Université Sorbonne Paris Nord
Laboratoire d'Informatique de Paris Nord - LIPN
Équipe: Algorithmes et Optimisation Combinatoire - AOC

TDIness and Multicuts

Thèse de doctorat
présentée par

Emiliano LANCINI

à l'École Doctorale Galilée

pour obtenir le grade de
Docteur en Informatique

Soutenue à Villetaneuse le 20 décembre 2019
devant le jury composé de:

Frédérique Bassino	Université Sorbonne Paris Nord	Président de jury
Mourad Baïou	Université Clermont Auvergne	Rapporteur
Denis Cornaz	Université Paris Dauphine	Rapporteur
Antoine Deza	McMaster University	Examinateur
Frédéric Meunier	École des Ponts ParisTech	Examinateur
Roberto Wolfler Calvo	Université Sorbonne Paris Nord	Directeur de thèse
Roland Grappe	Université Sorbonne Paris Nord	Co-encadrant de thèse
Mathieu Lacroix	Université Sorbonne Paris Nord	Co-encadrant de thèse

Abstract

In this thesis we study integer total dual integral (TDI) systems and box-totally dual integral (box-TDI) polyhedra associated with multicuts.

The first polyhedron we consider is the flow cone, that is, the cone generated by the incidence vectors of flows and edges of a graph. This cone is box-totally dual integral if and only if the graph series-parallel. We provide a system describing the flow cone made of inequalities associated with the multicuts of the graph. This system has integer coefficients, and we prove that it is totally dual integral if and only if the graph is series-parallel. Thereafter, when G is series-parallel, we provide the Schrijver system of the flow cone. This is the unique totally dual integral system with integer coefficients describing the flow cone such that the removal of any redundant constraint undermines its total dual integrality.

The second polyhedron we treat is the k-edge-connected spanning subgraph polyhedron, that is the convex hull of the k-edge-connected spanning subgraphs of a given graph. We first show that the connector polyhedron – corresponding to the case $k = 1$ – is box-totally dual integral for all graphs. Then, we prove that the k-edge-connected spanning subgraph polyhedron is box-totally dual integral for each $k \geq 2$ if and only if the graph is series-parallel.

The description of the k-edge-connected spanning subgraph polyhedron being dependent on the parity of k, we provide two distinct totally dual integral systems describing this polyhedron. When k is even, we provide a system with integer coefficients that is totally dual integral whenever the graph is series-parallel. When k is odd, we prove that the system known for describing this polyhedron for series-parallel graph is totally dual integral if and only if the graph is series-parallel.

Keywords: totally dual integral system, box-totally dual integral polyhedron, Schrijver system, flow cone, k-edge-connected graph, multicut, series-parallel graph, polyhedral study.

Résumé

Dans cette thèse nous nous intéressons aux systèmes total dual intégraux (TDI) et aux polyèdres total dual box-intégraux (box-TDI) en lien avec les multicoupes.

Dans un premier temps, nous considérons le cône des flots, c'est-à-dire le cône généré par les vecteurs d'incidence des flots et des arêtes du graphe. Ce cône est box-TDI si et seulement si le graphe associé est série-parallèle. En premier lieu, nous fournissons un système d'inegalités associées aux multicoupes du graphe qui décrit le cône des flots. Ce système est à coefficients entiers et il est TDI si et seulement si le graphe est série-parallèle. En outre, lorsque le graphe est série-parallèle, nous donnons une description du cône des flots à l'aide du système de Schrijver, qui est l'unique système TDI à coefficients entiers dont la suppression de n'importe quelle inégalité détruit le caractère TDI.

Deuxièmement, nous étudions le polyèdre des sous-graphes k-arête-connexes $P_k(G)$. Nous montrons que $P_1(G)$ est un polyèdre box-TDI pour n'importe quel graphe G. Ensuite, nous montrons que, pour chaque k fixé, $P_k(G)$ est box-TDI si et seulement si G est un graphe série-parallèle.

La description de $P_k(G)$ dépendant de la parité de k, nous étudions séparément les cas lorsque k est pair et impair. Pour k pair, nous fournissons un système à coefficients entiers qui est TDI si le graphe G est série-parallèle. Finalement, quand k est impair, nous montrons que le système à coefficients entiers qui décrit $P_k(G)$ est TDI si et seulement si G est série-parallèle.

Mots clés: système total dual intégral, polyèdre total dual box-intégral, système de Schrijver, cône des flots, graphe k-arête-connexe, multicoupe, graphe série-parallèle, étude polyédrale.

Acknowledgments

I would like to acknowledge the people that, directly or indirectly, contributed to this thesis and helped me in these three years.

Firstly and foremost, I would like thank my thesis advisors.

I am obliged to Professor Roberto Wolfler Calvo for the continuous support during my study. I am deeply grateful for his guidance and his advises, as well as for having given me the opportunity to make a doctorate. I sincerely believe that I was very lucky to have as thesis director a person with such outstanding human qualities.

I would like to express my great appreciation to Roland Grappe for his scientific help, brightening insights, and challenging questions. He greatly contributed to the results present in this work, and I deeply admire his unpaired quickness of thinking.

I wish to acknowledge the guidance provided by Mathieu Lacroix: his presence, expertise, and often tested patience were invaluable for me and I am extremely grateful for what he did during these years.

Besides my advisors, I would like to thank the reviewers of my thesis. I greatly thank Professor Mourad Baïou for his insights and commentaries on this work. I would like to equally thank Denis Cornaz for his useful and detailed commentaries; I am sincerely flattered by the great attention and effort he put in the reading of this manuscript.

I am grateful to Professor Antoine Deza and Professor Frédéric Meunier for having accepted to be part of my thesis committee. For the same reason, I thank LIPN head, Professor Frédérique Bassino.

I want to express my deepest gratitude to Michele Barbato. He has been an unofficial mentor, an appreciated co-author, and a very good friend.

I would like to also thank Mario Valencia-Pabon, for the interest and concern he manifested during my doctorate.

I express my gratitude to all the members of LIPN for the discussion and guidance they provided me, it has been a pleasure to be part of this lab. I sincerely thank Brigitte Guéveneux, for her precious help with bureaucratic and administrative matters.

I greatly thank Enrico, he has been my fellow in this journey since its beginning in the April of three years ago, and he animated it with never-lasting discussions and incisive thoughts.

I thank my lab mates Stefania and Marcos, for the ways they helped, challenged, and motivated me.

Furthermore, I want to thank all the wonderful people I met in these years, as well as all my friends scattered around the Europe.

I foremost thank Davide e Petra, how they made me feel part of their family is sincerely heartwarming.

I thank my parents, Maria and Sandro, and my brother, Eddi, for the fundamental support they give me.

To conclude, I want to thank the most Francesca, for being by my side during these years, for her graceful wisdom, and her splendid presence.

Contents

List of Figures

List of Examples

Glossary

In this glossary, we assume that: S, T are sets, s is an element of S, a and b are real numbers, i, j, m, n are integer numbers, $G = (V, E)$ and H are graphs, u, w are vertices of G and w' is a vertex of H, U, W are subsets of V, $\{V_1, \ldots, V_n\}$ is a partition of V, F is a subset of E, v is a vector, A is a matrix, M is a multicut, x is a vector in $\mathbb{Z}^E$.

Sets and Numbers

- 2^S - Power set of S.
- $|S|$ - Cardinality of S.
- $[a, b]$ - $\{x \in \mathbb{R} : a \leq x \leq b\}$.
- (a, b) - $\{x \in \mathbb{R} : a < x < b\}$.
- g.c.d.(m, n) - greatest common divisor of m and n.
- l.c.m.(m, n) - least common multiple of m and n.
- $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ - Sets of real, rational, and integer numbers.
- $S \Delta T$ - symmetric difference of S and T, that is $(S \cup T) \setminus (S \cap T)$.

Complexity

- $\mathcal{Co}$-$\mathcal{NP}$-complete - Class of hardest problems in $\mathcal{Co}$-$\mathcal{NP}$.
- $\mathcal{Co}$-$\mathcal{NP}$ - Class of problems that admits a polynomial certificate for negative instances.
- $\mathcal{NP}$ - Class of problems that admits a polynomial certificate for positive instances.

- $\mathcal{NP}$-complete - Class of hardest problems in $\mathcal{NP}$.
- $\mathcal{NP}$-hard - Class optimization of problems whose decision problem is $\mathcal{NP}$-complete.
- $\mathcal{P}$ - Class of polynomial problems.

Linear Algebra

- $\mathbf{0}$,$\mathbf{\frac{1}{2}}$, $\mathbf{1}, \mathbf{2}$ - Uniform vectors with all components equals to 0, $\frac{1}{2}$, 1, and 2.
- χ^S - Incidence vector of S.
- ξ_s - Element of the canonical base of $\mathbb{R}^S$.
- $v_{|S}$ - Restriction of v to the elements of S.
- $A^\top$, $v^\top$ - Transpose of A and v.
- $A^{(i,j)}$ - Minor of A obtained by removing the i^{th} row and j^{th} column.
- A_{ij} - Element of A at position i, j.
- $\text{aff}(S)$ - Affine hull of S.
- $\text{cone}(S)$ - Conic hull of S.
- $\text{conv}(S)$ - Convex hull of S.
- $\text{lattice}(S)$ - Lattice of S.
- $\text{Det}(A)$ - Determinant of A.

Graphs

- $\delta(U, W)$ - Set of edges having one endpoint in U and the other in W.
- $\delta(W)$ - Cut. Also, set of edges having exactly one endpoint in W.
- $\delta(V_1, \ldots, V_n)$ - Multicut defined by the partition $\{V_1, \ldots, V_n\}$.
- $\mathcal{E}_i$ - i^{th} ear of an ear decomposition of a graph.
- $\mathcal{B}_G$, $\mathcal{D}_G$, $\mathcal{M}_G$ - Sets of bonds, cuts, and multicuts of G.

- d_M - Order of the multicut M.
- $G^\star$ - Planar dual graph of G.
- $G[F]$, $G[W]$ - Subgraphs of G induced respectively by F and W.
- $G[x]$ - Graph $G[x] = (V, E')$, where E' is the family of edges of E with multiplicity x_e for every $e \in E$.
- $G_u \oplus_{w'} H$ - 1-Sum of G and H, obtained identifying u and w'.
- G_M - The reduced graph of M.
- K_n - Complete graph on n vertices.
- $K_{n,m}$ - Complete bipartite graph on n and m vertices.
- uw - Edge having u and w as endpoints.

Introduction

One of the most difficult, and yet common, tasks we face in our life is to choose. Every moment of our life is direct consequence of our and other people's choices. As a consequence, we spend time elaborating pro and contra of our decisions, even for the most trivial ones. In combinatorial optimization we study how to make optimal choices when we face a finite number of alternatives. These kind of problems are those that naturally rise in our everyday life. We can model the vast majority of the choices we make in terms of combinatorial optimization problems: which clothes a traveler should put in his luggage, how a small enterprise should plan the workers turnovers, or how an airline company should schedule its flights.

Many combinatorial optimization problems have a graphical representation as *graphs*, that allows us to concisely represent huge sets of solutions as object of the graph. The problems treated in this thesis fall in this category: indeed, we deal with problems associated with *multicuts*, a structure of graphs.

The number of possible solutions to these problems is such that the extensive exploration of all of them is not an option. Thus, more efficient methods should be found. This can be done by looking to the sets of solutions as geometrical objects: we can associate a *polyhedron* in a vector space to each combinatorial optimization problem. The analysis of this polyhedron allows us to efficiently tackle problems that seem hard to solve at a first sight. This approach, also called polyhedral combinatorial optimization (or polyhedral combinatorics) is one of the major approaches to understand and to solve combinatorial optimization problems: the study of the convex hull of the feasible solutions to a particular problem opens the door to polyhedral theory and advanced techniques.

In particular, we focus on two aspects of polyhedral combinatorial optimization: totally dual integral systems and box-totally dual integral polyhedra. The concept of total dual integrality dates back to the works of Edmonds, Giles, and Pulleyblank in the late '70s, and is strongly connected to min-max relations in combinatorial optimization. Box-total dual integrality arose thereafter as a generalization of total dual integrality, and was proved to have strong polyhedral properties by scholars like Cook and Schrijver.

In this thesis, we deal with the totally dual integral description of the flow cone and with the box-total dual integrality of the k-edge-connected spanning subgraph polyhedron.

Flows are classical objects in combinatorial optimization that model theoretical-oriented problems and have real-life applications. We characterize for which graphs a system based on multicuts describing the flow cone is totally dual integral. Then, we give the minimal totally dual integral system with integer coefficients that describes this cone.

The k-edge-connected spanning subgraph problem arises in the design of resistant networks. These networks are typical of transportation science and telecommunications. We characterize when the polyhedron of the k-edge-connected spanning subgraphs is box-TDI and we provide TDI systems based on multicuts that describe this polyhedron.

Overview of the Document

This thesis is structured in four chapters. In the first two chapters we present fundamental concepts and classical results of mathematical programming and total dual integrality. The last two chapters are dedicated to original results on total dual integrality of systems and polyhedra related with multicuts.

In Chapter 1, we give the fundamental definitions and notation used in this thesis, including those concerning mathematical programming, linear programming duality, matrices, and graph theory.

In Chapter 2, we formalize the concepts of total dual integrality and box-total dual integrality, and we present classical and recent results on these topics, focusing on those exploited in the rest of the thesis.

In Chapter 3, we study the flow cone. We provide a system with integer coefficients describing this cone that is TDI if and only if it is associated with a series-parallel graph. Moreover, we characterize the multicuts that are not disjoint union of two multicuts, and we provide the minimal totally dual integral system having integer coefficients for the flow cone of series-parallel graphs. The results of this chapter appear in [5].

In Chapter 4, we study the k-edge-connected spanning subgraph problem. We first analyze the connector polyhedron, corresponding to the case $k = 1$, and we prove that this polyhedron is box-TDI. Then, we show that the k-edge-connected spanning subgraph polyhedron is box-TDI if and only if the graph is series-parallel. We conclude by analyzing two systems describing the k-edge-connected spanning subgraph polyhedron. When k is even, we give a system that is TDI for series-parallel graphs. When k is odd, we show that the system given by Chopra [31] and Didi Biha and Mahjoub [50] is TDI for series-parallel graphs.

At the end of Chapters 3 and 4 we give an overview the results achieved, and we propose some questions and problems that could represent some future developments.

We conclude this thesis by summarizing the results achieved and by giving some perspectives of the work done.

Chapter 1

Preliminaries, Definitions, and Notation

This chapter provides the notations and basic definitions used throughout the thesis. In this thesis we study mathematical and geometrical properties of systems related with graph problems. Thus, we fix the notation and the definitions for linear algebra, mathematical programming, and graph theory.

Forewords. It is of great importance, for mathematics as known today, the correct definition of every concept. Sets, numbers, and relations are mathematical notions of common use in real life. However, a correct and formal definition of those objects is less trivial than it could appear.

Since a rigorous axiomatic treat on the true nature of these objects is out of our scope, we will assume as known the fundamental notions of the classical naive set theory. During our work we will pay attention not to use the notion of "set of all the sets" and similar problematic concepts.

We will try to highlight when the definitions and notations we use are non-standard or disputed.

1.1 Sets and Numbers

Let S be a set, we usually define S as $S = \{x : x \text{ has the property P}\}$. To say that x belongs to S, we write $x \in S$. To say that T is a subset of S, we write $T \subseteq S$; it holds that $S \subseteq S$, and we will denote that a T is a *proper subset* of S – that means T is not empty and has strictly less elements than S – by $T \subsetneq S$. We denote the empty set by $\emptyset$, and we assume that $\emptyset \subseteq S$ for every set S. The term *collection* is a synonym of set, but will be used to denote sets whose elements are sets. The *power set* of a set S is the collection of subsets of S, denoted by 2^S. The *cardinality* of S, denoted by $|S|$, is the number (possibly infinite) of elements belonging to S. Given two sets S and T, their *cartesian product* $S \times T$ is the set of ordered pairs $\{(s,t) : s \in S,\ t \in T\}$.

A *family* $\mathcal{F}$ of elements of a set S is a set where the elements of S may occur more than once. The *multiplicity* of an element s in $\mathcal{F}$ is the number of times s occurs in $\mathcal{F}$. A set S is *inclusionwise minimal* in a collection if there is no set T in the collection such that $T \subsetneq S$. A *partition* of a set S is a collection $\mathcal{C}$ of subsets of S such that each element of S belongs to exactly one set in $\mathcal{C}$. Given two sets S and U we define their *symmetric difference* by $S \Delta U = (S \cup U) \setminus (S \cap U)$.

We denote respectively the sets of real, rational, and integer numbers by $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$. Moreover, $\mathbb{R}_+$, $\mathbb{Q}_+$, and $\mathbb{Z}_+$ indicate respectively the sets $\{x \in \mathbb{R} : x \geq 0\}$, $\{x \in \mathbb{Q} : x \geq 0\}$, and $\{x \in \mathbb{Z} : x \geq 0\}$. We will say that a number x is *half-integer* when $2x \in \mathbb{Z}$.

Given a set of integer numbers $S = \{s_1, \dots, s_n\}$ the *greatest common divisor* (*g.c.d.*) is the biggest integer positive number d such that $\frac{s_i}{d} \in \mathbb{Z}$ for all $i = 1, \dots, n$. The *least common multiple* (*l.c.m.*) is the smallest integer positive number d such that $\frac{d}{s_i} \in \mathbb{Z}$ for all $i = 1, \dots, n$.

Given two real numbers a and b, we denote by $[a,b]$ the set $\{x \in \mathbb{R} : a \leq x \leq b\}$. Similarly, we denote[1] by (a,b) the set $\{x \in \mathbb{R} : a < x < b\}$.

1.2 Computational Complexity

In this thesis we treat problems of combinatorial optimization. These problems can be seen as “find the maximum value of a function among a finite set of solutions”. Those who pursued higher education in mathematics may think of combinatorial optimization problems as being “trivial”: when the number of solutions is finite, we can find the optimum by simply enumerating all the solutions. The challenge posed by these problems is not “if” we can find an optimal solution, but rather “how long will it take” to find it. The branch of theoretical computer science treating the computability is known as computational complexity theory. In this section, we informally present the basic concepts of this branch of computer science.

The complexity theory is based on the work of Cook [33], Edmonds [60], and Karp [103]. The aim of this field is to define whether a given problem is solvable in an “efficient way”

[1] Sometimes, this set is denoted by $]a,b[$.

or not. In this context, we have to stress the difference between a problem and an instance of it. A *problem* is a question having some input parameters, to which we want answer. A problem is defined by giving a general description of its parameters and by listing the properties that must be satisfied by a solution. An *instance* is obtained by giving a specific value to all its input parameters.

An *algorithm* is a set of rules for carrying out the calculation that finds the solution of every instance of a given problem. Each rule implies the execution of a certain number of operations one or more times, depending on the status of the computation. We informally say that an operation is *elementary* if its execution time is bounded by a constant. Some examples of operations we usually consider elementary are arithmetical operations, exchange of two data, and comparison between two objects of fixed size. The execution of an algorithm will result in a sequence of elementary operations. The *size* of an instance is the amount of data required to describe the instance itself.

Example 1.1: Problem, instance, and dimension.

A problem could be "Given a set of names, sort them in alphabetical order", while an instance of this problem can be the list of students of a class. The size of this instance will be the total number of characters in the list.

An algorithm solves a given problem in a certain number of elementary operations, depending on the size n of the instance. An algorithm is said to be $O(f(n))$ if the number of elementary operations necessary to solve an instance is upper bounded by $cf(n)$ for some real number c. When f is a polynomial in n we say that the algorithm is *polynomial*. Similarly, if $f(n)$ is a linear function, we say that the algorithm is *linear*.

We say that a problem belongs to the class $\mathscr{P}$ (Polynomial) if there exists a polynomial algorithm resolving every instance of the problem. A *decision problem* is a problem that has two possible answers: **yes** and **no**. Let $\mathcal{P}$ be a decision problem. The $\mathcal{Y}$-instances and the $\mathcal{N}$-instances of $\mathcal{P}$ are the sets of its instances having answer respectively **yes** and **no**. A certificate for a $\mathcal{Y}$-instance (resp. $\mathcal{N}$-instance) is an algorithmic proof that **yes** (resp. **no**) is the correct answer for that instance. We say $\mathcal{P}$ belongs to the class $\mathscr{NP}$ (nondeterministic polynomial) if all its $\mathcal{Y}$-instances admit a certificate whose correctness can be verified in polynomial time. It is immediate to see that the class $\mathscr{P}$ is contained in $\mathscr{NP}$. In the class $\mathscr{NP}$, we distinguish some problems that seem to be harder to solve than others, this particular set of problems is called $\mathscr{NP}$*-complete*. To determine whether a problem

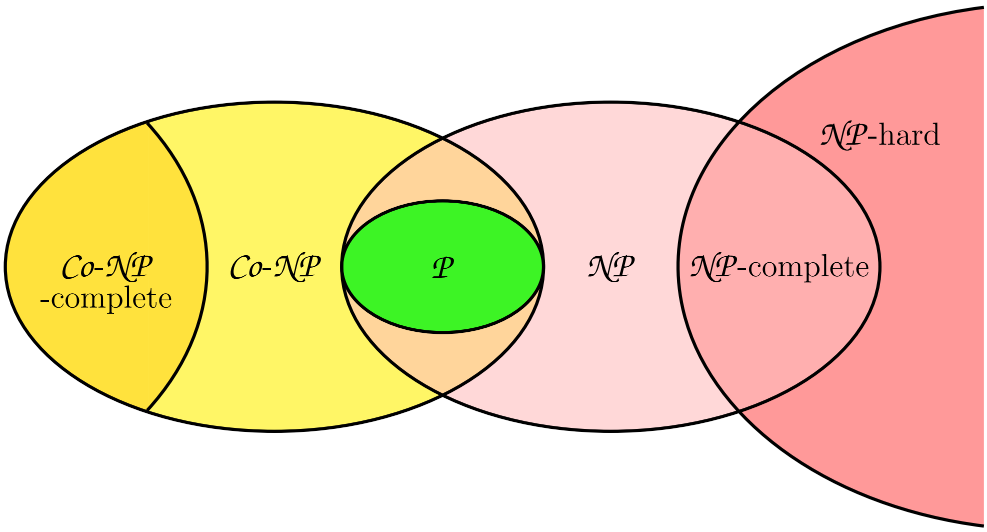


Figure 1.1: Some complexity classes. Note: we assume that $\mathcal{P} \neq \mathcal{NP}$.

is $\mathcal{NP}$-complete, we need the notion of polynomial reducibility. A decision problem $\mathcal{P}_1$ can be *polynomially reduced* to another decision problem $\mathcal{P}_2$, if there exists a polynomially computable function f such that for every instance $\mathcal{I}$ of $\mathcal{P}_1$, the answer is **yes** if and only if the answer of instance $f(\mathcal{I})$ for $\mathcal{P}_2$ is **yes**. A problem is $\mathcal{NP}$-complete if every other problem in $\mathcal{NP}$ can be polynomially reduced to it.

With every combinatorial optimization problem is associated a decision problem. Furthermore, each optimization problem whose decision problem is $\mathcal{NP}$-complete is said to be *$\mathcal{NP}$-hard*. The complementary class of $\mathcal{NP}$ is $\mathcal{C}o$-$\mathcal{NP}$: we say that a problem $\mathcal{P}$ belongs to the class $\mathcal{C}o$-$\mathcal{NP}$ if all its $\mathcal{N}$-instances admit a certificate whose correctness can be verified in polynomial time. Similarly to $\mathcal{NP}$, a problem is *$\mathcal{C}o$-$\mathcal{NP}$-complete* if every other problem in $\mathcal{C}o$-$\mathcal{NP}$ can be reduced to it in polynomial time. In Figure 1.1, we represent the classes of complexity we named in this section, under the assumption that $\mathcal{P} \neq \mathcal{NP}$. For an in-depth analysis of computational complexity theory and a wide collection of $\mathcal{P}$ and $\mathcal{NP}$ problems we refer to the landmark book of Garey and Johnson [78].

1.3 Linear Algebra

In this section we fix notation and give fundamental definitions for vectors, matrices, and all the fundamental concepts relative to linear algebra that we will use in the rest of the thesis.

1.3.1 Vectors and Vector Spaces

A *vector* is a finite ordered tuple of numbers disposed in a row (*row vector*) or in a column (*column vector*). The elements of a vector are called *entries* or *components*. We will denote the i^{th} component of a vector v by v_i. A vector with real entries is an element of $\mathbb{R}^n$, and we define the *vector space* of dimension n is the set of vectors of $\mathbb{R}^n$. When dealing with vectors, we sometimes refer to numbers as *scalar*.

We define three operations on vectors: sum, multiplication, and transposition. Given two vectors $v, w \in \mathbb{R}^n$, their sum is the vector $z = v + w \in \mathbb{R}^n$ whose i^{th} entry $z_i = v_i + w_i$, for all $i = 1, \ldots, n$. Given a row vector v and a column vector w with the same number of entries n, their scalar product[2] is the number

$$v \times w = \sum_{i=1}^{n} v_i w_i.$$

Given a vector $v \in \mathbb{R}^n$ and a scalar k, we denote by kv the vector of $\mathbb{R}^n$ having each component multiplied by k. The *transpose* of a row (respectively column) vector v is the column (resp. row) vector $v^\top$ with the same elements in the same order. Unless differently specified, we assume that all vectors we treat are column vectors.

Given a finite set S and the vector space $\mathbb{R}^{|S|}$, we will often consider the elements of $\mathbb{R}^{|S|}$ as indexed by the elements of S. Thus, we will write that a vector v belongs to $\mathbb{R}^S$, and we will denote its component associated with the element $s \in S$ by v_s. Given the set $T \subseteq S$, we denote by $\chi^T \in \{0,1\}^S$ the *incidence vector* of T, that is χ^T_s equals 1 if s belongs to T and 0 otherwise, for all $s \in S$. Given the set $\mathbb{R}^S$, the *canonical base* is the set of vectors $\chi^{\{s\}}$ for all $s \in S$, we denote[3] this set of vectors as ξ_s for all $s \in S$.

Let S be a set and $T \subseteq S$, and let $v \in \mathbb{R}^S$. We define the *restriction* of c to T the vector $v_{|T} \in \mathbb{R}^T$, that is

$$v_{|T_s} = v_s \text{ for all } s \in T.$$

1.3.2 Linear Functions and Linear Systems

Given a vector space $\mathbb{R}^n$, a *linear function* $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function such that, for all $v, w \in \mathbb{R}^n$ and $a \in \mathbb{R}$:

- $f(av) = af(v)$,

[2] we will usually omit the multiplication symbol $\times$.

[3] This notation is not standard, we decided to use the symbol ξ instead of the more common e or $\mathbf{e}$ to avoid confusion with the symbol used for the edges of a graph.

- $f(v + w) = f(v) + f(w)$.

We are interested in linear functions with real values, these functions are all of the form $f(v) = c^\top v$ for a given vector c.

We call *linear inequality* every inequality of the form $\alpha^\top x \leq \beta$, where $x \in \mathbb{R}^n$ is a variable, and $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ are given. Analogously, for a *linear equality*. A set of linear equalities and inequalities is called *linear system*. A linear system is *rational* if all the coefficients and the right-hand sides of the inequalities composing it are rational vectors and numbers. During the rest of the thesis, we will assume that all systems will be rational.

We will often deal with linear inequalities for which $\alpha \in \mathbb{R}^S$ is the incidence vector of a subset T of S. Formally, the inequality would be

$$\sum_{s \in T} x_s \leq \beta,$$

however, we will use the compact notation:

$$x(T) \leq \beta.$$

1.3.3 Linear Independence and Bases

Given a set of vectors $\{v^1, \ldots, v^m\} \subseteq \mathbb{R}^n$, a *linear combination* of these vectors is the weighted sum: $\lambda^1 v^1 + \cdots + \lambda^m v^m$, for some scalars $\lambda^1, \ldots, \lambda^m$. A vector is the linear combination of a set of vectors S if it can be obtained as linear combination of elements of S.

A set of vectors $\{v^1, \ldots, v^m\}$ is *linearly independent* if $\lambda^1 v^1 + \cdots + \lambda^m v^m = 0$ implies $\lambda^i = 0$ for all $i = 1, \ldots, m$. Equivalently, $v^1, \ldots, v^m$ are linearly independent if none of them is linear combination of the others. A set of vectors $v^1, \ldots, v^m$ *generates* a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ if all vectors in $\mathcal{V}$ are linear combinations of $v^1, \ldots, v^m$. A set of linearly independent vectors that generates a space is a *basis* of this space. The following proposition is well-known:

Proposition 1.1. *The dimension of a vector space equals the cardinality of a basis of the space.*

1.3.4 Affine Spaces

When studying polyhedra, it is natural to use the concept of affine hull. An *affine combination* of $S = \{v^1, \ldots, v^n\} \subseteq \mathbb{R}^n$ is the weighted sum $\lambda^1 v^1 + \cdots + \lambda^m v^m$ such that

$\lambda^m + \cdots + \lambda^m = 1$. One point $\bar{v}$ is *affinely independent* to a set S if there is no affine combination of elements of S that gives $\bar{v}$. Equivalently, the points $v^1, \ldots, v^m$ are affinely independent if and only if

$$\sum_{i=1}^{m} \lambda^i v^i = 0, \sum_{i=1}^{m} \lambda^i = 0,$$

implies that $\lambda^i = 0$ for all $i = 1, \ldots m$.

An *affine space* of $\mathbb{R}^n$ is a set closed under affine combinations, its *dimension* is one less than the cardinality of a maximal set of affinely independent points. Let A and $B \subseteq A$ be two affine spaces, then B is an *affine subspace* of A. If A and $B \subseteq A$ have the same dimension, then $A = B$. Given a set of points S, its *affine hull* $\text{aff}(S)$ is the set of affine combinations of elements of S.

1.3.5 Conic, Convex, and Integer Combinations

A *convex combination* of the vectors $v^1, \ldots, v^m$ is an affine combination $\lambda^1 v^1 + \cdots + \lambda^m v^m$ such that $\lambda^i \geq 0$ for all $i = 1, \ldots, m$. A set $S \subseteq \mathbb{R}^n$ is *convex* if it is closed under convex combinations of its elements. The *convex hull* of S is the smallest convex set containing S and it is denoted by $\text{conv}(P)$. Not surprisingly, the convex hull of a set coincides with the set of points $\{x \in \mathbb{R}^n : x \text{ is a convex combination of points of } S\}$.

A *conic combination* of $v^1, \ldots, v^m$ is a linear combination $\lambda^1 v^1 + \cdots + \lambda^m v^m$ such that $\lambda^i \geq 0$ for all $i = 1, \ldots, m$. We say that a set $C \subseteq \mathbb{R}^n$ is a *cone*, if there exists a $t \in \mathbb{R}^n$, and a set $S \subseteq \mathbb{R}^n$ closed under conic combinations, such that $C = \{x \in \mathbb{R}^n : x = t + s, \text{ for some } s \in S\}$. In a similar way, we define the *conic hull* of a set $S \subseteq \mathbb{R}^n$ as the smallest set containing S closed under conic combinations, or equivalently as the set of points $\text{cone}(S) = \{x \in \mathbb{R}^n : x \text{ is conic combination of points of } S\}$. If C is the conic hull of S then we say that C is *generated* by S, and we say that C is *finitely generated* when there exists a finite set T such that $C = \text{cone}(T)$. Given C conic hull of a set S, we call *polar cone* of C the cone $C^o = \{x \in \mathbb{R}^n : x^\top y \leq 0, \text{ for all } y \in C\}$.

An *integer combination* of $v^1, \ldots, v^m$ is a linear combination $\lambda^1 v^1 + \cdots + \lambda^m v^m$ such that $\lambda^i \in \mathbb{Z}$ for all $i = 1, \ldots, m$. We define the *lattice* of S, as the set of points that are integer combinations of $v^1, \ldots, v^m$.

1.3.6 Matrices

A *matrix* is a disposition of real numbers[4] in a rectangular array with a finite number of rows and columns. The elements of the matrix are called *entries*. The element of the matrix A arranged in the i^{th} row and j^{th} column is denoted by A_{ij}. A matrix with r rows and n columns is said to belong to $\mathbb{R}^{r\times n}$. We say that $r\times n$ is the *dimension* of the matrix[5]. In this context, we can see row and column vectors as matrices with respectively one row, and one column. A matrix is *square* if it has the same number of rows and columns. A square matrix $A \in \mathbb{R}^{n\times n}$ is *diagonal* if $A_{ij} = 0$ whenever $i \neq j$, for all $i, j = 1, \dots, n$.

The fundamental unary operation we can apply to a matrix is the transposition. Given $A \in \mathbb{R}^{r\times n}$, the *transpose* of A, denoted by $A^\top$, is the matrix of $\mathbb{R}^{n\times r}$ defined element-wise $A^\top_{ij} = A_{ji}$ for all $i = 1, \dots, n$, $j = 1, \dots, r$.

Matrices of the same dimension are equipped with the operation of sum: let A and B belong to $\mathbb{R}^{r\times n}$. Then $C = A + B$ is the matrix of $\mathbb{R}^{r\times n}$ such that $C_{ij} = A_{ij} + B_{ij}$ for all $i = 1, \dots, r$, $j = 1, \dots, n$. Given a scalar $k \in \mathbb{R}$ and a matrix $A \in \mathbb{R}^{r\times n}$, $kA \in \mathbb{R}^{r\times n}$ (rarely Ak) will denote the matrix whose ij^{th} entry is kA_{ij} for $i = 1, \dots, r$ and $j = 1, \dots, n$. Another important operation defined on matrices is the matricial product denoted by $\times$. Given two matrices $A \in \mathbb{R}^{r\times n}$ and $B \in \mathbb{R}^{n\times \ell}$, we can multiply them. Hence, $C \in \mathbb{R}^{r\times \ell} = A \times B$ is a matrix given by:

$$C_{ij} = \sum_{h=1}^{n} A_{ih}B_{hj} \quad \text{for all } i = 1, \dots, r \text{ and } j = 1, \dots, \ell.$$

Given a square matrix $A \in \mathbb{R}^{n\times n}$, the *determinant* of a A, denoted by $\mathrm{Det}(A)$, is the signed n-dimensional volume of the parallelotope defined by $\mathbf{0}$ and the column vectors of A. An $m \times m$*-minor* of a $r \times n$ matrix, with $m \leq r, n$, is the determinant of a square submatrix of dimension $m \times m$. We call *singular* a matrix whose determinant is 0. The (i, j)-minor of A is the determinant of $A^{(i,j)}$, that is the matrix obtained by removing the i^{th} row and the j^{th} column. There are many ways to compute the determinant of a $n \times n$ matrix, we provide the following formula[6]:

$$\mathrm{Det}(A) = -\sum_{i=1}^{n} \left((-1)^i A_{1i}\mathrm{Det}(A^{(1,i)})\right).$$

A related concept is that of rank: the *row rank* of a matrix is the maximum number of rows that are linearly independent when seen as row vectors. A matrix is full row rank

[4]We can enlarge the definition to *fields*, but it is not fundamental for this work.

[5]Note that $\times$ is just a typographical symbol in this case.

[6]The provided formula is probably the most understandable, but it is not efficient. There exist algorithms that compute the determinant of a given matrix in polynomial time.

when all its rows are linearly independent. A square submatrix is nonsingular if and only if it is full row rank.

We list some well-known classical operations on the rows of a matrix, called the *Gauss operations*:

i. exchanging two rows,

ii. summing k times a row to a different one, for a real number k,

iii. multiplying all the elements of a row for a real number $k \neq 0$.

These operations do not change the rank of the matrix, and only the first and the third operations affect the value of the determinant. If A' is obtained from A by performing once the first operation, then $\mathrm{Det}(A') = -\mathrm{Det}(A)$. If A' is obtained from A by performing once the third operation, then $\mathrm{Det}(A') = k\mathrm{Det}(A)$.

For further information about matrices we refer to any undergraduate book of linear algebra (e.g. [115]).

1.3.7 Polyhedra

One of the main topics of this thesis is the study of specific integrality conditions on polyhedra.

Let us consider the affine space $\mathbb{R}^n$ for a certain $n > 0$. A *hyperplane* is an affine subspace of dimension $n-1$. An hyperplane separates the space into two *half-spaces*. We consider the hyperplane as being a subset of each of the subspaces it separates.

Every hyperplane H can be expressed as $\{x \in \mathbb{R}^n : \alpha^\top x = \beta\}$ for some $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$. Similarly, the two half-spaces defined by H will be $\{x \in \mathbb{R}^n : \alpha^\top x \geq \beta\}$ and $\{x \in \mathbb{R}^n : \alpha^\top x \leq \beta\}$.

We define a *polyhedron* as the intersection of a finite number of halfspaces of $\mathbb{R}^n$. Thus, a polyhedron is the set of solutions of a linear system: $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, for a matrix $A \in \mathbb{R}^{\ell \times n}$ and $b \in \mathbb{R}^\ell$. In this case, we say that P is described by $Ax \leq b$. By definition, polyhedra are convex sets.

Remark. It is important to note that the linear system that describes a polyhedron is not unique.

A *polytope* is a polyhedron that is the convex combination of a finite number of points. We can consider polytopes as “bounded polyhedra”.

Observation 1.2. *A polyhedron $P \in \mathbb{R}^n$ is a polytope if and only if there exists a $c \in \mathbb{R}^n_+$ such that $P \subseteq \{x \in \mathbb{R}^n : -c \leq x \leq c\}$.*

A hyperplane H is called a *supporting hyperplane* of the polyhedron P if P is contained in one of the two half-spaces bounded by H and $P \cap H \neq \emptyset$. A *face* F of a polyhedron P is the set of points $F = P$ or $F = P \cap H$ for some supporting hyperplane H. A nonempty face that contains no other face of P is called *minimal face*. When such a face is a point, we call it *vertex* and we say that P is *pointed*. A face strictly contained in P that is contained in no other face is called *facet*.

Let $P = \{x : Ax \leq b\}$ be a polyhedron of $\mathbb{R}^n$ and F be a face of P. A matrix M is *face-defining* for F if M is full row rank and $\text{aff}(F)$ can be written $\{x : Mx = d\}$ for some d. A matrix is face-defining for P if it is face-defining for some of its faces. A face-defining matrix for a facet of P is called *facet-defining*.

Remark. The definition of face-defining matrix is independent from the description of P. On the other hand, all facet-defining inequalities for a given facet of a full-dimensional polyhedron are relatively multiple.

Observation 1.3. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron and let $F = \{x \in P : Bx = b\}$ be a face of P. If B has full-row rank and $n - \dim(F)$ rows, then B is face-defining for F.*

We say that a polyhedron is *integer* if every face contains an integer point. Similarly, a polyhedron is *rational* if every face contains a rational point. For pointed polyhedra these conditions are satisfied if and only if the vertices are integer/rational.

The following result relates integer polyhedra and supporting hyperplanes.

Theorem 1.4 ([63])**.** *A polyhedron P is integer if and only if each supporting hyperplane of P contains an integer point.*

A stronger requirement is the following: a polyhedron P is *box-integer* if $P \cap \{x : \ell \leq x \leq u\}$ is an integer polyhedron for all integer vectors ℓ and u. Figure 1.2 provides an example of an integer polytope that is not box-integer.

A *polyhedral cone* is a polyhedron that is also a cone. Polyhedral cones are finitely generated. A pointed polyhedral cone that has 0 as vertex is described by systems of the form $Ax \leq \mathbf{0}$.

The *dominant* of a polyhedron P is the set of points $\{x \in \mathbb{R}^n : x = y + z \ \forall y \in P, z \geq \mathbf{0}\}$.

A *dilation* of a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is the polyhedron $KP = \{x : A(K^{-1}x) \leq b\}$ for some diagonal matrix K such that $K_{ii} > 0$ for all $i = 1, \ldots, n$. A

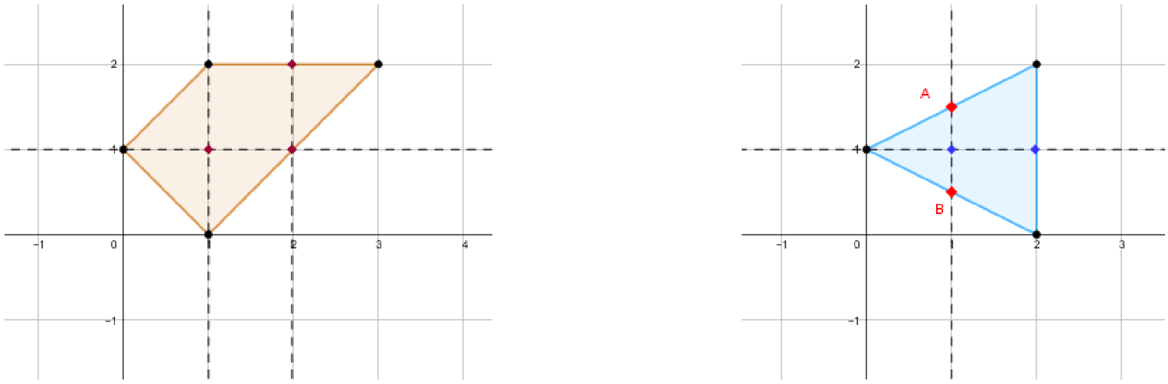


Figure 1.2: A box-integer polytope (left) and a non box-integer polytope (right): A and B are non integer points given by the intersection with an integer box-constraint.

uniform dilation is a dilation such that all the elements on the diagonal of K are equal; let k be the value of the entries of the diagonal of K. We denote such dilation by kP, and it is immediate to note that $kP = \{x : Ax \leq kb\}$.

1.3.8 Submodularity and Polymatroids

Let U be a set and $f : 2^U \to \mathbb{R}$ be a function. We say that f is a *set function* on U. A set function f is called *submodular* if:

$$f(T) + f(S) \geq f(T \cup S) + f(T \cap S) \quad \text{for all subsets } S \text{ and } T \text{ of } U.$$

Submodular set functions are associated with a special class of polyhedra, the polymatroids. A polymatroid is the ploytope described by:

$$\begin{cases} x(S) \leq f(S) & \text{for all subsets } S \text{ of } U, \\ x \geq \mathbf{0}, \end{cases}$$

for a submodular set function f on U. Similarly, an *extended polymatroid* is the polyhedron described by

$$x(S) \leq f(S) \quad \text{for all subsets } S \text{ of } U,$$

for some submodular set function f on U. Polymatroids were introduced by Edmonds [62], who gave also the following classical result (see also [135, Section 44.3]).

Theorem 1.5. *For any integer submodular set function f, the polymatroid P_f and the extended polymatroid EP_f are integer polyhedra.*

In Exemple 1.2 we introduce a well-known polymatroid, for the definitions of *graph* and *forest* we refer to Section 1.6

Example 1.2: The forest polytope.

Consider a graph $G = (V, E)$. In the following, we describe the convex hull of points in $\mathbb{R}^E$ that are incidence vectors of forests of G. We call this polytope the *forest polytope* of G.

Proposition 1.6. *Given a graph $G = (V, E)$, its forest polytope $\mathcal{F}$ is described by the following system.*

$$\begin{cases} x(F) \leq |V[F]| - 1 & \text{for all nonempty subsets } F \text{ of } E, & (1.1) \\ x \geq \mathbf{0}. & & (1.2) \end{cases}$$

Proof. The points $x \in \{0, 1\}^E$ represents subsets of edges of G. We denote by $G[x]$ the graph induced by such subset of edges. Inequalites (1.1) imply that a binary x does not belong to $\mathcal{F}$ if $G[x]$ contains a circuit. On the other hand, if $G[x]$ is forest of G, then x respects inequalities (1.1). Moreover, the right-hand side of (1.1) is a submodular set function, thus the polytope described by (1.1) and (1.2) is a polymatroid, that is integer by Theorem 1.5. ∎

1.4 Mathematical Programming

Mathematical programming is one of the most important branches of operational research. It concerns the maximization or minimization of a function, that represents the objective of our decision, under a set of constraints imposed by the nature of the problem being studied. The meaning of these constraints may vary: limited budget, logical implications, legal restrictions, and other kind of limitations could interfere with our aim. More generally, mathematical programming can be defined as a formal mathematical representation aimed at maximizing the output of a choice under the hypothesis of limited resources. When this representation uses only linear functions, we have a linear-programming model.

A mathematical optimization problem can be seen as $\max\{c(x) : x \in P\}$ for some *objective function* c and set P.

A point x is a *feasible solution* (or a *solution*) if it respects all the constraints. A solution x^* is an *optimum* or an *optimal solution* of $\min\{c(x) : x \in P\}$ if $c(x^*) = \min\{c(x) : x \in P\}$. Given a solution x, we say that a constraint is *active* for x if it is satisfied with equality. A constraint is active for a face F if the corresponding inequality is satisfied with equality by all points x in F. We say that a problem $\min\{c(x) : x \in P\}$ is *feasible* if it admits a feasible solution, and we say that it is *bounded* if there exists two real numbers u, ℓ such that $\ell \leq \min\{c(x) : x \in P\} \leq u$. We use the terms *unfeasible* and *unbounded* to denote the contrary of feasible and bounded.

1.4.1 Linear Programming

In this thesis we focus on *linear programming (LP)*. A problem is *linear* when both the constraints and the objective function are linear. The objective function is a linear function defined on this polyhedron, hence we can always define it as a scalar product between a *cost vector* c and its argument x. In this context, we use the terms *cost function*, *objective function*, and *cost vector* as synonyms. A linear programming algorithm finds a point in the polyhedron where this function has the largest (or smallest) value, if such a point exists.

The set of solutions – also called feasible region – of a linear program is given by the intersection of half-spaces, hence its feasible region is a convex polyhedron. We usually denote a linear program by $\max\{c^\top x : Ax \leq b, x \in \mathbb{R}^n\}$, for some $c \in \mathbb{R}^n, b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$. When possible, will omit $x \in \mathbb{R}^n$, to ease the notation. In this context the constraints are called *linear constraints*. Some constraints we find commonly in mathematical programming are the *nonnegativity constraints*, that are the constraints of the form $x \geq \mathbf{0}$.

Theorem 1.7. *Linear Programming is in $\mathcal{P}$.*

Indeed, there exists two algorithms that solve LP in polynomial time: the ellipsoid method [106] and Karmakar's algorithm [102]. Moreover, Tardos [148] proved that a linear program is solvable in polynomial time with respect to the size of the constraint matrix A. Thus, all the computational hardness of a linear program lies in the polyhedral structure of the program itself.

1.4.2 Combinatorial Optimization and Linear Integer Programming

A *combinatorial optimization problem* consists in minimizing (or maximizing) a function over a discrete set[7].

We are interested in combinatorial optimization problems that have linear objective function. Thus, our problems will be of the form $\max\{c^\top x : x \in S\}$, for some vector c and discrete set S. Moreover, we want to restrict our research to domains that are strictly

[7] The reader can think of a discrete set as finite.

correlated with linear programming. Indeed, the scope of our research is to study the polyhedral structure of conv(S) to solve the combinatorial optimization problem defined on S. Therefore, we require S to have some sort of convexity (that we lost when we assumed S to be discrete). We say that a set of integer points S is *pseudo-convex* if S is the set of integer points of conv(S). The interested reader can find some more results and possible definitions of discrete convexity in the work of Danilov and Koshevoi [45].

If we assume that our domain is pseudo convex, we reduce our scope to problems of the form $\max\{c^\top x : Ax \leq b, x \in \mathbb{Z}^n\}$, for some $c \in \mathbb{R}^n, b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$. The linear problem $\max\{c^\top x : Ax \leq b\}$ is the *linear relaxation* of our integer problem. Note that, for a given pseudo-convex set, there exist many different linear relaxations of the domain, as shown in Figure 1.3.

Problem like this are also called *integer linear programs*. We will use the terms combinatorial optimization problem and integer linear program as synonyms.

In general, solving such a combinatorial optimization problem is a hard task: many combinatorial optimization problems are $\mathcal{NP}$-hard [78]. Therefore, linear integer programs seem to be harder to solve than linear programs. This is partially due to the lack of "local properties": the domain being not convex, it is not trivial to move smoothly from one feasible solution to another. On the other hand, the convex hull of our domain possibly has an exponential number of facets, hence using linear programming techniques can still be not efficient. Nevertheless, there exist many combinatorial optimization problems with an exponential number of solutions that are treatable.

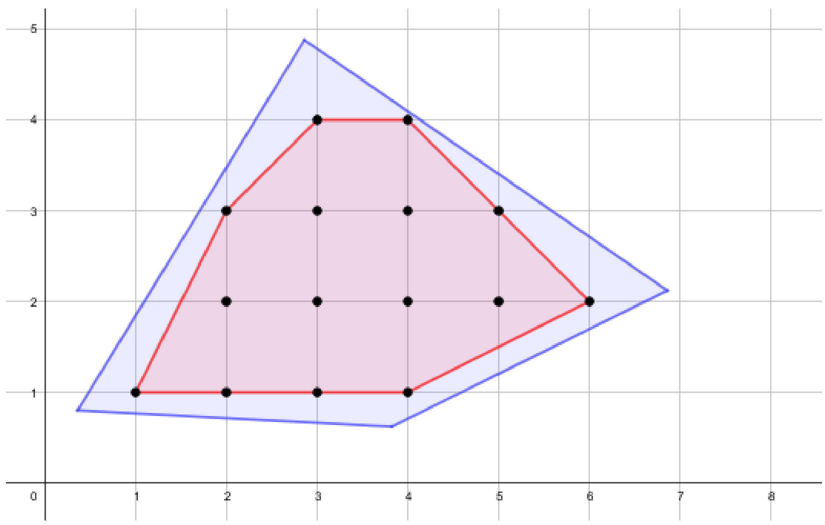


Figure 1.3: Two linear relaxations of a set of integer points.

Polyhedral approach. Our approach is based on linear programming. Let S be a discrete set, then consider the combinatorial optimization problem $\max\{c^\top x : x \in S\}$ and its linear relaxation $\max\{c^\top x : x \in \mathrm{conv}(S)\}$. We know that $\mathrm{conv}(S)$ is a polytope having elements of S as vertices, and we know that there exists a vertex attaining the optimal solution of $\max\{c^\top x :\in \mathrm{conv}(S)\}$. Thus, solving the linear relaxation of our problem gives us an optimal solution to $\max\{c^\top x : x \in S\}$.

As already mentioned, the criticality of this approach is the fact that $\mathrm{conv}(S)$ has possibly an exponential number of facets, and so the problem could be still untreatable. Nevertheless, this approach, proposed by Edmonds [61], turns out to be useful under some hypothesis. Indeed, Grötschel, Lovász, and Schrijver [86], Karp and Papadimitrou [104], and Padberg and Rao [119] showed under which assumptions we can use the ellipsoid method to solve a combinatorial optimization problem in polynomial time. In order to present the major result in this sense, we need the concept of separation problem. Given $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, its *separation problem* consists in deciding whether a point $\bar{x}$ belongs to P and, in negative case, provide an inequality of $Ax \leq b$ violated by $\bar{x}$. The following theorem is of great importance for combinatorial optimization.

Theorem 1.8 (Equivalence of separation and optimization)**.** *Let $\mathcal{A}$ be a class of inequality systems, then the optimization problem for $\mathcal{A}$ is solvable in polynomial time if and only if the separation problem for $\mathcal{A}$ is solvable in polynomial time.*

For the proof and some extensions of this result, we refer to [87].

1.4.3 Duality in Linear Programming

One of the most interesting aspects of linear programming is the existence, for each problem, of a corresponding *dual problem.* The relation occurring between a problem – named also *primal* – and its dual is so strict that we sometimes consider the couple primal-dual as a unique instance of a problem.

Given a primal problem $\mathcal{P}$ and its dual $\mathcal{D}$, there exists a bijection between constraints (resp. variables) of $\mathcal{P}$ and variables (resp. constraints) of $\mathcal{D}$. Moreover, the cost function of the one problem is the right-hand side of the other, and, if A is the constraint matrix of the primal, then $A^\top$ is constraint matrix of the dual. Given a LP problem $\mathcal{P}$, its dual is the following:

$$\max c^\top x \quad (\mathcal{P}) \qquad\qquad \min b^\top y \quad (\mathcal{D})$$

s.t. $\qquad\qquad$ s.t.

$$\{Ax \leq b \quad (1.3) \qquad\qquad \begin{cases} A^\top y = c \\ y \geq \mathbf{0} \end{cases} \quad (1.4)$$

Since we can always rewrite a LP problem to be in the same form of $\mathcal{P}$, the previous example is completely general. However, there is a set of rules we can apply, in order to deduce the dual of a LP problem, even if the system is not in the form as above. We summarized these rules in Table 1.1. A nice result is that the primal problem is the dual of its dual problem, hence there exists a one-to-one correspondence between problems and duals. The interrelation existing between these two problems is explained in the following result.

Theorem 1.9 (Strong Duality Theorem)**.** *Let* $\mathcal{P} : \max\{c^\top x : Ax \leq b\}$ *be a primal LP problem and let* $\mathcal{D} : \min\{b^\top y : y \geq \mathbf{0},\, y^\top A = c\}$ *be its dual. Then:*

- $\mathcal{P}$ *is unbounded if and only if* $\mathcal{D}$ *is unfeasible.*
- $\mathcal{P}$ *is unfeasible if and only if* $\mathcal{D}$ *is unbounded.*

If $\mathcal{P}$ *is feasible and bounded,* $\max\{c^\top x : Ax \leq b\} = \min\{b^\top y : y \geq \mathbf{0},\, y^\top A = c\}$.

Primal (Maximization)			Dual (Minimization)	
i^{th} constraint	$\leq b_i$	$\longleftrightarrow$	i^{th} variable	≥ 0
i^{th} constraint	$\geq b_i$	$\longleftrightarrow$	i^{th} variable	≤ 0
i^{th} constraint	$= b_i$	$\longleftrightarrow$	i^{th} variable	free
j^{th} variable	≥ 0	$\longleftrightarrow$	j^{th} constraint	$\geq c_j$
j^{th} variable	≤ 0	$\longleftrightarrow$	j^{th} constraint	$\leq c_i$
j^{th} variable	free	$\longleftrightarrow$	j^{th} constraint	$= c_j$

Table 1.1: Rules of LP duality

Another useful result is that of complementary slackness.

Theorem 1.10 (Complementary Slackness)**.** *If, in an optimal solution of a linear program, the value of the dual variable associated with a constraint is nonzero, then that constraint must be satisfied with equality. Further, if a constraint is satisfied with strict inequality, then its corresponding dual variable must be zero.*

Unfortunately, there is no one-to-one correspondence between optima of the primal and those of the dual. Nevertheless, Theorem 1.10 is a strong tool to derive an optimal solution of the primal from the optimal solution of the dual and vice versa.

1.5 Unimodular and Equimodular Matrices

In this section, we study some classes of matrices that play an important role in combinatorial optimization, in particular when studying total dual integrality, as we will see in the next chapter.

A *unimodular matrix* (*UM*) is a full row rank $r \times n$, integer matrix whose $r \times r$ minors have value $0, \pm 1$. A quite more specific definition is that of total unimodularity. A matrix is a *totally unimodular matrix* (*TUM*) if all its square submatrices have determinant equal to $0, \pm 1$. In particular all entries of a totally unimodular matrix are $0, \pm 1$.

Another possible way to generalize this concept is to consider when all the full dimensional determinants have the same value: an *equimodular matrix* (*EM*) is a full row rank $r \times n$ matrix whose non zero full dimensional determinants have the same absolute value. A further generalization is that of total equimodularity. A matrix is a *totally equimodular matrix* (*TEM*) if all its sets of linearly independent rows form an equimodular matrix.

Clearly, the most restrictive definition is that of total unimodularity, that implies all the others. Moreover, the discriminant between unimodular matrices and equimodular matrices is well-known (see *e.g.* [28]) and involves a remarkable property for integer linear programming.

Observation 1.11. *Let $A \in \mathbb{R}^{r \times n}$ be an equimodular matrix. If the g.c.d. of all the $r \times r$ determinants equals 1, then A is unimodular.*

In Example 1.3 we present some matrices belonging to the different classes we treat. Moreover, Figure 1.4 shows the inclusions between these classes of matrices.

We give an original result that characterizes totally unimodular matrices in terms of total equimodularity.

Proposition 1.12. *A matrix is totally unimodular if and only if it is unimodular and totally equimodular.*

Proof. Total unimodularity directly implies unimodularity and total equimodularity. On the other hand, suppose there exists a matrix $A \in \mathbb{Z}^{r \times n}$ that is unimodular and totally equimodular but not totally unimodular. By definition of unimodularity, all the entries of A are integer numbers.

Example 1.3: Some examples of matrices.

Here we list some examples of matrices belonging to the classes we mentioned. First, we highlight that there exist matrices that are equimodular but not unimodular, indeed the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

is equimodular but not unimodular. Moreover, the multiplication of a row by a constant number preserves both equimodularity and total equimodularity, but not unimodularity:

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}.$$

Furthermore, A and B are totally equimodular, unlike

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

that is unimodular (and hence equimodular) but not totally equimodular: the second row has both a 2 and a 1, so it is a non equimodular subset of rows.
We show in Proposition 1.12 that unimodular matrices that are also totally equimodular are totally unimodular.

Let now B be a square submatrix of A with determinant equal to $\pm k$ for some integer $k > 1$, and let R the subset of indices corresponding to the rows of B. Then, by total equimodularity of A, all non-singular square submatrices of A defined on R have determinant $\pm k$. Let us denote the set of these matrices by $\mathcal{B}$. Let $A' \in \mathbb{Z}^{r\times r}$ be a full row rank square submatrix of A, then:

$$\mathrm{Det}(A') = -\sum_{i=1}^{r} \left((-1)^i A'_{ji}\mathrm{Det}(A'^{(j,i)})\right) \quad \text{for some } j \notin R.$$

We can calculate recursively $\mathrm{Det}(A')$ by Laplace expansion on the rows not in R, obtaining:

$$\mathrm{Det}(A') = \sum_{B'\in\mathcal{B}} \left(\lambda_{B'}\mathrm{Det}(B')\right), \tag{1.5}$$

where $\lambda_{B'} \in \mathbb{Z}$ because it is a sum of products of elements of A'. Since $\mathrm{Det}(B') \in \{0, \pm k\}$ for all $B' \in \mathcal{B}$, we can rewrite (1.5):

$$\mathrm{Det}(A') = k \sum_{B' \in \mathcal{B}} \left(\lambda_{B'} \frac{\mathrm{Det}(B')}{k} \right). \tag{1.6}$$

This leads to a contradiction: $\mathrm{Det}(A') = \pm 1$, while the the right-hand side of (1.6) is k times an integer number. ■

1.5.1 Unimodular Matrices and Linear Programming

Unimodular matrices are related with very strong results in integer linear programming, and are a powerful tool to detect integer polyhedra.

Hoffman and Kruskal [99] proved that the coprimality of all full dimensional determinants implies integrality of the polyhedron:

Theorem 1.13. *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ and $A \in \mathbb{Z}^{r \times n}$. Then, the g.c.d. of all the $r \times r$ determinants of A equals 1 if and only if P is integer for every integer vector b.*

Since every unimodular matrix satisfies the condition of the theorem, we can deduce the following:

Corollary 1.14. *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ and $A \in \mathbb{Z}^{r \times n}$ be unimodular. Then, P is integer for all integer b.*

Hoffman and Kruskal themselves generalized this result in the same paper, introducing total unimodular matrices.

Theorem 1.15. *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ and $A \in \mathbb{Z}^{r \times n}$. Then, P is box-integer for every integer vector b if and only if A is totally unimodular.*

This result has very important consequences that we will explore in Chapter 2.

1.5.2 Recognition of Totally Unimodular Matrices

Here we discuss how we can recognize totally unimodular matrices, let us start from a trivial observation:

Observation 1.16. *Matrix A is totally unimodular if and only if $A^\top$ is.*

The following sufficient condition is credited to Heller, Tompkins, and Gale [94] (sometimes, also to Hoffman and Kruskal).

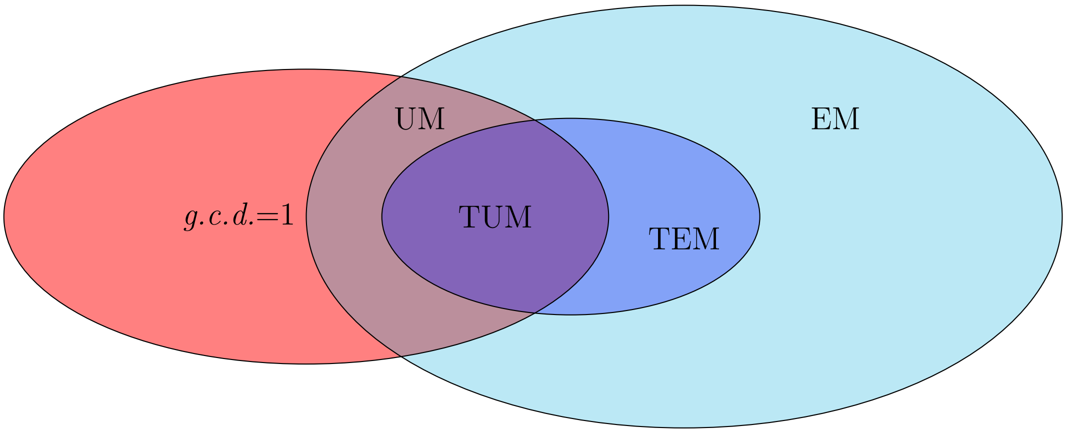


Figure 1.4: Classes of matrices.

Theorem 1.17. *Let A be a matrix such that:*

- *the entries of A are $0, \pm 1$,*
- *each row of A contains at most two nonzero elements,*
- *the columns of A can be partitioned into two sets A_1, A_2 such that two nonzero entries in a row are in the same set of columns if and only if they have different sign;*

then A is totally unimodular.

Corollary 1.18. *A matrix is totally unimodular if it contains at most one $+1$ and at most one -1 in each column.*

As an immediate consequence, we can highlight a well-known class of totally unimodular matrices:

Corollary 1.19. *The edge-incidence matrix of a bipartite graph is totally unimodular.*

Proof. It is sufficient to partition the columns of the matrix in V_1 and V_2 corresponding to the parts of the graph. Being the graph bipartite, no rows have two ones in either V_1 or V_2. ■

Camion [16] and Ghouila-Houri [80] provided two classical characterizations of totally unimodular matrices. For the first we need the following definition: a matrix is *Eulerian* if the sum of the elements on the rows and the sum of the elements on the columns are even numbers.

Theorem 1.20 (Camion's Characterization). *A $0/\pm1$ matrix is totally unimodular if and only if the sum of the elements in each Eulerian square submatrix is a multiple of* 4.

Theorem 1.21 (Ghouila-Houri's Characterization). *A matrix A is totally unimodular if and only if we can partition for every subset S of columns of A into two sets S_1 and S_2 such that*

$$\sum_{j\in S_1} A_{ij} - \sum_{\ell\in S_2} A_{i\ell} \in \{-1,0,1\} \quad \textit{for every row } i.$$

From this characterization stems Theorem 1.17: indeed, if a matrix satisfy the conditions of Theorem 1.17, we can just choose S_1 and S_2 of Theorem 1.21 respectively as $S\cap A_1$ and $S\cap A_2$.

Another sufficient consequence is based on the consecutive property: a 0/1 matrix has the *consecutive ones property* if, for every row i, $A_{ij} = A_{ij'} = 1$, with $j < j'$ implies that $A_{i\ell} = 1$ for all $j < \ell < j'$.

Corollary 1.22. *Every matrix with the consecutive ones property is totally unimodular.*

Seymour [140] characterized totally unimodular matrices in terms of network matrices. We avoid further details on this fundamental result that can be found in Seymour [140, 141] and Schrijver [134, Section 19]. A major outcome of [140] is the fact that there exists a polynomial-time algorithm to test whether a matrix is totally unimodular, as shown by Truemper [151].

Theorem 1.23. *The problem to decide if a given matrix is totally unimodular is in $\mathcal{P}$.*

1.5.3 Recognition of Equimodular Matrices

We will discuss the role of equimodular matrices in Chapter 2. We conclude this section providing some ways to check whether a matrix is equimodular. Here we show a characterization due to Heller [93] (see [28]).

Theorem 1.24. *Let $A \in \mathbb{R}^{r\times n}$ be full row rank. Then, the following are equivalent.*

- *A is equimodular.*
- *For each nonsingular full dimensional submatrix D of A, lattice(D) = lattice(A).*
- *For each nonsingular full dimensional submatrix D of A, $D^{-1}A$ is integer.*
- *For each nonsingular full dimensional submatrix D of A, $D^{-1}A$ has $0/\pm1$ entries.*

- *For each nonsingular full dimensional submatrix D of A, $D^{-1}A$ is totally unimodular.*
- *There exists a $r \times r$ submatrix D of A, such that $D^{-1}A$ is totally unimodular.*

Combining Theorems 1.23 and 1.24, we obtain the following result.

Corollary 1.25. *The problem to decide if a given matrix is equimodular is in $\mathcal{P}$.*

We give here some original simple observations we will use in Chapter 4.

Observation 1.26. *Let $A \in \mathbb{R}^{I \times J}$ be a full row rank matrix, $j \in J$, $\mathbf{c}$ be a column of A, and $\mathbf{v} \in \mathbb{R}^I$. If A is equimodular, then so are the following if they have full row rank:*
(i) $\begin{bmatrix} A & \mathbf{c} \end{bmatrix}$, *(ii)* $\begin{bmatrix} A \\ \pm\xi_j \end{bmatrix}$, *(iii)* $\begin{bmatrix} A & \mathbf{v} \\ \mathbf{0}^\top & \pm 1 \end{bmatrix}$, *(iv)* $\begin{bmatrix} A & \mathbf{0} \\ \pm\xi_j & \pm 1 \end{bmatrix}$.

Observation 1.27. *Gauss operations preserving the rank preserve equimodularity.*

Observation 1.28 ([28])**.** *Let F be a face of a polyhedron. If a face-defining matrix of F is equimodular, then so are all face-defining matrices of F.*

1.6 Graphs

Graphs are a convenient mean to model discrete data in mathematics, as well as a fruitful source of combinatorial optimization problems. Indeed, in this thesis we focused all problems we tackle are defined on graphs.

A *graph* G is a pair $G = (V, E)$, where V is the set of *vertices* of G, and E is a family[8] of unordered pairs of elements of V, called *edges*. The terms *node* and *arc* are sparsely used as synonyms respectively of vertex and edge.

Given an edge, the vertices composing it are called *endpoints*. We sometimes denote an edge e having v and w as endpoints as vw. The cardinalities of V and E are respectively called *order* and *size* of G. Two edges are *incident*[9] if they share an endpoint. An edge that is incident only to one vertex, is a *loop* and two edges sharing both endpoints are called *parallel edges*. A graph that have neither loops nor parallel edges is called *simple*. An edge and a vertex are *incident* one to the other if the latter is an endpoint of the first. We denote the set of edges incident to a vertex v by $\delta(v)$. The *degree* of a vertex is the

[8]Following our definition a graph can have multiple edges with the same endpoints. In literature, these objects are sometimes called *multigraphs*.

[9]Sometimes, the term *adjacent* is also used.

number of edges incident on it. Two vertices are *adjacent* if they are endpoints of a same edge.

We usually represent a graph by drawing points (the vertices) connected by lines (the edges).

A *path* is an alternating sequence of vertices and edges $P = \{v_1, e_1, v_2, \ldots, e_{k-1}, v_k\}$, such that two consecutive elements of the sequence are incident. We say that the *length* of a path is the number of vertices in the path[10].

A graph G is *connected* if for each pair of vertices there exists a path in G containing both of them. We can generalize this concept: a graph G is k*-connected* if the removal of any $k-1$ vertices does not disconnect it. Moreover, a graph is k*-edge-connected* if the removal of any $k-1$ edges does not disconnect it.

Observation 1.29. *A graph is k-connected (resp. k-edge-connected) if and only if there exist k vertex-disjoint (resp. edge-disjoint) paths between any two vertices.*

We will now define some basic elements of graphs, let $G = (V, E)$ be a graph. A subgraph of G is a graph $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$. A subgraph $H = (V', E')$ of G, is an *induced subgraph* if E' is the set of edges of E having both endpoints in V'. We denote the subgraph induced by V' by $G[V']$. An *edge-induced subgraph* is a subgraph $H = (V', E')$ where $E' \subseteq E$ and V' is the set of endpoints of the edges of E'.

A *contraction* of an edge e is the operation of replacing e and its endpoints by a single vertex. If H is obtained from G by contracting the edge e, we write that $H = G/e$. A *subdivision* of an edge e is the operation of replacing e and its endpoints by a path of length 2. A *deletion* is the removal of an edge or a vertex and all edges incident to it from the graph. The graph obtained by the removal of a vertex v is denoted by $G \setminus \{v\}$. The notation is the same for the edges. A graph H is a *topological minor* (or simply, a *minor*) of G if we can obtain H from G by a sequence of edge contraction, edge deletion, and vertex deletion. Given two graphs G and H and two vertices $v \in G$ and $w \in H$, we define the *1-Sum* $G_v \oplus_w H$ as the graph obtained by identifying the vertices v of G and w of H.

One of the objectives of graph theory is to characterize different classes of graphs in terms of forbidden minors and/or forbidden induced subgraphs. An interesting result in this sense, is due to Robertson and Seymour [126]: they proved what was previously known as Wagner's Conjecture.

Theorem 1.30 (Robertson-Seymour Theorem)**.** *Every class of graphs closed under minors can be defined by a finite set of forbidden minor graphs.*

[10]This choice is standard, but not universally accepted.

1.6.1 Structures of Graphs

We list here the objects that we will use in our research, together with some elementary observations.

A subset of edges that induces a graph whose vertices have even degree is called a *cycle*. With an abuse of notation, we indicate with the same letter the subgraph induced by the cycle. A *circuit* is a cycle inducing a connected graph whose vertices have degree 2. The *length* of a circuit is the number of edges forming it.

We say that an edge is a *chord* of a circuit C if it has as endpoints two vertices of C that are not adjacent in C. A circuit with no chord is said to be *chordless*. A *hole* of G is a chordless circuit of length at least 4. We say that a graph is *chordal* if it has no holes. Given two set of vertices of G, say W_1, W_2, we define by $\delta(W_1, W_2)$ the set of edges having one endpoint in W_1 and the other in W_2. A *cut* is a set of edges of G having exactly one endpoint in a subset of vertices W of V. We denote such a cut by $\delta(W)$[11]. The sets W and $V \setminus W$ are called *shores*. A *bond* is a cut containing no other nonempty cut. In the literature, a bond is sometimes called a *central cut*. We list some well-known facts.

Observation 1.31 ([10]). *For a graph G the following hold:*

- *Every bond intersects every circuit of G in an even number of edges.*
- *A nonempty cut is a bond if and only if both its shores induce connected subgraphs.*
- *Symmetric difference of two cuts is a cut.*

Given a partition $\{W_1, \dots, W_k\}$ of V, a *multicut* is the set of edges having endpoints into two distinct sets of the partition. Following the notation on δ, we denote this multicut by $\delta(W_1, \dots, W_k)$. For each multicut M there exists a unique partition of vertices $\{V_1, \dots, V_{d_M}\}$ such that $G[V_i]$ is connected for all $i = 1, \dots, d_M$. We call such $\{V_1, \dots, V_{d_M}\}$ *shores*, and we say that d_M is the *order* of M. The *reduced graph* of a multicut M is the graph G_M obtained by contracting all the edges of $E \setminus M$ (see Figure 1.5). Note that a multicut of G_M is also a multicut of G.

In Chapters 3 and 4 we will treat some systems whose constraints are associated with bonds, cuts, and multicuts. The following trivial observations will be useful for our results.

Observation 1.32. *Every cut is the disjoint union of bonds.*

[11] A cut $\delta(W)$ is the set of edges $\delta(W, V \setminus W)$, so we are omitting the complementary set.

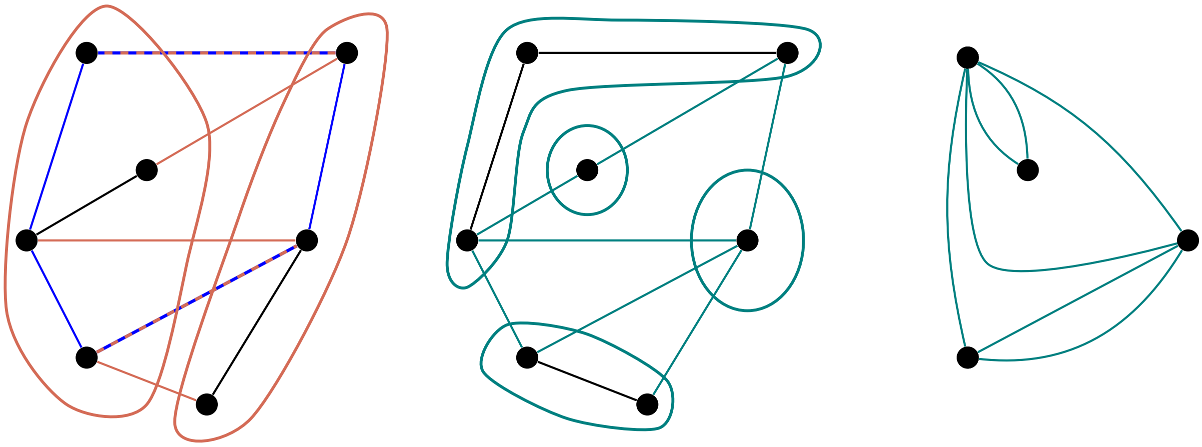


Figure 1.5: Left: a circuit C and a bond B.
Center-right: a multicut M of a graph G, and the reduced graph G_M.

Observation 1.33. *Let $M = \delta(V_1, \dots, V_{d_M})$ be a multicut, then*

$$\chi^M = \sum_{i=1}^{d_M} \frac{1}{2} \chi^{\delta(V_i)}.$$

Observation 1.34 ([40])**.** *A set of edges M of a graph G is a multicut if and only if $|M \cap C| \neq 1$ for all circuits C of G.*

We denote respectively by $\mathcal{M}_G$ and $\mathcal{B}_G$ the set of multicuts and the set of bonds of G.

A *flow* is a couple (C, e), where C is a circuit of G and e is an edge of C. In a flow (C, e), the edge e represents a demand and $C \setminus e$ represents the path satisfying this demand. With an abuse of notation, we say that the *incidence vector* of a flow (C, e) is the $0/\pm 1$ vector $\chi^{C \setminus e} - \chi^e$.

Observation 1.35. *For every flow F and bond B of a graph, $\left(\chi^F\right)^\top \chi^B \geq 0$.*

Some graph classes. A graph having no cycles is a *forest*, a connected forest is a *tree*. A graph is *bipartite* if we can partition its vertices into two subsets, say V, W such that every edge of G has one endpoint in V and the other in W. V and W are called *parts* of G. Bipartite graphs are characterized for not having odd-length cycles as subgraphs. The *complete graph on n vertices*, denoted by K_n is the graph having n vertices and one edge between each two distinct vertices. The *complete bipartite graph $K_{m,n}$* is the bipartite graph having a part V of m vertices, a part W of n vertices, and an edge connecting each vertex of V to each vertex of W.

In the following, we present some properties and characterizations of the graphs we study in this thesis. Every result unproved or unreferenced can be found in classical books on graph theory, *e.g.* the book of Diestel [52].

1.6.2 Planar Graphs

A graph is *planar* if it can be drawn onto the plane in a way such that no edges meet in any point other than their common endpoints. Such a drawing is called *planar embedding*. Note that the representation of a graph is independent from its planarity. Once a graph G is drawn onto the plane, it subdivides the plane in a set of *regions* delimited by edges and vertices. Let G be a planar graph, the *planar dual* of G is the graph having a vertex for each region of the planar embedding, and an edge between two vertices for each edge separating two regions. We denote the planar dual by $G^\star$. It is immediate to see that the planar dual of a planar graph is planar. In figure 1.6, we can see two planar graphs reciprocally dual.

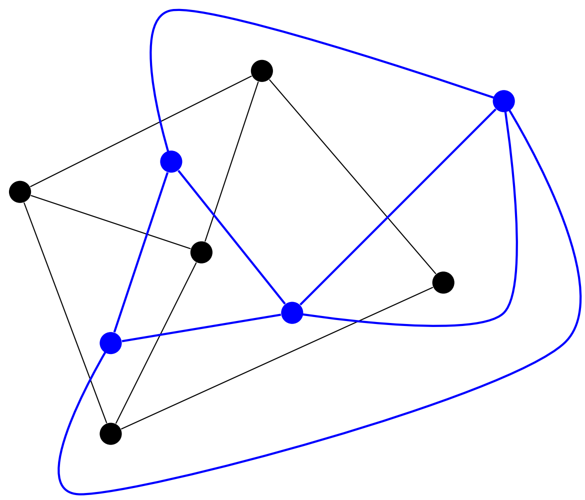


Figure 1.6: A graph (black) and its planar dual (blue).

Planar duality is indeed a powerful instrument that has various applications. In Chapter 3 we will use the following property:

Proposition 1.36. *Let G be a planar graph, there exists a one-to-one correspondence between the circuits (resp. bonds) of G and bonds (resp. circuits) of $G^\star$.*

Planar graphs have been characterized by Kuratowski [108].

Theorem 1.37 (Kuratowski's Theorem)**.** *A graph is planar if and only if it does not have neither K_5 nor $K_{3,3}$ as minors.*

1.6.3 Series-parallel Graphs

The class of graphs we study the most in this thesis is the class of series-parallel graphs.

A graph is *series-parallel* if it can be built starting from K_2, the graph with two vertices and one edge, by applying repeatedly the following operations:

- *(parallelization)* adding an edge parallel to an existing one,
- *(subdivision)* subdividing an edge,
- *(1-Sum)* 1-Summing two series-parallel graphs.

We refer to these building operations as *series-parallel operations*. In Figure 1.7, we show the three operations.

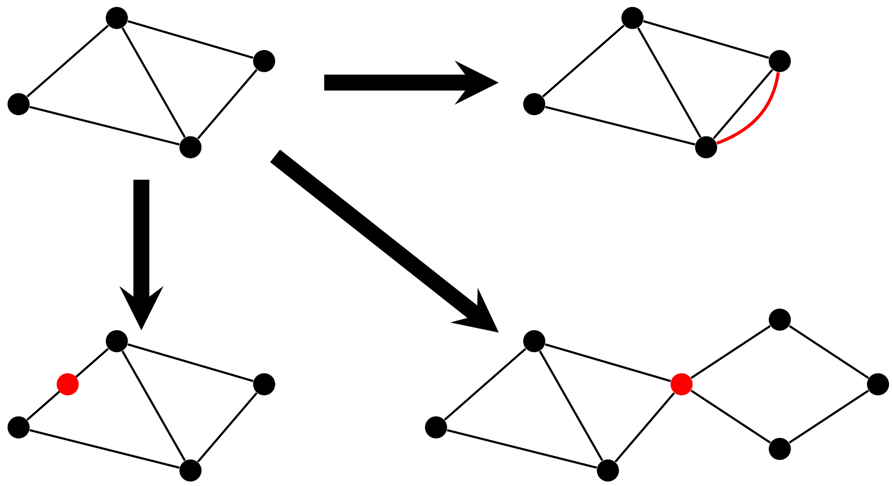


Figure 1.7: Series-parallel graphs: the series-parallel operations.

Equivalently, we can say that a graph is series-parallel if its 2-connected components can be built from the circuit of length 2, by repeatedly adding edges parallel to an existing one, and subdividing edges.

Series-parallel graphs are a remarkable class of graphs. They are fairly simple and their building characterizations allow easy proofs for many results. Moreover, many usually hard problems are treatable on series-parallel graphs. The most important characterization of series-parallel graphs is due to Duffin [59].

Theorem 1.38. *A graph is series-parallel if and only if it does not have K_4 (see Figure 1.8) as a minor.*

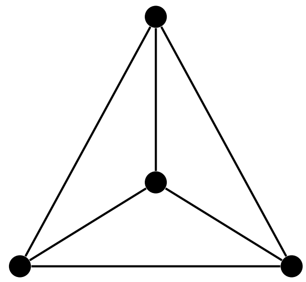


Figure 1.8: K_4, the smallest non-series-parallel graph.

Theorem 1.38 implies various properties, for instance the fact that series-parallel graphs are planar.

Corollary 1.39. *Every series-parallel graph is planar.*

Proof. To see that, we observe that K_4 is a subgraph of K_5, and it can be obtained from $K_{3,3}$ by contracting two edges incident to a same vertex. Combining Theorems 1.37 and 1.38 we obtain the result. ■

Moreover, K_4 being the planar dual of itself, the class of series-parallel graphs is closed under planar duality.

Observation 1.40. *A graph is series-parallel if and only if its planar dual is.*

In 1992, Eppstein [66] gave another powerful tool to build series-parallel graphs, based on the concept of ear decomposition. An *open nested ear decomposition* of a graph $G = (V, E)$ is a partition of E into $\mathcal{E}_0, \ldots, \mathcal{E}_m$ such that $\mathcal{E}_0$ is a circuit, and the *ears* $\mathcal{E}_i$, for $i = 1, \ldots m$, are paths with the following properties:

- both the end vertices of $\mathcal{E}_i$ lie in a single $\mathcal{E}_j$ for some $j < i$,
- no internal vertex of $\mathcal{E}_i$ is in $\mathcal{E}_j$ for all $j < i$,
- if two ears $\mathcal{E}_i$ and $\mathcal{E}_h$ have both their end vertices in the same ear $\mathcal{E}_j$, then a path in $\mathcal{E}_j$ between the extremities of $\mathcal{E}_i$ contains either both or none the endpoints of $\mathcal{E}_h$.

In Figure 1.9, we see an open nested ear decomposition of a series-parallel graph.

Series-parallel graphs are those that admits an open nested ear decomposition.

Theorem 1.41 ([66])**.** *A non-trivial 2-connected graph is series-parallel if and only if admits an open nested ear decomposition.*

There exist some other definitions of series-parallel graph, *e.g.* those used by Duffin [59] and Eppstein [66]. The definitions given in this thesis are consistent with the vast majority of the literature.

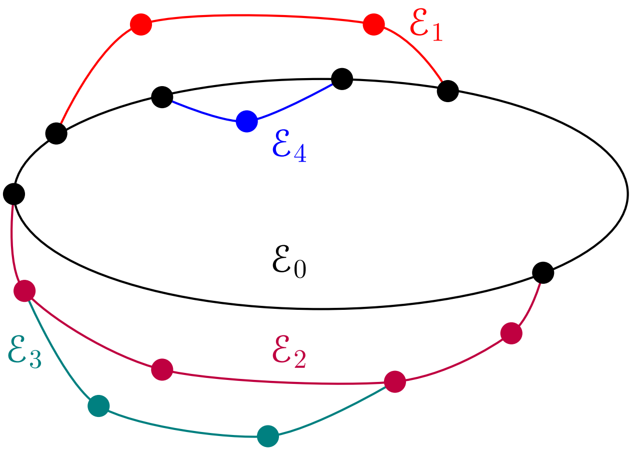


Figure 1.9: An open nested ear decomposition.

Problems on series-parallel graphs Many optimization problems on graphs are $\mathcal{NP}$-hard. However, specific problems, like the minimum spanning tree or the maximum matching problem, admit a polynomial-time algorithm. Moreover, problem usually hard to solve can become treatable on a specific class of graphs.

Takamizawa, Nishizeki, and Saito [147] provided a general characterization of linear-time solvable problems on series-parallel graphs. As a consequence, the following problems are solvable in linear time in series-parallel graphs:

- the minimum vertex cover problem ($\mathcal{NP}$-hard in general),
- the maximum outerplanar induced subgraph problem ($\mathcal{NP}$-hard in general),
- the maximum matching problem ($\mathcal{P}$, but non linear-time solvable in general),
- the maximum cut problem ($\mathcal{NP}$-hard in general),
- the Steiner tree problem ($\mathcal{NP}$-hard in general),

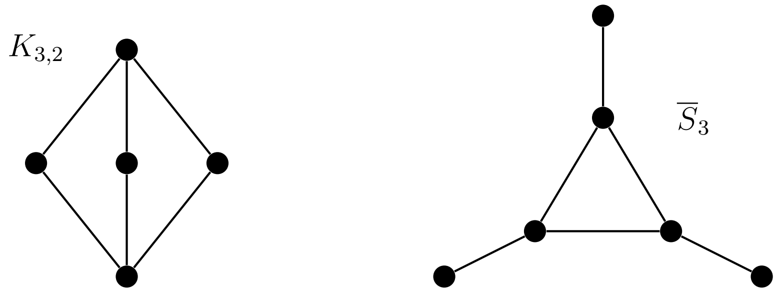


Figure 1.10: Two series-parallel graphs: $K_{3,2}$ and $\overline{S}_3$ (also known as net graph).

More generally, series-parallel graphs are a "docile" class for optimization problems: a wide spectrum of problems hard in general are treatable in this class [127, 142, 143, 149]. Moreover, for series-parallel graphs it is known the linear programming description of many polyhedra that are usually problematic to describe [10, 29, 30, 47, 49, 51, 84, 114, 122]. In this thesis we are interested in connectivity problems. In series-parallel graphs, many of these problem are easy to solve: like the minimum k-edge-connected spanning subgraph problem [50], the generalized Steiner tree problem [153], and the Steiner traveling salesman problem [43].

Lastly, the recognition of series-parallel graphs [66] can be done in polynomial time.

Chapter 2

Total Dual Integrality in Combinatorial Optimization

In this chapter we study a classical concept of linear programming: total dual integrality. Total dual integrality is a powerful tool used to study linear programming relaxations of combinatorial problems.

2.1 Total Dual Integrality

A rational system of linear inequalities $Ax \leq b$ is *totally dual integral* (*TDI*) if the minimization problem in the linear programming duality:

$$\max\{cx : Ax \leq b\} = \min\{yb : y \geq \mathbf{0}, yA = c\}$$

admits an integral optimal solution for each integral vector c such that the maximum is finite.

A motivation for the study of TDI systems is given by the fact that totally dual integral systems can be used to prove integrality of the polyhedra they describe. Some initial results on this topic were provided by Fulkerson [76] and Hoffman [98], and where then unified by Edmonds and Giles [63] in the following theorem.

Theorem 2.1. *If $Ax \leq b$ is a TDI system and b is integral, then $\{x : Ax \leq b\}$ is an integer polyhedron.*

A first fundamental remark is that that total dual integrality is a property related to systems, and not to polyhedra. Indeed, a polyhedron can be described by various systems, which can or cannot share the property of being TDI, as showed by Example 2.1.

Example 2.1: A TDI and a non-TDI systems.

Consider the two following systems:

$$\begin{cases} x_1 + x_2 \geq 0 \\ x_1 + x_3 \geq 0 \\ x_2 + x_3 \geq 0 \end{cases} \tag{2.1}$$

$$\begin{cases} x_1 + x_2 \geq 0 \\ x_1 + x_3 \geq 0 \\ x_2 + x_3 \geq 0 \\ x_1 + x_2 + x_3 \geq 0 \end{cases} \tag{2.2}$$

Clearly, Systems (2.1) and (2.2) describe the same polyhedron. System (2.2) is TDI, however the same does not hold for System (2.1). Indeed, the dual of $\max\{\mathbf{1}^\top x : x$ is a solution to $(2.1)\}$ is:

$$\begin{aligned} &\max \mathbf{0} \\ &\text{s.t.} \\ &\begin{cases} y_1 + y_2 = 1 \\ y_1 + y_3 = 1 \\ y_2 + y_3 = 1 \\ y \geq \mathbf{0}. \end{cases} \end{aligned}$$

Whose unique solution is $y = \frac{\mathbf{1}}{\mathbf{2}}$.

Interestingly, the given example is not a special case: for a given polyhedron we can always find a TDI system and a non-TDI system describing it. Here, we anticipate a result due to Giles and Pulleyblank [82]. We will analyze in depth this theorem and its consequences in Section 2.1.4.

Theorem 2.2. *Every rational polyhedron is described by a TDI system $Ax \leq b$, with A integer. Moreover, b can be chosen integer if and only if the polyhedron is integer.*

We can combine Theorems 2.1 and 2.2 as follows:

Corollary 2.3. *A polyhedron P is integer if and only if there exists a TDI system $Ax \leq b$ with b integer describing it.*

Another important reason for studying total dual integrality lies in the fact that duality in linear programming can be interpreted as a min-max relation between two different problems. We discuss this property in details later in this chapter.

Example 2.2: Kőnig's Theorem.

For a graph, a *matching* is a set of pairwise non incident edges. We denote by $\nu(G)$ the cardinality of the largest matching of a graph $G = (V, E)$. Similarly, a *vertex cover* is a set of vertices such that all the edges of the graph are incident to at least one vertex of the set; we denote by $\tau(G)$ the cardinality of the smallest vertex cover of G.

Theorem 2.4 (Kőnig's Theorem). *If G is a bipartite graph, $\nu(G) = \tau(G)$.*

This result is a good example of the min-max relations we can obtain by total dual integrality. It is easy to see that $\nu(G) \leq \tau(G)$, since at least one endpoint for each edge of a matching is required in order to cover all the edges of the matching, and so all the edges of G. This relation can be explained also by linear programming duality:

$$\max\{\mathbf{1}^\top y : y \geq \mathbf{0},\ y\,(\delta(v)) \leq \mathbf{1}\ \forall v \in V\} \leq \min\{\mathbf{1}^\top x : x \geq \mathbf{0},\ x_v + x_u \geq \mathbf{1}\ \forall vu \in E\} \tag{2.3}$$

Without further hypothesis, the minimization problem in (2.3) does not admit an integer optimal solution. On the contrary, the matrix of constraints of the dual is totally unimodular when the graph is bipartite by Corollary 1.19. Hence, the dual problem have integer optimum by Theorem 1.15, and the optimum of the two combinatorial problems is equal. Moreover, we can extend this result to the weighted case and deduce the result known as Egerváry's Theorem.

Other well-known results that can be proved by total dual integrality are the Max-flow Min-cut Theorem [65, 69], the Nash-Williams Orientation Theorem [116] (see [73, 75]), and the Cunningham-Marsh Formula [44]. The interested reader can found the proof of these results in terms of total dual integrality in the books of Schrijver: [134, Section 7.10] for the Max-flow Min-cut Theorem, [134, Section 22.2] for the Nash-Williams Orientation Theorem, and [135, Section 25.3] for the Cunningham-Marsh formula. All of these results were first proved without the use of total dual integrality. In the following of this chapter, we provide some results strongly relying on total dual integrality.

Directly proving that a system is TDI is not trivial. Indeed, to determine if a problem has an integer optimal solution for every integer right-hand side of the constraints can be quite a hard task. In Chapter 4, we show an example of this kind of proof. In the following,

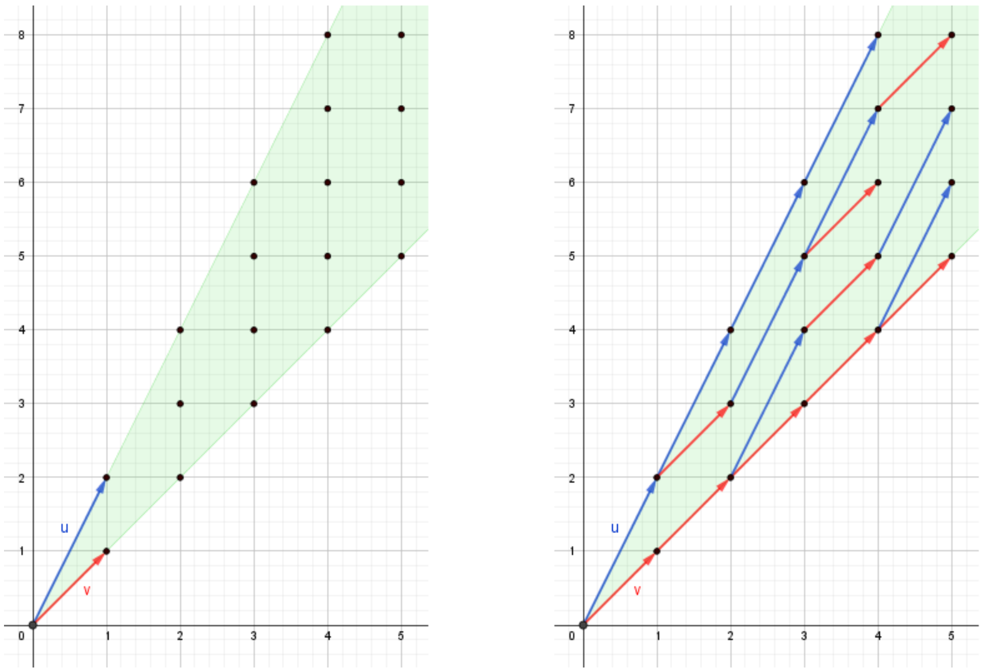


Figure 2.1: A visual representation: u and v form a Hilbert basis

we will analyze some of the tools we can use to determine whether a system is TDI.

2.1.1 Hilbert Bases

Totally dual integral systems are characterized in terms of Hilbert bases. A set $\{v^1, \ldots, v^k\}$ of vectors is a *Hilbert basis*[1] if each integer vector in their conic hull can be expressed as a nonnegative integer combination of $v^1, \ldots, v^k$. We give in figure 2.1 a representation of an Hilbert basis.

The following central result was proved by Giles and Pulleyblank [82].

Theorem 2.5. *A system $Ax \leq b$ is TDI if and only if for every face F of $P = \{x : Ax \leq b\}$, the rows of A associated with tight constraints for F form a Hilbert basis.*

The statement of Theorem 2.5 is still valid if we consider only minimal faces, as stated by the following.

[1] The term *Hilbert basis* is sometimes used to indicate different (even if related) concepts in combinatorics, as testified by [83, 109, 138]. The reader should pay attention to the fact that Theorem 2.5 is not valid if we use one of the alternative definitions. We give further details on this topic in Section 3.4.

Corollary 2.6. *A system $Ax \leq b$ is TDI if and only if for every minimal face F of $P = \{x : Ax \leq b\}$, the rows of A associated with tight constraints for F form a Hilbert basis.*

Corollary 2.6 gives a characterization of Hilbert bases in terms of total dual integrality, and, conversely, provides a simple result for proving TDIness of systems describing polyhedral cones.

Corollary 2.7. *A system $Ax \leq 0$ is TDI if and only if the rows of A form a Hilbert basis.*

2.1.2 Sufficient Conditions for TDIness

The characterization given in Theorem 2.5 is not the only tool we can use to prove TDIness.

The first alternative result, that we implicitly used for proving Theorem 2.4, relies on totally unimodular matrices. As shown in Section 1.5, totally unimodular matrices describe integer polyhedra. The following result is a direct consequence of Theorem 1.15 and Observation 1.16.

Theorem 2.8. *Let A be a totally unimodular matrix, then both problems in the linear programming duality:*

$$\max\{cx : Ax \leq b\} = \min\{yb : yA = c, y \geq \mathbf{0}\},$$

admit integer optimal solutions, for all integer vectors b and c.

In other words, every system whose constraint matrix is totally unimodular is TDI.

As totally unimodular matrices are well-characterized and recognizable in polynomial time, Theorem 2.8 is one of the principal instruments used in the literature for proving total dual integrality of systems.

The following characterization of Schrijver and Seymour [136] can be used to prove total dual integrality using combinatorial arguments.

Theorem 2.9. *The system $Ax \leq b$ is TDI if and only if*

$$\min\{yb : yA = c, y \geq \mathbf{0}, y \text{ half-integer}\} \tag{2.4}$$

is finite and attained by an integer vector, for every integer vector c such that the primal problem is finite.

Different other sufficient conditions have been given during the years. An efficient strategy is the one of analyzing the case when the constraints are incidence vectors of sets of a collection. Schrijver [131] gave general condition for a collection and a set function to describe a TDI system. This result generalizes different previous works [63, 72, 85]; a recollection of these results can be found in [132] and in the book of Schrijver [135, Volume B]. We will analyze further these results in terms of box-TDIness and polymatroids [62] in the Section dedicated to box-TDIness. O'Shea and Sebő [117, 118] characterized totally dual integral systems in terms of polynomial ideals. Some sufficient conditions on the constraint matrix conditions were given by Conforti and Cornuéjols [32]. A necessary condition for a system to be TDI is given by Gerards and Sebő [79].

2.1.3 TDI Systems and Operations

As we have seen in the example provided at the beginning of this section, total dual integrality is not a property shared by different systems describing the same polyhedron. Indeed, TDIness has a peculiar behavior under operations that we usually consider "safe". Here, we study some of the operations that preserve or disrupt total dual integrality. All the unreferenced results are taken by [134, Section 22.5]. The interested reader can find some more insights and results in Cook [35].

The multiplication of (some of) the constraints defining a TDI system by a rational does not, in general, preserve the total dual integrality, even if it does not change the described polyhedron. The following elementary example should convince the reader of this fact.

Example 2.3: Losing TDIness by multiplication.

Consider the system $x_1 \leq 1, x_2 \leq 1$. It is TDI because its constraint matrix is totally unimodular. However, if we multiply one of the constraint by 2 and consider the duality $\min\{x_1 + x_2 : 2x_1 \leq 2, x_2 \leq 1\} = \max\{2y_1 + y_2 : 2y_1 = 1, y_2 = 1, y \geq \mathbf{0}\}$, we see that the only feasible dual solution is not integer.

Conversely, dividing constraints of a TDI system by a positive integer preserves the total dual integrality of the system.

Observation 2.10 (Division by an integer)**.** *Let $Ax \leq b$ be a totally dual integral system, and let $k \in \mathbb{Z}, k > 0$. The system obtained by dividing both sides of a constraint of $Ax \leq b$ by k is TDI.*

Multiplying the right hand side of a TDI system by a positive rational gives a totally dual integral system.

Observation 2.11 (Multiplication of the right hand side)**.** *Let $Ax \leq b$ and $k \in \mathbb{Q}, k > 0$. Then $Ax \leq b$ is TDI if and only if $Ax \leq kb$ is.*

Total dual integrality is preserved under the addition of slack variables.

Proposition 2.12 (Addition and removal of slack variables)**.** *The system $Ax \leq b, ax \leq p$, where a is an integer vector, is TDI if and only if the system $Ax \leq b,\ ax + t = p,\ t \geq 0$, where t is a new variable, is TDI.*

Another operation that does not change the TDIness of a system is the addition of a column identical to an existing one.

Observation 2.13 (Addition of an identical column)**.** *Let $Ax \leq b$ be a system, and let α be a column of A. Then the system $Ax + \alpha y \leq b$, where $y \in \mathbb{R}$ is a new variable, is TDI if and only if $Ax \leq b$ is.*

Adding redundant constraints to a TDI system preserves it TDIness.

Proposition 2.14 (Addition of redundant constraints)**.** *Let $Ax \leq b$ be a TDI system, and $ax \leq p$ be a constraint respected by all the points $\{x : Ax \leq b\}$. Then the system $Ax \leq b, ax \leq p$ is TDI.*

2.1.4 Obtaining TDI Systems

Theorem 2.2 assures that it is always possible to find a TDI system that describes a given polyhedron. In the present section, we show how we can obtain a TDI system describing a polyhedron P, assuming we know a non TDI system describing it.

There exist substantially two ways to obtain a TDI system from a non TDI system. The first consists in adding redundant constraints, while the other consists in dividing the existing constraints for a sufficiently large integer.

Division. Given a system $Ax \leq b$, we can always obtain a TDI system describing the same set of points by simply dividing the inequalities by a sufficiently large integer.

Theorem 2.15 ([82])**.** *For each rational system $Ax \leq b$ there exists a natural number k such that $k^{-1}(Ax) \leq k^{-1}b$ is TDI.*

Example 2.4: Obtaining TDIness from division.

By dividing both sides of System (2.1) by 2, we obtain a TDI system.

Addition of constraints. The second technique we could use to obtain a TDI system is to add a set of redundant constraints to a TDI system. In Example 2.1 we provided a case where the addition of a redundant constraint to a non TDI system resulted in a TDI system. The following proposition can be seen as a consequence of Theorem 2.5 and the fact that every rational polyhedral cone admits a finite Hilbert basis.

Theorem 2.16 ([82])**.** *Let $Ax \leq b$ be a non TDI system, then there exists a TDI system $A'x \leq b'$ obtained by adding redundant constraints to $Ax \leq b$.*

Depending on the objective, the two techniques have different applications. If we are looking for a min-max relation between non integer objects, if we are looking for the $\frac{1}{k}$-integrality of a polyhedron, or if we want a bound on the gap between two combinatorial values, dividing a system by an integer number can lead to good results. On the other hand, for purely combinatorial applications, like proving integrality of a polyhedron, or describing a min-max relation between combinatorial objects, we usually look for an integer TDI system.

As already shown for Systems (2.1) and (2.2), the removal of a redundant constraint can disrupt the TDIness of a system. This justifies the interest on minimal integer TDI systems.

2.1.5 Total Dual Integrality and Min-max Relations

Total dual integrality is strongly related to min-max relations. Many combinatorial min-max relations stem from the fact that certain linear programs admit integer optima. When this happens, we can use Theorem 1.9 to deduce a min-max relation. Indeed, we can always set up the following chain of inequalities:

$$\begin{aligned} \max\{c^\top x : Ax \leq b, x \in \mathbb{Z}^n\} &\leq^{(1)} \max\{c^\top x : Ax \leq b, x \in \mathbb{R}^n\} =^{(2)} \\ = \min\{yb : yA = c, \mathbb{R}^m_+\} &\leq^{(3)} \min\{yb : yA = c, y \in \mathbb{Z}^m_+\}. \end{aligned} \tag{2.5}$$

Inequalities (2.5) give a bound on the value of the optimal solutions to a combinatorial optimization problem, that is the optimal value of another optimization problem. Equation (2) is consequence of the Strong Duality Theorem 1.9. When the system $Ax \leq b$ describes an integer polyhedron, inequality (1) is attained to equality, hence strengthening the bound. Moreover, inequality (3) is an equality whenever $Ax \leq b$ is TDI. To conclude, if b is integer and $Ax \leq b$ is TDI, all the elements of (2.5) are equal by Theorem 2.1.

Moreover, we can insert a term in Chain (2.5):

$$\min\{yb : yA = c, \mathbb{R}^m_+\} \leq \min\{yb : yA = c, 2y \in \mathbb{Z}^m_+\} \leq \min\{yb : yA = c, y \in \mathbb{Z}^m_+\}. \quad (2.6)$$

We can combine Theorem 2.9 and Theorem 2.1 to see that, if the last inequality of (2.6) is respected with equality, then all terms of (2.5) are equal.

Among the TDI systems, those having integer coefficients play a main role, since they give min-max relations between combinatorial objects. However, the set $P = \{x \in \mathbb{Z}^n : Ax \leq b\}$ is described by different systems with integer coefficients, each of which determines a different dual problem. Hence, the min-max relation we can deduce by TDIness is not unique. Indeed, if $Ax \leq b$ is a TDI system describing P, there could exist some redundant constraints whose removal does not destroys the TDIness of the system. Since every constraint of the primal problem is associated with a variable of the dual, when we remove these "useless" constraints we obtain a min-max relation between smaller sets of objects. This motivates the interest for minimal integer TDI systems.

2.1.6 The Schrijver System

A system $Ax \leq b$ is *minimally TDI* if removing any redundant constraint leads to a non TDI system. Equivalently, $Ax \leq b$ is minimally TDI if and only if each of its constraints determines a supporting hyperplane of $\{x : Ax \leq b\}$ and is not a nonnegative integer combination of other constraints in $Ax \leq b$.

Theorem 2.17 ([128])**.** *Let P be a full-dimensional polyhedron. There exists a unique minimal TDI system $Ax \leq b$, with A integer describing P. Moreover P is integer if and only if b can be chosen integer.*

The minimal TDI system having integer coefficients was named [39] the *Schrijver system* of P. As mentioned by Pulleyblank [124], it is not uncommon that the minimal integral system and the Schrijver system of a polyhedron coincide. This is the case of the matching polytope and matroid polyhedra. However, this is not true in general, as shown by Cook [34] and Pulleyblank [124] for the b-matching polyhedron, and by Sebő [137] for the t-join polyhedron.

2.1.7 Literature Analysis

Totally dual integral systems were introduced by Edmonds and Giles in the late ’70s [63], even if we can recognize early stage of this concept in the work of Edmonds [62], Fulkerson [76], and Hoffman [98]. TDIness, characterizing integer polyhedra [82, 128] and min-max combinatorial relations [129], rapidly spread as a fundamental tool of mathematical programming.

A remarkable result is the work of Cunningham and Marsh [44], where they provide a TDI system describing the matching polytope. Schrijver and Seymour [136] and Schrijver [130] proposed alternative proofs for this result. For more references on TDIness we refer to the survey of Pulleyblank [125], as well as the books of Schrijver [134, 135].

Theorem 2.17 in [128] brought interest also on minimal integer TDI systems [34, 39, 124, 137]. Total dual integrality has been exploited also for generalizing Caratheodory’s Theorem [18] to integer vectors [37, 138].

Even if TDIness is nowadays considered a classical concept, it is still a powerful tool and active area of research [20, 22, 23, 40, 100, 111, 154].

Total dual integrality has recently been extended to different contexts than the linear programming. In [144] De Carli Silva and Tunçel, propose a generalization of total dual integrality to semidefinite programming, and restate some classical theorems for this definition. In [57] and [146] total dual integrality is extended to linear complementarity problems.

2.2 Box-Total Dual Integrality

A stronger concept than total dual integrality is that of box-total dual integrality. A system $Ax \leq b$ is *box-totally dual integral* (*box-TDI*) if the system $Ax \leq b, \ell \leq x \leq u$ is TDI for all rational vectors with possibly infinite components, ℓ and u. We call *box constraints* the constraints $\ell \leq x \leq u$. There exists no result analogous to Theorem 2.2 for box-TDIness, this motivates the following definition: a polyhedron that can be described by a box-TDI system is called a *box-TDI polyhedron.* Example 2.5 presents a non box-TDI polyhedron. The definition of box-TDI polyhedra is consistent with the following result, due to Cook [36]:

Theorem 2.18. *Every TDI system describing a box-TDI polyhedron is box-TDI.*

Theorem 2.18 highlights the first fundamental difference between TDIness and box-TDIness: while TDIness is by all means a property of systems, box-TDIness is essentially a polyhedral property. As TDIness was related to integrality, so is box-TDIness: we recall that a polyhedron is box-integer if it remains integer when it is intersected with a set of integer box constraints. The following result is a consequence of Theorem 2.2.

Proposition 2.19. *Let P be an integer box-TDI polyhedron. Then P is box-integer.*

Example 2.5: A simple non-box-TDI polyhedron.

Let us consider the polyhedron $P = \{x \in \mathbb{R}^2 : x_1 - 2x_2 = 0\}$. The following system describes P and is TDI.

$$x_1 - 2x_2 = 0. \tag{2.7}$$

On the other hand the polyhedron $P' = P \cap \{x : 0 \leq x_1 \leq 1\}$ is not an integer polyhedron since it has $(1, \frac{1}{2})$ as a vertex. This polyhedron is described by:

$$\begin{cases} x_1 - 2x_2 = 0, \\ x \geq \mathbf{0}, \\ x \leq \mathbf{1}. \end{cases} \tag{2.8}$$

System (2.8) is the intersection of System (2.7) with the integer box-constraint $\mathbf{0} \leq x \leq \mathbf{1}$. System (2.8) is not TDI, because P' is not integer and Theorem 2.1. Thus, System (2.7) is a TDI system describing P that is not box-TDI, hence P is not box-TDI by Proposition 2.19.

2.2.1 Recognizing Box-Total Dual Integrality

In the following we provide some conditions for systems and polyhedra to be box-TDI.

Box property, normal cones, and tangent cones. We can extend the concept of Hilbert basis in order to deal with box-TDIness. We say that a set S of vectors in $\mathbb{R}^n$ is a *box-Hilbert basis* if $S \cup_{i \in I} (\pm)\xi^i$ is a Hilbert basis for all $I \subseteq \{1, \dots, n\}$, where we recall that ξ^i is the i^{th} element of the canonical basis of $\mathbb{R}^n$. The definition of box-Hilbert basis is due to Cook [36], who characterized box-TDI systems in terms of box-Hilbert basis, in a way pretty similar to Theorem 2.5.

Theorem 2.20. *A linear system $Ax \leq b$ is box-TDI if and only if for each face F of $\{x : Ax \leq b\}$ the set of constraints tight for F in $Ax \leq b$ is a box-Hilbert basis.*

Despite its structure resembling that of Theorem 2.5, Theorem 2.20 is sparsely used, this is possibly due to the intrinsic complexity of the definition of box-Hilbert basis. In the same paper, Cook proposed the following characterization:

Theorem 2.21. *A polyhedron P is box-TDI if and only if for each rational vector c, there exists an integral vector $\bar{c}$ such that $\lfloor c \rfloor \leq \bar{c} \leq \lceil c \rceil$ and such that each optimal solution of* $\max\{cx : x \in P\}$ *is also an optimal solution of* $\max\{\bar{c}x : x \in P\}$.

This result, as well as the characterization we present below, is strictly related with the concept of tangent and normal cones. Given a polyhedron $P = \{x : Ax \leq b\}$ and a nonempty face $F = \{x \in P : A_F x = b_F\}$, for $A_F x \leq b_F$ subset of rows of $Ax \leq b$, we define the *tangent cone* associated with F the cone $C_F = \{x : A_F x \leq b_F\}$. When F is a minimal face of P we say that C_F is a *minimal tangent cone*. Similarly, the *normal cone* for F is the cone generated by the rows of A_F.

The formulation of Theorem 2.21 we presented above is the one proposed by Schrijver [134]; the original formulation was given in [36] in terms of box property: a cone C has the *box property* if for all rational vector v in C there exists an integer vector w in C, $\lfloor v \rfloor \leq w \leq \lceil v \rceil$. Then, Theorem 2.21 can be restated as "*A polyhedron P is box-TDI if and only if, for each of its faces F, the cone generated by the tight constraints for F has the box property*".

In a recent paper [28], Chervet, Grappe, and Robert gave a result similar to Theorem 2.21 based on tangent cones.

Theorem 2.22. *A polyhedron P is box-TDI if and only if every minimal tangent cone of P is box-TDI.*

Theorems 2.21 and 2.22 are in some sense complementary. Indeed, the tangent cone and the normal cone of a face are polar one of the other; as we will see later in this section, polarity behaves nicely with respect to box-TDIness of cones.

Box-TDIness and Matrices. As we have already noted, totally unimodular matrices play a central role in system with high requirements of integrality. Thus, it is unsurprising that totally unimodular matrices are strongly related with box-TDI systems. We restate Theorem 1.15 in terms of box-total dual integrality.

Theorem 2.23 ([99])**.** *A rational matrix A is totally unimodular if and only if $Ax \leq b$ is a box-TDI system for every rational vector b.*

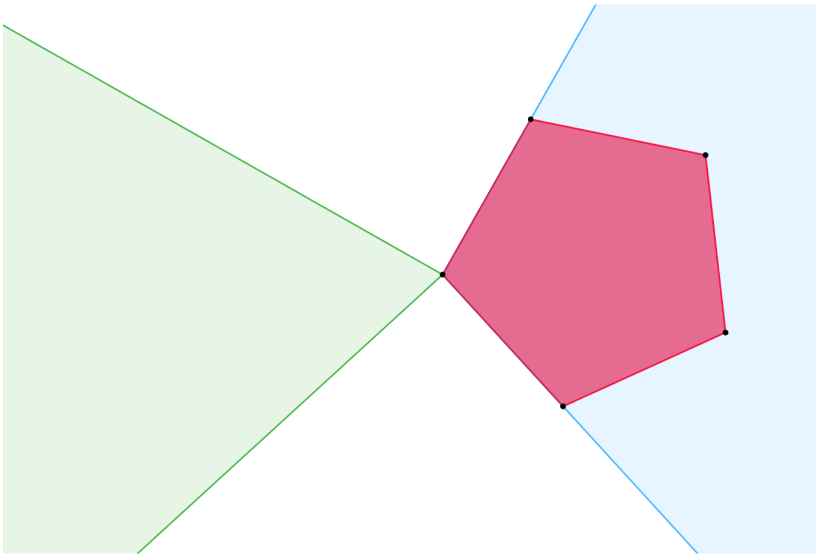


Figure 2.2: The **normal cone** and the **(minimal) tangent cone** for a vertex of a polyhedron.

The vast majority of box-TDI systems found until the 2000's are related with totally unimodular matrices, these results were summarized by Schrijver with the following result.

Theorem 2.24 (Theorem 5.35 of [135])**.** *Let $Ax \leq b$ be a system of linear inequalities, with A an $m \times n$ matrix. Suppose that for each $c \in \mathbb{R}^n$, $\max\{c^\top x : Ax \leq b\}$ has (if finite) an optimum dual solution $y \in \mathbb{R}^m_+$ such that the rows of A corresponding to positive components of y form a totally unimodular submatrix of A. Then $Ax \leq b$ is box-TDI.*

This result includes a wide spectrum of sufficient conditions for total dual integrality. Indeed, for various classical min–max theorems (*e.g.* the Lucchesi–Younger Theorem [110]) the proofs are divided in the following steps: first it is shown that the active constraints in the optimum of the problem can be chosen to be "nice"; second, these nice constraints form a totally unimodular matrix. Here "nice" can be declined as "cross-free", "laminar", and similar cases.

Whenever we look for box-TDI systems, Theorem 2.18 allows us to break the task in two distinct passages: the recognition of box-TDIness of the polyhedron, and the search for a TDI system describing such polyhedron. Recently, Chervet, Grappe, and Robert [28], characterized box-TDI polyhedra in terms of principal box-integrality and equimodularity of constraint matrices.

A polyhedron P is *principally box-integer* if kP is box-integer for all $k \in \mathbb{Z}_+$ such that kP is integer.

Theorem 2.25. *Let P be a rational polyhedron. Then the following are equivalent.*

i. P is a box-TDI polyhedron.

ii. P is principally box-integer.

iii. Every face of P has an equimodular face-defining matrix.

iv. Every face defining matrix of P is equimodular.

v. Every face of P has a totally unimodular face-defining matrix.

Theorem 2.25 has the peculiarity of giving necessary and sufficient conditions for box-TDIness that are independent from the description of the polyhedron itself. Indeed, face-defining matrices my be completely unrelated with the known descriptions of the polyhedron. Thanks to this result, we can prove the box-TDIness of a system by first showing the box-TDIness of the polyhedron, and then by proving that the system is TDI.

From Theorem 2.25 we can deduce the following result originally proved by Edmonds and Giles [64].

Corollary 2.26. *Every box-TDI polyhedron can be described by a $0/\pm 1$-matrix.*

In Example 2.6, we use Theorem 2.25, to show that there exist some box-integer polyhedra that are not box-TDI.

Example 2.6: A box-integer polyhedron that is not box-TDI.

As we saw in Proposition 2.19, every integer box-TDI polyhedron is box-integer. The contrary does not hold: an integer 0/1-polytope is box-integer, but this is not forcibly true for all its dilations. The following counterexample of a box-integer polyhedron that is not box-TDI is taken from [28]. Consider the following set of points of $\mathbb{R}^5$: $\{\mathbf{0}, (1,1,0,0,0), (1,0,1,0,0), (1,0,0,1,0), (1,1,1,1,1)\}$, and let P be the convex hull of these points. P is a 0/1 polytope, hence it is box-integer. However, it can be checked that $(2,1,1,1,\frac{1}{2})$ is a fractional vertex of $2P \cap \{x_2 = x_3 = x_4 = 1\}$.

We have a condition similar to Theorem 2.23 for totally equimodular matrices:

Theorem 2.27 ([28])**.** *A rational matrix A is totally equimodular if and only if $\{x : Ax \leq b\}$ is a box-TDI polyhedron for all rational b.*

Box-integer Cones. The characterizations valid for rational polyhedra can be reinforced when treating polyhedral cones.

Observation 2.28 ([28]). *Let $C = \{x : Ax \geq \mathbf{0}\}$ be a cone. Then C is box-TDI if and only if it is box-integer.*

Moreover, a cone is box-TDI if and only if its polar cone is.

Proposition 2.29 ([28]). *Let $C = \{x : Ax \geq \mathbf{0}\}$ be a cone, and let C^o be its polar. Then C is box-TDI if and only if C^o is.*

The ESP property. The ESP property is a combinatorial property associated with box-TDIness. It was introduced by Chen, Chen, and Zang in [21], and it is involved in many recent results on box-TDIness.

Let $Ax \leq b, x \geq \mathbf{0}$ be a rational system, and let R and S be respectively the set of indices of the rows and of the columns of A. For any collection Λ of elements of R, and any element s of S, we denote by $b(\Lambda) = \sum_{r\in\Lambda} b_r$ and $d_\Lambda(s) = \sum_{r\in\Lambda} A_{r,s}$. An *equitable subpartition* of Λ consists of two collections Λ_1 and Λ_2 of elements of R such that:

1. $b(\Lambda_1) + b(\Lambda_2) \leq b(\Lambda)$;
2. $d_{\Lambda_1\cup\Lambda_2}(s) \geq d_\Lambda(s)$ for all $s \in S$;
3. $d_{\Lambda_i}(s) \geq \left\lfloor \frac{d_\Lambda(s)}{2} \right\rfloor$ for all $s \in S$, for $i = 1, 2$.

We say that $Ax \leq b, x \geq \mathbf{0}$ is *equitably subpartitionable* (*ESP*) if every collection of R admits an equitable subpartition.

The link between ESP systems and box TDIness is the following:

Theorem 2.30 ([54]). *Every ESP system $Ax \leq b, x \geq \mathbf{0}$, with A integer, is box-TDI.*

One of the great improvements of Theorem 2.30 with respect to some of the previous results is that the ESP property is essentially combinatorial, on the contrary of Theorem 2.18, that often leads us to use at the same time combinatorial and polyhedral arguments.

Polymatroids. Theorem 2.23 is one of the most used tools to prove box-TDIness of polyhedra: as pointed out by Chen, Hu, and Zang [26]: "*Almost all known box-TDI systems can be verified via totally unimodular matrices* [. . .] *or the ESP property*". In this paragraph we deal with polymatroids and some related classes of polyhedra whose box-total dual integrality can be proved via total unimodular matrices.

We recall that a polymatroid is a the polyhedron defined by $\{x \in \mathbb{R}^U : x \geq \mathbf{0}, x(S) \leq f(S) \text{ for all } S \subseteq U\}$ for a finite set U and submodular set function f. The following result is due to Edmonds [62]:

Theorem 2.31. *For a finite set U and a submodular set function f the system $\{x(S) \leq f(S) \text{ for all } S \subseteq U\}$ is box-TDI.*

This result has been generalized to various classes of collections, less rich than the power set. Schrijver [135, section 44-49] gave a deep analysis of the results in this topic.

2.2.2 Operations on Box-TDI Polyhedra

In this section we will consider what are the effects of some simple operations on box-TDI polyhedra.

Faces, dominants, and variables. One of the interesting properties of box-TDIness is its stability with respect to the passage to faces and dominants.

Theorem 2.32. *All faces of a box-TDI polyhedron are box-TDI.*

Theorem 2.33 ([36])**.** *The dominant of a box-TDI polyhedron is box-TDI.*

Another operation preserving box-TDIness is the copy of a column.

Theorem 2.34 ([64], cf. [36])**.** *If $Ax \leq b$ is box-TDI, then $ax_0 + Ax \leq b$ is box-TDI again, where a is a column of A and x_0 is a new variable.*

Moreover, this is valid also under the existence of nonnegativity inequalities.

Lemma 2.35 ([25])**.** *Let $Ax \geq b, x \geq \mathbf{0}$ be a (box-)TDI system, and let B be obtained from A by adding a column identical to an existing one. Then $Bx \geq b, x \geq \mathbf{0}$ is (box-)TDI.*

Projections. The projection onto subsets of variables preserve the box-TDIness of a polyhedron, this result can be found in Schrijver [134, Section 22.5]:

Proposition 2.36. *Let $P = \{x \in \mathbb{R}^n, y \in \mathbb{R} : A(x, y) \leq b\}$ be a box-TDI polyhedron. Then, $P' = \{x \in \mathbb{R}^n : \exists y : (x, y) \in P\}$ is a box-TDI polyhedron.*

Translation. Integrality of polyhedra is not preserved under rational translations. On the contrary, box-TDIness is preserved.

Observation 2.37. *If a polyhedron is box-TDI, then so are all its rational translations.*

Intersection. Box-Total dual integrality is not preserved under intersection, as proved by Example 2.7.

Example 2.7: Intersection of polyhedra does not preserve box-TDIness.

Consider the following polyhedra:

$$P = \{x \in \mathbb{R}^4 : \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} x \geq \mathbf{0}\} \text{ and } Q = \{x \in \mathbb{R}^4 : \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} x \geq \mathbf{0}\}$$

Both the constraint matrices of P and Q are totally unimodular, hence P and Q are box-TDI polyhedra. On the contrary,

$$P \cap Q = \{x \in \mathbb{R}^4 : \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} x \geq \mathbf{0}\},$$

Let A be the constraint matrix of $P \cap Q$. Since there exists at least two points respecting all the constraints to equality, A is face-defining for $P \cap Q$, and it is not equimodular. Therefore $P \cap Q$ is not a box-TDI polyhedron.

However there exists a notable example of class of box-TDI polyhedra closed under intersection: the class of polymatroids. The following result is due to Edmonds [62].

Theorem 2.38. *The intersection of any two (extended) polymatroid is box-TDI.*

Dilations. Uniform dilations preserve box-TDIness, as we can deduce also from Theorem 2.25. Moreover, if $Ax \leq b$ is a box-TDI system, then $Ax \leq kb$ is box-TDI for all rational $k > 0$. On the other hand, dilations generally do not preserve box-TDIness, even when they preserve integrality.

2.2.3 History and Notable Examples of Box-TDIness

The concept of box-total dual integrality was introduced by Edmonds and Giles in the late '70s [63] and received a peak of interest in the following years as testified, among the others, by the works of Cook [36] and Schrijver [129]. We refer to classical surveys such as: Edmonds and Giles [64], Pulleyblank [123, 125], and Schrijver [132, 133]. Later, Frank and Király [74] analyzed polyhedral results related with submodular functions and polymatroids. Schrijver presented an exhaustive analysis of the topic in its books [134, 135], that inspired the vast majority of the material presented in this Chapter.

While total dual integrality received a lot of attention, box-total dual integrality appears to be somehow less studied. The reasons of this difference lie in the difficulty of providing box-TDI systems for polyhedra. Historically, box-TDI polyhedra and systems were strictly connected with totally unimodular matrices. The lack of alternative efficient tools for treating box-TDIness has perhaps demotivated the research of box-TDI systems until recent times, when some theoretical results shed new interest on box-TDIness. As an example, in 2009 Chen, Ding, and Zang [25] still relied on Theorem 2.21 to show a box-TDI system describing the 2-edge-connected spanning subgraph polyhedron for series-parallel graphs.

Starting from an idea of Ding and Zang [55], Chen, Chen, and Zang [21] introduced the ESP property. Using this result, Ding, Tan, and Zang [54] characterized the graphs for which the TDI system of Cunningam and Marsh [44] describing the matching polytope is actually box-TDI. Moreover, Chen, Ding, and Zang [24] exploited the ESP property to characterize box-Mengerian matroid ports. Chen, Hu, and Zang [26] summarized the results achieved by the scientific community in the decade between 2003 (when Schrijver published his monograph [135]) and 2013, focusing on the results related to hypergraphs. Cornaz, Grappe, and Lacroix [41] provided several box-TDI systems in series-parallel graphs. New insights on box-TDI polyhedra, along with new geometric characterizations, have been recently announced by Chervet, Grappe, and Robert [28]. In Example 2.8, we show a remarkable case of a topic of research related with box-TDI polyhedra.

2.3 Hardness of Recognising (Box-)TDIness

TDI systems and Hilbert bases. The recognition of TDI systems is a topic that interested many academics. Let $Ax \leq b$ be a rational system with A integer, how complex is to determine whether it is TDI or box-TDI?

Generally one can see directly [134, Section 22.9] that the problems "Does $Ax \leq b$ describe an integer polyhedron?", "Is $Ax \leq b$ TDI?", and "Is $Ax \leq b$ box-TDI?" belong to $\mathcal{C}o\text{-}\mathcal{NP}$. Cook [36] showed that the problem "Does $Ax \leq b$ describe a box-TDI polyhedron?" is in $\mathcal{C}o\text{-}\mathcal{NP}$ too. Papadimitriou and Yannakakis [121] proved that decide whether a system describes an integer polyhedron is a $\mathcal{C}o\text{-}\mathcal{NP}$-complete problem. The recognition of TDI and box-TDI systems has been proved to be $\mathcal{C}o\text{-}\mathcal{NP}$-complete by Ding, Feng, and Zang [53]. Pap [120] proved that the result holds even under the assumptions that the system has only binary coefficients, and that the defined polyhedron is a cone.

Example 2.8: Box-perfect graphs.

A graph $G = (V, E)$ is said to be perfect if, for all induced subgraphs H, the size of a maximum independent set of H – denoted by $\alpha(H)$ – equals the minimum number of cliques which cover all the vertices of H. We will denote the latter by $\theta(H)$. We can express this condition by linear programming duality.

$$\max \sum_{v \in V} c_v x_v \qquad (\mathcal{P})$$

s.t.

$$\begin{cases} \sum_{v \in K} x_v \leq 1 & \forall \text{ clique } K \text{ of } G \\ x_v \geq \mathbf{0} \end{cases} \tag{2.9}$$

$$\min \sum_{K \text{ clique of } G} y_K \qquad (\mathcal{D})$$

s.t.

$$\begin{cases} \sum_{K : v \in K} y_K \geq c_v & \forall v \in V \\ y_v \geq \mathbf{0} \end{cases} \tag{2.10}$$

Problems $\mathcal{P}$ and $\mathcal{D}$ are one the dual of the other. Integer solutions to $\mathcal{P}$ for $c \in \{0, 1\}$ are independent sets of the subgraph induced by $V' = \{v \in V : c_v = 1\}$. Similarly an integer solution of $\mathcal{D}$ is a clique cover of the subgraph induced by V'. Therefore, by linear programming duality, $\alpha(H) \leq \theta(H)$ for all induced subgraphs H of G. Cameron and Edmonds [15] characterized perfection of G in terms of TDIness of (2.9).

Proposition 2.39. *A graph G is perfect if and only if System* (2.9) *is TDI.*

Motivated by Proposition 2.39, Cameron and Edmonds [13, 15] introduced the concept of box-perfect graphs. A graph G is *box-perfect* if System (2.9) associated with G is box-TDI. The lack of a combinatorial characterization of box-perfect graphs alimented a fruitful stream of research. In [15], Cameron and Edmonds prove that p-Comparability graphs and Cocomparatibility graphs are box-perfect. In [14], Cameron shows the effects of some graph operations on box-perfection, including the fact that box-perfection is not preserved under the passage to the complement. Moreover, she proves that totally unimodular graphs – that is, the class of graphs having a totally unimodular clique matrix – are box-perfect. Ding, Zang, and Zhao [56] introduced new classes of box-perfect graphs.

If we assume the rank of A as fixed, Cook, Lovasz, and Schrijver [38] proved that we can say whether $Ax \leq b$ is TDI in polynomial time; their work extended previous results of Chandrasekaran and Shirali [19] and Giles and Orin [81]. Starting from a characterization of Hilbert bases of Sebő [138], Dueck, Hoşten, and Sturmfelsn [58] proved that the same problem belongs to $\mathcal{P}$ in case of fixed codimension.

By Corollary 2.7, we deduce equivalent results for Hilbert bases. Complementary results on Hilbert bases of cones can be found in [95, 96, 97]. It is an open question to decide if "Is $Ax \leq \mathbf{0}$ box-TDI?" and "Is $\{x : Ax \leq \mathbf{0}\}$ a box-TDI cone?" are solvable in polynomial time. Chervet, Grappe, and Robert [28], showed that these problems belong to $\mathcal{Co}$-$\mathcal{NP}$.

Matrices. By the characterization of equimodular matrices in terms of totally unimodular matrices (see Theorem 1.24), it turns out that we can tell if a matrix is equimodular in polynomial time. To decide whether a matrix is totally equimodular is a $\mathcal{Co}$-$\mathcal{NP}$ problem, but it is not known nether if it is polynomial nor if it is $\mathcal{Co}$-$\mathcal{NP}$-complete.

In Table 2.1, we summarize the result stated in this section.

Problem	**Class**	**Reference**
Does $Ax \leq b$ describe an integer polyhedron?	$\mathcal{Co}$-$\mathcal{NP}$-complete	[121]
Is $Ax \leq b$ TDI?	$\mathcal{Co}$-$\mathcal{NP}$-complete	[53]
Is $Ax \leq b$ box-TDI?	$\mathcal{Co}$-$\mathcal{NP}$-complete	[53]
Is $\{x : Ax \leq b\}$ a box-TDI polyhedron?	$\mathcal{Co}$-$\mathcal{NP}$	[36]
Is $Ax \leq \mathbf{0}$ TDI?	$\mathcal{Co}$-$\mathcal{NP}$-complete	[120]
Is $\{x : Ax \leq \mathbf{0}\}$ a box-TDI cone?	$\mathcal{Co}$-$\mathcal{NP}$	[28]
Is A equimodular?	$\mathcal{P}$	[28]
Is A totally equimodular?	$\mathcal{Co}$-$\mathcal{NP}$	[28]

Table 2.1: Hardness of problems related with TDIness

Chapter 3

The Schrijver System of the Flow Cone

3.1 Flows, Cuts, and Related Polyhedra

In this chapter we are interested in TDI, box-TDI, and Schrijver systems describing the flow cone of series-parallel graphs. Given a graph $G = (V, E)$, we recall that a *flow* of G is a couple (C, e) with C a circuit of G and e an edge of C. The *flow cone* of G is the cone generated by the flows of G and the unit vectors ξ_e of $\mathbb{R}^E$.

Flows and cuts are some of the most known and studied objects of combinatorial optimization. Indeed, different famous min max relations involve or are expressible as linear programming duality between (multi)cuts and (multi)flows. When G is series-parallel, different systems describing polyhedra related with cuts and flows, are known to be TDI (see *e.g.* [135, Corollary 29.9c] and [40]), moreover, Cornaz, Grappe, and Lacroix [41] recently proved that various of these systems are in fact box-TDI when G is series-parallel.

When G has no K_5-minor, the flow cone of G is the polar of the cut cone and is described by $x(C) \geq 0$, for all cuts C of G [139]. Chervet, Grappe, and Robert [28] proved that the flow cone is a box-TDI polyhedron if and only if the graph is series-parallel. Moreover, they provided the following box-TDI system:

$$\frac{1}{2}x(B) \geq 0 \quad \text{for all bonds } B \text{ of } G. \tag{3.1}$$

Quoting them, they "*leave open the question of finding a box-TDI system with integer coefficients, which exists by [134, Theorem 22.6(i)] and [36, Corollary 2.5].*"

Contribution. The goal of this chapter is to answer the question of [28] mentioned above.

We first prove that

$$x(M) \geq 0 \quad \text{for all multicuts } M \text{ of } G, \tag{3.2}$$

is a TDI system describing the flow cone if and only if the graph is series-parallel. As the flow cone is a box-TDI polyhedron for such graphs, this implies that System (3.2) is a box-TDI system if and only if the graph is series-parallel. We then refine this result by providing the corresponding Schrijver system, which is composed of the so-called chordal multicuts—see Corollary 3.6. This completely answers the question of [28].

We conclude the chapter with some results and insights for future direction of research.

3.2 An Integer TDI System for the Flow Cone for Series-parallel Graphs

We aim to find an integer TDI system describing $\{x \in \mathbb{R}^E : x(B) \geq 0$ for all bonds B of $G\}$ when G is series-parallel. This is equivalent to find an integer Hilbert basis for the cone generated by the cuts of G. The set of cuts of K_3 is not a Hilbert basis, however, Chervet, Grappe and Robert [28], proved that the set of $\frac{1}{2}$-incidence vectors of bonds of series-parallel graphs are a Hilbert basis. A natural candidate solution to our problem is the set of multicuts of G. Indeed, a multicut $\delta(V_1, \ldots, V_d)$ is the $\frac{1}{2}$-sum of the bonds $\frac{1}{2}\delta(V_1) + \cdots + \frac{1}{2}\delta(V_d)$.

3.2.1 A Characterization

The following result characterizes series-parallel graphs as the class of graphs for which the multicuts form a Hilbert basis.

Theorem 3.1. *The multicuts of a graph form a Hilbert basis if and only if the graph is series-parallel.*

Proof. First, let us show that the incidence vectors of the multicuts of a non series-parallel graph do not form a Hilbert basis. Suppose that $G = (V, E)$ has K_4 as a minor. Then, V can be partitioned into four sets $\{V_1, \ldots, V_4\}$ such that V_i induces a connected subgraph and at least one edge connects each pair V_i, V_j for $i, j = 1, \ldots, 4$ $i \neq j$. Let $M = \delta(V_1, V_2, V_3, V_4)$. We subdivide M into $E_1, \ldots, E_6$ as in Figure 3.1.

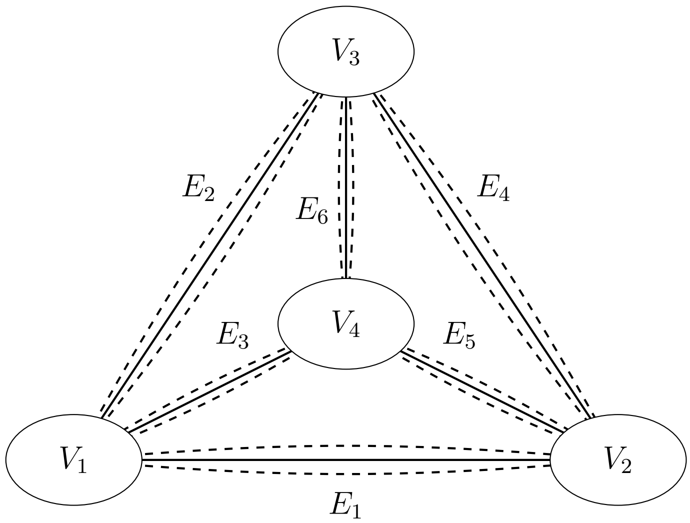


Figure 3.1: Here, edges represent sets of edges of G having endpoints in distinct V_i's.

Let us define $w \in \mathbb{Z}^E$ as follows:

$$w_e = \begin{cases} 2 & \text{if } e \in E_1, \\ 1 & \text{if } e \in E_2, \dots, E_6, \\ 0 & \text{otherwise.} \end{cases}$$

Since $w = \frac{1}{2}\delta(V_1) + \frac{1}{2}\delta(V_2) + \frac{1}{2}\delta(V_1 \cup V_3) + \frac{1}{2}\delta(V_1 \cup V_4)$, it belongs to the cut cone of G. So it suffices now to prove that K_4^+, the graph obtained from K_4 by duplicating one edge, is not the sum of multicuts of K_4. Observe that multicuts of K_4 have 3, 4, 5, or 6 edges. Since K_4^+ has 7 edges, it should be the sum of two multicuts, with respectively 3 and 4 edges. That is, K_4^+ is the sum of a $3-$star and a cycle of length 4 of K_4, a contradiction.

For the other direction, remark that each multicut of a series-parallel graph is the disjoint union of multicuts of its 2-connected components. Since they belong to disjoint spaces, if the set of multicuts of each 2-connected component forms a Hilbert basis, then so does their union. Hence, it is enough to prove that the multicuts of a 2-connected series-parallel graph form a Hilbert basis. From now on, assume the graph to be 2-connected.

We prove the result by induction on the number of edges of G. When G has a single edge, that is $G = (\{u, v\}, \{e\})$, the only nonempty multicut is $\{e\}$, and its incidence vector forms a Hilbert basis.

Otherwise, by construction of 2-connected series-parallel graphs, G is obtained either by adding a parallel edge to or by subdividing an edge of a 2-connected series-parallel graph $H = (W, F)$. By the induction hypothesis, $\mathcal{M}_H$ is a Hilbert basis.

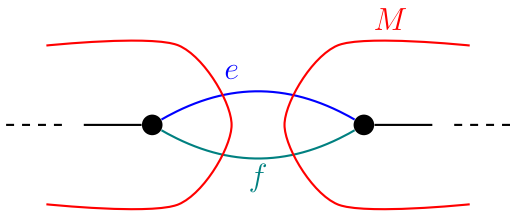


Figure 3.2: Multicuts and parallel edges. The removal of f from the graph does not change the structure of the multicut M containing e.

Suppose first that G is obtained from H by adding an edge f parallel to an edge e of F. A subset of edges M of H containing (respectively not containing) e is a multicut if and only if $M \cup f$ (respectively M) is a multicut of G (see Figure 3.2). Thus, the incidence vector of each multicut of G is obtained by copying the component associated with e in the component of f. Since the incidence vectors of the multicuts of H are a Hilbert basis, so are the incidence vectors of the multicuts of G.

Suppose now that G is obtained from H by subdividing an edge $e \in F$. We denote by u the new vertex and by f and g the edges adjacent to it. A multicut M of G can be expressed as the half-sum of the bonds of G. Since System (3.1) is TDI in series-parallel graphs [28, end of Section 6.4], the set of vectors $\{\frac{1}{2}\chi^B : B \in \mathcal{B}_G\}$ is a Hilbert basis.

Let a be an integer vector in the cut cone. We claim that a is an integer conic combination of the incidence vectors of the multicuts of G. There exist $\lambda_B \in \frac{1}{2}\mathbb{Z}_+$ for all $B \in \mathcal{B}_G$ such that $a = \sum_{B \in \mathcal{B}_G} \lambda_B \chi^B$. The vector a is an integer combination of multicuts of G if and only if $a - \lfloor \lambda_{\delta(u)} \rfloor \chi^{\delta(u)}$ is, thus we may assume that $\lambda_{\delta(u)} \in \{0, \frac{1}{2}\}$. Define $b \in \mathbb{Z}^F$ by:

$$b_h = \begin{cases} a_f + a_g - 2\lambda_{\delta(u)} & \text{if } h = e, \\ a_h & \text{otherwise,} \end{cases} \qquad \text{for all } h \in F.$$

Remark that $(B \setminus e) \cup f$ and $(B \setminus e) \cup g$ are bonds of G whenever B is a bond of H containing e (see Figure 3.3). Moreover, a bond B of H which does not contain e is a bond of G.

Since $\delta(u)$ is the unique bond of G containing both f and g, we have:

$$b = \sum_{B \in \mathcal{B}_H : e \in B} (\lambda_{(B \setminus e) \cup f} + \lambda_{(B \setminus e) \cup g}) \chi^B + \sum_{B \in \mathcal{B}_H : e \notin B} \lambda_B \chi^B.$$

Thus, b belongs to the cut cone of H. Moreover, as $\lambda_{\delta(u)}$ is half-integer, b is integer. By the induction hypothesis, $\mathcal{M}_H$ is a Hilbert basis, hence there exist $\mu_M \in \mathbb{Z}_+$ for all $M \in \mathcal{M}_H$ such that $b = \sum_{M \in \mathcal{M}_H} \mu_M \chi^M$. Consider the family $\mathcal{N}$ of multicuts of H where each multicut M of H appears μ_M times.

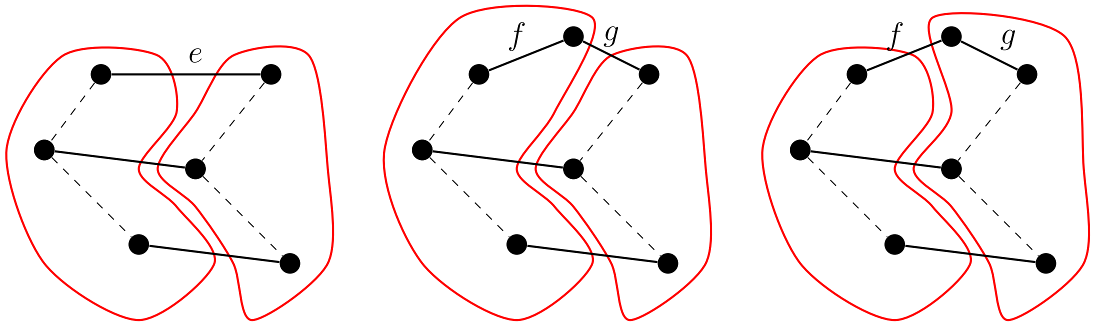


Figure 3.3: A bond containing e after the subdivision of e contains either f or g.

Suppose first that $\lambda_{\delta(u)} = 0$. Then, $a_f + a_g$ multicuts of $\mathcal{N}$ contain e. Let $\mathcal{P}$ be a family of a_f multicuts of $\mathcal{N}$ containing e and $\mathcal{Q} = \{F \in \mathcal{N} : e \in F\} \setminus \mathcal{P}$. Then, we have

$$a = \sum_{M \in \mathcal{N} : e \notin M} \chi^M + \sum_{M \in \mathcal{P}} \chi^{(M \setminus e) \cup f} + \sum_{M \in \mathcal{Q}} \chi^{(M \setminus e) \cup g},$$

hence a is a nonnegative integer combination of multicuts of G. For a graphical visualization of this passage see Exemple 3.1.

Suppose now that $\lambda_{\delta(u)} = \frac{1}{2}$. Then, $a_f + a_g - 1$ multicuts of $\mathcal{N}$ contain e. Let $\mathcal{P}$ be a family of $a_f - 1$ multicuts of $\mathcal{N}$ containing e, let $\mathcal{Q}$ be a family of $a_g - 1$ multicuts in $\{F \in \mathcal{N} : e \in F\} \setminus \mathcal{P}$, and denote by N the unique multicut of $\mathcal{N}$ containing e which is not in $\mathcal{P} \cup \mathcal{Q}$. Then, we have

$$a = \sum_{M \in \mathcal{N} : e \notin M} \chi^M + \sum_{M \in \mathcal{P}} \chi^{(M \setminus e) \cup f} + \sum_{M \in \mathcal{Q}} \chi^{(M \setminus e) \cup g} + \chi^{N \setminus e \cup \{f,g\}}.$$

Hence a is a nonnegative integer combination of multicuts of G. This proves that $\mathcal{M}_G$ is a Hilbert basis. ■

3.2.2 An Integer Box-TDI System for the Flow Cone

Combining the box-TDIness of the flow cone, Corollary 2.7, and Theorem 3.1 yields a box-TDI system for the flow cone of a series-parallel graph with only integer coefficients. This provides a first answer to the question of [28].

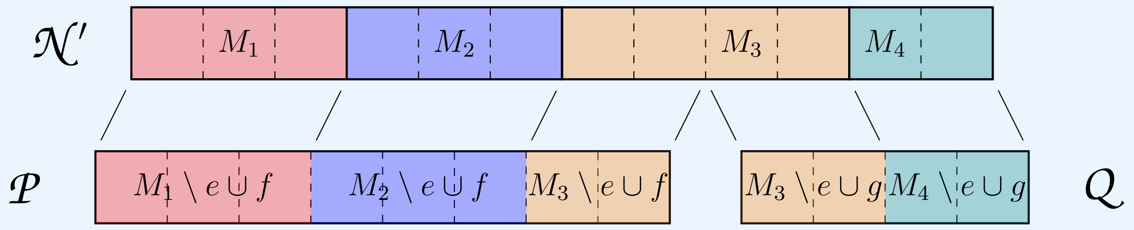
Corollary 3.2. *The following statements are equivalent:*

i. G is a series-parallel graph,

ii. System (3.2) *is TDI,*

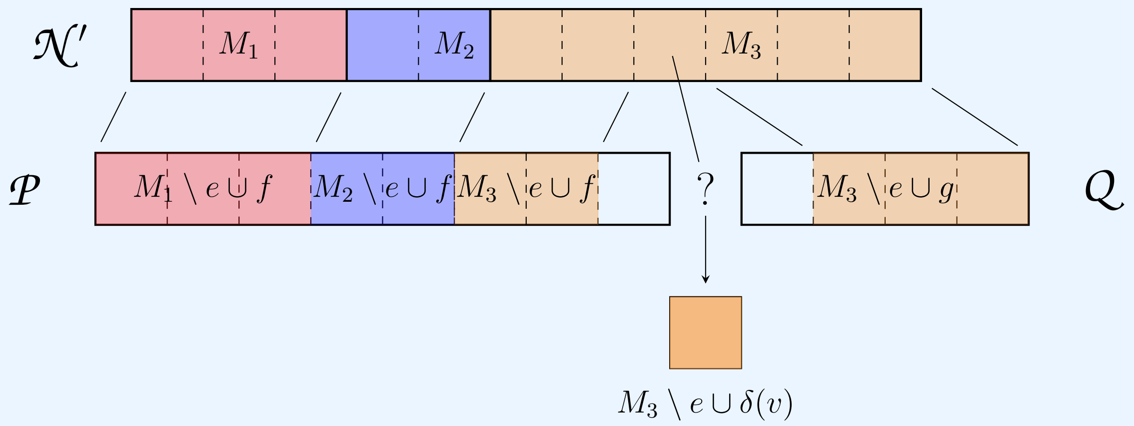
Example 3.1: Intuition for the last passage of Theorem 3.1.

In the last part of the proof, we defined three families of multicuts $\mathcal{N}$, $\mathcal{P}$, and $\mathcal{Q}$. Here we explain the intuitive idea of that passage.
For the sake of brevity, let $\mathcal{N}'$ be the subfamily of multicuts of $\mathcal{N}$ containing e.



When $\lambda_{\delta(v)} = 0$, we have that $\sum_{M \in \mathcal{N}'} \mu_M = b_e = a_f + a_g$. Hence, we can "distribute in an integer way" the multicuts of $\mathcal{N}'$ among two families $\mathcal{P}$ and $\mathcal{Q}$ such that $\sum_{M \in \mathcal{P}} \mu_M = a_f$ and $\sum_{M \in \mathcal{Q}} \mu_M = a_g$.

When $\lambda_{\delta(v)} = \frac{1}{2}$, the procedure is just a bit more complicated.



Now, the multicuts in $\mathcal{N}'$ are not sufficient to cover both the requests a_f and a_g. In this case we "integrally distribute" all the multicuts but one among $\mathcal{P}$ and $\mathcal{Q}$. The only multicut M not assigned will become $M \setminus e \cup \delta(v)$, hence contributing to both a_f and a_g.

iii. System (3.2) *is box-TDI.*

Proof. (*i.*$\Leftrightarrow$*ii.*) This equivalence follows by combining Corollary 2.7 and Theorem 3.1.

(*ii.*$\Leftrightarrow$*iii.*) If G is series-parallel, then System (3.1) is box-TDI [28, end of Section 6.4]. Hence, the flow cone of G is box-TDI. Since a TDI system describing a box-TDI polyhedron is a box-TDI system [36], point *ii.* implies point *iii.*. A box-TDI system being TDI by definition, point *iii.* implies point *ii.*. ■

3.3 The Schrijver System for the Flow Cone

3.3.1 A Minimal Integer Hilbert Basis

Theorem 3.1 provides the set of graphs whose multicuts form a Hilbert basis. The following theorem refines this result by characterizing the multicuts which form the minimal Hilbert basis for this class of graphs.

A multicut is *chordal* when its reduced graph is 2-connected and chordal. Note that bonds are chordal multicuts.

Lemma 3.3. *Let C be a circuit of length at least 4 in a series-parallel graph G. Then, there exists a pair of vertices nonadjacent in $G[C]$ whose removal disconnects G.*

Proof. We can assume that there are two nonadjacent vertices u and v of $G[C]$ such that there exists a path P between u and v that has no internal vertex in C. Indeed, otherwise, removing any two nonadjacent vertices of $G[C]$ would disconnect G.

Let us show that removing u and v disconnects G. Denote by Q and R the two paths of C between u and v. By contradiction, suppose that $G \setminus \{u, v\}$ is connected. Then, there exists a path containing neither u nor v between an internal vertex of R and an internal vertex of either P or Q. Let S be a minimal path of this kind. Then, no internal vertex of S belongs to P, Q, or R, and the subgraph composed of P, Q, R and S is a subdivision of K_4. This contradicts the hypothesis that G is series-parallel. ■

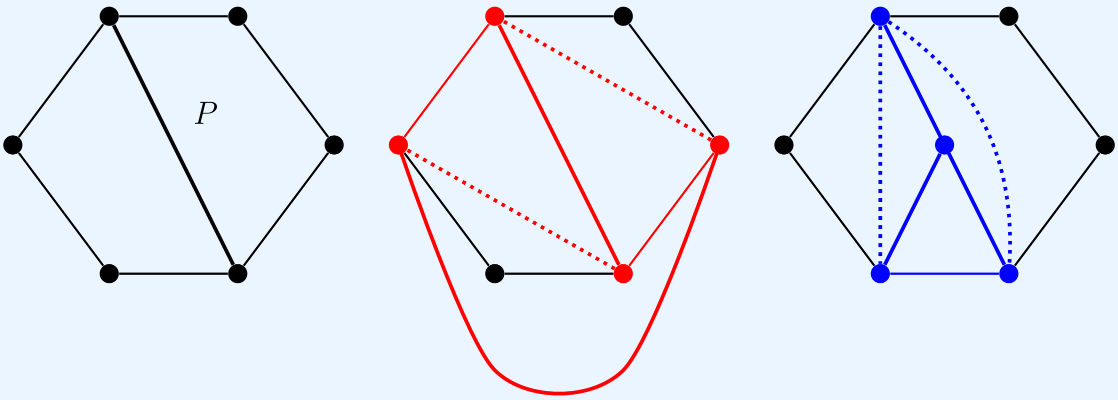
Theorem 3.4. *The chordal multicuts of a series parallel graph form a minimal integer Hilbert basis.*

Proof. Let $G = (V, E)$ be a series-parallel graph. By Theorem 3.1, the multicuts of G form an integer Hilbert basis. Hence, the minimal integer Hilbert basis is composed of the multicuts which are not disjoint union of other multicuts. These multicuts are characterized in the following lemma.

Lemma 3.5. *A multicut of a series-parallel graph G is chordal if and only if it can not be expressed as the disjoint union of other nonempty multicuts.*

Proof. Let M be a multicut of G. Recall that every multicut of G_M is a multicut of G. Beside, since the disjoint union of multicuts is a multicut, a disjoint union of nonempty multicuts is actually the disjoint union of two nonempty multicuts.

Example 3.2: Intuition for Lemma 3.3.



The path P divides the circuit C of a series-parallel graph into two parts. No external path connects vertices belonging to different parts, as otherwise there would exist a K_4-minor in G (highlighted in red). Moreover, no external path connects an internal vertex of P to a vertex of $C \setminus P$, as otherwise there would exist a K_4-minor in G (in blue). Thus, the removal of the extremities of P disconnects the graph.

We first prove that, if G_M is 2-connected and chordal, then M is not the disjoint union of two nonempty multicuts. By contradiction, suppose that G_M is 2-connected and chordal, and $M = M_1 \cup M_2$ where M_1, M_2 are disjoint multicuts of G_M. If C is a circuit of length at most three in G_M, then $C \subseteq M_i$ for some $i = 1, 2$. Indeed, the edges of C are partitioned by M_1 and M_2, and a multicut and a circuit intersect in either none or at least two edges by Observation 1.34.

Since G_M is 2-connected and M_i is nonempty for $i = 1, 2$, there exists at least a circuit containing edges of both M_1 and M_2, as stated by a well-known theorem of Whitney (see *e.g.* [9, Theorem 3.2] and [155]). Let C be such a circuit of smallest length. Then, C has length at least 4, as otherwise it would be contained in one of M_1 and M_2. Since G_M is chordal, there exists a chord c of C. Denote by P_1 and P_2 the two paths of C between the endpoints of c. For $i = 1, 2$, the circuit $P_i \cup \{c\}$ is strictly shorter than C. Since C is the shortest circuit intersecting both M_1 and M_2, we get that $P_i \cup \{c\} \subseteq M_i$ for $i = 1, 2$. But then $c \in M_1 \cap M_2$, a contradiction.

To prove the other direction, first suppose that G_M is not 2-connected. Then, the set of edges of each 2-connected component of G_M is a multicut of G, and M is the disjoint union of these multicuts. Now, suppose that G_M is not chordal, that is, G_M contains a chordless circuit C of length at least 4.

By Lemma 3.3, there exist two vertices u and v of C, nonadjacent in $G[C]$, whose removal disconnects G. Denote by $V_1, \dots, V_k$ the sets of vertices of the connected components of $G \setminus \{u, v\}$, and let $G_i = G[V_i \cup \{u, v\}]$, for $i = 1, \dots, k$. Note that, since C is chordless, $E(G_i) \cap E(G_j) = \emptyset$ for all distinct i and j. Thus, M is the disjoint union of $E(G_1), \dots, E(G_k)$.

Let us prove that $E(G_i)$ is a multicut of G_M, for $i = 1, \dots, k$. Consider a circuit D of G_M. If D is contained in one of the G_i's, then $|D \cap G_j| \neq 1$ for $j = 1, \dots, k$. Otherwise, D is the union of two paths from u to v, these paths being contained in two different G_i's. Without loss of generality, let these paths be $P_1 \in G_1$ and $P_2 \in G_2$. Then, we have $D \cap G_i = P_i$ if $i = 1, 2$, and $\emptyset$ otherwise. Since C has no chord, the shortest path from u to v in each G_i is of length at least two, hence $|P_i| \geq 2$. Therefore $|D \cap G_i| \neq 1$ for $i = 1, \dots, k$.

Therefore, $E(G_i)$ is a multicut of G_M, and hence of G, for $i = 1, \dots, k$. Hence, M is the disjoint union of multicuts of G. ■

Therefore, we conclude that chordal multicuts of a series-parallel graphs form a minimal Hilbert basis. ■

3.3.2 The Schrijver System of the Flow Cone in Series-Parallel Graphs

Corollary 3.2 provides an integer box-TDI description of the flow cone in series-parallel graphs. However, this box-TDI description is not minimal: there are redundant inequalities whose removal does not disrupt the box-TDIness. Here, we provide the minimal integer box-TDI system for this cone. This completely answers the question of [28, end of Section 6.4].

Corollary 3.6. *The Schrijver system for the flow cone of a series-parallel graph G is the following:*

$$x(M) \geq 0 \quad \textit{for all chordal multicuts } M \textit{ of } G. \tag{3.3}$$

Moreover, this system is box-TDI.

Proof. By Corollary 2.7 and Theorem 3.4, System (3.3) is TDI. Since every bond is a chordal multicut, this system describes the flow cone for series parallel graphs. Therefore, by [36, Corollary 2.5] and by the flow cone being box-TDI for series-parallel graphs, System (3.3) is box-TDI. ■

3.3.3 Cone of Conservative Functions

By planar duality, Corollary 3.6 implicitly provides the Schrijver system for the cone of conservative functions [135, Corollary 29.2h] in series-parallel graphs.

If we denote by $\mathcal{H}$ the class of planar dual graphs of chordal, two-connected, series-parallel graphs, we can state the following.

Corollary 3.7. *The Schrijver system for the cone of conservative functions of a series-parallel graph G is the following:*

$$x(H) \geq 0 \quad \textit{for all subgraph } H \textit{ of } G \textit{ belonging to } \mathcal{H}.$$

Moreover, this system is box-TDI.

We do not know an explicit characterization of $\mathcal{H}$, however, we can give some necessary conditions.

Observation 3.8. *Let $G = (V, E) \in \mathcal{H}$. Then G is series-parallel and two-connected, Moreover, we have that $V = \{u, \bar{V}\}$ such that:*

1. *$G[\bar{V}]$ does not have neither loops nor circuits of length 3 or more,*

2. *u is adjacent to all vertices of $\bar{V}$ that are adjacent to less than 2 vertices of $\bar{V}$.*

Note that Condition 1 above can be restated as "$G[\bar{V}]$ is a tree with possibly parallel edges".

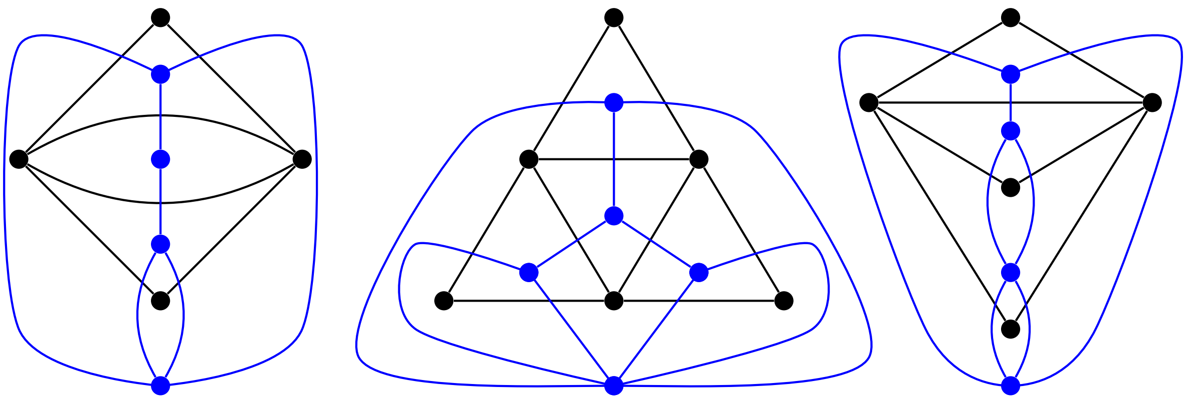


Figure 3.4: Graphs in $\mathcal{H}$ and their planar dual graphs.

3.4 Related Results and Perspectives

By (3.1), we deduce that the set of bonds is a $\frac{1}{2}$-Hilbert basis for all series-parallel graphs. Asking for which class of graphs the sets of bonds/cuts/multicuts form a Hilbert basis is an interesting question. As an example, Laurent [109], and Goddyn, Huynh, and Deshpande [83], studied the graphs for which

$$\operatorname{cone}(\mathcal{D}_G) \cap \operatorname{lattice}(\mathcal{D}_G) = \operatorname{intcone}(\mathcal{D}_G), \tag{3.4}$$

where $\operatorname{intcone}(\mathcal{D}_G)$ denotes the set of nonnegative integer combinations of elements of $\mathcal{D}_G$. These authors call *Hilbert basis* a set of vectors that respects condition (3.4). The two different definitions of Hilbert basis lead to different results, as we show in Example 3.3. Hence, we call Hilbert basis only the sets of vectors that respects the definition given in Section 2.1.1.

Example 3.3: Different "Hilbert bases" have different properties.

Consider the graph K_3. The set of cuts for this graph is given by the couples of edges. Thus, the set $\mathcal{D}_{K_3}$ in this case is $\{(1,1,0),(1,0,1),(0,1,1)\}$. By the result of Laurent [109], $\operatorname{cone}(\mathcal{D}_{K_3}) \cap \operatorname{lattice}(\mathcal{D}_{K_3}) = \operatorname{intcone}(\mathcal{D}_{K_3})$. However, vector $(1,1,1)$ is the $\frac{1}{2}$-sum of the elements of $\mathcal{D}_{K_3}$, and it can not be obtained as integer positive combination of $\{(1,1,0),(1,0,1),(0,1,1)\}$. Thus, $\mathcal{D}_{K_3}$ is not a Hilbert basis. Moreover, we can see from Example 2.1 that the system described by $x(B) \geq 0$ for all $B \in \mathcal{D}_{K_3}$ is not TDI.

A Hilbert Basis of Bonds

In order to find some nice characterization, we can instead ask when $\frac{1}{2}$ the incident vectors of cuts of a graph form a Hilbert basis. Here we show that the circuit cover property, studied by Alspach, Goddyn, and Zhang [2] and Seymour [139], is a strictly related condition. A graph G has the *circuit cover property* if $p \in \mathbb{Z}^E$ is an integer conic combination of incidence vectors of circuits of G whenever p satisfies:

- $p(C \setminus e) \geq p_e$ for all circuit C and $e \in C$, and
- $p(D)$ is even, for all cuts D of G.

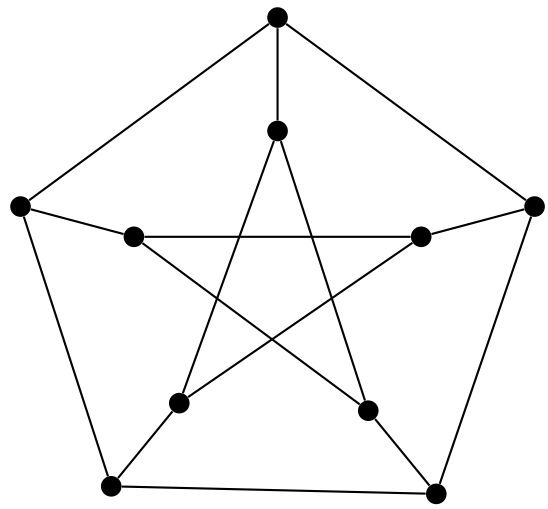


Figure 3.5: The Petersen graph P_{10}.

Such a p is called *admissible weight.*

Seymour proved that every planar graph has the circuit cover property. Moreover Alspach, Goddyn, and Zhang [2] proved the following:

Theorem 3.9. *A graph G has the circuit cover property if and only if G has no Petersen graph P_{10} as a minor (see Figure 3.5).*

We can translate the condition on p to be admissible for G into a condition on the planar dual of G.

Lemma 3.10. *Let G be a planar graph and let $G^\star$ be its planar dual. If $w \in \mathbb{Z}^E$ is an integer vector in* $\text{cone}(\mathcal{D}_G)$, *then $2w$ is an admissible weight for $G^\star$.*

Proof. Each circuit intersects each bond in either none or at least two edges. Thus, the first condition for $2w$ is satisfied whenever w belongs to the cone generated by the circuits of $G^\star$. This cone is the cut-cone of G. The second condition is stems holds because each component of $2w$ is even. ■

The following result directly stems from Lemma 3.10.

Observation 3.11. *The set of vectors $\frac{1}{2}\chi^B$ for $B \in \mathcal{B}_G$ is a Hilbert basis if G is planar. Thus, System* (3.1) *is TDI when G is planar.*

It is interesting to note that the Hilbert basis given is minimal. Observation 3.11 gives only a sufficient condition on the graph G. Numerical tests for small examples of nonplanar graphs suggest that K_5 is the minimal graph for which $\frac{1}{2}\chi^B$ for $B \in \mathcal{B}_G$ is not an Hilbert basis. This is consistent with related results on the same class of graphs, *e.g.* the description of Seymour of the cut cone. Therefore, we propose the following conjecture:

Conjecture 3.12. *The set of vectors $\frac{1}{2}\chi^B$ for $B \in \mathcal{B}_G$ is a Hilbert basis when G has no K_5 minor.*

A TDI System when G is not Series-parallel

The second question that arises naturally after this work is the following:

Open Problem 3.13. *What is the integer TDI system for the flow cone when G has a K_4-minor?*

Even if we restrict our search to planar graphs, we have very little guesses. Using SageMath [150] we experimentally checked that, for K_4, a minimal Hilbert basis for the cut cone is given by multicuts and the set of vectors $\{\mathbf{1}+\xi_e$ for each edge $e \in K_4\}$. However, we failed to discern a pattern when looking at other small non series-parallel graphs.

3.5 Conclusions

In this chapter we analyzed the TDI description of the flow cone. In Section 3.2, we provided an integer TDI system describing this cone if and only if G is series-parallel. From this work we can deduce that, for series-parallel graphs, the multicut partitioning problem admits an integer solution whenever it is feasible. This could be a starting point for results analogous to those about the cut packing problem (see *e.g.* [1, 8, 17]). Moreover, in Theorem 3.1, we implicitly provided an operation to build a Hilbert basis starting from an existing one.

Corollary 3.14. *Let A, A', and A'' be three matrices as follows:*

$$A = \begin{bmatrix} A_0 & \mathbf{0} \\ A_1 & \mathbf{1} \end{bmatrix}, \qquad A' = \begin{bmatrix} \frac{1}{2}A_0 & \mathbf{0} & \mathbf{0} \\ \frac{1}{2}A_1 & \boldsymbol{\frac{1}{2}} & \mathbf{0} \\ \frac{1}{2}A_1 & \mathbf{0} & \boldsymbol{\frac{1}{2}} \\ \mathbf{0}^\top & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \qquad A'' = \begin{bmatrix} A_0 & \mathbf{0} & \mathbf{0} \\ A_1 & \mathbf{1} & \mathbf{0} \\ A_1 & \mathbf{0} & \mathbf{1} \\ A_1 & \mathbf{1} & \mathbf{1} \\ \mathbf{0}^\top & 1 & 1 \end{bmatrix}.$$

Then, if the rows of A and A' are two Hilbert bases, then so are the rows of A''.

This operation recalls the *parallel extension* used in a slightly different context by Chen, Ding, and Zang in [25].

In Section 3.3, we strengthened the result by providing the minimal integer TDI system describing the flow cone in series-parallel graphs.

Chapter 4

Box-TDIness and Edge-Connectivity

In this chapter, we study dual integrality properties of systems and polyhedra associated with the k-edge-connected spanning subgraph problem.

Our interest is twofold. First, we prove that $P_k(G)$, that is the convex hull of the k-edge-connected spanning subgraphs of G, is a box-TDI polyhedron if and only if G is a series-parallel graph. Secondly, we provide a TDI system with integer coefficients describing $P_k(G)$ for this class of graphs. Moreover, we deal with the case $k = 1$ separately, as in this case the results we provide are valid for all graphs.

The k-edge-connected spanning subgraph problem has been studied under two different assumptions: indeed, we can choose to either allow or forbid that each edge of G can be "taken multiple times". We extensively treat the case where each edge can be taken multiple times. Nevertheless, our results hold under both assumptions. Indeed, we can impose that each edge can be taken at most once by adding the box-constraint $x \leq \mathbf{1}$. By definition of box-TDIness, this operation preserves the box-TDIness of systems and polyhedra treated.

4.1 Connected Subgraph Problems

The following problem is classical in combinatorial optimization: given a graph G with costs on the edges, find the connected subgraph of G of minimum cost that covers all the vertices of G. If we assume that all the edges have positive cost, the problem is known as the *minimum spanning tree problem.* The minimum spanning tree problem has a classical application in energy distribution and telecommunications. Indeed, Otakar Borůvka, one of the first scientists that studied the problem, was motivated by the study of an efficient way to build an electrical network in Moravia. Nowadays, connected subgraphs model a vast category of problems that are related with physical networks, like in telecommunications and transportation science.

Telecommunication networks have to be resistant to failures. In particular, we want these networks to remain operative even if some of their elements suddenly stop working. Therefore, one of the stages when designing telecommunication networks is to define a network that remains connected even when a certain number of links is removed. This leads to the k-edge-connected spanning subgraph problem, a generalization of the minimum spanning tree problem.

4.1.1 The k-edge-connected Spanning Subgraph Problem

Given a graph $G = (V, E)$, a *k-edge-connected spanning subgraph* of G is a graph $H = (V, F)$, where F is a family of edges of E, such that H is a k-edge-connected graph, spanning all the vertices of G. It is important to note that we are allowed to take each edge more than once, this can be interpreted as the addition of multiple parallel edges between two nodes. We present an example of a 3-edge-connected spanning subgraph in Figure 4.1 where the edges taken multiple times are represented by parallel edges.

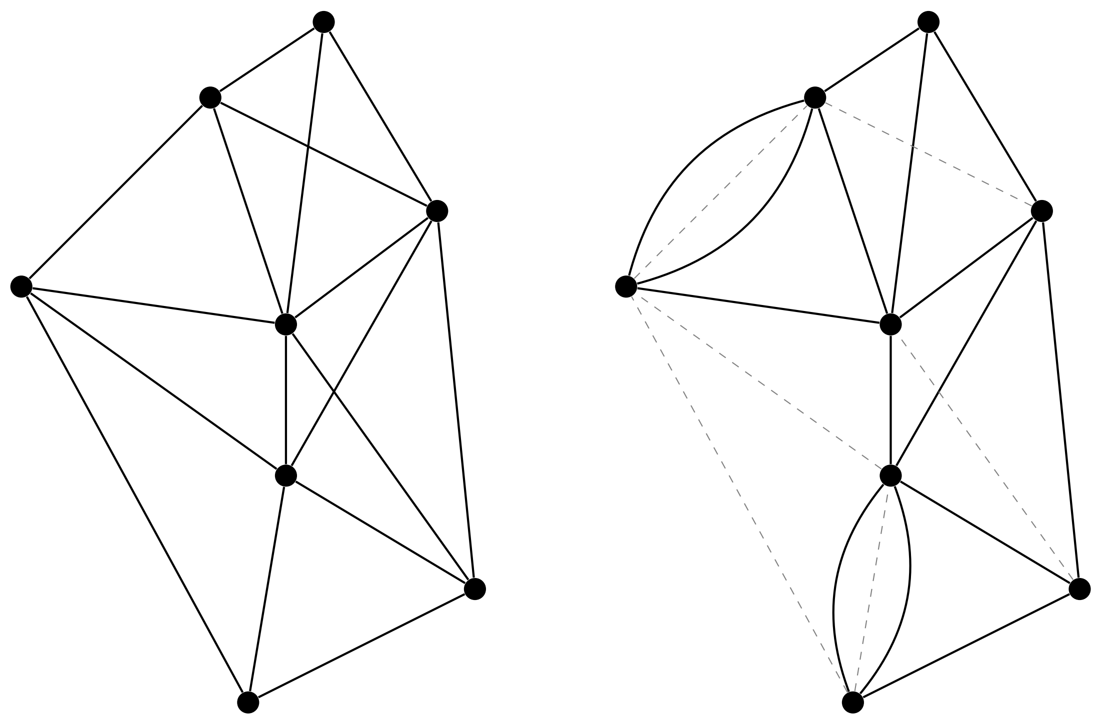


Figure 4.1: A graph and a 3-edge-connected spanning subgraph.

We are interested in the *k-edge-connected spanning subgraph problem* (*k-ECSSP*): given a graph $G = (V, E)$, and an edge cost function $c \in \mathbb{R}^E$, find the k-edge-connected spanning subgraph $H = (V, F)$ that minimizes $c^\top x$, where $x \in \mathbb{R}^E$ is the vector in which x_e is the multiplicity of e in F, for all edges e in E.

We denote by $P_k(G)$ the *k-edge-connected spanning subgraph polyhedron*, that is the convex hull of the incidence vectors of k-edge-connected spanning subgraphs of G. This polyhedron is the set of solutions of the k-ECSSP.

The k-ECSSP is $\mathcal{NP}$-hard for every fixed $k \geq 2$ [78]. Literature on the k-ECSSP has often focused on special cases of the problem. The most studied one is the 2-ECSSP which is a relaxation of the well-known Traveling Salesman Problem; see [67]. In [153], Winter introduced a linear-time algorithm solving the 2-ECSSP on series-parallel graphs.

Different algorithms have been devised in order to deal with the k-ECSSP, such as branch-and-cut procedures [7, 42], approximation algorithms [27, 77], cutting plane algorithms [90], and heuristics [107]. By exploiting a polynomial separation algorithm of Barahona [3] and their polyhedral description, Didi Biha and Mahjoub [50] proved that, when G is series-parallel, the k-ECSSP is solvable in polynomial time.

4.1.2 Some Related Problems

When we design robust networks we can consider many possible “reliability requirements”. In this sense, a brief overview of some problems related to the design of survivable networks could be profitable for a more complete comprehension of the subject.

Node-connectivity. An alternative connectivity requirement can be set on the minimum number of vertices necessary to disconnect the subgraph. This leads to the k-connected spanning subgraph problem. It should be noted that every k-connected spanning subgraph is k-edge-connected, hence node-connectivity is a stronger requirement than edge-connectivity.

Survivable network design. A generalization of the k-edge-connected spanning subgraph problem is the *edge survivable network design problem*: given a graph G and a requirement $r(v)$ for each vertex v of G, find the subgraph of G such that between each couple of vertices v, w there exist $\min(r(v), r(w))$ edge-disjoint paths. Clearly, when r is uniform, we obtain again the k-ECSSP. At the same time this problem is a generalization of many other well-known problems. We can further generalize this problem by adding a requirement on the robustness of the network after the removal of a set of nodes.

These problems were introduced by Grötschel, Monma, and Stoer in [89], and further studied by Grötschel and Monma [88], and Grötschel, Monma and Stoer [90, 91, 92]. For a more detailed survey, we refer to the book of Stoer [145]. Among the generalizations of the k-edge-connected spanning subgraph problem contained in the survivable network design problem, we mention the generalized Steiner tree problems [153] and the k-edge-connected star problem [71].

L-hop-constraints. If we consider that the condition on the k-edge-connectivity can be seen as the existence of k edge-disjoint paths between each couple of vertices, we can add some constraints on the length L of these paths. This leads to the k-edge-connected L-hop-constrained spanning subgraph problem [11, 46, 101].

For a more exhaustive analysis of the design of survivable networks, we address the reader to [70, 105, 145].

4.2 Case $k = 1$: the Connector Polyhedron

Before dealing with the k-edge-connected spanning subgraph polyhedron in the general case, we give some TDIness and box-TDIness results for the case $k = 1$. $P_1(G)$ is also known as the *connector polyhedron*, or the *connector polytope*[1] when we impose $x \leq \mathbf{1}$ [135]. In Figure 4.2, we present the forest polytope, the tree polytope, and the connector polytope of K_3.

Proposition 4.1. *$P_1(G)$ is a box-TDI polyhedron for all graphs G.*

Proof. We already saw in Example 1.2 that the forest polytope of $G = (V, E)$ – that is the convex hull of incidence vectors of forests of G – is the polymatroid described by:

$$\begin{cases} x(F) \leq |V[F]| - 1 & \text{for all } F \subseteq E, \\ x \geq \mathbf{0}. \end{cases} \tag{4.1}$$

By Theorem 2.31, System (4.1) is box-TDI. The spanning tree polytope, which is the convex hull of the incidence vectors of the spanning trees of G, is box-TDI by Theorem 2.32 because it is the face of the forest polytope defined by $x(E) \leq |V| - 1$. Remark that adding edges to a spanning tree of G, we obtain a connected spanning subgraph of G. Moreover, every integer point of $P_1(G)$ is a connected spanning subgraph of G, and hence can be

[1] Not to be confused with the *connected subgraph polytope*, that is the convex hull of sets of edges of G that *induce* a connected subgraph. (see *e.g.* [48]).

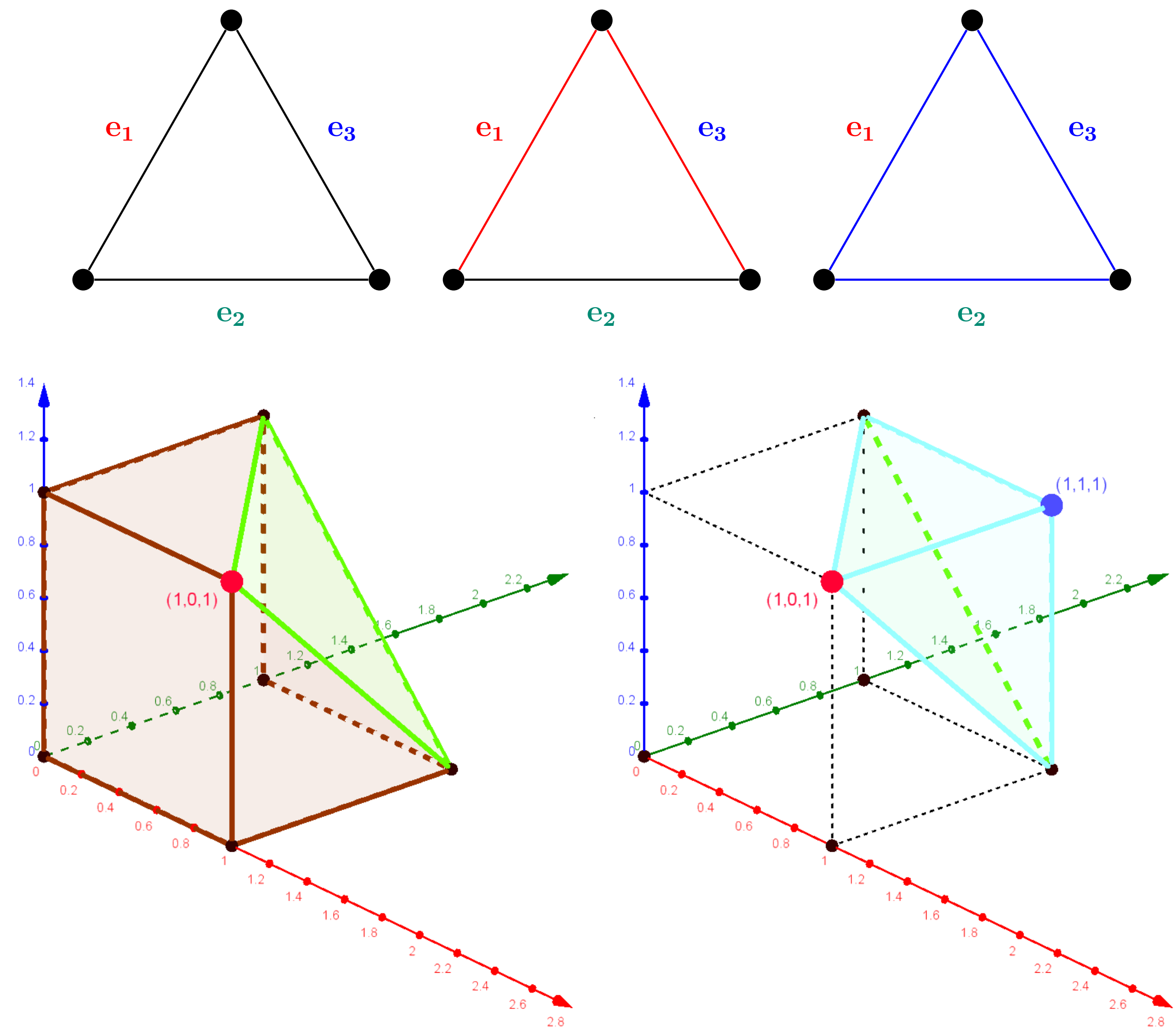


Figure 4.2: A graph G, a tree, and a 1-connected spanning subgraph.
Below, the forest polytope, the tree polytope, and the connector polytope of G.
Note that the tree polytope is a face of both the forest and the connector polytopes.

obtained from a spanning tree of G by adding edges. Therefore, $P_1(G)$ is the dominant of the spanning tree polytope of G. By Theorem 2.33, $P_1(G)$ is a box-TDI polyhedron. ■

It is natural to ask whether there exists an integer TDI system describing such polyhedron. The answer is affirmative: the integer system provided by Fulkerson [76] is nowadays known for being TDI [135, Theorem 50.8].

Theorem 4.2. *For a graph G, the connected spanning subgraph polyhedron is described by the following box-TDI system:*

$$\begin{cases} x(M) \geq d_M - 1 & \text{for all multicuts } M \text{ of } G, \\ x \geq \mathbf{0}. \end{cases} \tag{4.2}$$

System (4.2) is not minimally TDI, since every constraint associated with a multicut of order 3 that is disjoint union of two bonds does not contribute to the TDIness of the system. An example of such multicut is given in Figure 4.3. However, the same does not hold when the multicut has order 4 or more. At the best of our knowledge, the Schrijver system for this polyhedron is not known.

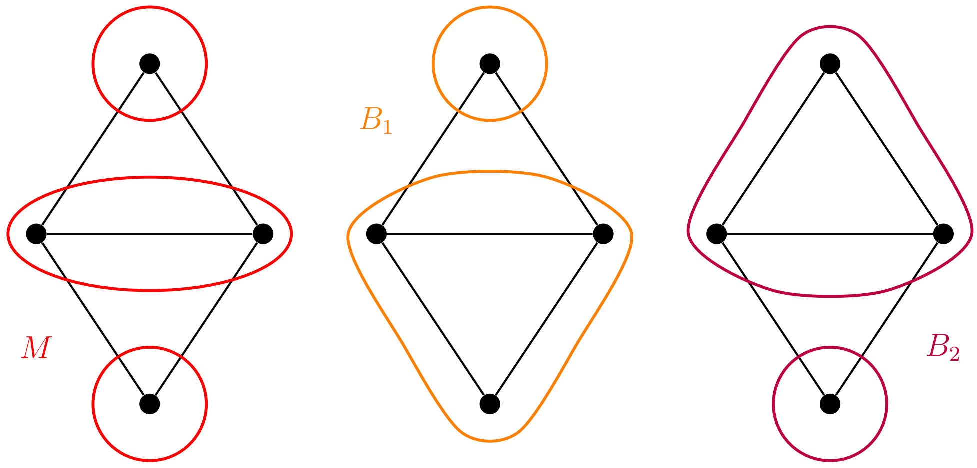


Figure 4.3: A multicut M that is given by disjoint union of two bonds B_1 and B_2.

4.3 The k-edge-connected Spanning Subgraph Polyhedron

In this section, we give some known results for $P_k(G)$ when $k \geq 2$. We study separately the cases k even and k odd. In order to better distinguish these cases, and to ease the notation, we will write that $k = 2h$ when k is even, and $k = 2h + 1$ when k is odd.

In 1985, Cornuéjols, Fonlupt, and Naddef [43] showed that the following system

$$\begin{cases} x(D) \geq 2 & \text{for all cuts } D \text{ of } G, \\ x \geq \mathbf{0} \end{cases} \tag{4.3}$$

describes $P_2(G)$ when G is a series-parallel graph. More generally, Vandenbussche and Nemhauser [152] characterized in terms of forbidden minors the class of graphs for which system (4.3) describes $P_2(G)$.

Chopra [31] provided a relaxation of $P_k(G)$ for all k odd and introduced a set of facet-defining inequalities, the so-called *LOP-inequalities*. Didi Biha and Mahjoub [50] provided a complete description of $P_k(G)$ for all k, when G is series-parallel.

Theorem 4.3. *Let G be a series-parallel graph and h be a positive integer. Then $P_{2h}(G)$ is described by:*

$$(4.4) \quad \begin{cases} x(D) \geq 2h \quad \text{for all cuts } D \text{ of } G, & (4.4a) \\ x \geq \mathbf{0}, & (4.4b) \end{cases}$$

and $P_{2h+1}(G)$ is described by:

$$(4.5) \quad \begin{cases} x(M) \geq (h+1)d_M - 1 \quad \text{for all multicuts } M \text{ of } G, & (4.5a) \\ x \geq \mathbf{0}. & (4.5b) \end{cases}$$

We call Constraints (4.5a) *partition constraints*. Some authors (like [6, 31]) call these constraints *outerplanar inequalities* or *SP-partition inequalities*.

What if we impose $x \leq \mathbf{1}$? As we mentioned in the introduction of the chapter, $P_k(G)$ has been a subject of research also under the assumption that we can take each edge at most once. Grötschel and Monma [88] describe several basic facets of $P_k(G) \cap \{x : x \leq \mathbf{1}\}$. Further polyhedral results for the 2-ECSSP have been obtained by Boyd and Hao [12] and Mahjoub [112, 113]. Moreover, Fonlupt and Mahjoub [68] extensively studied the extremal points of the k-edge-connected spanning subgraph polytope and presented the class of graphs for which this polytope is described by System (4.3) plus the inequalities $x \leq \mathbf{1}$. Barahona and Mahjoub [4] proved that, when G is a Halin graph, $P_2(G)$ is described by (4.3) plus the inequalities $x \leq \mathbf{1}$ and the so-called *odd-wheels inequalities*.

Box-TDIness. In 2006, Chen, Ding, and Zang [25] provided a box-TDI result for $P_2(G)$ for series-parallel graphs.

Theorem 4.4. *The system:*

$$\begin{cases} \frac{1}{2}x(D) \geq 1 & \text{for all cuts } D \text{ of } G, \\ x \geq \mathbf{0} \end{cases} \tag{4.6}$$

is box-TDI if and only if G is a series-parallel graph.

As mentioned in Chapter 2, if we multiply both sides of System (4.6), we potentially undermine the total dual integrality of the system. At the same time, Theorem 4.4 alone is not sufficient to state that $P_2(G)$ is a box-TDI polyhedron if and only if G is series-parallel.

4.4 Preliminary Results

In this section, we collect the technical results we will use in the rest of the chapter, the majority of these results are already known in the literature.

First, we give an alternative description of $P_k(G)$ when k is even and G is series-parallel. By combining the description of $P_{2h}(G)$ given in Theorem 4.3 and Observation 1.33, we deduce the following.

Observation 4.5. *Let G be a series-parallel graph and $h \in \mathbb{Z}$, $h \geq 1$. The polyhedron $P_{2h}(G)$ is described by:*

$$(4.7) \quad \begin{cases} x(M) \geq hd_M & \text{for all multicuts } M \text{ of } G, \quad (4.7a) \\ x \geq \mathbf{0} & (4.7b) \end{cases}$$

Multicuts play a central role in the rest of the chapter, hence we will say that a multicut is *tight* for a point x of $P_k(G)$ when the corresponding partition constraint (either (4.7a) or (4.5a), depending on the parity of k) is satisfied with equality by x. Similarly, we will say that a multicut M is *active* for a solution y of the dual of System (4.5) when $y_M > 0$.

The following observation states that for all multicuts that strictly contain $\delta(v)$ for a vertex v of degree 2 and are tight for a point of $P_k(G)$, v and the vertices adjacent to it belong to three different shores.

Observation 4.6. *Let $G = (V, E)$ be a simple series-parallel graph, let M be a multicut of G strictly containing $\delta(v) = \{f, g\}$, and let $k \geq 2$. If M is tight for a point of $P_k(G)$, then both $M \setminus f$ and $M \setminus g$ are multicuts of G of order $d_M - 1$.*

Proof. Let $M \supsetneq \delta(v)$ be a multicut of G such that $M \setminus f$ is not a multicut. Then, $M = \delta(v, V_2, \ldots, V_{d_M})$ and the two vertices of G adjacent to v belong to the same shore, say V_2. Then, we have that $M \setminus \delta(v) = \delta(V_2 \cup v, \dots, V_{d_M})$, hence it is a multicut with order $d_{M\setminus\delta(v)} = d_M - 1$. then, if $k = 2h+1$ for an integer $h \geq 1$, we have:

$$(h+1)d_M - 1 < (h+1)(d_M - 1) - 1 + (h+1)d_{\delta(v)} - 1 \leq x(M \setminus \delta(v)) + x(\delta(v)) = x(M), \quad (4.8)$$

else, if $k = 2h$ for an integer $h \geq 1$:

$$hd_M < h(d_M - 1) + hd_{\delta(v)} \leq x(M \setminus \delta(v)) + x(\delta(v)) = x(M). \quad (4.9)$$

Since for both (4.8) and (4.9) the first term is the right hand side of the partition constraint associated with M, this multicut is not tight for any point $x \in P_k(G)$. ■

Proposition 4.7. *Let $G = (V, E)$ be a simple 2-connected series parallel graph, and let $|E| \geq 2$. Then, at least one of the following holds:*

(a) there are in G at least two adjacent vertices of degree 2;

(b) there is in G at least a vertex of degree 2 that belongs to a circuit of length exactly 3;

(c) there exist in G two vertices of degree 2 that belong to the same circuit of length 4.

Proof. Consider the open nested ear decomposition of G, $\mathcal{E}_1, \dots, \mathcal{E}_m$. Let $\mathcal{E}_h$ be an ear with extremities on $\mathcal{E}_j$ such that no other ear has extremities on an internal vertex of $\mathcal{E}_h$ and such that there exists a path $P_h \subseteq \mathcal{E}_j$ between the endpoints of $\mathcal{E}_h$ such that no ear has extremities on an internal vertex of P_h. This ear exists by the definition of open nested ear decomposition. Since no internal vertex of $\mathcal{E}_h$ and P_h is the extremity of an ear, they are all degree 2 vertices. Hence, either condition (a) is realized, or both $\mathcal{E}_h$ and P_h have less than 2 internal vertices. In the second case, by G being simple, at least one among $\mathcal{E}_h$ and P_h has one internal vertex, hence at least one among (b) and (c) is satisfied. ■

Theorem 4.8 ([50])**.** *Let x be a point of $P_{2h+1}(G)$, and let $M = \delta(V_1, \dots, V_{d_M})$ be a tight multicut for x. Then, the following hold:*

1. if $d_M \geq 3$, then $x(\delta(V_i, V_j)) \leq h + 1$ for all $i \neq j \in \{1, \dots, d_M\}$.

2. G_M is 2-connected.

Lemma 4.9. *Let $G = (V, E)$ be a simple series-parallel graph, $v \in V$ be a vertex of degree 2, and M be a multicut such that $|M \cap \delta(v)| = 1$. Then, $M \cup \delta(v)$ and $M \Delta \delta(v)$ are multicuts. Moreover, $d_{M\cup\delta(v)} = d_M + 1$, and $d_{M\Delta\delta(v)} = d_M$.*

Proof. Let $M = \delta(V_1, \dots, V_{d_M})$, $\delta(v) = \{uv, vw\}$, $v, w \in V_1$, and $u \in V_2$. Then $M \cup vw = \delta(v, V_1 \setminus \{v\}, \dots, V_{d_M})$, and $M \Delta \delta(v) = \delta(V_1 \setminus \{v\}, V_2 \cup \{v\}, \dots, V_{d_M})$. ■

Observation 4.10. *Let e be an edge of K_4. Then, the following inequality is facet-defining for $P_{2h+1}(K_4)$:*

$$\sum_{f \in E \setminus e} x_f + 2x_e \geq 4(h+1) - 1. \tag{4.10}$$

Chopra [31] called the inequalities like (4.10) *LOP-inequalities*, and provided a sufficient condition for such inequalities to be facet-defining for $P_{2h+1}(G)$.

The two following results give sufficient conditions for an inequality to be facet-defining for $P_{2h+1}(G)$.

Theorem 4.11 ([31])**.** *Let $G = (V, E)$ be a graph, M be a multicut of G, and $a^\top x \geq b$ be a facet-defining inequality for $P_{2h+1}(G_M)$. Then $\tilde{a}^\top x \geq b$ is a facet defining inequality for $P_{2h+1}(G)$, where $\tilde{a}$ is defined as:*

$$\tilde{a}_e = \begin{cases} a_e & \text{if } e \in M, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } e \in E.$$

Theorem 4.12 ([31])**.** *Let $G = (V, E)$ and $G' = G \setminus e_0$ for a certain edge e_0 parallel to an edge of $e_1 \in E$. Let $a^\top x \geq b$ be a facet-defining inequality for $P_{2h+1}(G')$. Then $\tilde{a}^\top x \geq b$ is facet defining for $P_{2h+1}(G)$, where $\tilde{a}$ is defined as:*

$$\tilde{a}_e = \begin{cases} a_{e_1} & \text{if } e = e_0, \\ a_e & \text{otherwise,} \end{cases} \qquad \text{for all } e \in E.$$

4.5 Box-TDIness of $P_k(G)$

In this section, we extend Proposition 4.1 by characterizing for which graphs $P_k(G)$ is a box-TDI polyhedron when $k > 1$. We first show that, for these k, $P_k(G)$ is not box-TDI if G is not series-parallel.

Lemma 4.13. *For $k \geq 2$, if $G = (V, E)$ contains a K_4-minor, then $P_k(G)$ is not box-TDI.*

Proof. G having a K_4-minor, there exists a multicut M such that G_M is a K_4, possibly with parallel edges. When k is odd, by Observation 4.10, Constraint (4.10) is facet-defining for $P_k(K_4)$. Combining Theorems 4.11 and 4.12, we deduce that there exists a facet-defining inequality:

$$\mathbf{1}^\top x(E') + \mathbf{2}^\top x(E'') \geq b, \tag{4.11}$$

where E' and E'' are two disjoint nonempty subsets of E and $b \in \mathbb{Q}$. Note that the coefficient vector of Constraint (4.11) is $[\mathbf{0}^\top, \mathbf{1}^\top, \mathbf{2}^\top] \in \mathbb{R}^E$, hence it is not equimodular. Thus, $P_k(G)$ is not box-TDI by Theorem 2.25.

We now prove the case when k is even. Since G has a K_4-minor, there exists a partition $\{V_1, \ldots, V_4\}$ of V such that $G[V_i]$ is connected and $\delta(V_i, V_j) \neq \emptyset$ for all $i < j \in \{1, \ldots, 4\}$. We now prove that the matrix T whose three rows are $\chi^{\delta(V_i)}$ for $i = 1, 2, 3$ is a face-defining matrix for $P_k(G)$ which is not equimodular. This will end the proof by Theorem 2.25.

Let e_{ij} be an edge in $\delta(V_i, V_j)$ for all $i < j \in \{1, \ldots, 4\}$. The submatrix of T formed by the columns associated with edges e_{ij} is the following:

$$\begin{array}{c c}
 & \begin{array}{cccccc} e_{12} & e_{13} & e_{23} & e_{14} & e_{24} & e_{34} \end{array} \\
\begin{array}{c} \chi^{\delta(V_1)} \\ \chi^{\delta(V_2)} \\ \chi^{\delta(V_3)} \end{array} & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}
\end{array}$$

The matrix T is not equimodular as the first three columns form a matrix of determinant -2 whereas the last three ones have determinant 1.

To show that T is face-defining, we exhibit $|E| - 2$ affinely independent points of $P_k(G)$ satisfying $x(\delta(V_i)) = k$ for $i = 1, 2, 3$ as follows. Let $D_1 = \{e_{12}, e_{14}, e_{23}, e_{34}\}$, $D_2 = \{e_{12}, e_{13}, e_{24}, e_{34}\}$, $D_3 = \{e_{13}, e_{14}, e_{23}, e_{24}\}$ and $D_4 = \{e_{14}, e_{24}, e_{34}\}$. First, we define the points $S_j = \sum_{i=1}^4 k\chi^{E[V_i]} + \frac{k}{2}\chi^{D_j}$, for $j = 1, 2, 3$, and $S_4 = \sum_{i=1}^4 k\chi^{E[V_i]} + k\chi^{D_4}$. Note that they are affinely independent.

Now, for each edge $e \notin \{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$, we construct the point S_e as follows. When $e \in E[V_i]$ for some $i = 1, \ldots, 4$, we define $S_e = S_4 + \chi^e$. Adding the point S_e maintains affine independence as S_e is the only point not satisfying $x_e = k$. When $e \in \delta(V_i, V_j)$ for some i, j, we define $S_e = S_\ell - \chi^{e_{ij}} + \chi^e$, where S_ℓ is S_1 if $e \in \delta(V_1, V_4) \cup \delta(V_2, V_3)$ and S_2 otherwise. Affine independence comes because S_e is the only point involving e. ■

The following theorem is characterizes when $P_k(G)$ is box-TDI. When k is even, the result stems from [25]. When k is odd, we show by induction on the number of edges of G that each face-defining matrix of $P_k(G)$ is equimodular, and we use the characterization given in Theorem 2.25.

Theorem 4.14. *For $k \geq 2$, $P_k(G)$ is a box-TDI polyhedron if and only if G is series-parallel.*

Proof. Necessity stems from Lemma 4.13. Let us now prove sufficiency. For $P_2(G)$, Chen, Ding, and Zang [25] proved that System (4.6) is box-TDI. This implies box-TDIness for all even k: multiplying the right-hand side of a box-TDI system by a positive rational preserves its box-TDIness [134, Section 22.5]. System (4.4) can be obtained from (4.6) by multiplying its left-hand side by 2 and its right-hand side by k. Thus $P_k(G)$ is a box-TDI polyhedron when G is series-parallel and k is an even integer.

The rest of the proof is dedicated to the case where $k = 2h+1$ for some integer $h \geq 1$. For this purpose, we prove that every face of $P_{2h+1}(G)$ admits an equimodular face-defining matrix. The characterization of box-TDIness given in Theorem 2.25 concludes. We proceed by induction on the number of edges of G.

As a base-case of the induction we consider the series-parallel graph G consisting of two vertices connected by a single edge. Then, $P_{2h+1}(G) = \{x \in \mathbb{R}_+ : x \geq 2h+1\}$ is box-TDI.

(1-sum) Let G be the 1-sum of two series-parallel graphs $G^1 = (W^1, E^1)$ and $G^2 = (W^2, E^2)$. By induction, there exist two box-TDI systems $A^1 y \geq b^1$ and $A^2 z \geq b^2$ describing respectively $P_{2h+1}(G^1)$ and $P_{2h+1}(G^2)$. If v is the vertex of G obtained by the identification, $G \setminus v$ is not connected, hence, by Statement 2 of Theorem 4.8, a multicut M of G is tight for a face of $P_{2h+1}(G)$ only if $M \subseteq E^i$ for some $i = 1, 2$. It follows that for every face F of $P_{2h+1}(G)$ there exist two faces F^1 and F^2 of $P_{2h+1}(G^1)$ and $P_{2h+1}(G^2)$ respectively, such that $F = F^1 \times F^2$. Then $P_{2h+1}(G) = \{(y,z) \in \mathbb{R}_+^{E^1} \times \mathbb{R}_+^{E^2} : A^1 y \geq b^1, A^2 z \geq b^2\}$ and so it is box-TDI.

(Parallelization) Let now G be obtained from a series-parallel graph H by adding an edge g parallel to an edge f of H and suppose that $P_{2h+1}(H)$ is box-TDI. Note that $P_{2h+1}(G)$ is obtained from $P_{2h+1}(H)$ by duplicating f's column and adding $x_g \geq 0$. Hence, by Lemma 2.35, $P_{2h+1}(G)$ is a box-TDI polyhedron.

(Subdivision) Let $G = (V, E)$ be obtained by subdividing an edge uw of a series-parallel graph $G' = (V', E')$ into a path of length two uv, vw. By contradiction, suppose there exists a non-empty face $F = \{x \in P_{2h+1}(G) : A_F x = b_F\}$ such that A_F is a face-defining matrix of F which is not equimodular. Take such a face with maximum dimension. Then, every face-defining submatrix of A_F is equimodular. We may assume that A_F is defined by the partition constraints (4.5a) associated with the set of multicuts $\mathcal{M}_F$ and the nonnegativity constraints associated with the set of edges $\mathcal{E}_F$.

Claim 4.14.1. *$\mathcal{E}_F = \emptyset$.*

Proof. Suppose there exists an edge $e \in \mathcal{E}_F$. Let $H = G \setminus e$ and let $A_{F_H} x = b_{F_H}$ be the system obtained from $A_F x = b_F$ by removing the column and the nonnegativity constraint associated with e. The matrix A_F being of full row rank, so is A_{F_H}. Since $M \setminus e$ is a multicut of H for all M in $\mathcal{M}_F$, the set $F_H = \{x \in P_{2h+1}(H) : A_{F_H} x = b_{F_H}\}$ is a face of $P_{2h+1}(H)$. Moreover, deleting e's coordinate of aff(F) gives aff(F_H) so A_{F_H} is face-defining for F_H. By the induction hypothesis, A_{F_H} is equimodular, and hence so is A_F by Observation 1.26. ∎

Claim 4.14.2. *For all $e \in \{uv, vw\}$, at least one multicut of $\mathcal{M}_F$ different from $\delta(v)$ contains e.*

Proof. Suppose that uv belongs to no multicut of $\mathcal{M}_F$ different from $\delta(v)$.

First, suppose that $\delta(v)$ does not belong to $\mathcal{M}_F$. Then, the column of A_F associated with uv is zero. Let A'_F be the matrix obtained from A_F by removing this column. Every multicut of G not containing uv is a multicut of G' (relabelling vw by uw), so the rows of A'_F are associated with multicuts of G'. Thus, $F' = \{x \in P_k(G') : A'_F x = b_F\}$ is a face of $P_{2h+1}(G')$. Removing uv's coordinate from the points of F gives a set of points of F' of affine dimension at least $\dim(F) - 1$. Since A'_F has the same rank of A_F and one column less than A_F, then A'_F is face-defining for F' by Observation 1.3. By induction hypothesis, A'_F is equimodular, hence so is A_F.

Suppose now that $\delta(v)$ belongs to $\mathcal{M}_F$. Then, the column of A_F associated with uv has zeros in each row but $\chi^{\delta(v)}$. Let $A^\star_F x = b^\star_F$ be the system obtained from $A_F x = b_F$ by removing the row associated with $\delta(v)$. Then $F^\star = \{x \in P_k(G) : A^\star_F x = b^\star_F\}$ is a face of $P_k(G)$ of dimension $\dim(F) + 1$. Indeed, it contains F and $z + \alpha\chi^{uv}$ for every point z of F and $\alpha > 0$. Hence, $A^\star_F$ is face-defining for $F^\star$. This matrix is equimodular by the maximality assumption on F, and so is A_F by Observation 1.26. ∎

Claim 4.14.3. $|M \cap \delta(v)| \neq 1$ *for every multicut $M \in \mathcal{M}_F$.*

Proof. Suppose there exists a multicut M tight for F such that $|M \cap \delta(v)| = 1$. Without loss of generality, suppose that M contains uv and not vw. Then, $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} \geq x_{uv}\}$ because of the partition inequality (4.5a) associated with the multicut $M\Delta\delta(v)$. Moreover, the partition inequality associated with $\delta(v)$ and the integrality of $P_{2h+1}(G)$ imply $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} \geq h+1\}$. The proof is divided into two cases.

Case 1. $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} = h+1\}$. We prove this case by exhibiting an equimodular face-defining matrix for F. By Observation 1.28, this implies that A_F equimodular, which contradicts the assumption on F.

Equality $x_{vw} = h+1$ can be expressed as a linear combination of rows of $A_F x = b_F$. Let $A'_F x = b'_F$ denote the system obtained by replacing a row of $A_F x = b_F$ by $x_{vw} = h+1$ in such a way that the underlying affine space remains unchanged. Denote by $\mathcal{N}$ the set of multicuts of $\mathcal{M}_F$ containing vw but not uv. If $\mathcal{N} \neq \emptyset$, then let N be in $\mathcal{N}$. We now modify the system $A'_F x = b'_F$ by performing the following operations.

1. Every row associated with a multicut M strictly containing $\delta(v)$ is replaced by the partition constraint (4.5a) associated with $M \setminus vw$ set to equality.

2. Whenever $\delta(v) \in \mathcal{M}_F$, replace the row associated with $\delta(v)$ by the box constraint $x_{uv} = h$.

3. Replace every row associated with $M \in \mathcal{N} \setminus N$ by the partition constraint (4.5a) associated with $M \Delta \delta(v)$ set to equality.

4. Whenever $\mathcal{N} \neq \emptyset$, replace the row associated with N by the box constraint $x_{uv} = h + 1$.

These operations do not modify the underlying affine space. Indeed, in Operation 1, $M \setminus vw$ is tight for F because of Observation 4.6 and $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} = h + 1\}$. Operation 2 is applied only if $F \subseteq \{x \in P_{2h+1}(G) : x_{uv} = h\}$. Operations 3 and 4 are applied only if $\mathcal{N} \neq \emptyset$, which implies that $F \subseteq \{x \in P_{2h+1}(G) : x_{uv} = h + 1\}$ because of the constraint (4.5a) associated with $N \Delta \delta(v)$ and $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} \geq x_{uv}\}$. Note that Operations 2 and 4 cannot be applied both, hence the rank of the matrix remains unchanged.

Let $A''_F x = b''_F$ be the system obtained by removing the row $x_{vw} = h+1$ from $A'_F x = b'_F$. By construction, $A''_F x = b''_F$ is composed of constraints (4.5a) set to equality and possibly $x_{uv} = h$ or $x_{uv} = h + 1$. Moreover, the column of A''_F associated with vw is zero. Let $F'' = \{x \in P_{2h+1}(G) : A''_F x = b''_F\}$. For every point z of F and $\alpha \geq 0$, $z + \alpha\chi^{vw}$ belongs to F'' because the column of A''_F associated with vw is zero, and $z + \alpha\chi^{vw} \in P_{2h+1}(G)$. This implies that $\dim(F'') \geq \dim(F) + 1$.

If F'' is a face of $P_{2h+1}(G)$, then A''_F is face-defining for F'' by Observation 1.3 and by A'_F being face-defining for F. By the maximality assumption on F, A''_F is equimodular, and hence so is A'_F by Observation 1.26.

Otherwise, by construction, $F'' = F^\star \cap \{x \in \mathbb{R}^E : x_{uv} = t\}$ where $F^\star$ is a face of $P_{2h+1}(G)$ strictly containing F and $t \in \{h, h + 1\}$. Therefore, there exists a face-defining matrix of F'' given by a face-defining matrix of $F^\star$ and the row χ^{uv}. Such a matrix is equimodular by the maximality assumption of F and Observation 1.26. Hence, A''_F is equimodular by Observation 1.28, and so is A'_F by Observation 1.26.

Case 2. $F \not\subseteq \{x \in P_{2h+1}(G) : x_{vw} = h+1\}$. Thus, there exists $z \in F$ such that $z_{vw} > h+1$. By Claim 4.14.2, there exists a multicut $N \neq \delta(v)$ containing vw which is tight for F. By Theorem 4.8-1, the existence of z implies that N is a bond. Thus, $uv \notin N$ and $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} = x_{uv}\}$. Consequently, $L = N \Delta \delta(v)$ is also a bond tight for F. Moreover, N is the unique multicut tight for F containing vw. Suppose indeed that there exists a multicut B containing vw tight for F. Then, B is a bond by Theorem 4.8-1 and the existence of z. Moreover, $B \Delta N$ is a multicut not containing vw. This implies that no point x of F satisfies the partition constraint associated with $B\Delta N$ because $x(B\Delta N) = x(B)+x(N)-2x(B\cap N) = 2(2h+1)-2x(B\cap N) \leq 4h+2-2x_e \leq 2h$, a contradiction.

Consider the matrix $A_F^\star$ obtained from A_F by removing the row associated with N. Matrix $A_F^\star$ is a face-defining matrix for a face $F^\star \supseteq F$ of $P_{2h+1}(G)$ because $F^\star$ contains F and $z + \alpha\chi^{uv}$ for every point z of F and $\alpha > 0$. By the maximality assumption, the matrix $A_F^\star$ is equimodular. Let B_F be the matrix obtained from A_F by replacing the row χ^N by the row $\chi^N - \chi^L$. Then, B_F is face-defining for F. Moreover, B_F is equimodular by Observation 1.26 — a contradiction. ∎

Let $A'_F x = b'_F$ be the system obtained from $A_F x = b_F$ by removing uv's column from A_F and subtracting $h+1$ times this column to b_F. We now show that $\{x \in P_{2h+1}(G') : A'_F x = b'_F\}$ is a face of $P_{2h+1}(G')$ if $\delta(v) \notin \mathcal{M}_F$, and $P_{2h+1}(G') \cap \{x : x_{uw} = h\}$ otherwise. Indeed, consider a multicut M in $\mathcal{M}_F$. If $M = \delta(v)$, then the row of $A'_F x = b'_F$ induced by M is nothing but $x_{uw} = h$. Otherwise, by Observation 4.6 and Claim 4.14.3, the set $M \setminus uv$ is a multicut of G' (relabelling vw by uw) of order d_M if $uv \notin M$ and $d_M - 1$ otherwise. Thus, the row of $A'_F x = b'_F$ induced by M is the partition constraint (4.5a) associated with $M \setminus uv$ set to equality.

By construction, A'_F has full row rank and one column less than A_F. We prove that A'_F is face-defining by exhibiting $\dim(F)$ affinely independent points of $P_{2h+1}(G')$ satisfying $A'_F x = b'_F$. Because of the integrality of $P_{2h+1}(G)$, there exist $n = \dim(F) + 1$ affinely independent integer points $z^1, \dots, z^n$ of F. By Claim 4.14.3, every multicut in $\mathcal{M}_F$ contains either both uv and vw or none of them. Then, Claim 4.14.2 and Theorem 4.8-1 imply that $F \subseteq \{x \in \mathbb{R}^E : x_{uv} \leq h+1, x_{vw} \leq h+1\}$. Combined with the partition inequality $x_{uv} + x_{vw} \geq 2h+1$ associated with $\delta(v)$, this implies that at least one of z^i_{uv} and z^i_{vw} is equal to $h+1$ for $i = 1, \dots, n$. Since exchanging the uv and vw coordinates of any point of F gives a point of F by Claim 4.14.3, the hypotheses on $z^1, \dots, z^n$ are preserved under the assumption that $z^i_{uv} = h+1$ for $i = 1, \dots, n-1$. Let $y^1, \dots, y^{n-1}$ be the points obtained from $z^1, \dots, z^{n-1}$ by removing uv's coordinate. Since every multicut of G' is a multicut of G with the same order, $y^1, \ldots, y^{n-1}$ belong to $P_{2h+1}(G')$. By construction, they satisfy $A'_F x = b'_F$ so they belong to a face of $P_{2h+1}(G')$ or $P_{2h+1}(G') \cap \{x : x_{uw} = h\}$. This implies that A'_F is a face-defining matrix of $P_{2h+1}(G')$ if $\delta(v) \notin \mathcal{M}_F$, and $P_{2h+1}(G') \cap \{x : x_{uw} = h\}$ otherwise.

By induction, $P_{2h+1}(G')$ is a box-TDI polyhedron and hence so is $P_{2h+1}(G') \cap \{x : x_{uw} = h\}$. Hence, A'_F is equimodular by Theorem 2.25. Since the columns of A_F associated with uv and vw are equal, Observation 1.26 implies that A_F is equimodular — a contradiction to its assumption of non-equimodularity. ■

4.6 An integer TDI system - Case k even

In Section 4.5, we proved that $P_k(G)$ is a box-TDI polyheron if and only if G is series-parallel. When k is an even integer, the result stems from the work of Chen, Ding, and Zang [25], who also gave a TDI system describing $P_2(G)$. However, their system does not have integer coefficients. In this section and in the following one, we provide a TDI system for $P_k(G)$ having integer coefficients. In particular, this section is devoted to the case $k = 2h$ for some integer $h \geq 1$.

Given $\bar{x} \in \mathbb{Z}_+^E$, we denote by $G[\bar{x}]$ the graph induced by the edges $e \in E$ taken $\bar{x}_e$ times.

Theorem 4.15. *For a series-parallel graph G and $h \in \mathbb{Z}_+$, System* (4.7) *is TDI.*

Proof. We only prove the case $h = 1$ since multiplying the right hand side of a system by a positive constant preserves its TDIness [134, Section 22.5].

The proof is done by induction on the number of edges of the graph $G = (V, E)$. As a base-case of the induction we consider the series-parallel graph G consisting of two vertices connected by a single edge ℓ. Then, System (4.7) is $x_\ell \geq 2, x_\ell \geq 0$ and is TDI.

(1-sum) Let G be the 1-sum of two series-parallel graphs $G^1 = (W^1, E^1)$ and $G^2 = (W^2, E^2)$. We prove the TDIness of System (4.7) associated with G using Corollary 2.6. More precisely, we prove that for any vertex $\bar{x}$ of $P_2(G)$, the set of vectors $\{\chi^M : M \in \mathcal{M}_{\bar{x}}\} \cup \{\chi^e : \bar{x}_e = 0\}$ is a Hilbert basis.

By construction, we have $\bar{x} = (\bar{x}^1, \bar{x}^2)$ where $\bar{x}^i \in P_2(G^i)$ for $i = 1, 2$. Moreover, for each multicut $M \in \mathcal{M}_{\bar{x}}$, the graph obtained from $G[\bar{x}]$ by contracting the edges of $E \setminus M$ is a circuit. Indeed, it is 2-edge-connected since $G[\bar{x}]$ is, and it has $\bar{x}(M) = d_M$ edges and d_M vertices. Therefore M is either a multicut of G^1 tight for $\bar{x}^1$ or one of G^2 tight for $\bar{x}^2$.

By induction, Systems (4.7) associated with G^1 and G^2 are TDI. Thus, $\{\chi^M : M \in \mathcal{M}_{\bar{x}} \cap \mathcal{M}(G^i)\} \cup \{\chi^e : e \in E^i, \bar{x}_e = 0\}$ is a Hilbert basis for $i = 1, 2$ by Corollary 2.6. Since they belong to disjoint spaces, their union is a Hilbert basis. By Corollary 2.6, System (4.7) is TDI.

(Parallelization) Let now G be obtained from a series-parallel graph H by adding an edge g parallel to an edge f of H. System (4.7) associated with G is obtained from that associated with H by duplicating f's column in constraints (4.7a) and adding the nonnegativity constraint $x_g \geq 0$. By Lemma 2.35, System (4.7) is TDI.

(Subdivision) Let $G = (V, E)$ be obtained by subdividing an edge uw of a series-parallel graph $G' = (V', E')$ into a path of length two uv, vw. We prove the TDIness of System (4.7) associated with G using Corollary 2.6. More precisely, we prove that for any vertex z of $P_2(G)$, the set of vectors $\mathcal{M}_z \cup \mathcal{E}_z$ forms a Hilbert basis.

Without loss of generality, suppose $z_{uv} \geq z_{vw}$. Define $z' \in \mathbb{Z}^{E'}$ by $z'_{uw} = z_{vw}$ and $z'_e = z_e$ for all edges e in $E' \cap E$. Remark that z' belongs to $P_2(G')$ since $G'[z']$ is obtained by contracting the edge uv in $G[z]$ which preserves 2-edge-connectivity.

Remark that for all $e \in E$, $z_e \in \{0, 1, 2\}$. Indeed, $z_e > 0$ implies that e is in a cut tight for z since z is a vertex of $P_2(G)$ which is an integer polyhedron described by System (4.4). As $z_{uv} \geq z_{vw}$, the partition constraint (4.7a) associated with $\delta(v)$ implies that $z_{uv} \in \{1, 2\}$. We now consider two different cases depending on the value of z_{uv}.

Case i. $z_{uv} = 2$.

First, note that there exists a unique $N \in \mathcal{M}_z$ containing uv and that N is a bond. Indeed, if a multicut $M = \delta(V_1, \ldots, V_{d_M}) \in \mathcal{M}_z$ satisfies $d_M \geq 3$ and $uv \in \delta(V_1, V_2)$, then the multicut $M' = \delta(V_1 \cup V_2, V_3, \ldots, V_{d_M})$ is such that $z(M') \leq z(M) - 2 < d_M - 1 = d_{M'}$. Hence, the partition constraint (4.7a) associated with M' is violated – a contradiction. Moreover, every multicut M with $d_M = 2$ is a bond. Suppose now that there exist two bonds B_1 and B_2 in $\mathcal{M}_z$ containing uv. Then, $z(B_1 \Delta B_2) \leq z(B_1) + z(B_2) - 2z_{uv} = 0$ – a contradiction. This implies that $\mathcal{M}_z = \mathcal{M}_{z'} \cup N$ as a multicut M not containing uv is tight for z if and only if it is tight for z'. By induction and Theorem 2.5, $\mathcal{M}_{z'} \cup \mathcal{E}_{z'}$ is a Hilbert basis. As $\mathcal{E}_z = \mathcal{E}_{z'}$ and N is the only member of $\mathcal{M}_z \cup \mathcal{E}_z$ containing uv, $\mathcal{M}_z \cup \mathcal{E}_z$ is also a Hilbert basis.

Case ii. $z_{uv} = 1$.

Let $\mathbf{v}$ be any integer point of the cone generated by $\mathcal{M}_z \cup \mathcal{E}_z$. We prove that $\mathbf{v}$ can be expressed as an integer nonnegative combination of vectors $\mathcal{M}_z \cup \mathcal{E}_z$ which implies that $\mathcal{M}_z \cup \mathcal{E}_z$ is a Hilbert basis.

Let $\mathcal{B}_z$ be the set of bonds of $\mathcal{M}_z$. Since System (4.6) is TDI in series-parallel graphs, the set of vectors $\{\frac{1}{2}\chi^B : B \in \mathcal{B}_z\} \cup \mathcal{E}_z$ forms a Hilbert basis by Theorem 2.5. Then, there exist $\lambda_B \in \frac{1}{2}\mathbb{Z}_+$ for all $B \in \mathcal{B}_z$ and $\mu_e \in \mathbb{Z}_+$ for all $e \in \mathcal{E}_z$ such that $\mathbf{v} = \sum_{B \in \mathcal{B}_z} \lambda_B \chi^B + \sum_{e \in \mathcal{E}_z} \mu_e \chi^e$.

Since $z_{uv} \geq z_{vw}$, the partition inequality (4.7a) associated with $\delta(v)$ implies that $z_{vw} = 1$ and $\delta(v) \in \mathcal{M}_z$. The vector $\mathbf{v}$ is an integer combination of vectors of $\mathcal{M}_z \cup \mathcal{E}_z$ if and only if $\mathbf{v} - \lfloor \lambda_{\delta(v)} \rfloor \chi^{\delta(v)}$ is, thus we may assume that $\lambda_{\delta(v)} \in \{0, \frac{1}{2}\}$. Define $\mathbf{w} \in \mathbb{Z}^{E'}$ by:

$$\mathbf{w}_e = \begin{cases} \mathbf{v}_{uv} + \mathbf{v}_{vw} - 2\lambda_{\delta(v)} & \text{if } e = uw, \\ \mathbf{v}_e & \text{otherwise.} \end{cases}$$

Remark that $(B \setminus uw) \cup uv$ and $(B \setminus uw) \cup vw$ are bonds of $\mathcal{M}_z$ whenever B is a bond of $\mathcal{M}_{z'}$ containing uw because $z'_{uw} = z_{uv} = z_{vw} = 1$. Moreover, a bond B of $\mathcal{M}_{z'}$ which does not contain uw is a bond of $\mathcal{M}_z$. Since $\delta(v)$ is the unique bond of G containing both uv and vw, and $\mathcal{E}_z = \mathcal{E}_{z'}$, we have:

$$\mathbf{w} = \sum_{B \in \mathcal{B}_{z'} : uw \in B} (\lambda_{(B \setminus uw) \cup uv} + \lambda_{(B \setminus uw) \cup vw}) \chi^B + \sum_{B \in \mathcal{B}_{z'} : uw \notin B} \lambda_B \chi^B + \sum_{e \in \mathcal{E}_{z'}} \mu_e \chi^e.$$

Thus, $\mathbf{w}$ belongs to the cone of $\mathcal{M}_{z'} \cup \mathcal{E}_{z'}$. By the induction hypothesis, $\mathcal{M}_{z'} \cup \mathcal{E}_{z'}$ is a Hilbert basis, hence there exist $\lambda'_M \in \mathbb{Z}_+$ for all $M \in \mathcal{M}_{z'}$ and $\mu'_e \in \mathbb{Z}_+$ for all $e \in \mathcal{E}_{z'}$ such that $\mathbf{w} = \sum_{M \in \mathcal{M}_{z'}} \lambda'_M \chi^M + \sum_{e \in \mathcal{E}_{z'}} \mu'_e \chi^e$.

Consider the family $\mathcal{N}$ of multicuts of $\mathcal{M}_{z'}$ where each multicut M of $\mathcal{M}_{z'}$ appears λ'_M times. Suppose first that $\lambda_{\delta(v)} = 0$. Then, $\mathbf{v}_{uv} + \mathbf{v}_{vw}$ multicuts of $\mathcal{N}$ contain uw. Let $\mathcal{P}$ be a family of $\mathbf{v}_{uv}$ multicuts of $\mathcal{N}$ containing uw and $\mathcal{Q} = \{F \in \mathcal{N} : uw \in F\} \setminus \mathcal{P}$. Then, we have

$$\mathbf{v} = \sum_{M \in \mathcal{N} : uw \notin M} \chi^M + \sum_{M \in \mathcal{P}} \chi^{(M \setminus uw) \cup uv} + \sum_{M \in \mathcal{Q}} \chi^{(M \setminus uw) \cup vw} + \sum_{e \in \mathcal{E}_{z'}} \mu'_e \chi^e. \tag{4.12}$$

Suppose now that $\lambda_{\delta(v)} = \frac{1}{2}$. Then, $\mathbf{v}_{uv} + \mathbf{v}_{vw} - 1$ multicuts of $\mathcal{N}$ contain uw. Let $\mathcal{P}$ be a family of $\mathbf{v}_{uv} - 1$ multicuts of $\mathcal{N}$ containing uw, let $\mathcal{Q}$ be a family of $\mathbf{v}_{vw} - 1$ multicuts in $\{F \in \mathcal{N} : uw \in F\} \setminus \mathcal{P}$, and denote by N the unique multicut of $\mathcal{N}$ containing uw which is not in $\mathcal{P} \cup \mathcal{Q}$. Then, we have

$$\mathbf{v} = \sum_{M \in \mathcal{N} : uw \notin M} \chi^M + \sum_{M \in \mathcal{P}} \chi^{(M \setminus uw) \cup uv} + \sum_{M \in \mathcal{Q}} \chi^{(M \setminus uw) \cup vw} + \chi^{N \cup \delta(v) \setminus uw} + \sum_{e \in \mathcal{E}_{z'}} \mu'_e \chi^e. \tag{4.13}$$

Every $M \in \mathcal{M}_{z'}$ not containing uw is in $\mathcal{M}_z$. For every $M \in \mathcal{M}_{z'}$ containing uw, $(M \setminus uw) \cup uv$, $(M \setminus uw) \cup vw$ and $(M \setminus uw) \cup \delta(v)$ belong to $\mathcal{M}_z$ since $z'_{uw} = z_{uv} = z_{vw} = 1$. Since $\mathcal{E}_z = \mathcal{E}_{z'}$, then $\mathbf{v}$ is a nonnegative integer combination of vectors of $\mathcal{M}_z \cup \mathcal{E}_z$ in both (4.12) and (4.13). This proves that $\mathcal{M}_z \cup \mathcal{E}_z$ is a Hilbert basis. ■

Theorem 4.15 and Lemma 4.13 characterize the box-TDIness of System (4.7) as follows.

Corollary 4.16. *System* (4.7) *is box-TDI if and only if* G *is series-parallel.*

Theorem 4.15 does not give a necessary condition for System (4.7) to be TDI. On one hand, we know that System (4.7) is not TDI whenever the set of the solutions is a non integer polyhedron. On the other, we directly computed that System (4.7) is TDI for K_4. Since System (4.7) describes an integer polyhedron if and only if the underlying graph G has not the graphs in Figure 4.4 as minors (see [152]), that are "similar" to K_4, we conjecture that for this class of graphs System (4.7) is TDI.

Conjecture 4.17. *System* (4.7) *is TDI if and only if the two graphs in Figure 4.4 are not minors of* G*.*

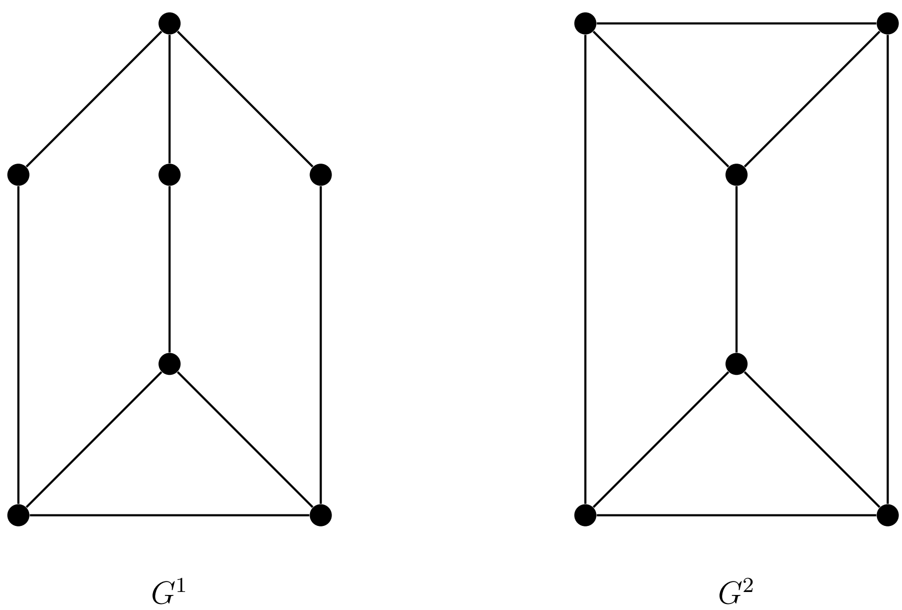


Figure 4.4: Minimal graphs for which System (4.3) is not integer

Why System (4.7) **is not TDI for** G^1**?** The attentive reader probably noticed that G^1 can be obtained from K_4 by subdividing three edges incident to a same vertex. In the proof of Theorem 4.15 however, we showed that this operation preserves the TDIness of System (4.7). We remark that this is not a contradiction: in our proof we implicitly exploited the fact that the polyhedron described by System (4.7) is integer. When this hypothesis does not hold, the proof given is no more valid because we do not explore all the vertices of the polyhedron.

4.7 An integer TDI system - Case k odd

In this section, we prove that System (4.5) is TDI if G is a series-parallel graph.

Theorem 4.18. *Let G be a series-parallel graph and h a positive integer. Then, System* (4.5) *is TDI.*

Proof. We prove the result by contradiction. Let $G = (V, E)$ be a series-parallel graph such that System (4.5) is not TDI. By definition of TDIness, there exists $c \in \mathbb{Z}^E$ such that $\mathcal{D}_{(G,c)}$:

$$\max \sum_{M \in \mathcal{M}_G} b_M y_M$$

s.t.

$$\begin{cases} \displaystyle\sum_{M \in \mathcal{M}_G : e \in M} y_M \le c_e & \text{for all } e \in E \qquad (4.14a) \\ y_M \ge 0 & \text{for all } M \in \mathcal{M}_G \qquad (4.14b) \end{cases}$$

is feasible, bounded, and does not admit an integer optimal solution, where we defined b_M as $b_M = (h+1)d_M - 1$ for all $M \in \mathcal{M}_G$. Without loss of generality, we assume that:

(i) G has a minimum number of edges,

(ii) $\sum_{e \in E} c_e$ is minimum with respect to (i).

Note that assumption (ii) is not restrictive. Indeed, if $c \ge \mathbf{0}$, the point $\mathbf{0}$ is a solution to $\mathcal{D}_{(G,c)}$. Therefore, $\mathcal{D}_{(G,c')}$ is feasible for all $\mathbf{0} \le c' \le c$. Moreover, if $c_e < 0$ for some edge e that belongs to a multicut $\mathcal{D}_{(G,c)}$ is not feasible. On the other hand, if $c \ge \mathbf{0}$ is finite, then $\mathcal{D}_{(G,c)}$ is bounded.

We recall that a multicut M is *active* for a solution y to $\mathcal{D}_{(G,c)}$ if $y_M > 0$. Note that, by complementary slackness, a multicut is active for an optimal solution to $\mathcal{D}_{(G,c)}$ only if it is tight for an optimal solution to the primal problem. In particular, if a multicut is tight for no point of $P_{2h+1}(G)$, then it is not active for every optimal solution to $\mathcal{D}_{(G,c)}$. Thus, we will use Observation 4.6 and Theorem 4.8 to deduce properties of the optimal solutions to $\mathcal{D}_{(G,c)}$.

Claim 4.18.1. *G is simple, 2-connected and different from K_2.*

Proof. By definition, multicuts do not contain loops hence, by minimality assumption (i), G has no loop. Suppose by contradiction that there exist two parallel edges e_1 and e_2 and

$c_{e_1} \le c_{e_2}$. Since a multicut contains either both e_1 and e_2 or none of them, the inequality (4.14a) associated with e_2 is redundant because $c_{e_1} \le c_{e_2}$. This contradicts minimality assumption (i), so G is simple.

Assume by contradiction that G is not 2-connected. Then G is the 1-Sum of two distinct graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. By Statement (2) of Theorem 4.8, the multicuts of G that intersect both E_1 and E_2 are not tight for the points of $P_{2h+1}(G)$, by complementary slackness (Theorem 1.10), these multicuts are not active for the optimal solutions to $\mathcal{D}_{(G,c)}$. Hence, every optimal solution y to $\mathcal{D}_{(G,c)}$ is of the form:

$$y_M = \begin{cases} y^1_M & \text{if } M \in \mathcal{M}_{G_1}, \\ y^2_M & \text{if } M \in \mathcal{M}_{G_2}, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } M \in \mathcal{M}_G,$$

where y^i is an optimal solution to $\mathcal{D}_{(G_i, c_{|E_i})}$ for $i = 1, 2$. By minimality assumption (i), there exists an integer optimal solution $\bar{y}^i$ to $\mathcal{D}_{(G_i, c_{|E_i})}$ for $i = 1, 2$, implying that $(\bar{y}^1, \bar{y}^2)$ is an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction.

Finally, if $G = K_2$, $\mathcal{M}_G$ contains only one multicut, say $\{e\}$, and the optimal solution to $\mathcal{D}_{(G,c)}$ is $y^*_{\{e\}} = c_e$ which is integer. ■

From the definition of series-parallel graphs, Claim 4.18.1 implies that G contains at least one degree 2 vertex. Let $\widehat{V}$ be the set of vertices of degree 2 in G.

Claim 4.18.2. *For all edges $e \in E$, $c_e \ge 1$.*

Proof. By hypothesis, c is integer and $\mathcal{D}_{(G,c)}$ has an optimal solution, say y^*. Since $y^* \ge 0$, then $c \ge 0$ by inequalities (4.14a). Suppose by contradiction that there exists an edge $e \in E$ with $c_e = 0$. Set $G' = G/e$ and $c' = c_{|E \setminus e}$. The active multicuts for y^* do not contain the edge e so they are multicuts of G' since $\mathcal{M}_{G'} = \{M \in \mathcal{M}_G | e \notin M\}$. Hence, the point $y' \in \mathbb{R}^{\mathcal{M}_{G'}}$ defined by $y'_M = y^*_M$ for all $M \in \mathcal{M}_{G'}$ is a solution to $\mathcal{D}_{(G',c')}$.

By minimality assumption (i), there exists an integer optimal solution $\tilde{y}$ to $\mathcal{D}_{(G',c')}$. Extending $\tilde{y}$ to a point of $\mathbb{Z}^{\mathcal{M}_G}$ by setting to 0 the missing components gives an integer solution to $\mathcal{D}_{(G,c)}$ with cost $b^\top \tilde{y} \ge b^\top y' = b^\top y^*$. This is an integer optimal solution to $\mathcal{D}_{(G,c)}$ since y^* is optimal, a contradiction with the hypothesis that $\mathcal{D}_{(G,c)}$ has no integer optimal solution. ■

Claim 4.18.3. *For every optimal solution y to $\mathcal{D}_{(G,c)}$, $0 \le y_M < 1$ for all $M \in \mathcal{M}_G$.*

Proof. By contradiction, suppose that y^* is an optimal solution to $\mathcal{D}_{(G,c)}$ such that there exists a multicut M such that $y^*_M \ge 1$. Therefore, the point y' defined by $y' = y^* - \xi_M$ is a solution to $\mathcal{D}_{(G,c')}$ where $c' = c - \chi^M$. By minimality assumption (ii), $\mathcal{D}_{(G,c')}$ admits an integer optimal solution y''. The point $\tilde{y}$ defined by $\tilde{y} = y'' + \xi_M$ is an integer solution to $\mathcal{D}_{(G,c)}$ and we have:

$$b^\top \tilde{y} = b^\top y'' + b_M \geq b^\top y' + b_M = b^\top y^*.$$

Therefore $\tilde{y}$ is an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction. ■

Claim 4.18.4. *Let $v \in \widehat{V}$, $\delta(v) = \{e_1, e_2\}$, y be an optimal solution to $\mathcal{D}_{(G,c)}$, and M_1 be an active multicut for y such that $M_1 \cap \delta(v) = e_1$. Then, if $y_{\delta(v)} > 0$, there exists no multicut M_2 such that $y_{M_2} > 0$ and $M_2 \cap \delta(v) = e_2$.*

Proof. We prove the result by contradiction. Assume that there exists a M_2 such that $y_{M_2} > 0$ and $M \cap \delta(v) = e_2$, and assume that M_1 and $\delta(v)$ are active for y. By Lemma 4.9, $M_i' = M_i \cup \delta(v)$ is a multicut of G such that $d_{M_i'} = d_{M_i} + 1$ for $i = 1, 2$. Let $\varepsilon > 0$ such that $\varepsilon \leq \min\left(y_{M_1}, y_{M_2}, y_{\delta(v)}\right)$. Then, the point:

$$y' = y - \varepsilon\left(\chi^{M_1} + \chi^{M_2} + \chi^{\delta(v)}\right) + \varepsilon\left(\chi^{M_1'} + \chi^{M_2'}\right)$$

is a solution to $\mathcal{D}_{(G,c)}$, and we have $b^\top y' = b^\top y + \varepsilon$, thus y is not optimal, a contradiction. ■

Claim 4.18.5. *For every optimal solution to $\mathcal{D}_{(G,c)}$, the constraints* (4.14a) *associated with the edges incident to a degree 2 vertex are tight.*

Proof. We prove the result by contradiction. Suppose there exist an optimal solution y^* to $\mathcal{D}_{(G,c)}$ and a vertex v with $\delta(v) = \{e_1, e_2\}$ such that the inequality (4.14a) associated with e_1 is not tight and let, for $i = 1, 2$, s_i be the slack of the constraint associated with e_i, that means:

$$s_i = c_{e_i} - \sum_{M \in \mathcal{M}_G : e_i \in M} y^*_M$$

Inequality (4.14a) associated with e_2 is tight, as otherwise there exists a $\eta > 0$, $\eta < \min(s_1, s_2)$, such that $y^* + \eta\xi_{\delta(v)}$ is a solution to $\mathcal{D}_{(G,c)}$, a contradiction with the optimality of y^*. Hence, Claims 4.18.2 and 4.18.3 imply that there are at least two distinct multicuts M_1 and M_2 active for y^* and containing e_2. Let ε be a rational number, $0 < \varepsilon \leq \min(y^*_{M_1}, y^*_{M_2}, s_1)$. For $i = 1, 2$, $e_1 \in M_i$, as otherwise $y' = y^* + \varepsilon(\xi_{M_i \cup e_1} - \xi_{M_i})$ is a solution to $\mathcal{D}_{(G,c)}$. This solution is such that $b^\top y' = b^\top y^* + \varepsilon(h+1) > b^\top y^*$, a contradiction with the optimality of y^*, for $i = 1, 2$. Since both M_1 and M_2 contain $\delta(v)$ and are distinct, at least one of them, say M_1, strictly contains $\delta(v)$. Then, $y'' = y^* + \varepsilon(-\xi_{M_1} + \xi_{M_1 \setminus e_2} + \xi_{\delta(v)})$ is a solution to $\mathcal{D}_{(G,c)}$ because $M_1 \setminus e_2$ belongs to $\mathcal{M}_G$ by Observation 4.6. Then, $b^\top y' = b^\top y^* + \varepsilon(-b_{M_1} + b_{M_1} - (h+1) + 2h + 1) > b^\top y^*$, a contradiction. ■

Given a solution y to $\mathcal{D}_{(G,c)}$, we define for each vertex $v \in \widehat{V}$ the set $\mathcal{A}_v^y$ as the set of multicuts active for y that strictly contain $\delta(v)$. Moreover we define the value α_v^y as:

$$\alpha_v^y = \sum_{M \in \mathcal{A}_v^y} y_M. \tag{4.15}$$

Claim 4.18.6. *For every optimal solution y to $\mathcal{D}_{(G,c)}$, we have $0 < \alpha_v^y < 1$ for all $v \in \widehat{V}$.*

Proof. Suppose by contradiction that there exist an optimal solution y^* to $\mathcal{D}_{(G,c)}$ and a vertex v of $\widehat{V}$ such that either $\alpha_v^{y^*} \geq 1$ or $\alpha_v^{y^*} = 0$. Denote the two edges incident to v by e_1 and e_2 in such a way that $c_{e_1} \leq c_{e_2}$.

Suppose first that $\alpha_v^{y^*} \geq 1$. By Claim 4.18.3, there exist at least two multicuts in $\mathcal{A}_v^{y^*}$. Let $\mathcal{A}_v^{y^*} = \{M_1, \ldots, M_n\}$, by Observation 4.6, for all $i = 1, \ldots, n$, $M_i' = M_i \setminus e_1$ is a multicut of G with $d_{M_i'} = d_{M_i} - 1$. Let $c' = c - \xi_{e_1}$. By $\alpha_v^{y^*} \geq 1$, there exist ϵ_i for all $i = 1, \ldots, n$, such that $0 \leq \epsilon_i \leq y^*_{M_i}$ and $\sum_{i=1}^n \epsilon_i = 1$. The point y^1 defined by:

$$y^1 = y^* + \sum_{i=1}^n \left(-\epsilon_i \xi_{M_i} + \epsilon_i \xi_{M_i'}\right)$$

is a solution to $\mathcal{D}_{(G,c')}$. By definition of b, we have:

$$b^\top y^1 = b^\top y^* - h - 1. \tag{4.16}$$

By minimality assumption (ii), $\mathcal{D}_{(G,c')}$ admits an integer optimal solution, say y^2. This solution satisfies with equality the capacity constraint (4.14a) associated with e_2 as otherwise $y^2 + \xi_{\delta(v)}$ would be a solution to $\mathcal{D}_{(G,c)}$ with cost $b^\top y^2 + b_{\delta(v)} \geq b^\top y^1 + 2h + 1$, contradicting the assumption that y^* is optimal by (4.16) and $h \geq 1$. Hence, there exists a multicut $\bar{M}$ active for y^2 containing e_2 but not e_1 since $c'_{e_1} \leq c'_{e_2} - 1$. By definition, $\bar{M} \cup e_1$ is a multicut of G with order $d_{\bar{M}} + 1$. Define $y^3 \in \mathbb{Z}^{\mathcal{M}_G}$ by:

$$y^3_M = y^2 - \chi^{\bar{M}} + \chi^{\bar{M} \cup e_1}$$

By definition of c' and y^2, the point y^3 is an integer solution to $\mathcal{D}_{(G,c)}$. Therefore, by (4.16), y^2 being optimal in $\mathcal{D}_{(G,c')}$ and by definition of y^3, we have:

$$b^\top y^* = b^\top y^1 + h + 1 \leq b^\top y^2 + h + 1 \leq b^\top y^3.$$

Thus, y^3 is an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction.

Suppose now that $\alpha_v^{y^*} = 0$. First, note that $\delta(v)$ is not an active multicut for y^*. Otherwise by Claims 4.18.2, 4.18.3 and 4.18.5, there would be a multicut containing e_1

and not e_2, say N_1, and a multicut containing e_2 and not e_1, say N_2, which are both active for y^*. This contradicts Claim 4.18.4. This, and the definition of $\alpha_v^{y^*}$ imply that no active multicut contains $\delta(v)$.

By Lemma 4.9, if a multicut M contains e_2 but not e_1, then $M\Delta\delta(v)$ is a multicut with the same order and $b_M = b_{M\Delta\delta(v)}$. Hence, we can define the point $y^4 \in \mathbb{Q}^{\mathcal{M}_G}$:

$$y^4_M = \begin{cases} 0 & \text{if } e_1 \in M, \\ y^*_M + y^*_{M\Delta\delta(v)} & \text{if } e_1 \notin M \text{ and } e_2 \in M, \\ y^*_M & \text{otherwise,} \end{cases} \qquad \text{for all } M \in \mathcal{M}_G,$$

which is a solution to $\mathcal{D}_{(G,\hat{c})}$, where $\hat{c}$ is defined by:

$$\hat{c}_e = \begin{cases} c_{e_1} + c_{e_2} & \text{if } e = e_2, \\ 0 & \text{if } e = e_1, \\ c_e & \text{otherwise,} \end{cases} \qquad \text{for all } e \in E.$$

By construction, we have:

$$b^\top y^4 = b^\top y^*. \tag{4.17}$$

By Claim 4.18.2, $\mathcal{D}_{(G,\hat{c})}$ admits an integer optimal solution, say y^5. Let $\mathcal{S}$ be the family of active multicuts for y^5 containing e_2, where each multicut M appears y^5_M times in $\mathcal{S}$. We have $|\mathcal{S}| > c_{e_2}$ as otherwise y^5 would be an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction.

We now construct from y^5 an integer solution y^6 to $\mathcal{D}_{(G,c)}$ with the same cost by replacing e_2 by e_1 in some active multicuts for y^5. More formally, since $|\mathcal{S}| \geq c_{e_1}$, there exists $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| = c_{e_1}$. By Lemma 4.9, $M\Delta\delta(v)$ is a multicut of G for all $M \in \mathcal{S}'$ and $b_M = b_{M\Delta\delta(v)}$. Let $y^6 \in \mathbb{Z}^{\mathcal{M}_G}$ be the point defined by:

$$y^6 = y^5 + \sum_{M\in\mathcal{S}'} \left(\xi_{M\Delta\delta(v)} - \xi_M\right) \tag{4.18}$$

By construction, we have:

$$b^\top y^6 = b^\top y^5. \tag{4.19}$$

Remark that for each $M \in \mathcal{S}'$, adding $\xi_{M\Delta\delta(v)} - \xi_M$ to a point of $\mathbb{R}^{\mathcal{M}_G}$ increases (resp. decreases) by 1 the left-hand side of the inequality (4.14a) associated with e_1 (resp. e_2) while not changing the left-hand side of the inequalities (4.14a) associated with the edges of $E \setminus \{e_1, e_2\}$. Therefore, by definition of $\hat{c}$, y^6 is a solution to $\mathcal{D}_{(G,c)}$. By (4.19), y^5 being optimal and (4.17), we have:

$$b^\top y^6 = b^\top y^5 \geq b^\top y^4 = b^\top y^*.$$

Therefore y^6 is an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction. ∎

The previous result implies that for each optimal solution y and for each $v \in \widehat{V}$ there exists at least one multicut strictly containing $\delta(v)$ that is active for y. For the following claims we need to define a subset of optimal solutions to $\mathcal{D}_{(G,c)}$: let $\mathfrak{D}_v$ be the set, possibly empty, of optimal solutions to $\mathcal{D}_{(G,c)}$ for which $\delta(v)$ is not active. $\mathfrak{D}_v$ is a face of the polytope of the solutions, hence, if it is not empty, there exists a solution y in $\mathfrak{D}_v$ such that $\alpha_v^y \geq \alpha_v^z$ for all solutions $z \in \mathfrak{D}_v$.

The following claim presents the structure of a specific optimal solution to $\mathcal{D}_{(G,c)}$ for which $\delta(v)$ is not active. Figure 4.5 gives an insight on the structure of this solution.

Claim 4.18.7. *Let $v \in \widehat{V}$ with $\delta(v) = \{e_1, e_2\}$ and let y^* in $\mathfrak{D}_v$ maximizing $\alpha_v^{y^*}$.*

The only multicuts active for y^* that intersect $\delta(v)$ are two bonds $F \cup e_1$ and $F \cup e_2$, and a multicut $F \cup \{e_1, e_2\}$ of order 3, where F is a subset of $E \setminus \delta(v)$ that does not contain nonempty multicuts.

Proof. By Claim 4.18.6, there exists at least one multicut strictly containing $\delta(v)$ which is active for y^*, say M_0. By $y^*_{\delta(v)} = 0$ and Claim 4.18.5, there exists at least one multicut active for y^* which contains e_i and not $\delta(v) \setminus e_i$, for $i = 1, 2$. Let M_i be such a multicut with maximum order.

We claim that $M_j \setminus \delta(v)$ does not contain multicuts for $j = 0, 1, 2$. First, we prove that $M_0 \setminus \delta(v)$ does not contain multicuts. Let $M_0 = \delta(v, V_2, V_3, \ldots, V_{d_{M_0}})$, note that, by Observation 4.6 and complementary slackness, the two vertices adjacent to v belong to two different shores, say V_2 and V_3. By contradiction, let M_0' be the maximal multicut contained in $M_0 \setminus \delta(v)$. We have that $d_{M_0'} = d_{M_0} - 2$ because $M_0 = \delta(v \cup V_2 \cup V_3, \ldots, V_{d_{M_0}})$. For $i = 1, 2$, $M_i' = M_i \cup \delta(v)$ is a multicut with order $d_{M_i'} = d_{M_i} + 1$. Let $0 < \varepsilon < y^*_{M_\ell}$ for $\ell = 0, 1, 2$. Then, let $y' \in \mathbb{R}^{\mathcal{M}_G}$ be the point defined by:

$$y' = y^* - \varepsilon\xi_{M_0} + \varepsilon\xi_{M_0'} + \varepsilon \sum_{i=1,2;i} \left(-\xi_{M_i} + \xi_{M_i'}\right).$$

By construction, y' is a solution to $\mathcal{D}_{(G,c)}$ such that $b^\top y^* = b^\top y'$. Hence y' is an optimal solution, but we have $\alpha_v^{y'} = \alpha_v^{y^*} + \varepsilon$ because $\delta(v) \subsetneq M_i'$ for $i = 1, 2$. This contradicts the maximality of $\alpha_v^{y^*}$. Therefore $M_0 \setminus \delta(v)$ does not contain nonempty multicuts. This implies that $d_{M_0} = 3$.

Now, we show that $M_1 \setminus \delta(v)$ does not contain nonempty multicuts. The result for M_2 stems by symmetry. By contradiction, let M_1' be the maximal multicut contained in

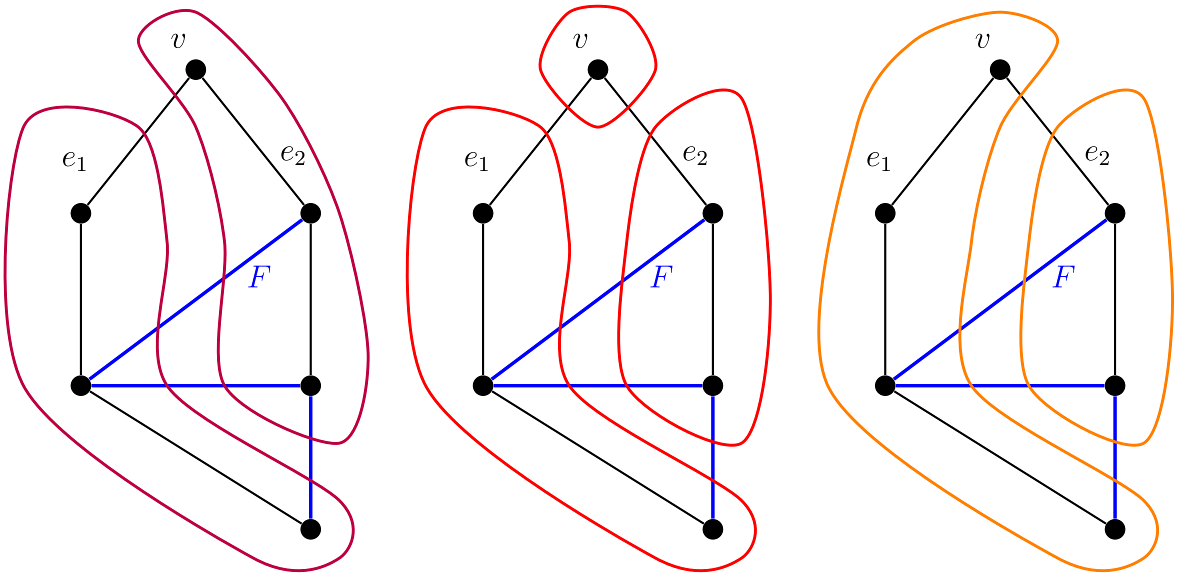


Figure 4.5: Visual representation of Claim 4.18.7: the only multicuts active for y^* that intersect $\delta(v)$ are $F \cup e_1$, $F \cup \{e_1, e_2\}$, and $F \cup e_2$ for some set F not containing nonempty multicuts.

$M_1 \setminus \delta(v)$. Note that $d_{M_1'} = d_{M_1} - 1$. Let $M_2' = M_2 \cup \delta(v)$, it is a multicut with order $d_{M_2'} = d_{M_2} + 1$. Let $0 < \varepsilon < y^*_{M_\ell}$ for $\ell = 1, 2$. Then, let $y' \in \mathbb{R}^{\mathcal{M}_G}$ be the point defined by:

$$y' = y^* - \varepsilon\xi_{M_1} + \varepsilon\xi_{M_1'} - \varepsilon\xi_{M_2} + \varepsilon\xi_{M_2'}.$$

By construction, y' is a solution to $\mathcal{D}_{(G,c)}$ such that $b^\top y^* = b^\top y'$. Hence y' is an optimal solution, but we have $\alpha_v^{y'} = \alpha_v^{y^*} + \varepsilon$ because $\delta(v) \subsetneq M_2'$. This contradicts the maximality of $\alpha_v^{y^*}$.

Therefore $M_i \setminus \delta(v)$ does not contain multicuts for $i = 1, 2$. This implies that $d_{M_1} = d_{M_2} = 2$.

We now prove that there exists a set F such that $M_0 = F \cup \delta(v)$, and $M_i = F \cup e_i$ for $i = 1, 2$. Remark that $M_1 \cup M_2$ is a multicut so $y'' = y^* + \varepsilon(\xi_{M_1 \cup M_2} - \xi_{M_1} - \xi_{M_2})$ is a solution to $\mathcal{D}_{(G,c)}$. The optimality of y^* implies $d_{M_1 \cup M_2} \leq 3$. Since M_1 and M_2 are distinct bonds, there exists $F \subseteq E \setminus \delta(v)$ such that $M_i = F \cup e_i$, for $i = 1, 2$.

Finally, let $N_0 = M_0 \setminus e_2$ and $N_1 = M_1 \cup e_2$. Note that $\tilde{y} = y^* + \varepsilon(\xi_{N_0} - \xi_{M_0} + \xi_{N_1} - \xi_{M_1})$ is an optimal solution to $\mathcal{D}_{(G,c)}$ for which $\delta(v)$ is not active. Moreover, N_0 and M_2 are bonds active for $\tilde{y}$ since $d_{M_0} = 3$. This implies that $N_0 = F \cup e_1$, and hence $M_0 = F \cup \delta(v)$. ■

Claim 4.18.8. *Let $v \in \widehat{V}$ and y be an optimal solution to $\mathcal{D}_{(G,c)}$. Then,* *i. if* $y_{\delta(v)} = 0$*, then* $c_e = 1$ *for all* $e \in \delta(v)$*,*

ii. if $y_{\delta(v)} > 0$*, then* $\alpha_v^y + y_{\delta(v)} = 1$*, and there exists* $e \in \delta(v)$ *such that* $c_e = 1$*.*

Proof. ($i.$) First suppose that $y_{\delta(v)} = 0$. Let y' be a point of $\mathfrak{D}_v$ having $\alpha_v^{y'}$ maximum, such point exists because $\mathfrak{D}_v \neq \emptyset$. Then, by Claim 4.18.7, there exist exactly two active multicuts for y' containing e_i for $i = 1, 2$. Combining Claims 4.18.3 and 4.18.5, and the integrality of c, we obtain that $c_{e_i} = 1$ for all $i = 1, 2$.

($ii.$) Let now $y_{\delta(v)} > 0$. By Claim 4.18.4, there exists an edge $e \in \delta(v)$ such that every multicut active for y that contains e contains $\delta(v)$. Hence, the constraint (4.14a) associated with e is:

$$c_e \geq \sum_{M : e \in M} y_M^* = y_{\delta(v)}^* + \sum_{M \in \mathcal{A}_v^{y^*}} y_M^* = y_{\delta(v)}^* + \alpha_v^{y^*}. \tag{4.20}$$

By Claim 4.18.5, the constraint (4.14a) associated with e is tight. Thus, $y_{\delta(v)}^* + \alpha_v^{y^*} = c_e$, and by Claims 4.18.3 and 4.18.6, and c_e being integer, we have that $c_e = 1$. ■

The next (and last) three claims of the proof state some deduce some structural property of the graph G. In particular we focus our attention on the vertices of $\widehat{V}$. We give some insights on the proof of the following claim in Example 4.1.

Claim 4.18.9. *Vertices of degree 2 are not adjacent in* G*.*

Proof. Assume by contradiction that there exist two adjacent vertices v_1 and v_2 in $\widehat{V}$, and denote $\delta(v_i) = \{e_0, e_i\}$ for $i = 1, 2$.

We first claim that $\delta(v_i)$ is active for all optimal solutions to $\mathcal{D}_{(G,c)}$. We prove that $\delta(v_1)$ is active for all optimal solutions, the result for $\delta(v_2)$ is obtained by symmetry. By contradiction, suppose that $\mathfrak{D}_{v_1} \neq \emptyset$. Among the solutions belonging to $\mathfrak{D}_{v_1}$, let y^1 be one having $\alpha_{v_1}^y$ maximum.

Then, by Claim 4.18.7, all the active multicuts for y^1 intersecting $\delta(v_1)$ are $M_0 = F \cup \delta(v_1)$, $B_1 = F \cup e_1$, and $B_0 = F \cup e_0$, where B_i are bonds for $i = 0, 1$, and $F \subseteq E \setminus \delta(v)$ contains no nonempty multicut. By Claim 4.18.6, there exists a multicut M active for y^1 strictly containing $\delta(v_2)$. By $\delta(v_1) \cap \delta(v_2) = e_0$, M intersects $\delta(v_1)$, hence $M = M_0$, $F = \{e_2\}$ and $B_0 = \delta(v_2)$.

As $\delta(v_1) = 0$, by Claim 4.18.8-($i.$), $c_{e_0} = c_{e_1} = 1$. By Claim 4.18.5, the constraints associated with e_0 and e_1 are tight. Since $\mathcal{A}_{v_1}^{y^1} = \{M_0\}$, we have:

$$c_{e_i} = y_{M_0}^1 + y_{B_i}^1 = 1 \quad \text{for } i = 0, 1. \tag{4.21}$$

Let $\{M_1, \dots, M_n\}$ be the set of active multicuts for y^1 such that $M_i \cap \{e_0, e_1, e_2\} = e_2$, for $i = 1, \dots, n$. The constraint (4.14a) associated with e_2 is tight, hence:

$$c_{e_2} = y^1_{M_0} + y^1_{B_0} + y^1_{B_1} + \sum_{i=1}^{n} y^1_{M_i} = 1 + y^1_{B_0} + \sum_{i=1}^{n} y^1_{M_i}. \tag{4.22}$$

By Claim 4.18.3 and $c_{e_2} \in \mathbb{Z}$, we have that $\{M_1, \dots, M_n\} \neq \emptyset$. Moreover, note that $c_{e_2} \geq 2$. Thus, combining (4.21) and (4.22), we have:

$$\sum_{i=1}^{n} y^1_{M_i} = c_{e_2} - 1 - y^1_{B_0} \geq y^1_{M_0}. \tag{4.23}$$

Then, there exist $\epsilon_1, \dots, \epsilon_n$ such that $0 \leq \epsilon_i \leq y^1_{M_i}$ for $i = 1, \dots, n$, and

$$\sum_{i=1}^{n} \epsilon_i = y^1_{M_0}.$$

We have that, for $i = 1, \dots, n$, $M_i \cup e_0$ is a multicut with order $d_{M_i} + 1$, hence we can consider the following solution:

$$y^2 = y^1 - \left(y^1_{M_0} \xi_{M_0} + \sum_{i=1}^{n} (\epsilon_i \xi_{M_i}) \right) + \left(y^1_{M_0} \xi_{M_0 \setminus e_0} + \sum_{i=1}^{n} (\epsilon_i \xi_{M_i \cup e_0}) \right). \tag{4.24}$$

We have that $b^\top y^1 = b^\top y^2$, but $\alpha^{y^2}_{v_1} = 0$, a contradiction with Claim 4.18.6. By symmetry, we deduce that both $\delta(v_1)$ and $\delta(v_2)$ are active for all optimal solutions to $\mathcal{D}_{(G,c)}$.

By Claim 4.18.4, we have that, for every optimal solution y to $\mathcal{D}_{(G,c)}$ and multicut $M \in \mathcal{M}_G$, if $y_M > 0$ and $e_i \in M$, then $e_0 \in M$, for $i = 1, 2$.

Let y^* be the optimal solution to $\mathcal{D}_{(G,c)}$ maximizing $\alpha^y_{v_1}$. We have that $\mathcal{A}^{y^*}_{v_2} \subseteq \mathcal{A}^{y^*}_{v_1}$ and all the multicuts in $\mathcal{A}^{y^*}_{v_2}$ have order at most 3. Otherwise, let $M \in \mathcal{A}^{y^*}_{v_2} \setminus \mathcal{A}^{y^*}_{v_1}$ (resp. M such that $d_M \geq 4$), and $0 < \varepsilon < \min(y^*_M, y^*_{\delta(v_1)})$. The solution

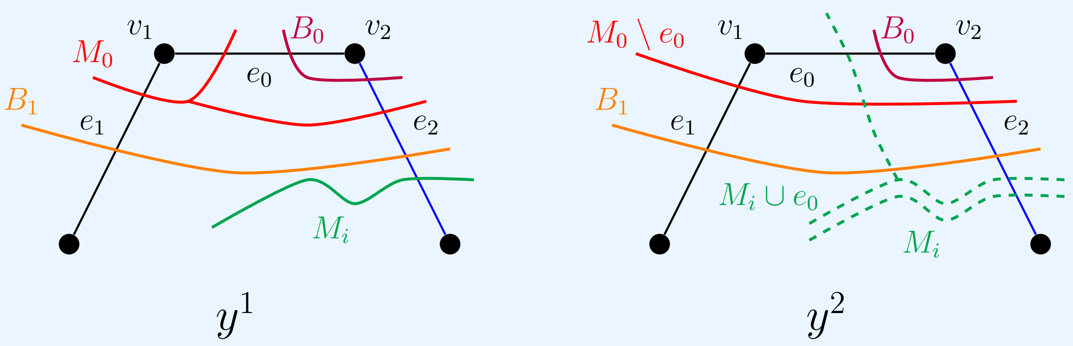
$$y^3 = y^* - \varepsilon(\xi_M + \xi_{\delta(v_1)}) + \varepsilon(\xi_{M \setminus e_2} + \xi_{\delta(v_1) \cup e_2})$$

is optimal, but $\alpha^{y^3}_{v_1} = \alpha^{y^*}_{v_1} + \varepsilon$ by the choice of M, a contradiction with the choice of y^*. Thus, $\bar{M} = \{e_0, e_1, e_2\}$ is the only multicut in $\mathcal{A}^{y^*}_{v_2}$.

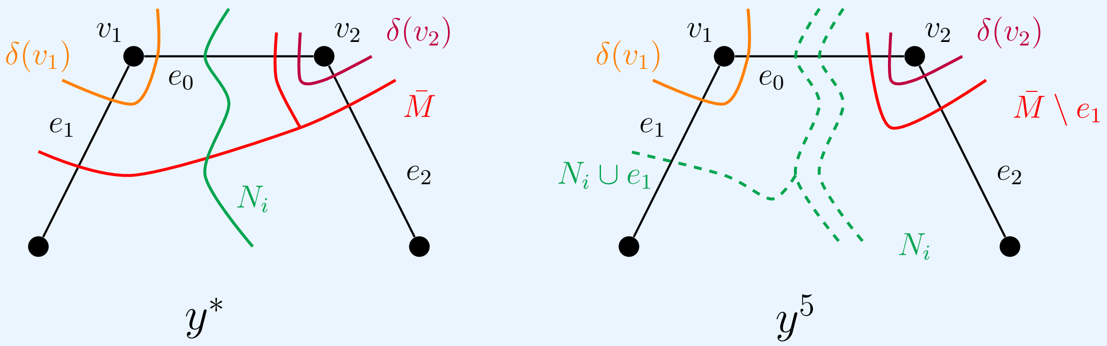
Let $\{N_1, \dots, N_m\}$ be the set of active multicuts for y^* such that $N_i \cap \{e_0, e_1, e_2\} = e_0$. The constraint associated with e_0 is tight by Claim 4.18.5, hence, by $\mathcal{A}^{y^*}_{v_2} \subseteq \mathcal{A}^{y^*}_{v_1}$, we have:

$$c_{e_0} = \alpha^{y^*}_{v_1} + y^*_{\delta(v_1)} + y^*_{\delta(v_2)} + \sum_{i=1}^{m} y^*_{N_i}. \tag{4.25}$$

Example 4.1: Hints for Claim 4.18.9.



Case $\delta(v_1)$ not active. We change the solution by substituting M_0 with $M_0 \setminus e_0 = B_1$, and by partially substituting M_i with $M_i \cup e_0$ for $i = 1, \ldots, n$.



Last passage. We change y^* by substituting $\bar{M}$ with $\bar{M} \setminus e_1 = \delta(v_2)$, and by partially substituting N_i with $N_i \cup e_1$ for $i = 1, \ldots, m$.

By Claim 4.18.8-(*ii.*) applied to v_1, we have $y^*_{\delta(v_1)} + \alpha^{y^*}_{v_1} = 1$, and so:

$$c_{e_0} = 1 + y^*_{\delta(v_2)} + \sum_{i=1}^{m} y^*_{N_i}. \tag{4.26}$$

By $\mathcal{A}^{y^*}_{v_2} = \bar{M}$ and Claim 4.18.8-(*ii.*) applied to v_2, we have $y^*_{\delta(v_2)} + y^*_{\bar{M}} = 1$, hence:

$$c_{e_0} = 2 - y^*_{\bar{M}} + \sum_{i=1}^{m} y^*_{N_i}. \tag{4.27}$$

By c_{e_0} being integer, $y^*_{\bar{M}} < 1$ by Claim 4.18.3, and (4.27), we have:

$$\sum_{i=1}^{m} y^*_{N_i} \geq y^*_{\bar{M}}. \tag{4.28}$$

Hence, let $\lambda_1, \dots, \lambda_m$ be such that $0 \le \lambda_i \le y^*_{N_i}$ for $i = 1, \dots, m$, and $\sum_{i=1}^m \lambda_i = y^*_{\bar{M}}$. Remark that $\delta(v_2) = \bar{M} \setminus e_1$. Then, the point:

$$y^5 = y^* - \left(y^*_{\bar{M}} \xi_{\bar{M}} + \sum_{i=1}^m \lambda_i \xi_{N_i} \right) + \left(y^*_{\bar{M}} \xi_{\delta(v_2)} + \sum_{i=1}^m \lambda_i \xi_{N_i \cup e_1} \right)$$

is a solution to $\mathcal{D}_{(G,c)}$, and it is optimal by definition of b. Moreover,

$$y^5_{\delta(v_2)} = y^*_{\bar{M}} + y^*_{\delta(v_2)} = 1,$$

a contradiction with Claim 4.18.3. ■

The following claim forbids a circuit of length 3 to contain a vertex of $\widehat{V}$, we informally present the proof of this claim along with some pictures in Example 4.2.

Claim 4.18.10. *A degree 2 vertex does not belong to a circuit of length 3 in G.*

Proof. Assume by contradiction that in G there exist a vertex $v \in \widehat{V}$ and a circuit $\{e_1, e_2, e_3\}$ such that $\delta(v) = \{e_1, e_2\}$. By Observation 1.34, a multicut contains e_3 only if it intersects $\delta(v)$. On the other hand, by Observation 4.6 and complementary slackness, for each M active multicut for an optimal solution, $M \neq \delta(v)$, $e_3 \in M$, we have that $M \cap \delta(v) \neq \emptyset$. Thus, for every optimal solution y to $\mathcal{D}_{(G,c)}$, we have:

$$\sum_{M: e_3 \in M} y_M = \sum_{M: e_1 \in M, M \neq \delta(v)} y_M + \sum_{M: e_2 \in M, M \neq \delta(v)} y_M - \alpha^y_v. \tag{4.29}$$

Let y^* be an optimal solution to $\mathcal{D}_{(G,c)}$. By the constraint (4.14a) associated with e_3, (4.29), and Claim 4.18.5, we have:

$$c_{e_3} \ge \sum_{M: e_3 \in M} y^*_M = c_{e_1} + c_{e_2} - 2y^*_{\delta(v)} - \alpha^{y^*}_v. \tag{4.30}$$

By Claims 4.18.6 and 4.18.8-($ii.$), we have that $2y^*_{\delta(v)} + \alpha^{y^*}_v < 2$. Thus, we deduce from (4.30) that $c_{e_3} \ge c_{e_1} + c_{e_2} - 1$.

Define $G' = G \setminus e_3$ and $c' = c_{|E \setminus e_3}$. Note that for each multicut $M \in \mathcal{M}_G$, $M \setminus e_3$ is a multicut of G' with order at least d_M. Hence, y^* induces a solution to $\mathcal{D}_{(G',c')}$ of cost at least $b^\top y^*$. By minimality assumption (i), there exists an integer optimal solution y' to $\mathcal{D}_{(G',c')}$, and we have $b^\top y' \ge b^\top y^*$.

Let $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) be the set of multicuts $M = \delta(V_1, \dots, V_{d_M})$ of G' active for y' such that the endpoints of e_3 belong (resp. do not belong) to a same V_i for some $i \in \{1, \dots, d_M\}$.

For each $M \in \mathcal{M}_1$ (resp. $M \in \mathcal{M}_2$), M (resp. $M \cup e_3$) is a multicut of G with the same order. Hence,

$$y'' = \sum_{M \in \mathcal{M}_1} y'_M \xi_M + \sum_{M \in \mathcal{M}_2} y'_M \xi_{M \cup e_3}$$

is a point of $\mathbb{Z}_+^{\mathcal{M}_G}$ with $b^\top y'' = b^\top y'$. Thus, $b^\top y'' \geq b^\top y^*$, and y'' is not a solution to $\mathcal{D}_{(G,c)}$. By definition, y'' respects every constraint of $\mathcal{D}_{(G,c)}$ except for the constraint (4.14a) associated with e_3, hence this constraint is violated by y''.

By definition of y'', we have:

$$\sum_{M: e_3 \in M} y''_M = \sum_{M: e_1 \in M, M \neq \delta(v)} y''_M + \sum_{M: e_2 \in M, M \neq \delta(v)} y''_M - \alpha_v^{y''}. \tag{4.31}$$

Therefore, by the inequality (4.14a) associated with e_3, Equation (4.31), and the inequalities (4.14a) associated with e_1 and e_2, we have:

$$c_{e_3} < \sum_{M: e_3 \in M} y''_M = \sum_{M: e_1 \in M} y''_M + \sum_{M: e_2 \in M} y''_M - \alpha_v^{y''} - 2y''_{\delta(v)} \leq c_{e_1} + c_{e_2} - \alpha_v^{y''} - 2y''_{\delta(v)}. \tag{4.32}$$

Thus, by Equation (4.30), we have $\alpha_v^{y''} + 2y''_{\delta(v)} < \alpha_v^{y^*} + 2y^*_{\delta(v)} < 2$. By $c_{e_3} \geq c_{e_2} + c_{e_1} - 1$, the integrality of y'', and Equation (4.32), we have that $\alpha_v^{y''} = y''_{\delta(v)} = 0$, and so $c_{e_3} = c_{e_1} + c_{e_2} - 1$. Hence, by the integrality of y'':

$$c_{e_3} + 1 = \sum_{M: e_3 \in M} y''_M = \sum_{M: e_1 \in M} y''_M + \sum_{M: e_2 \in M} y''_M = c_{e_1} + c_{e_2}. \tag{4.33}$$

For $i = 1, 2$, since $c_{e_i} \geq 1$, there exists a multicut M_i active for y'' such that $M_i \cap \delta(v) = e_i$.

We claim that the constraint (4.14a) associated with e_3 is not tight for y^*. By $c_{e_3} = c_{e_1} + c_{e_2} - 1$, (4.30), and Claim 4.18.6, stems that $\delta(v)$ is active for y^*. Hence, by Claim 4.18.8-(*ii.*), we have:

$$\alpha_v^{y^*} + y^*_{\delta(v)} = 1. \tag{4.34}$$

By tightness of the constraint associated with e_1, and (4.34), there exists an edge in $\delta(v)$, say e_1, such that all the active multicuts containing e_1 contain e_2. Moreover, by Claim 4.18.5, the constraint associated with e_2 is tight for y^*, hence:

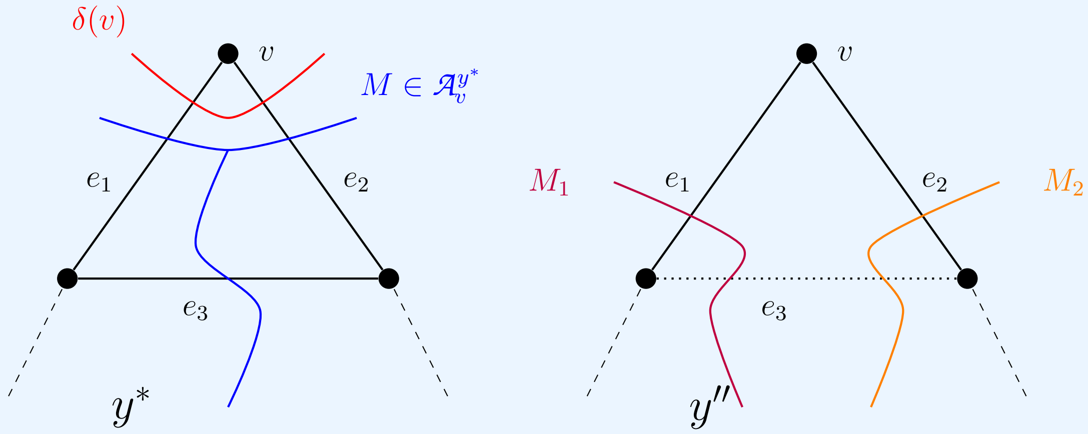
$$\sum_{M: M \cap \delta(v) = e_2} y^*_M + \alpha_v^{y^*} + y^*_{\delta(v)} = \sum_{M: M \cap \delta(v) = e_2} y^*_M + 1 = c_{e_2}. \tag{4.35}$$

In particular, $\sum_{M: M \cap \delta(v) = e_2} y^*_M$ is integer. By Observation 1.34, a multicut active for y^* contains e_2 but not e_1 if and only if it contains e_3. Thus, the constraint associated with e_3 is not tight for y^*, because c_{e_3} is integer and $\alpha_v^{y^*}$ is not, so:

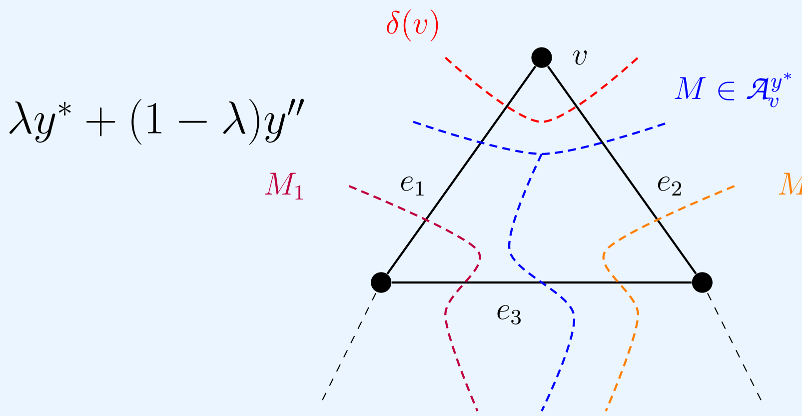
$$c_{e_3} > \sum_{M: e_3 \in M} y^*_M = \sum_{M: M \cap \delta(v) = e_2} y^*_M + \alpha_v^{y^*}.$$

Example 4.2: Idea of Claim 4.18.10.

We consider two points y^* and y'', where the first is an optimal solution to $\mathcal{D}_{(G,c)}$, and the latter violates only the constraint associated with e_3 and has (at least) optimal value.



In the proof we show that the constraint associated with e_3 is not tight for y^* and that there exist four multicuts as above that are active for the two points. By construction of y'' and y^*, there is a $\lambda \in (0,1)$ such that $\lambda y^* + (1-\lambda)y''$ is an optimal solution to $\mathcal{D}_{(G,c)}$. The multicuts active for this solution are the union of those active for y^* and those active for y''. This contradicts Claim 4.18.4.



The point y'' respects all the constraints of $\mathcal{D}_{(G,c)}$ except the one associated with e_3, and this constraint is not tight for y^*. Therefore, there exists $\lambda \in (0,1)$, such that $\tilde{y} = \lambda y^* + (1-\lambda)y''$ is a solution to $\mathcal{D}_{(G,c)}$. Moreover, $\tilde{y}$ is optimal because $b^\top y^* \leq b^\top y''$.

All multicuts active for at least one between y^* and y'' are active for $\tilde{y}$. Since $\delta(v)$ is active for y^* and M_1, M_2 are active for y'', the three multicuts M_1, M_2, and $\delta(v)$ are active for $\tilde{y}$, a contradiction with Claim 4.18.4. ∎

Claim 4.18.11. *In G no circuit of length 4 contains two or more vertices of $\widehat{V}$.*

Proof. Assume by contradiction that there exists a circuit $C = \{e_1, \ldots, e_4\}$ in G such that $v_1, v_2 \in \widehat{V}$ belong to C. By Claim 4.18.9, v_1 and v_2 are not adjacent, hence we assume that $\delta(v_1) = \{e_1, e_2\}$ and $\delta(v_2) = \{e_3, e_4\}$. Let v_3 and v_4 be the remaining vertices of C.

We have that $\delta(v_1)$ is active for all optimal solutions to $\mathcal{D}_{(G,c)}$. Indeed, let y' be a solution maximizing $\alpha_{v_1}^{y'}$ among all the solutions in $\mathfrak{D}_{v_1}$. By Statement 2 of Theorem 4.8, for every multicut M in $\mathcal{A}_{v_2}^{y'}$, we have $M = \delta\{v_2, V_2, \ldots, V_{d_M}\}$, with v_3 and v_4 belonging to different V_i's, hence $M \cap \delta(v_1) \neq \emptyset$. However, by Claim 4.18.7 applied to v_1, $M \setminus \delta(v_1)$ can not contain any multicut, a contradiction. Exchanging the role of v_1 and v_2, we deduce that $\delta(v_2)$ is active for all optimal solutions to $\mathcal{D}_{(G,c)}$.

Without loss of generality, there exists an optimal solution y such that $\alpha_{v_1}^y \geq \alpha_{v_2}^y$. Then, we can build an optimal solution to $\mathcal{D}_{(G,c)}$, say y^*, such that $\mathcal{A}_{v_1}^{y^*} \supseteq \mathcal{A}_{v_2}^{y^*}$. Indeed, suppose there exist $\{M_1, \ldots, M_n\} \in \mathcal{A}_{v_2}^y \setminus \mathcal{A}_{v_1}^y$. Then, since $\alpha_{v_1}^y \geq \alpha_{v_2}^y$, there exist $\{N_1, \ldots, N_m\}$ in $\mathcal{A}_{v_1}^y \setminus \mathcal{A}_{v_2}^y$, such that:

$$\sum_{i=1}^{n} y_{M_i} \leq \sum_{j=1}^{m} y_{N_j}. \tag{4.36}$$

Note that, by Statement 2 of Theorem 4.8 and complementary slackness, v_3 and v_4 belong to different shores of N_j for each $j = 1, \ldots, m$, thus $N_j \cap \delta(v_2) \neq \emptyset$. Moreover, since $N_j \notin \mathcal{A}_{v_2}^y$, we have $|N_j \cap \delta(v_2)| = 1$, for all $j = 1, \ldots, m$. Furthermore, by $\delta(v_2)$ being active and Claim 4.18.4, there exists an edge in $\delta(v_2)$, say e_3, such that $N_j \cap \delta(v_2) = e_3$ for all $j = 1, \ldots, m$. Hence, there exist $\epsilon_1, \ldots, \epsilon_m$ such that $0 \leq \epsilon_j \leq y_{N_j}$, for $j = 1, \ldots, m$, and

$$\sum_{i=1}^{n} y_{M_i} = \sum_{j=1}^{m} \epsilon_j. \tag{4.37}$$

Therefore, let

$$y^* = y - \left(\sum_{i=1}^{n} y_{M_i} \xi_{M_i} + \sum_{j=1}^{m} y_{N_j} \xi_{N_j} \right) + \left(\sum_{i=1}^{n} y_{M_i} \xi_{M_i \setminus e_4} \sum_{j=1}^{m} y_{N_j} \xi_{N_j \cup e_4} \right). \tag{4.38}$$

The point so defined is a solution to $\mathcal{D}_{(G,c)}$, $b^\top y^* = b^\top y$, and $\mathcal{A}_{v_1}^{y^*} \supseteq \mathcal{A}_{v_2}^{y^*}$. Moreover, this implies that all the multicuts in $\mathcal{A}_{v_2}^{y^*}$ have order at least 4 because of Statement 2 of Theorem 4.8.

Let $\mathcal{A}_{v_2}^{y^*} = \{M'_1 \ldots, M'_p\}$. By $\mathcal{A}_{v_2}^{y^*} \subseteq \mathcal{A}_{v_1}^{y^*}$, we have that, for $i = 1, \ldots, p$, $M'_i = \delta(v_1, v_2, V_3^i, V_4^i, \ldots, V_{d_{M'_i}}^i)$, where V_3^i and V_4^i contain respectively v_3 and v_4. We define, for each $i = 1, \ldots, p$, the multicut M''_i as the maximal multicut contained in $M'_i \setminus \delta(v_2)$, that means $M''_i = \delta(v_1, v_2 \cup V_3^i \cup V_4^i, \ldots, V_{d_{M'_i}}^i)$. Since $\delta(v_2)$ is active for y^*, by Claim 4.18.8-(*ii.*), we have $\alpha_{v_2}^{y^*} + y^*_{\delta(v_2)} = 1$. Then, the point $y^1 \in \mathbb{Q}^{\mathcal{M}_G}$ defined by:

$$y^1 = y^* - \left(y^*_{\delta(v_2)} \xi_{\delta(v_2)} + \sum_{i=1}^{p} y^*_{M'_i} \xi_{M'_i} \right) + \left(\sum_{i=1}^{p} y^*_{M'_i} \xi_{M''_i} \right),$$

is a solution to $\mathcal{D}_{(G,c')}$, where $c' = c - \chi^{\delta(v_2)}$.

By $d_{M''_i} = d_{M'_i} - 2$ for all $i = 1, \ldots, p$, and $y^*_{\delta(v_2)} + \alpha_{v_2}^{y^*} = 1$, we have:

$$b^\top y^1 = b^\top y^* - \alpha_{v_2}^{y^*}(2h-2) - y^*_{\delta(v_2)}(2h-1) = b^\top y^* - (2h-1) - \alpha_{v_2}^{y^*}. \tag{4.39}$$

By minimality assumption (ii), $\mathcal{D}_{(G,c')}$ admits an integer optimal solution, say y^2. The point $y^3 \in \mathbb{Z}^{\mathcal{M}_G}$ defined by $y^3 = y^2 + \xi_{\delta(v_2)}$ is a solution to $\mathcal{D}_{(G,c)}$ such that:

$$b^\top y^3 = b^\top y^2 + 2h + 1. \tag{4.40}$$

Therefore, by (4.39), the optimality of y^2, and (4.40), we have:

$$b^\top y^* = b^\top y^1 + 2h + 1 + \alpha_{v_2}^{y^*} \le b^\top y^2 + 2h + 1 + \alpha_{v_2}^{y^*} = b^\top y^3 + \alpha_{v_2}^{y^*}. \tag{4.41}$$

By integrality of $P_{2h+1}(G)$ and duality, we have that $b^\top y^* \in \mathbb{Z}$. Furthermore, y^3 is integer by construction, so $b^\top y^3 \in \mathbb{Z}$. Then, by (4.41) and Claim 4.18.6, we have that $b^\top y^* \le b^\top y^3$, and so y^3 is an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction. ■

Claims 4.18.1, 4.18.9, 4.18.10, 4.18.11 and Proposition 4.7 imply that G is not series-parallel, a contradiction. ■

The following corollary summarizes the results we achieved for $P_k(G)$, when k is odd.

Corollary 4.19. *System* (4.5) *is box-TDI if and only if G is series-parallel.*

Proof. When G is not series-parallel, System (4.5) does not describe an integer polyhedron by Observation 4.10 and Theorems 4.11 and 4.12. Thus System (4.5) is not TDI by Theorem 2.1.

Whenever G is series-parallel, $P_k(G)$ is box-TDI by Theorem 4.14 and System (4.5) is TDI by Theorem 4.18. Theorem 2.18 concludes. ■

The results presented in this section finalize our contribution to the topic: Theorems 4.14, 4.15, and 4.18, show that, when G is series-parallel, $P_k(G)$ is a box-TDI polyhedron and provide a TDI system with integer coefficients describing $P_k(G)$.

4.8 Conclusions and Perspectives

In this chapter we studied the box-TDIness of $P_k(G)$, that is the convex hull of the k-edge-connected spanning subgraphs of G.

We first used the properties of polymatroids and some classical results on box-TDIness to prove that $P_1(G)$ is box-TDI for all choices of G, and we reported an already known integer totally dual integral system that describes this polyhedron.

Then, we proved that $P_k(G)$ is a box-TDI polyhedron if and only if G is a series-parallel graph. This result is an extension of the work of Chen, Ding, and Zang [25], that analyzed the case $k = 2$. The box-total dual integrality of $P_k(G)$ immediately stems from their result, when G is series-parallel and k is even. For the case k odd, we started from scratch and relied on the recent characterization of box-TDI polyhedra given in [28].

Since no integer box-TDI system describing $P_k(G)$ was previously known, we provided a TDI system with integer coefficients describing $P_k(G)$ for all k even.

We concluded by showing that the system describing $P_k(G)$ when $k = 2h + 1$ for some integer $h \geq 1$ is TDI when G is series-parallel.

It has to be remarked that all the results achieved in this chapter are valid also under the restrictive assumption that in a k-edge-connected spanning subgraph each edge can appear at most once.

Even if we consider the topic extensively explored, there are some questions that are unanswered.

The polyhedron $P_k(G)$ is contained in the first orthant $\{x \in \mathbb{R}^E : x \geq \mathbf{0}\}$. However, it seems reasonable that the removal of the nonnegativity constraints does not undermine the box-TDIness of $P_k(G)$ or the TDIness of the systems that describe it. If this intuition is truthful, we could unify the results of Chapters 3 and 4. Indeed, in this case the flow cone would correspond to $P_0(G)$.

In this chapter, we did not investigate the minimality of the systems we treated. Hence, a future development could be to find the Schrijver systems describing $P_k(G)$ and $P_k(G) \cap \{x \leq \mathbf{1}\}$ when G is series-parallel, as well as the Schrijver system of $P_1(G)$ for a generic graph G.

Conclusions

In this thesis we explored box-total dual integrality of some systems and polyhedra associated with multicuts and graph connectivity. We focused on two polyhedra: the flow cone and the k-edge-connected spanning subgraph polyhedron.

In Chapter 3, we studied the flow cone. For this polyhedron, we provided a system with integer coefficients that is totally dual integral if and only if the associated graph is series-parallel. The flow cone being a box-totally dual integral polyhedron for this class of graphs, the system given is box-totally dual integral.

Then, we characterized the multicuts of series-parallel graphs that are not disjoint union of other multicuts. This gave us the minimal TDI system with integer coefficients – also known as Schrijver System – describing the flow cone of series-parallel graphs.

We also provided a TDI system with half-integer coefficients describing the flow cone of planar graphs, and we proposed some possible direction of research in this field.

In Chapter 4, we studied the k-edge-connected spanning subgraph polyhedron $P_k(G)$. First, we showed that $P_1(G)$ is a box-TDI polyhedron for all graphs G, and we reported a TDI system with integer coefficients describing it. Then, we proved that for each fixed $k \geq 2$, $P_k(G)$ is a box-TDI polyhedron if and only if G is a series-parallel graph. This result extends the work of Chen, Ding, and Zang [25]. The description of $P_k(G)$ being dependent to the parity of k, we studied separately the systems that describe $P_k(G)$ depending whether k is even or odd.

In the case k even, we reinforced the results of Chen, Ding, and Zang [25] by providing a TDI system with integer coefficients for series-parallel graphs. This system being TDI for a larger class of graphs, this leaves open the question on which is the maximal class of graphs for which the system given is TDI.

In the case k odd, we showed that the integer system describing $P_k(G)$ provided by Chopra [31] and Didi Biha and Mahjoub [50] is TDI if and only if G is series-parallel.

From a more general point of view, this thesis presents an interesting methodology to deal with box-totally dual integral systems.

Until recent times, the most efficient way to prove that a system is box-TDI was either to prove that the set of active constraints for each face form a totally unimodular matrix, or to use Theorem 2.21. The relatively recent introduction of the ESP property gave a boost to the research of box-TDI systems and polyhedra. Nevertheless, proving the box-TDIness of a system was substantially equivalent to proving the box-TDIness of a polyhedron.

On the contrary, the recent characterization of box-TDI polyhedra in terms of equimodular face-defining matrices allows us to change perspective: instead of directly proving the box-TDIness of a system, we can now use a "*divide et impera*" approach and prove separately the TDIness of the system and the box-TDIness of the polyhedron described. This potentially simplify the whole process, especially when a TDI system is already known.

We followed this approach in Chapter 4: we first proved the box-TDIness of the k-edge-connected spanning subgraph polyhedron, and then showed a TDI system describing it. The systems obtained this way are box-TDI.

Perspectives

In Chapters 3 and 4, we suggested some open problems related with our results. We discuss here more general extensions of the work done in this thesis.

The results we achieved are on polyhedra defined by multicuts. Indeed, the polyhedra we proved to be box-TDI are of the kind

$$P_b = \{x \in \mathbb{R}^E_+ : x(M) \geq b_M \text{ for all multicuts } M \text{ of } G = (V, E)\},$$

for some $b \in \mathbb{Z}^{\mathcal{M}_G}$ and G series-parallel graph. It could be interesting explore which properties must have b to imply the box-TDIness of P_b. The problem is not trivial, indeed we can easily find some examples of b such that P_b is not box-TDI.

Furthermore, we could consider if similar problems on series-parallel graphs are defined on box-TDI polyhedra.

In a more general way, we aim to use the "*divide et impera*" approach we followed in Chapter 4, to prove the box-TDIness of systems and polyhedra that resulted hard to treat otherwise.

Bibliography

[1] Ageev, A. A. On finding the maximum number of disjoint cuts in seymour graphs. In *European Symposium on Algorithms* (1999), Springer, pp. 490–497.

[2] Alspach, B., Goddyn, L., and Zhang, C. Q. Graphs with the circuit cover property. *Transactions of the American Mathematical Society 344*, 1 (1994), 131–154.

[3] Barahona, F. Separating from the dominant of the spanning tree polytope. *Operations research letters 12*, 4 (1992), 201–203.

[4] Barahona, F., and Mahjoub, A. R. On two-connected subgraph polytopes. *Discrete Mathematics 147*, 1-3 (1995), 19–34.

[5] Barbato, M., Grappe, R., Lacroix, M., Lancini, E., and Wolfler Calvo, R. The Schrijver system of the flow cone in series-parallel graphs. *To appear in Discrete Applied Mathematics* (2020).

[6] Bendali, F., Diarrassouba, I., Biha, M. D., Mahjoub, A. R., and Mailfert, J. The k-edge connected subgraph problem: Valid inequalities and branch-and-cut. In *2007 6th International Workshop on Design and Reliable Communication Networks* (2007), IEEE, pp. 1–8.

[7] Bendali, F., Diarrassouba, I., Mahjoub, A. R., Didi Biha, M., and Mailfert, J. A branch-and-cut algorithm for the k-edge connected subgraph problem. *Networks 55*, 1 (2010), 13–32.

[8] Bergner, M., Lübbecke, M. E., and Witt, J. T. A branch-price-and-cut algorithm for packing cuts in undirected graphs. In *International Symposium on Experimental Algorithms* (2014), Springer, pp. 34–45.

[9] Bondy, J. A., and Murty, U. S. R. *Graph theory with applications*, vol. 290. Macmillan London, 1976.

[10] Borne, S., Fouilhoux, P., Grappe, R., Lacroix, M., and Pesenau, P. Circuit and bond polytopes on series-parallel graphs. *Discrete Optimization 17* (2015), 55–68.

[11] Botton, Q., Fortz, B., Gouveia, L., and Poss, M. Benders decomposition for the hop-constrained survivable network design problem. *INFORMS journal on computing 25*, 1 (2013), 13–26.

[12] Boyd, S. C., and Hao, T. An integer polytope related to the design of survivable communication networks. *SIAM Journal on Discrete Mathematics 6*, 4 (1993), 612–630.

[13] Cameron, K. *Polyhedral and algorithmic ramifications of antichains.* PhD thesis, University of Waterloo, 1983.

[14] Cameron, K. A min-max relation for the partial q-colouring of a graph. part ii: Box perfection. *Graph Colouring and Variations 39* (1989), 15.

[15] Cameron, K., and Edmonds, J. Coflow polyhedra. *The Julius Petersen graph theory centennial 6* (1992), 331.

[16] Camion, P. Characterization of totally unimodular matrices. *Proceedings of the American Mathematical Society 16*, 5 (1965), 1068–1073.

[17] Caprara, A., Panconesi, A., and Rizzi, R. Packing cuts in undirected graphs. *Networks: An International Journal 44*, 1 (2004), 1–11.

[18] Carathéodory, C. Über den variabilitätsbereich der koeffizienten von potenzreihen, die gegebene werte nicht annehmen. *Mathematische Annalen 64*, 1 (1907), 95–115.

[19] Chandrasekaran, R. Polynomial algorithms for totally dual integral systems and extensions. In *North-Holland Mathematics Studies*, vol. 59. Elsevier, 1981, pp. 39–51.

[20] Chen, Q., Chen, X., and Zang, W. A polyhedral description of kernels. *Mathematics of Operations Research 41*, 3 (2016), 969–990.

[21] Chen, X., Chen, Z., and Zang, W. A unified approach to box-mengerian hypergraphs. *Mathematics of Operations Research 35*, 3 (2010), 655–668.

[22] Chen, X., Chen, Z., and Zang, W. Total dual integrality in some facility location problems. *SIAM Journal on Discrete Mathematics 26*, 3 (2012), 1022–1030.

[23] Chen, X., Diao, Z., Hu, X., and Tang, Z. Total dual integrality of triangle covering. In *International Conference on Combinatorial Optimization and Applications* (2016), Springer, pp. 128–143.

[24] Chen, X., Ding, G., and Zang, W. A characterization of box-mengerian matroid ports. *Mathematics of Operations Research 33*, 2 (2008), 497–512.

[25] Chen, X., Ding, G., and Zang, W. The box-TDI system associated with 2-edge connected spanning subgraphs. *Discrete Applied Mathematics 157*, 1 (2009), 118–125.

[26] Chen, X., Hu, X., and Zang, W. Dual integrality in combinatorial optimization. *Handbook of Combinatorial Optimization* (2013), 995–1063.

[27] Cheriyan, J., and Thurimella, R. Approximating minimum-size k-connected spanning subgraphs via matching. *SIAM Journal on Computing 30*, 2 (2000), 528–560.

[28] Chervet, P., Grappe, R., and Robert, L.-H. Principally box-integer polyhedra and equimodular matrices. *To appear in Mathematical Programming* (2019).

[29] Chopra, S. The equivalent subgraph and directed cut polyhedra on series-parallel graphs. *SIAM Journal on Discrete Mathematics 5*, 4 (1992), 475–490.

[30] Chopra, S. The graph partitioning polytope on series-parallel and 4-wheel free graphs. *SIAM journal on discrete mathematics 7*, 1 (1994), 16–31.

[31] Chopra, S. The k-edge-connected spanning subgraph polyhedron. *SIAM Journal on Discrete Mathematics 7*, 2 (1994), 245–259.

[32] Conforti, M., and Cornuéjols, G. Balanced 0,±1-matrices, bicoloring and total dual integrality. *Mathematical Programming 71*, 3 (1995), 249–258.

[33] Cook, S. A. The complexity of theorem-proving procedures. In *Proceedings of the third annual ACM symposium on Theory of computing* (1971), ACM, pp. 151–158.

[34] Cook, W. A minimal totally dual integral defining system for the b-matching polyhedron. *SIAM Journal on Algebraic Discrete Methods 4*, 2 (1983), 212–220.

[35] Cook, W. Operations that preserve total dual integrality. *Operations Research Letters 2*, 1 (1983), 31–35.

[36] Cook, W. On box totally dual integral polyhedra. *Mathematical Programming 34*, 1 (1986), 48–61.

[37] Cook, W., Fonlupt, J., and Schrijver, A. An integer analogue of carathéodory's theorem. *J. Comb. Theory, Ser. B 40*, 1 (1986), 63–70.

[38] Cook, W., Lovász, L., and Schrijver, A. A polynomial-time test for total dual integrality in fixed dimension. In *Mathematical programming at Oberwolfach II*. Springer, 1984, pp. 64–69.

[39] Cook, W., and Pulleyblank, W. R. Linear systems for constrained matching problems. *Mathematics of Operations Research 12*, 1 (1987), 97–120.

[40] Cornaz, D. Max-multiflow/min-multicut for G+H series-parallel. *Discrete Mathematics 311*, 17 (2011), 1957 – 1967.

[41] Cornaz, D., Grappe, R., and Lacroix, M. Trader multiflow and box-TDI systems in series–parallel graphs. *Discrete Optimization 31* (2019), 103–114.

[42] Cornaz, D., Magnouche, Y., and Mahjoub, A. R. On minimal two-edge-connected graphs. In *Control, Decision and Information Technologies (CoDIT), 2014 International Conference on* (2014), IEEE, pp. 251–256.

[43] Cornuéjols, G., Fonlupt, J., and Naddef, D. The traveling salesman problem on a graph and some related integer polyhedra. *Mathematical programming 33*, 1 (1985), 1–27.

[44] Cunningham, W. H., and Marsh, A. A primal algorithm for optimum matching. In *Polyhedral Combinatorics*. Springer, 1978, pp. 50–72.

[45] Danilov, V. I., and Koshevoy, G. A. Discrete convexity and unimodularity—i. *Advances in Mathematics 189*, 2 (2004), 301–324.

[46] Diarrassouba, I., Gabrel, V., Mahjoub, A. R., Gouveia, L., and Pesneau, P. Integer programming formulations for the k-edge-connected 3-hop-constrained network design problem. *Networks 67*, 2 (2016), 148–169.

[47] Didi Biha, M., Kerivin, H., and Mahjoub, A. R. Steiner trees and polyhedra. *Discrete Applied Mathematics 112*, 1-3 (2001), 101–120.

[48] Didi Biha, M., Kerivin, H. L., and Ng, P. H. Polyhedral study of the connected subgraph problem. *Discrete Mathematics 338*, 1 (2015), 80–92.

[49] Didi Biha, M., and Mahjoub, A. The k-edge connected subgraph problem i: Polytopes and critical extreme points. *Linear algebra and its applications 381* (2004), 117–139.

[50] Didi Biha, M., and Mahjoub, A. R. k-edge connected polyhedra on series-parallel graphs. *Operations research letters 19*, 2 (1996), 71–78.

[51] Didi Biha, M., and Mahjoub, A. R. Steiner k-edge connected subgraph polyhedra. *Journal of Combinatorial Optimization 4*, 1 (2000), 131–144.

[52] Diestel, R. Graph theory. 2005. *Grad. Texts in Math 101* (2005).

[53] Ding, G., Feng, L., and Zang, W. The complexity of recognizing linear systems with certain integrality properties. *Mathematical Programming 114*, 2 (2008), 321–334.

[54] Ding, G., Tan, L., and Zang, W. When is the matching polytope box-totally dual integral? *Mathematics of Operations Research 43*, 1 (2017), 64–99.

[55] Ding, G., and Zang, W. Packing cycles in graphs. *Journal of Combinatorial Theory, Series B 86*, 2 (2002), 381–407.

[56] Ding, G., Zang, W., and Zhao, Q. On box-perfect graphs. *Journal of Combinatorial Theory, Series B 128* (2018), 17–46.

[57] Dubey, D., and Neogy, S. Total dual integrality and integral solutions of the linear complementarity problem. *Linear Algebra and its Applications 557* (2018), 359–374.

[58] Dueck, P., Hoşten, S., and Sturmfels, B. Normal toric ideals of low codimension. *Journal of Pure and Applied Algebra 213*, 8 (2009), 1636–1641.

[59] Duffin, R. J. Topology of series-parallel networks. *Journal of Mathematical Analysis and Applications 10*, 2 (1965), 303–318.

[60] Edmonds, J. Covers and packings in a family of sets. *Bulletin of the American Mathematical Society 68*, 5 (1962), 494–499.

[61] Edmonds, J. Maximum matching and a polyhedron with 0, 1-vertices. *Journal of research of the National Bureau of Standards B 69*, 125-130 (1965), 55–56.

[62] Edmonds, J. Submodular functions, matroids, and certain polyhedra. *Combinatorial structures and their applications* (1970), 69–87.

[63] Edmonds, J., and Giles, R. A min-max relation for submodular functions on graphs. In *Annals of Discrete Mathematics*, vol. 1. Elsevier, 1977, pp. 185–204.

[64] Edmonds, J., and Giles, R. Total dual integrality of linear inequality systems. In *Progress in Combinatorial Optimization*, W. R. PULLEYBLANK, Ed. Academic Press, 1984, pp. 117 – 129.

[65] Elias, P., Feinstein, A., and Shannon, C. A note on the maximum flow through a network. *IRE Transactions on Information Theory 2*, 4 (1956), 117–119.

[66] Eppstein, D. Parallel recognition of series-parallel graphs. *Information and Computation 98*, 1 (1992), 41 – 55.

[67] Erickson, R. E., Monma, C. L., and Veinott Jr, A. F. Send-and-split method for minimum-concave-cost network flows. *Mathematics of Operations Research 12*, 4 (1987), 634–664.

[68] Fonlupt, J., and Mahjoub, A. R. Critical extreme points of the 2-edge connected spanning subgraph polytope. *Mathematical programming 105*, 2-3 (2006), 289–310.

[69] Ford, L., and Fulkerson, D. Maximal flow through a network. *Canadian Journal of Mathematics 8* (1956), 399 – 404.

[70] Fortz, B. *Design of survivable networks with bounded rings.* Springer Science & Business Media, 2013.

[71] Fouilhoux, P., Karasan, O. E., Mahjoub, A. R., Özkök, O., and Yaman, H. Survivability in hierarchical telecommunications networks. *Networks 59*, 1 (2012), 37–58.

[72] Frank, A. Kernel systems of directed graphs. *Acta Sci. Math.(Szeged) 41*, 1-2 (1979), 63–76.

[73] Frank, A. On the orientation of graphs. *Journal of Combinatorial Theory, Series B 28*, 3 (1980), 251–261.

[74] Frank, A., and Király, T. A survey on covering supermodular functions. In *Research Trends in Combinatorial Optimization*. Springer, 2009, pp. 87–126.

[75] Frank, A., and Tardos, É. Matroids from crossing families. In *Finite and Infinite Sets*. Elsevier, 1984, pp. 295–304.

[76] Fulkerson, D. R. Blocking and anti-blocking pairs of polyhedra. *Mathematical programming 1*, 1 (1971), 168–194.

[77] Gabow, H. N., Goemans, M. X., Tardos, É., and Williamson, D. P. Approximating the smallest k-edge connected spanning subgraph by lp-rounding. *Networks: An International Journal 53*, 4 (2009), 345–357.

[78] Garey, M. R., and Johnson, D. S. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., New York, NY, USA, 1979.

[79] Gerards, A. M. H., and Sebő, A. Total dual integrality implies local strong unimodularity. *Mathematical Programming 38*, 1 (Jul 1987), 69–73.

[80] Ghouila-Houri, A. Caractérisation des matrices totalement unimodulaires. *Comptes Redus Hebdomadaires des Séances de l'Académie des Sciences (Paris) 254* (1962), 1192–1194.

[81] Giles, F., and Orlin, J. Verifying total dual integrality, 1981.

[82] Giles, F. R., and Pulleyblank, W. R. Total dual integrality and integer polyhedra. *Linear algebra and its applications 25* (1979), 191–196.

[83] Goddyn, L., Huynh, T., and Deshpande, T. On hilbert bases of cuts. *Discrete Mathematics 339*, 2 (2016), 721–728.

[84] Grappe, R., and Lacroix, M. The st-bond polytope on series-parallel graphs. *RAIRO-Operations Research 52*, 3 (2018), 923–934.

[85] Gröflin, H., and Hoffman, A. J. Lattice polyhedra ii: generalization, constructions and examples. In *North-Holland Mathematics Studies*, vol. 65. Elsevier, 1982, pp. 189–203.

[86] Grötschel, M., Lovász, L., and Schrijver, A. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica 1*, 2 (1981), 169–197.

[87] Grötschel, M., Lovász, L., and Schrijver, A. *Geometric algorithms and combinatorial optimization*. Springer-Verlag, 1988.

[88] Grötschel, M., and Monma, C. L. Integer polyhedra arising from certain network design problems with connectivity constraints. *SIAM Journal on Discrete Mathematics 3*, 4 (1990), 502–523.

[89] Grötschel, M., Monma, C. L., and Stoer, M. Polyhedral approaches to network survivability. In *Reliability of computer and Communication Networks* (1989), pp. 121–142.

[90] Grötschel, M., Monma, C. L., and Stoer, M. Computational results with a cutting plane algorithm for designing communication networks with low-connectivity constraints. *Operations Research 40*, 2 (1992), 309–330.

[91] Grötschel, M., Monma, C. L., and Stoer, M. Facets for polyhedra arising in the design of communication networks with low-connectivity constraints. *SIAM Journal on Optimization 2*, 3 (1992), 474–504.

[92] Grötschel, M., Monma, C. L., and Stoer, M. Polyhedral and computational investigations for designing communication networks with high survivability requirements. *Operations Research 43*, 6 (1995), 1012–1024.

[93] Heller, I. On linear systems with integral valued solutions. *Pacific Journal of Mathematics 7*, 3 (1957), 1351–1364.

[94] Heller, I., and Tompkins, C. An extension of a theorem of dantzig's. *Linear inequalities and related systems 38* (1956), 247–254.

[95] Hemmecke, R. On the computation of hilbert bases of cones. In *Mathematical software*. World Scientific, 2002, pp. 307–317.

[96] Henk, M., and Weismantel, R. On hilbert bases of polyhedral cones.

[97] Henk, M., and Weismantel, R. The height of minimal hilbert bases. *Results in mathematics 32*, 3-4 (1997), 298–303.

[98] Hoffman, A. A generalization of max flow—min cut. *Mathematical Programming 6*, 1 (1974), 352–359.

[99] Hoffman, A. J., and Kruskal, J. B. Integral boundary points of convex polyhedra. *Linear inequalities and related systems 38* (1956), 223–246.

[100] Hoşten, S., and Thomas, R. R. Gomory integer programs. *Mathematical Programming 96*, 2 (2003), 271–292.

[101] Huygens, D., Mahjoub, A. R., and Pesneau, P. Two edge-disjoint hop-constrained paths and polyhedra. *SIAM Journal on Discrete Mathematics 18*, 2 (2004), 287–312.

[102] Karmarkar, N. A new polynomial-time algorithm for linear programming. In *Proceedings of the sixteenth annual ACM symposium on Theory of computing* (1984), ACM, pp. 302–311.

[103] Karp, R. M. Reducibility among combinatorial problems. In *Complexity of computer computations*. Springer, 1972, pp. 85–103.

[104] Karp, R. M., and Papadimitriou, C. H. On linear characterizations of combinatorial optimization problems. *SIAM Journal on Computing 11*, 4 (1982), 620–632.

[105] Kerivin, H., and Mahjoub, A. R. Design of survivable networks: A survey. *Networks: An International Journal 46*, 1 (2005), 1–21.

[106] Khachiyan, L. G. A polynomial algorithm in linear programming. In *Doklady Academii Nauk SSSR* (1979), vol. 244, pp. 1093–1096.

[107] Ko, C., and Monma, C. Heuristic methods for designing highly survivable communication networks. In *Technical report*. Bell Communications Research Chichester, 1989.

[108] Kuratowski, C. Sur le probleme des courbes gauches en topologie. *Fundamenta mathematicae 15*, 1 (1930), 271–283.

[109] Laurent, M. Hilbert bases of cuts. *Discrete Mathematics 150*, 1 (1996), 257 – 279.

[110] Lucchesi, C. L., and Younger, D. H. A minimax theorem for directed graphs. *Journal of the London Mathematical Society 2*, 3 (1978), 369–374.

[111] Magnouche, Y., and Martin, S. The multi-terminal vertex separator problem: Polytope characterization and TDI-ness. In *International Symposium on Combinatorial Optimization* (2016), Springer, pp. 320–331.

[112] Mahjoub, A. R. Two-edge connected spanning subgraphs and polyhedra. *Mathematical Programming 64*, 1-3 (1994), 199–208.

[113] Mahjoub, A. R. On perfectly two-edge connected graphs. *Discrete Mathematics 170*, 1-3 (1997), 153–172.

[114] Martínez, F. J. Z. Series–parallel graphs are windy postman perfect. *Discrete Mathematics 308*, 8 (2008), 1366–1374.

[115] Meyer, C. D. *Matrix analysis and applied linear algebra*, vol. 71. Siam, 2000.

[116] Nash-Williams, C. S. J. Well-balanced orientations of finite graphs and unobtrusive odd-vertex-pairings. *Recent Progress in Combinatorics* (1969), 133–149.

[117] O'Shea, E., and Sebő, A. Characterizations of total dual integrality. In *International Conference on Integer Programming and Combinatorial Optimization* (2007), Springer, pp. 382–396.

[118] O'Shea, E., and Sebö, A. Alternatives for testing total dual integrality. *Mathematical programming 132*, 1-2 (2012), 57–78.

[119] Padberg, M., and Rao, M. The russian method for linear programming iii: Bounded integer programming. Tech. rep., Technical Report 81-39, New York University, Graduate School of Business and . . . , 1981.

[120] Pap, J. Recognizing conic TDI systems is hard. *Mathematical programming 128*, 1-2 (2011), 43–48.

[121] Papadimitriou, C. H., and Yannakakis, M. On recognizing integer polyhedra. *Combinatorica 10*, 1 (1990), 107–109.

[122] Prodon, A. A polyhedron for steiner trees in series-parallel graphs. *Département de Mathématiques, École Polytechnique Fédérale de Lausanne* (1985).

[123] Pulleyblank, W. Chapter v polyhedral combinatorics. In *Optimization*, vol. 1 of *Handbooks in Operations Research and Management Science*. Elsevier, 1989, pp. 371 – 446.

[124] Pulleyblank, W. R. Total dual integrality and b-matchings. *Operations Research Letters 1*, 1 (1981), 28–30.

[125] Pulleyblank, W. R. Polyhedral combinatorics. In *Mathematical Programming the State of the Art*. Springer, 1983, pp. 312–345.

[126] Robertson, N., and Seymour, P. Graph minors. xx. wagner's conjecture. *Journal of Combinatorial Theory, Series B 92*, 2 (2004), 325 – 357. Special Issue Dedicated to Professor W.T. Tutte.

[127] Scheffler, P. *A practical linear time algorithm for disjoint paths in graphs with bounded tree-width*. TU, Fachbereich 3, 1994.

[128] Schrijver, A. On total dual integrality. *Linear algebra and its applications 38* (1981), 27–32.

[129] Schrijver, A. Min-max relations for directed graphs. In *Bonn Workshop on Combinatorial Optimization (Bonn, 1980)* (1982), vol. 16, pp. 261–280.

[130] Schrijver, A. Short proofs on the matching polyhedron. *Journal of Combinatorial Theory, Series B 34*, 1 (1983), 104–108.

[131] Schrijver, A. Proving total dual integrality with cross-free families—a general framework. *Mathematical Programming 29*, 1 (1984), 15–27.

[132] Schrijver, A. Total dual integrality from directed graphs, crossing families, and sub-and supermodular functions. In *Progress in combinatorial optimization*. Elsevier, 1984, pp. 315–361.

[133] Schrijver, A. Polyhedral combinatorics. In *Handbook of combinatorics*, M. Grotschel, R. Graham, and L. Lovasz, Eds. Elsevier Science, 1995, pp. 1649–1704.

[134] Schrijver, A. *Theory of linear and integer programming*. John Wiley & Sons, 1998.

[135] Schrijver, A. *Combinatorial optimization: polyhedra and efficiency*, vol. 24. Springer Science & Business Media, 2003.

[136] Schrijver, A., and Seymour, P. A proof of total dual integrality of matching polyhedra. *Stichting Mathematisch Centrum. Zuivere Wiskunde*, ZN 79/77 (1977).

[137] Sebő, A. The schrijver system of odd join polyhedra. *Combinatorica 8*, 1 (1988), 103–116.

[138] Sebő, A. Hilbert bases, Caratheodory's theorem and combinatorial optimization. In *Proceedings of the 1st Integer Programming and Combinatorial Optimization Conference* (1990), University of Waterloo Press, pp. 431–455.

[139] Seymour, P. D. Sums of circuits. In *Graph theory and related topics* (1979), J. A. Bondy and U. S. R. Murty, Eds., vol. 1, Academic Press, pp. 341–355.

[140] Seymour, P. D. Decomposition of regular matroids. *Journal of combinatorial theory, Series B 28*, 3 (1980), 305–359.

[141] Seymour, P. D. Applications of the regular matroid decomposition. *Matroid theory (Szeged, 1982) 40* (1985), 345–357.

[142] Seymour, P. D. Colouring series-parallel graphs. *Combinatorica 10*, 4 (1990), 379–392.

[143] Shen, S., and Smith, J. C. Polynomial-time algorithms for solving a class of critical node problems on trees and series-parallel graphs. *Networks 60*, 2 (2012), 103–119.

[144] Silva, M. K., and Tunçel, L. A notion of total dual integrality for convex, semidefinite, and extended formulations. *arXiv preprint arXiv:1801.09155* (2018).

[145] Stoer, M. *Design of survivable networks.* Springer, 2006.

[146] Sumita, H., Kakimura, N., and Makino, K. Total dual integrality of the linear complementarity problem. *Annals of Operations Research* (2018), 1–23.

[147] Takamizawa, K., Nishizeki, T., and Saito, N. Linear-time computability of combinatorial problems on series-parallel graphs. *Journal of the ACM (JACM) 29*, 3 (1982), 623–641.

[148] Tardos, E. A strongly polynomial algorithm to solve combinatorial linear programs. *Operations Research 34*, 2 (1986), 250–256.

[149] Telle, J. A., and Proskurowski, A. Algorithms for vertex partitioning problems on partial k-trees. *SIAM Journal on Discrete Mathematics 10*, 4 (1997), 529–550.

[150] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 8.8)*, 2019. `https://www.sagemath.org`.

[151] Truemper, K. A decomposition theory for matroids. v. testing of matrix total unimodularity. *Journal of Combinatorial Theory, Series B 49*, 2 (1990), 241–281.

[152] Vandenbussche, D., and Nemhauser, G. L. The 2-edge-connected subgraph polyhedron. *Journal of combinatorial optimization 9*, 4 (2005), 357–379.

[153] Winter, P. Generalized steiner problem in series-parallel networks. *Journal of Algorithms 7*, 4 (1986), 549–566.

[154] Yonebayashi, Y. An algorithmic approach to dual integrality of matching and extensions. *Discrete Optimization* (2018).

[155] Zhao, K. A simple proof of Whitney's theorem on connectivity in graphs. *Mathematica Bohemica 136*, 1 (2011), 25–26.