^3},\\
\left|\int B_1(q_1,q_2)\chi\left(\frac{y_iy_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho dy\right| &\leq \frac{C}{s^3},\\
\left|\int R_1(y,s)\chi\left(\frac{y_iy_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho dy\right| &\leq \frac{C}{s^3},
\end{aligned}$$

provided that $s \geq s_4(A)$. Then, (2.125) is derived by adding all the above estimates.

Step 2: In this part, we will concentrate on the proofs of items (*iii*) and (*iv*). We now rewrite (2.67) in its integral form: for each $s \geq \tau$

$$\left\{\begin{array}{lll}
q_1(s) &=& \mathcal{K}(s,\tau)q_1(\tau) + \int_\tau^s \mathcal{K}(s,\sigma)\left[(V_{1,1}q_1)(\sigma) + (V_{1,2}q_2)(\sigma) + B_1(q_1,q_2)(\sigma) + R_1(\sigma)\right]d\sigma\\
&=& \sum_{i=1}^5 \vartheta_{1,i}(s,\tau),\\
q_2(s) &=& \mathcal{K}(s,\tau)q_2(\tau) + \int_\tau^s \mathcal{K}(s,\sigma)\left[(V_{2,1}q_1)(\sigma) + (V_{2,2}q_2)(\sigma) + B_2(q_1,q_2)(\sigma) + R_2(\sigma)\right]d\sigma\\
&=& \sum_{i=1}^5 \vartheta_{2,i}(s,\tau),
\end{array}\right. \tag{2.132}$$

where $\{\mathcal{K}(s,\tau)\}_{s\geq\tau}$ is the fundamental solution associated to $\mathcal{L} + V$ and defined by

$$\left\{\begin{array}{l}
\partial_s\mathcal{K}(s,\tau) = (\mathcal{L}+V)\mathcal{K}(s,\tau), \quad \forall s > \tau,\\
\mathcal{K}(\tau,\tau) = Id.
\end{array}\right. \tag{2.133}$$

Let us now introduce some notations:

$$\vartheta_{1,1}(s,\tau) = \mathcal{K}(s,\tau)q_1(\tau), \quad \vartheta_{1,2}(s,\tau) = \int_\tau^s \mathcal{K}(s,\sigma)(V_{1,1}q_1)(\sigma)d\sigma,$$
$$\vartheta_{1,3}(s,\tau) = \int_\tau^s \mathcal{K}(s,\sigma)(V_{1,2}q_2)(\sigma)d\sigma, \quad \vartheta_{1,4}(s,\tau) = \int_\tau^s \mathcal{K}(s,\sigma)(B_1(q_1,q_2))(\sigma)d\sigma,$$
$$\vartheta_{1,5}(s,\tau) = \int_\tau^s \mathcal{K}(s,\sigma)(R_1(.,\sigma))d\sigma,$$

and

$$\vartheta_{2,1}(s,\tau) = \mathcal{K}(s,\tau)(q_2(\tau)), \quad \vartheta_{2,2}(s,\tau) = \int_\tau^s \mathcal{K}(s,\sigma)(V_{2,1}q_1)(\sigma)d\sigma,$$
$$\vartheta_{2,3}(s,\tau) = \int_\tau^s \mathcal{K}(s,\sigma)(V_{2,2}q_2)(\sigma)d\sigma, \quad \vartheta_{2,4}(s,\tau) = \int_\tau^s \mathcal{K}(s,\sigma)(B_2(q_1,q_2))(\sigma)d\sigma,$$
$$\vartheta_{2,5}(s,\tau) = \int_\tau^s \mathcal{K}(s,\sigma)(R_2(.,\sigma))d\sigma.$$

From (2.132), we can see the strong influence of $\mathcal{K}$. For that reason, we will study the dynamics of that operator:

Lemme 2.14 (A priori estimates of the linearized operator)**.** *For all $\rho^* \geq 0$, there exists $s_5(\rho^*) \geq 1$ such that $\sigma \geq s_5(\rho^*)$ the following holds: If we have $v \in L^2_\rho(\mathbb{R}^N)$, satisfying*

$$\sum_{m=0}^{2} |v_m| + \left\| \frac{v_-}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} + \|v_e\|_{L^\infty(\mathbb{R}^N)} < \infty, \tag{2.134}$$

where the above components are introduced in (2.80)*, then, for all $s \in [\sigma, \sigma+\rho^*]$, the function $\theta(s) = \mathcal{K}(s,\sigma)v$ satisfies*

$$\begin{aligned} \left\| \tfrac{\theta_-(y,s)}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \leq & \frac{Ce^{s-\sigma}\left((s-\sigma)^2+1\right)}{s} \left(|v_0| + |v_1| + \sqrt{s}|v_2|\right) \\ & + Ce^{-\frac{(s-\sigma)}{2}} \left\| \tfrac{v_-}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} + C\frac{e^{-(s-\sigma)^2}}{s^{\frac{3}{2}}} \|v_e\|_{L^\infty(\mathbb{R}^N)}, \end{aligned} \tag{2.135}$$

and

$$\|\theta_e(y,s)\|_{L^\infty(\mathbb{R}^N)} \leq Ce^{s-\sigma}\left(\sum_{l=0}^{2} s^{\frac{l}{2}}|v_l| + s^{\frac{3}{2}} \left\| \frac{v_-}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \right) + Ce^{-\frac{s-\sigma}{p}} \|v_e\|_{L^\infty(\mathbb{R}^N)}. \tag{2.136}$$

Proof. The proof of this result was given by Bricmont and Kupiainen [1] in the one dimensional case. Later, it was extended to the higher dimensional case by Nguyen and Zaag [18]. We kindly refer interested readers to Lemma 2.9 in [18] for details of the proof. □

We now use Lemmas 2.14, 2.8, 2.18, 2.19 and 2.20 to deduce the following Lemma which implies Proposition 2.13.

Lemme 2.15. *For all $A \geq 1, \rho^* \geq 0$, there exists $s_6(A,\rho^*) \geq 1$ such that $\forall s_0 \geq s_6(A,\rho^*)$ and $q(s) \in S_A(s), \forall s \in [\tau, \tau+\rho^*]$ where $\tau \geq s_0$. Then, we have the following properties: for all $s \in [\tau, \tau+\rho^*]$,* *i) (The linear term $\vartheta_{1,1}(s,\tau)$ and $\vartheta_{2,1}(s,\tau)$)*

$$\begin{aligned}
\left\|\frac{(\vartheta_{1,1}(s,\tau))_-}{1+|y|^3}\right\|_{L^\infty} &\leq Ce^{-\frac{s-\tau}{2}}\left\|\frac{q_{1,-}(.,\tau)}{1+|y|^3}\right\|_{L^\infty}+\frac{Ce^{-(s-\tau)^2}}{s^{\frac{3}{2}}}\|q_{1,e}(\tau)\|_{L^\infty}+\frac{C}{s^2},\\
\|(\vartheta_{1,1}(s,\tau))_e\|_{L^\infty} &\leq Ce^{-\frac{s-\tau}{p}}\|q_{1,e}(\tau)\|_{L^\infty}+Ce^{s-\tau}s^{\frac{3}{2}}\left\|\frac{q_{1,-}(.,\tau)}{1+|y|^3}\right\|_{L^\infty}+\frac{C}{\sqrt{s}},\\
\left\|\frac{(\vartheta_{2,1}(s,\tau))_-}{1+|y|^3}\right\|_{L^\infty} &\leq Ce^{-\frac{s-\tau}{2}}\left\|\frac{q_{2,-}(.,\tau)}{1+|y|^3}\right\|_{L^\infty}+\frac{Ce^{-(s-\tau)^2}}{s^{\frac{3}{2}}}\|q_{2,e}(\tau)\|_{L^\infty}+\frac{C}{s^{\frac{p_1+5}{2}}},\\
\|(\vartheta_{2,1}(s,\tau))_e\|_{L^\infty} &\leq Ce^{-\frac{s-\tau}{p}}\|q_{2,e}(\tau)\|_{L^\infty}+Ce^{s-\tau}s^{\frac{3}{2}}\left\|\frac{q_{2,-}(.,\tau)}{1+|y|^3}\right\|_{L^\infty}+\frac{C}{s^{\frac{p_1+2}{2}}},
\end{aligned}$$

where $L^\infty = L^\infty(\mathbb{R}^N)$.

ii) The quadratic term $\vartheta_{1,2}(s,\tau)$ and $\vartheta_{2,2}(s,\tau)$

$$\begin{aligned}
\left\|\frac{(\vartheta_{1,2}(s,\tau))_-}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C(s-\tau)}{s^2}, \quad \|(\vartheta_{1,2}(s,\tau))_e\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C(s-\tau)}{s^{\frac{1}{2}}},\\
\left\|\frac{(\vartheta_{2,2}(s,\tau))_-}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C(s-\tau)}{s^{\frac{p_1+5}{2}}}, \quad \|(\vartheta_{2,2}(s,\tau))_e\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C(s-\tau)}{s^{\frac{p_1+2}{2}}}.
\end{aligned}$$

iii) The correction terms $\vartheta_{1,3}(s,\tau)$ and $\vartheta_{2,3}(s,\tau)$

$$\begin{aligned}
\left\|\frac{(\vartheta_{1,3}(s,\tau))_-}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C(s-\tau)}{s^2}, \quad \|(\vartheta_{1,3}(s,\tau))_e\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C(s-\tau)}{s^{\frac{1}{2}}},\\
\left\|\frac{(\vartheta_{2,3}(s,\tau))_-}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C(s-\tau)}{s^{\frac{p_1+5}{2}}}, \quad \|(\vartheta_{2,3}(s,\tau))_e\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C(s-\tau)}{s^{\frac{p_1+2}{2}}}.
\end{aligned}$$

iv) The correction terms $\vartheta_{1,4}(s,\tau)$ and $\vartheta_{2,4}(s,\tau)$

$$\begin{aligned}
\left\|\frac{(\vartheta_{1,3}(s,\tau))_-}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C(s-\tau)}{s^2}, \quad \|(\vartheta_{1,3}(s,\tau))_e\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C(s-\tau)}{s^{\frac{1}{2}}},\\
\left\|\frac{(\vartheta_{2,3}(s,\tau))_-}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C(s-\tau)}{s^{\frac{p_1+5}{2}}}, \quad \|(\vartheta_{2,3}(s,\tau))_e\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C(s-\tau)}{s^{\frac{p_1+2}{2}}}.
\end{aligned}$$

v) The correction terms $\vartheta_{1,5}(s,\tau)$ and $\vartheta_{2,5}(s,\tau)$

$$\begin{aligned}
\left\|\frac{(\vartheta_{1,3}(s,\tau))_-}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C(s-\tau)}{s^2}, \quad \|(\vartheta_{1,3}(s,\tau))_e\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C(s-\tau)}{s^{\frac{1}{2}}},\\
\left\|\frac{(\vartheta_{2,3}(s,\tau))_-}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C(s-\tau)}{s^{\frac{p_1+5}{2}}}, \quad \|(\vartheta_{2,3}(s,\tau))_e\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C(s-\tau)}{s^{\frac{p_1+2}{2}}}.
\end{aligned}$$

Proof. The result is implied from the definition of the shrinking set $V_A(s)$ and Lemma 2.8 and the bounds for $V, V_{j,k}, B_1, B_2, R_1, R_2$ with $j,k \in \{1,2\}$ which are shown in Lemmas 2.18, 2.19 and 2.20. For details in a quite similar case, see Lemma 4.20 in Tayachi and Zaag [24]. □

We now come back to the proof of Proposition (2.13): In fact, the conclusion of (*iii*) and (*iv*) of Proposition 2.13 follows by using formula (2.132) and Lemma (2.15). This concludes the proof of Proposition 2.13. □

2.4.2 Conclusion of the proof of Proposition 2.11

In this subsection, we will give prove a Proposition which implies Proposition 2.11 directly. More precisely, this is our statement:

Proposition 2.16. *There exists $A_7 \geq 1$ such that for all $A \geq A_7$, there exists $s_7(A) \geq 1$ such that for all $s_0 \geq s_7(A)$, we have the following properties: If the following conditions hold:*

a) $(q_1, q_2)(s_0) = (\phi_1, \phi_2)$ *with* $(d_0, d_1) \in \mathcal{D}_{A,s_0}$,

b) For all $s \in [s_0, s_1]$ *we have* $(q_1, q_2)(s) \in V_A(s)$.

Then, for all $s \in [s_0, s_1]$, *we have*

$$\forall i, j \in \{1, \cdots, N\}, \quad |q_{1,i,j}(s)| \leq \frac{A^2 \ln s}{2s^2}, \tag{2.137}$$

$$\left\| \frac{q_{1,-}(y,s)}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A}{2s^2}, \quad \|q_{1,e}(s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^2}{2\sqrt{s}}, \tag{2.138}$$

$$\left\| \frac{q_{2,-}(y,s)}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^2}{2s^{\frac{p_1+5}{2}}}, \quad \|q_{2,e}(s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^3}{2s^{\frac{p_1+2}{2}}}. \tag{2.139}$$

Note that $\mathcal{D}_{A,s_0}$ is introduced in Lemma 2.9 and (ϕ_1, ϕ_2) is defined as in Definition (2.2).

Proof. The proof relies on Propostion 2.13 and is quiet similar to Proposition 4.7 in Merle and Zaag [15]. For that reason, we only give a short proof to (2.137).

We use (2.125) to deduce that

$$\left| \int_{s_0}^{s} (\tau^2 q_{1,j,k}(\tau)) d\tau \right| \leq CA(\ln(s) - \ln(s_0)), \forall j, k \in \{1, ..., N\},$$

this yields

$$|q_{1,j,k}(s)| \leq CAs^{-2} \ln s \leq \frac{A^2 \ln s}{2s^2},$$

if $A \geq A_7$ large enough and $s \geq s_7(A)$. Then, (2.137) follows. This also finishes the proof of Proposition 2.16. □

Conclusion of the proof of Proposition 2.11

Proof. From Proposition 2.16, if $(q_1, q_2)(s_1) \in \partial V_A(s_1)$ then:

$$\left(q_{1,0}, (q_{1,j})_{1 \leq j \leq N}, q_{2,0}, (q_{2,j})_{1 \leq j \leq N}, (q_{2,j,k})_{1 \leq j,k \leq N}\right)(s_1) \in \partial \hat{V}_A(s_1). \tag{2.140}$$

This concludes item (i) of Proposition 2.11.

The proof of item (ii) of Proposition 2.11: In fact, thanks to (2.140), we derive the two following situations:

\+ The first situation: Either there exists $\epsilon_0 \in \{-1, 1\}$ such that $q_{1,0}(s_1) = \epsilon_0 \frac{A}{s_1^2}$; or there exist $j_0 \in \{1, ..., N\}$ and $\epsilon_0 \in \{-1, 1\}$ such that $q_{1,j_0} = \epsilon_0 \frac{A}{s_1^2}$; or exists $\epsilon_0 \in \{-1, 1\}$ such that $q_{2,0} = \epsilon_0 \frac{A^2}{s_1^{p_1+2}}$; or there exist $j_0 \in \{1, ..., N\}$ and $\epsilon_0 \in \{-1, 1\}$ such that $q_{2,j_0}(s_1) = \epsilon_0 \frac{A^2}{s_1^{p_1+2}}$.

Without loss of generality, we can suppose that $q_{1,0} = \epsilon_0 \frac{A}{s_1^2}$ (the other cases are quiet similar). Then, by using (2.123), we can prove that the sign of $q'_{1,0}(s_1)$ is oppsite to the sign of $\left(\epsilon_0 \frac{A}{s_1^2}\right)'$. In other words,

$$\epsilon_0 \left(q_{1,0} - \epsilon_0 \frac{A}{s^2} \right)' (s_1) > 0.$$

+ The second situation: There exist $j_0, k_0, \epsilon_0 \in -1, 1$ and $\epsilon_0 \in \{-1, 1\}$ such that $q_{2,j_0,k_0}(s_1) = \epsilon_0 \frac{A^5 \ln s}{s_1^{p_1+2}}$, by using (2.126), we can prove that

$$\epsilon_0 \left(q_{2,j_0,k_0}(s) - \epsilon_0 \frac{A^5 \ln s}{s^{p_1+2}} \right)' (s_1) > 0.$$

From two situations in the above, we deduce that there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$

$$(q_{1,0}, (q_{1,j})_{1 \le j \le N}, q_{2,0}, (q_{2,j})_{1 \le j \le N}, (q_{2,j,k})_{1 \le j,k \le N})\,(s_1 + \delta) \notin \hat{V}_A(s_1 + \delta).$$

provided that $A \ge A_3$ and $s_0 \ge s_3(A)$. Then, the item (ii) of Proposition follows. Thus, we derive the conclusion of Proposition 2.11. □

2.5 Appendix

In this appendix, we state and prove several technical and and straightforward results need in our paper.

We first give a Taylor expansion of the quadratic terms defined in (2.29) and (2.30).

Lemme 2.17 (Asymptotics of $\bar{B}_1$ and $\bar{B}_2$)**.** *Let us consider $\bar{B}_1(\bar{w}_1, w_2)$ and $\bar{B}_2(\bar{w}_1, w_2)$, defined in* (2.29) *and* (2.30)*, respectively. Then, the following holds*

$$\bar{B}_1(\bar{w}_1, w_2) = \frac{p}{2\kappa} \bar{w}_1^2 + O(|\bar{w}_1|^3 + |w_2|^2), \tag{2.141}$$

$$\bar{B}_2(\bar{w}_1, w_2) = \frac{p}{\kappa} \bar{w}_1 w_2 + O\left(|\bar{w}_1|^2 |w_2|\right) + O\left(|w_2|^3\right), \tag{2.142}$$

as $(\bar{w}_1, w_2) \to (0, 0)$.

Proof. In fact, bearing in mind that $p \in \mathbb{N}$. Then, by using the Newton binomial formula, we derive the following:

$$(\bar{w}_1 + \kappa + i w_2)^p = (\bar{w}_1 + \kappa)^p + ip(\bar{w}_1 + \kappa)^{p-1} w_2 + p(p-1)(\bar{w}_1 + \kappa)^{p-2} w_2^2 + G(\bar{w}_1, w_2),$$

where

$$|G(\bar{w}_1, w_2)| \le C |w_2|^3, \quad \forall |\bar{w}_1| + |w_2| \le 1.$$

This gives us

$$\text{Re } ((\bar{w}_1 + \kappa + i w_2)^p) = (\bar{w}_1 + \kappa)^p + p(p-1)(\bar{w}_1 + \kappa)^{p-2} w_2^2 + \text{ Re } (G), \tag{2.143}$$

$$\text{Im } ((\bar{w}_1 + \kappa + i w_2)^p) = p(\bar{w}_1 + \kappa)^{p-1} w_2 + \text{ Im } (G). \tag{2.144}$$

Moreover, we apply again the Newton binomial formula to $(\kappa+\bar{w}_1)^p, (\kappa+\bar{w}_1)^{p-1}$ and we get

$$
\begin{aligned}
(\kappa+\bar{w}_1)^p &= \kappa^p + \frac{p}{p-1}\bar{w}_1 + \frac{p}{2\kappa}\bar{w}_1^2 + O(|\bar{w}_1|^3), && (2.145)\\
(\kappa+\bar{w}_1)^{p-1} &= \frac{1}{p-1} + \frac{1}{\kappa}\bar{w}_1 + O(|\bar{w}_1|^2). && (2.146)
\end{aligned}
$$

Thus, (2.141) follows by (2.143) and (2.145). Moreover, (2.142) follows by (2.144) and (2.146). □

Now, we give an expansion of the potentials defined in (2.68) and (2.69) - (2.72). The following is our statement:

Lemme 2.18 (The potential functions V and $V_{j,k}$ with $j,k\in\{1,2\}$)**.** *We consider* $V, V_{1,1}, V_{1,2}, V_{2,1}$ *and* $V_{2,2}$ *as defined in* (2.68) *and* (2.69) - (2.72). *Then, the following holds:*

(i) For all $s\geq 1$ *and* $y\in\mathbb{R}^N$, *we have* $|V(y,s)|\leq C$,

$$|V(y,s)| \leq \frac{C(1+|y|^2)}{s}, \tag{2.147}$$

and

$$V(y,s) = -\frac{(|y|^2-2N)}{4s} + \tilde{V}(y,s), \tag{2.148}$$

where $\tilde{V}$ *satisfies*

$$|\tilde{V}(y,s)| \leq C\frac{(1+|y|^4)}{s^2}, \forall s\geq 1, |y|\leq 2K\sqrt{s}. \tag{2.149}$$

(ii) For all $s\geq 1$ *and* $y\in\mathbb{R}^N$, *the potential functions* $V_{j,k}$ *with* $j,k\in\{1,2\}$ *satisfy*

$$
\begin{aligned}
|V_{1,1}(y,s)| + |V_{2,2}(y,s)| &\leq \frac{C(1+|y|^4)}{s^4},\\
|V_{1,2}(y,s)| + |V_{2,1}(y,s)| &\leq \frac{C(1+|y|^2)}{s^2}.
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
\|V_{1,1}\|_{L^\infty(\mathbb{R}^N)} + \|V_{2,2}\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C}{s^2},\\
\|V_{1,2}\|_{L^\infty(\mathbb{R}^N)} + \|V_{2,1}\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C}{s}.
\end{aligned}
$$

Proof. We see that item (ii) is derived directly from the defintions of $V_{j,k}, j,k\in\{1,2\}$. In addition to that, the proof of (i) is quite similar to Lemma B.1, page 1270 in [18]. □

Now, we give some Taylor expansions of B_1 and B_2, introduced in (2.73) and (2.74), respectively.

Lemme 2.19 (The quadratic terms $B_1(q_1,q_2)$ and $B_2(q_1,q_2)$)**.** *Let us consider* $B_1(q_1,q_2)$ *and* $B_2(q_1,q_2)$, *defined in* (2.73) *and* (2.74) *respectively. For all* $A\geq 1$, *there exists* $s_8(A)\geq 1$ *such that for all* $s\geq s_8(A)$, *if* $(q_1,q_2)(s)\in V_A(s)$, *then*

$$
\begin{aligned}
|B_1(q_1,q_2)| &\leq C\left(|q_1|^2+|q_2|^2\right), && (2.150)\\
|B_2(q_1,q_2)| &\leq C\left(\frac{|q_1|^2}{s} + |q_1.q_2| + |q_2|^2\right). && (2.151)
\end{aligned}
$$

Proof. Let us recall the two functions $F_1(u_1, u_2)$ and $F_2(u_1, u_2)$ which are defined in (2.24). As a matter of fact, they belong to $C^\infty(\mathbb{R}^2)$. Then, by using Taylor expansion, we obtain

$$\begin{aligned} F_1(\Phi_1 + q_1, \Phi_2 + q_2) &= \sum_{j,k\le p} \frac{1}{j!k!} \partial^{j+k}_{u_1^j u_2^k} F_1(\Phi_1, \Phi_2) q_1^j q_2^k, \\ F_2(\Phi_1 + q_1, \Phi_2 + q_2) &= \sum_{j,k\le p} \frac{1}{j!k!} \partial^{j+k}_{u_1^j u_2^k} F_2(\Phi_1, \Phi_2) q_1^j q_2^k. \end{aligned}$$

Thus, (2.150) and (2.151) follow by definitions of B_1, B_2 and the definition of $V_A(s)$. □

In the following lemma, we give various estimates involving rest terms R_1 and R_2, defined in (2.75) and (2.76), respectively.

Lemme 2.20 (Rest terms R_1, R_2)**.** *For all $s \ge 1$, let us consider R_1, R_2 defined in* (2.75) *and* (2.76)*. Then,*

(i) For all $s \ge 1$ and $y \in \mathbb{R}^N$

$$\begin{aligned} R_1(y,s) &= \frac{c_{1,p}}{s^2} + \tilde{R}_1(y,s), \\ R_2(y,s) &= \frac{c_{2,p}}{s^3} + \tilde{R}_2(y,s), \end{aligned}$$

where $c_{1,p}$ and $c_{2,p}$ are constants depended on p and $\tilde{R}_1, \tilde{R}_2$ satisfy: for all $|y| \le 2K\sqrt{s}$

$$\begin{aligned} |\tilde{R}_1(y,s)| &\le \frac{C(1+|y|^4)}{s^3}, \\ |\tilde{R}_2(y,s)| &\le \frac{C(1+|y|^6)}{s^4}. \end{aligned}$$

(ii) Moreover, we have for all $s \ge 1$

$$\begin{aligned} \|R_1(.,s)\|_{L^\infty(\mathbb{R}^N)} &\le \frac{C}{s}, \\ \|R_2(.,s)\|_{L^\infty(\mathbb{R}^N)} &\le \frac{C}{s^2}. \end{aligned}$$

Proof. The proofs for R_1 and R_2 are quite similar. For that reason, we only give the proof of the estimates on R_2. This means that we need to prove the following estimates:

$$R_2(y,s) = -\frac{N(N+4)\kappa}{(p-1)s^3} + \tilde{R}_2(y,s), \tag{2.152}$$

where

$$|\tilde{R}_2(y,s)| \le C\frac{(1+|y|^6)}{s^4}, \forall |y| \le 2K\sqrt{s},$$

and

$$\|R_2(.,s)\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{s^2}. \tag{2.153}$$

We recall the definition of $R_2(y,s)$ in (2.76) as follows

$$R_2(y,s) = \Delta\Phi_2 - \frac{1}{2}y\cdot\nabla\Phi_2 - \frac{\Phi_2}{p-1} + F_2(\Phi_1,\Phi_2) - \partial_s\Phi_2.$$

Then, we can rewrite R_2

$$R_2(y,s) = \Delta\Phi_2 - \frac{1}{2}y\cdot\nabla\Phi_2 - \frac{\Phi_2}{p-1} + p\Phi_1^{p-1}\Phi_2 - \partial_s\Phi_2 + R_2^*(y,s),$$

where

$$R_2^*(y,s) = F_2(\Phi_1,\Phi_2) - p\Phi_1^{p-1}\Phi_2.$$

Using the definition of F_2 in (2.24), and the defintions of Φ_1,Φ_2 in (2.64) and (2.65), we derive that

$$|R_2^*(y,s)| \le \frac{C(1+|y|^6)}{s^4}, \quad \forall |y|\le 2K\sqrt{s},$$

and

$$\|R_2^*(.,s)\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{s^2}.$$

In addition to that, we introduce $\bar{R}_2$ by

$$\bar{R}_2(y,s) = \Delta\Phi_2 - \frac{1}{2}y\cdot\nabla\Phi_2 - \frac{\Phi_2}{p-1} + p\Phi_1^{p-1}\Phi_2 - \partial_s\Phi_2.$$

Then, we may obtain the conclusion if the following two estimates hold:

$$\left|\bar{R}_2(y,s) + \frac{N(N+4)\kappa}{(p-1)s^3}\right| \le \frac{C(1+|y|^6)}{s^4}, \tag{2.154}$$

$$\|\bar{R}_2(.,s)\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{s^2}. \tag{2.155}$$

+ The proof of (2.154)*:* We first aim at expanding $\Delta\Phi_2$ in a polynomial in y of order less than 4 via the Taylor expansion. Indeed, $\Delta\Phi_2$ is given by

$$\begin{aligned}\Delta\Phi_2 &= \frac{2N}{s^2}\left(p-1+\frac{(p-1)^2|y|^2}{4ps}\right)^{-\frac{p}{p-1}} - \frac{(p-1)|y|^2}{s^3}\left(p-1+\frac{(p-1)^2}{4p}\frac{|y|^2}{s}\right)^{-\frac{2p-1}{p-1}}\\ &- \frac{(N+2)(p-1)|y|^2}{2s^3}\left(p-1+\frac{(p-1)^2}{4p}\frac{|y|^2}{s}\right)^{-\frac{2p-1}{p-1}}\\ &+ \frac{(2p-1)(p-1)^2|y|^4}{4ps^4}\left(p-1+\frac{(p-1)^2}{4p}\frac{|y|^2}{s}\right)^{-\frac{3p-2}{p-1}}.\end{aligned}$$

Besides that, we use Taylor expansion in the variable $z=\frac{|y|}{\sqrt{s}}$ to function $\left(p-1+\frac{(p-1)^2}{4p}\frac{|y|^2}{s}\right)^{-\frac{p}{p-1}}$ in the domain where $|z|\le 2K$ and this yields the following:

$$\left|\left(p-1+\frac{(p-1)^2|y|^2}{4ps}\right)^{-\frac{p}{p-1}} - \frac{\kappa}{p-1} + \frac{\kappa}{4(p-1)}\frac{|y|^2}{s}\right| \le \frac{C(1+|y|^4)}{s^2}, \quad \forall |y|\le 2K\sqrt{s}.$$

which yields

$$\left|\frac{2N}{s^2}\left(p-1+\frac{(p-1)^2|y|^2}{4ps}\right)^{-\frac{p}{p-1}}-\frac{2N\kappa}{(p-1)s^2}+\frac{N\kappa|y|^2}{2(p-1)s^3}\right|\leq\frac{C(1+|y|^6)}{s^4},$$

for all $|y|\leq 2K\sqrt{s}$.

It is similar to estimate the other terms in the formula of $\Delta\Phi_2$, defined in the above. Therefore, we finally obtain

$$\left|\Delta\Phi_2-\frac{2N\kappa}{(p-1)s^2}+\frac{N\kappa|y|^2}{(p-1)s^3}+2\frac{k|y|^2}{(p-1)s^3}\right|\leq\frac{C(1+|y|^6)}{s^4},\quad\forall|y|\leq 2K\sqrt{s}.\tag{2.156}$$

As we did for $\Delta\Phi_2$, we estimate similarly the other terms in formula of $\bar{R}_2$, for all $|y|\leq 2K\sqrt{s}$:

$$\left|-\frac{1}{2}y\cdot\nabla\Phi_2+\frac{\kappa|y|^2}{(p-1)s^2}-\frac{\kappa|y|^4}{4(p-1)s^3}-\frac{\kappa|y|^4}{4(p-1)s^3}\right|\leq\frac{C(1+|y|^6)}{s^4},\tag{2.157}$$

$$\left|-\frac{\Phi_2}{p-1}+\frac{\kappa|y|^2}{(p-1)^2s^2}-\frac{\kappa|y|^4}{4(p-1)^2s^3}-\frac{2N\kappa}{(p-1)^2s^2}\right|\leq\frac{C(1+|y|^6)}{s^4},\tag{2.158}$$

$$\left|p\Phi_1^{p-1}\Phi_2+T(y,s)\right|\leq\frac{C(1+|y|^6)}{s^4},\tag{2.159}$$

$$\left|-\partial_s\Phi_2-\frac{2\kappa|y|^2}{(p-1)s^3}+\frac{4N\kappa}{(p-1)s^3}\right|\leq\frac{C(1+|y|^6)}{s^4},\tag{2.160}$$

where

$$T(y,s)=-\frac{p\kappa|y|^2}{(p-1)^2s^2}+\frac{(2p-1)\kappa|y|^4}{4(p-1)^2s^3}-\frac{N\kappa|y|^2}{(p-1)s^3}+\frac{2pN\kappa}{(p-1)^2s^2}+\frac{N^2\kappa}{(p-1)s^3}$$

Thus, by an addition (2.156), (2.157), (2.158), (2.159) and (2.160), we obtain the following

$$\left|\bar{R}_2(y,s)+\frac{N(N+4)\kappa}{(p-1)s^3}\right|\leq\frac{C(1+|y|^6)}{s^4},\quad\forall|y|\leq 2K\sqrt{s},$$

this concludes (2.154).
+ The proof (2.155)*:* We rewrite Φ_1,Φ_2 as follows

$$\Phi_1(y,s)=R_{1,0}(z)+\frac{N\kappa}{2ps}\text{ and }\Phi_2(y,s)=\frac{1}{s}R_{2,1}(z)-\frac{2N\kappa}{(p-1)s^2}\text{ where }z=\frac{y}{\sqrt{s}},$$

where $R_{1,0}$ and $R_{2,1}$ are defined in (2.56) and (2.60), respectively. In addition to that, we rewrite $\bar{R}_2$ in terms of $R_{1,0}$ and $R_{2,1}$, and we note that $R_{1,0}$ and $R_{2,1}$ satisfy (2.52) and (2.54). Then, it follows that

$$|\bar{R}_2(y,s)|\leq\frac{C}{s^2},\forall y\in\mathbb{R}^N.$$

Hence, (2.155) follows. This concludes the proof of this Lemma. □

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Chapter 3

A blowup solution of a complex semilinear heat equation with an non integer power [1]

G. K. Duong

Abstract: *In this paper, we consider the following semi-linear complex heat equation*

$$\partial_t u = \Delta u + u^p, u \in \mathbb{C}$$

in $\mathbb{R}^N$, *with an arbitrary power* p, $p > 1$. *We construct for this equation a complex solution* $u = u_1 + iu_2$, *which blows up in finite time* T *and only at one blowup point* a. *Moreover, we also describe the asymptotic behaviors of the solution by the following final profiles:*

$$\begin{aligned} u(x,T) &\sim \left[\frac{(p-1)^2|x-a|^2}{8p|\ln|x-a||}\right]^{-\frac{1}{p-1}}, \\ u_2(x,T) &\sim \frac{2p}{(p-1)^2}\left[\frac{(p-1)^2|x-a|^2}{8p|\ln|x-a||}\right]^{-\frac{1}{p-1}}\frac{1}{|\ln|x-a||} > 0, \text{ as } x \to a. \end{aligned}$$

In addition to that, since we also have $u_1(0,t) \to +\infty$ *and* $u_2(0,t) \to -\infty$ *as* $t \to T$, *the blowup in the imaginary part shows a new phenomenon unknown for the standard heat equation in the real case: a non constant sign near the singularity, with the existence of a vanishing surface for the imaginary part, shrinking to the origin. In our work, we have succeeded to extend for any power* p *where the non linear term* u^p *is not continuous if* p *is irrational. In particular, the solution which we have constructed has a positive real part. We study our equation as a system of the real part and the imaginary part* u_1 *and* u_2. *Our work relies on two main arguments: the reduction of the problem to a finite dimensional one and a topological argument based on the index theory to get the conclusion.*

Mathematics Subject Classification: *35K55, 35K57 35K50, 35B44 (Primary); 35K50, 35B40 (Secondary).*

Keywords: *Blowup solution, Blowup profile, Semilinear complex heat equation, non variation heat equation.*

[1] This work was accepted for publication in *J. Differential Equations*, (2019).
Doi: 10.1016/j.jde.2019.05.024

3.1 Introduction

3.1.1 Earlier work

In this work, we are interested in the following complex-valued semilinear heat equation

$$\begin{cases} \partial_t u &= \Delta u + F(u), t \in [0,T), \\ u(0) &= u_0 \in L^\infty(\mathbb{R}^N), \end{cases} \tag{3.1}$$

where $F(u) = u^p$, and $u(t) : \mathbb{R}^N \to \mathbb{C}$, $L^\infty(\mathbb{R}^N) := L^\infty(\mathbb{R}^N, \mathbb{C})$ and $p > 1$.

Typically, when $p = 2$, model (3.1) becomes the following

$$\begin{cases} \partial_t u &= \Delta u + u^2, t \in [0,T), \\ u(0) &= u_0 \in L^\infty(\mathbb{R}^N). \end{cases} \tag{3.2}$$

This model is connected to the viscous Constantin-Lax-Majda equation with a viscosity term, which is a one dimensional model for the vorticity equation in fluids. For more details, the readers are addressed to the following works: Constantin, Lax, Majda [2]; Guo, Ninomiya, Shimojo and Yanagida [8]; Okamoto, Sakajo and Wunsch [24]; Sakajo [25] and [26]; Schochet [27].

The local Cauchy problem for model (3.1) can be solved in $L^\infty(\mathbb{R}^N)$ when p is integer, thanks to a fixed-point argument. However, if p is an irrational number, then, the local Cauchy problem has not been solved yet, up to our knowledge. In my point of view, this probably comes from the discontinuity of $F(u)$ on $\{u \in \mathbb{R}^*_-\}$ and this challenge is also one of the main difficulties of the paper. As a matter of fact, we solve the Cauchy problem in Appendix 3.5 for data $u_0 \in L^\infty(\mathbb{R}^N)$, with $\text{Re}(u_0) \geq \lambda$, for some $\lambda > 0$. Accordingly, a maximal solution may be global in time or may exist only for $t \in [0,T)$, for some $T > 0$. In that case, we have to options:

(*i*) Either $\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \to +\infty$ as $t \to T$.

(*ii*) Or $\min_{x\in\mathbb{R}^N} \text{Re}(u(x,t)) \to 0$ as $t \to T$.

In this paper, we are interested in case (*i*) which is referred to as *finite-time blow-up* in the sequel.

In addition to that, a blowup solution u is called *Type I* if

$$\limsup_{t\to T} (T-t)^{\frac{1}{p-1}} \|u(.,t)\|_{L^\infty(\mathbb{R}^N)} < +\infty.$$

Otherwise, the solution u is called *Type II*.

In addition to that, T is called the blowup time of u and a point $a \in \mathbb{R}^N$ is called a blowup point if and only if there exist sequences $\{(a_j, t_j)\} \to (a, T)$ as $j \to +\infty$ such that

$$|u_1(a_j,t_j)| + |u_2(a_j,t_j)| \to +\infty \text{ as } j \to +\infty.$$

In our work, we are interested in constructing a blowup solution of (3.1) which is of *Type I*.

Let us quickly mention some typical works for this situation (for more details, the readers can see the introduction in Duong [5] where has treated for the integer case).

(i) **For the real case:** We first mention to Bricmont and Kupiainen [1], the authors have constructed a real positive solution to the following equation

$$\partial_t u = \Delta u + |u|^{p-1}u, p > 1, \tag{3.3}$$

which blows up in finite time T, only at the origin and they have also derived the profile of the solution such that

$$\left\| (T-t)^{\frac{1}{p-1}} u(.,t) - f_0 \left(\frac{.}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{1 + \sqrt{|\ln(T-t)|}},$$

where f_0 is defined by

$$f_0(z) = \left(p - 1 + \frac{(p-1)^2}{4p}|z|^2 \right)^{-\frac{1}{p-1}}. \tag{3.4}$$

In addition to that, Herrero and Velázquez derived in [13], the same result with a different method. Particularly, Merle and Zaag gave in [17], a proof which is simpler than the one in [1] and proposed the following two-step method (see also the note [15]):

- Reduction of the infinite dimensional problem to a finite dimensional one.
- Solution of the finite dimensional problem thanks to a topological argument based on Index theory.

Moreover, they also proved the stability of the blowup profile for (3.3). In addition to that, we would like to mention that this method has been successful in various situations such as: Ebde and Zaag [6]; Tayachi and Zaag [28] and Ghoul; Nguyen and Zaag [9]; [10] (with a gradient term) and [11]. We would like to mention Nguyen and Zaag [21] who have considered the following quasi-critical double source equation

$$\partial_t u = \Delta u + |u|^{p-1}u + \frac{\mu |u|^{p-1}u}{\ln^a(2+u^2)}.$$

Besides that, we have Duong, Nguyen and Zaag [4], the authors have considered the following non scale invariant equation

$$\partial_t u = \Delta u + |u|^{p-1}u \ln^\alpha(2+u^2).$$

(ii) **For the complex case:** The blowup problem for the complex-valued parabolic equations has been studied intensively by many authors, in particular for the Complex Ginzburg Landau equation (CGL)

$$\partial_t u = (1+i\beta)\Delta u + (1+i\delta)|u|^{p-1}u. \tag{3.5}$$

This is the case of an ealier work of Zaag in [29] for equation (3.5) when $\beta = 0$ and δ small enough. Later, Masmoudi and Zaag generalized in [18] the result of [29] and constructed a blowup solution for (3.5) with a super critical condition $p - \delta^2 - \beta\delta - \beta\delta p > 0$.

Recently, Nouaili and Zaag in [23] have constructed a blowup solution for equation (3.5), for a critical case where $\beta = 0$ and $p = \delta^2$.

In addtiion to that, there are many works for equation (3.1), in particular for equation (3.2). We mention Nouaili and Zaag [22], these authors have constructed for equation (3.2), a complex solution $u = u_1 + iu_2$ which blows up in finite time T only at the origin. Note that the real and the imaginary parts blow up simultaneously. In particular, [22] leaves unanswered the question of the derivation of the profile of the imaginary part, and this is precisely our aim in this paper, not only for equation (3.2), but also for equation (3.1) for all $p > 1$.

Besides that, we would like to mention also some classification results, proven by Harada in [12], for blowup solutions of (3.2) which satisfy some reasonable assumptions. In particular, in that works, we are able to derive a sharp blowup profile for the imaginary part of the solution. However, [12] is limited with $p = 2$.

Recently, we mention Duong [5], the author has treated for the cases where p takes an arbitrary integer value.

3.1.2 Statement of the result

As we mentioned in the previous section, we only have treated in [5] the case where $p \in \mathbb{N}, p \geq 2$ which the handling of the nonlinear term is much easier. In this work, we do better and give a proof which holds also for the cases where $p \notin \mathbb{N}$. We believe we made an important achievement, we acknowledge that we left unanswered the case where $p > 1$ and $p \notin \mathbb{N}$. From the limitation of the mentioned works in the previous section, it motivates us to study model (3.1) in general even for an irrational number. More precisely, the following theorem is our main result:

Theorem 3.1 (Existence of a blowup solution for (3.1) and a sharp discription of its profile). *For each $p > 1$ and $p_1 \in \left(0, \min\left(\frac{p-1}{4}, \frac{1}{2}\right)\right)$, there exists $T_1(p, p_1) > 0$ such that for all $T \leq T_1$, there exist initial data $u(0) = u_1(0) + iu_2(0)$ such that equation (3.1) has a unique solution u on $\mathbb{R}^N \times [0, T)$ satisfying the following:*

i) The solution u blows up in finite time T only at the origin and $Re(u) > 0$ on $\mathbb{R}^N \times [0, T)$. Moreover, it satisfies the following

$$\left\| (T-t)^{\frac{1}{p-1}} u(.,t) - f_0\left(\frac{.}{\sqrt{(T-t)|\ln(T-t)|}}\right) \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{1 + \sqrt{|\ln(T-t)|}}, \quad (3.6)$$

and

$$\left\| (T-t)^{\frac{1}{p-1}} |\ln(T-t)| u_2(.,t) - g_0\left(\frac{.}{\sqrt{(T-t)|\ln(T-t)|}}\right) \right\|_{L^\infty(\mathbb{R}^N)} \quad (3.7)$$

$$\leq \frac{C}{1 + |\ln(T-t)|^{\frac{p_1}{2}}},$$

where f_0 is defined in (3.4) and g_0 is defined as follows

$$g_0(z) = \frac{|z|^2}{\left(p - 1 + \frac{(p-1)^2}{4p}|z|^2\right)^{\frac{p}{p-1}}}. \quad (3.8)$$

ii) *There exists a complex function* u^* *in* $C^2(\mathbb{R}^N\backslash\{0\})$ *such that* $u(t) \to u^* = u_1^* + iu_2^*$ *as* $t \to T$, *uniformly on compact sets of* $\mathbb{R}^N\backslash\{0\}$, *and we have the following asymptotic behaviors:*

$$u^*(x) \sim \left[\frac{(p-1)^2|x|^2}{8p|\ln|x||}\right]^{-\frac{1}{p-1}}, \text{ as } x \to 0, \tag{3.9}$$

and

$$u_2^*(x) \sim \frac{2p}{(p-1)^2}\left[\frac{(p-1)^2|x|^2}{8p|\ln|x||}\right]^{-\frac{1}{p-1}} \frac{1}{|\ln|x||}, \text{ as } x \to 0. \tag{3.10}$$

Remark 3.2. *We remark that the condition on the parameter* $p_1 < \min\left(\frac{p-1}{4}, \frac{1}{2}\right)$ *comes from the definition of the set* $V_A(s)$ *(see in item* (i) *of Definition 3.1), Proposition 3.18 and Lemma 3.26. Indeed, this condition ensures that the projections of the quadratic term* $B_2(q_2, q_2)$ *on the negative and outer parts are smaller than the conditions in* $V_A(s)$*. Then, we can conclude* (3.132) *and* (3.134) *by using Lemma 3.26 and definition of* $V_A(s)$*.*

Remark 3.3. *We can show that the constructed solution in the above Theorem satisfies the following asymptotic behaviors:*

$$u(0,t) \sim \kappa(T-t)^{-\frac{1}{p-1}}, \tag{3.11}$$

$$u_2(0,t) \sim -\frac{2N\kappa}{(p-1)}\frac{(T-t)^{-\frac{1}{p-1}}}{|\ln(T-t)|^2}, \tag{3.12}$$

as $t \to T$, *(see* (3.91) *and* (3.92)*). Therefore, we deduce that* u *blows up at time* T *only at* 0*. Note that, the real and imaginary parts simultaneously blow up. Moreover, from item* (ii) *of Theorem 3.1, the blowup speed of* u_2 *is softer than* u_1 *because of the quantity* $\frac{1}{|\ln|x||}$ *(see* (3.9) *and* (3.10)*).*

Remark 3.4 (A strong singularity of the imaginary part)**.** *We observe from* (3.10) *and* (3.12) *that there is a strong sigularity at the neighborhood of* a *as* $t \to T$; *when* x *close to* 0, *we have* $u_2(x,t)$ *which becomes large and positive as* $t \to T$, *however, we always have* $u_2(0,t) \to -\infty$ *as* $t \to T$. *Thus the imaginary part has no constant sign near the singularity. In particular, if* t *is near* T, *there exists* $b(t) > 0$ *in* $\mathbb{R}^N$ *and* $b(t) \to 0$ *as* $t \to T$ *such that at time* t, $u_2(.,t)$ *vanishes on some surface close to the sphere of center* 0 *and radius* $b(t)$*. Therefore, we don't have* $|u_2(x,t)| \to +\infty$ *as* $(x,t) \to (0,T)$*. This non constant property for the imaginary part is very surprising to us. In the frame work of semilinear heat equation, such a property can be encountered for phase invariant complex equations, such as the Complex Ginzburg-Landau (CGL) equation (see Zaag in [29], Masmoudi and Zaag in [18], Nouaili-Zaag [23]). As for complex parabolic equation with no phase invariance, this is the first time such a sign change in available, up to our knowledge. We would like to mention that such a sign change near the singularity was already observed for the semilinear wave equation non characteristic blowup point (see Merle and Zaag in [19], [20] and Côte and Zaag in [3]).*

Remark 3.5. *For each* $a \in \mathbb{R}^N$, *by using the translation* $u_a(.,t) = u(.-a,t)$, *we can prove that* u_a *also satisfies equation* (3.1) *and the solution blows up at time* T *only at the point* a*. We can derive that* u_a *satisfies all estimates* (3.6) - (3.10) *by replacing* x *by* $x - a$*.*

Remark 3.6. *In Theorem* (3.1)*, the initial data* $u(0)$ *is given exactly as follows*

$$u(0) = u_1(0) + iu_2(0),$$

where

$$\begin{aligned} u_1(x,0) &= T^{-\frac{1}{p-1}} \left\{ \left(p - 1 + \frac{(p-1)^2|x|^2}{4pT|\ln T|} \right)^{-\frac{1}{p-1}} + \frac{N\kappa}{2p|\ln T|} \right. \\ &+ \left. \frac{A}{|\ln T|^2} \left(d_{1,0} + d_{1,1} \cdot \frac{x}{\sqrt{T}} \right) \chi_0 \left(\frac{16|x|}{K_0\sqrt{T|\ln T|}} \right) \right\} \chi_0 \left(\frac{|x|}{\sqrt{T|\ln T|}} \right) \\ &+ U^*(x) \left(1 - \chi_0 \left(\frac{|x|}{\sqrt{T|\ln T|}} \right) \right) + 1, \\ u_2(x,0) &= T^{-\frac{1}{p-1}} \chi_0 \left(\frac{|x|}{\sqrt{T|\ln T|}} \right) \left\{ \frac{|x|^2}{T|\ln T|^2} \left(p - 1 + \frac{(p-1)^2|x|^2}{4pT|\ln T|} \right)^{-\frac{p}{p-1}} - \frac{2N\kappa}{(p-1)|\ln T|^2} \right. \\ &+ \left[\frac{A^2}{|\ln T|^{p_1+2}} \left(d_{2,0} + d_{2,1} \cdot \frac{x}{\sqrt{T}} \right) + \frac{A^5 \ln(|\ln(T)|)}{|\ln T|^{p_1+2}} \left(\frac{1}{2} \frac{x^{\mathfrak{T}}}{\sqrt{T}} \cdot d_{2,2} \cdot \frac{x}{\sqrt{T}} - Tr(d_{2,2}) \right) \right] \\ &\times \left. \chi_0 \left(\frac{2|x|}{K_0\sqrt{T|\ln T|}} \right) \right\}. \end{aligned}$$

where $\kappa = (p-1)^{-\frac{1}{p-1}}, K_0$ *and* A *are positive constants fixed large enough;* $d_1 = (d_{1,0}, d_{1,1})$ *and* $d_2 = (d_{2,0}, d_{2,1}, d_{2,2})$ *are parameters which we fine tune in our proof; and* $\chi_0 \in C_0^\infty[0, +\infty)$ *satisfies* $\|\chi_0\|_{L^\infty(\mathbb{R}^N)} \leq 1$*,* $supp \chi_0 \subset [0, 2]$ *and* $\chi_0(x) = 1$ *for all* $|x| \leq 1$ *and* U^* *is given in* (3.86) *which is related to the final profile, given in* (3.9)*.*

Note that when $p \in \mathbb{N}$*, we took in [5] a simpler expression for initial data, not in involving the final profile* U^**, nor the* $(+1)$ *term in* $u_1(0)$*. In particular, adding this* $(+1)$ *term in our idea to ensure that the real part of the solution straps positive.*

Remark 3.7. *We see in* (3.17) *that the equation satisfied by of* u_2 *is almost "linear" in* u_2*. Hence, given an arbitrary* $c_0 \neq 0$*, we can change a little in our proof to construct a solution* $u_{c_0} = u_{1,c_0} + iu_{2,c_0}$ *in* $t \in [0, T)$*, which blows up in finite time* T *only at the origin such that* (3.6) *and* (3.9) *hold and the following holds*

$$\left\| (T-t)^{\frac{1}{p-1}} |\ln(T-t)| u_{2,c_0}(.,t) - c_0 g_0 \left(\frac{.}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{|\ln(T-t)|^{\frac{p_1}{2}}}, \tag{3.13}$$

and

$$u_2^*(x) \sim \frac{2pc_0}{(p-1)^2} \left[\frac{(p-1)^2|x|^2}{8p|\ln|x||} \right]^{-\frac{1}{p-1}} \frac{1}{|\ln|x||}, \text{ as } x \to 0, \tag{3.14}$$

Remark 3.8. *As in the case* $p = 2$ *treated by Nouaili and Zaag [22], and we also mentioned we suspect the behavior in Theorem 3.1 to be unstable. This is due to the fact that the number of parameters in the initial data we consider below in Definition 3.2 (see also Remark 3.6 above) is higher than the dimension of the blowup parameters which is* $N + 1$ *(*N *for the blowup points and* 1 *for the blowup time).*

Besides that, we can use the technique of Merle [14] to construct a solution which blows up at arbitrary given points. More precisely, we have the following Corollary:

Corollary 3.9 (Blowing up at k distinct points)**.** *For any given points, $x_1, ..., x_k$, there exists a solution of* (3.1) *which blows up exactly at $x_1, ..., x_k$. Moreover, the local behavior at each blowup point x_j is also given by* (3.6), (3.7), (3.9), (3.10) *by replacing x by $x - x_j$ and $L^\infty(\mathbb{R}^N)$ by $L^\infty(|x - x_j| \leq \epsilon_0)$, for some $\epsilon_0 > 0$.*

3.1.3 The strategy of the proof

From the singularity of the nonlinear term $F(u) = u^p$ when p is irrational, we can not apply the techniques we used in [5] where $p \in \mathbb{N}$ (also used in [17], [**?**], ...). We need to modify this method. We see that, although our nonlinear term in not continuous in $\mathbb{C}$, it is continuous in the following half plane

$$\{z \in \mathbb{C} \quad |\mathrm{Re}(z) > 0\}.$$

Relying on this property, our problem will be derived by using the techniques which were used in [5] and the fine control of the positivity of the real part. In fact, the control of the positivity follows from ideas given in Merle and Zaag [16] (see also Ghoul, Nguyen and Zaag in [10] where inherited ideas from [16]) which helps us to construct initial data.

In addition to that, we also define a shrinking set $S(t)$ (see in Definition 3.1) which allows a very fine control of the positivity of the real part. More precisely, it is proceed to control our solution on three regions $P_1(t), P_2(t)$ and $P_3(t)$ which are given in subsection 3.3.2 and which we recall here:

- $P_1(t)$, called the blowup region, i.e $|x| \leq K_0\sqrt{(T-t)|\ln(T-t)|}$: We control our solution as a perturbation of the intermediate blowup profiles (for $t \in [0, T)$) f_0 and g_0 given in (3.6) and (3.7), respectively.

- $P_2(t)$, called the intermediate region, i.e $\frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|} \leq |x| \leq \epsilon_0$: In this region, we will control our solution by control the rescaled function U of u (see more (3.74)) to approach $\hat{U}_{K_0}(\tau)$ (see in (3.79)), by using a classical parabolic estimates. Roughly speaking, we control our solution as a perturbation of the final profiles for $t = T$ given in (3.9) and (3.10).

- $P_3(t)$, called the regular region, i.e $|x| \geq \frac{\epsilon_0}{4}$: In this region, we control the solution as a perturbation of initial data ($t = 0$). Indeed, T will be chosen small by the end of the proof.

Fixing some constants involved in the definition $S(t)$, we can prove that our problem will be solved by the control of the solution in $S(t)$. Moreover, we prove via a priori estimates in the different regions P_1, P_2, P_3 that the control is reduced to the control of a finite dimensional components of the solution. Finally, we may apply the techniques in [5] to get our conclusion.

We will organize our paper as follows:

- In Section 3.2: We give a formal approach to explain how the profiles given in Theorem 3.1, appear naturally. Moreover, we also approach our problem through two independent directions: *Inner expansion* and *Outer expansion*, in order to show that our profiles are reasonable.

- In Section 3.3: We give a formulation for our problem (see equation (3.56)) and, step by step we give the rigorous proof for Theorem 3.1, assuming some technical estimates.

- In Section 3.4, we prove the techical estimates assumed in Section 3.3.

3.2 Derivation of the profile (formal approach)

In this section, we would like to give a formal approach to our problem which explains how we derive the profiles for the solution of equation (3.1), given in Theorem (3.1), as well the asymptotic behaviors of the solution. In particular, we would like to mention that the main difference between the case $p \in \mathbb{N}$ and $p \notin \mathbb{N}$ resides in the way we handle the nonlinear term u^p. For that reason, we will give a lot of care for the estimates involving the nonlinear term, and go quickly while giving estimates related to other terms, kindly refering the reader to [5] where the case $p \in \mathbb{N}$ was treated.

3.2.1 Modeling the problem

In this part, we will give definitions and special symbols important for our work and explain how f_0 and g_0 arise as the blowup profiles for the solution of equation (3.1) as stated in (3.6) and (3.7). Our aim in this section is to give solid (though formal) hints for the existence of a solution $u(t) = u_1(t) + iu_2(t)$ to equation (3.1) such that

$$\lim_{t \to T} \|u(.,t)\|_{L^\infty(\mathbb{R}^N)} = +\infty, \tag{3.15}$$

and u obeys the profiles in (3.6) and (3.7), for some $T > 0$. As we have pointed out in the introduction, we are interested in the case where

$$p \notin \mathbb{N},$$

noting that in this case, we already have a difficulty to properly define the nonlinear term u^p as a continuous term. In order to overcome this difficulty, we will restrict ourselves to the case where

$$\operatorname{Re}(u) > 0. \tag{3.16}$$

Our main challenge in this work will be to show that (3.16) is propagated by the flow, at least for the initial data we are suggesting (see Definition 3.2 below). Therefore, under the condition (3.16), by using equation (3.1), we deduce that u_1, u_2 solve:

$$\left\{ \begin{array}{rcl} \partial_t u_1 &=& \Delta u_1 + F_1(u_1,u_2), \\ \partial_t u_2 &=& \Delta u_2 + F_2(u_1,u_2). \end{array} \right. \tag{3.17}$$

where $F_1(0,0) = F_2(0,0) = 0$ and for all $(u_1,u_2) \neq 0$ we have

$$\left\{ \begin{array}{rcl} F_1(u_1,u_2) &=& \operatorname{Re}\left[(u_1+iu_2)^p\right] = |u|^p \cos\left[p \operatorname{Arg}\,(u_1,u_2)\right], \\ F_2(u_1,u_2) &=& \operatorname{Im}\left[(u_1+iu_2)^p\right] = |u|^p \sin\left[p \operatorname{Arg}\,(u_1,u_2)\right], \end{array} \right. \tag{3.18}$$

with $|u| = (u_1^2+u_2^2)^{\frac{1}{2}}$ and $\operatorname{Arg}(u_1,u_2), u_1 > 0$ is defined as follows:

$$\operatorname{Arg}(u_1,u_2) = \arcsin\left[\frac{u_2}{\sqrt{u_1^2+u_2^2}}\right]. \tag{3.19}$$

Note that, in the case where $p \in \mathbb{N}$, we had the following simple expressions for F_1, F_2

$$\left\{\begin{array}{lll} F_1(u_1,u_2) & = & \mathrm{Re}\left[(u_1+iu_2)^p\right] = \sum_{j=0}^{\left[\frac{p}{2}\right]} C_p^{2j}(-1)^j u_1^{p-2j}u_2^{2j}, \\ F_2(u_1,u_2) & = & \mathrm{Im}\left[(u_1+iu_2)^p\right] = \sum_{j=0}^{\left[\frac{p-1}{2}\right]} C_p^{2j+1}(-1)^j u_1^{p-2j-1}u_2^{2j+1}. \end{array}\right. \quad (3.20)$$

Of course, both expressions (3.18) and (3.20) coincide when $p \in \mathbb{N}$. In fact, we will follow our strategy in [5] for $p \in \mathbb{N}$ and focus mainly on how we handle the nonlinear terms, since we have a different expression when $p \notin \mathbb{N}$.

Let us introduce *the similarity-variables* for $u = u_1 + iu_2$ as follows:

$$w(y,s) = (T-t)^{\frac{1}{p-1}}u(x,t), y = \frac{x}{\sqrt{T-t}}, s = -\ln(T-t) \text{ and } w = w_1 + iw_2. \quad (3.21)$$

Then, w_1 and w_2 are real functions. Moreover, by using (3.17), we obtain a system satisfied by (w_1, w_2), for all $y \in \mathbb{R}^N$ and $s \geq -\ln T$ as follows:

$$\left\{\begin{array}{lll} \partial_s w_1 & = & \Delta w_1 - \frac{1}{2}y\cdot\nabla w_1 - \frac{w_1}{p-1} + F_1(w_1,w_2), \\ \partial_s w_2 & = & \Delta w_2 - \frac{1}{2}y\cdot\nabla w_2 - \frac{w_2}{p-1} + F_2(w_1,w_2). \end{array}\right. \quad (3.22)$$

Then note that studying the asymptotic behavior of $u_1 + iu_2$ as $t \to T$ is equivalent to studying the asymptotic behavior of $w_1 + iw_2$ in long time. We are first interested in the set of constant solutions of (3.22), denoted by

$$\mathcal{S} = \{(0,0)\} \cup \left\{\kappa\left(\cos\left(\frac{2k\pi}{p-1}\right), \sin\left(\frac{2k\pi}{p-1}\right)\right) \text{ where } \kappa = (p-1)^{-\frac{1}{p-1}}, \text{ and } k \in \mathbb{N}\right\}.$$

We remark that $\mathcal{S}$ is infinity if p is irrational. However, from the transformation (3.21), we slightly precise our goal in (3.15) by requiring in addition that

$$(w_1, w_2) \to (\kappa, 0) \text{ as } s \to +\infty.$$

Introducing $w_1 = \kappa + \bar{w}_1$, our goal because to get

$$(\bar{w}_1, w_2) \to (0,0) \text{ as } s \to +\infty.$$

From (3.22), we deduce that $\bar{w}_1, w_2$ satisfy the following system

$$\left\{\begin{array}{lll} \partial_s \bar{w}_1 & = & \mathcal{L}\bar{w}_1 + \bar{B}_1(\bar{w}_1, w_2), \\ \partial_s w_2 & = & \mathcal{L}w_2 + \bar{B}_2(\bar{w}_1, w_2), \end{array}\right. \quad (3.23)$$

where

$$\mathcal{L} = \Delta - \frac{1}{2}y\cdot\nabla + Id, \quad (3.24)$$

$$\bar{B}_1(\bar{w}_1, w_2) = F_1(\kappa + \bar{w}_1, w_2) - \kappa^p - \frac{p}{p-1}\bar{w}_1, \quad (3.25)$$

$$\bar{B}_2(\bar{w}_1, w_2) = F_2(\kappa + \bar{w}_1, w_2) - \frac{p}{p-1}w_2. \quad (3.26)$$

It is important to study the linear operator $\mathcal{L}$ and the asymptotic behaviors of $\bar{B}_1$ and $\bar{B}_2$ as $(\bar{w}_1, w_2) \to (0,0)$ which will appear as "quadratic" terms.

- *The properties of $\mathcal{L}$:*

In fact, $\mathcal{L}$ plays an important role in our analysis. It is easy to find an analysis space such that operator is self-adjoint. Indeed, $\mathcal{L}$ is self-adjoint in $\mathcal{D}(\mathcal{L}) \subset L^2_\rho(\mathbb{R}^N)$, where $L^2_\rho(\mathbb{R}^N)$ is the weighted space associated to the weight ρ defined by

$$\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{\frac{N}{2}}} = \prod_{j=1}^{N} \rho(y_j), \text{ with } \rho(\xi) = \frac{e^{-\frac{|\xi|^2}{4}}}{(4\pi)^{\frac{1}{2}}}, \tag{3.27}$$

and the spectrum set of $\mathcal{L}$

$$\text{spec}(\mathcal{L}) = \left\{1 - \frac{m}{2}, m \in \mathbb{N}\right\}. \tag{3.28}$$

Moreover, we can find eigenfunctions which correspond to each eigenvalue $1 - \frac{m}{2}, m \in \mathbb{N}$:

- The one space dimensional case: the eigenfunction corresponding to the eigenvalue $1 - \frac{m}{2}$ is h_m, the rescaled Hermite polynomial given as follows

$$h_m(y) = \sum_{j=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^j m! y^{m-2j}}{j!(m-2j)!}. \tag{3.29}$$

In particular, we have the following orthogonality property:

$$\int_{\mathbb{R}} h_i h_j \rho dy = i! 2^i \delta_{i,j}, \quad \forall (i,j) \in \mathbb{N}^2.$$

- The higher dimensional case: $N \geq 2$, the eigenspace $\mathcal{E}_m$, corresponding to the eigenvalue $1 - \frac{m}{2}$ is defined as follows:

$$\mathcal{E}_m = \left\langle h_\beta(y) = h_{\beta_1}(y_1)...h_{\beta_N}(y_N) \middle| |\beta| = \sum_{i=1}^{N} \beta_i = m \text{ and } \beta = (\beta_1, ..., \beta_N) \in \mathbb{N}^N \right\rangle. \tag{3.30}$$

Accordingly, we can represent an arbitrary function $r \in L^2_\rho(\mathbb{R}^N)$ as follows:

$$r(y) = \sum_{\beta, \beta \in \mathbb{N}^N} r_\beta h_\beta(y),$$

where r_β is the projection of r on h_β for any $\beta \in \mathbb{N}^N$ which is defined as follows:

$$r_\beta = \mathbb{P}_\beta(r) = \int r k_\beta \rho dy, \forall \beta \in \mathbb{N}^n, \tag{3.31}$$

with

$$k_\beta(y) = \frac{h_\beta}{\|h_\beta\|^2_{L^2_\rho}}. \tag{3.32}$$

- *The asymptotic behaviors of $\bar{B}_1(\bar{w}_1, w_2), \bar{B}_2(\bar{w}_1, w_2)$:* The following holds:

$$\bar{B}_1(\bar{w}_1, w_2) = \frac{p}{2\kappa} \bar{w}_1^2 + O(|\bar{w}_1|^3 + |w_2|^2), \tag{3.33}$$

$$\bar{B}_2(\bar{w}_1, w_2) = \frac{p}{\kappa} \bar{w}_1 w_2 + O\left(|\bar{w}_1|^2 |w_2|\right) + O\left(|w_2|^3\right), \tag{3.34}$$

as $(\bar{w}_1, w_2) \to (0,0)$. Note that although we have here the expressions of nonlinear terms F_1 and F_2 which are different from the case $p \in \mathbb{N}$ (see (3.18) and (3.20)), the expressions coincide, since we have $w \sim \kappa = (p-1)^{-\frac{1}{p-1}}$ in all case (see Lemma 3.24 below).

3.2.2 Inner expansion

In this part, we study the asymptotic behaviors of the solution in $L^2_\rho(\mathbb{R}^N)$. Moreover, for simplicity we suppose that $N = 1$, and we recall that we aim at constructing a solution of (3.23) such that $(\bar{w}_1, w_2) \to (0, 0)$. Note first that the spectrum of $\mathcal{L}$ contains two positive eigenvalues $1, \frac{1}{2}$, a neutral eigenvalue 0 and all the other ones are strictly negative. So, in the representation of the solution in $L^2_\rho(\mathbb{R})$, it is reasonable to think that the part corresponding to the negative spectrum is easily controlled. Imposing a symmetry condition on the solution with respect of y, it is reasonable to look for a solution $\bar{w}_1, w_2$ of the form:

$$\begin{aligned}\bar{w}_1 &= \bar{w}_{1,0}h_0 + \bar{w}_{1,2}h_2,\\ w_2 &= w_{2,0}h_0 + w_{2,2}h_2.\end{aligned}$$

From the assumption that $(\bar{w}_1, w_2) \to (0, 0)$, we see that $\bar{w}_{1,0}, \bar{w}_{1,2}, w_{2,0}, w_{2,2} \to 0$ as $s \to +\infty$. We see also that we can understand the asymptotic behaviors of $\bar{w}_1$ and w_2 in $L^2_\rho(\mathbb{R}^N)$ from the study of the asymptotic behaviors of $\bar{w}_{1,0}, \bar{w}_{1,2}, w_{2,0}$ and $w_{2,2}$.

We now project equation (3.23) on h_0 and h_2. Using the asymptotic behaviors of $\bar{B}_1$ and $\bar{B}_2$ in (3.33) and (3.34), we get the following ODEs for $\bar{w}_{1,0}, \bar{w}_{1,2}, w_{2,0}$ and $w_{2,2}$

$$\begin{aligned}\partial_s \bar{w}_{1,0} &= \bar{w}_{1,0} + \frac{p}{2\kappa}\left(\bar{w}_{1,0}^2 + 8\bar{w}_{1,2}^2\right) && (3.35)\\ &+ O(|\bar{w}_{1,0}|^3 + |\bar{w}_{1,2}|^3) + O(|w_{2,0}|^2 + |w_{2,2}|^2),\\ \partial_s \bar{w}_{1,2} &= \frac{p}{\kappa}\left(\bar{w}_{1,0}\bar{w}_{1,2} + 4\bar{w}_{1,2}^2\right) && (3.36)\\ &+ O(|\bar{w}_{1,0}|^3 + |\bar{w}_{1,2}|^3) + O(|w_{2,0}|^2 + |w_{2,2}|^2),\\ \partial_s w_{2,0} &= w_{2,0} + \frac{p}{\kappa}\left[\bar{w}_{1,0}w_{2,0} + 8\bar{w}_{1,2}w_{2,2}\right] && (3.37)\\ &+ +O((|\bar{w}_{1,0}|^2 + |\bar{w}_{1,2}|^2)(|w_{2,0}| + |w_{2,2}|)) + O(|w_{2,0}|^3 + |w_{2,2}|^3),\\ \partial_s w_{2,2} &= \frac{p}{\kappa}\left[\bar{w}_{1,0}w_{2,2} + \bar{w}_{1,2}w_{2,0} + 8\bar{w}_{1,2}w_{2,2}\right] && (3.38)\\ &+ O((|\bar{w}_{1,0}|^2 + |\bar{w}_{1,2}|^2)(|w_{2,0}| + |w_{2,2}|)) + O(|w_{2,0}|^3 + |w_{2,2}|^3).\end{aligned}$$

Assuming that

$$\bar{w}_{1,0}, w_{2,0}, w_{2,2} \ll \bar{w}_{1,2}, \tag{3.39}$$

and

$$\bar{w}_{1,0}, w_{2,0}, w_{2,2} \lesssim \frac{1}{s^2}, \tag{3.40}$$

as $s \to +\infty$.

Similarly in Duong [5] where the author have treated for $p \in \mathbb{N}$, we also obtain the following asymptotic behaviors of $\bar{w}_{1,0}, \bar{w}_{1,2}, w_{2,0}$ and $w_{2,2}$

$$\begin{aligned}\bar{w}_{1,0} &= O\left(\frac{1}{s^2}\right),\\ \bar{w}_{1,2} &= -\frac{\kappa}{4ps} + O\left(\frac{\ln s}{s^2}\right),\\ w_{2,0} &= O\left(\frac{1}{s^3}\right),\\ w_{2,2} &= \frac{c_{2,2}}{s^2} + O\left(\frac{\ln s}{s^3}\right) \quad \text{for some } c_{2,2} \in \mathbb{R},\end{aligned}$$

as $s \to +\infty$ under the assumptions in (3.39) and (3.40).

Thus, we have

$$
\begin{aligned}
w_1 &= \kappa - \frac{\kappa}{4ps}(y^2 - 2) + O\left(\frac{1}{s^2}\right), && (3.41)\\
w_2 &= \frac{c_{2,2}}{s^2}(y^2 - 2) + O\left(\frac{\ln s}{s^3}\right), && (3.42)
\end{aligned}
$$

in $L^2_\rho(\mathbb{R})$ for some $c_{2,2}$ in $\mathbb{R}$. Note that, by using parabolic regularity, we can derive that the asymptotic behaviors (3.41) *and* (3.42) also hold for all $|y| \leq K$, where K is an arbitrary positive constant.

3.2.3 Outer expansion

As for the inner expansion, we here also assume that $N = 1$. We see that asymptotic behaviors (3.41) and (3.42) can not give us a shape, since they hold uniformly on compact sets (where we only see the constant solution $(\kappa, 0)$) and not in larger sets. Fortunately, we observe from (3.41) and (3.42) that the profile may be based on the following variable:

$$
z = \frac{y}{\sqrt{s}}. \tag{3.43}
$$

This motivates us to look for solutions of the form:

$$
\begin{aligned}
w_1(y, s) &= \sum_{j=0}^{\infty} \frac{R_{1,j}(z)}{s^j},\\
w_2(y, s) &= \sum_{j=1}^{\infty} \frac{R_{2,j}(z)}{s^j}.
\end{aligned}
$$

Note that, our purpose is to construct a solution where the real part is positive. So, it is reasonable to assume that $w_1 > 0$ and it follows that $R_{1,0}(z) > 0$ for all $z \in \mathbb{R}$. Besides that, we also assume that $R_{1,j}, R_{2,j}$ are smooth and have bounded derivatives. From the definitions of F_1 and F_2, given in (3.18), we have the following

$$
\begin{aligned}
\left| F_1\left(\sum_{j=0}^{\infty} \frac{R_{1,j}(z)}{s^j}, \sum_{j=1}^{\infty} \frac{R_{2,j}(z)}{s^j}\right) - R_{1,0}^p(z) - \frac{pR_{1,0}^{p-1}(z)R_{1,1}(z)}{s}\right| &\leq \frac{C(z)}{s^2},\\
\left| F_2\left(\sum_{j=0}^{\infty} \frac{R_{1,j}(z)}{s^j}, \sum_{j=1}^{\infty} \frac{R_{2,j}(z)}{s^j}\right) - \frac{pR_{1,0}^{p-1}(z)R_{2,1}(z)}{s} \right.\\
\left. - \frac{1}{s^2}\left(pR_{1,0}^{p-1}(z)R_{2,2} + p(p-1)R_{1,0}^{p-2}(z)R_{1,1}(z)R_{2,1}(z)\right)\right| &\leq \frac{C(z)}{s^3}.
\end{aligned}
$$

Thus, for each $z \in \mathbb{R}$, by using system (3.22), taking $s \to +\infty$, we obtain the following system:

$$
\begin{aligned}
0 &= -\frac{1}{2}R'_{1,0}(z)\cdot z - \frac{R_{1,0}(z)}{p-1} + R^p_{1,0}(z), && (3.44)\\
0 &= -\frac{1}{2}zR'_{1,1}(z) - \frac{R_{1,1}}{p-1}(z) + pR^{p-1}_{1,0}(z)R_{1,1}(z) + R''_{1,0}(z) + \frac{zR'_{1,0}(z)}{2}, && (3.45)\\
0 &= -\frac{1}{2}R'_{2,1}(z)\cdot z - \frac{R_{2,1}}{p-1}(z) + pR^{p-1}_{1,0}(z)R_{2,1}(z), && (3.46)\\
0 &= -\frac{1}{2}R'_{2,2}(z).z - \frac{R_{2,2}(z)}{p-1} + pR^{p-1}_{1,0}(z)R_{2,2}(z) + R''_{2,1}(z) + R_{2,1}(z) && (3.47)\\
&+ \frac{1}{2}R'_{2,1}(z)\cdot z + p(p-1)R^{p-2}_{1,0}(z)R_{1,1}(z)R_{2,1}(z).
\end{aligned}
$$

This system is quite similar to [5] (where $p \in \mathbb{N}$), and we can find the formulas of $R_{1,0}, R_{1,1}, R_{2,1}$ and $R_{2,2}$ as follows:

$$
\begin{aligned}
R_{1,0}(z) &= \left(p-1+b|z|^2\right)^{-\frac{1}{p-1}}, && (3.48)\\
R_{1,1}(z) &= \frac{(p-1)}{2p}(p-1+bz^2)^{-\frac{p}{p-1}} && (3.49)\\
&- \frac{p-1}{4p}z^2\ln(p-1+bz^2)(p-1+bz^2)^{-\frac{p}{p-1}},\\
R_{2,1}(z) &= \frac{z^2}{(p-1+bz^2)^{\frac{p}{p-1}}}, && (3.50)\\
R_{2,2}(z) &= -2(p-1+bz^2)^{-\frac{p}{p-1}} + H_{2,2}(z), && (3.51)
\end{aligned}
$$

where $b = \frac{(p-1)^2}{4p}$ and

$$
\begin{aligned}
H_{2,2}(z) &= C_{2,1}(p)z^2(p-1+bz^2)^{-\frac{2p-1}{p-1}} + C_{2,2}(p)z^2\ln(p-1+bz^2)(p-1+bz^2)^{-\frac{p}{p-1}}\\
&+ C_{2,3}(p)z^2\ln(p-1+bz^2)(p-1+bz^2)^{-\frac{2p-1}{p-1}},
\end{aligned}
$$

for some $C_{2,1}, C_{2,2}$ and $C_{2,3}$ in $\mathbb{R}$.

3.2.4 Matching asymptotic behaviors

By comparing the inner expansion and the outer expansions and fixing several constants, we then have the following profiles for w_1 and w_2

$$
\begin{cases}
w_1(y,s) \sim \Phi_1(y,s),\\
w_2(y,s) \sim \Phi_2(y,s),
\end{cases} \qquad (3.52)
$$

where

$$
\begin{aligned}
\Phi_1(y,s) &= \left(p-1+\frac{(p-1)^2}{4p}\frac{|y|^2}{s}\right)^{-\frac{1}{p-1}} + \frac{N\kappa}{2ps}, && (3.53)\\
\Phi_2(y,s) &= \frac{|y|^2}{s^2}\left(p-1+\frac{(p-1)^2}{4p}\frac{|y|^2}{s}\right)^{-\frac{p}{p-1}} - \frac{2N\kappa}{(p-1)s^2}, && (3.54)
\end{aligned}
$$

for all $(y,s) \in \mathbb{R}^N \times (0,+\infty)$. In the next section, we will give a rigorous proof for the existence of a solution (w_1, w_2) of equation (3.22) satisfying (3.52).

3.3 Existence of a blowup solution in Theorem 3.1

In Section 3.2, we adopted a formal approach in order to justify how profiles f_0 and g_0 arise as blowup profiles for the solution of equation (3.1), given in Theorem 3.1. In this section, we give a rigorous proof to justify the existence of a solution approaching those profiles.

3.3.1 Formulation of the problem

In this subsection, we aim at giving a complete formulation of our problem in order to justify the formal approach which is given in the previous section. We introduce

$$\left\{\begin{array}{lcl} w_1 & = & \Phi_1 + q_1, \\ w_2 & = & \Phi_2 + q_2, \end{array}\right. \tag{3.55}$$

where Φ_1, Φ_2 are defined in (3.53) and (3.54), respectively. Then, by using (3.22), we derive the following system, satisfied by (q_1, q_2)

$$\partial_s \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \mathcal{L} + V & 0 \\ 0 & \mathcal{L} + V \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} V_{1,1} & V_{1,2} \\ V_{2,1} & V_{2,2} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} B_1(q_1,q_2) \\ B_2(q_1,q_2) \end{pmatrix} + \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}, \tag{3.56}$$

where linear operator $\mathcal{L}$ is defined in (3.24) and:

- Potential functions $V, V_{1,1}, V_{1,2}, V_{2,1}$ and $V_{2,2}$ are defined as follows

$$\begin{aligned}
V(y,s) &= p\left(\Phi_1^{p-1} - \frac{1}{p-1}\right), &(3.57)\\
V_{1,1}(y,s) &= \partial_{u_1} F_1(u_1,u_2)|_{(u_1,u_2)=(\Phi_1,\Phi_2)} - p\Phi_1^{p-1}, &(3.58)\\
V_{1,2}(y,s) &= \partial_{u_2} F_1(u_1,u_2)|_{(u_1,u_2)=(\Phi_1,\Phi_2)}, &(3.59)\\
V_{2,1}(y,s) &= \partial_{u_1} F_2(u_1,u_2)|_{(u_1,u_2)=(\Phi_1,\Phi_2)}, &(3.60)\\
V_{2,2}(y,s) &= \partial_{u_2} F_2(u_1,u_2)|_{(u_1,u_2)=(\Phi_1,\Phi_2)} - p\Phi_1^{p-1}. &(3.61)
\end{aligned}$$

- Quadratic terms $B_1(q_1,q_2)$ and $B_2(q_1,q_2)$ are defined as follows:

$$\begin{aligned}
B_1(q_1,q_2) &= F_1(\Phi_1+q_1, \Phi_2+q_2) - F_1(\Phi_1,\Phi_2) - \partial_{u_1} F_1(u_1,u_2)|_{(u_1,u_2)=(\Phi_1,\Phi_2)} q_1 &(3.62)\\
&- \partial_{u_2} F_1(u_1,u_2)|_{(u_1,u_2)=(\Phi_1,\Phi_2)} q_2, \\
B_2(q_1,q_2) &= F_2(\Phi_1+q_1, \Phi_2+q_2) - F_2(\Phi_1,\Phi_2) - \partial_{u_1} F_2(u_1,u_2)|_{(u_1,u_2)=(\Phi_1,\Phi_2)} q_1 \\
&- \partial_{u_2} F_2(u_1,u_2)|_{(u_1,u_2)=(\Phi_1,\Phi_2)} q_2. &(3.63)
\end{aligned}$$

- Rest terms $R_1(y,s)$ and $R_2(y,s)$ are defined as follows:

$$\begin{aligned}
R_1(y,s) &= \Delta\Phi_1 - \frac{1}{2} y\cdot\nabla\Phi_1 - \frac{\Phi_1}{p-1} + F_1(\Phi_1,\Phi_2) - \partial_s\Phi_1, &(3.64)\\
R_2(y,s) &= \Delta\Phi_2 - \frac{1}{2} y\cdot\nabla\Phi_2 - \frac{\Phi_2}{p-1} + F_2(\Phi_1,\Phi_2) - \partial_s\Phi_2, &(3.65)
\end{aligned}$$

where F_1, F_2 are defined in (3.18).

By the linearization around Φ_1, Φ_2, our problem is reduced to constructing a solution (q_1, q_2) of system (3.56), satisfying

$$\|q_1(., s)\|_{L^\infty(\mathbb{R}^N)} + \|q_2(., s)\|_{L^\infty(\mathbb{R}^N)} \to 0 \text{ as } s \to +\infty.$$

Looking at system (3.56), we already know some of the main properties of $\mathcal{L}$ (see page 121). As for potentials $V_{j,k}$ where $j, k \in \{1, 2\}$, they admit the following asymptotic behaviors:

$$\begin{aligned} \|V_{1,1}(., s)\|_{L^\infty(\mathbb{R}^N)} + \|V_{2,2}(., s)\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C}{s^2}, \\ \|V_{1,2}(., s)\|_{L^\infty(\mathbb{R}^N)} + \|V_{2,1}(., s)\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C}{s}, \forall s \geq 1, \end{aligned}$$

(see Lemma 3.25 below).

Regarding B_1 and B_2 which are considered as “quadratic” terms, we have in fact the following estimates

$$\begin{aligned} \|B_1(q_1, q_2)\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{CA^4}{s^{\frac{\min(2,p)}{2}}}, \\ \|B_2(q_1, q_2)\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{CA^2}{s^{1+\min\left(\frac{p-1}{4}, \frac{1}{2}\right)}}, \end{aligned}$$

provided that q_1 and q_2 are small in some senses (see Lemma 3.26 below).

In addition to that, we also mention R_1 and R_2 which are considered as rest terms, satisfying in fact the following asymptotic behaviors

$$\begin{aligned} \|R_1(., s)\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C}{s}, \\ \|R_2(., s)\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C}{s^2}, \end{aligned}$$

(see Lemma 3.27 below).

As a matter of fact, the dynamic of equation (3.56) will mainly depends on the main linear operator

$$\begin{pmatrix} \mathcal{L} + V & 0 \\ 0 & \mathcal{L} + V \end{pmatrix},$$

and the effects of the other terms will be less important except on the zero mode of this equation. For that reason, we need to understand the dynamics of $\mathcal{L} + V$. Since the spectral properties of $\mathcal{L}$ were already introduced in Section 3.2.1, we will focus here on the effect of V.

i) Effect of V inside the blowup region $\{|y| \leq K_0\sqrt{s}\}$ with $K_0 > 0$: It satisfies the following estimate:

$$V \to 0 \text{ in } L^2_\rho(|y| \leq K_0\sqrt{s}) \text{ as } s \to +\infty,$$

which means that the effect of V will be negligible with respect of the effect of $\mathcal{L}$, except perhaps on the null mode of $\mathcal{L}$ (see item (*ii*) of Proposition 3.18 below).

ii) Effect of V outside the blowup region: For each $\epsilon > 0$, there exist $K_\epsilon > 0$ and $s_\epsilon > 0$ such that

$$\sup_{\frac{y}{\sqrt{s}} \geq K_\epsilon, s \geq s_\epsilon} \left| V(y, s) - \left(-\frac{p}{p-1} \right) \right| \leq \epsilon.$$

Since 1 is the biggest eigenvalue of $\mathcal{L}$ (see (3.28)), the operator $\mathcal{L}+V$ behaves as one with with a fully negative spectrum outside blowup region $\{|y| \geq K_\epsilon\sqrt{s}\}$, which makes the control of the solution in this region easy.

Since the asymptotic behavior of potential V inside and outside the blowup region is different, we will consider the dynamics of the solution for $|y| \leq 2K_0\sqrt{s}$ and for $|y| \geq K_0\sqrt{s}$ separately for some K_0 to be fixed large. For that purpose, we introduce the following cut-off function

$$\chi(y,s) = \chi_0\left(\frac{|y|}{K_0\sqrt{s}}\right), \tag{3.66}$$

where χ_0 is defined as a cut-off function

$$\chi_0 \in C_0^\infty[0,+\infty), \chi_0(x) = \begin{cases} 1 & \text{for } x \leq 1, \\ 0 & \text{for } x \geq 2, \end{cases} \quad \text{and } \|\chi_0\|_{L^\infty(\mathbb{R}^N)} \leq 1. \tag{3.67}$$

Hence, it is reasonable to consider separately the solution in the blowup region $\{|y| \leq 2K_0\sqrt{s}\}$ and in the regular region $\{|y| \geq K_0\sqrt{s}\}$. More precisely, let us define the following notation for all functions r in $L^\infty(\mathbb{R}^N)$ as follows

$$r = r_b + r_e \text{ with } r_b = \chi r \text{ and } r_e = (1-\chi)r. \tag{3.68}$$

Note in particular that $\operatorname{supp}(r_b) \subset \mathbb{B}(0, 2K_0\sqrt{s})$ and $\operatorname{supp}(r_e) \subset \mathbb{R}^N \setminus \mathbb{B}(0, K_0\sqrt{s})$. Besides that, we also expand r_b in $L^2_\rho(\mathbb{R}^N)$ according to the spectrum of $\mathcal{L}$ (see Section 3.2.1 above):

$$r_b(y) = r_0 + r_1\cdot y + \frac{1}{2}y^{\mathcal{T}}\cdot r_2 \cdot y - \operatorname{Tr}(r_2) + r_-(y), \tag{3.69}$$

where r_0 is a scalar, r_1 is a vector in $\mathbb{R}^N$ and r_2 is a $N\times N$ matrix defined by

$$\begin{aligned} r_0 &= \int_{\mathbb{R}^N} r_b \rho(y)dy, \\ r_1 &= \int_{\mathbb{R}^N} r_b \frac{y}{2}\rho(y)dy, \\ r_2 &= \left(\int_{\mathbb{R}^N} r_b\left(\frac{1}{4}y_jy_k - \frac{1}{2}\delta_{j,k}\right)\rho(y)dy\right)_{1\leq j,k\leq N}, \end{aligned}$$

with $\operatorname{Tr}(r_2)$ being the trace of matrix r_2. The reader should keep in mind that r_0, r_1, r_2 are only the coordinates of r_b, not for r. Note that r_m is the projection of r_b on the eigenspace of $\mathcal{L}$ corresponding to the eigenvalue $\lambda = 1 - \frac{m}{2}$. Accordingly, r_- is the projection of r_b on the negative part of the spectrum of $\mathcal{L}$. As a consequence of (3.68) and (3.69), we see that every $r \in L^\infty(\mathbb{R}^N)$ can be decomposed into 5 components as follows:

$$r = r_b + r_e = r_0 + r_1\cdot y + \frac{1}{2}y^{\mathcal{T}}\cdot r_2\cdot y - \operatorname{Tr}(r_2) + r_- + r_e. \tag{3.70}$$

3.3.2 The shrinking set

According to (3.21) and (3.55), our goal is to construct a solution (q_1, q_2) of system (3.56) such that they satisfy the following estimates:

$$\|q_1(.,s)\|_{L^\infty(\mathbb{R}^N)} + \|q_2(.,s)\|_{L^\infty(\mathbb{R}^N)} \to 0 \text{ as } s \to +\infty. \tag{3.71}$$

Here, we aim at constructing a shrinking set to 0. Then, the control of $(q_1, q_2) \to 0$, will be a consequence of the control of (q_1, q_2) in this shrinking set. In addition to that, we have to control the solution q_1 so that

$$w_1 = q_1 + \Phi_1 > 0, \tag{3.72}$$

(this is equivalent to have $u_1 > 0$) and it is one of the main difficults in our analysis. As a matter of fact, the shrinking sets which were constructed in [17] by Merle and Zaag or even in [5], are not sharp enough to ensure (3.72). In other words, our set has to shrink to 0 as $s \to +\infty$ and ensure that the real part of the solution to (3.22) is always positive. In fact, the positivity is the first thing to be solved. For the control of the positivity of the real part, we rely on the ideas, given by Merle and Zaag [16] for the control of the solution of the following equation:

$$\partial_t u = \Delta u - \eta \frac{|\nabla u|^2}{u} + |u|^{p-1} u, u \in \mathbb{R}. \tag{3.73}$$

In [16], the authors needed a sharp control of u and $|\nabla u|$ near zero, in order to bound the term $\frac{|\nabla u|^2}{u}$. Here, we will use their ideas in order to control u_1 near zero and ensure its positivity. As in [16], we will control the solution differently in 3 overlapping regions defined as follows:

For $K_0 > 0, \alpha_0 > 0, \epsilon_0 > 0, t \in [0, T)$ and $s = -\ln(T-t)$, we introduce a cover of $\mathbb{R}^N$ as follows

$$\mathbb{R}^N \subset P_1(t) \cup P_2(t) \cup P_3(t),$$

where

$$\begin{aligned}
P_1(t) &= \{x| \quad |x| \le K_0 \sqrt{(T-t)|\ln(T-t)|}\} = \{x| \quad |y| \le K_0 \sqrt{s}\} = \{x| \quad |z| \le K_0\}, \\
P_2(t) &= \left\{x| \quad \frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|} \le |x| \le \epsilon_0 \right\} = \left\{x| \quad \frac{K_0}{4}\sqrt{s} \le |y| \le \epsilon_0 e^{\frac{s}{2}} \right\} \\
&= \left\{x| \quad \frac{K_0}{4} \le |z| \le \frac{\epsilon_0}{\sqrt{s}} e^{\frac{s}{2}} \right\}, \\
P_3(t) &= \left\{x| \quad |x| \ge \frac{\epsilon_0}{4}\right\} = \left\{x| \quad |y| \ge \frac{\epsilon_0 e^{\frac{s}{2}}}{4}\right\} = \left\{x| \quad |z| \ge \frac{\epsilon_0}{4\sqrt{s}} e^{\frac{s}{2}}\right\},
\end{aligned}$$

with

$$y = \frac{x}{\sqrt{T-t}} \text{ and } z = \frac{y}{\sqrt{s}} = \frac{x}{\sqrt{(T-t)|\ln(T-t)|}}.$$

In the following, let us explain how we derive the positivity condition from the various estimate we impose on the solution in the 3 regions. Then

a) In $P_1(t)$, the *blowup region*: In this region, we control the positivity of u_1 by controlling the positivity of w_1 (see the similarity variables given in (3.21)). More precisely, as we mentioned in Subsection 3.1.3, w will be controlled as a pertubation of the profiles Φ_1, Φ_2 ((3.53) and (3.54)). By using the positivity of Φ_1 and a good estimate of the distance of w_1 to these profiles, we may deduce the positivity of w_1, which leads to the positivity of u_1.

b) In $P_2(t)$, the *intermediate region*: In this region, we control u via a rescaled function U of u as follows:

$$U(x,\xi,\tau) = (T - t(x))^{-\frac{1}{p-1}} u(x + \xi\sqrt{T - t(x)}, t(x) + \tau(T - t(x))), \tag{3.74}$$

where $t(x)$ is uniquely defined for $|x|$ small enough by

$$|x| = \frac{K_0}{4}\sqrt{(T - t(x))\,|\ln(T - t(x))|}. \tag{3.75}$$

We also introduce

$$\theta(x) = T - t(x). \tag{3.76}$$

We see that, on the domain $(\xi,\tau) \in \mathbb{R}^N \times \left[-\frac{t(x)}{T-t(x)}, 1\right)$, U satisfies the following equation:

$$\partial_\tau U = \Delta_\xi U + U^p. \tag{3.77}$$

By using classical parabolic estimates on U, we can prove the following the rescaled U at time $\tau(x,t)$, has a behavior similar to $\hat{U}_{K_0}(\tau(x,t))$, for all $|\xi| \leq \alpha_0\sqrt{|\ln(T - t(x)|}$ where

$$\tau(x,t) = \frac{t - t(x)}{T - t(x)},$$

and $\hat{U}_{K_0}(\tau)$ is unique solution of the following ODE

$$\begin{cases} \partial_\tau \hat{U}_{K_0} &= \hat{U}^p_{K_0}(\tau), \\ \hat{U}_{K_0}(0) &= \left(p - 1 + \frac{(p-1)^2 K_0^2}{64p}\right)^{-\frac{1}{p-1}}. \end{cases} \tag{3.78}$$

In particular, we can solve (3.78) with an explicit solution:

$$\hat{U}_{K_0}(\tau) = \left((p-1)(1-\tau) + \frac{(p-1)^2 K_0^2}{64p}\right)^{-\frac{1}{p-1}}, \forall \tau \in [0,1). \tag{3.79}$$

Then, by using the positivity of $\hat{U}_{K_0}$, we derive that $u_1 > 0$, in this region.

c) In $P_3(t)$, the *regular region*: We control the solution in this region as a perturbation of initial data, thanks to the well-posedness property of the Cauchy problem for equation (3.1), to derive that our solution is close to initial data, (in fact, T will be taken small enough). Therefore, if initial data is strictly larger than some constant, we will derive the positivity of u_1.

The above strategy makes the real part of our solution becomes positive. Therefore, it remains to control the solution in order to get

$$\|q_1(.,s)\|_{L^\infty(\mathbb{R}^N)} + \|q_2(.,s)\|_{L^\infty(\mathbb{R}^N)} \to +\infty,$$

(see (3.55)). This part is in fact quite similar to the integer case, done in [5].

From the above arguments, we give in the following our definition of the shrinking set.

Definition 3.1 (A shrinking set to 0). *For all* $T > 0, K_0 > 0, \alpha_0 > 0, \epsilon_0 > 0, A > 0, \delta_0 > 0, \eta_0 > 0, p_1 \in \left(0, \min\left(\frac{p-1}{4}, \frac{1}{2}\right)\right)$ *and* $t \in [0, T)$, *we define the set* $S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t) \subset C([0, t], L^\infty(\mathbb{R}^N))$ *(or* $S(t)$ *for short) as follows:* $u = u_1 + iu_2 \in S(t)$ *if the following condition hold:*

(*i*) *Control in the blowup region* $P_1(t)$: *We have* $(q_1, q_2)(s) \in V_{p_1,K_0,A}(s)$ *where* $s = -\ln(T-t)$, (q_1, q_2) *is defined as in* (3.55) *and* $V_{p_1,K_0,A}(s) = V_A(s) \in (L^\infty(\mathbb{R}^N))^2$ *is the set of all function* $(q_1, q_2) \in (L^\infty(\mathbb{R}^N))^2$ *such that the following holds:*

$$|q_{1,0}(s)| \le \frac{A}{s^2} \quad \text{and} \quad |q_{2,0}(s)| \le \frac{A^2}{s^{p_1+2}},$$
$$|q_{1,j}(s)| \le \frac{A}{s^2} \quad \text{and} \quad |q_{2,j}(s)| \le \frac{A^2}{s^{p_1+2}}, \forall 1 \le j \le N,$$
$$|q_{1,j,k}(s)| \le \frac{A^2 \ln s}{s^2} \quad \text{and} \quad |q_{2,j,k}(s)| \le \frac{A^5 \ln s}{s^{p_1+2}}, \forall 1 \le j, k \le N,$$
$$\left\|\frac{q_{1,-}(y,s)}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} \le \frac{A}{s^2} \quad \text{and} \quad \left\|\frac{q_{2,-}(y,s)}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} \le \frac{A^2}{s^{\frac{p_1+5}{2}}},$$
$$\|q_{1,e}(.,s)\|_{L^\infty(\mathbb{R}^N)} \le \frac{A^2}{\sqrt{s}} \quad \text{and} \quad \|q_{2,e}(.,s)\|_{L^\infty(\mathbb{R}^N)} \le \frac{A^3}{s^{\frac{p_1+2}{2}}},$$

where the coordinates of q_1 *and* q_2 *are introduced in* (3.70) *with* $r = q_1$ *or* $r = q_2$.

(*ii*) *Control in the intermediate region* $P_2(t)$: *For all* $|x| \in \left[\frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|}, \epsilon_0\right]$, $\tau(x,t) = \frac{t-t(x)}{T-t(x)}$ *and* $|\xi| \le \alpha_0\sqrt{|\ln(T-t(x))|}$, *we have*

$$\left|U(x,\xi,\tau(x,t)) - \hat{U}_{K_0}(\tau(x,t))\right| \le \delta_0,$$

where $\hat{U}_{K_0}$ *defined in* (3.79).

iii *Control in the regular region* $P_3(t)$: *For all* $|x| \ge \frac{\epsilon_0}{4}$,

$$|u(x,t) - u(x,0)| \le \eta_0, \forall i = 0, 1.$$

Finally, we also define the set $S^*(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0) \subset C([0,T), L^\infty(\mathbb{R}^N))$ *as the set of all* $u \in C([0,T), L^\infty(\mathbb{R}^N)$ *such that*

$$u \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t), \forall t \in [0,T).$$

The following lemma, we show the estimates of the fuction being in $V_A(s)$ and this lemma is given in [5]:

Lemme 3.10. *For all* $A \ge 1, s \ge 1$, *if we have* $(q_1, q_2) \in V_A(s)$, *then the following estimates hold:*

(*i*) *We have*

$$\|q_1\|_{L^\infty(\mathbb{R}^N)} \le \frac{CA^2}{\sqrt{s}} \text{ and } \|q_2\|_{L^\infty(\mathbb{R}^N)} \le \frac{CA^3}{s^{\frac{p_1+2}{2}}}.$$

(ii) For all $y \in \mathbb{R}^N$, we have

$$|q_{1,b}(y)| \leq \frac{CA^2 \ln s}{s^2}(1+|y|^3), \quad |q_{1,e}(y)| \leq \frac{CA^2}{s^2}(1+|y|^3) \text{ and } |q_1| \leq \frac{CA^2 \ln s}{s^2}(1+|y|^3),$$

and

$$|q_{2,b}(y)| \leq \frac{CA}{s^{\frac{p_1+5}{2}}}(1+|y|^3), \quad |q_{2,e}(y)| \leq \frac{CA^3}{s^{\frac{p_1+5}{2}}}(1+|y|^3) \text{ and } |q_2| \leq \frac{CA^3}{s^{\frac{p_1+5}{2}}}(1+|y|^3).$$

and

where C will henceforth be an constant which depends only on K_0.

Proof. See Lemma 3.2, given in [5]. □

As matter of fact, if $u \in S_(t)$ then, from item (i) of Lemma 3.10, the similarity variables (3.21) and (3.55), we derive the following

$$\left\| (T-t)^{\frac{1}{p-1}} u(.,t) - f_0\left(\frac{.}{\sqrt{(T-t)|\ln(T-t)|}}\right)\right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{CA^2}{1+\sqrt{|\ln(T-t)|}}, \quad (3.80)$$

and

$$\left\| (T-t)^{\frac{1}{p-1}} |\ln(T-t)| u_2(.,t) - g_0\left(\frac{.}{\sqrt{(T-t)|\ln(T-t)|}}\right)\right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{CA^3}{1+|\ln(T-t)|^{\frac{p_1}{2}}} \quad (3.81)$$

We see in the definition of $S(t)$ that there are many parameters, so the dependence of the constants on them is very important in our analysis. We would like to mention that, we use the notation C for these constants which depend at most on K_0. Otherwise, if the constant depends on $K_0, A_1, A_2, ...$ we will write $C(A_1, A_2, ...)$.

We now prove in the following lemma the positivity of $\text{Re}(u)$ at time t if u belongs to $S(t)$ (this is a crucial estimate in our argument):

Lemme 3.11 (The positivity of the real part of functions trapped in $S(t)$)**.** *For all $K_0, A \geq 1$ $\alpha_0 > 0, \delta_0 < \frac{\hat{U}(0)}{2}, \eta_0 < \frac{1}{2}$, there exists $\epsilon_1(K_0) > 0$ such that for all $\epsilon_0 \leq \epsilon_1$ there exists $T_1(A, K_0, \epsilon_0)$ such that for all $T \leq T_1$ the following holds: if $u \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t)$ for all $t \in [0, t_1]$ for some $t_1 \in [0, T)$, and $Re(u(0)) \geq 1$ for all $|x| \geq \frac{\epsilon_0}{4}$, then*

$$Re(u(x,t)) \geq \frac{1}{2}, \forall (x,t) \in \mathbb{R}^N \times \in [0, t_1].$$

Proof. We write that $u = u_1 + iu_2$, with $\text{Re}(u) = u_1$. Then, we estimate u_1 on the 3 regions $P_1(t), P_2(t)$ and $P_3(t)$.

\+ *The estimate in $P_1(t)$:* We use the fact that $(q_1, q_2) \in V_A(s)$ together with item (i) in Lemma 3.10, and the definition (3.55) of q_1 and the definition of Φ_1 given in (3.53), to derive the following: for all $|y| \leq K_0\sqrt{s}$,

$$\left| w_1(y,s) - f_0\left(\frac{y}{\sqrt{s}}\right)\right| \leq \frac{CA^2}{\sqrt{s}}.$$

Using the definition (3.53) of Φ_1, we write for all $|y| \leq 2K_0\sqrt{s}$

$$\begin{aligned} w_1(y,s) &\geq f_0\left(\frac{y}{\sqrt{s}}\right) - \frac{CA^2}{\sqrt{s}} \\ &\geq \left(p-1+\frac{(p-1)^2}{4p}K_0^2\right)^{-\frac{1}{p-1}} - \frac{CA^2}{\sqrt{s}}, \end{aligned}$$

By definition (3.21) of the similarity variables, we implies that

$$(T-t)^{\frac{1}{p-1}}u_1(x,t) \geq \left(p-1+\frac{(p-1)^2}{4p}K_0^2\right)^{-\frac{1}{p-1}} - \frac{CA^2}{\sqrt{|\ln(T-t)|}},$$

for all $|x| \leq K_0\sqrt{(T-t)|\ln(T-t)|}$.
Therefore,

$$u_1(x,t) \geq (T-t)^{-\frac{1}{p-1}}\left[\left(p-1+\frac{(p-1)^2}{4p}K_0^2\right)^{-\frac{1}{p-1}} - \frac{CA^2}{\sqrt{|\ln(T-t)|}}\right] \geq \frac{1}{2},$$

provided that $T \leq T_{1,1}(K_0, A)$.

+ *The estimate in* $P_2(t)$*:* Since we have $u \in S(t)$, using item (ii) in the Definition 3.1, we derive that: for all $x \in \left[\frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|}, \epsilon_0\right]$

$$\left|U(x,0,\tau(x,t)) - \hat{U}_{K_0}(\tau(x,t))\right| \leq \delta_0,$$

where $\tau(x,t) = \frac{t-t(x)}{T-t(x)}$. In particular, by using the definition of $t(x)$ given in (3.75) and the fact that

$$|x| \geq \frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|},$$

we have $\tau(x,t) \in [0,1)$. Therefore,

$$\begin{aligned} U_1(x,0,\tau(x,t)) &\geq \hat{U}_{K_0}(\tau(x,t)) - \delta_0 \\ &\geq \hat{U}_{K_0}(0) - \delta_0 \\ &\geq \frac{1}{2}\hat{U}_{K_0}(0) = \frac{1}{2}\left(p-1+\frac{(p-1)^2}{4p}\frac{K_0^2}{16}\right)^{-\frac{1}{p-1}}, \end{aligned}$$

provided that $\delta_0 \leq \frac{1}{2}\hat{U}_{K_0}(0)$. By definition (3.74) of U, this implies that

$$(T-t(x))^{\frac{1}{p-1}}u_1(x,t) = U_1(x,0,\tau(x,t)) \geq \frac{1}{2}\left(p-1+\frac{(p-1)^2}{4p}\frac{K_0^2}{16}\right)^{-\frac{1}{p-1}}.$$

Using the definition of $t(x)$ in (3.75) we write

$$T-t(x) \sim \frac{8}{K_0^2}\frac{|x|^2}{|\ln|x||}, \text{ as } |x| \to 0.$$

Therefore, there exists $\epsilon_{1,1}(K_0) > 0$ such that for all $\epsilon_0 \leq \epsilon_{1,1}$, and for all $|x| \leq \epsilon_0$, we have

$$(T-t(x))^{-\frac{1}{p-1}}\frac{1}{2}\left(p-1+\frac{(p-1)^2}{4p}\frac{K_0^2}{16}\right)^{-\frac{1}{p-1}} \geq \frac{1}{2}.$$

Then, we conclude that for all $|x| \in \left[\frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|}, \epsilon_0\right]$, we have

$$u_1(x,t) \geq \frac{1}{2},$$

provided that $T \leq T_{2,1}(\epsilon_0, K_0)$.

+ *The estimate in $P_3(t)$:* It is very easy to control our solution in this region. Indeed, item (iii) of Definition 3.1, we have for all $|x| \geq \frac{\epsilon_0}{4}$

$$u_1(x,t) \geq \text{ Re}(u)(x,0) - \eta_0 \geq 1 - \frac{1}{2} = \frac{1}{2},$$

provided that $\eta_0 \leq \frac{1}{2}$. This concludes the proof of Lemma 3.11. □

Thanks to Lemma 3.11, we can handle the singularity of the nonlinear term u^p when our solution is in $S(T, A, \alpha_0, \epsilon_0, A, \delta_0, \eta_0)$. In addition to that, from item (i) of Lemma 3.11, (3.80) and (3.81) our problem is reduced to finding parameters $T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0$, and constructing initial data $u(0) \in L^\infty(\mathbb{R}^N)$ such that the solution u of equation (3.1), exists on $[0,T)$ and satisfies

$$u \in S^*(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0). \tag{3.82}$$

3.3.3 Preparing initial data and the existence of a solution trapped in S(t)

In this subsection, we would like to define initial data $u(0)$, which depend on some parameters to be fine-tuned in order to get a good solution. The following is our definition:

Definition 3.2 (Preparing of initial data). *For each $A \geq 1, T > 0$, $d_1 = (d_{1,0}, d_{1,1}) \in \mathbb{R}^1 \times \mathbb{R}^N$, and $d_2 = (d_{2,0}, d_{2,1}, d_{2,1}) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2}$, we introduce the following functions defined at $s_0 = -\ln T$:*

$$\begin{aligned}
\phi_{1,K_0,A,d_1}(y, s_0) &= \frac{A}{s_0^2}(d_{1,0} + d_{1,1}\cdot y)\chi_0\left(\frac{16|y|}{K_0\sqrt{s_0}}\right),\\
\phi_{2,K_0,A,d_2}(y, s_0) &= \left(\frac{A^2}{s_0^{p_1+2}}(d_{2,0} + d_{2,1}\cdot y) + \frac{A^5 \ln s_0}{s_0^{p_1+2}}\left(\frac{1}{2}y^{\mathcal{T}}\cdot d_{2,2}\cdot y - \text{Tr }(d_{2,2})\right)\right)\chi_0\left(\frac{16|y|}{K_0\sqrt{s_0}}\right).
\end{aligned}$$

We also define initial data $u_{K_0,A,d_1,d_2}(0) = u_{1,K_0,A,d_1}(0) + iu_{2,K_0,A,d_2}(0)$ for equation (3.1) as follows:

$$\begin{aligned}
u_{1,K_0,A,d_1}(x,0) &= T^{-\frac{1}{p-1}}\left\{\phi_{1,K_0,A,d_1}\left(\frac{x}{\sqrt{T}}, -\ln T\right) + \Phi_1\left(\frac{x}{\sqrt{T}}, -\ln T\right)\right\}\chi_1(x) \qquad (3.83)\\
&+ U^*(x)(1-\chi_1(x)) + 1,\\
u_{2,K_0,A,d_2}(x,0) &= T^{-\frac{1}{p-1}}\left\{\phi_{2,K_0,A,d_2}\left(\frac{x}{\sqrt{T}}, -\ln T\right) + \Phi_2\left(\frac{x}{\sqrt{T}}, -\ln T\right)\right\}\chi_1(x), \qquad (3.84)
\end{aligned}$$

where Φ_1 and Φ_2 are defined in (3.53), (3.54) *and $\chi_1(x)$ is defined as follows*

$$\chi_1(x) = \chi_0\left(\frac{|x|}{\sqrt{T}|\ln T|}\right), \tag{3.85}$$

with χ_0 *defined in* (3.67)*, and* $U^* \in C^1(\mathbb{R}^N \backslash \{0\}, \mathbb{R})$ *is defined for all* $x \in \mathbb{R}^N, x \neq 0$

$$U^*(x) = \begin{cases} \left[\frac{(p-1)^2|x|^2}{8p|\ln|x||}\right]^{-\frac{1}{p-1}} & \text{if } |x| \leq C^*, \\ \frac{1}{1+|x|^2} & \text{if } |x| \geq 1, \\ U^*(x) > 0 & \text{for all } x \neq 0, \end{cases} \tag{3.86}$$

where C^* *is a fixed constant strictly less than* 1 *enough, and* U^* *satisfies the following property: for each* $\epsilon_0 \leq \frac{C^*}{2}$ *we have*

$$U^*(x) \leq U^*(\epsilon_0), \text{ for all } |x| \geq \epsilon_0. \tag{3.87}$$

Remark 3.12. *Roughly speaking, the critical data we done here are superposition of two items:*

- $T^{-\frac{1}{p-1}} \{\phi_1 + \Phi_1\}$ *in* $P_1(0)$
- U^* *in* $P_2(0)$.

The first form is well-known in previous construction problems. As for the second, we borrowed it from Merle and Zaag in [16]. Note that U^* *is the candidate for the final profile of the real part, as we can see from own main result in Theorem 3.1. More crucially, we draw your attention to the fact that in comparision with [16], we add here* $+1$ *to the expression in* (3.83)*, and this term will allow us to have the initial condition*

$$Re(u(0)) \geq 1,$$

which is essential to make the nonlinear term u^p *well-defined, and the Cauchy problem solvable (see Appendix 3.5). This is an important idea of ours.*

From the above definition, we show in the following lemma some rough properties of the initial data.

Lemme 3.13. *For all* $K_0 \geq 1, A \geq 1$, $|d_1| \leq 2, |d_2| \leq 2$, *and for all* $\epsilon_0 \leq \frac{C^*}{2}$ *(where* C^* *is introduced in* (3.87)*), there exists* $T_2(\epsilon_0, K_0, A) > 0$ *such that for all* $T \leq T_2$*, if* $u(0) = u_{K_0,A,d_1,d_2}(0)$ *is defined as in Definition 3.2, then the following holds:*

(*i*) *The initial data belongs to* $L^\infty(\mathbb{R}^N)$ *and satisfies the following*

$$\|u(.,0)\|_{L^\infty(|x| \geq \epsilon_0)} \leq 1 + \left(\frac{(p-1)^2|\epsilon_0|^2}{8p|\ln \epsilon_0|}\right)^{-\frac{1}{p-1}}.$$

(*ii*) *The real part of the initial data, Re*$(u(0))$ *is positive. In particular,*

$$Re(u(x,0)) \geq 1, \forall x \in \mathbb{R}^N.$$

Proof.

(i) It is obvious to see that the initial data belongs to $L^\infty(\mathbb{R}^N)$ with the assumptions in this Lemma. It remains to prove the estimate in item (i). We now take $\epsilon_0 \le \frac{C^*}{2}$, and we use definition of χ_1 in (3.85) to deduce that $\mathrm{supp}(\chi_1) \subset \{|x| \le 2\sqrt{T}|\ln T|\}$. Moreover, we have

$$\sqrt{T}|\ln T| \to 0 \text{ as } T \to 0.$$

Then, we have

$$\sqrt{T}|\ln T| \le \frac{\epsilon_0}{4},$$

provided that $T \le T_{2,1}(\epsilon_0)$. Hence,

$$\mathrm{supp}(\chi_1) \subset \{|x| \le \frac{\epsilon_0}{2}\},$$

Hence, it follows the definition of $u(0)$ that: for all $|x| \ge \epsilon_0$, we have

$$u(x,0) = U^*(x) + 1,$$

Using (3.87), our result follows.

(ii) We see in the definition of $u(0)$ that we have $\mathrm{supp}(\phi_{1,K_0,A,d_1}) \subset \{|y| \le \frac{K_0}{8}\sqrt{s_0}\}$ and we have the following

$$\|\phi_{1,K_0,A,d_1}\left(\frac{x}{\sqrt{T}}, -\ln T\right)\|_{L^\infty(\mathbb{R}^N)} \le \frac{CA}{|\ln T|^{\frac{3}{2}}}.$$

In addition to that, in the region $\{|x| \le \frac{K_0}{8}\sqrt{T|\ln T|}\}$, the function $\Phi_1\left(\frac{x}{\sqrt{T}}, -\ln T\right)$ is bounded from below by a positive constant which depends only on K_0. Therefore, there exists $T_{2,2}(A, K_0) > 0$ such that for all $T \le T_{2,2}$ for all $|x| \le \frac{K_0}{8}\sqrt{T|\ln T|}$ we have

$$\phi_{1,K_0,A,d_1}\left(\frac{x}{\sqrt{T}}, -\ln T\right) + \Phi_1\left(\frac{x}{\sqrt{T}}, -\ln T\right) > 0.$$

Therefore: for all $|x| \le \frac{K_0}{8}\sqrt{T|\ln T|}$, we have

$$\mathrm{Re}(u(x,0)) \ge 1.$$

Now, if $|x| \ge \frac{K_0}{8}\sqrt{T|\ln T|}$, then we have $\phi_{1,K_0,A,d_1}(y,s_0) = 0$. Since $\Phi_1(y,s_0) > 0$ from (3.53) and $U^*(x) > 0$ from (3.87), we directly see from the definition (3.83) for $\mathrm{Re}(u(0))$ that

$$\mathrm{Re}(u(x,0)) \ge 1.$$

This concludes the proof of Lemma 3.13. $\square$

Following the above lemma, we will prove that there exists a domain $\mathcal{D}_{K_0,A,s_0}$, with $s_0 = -\ln T$ such that for all $(d_1,d_2) \in \mathcal{D}_{K_0,A,s_0}$, the initial $u_{K_0,A,d_1,d_2}(0)$ is trapped in

$$S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, 0) = S(0).$$

In particular, we show that the initial data strictly satisfies almost the conditions of $S(0)$ except a few of the conditions in item (i) of Definition 3.1. More precisely, these conditions concern the following modes

$$(q_{1,0}, (q_{1,j})_{1\le j\le N}, q_{2,0}, (q_{2,j})_{1\le j\le N}, (q_{2,j,k})_{1\le j,k\le N})(s_0).$$

The following is our lemma:

Lemme 3.14 (Control of initial data)**.** *There exists $K_3 \geq 1$ such that for all each $K_0 \geq K_3, A \geq 1$ and $\delta_1 > 0$, there exists $\alpha_3(K_0, \delta_1)$ such that for all $\alpha_0 \leq \alpha_3$, there exists $\epsilon_3(K_0, \alpha_0, \delta_1) > 0$ such that for all $\epsilon_0 \leq \epsilon_3, \eta_0 > 0$, there exists $T_3(K_0, \alpha_0, \epsilon_1, A, \delta_1, \eta_0) > 0$ such that for all $T \leq T_3$ and $s_0 = -\ln T$, there exists $\mathcal{D}_{K_0,A,s_0} \subset [-2,2]^{N^2+2N+2}$ such that the following holds: if $u(0) = u_{K_0,A_0,d_1,d_2}(0)$ (see Definition 3.2), then*
(I) For all $(d_1, d_2) \in \mathcal{D}_{K_0,A,s_0}$, we have $u(0) \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_1, \eta_0, 0)$. In particular, we have:

- *(i) Estimates in $P_1(0)$: We have $(q_1, q_2)(s_0) \in V_A(s_0)$ where $(q_1, q_2)(s_0)$ are defined in (3.21) and (3.55). Moreover, we have also the following strictly estimates:*

$$\begin{aligned} |q_{1,j,k}(s_0)| &\leq \frac{A^2 \ln s_0}{2s_0^2}, \forall 1 \leq j,k \leq N \\ \left\| \frac{q_{1,-}(.,s_0)}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A}{2s_0^2} \quad &and \quad \left\| \frac{q_{2,-}(.,s_0)}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^2}{2s_0^{\frac{p_1+5}{2}}}, \\ \|q_{1,e}(.,s_0)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^2}{2\sqrt{s_0}} \quad &and \quad \|q_{2,e}(.,s_0)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^3}{2s_0^{\frac{p_1+2}{2}}}. \end{aligned}$$

- *(ii) Estimates in $P_2(0)$: For all $|x| \in \left[\frac{K_0}{4}\sqrt{T|\ln T|}, \epsilon_0\right]$, $\tau_0(x) = \frac{-t(x)}{\theta(x)}$ with $\theta(x) = T - t(x)$ and $|\xi| \leq \alpha_0 \sqrt{|\ln(T-t(x))|}$, we have*

$$|U(x, \xi, \tau_0(x)) - \hat{U}_{K_0}(\tau_0(x))| \leq \delta_1,$$

where $U(x, \xi, \tau)$ is defined in (3.74) and $\hat{U}_{K_0}(\tau)$ is defined in (3.79).

(II) There exists a mapping Ψ_1 such that

$$\begin{aligned} \Psi_1 : \mathbb{R}^{N^2+2N+2} &\rightarrow \mathbb{R}^{N^2+2N+2} \\ (d_1, d_2) &\mapsto \Psi_1(d_1, d_2), \end{aligned}$$

where

$$\Psi_1(d_1, d_2) = (q_{1,0}, (q_{1,j})_{1j \leq N}, q_{2,0}, (q_{2,j})_{1 \leq j \leq N}, (q_{2,j,k})_{1 \leq j,k \leq N})(s_0),$$

and Ψ_1 is linear, one to one from $\mathcal{D}_{K_0,A,s_0}$ to $\hat{V}_A(s_0)$, where

$$\hat{V}_A(s) = \left[-\frac{A}{s^2}, \frac{A}{s^2}\right]^{1+N} \times \left[-\frac{A^2}{s^{p_1+2}}, \frac{A^2}{s^{p_1+2}}\right]^{1+N} \times \left[-\frac{A^5 \ln s}{s^{p_1+2}}, \frac{A^5 \ln s}{s^{p_1+2}}\right]^{N^2}. \tag{3.88}$$

Moreover, we have

$$\Psi_1(\partial \mathcal{D}_{K_0,A,s_0}) \subset \partial \hat{V}_A(s_0),$$

and

$$deg\left(\Psi_1|_{\mathcal{D}_{K_0,A,s_0}}\right) \neq 0. \tag{3.89}$$

Proof. If we forget about the terms involving U^* and the $+1$ term in our definition (3.83) - (3.84) of initial data, then we are exactly in the framework of the integer case, treated in Duong [5] (see Lemma 3.4 in [5]). Therefore, when p is not integer, we only need to understand the effect of U^* and the $+1$ term in order to complete the proof. The argument is only technical. For that reason, we leave it to Appendix 3.7. □

Now, we give a key-proposition for our argument. More precisely, in the following proposition, we prove the existence of a solution of equation (3.56) trapped in the shrinking set:

Proposition 3.15 (Existence of a solution trapped in $S^*(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0)$)**.** *We can chose the parameters $T, K_0, \alpha_0, \epsilon_0, A$, δ_0 and η_0 such that there exists $(d_1, d_2) \in \mathbb{R}^{N^2+2N+2}$ such that the solution u of equation*(3.1) *with initial data given in Definition 3.2, exists on* $[0, T)$ *and satisfies*

$$u \in S^*,$$

where $S^* = S^*(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0)$ is defined in Definition 3.1.

Proof. The proof of this Proposition is given 2 steps:

- The first step: We reduce our problem to a finite dimensional one. In other words, we aim at proving that the control of $u(t)$ in the shrinking set $S(t)$ reduces to the control of the components

$$(q_{1,0}, (q_{1,j})_{1\le j\le N}, q_{2,0}, (q_{2,j})_{1\le j\le N}, (q_{2,j,k})_{1\le j,k\le N})(s)$$

  in $\hat{V}_A(s)$, defined in (3.88).

- The second step: We get the conclusion of Proposition 3.15 by using a topological argument in finite dimension.

- *Step 1: Reduction to a finite dimensional problem:* Using *a priori estimates*, our problem will be reduced to the control of a finite number of components.

Proposition 3.16 (Reduction to a finite dimensional problem)**.** *There exist parameters $K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0$ and $T > 0$ such that the following holds:*

(a) *Assume that initial data $u(0) = u_{K_0,A,d_1,d_2}(0)$ is given in Definition 3.2 with $(d_1, d_2) \in \mathcal{D}_{K_0,A,s_0}$*

(b) *Assume furthermore that the solution u of equation* (3.1) *satisfies:*

$$u \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t),$$

for all $t \in [0, t_*]$, for some $t_* \in [0, T)$ and

$$u \in \partial S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t_*).$$

Then, we have:

(i) *(Reduction to finite dimensions): It holds that*

$$(q_{1,0}, (q_{1,j})_{1\le j\le N}, q_{2,0}, (q_{2,j})_{1\le j\le N}, (q_{2,j,k})_{1\le j,k\le N})(s_*) \in \partial\hat{V}_A(s_*),$$

where the above components are of $(q_1, q_2)(s)$, defined in (3.21), *and* (3.55), *$\hat{V}_A(s)$ is defined as in* (3.88) *and $s_* = -\ln(T - t_*)$.*

(ii) (Transverse outgoing crossing): There exists $\nu_0 > 0$ *such that*

$$(q_{1,0}, (q_{1,j})_{1\le j\le N}, q_{2,0}, (q_{2,j})_{1\le j\le N}, (q_{2,j,k})_{1\le j,k\le N})(s_* + \nu) \notin \hat{V}_A(s_* + \nu), \qquad (3.90)$$

for all $\nu \in (0, \nu_0)$*. This implies that there exists* $\nu_1 > 0$ *such that* u *exists on* $[0, t_* + \nu_1)$ *and for all* $\nu \in (0, \nu_1)$

$$u(t_* + \nu) \notin S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t_* + \nu).$$

The proof of this Lemma uses techniques given in [16] which were developed from [1] and [17] in the real case. However, it is true that our shrinking set involves more conditions than the shrinking set used in [1], [5], [16] and [17]. In fact, the additional conditions are useful to ensure that our solution always stays positive. In particular, the set $V_A(s)$ plays an important role. Indeed, as for the integer case in [5], only the nonnegative modes $(q_{1,0}, (q_{1,j})_{1\le j\le N}, q_{2,0}, (q_{2,j})_{1\le j\le N}, (q_{2,j,k})_{1\le j,k\le N})(s_*)$ may touch the boundary of $\hat{V}_A(s_*)$ and leave in short time later. However, the control of the solution with the positive real part is also our highlight and of course it is the main difficulty in our work. This proposition makes the heart of the paper and needs many steps to be proved. For that reason, we dedicate a whole section to its proof (Section 3.4 below). Let us admit it here, and get to the conclusion of Proposition 3.15 in the second step.

- *Step 2: Conclusion of Proposition 3.15 by a topological argument.* In this step, we give the proof of Proposition 3.15 assuming that Proposition 3.16 holds. In fact, we aim at proving the existence of a parameter $(d_1, d_2) \in \mathcal{D}_{K_0,A,s_0}$ such that the solution u of equation (3.1) with initial data $u_{K_0,A,d_1,d_2}(0)$ (given in Definition 3.2), exists on $[0, T)$ and satisfies

$$u \in S^*(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0),$$

where the parameters will be suitably chosen. Our argument is analogous to the argument of Merle and Zaag [17]. For that reason, we only give a brief proof. Let us fix $T, K_0, \delta_0, \alpha_0, \epsilon_0, A, \alpha_0, \eta_0$ such that Lemma 3.14, Proposition 3.16 and Lemma 3.11 hold. Then, for all $(d_1, d_2) \in \mathcal{D}_{K_0,A,s_0}$ and from Lemma 3.14 we have the initial data

$$u_{K_0,A,d_1,d_2}(0) \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, 0).$$

Thanks to Lemmas 3.11 and 3.14, for each $(d_1, d_2) \in \mathcal{D}_{K_0,A,s_0}$ we can define $t_*(d_1, d_2) \in [0, T)$ as the maximum time such that the solution u_{d_1,d_2} of equation (3.1), with initial data $u_{K_0,A,d_1,d_2}(0)$ trapped in $S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t)$ for all $t \in [0, t_*(d_1, d_2))$. We have the two following cases:
+ Case 1: If there exists (d_1, d_2) such that $t_*(d_1, d_2) = T$ then our problem is solved
+ Case 2: For all $(d_1, d_2) \in \mathcal{D}_{K_0,A,s_0}$, we have

$$t_*(d_1, d_2) < T.$$

By contradiction, we can prove that the second case can not occur. Indeed, if it is true, by using the continuity of the solution u in time and the definition of $t_* = t_*(d_1, d_2)$, we can deduce that $u \in \partial S(t_*)$. Using item (i) of Proposition 3.16, we derive

$$(q_{1,0}, (q_{1,j})_{1\le j\le N}, q_{2,0}, (q_{2,j})_{1\le j\le N}, (q_{2,j,k})_{1\le j,k\le N})(s_*) \in \partial\hat{V}_A(s_*),$$

where $s_* = -\ln(T - t_*)$. Then, the following mapping Γ is well-defined

$$\begin{array}{rcl}\Gamma : \mathcal{D}_{K_0,A,s_0} & \to & \partial\left([-1,1]^{N^2+2N+2}\right)\\ (d_1, d_1) & \mapsto & \Gamma(d_1, d_2),\end{array}$$

where

$$\Gamma(d_1,d_2) = \left(\frac{s_*^2}{A}(q_{1,0},(q_{1,j})_{1\le j\le N})(s_*), \frac{s_*^{p_1+2}}{A^2}(q_{2,0},(q_{2,j})_{1\le j\le N})(s_*), \frac{s_*^{p_1+2}}{A^5\ln s_*}(q_{2,j,k})_{1\le j,k\le N}(s_*)\right),$$

and $s_* = s_*(d_1, d_2) = -\ln(T - t_*(d_1, d_2))$.

Moreover, it satisfies the two following properties:

(i) Γ is continuous from $\mathcal{D}_{K_0,A,s_0}$ to $\partial\left([-1,1]^{N^2+2N+2}\right)$. This is a consequence of item (ii) in Proposition (3.16).

(ii) The degree of the restriction $\Gamma\mid_{\partial\mathcal{D}_{K_0,A,s_0}}$ is non zero. Indeed, again by item (ii) in Proposition 3.16, we have

$$s_*(d_1, d_2) = s_0,$$

in this case. Applying (3.89), we get the conclusion.

In fact, such a mapping Γ can not exist by Index theorem and this is a contradiction. Thus, Proposition 3.15 follows, assuming that Proposition 3.16 holds (see Section 3.4 for the proof of latter). $\square$

3.3.4 The proof of Theorem 3.1

In this section, we aim at giving the proof of Theorem 3.1 by using Proposition 3.15.

The proof of Theorem 3.1: Except for the treatment of the nonlinear term, this part is quite similar to what we did in [5] when p is integer. Nevertheless, for the reader's convenience, we give the proof here, insisting on the way we handle the nonlinear term.

+ *The proof of item (i) of Theorem 3.1:* Using Proposition 3.15, there exists $(d_1, d_2) \in \mathbb{R}^{N^2+2N+2}$ such that the solution u of equation (3.1) with initial data $u_{K_0,A,d_1,d_2}(0)$ (given in Definition 3.2), exists on $[0, T)$ and satisfies:

$$u \in S^*(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0).$$

Thanks to item (i) in Definition 3.1, item (i) of Lemma 3.10, and definitions (3.21) and (3.55) of (w_1, w_2) and (q_1, q_2), respectively, we conclude (3.6) and (3.7). In addition to that, we have $\mathrm{Re}(u) > 0$. Moreover, we use again the definition of $V_A(s)$ to conclude the following asymptotic behaviors:

$$u(0,t) \sim \kappa(T-t)^{-\frac{1}{p-1}}, \tag{3.91}$$

$$u_2(0,t) \sim -\frac{2N\kappa}{(p-1)}\frac{(T-t)^{-\frac{1}{p-1}}}{|\ln(T-t)|^2}, \tag{3.92}$$

as $t \to T$, which means that u blows up at time T and the origin is a blowup point. Moreover, the real and imaginary parts simultaneously blow up . It remains to prove that for all $x \neq 0$, x is not a blowup point of u. The following Lemma allows us to conclude.

Lemme 3.17 (No blow-up under some threshold; Giga and Kohn [7]). *For all $C_0 > 0, 0 \le T_1 < T$ and $\sigma > 0$ small enough, there exists $\epsilon_0(C_0, T, \sigma) > 0$ such that if $u(\xi, \tau)$ satisfies the following estimates for all $|\xi| \le \sigma, \tau \in [T_1, T)$:*

$$|\partial_\tau u - \Delta u| \le C_0 |u|^p,$$

and

$$|u(\xi, \tau)| \le \epsilon_0 (1-\tau)^{-\frac{1}{p-1}}.$$

Then, u does not blow up at $\xi = 0, \tau = T$.

Proof. See Theorem 2.1 in Giga and Kohn [7]. Although the proof of [7] was given in the real case, it extends naturally to the complex valued case. □

We next use Lemma 3.17 to conclude that u does not blow up at $x_0 \neq 0$. Indeed, let us consider $x_0 \neq 0$. Then, we use (3.6) to deduce the following:

$$\sup_{|x-x_0|\le \frac{|x_0|}{2}} (T-t)^{\frac{1}{p-1}} |u(x,t)| \le \left| f_0 \left(\frac{\frac{|x_0|}{2}}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right| + \frac{C}{\sqrt{|\ln(T-t)|}} \to 0, \quad (3.93)$$

as $t \to T$. Applying Lemma 3.17 to $u(x - x_0, t)$, with some σ small enough such that $\sigma \le \frac{|x_0|}{2}$, and T_1 close enough to T, we see that $u(x - x_0, t)$ does not blow up at time T and $x = 0$. Hence, x_0 is not a blow-up point of u. This concludes the proof of item (i) in Theorem 3.1.

\+ *The proof of item (ii) of Theorem 3.1:* Here, we use the argument of Merle in [14] to deduce the existence of $u^* = u_1^* + iu_2^*$ such that $u(t) \to u^*$ as $t \to T$ uniformly on compact sets of $\mathbb{R}^N \backslash \{0\}$. In addition to that, we use the techniques in Zaag [30], Masmoudi and Zaag [18], Tayachi and Zaag [28] for the proofs of (3.9) and (3.10).

Indeed, for all $x_0 \in \mathbb{R}^N, x_0 \neq 0$, we deduce from (3.6), (3.7) that not only (3.93) holds but also the following is satisfied

$$\begin{aligned} \sup_{|x-x_0|\le \frac{|x_0|}{2}} (T-t)^{\frac{1}{p-1}} |\ln(T-t)||u_2(x,t)| &\le \left| \frac{9|x_0|^2}{4(T-t)|\ln(T-t)|} f_0^p \left(\frac{\frac{|x_0|}{2}}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right| \\ &+ \frac{C}{|\ln(T-t)|^{\frac{p_1}{2}}} \to 0, \text{ as } t \to T. \end{aligned} \quad (3.94)$$

We now consider x_0 such that $|x_0|$ is small enough, and K to be fixed later. We define $t_0(x_0)$ by

$$|x_0| = K\sqrt{(T - t_0(x_0))|\ln(T - t_0(x_0))|}. \quad (3.95)$$

Note that $t_0(x_0)$ is unique when $|x_0|$ is small enough and $t_0(x_0) \to T$ as $x_0 \to 0$.

We introduce rescaled functions $U(x_0, \xi, \tau)$ and $V_2(x_0, \xi, \tau)$ as follows:

$$U(x_0, \xi, \tau) = (T - t_0(x_0))^{\frac{1}{p-1}} u(x,t). \quad (3.96)$$

and

$$V_2(x_0, \xi, \tau) = |\ln(T - t_0(x_0))| U_2(x_0, \xi, \tau), \quad (3.97)$$

where $U_2(x_0,\xi,\tau)$ is defined by

$$U(x_0,\xi,\tau) = U_1(x_0,\xi,\tau) + iU_2(x_0,\xi,\tau),$$

and

$$(x,t) = \left(x_0 + \xi\sqrt{T - t_0(x_0)}, t_0(x_0) + \tau(T - t_0(x_0))\right), \text{ and } (\xi,\tau) \in \mathbb{R}^N \times \left[-\frac{t_0(x_0)}{T - t_0(x_0)}, 1\right). \tag{3.98}$$

We can see that with these notations, we derive from item (i) in Theorem 3.1 the following estimates for initial data at $\tau = 0$ of U and V_2

$$\sup_{|\xi| \le |\ln(T - t_0(x_0))|^{\frac{1}{4}}} |U(x_0,\xi,0) - f_0(K_0)| \le \frac{C}{1 + (|\ln(T - t_0(x_0))|^{\frac{1}{4}})} \to 0, \tag{3.99}$$

$$\sup_{|\xi| \le |\ln(T - t_0(x_0))|^{\frac{1}{4}}} |V_2(x_0,\xi,0) - g_0(K_0)| \le \frac{C}{1 + (|\ln(T - t_0(x_0))|^{\gamma_1})} \to 0, \tag{3.100}$$

as $x_0 \to 0$ and note that f_0 and g_0 are defined as in (3.4) and (3.8) respectively, and $\gamma_1 = \min\left(\frac{1}{4}, \frac{p_1}{2}\right)$.

Moreover, using equations (3.17), we derive the following equations for U, V_2: for all $\xi \in \mathbb{R}^N, \tau \in [0,1)$

$$\partial_\tau U = \Delta_\xi U + U^p, \tag{3.101}$$

$$\partial_\tau V_2 = \Delta_\xi V_2 + |\ln(T - t_0(x_0))| F_2(U_1,U_2), \tag{3.102}$$

where F_2 is defined in (3.18).

Besides that, from (3.93) and (3.101), we can apply Lemma 3.17 to U when $|\xi| \le |\ln(T - t_0(x_0))|^{\frac{1}{4}}$ and obtain:

$$\sup_{|\xi| \le \frac{1}{2}|\ln(T - t_0(x_0))|^{\frac{1}{4}}, \tau\in[0,1)} |U(x_0,\xi,\tau)| \le C. \tag{3.103}$$

Then, we aim at proving for $V_2(x_0,\xi,\tau)$ that

$$\sup_{|\xi| \le \frac{1}{16}|\ln(T - t_0(x_0))|^{\frac{1}{4}}, \tau\in[0,1)} |V_2(x_0,\xi,\tau)| \le C. \tag{3.104}$$

+ *The proof for* (3.104)*:* We first use (3.103) to derive the following rough estimate:

$$\sup_{|\xi| \le \frac{1}{2}|\ln(T - t_0(x_0))|^{\frac{1}{4}}, \tau\in[0,1)} |V_2(x_0,\xi,\tau)| \le C|\ln(T - t_0(x_0))|. \tag{3.105}$$

We first introduce ψ a cut-off function $\psi \in C_0^\infty(\mathbb{R}^N), 0 \le \psi \le 1, supp(\psi) \subset B(0,1), \psi = 1$ on $B(0,\frac{1}{2})$. Introducing

$$\psi_1(\xi) = \psi\left(\frac{2\xi}{|\ln(T - t_0(x_0))|^{\frac{1}{4}}}\right) \text{ and } V_{2,1}(x_0,\xi,\tau) = \psi_1(\xi)V_2(x_0,\xi,\tau). \tag{3.106}$$

Then, we deduce from (3.102) an equation satisfied by $V_{2,1}$

$$\partial_\tau V_{2,1} = \Delta_\xi V_{2,1} - 2\ \mathrm{div}(V_2 \nabla \psi_1) + V_2 \Delta \psi_1 + |\ln(T - t_0(x_0))| \psi_1 F_2(U_1, U_2). \qquad (3.107)$$

Hence, we can write $V_{2,1}$ with an integral equation as follows

$$\begin{aligned} V_{2,1}(\tau) &= e^{\Delta \tau}(V_{2,1}(0)) + \int_0^\tau e^{(\tau-\tau')\Delta} \{-2\ \mathrm{div}\ (V_2 \nabla \psi_1) + V_2 \Delta \psi_1 \\ &+ |\ln(T - t_0(x_0))| \psi_1 F_2(U_1, U_2))(\tau')\}\, d\tau'. \end{aligned} \qquad (3.108)$$

Besides that, using (3.103) and (3.105) and the fact that

$$|\nabla \psi_1| \leq \frac{C}{|\ln(T - t_0(x_0))|^{\frac{1}{4}}} \quad \text{and} \quad |\Delta \psi_1| \leq \frac{C}{|\ln(T - t_0(x_0))|^{\frac{1}{2}}},$$

we deduce that

$$\begin{aligned} \left| \int_0^\tau e^{(\tau-\tau')\Delta} \left(-2\ \mathrm{div}\ (V_2 \nabla \psi_1)\right) d\tau' \right| &\leq C \int_0^\tau \frac{\|V_2 \nabla \psi_1\|_{L^\infty(\mathbb{R}^N)}(\tau')}{\sqrt{\tau - \tau'}} d\tau' \leq C|\ln(T - t_0(x_0))|^{\frac{3}{4}}, \\ \left| \int_0^\tau e^{(\tau-\tau')\Delta} \left(V_2(\tau') \Delta \psi_1\right) d\tau' \right| &\leq C \int_0^\tau \|V_2 \Delta \psi_1\|_\infty(\tau') d\tau' \leq C|\ln(T - t_0(x_0))|^{\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} &\left| \int_0^\tau e^{(\tau-\tau')\Delta} \left(\psi_1 |\ln(T - t_0(x_0))| F_2(U_1, U_2)(\tau')\right) d\tau' \right| \\ &\leq C \int_0^\tau \||\ln(T - t_0(x_0))| \psi_1 F_2(U_1, U_2)\|_{L^\infty(\mathbb{R}^N)}(\tau') d\tau'. \end{aligned}$$

Since the last term in (3.108) involves the nonlinear term $F_2(U_1, U_2)$, we need to handle it differently from the case where p is integer: using the definition (3.18) of F_2, and (3.103) and the fact that U_1 is positive, we write from for all $|\xi| \leq \frac{1}{2}|\ln(T - t_0(x_0))|^{\frac{1}{4}}, \tau \in [0, 1)$ we have

$$\begin{aligned} |\psi_1 \ln(T - t_0(x_0)) F_2(U_1, U_2)(\tau)| &\leq C \left(U_1^2 + U_2^2\right)^{\frac{p-1}{2}} |\psi_1 \ln(T - t_0(x_0)) U_2(\tau)| \\ &\leq C\|V_{2,1}(\tau)\|_{L^\infty(\mathbb{R}^N)}. \end{aligned}$$

Hence, from (3.108) and the above estimates, we derive

$$\|V_{2,1}(\tau)\|_{L^\infty(\mathbb{R}^N)} \leq C|\ln(T - t_0(x_0))|^{\frac{3}{4}} + C \int_0^\tau \|V_{2,1}(\tau')\|_{L^\infty(\mathbb{R}^N)} d\tau'.$$

Thanks to Gronwall Lemma, we deduce that

$$\|V_{2,1}(\tau)\|_{L^\infty(\mathbb{R}^N)} \leq C|\ln(T - t_0(x_0))|^{\frac{3}{4}}, \forall \tau \in [0, 1),$$

which yields

$$\sup_{|\xi| \leq \frac{1}{4}|\ln(T - t_0(x_0))|^{\frac{1}{4}}, \tau \in [0,1)} |V_2(x_0, \xi, \tau)| \leq C|\ln(T - t_0(x_0))|^{\frac{3}{4}}. \qquad (3.109)$$

We apply iteratively for

$$V_{2,2}(x_0,\xi,\tau) = \psi_2(\xi)V_2(x_0,\xi,\tau) \text{ where } \psi_2(\xi) = \psi\left(\frac{4\xi}{|\ln(T-t_0(x_0))|^{\frac{1}{4}}}\right).$$

Similarly, we deduce that

$$\sup_{|\xi|\le \frac{1}{8}|\ln(T-t_0(x_0))|^{\frac{1}{4}},\tau\in[0,1)} |V_2(x_0,\xi,\tau)| \le C|\ln(T-t_0(x_0))|^{\frac{1}{2}}.$$

We apply this process a finite number of steps to obtain (3.104). We now come back to our problem, and aim at proving that:

$$\sup_{|\xi|\le \frac{1}{16}|\ln(T-t_0(x_0))|^{\frac{1}{4}},\tau\in[0,1)} \left|U(x_0,\xi,\tau) - \hat{U}_{K_0}(\tau)\right| \le \frac{C}{1+|\ln(T-t_0(x_0))|^{\gamma_2}}, \quad (3.110)$$

$$\sup_{|\xi|\le \frac{1}{32}|\ln(T-t_0(x_0))|^{\frac{1}{4}},\tau\in[0,1)} \left|V_2(x_0,\xi,\tau) - \hat{V}_{2,K_0}(\tau)\right| \le \frac{C}{1+|\ln(T-t_0(x_0))|^{\gamma_3}}, \quad (3.111)$$

where γ_2, γ_3 are positive small enough and $(\hat{U}_{K_0}, \hat{V}_{2,K_0})(\tau)$ is the solution of the following system:

$$\partial_\tau \hat{U}_{K_0} = \hat{U}^p_{K_0}, \quad (3.112)$$

$$\partial_\tau \hat{V}_{2,K_0} = p\hat{U}^{p-1}_{K_0}\hat{V}_{2,K_0}. \quad (3.113)$$

with initial data at $\tau = 0$

$$\hat{U}_{K_0}(0) = f_0(K_0),$$
$$\hat{V}_{2,K_0}(0) = g_0(K_0).$$

given by

$$\hat{U}_{K_0}(\tau) = \left((p-1)(1-\tau) + \frac{(p-1)^2K_0^2}{4p}\right)^{-\frac{1}{p-1}}, \quad (3.114)$$

$$\hat{V}_{2,K_0}(\tau) = K_0^2\left((p-1)(1-\tau) + \frac{(p-1)^2K_0^2}{4p}\right)^{-\frac{p}{p-1}}. \quad (3.115)$$

for all $\tau \in [0,1)$. The proof of is cited to Section 5 of Tayachi and Zaag [28] and here we will use (3.110) to prove (3.111). For the reader's convenience, we give it here. Let us consider

$$\mathcal{V}_2 = V_2 - \hat{V}_{2,K_0}(\tau). \quad (3.116)$$

Using (3.104), we deduce the following

$$\sup_{|\xi|\le \frac{1}{16}|\ln(T-t_0(x_0))|^{\frac{1}{4}},\tau\in[0,1)} |\mathcal{V}_2| \le C. \quad (3.117)$$

In addition to that, from (3.102) we write an equation on $\mathcal{V}_2$ as follows:

$$\partial_\tau \mathcal{V}_2 = \Delta\mathcal{V}_2 + p\hat{U}^{p-1}_{K_0}\mathcal{V}_2 + p(U_1^{p-1} - \hat{U}^{p-1}_{K_0})V_2 + \mathcal{G}_2(x_0,\xi,\tau), \quad (3.118)$$

where

$$\mathcal{G}_2(x_0,\xi,\tau) = |\ln(T-t_0(x_0))| \left(F_2(U_1,U_2) - pU_1^{p-1}U_2\right).$$

As for the last term in (3.118), we need here to carefully handle this expression, sine it involves a nonlinear term, which needs a treatment different from the case where p is integer. From the definition (3.18) of F_2, we have

$$\begin{aligned}\left|F_2(U_1,U_2) - pU_1^{p-1}U_2\right| &\leq \left|pU_2\left((U_1^2+U_2^2)^{\frac{p-1}{2}} - U_1^{p-1}\right)\right| \\ &+ \left|(U_1^2+U_2^2)^{\frac{p}{2}}\left\{\sin\left(p\arcsin\left(\frac{U_2}{\sqrt{U_1^2+U_2^2}}\right)\right) - \frac{pU_2}{\sqrt{U_1^2+U_2^2}}\right\}\right|.\end{aligned}$$

And we deduce from (3.104) and (3.110) with $\epsilon_0 > 0$ small enough that

$$\left|F_2(U_1,U_2) - pU_1^{p-1}U_2\right| \leq C|U_2|^3,$$

Plugging the above estimate and using (3.97) and (3.104), we have the following

$$\sup_{|\xi|\leq\frac{1}{16}|\ln(T-t_0)|^{\frac{1}{4}},\tau\in[0,1)} |\mathcal{G}_2(x_0,\xi,\tau)| \leq \frac{C}{|\ln(T-t_0(x_0))|^2}. \quad (3.119)$$

Introducing

$$\bar{\mathcal{V}}_2 = \psi_*(\xi)\mathcal{V}_2,$$

where

$$\psi_* = \psi\left(\frac{16\xi}{|\ln(T-t_0(x_0))|^{\frac{1}{4}}}\right),$$

and ψ is the cut-off function which has been introduced above. We also note that $\nabla\psi_*, \Delta\psi_*$ satisfy the following estimates

$$\|\nabla_\xi\psi_*\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{|\ln(T-t_0(x_0))|^{\frac{1}{4}}} \text{ and } \|\Delta_\xi\psi_*\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{|\ln(T-t_0(x_0))|^{\frac{1}{2}}}. \quad (3.120)$$

In particular, $\bar{\mathcal{V}}_2$ satisfies

$$\partial_\tau\bar{\mathcal{V}}_2 = \Delta\bar{\mathcal{V}}_2 + p\hat{U}_{K_0}^{p-1}(\tau)\bar{\mathcal{V}}_2 - 2\text{ div }(\mathcal{V}_2\nabla\psi_*) + \mathcal{V}_2\Delta\psi_* + p(U_1^{p-1} - \hat{U}_{K_0}^{p-1})\psi_*V_2 + \psi_*\mathcal{G}_2, \quad (3.121)$$

By Duhamel principal, we derive the following integral equation

$$\begin{aligned}\bar{\mathcal{V}}_2(\tau) = e^{\tau\Delta}(\bar{\mathcal{V}}_2(\tau)) &+ \int_0^\tau e^{(\tau-\tau')\Delta}\left\{p\hat{U}_{K_0}^{p-1}\bar{\mathcal{V}}_2 - 2\text{ div }(\mathcal{V}_2\nabla\psi_*) + \mathcal{V}_2\Delta\psi_*\right. \quad (3.122)\\ &+ \left. p(U_1^{p-1} - \hat{U}_{K_0}^{p-1})\psi_*V_2 + \psi_*\mathcal{G}_2\right\}(\tau')d\tau'.\end{aligned}$$

Besides that, we use (3.110), (3.114), (3.117), (3.120), (3.119) to derive the following estimates: for all $\tau \in [0,1)$

$$\begin{aligned}
|\hat{U}_{K_0}(\tau)| &\leq C,\\
\|\mathcal{V}_2\nabla\psi_*\|_{L^\infty(\mathbb{R}^N)}(\tau) &\leq \frac{C}{|\ln(T-t_0(x_0))|^{\frac{1}{4}}},\\
\|\mathcal{V}_2\Delta\psi_*\|_{L^\infty(\mathbb{R}^N)}(\tau) &\leq \frac{C}{|\ln(T-t_0(x_0))|^{\frac{1}{2}}},\\
\left\|\left(U_1^{p-1}-\hat{U}_{K_0}^{p-1}\right)\psi_*\right\|_{L^\infty(\mathbb{R}^N)}(\tau) &\leq \frac{C}{|\ln(T-t_0(x_0))|^{\gamma_2}},\\
\|\mathcal{G}_2\psi_*\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C}{|\ln(T-t_0(x_0))|^{2}}.
\end{aligned}$$

where γ_2 given in (3.110). Hence, we derive from the above estimates that: for all $0 \leq \tau' < \tau < 1$

$$\begin{aligned}
|e^{(\tau-\tau')\Delta}p\hat{U}_{K_0}^{p-1}\bar{\mathcal{V}}_2(\tau')| &\leq C\|\bar{\mathcal{V}}_2(\tau')\|,\\
|e^{(\tau-\tau')\Delta}(\mathrm{div}(\mathcal{V}_2\nabla\psi_*))| &\leq C\frac{1}{\sqrt{\tau-\tau'}}\frac{1}{|\ln(T-t_0(x_0))|^{\frac{1}{4}}},\\
|e^{(\tau-\tau')\Delta}(\mathcal{V}_2\Delta\psi_*)| &\leq \frac{C}{|\ln(T-t_0(x_0))|^{\frac{1}{2}}},\\
|e^{(\tau-\tau')\Delta}(p(U_1^{p-1}-\hat{U}_{K_0}^{p-1})\psi_* V_2)(\tau')| &\leq \frac{C}{|\ln(T-t_0(x_0))|^{\gamma_2}},\\
|e^{(\tau-\tau')\Delta}(\psi_*\mathcal{G}_2)(\tau')| &\leq \frac{C}{|\ln(T-t_0(x_0))|}.
\end{aligned}$$

Plugging into (3.122), we obtain

$$\|\bar{\mathcal{V}}_2(\tau)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{|\ln(T-t_0(x_0))|^{\gamma_3}} + C\int_0^\tau \|\bar{\mathcal{V}}_2(\tau')\|_{L^\infty(\mathbb{R}^N)}d\tau',$$

where $\gamma_3 = \min(\frac{1}{4}, \gamma_2)$. Then, thanks to Gronwall inequality, we get

$$\|\bar{\mathcal{V}}_2\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{|\ln(T-t_0(x_0))|^{\gamma_3}}.$$

Hence, (3.111) follows. Finally, we easily find the asymptotics of u^* and u_2^* as follows, thanks to the definition of U and V_2 and to estimates (3.110) and (3.111):

$$u^*(x_0) = \lim_{t\to T} u(x_0,t) = (T-t_0(x_0))^{-\frac{1}{p-1}} \lim_{\tau\to 1} U(x_0,0,\tau) \sim (T-t_0(x_0))^{-\frac{1}{p-1}} \left(\frac{(p-1)^2}{4p}K_0^2\right)^{-\frac{1}{p-1}}, \tag{3.123}$$

$$\begin{aligned}
u^*(x_0) = \lim_{t\to T} u(x_0,t) &= (T-t_0(x_0))^{-\frac{1}{p-1}} \lim_{\tau\to 1} U(x_0,0,\tau)\\
&\sim (T-t_0(x_0))^{-\frac{1}{p-1}} \left(\frac{(p-1)^2}{4p}K_0^2\right)^{-\frac{1}{p-1}},
\end{aligned} \tag{3.124}$$

and

$$\begin{aligned} u_2^*(x_0) = \lim_{t\to T} u_2(x_0,t) &= \frac{(T-t_0(x_0))^{-\frac{1}{p-1}}}{|\ln(T-t_0(x_0))|} \lim_{\tau\to 1} V_2(x_0,0,\tau) \\ &\sim \frac{(T-t_0(x_0))^{-\frac{1}{p-1}}}{|\ln(T-t_0(x_0))|} \left(\frac{(p-1)^2}{4p}\right)^{-\frac{p}{p-1}} (K_0^2)^{-\frac{1}{p-1}}. \quad (3.125) \end{aligned}$$

Using the relation (3.95), we find that

$$T - t_0(x_0) \sim \frac{|x_0|^2}{2K_0^2 |\ln|x_0||} \text{ and } \ln(T-t_0(x_0)) \sim 2\ln(|x_0|), \quad \text{as } x_0 \to 0. \quad (3.126)$$

Plugging (3.126) into (3.123) and (3.125), we get the conclusion of item (ii) of Theorem 3.1.

This concludes the proof of Theorem 3.1 assuming that Proposition 3.16 holds. Naturally, we need to prove this proposition on order to finish the argument. This will be done in the next section.

3.4 The proof of Proposition 3.16

This section is devoted to the proof of Proposition 3.16, which is considered as central in our analysis. We would like to proceed into two parts:

+ In the first part, we derive *a priori estimates* on u in every component $P_j(t)$ where $j = 1, 2$ or 3.

+ In the second part, we use a priori estimates to derive new bounds which improve all the bounds in Definition 3.1, except for the non-negative modes

$$(q_{1,0}, (q_{1,j})_{1\le j\le N}, q_{2,0}, (q_{2,j})_{1\le j\le N}, (q_{2,j,k})_{1\le j,k\le N}).$$

This means that the problem is reduced to the control of these components, which is the conclusion of item (i) of Proposition 3.16. As for item (ii) of Proposition 3.16 is just a direct consequence of the dynamics of these modes.

3.4.1 A priori estimates in $P_1(t), P_2(t)$ and $P_3(t)$

In this section, we aim at giving a *priori estimates* to the solution $u(t)$ on $P_1(t), P_2(t)$ and $P_3(t)$ which are important to get the conclusion of Proposition 3.16:

+ *A priori estimates in $P_1(t)$:* Here we give in the following proposition some estimates relevant to the region $P_1(t)$:

Proposition 3.18. *For all $A, K_0 \ge 1$ and $\epsilon_0 > 0, \alpha_0 > 0, \delta_0 > 0, \eta_0 > 0$, there exists $T_4(K_0, A, \epsilon_0)$ such that for all $T \le T_4$, if u is a solution of equation* (3.1) *on* $[0, t_1]$ *for some $t_1 \in [0, T)$ and $u \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t)$ for all $t \in [0, t_1]$, then, the following holds: for all $s_0 \le \tau \le s \le s_1$ with $s_1 = \ln(T - t_1)$, we have:*

(i) *(ODE satisfied by the positive modes) For all $j \in \{1, ..., N\}$, we have*

$$\left|q_{1,0}'(s) - q_{1,0}(s)\right| + \left|q_{1,j}'(s) - \frac{1}{2} q_{1,j}(s)\right| \le \frac{C}{s^2}, \forall 1 \le j \le N, \quad (3.127)$$

and

$$\left|q_{2,0}'(s) - q_{2,0}(s)\right| + \left|q_{2,j}'(s) - \frac{1}{2}q_{2,j}(s)\right| \leq \frac{C}{s^{p_1+2}}, \forall 1 \leq j \leq N. \tag{3.128}$$

(ii) *(ODE satisfied by the null modes) For all* $1 \leq j,k \leq N$, *we have*

$$\left|q_{1,j,k}'(s) + \frac{2}{s}q_{1,j,k}(s)\right| \leq \frac{CA}{s^3}, \tag{3.129}$$

and

$$\left|q_{2,j,k}'(s) + \frac{2}{s}q_{2,j,k}(s)\right| \leq \frac{CA^2 \ln s}{s^{p_1+3}}. \tag{3.130}$$

(iii) *(Control of the negative part) We have the following estimates*

$$\begin{aligned} \left\|\frac{q_{1,-}(.,s)}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} &\leq Ce^{-\frac{s-\tau}{2}}\left\|\frac{q_{1,-}(.,\tau)}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} \\ &+ C\frac{e^{-(s-\tau)^2}}{s^{\frac{3}{2}}}\|q_{1,e}(.,\tau)\|_{L^\infty(\mathbb{R}^N)} + \frac{C(1+s-\tau)}{s^2}, \end{aligned} \tag{3.131}$$

and

$$\begin{aligned} \left\|\frac{q_{2,-}(.,s)}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} &\leq Ce^{-\frac{s-\tau}{2}}\left\|\frac{q_{2,-}(.,\tau)}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} \\ &+ C\frac{e^{-(s-\tau)^2}}{s^{\frac{3}{2}}}\|q_{2,e}(.,\tau)\|_{L^\infty(\mathbb{R}^N)} + \frac{C(1+s-\tau)}{s^{\frac{p_1+5}{2}}}. \end{aligned} \tag{3.132}$$

(v) *(Control of the outer part) We have the following estimates*

$$\begin{aligned} \|q_{1,e}(.,s)\|_{L^\infty(\mathbb{R}^N)} &\leq Ce^{-\frac{(s-\tau)}{p}}\|q_{1,e}(.,\tau)\|_{L^\infty(\mathbb{R}^N)} \\ &+ Ce^{s-\tau}s^{\frac{3}{2}}\left\|\frac{q_{1,-}(.,\tau)}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} + \frac{C(1+s-\tau)e^{s-\tau}}{\sqrt{s}}, \end{aligned} \tag{3.133}$$

and

$$\begin{aligned} \|q_{2,e}(.,s)\|_{L^\infty(\mathbb{R}^N)} &\leq Ce^{-\frac{(s-\tau)}{p}}\|q_{2,e}(.,\tau)\|_{L^\infty(\mathbb{R}^N)} \\ &+ Ce^{s-\tau}s^{\frac{3}{2}}\left\|\frac{q_{2,-}(.,\tau)}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^N)} + \frac{C(1+s-\tau)e^{s-\tau}}{s^{\frac{p_1+2}{2}}}. \end{aligned} \tag{3.134}$$

Proof. By using the fact that $u(t) \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t)$ for all $t \in [0, t_1]$, we derive that $(q_1, q_2)(s) \in V_A(s)$ for all $s \in [s_0, s_1]$ and $(q_1, q_2)(s)$ satisfies equation (3.56). In addition to that, we deduce also the fact that $q_1(s) + \Phi_1(s) \geq \frac{e^{-\frac{s}{p-1}}}{2}$ for all $s \in [s_0, s_1]$ (see Lemma 3.11). Although potential terms $V_{j,k}, j,k \in \{1,2\}$, quadratic terms B_1, B_2 and rest terms R_1, R_2 (see equation (3.56)) are different from the case where p is integer, they behavior as in that case (see Lemmas 3.25, 3.26, 3.27 below). Thus, the result is derived from the projection of equation (3.56) and the dynamics of the operator $\mathcal{L} + V$. For that reason, we kindly refer the the reader to the proof of Lemma 4.2 given in [5] for the case where p is integer. □

+ *A priori estimates in* $P_2(t)$*:*

In this step, we aim at proving the following lemma which gives a priori estimates on u in $P_2(t)$. The following is our main result:

Lemme 3.19. *For all* $K_0 \geq 1, \delta_1 \in (0,1), \xi_0 \geq 1, \Lambda_5 > 0, \lambda_5 > 0$, *the following holds: If* $U(\xi,\tau)$ *a solution of equation* (3.101), *for all* ξ *and* $\tau \in [\tau_1,\tau_2]$ *with* $0 \leq \tau_1 \leq \tau_2 \leq 1$, *such that for all* $\tau \in [\tau_1,\tau_2]$ *and for all* $\xi \in [-2\xi_0, 2\xi_0]$, *we have*

$$|U(\xi,\tau)| \leq \Lambda_5 \text{ and } Re\,(U(\xi,\tau)) \geq \lambda_5 \text{ and } \left|U(\xi,\tau_1) - \hat{U}_{K_0}(\tau_1)\right| \leq \delta_1, \tag{3.135}$$

then, there exists $\epsilon = \epsilon(K_0,\Lambda_5,\lambda_5,\delta_1,\xi_0)$ *such that for all* $\xi \in [-\xi_0,\xi_0]$ *and for all* $\tau \in [\tau_1,\tau_2]$ *we have*

$$\left|U(\xi,\tau) - \hat{U}(\tau)\right| \leq \epsilon,$$

where $\hat{U}_{K_0}(\tau)$ *is given* (3.79). *In particular, we have* $\epsilon(K_0,\Lambda_5,\lambda_5,\delta_1,\xi_0) \to 0$ *as* $(\delta_1,\xi_0) \to (0,+\infty)$.

Proof. We introduce ψ as a cut-off function in $C_0^\infty(\mathbb{R})$ which satisfies the following:

$$\psi(x) = 0 \text{ if } |x| \geq 2, |\psi(x)| \leq 1 \text{ for all } x \text{ and } \psi(x) = 1 \text{ for all } |x| \leq 1,$$

and we also define ψ_1 as follows

$$\psi_1(\xi) = \psi\left(\frac{|\xi|}{\xi_0}\right).$$

Then, we have $\psi_1 \in C_0^\infty(\mathbb{R}^N)$, and $\text{supp}(\psi_1) \subset \{|\xi| \text{ such that } |\xi| \leq 2\xi_0\}$ and $\psi_1(\xi) = 1$ for all $|\xi| \leq \xi_0$. In addition to that, we let

$$V_1(\xi,\tau) = \psi_1(\xi)\left(U(\xi,\tau) - \hat{U}_{K_0}(\tau)\right), \forall \tau \in [\tau_1,\tau_2], \xi \in \mathbb{R}^N.$$

Thanks to equation (3.101), we derive that V_1 satisfies the following equation:

$$\partial_\tau V_1 = \Delta_\xi V_1 - 2\,\text{div}\,(U\nabla\psi_1) + U\Delta\psi_1 + \psi_1(\xi)\left(U^p - \hat{U}^p\right). \tag{3.136}$$

Therefore, we can write $V_1(\xi,\tau)$ under the following integral equation

$$V_1(\tau) = e^{(\tau-\tau_1)\Delta}(V_1(\tau_1)) + \int_{\tau_1}^{\tau} e^{(\tau-\tau')\Delta}\left(-2\,\text{div}\,(U\nabla\psi_1) + U\Delta\psi_1 + \psi_1\left(U^p - \hat{U}^p\right)\right)(\tau')d\tau'. \tag{3.137}$$

In addition to that, we have the following fact from (3.135) (in particular the estimate $\text{Re}(U(\xi,\tau)) \geq \lambda_5$ in (3.135) is crucial for the 4^{th} term in (3.137)): for all $\tau \in [\tau_1,\tau_2]$

$$\begin{aligned}
\|V_1(\tau_1)\|_{L^\infty(\mathbb{R}^N)} &\leq \delta_1, \\
\|U\nabla\psi_1\|_{L^\infty(\mathbb{R}^N)}(\tau) &\leq \frac{C(\Lambda_5)}{\xi_0}, \\
\|U\Delta\psi_1\|_{L^\infty(\mathbb{R}^N)}(\tau) &\leq \frac{C(\Lambda_5)}{\xi_0^2}, \\
\left\|\psi_1(U^p - \hat{U}^p)\right\|_{L^\infty(\mathbb{R}^N)}(\tau) &\leq C(K_0,\Lambda_5,\lambda_5)\|V_1\|_{L^\infty(\mathbb{R}^N)}(\tau),
\end{aligned}$$

which yields when $\tau_1 \le \tau' < \tau \le \tau_2$,

$$\begin{aligned}
\left\| e^{(\tau-\tau_1)\Delta}(V_1(\tau_1)) \right\| &\le \delta_1, \\
\left\| e^{(\tau-\tau')\Delta}(\text{div } (U\nabla\psi_1)(\tau')) \right\|_{L^\infty(\mathbb{R}^N)} &\le \frac{C(\Lambda_5)}{\xi_0}\frac{1}{\sqrt{\tau-\tau'}}, \\
\left\| e^{(\tau-\tau')\Delta}(U\Delta\psi_1(\tau')) \right\|_{L^\infty(\mathbb{R}^N)} &\le \frac{C(\Lambda_5)}{\xi_0^2}, \\
\left\| e^{(\tau-\tau')\Delta}(\psi_1(U^p - \hat{U}^p)(\tau')) \right\|_{L^\infty(\mathbb{R}^N)} &\le C(K_0,\Lambda_5,\lambda_5)\|V_1\|_{L^\infty(\mathbb{R}^N)}(\tau').
\end{aligned}$$

Plugging into (3.137), we have for all $\tau \in [\tau_1, \tau_2]$

$$\|V_1(\tau)\|_{L^\infty(\mathbb{R}^N)} \le C(K_0,\Lambda_5,\lambda_5)\left(\delta_1 + \frac{1}{\xi_0}\right) + C(K_0,\Lambda_5,\lambda_5)\int_{\tau_1}^{\tau} \|V_1(\tau')\|_{L^\infty(\mathbb{R}^N)}\, d\tau'.$$

Thanks to Gronwall lemma, we obtain the following

$$\|V_1(\tau)\|_{L^\infty(\mathbb{R}^N)} \le C(K_0,\Lambda_5,\lambda_5)\left(\delta_1 + \frac{1}{\xi_0}\right), \forall \tau \in [\tau_1, \tau_2].$$

Since $V_1(\tau) = U(\tau) - \hat{U}(\tau)$ for all $\xi \in [-\xi_0, \xi_0]$ and for all $\tau \in [\tau_1, \tau_2]$, this concludes our lemma. □

\+ *A proiori estimates in $P_3(t)$:* We aim at proving the following lemma which gives a priori estimates on u in $P_3(t)$.

Lemme 3.20 (A priori estimates in $P_3(t)$)**.** *For all $K_0 \ge 1, A \ge 1, \eta > 0, \epsilon_0 > 0, \sigma \ge 1$ and $|d_1| + |d_2| \le 2$, there exists $T_6(K_0, A, \epsilon_0, \eta, \sigma) > 0$, such that for all $T \le T_6$ the following holds: If u is a solution of equation* (3.1) *for all $t \in [0, t_*]$ for some $t_* \in [0, T)$ with initial data $u(0) = u_{K_0,A,d_1,d_2}(0)$ (see Definition 3.2) and*

$$|u(x,t)| \le \sigma, \forall |x| \in \left[\frac{\epsilon_0}{8}, +\infty\right), t \in [0, t_*], \tag{3.138}$$

then,

$$|u(x,t) - u(x,0)| \le \eta, \forall |x| \ge \frac{\epsilon_0}{4}, t \in [0, t_*].$$

Proof. We introduce ψ, a cut-off function in $C^\infty(\mathbb{R})$ defined as follows

$$\psi(r) = 0 \text{ if } |r| \le \frac{1}{2}, \quad \psi(r) = 1 \text{ for all } |r| \ge 1 \text{ and } |\psi(r)| \le 1 \text{ for all } r,$$

and we also introduce $\psi_{\epsilon_0} \in C^\infty(\mathbb{R}^N)$ as follows

$$\psi_{\epsilon_0}(x) = \psi\left(\frac{4|x|}{\epsilon_0}\right).$$

Then, $\psi_{\epsilon_0} \in C^\infty(\mathbb{R}^N)$, and $\psi_{\epsilon_0}(x) = 1$ for all $|x| \ge \frac{\epsilon_0}{4}$ and $\psi_{\epsilon_0} = 0$ for all $|x| \le \frac{\epsilon_0}{8}$. We define as well

$$v = \psi_{\epsilon_0} u.$$

Thanks to equation (3.1), we derive an equation satisfied by v

$$\partial_t v = \Delta v - 2\,\mathrm{div}(u\nabla\psi_{\epsilon_0}) + u\Delta\psi_{\epsilon_0} + \psi_{\epsilon_0}u^p = \Delta v - 2\mathrm{div}\left(u\nabla\psi_{\epsilon_0}\right) + G(u), \tag{3.139}$$

where

$$G(u) = u\Delta\psi_{\epsilon_0} + \psi_{\epsilon_0}u^p.$$

Using (3.138), we get

$$\|G(t,u(t))\|_{L^\infty(\mathbb{R}^N)} \leq C(\sigma,\epsilon_0), \forall t\in[0,t_*].$$

By Duhamel formula, we derive

$$v(t) = e^{t\Delta}(v(0)) + \int_0^t e^{(t-s)\Delta}(G(s,u(s)))ds, \tag{3.140}$$

which yields

$$v(t) - v(0) = e^{t\Delta}(v(0)) - v(0) + \int_0^t e^{(t-s)\Delta}(G(s,u(s)))ds.$$

Thus,

$$\|v(t)-v(0)\|_{L^\infty(\mathbb{R}^N)} \leq \|e^{t\Delta}(v(0)) - v(0)\|_{L^\infty(\mathbb{R}^N)} + \left\|\int_0^t e^{(t-s)\Delta}(G(s,u(s)))ds\right\|_{L^\infty(\mathbb{R}^N)}.$$

In addition to that, if $T\leq T_{6,1}(\epsilon_0)$, we have $\chi_1(x)=0$, for all $|x|\geq\frac{\epsilon_0}{8}$, where χ_1 defined in (3.87) is involved in Definition 3.1 of initial data $u(0)$. As a matter of fact, from the definition of $u(0)$, we deduce from this fact that

$$v(0) = \psi_{\epsilon_0}\left(U^* + 1\right).$$

Since $\Delta v(0)\in L^\infty(\mathbb{R}^N)$, it follows that

$$\left\|e^{t\Delta}(v(0)) - v(0)\right\|_{L^\infty(\mathbb{R}^N)} \to 0 \text{ as } t\to 0.$$

Besides that, we have also

$$\left\|\int_0^t e^{(t-s)\Delta}(G(s,u(s)))ds\right\|_{L^\infty(\mathbb{R}^N)} \to 0 \text{ as } t\to 0.$$

Therefore, for all $t\in[t_0,t_*]$ we have

$$\|v(t)-v(0)\|_{L^\infty(\mathbb{R}^N)} \leq \eta,$$

provided that $T\leq T_{6,2}(K_0,A,\epsilon_0,\eta,\sigma)$. This concludes our lemma. □

Finally, we need the following Lemma to get the conclusion of our proof:

Lemme 3.21. *There exists $K_7\geq 1$ such that for all $K_0\geq K_7, A\geq 1$, and $\delta_1>0$, there exists $\alpha_7(K_0,A,\delta_1)>0$ such that for all $\alpha_0\leq\alpha_7$, there exists $\epsilon_7(K_0,\alpha_0,A,\delta_1)>0$ such that for all $\epsilon_0\leq\epsilon_7$ there exist $\delta_7(\delta_1)>0, T_7(K_0,\epsilon_0,A,\delta_1)>0$, $\eta_7(K_0,\epsilon_0,A)>0$ such that for all $\delta_0\leq\delta_7,\eta_0\leq\eta_7$ and for all $T\leq T_7$ if $u\in S(T,K_0,\alpha_0,\epsilon_0,A,\delta_0,\eta_0,t)$ for all $t\in[0,t_*]$, for some $t_*\in[0,T)$, then the following holds:*

$$\textit{whenever } |x|\in\left[\frac{K_0}{4}\sqrt{(T-t_*)|\ln(T-t_*)|},\epsilon_0\right]$$

(*i*) *For all* $|\xi| \leq 2\alpha_0\sqrt{|\ln(T-t(x))|}$ *and for all*

$$\tau \in \left[\max\left(0, \frac{-t(x)}{T-t(x)}\right), \frac{t_* - t(x)}{T-t(x)}\right],$$

if $U(x,\xi,\tau)$ *satisfies equation* (3.101)*, then*

$$|U(x,\xi,\tau)| \leq C_7^*(p) \text{ and } Re\left(U(\xi,\tau)\right) \geq C_7^{**}(K_0,p),$$

where $U(\xi,\tau)$ *is defined as in* (3.74)*,* $t(x)$ *is defined in* (3.75)*, and* C_7^* *depends only on the parameter* p *and* $C_7^{**}(K_0,p)$ *depends on parameters* K_0 *and* p*.*

(*ii*) *For all* $|\xi| \leq 2\alpha_0\sqrt{|\ln(T-t(x))|}$*, if we define*

$$\tau_0(x) = \max\left(0, \frac{-t(x)}{T-t(x)}\right), \tag{3.141}$$

then, we have

$$|U(x,\xi,\tau_0) - \hat{U}_{K_0}(\tau_0)| \leq \delta_1.$$

Proof. The idea of the proof relies on the argument in Lemma 2.6, given in [16].
+ *The proof of item* (*i*)*:* We aim at proving that for all $|x| \in \left[\frac{K_0}{4}\sqrt{(T-t_*)|\ln(T-t_*)|}, \epsilon_0\right]$, $|\xi| \leq 2\alpha_0\sqrt{|\ln(T-t(x))|}$ and $t \in [\max(0,t(x)), t_*]$, we have

$$|U(x,\xi,\tau(x,t))| \leq C_7^*, \tag{3.142}$$

and

$$\text{Re}\left(U(\xi,\tau)\right) \geq C_7^{**}, \tag{3.143}$$

where $\tau(x,t) = \frac{t-t(x)}{T-t(x)}$ and $C_7^*, C_7^{**} > 0$. Let us introduce a parameter $\delta > 0$ to be fixed later in our proof, small enough (note that δ has nothing to do with the parameters δ_0, δ_1 in the statement of our lemma). We observe that if we have $\alpha_0 \leq \alpha_{1,7}(K_0,\delta)$ for some $\alpha_{1,7} > 0$ and small enough, then for all $|\xi| \leq 2\alpha_0\sqrt{|\ln(T-t(x))|}$, we have

$$(1-\delta)|x| \leq |x + \xi\sqrt{T-t(x)}| \leq (1+\delta)|x|. \tag{3.144}$$

We also recall the definition of rescaled function $U(x,\xi,\tau(x,t))$ as follows

$$U(x,\xi,\tau) = (T-t(x))^{\frac{1}{p-1}} u(x+\xi\sqrt{T-t(x)}, t(x)+\tau(T-t(x))).$$

Introducing $X = x + \xi\sqrt{T-t(x)}$, we write

$$U(x,\xi,\tau(x,t)) = (T-t(x))^{\frac{1}{p-1}} u(X,t).$$

We here consider 3 cases:
+ *Case 1:* We consider the case where

$$|X| \leq \frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|}.$$

Using the fact that $u \in S(t)$, in particular item (i) of Definition 3.1, we see that Lemma 3.10 and (3.80) hold, hence

$$\left|(T-t)^{\frac{1}{p-1}}u(X,t) - f_0\left(\frac{X}{\sqrt{(T-t)|\ln(T-t)|}}\right)\right| \le \frac{CA^3}{\sqrt{1+|\ln(T-t)|}}.$$

Then, we derive the following

$$\begin{aligned}
|U(x,\xi,\tau(x,t))| &\le \left(\frac{T-t}{T-t(x)}\right)^{-\frac{1}{p-1}}\left(f_0(0) + \frac{CA^3}{\sqrt{1+|\ln(T-t)|}}\right)\\
&= \left(\frac{T-t}{T-t(x)}\right)^{-\frac{1}{p-1}}\left(\kappa + \frac{CA^3}{\sqrt{1+|\ln(T-t)|}}\right), \qquad (3.145)\\
\mathrm{Re}(U(x,\xi,\tau(x,t))) &\ge \left(\frac{T-t}{T-t(x)}\right)^{-\frac{1}{p-1}}\left(f_0(0) - \frac{CA^3}{\sqrt{1+|\ln(T-t)|}}\right)\\
&= \left(\frac{T-t}{T-t(x)}\right)^{-\frac{1}{p-1}}\left(\kappa - \frac{CA^3}{\sqrt{1+|\ln(T-t)|}}\right). \qquad (3.146)
\end{aligned}$$

Besides that, we deduce the following from (3.144) and the following fact

$$|X| \le \frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|},$$

that

$$|x| \le \frac{K_0}{4(1-\delta)}\sqrt{(T-t)|\ln(T-t)|}.$$

In addition to that, we have that the function $T-t(x)$ is an increasing function if $|x|$ small enough. Therefore,

$$T-t(x) \le T-t\left(\frac{K_0}{4(1-\delta)}\sqrt{(T-t)|\ln(T-t)|}\right). \qquad (3.147)$$

As a matter of fact, we have the following asymptotic behavior of $\theta(x) = T-t(x)$

$$\ln\theta(x) \sim 2\ln|x| \text{ and } \theta(x) \sim \frac{8}{K_0^2}\frac{|x|^2}{|\ln|x||} \text{ as } |x| \to 0. \qquad (3.148)$$

Plugging (3.148) in (3.147), we obtain the following

$$T-t(x) \le T-t\left(\frac{K_0}{4(1-\delta)}\sqrt{(T-t)|\ln(T-t)|}\right) \sim \frac{8K_0^2(T-t)|\ln(T-t)|}{K_0^2 16(1-\delta)^2\frac{1}{2}|\ln(T-t)|} = \frac{(T-t)}{(1-\delta)^2}.$$

In particular, from $t \in [\max(0,t(x)), t_*]$, we have the following

$$T-t(x) \ge T-t.$$

Plugging into (3.145) and (3.146), we obtain

$$|U(x,\xi,\tau)| \le C^*_{1,7}(p,\delta),$$

and

$$\mathrm{Re}(U(x,\xi,\tau(x,t))) \geq C^{**}_{1,7}(p,\delta),$$

provided that δ is small enough, $K_0 \geq K_{1,7}(\delta)$ which is large enough and $T \leq T_{1,7}(K_0, A)$. Note that $C^*_{1,7}(p,\delta)$ and $C^{**}_7(p,\delta)$ depend on δ and p, in particular, $C^*_{1,7}(\delta,p)$ is bounded when $\delta \to 0$.

+ *The second case:* We consider the case where

$$|X| \in \left[\frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|}, \epsilon_0\right].$$

By using the definition of $U(x,\xi,\tau(x,t))$, we deduce that

$$U(x,\xi,\tau(x,t)) = \left(\frac{T-t(x)}{T-t(X)}\right)^{\frac{1}{p-1}} U(X,0,\tau(X,t)).$$

However, using the fact that $u \in S(t)$, in particular item (ii) of Definition 3.1, we have

$$|U(X,0,\tau(X,t))| \leq \delta_0 + \hat{U}(1).$$

In addition to that, we use (3.144), the definition of $t(x)$ and the fact that

$$|X| \geq \frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|}$$

to derive the following

$$1 \leq \frac{T-t(x)}{T-t(X)} \leq 2,$$

provided that $\delta > 0$, small enough. Therefore, we have

$$|U(x,\xi,\tau(x,t))| \leq 2^{\frac{1}{p-1}}\left(\delta_0 + \hat{U}_{K_0}(1)\right) \leq \frac{1}{2},$$

and

$$\mathrm{Re}(U(x,\xi,\tau(x,t))) \geq \hat{U}_{K_0}(0) - \delta_0 \geq \frac{1}{2}\hat{U}_{K_0}(0),$$

provided that $\delta_0 \leq \frac{1}{2}\hat{U}_{K_0}(0)$ and $K_0 \geq K_{2,7}$.

+ *The third case:* We consider the case where $|X| \geq \epsilon_0$. Using the fact that $u \in S(t)$, in particular item (iii) of Definition 3.1, we have

$$\begin{aligned}
|U(x,\xi,\tau(x,t))| &= (T-t(x))^{\frac{1}{p-1}}|u(X,t)| \leq (T-t(x))^{\frac{1}{p-1}}(|u(X,0)| + \eta_0),\\
\mathrm{Re}\left(U(x,\xi,\tau(x,t))\right) &= (T-t(x))^{\frac{1}{p-1}}\mathrm{Re}(u(X,t)) \geq (T-t(x))^{\frac{1}{p-1}}\left(\mathrm{Re}(u(X,0)) - \eta_0\right).
\end{aligned}$$

Using the definition (3.83), we have for all $|X| \geq \epsilon_0$

$$u(X,0) = U^*(X) + 1,$$

provided that $T \leq T_{2,7}(\epsilon_0)$. In addition to that, we have the following fact

$$\begin{aligned}
T - t(x) &\sim \frac{16|x|^2}{K_0^2|\ln|x||},\\
u(X,0) &\sim U^*(X) = \left[\frac{(p-1)^2|x|^2}{8p|\ln|x||}\right]^{-\frac{1}{p-1}},
\end{aligned}$$

as $(X,x)\to(0,0)$, and in particular, from (3.144), we have

$$(1-\delta)|x|\le|X|\le(1+\delta)|x|.$$

Therefore, we have

$$\begin{aligned}|U(x,\xi,\tau(x,t))|&\le C^*_{2,7}(\delta),\\ \mathrm{Re}(U(x,\xi,\tau(x,t)))&\ge C^{**}_{2,7}(K_0,\delta),\end{aligned}$$

provided that $K_0\ge K_{3,7},\eta_0\le\eta_{1,7}(\delta)$ and δ is small. We conclude item (i).
The proof of item (ii): We aim at proving that for all $|\xi|\le 2\alpha_0\sqrt{|\ln\theta(x)|}$ and $\tau_0(x)=\max\left(0,-\frac{t(x)}{\theta(x)}\right)$, we have

$$\left|U(x,\xi,\tau_0(x))-\hat U_{K_0}(\tau_0(x))\right|\le\delta_1. \tag{3.149}$$

Considering 2 cases for the proof of (3.149):
+ Case 1: We consider the case where

$$|x|\le\frac{K_0}{4}\sqrt{T|\ln T|},$$

then, we deduce from the definition of $t(x)$ given by (3.75) that $t(x)\le 0$. Thus, by definition (3.141), we have

$$\tau_0(x)=\frac{-t(x)}{\theta(x)}.$$

Therefore, (3.149) directly follows item (ii) of Lemma 3.14 with $K_0\ge K_{4,7},\alpha_0\le\alpha_{3,7},\epsilon_0\le\epsilon_{3,7}$ (see in Lemma 3.14)
+ Case 2: We consider the case where

$$|x|\ge\frac{K_0}{4}\sqrt{T|\ln T|},$$

which yields $t(x)\ge 0$. Thus, by definition (3.141), we have

$$\tau_0(x)=0.$$

We let $X=x+\xi\sqrt{\theta(x)}$. According to the definitions of $U,\hat U_{K_0}$ which are given by (3.74) and (3.79), we write

$$\begin{aligned}\left|U(x,\xi,0)-\hat U_{K_0}(0)\right|&=\left|\theta^{-\frac{1}{p-1}}(x)u\left(X,t(x)\right)-\left((p-1)+\frac{(p-1)^2}{4p}\frac{K_0^2}{16}\right)^{-\frac{1}{p-1}}\right|\\
=&\left|\theta^{-\frac{1}{p-1}}(x)u\left(X,t(x)\right)-\left((p-1)+\frac{(p-1)^2}{4p}\frac{|X|^2}{\theta(x)|\ln\theta(x)|}\right)^{-\frac{1}{p-1}}\right.\\
+&\left.\left((p-1)+\frac{(p-1)^2}{4p}\frac{|X|^2}{\theta(x)|\ln\theta(x)|}\right)^{-\frac{1}{p-1}}-\left((p-1)+\frac{(p-1)^2}{4p}\frac{K_0^2}{16}\right)^{-\frac{1}{p-1}}\right|\\
\le&\ (I)+(II),\end{aligned}$$

where $\theta(x) = T - t(x)$, and

$$
\begin{aligned}
(I) &= \left|\theta^{-\frac{1}{p-1}}(x)u\left(X, t(x)\right) - \left((p-1) + \frac{(p-1)^2}{4p}\frac{|X|^2}{\theta(X)|\ln\theta(X)|}\right)^{-\frac{1}{p-1}}\right|,\\
(II) &= \left|\left((p-1) + \frac{(p-1)^2}{4p}\frac{|X|^2}{\theta(X)|\ln\theta(X)|}\right)^{-\frac{1}{p-1}} - \left((p-1) + \frac{(p-1)^2}{4p}\frac{K_0^2}{16}\right)^{-\frac{1}{p-1}}\right|.
\end{aligned}
$$

Since

$$
|X| \le (1+\delta)|x| \le \frac{(1+\delta)K_0}{4}\sqrt{(T-t(x))|\ln(T-t(x))|} \le K_0\sqrt{(T-t(x))|\ln(T-t(x))|},
$$

Using item (i) of Definition 3.1, taking $t = t(x)$, we write

$$
(I) \le \frac{C(K_0)A^2}{\sqrt{|\ln(T-t(x))|}} \le \frac{C(K_0)A^2}{\sqrt{|\ln T|}} \le \frac{\delta_1}{2},
$$

provided that $T \le T_{4,7}(K_0, A, \delta_1)$. Besides that, from (3.144) we have

$$
(1-\delta)^2\frac{K_0^2}{16} \le \frac{|X|^2}{\theta(X)\left|\ln\theta(X)\right|} \le (1+\delta)^2\frac{K_0^2}{16}.
$$

This yields

$$
(II) \le \frac{\delta_1}{2},
$$

provided that δ is small enough. Then, (3.149) follows. Finally, we fix $\delta > 0$ small enough and we conclude our lemma. □

3.4.2 The conclusion of Proposition 3.16

It this subsection, we would like to conclude the proof of Proposition 3.16. As we mentioned earlier, in the analysis of the shrinking set $S(t)$, the heart is the set $V_A(s)$ (see item (i) of Definition 3.1 of $S(t)$). So, let us first give an important argument related the analysis of $V_A(s)$; the reduction to finite dimensions. More precisely, we prove that if the solution (q_1, q_2) of equation (3.56) satisfies $(q_1, q_2)(s) \in V_A(s)$ for all $s \in [s_0, s_*]$ and $(q_1, q_2)(s_*) \in \partial V_A(s_*)$ for some $s_* \in [s_0, +\infty)$ with $s_0 = -\ln T$, then, we can directly derive that

$$
(q_{1,0}, (q_{1,j})_{j\le n}, q_{2,0}, (q_{2,j})_{j\le n}, (q_{2,j,k})_{j,k\le n})(s_*) \in \hat{\partial} V_A(s_*),
$$

where $\hat{V}_A(s_*)$ is defined in (3.88). After that, we will use the dynamic of these modes to derive that they will leave $\hat{V}_A$ after that. The following is our statement

Proposition 3.22 (A reduction to finite dimensional problem)**.** *There exists $A_8 \ge 1, K_8 \ge 1$ such that for all $A \ge A_8, K_0 \ge K_8$, there exists $s_8(A, K_0) \ge 1$ such that for all $s_0 \ge s_8(A, K_0)$, we have the following properties: If the following conditions hold:*

a) We take the initial data $(q_1, q_2)(s_0)$ are defined by $u_{A,K_0,d_1,d_2}(0)$ with $s_0 = -\ln T$ (see Definition 3.2, (3.21) *and* (3.55)*) and $(d_0, d_1) \in \mathcal{D}_{K_0,A,s_0}$ (see in Lemma* (3.14)*).*

b) *For all* $s \in [s_0, s_1]$, *the solution* (q_1, q_2) *of equation* (3.56) *satisfies:* $(q_1, q_2)(s) \in V_A(s)$ *and* $q_1(s) + \Phi_1(s) \geq \frac{1}{2} e^{-\frac{s}{p-1}}$.

Then, for all $s \in [s_0, s_1]$, *we have*

$$\begin{aligned} \forall i, j \in \{1, \cdots, n\}, \quad |q_{1,i,j}(s)| &\leq \frac{A^2 \ln s}{2s^2}, & (3.150) \\ \left\| \frac{q_{1,-}(.,s)}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{A}{2s^2}, \quad \|q_{1,e}(s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^2}{2\sqrt{s}}, & (3.151) \\ \left\| \frac{q_{2,-}(.,s)}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{A^2}{2s^{\frac{p_1+5}{2}}}, \quad \|q_{2,e}(s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^3}{2s^{\frac{p_1+2}{2}}}. & (3.152) \end{aligned}$$

Proof. The proof is quite similar to Proposition 4.4 in [5]. Indeed, the proof is a consequence of Proposition 3.18, exactly as in [5]. Thus, we omit the proof and refer the reader to [5]. □

Here, we give the conclusion of the proof of Proposition 3.16:

Conclusion of the proof of Proposition 3.16: We first choose the parameters $K_0, A, \alpha_0, \epsilon_0, \delta_0, \delta_1, \eta_0, \eta$ and $T > 0$ such that all the above Lemmas and Propositions which are necessary to the proof, are satisfied. In particular, we also note that the parameters $\delta_1$and η which are introduced in Lemma 3.14 and Lemma 3.20, will be small enough ($\delta_1 \ll \delta_0$ and $\eta \ll \eta_0$). Finally, we fix the constant T small enough, depending on all the above parameters, then we conclude our Proposition. We now assume the solution u of equation (3.1) with initial data $u_{K_0,A,d_1,d_2}(0)$, defined in Definition 3.2, satisfies the following

$$u \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t) = S(t),$$

for all $t \in [0, t_*]$ for some $t_* \in [0, T)$ and

$$u \in \partial S(t_*).$$

We aim at proving that

$$(q_1, q_2)(s_*) \in \partial V_A(s_*), \tag{3.153}$$

where $s_* = \ln(T - t_*)$. Indeed, by contradiction, we suppose that (3.153) is not true, then, by using Definition 3.1 of $S(t)$, we derive the following:

(I) Either, there exist x_*, ξ_* which satisfy

$$\begin{aligned} |x_*| &\in \left[\frac{K_0}{4} \sqrt{(T-t_*)|\ln(T-t_*)|}, \epsilon_0 \right], \\ |\xi_*| &\leq \alpha_0 \sqrt{|\ln(T - t(x_*))|}. \end{aligned}$$

and

$$|U(x_*, \xi_*, \tau(x_*, t_*)) - \hat{U}(\tau(x_*, \tau_*))| = \delta_0.$$

(II) Or, there exists x^* such that $|x^*| \geq \frac{\epsilon_0}{4}$ and

$$|u(x^*, t_*) - u(x^*, 0)| = \eta_0.$$

We would like to prove that (I) and (II) can not occur. Indeed, if the first case occurs, then, letting $\tau_0(x_*) = \max\left(-\frac{t(x_*)}{\theta(x_*)}, 0\right)$, it follows from Lemma 3.21 that: For all $|\xi| \leq 2\alpha_0 \sqrt{|\ln(T - t(x_*))|}$, we have

$$\left| U(x_*, \xi, \tau_0(x_*)) - \hat{U}(\tau_0(x_*)) \right| \leq \delta_1,$$

and for all $\tau \in \left[\max\left(-\frac{t(x_*)}{T-t(x_*)}, \frac{t_*-t(x_*)}{T-t(x_*)}\right)\right]$, we have

$$\begin{array}{rcl} |U(x_*, \xi, \tau(x_*))| & \leq & C_7^*, \\ \mathrm{Re}(U(x_*, \xi, \tau(x_*))) & \geq & C_7^{**}, \end{array}$$

where C_7^*, C_7^{**} are given in Lemma 3.21.

Then, we apply Lemma 3.19, with $\xi_0 = \alpha_0 \sqrt{|\ln(T - t(x_*))|}, \tau_1 = \tau_0(x_*), \tau_2 = \frac{t_*-t(x_*)}{T-t(x_*)}, \lambda_5 = C_7^{**}$ and $\Lambda_5 = C_7^*$, to derive that: for all $\xi \in [-\xi_0, \xi_0]$

$$\left| U(x_*, \xi, \tau(x_*, t_*)) - \hat{U}(\tau(x_*, t_*)) \right| \leq C(K_0, \Lambda_5 \lambda_5, \delta_1, \xi_0),$$

where $C(K_0, \Lambda_5, \lambda_5, \delta_1, \xi_0) \to 0$ as $(\delta_1, \xi_0) \to (0, +\infty)$. Taking $(\delta_1, \xi_0) \to (0, +\infty)$ and note that $\xi_0 \to +\infty$ as $\epsilon_0 \to 0$, we write

$$\left| U(x_*, \xi_*, \tau(x_*, t_*)) - \hat{U}(\tau(x_*, t_*)) \right| \leq \frac{\delta_0}{2},$$

this is a contradiction.

If (II) occurs, we have for all $|x| \in \left[\frac{\epsilon_0}{8}, +\infty\right)$

$$|u(x,t)| \leq C(\epsilon_0, A, \delta_0, \eta_0), \forall t \in [0, t_*].$$

Indeed, we consider the two following cases:

+ The case where $|x| \geq \frac{\epsilon_0}{4}$, using item (iii) if the definition of $S(t)$, we derive the following

$$|u(x,t)| \leq |u(x,0)| + \eta_0 \leq C(A, \eta_0), \forall t \in [0, t_*].$$

+ The case where $|x| \in \left[\frac{\epsilon_0}{8}, \frac{\epsilon_0}{4}\right]$, using item (ii) in the definition of $S(t)$, we have

$$|u(x,t)| \leq C(\delta_0) \left(T - t(x)\right)^{-\frac{1}{p-1}} \leq C(\epsilon_0, \delta_0), \forall t \in [0, t_*].$$

Then, we apply Lemma 3.20 with $\eta \leq \frac{\eta_0}{2}$ and $\sigma = C(\epsilon_0, A, \delta_0, \eta_0)$, to derive the following

$$|u(x^*, t_*) - u(x^*, 0)| \leq \frac{\eta_0}{2}.$$

Therefore, (II) can not occurs. Thus, (3.153) follows. In addition to that, from (3.153), Proposition 3.18 and Lemma 3.22, we conclude the proof of item (i) of Proposition 3.16. Since, item (ii) follows from item (i) (see for instance the proof of Proposition 3.6, given in [5]). This concludes the proof of Proposition 3.16.

3.5 Cauchy problem for equation (3.1)

In this section, we give a proof to a local Cauchy problem in time.

Lemme 3.23 (A local Cauchy problem for a complex heat equation)**.** *Let u_0 be any function in $L^\infty(\mathbb{R}^N)\left(\mathbb{R}^N, \mathbb{C}\right)$ such that*

$$Re(u_0(x)) \geq \lambda, \forall x \in \mathbb{R}^N, \tag{3.154}$$

for some constant $\lambda > 0$. Then, there exists $T_1 > 0$ such that equation (3.1) *with initial data u_0, has a unique solution on $(0, T_1]$. Moreover, $u \in C\left((0, T_1], L^\infty(\mathbb{R}^N)\right)$ and*

$$Re(u(t)) \geq \frac{\lambda}{2}, \forall (t,x) \in [0, T_1] \times \mathbb{R}^N.$$

Proof. The proof relies on a fixed-point argument. Indeed, we consider the space

$$X = C\left((0, T_1], L^\infty(\mathbb{R}^N)(\mathbb{R}^N, \mathbb{C})\right).$$

It is easy to check that X is an Banach space with the following norm

$$\|u\|_X = \sup_{t\in(0,T_1]} \|u(t)\|_{L^\infty(\mathbb{R}^N)}, \forall u = (u(t))_{t\in(0,T_1]} \in X.$$

We also introduce the closed set $B_\lambda^+\left(0, 2\|u_0\|_{L^\infty(\mathbb{R}^N)}\right) \subset X$ defined as follows

$$\begin{aligned} B_\lambda^+\left(0, 2\|u_0\|_{L^\infty(\mathbb{R}^N)}\right) &= \left\{u \in X \text{ such that } \|u\|_X \le 2\|u_0\|_{L^\infty(\mathbb{R}^N)}\right\} \\ &\cap \left\{u \in X | \forall t \in (0, T_1], \mathrm{Re}(u(t,x)) \ge \frac{\lambda}{2} \text{ a. e}\right\}. \end{aligned}$$

Let $\mathcal{Y}$ be the following mapping

$$\mathcal{Y} : B_\lambda^+\left(0, 2\|u_0\|_{L^\infty(\mathbb{R}^N)}\right) \to X,$$

where $\mathcal{Y}(u) = (\mathcal{Y}(u)(t))_{t\in(0,T_1]}$ is defined by

$$\mathcal{Y}(u)(t) = e^{t\Delta}(u_0) + \int_0^t e^{(t-s)\Delta}(u^p(s))ds. \tag{3.155}$$

Note that, when $u \in B_\lambda^+\left(0, 2\|u_0\|_{L^\infty(\mathbb{R}^N)}\right)$, u^p is well defined as in (3.18) and (3.19). We claim that there exists $T^* = T^*(\|u_0\|_{L^\infty(\mathbb{R}^N)}, \lambda) > 0$ such that for all $0 < T_1 \le T^*$, the following assertion hold:

(i) The mapping is reflexive on $B_\lambda^+\left(0, 2\|u_0\|_{L^\infty(\mathbb{R}^N)}\right)$, meaning that

$$\mathcal{Y} : B_\lambda^+\left(0, 2\|u_0\|_{L^\infty(\mathbb{R}^N)}\right) \to B_\lambda^+\left(0, 2\|u_0\|_{L^\infty(\mathbb{R}^N)}\right).$$

(ii) The mapping $\mathcal{Y}$ is a contraction mapping on $B_\lambda^+\left(0, 2\|u_0\|_{L^\infty(\mathbb{R}^N)}\right)$:

$$\|\mathcal{Y}(u_1) - \mathcal{Y}(u_2)\|_X \le \frac{1}{2}\|u_1 - u_2\|_X,$$

for all $u_1, u_2 \in B_\lambda^+\left(0, 2\|u_0\|_{L^\infty(\mathbb{R}^N)}\right)$.

The proof for (i)*:* By observe that, by using the regular property of operator $e^{t\Delta}$, we conclude that $\mathcal{Y}(u) \in C\left((0, T_1], L^\infty(\mathbb{R}^N)(\mathbb{R}^N, \mathbb{C}) \cap C(\mathbb{R}^N, \mathbb{C})\right)$. Besides that, for all $u \in B_\lambda^+\left(0, 2\|u_0\|_{L^\infty(\mathbb{R}^N)}\right)$ we derive from (3.155) that for all $t \in (0, T_1]$

$$\begin{aligned} \|\mathcal{Y}(u)(t)\|_{L^\infty(\mathbb{R}^N)} &= \left\|e^{t\Delta}(u_0) + \int_0^t e^{(t-s)\Delta}(u^p(s))ds\right\|_{L^\infty(\mathbb{R}^N)} \\ &\le \left\|e^{t\Delta}(u_0)\right\|_{L^\infty(\mathbb{R}^N)} + \left\|\int_0^t e^{(t-s)\Delta}(u^p(s))ds\right\|_{L^\infty(\mathbb{R}^N)} \\ &\le \|u_0\|_{L^\infty(\mathbb{R}^N)} + t2^p\|u_0\|^p_{L^\infty(\mathbb{R}^N)}. \end{aligned}$$

Hence, if we take $T_1 \leq \frac{1}{2^p \|u_0\|^{p-1}_{L^\infty(\mathbb{R}^N)}}$ then we have

$$\|\mathcal{Y}(u)\|_X = \sup_{t\in(0,T_1]} \|\mathcal{Y}(u)\|_{L^\infty(\mathbb{R}^N)} \leq 2\|u_0\|_{L^\infty(\mathbb{R}^N)}.$$

Now, let us note from (3.154) that

$$\mathrm{Re}\left(e^{t\Delta}(u_0)\right) = e^{t\Delta}\left(\mathrm{Re}(u_0)\right) \geq e^{t\Delta}\left(\lambda\right) = \lambda.$$

Therefore, from (3.155) for all $(t,x) \in (0,T_1] \times \mathbb{R}^N$

$$\mathrm{Re}(\mathcal{Y}(u)(t,x)) \geq \lambda - \left\| \int_0^t e^{(t-\tau)\Delta}(u^p)(\tau)d\tau \right\|_{L^\infty(\mathbb{R}^N)}.$$

Note that,

$$\left\| \int_0^t e^{(t-\tau)\Delta}(u^p)(\tau)d\tau \right\|_{L^\infty(\mathbb{R}^N)} \leq t2^p\|u_0\|^p_{L^\infty(\mathbb{R}^N)}.$$

So, if $T_1 \leq \frac{\lambda}{2^{p+1}\|u_0\|_{L^\infty(\mathbb{R}^N)}}$, then for all $t \in (0,T_1] \times \mathbb{R}^N$

$$\mathrm{Re}(\mathcal{Y}(u)(t,x)) \geq \frac{\lambda}{2}.$$

Therefore,

$$\mathcal{Y}(u) \in B^+_\lambda\left(0, 2\|u_0\|_{L^\infty(\mathbb{R}^N)}\right).$$

The proof of (ii): We first recall that the function $G(u) = u^p, u \in \mathbb{C}$ is analytic on

$$\left\{u \in \mathbb{C} \text{ such that } \mathrm{Re}(u) \geq \frac{\lambda}{2}\right\}.$$

Then, there exists $C_2(\|u_0\|_{L^\infty(\mathbb{R}^N)}, \lambda) > 0$ such that

$$\begin{aligned} \|\mathcal{Y}(u_1) - \mathcal{Y}(u_2)\|_X &= \sup_{t\in(0,T_1]} \left\| \int_0^t e^{(t-s)\Delta}\left(u_1^p - u_2^p\right)(s)ds \right\|_{L^\infty(\mathbb{R}^N)} \\ &\leq T_1 C_2 \sup_{t\in(0,T_1]} \|u_1 - u_2\|_{L^\infty(\mathbb{R}^N)}. \end{aligned}$$

Then, if we impose

$$T_1 \leq \frac{1}{2C_2},$$

(ii) follows.

We now choose $T^* = \min\left(\frac{1}{2^p\|u_0\|^{p-1}_{L^\infty(\mathbb{R}^N)}}, \frac{\lambda}{2^{p+1}\|u_0\|^p_{L^\infty(\mathbb{R}^N)}}, \frac{1}{2C_2}\right)$. Then, for all $T_1 \leq T^*$, item (i) and (ii) hold. Thanks to a Banach fixed-point argument, there exists a unique $u \in B^+_\lambda\left(0, 2\|u_0\|_{L^\infty(\mathbb{R}^N)}\right)$ such that

$$\mathcal{Y}(u)(t) = u(t), \forall t \in (0, T_1],$$

and we easily check that $u(t)$ satisfies equation (3.1) for all $(0,T_1]$ with $u(0) = u_0$. Moreover, from the definition of $B^+_\lambda\left(0, 2\|u_0\|_{L^\infty(\mathbb{R}^N)}\right)$ we have

$$\mathrm{Re}(u)(t,x) \geq \frac{\lambda}{2}.$$

This concludes the proof of Lemma 3.23. □

3.6 Some Taylor expansions

In this section appendix, we state and prove several technical and straightforward results needed in our paper.

Lemme 3.24 (Asymptotics of $\bar{B}_1, \bar{B}_2$)**.** *We consider* $\bar{B}_1(\bar{w}_1, w_2)$ *as in* (3.25), (3.26). *Then, the following holds:*

$$\bar{B}_1(\bar{w}_1, w_2) = \frac{p}{2\kappa}\bar{w}_1^2 + O(|\bar{w}_1|^3 + |w_2|^2), \quad (3.156)$$

$$\bar{B}_2(\bar{w}_1, w_2) = \frac{p}{\kappa}\bar{w}_1 w_2 + O\left(|\bar{w}_1|^2|w_2|\right) + O\left(|w_2|^3\right). \quad (3.157)$$

as $(\bar{w}_1, w_2) \to (0,0)$.

Proof. The proof of (3.156) is quite the same as the proof of (3.157). So, we only prove (3.157), hoping the reader will have no problem to check (3.156) if necessary. Since, $\kappa = (p-1)^{-\frac{1}{p-1}} > 0$, we derive $\kappa + \bar{w}_1 > 0$ when $\bar{w}_1$ is near 0, so we can write $B_2(\bar{w}_1, w_2)$ as follows

$$\bar{B}_2(\bar{w}_1, w_2) = \left((\kappa+\bar{w}_1)^2 + w_2^2\right)^{\frac{p}{2}} \sin\left[p \arcsin\left(\frac{w_2}{\sqrt{(\kappa+\bar{w}_1)^2 + w_2^2}}\right)\right] - \frac{p}{p-1} w_2,$$

as $\bar{w}_1 \to 0$. Thus,

$$\begin{aligned} B_2(\bar{w}_1, w_2) &= \left((\kappa+\bar{w}_1)^2 + w_2^2\right)^{\frac{p}{2}} \frac{p w_2}{\sqrt{(\kappa+\bar{w}_1)^2 + w_2^2}} - \frac{p}{p-1} w_2 \\ &+ \left((\kappa+\bar{w}_1)^2 + w_2^2\right)^{\frac{p}{2}} \left\{\sin\left[p\arcsin\left(\frac{w_2}{\sqrt{(\kappa+\bar{w}_1)^2 + w_2^2}}\right)\right] - \frac{p w_2}{\sqrt{(\kappa+\bar{w}_1)^2 + w_2^2}}\right\} \\ &= \left((\kappa+\bar{w}_1)^2 + w_2^2\right)^{\frac{p-1}{2}} p w_2 - \frac{p}{p-1} w_2 \\ &+ \left((\kappa+\bar{w}_1)^2 + w_2^2\right)^{\frac{p}{2}} \left\{\sin\left[p\arcsin\left(\frac{w_2}{\sqrt{(\kappa+\bar{w}_1)^2 + w_2^2}}\right)\right] - \frac{p w_2}{\sqrt{(\kappa+\bar{w}_1)^2 + w_2^2}}\right\} \\ &= (I) + (II). \end{aligned}$$

In addition to that, we have the fact

$$\sin(px) - px = O(|x|^3),$$
$$\frac{w_2}{\sqrt{(\kappa+\bar{w}_1)^2 + w_2^2}} = O(|w_2|),$$

as $x \to 0$ and $(\bar{w}_1, w_2) \to (0,0)$. Plugging these estimates in (II), we obtain

$$(II) = O(|w_2|^3).$$

as $(\bar{w}_1, w_2) \to (0,0)$. For (I), we use a Taylor expansion for $((\kappa+\bar{w}_1)^2 + w_2^2)$, around $(\bar{w}_1, w_2) = (0,0)$:

$$((\kappa+\bar{w}_1)^2 + w_2^2)^{\frac{p}{2}} = \frac{1}{p-1} + \frac{(p-1)}{\kappa(p-1)}\bar{w}_1 + O(|\bar{w}_1|^2) + O(|w_2|^2).$$

Plugging this in (I), we derive the following:

$$(I) = \frac{p}{\kappa}\bar{w}_1 w_2 + O(|\bar{w}_1|^2 w_2) + O(|w_2|^3),$$

as $(\bar{w}_1, w_2) \to (0,0)$. From the estimates of (I) and (II), we conclude the Lemma. □

In the following lemma, we aim at giving some bounds and expansions of V and $V_{i,j}, j,k \in \{1,2\}$

Lemme 3.25 (The potential functions V and $V_{j,k}$ with $j,k \in \{1,n\}$). *We consider $V, V_{1,1}, V_{1,2}$, $V_{2,1}$ and $V_{2,2}$ defined in* (3.57) *and* (3.58) - (3.61). *Then, the following holds:*

(i) For all $s \geq 1$ and $y \in \mathbb{R}^N$, we have $|V(y,s)| \leq C$,

$$|V(y,s)| \leq \frac{C(1+|y|^2)}{s}, \tag{3.158}$$

and

$$V(y,s) = -\frac{(|y|^2 - 2N)}{4s} + \tilde{V}(y,s), \tag{3.159}$$

where $\tilde{V}$ satisfies

$$|\tilde{V}(y,s)| \leq C\frac{(1+|y|^4)}{s^2}, \forall s \geq 1, |y| \leq 2K_0\sqrt{s}. \tag{3.160}$$

(ii) Potential functions $V_{j,k}$ with $j,k \in \{1,2\}$ satisfy the following estimates

$$\begin{aligned} \|V_{1,1}\|_{L^\infty(\mathbb{R}^N)} + \|V_{2,2}\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C}{s^2}, \\ \|V_{1,2}\|_{L^\infty(\mathbb{R}^N)} + \|V_{2,1}\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C}{s}, \end{aligned}$$

and

$$\begin{aligned} |V_{1,1}(y,s)| + |V_{2,2}(y,s)| &\leq \frac{C(1+|y|^4)}{s^4}, \\ |V_{1,2}(y,s)| + |V_{2,1}(y,s)| &\leq \frac{C(1+|y|^2)}{s^2}, \end{aligned}$$

for all $s \geq 1$ and $y \in \mathbb{R}^N$.

Proof. We note that the proof of (i) was given in Lemma B.1, page 1270 in [21]. So, it remains to prove item (ii). Moreover, the technique for these estimates is the same, so we only give the proof to the following estimates:

$$\|V_{1,1}\|_{L^\infty(\mathbb{R}^N)} + \|V_{2,2}\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{s^2}, \tag{3.161}$$

$$|V_{1,1}(y,s)| + |V_{2,2}(y,s)| \leq \frac{C(1+|y|^4)}{s^4}. \tag{3.162}$$

\+ *The proof of* (3.161)*:* We recall the expressions of $V_{1,1}$ and $V_{2,2}$:

$$\begin{aligned} V_{1,1} &= \partial_{u_1} F_1(u_1,u_2)|_{(u_1,u_2)=(\Phi_1,\Phi_2)} - p\Phi_1^{p-1}, \\ V_{2,2} &= \partial_{u_2} F_2(u_1,u_2)|_{(u_1,u_2)=(\Phi_1,\Phi_2)} - p\Phi_1^{p-1}, \end{aligned}$$

where Φ_1, Φ_2 are given by (3.58) and (3.61). Hence, we can rewrite $V_{1,1}$ and $V_{2,2}$ as follows

$$\begin{aligned} V_{1,1} &= p(u_1^2+u_2^2)^{\frac{p-2}{2}} \left(u_1 \cos\left[p \arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}} \right) \right] + u_2 \sin\left[p \arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}} \right) \right] \right) \\ &- p\Phi_1^{p-1}, \\ V_{2,2} &= p(u_1^2+u_2^2)^{\frac{p-2}{2}} \left(u_1 \cos\left[p \arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}} \right) \right] \right) + u_2 \sin\left[p \arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}} \right) \right] \\ &- p\Phi_1^{p-1}, \end{aligned}$$

We first estimate to $V_{1,1}$, from the above equalities, we decompose $V_{1,1}$ into the following

$$V_{1,1} = V_{1,1,1} + V_{1,1,2} + V_{1,1,3}, \tag{3.163}$$

where

$$\begin{aligned} V_{1,1,1} &= p\left(\Phi_1^2+\Phi_2^2\right)^{\frac{p-2}{2}} \Phi_1 - p\Phi_1^{p-1}, \\ V_{1,1,2} &= p\left(\Phi_1^2+\Phi_2^2\right)^{\frac{p-2}{2}} \Phi_1 \left(\cos\left[p \arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}} \right) \right] - 1 \right), \\ V_{1,1,3} &= p(\Phi_1^2+\Phi_2^2)^{\frac{p-2}{2}} \Phi_2 \sin\left[p \arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}} \right) \right]. \end{aligned}$$

As matter of fact, from the definitions of Φ_1, Φ_2, we have the following

$$\left\| \frac{\Phi_2(.,s)}{\Phi_1(.,s)} \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{s}, \tag{3.164}$$

$$\|\Phi_1(.,s)\|_{L^\infty(\mathbb{R}^N)} \leq C, \tag{3.165}$$

$$\|\Phi_2(.,s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{s}, \tag{3.166}$$

for all $s \geq 1$ and

$$|\cos(p \arcsin x) - 1| \leq C|x|^2, \tag{3.167}$$

$$|\sin(p \arcsin x) - px| \leq C|x|^3, \tag{3.168}$$

for all $|x| \leq 1$. By using (3.164), (3.165), (3.166), (3.167) and (3.168), we get the following bound for $V_{1,1,2}$ and $V_{1,1,3}$

$$\|V_{1,1,2}(.,s)\|_{L^\infty(\mathbb{R}^N)} + \|V_{1,1,3}(.,s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{s^2}. \tag{3.169}$$

For $V_{1,1,1}$, using (3.164), we derive

$$|V_{1,1,1}| = \left| p\Phi_1^{p-1} \left(\left(1 + \frac{\Phi_2^2}{\Phi_1^2} \right)^{\frac{p-2}{2}} - 1 \right) \right| \leq \frac{C}{s^2}.$$

This gives the following

$$\|V_{1,1}(.,s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{s^2}.$$

We can apply the technique to $V_{2,2}$ to get a similar estimate as follows

$$\|V_{2,2}(.,s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{s^2}.$$

Then, (3.161) follows.

+ *The proof of* (3.162)*:* We can see that on the domain $\{|y| \geq K_0\sqrt{s}\}$, we have the following fact

$$\frac{1+|y|^4}{s^4} \geq \frac{C}{s^2},$$

and in particular, (3.161) holds. Thus, for all $|y| \geq K_0\sqrt{s}$, we have

$$|V_{1,1}(y,s)| + |V_{2,2}(y,s)| \leq \frac{C(|y|^4+1)}{s^4}.$$

Therefore, it is sufficient to give the estimate on the domain $\{|y| \leq 2K_0\sqrt{s}\}$. In fact, on this domain there exists $C(K_0) > 0$ such that

$$\frac{1}{C} \leq \Phi_1(y,s) \leq C.$$

In addition to that, using the definition of Φ_2 given by (3.53), we derive the following

$$|\Phi_2(y,s)| \leq C\frac{(|y|^2+1)}{s^2}, \forall (y,s) \in \mathbb{R}^N \times [1,+\infty). \tag{3.170}$$

Then, from (3.163) we have

$$\begin{aligned} |V_{1,1,2}(y,s)| &\leq |\Phi_2(y,s)|^2 \leq C\frac{(1+|y|^4)}{s^4}, \\ |V_{1,1,3}(y,s)| &\leq |\Phi_2(y,s)|^2 \leq C\frac{(1+|y|^4)}{s^4}. \end{aligned}$$

We now estimate $V_{1,1,1}$, thanks to a Taylor expansion of $(\Phi_1^2+\Phi_2^2)^{\frac{p-2}{2}}$, around Φ_2

$$\left|(\Phi_1^2+\Phi_2^2)^{\frac{p-2}{2}} - \Phi_1^{p-2}\right| \leq C|\Phi_2|^2.$$

This directly yields

$$|V_{1,1,1}(y,s)| \leq C(K_0)|\Phi_2|^2 \leq C\frac{(1+|y|^4)}{s^4}.$$

So,

$$|V_{1,1}(y,s)| \leq C\frac{(1+|y|^4)}{s^4}, \forall y \in \mathbb{R}^N.$$

Moreover, we can proceed similarly for $V_{2,2}$, and get

$$|V_{2,2}(y,s)| \leq C\frac{(1+|y|^4)}{s^4}, \forall y \in \mathbb{R}^N.$$

Thus, (3.162) follows. $\square$

Now, we give some estimates on quadratic terms $B_1(q_1, q_2)$ and $B_2(q_1, q_2)$

Lemme 3.26 (The terms $B_1(q_1, q_2)$ and $B_2(q_1, q_2)$)**.** *Let us consider $B_1(q_1, q_2)$ and $B_2(q_1, q_2)$, defined in* (3.62) *and* (3.63)*, respectively. For all $A \geq 1$, there exists $s_9(A) \geq 1$ such that for all $s_0 \geq s_9(A)$, if $(q_1, q_2)(s) \in V_A(s)$ and $q_1(s) + \Phi_1(s) \geq \frac{1}{2} e^{-\frac{s}{p-1}}$ for all $s \in [s_0, s_1]$, then, the following holds: for all $s \in [s_0, s_1]$,*

$$
\begin{aligned}
|\chi(y,s) B_1(q_1, q_2)| &\leq C \left(|q_1|^2 + |q_2|^2 \right), && (3.171)\\
|\chi(y,s) B_2(q_1, q_2)| &\leq C \left(\frac{|q_1|^2}{s} + |q_1||q_2| + |q_2|^2 \right), && (3.172)\\
\|B_1(q_1, q_2)\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{CA^4}{\frac{\min(2,p)}{2}}, && (3.173)\\
\|B_2(q_1, q_2)\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{CA^2}{s^{1+\min\left(\frac{p-1}{4}, \frac{1}{2}\right)}}, && (3.174)
\end{aligned}
$$

where $\chi(y,s)$ is defined as in (3.66)*.*

Proof. We first would like to note that the condition $q_1(s) + \Phi_1(s) \geq \frac{1}{2} e^{-\frac{s}{p-1}}$ is to ensure that the real part $w_1 = q_1(s) + \Phi_1(s) > 0$. Then, (3.16) holds and functions F_1 and F_2 which are involved in the definitions of B_1 and B_2 are well-defined (see (3.18)). For the proof of Lemma 3.26, we only prove for (3.172) and (3.174), because the other ones follow similarly.

\+ *The proof for* (3.172)*:* Using the fact that the support of $\chi(y,s)$ is $\{|y| \leq 2K_0\sqrt{s}\}$, it is enough to prove (3.172) for all $\{|y| \leq 2K_0\sqrt{s}\}$. Since we have $(q_1, q_2) \in V_A(s)$, we derive from item (ii) of Lemma 3.10 and the definition of Φ_1, Φ_2 that

$$
\frac{1}{C} \leq q_1 + \Phi_1 \leq C, \quad |q_2 + \Phi_2| \leq \frac{C}{s}.
$$

and

$$
|q_1| \leq \frac{CA}{\sqrt{s}}, \quad |q_2| \leq \frac{CA^2}{s^{\frac{p_1+2}{2}}}, \forall |y| \leq 2K_0\sqrt{s}. \tag{3.175}
$$

In addition to that, we write $B_2(q_1, q_2)$ as follows:

$$
\begin{aligned}
B_2(q_1, q_2) &= F_2\left(\Phi_1 + q_1, \Phi_2 + q_2\right) - F_2(\Phi_1, \Phi_2) - \partial_{u_1} F_2(q_1 + \Phi_1, q_2 + \Phi_2) q_1\\
&- \partial_{u_2} F_2(q_1 + \Phi_1, q_2 + \Phi_2) q_2.
\end{aligned}
$$

where

$$
F_2(u_1, u_2) = \left(u_1^2 + u_2^2\right)^{\frac{p}{2}} \sin\left[p \arcsin\left(\frac{u_2}{\sqrt{u_1^2 + u_2^2}} \right) \right].
$$

Using a Taylor expansion for the function $F_2(q_1 + \Phi_1, q_2 + \Phi_2)$ at $(q_1, q_2) = (0, 0)$, we derive the following

$$
\begin{aligned}
F_2(q_1 + \Phi_1, q_2 + \Phi_2) &= \sum_{j+k \leq 4} \frac{1}{j!k!} \partial^{j+k}_{q_1^j q_2^k} \left(F_2(q_1 + \Phi_1, q_2 + \Phi_2) \right) \Big|_{(q_1,q_2)=(0,0)} q_1^j q_2^k +\\
&+ \sum_{j+k=5} G_{j,k}(q_1, q_2) q_1^j q_2^k,
\end{aligned}
$$

where

$$G_{j,k}(q_1,q_2) = \frac{5}{j!k!}\int_0^1 (1-t)^4 \partial^5_{q_1^j q_2^k}(F_2(\Phi_1+tq_1,\Phi_2+tq_2))dt.$$

In particular, we have

$$|G_{j,k}(q_1,q_2)| \le C, \forall j+k=5.$$

As a matter of fact, we have

$$\partial^{j+k}_{q_1^j q_2^k}(F_2(q_1+\Phi_1,q_2+\Phi_2))\big|_{(q_1,q_2)=(0,0)} = \partial^{j+k}_{u_1^j u_2^k}F_2(u_1,u_2)\big|_{(u_1,u_2)=(0,0)} \tag{3.176}$$

Therefore, from (3.175), we have

$$\begin{aligned}
&\left|F_2(q_1+\Phi_1,q_2+\Phi_2) - \sum_{j+k\le 5}\frac{1}{j!k!}\partial^{j+k}_{u_1^j u_2^k}F_2(u_1,u_2)\big|_{(u_1,u_2)=(\Phi_1,\Phi_2)}\, q_1^j q_2^k\right| \\
\le\ & C\sum_{j=0}^{5}|q_1^j q_2^{5-j}| \le C\left(\frac{|q_1|^2}{s} + |q_1||q_2| + |q_2|^2\right).
\end{aligned}$$

In addition to that, we have the following fact,

$$|\partial^{j+k}_{u_1^j u_2^k}F_2(u_1,u_2)\big|_{(u_1,u_2)=(\Phi_1,\Phi_2)}| \le C, \forall j+k\le 4,$$

and for all $1\le j\le 4$, we have

$$\left|\partial^j_{u_1^j}F_2(u_1,u_2)\right|_{(u_1,u_2)=(\Phi_1,\Phi_2)} \le \frac{C}{s}.$$

This concludes (3.172).

The proof of (3.174)*:* We rewrite $B_2(q_1,q_2)$ explicitly as follows:

$$\begin{aligned}
B_2(q_1,q_2) &= \left((q_1+\Phi_1)^2+(q_2+\Phi_2)^2\right)^{\frac{p}{2}}\sin\left[p\arcsin\left(\frac{q_2+\Phi_2}{\sqrt{(q_1+\Phi_1)^2+(q_2+\Phi_2)^2}}\right)\right] \\
-\ & (\Phi_1^2+\Phi_2^2)^{\frac{p}{2}}\sin\left[p\arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}}\right)\right] \\
-\ & p\left(\Phi_1^2+\Phi_2^2\right)^{\frac{p-2}{2}}\left(\Phi_1\sin\left[p\arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}}\right)\right] - \Phi_2\cos\left[p\arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}}\right)\right]\right)q_1 \\
-\ & p\left(\Phi_1^2+\Phi_2^2\right)^{\frac{p-2}{2}}\left(\Phi_2\sin\left[p\arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}}\right)\right] + \Phi_1\cos\left[p\arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}}\right)\right]\right)q_2.
\end{aligned}$$

Then, we decompose $B_2(q_1,q_2)$ as follows:

$$B_2(q_1,q_2) = B_{2,1}(q_1,q_2)+B_{2,2}(q_1,q_2)+B_{2,3}(q_1,q_2)+B_{2,4}(q_1,q_2)+B_{2,5}(q_1,q_2)+B_{2,6}(q_1,q_2),$$

where

$$\begin{aligned}
B_{2,1}(q_1,q_2) &= p(q_2+\Phi_2)\left((q_1+\Phi_1)^2+(q_2+\Phi_2)^2\right)^{\frac{p-1}{2}} - p(\Phi_1^2+\Phi_2^2)^{\frac{p-1}{2}}\Phi_2 \qquad (3.177)\\
&- p\left(\Phi_1^2+\Phi_2^2\right)^{\frac{p-2}{2}}\Phi_1 q_2,\\
B_{2,2}(q_1,q_2) &= ((q_1+\Phi_1)^2+(q_2+\Phi_2)^2)^{\frac{p}{2}}\left\{\sin\left[p\arcsin\left(\frac{q_2+\Phi_2}{\sqrt{(q_1+\Phi_1)^2+(q_2+\Phi_2)^2}}\right)\right]\right.\\
&- \left.\frac{p(q_2+\Phi_2)}{\sqrt{(q_1+\Phi_1)^2+(q_2+\Phi_2)^2}}\right\}, \qquad (3.178)\\
B_{2,3}(q_1,q_2) &= \left(\Phi_1^2+\Phi_2^2\right)^{\frac{p}{2}}\left(\frac{p\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}} - \sin\left[p\arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}}\right)\right]\right), \qquad (3.179)\\
B_{2,4}(q_1,q_2) &= p(\Phi_1^2+\Phi_2^2)^{\frac{p-2}{2}}\Phi_1\left(1-\cos\left[p\arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}}\right)\right]\right)q_2, \qquad (3.180)\\
B_{2,5}(q_1,q_2) &= p\left(\Phi_1^2+\Phi_2^2\right)^{\frac{p-2}{2}}\left\{\Phi_2\cos\left[p\arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}}\right)\right]\right.\\
&- \left.\Phi_1\sin\left[p\arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}}\right)\right]\right\}q_1, \qquad (3.181)\\
B_{2,6}(q_1,q_2) &= -p(\Phi_1^2+\Phi_2^2)^{\frac{p-2}{2}}\Phi_2\sin\left[p\arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}}\right)\right]q_2. \qquad (3.182)
\end{aligned}$$

we prove that: for all $y\in\mathbb{R}^N$:

$$|B_{2,j}(q_1,q_2)| \le \frac{CA^2}{s^{1+\min\left(\frac{p-1}{4},\frac{1}{2}\right)}}, \forall j=1,...,6.$$

We now aim at an estimate on $B_{2,1}(q_1,q_2)$: We first need to prove the following:

$$\left|\left((\Phi_1+q_1)^2+(\Phi_2+q_2)^2\right)^{\frac{p-1}{2}} - (\Phi_1^2+\Phi_2^2)^{\frac{p-1}{2}}\right| \le C\,|Z|^{\min\left(\frac{p-1}{2},1\right)}, \qquad (3.183)$$

where

$$|Z| = 2q_1\Phi_1+2q_2\Phi_2+q_1^2+q_2^2.$$

Note that Z is bounded. On the other hand, we have $(\Phi_1+q_1)^2+(\Phi_2+q_2)^2)^{\frac{p-1}{2}} = (\Phi_1^2+\Phi_2^2+Z)^{\frac{p-1}{2}}$. Then, if $\frac{p-1}{2}\ge 1$, using a Taylor expansion of the function $(\Phi_1^2+\Phi_2^2+Z)^{\frac{p-1}{2}}$ around $Z_0=0$ (note that $\Phi_1^2+\Phi_2^2$ is uniformly bounded), we obtain the following:

$$\left|\left((\Phi_1+q_1)^2+(\Phi_2+q_2)^2\right)^{\frac{p-1}{2}} - (\Phi_1^2+\Phi_2^2)^{\frac{p-1}{2}}\right| \le C\,|Z|,$$

which yields (3.183). If $\frac{p-1}{2}<1$, then, we have

$$\left|\left((\Phi_1+q_1)^2+(\Phi_2+q_2)^2\right)^{\frac{p-1}{2}} - (\Phi_1^2+\Phi_2^2)^{\frac{p-1}{2}}\right| = \left(\Phi_1^2+\Phi_2^2\right)^{\frac{p-1}{2}}\left|(1+\xi)^{\frac{p-1}{2}}-1\right|,$$

where

$$\xi = \frac{Z}{\Phi_1^2+\Phi_2^2}.$$

In particular, we have $\xi \geq -1$. In addition to that, we have the following fact: for all $\xi \geq -1$

$$\left|(1+\xi)^{\frac{p-1}{2}} - 1\right| \leq C\,|\xi|^{\frac{p-1}{2}} \tag{3.184}$$

Therefore, (3.184) gives the following

$$\left|\left((\Phi_1+q_1)^2+(\Phi_2+q_2)^2\right)^{\frac{p-1}{2}} - \left(\Phi_1^2+\Phi_2^2\right)^{\frac{p-1}{2}}\right| \leq C\left(\Phi_1^2+\Phi_2^2\right)^{\frac{p-1}{2}}\left|\frac{Z}{\Phi_1^2+\Phi_2^2}\right|^{\frac{p-1}{2}} \leq C\,|Z|^{\frac{p-1}{2}}.$$

Then, (3.183) follows. Using $(q_1,q_2)(s) \in V_A(s)$ and $Z = 2\Phi_1 q_1 + 2\Phi_2 q_2 + q_1^2 + q_2^2$, we write

$$\|Z\|_{L^\infty(\mathbb{R}^N)} \leq \frac{CA^2}{\sqrt{s}}, \forall s \geq 1.$$

So, we deduce from (3.183) that

$$\|p\Phi_2\left(\left((\Phi_1+q_1)^2+(\Phi_2+q_2)^2\right)^{\frac{p-1}{2}}\right) - p\Phi_2(\Phi_1^2+\Phi_2^2)^{\frac{p-1}{2}}\|_{L^\infty(\mathbb{R}^N)} \leq \frac{CA^2}{s^{1+\min\left(\frac{p-1}{4},\frac{1}{2}\right)}}. \tag{3.185}$$

Using (3.183), we have the following

$$\left\|\left((\Phi_1+q_1)^2+(\Phi_2+q_2)^2\right)^{\frac{p-1}{2}} - (\Phi_1^2+\Phi_2^2)^{\frac{p-2}{2}}\Phi_1\right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{CA^2}{s^{\min\left(\frac{p-1}{4},\frac{1}{2}\right)}}. \tag{3.186}$$

Indeed, we have

$$\begin{aligned}
& \left|\left((\Phi_1+q_1)^2+(\Phi_2+q_2)^2\right)^{\frac{p-1}{2}} - (\Phi_1^2+\Phi_2^2)^{\frac{p-2}{2}}\Phi_1\right| \\
\leq & \left|\left((\Phi_1+q_1)^2+(\Phi_2+q_2)^2\right)^{\frac{p-1}{2}} - (\Phi_1^2+\Phi_2^2)^{\frac{p-1}{2}}\right| \\
+ & \left|(\Phi_1^2+\Phi_2^2)^{\frac{p-1}{2}} - (\Phi_1^2+\Phi_2^2)^{\frac{p-2}{2}}\Phi_1\right| \\
\leq & \frac{CA^2}{s^{\frac{\min\left(\frac{p-1}{2},1\right)}{2}}} + \frac{C}{s^2}.
\end{aligned}$$

Then, (3.186) holds.

On the other hand, using (3.186) and the following

$$\|q_2(.,s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{CA^3}{s^{\frac{p_1+2}{2}}}, p_1 > 0,$$

we conclude that

$$\left\|pq_2\left((\Phi_1+q_1)^2+(\Phi_2+q_2)^2\right)^{\frac{p-1}{2}} - (\Phi_1^2+\Phi_2^2)^{\frac{p-2}{2}}\Phi_1\right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{CA^2}{s^{1+\min\left(\frac{p-1}{4},\frac{1}{2}\right)}}, \tag{3.187}$$

provided that $s \geq s_{1,9}(A)$. From (3.185) and (3.187), we have

$$\|B_{2,1}(q_1,q_2)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{CA^2}{s^{1+\min\left(\frac{p-1}{4},\frac{1}{2}\right)}}. \tag{3.188}$$

We next give a bound to $B_{2,2}(q_1,q_2)$: Using the following fact

$$|\sin(p\arcsin x) - px| \le C|x|^3, \forall |x| \le 1,$$

we derive the following

$$\begin{aligned}&\left|\sin\left[p\arcsin\left(\frac{q_2+\Phi_2}{\sqrt{(q_1+\Phi_1)^2+(q_2+\Phi_2)^2}}\right)\right] - \frac{p(q_2+\Phi_2)}{\sqrt{(q_1+\Phi_1)^2+(q_2+\Phi_2)^2}}\right|\\ \le\ & C\frac{|(q_2+\Phi_2)|^3}{((q_1+\Phi_1)^2+(q_2+\Phi_2)^2)^{\frac{3}{2}}}.\end{aligned}$$

Plugging the above estimate into $B_2(q_1,q_2)$, we deduce the following

$$|B_{2,2}(q_1,q_2)| \le C\left((q_1+\Phi_1)^2+(\Phi_2+q_2)^2\right)^{\frac{p-3}{2}}|q_2+\Phi_2|^3,$$

which yields

$$|B_{2,2}(q_1,q_2)| \le C|q_2+\Phi_2|^{\min(p,3)},$$

Using $(q_1,q_2)\in V_A(s)$, it gives the following

$$|q_2+\Phi_2| \le \frac{C}{s},$$

provided that $s \ge s_{2,9}(A)$. Then,

$$\|B_{2,2}(q_1,q_2)\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{s^{\min(p,3)}}. \tag{3.189}$$

It is similar to estimate to $B_{2,3}(q_1,q_2)$

$$\|B_{2,3}(q_1,q_2)\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{s^3}. \tag{3.190}$$

We estimate to $B_{2,4}(q_1,q_2)$, using the following

$$|1-\cos(p\arcsin x)| \le C|x|^2, \forall |x| \le 1,$$

we write

$$|B_{2,4}(q_1,q_2)| \le C\left\|\frac{\Phi_2}{\Phi_1}\right\|^2_{L^\infty(\mathbb{R}^N)}\|q_2\|_{L^\infty(\mathbb{R}^N)} \le \frac{CA^3}{s^3}.$$

Then, we derive that

$$\|B_{2,4}(q_1,q_2)\|_{L^\infty(\mathbb{R}^N)} \le \frac{CA^3}{s^3}. \tag{3.191}$$

We also estimate to $B_{2,5}, B_{2,6}$ as follows:

$$\|B_{2,5}(q_1,q_2)\|_{L^\infty(\mathbb{R}^N)} \le \frac{CA^2}{s^{\frac{3}{2}}}, \tag{3.192}$$

$$\|B_{2,6}(q_1,q_2)\|_{L^\infty(\mathbb{R}^N)} \le \frac{CA^3}{s^2}. \tag{3.193}$$

Thus, from (3.188), (3.189), (3.190), (3.191), (3.192) and (3.193), we conlude (3.174), provided that $s \ge s_{3,9}(A)$. □

In the following Lemma, we aim at giving estimates to the rest terms R_1, R_2 :

Lemme 3.27 (The rest terms R_1, R_2)**.** *For all $s \geq 1$, let us consider R_1 and R_2 , defined in* (3.64) *and* (3.65)*, respectively. Then,*

(i) For all $s \geq 1$ and $y \in \mathbb{R}^N$, we have

$$\begin{aligned} R_1(y,s) &= \frac{c_{1,p}}{s^2} + \tilde{R}_1(y,s), \\ R_2(y,s) &= \frac{c_{2,p}}{s^3} + \tilde{R}_2(y,s), \end{aligned}$$

where $c_{1,p}$and $c_{2,p}$ are constants depended on p and $\tilde{R}_1, \tilde{R}_2$ satisfy

$$\begin{aligned} |\tilde{R}_1(y,s)| &\leq \frac{C(1+|y|^4)}{s^3}, \\ |\tilde{R}_2(y,s)| &\leq \frac{C(1+|y|^6)}{s^4}, \end{aligned}$$

for all $|y| \leq 2K_0\sqrt{s}$.

(ii) Moreover, we have for all $s \geq 1$

$$\begin{aligned} \|R_1(.,s)\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C}{s}, \\ \|R_2(.,s)\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C}{s^2}. \end{aligned}$$

Proof. The proof for R_1 is quite the same as the proof for R_2. For that reason, we only give the proof of the estimates on R_2. This means that, we need to prove the following estimates:

$$R_2(y,s) = -\frac{N(N+4)\kappa}{(p-1)s^3} + \tilde{R}_2(y,s), \tag{3.194}$$

with

$$|\tilde{R}_2(y,s)| \leq \frac{C(1+|y|^6)}{s^4}, \forall |y| \leq 2K_0\sqrt{s}.$$

and

$$\|R_2(.,s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{s^2}. \tag{3.195}$$

We recall the definition of $R_2(y,s)$:

$$R_2(y,s) = \Delta\Phi_2 - \frac{1}{2}y\cdot\nabla\Phi_2 - \frac{\Phi_2}{p-1} + F_2(\Phi_1,\Phi_2) - \partial_s\Phi_2,$$

Then, we can rewrite R_2 as follows

$$R_2(y,s) = \Delta\Phi_2 - \frac{1}{2}y\cdot\nabla\Phi_2 - \frac{\Phi_2}{p-1} + p\Phi_1^{p-1}\Phi_2 - \partial_s\Phi_2 + R_2^*(y,s),$$

where

$$R_2^*(y,s) \;=\; \left(\Phi_1^2+\Phi_2^2\right)^{\frac{p}{2}} \sin\left[p \arcsin\left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}}\right)\right] - p\Phi_1^{p-1}\Phi_2.$$

Using the defintions of Φ_1, Φ_2 given in (3.53) and (3.54), we obtain the following:

$$\begin{aligned} |R_2^*(y,s)| &\leq \left| (\Phi_1^2+\Phi_2^2)^{\frac{p}{2}} \left\{ \sin \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}} \right) \right] - p \frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}} \right\} \right| \\ &+ \left| p\Phi_2((\Phi_1^2+\Phi_2^2)^{\frac{p-1}{2}} - \Phi_1^{p-1}) \right|. \end{aligned}$$

It is similar to the proofs of estimations given in the proof of Lemma 3.26, we can prove the following

$$\begin{aligned} |R_2^*(y,s)| &\leq \frac{C(1+|y|^6)}{s^4}, \quad \forall |y| \leq 2K_0\sqrt{s}, \\ &\text{and} \\ \|R_2^*(.,s)\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C}{s^2}. \end{aligned}$$

In addition to that, we introduce $\bar{R}_2$ as follows:

$$\bar{R}_2(y,s) = \Delta\Phi_2 - \frac{1}{2} y \cdot \nabla\Phi_2 - \frac{\Phi_2}{p-1} + p\Phi_1^{p-1}\Phi_2 - \partial_s \Phi_2.$$

Then, we aim at proving the following:

$$\left| \bar{R}_2(y,s) + \frac{N(N+4)\kappa}{(p-1)s^3} \right| \leq \frac{C(1+|y|^6)}{s^4}, \text{ for all } |y| \leq 2K_0\sqrt{s}, \tag{3.196}$$

$$\|\bar{R}_2(.,s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{s^2}. \tag{3.197}$$

+ The proof of (3.196)*:* We first aim at expanding $\Delta\Phi_2$ in a polynomial in y of order less than 4 via the Taylor expansion. Indeed, $\Delta\Phi_2$ is given by

$$\begin{aligned} \Delta\Phi_2 &= \frac{2N}{s^2}\left(p-1+\frac{(p-1)^2|y|^2}{4ps}\right)^{-\frac{p}{p-1}} - \frac{(p-1)|y|^2}{s^3}\left(p-1+\frac{(p-1)^2}{4p}\frac{|y|^2}{s}\right)^{-\frac{2p-1}{p-1}} \\ &- \frac{(N+2)(p-1)|y|^2}{2s^3}\left(p-1+\frac{(p-1)^2}{4p}\frac{|y|^2}{s}\right)^{-\frac{2p-1}{p-1}} \\ &+ \frac{(2p-1)(p-1)^2|y|^4}{4ps^4}\left(p-1+\frac{(p-1)^2}{4p}\frac{|y|^2}{s}\right)^{-\frac{3p-2}{p-1}}. \end{aligned}$$

Besides that, we make a Taylor expansion in the variable $z = \frac{|y|}{\sqrt{s}}$ for $\left(p-1+\frac{(p-1)^2}{4p}\frac{|y|^2}{s}\right)^{-\frac{p}{p-1}}$ when $|z| \leq 2K$, and we get

$$\left| \left(p-1+\frac{(p-1)^2|y|^2}{4ps}\right)^{-\frac{p}{p-1}} - \frac{\kappa}{p-1} + \frac{\kappa}{4(p-1)}\frac{|y|^2}{s} \right| \leq \frac{C(1+|y|^4)}{s^2} \quad \forall |y| \leq 2K\sqrt{s}.$$

which yields

$$\left| \frac{2N}{s^2}\left(p-1+\frac{(p-1)^2|y|^2}{4ps}\right)^{-\frac{p}{p-1}} - \frac{2N\kappa}{(p-1)s^2} + \frac{N\kappa|y|^2}{2(p-1)s^3} \right| \leq \frac{C(1+|y|^6)}{s^4}, \quad \forall |y| \leq 2K\sqrt{s}.$$

It is similar to estimate the other termes in $\Delta\Phi_2$ as the above. Finally, we obtain

$$\left|\Delta\Phi_2 - \frac{2N\kappa}{(p-1)s^2} + \frac{N\kappa|y|^2}{(p-1)s^3} + 2\frac{k|y|^2}{(p-1)s^3}\right| \leq \frac{C(1+|y|^6)}{s^4}, \forall |y| \leq 2K\sqrt{s}. \quad (3.198)$$

As we did for $\Delta\Phi_2$, we estimate similarly the other terms in $\bar{R}_2$: for all $|y| \leq 2K\sqrt{s}$

$$\left|-\frac{1}{2}y\cdot\nabla\Phi_2 + \frac{\kappa|y|^2}{(p-1)s^2} - \frac{\kappa|y|^4}{4(p-1)s^3} - \frac{\kappa|y|^4}{4(p-1)s^3}\right| \leq \frac{C(1+|y|^6)}{s^4}, \quad (3.199)$$

$$\left|-\frac{\Phi_2}{p-1} + \frac{\kappa|y|^2}{(p-1)^2s^2} - \frac{\kappa|y|^4}{4(p-1)^2s^3} - \frac{2N\kappa}{(p-1)^2s^2}\right| \leq \frac{C(1+|y|^6)}{s^4}, \quad (3.200)$$

$$\left|p\Phi_1^{p-1}\Phi_2 + T(y)\right| \leq \frac{C(1+|y|^6)}{s^4}, \quad (3.201)$$

$$\left|-\partial_s\Phi_2 - \frac{2\kappa|y|^2}{(p-1)s^3} + \frac{4N\kappa}{(p-1)s^3}\right| \leq \frac{C(1+|y|^6)}{s^4}, \quad (3.202)$$

where

$$T(y) = -\frac{p\kappa|y|^2}{(p-1)^2s^2} + \frac{(2p-1)\kappa|y|^4}{4(p-1)^2s^3} - \frac{N\kappa|y|^2}{(p-1)s^3} + \frac{2pN\kappa}{(p-1)^2s^2} + \frac{N^2\kappa}{(p-1)s^3}.$$

Thus, we use (3.198), (3.199), (3.200), (3.201) and (3.202) to deduce the following

$$\left|\bar{R}_2(y,s) + \frac{N(N+4)\kappa}{(p-1)s^3}\right| \leq \frac{C(1+|y|^6)}{s^4}, \quad \forall |y| \leq 2K\sqrt{s},$$

and (3.196) follows
+ The proof (3.197)*:* We rewrite Φ_1, Φ_2 as follows

$$\Phi_1(y,s) = R_{1,0}(z) + \frac{N\kappa}{2ps} \text{ and } \Phi_2(y,s) = \frac{1}{s}R_{2,1}(z) - \frac{2N\kappa}{(p-1)s^2} \text{ where } z = \frac{y}{\sqrt{s}},$$

where $R_{1,0}$ and $R_{2,1}$ are defined in (3.48) and (3.50), respectively. In addition to that, we rewrite $\bar{R}_2$ in terms of $R_{1,0}$ and $R_{2,1}$, and we note that $R_{1,0}$ and $R_{2,1}$ satisfy (3.44) and (3.46). Then, it follows that

$$|\bar{R}_2(y,s)| \leq \frac{C}{s^2}, \forall y \in \mathbb{R}^N.$$

Hence, (3.197) follows. This concludes the proof of this Lemma. □

3.7 Preparation of initial data

Here, we here give the proof of Lemma 3.14. We can see that part (II) directly follows from item (i) of part (II). The techniques of the proof are given in [16] and [28]. Although those papers are written in the real-valued case, unlike ours, where we handle the complex-valued case, we reduce in fact to the real case, for the real and the imaginary parts. In addition to that, the set $\mathcal{D}_{K_0,A,s_0}$ is the product of two parts, the first one depends only on d_1, and the other one depends only on d_2. Moreover, the real part is almost the same as initial data in the Vortex model, treated in [16], except for the new term 1, but this term is very small after changing to similarity variables: $e^{-\frac{s}{p-1}}$. In fact, handling the imaginary part is easier than handling the real part. For those reasons, we kindly refer the reader to Lemma 2.4 in [16] and Proposition 4.5 in [28] for the proof of item (i) of (I) and (II). So, we only prove that our initial data satisfies item (ii) in definition of $S(0)$ (item (iii) is obvious).

Let us consider $T>0, K_0, \alpha_0, \epsilon_0$ and δ_1 which will be suitably chosen later. In fact, we aim at proving the following: For all $|x| \in \left[\frac{K_0}{4}\sqrt{T|\ln T|}, \epsilon_0\right]$, and $|\xi| \leq 2\alpha_0\sqrt{|\ln(T-t(x))|}$ and $\tau_0(x) = -\frac{t(x)}{T-t(x)}$, we have

$$\left|U(x,\xi,\tau_0(x)) - \hat{U}(\tau_0(x))\right| \leq \delta_1. \tag{3.203}$$

We now introduce some necessary notations for our proof,

$$\theta_0 = T, \quad r(0) = \frac{K_0}{4}\sqrt{\theta_0|\ln(\theta_0)|} \text{ and } R(0) = \theta_0^{\frac{1}{2}}|\ln\theta_0|^{\frac{p}{2}}. \tag{3.204}$$

Then, we have the following asymptotic behaviors:

$$\theta(r(0)) \sim \theta_0, \qquad \theta(R(0)) \sim \frac{16}{K_0^2}\theta_0|\ln\theta_0|, \quad \theta(2R(0)) \sim \frac{64}{K_0^2}\theta_0|\ln\theta_0|^{p-1}, \tag{3.205}$$

$$\ln\theta(r(0)) \sim \ln\theta(R(0)) \sim \ln\theta(2R(T)). \tag{3.206}$$

In addition to that, if $\alpha_0 \leq \frac{K_0}{16}$ and $\epsilon_0 \leq \frac{2}{3}C^*$, where C^* is introduced in (3.87), then, from the definition (3.75) and $|x| \in [r(0), \epsilon_0]$, and for all $|\xi| \leq 2\alpha_0\sqrt{|\ln\theta(x)|}$, with $\theta(x) = T - t(x)$, we have

$$\left|\xi\sqrt{\theta(x)}\right| \leq \frac{1}{2}|x|,$$

which yields

$$\frac{r(0)}{2} \leq \frac{|x|}{2} = |x| - \frac{|x|}{2} \leq |x + \xi\sqrt{\theta(x)}| \leq \frac{3}{2}|x| \leq \frac{3}{2}\epsilon_0 \leq C^*. \tag{3.207}$$

Hence, using (3.74), (3.2) and the definition of χ_1 and the fact that $|\xi| \leq 2\alpha_0\sqrt{|\ln\theta(x)|}$, we can write

$$U(x,\xi,\tau_0) = U_1(x,\xi,\tau_0) + iU_2(x,\xi,\tau_0),$$

where

$$\begin{aligned}
U_1(x,\xi,\tau_0) &= (I)\chi_1(x+\xi\sqrt{\theta(x)}) + (II)(1-\chi_1(x+\xi\sqrt{\theta(x)})) + (III),\\
(I) &= \left(\frac{\theta(x)}{\theta_0}\right)^{\frac{1}{p-1}}\Phi_1\left(\frac{x+\xi\sqrt{\theta(x)}}{\sqrt{T}}, |\ln(T)|\right),\\
(II) &= (\theta(x))^{\frac{1}{p-1}}U^*\left(x+\xi\sqrt{\theta(x)}\right),\\
(III) &= (\theta(x))^{\frac{1}{p-1}},\\
U_2(x,\tau,\tau_0) &= \left(\frac{\theta(x)}{\theta_0}\right)^{\frac{1}{p-1}}\Phi_2\left(\frac{x+\xi\sqrt{\theta(x)}}{\sqrt{T-t_0}}, |\ln(T-t_0)|\right).
\end{aligned}$$

Then, the conclusion of the proof of (3.203) will follow from the 4 following estimates:

$$\left|(I)-\hat U(\tau_0)\right| \le \frac{\delta_1}{4}, \text{ for all } |x|\in\left[r(0),\frac{200}{99}R(0)\right] \text{ and } |\xi|\le 2\alpha_0\sqrt{|\ln\theta(x)|} \quad (3.208)$$

$$\left|(II)-\hat U(\tau_0)\right| \le \frac{\delta_1}{4}, \text{ for all } |x|\in[r(0),\epsilon_0] \text{ and } |\xi|\le 2\alpha_0\sqrt{|\ln\theta(x)|}, \quad (3.209)$$

$$|(III)| \le \frac{\delta_1}{4}, \text{ for all } |x|\in[r(0),\epsilon_0] \text{ and } |\xi|\le 2\alpha_0\sqrt{|\ln\theta(x)|}, \quad (3.210)$$

$$|U_2(x,\xi,\tau_0)| \le \frac{\delta_1}{4}, \text{ for all } |x|\in\left[r(0),\frac{200}{99}R(0)\right] \text{ and } |\xi|\le 2\alpha_0\sqrt{|\ln\theta(x)|} \quad (3.211)$$

In fact, it is very easy to estimate for (3.210) for ϵ_0 small enough.

We now estimate (3.211): We rewrite $U_2(x,\xi,\tau_0)$ by using (3.84) as follows:

$$\begin{aligned}
|U_2(x,\xi,\tau_0)| &= U_2\left(x,\xi,\frac{-t(x)}{T-t(x)}\right)\\
&= \left(\frac{\theta_0}{\theta(x)}\right)^{-\frac{1}{p-1}}\frac{|x+\xi\sqrt{\theta(x)}|^2}{T|\ln T|}\left(p-1+\frac{|x+\xi\sqrt{\theta(x)}|^2}{T|\ln T|}\right)^{-\frac{p}{p-1}}\frac{1}{|\ln T|}\\
&\le \frac{C}{|\ln T|}\left((p-1)\frac{\theta_0}{\theta(x)}+\frac{(p-1)^2}{4p}\frac{|x+\xi\sqrt{x}|^2}{\theta(x)|\ln(\theta_0)|}\right)^{-\frac{1}{p-1}}.
\end{aligned}$$

In addition to that, for all $|x|\in\left[r(0),\frac{200}{99}R(0)\right]$ and $|\xi|\le 2\alpha_0\sqrt{|\ln\theta(x)|}$, we have

$$\frac{|x+\xi\sqrt{x}|^2}{\theta(x)|\ln(\theta_0)|}\ge\frac{1}{CK_0^2},$$

which yields

$$|U_2(x,\xi,\tau_0)|\le\frac{CK_0^{\frac{2}{p-1}}}{|\ln T|}\le\frac{\delta_1}{4},$$

provided that $T\le T_{1,3}(K_0,\delta_1,\alpha_0)$ and for all $|x|\in\left[r(0),\frac{200}{99}R(0)\right]$.

Estimate of (3.208): We derive from the definition of Φ_1 in (3.53) and the definition of $\hat U(\tau)$ in (3.113) that

$$\begin{aligned}
\left|(I)-\hat U\left(\frac{-t(x)}{\theta(x)}\right)\right| &= \left|\left((p-1)\left(\frac{\theta_0}{\theta(x)}\right)+\frac{(p-1)^2}{4p}\frac{\left|x+\xi\sqrt{\theta(x)}\right|^2}{\theta(x)|\ln\theta_0|}\right)^{-\frac{1}{p-1}}\right.\\
&- \left.\left((p-1)\left(\frac{\theta_0}{\theta(x)}\right)+\frac{(p-1)^2}{4p}\frac{K_0^2}{16}\right)^{-\frac{1}{p-1}}\right|
\end{aligned}$$

In addition to that, from (3.75), we have

$$(1-2\alpha_0)^2\frac{K_0^2}{16}\frac{|\ln\theta(x)|}{|\ln\theta_0|}\le\frac{\left|x+\xi\sqrt{\theta(x)}\right|^2}{\theta(x)|\ln\theta_0|}\le(1+2\alpha_0)^2\frac{K_0^2}{16}\frac{|\ln\theta(x)|}{|\ln\theta_0|}, \quad (3.212)$$

for all $|\xi| \leq 2\alpha_0\sqrt{|\ln\theta(x)|}$.
Using the monotonicity of $\theta(x)$, we have the fact that for all $|x| \in \left[r(0), \frac{200}{99}R(0)\right]$

$$\frac{|\ln r(0)|}{|\ln\theta_0|} \leq \frac{|\ln\theta(x)|}{|\ln\theta_0|} \leq \frac{|\ln R(0)|}{|\ln\theta_0|}.$$

Thanks to (3.205), we derive

$$\frac{|\ln\theta(x)|}{|\ln\theta_0|} \sim 1 \text{ as } T \to 0. \tag{3.213}$$

This yields

$$\left|(I) - \hat{U}\left(\frac{-t(x)}{\theta(x)}\right)\right| \leq C(K_0)\left|\frac{|x+\xi\sqrt{\theta(x)}|^2}{\theta(x)|\ln\theta_0|} - \frac{K_0^2}{16}\right| \to 0$$

uniformly for all $|x| \in \left[r(0), \frac{200}{99}R(0)\right], |\xi| \leq 2\alpha_0\sqrt{|\ln\theta(x)|}$ as $\alpha_0 \to 0$ and $T \to 0$. Hence, there exists $\alpha_{2,3}(K_0,\delta_1)$ and $T_{2,3}(K_0,\delta_1)$ such that

$$\left|(I) - \hat{U}\left(\frac{-t(x)}{\theta(x)}\right)\right| \leq \frac{\delta_1}{4}, \forall |x| \in \left[r(0), \frac{200}{99}R(0)\right] \text{ and } |\xi| \leq 2\alpha_0\sqrt{|\ln\theta(x)|},$$

provided that $\alpha_0 \leq \alpha_{2,3}$ and $T \leq T_{2,3}$. This concludes the proof of (3.208).

Estimate (3.209): Let $|x| \in \left[\frac{99}{100}R(0), \epsilon_0\right]$. We use the definition of U^* to rewrite (II) as follows

$$\begin{aligned}
(II) &= \left(\frac{(p-1)^2}{8p}\frac{\left|x+\xi\sqrt{\theta(x)}\right|^2}{\theta(x)|\ln(x+\xi\sqrt{\theta(x)})|}\right)^{-\frac{1}{p-1}} = \left(\frac{(p-1)^2}{8p}\frac{\left|\frac{K_0}{4}\sqrt{|\ln\theta(x)|}+\xi\right|^2}{|\ln(x+\xi\sqrt{\theta(x)})|}\right)^{-\frac{1}{p-1}}\\
&= \left(\frac{(p-1)^2K_0^2}{64} + \frac{(p-1)^2}{8p}\left(\frac{\left|\frac{K_0}{4}\sqrt{|\ln\theta(x)|}+\xi\right|^2}{|\ln(x+\xi\sqrt{\theta(x)})|} - \frac{K_0^2}{8}\right)\right)^{-\frac{1}{p-1}}.
\end{aligned}$$

Then,

$$\begin{aligned}
\left|(II) - \hat{U}\left(\frac{t_0-t(x)}{\theta(x)}\right)\right| &= \left|\left(\frac{(p-1)^2K_0^2}{64} + \frac{(p-1)^2}{8p}\left(\frac{\left|\frac{K_0}{4}\sqrt{|\ln\theta(x)|}+\xi\right|^2}{|\ln(x+\xi\sqrt{\theta(x)})|} - \frac{K_0^2}{8}\right)\right)^{-\frac{1}{p-1}}\right.\\
&\quad - \left.\left(\frac{(p-1)^2K_0^2}{64p} + (p-1)\frac{\theta_0}{\theta(x)}\right)^{-\frac{1}{p-1}}\right|\\
&\leq C(K_0)((II_1) + (II_2)),
\end{aligned}$$

where

$$\begin{aligned}
(II_1) &= \left|\frac{\left|\frac{K_0}{4}\sqrt{|\ln\theta(x)|}+\xi\right|^2}{|\ln(x+\xi\sqrt{\theta(x)})|} - \frac{K_0^2}{8}\right|,\\
(II_2) &= (p-1)\frac{\theta_0}{\theta(x)}.
\end{aligned}$$

Let us give a bound to (II_1): Because $|\xi| \le 2\alpha_0\sqrt{|\ln\theta(x)|}$, we have

$$\begin{aligned} |(II_1)| &\le \left| \frac{\left|\frac{K_0}{4}\sqrt{|\ln\theta(x)|} + 2\alpha_0\sqrt{\ln\theta(x)}\right|^2}{|\ln|x + 2\alpha_0\sqrt{\theta(x)|\ln\theta(x)|}||} - \frac{K_0^2}{8} \right| \\ &= \left| \frac{\ln\theta(x)}{|\ln|x + \frac{\alpha_0 K_0 |x|}{2}||} \left(\frac{K_0}{4} + 2\alpha_0\right)^2 - \frac{K_0^2}{8} \right|. \end{aligned}$$

Using the fact that

$$\ln\theta(x) = \ln(T - t(x)) \sim 2\ln|x|,$$

and

$$|\ln(|x + 2\alpha_0\sqrt{\theta(x)\ln\theta(x)}|)| = |\ln|x + \frac{K_0}{2}|x||| \sim |\ln|x||,$$

as $|x| \to 0$, we derive that, there exists $\alpha_{3,3}(K_0, \delta_1)$ such that for all $\alpha_0 \le \alpha_{3,3}$, there exists $\epsilon_{3,3}(K_0, \alpha_0, \delta_1)$ such that for all $\epsilon_0 \le \epsilon_{3,3}$, for all $x \in \left[\frac{99}{100}R(0), \epsilon_0\right]$ and for all $|\xi| \le 2\alpha_0\sqrt{|\ln\theta(x)|}$, we obtain

$$|(II_1)| \le \frac{\delta_1}{2}.$$

It remains to give a bound for (II_2). From (3.205), the fact that $|x| \ge \frac{99}{100}R(0)$ and the monotonicity of $\theta(x)$, we have

$$|(II_2)| \le \left| \frac{\theta(0)}{\theta\left(\frac{99}{100}R(0)\right)} \right| \le C|\ln\theta(0)|^{-(p-1)} \le \frac{\delta_1}{2},$$

provided that $T \le T_{4,3}(K_0, \delta_1)$. This gives (3.203), and concludes the proof of Lemma 3.14.

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Chapter 4

Profile of a touch-down solution to a nonlocal MEMS Model [1].

G. K. Duong and H. Zaag

Abstract: *In this paper, we are interested in the mathematical model of MEMS devices which is presented by the following equation on* $(0, T) \times \Omega$:

$$\partial_t u = \Delta u + \frac{\lambda}{(1-u)^2 \left(1 + \gamma \int_\Omega \frac{1}{1-u} dx\right)^2} \text{ and } 0 \le u < 1,$$

where Ω *is a* C^2 *bounded domain in* $\mathbb{R}^N$ *and* $\lambda, \gamma > 0$*. In this work, we have succeeded to construct a solution which quenches in finite time* T *only at one interior point* $a \in \Omega$*. In particular, we give a description of the quenching behavior according to the following final profile*

$$1 - u(x, T) \sim \theta^* \left[\frac{|x-a|^2}{|\ln |x-a||}\right]^{\frac{1}{3}} \text{ for some } \theta^* > 0 \text{ as }, x \to a.$$

The construction relies on some connection between the quenching phemonenon and the blowup phenomenon. More precisely, we change our problem to the construction of a blowup solution for a related PDE and describe its asymptotic behaviors. The method is inspired by the work of Merle and Zaag [14] with a suitable modification. In addition to that, the proof relies on two main steps: A reduction to a finite dimensional problem and a topological argument based on Index theory. The main difficulty and novelty of this work is that we handle the nonlocal integral term in the above equation. The interpretation of the finite dimensional parameters in terms of the blowup point and the blowup time allows to derive the stability of the constructed solution with respect to initial data.

Mathematics Subject Classification: *35K50, 35B40 (Primary); 35K55, 35K57 (Secondary).*

Keywords: *Blowup solution, Blowup profile, MEMS model, touch-down phenomenon, asymptotic behavior.*

[1]This work was accepted for publication in *Math. Models Methods Appl. Sci* (2019).
Doi: 10.1142/S0218202519500222

4.1 Introduction.

We are interested in the motion of some elastic membranes which is usually found in Micro-Electro Mechanical Systems (MEMS) devices, which are available in a variety of electronic devices such as: microphones: transducers; sensors; actuators and so on. Described briefly, MEMS devices contain an elastic membrane which is hanged above a rigid ground plate connected in series with a fixed voltage source and a fixed capacitor. For more details on the physical background and possible applications, we refer the reader to [4], [10], [18] and [19].

For a MEMS device (in [9] and [10]), the distance between the rigid ground plate and the elastic membrane changes with time. It is referred to as the *deflection* of the membrane. Here, we assume that this distance is very small compared to the device. In fact, we can fully describe the behavior of the deflection by the following hyperbolic equation

$$\begin{cases} \varepsilon^2\partial_{tt}u + \partial_t u &= \Delta u + \dfrac{\lambda f(x,t)}{(1-u)^2\left(1+\gamma\displaystyle\int_\Omega \frac{1}{1-u}dx\right)^2}, \quad x \in \Omega, t > 0, \\ u(x,t) &= 0, x \in \partial\Omega, t > 0, \\ u(x,0) &= u_0(x), x \in \bar{\Omega}. \end{cases} \tag{4.1}$$

where Ω is considered as the domain of the rigid plate, u is the deflection of the membrane to the plate, $\lambda > 0, \gamma > 0$ and f is continuous. Here, the distance between the rest position of the membrane and the rigid plate is normalized to 1. When the device is under voltage, u will vary in the interval $[0,1)$. In addition to that, the parameter λ represents the ratio of the reference electrostatic force to the reference elastic force and ε is the ratio of the interaction of the inertial and damping terms in our model. Moreover, the function f represents the varying dielectric properties of the membrane, see [7] for more details.

In fact, we are interested in a simpler case of (4.1) considered in the following parabolic equation:

$$\begin{cases} \partial_t u &= \Delta u + \dfrac{\lambda}{(1-u)^2\left(1+\gamma\displaystyle\int_\Omega \frac{1}{1-u}dx\right)^2}, \quad x \in \Omega, t > 0, \\ u(x,t) &= 0, x \in \partial\Omega, t > 0, \\ u(x,0) &= u_0(x), x \in \Omega. \end{cases} \tag{4.2}$$

Moreover, we are also interested in the following generalization of problem (4.2):

$$\begin{cases} \partial_t u &= \Delta u + \dfrac{\lambda}{(1-u)^p\left(1+\gamma\displaystyle\int_\Omega \frac{1}{1-u}dx\right)^q}, \quad x \in \Omega, t > 0, \\ u(x,t) &= 0, x \in \partial\Omega, t > 0, \\ u(x,0) &= u_0(x), x \in \Omega, \end{cases} \tag{4.3}$$

where $p, q > 0$. Introducing

$$Q_T = (0,T) \times \Omega, \text{ where } T > 0, \tag{4.4}$$

we say that u is a *classical solution* of (4.2) (in the sense of Proposition 1.2.2 page 13 in Kavallaris and Suzuki [11]) if u is a function in $C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ that satisfies (4.2) at every point in Q_T as well as the boundary and initial conditions, with

$$u(x,t) \in [0,1), \forall x \in \Omega, t \in (0,T).$$

According to the above mentioned reference in [11], the local Cauchy problem of (4.2) is solved. Then, either our solution is global in time or there exists $T > 0$ such that

$$\liminf_{t \to T} \left[\min_{x \in \bar{\Omega}} \{1 - u(t,x)\} \right] = 0. \tag{4.5}$$

We can see that if the above condition occurs, the right-hand side of (4.2) may become singular. This phenomenon is referred to as *touch-down* in finite time T in reference to the physical phenomenon, where the membrane "touches" the rigid ground plate which is placed below. In fact, in our setting, we follow the literature and place the regid plate at $u = 1$, above the membrane which is located at $u(x,t)$. Note that in case of *touch-down*, the MEMS device breaks down.

Mathematically, we may refer to the behavior in (4.5) as finite-time quenching. Moreover, $a \in \Omega$ is a quenching point if and only if there exist sequences $(a_n, t_n) \in \Omega \times (0,T)$ such that

$$u(a_n, t_n) \to 1, \text{ as } n \to +\infty.$$

The *touch-down* phenomenon has been strongly studied in recent decades. In one space dimension, we would like to mention the paper by Guo, Hu and Wang in [6] who gave a sufficient condition for quenching, and also a lower bound on the quenching final profile (see Remark 4.5 below). There is also the paper by Guo and Hu in [5] who find a constant limit for the similarity variables version valid only on compact sets, and yielding the quenching rate.

In higher dimensions, let us for example mention the following result by Guo and Kavallaris [7]:

Consider Ω such that $|\Omega| \le \frac{1}{2}$. Then, for all $\lambda > 0$ fixed and $\gamma > 0$, there exist initial data with a small energy such that problem (4.2) *has a solution which quenches in finite time.*

In our paper, we are interested in proving a general quenching result with no restriction on any $\lambda > 0, \gamma > 0$ and C^2 bounded domain Ω. In fact, we do much better than [5] and [6], and give a sharp description of the asymptotic behavior of the solution near the quenching region. The following is the main result:

Theorem 4.1 (Existence of a *touch-down* solution)**.** *Let us consider $\lambda > 0, \gamma > 0$ and Ω a C^2 bounded domain in $\mathbb{R}^N$, containing the origin. Then, there exist initial data $u_0 \in C^\infty(\bar{\Omega})$ such that the solution of* (4.2) *quenches in finite time $T = T(u_0) > 0$ only at the origin. In particular, the following holds:*

(*i*) *The intermediate profile: For all $t \in [0,T)$*

$$\left\| \frac{(T-t)^{\frac{1}{3}}}{1-u(.,t)} - \theta^* \left(3 + \frac{9}{8} \frac{|.|^2}{\sqrt{(T-t)|\ln(T-t)|}} \right)^{-\frac{1}{3}} \right\|_{L^\infty(\Omega)} \le \frac{C}{\sqrt{|\ln(T-t)|}}, \tag{4.6}$$

for some $\theta^* = \theta^*(\lambda, \gamma, \Omega, T) > 0$.

(*ii*) *The final profile: There exists* $u^* \in C^2(\Omega) \cap C(\bar{\Omega})$ *such that* u *uniformly converges to* u^* *as* $t \to T$, *and*

$$1 - u^*(x) \sim \theta^* \left[\frac{9}{16} \frac{|x|^2}{|\ln|x||} \right]^{\frac{1}{3}} \quad \text{as } x \to 0. \tag{4.7}$$

Remark 4.2. *Note that when* $\gamma = 0$, *our problem coincides with the work of Filippas and Guo [3] and also Merle and Zaag [14]. Our paper is then meaningful when* $\gamma \neq 0$, *and the whole issue is how to control the non local term. Note that [3] derived the final quenching profile, however, only in one space dimension, whereas [14] constructed a quenching solution in higher dimensions, proved its stability with respect to initial data, and gave its intermediate and final profiles.*

Remark 4.3. *For simplicity, we choose to write our result when the solution quenches at the origin. Of course, we can make it quenches at any arbitrary* $a \in \Omega$, *simply replace* x *by* $x - a$ *in the statement.*

Remark 4.4. *In Theorem 4.1, we can describe the evolution of our solution at* $x = 0$ *as follows:*

$$1 - u(0,t) \sim \frac{\sqrt[3]{3}}{\theta^*}(T-t)^{\frac{1}{3}}, \quad \text{as } t \to T.$$

Remark 4.5. *From* (4.7), *we see that the final profile* u^* *has a cusp at the origin which is equivalent to*

$$\frac{C_0 |x|^{\frac{2}{3}}}{|\ln|x||^{\frac{1}{3}}}.$$

This description is in fact much better than the result of Guo, Hu and Wang in [6] who gave some sufficient conditions for quenching in one space dimension, and proved the existence of a cusp at the quenching point bounded from below by $C(\beta)|x|^\beta$ *for any* $\beta \in \left(\frac{2}{3}, 1\right)$, *which is less accurate than our estimate* (4.7).

Remark 4.6. *Note that we can explicitly write the formula of the initial data*

$$u(x,0) = \frac{\bar{u}(x,0)}{\bar{u}(x,0)+1}, \tag{4.8}$$

where

$$\bar{u}(x,0) = \frac{\bar{\theta}(0)}{\lambda^{\frac{1}{3}}} U(x,0),$$

with

$$U(x,0) = T^{-\frac{1}{3}} \left[\varphi(\frac{x}{\sqrt{T}}, -\ln T) + (d_0 + d_1 \cdot z)\, \chi_0 \left(\frac{32|z|}{K_0} \right) \right] \chi_1(x) + (1 - \chi_1(x)) H^*(x),$$

$$z = \frac{x}{\sqrt{T|\ln T|}},$$

$$\chi_1(x) = \chi_0 \left(\frac{|x|}{\sqrt{T}|\ln T|} \right),$$

and $\bar{\theta}(0)$ *is the unique positive solution of the following equation*

$$\bar{\theta}(0) = \left(1 + \gamma|\Omega| + \frac{\gamma}{\sqrt[3]{\lambda}}\bar{\theta}(0)\int_\Omega U(0)dx\right)^{\frac{2}{3}},$$

and note that χ_0, φ *and* H^* *are defined in* (4.28), (4.33) *and* (4.62), *respectively. Here,* T *is small enough and parameters* d_0 *and* d_1 *are fine-tuned in order to get the desired behavior.*

Remark 4.7 (An open question). *How big can* θ^* *be? This question is related to the work of Merle and Zaag in [14] (see the Theorem on page 1499), which corresponds to the case where* $\gamma = 0$. *For that case, the answer is* $\theta^* = \frac{1}{\sqrt[3]{\lambda}}$. *It is very interesting to answer the question in the general case. By a glance to* (4.18), (4.86) *and* (4.87), *we know that* θ^* *is strictly greater than* $\frac{(1+\gamma|\Omega|)^{\frac{2}{3}}}{\sqrt[3]{\lambda}}$. *Let us define*

$$\mathcal{T}_{max} = \left(\frac{(1+\gamma|\Omega|)^{\frac{2}{3}}}{\sqrt[3]{\lambda}}, +\infty\right),$$

and

$$\mathcal{T} = \left\{\theta^* \in \mathbb{R} \text{ such that } (4.6) \text{ holds with } u \text{ a positive solution to } (4.2), \text{ for some } T > 0\right\}.$$

Then, by a fine modification in the proof, we can construct a solution such that θ^* *arbitrarily takes large values in* $\mathcal{T}_{max}$. *In particular, we can prove that* $\mathcal{T}$ *is a dense subset of* $\mathcal{T}_{max}$. *We would like to make the following conjecture*

$$\mathcal{T} = \mathcal{T}_{max}.$$

Now, we would like to mention that our proof of Theorem 4.1 holds in a more general setting. More precisely, if we consider problem (4.3) in the following regime

$$N - \frac{2}{p+1} > 0, \text{ and } q > 0 \text{ and } N \geq 1, \tag{4.9}$$

then, Theorem 4.1 changes as follows:

Theorem 4.8 (Existence of a *touch-down* solution to (4.3)). *Consider* $\lambda, \gamma > 0$, *and* Ω *a* C^2 *bounded domain in* $\mathbb{R}^N$ *and condition* (4.9) *holds. Then, there exist initial data* $\hat{u}_0$ *in* $C^\infty(\bar{\Omega})$ *such that the solution of equation* (4.3) *touches down in finite time only at the origin. In particular, the following holds:*

(*i*) *The intermediate profile, for all* $t \in [0, T)$

$$\left\|\frac{(T-t)^{\frac{1}{p+1}}}{1-u(.,t)} - \hat{\theta}^*\left(p+1+\frac{(p+1)^2}{4p}\frac{|.|^2}{\sqrt{(T-t)|\ln(T-t)|}}\right)^{-\frac{1}{p+1}}\right\|_{L^\infty(\Omega)} \leq \frac{C}{\sqrt{|\ln(T-t)|}}, \tag{4.10}$$

for some $\hat{\theta}^*(\lambda, \gamma, \Omega, p, q) > 0$.

(ii) The exists $\hat{u}^* \in C^2(\Omega) \cap C(\bar{\Omega})$ *such that* u *uniformly converges to* $\hat{u}^*$ *as* $t \to T$, *and*

$$1 - \hat{u}^*(x) \sim \hat{\theta}^* \left[\frac{(p+1)^2}{8p} \frac{|x|^2}{|\ln|x||} \right]^{\frac{1}{p+1}} \quad as\ x \to 0. \tag{4.11}$$

Remark 4.9. *We don't give the proof of Theorem 4.8 here because the techniques are the same as for Theorem 4.1. In fact, for simplicity, we will only give the proof for the MEMS case*

$$p = q = 2,$$

considered in equation (4.2) *and Theorem 4.1. Of course, all our estimates can be carried on for the general case.*

In addition to that, we can apply the techniques of Merle in [12] to create a solution which quenches at arbitrary given points.

Corollary 4.10. *For any* k *points* $a_1, a_2, ..., a_k$ *in* Ω, *there exist initial data such that* (4.3) *has a solution which quenches exactly at* $a_1, ..., a_k$. *Moreover, the local behavior at each* a_i *is also given by* (4.10), (4.11) *by replacing* x *by* $x - a_i$ *and* $L^\infty(\Omega)$ *by* $L^\infty(|x - a_i| \leq \omega_0)$, *for some* $\omega_0 > 0$, *small enough.*

As a consequence of our techniques, we can derive the stability of the quenching solution which we constructed in Theorem 4.8 under the perturbations of initial data.

Theorem 4.11 (Stability of the constructed solution)**.** *Let us consider* $\hat{u}$, *the solution which we constructed in Theorem 4.8 and we also define* $\hat{T}$ *as the quenching time of the solution and* $\hat{\theta}^*$ *as the coefficient in front of the profiles* (4.10) *and* (4.11). *Then, there exists an open subset* $\hat{\mathfrak{U}}_0$ *in* $C_{0,+}(\bar{\Omega})$, *containing* $\hat{u}(0)$ *such that for all initial data* $u_0 \in \hat{\mathfrak{U}}_0$, *equation* (4.3) *has a unique solution* u *quenching in finite time* $T(u_0)$ *at only one quenching point* $a(u_0)$. *Moreover, the asymptotic behaviors* (4.10) *and* (4.11) *hold by replacing* $\hat{u}(x,t)$ *by* $u(x - a(u_0), t)$, *and* $\hat{\theta}^*$ *by some* $\theta^*(u_0)$. *Note that, we have*

$$(a(u_0), T(u_0), \theta^*(u_0)) \to (0, \hat{T}, \hat{\theta}), \ \ as \ \|u_0 - \hat{u}_0\|_{C(\bar{\Omega})} \to 0.$$

Let us now comment on the organization of the paper. As we have stated earlier, Theorem 4.1 is a special case of Theorem 4.8. For simplicity in the notations, we only prove Theorem 4.1. The interested reader may derive the general case by inspection. Moreover, we don't prove Corollary 4.10 and Theorem 4.11, since the former follows from Theorem 4.8 and the techniques of Merle in [12], and the latter follows also from Theorem 4.8 by the method of Merle and Zaag in [15]. In conclusion, we only prove Theorem 4.1 in this paper.

The paper is organized as follows:

- In Section 2, we give a different formulation of the problem, and show how the profile in (4.6) arises naturally.
- In Section 3, we give the proof without technical details.
- In Section 4, we prove the technical details.

Some appendices are added at the end.

4.2 Setting of the problem

4.2.1 Our main idea

We aim in this subsection at explaining our key idea in this paper. The rigorous proof will be given later. Introducing

$$\alpha(t) = \frac{\lambda}{\left(1 + \gamma \int_\Omega \frac{1}{1-u(t)} dx\right)^2}, \tag{4.12}$$

we rewrite (4.2) as the following

$$\partial_t u = \Delta u + \frac{\alpha(t)}{(1-u)^2}. \tag{4.13}$$

Under this general form, we see our equation (4.2) as a step by step generalization, starting from a much simpler context:

- **Problem 1: Case where** $\alpha(t) \equiv \alpha_0$. This case was considered by Merle and Zaag in [14] where, the authors constructed a solution u_{α_0} satisfying

$$u_{\alpha_0}(x,t) \to 1 \text{ as } (x,t) \to (x_0, T),$$

for some $T > 0$, and $x_0 \in \Omega$. In particular, they gave a sharp description for the quenching profile. Technically, the authors in that work introduced

$$\bar{u} = \frac{1}{1-u} - 1 = \frac{u}{1-u},$$

and constructed a blowup solution for the following equation derived from (4.13):

$$\partial_t \bar{u} = \Delta \bar{u} - 2\frac{|\nabla \bar{u}|^2}{\bar{u}} + \alpha_0 \bar{u}^4, \text{ with } \alpha(t) \equiv \alpha_0, \tag{4.14}$$

(see equation (III), page 1500 in [14] for more details).

- **Problem 2: Case where** $0 < \alpha_1 \leq \alpha(t) \leq \alpha_2$ **for all** $t > 0$ **for some** $0 < \alpha_1 < \alpha_2$. This case is indeed a reasonable generalization which follows with no difficulty from the stategy of [14] for **Problem 1**.

- **Problem 3: Equation** (4.2). Our idea here is to see (4.2) as a coupled system between **Problem 2** and (4.12):

$$\begin{cases} \partial_t u &= \Delta u + \frac{\alpha(t)}{(1-u)^2}, \\ \alpha(t) &= \frac{\lambda}{\left(1+\gamma \int_\Omega \frac{1}{1-u} dx\right)^2}. \end{cases}$$

A simple idea would be to try a kind of fixed-point argument starting from some solution to **Problem 1**, then defining $\alpha(t)$ according to (4.12) defined with this solution, then solving **Problem 2** with this $\alpha(t)$, then defining a new $\alpha(t)$ with the new solution, and so forth.

In order to make this method to work, one has to check whether the iterated $\alpha(t)$ stay away from 0 and $+\infty$, as requested in the context of **Problem 2**. We checked whether this holds when u solves **Problem 1**. Fortunately, this was the case, and this gave us a serious hint to treat our equation (4.2) as a perturbation of **Problem 1**.

In fact, our proof uses no interation, and we diredly apply the stategy of Merle and Zaag in [14] to control the various terms (including the nonlocal term), in order to find a solution which stays near the desired behavior.

4.2.2 Formulation of the problem

In this section, we aim at setting the mathematical framework of our problem. The rigorous proof will be given later. Our aim is to construct a solution for equation (4.2), defined for all $(x,t) \in \Omega \times [0,T)$, for some $T > 0$ with $0 \leq u(x,t) < 1$, and

$$u(x,t) \to 1 \text{ as } (x,t) \to (x_0,T),$$

for some $x_0 \in \Omega$. Without loss of generality, we assume that

$$x_0 = 0 \in \Omega.$$

Introducing,

$$\bar{u} = \frac{1}{1-u} - 1 = \frac{u}{1-u} \in [0,+\infty), \tag{4.15}$$

we derive from (4.2) the following equation on $\bar{u}$

$$\begin{cases} \partial_t \bar{u} &= \Delta \bar{u} - 2\frac{|\nabla \bar{u}|^2}{\bar{u}+1} + \dfrac{\lambda(\bar{u}+1)^4}{(1+\gamma|\Omega| + \gamma \int_\Omega \bar{u}dx)^2}, x \in \Omega, t > 0, \\ \bar{u}(x,t) &= 0, x \in \partial\Omega, t > 0, \\ \bar{u}(x,0) &= \bar{u}_0(x), \quad x \in \bar{\Omega}. \end{cases} \tag{4.16}$$

Our aim becomes then to construct a blowup solution for equation (4.16) such that

$$\bar{u}(0,t) \to +\infty \text{ as } t \to T.$$

In order to see our equation as a (not so small) perturbation of the standard case in (4.14), we suggest to make one more scaling by introducing

$$U(x,t) = \frac{\lambda^{\frac{1}{3}}}{\bar{\theta}(t)} \bar{u}(x,t), \quad U(x,t) \geq 0, \quad \forall (x,t) \in \Omega \times [0,T), \tag{4.17}$$

with

$$\bar{\theta}(t) = \left(1 + \gamma|\Omega| + \gamma \int_\Omega \bar{u}(t)dx\right)^{\frac{2}{3}}. \tag{4.18}$$

Then, thanks to equation (4.16), we deduce the following equation to be satisfied by U:

$$\begin{cases} \partial_t U &= \Delta U - 2\frac{|\nabla U|^2}{U + \frac{\lambda^{\frac{1}{3}}}{\bar{\theta}(t)}} + \left(U + \frac{\lambda^{\frac{1}{3}}}{\bar{\theta}(t)}\right)^4 - \frac{\bar{\theta}'(t)}{\bar{\theta}(t)} U, x \in \Omega, t > 0, \\ U(x,t) &= 0, x \in \partial\Omega, t > 0, \\ U(x,0) &= U_0(x), x \in \bar{\Omega}. \end{cases} \tag{4.19}$$

Note that in the blowup regime, which is our focus, U is large and equation (4.19) appears indeed as a perturbation of equation (4.14).
Introducing the following notation

$$\bar{\mu}(t) \quad = \quad \int_{\Omega} U(t)dx, \tag{4.20}$$

we may rewrite (4.18) as the following equation

$$\bar{\theta}(t) = \left(1 + \gamma|\Omega| + \frac{\gamma}{\lambda^{\frac{1}{3}}}\bar{\theta}(t)\bar{\mu}(t)\right)^{\frac{2}{3}}. \tag{4.21}$$

This implies that $\bar{\theta}(t)$ solves the following cubic equation

$$\theta^3(t) = \left(1 + \gamma|\Omega| + \frac{\gamma}{\lambda^{\frac{1}{3}}}\bar{\theta}(t)\bar{\mu}(t)\right)^2 = (A + B(t)\bar{\theta}(t))^2 = A^2 + 2AB(t)\bar{\theta}(t) + B^2(t)\bar{\theta}^2(t), \tag{4.22}$$

where

$$A = 1 + \gamma|\Omega| \text{ and } B(t) = \frac{\gamma}{\lambda^{\frac{1}{3}}}\bar{\mu}(t).$$

Since it happens that $\bar{\theta}(t)$ is the unique positive solution of (4.22), we may solve (4.22) and express $\bar{\theta}(t)$ in terms of $\bar{\mu}(t)$ as follows

$$\begin{aligned}\bar{\theta}(t) \quad &= \quad \frac{\sqrt[3]{27A^2 + 3\sqrt[3]{3}\sqrt{27A^2 + 4A^3B^3(t)} + 18AB^3(t) + 2B^6(t)}}{3\sqrt[3]{2}} + \frac{B^3(t)}{3} \\ &+ \quad \frac{\sqrt[3]{2}(6AB(t) + B^4(t))}{\sqrt[3]{27A^2 + 3\sqrt[3]{3}\sqrt{27A^2 + 4A^3B^3(t)} + 18AB^3(t) + 2B^6(t)}}.\end{aligned} \tag{4.23}$$

Particularly, we show here the equivalence between equation (4.16) and (4.19).

Lemme 4.12 (Equivalence between (4.16) and (4.19))**.** *Consider* $\lambda > 0, \gamma > 0$ *and* Ω *a bounded domain in* $\mathbb{R}^N$*. Then, the following holds:*

(i) *We consider* $\bar{u}$ *a solution of equation* (4.16) *on* $[0, T)$, *for some* $T > 0$ *and introduce*

$$U(t) = \frac{\lambda^{\frac{1}{3}}}{\bar{\theta}(t)}\bar{u}(t),$$

where $\bar{\theta}(t) = \left(1 + \gamma|\Omega| + \gamma\int_{\Omega}\bar{u}(t)dx\right)^{\frac{2}{3}}$. *Then,* U *is a solution of equation* (4.19) *on* $[0, T)$.

(ii) *Otherwise, we consider* U *a solution of equation* (4.19) *on* $[0, T)$, *for some* $T > 0$ *and introduce*

$$\bar{u}(t) = \frac{\bar{\theta}(t)}{\lambda^{\frac{1}{3}}}U(t), \forall t \in [0, T),$$

where $\bar{\theta}(t)$ *is defined as in relation* (4.21), *then* $\bar{u}$ *is the solution of equation* (4.16) *on* $[0, T)$. *In particular, the uniqueness of the solution is preserved.*

Proof. The proof is easily deduced from the definition in this lemma. We kindly ask the reader to self-check. □

Remark 4.13. *From settings* (4.15) *and* (4.17) *and the local well-posedness of equation* (4.2) *in the sense of classical solutions (see Proposition 1.2.2 at page 12 in Kavallaris and Suzuki [11]), we can derive the local existence and uniqueness of classical solutions of equations* (4.16) *and* (4.19)*. Since the nonnegativity is preserved for these equations, we will assume that* $\bar{u}$ *and* U *are nonegative.*

Thanks to Lemma 4.12, our problem is reduced to constructing a nonnegative solution to (4.19), which blows up in finite time only at the origin. We also aim at describing its asymptotic behaviors at the singularity.
Since we defined U in (4.17) on purpose so that (4.19) appears as a perturbation of equation (4.14) for U large, it is reasonable to make the following hypotheses:

(*i*) $1 \leq \bar{\theta}(t) \leq C_0$ for some $C_0 > 0$. Note that from (4.21), we have $\bar{\theta}(t) \geq 1$.

(*ii*) $|\bar{\theta}'(t)| \ll U^3(t)$ when U large.

It is then reasonable to expect for equation (4.19) the same profile as the one constructed in [14] for equation (4.14). So, it is natural to follow that work by introducing the following *Similarity-Variables*:

$$W(y,s) = (T-t)^{\frac{1}{3}} U(x,t), \text{ and } s = -\ln(T-t) \text{ and } y = \frac{x}{\sqrt{T-t}}. \tag{4.24}$$

Using equation (4.19), we write the equation of W in (y,s) as follows

$$\begin{cases} \partial_s W &= \Delta W - \frac{1}{2} y \cdot \nabla W - \frac{W}{3} - 2\frac{|\nabla W|^2}{W + \frac{\lambda^{\frac{1}{3}} e^{-\frac{s}{3}}}{\theta(s)}} + \left(W + \frac{\lambda^{\frac{1}{3}} e^{-\frac{s}{3}}}{\theta(s)}\right)^4 - \frac{\theta'(s)}{\theta(s)} W, \\ W(y,s) &= 0, y \in \partial\Omega_s, s > -\ln T, \\ W(y, -\ln T) &= W_0(y), y \in \bar{\Omega}_s, \end{cases} \tag{4.25}$$

where

$$\theta(s) = \bar{\theta}(t(s)) = \bar{\theta}(T - e^{-s}), \tag{4.26}$$

and

$$\Omega_s = e^{\frac{s}{2}} \Omega, \tag{4.27}$$

with $\bar{\theta}$ satisfies (4.21) and (4.23).

We observe in equation (4.25) that Ω_s changes as $s \to +\infty$. This is a major difficulty in comparison with the situation where $\Omega = \mathbb{R}^N$. In order to overcome this difficulty, we intend to introduce some cut-off of the solution, so that we reduce to the case $\Omega = \mathbb{R}^N$. Of course, there is a price to pay, in the sense that we will need to handle some cut-off terms. Our model for this will be the work made by Mahmoudi, Nouaili and Zaag in [13] for the construction of a blowup solution to the semilinear heat equation defined on a certain circle. Let us note that the situation with Ω bounded was already mentioned in [14]. However, the authors in that work avoided the problem by giving the proof only in the case where $\Omega = \mathbb{R}^N$. In this work, we are happy to handle the case with a bounded Ω, following the ideas of Mahmoudi, Nouaili and Zaag in [13]. Let us mention that Velázquez was also faced in [22] by the question of reducing a problem defined on a bounded interval to a problem considered on the whole real line. He made the reduction thanks to the extension of the solution defined on a interval to another solution defined on the whole line, thanks to some truly 1-d techniques. In our case, given that we work in higher dimensions, we use a different method, based on the localization of the equation, thanks to some cut-off functions.

More precisely, we introduce the following cut-off function $\chi_0 \in C_0^\infty([0,+\infty))$, satisfying

$$supp(\chi_0) \subset [0,2], \quad 0 \leq \chi_0(x) \leq 1, \forall x \text{ and } \chi_0(x) = 1, \forall x \in [0,1]. \tag{4.28}$$

Then, we define the following function

$$\psi_{M_0}(y,s) = \chi_0\left(M_0 y e^{-\frac{s}{2}}\right), \text{ for some } M_0 > 0. \tag{4.29}$$

Let us introduce

$$w(y,s) = \begin{cases} W(y,s)\psi_{M_0}(y,s) & \text{if} \quad y \in \Omega_s, \\ 0 & \text{otherwise.} \end{cases} \tag{4.30}$$

We remark that w is defined on $\mathbb{R}^N$ and $s \geq -\ln T$ and $w \equiv 0$ whenever $|y| \geq \frac{2}{M_0} e^{\frac{s}{2}}$. Note that M_0 will be fixed large enough together with others parameters at the end of our proof.

Using equation (4.25), we derive from (4.21) the equation satisfied by w as follows

$$\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{3} w - 2\frac{|\nabla w|^2}{w + \frac{\lambda^{\frac{1}{2}} e^{-\frac{s}{3}}}{\theta(s)}} + \left(w + \frac{\lambda^{\frac{1}{3}} e^{-\frac{s}{3}}}{\theta(s)}\right)^4 - \frac{\theta'(s)}{\theta(s)} w + F(w,W), \tag{4.31}$$

where $F(w,W)$ encapsulates the cut-off terms and is defined as follows

$$F(w,W) = \begin{cases} W\left[\partial_s \psi_{M_0} - \Delta\psi_{M_0} + \frac{1}{2} y \cdot \nabla \psi_{M_0}\right] - 2\nabla\psi_{M_0} \cdot \nabla W \\ +2\frac{|\nabla w|^2}{w + \frac{\lambda^{\frac{1}{2}} e^{-\frac{s}{3}}}{\theta(s)}} - 2\frac{|\nabla W|^2 \psi_{M_0}}{W + \frac{\lambda^{\frac{1}{2}} e^{-\frac{s}{3}}}{\theta(s)}} + \psi_{M_0}\left(W + \frac{\lambda^{\frac{1}{3}} e^{-\frac{s}{3}}}{\theta(s)}\right)^4 - \left(w + \frac{\lambda^{\frac{1}{3}} e^{-\frac{s}{3}}}{\theta(s)}\right)^4 \\ \text{if } y \in \Omega e^{\frac{s}{2}}, \\ 0 \text{ otherwise.} \end{cases} \tag{4.32}$$

We remark that $F \equiv 0$ on the region $\{y \in \mathbb{R}^N \mid |y| \leq \frac{1}{M_0} e^{\frac{s}{2}} \text{ or } |y| \geq \frac{2}{M_0} e^{\frac{s}{2}}\}$ and that we have from the conditions (i) and (ii) on $\bar{\theta}(t)$ on page 190 that

$$1 \leq \theta(s) \leq C_0, \text{ and } |\theta'(s)| \ll W^3(y,s).$$

Making the further assumption that

$$\theta'(s) \to 0,$$

we see that equation (4.35) is almost the same as equation (15) at page 1502 in [14] at least when $|y| \leq \frac{e^{\frac{s}{2}}}{M_0}$. Hence, it is reasonable to expect for equation (4.31) the same profile as the authors found in [14] for equation (15) in that work, namely

$$\varphi(y,s) = \left(3 + \frac{9}{8}\frac{|y|^2}{s}\right)^{-\frac{1}{3}} + \frac{(3)^{-\frac{1}{3}} N}{4s}, \tag{4.33}$$

(note that, this profile was also defined in [14] for a general $p > 2$, and that here we need to take $p = 4$ and $a = 2$, hence $\kappa = (3)^{-\frac{1}{3}}$). In particular, we would like to construct w as a perturbation of φ. So, we introduce the following function

$$q = w - \varphi. \tag{4.34}$$

Using equation (4.25), we easily write the equation of q

$$\partial_s q = (\mathcal{L} + V)q + T(q) + B(q) + N(q) + R(y,s) + F(w,W), \tag{4.35}$$

where

$$\mathcal{L} = \Delta - \frac{1}{2} y \cdot \nabla + Id, \tag{4.36}$$

$$V(y,s) = 4\left(\varphi^3(y,s) - \frac{1}{3}\right), \tag{4.37}$$

$$T(q,\theta(s)) = -2\frac{|\nabla q + \nabla \varphi|^2}{q + \varphi + \frac{\lambda^{\frac{1}{3}} e^{-\frac{s}{3}}}{\theta(s)}} + 2\frac{|\nabla \varphi|^2}{\varphi + \frac{\lambda^{\frac{1}{3}} e^{-\frac{s}{3}}}{\theta(s)}}, \tag{4.38}$$

$$B(q) = \left(q + \varphi + \frac{\lambda^{\frac{1}{3}} e^{-\frac{s}{3}}}{\theta(s)}\right)^4 - \varphi^4 - 4\varphi^3 q, \tag{4.39}$$

$$R(y,s) = -\partial_s \varphi + \Delta \varphi - \frac{1}{2} y \cdot \nabla \varphi - \frac{\varphi}{3} + \varphi^4 - 2\frac{|\nabla \varphi|^2}{\varphi + \frac{\lambda^{\frac{1}{3}} e^{-\frac{s}{3}}}{\theta(s)}}, \tag{4.40}$$

$$N(q) = -\frac{\theta'(s)}{\theta(s)}(q + \varphi), \tag{4.41}$$

with $\theta(s)$ defined in (4.26) and $F(w,W)$ given in (4.32).

In particular, we assume that U and q have good conditions such that Lemmas 4.36, 4.37, 4.38, 4.39 and 4.40 hold. Then, it is easy to see that all terms in the right-hand side of (4.35) become very small, except for $(\mathcal{L} + V)q$. As a matter of fact, this term plays the most important role in our analysis. Therefore, we show here some main properties on the linear operator $\mathcal{L}$ and the potential V (see more details in [1], [2]):

- *Operator* $\mathcal{L}$*:* This operator is self-adjoint in $\mathcal{D}(\mathcal{L}) \subset L^2_\rho(\mathbb{R}^N)$, where $L^2_\rho(\mathbb{R}^N)$ is defined as follows

$$L^2_\rho(\mathbb{R}^N) = \left\{ f \in L^2_{loc}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |f(y)|^2 \rho(y) dy < +\infty \right\},$$

and

$$\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{\frac{N}{2}}}.$$

This is the spectrum set of operator $\mathcal{L}$

$$\text{Spec}(\mathcal{L}) = \left\{1 - \frac{m}{2} \mid m \in \mathbb{N}\right\}.$$

The eigenspace which corresponds to $\lambda_m = 1 - \frac{m}{2}$ is given by

$$\mathcal{E}_m = \langle h_{m_1}(y_1).h_{m_2}(y_2)....h_{m_N}(y_N) \mid m_1 + ... + m_N = m \rangle,$$

where h_{m_i} is the (rescaled) Hermite polynomial in one dimension.

- *Potential* V*:* It has two important properties:

(i) The potential $V(.,s) \to 0$ in $L^2_\rho(\mathbb{R}^N)$ as $s \to +\infty$: In particular, in the region $|y| \leq K_0\sqrt{s}$ (the singular domain), V has some weak perturbations on the effect of operator $\mathcal{L}$.

(ii) $V(y,s)$ is almost a constant on the region $|y| \geq K_0\sqrt{s}$: For all $\epsilon > 0$, there exists $\mathcal{C}_\epsilon > 0$ and s_ϵ such that

$$\sup_{s \geq s_\epsilon, \frac{|y|}{\sqrt{s}} \geq \mathcal{C}_\epsilon} \left| V(y,s) - \left(-\frac{4}{3}\right)\right| \leq \epsilon.$$

Note that $-\frac{4}{3} < -1$ and that the largest eigenvalue of $\mathcal{L}$ is 1. Hence, roughly speaking, we may assume that $\mathcal{L} + V$ admits a strictly negative spectrum. Thus, we can easily control our solution in the region $\{|y| \geq K_0\sqrt{s}\}$ with K_0 large enough.

From these properties, it appears that the behavior of $\mathcal{L} + V$ is not the same inside and outside of the singular domain $\{|y| \leq K_0\sqrt{s}\}$. Therefore, it is natural to decompose every $r \in L^\infty(\mathbb{R}^N)$ as follows:

$$r(y) = r_b(y) + r_e(y) \equiv \chi(y,s)r(y) + (1 - \chi(y,s))r(y), \tag{4.42}$$

where $\chi(y,s)$ is defined as follows

$$\chi(y,s) = \chi_0\left(\frac{|y|}{K_0\sqrt{s}}\right), \tag{4.43}$$

and χ_0 is given in (4.28). From the above decomposition, we immediately have the following:

$$\begin{array}{rcl} \text{Supp } (r_b) & \subset & \{|y| \leq 2K_0\sqrt{s}\}, \\ \text{Supp } (r_e) & \subset & \{|y| \geq K_0\sqrt{s}\}. \end{array}$$

In addition to that, we are interested in expanding r_b in $L^2_\rho\left(\mathbb{R}^N\right)$ according to the basis which is created by the eigenfunctions of operator $\mathcal{L}$:

$$\begin{array}{rcl} r_b(y) & = & r_0 + r_1 \cdot y + y^T \cdot r_2 \cdot y - 2 \ \text{Tr}(r_2) + r_-(y), \\ & \text{or} & \\ r_b(y) & = & r_0 + r_1 \cdot y + r_\perp(y), \end{array}$$

where

$$r_i = \left(P_\beta(r_b)\right)_{\beta \in \mathbb{N}^N, |\beta| = i}, \forall i \geq 0, \tag{4.44}$$

with $P_\beta(r_b)$ being the projection of r_b on the eigenfunction h_β defined as follows:

$$P_\beta(r) = \int_{\mathbb{R}^N} r_b \frac{h_\beta}{\|h_\beta\|_{L^2_\rho(\mathbb{R}^N)}} \rho dy, \forall \beta \in \mathbb{N}^N, \tag{4.45}$$

and

$$r_\perp = P_\perp(r) = \sum_{\beta \in \mathbb{N}^N, |\beta| \geq 2} h_\beta P_\beta(r_b), \tag{4.46}$$

and

$$r_- = \sum_{\beta \in \mathbb{R}^N, |\beta| \geq 3} h_\beta P_\beta(r_b). \quad (4.47)$$

In other words, $r_\perp$ is the part of r_b which is orthogonal to the eigenfunctions corresponding to eigenvalues 0 and 1 and r_- is orthogonal to the eigenfunctions corresponding to eigenvalues $1, \frac{1}{2}$ and 0. We should note that r_0 is a scalar, r_1 is a vector and r_2 is a square matrix of order n and that they are the components of r_b not r. Finally, we write r as follows

$$r(y) = r_0 + r_1 \cdot y + y^T \cdot r_2 \cdot y - 2\,\mathrm{Tr}(r_2) + r_-(y) + r_e(y), \quad (4.48)$$

or

$$r(y) = r_0 + r_1 \cdot y + r_\perp(y) + r_e(y). \quad (4.49)$$

A summary of our problem: Even though we created many extra functions from U to q, we always concentrate on solution U to equation (4.19). More precisely, we aim at constructing U blowing up in finite time. Then, we will use equation (4.35) as a crucial formulation in our proof. Indeed, in order to control U blowing up in finite time, it is enough to control the transform function q of U (see definitions (4.24), (4.30) and (4.34)) satisfying

$$\|q(.,s)\|_{L^\infty(\mathbb{R}^N)} \to 0, \text{ as } s \to +\infty. \quad (4.50)$$

4.3 The proof of the existence result assuming technical details

In this section, we aim at giving a proof without technical details to Theorem 4.1. We would like to summarize the structure of this section as follows:

- *Construction of a shrinking set:* We rely here on the ideas of the Merle and Zaag's work in [14] to introduce a shrinking set that will guarantee the convergence to zero for q defined in (4.34). This set will constrain our solution as we want. Once our solution is trapped in, we may show the main asymptotic behavior of our solution. In particular, (4.50) holds and our result follows.

- *Preparation of initial data:* We introduce a family of initial data to equation (4.19) depending on some finite set parameters. As a matter of fact, we will choose these parameters such that our solution belongs to the shrinking set for all $t \in [0, T)$.

- *The existence of a trapped solution:* Using a reduction to a finite dimensional problem (corresponding to the finite parameters introduced in our initial data) and a topological argument, we can derive the existence of a blowup solution in finite time, trapped in the shrinking set. More precisely, we show in this part that there exist initial data in that family of initial data such that our solution is completely confined in the shrinking set.

- *The conclusion of Theorem 4.1:* Finally, we rely on the existence of a blowup solution, trapped in the shrinking set to get the conclusion of Theorem 4.1.

4.3.1 Shrinking set

In order to control the solution U blowing up in finite time and satisfying (4.50), we adopt the general ideas given by Merle and Zaag in [14]. For each $K_0 > 0, \epsilon_0 > 0, \alpha_0 > 0$ and

$t \in [0, T)$ with $T > 0$, we define

$$P_1(t) = \left\{ x \in \mathbb{R}^N \mid |x| \leq K_0 \sqrt{(T-t)|\ln(T-t)|} \right\}, \tag{4.51}$$

$$P_2(t) = \left\{ x \in \mathbb{R}^N \mid \frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|} \leq |x| \leq \epsilon_0 \right\}, \tag{4.52}$$

$$P_3(t) = \left\{ x \in \mathbb{R}^N \mid |x| \geq \frac{\epsilon_0}{4} \right\}. \tag{4.53}$$

As a matter of fact, we have

$$\Omega \subset \mathbb{R}^N = P_1(t) \cup P_2(t) \cup P_3(t), \text{ for all } t \in [0, T).$$

We aim at controlling our problem on $P_1(t), P_2(t)$ and $P_3(t)$ as follows:

- On region $P_1(t)$(*blowup region*): We control w (see (4.24)) instead of U. More precisely, we show that w is a perturbation of the profile φ (the blowup profile, introduced in (4.33)). Then, (4.50) will follow from the control of w.

- On region $P_2(t)$(*intermidiate region*): We control a rescaled function $\mathcal{U}$ instead of U. More precisely, $\mathcal{U}$ is defined as follows: For all $x \in P_2(t), \xi \in (T - t(x))^{-\frac{1}{2}}(\bar{\Omega} - x)$ and $\tau \in \left[-\frac{t(x)}{T-t(x)}, 1\right)$, we define

$$\mathcal{U}(x, \xi, \tau) = (T - t(x))^{\frac{1}{3}}\, U\left(x + \xi\sqrt{T - t(x)}, (T - t(x))\tau + t(x)\right), \tag{4.54}$$

where $t(x)$ is defined as the solution of the following equation

$$|x| = \frac{K_0}{4}\sqrt{(T - t(x))|\ln(T - t(x))|} \text{ and } t(x) < T. \tag{4.55}$$

We remark that if ϵ_0 is small enough, then $t(x)$ is well defined for all x in $P_2(t)$. In addition to that, using (4.55), we have the following asymptotic

$$t(x) \to T, \text{ as } x \to 0.$$

For convenience, we introduce

$$\varrho(x) = T - t(x). \tag{4.56}$$

Then, the following holds

$$\varrho(x) \to 0 \text{ as } x \to 0.$$

As a matter of fact, using (4.19), we write the equation satisfied by $\mathcal{U}$ in $(\xi, \tau) \in \varrho^{-\frac{1}{2}}(x)(\bar{\Omega} - x) \times \left[-\frac{t(x)}{\varrho(x)}, 1\right)$ as follows:

$$\partial_\tau \mathcal{U} = \Delta_\xi \mathcal{U} - 2\frac{|\nabla \mathcal{U}|^2}{\mathcal{U} + \frac{\lambda^{\frac{1}{3}}\varrho^{\frac{1}{3}}(x)}{\tilde{\theta}(\tau)}} + \left(\mathcal{U} + \frac{\lambda^{\frac{1}{3}}\varrho^{\frac{1}{3}}(x)}{\tilde{\theta}(\tau)}\right)^4 - \frac{\tilde{\theta}'_\tau(\tau)}{\tilde{\theta}(\tau)}\mathcal{U}, \tag{4.57}$$

where

$$\tilde{\theta}(\tau) = \bar{\theta}(\tau\varrho(x) + t(x)), \tag{4.58}$$

with $\bar{\theta}(t)$ defined in (4.56). We now consider the following domain

$$|\xi| \leq \alpha_0\sqrt{|\ln(\varrho(x))|} \text{ and } \tau \in \left[-\frac{t(x)}{\varrho(x)}, 1\right).$$

When $\tau = 0$, we are in region $P_1(t(x))$, in fact (note that $P_1(t(x))$ and $P_2(t(x))$ have some overlapping by definition). From our constraints in $P_1(t(x))$, we derive that $\mathcal{U}(x, \xi, 0)$ is flat in the sense that

$$\mathcal{U}(x,\xi,0) \sim \left(3 + \frac{9}{8}\frac{K_0^2}{16}\right)^{-\frac{1}{3}}.$$

Our idea is to show that this flatness will be conserved for all $\tau \in [0, 1)$ (that is for all $t \in [t(x), T)$), in the sense that the solution will not depend that much on space. In one word, $\mathcal{U}$ is regarded as a perturbation of $\hat{\mathcal{U}}(\tau)$, where $\hat{\mathcal{U}}(\tau)$ is defined as follows

$$\begin{cases} \partial_\tau \hat{\mathcal{U}}(\tau) &= \hat{\mathcal{U}}^4(\tau), \\ \hat{\mathcal{U}}(0) &= \left(3 + \frac{9}{8}\frac{K_0^2}{16}\right)^{-\frac{1}{3}}. \end{cases} \tag{4.59}$$

Note that, we can give an explicit formula to the solution of equation (4.59)

$$\hat{\mathcal{U}}(\tau) = \left(3(1-\tau) + \frac{9}{8}\frac{K_0^2}{16}\right)^{-\frac{1}{3}}. \tag{4.60}$$

- On region $P_3(t)$(*regular region*): Thanks to the well-posedness property of the Cauchy problem for equation (4.35), we control the solution U as a perturbation of initial data $U(0)$. Indeed, the blowup time T will be chosen small in our analysis.

Relying on those ideas, we give in the following the definition of our shrinking set:

Definition 4.1 (Definition of $S(t)$). *Let us consider $T > 0, K_0 > 0, \epsilon_0 > 0, \alpha_0 > 0, A > 0, \delta_0 > 0, C_0 > 0, \eta_0 > 0$ and $t \in [0, T)$. Then, we introduce the following set*

$$S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t) \quad (S(t) \text{ for short}),$$

as a subset of $C^{2,1}(\Omega \times (0,t)) \cap C(\bar{\Omega} \times [0,t])$, containing all functions U satisfying the following conditions:

(i) ***Estimates in** $P_1(t)$: We have $q(s) \in V_{K_0,A}(s)$, where $q(s)$ is introduced in (4.34), $s = -\ln(T-t)$ and $V_{K_0,A}(s)$ is a subset of all function r in $L^\infty(\mathbb{R}^N)$, satisfying the following estimates:*

$$\begin{aligned} |r_i| &\le \frac{A}{s^2}, (i = 0,1), \text{ and } |r_2| \le \frac{A^2 \ln s}{s^2}, \\ |r_-(y)| &\le \frac{A^2}{s^2}(1 + |y|^3), \text{ and } \|r_e\|_{L^\infty(\mathbb{R}^N)} \le \frac{A^2}{\sqrt{s}}, \\ |(\nabla r)_\perp| &\le \frac{A}{s^2}(1 + |y|^3), \forall y \in \mathbb{R}^N, \end{aligned}$$

where the definitions of $r_i, r_-, (\nabla r)_\perp$ are given in (4.44), (4.46) and (4.47), respectively.

(ii) ***Estimates in** $P_2(t)$: For all $|x| \in \left[\frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|}, \epsilon_0\right]$, $\tau(x,t) = \frac{t-t(x)}{\varrho(x)}$ and $|\xi| \le \alpha_0\sqrt{|\ln \varrho(x)|}$, we have the following*

$$\begin{aligned} \left|\mathcal{U}(x,\xi,\tau(x,t)) - \hat{\mathcal{U}}(\tau(x,t))\right| &\le \delta_0, \\ |\nabla_\xi \mathcal{U}(x,\xi,\tau(x,t))| &\le \frac{C_0}{\sqrt{|\ln \varrho(x)|}}, \end{aligned}$$

where $\mathcal{U}$, $\hat{\mathcal{U}}$ *and* $\varrho(x)$ *are given* (4.54), (4.56) *and* (4.60), *respectively.*

(*iii*) ***Estimates in*** $P_3(t)$***:*** *For all* $x \in \{|x| \geq \frac{\epsilon_0}{4}\} \cap \Omega$, *we have*

$$\begin{aligned}|U(x,t) - U(x,0)| &\leq \eta_0,\\ \left|\nabla U(x,t) - \nabla e^{t\Delta} U(x,0)\right| &\leq \eta_0.\end{aligned}$$

In addition to that, we would like to introduce the set $S^*(K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, T)$ as follows:

Definition 4.2. *For all* $T > 0, K_0 > 0, \epsilon_0 > 0, \alpha_0 > 0, A > 0, \delta_0 > 0, C_0 > 0,$ *and* $\eta_0 > 0$, *we introduce* $S^*(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0)$ *(*$S^*(T)$ *for short) as the subset of all functions* U *in* $C^{2,1}(\Omega \times (0,T)) \cap C(\bar{\Omega} \times [0,T))$, *satisfying the following: for all* $t \in [0,T)$, *we have*

$$U \in S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t).$$

Remark 4.14. *The shrinking set* $S(t)$ *is inspired by the work of Merle and Zaag in [14]. However, we've made two major changes:*

- A simplification, by removing an unnecessary condition on the second derivative in space in region $P_2(t)$.

- A smart change in region $P_3(t)$, *by replacing* ∇U *by* $\nabla e^{t\Delta} U(0)$. *This change is crucial since we are working on a bounded domain* Ω.

Remark 4.15. *The conditions in* P_2 *and* P_3 *in Definition 4.1 are designed to make our solution more regular and these conditions help us to control* U *in* P_1 *and* $q(s) \in V_{K_0,A}(s)$. *Finally, the main purpose is to satisfy* (4.50). *In other words, the control* U *in* P_1 *is the main issue.*

Remark 4.16. *In our paper, we use a lot of parameters to control our solution. However, they will be fixed at the end of the proof. In addition to that, we would like to give some conventions on the universal constant in our paper: We use* C *for universal constants which depend only* $N, \Omega, \gamma, \lambda$ *and we write* $C(K_0, \epsilon_0, ...)$ *for constants which depend* $K_0, \epsilon_0, ...,$ *respectively.*

As we mentioned in Remark 4.15, we would like to show here some estimates of the sizes of q and ∇q, where q is the transformed function of U when $U \in S(t)$.

Lemme 4.17 (Sizes of q and ∇q)**.** *Let us consider* $K_0 \geq 1$ *and* $\epsilon_0 > 0$. *Then, there exist* $T_1(K_0, \epsilon_0)$ *and* $\eta_1(\epsilon_0)$ *such that for all* $\alpha_0 > 0, A > 0, \delta_0 \leq \frac{1}{2}\hat{\mathcal{U}}(0)$ *(see* (4.60)*),* $C_0 > 0, \eta_0 \leq \eta_1, T \leq T_1$ *and* $t \in [0,T)$*: if* $U \in S(K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$, *then, the following holds:*

(*i*) *The estimates on* q*: For all* $y \in \mathbb{R}^N$ *and* $s = -\ln(T-t)$, *we have*

$$|q(y,s)| \leq \frac{C(K_0)A^2}{\sqrt{s}} \text{ and } |q(y,s)| \leq \frac{C(K_0)A^2 \ln s}{s^2}(1 + |y|^3).$$

(*ii*) *The estimates on* ∇q*: For all* $y \in \mathbb{R}^N$, *we have*

$$|\nabla q(y,s)| \leq \frac{C(K_0, C_0)A^2}{\sqrt{s}}, \quad |\nabla q(y,s)| \leq \frac{C(K_0, C_0)A^2 \ln s}{s^2}(1 + |y|^3),$$

and

$$|(1 - \chi(y,s))\nabla q(y,s)| \leq \frac{C(K_0)}{\sqrt{s}}.$$

Proof. The conclusion directly follows from the definition of the shrinking set $S(t)$ and $V_{K_0,A}(s)$. In addition to that, these definitions are almost the same as in [14]. Therefore, we kindly refer the reader to see Lemma B.1 at page 1537 in [14]. $\square$

4.3.2 Initial data

In this subsection, we will concentrate on introducing our initial data to equation (4.19) so that it is trapped in $S(0)$. In order to do that, we first introduce the following cut-off function:

$$\chi_1(x) = \chi_0\left(\frac{|x|}{\sqrt{T}|\ln T|}\right), \tag{4.61}$$

where χ_0 is given in (4.28). In addition to that, we introduce H^* as a function in $C_0^\infty(\mathbb{R}^N \setminus \{0\})$ satisfying

$$H^*(x) = \begin{cases} \left[\frac{9}{16}\frac{|x|^2}{|\ln|x||}\right]^{-\frac{1}{3}}, & \forall |x| \le \min\left(\frac{1}{4}d(0,\partial\Omega), \frac{1}{2}\right), x \neq 0, \\ 0, \quad \forall |x| \ge \frac{1}{2}d(0,\partial\Omega), & \end{cases} \tag{4.62}$$

and for all $x \in \mathbb{R}^N, x \neq 0$, the following condition holds

$$0 \le H^*(x) \le \left[\frac{9}{16}\frac{|x|^2}{|\ln|x||}\right]^{-\frac{1}{3}}.$$

We now give the definition of our initial data corresponding to equation (4.19): For all $(d_0, d_1) \in \mathbb{R}^{1+N}$, we define

$$\begin{aligned} U_{d_0,d_1}(x,0) &= T^{-\frac{1}{3}}\left[\varphi\left(\frac{x}{\sqrt{T}}, -\ln s_0\right) + (d_0 + d_1\cdot z)\chi_0\left(\frac{|z|}{\frac{K_0}{32}}\right)\right]\chi_1(x) \\ &+ H^*(x)\left(1-\chi_1(x)\right), \end{aligned} \tag{4.63}$$

where $z = \dfrac{x}{\sqrt{T|\ln T|}}$ and note that φ, χ_0, χ_1 and H^* are defined as in (4.33), (4.28), (4.61) and (4.62), respectively.

From (4.63), we would like to give the definition of initial data corresponding to equation (4.35), $q_{d_0,d_1}(s_0)$ with $s_0 = -\ln T$:

$$q_{d_0,d_1}(y,s_0) = e^{-\frac{s_0}{3}}U_{d_0,d_1}\left(ye^{-\frac{s_0}{3}}, 0\right)\psi_{M_0}(y,s_0) - \varphi(y,s_0), \tag{4.64}$$

where and ψ_{M_0}, φ and U_{d_0,d_1} are introduced in (4.29), (4.33) and (4.63), respectively.

Remark 4.18. *We would like to explain in brief how our initial data U_{d_0,d_1} has naturally the form shown in* (4.63)*. As we mentiond at the beginning of this section, our purpose is to control initial data in $S(0)$. More precisely, our inital data have to satisfy items (i) and (ii) in Definition 4.1. As a matter of fact, when T is small enough, the second term in the right hand side of* (4.63) *is zero on $P_1(0)$. Then, our initial data has only the first term and we adopt the idea given in [15] (see also [14], [8]), we use d_0, d_1 in order to control $q(s_0)$ in $V_{K_0,A}(s_0)$. In addition to that, we would like to mention that Proposition 4.24 below states that when q is trapped in $V_{K_0,A}(s)$, it has only two components $(q_0, q_1)(s)$ which may attain their upper bound, the others being strictly less than their upper bound specified in the definition of $V_{K_0,A}(s)$. This is indeed the reason to use (d_0, d_1) in our initial data. More precisely, these $1 + n$ parameters allows us to a reduction to a finite dimensional problem. We now mention the control in P_2. In that region, $|x|$ is small enough and we may consider that U is near the final profile*

$$\left[\frac{9}{16}\frac{|x|^2}{|\ln|x||}\right]^{-\frac{1}{3}}.$$

As a matter of fact, it is reasonable to introduce H^* as the main asymptotic of our initial data in $P_2(0)$. Using some priori estimates, we can derive good estimates in $P_2(0)$. More precisely, the following proposition is our statement:

Proposition 4.19 (Preparation of initial data)**.** *There exists $K_2 > 0$ such that for all $K_0 \geq K_2$ and $\delta_2 > 0$, there exist $\alpha_2(K_0, \delta_2) > 0$ and $C_2(K_0) > 0$ such that for every $\alpha_0 \in (0, \alpha_2]$, there exists $\epsilon_2(K_0, \delta_2, \alpha_0) > 0$ such that for every $\epsilon_0 \in (0, \epsilon_2]$ and $A \geq 1$, there exists $T_2(K_0, \delta_2, \epsilon_0, A, C_2) > 0$ such that for all $T \leq T_2$ and $s_0 = -\ln T$. The following holds:*

(I) We can find a set $\mathcal{D}_A \subset [-2, 2] \times [-2, 2]^N$ such that if we define the following mapping

$$\begin{aligned} \Gamma : \mathbb{R} \times \mathbb{R}^N &\to \mathbb{R} \times \mathbb{R}^N \\ (d_0, d_1) &\mapsto (q_0, q_1)(s_0), \end{aligned}$$

then, Γ is affine, one to one from $\mathcal{D}_A$ to $\hat{\mathcal{V}}_A(s_0)$, where $\hat{\mathcal{V}}_A(s)$ is defined as follows

$$\hat{\mathcal{V}}_A(s) = \left[-\frac{A}{s^2}, \frac{A}{s^2}\right]^{1+N}. \tag{4.65}$$

Moreover, we have

$$\Gamma\,|_{\partial\mathcal{D}_A} \subset \partial\hat{\mathcal{V}}_A(s_0),$$

and

$$deg\left(\Gamma\,|_{\partial\mathcal{D}_A}\right) \neq 0, \tag{4.66}$$

where q_0, q_1 are defined as in (4.48)*, considered as the components of $q_{d_1,d_1}(s_0)$, which is a transform function of $U_{d_0,d_1}(0)$, given in* (4.34)*.*

(II) We now consider $(d_0, d_1) \in \mathcal{D}_A$. Then, initial data $U_{d_0,d_1}(0)$ belongs to

$$S(K_0, \epsilon_0, \alpha_0, A, \delta_2, C_2, 0, 0) = S(0),$$

where $S(0)$ is defined in Definition 4.1. Moreover, the following estimates hold (*i*) *Estimates in* $P_1(0)$: *We have* $q_{d_0,d_1}(s_0) \in \mathcal{V}_{K_0,A}(s_0)$ *and*

$$|q_0(s_0)| \leq \frac{A}{s_0^2}, \quad |q_{1,j}(s_0)| \leq \frac{A}{s_0^2}, \quad |q_{2,i,j}(s_0)| \leq \frac{\ln s_0}{s_0^2}, \forall i,j \in \{1,...,N\},$$

$$|q_-(y,s_0)| \leq \frac{1}{s_0^2}(|y|^3+1), \quad |\nabla q_\perp(y,s_0)| \leq \frac{1}{s_0^2}(|y|^3+1), \forall y \in \mathbb{R}^N,$$

and

$$q_e \equiv 0,$$

where the components of $q_{d_0,d_1}(s_0)$ *are defined in* (4.46).

(*ii*) *Estimates in* $P_2(0)$: *For every* $|x| \in \left[\frac{K_0}{4}\sqrt{T|\ln T|}, \epsilon_0\right]$, $\tau_0(x) = -\frac{t(x)}{\varrho(x)}$ *and* $|\xi| \leq \alpha_0\sqrt{|\ln \varrho(x)|}$, *we have*

$$\left|\mathcal{U}(x,\xi,\tau_0(x)) - \hat{\mathcal{U}}(\tau_0(x))\right| \leq \delta_2 \text{ and } |\nabla_\xi \mathcal{U}(x,\xi,\tau_0(x))| \leq \frac{C_2}{\sqrt{|\ln \varrho(x)|}},$$

where $\mathcal{U}, \hat{\mathcal{U}}$, *and* $\varrho(x)$ *are defined as in* (4.54), (4.60) *and* (4.56), *respectively.*

Proof. The proof of Proposition 4.19 will be given in Appendix 4.5. We now assume that this proposition holds and continue to get to the conclusion of Theorem 4.1. □

4.3.3 Existence of a solution trapped in $S^*(T)$

In this subsection, we would like to derive the existence of a blowup solution U to equation (4.19), trapped in $S^*(T)$. As we said earlier, our proof will be a (non trivial) adaptation of the proof designed by Merle and Zaag in [14] for the more standard case (4.14). However, in comparision with (4.14), we observe in equation (4.19) a new feature, the nonlocal term involving $\bar{\theta}(t)$. As a matter of fact, it is important to study this term and it derivative. In particular, in the works which we used to make the main idea for our work (such as [14], [15], [8]), the authors only studied for constant coefficients parabolic equations. Hence, it makes a main highlight in our work. For that reason, we show here some estimates on $\bar{\theta}(t)$ (also on $\bar{\mu}(t)$). The following is our statement:

Proposition 4.20 (Some estimates of $\bar{\theta}(t)$ and $\bar{\mu}(t)$)**.** *Let us consider* $\lambda > 0, \gamma > 0$ *and* Ω *a* C^2 *bounded domain. Then, there exists* $K_3 > 0$ *such that for all* $K_0 \geq K_3, \delta_0 > 0$, *there exist* $\alpha_3(K_0,\delta_0) > 0$ *such that for all* $\alpha_0 \leq \alpha_3$, *there exists* $\epsilon_3(K_0,\delta_0,\alpha_0) > 0$ *such that for all* $\epsilon_0 \leq \epsilon_3$ *and* $A \geq 1, C_0 > 0, \eta_0 > 0$, *there exists* $T_3 > 0$ *such that for all* $T \leq T_3$ *the following holds: Assuming* U *is a non negative solution of equation* (4.19) *on* $[0,t_1]$, *for some* $t_1 < T$, $U \in S(T,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0,\eta_0,t) = S(t)$ *for all* $t \in [0,t_1]$ *and* $U(0) = U_{d_0,d_1}(0)$, *given in* (4.63) *with some* $(d_0,d_1) \in \mathbb{R}^{1+N}$ *satisfied that* $|d_0|+|d_1| \leq 2$, *the following statements hold:*

(*i*) *For all* $t \in [0,t_1]$, $\bar{\mu}(t)$ *and* $\bar{\theta}(t)$ *are positive and these estimates hold*

$$0 \leq \bar{\mu}(t) \leq C, \tag{4.67}$$
$$1 \leq \bar{\theta}(t) \leq C. \tag{4.68}$$

Moreover, for all $t \in (0,t_1)$, *we have*

$$|\bar{\mu}'(t)| \leq C(T-t)^{\frac{3N-8}{6}}|\ln(T-t)|^N, \tag{4.69}$$
$$\left|\bar{\theta}'(t)\right| \leq C(T-t)^{\frac{3N-8}{6}}|\ln(T-t)|^N. \tag{4.70}$$

(ii) In particular, if $U \in S(t)$ for all $t \in [0,T)$, then $\bar{\mu}(t)$ and $\bar{\theta}(t)$ converge respectively to $\bar{\mu}_T$ and $\bar{\theta}_T \in \mathbb{R}_+^*$ as $t \to T$.

Remark 4.21. *Although we know from item (ii) that $\bar{\theta}(t)$ converges to $\bar{\theta}_T$, we don't know how big is $\bar{\theta}_T$. In particular, the dependence of these constants on γ, λ, Ω and $T, \epsilon_0, ...,$ is not clear yet.*

Proof. We can see that item (ii) is a direct consequence of (i). So, we give only the proof of item (i). Using (4.20) and the fact that $U(t) \geq 0$ for all t, we derive the following

$$\bar{\mu}(t) = \int_\Omega U(t)dx \geq 0.$$

In addition to that, we write

$$\bar{\mu}(t) \leq \int_\Omega U(t)dx \leq \int_{P_1(t)} U(t)dx + \int_{P_2(t)} U(t)dx + \int_{P_3(t)} U(t)dx, \tag{4.71}$$

where $P_1(t), P_2(t), P_3(t)$ are given in (4.51), (4.52) and (4.53), respectively. Remembering $\varrho(x)$, defined in (4.56), we see that the following holds

$$\varrho(x) \sim \frac{8}{K_0^2}\frac{|x|^2}{|\ln|x||} \text{ as } x \to 0.$$

In particular, using Definition 4.1, we get the following estimates: for all $t \in [0, t_1]$

$$\begin{aligned}
\text{On } P_1(t), |U(x,t)| &\leq (T-t)^{-\frac{1}{3}}\left[\frac{CA^2}{\sqrt{|\ln(T-t)|}} + |\varphi(0, -\ln(T-t))|\right] \leq C(T-t)^{-\frac{1}{3}},\\
\text{On } P_2(t), |U(x,t)| &\leq \varrho^{-\frac{1}{3}}(x)\left[\hat{\mathcal{U}}(\tau(x,t)) + \delta_0\right] \leq C\left[\frac{|x|^2}{|\ln|x||}\right]^{-\frac{1}{3}},\\
\text{On } P_3(t), |U(x,t)| &\leq |U(x,0)| + \eta_0 \leq |U(x,0)| + 1,
\end{aligned}$$

provided that $K_0 \geq K_{3,1}$ $\delta_0 \leq 1$ and $\eta_0 \leq 1$. Integrating U on each $P_i(t), i = 1,2,3$, we obtain the following

$$\begin{aligned}
\int_{P_1(t)} U(t)dx &\leq C(K_0)(T-t)^{\frac{N}{2}-\frac{1}{3}}|\ln(T-t)|^{\frac{N}{2}},\\
\int_{P_2(t)} U(t)dx &\leq C\int_{|x|\leq\epsilon_0}\left[\frac{|x|^2}{|\ln|x||}\right]^{-\frac{1}{3}} dx \leq C\epsilon_0^{N-\frac{2}{3}}|\ln\epsilon_0|^{\frac{1}{3}},\\
\int_{P_3(t)} U(t) &\leq \left[\int_{\frac{\epsilon_0}{4}\leq|x|, x\in\Omega}[H^*+1]\,dx\right],
\end{aligned}$$

where H^* is defined in (4.62). Using (4.71) and the above estimates, it is easy to obtain the following estimate

$$\mu(t) \leq C, \text{ for all } t \in [0, t_1],$$

provided that $K_0 \geq K_{3,2}(\lambda,\gamma), \epsilon_0 \leq \epsilon_{3,1}(\lambda,\gamma), \eta_0 \leq \eta_{3,1}(\lambda,\gamma)$ and $T \leq T_{3,1}(K_0,\lambda,\gamma)$. This yields (4.67) and (4.68) also follows by using (4.21) and (4.67). We now give a proof to (4.69). Integrating two sides of equation (4.19), we get the following ODE

$$\bar{\mu}'(t) + \frac{\bar{\theta}'(t)}{\bar{\theta}(t)}\bar{\mu}(t) = \int_\Omega \Delta U(t)dx + \int_\Omega\left(\left(U(t) + \frac{\lambda^{\frac{1}{3}}}{\bar{\theta}(t)}\right)^4 - 2\frac{|\nabla U(t)|^2}{U(t) + \frac{\lambda^{\frac{1}{3}}}{\bar{\theta}(t)}}\right)dx. \tag{4.72}$$

We aim at proving the following estimate

$$\left|\int_\Omega \left(\left(U(t)+\frac{\lambda^{\frac{1}{3}}}{\bar\theta(t)}\right)^4 - 2\frac{|\nabla U(t)|^2}{U(t)+\frac{\lambda^{\frac{1}{3}}}{\bar\theta(t)}}\right)dx\right| \le C(T-t)^{\frac{3n-8}{6}}|\ln(T-t)|^n. \tag{4.73}$$

In order to do so, we first prove that

$$\int_\Omega U^4(t)dx \le C(T-t)^{\frac{3N-8}{6}}|\ln(T-t)|^N, \tag{4.74}$$

$$\int_\Omega \frac{|\nabla U(t)|^2}{U(t)+\frac{\lambda^{\frac{1}{3}}}{\bar\theta(t)}}dx \le C(T-t)^{\frac{3N-8}{6}}|\ln(T-t)|^N. \tag{4.75}$$

The techniques of proofs (4.74) and (4.75) are the same. Therefore, we only give here the proof of (4.75). Let us consider

$$I(x,t) = \frac{|\nabla U(x,t)|^2}{U(x,t)+\frac{\lambda^{\frac{1}{3}}}{\bar\theta(t)}}.$$

Then,

$$\int_\Omega I(x,t)dx \le \int_{P_1(t)} I(x,t)dx + \int_{P_2(t)} I(x,t)dx + \int_{P_3(t)} I(x,t)dx.$$

Now we claim the following lemma:

Lemme 4.22. *Under the hypothesis in Proposition 4.20, for all $t \in (0,t_1]$, the following estimates hold:*

$$\text{On } P_1(t) : I(x,t) \le C(K_0)(T-t)^{-\frac{4}{3}}, \tag{4.76}$$

$$\text{On } P_2(t) : I(x,t) \le C(K_0)\varrho^{-\frac{4}{3}}(x) \le C(K_0)\left[\frac{|x|^2}{|\ln|x||}\right]^{-\frac{4}{3}}, \tag{4.77}$$

$$\text{On } P_3(t) : I(x,t) \le C\left(|\nabla U(x,0)|^2+\eta_0^2\right) = C(|\nabla H^*(x)|+\eta_0^2). \tag{4.78}$$

Proof. From the definition of $S(t)$, we easily derive (4.78). So, we only give here the proofs of (4.76) and (4.77). We now start with (4.76). Let us consider $x \in P_1(t)$ and we use the condition of U in $P_1(t)$ to get the following

$$\frac{1}{C(K_0)}(T-t)^{-\frac{1}{3}} \le U(x,t) \le C(K_0)(T-t)^{-\frac{1}{3}}. \tag{4.79}$$

In addition to that, thanks to item (ii) in Lemma 4.17, we get

$$|\nabla_y W\left(\frac{x}{\sqrt{T-t}}, -\ln(T-t)\right)| \le \frac{C(K_0)A^2}{\sqrt{|\ln(T-t)|}},$$

which yields

$$|\nabla U(x,t)| \le C(K_0)(T-t)^{-\frac{5}{6}}. \tag{4.80}$$

Then, (4.76) follows by (4.79) and (4.80).

We now consider $x \in P_2(t)$. It is easy to derive from item (ii) in Definition 4.1 that

$$\begin{aligned} \frac{1}{C(K_0)}\varrho^{\frac{1}{3}}(x) &\leq U(x,t) \leq C(K_0)\varrho^{\frac{1}{3}}(x), \\ |\nabla U(x,t)| &\leq C\varrho^{-\frac{5}{6}}(x), \end{aligned}$$

provided that $\delta_0 \leq \delta_{3,1}$ and $\epsilon_0 \leq \epsilon_{3,2}$. This gives (4.77) and concludes the proof of Lemma 4.22. $\square$

We now continue the proof of Proposition 4.20. Considering $t \in (0, t_1)$ and taking the integral on two sides of (4.76), we write

$$\begin{aligned} \int_{P_1(t)} |I(x,t)|dx &\leq C(K_0)\int_{|x|\leq K_0\sqrt{(T-t)|\ln(T-t)|}} (T-t)^{-\frac{4}{3}}dx \\ &\leq C(K_0)(T-t)^{\frac{N}{2}-\frac{4}{3}}\left|\ln(T-t)\right|^{\frac{N}{2}}. \end{aligned}$$

Integrating the two sides of (4.77) and using the following fact

$$\varrho(x) \sim \frac{8}{K_0^2}\frac{|x|^2}{|\ln|x||} \text{ as } x \to 0,$$

we obtain the following

$$\int_{P_2(t)} |I(x,t)| \leq C(K_0)\left[\epsilon_0^{N-\frac{8}{3}}|\ln\epsilon_0|^{\frac{4}{3}} - ((T-t)|\ln(T-t)|)^{\frac{3N-8}{6}}|\ln((T-t)|\ln(T-t)|)|^{\frac{4}{3}}\right].$$

In addition to that, from (4.78), we have

$$\int_{P_3(t)} |I(x,t)|dx \leq C.$$

Hence, (4.75) holds.
In addition to that, using (4.163), we can derive that

$$\int_\Omega \Delta U(t)dx < \infty, \forall t \in (0,t_1).$$

Therefore, we have

$$\lim_{\upsilon\to 0}\int_{\{x,d(x,\partial\Omega)>\upsilon\}} \Delta U dx = \int_\Omega \Delta U(t)dx.$$

Moreover, for all $\upsilon > 0$ small enough and from item (iii) of Definition (4.1), we have

$$\left|\int_{\{x,d(x,\partial\Omega)>\upsilon\}} \Delta U dx\right| = \left|\int_{\partial\{x,d(x,\partial\Omega)>\upsilon\}} \nu(x)\cdot\nabla U(x,t)dS\right| \leq C. \tag{4.81}$$

This implies that

$$\int_\Omega \Delta U(t)dx \leq C. \tag{4.82}$$

Hence, from (4.72), (4.73) and (4.82), we derive the following

$$\left|\bar{\mu}'(t) + \frac{\bar{\theta}'(t)}{\bar{\theta}(t)}\bar{\mu}(t)\right| \leq C(T-t)^{\frac{3N-8}{6}} |\ln(T-t)|^N. \quad (4.83)$$

In addition to that, from the relation between $\bar{\mu}$ and $\bar{\theta}$ in (4.21), we write

$$\frac{\bar{\theta}'(t)}{\bar{\theta}(t)} = \frac{2\gamma}{3\lambda^{\frac{1}{3}}}\left(1 - \frac{\frac{2\gamma}{3\lambda^{\frac{1}{3}}}\bar{\mu}(t)}{\left(1+\gamma|\Omega| + \frac{\gamma}{\lambda^{\frac{1}{3}}}\bar{\theta}(t)\bar{\mu}(t)\right)^{\frac{1}{3}}}\right)^{-1}\left(1+\gamma|\Omega| + \frac{\gamma}{\lambda^{\frac{1}{3}}}\bar{\theta}(t)\bar{\mu}(t)\right)^{-\frac{1}{3}}\mu'(t).$$

We also have the fact that

$$\sqrt{\bar{\theta}(t)} \geq \frac{\gamma}{\lambda^{\frac{1}{3}}}\bar{\mu}(t),$$

which yields that

$$1 \leq \left(1 - \frac{\frac{2\gamma}{3\lambda^{\frac{1}{3}}}\bar{\mu}(t)}{\left(1+\gamma|\Omega| + \frac{\gamma}{\lambda^{\frac{1}{3}}}\bar{\theta}(t)\bar{\mu}(t)\right)^{\frac{1}{3}}}\right)^{-1} \leq 3.$$

Hence, $\bar{\theta}'(t)$ and $\bar{\mu}'(t)$ have the same sign and we can use (4.83) to conclude that

$$|\bar{\mu}'(t)| \leq C(T-t)^{\frac{3N-8}{6}} |\ln(T-t)|^N. \quad (4.84)$$

This yields (4.69) and (4.70). Thus, we get the conclusion of the proof of Proposition 4.20. ☐

Proposition 4.23 (Existence of a solution to equation (4.19), confined in S^*)**.** *We can find parameters $T > 0, K_0 > 0, \epsilon_0 > 0, \alpha_0 > 0, A > 0, \delta_0 > 0, C_0 > 0, \eta_0 > 0$ such that there exist $(d_0, d_1) \in \mathbb{R} \times \mathbb{R}^N$ such that with initial data $U_{d_0,d_1}(0)$(given in (4.63)), the solution U of equation (4.19) exists on $\Omega \times [0, T)$ and*

$$U \in S^*(T),$$

where $S^*(T) = S^*(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0)$, given in (4.2).

Proof. As a matter of fact, this Proposition plays a central role in our problem. In other words, it will imply Theorem 4.1 (see subsection 4.3.4 below). The proof of this Proposition will be presented in two steps:

- *First step:* We use a reduction of our problem to a finite dimensional one. More precisely, we prove that the control U in $S(t)$ for all $t \in [0, T)$ is reduced to the control of $(q_0, q_1)(s)$ in $\hat{\mathcal{V}}_A(s)$ (see Proposition 4.24 below).
- *Second step:* In this step, we aim at proving that there exist $(d_0, d_1) \in \mathbb{R}^{1+N}$ such that $U \in S^*(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, T)$ with suitable parameters. Then, the conclusion follows from a topological argument based on Index theory.

We now give two main steps with more technical details:

a) Reduction to a finite dimensional problem: In this step, we derive that the control of $U \in S(t)$ with $t \in [0, T)$ is reduced to the control of the transform function $q(s)$ such that two first components $(q_0, q_1)(s)$ are trapped in $\hat{\mathcal{V}}_A(s)$ (see (4.65)), where $s = -\ln(T-t)$. More precisely, the following proposition is our statement:

Proposition 4.24 (Reduction to a finite dimensional problem). *There exist $T > 0, K_0 > 0, \epsilon_0 > 0, \alpha_0 > 0, A > 0, \delta_0 > 0, C_0 > 0$ and $\eta_0 > 0$ such that the following holds: We consider U a solution of equation* (4.19) *that exists on $[0, t_1]$, for some $t_1 < T$, with initial data $U_{d_0,d_1}(0)$ given in* (4.63), *for some $(d_0, d_1) \in \mathcal{D}_A$. We also assume that we have $U \in S(t)$ for all $\forall t \in [0, t_1]$ and $U \in \partial S(t_1)$ (see the definition of $S(t) = S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$ in Definition 4.1 and the set $\mathcal{D}_A$ given in Proposition 4.19). Then, the following statements hold:*

(i) *We have $(q_0, q_1)(s_1) \in \partial\hat{\mathcal{V}}_A(s_1)$, where $(q_0, q_1)(s)$ are components of $q(s)$ given in* (4.48) *and $q(s)$ is the transform function of U defined in* (4.34) *and $s_1 = \ln(T - t_1)$.*

(ii) *There exists $\nu_0 > 0$ such that for all $\nu \in (0, \nu_0)$, we have*

$$(q_0, q_1)(s_1 + \nu) \notin \hat{\mathcal{V}}_A(s_1 + \nu).$$

Consequently, there exists $\nu_1 > 0$ such that

$$U \notin S(t_1 + \nu), \forall \nu \in (0, \nu_1).$$

The idea of the proof is inspired (in a non trivial way) by the ideas given by Merle and Zaag in [14]. Since the proof is long and technical, we leave it to Section 4.4. Therefore, we assume here that Proposition 4.24 holds and go forward to the conclusion of Proposition 4.23.

b) Topological argument and the conclusion of Proposition 4.23: In this step, by using Proposition 4.24 and a topological argument based on Index theory, we conclude Proposition 4.23. More precisely, we prove that there exist $T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0$ and $(d_0, d_1) \in \mathcal{D}_A$ such that with initial data $U_{d_0,d_1}(0)$ (defined in (4.63)), the solution of equation (4.19) exists on $[0, T)$ and belongs to $S^*(T)$ where $S^*(T)$ is defined in Definition 4.2. Indeed, let us consider parameters $T > 0, K_0 > 0, \epsilon_0 > 0, \alpha_0 > 0, A > 0, \delta_0 > 0, C_0$ and $\eta_0 > 0$ such that Propositions 4.19 and 4.24 hold. Using Proposition 4.19, we have the following

$$\forall (d_0, d_1) \in \mathcal{D}_A, \quad U_{d_0,d_1}(0) \in S(0).$$

In particular, it follows from Proposition 1.2.2 page 12 in Kavallaris and Suzuki [11] together with Lemma 4.12 that equation (4.19) is locally in time well-posed in $C^{2,1}(\Omega \times (0, T_0)) \subset C(\bar{\Omega} \times [0, T_0])$, for some $T_0 > 0$. Therefore, for every $(d_0, d_1) \in \mathcal{D}_A$, we define $t^*(d_0, d_1) \in [0, T)$ as the maximum time, satisfying

$$U_{d_0,d_1} \in S(t), \forall t \in [0, t^*(d_0, d_1)),$$

where U_{d_0,d_1} is the solution of (4.19) corresponding to initial data $U_{d_0,d_1}(0)$, introduced in (4.63). Then, we have two possible cases:

a) Either $t^*(d_0, d_1) = T$ for some $(d_0, d_1) \in \mathcal{D}_A$, then, we get the conclusion of the proof.

b) Or $t^*(d_0, d_1) < T$, for all $(d_0, d_1) \in \mathcal{D}_A$. This case in fact never occurs, as we will show in the following.

Indeed, assuming by contradiction that case *b)* hold and using the continuity of the solution in time and the definition of the maximal time $t^*(d_0, d_1)$, we have

$$U_{d_0,d_1}(t^*(d_0, d_1)) \in \partial S(t^*(d_0, d_1)).$$

Thanks to the finite dimensional reduction property given in item (i) of Proposition 4.24, we derive the following

$$(q_0, q_1)\,(s_*(d_1, d_2)) \in \partial\hat{\mathcal{V}}_A(s_*(d_0, d_1)),$$

where q_0, q_1 are defined in (4.48) as the components of q_{d_0,d_1}, which is a transformed function of U_{d_0,d_1} (see (4.34)) and $s_*(d_0, d_1) = -\ln(T - t^*(d_0, d_1))$. Then, we may define the following mapping

$$\begin{array}{rcl}\Lambda : \mathcal{D}_A & \to & \left([-1,1] \times [-1,1]^N\right)\\ (d_0, d_1) & \mapsto & \dfrac{s_*^2(d_0, d_1)}{A}\,(q_0, q_1)\,(s_*(d_0, d_1)).\end{array}$$

From the definition of $t^*(d_1, d_2)$, the components (q_0, q_1) and the transversal crossing property given in item (ii) in Proposition 4.24, we see that Λ is continuous on $\mathcal{D}_A$. In addition to that, from item (i) of Proposition 4.19, we can derive that for all $(d_0, d_1) \in \partial\mathcal{D}_A$

$$(q_0, q_1)\,(s_0) \in \partial\hat{\mathcal{V}}_A(s_0), \quad s_0 = -\ln T.$$

However, using item (ii) of Proposition 4.24 again and the definition of $t^*(d_0, d_1)$ we deduce that

$$t^*(d_0, d_1) = 0,$$

which yields

$$s_*(d_0, d_1) = s_0 \text{ and } \Lambda(d_0, d_1) = \frac{s_0^2}{A}\Gamma(d_0, d_1),$$

where Γ is defined in item (I) of Proposition 4.19. Hence, thanks to (4.66), we conclude

$$\deg\,\left(\Lambda\,|_{\mathcal{D}_A}\right) \neq 0.$$

In fact, such a mapping Λ can not exist by using Index theory. Hence, case $b)$ doesn't occur only case $a)$ occurs. Thus, the conclusion of Proposition 4.23 follows. □

4.3.4 The conclusion of Theorem 4.1

In this subsection, we would like to give a complete proof of Theorem 4.1. We now consider the solution U which has been constructed in Proposition 4.23. Then, U exists on $[0, T)$ and

$$U(t) \in S(t), \forall t \in [0, T).$$

Using item (i) in Definition 4.1, we have the following

$$q \text{ exists on } [-\ln T, +\infty) \text{ and } \|q(., s)\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{\sqrt{s}}, \forall s \in [-\ln T, +\infty), \tag{4.85}$$

for some constant $C > 0$. Thanks to (4.15), (4.17), (4.24) and (4.30), we have

$$\left\| \frac{(T-t)^{\frac{1}{3}}\lambda^{\frac{1}{3}}}{\bar{\theta}(t)(1-u(.,t))} - \left(3 + \frac{9}{8}\frac{|.|^2}{(T-t)|\ln(T-t)|}\right)^{-\frac{1}{3}} \right\|_{L^\infty(\Omega)} \le \frac{C}{\sqrt{|\ln(T-t)|}}. \tag{4.86}$$

Using (4.68) and (4.70), we can derive that $\bar{\theta}(t)$ converges to $\theta_T > 0$ with

$$\left|\bar{\theta}(t) - \theta_T\right| \le C(T-t)^{\frac{1}{12}}, \forall t \in [0,T).$$

This implies that (4.6) hold with

$$\theta^* = \frac{\theta_T}{\lambda^{\frac{1}{3}}}. \tag{4.87}$$

Thus, item (i) of Theorem 4.1 follows.

We now prove that u quenches only at 0. Indeed, from the above estimate, we can derive that 0 is a quenching point of u. Now, we aim at proving that $x \in \Omega\backslash\{0\}$ are not quenching points of u. In fact, relying on relations (4.15) and (4.17), it is enough to prove the following Lemma:

Proposition 4.25. *The solution U satisfies the following statements:*

(i) For all $x \in \Omega \setminus \{0\}$, there exits $\nu(x) > 0$ such that

$$\limsup_{t\to T} \sup_{|x'-x|\le \nu(x)} U(x',t) < +\infty. \tag{4.88}$$

(ii) For all $x \in \Omega\backslash\{0\}$, $\lim_{t\to T} U(x,t)$ exists. In particular, if we define for all $x \in \Omega\diagdown\{0\}$

$$U^*(x) = \lim_{t\to T} U(x,t),$$

then $u^* \in C(\bar{\Omega} \setminus \{0\})$, and $U(t)$ uniformly converges to u^* on every compact subset of $\bar{\Omega} \setminus \{0\}$. In particular, we have the following asymptotic behavior

$$U^*(x) \sim \left[\frac{9}{32}\frac{|x|^2}{|\ln|x||}\right]^{-\frac{1}{3}}, \text{ as } x \to 0. \tag{4.89}$$

Proof. We consider U the solution constructed in Proposition 4.23. The proof will be given in two parts:

- *The proof of item (i):* The proof follows from the definition of shrinking set $S(t)$. Let us consider two cases: $|x| > \frac{\epsilon_0}{4}, x \in \Omega$ and $|x| \le \frac{\epsilon_0}{4}, x \in \Omega$.

+ The case where $|x| > \frac{\epsilon_0}{4}, x \in \Omega$: Using item (iii) of Definition 4.1, we conclude that for all $t \in [0,T)$,

$$U(x,t) \le U(x,0) + \eta_0 < +\infty.$$

Then, (4.88) follows.

+ The case where $|x| \le \frac{\epsilon_0}{4}, x \in \Omega$: For every x in that region, we can find t_x close to T such that $|x| \in \left[\frac{K_0}{4}\sqrt{(T-t_x)|\ln(T-t_x)|}, \epsilon_0\right]$. Moreover, if we have $t \in [t_x, T)$, we derive then $|x| \in \left[\frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|}, \epsilon_0\right]$. Considering $t \in [t_x, T)$ and using item (ii) in Definition 4.1, we derive the following

$$U(x+\xi\sqrt{\varrho(x)}, t) \le \varrho^{-\frac{1}{3}}(x)\left[\hat{\mathcal{U}}(\tau(x,t)) + \delta_0\right], \forall |\xi| \le \alpha_0\sqrt{|\ln \varrho(x)|}.$$

This estimate directly implies (4.88).

- *The proof of item (ii):* By using parabolic regularity and the technique given by Merle in [12], item (i) and Lemma 4.42, we may derive that there exists a function $U^* \in C(\bar{\Omega}\backslash\{0\})$ such that $U(x,t) \to U^*(x)$, as $t \to T$, for all $x \in \bar{\Omega}, x \neq 0$. Moreover, one can prove that the convergence is uniform on every compact subset of $\bar{\Omega}\backslash\{0\}$. It remains to give asymptotic behavior (4.89). We consider $x_0 \in \Omega$ such that $|x_0|$ is small enough. We first introduce the following functions: $\mathcal{U}(x_0,\xi,\tau)$ is defined in (4.54) and

$$\mathcal{V}(x_0,\xi,\tau) \quad = \quad \nabla_\xi \mathcal{U}(x_0,\xi,\tau), \tag{4.90}$$

where $\xi \in \varrho^{-\frac{1}{3}}(x_0)(\Omega - x_0) \subset \mathbb{R}^N$ and $\tau \in \left[-\frac{t(x_0)}{\varrho(x_0)}, 1\right)$, where $t(x_0)$ and $\varrho(x_0)$ are defined as in (4.55) and (4.56), respectively. We aim at proving the following estimates:

$$\sup_{\tau\in[0,1),|\xi|\le|\ln(\varrho(x_0))|^{\frac{1}{4}}} \left|\mathcal{U}(x_0,\xi,\tau) - \hat{\mathcal{U}}(\tau)\right| \le \frac{C}{|\ln(\varrho(x_0))|^{\frac{1}{4}}}, \tag{4.91}$$

$$\sup_{\tau\in[0,1),|\xi|\le 2|\ln(\varrho(x_0))|^{\frac{1}{4}}} |\mathcal{V}(x_0,\xi,\tau)| \le \frac{C}{|\ln(\varrho(x_0))|^{\frac{1}{4}}}, \tag{4.92}$$

and

$$\sup_{\tau\in[\tau_0,1),|\xi|\le\frac{1}{2}|\ln(\varrho(x_0))|^{\frac{1}{4}}} |\partial_\tau \mathcal{U}(x_0,\xi,\tau)| \le C(x_0), \tag{4.93}$$

for some $\tau_0 \in (0,1)$, fixed, and we also recall that $\hat{\mathcal{U}}(\tau)$ is introduced in (4.60).

We see that (4.92) follows from the fact that $U \in S(t), \forall t \in [0,T)$ and item (ii) of Definition 4.1. Thus, we only need to give the proofs of (4.91) and (4.93).

- *The proof of* (4.91)*:* We write here the equation of $\mathcal{U}$ from (4.59)

$$\partial_\tau \mathcal{U} = \Delta_\xi \mathcal{U} - 2\frac{|\nabla_\xi \mathcal{U}|^2}{\mathcal{U} + \frac{\lambda^{\frac{1}{3}}\varrho^{\frac{1}{3}}(x_0)}{\tilde{\theta}(\tau)}} + \left(\mathcal{U} + \frac{\lambda^{\frac{1}{3}}\varrho^{\frac{1}{3}}(x_0)}{\tilde{\theta}(\tau)}\right)^4 - \frac{\tilde{\theta}'(\tau)}{\tilde{\theta}(\tau)}\mathcal{U}, \tag{4.94}$$

where $\tilde{\theta}(\tau) = \bar{\theta}(\tau\varrho(x_0) + t(x_0))$ is given in (4.58). From (4.86) with $t = t(x_0)$, we derive that

$$\sup_{|\xi|\le 6|\ln(\varrho(x_0))|^{\frac{1}{4}}} \left|\mathcal{U}(x_0,\xi,0) - \hat{\mathcal{U}}(0)\right| \quad \le \quad \frac{C}{|\ln(\varrho(x_0))|^{\frac{1}{4}}}. \tag{4.95}$$

In addition to that, from item (ii) of Definition 4.1, we have for all $|\xi| \le 6|\ln\varrho(x_0)|^{\frac{1}{4}}$ and $\tau \in [0,1)$:

$$\mathcal{U}(x_0,\xi,\tau) \ge \frac{1}{2}\hat{\mathcal{U}}(0), \tag{4.96}$$

$$\mathcal{U}(x_0,\xi,\tau) \le \frac{3}{2}\hat{\mathcal{U}}(1), \tag{4.97}$$

provided that $\delta_0 \le \frac{1}{2}\hat{\mathcal{U}}(0)$. We now consider $\mathbb{U}(\xi,\tau)$ as follows

$$\mathbb{U}(\xi,\tau) = \mathcal{U}(x_0,\xi,\tau) - \hat{\mathcal{U}}(\tau), \text{ where } \xi \in \varrho^{-\frac{1}{3}}(x_0)(\Omega - x_0) \text{ and } \tau \in [0,1).$$

We then derive an equation satisfied by $\mathbb{U}$

$$\partial_\tau \mathbb{U} = \Delta_\xi \mathbb{U} + G_1 + G_2, \tag{4.98}$$

where G_1, G_2 are defined as follows

$$\begin{aligned} G_1(\xi,\tau) &= -2\frac{|\nabla \mathcal{U}|^2}{\mathcal{U} + \frac{\lambda^{\frac{1}{3}}\varrho^{\frac{1}{3}}(x_0)}{\tilde{\theta}(\tau)}} - \frac{\tilde{\theta}'(\tau)}{\tilde{\theta}(\tau)}\mathcal{U}, \\ G_2(\xi,\tau) &= \left(\mathcal{U} + \frac{\lambda^{\frac{1}{3}}\varrho^{\frac{1}{3}}(x_0)}{\tilde{\theta}(\tau)}\right)^4 - \hat{\mathcal{U}}^4(\tau). \end{aligned}$$

Next, we derive from the definition of $\tilde{\theta}(\tau)$, Proposition 4.20 and the fact that for all $\tau \in (0,1)$,

$$\left|\tilde{\theta}'(\tau)\right| \leq C\varrho^{\frac{1}{12}}(x_0)(1-\tau)^{-\frac{11}{12}},$$

and

$$1 \leq \tilde{\theta}(\tau) \leq C.$$

Hence, from (4.92), (4.96) and (4.97), we deduce that for all $\tau \in [0,1)$, $|\xi| \leq 2|\ln \varrho(x_0)|^{\frac{1}{4}}$

$$\begin{aligned} |G_1(\xi,\tau)| &= \left| -2\frac{|\nabla \mathcal{U}(x_0,\xi,\tau)|^2}{\mathcal{U}(x_0,\xi,\tau) + \frac{\lambda^{\frac{1}{3}}\varrho^{\frac{1}{3}}(x_0)}{\tilde{\theta}(\tau)}} - \frac{\tilde{\theta}'(\tau)}{\tilde{\theta}(\tau)}\left(\mathcal{U}(x_0,\xi,\tau)\right)\right| \\ &\leq \frac{C}{|\ln \varrho(x_0)|^{\frac{1}{4}}}\left((1-\tau)^{-\frac{11}{12}} + 1\right). \end{aligned}$$

In addition to that, we derive from (4.97) that

$$|G_2(\xi,\tau)| \leq C|\mathbb{U}(x_0,\xi,\tau)| + \frac{C}{|\ln \varrho(x_0)|^{\frac{1}{4}}}.$$

We now recall the cut-off function χ_0, defined as in (4.28) , then, we introduce

$$\phi_1(\xi) = \chi_0\left(\frac{|\xi|}{|\ln(\varrho(x_0))|^{\frac{1}{4}}}\right).$$

As a matter of fact, we have some rough estimates on ϕ_1

$$\|\nabla_\xi \phi_1\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{|\ln(\varrho(x_0))|^{\frac{1}{4}}} \text{ and } \|\Delta_\xi \phi_1\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{|\ln(\varrho(x_0))|^{\frac{1}{2}}}. \tag{4.99}$$

Let us define $\mathbb{U}_1(\xi,\tau) = \phi_1(\xi)\mathbb{U}(\xi,\tau)$, for all $\xi \in \mathbb{R}^N$ and $\tau \in [0,1)$. Then, $\mathbb{U}_1$ satisfies the following equation

$$\partial_t \mathbb{U}_1 = \Delta \mathbb{U}_1 - 2\nabla\phi_1 \cdot \nabla \mathbb{U} - \Delta\phi_1 \mathbb{U} + \phi_1 G_1(\xi,\tau) + \phi_1 G_2(\xi,\tau).$$

Using Duhamel principal, we write an integral equation satisfied by $\mathbb{U}_1$

$$\mathbb{U}_1(\tau) = e^{\tau\Delta}\mathbb{U}_1(0) + \int_0^\tau e^{(\tau-\sigma)\Delta}\left[-2\nabla\phi_1 \cdot \nabla\mathbb{U} - \Delta\phi_1\mathbb{U} + \phi_1 G_1 + \phi_1 G_2\right](\sigma)d\sigma.$$

This implies that for all $\tau \in [0,1)$, we have

$$\|\mathbb{U}_1(.,\tau)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{|\ln(\varrho(x_0))|^{\frac{1}{4}}} + C\int_0^\tau \|\mathbb{U}_1(.,\sigma)\|_{L^\infty(\mathbb{R}^N)} d\sigma.$$

Thanks to Granwall inequality, we get the following

$$\|\mathbb{U}_1(.,\tau)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{|\ln(\varrho(x_0))|^{\frac{1}{4}}}, \forall \tau \in [0,1)\,,$$

which yields (4.91).

Using (4.93), we can derive that the limit $\lim_{\tau\to 1} \mathcal{U}(x_0,0,\tau)$ exists. In addition to that, we derive from (4.91) that

$$U^*(x_0) = \lim_{\tau\to 1} \frac{\mathcal{U}(x_0,0,\tau)}{\varrho^{\frac{1}{3}}(x_0)} \sim \left(\frac{9}{8}\frac{K_0^2}{16}\varrho(x_0)\right)^{-\frac{1}{3}} \sim \left[\frac{9}{16}\frac{|x_0|^2}{|\ln|x_0||}\right]^{-\frac{1}{3}} \text{ as } x_0 \to 0.$$

This is the conclusion of (4.89). So, we get the proof in Proposition 4.25 and we also get the complete conclusion of Theorem 4.1. □

4.4 Reduction to a finite dimensional problem

This section plays a central role in our analysis. In fact, it is devoted to the proof of Proposition 4.24. More precisely, this section has two parts:

- In the first subsection, we prove *a priori estimates* on U in $P_1(t), P_2(t)$ and $P_3(t)$ when U is trapped in $S(t)$.

- The second subsection is devoted to the conclusion of Proposition 4.24. In fact, we use the first subsection to derive that U satisfies almost all the conditions in $S(t)$ with strict bounds, except for the bounds on $q_0(s)$ and $q_1(s)$, with $s = -\ln(T-t)$. This means that in order to control U in $S(t)$, we need to control only $(q_0,q_1)(s)$ in $\hat{\mathcal{V}}_A(s)$, defined in (4.65). In addition to that, we also prove the outgoing transversal crossing property. It means that if the solution U touches the boundary of $S(t_1)$ for some $t_1 \in (0,T)$, then, U will be outside $S(t)$ for all $t \in (t_1, t_1+\nu)$ with ν small enough. In one word, this is the reduction to a finite dimensional problem: the control of two components $(q_0,q_1)(s)$ in $\hat{\mathcal{V}}_A(s)$.

4.4.1 A priori estimates

We proceed in 3 steps: a, b and c), respectively devoted to parts $P_1(t), P_2(t)$ and $P_3(t)$.

a) We aim in the following Proposition at proving a priori estimates for U in $P_1(t)$:

Lemme 4.26. *There exists $K_4 > 0, A_4 > 0$ such that for all $K_0 \geq K_4, A \geq A_4$ and $l^* > 0$ there exists $T_4(K_0,A,l^*)$ such that for all $\epsilon_0 > 0, \alpha_0 > 0, \delta_0 > 0, \eta_0 > 0, C_0 > 0, T \leq T_4$ and for all $l \in [0,l^*]$, the following holds: Assume that we have the following conditions:*

- *We consider initial data $U(0) = U_{d_0,d_1}(0)$, given in (4.63) and $(d_0,d_1) \in \mathcal{D}_A$, given in Proposition 4.19 such that $(q_0,q_1)(s_0)$ belongs to $\hat{\mathcal{V}}_A(s_0)$, where $s_0 = -\ln T$, $\hat{\mathcal{V}}_A(s)$ is defined in (4.65) and q_0, q_1 are components of $q_{d_0,d_1}(s_0)$, a transform function of U, defined in (4.34).*

- *We have* $U \in S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$ *for all* $t \in [T - e^{-\sigma}, T - e^{-(\sigma+l)}]$, *for some* $\sigma \geq s_0$ *and* $l \in [0, l^*]$.

Then, the following estimates hold:

(*i*) *For all* $s \in [\sigma, \sigma + l]$, *we have*

$$|q_0'(s) - q_0(s)| + \left| q_{1,i}'(s) - \frac{1}{2} q_{1,i}(s) \right| \leq \frac{C}{s^2}, \forall i \in \{1, ..., N\}, \tag{4.100}$$

and

$$\left| q_{2,i,j}'(s) + \frac{2}{s} q_{2,i,j}(s) \right| \leq \frac{CA}{s^3}, \forall i, j \in \{1, ..., N\}, \tag{4.101}$$

where $q_1 = (q_{1,j})_{1 \leq i \leq N}, q_2 = (q_{2,i,j})_{1 \leq i,j \leq N}$ *and* q_1, q_2 *are defined in* (4.44).

(*ii*) *Control of* $q_-(s)$: *For all* $s \in [\sigma, \sigma + l]$ *and* $y \in \mathbb{R}^N$, *we have the two following cases:*

- *The case where* $\sigma \geq s_0$:

$$|q_-(y, s)| \leq C \left(A e^{-\frac{s-\sigma}{2}} + A^2 e^{-(s-\sigma)^2} + (s - \sigma) \right) \frac{(1 + |y|^3)}{s^2}, \tag{4.102}$$

- *The case where* $\sigma = s_0$

$$|q_-(y, s)| \leq C(1 + (s - \sigma)) \frac{(1 + |y|^3)}{s^2}. \tag{4.103}$$

(*iii*) *Control of the gradient term of* q: *For all* $s \in [\sigma, \sigma + l], y \in \mathbb{R}^N$, *we have the two following cases:*

- *The case where* $\sigma \geq s_0$:

$$|(\nabla q)_\perp(y, s)| \leq C \left(A e^{-\frac{s-\sigma}{2}} + e^{-(s-\sigma)^2} + (s - \sigma) + \sqrt{s - \sigma} \right) \frac{(1 + |y|^3)}{s^2}, \tag{4.104}$$

- *The case where* $\sigma = s_0$

$$|(\nabla q)_\perp(y, s)| \leq C \left(1 + (s - \sigma) + \sqrt{s - \sigma} \right) \frac{(1 + |y|^3)}{s^2}. \tag{4.105}$$

(*iii*) *Control of the outside part* q_e: *For all* $s \in [\sigma, \sigma + \lambda]$, *we have the two following cases:*

- *The case where* $\sigma \geq s_0$:

$$\|q_e(., s)\|_{L^\infty(\mathbb{R}^N)} \leq C \left(A^2 e^{-\frac{s-\sigma}{2}} + A e^{(s-\sigma)} + 1 + (s - \sigma) \right) \frac{1}{\sqrt{s}}, \tag{4.106}$$

- *The case where* $\sigma = s_0$

$$\|q_e(., s)\|_{L^\infty(\mathbb{R}^N)} \leq C \left(1 + (s - \sigma) \right) \frac{1}{\sqrt{s}}. \tag{4.107}$$

Proof. The proof of this proposition relies completely on techniques given by Merle and Zaag in [14]. As a matter of fact, the equation (4.35) is quite the same as in that paper if we ignore some perturbations which will be very small in our analysis. More precisely, thanks to Lemmas 4.35, 4.36, 4.37, 4.38, 4.39 and 4.40, we assert that the techniques in [14] hold in our case. Hence, we kindly refer the reader to Lemma 3.2 at page 1523 in [14] for more details. □

This implies a priori estimates in $P_1(t)$ as follows:

Proposition 4.27 (A priori estimates in $P_1(t)$)**.** *There exist $K_5 \geq 1$ and $A_5 \geq 1$ such that for all $K_0 \geq K_5, A \geq A_5, \epsilon_0 > 0, \alpha_0 > 0, \delta_0 \leq \frac{1}{2}\hat{\mathcal{U}}(0), C_0 > 0, \eta_0 > 0$, there exists $T_5(K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0)$ such that for all $T \leq T_5$, the following holds: If U a nonnegative solution of equation* (4.19) *satisfying $U \in S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$ for all $t \in [0, t_5]$ for some $t_5 \in [0, T)$, and initial data $U(0) = U_{d_0,d_1}$ given in* (4.63) *for some $d_0, d_1 \in \mathcal{D}_A$ given in Proposition 4.19, then, for all $s \in [-\ln T, -\ln(T - t_5)]$, we have the following:*

$$
\begin{aligned}
\forall i, j \in \{1, \cdots, n\}, \quad |q_{2,i,j}(s)| &\leq \frac{A^2 \ln s}{2s^2}, \\
\left\| \frac{q_{,-}(.,s)}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{A}{2s^2}, \left\| \frac{(\nabla q(.,s))_\perp}{1+|y|^3} \right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A}{2s^2} \text{ and } \|q_e(s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^2}{2\sqrt{s}},
\end{aligned}
$$

where q is a transformed function of U given in (4.34).

Proof. The proof is a consequence of Lemma 4.26. In particular, the proof is the same as in the work of Merle and Zaag in [15]. Hence, we refer the reader to Proposition 3.7, page 157 in that work. □

b) We now show a priori estimates on U in $P_2(t)$. We start with the following lemma:

Lemme 4.28 (A priori estimates in the intermediate region)**.** *There exists K_6 and $A_6 > 0$, such that for all $K_0 \geq K_6, A \geq A_6, \delta_6 > 0$, there exists $\alpha_6(K_0, \delta_6) > 0, C_6(K_0, A) > 0$ such that for all $\alpha_0 \leq \alpha_6, C_0 > 0$, there exists $\epsilon_6(\alpha_0, A, \delta_6, C_0)$ such that for all $\epsilon_0 \leq \epsilon_6$, there exists $T_6(\epsilon_0, A, \delta_6, C_0)$ and $\eta_6(\epsilon_0, A, \delta_0, C_0) > 0$ such that for all $T \leq T_6, \eta_0 \leq \eta_6, \delta_0 \leq \frac{1}{2}\left(3 + \frac{9}{8}\frac{K_0^2}{16}\right)^{-\frac{1}{3}}$, the following holds: if $U \in S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$ for all $t \in [0, t_*]$, for some $t_* \in [0, T)$, then, for all $|x| \in \left[\frac{K_0}{4}\sqrt{(T - t_*)|\ln(T - t_*)|}, \epsilon_0\right]$, we have:*

(*i*) *For all $|\xi| \leq \frac{7}{4}\alpha_0\sqrt{|\ln \varrho(x)|}$ and $\tau \in \left[\max\left(0, -\frac{t(x)}{\varrho(x)}\right), \frac{t_* - t(x)}{\varrho(x)}\right]$, the transformed function $\mathcal{U}(x, \xi, \tau)$ defined in* (4.54) *satisfies the following:*

$$
|\nabla_\xi \mathcal{U}(x, \xi, \tau)| \leq \frac{2C_0}{\sqrt{|\ln \varrho(x)|}}, \tag{4.108}
$$

$$
\mathcal{U}(x, \xi, \tau) \geq \frac{1}{4}\left(3 + \frac{9}{8}\frac{K_0^2}{16}\right)^{-\frac{1}{3}}, \tag{4.109}
$$

$$
|\mathcal{U}(x, \xi, \tau)| \leq 4. \tag{4.110}
$$

(ii) For all $|\xi| \leq 2\alpha_0\sqrt{|\ln \varrho(x)|}$ *and* $\tau_0 = \max\left(0, -\frac{t(x)}{\varrho(x)}\right)$*: we have*

$$\left|\mathcal{U}(x,\xi,\tau_0) - \hat{\mathcal{U}}(\tau_0)\right| \leq \delta_6 \text{ and } |\nabla_\xi \mathcal{U}(x,\xi,\tau_0)| \leq \frac{C_6}{\sqrt{|\ln \varrho(x)|}}.$$

Proof. We leave the proof to Appendix 4.6. □

Using the above lemma, we now give a priori estimates in $P_2(t)$. The following is our statement:

Proposition 4.29 (A priori estimates in $P_2(t)$)**.** *There exists* $K_7 > 0$ *and* $A_7 > 0$ *such that for all* $K_0 \geq K_7, A \geq A_7$*, there exists* $\delta_7 \leq \frac{1}{2}\hat{\mathcal{U}}(0)$ *and* $C_7(K_0, A)$ *such that for all* $\delta_0 \leq \delta_7, C_0 \geq C_7$ *there exists* $\alpha_7(K_0, \delta_0)$ *such that for all* $\alpha_0 \leq \alpha_7$*, there exist* $\epsilon_7(K_0, \delta_0, C_0) > 0$ *such that for all* $\epsilon_0 \leq \epsilon_7$*, there exists* $T_7(\epsilon_0, A, \delta_0, C_0) > 0$ *such that for all* $T \leq T_7$ *the following holds: We assume that we have* $U \in S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, t)$ *for all* $t \in [0, t_7]$ *for some* $t_7 \in [0, T)$*, then, for all* $|x| \in \left[\frac{K_0}{4}\sqrt{(T-t_*)|\ln(T-t_*)|}, \epsilon_0\right]$, $|\xi| \leq \alpha_0\sqrt{|\ln \varrho(x)|}$ *and* $\tau \in \left[\max\left(-\frac{t(x)}{\varrho(x)}, 0\right), \frac{t_7 - t(x)}{\varrho(x)}\right]$*, we have*

$$\left|\mathcal{U}(x,\xi,\tau_*) - \hat{\mathcal{U}}(x,\xi,\tau_*)\right| \leq \frac{\delta_0}{2} \text{ and } |\nabla \mathcal{U}(x,\xi,\tau)| \leq \frac{C_0}{2\sqrt{|\ln \varrho(x)|}},$$

where $\varrho(x) = T - t(x)$*.*

Proof. We leave the proof to Appendix 4.7. □

Remark 4.30. *Unlike what Merle and Zaag did in [14], we don't require any condition in* $\nabla^2\mathcal{U}$ *in* $P_2(t)$ *(see Definition 4.1), as we have aldready stated in Remark 4.14. Accordingly, our a priori estimates in* $P_2(t)$ *will be simpler than those of [14], as one may see from the proof given in Appendix C.*

c) We now give a priori estimates on U in $P_3(t)$:

Proposition 4.31 (A priori estimates in P_3)**.** *Let us consider* $K_0 > 0, \epsilon_0 > 0, \alpha_0 > 0, A > 0, \delta_0 \in [0, \frac{1}{2}\hat{\mathcal{U}}(0)], C_0 > 0, \eta_0 > 0$*. Then, there exists* $T_8(\eta_0) > 0$ *such that for all* $T \leq T_8$*, the following holds: We assume that* U *is a nonnegative solution of* (4.19) *on* $[0, t_8]$ *for some* $t_8 < T$*, and* $U \in S(K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$ *for all* $t \in [0, t_8]$ *and initial data* $U(0) = U_{d_0,d_1}$ *given in* (4.63) *with* $|d_0|, |d_1| \leq 2$*. Then, for all* $|x| \geq \frac{\epsilon_0}{4}$ *and* $t \in (0, t_8]$*,*

$$|U(x,t) - U(x,0)| \leq \frac{\eta_0}{2}, \tag{4.111}$$

$$\left|\nabla U(x,t) - \nabla e^{t\Delta}U(x,0)\right| \leq \frac{\eta_0}{2}. \tag{4.112}$$

Remark 4.32. *As we have mentioned in Remark 4.14, we draw the attention of the reader to the change we have made with respect to the work of Merle and Zaag in [14]: We compare* $\nabla U(t)$ *to* $\nabla e^{t\Delta}U(0)$ *and not to* $\nabla U(0)$ *in [14] and this is crucial, since we are working on a bounded domain.*

Following the remark, we have just stated, we give in the following a crucial parabolic estimate for the free Dirichlet heat semi-group in Ω:

Lemme 4.33 (A parabolic regularity on the linear problem)**.** *Let us consider initial data U_{d_0,d_1}, given in* (4.63)*, for some* $|d_0|, |d_1| \leq 2$*. If we define*

$$L(t) = e^{t\Delta} U_{d_0,d_1}, t \in (0,T].$$

Then, $L(t) \in C(\bar{\Omega} \times [0,T]) \cap C^\infty(\Omega \times (0,T])$*. Moreover, the following holds*

$$\|\nabla_x L(t)\|_{L^\infty(\mathbb{R}^N)\left(|x| \geq \frac{\epsilon_0}{8}, x \in \Omega\right)} \leq C(\epsilon_0), \forall [0,T], \tag{4.113}$$

where ϵ_0 *introduced in the definition of* U_{d_0,d_1}.

Proof. See Appendix 4.9 □

The proof of Proposition 4.31. We rewrite the equation satisfied by U as follows

$$\partial_t U = \Delta U + G(U),$$

where

$$G(U) = -2 \frac{|\nabla U|^2}{U + \frac{\lambda^{\frac{1}{3}}}{\bar{\theta}(t)}} + \left(U + \frac{\lambda^{\frac{1}{3}}}{\bar{\theta}(t)} \right)^4 - \frac{\bar{\theta}'(t)}{\bar{\theta}(t)} U.$$

We remark that in order to get the conclusion, it is enough to prove that for all $x \in \Omega, |x| \geq \frac{\epsilon_0}{4}$ and $t \in (0, t_8]$, we have the following estimates

$$|U_1(x,t) - U_1(x,0)| \leq \frac{\eta_0}{2}, \tag{4.114}$$

$$\left| \nabla U_1(x,t) - \nabla e^{t\Delta} U_1(x,0) \right| \leq \frac{\eta_0}{2}, \tag{4.115}$$

where $U_1(x,t) = \exp\left(\int_0^t \frac{\bar{\theta}'(s)}{\bar{\theta}(s)} ds \right) U(x,t)$. Using the equation satisfied by U, we may derive an equation satisfied by U_1 as follows:

$$\partial_t U_1 = \Delta U_1 + G_1, \tag{4.116}$$

where $G_1(t) = \exp\left(\int_0^t \frac{\bar{\theta}'(s)}{\bar{\theta}(s)} ds \right) \left[-2 \frac{|\nabla U|^2}{U + \frac{\lambda^{\frac{1}{3}}}{\bar{\theta}(t)}} + \left(U + \frac{\lambda^{\frac{1}{3}}}{\bar{\theta}(t)} \right)^4 \right]$. In particular, from the fact that $U \in S(t)$ and Proposition 4.20, we can derive the following

$$\left| \exp\left(\pm \int_0^t \frac{\bar{\theta}'(s)}{\bar{\theta}(s)} \right) ds \right| \leq 2.$$

Moreover, from item (*iii*) of Definition 4.1 and Lemma 4.33, we derive the following:

$$|G_1(x,t)| \leq C(K_0, \epsilon_0, \eta_0), \forall |x| \geq \frac{\epsilon_0}{8} \text{ and } \forall t \in (0, t_8].$$

In the following, we first prove (4.114) then (4.115).

+ The proof of (4.114)*:* We consider a cut-off function $\chi_2 \in C_0^\infty(\bar{\Omega})$ such that $\chi_2 = 1$ for all $|x| \geq \frac{\epsilon_0}{6}, x \in \bar{\Omega}$ and $\chi_2 = 0$ for all $|x| \leq \frac{\epsilon_0}{8}$ and $|\nabla \chi_2| + |\Delta \chi_2| \leq C(\epsilon_0)$. If we define $U_2 = U_1 \chi_2$, then U_2 satisfies the following

$$\partial_t U_2 = \Delta U_2 + G_2,$$

where

$$G_2(U) = -2\nabla U_1 \cdot \nabla \chi_2 - \Delta \chi_2 U_1 - \chi_2 G_1.$$

Using the estimate of G_1 and the following fact

$$|\nabla U_1(x,t)| + |U_1(x,t)| \leq C(K_0, \epsilon_0, \eta_0), \forall |x| \geq \frac{\epsilon_0}{8} \text{ and } t \in [0, t_8],$$

which is a consequence of the fact that $U \in S(t)$ (particularly items (i) and (iii) in Definition 4.1), we conclude the following

$$\|G_2(x,t)\|_{L^\infty(\Omega)} \leq C(K_0, \epsilon_0, C_0, \eta_0), \forall |x| \geq \frac{\epsilon_0}{8} \text{ and } \forall t \in [0, t_8].$$

We now use a Duhamel formula to write U_2 as follows

$$U_2(t) = e^{t\Delta} U_2(0) + \int_0^t e^{(t-\tau)\Delta} \left(G_2(U(\tau))\right) d\tau, \tag{4.117}$$

where $e^{t\Delta}$ stands for the Dirichlet heat semi-group on Ω (see more in Appendix 4.9). In particular, we have for all $U_0 \in L^\infty(\Omega)$,

$$\left\|e^{t\Delta} U_0\right\|_{L^\infty(\Omega)} \leq \|U_0\|_{L^\infty(\Omega)}.$$

Therefore,

$$\begin{aligned} |U_2(t) - U_2(0)| &\leq \left|U_2(t) - e^{t\Delta} U_2(0)\right| + \left|e^{t\Delta} U_2(0) - U_2(0)\right| \\ &\leq \left|\int_0^t e^{(t-s)\Delta} G_2(s) ds\right| + \left|e^{t\Delta} U_2(0) - U_2(0)\right| \\ &\leq C(K_0, \epsilon_0, C_0, \eta_0) T + \left\|e^{t\Delta}(U_2(0)) - U_2(0)\right\|_{L^\infty(\Omega)}. \end{aligned}$$

In addition to that, because $U_2(0)$ is smooth and has a compact support in Ω, we can prove that

$$\left\|e^{t\Delta}(U_2(0)) - U_2(0)\right\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to 0,$$

which yields the fact that

$$\|U_2(t) - U_2(0)\|_{L^\infty(\Omega)} \leq \frac{\eta_0}{2},$$

provided that $T \leq T_{8,1}(K_0, \epsilon_0, C_0, \eta_0)$. This concludes the proof of (4.114).

+ *The proof of* (4.115)*:* We derive from (4.117) the following fact:

$$\nabla U_2(t) = \nabla e^{t\Delta} U_2(0) + \int_0^t \nabla e^{(t-\tau)\Delta} G_2(\tau) d\tau.$$

This implies that

$$\left|\nabla U_2(t) - \nabla e^{t\Delta} U_1(0)\right| \leq \left|\nabla e^{t\Delta} U_2(0) - \nabla e^{t\Delta} U_1(0)\right| + \left|\int_0^t \nabla e^{(t-\tau)\Delta} G_2(\tau) d\tau\right|.$$

Using (4.146) and Lemma (4.41) in the below, we derive that

$$\left|\int_0^t \nabla e^{(t-\tau)\Delta} G_2(\tau) d\tau\right| \leq C(K_0, \epsilon_0, C_0, \eta_0) \int_0^t \frac{1}{\sqrt{t-\tau}} d\tau \leq C(K_0, \epsilon_0, C_0, \eta_0) \sqrt{T}.$$

In order to finish the proof, it is enough to prove that for all $|x| \geq \frac{\epsilon_0}{4}$, we have

$$\left|\nabla e^{t\Delta} U_2(0) - \nabla e^{t\Delta} U_1(0)\right| \leq \frac{\eta_0}{4}, \tag{4.118}$$

provided that $T \leq T_{8,2}$. Indeed, using Dirichlet heat semi-group and Lemma 4.41 below, we may write the following:

$$\begin{aligned}
\left|\nabla e^{t\Delta} U_2(0) - \nabla e^{t\Delta} U_1(0)\right| &= \left|\int_\Omega \nabla_x G(x,y,t,0)(1-\chi_2(y)) U_{d_0,d_1}(y) dy\right| \\
&\leq C \int_{|y| \leq \frac{\epsilon_0}{6}} \frac{\exp\left(-\frac{|x-y|^2}{t}\right)}{t^{\frac{N+1}{2}}} |U_{d_0,d_1}(y)| dy \\
&\leq C \int_{|y| \leq \frac{\epsilon_0}{6}} \exp\left(-\frac{|x-y|^2}{t}\right) \left(\frac{|x-y|}{\sqrt{t}}\right)^{N+2} \frac{\sqrt{t}}{|x-y|^{N+2}} |U_{d_0,d_1}(y)| dy \\
&\leq C(\epsilon)\sqrt{t} \int_{|y| \leq \frac{\epsilon_0}{6}} |U_{d_0,d_1}(y)| dy \\
&\leq C(\epsilon_0)\sqrt{t} \|U_{d_0,d_1}\|_{L^1(\Omega)} \leq C(\epsilon_0)\sqrt{T}.
\end{aligned}$$

This yields (4.118), provided that $T \leq T_{8,3}(\epsilon_0)$. In particular, from the definitions of U_1 and U_2, we can derive (4.115). Finally, we get the conclusion of Proposition 4.31. □

4.4.2 The conclusion of the proof of Proposition 4.24

It this part, we aim at giving a complete proof to Proposition 4.24:

The proof of Proposition 4.24 . We first choose parameters $K_0, \epsilon_0 > 0, \alpha_0 > 0, A > 0, \delta_0 > 0, \delta_1 > 0, C_0 > 0, \eta_0 > 0$ and $T > 0$ such that Propositions 4.19, 4.27, 4.29 and 4.31 hold. In particular, the constant T will be fixed small later. Then, the conclusion of the proof follows as we will show in the following. We now consider U, a solution of equation (4.19), with initial data $U_{d_0,d_1}(0)$, defined in Definition 4.63 and satisfying the following:

$$U \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, C_0, \eta_0, t) = S(t),$$

for all $t \in [0, t_*]$ for some $t_* \in (0, T)$ and

$$u \in \partial S(t_*).$$

(i) Using Propositions 4.27, 4.29 and 4.31, we can derive that

$$(q_1, q_2)(s_*) \in \partial \hat{V}_A(s_*), \tag{4.119}$$

where $s_* = \ln(T - t_*)$.

(ii) Using item (i), we derive that either

$$|q_0(s_*)| = \frac{A}{s_*^2},$$

or there exists $j_0 \in \{1, ..., N\}$ such that

$$|q_{1,j_0}(s_*)| = \frac{A}{s_*^2}.$$

Then, without loss of generality, we can suppose that the first case occurs, because the argument is the same for other cases. Hence, using (4.100) in Lemma 4.26, we see that

$$|q_0'(s) - q_0(s)| \leq \frac{C}{s^2}.$$

Therefore, we obtain that the sign of $q_0'(s_*)$ is opposite to the sign of

$$\frac{d}{ds}\left(\epsilon_0 \frac{A}{s^2}\right)(s_*),$$

provided that $A \geq 2C$, where $\epsilon_0 = \pm 1$ and $q_0(s_*) = \epsilon_0 \frac{A}{s_*^2}$. This means that the flow of q_0 is transverse outgoing on the bounds of the shrinking set

$$-\frac{A}{s^2} \leq q_0(s) \leq \frac{A}{s^2}.$$

It follows then that $(q_0, q_1)(s)$ leaves $\hat{\mathcal{V}}(s)$ at s_*. Thus, we conclude item (ii). Finally, we get the conclusion of Proposition 4.24 $\square$

4.5 Preparation of initial data

In this section, we give the proof of Proposition 4.19. More precisely, we aim at proving the following lemma which directly implies Proposition 4.19:

Lemme 4.34. *There exists $K_2 > 0$ such that for all $K_0 \geq K_2, \delta_2 > 0$, there exist $\alpha_2(K_0, \delta_2) > 0, C_2 > 0$ such that for all $\alpha_0 \in (0, \alpha_2]$ there exists $\epsilon_2(K_0, \delta_2, \alpha_0) > 0$ such that for all $\epsilon_0 \in (0, \epsilon_2]$ and $A \geq 1$, there exists $T_2(K_0, \delta_2, \epsilon_0, A, C_2) > 0$ such that for all $T \in (0, T_2]$, there exists a subset $\mathcal{D}_A \subset [-2, 2]^{1+N}$ such that the following properties hold: Asumme that initial data $U_{d_0,d_1}(0)$ is given as in (4.63), then:*

A) For all $(d_0, d_1) \in \mathcal{D}_A$, we have initial data

$$U(0) = U_{d_0,d_1}(0) \in S(T, K_0, \epsilon_0, \alpha_0, A, \delta_2, C_2, 0, 0).$$

In particular, we have the following:

(i) Estimates in $P_1(0)$: we have the transformed function $q(s_0)$ of $U_{d_0,d_1}(0)$, trapped in $V_{K_0,A}(s_0)$, where $s_0 = -\ln T$ and we have also the following estimates:

$$\left|q_0(s_0) - \frac{Ad_0}{s_0^2}\right| + \left|q_{1,j}(s_0) - \frac{Ad_{1,j}}{s_0^2}\right| \leq Ce^{-s_0}, \text{ for all } j \in \{1, ..., n\},$$

$$|q_{2,i,j}(s_0)| \leq \frac{\ln s_0}{s_0^2}, \text{ for all } i, j \in \{1, ..., n\},$$

$$|q_-(y, s_0)| \leq \frac{1}{s_0^2}(1 + |y|^3), \quad |(\nabla_y q)_\perp(y, s_0)| \leq \frac{1}{s_0^2}(1 + |y|^3), \text{ for all } y \in \mathbb{R}^N,$$

and

$$q_e(s_0) \equiv 0,$$

where the components of q are defined in (4.49).

(ii) Estimates in $P_2(0)$*: For all* $|x| \in \left[\frac{K_0}{4}\sqrt{T|\ln T|}, \epsilon_0\right]$ *and and* $|\xi| \le \alpha_0\sqrt{|\ln \varrho(x)|}$*, we have*

$$\left|\mathcal{U}(x,\xi,\tau_0(x)) - \hat{\mathcal{U}}(\tau_0(x))\right| \le \delta_2, \text{ and } |\nabla_\xi \mathcal{U}(x,\xi,\tau_0(x))| \le \frac{C_2}{\sqrt{|\ln \varrho(x)|}},$$

where $\tau_0(x) = -\frac{t(x)}{\varrho(x)}$ *and* $\mathcal{U}, \hat{\mathcal{U}}, t(x), \varrho(x)$ *are given in* (4.54)*,* (4.55)*,* (4.56) *and* (4.60)*.*

B) We have the following facts

$$(d_0,d_1) \in \mathcal{D}_A \text{ if and only if } (q_0,q_1)(s_0) \in \hat{\mathcal{V}}_A(s_0)$$

and

$$(d_0,d_1) \in \partial\mathcal{D}_A \text{ if and only if } (q_0,q_1)(s_0) \in \partial\hat{\mathcal{V}}_A(s_0),$$

where $\hat{\mathcal{V}}_A(s)$ *given in* (4.65)*.*

Proof. We see that part $B)$ directly follows from item (i) of part $A)$. In addition to that, our definition is almost the same as in [21] (see also Ghoul, Nguyen and Zaag [8]; Merle and Zaag [14] and [15]). So, we kindly refer the reader to see the proofs of the existence of the set $\mathcal{D}_A$, item i in $A)$ and part $B)$ in Proposition 4.5 in [21]. Here we only give the proof of item (ii) in part $A)$. We now consider $T > 0, K_0 > 0, \epsilon_0 > 0, \alpha_0 > 0, \delta_2 > 0, C_2 > 0, \eta_0 > 0$. We aim at proving that if these constants are suitably chosen, then for all $x \in \left[\frac{K_0}{4}\sqrt{T|\ln T|}, \epsilon_0\right]$ and $|\xi| \le 2\alpha_0\sqrt{|\ln \varrho(x)|}$, where $\varrho(x)$ given in (4.55), we have the following

$$\left|\mathcal{U}(x,\xi,\tau_0(x)) - \hat{\mathcal{U}}(\tau_0(x))\right| \le \delta_2, \quad |\nabla_\xi \mathcal{U}(x,\xi,\tau_0(x))| \le \frac{C_2}{\sqrt{|\ln \varrho(x)|}}.$$

We observe from the definition of $t(x)$ given in (4.55) that if $\alpha_0 \le \alpha_{2,1}$ and $\epsilon_0 \le \epsilon_{2,1}$, then, for all $x \in \left[\frac{K_0}{4}\sqrt{T\ln T}, \epsilon_0\right]$ and $|\xi| \le 2\alpha_0\sqrt{|\ln \varrho(x)|}$, we have

$$\left|\xi\sqrt{\varrho(x)}\right| \le \frac{|x|}{2},$$

which yields

$$\frac{r_0}{2} \le \frac{|x|}{2} \le \left|x + \xi\sqrt{T(x)}\right| \le \frac{3}{2}|x|, \text{ with } r_0 = \frac{K_0}{4}\sqrt{T|\ln T|}. \tag{4.120}$$

Hence, for all $x \in \left[\frac{K_0}{4}\sqrt{T|\ln T|}, \epsilon_0\right]$, we have

$$\chi\left(16(x+\xi\sqrt{\varrho(x)})\sqrt{T}, -\ln T\right)\chi_1(x+\xi\sqrt{\varrho(x)}) = 0,$$

where χ and χ_1 are defined in (4.43) and (4.61), respectively. Therefore, from (4.63) and the definition of $\mathcal{U}$ in (4.54), we may derive that for all $x \in \left[\frac{K_0}{4}\sqrt{T|\ln T|}, \epsilon_0\right]$ and $|\xi| \le 2\alpha_0\sqrt{|\ln \varrho(x)|}$,

$$\mathcal{U}(x,\xi,\tau_0) = (I)\chi_1\left(x+\xi\sqrt{\varrho(x)}\right) + (II)\left(1-\chi_1(x+\xi\sqrt{\varrho(x)})\right),$$

where

$$(I) \quad = \quad \left(\frac{\varrho(x)}{T}\right)^{\frac{1}{3}} \left(3 + \frac{9}{8}\frac{|x+\xi\sqrt{\varrho(x)}|^2}{T|\ln T|}\right)^{-\frac{1}{3}},$$

and

$$(II) \quad = \quad \varrho^{\frac{1}{3}}(x) H^*(x+\xi\sqrt{\varrho(x)}),$$

with $H^*(x)$ given in (4.62). In addition to that, from the definition of $\varrho(x)$, given in (4.56), we obtain the following asymptotics

$$\ln\varrho(x) \sim 2\ln|x| \text{ and } \varrho(x) \sim \frac{8}{K_0^2}\frac{|x|^2}{|\ln|x||} \text{ as } |x| \to 0. \tag{4.121}$$

Besides that, we introduce $r_0 = \frac{K_0}{4}\sqrt{T|\ln T|}$ and $R_0 = \sqrt{T}|\ln T|$. Then, the following holds

$$\varrho(r_0) \sim T, \text{ and } \varrho(R_0) \sim \frac{16}{K_0^2}T|\ln T| \text{ and } \varrho(2R_0) \sim \frac{64}{K_0^2}T|\ln T| \text{ as } T \to 0. \tag{4.122}$$

We aim in the following at giving some estimates on $\mathcal{U}(x,\xi,\tau_0(x))$ and $\nabla_\xi\mathcal{U}(x,\xi,\tau_0(x))$.

- *Estimate on* $\mathcal{U}$*:* From the definition of the cut-off function χ_1 given in (4.61), it is enough to prove that for all $|x| \in \left[r_0, (2+\frac{1}{100})R_0\right]$ and $|\xi| \le 2\alpha_0\sqrt{|\ln\varrho(x)|}$, we have

$$\left|I_1 - \hat{\mathcal{U}}(\tau_0)\right| \le \frac{\delta_2}{2}, \tag{4.123}$$

on one hand and also that for all $|x| \in \left[\frac{99}{100}R_0, \epsilon_0\right]$ and $|\xi| \le 2\alpha_0\sqrt{|\ln\varrho(x)|}$, we have

$$\left|I_2 - \hat{\mathcal{U}}(\tau_0)\right| \le \frac{\delta_2}{2}, \tag{4.124}$$

on the other hand. Indeed, let us start with the proof of (4.123): We consider $|x| \in \left[r_0, (2+\frac{1}{100})R_0\right]$ and $|\xi| \le 2\alpha_0\sqrt{|\ln\varrho(x)|}$. Then, we write the following:

$$\left|I_1 - \hat{\mathcal{U}}(\tau_0(x))\right| = \left|\left(3\frac{T}{\varrho(x)} + \frac{9}{8}\frac{|x+\xi\sqrt{\varrho(x)}|^2}{\varrho(x)|\ln T|}\right)^{-\frac{1}{3}} - \left(3\frac{T}{\varrho(x)} + \frac{9}{8}\frac{K_0^2}{16}\right)^{-\frac{1}{3}}\right|$$

$$= \left|\left(3\frac{T}{\varrho(x)} + \frac{9}{8}\frac{K_0^2}{16} + \frac{9}{8}\left[\frac{|x+\xi\sqrt{\varrho(x)}|^2}{\varrho(x)|\ln T|} - \frac{K_0^2}{16}\right]\right)^{-\frac{1}{3}} - \left(3\frac{T}{\varrho(x)} + \frac{9}{8}\frac{K_0^2}{16}\right)^{-\frac{1}{3}}\right|.$$

In addition to that, we have

$$\begin{aligned}\frac{|x+\xi\sqrt{\varrho(x)}|^2}{\varrho(x)|\ln T|} - \frac{K_0^2}{16} &= \frac{|x|^2}{\varrho(x)|\ln T|}\left(1 + 2\frac{x\cdot\xi}{|x|^2}\sqrt{\varrho(x)} + \frac{|\xi|^2\varrho(x)}{|x|^2}\right) - \frac{K_0^2}{16}\\ &= \frac{K_0^2}{16}\frac{|\ln\varrho(x)|}{|\ln T|}\left(1 + 2\frac{x\cdot\xi}{|x|^2}\sqrt{\varrho(x)} + \frac{|\xi|^2\varrho(x)}{|x|^2}\right) - \frac{K_0^2}{16}.\end{aligned}$$

Besides that, we also have the following:

$$\left|\frac{x\cdot\xi}{|x|^2}|\sqrt{\varrho(x)}|\right| \le 4\alpha_0,$$

$$\left|\frac{|\xi|^2}{|x|^2}\varrho(x)\right| \le 4\alpha_0^2.$$

Moreover, for all $|x| \in \left[r_0, \left(2+\frac{1}{100}\right) R_0\right]$, we derive from (4.122) that

$$\frac{|\ln \varrho(x)|}{|\ln T|} \sim 1, \text{ as } T \to 0.$$

So, the following holds

$$\left| \frac{|x+\xi\sqrt{\varrho(x)}|^2}{\varrho(x)|\ln T|} - \frac{K_0^2}{16} \right| \to 0,$$

as $(\alpha_0, T) \to (0,0)$. From this fact, we can derive that if $T \leq T_{2,1}(K_0, \delta_2), \alpha_0 \leq \alpha_{2,2}(K_0, \delta_2)$, we have

$$\begin{aligned} \left| I_1 - \hat{\mathfrak{U}}(\tau_0(x)) \right| &= \left| \left(3\frac{T}{\varrho(x)} + \frac{9}{8} \frac{|x+\xi\sqrt{\varrho(x)}|^2}{\varrho(x)|\ln T|} \right)^{-\frac{1}{3}} - \left(3\frac{T}{\varrho(x)} + \frac{9}{8}\frac{K_0^2}{16} \right)^{-\frac{1}{3}} \right| \\ &\leq C(K_0) \left| \frac{|x+\xi\sqrt{T(x)}|^2}{T(x)|\ln T|} - \frac{K_0^2}{16} \right| \leq \frac{\delta_1}{2}. \end{aligned}$$

This concludes the proof of (4.123).

We now aim at proving (4.124). We consider $|x| \in \left[\frac{99}{100}R_0, \epsilon_0\right]$ and $|\xi| \leq 2\alpha_0\sqrt{|\ln \varrho(x)|}$. Using the definition of (II), we write as follows

$$\left|(II) - \hat{\mathfrak{U}}(\tau_0(x))\right| = \left| \left(\frac{9}{16} \frac{\left|x+\xi\sqrt{\varrho(x)}\right|^2}{\varrho(x)|\ln|x+\xi\sqrt{\varrho(x)}||} \right)^{-\frac{1}{3}} - \left(3\frac{T}{\varrho(x)} + \frac{9}{8}\frac{K_0^2}{16} \right)^{-\frac{1}{3}} \right|$$

$$= \left| \left(\frac{9}{8}\frac{K_0^2}{16} + \frac{9}{16} \left(\frac{\left|x+\xi\sqrt{\varrho(x)}\right|^2}{\varrho(x)|\ln|x+\xi\sqrt{\varrho(x)}||} - \frac{K_0^2}{8} \right) \right)^{-\frac{1}{3}} - \left(\frac{9}{8}\frac{K_0^2}{16} + 3\frac{T}{\varrho(x)} \right)^{-\frac{1}{3}} \right|.$$

Besides that, the function $\varrho(x)$ is radial in x, and increasing in $|x|$ when $|x|$ is small enough. Then, for all $\epsilon_0 \leq \epsilon_{2,1}$ and $|x| \in \left[\frac{99}{100}R_0, \epsilon_0\right]$, we have

$$\left|\frac{T}{\varrho(x)}\right| \leq \left|\frac{T}{\varrho\left(\frac{99}{100}R_0\right)}\right| \leq C(K_0)|\ln T|^{-1} \to 0 \text{ as } T \to 0. \tag{4.125}$$

In addition to that, we have

$$\begin{aligned} \frac{\left|x+\xi\sqrt{\varrho(x)}\right|^2}{\varrho(x)|\ln|x+\xi\sqrt{\varrho(x)}||} - \frac{K_0^2}{8} &= \frac{1}{\varrho(x)|\ln|x+\xi\sqrt{\varrho(x)}||} \left[|x|^2 + 2x\cdot\xi\sqrt{\varrho(x)} + |\xi|^2\varrho(x)\right] \\ &- \frac{K_0^2}{8} \\ &= \frac{K_0^2}{16}\left[\frac{|\ln \varrho(x)|}{|\ln|x+\xi\sqrt{\varrho(x)}||} - 2 + 4\alpha_0 \frac{|\ln \varrho(x)|}{|\ln|x+\xi\sqrt{\varrho(x)}||} \right. \\ &+ \left. 4\alpha_0^2 \frac{|\ln \varrho(x)|}{|\ln|x+\xi\sqrt{\varrho(x)}||} \right]. \end{aligned}$$

In particular, we have the following fact

$$\begin{aligned} \ln \varrho(x) &\sim 2\ln|x|, \text{ as } |x|\sim 0,\\ \frac{1}{|\ln|x+\xi\sqrt{\varrho(x)}||} &\sim \frac{1}{|\ln|x||}, \text{ as } \alpha_0\to 0. \end{aligned}$$

This yields

$$\left|\frac{\left|x+\xi\sqrt{\varrho(x)}\right|^2}{\varrho(x)|\ln|x+\xi\sqrt{\varrho(x)}||}-\frac{K_0^2}{8}\right|\to 0 \text{ as } (\epsilon_0,\alpha_0)\to(0,0). \tag{4.126}$$

From (4.125) and (4.126), we derive that

$$\left|(II)-\hat{\mathcal{U}}(\tau_0(x))\right|\le C(K_0)\left[\left|\frac{\left|x+\xi\sqrt{\varrho(x)}\right|^2}{\varrho(x)|\ln|x+\xi\sqrt{\varrho(x)}||}-\frac{K_0^2}{8}\right|+\frac{T}{\varrho(x)}\right]\le\frac{\delta_2}{2},$$

provided that $\alpha\le\alpha_{2,3}(K_0,\delta_2),\epsilon_0\le\alpha_{2,2}(K_0,\delta_2,\alpha_0)$ and $T\le T_{2,3}$. Thus, (4.124) holds. Finally, we get the conclusion that for all $|x|\in\left[\frac{K_0}{4}\sqrt{T|\ln T|},\epsilon_0\right]$ and $|\xi|\le 2\alpha_0\sqrt{|\ln\varrho(x)|}$, we have

$$\left|\mathcal{U}(x,\xi,\tau_0(x))-\hat{\mathcal{U}}(\tau_0(x))\right|\le\delta_2.$$

- *Estimate on $\partial_\xi\mathcal{U}$:* From the definition of $\mathcal{U}(x,\xi,\tau_0(x))=\mathcal{U}\left(x,\xi,-\frac{t(x)}{\varrho(x)}\right)$ given in (4.54) and expression (4.63) of initial data, we decompose $\nabla_\xi\mathcal{U}$ as follows

$$\partial_\xi\mathcal{U}(x,\xi,\tau_0(x))=B_1+B_2+B_3,$$

where

$$\begin{aligned} B_1 &= \left[-\frac{3}{4}\frac{\varrho^{\frac{5}{6}}(x)}{T^{\frac{4}{3}}|\ln T|}(x+\xi\sqrt{\varrho(x)})\left(3+\frac{9}{8}\frac{\left|x+\xi\sqrt{\varrho(x)}\right|^2}{T|\ln T|}\right)^{-\frac{4}{3}}\right]\chi_1(x+\xi\sqrt{\varrho(x)}),\\ B_2 &= \varrho^{\frac{5}{6}}(x)\nabla H^*(x+\xi\sqrt{\varrho(x)})\left(1-\chi_1(x+\xi\sqrt{\varrho(x)})\right),\\ B_3 &= \left[\left(\frac{\varrho(x)}{T}\right)^{\frac{1}{3}}\left(3+\frac{9}{8}\frac{|x+\xi\sqrt{\varrho(x)}|^2}{T|\ln T|}\right)^{-\frac{1}{3}}+\frac{3^{-\frac{1}{3}}N}{8|\ln T|}\frac{\varrho^{\frac{1}{3}}(x)}{T^{\frac{1}{3}}}-\varrho^{\frac{1}{3}}(x)H^*(x+\xi\sqrt{\varrho(x)})\right]\\ &\times \sqrt{\varrho(x)}\nabla\chi_1(x+\xi\sqrt{\varrho(x)}). \end{aligned}$$

It is enough to prove the following estimates:

- Estimate of B_1 : For all $|x|\in\left[r_0;(2+\frac{1}{100})R_0\right]$ and $|\xi|\le 2\alpha_0\sqrt{|\ln\varrho(x)|}$ we have

$$|B_1|\le\frac{C(K_0)}{\sqrt{|\ln\varrho(x)|}}. \tag{4.127}$$

- Estimate of B_2 : For all $|x| \in \left[\frac{99}{100}R_0, \epsilon_0\right]$ and $|\xi| \le 2\alpha_0\sqrt{|\ln \varrho(x)|}$, we have

$$|B_2| \le \frac{C(K_0)}{\sqrt{|\ln \varrho(x)|}}. \tag{4.128}$$

- Estimate of B_3 : For all $|x| \in \left[\frac{99}{100}R_0, (2+\frac{1}{100})R_0\right]$ and $|\xi| \le 2\alpha_0\sqrt{|\ln \varrho(x)|}$, we have

$$|B_3| \le \frac{C(K_0)}{\sqrt{|\ln \varrho(x)|}}. \tag{4.129}$$

We now start the proof:

- *Estimate of* B_1*:* We have the fact that for all $|z| \ge 1$

$$\left(3 + \frac{9}{8}|z|^2\right)^{-\frac{4}{3}} \le C|z|^{-\frac{8}{3}}.$$

Then,

$$\begin{aligned} |B_1| &\le C\frac{\varrho^{\frac{5}{6}}(x)}{T^{\frac{4}{3}}|\ln T|}\frac{T^{\frac{4}{3}}|\ln T|^{\frac{4}{3}}}{|x+\xi\sqrt{\varrho(x)}|^{\frac{5}{3}}} \\ &\le C\frac{\varrho^{\frac{5}{6}}(x)|\ln T|^{\frac{1}{3}}}{|x+\xi\sqrt{\varrho(x)}|^{\frac{5}{3}}}. \end{aligned}$$

Using (4.120), we obtain the following:

$$|B_1| \le C\frac{\varrho^{\frac{5}{6}}(x)|\ln T|^{\frac{1}{3}}}{|x|^{\frac{5}{3}}}.$$

In addition to that, for all $|x| \in \left[r_0, (2+\frac{1}{100})R_0\right]$, we have

$$|\ln \varrho(x)| \sim |\ln T|, \text{ as } T \to 0.$$

Then, we have

$$|B_1| \le \frac{C}{K_0^2}\frac{\varrho^{\frac{5}{6}}(x)|\ln T|^{\frac{1}{3}}}{\varrho^{\frac{5}{6}}(x)|\ln \varrho(x)|^{\frac{5}{6}}} \le \frac{C}{\sqrt{|\ln \varrho(x)|}},$$

provided that $K_0 \ge K_{2,3}, T \le T_{2,4}$. This yields (4.127).

- *Estimate of* B_2*:* From the definition of $H^*(x)$, when $|x| \le \epsilon_0$, ϵ_0 small enough, we have

$$H^*(x) = \left[\frac{9}{16}\frac{|x|^2}{|\ln|x||}\right]^{-\frac{1}{3}}.$$

This implies

$$|\nabla H^*(x)| \le C\frac{|\ln|x||^{\frac{1}{3}}}{|x|^{\frac{5}{3}}}.$$

Hence,

$$|B_2| \le C\frac{\varrho^{\frac{5}{6}}(x)|\ln|x||^{\frac{1}{3}}}{|x|^{\frac{5}{3}}} \le C\frac{|\ln|x||^{\frac{1}{3}}}{|\ln \varrho(x)|^{\frac{1}{3}}}\frac{1}{\sqrt{|\ln \varrho(x)|}},$$

on one hand. On the other hand, we have the following

$$|\ln \varrho(x)| \sim 2|\ln |x||, \text{ as } x \to 0.$$

Thus, (4.128) holds provided that $\epsilon_0 \leq \epsilon_{2,4}(K_0)$.

- *Estimate of B_3:* We first use the definition of χ_1 in (4.61) to write

$$|\nabla_x \chi_1(x)| \leq \frac{C}{\sqrt{T}|\ln T|}.$$

We now consider $|x| \in \left[\frac{99}{100}R_0, (2+\frac{1}{100})R_0\right]$ and $|\xi| \leq 2\alpha_0\sqrt{|\ln \varrho(x)|}$. We define

$$B_4 = \left(3T + \frac{9}{8}\frac{|x+\xi\sqrt{\varrho(x)}|^2}{|\ln T|}\right)^{-\frac{1}{3}} + \frac{3^{-\frac{1}{3}}N}{8T^{\frac{1}{3}}|\ln T|} - H^*(x+\xi\sqrt{\varrho(x)}).$$

Then,

$$B_3 = B_4 \varrho^{\frac{5}{6}}(x)\nabla\chi_1\left(x+\xi\sqrt{\varrho(x)}\right).$$

Estimates on B_4: Using the fact that $|x| \in \left[\frac{99}{100}R_0, (2+\frac{1}{100})R_0\right]$, $|\xi| \leq 2\alpha_0\sqrt{|\ln \varrho(x)|}$ and (4.120), we can derive that

$$\frac{1}{C}T|\ln T| \leq \frac{|x+\xi\sqrt{\varrho(x)}|^2}{|\ln T|} \leq CT|\ln T|,$$
$$\frac{1}{C}T|\ln T| \leq \frac{|x+\xi\sqrt{\varrho(x)}|^2}{|\ln|x+\xi\sqrt{\varrho(x)}||} \leq CT|\ln T|.$$

This implies that

$$|B_4| \leq C(T|\ln T|)^{-\frac{1}{3}}.$$

Hence, we estimate B_3:

$$\begin{aligned} |B_3| &\leq C(T|\ln T|)^{-\frac{1}{3}}\varrho^{\frac{5}{6}}(x)\nabla_x\chi_1\left(x+\xi\sqrt{\varrho(x)}\right) \\ &\leq C\frac{\varrho^{\frac{5}{6}}(x)}{T^{\frac{5}{6}}}\frac{1}{|\ln T|^{\frac{4}{3}}} \end{aligned}$$

In addition to that, for all $|x| \in \left[\frac{99}{100}R_0, (2+\frac{1}{100})R_0\right]$, we use (4.122) to deduce that

$$|\varrho(x)| \leq CT|\ln T|,$$

and we also have the following fact

$$|\ln \varrho(x)| \sim |\ln T|, \text{ as } T \to 0.$$

So, we conclude that

$$|B_3| \leq \frac{C}{\sqrt{|\ln \varrho(x)|}}$$

provided that $K_0 \geq K_{2,4}, \epsilon_0 \leq \epsilon_{2,5}(K_0, \alpha_0)$ and $T \leq T_{2,5}(K_0)$. Thus, we get the conclusion of (4.129). Finally, the conclusion of Lemma 4.34 follows. □

4.6 A priori estimates in the intermediate region

In this section, we aim at giving the proof of Lemma 4.28. Because our definitions are the same as in [14], estimates in this Proposition follow in the same way as in that work. Hence, we kindly refer the reader to Lemma 2.6 in page 1515 in that work for the proof of (4.108) and item (*ii*). It happens that, although the authors in [14] gave a statement which is similar to (4.109), they did not gave the proof. For that reason, we give here the proof of (4.109) and (4.110).

- *The proof of* (4.109)*:* Let us consider $|x| \in \left[\frac{K_0}{4}\sqrt{(T-t_*)|\ln(T-t_*)|}, \epsilon_0\right]$,$|\xi| \leq \frac{7}{4}\alpha_0\sqrt{|\ln \varrho(x)|}$ and $\tau \in \left[\max\left(0, -\frac{t(x)}{\varrho(x)}\right), \frac{t_*-t(x)}{\varrho(x)}\right]$. As a matter of fact, there exists $t \in [0, t_*]$ such that

$$\tau = \frac{t - t(x)}{\varrho(x)}.$$

Let us define

$$X = x + \xi\sqrt{\varrho(x)}.$$

We aim at considering the three following cases:

+ The case where $|X| \leq \frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|}$. We write

$$\mathcal{U}(x,\xi,\tau) = \varrho^{\frac{1}{3}}(x)U(X,t).$$

We have the fact that $X \in P_1(t)$. Then, using item (i) in Definition 4.1 together with item (i) in Lemma 4.17, we get

$$\begin{aligned}
(T-t)^{-\frac{1}{3}}U(X,t) &= W(Y,s), \text{ where } Y = \frac{X}{\sqrt{T-t}}, s = -\ln(T-t) \\
&\geq \left(3 + \frac{|X|^2}{(T-t)|\ln(T-t)|}\right)^{-\frac{1}{3}} - \frac{CA^2}{\sqrt{s}} \\
&\geq \frac{1}{2}\left(3 + \frac{9}{8}\frac{K_0^2}{16}\right)^{-\frac{1}{3}},
\end{aligned}$$

provided that $T \leq T_{6,1}(K_0, A)$. This yields

$$\mathcal{U}(x,\xi,\tau) \geq \left(\frac{\varrho(x)}{T-t}\right)^{-\frac{1}{3}}\frac{1}{2}\left(3 + \frac{9}{8}\frac{K_0^2}{16}\right)^{-\frac{1}{3}}.$$

In addition to that, the function $|x| \mapsto \varrho(x)$ is increasing when $|x|$ is small enough. This implies that

$$\varrho(x) \leq \varrho\left(\frac{K_0}{4(1-\frac{7}{4}\alpha_0)}\sqrt{(T-t)|\ln(T-t)|}\right).$$

From (4.55), (4.56) and (4.121), we derive that

$$\varrho\left(\frac{K_0}{4(1-\frac{7}{4}\alpha_0)}\sqrt{(T-t)|\ln(T-t)|}\right) \sim \frac{8}{K_0^2}\frac{K_0^2}{16(1-\frac{7}{4}\alpha_0)^2}\frac{2(T-t)|\ln(T-t)|}{|\ln(T-t)|} = \frac{(T-t)}{(1-\frac{7}{4}\alpha_0)^2},$$

as $T \to 0$. Hence, we have

$$\frac{\varrho(x)}{T-t} \leq 4,$$

provided that $\alpha_0 \leq \frac{2}{7}, T \leq T_{6,2}$. Finally, we get

$$\mathcal{U}(x,\xi,\tau) \geq \frac{1}{4}\left(3+\frac{8}{9}\frac{K_0^2}{16}\right)^{-\frac{1}{3}}.$$

+ The case where $|X| \in \left[\frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|}, \epsilon_0\right]$. In other words, we have $X \in P_2(t)$. We write as follows

$$\mathcal{U}(x,\xi,\tau) = \varrho(x)U(X,t).$$

In addition to that, using item (ii) in the definition of $S(t)$ (see Definition 4.1), we get the following:

$$U(X,t) = \varrho^{-\frac{1}{3}}(X)\mathcal{U}(X,0,\frac{t-t(X)}{\varrho(X)}) \geq \varrho^{-\frac{1}{3}}(X)\frac{1}{2}\left(3+\frac{9}{8}\frac{K_0^2}{16}\right)^{-\frac{1}{3}},$$

provided that $\delta_0 \leq \frac{1}{2}\left(3+\frac{9}{8}\frac{K_0^2}{16}\right)^{-\frac{1}{3}}$. In particular, using the fact that

$$(1-\frac{7}{4}\alpha_0)|x| \leq |X| \leq (1+\frac{7}{4}\alpha_0)|x|. \tag{4.130}$$

Then, we get

$$\left(\frac{\varrho(x)}{\varrho(X)}\right)^{\frac{1}{3}} \geq \frac{1}{2},$$

provided that $\alpha_0 \leq \alpha_{7,2}(K_0)$ and $|x| \leq \epsilon_{7,2}(K_0,\alpha_0)$. This yields that

$$\mathcal{U}(x,\xi,\tau) \geq \frac{1}{4}\left(3+\frac{9}{8}\frac{K_0^2}{16}\right)^{-\frac{1}{3}}.$$

+ The case where $|X| \geq \epsilon_0$. This means $X \in P_3(t)$. We first have the following fact

$$\mathcal{U}(x,\xi,\tau) = \varrho^{\frac{1}{3}}(x)U(X,t) \geq \frac{1}{2}\varrho^{\frac{1}{3}}(x)U(X,0),$$

provided that $\eta_0 \leq \frac{1}{2}$ and $\epsilon_0 \leq \epsilon_{6,3}$. We remark also that $|X| \leq (1+\frac{7}{4}\alpha_0)|x| \leq (1+\frac{7}{4}\alpha_0)\epsilon_0 \leq \frac{3}{2}\epsilon_0$. Then,

$$U(X,0) = \left[\frac{9}{16}\frac{|X|^2}{|\ln|X||}\right]^{-\frac{1}{3}}.$$

Moreover, using (4.121) and (4.130), we get

$$\varrho^{\frac{1}{3}}(x)U(X,0) \geq \frac{1}{\sqrt[3]{2}}\left[\frac{9}{8}\frac{K_0^2}{16}\right]^{-\frac{1}{3}} \geq \frac{1}{2}\left(3+\frac{9}{8}\frac{K_0^2}{16}\right)^{-\frac{1}{3}},$$

provided that $\alpha_0 \leq \alpha_{6,4}, \epsilon_0 \leq \epsilon_{6,3}$.

As a matter of fact, we obtain the following

$$\mathcal{U}(x,\xi,\tau) \geq \frac{1}{4}\left(3+\frac{9}{8}\frac{K_0^2}{16}\right)^{-\frac{1}{3}}.$$

This completely concludes the proof of (4.109).

- *The proof of* (4.110)*:* The idea of the proof is similar to the first one. We also consider three cases

+ The case where $|X| \le \frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|}$. This implies that $X \in P_1(t)$. We write here

$$\mathcal{U}(x,\xi,\tau) = \varrho^{\frac{1}{3}}(x)U(X,t).$$

Using item (i) in the definition of $S(t)$(see Definition 4.1), together with item (i) in Lemma 4.17, we derive that

$$|U(X,t)| \le (T-t)^{-\frac{1}{3}}\left[\left(3+\frac{9}{8}\frac{|X|^2}{(T-t)|\ln(T-t)|}\right)^{-\frac{1}{3}} + \frac{CA^2}{\sqrt{|\ln(T-t)|}}\right] \le 2(T-t)^{-\frac{1}{3}},$$

provided that $T \le T_{6,5}$. In addition to that, from the following fact

$$\frac{K_0}{4}\sqrt{\varrho(X)|\ln\varrho(X)|} = |X| \le \frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|},$$

this yields that

$$\varrho(X) \le T-t.$$

Then,

$$\mathcal{U}(x,\xi,\tau) \le 2\left(\frac{\varrho(x)}{\varrho(X)}\right)^{\frac{1}{3}}.$$

On the other hand, using (4.130), we can derive

$$\frac{\varrho(x)}{\varrho(X)} \le 2, \tag{4.131}$$

provided that $\alpha_0 \le \alpha_{6,4}$. This also yields that

$$\mathcal{U}(x,\xi,\tau) \le 4.$$

+ The case where $|X| \in \left[\frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|}, \epsilon_0\right]$. This means $X \in P_2(t)$. We write

$$\mathcal{U}(x,\xi,\tau) = \varrho^{\frac{1}{3}}(x)\varrho^{-\frac{1}{3}}(X)\mathcal{U}(X,0,\frac{t-t(X)}{\varrho(X)}).$$

Hence, we derive from item (ii) of Definition 4.1, the fact that $U \in S(t)$ and (4.130) that

$$\mathcal{U}(x,\xi,\tau) \le \left(\frac{\varrho(x)}{\varrho(X)}\right)^{\frac{1}{3}}\mathcal{U}(X,0,\frac{t-t(X)}{\varrho(X)}) \le 4,$$

provided that $K_0 \ge K_{6,2}, \alpha_0 \le \alpha_{6,4}(K_0), \delta_0 \le \delta_{6,1}$.

+ The case where $|X| \ge \epsilon_0$. The result follows from item (iii) of Definition 4.1.

Hence, (4.110) follows. Finally, we get the conclusion of Lemma 4.28.

4.7 A priori estimate on $P_2(t)$

In this section, we aim at giving the proof of Proposition 4.29

The proof of Proposition 4.29 . We first choose parameters $K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, \delta_6$ such that Lemma 4.28 holds. Then, items (i) and (ii) in that Lemma hold. We would like to prove that: for all

$$|x| \in \left[\frac{K_0}{4}\sqrt{(T-t_7)|\ln(T-t_7)|}, \epsilon_0\right], |\xi| \leq \alpha_0\sqrt{|\ln \varrho(x)|},$$

and

$$\tau \in \left[\max\left(0, -\frac{t(x)}{\varrho(x)}\right), \frac{t_7 - t(x)}{\varrho(x)}\right] = [\tau_0, \tau_7],$$

the following holds

$$\left|\mathcal{U}(x,\xi,\tau) - \hat{\mathcal{U}}(\tau)\right| \leq \frac{\delta_0}{2}, \tag{4.132}$$

$$|\nabla_\xi \mathcal{U}(x,\xi,\tau)| \leq \frac{C_0}{2\sqrt{|\ln \varrho(x)|}}. \tag{4.133}$$

We first recall equation (4.57)

$$\partial_\tau \mathcal{U} = \Delta_\xi \mathcal{U} - 2\frac{|\nabla \mathcal{U}|^2}{\mathcal{U} + \frac{\lambda^{\frac13}\varrho^{\frac13}(x)}{\tilde{\theta}(\tau)}} + \left(\mathcal{U} + \frac{\lambda^{\frac13}\varrho^{\frac13}(x)}{\tilde{\theta}(\tau)}\right)^4 - \frac{\tilde{\theta}'_\tau(\tau)}{\tilde{\theta}(\tau)}\mathcal{U}.$$

- *The proof of* (4.132)*:* We first introduce the following function

$$\mathcal{Z}(\xi,\tau) = \mathcal{U}(x,\xi,\tau) - \hat{\mathcal{U}}(\tau).$$

Using (4.57), we write the following equation

$$\partial_\tau \mathcal{Z} = \Delta \mathcal{Z} + \left(\mathcal{U} + \frac{\tilde{\theta}(\tau)\varrho^{\frac13}(x)}{\lambda^{\frac13}}\right)^4 - \hat{\mathcal{U}}^4(\tau) + G(\xi,\tau),$$

where

$$G(\xi,\tau) = -2\frac{|\nabla \mathcal{U}|^2}{\mathcal{U} + \frac{\lambda^{\frac13}\varrho^{\frac13}(x)}{\tilde{\theta}(\tau)}} - \frac{\tilde{\theta}'_\tau(\tau)}{\tilde{\theta}(\tau)}\mathcal{U}.$$

Using Proposition 4.20 and the definition of $\tilde{\theta}(\tau)$ in (4.58), we derive that

$$\left|\tilde{\theta}'(\tau)\right| \leq C\varrho^{\frac{1}{12}}(x)(1-\tau)^{-\frac{11}{12}}. \tag{4.134}$$

Hence, from Lemma 4.29, we derive the following: for all $|\xi| \leq \frac74\alpha_0\sqrt{|\ln \varrho(x)|}$ and $\tau \in [\tau_0, \tau_7]$,

$$|G(\xi,\tau)| \leq \frac{C}{|\ln \varrho(x)|^{\frac12}}\left((1-\tau)^{-\frac{11}{12}} + 1\right),$$

provided that $|x| \le \epsilon_{7,2}(K_0, \delta_0)$. In particular,

$$\left| \left(\mathcal{U} + \frac{\tilde{\theta}(\tau)\varrho^{\frac{1}{3}}(x)}{\lambda^{\frac{1}{3}}} \right)^4 - \hat{\mathcal{U}}^4(\tau) \right| \le C \left(|\mathcal{Z}| + \varrho^{\frac{1}{3}}(x) \right).$$

We here define $\chi_1(\xi) = \chi_0 \left(\frac{|\xi|}{\sqrt{|\ln \varrho(x)|}} \right)$, where $\chi_0 \in C_0^\infty(\mathbb{R}), \chi_0(x) = 1, \forall |x| \le \frac{5}{4}, \chi_0(x) = 0, \forall |x| \ge \frac{7}{4}$, and $0 \le \chi_0 \le 1$. As a matter of fact, we have the following estimates

$$|\nabla \chi_1| \le \frac{C}{\sqrt{|\ln \varrho(x)|}} \text{ and } \left|\nabla^2 \chi_1\right| \le \frac{C}{|\ln \varrho(x)|}. \tag{4.135}$$

Introducing

$$\mathcal{Z}_1(\xi, \tau) = \chi_2(\xi) \mathcal{Z}(\xi, \tau),$$

we then write an equation satisfied by $\mathcal{Z}_1$

$$\partial_\tau \mathcal{Z}_1 = \Delta \mathcal{Z}_1 + G_1(\xi, \tau),$$

where G_1 satisfies the following: for all $|\xi| \le \frac{7}{4}\alpha_0 \sqrt{|\ln \varrho(x)|}$

$$|G_1(x, \xi, \tau)| \le C(|\mathcal{Z}_1| + \frac{1}{|\ln \varrho(x)|^{\frac{1}{2}}} \left((1-\tau)^{-\frac{11}{12}} + 1 \right),$$

Using Duhamel's principal, we derive the following

$$\begin{aligned} \|\mathcal{Z}_1(\tau)\|_{L^\infty(\mathbb{R}^N)} &\le \left(\delta_6 + \frac{C}{|\ln \varrho(x)|^{\frac{1}{2}}} \right) + C \int_{\tau_0}^{\tau} \|\mathcal{Z}_1(s)\|_{L^\infty(\mathbb{R}^N)} ds \\ &\le 2\delta_6 + C \int_0^\tau \|\mathcal{Z}_1(s)\|_{L^\infty(\mathbb{R}^N)} ds. \end{aligned}$$

Using Gronwall's inequality, we get the following

$$\|\mathcal{Z}_1\|_{L^\infty(\mathbb{R}^N)}(\mathbb{R}^N) \le 2C\delta_6.$$

In particular, if we choose $C_0 \ge 4C\delta_6$, then (4.132) follows.

- *The proof of* (4.133)*:* We rely on the idea as for the proof of (4.132). We consider $\mathcal{Z}_2(\xi, \tau) = \chi_1 \mathcal{U}(x, \xi, \tau) \exp\left(\int_{\tau_0}^{\tau} \frac{\tilde{\theta}'(s)}{\tilde{\theta}(s)} ds \right)$, where χ_1 given in the proof of (4.132). Then, we can derive an equation satisfied by $\mathcal{Z}_2$ as follows

$$\partial_\tau \mathcal{Z}_2 = \Delta \mathcal{Z}_2 + \chi_1 \mathcal{U}^4 \exp\left(\int_{\tau_0}^{\tau} \frac{\tilde{\theta}'(s)}{\tilde{\theta}(s)} ds \right) + G_2(\xi, \tau), \tag{4.136}$$

where G_2 defined by

$$\begin{aligned} G_2(\xi, \tau) &= \exp\left(\int_{\tau_0}^{\tau} \frac{\tilde{\theta}'(s)}{\tilde{\theta}(s)} ds \right) \left[-2\nabla\chi_1 \cdot \nabla \mathcal{U} - \Delta \chi_1 U - \frac{\chi_1 |\nabla \mathcal{U}|^2}{\mathcal{U} + \frac{\lambda^{\frac{1}{3}} \varrho^{\frac{1}{3}}(x)}{\tilde{\theta}(\tau)}} \right. \\ &+ \left. \chi_1 \left(\mathcal{U} + \frac{\lambda^{\frac{1}{3}} \varrho^{\frac{1}{3}}(x)}{\tilde{\theta}(\tau)} \right)^4 - \chi_1 \mathcal{U}^4 \right]. \end{aligned}$$

In particular, from (4.134), we can get the following fact

$$\left|\exp\left(\pm\int_{\tau_0}^{\tau}\frac{\tilde{\theta}'(s)}{\tilde{\theta}(s)}ds\right)\right|\le 2, \forall \tau\in[\tau_0,\tau_7], \tag{4.137}$$

as $|x|\le\epsilon_{8,1}$. Then, using the results in Lemma 4.28, we can deduce the following

$$\|G_2(.,\tau)\|_{L^\infty(\mathbb{R}^N)}\le\frac{C}{|\ln\varrho(x)|}, \forall\tau\in[\tau_0,\tau_7],$$

provided that $|x|\le\epsilon_{8,2}(K_0)$. We write $\mathcal{Z}_2$ in the following integral equation

$$\mathcal{Z}_2(\tau)=e^{(\tau-\tau_0)\Delta}\mathcal{Z}_2(\tau_0)+\int_{\tau_0}^{\tau}e^{(\tau-s)\Delta}\left[\chi_1\mathcal{U}^4(\sigma)\exp\left(\int_{\tau_0}^{s}\frac{\tilde{\theta}'(\sigma)}{\tilde{\theta}(\sigma)}d\sigma\right)+G_2(s)\right]ds. \tag{4.138}$$

We now aim at proving the following estimates:

$$\left\|\nabla e^{(\tau-\tau_0)\Delta}\mathcal{Z}_2(\tau_0)\right\|_{L^\infty(\mathbb{R}^N)} \le \frac{C_6+C}{\sqrt{|\ln\varrho(x)|}}, \tag{4.139}$$

$$\begin{aligned}\left\|\nabla e^{(\tau-s)\Delta}\left(\chi_1\mathcal{U}^4(s)\exp\left(\int_0^s\frac{\tilde{\theta}'(\sigma)}{\tilde{\theta}(\sigma)}d\sigma\right)\right)\right\|_{L^\infty(\mathbb{R}^N)} &\le C\|\nabla\mathcal{Z}_2\|_{L^\infty(\mathbb{R}^N)}\\ &+\frac{C}{\sqrt{|\ln\varrho(x)|}}.\end{aligned} \tag{4.140}$$

\+ *The proof of* (4.139)*:* We write $e^{(\tau-\tau_0)\Delta}\mathcal{Z}_2(\tau_0)$ as follows

$$e^{(\tau-\tau_0)\Delta}\mathcal{Z}_2(\tau_0)(\xi,\tau)=\int_{\mathbb{R}^N}\frac{e^{\left(-\frac{|\xi-\xi'|^2}{4(\tau-s)}\right)}}{(4\pi(\tau-s))^{\frac{n}{2}}}\chi_1(\xi')\mathcal{U}(x,\xi',\tau_0(x))\exp\left(\int_0^s\frac{\tilde{\theta}'(\sigma)}{\tilde{\theta}(\sigma)}d\sigma\right)d\xi'.$$

This yields

$$\begin{aligned}\left|\nabla_\xi e^{(\tau-\tau_0)\Delta}\mathcal{Z}_2(\tau_0)(\xi,\tau)\right| &= \left|\int_{\mathbb{R}^N}\frac{\nabla_\xi e^{\left(-\frac{|\xi-\xi'|^2}{4(\tau-s)}\right)}}{(4\pi(\tau-s))^{\frac{n}{2}}}\chi_1(\xi')\mathcal{U}(x,\xi',\tau_0)\exp\left(\int_{\tau_0}^s\frac{\tilde{\theta}'(\sigma)}{\tilde{\theta}(\sigma)}d\sigma\right)d\xi'\right|\\
&= \left|\int_{\mathbb{R}^N}\frac{\nabla_{\xi'} e^{\left(-\frac{|\xi-\xi'|^2}{4(\tau-s)}\right)}}{(4\pi(\tau-s))^{\frac{n}{2}}}\chi_1(\xi')\mathcal{U}(x,\xi',\tau_0)\exp\left(\int_{\tau_0}^s\frac{\tilde{\theta}'(\sigma)}{\tilde{\theta}(\sigma)}d\sigma\right)d\xi'\right|\\
&\le \left|\int_{\mathbb{R}^N}\frac{e^{\left(-\frac{|\xi-\xi'|^2}{4(\tau-s)}\right)}}{(4\pi(\tau-s))^{\frac{n}{2}}}\nabla_{\xi'}\chi_1(\xi')\mathcal{U}(x,\xi',\tau_0)\exp\left(\int_{\tau_0}^s\frac{\tilde{\theta}'(\sigma)}{\tilde{\theta}(\sigma)}d\sigma\right)d\xi'\right|\\
&+ \left|\int_{\mathbb{R}^N}\frac{e^{\left(-\frac{|\xi-\xi'|^2}{4(\tau-s)}\right)}}{(4\pi(\tau-s))^{\frac{n}{2}}}\chi_1(\xi')\nabla_{\xi'}\mathcal{U}(x,\xi',\tau_0)\exp\left(\int_{\tau_0}^s\frac{\tilde{\theta}'(\sigma)}{\tilde{\theta}(\sigma)}d\sigma\right)d\xi'\right|.\end{aligned}$$

Thus, using the above estimate, the result of item (*ii*) in Lemma 4.28 and (4.135), we can conclude (4.139).

+ *The proof of* (4.140)*:*

$$\begin{aligned}
&\left|\nabla e^{(\tau-s)\Delta}\left(\chi_1\mathcal{U}^4(s)\exp\left(\int_{\tau_0}^{s}\frac{\tilde\theta'(\sigma)}{\tilde\theta(\sigma)}d\sigma\right)\right)(\xi,s)\right|\\
= &\left|\int_{\mathbb{R}^N}\frac{\nabla_\xi e^{\left(-\frac{|\xi-\xi'|^2}{4(\tau-s)}\right)}}{(4\pi(\tau-s))^{\frac{n}{2}}}\chi_1(\xi')\mathcal{U}^4(x,\xi',\tau)\exp\left(\int_{\tau_0}^{s}\frac{\tilde\theta'(\sigma)}{\tilde\theta(\sigma)}d\sigma\right)d\xi'\right|\\
= &\left|\int_{\mathbb{R}^N}\frac{\nabla_{\xi'} e^{\left(-\frac{|\xi-\xi'|^2}{4(\tau-s)}\right)}}{(4\pi(\tau-s))^{\frac{n}{2}}}\chi_1(\xi')\mathcal{U}^4(x,\xi',\tau)\exp\left(\int_{\tau_0}^{s}\frac{\tilde\theta'(\sigma)}{\tilde\theta(\sigma)}d\sigma\right)d\xi'\right|\\
\le &\left|\int_{\mathbb{R}^N}\frac{e^{\left(-\frac{|\xi-\xi'|^2}{4(\tau-s)}\right)}}{(4\pi(\tau-s))^{\frac{n}{2}}}\nabla_{\xi'}\chi_1(\xi')\mathcal{U}^4(x,\xi',\tau)\exp\left(\int_{\tau_0}^{s}\frac{\tilde\theta'(\sigma)}{\tilde\theta(\sigma)}d\sigma\right)d\xi'\right|\\
+ &\left|\int_{\mathbb{R}^N}\frac{e^{\left(-\frac{|\xi-\xi'|^2}{4(\tau-s)}\right)}}{(4\pi(\tau-s))^{\frac{n}{2}}}\chi_1(\xi')4\mathcal{U}^3(x,\xi',\tau)\nabla_{\xi'}\mathcal{U}(x,\xi',s)\exp\left(\int_{\tau_0}^{s}\frac{\tilde\theta'(\sigma)}{\tilde\theta(\sigma)}d\sigma\right)d\xi'\right|.
\end{aligned}$$

In particular, we have the following fact

$$\begin{aligned}
\nabla_{\xi'}(\mathcal{Z}_2)(\xi',s) &= \nabla_{\xi'}\left(\chi_1(\xi')\mathcal{U}(x,\xi',s)\exp\left(\int_{\tau_0}^{s}\frac{\tilde\theta'(\sigma)}{\tilde\theta(\sigma)}d\sigma\right)\right)\\
&= \nabla_{\xi'}\chi(\xi')\mathcal{U}(x,\xi',s)\exp\left(\int_{\tau_0}^{s}\frac{\tilde\theta'(\sigma)}{\tilde\theta(\sigma)}d\sigma\right)+\chi(\xi')\nabla_{\xi'}\mathcal{U}(x,\xi',s)\exp\left(\int_{\tau_0}^{s}\frac{\tilde\theta'(\sigma)}{\tilde\theta(\sigma)}d\sigma\right).
\end{aligned}$$

Then, using (4.110), (4.135) and the definition of $\mathcal{Z}_2(s)$, we get the following

$$\left|\nabla e^{(\tau-s)\Delta}\left(\chi_1\mathcal{U}^4(s)\exp\left(\int_{\tau_0}^{s}\frac{\tilde\theta'(\sigma)}{\tilde\theta(\sigma)}d\sigma\right)\right)(\xi,s)\right| \le C\|\nabla\mathcal{Z}_2(s)\|_{L^\infty(\mathbb{R}^N)}+\frac{C}{\sqrt{|\ln\varrho(x)|}},$$

which yields (4.140).

We now come back to the proof of (4.133). We use (4.138), (4.139) and (4.140) to obtain the following

$$\|\nabla\mathcal{Z}_2(\tau)\|_{L^\infty(\mathbb{R}^N)} \le \frac{C_6+C}{\sqrt{|\ln\varrho(x)|}}+C\int_{\tau_0}^{\tau}\|\nabla\mathcal{Z}_2(s)\|_{L^\infty(\mathbb{R}^N)}.$$

Thanks to Gronwall's inequality, we derive the following

$$\|\nabla\mathcal{Z}_2(\tau)\|_{L^\infty(\mathbb{R}^N)} \le \frac{C(C_6)}{\sqrt{|\ln\varrho(x)|}}.$$

In addition to that, from the definition of $\mathcal{Z}_2$, we deduce that for all $|\xi|\le\alpha_0\sqrt{|\ln\varrho(x)|}$,

$$\mathcal{Z}_2(\xi,\tau)=\mathcal{U}(x,\xi,\tau)\exp\left(\int_{\tau_0}^{\tau}\frac{\tilde\theta'(\sigma)}{\tilde\theta(\sigma)}d\sigma\right).$$

This implies that

$$|\nabla_\xi \mathcal{U}(x,\xi,\tau)| \leq \frac{2C(C_6)}{\sqrt{|\ln \varrho(x)|}}.$$

Finally, if we take $C_0 \geq 4C(C_6)$, then

$$|\nabla_\xi \mathcal{U}(x,\xi,\tau)| \leq \frac{C_0}{2\sqrt{|\ln \varrho(x)|}},$$

which implies (4.133).

□

4.8 Some bounds on terms in equation (4.35)

In this section, we give essential ingredients for the proof of Lemma 4.26. More precisely, we will estimate some functions involved in equation (4.35): V, J, B, R, N and F. In fact, as we explained in the proof Section right after Lemma 4.26, we choose not to prove Lemma 4.26, in order to avoid lenghy estimates aldready mentioned by Merle and Zaag in [14]. The interested reader may use our estimates in this section and follow the proof of Lemma 3.2 on page 1523 in [14] in order to check the argument.

Let us first give some estimates on $V(y,s)$:

Lemme 4.35 (Expansion and bounds on the potential V)**.** *We consider V defined in* (4.37)*. Then, the following holds: V is bounded on $\mathbb{R}^N \times [1,+\infty)$ and for all $s \geq 1$*

$$|V(y,s)| \leq C\frac{(1+|y|^2)}{s}, \forall y \in \mathbb{R}^N,$$

and

$$V(y,s) = -\frac{(|y|^2 - 2N)}{4s} + \tilde{V}(y,s),$$

where $\tilde{V}$ satisfies the following

$$\left|\tilde{V}(y,s)\right| \leq C(K_0)\frac{(1+|y|^4)}{s^2}, \forall |y| \leq K_0\sqrt{s}.$$

Proof. The proof is easily derived from the explicit formula of V. We kindly refer the readers to self-chek or see Lemma B.1, page 1270 in [16] with $p = 4$. □

We now give a bound on the quadratic term $B(q)$.

Lemme 4.36 (A bound on $B(q)$)**.** *Let us consider $B(q)$ defined in* (4.39)*. If $\theta(s) \geq 1$, for all s and $|q| \leq 1$, then, the following holds*

$$|B(q)| \leq C(K_0)\left(|q|^2 + e^{-\frac{s}{3}}\right).$$

Proof. By using Newton binomial formula, the conclusion directly follows. □

Next, we aim at giving some bounds on $J(q,\theta(s))$. The following is our statement:

Lemme 4.37 (Bound on $J(q,\theta(s))$)**.** *For all $K_0 > 0, A \geq 1$ and $\epsilon_0 > 0$, there exist $\eta_9(\epsilon_0)$ and $T_9(K_0,\epsilon_0,A)$ such that for all $\alpha_0 > 0, C_0 > 0$ and $T \leq T_9$, $\delta_0 \leq \frac{1}{2}\hat{\mathfrak{U}}(0)$ and $\eta_0 \leq \eta_9$, the following holds: If $U \in S(T,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0,\eta_0,t)$ for some $t \in [0,T)$, then, for all $|y| \leq 2K_0\sqrt{s}$, $s = -\ln(T-t)$, we have the following estimates:*

$$\left|\left(T(q,\theta(s)) + 4\frac{\nabla\varphi\cdot\nabla q}{\varphi + \frac{\lambda^{\frac{1}{3}}e^{-\frac{s}{3}}}{\theta(s)}}\right)\right| \leq C(K_0,A)\left(\frac{|y|^2}{s^2}|q| + s^{-1}|q|^2 + |\nabla q|^2\right), \quad (4.141)$$

$$|T(q,\theta(s))| \leq C(K_0,A)\left(\frac{|q|}{s} + \frac{|\nabla q|}{\sqrt{s}}\right), \quad (4.142)$$

where q is a transformed function of U given in (4.34) *and $T(q,\theta(s))$ is defined in* (4.38). *In particular, for all $y \in \mathbb{R}^N$, we have*

$$|(1-\chi(y,s))T(q,\theta(s))| \leq C(K_0,C_0)\min\left(\frac{1}{s}, \frac{|y|^3}{s^{\frac{5}{2}}}\right). \quad (4.143)$$

Proof. The techniques of the proof of estimates (4.141), (4.142) and (4.143) are the same. Although, function $J(q,\theta)$ is our work has some differences from the work of Merle and Zaag in [14], we assert that the proof still holds with our model. In order to show this argument, we kindly ask to refer the reader to check Lemma B.4 in that work. For that reason, we only give the proof of (4.141) and (4.142) here, and we leave the proof of (4.143) for the reader to be done similarly as for comparison Lemma B.4 in [14]. We now consider $|y| \leq 2K_0\sqrt{s}$, and introduce $G(h) = -2\frac{|\nabla\varphi + h\nabla q|^2}{\varphi + \frac{\lambda^{\frac{1}{3}}e^{-\frac{s}{3}}}{\theta(s)} + hq} + 2\frac{|\nabla\varphi|^2}{\varphi + \frac{\lambda^{\frac{1}{3}}e^{-\frac{s}{3}}}{\theta(s)}}$, $h \in [0,1]$. Then, we have the following:

$$G_h'(h) = \frac{2q\,|\nabla\varphi + h\nabla q|^2}{\left(\varphi + \frac{\lambda^{\frac{1}{3}}e^{-\frac{s}{3}}}{\theta(s)} + hq\right)^2} - 4\frac{\nabla q(\nabla\varphi + h\nabla q)}{\varphi + \frac{\lambda^{\frac{1}{3}}e^{-\frac{s}{3}}}{\theta(s)} + hq},$$

$$G_h''(h) = -4q^2\frac{|\nabla\varphi + h\nabla q|^2}{\left(\varphi + \frac{\lambda^{\frac{1}{3}}e^{-\frac{s}{3}}}{\theta(s)} + hq\right)^3} + 8q\frac{\nabla q(\nabla\varphi + h\nabla q)}{\left(\varphi + \frac{\lambda^{\frac{1}{3}}e^{-\frac{s}{3}}}{\theta(s)} + hq\right)^2} - 4\frac{|\nabla q|^2}{\varphi + \frac{\lambda^{\frac{1}{3}}e^{-\frac{s}{3}}}{\theta(s)} + hq}.$$

Using a Taylor expansion of $G(h)$ on $[0,1]$, at $h = 0$, we get the following:

$$G(1) = G(0) + G'(0) + \int_0^1 (1-h)G''(h)dh.$$

Using the following facts

$$G(1) = J(q,\theta(s)), G(0) = 0,$$

we write the following

$$T(q,\theta(s)) = \left(\frac{2q|\nabla\varphi|^2}{\left(\varphi + \frac{\lambda^{\frac{1}{3}}e^{-\frac{s}{3}}}{\theta(s)}\right)^2} - \frac{4\nabla\varphi\cdot\nabla q}{\varphi + \frac{\lambda^{\frac{1}{3}}e^{-\frac{s}{3}}}{\theta(s)}}\right) + \int_0^1 (1-h)G_h''(h)dh.$$

From the definition of φ given in (4.33), we can derive that for all $s \geq 1$ and $y \in \mathbb{R}^N$, we have

$$\frac{|\nabla\varphi(y,s)|^2}{\varphi^2(y,s)} \leq C\frac{|y|^2}{s} \text{ and } |\nabla\varphi(y,s)| \leq Cs^{-\frac{1}{2}}.$$

In addition to that, using Lemma 4.17, we can prove that there exists $s_9(A, K_0)$ such that for all $s \geq s_0 \geq s_9$, $h \in [0,1]$ and $|y| \leq 2K_0\sqrt{s}$, we have the following

$$|F''(h)(y,s)| \leq C(A,K_0)\left(\frac{|q|^2}{s} + |\nabla q|^2\right) \leq C(A,K_0)\left(\frac{|q|}{s} + \frac{|\nabla q|}{\sqrt{s}}\right).$$

Thus, (4.141) and (4.142) follow. □

We now aim at giving some estimates on R. The following is our statement:

Lemme 4.38 (Bounds on R)**.** *Let us consider R defined in* (4.40). *We assume that $\theta(s) \geq 1$, for all $s \geq 1$. Then, for all $s \geq 1$ and $y \in \mathbb{R}^N$, the following holds:*

$$\left|R(y,s) - \frac{c_1}{s^2}\right| \leq C\frac{(1+|y|^3)}{s^3},$$

and

$$|\nabla R(y,s)| \leq C\frac{(1+|y|^3)}{s^3}.$$

In particular,

$$\|R(.,s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{s}.$$

Proof. The function R, in our work is different from the definiton in [14] (up to a very small difference). Hence, the proof of [14] holds in our case with minor adaptation. Accordingly, we kindly refer the reader to check Lemma B.5 page 1541 in that work. □

We now give some estimates on N. The control of this term is a new contributation of our study. In addition to that, it is a direct consequence of Proposition 4.20 on the control $\bar{\theta}(t)$. The following is our statement:

Lemme 4.39 (Bound on $N(q,\theta(s))$)**.** *There exists $K_{10} > 0$ such that for all $K_0 \geq K_{10}, A > 0$ and $\delta_0 \leq \frac{1}{2}\left(3 + \frac{9}{8}\frac{K_0^2}{16}\right)^{-\frac{1}{3}}$, there exist $\alpha_{10}(K_0,\delta_0) > 0$ and $C_{10}(K_0) > 0$ such that for every $\alpha_0 \in (0,\alpha_{10}]$ we can find $\epsilon_{10}(K_0,\delta_0,\alpha_0) > 0$ such that for every $\alpha_0 \in (0,\epsilon_{10}], \eta_0 \leq 1$, there exists $T_{10}(K_0) > 0$ such that all for all $T \leq T_{10}$, the following holds: Assume that U is a nonnegative solution of equation* (4.19) *on $[0,t_{10}]$ for some $t_{10} \leq T_{10}$, and initial data $U(0) = U_{d_0,d_1}$ given in* (4.63) *for some $(d_0,d_1) \in \mathbb{R}\times\mathbb{R}^N$, satisfying $|d_0|,|d_1| \leq 2$, and $U \in S(T,K_0,\epsilon_0,\alpha_0,A,\delta_0,C_0,\eta_0,t)$ for all $t \in [0,t_{10}]$. Then, for all $s = -\ln(T-t)$ with $t \in [0,t_{10}]$, the following estimate holds:*

$$\|N(q,\theta(s))\|_{L^\infty(\mathbb{R}^N)} \leq \frac{1}{s^{2019}},$$

where $N(q,\theta(s))$ is defined in (4.41).

Proof. Using the fact that U is in $S(t)$, item (i) in Definition 4.1 and item (i) of Lemma 4.17, we derive that

$$\|(q+\varphi)(.,s)\|_{L^\infty(\mathbb{R}^N)} \leq C.$$

Hence, it is enough to find a bound on the following quanlity

$$\frac{\theta'(s)}{\theta(s)}.$$

(see in definition (4.41)). As a matter of fact, using Proposition 4.20, it is clear to have the following

$$\left|\frac{\theta'(s)}{\theta(s)}\right| = \left|\frac{\bar{\theta}'(t)}{\bar{\theta}(t)}\right| \left|\frac{dt}{ds}\right| \leq Ce^{\frac{8-3N-6}{6}s}|s^N|.$$

Hence, there exists s_{10} large enough such that for all $s \geq s_0 \geq s_{10}$, we can write

$$\|N(q, \theta(s))\|_{L^\infty(\mathbb{R}^N)} \leq Ce^{-\frac{s}{6}}|s|^N \leq \frac{1}{s^{2019}},$$

which yields the conclusion of the proof. □

Finally, we give a bound on $F(w, W)$. As a matter of fact, this is an important bridge that connects the problems in $\mathbb{R}^N$ and in a bounded domain. In other words, it is created by the localization around blowup region. Fortunately, this term is controled as a small perturbation in our analysis. More precisely, the following is our statement:

Lemme 4.40 (Bound on $F(w, W)$). *Let us consider $F(w, W)$, defiend in* (4.32)*. Then, there exists $\epsilon_{11} > 0$ such that $K_0 > 0, \epsilon_0 \leq \epsilon_{11}, \alpha_0 > 0, A > 0, \delta_0 > 0, C_0 > 0, \eta_0 > 0$, there exists $T_{11} > 0$ such that for all $T \leq T_{11}$, the following holds: Assuming that $U \in S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$, for all $t \in [0, t_{11}]$, for some $t_{11} \in [0, T)$, then, we have*

$$\|F(w, W)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{1}{s^{2019}},$$

where $s = -\ln(T - t)$.

Proof. From the definition of F, it is enough to consider $|y| \in \left[\frac{e^{\frac{s}{2}}}{M_0}e^{\frac{s}{2}}, \frac{2e^{\frac{s}{2}}}{M_0}\right]$. We now take $\epsilon_0 \leq \frac{1}{2M_0}$, then, this domain corresponds to the region $P_3(t)$ where our solution U is regared as a perturbation of initial data. Using the fac that U is in $S(t)$, then, we can derive from item (iii) in Definition 4.1 that

$$\begin{aligned} |W(y, s)| &\leq C(K_0)e^{-\frac{s}{3}}, \\ |\nabla_y W(y, s)| &\leq C(K_0)e^{-\frac{s}{3}}. \end{aligned}$$

In addition to that, from definition (4.30), we deduce that

$$\begin{aligned} |w(y, s)| &\leq C(K_0, M_0)e^{-\frac{s}{3}}, \\ |\nabla w(y, s)| &\leq C(K_0, M_0)e^{-\frac{s}{3}}. \end{aligned}$$

On the other hand, using the definition of ψ_{M_0} given in (4.29), we get the following

$$\left|\partial_s \psi_{M_0} - \Delta \psi_{M_0} + \frac{1}{2} y \cdot \nabla \psi_{M_0}\right| \leq C(M_0).$$

In fact, using the above estimate, we can get the conclusion if $s \geq s_0(K_0)$. □

4.9 The Dirichlet heat semi-group on Ω

In this section, we aim at giving some main properties of the Dirichlet heat semi-group $\left(e^{t\Delta}\right)_{t>0}$ (see more details in [20] or chapter 16 in [17]). In particular, we prove the parabolic regularity estimate of Lemma 4.33. We consider the following equation

$$\left\{\begin{array}{rcl} \partial_t U - \Delta U &=& 0 \text{ in } \Omega \times (0,T), \\ U &=& 0 \text{ in } \partial\Omega \times (0,T), \\ U(x,0) &=& U_0(x) \text{ in } \bar{\Omega}. \end{array}\right. \tag{4.144}$$

In particular, one can prove that there exists $G(x,y,t,\tau), t \geq \tau$ nonnegative, symemtric in x, y, i.e $G(x,y,t,\tau) = G(y,x,t,\tau)$ and defined in $\Omega \times \Omega \times (0,T) \times [0,T)$ with the following condition

$$\left\{\begin{array}{rcl} (\partial_t - \Delta) G(x,y,t,\tau) &=& \delta(x-y)\delta(t-\tau), \\ G(x,y,\tau,\tau) &=& 0 \text{ and } G(x,y,t,\tau) = 0 \text{ if } x \in \partial\Omega. \end{array}\right. \tag{4.145}$$

Moreover, for all $f \in L^\infty(\Omega)$, we have

$$(e^{t\Delta} f)(x) = \int_\Omega G(x,y,t,0) f(y) dy. \tag{4.146}$$

Hence, we can write the solution of equation (4.144) as follows

$$U(t) = e^{t\Delta}(U_0).$$

We now consider furthermore the following non-homogeneous equation

$$\left\{\begin{array}{rcl} \partial_t U - \Delta U &=& F \text{ in } \Omega \times (0,T), \\ U &=& 0 \text{ in } \partial\Omega \times (0,T), \\ U(x,0) &=& U_0(x) \text{ in } \bar{\Omega}. \end{array}\right. \tag{4.147}$$

If $F \in C(\Omega \times (0,T)), u_0 \in C(\Omega)$ and Ω is C^2, bounded domain in $\mathbb{R}^N$. Then, we can prove that there locally exists a classical solution of problem (4.147). Then, by using Duhamel principal, the solution satisfies the following integral equation

$$U(t) = e^{t\Delta}(U_0) + \int_0^t e^{(t-s)\Delta} F(s) ds.$$

Sometimes, we also call $G(x,y,t,\tau)$ the Green function. Let us give in the following the main properties of the Green function:

Lemme 4.41. *Let us consider the Green function called $G(x,y,t,\tau)$ above. Then, the following holds: for all $(x,y,t,\tau) \in \Omega \times \Omega \times (0,T) \times [0,T)$ and integer numbers r, s, we have*

$$\left|\partial_t^r \partial^s_{x_1^{s_1} \dots x_N^{s_N}} G(x,y,t,\tau)\right| \leq C(t-\tau)^{-\frac{N+2r+s}{2}} \exp\left(-c(\Omega)\frac{|x-y|^2}{t-\tau}\right).$$

Proof. We kindly refer the reader to see Theorem 16.3, page 413 in [17]. □

We now prove in the following Lemma 4.33

The proof of Lemma 4.33. From the defintion of the semigroup $e^{t\Delta}$, it is easy to derive that $L(t) \in C(\bar{\Omega} \times [0,T]) \cap C^{\infty}(\Omega \times (0,T])$. Hence, it is enough to give the proof of (4.113). Indeed, we first derive the support of $U_{d_0,d_1} = \{|x| \leq \frac{1}{2} d(0,\partial\Omega)\}$. We now consider two following regions:

$$\begin{aligned} \Omega_1 &= \{\frac{\epsilon_0}{8} \leq |x| \leq \frac{7}{8} d(0,\partial\Omega)\},\\ \Omega_2 &= \{|x| > \frac{3}{4} d(0,\partial\Omega)\} \cap \Omega. \end{aligned}$$

In addition to that, we can write $L_1(t)$ as follows

$$L(x,t) = \int_{\Omega} G(x,y,t,0) U_{d_0,d_1}(y) dy = \int_{\{|y| \leq \frac{1}{2} d(0,\partial\Omega)\}} G_{\Omega}(x,y,t,0) U_{d_0,d_1}(y) dy, \tag{4.148}$$

which yields

$$\nabla L(x,t) = \int_{\{|y| \leq \frac{1}{2} d(0,\partial\Omega)\}} \nabla_x G(x,y,t,0) U_{d_0,d_1}(y) dy. \tag{4.149}$$

- We consider the case where $x \in \Omega_2$: Thanks to Lemma 4.41 and (4.149), we have

$$\begin{aligned} |\nabla L(x,t)| &\leq \int_{\{|y| \leq \frac{1}{2} d(0,\partial\Omega)\}} |\nabla_x G(x,y,t,0)| |U_{d_0,d_1}(y)| dy \\ &\leq \int_{\{|y| \leq \frac{1}{2} d(0,\partial\Omega)\}} \frac{C \exp\left(-c_{\Omega} \frac{|x-y|^2}{t}\right)}{t^{\frac{N+1}{2}}} |U_{d_0,d_1}(y)| dy \\ &\leq C \int_{\{|y| \leq \frac{1}{2} d(0,\partial\Omega)\}} \exp\left(-c_{\Omega} \frac{|x-y|^2}{t}\right) \frac{|x-y|^{N+1}}{t^{\frac{N+1}{2}}} \frac{|U_{d_0,d_1}(y)|}{|x-y|^{N+1}} dy \\ &\leq C \int_{\{|y| \leq \frac{1}{2} d(0,\partial\Omega)\}} \frac{|U_{d_0,d_1}(y)|}{|x-y|^{N+1}} dy \end{aligned}$$

Because $x \in \Omega_2$, we have the following fact

$$\frac{1}{|x-y|^{N+1}} \leq C.$$

This yields the following

$$|\nabla L(x,t)| \leq C \int_{\{|y| \leq \frac{1}{4} d(0,\partial\Omega)\}} |U_{d_0,d_1}(y)|\, dy.$$

In addition to that, using (4.63), we have the following

$$\begin{aligned} &\int_{|y| \leq \frac{1}{2} d(0,\partial\Omega)} |U_{d_0,d_1}(y)| dy = \int_{|y| \leq 2\sqrt{T} |\ln T|} |U_{d_0,d_1}(y)| dy + \int_{2\sqrt{T} |\ln T| \leq |y| \leq \frac{1}{2} d(0,\partial\Omega)} |U_{d_0,d_1}(y)| dy \\ = &\int_{|y| \leq 2\sqrt{T} |\ln T|} T^{-\frac{1}{3}} \left| \varphi\left(\frac{y}{\sqrt{T}}, -\ln s_0\right) + (d_0 + d_1 \cdot \frac{y}{\sqrt{T|\ln T|}}) \chi_0 \left(\frac{|y|}{\sqrt{T|\ln T|} \frac{K_0}{32}} \right) \right| \chi_1(y) dy \\ + &\int_{2\sqrt{T} |\ln T| \leq |y| \leq \frac{1}{2} d(0,\partial\Omega)} |(1 - \chi_1(y)) H^*(y)| dy \leq C, \end{aligned}$$

which yields

$$|\nabla L(x,t)| \leq C, \text{ for all in } \Omega_2, \tag{4.150}$$

It is similar to prove the following estimate

$$|\nabla^2 L(x,t)| \leq C, \text{ for all in } \Omega_2. \tag{4.151}$$

- We consider the case where $x \in \Omega_1$: Let us define $\phi(x)$ as a function in $C_0^\infty\left(\mathbb{R}^N\right)$ and satisfying the following conditions

$$\begin{aligned}
\phi(x) &= 0 \text{ if } |x| \geq \frac{11}{12} d(0,\partial\Omega),\\
\phi(x) &= 1 \text{ if } |x| \leq \frac{7}{8} d(0,\partial\Omega).
\end{aligned}$$

Then, we also introduce the following function

$$L_1(x,t) = \phi(x)\nabla L(x,t).$$

We now write an equation satisfied by L_1

$$\left\{\begin{aligned}
\partial_t L_1 - \Delta L_1 &= -2\nabla\phi \cdot \nabla^2 L - \Delta\phi\nabla L \text{ in } \Omega\times(0,T),\\
L_1 &= 0 \text{ in } \partial\Omega\times(0,T),\\
L_1(x,0) &= \phi\nabla L(0) = \phi\nabla U_{d_0,d_1} \text{ in } \bar{\Omega}.
\end{aligned}\right. \tag{4.152}$$

Using Duhamel's formula, we get

$$L_1(t) = e^{t\Delta}L_1(0) + \int_0^t e^{(t-s)\Delta}\left[-2\nabla\phi\cdot\nabla^2 L - \Delta\phi\nabla L\right](s)ds. \tag{4.153}$$

We now aim at proving the following fact

$$\|e^{(t-s)\Delta}(\Delta\phi\nabla L)(s)\|_{L^\infty(\Omega)} \leq C\|L_1(s)\|_{L^\infty(\Omega)} + C,, \tag{4.154}$$

$$\|e^{(t-s)\Delta}(\nabla\phi\cdot\nabla^2 L)(s)\|_{L^\infty(\Omega)} \leq \frac{C\|L_1(s)\|_{L^\infty(\Omega)}}{\sqrt{t-s}} + C\left(1+\frac{1}{\sqrt{t-s}}\right),. \tag{4.155}$$

- *The proof of* (4.154)*:* We have the following fact

$$\begin{aligned}
|\Delta\phi\nabla L| &= |I_{\{|x|\leq \frac{7}{8}d(0,\partial\Omega)\}}\Delta\phi\nabla L| + |I_{\{|x|> \frac{7}{8}d(0,\partial\Omega)\}}\Delta\phi\nabla L|\\
&\leq C|\phi\nabla L| + C = C|L_1| + C.
\end{aligned}$$

Then, by using the monotonicity of the operator $e^{(t-s)\Delta}$, we derive directly (4.154).

- *The proof of* (4.155)*:* From the definition of operator $e^{(t-s)\Delta}$, we can write the following

$$e^{(t-s)\Delta}(\nabla\phi\cdot\nabla^2 L(s)) = \int_\Omega G(x,y,t,s)\nabla\phi(y)\cdot\nabla^2 L(y,s)dy.$$

We consider $j \in \{1,...,n\}$, and integrate by part, we get the following

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^n G(x,y,t,s)\partial_{y_i}\phi(y)\partial^2_{y_iy_j}Ldy\\
= &-\int_\Omega(\nabla_y G(x,y,t,s)\cdot\nabla\phi + G(x,y,t,s)\Delta\phi)\partial_{y_j}L(y,s)dy\\
= &-\int_\Omega \nabla_y G(x,y,t,s)\cdot\nabla\phi\partial_{y_j}L(y,s)dy\\
- &\int_\Omega G(x,y,t,s)\Delta\phi\partial_{y_j}L(y,s)dy.
\end{aligned}$$

Using the defintion of ϕ in the above and (4.150), we have the following fact:

$$\begin{aligned}|\nabla L| &= |I_{\{|x|\le \frac{7}{8}d(0,\partial\Omega)\}}\nabla L| + |I_{\{|x|>\frac{7}{8}d(0,\partial\Omega)\}}\nabla L| \\ &= |I_{\{|x|\le \frac{7}{8}d(0,\partial\Omega)\}}\phi(x)\nabla L| + |I_{\{|x|>\frac{7}{8}d(0,\partial\Omega)\}}\nabla L| \\ &\le |L_1| + C.\end{aligned}$$

Then,

$$\begin{aligned}\left|\int_\Omega \sum_{i=1}^n G(x,y,t,s)\partial_{y_i}\phi(y)\partial^2_{y_iy_j}L(y)dy\right| &\le (\|L_1(s)\|_{L^\infty(\Omega)} + C)\left|\int_\Omega \nabla_y G(x,y,t,s)\cdot\nabla\phi dy\right| \\ &+ (\|L_1(s)\|_{L^\infty(\Omega)} + C)\left|\int_\Omega G(x,y,t,s)\Delta\phi dy\right| \\ &\le (\|L_1(s)\|_{L^\infty(\Omega)} + C)\left[\frac{C}{\sqrt{t-s}} + C\right],\end{aligned}$$

which implies (4.155). We now use (4.153), (4.154) and (4.155) to deduce the following

$$\begin{aligned}\|L_1(t)\|_{L^\infty(\Omega)} &\le C\|\nabla U_{d_0,d_1}\|_{L^1(\Omega)} \\ &+ \int_0^t \left[C\left(1+\frac{1}{\sqrt{t-s}}\right)\|L_1(s)\|_{L^\infty(\Omega)} + C\left(1+\frac{1}{\sqrt{t-s}}\right)\right]ds.\end{aligned} \tag{4.156}$$

Using Gronwall's lemma, we obtain the following estimate

$$\|L_1(t)\|_{L^\infty(\Omega)} \le C\|\nabla U_{d_0,d_1}\|_{L^1(\Omega)}.$$

We admit the following fact which we will be proved at the end:

$$\|\nabla U_{d_0,d_1}\|_{L^1(\Omega)} \le CT^{-\frac{1}{2}} + C(\epsilon_0). \tag{4.157}$$

This estimate gives a rough estimation on L_1 as follows

$$\|L_1(t)\|_{L^\infty(\Omega)} \le CT^{-\frac{1}{2}} + C(\epsilon_0). \tag{4.158}$$

Let us improve this estimate. We come back to identity (4.153) and consider the set of all $x \in \Omega$ such that $|x| \ge \frac{\epsilon_0}{8}$. By using the definition of U_{d_0,d_1} in (4.63), we first prove the following fact

$$\|e^{t\Delta}(\nabla U_{d_0,d_1})\|_{L^\infty(|x|\ge\frac{\epsilon_0}{8}, x\in\Omega)} \le C(\epsilon_0). \tag{4.159}$$

Indeed, we write $e^{t\Delta}(\nabla U_{d_0,d_1})$ as follows

$$\begin{aligned}e^{t\Delta}(\nabla U_{d_0,d_1}) &= \int_\Omega G(x,y,t,0)\nabla_y U_{d_0,d_1}(y)dy = \int_{|y|\le\frac{1}{2}d(0,\partial\Omega)} G(x,y,t,0)\nabla_y U_{d_0,d_1}(y)dy \\ &= \int_{|y|\le\frac{\epsilon_0}{16}} G(x,y,t,0)\nabla_y U_{d_0,d_1}(y)dy + \int_{\frac{\epsilon_0}{16}\le|y|\le\frac{1}{2}d(0,\partial\Omega)} G(x,y,t,0)\nabla_y U_{d_0,d_1}(y)dy \\ &= I_1 + I_2.\end{aligned}$$

+ Bound on I_1: Using integration by parts, we get the following:

$$I_1 = -\int_{|y|\le\frac{\epsilon_0}{16}} \nabla_y G(x,y,t,0)U_{d_0,d_1}(y)dy + \int_{|y|=\frac{\epsilon_0}{16}} G(x,y,t,0)U_{d_0,d_1}(y)\eta(y)dS.$$

From Lemma 4.41, we derive that

$$\begin{aligned}
|I_1(x,t)| &\leq \int_{|y|\leq \frac{\epsilon_0}{16}} \frac{\exp\left(-c_\Omega \frac{|x-y|^2}{t}\right)}{t^{\frac{N+1}{2}}} |U_{d_0,d_1}(y)|dy + C(\epsilon_0)\\
&\leq \int_{|y|\leq \frac{\epsilon_0}{16}} \exp\left(-c_\Omega \frac{|x-y|^2}{t}\right) \frac{|x-y|^{N+1}}{t^{\frac{N+1}{2}}} \frac{1}{|x-y|^{N+1}} |U_{d_0,d_1}(y)|dy + C(\epsilon_0)\\
&\leq C(\epsilon_0)\|U_{d_0,d_1}\|_{L^1(\Omega)} + C(\epsilon_0) \leq C_1(\epsilon_0).
\end{aligned}$$

+ Bound on I_2: It is easy to prove that

$$\|\nabla U_{d_0,d_1}(.)\|_{L^\infty(\frac{\epsilon_0}{16}\leq|y|\leq\frac{1}{2}d(0,\partial\Omega))} \leq C(\epsilon_0).$$

This yields directly that

$$|I_2(x,t)| \leq C(\epsilon_0)\int_{\frac{\epsilon_0}{16}\leq|y|\leq\frac{1}{2}d(0,\partial\Omega)} G(x,y,t,0)dy \leq C(\epsilon_0).$$

Hence, we get the conclusion the proof of (4.159). Using (4.156), (4.158) and (4.159), we get the following: for all $|x| \geq \frac{\epsilon_0}{8}, x \in \Omega$

$$|L_1(x,t)| \leq C(\epsilon_0) + C\int_0^t \left(1 + \frac{1}{\sqrt{t-s}}\right) T^{-\frac{1}{2}} ds \leq C(\epsilon_0),$$

provided that $T < 1$. This yields that for all $x \in \Omega_1$

$$|\nabla L(x,t)| \leq C(\epsilon_0). \tag{4.160}$$

Finally, (4.113) follows from (4.150) and (4.160), which will conclude the proof of Lemma 4.33. However, in order to finish the proof we need to prove (4.157): Indeed, from the definition of U_{d_0,d_1} given in (4.63), we write

$$\nabla_x U_{d_0,d_1}(x) = I_1(x) + I_2(x) + I_3(x) + I_4(x),$$

where

$$\begin{aligned}
I_1 &= T^{-\frac{1}{3}}\left[-\frac{3}{4}\left(3 + \frac{9}{8}\frac{|x|^2}{T|\ln T|}\right)^{-\frac{4}{3}} \frac{x}{T|\ln T|} + \frac{d_1}{\sqrt{T|\ln T|}}\chi_0\left(\frac{x}{\sqrt{T|\ln T|}}\right)\right.\\
&+ \left.\frac{d_1\cdot x}{\sqrt{T|\ln T|}}\chi_0'\left(\frac{x}{\sqrt{T|\ln T|}}\right)\frac{x}{|x|\sqrt{T|\ln T|}}\right]\chi_0\left(\frac{|x|}{\sqrt{T|\ln T|}}\right),\\
I_2 &= T^{-\frac{1}{3}}\left[\left(3 + \frac{9}{8}\frac{|x|^2}{T|\ln T|}\right)^{-\frac{1}{3}} + \left(d_0 + \frac{d_1\cdot x}{\sqrt{T|\ln T|}}\right)\chi_0\left(\frac{|x|}{\sqrt{T|\ln T|}}\right)\right]\\
&\times \chi_0'\left(\frac{|x|}{\sqrt{T}|\ln T|}\right)\frac{x}{|x|\sqrt{T}|\ln T|},\\
I_3 &= \left(1 - \chi_0\left(\frac{x}{\sqrt{T}|\ln T|}\right)\right)\nabla H^*(x),\\
I_4 &= -\chi_0'\left(\frac{x}{\sqrt{T}|\ln T|}\right)\frac{x}{|x|\sqrt{T}|\ln T|}H^*(x).
\end{aligned}$$

As a matter of fact, we have the following

$$\|\nabla U_{d_0,d_1}\|_{L_1} \leq \int_\Omega |I_1(x)|dx + \int_\Omega |I_2(x)|dx + \int_\Omega |I_3(x)|dx + \int_\Omega |I_4(x)|dx.$$

In particular, we have

$$\begin{aligned}
\mathrm{Supp}(I_1) &\subset \{|x| \leq 2\sqrt{T}|\ln T|\},\\
\mathrm{Supp}(I_2) &\subset \{\sqrt{T}|\ln T| \leq |x| \leq 2\sqrt{T}|\ln T|\},\\
\mathrm{Supp}(I_3) &\subset \{\sqrt{T}|\ln T| \leq |x| \leq \frac{1}{2}d(0,\partial\Omega)\},\\
\mathrm{Supp}(I_4) &\subset \{\sqrt{T}|\ln T| \leq |x| \leq 2\sqrt{T}|\ln T|\}.
\end{aligned}$$

By some simple upper bounds on I_1 and I_2, we can derive that

$$\int_\Omega |I_1(x)|dx \leq CT^{-\frac{1}{2}} + C \text{ and } \int_\Omega |I_2(x)|dx \leq CT^{-\frac{1}{2}} + C.$$

We now aim at estimating I_3 and I_4.

+ Estimate on I_3: We write as follows

$$\begin{aligned}
\int_\Omega |I_3|(x)dx &= \int_{\sqrt{T}|\ln T| \leq |x| \leq \min\left(\frac{1}{2},\frac{1}{4}d(0,\partial\Omega)\right)} |I_3(x)|dx + \int_{\min\left(\frac{1}{2},\frac{1}{4}d(0,\partial\Omega)\right) \leq |x| \leq \frac{1}{2}d(0,\partial\Omega)} |I_3(x)|dx\\
&\leq \int_{\sqrt{T}|\ln T| \leq |x| \leq \min\left(\frac{1}{2},\frac{1}{4}d(0,\partial\Omega)\right)} |I_3(x)|dx + C.
\end{aligned}$$

In addition to that,

$$\begin{aligned}
\int_{\sqrt{T}|\ln T| \leq |x| \leq \min\left(\frac{1}{2},\frac{1}{4}d(0,\partial\Omega)\right)} |I_3(x)|dx &\leq C\int_{\sqrt{T}|\ln T| \leq |x| \leq \min\left(\frac{1}{2},\frac{1}{4}d(0,\partial\Omega)\right)} |x|^{-\frac{4}{3}}|\ln|x||^{\frac{1}{3}}dx\\
&\leq CT^{-\frac{1}{2}} + C.
\end{aligned}$$

This implies that

$$\int_\Omega |I_3(x)|dx \leq CT^{-\frac{1}{2}} + C.$$

+ Estimate on I_4: We have

$$\int_{\sqrt{T}|\ln T| \leq |x| \leq 2\sqrt{T}|\ln T|} |I_4(x)|dx \leq \frac{C}{\sqrt{T|\ln T|}} \int_{\sqrt{T}|\ln T| \leq |x| \leq 2\sqrt{T}|\ln T|} |\ln|x||^{\frac{1}{3}}|x|^{-\frac{2}{3}}dx \leq CT^{-\frac{1}{2}}.$$

From the above estimates, we can conclude (4.157). We also finish the proof of Lemma 4.33. □

4.10 Some Parabolic estimates

In this section, we aim at giving some estimates on $U, \nabla U$ and $\nabla^2 U$. More precisely, the following is our statement:

Lemme 4.42 (Parabolic estimates on U)**.** *We consider U a solution to equation* (4.19) *and $U \in S(T, K_0, \epsilon_0, \alpha_0, A, \delta_0, C_0, \eta_0, t)$, for all $t \in [0, t_1]$ for some $t_1 \leq T$. Then, the following estimates follows: for all $t \in [0, t_1)$*

$$\begin{aligned} \|U(.,t)\|_{L^\infty(\Omega)} &\leq C(K_0, A)(T-t)^{-\frac{1}{3}}, & (4.161)\\ \|\nabla U(.,t)\|_{L^\infty(\Omega)} &\leq C(K_0, A)\frac{(T-t)^{-\frac{5}{6}}}{|\ln(T-t)|^{\frac{1}{2}}}, & (4.162)\\ \|\nabla^2 U(.,t)\|_{L^\infty(\Omega)} &\leq C(K_0, A)(T-t)^{-c}, & (4.163)\end{aligned}$$

for some constant $c = c(K_0, A) > 0$.

In particular, we have the following local convergence: We assume furthermore that $U \in S(t)$, for all $t < T$. Then, for all $x \in \Omega$ there exist $R_x > 0, t_x \in [0, T)$ such that the following holds

$$\|\partial_t U(.,t)\|_{L^\infty(B(x,R_x))} \leq C(K_0, A, T, x), \forall t \in [t_x, T). \tag{4.164}$$

Remark 4.43. *We would like to remark that from* (4.164) *and the definition of the shrinking set $S(t)$ (see Definition 4.1), we ensure for all $x_0 \in \Omega \backslash \{0\}$, $U(x_0, t)$ is convergent as $t \to T$.*

Proof. We see that estimates (4.161) and (4.162) directly follow from the definition of the shrinking set and Lemma 4.17. For that reason, we only give here the proofs of (4.163) and (4.164).

- *The proof of* (4.163)*:* From (4.15), (4.17), we consider u defined as follows:

$$u(x,t) = 1 - \frac{1}{1 + \frac{\lambda^{\frac{1}{3}} U(x,t)}{\theta(t)}}. \tag{4.165}$$

Then, u satisfies (4.2) and $u(0)$ is in $C_0^\infty(\Omega)$. We now derive an equation satisfied by $\nabla^2 u$ as follows:

$$\partial_t \nabla^2 u = \Delta(\nabla^2 u) + H_1 \nabla^2 u + H_2, \tag{4.166}$$

where $H_1 = \frac{2\lambda}{\theta^3(t)} \frac{1}{(1-u)^3}$ and $H_2 = (H_{2,i,j})_{1 \leq i,j \leq N}$ is a square matrix with

$$H_{2,i,j} = 6\frac{\partial_{y_i} u \partial_{y_j} u}{(1-u)^4}.$$

Using the definition of u, (4.68) and two estimates (4.161) and (4.162), we can derive the following fact: for all $t \in [0, T)$,

$$\begin{aligned} \|H_1(t)\|_{L^\infty(\Omega)} &\leq C(K_0, A)(T-t)^{-1},\\ \|H_2(t)\|_{L^\infty(\Omega)} &\leq C(K_0, A)(T-t)^{-\frac{5}{3}}.\end{aligned}$$

We write $\nabla^2 u$ under the integral equation following

$$\nabla^2 u(t) = e^{t\Delta}(\nabla^2 u(0)) + \int_0^t e^{(t-s)\Delta}\left[H_1(s)\nabla^2 u + H_2(s)\right](s)ds.$$

This implies that

$$\|\nabla^2 u(t)\|_{L^\infty(\Omega)} \leq \|e^{t\Delta}(\nabla^2 u(0))\|_{L^\infty(\Omega)} + C(K_0, A)\int_0^t \left(\frac{1}{T-s}\|\nabla^2 u(s)\|_{L^\infty(\Omega)} + (T-s)^{-\frac{5}{3}}\right) ds.$$

Besides that, we can prove that there exists $c_1 > 0$ such that

$$\|e^{t\Delta}(\nabla^2 u(0))\|_{L^\infty(\Omega)} \leq C(T-t)^{-c_1}.$$

Thanks to Growall's lemma, we get the following

$$\|\nabla^2 u\|_{L^\infty(\Omega)} \leq C(K_0, A)(T-t)^{-c_2}, \text{ with some constant } c_2 > 0.$$

Finally, from the relation between u and U, we can get the conclusion of (4.163).

- *The proof of* (4.164)*:* By using the definitions (4.52) and (4.52) of $P_2(t)$ and $P_3(t)$, respectively, if we consider an arbitrary $x \in \Omega \setminus \{0\}$, then, there exist t_x, r_x such that

$$\text{the ball of radius } r_x, \text{ centred } x \quad B(x, r_x) \in P_2(t) \cup P_3(t), \forall t \in [t_x, T).$$

Then, using the definition of the shrinking set $S(t)$, given in Definition 4.1 and the fact that $u \in S(t)$ for all $t \in [t_x, T)$, we derive that there exists $C(K_0, x)$ such that for all $t \in [t_x, T)$, we have

$$\|U(.,t)\|_{L^\infty(\mathbb{R}^N)(B(x,r_x))} \leq C(K_0, x). \tag{4.167}$$

In addition to that, we derive from Proposition 4.20, we have

$$1 \leq \bar{\theta}(t) \leq C, \text{ and } |\bar{\theta}'(t)| \leq C(T-t)^{\frac{3n-8}{6}} |\ln(T-t)|^n \leq (T-t)^{-\frac{11}{12}}, \tag{4.168}$$

for all $t \in [t_x, T)$.

We recall u, defined in (4.165). We now derive an equation satisfied by $\partial_t u$

$$\partial_t(\partial u) = \Delta \partial_t u + H_1 \partial_t u + H_3(t), \tag{4.169}$$

where

$$\begin{aligned} H_1(t) &= \frac{2\lambda}{\bar{\theta}^3(t)} \frac{1}{(1-u)^3}, \\ H_3(t) &= -\frac{3\lambda}{(1-u)^2} \frac{\bar{\theta}'(t)}{\bar{\theta}^4(t)}. \end{aligned}$$

We then introduce the following cut-off function : $\phi \in C_0^\infty(\mathbb{R}^N)$ which satisfying

$$\phi(z) = 1 \text{ if } |z - x| \leq \frac{r_x}{2}, \text{ and } \phi(z) = 0 \text{ if } |z - x| \geq \frac{3}{4} r_x \text{ and } 0 \leq \phi(z) \leq 1, \forall z \in \mathbb{R}^N.$$

Particularly, we also define

$$v(z,t) = \phi(z) \partial_t u(z,t) \text{ for all } z \in \mathbb{R}^N.$$

Using (4.169), we can derive an equation satisfied by $v(t)$ as follows

$$\partial_t v = \Delta v - 2\text{div}(\nabla \phi \partial_t u) + \Delta \phi \partial_t u + H_1 v(t), \tag{4.170}$$

Using (4.163), (4.165) (4.167) and the fact that U is nonnegative solution, we can deduce that

$$\|\nabla \phi \partial_t u(t)\|_{L^\infty(\mathbb{R}^N)} \leq C(K_0, A, x)(T-t)^{-c},$$

and

$$\|\Delta\phi\partial_t u(t)\|_{L^\infty(\mathbb{R}^N)} \leq C(K_0, A, x)(T-t)^{-c}.$$

Moreover, we canderive from (4.167) and (4.168) that

$$\|I_{\{|z-x|\leq r_x\}} H_1(t)\|_{L^\infty(\mathbb{R}^N)} \leq C(K_0, A, x),$$

and

$$\|H_3(t)\|_{L^\infty(\mathbb{R}^N)} \leq C(K_0, x)(T-t)^{-\frac{11}{12}}.$$

We now deduce from (4.170) that v satisfies the following integral equation

$$v(t) = e^{(t-t_x)\Delta} v(t_x) + \int_{t_x}^{t} e^{(t-s)\Delta}\left[-2\text{div}(\nabla\phi\partial_t u) + \Delta\phi\partial_t u + H_1 v(s)\right] ds,$$

where $e^{t\Delta}$ stands for the heat semigroup on $\mathbb{R}^N$. Then, we get the following

$$|v(t)| \leq C(K_0, A, x)(1 + (T-t)^{-c+1}) + 2\left|\int_{t_x}^{t} e^{(t-s)\Delta}\text{div}(\nabla\phi\partial_t u) ds\right|.$$

In particular, we have

$$\left|e^{(t-s)\Delta}\text{div}(\nabla\phi\partial_t u)\right| \leq \frac{C}{\sqrt{t-s}}\|\nabla\phi\partial_t u\|_{L^\infty(\mathbb{R}^N)} \leq C(K_0, x)\frac{(T-t)^{-c}}{\sqrt{t-s}}.$$

This implies that

$$|v(t)| \leq C(K_0, A, x)(1 + (T-t)^{-c+1}) + C(K_0, A, x)\int_{t_x}^{t} \frac{(T-s)^{-c}}{(t-s)^{\frac{1}{2}}} ds.$$

+ If $-c + \frac{1}{4} \geq 0$. This give us that

$$|v(t)| \leq C(K_0, A, x),$$

which yields the conclusion of our proof.

+ Otherwise, we use the above estimate to derive that

$$|v(t)| \leq C(K_0, A, T, x)(T-t_x)^{-c+\frac{1}{4}}.$$

We can see that by using a parabolic estimate as we have done. We can improve our estimate on $|v(t)|$ from $C(K_0, A, x)(T-t)^{-c}$ to $C(K_0, A, x)(T-t)^{-c+\frac{1}{4}}$. Hence, we can repeat with a finite steps to get the conclusion of the proof. We kindly refer the reader to check this argument.

□

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