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Dynamique en temps long et en temps fini de l'équation de Schrödinger non-linéaire en dehors d'un obstacle

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Résumé

Cette thèse est consacrée à l'étude de la dynamique de l'équation de Schrödinger non-linéaire (NLS) focalisante en dehors d'un obstacle compact et convexe, avec des conditions de Dirichlet au bord de l'obstacle. Nous nous intéressons à l'étude du comportement asymptotique des solutions en temps long et en temps fini. Dans cette thèse, nous prouvons l'existence de ces trois types de solutions: ondes solitaires (solitons), solutions "explosives " (formant des singularités en temps fini) et des solutions dispersives (des solutions globales et se comportant asymptotiquement comme des solutions linéaires) pour l'équation NLS en dehors d'un obstacle convexe.

Dans la première partie de la thèse, nous construisons des ondes solitaires, se rapprochant, en temps long, des ondes solitaires existant pour l'équation NLS posée sur l'espace euclidien sans obstacle. Ainsi, ces ondes montre l'optimalité d'un seuil d'énergie en dessous duquel toutes les solutions de l'équation sont globales et ont un comportement asymptotique linéaire.

Dans la deuxième partie de la thèse, nous prouvons l'existence des solutions explosives pour l'équation NLS à l'extérieur d'une boule. Nous prouvons que les solutions d'énergie négative et de variance finie forment une singularité en temps fini (ceci été conjecturé mais non démontré). Dans certains cas, nous étudions également le comportement des solutions sous le seuil masse-énergie mentionné ci-dessus.

Dans la troisième partie de la thèse, nous étudions la dynamique de l'équation NLS en dehors d'un obstacle convexe exactement au seuil masse-énergie, c'est à dire lorsque la masse et l'énergie de la donnée initiale sont égales à la masse et à l'énergie du soliton. Nous montrons que la solution est globale en temps et se disperse en temps long.

Dans la dernière partie de la thèse, nous présentons des simulations numériques pour l'équation NLS en dehors d'un obstacle compact et convexe. Nous étudions l'interaction entre les solutions de type ondes solitaires (solitons) se déplaçant à différentes vitesses vers l'obstacle sous différents angles. Ainsi, nous montrons que la présence de l'obstacle modifie globalement le comportement des solutions.

Mots clés : Équation de Schrödinger non-linéaire, obstacle, onde solitaire, solution explosive, dispersion.

Abstract

The main objective of this thesis is to study the dynamics of the focusing nonlinear Schrödinger equation (NLS) in the exterior of a compact and strictly convex obstacle, with Dirichlet boundary conditions. We study the asymptotic behavior of the solution for large times and finite time. We prove the existence of these types of solutions: solitary wave solutions (solitons), blow-up solutions (solutions with finite time of existence), and scattering solutions (global and behaving asymptotically as linear solutions), for the NLS equation in the exterior of a convex obstacle.

We first construct solitary wave solutions for the NLS in the exterior of a strictly convex obstacle. These solutions behave asymptotically as solitary waves on $\mathbb{R}^3$ for large times and satisfy Dirichlet boundary conditions. These soliton solutions prove the optimality of the mass-energy threshold for global existence and scattering.

Secondly, we prove the existence of blow-up solutions for the NLS in the exterior of a ball. We prove that finite variance, negative energy solutions break down in finite time. In some cases, we also study the behavior of solutions under the mass-energy threshold mentioned above.

Next, we study the dynamics of the focusing $3d$ cubic NLS equation in the exterior of a strictly convex obstacle at exactly the mass-energy threshold (i.e., if the initial data has the mass-energy equal to that of a soliton solution). In this case, we prove that the solution is global in time and scatters in both time directions.

Finally, we present numerical simulations for the focusing nonlinear Schrödinger equation in the exterior of a smooth, compact, strictly convex obstacle, with Dirichlet boundary conditions. We study the interaction between solitary wave solutions (solitons) traveling with different velocities towards the obstacle at different angles, and show how the obstacle changes the overall behavior of solutions.

Keywords : Nonlinear Schrödinger equation, obstacle, solitary wave, blow-up, scattering.

Contents

Chapter I

Introduction

The subject of this thesis is to study the dynamics of the nonlinear Schrödinger equation with a focusing nonlinearity, which is typically of power type, in the exterior of a compact and strictly convex obstacle Θ with Dirichlet boundary conditions in dimension $d \geq 2$.

We consider $\Omega = \mathbb{R}^d \setminus \Theta$ the exterior of the obstacle Θ and the Cauchy problem

$$\begin{cases} i\partial_t u + \Delta_\Omega u = -|u|^{p-1}u & \forall (t,x) \in \mathbb{R} \times \Omega, \\ u(t_0,x) = u_0(x) & \forall x \in \Omega, \\ u(t,x) = 0 & \forall (t,x) \in \mathbb{R} \times \partial\Omega. \end{cases} \tag{ NLS$_\Omega$}$$

We are interested in the asymptotic behavior of the solutions that exist on long and finite time intervals (such as global existence, scattering and blow-up), of the nonlinear Schrödinger equation (NLS_Ω) .

The linear Schrödinger equation was introduced by the Austrian physicist Erwin Schrödinger in 1925. It is a fundamental equation in non-relativistic quantum mechanics and it is the analogue of Hamilton's laws in non-relativistic classical mechanics. The Schrödinger's equation can be used to describe the state of quantum particles under some forces, such as the ones exercised by the electrons present in an atom or a molecule. The nonlinear Schrödinger equation, or NLS equation for short, appears in many physics models, for instance, in laser propagation and Bose-Einstein condensate. A typical application of the NLS equation is the description of the wave motion and interaction in plasma physics, as well as modeling connected with the fluid and air dynamics.

The main motivation of this PhD work is to understand the influence of a smooth, compact and convex obstacle on the global existence and long time asymptotic dynamics of the solutions of the focusing NLS_Ω equation. The study of the wave-type equations in the exterior of obstacles of different geometrical characteristics and shapes have started in 1961 with Morawetz result for the local energy decay of the solutions of the linear wave equation outside an obstacle in [71],[72], [64] and [65]. That study was restricted to a star-shape obstacle. However, those were the first works on the influence of the geometry of the underlying space on the dynamics of the equation. Morawetz result was generalized to different types of obstacles ranging from Ivrii work on almost star-shaped obstacle in [51] in 1969, to the result of Morawetz, Ralston and Strauss in 1977 on non-trapping obstacles in [74], [75]. During that period, different results were obtained for other types of obstacles by Bloom in 1974 − 1976 [12], [11], [13], by Strauss in [84], by Morawetz in [73] and in 1987 by Liu [67], [68]. Let us mention that apart from the works mentioned above for fixed obstacle, the problem was also studied for moving obstacles by J. Cooper and W. Strauss, in [23].

Before stating the main results, we briefly present the local and global theory of the nonlinear Schrödinger equation in both the Euclidean space $\mathbb{R}^d$ and an exterior domain Ω. We first recall some well-known properties of both equations such as conservation laws, scaling and invariances. We present the Cauchy problem theory for the nonlinear Schrödinger equations in Euclidean space $\mathbb{R}^d$ and in the exterior domain from historical perspective, starting by reviewing different works on the Strichartz estimates in exterior domain, which are the key results used to study the local well-posedness. Finally, we present our main results with outlines of the proofs.

1 Nonlinear Schrödinger equation in the whole Euclidean space

1.1 Preliminaries

We consider the nonlinear Schrödinger equation in $\mathbb{R}^d$ of the form

$$(\text{NLS}_{\mathbb{R}^d})\begin{cases} i\partial_t u + \Delta_{\mathbb{R}^d} u = -|u|^{p-1}u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(t_0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{I.1}$$

where $t_0 \in \mathbb{R}$ is the initial time, ∂_t is the derivative with respect to the time variable, $\Delta_{\mathbb{R}^d}$ is the Laplacian operator in the space variable, $\Delta_{\mathbb{R}^d} u = \sum_{j=1}^{d} \partial_{x_j}^2 u$, $u_0 \in H^1(\mathbb{R}^d)$ is the initial data and $p > 1$ for $d = 1, 2$ and $p < \frac{d+2}{d-2}$ for $d \geq 3$. Here, u is a complex-valued function,

$$\begin{aligned} u : [T_0, +\infty) \times \Omega &\longrightarrow \mathbb{C} \\ (t, x) &\longmapsto u(t, x). \end{aligned}$$

The ($\mathrm{NLS}_{\mathbb{R}^d}$) equation combines the dispersive behavior of the linear part of the equation with a nonlinearity. The dynamics of the equation depends on the sign of the nonlinearity. We are interested in the focusing (i.e., negative sign in front of the nonlinearity) NLS equation, which has a richer and more involved dynamics due to the opposite effects of the nonlinearity and the dispersion by Laplacian. In other words, the sign of the nonlinear term counterbalances the dispersive part of the equation.

The NLS equation posed on the whole Euclidean space $\mathbb{R}^d$ is invariant by the scaling transformation, that is,

$$u(t, x) \longmapsto \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t)\ , \quad \text{for } \lambda > 0. \tag{I.2}$$

This scaling identifies the critical Sobolev space $\dot{H}^{s_c}_x$, where the critical regularity s_c is given by $s_c := \frac{d}{2} - \frac{2}{p-1}$. The case when $s_c = 0$, i.e., $p = 1 + \frac{4}{d}$, is referred to as the mass-critical or L^2-critical and the case when $s_c = 1$, i.e., $p = \frac{d+2}{d-2}$ is called the energy-critical or $\dot{H}^1$-critical.

Definition 1.1. *Let I be a time interval such that $t_0 \in I$ and $u_0 \in H^1(\mathbb{R}^d)$. Let $p > 1$ for $d = 1, 2$ and $p < \frac{d+2}{d-2}$ for $d \geq 3$. A function $u \in C(I, H^1(\mathbb{R}^d))$ is a strong solution of* ($\mathrm{NLS}_{\mathbb{R}^d}$) *on I if and only if u satisfies the following Duhamel formula*

$$u(t, x) = S(t)u_0 + i \int_0^t S(t-s)|u(s)|^{p-1} u(s)\, ds, \qquad \textit{for all } \ t \in I, \tag{I.3}$$

where $S(t)$ is the free Schrödinger operator $S(t) := e^{it\Delta_{\mathbb{R}^3}}$ given by

$$S(t)u_0(x) := \left(\frac{1}{4\pi i t}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} u_0(y) dy.$$

1.2 Cauchy problem for the $\mathrm{NLS}_{\mathbb{R}^d}$ equation

On the whole space $\mathbb{R}^d$, the Cauchy problem for the $\mathrm{NLS}_{\mathbb{R}^d}$ equation has been successfully studied in both defocusing and focusing cases.

The Cauchy problem in $H^1(\mathbb{R}^d)$ for the $\mathrm{NLS}_{\mathbb{R}^d}$ equation on the Euclidean space $\mathbb{R}^d$ have been quite extensively investigated in many cases after seminal works, by Ginibre and Velo in [35, 36] and T. Kato in [53], see also [37], [87], [18], [20], [19]. Local existence and uniqueness are usually proved by contraction mapping methods via Strichartz estimates. These estimates were first obtained in [85] as a consequence of a Fourier restriction theorem, later Strichartz's estimates were generalized in [37]. Moreover, in [93] Yajima generalized the Schrichartz estimates for inhomogeneous case, see also [18]. We refer to [54] for the end point case, which is still an open question for the problem in an exterior domain.

Definition 1.2. *We say a pair (q, r) is L^2-admissible if*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \qquad \textit{where} \quad 2 \leq r \leq \frac{2d}{d-2} \quad (2 \leq r \leq \infty, \ \textit{if}\, d = 1, \ 2 \leq r < \infty, \ d = 2). \tag{I.4}$$

We consider the integral equation (I.3) with $u_0 \in H^1(\mathbb{R}^d)$ and the nonlinearity $p > 1$ for $d = 1, 2$ and $p < \frac{d+2}{d-2}$ for $d \geq 3$, i.e., we are considering the energy-subcritical cases.

Theorem 1.3. *Let $u_0 \in H^1(\mathbb{R}^d)$. Then there exists $T > 0$, depending on $\|u_0\|_{H^1}, p, d$, and a unique solution $u(t)$ of (NLS$_{\mathbb{R}^d}$) on the time interval $[-T, T]$ with*

$$u \in C([-T, T], H^1(\mathbb{R}^d)) \cap L^q([-T, T], W^{1,r}(\mathbb{R}^d)),$$

where (q, r) is the L^2-admissible pair $(\frac{4(p+1)}{(d-2)(p-1)}, \frac{d(p+1)}{d+p-1})$ for $d \geq 3$, and (q, r) is an L^2-admissible pair with $r \in [2, \infty)$, for $d = 2$ and $r \in [2, \infty]$, for $d = 1$.

Furthermore, the solution u can be extended to a maximal interval of existence $[0, T_+)$ and the following alternative holds:
either $T_+ = +\infty$ (the solution is global) or $T_+ < +\infty$ (the solution blows up in finite time) with

$$\lim_{t \to T_+} \|\nabla u(t, \cdot)\|_{L^2} = +\infty.$$

The proof proceeds in two main steps: first, one proves the existence and uniqueness of a solution using contraction mapping in a natural Banach space for a small time interval and using the fact that the time interval is uniform on bounded sets of H^1, the solution is extended to a maximal interval of existence.

The solutions of ($\text{NLS}_{\mathbb{R}^d}$) satisfy the mass, energy and momentum conservation laws:

$$\begin{aligned} M_{\mathbb{R}^d}[u(t)] &:= \int_{\mathbb{R}^d} |u(t,x)|^2 dx = M_{\mathbb{R}^d}[u_0], \\ E_{\mathbb{R}^d}[u(t)] &:= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(t,x)|^2 dx - \frac{1}{p+1}\int_{\Omega} |u(t,x)|^{p+1}\, dx = E_{\mathbb{R}^d}[u_0], \\ P_{\mathbb{R}^d}[u(t)] &:= \operatorname{Im}\int_{\mathbb{R}^d} \nabla u(t,x)\, \overline{u}(t,x)\, dx = P_{\mathbb{R}^d}[u_0]. \end{aligned}$$

Furthermore, the ($\text{NLS}_{\mathbb{R}^d}$) equation enjoys several invariances: if $u(t,x)$ is a solution, then

- Space translation invariance: $u(t,x+x_0)$ is also a solution, $x_0 \in \mathbb{R}^d$.
- Time translation invariance: $u(t+t_1,x)$ is also a solution, for $t_1 \in \mathbb{R}$.
- Phase rotation invariance: $e^{i\theta_0}u(t,x)$ is also solution, for $\theta_0 \in \mathbb{R}$.
- Galilean invariance: $e^{i\frac{x\cdot v}{2}}e^{-i\frac{|v|^2}{2}\,t}\, u(t,x-t\,v)$ is also a solution, for $v \in \mathbb{R}^d$.
- In the case when $p = 1+\frac{4}{n}$, pseudo-conformal invariance: $\frac{1}{|t|^{\frac{d}{2}}}\overline{u(\frac{x}{t},\frac{1}{t})}e^{i\frac{|x|^2}{4t}}$ is a solution for $t \neq 0$.

Consider solitary wave solution of the NLS equation on the whole Euclidean space $\mathbb{R}^d$,

$$u(t,x) = e^{it\,\omega}Q_\omega(x), \tag{I.5}$$

where Q_ω is a solution of the nonlinear elliptic equation:

$$\begin{cases} -\Delta Q_\omega + \omega\, Q_\omega = |Q_\omega|^{p-1}\, Q_\omega, \\ Q_\omega \in H^1(\mathbb{R}^3). \end{cases} \tag{I.6}$$

Applying a Galilean transform to the soliton solution $e^{it\,\omega}Q_\omega(x)$ of the NLS equation on $\mathbb{R}^d$, we obtain a soliton solution, moving on the line $x = tv$ with velocity $v \in \mathbb{R}^d$,

$$\tilde{u}(t,x) = e^{i(\frac{1}{2}(x\cdot v)-\frac{1}{4}|v|^2 t+t\,\omega)}Q_\omega(x-t\,v). \tag{I.7}$$

The soliton (I.7) is a global solution of the focusing NLS equation on $\mathbb{R}^d$, but it is not a soliton solution of (NLS_Ω) since this soliton solution does not satisfies Dirichlet boundary conditions.

At least 3 types of solutions for the $\mathrm{NLS}_{\mathbb{R}^d}$ equation are known:

- Scattering solution, i.e., global solutions, which behave asymptotically as linear solutions: a global solution u scatters in H^1 if there exists a unique $u_+ \in H^1$ such that
$$\lim_{t\to\infty} \left\| u(t) - e^{it\Delta_{\mathbb{R}^3}} u_+ \right\|_{H^1} = 0.$$
- Blow-up solution, i.e., solutions with finite time of existence $T_+ < \infty$ (respectively, $T_- < \infty$). As a consequence of the blow-up alternative mentioned above,
$$\lim_{t\to T_+} \|u(t,\cdot)\|_{H^1} = \infty, \qquad \text{respectively, } \lim_{t\to T_-} \|u(t,\cdot)\|_{H^1} = \infty.$$
- Solitary wave solution (soliton) such as (I.5), which neither scatters, nor blows up in finite time, a truly nonlinear structure.

Let us recall briefly some known results on the characterization of the behavior of the solution of the ($\mathrm{NLS}_{\mathbb{R}^d}$) equation:

In [55], F. Merle and C. Kenig studied the behavior of the radial solutions to the $\mathrm{NLS}_{\mathbb{R}^d}$ equation in the energy-critical case for dimension $d = 3, 4, 5$. They established a dichotomy for scattering vs. blow-up solutions for data with energy below the energy of the stationary solution W, i.e., $E_{\mathbb{R}^d}[u] < E_{\mathbb{R}^d}[W]$. The criterion is given by the initial data gradient comparison to that of the stationary solution W.

Let $u_0 \in \dot{H}^1(\mathbb{R}^d),\ d = 3, 4, 5$, radial such that $E_{\mathbb{R}^d}[u_0] < E_{\mathbb{R}^d}[W]$.

1. If $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, then the solution $u(t)$ is global and scatters in $\dot{H}^1(\mathbb{R}^d)$.
2. If $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$, then the solution $u(t)$ blows up in finite time.

In the spirit of the above result on the classification of solutions behavior in the energy-critical NLS equation, J. Holmer and S. Roudenko in [45] studied the behavior of the radial solutions of the $3d$ cubic $\mathrm{NLS}_{\mathbb{R}^3}$ equation, i.e., mass-supercritical and energy-subcritical, whenever the initial data satisfies a mass-energy criterion given by the ground state threshold. The criterion is expressed in terms of the scale-invariant quantities $\|u\|_{L^2} \|\nabla u\|_{L^2}$ and $M[u]E[u]$. Let u_0 in $H^1(\mathbb{R}^d)$, such that
$$E_{\mathbb{R}^3}[u_0] M_{\mathbb{R}^3}[u_0] < E_{\mathbb{R}^3}[Q] M_{\mathbb{R}^3}[Q].$$

1. If $\|u_0\|_{L^2(\mathbb{R}^3)} \|\nabla u_0\|_{L^2(\mathbb{R}^3)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}$, then the solution $u(t)$ exists globally and scatters in H^1.
2. If $\|u_0\|_{L^2(\mathbb{R}^3)} \|\nabla u_0\|_{L^2(\mathbb{R}^3)} > \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}$, and if either u_0 is radial or has a finite variance, i.e., $|x|u_0 \in L^2(\mathbb{R}^3)$, then the solution $u(t)$ blows up in finite time.

This result was extended later by the previous authors with T. Duyckaerts to the non-radial setting in [29]. Moreover, the same result was later extended to arbitrary space dimensions and focusing mass-supercritical power nonlinearities by T. Cazenave, J. Xie and D. Fang, see [32] and by C. Guevara in [43], see also [46] for "weak" blow-up.

Following the work of T. Duyckaerts and F. Merle [30] in the energy-critical case, T. Duyckaerts and S. Roudenko [31] classified the behavior of solutions to the $3d$ cubic $\mathrm{NLS}_{\mathbb{R}^3}$ equation exactly at the mass-energy threshold, namely, when $E_{\mathbb{R}^3}[u_0]M_{\mathbb{R}^3}[u_0] = E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q]$, with $H^1(\mathbb{R}^3)$ initial data satisfying the mass-gradient bound (as in part 1 and 2 above), here, Q is the ground state solution of the nonlinear elliptic equation (I.6).

At the mass-energy level, the NLS equation has a richer dynamics for the long time behavior of solutions compared to the results mentioned above under the threshold. In [31], the authors proved the existence of special solutions, denoted by Q^+ and Q^-. These solutions approache the soliton, up to symmetries, in one time direction, that is, there exits $e_0 > 0$ such that

$$\forall t \geq 0 \qquad \left\| Q^\pm - e^{it}Q \right\|_{H^1(\mathbb{R}^3)} \leq ce^{-e_0 t} \tag{I.8}$$

The behavior of $Q^\pm$ in the opposite time direction is different: Q^- scatters for negative time but Q^+ has finite time of existence. The existence of these special solutions is derived from the existence of the two real nonzero eigenvalues for the linearized operator around the soliton $e^{it}Q$. Moreover, these special solutions have the same mass-energy of the soliton, $\|\nabla Q^-\|_{L^2(\mathbb{R}^3)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)}$ and $\|\nabla Q^+\|_{L^2(\mathbb{R}^3)} > \|\nabla Q\|_{L^2(\mathbb{R}^3)}$.

Let $u_0 \in H^1(\mathbb{R}^d)$ such that

$$E_{\mathbb{R}^3}[u_0]M_{\mathbb{R}^3}[u_0] = E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q]$$

and

1. if $\|u_0\|_{L^2(\mathbb{R}^3)} \|\nabla u_0\|_{L^2(\mathbb{R}^3)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}$, then the corresponding solution $u(t)$ of ($\mathrm{NLS}_{\mathbb{R}^3}$) is global and either scatters or $u = Q^-$, up to the symmetries;

2. if $\|u_0\|_{L^2(\mathbb{R}^3)} \|\nabla u_0\|_{L^2(\mathbb{R}^3)} = \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}$, then $u = e^{it}Q$, up to the symmetries;

3. if $\|u_0\|_{L^2(\mathbb{R}^3)} \|\nabla u_0\|_{L^2(\mathbb{R}^3)} > \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}$, and if either u_0 is radial or $|x|u_0 \in L^2(\mathbb{R}^3)$, then either u has a finite time of existence or $u = Q^+$, up to the symmetries.

2 Nonlinear Schrödinger equation in the exterior of an obstacle

2.1 Preliminaries

We now consider the focusing nonlinear Schrödinger equation in the exterior of a compact and strictly convex obstacle Θ with Dirichlet boundary conditions in dimension $d \geq 2$

$$\begin{cases} i\partial_t u + \Delta_\Omega u = -|u|^{p-1}u, & (t,x) \in \mathbb{R} \times \Omega, \\ u(t_0,x) = u_0(x), & x \in \Omega, \\ u(t,x) = 0, & \forall (t,x) \in \mathbb{R} \times \partial\Omega, \end{cases} \tag{NLS$_\Omega$}$$

where $\Omega = \mathbb{R}^d \setminus \Theta$, Δ_Ω is the Dirichlet Laplace operator on Ω, which is a self-adjoint operator on $L^2(\Omega)$ with form domain $H_0^1(\Omega)$ and $p > 1$ for $d = 1, 2$ or $p < \frac{d+2}{d-2}$ for $d \geq 3$.

We denote by $H_0^1 := H_0^1(\Omega)$ the energy space, which is also the domain of the square root of $-\Delta_\Omega$. We take initial data $u_0 \in H_0^1(\Omega)$.

We first recall the definition of the Sobolev spaces on the domain Ω associated with powers of the Dirichlet Laplacian Δ_Ω.

Definition 2.1. *For $s \geq 0$, $1 < p < \infty$, let $\dot{H}_D^{s,p}(\Omega)$ and $H_D^{s,p}(\Omega)$ denote the completions of $C_c^\infty(\Omega)$ under the norms*

$$\|f\|_{\dot{H}_D^{s,p}} := \left\|(-\Delta_\Omega)^{\frac{s}{2}} f\right\|_{L^p} \qquad \text{and} \qquad \|f\|_{H_D^{s,p}} := \left\|(1-\Delta_\Omega)^{\frac{s}{2}} f\right\|_{L^p}. \tag{I.9}$$

When $p = 2$, we write $\dot{H}_D^s(\Omega)$ for $\dot{H}_D^{s,2}(\Omega)$, and $H_D^s(\Omega)$ for $H_D^{s,2}(\Omega)$.

It is well known that $H_0^{s,p}(\Omega) = H_D^{s,p}(\Omega)$ for $0 < s < \frac{1}{p}$ and for $\frac{1}{p} < s < 1 + \frac{1}{p}$. In particular, for $s = 1$ and $p = 2$ we have $H_0^1(\Omega) = H_D^1(\Omega)$.

In [57], the authors proved the following equivalence of Sobolev space norms under some restriction of the regularity s.

Theorem A (Equivalence of Sobolev spaces,[57])**.** *Let $d \geq 3$. Assume $1 < q < \infty$ and $0 < s < \min\{1 + \frac{1}{q}, \frac{d}{q}\}$. Then*

$$\left\|(-\Delta_{\mathbb{R}^d})^{\frac{s}{2}} f\right\|_{L^q} \sim_{q,s} \left\|(-\Delta_\Omega)^{\frac{s}{2}} f\right\|_{L^q} \quad \forall f \in C_c^\infty(\Omega). \tag{I.10}$$

The solutions of (NLS$_\Omega$) satisfy the mass and energy conservation laws:

$$M_\Omega[u(t)] := \int_\Omega |u(t,x)|^2 dx = M_\Omega[u_0].$$
$$E_\Omega[u(t)] := \frac{1}{2}\int_\Omega |\nabla u(t,x)|^2 dx - \frac{1}{p+1}\int_\Omega |u(t,x)|^{p+1}\, dx = E_\Omega[u_0].$$

The momentum

$$P_\Omega[u(t)] := \operatorname{Im} \int_\Omega \nabla u(t,x)\, \overline{u}(t,x)\, dx$$

is not conserved for (NLS$_\Omega$) equation with Dirichlet boundary conditions because of the following boundary term,

$$\frac{d}{dt} P_\Omega[u(t)] = \operatorname{Im} \int_{\partial\Omega} |\nabla u(t,x)|^2\, \vec{n}\, dx,$$

where $\vec{n}$ is the outward normal vector of Ω.

Since the presence of the obstacle does not change the intrinsic dimensionality of the problem, so that, the scaling given in (I.2) identifies the same critical Sobolev space $\dot{H}_D^{s_c}$, for NLS$_\Omega$ equation, where $s_c := \frac{d}{2} - \frac{2}{p-1}$. Moreover, for $u_0 \in \dot{H}_D^s$ the NLS$_\Omega$ equation is sub-critical for $s > s_c$ and super-critical for $s < s_c$.

Definition 2.2. *Let I be a time interval such that $t_0 \in I$ and $u_0 \in H_0^1(\Omega)$. Let $1 < p < \frac{d+2}{d-2}$ for $d \geq 3$. A function $u \in C(I, H_0^1(\Omega))$ is a strong solution of* (NLS$_\Omega$) *on I if and only if u satisfies the following Duhamel formula*

$$u(t,x) = e^{it\Delta_\Omega} u_0 + i\int_0^t e^{i(t-s)\Delta_\Omega} |u(s)|^{p-1} u(s)\, ds, \qquad \textit{for all } t \in I, \tag{I.11}$$

where Δ_Ω is the Dirichlet Laplacian.

The presence of an obstacle breaks down several invariances of the equation, such as, space translation, Galilean and pseudo-conformal invariance. However, the NLS_Ω equation enjoys the time translation and phase rotation invariances.

2.2 Cauchy problem for the NLS_Ω equation

On exterior domain, the local well-posedness relies also on Strichartz estimates. The Cauchy problem in $H_0^1(\Omega)$ for the NLS_Ω equation in the exterior of obstacle is now well understood.

In [15], the authors proved Strichartz estimates with a loss of " $\frac{1}{2}$-derivative" on the exterior of a non-trapping obstacle using local smoothing estimates. This Strichartz estimates turned into a local existence result in the energy class with a restriction in nonlinearity power due to the resulting loss. For example, in dimension $d = 3$, the authors proved the well-posedness only for the sub-cubic ($p < 3$) nonlinearity. Nevertheless, they proved the global existence for the cubic NLS_Ω equation, provided $\|u_0\|_{H_0^1(\Omega)}$ is sufficiently small.

The well-posedness for the cubic nonlinearities was later improved in [4], using new Strichartz estimates obtained by combining the smoothing effect used above with a semiclassical Strichartz estimate on small intervals of time in [5]. The same result was obtained in [47] for (NLS_Ω) on the exterior of a ball, in dimension $d = 3$ using a precise smoothing effects near the boundary of a ball.

Later, in [78], Planchon and Vega obtained a scale invariant norm $L^4_{t,x}$ Strichartz estimate in dimension $d = 3$. (This estimate has a loss of a $\frac{1}{4}$-derivative). They used bilinear Virial identities for the nonlinear Schrödinger equation, which allowed them to extend bilinear improvements to Strichartz inequalities as follows.

Let $S(t)$ denote the linear flow for the Schrödinger equation on an exterior of non-trapping domain Ω and let $s \geq 0$. Then

$$\|S(t)u_0\|_{L_t^4 \dot{H}_D^{s,4}(\Omega)} \leq \|u_0\|_{\dot{H}_D^{s+\frac{1}{4}}(\Omega)}.$$

This estimate proved the local well-posedness for the energy-subcritical equation in dimension $d = 3$, i.e., $1 < p < 5$, see Theorem 3.4 in [78]. However, this estimate barely misses the $L^4_t L^\infty_x$ control for H^1_0 initial data, hence, the restriction to H^1-subcritical nonlinearity.

After that, in [48] O. Ivanovici proved the Strichartz estimates for (NLS_Ω) except the end point case, using the Melrose and Taylor parametrix. The authors proved also that the quintic ($p = 5$) (NLS_Ω) equation is locally well-posed in $H^1_0(\Omega)$ and globally well-posed in time for initial data $u_0 \in H^1_0(\Omega)$ sufficiently small. Moreover, a local Cauchy theory and scattering results are obtained for the defocusing (NLS_Ω) equation in the exterior of a strictly convex obstacle in dimension $d = 3$, in the inter-critical case, i.e., for $\frac{7}{3} < p < 5$ using the following Strichartz estimates.

Theorem B ([48])**.** *Let $d \geq 2$, $\Omega \subset \mathbb{R}^d$ be the exterior of a smooth compact strictly convex obstacle. Let $q, \tilde{q} > 2$ and $2 \leq r, \tilde{r} \leq \infty$ satisfy the scaling conditions: $\frac{2}{q} + \frac{d}{r} = \frac{d}{2} = \frac{2}{\tilde{q}} + \frac{d}{\tilde{r}}$ Then*

$$\|u\|_{L^q_t L^r_x} := \left\| e^{it\Delta} u_0 + i \int_0^t e^{i(t-s)\Delta} F(s) ds \right\|_{L^q_t L^r_x} \leq C_s \left(\|u_0\|_{L^2(\Omega)} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \right). \tag{I.12}$$

In [50], O. Ivanovici and F. Planchon extended the result of [48] for the quintic (NLS_Ω) equation outside a non-trapping obstacle in $\mathbb{R}^3$, using a smoothing effect to estimate $L^2_t L^5_x$ for the linear equation. Moreover, they proved that (NLS_Ω) equation is also locally well-posed in $\dot{H}^{s_c}_D$, with $s_c = \frac{3}{2} - \frac{2}{p-1}$ and $3 + \frac{2}{5} < p < 5$.
In [63], we prove that the (NLS_Ω) equation is locally well-posed in $H^{s_c}_D(\Omega)$, for $0 < s_c < 1$ (cf. Theorem 2.3).

Let us mention that apart from the works cited above, the NLS_Ω equation in the exterior of a star-shaped obstacle in dimension $d = 2$ was studied by Blair, Smith and Sogge in [10], F. Planchon and L. Vega in [79]. This result was generalized by F.A. Shakra in [1] for the $2d$ NLS_Ω outside of a non-trapping obstacle. In [50], O. Ivanovici and F. Planchon proved scattering results for the defocusing NLS_Ω equation in the exterior of a star-shaped compact obstacle for $3 \leq p < 5$, in dimension $d = 3$. We also refer to [66] for global well-posedness and scattering results for the defocusing (NLS_Ω) in the exterior of balls with radial data. Let us also mention the recent works on dispersive estimates outside one or several strictly convex obstacles of

O. Ivanovici and G. Lebeau in [49], as well as D. Lafontaine in [61],[62].

Theorem C. *Let $d \geq 2$, $\Omega \subset \mathbb{R}^d$, be the exterior of a smooth compact strictly convex obstacle. Consider $0 < s_c < 1$, i.e., $1 + \frac{4}{d} < p < \frac{d+2}{d-2}$ and let $u_0 \in H_0^1(\Omega)$. Then there exists $T := T(u_0) > 0$ such that a solution $u \in C([-T,T],\Omega)$ is a strong solution to* (NLS_Ω) *equation.*

The local existence and uniqueness in $H_0^1(\Omega)$ is carried out by classical methods, using a fixed point argument via Strichartz estimates. The proof is very similar to the one for the NLS equation posed on the whole Euclidean space. Moreover, the Cauchy problem for the NLS_Ω equation is also well-posed in $H^2(\Omega) \cap H_0^1(\Omega)$ in dimension $d = 3$ for $p > 2$. The local existence of solutions for the NLS_Ω equation in $H^2 \cap H_0^1(\Omega)$ can be established using the fact that H^2 is an algebra and the following continuous embedding holds for any smooth domain $\Omega \subset \mathbb{R}^3$, $H^2(\Omega) \subset L^\infty(\Omega)$, see [15, Proposition 2.1]. Thus, we don't have to control the nonlinearity growth but just need the regularity of the nonlinear term.

Proposition D. *Assume $d = 3$ and $p > 2$. Let $u_0 \in H^2 \cap H_0^1(\Omega)$. Then there exists $T > 0$ and a unique solution $u(t)$ of* (NLS_Ω) *with $u \in C([-T,T], H^2 \cap H_0^1(\Omega))$.*

It is classical that the solution u can be extended to a maximal time existence interval $I = (-T_-, T_+)$ and the following alternative holds:
either $T_+ = \infty$ (respectively, $T_- = \infty$) or $T_+ < \infty$ (respectively, $T_- < \infty$) and

$$\lim_{t \to T_+} \|u(t,\cdot)\|_{H_0^1(\Omega)} = \infty, \quad \text{respectively, } \lim_{t \to T_-} \|u(t,\cdot)\|_{H_0^1(\Omega)} = \infty.$$

We next state our results for the local well-posedness for the NLS_Ω equation in the exterior of a convex obstacle in the critical Sobolev space $\dot{H}_D^{s_c}(\Omega)$ for $d = 3$ and $0 < s_c < 1$, i.e., $\frac{7}{3} < p < 5$. For that, we use the fractional chain rule in the exterior of a compact convex obstacle given in [57].

Theorem 2.3 (Well-posedness in $H_D^{s_c}(\Omega)$) **).** *Let $0 < s_c < 1$ (or $\frac{7}{3} < p < 5$.)*
Let $u_0 \in H_D^{s_c}(\Omega)$. Then there exists a unique solution $u(t)$ of (NLS_Ω) *with initial data u_0 defined on $[0,T]$ for some $T > 0$, such that*

$$u \in C([0,T], H_D^{s_c}(\Omega)) \cap L^q([0,T], H_D^{s_c,r}),$$

where $(q, r) = \left(p + 1, \frac{6(p+1)}{3p-1}\right)$.

Furthermore, the solution u *can be extended to a maximal interval of existence* $[0, T_+)$ *and the following alternative holds:*
either $T_+ = +\infty$ *(the solution is global) or* $T_+ < +\infty$ *(the solution blows up in finite time) and*

$$\lim_{t \to T_+} \|u(t, \cdot)\|_{H_D^{s_c}(\Omega)} = +\infty.$$

After establishing the local existence one can ask whether it is possible to characterize the behavior of solutions for the NLS_Ω equation, which is exactly the goal of this thesis. This question has been extensively studied for (NLS) on the whole space $\mathbb{R}^d$.

In the exterior of a convex obstacle, R. Killip, M. Visan and X. Zhang proved in [58] that the threshold for global existence and scattering is the same as for the cubic equation on $\mathbb{R}^3$, see also [94] for $\frac{7}{3} < p < 5$. Assume $p = 3$ and $d = 3$.

Let $u_0 \in H_0^1(\Omega)$ such that

$$E_\Omega[u_0]M_\Omega[u_0] < E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q] \quad \text{and} \quad \|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}. \quad \text{(I.13)}$$

Then the solution $u(t)$ is globally defined and scatters in $H_0^1(\Omega)$.

The existence of blow-up solutions is not treated in the results mentioned above. In [21], the authors revisited the scattering result by utilizing Dodson and Murphy's approach [27], [28] and the dispersive estimate established by O. Ivanovici and G. Lebeau in [49]. Furthermore, in [59] the authors proved global existence and scattering for the quintic defocusing (NLS_Ω) equation for all initial data in $\dot{H}_0^1(\Omega)$.

3 Main results

Recall that, the ($NLS_{\mathbb{R}^d}$) equation posed on $\mathbb{R}^d$ has at least 3-types of solution: scattering, blow-up and solitary wave solutions. In our main results, we prove the existence of a solitary wave solution, blow-up solution and we studied the scattering threshold for the (NLS_Ω) equation.

In Chapter II, we prove the existence of a family of solitary wave solutions to the NLS_Ω equation in the exterior of strictly convex obstacle, which behaves asymptotically as a solitary wave solution in $\mathbb{R}^d$, for long time. These solitary wave solutions prove the optimality of the scattering threshold for the (NLS_Ω) equation, for an arbitrary small velocity.

In Chapter III, we prove the existence of blow-up solutions for the NLS_Ω equation in the exterior of the unit ball of $\mathbb{R}^d$. We also study the behavior of the solutions that satisfy a symmetry assumption under the mass-energy threshold. We prove that the blow-up criterion is the same as for the problem posed on the Euclidean space given by the work of J. Holmer and S. Roudenko in [45].

In Chapter IV, we study the behavior of the solution of the cubic NLS_Ω equation at the mass-energy threshold. We prove that if the initial data satisfy $M_\Omega[u_0]E_\Omega[u_0] = E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q]$ and $\|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}$, then the corresponding solution is globally defined and scatters in both time direction. This chapter is a joint work with Thomas Duyckaerts and Svetlana Roudenko.

In Chapter V, we present numerical simulations of solutions to the two dimensional focusing NLS_Ω equation with Dirichlet boundary conditions. We study the interaction between the solitary wave solutions traveling with different velocities v and the obstacle. We compare these simulations to the one for the NLS equation in $\mathbb{R}^2$ in order to study the influence of an obstacle on the dynamics of the NLS_Ω equation and how the obstacle changes the overall behavior of solutions.

3.1 Existence of solitary waves outside an obstacle

In Chapter II, we consider the focusing L^2-supercritical ($\frac{7}{3} < p < 5$) Schrödinger equation in the exterior of a smooth, compact, strictly convex obstacle $\Theta \subset \mathbb{R}^3$. Let $T_0 > 0$ and $\Omega = \mathbb{R}^3 \setminus \Theta$.

Consider

$$\begin{cases} i\partial_t u + \Delta_\Omega u = -|u|^{p-1}u, & (t,x) \in [T_0,+\infty)\times\Omega,\\ u(T_0,x) = u_0(x), & x\in\Omega,\\ u(t,x)=0, & (t,x)\in[T_0,+\infty)\times\partial\Omega. \end{cases}$$

The soliton solutions for NLS equation are constructed using Galilean invariance, a transformation, specific to the equation on $\mathbb{R}^d$ and is not valid for the (NLS_Ω) equation outside the obstacle. We construct a solution $u(t)$ for the focusing (NLS_Ω) equation outside of a strictly convex obstacle. This solution behaves asymptotically as a solitary wave on $\mathbb{R}^3$ for large time and satisfies Dirichlet boundary conditions, such that

$$\left\| u(t,x) - e^{i(\frac{1}{2}(x\cdot v)-\frac{1}{4}|v|^2 t\,\omega)} Q_\omega(x-tv)\Psi(x) \right\|_{H_0^1(\Omega)} \longrightarrow 0, \text{ as } t\to+\infty, \tag{I.14}$$

where Ψ is a C^∞-function such that $\Psi=0$ near Θ and $\Psi=1$ for $|x|>>1$ and v is the velocity. These soliton solutions prove the optimality of the threshold for global existence and scattering given in [58].

When the velocity of the solitary wave is high, we prove the existence of such a solution for (NLS_Ω) equation using a classical fixed-point argument.

Theorem 3.1. *Assume* $2\le p<5$.
Let $\Omega=\mathbb{R}^3\setminus\Theta$, *where* Θ *is any smooth compact obstacle, and let* Q_ω *be any solution of* (I.6). *Let* $\omega, T_0>0$. *Then there exists* $V_0 := V_0(\omega)\gg 1$ *with the following property. Let* $v\in\mathbb{R}^3$ *be the velocity such that* $|v|>V_0$.
Then there exists $\delta>0$ *and a functions* r_ω *defined on* $[T_0,+\infty)\times\Omega$ *satisfying*

$$\forall t\in[T_0,+\infty) \qquad \|r_\omega(t)\|_{H^2\cap H_0^1(\Omega)} \le C_\omega\,|v|^3\, e^{-\delta\sqrt{\omega}|v|t},$$

such that $u(t,x) = e^{i(\frac{1}{2}(x.v)-\frac{1}{4}|v|^2t+t\,\omega)}Q_\omega(x-tv)\Psi(x)+r_\omega(t,x), \quad (t,x)\in[T_0,+\infty)\times\Omega,$
is a solution of (NLS_Ω).

In this Theorem, Q_ω is any solution of the nonlinear elliptic equation (I.6), does not have to be the ground state. The contraction mapping argument, used here, requires the assumption of high velocity, which does not allow us to show the optimality of the threshold for scattering.

For an arbitrary nonzero velocity, we use a compactness argument similar to the one that was introduced by F.Merle in 1990 to construct solutions of NLS blowing up at several blow-up points together with a topological argument using Brouwer's theorem to control the unstable direction of the linearized operator at soliton.

Theorem 3.2. *Assume* $\frac{7}{3} < p < 5$. *Let* $v \in \mathbb{R}^3 \backslash \{0\}$ *be the velocity,* $\omega > 0$. *Then there exists* $\delta > 0$, $T_0 > 0$ *and a function* r_ω *defined on* $[T_0, +\infty) \times \Omega$ *satisfying*

$$\|r_\omega(t)\|_{H_0^1(\Omega)} \leq e^{-\delta\sqrt{\omega}|v|t}, \qquad t \in [T_0, +\infty),$$

such that,

$$u(t,x) = e^{i(\frac{1}{2}(x.v)-\frac{1}{4}|v|^2t+t\,\omega)}Q_\omega(x-tv)\Psi(x) + r_\omega(t,x), \quad (t,x) \in [T_0, +\infty) \times \Omega,$$

is a solution of (NLS_Ω).

Unlike Theorem 3.1, here, Q is the ground state solution (radial, positive and exponentially decaying) to the nonlinear elliptic equation (I.6). Nevertheless, we prove the existence of a solitary wave solution to (NLS_Ω) equation for an arbitrary nonzero velocity v, which proves the optimality of the scattering threshold in [58] and [94] for small velocity.

The proof of Theorem 3.2 relies on a compactness argument similar to the main argument used in [70], [69] and [24], that utilizes the structure of an operator $\mathcal{L}$ linearized around the ground state soliton. In the L^2-supercritical case, it is well known that there exist two eigenfunctions of this linearized operator (see [91], [82], [40]) denoted by $\mathcal{Y}^+$ and $\mathcal{Y}^-$:

$$\mathcal{L}\mathcal{Y}^\pm = \pm e_0 \mathcal{Y}^\pm,$$

where e_0 and $-e_0$ are simple real eigenvalues of $\mathcal{L}$.

We consider a sequence of large times $\{T_n\}$ and we define a solution $u_n(t)$ with large initial data $u(T_n)$ expressed in terms of the eigenfunctions $\mathcal{Y}^+$ and $\mathcal{Y}^-$ of the linearized operator such that a suitable uniform estimate holds for any $t \in [t_0, T_n]$,

$$\|u_n(t) - e^{i(\frac{1}{2}(x\cdot v)-\frac{1}{4}|v|^2t}Q(x-tv)\Psi(x)\|_{H_0^1(\Omega)} \leq Ce^{-\delta\sqrt{\omega}|v|t}.$$

Then by a compactness argument, we construct a solution $u(t)$ of (NLS_Ω) such that (I.14) hold. To prove the uniform estimate claimed above, we use a modulation argument for large times in the phase and translation parameters to obtain some orthogonality conditions and a bootstrap argument with the coercivity property of the linearized operator to control the modulation parameters and others terms expressed in terms of the eigenfunctions. The linearized operator is positive-definite up to the four directions Q, ∇Q and two other directions expressed in terms of the eigenfunctions $\mathcal{Y}^{\pm}$. The directions Q and ∇Q are controlled by the orthogonality conditions given by the modulation. The direction $\mathcal{Y}^{+}$ is stable, in the sense that it can be controlled, however, the other direction $\mathcal{Y}^{-}$ is unstable and cannot be controlled by a scaling argument, even if we introduce an extra parameter in the modulation. We, therefore, have to use a topological argument to control this unstable direction and to conclude the proof of the uniform estimate on $[t_0, T_n]$.

The solitary waves (I.14) prove the optimality of the threshold (I.13) for global existence and scattering solutions. Indeed, the solution $u(t)$ of (NLS_Ω) is global, does not scatter for positive time direction, and we have

$$E_\Omega[u] = \frac{|v|^2}{8}\int |Q|^2 + E_{\mathbb{R}^3}[Q]\,. \tag{I.15}$$

Since the velocity v can be taken arbitrarily small, we have proved that for all $\varepsilon > 0$ there exists a solution $u_\varepsilon(t)$ of (NLS_Ω), which is global and does not scatter for positive time such that

$$M_\Omega[u_\varepsilon] = M_{\mathbb{R}^3}[Q]\,, \quad \sup_{t\geq T_0} \|\nabla u_\varepsilon(t)\|_{L^2(\Omega)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)} + \varepsilon$$

and

$$E_\Omega[u_\varepsilon] < E_{\mathbb{R}^3}[Q] + \varepsilon\,.$$

All results obtained up to now for the equation (NLS_Ω) are for global existence and scattering solutions but the existence of blow-up solutions was conjectured but not yet demonstrated. This is the object of the next chapter.

3.2 Existence of blow-up solutions outside an obstacle

In Chapter III, we prove that finite variance, negative energy solutions to the NLS_Ω equation in the exterior of the unit ball of $\mathbb{R}^d$ with Dirichlet boundary conditions break down in finite time.

The proof of the existence of blow-up solutions for ($\mathrm{NLS}_{\mathbb{R}^d}$) given by Glassey in [38], is based on a convexity argument for the variance, namely, the second derivative of the variance for ($\mathrm{NLS}_{\mathbb{R}^d}$) equation is

$$\frac{1}{16}\frac{d^2}{dt^2}\int_{\mathbb{R}^d}|x|^2|u|^2 = E_{\mathbb{R}^d}[u] - \frac{1}{2}\left(\frac{d}{2} - \frac{d+2}{p+1}\right)\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^d)}.$$

Now, if $p > 1 + \frac{d}{4}$ and $E_{\mathbb{R}^d}[u] < 0$, then $u(t)$ blows up in finite time.

This proof does not adapt directly to the case of an exterior domain because the boundary term in the Virial identity above does not have a favorable sign, that is,

$$\frac{1}{16}\frac{d^2}{dt^2}\int_{\Omega}|x|^2|u(t,x)|^2 = E_{\Omega}[u] - \frac{1}{2}\left(\frac{d}{2} - \frac{d+2}{p+1}\right)\int_{\Omega}|u|^{p+1}\,dx - \frac{1}{4}\int_{\partial\Omega}|\nabla u|^2\,(x\cdot\vec{n})\,d\sigma(x), \quad \text{(I.16)}$$

where $\vec{n}$ is the unit outward normal vector. One can see that the last term is positive, indeed,

$$x\cdot\vec{n} \leq 0, \ \text{ for all } \ x \in \partial\Omega = \partial B(0,1).$$

To solve this problem, we define a new *shifted* variance quantity, which allows us to control the boundary term.

Theorem 3.3. *Assume* $\Theta = B(0,1)$ *and* $p \geq 5$.

- *For* $d = 2$, *let* $u_0 \in H^1_0(\Omega)$ *such that* $E[u_0] + \frac{1}{8}M[u_0] < 0$ *and* $|x|u_0 \in L^2(\Omega)$.
- *For* $d = 3$, *let* $u_0 \in H^2 \cap H^1_0(\Omega)$ *such that* $E[u_0] < 0$ *and* $|x|u_0 \in L^2(\Omega)$.

Let $u(t)$ *be the corresponding solution of* (NLS_Ω) *with the maximal time interval* I *of existence. Then the length of* I *is finite and the solution* $u(t)$ *blows up in finite time.*

In this result, we use the following variance quantity:

$$\mathcal{V}(u(t)) := \int_{\Omega}\left(|x|^2 - 2|x| + 10\right)|u(t,x)|^2\,dx.$$

The proof consists of computing the second derivative of this variance. The second derivative of $\int_\Omega -2|x||u(t,x)|^2dx$ allows us to cancel the boundary terms in (I.16) and the last (constant) term is used to obtain a positive quantity.

Theorem 3.4. *Assume* $\Theta = B(0,1)$ *and* $p \geq 1 + \frac{4}{d}$.

- *For* $d = 2$, *let* $u_0 \in H_0^1(\Omega)$ *such that* $u_0(-x_1, x_2) = u_0(x_1, -x_2) = -u_0(x_1, x_2)$.
- *For* $d = 3$, *let* $u_0 \in H^2 \cap H_0^1(\Omega)$ *such that* $u_0(-x_1, x_2, x_3) = u_0(x_1, -x_2, x_3) = u_0(x_1, x_2, -x_3) = -u_0(x_1, x_2, x_3)$.

Let $u(t)$ *be the corresponding solution of* (NLS_Ω) *with the maximal time interval* I *of existence. If* $E[u] < 0$ *and* $|x|u_0 \in L^2(\Omega)$, *then the length of* I *is finite, and thus, the solution* $u(t)$ *blows up in finite time.*

The modified variance quantity used in this theorem is defined as follows: let $C > 0$ be a positive constant (to be specified in Chapter III), denote

$$\mathcal{V}(u(t)) := \int_\Omega \left(|x|^2 - C|x_1| + C|x_2| + C^2\right) |u(t,x)|^2 \, dx.$$

The second derivative of the additional terms allows us to control the boundary term in (I.16) using the symmetry assumption with a good choice of the constant C.

Let us mention that, Theorems 3.3 and 3.4 remain true for (NLS_Ω) outside a ball of radius $r > 1$ and centred at any point x_0. One would have to use a symmetry around x_0 instead of the origin. We can generalize these theorems for any dimension $d \geq 4$, whenever an appropriate well-posedness theory is available. In dimension $d \geq 4$, for Theorem 3.4 we should suppose d symmetries,

$$u_0(x_1, .., -x_i, .., x_d) = -u_0(x_1, .., x_i, .., x_d), \qquad \text{for } \; i = 1, 2, ..., d.$$

Further details are given in Chapter III.

Moreover, we give an explicit blow-up solution $u(t)$ for (NLS_Ω) in the mass-critical case ($p = 1 + \frac{4}{d}$). This solution is similar to the one constructed in [14] for the NLS equation inside of a domain in $\mathbb{R}^2$, using pseudo-conformal transformation, that is,

$$u(t,x) := \frac{1}{(T-t)} Q\left(\frac{x - x_0}{(T-t)}\right) \Psi(x) e^{i\left(\frac{4-(x-x_0)^2}{4(T-t)}\right)} + r(t,x),$$

where $r(t,x)$ is a smooth function defined on $[0,T) \times \Omega$ and exponentially decaying.

In some cases, we also study the behavior of solutions under the mass-energy threshold mentioned above. We prove that the blow-up criterion for the NLS_Ω equation with symmetric initial data is the same as for the problem posed on Euclidean space given by previous work of Holmer and Roudenko in [45]. In particular, we prove that if the initial data $u_0 \in H_0^1(\Omega)$ satisfies the following symmetry,

$$u_0(x_1, .., x_i, .., x_d) = -u_0(x_1, .., -x_i, .., x_d), \quad \text{for } \; i = 1, 2, .., d, \tag{I.17}$$

and

$$M_\Omega[u_0]E_\Omega[u_0] < M_{\mathbb{R}^d}[Q]E_{\mathbb{R}^d}[Q],$$
$$\|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} > \|Q\|_{L^2(\mathbb{R}^d)} \|\nabla Q\|_{L^2(\mathbb{R}^3)},$$

then $u(t)$ blows up in finite time.

3.3 Existence of scattering solution at the mass-energy threshold

In Chapter IV, we study the dynamics of the $3d$ focusing cubic NLS_Ω equation in the exterior of a strictly convex obstacle exactly at the mass-energy threshold, namely, when $E_\Omega[u_0]M_\Omega[u_0] = E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q]$, with $H_0^1(\Omega)$ initial data satisfying the initial mass-gradient bound $\|\nabla u_0\|_{L^2} \|u_0\|_{L^2} < \|\nabla Q\|_{L^2} \|Q\|_{L^2}$. Here, Q is the ground state solution of the nonlinear elliptic equation (I.6). In this case, we prove that the solution $u(t)$ is global in time and scatters in both time directions.

The description of the dynamics of the $\mathrm{NLS}_{\mathbb{R}^3}$ equation, at the mass-energy level, is connected to the behavior of the specific solutions Q^+ and Q^-, which may scatter or blow-up for negative time and converge to a soliton solution for positive time direction, i.e., (I.8) hold for $t \geq 0$. However, the NLS_Ω equation does not admit analogues of these special solutions. Indeed, these specific solutions have to converge to Q for large times. However, there is no function in $H_0^1(\Omega)$ such that (I.8) holds on Ω; since $Q_{|\Omega}$ does not satisfy the Dirichlet boundary condition.

Furthermore, one can easily see that in the presence of an obstacle there is no function $u \in H_0^1(\Omega)$ such that $E_\Omega[u]M_\Omega[u] = E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q]$ and $\|\nabla u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} = \|\nabla Q\|_{L^2(\mathbb{R}^3)} \|Q\|_{L^2(\mathbb{R}^3)}$. Indeed, if we extend u by 0 on the obstacle, then by the characterization of Q on $\mathbb{R}^3$, the function u must be equal to Q up to the symmetries, which does not satisfy the Dirichlet boundary condition.

Theorem 3.5. *Let $u_0 \in H^1_0(\Omega)$ be such that*

$$M_\Omega[u_0]E_\Omega[u_0] = M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q] \quad \textit{and} \; \|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)} . \tag{I.18}$$

Then the corresponding solution $u(t)$ scatters in both time directions.

The proof of Theorem 3.5 is based on the approach of T. Duyckaerts and F. Merle in [30] and T. Duyckaerts with S. Roudenko in [31]. We use a refinement of the concentration-compactness argument established by the profile decomposition method given in the work of Killip, Visan and Zhang in [58] for the NLS_Ω equation outside a convex obstacle.

We first identify a quadratic truncated form associated to the linearized operator $\mathcal{L}$ and we prove that this form is positive on a subspace of H^1. We use a modulation in the phase rotation and in space translation parameters near the truncated ground state solution, in order to obtain certain orthogonality conditions. To control the modulation parameters on some time interval, we use the mass and energy conservation laws with the orthogonality conditions and the coercivity property of the linearized operator. We next prove that the extension $\underline{u}$ of a non-scattering solution u to (NLS_Ω) equation satisfying (I.18), is compact in H^1, up to a translation parameter $x(t)$ in space, which we identify with the translation parameter given by the modulation on certain time interval.

In [31], the authors use the momentum conservation laws with the Galilean transformation to control the translation parameters $x(t)$. In particular, they consider a solution with zero momentum. This argument does not adapt to our setting, in an exterior domain the (NLS_Ω) equation does not conserve the momentum.

To prove that the space translation $x(t)$ is bounded, we first approach $x(t)$ by an auxiliary translation parameters given by previous work mentioned above on $\mathbb{R}^3$. Next, we use the local virial identity with the estimates on the modulation parameters to get a spatial control. Combining the compactness properties with the control of the space translation parameter $x(t)$, we obtain that the non-scattering solution $u(t)$ to (NLS_Ω) equation, satisfying (I.18), is compact in H^1. We prove that the parameter $\delta(t) := \left| \|\nabla Q\|_{L^2} - \|\nabla u\|_{L^2} \right|$ converges to 0 in mean. Finally, we conclude the proof of Theorem 3.5, using the compactness argument with the convergence in mean of $\delta(t)$.

3.4 Numerical simulations

In Chapter V, we develop numerical methods to study the behavior of solutions to the NLS_Ω equation in the exterior of a ball. Our goal is to understand numerically the interaction between a solitary wave solution traveling with a velocity v and the obstacle. We call the interaction *weak* if the soliton solution preserves the same shape after the collision with the obstacle, and we call it *strong* if the soliton doesn't preserve the same shape and splits into several bumps or behaves as a sum of several solitary waves in a long run. We first study the dependence on the distance between the obstacle and the soliton. In order to study how different interactions affect the evolution of the solution, we need to take the initial data u_0 at a minimal distance to the obstacle, otherwise, the solution will have a similar behavior to the problem without an obstacle. For example, if we consider the initial data with large mass and large distance to the obstacle, then this solution blows up in finite time before reaching the obstacle, so that, there is no interaction present, even if we vary all parameters, which the solution depends on. This is not an interesting scenario for our purpose.

We first consider different examples, where a solution blows up in finite time with specific parameters in our computational domain with obstacle, such that there is no interaction between the soliton and the obstacle. Next, we study the behavior of the same solution such that the solution has either a weak or a strong interaction, depending on different parameters (for example, distance, velocity direction and translation parameters).

According to our numerical simulations, the solitary wave amplitudes decrease at the collision or interaction (even a small interaction) between the soliton and the obstacle. This is explained by the appearance of a reflection residue due to the presence of the obstacle with Dirichlet boundary conditions. After the collision, our numerical results show that, if there is a weak or small interaction, then the solitary wave is transmitted almost completely with a little backward reflection, and if there is a strong interaction, then the solution does not preserve the shape of the original solitary waves but it will split into several waves and will behave as a sum of two or more solitons with backward reflection. We also observe that the leading reflected wave has a dispersive behavior. The reflection phenomenon, the loss of the amplitude, and the shape of the soliton make the existence of blow-up solutions more challenging. Nevertheless, we have confirmed numerically the existence of blow-up solutions after the collision for the NLS_Ω equation in both cases of a weak and strong interaction with the obstacle.

Recall that the behavior of the focusing L^2-critical $\text{NLS}_{\mathbb{R}^d}$ equation on the whole space $\mathbb{R}^d$ was studied by Weinstein in [90]. The author showed that the solution behavior depends on the L^2-norm or the mass of the ground state solution of the following elliptic equation

$$\begin{cases} -\Delta Q + Q = |Q|^2 Q, \\ Q \in H^1(\mathbb{R}^2). \end{cases}$$

We numerically justify that the following sharp threshold for the focusing cubic NLS_Ω equation, in dimension $d = 2$, is the same as the one given in [90].

The behavior of the solutions splits into two possible scenarios:

- if $M_\Omega[u] < M_{\mathbb{R}^2}[Q]$, then the solution exists globally in time.
- if $M_\Omega[u] \geq M_{\mathbb{R}^2}[u]$, then the solution may blow-up in finite time.

4 Conclusion and perspective

In this thesis, we have obtained new results on the asymptotic behavior of solutions to the NLS_Ω equation outside a convex obstacle. First, we proved the existence of solitary wave solutions (global solutions). Secondly, we proved the existence of blow-up solutions, which was an open question for some times. We studied the scattering threshold at the mass-energy level. Despite the complexity of the geometry of the space, we found that the NLS_Ω equation has a distinct dynamics at the mass-energy threshold. Moreover, we provide numerical simulations of solutions to the equation and study the influence of geometry on the behavior of solutions. We show different simulations proving numerically that the obstacle has a substantial influence on the behavior of solutions.

Classification of solutions for the NLS_Ω equation. One of my current projects is to study the well-posedness and global existence for the focusing NLS_Ω equation in the energy-critical case. In [59], the authors proved the well-posedness and global existence for the quintic ($p = 5$) defocusing (NLS_Ω) in dimension 3. We want to extend this result to higher dimensions and to study the same questions for the focusing (NLS_Ω) equation. The energy is the highest regularity conservation law, this has important consequence for the local and global theories. We would also like to investigate the local well-posedness in the energy supercritical case for

(NLS_Ω) equation.

We would like to study the behavior of solutions to the focusing NLS_Ω equation in the energy-critical case. We expect the same scattering/blow-up dichotomy holds as for the $\mathrm{NLS}_{\mathbb{R}^d}$ equation, under a natural energy threshold (given by the energy of an explicit stationary solution). This was proved on the whole Euclidean space in [55] for $d \in \{3, 4, 5\}$. Working outside an obstacle breaks down many tools and results related to the long-time behavior of the propagator. This result would be an analog, for the energy-critical case, of the one in [58], however, the energy-critical problem will be much more difficult than in $\mathbb{R}^d$ compared to the subcritical case.

Construction of multi-soliton solutions for the NLS_Ω equation. In the whole Euclidean space, the construction of multi-soliton solutions for $\mathrm{NLS}_{\mathbb{R}^d}$ equation was studied by several authors, who contributed to a large literature on the problem. In [70], F. Merle first established the existence of multi-solitons blowing up at several blow-up points. This result was extended for several cases depending on the criticality of the equation with respect to the conserved mass and energy, see for example [69], [24], [77]. We would like to construct global multi-soliton solutions for the NLS_Ω equation outside a convex, smooth and compact obstacle for the L^2 sub/super-critical and critical nonlinearities.

Chapter II

Construction of a solitary wave solution of the nonlinear focusing Schrödinger equation outside a strictly convex obstacle in the L^2-supercritical case

Abstract.
We consider the focusing L^2-supercritical Schrödinger equation in the exterior of a smooth, compact, strictly convex obstacle $\Theta \subset \mathbb{R}^3$. We construct a solution behaving asymptotically as a solitary wave on $\mathbb{R}^3$, for large times. When the velocity of the solitary wave is high, the existence of such a solution can be proved by a classical fixed point argument. To construct solutions with arbitrary nonzero velocity, we use a compactness argument similar to the one that was introduced by F. Merle in 1990 to construct solutions of the NLS equation blowing up at several points together with a topological argument using Brouwer's theorem to control the unstable direction of the linearized operator at the soliton. These solutions are arbitrarily close to the scattering threshold given by a previous work of R. Killip, M. Visan and X. Zhang, which is the same as the one on the whole Euclidean space given by S. Roundenko and J. Holmer in the radial case and by the previous authors with T. Duyckaerts in the non-radial case.

1 Introduction

We consider the focusing nonlinear Schrödinger equation in the exterior of a smooth compact strictly convex obstacle $\Theta \subset \mathbb{R}^3$ with Dirichlet boundary conditions:

Let $\Omega = \mathbb{R}^3 \setminus \Theta,\ T_0 > 0$ and $u_0 \in H_0^1(\Omega)$,

$$(\mathrm{NLS}_\Omega)\begin{cases} i\partial_t u + \Delta_\Omega u = -|u|^{p-1}u & \forall (t,x) \in [T_0, +\infty) \times \Omega, \\ u(T_0, x) = u_0(x) & \forall x \in \Omega, \\ u(t,x) = 0 & \forall (t,x) \in [T_0, +\infty) \times \partial\Omega. \end{cases} \quad \text{(II.1)}$$

Recall that, the scaling given in (I.2) identifies the critical Sobolev space $\dot{H}_D^{s_c}$, for the (NLS_Ω) equation, where the critical regularity s_c is given by $s_c := \frac{3}{2} - \frac{2}{p-1}$.

Throughout this Chapter, we will take $\frac{7}{3} < p < 5$. Since the presence of the obstacle does not change the intrinsic dimensionality of the problem, we may regard the NLS_Ω equation as being inter-critical, i.e., $\dot{H}_0^1(\Omega)$-subcritical and $L^2(\Omega)$-supercritical

We recall the definition of the Sobolev spaces on the domain Ω associated with powers of the Dirichlet Laplacian Δ_Ω.

Definition 1.1. *For $s \geq 0,\ 1 < p < \infty$, let $\dot{H}_D^{s,p}(\Omega)$ and $H_D^{s,p}(\Omega)$ denotes the completions of $C_c^\infty(\Omega)$ under the norms*

$$\|f\|_{\dot{H}_D^{s,p}} := \left\|(-\Delta_\Omega)^{\frac{s}{2}} f\right\|_{L^p} \qquad \text{and} \qquad \|f\|_{H_D^{s,p}} := \left\|(1-\Delta_\Omega)^{\frac{s}{2}} f\right\|_{L^p} \quad \text{(II.2)}$$

When $p = 2$, we write $\dot{H}_D^s(\Omega)$ for $\dot{H}_D^{s,2}(\Omega)$ and $H - D^s(\Omega)$ for $H_D^{s,2}(\Omega)$.

It is well known that $H_0^{s,p}(\Omega) = H_D^{s,p}(\Omega)$ for $0 < s < \frac{1}{p}$ and for $\frac{1}{p} < s < 1 + \frac{1}{p}$. In particular, for $s = 1$ and $p = 2$ we have $H_0^1(\Omega) = H_D^1(\Omega)$.

In [57], the authors proved the following equivalence Sobolev space norm under some restriction of the regularity s.

Assume $1 < q < \infty$ and $0 < s < \min\{1 + \frac{1}{q}, \frac{3}{q}\}$ Then

$$\left\|(-\Delta_{\mathbb{R}^3})^{\frac{s}{2}} f\right\|_{L^q} \sim_{q,s} \left\|(-\Delta)^{\frac{s}{2}} f\right\|_{L^q} \quad \forall f \in C_c^\infty(\Omega). \quad \text{(II.3)}$$

The study of the NLS_Ω equation outside an obstacle was initiated by N. Burq, P. Gérard and N. Tzvetkov in [15], who proved local well-posedness assuming that the obstacle is non-trapping, under some restrictions on p. In particular, in dimension $d = 3$, the authors proved well-posedness for sub-cubic, i.e., $p < 3$, nonlinearity. Nevertheless, they proved the global existence for the cubic NLS_Ω equation, provided $\|u_0\|_{H^1_0(\Omega)}$ is sufficiently small. In [78], L. Vega and F. Planchon proved under the same non-trapping assumption that the NLS_Ω equation, in dimension $d = 3$, is locally well-posed for $1 < p < 5$, see also [48]. After that, F. Planchon and O. Ivanovici extended the result to the quintic Schrödinger equation outside a non-trapping domain, see [50].

Local well-posedness in the critical Sobolev space $\dot{H}^{s_c}_D(\Omega)$ was obtained by O. Ivanovici and F. Planchon in [50], for $3+\frac{2}{5} < p < 5$. However, we prove that the NLS_Ω equation is well-posed in $H^{s_c}_D(\Omega)$ for $0 < s_c < 1$, i.e., $\frac{7}{3} < p < 5$, using the fractional chain rule in the exterior of a compact convex obstacle given in [57]. We refer to Subsection 2.1 for the proof details of the following Theorem.

Theorem 1.2 (Well posedness in $H^{s_c}_D(\Omega)$)). *Let $0 < s_c < 1$, i.e., $\frac{7}{3} < p < 5$. Let $u_0 \in H^{s_c}_D(\Omega)$ then there exists a unique solution $u(t,x)$ of (NLS_Ω) with initial data u_0 defined on $[0,T]$ for some $T > 0$, such that*

$$u \in C([0,T], H^{s_c}_D(\Omega)) \cap L^q([0,T], H^{s_c,r}_D),$$

where $(q,r) = \left(p+1, \frac{6(p+1)}{3p-1}\right)$.

Furthermore, the solution u can be extended to a maximal interval of existence $[0,T_+)$ and the following alternative holds,
Either $T_+ = +\infty$ (the solution is global) or $T_+ < +\infty$ (the solution blows up in finite time) and

$$\lim_{t\to T_+} \|u(t,\cdot)\|_{H^{s_c}_D(\Omega)} = +\infty.$$

Consider a solitary waves solution of (NLS_Ω), with $\Omega = \mathbb{R}^3$, that is, $u(t,x) = e^{it\omega}Q_\omega(x)$, where Q_ω is a solution of the nonlinear elliptic equation:

$$\begin{cases} -\Delta Q_\omega + \omega\, Q_\omega = |Q_\omega|^{p-1} Q_\omega, \\ Q_\omega \in H^1(\mathbb{R}^3). \end{cases} \tag{II.4}$$

This elliptic equation admits solutions if and only if $\omega > 0$. In this Chapter, we will denote by Q_ω the ground state, which is the unique radial positive solution of (IV.1).

We recall that Q_ω is smooth and exponentially decaying at infinity and characterized as the unique minimizer for the Gagliardo-Nirenberg inequality up to scaling, space translation and phase shift, see [60].

The (NLS) equation posed on the whole Euclidean space $\mathbb{R}^3$, also enjoys Galilean invariance. If $u(t,x)$ is solution, then $u(t,x-vt)\,e^{i(\frac{x\cdot v}{2}-\frac{|v|^2}{4}t)}$ is also a solution, for $v \in \mathbb{R}^3$.

Applying a Galilean transform to the solution $e^{it\omega}Q_\omega(x)$ of the (NLS) on $\mathbb{R}^3$, we obtain a soliton solution, moving on the line $x = tv$ with velocity $v \in \mathbb{R}^3$:

$$u(t,x) = e^{i(\frac{1}{2}(x\cdot v)-\frac{1}{4}|v|^2t+t\,\omega)}Q_\omega(x-t\,v). \tag{II.5}$$

The soliton (II.5) is a global solution of the focusing nonlinear Schrödinger equation (NLS) posed on the whole space, but is not a solution of (NLS_Ω). Our goal is to construct solitary wave of the (NLS_Ω) satisfying Dirichlet boundary conditions and behaving asymptotically as the solitary wave in (II.5), as $t \longrightarrow +\infty$.

The main result of this Chapter is the following.

Theorem 1.3. *Assume* $\frac{7}{3} < p < 5$.
Let Ψ *be a* C^∞ *function such that:* $\begin{cases} \Psi = 0 & \textit{near } \Theta, \\ \Psi = 1 & \textit{if } |x| \gg 1. \end{cases}$
Let $v \in \mathbb{R}^3\backslash\{0\}$ *be the velocity,* $\omega > 0$. *Then there exists* $\delta > 0$, $T_0 > 0$ *and a function* r_ω, *defined on* $[T_0,+\infty)\times\Omega$, *satisfying*

$$\|r_\omega(t)\|_{H_0^1(\Omega)} \leq e^{-\delta\sqrt{\omega}|v|t} \qquad \forall t \in [T_0,+\infty),$$

such that

$$u(t,x) = e^{i(\frac{1}{2}(x\cdot v)-\frac{1}{4}|v|^2t+t\,\omega)}Q_\omega(x-tv)\Psi(x) + r_\omega(t,x), \quad \forall (t,x) \in [T_0,+\infty)\times\Omega,$$

is a solution of (NLS_Ω).

Remark 1.4. *Theorem 1.3 can be generalized for any dimension* $d \geq 3$. *Moreover, this result* *can be extended to the subcritical case $1 < p < \frac{7}{3}$, which is easier to prove due to the stability of solitons.*

Remark 1.5. *One can prove Theorem 1.3 for a non-trapping obstacles using the same arguments. The restriction to a strictly convex obstacle is purely technical. In section 2, we use the fact that the NLS_Ω equation is well-posed on $H^{s_c}_D(\Omega)$, for $0 < s_c < 1$, with $s_c = \frac{3}{2} - \frac{3}{p-1}$ (Cf. Theorem 1.2). For that we need to use a Strichartz estimate from [48] (Cf. Theorem B) and some fractional rules given by [57] for a strictly convex obstacle (Cf. Proposition C). Because of this, we shall suppose that the obstacle Θ is strictly convex.*

In the spirit of the works of C. Kenig and F. Merle on the energy-critical equations in [55] and [56], J Holmer and S. Roudenko have studied in [45] the behavior (i.e., scattering and global existence) of the solutions of the focusing radial cubic (i.e., $p = 3$) nonlinear Schrödinger equation on $\mathbb{R}^3$, whenever the initial data satisfies a smallness criterion given by the ground state threshold. The criterion is expressed in terms of the scale-invariant quantities $\|u_0\|^{1-s}_{L^2(\mathbb{R}^3)} \|\nabla u_0\|^{s}_{L^2(\mathbb{R}^3)}$ and $M^{1-s}_{\mathbb{R}^3}[u]E^s_{\mathbb{R}^3}[u]$. This result was later extended to the non-radial case in [29] and to arbitrary space dimensions and focusing inter-critical power nonlinearities by T. Cazenave, J. Xie and D. Fang, see [32] and, by C. Guevara in [43].

Theorem A ([45],[29],[32],[43])**.** *Let $s = \frac{3}{2} - \frac{2}{p-1}$ and $\frac{7}{3} < p < 5$. Let $u_0 \in H^1(\mathbb{R}^3)$ satisfy*

$$\|u_0\|^{1-s}_{L^2(\mathbb{R}^3)} \|\nabla u_0\|^{s}_{L^2(\mathbb{R}^3)} < \|Q\|^{1-s}_{L^2(\mathbb{R}^3)} \|\nabla Q\|^{s}_{L^2(\mathbb{R}^3)}, \tag{II.6}$$

$$M_{\mathbb{R}^3}[u_0]^{1-s} E_{\mathbb{R}^3}[u_0]^s < M_{\mathbb{R}^3}[Q]^{1-s} E_{\mathbb{R}^3}[Q]^s. \tag{II.7}$$

Then u scatters in $H^1(\mathbb{R}^3)$.

Theorem A remains true for (NLS_Ω) in the exterior of a strictly convex obstacle in three dimension. Indeed, R. Killip, M. Visan and X. Zhang had proved in [58] that the threshold for global existence and scattering is the same as for the cubic NLS equation on $\mathbb{R}^3$. Later, K. Yang in [94] extended this result for $\frac{7}{3} < p < 5$.

The solitary waves constructed in the main Theorem 1.3 prove the optimality of the threshold for scattering given in [94, Theorem 1.3]. Indeed, the solution u of (NLS_Ω) is global, does not scatter for positive time direction and we have

$$E_\Omega[u] = \frac{|v|^2}{8} \int_{\mathbb{R}^3} |Q|^2 + E_{\mathbb{R}^3}[Q]\,. \tag{II.8}$$

Since, the velocity v can be taken arbitrary small, we have proved that for all $\varepsilon > 0$ there exists a solution u_ε of (NLS_Ω), which is global and does not scatter for positive time such that

$$M_\Omega[u_\varepsilon] = M_{\mathbb{R}^3}[Q]\,, \quad \sup_{t\geq T_0} \|\nabla u_\varepsilon(t)\|^s_{L^2(\Omega)} < \|\nabla Q\|^s_{L^2(\mathbb{R}^3)} + \varepsilon$$

and

$$E_\Omega[u_\varepsilon]^s < E_{\mathbb{R}^3}[Q]^s + \varepsilon\,.$$

The proof of Theorem 1.3 relies on a compactness argument that uses the structure of the linearized operator around the ground state soliton. If the velocity v is large enough, we can use a simple fixed point theorem to construct a soliton solution of (NLS_Ω).

Theorem 1.6. *Assume* $2 \leq p < 5$.
Let $\Omega = \mathbb{R}^3 \setminus \Theta$, *where* Θ *is any smooth compact obstacle and* Q_ω *be any solution of* (IV.1).
Let Ψ *be a* C^∞ *function such that:* $\begin{cases} \Psi = 0 & \text{near } \Theta, \\ \Psi = 1 & \text{if } |x| \gg 1. \end{cases}$
Let $\omega, T_0 > 0$. *Then there exists* $V_0 := V_0(\omega) \gg 1$ *with the following property. Let* $v \in \mathbb{R}^3$ *be the velocity such that* $|v| > V_0$.

Then there exists $\delta > 0$ *and a functions* r_ω *defined on* $[T_0, +\infty) \times \Omega$ *satisfying*

$$\forall t \in [T_0, +\infty) \qquad \|r_\omega(t)\|_{H^2\cap H^1_0(\Omega)} \leq C_\omega\, |v|^3\, e^{-\delta\sqrt{\omega}|v|t},$$

such that $u(t,x) = e^{i(\frac{1}{2}(x\cdot v) - \frac{1}{4}|v|^2 t + t\,\omega)} Q_\omega(x - tv)\Psi(x) + r_\omega(t,x), \quad \forall (t,x) \in [T_0, +\infty) \times \Omega,$
is a solution of (NLS_Ω).

Unlike Theorem 1.3, Q_ω is any solution of the nonlinear elliptic equation (IV.1) (not necessarily the ground state) and $\Theta \subset \mathbb{R}^3$ does not have to be convex, which makes Theorem 1.3 more general for high velocity. However, we can see in (II.8) that the choice of high velocity does not allow us to use Theorem 1.6 to show the optimality of the threshold for scattering in [58] and [94]. Let us mention that, this result can be extended for any dimension $d \geq 3$. We will give the proof of the Theorem 1.6 for the cubic case $p = 3$. The proof for general $p \in [2, 5)$ is very similar, see Remark 4.1.

Let us mention that apart of the works cited above, the NLS_Ω equation outside convex obstacle was also studied by O. Ivanovivi and G. Lebeau in [49]. The NLS_Ω equation in the exterior of star-shaped obstacle in dimension $d = 2$ was studied by Blair, Smith and Sogge in [10] and by F. Planchon and Luis Vega in [79]. This result was generalized by Farah Abou Shakra in [1] for 2D (NLS_{Ω}) outside non-trapping. We also refer to [66], for global well-posedness and scattering result for the defocusing (NLS_{Ω}) in the exterior of balls with radial data. Let us also mention the recent works on dispersive estimates outside two or several strictly convex obstacles of D. Lafontaine in [61], [62].

We end this section by giving an outline of the proofs of the two theorems above.

Outline of the proof of Theorem 1.3.
The structure of the proof is similar to the one for construction of multi-soliton for (NLS) on $\mathbb{R}^d$ in the subcritical case in [69] with an additional argument coming from [24], which allows us to handle the supercritical character of the non-linearity. The compactness argument used in this Chapter is similar to the main argument used in [69],[24], and [70].

Note that, even though we use some similar arguments, a large part of the proof of Theorem 1.3 is different. This is due to of the presence of the obstacle Θ, which makes the calculations more complicated.

Recall that the soliton $Q_{\omega}(x - t\,v)e^{i(\frac{1}{2}(x\cdot v)-\frac{1}{4}|v|^2t+t\,\omega)}$ is an exact solution of the (NLS) on the whole space $\mathbb{R}^3$. Therefore, the proof consists in the construction of a smooth correction $r_{\omega}(t,x)$ with some uniform estimates, such that $R(t,x) + r_{\omega}(t,x)$ is a solution of the equation (NLS_{Ω}), where $R(t,x) = e^{i(\frac{1}{2}(x\cdot v)-\frac{1}{4}|v|^2t+t\,\omega)}Q_{\omega}(x - t\,v)\,\Psi(x)$.

The Chapter is organized as follows. We next review some properties of the ground state Q in §2.2 and we recall some spectral properties of the linearized Schrödinger operator around the soliton $e^{it}Q$, in §2.3.

In the subcritical case, Cazenave and Lions [17], Weinstein [92] proved that the solitary waves are stable when $1 < p < \frac{7}{3}$, which means that the nonlinearity has an L^2-subcritical growth. From [92], there exits $\lambda > 0$ such that for any real-valued function $h \in H^1$,

$$(h, Q_{\omega}), (h, \nabla Q_{\omega}) = 0 \Longrightarrow \int \{|\nabla h|^2 + \omega\,|h|^2 - pQ_{\omega}^{p-1}\,|Q_{\omega}|^2\} \geq \lambda\,\|h\|_{H^1}^2\,. \tag{II.9}$$

In [69], the authors use some modulation in the scaling, phase and translation parameters, to control these two direction.

In the supercritical case, it is well known that the soliton is unstable, see [42]. Indeed, for $\frac{7}{3} < p < 5$, there exists two eigenfunctions of the linearized operator around the ground state Q, see, e.g., Weinstein [91], Schlag [82], Grillakis [40] and denoted by $\mathcal{Y}^{\pm}$. Thus, the above property (II.9) of the linearized operator does not hold, but an effective coercivity property can be expressed in terms of the eigenfunctions $\mathcal{Y}^{\pm}$, see Lemma 2.7.

In §2.4, we suppose that there exists a solution u_n of (NLS_Ω) for $t \in [T_0, T_n]$ that satisfies some uniform estimate with initial data $u_n(T_n)$ and $\{T_n\}$ is an increasing sequence of times. Then by compactness argument we construct a solution u of (NLS_Ω) for $[T_0, +\infty)$, with initial data $u(T_0)$ and $T_0 > 0$, which concludes the proof of Theorem 1.3.

In Section 3, we prove the existence of the solution u_n and the uniform estimate assumed in the previous section. For this, we use a modulation in the phase and translation parameters in the decomposition of the solution for large time to obtain orthogonality conditions. Next, we define a maximal time interval on which suitable exponential estimates of the modulation parameters hold, as well as, the uniform estimate used in §2.4 and others terms expressed in terms of the eigenfunctions $\mathcal{Y}^+$ and $\mathcal{Y}^-$. In order to conclude the bootstrap argument, we improve theses estimates using a coercivity property of the linearized operator. Indeed, the linearized operator $(\mathcal{L}\ \cdot, \cdot)$ is positive definite up to the four directions: Q, $\partial_x Q$ and $\mathcal{Y}^{\pm}$, see [30] and [31]. As in the subcritical case, the two directions $Q_\omega, \nabla Q_\omega$ are still to be controlled by the modulation with respect to the translation and phase parameters. The direction $\mathcal{Y}^+$ is stable in the sense that can be controlled, however, the other direction $\mathcal{Y}^-$ is unstable and cannot be controlled by a scaling argument, even if we introduce an extra parameter in the modulation. Therefore, we have to use a topological argument to control this unstable direction and to conclude the proof of the uniform estimate on $[T_0, T_n]$.

Outline of the proof of Theorem 1.6.

In Section 4, we prove Theorem 1.6 using a fixed point argument similar to the one used in [14], and in [39], to construct a solution blowing up in finite time at a fixed point, or at several blow-up points, in the interior of a bounded domain. However, even though we use a similar argument a large part the proof is different, due to the fact that the solution that we construct is global, and that the bounded domain is replaced by the exterior of an obstacle.

We prove, constructing a contraction mapping, the existence of a smooth correction $r_\omega(t,x)$ such that $u(t,x) = R(t,x) + r_\omega(t,x)$ is a solution of (NLS_Ω), where

$$R(t,x) = e^{i(\frac{1}{2}(x\cdot v) - \frac{1}{4}|v|^2 t + t\,\omega)} Q_\omega(x - t\,v)\Psi(x)\,.$$

We have

$$(i\partial_t + \Delta)R(t,x) = -\Psi(x)\left|H(t,x)\right|^2 H(t,x) + 2\nabla\Psi(x)\nabla H(t,x) + \Delta\Psi(x)H(t,x),$$

where $H(t,x) = e^{i(\frac{1}{2}(x\cdot v) - \frac{1}{4}|v|^2 t + t\,\omega)} Q_\omega(x - t\,v)$.

We look for $r_w \in C([T_0,+\infty), H^2(\Omega)\cap H_0^1(\Omega))$ such that

$$\begin{cases} i\partial_t r_\omega + \Delta r_\omega = -\left|R + r_\omega\right|^2 (R + r_\omega) + \Psi\left|H\right|^2 H - 2\nabla\Psi\nabla H - \Delta\Psi H\,, \\ \qquad\qquad r_\omega(t) \longrightarrow 0 \quad t \longrightarrow +\infty \quad \text{in} \quad H^2(\Omega)\cap H_0^1(\Omega). \end{cases} \tag{II.10}$$

We shall look for solutions of (NLS_Ω) in the following space

$$E = \left\{ r_\omega \in C\left([T_0,+\infty), H^2(\Omega)\cap H_0^1(\Omega)\right),\ \left\|r_\omega\right\|_E < \infty \right\}\,,$$
$$\left\|r_\omega\right\|_E = \sup_{t\geq T_0}\left\{ e^{\delta\sqrt{\omega}|v|t}\left(\frac{1}{|v|^3}\left\|r_\omega\right\|_{H^2(\Omega)} + \left\|r_\omega\right\|_{L^2(\Omega)}\right)\right\}.$$

Let

$$\begin{aligned} \Phi : (B_E, d_E) &\longrightarrow (B_E, d_E) \\ r_\omega \longmapsto \Phi(r_\omega) &= -i\int_t^{+\infty} S(t-\tau)\left(\left|R + r_\omega\right|^2 (R + r_\omega) + \Psi\left|H\right|^2 H - 2\nabla\Psi\nabla H - \Delta\Psi H\right) d\tau, \end{aligned}$$

where S(t) is the unitary group of the linear Schrödinger equation on Ω with Dirichlet boundary conditions, $B_E := B_E(0,1) = \{h \in E,\ \left\|h\right\|_E \leq 1\}$ and $d_E(h,g) = \left\|h - g\right\|_E$. One can check that (B_E, d_E) is a complete metric space.

Our goal is to solve the integral formulation of (II.10) by a fixed point argument. Using the high velocity assumption, we prove that Φ is stable on B_E and it is a contraction mapping. Thus, by the fixed point theorem, we conclude that there exists a unique solution r_ω of (II.10) on E.

Appendix Section 5 contains the proof of the coercivity property of the linearized Schrödinger operator, the local existence of the equation in the critical space $H^{s_c}_D(\Omega)$, for $0 < s_c < 1$, the modulation for time independent function and other technical results.
Appendix Section 6 contains the computation of some estimates used on the proof of Theorem 1.6.

Notation:
If a and b are two functions of t and if b is positive, we write $a = O(b)$ when there exists a constant $C > 0$ independent of t such that $|a(t)| \leq C\, b(t)$ for all t.
For $h \in \mathbb{C}$, we denote $h_1 = \operatorname{Re} h$ and $h_2 = \operatorname{Im} h$.
Throughout this Chapter, C denotes a positive constant independent of t, that may change from line to line and may depend on ω and Ω.
We denote by $|\cdot|$ a $\mathbb{R}^d$-norm with $d = 1, 2, 3$.
For simplicity, we will write $\Delta := \Delta_\Omega$.
Denote by $(\cdot, \cdot)$, the real L^2-scalar product,

$$(f, g) = \operatorname{Re} \int f\, \overline{g} = \int \operatorname{Re} g\ \operatorname{Re} f + \int \operatorname{Im} g\ \operatorname{Im} f\ .$$

2 Construction of the solution assuming uniform estimates

2.1 Well posedness in $H^{s_c}_D(\Omega)$

In this subsection , we prove Theorem 1.2 and we will only prove the local existence statement. The construction of a maximal solution is standard and we omit it. Let us recall that the usual Strichartz estimates are also available outside a convex obstacle, see [57] and [48]:

Theorem B. *Let $d \geq 2$, $\Omega \subset \mathbb{R}^d$ be the exterior of a smooth compact strictly convex obstacle. Let $q, \tilde{q} > 2$ and $2 \leq r, \tilde{r} \leq \infty$ satisfy the scaling conditions: $\frac{2}{q} + \frac{d}{r} = \frac{d}{2} = \frac{2}{\tilde{q}} + \frac{d}{\tilde{r}}$
Then*

$$\left\| e^{it\Delta} u_0 \pm i \int_0^t e^{i(t-s)\Delta} F(s) ds \right\|_{L^q_t L^r_x} \leq C_s \left(\|u_0\|_{L^2(\Omega)} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \right). \tag{II.11}$$

To estimate the nonlinearity $|u|^{p-1} u$ in $H^{s_c}_D(\Omega)$, we have to use some fractional estimate. We refer to [57], for the following Proposition.

Proposition C. *(Fractional chain rule)*
Suppose $G \in C^1(\mathbb{C})$, $s \in (0,1]$, and $1 < p, p_1, p_2 < \infty$ are such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $0 < s < \min\left(1 + \frac{1}{p_2}, \frac{3}{p_2}\right)$. Then there exists $C > 0$ such that

$$\left\|(-\Delta_\Omega)^{\frac{s}{2}} G(f)\right\|_{L^p(\Omega)} \le C \left\|G'(f)\right\|_{L^{p_1}(\Omega)} \left\|(-\Delta_\Omega)^{\frac{s}{2}} f\right\|_{L^{p_2}(\Omega)}, \tag{II.12}$$

Uniformly for $f \in C_c^\infty(\Omega)$.

Remark 2.1. *For the sake of simplicity, we will write the Dirichlet Laplacian as Δ instead of Δ_Ω.*

For the proof of Theorem 1.2, we claim the following result .

Claim 2.2 (Hölder's inequalities)**.** *Let $(p+1, \frac{6(p+1)}{3p-1})$ be admissible pair, i.e., $\frac{2}{p+1} + \frac{3(3p-1)}{6(p+1)} = \frac{3}{2}$. Let $u, v \in L^\infty H_D^{s_c} \cap L^{p+1} H_D^{s_c, \frac{6(p+1)}{3p-1}}([0,T] \times \Omega)$*
Then,

$$\left\||u|^{p-1} u\right\|_{L^{\frac{p+1}{p}} L^{\frac{6(p+1)}{3p+7}}} \le C \|u\|^{p-1}_{L^{p+1} L^{\frac{3(p-1)(p+1)}{4}}} \|u\|_{L^{p+1} L^{\frac{6(p+1)}{3p-1}}}. \tag{II.13}$$

$$\left\|-(\Delta)^{\frac{s_c}{2}} |u|^{p-1} u\right\|_{L^{\frac{p+1}{p}} L^{\frac{6(p+1)}{3p+7}}} \le C \|u\|^{p-1}_{L^{p+1} H_D^{s_c, \frac{6(p+1)}{3p-1}}} \left\|(-\Delta)^{\frac{s_c}{2}} u\right\|_{L^{p+1} L^{\frac{6(p+1)}{3p-1}}}. \tag{II.14}$$

$$\begin{aligned}\left\||u|^{p-1} u - |v|^{p-1} v\right\|_{L^{\frac{p+1}{p}} L^{\frac{6(p+1)}{3p+7}}} \le C \|u - v\|_{L^{p+1} L^{\frac{6(p+1)}{3p-1}}} \\ \left(\|u\|^{p-1}_{L^{p+1} H_D^{s_c, \frac{6(p+1)}{3p-1}}} + \|v\|^{p-1}_{L^{p+1} H_D^{s_c, \frac{6(p+1)}{3p-1}}}\right)\end{aligned} \tag{II.15}$$

Proof. Note that $p+1 > 2$, since $\frac{7}{3} < p < 5$, and $\left(\frac{p+1}{p}, \frac{6(p+1)}{3p+7}\right)$ is the dual of the L^2-admissible pair $(p+1, \frac{6(p+1)}{3p-1})$.

For the first estimate it suffices to apply Hölder's inequality, we obtain

$$\begin{aligned}\left\||u|^{p-1} u\right\|_{L^{\frac{p+1}{p}} L^{\frac{6(p+1)}{3p+7}}} &\le C_s \left\||u|^{p-1}\right\|_{L^{p+1} L^{\frac{3(p+1)}{4}}} \|u\|_{L^{p+1} L^{\frac{6(p+1)}{3p-1}}} \\ &\le C \|u\|^{p-1}_{L^{p+1} L^{\frac{3(p-1)(p+1)}{4}}} \|u\|_{L^{p+1} L^{\frac{6(p+1)}{3p-1}}}.\end{aligned}$$

Next, we prove the estimate (II.14), using the fractional chain rule (II.12) with Hölder's inequality in time.

$$\left\|(-\Delta)^{\frac{s_c}{2}} |u|^{p-1} u\right\|_{L^{\frac{6(p+1)}{3p+7}}} \leq C \left\||u|^{p-1}\right\|_{L^{\frac{3(p+1)}{4}}} \left\|(-\Delta)^{\frac{s_c}{2}} u\right\|_{L^{\frac{6(p+1)}{3p-1}}}, \tag{II.16}$$

provided for $0 < s_c < \min\{1 + \frac{(3p-1)}{6(p+1)}, \frac{3(3p-1)}{6(p+1)}\}$. Since we consider $0 < s_c < 1$, i.e., $\frac{7}{3} < p < 5$, then the condition $0 < s_c < \min\{1 + \frac{(3p-1)}{6(p+1)}, \frac{3(3p-1)}{6(p+1)}\}$ is satisfied.

Using the above estimate (II.16) with Hölder inequality in time, we have

$$\begin{aligned}\left\|(-\Delta)^{\frac{s_c}{2}} |u|^{p-1} u\right\|_{L^{\frac{p+1}{p}} L^{\frac{6(p+1)}{3p+7}}} &\leq C \left\||u|^{p-1}\right\|_{L^{\frac{p+1}{p-1}} L^{\frac{3(p+1)}{4}}} \left\|(-\Delta_\Omega)^{\frac{s_c}{2}} u\right\|_{L^{p+1} L^{\frac{6(p+1)}{3p-1}}} \\ &\leq C \|u\|^{p-1}_{L^{p+1} L^{\frac{3(p-1)(p+1)}{4}}} \left\|(-\Delta_\Omega)^{\frac{s_c}{2}} u\right\|_{L^{p+1} L^{\frac{6(p+1)}{3p-1}}}\end{aligned}$$

By the equivalence of Sobolev norms (II.3) with Sobolev inequality we obtain

$$\left\|(-\Delta)^{\frac{s_c}{2}} |u|^{p-1} u\right\|_{L^{\frac{p+1}{p}} L^{\frac{6(p+1)}{3p+7}}} \leq C \|u\|^{p-1}_{L^{p+1} H_D^{s_c, \frac{6(p+1)}{3p-1}}} \left\|(-\Delta_\Omega)^{\frac{s_c}{2}} u\right\|_{L^{p+1} L^{\frac{6(p+1)}{3p-1}}}.$$

Now, let us prove the last estimate. We use the following elementary inequality

$$\forall (\xi, \zeta) \in \mathbb{C}^2, \qquad \left| |\xi|^{p-1} \xi - |\zeta|^{p-1} \zeta \right| \leq C_p \left(|\xi|^{p-1} + |\xi|^{p-1} \right) |\xi - \zeta| \tag{II.17}$$

As a consequence, fixing t, we deduce

$$\left\| |u|^{p-1} u - |v|^{p-1} v \right\|_{L^{\frac{6(p+1)}{3p+7}}} \leq C_p \left\| |u - v| \left(|u|^{p-1} + |v|^{p-1} \right) \right\|_{L^{\frac{6(p+1)}{3p+7}}}$$

Taking the $L^{\frac{p+1}{p}}$-norm in time and using Hölder inequality, we obtain

$$\begin{aligned}\left\| |u|^{p-1} u - |v|^{p-1} v \right\|_{L^{\frac{p+1}{p}} L^{\frac{6(p+1)}{3p+7}}} &\leq C_p \|u - v\|_{L^{p+1} L^{\frac{6(p+1)}{3p-1}}} \left\| |u|^{p-1} + |v|^{p-1} \right\|_{L^{\frac{p+1}{p-1}} L^{\frac{3(p+1)}{4}}} \\ &\leq C_p \|u - v\|_{L^{p+1} L^{\frac{6(p+1)}{3p-1}}} \left(\|u\|^{p-1}_{L^{p+1} L^{\frac{3(p-1)(p+1)}{4}}} + \|v\|^{p-1}_{L^{p+1} L^{\frac{3(p-1)(p+1)}{4}}} \right).\end{aligned}$$

By the equivalence of Sobolev norms (II.3) and Sobolev inequality, we deduce (II.15). This concludes the proof of Claim 2.2. □

For $T > 0$ and $M > 0$ to be specified later, let B_T be the ball of $X = C([0,T], H_D^{s_c}) \cap L_t^{p+1} H_D^{s_c, \frac{6(p+1)}{3p-1}}$, with radius $M > 0$ and center 0, i.e., the set of functions

$$u \in X = C([0,T], H_D^{s_c}) \cap L_t^{p+1} H_D^{s_c, \frac{6(p+1)}{3p-1}}$$

such that

$$\|u\|_{L^\infty H_D^{s_c}} \leq M \text{ and } \|u\|_{L^{p+1} H_D^{s_c, \frac{6(p+1)}{3p-1}}} \leq M.$$

Denote

$$d_B(u,v) = \|u - v\|_{L^\infty L^2} + \|u - v\|_{L^{p+1} L^{\frac{6(p+1)}{3p-1}}}$$

Lemma 2.3. (B_T, d_B) *is a complete metric space.*

Proof. It is an immediate consequence of the easy fact that B_T is a closed subset of the following Banach space

$$Y := C([0,T], L^2) \cap L^{p+1} L^{\frac{6(p+1)}{3p-1}}.$$

Let $(u_n)_n$ be a sequence of elements of B, which converges, for the Y norm, to $u \in Y$. One can prove that $u \in L_t^\infty H_D^{s_c} \cap L_t^{p+1} H_D^{s_c, \frac{6(p+1)}{3p-1}}$, and $\|u\|_{L^\infty H_D^{s_c}} \leq M$ and $\|u\|_{L^{p+1} H_D^{s_c, \frac{6(p+1)}{3p-1}}} \leq M$, using the fact that $L^{p+1} L^{\frac{6(p+1)}{3p-1}}$ is a reflexive space and the standard property of the weak convergence. □

For $v \in B_T$ we define $\Phi(v)(t) := e^{it\Delta} u_0 + D(v)(t)$, where $D(v)$ is the Duhamel term given by

$$D(v)(t) := -i \int_0^t e^{i(t-s)\Delta} |v(s)|^{p-1} v(s) ds.$$

- Step 1 : Stability of B_T.

  We will prove that: for $v \in B_T \Longrightarrow \Phi(v) \in B_T$, for a good choice of M and T.
  We have

$$\left\| e^{it\Delta} u_0 \right\|_{L^\infty H_D^{s_c}(I \times \Omega)} = \|u_0\|_{H_D^{s_c}(\Omega)} \leq \frac{M}{2}.$$

  If the following conditions satisfied

$$2 \|u_0\|_{H_D^{s_c}(\Omega)} \leq M, \tag{II.18}$$

Using Strichartz estimate (recall that $(p+1, \frac{6(p+1)}{3p-1})$ is an admissible pair) we obtain

$$\left\|e^{it\Delta}u_0\right\|_{L^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}(I\times\Omega)} \leq C_s \left\|(1-\Delta)^{\frac{s_c}{2}}u_0\right\|_{L^2(\Omega)} = C_s \|u_0\|_{H_D^{s_c}(\Omega)} \leq \frac{M}{2}.$$

If M is chosen so that

$$M \geq 2C_s \|u_0\|_{H_D^{s_c}(\Omega)}. \tag{II.19}$$

Take $T>0$, such that we have

$$\max\left(\left\|e^{it\Delta}u_0\right\|_{L^\infty H_D^{s_c}([0,T]\times\Omega)}, \left\|e^{it\Delta}u_0\right\|_{L^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}([0,T]\times\Omega)}\right) \leq \frac{M}{2}, \tag{II.20}$$

if (II.18), (II.19) are satisfied.

We next treat the Duhamel term.

$$\|D(v)\|_{L^\infty H_D^{s_c}([0,T]\times\Omega)} = \left\|\int_0^t e^{i(t-\sigma)\Delta}|u(\sigma)|^{p-1}u(\sigma)d\sigma\right\|_{L^\infty H_D^{s_c}([0,T]\times\Omega)}$$

We use Strichartz estimate, Claim 2.2 and Sobolev inequality to obtain

$$\begin{aligned}
\|D(v)\|_{L^\infty H_D^{s_c}} &\leq C_s \left\|(1-\Delta)^{\frac{s_c}{2}}|u|^{p-1}u\right\|_{L^{\frac{p+1}{p}}L^{\frac{6(p+1)}{3p+7}}} \\
&\leq C\|u\|_{L^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}} \left\||u|^{p-1}\right\|_{L^{\frac{p+1}{p-1}}L^{\frac{3(p+1)}{4}}} \\
&\leq C\|u\|_{L^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}} \|u\|^{p-1}_{L^{p+1}L^{\frac{3(p-1)(p+1)}{4}}} \\
&\leq C_1\|u\|_{L^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}} \|u\|^{p-1}_{L^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}}
\end{aligned}$$

We can obtain the same thing for $L^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}([0,T]\times\Omega)$-norm of the Duhamel term.

$$\begin{aligned}
\|D(v)\|_{L^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}} &= \left\|\int_0^t e^{i(t-s)\Delta}(1-\Delta)^{\frac{s_c}{2}}\,|u(s)|^{p-1}\,u(s)ds\right\|_{L^{p+1}L^{\frac{6(p+1)}{3p-1}}} \\
&\leq C_s \left\|(1-\Delta)^{\frac{s_c}{2}}\,|u|^{p-1}\,u\right\|_{L^{\frac{p+1}{p}}L^{\frac{6(p+1)}{3p+7}}} \\
&\leq C\,\|u\|_{L^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}}\left\||u|^{p-1}\right\|_{L^{\frac{p+1}{p-1}}L^{\frac{3(p+1)}{4}}} \\
&\leq C\,\|u\|_{L^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}}\,\|u\|^{p-1}_{L^{p+1}L^{\frac{3(p-1)(p+1)}{4}}} \\
&\leq C_2\,\|u\|_{L^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}}\,\|u\|^{p-1}_{L^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}}
\end{aligned}$$

Finally, we have obtained

$$\|D(v)\|_{L^\infty H_D^{s_c}([0,T]\times\Omega)}+\|D(v)\|_{L^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}([0,T]\times\Omega)}\leq\frac{M}{2}.$$

If the following conditions are satisfied

$$\max\{C_1,C_2\}M^{p-1}\leq\frac{1}{2}. \tag{II.21}$$

- Step 2: Contraction property

  Let $u,v\in B_T$. Using Claim 2.2, we have

$$\begin{aligned}
\|\Phi(u)-\Phi(v)\|_{L^\infty L^2\cap L^{p+1}L^{\frac{6(p+1)}{3p-1}}} &= \|D(u)-D(v)\|_{L^\infty L^2\cap L^{p+1}L^{\frac{6(p+1)}{3p-1}}} \\
&\leq C_s\left\||u|^{p-1}\,u-|v|^{p-1}\,v\right\|_{L^{\frac{p+1}{p}}L^{\frac{6(p+1)}{3p+7}}} \\
&\leq C\ \|u-v\|_{L^{p+1}L^{\frac{6(p+1)}{3p-1}}}\left(\|u\|^{p-1}_{L^{p+1}L^{\frac{3(p-1)(p+1)}{4}}}+\|v\|^{p-1}_{L^{p+1}L^{\frac{3(p-1)(p+1)}{4}}}\right) \\
&\leq C\ \|u-v\|_{L^{p+1}L^{\frac{6(p+1)}{3p-1}}}\left(\|u\|^{p-1}_{L^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}}+\|v\|^{p-1}_{L^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}}\right) \\
&\leq C_3\,M^{p-1}\,\|u-v\|_{L^{p+1}L^{\frac{6(p+1)}{3p-1}}}\ ,
\end{aligned}$$

which yields,

$$d_B(\Phi(u)-\Phi(v))\leq C_3\ M^{p-1}\,d_B(u,v).$$

And this prove that Φ is a contraction if the following condition is satisfied

$$C_3\, M^{p-1} < 1. \tag{II.22}$$

Then by fixed point theorem there exist a unique solution $u(t,x)$ to the NLS_Ω equation. We have proved that Φ is a contraction on the metric space (B_T, d_B) if the conditions (II.20), (II.21) and (II.22) on M and T hold. Indeed, we take $\|u_0\|_{H^{s_c}_D}$ small so that

$$\left(2\, \|u_0\|_{H^{s_c}_D}\right)^{p-1} \leq \frac{1}{2^p \max\{C_1, C_2, C_3\} \max\{C_s, 1\}^{p-1}}.$$

We can take M as

$$M = 2\max\{C_s, 1\}\, \|u_0\|_{H^{s_c}_D}.$$

Thus, the condition (II.18), (II.19) are satisfied. Moreover, we have

$$M^{p-1} \leq \frac{1}{2\max\{C_1, C_2, C_3\}}.$$

Then, the condition (II.21) and (II.22) are satisfied. Note that, we can take T such that (II.20) hold.

It remains to check that $u \in C([0,T], H^{s_c}_D(\Omega))$, which will be done in step 3 and in step 4 we prove also the uniqueness of u among the $C([0,T], H^{s_c}_D(\Omega))$ solutions.

- Step 3 : Continuity

$$u(t) = e^{it\Delta}u_0 + i\int_0^t e^{i(t-s)\Delta}\, |u(s)|^{p-1}\, u(s)ds.$$

It is well known that the function: $t \longmapsto e^{it\Delta}u_0$ is in $C([0,T], H^{s_c}_D(\Omega))$.
Next, we recall that from Step 1 and Step 2 that

$$t \longmapsto |u(t)|^{p-1}\, u(t) \in L^{\frac{p+1}{p}} H_D^{s_c, \frac{6(p+1)}{3p+7}}([0,T] \times \Omega).$$

By Strichartz inequality, we have that the Duhamel term $D(u) \in C([0,T], H^{s_c}_D(\Omega))$.
Thus, we get $u = e^{i\Delta}u_0 + D(u)\ \in C([0,T], H^{s_c}_D(\Omega))$.

- Step 4: Uniqueness.

Let u and v be two solutions in $C([0,T],H_D^{s_c}(\Omega))$ with the same initial data u_0. Then

$$u(t)-v(t)=i\int_0^t e^{i(t-s)\Delta}\left(|u(s)|^{p-1}u(s)-|v(s)|^{p-1}v(s)\right)ds,$$

by Strichartz inequality, if $\theta>0$

$$\begin{aligned}\|u-v\|_{L_\theta^{p+1}L^{\frac{6(p+1)}{3p-1}}} &\leq C\left\||u|^{p-1}u-|v|^{p-1}v\right\|_{L_\theta^{\frac{p+1}{p}}L^{\frac{6(p+1)}{3p+7}}}\\ &\leq C_4\|u-v\|_{L_\theta^{p+1}L^{\frac{6(p+1)}{3p-1}}}\left(\|u\|^{p-1}_{L_\theta^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}}+\|v\|^{p-1}_{L_\theta^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}}\right)\end{aligned}$$

Choosing $\theta>0$ small enough, so that

$$C_4\left(\|u\|^{p-1}_{L_\theta^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}}+\|v\|^{p-1}_{L_\theta^{p+1}H_D^{s_c,\frac{6(p+1)}{3p-1}}}\right)<1.$$

We deduce that $\|u-v\|_{L_\theta^{p+1}L^{\frac{6(p+1)}{3p-1}}}=0$, then $u=v$ in $[0,\theta]$. Iterating this argument, we obtain $u=v$ in $[0,T]$.

2.2 Properties of the ground state

We recall some well-known properties of the ground state and we refer the reader to [90], [60], [86, Appendix B] and [45] for more details.

Proposition 2.4 (Exponential decay of Q)**.** *Let Q be a solution of (IV.1) with $\omega=1$, then the following properties hold:*
1) *$Q\in H^{3,p}(\mathbb{R}^3)$ for every $2\leq p<+\infty$. In particular, $Q\in C^2$ and $|D^\beta Q(x)|\longrightarrow 0$, as $|x|\longrightarrow\infty$, for all $|\beta|\leq 2$.*
2) *there exists $\delta>0$ such that*

$$e^{\delta|x|}\left(|Q(x)|+|\nabla Q(x)|+\left|\nabla^2 Q(x)\right|\right)\in L^\infty(\mathbb{R}^3).$$

Proof. See [6] and [16, chapter 8] for the proof. □

We can deduce $Q_\omega(x)$ from $Q(x):Q_\omega(x)=\omega^{\frac{1}{p-1}}Q(\sqrt{\omega}x)$.

Then, there exits $C > 0$ and $\delta > 0$ such that

$$|Q_\omega(x)| + |\nabla Q_\omega(x)| + \left|\nabla^2 Q_\omega(x)\right| \leq Ce^{-\delta\sqrt{\omega}|x|}. \tag{II.23}$$

2.3 Spectral theory of the linearized operator

Consider a solution u of the nonlinear Schrödinger equations close to the soliton $e^{it}Q$. Let $h \in \mathbb{C}$ such that $h = h_1 + ih_2$.
We can write $u(t,x)$ as, $u(t,x) = e^{it}\left(Q(t,x) + h(t,x)\right)$. Note that h is the solution of the following equation,

$$\partial_t h + \mathcal{L}h = S(h), \qquad \mathcal{L} := \begin{pmatrix} O & -L^- \\ L^+ & 0 \end{pmatrix},$$

where $S(h)$ contains the nonlinear terms on h and the self-adjoint operators L^- and L^+ are defined by:

$$L^+ h_1 = -\Delta h_1 + h_1 - pQ^{p-1}h_1 \qquad \text{and} \qquad L^- h_2 = -\Delta h_2 + h_2 - Q^{p-1}h_2.$$

In all of the sequel, we assume $\frac{7}{3} < p < 5$. The spectral properties of the linearized operator $\mathcal{L}$ around the ground state are well-known and we refer to [91], [41] and [82] for the following Proposition.

Proposition 2.5. *Let $\sigma(\mathcal{L})$ be the spectrum of the operator $\mathcal{L}$ defined on $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and let $\sigma_{ess}(\mathcal{L})$ be its essential spectrum. Then*

$$\sigma_{ess}(\mathcal{L}) = \left\{i\xi : \ \xi \in \mathbb{R}, \ |\xi| \geq 1\right\}, \quad \sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, e_0\} \quad \textit{with } e_0 > 0.$$

Moreover, e_0 and $-e_0$ are simple eigenvalues of $\mathcal{L}$ with eigenfunctions $\mathcal{Y}^+$ and $\mathcal{Y}^-$,

$$\mathcal{L}\mathcal{Y}^\pm = \pm e_0 \mathcal{Y}^\pm,$$

and $\overline{\mathcal{Y}^+} = \mathcal{Y}^-$. Furthermore $\mathcal{Y}^+, \mathcal{Y}^- \in \mathcal{S}(\mathbb{R}^3)$, in fact, there exists $\delta > 0$ and $C > 0$ such that

$$\left|\mathcal{Y}^\pm\right| + \left|\nabla \mathcal{Y}^\pm\right| \leq Ce^{-\delta|x|}.$$

Remark 2.6. *The null-space of L^+ is spanned by $\partial_{x_1}Q$, $\partial_{x_2}Q$ and $\partial_{x_3}Q$ and the null-space of L^- is spanned by Q.*

Moreover, the operators L^+ and L^- satisfy the following coercivity property for the L^2-supercritical case.

Lemma 2.7 (Coercivity)**.** *There exists $C > 0$ such that for all $h = h_1 + ih_2 \in H^1(\mathbb{R}^3)$, we have*

$$\|h\|_{H^1}^2 \leq C\Big[\big(L^+h_1, h_1\big) + \big(L^-h_2, h_2\big) + \sum_{j=1}^{3} \Big(\int \partial_{x_j} Q\, h_1\Big)^2 + \Big(\int Qh_2\Big)^2 \tag{II.24}$$

$$+ \Big(\mathrm{Im} \int \mathcal{Y}^+ \bar{h}\Big)^2 + \Big(\mathrm{Im} \int \mathcal{Y}^- \bar{h}\Big)^2 \Big].$$

Proof. The proof of this result is well known and for the sake of completeness, we will give it in Appendix, Subsection 5.1. □

Remark 2.8. *The scalar products (L^+h_1, h_1) and (L^-h_2, h_2) must be understood in the sense of the quadratic form $\int |\nabla h_1|^2 + |\nabla h_2|^2 + |h|^2 - \int pQ^{p-1}h_1^2 - \int Q^{p-1}h_2^2$.*

Moreover, Lemma 2.7 is still valid with $h \in H_0^1(\Omega)$. Indeed, h can be extended to a $H^1(\mathbb{R}^3)$ function by letting $h(x) = 0$ for $x \in \Theta$.

Finally, we extend the Proposition 2.5 to the linearized operator $\mathcal{L}_\omega$ around the soliton $e^{it\omega}Q_\omega$, by a simple scaling argument.

Corollary 2.9 ([22])**.** *Let $\omega > 0$ and $h \in \mathbb{C}$ such that $h = h_1 + h_2$. The linearized operator $\mathcal{L}_\omega$ is defined by*

$$\mathcal{L}_\omega h = -L_\omega^-\, h_2 + i\, L_\omega^+\, h_1,$$

where,

$$L_\omega^+\, h_1 = -\Delta h_1 + \omega h_1 - pQ_\omega^{p-1} h_1 \qquad \text{and} \qquad L_\omega^-\, h_2 = -\Delta h_2 + \omega h_2 - Q_\omega^{p-1} h_2.$$

Moreover, the spectrum $\sigma(\mathcal{L}_\omega)$ of $\mathcal{L}$ satisfies

$$\sigma(\mathcal{L}_\omega) \cap \mathbb{R} = \{-e_\omega, 0, e_\omega\}, \quad \text{where} \;\; e_\omega = \omega^{\frac{3}{2}} e_0 > 0.$$

Furthermore, e_ω and $-e_\omega$ are simple eigenvalues of $\mathcal{L}_\omega$ with eigenfunctions $\mathcal{Y}_\omega^+$ and $\mathcal{Y}_\omega^-$

$$\mathcal{L}_\omega \mathcal{Y}_\omega^\pm = \pm e_\omega \mathcal{Y}_\omega^\pm,$$

where,

$$\mathcal{Y}_\omega^\pm(x) = \omega^{\frac{1}{4}} \mathcal{Y}^\pm(\sqrt{\omega} x) \quad \text{and} \quad \mathcal{Y}_\omega^+ = \overline{\mathcal{Y}_\omega^-}.$$

Remark 2.10. *The null-space of L_ω^+ is spanned by $\partial_{x_1}Q_\omega$, $\partial_{x_2}Q_\omega$ and $\partial_{x_3}Q_\omega$ and the null-space of L_ω^- is spanned by Q_ω.*

2.4 Compactness argument

Denote:

$$R(t,x) = Q_\omega(x - t\,v)\Psi(x)e^{i\varphi(t,x)}$$
$$Y_\pm(t,x) = \mathcal{Y}_\omega^\pm(x - tv)\Psi(x)e^{i\varphi(t,x)},$$

where, $\varphi(t,x) = \frac{1}{2}(x\cdot v) - \frac{1}{4}|v|^2 t + t\,\omega$.
Let $T_n \to \infty$, $n \in \mathbb{N}$, be an increasing sequence of times.

Proposition 2.11. *There exists $n_0 \geq 0$, $T_0 > 0$ and $C > 0$ (independent of n) such that the following holds. For each $n \geq n_0$ there exists $\lambda_n := (\lambda_n^\pm)_n \in \mathbb{R}^2$ such that*

$$|\lambda_n| \leq e^{-\delta\sqrt{\omega}|v|T_n},$$

and the solution u_n of

$$\begin{cases} i\partial_t u_n + \Delta u_n = -|u_n|^{p-1}u_n, \\ u_n(T_n) = R(T_n) + i\,\lambda_n^\pm Y_\pm(T_n), \end{cases} \tag{II.25}$$

is defined on the interval time $[T_0, T_n]$ and satisfies

$$\forall t \in [T_0, T_n] \qquad \|u_n(t) - R(t)\|_{H_0^1(\Omega)} \leq Ce^{-\delta\sqrt{\omega}|v|t}. \tag{II.26}$$

Proof. We will assume this proposition to prove Theorem 1.3, postponing the proof of it to Section 3. □

Now, we will start the proof of the Theorem 1.3 assuming the main Proposition 2.11. The proof is based on a compactness argument and the uniform estimate (II.26).
Renumbering the indices, we can take $n_0 = 0$ in Proposition 2.11.

Proof of Theorem 1.3 assuming Proposition 2.11. The proof proceeds in several steps.

- Step 1: Compactness argument. The Proposition 2.11 implies that there exists a sequence $u_n(t)$ of solution defined on $[T_0, T_n]$ such that

$$\forall n \in \mathbb{N},\ \forall t \in [T_0, T_n], \quad \|u_n(t) - R(t)\|_{H_0^1(\Omega)} \leq Ce^{-\delta\sqrt{\omega}|v|t}.$$

Lemma 2.12.

$$\lim_{M\to+\infty}\sup_{n\in\mathbb{N}}\int_{|x|\geq M} u_n^2(T_0,x)\,dx = 0.$$

Proof. The proof of the lemma is the same as in [69] for the construction of multi-soliton solutions of (NLS) for the subcritical case on $\mathbb{R}^d$. We give it for the sake of completeness.

Let $\varepsilon > 0$ and $T_\varepsilon \geq T_0$ such that: $C^2 e^{-2\delta\sqrt{\omega}|v|T_\varepsilon} < \varepsilon$, where C and δ are the same constants as in the Proposition 2.11.
For n large enough, so that $T_n \geq T_\varepsilon$ and due to (II.26), we have

$$\int_\Omega |u_n(T_\varepsilon) - R(T_\varepsilon)|^2\,dx \leq C^2 e^{-2\delta\sqrt{\omega}|v|T_\varepsilon} \leq \varepsilon.$$

Let $M(\varepsilon) > 0$ such that

$$\int_{|x|\geq M(\varepsilon)} |R(T_\varepsilon)|^2\,dx < \varepsilon,$$

by direct computation,

$$\int_{|x|\geq M(\varepsilon)} |u_n(T_\varepsilon)|^2\,dx \leq 4\varepsilon.$$

Now consider a C^1 cut-off function $f:\mathbb{R}\longrightarrow[0,1]$ such that

$$f\equiv 0 \quad \text{on }]-\infty,1]; \qquad 0 < f' < 2 \;\text{ on } (1,2); \qquad f\equiv 1 \quad \text{on } \;(2,+\infty).$$

For $K_\varepsilon > 0$ to be specified later, we can check that

$$\frac{d}{dt}\int_\Omega |u_n(t)|^2 f\left(\frac{|x|-M(\varepsilon)}{K_\varepsilon}\right)dx = \frac{-2}{K_\varepsilon}\operatorname{Im}\int_\Omega u_n(t)\left(\nabla\overline{u_n}.\frac{x}{|x|}\right)f'\left(\frac{|x|-M(\varepsilon)}{K_\varepsilon}\right)dx. \tag{II.27}$$

From Proposition 2.11, $\exists\,\alpha>0$, $\forall n$ and $\forall t\geq T_0$, $\|u_n(t)\|^2_{H^1_0}\leq\alpha$. Using (II.27) we get

$$\left|\frac{d}{dt}\int_\Omega |u_n(t)|^2 f\left(\frac{|x|-M(\varepsilon)}{K_\varepsilon}\right)\right| \leq \frac{4}{K_\varepsilon}\|u_n(t)\|^2_{H^1_0} \leq \frac{4}{K_\varepsilon}\alpha.$$

Now, we choose $K_\varepsilon > 0$ independently of n such that

$$K_\varepsilon \geq \left(\frac{T_\varepsilon - T_0}{\varepsilon}\right)4\,\alpha,$$

which yields

$$\left|\frac{d}{dt}\int_\Omega |u_n(t)|^2 f\left(\frac{|x|-M(\varepsilon)}{K_\varepsilon}\right)\right| \leq \frac{\varepsilon}{T_\varepsilon - T_0}.$$

Integrating on the time interval $[T_0, T_\varepsilon]$, we get

$$\begin{aligned}
&\int_\Omega |u_n(T_0)|^2 f\left(\frac{|x|-M(\varepsilon)}{K_\varepsilon}\right) dx - \int_\Omega |u_n(T_\varepsilon)|^2 f\left(\frac{|x|-M(\varepsilon)}{K_\varepsilon}\right) dx \\
&\leq \int_{T_0}^{T_\varepsilon} \left| \frac{d}{dt} \int |u_n(t)|^2 f\left(\frac{|x|-M(\varepsilon)}{K_\varepsilon}\right) dx \right| dt \\
&\leq \varepsilon.
\end{aligned}$$

Hence, $$\int_\Omega |u_n(T_0)|^2 f\left(\frac{|x|-M(\varepsilon)}{K_\varepsilon}\right) dx \leq \varepsilon + \int_\Omega |u_n(T_\varepsilon)|^2 f\left(\frac{|x|-M(\varepsilon)}{K_\varepsilon}\right) dx.$$

Due to the properties of f, we have

$$\begin{aligned}
\int_{|x|>2K_\varepsilon+M(\varepsilon)} |u_n(T_0)|^2 dx &\leq \int_\Omega |u_n(T_0)|^2 f\left(\frac{|x|-M(\varepsilon)}{K_\varepsilon}\right) dx \\
&\leq \varepsilon + \int_\Omega |u_n(T_\varepsilon)|^2 f\left(\frac{|x|-M(\varepsilon)}{K_\varepsilon}\right) dx \\
&\leq \varepsilon + \int_{|x|\geq M(\varepsilon)} |u_n(T_\varepsilon)|^2 dx \\
&\leq \varepsilon + 4\varepsilon = 5\varepsilon.
\end{aligned}$$

This concludes the proof of the lemma. □

By the main proposition, we have

$$\|u_n(T_0)\|_{H_0^1(\Omega)} \leq \alpha.$$

Since H_0^1 is a Hilbert space, there exists a subsequence of $(u_n(t))_n$ that we still denote by $(u_n(t))_n$ to simplify notation and $\mathcal{U}_0 \in H_0^1(\Omega)$ such that

$$u_n(T_0) \rightharpoonup \mathcal{U}_0 \quad \text{in} \quad H_0^1(\Omega), \quad \text{as} \quad n \longrightarrow +\infty.$$

By the compactness of the embedding of $H_0^1(\{|x| \leq A\})$ into $L^2(\{|x| \leq A\})$, we have

$$u_n(T_0) \longrightarrow \mathcal{U}_0 \quad \text{in} \quad L^2_{loc}(\Omega)\,.$$

By Lemma 2.12, we get $u_n(T_0) \longrightarrow \mathcal{U}_0 \quad \text{in} \quad L^2(\Omega)$.

Now using the following interpolation inequality

$$\forall s \in (0,1), \quad \|u_n(t) - \mathcal{U}_0\|_{H^s_D(\Omega)} \leq \|u_n(t) - \mathcal{U}_0\|^{1-s}_{L^2(\Omega)} \, \|u_n(t) - \mathcal{U}_0\|^{s}_{H^1_0(\Omega)},$$

we obtain,

$$u_n(T_0) \longrightarrow \mathcal{U}_0 \quad \text{in} \quad H^s_D(\Omega), \ \forall s \in (0,1). \tag{II.28}$$

- Step 2: Construction of the solution. Due to Theorem 1.2, the equation (NLS_Ω) is well-posed in $H^{s_c}_D(\Omega)$, for $0 < s_c < 1$.
  Let $\widetilde{u}$ be the maximal solution of

$$\begin{cases} i\partial_t \widetilde{u} + \Delta \widetilde{u} = -|\widetilde{u}|^{p-1}\widetilde{u} & \forall (t,x) \in [T_0, \widetilde{T}) \times \Omega, \\ \widetilde{u}(T_0, x) = \mathcal{U}_0 & \forall x \in \Omega, \\ \widetilde{u}(t,x) = 0 & \forall (t,x) \in [T_0, \widetilde{T}) \times \partial\Omega. \end{cases} \tag{II.29}$$

By (II.28) we have

$$u_n(T_0) \longrightarrow \mathcal{U}_0 = \widetilde{u}(T_0, x) \quad \text{in} \quad H^{s_c}_D(\Omega), \quad \text{for} \ \ 0 < s_c < 1. \tag{II.30}$$

For n large enough, $u_n(t)$ is defined for all $t \in [T_0, \widetilde{T})$ and by the continuity of the flow we have

$$u_n(t) \longrightarrow \widetilde{u}(t) \quad \text{in} \quad H^{s_c}_D(\Omega), \quad \text{for} \ \ 0 < s_c < 1.$$

Due to the main Proposition 2.11, we know that for n large enough $u_n(t)$ is uniformly bounded in $H^1_0(\Omega)$. Then necessarily,

$$\forall t \in [T_0, \widetilde{T}), \quad u_n(t) \rightharpoonup \widetilde{u}(t) \quad \text{in} \quad H^1_0(\Omega).$$

Using the property of weak convergence and by the main proposition, it follows that

$$\forall t \in [T_0, \widetilde{T}), \quad \|\widetilde{u}(t) - R(t)\|_{H^1_0(\Omega)} \leq \liminf \|u_n(t) - R(t)\|_{H^1_0(\Omega)} \leq C e^{-\delta\sqrt{\omega}|v|t}.$$

In particular, we deduce that, $\widetilde{u}$ is bounded in $H^1_0(\Omega)$. Due to the blow-up alternative, we get $\widetilde{T} = +\infty$. Finally, we have $\widetilde{u} \in C([T_0, +\infty), H^1_0(\Omega))$ and by (II.26) in Proposition 2.11,

$$\forall t \in [T_0, +\infty), \quad \|\widetilde{u}(t) - R(t)\|_{H^1_0(\Omega)} \leq e^{-\delta\sqrt{\omega}|v|t},$$

which concludes the proof of Theorem 1.3.

□

3 Proof of the uniform estimate

3.1 Bootstrap and topological arguments

In this section, we prove the main Proposition 2.11. We use some modulation in the phase and translation parameters in the decomposition of the solution to obtain the orthogonality conditions. Next, we use a bootstrap argument to control these parameters and some scalar product that are related to the size of the soliton. Finally, to conclude the proof we use a topological argument to control the unstable direction.

Remark 3.1. *In this section, to simplify notations we will write r instead of r_ω and we will drop the index n for most variables. Hence, we will write u for u_n, $\lambda^\pm$ for $\lambda_n^\pm$, etc. Only the sequence of times will be written with the index n. As Proposition 2.11 is proved for given n, this should not be a source of confusion. We possibly drop the first terms of the sequence T_n, so that, for all n, T_n is large enough for our purposes.*

3.1.1 Modulated final data

Lemma 3.2 (modulation for time independent function)**.** *There exists $C, \epsilon > 0$ such that the following holds.*
Given $\alpha \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$. If $u(x) \in L^2$ is such that

$$\|u - R\|_{L^2} \leq \epsilon.$$

Then there exists modulation parameters $y = (y_i)_i \in \mathbb{R}^3$ and $\mu \in \mathbb{R}$, such that setting

$$r(x) = u(x) - \widetilde{R}(x),$$

the following holds

$$\|r\|_{L^2} + |y| + |\mu| \leq C \|u - R\|_{L^2},$$

and

$$\mathrm{Re} \int r(x) \partial_{x_j} \widetilde{Q}_\omega(x) \Psi(x) e^{-i\left(\frac{1}{2}(x\cdot v)+\theta\right)} e^{-i\mu} dx = \mathrm{Im} \int r(x) \overline{\widetilde{R}}(x) dx = 0,\ j = 1, 2, 3,$$

where,

$$\begin{aligned} R(x) &= Q_\omega(x-\alpha)\Psi(x)e^{i\,(\frac{1}{2}(x\cdot v)+\theta)}, \\ \widetilde{Q}_\omega(x) &= Q_\omega(x-\alpha-y), \\ \widetilde{R}(x) &= \widetilde{Q}_\omega(x)\Psi(x)e^{i\,(\frac{1}{2}(x\cdot v)+\theta)}e^{i\mu}. \end{aligned}$$

Furthermore, $u \longmapsto (r,y,\mu)$ is a smooth C^1-diffeomorphism.

Proof. see Appendix, Subsection 5.2.

□

Note that the previous lemma applies to time independent functions. A consequence of this modulation in the decomposition of fixed u is the the following result on a solution $u(t)$ of (II.25).

Corollary 3.3. *There exists $C, \epsilon > 0$ such that the following holds for all $t \in [T, T_n]$, for $T > T_0$, if $u(t,\cdot) \in L^2_x$ satisfies*

$$\|u(t) - R(t)\|_{L^2} \leq \epsilon.$$

Then there exits a C^1-functions $y : [T, T_n] \longrightarrow \mathbb{R}^3$ and $\mu : [T, T_n] \longrightarrow \mathbb{R}$ such that if we set

$$r(t,x) = u(t,x) - \widetilde{R}(t,x),$$

the following holds

$$\|r(t)\|_{L^2} + |y(t)| + |\mu(t)| \leq C\,\|u(t) - R(t)\|_{L^2}\,,$$

and

$$\mathrm{Re}\int r(t,x)\partial_{x_j}\widetilde{Q}_\omega(t,x)\Psi(x)e^{-i\,(\frac{1}{2}(x\cdot v)+\theta(t))}e^{-i\mu(t)}dx = 0 \;\; j = 1,2,3, \tag{II.31}$$

$$\mathrm{Im}\int r(t,x)\overline{\widetilde{R}}(t,x)dx = 0, \tag{II.32}$$

where,

$$\begin{aligned} &R(t,x) = Q_\omega(x-\alpha(t))\,\Psi(x)e^{i\,(\frac{1}{2}(x\cdot v)+\theta(t))}\,,\;\; \textit{with } \alpha(t) := t\,v \;\textit{ and }\; \theta(t) := -\frac{1}{4}|v|^2 t + t\,\omega\,. \\ &\widetilde{Q}_\omega(t,x) = Q_\omega(x-\alpha(t)-y(t))\,. \\ &\widetilde{R}(t,x) = \widetilde{Q}_\omega(t,x)\Psi(x)e^{i\,(\frac{1}{2}(x\cdot v)+\theta(t))}e^{i\mu(t)}\,. \end{aligned}$$

Proof. For small λ, the solution $u(t)$ is close to the soliton $R(t)$ for t close to T_n. Assume that $u(t)$ satisfies (3.3) on $[T, T_n]$. Applying Lemma 3.2 to u(t) for any $t \in [T, T_n]$ and since the map

$t \longmapsto u(t)$ is continuous in H_0^1, we obtain the existence of continuous functions $y : [T, T_n] \longrightarrow \mathbb{R}^3$ and $\mu : [T, T_n] \longrightarrow \mathbb{R}$ such that (II.31) and (II.32) hold. □

Notation: $u(t)$ is defined and modulable around $R(t)$ for t close to T_n, in the sense of the previous Corollary.

$$R(t,x) = Q_\omega(x - tv)\Psi(x)e^{i\varphi(t,x)}\,, \quad \text{where} \quad \varphi(t,x) = \frac{1}{2}x\cdot v - \frac{1}{4}|v|^2 t + t\,\omega\,.$$
$$\widetilde{Q}_\omega(t,x) = Q_\omega(x - tv - y(t))\,.$$
$$\widetilde{R}(t,x) = \widetilde{Q}_\omega(t,x)\Psi(x)e^{i\widetilde{\varphi}(t,x)}\,, \quad \text{where} \quad \widetilde{\varphi}(t,x) = \frac{1}{2}x\cdot v - \frac{1}{4}|v|^2 t + t\,\omega + \mu(t).$$
$$\widetilde{\mathcal{Y}}_\omega^{\mp}(t,x) = \mathcal{Y}_\omega^{\mp}(x - tv - y(t)).$$
$$\widetilde{Y}_{\mp}(t,x) = \widetilde{\mathcal{Y}}_\omega^{\mp}(t,x)\Psi(x)e^{i\widetilde{\varphi}(t,x)} \quad \text{and} \quad \alpha^{\pm}(t) = \operatorname{Im}\int \widetilde{Y}_{\mp}(t,x)\overline{r}(t,x)dx.$$
$$\widetilde{L}_\omega^{+}h_1 = -\Delta h_1 + \omega h_1 - p\widetilde{Q}_\omega^{p-1}h_1 \quad \text{and} \quad \widetilde{L}_\omega^{-}h_2 = -\Delta h_2 + \omega h_2 - \widetilde{Q}_\omega^{p-1}h_2.$$

Lemma 3.4 (Modulated final data)**.** *There exists $C > 0$ (independent of n) such that for all $\alpha^+ \in B_{\mathbb{R}}(e^{-\delta\sqrt{\omega}|v|T_n})$ there exists a unique λ such that*

$$|\lambda| \le C\left|\alpha^+\right|,$$

and the modulation parameters $(r(T_n), y(T_n), \mu(T_n))$ of $u(T_n)$ satisfy

$$\begin{cases} \alpha^+(T_n) &= \alpha^+, \\ \alpha^-(T_n) &= 0. \end{cases} \tag{II.33}$$

Proof. *See Appendix, Subsection 5.3.* □

Let T_0 to be specified later, independent of n. Let α^+ to be chosen, λ be given by Lemma 3.4 and let u be the corresponding solution of (II.25). We now define the maximal time interval $[T(\alpha^+), T_n]$, on which suitable exponential estimates hold.

Definition 3.5. *Let $T(\alpha^+)$ be the infimum of $T \ge T_0$ such that the following properties hold for all $t \in [T, T_n]$:*
Closeness to $R(t)$:

$$\|u(t) - R(t)\|_{H_0^1} \le \varepsilon\,.$$

In particular, this ensures that $u(t)$ is modulable around $R(t)$ in the sense of Lemma 3.2.

Estimates on the modulation parameters: There exists $M>0$ and $M'>0$ to be specified later,

$$\|r(t)\|_{H^1_0} \le Me^{-\delta\sqrt{\omega}|v|t} \tag{II.34}$$

$$|y(t)| \le M'e^{-\delta\sqrt{\omega}|v|t} \tag{II.35}$$

$$|\mu(t)| \le M'e^{-\delta\sqrt{\omega}|v|t} \tag{II.36}$$

$$\left|\alpha^{\pm}(t)\right| \le e^{-\delta\sqrt{\omega}|v|t}. \tag{II.37}$$

Note that, if for all n we can find α^+ such that $T(\alpha^+) = T_0$ then the Proposition 2.11 is proved. It remains to prove the existence of such value of α^+.

Denote $h(t,x) = e^{-i\widetilde{\varphi}(t,x)}r(t,x)$. Recall that,

$$\begin{aligned} u(t,x) &= \widetilde{R}(t,x) + r(t,x) \\ &= e^{i\widetilde{\varphi}(t,x)}(\widetilde{Q}_\omega(t,x)\Psi(x) + h(t,x)). \end{aligned}$$

Lemma 3.6. *Let $t \in [T(\alpha^+), T_n]$ and let $C, \delta > 0$. We have*

$$\begin{aligned} &i\partial_t h + \Delta h - \omega h + (\frac{p+1}{2})\widetilde{Q}_\omega^{p-1}\Psi^{p-1}h + (\frac{p-1}{2})\widetilde{Q}_\omega^{p-1}\Psi^{p-1}\overline{h} + i\,v.\nabla h - \frac{d\mu(t)}{dt}h \\ &+\widetilde{Q}_\omega^{p}\Psi(\Psi^{p-1}-1) + 2\nabla\widetilde{Q}_\omega\nabla\Psi + \widetilde{Q}_\omega\Delta\Psi + i\,v\,\widetilde{Q}_\omega\nabla\Psi - i\,\frac{dy(t)}{dt}\nabla\widetilde{Q}_\omega\Psi - \frac{d\mu(t)}{dt}\widetilde{Q}_\omega\Psi + \beta(t,x) = 0, \end{aligned} \tag{II.38}$$

where $\beta(t,x)$ is a remainder terms on h.

$$\left|\frac{d\mu(t)}{dt}\right| + \left|\frac{dy(t)}{dt}\right| \le C\,\|h(t)\|^2_{H^1_0} + Ce^{-2\delta\sqrt{\omega}|v|t}. \tag{II.39}$$

$$\left|\frac{d\alpha^{\pm}(t)}{dt} \pm e_\omega\alpha^{\pm}(t)\right| \le C\,\|h(t)\|^3_{H^1_0} + Ce^{-2\delta\sqrt{\omega}|v|t}. \tag{II.40}$$

Proof. For the equation (II.38) of h it suffices to plug the above expression of $u(t,x)$ on the nonlinear Schrödinger equation : $i\partial_t u + \Delta u = -|u|^{p-1}u$. Using, the elliptic equation (IV.1) of Q_ω and the Taylor expansion for the nonlinear term, we get (II.38), with $\|\beta(t)\|_{L^2} \le C\,\|h(t)\|^2_{H^1_0}$. For the proof of (II.39) and (II.40), we claim the following estimates.

Claim 3.7.

$$\operatorname{Im}\int \partial_t\overline{h}(t,x)\widetilde{Q}_\omega(t,x)\Psi(x)dx = \sum_{k=1}^{3}\operatorname{Im}\int \overline{h}(t,x)(v_k + \frac{dy_k}{dt}(t))\,\partial_{x_k}\widetilde{Q}_\omega(t,x)\Psi(x)dx.$$

$$\mathrm{Re}\int \partial_t \overline{h}(t,x)\partial_{x_j}\widetilde{Q}_\omega(t,x)\,\Psi(x)dx = \sum_{k=1}^{3}\mathrm{Re}\int \overline{h}(t,x)(v_k+\frac{dy_k}{dt}(t))\partial_{x_k}\partial_{x_j}\widetilde{Q}_\omega(t,x)\Psi(x)dx\,, j=1,2,3.$$

Proof. It is just a consequence of the orthogonality conditions in Lemma 3.2. So, we have

$$\mathrm{Re}\int h(t,x)\partial_{x_j}\widetilde{Q}_\omega(t,x)\,\Psi(x)dx = \mathrm{Im}\int h(t,x)\widetilde{Q}_\omega(t,x)\Psi(x)dx = 0,\ \ j=1,2,3.$$

Differentiating each equality with respect to the time variable t, the Claim 3.7 follows. □

Now let us estimate $\frac{dy}{dt}(t)$ and $\frac{d\mu(t)}{dt}$ in (II.39). Multiply by $\partial_{x_j}\widetilde{Q}_\omega\Psi$ and take the imaginary part of the equation (II.38). Using the Claim 3.7 and the fact that Q_ω is radial, so that

$$\begin{cases} Q_\omega(x_1,x_2,x_3) = Q_\omega(-x_1,x_2,x_3), \\ \partial_{x_1}Q_\omega(x_1,x_2,x_3) = -\partial_{x_1}Q_\omega(-x_1,x_2,x_3). \end{cases} \tag{II.41}$$

which yields

$$\int \partial_{x_1}Q_\omega(x_1,x_2,x_3)Q_\omega(x_1,x_2,x_3)\,dx = -\int \partial_{x_1}Q_\omega(x_1,x_2,x_3)Q_\omega(x_1,x_2,x_3)\,dx.$$

Hence

$$\int \partial_{x_j}Q_\omega(t,x)\;Q_\omega(t,x)\,dx = 0, \qquad \text{for } j=1,2,3.$$

We obtain the following equality on $\frac{dy(t)}{dt}$.

$$\begin{aligned}\frac{dy_j(t)}{dt}\|\partial_{x_j}\widetilde{Q}_\omega\Psi\|_{L^2}^2 = &\underbrace{\int h_1(t,x)\,\frac{dy(t)}{dt}.\nabla(\partial_{x_j}\widetilde{Q}_\omega(t,x))\Psi(x)dx}_{\mathrm{I_h^y}} - \underbrace{\frac{d\mu(t)}{dt}\int h_2(t,x)\partial_{x_j}\widetilde{Q}_\omega(t,x)\Psi(x)dx}_{\mathrm{I_h^\mu}} \\ &\underbrace{-\int \widetilde{L}_\omega^- h_2(t,x)\partial_{x_j}\widetilde{Q}_\omega(t,x)\Psi(x)dx + \int h_2(t,x)\widetilde{Q}_\omega^{p-1}(t,x)(\Psi^{p-1}(x)-1)dx}_{\mathrm{I_h^1}} \\ &\underbrace{+\int h_1(t,x)\,\partial_{x_j}\widetilde{Q}_\omega(t,x)\,v.\nabla\Psi(x)dx}_{\mathrm{I_h^2}} + O(\|h(t)\|_{H_0^1}^2).\end{aligned}$$

Taking the scalar product with $\widetilde{Q}_\omega(x)\Psi$ and the equation (II.38) on h. Using the same argument as above, we get the following equality on $\frac{d\mu(t)}{dt}$.

$$\frac{d\mu(t)}{dt}\|\widetilde{Q}_\omega\Psi\|_{L^2}^2 = \underbrace{\int h_2(t,x)\frac{dy(t)}{dt}.\nabla\widetilde{Q}_\omega(t,x)\Psi(x)dx}_{\mathrm{J_h^y}} - \underbrace{\int\frac{d\mu(t)}{dt}h_1(t,x)\widetilde{Q}_\omega(t,x)\Psi(x)dx}_{\mathrm{J_h^\mu}}$$
$$- \underbrace{\int \widetilde{L}_\omega^+ h_1(t,x)\widetilde{Q}_\omega(t,x)\Psi(x)dx + \int p\widetilde{Q}_\omega^{p-1}(t,x)h_1(t,x)(\Psi^{p-1}(x)-1)dx}_{\mathrm{J_h^1}}$$
$$- \underbrace{\int h_2(t,x)\widetilde{Q}_\omega(t,x)v.\nabla\Psi(x)dx}_{\mathrm{J_h^2}} + \underbrace{\int \widetilde{Q}_\omega^{p+1}(t,x)\Psi^2(x)(\Psi^{p-1}(x)-1)}_{\mathrm{J_1}}$$
$$+ \underbrace{\int \widetilde{Q}_\omega^2(t,x)\Delta\Psi(x)\Psi(x)dx}_{\mathrm{J_2}} + O\left(\|h(t)\|_{H_0^1}^2\right).$$

Summing the absolute values of the two equalities above and using the fact that

$$\|\widetilde{Q}_\omega\Psi\|_{L^2}^2 = \|Q_\omega\|_{L^2}^2 + O(e^{-2\delta\sqrt{\omega}|v|t}) \quad \text{and} \quad \|\nabla\widetilde{Q}_\omega\Psi\|_{L^2}^2 = \|\nabla Q_\omega\|_{L^2}^2 + O(e^{-2\delta\sqrt{\omega}|v|t}),$$

We obtain the left hand side on the estimate (II.39) Next, we have to estimate the right hand side in both equalities.

$$\begin{aligned}|\mathrm{I_h^y}| := \left|\int h_1(t,x)\frac{dy(t)}{dt}.\nabla(\partial_{x_j}\widetilde{Q}_\omega(x))\Psi(x)dx\right| &\leq C\left|\frac{dy(t)}{dt}\right|\|h(t)\|_{L^2}\\ &\leq C_1\left|\frac{dy(t)}{dt}\right| Me^{-\delta\sqrt{\omega}|v|T_0}\\ &\leq \frac{1}{10}\left|\frac{dy(t)}{dt}\right|\left\|\partial_{x_j}Q_\omega\right\|_{L^2}^2.\end{aligned}$$

Provided

$$Me^{-\delta\sqrt{\omega}|v|T_0} \leq \frac{1}{10\,C_1}\left\|\partial_{x_j}Q_\omega\right\|_{L^2}^2,\ j=1,2,3. \tag{II.42}$$

$$\begin{aligned}|\mathrm{I_h^\mu}| := \left|\frac{d\mu(t)}{dt}\int h_2(t,x)\partial_{x_j}\widetilde{Q}_\omega(x)\Psi(x)dx\right| &\leq C\left|\frac{d\mu(t)}{dt}\right|\|h(t)\|_{L^2}\\ &\leq C_2\left|\frac{d\mu(t)}{dt}\right| Me^{-\delta\sqrt{\omega}|v|T_0}\\ &\leq \frac{1}{10}\left|\frac{d\mu(t)}{dt}\right|\|Q_\omega\|_{L^2}^2,\end{aligned}$$

if the following condition is satisfied,

$$Me^{-\delta\sqrt{\omega}|v|T_0} \leq \frac{1}{10C_2} \|Q_\omega\|_{L^2}^2 . \tag{II.43}$$

$$\begin{aligned} |\mathrm{J}_\mathrm{h}^\mathrm{y}| := \left| \int h_2(t,x) \frac{dy(t)}{dt} . \nabla \widetilde{Q}_\omega(x) \Psi(x) dx \right| &\leq C \left| \frac{dy(t)}{dt} \right| \|h(t)\|_{L^2} \\ &\leq C \left| \frac{dy(t)}{dt} \right| Me^{-\delta\sqrt{\omega}|v|T_0} \\ &\leq \frac{1}{10} \left| \frac{dy(t)}{dt} \right| \left\| \partial_{x_j} Q_\omega \right\|_{L^2}^2 , \end{aligned}$$

if the condition (II.42) holds.

$$\begin{aligned} |\mathrm{J}_\mathrm{h}^\mu| := \left| \frac{d\mu(t)}{dt} \int h_1(t,x) \widetilde{Q}_\omega(x) \Psi(x) dx \right| \leq C \left| \frac{d\mu(t)}{dt} \right| \|h(t)\|_{L^2} \leq C \left| \frac{d\mu(t)}{dt} \right| Me^{-\delta\sqrt{\omega}|v|T_0} \\ \leq \frac{1}{10} \left| \frac{d\mu(t)}{dt} \right| \|Q_\omega\|_{L^2}^2 , \end{aligned}$$

if the condition (II.43) is verified.
We next treat the terms $\mathrm{I}_\mathrm{h} := \mathrm{I}_\mathrm{h}^1 + \mathrm{I}_\mathrm{h}^2$ and $\mathrm{J}_\mathrm{h} := \mathrm{J}_\mathrm{h}^1 + \mathrm{J}_\mathrm{h}^2$ that depend on h. We will estimate the main integral for both terms, where appear the self-adjoint operator $\widetilde{L}_\omega^+$ and $\widetilde{L}_\omega^-$.

$$\begin{aligned} \left| \int \widetilde{L}_\omega^- h_2(t,x) \partial_{x_j} \widetilde{Q}_\omega(x) \Psi(x) dx \right| = \left| \int h_2(t,x) \widetilde{L}_\omega^- \left(\partial_{x_j} \widetilde{Q}_\omega(x) \Psi(x) \right) dx \right| \\ \leq C \|h(t)\|_{H_0^1} . \end{aligned}$$

Similarly, we can estimate the integral on $\widetilde{L}_\omega^+$. We obtain

$$|\mathrm{I}_\mathrm{h}| + |\mathrm{J}_\mathrm{h}| \leq C \|h\|_{L^2} .$$

Finally, we have to estimate J_1 and J_2. Using the exponential decay of Q and the fact that $\Delta\Psi$ and $(\Psi^{p-1} - 1)$ have a compact support, we get

$$\begin{aligned} |\mathrm{J}_1 + \mathrm{J}_2| := \left| \int \widetilde{Q}_\omega^{p+1}(x) \Psi(x)^2 (\Psi^{p-1} - 1) + \int \widetilde{Q}_\omega^2(x) \Delta\Psi\Psi dx \right| \\ \leq Ce^{-2\delta\sqrt{\omega}|v|t}. \end{aligned}$$

We have proved the estimate (II.39), if conditions (II.42) and (II.43) on M hold. For T_0 large enough,

$$Me^{-\delta\sqrt{\omega}|v|T_0} \leq \frac{1}{10C'} \min\left(\left\| \partial_{x_j} Q_\omega \right\|_{L^2}^2, \|Q_\omega\|_{L^2}^2 \right), \tag{II.44}$$

where $C' = \max(C_1, C_2)$.

Next, we have to prove the last estimate (II.40). Let us recall that

$$\alpha^{\pm}(t) = \operatorname{Im} \int \overline{r}(t,x) \widetilde{Y}_{\mp}(t,x) dx = \operatorname{Im} \int \overline{h}(t,x) \widetilde{\mathcal{Y}}_\omega^{\mp}(t,x) \Psi(x) dx.$$

$$\begin{aligned} \frac{d}{dt}\alpha^{\pm}(t) = &- \underbrace{\operatorname{Im} \int \overline{h}(t,x) \frac{dy(t)}{dt} . \nabla \widetilde{\mathcal{Y}}_\omega^{\mp}(t,x) \Psi(x) dx}_{\mathrm{I}_1} - \underbrace{\operatorname{Im} \int \overline{h}(t,x)\, v . \nabla \widetilde{\mathcal{Y}}_\omega^{\mp}(t,x) \Psi(x) dx}_{\mathrm{I}_2} \\ &+ \underbrace{\operatorname{Im} \int \partial_t \overline{h}(t,x) \widetilde{\mathcal{Y}}_\omega^{\mp}(t,x) \Psi(x) dx}_{\mathrm{I}_3}. \end{aligned}$$

Due to (II.39) and the exponential decay properties of the eigenfunctions of the linearized operator. We get

$$|\mathrm{I}_1| = \left| \operatorname{Im} \int \overline{h}(t,x) \frac{dy(t)}{dt} . \nabla \widetilde{\mathcal{Y}}_\omega^{\mp}(t,x) \Psi(x) dx \right| \leq C \left| \frac{dy(t)}{dt} \right| \|h\|_{L^2} \leq C \|h(t)\|_{H_0^1}^3 + + Ce^{-2\delta\sqrt{\omega}|v|t}.$$

One can check that the second integral I_2 will be simplified with a term from I_3.

Now, let us estimate I_3. For this we have to use the equation (II.38) of h. One can see that the main terms is the following

$$\partial_t \overline{h} = -i \Delta \overline{h} + i \omega \overline{h} - i (\frac{p+1}{2}) \widetilde{Q}_\omega^{p-1} \Psi^{p-1} \overline{h} - i (\frac{p-1}{2}) \widetilde{Q}_\omega^{p-1} \Psi^{p-1} h + f$$

Where f contains all others terms of the equation (II.38). Let $h = h_1 + ih_2$,

$$\begin{aligned} -i \Delta \overline{h} + i \omega \overline{h} - i (\frac{p+1}{2}) \widetilde{Q}_\omega^{p-1} \Psi^{p-1} \overline{h} - i (\frac{p-1}{2}) \widetilde{Q}_\omega^{p-1} \Psi^{p-1} h &= i \widetilde{L}_\omega^{+} h_1 + \widetilde{L}_\omega^{-} h_2 + \widetilde{Q}_\omega^{p-1} h_2 (1 - \Psi^{p-1}) \\ &+ i\, p \widetilde{Q}_\omega^{p-1} h_1 (1 - \Psi^{p-1}). \end{aligned}$$

Multiplying (II.38) by $\widetilde{\mathcal{Y}}_\omega^{\mp}(t,x)\Psi(x)$ and take the imaginary part, we obtain I_3 on the left hand side. The terms containing the linearized operator will be treated later. To estimate the other terms, we use the fact that Q_ω and $\mathcal{Y}_\omega^{\mp}$ are radial, exponentially decaying at infinity and the compact support of $\nabla\Psi$ and $(1 - \Psi^{p-1})$. Also, we have to use the estimate (II.39) to obtain the right hand side of the estimate (II.40).

To complete the proof we have to compute the terms of the linearized operator.
Let $y_1^{\mp}(t,x) = \mathrm{Re}\left(\widetilde{\mathcal{Y}}_\omega^{\mp}(t,x)\right)$ and $y_2^{\mp}(t,x) = \mathrm{Im}\left(\widetilde{\mathcal{Y}}_\omega^{\mp}(t,x)\right)$. Thus,

$$\begin{cases} \widetilde{L}_\omega^{+} y_1^{\mp} = \mp e_\omega y_2^{\mp}, \\ \widetilde{L}_\omega^{-} y_2^{\mp} = \pm e_\omega y_1^{\mp}. \end{cases} \tag{II.45}$$

Recall that $\widetilde{L}^{\pm}$ are self-adjoint operator.

$$\begin{aligned} \mathrm{Im}\int (i\,\widetilde{L}_\omega^{+} h_1 + \widetilde{L}_\omega^{-} h_2)(y_1^{\mp} + i y_2^{\mp})\Psi dx &= \mathrm{Im}\int i\,(\widetilde{L}_\omega^{+} h_1) y_1^{\mp}\Psi + i\,(\widetilde{L}_\omega^{-} h_2) y_2^{\mp}\Psi dx \\ &= \mathrm{Im}\int i\,h_1(\widetilde{L}_\omega^{+} y_1^{\mp}\Psi) + i\,h_2(\widetilde{L}_\omega^{-} y_2^{\mp}\Psi) dx \\ &= \mathrm{Im}\int i\,h_1(\mp e_\omega y_2^{\mp}\Psi) + i\,h_2(\pm e_\omega y_1^{\mp}\Psi) dx + O(e^{-2\delta\sqrt{\omega}|v|t}) \\ &= \mp e_\omega\,\mathrm{Im}\int \overline{h}\,\widetilde{\mathcal{Y}}_\omega^{\mp}\,\Psi dx + O(e^{-2\delta\sqrt{\omega}|v|t}) \\ &= \mp e_\omega \alpha^{\pm}(t,x) + O(e^{-2\delta\sqrt{\omega}|v|t}). \end{aligned}$$

This concludes the proof of the Lemma 3.6 □

3.1.2 Control of the modulation parameters

We claim the following estimates of $v(t), \mu$ and y on $[T(\alpha^+), T_n]$.

Lemma 3.8 (Control of v, y and μ.)**.** *For T_0 large enough independent of n and $\forall\,\alpha^+$ such that*

$$\left|\alpha^+\right| \le e^{-\delta\sqrt{\omega}|v|T_n}.$$

the following holds

$$\forall t \in [T(\alpha^+), T_n], \qquad \|u(t) - R(t)\|_{H_0^1} \le C e^{-\delta\sqrt{\omega}|v|t} \le \frac{\epsilon}{2} \tag{II.46}$$

$$\|r(t)\|_{H_0^1} \le \frac{M}{2} e^{-\delta\sqrt{\omega}|v|t} \tag{II.47}$$

$$|\mu(t)| + |y(t)| \le \frac{M'}{2} e^{-\delta\sqrt{\omega}|v|t}. \tag{II.48}$$

We postpone the proof of Lemma 3.8 to the end of this section.

3.1.3 Control of the stable direction

Lemma 3.9. *For T_0 large enough, independent of n and $\forall\, \alpha^+$ such that $|\alpha^+| \leq e^{-\delta\sqrt{\omega}|v|T_n}$. The following holds*

$$\forall t \in [T(\alpha^+), T_n], \qquad \left|\alpha^-(t)\right| \leq \frac{1}{2}\, e^{-\delta\sqrt{\omega}|v|t}.$$

Proof.

$$\frac{d}{dt}\left(\alpha^-(t)e^{-e_\omega t}\right) = \left(\frac{d}{dt}\alpha^-(t) - e_\omega \alpha^-(t)\right)e^{-e_\omega t}.$$

Due to (II.40) and (II.47), we have

$$\left|\frac{d}{dt}\left(\alpha^-(t)e^{-e_\omega t}\right)\right| \leq \left(C\frac{M^3}{8}e^{-2\delta\sqrt{\omega}|v|t} + Ce^{-\delta\sqrt{\omega}|v|t}\right)e^{-\delta\sqrt{\omega}|v|t}e^{-e_\omega t}.$$

Then, we obtain by integration on $[t, T_n]$ and using that $\alpha^-(T_n) = 0$,

$$\left|\alpha^-(t)\right| \leq \left(C_3\frac{M^3}{8}e^{-2\delta\sqrt{\omega}|v|t} + C_4e^{-\delta\sqrt{\omega}|v|t}\right)e^{-\delta\sqrt{\omega}|v|t}.$$

Hence,

$$\forall t \in [T(\alpha^+), T_n], \qquad \left|\alpha^-(t)\right| \leq \frac{1}{2}e^{-\delta\sqrt{\omega}|v|t}.$$

If the following conditions are satisfied

$$C_3\frac{M^3}{8}e^{-2\delta\sqrt{\omega}|v|T_0} \leq \frac{1}{4}, \tag{II.49}$$

$$C_4e^{-\delta\sqrt{\omega}|v|T_0} \leq \frac{1}{4}. \tag{II.50}$$

□

3.1.4 Control of the unstable direction by a topological argument

Finally, we have to control $\alpha^+(t)$. For this, we will provide the existence of a suitable value of α^+.

Lemma 3.10. *For $\delta > 0$ small enough and T_0 large enough, there exists α^+ such that $|\alpha^+| \leq e^{-\delta\sqrt{\omega}|v|t}$ and $T(\alpha^+) = T_0$.*

Proof. We argue by contradiction. Assume that, $\forall\, \alpha^+$ such that $|\alpha^+| \leq e^{-\delta\sqrt{\omega}|v|t}$, one has $T(\alpha^+) > T_0$.

From Lemma 3.8 and 3.9 we have

$$\begin{aligned}
\left\|u(T(\alpha^+)) - R(T(\alpha^+))\right\|_{H_0^1} &\leq \frac{\varepsilon}{2} \\
\left\|r(T(\alpha^+))\right\|_{H_0^1} &\leq \frac{M}{2} e^{-\delta\sqrt{\omega}|v|T(\alpha^+)} \\
\left|y(T(\alpha^+))\right| + \left|\mu(T(\alpha^+))\right| &\leq \frac{M'}{2} e^{-\delta\sqrt{\omega}|v|T(\alpha^+)} \\
\left|\alpha^-(T(\alpha^+))\right| &\leq \frac{1}{2} e^{-\delta\sqrt{\omega}|v|T(\alpha^+)}.
\end{aligned}$$

By the definition of $T(\alpha^+)$ and the continuity of the flow, one must have

$$\left|\alpha^+(T(\alpha^+))\right| = e^{-\delta\sqrt{\omega}|v|T(\alpha^+)}.$$

Let $T < T(\alpha^+)$ be close enough to $T(\alpha^+)$ so that the solution $u(t)$ and its modulation are well-defined on $[T, T_n]$.

For $t \in [T, T_n]$, let $\quad \mathcal{N}(\alpha^+(t)) = \mathcal{N}(t) = \left|e^{\delta\sqrt{\omega}|v|t}\alpha^+(t)\right|^2.$

$$\frac{d}{dt}\mathcal{N}(t) = e^{2\delta\sqrt{\omega}|v|t}\left[2\delta\sqrt{\omega}|v|\ \alpha^+(t) + 2\frac{d}{dt}\alpha^+(t)\right]\alpha^+(t) \tag{II.51}$$

Multiplying by $2\left|\alpha^+(t)\right|$ the estimate (II.40), we obtain

$$\left|2\alpha^+(t)\frac{d}{dt}\alpha^+(t) + 2e_\omega\alpha^+(t)^2\right| \leq C\left|\alpha^+(t)\right|\left(\|h(t)\|^3_{H_0^1} + e^{-2\delta\sqrt{\omega}|v|t}\right),$$

which yields

$$\frac{d}{dt}\left|\alpha(t)\right|^2 + 2e_\omega\left|\alpha(t)\right|^2 \leq C\left|\alpha^+(t)\right|\left(\|h(t)\|^3_{H_0^1} + e^{-2\delta\sqrt{\omega}|v|t}\right)$$

By (II.51), it follows that

$$\frac{d}{dt}\mathcal{N}(t) = e^{2\delta\sqrt{\omega}|v|t}[2\delta\sqrt{\omega}|v| - 2e_\omega]|\alpha^+(t)|^2 + O\left(e^{2\delta\sqrt{\omega}|v|t}\left|\alpha^+(t)\right|(\|h(t)\|^3_{H_0^1} + e^{-2\delta\sqrt{\omega}|v|t})\right).$$

Due to (II.47) we have

$$e^{2\delta\sqrt{\omega}|v|t}\left|\alpha^+(t)\right|(\|h(t)\|^3_{H_0^1} + e^{-2\delta\sqrt{\omega}|v|t}) \leq C\sqrt{\mathcal{N}(t)}\left(\frac{M^3}{8}e^{-2\delta\sqrt{\omega}|v|t} + e^{-\delta\sqrt{\omega}|v|t}\right).$$

Let $\delta > 0$ such that $2e_\omega - 2\delta\sqrt{\omega}|v| \geq e_\omega$, so that

$$\frac{d}{dt}\mathcal{N}(t) \leq -e_\omega \mathcal{N}(t) + \left(C_5 \frac{M^3}{8} e^{-2\delta\sqrt{\omega}|v|t} + C_6 e^{-\delta\sqrt{\omega}|v|t} \right) \sqrt{\mathcal{N}(t)}.$$

We consider the above estimate at $t = T(\alpha^+) \geq T_0$, so large such that

$$C_5 \frac{M^3}{8} e^{-2\delta\sqrt{\omega}|v|T_0} \leq \frac{1}{4} e_\omega, \tag{II.52}$$

$$C_6 e^{-\delta\sqrt{\omega}|v|T_0} \leq \frac{1}{4} e_\omega. \tag{II.53}$$

Using that $\mathcal{N}(T(\alpha^+)) = 1$, we get

$$\forall \alpha^+ \in B(e^{-\delta\sqrt{\omega}|v|T_n}), \qquad \frac{d}{dt}\mathcal{N}(T(\alpha^+)) \leq -\frac{1}{2} e_\omega. \tag{II.54}$$

From (II.54), a standard argument says that the map: $\alpha^+ \longmapsto T(\alpha^+)$ is continuous.

Indeed, by (II.54), $\forall \varepsilon > 0$, $\exists \eta > 0$ such that

$$\mathcal{N}(T(\alpha^+) - \varepsilon) > 1 + \eta,$$

and

$$\mathcal{N}(t) < 1 - \eta, \qquad \forall t \in [T(\alpha^+) + \varepsilon, T_n] \quad \text{(possibly empty)}.$$

By continuity of the flow of the (NLS) equation, it follows that $\exists \theta > 0$ such that, for all $\|\tilde{\alpha}^+ - \alpha^+\| \leq \theta$, the corresponding $\tilde{\alpha}^+(t)$ satisfies

$$|\mathcal{N}(\tilde{\alpha}^+(t)) - \mathcal{N}(\alpha^+(t))| \leq \frac{\eta}{2} \qquad \forall t \in [T(\alpha^+) - \varepsilon, T_n].$$

In particular, $T(\alpha^+) - \varepsilon < T(\tilde{\alpha}^+) < T(\alpha^+) + \varepsilon$.
Now we consider the continuous map

$$\begin{aligned} P : B_{\mathbb{R}}(e^{-\delta\sqrt{\omega}|v|T_n}) &\longrightarrow S_{\mathbb{R}}(e^{-\delta\sqrt{\omega}|v|T_n}) \\ \alpha^+ &\longmapsto e^{-\delta\sqrt{\omega}|v|(T_n - T(\alpha^+))}\, \alpha^+(T(\alpha^+)) \end{aligned}$$

Let $\alpha^+ \in S_{\mathbb{R}}(e^{-\delta\sqrt{\omega}|v|T_n})$, from (II.54) it follows that $T(\alpha^+) = T_n$ and $P(\alpha^+) = \alpha^+$, which means that $P|_{S_{\mathbb{R}}(e^{-\delta\sqrt{\omega}|v|T_n})} = Id$. But this contradicts Brouwer's fixed point theorem.
So, $\exists \alpha^+ \in B_{\mathbb{R}}(e^{-\delta\sqrt{\omega}|v|T_n})$ such that $T(\alpha^+) = T_0$. □

3.2 Estimate on the modulation parameters

Proof. This section is devoted to the proof of the Lemma 3.8. For that, we claim the following results which will be proved at the end of the proof.
Let us recall that $\widetilde{R}(t,x) = e^{i\widetilde{\varphi}(t,x)}\widetilde{Q}_\omega(t,x)\Psi(x)$.

Claim 3.11.

$$\left|\frac{d}{dt}\left(E(\widetilde{R}(t)) + (\frac{\omega}{2} + \frac{|v|^2}{8})M(\widetilde{R}(t)) - \frac{v}{2}P(\widetilde{R}(t))\right)\right| \leq Ce^{-2\delta\sqrt{\omega}|v|t} + M^2e^{-3\delta\sqrt{\omega}|v|t}. \tag{II.55}$$

Claim 3.12.

$$\Bigg|\left[E(u(t)) + (\frac{\omega}{2} + \frac{|v|^2}{8})M(u(t)) - \frac{v}{2}P(u(t))\right] - \left[E(\widetilde{R}(t)) + (\frac{\omega}{2} + \frac{|v|^2}{8})M(\widetilde{R}(t)) - \frac{v}{2}P(\widetilde{R}(t))\right] \\ - \frac{1}{2}\left[(\widetilde{L}^+_\omega h_1(t), h_1(t)) + (\widetilde{L}^-_\omega h_2(t), h_2(t))\right]\Bigg| \leq CMe^{-2\delta\sqrt{\omega}|v|t} + CM^2e^{-3\delta\sqrt{\omega}|v|t}. \tag{II.56}$$

Claim 3.13. *There exists $C > 0$ such that,*

$$\|h(t)\|^2_{H^1_0} \leq C\left[\left(\widetilde{L}^+_\omega h_1(t), h_1(t)\right) + \left(\widetilde{L}^-_\omega h_2(t), h_2(t)\right) + \left(\alpha^\pm(t)\right)^2 + M^2e^{-4\delta\sqrt{\omega}|v|t}\right] \tag{II.57}$$

Now, we start the proof of Lemma 3.8. Let $t \in [T(\alpha^+), T_n]$, integrating (II.55) on $[t, T_n]$ we get

$$\left|\left[E(\widetilde{R}(T_n)) + (\frac{\omega}{2} + \frac{|v|^2}{8})M(\widetilde{R}(T_n)) - \frac{v}{2}P(\widetilde{R}(T_n))\right] - \left[E(\widetilde{R}(t)) + (\frac{\omega}{2} + \frac{|v|^2}{8})M(\widetilde{R}(t)) - \frac{v}{2}P(\widetilde{R}(t))\right]\right| \\ \leq Ce^{-2\delta\sqrt{\omega}|v|t} + M^2e^{-3\delta\sqrt{\omega}|v|t}.$$

From the above estimate and (II.56), we have

$$\left|\left[\left(\widetilde{L}^+_\omega h_1(T_n), h_1(T_n)\right) + \left(\widetilde{L}^-_\omega h_2(T_n), h_2(T_n)\right)\right] - \left[\left(\widetilde{L}^+_\omega h_1(t), h_1(t)\right) + \left(\widetilde{L}^-_\omega h_2(t), h_2(t)\right)\right]\right| \\ \leq CMe^{-2\delta\sqrt{\omega}|v|t} + CM^2e^{-3\delta\sqrt{\omega}|v|t}. \tag{II.58}$$

From Lemma 3.2 and Lemma 3.4 we have

$$\left|\left(\widetilde{L}^+_\omega h_1(T_n), h_1(T_n)\right) + \left(\widetilde{L}^-_\omega h_2(T_n), h_2(T_n)\right)\right| \leq C\,\|h(T_n)\|^2_{H^1_0} \leq C\,|\lambda|^2 \leq Ce^{-2\delta\sqrt{\omega}|v|t}. \tag{II.59}$$

We deduce from (II.58), (II.59) and the Claim 3.13 that

$$\begin{aligned}\|h(t)\|^2_{H^1_0} &\leq C(\widetilde{L}^+_\omega h_1(t), h_1(t)) + C(\widetilde{L}^-_\omega h_2(t), h_2(t)) + C\left(\alpha^\pm(t)\right)^2 + CM^2 e^{-4\delta\sqrt{\omega}|v|t}\\ &\leq C_7 M e^{-2\delta\sqrt{\omega}|v|t} + C_8 M^2 e^{-3\delta\sqrt{\omega}|v|t}.\end{aligned}$$

If T_0 satisfies

$$C_7 e^{-2\delta\sqrt{\omega}|v|T_0} \leq \frac{1}{4}, \tag{II.60}$$

$$C_8 M e^{-3\delta\sqrt{\omega}|v|T_0} \leq \frac{1}{4}. \tag{II.61}$$

Then, we have

$$\|h(t)\|_{H^1_0} \leq \frac{M}{2} e^{-\delta\sqrt{\omega}|v|t},$$

provided conditions (II.44), (II.49), (II.50), (II.53), (II.52), (II.60) and (II.61) on M and T_0 hold. However it is easy to find T_0 and M satisfying these conditions. We take T_0 large enough such that

$$\max(C_4, C_6, C_7) e^{-\delta\sqrt{\omega}|v|T_0} \leq \frac{1}{4}\min(1, e_\omega), \tag{II.62}$$

and we take M such that

$$M e^{-\delta\sqrt{\omega}|v|T_0} \leq \frac{1}{10C'} \min\left(\left\|\partial_{x_j} Q\Psi\right\|^2_{L^2}, \|Q\Psi\|^2_{L^2}\right), \tag{II.63}$$

$$\max(C_3, C_5)\frac{M^3}{8} e^{-2\delta\sqrt{\omega}|v|T_0} \leq \frac{1}{4}\min(1, e_\omega). \tag{II.64}$$

$$C_8 M e^{-\delta\sqrt{\omega}|v|T_0} \leq \frac{1}{4} \tag{II.65}$$

From Lemma 3.6 we have

$$\begin{aligned}\left|\frac{d\mu(t)}{dt}\right| + \left|\frac{dy(t)}{dt}\right| &\leq C\,\|h(t)\|^2_{H^1_0} + Ce^{-2\delta\sqrt{\omega}|v|t}\\ &\leq C\frac{M^2}{4} e^{-2\delta\sqrt{\omega}|v|t} + Ce^{-2\delta\sqrt{\omega}|v|t}.\end{aligned}$$

We integrate the above estimate on some time interval $[t, T_n]$, for $t \in [T(\alpha^+), T_n]$.

$$|\mu(t)| + |y(t)| \leq |\mu(T_n)| + |y(T_n)| + C\frac{M^2}{4} e^{-2\delta\sqrt{\omega}|v|t} + Ce^{-2\delta\sqrt{\omega}|v|t}.$$

Furthermore, due to the definition of $T(\alpha^+)$ we get

$$|\mu(t)| + |y(t)| \leq C_1' e^{-2\delta\sqrt{\omega}|v|t} + C_2' \frac{M^2}{4} e^{-2\delta\sqrt{\omega}|v|t}.$$

Then, we can deduce that $\quad |\mu(t)| + |y(t)| \leq \frac{M'}{2} e^{-\delta\sqrt{\omega}|v|t}.$

Provided, for T_0 large enough

$$C_1' e^{-\delta\sqrt{\omega}|v|T_0} \leq \frac{M'}{4}, \tag{II.66}$$

and we take M' such that

$$C_2' \frac{M^2}{4} e^{-2\delta\sqrt{\omega}|v|T_0} \leq \frac{M'}{4}. \tag{II.67}$$

Finally, we obtain

$$\begin{aligned}
\|u(t) - R(t)\|_{H_0^1} &\leq \left\|R(t) - \widetilde{R}(t)\right\|_{H_0^1} + \|h(t)\|_{H_0^1} \\
&\leq C\,|y(t)| + \|h(t)\|_{H_0^1} \\
&\leq C e^{-\delta\sqrt{\omega}|v|t} \\
&\leq \frac{\varepsilon}{2},
\end{aligned}$$

which concludes the proof of Lemma 3.8, by taking T_0 large enough. □

Proof of Claim 3.11. Recall that $\widetilde{R}(t,x) = e^{i\widetilde{\varphi}(t,x)} \widetilde{Q}_\omega(t,x)\Psi(x)$.

$$\nabla\widetilde{R}(t,x) = [i\frac{v}{2}\widetilde{Q}_\omega\Psi + \nabla(\widetilde{Q}_\omega\Psi)]\, e^{i\widetilde{\varphi}(t,x)}, \qquad \left|\nabla\widetilde{R}(t,x)\right|^2 = \frac{|v|^2}{4}\widetilde{Q}_\omega^2\Psi^2 + \left|\nabla(\widetilde{Q}_\omega\Psi)\right|^2.$$

$$\begin{aligned}
E(\widetilde{R}(t)) &= \frac{1}{2}\int \left|\nabla \widetilde{R}(t)\right|^2 dx - \frac{1}{p+1}\int \widetilde{Q}_\omega^{p+1}\Psi^{p+1}dx,\\
\frac{d}{dt}E(\widetilde{R}(t)) &= \frac{1}{2}\frac{d}{dt}\left[\int \frac{|v|^2}{4}\widetilde{Q}_\omega^2\Psi^2 + \left|\nabla(\widetilde{Q}_\omega\Psi)\right|^2 dx - \frac{1}{p+1}\int \widetilde{Q}_\omega^{p+1}\Psi^{p+1}dx\right]\\
&= \frac{1}{2}\int \frac{|v|^2}{4}2(-v-\frac{dy(t)}{dt})\nabla\widetilde{Q}_\omega\,\widetilde{Q}_\omega\Psi^2 + 2(-v-\frac{dy(t)}{dt}).\nabla(\nabla(\widetilde{Q}_\omega\Psi))\nabla(\widetilde{Q}_\omega\Psi)dx\\
&- \frac{1}{p+1}\int (p+1)(-v-\frac{dy(t)}{dt})\nabla\widetilde{Q}_\omega\,\widetilde{Q}_\omega^p\Psi^{p+1}dx\\
&= (-v-\frac{dy(t)}{dt})\left[\int \frac{|v|^2}{4}\nabla\widetilde{Q}_\omega\,\widetilde{Q}_\omega\,\Psi^2 + \nabla(\nabla\widetilde{Q}_\omega\Psi)\nabla(\widetilde{Q}_\omega\Psi)dx - \int \nabla\widetilde{Q}_\omega\,\widetilde{Q}_\omega^p\,\Psi^{p+1}\right],\\
\text{where,}\quad & (-v-\frac{dy(t)}{dt}).\nabla((\nabla\widetilde{Q}_\omega\Psi))\nabla(\widetilde{Q}_\omega\Psi) = \sum_{k=1}^{3}\sum_{j=1}^{3}\left(-v_k - \frac{dy_k(t)}{dt}\right)\partial_{x_k}\partial_{x_j}(\widetilde{Q}_\omega\Psi)\partial_{x_j}(\widetilde{Q}_\omega\Psi).\\
M(\widetilde{R}(t)) &= \int \left|\widetilde{R}(t)\right|^2 dx,\\
\frac{d}{dt}M(\widetilde{R}(t)) &= \frac{d}{dt}\int \widetilde{Q}_\omega^2\Psi^2 dx = 2(-v-\frac{dy(t)}{dt})\int \nabla\widetilde{Q}_\omega\,\widetilde{Q}_\omega\,\Psi^2 dx.\\
P(\widetilde{R}(t)) &= \operatorname{Im}\int \nabla\widetilde{R}(t)\overline{\widetilde{R}}(t)dx,\\
\frac{d}{dt}P(\widetilde{R}(t)) &= \frac{d}{dt}\left(\frac{v}{2}\int \widetilde{Q}_\omega^2\,\Psi^2 dx\right) = v\int (-v-\frac{dy(t)}{dt})\nabla\widetilde{Q}_\omega\,\widetilde{Q}_\omega\Psi^2 dx.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\frac{d}{dt}\left[E(\widetilde{R}(t)) + (\frac{\omega}{2}+\frac{|v|^2}{8})M(\widetilde{R}(t)) - \frac{v}{2}P(\widetilde{R}(t))\right] &= \frac{\omega}{2}(-v-\frac{dy(t)}{dt})\int \nabla\widetilde{Q}_\omega\,\widetilde{Q}_\omega\Psi^2\\
&+ (-v-\frac{dy(t)}{dt})\left[\int \nabla(\nabla\widetilde{Q}_\omega\,\Psi)\nabla(\widetilde{Q}_\omega\Psi)dx - \int \nabla\widetilde{Q}_\omega\,\widetilde{Q}_\omega^p\,\Psi^{p+1}\right].
\end{aligned}$$

For the first integral, we have

$$\left|\int \nabla\widetilde{Q}_\omega\,\widetilde{Q}_\omega\Psi^2\right| = \left|\frac{1}{2}\int \nabla\widetilde{Q}_\omega^2\,\Psi^2\right| = \left|\int \widetilde{Q}_\omega^2\nabla\Psi\,\Psi\right| \le C\,e^{-2\delta\sqrt{\omega}|v|t}.$$

Using (II.39) and the fact that the support of the derivatives of Ψ is compact. Furthermore, in the second integral, we have some terms with Ψ which doesn't have a compact support. For this terms, we have to use the fact that Q_ω is a radial function, concluding the proof of the Claim 3.11 □

Proof of Claim 3.12. Recall that,

$$u(t,x) = e^{i\widetilde{\varphi}(t,x)}\left(\widetilde{Q}_\omega(t,x)\Psi(x) + h(t,x)\right).$$

$$\begin{aligned} E(u(t)) &= E(e^{i\widetilde{\varphi}}\left[\widetilde{Q}_\omega\Psi + h\right]) \\ &= \frac{1}{2}\int_\Omega \left|\nabla\left(e^{i\widetilde{\varphi}}\left(\widetilde{Q}_\omega\Psi + h\right)\right)\right|^2 dx - \frac{1}{p+1}\int_\Omega \left|\widetilde{Q}_\omega\Psi + h\right|^{p+1} dx. \end{aligned}$$

Using Taylor expansion,

$$\begin{aligned} \left|\widetilde{Q}_\omega\Psi + h\right|^{p+1} &= \widetilde{Q}_\omega^{p+1}\,\Psi^{p+1} + \left(\frac{p+1}{2}\right)\widetilde{Q}_\omega^{p}\,\Psi^{p}\,(h+\overline{h}) \\ &+ \frac{1}{2}\left(\frac{p+1}{2}\right)\left(\frac{p-1}{2}\right)\widetilde{Q}_\omega^{p-1}\,\Psi^{p-1}\left(h^2+\overline{h}^2\right) + \left(\frac{p+1}{2}\right)^2\widetilde{Q}_\omega^{p-1}\,\Psi^{p-1}h\,\overline{h} + \beta(t,x). \end{aligned}$$

and

$$\begin{aligned} \left|\nabla\left(e^{i\widetilde{\varphi}}(\widetilde{Q}_\omega\Psi + h)\right)\right|^2 &= \left|e^{i\widetilde{\varphi}}\left(i\frac{v}{2}(\widetilde{Q}_\omega\Psi + h) + (\nabla(\widetilde{Q}_\omega\,\Psi) + \nabla h)\right)\right|^2 \\ &= \frac{|v|^2}{4}\left|\widetilde{Q}_\omega\Psi + h\right|^2 - v\,\nabla(\widetilde{Q}_\omega\Psi)h_2 + v\,\widetilde{Q}_\omega\Psi\nabla h_2 + v(h_1\nabla h_2 - h_2\nabla h_1) \\ &+ \left|\nabla(\widetilde{Q}_\omega\Psi)\right|^2 + 2\nabla(\widetilde{Q}_\omega\Psi)\nabla h_1 + |\nabla h|^2. \end{aligned}$$

Here and until the end the proof: $\int$ denote the integral over Ω.

We have

$$\begin{aligned} E(u(t)) - E(\widetilde{R}(t)) &= \frac{|v|^2}{4}\int\widetilde{Q}_\omega\Psi h_1 + \frac{|v|^2}{8}\int|h|^2 + \frac{1}{2}\int|\nabla h|^2 + \int\nabla(\widetilde{Q}_\omega\Psi).\nabla h_1 - \int\widetilde{Q}_\omega^{p}\Psi^{p}h_1 \\ &- \int v.\nabla(\widetilde{Q}_\omega\Psi)h_2 + \int\frac{v}{2}.(h_1\nabla h_2 - h_2\nabla h_1) - \frac{p}{2}\int\widetilde{Q}_\omega^{p-1}\Psi^{p-1}h_1^2 \\ &+ \frac{1}{2}\int\widetilde{Q}_\omega^{p-1}\Psi^{p-1}h_2^2 + \beta(t,x). \end{aligned}$$

$$\begin{aligned} \left(\frac{\omega}{2} + \frac{|v|^2}{8}\right)\left(M(u(t)) - M(\widetilde{R}(t))\right) &= \left(\omega + \frac{|v|^2}{4}\right)\int\widetilde{Q}_\omega\Psi h_1 + \left(\frac{\omega}{2} + \frac{|v|^2}{8}\right)\int|h|^2. \\ -\frac{v}{2}.\left(P(u(t)) - P(\widetilde{R}(t))\right) &= -\frac{|v|^2}{2}\int\widetilde{Q}_\omega\Psi h_1 - \frac{|v|^2}{4}\int|h|^2 - \int\frac{v}{2}.(h_1\nabla h_2 + h_2\nabla h_1) \\ &+ \int v.\nabla(\widetilde{Q}_\omega\Psi)h_2. \end{aligned}$$

Then we have,

$$\Big[E(u(t))+(\frac{\omega}{2}+\frac{|v|^2}{8})M(u(t))-\frac{v}{2}P(u(t))\Big]-\Big[E(\widetilde{R}(t))+(\frac{\omega}{2}+\frac{|v|^2}{8})M(\widetilde{R}(t))-\frac{v}{2}P(\widetilde{R}(t))\Big]$$
$$=\frac{1}{2}\Big[(\widetilde{L}^+_\omega h_1,h_1)+(\widetilde{L}^-_\omega h_2,h_2)\Big]-\frac{p}{2}\int\widetilde{Q}^{p-1}_\omega h_1^2(\Psi^{p-1}-1)-\frac{1}{2}\int\widetilde{Q}^{p-1}_\omega h_2^2(\Psi^{p-1}-1)$$
$$+\int-\Delta(\widetilde{Q}_\omega\Psi)h_1dx-\int\widetilde{Q}^p_\omega\Psi^p h_1+\int\omega\widetilde{Q}_\omega\Psi h_1+\beta(t,x)$$
$$=\frac{1}{2}\Big[(\widetilde{L}^+_\omega h_1,h_1)+(\widetilde{L}^-_\omega h_2,h_2)\Big]-\frac{p}{2}\int\widetilde{Q}^{p-1}_\omega h_1^2(\Psi^{p-1}-1)-\frac{1}{2}\int\widetilde{Q}^{p-1}_\omega h_2^2(\Psi^{p-1}-1)$$
$$+\int\underbrace{(-\Delta\widetilde{Q}_\omega+\omega\widetilde{Q}_\omega-\widetilde{Q}^p_\omega)}_{=0}\Psi\,h_1-2\int\nabla\widetilde{Q}_\omega\nabla\Psi h_1-\int\widetilde{Q}_\omega\Delta\Psi h_1-\int\widetilde{Q}^p_\omega\Psi(\Psi^{p-1}-1)h_1+\beta(t,x).$$

Using the fact that $\nabla\Psi,\Delta\Psi$ and $(\Psi^{p-1}-1)$ has a compact support, to conclude the proof of Claim 3.12. □

Proof of Claim 3.13. The proof of (II.57) is a standard consequence of Lemma 2.7 and the following orthogonality conditions, $\text{Re}\int\partial_{x_j}\widetilde{Q}_\omega\,\Psi\,\overline{h}\,dx=0,\text{Im}\int\widetilde{Q}_\omega\,\Psi\,\overline{h}\,dx=0.$

Due to (II.24), there exits $C>0$ such that

$$\|h(t)\|^2_{H^1_0}\leq C\Big[\Big(\widetilde{L}^+_\omega h_1(t,x),h_1(t,x)\Big)+\Big(\widetilde{L}^-_\omega h_2(t,x),h_2(t,x)\Big)+\sum_{j=1}^{3}\Big(\int\partial_{x_i}\widetilde{Q}_\omega(t,x)h_1(t,x)dx\Big)^2$$
$$+\Big(\int\widetilde{Q}_\omega(t,x)h_2(t,x)dx\Big)^2+\Big(\text{Im}\int\widetilde{\mathcal{Y}}^\mp_\omega(t,x)\,\overline{h}(t,x)dx\Big)^2\Big]$$

Using the orthogonality conditions, we get

$$\int\partial_{x_j}\widetilde{Q}_\omega h_1=\int\partial_{x_j}\widetilde{Q}_\omega(1-\Psi)h_1\quad\text{and}\quad\int\widetilde{Q}_\omega h_2=\int\widetilde{Q}(1-\Psi)h_2\,.$$

Due to the exponential decay of Q and the compact support of $(1-\Psi)$, we have

$$\Big|\int\partial_{x_j}\widetilde{Q}_\omega(t)h_1(t)\Big|\leq CMe^{-2\delta\sqrt{\omega}|v|t}\quad\text{and}\quad\Big|\int\widetilde{Q}_\omega(t)h_2(t)\Big|\leq CMe^{-2\delta\sqrt{\omega}|v|t}$$

$$\text{Im}\int\widetilde{\mathcal{Y}}^\mp_\omega(t,x)\,\overline{h}(t,x)=\alpha^\pm(t)+\text{Im}\int\widetilde{\mathcal{Y}}^\mp_\omega(t,x)\,\overline{h}(t,x)(1-\Psi(x))dx$$
$$=\alpha^\pm(t)+O\left(Me^{-2\delta\sqrt{\omega}|v|t}\right)$$

This concludes the proof of the Claim 3.13. □

4 Fixed point theorem

Proof. This section is devoted to the proof of Theorem 1.6.
Recall that, if $\Theta = \emptyset$ then $H(t,x) = e^{i\varphi(t,x)}Q_\omega(x - tv)$, where $\varphi(t,x) = \frac{1}{2}(x \cdot v) - \frac{1}{4}|v|^2 t + t\omega$, is an exact soliton solution of (NLS).

Let $R(t,x) = e^{i\varphi(t,x)}Q_\omega(x-tv)\Psi(x)$. Write

$$(i\partial_t + \Delta)R = -\Psi |H|^2 H + 2\nabla\Psi\nabla H + \Delta\Psi H.$$

We look for $r_\omega \in C([T_0, +\infty), H^2(\Omega) \cap H_0^1(\Omega))$ such that

$$\begin{cases} i\partial_t r_\omega + \Delta r_\omega = -|R + r_\omega|^2 (R + r_\omega) + \Psi |H|^2 H - 2\nabla\Psi\nabla H - \Delta\Psi H, \\ r_\omega(t) \longrightarrow 0 \ \text{ as } \ t \longrightarrow +\infty \ \text{ in } H^2(\Omega) \cap H_0^1(\Omega). \end{cases} \tag{II.68}$$

Set

$$\begin{aligned} A_0(t,x) &= \Psi(x)(1 - \Psi^2(x))|H(t,x)|^2 H(t,x) - 2\nabla\Psi(x)\nabla H(t,x) - \Delta\Psi(x)H(t,x), \\ A_1(r_\omega(t,x)) &= -R(t,x)^2\overline{r_\omega}(t,x) - 2|R(t,x)|^2 r_\omega(t,x), \\ A_2(r_\omega(t,x)) &= -\overline{R}(t,x)r_\omega^2(t,x) - 2R(t,x)|r_\omega(t,x)|^2, \\ A_3(r_\omega(t,x)) &= -|r_\omega(t,x)|^2 r_\omega(t,x). \end{aligned}$$

We shall look for solutions of (II.68) in this space:

$$E = \{r_\omega \in C\left([T_0, +\infty), H^2(\Omega) \cap H_0^1(\Omega)\right), \ \|r_\omega\|_E < \infty\},$$

such that

$$\|r_\omega\|_E = \sup_{t \geq T_0} \left\{ e^{\delta\sqrt{\omega}|v|t} \left(\frac{1}{|v|^3} \|r_\omega\|_{H^2(\Omega)} + \|r_\omega\|_{L^2(\Omega)} \right) \right\}.$$

Let

$$\begin{aligned} \Phi : (B_E, d_E) &\longrightarrow (B_E, d_E) \\ r_\omega &\longmapsto \Phi(r_\omega) = -i\int_t^{+\infty} S(t-\tau)A_0(\tau)d\tau - i\sum_{k=1}^{3}\int_t^{+\infty} S(t-\tau)A_k(r_\omega(\tau))d\tau. \end{aligned}$$

Where $B_E = B_E(0,1) = \{h \in E, \ \|h\|_E \leq 1\}$ and $d_E(h,g) = \|h - g\|_E$.
One can check that (B_E, d_E) is a complete metric space.
Here S(t) is the unitary group of the linear Schrödinger equation with Dirichlet boundary con-

ditions.

Denote,

$$\mathrm{J}_0(\mathrm{t}) = \int_t^{+\infty} S(t-\tau)A_0(\tau)d\tau,$$
$$\mathrm{J}_\mathrm{k}(\mathrm{r}_\omega(\mathrm{t})) = \int_t^{+\infty} S(t-\tau)A_k(r_\omega(\tau))d\tau, \quad k=1,2,3.$$

Remark 4.1. *For $2 \leq p < 5$, the proof is also based on a fixed point theorem as the cubic case. Indeed, we have to use Taylor expansion for the non-linearity $|R+r_\omega|^{p-1}(R+r_\omega)$ and we can divide the function Φ in three integrals, one for the terms that are independent on r_ω, the other for the linear terms on r_ω and the last one for the nonlinear terms on r_ω. Finally, we use the same space E and norm to prove that Φ is a contraction mapping for high velocity.*

In step 1, we will prove that the ball B_E is stable by Φ and in the second step we will prove that Φ is a contraction mapping on the complete metric space (B_E, d). Finally, in step 3 we will conclude by fixed point theorem the existence of the solution of the (NLS_Ω).

- Step 1 : Stability of B_E by Φ.

Lemma 4.2. *There exists $C_\omega > 0$ and $\delta > 0$ such that,*

$$\|\mathrm{J}_0\|_E \leq \frac{C_\omega}{|v|} \tag{II.69}$$
$$\|\mathrm{J}_1(\mathrm{r}_\omega)\|_E \leq \frac{C_\omega}{|v|}\|r_\omega\|_E \tag{II.70}$$
$$\|\mathrm{J}_2(\mathrm{r}_\omega)\|_E \leq C_\omega |v|^4 e^{-\delta\sqrt{\omega}|v|T_0}\|r_\omega\|_E^2 \tag{II.71}$$
$$\|\mathrm{J}_3(\mathrm{r}_\omega)\|_E \leq C_\omega |v|^5 e^{-2\delta\sqrt{\omega}|v|T_0}\|r_\omega\|_E^3 \tag{II.72}$$
$$\forall r_\omega \in B_E, \quad \|\Phi(r_\omega)\|_E \leq 1. \tag{II.73}$$

Proof. 1. **Estimate For J_0.**
Recall that $A_0(t,x) = \Psi(x)(1-\Psi^2(x))|H(t,x)|^2 H(t,x) - 2\nabla\Psi(x)\nabla H(t,x) - \Delta\Psi(x)H(t,x)$, where $H(t,x) = Q_\omega(x-tv)e^{i(\frac{1}{2}(x\cdot v)-\frac{|v|^2}{4}t+t\omega)}$.
Let us prove that there exists $C_\omega > 0$ such that,

$$\|A_0(t)\|_{H^2} \leq C_\omega |v|^3 e^{-\delta\sqrt{\omega}|v|t}, \quad \forall t \in [T_0, +\infty). \tag{II.74}$$

It suffices to estimate the L^2 norm of A_0 and $\nabla^2 A_0$, due to the following elementary interpolation inequality, if $f \in H^2$,

$$\|\nabla f\|_{L^2} \leq \left\|\nabla^2 f\right\|_{L^2}^{\frac{1}{2}} \|f\|_{L^2}^{\frac{1}{2}}. \tag{II.75}$$

We will use the fact that $\Psi(1-\Psi^2)$, $\nabla\Psi$ and $\Delta\Psi$ have a compact support. We will suppose that their support is include in $\{|x| < M\}$, for some $M > 0$.

Let $x \in \operatorname{supp}\{\Psi(1-\Psi^2)\} \subset \{|x| < M\}$ then $\{\, t\,|v| - M \leq |x - tv|\,\}$.
By (II.23), we have

$$\begin{cases} |Q_\omega(x-tv)| \leq C_\omega e^{\delta\sqrt{\omega}M} e^{-\delta\sqrt{\omega}|v|t}, \\ |\nabla Q_\omega(x-tv)| \leq C_\omega e^{\delta\sqrt{\omega}M} e^{-\delta\sqrt{\omega}|v|t}. \end{cases} \tag{II.76}$$

Then,

$$\|A_0\|_{L^2} \leq C_\omega\,|v|\,e^{-\delta\sqrt{\omega}|v|t}.$$

Now, let us estimate $\nabla^2 A_0$.

Recall that $A_0 = \Psi(1-\Psi^2)\,|H|^2 H - 2\sum_{k=1}^{3} \partial_{x_k}\Psi\partial_{x_k}H - \ \Delta\Psi H.$

$$\begin{aligned} \partial_{x_j}\partial_{x_i}A_0(t,x) &= \partial_{x_j}\partial_{x_i}\left[\Psi(1-\Psi^2)\right]|H|^2 H + \partial_{x_i}\left[\Psi(1-\Psi^2)\right]\partial_{x_j}\left[|H|^2 H\right] \\ &+ \partial_{x_j}\left[\Psi(1-\Psi^2)\right]\partial_{x_i}\left[|H|^2 H\right] + \left[\Psi(1-\Psi^2)\right]\partial_{x_j}\partial_{x_i}\left[|H|^2 H\right] \\ &- 2\left(\sum_{k=1}^{3}\partial_{x_j}\partial_{x_i}\left[\partial_{x_k}\Psi\right]\ \partial_{x_k}H + \partial_{x_i}\left[\partial_{x_k}\Psi\right]\partial_{x_j}\partial_{x_k}H\right) \\ &- 2\left(\sum_{k=1}^{3}\partial_{x_j}\partial_{x_k}\Psi\ \partial_{x_i}\left[\partial_{x_k}H\right] + \partial_{x_k}\Psi\ \ \partial_{x_j}\partial_{x_i}\left[\partial_{x_k}H\right]\right) \\ &- \partial_{x_j}\partial_{x_i}\left[\Delta\Psi\right]H - \partial_{x_i}\left[\Delta\Psi\right]\partial_{x_j}H - \partial_{x_j}\left[\Delta\Psi\right]\partial_{x_i}H - \left[\Delta\Psi\right]\partial_{x_j}\partial_{x_i}H \end{aligned}$$

Claim 4.3.

$$\left|\nabla^{4-k}\Psi(x)\nabla^k H(t,x)\right| \leq C_\omega\,|v|^k\,e^{-\delta\sqrt{\omega}|v|t},\ \ \textit{where } k = 1,2,3.$$

$$\left|\nabla^{2-k}\left(\Psi(x)(1-\Psi^2(x))\right)\nabla^k(|H(t,x)|^2 H(t,x))\right| \leq C_\omega\,|v|^k\,e^{-\delta\sqrt{\omega}|v|t},\ \ \textit{where } k = 1,2.$$

Where,

$$\nabla^3 f \ \nabla^1 g = \sum_{k=1}^{3} \left(\partial_{x_i x_j x_k} f\right) \ (\partial_{x_k} g) \,, \qquad \nabla^2 f \ \nabla^2 g = \sum_{k=1}^{3} \left(\partial_{x_j x_k} f\right) (\ \partial_{x_i x_k} g) \,,$$

$$(\nabla^2 f) \ g = \left(\partial_{x_i x_j} f\right) g \,, \qquad \nabla^1 f \ \nabla^1 g = (\partial_{x_i} f) \left(\partial_{x_j} g\right).$$

Proof. We postpone the proof of Claim 4.3 to Appendix, Section 6. □

By the Claim 4.3, we have

$$\left\|\nabla^2 A_0(t)\right\|_{L^2} \leq C_\omega \left|v\right|^3 e^{-\delta\sqrt{\omega}|v|t}.$$

This concludes the proof of (II.74).
Thus, we obtain

$$\|\mathrm{J}_0(\mathrm{t})\|_{H^2} \leq \int_t^{+\infty} \|A_0(\tau)\|_{H^2} \, d\tau \leq C_\omega \left|v\right|^2 e^{-\delta\sqrt{\omega}|v|t},$$

$$\|\mathrm{J}_0\|_E \leq \frac{C_\omega}{|v|}.$$

2. **Estimate for** J_1**.**
Recall that $A_1(r_\omega(t,x)) = -R(t,x)\overline{r}_\omega(t,x) - 2\left|R(t,x)\right|^2 r_\omega(t,x)$.
Using the elementary interpolation inequality (II.75), we have

$$\begin{aligned}\|\mathrm{J}_1(\mathrm{r}_\omega(\mathrm{t}))\|_{H^2} &\leq \int_t^{+\infty} \|A_1(r_\omega(\tau))\|_{H^2} \, d\tau \\ &\leq C \int_t^{+\infty} \|A_1(r_\omega(\tau))\|_{L^2} \, d\tau + C \int_t^{+\infty} \left\|\nabla^2 A_1(r_\omega(\tau))\right\|_{L^2} d\tau.\end{aligned}$$

Let us prove that there exists $C_\omega > 0$ such that

$$\int_t^{+\infty} \|A_1(r_\omega(\tau))\|_{L^2} \, d\tau \leq \frac{C_\omega}{|v|} e^{-\delta\sqrt{\omega}|v|t} \|r_\omega\|_E \,, \tag{II.77}$$

$$\int_t^{+\infty} \left\|\nabla^2 A_1(r_\omega(\tau))\right\|_{L^2} d\tau \leq C_\omega \left|v\right|^2 e^{-\delta\sqrt{\omega}|v|t} \|r_\omega\|_E \,. \tag{II.78}$$

$$\begin{aligned}\int_t^{+\infty} \|A_1(r_\omega(\tau))\|_{L^2} \, d\tau &\leq C \int_t^{+\infty} \|r_\omega(\tau)\|_{L^2} \, d\tau \leq C \int_t^{+\infty} e^{-\delta\sqrt{\omega}|v|\tau} d\tau \, \|r_\omega\|_E \\ &\leq \frac{C_\omega}{|v|} e^{-\delta\sqrt{\omega}|v|t} \|r_\omega\|_E \,.\end{aligned}$$

This prove the first estimate. Now, let us look to the second estimate

$$\begin{aligned}\int_t^{+\infty} \left\|\nabla^2 A_1(r_\omega(\tau))\right\|_{L^2} d\tau \leq\ & C \underbrace{\int_t^{+\infty} \left\|\nabla^2 R^2(\tau)\right\|_{L^\infty} \|r_\omega(\tau)\|_{L^2} d\tau}_{\mathrm{I}_1} \\ & + C \underbrace{\int_t^{+\infty} \left\|\nabla R^2(\tau)\right\|_{L^\infty} \|\nabla r_\omega(\tau)\|_{L^2} d\tau}_{\mathrm{I}_2} \\ & + C \underbrace{\int_t^{+\infty} \left\|R^2(\tau)\right\|_{L^\infty} \left\|\nabla^2 r_\omega(\tau)\right\|_{L^2} d\tau}_{\mathrm{I}_3} .\end{aligned}$$

It is easy to see that

$$\begin{aligned}|\mathrm{I}_1| &\leq C_\omega |v| e^{-\delta\sqrt{\omega}|v|t} \|r_\omega\|_E ,\\ |\mathrm{I}_3| &\leq C_\omega |v|^2 e^{-\delta\sqrt{\omega}|v|t} \|r_\omega\|_E .\end{aligned}$$

For I_2 we use the elementary interpolation inequality (II.75),

$$\|\nabla r_\omega(\tau)\|_{L^2} \leq \left\|\nabla^2 r_\omega(\tau)\right\|_{L^2}^{\frac{1}{2}} \|r_\omega\|_{L^2}^{\frac{1}{2}} .$$

Thus we get,

$$|\mathrm{I}_2| \leq C_\omega |v|^{\frac{3}{2}} e^{-\delta\sqrt{\omega}|v|t} \|r_\omega(\tau)\|_E .$$

And this concludes the proof of the estimates (II.77) and (II.78).
Due to (II.77), (II.78) and the fact that $|v| > 1$ we have

$$\|\mathrm{J}_1(\mathrm{r}_\omega)\|_{H^2} \leq C_\omega |v|^2 e^{-\delta\sqrt{\omega}|v|t} \|r_\omega\|_E .$$

Then

$$\|\mathrm{J}_1(\mathrm{r}_\omega)\|_E \leq \frac{C_\omega}{|v|} \|r_\omega\|_E .$$

3. **Estimate for** J_2**.** Recall that $A_2(r_\omega(t,x)) = -\overline{R}(t,x) r_\omega^2(t,x) - 2R(t,x) |r_\omega(t,x)|^2$.

$$\|\mathrm{J}_2(\mathrm{r}_\omega(\mathrm{t})\|_{H^2} \leq C \int_t^{+\infty} \|A_2(r_\omega(\tau)\|_{H^2} d\tau.$$

Using the fact that H^2 is an algebra we obtain

$$\begin{aligned}\|\mathrm{J}_2(\mathrm{r}_\omega(\mathrm{t})\|_{H^2} &\leq C\int_t^{+\infty}\|R(\tau)\|_{H^2}\|r_\omega(\tau)\|_{H^2}^2\,d\tau\\&\leq C_\omega\,|v|^2\int_t^{+\infty}|v|^6\,e^{-2\delta\sqrt{\omega}|v|\tau}d\tau\,\|r_\omega\|_E^2\\&\leq C_\omega\,|v|^7\,e^{-2\delta\sqrt{\omega}|v|t}\,\|r_\omega\|_E^2\,.\end{aligned}$$

Then

$$\|\mathrm{J}_2(\mathrm{r}_\omega)\|_E\leq C_\omega\,|v|^4\,e^{-\delta\sqrt{\omega}|v|T_0}\,\|r_\omega\|_E^2\,.$$

4. **Estimate for** J_3**.**

We have $A_3(r_\omega(t,x)) = -\,|r_\omega(t,x)|^2\,r_\omega(t,x)$.

$$\begin{aligned}\|\mathrm{J}_3(\mathrm{r}_\omega(\mathrm{t}))\|_{H^2} &\leq \int_t^{+\infty}\|A_3(r_\omega(\tau))\|_{H^2}^3\,d\tau\leq\int_t^{+\infty}\|r_\omega(\tau)\|_{H^2}^3\,d\tau\\&\leq\int_t^{+\infty}|v|^9\,e^{-3\delta\sqrt{\omega}|v|\tau}d\tau\,\|r_\omega\|_E^3\\&\leq C_\omega\,|v|^8\,e^{-3\delta\sqrt{\omega}|v|t}\,\|r_\omega\|_E^3\end{aligned}$$

This implies that

$$\|\mathrm{J}_3(\mathrm{r}_\omega)\|_E\leq C_\omega\,|v|^5\,e^{-2\delta\sqrt{\omega}|v|T_0}\,\|r_\omega\|_E^3\,.$$

5. **Stability of** Φ**.**

Recall that $\Phi(r_\omega(t,x)) = -i\,\mathrm{J}_0(\mathrm{t}) - \mathrm{i}\sum_{\mathrm{k}=1}^{3}\mathrm{J}_\mathrm{k}(\mathrm{r}_\omega(\mathrm{t},\mathrm{x}))$.

Using the fact that the velocity v is large enough in each estimate (II.69),(II.70),(II.71) and (II.72), we get

$$\forall r_\omega\in B_E(0,1),\quad\|\Phi(r_\omega)\|_E\leq\|\mathrm{J}_0\|_E+\sum_{k=1}^{3}\|\mathrm{J}_\mathrm{k}(\mathrm{r}_\omega)\|_E\leq 1.$$

□

- Step 2 : Contraction mapping. Let $f, g \in B_E(0,1)$

$$\begin{aligned}\|\Phi(f(t)) - \Phi(g(t))\|_{H^2} \leq & \Big\| \underbrace{\int_t^{+\infty} S(t-\tau)\left(A_1(f(\tau)) - A_1(g(\tau))\right) d\tau}_{\mathrm{J_1(f)-J_1(g)}} \Big\|_{H^2} \\ & + \Big\| \underbrace{\int_t^{+\infty} S(t-\tau)\left(A_2(f(\tau)) - A_2(g(\tau))\right) d\tau}_{\mathrm{J_2(f)-J_2(g)}} \Big\|_{H^2} \\ & + \Big\| \underbrace{\int_t^{+\infty} S(t-\tau)\left(A_3(f(\tau)) - A_3(g(\tau))\right) d\tau}_{\mathrm{J_3(f)-J_3(g)}} \Big\|_{H^2}.\end{aligned}$$

Lemma 4.4. *For all $T_0 > 0, \omega > 0$, there exists $V_0 > 0$ such that for $|v| > V_0$, for all $f, g \in B_E$, we have*

$$d_E(\Phi(f) - \Phi(g)) \leq \frac{1}{2}\, d_E(f-g).$$

Proof. Due to Lemma 4.2 we have

$$\|\mathrm{J_1(f) - J_1(g)}\|_E \leq \frac{C_\omega}{|v|} \|f-g\|_E\,,$$

Let $V_0 > 0$ large enough to be chosen below such that for $|v| > V_0$, we have

$$\frac{1}{|v|} \leq \frac{1}{8}. \tag{II.79}$$

Then,

$$\|\mathrm{J_1(f) - J_1(g)}\|_E \leq \frac{1}{8}\ \|f-g\|_E\,. \tag{II.80}$$

Recall that $A_2(h(t,x)) = -\overline{R}(t,x))h^2(t,x) - 2R(t,x)\,|h(t,x)|^2$.

$$\begin{aligned}\|\mathrm{J_2(f(t)) - J_2(g(t))}\|_{H^2} &\leq C \int_t^{+\infty} \|R(\tau)\|_{H^2} \left\|f^2(t) - g^2(t)\right\|_{H^2} d\tau \\ &\leq C_\omega\,|v|^2 \int_t^{+\infty} |v|^6\, e^{-2\delta\sqrt{\omega}|v|\tau} d\tau\,(\|f\|_E + \|g\|_E)\,\|f-g\|_E \\ &\leq C_\omega\,|v|^7\, e^{-\delta\sqrt{\omega}|v|T_0} e^{-\delta\sqrt{\omega}|v|t}\,(\|f\|_E + \|g\|_E)\,\|f-g\|_E\,.\end{aligned}$$

This implies that

$$\|\mathrm{J_2(f(t)) - J_2(g(t))}\|_E \leq C_\omega\,|v|^4\, e^{-\delta\sqrt{\omega}|v|T_0}\,(\|f\|_E + \|g\|_E)\,\|f-g\|_E\,.$$

Since the velocity v is large enough we have

$$\forall f, g \in B_E(0,1), \quad C_\omega |v|^4 e^{-\delta\sqrt{\omega}|v|T_0} \left(\|f\|_E + \|g\|_E\right) \leq \frac{1}{8}, \tag{II.81}$$

then

$$\|\mathrm{J}_2(\mathrm{f}) - \mathrm{J}_3(\mathrm{g})\|_E \leq \frac{1}{8} \|f-g\|_E. \tag{II.82}$$

Recall that, $A_3(h(t,x)) = -h(t,x)\,|h(t,x)|^2$.

$$\begin{aligned}
\|\mathrm{J}_3(\mathrm{f(t)}) - \mathrm{J}_3(\mathrm{g(t)})\|_{H^2} &\leq \int_t^{+\infty} \left\| |f(\tau)|^2 f(\tau) - |g(\tau)|^2 g(\tau) \right\|_{H^2} d\tau \\
&\leq \int_t^{+\infty} \left\| \bar{f}(\tau)\left(f^2(\tau) - g^2(\tau)\right) + g^2(\tau)\left(\bar{f}(\tau) - \bar{g}(\tau)\right)\right\|_{H^2} d\tau \\
&\leq C \int_t^{+\infty} \|f(\tau) - g(\tau)\|_{H^2} \left(\|f(\tau)\|_{H^2}^2 + \|g(\tau)\|_{H^2}^2\right) d\tau \\
&\leq C \int_t^{+\infty} |v|^9 e^{-3\delta\sqrt{\omega}|v|\tau} d\tau \left(\|f\|_E^2 + \|g\|_E^2\right) \|f-g\|_E \\
&\leq C_\omega |v|^8 e^{-2\delta\sqrt{\omega}|v|T_0} e^{-\delta\sqrt{\omega}|v|t} \left(\|f\|_E^2 + \|g\|_E^2\right) \|f-g\|_E.
\end{aligned}$$

Hence

$$\|\mathrm{J}_3(\mathrm{f(t)}) - \mathrm{J}_3(\mathrm{g(t)})\|_E \leq C_\omega |v|^5 e^{-2\delta\sqrt{\omega}|v|T_0} \left(\|f\|_E^2 + \|g\|_E^2\right) \|f-g\|_E.$$

Due to the choice of the high velocity v we have

$$\forall f, g \in B_E(0,1), \quad C_\omega |v|^5 e^{-2\delta\sqrt{\omega}|v|T_0} \left(\|f\|_E^2 + \|g\|_E^2\right) \leq \frac{1}{8}, \tag{II.83}$$

and thus

$$\|\mathrm{J}_3(\mathrm{f}) - \mathrm{J}_3(\mathrm{g})\|_E \leq \frac{1}{8} \|f-g\|_E. \tag{II.84}$$

The inequalities (II.79), (II.81), (II.83) specify how large V_0 needs to be taken and from (II.80), (II.82) and (II.84) we have

$$\forall f, g \in B_E(0,1), \ \|\Phi(f) - \Phi(g)\|_E \leq \frac{1}{2} \|f-g\|_E.$$

Thus Φ is a contraction mapping for v large enough.

□

- Step 3: Conclusion.

  Due to steps 1 and 2, Φ is a contraction mapping for high velocity on the complete metric space (B_E, d_E). By the fixed point Theorem there exists a unique solution,

$$r_\omega(t,x) = \Phi(r_\omega(t,x)) = -i\,\mathrm{J}_0(\mathrm{t}) - \mathrm{i}\sum_{\mathrm{k}=1}^{3} \mathrm{J}_\mathrm{k}(\mathrm{r}_\omega(\mathrm{t},\mathrm{x})),$$

such that

$$\|r_\omega(t)\|_{H^2} \leq C_\omega\,|v|^3\,e^{-\delta\sqrt{\omega}|v|t} \quad \forall t \in [T_0, +\infty),$$

which concludes the proof of Theorem 1.6.

□

Appendix

5 Proof of some Technical results

5.1 Proof of Lemma 2.7

Recall that for all $f \in H^1 \backslash \{\lambda Q\,;\ \lambda \in \mathbb{R}\}$ real valued, we have $\int (L^- f) f > 0$. Denote $y_1 = \mathrm{Re}(\mathcal{Y}^+)$ and $y_2 = \mathrm{Im}(\mathcal{Y}^+)$. Since y_2 is not colinear to Q, we have

$$-\operatorname{Im} \int \mathcal{Y}^+ \overline{\mathcal{Y}^-} = 2 \int y_1 y_2 = \frac{2}{e_0} \int -(L^- y_2) y_2 \neq 0. \tag{II.85}$$

Let $h \in H^1$ such that $h = h_1 + i h_2$, we can write h as,

$$h = h^\perp + g,$$

where,

$$\begin{cases} h^\perp \in G^\perp = \{h \in H^1,\ (h, iQ) = (h, i\mathcal{Y}^\pm) = (h, \partial_{x_j} Q) = 0,\ j = 1,2,3\}, \\ g \in Span\{i\mathcal{Y}^+, i\mathcal{Y}^-, (\partial_{x_j} Q)_{j=1,2,3}, i\,Q\}. \end{cases}$$

Denote by:

$$\begin{cases} \phi_1 = \mathcal{Y}^+, & \mu_1 = i\mathcal{Y}^-. \\ \phi_2 = \mathcal{Y}^-, & \mu_2 = i\mathcal{Y}^+. \\ \phi_k = \partial_{x_{k-2}} Q, & \mu_k = \partial_{x_{k-2}} Q \quad k = 3,4,5. \\ \phi_6 = i\,Q, & \mu_6 = iQ - \mu_1(\phi_1, iQ) - \mu_2(\phi_2, iQ). \end{cases} \tag{II.86}$$

Next, one can verify that: $(\phi_j, \mu_k) = \zeta_j\, \delta_j^k$, by (II.85) we have $\zeta_1, \zeta_2 \neq 0$ and it is clear that

$\zeta_j \neq 0,\ \forall j \in [\![3;6]\!]$. This implies that $(\phi_j, \mu_j)_j$ is a biorthogonal family then we can write g as the following

$$g = \sum_{j=1}^{6} \frac{(h,\mu_j)}{\zeta_j}\phi_j = \frac{1}{\zeta_1}(h, i\,\mathcal{Y}^-)\,\mathcal{Y}^+ + \frac{1}{\zeta_2}(h, i\,\mathcal{Y}^+)\,\mathcal{Y}^- + \sum_{j=1}^{3}\frac{1}{\zeta_{j+2}}(h, \partial_{x_j}Q)\,\partial_{x_j}Q$$
$$+ \frac{1}{\zeta_6}\Big((h, iQ) - (h, i\mathcal{Y}^-)(\mathcal{Y}^+, iQ) - (h, i\mathcal{Y}^+)(\mathcal{Y}^+, iQ)\Big)\, i\,Q.$$

We refer to [31, Proposition 2.7] for the following coercivity property of $\mathcal{L}$. There exists a constant $c > 0$ such that

$$\forall h \in G^{\perp}, \qquad \Phi(h) \geq c\,\|h\|_{H^1}^2\,,$$

where, $\Phi(h) = \frac{1}{2}(L^+h_1, h_1) + \frac{1}{2}(L^-h_2, h_2)$. Next, we have

$$\begin{aligned}\|h\|_{H^1}^2 = \|h^{\perp} + g\|_{H^1}^2 \leq c\|h^{\perp}\|_{H^1}^2 + c\,\|g\|_{H^1}^2 &\\ \leq C\,\Phi(h) + C\left(\mathrm{Im}\int \mathcal{Y}^+\,\overline{h}\right)^2 + C\left(\mathrm{Im}\int \mathcal{Y}^-\,\overline{h}\right)^2 + C\left(\int Q\,h_2\right)^2 &\\ + C\sum_{j=1}^{3}\left(\int \partial_{x_j}Q\,h_1\right)^2. &\end{aligned}$$

5.2 Proof of Lemma 3.2

Let $\rho = u - R$ and let

$$\begin{aligned}\Phi : L^2 \times \mathbb{R}^3 \times \mathbb{R} &\longrightarrow \mathbb{R}^4\\ (\rho\ ,\ y\ ,\ \mu) &\longrightarrow \left(\mathrm{Re}\int(\rho + R - \widetilde{R})\nabla\widetilde{Q}_\omega \Psi e^{-i\left(\frac{1}{2}(x\cdot v)+\theta\right)}\,e^{-i\mu}dx\ ,\ \mathrm{Im}\int(\rho + R - \widetilde{R})\overline{\widetilde{R}}\,dx\right).\end{aligned}$$

Denote:

$$\Phi_1(\rho, y, \mu) = \mathrm{Re}\int(\rho + R - \widetilde{R})\nabla\widetilde{Q}_\omega\Psi e^{-i\left(\frac{1}{2}(x\cdot v)+\theta\right)}\,e^{-i\mu}dx,$$
$$\Phi_2(\rho, y, \mu) = \mathrm{Im}\int(\rho + R - \widetilde{R})\overline{\widetilde{R}}.$$

- Step 1: Computation of $d_{(y,\mu)}\Phi_1$. Let $z \in \mathbb{R}^3, l \in \mathbb{R}$.

$$\left(d_y\Phi_1(\rho,y,\mu)\,.z\right)_j = \mathrm{Re}\Bigg(z_j\int \partial_{x_j}\widetilde{Q}_\omega\partial_{x_j}\widetilde{Q}_\omega\Psi^2\,dx + \sum_{k\neq j} z_k\int \partial_{x_k}\widetilde{Q}_\omega\partial_{x_j}\widetilde{Q}_\omega\Psi^2\,dx - \sum_{k=1}^{3}\int(\rho+R-\widetilde{R})\partial_{x_k}\partial_{x_j}\widetilde{Q}_\omega\Psi e^{-i\left(\frac{1}{2}(x\cdot v)+\theta\right)}\,e^{-i\mu}\,z_k\,dx\Bigg). \quad \text{(II.87)}$$

$$\left(d_\mu\Phi_1(\rho,y,\mu).l\right)_j = \mathrm{Re}\left(i\int l\,(\rho+R-\widetilde{R})\partial_{x_j}\widetilde{Q}_\omega\Psi e^{-i\left(\frac{1}{2}(x\cdot v)+\theta\right)}e^{-i\mu}\,dx\right). \quad \text{(II.88)}$$

Claim 5.1.

$$\left(d_y\Phi_1(\rho,y,\mu)\,.z\right)_j = z_j\left\|\partial_{x_j}\widetilde{Q}_\omega\Psi\right\|_{L^2}^2 + O\left(|z|\left(\|\rho\|_{L^2}+|y|\right)\right). \quad \text{(II.89)}$$

$$d_y\Phi_1(0,0,0) = diag\left(\left\|\partial_{x_j}\widetilde{Q}_\omega\Psi\right\|_{L^2}^2\right). \quad \text{(II.90)}$$

$$d_\mu\Phi_1(\rho,y,\mu).l\; = O(|l|\left(\|\rho\|_{L^2}+|y|\right)). \quad \text{(II.91)}$$

$$d_\mu\Phi_1(0,0,0) = 0. \quad \text{(II.92)}$$

Proof. For the first estimate we have

$$\left|\int \rho\,\partial_{x_k,x_j}\widetilde{Q}_\omega\Psi e^{-i\left(\frac{1}{2}(x\cdot v)+\theta\right)}e^{-i\mu}\right| \leq C\,\|\rho\|_{L^2},$$

and

$$\begin{aligned}\int(R-\widetilde{R})\partial_{x_k,x_j}\widetilde{Q}_\omega\Psi dx &= \int\int_0^1 \frac{d}{dt}R(x-ty)dt\,\partial_{x_k,x_j}\widetilde{Q}_\omega\Psi dx\\ &= \int\int_0^1 y\nabla R(x-ty)dt\,\partial_{x_k,x_j}\widetilde{Q}_\omega\Psi dx,\end{aligned}$$
$$\left|\int(R-\widetilde{R})\partial_{x_k,x_j}\widetilde{Q}_\omega\Psi dx\right| \leq C\,|y|\,.$$

This implies that

$$\mathrm{Re}\left(\sum_{k=1}^{3}\int z_k(\rho+R-\widetilde{R})\partial_{x_k}\partial_{x_j}\widetilde{Q}_\omega\Psi e^{-i\left(\frac{1}{2}(x\cdot v)+\theta\right)}e^{-i\mu}\,dx\right) = O\left(|z|\left(\|\rho\|_{L^2}+|y|\right)\right). \quad \text{(II.93)}$$

Since Q_ω is radial, we have

$$\forall k\neq j, \qquad \int \partial_{x_k}Q\,\partial_{x_j}Q\,dx = 0,$$

which yields, for $k \neq j$

$$\int \partial_{x_k} \widetilde{Q}_\omega \partial_{x_j} \widetilde{Q}_\omega \, \Psi^2 \, dx \leq \int \partial_{x_k} \widetilde{Q}_\omega \partial_{x_j} \widetilde{Q}_\omega \, dx \ = 0.$$

Then

$$\mathrm{Re} \left(\sum_{k \neq j} z_k \int \partial_{x_k} \widetilde{Q}_\omega \partial_{x_j} \widetilde{Q}_\omega \Psi^2 \, dx \right) = 0. \tag{II.94}$$

The estimate (II.89) it is a consequence of (II.93) and (II.94). Applying (II.89) at point $(0,0,0)$, we get (II.90).

Due to (II.88), we have

$$\left(d_\mu \Phi_1(\rho, y, \mu).l\right)_j = \mathrm{Re} \left(i \int l \, (\rho + R - \widetilde{R}) \partial_{x_j} \widetilde{Q}_\omega \Psi e^{-i\left(\frac{1}{2}(x \cdot v) + \theta\right)} e^{-i\mu} \right).$$

Then

$$d_\mu \Phi_1(\rho, y, \mu).l = \mathrm{Im} \int l \, (\rho + R - \widetilde{R}) \partial_{x_j} \widetilde{Q}_\omega \Psi e^{-i\left(\frac{1}{2}(x \cdot v) + \theta\right)} e^{-i\mu} dx$$

Similarly to the proof of the estimate (II.93), we have

$$d_\mu \Phi_1(\rho, y, \mu).l = O(|l| \left(\|\rho\|_{L^2} + |y|\right)).$$

Finally, due to the above equality it is easy to see that

$$d_\mu \Phi_1(0,0,0) = 0,$$

which concludes the proof of the Claim 5.1 □

- Step 2: Computation of $d_{(y,\mu)} \Phi_2$.
  Recall that

$$\Phi_2(\rho, y, \mu) = \mathrm{Im} \int (\rho + R - \widetilde{R}) \, \overline{\widetilde{R}}.$$

$$d_y \Phi_2(\rho, y, \mu).l = -\,\mathrm{Im} \left(\sum_{j=1}^{3} \int l_j (\rho + R - \widetilde{R}) \partial_{x_j} \widetilde{Q}_\omega \Psi e^{-i\left(\frac{1}{2}(x \cdot v) + \theta\right)} e^{-i\mu)} \right). \tag{II.95}$$

$$d_\mu \Phi_2(\rho, y, \mu).q = -\int q \, \widetilde{Q}_\omega^2 \Psi^2 - \mathrm{Re} \int \ q(\rho + R - \widetilde{R}) \, \overline{\widetilde{R}}. \tag{II.96}$$

Claim 5.2. *Let* $l \in \mathbb{R}^3, q \in \mathbb{R}$.

$$d_y\Phi_2(\rho, y, \mu).l = O(|l|\,(\|\rho\|_{L^2} + |y|)). \tag{II.97}$$

$$d_y\Phi_2(0,0,0) = 0. \tag{II.98}$$

$$d_\mu\Phi_2(\rho, y, \mu).q = -\int q\,\widetilde{Q}_\omega^2\Psi^2 + O(|q|\,(\|\rho\|_{L^2} + |y|)). \tag{II.99}$$

$$d_\mu\Phi_2(0,0,0) = -\left\|\widetilde{Q}_\omega\Psi^2\right\|_{L^2}. \tag{II.100}$$

Proof. Using the same argument as in the proof of Claim 5.1, we obtain

$$\mathrm{Im}\left(\sum_{j=1}^{3}\int l_j(\rho + R - \widetilde{R})\partial_{x_j}\widetilde{Q}_\omega\Psi e^{-i\left(\frac{1}{2}(x\cdot v)+\theta\right)}e^{-i\mu}\right) = O(|l|\,(\|\rho\|_{L^2} + |y|)).$$

Due to (II.95), we obtain the first estimate. Applying (II.97) at point $(0,0,0)$, we obtain

$$d_y\Phi_2(0,0,0) = 0.$$

Similarly to the proof of $d_{y,\mu}\phi_1$, we have

$$\mathrm{Re}\int q(\rho + R - \widetilde{R})\overline{\widetilde{R}} = O(|q|\,(\|\rho\|_{L^2} + |y|)).$$

Using the above estimate and (II.96), we get

$$d_\mu\Phi_2(\rho, y, \mu).q = -\int q\,\widetilde{Q}_\omega^2\Psi^2 + O(|q|\,(\|\rho\|_{L^2} + |y|)).$$

Then

$$d_\mu\Phi_2(0,0,0) = -\left\|\widetilde{Q}_\omega\Psi^2\right\|_{L^2}.$$

This concludes the proof of the Claim 5.2 □

- Step 3: Conclusion
  From Step 1 and Step 2 we get

$$d_{(y,\mu)}\Phi(\rho, y, \mu) = \begin{pmatrix} \left\|\partial_{x_1}\widetilde{Q}_\omega\Psi\right\|_{L^2}^2 & 0 & 0 & 0 \\ 0 & \left\|\partial_{x_2}\widetilde{Q}_\omega\Psi\right\|_{L^2}^2 & 0 & 0 \\ 0 & 0 & \left\|\partial_{x_3}\widetilde{Q}_\omega\Psi\right\|_{L^2}^2 & 0 \\ 0 & 0 & 0 & -\left\|\widetilde{Q}_\omega\Psi\right\|_{L^2} \end{pmatrix} + O(\|\rho\|_{L^2} + |y|).$$

We can deduce that $d_{(y,\mu)}\Phi(0,0,0)$ is invertible and we have $\Phi(0,0,0) = 0$.

Then, by the Implicit function theorem, there exists $\varepsilon_0 > 0,\ \varepsilon_0 \leq \eta$ and a C^1-function

$$\begin{aligned} g : B_{L^2}(0,\varepsilon) &\longrightarrow B_{\mathbb{R}^4}(0,\eta) \\ \rho &\longmapsto g(\rho) = ((y(\rho), \mu(\rho)) \end{aligned}$$

such that $\Phi(\rho, y, \mu) = 0$ in $B_{L^2}(0,\varepsilon) \times g(B_{L^2}(0;\varepsilon))$ is equivalent to $(y,\mu) = g(\rho)$.

Finally we set

$$r := r(\rho) = \rho + R - \widetilde{R}(\cdot - y(\rho))e^{i\mu(\rho)}.$$

5.3 Proof of Lemma 3.4

$$\begin{aligned} \sigma : \mathbb{R}^2 &\longrightarrow H_0^1 \\ \lambda := \lambda^{\pm} &\longmapsto i\left(\lambda^+ Y_+(T_n) + \lambda^- Y_-(T_n)\right), \end{aligned} \qquad \begin{aligned} \Gamma : B_{H_0^1}(\varepsilon) &\longrightarrow H_0^1 \times \mathbb{R}^3 \times \mathbb{R} \\ \rho &\longmapsto (r\,,\, y\,,\, \mu). \end{aligned}$$

Where, (r, y, μ) is the modulation of $u(T_n)$ around $R(T_n)$ and $B_{H_0^1}(\varepsilon)$ is a ball of radius $\varepsilon > 0$ which is defined in the proof of the Lemma 3.2.

$$\begin{aligned} \Lambda : H_0^1 \times \mathbb{R}^3 \times \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ (r, y, \mu) &\longmapsto \left(\alpha^+(T_n) = \operatorname{Im}\int \widetilde{Y}_-(T_n, x)\,\overline{r}(T_n, x)dx\,,\ \alpha^-(T_n) = \operatorname{Im}\int \widetilde{Y}_+(T_n, x)\,\overline{r}(T_n, x)dx\right). \end{aligned}$$

We have, $\sigma(0) = 0$, $\Gamma(0) = (0,0,0)$, $\Lambda(0,0,0) = (0,0)$.
Denote: $\Theta = \Lambda \circ \Gamma \circ \sigma$.
Now let us prove that Θ is a diffeomorphism on a $\mathcal{V}_0$ a neighbourhood of $0 \in \mathbb{R}^2$ by computing $d\Theta = d\Lambda \circ d\Gamma \circ d\sigma$.

Firstly, we have that $d\sigma(\lambda) = \sigma$, for all $\lambda \in \mathbb{R}^2$. Secondly, let $l \in H_0^1, z \in \mathbb{R}^3, q \in \mathbb{R}$ such that

$$d\Lambda(r,y,\mu).(l,z,q) = \Big(\operatorname{Im} \int \widetilde{Y}_-(x)\,\bar{l}(x) - \sum_{j=1}^{3} z_j \partial_{x_j} \widetilde{Y}_-(x)\,\overline{r}(x) + iq\widetilde{Y}_-(x)\,\overline{r}(x)\,dx\;,$$
$$\operatorname{Im} \int \widetilde{Y}_+(x)\,\bar{l}(x) - \sum_{j=1}^{3} z_j \partial_{x_j} \widetilde{Y}_+(x)\,\overline{r}(x) + iq\widetilde{Y}_+(x)\,\overline{r}(x)\,dx \Big).$$

Finally, we have to compute $d\,\Gamma$. Let Φ and g defined as in the proof of the Lemma 3.2 for $R(t_n)$. Then, we obtain

$$\Gamma(\rho) = \left(\rho + R(T_n) - \widetilde{R}(T_n, \cdot - y(\rho)), y(\rho), \mu(\rho)\right).$$
$$d\Gamma(\rho).l = \left(l + \nabla R(T_n, \cdot - y(\rho))e^{i\mu(\rho)} dy(\rho).l + iR(\cdot - y(\rho))e^{i\mu(\rho)} d\mu(\rho).l\;,\; dy(\rho)\;,\; d\mu(\rho)\right). \tag{II.101}$$

we have

$$\Phi(\rho, y(\rho), \mu(\rho)) = 0 \Longrightarrow \begin{cases} \Phi_1(\rho, y(\rho), \mu(\rho)) & = 0 \\ \Phi_2(\rho, y(\rho), \mu(\rho)) & = 0 \end{cases} \Longrightarrow \begin{cases} d_1\Phi_1 + d_2\Phi_1\, dy(\rho) + d_3\Phi_1\, d\mu(\rho) = 0 \\ d_1\Phi_2 + d_2\Phi_2\, dy(\rho) + d_3\Phi_2\, d\mu(\rho) = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} dy(\rho) = (d_2\Phi_1)^{-1}\left[-(d_1\Phi_1) - (d_3\Phi_1)d\mu(\rho)\right] \\ d\mu(\rho) = (d_3\Phi_2)^{-1}(d_2\Phi_2)(d_2\Phi_1)^{-1}(d_1\Phi_1) - (d_3\Phi_2)^{-1}(d_1\Phi_2) - (d_3\Phi_1)^{-1}(d_1\Phi_1) \\ \qquad\qquad + (d_3\Phi_1)^{-1}(d_2\Phi_1)(d_2\Phi_2)^{-1}(d_1\Phi_2). \end{cases} \tag{II.102}$$

Recall that

$$d\Theta(\lambda).\tilde{\lambda} = d\Lambda(d\,\Gamma(\sigma(\lambda)))\,.\,d\,\Gamma(\sigma(\lambda))\,.\,\sigma(\tilde{\lambda}),$$

by (II.101) we get

$$d\,\Gamma(\sigma(\lambda)).\sigma(\tilde{\lambda}) = \Big(\sigma(\tilde{\lambda}) + \nabla R(T_n, \cdot - y(\sigma(\lambda))e^{i\mu(\sigma(\lambda))} dy(\sigma(\lambda)).\sigma(\tilde{\lambda})$$
$$+\, i\, R(T_n, \cdot - y(\sigma(\lambda)))e^{i\mu(\sigma(\lambda))} d\mu(\sigma(\lambda)).\sigma(\tilde{\lambda})\;,\; dy(\sigma(\lambda)).\sigma(\tilde{\lambda})\;,\; d\mu(\sigma(\lambda)).\sigma(\tilde{\lambda})\Big).$$

We claim the following estimate which will be proved at the end of this proof.

Claim 5.3. *Let $\delta > 0$ such that*

$$dy(\sigma(\lambda)).\sigma(\tilde{\lambda}) = O\left((e^{-\delta\sqrt{\omega}|v|T_n} + |\lambda|)|\tilde{\lambda}|\right)$$
$$d\mu(\sigma(\lambda)).\sigma(\tilde{\lambda}) = O\left((e^{-\delta\sqrt{\omega}|v|T_n} + |\lambda|)|\tilde{\lambda}|\right).$$

By the claim above we have

$$d\Gamma(\sigma(\lambda)).\sigma(\tilde{\lambda}) = \left(\sigma(\tilde{\lambda})\,,0\,,0\right) + O\left((e^{-\delta\sqrt{\omega}|v|T_n} + |\lambda|)|\tilde{\lambda}|\right).$$

Using the expression of $d\Lambda$, we get

$$d\Theta(\lambda).\tilde{\lambda} = d\Lambda(\sigma(\lambda)).(\sigma(\tilde{\lambda}), 0, 0) + O\left((e^{-\delta\sqrt{\omega}|v|T_n} + |\lambda|)|\tilde{\lambda}|\right),$$
$$d\Theta(\lambda) = \mathcal{M} + O\left(e^{-\delta\sqrt{\omega}|v|T_n} + |\lambda|\right).$$

Where $\mathcal{M}$ is a matrix such that

$$\mathcal{M} = \begin{pmatrix} \mathrm{Re}\int \widetilde{Y}_-(T_n,x)\overline{Y}_+(T_n,x)dx & \mathrm{Re}\int \widetilde{Y}_-(T_n,x)\overline{Y}_-(T_n,x)dx \\ \mathrm{Re}\int \widetilde{Y}_+(T_n,x)\overline{Y}_+(T_n,x)dx & \mathrm{Re}\int \widetilde{Y}_+(T_n,x)\overline{Y}_-(T_n,x)dx \end{pmatrix}$$

Since $\mathcal{Y}_+$ and $\mathcal{Y}_-$ are linearly independent, then the following matrix is invertible

$$\mathcal{A} = \begin{pmatrix} \mathrm{Re}\int Y_-(T_n,x)\overline{Y}_+(T_n,x)dx & \mathrm{Re}\int Y_-(T_n,x)\overline{Y}_-(T_n,x)dx \\ \mathrm{Re}\int Y_+(T_n,x)\overline{Y}_+(T_n,x)dx & \mathrm{Re}\int Y_+(T_n,x)\overline{Y}_-(T_n,x)dx \end{pmatrix}$$

And we have

$$\left|\mathrm{Re}\int \left(\widetilde{Y}_-(T_n,x) - Y_-(T_n,x)\right)\overline{Y}_+(T_n,x)\,dx\right| \leq C\,|y| \leq C\,|\lambda|.$$

We deduce that $\mathcal{M}$ is invertible, thus $d\Theta$ is invertible on a some ball $B_{\mathbb{R}^2}(\beta)$. This implies that Θ is a diffeomorphism from the ball $B_{\mathbb{R}^2}(\beta)$ ($\beta > 0$ independent of n for n large enough) to some neighborhood $\mathcal{U}$ of $0 \in \mathbb{R}^2$.

Let $\eta > 0$ be such that $B_{\mathbb{R}^2}(\eta) \subset \mathcal{U}$. Then, for any $\alpha^+ \in B_{\mathbb{R}}(\eta)$, there exist a unique $\lambda = \lambda(\alpha^+) \in B_{\mathbb{R}^2}(\beta)$ such that

$$\Theta(\lambda(\alpha^+)) = (\alpha^+\,,\,0) \quad \textit{and} \quad \left|\lambda(\alpha^+)\right| \leq C\left|\alpha^+\right|.$$

And this concludes the proof of Lemma 3.4.

Proof of Claim 5.3. From (II.102), we have

$$
\begin{aligned}
dy(\rho) &= (d_2\Phi_1)^{-1}\left[-(d_1\Phi_1) - (d_3\Phi_1)d\mu(\rho)\right],\\
d\mu(\rho) &= (d_3\Phi_2)^{-1}(d_2\Phi_2)(d_2\Phi_1)^{-1}(d_1\Phi_1) - (d_3\Phi_2)^{-1}(d_1\Phi_2) - (d_3\Phi_1)^{-1}(d_1\Phi_1)\\
&\quad - (d_3\Phi_1)^{-1}(d_2\Phi_1)(d_2\Phi_2)^{-1}(d_1\Phi_2).
\end{aligned}
$$

Remark that it suffices to prove that

$$
\begin{aligned}
d_1\Phi_1.\sigma(\tilde\lambda) &= O\left(|\lambda\,\tilde\lambda|\right)\\
d_1\Phi_2.\sigma(\tilde\lambda) &= O\left((e^{-\delta\sqrt{\omega}|v|T_n} + |\lambda|)|\tilde\lambda|\right).
\end{aligned}
$$

Letting $l \in H_0^1$, we have

$$
\begin{aligned}
d_1\Phi_1(\rho, y, \mu).l &= \mathrm{Re}\int l(x)\,\nabla\widetilde{Q}_\omega(T_n, x)\Psi(x)e^{-i\widetilde{\varphi}(T_n,x)}dx,\\
d_1\Phi_2(\rho, y, \mu).l &= \mathrm{Im}\int l(x)\,\overline{\widetilde{R}}(T_n, x)dx.
\end{aligned}
$$

Recall that $\sigma(\tilde\lambda) = i\left(\tilde\lambda^+ Y_+(T_n, x) + \tilde\lambda^- Y_-(T_n, x)\right)$.

$$
\begin{aligned}
d_1\Phi_1.\sigma(\tilde\lambda) &= \mathrm{Re}\int i\left(\tilde\lambda^+ Y_+ + \tilde\lambda^- Y_-\right)\nabla\widetilde{Q}_\omega\Psi e^{-i\widetilde{\varphi}}dx\\
&= \mathrm{Im}\Bigg[e^{-i\mu}\tilde\lambda^+\underbrace{\int \mathcal{Y}_\omega^+\,\nabla\widetilde{Q}_\omega\Psi dx}_{\mathrm{I}_1} + e^{-i\mu}\tilde\lambda^-\underbrace{\int \mathcal{Y}_\omega^-\,\nabla\widetilde{Q}_\omega\Psi dx}_{\mathrm{I}_2}\Bigg].
\end{aligned}
$$

$$
\mathrm{I}_1 + \mathrm{I}_2 = \int \mathcal{Y}_\omega^+\,\nabla Q_\omega\Psi dx + \int \mathcal{Y}_\omega^-\,\nabla Q_\omega\Psi dx + O(|y|).
$$

Since $\mathcal{Y}_\omega^\pm$ and Q_ω are radial, we have

$$
\int \mathcal{Y}_\omega^\pm\,\nabla Q_\omega\Psi\,dx \le \int \mathcal{Y}_\omega^\pm\,\nabla Q_\omega\,dx = 0,
$$

and using $|y| \le |\lambda|$ we get

$$d_1\Phi_1.\sigma(\tilde{\lambda}) = O\left(|\lambda\tilde{\lambda}|\right).$$

Denote $y_1 = \mathrm{Re}\left(\mathcal{Y}^+_\omega\right) = \mathrm{Re}\left(\mathcal{Y}^-_\omega\right)$ and $y_2 = \mathrm{Im}\left(\mathcal{Y}^+_\omega\right) = -\,\mathrm{Im}\left(\mathcal{Y}^-_\omega\right)$.
Recall that $\mathcal{L}_\omega \mathcal{Y}^\pm_\omega = \pm e_\omega \mathcal{Y}^\pm_\omega$.

$$\begin{aligned} d_1\Phi_2.\sigma(\tilde{\lambda}) &= \mathrm{Im}\int i\left(\tilde{\lambda}^+ Y_+ + \tilde{\lambda}^- Y_-\right)\widetilde{Q}_\omega \Psi\, e^{-i\widetilde{\varphi}} dx \\ &= \mathrm{Re}\left[e^{-i\mu}\tilde{\lambda}^+ \underbrace{\int \mathcal{Y}^+_\omega\, \widetilde{Q}\,\Psi\, dx}_{\mathrm{J}_1} + e^{-i\mu}\tilde{\lambda}^- \underbrace{\int -\mathcal{Y}^-_\omega\,\widetilde{Q}\,\Psi\, dx}_{\mathrm{J}_2}\right]. \end{aligned}$$

$$\begin{aligned} \mathrm{J}_1 + \mathrm{J}_2 &= \int \left(-L^-_\omega y_2 + i\, L^+_\omega y_1\right)\widetilde{Q}_\omega \Psi dx + \int -\left(L^-_\omega y_2 + i\, L^+_\omega y_1\right)\widetilde{Q}_\omega \Psi dx \\ &= -2\, i\int L^-_\omega y_2\, (\widetilde{Q}_\omega \Psi)\, dx. \end{aligned}$$

Since L^-_ω is a self-adjoint operator.

$$\mathrm{J}_1 + \mathrm{J}_2 = -2\, i\int y_2\, L^-_\omega (Q_\omega \Psi)\, dx + O(|y|).$$

Using the fact that $\partial_{x_j}\Psi$ has a compact support, $L^-_\omega(Q_\omega) = 0$ and $|y| \leq |\lambda|$ we get

$$d_1\Phi_2.\sigma(\tilde{\lambda}) = O\left((e^{-\delta\sqrt{\omega}|v|T_n} + |\lambda|)|\tilde{\lambda}|\right).$$

This concludes the proof of the Claim 5.3.

□

6 Computation of some estimates

Proof of Claim 4.3. Using (II.76) and the compact support of $\nabla^k\Psi$, we obtain the first estimate.
Let us prove the second inequality.

Notice that $F : z \longmapsto |z|^2 z = z^2\bar{z}$ is differentiable on $\mathbb{C}$ and

$$\frac{dF}{dz}(z) = 2\left|z\right|^2, \qquad \frac{dF}{(dz)^2}(z) = 2\bar{z}, \qquad \frac{dF}{d\bar{z}}(z) = z^2, \qquad \frac{dF}{(d\bar{z})^2}(z) = 0$$
$$\frac{d^2F}{dzd\bar{z}}(z) = \frac{d^2F}{d\bar{z}dz}(z) = 0.$$

Since $x \longmapsto H(t,x)$ is smooth. Then we have,

$$\begin{aligned}\nabla\left(\left|H(t,x)\right|^2 H(t,x)\right) &= \nabla H(t,x)\nabla_z F(H(t,x)) + \nabla\overline{H}(t,x)\nabla_{\bar{z}} F(H(t,x))\\ \nabla^2\left(\left|H(t,x)\right|^2 H(t,x)\right) &= \nabla^2 H(t,x)\nabla_z F(H(t,x)) + \nabla^1 H(t,x)\nabla^1 H(t,x)\nabla_{zz} F(H(t,x))\\ &\quad + \nabla^2\overline{H}(t,x)\nabla_{\bar{z}} F(H(t,x)),\end{aligned}$$

where, $\nabla f = \left(\partial_{x_i} f\right)_i,\ i = 1,2,3.$
Using again the fact that $\nabla^k\Psi$ has a compact support and the exponential decay of Q_ω to conclude the proof. □

Chapter III

On blow-up solutions to the nonlinear Schrödinger equation on the exterior of the unit ball

Abstract: In this chapter, we consider the Schrödinger equation with a mass-supercritical focusing nonlinearity, in the exterior of the unit ball of $\mathbb{R}^d$ with Dirichlet boundary conditions. We prove that solutions with negative energy blow up in finite time. Assuming furthermore that the nonlinearity is energy-subcritical, we also prove (under additional symmetry conditions) blow-up with the same optimal ground-state criterion as in the work of Holmer and Roudenko [44] on $\mathbb{R}^d$. The classical proof of Glassey, based on the concavity of the variance, fails in the exterior of an obstacle because of the appearance of the boundary terms with an unfavorable sign in the second derivative of the variance. The main idea of our proof is to introduce a new modified variance, which is bounded from below and strictly concave for the solutions that we consider.

1 Introduction

We consider the focusing nonlinear Schrödinger equation with Dirichlet boundary conditions.

$$\begin{cases} i\partial_t u + \Delta_\Omega u = -|u|^{p-1}u & (t,x) \in \mathbb{R} \times \Omega, \\ u(t_0,x) = u_0(x) & x \in \Omega, \\ u(t,x) = 0 & (t,x) \in \mathbb{R} \times \partial\Omega, \end{cases} \qquad (\text{NLS}_\Omega)$$

where $\Omega = \mathbb{R}^d \setminus \Theta$ is the exterior of the unit ball on $\mathbb{R}^d$.

Recall from the introduction that, the local well-posedness for the (NLS_Ω) equation in the exterior of a smooth, compact, convex domain was studied in several articles and it is now well understood in many cases. Local existence and uniqueness are usually proved by contraction mapping methods via Strichartz estimates. In [48], O. Ivanovici proved the Strichartz estimates for (NLS_Ω) except the end point case, using the Melrose and Taylor parametrix, see also [10], [15], [57]. The local well-posedness in $H_0^1(\Omega)$, for $1 < p < 5$ in dimension $d = 3$, for (NLS_Ω) equation in the exterior of a non-trapping domain was obtained by L. Vega and F. Planchon in [78]. F. Planchon and O. Ivanovici [50] extended the result to the quintic Schrödinger equation outside a non-trapping domain, see also [1] for $d = 2$, [58, 94] for global existence for $d = 3$ and [59] for the defocusing case (Cf. Proposition A below for a precise local well posedness statement needed for our purpose).

In [58], the authors proved global existence and scattering of solutions for the focusing 3d cubic (NLS_Ω) equation, whenever the initial data satisfies a smallness criterion given by the ground state threshold, see also [94] for $\frac{7}{3} < p < 5, d = 3$. The criterion is expressed in terms of the scale-invariant quantities $\|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)}$ and $M_\Omega[u]E_\Omega[u]$. Moreover, in [21] the authors revisited the proof of scattering using Dodson and Murphy's approach [27], [28] and the dispersive estimate established in [49]. In [63], we construction a solitary wave solution for (NLS_Ω) behaving asymptotically as a soliton on $\mathbb{R}^3$, as large time. This solution is global, does not scatter and prove the optimality of the threshold for scattering given above.

All results mentioned above for (NLS_Ω) concern global solutions but the existence of blow-up solutions is still an open question which is the purpose of this paper. Before stating our blow-up results, let us recall the proof of the classical blow-up criterion of Vlasov-Petrishev-Talanov [83], Zakharov [88] and Glassey [38] which states that finite variance and negative energy solution break down in finite time. This proof is a convexity argument on the variance $V(t)$ defined as the following,

$$V(t) := V(u(t)) = \int_{\mathbb{R}^d} |x|^2 |u(t,x)|^2 dx.$$

Assuming $V(0) < \infty$, the following virial identity holds:

$$\frac{1}{16}\frac{d^2}{dt^2}V(t) = E_{\mathbb{R}^d}[u] - \frac{1}{2}\left(\frac{d}{2} - \frac{d+2}{p+1}\right)\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^d)},$$

If $p > 1 + \frac{d}{4}$ and $E_{\mathbb{R}^d}[u] < 0$ then u blows up in finite time. As proved in [44], in the energy subcritical case ($d \leq 2$ or $p < 1 + \frac{4}{d-2}$), the assumption $E_{\mathbb{R}^d}[u] < 0$ can be weakened to a condition on the initial data which can be formulated in term of the ground state (see Theorem 1.4 below).

This proof does not adapt directly to the case of an exterior domain because the boundary term in the virial identity does not have a favorable sign,

$$\frac{1}{16}\frac{d^2}{dt^2}V(u(t)) = E_\Omega[u] - \frac{1}{2}\left(\frac{d}{2} - \frac{d+2}{p+1}\right)\int_\Omega |u|^{p+1}\,dx - \frac{1}{4}\int_{\partial\Omega} |\nabla u|^2\,(x.\vec{n})\,d\sigma(x),$$

where $\vec{n}$ is the unit outward normal vector. One can see that $x.\vec{n} \leq 0$ on $\partial\Omega$. For that we will define a new shifted variance $\mathcal{V}(t)$ which allows us to control the boundary term and to prove the existence of blow-up solution for (NLS_Ω) equation. In the energy subcritical case, with an additional symmetry assumption on the initial data, we prove blow-up with the sufficient condition obtained in [44] on the Euclidean space.
Before stating our main results, we recall the needed local well-posedness property for (NLS_Ω) equation posed outside a convex obstacle.

Proposition A. *Assume $p \geq 3$ if $d = 2$ and $1 + \frac{4}{d} \leq p < \frac{d+2}{d-2}$ if $d \geq 3$. Let $u_0 \in H^1_0(\Omega)$ then there exists $T > 0$ and a unique solution $u(t)$ of* (NLS_Ω) *equation with $u \in C([-T,T], H^1_0(\Omega))$ Assume $d = 3$ and $p > 2$. Let $u_0 \in H^2 \cap H^1_0(\Omega)$ then there exists $T > 0$ and a unique solution $u(t)$ of* (NLS_Ω) *equation with $u \in C([-T,T], H^2 \cap H^1_0(\Omega))$.*

The local existence and uniqueness in $H^1_0(\Omega)$ is carried out by classical methods, using fixed point argument via Strichartz estimates. The proof is very similar to the one for (NLS) equation posed on the whole Euclidean space. Moreover, the local existence of solutions for (NLS_Ω) equation in $H^2 \cap H^1_0(\Omega)$ can be established using the fact that H^2 is an algebra and the following continuous embedding for any smooth domain $\Omega \subset \mathbb{R}^3$, $H^2(\Omega) \subset L^\infty(\Omega)$, see [15, Proposition

2.1]. Thus we don't have to control the nonlinearity growth but we just need regularity of the nonlinear term.

It is classical that the solution u can be extended to a maximal time interval $I = (-T_-, T_+)$ of existence and the following alternative hold:
Either $T_+ = \infty$ (respectively $T_- = \infty$) or $T_+ < \infty$ (respectively $T_- < \infty$) and

$$\lim_{t\to T_+} \|u(t,\cdot)\|_{H_0^1(\Omega)} = \infty, \quad \text{respectively} \lim_{t\to T_-} \|u(t,\cdot)\|_{H_0^1(\Omega)} = \infty.$$

Now we state our main results.

Theorem 1.1. *Assume $\Theta = B(0,1)$ and $p \geq 5$.*

- *for $d = 2$, let $u_0 \in H_0^1(\Omega)$ such that $E[u_0] + \frac{1}{8}M[u_0] < 0$ and $|x|u_0 \in L^2(\Omega)$,*
- *for $d = 3$, let $u_0 \in H^2 \cap H_0^1(\Omega)$ such that $E[u_0] < 0$ and $|x|u_0 \in L^2(\Omega)$,*

and let u be the corresponding solution of (NLS_Ω) *with maximal time interval I of existence then the length of I is finite and the solution u blows up in finite time.*

Theorem 1.2. *Assume $\Theta = B(0,1)$ and $p \geq 1 + \frac{4}{d}$.*

- *for $d = 2$, let $u_0 \in H_0^1(\Omega)$ such that $u_0(-x_1, x_2) = u_0(x_1, -x_2) = -u_0(x_1, x_2)$,*
- *for $d = 3$, let $u_0 \in H^2 \cap H_0^1(\Omega)$ such that $u_0(-x_1, x_2, x_3) = u_0(x_1, -x_2, x_3) = u_0(x_1, x_2, -x_3) = -u_0(x_1, x_2, x_3)$,*

and let u be the corresponding solution of (NLS_Ω) *with maximal time interval I of existence. If $E[u] < 0$ and $|x|u_0 \in L^2(\Omega)$, then the length of I is finite and thus the solution u blows up in finite time.*

Remark 1.3. *The Theorems 1.1 and 1.2 remain true for* (NLS_Ω) *outside a ball of radius $r > 1$ and centred at any point x_0. One would have to use a symmetry around x_0 instead of the origin. We can also generalized these theorems for any dimension $d \geq 4$, whenever an appropriate well-posedness theory is available. In dimension $d \geq 4$, for Theorem 1.2 we should suppose d symmetries,*

$$u_0(x_1, .., -x_i, .., x_d) = -u_0(x_1, .., x_i, .., x_d), \quad \text{for } i = 1, 2, ..., d.$$

More details are given in sections 4 and 5.

Now we introduce the concept of ground state. Let Q be the solution of the following nonlinear elliptic equation

$$-Q+\Delta Q+|Q|^{p-1}Q=0, \quad Q=Q(x), \quad x\in\mathbb{R}^d. \tag{III.1}$$

This nonlinear equation has infinite number of solutions in $H^1(\mathbb{R}^d)$. Among these there is exactly one solution of minimal mass up to scaling, space translation and phase shift, called the ground state solution. It is real-valued, positive, radial, smooth and exponentially decaying, see [60]. We henceforth denote by Q, this ground state solution.

Theorem 1.4. *Assume* $\Theta=B(0,1)$ *and* $s_c=\frac{d}{2}-\frac{2}{p-1}$. *Let* $u_0\in H_0^1(\Omega)$ *and let* u *be the corresponding solution of* (NLS_Ω) *with maximal time interval* I *of existence such that*

- *for* $d=2$, $u_0(-x_1,x_2)=u_0(x_1,-x_2)=-u_0(x_1,x_2)$ *and* $s_c>0$, *i.e.,* $p>3$.
- *for* $d\geq 3$, $u_0(x_1,..,-x_i,..,x_d)=-u_0(x_1,..,x_i,..,x_d)$, *for* $i=1,2,...,d$. *and* $0<s_c<1$, *i.e.,* $1+\frac{4}{d}<p<\frac{d+2}{d-2}$.

Suppose that,

$$M_\Omega[u_0]^{\frac{1-s_c}{s_c}}E_\Omega[u_0]<M_{\mathbb{R}^d}[Q]^{\frac{1-s_c}{s_c}}E_{\mathbb{R}^d}[Q]. \tag{III.2}$$

If (III.2) *holds and*

$$\|u_0\|^{1-s_c}_{L^2(\Omega)}\|\nabla u_0\|^{s_c}_{L^2(\Omega)}>\|Q\|^{1-s_c}_{L^2(\mathbb{R}^d)}\|\nabla Q\|^{s_c}_{L^2(\mathbb{R}^d)}. \tag{III.3}$$

Then for $t\in I$,

$$\|u_0\|^{1-s_c}_{L^2(\Omega)}\|\nabla u(t)\|^{s_c}_{L^2(\Omega)}>\|Q\|^{1-s_c}_{L^2(\mathbb{R}^d)}\|\nabla Q\|^{s_c}_{L^2(\mathbb{R}^d)}. \tag{III.4}$$

Furthermore, if $|x|u_0\in L^2(\Omega)$ *then the length of* I *is finite and thus the solution blows up in finite time.*

Let us mention that in the L^2-critical case we can find an almost explicit blow-up solution for (NLS_Ω) equation using pseudo-conformal transformation. In this case, we can construct a blow-up solution for (NLS_Ω) equation by adapting the argument of N. Burq, P.Gérard and N. Tzvetkov in [14], for (NLS) equation inside a domain.

Assume $p=1+\frac{4}{d}$. Let Ψ be a $\mathcal{C}^\infty$-function such that $\Psi=0$ near Θ and $\Psi=1$ for $|x|>>1$ and let Q be any solution of the nonlinear elliptic equation (III.1), (it does not have to be the ground state) then there exists $T>0$ and a smooth function $r(t,x)$ defined on $[0,T)\times\Omega$ and exponentially decaying as $t \to T$ such that

$$u(t,x) := \frac{1}{(T-t)} Q\left(\frac{x-x_0}{(T-t)}\right)\Psi(x)e^{i(\frac{4-(x-x_0)^2}{4(T-t)})} + r(t,x)$$

is solution for (NLS_Ω) satisfying the Dirichlet boundary conditions, which blow-up in finite time T. The proof is similar to one given in [14] for (NLS) equation inside a domain in $\mathbb{R}^2$. We need to construct the smooth correction r such that u is a solution of (NLS_Ω) satisfying Dirichlet boundary conditions. To achieve this, one define a contraction mapping using the Duhamel formula on the complete metric space (E, d) such that

$$E := \{f \in \mathcal{C}([0,T), H^2(\Omega) \cap H_0^1(\Omega)); \|f\|_E < \infty\},$$

equipped with the norm

$$\|f\|_E := \sup_{t\in[0,T)} \{e^{\frac{1}{2(T-t)}} \|f\|_{L^2(\Omega)} + e^{\frac{1}{3(T-t)}} \|f\|_{H^2(\Omega)}\}.$$

The existence of the smooth correction r follows from the fixed point theorem.

We give a review of some properties related to the ground state Q, in section 2 and we prove Pohozaev's identities outside an obstacle in section 3, In section 4, we prove the existence of blow-up solution for $p \geq 5$ using a convexity argument on the modified variance. In section 5, we study the existence of symmetric blow-up solution for $p > 1 + \frac{4}{d}$ using a different variance. Finally, in section 6, we study the behavior of the solutions, in particular, the blow-up criteria for the solutions with initial data beyond the ground state threshold.

2 Properties of the Ground State

Weinstein [90] proved that the sharp constant C_{GN} in the Gagliardo-Nirenberg estimate

$$\|f\|_{L^{p+1}}^{p+1} \leq C_{GN} \|\nabla f\|_{L^2}^{\frac{d(p-1)}{2}} \|f\|_{L^2}^{2-\frac{(d-2)(p-1)}{2}} \tag{III.5}$$

is attained at the function Q (the ground state described in the introduction), i.e.,

$$C_{GN} = \frac{\|Q\|_{L^{p+1}(\mathbb{R}^d)}^{p+1}}{\|\nabla Q\|_{L^2(\mathbb{R}^d)}^{\frac{d(p-1)}{2}} \|Q\|_{L^2(\mathbb{R}^d)}^{2-\frac{(d-2)(p-1)}{2}}}.$$

Multiplying (III.1) by Q and integrating by parts, we obtain

$$-\|Q\|^2_{L^2(\mathbb{R}^d)} - \|\nabla Q\|_{L^2(\mathbb{R}^d)} + \|Q\|^{p+1}_{L^{p+1}(\mathbb{R}^d)} = 0. \tag{III.6}$$

Multiplying (III.1) by $x.\nabla Q$ and integrating by parts, we obtain the following identity

$$\frac{d}{2}\|Q\|^2_{L^2(\mathbb{R}^d)} + \frac{d-2}{2}\|\nabla Q\|^2_{L^2(\mathbb{R}^d)} - \frac{d}{p+1}\|Q\|^{p+1}_{L^{p+1}(\mathbb{R}^d)} = 0. \tag{III.7}$$

These two identities (III.6) and (III.7) enable us to obtain these relations

$$\|\nabla Q\|^2_{L^2(\mathbb{R}^d)} = \frac{d(p-1)}{(d+2)-p(d-2)}\|Q\|^2_{L^2(\mathbb{R}^d)}$$
$$\|Q\|^{p+1}_{L^{p+1}(\mathbb{R}^d)} = \frac{2(p+1)}{d(p-1)}\|\nabla Q\|^2_{L^2(\mathbb{R}^d)}$$

and thus, reexpress

$$C_{GN} = \left(\frac{2(p+1)}{d(p-1)}\|\nabla Q\|_{L^2(\mathbb{R}^d)}\|Q\|^{\frac{4-(d-2)(p-1)}{d(p-1)-4}}_{L^2(\mathbb{R}^d)}\right)^{-\frac{d(p-1)-4}{2}} \tag{III.8}$$

We also compute

$$E_{\mathbb{R}^d}[Q] := \frac{1}{2}\|\nabla Q\|^2_{L^2(\mathbb{R}^d)} - \frac{1}{p+1}\|Q\|^{p+1}_{L^{p+1}(\mathbb{R}^d)} = \frac{d(p-1)-4}{2d(p-1)}\|\nabla Q\|^2_{L^2(\mathbb{R}^d)}. \tag{III.9}$$

3 Pohozaev's identities outside an obstacle

This section is devoted to the proof of the Pohozaev's Identity in exterior domain. In the following Proposition Ω can be the exterior of any regular obstacle.

Proposition 3.1 (Pohozaev's identity)**.** *Let $u \in H^2 \cap H^1_0(\Omega)$, $|x|\, u \in L^2(\Omega)$ then we have,*

$$\operatorname{Re}\int_\Omega \Delta\overline{u}\,(\frac{d}{2}u + x.\nabla u)\,dx = -\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2\ (x.\vec{n})\,d\sigma(x). \tag{III.10}$$

$$\operatorname{Re}\int_\Omega \Delta\overline{u}\left(\nabla u.\frac{x}{|x|} + (\frac{d-1}{2})\frac{u}{|x|}\right)dx = -\frac{(d-1)(d-3)}{4}\int_\Omega \frac{|u|^2}{|x|^3}\,dx - \int_\Omega \frac{|\nabla u|^2}{|x|}\,dx + \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2\frac{x.\vec{n}}{|x|}\,d\sigma(x), \tag{III.11}$$

where $|\not\nabla u|^2 := |\nabla u|^2 - \left|\frac{x}{|x|}.\nabla u\right|^2$ *and* $\vec{n}$ *is the outward unit normal vector.*

Proof. Using integration by parts and the fact that u satisfies Dirichlet boundary condition (i.e., $u = 0$ on $\partial\Omega$), we obtain

$$\begin{aligned}
\operatorname{Re}\int_\Omega \Delta\overline{u}\,(x.\nabla u)\,dx &= -\sum_{j=1}^d \operatorname{Re}\int_\Omega \partial_{x_j}\overline{u}\,\partial_{x_j}u\,dx - \sum_{j,k=1}^d \operatorname{Re}\int_\Omega x_k\,\partial_{x_j}\partial_{x_k}u\,\partial_{x_j}\overline{u}\,dx \\
&+ \int_{\partial\Omega} |\nabla u|^2\ (x.\vec{n})\,d\sigma(x) \\
&= -\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\sum_{k=1}^d \int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\sum_{k=1}^d \int_{\partial\Omega} |\nabla u|^2\,x_k n_k\,d\sigma(x) \\
&+ \int_{\partial\Omega} |\nabla u|^2\ (x.\vec{n})\,d\sigma(x) \\
&= -\int_\Omega |\nabla u|^2\,dx - \frac{d}{2}\operatorname{Re}\int_\Omega \Delta\overline{u}\,u\,dx + \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2\ (x.\vec{n})\,d\sigma(x).
\end{aligned}$$

This conclude the proof of (III.10). Now let us prove (III.11), using the same argument as above and the fact that,

$$\partial_{x_k}\left(\frac{x_j}{|x|}\right) = \begin{cases} \frac{1}{|x|} - \frac{x_j^2}{|x|^3} & \text{if } j = k, \\ \\ -\frac{x_j\,x_k}{|x|^3} & \text{if } j \neq k, \end{cases}$$

we obtain

$$\begin{aligned}
\operatorname{Re}\int_\Omega \Delta\overline{u}\ \nabla u.\frac{x}{|x|}\,dx &= -\frac{1}{2}\left[\sum_{j,k=1}^d \int_\Omega \partial_{x_k}\overline{u}\ \partial_{x_k}\partial_{x_j}u\ \frac{x_j}{|x|} + \partial_{x_k}u\ \partial_{x_k}\partial_{x_j}\overline{u}\ \frac{x_j}{|x|}\,dx\right] \\
&- \operatorname{Re}\left[\sum_{j=1}^d \int_\Omega \partial_{x_j}\overline{u}\ \partial_{x_j}u\left(\frac{1}{|x|} - \frac{x_j^2}{|x|^3}\right)dx\right] + \operatorname{Re}\left[\sum_{\substack{j,k=1\\ j\neq k}}^d \int_\Omega \partial_{x_k}\overline{u}\ \partial_{x_j}u\ \frac{x_j\,x_k}{|x|^3}\,dx\right] \\
&+ \int_{\partial\Omega} |\nabla u|^2\,\frac{(x.\vec{n})}{|x|}d\sigma(x) \\
&= -\frac{1}{2}\left[\sum_{j=1}^d \int_\Omega \frac{x_j}{|x|}\partial_{x_j}\left(|\nabla u|^2\right)\,dx\right] - \int_\Omega \frac{|\nabla u|^2}{|x|}\,dx + \int_\Omega \left|\frac{x}{|x|}.\nabla u\right|^2 \frac{1}{|x|}\,dx \\
&+ \int_{\partial\Omega} |\nabla u|^2\,\frac{(x.\vec{n})}{|x|}d\sigma(x) \\
&= \left(\frac{d-1}{2}\right)\int_\Omega \frac{|\nabla u|^2}{|x|}\,dx - \int_\Omega \frac{|\not\nabla u|^2}{|x|}\,dx + \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2\,\frac{(x.\vec{n})}{|x|}d\sigma(x)
\end{aligned}$$

$$\operatorname{Re}\int_\Omega \Delta\overline{u}\,\nabla u.\frac{x}{|x|}\,dx = \left(\frac{d-1}{2}\right)\operatorname{Re}\int_\Omega -\Delta\overline{u}\,u\,\frac{1}{|x|}\,dx + \left(\frac{d-1}{2}\right)\operatorname{Re}\sum_{k=1}^{d}\int_\Omega \partial_{x_k}\overline{u}\,u\,\frac{x_k}{|x|^3}\,dx$$
$$-\int_\Omega \frac{|\nabla u|^2}{|x|}\,dx + \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2\,\frac{(x.\vec{n})}{|x|}d\sigma(x).$$

Using the fact that,

$$\left(\frac{d-1}{2}\right)\operatorname{Re}\sum_{k=1}^{d}\int_\Omega \partial_{x_k}\overline{u}\,u\,\frac{x_k}{|x|^3}\,dx = -\frac{(d-1)(d-3)}{4}\int_\Omega \frac{|u|^2}{|x|^3}\,dx,$$

we obtain (III.11), which concludes the proof of Proposition 3.1. □

4 Existence of blow-up solution

This section is devoted to the proof of Theorem 1.1. We assume $d \in \{2,3\}$. Nevertheless, the computations below still valid for $d \geq 4$ if an appropriate Cauchy theory is available.

Denote:

$$\Upsilon_1(u(t)) := \int_\Omega |x|\,|u(t,x)|^2\,dx, \qquad \Upsilon_2(u(t)) := \int_\Omega |x|^2\,|u(t,x)|^2\,dx.$$

We will start by proving the following virial identities in the exterior of a convex obstacle, in particular in the exterior of a ball which is needed in the proof of Theorem 1.1.

Proposition 4.1. *Assume that Ω is the exterior of a convex obstacle. Let $u_0 \in H^2 \cap H^1_0(\Omega)$, $|x|\,u_0 \in L^2(\Omega)$ and let u be the corresponding solution of* (NLS_Ω) *equation. Then*

$$\frac{d}{dt}\Upsilon_2(u(t)) = 4\ \operatorname{Im}\int_\Omega \overline{u}(t,x)\;x.\nabla u(t,x)dx. \tag{III.12}$$
$$\frac{1}{16}\frac{d^2}{dt^2}\Upsilon_2(u(t)) = E_\Omega[u] - \frac{1}{2}\left(\frac{d}{2}-\frac{d+2}{p+1}\right)\int_\Omega |u|^{p+1}\,dx - \frac{1}{4}\int_{\partial\Omega}|\nabla u|^2\;(x.\vec{n})\,d\sigma(x). \tag{III.13}$$

And

$$\frac{d}{dt}\Upsilon_1(u(t)) = 2\ \operatorname{Im}\sum_{j=1}^{d}\int_\Omega \overline{u}(t,x)\;\frac{x_j}{|x|}.\partial_{x_j}u(t,x)dx. \tag{III.14}$$

$$\frac{1}{16}\frac{d^2}{dt^2}\Upsilon_1(u(t)) = \frac{(d-1)(d-3)}{16}\int_\Omega \frac{|u|^2}{|x|^3}\,dx - \frac{(d-1)(p-1)}{8(p+1)}\int_\Omega \frac{|u|^{p+1}}{|x|}\,dx + \frac{1}{4}\int_\Omega \frac{|\not\nabla u|^2}{|x|}\,dx - \frac{1}{8}\int_{\partial\Omega} |\nabla u|^2\,\frac{x.\vec{n}}{|x|}\,d\sigma(x). \quad \text{(III.15)}$$

where $|\not\nabla u|^2 := |\nabla u|^2 - \left|\frac{x}{|x|}.\nabla u\right|^2$ *and* $\vec{n}$ *is the outward unit normal vector.*

Remark 4.2. *Recall that, in Theorem 1.1 we assume that* Ω *is the exterior of the unit ball. We denotes by* $\vec{n}$ *the outward unit normal vector, i.e., the normal vector exterior to* Ω*, so that* $|x| = 1$ *and* $x.\vec{n} = -1$ *on* $\partial\Omega = \partial B(0,1)$*.*

Proof. Multiplying the equation by $|x|^2\overline{u}$ and taking the imaginary part we get,

$$\operatorname{Im}\int_\Omega i\partial_t u|x|^2\overline{u}dx + \operatorname{Im}\int_\Omega \Delta u|x|^2\overline{u}dx = -\operatorname{Im}\int_\Omega |u|^{p-1}u|x|^2\overline{u}dx = 0.$$

Which yields,

$$\frac{1}{2}\frac{d}{dt}\Upsilon_2(u(t)) = \frac{1}{2}\frac{d}{dt}\int_\Omega |x|^2\,|u(t,x)|^2\,dx = -\operatorname{Im}\int_\Omega |x|^2\Delta u\,\overline{u}\,dx.$$

Integration by parts ensure

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |x|^2|u(t,x)|^2dx = 2\sum_{k=1}^{d}\operatorname{Im}\int_\Omega x_k\partial_{x_k}u\,\overline{u}dx = 2\operatorname{Im}\int_\Omega \overline{u}\,x.\nabla u\,dx.$$

This implies (III.12). Now let us compute the second derivative of Υ_2.

$$\begin{aligned}
\frac{d^2}{dt^2}\Upsilon_2(u(t)) :&= 4\frac{d}{dt}\operatorname{Im}\int_\Omega \overline{u}\,x.\nabla u\,dx\\
&= 4\left(\operatorname{Im}\int_\Omega \partial_t\overline{u}\,x.\nabla u dx + \operatorname{Im}\int_\Omega \overline{u}\,x.\nabla(\partial_t u)dx\right)\\
&= 4\left(\operatorname{Im}\int_\Omega (-i\Delta\overline{u} - i|u|^{p-1}\overline{u})\,x.\nabla u\,dx + \operatorname{Im}\int_\Omega \overline{u}\,x.\nabla(i\Delta u + i|u|^{p-1}u)dx\right)\\
&= 4\Bigg[\underbrace{\operatorname{Re}\int_\Omega -\Delta\overline{u}\,x.\nabla u\,dx}_{\mathrm{I}_1} + \underbrace{\operatorname{Re}\int_\Omega \overline{u}\,x.\nabla(\Delta u)\,dx}_{\mathrm{I}_2} + \underbrace{\operatorname{Re}\int_\Omega -|u|^{p-1}\overline{u}\,x.\nabla u\,dx}_{\mathrm{I}_3}\\
&\quad + \underbrace{\operatorname{Re}\int_\Omega \overline{u}\,x.\nabla(|u|^{p-1}u)\,dx}_{\mathrm{I}_4}\Bigg]
\end{aligned}$$

$$\begin{aligned}\mathrm{I}_2 :&= \mathrm{Re}\sum_{k=1}^{d}\int_{\Omega}\overline{u}\; x_k\partial_{x_k}\Delta u dx = \sum_{k=1}^{d}\mathrm{Re}\int_{\Omega}-\partial_{x_k}(\overline{u}x_k)\Delta u\, dx + \mathrm{Re}\sum_{k=1}^{d}\int_{\partial\Omega}\overline{u}\,\Delta u\;(x_k n_k)\, d\sigma(x)\\ &= \mathrm{Re}\int_{\Omega}-\nabla\overline{u}.x\,\Delta u\, dx - d\,\mathrm{Re}\int_{\Omega}\overline{u}\,\Delta u\, dx\end{aligned}$$

$$\mathrm{I}_1 + \mathrm{I}_2 := -2\,\mathrm{Re}\int_{\Omega}\Delta\overline{u}\;(x.\nabla u + \frac{d}{2}\,u)\, dx$$

Using Pohozaev's Identity (III.10), we get

$$\mathrm{I}_1 + \mathrm{I}_2 := 2\int_{\Omega}|\nabla u|^2\, dx - \int_{\partial\Omega}|\nabla u|^2\;\;(x.\vec{n})\, d\sigma(x).$$

$$\mathrm{I}_4 := \mathrm{Re}\int_{\Omega}\overline{u}\, x.\nabla(|u|^{p-1}u)\; dx = -\,\mathrm{Re}\int_{\Omega}\nabla\overline{u}.x\;|u|^{p-1}\,u\, dx - d\int_{\Omega}|u|^{p+1}\, dx.$$

Using the fact that

$$\nabla(|u|^{p+1}) = (p+1)\,|u|^{p-1}\;\;\mathrm{Re}\,(\overline{u}\,\nabla u)\,, \tag{III.16}$$

we obtain

$$\begin{aligned}\mathrm{I}_3 + \mathrm{I}_4 &= -2\,\mathrm{Re}\int_{\Omega}|u|^{p-1}\overline{u}\; x.\nabla u\, dx - d\int_{\Omega}|u|^{p+1}\, dx\\ &= -\frac{2}{p+1}\,\mathrm{Re}\int_{\Omega}x.\nabla(|u|^{p+1})\, dx - d\int_{\Omega}|u|^{p+1}\, dx\\ &= \left(\frac{2\,d}{p+1} - d\right)\int_{\Omega}|u|^{p+1}\, dx.\end{aligned}$$

Which yields

$$\frac{d^2}{dt^2}\Upsilon_2(u(t)) = 8\int_{\Omega}|\nabla u|^2\, dx + \left(\frac{8\,d}{p+1} - 4d\right)\int_{\Omega}|u|^{p+1}\, dx - 4\int_{\partial\Omega}|\nabla u|^2\,(x.\vec{n})\, d\sigma(x).$$

Thus

$$\frac{1}{16}\frac{d^2}{dt^2}\Upsilon_2(u(t)) = E_{\Omega}[u] - \frac{1}{2}\left(\frac{d}{2} - \frac{d+2}{p+1}\right)\int_{\Omega}|u|^{p+1}\, dx - \frac{1}{4}\int_{\partial\Omega}|\nabla u|^2\;\;(x.\vec{n})\, d\sigma(x).$$

This concludes the proof of (III.13). Now let us compute the first derivative of Υ_1. Similarly, multiplying the equation by $|x|\overline{u}$ and taking the imaginary part we get,

$$\frac{1}{2}\frac{d}{dt}\Upsilon_1(u(t)) = \frac{d}{dt}\int_{\Omega}|x||u(t,x)|^2 dx = \mathrm{Im}\int_{\Omega}-\Delta u\,|x|\,\overline{u}\, dx = \mathrm{Im}\sum_{j=1}^{d}\int_{\Omega}\frac{x_j}{|x|}\partial_{x_j}u\;\overline{u}\, dx.$$

Thus, we obtain (III.14)

$$\frac{d}{dt}\Upsilon_1(u(t)) = 2 \ \mathrm{Im}\sum_{j=1}^{d}\int_{\Omega}\overline{u}(t,x)\ \frac{x_j}{|x|}.\partial_{x_j}u(t,x)dx = 2\ \mathcal{W}(u(t)).$$

For the second derivative of Υ_1, using the (NLS_Ω) we get

$$\begin{aligned}\frac{d^2}{dt^2}\Upsilon_1(u(t)) &= 2\,\mathrm{Im}\sum_{j=1}^{d}\int_{\Omega}\partial_t\overline{u}\frac{x_j}{|x|}\partial_{x_j}u\,dx + 2\,\mathrm{Im}\sum_{j=1}^{d}\int_{\Omega}\overline{u}\,\frac{x_j}{|x|}\partial_{x_j}(\partial_t u)\,dx\\ &= \underbrace{2\,\mathrm{Re}\sum_{j=1}^{d}\int_{\Omega}-\Delta\overline{u}\,\frac{x_j}{|x|}\,\partial_{x_j}u\,dx}_{\mathrm{J}_1} + \underbrace{2\,\mathrm{Re}\sum_{j=1}^{d}\int_{\Omega}\overline{u}\,\frac{x_j}{|x|}\,\partial_{x_j}(\Delta u)\,dx}_{\mathrm{J}_2}\\ &\quad+\underbrace{2\,\mathrm{Re}\sum_{j=1}^{d}\int_{\Omega}-|u|^{p-1}\,\overline{u}\,\frac{x_j}{|x|}\,\partial_{x_j}u\,dx}_{\mathrm{J}_3} + \underbrace{2\,\mathrm{Re}\sum_{j=1}^{d}\int_{\Omega}\overline{u}\,\frac{x_j}{|x|}\,\partial_{x_j}(|u|^{p-1}u)\,dx}_{\mathrm{J}_4}\end{aligned}$$

We will compute each integral apart, using again integration by parts and the Dirichlet boundary condition, i.e., $u = 0$ on $\partial\Omega$.

$$\begin{aligned}\mathrm{J}_2 := 2\,\mathrm{Re}\sum_{j=1}^{d}\int_{\Omega}\overline{u}\,\frac{x_j}{|x|}\,\partial_{x_j}(\Delta u)\,dx &= 2\,\mathrm{Re}\sum_{j=1}^{d}\int_{\Omega}-\partial_{x_j}\left(\overline{u}\frac{x_j}{|x|}\right)\Delta u\,dx\\ &= -2\,\mathrm{Re}\int_{\Omega}\nabla\overline{u}.\frac{x}{|x|}\,\Delta u\,dx - 2\,\mathrm{Re}\int_{\Omega}\overline{u}\,\frac{(d-1)}{|x|}\,\Delta u\,dx\end{aligned}$$

$$\begin{aligned}\mathrm{J}_1+\mathrm{J}_2 := -4\,\mathrm{Re}\int_{\Omega}\Delta\overline{u}\,\nabla u.\frac{x}{|x|}\ dx - 2\,\mathrm{Re}\int_{\Omega}\Delta\overline{u}\,\frac{(d-1)}{|x|}\,u\,dx\\ = -4\left[\mathrm{Re}\int_{\Omega}\Delta\overline{u}\left(\nabla u.\frac{x}{|x|}+\frac{(d-1)}{2}\,u\,\frac{1}{|x|}\right)\,dx\right]\end{aligned}$$

Using (III.11), we get

$$\begin{aligned}\mathrm{J}_1+\mathrm{J}_2 &= -4\left[-\frac{(d-1)(d-3)}{4}\int_{\Omega}\frac{|u|^2}{|x|^3}\,dx - \int_{\Omega}\frac{|\nabla\!\!\!/ u|^2}{|x|}\,dx + \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2\,\frac{x.\vec{n}}{|x|}\,d\sigma(x)\right]\\ &= (d-1)(d-3)\int_{\Omega}\frac{|u|^2}{|x|^3}\,dx + 4\int_{\Omega}\frac{|\nabla\!\!\!/ u|^2}{|x|}\,dx - 2\int_{\partial\Omega}|\nabla u|^2\,\frac{x.\vec{n}}{|x|}\,d\sigma(x).\end{aligned}$$

$$\begin{aligned}
\mathrm{J}_4 :&= 2\,\mathrm{Re}\sum_{j=1}^{d}\int_\Omega \overline{u}\,\frac{x_j}{|x|}\,\partial_{x_j}(|u|^{p-1}u)\,dx\\
&= -2\,\mathrm{Re}\sum_{j=1}^{d}\int_\Omega \partial_{x_j}\overline{u}\,\frac{x_j}{|x|}|u|^{p-1}u\,dx - 2\,\mathrm{Re}\sum_{j=1}^{d}\int_\Omega \overline{u}\,\partial_{x_j}\left(\frac{x_j}{|x|}\right)|u|^{p-1}u\,dx\\
&= -2\,\mathrm{Re}\int_\Omega \nabla\overline{u}.\frac{x}{|x|}\,|u|^{p-1}u\,dx - 2(d-1)\int_\Omega \frac{|u|^{p+1}}{|x|}\,dx
\end{aligned}$$

Due to (III.16), we have

$$\begin{aligned}
\mathrm{J}_3+\mathrm{J}_4 :&= -4\,\mathrm{Re}\int_\Omega \nabla\overline{u}.\frac{x}{|x|}|u|^{p-1}u\,dx - 2(d-1)\int_\Omega \frac{|u|^{p+1}}{|x|}\,dx\\
&= \frac{-4}{p+1}\int_\Omega \frac{x}{|x|}.\nabla(|u|^{p+1})\,dx - 2(d-1)\int_\Omega \frac{|u|^{p+1}}{|x|}\,dx\\
&= \frac{-2(d-1)(p-1)}{p+1}\int_\Omega \frac{|u|^{p+1}}{|x|}\,dx.
\end{aligned}$$

Summing all terms, we get

$$\begin{aligned}
\frac{d^2}{dt^2}\Upsilon_1(u(t)) &= \frac{d^2}{dt^2}\int_\Omega |x|\,|u|^2\,dx\\
&= (d-1)(d-3)\int_\Omega \frac{|u|^2}{|x|^3}\,dx + 4\int_\Omega \frac{|\nabla u|^2}{|x|}\,dx - \frac{2(d-1)(p-1)}{p+1}\int_\Omega \frac{|u|^{p+1}}{|x|}\,dx\\
&\quad - 2\int_{\partial\Omega} |\nabla u|^2\,\frac{x.\vec{n}}{|x|}\,d\sigma(x).
\end{aligned}$$

This concludes the proof of Proposition 4.1. □

Proof of Theorem 1.1. Let $u_0 \in H^2\cap H^1_0(\Omega)$ (we will later relax the assumption to $u_0 \in H^1_0(\Omega)$ if $d=2$), $|x|u_0 \in L^2(\Omega)$, $E[u_0]+\frac{1}{8}M[u_0] < 0$ if d=2 and $E[u] < 0$ if $d \geq 3$. Let u be the corresponding solution of (NLS_Ω) outside the unit ball $B(0,1)$, with maximal time interval I of existence. Define the variance used in this proof:

$$\mathcal{V}(u(t)) := \int_\Omega \Big(|x|^2 - 2|x| + 10\Big)\,|u(t,x)|^2\,dx.$$

From Proposition 4.1 we have

$$\begin{aligned}
\frac{1}{16}\frac{d^2}{dt^2}\mathcal{V}(u(t)) &= E_\Omega[u] - \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{|x|}\,dx - \frac{1}{2}\left(\frac{d}{2}-\frac{d+2}{p+1}\right)\int_\Omega |u|^{p+1}\,dx + \frac{(d-1)(p-1)}{4(p+1)}\int_\Omega \frac{|u|^{p+1}}{|x|}\,dx\\
&- \frac{(d-1)(d-3)}{8}\int_\Omega \frac{|u|^2}{|x|^3}\,dx - \frac{1}{4}\int_{\partial\Omega}|\nabla u|^2\,(x.\vec{n})\,d\sigma(x) + \frac{1}{4}\int_{\partial\Omega}|\nabla u|^2\,\frac{x.\vec{n}}{|x|}\,d\sigma(x). \quad \text{(III.17)}
\end{aligned}$$

Let us control first the boundary terms. Recall that Ω is the exterior of the unit ball, so that $x \cdot \vec{n} = -1$ and $|x| = 1$ for $x \in \partial\Omega$.

$$-\frac{1}{4}\int_{\partial\Omega} |\nabla u|^2 \, (x.\vec{n})\, d\sigma(x) + \frac{1}{4}\int_{\partial\Omega} |\nabla u|^2 \, \frac{x.\vec{n}}{|x|}\, d\sigma(x) = \left(\frac{1}{4} - \frac{1}{4}\right)\int_{\partial\Omega} |\nabla u|^2 \, (x.\vec{n})\, d\sigma(x) = 0.$$

Now, we will estimate the nonlinear terms. Using the fact that $p \geq 5$, we have

$$\begin{aligned}
-\frac{1}{2}\left(\frac{d}{2} - \frac{d+2}{p+1}\right)\int_{\Omega} |u|^{p+1}\, dx &+ \frac{(d-1)(p-1)}{4(p+1)}\int_{\Omega} \frac{|u|^{p+1}}{|x|}\, dx \\
&\leq \left[-\frac{1}{2}\left(\frac{d}{2} - \frac{d+2}{p+1}\right) + \frac{(d-1)(p-1)}{4(p+1)}\right]\int_{\Omega} |u|^{p+1}\, dx \\
&= -\left(\frac{p-5}{4(p+1)}\right)\int_{\Omega} |u|^{p+1}\, dx \leq 0.
\end{aligned}$$

Finally, for all $d \neq 2$ one can see that,

$$\frac{-(d-1)(d-3)}{8}\int_{\Omega} \frac{|u|^2}{|x|^3}\, dx \leq 0. \tag{III.18}$$

In particular, for $d = 3$ we have $\frac{-(d-1)(d-3)}{8}\displaystyle\int_{\Omega} \frac{|u|^2}{|x|^3}\, dx = 0$.
For $d = 2$, we use the fact that, $E_{\Omega}[u] + \frac{1}{8}M_{\Omega}[u] < 0$ and $\Theta = B(0,1)$. Indeed,

$$E_{\Omega}[u] + \frac{1}{8}\int_{\Omega} \frac{|u|^2}{|x|^3}\, dx \leq E_{\Omega}[u] + \frac{1}{8}M_{\Omega}[u] < 0.$$

This implies that the second derivative of the variance is bounded by a negative constant, for all $t \in I$.

$$\frac{d^2}{dt^2}\mathcal{V}(u(t)) \leq -A, \quad \text{where} \quad \text{-A=}\begin{cases} E_{\Omega}[u] + \frac{1}{8}M_{\Omega}[u] < 0 & \text{if } d = 2, \\ E_{\Omega}[u] < 0 & \text{if } d = 3. \end{cases}$$

Moreover, integrating twice over t, we have that

$$\mathcal{V}(u(t)) \leq -At^2 + Bt + C, \text{ where } B = \frac{d}{dt}\mathcal{V}(u_0) \text{ and } C = \mathcal{V}(u_0). \tag{III.19}$$

By density (III.19) remains true, if $d = 2$, assuming that $u_0 \in H_0^1(\Omega)$ and $|x|u_0 \in L^2(\Omega)$. Due to (III.19), there exists T^* such that $\mathcal{V}(u(T^*)) < 0$, which is a contradiction. Then the length of the maximal time interval of existence I is finite and one can prove that the solution u blows up in finite time. This concludes the proof of Theorem 1.1. □

5 Existence of blow-up symmetric solution

In this section we prove Theorem 1.2. Assume $d = 2$ and $\Theta = B(0,1)$.
The variance identity here is the following: Let $C > 0$ be a positive constant to be specified later,

$$\mathrm{V}(u(t)) := \int_\Omega \left(|x|^2 - C|x_1| - C|x_2| + C^2\right) |u(t,x)|^2 \, dx$$

Denote

$$\Gamma_1 := \int_\Omega |x_1||u(t,x)|^2 dx, \qquad \Gamma_2 := \int_\Omega |x_2||u(t,x)|^2 dx.$$

Proposition 5.1. *Let $u_0 \in H^2 \cap H_0^1(\Omega)$ and $|x|\, u_0 \in L^2(\Omega)$ such that $u_0(-x_1,x_2) = u_0(x_1,-x_2) = -u_0(x_1,x_2)$, and let u be the corresponding solution of (NLS_Ω) equation. Then*

$$\frac{d}{dt}\Gamma_1(u(t)) = 8 \,\mathrm{Im} \int_{\Omega^{++}} \partial_{x_1} u(t,x)\, \overline{u}(t,x)\, dx, \tag{III.20}$$
$$\frac{d^2}{dt^2}\Gamma_1(u(t)) = 8 \int_{\partial\Omega^{++}} |\nabla u(t,x)|^2 \, |n_1| \, dx, \tag{III.21}$$

and

$$\frac{d}{dt}\Gamma_2(u(t)) = 8 \,\mathrm{Im} \int_{\Omega^{++}} \partial_{x_2} u(t,x)\, \overline{u}(t,x) dx, \tag{III.22}$$
$$\frac{d^2}{dt^2}\Gamma_2(u(t)) = 8 \int_{\partial\Omega^{++}} |\nabla u(t,x)|^2 \, |n_2| \, dx. \tag{III.23}$$

Where

$$\Omega := \bigcup \Omega^{\pm\pm} := \Omega^+ \cup \Omega^- \quad \text{and} \quad \Omega^{\pm\pm} := \{x_1 \in \mathbb{R}^\pm \text{ and } x_2 \in \mathbb{R}^\pm\},$$
$$\Omega^\pm := \{x_1 \in \mathbb{R}^\pm \text{ and } x_2 \in \mathbb{R}\} \quad \text{and} \quad \Omega_\pm := \{x_1 \in \mathbb{R} \text{ and } x_2 \in \mathbb{R}^\pm\}.$$

Remark 5.2. *Let us mention that, due to the continuity of the flow, the symmetry properties of u_0, i.e., $u_0(-x_1,x_2) = u_0(x_1,-x_2) = -u_0(x_1,x_2)$ are conserved, that is,*

$$u(t,-x_1,x_2) = u(t,x_1,-x_2) = -u(t,x_1,x_2).$$

Furthermore, one can see that u satisfies the Dirichlet boundary conditions on each set defined above, $\Omega^{\pm\pm}, \Omega^\pm$.

Proof. Multiply the equation by $|x_1|\bar{u}$ and take the imaginary part to get:

$$\operatorname{Im}\int_\Omega i\partial_t u|x_1|\overline{u}\,dx + \operatorname{Im}\int_\Omega \Delta u|x_1|\overline{u}\,dx = -\underbrace{\operatorname{Im}\int_\Omega |u|^{p-1}u|x_1|\overline{u}\,dx}_{=0}$$

which yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |x_1||u(t,x)|^2\,dx = -\operatorname{Im}\int_\Omega |x_1|\Delta u\,\overline{u}\,dx$$
$$\frac{d}{dt}\Gamma_1(u(t)) := \frac{d}{dt}\int_\Omega |x_1||u(t,x)|^2dx = -2\operatorname{Im}\int_\Omega |x_1|\,\Delta u\,\overline{u}\,dx.$$

Integration by parts ensures,

$$\begin{aligned}\operatorname{Im}\int_\Omega |x_1|\Delta u\,\overline{u}\,dx &= -\operatorname{Im}\int_\Omega \partial_{x_1}(|x_1|)\,\partial_{x_1}u\,\overline{u}\,dx\\ &= -\left(\operatorname{Im}\int_{\Omega^+}\partial_{x_1}u\,\overline{u}\,dx - \operatorname{Im}\int_{\Omega^-}\partial_{x_1}u\,\overline{u}\,dx\right)\\ &= -4\ \operatorname{Im}\int_{\Omega^{++}}\partial_{x_1}u\,\overline{u}\,dx,\end{aligned}$$

which yields (III.20). Now, let us compute the second derivative of $\Gamma_1(u(t))$.
Denote:

$$\alpha(t) := \operatorname{Im}\int_{\Omega^{++}}\overline{u}(t,x)\ x.\nabla u(t,x)dx, \quad \beta(t) := \operatorname{Im}\int_{\Omega^{++}}\overline{u}(t,x)\,(x-e_1).\nabla u(t,x)dx.$$

Thus, we have

$$\frac{d}{dt}\Gamma_1(u(t)) = 8(\alpha(t)-\beta(t)).$$

Due to the symmetry properties of u, one can see that $\alpha(t)$ is equal to $\frac{1}{16}\frac{d}{dt}\Upsilon_2(t)$ and $\beta(t)$ is equal to (III.12) applied to $(x-e_1)$, where $e_1=(1,0)$. By (III.13), we obtain,

$$\frac{d}{dt}\alpha(t) = 4E_+[u] - 2\left(\frac{d}{2}-\frac{d+2}{p+1}\right)\int_{\Omega^{++}}|u|^{p+1}\,dx - \int_{\partial\Omega^{++}}|\nabla u|^2\ (x.\vec{n})\,d\sigma(x),$$
$$\frac{d}{dt}\beta(t) = 4E_+[u] - 2\left(\frac{d}{2}-\frac{d+2}{p+1}\right)\int_{\Omega^{++}}|u|^{p+1}\,dx - \int_{\partial\Omega^{++}}|\nabla u|^2\ ((x-e_1).\vec{n})\,d\sigma(x).$$

where, $E_+[u] = \dfrac{1}{2}\|\nabla u\|_{L^2(\Omega^{++})} - \dfrac{1}{p+1}\displaystyle\int_{\Omega^{++}}|u|^{p+1}dx$. Taking the difference between these two equalities, we get

$$\frac{d}{dt}\left(\alpha(t)-\beta(t)\right)=-\int_{\partial\Omega^{++}}|\nabla u|^2\,(e_1.\vec{n})\,dx=-\int_{\partial\Omega^{++}}|\nabla u|^2\,n_1\,dx,$$

which yields

$$\frac{d^2}{dt^2}\Gamma_1(u(t)):=-8\int_{\partial\Omega^{++}}|\nabla u|^2\,n_1\,dx.$$

As $\vec{n}$ is the unit outward normal vector, thus we have, $n_1\leq 0$ on $\partial\Omega^{++}$. Then, we obtain

$$\frac{d^2}{dt^2}\Gamma_1(u(t))=8\int_{\partial\Omega^{++}}|\nabla u|^2\,|n_1|\,dx.$$

Recall that

$$\Gamma_2:=\int_\Omega |x_2||u(t,x)|^2dx.$$

We multiply the equation by $|x_2|\bar{u}$ and take the imaginary part to get:

$$\operatorname{Im}\int_\Omega i\partial_t u|x_2|\overline{u}\,dx+\operatorname{Im}\int_\Omega \Delta u|x_2|\overline{u}\,dx=-\operatorname{Im}\int_\Omega |u|^{p-1}u|x_2|\overline{u}\,dx=0.$$

Which yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |x_2||u(t,x)|^2\,dx=-\operatorname{Im}\int_\Omega |x_2|\Delta u\,\overline{u}\,dx$$

$$\frac{d}{dt}\Gamma_2(u(t)):=\frac{d}{dt}\int_\Omega |x_2||u(t,x)|^2dx=-2\operatorname{Im}\int_\Omega |x_2|\,\Delta u\,\overline{u}\,dx.$$

Integration by parts ensures,

$$\begin{aligned}\operatorname{Im}\int_\Omega |x_2|\Delta u\,\overline{u}\,dx&=-\operatorname{Im}\int_\Omega \partial_{x_2}(|x_2|)\,\partial_{x_2}u\,\overline{u}\,dx\\&=-\left(\operatorname{Im}\int_{\Omega_+}\partial_{x_2}u\,\overline{u}\,dx-\operatorname{Im}\int_{\Omega_-}\partial_{x_2}u\,\overline{u}\,dx\right)\\&=-4\operatorname{Im}\int_{\Omega^{++}}\partial_{x_2}u\,\overline{u}\,dx,\end{aligned}$$

which yields (III.22). The proof of $\frac{d^2}{dt^2}\Gamma_2(t)$ is similar to proof of $\frac{d^2}{dt^2}\Gamma_1(t)$, the only difference is to apply (III.12) and (III.13) to $(x-e_2)$, where $e_2=(0,1)$ instead of $(x-e_1)$. Thus, we get

$$\frac{d^2}{dt^2}\Gamma_2(u(t)):=-8\int_{\partial\Omega^{++}}|\nabla u|^2\,n_2\,dx,$$

As $\vec{n}$ is the unit outward normal vector, we have $n_2 \leq 0$ on $\partial\Omega^{++}$. Then we obtain,

$$\frac{d^2}{dt^2}\Gamma_2(u(t)) := 8\int_{\partial\Omega^{++}} |\nabla u|^2 \, |n_2| \, dx.$$

This concludes the proof of Proposition 5.1. □

Proof of Theorem 1.2. Let $u_0 \in H_0^1(\Omega), |x|u_0 \in L^2(\Omega)$ and $E_\Omega[u_0] < 0$ such that $u_0(-x_1, x_2) = u_0(x_1, -x_2) = u_0(x_1, x_2)$ and let u be solution of (NLS_Ω) equation. From Proposition 5.1 and 4.1, we deduce the second derivative of the variance for $d = 2$,

$$\frac{d^2}{dt^2}\mathrm{V}(u(t)) = 16E_\Omega[u] - 8\left(\frac{p-3}{p+1}\right)\int_\Omega |u|^{p+1}\, dx \\ + \int_{\partial\Omega^{++}} |\nabla u|^2 \, \Big[16|x.\vec{n}| - 8C\left(|n_1| + |n_2|\right)\Big]\, d\sigma(x). \quad \text{(III.24)}$$

Using the fact that $p \geq 3$ and $E_\Omega[u] < 0$, one can see that the first two terms are negative, i.e., $16E_\Omega[u] - 8\left(\frac{p-3}{p+1}\right)\int_\Omega |u|^{p+1}dx \leq 0$. Choosing $C > 0$ such that

$$C \geq \max_{x\in\partial\Omega}\left(\frac{2|x.\vec{n}|}{|n_1| + |n_2|}\right) = \frac{2}{|n_1| + |n_2|},$$

this implies that,

$$\frac{d^2}{dt^2}\mathrm{V}(u(t)) \leq -A, \quad \text{where} \;\; A > 0.$$

Using the same argument as in the first proof, one can prove that the length of the maximal time interval of existence I is finite. Therefore, the solution u blows up in finite time and this concludes the proof of Theorem 1.2 in dimension 2. For any dimension $d \geq 3$, we should suppose that, $u_0(x_1, .., x_i, .., x_d) = -u_0(x_1, .., -x_i, .., x_d)$, for $i = 1, 2, ..., d$ and using the following variance:

$$\mathrm{V}(u(t)) = \int_\Omega \left(|x|^2 - C\sum_{i=1}^{d}|x_i| + C^2\right)|u(t,x)|^2 \, dx.$$

One can check that,

$$\frac{d^2}{dt^2}\mathrm{V}(u(t)) = 16E_\Omega[u] - 8\left(\frac{d}{2} - \frac{d+2}{p+1}\right)\int_\Omega |u|^{p+1}\, dx \\ + \int_{\partial\{x_i\geq 0,\, 1\leq i\leq d\}} |\nabla u|^2 \, \left[2^{d+2}|x.\vec{n}| - 2^{d+1}C\sum_{i=1}^{d}|n_i|\right] d\sigma(x) \quad \text{(III.25)}$$

Using the fact that $p \geq 1 + \frac{4}{d}$, $E_\Omega[u] < 0$ and choosing C such that

$$C \geq 2 \max_{x \in \partial\Omega} |x.\vec{n}| \left(\sum_{i=1}^{d} |n_i| \right)^{-1} = 2 \left(\sum_{i=1}^{d} |n_i| \right)^{-1},$$

we get

$$\frac{d^2}{dt^2} \mathrm{V}(u(t)) \leq -A, \quad \text{where} \quad A > 0.$$

Then u blows up in finite time for any dimension $d \geq 3$ and this concludes the proof of Theorem 1.2. □

6 Ground state threshold for blow-up

This section is devoted to the proof of Theorem 1.4. Assume that $d = 2$ and $p > 3$.
Let $u_0 \in H_0^1(\Omega)$ and $|x|\, u_0 \in L^2(\Omega)$ such that, $u_0(-x_1, x_2) = u_0(x_1, -x_2) = -u_0(x_1, x_2)$. Let u be the corresponding solution of (NLS$_\Omega$) equation. Moreover, we consider the same modified variance as in the proof of Theorem 1.2.

$$\mathrm{V}(u(t)) := \int_\Omega \left(|x|^2 - C|x_1| - C|x_2| + C^2 \right) |u(t,x)|^2 \, dx.$$

Lemma 6.1. *Let $u_0 \in H_0^1(\Omega)$ satisfy*

$$M_\Omega[u_0]^{1-s_C} E_\Omega[u_0]^{s_c} < M_{\mathbb{R}^2}[Q]^{1-s_c} E_{\mathbb{R}^2}[Q]^{s_c}. \tag{III.26}$$

$$\|u_0\|_{L^2(\Omega)}^{1-s_c} \|\nabla u_0\|_{L^2(\Omega)}^{s_c} > \|Q\|_{L^2(\mathbb{R}^2)}^{1-s_c} \|\nabla Q\|_{L^2(\mathbb{R}^2)}^{s_c}. \tag{III.27}$$

Then the corresponding solution u to (NLS$_\Omega$) *satisfy,*

$$\forall t \in I, \qquad \|u_0\|_{L^2(\Omega)}^{1-s_c} \|\nabla u(t)\|_{L^2(\Omega)}^{s_c} > \|Q\|_{L^2(\mathbb{R}^2)}^{1-s_c} \|\nabla Q\|_{L^2(\mathbb{R}^2)}^{s_c}. \tag{III.28}$$

Proof. The proof of the lemma is the same as in [44], [45] for the proof of blow-up solutions of (NLS) equation on $\mathbb{R}^d$. We give it for the sake of completeness and for the convenience of the reader. The key point is that a function $f \in H_0^1(\Omega)$ extended by 0 outside Ω can be identified to an element of $H^1(\mathbb{R}^2)$. Thus, it satisfies the same Gagliardo-Nirenberg inequality as an element of $H^1(\mathbb{R}^2)$.

Multiplying the energy by $M_\Omega[u]^{\frac{1-s_c}{s_c}}$ and applying Gagliardo-Nirenberg's inequality for $d=2$, we have,

$$\begin{aligned}
M_\Omega[u]^{\frac{1-s_c}{s_c}} E_\Omega[u] &= \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)}\|u\|^{2\frac{(1-s_c)}{s_c}}_{L^2(\Omega)} - \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\Omega)}\|u\|^{2\frac{(1-s_c)}{s_c}}_{L^2(\Omega)} \\
&\geq \frac{1}{2}\left(\|\nabla u\|_{L^2(\Omega)}\|u\|^{\frac{1-s_c}{s_c}}_{L^2(\Omega)}\right)^2 - \frac{C_{GN}}{p+1}\left(\|\nabla u\|_{L^2(\Omega)}\|u\|^{\frac{1-s_c}{s_c}}_{L^2(\Omega)}\right)^{p-1} \\
&\geq f\left(\|\nabla u\|_{L^2(\Omega)}\|u\|^{\frac{1-s_c}{s_c}}_{L^2(\Omega)}\right),
\end{aligned}$$

where $f(x)=\frac{1}{2}x^2-\frac{C_{GN}}{p+1}x^{p-1}$. Then, $f'(x)=x-\frac{C_{GN}(p-1)}{p+1}x^{p-2}$, and thus, $f'(x)=0$ for $x_0=0$ and $x_1=\left(\frac{C_{GN}(p-1)}{p+1}\right)^{-\frac{1}{p-3}}=\|\nabla Q\|_{L^2(\mathbb{R}^2)}\|Q\|^{\frac{1-s_c}{s_c}}_{L^2(\mathbb{R}^2)}$ by (III.8). Since (III.5) is attained at ground state Q then we have, $f(\|\nabla Q\|_{L^2(\mathbb{R}^2)}\|Q\|^{\frac{1-s_c}{s_c}}_{L^2(\mathbb{R}^2)})=M_{\mathbb{R}^2}[Q]^{\frac{1-s_c}{s_c}}E_{\mathbb{R}^2}[Q]$, we also have $f(0)=0$. Thus, the function f is increasing on $(0,x_1)$ and decreasing on (x_1,∞). Using the energy conservation, we get

$$f\left(\|\nabla u\|_{L^2(\Omega)}\|u\|^{\frac{1-s_c}{s_c}}_{L^2(\Omega)}\right) \leq M_\Omega[u]^{\frac{1-s_c}{s_c}}E_\Omega[u(t)] < f(x_1). \tag{III.29}$$

If condition (III.27) holds, i.e., $\|u_0\|^{1-s_c}_{L^2(\Omega)}\|\nabla u_0\|^{s_c}_{L^2(\Omega)} > x_1 = \|Q\|^{1-s_c}_{L^2(\mathbb{R}^2)}\|\nabla Q\|^{s_c}_{L^2(\mathbb{R}^2)}$, then by (III.29) and the continuity of $\|\nabla u(t)\|_{L^2(\Omega)}$ in time we obtain (III.28) for all time $t\in I$.

□

Moreover, if the conditions (III.26) and (III.27) holds, then there exists $\delta_1>0$ such that

$$M_\Omega[u_0]^{1-s_c}E_\Omega[u_0]^{s_c} < (1-\delta_1)M_{\mathbb{R}^2}[Q]^{1-s_c}E_{\mathbb{R}^2}[Q]^{s_c}. \tag{III.30}$$

Thus, there exists $\delta_2 := \delta_2(\delta_1)>0$ such that

$$\forall t\in I, \qquad \|u_0\|^{1-s_c}_{L^2(\Omega)}\|\nabla u(t)\|^{s_c}_{L^2(\Omega)} > (1+\delta_2)\|Q\|^{1-s_c}_{L^2(\mathbb{R}^2)}\|\nabla Q\|^{s_c}_{L^2(\mathbb{R}^2)}. \tag{III.31}$$

Now let us prove that

$$\begin{aligned}
\frac{d^2}{dt^2}\mathrm{V}(u(t)) \leq 8(p-1)E_\Omega[u] - 4(p-3)\|\nabla u\|^2_{L^2(\Omega)} \\
+\int_{\partial\Omega^{++}}|\nabla u|^2\left[16|x.\vec{n}|-8C\left(|n_1|+|n_2|\right)\right]d\sigma(x).
\end{aligned} \tag{III.32}$$

From (III.24), we have

$$\begin{aligned}\frac{d^2}{dt^2}\mathrm{V}(u(t)) &= 16E_\Omega[u] - 8\left(\frac{p-3}{p+1}\right)\int_\Omega |u|^{p+1}\,dx + \int_{\partial\Omega^{++}} |\nabla u|^2 \left[16|x.\vec{n}| - 8C\left(|n_1|+|n_2|\right)\right] d\sigma(x) \\ &\leq 8\left\|\nabla u\right\|_{L^2}^2 - \frac{8(p-1)}{p+1}\int_\Omega |u|^{p+1}dx + \int_{\partial\Omega^{++}} |\nabla u|^2 \left[16|x.\vec{n}| - 8C\left(|n_1|+|n_2|\right)\right] d\sigma(x) \\ &\leq 8(p-1)E[u] - 4(p-3)\left\|\nabla u\right\|_{L^2(\Omega)}^2 + \int_{\partial\Omega^{++}} |\nabla u|^2 \left[16|x.\vec{n}| - 8C\left(|n_1|+|n_2|\right)\right] d\sigma(x).\end{aligned}$$

Multiplying (III.32) by $M_\Omega[u]^{\frac{1-s_c}{s_c}}$ and using (III.9) for $d=2$ with the two refined inequalities (III.30) and (III.31), we have

$$\begin{aligned}M_\Omega[u]^{\frac{1-s_c}{s_c}}\frac{d^2}{dt^2}\mathrm{V}(u(t)) &\leq \left(8(p-1)E_\Omega[u] - 4(p-3)\left\|\nabla u\right\|_{L^2(\Omega)}^2\right) M_\Omega[u]^{\frac{1-s_c}{s_c}} \\ &+ \int_{\partial\Omega^{++}} |\nabla u|^2 \left[16|x.\vec{n}| - 8C\left(|n_1|+|n_2|\right)\right] d\sigma(x) M_\Omega[u]^{\frac{1-s_c}{s_c}} \\ &\leq 8(p-1)(1-\delta_1)E_{\mathbb{R}^2}[Q]M_{\mathbb{R}^2}[Q]^{\frac{1-s_c}{s_c}} - 4(p-3)(1+\delta_2)\left\|\nabla Q\right\|_{L^2(\mathbb{R}^2)}^2 M_{\mathbb{R}^2}[Q]^{\frac{1-s_c}{s_c}} \\ &+ \int_{\partial\Omega^{++}} |\nabla u|^2 \left[16|x.\vec{n}| - 8C\left(|n_1|+|n_2|\right)\right] d\sigma(x) M_\Omega[u]^{\frac{1-s_c}{s_c}} \\ &\leq 8(p-1)(1-\delta_1)\frac{(p-3)}{2(p-1)}\left\|\nabla Q\right\|_{L^2(\mathbb{R}^2)}^2 M_{\mathbb{R}^2}[Q]^{\frac{1-s_c}{s_c}} \\ &- 4(p-3)(1+\delta_2)\left\|\nabla Q\right\|_{L^2(\mathbb{R}^2)}^2 M_{\mathbb{R}^2}[Q]^{\frac{1-s_c}{s_c}} \\ &+ \int_{\partial\Omega^{++}} |\nabla u|^2 \left[16|x.\vec{n}| - 8C\left(|n_1|+|n_2|\right)\right] d\sigma(x) M_\Omega[u]^{\frac{1-s_c}{s_c}},\end{aligned}$$

which yields

$$\begin{aligned}M_\Omega[u]^{\frac{1-s_c}{s_c}}\frac{d^2}{dt^2}\mathrm{V}(u(t)) &\leq \Big[4(p-3) - 4(p-3)\Big]\left\|\nabla Q\right\|_{L^2(\mathbb{R}^2)}^2 M_{\mathbb{R}^2}[Q]^{\frac{1-s_c}{s_c}} \\ &- \Big[4(p-3)\delta_1 + 4(p-3)\delta_2\Big]\left\|\nabla Q\right\|_{L^2(\mathbb{R}^2)}^2 M_{\mathbb{R}^2}[Q]^{\frac{1-s_c}{s_c}} \\ &+ \int_{\partial\Omega^{++}} |\nabla u|^2 \left[16|x.\vec{n}| - 8C\left(|n_1|+|n_2|\right)\right] d\sigma(x) M_\Omega[u]^{\frac{1-s_c}{s_c}}.\end{aligned}$$

Using the fact that $p>3$ and choosing $C \geq \frac{2|x.\vec{n}|}{|n_1|+|n_2|} = \frac{2}{|n_1|+|n_2|}$ imply that the second derivative of the variance is bounded by a negative constant, for all $t \in I$,

$$\frac{d^2}{dt^2}\mathrm{V}(u(t)) \leq -A, \quad \text{where } A > 0.$$

Thus the maximal time interval of existence I is finite and the solution u blows up in finite time. This concludes the proof of Theorem 1.4 in dimension 2.

Next, we will give the proof for dimension $d \geq 3$. For that, we suppose that

$$u_0(x_1, .., -x_i, .., x_d) = -u_0(x_1, .., x_i, .., x_d), \quad \text{for } \ i = 1, 2, ..., d.$$

Using the following variance

$$\mathrm{V}(u(t)) = \int_{\Omega} \left(|x|^2 - C \sum_{i=1}^{d} |x_i| + C^2 \right) |u(t,x)|^2 \, dx,$$

we have,

$$\frac{d^2}{dt^2}\mathrm{V}(u(t)) = 4d(p-1)E_{\Omega}[u] - (2d(p-1)-8) \|\nabla u\|^2_{L^2(\Omega)} + \int_{\partial\{x_i \geq 0,\, 1 \leq i \leq d\}} |\nabla u|^2 \left[2^{d+2}|x.\vec{n}| - 2^{d+1} C \sum_{i=1}^{d} |n_i| \right] d\sigma(x). \quad \text{(III.33)}$$

Using the same argument as above, one can check that Lemma III.26 remains true for $d \geq 3$, see [44], [43]. If the conditions (III.26) and (III.27) hold then there exists $\delta_1 > 0$, $\delta_2(\delta_1) > 0$ such that (III.30) and (III.31) are valid for $d \geq 3$. Multiplying (III.33) by $M_{\Omega}[u]^{\frac{1-sc}{sc}}$ and using (III.9) with the two refined inequalities (III.30) and (III.31), we have

$$\begin{aligned} M_{\Omega}[u]^{\frac{1-sc}{sc}} \frac{d^2}{dt^2}\mathrm{V}(u(t)) &\leq 4d(p-1)(1-\delta_1)E_{\mathbb{R}^d}[Q]M_{\mathbb{R}^d}[Q]^{\frac{1-sc}{sc}} \\ &- (2d(p-1)-8)(1+\delta_2) \|\nabla Q\|^2_{L^2\left(\mathbb{R}^d\right)} M_{\mathbb{R}^d}[Q]^{\frac{1-sc}{sc}} \\ &+ \int_{\partial\{x_i \geq 0,\, 1 \leq i \leq d\}} |\nabla u|^2 \left[2^{d+2}|x.\vec{n}| - 2^{d+1} C \sum_{i=1}^{d} |n_i| \right] d\sigma(x) M_{\Omega}[u]^{\frac{1-sc}{sc}} \\ &\leq (2d(p-1)-8)(1-\delta_1) \|\nabla Q\|^2_{L^2\left(\mathbb{R}^d\right)} M_{\mathbb{R}^d}[Q]^{\frac{1-sc}{sc}} \\ &- (2d(p-1)-8)(1+\delta_2) \|\nabla Q\|^2_{L^2\left(\mathbb{R}^d\right)} M_{\mathbb{R}^d}[Q]^{\frac{1-sc}{sc}} \\ &+ \int_{\partial\{x_i \geq 0,\, 1 \leq i \leq d\}} |\nabla u|^2 \left[2^{d+2}|x.\vec{n}| - 2^{d+1} C \sum_{i=1}^{d} |n_i| \right] d\sigma(x) M_{\Omega}[u]^{\frac{1-sc}{sc}}, \end{aligned}$$

which yields

$$\begin{aligned} M_{\Omega}[u]^{\frac{1-sc}{sc}} \frac{d^2}{dt^2}\mathrm{V}(u(t)) &\leq -\Big[(2d(p-1)-8)\delta_1 + (2d(p-1)-8)\delta_2 \Big] \|\nabla Q\|^2_{L^2\left(\mathbb{R}^d\right)} M_{\mathbb{R}^d}[Q]^{\frac{1-sc}{sc}} \\ &+ \int_{\partial\{x_i \geq 0,\, 1 \leq i \leq d\}} |\nabla u|^2 \left[2^{d+2}|x.\vec{n}| - 2^{d+1} C \sum_{i=1}^{d} |n_i| \right] d\sigma(x) \, M_{\Omega}[u]^{\frac{1-sc}{sc}}. \end{aligned}$$

Thus
$$\frac{d^2}{dt^2}\mathrm{V}(u(t)) \leq -A, \quad \text{where} \;\; A > 0.$$

Provided $p > 1 + \frac{4}{d}$ and $C \geq 2\left(\sum_{i=1}^{d} |n_i|\right)^{-1}$. Then u blows up in finite time and this concludes the proof of Theorem 1.4.

Chapter IV

Scattering at the threshold

Abstract. In this Chapter, joint work with Thomas Duyckaerts and Svetlana Roundenko, we study the dynamics of the focusing $3d$ cubic NLS_Ω equation in the exterior of a strictly convex obstacle at exactly the mass-energy threshold, namely, when $E_\Omega[u_0]M_\Omega[u_0] = E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q]$, with $H_0^1(\Omega)$ initial data satisfying an initial mass-gradient bound, where Q is the ground state solution of the nonlinear elliptic equation (I.6). In this case, we prove that the solution is globally defined and scatter in both time direction.

1 Introduction

We consider the focusing nonlinear Schrödinger equation in the exterior of a smooth compact strictly convex obstacle $\Theta \subset \mathbb{R}^3$ with Dirichlet boundary conditions:

$$\begin{cases} i\partial_t u + \Delta_\Omega u = -|u|^2 u & \forall (t,x) \in \mathbb{R} \times \Omega, \\ u(t_0, x) = u_0(x) & \forall x \in \Omega, \\ u(t,x) = 0 & \forall (t,x) \in \mathbb{R} \times \partial\Omega. \end{cases} \tag{NLS$_\Omega$}$$

Where $\Omega = \mathbb{R}^3 \setminus \Theta$, Δ_Ω is the Dirichlet Laplace operator on Ω and $t_0 \in \mathbb{R}$ is the initial time and we take the initial data $u_0 \in H_0^1(\Omega)$.

The scaling given in (I.2) identifies the critical Sobolev space $\dot{H}_x^{\frac{1}{2}}$, for the cubic (NLS$_\Omega$) equation in dimension $d = 3$. We may regard (NLS$_\Omega$) equation as being $H^1(\Omega)$-subcritical and $L^2(\Omega)$-supercritical.

Recall from the Introduction that the cubic (NLS$_\Omega$) equation is locally well-posed in $H_0^1(\Omega)$ in dimension $d = 3$.

In this Chapter, we will study the global well-posedness and scattering of the solution. Let us first recall some earlier results for global existence and scattering, see [78], [10] and [58]. if u has a finite Strichartz norm then u scatters forward in time or if the initial data is sufficiently small in $H_0^1(\Omega)$ then the corresponding solution $u(t)$ is global and scatter in both time directions, that is,

$$\exists u_+ \in H_0^1(\Omega), \quad \text{such that} \quad \lim_{t \longrightarrow \pm\infty} \left\| u(t) - e^{it\Delta_\Omega} u_+ \right\|_{H_0^1(\Omega)} = 0.$$

Recall that the global existence and scattering for large data was studied for the (NLS) equation posed on the whole Euclidean space in several articles. The authors have studied the behavior (i.e., global existence and scattering) of the solutions of the focusing cubic (NLS) equation on $\mathbb{R}^3$, whenever the initial data satisfies a smallness criterion given by the ground state solution of the following nonlinear elliptic equation

$$\begin{cases} -\Delta Q_\omega + \omega\, Q_\omega = |Q_\omega|^2\, Q_\omega, \\ Q_\omega \in H^1(\mathbb{R}^3). \end{cases} \tag{IV.1}$$

In this Chapter, we will denote by $Q := Q_\omega$ the ground state which is the unique radial positive solution of (IV.1). We recall that Q is smooth and exponentially decaying at infinity and characterized as the unique minimizer for the Gagliardo-Nirenberg inequality up to scaling, space translation and phase shift, see [60].

The question for global existence and scattering was studied in [58] for the focusing cubic (NLS_Ω) outside a strictly convex obstacle.

Theorem A. *Let $u_0 \in H^1(\mathbb{R}^3)$ satisfy*

$$\|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}, \tag{IV.2}$$

$$M_\Omega[u_0]E_\Omega[u_0] < M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q]. \tag{IV.3}$$

Then u scatters in both time directions.

The purpose of this Chapter is to study the behavior of solutions to (NLS_Ω) at the mass-energy threshold, i.e., when

$$E_\Omega[u]M_\Omega[u] = E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q]. \tag{IV.4}$$

In [31] T. Duyckaerts and S. Roudenko have described the behavior of the solutions of the (NLS) equation at the mass-energy threshold. At this mass-energy level, the (NLS) equation has a richer dynamics for the long time behavior of the solutions compared to the result mentioned above. The authors proved the existence of special solutions, denoted by Q^+ and Q^-. Theses solutions approaches the soliton, up to symmetries, in one time direction, that is, there exits $e_0 > 0$ such that

$$\forall t \geq 0 \qquad \left\|Q^\pm - e^{it}Q\right\|_{H^1(\mathbb{R}^3)} \leq ce^{-e_0 t} \tag{IV.5}$$

The behavior of $Q^\pm$ on the opposite time direction is completely different, Q^- scatters for negative time but Q^+ has finite time of existence. The existence of these special solutions is derived from the existence of the two real nonzero eigenvalues for the Linearized operator around the soliton $e^{it}Q$. Moreover, these special solutions have the same mass-energy of the soliton, $\|\nabla Q^-\|_{L^2(\mathbb{R}^3)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)}$ and $\|\nabla Q^+\|_{L^2(\mathbb{R}^3)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)}$. Thus, if we consider an initial data u_0 such that (IV.4) holds and $\|u_0\|_{L^2(\mathbb{R}^3)} \|\nabla u_0\|_{L^2(\mathbb{R}^3)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}$ (Resp. if $\|u_0\|_{L^2(\mathbb{R}^3)} \|\nabla u_0\|_{L^2(\mathbb{R}^3)} > \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}$) then the corresponding solution u of (NLS) is global and either scatter or $u = Q^-$, up to the symmetries (Resp. either u has a finite time of existence or $u = Q^+$, up to the symmetries.) Moreover, if the gradient of u_0 is equal to the gradient of Q then u is equal to Q up to the symmetries.

Note that, the (NLS_Ω) equation does not admit an analogue of these special solutions. In the presence of the obstacle there are no function in $H_0^1(\Omega)$ such that (IV.5) hold. Even, if we extend a function by zero on the obstacle, by the previous work on $\mathbb{R}^3$, this function has to converge to Q for large time. However, this function does not satisfy Dirichlet Boundary conditions. The same remark also hold if a function $u \in H_0^1(\Omega)$ satisfy (IV.4) and $\|\nabla u_0\|_{L^2(\Omega)} \|u_0\|_{L^2(\Omega)} = \|\nabla Q\|_{L^2(\mathbb{R}^3)} \|Q\|_{L^2(\mathbb{R}^3)}$. By extending u_0 by 0 on the obstacle then the solution u must be equal to Q up to the symmetries which does not obey Dirichlet Boundary conditions.

Now let us state our result.

Theorem 1.1. *Let $u_0 \in H_0^1(\Omega)$ and let u be the corresponding solution such that if u_0 satisfy*

$$M_\Omega[u]E_\Omega[u] = M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q] \quad \text{and } \|u_0\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}. \tag{IV.6}$$

Then u scatter for positive time direction.

The techniques used in this result is essentially based on the approach of the result in the Euclidean setting of the first author and F. Merle in [30] and [31], which also employed methods of C. Kenig and F. Merle in [55] and in [45], [29]. In particular, the scattering is established using concentration-compactness argument that requires a profile decomposition method given by R. Killip, M. Visan and X. Zhang for the problem in the exterior of a convex obstacle in [58] (for energy-crticial) and in [59] (for the energy-sub-critical). In [31], the translation parameter is controlled using conservation of the momentum, leading ultimately to the fact that $e^{-it}u$ converges exponentially to Q. This conservation law is not available for the NLS_Ω equation, and we must achieve this control through a new intricate limiting argument, that relies among other things on the uniqueness theorem in [29].

The Chapter is organized as follow follows, in section 2, we recall some known properties of the ground state, coercivity property associated to the linearized operator under some orthogonality conditions. In §2, we recall Strichartz estimate, stability theory and the profile decomposition for the (NLS_Ω) equation outside a strictly convex obstacle. In §3.2, we use a modulation in the phase rotation and in space translation parameters near the ground state solution truncated, in order to obtain some orthogonality conditions. In §, we use the profile decomposition to prove a compactness property, which yields to the existence of a continuous translation parameters $x(t)$ such that the extension of a non-scattering solution $\underline{u}(t, x + x(t))$, that satisfies (IV.6) is compact in H^1. In §4.2, we control the space translation $x(t)$ using an auxiliary translation parameters given by a modulation on $\mathbb{R}^3$, local virial identity with the estimation on the modulation parameters. In §4.3, we prove that the parameter $\delta(t) = |\|\nabla Q\|_{L^2} - \|\nabla u\|_{L^2}|$ converge to 0 in mean. Finally, we conclude the proof of Theorem 1.1 using compactness properties with the control of the space translation parameter $x(t)$ and the convergence in mean.

Notation:

Define Ψ as a C^∞ function such that: $\begin{cases} \Psi = 0 & near\ \Theta, \\ \Psi = 1 & if\ |x| \gg 1. \end{cases}$

We write $a = O(b)$, when a and b are two quantities, and there exists a positive constant C independent of parameters, such that $|a| \leq C\,b$, and $a \approx b$, when $a = O(b)$ and $b = O(a)$.

For $h \in \mathbb{C}$, we denote $h_1 = \operatorname{Re} h$ and $h_2 = \operatorname{Im} h$.

Throughout this paper, C denotes a large positive constant and c is a small positive constant, that may change from line to line; both do not depend on parameters. We denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^3$.

For simplicity, we write $\Delta = \Delta_\Omega$.

The real L^2-scalar product $(\cdot,\cdot)$ means

$$(f, g) = \operatorname{Re} \int f\,\overline{g} = \int \operatorname{Re} g\ \operatorname{Re} f + \int \operatorname{Im} g\ \operatorname{Im} f\ .$$

2 Preliminaries

2.1 Properties of the ground state

We recall here some well-known properties of the ground state. We refer the reader to to [90], [60] , [86, Appendix B] and [45] for more details. Consider the following nonlinear elliptic equation on $\mathbb{R}^3$

$$-Q + \Delta Q + |Q|^2 Q = 0. \tag{IV.7}$$

The ground state is characterized as the unique positive, radial solution of (IV.7). It is also (up to standard transformations) the unique minimizer of the Gagliardo-Niremberg inequality: if $u \in H^1$,

$$\|u\|^4_{L^4(\mathbb{R}^3)} \leq C_{GN}\, \|\nabla u\|^3_{L^2(\mathbb{R}^3)}\, \|u\|_{L^2(\mathbb{R}^3)}\,, \quad \|Q\|^4_{L^4(\mathbb{R}^3)} = C_{GN}\, \|\nabla Q\|^3_{L^2(\mathbb{R}^3)}\, \|Q\|_{L^2(\mathbb{R}^3)}\,. \tag{IV.8}$$

Moreover

$$\|u\|_{L^4(\mathbb{R}^3)}^4 = C_{GN}\,\|\nabla u\|_{L^2(\mathbb{R}^3)}^3\,\|u\|_{L^2(\mathbb{R}^3)} \Longrightarrow \exists\lambda_0\in\mathbb{C}, \exists\mu_0\in\mathbb{R}, \exists x_0\in\mathbb{R}^3 : u(x)=\lambda_0 Q(\mu_0(x+x_0)). \tag{IV.9}$$

We also have the Pohozaev identities:

$$\|Q\|_{L^4(\mathbb{R}^3)}^4 = 4\,\|Q\|_{L^2(\mathbb{R}^3)}^2 \quad \text{and} \quad \|\nabla Q\|_{L^2(\mathbb{R}^3)}^2 = 3\,\|Q\|_{L^2(\mathbb{R}^3)}^2. \tag{IV.10}$$

As a consequence of (IV.8), one has

Proposition 2.1. *There exists a function $\varepsilon(\eta)$, defined for small $\eta>0$ such that $\lim_{\eta\to 0}\varepsilon(\eta)=0$ and*

$$\forall u\in H^1(\mathbb{R}^3),\quad \Big|\,\|u\|_{L^4(\mathbb{R}^3)} - \|Q\|_{L^4(\mathbb{R}^3)}\Big| + \Big|\|u\|_{L^2(\mathbb{R}^3)} - \|Q\|_{L^2(\mathbb{R}^3)}\Big| + \Big|\|\nabla u\|_{L^2(\mathbb{R}^3)} - \|\nabla Q\|_{L^2(\mathbb{R}^3)}\Big| \le \eta$$
$$\Longrightarrow \exists\,\theta_0\in\mathbb{R} \text{ and } \exists\,x_0\in\mathbb{R}^3,\quad \Big\|u - e^{i\theta_0}Q(\cdot - x_0)\Big\|_{H^1(\mathbb{R}^3)} \le \varepsilon(\eta). \tag{IV.11}$$

Next, we recall some known properties on the decay of Q, see [34], [6] and [16, chapter 8].

Proposition 2.2 (Exponential decay of Q)**.** *Let Q be the ground state solution of (IV.7), then there exists $a, C>0$ such that for $|x|>1$,*

$$\left|Q(x) - \frac{a}{|x|}e^{-|x|}\right| \le \frac{Ce^{-|x|}}{|x|^{3/2}}.$$

Moreover,

$$\left|\nabla Q(x) + \nabla^2 Q(x)\right| \le C\frac{e^{-|x|}}{|x|}.$$

Lemma 2.3. *Let Q be the ground state solution of (IV.7), $M>0$ large, $X\in\mathbb{R}^3$ and let g be a L^1-function. Then for $k>0$, we have*

$$|X|\ge 2M \implies \int_{|x|\le M}\left(Q^k(x-X) + |\nabla Q(x-X)|^k\right) g(x)\,dx = O\left(\frac{e^{-k|X|}}{|X|^k}\right), \tag{IV.12}$$

where $O(\cdot)$ depends on k, g and M.

Furthermore, there exists $c_M>0$ such that

$$\int_{|x|\le M} Q^k(x-X)\,dx \ge c_M\,\frac{e^{-k|X|}}{|X|^k}. \tag{IV.13}$$

Proof. First, note that

$$\frac{1}{2}|X| < |X| - M < |x - X|, \quad \text{if } |X| \geq 2M.$$

This implies that, for $|X| \geq 2M$ we have

$$e^{-|x-X|} \leq e^{M}e^{-|X|} \leq c\, e^{-|X|} \quad \text{and} \quad \frac{1}{2\,|x-X|} \leq \frac{1}{|X|}.$$

Using the exponential decay of Q in Proposition 2.2, we obtain,

$$\int_{|x|\leq M} Q^k(x-X)g(x)\,dx = O\left(\frac{e^{-k|X|}}{|X|^k}\right), \quad \text{for } k > 0.$$

Similarly, we get

$$\int_{|x|\leq M} |\nabla Q(x-X)|^k\, g(x)\,dx = O\left(\frac{e^{-k|X|}}{|X|^k}\right), \quad \text{for } k > 0.$$

The proof of (IV.13) is similar by applying again Proposition 2.2 and we omit it. □

Let $u \in H_0^1(\Omega)$ and denote $\underline{u} \in H^1(\mathbb{R}^3)$ such that

$$\underline{u}(x) = \begin{cases} u(x) & \forall x \in \ \Omega, \\ 0 & \forall x \in \ \Omega^c. \end{cases} \tag{IV.14}$$

Remark 2.4. *We denote by* $M_{\mathbb{R}^3}[\underline{u}] = \|\underline{u}\|^2_{L^2(\mathbb{R}^3)}$ *and* $E_{\mathbb{R}^3}[\underline{u}] = \frac{1}{2}\|\nabla \underline{u}\|^2_{L^2(\mathbb{R}^3)} - \frac{1}{p+1}\|\underline{u}\|^{p+1}_{L^{p+1}(\mathbb{R}^3)}$. *Note that, we have* $M_\Omega[u] = M_{\mathbb{R}^3}[\underline{u}]$ *and* $E_\Omega[u] = E_{\mathbb{R}^3}[\underline{u}]$. *To simplify notations we will drop the index* Ω *of the mass and the energy of the* (NLS$_\Omega$) *equation, so that, we will write* $M[u]$ *and* $E[u]$ *instead of* $M_\Omega[u]$ *and* $E_\Omega[u]$.

Assume that $\underline{u}$ satisfies the left-hand side of (IV.11). Then there exists $x_0 \in \mathbb{R}^3$ and $\theta_0 \in \mathbb{R}$ such that

$$\left\|\underline{u} - e^{i\theta_0}Q(\cdot - x_0)\right\|_{H^1} \leq \varepsilon(\eta).$$

Which yields

$$\frac{1}{C}\frac{e^{-|x_0|}}{|x_0|} \leq \|Q(x-x_0)|_{H^1(\Omega^c)} \leq \varepsilon(\eta) \tag{IV.15}$$

This implies that $|x_0|$ is large (depending on $\varepsilon(\eta)$).

2.2 Coercivity property

We next recall some known properties of the the linearized operator on $\mathbb{R}^3$. Consider a solution u of (NLS) close to $e^{it}Q$ and write u as the following

$$u(t,x) = e^{it}\left(Q(x) + \hbar(t,x)\right).$$

Note that $\hbar$ is the solution of the equation

$$\partial_t \hbar + \mathcal{L}\hbar = \mathcal{R}(\hbar), \qquad \mathcal{L}\hbar = -\mathcal{L}_- \hbar_2 + i\mathcal{L}_+ \hbar_1.$$

Where

$$\mathcal{L}_+ \hbar_1 := -\Delta \hbar_1 + \hbar_1 - 3Q^2 \hbar_1, \qquad \mathcal{L}_- \hbar_1 := -\Delta \hbar_2 + \hbar_2 - Q^2 \hbar_2,$$
$$\mathcal{R}(\hbar) := iQ(2|\hbar|^2 + \hbar^2) + i|\hbar|^2 \hbar.$$

Define $\Phi(\hbar)$, a linearized energy on $\mathbb{R}^3$, by

$$\Phi(\hbar) := \frac{1}{2}\int |\hbar|^2 + \frac{1}{2}|\nabla \hbar|^2 - \frac{1}{2}\int Q^2(3\hbar_1^2 + \hbar_2^2). \tag{IV.16}$$

We next define a subspace of H^1 where Φ is positive.

$$\mathcal{G} := \left\{ \hbar \in H^1 \backslash \int \partial_{x_j} Q \hbar_1 = 0, \int Q\hbar_2 = 0,\ j = 1,2,3 \right\}.$$

Then by [31], there exists $c > 0$ such that

$$\forall \hbar \in \mathcal{G}, \quad \Phi(\hbar) \geq c \left\|\hbar\right\|_{H^1}^2. \tag{IV.17}$$

Let $h \in H^1(\mathbb{R}^3)$. Define

$$\Phi_\Psi(h) := \frac{1}{2}\int |\nabla h|^2 - \frac{1}{2}\int Q^2 \Psi^2(\cdot + X)(3h_1^2 + h_2^2) + \frac{1}{2}\int |h|^2. \tag{IV.18}$$

Where Ψ is defined in the end of the introduction.

Lemma 2.5. *There exists $c > 0$ and $X > 0$ large such that for all $h \in H^1$ satisfying the* *following orthogonality relations*

$$\mathrm{Re}\int \Delta(Q(x)\Psi(x+X))h(x+X)\,dx = 0, \quad \mathrm{Im}\int Q(x)\Psi(x+X)h(x+X)\,dx = 0.$$
$$\mathrm{Re}\int \partial_{x_k}(Q(x)\Psi(x+X))h(x+X)\,dx = 0, \quad k = 1,2,3.$$

Then

$$\Phi_\Psi(h(\cdot+X)) \geq c\,\|h\|_{H^1}^2\,. \tag{IV.19}$$

Proof. Write $h(\cdot+X) = \tilde{h}(\cdot+X) + r(\cdot+X)$, where

$$\tilde{h}(\cdot+X) \in \left\{ f \in H^1 \backslash\, \mathrm{Re}\int \Delta Q\, f = \mathrm{Im}\int Qf = \mathrm{Re}\int \partial_{x_k} Q\, f = 0,\ k = 1,2,3 \right\}$$
$$r(\cdot+X) \in \mathrm{span}\{iQ, \Delta Q, \partial_{x_1}Q, \partial_{x_2}Q, \partial_{x_3}Q\}$$

By (IV.16) and (IV.17), we have $\Phi(\tilde{h}(\cdot+X)) \geq c\|\tilde{h}\|_{H^1}^2$. Write r as the following

$$r(\cdot+X) = \sum_{k=1}^{3} \alpha_k \partial_{x_k} Q + \beta iQ + \gamma\Delta Q.$$

Taking the real L^2-scalar product in $\mathbb{R}^3$ of r with iQ and using orthogonality condition with Lemma 2.2 we get

$$\beta = \frac{1}{\|Q\|_{L^2}^2}(r(\cdot+X), iQ) = \frac{1}{\|Q\|_{L^2}^2}((h(\cdot+X) - \tilde{h}(\cdot+X), iQ)$$
$$= \frac{1}{\|Q\|_{L^2}^2}\left(\mathrm{Im}\int h(x+X)Q(x)\,dx - \mathrm{Im}\int \tilde{h}(x+X)Q(x)\,dx.\right)$$

By the definition of $\tilde{h}$, we have $\mathrm{Im}\int \tilde{h}(x+X)Q(x)dx = 0$. Using the orthogonality conditions and the exponential decay of Q with Lemma 2.3, we obtain

$$\beta = \frac{1}{\|Q\|_{L^2}^2}\,\mathrm{Im}\int h(x+X)Q(x)dx$$
$$= \frac{1}{\|Q\|_{L^2}^2}\,\mathrm{Im}\int h(x+X)Q(x)\Psi(x+X)\,dx - \frac{1}{\|Q\|_{L^2}^2}\,\mathrm{Im}\int h(x+X)Q(x)(\Psi(x+X)-1)dx$$
$$= O(e^{-|X|}\,\|h\|_{H^1}).$$

Similarly by taking the scalar product of r with ΔQ and $\partial_{x_k}Q$ and using orthogonality condition with Lemma 2.2 we obtain $\gamma = \alpha_k = O(e^{-|X|}\,\|h\|_{H^1})$.

Thus

$$\begin{aligned} \|r\|_{H^1} &\leq Ce^{-|X|}\|h\|_{H^1} \\ |\Phi_\Psi(r(\cdot+X))| &\leq e^{-2|X|}\|h\|_{H^1}^2 \end{aligned}$$

We have,

$$\Phi_\Psi(h(\cdot+X)) = \Phi_\Psi(\tilde{h}(\cdot+X)) + \Phi_\Psi(r(\cdot+X)) + 2B_\Psi(\tilde{h}(\cdot+X), r(\cdot+X)),$$

where

$$\begin{aligned} B_\Psi(f,g) :&= \frac{1}{2}\int \nabla f_1(x)\nabla g_1(x) + f_1(x)\, g_1(x) - 3Q^2(x)\Psi^2(x+X)f_1(x)\, g_1(x)\, dx \\ &+ \frac{1}{2}\int \nabla f_2(x)\nabla g_2(x) + f_2(x) g_2(x) - Q^2(x)\Psi^2(x+X)f_2(x))\, g_2(x)\, dx. \end{aligned}$$

and

$$\left|B_\Psi(\tilde{h}(\cdot+X), r(\cdot+X))\right| \leq e^{-|X|}\|h\|_{H^1}.$$

Then,

$$\Phi_\Psi(h(\cdot+X)) = \Phi(\tilde{h}(\cdot+X)) + O\left(e^{-c|X|}\|h\|_{H^1}\right) \geq C\|h\|_{H^1}^2.$$

This implies that, there exists $c, R > 0$ such that for $|X| > R$

$$\Phi_\Psi(h(\cdot+X)) \geq c\|h\|_{H^1}^2.$$

□

2.3 Cauchy theory and profile decomposition

Next, we recall results needed in Section 4.1 to prove the compactness property, up to a space translation, of a critical solution of (NLS_Ω) equation using a profile decomposition. We will use the same notations as in [58]. Without loss of generality, we assume that $0 \in \Theta = \Omega^c$ and $\Theta \subset B(0,1)$.

We define χ a smooth cutoff function in $\mathbb{R}^3$

$$\chi(x) := \begin{cases} 1 & |x| \leq \frac{1}{4}, \\ \\ 0 & |x| > \frac{1}{2}. \end{cases}$$

We define spaces $S^k(I)$, $k = 0, 1$, as follows, in order to avoid the endpoint of Strichartz estimate for exterior domain, see Theorem 2.6 below.

$$S^0(I) := L_t^\infty L_x^2(I \times \Omega) \cap L_t^{\frac{5}{2}} L_x^{\frac{30}{7}}(I \times \Omega),$$
$$S^1(I) : = \{u : I \times \Omega \longrightarrow \mathbb{C} \mid u, (-\Delta_\Omega)^{\frac{1}{2}} u \in S^0(I)\}.$$

By interpolations,

$$\|u\|_{L_t^q L_x^r(I\times\Omega)} \leq \|u\|_{S^0(I)}, \qquad \text{for all } \frac{2}{q} + \frac{3}{r} = \frac{3}{2}, \text{ with } \frac{5}{2} \leq q \leq \infty.$$

Similar estimate hold for $S^1(I)$. We will, in particular, use (q, r) equal to $(5, \frac{30}{11})$ and $(\infty, 2)$.

One particular Strichartz space we use is

$$X^1(I) := L_t^5 H_0^{1, \frac{30}{11}}(I \times \Omega).$$

Note that, by Sobolev embedding, there exists $C > 0$ such that $\|f\|_{L^5_{t,x}(I\times\Omega)} \leq C \|f\|_{X^1(I)}$.

Note that, we define $S^0(I)$, respectively $S^1(I)$, such that only all Strichartz pairs used in this paper are covered. Nevertheless, one might also consider $q \geq 2 + \varepsilon$, for all $\varepsilon > 0$ to avoid the endpoint.

Furthermore, we define $N^0(I)$ as the corresponding dual of $S^0(I)$ and

$$N^1(I) := \{u : I \times \Omega \longrightarrow \mathbb{C} \mid u, (-\Delta_\Omega)^{\frac{1}{2}} u \in N^0(I)\}.$$

So that, we have

$$\|u\|_{N^0(I)} \leq \|u\|_{L_t^{q'} L_x^{r'}(I\times\Omega)} \qquad \text{for all} \quad \frac{2}{q}+\frac{3}{r}=\frac{3}{2}, \text{ with } \quad \frac{5}{2}\leq q\leq\infty,$$

$$\frac{1}{q}+\frac{1}{q'}=1, \quad \text{and} \quad \frac{1}{r}+\frac{1}{r'}=1.$$

In particular, we will use $(q', r') = (\frac{5}{3}, \frac{30}{23})$ the Hölder dual to the Strichartz pairs $(q,r)=(\frac{5}{2},\frac{30}{7})$. One can get similar estimate for $N^1(I)$ using the same pair, see Theorem 2.6.

Next, we recall the Strichartz estimates with the above notations as follows:

Theorem 2.6 (Strichartz estimate, [48]). *Let I be a time interval and $t_0 \in I$. Let $u_0 \in H_0^1(\Omega)$ then there exists a constant $C>0$ such that the solution $u(t,x)$ to the nonlinear Schrödinger equation on $\Omega\times\mathbb{R}$ with Dirichlet boundary conditions*

$$\begin{cases} i\partial_t u + \Delta_\Omega u = f \quad \text{on } \mathbb{R}\times\Omega. \\ u(0,x)=u_0(x) \\ u_{|\partial\Omega}=0 \end{cases}$$

satisfies

$$\|u\|_{S^0(I)} \leq C\left(\|u_0\|_{L^2(\Omega)} + \|f\|_{N^0(I)}\right),$$

and

$$\|u\|_{S^1(I)} \leq C\left(\|u_0\|_{H_0^1(\Omega)} + \|f\|_{N^1(I)}\right).$$

In particular,

$$\|u\|_{X^1(I\times\Omega)} \leq C\left(\|u_0\|_{H_0^1(\Omega)} + \|f\|_{L^{\frac{5}{3}} H_0^{1,\frac{30}{23}}(I\times\Omega)}\right).$$

Proposition 2.7 (Local Smoothing, Corollary 2.14, [59]). *Given $u_0\in H_0^1(\Omega)$, we have*

$$\left\|\nabla e^{it\Delta_\Omega}u_0\right\|_{L^{\frac{5}{2}}L^{\frac{30}{17}}(|t-\tau|\leq T,\,|x-z|\leq R)} \leq R^{\frac{31}{60}}T^{\frac{1}{5}}\left\|e^{it\Delta_\Omega}u_0\right\|^{\frac{1}{6}}_{L^5_{t,x}(R\times\Omega)}\|u_0\|^{\frac{5}{6}}_{H_0^1(\Omega)},$$

uniformly in u_0 and the parameters $R,T>0$, $z\in\mathbb{R}^3$ and $\tau\in\mathbb{R}$.

Lemma 2.8 (Stability,[58]). *Let $I\subset\mathbb{R}$ be a time interval and let $\tilde{u}$ be an approximate solution to (NLS_Ω) on $I\times\Omega$ in the sense that*

$$i\partial_t\tilde{u}+\Delta_\Omega\tilde{u} = -|\tilde{u}|^2\tilde{u}+e, \quad \textit{for some function } e$$

Assume that

$$\|\tilde{u}\|_{L^\infty H_0^1(I\times\Omega)} \leq \mathcal{E} \quad and \quad \|\tilde{u}\|_{L^5_{t,x}(I\times\Omega)} \leq L$$

for some positive constants $\mathcal{E}$ *and* L*. Let* $t_0 \in I$ *and* $u_0 \in H_0^1(\Omega)$ *and assume the smallness conditions*

$$\|\tilde{u}(t_0) - u(t_0)\|_{H_0^1(\Omega)} \leq \varepsilon \quad and \quad \|e\|_{N^1(I)} \leq \varepsilon$$

for some $0 < \varepsilon < \varepsilon_1 = \varepsilon_1(\mathcal{E}, L)$*. then there exists a unique solution* $u : I \times \Omega \longrightarrow \mathbb{C}$ *to (*NLS_Ω*) with initial data* $u(t_0) = u_0$ *satisfying*

$$\|u - \tilde{u}\|_{X^1(I\times\Omega)} \leq C(\mathcal{E}, L)\varepsilon.$$

Theorem 2.9 (Linear profile decomposition in $H_0^1(\Omega)$, Theorem 3.2 [58])**.** *Let* $\{f_n\}$ *be a bounded sequence in* $H_0^1(\Omega)$*. After passing to a subsequence, there exist* $J^* \in \{0, 1, 2, ...\infty\}$, $\{\phi_n^j\}_{j=1}^{J^*} \subset H_0^1(\Omega)$, $\{t_n^j\}_{j=1}^{J^*} \subset \mathbb{R}$ *such that, for each* j *either* $t_n^j \equiv 0$ *or* $t_n^j \to \mp\infty$ *and* $\{x_n^j\}_{j=1}^{J^*} \subset \Omega$ *conforming to one of the following two cases for each* j *:*
Case 1: $x_n^j = 0$ *and there exists* $\phi^j \in H_0^1(\Omega)$ *so that* $\phi_n^j := e^{it_n^j\Delta_\Omega}\phi^j$*.*
Case 2: $|x_n^j| \to \infty$ *and there exists* $\phi^j \in H^1(\mathbb{R}^3)$ *so that*

$$\phi_n^j := e^{it_n^j\Delta_\Omega}[(\chi_n^j\phi^j)(x - x_n^j)], \;\; with \quad \chi_n^j(x) := \chi\left(\frac{x}{|x_n^j|}\right).$$

Moreover, for any finite $0 \leq J \leq J^*$ *we have the decomposition*

$$f_n = \sum_{j=1}^{J} \phi_n^j + \omega_n^J$$

with $\omega_n^J \in H_0^1(\Omega)$ *satisfying*

$$\lim_{J\to J^*} \limsup_{n\to\infty} \left\|e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5_{t,x}(\mathbb{R}\times\Omega)} = 0, \tag{IV.20}$$

$$\lim_{n\to\infty} \left\{M[f_n] - \sum_{j=1}^{J} M[\phi_n^j] - M[\omega_n^J]\right\} = 0, \tag{IV.21}$$

$$\lim_{n\to\infty} \left\{E[f_n] - \sum_{j=1}^{J} E[\phi_n^j] - E[\omega_n^J]\right\} = 0, \tag{IV.22}$$

$$[e^{-it_n^J\Delta_\Omega}\omega_n^J](x + x_n^J) \rightharpoonup 0 \quad weakly\ in\ H^1(\mathbb{R}^3), \tag{IV.23}$$

$$\lim_{n\to\infty} \left|x_n^j - x_n^k\right| + \left|t_n^j - t_n^k\right| = \infty \quad for\ each\ j \neq k. \tag{IV.24}$$

Theorem 2.10 ([58])**.** *Let* $\{t_n\} \subset \mathbb{R}$ *be such that* $t_n \equiv 0$ *or* $t_n \to \pm\infty$*. Let* $\{x_n\} \subset \Omega$ *be such* *that $|x_n|$ tends to ∞, as n goes to ∞. Assume $\phi \in H^1(\mathbb{R}^3)$ satisfies*

$$\|\nabla\phi\|_{L^2(\mathbb{R}^3)}\|\phi\|_{L^2(\mathbb{R}^3)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)}\|Q\|_{L^2(\mathbb{R}^3)} \tag{IV.25}$$

$$M_{\mathbb{R}^3}[\phi]E_{\mathbb{R}^3}[\phi] < M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q]. \tag{IV.26}$$

Define

$$\phi_n := e^{it_n\Delta_\Omega}\left[(\chi_n\phi)(x-x_n)\right], \text{ with } \chi_n(x) := \chi\left(\frac{x}{|x_n|}\right)$$

Then, for n sufficiently large, there exists a global solution v_n to (NLS$_\Omega$) *with initial data $v_n(0) := \phi_n$, which satisfies*

$$\|v_n\|_{L^5_{t,x}(\mathbb{R}\times\Omega)} \leq C\left(\|\phi\|_{H^1(\mathbb{R}^3)}\right).$$

Furthermore, for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ and $\psi_\varepsilon \in C_c(\mathbb{R}\times\mathbb{R}^3)$ such that, for all $n \geq N_\varepsilon$

$$\|\underline{v}_n(t-t_n, x+x_n) - \psi_\varepsilon(t,x)\|_{L^5H^{1,\frac{30}{11}}(\mathbb{R}\times\mathbb{R}^3)} < \varepsilon. \tag{IV.27}$$

Remark 2.11. *Note that, we have made a slight modification in the notation of the above result, in order, to keep the consistent notations in this paper. We denote $\underline{v}_n$ the extension of the solution v_n by 0 on Ω^c, such that $\underline{v}_n \in H^1(\mathbb{R}^3)$. Let us mention that ϕ_n is well defined in $H^1_0(\Omega)$, indeed, by the definition of χ_n and as $|x_n| \to \infty$, we have*

$$\text{If } x \in \partial\Omega, \quad \text{then} \quad \chi_n(x-x_n) \longrightarrow 0, \qquad \text{as } n \to +\infty.$$

Moreover, one can check that the energy-mass assumption (IV.26) *is equivalent to one given in [58, Theorem 4.1] using the following identity.*

$$\left\{u_0 : E[u_0]M[u_0] < E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q]\right\} := \bigcup_{0<\lambda<\infty}\left\{u_0 : E[u_0] + \lambda M[u_0] < 2\sqrt{\lambda E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q]}\right\}.$$

3 Modulation

Let $u \in H^1_0(\Omega)$ and define

$$\delta(u) := \left|\int_{\mathbb{R}^3}|\nabla Q|^2 - \int_\Omega |\nabla u|^2\right|.$$

Assume that,

$$M[u] = M_{\mathbb{R}^3}[Q], \quad E[u] = E_{\mathbb{R}^3}[Q]. \tag{IV.28}$$

Lemma 3.1. *Let $u \in H^1_0(\Omega)$ satisfying* (IV.28) *and $\delta(u)$ small enough. Then there exists*

$X_0 \in \mathbb{R}^3$ large and $\theta_0 \in \mathbb{R}$ such that

$$e^{-i\theta_0}u(x) = Q(x - X_0)\Psi(x) + h(x), \tag{IV.29}$$

with $\|h\|_{H_0^1(\Omega)} \leq \tilde{\varepsilon}(\delta(u))$ where $\tilde{\varepsilon}(\delta(u)) \to 0$, as $\delta(u) \to 0$.

Proof. Let $\underline{u} \in H^1(\mathbb{R}^3)$ defined as above (IV.14) and observe that $\delta(u) = \delta(\underline{u})$. By Proposition 2.1, since

$$M[u] = M_{\mathbb{R}^3}[\underline{u}] = M_{\mathbb{R}^3}[Q], \quad E[u] = E_{\mathbb{R}^3}[\underline{u}] = E_{\mathbb{R}^3}[Q]. \tag{IV.30}$$

and $\delta(\underline{u})$ is small enough then there exits $\theta_0 \in \mathbb{R}$ and $X_0 \in \mathbb{R}^3$ such that

$$e^{-i\theta_0}\underline{u}(x) = Q(x - X_0) + \tilde{h}(x)$$

with $\|\tilde{h}\|_{H^1(\mathbb{R}^3)} \leq \tilde{\varepsilon}(\delta(\underline{u}))$, where $\tilde{\varepsilon}(\delta(\underline{u})) \longrightarrow 0$ as $\delta(\underline{u}) \longrightarrow 0$.

Moreover, if $x \in \Omega^c$ then $\underline{u}(x) = 0$ on Ω^c, which implies that

$$x \in \Omega^c \quad \Longrightarrow \quad Q(x - X_0) + \tilde{h}(x) = 0, \tag{IV.31}$$

and for $\delta(\underline{u})$ small enough, by (IV.15), $|X_0|$ is large such that

$$\frac{e^{-|X_0|}}{|X_0|} \leq C\,\tilde{\varepsilon}(\delta(\underline{u})).$$

We write,

$$\begin{aligned} e^{-i\theta_0}\underline{u}(x) &= Q(x - X_0)\Psi(x) + (1 - \Psi(x))Q(x - X_0)) + \tilde{h}(x) \\ &= Q(x - X_0)\Psi(x) + h(x). \end{aligned}$$

Using the fact that $(1 - \Psi)$ has a compact support, Q having an exponential decay, $|X_0|$ being large, and Lemma 2.3, we get

$$\|h\|_{H^1(\mathbb{R}^3)} \leq \tilde{\varepsilon}(\delta(\underline{u})) + Ce^{-|X_0|} \leq \tilde{\varepsilon}(\delta(\underline{u})).$$

By (IV.31) and the definition of Ψ, we have

$$h(x) = 0, \quad \text{if } \ x \in \Omega^c.$$

Thus, $h(x) = 0$ on $\partial\Omega$ and $h(x) \in H_0^1(\Omega)$, which concludes the proof. $\square$

Lemma 3.2. *There exists $\delta_0 > 0$ and a positive function $\varepsilon(\delta)$ defined for $0 < \delta \leq \delta_0$, which tends to 0 when $\delta \to 0$, such that for any $u \in H_0^1(\Omega)$ satisfying (IV.28) and $\delta(u) < \delta_0$, there exists a couple $(\mu, X) \in \mathbb{R} \times \mathbb{R}^3$ such that the following hold*

$$\left\| u(x) - Q(x-X)\Psi(x)e^{i\mu} \right\|_{H_0^1(\Omega)} \leq \varepsilon(\delta), \tag{IV.32}$$

and

$$\operatorname{Re} \int_\Omega u(x)\, \partial_{x_k}(Q(x-X)\Psi(x))e^{-i\mu}\, dx = 0, \qquad k = 1,2,3, \tag{IV.33}$$

$$\operatorname{Im} \int_\Omega u(x)\, Q(x-X)\Psi(x)e^{-i\mu} dx = 0. \tag{IV.34}$$

The parameters μ and X are unique in $\mathbb{R}/\pi\mathbb{Z} \times \mathbb{R}^3$ and the mapping $u \to (\mu, X)$ is C^1.

Proof. Let

$$\begin{aligned} \Phi : H_0^1(\Omega) \times \mathbb{R}^3 \times \mathbb{R} &\longrightarrow \mathbb{R}^4 \\ (u\ ,\ X\ ,\ \mu) &\longmapsto (\Phi_k(u, X, \mu))_{1\leq k \leq 4}\,, \end{aligned}$$

where

$$\begin{aligned} \Phi_k(u, X, \mu) &:= \operatorname{Re} \int_\Omega u(x)\, \partial_{x_k}(Q(x-X)\Psi(x))e^{-i\mu}\, dx,\ k = 1,2,3, \\ \Phi_4(u, X, \mu) &:= \operatorname{Im} \int_\Omega u(x)\, Q(x-X)\Psi(x)\, e^{-i\mu} dx. \end{aligned}$$

Let $X_0 \in \mathbb{R}^3$. Note that $\Phi(Q(\cdot - X_0)\Psi, X_0, 0) = 0$, indeed, using integration by parts we get

$$\begin{aligned} \Phi_k(Q(\cdot - X_0)\Psi, X_0, 0) &= \operatorname{Re} \int_\Omega Q(x-X_0)\Psi(x)\partial_{x_k}(Q(x-X_0)\Psi(x))\, dx \\ &= \frac{1}{2} \operatorname{Re} \int_\Omega \partial_{x_k}((Q(x-X_0)\Psi(x))^2)\, dx = 0. \\ \Phi_4(Q(\cdot - X_0)\Psi, X_0, 0) &= \operatorname{Im} \int_\Omega Q(x-X_0)^2\Psi(x)^2\, dx = 0. \end{aligned}$$

- Step 1: Computation of $d_{(X,\mu)}\Phi_k$.

$$\frac{\partial}{\partial X_j}\Phi_k(u, X, \mu) = -\operatorname{Re} \int_\Omega e^{-i\mu} u(x)\partial_{x_k}(\partial_{x_j} Q(x-X)\Psi(x))\, dx$$

Integrating by parts, we obtain

$$\frac{\partial}{\partial X_j}\Phi_k(Q(\cdot - X_0)\Psi, X_0, 0) = \operatorname{Re}\int_\Omega \partial_{x_j} Q(x-X_0)\Psi(x)\partial_{x_k}(Q(x-X_0)\Psi(x))\,dx.$$

If $k = j$, we have

$$\begin{aligned}\frac{\partial}{\partial X_j}\Phi_k(Q(\cdot - X_0)\Psi, X_0, 0) &= \operatorname{Re}\int_\Omega (\partial_{x_j} Q(x-X_0))^2\,dx + \operatorname{Re}\int_\Omega (\partial_{x_j} Q(x-X_0))^2(\Psi(x)^2-1)\,dx \\ &\quad + \operatorname{Re}\int_\Omega Q(x-X_0)\partial_{x_j}Q(x-X_0)\Psi(x)\partial_{x_j}\Psi(x)\,dx.\end{aligned}$$

Using the fact that $\partial_{x_j}\Psi$ has a compact support and the exponential decay of Q, we obtain

$$\begin{aligned}\frac{\partial}{\partial X_j}\Phi_k(Q(\cdot - X_0)\Psi, X_0, 0) &= \left\|\partial_{x_j} Q\right\|^2_{L^2(\mathbb{R}^3)} + O(e^{-2|X_0|}) \\ &= \frac{1}{3}\left\|\nabla Q\right\|^2_{L^2(\mathbb{R}^3)} + O(e^{-2|X_0|}).\end{aligned}$$

If $k \neq j$, then

$$\begin{aligned}\frac{\partial}{\partial X_j}\Phi_k(Q(\cdot - X_0)\Psi, X_0, 0) &= \operatorname{Re}\int_\Omega \partial_{x_j} Q(x-X_0)\Psi(x)\partial_{x_k}(Q(x-X_0)\Psi(x))\,dx \\ &= \operatorname{Re}\int_\Omega \partial_{x_j} Q(x-X_0)\partial_{x_k}Q(x-X_0)\Psi^2(x)\,dx \\ &+ \operatorname{Re}\int_\Omega \partial_{x_j} Q(x-X_0)\Psi(x)Q(x-X_0)\partial_{x_k}\Psi(x)dx \\ &= \operatorname{Re}\int_\Omega \partial_{x_j} Q(x-X_0)\partial_{x_k}Q(x-X_0)dx \\ &+ \operatorname{Re}\int_\Omega \partial_{x_j} Q(x-X_0)\partial_{x_k}Q(x-X_0)(\Psi(x)^2-1)dx \\ &+ \operatorname{Re}\int_\Omega \partial_{x_j} Q(x-X_0)\Psi(x)Q(x-X_0)\partial_{x_k}\Psi(x)dx.\end{aligned}$$

Using the same argument as above and the fact that Q is radial ($\int \partial_{x_j} Q\partial_{x_k} Q = 0$, if $k \neq j$), we obtain

$$\frac{\partial}{\partial X_j}\Phi_k(Q(\cdot - X_0)\Psi, X_0, 0) = O(e^{-2|X_0|}).$$

Next, we compute $\frac{\partial}{\partial\mu}\Phi_k(u,X,\mu)$:

$$\frac{\partial}{\partial\mu}\Phi_k(u,X,\mu) = \mathrm{Re}\int_\Omega -ie^{-i\mu}u(x)\partial_{x_k}(Q(x-X)\Psi(x))dx.$$

$$\frac{\partial}{\partial\mu}\Phi_k(Q(\cdot-X_0)\Psi,X_0,0) = \mathrm{Im}\int_\Omega Q(x-X_0)\Psi(x)\partial_{x_k}(Q(x-X_0)\Psi(x))\,dx = 0$$

- Step 2 : Computation of $d_{(X,\mu)}\Phi_4$.

$$\frac{\partial}{\partial X_j}\Phi_4(u,X,\mu) = -\,\mathrm{Im}\int_\Omega e^{-i\mu}u(x)(\partial_{x_j}Q(x-X)\Psi(x))\,dx$$

We get

$$\frac{\partial}{\partial X_j}\Phi_4(Q(\cdot-X_0)\Psi,X_0,0) = -\,\mathrm{Im}\int_\Omega Q(x-X_0)\Psi(x)\partial_{x_j}(Q(x-X_0)\Psi(x))\,dx = 0.$$

$$\begin{aligned}\frac{\partial}{\partial\mu}\Phi_4(u,X,\mu) &= \mathrm{Im}\int_\Omega -ie^{-i\mu}u(x)Q(x-X)\Psi(x)\,dx\\ \frac{\partial}{\partial\mu}\Phi_4(Q(\cdot-X_0)\Psi,X_0,0) &= -\int_\Omega Q(x-X_0)^2\Psi(x)^2\\ &= -\int_\Omega Q(x-X_0)^2 - \int_\Omega Q(x-X_0)^2(\Psi(x)^2-1)\\ &= -\,\|Q\|^2_{L^2(\mathbb{R}^3)} + O(e^{-2|X_0|}).\end{aligned}$$

- Step 3: Conclusion.
  Combining Step 1 and Step 2 we get

$$d_{(X,\mu)}\Phi(Q(\cdot-X_0)\Psi,X_0,0) = \begin{pmatrix} \frac{1}{3}\|\nabla Q\|^2_{L^2(\mathbb{R}^3)} & 0 & 0 & 0\\ 0 & \frac{1}{3}\|\nabla Q\|^2_{L^2(\mathbb{R}^3)} & 0 & 0\\ 0 & 0 & \frac{1}{3}\|\nabla Q\|^2_{L^2(\mathbb{R}^3)} & 0\\ 0 & 0 & 0 & -\,\|Q\|^2_{L^2(\mathbb{R}^3)}\end{pmatrix} + O(e^{-2|X_0|}).$$

We can deduce that $d_{(X,\mu)}\Phi$ is invertible at $(Q(\cdot-X_0)\Psi(\cdot),X_0,0)$, if $|X_0|$ is large. Then, by the implicit function theorem there exists $\epsilon_0,\eta_0>0$ such that for $u\in H^1_0(\Omega)$,

$$\|u(\cdot)-Q(\cdot-X_0)\Psi(\cdot)\|^2_{H^1_0(\Omega)} < \epsilon_0 \Longrightarrow \exists!(X,\mu),\quad |\mu|+|X-X_0|\le\eta_0 \text{ and } \Phi(u,X,\mu)=0.$$

□

Let u be a solution of (NLS_Ω) satisfying (IV.28). In the sequel we will write

$$\delta(t) := \delta(u(t)).$$

Let $D_{\delta_0} = \{t \in I; \delta(t) < \delta_0\}$, where I is the maximal time interval of existence of u.
By Lemma 3.2, we can define C^1 functions $X(t)$ and $\mu(t)$, for $t \in D_{\delta_0}$. We will rather work with the parameter $\theta(t) := \mu(t) - t$. Write

$$e^{-i\theta(t)-it}u(t,x) := (1+\rho(t))Q(x-X(t))\Psi(x) + h(t,x), \tag{IV.35}$$

where $h(x) \in H_0^1(\Omega)$ and

$$\rho(t) := \mathrm{Re}\, \frac{e^{-i\theta(t)-it}\int \nabla\left(Q(x-X(t))\Psi(x)\right)\cdot \nabla \underline{u}(t,x)dx}{\int \left|\nabla\left(Q(x-X(t))\Psi(x)\right)\right|^2 dx} - 1.$$

This implies that

$$e^{-i\theta(t)-it}\underline{u}(t,x+X(t)) := (1+\rho(t))Q(x)\Psi(x+X(t)) + \underline{h}(t,x+X(t)), \tag{IV.36}$$

where $\underline{h}(x) \in H^1(\mathbb{R}^3)$ is define by

$$\underline{h}(t,x) := \begin{cases} h(t,x) & \forall x \in \Omega, \\ 0 & \forall x \in \Omega^c. \end{cases}$$

One can see that $\rho(t)$ is chosen such that h satisfies the following orthogonality condition

$$\mathrm{Re}\int_\Omega \Delta(Q(x-X(t))\Psi(x))h(t,x)\,dx = \mathrm{Re}\int \Delta(Q(x)\Psi(x+X(t)))\underline{h}(t,x+X((t))\,dx = 0. \tag{IV.37}$$

By Lemma 3.2, h also satisfies the following orthogonality conditions

$$\mathrm{Im}\int_\Omega h(t,x)Q(x-X(t))\Psi(x)\,dx = \mathrm{Im}\int \underline{h}(t,x+X(t))Q(x)\Psi(x+X(t))\,dx = 0. \tag{IV.38}$$

$$\operatorname{Re}\int_\Omega h(t,x)\partial_{x_k}(Q(x-X(t))\Psi(x))\,dx = \\ \operatorname{Re}\int \underline{h}(t,x+X(t))\partial_{x_k}(Q(x)\Psi(x+X(t)))\,dx = 0, \quad k=1,2,3. \quad \text{(IV.39)}$$

In the following lemma, to simplify notation, we denote $f(\cdot + X)$ by $f_X(\cdot)$ for any function f. If f is a complex function, then we denote by $f_{1X}(\cdot)$ the real part of f_X and by $f_{2X}(\cdot)$ the imaginary part.

Lemma 3.3. *Let $u(t)$ be a solution of* (NLS$_\Omega$) *satisfying* (IV.28)*. Then the following estimates hold for $t \in D_{\delta_0}$,*

$$|\rho(t)| + O\left(\frac{e^{-2|X(t)|}}{|X(t)|^2}\right) \approx \left|\int Q\Psi_X \underline{h}_{1X} dx\right| + O\left(\frac{e^{-2|X(t)|}}{|X(t)|^2}\right) \approx \delta(t) + O\left(\frac{e^{-2|X(t)|}}{|X(t)|^2}\right) \\ \approx \|h(t)\|_{H_0^1(\Omega)} + O\left(\frac{e^{-|X(t)|}}{|X(t)|}\right). \quad \text{(IV.40)}$$

Proof. Let $\tilde{\delta}(t) := |\rho(t)| + \|\underline{h}\|_{H^1} + \delta(t)$, which is small if $\delta(t)$ is small. By the expansion of u in (IV.36) we have $e^{-i\theta(t)-it}\underline{u}(t,x+X(t)) := (1+\rho(t))Q(x)\Psi_X(x) + \underline{h}_X(t,x)$, thus if $x + X(t) \in \Omega$, $\underline{u}(t,x+X(t)) = u(t,x+X(t))$, otherwise $\underline{u}(t,x+X(t)) = 0$.

- Step 1: Approximation of $|\rho|$ using mass conservation.
  Since $M[u] = M_{\mathbb{R}^3}[\underline{u}] = M_{\mathbb{R}^3}[Q\Psi_X + \rho Q\Psi_X + \underline{h}_X] = M_{\mathbb{R}^3}[Q]$, we have,

$$\int \left(Q^2(\Psi_X^2 - 1) + 2\rho\, Q^2\Psi_X^2 + 2\rho\, Q\Psi_X \underline{h}_{1X} + \rho^2 Q^2 \Psi_X^2 + 2Q\Psi_X\underline{h}_{1X} + |\underline{h}_X|^2\right) dx = 0. \quad \text{(IV.41)}$$

  Using (IV.41) and Lemma 2.3, we obtain

$$2|\rho|\left|\int Q^2\,\Psi_X^2\right| = \left|2\int Q\Psi_X\underline{h}_{1X} + \int Q^2(\Psi_X^2 - 1) + 2\rho\int Q\Psi_X\underline{h}_{1X}dx + \rho^2\int Q^2\Psi_X^2 + \int |\underline{h}_X|^2\,dx\right| \\ = 2\left|\int Q\Psi_X\underline{h}_{1X}\,dx\right| + O\left(\tilde{\delta}^2 + \frac{e^{-2|X(t)|}}{|X(t)|^2}\right),$$

  which yields

$$|\rho| = \frac{1}{M[Q]}\left|\int Q\Psi_X\underline{h}_{1X}\,dx\right| + O\left(\tilde{\delta}^2 + \frac{e^{-2|X(t)|}}{|X(t)|^2}\right) \quad \text{(IV.42)}$$

- Step 2: Approximation of $|\rho|$ in terms of δ.

By the definition of $\delta(t)$, we have

$$\begin{aligned}\delta(t) &= \left|\int \left|\nabla(Q\Psi_X + \rho Q\Psi_X + \underline{h}_X)\right|^2 dx - \int |\nabla Q|^2 dx\right| \\ &= \left|\int |\nabla(Q\Psi_X)|^2 + 2\rho\,|\nabla(Q\Psi_X)|^2 + \rho^2\,|\nabla(Q\Psi_X)|^2 + 2\rho\nabla(Q\Psi_X).\nabla\underline{h}_{1X}\right. \\ &\quad \left. + 2\nabla(Q\Psi_X).\nabla\underline{h}_{1X} + \left|\nabla\underline{h}_X\right|^2 - \int |\nabla Q|^2 dx\right|.\end{aligned}$$

Using integration by parts and the orthogonality condition (IV.37), we get

$$\delta(t) = \left|\int |\nabla Q|^2 (\Psi_X^2 - 1) + 2\,\nabla Q\cdot\nabla\Psi_X Q\Psi_X + Q^2\,|\nabla\Psi_X|^2\right. \\ \left. + (2\rho+\rho^2)\int |\nabla(Q\Psi_X)|^2 + \int |\nabla \mathfrak{b}|^2\right|.$$

Using the fact that $(\Psi^2 - 1)$, $\nabla\Psi$ have a compact support and by Lemma 2.3 we get,

$$|\rho| = \frac{\delta}{2\,\|\nabla Q\|^2_{L^2(\mathbb{R}^3)}} + O\left(\tilde{\delta}^2 + \frac{e^{-2|X(t)|}}{|X(t)|^2}\right). \tag{IV.43}$$

- Step 3: Energy and Mass conservation.
  We denote: $g = \rho Q\Psi_X + \underline{h}_X$. Since $E_{\mathbb{R}^3}[\underline{u}] = E_{\mathbb{R}^3}[Q\Psi_X + g] = E_{\mathbb{R}^3}[Q]$, we have

$$\frac{1}{2}\int |\nabla(Q\Psi_X)|^2 - \frac{1}{2}\int |\nabla Q|^2 - \frac{1}{4}\int Q^4\Psi_X^4 + \frac{1}{4}\int Q^4 + \int \nabla(Q\Psi_X).\nabla g_1 - \int Q^3\Psi_X^3\, g_1 \tag{IV.44}$$

$$+\frac{1}{2}\int |\nabla g|^2 - \frac{1}{2}\int Q^2\Psi_X^2(3g_1^2 + g_2^2) - \int Q\Psi_X|g|^2 g_1 - \frac{1}{4}|g|^4 = 0 \tag{IV.45}$$

First, we estimate (IV.44), for that we denote:

$$\begin{aligned}A_0 &:= \frac{1}{2}\int |\nabla(Q\Psi_X)|^2 - \frac{1}{2}\int |\nabla Q|^2 - \frac{1}{4}\int Q^4\Psi_X^4 + \frac{1}{4}\int Q^4. \\ A_L(g) &:= \int \nabla(Q\Psi_X).\nabla g_1 - \int Q^3\Psi_X^3 g_1.\end{aligned}$$

In this step we show,

$$A_0 = O\left(\frac{e^{-2|X(t)|}}{|X(t)|^2}\right). \tag{IV.46}$$

$$A_L(g) = \frac{1}{2}\int |g|^2 - 2\int \nabla Q.\nabla\Psi_{_X}\, g_1 - \int Q\Delta\Psi_{_X}\, g_1 - \int Q^3\Psi_{_X}(\Psi_{_X}^2 - 1)g_1 + O\left(\frac{e^{-2|X(t)|}}{|X(t)|^2}\right). \tag{IV.47}$$

Using the fact that $\nabla\Psi, (\Psi^2 - 1)$ and $(\Psi^4 - 1)$ have a compact support and Lemma 2.3, we have

$$A_0 = \frac{1}{2}\int |\nabla(Q\Psi_{_X})|^2 - \frac{1}{2}\int |\nabla Q|^2 - \frac{1}{4}\int Q^4\Psi_{_X}^4 + \frac{1}{4}\int Q^4 = O\left(\frac{e^{-2|X(t)|}}{|X(t)|^2}\right).$$

Next, we will compute $A_L(g)$. Using integration by parts, we obtain the following equalities,

$$\int \nabla(Q\Psi_{_X}).\nabla g_1 = -\int \Delta(Q\Psi_{_X})g_1 = -\int \Delta Q\, \Psi_{_X} g_1 - 2\int \nabla Q.\nabla\Psi_{_X} g_1 - \int Q\Delta\Psi_{_X} g_1.$$

$$-\int Q^3\Psi_{_X}^3 g_1 = -\int Q^3\Psi_{_X}\, g_1 - \int Q^3\Psi_{_X}(\Psi_{_X}^2 - 1)g_1.$$

Using the equation (IV.7) of Q we have

$$\int \Delta Q\, \Psi_{_X} g_1 + \int Q^3\Psi_{_X} g_1 = \int Q\Psi_{_X}\, g_1.$$

Which yields

$$A_L(g) = -\int Q\Psi_{_X} g_1 - 2\int \nabla Q.\nabla\Psi_{_X} g_1 - \int Q\Delta\Psi_{_X}\, g_1 - \int Q^3\Psi_{_X}(\Psi_{_X}^2 - 1)g_1.$$

Since $M[u] = M[\underline{u}] = M[Q\Psi_{_X} + g] = M[Q]$ we have,

$$\int Q^2(\Psi_{_X}^2 - 1) + 2\int Q\Psi_{_X} g_1 + \int |g|^2 = 0$$

$$-\int Q\Psi_{_X} g_1 = \frac{1}{2}\int |g|^2 + O\left(\frac{e^{-2|X(t)|}}{|X(t)|^2}\right).$$

This implies

$$A_L(g) = \frac{1}{2}\int |g|^2 - 2\int \nabla Q.\nabla \Psi_{_X}\, g_1 - \int Q\Delta\Psi_{_X}\, g_1 - \int Q^3\Psi_{_X}(\Psi_{_X}^2 - 1)g_1 + O\left(\frac{e^{-2|X(t)|}}{|X(t)|^2}\right).$$

- Step 4: Approximation of $\|h\|_{H_0^1(\Omega)}$.
  Recall that $g = \rho\, Q\Psi_{_X} + \underline{h}_{_X}$. In this step we prove

$$\|h\|_{H_0^1(\Omega)} = O\left(|\rho| + \tilde{\delta}^{\frac{3}{2}} + \frac{e^{-|X(t)|}}{|X(t)|} + \frac{e^{-|X(t)|}}{|X(t)|}\tilde{\delta}^{\frac{1}{2}}\right)$$

Summing up all terms (IV.45), (IV.46) and (IV.47), we obtain

$$\begin{aligned}&\frac{1}{2}\int |\rho\, Q\Psi_{_X} + \underline{h}_{_X}|^2 - 2\int \nabla Q.\nabla\Psi_{_X}(\rho\, Q\Psi_{_X} + \underline{h}_{1_X}) - \int Q\Delta\Psi_{_X}(\rho\, Q\Psi_{_X} + \underline{h}_{1_X})\\ &- \int Q^3\Psi_{_X}(\Psi_{_X}^2 - 1)(\rho Q\Psi_{_X} + \underline{h}_{1_X}) + \frac{1}{2}\int \left|\nabla(\rho\, Q\Psi_{_X} + \underline{h}_{_X})\right|^2 - \frac{1}{2}\int Q^2\Psi_{_X}^2(3(\rho\, Q\Psi_{_X} + \underline{h}_{1_X})^2 + \underline{h}_{2_X}^2)\\ &- \int Q\Psi_{_X}|\rho\, Q\Psi_{_X} + \underline{h}_{_X}|^2(\rho\, Q\Psi_{_X} + \underline{h}_{1_X}) - \frac{1}{4}\int |\rho\, Q\Psi_{_X} + \underline{h}_{_X}|^4 + O\left(\frac{e^{-2|X(t)|}}{|X(t)|^2}\right) = 0\end{aligned}$$

Denote:

$$\begin{aligned}B_L(\underline{h}) &:= -2\int \nabla Q.\nabla\Psi_{_X}(\rho\, Q\Psi_{_X} + \underline{h}_{1_X}) - \int Q\Delta\Psi_{_X}(\rho\, Q\Psi_{_X} + \underline{h}_{1_X})\\ &\quad - \int Q^3\Psi_{_X}(\Psi_{_X}^2 - 1)(\rho Q\Psi_{_X} + \underline{h}_{1_X}).\\ B_{NL}^1(\underline{h}) &:= \frac{1}{2}\int |\rho\, Q\Psi_{_X} + \underline{h}_{_X}|^2 + \frac{1}{2}\int \left|\nabla(\rho\, Q\Psi_{_X} + \underline{h}_{_X})\right|^2.\\ B_{NL}^2(\underline{h}) &:= -\frac{1}{2}\int Q^2\Psi_{_X}^2(3(\rho\, Q\Psi_{_X} + \underline{h}_{1_X})^2 + \underline{h}_{2_X}^2) - \int Q\Psi_{_X}|\rho\, Q\Psi_{_X} + \underline{h}_{_X}|^2(\rho\, Q\Psi_{_X} + \underline{h}_{1_X})\\ &\quad - \frac{1}{4}\int |\rho\, Q\Psi_{_X} + \underline{h}_{_X}|^4.\end{aligned}$$

Next, we estimate each term. Using the fact that $\nabla\Psi$, $\Delta\Psi$ and $(\Psi^2 - 1)$ have compact supports and Lemma 2.3, we obtain

$$\begin{aligned}B_L(\underline{h}) &= -\int (2\nabla Q.\nabla\Psi_{_X} + Q\Delta\Psi_{_X})(\rho\, Q\Psi_{_X} + \underline{h}_{1_X}) - \int Q^3\Psi_{_X}(\Psi_{_X}^2 - 1)(\rho Q\Psi_{_X} + \underline{h}_{1_X})\\ &= O\left(|\rho|\frac{e^{-2|X(t)|}}{|X(t)|^2} + \frac{e^{-|X(t)|}}{|X(t)|}\|\underline{h}\|_{H^1}\right) + O\left(|\rho|\frac{e^{-4|X(t)|}}{|X(t)|^4} + \|\underline{h}\|_{H^1}\frac{e^{-3|X(t)|}}{|X(t)|^3}\right).\end{aligned}$$

Using the orthogonality condition (IV.37), we get

$$\begin{aligned}
B^1_{NL}(\underline{h}) &= \frac{1}{2}\int |\rho Q\Psi_X + \underline{h}_X|^2 + \frac{1}{2}\int \left|\nabla(\rho Q\Psi_X + \underline{h}_X)\right|^2. \\
&= \frac{\rho^2}{2}\int Q^2\Psi_X^2 + \rho\int Q\Psi_X\,\underline{h}_{1X} + \frac{1}{2}\int |\underline{h}_X|^2 + \frac{\rho^2}{2}\int |\nabla(Q\Psi_X)|^2 + \rho\int \nabla(Q\Psi_X).\nabla\underline{h}_{1X} \\
&+ \frac{1}{2}\int |\nabla\underline{h}_X|^2 \\
&= \rho\int Q\Psi_X\underline{h}_{1X} + \frac{1}{2}\int |\underline{h}|^2 + \frac{1}{2}\int |\nabla\underline{h}|^2 + O(|\rho|^2).
\end{aligned}$$

$$\begin{aligned}
B^2_{NL}(\underline{h}) = &-\frac{1}{2}\int Q^2\Psi_X^2(3\underline{h}_{1X}^2 + \underline{h}_{2X}^2) - \frac{1}{4}\int |\underline{h}_X|^4 - \rho\int Q\Psi_X|\underline{h}_X|^2\underline{h}_{1X} - \int Q\Psi_X|\underline{h}_X|^2\underline{h}_{1X} \\
&- \frac{\rho^2}{2}\int Q^2\Psi_X^2(3\underline{h}_{1X}^2 + \underline{h}_{2X}^2) - \rho\int Q^2\Psi_X^2|\underline{h}_X|^2 - 2\rho\int Q^2\Psi_X^2\,\underline{h}_{1X}^2 - \rho^3\int Q^3\Psi_X^3\,\underline{h}_{1X} \\
&- 3\rho^2\int Q^3\Psi_X^3\,\underline{h}_{1X} - 3\rho\int Q^3\Psi_X^3\,\underline{h}_{1X} - \frac{\rho^4}{4}\int Q^4\Psi_X^4 - \rho^3\int Q^4\Psi_X^4 - \frac{3\rho^2}{2}\int Q^4\Psi_X^4.
\end{aligned}$$

By the equation (IV.1) and using again the orthogonality condition (IV.37), we have

$$\begin{aligned}
-3\rho\int Q^3\Psi_X^3\,\underline{h}_{1X} &= -3\rho\int Q\Psi_X\underline{h}_{1X} - 3\rho\int Q\Psi_X^2(\Psi_X - 1)\underline{h}_{1X} - 6\rho\int \nabla Q.\nabla\Psi_X\,\underline{h}_{1X} \\
&- 3\rho\int \Delta\Psi_X Q\,\underline{h}_{1X} = -3\rho\int Q\Psi_X\,\underline{h}_{1X} + O\left(|\rho|\frac{e^{-|X(t)|}}{|X(t)|}\,\|\underline{h}\|_{H^1}\right).
\end{aligned}$$

Using the fact that

$$\begin{aligned}
&\rho\int Q\Psi_X|\underline{h}_X|^2\underline{h}_{1X} = O(|\rho|\,\|\underline{h}\|_{H^1}^3) \\
&\frac{\rho^2}{2}\int Q^2\Psi_X^2(3\underline{h}_{1X}^2 + \underline{h}_{2X}^2) - \rho\int Q^2\Psi_X^2|\underline{h}_X|^2 - 2\rho\int Q^2\Psi_X^2\underline{h}_{1X}^2 = O(|\rho|^2\,\|\underline{h}\|_{H^1}^2 + |\rho|\,\|\underline{h}\|_{H^1}^2) \\
&-\rho^3\int Q^3\Psi_X^3\underline{h}_{1X} - 3\rho^2\int Q^3\Psi_X^3\underline{h}_{1X} = O(|\rho|^3\,\|\underline{h}\|_{H^1} + |\rho|^2\,\|\underline{h}\|_{H^1})
\end{aligned}$$

and

$$-\frac{\rho^4}{4}\int Q^4\Psi_X^4 - \rho^3\int Q^4\Psi_X^4 - \frac{3\rho^2}{2}\int Q^4\Psi_X^4 = O(|\rho|^4 + |\rho^2|).$$

we obtain

$$\begin{aligned}
B^2_{NL} = &-\frac{1}{2}\int Q^2\Psi_X^2(3\underline{h}_{1X}^2 + \underline{h}_{2X}^2) - \int Q\Psi_X|\underline{h}_X|^2\underline{h}_{1X} - \frac{1}{4}\int |\underline{h}|^4 - 3\rho\int Q\Psi_X\underline{h}_{1X} \\
&+ O\left(|\rho|\,\|\underline{h}\|_{H^1}^2 + +|\rho|\frac{e^{-|X(t)|}}{|X(t)|}\,\|\underline{h}\|_{H^1} + |\rho|^2\right).
\end{aligned}$$

Thus,

$$\begin{aligned} B_L(\underline{h}) + B^1_{NL}(\underline{h}) + B^2_{NL}(\underline{h}) &= \frac{1}{2}\int |\underline{h}|^2 - \frac{1}{2}\int Q^2\Psi_X^2(3\underline{h}_{1X}^2 + \underline{h}_{2X}^2) + \frac{1}{2}\int |\nabla \underline{h}|^2 \\ &\quad - \frac{1}{4}\int |\underline{h}|^4 - \int Q\Psi_X |\underline{h}_X|^2 \underline{h}_{1X} - 2\rho \int Q\Psi_X \underline{h}_{1X} \\ &= O\bigg(|\rho|\, \|\underline{h}\|^2_{H^1} + |\rho|^2 + \frac{e^{-2|X(t)|}}{|X(t)|^2} + \frac{e^{-|X(t)|}}{|X(t)|} \|\underline{h}\|_{H^1} \bigg). \end{aligned} \tag{IV.48}$$

Recall that, from (IV.18) we have

$$\Phi_\Psi(\underline{h}) := \frac{1}{2}\int |\nabla \underline{h}|^2 - \frac{1}{2}\int Q^2\Psi_X^2(3\underline{h}_{1X}^2 + \underline{h}_{2X}^2) + \frac{1}{2}\int |\underline{h}|^2.$$

By (IV.48), one can see that,

$$\begin{aligned} \Phi_\Psi(\underline{h}_X) &= \frac{1}{4}\int |\underline{h}|^4 + \int Q\Psi_X |\underline{h}_X|^2 \underline{h}_{1X} + 2\rho\int Q\Psi_X \underline{h}_{1X} \\ &\quad + O\left(|\rho|\, \|\underline{h}\|^2_{H^1} + |\rho|^2 + \frac{e^{-2|X(t)|}}{|X(t)|^2} + \frac{e^{-|X(t)|}}{|X(t)|} \|\underline{h}\|_{H^1} \right). \end{aligned}$$

Thus,

$$\left|\Phi_\Psi(\underline{h}_X)\right| \le C\bigg(\|\underline{h}\|^3_{H^1} + 2|\rho| \left|\int Q\Psi_X \underline{h}_{1X}\right| + |\rho|^2 + \frac{e^{-2|X(t)|}}{|X(t)|^2} + \frac{e^{-|X(t)|}}{|X(t)|} \|\underline{h}\|_{H^1} \bigg).$$

By the coercivity property (IV.19) we obtain

$$\|\underline{h}\|_{H^1} = O\left(|\rho| + \tilde{\delta}^{\frac{3}{2}} + \frac{e^{-|X(t)|}}{|X(t)|} + \left|\int Q\Psi_X \underline{h}_{1X}\right| \right).$$

By (IV.42), we deduce

$$\|h\|_{H^1_0(\Omega)} = \|\underline{h}\|_{H^1(\mathbb{R}^3)} = O\left(|\rho| + \tilde{\delta}^{\frac{3}{2}} + \frac{e^{-|X(t)|}}{|X(t)|} + \frac{e^{-|X(t)|}}{|X(t)|}\tilde{\delta}^{\frac{1}{2}} \right) \tag{IV.49}$$

and thus by (IV.43), we get

$$\tilde{\delta} = O\left(|\rho| + \frac{e^{-|X(t)|}}{|X(t)|} \right),$$

which implies (IV.40) and this concludes the proof of Lemma 3.3.

□

Lemma 3.4. *Under the assumptions of Lemma 3.3, for all* $t \in D_{\delta_0}$:

$$|\rho'(t)| + |X'(t)| + |\theta'(t)| = O\left(\delta + \frac{e^{-|X(t)|}}{|X(t)|}\right) \tag{IV.50}$$

Proof. Let $\delta^*(t) := \delta(t) + |\rho'(t)| + |X'(t)| + |\theta'(t)|$. Using the equation (NLS_Ω) , Lemma 2.3 and Lemma 3.3 we obtain,

$$\begin{aligned} i\partial_t h + \Delta h + i\rho' Q_{-X}\Psi - iX' \cdot \nabla Q_{-X}\Psi - \theta' Q_{-X}\Psi \\ = O\left(\delta + \frac{e^{-|X(t)|}}{|X(t)|} + \delta^*(\delta + \frac{e^{-|X(t)|}}{|X(t)|})\right) \text{ in } L^2. \end{aligned} \tag{IV.51}$$

By the orthogonality conditions (IV.38), (IV.39) and Lemma 3.3, we have

$$\mathrm{Im}\int_\Omega \partial_t h\, Q_{-X}\Psi dx = \mathrm{Im}\int_\Omega h\, X' \cdot \nabla Q_{-X}\Psi\, dx = O\left(\delta^*(\delta + \frac{e^{-|X(t)|}}{|X(t)|})\right), \tag{IV.52}$$

$$\begin{aligned} \mathrm{Re}\int_\Omega \partial_t h\, \partial_{x_k}(Q_{-X}\Psi)dx &= \sum_{j=1}^{3}\mathrm{Re}\int_\Omega h\, X_j'(\partial_{x_k}(\partial_{x_j}Q\Psi))\, dx \\ &= O\left(\delta^*(\delta + \frac{e^{-|X(t)|}}{|X(t)|})\right),\ k = 1,2,3, \end{aligned} \tag{IV.53}$$

$$\mathrm{Re}\int_\Omega \partial_t h\, \Delta(Q_{-X}\Psi)dx \;=\; \sum_{j=1}^{3}\mathrm{Re}\int_\Omega h\, X_j'\Delta(\partial_{x_j}Q_{-X}\Psi)\, dx \;=\; O\left(\delta^*(\delta + \frac{e^{-|X(t)|}}{|X(t)|})\right). \tag{IV.54}$$

Multiplying (IV.51) by $Q_{-X}\Psi$, integrating the real part, using (IV.52), the orthogonality condition (IV.37), and then integrating by parts, we get

$$|\theta'| = O\left(\delta + \frac{e^{-|X(t)|}}{|X(t)|} + \delta^*(\delta + \frac{e^{-|X(t)|}}{|X(t)|})\right) \tag{IV.55}$$

Similarly, multiplying (IV.51) by $\partial_{x_j}(Q_{-X}\Psi)$, $j \in 1,2,3$, integrating the imaginary part, using (IV.53) and Lemma 3.3, we obtain

$$|X_j'(t)| = O\left(\delta + \frac{e^{-|X(t)|}}{|X(t)|} + \delta^*(\delta + \frac{e^{-|X(t)|}}{|X(t)|})\right), \qquad j = 1,2,3. \tag{IV.56}$$

Multiplying (IV.51) by $\Delta(Q_{-X}\Psi)$, integrating the imaginary part, and using (IV.54) and

Lemma 3.3, we get

$$|\rho'| = O\left(\delta + \frac{e^{-|X(t)|}}{|X(t)|} + \delta^*(\delta + \frac{e^{-|X(t)|}}{|X(t)|})\right) \tag{IV.57}$$

Summing up (IV.55), (IV.56) and (IV.57) we obtain

$$\delta^* = O\left(\delta + \frac{e^{-|X(t)|}}{|X(t)|} + \delta^*(\delta + \frac{e^{-|X(t)|}}{|X(t)|})\right)$$

which concludes the proof by choosing δ_0 sufficiently small. □

4 Scattering

In this section, we prove Theorem 1.1. We start proving, in §4.1 that the extension $\underline{u}$ of a non-scattering solution u to (NLS$_\Omega$) equation satisfying (IV.2) and (IV.3) is compact in H^1 up to a translation parameter $x(t)$ in space. In §4.2, we prove that the space translation $x(t)$ is bounded by approaching it by an auxiliary translation parameters given by previous work on $\mathbb{R}^3$, in [31]. Moreover, we use a local virial identity with the estimates in Section 3 of the modulation parameters to get a spacial control and to conclude the proof of Proposition 4.1. In §4.3, we prove that the parameter $\delta(t)$ converge to 0 in mean. Finally, combining the results of earlier section, the compactness properties with the control of the space translation parameter $x(t)$ and the convergence in mean, we obtain a contradiction of the existence of a non-scattering solution, therefore, we conclude the proof of Theorem 1.1.

4.1 Compactness properties

Proposition 4.1. *Let u be a solution of (NLS$_\Omega$) such that*

$$M[u] = M_{\mathbb{R}^3}[Q], \quad E[u] = E_{\mathbb{R}^3}[Q], \quad \|u_0\|_{L^2(\Omega)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)}. \tag{IV.58}$$

which does not scatters for positive times. Then there exists a continuous function $x(t)$ such that

$$K := \{\underline{u}(x + x(t), t),\, t \in [0, +\infty)\} \tag{IV.59}$$

has a compact closure in $H^1(\mathbb{R}^3)$.

Proof. It is sufficient to show that for every sequence of time $\tau \geq 0$, there exists (extracting if necessary) a sub-sequence x_n such that $u(x + x_n, \tau_n)$ has a limit in $H^1_0(\Omega)$.

By the profile decomposition in Theorem 2.9, we have

$$u_n := u(x,\tau_n) = \sum_{j=1}^{J} \phi_n^j(x) + \omega_n^J(x). \tag{IV.60}$$

We need to show that $J^* = 1, \omega_n^1 \to 0$ in $H_0^1(\Omega)$ and $t_n^j \equiv 0$. By the Pythagorean expansion properties of the profile decomposition we have,

$$\sum_{j=1}^{J} \lim_{n\to\infty} M[\phi_n^j] + \lim_{n\to\infty} M[\omega_n^J] = \lim_{n\to\infty} M[u_n] = M[Q], \tag{IV.61}$$

$$\sum_{j=1}^{J} \lim_{n\to\infty} E[\phi_n^j] + \lim_{n\to\infty} E[\omega_n^J] = \lim_{n\to\infty} E[u_n] = E[Q]. \tag{IV.62}$$

Scenario I: More than one profile are nonzero, i.e., $J^* \geq 2$. Thus, there exists an $\varepsilon > 0$ such that for all j,

$$M[\phi_n^j]E[\phi_n^j] \leq M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q] - \varepsilon \tag{IV.63}$$

$$\left\|\phi_n^j\right\|_{L^2(\Omega)} \left\|\nabla\phi_n^j\right\|_{L^2(\Omega)} \leq \|Q\|_{L^2(\mathbb{R}^3} \|\nabla Q\|_{L^2(\mathbb{R}^3)} - \varepsilon \tag{IV.64}$$

Recall that by [58, Theorem 3.2], if $v_0 \in H_0^1(\Omega)$ satisfies

$$\|v_0\|_{L^2(\Omega)} \|\nabla v_0\|_{L^2(\Omega)} < \|Q\|_{L^2(\mathbb{R}^3)} \|\nabla Q\|_{L^2(\mathbb{R}^3)}, \tag{IV.65}$$

$$M[v_0]E[v_0] < M_{\mathbb{R}^3}[Q]E_{\mathbb{R}^3}[Q], \tag{IV.66}$$

then the corresponding solution $v(t)$ of (NLS_Ω) scatters in both time directions.

• Suppose j is as in Case 1 (Theorem 2.9), i.e., $x_n^j = 0$ for all n :
When $t_n^j \equiv 0$, we define v^j as the solution to (NLS_Ω) with initial data $v^j(0) = \phi^j$. Then by (IV.65) and (IV.66), v^j is a global and scattering solution.
When $t_n^j \to \pm\infty$, we define v^j as the solution to (NLS_Ω) which scatters to $e^{it\Delta_\Omega}\phi^j$ as $t \to \pm\infty$:

$$\lim_{t\to\pm\infty} \left\|v^j(t) - e^{it\Delta_\Omega}\phi^j\right\|_{H_0^1(\Omega)} = 0.$$

In both cases, we have

$$\lim_{n\to\infty} \left\|v^j(t_n^j) - \phi_n^j\right\|_{H_0^1(\Omega)} = 0. \tag{IV.67}$$

Thus, by (IV.65) and (IV.66), v^j satisfies (IV.65) and (IV.66) and we see that v^j is a global solution with finite scattering size.

Therefore, we can approximate v^j in $L^5H^{1,\frac{30}{11}}(\mathbb{R}\times\Omega)$ by $C_c^\infty(\mathbb{R}\times\mathbb{R}^3)$ functions. More precisely, for any $\varepsilon>0$, there exists $\psi_\varepsilon^j\in C_c^\infty(\mathbb{R}\times\mathbb{R}^3)$ such that

$$\left\|v^j-\psi_\varepsilon^j\right\|_{L^5H^{1,\frac{30}{11}}(\mathbb{R},\Omega)}\leq\frac{\varepsilon}{2}.$$

Let $v_n^j(t,x)=v^j(t+t_n^j,x)$. Then from above v_n^j is a global and scattering solution and by changing variables in time, for any $\varepsilon>0$, there exists $\psi_\varepsilon^j\in C_c^\infty(\mathbb{R}\times\mathbb{R}^3)$ such that, for n sufficiently large, we have

$$\left\|v_n^j(t,x)-\psi_\varepsilon^j(t+t_n^j,x)\right\|_{L^5H^{1,\frac{30}{11}}(\mathbb{R}\times\Omega)}<\varepsilon. \tag{IV.68}$$

• Suppose j is as in Case 2 (Theorem 2.9):
We apply Theorem 2.10 to obtain a global solution v_n^j with $v_n^j(0)=\phi_n^j$. Furthermore, this solution has finite scattering size and satisfies, for n sufficiently large,

$$\left\|\underline{v}_n^j(t,x)-\psi_\varepsilon^j(t+t_n^j,x-x_n^j)\right\|_{L^5H^{1,\frac{30}{11}}(\mathbb{R},\mathbb{R}^3)}<\varepsilon. \tag{IV.69}$$

In all cases, we can find $\psi_\varepsilon^j\in C_c^\infty$ such that (IV.69) holds and, there exists $C_j>0$, independent of n, such that

$$\|v_n^j\|_{X^1(\mathbb{R}\times\Omega)}\leq C_j\,. \tag{IV.70}$$

Note that for large j, by the small data theory we have, $\|v_n^j\|_{X^1(\mathbb{R}\times\Omega)}\lesssim\|\phi_n^j\|_{H_0^1(\Omega)}$. Combining this with (IV.61), (IV.62), we deduce

$$\limsup_{n\to+\infty}\sum_{j=1}^{J}\left\|v_n^j\right\|_{X^1(\mathbb{R}\times\Omega)}^2\leq C,\qquad\text{uniformly for finite } J\leq J^\star. \tag{IV.71}$$

We first prove the asymptotic decoupling of the nonlinear profile using the orthogonality properties (IV.24).

Lemma 4.2 (Decoupling of nonlinear profiles)**.** *For $k\neq j$, we have*

$$\lim_{n\to+\infty}\left\|v_n^jv_n^k\right\|_{L^{\frac52}H_0^{1,\frac{15}{11}}(\mathbb{R}\times\Omega)}+\left\|\nabla v_n^j\nabla v_n^k\right\|_{L^{\frac52}L^{\frac{15}{11}}(\mathbb{R}\times\Omega)}+\left\|v_n^jv_n^k\right\|_{L^{\frac52}L^{\frac{30}{17}}(\mathbb{R}\times\Omega)}+\left\|\nabla v_n^jv_n^k\right\|_{L^{\frac52}L^{\frac{30}{17}}(\mathbb{R}\times\Omega)}=0. \tag{IV.72}$$

Proof. We only prove $\|v_n^jv_n^k\|_{L^{\frac52}H_0^{1,\frac{15}{11}}(\mathbb{R}\times\Omega)}+\|v_n^jv_n^k\|_{L^{\frac52}L^{\frac{30}{17}}(\mathbb{R}\times\Omega)}=o_n(1)$. The other proofs are analogous.

Recall that by (IV.69), for any ε, there exists $N_\varepsilon \in \mathbb{N}$ and $\psi_\varepsilon^j, \psi_\varepsilon^j \in C_c^\infty(\mathbb{R}\times\mathbb{R}^3)$ such that for all $n \geq \mathbb{N}_\varepsilon$ we have

$$\left\|\underline{v}_n^k(t,x) - \psi_\varepsilon^k(t+t_n^k, x-x_n^k)\right\|_{L^5 H^{1,\frac{30}{11}}(\mathbb{R},\mathbb{R}^3)} + \left\|\underline{v}_n^j(t,x) - \psi_\varepsilon^j(t+t_n^j, x-x_n^j)\right\|_{L^5 H^{1,\frac{30}{11}}(\mathbb{R},\mathbb{R}^3)} < \varepsilon. \quad \text{(IV.73)}$$

Using (IV.24), one can see that the supports of $\psi_\varepsilon^j(t,x)$ and $\psi_\varepsilon^k(\cdot+t_n^k-t_n^j, \cdot - x_n^k + x_n^j)$ are disjoint for n sufficiently large (if j,k conforms to case 1, then $\psi_\varepsilon^j(\cdot,\cdot)$ and $\psi_\varepsilon^k(\cdot+t_n^k-t_n^j,\cdot)$ have disjoint time supports), and similarly for the derivatives. Hence

$$\lim_{n\to+\infty}\left\|\psi_\varepsilon^j(t,x)\,\psi_\varepsilon^k(\cdot+t_n^k-t_n^j,\cdot-x_n^k+x_n^j)\right\|_{L^{\frac{5}{2}}H^{1,\frac{15}{11}}(\mathbb{R}\times\mathbb{R}^3)} = 0. \quad \text{(IV.74)}$$

$$\lim_{n\to+\infty}\left\|\psi_\varepsilon^j(t,x)\,\psi_\varepsilon^k(\cdot+t_n^k-t_n^j,\cdot-x_n^k+x_n^j)\right\|_{L^{\frac{5}{2}}H^{1,\frac{30}{17}}(\mathbb{R}\times\mathbb{R}^3)} = 0. \quad \text{(IV.75)}$$

Combining (IV.73), (IV.74) and (IV.70), we have

$$\begin{aligned}\left\|v_n^j v_n^k\right\|_{L^{\frac{5}{2}}H^{1,\frac{15}{11}}(\mathbb{R}\times\Omega)} &\leq \left\|v_n^j - \psi_\varepsilon^j(\cdot+t_n^j,\cdot-x_n^j)\right\|_{L^5H^{1,\frac{30}{11}}(\mathbb{R}\times\mathbb{R}^3)}\left\|v_n^k\right\|_{L^5H^{1,\frac{30}{11}}(\mathbb{R},\mathbb{R}^3)}\\ &+\left\|\psi_\varepsilon^j\right\|_{L^5H^{1,\frac{30}{11}}(\mathbb{R},\mathbb{R}^3)}\left\|v_n^k - \psi_\varepsilon^k(\cdot+t_n^k,\cdot-x_n^k)\right\|_{L^5H^{1,\frac{30}{11}}(\mathbb{R},\mathbb{R}^3)}\\ &+\left\|\psi_\varepsilon^j(t,x)\,\psi_\varepsilon^k(\cdot+t_n^k-t_n^j,\cdot-x_n^k+x_n^j)\right\|_{L^{\frac{5}{2}}H^{1,\frac{15}{11}}(\mathbb{R}\times\mathbb{R}^3)} \leq C\varepsilon,\end{aligned}$$

provided n is large enough, since the last term goes to 0 as n goes to infinity.

Next, we estimate $\left\|v_n^j v_n^k\right\|_{L^{\frac{5}{2}}L^{\frac{30}{17}}(\mathbb{R}\times\Omega)}$.

$$\begin{aligned}\left\|v_n^j v_n^k\right\|_{L^{\frac{5}{2}}L^{\frac{30}{17}}(\mathbb{R}\times\Omega)} &\leq \left\|\underline{v}_n^j - \psi_\varepsilon^j(\cdot+t_n^j,\cdot-x_n^j)\right\|_{L^5_{t,x}(\mathbb{R}\times\mathbb{R}^3)}\left\|\underline{v}_n^k\right\|_{L^5L^{\frac{30}{11}}(\mathbb{R}\times\mathbb{R}^3)}\\ &+\left\|\psi_\varepsilon^j\right\|_{L^5L^{\frac{30}{11}}}\left\|\underline{v}_n^k - \psi_\varepsilon^k(\cdot+t_n^k,\cdot-x_n^k)\right\|_{L^5_{t,x}(\mathbb{R}\times\mathbb{R}^3)}\\ &+\left\|\psi_\varepsilon^j(t,x)\,\psi_\varepsilon^k(\cdot+t_n^k-t_n^j,\cdot-x_n^k+x_n^j)\right\|_{L^{\frac{5}{2}}H^{1,\frac{30}{17}}(\mathbb{R}\times\mathbb{R}^3)}\end{aligned}$$

Using (IV.73), (IV.75) and (IV.70) and Sobolev embedding $\|\cdot\|_{L^5_{t,x}} \leq C\,\|\cdot\|_{L^5H^{1,\frac{30}{11}}}$, we obtain

$$\begin{aligned}\left\|v_n^j v_n^k\right\|_{L^{\frac{5}{2}}L^{\frac{30}{17}}(\mathbb{R}\times\Omega)} &\leq \left\|v_n^j - \psi_\varepsilon^j(\cdot+t_n^j,\cdot-x_n^j)\right\|_{L^5H^{1,\frac{30}{11}}(\mathbb{R},\mathbb{R}^3)}\left\|v_n^k\right\|_{L^5H^{1,\frac{30}{11}}(\mathbb{R},\mathbb{R}^3)}\\ &+\left\|\psi_\varepsilon^j\right\|_{L^5H^{1,\frac{30}{11}}(\mathbb{R},\mathbb{R}^3)}\left\|v_n^k - \psi_\varepsilon^k(\cdot+t_n^k,\cdot-x_n^k)\right\|_{L^5H^{1,\frac{30}{11}}(\mathbb{R},\mathbb{R}^3)}\\ &+\left\|\psi_\varepsilon^j(t,x)\,\psi_\varepsilon^k(\cdot+t_n^k-t_n^j,\cdot-x_n^k+x_n^j)\right\|_{L^{\frac{5}{2}}H^{1,\frac{30}{17}}(\mathbb{R}\times\mathbb{R}^3)} \leq C\varepsilon,\end{aligned}$$

provided n is large enough, which concludes the proof of Lemma 4.2. □

We return to the proof of Proposition 4.1. As a consequence of the asymptotic decoupling of the nonlinear profile in Lemma 4.2, we have

$$\limsup_{n\to\infty} \| \sum_{j=1}^{J} v_n^j \|_{X^1(\mathbb{R}\times\Omega)} \leq C \tag{IV.76}$$

uniformly for finite $J \leq J^*$. Indeed, by (IV.71) and (IV.72) and we obtain

$$\begin{aligned}\left\| \sum_{j=1}^{J} v_n^j \right\|^2_{L^5 L^{\frac{30}{11}}(\mathbb{R}\times\Omega)} = \left\| \left(\sum_{j=1}^{J} v_n^j \right)^2 \right\|_{L^{\frac{5}{2}} L^{\frac{15}{11}}(\mathbb{R}\times\Omega)} &\leq \sum_{j=1}^{J} \left\| v_n^j \right\|^2_{L_t^5 L_x^{\frac{30}{11}}(\mathbb{R}\times\Omega)} + C(J) \sum_{j\neq k} \left\| v_n^j v_n^k \right\|_{L^{\frac{5}{2}} L^{\frac{15}{11}}(\mathbb{R}\times\Omega)} \\ &\leq C + o_n(1).\end{aligned}$$

Similarly,

$$\begin{aligned}\left\| \sum_{j=1}^{J} \nabla v_n^j \right\|^2_{L_t^5 L_x^{\frac{30}{11}}(\mathbb{R}\times\Omega)} = \left\| \left(\sum_{j=1}^{J} \nabla v_n^j \right)^2 \right\|_{L^{\frac{5}{2}} H^{1,\frac{15}{11}}(\mathbb{R}\times\Omega)} &\leq \sum_{j=1}^{J} \left\| \nabla v_n^j \right\|^2_{L_t^5 L^{\frac{30}{11}}(\mathbb{R}\times\Omega)} \\ &+ C(J) \sum_{j\neq k} \left\| \nabla v_n^j \nabla v_n^k \right\|_{L^{\frac{5}{2}} L^{\frac{15}{11}}(\mathbb{R}\times\Omega)} \leq C.\end{aligned}$$

This completes the proof of (IV.76). Using similar argument, one can check that for given $\eta > 0$, there exists $J' := J'(\eta)$ such that

$$\forall J \geq J', \quad \limsup_{n\to\infty} \| \sum_{j=J'}^{J} v_n^j \|_{X^1(\mathbb{R}\times\Omega)} \leq \eta. \tag{IV.77}$$

For each n and J, we define an approximate solution u_n^J to (NLS_Ω) by

$$u_n^J = \sum_{j=1}^{J} v_n^j + e^{it\Delta_\Omega} \omega_n^J \tag{IV.78}$$

Before continuing with the rest of the proof of Proposition 4.1, we claim the following statements hold true.

Claim 4.3.

$$\lim_{n\to\infty} \left\| u_n^J(0) - u_n(0) \right\|_{H_0^1(\Omega)} = 0.$$

Claim 4.4.

$$\exists C>0,\ \forall J,\quad \limsup_{n\to\infty}\left\|u_n^J\right\|_{X^1(\mathbb{R}\times\Omega)}\leq C\,.$$

Claim 4.5.

$$\lim_{J\to J^*}\limsup_{n\to\infty}\left\|i\partial_t u_n^J+\Delta_\Omega u_n^J+\left|u_n^J\right|^2 u_n^J\right\|_{N^1(\mathbb{R})}=0.$$

Applying Lemma 2.8 we get that u_n is a global solution with finite scattering size which yields a contradiction, showing that there is only one profile. Hence, scenario I cannot occur.

Proof of Claim 4.3 . Using (IV.67) if j conforms to case 1 or the fact that $v_n^j(0)=\phi_n^j$ if j conforms to case 2, the decomposition of u_n in (IV.60) and u_n^J in (IV.78), we obtain

$$\left\|u_n^J(0)-u_n(0)\right\|_{H_0^1(\Omega)}\leq\sum_{j=1}^{J}\left\|v_n^j(0)-\phi_n^j\right\|_{H_0^1(\Omega)}\longrightarrow 0,\ \text{as } n\to\infty. \tag{IV.79}$$

□

Proof of Claim 4.4 . Using (IV.76), Strichartz estimate with (IV.20), we obtain

$$\limsup_{n\to\infty}\left\|u_n^J\right\|_{X^1(\mathbb{R}\times\Omega)}\leq\limsup_{n\to\infty}\|\sum_{j=1}^{J}v_n^j\|_{X^1(\mathbb{R}\times\Omega)}+\limsup_{n\to+\infty}\left\|\omega_n^J\right\|_{H_0^1(\Omega)}\leq C.$$

□

Proof of Claim 4.5 . Let $F(z)=-|z|^2z$, recall $\sum_{j=1}^J v_n^j=u_n^J-e^{it\Delta_\Omega}\omega_n^J$, and write

$$\begin{aligned}(i\partial_t+\Delta_\Omega)u_n^J-F(u_n^J)&=\sum_{j=1}^{J}F(v_n^j)-F(u_n^J)\\&=\sum_{j=1}^{J}F(v_n^j)-F(\sum_{j=1}^{J}v_n^j)+F(u_n^J-e^{it\Delta_\Omega}\omega_n^J)-F(u_n^J).\end{aligned}$$

We have

$$\left|\sum_{j=1}^{J}F(v_n^j)-F(\sum_{j=1}^{J}v_n^j)\right|\leq C\sum_{j\neq k}|v_n^j|^2|v_n^k|. \tag{IV.80}$$

Taking the derivatives, we get

$$\left|\nabla\left\{\sum_{j=1}^{J}F(v_n^j)-F(\sum_{j=1}^{J}v_n^j)\right\}\right|\leq C\sum_{j\neq k}|\nabla v_n^j||v_n^j||v_n^k|+C\sum_{j\neq k}|v_n^j|^2|\nabla v_n^k|,$$

which yields

$$\left\| \sum_{j=1}^{J} F(v_n^j) - F(\sum_{j=1}^{J} v_n^j) \right\|_{L^{\frac{5}{3}} L^{\frac{30}{23}}} \leq C \left(\sum_{j \neq k} \left\| v_n^j \right\|_{L^5_{t,x}} \left\| v_n^j v_n^k \right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} \right),$$

$$\begin{aligned} \left\| \nabla \left\{ \sum_{j=1}^{J} F(v_n^j) - F(\sum_{j=1}^{J} v_n^j) \right\} \right\|_{L^{\frac{5}{3}} L^{\frac{30}{23}}} &\leq C \left(\sum_{j \neq k} \left\| v_n^j v_n^k \nabla v_n^j \right\|_{L^{\frac{5}{3}} L^{\frac{30}{23}}} + \sum_{j \neq k} \left\| |v_n^j|^2 \nabla v_n^k \right\|_{L^{\frac{5}{3}} L^{\frac{30}{23}}} \right) \\ &\leq C \sum_{j \neq k} \left\| v_n^j \right\|_{L^5_{t,x}} \left(\left\| \nabla v_n^j v_n^k \right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} + \left\| v_n^j \nabla v_n^k \right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} \right), \end{aligned}$$

which goes to 0 as $n \to \infty$, in view of Lemma 4.2 and (IV.70). In addition,

$$\left\| F(u_n^J - e^{it\Delta_\Omega} \omega_n^J) - F(u_n^J) \right\|_{L^{\frac{5}{3}} H^{1,\frac{30}{23}}} \leq \left\| F(u_n^J - e^{it\Delta_\Omega} \omega_n^J) - F(u_n^J) \right\|_{L^{\frac{5}{3}} L^{\frac{30}{23}}} \tag{IV.81}$$

$$+ \left\| \nabla \left(F(u_n^J - e^{it\Delta_\Omega} \omega_n^J) - F(u_n^J) \right) \right\|_{L^{\frac{5}{3}} L^{\frac{30}{23}}}. \tag{IV.82}$$

We estimate the differences as

$$\left| F(u_n^J - e^{it\Delta_\Omega} \omega_n^J) - F(u_n^J) \right| \leq C \left(\left| e^{it\Delta_\Omega} \omega_n^J \right|^2 \left| e^{it\Delta_\Omega} \omega_n^J \right| + \left| u_n^J \right|^2 \left| e^{it\Delta_\Omega} \omega_n^J \right| \right),$$

$$\begin{aligned} \left| \nabla \left\{ F(u_n^J - e^{it\Delta_\Omega} \omega_n^J) - F(u_n^J) \right\} \right| \leq C \Big(& \left| e^{it\Delta_\Omega} \omega_n^J \right|^2 \left| \nabla e^{it\Delta_\Omega} \omega_n^J \right| + \left| \nabla u_n^J \right| \left| u_n^J \right| \left| e^{it\Delta_\Omega} \omega_n^J \right| \\ & + \left| \nabla u_n^J \right| \left| \nabla e^{it\Delta_\Omega} \omega_n^J \right|^2 + \left| u_n^J \right|^2 \left| \nabla e^{it\Delta_\Omega} \omega_n^J \right| \Big). \end{aligned}$$

Using Claim 4.4, Hölder and Sobolev inequalities, we get

$$\begin{aligned} \text{(IV.81)} &\leq \left\| e^{it\Delta_\Omega} \omega_n^J \right\|_{L^5_{t,x}} \left[\left\| u_n^J \right\|_{L^5 L^{\frac{30}{11}}} \left\| u_n^J \right\|_{L^5_{t,x}} + \left\| e^{it\Delta_\Omega} \omega_n^J \right\|_{L^5 L^{\frac{30}{11}}} \left\| e^{it\Delta_\Omega} \omega_n^J \right\|_{L^5_{t,x}} \right] \\ &\leq \left\| e^{it\Delta_\Omega} \omega_n^J \right\|_{L^5_{t,x}} \left[\left\| u_n^J \right\|_{X^1}^2 + \left\| e^{it\Delta_\Omega} \omega_n^J \right\|_{X^1}^2 \right] + \left\| e^{it\Delta_\Omega} \omega_n^J \right\|_{L^5_{t,x}}^2 \left\| u_n^J \right\|_{X^1} \\ &\leq C \left\| e^{it\Delta_\Omega} \omega_n^J \right\|_{L^5_{t,x}}, \end{aligned}$$

which converges to 0 as $n \to \infty$ and $J \to \infty$. Similarly,

$$\begin{aligned} \text{(IV.82)} \leq & \left\| \nabla u_n^J u_n^J \right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} \left\| e^{it\Delta_\Omega} \omega_n^J \right\|_{L^5_{t,x}} + \left\| \nabla u_n^J \right\|_{L^5 L^{\frac{30}{11}}} \left\| \left| e^{it\Delta_\Omega} \omega_n^J \right|^2 \right\|_{L^{\frac{5}{2}}_{t,x}} \\ & + \left\| \nabla (e^{it\Delta_\Omega} \omega_n^J) \right\|_{L^5 L^{\frac{30}{11}}} \left\| \left| e^{it\Delta_\Omega} \omega_n^J \right|^2 \right\|_{L^{\frac{5}{2}}_{t,x}} + \left\| u_n^J \right\|_{L^5_{t,x}} \left\| u_n^J \nabla e^{it\Delta_\Omega} \omega_n^J \right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} \end{aligned}$$

$$(\text{IV.82}) \leq \left\|\nabla u_n^J\right\|_{L^5 L^{\frac{30}{11}}} \left[\left\|u_n^J\right\|_{L^5_{t,x}} \left\|e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5_{t,x}} + \left\|e^{it\Delta_\Omega}\omega_n^J\right\|^2_{L^5_{t,x}}\right]$$
$$+ \left\|\nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5 L^{\frac{30}{11}}} \left\|e^{it\Delta_\Omega}\omega_n^J\right\|^2_{L^5_{t,x}} + \left\|u_n^J\right\|_{L^5_{t,x}} \left\|u_n^J \nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}}.$$

Thus, it remains to show that

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left\|u_n^J \nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} = 0. \tag{IV.83}$$

Recall that $u_n^J = \sum_{j=1}^J v_n^j + e^{it\Delta_\Omega}\omega_n^J$. Then

$$\left\|u_n^J \nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} \leq \left\|\sum_{j=1}^J v_n^j \nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} + \left\|e^{it\Delta_\Omega}\omega_n^J \nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}}$$
$$\leq \left\|\sum_{j=1}^J v_n^j \nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} + \left\|e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5_{t,x}} \left\|\nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5 L^{\frac{30}{11}}}.$$

Hence, Claim 4.5 holds if

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left\|\sum_{j=1}^J v_n^j \nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} = 0.$$

From (IV.77), we have $\forall \eta > 0, \exists J' = J'(\eta)$ such that

$$\forall J \geq J', \quad \limsup_{n\to\infty} \left\|\sum_{j=J'}^J v_n^j\right\|_{X^1} < \eta.$$

Thus, we have

$$\limsup_{n\to\infty} \left\|\left(\sum_{j=J'}^J v_n^j\right) \nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} \leq \limsup_{n\to\infty} \left\|\sum_{j=J'}^J v_n^j\right\|_{X^1} \left\|\nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5_{t,x}} \leq \eta,$$

where η is arbitrary and $J' = J'(\eta)$ as in (IV.77). Thus, to prove (IV.83) it suffices to show that

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left\|v_n^j \nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}} L^{\frac{30}{17}}} = 0 \quad \text{for all } 1 \leq j \leq J'. \tag{IV.84}$$

We approximate v_n^j by $C_c^\infty(R\times\mathbb{R}^3)$ functions ψ_ε^j obeying (IV.69) with support in $[-T,T]\times\{|x| \leq$

$R\}$. From Proposition 2.7 and (IV.20), we deduce

$$\begin{aligned}\left\|v_n^j \nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}}L^{\frac{30}{17}}} &\le \left\|v_n^j - \psi_\varepsilon^j(\cdot + t_n^j, \cdot - x_n^j)\right\|_{L^5_{t,x}} \left\|\nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5 L^{\frac{30}{11}}} \\ &+ \left\|\psi_\varepsilon^j\right\|_{L^\infty_{t,x}} \left\|\nabla e^{it\Delta_\Omega}\omega_n^J\right\|_{L^{\frac{5}{2}}L^{\frac{30}{17}}(\{|t|\le T,\, |x|\le R\})} \\ &\le C\varepsilon + CR^{\frac{31}{60}}T^{\frac{1}{5}} \left\|e^{it\Delta_\Omega}\omega_n^J\right\|_{L^5_{t,x}}^{\frac{1}{6}} \left\|\omega_n^J\right\|_{H_0^1(\Omega)}^{\frac{5}{6}}.\end{aligned}$$

By taking the limit and choosing ε small, we obtain (IV.83). Hence, Claim 4.5 holds. ☐

Scenario II : Only one nonzero profile. By (IV.60)

$$u_n := u(x, \tau_n) = \phi_n^1 + \omega_n^1,$$

with

$$\lim_{n\to\infty} \left\|\omega_n^1\right\|_{H_0^1(\Omega)} = 0, \tag{IV.85}$$

If not, there exists $\varepsilon > 0$ such that $\forall n$,

$$E[\phi_n^1]M[\phi_n^1] \le E_{\mathbb{R}^3}[Q]M_{\mathbb{R}^3}[Q] - \varepsilon,$$

and one can show by the previous argument that u scatters in $H_0^1(\Omega)$.

It remains to show that t_n^1 is bounded and this will prove the convergence, up to a subsequence.

- If $t_n^1 \to +\infty$ (similarly $t_n^1 \to -\infty$) and ϕ_n^1 conforms to Case 1, i.e., $\phi_n^1 = e^{it_n^1\Delta_\Omega}\phi^1$.

$$\begin{aligned}\left\|e^{it\Delta_\Omega}u_n\right\|_{L^5_{t,x}([0,+\infty)\times\Omega)} &= \left\|e^{it\Delta_\Omega}\phi_n^1 + e^{it\Delta_\Omega}\omega_n^1\right\|_{L^5_{t,x}([0,+\infty)\times\Omega)} \\ &\le \left\|e^{i(t+t_n^1)\Delta_\Omega}\phi^1\right\|_{L^5_{t,x}([0,+\infty)\times\Omega)} + \left\|\omega_n^1\right\|_{H_0^1(\Omega)} \\ &\le \left\|e^{it\Delta_\Omega}\phi^1\right\|_{L^5_{t,x}([t_n^1,+\infty)\times\Omega)} + \left\|\omega_n^1\right\|_{H_0^1(\Omega)},\end{aligned}$$

  which goes to 0 as n goes to ∞, showing that u_n scatters for positive (similarly negative) time, a contradiction.

- If $t_n^1 \to +\infty$ (similarly $t_n^1 \to -\infty$) and ϕ_n^1 conforms to Case 2, i.e.,

$$\phi_n^1 = e^{it_n^1\Delta_\Omega}[(\chi_n^1\phi^1)(x - x_n^1)], \quad \text{where } \chi_n^1 := \chi\left(\frac{x}{|x_n^1|}\right).$$

We first prove that

$$\lim_{n\to+\infty}\left\|e^{it\Delta_{\Omega_n}}(\chi_n^1\phi^1)-e^{it\Delta_{\mathbb{R}^3}}(\chi_n^1\phi^1)\right\|_{L^5_{t,x}((0,+\infty)\times\mathbb{R}^3)}=0, \tag{IV.86}$$

where $\Omega_n:=\Omega-\{x_n\}$. Indeed, by a density argument, for any $\varepsilon>0$, there exist $\psi_\varepsilon\in\mathcal{C}_c^\infty(\mathbb{R}^3)$ such that

$$\left\|\phi^1-\psi_\varepsilon\right\|_{H^1(\mathbb{R}^3)}\leq\frac{\varepsilon}{4}. \tag{IV.87}$$

By the definition of χ_n, as $|x_n|\longrightarrow+\infty$, for any $\varepsilon>0$ there exists $N_\varepsilon\in\mathbb{N}$ such that,

$$\forall n\geq N_\varepsilon,\quad \left\|\chi_n^1\phi^1-\phi^1\right\|_{H^1(\mathbb{R}^3)}\leq\frac{\varepsilon}{4}. \tag{IV.88}$$

Using (IV.87) and (IV.88), we have

$$\forall n\geq N_\varepsilon,\quad \left\|\chi_n^1\phi^1-\psi_\varepsilon\right\|_{H^1(\mathbb{R}^3)}\leq\frac{\varepsilon}{2}.$$

Combining this with the Strichartz inequality, we obtain for large n

$$\left\|e^{it\Delta_{\Omega_n}}\left(\chi_n^1\phi^1-\psi_\varepsilon\right)\right\|_{L^5_{t,x}((0,+\infty)\times\mathbb{R}^3)}+\left\|e^{it\Delta_{\mathbb{R}^3}}\left(\chi_n^1\phi^1-\psi_\varepsilon\right)\right\|_{L^5_{t,x}((0,+\infty)\times\mathbb{R}^3)}\leq\frac{\varepsilon}{2}. \tag{IV.89}$$

From [58, Proposition 2.13], as $|x_n|\longrightarrow+\infty$, we have for large n

$$\left\|e^{it\Delta_{\Omega_n}}\psi_\varepsilon-e^{it\Delta_{\mathbb{R}^3}}\psi_\varepsilon\right\|_{L^5_{t,x}((0,\infty)\times\mathbb{R}^3)}\leq\frac{\varepsilon}{2}. \tag{IV.90}$$

which yields (IV.86). We now have

$$\begin{aligned}
\left\|e^{it\Delta_\Omega}u_n\right\|_{L^5_{t,x}([0,+\infty)\times\Omega)}&=\left\|e^{it\Delta_\Omega}\phi_n^1+e^{it\Delta_\Omega}\omega_n^1\right\|_{L^5_{t,x}([0,+\infty)\times\Omega)}\\
&\leq\left\|e^{i(t+t_n^1)\Delta_\Omega}(\chi_n^1\phi^1)(x-x_n^1)\right\|_{L^5_{t,x}([0,+\infty)\times\Omega)}+\left\|\omega_n^1\right\|_{H_0^1(\Omega)}\\
&\leq\left\|e^{it\Delta_\Omega}(\chi_n^1\phi^1)(x-x_n^1)\right\|_{L^5_{t,x}([t_n^1,+\infty)\times\Omega)}+\left\|\omega_n^1\right\|_{H_0^1(\Omega)}\\
&\leq\left\|e^{it\Delta_\Omega}(\chi_n^1\phi^1)(x-x_n^1)\right\|_{L^5_{t,x}([t_n^1,+\infty)\times\Omega)}+\left\|\omega_n^1\right\|_{H_0^1(\Omega)}\\
&\leq\left\|e^{it\Delta_{\Omega_n}}(\chi_n^1\phi^1)-e^{it\Delta_{\mathbb{R}^3}}(\chi_n^1\phi^1)\right\|_{L^5_{t,x}((t_n^1,+\infty)\times\mathbb{R}^3)}\\
&+\left\|e^{it\Delta_{\mathbb{R}^3}}(\chi_n^1\phi^1)\right\|_{L^5_{t,x}((t_n^1,+\infty)\times\mathbb{R}^3)}+\left\|\omega_n^1\right\|_{H_0^1(\Omega)},
\end{aligned}$$

which goes to 0 as n goes to ∞, by (IV.86) and the monotone convergence theorem, showing that u_n scatters for positive (respectively, negative) time, a contradiction. This completes the proof of Proposition 4.1.

□

Corollary 4.6. *Let u be as in Proposition 4.1. Then one can choose the continuous function $x(t)$ such that $X(t) = x(t)$ for all $t \in D_{\delta_0}$, and the set K has a compact closure in $H^1(\mathbb{R}^3)$.*

Proof. Recall that by the definition of D_{δ_0}, the modulation parameters $X(t), \theta(t)$ and $\alpha(t)$ are well defined for all $t \in D_{\delta_0}$. Let $x(t)$ be the translation parameter given by Proposition 4.1. Let $R_0 > 0$. Then by the decomposition of u in (IV.36), Proposition 3.3 and the fact $\Psi(x) = 1$ for $|x|$ large, there exists $C_\star > 0$ such that

$$\forall t \in D_{\delta_0}, \qquad \int_{|x|\le R_0} |\nabla Q|^2 + |Q|^2 - C_\star \left(\delta(t) + \frac{e^{-|X(t)|}}{|X(t)|}\right) \le \int_{|x-X(t)|\le R_0} |\nabla \underline{u}|^2 + |\underline{u}|^2.$$

Taking δ_0 small if necessary, there exists $\varepsilon_0 > 0$ such that

$$\forall t \in D_{\delta_0}, \qquad \int_{|x+x(t)-X(t)|\le R_0} |\nabla \underline{u}(t, x + x(t))|^2 + |\underline{u}(t, x + x(t))|^2 \ge \varepsilon_0 > 0.$$

Using the fact that K has a compact closure in $H^1(\mathbb{R}^3)$, we get that $|x(t) - X(t)|$ is bounded. Thus, one can modify $x(t)$ such that $\overline{K}$ remains compact and for all t in D_{δ_0}, $x(t) = X(t)$. □

4.2 Control of the translation parameters

Proposition 4.7. *Consider a solution u of* (NLS$_\Omega$) *such that*

$$M[u] = M_{\mathbb{R}^3}[Q], \; E[u] = E_{\mathbb{R}^3}[Q], \; \|\nabla u_0\|_{L^2(\Omega)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)} \tag{IV.91}$$

and

$$K := \{\underline{u}(t, x + x(t)); t \ge 0\} \tag{IV.92}$$

has a compact closure in $H^1(\mathbb{R}^3)$. Then $x(t)$ is bounded.

We will start proving the following lemma.

Lemma 4.8. *Let u be as in the Proposition 4.7. Let $\{t_n\}$ be a sequence of time, such that $t_n \longrightarrow +\infty$. Then $|x(t_n)| \longrightarrow +\infty$ as $n \to +\infty$, if and only if $\delta(t_n) \longrightarrow 0$ as n goes to $+\infty$.*

Proof. We first prove that $\delta(t_n) \longrightarrow 0$ implies that $|x(t_n)| \longrightarrow +\infty$ as $n \to +\infty$. If not, $x(t_n)$ converges (after extraction) to x_∞ in $\mathbb{R}^3$. By the compactness of the closure of K, $\underline{u}(t_n, \cdot + x(t_n))$ converges in $H^1(\mathbb{R}^3)$ to some $v_0(\cdot - x_\infty) \in H^1(\mathbb{R}^3)$. By the assumption (IV.91) and the fact that $\delta(t_n) \longrightarrow 0$, $E_{\mathbb{R}^3}[v_0] = E_{\mathbb{R}^3}[Q], M_{\mathbb{R}^3}[v_0] = M_{\mathbb{R}^3}(Q)$ and $\|\nabla v_0\|_{L^2(\mathbb{R}^3)} = \|\nabla Q\|_{L^2(\mathbb{R}^3)}$. By Proposition 2.1, there exist $\theta_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^3$ such that $v_0 = e^{i\theta_0}Q(\cdot - x_0)$. On the other hand, if $x + x(t_n) \in \Omega$, then $u(t_n, x + x(t_n))$ converges in $H^1_0(\Omega)$, as $H^1_0(\Omega)$ is a close subspace of $H^1(\mathbb{R}^3)$. Thus, the restriction of $v_0(\cdot - x_\infty)$ to Ω belongs to $H^1_0(\Omega)$, which contradicts the fact that $e^{i\theta_0}Q(\cdot + x_\infty - x_0) \notin H^1_0(\Omega)$.

Next, we prove that $|x(t_n)| \longrightarrow +\infty$ as $n \to +\infty$ implies that $\delta(t_n) \longrightarrow 0$ as n goes to $+\infty$. We argue by contradiction, assuming (after extraction) that

$$\delta(t_n) \xrightarrow[n\to+\infty]{} \delta_\infty > 0 \quad \text{and} \quad t_n \xrightarrow[n\to+\infty]{} t_\infty \in \mathbb{R} \cup \{\pm\infty\}.$$

By the continuity of $x(t)$, using $|x(t_n)| \to +\infty$, we must have $t_\infty \in \{\pm\infty\}$.
Assume, say, $t_\infty = +\infty$, and let $\varphi_\infty = \lim_{n\to+\infty} \underline{u}(t_n, x + x(t_n))$ in $H^1(\mathbb{R}^3)$ (after extraction). We have

$$E_{\mathbb{R}^3}[\varphi_\infty] = E_{\mathbb{R}^3}[Q], \quad M_{\mathbb{R}^3}[\varphi_\infty] = M_{\mathbb{R}^3}[Q], \quad \int_{\mathbb{R}^3} |\nabla \varphi_\infty|^2 = \int_{\mathbb{R}^3} |\nabla Q|^2 - \delta_\infty < \int_{\mathbb{R}^3} |\nabla Q|^2.$$

Let φ be the solution of (NLS) with the initial datum φ_∞ at $t = 0$. By [31], φ is global and one of the following holds:

1. φ scatters in both time directions.
2. $\exists\, \tau, \theta \in \mathbb{R}$ and $\varepsilon \in \{\pm 1\}$ such that $\varphi(t) = e^{i\theta}U_-(\varepsilon\, t + \tau)$, where $U_-(t) \xrightarrow[t\to+\infty]{} Q$ and U_- scatters for negative time.

In case (1) or in the case (2) with $\varepsilon = -1$, one can prove by approximation, following the proof of Theorem 4.1 in [58], that u scatters for positive time.
In case (2) with $\varepsilon = +1$, we obtain for large n, with the same argument

$$\|u\|_{S(-\infty,t_n)} \leq C\ \|U_-\|_{S(-\infty,t_\infty)}, \qquad \text{where } C \text{ is a fixed constant.}$$

Letting n go to $+\infty$, we see that u has a has a finite Strichartz norm, thus, u scatters also in both time directions, which contradict the fact that u satisfies (IV.92) and (IV.91). ☐

Lemma 4.9. *Let $X(t)$ be as in (IV.35). Taking a smaller δ_0 if necessary, there exists $C > 0$ such that*

$$\frac{e^{-|X(t)|}}{|X(t)|} \leq C\delta(t) \quad \textit{for any}\ \ t \in D_{\delta_0}. \tag{IV.93}$$

Proof. Note that, by Proposition 4.1, taking a smaller δ_0 if necessary, we can assume $|X(t)| \geq C$ for an arbitrarily large constant $C > 0$. The proof now consists in 3 steps.

- Step 1: Estimate of $\delta(t)$ with respect to an auxiliary modulation parameter $X_1(t)$ on $\mathbb{R}^3$. Let $\underline{u}(t) \in H^1(\mathbb{R}^3)$ be the extension of u to $\mathbb{R}^3$ defined as in (IV.14), we then have

$$M_{\mathbb{R}^3}[\underline{u}] = M_{\mathbb{R}^3}[Q], \quad E_{\mathbb{R}^3}[\underline{u}] = E_{\mathbb{R}^3}[Q], \quad \text{and} \quad \int_{\mathbb{R}^3} |\nabla \underline{u}|^2 < \int_{\mathbb{R}^3} |\nabla Q|^2 . \tag{IV.94}$$

Arguing as in Section 3, but on the whole space $\mathbb{R}^3$, see [31, Lemma 4.1 and 4.2], there exist $\theta_1(t)$ and $X_1(t)$, C^1 functions of t, such that

$$e^{-i\theta_1(t)-it}\underline{u}(t, x + X_1(t)) = (1 + \rho_1(t))Q(x) + \widetilde{h}(t, x), \tag{IV.95}$$

where

$$\rho_1(t) = \operatorname{Re} \frac{e^{-i\theta_1 - it} \int_{\mathbb{R}^3} \nabla \underline{u}(t, x + X_1(t)).\nabla Q(x) dx}{\|\nabla Q\|^2_{L^2(\mathbb{R}^3)}} - 1, \tag{IV.96}$$

$$|\rho_1(t)| \approx \left| \int_{\mathbb{R}^3} Q\, \widetilde{h}\, dx \right| \approx \left\| \widetilde{h} \right\|_{H^1(\mathbb{R}^3)} \approx \delta(t). \tag{IV.97}$$

In this step we prove:

$$\frac{e^{-|X_1(t)|}}{|X_1(t)|} \leq C\delta(t). \tag{IV.98}$$

By (IV.95), $x \in \Omega^c$ implies $(1 + \rho_1(t))Q(x - X_1(t)) + \widetilde{h}(t, x - X_1(t)) = 0$, i.e.,

$$\left\| (1 + \rho_1(t))Q(x - X_1(t)) + \widetilde{h}(t, x - X_1(t)) \right\|_{L^2(\Omega^c)} = 0.$$

By (IV.97), we have

$$\int_{\Omega^c} |Q(x - X_1(t))|^2\, dx \leq C\, \delta(t)^2. \tag{IV.99}$$

By (IV.15), one can see that $|X_1(t)|$ is large. For $x \in \Omega^c$, we have

$$\frac{1}{2}|X_1(t)| \leq |x - X_1(t)| \leq 2|X_1(t)|.$$

From Lemma 2.2, we have

$$Q(x) = \frac{e^{-|x|}}{|x|} \left(a + O(\frac{1}{|x|^{\frac{1}{2}}}) \right), \quad \text{for some } a > 0.$$

Using (IV.99), we obtain (IV.98).

- Step 2: Comparison of $X(t)$ and $X_1(t)$.
  We prove that there exists $C > 0$ such that

$$|X(t) - X_1(t)| \leq C \quad \forall t \in D_{\delta_0}. \tag{IV.100}$$

We fix $t \in D_{\delta_0}$. We can assume

$$|X(t) - X_1(t)| \geq 1, \tag{IV.101}$$

or else we are done.
Let $x \in \Omega$, by (IV.95) and (IV.36), we have

$$\begin{aligned} u(t,x) &= e^{i\theta(t)+it}(1+\rho(t))Q(x-X(t))\Psi(x) + e^{i\theta(t)+it}h(t,x) \\ &= e^{i\theta_1(t)+it}(1+\rho_1(t))Q(x-X_1(t)) + e^{i\theta_1(t)+it}\widetilde{h}(t,x). \end{aligned}$$

Using (IV.97) and Proposition 3.3 , we have

$$\int_{|x-X(t)|<1} \left|Q(x-X(t))\Psi(x)e^{i\theta(t)} - Q(x-X_1(t))e^{i\theta_1(t)}\right|^2 \leq C\left(\delta^2(t) + \frac{e^{-2|X(t)|}}{|X(t)|^2}\right).$$

Recall that $|X_1(t)|$ and $|X(t)|$ are large and $\Psi(x) = 1$ for large $|x|$.

$$\begin{aligned} \int_{|x|<1} |Q(x)|^2 dx &\leq C\int_{|x-X(t)|<1} |Q(x-X_1(t))|^2 dx + C\delta^2(t) + C\,\frac{e^{-2|X(t)|}}{|X(t)|^2} \\ &\leq \int_{|x-X(t)|<1} \frac{e^{-2|x-X_1(t)|}}{|x-X_1(t)|^2}\,dx + C\delta^2(t) + C\,\frac{e^{-2|X(t)|}}{|X(t)|^2}. \end{aligned}$$

Using the fact that $|x - X_1(t)| \geq |X(t) - X_1(t)| - |x - X(t)| \geq |X(t) - X_1(t)| - 1$, in the support of the integral in the last line, we obtain

$$\int_{|x|<1} |Q(x)|^2 dx \leq C\frac{e^{-2|X(t)-X_1(t)|}}{|X(t)-X_1(t)|^2} + C\,\delta^2(t) + C\,\frac{e^{-2|X(t)|}}{|X(t)|^2}.$$

Recall that, by Lemma 4.8 if $|X(t)|$ is large, then $\delta(t)$ and $\frac{e^{-2|X(t)|}}{|X(t)|^2}$ are small. By (IV.101), we get

$$\frac{1}{2}\int_{|x|<1} |Q(x)|^2\,dx \leq C\,\frac{e^{-2|X(t)-X_1(t)|}}{|X(t)-X_1(t)|^2} \leq Ce^{-2|X(t)-X_1(t)|},$$

which yields

$$|X(t) - X_1(t)| \leq C - \log\left(\frac{1}{2}\int_{|x|<1} |Q(x)|^2\, dx\right).$$

Thus, $|X(t) - X_1(t)|$ is bounded.

- Step 3: Conclusion of the proof.
  From Step 2 we have $|X(t) - X_1(t)| \leq C$, and since $|X(t)|$ is large, we have

$$\frac{1}{2}|X(t)| \leq |X(t)| - |X(t) - X_1(t)| \leq |X_1(t)| \leq |X_1(t) - X(t)| + |X(t)| \leq 2|X(t)|. \quad \text{(IV.102)}$$

By Step 1, we get $\delta^2(t) \geq C\,\frac{e^{-2|X_1(t)|}}{|X_1(t)|^2}$, which implies

$$\delta^2(t) \geq C\,\frac{e^{-2|X(t)|}}{|X(t)|^2},$$

concluding the proof of Lemma 4.9.

□

Lemma 4.10. *Let u be a solution of* (NLS$_\Omega$) *satisfying the assumptions of the Proposition 4.7. Then there exists a constant $C > 0$ such that if $0 \leq \sigma \leq \tau$*

$$\int_\sigma^\tau \delta(t) \leq C\left[1 + \sup_{t\in[\sigma,\tau]} |x(t)|\right](\delta(\sigma) + \delta(\tau)) \quad \text{(IV.103)}$$

Proof. Let φ be a smooth radial function such that

$$\varphi(x) := \begin{cases} |x|^2 & \text{if} \quad |x| \leq 1, \\ 0 & \text{if} \quad |x| \geq 2. \end{cases}$$

Consider the localized variance,

$$\mathcal{Y}_R(t) = \int_\Omega R^2 \varphi\left(\frac{x}{R}\right) |u(t,x)|^2\, dx,$$

where R is large positive constant, to be specified later. Then,

$$\mathcal{Y}_R'(t) = 2R\,\mathrm{Im}\int_\Omega \bar{u}\,\nabla\varphi\left(\frac{x}{R}\right).\nabla u\, dx, \qquad |\mathcal{Y}_R'(t)| \leq C\,R.$$

Furthermore,

$$\mathcal{Y}_R''(t) = 8\int |\nabla u|^2 - 6\int |u|^4 + A_R(u(t)) - 2\int_{\partial\Omega} |\nabla u|^2\, x.\vec{n}d\sigma(x),$$

where $\vec{n}$ is the outward normal vector and

$$\begin{aligned} A_R(u(t)) := 4\sum_{j\neq k}\int_\Omega \frac{\partial^2\varphi}{\partial x_j\partial x_k}\left(\frac{x}{R}\right)\frac{\partial u}{\partial x_j}\frac{\partial \bar{u}}{\partial x_k} + 4\sum_j \int_\Omega \left(\frac{\partial^2\varphi}{\partial x_j^2}\left(\frac{x}{R}\right) - 2\right)\left|\partial_{x_j} u\right|^2 \\ - \frac{1}{R^2}\int_\Omega |u|^2\Delta^2\varphi\left(\frac{x}{R}\right) - \int_\Omega \left(\Delta\varphi\left(\frac{x}{R}\right) - 6\right)|u|^4. \end{aligned} \tag{IV.104}$$

As $\partial\Omega$ is convex and $0 \in \Omega$ one can see that $x.\vec{n} \leq 0$, for all $x \in \partial\Omega$. Thus,

$$-2\int_{\partial\Omega} |\nabla u|^2\, x.\vec{n}\, d\sigma(x) = 2\int_{\partial\Omega} |\nabla u|^2\, |x.\vec{n}|\, d\sigma(x).$$

Using the fact $\|Q\|_{L^4}^4 = \frac{4}{3}\|\nabla u\|_{L^2}^2$ and $E[u] = E_{\mathbb{R}^3}[Q]$ we have, $8\|\nabla u\|_{L^2}^2 - 6\|u\|_{L^4}^4 = 4\delta(t)$, which yields,

$$\mathcal{Y}_R''(t) = 4\delta(t) + A_R(u(t)) + 2\int_{\partial\Omega} |\nabla u|^2\, |x.\vec{n}|\, d\sigma(x), \tag{IV.105}$$

- Step 1: Bound on A_R.
  In this step we prove: for $\varepsilon > 0$, there exists a constant $R_\varepsilon > 0$ such that

$$\forall t \geq 0,\ R \geq R_\varepsilon(1 + |x(t)|) \Longrightarrow |A_R(u(t))| \leq \varepsilon\delta(t). \tag{IV.106}$$

We distinguish two cases: δ small or not. In the first case, we will use the estimate on the modulation parameters in Section 3. Consider $\delta_0 > 0$, as in the previous Section, such that the modulation parameters, $\Theta(t), X(t), \rho(t)$ are well defined for all $t \in D_{\delta_0}$. Let δ_1 to be specified later such that $0 < \delta_1 < \delta_0$. Assume that $t \in D_{\delta_1}$. Let $g_{-X} = \rho Q_{-X}\Psi + h$, then from Proposition 3.3 with Lemma 4.9 and (IV.35), we have

$$u(t,x) = e^{i\theta(t)+it}Q(x - X(t))\Psi(x) + g(t, x - X(t))e^{i\theta(t)+it} \text{ and } \|g\|_{H_0^1(\Omega)} \leq C\delta(t). \tag{IV.107}$$

For the equation on $\mathbb{R}^3$, $\mathcal{Y}_R$ is defined by integration on $\mathbb{R}^3$ (not on Ω). The corresponding A_R is also defined with an integration on $\mathbb{R}^3$ instead of Ω. A crucial point is that when R is large, by the property of φ, all the integrands in the definition (IV.104) of A_R are 0 close to the obstacle, so that it can be integrated over $\mathbb{R}^3$ instead of Ω. Note that it only works if R is large enough. If θ_0 and x_0 are fixed, $e^{i\theta_0+it}Q(\cdot + x_0)$ is a solution of (NLS) such that the corresponding $\mathcal{Y}_R(t)$ does not depend on t and also $\delta(t) = 0$. Thus, $A_R(e^{i\theta_0+it}Q(\cdot + x_0)) = 0$ for all R and t.

Using the change of variable $y = x - X(t)$ in (IV.104), we get

$$\begin{aligned}|A_R(u(t))| &= \left|A_R(u(t)) - A_R(e^{i\theta(t)+it}Q(x-X(t))\right| \\ &\leq C\int_{|y+X(t)|\geq R}\Big(|\nabla Q(y)||\nabla g(y)| + |\nabla g(y)|^2 + |Q(y)||g(y)| + |Q(y)||g(y)|^3 \\ &\qquad + |g(y)|^2 + |g(y)|^4\Big)dy \\ &\leq \int_{|y+X(t)|\geq R}\frac{e^{-|y|}}{|y|}\left(|\nabla g(y)| + |g(y)| + |g(y)|^3\right) + |\nabla g(y)|^2 + |g(y)|^2 + |g(y)|^4 dy.\end{aligned}$$

By (IV.107), we have $\|g\|_{H_0^1(\Omega)} \leq C\delta(t)$, which yields

$$\begin{aligned}R \geq R_0 + |X(t)| \Longrightarrow |A_R(u(t))| &\leq C\left[e^{-R_0}(\delta(t)+\delta(t)^3) + \delta(t)^2 + \delta(t)^4\right] \\ &\leq C\left[e^{-R_0} + e^{-R_0}\delta(t)^2 + \delta(t) + \delta(t)^3\right]\delta(t) \\ &\leq \varepsilon\delta(t),\end{aligned}$$

provided $R_0 > 0$ is such that $Ce^{-R_0} \leq \frac{\varepsilon}{2}$ and δ_1 is such that $Ce^{-R_0}\delta_1^2 + \delta_1 + \delta_1^3 \leq \frac{\varepsilon}{2}$. Since $0 < \delta_1 < \delta_0$ and $x(t) = X(t)$ on D_{δ_0}, we obtain (IV.106) for $\delta(t) < \delta_1$.
Now consider the second case, i.e., $\delta(t) \geq \delta_1$. By (IV.104), we have

$$|A_R(u(t))| \leq C\int_{|x-x(t)|\geq R-|x(t)|}|\nabla u(t)|^2 + |u(t)|^4 + |u(t)|^2 dx.$$

By the compactness of K, there exists $R_1 > 0$ such that

$$R \geq |x(t)| + R_1 \text{ and } \delta(t) \geq \delta_1 \Longrightarrow |A_R(u(t))| \leq \varepsilon\delta_1 \leq \varepsilon\delta(t), \tag{IV.108}$$

which concludes the proof of (IV.106) and completes Step 1.

- Step 2: Conclusion of the proof.
  By (IV.107), we get that there exists $R_2 > 0$ such that,

$$R \geq R_2(1 + |x(t)|) \Longrightarrow \left|\mathcal{Y}_R''(t)\right| \geq 2\delta(t).$$

Let $R = R_2(1 + \sup_{\sigma\leq t\leq\tau}|x(t)|)$ then,

$$2\int_\sigma^\tau \delta(t)dt \leq \int_\sigma^\tau \mathcal{Y}_R''(t)\,dt \leq \mathcal{Y}_R'(\tau) - \mathcal{Y}_R'(\sigma). \tag{IV.109}$$

If $\delta(t) < \delta_0$, then by Step 1, changing the variable $y = x - X(t)$ and since $\Psi(x) = 1$ for large $|x|$, we obtain

$$\begin{aligned}\mathcal{Y}_R'(t) &= 2R\,\mathrm{Im}\int \bar{g}(y)\,\nabla\varphi\left(\frac{y+X(t)}{R}\right)\cdot\nabla\left(Q(y)\Psi(y+X(t))\right)\\ &+ 2R\,\mathrm{Im}\int Q(y)\Psi(y+X(t))\nabla\varphi\left(\frac{y+X(t)}{R}\right)\cdot\nabla g(y)\,dy\\ &+ 2R\,\mathrm{Im}\int \bar{g}(y)\nabla\varphi\left(\frac{y+X(t)}{R}\right)\cdot\nabla g(y)dy,\end{aligned}$$

which yields

$$\left|\mathcal{Y}_R'(t)\right| \le CR(\delta(t)+\delta(t)^2) \le CR\delta(t).$$

This inequality is also valid for $\delta(t) \ge \delta_0$, by straightforward estimates. Using (IV.109), we obtain

$$\begin{aligned}\int_\sigma^\tau \delta(t)dt &\le C\,R(\delta(\sigma)+\delta(\tau))\\ &\le C\,R_2\left(1+\sup_{\sigma\le t\le\tau}|x(\tau)|\right)(\delta(\sigma)+\delta(\tau)).\end{aligned}$$

This concludes the proof of Lemma 4.10.

□

Lemma 4.11. *There exists a constant $C > 0$ such that*

$$\forall \sigma,\tau > 0 \quad \textit{with} \quad \sigma+1\le\tau, \quad |x(\tau)-x(\sigma)| \le C\int_\sigma^\tau \delta(t)dt \tag{IV.110}$$

Proof. Let $\delta_0 > 0$ be as in Section 3. Let us first show that there exists $\delta_1 > 0$ such that,

$$\forall \tau \ge 0 \quad \inf_{t\in[\tau,\tau+2]}\delta(t) \ge \delta_1 \quad \text{or} \quad \sup_{t\in[\tau,\tau+2]}\delta(t) < \delta_0. \tag{IV.111}$$

If not, there exist $t_n, t_n' \ge 0$ such that

$$\delta(t_n) \xrightarrow[n\to+\infty]{} 0, \quad \delta(t_n') \ge \delta_0, \quad |t_n - t_n'| \le 2, \tag{IV.112}$$

extracting a subsequence if necessary, we may assume

$$\lim_{n\to+\infty} t_n - t_n' = \tau \in [-2,2]. \tag{IV.113}$$

Note that if t'_n goes to $+\infty$, then $|x(t'_n)|$ converges (after extraction) to a limit $X_0 \in \mathbb{R}^3$. If not $|x(t'_n)| \longrightarrow +\infty$ and by Lemma 4.8, $\delta(t'_n) \longrightarrow 0$, which contradicts (IV.112).
By the compactness of K, we have

$$\underline{u}(t'_n, \cdot + x(t'_n)) \xrightarrow[n \longrightarrow +\infty]{} w_0 \in H^1(\mathbb{R}^3).$$

Denote $v_0(x) = w_0(x - X_0)$. We have

$$\underline{u}(t'_n, \cdot + x(t'_n)) \xrightarrow[n \longrightarrow +\infty]{} v_0(\cdot + X_0) \in H^1(\mathbb{R}^3). \tag{IV.114}$$

Thus,

$$\underline{u}(t'_n) \xrightarrow[n \longrightarrow +\infty]{} v_0 \in H^1(\mathbb{R}^3).$$

In particular, $v_0 = 0$ on Ω^c and we obtain,

$$u(t'_n) \xrightarrow[n \longrightarrow +\infty]{} v_0 \in H^1_0(\Omega). \tag{IV.115}$$

Since $\delta(t'_n) = \int |\nabla Q|^2 - \int |\nabla \underline{u}(t'_n, \cdot + x(t'_n))|^2 \geq \delta_0 > 0$, we have

$$\|\nabla v_0\|_{L^2(\Omega)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)}.$$

Let $v(t)$ be a solution of (NLS$_\Omega$) with initial data v_0 at $t = 0$ and maximal time of existence I. Then by continuity of the flow of the NLS$_\Omega$ equation, we have for all $t \in I$,

$$\|\nabla v(t)\|_{L^2(\Omega)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)}. \tag{IV.116}$$

As a consequence, $I = \mathbb{R}$ and by continuity of the flow of the NLS$_\Omega$ equation, (IV.113) and (IV.115), we have

$$u(t_n) \xrightarrow[n \longrightarrow +\infty]{} v(\tau) \in H^1_0(\Omega).$$

Since $\delta(t_n) \to 0$, $\|\nabla v(\tau)\|_{L^2(\Omega)} = \|\nabla Q\|_{L^2(\mathbb{R}^3)}$, which contradicts (IV.116).

Now, we prove (IV.110) with an additional condition that $\tau < \sigma + 2$. By (IV.111), we may assume that

$$\inf_{t \in [\sigma, \tau]} \delta(t) \geq \delta_1 \quad \text{or} \quad \sup_{t \in [\sigma, \tau]} \delta(t) < \delta_0.$$

In the first case, we have $\int_\sigma^\tau \delta(t) \geq \delta_1$ and by a straightforward consequence of the compactness of K and the continuity of the flow of (NLS$_\Omega$) equation, we have

$$\exists C > 0,\ \forall t, s \geq 0, \quad |t-s| \leq 2 \Longrightarrow |X(t) - X(s)| \leq \frac{C}{\delta_1} \int_\sigma^\tau \delta(t) dt.$$

In the second case, by Corollary 4.6 we have, $\forall t \in D_{\delta_0},\ x(t) = X(t)$, and from Lemmas 3.4 and 4.9, we have

$$|X'(t)| \leq C\delta(t). \tag{IV.117}$$

Thus, (IV.110) follows from the time integration of (IV.117) for $\tau < \sigma + 2$.

To conclude the proof of Lemma 4.11, we divide $[\sigma, \tau]$ into intervals of length at least 1 and at most 2 and combine together the previous inequalities to get (IV.110). ☐

Proof of the Proposition 4.7. We argue by contradiction. Assume that there exists $\tau_n \longrightarrow +\infty$ such that $|x(\tau_n)| \longrightarrow +\infty$ and $|x(\tau_n)| = \sup_{t\in[0,\tau_n]} |x(t)|$. By Lemma 4.8, $\delta(\tau_n) \xrightarrow[n\to+\infty]{} 0$. Let N_0 be such that $C\delta(\tau_n) \leq \frac{1}{100}$ for all $n \geq N_0$. By Lemmas 4.10 and 4.11 we have

$$\begin{aligned} |x(\tau_n) - x(\tau_{N_0})| &\leq C \int_{\tau_{N_0}}^{\tau_n} \delta(t) dt \\ &\leq C(1 + |x(\tau_n)|)(\delta(\tau_{N_0}) + \delta(\tau_n)), \end{aligned}$$

hence,

$$|x(\tau_n)| \leq C|x(\tau_{N_0})|,$$

which gives a contradiction. This concludes the proof of Proposition 4.7. ☐

4.3 Convergence in mean

Lemma 4.12. *Consider a solution $u(t)$ of* (NLS$_\Omega$) *satisfying assumptions of Proposition 4.7. Then*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T \delta(t) dt = 0. \tag{IV.118}$$

Corollary 4.13. *Under the assumptions of Proposition 4.7, there exists a sequence of times t_n such that $t_n \to +\infty$ and*

$$\lim_{n\longrightarrow+\infty} \delta(t_n) = 0.$$

Proof of Lemma 4.12. Let φ be a smooth radial function such that

$$\varphi(x) := \begin{cases} |x|^2 & \text{if} \quad |x| \leq 1, \\ 0 & \text{if} \quad |x| \geq 2. \end{cases}$$

Consider the localized variance,

$$\mathcal{Y}_R(t) = \int_\Omega R^2\varphi\left(\frac{x}{R}\right)|u(t,x)|^2\,dx.$$

where R is large positive constant, to be specified later. Then,

$$\mathcal{Y}'_R(t) = 2R\,\mathrm{Im}\int_\Omega \bar{u}\,\nabla\varphi\left(\frac{x}{R}\right).\nabla u\,dx, \qquad |\mathcal{Y}'_R(t)| \le C\,R. \tag{IV.119}$$

Furthermore, as in the proof of Lemma 4.10 we have,

$$\mathcal{Y}''_R(t) = 4\delta(t) + A_R(u(t)) + 2\int_{\partial\Omega}|\nabla u|^2\,|x.\vec{n}|d\sigma(x), \tag{IV.120}$$

where

$$\begin{aligned} A_R(u(t)) := 4\sum_{j\neq k}\int_\Omega \frac{\partial^2\varphi}{\partial x_j\partial x_k}\left(\frac{x}{R}\right)\frac{\partial u}{\partial x_j}\frac{\partial \bar{u}}{\partial x_k} + 4\sum_j\int_\Omega\left(\frac{\partial^2\varphi}{\partial x_j^2}\left(\frac{x}{R}\right)-2\right)\left|\partial_{x_j}u\right|^2 \\ -\frac{1}{R^2}\int_\Omega |u|^2\Delta^2\varphi\left(\frac{x}{R}\right) - \int_\Omega\left(\Delta\varphi\left(\frac{x}{R}\right)-6\right)|u|^4. \end{aligned} \tag{IV.121}$$

If $|y|\le 1, (\Delta^2\varphi)(y) = 0, \partial^2_{x_j}\varphi(y) = 2$ and $\Delta\varphi(y) = 6$. Thus,

$$|A_R(u(t))| \le C\int_{|x|\ge R}|\nabla u|^2 + |u|^4 + \frac{1}{R^2}|u|^2. \tag{IV.122}$$

Let $x(t)$ as in Corollary 4.6 and K defined as (IV.59). Let $\varepsilon > 0$. By the compactness of K and Proposition 4.7, there exists $R_0(\varepsilon) > 0$ such that

$$\forall t\ge 0, \qquad \int_{|x-X(t)|\ge R_0(\varepsilon)}|\nabla u|^2 + |u|^2 + |u|^4 \le \varepsilon. \tag{IV.123}$$

Furthermore $x(t)$ is bounded thus $\frac{x(t)}{t}\xrightarrow[t\to+\infty]{} 0$. There exists $t_0(\varepsilon)$ such that

$$\forall t\ge t_0(\varepsilon),\ |x(t)|\le\varepsilon t.$$

Let $T\ge t_0(\varepsilon),\ R = \varepsilon T + R_0(\varepsilon) + 1\quad$ for $\ t\in[t_0(\varepsilon),T]$. Next, we use the fact that $|x(t)|\le\varepsilon T$ and $R_0(\varepsilon)+\varepsilon T\le R$, to get

$$\begin{aligned} \int_{|x|\ge R}|\nabla u|^2 + |u|^4 + \frac{1}{R^2}|u|^2 &\le \int_{|x-x(t)|+|x(t)|\ge R}|\nabla u|^2 + |u|^4 + \frac{1}{R^2}|u|^2 \\ &\le \int_{|x-x(t)|\ge R_0(\varepsilon)}|\nabla u|^2 + |u|^4 + \frac{1}{R^2}|u|^2 \le \varepsilon. \end{aligned} \tag{IV.124}$$

By (IV.119), we have

$$\int_{t_0(\varepsilon)}^{T} \mathcal{Y}_R''(t)dt \leq \left|\mathcal{Y}_R'(T)\right| + \left|\mathcal{Y}_R'(t_0(\varepsilon))\right| \leq C\,R.$$

From (IV.120), (IV.122) and (IV.124) we have

$$\int_{t_0(\varepsilon)}^{T} \delta(t)dt \leq C(R+T\varepsilon) \leq CR_0(\varepsilon) + \varepsilon T + 1,$$

where $C > 0$, independent of T and ε.
This yields

$$\frac{1}{T}\int_0^T \delta(t)dt \leq \frac{1}{T}\int_0^{t_0(\varepsilon)} \delta(t)\,dt + \frac{C}{T}(R_0(\varepsilon)+1) + C\varepsilon.$$

Taking first limsup as $T \to +\infty$, and letting ε tend to 0, we obtain (IV.118). □

Proposition 4.14. *Let u be a solution of* (NLS$_\Omega$) *such that*

$$M[u] = M_{\mathbb{R}^3}[Q],\ E[u] = E_{\mathbb{R}^3}[Q],\ \|\nabla u_0\|_{L^2(\Omega)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)} \tag{IV.125}$$

and $K = \{u(t); t \geq 0\}$ has a compact closure in $H_0^1(\Omega)$. Then $u \equiv 0$.

Proof. If not, there exists a solution $u \neq 0$ such that the assumptions of this Proposition are satisfied. From Lemma 4.12, there exists t_n such that $t_n \longrightarrow +\infty$ and $\delta(t_n)$ tends to 0. By the compactness of the closure of K, $u(t_n)$ converges in $H_0^1(\Omega)$ to some $v_0 \in H_0^1(\Omega)$ and the fact that $\delta(t_n)$ tends to 0 implies that $E[v_0] = E_{\mathbb{R}^3}[Q], M[v_0] = M_{\mathbb{R}^3}[Q]$ and $\|\nabla v_0\|_{L^2(\Omega)} = \|\nabla Q\|_{L^2(\mathbb{R}^3)}$. Thus, $v_0 = e^{i\theta_0}Q(x-x_0) \notin H_0^1(\Omega)$, for some parameters $\theta_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^3$, which contradicts the fact that $v_0 \in H_0^1(\Omega)$. □

Chapter V

Numerical simulations of solitary waves behavior to the nonlinear Schrödinger equation outside an obstacle

Abstract.
In this Chapter, joint work with Thomas Duyckaerts, Svetlana Roundenko and Kai Yang, we perform numerical simulations of the $2d$ focusing nonlinear Schrödinger equation in the exterior of a smooth, compact, strictly convex obstacle, with Dirichlet boundary conditions. We study the interaction between solitary wave solutions (solitons) traveling towards the obstacle with different velocities and with different angles, and show how the obstacle changes the overall behavior of solutions.

1 Introduction

We present numerical computations of solutions to the $2d$ focusing nonlinear Schrödinger equation (NLS_Ω) outside a strictly convex obstacle with Dirichlet boundary conditions:

$$(\text{NLS}_\Omega) \quad \begin{cases} i\partial_t u + \Delta_\Omega u = -|u|^{p-1}u & (t,x) \in \mathbb{R} \times \Omega, \\ u(t_0,x) = u_0(x) & \forall x \in \Omega, \\ u(t,x) = 0 & (t,x) \in \mathbb{R} \times \partial\Omega. \end{cases} \tag{V.1}$$

Here, Ω is an exterior domain in $\mathbb{R}^2$ and Δ_Ω is the Dirichlet Laplacian defined by $\Delta_\Omega := \partial_x^2 + \partial_y^2$, $(x,y) \in \mathbb{R}^2$ is a space variable.

Numerically, the computational domain has to be bounded and one must consider a large domain, which requires more careful coding techniques in order to achieve the desired accuracy and to handle the multidimensional calculations. In particular, the $2d$ discretized space matrix (especially Laplacian matrix) with a refined mesh. A typical numerical solution consists of imposing a boundary condition on an artificial boundary, which does not affect the solution in the interior domain as it does not generate any undesirable artifacts such as reflected waves. To bound the computational domain, we also impose Dirichlet boundary conditions.

The nonlinear Schrödinger equation outside obstacle conserve both mass and energy:

$$M[u(t)] = \int_\Omega |u(t,x)|^2 dx = M[u_0]. \tag{V.2}$$

$$E[u(t)] = \int_\Omega |\nabla u(t,x)|^2 \, dx - \frac{1}{p+1}\int_\Omega |u(t,x)|^{p+1}\, dx = E[u_0]. \tag{V.3}$$

The NLS_Ω equation is well-posed in $H_0^1(\Omega)$ in dimension $d = 2$, see Proposition A for more details. Moreover, the (NLS_Ω) equation is invariant by the scaling transformation $u(t,x) \longrightarrow \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2)$, therefore, the critical regularity is given by $s_c = \frac{p-3}{p-1}$. In our simulations, we consider the cubic, i.e., $p = 3$ and the quintic, i.e., $p = 5$, NLS_Ω equations. The first case is referred to as the mass-critical or L^2-critical, since $s_c = 0$, and the second case is called L^2-supercritical case ($s_c = \frac{1}{2}$).

Various numerical methods are employed in order to approximate the nonlinear Schrödinger equation ranging from explicit and implicit schemes to finite Fourier transform or pseudo-spectral methods. It is popular to use the time discretization via different methods, for example, the Crank-Nicholson scheme [25], Runge-Kutte type [2], [3], [52], symplectic and splitting type, [81], [80] and [89], [9] and relaxation methods [7] and [8].

In this work, we use the well-known Crank-Nicholson scheme for the time discretization for the NLS_Ω equation. The time discretization is based on a time centering method $\frac{u^{n+1}+u^n}{2}$. The Crank-Nicolson-type schemes provide a high order method that preserves both mass and energy, however, it requires more steps and time in solving the nonlinear terms for the NLS_Ω equation.

For the space discretization it is possible to use either finite difference or finite element method. In our simulations, we consider the NLS_Ω equation outside of a ball (in $2d$ it is a disk) and we define also the computational domain as a disk as well. The space discretization is given by the finite difference method in polar coordinates (r, θ) such that $0 \leq \theta \leq 2\pi$ and $r_0 \leq r \leq R_0$, where r_0 is the radius of the obstacle (a white disk in Figure V.1) and R_0 is the radius of the computational domain Ω (a large blue disk in Figure V.1). Therefore, we use the following domain in our simulations:

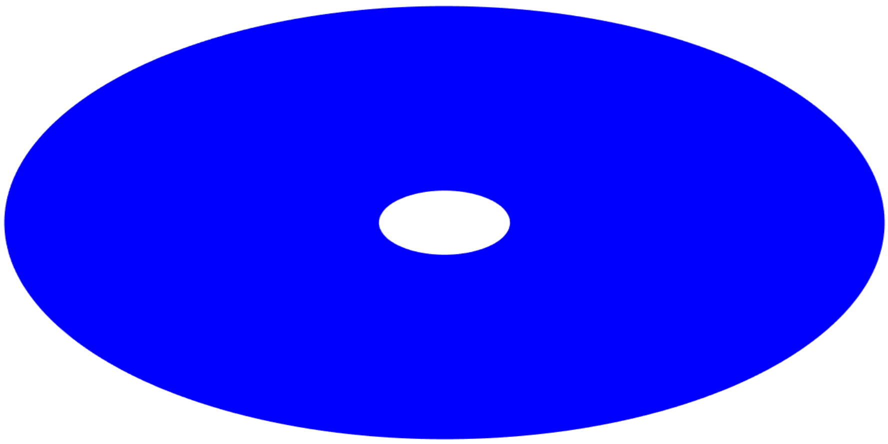


Figure V.1 – The computational domain Ω.

Let's consider the semidiscretization in time. Let T_{max} be the existence time of the solution and $T_{\delta t} < T_{max}$ be the computational time. We use N points for the time discretization, thus, defining a time step $\delta t = T_{\delta_t}/N$. We discretize the NLS_Ω equation at times $t_n = n\delta t$, $n = 1, .., N$. We define the variable u^n and u^{n+1}, which are the approximations of u at time t_n and t_{n+1}, i.e., $u^n = u(t_n)$ and $u^{n+1} = u(t_{n+1})$. We use the following scheme

$$i\frac{u^{n+1}-u^n}{\delta t}+\frac{1}{2}\Delta u^{u+1}+\frac{1}{2}\Delta u^n=-F(u^{n+1},u^n), \tag{V.4}$$

where F is the nonlinear term $|u|^{p-1}u$ approximated by

$$F(u^{n+1},u^n):=\frac{2}{p+1}\frac{|u^{n+1}|^{p+1}-|u^n|^{p+1}}{|u^{n+1}|^2-|u^n|^2}\frac{u^{n+1}+u^n}{2}.$$

Note that, u^n is the known variable and for $n=0$, $u^0:=u_0$ is the given initial condition. We compute the evolution $u^n\longrightarrow u^{n+1}$ by solving the above system (V.4). For that, we use the Newton iteration to solve the nonlinear implicit system (V.4), that is,

$$\begin{cases}(u^{n+1})^{k+1}=(u^{n+1})^k-J^{-1}\cdot G((u^{n+1})^k),\\ (u^{n+1})^0=1.001\cdot u^n,\end{cases} \tag{V.5}$$

where $G(u^{n+1})=u^{n+1}-u^n+\frac{\delta t}{2i}\Delta u^{n+1}+\frac{\delta t}{2i}\Delta u^n+F(u^{n+1},u^n)$ and J is the Jacobian of G .

As a stopping criterion for (V.5), we compute $\left\|(u^{n+1})^{k+1}-(u^{n+1})^k\right\|_{L^\infty}$ with a tolerance usually equal to 10^{-15}, which is close to the machine epsilon. In order, to reach the blow-up time (or the closest time), we slightly decrease the tolerance according to the examples treated.

Remark 1.1. *The Crank-Nicholson scheme* (V.4) *conserves the following discretized quantities:*
It preserves the discretized mass (or the L^2-norm) and the discretized energy, which is the discrete analogue of the mass and energy conservation property of (V.2) *and* (V.3)*.*

If we consider the rectangular coordinate (x,y), we use the time average $u^{n+\frac{1}{2}}=\frac{u^{n+1}+u^n}{2}$, i.e., $u^{n+1}=2u^{n+\frac{1}{2}}-u^n$, take the inner product of (V.4) *with $-i\ \bar{u}^{n+\frac{1}{2}}$, and the real parts of each components of equality, we obtain*

$$M[u^n]=\|u^n\|_{l^2(\mathbb{N})}=h^2\sum_{i,j=0}^{N}|u^n_{i,j}|^2=M[u^0],\quad for\, n\geq 0, \tag{V.6}$$

where $u^n_{i,j}:=u^n(x_i,y_j)$ and $h:=x_{j+1}-x_j=y_{j+1}-y_j$.
In polar coordinate (r,θ), it will be preserved as follows,

$$M[u^n]=\sum_{i,j=0}^{N}|u^n_{i,j}|^2\,r_i\,dr\,d\theta=M[u^0],\quad for\, n\geq 0, \tag{V.7}$$

where $u^n_{i,j} := u^n(r_i, \theta_j)$ *and* $r_i = r(i)$, $dr := r_{i+1} - r_i$ *and* $d\theta = \theta_{j+1} - \theta_j$.

Similarly, for the conservation of the discrete energy in rectangular coordinate (x, y), *we take the inner product of* (V.4) *with* $\bar{u}^{n+1} - \bar{u}^n$ *and taking the imaginary part, we get*

$$E[u^n] = h^2 \left(\sum_{i,j=0}^{N} \left| D\, u^n_{i,j} \right|^2 - \frac{1}{4} |u^n_{i,j}|^4 \right) = E[u_0], \quad \textit{for}\, n \geq 0, \tag{V.8}$$

In polar coordinate (r, θ), *it will be preserved as follows,*

$$E[u^n] = \sum_{i,j=0}^{N} \left(\left| D\, u^n_{i,j} \right|^2 - \frac{1}{4} |u^n_{i,j}|^4 \right) r_i\, dr\, d\theta = E[u_0], \quad \textit{for}\, n \geq 0, \tag{V.9}$$

where D *is the standard second-order approximation of the gradient with finite difference method. For brevity, we omit the above prove and further discussion on polar coordinate representation of discretized energy.*

Note that, in our simulations the mass is well preserved, since the relative mass-error is bounded by 10^{-14}, *at least by the end of the simulations at* $T = 20$ *with time step* 10^{-2}, *as shown in Figure V.2 and V.3. The evolution of the relative mass error is*

$$\max_{0 \leq k \leq n} (M[u^k]) - \min_{0 \leq k \leq n} (M[u]^k). \tag{V.10}$$

$$\frac{M[u^n] - M[u_0]}{M[u_0]}. \tag{V.11}$$

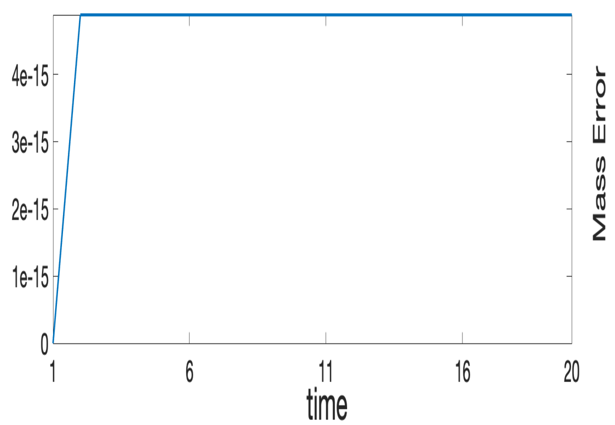


Figure V.2 – Evolution of the relative mass error (V.10) to the scheme (V.4) in $2d$ and $p = 3$.

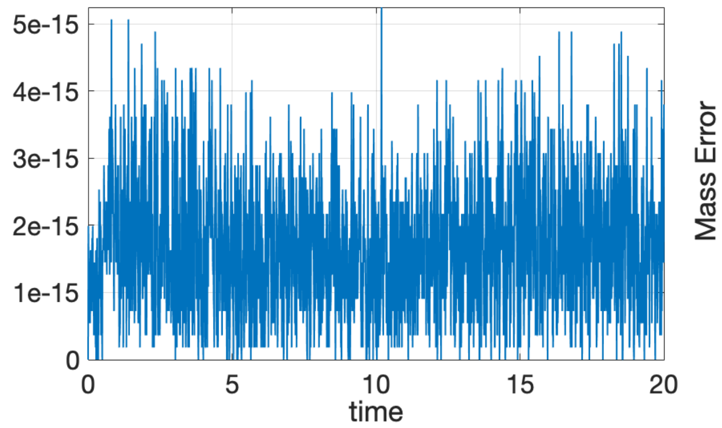


Figure V.3 – Evolution of the relative mass error (V.11) to the scheme (V.4) in $2d$ and $p = 3$.

However, the energy is also almost preserved, since the relative energy-error is 10^{-4} *(for the moving solution the energy error is not stable).*

Recall that we consider the NLS_Ω equation with Dirichlet boundary conditions for the obstacle, i.e., $u(t, r_0) := u^n(r_1, \theta_j) = 0,\ 1 \leq j \leq N$, and for the boundary of the computations domain, i.e., $u(t, R) := u^n(r_N, \theta_j),\ 1 \leq j \leq N$. Thus, to solve the above system, we should consider an initial condition such that u^0 satisfies Dirichlet boundary conditions. We typically consider a shifted Gaussian as initial data, therefore, we define the translation parameters (x_c, y_c) such that u^0 is smooth and vanishes to 0 near both the obstacle and the boundary of the computational domain.

We denote by

$$d^* := \min_{x \in \Omega} d(x, \Omega^c), \tag{V.12}$$

the distance between the initial data u^0 and the obstacle such that u^0 is well-defined. Note that, if we consider the initial condition with $d >> d^*$, then the presence of the obstacle does not affect the behavior of the solution. For example, if we consider u^0 with a large mass such that $d >> d^*$, then the solution will blow-up in finite time before the obstacle for all velocity directions, see Figure V.6 and Figure V.8 for different situations. Therefore, there is no interaction between the obstacle and the solution. In this case, numerically the soliton behaves as a solution posed on a computational domain without obstacle, see Figure V.5 and Figure V.4.

For the purpose of this work and in order to interpret the influence and the interaction of a solitary wave solution and the obstacle, we always consider that the distance d to be the minimal distance d^* such that if we make any slight modification of the velocity direction or the translation parameters, then there will be at least a weak or small interaction.

Our goal is to understand the interaction between a solitary wave (traveling with a velocity v) and the obstacle, as well as the influence of the obstacle on the behavior of the NLS_Ω equation numerically. According to our numerical simulations, the solitary wave amplitudes decrease at the collision or any interaction (even a small interaction) between the soliton and the obstacle. This is explained by the appearance of a reflection phenomenon due to the presence of the obstacle with Dirichlet boundary conditions. After the collision, our numerical results show that, if there is a weak or small interaction, then the solitary wave is transmitted almost completely with little backward reflection. If there is a strong interaction, then the solution does not preserve the shape of the original solitary wave but it will split the original wave into several waves and behaves as a sum of two or more solitons with backward reflection. We also observe that the leading reflected wave has a dispersive behaviour. The reflection phenomenon, the loss of the amplitude and the shape of the soliton make the existence of blow-up solution more challenging. Nevertheless, we have confirmed numerically the existence of blow-up solutions after the collision for the NLS_Ω equation in cases of weak interaction with the obstacle.

2 The $\mathrm{NLS}_{\mathbb{R}^2}$ equation

In this section, we give different numerical simulations of the focusing nonlinear Schrödinger equation on the whole Euclidean space $\mathbb{R}^2$. For that, we consider a bounded computational domain without obstacle and we impose Dirichlet boundary conditions on the artificial boundary of the bounded domain, which does not affect the solution in the interior domain. In order to approximate the $\mathrm{NLS}_{\mathbb{R}^2}$ equation, we use the same time discretization, i.e., the implicit Crank-Nicolson scheme given in (V.4) with applying the Newton iteration to solve the nonlinear terms. In this section we give various examples that will be considered in the next sections in order to study the influence of the obstacle on the behavior of the solutions.

Remark 2.1. *In this section, we state examples in rectangular coordinates (x, y), however, in our implementations for NLS_Ω equation we convert into polar coordinates (r, θ).*

Consider the focusing cubic ($NLS_{\mathbb{R}^2}$) equation, i.e., L^2-critical, with the following initial condition

$$u^0(x,y) := u(0,x,y) = A_0\, e^{-\frac{1}{2}(x-x_x)^2+(y-y_c)^2)}\; e^{i(\frac{1}{2}(v_x\cdot x+v_y\cdot y))}, \tag{V.13}$$

where $v := (v_x, v_y)$ is the velocity vector and (x_c, y_c) is the translation parameters. Note that, here all parameters are the same as in the Section 5.2, in particular, see Figure V.20 for velocity direction. In both cases, the solution for the $NLS_{\mathbb{R}^2}$ equation blows up in finite time. However, a snapshot of the solution u to $NLS_{\mathbb{R}^2}$ equation blows up in finite time at $t = 0.75$ but the same initial condition has a different evolution for the NLS_Ω equation, see Figure V.22.

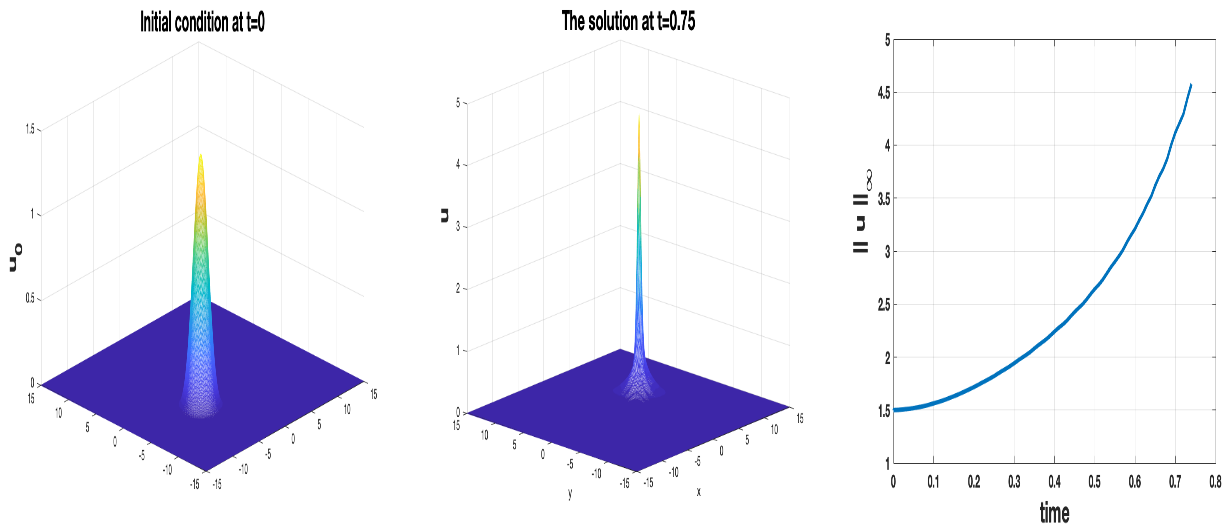


Figure V.4 – Solution to $2d$ cubic $NLS_{\mathbb{R}^2}$ with initial data u_0 at time $t = 0.75$ moving on the line $y = \frac{2}{3}x$ on the left (two first subplots) and the L^∞-norm of the solution depending on time on the right subplot.

Next, we consider the focusing quintic $NLS_{\mathbb{R}^2}$ equation, with the initial data u_0 as in (V.13) with $v = (v_x, 0)$ and $(x_c = -4.5, y_c = 0)$. Let us mention that, all parameters here are the same as in Section 6.1, in particular, see Figure V.24. In this case, the solution u to the $NLS_{\mathbb{R}^2}$ equation, which is moving on line $y = 0$ blows up in finite time at $t = 0.64$ and it is plotted in the following figure with the L^∞-norm. On the other hand, the solution to the NLS_Ω equation does not blow-up in finite time, it has a completely new dynamics and the solitary waves does not even preserve it shape after the collision.

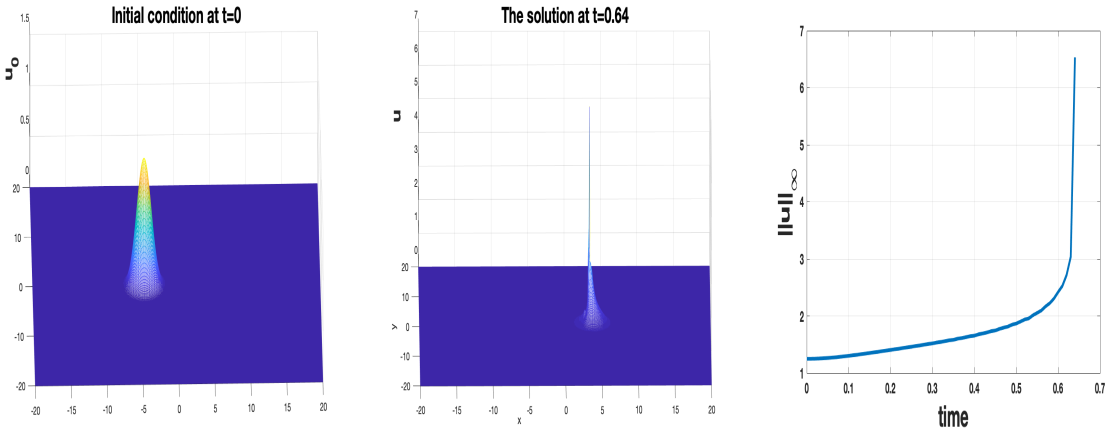


Figure V.5 – Solution to $2d$ quintic $\mathrm{NLS}_{\mathbb{R}^2}$ with initial data u_0 at time $t = 0.64$ moving on the line $y = 0$ on the left (two first subplots) and the L^∞-norm of the solution depending on time on the right subplot.

3 Dependence on the distance

We consider the $2d$ cubic and quintic NLS_Ω equations ($p = 3, 5$). Our goal in this section is to consider a solution with initial condition u^0 such that the distance d between the obstacle and the initial data is large than the minimal distance d^*. We take a shifted Gaussian initial condition similar to (V.13),

$$u^0(x, y) := u(0, x, y) = A_0\, e^{-\frac{1}{2}(x-x_0)^2+(y-y_0)^2}, \tag{V.14}$$

where (x_0, y_0) are the translation parameters such that $d >> d^*$. Therefore, $u(0, x, y)$ is smooth and satisfies Dirichlet boundary conditions. We will also take $u(0, x, y)$ with the following phase $e^{i(\frac{1}{2}(v_x \cdot x+v_y \cdot y))}$, where $v := (v_x, v_y)$ is the velocity vector, which governs the movement of this initial condition, see Figure V.6.

We consider the cubic (NLS_Ω) equation and we take the initial condition (V.14) with large mass and $d >> d^*$. Then the corresponding solution to (V.4) blows up in finite time before the obstacle for any direction of the velocity vector v, see Figure V.7. Note that, we study the case when $d \equiv d^*$ and the solution concentrate in their blow-up core after the obstacle, for the same initial data and for different velocity direction. We will also study the influence of the obstacle when there is an interaction between the traveling wave and the obstacle. In Section 5.2, we consider the weak interaction for the cubic (NLS_Ω) equation, L^2-critical case and in Section 6.2 we study the strong interaction. Moreover, we will see that in these cases the solution may have a different behavior for long time.

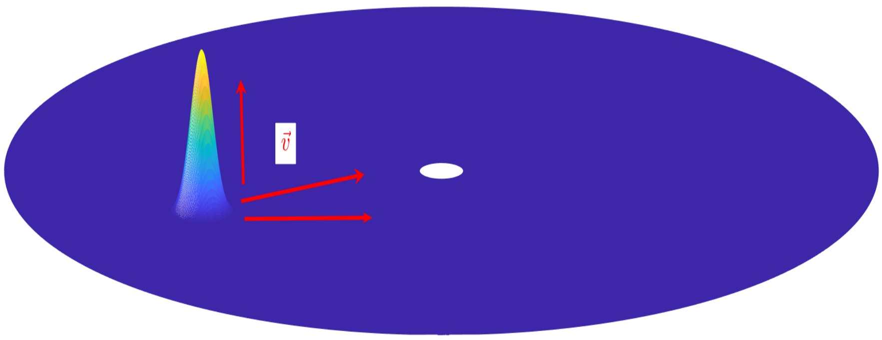


Figure V.6 – If the solition is far from the obstacle ($d >> d^*$), then the blow-up occurs in any direction of the initial velocity $\vec{v}$ shown on the picture.

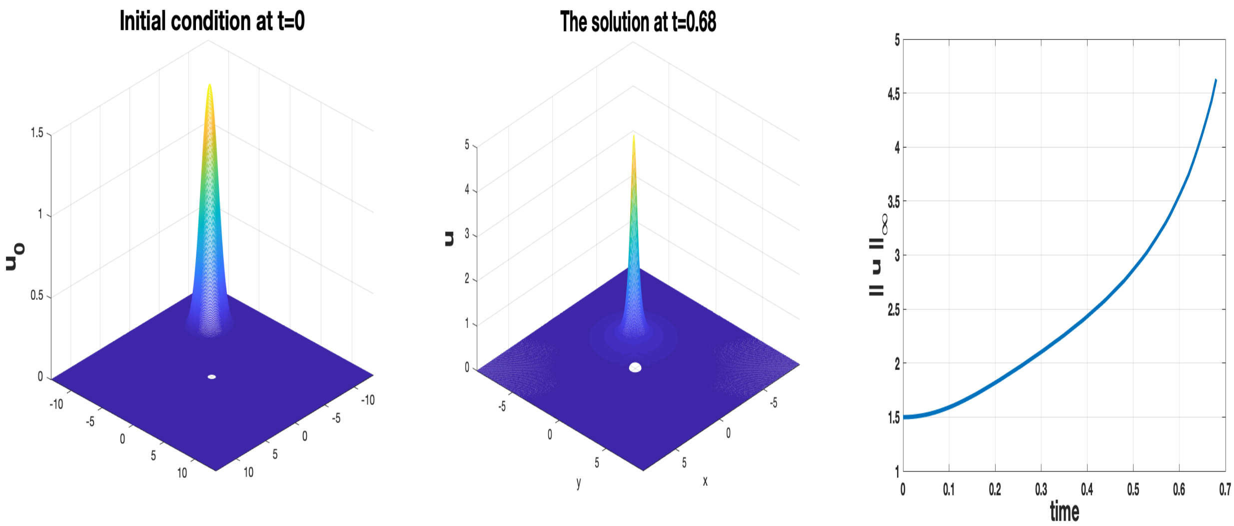


Figure V.7 – A snapshot of the initial data u_0 and the solitary wave solution u to (V.4) at time $t = 0.68$ with $d >> d^*$ on the left, the L^∞-norm depending in time.

Next, we consider the $2d$ quintic NLS_Ω equation and we take the initial condition (V.14) with large mass and $d >> d^*$. In the following scenario, we fixed all parameters (A_0 and $v = (v_x, 0)$) except the translation parameters (x_0, y_0). We vary the vertical translation parameters y_0, as shown or demonstrated in the following picture.

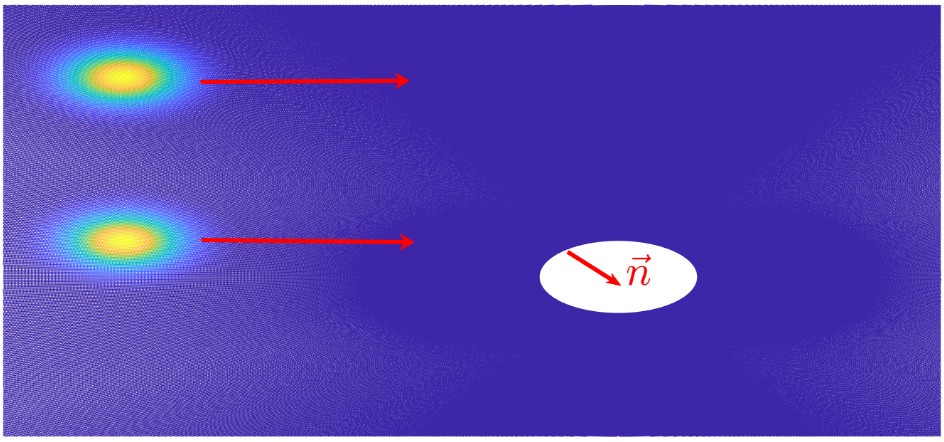


Figure V.8 – The direction of solitary waves moving in different lines $y_0 = 5$ and $y_0 = 2$ with $d >> d^*$.

A snapshot of the corresponding solution to (V.4) is plotted in Figure V.9. As in the previous example, the solution blows up in finite time before the obstacle for x_0 large. Let us mention that, we consider $d \equiv d^*$ for the same situation with the same initial data u_0, i.e., we fix the variables $A_0 = 1.25$ and velocity $v = (15, 0)$. In Section 5.1 and 6.1, we study the weak and strong interaction for the quintic NLS_Ω equation (L^2-supercritical case).

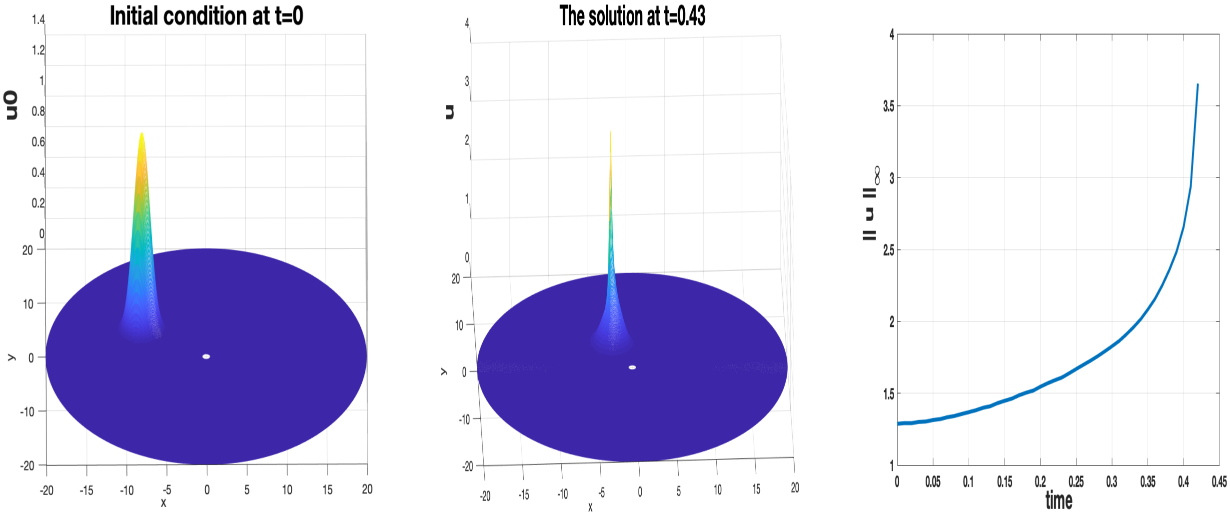


Figure V.9 – Solution to (V.4) with initial condition u_0 and $d >> d^*$; the initial data is on the left subplot, the solution that blows up in finite time at $t = 0.43$ before the obstacle on the middle subplot. Right: the time dependence of the L^∞-norm.

4 Perturbations of the soliton

We consider the Cauchy problem of the L^2-critical (or mass-critical) $2d$ NLS_Ω equation with Dirichlet boundary conditions.

$$\begin{cases} i\partial_t u + \Delta u = -|u|^2 u, & (t,x) \in \Omega \times \mathbb{R}, \\ u_{|\partial\Omega} = 0. \end{cases} \tag{V.15}$$

Here, Ω is defined as in Figure V.1. The behavior of the solutions to the focusing mass-critical NLS_Ω equation on the whole Euclidean space $\mathbb{R}^d$, for example, $p = 3$ in dimension $d = 2$, was first studied by Weinstein in [90]. He showed a sharp threshold for global existence using Gagliardo-Nirenberg inequality combined with the energy conservation,

$$\|\nabla u\|_{L^2}^2 \leq \left(1 - \frac{\|u\|_{L^2}^2}{\|Q\|_{L^2}^2}\right)^{-1} E[u],$$

which implies that (i) if $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then an H^1 solution exists globally in time and (ii) if $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$, then the solution may blow up in finite time. Moreover, in the first case it was recently proven that for $u_0 \in L^2$, the corresponding solution is global and scatters in $L^2(\mathbb{R}^d)$, see [26]. We recall that, Q is the ground state solution (positive, smooth and vanishing at ∞) of the following nonlinear elliptic equation.

$$-\Delta Q + Q = Q^3. \tag{V.16}$$

Recall that the ground state solution Q of (V.16) is radially symmetric and exponentially decaying at infinity. Note that, in dimension $d \geq 2$, the ground state solution is not explicit but its properties are well-known. There are various numerical methods, for example, renormalization method and shooting method, that produce the ground state numerically, see [33, chapter 28] and [76].

For the purpose of this work, we are interested in real positive and symmetric solutions to (x_c, y_c). We defined the translation parameters (x_c, y_c) such that the distance d between the initial data u_0 and the obstacle is minimal, i.e., u_0 is smooth and satisfies Dirichlet boundary conditions. We start investigating evolution of solutions to the NLS_Ω equation by considering initial data of perturbed solitons, of the form

$$u_0 \equiv u(x,y,0) = \lambda\, Q(x - x_c, y - y_c), \quad \lambda \in \mathbb{R},$$

where Q is numerically constructed soliton ground state solution to (V.16) shifted by (x_c, y_c).

The perturbation of the soliton solution to (NLS_Ω) with large mass initial condition, for example, $\lambda = 1.1$, leads to a blow up solution at time $t = 0.84$ with the diverging L^∞-norm. Recall that we use Newton iteration to solve the implicit scheme (V.4) and to reach the desired accuracy. Thus, it is difficult to approach the blow-up time while maintaining the convergence of the Newton iteration (V.5). For that, we need to run the code with more refined mesh in order to maintain the convergence of (V.5). This is challenging to handle in the 2D non-radial case. We decide that a solution blows up in finite time when its height ($L(\infty$-norm) is 3 times higher than that of the initial data (otherwise, the iterations of (V.5) take long time to converge or may not converge at all).

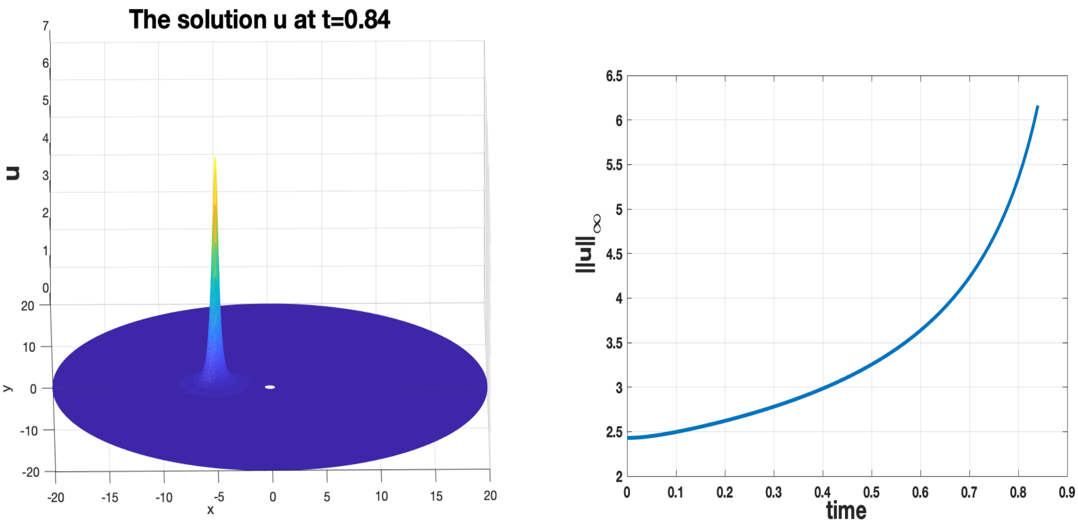


Figure V.10 – Solution to (NLS_Ω) with $u(x, y, 0) = 1.1\, Q(x - 4.5, y)$ at $t = 0.84$ on the left and its L^∞ norm depending on time on the right.

We next consider the initial condition of the perturbed soliton, $u(0, x, y) = 0.9\, Q(x - 4.5, y)$. A snapshot of the corresponding solution of the NLS_Ω equation at time $t = 1.5$ and the $L^\infty-$norm depending on time can be seen in Figure V.11. In the present situation, we see that $L^\infty-$norm is monotonically decreasing with a definite negative slope. Therefore, we conclude that this solution disperses in long run, as expected for perturbations with smaller mass than the soliton. Nevertheless, we run this example with the same initial condition for longer times and the next Figure V.12 shows that the L^∞-norm keeps decreasing toward 0.

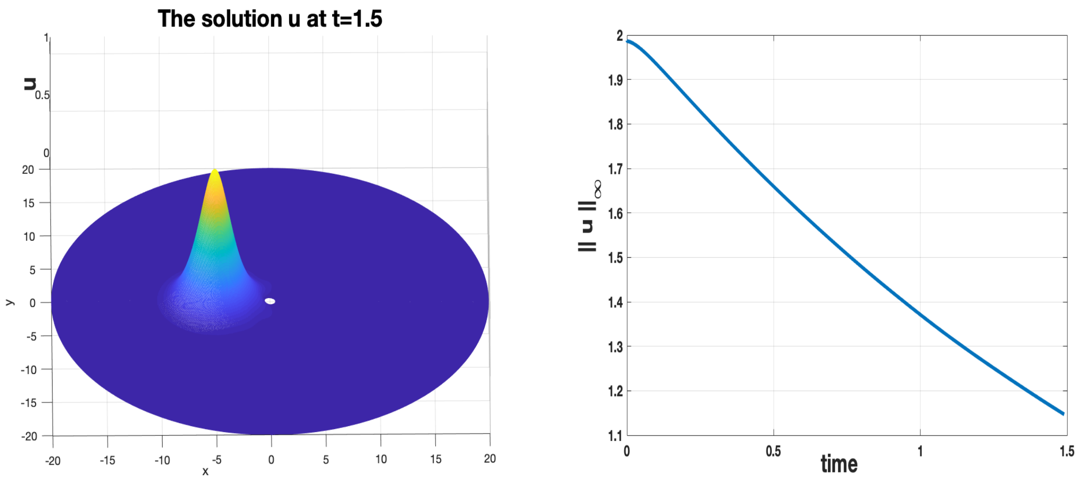


Figure V.11 – Solution to (NLS_Ω) with $u(x, y, 0) = 0.9\, Q(x - 4.5, y)$ at $t = 1.5$ (left) and L^∞ norm for $0 < t < 1.5$ (right).

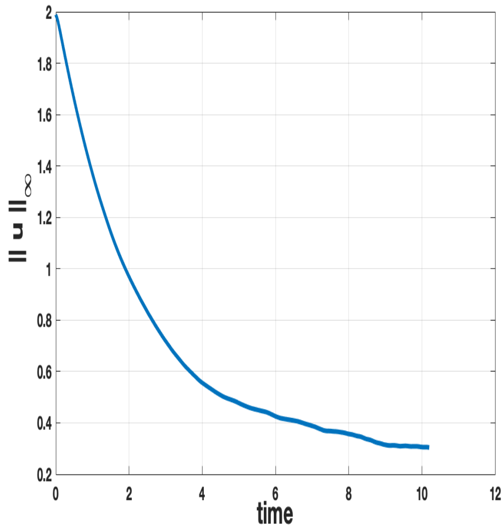


Figure V.12 – The L^∞-norm for the solution in Figure V.11 for $0 < t < 10$.

5 Weak interaction between soliton and obstacle

5.1 The L^2-supercritical case

We consider the $2d$ quintic NLS_Ω equation ($p = 5$), which is L^2-supercritical. Our goal is to study the interaction between the obstacle and the solution. For that, we take a shifted Gaussian $u_0 = A_0\, e^{-\frac{1}{2}((x-x_c)^2+(y-y_c)^2)}$ as initial condition, where (x_c, y_c) are the translation parameters. To make the solution move or travel, we multiply the initial data by the phase

$e^{i(\frac{1}{2}(v_x \cdot x + v_y \cdot y))}$, where $v = (v_x, v_y)$ is the velocity vector. Thus, the initial condition is

$$u_0 \equiv u(0,x,y) := A_0\, e^{-\frac{1}{2}((x-x_c)^2+(y-y_c)^2)}\, e^{i(\frac{1}{2}(v_x . x + v_y . y))}. \tag{V.17}$$

In the following simulation, we fix A_0, the velocity v and we vary the translation parameters. Recall that, we choose x_c and y_c such that u_0 is smooth and satisfies Dirichlet boundary conditions, see Figure V.8.

We start with an example, where there is no interaction in order to compare the behavior of the solution for different scenarios later, especially when there will be a strong interaction. For that, we consider the initial data u_0 from (V.17) with

$$A_0 = 1.25, \quad x_c = -4.5, \quad y_c = 5, \quad \text{and} \quad v = (15, 0), \tag{V.18}$$

which can be seen on the left of Figure V.13. The middle subplot shows that the corresponding solution of (V.4) blows up in finite time at $t = 0.57$ with the diverging L^∞-norm. Snapshots of the solution in time are plotted in Figure V.14. We observe that the solution blows up in finite time and there is no interaction between the solution and the obstacle.

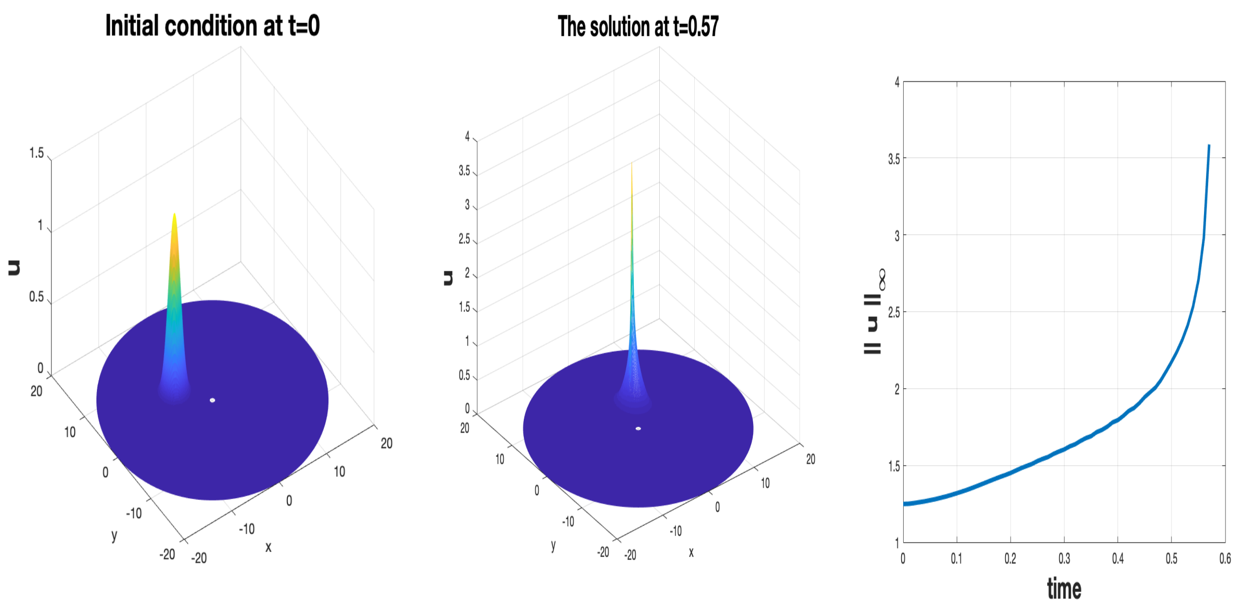


Figure V.13 – Solution to (V.4) with u_0 from (V.18) (left) close to blow-up time (middle), time dependence of the L^∞-norm (right).

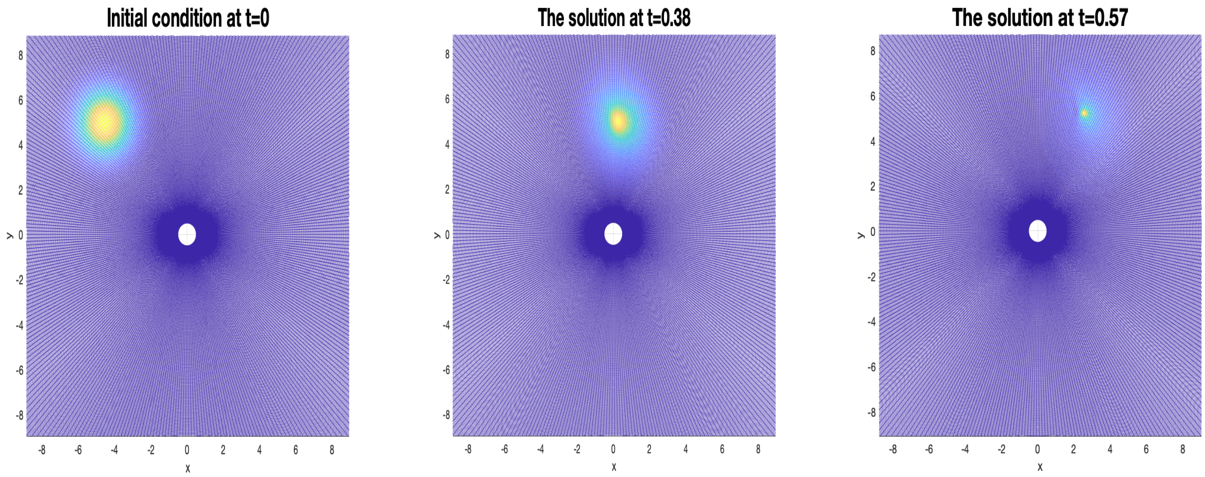


Figure V.14 – Snapshots of the evolution of u_0 from (V.18) in time $t = 0$, $t = 0.38$ and $t = 0.57$.

Next, we take the same initial data u_0 as in the previous example (V.18) with $y_c = 2$ as shown in Figure V.8. In this case, we expect that the traveling wave solution has some weak interaction with the obstacle.

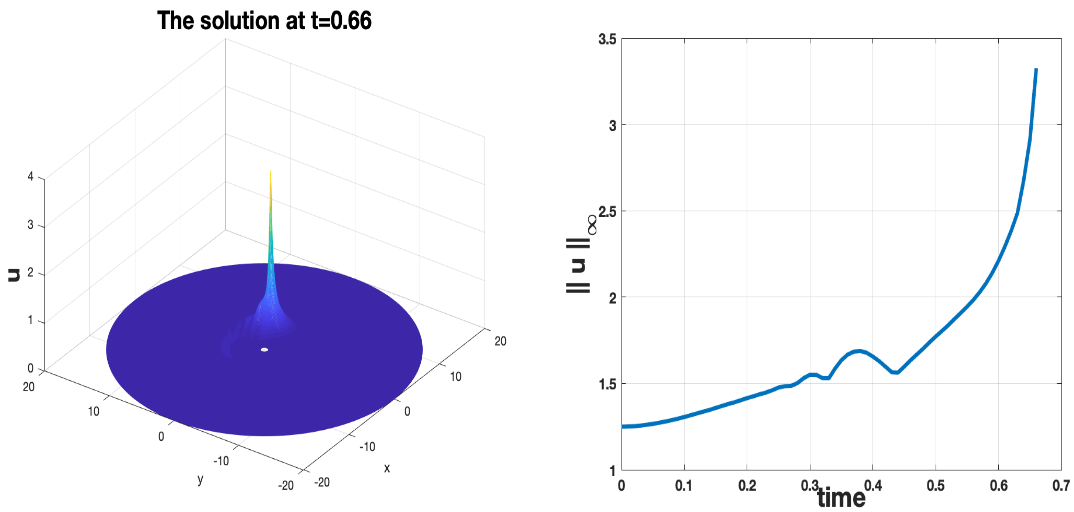


Figure V.15 – Solution to (V.4) with u_0 from (V.18) with $y_c = 2$. Left: snapshot at time $t = 0.66$. Right: the time dependence of the L^∞-norm.

We observe that with this weak interaction the solution still blows up in finite at time $t = 0.66$ but the blow-up time is delayed compared to the previous case, where there was no interaction between the solution and the obstacle, see Figure V.15. Moreover, we observe that there is a slight perturbation of the growth of the L^∞-norm: at the collision, the amplitude of the solution starts decreasing but after the weak interaction, the solution is back to the concentration leading to the blow-up. This can be explained by the appearance of small reflected waves after the collision, which scatter at the end of the simulation. They can be seen in the snapshots of the solution in Figure V.16 with the view onto xy-plane and zooming near the obstacle.

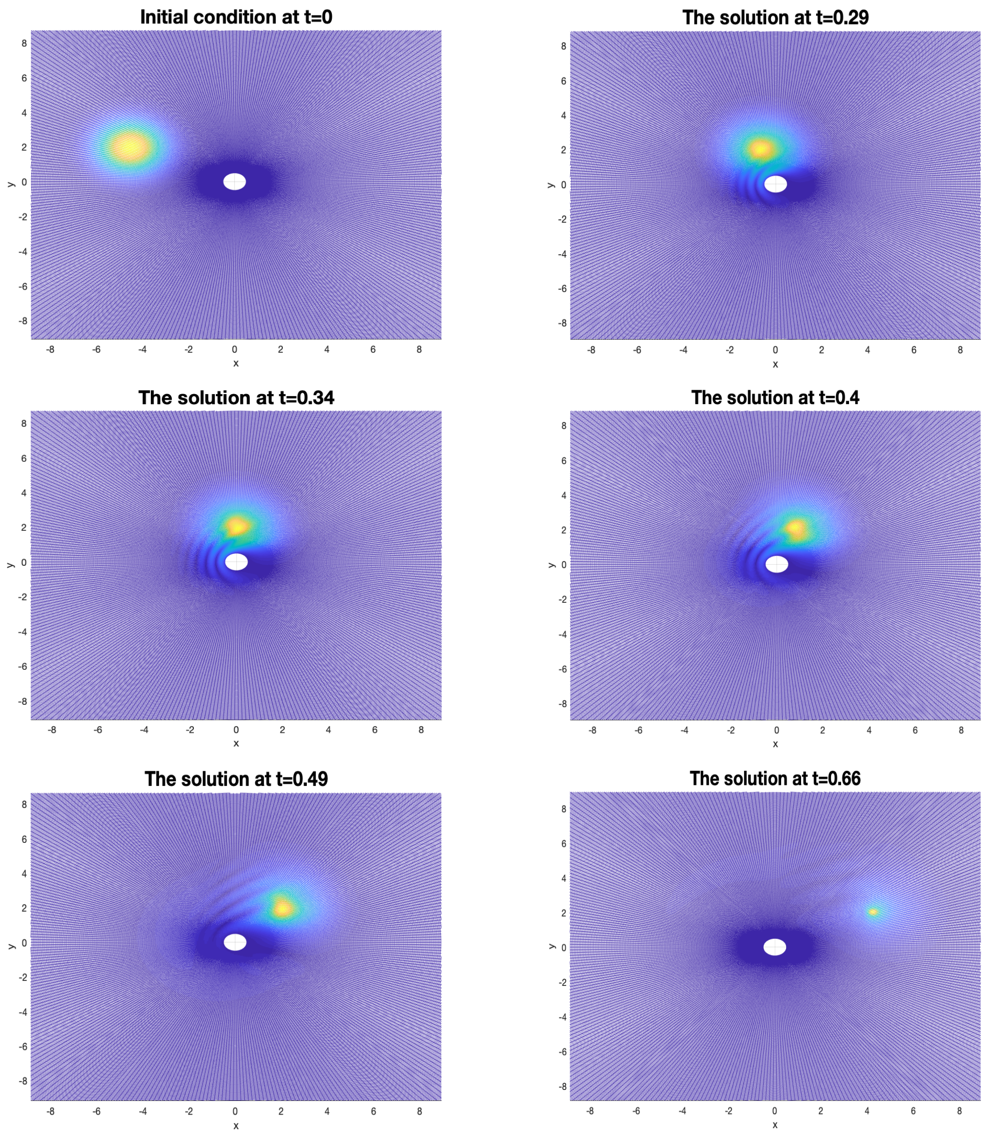


Figure V.16 – Snapshots of the time evolution of the solution u with initial u_0 from (V.18) with $y_c = 2$, which eventually blows up in finite time.

5.2 The L^2-critical case

We consider the cubic NLS_Ω equation ($p = 3$), which is L^2-critical. We study the initial condition,

$$u_0 = A_0\, e^{-\frac{1}{2}((x-x_c)^2+(y-y_c)^2)}\, e^{i\,(\frac{1}{2}(v_x\cdot x+v_y\cdot y))}, \tag{V.19}$$

where (x_c, y_c) are translation parameters, $v = (v_x, v_y)$ is the velocity vector and A_0 is the amplitude. In the following simulation, we fix $A_0 = 1.5$, (x_c, y_c) and vary the direction of the velocity vector. We choose $x_c = -4.5$ and $y_c = -4$ such that u_0 is smooth and satisfies Dirichlet boundary conditions. In the following simulation, we consider the following two scenarios:

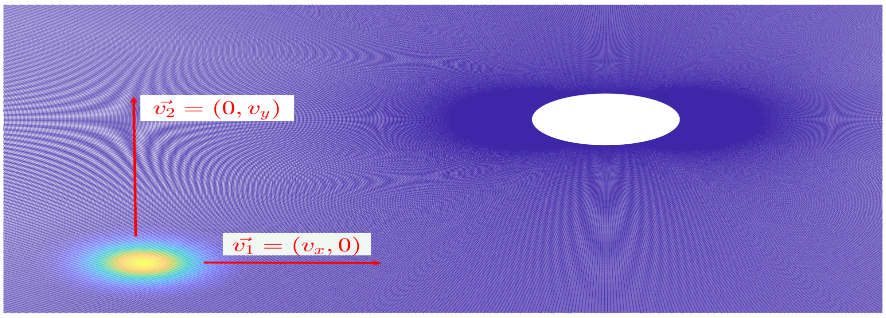


Figure V.17 – The direction of the movement of the solution in next two examples.

We start with the initial datum u_0 described above with $\vec{v_1} = (v_x, 0)$, $v_x = 15$. We observe that the solution blows up at time $t = 0.62$. It does not interact with the obstacle; its behavior is the same as it would be of a solitary wave on the whole space.

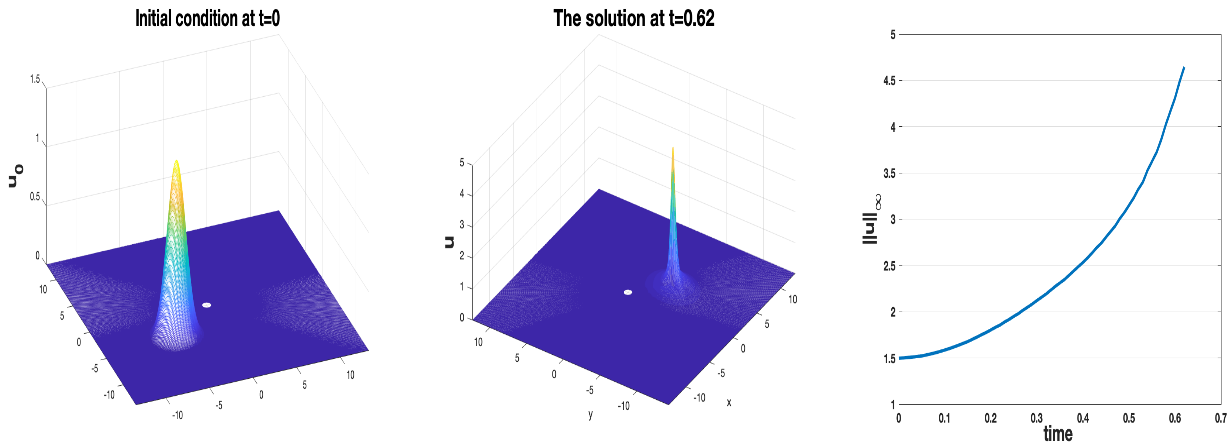


Figure V.18 – The initial condition u_0 from (V.19) (left); the time evolution at $t = 0.62$ (middle); the time dependence of the L^∞-norm (right).

Next, we consider the same initial condition (V.19) with the velocity vector v_2, which is perpendicular to the direction in the previous case as shown in Figure V.17. The velocity vector is $v_2 = (v_x, v_y) = (0, 15)$. We observe that the solution blows up at the same $t = 0.62$.

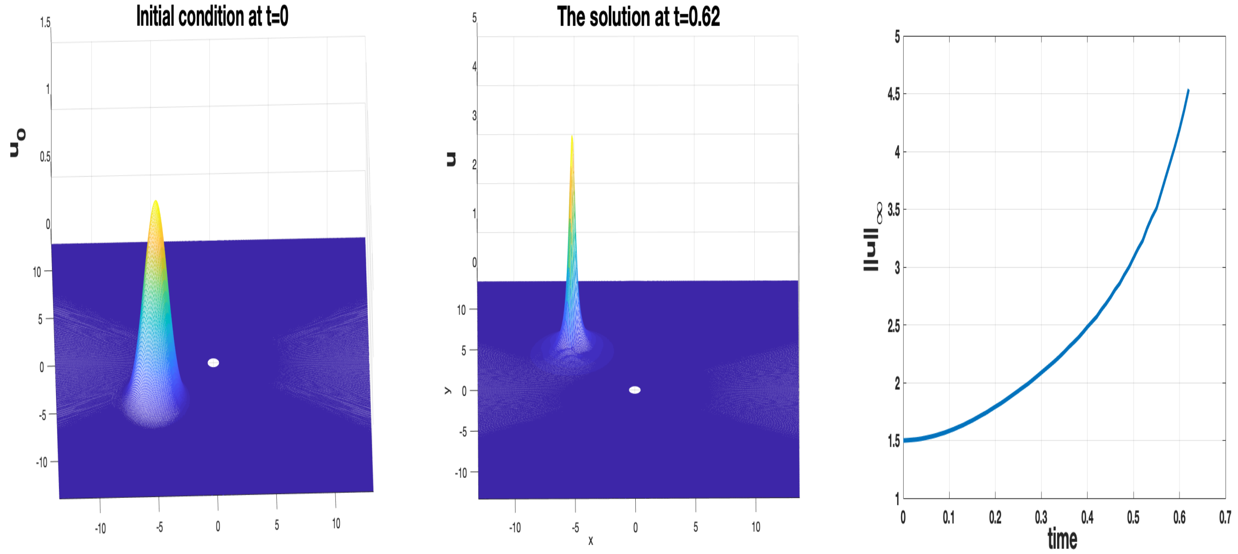


Figure V.19 – The initial data u_0 (V.19) (left); its time evolution to (V.4) at $t = 0.62$ (middle); the time dependence of the L^∞-norm (right).

In our third example, we take the initial datum u_0 (V.19) with the velocity $\vec{v}$ that has a different direction but has the same magnitude $|\vec{v}|$, as in the previous two examples: we choose $v_1 = (v_x, v_y)$ and $v_2 = (v_y, v_x)$ as shown on Figure V.20.

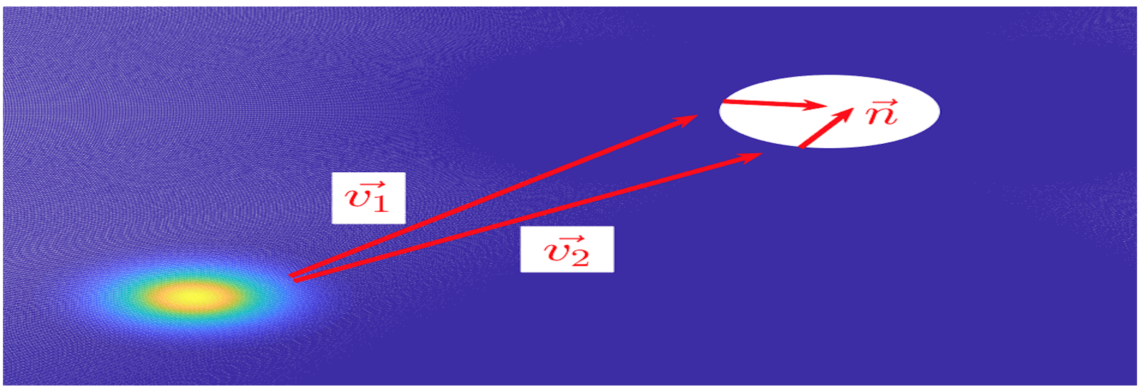


Figure V.20 – The direction of the movement of the solution.

We choose $v_1 = (10, 15)$ such that the solution has a small interaction with the obstacle. After the collision, we observe that the solution has almost the same behavior (as the previous example with weak interaction see Figure V.15), i.e., it blows up with slightly dispersive reflection part, preserving the shape of the soliton, similar to two previous two cases. The solution blows up in finite time $t = 0.76$ after the interaction with the obstacle but the blow-up time is longer than in the previous cases (compare to $t = 0.62$). Moreover, we see that at the collision time the L^∞-norm has again a slight perturbation (or a small oscillation), however, it continues to increase and is perturbed less in the overall growing of the L^∞-norm, compared to the perturbation in the previous case shown in Figure V.15.

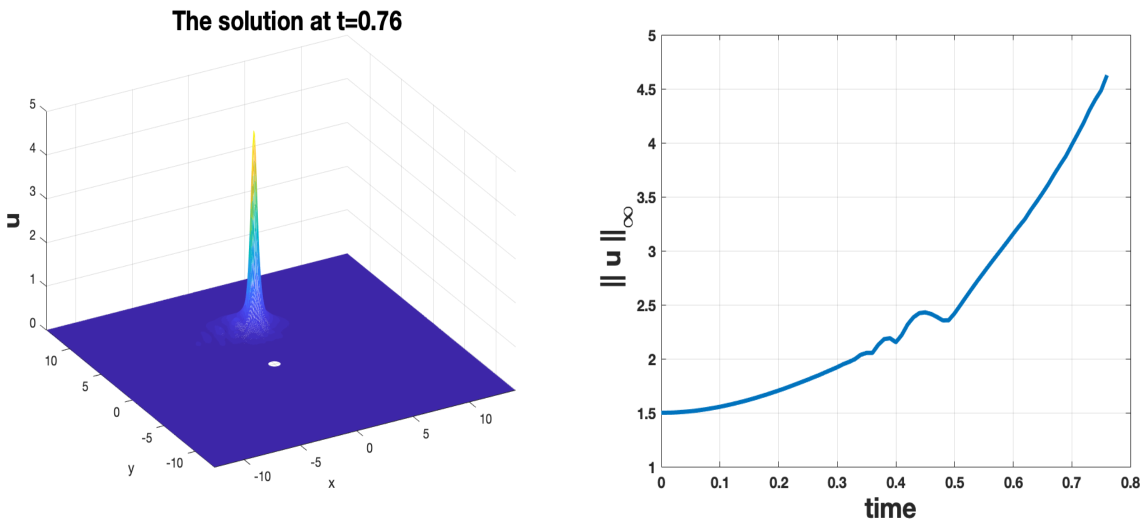


Figure V.21 – Solution to (V.4) with initial condition u_0 (V.19) and velocity v_1 at time $t = 0.76$, moving on the line $y = \frac{3}{2}x$ (left); the time dependence of L^∞-norm (right).

In our 4th example, we take the same initial condition u_0 (V.19) but with the velocity vector $v = v_2 = (15, 10)$. A snapshot of the corresponding time evolution at time $t = 1.25$ is plotted on the left of Figure V.22. The right subplot shows the L^∞-norm depending on time, which appears to diverge at the beginning of the simulation but after the collision it starts to be monotonically decreasing. This solution disperses, or in other words, is a scattering solution. Thus, the obstacle arrests the blow-up. This is a **different behavior** compared to the previous examples, where the solutions were transmitted almost with the same shape after the interaction and the soliton core was preserved. Unlike the previous examples, the collision of the solution with the obstacle here creates reflected waves, which then disperse the solution. The reflection causes the loss of the mass in the main part of the solution, which arrests the blow-up in finite time unlike the examples above, where the reflection does not affect the blow-up of the solution and only influences (delays) the blow-up time. In this case the interaction between the soliton and the obstacle has a substantial influence on the behavior of the solution, which is a completely new dynamics compared to the dynamics on the whole space. We provide snapshots of the behavior of the solution for different time steps, see Figure V.23.

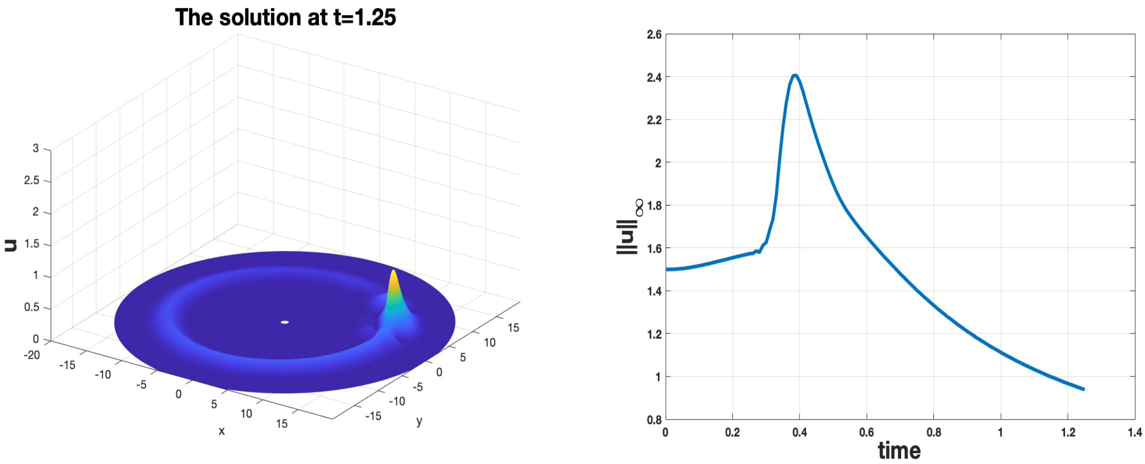


Figure V.22 – Solution to (V.4) with initial condition u_0 and velocity v_2 at time $t = 1.25$ moving on the line $y = \frac{2}{3}x$ (left), the time dependence of the L^∞-norm of the solution (right).

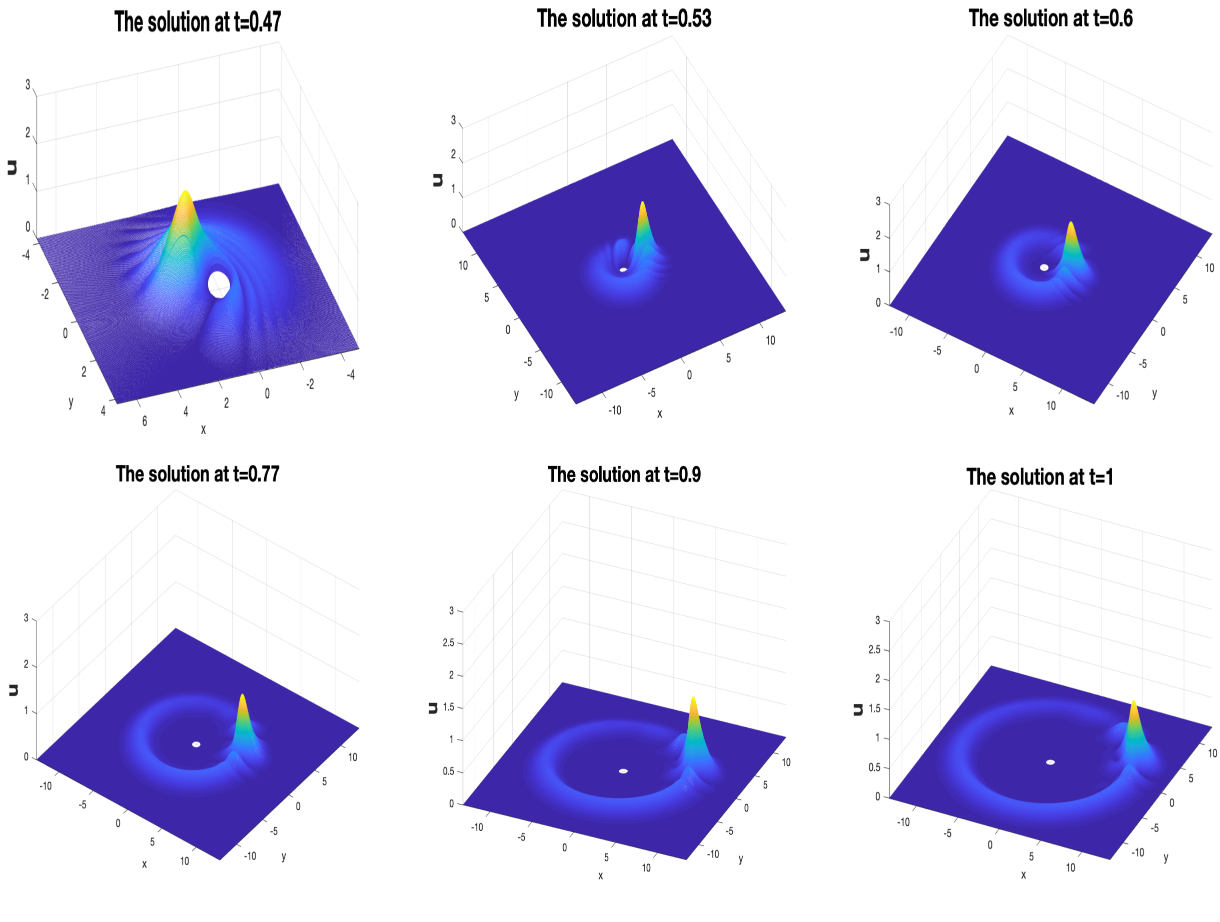


Figure V.23 – Snapshots of the behavior of the solution u to (V.4) with $v_2 = (15, 10)$ moving on the line $y = \frac{2}{3}x$.

6 Strong interaction between soliton and obstacle

6.1 The L^2-supercritical case

We consider the $2d$ quintic NLS$_\Omega$ equation ($p = 5$), which is L^2-supercritical. Our goal is to investigate the strong interaction between the obstacle and the solution. We call it a textitstrong interaction, if the solution is moving in the same direction as the outward normal vector of the obstacle, see Figure V.24.

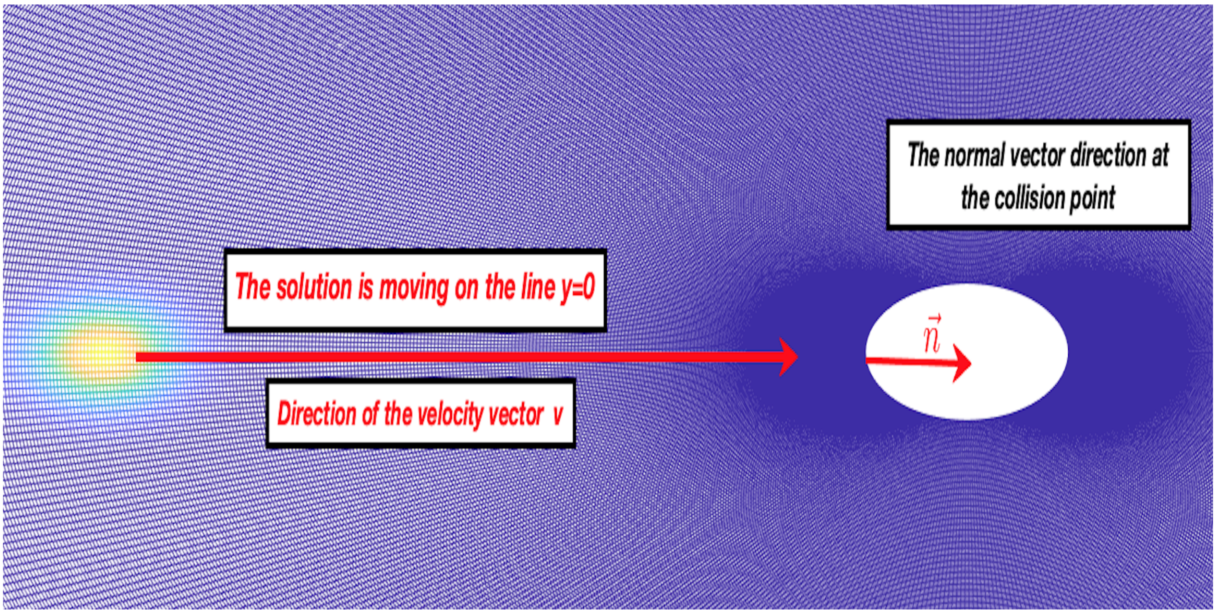


Figure V.24 – The direction of movement of the solution, on the line $y = 0$ with outward normal vector.

We consider the same initial data, i.e., the shifted Gaussian (V.17), with the same phase as for the quintic NLS$_\Omega$ equation described in subsection 5.1 with $y_c = 0$ ($A_0 = 1.25$, $x_c = -4.5$ and $v = (v_x, 0)$ are fixed parameters). In the present situation, the solution is moving on the line $y = 0$, i.e., in the same direction of the outward normal vector to the obstacle. The solitary wave hits the obstacle straight on causing a strong interaction between the wave and the obstacle. In this case, the solution scatters and does not preserve the shape of the original solitary wave. After the collision, the solitary wave solution splits into several waves and behaves as a sum of two or more solitons with a backward reflection. We observe also that the leading reflected wave has a dispersive behavior. Moreover, one can see that the presence of the obstacle prevents completely blow-up. Before the interaction, the L^∞-norm of the solution starts increasing, indicating a possible blow-up behavior, however, after the collision time the amplitude of the solution decreases toward 0, which confirms the dispersion of the solution in a long run (scattering).

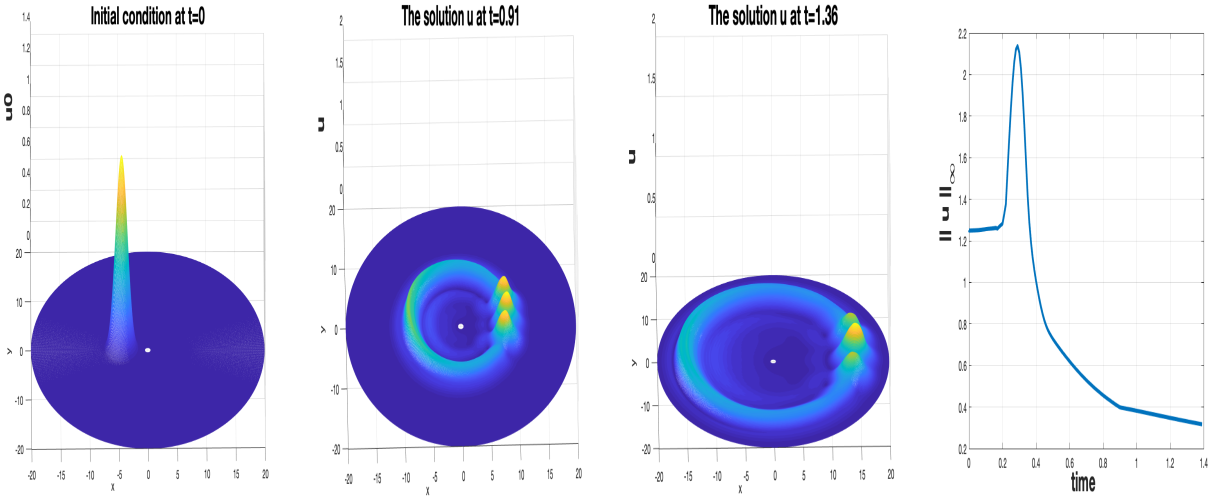


Figure V.25 – Solution to (V.4) with the initial data u_0 (V.17) the first subplot is the initial data at $t = 0$; next subplots are the solution u in time moving on the line $y = 0$; the last subplot is the time dependence of the L^∞-norm of the solution.

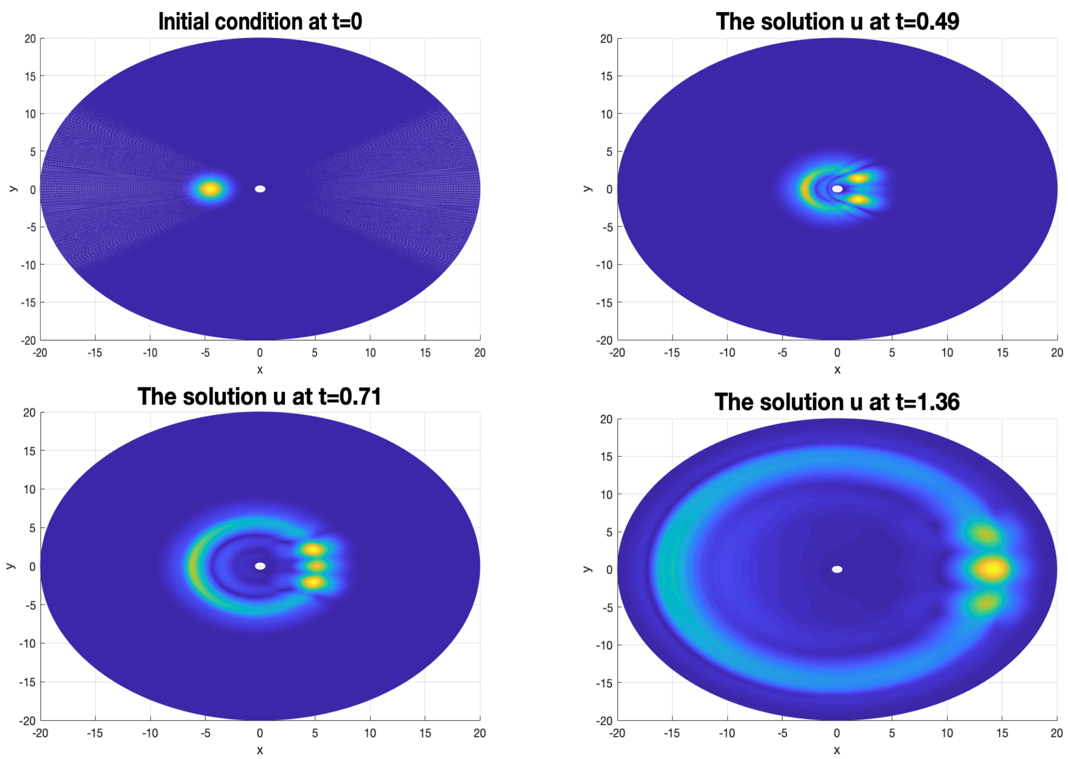


Figure V.26 – Snapshots of the behavior of the solution u to (V.4) with $(X, Y)-$view.

6.2 The L^2-critical case

We study a strong interaction as in the previous subsection 6.1 in the L^2-critical, that is,

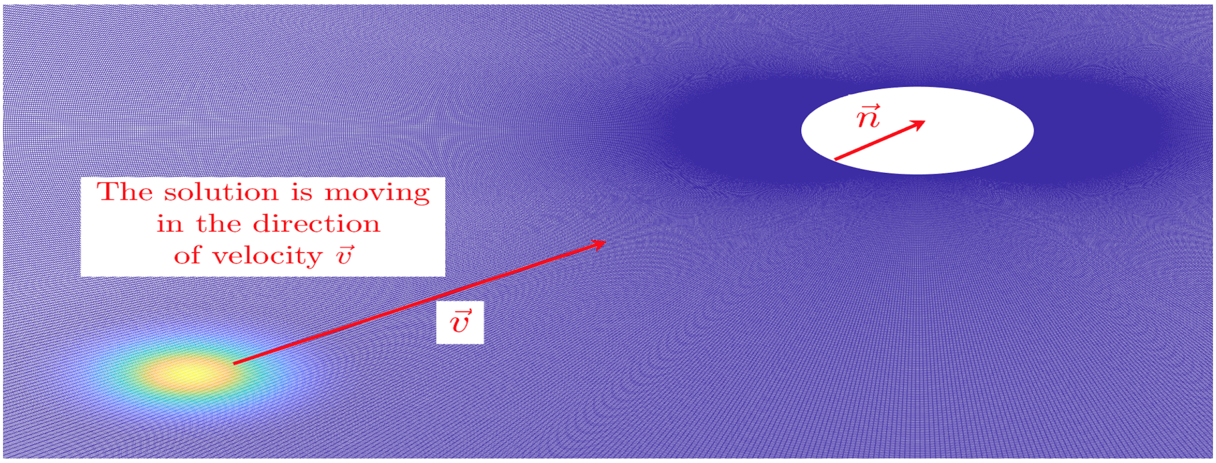


Figure V.27 – The direction of movement of the solution u on the line $y = x$ and the same direction of the outward normal vector.

We consider the cubic NLS_Ω equation with the same initial condition (V.17) and we take the velocity $v = (15, 15)$ ($A_0 = 1.5$, $x_c = -4.5$ and $y_c = -4.5$ are fixed parameters) such that that the u_0 is smooth and satisfies Dirichlet boundaries condition. Note that the solution u is moving in the line $y = x$, i.e., in the same direction of the normal vector as described in Figure V.27.

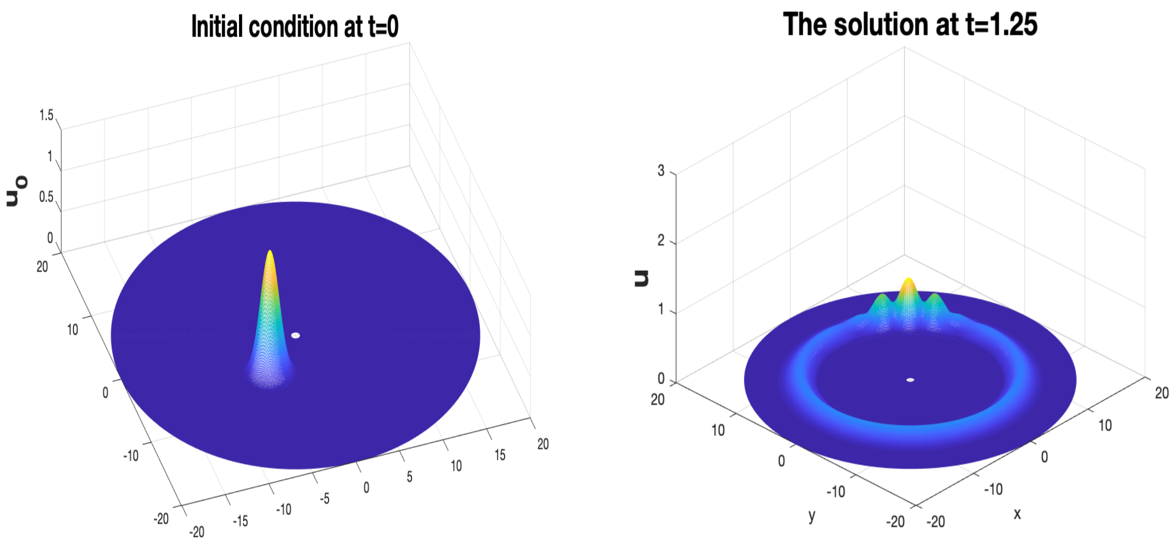


Figure V.28 – The initial data u_0 at $t = 0$ (left); the corresponding solution u to (V.4) at time $t = 1.25$ (right).

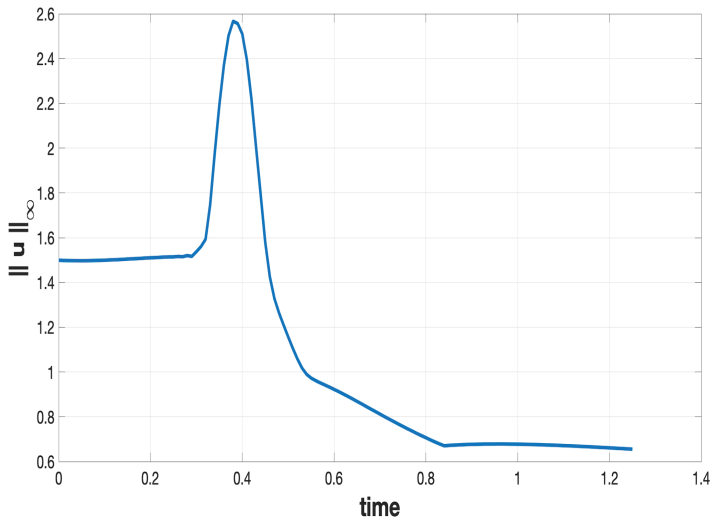
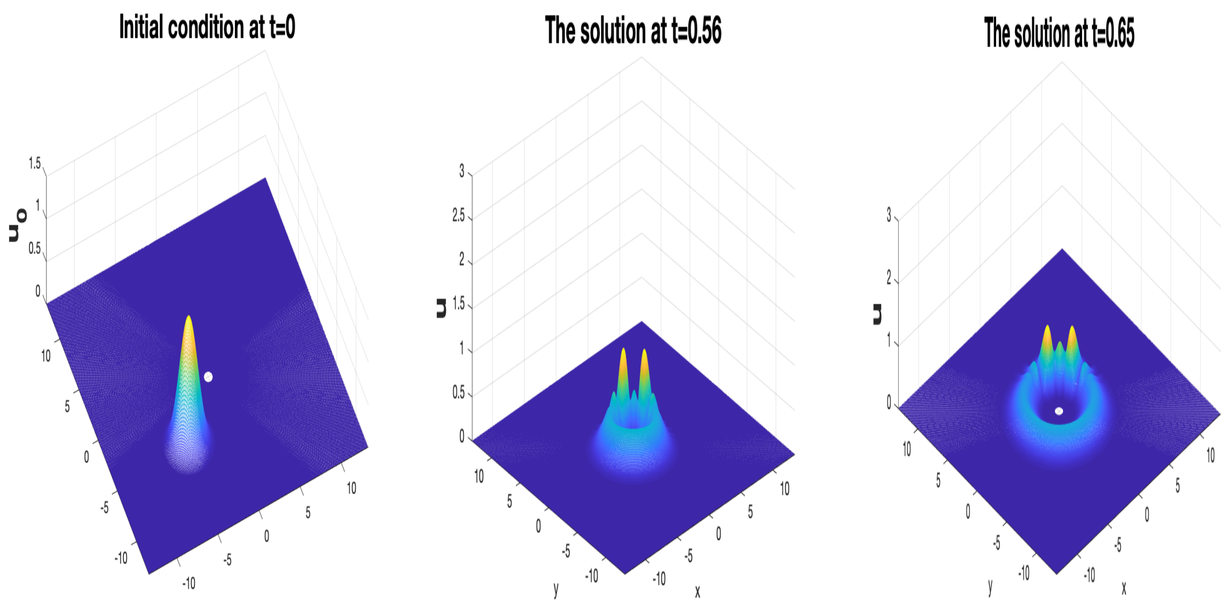


Figure V.29 – the time dependence of L^∞ norm.

We observe that the strong interaction has a substantial influence on the dynamics of the solution. Recall that, in the weak interaction case for a similar example in subsection 5.2 (L^2-critical) the solution blows up in finite time. However, we observe a scattering behavior here, the solution splits into several soliton or bumps with a dispersive backward reflected waves. Therefore, we confirms that the strong interaction has a substantial influence on the dynamics of the equation and it transforms such a blow-up behavior into scattering.



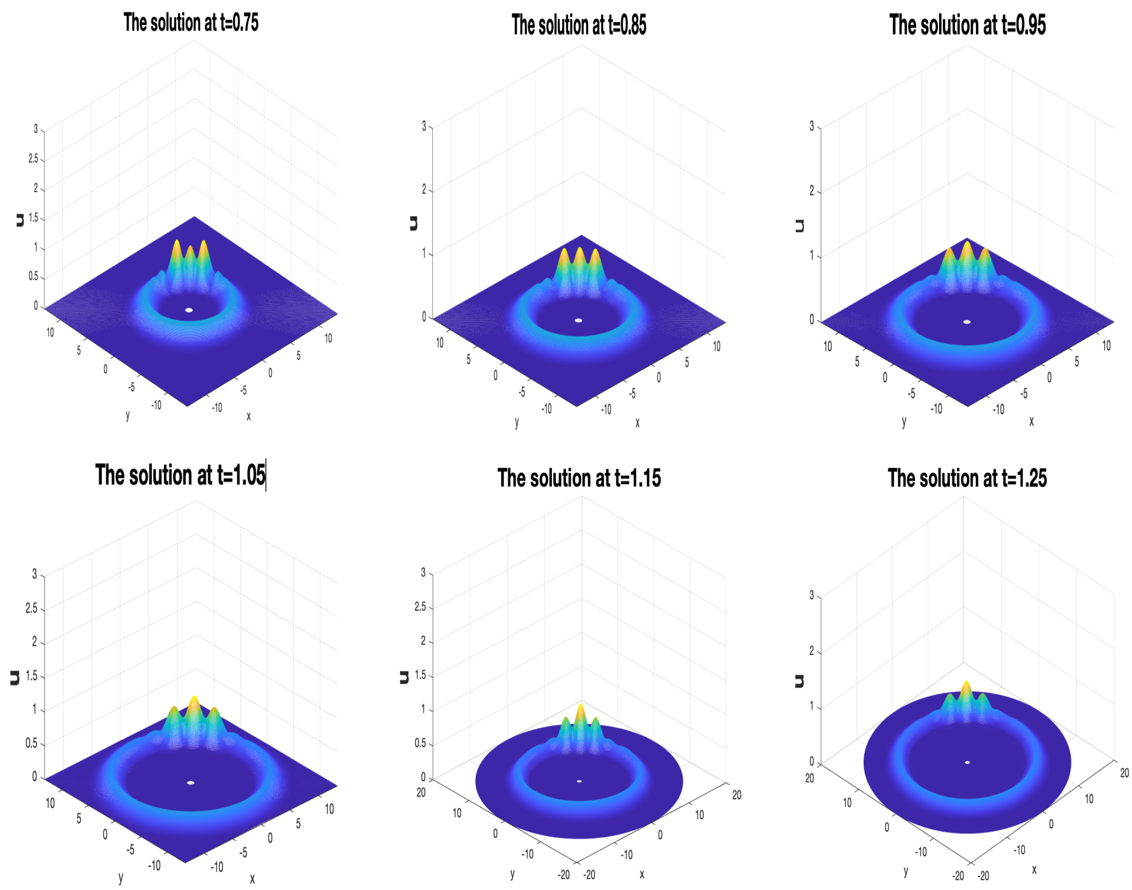


Figure V.30 – Snapshots of the behavior of the solution u.

7 Conclusion and future projects

According to our initial simulations, we observe that the interaction between a solitary wave and the obstacle has an influence on the behavior of the solution to the NLS$_\Omega$ equation, which depends on the direction of the velocity of vector $\vec{v}$, distance and translation parameters (x_c, y_c), which yield either strong, weak or no interaction. We observed in Sections 5.1 and 5.2 that even a small interaction between the obstacle and the solution has an influence on the dynamics of the equation (at the least, on the blow-up time). Moreover, we conclude that the strong interaction has a significant influence on the behavior of the solution and on the shape of the soliton, which splits it into several bumps. In this case, the appearance of reflection waves due to the presence of the obstacle with Dirichlet boundary conditions prevents the solution from blowing up in finite time. Furthermore, this backward reflection has always a dispersive behavior, which might be an indication that the solution behaves like a multi-soliton solution in a long run after a strong collision. For a weak interaction, i.e., when the solution preserves the same shape as a soliton, the solution behaves as either a solitary wave solution, constructed in Chapter 2 (which exist for are positive times), or as the one given in Chapter 3 as a blow-up solution.

7.1 Dependance on the angle

A possible future project would be to investigate the dependence of the angle θ between the direction of the velocity vector v and the normal n, as shown in Figure V.31. We think that this may be an important future direction in understanding the interaction of a soliton and an obstacle.

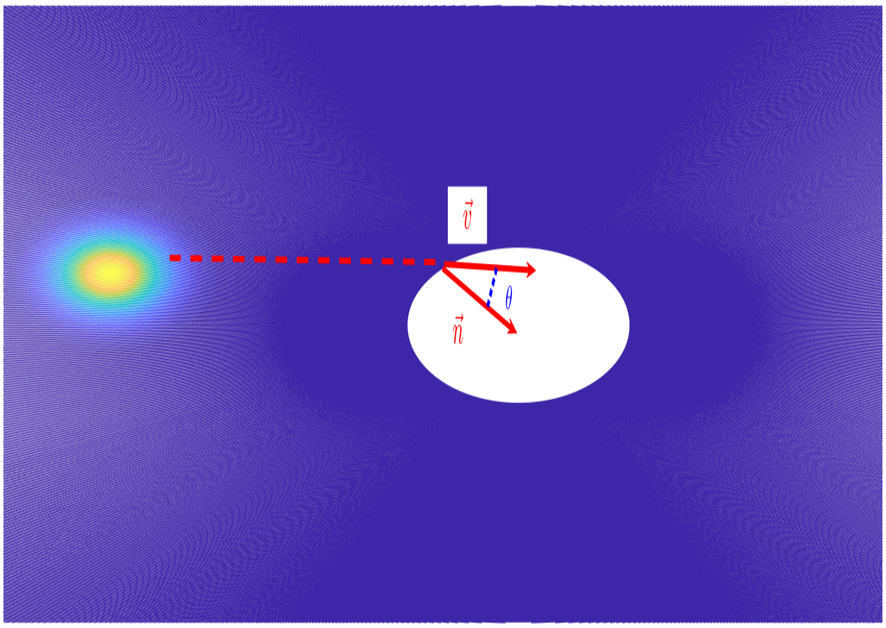


Figure V.31 – Dependence on the velocity vector $\vec{v}$ and the angle θ, and the outward normal vector $\vec{n}$.

7.2 Negative time behavior of soliton solution of the NLS_Ω equation

Another future investigation could be a construction of a non-scattering solution $u(t)$ as the author did in [63], based on estimates as $t \to +\infty$, which currently do not give any information about the behavior of $u(t)$ in the past time. The presence of the obstacle breaks down the space translation invariance. Hence, the solution behavior for positive and negative time direction might not be symmetric.

It would be interesting to study the solutions behavior for negative time. The solution $u(t)$ of the NLS_Ω equation can have at least the three known conceivable dynamics (scattering, blow-up, global existence) for the past time direction. Additionally, one might think that the solution can have a dynamics similar to the one described above in the numerical results. However, numerically, as the solution behavior acts as the sum of several solitons after a strong collision, one can see that the negative time behavior is close to the case of considering initial data as a multi-soliton, which travel with a different velocity vector in the direction toward the obstacle.

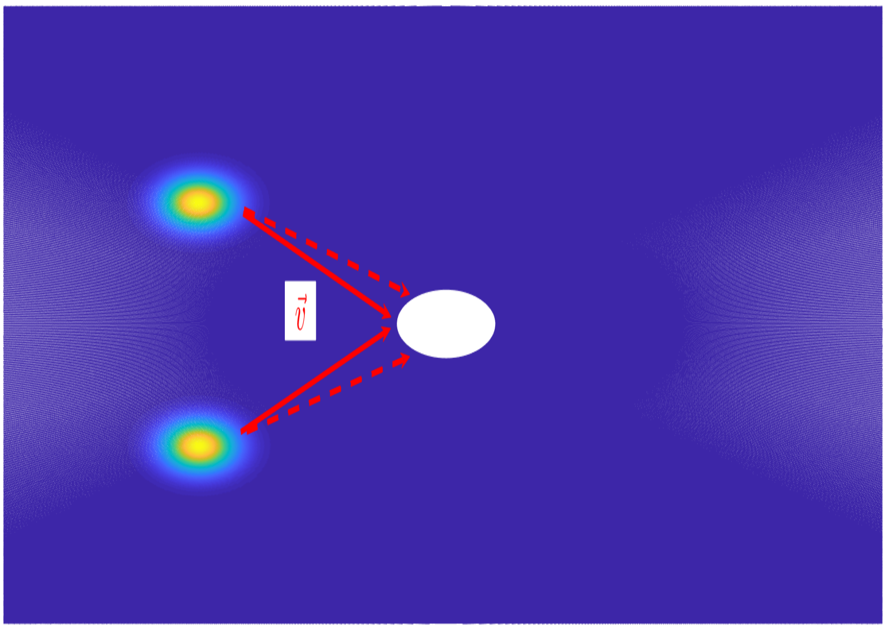


Figure V.32 – Multi-soliton behavior with various velocities v.

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